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G.W. Sands
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4

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Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

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THE OLYMPIAD CORNER
No. 102
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The two Olympiads which we present this issue were sent to us by Bruce Shawyer, Memorial University, St. John's, Newfoundland.

1987 BULGARIAN OLYMPIAD (final round)
Sofia, May 16–17, 1987

1. Let $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$, $n \geq 3$, be a polynomial with real coefficients such that $a_{n-1}/a_n > n + 1$. Prove that if $a_{n-2} = 0$ then at least one of the roots of $f(x)$ belongs to the open interval

$$\left(-\frac{1}{2}, \frac{1}{n+1}\right).$$

2. A polygon P is given. Two rotations ρ_i with centers O_i and angles ω_i, $0 < \omega_i < 2\pi$ $(i = 1,2)$, map P into itself. Prove that the number ω_1/ω_2 is rational.

3. The base of a pyramid $MABCD$ is a square $ABCD$, $MA = MD$, $MA^2 + AB^2 = MB^2$, and the area of the triangle ADM is equal to 1. Find the radius of the largest sphere that can be put into the given pyramid.

4. The sequence $x_1, x_2, \ldots$ is defined by the equalities $x_1 = x_2 = 1$ and

$$x_{n+2} = 14x_{n+1} - x_n - 4, \quad n \geq 1.$$

Prove that each member of the given sequence is a perfect square.

5. From a point E on the median AD of the triangle ABC, the perpendicular EF is dropped to the side BC. From a point M, lying on EF, two perpendiculars MN and MP are dropped to the sides AC and AB respectively. If the points N, E, and P are collinear show that the point M lies on the internal bisector of the angle BAC.

6. Let Δ be the set of all triangles inscribed in a given (fixed) circle for which the measures of the angles are integer degrees different from

45°, 90°, and 135°. For each $\tau \in \Delta$, we denote by $f(\tau)$ the triangle whose vertices are the points of intersection of the altitudes of τ with the given circle.

(a) Prove that for each τ there is an integer n such that among

$$\tau\ ,\ f(\tau)\ ,\ f^2(\tau) = f(f(\tau))\ ,\ \ldots\ ,\ f^n(\tau) = f(f^{n-1}(\tau))$$

there are two congruent triangles.

(b) Find the least value of n for which the condition (a) holds for all $\tau \in \Delta$.

*

SWEDISH MATHEMATICAL COMPETITION (final round)

November 22, 1986

1. Prove that the polynomial

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 3/4$$

has no real root.

2. The diagonals AC and BD of the quadrilateral $ABCD$ intersect at the interior point O. The areas of the triangles AOB and COD are s_1 and s_2, respectively, and the area of the quadrilateral is s. Prove that

$$\sqrt{s_1} + \sqrt{s_2} \leq \sqrt{s}.$$

Also prove that equality holds if and only if the lines AB and CD are parallel.

3. Let n be a positive integer greater than or equal to 3, and let S be the set of all pairs (a,b) of positive integers such that $1 \leq a < b \leq n$. Prove that the sets

$$\{(a,b) \in S : b < 2a\}$$

and

$$\{(a,b) \in S : b > 2a\}$$

have the same number of elements.

4. Show that the only positive solution of the system

$$x + y^2 + z^3 = 3$$
$$y + z^2 + x^3 = 3$$
$$z + x^2 + y^3 = 3$$

is $x = y = z = 1$.

5. In the rectangular array

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

of $m \times n$ real numbers, the difference between the maximum and the minimum element in each row is at most d, where $d > 0$. Each column is then rearranged in decreasing order so that the maximum element of the column occurs in the first row, and the minimum element occurs in the last row. Show that in the rearranged array the difference between the maximum and the minimum elements in each row is still at most d.

6. The union of a finite number of intervals cover the interval [0,1]. Show that one can choose among these intervals pairwise disjoint intervals of total length at least 1/2.

* * *

Please send in your nice solutions. I also remind the readers that as the Olympiad season is soon on us, it is the time to collect national and regional contests and send them in to me for use in this column.

* * *

Murray Klamkin writes with two comments on problems discussed in the December number.

Finland 2. [1987: 246; 1988: 299]

Does there exist a second degree polynomial $p(x,y)$ in two variables such that every non–negative integer n equals $p(k,m)$ for one and only one ordered pair (k,m) of non–negative integers?

Comment by Murray S. Klamkin, The University of Alberta, Edmonton.

One should note Léo Sauvé's comments to *Crux* 855 [1984: 300–303]. There he discusses the history of the problem of finding a polynomial $p_n(k_1,\dots,k_n)$ which is a bijection of $\mathbb{N}^n$ onto $\mathbb{N}$, the set of natural numbers. He singled out the solution of Leroy F. Meyers of The Ohio State University. Meyers first gave a polynomial bijection g_n of $\mathbb{N}_0^n$ onto $\mathbb{N}_0$, the set of all *nonnegative* integers. Explicitly

$$g_n(x_1,\dots,x_n) = \sum_{j=1}^{n} \binom{x_1+\cdots+x_j+j-1}{j},$$

which is of least degree n. Then $p_n = 1 + g_n$ will do.

France 1. [1987: 246; 1988: 300]

Let $t_1,t_2,\ldots,t_n$ be n real numbers satisfying $0 < t_1 \le t_2 \le \ldots \le t_n < 1$. Prove that

$$(1 - t_n)^2\left[\frac{t_1}{(1 - t_1^2)^2} + \frac{t_2^2}{(1 - t_2^3)^2} + \cdots + \frac{t_n^n}{(1 - t_n^{n+1})^2}\right] < 1.$$

Comment by Murray S. Klamkin, The University of Alberta, Edmonton.

One can obtain a better upper bound using the majorization inequality (see [1] or [2]).

The rth term A_r of the left–hand side satisfies

$$A_r = \frac{(1 - t_n)^2 t_r^r}{(1 - t_r^{r+1})^2} \le \frac{(1 - t_r)^2 t_r^r}{(1 - t_r^{r+1})^2} = \frac{t_r^r}{(1 + t_r + \cdots + t_r^r)^2}. \tag{1}$$

It follows from the majorization inequality that, since x^y is a convex function in y for x fixed, $0 < x < 1$, and the vector $(2r,2r-2,\ldots,2,0)$ majorizes the vector $(r,r,\ldots,r)$, we have

$$1 + x^2 + x^4 + \cdots + x^{2r} \ge (r + 1)x^r$$

or

$$\frac{x^r}{1 + x^2 + x^4 + \cdots + x^{2r}} \le \frac{1}{r + 1}.$$

Hence,

$$\frac{x^{2r}}{(1 + x^2 + \cdots + x^{2r})^2} \le \frac{1}{(r + 1)^2}.$$

Letting $t_r = x^2$ we obtain from (1)

$$A_r \le \frac{1}{(r + 1)^2}.$$

Finally,

$$\sum_{r=1}^{n} A_r \le \sum_{r=1}^{\infty} \frac{1}{(r + 1)^2} = \frac{\pi^2}{6} - 1 < 1.$$

References:

[1] D.S. Mitrinovic, *Analytic Inequalities*, Springer–Verlag, Heidelberg, 1970, p. 112.

[2] Solution of *Crux* 1216 [1988: 120].

* * *

We next continue with solutions to the (unused) problems from the 28th I.M.O. in Havana that were given in the October 1987 number of this column.

Iceland 1. [1987: 247]

Let S_1 and S_2 be two spheres with distinct radii which touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally as well as the two neighbouring solid spheres. What are the possible values of n?

Solution by George Evagelopoulos, law student, Athens, Greece.

Denote by r, R the radii, and A, B the centers of the spheres S_1, S_2, respectively, and suppose that $r < R$. Denote by s the common radius of the solid spheres and let C be the center of one of them. Let D be the foot of the perpendicular from C to the line through A and B. The centers of the solid spheres form a regular n-gon with sides $2s$ and center D; the radius of its circumscribed circle is $|CD|$. It follows that

$$\sin\left(\frac{\pi}{n}\right) = \frac{s}{|CD|}. \tag{1}$$

Using Heron's formula on the triangle ABC we obtain

$$|CD|(r + R) = 2\sqrt{(r + R + s)rRs}.$$

From this we obtain

$$\frac{s^2}{|CD|^2} = \frac{(r + R)^2 s}{4(r + R + s)rR}. \tag{2}$$

We may now choose the unit of length such that $r + R = 2$. Then (1) and (2) imply

$$\frac{1}{\sin^2(\pi/n)} = rR(1 + 2/s). \tag{3}$$

The line L through the vertex of the cone and the point of tangency of the cone and the solid sphere with center C meets the line AB at the vertex of the cone. Let ω denote the angle between these two lines. Then, as this line is also tangent to S_1 and S_2 we easily see that

$$R - r = (R + r)\sin\omega = 2\sin\omega.$$

This implies

$$r = 1 - \sin\omega,\ R = 1 + \sin\omega$$

and thus

$$rR = \cos^2\omega. \tag{4}$$

From (3) and (4) we get

$$\frac{1}{\sin^2(\pi/n)} = (1 + 2/s)\cos^2\omega. \tag{5}$$

Let E be the foot of the perpendicular from A to the perpendicular from B to L. Let E_1 be the foot of the perpendicular from C to the line through A and E. Let F be the foot of the perpendicular from C to the perpendicular from B to L. Then $|CF| = |E_1E|$. Now applying the theorem of Pythagoras to the triangles ACE_1 and BCF we obtain

$$|AE_1|^2 = (r + s)^2 - (r - s)^2 = 4rs,$$
$$|CF|^2 = (R + s)^2 - (R - s)^2 = 4Rs,$$

and so

$$|AE| = |AE_1| + |CF| = 2\sqrt{s}(\sqrt{R} + \sqrt{r}).$$

Thus

$$2\sqrt{s}(\sqrt{R} + \sqrt{r}) = (R + r)\cos\omega = 2\cos\omega$$

or, from (4),

$$s = \frac{\cos^2\omega}{2(1 + \sqrt{Rr})} = \frac{\cos^2\omega}{2(1 + \cos\omega)}$$

so that

$$\frac{2}{s} = \frac{4(1 + \cos\omega)}{\cos^2\omega}.$$

From (5) we get

$$\frac{1}{\sin^2(\pi/n)} = \cos^2\omega + 4(1 + \cos\omega)$$

and thus

$$2 + \cos\omega = \frac{1}{\sin(\pi/n)}.$$

Now as $0 < \cos\omega < 1$ we obtain

$$1/3 < \sin\pi/n < 1/2.$$

But

$$1/2 = \sin\pi/6 > \sin\pi/7 > \sin\pi/8 > \sin\pi/9 > 1/3 > \sin\pi/10.$$

Thus the possible values of n are 7, 8 and 9.

<u>Morocco 1</u>. [1987: 247]

Let $\theta_1,\theta_2,\ldots,\theta_n$ be real numbers such that

$$\sin\theta_1 + \sin\theta_2 + \cdots + \sin\theta_n = 0.$$

Prove that

$$|\sin\theta_1 + 2\sin\theta_2 + \cdots + n\sin\theta_n| \le [n^2/4],$$

where $[X]$ is the integer part of X.

Solutions by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario and by Murray S. Klamkin, The University of Alberta.

By letting $x_i = \sin\theta_i$, it clearly suffices to prove that if $x_1, x_2, \ldots, x_n$ are real numbers such that

$$|x_i| \le 1 \quad \text{for all} \quad i = 1,2,\ldots,n \quad \text{and} \quad \sum_{i=1}^{n} x_i = 0, \qquad (*)$$

then

$$\left|\sum_{i=1}^{n} ix_i\right| \le \left[\frac{n^2}{4}\right].$$

Since the condition (*) is unchanged if we replace x_i by $-x_i$, it suffices to show that

$$\sum_{i=1}^{n} ix_i \le \left[\frac{n^2}{4}\right].$$

In fact we prove a bit more.

Claim. Let $a_1, \ldots, a_n$ be a given increasing sequence of positive numbers and $x_1, \ldots, x_n$ be real numbers satisfying (*). Put

$$S = \sum_{i=1}^{n} a_i x_i.$$

Then

$$S \le \begin{cases} a_{m+1} + a_{m+2} + \cdots + a_{2m} - a_1 - a_2 - \cdots - a_m & \text{if } n = 2m, \\ a_{m+2} + a_{m+3} + \cdots + a_{2m+1} - a_1 - a_2 - \cdots - a_m & \text{if } n = 2m+1. \end{cases}$$

To establish the claim assume that $0 < a_1 \le a_2 \le \cdots \le a_n$ and (*) holds.

Case 1: $n = 2m$. Eliminating x_m we have

$$\begin{aligned} S = {} & (a_1 - a_m)x_1 + (a_2 - a_m)x_2 + \cdots + (a_{m-1} - a_m)x_{m-1} \\ & + (a_{m+1} - a_m)x_{m+1} + (a_{m+2} - a_m)x_{m+2} + \cdots + (a_{2m} - a_m)x_{2m}. \end{aligned}$$

Now this is clearly maximized with

$$x_1 = x_2 = \cdots = x_{m-1} = -1, \quad x_{m+1} = x_{m+2} = \cdots = x_{2m} = 1$$

giving

$$S \le a_{m+1} + a_{m+2} + \cdots + a_{2m} - a_1 - a_2 - \cdots - a_m.$$

Case 2: $n = 2m + 1$. Eliminating x_{m+1} we have

$$\begin{aligned} S = {} & (a_1 - a_{m+1})x_1 + (a_2 - a_{m+1})x_2 + \cdots + (a_m - a_{m+1})x_m \\ & + (a_{m+2} - a_{m+1})x_{m+2} + \cdots + (a_{2m+1} - a_{m+1})x_{2m+1}. \end{aligned}$$

It is again clear that this is maximized with

$$x_1 = x_2 = \cdots = x_m = -1, \quad x_{m+2} = \cdots = x_{2m+1} = 1$$

giving

$$S \le a_{m+2} + a_{m+3} + \cdots + a_{2m+1} - a_1 - a_2 - \cdots - a_m.$$

The problem at hand corresponds to taking $a_i = i$. In case $n = 2m$ we get

$$S \le m^2 = n^2/4 = [n^2/4],$$

while in case $n = 2m + 1$ we get

$$S \le (m + 1)m = [n^2/4].$$

Poland 1. [1987: 248]

Find the number of partitions of the set $\{1,2,\ldots,n\}$ into three subsets A_1, A_2, A_3, some of which may be empty, such that the following conditions are satisfied:

(i) after the elements of each subset have been put in ascending order, every two consecutive elements of any subset have different parity;

(ii) if A_1, A_2 and A_3 are all non–empty, then in exactly one of them the smallest number is even.

Remark. A partition is determined by a family of sets A_1, A_2, A_3 such that

$$A_1 \cup A_2 \cup A_3 = \{1,2,\ldots,n\} \text{ and } A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \phi;$$

another ordering of the sets, e.g. A_2, A_3, A_1, gives the *same* partition as A_1, A_2, A_3.

Solution by George Evagelopoulos, law student, Athens, Greece.

We are free to assign numbers to the sets A_1, A_2, A_3 of the partition in the following way: let A_1 be the set which contains 1 and let A_2 and A_3 be determined by the demand that the minimum element of $\{1,\ldots,n\}\backslash A_1$ belongs to A_2 (when the partition is not $\{\{1,\ldots,n\},\phi,\phi\}$).

To count the partitions we imagine constructing partitions satisfying (i) and (ii) by successively allocating the numbers to the subsets so as to obey the rules. The number 1 must be put in A_1. We shall show that for each subsequent number there are exactly two possibilities. As long as A_2 remains empty, the choice is to put the next number in A_1 or in A_2. Suppose now that j is the least number of A_2. We now argue by induction on k, $1 \le k \le n - j$, that at stage $j + k$ there are exactly two sets $A_{p(1)}$, $A_{p(2)}$, (where p is some permutation of $\{1,2,3\}$), into which $j + k$ can be placed, according to the parity of $j + k$. This is because the parity of the next possible element of each of A_1, A_2, A_3 is fixed. With $k = 1$ this follows since $j + 1$ may not be placed into A_1 (as it is the same parity as $j - 1$), while it can be placed in either A_2 or A_3 to satisfy (i) and (ii). If the statement holds for $k < n - j$ and $k + j$ can be added to either $A_{p(1)}$ or $A_{p(2)}$ but not to the third set $A_{p(3)}$, suppose $k + j$ is placed in $A_{p(1)}$ (without loss of generality). Now $k + j + 1$ cannot be placed in $A_{p(2)}$, but it can be added to $A_{p(1)}$ or $A_{p(3)}$ since the allowable parity for $A_{p(1)}$ has changed but for $A_{p(3)}$ has remained the same. This completes the induction.

Thus there are 2^{n-1} possible partitions of this kind.

Rumania 1. [1987: 248]

Show that the numbers 1,2,...,1987 can be coloured using 4 colours so that no arithmetical progression with 10 terms has all its members coloured the same.

Solution by George Evagelopoulos, law student, Athens, Greece, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo.

The number of 4–colourings of the set $M = \{1,2,...,1987\}$ is 4^{1987}. Let A be the number of arithmetical progressions in M with 10 terms. The number of colourings which contain a monochromatic 10–term arithmetical progression is less than $4A\cdot 4^{1987-10}$. If we can show $A\cdot 4^{1987-9} < 4^{1987}$, i.e. $A < 4^9$, then there must be a colouring of the desired type.

If the first term of an element of A is k and the difference is equal to d then $1 \le k \le 1978$ and $d \le [(1987 - k)/9]$. Hence

$$A = \sum_{k=1}^{1978} \left[\frac{1987 - k}{9} \right] < \frac{1986 + 1985 + \cdots + 9}{9}$$

$$= \frac{1995\cdot 989}{9} < \frac{2^{11}2^{10}}{2^3} = 4^9,$$

as required.

Spain 1. [1987: 248]

Determine, with justification, the integer solutions of the equation

$$3z^2 = 2x^3 + 385x^2 + 256x - 58195.$$

Partial solution by Gillian Nonay and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We have been unable to determine *all* the integer solutions of the given diophantine equation. However, using a computer, we have found the following twelve solutions in the range $x \le 2000$, $z \ge 0$:

(x,z) = $(-191,28)$, $(-157,742)$, $(-67,592)$, $(-49,454)$, $(-23,196)$, $(-19,144)$, $(19,182)$, $(23,242)$, $(61,784)$, $(103,1442)$, $(521,11364)$, $(817,21196)$.

We now show that if (x,z) is a solution, then

(a) x must be odd (and hence z must be even),

(b) $x \equiv 1$ or $5 \pmod 6$,

(c) $x \geq -191$.

To prove (a), suppose x were even. Then clearly z is odd. Substitution of $x = 2y$, $z = 2t + 1$ into the given equation yields

$$3(2t + 1)^2 = 16y^3 + 1540y^2 + 512y - 58195,$$

which simplifies to

$$6t^2 + 6t = 8y^3 + 770y^2 + 256y - 29099.$$

This is a contradiction since the left side is even, whereas the right side is odd.

Next, since $3 \nmid 58195$, $x \not\equiv 0 \bmod 3$, and thus (b) follows from (a).

Finally, suppose $x < 0$. Then $2x^3 + 385x^2 + 256x - 58195 = 3z^2 \geq 0$, thus $2x^3 + 385x^2 > 0$, so $2x + 385 > 0$ and $x \geq -192$. Thus $x \geq -191$, by (a).

Editor's note. Can any of our readers complete the solution?

<u>U.S.A. 1</u>. [1987: 248]

Let $r > 1$ be a real number, and let n be the largest integer less than r. Consider an arbitrary real number x with $0 \leq x \leq n/(r - 1)$. By a base r expansion of x, we mean a representation of x in the form

$$x = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots ,$$

where the a_i are integers with $0 \leq a_i < r$. You may assume, without proof, that every number x in the interval $0 \leq x \leq n/(r - 1)$ has at least one base r expansion.

Prove that, if r is not an integer, then there exists a number p as above which has infinitely many distinct base r expansions.

Solutions by George Evagelopoulos, law student, Athens, Greece, and (with a slight variation) by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo.

In the usual way represent

$$x = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots$$

by $.a_1a_2a_3\ldots$. If r is not an integer and n is the largest integer less than r, then

$$.nnn\ldots = n(r^{-1} + r^{-2} + \cdots) = \frac{n}{r - 1} > 1.$$

So there is a least positive integer k such that

$$\underbrace{.nn\ldots n}_{k \text{ digits}} = n(r^{-1} + \cdots + r^{-k}) > 1.$$

We claim that

$$p = \frac{1}{r^{k+1} - 1} = .\underbrace{00\ldots0}_{}1\underbrace{00\ldots0}_{k \text{ zeros}}1\ldots$$

has infinitely many representations.

First note that

$$p + 1 < .\underbrace{nn\ldots n}_{}1\underbrace{0\ldots0}_{k \text{ digits}}10\ldots01\ldots$$

$$< .nnn\ldots = \frac{n}{r - 1}\,.$$

We may thus assume that there are digits $a_1, a_2, \ldots$, each between 0 and n, such that

$$1 + p = .a_1a_2a_3\ldots$$

and so

$$\frac{1 + p}{r^m} = .\underbrace{0\ldots0}_{m \text{ zeros}}a_1a_2\ldots$$

for any integer $m > 0$. Also

$$1 + p = 1 + .0\ldots010\ldots010\ldots01\ldots$$

and thus

$$\frac{1 + p}{r^m} = .\underbrace{00\ldots0}_{m\text{-}1 \text{ zeros}}1\underbrace{00\ldots0}_{k \text{ zeros}}10\ldots01\ldots\ .$$

It follows that $p = .0\ldots010\ldots010\ldots010\ldots$ can be represented in any of the forms

$$.\underbrace{0\ldots0}_{k+1 \text{ zeros}}a_1a_2a_3\ldots\ ,$$

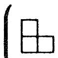
$$.\underbrace{0\ldots0}_{k}1\underbrace{0\ldots0}_{k+1}a_1a_2a_3\ldots\ ,$$

$$.\underbrace{0\ldots0}_{k}1\underbrace{0\ldots0}_{k}1\underbrace{0\ldots0}_{k+1}a_1a_2a_3\ldots\ ,$$

$$\vdots$$

Thus p has infinitely many distinct base r expansions.

<u>Vietnam 1</u>. [1987: 249]

Can a rectangular courtyard of dimension $m \times n$ be covered with tiles composed of 1×1 squares in the form of an L() if

(a) $m \times n = 1985 \times 1987$?

(b) $m \times n = 1987 \times 1989$?

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) The answer is no since neither 1985 nor 1987 is divisible by 3.

(b) Yes. Note that $1987 = 283 \cdot 7 + 6$ and $1989 = 221 \cdot 9$, so that the entire courtyard can be divided up into 6×9 and 7×9 lots. Figure 1 below shows that a 2×3 (and hence a 6×9) lot can be covered by L–trominos, while Figure 2 shows that a 7×9 lot can be covered by L–trominos.

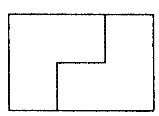


Figure 1

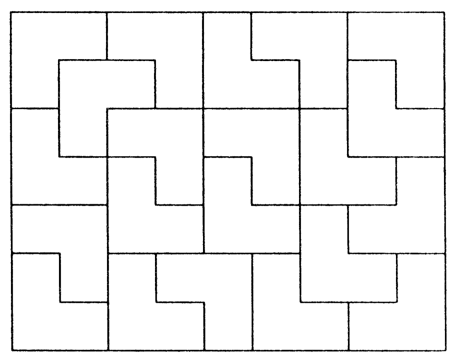


Figure 2

<u>West Germany 1</u>. [1987: 249]

How many words with n digits can be formed from the alphabet $\{0,1,2,3,4\}$, if neighbouring digits must differ by exactly one?

Solutions by George Evagelopoulos, law student, Athens, Greece; Murray S. Klamkin, The University of Alberta, Edmonton; and also by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo.

[*Editor's note.* The following solution is that of Shan and Wang, but the others were very similar.]

Let w_n denote the total number of such words. Let a_n, b_n, and c_n denote the number of those which end in 0, 1, and 2, respectively. Then by subtracting each digit of a word from 4 we see that the number of those words ending in 3 and 4 are b_n and a_n, respectively, and hence

$$w_n = 2a_n + 2b_n + c_n.$$

Furthermore, it is readily seen that the following equations hold:

$$a_{n+1} = b_n$$
$$b_{n+1} = a_n + c_n$$
$$c_{n+1} = 2b_n.$$

From these equations we easily deduce the recurrence relation

$$b_{n+1} = b_{n-1} + 2b_{n-1} = 3b_{n-1}$$

which, together with the obvious initial values $b_1 = 1$ and $b_2 = 2$, immediately yields

$$b_n = \begin{cases} 3^k & \text{if } n = 2k + 1 , \\ 2\cdot 3^{k-1} & \text{if } n = 2k . \end{cases}$$

Therefore, for all $n \geq 2$,

$$\begin{aligned} w_n &= 2a_n + 2b_n + c_n = 2b_{n-1} + 2b_n + 2b_{n-1} = 4b_{n-1} + 2b_n \\ &= \begin{cases} 4\cdot 2\cdot 3^{k-1} + 2\cdot 3^k = 14\cdot 3^{k-1} & \text{if } n = 2k + 1 , \\ 4\cdot 3^{k-1} + 2\cdot 2\cdot 3^{k-1} = 8\cdot 3^{k-1} & \text{if } n = 2k \end{cases} \\ &= \begin{cases} 14\cdot 3^{(n-3)/2} , & n \text{ odd} , \\ 8\cdot 3^{(n-2)/2} , & n \text{ even} . \end{cases} \end{aligned}$$

Finally, $w_1 = 5$, as is obvious.

Yugoslavia 1. [1987: 249]

Find the least number k such that for any $a \in [0,1]$ and any natural number n,

$$a^k(1 - a)^n < \frac{1}{(n + 1)^3}$$

is valid.

Solution by George Evagelopoulos, law student, Athens, Greece.

We show $k = 4$.

From the arithmetic–geometric mean inequality,

$$\sqrt[n+k]{a^k[(k/n)(1 - a)]^n} \leq \frac{k\cdot a + n[(k/n)(1 - a)]}{k + n} = \frac{k}{k + n} .$$

So we get

$$a^k(1-a)^n \le \frac{k^k n^n}{(n+k)^{n+k}}$$

where the equality holds only for $a = k(1-a)/n$, i.e.

$$a = \frac{k}{n+k}.$$

So we must find the least k such that for any n

$$\frac{k^k n^n}{(n+k)^{n+k}} < \frac{1}{(n+1)^3}$$

holds true.

For the pairs (k,n) equal to (1,1), (2,1) and (3,3) the inequality fails. Thus we have $k \ge 4$. We prove that $k = 4$ works, i.e.

$$4^4 \cdot n^n(n+1)^3 < (n+4)^{n+4}.$$

It is easy to check the inequality for $n = 1, 2, 3$, and so we assume $n \ge 4$. The desired inequality now follows from another application of the arithmetic–geometric mean inequality:

$$\begin{aligned}\sqrt[n+4]{4^4 n^n (n+1)^3} &= \sqrt[n+4]{16\cdot(2n)(2n)(2n)(2n)n^{n-4}(n+1)^3}\\ &\le \frac{16 + 8n + n(n-4) + 3(n+1)}{n+4}\\ &= \frac{n^2 + 7n + 19}{n+4} < \frac{n^2 + 8n + 16}{n+4} = n + 4.\end{aligned}$$

*   *   *

Next issue we will continue with solutions to problems posed at the 28th I.M.O. (Havana). Don't forget to send me your solutions and problem sets.

*   *   *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **September 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

1411. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ΔABC is acute angled with sides a, b, c and has circumcircle Γ with centre O. The inner bisector of $\angle A$ intersects Γ for the second time in A_1. D is the projection on AB of A_1. L and M are the midpoints of CA and AB respectively. Show that

(i) $AD = \frac{1}{2}(b + c)$;

(ii) $A_1D = OM + OL$.

1412. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find all positive integers n such that

$$(n - 36)(n - 144) - 4964$$

is the square of an integer.

1413. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $0 < x,y,z < 1$ let

$$u = z(1 - y), \qquad v = x(1 - z), \qquad w = y(1 - x).$$

Prove that

$$(1 - u - v - w)\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) \geq 3.$$

When does equality occur?

1414. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum value of the sum

$$\sqrt{\tan\frac{B}{2}\tan\frac{C}{2} + \lambda} + \sqrt{\tan\frac{C}{2}\tan\frac{A}{2} + \lambda} + \sqrt{\tan\frac{A}{2}\tan\frac{B}{2} + \lambda}$$

where A, B, C are the angles of a triangle and λ is a nonnegative constant. (The case $\lambda = 5$ is item 2.37 of O. Bottema et al, *Geometric Inequalities*.)

1415. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

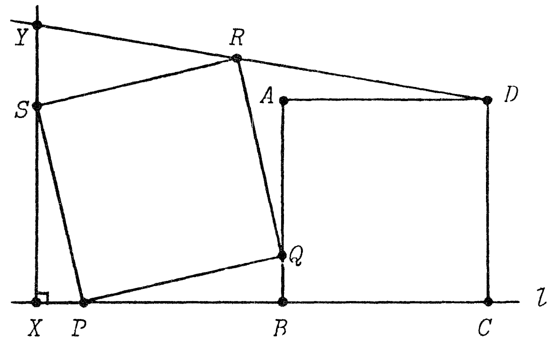
Given the system of differential equations

$$\begin{aligned}
\dot{x}_1 &= -(c_{12} + c_{13})x_1 + c_{12}x_2 + c_{13}x_3 \\
\dot{x}_2 &= c_{21}x_1 - (c_{21} + c_{23})x_2 + c_{23}x_3 \\
\dot{x}_3 &= c_{31}x_1 + c_{32}x_2 - (c_{31} + c_{32})x_3,
\end{aligned}$$

where the c's are positive constants, show that $\lim_{t\to\infty} x_i(t)$ is a weighted average, independent of i, of the initial values $x_1(0)$, $x_2(0)$, $x_3(0)$.

1416. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

In the figure, the unit square ABCD and the line l are fixed, and the unit square $PQRS$ rotates with P and Q lying on l and AB respectively. X is the foot of the perpendicular from S to l. Find the position of point Q so that the length XY is a maximum.



1417. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

O, A, B, C, D are five points in space, no four on the same plane, with $\angle AOB = \angle COD = 90^\circ$. Let p be the line through O intersecting AC and BD, and let q be the line through O intersecting AD and BC. Prove that $p \perp q$.

1418. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

Given that

$$\frac{\cos^3\theta}{\cos(\alpha - 3\theta)} = \frac{\sin^3\theta}{\sin(\alpha - 3\theta)} = m,$$

show that $\cos\alpha = 2m - m^{-1}$.

1419. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Points A, F_1, F_2, B, D_1, D_2, C, E_1, E_2 lie (in that order) on a circle, so that chords AD_1, AD_2; BE_1, BE_2; CF_1, CF_2 trisect angles A, B, C, respectively, of ΔABC. Let $D_1E_2 \cap D_2F_1 = P$, $E_1F_2 \cap E_2D_1 = Q$, $F_1D_2 \cap F_2E_1 = R$.

(a) Prove that ΔPQR is equilateral.

(b) Show that ΔPQR has side length

$$\frac{m_e - m}{3},$$

where m, m_e are the side lengths of the equilateral Morley triangles formed by the interior and exterior trisectors, respectively, of the angles of ΔABC.

1420. *Proposed by Shailesh Shirali, Rishi Valley School, India.*

If a, b, c are positive integers such that

$$0 < a^2 + b^2 - abc \leq c,$$

show that $a^2 + b^2 - abc$ is a perfect square. (This is a generalization of problem 6 of the 1988 I.M.O. [1988: 197].)

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

718. [1982: 49; 1983: 82] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is an acute-angled triangle with circumcenter O. The lines AO, BO, CO intersect BC, CA, AB in A_1, B_1, C_1, respectively. Show that

$$OA_1 + OB_1 + OC_1 \geq \frac{3R}{2},$$

where R is the circumradius.

IV. *Further generalization by M.S. Klamkin, University of Alberta.*

Let $A_0A_1 \cdots A_n$ be an n-simplex with interior point P, and for $i = 0,1,\ldots,n$ let the lines A_iP intersect the face opposite to A_i in the point B_i. Then if the barycentric representation of P is

$$\mathbf{P} = x_0\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n,$$

where $x_0 + x_1 + \cdots + x_n = 1$, $0 < x_i < 1$, we have [1987: 274–275] that

$$\mathbf{B}_i = \frac{\mathbf{P} - x_i\mathbf{A}_i}{1 - x_i}, \quad \overline{A_iB_i} = \frac{\overline{A_iP}}{1 - x_i},$$

and so

$$\overline{PB_i} = \frac{x_i}{1 - x_i}\,\overline{A_iP}.$$

Let $F(t)$ be an increasing convex function. Then since $t/(1 - t)$ is convex for $0 \leq t < 1$ we get

$$\sum_{i=0}^{n} F(\overline{PB_i}/\overline{A_iP}) = \sum_{i=0}^{n} F\left(\frac{x_i}{1 - x_i}\right) \geq (n + 1)F\left(\frac{1}{n + 1}\sum_{i=0}^{n}\frac{x_i}{1 - x_i}\right)$$

$$\geq (n + 1)F\left(\frac{1/(n + 1)}{1 - 1/(n + 1)}\right) = (n + 1)F(1/n). \tag{1}$$

In particular let $F(t) = t^m$, $m \geq 1$, to give

$$\sum_{i=0}^{n} (\overline{PB_i}/\overline{A_iP})^m \geq \frac{n + 1}{n^m}.$$

(This is also valid for $m \leq -1$ since $\left(\frac{t}{1 - t}\right)^m$ is convex in this range of m.) If P is the circumcenter O, we get

$$\sum_{i=0}^{n} (\overline{OB_i})^m \ge (n + 1)\left(\frac{R}{n}\right)^m.$$

If $F(t)$ is a decreasing concave function then inequality (1) is reversed.

Similarly, we have

$$\sum_{i=0}^{n} F(\overline{A_iB_i}/\overline{A_iP}) = \sum_{i=0}^{n} F\left(\frac{1}{1 - x_i}\right) \ge (n + 1)F\left(\frac{1}{n + 1}\sum_{i=0}^{n}\frac{1}{1 - x_i}\right)$$
$$\ge (n + 1)F\left(\frac{n + 1}{n}\right)$$

for F convex increasing,

$$\sum_{i=0}^{n} F(\overline{A_iP}/\overline{A_iB_i}) = \sum_{i=0}^{n} F(1 - x_i) \ge (n + 1)F\left(\frac{n}{n + 1}\right)$$

for F convex, and

$$\sum_{i=0}^{n} F(\overline{PB_i}/\overline{A_iB_i}) = \sum_{i=0}^{n} F(x_i) \ge (n + 1)F\left(\frac{1}{n + 1}\right)$$

for F convex.

* * *

1238. [1987: 119; 1988: 180] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Let $A = a^4$ where a is a positive integer. Find all positive integers x such that

$$A^{15x+1} \equiv A \bmod 6814407600,$$

or prove that there are none.

II. *Comment by David Singmaster, The Polytechnic of the South Bank, London.*

This is asking to show that

$$a^{60x+4} \equiv a^4 \bmod m \tag{1}$$

holds for all x and a when $m = 6814407600$. In my paper "A maximal generalization of Fermat's theorem", *Mathematics Magazine* 39 (1966) 103–107, I showed that

$$a^{r+s} \equiv a^s \bmod m$$

holds for all a if and only if $\lambda(m)|r$ and $N(m) \le s$. Here

$$N\left(\prod p_i^{n_i}\right) = \max\{n_i\},$$

and $\lambda(m)$ is Carmichael's λ–function, defined by

$$\lambda(2) = 1\ ,\ \lambda(4) = 2\ ,$$
$$\lambda(2^n) = 2^{n-2} \quad \text{for} \quad n > 2\ ,$$
$$\lambda(p^n) = \phi(p^n) = p^{n-1}(p-1) \quad \text{for prime } p > 2,$$

and

$$\lambda\left(\prod p_i^{n_i}\right) = \mathrm{lcm}\{\lambda(p_1^{n_1}),\lambda(p_2^{n_2}),\cdots\}$$

(ϕ being Euler's ϕ–function). This shows that (1) holds for all x and a if $\lambda(m)|60$ and $N(m) \leq 4$, which follow from the factorization of m given on [1988: 180].

We can invert the theorem as follows. Given r and s, then

$$a^{r+s} \equiv a^s \bmod m$$

holds if and only if $m|M$ where M is the largest integer such that $\lambda(M)|r$ and $N(M) \leq s$. It is straightforward to determine M from this characterization – it is the product of terms p^n, where p is a prime such that $p - 1|r$. We now determine the exponents n. Suppose $p^\ell \| r$ (that is, ℓ is the largest exponent such that $p^\ell|r$). For odd primes p, n is the smaller of s and $\ell + 1$. When $p = 2$, n is the smaller of s and $\ell + 2$, except for $\ell = 0$, when n is the smaller of s and 1. In the given case $r = 60$, $s = 4$, this yields $M = 6814407600$, a result proved on [1988: 180].

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<u>1260</u>. [1987: 181; 1988: 236] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokaisi, Aichi, Japan.*

Let ABC be a triangle with angles B and C acute, and let H be the foot of the perpendicular from A to BC. Let O_1 be the circle internally tangent to the circumcircle O of ΔABC and touching the segments AH and BH. Let O_3 be the circle similarly tangent to O, AH and CH. Finally let O_2 be the incircle of ΔABC, and denote the radii of O_1, O_2, O_3 by r_1, r_2, r_3, respectively.

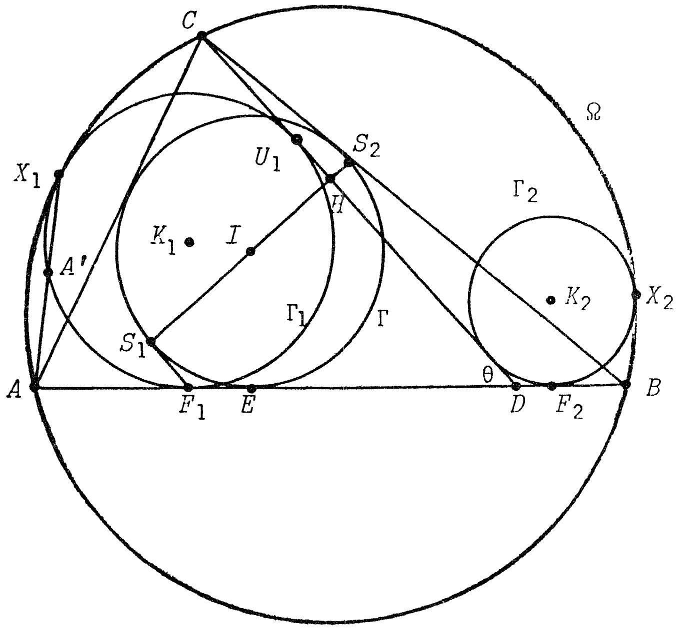
(a) Show that $r_2 = \dfrac{r_1 + r_3}{2}$.

(b) Show that the centers of O_1, O_2, O_3 are collinear.

Comment by G.R. Veldkamp, De Bilt, The Netherlands.

Regarding V. Thébault's 1938 generalization of this problem [1988: 237], Dr. H. Streefkerk gave three solutions in 1973 in the Dutch journal *Nieuw Tijdschrift voor Wiskunde* (vol. 60 pp.240–253). I gave a solution in *N.T.v.W.* 61 (1973) 86–89. A further solution was given by Mrs. B.C. Dijkstra–Kluyver and H. Streefkerk in *N.T.v.W.* 61 (1974) 172–173. All this was about ten years before the solution of K.B. Taylor. Below is a somewhat simplified version of my 1973 solution.

Theorem. D is an arbitrary point on the side c of a triangle ABC. Γ_1 (centre K_1, radius r_1) is the circle internally tangent to the circumcircle Ω at X_1 and touching the segments CD and AD. Γ_2 (K_2, r_2) is analogously defined. Γ (I, r) is the incircle of ΔABC. Then K_1, K_2 and I are collinear.



Proof. Ω is the image of Γ_1 under a dilatation $(X_1, R/r_1)$ where R is the circumradius. If A' is the second intersection of AX_1 and Γ_1 we have therefore $\overrightarrow{X_1A} = k\overrightarrow{X_1A'}$ with $k = R/r_1$. Hence

$$\overrightarrow{AA'} = \frac{k-1}{k}\cdot\overrightarrow{AX_1}.$$

The power of A with respect to Γ_1 is therefore

$$(\overrightarrow{AA'},\overrightarrow{AX_1}) = \frac{k-1}{k}\cdot\overline{AX_1}^{\,2}.$$

It follows that

$$\frac{k-1}{k}\cdot\overline{AX_1}^{\,2} = \overline{AF_1}^{\,2}.$$

Hence $\overline{AX_1} = m\overline{AF_1}$ where $m = k^{1/2}(k-1)^{-1/2}$. We have analogously

$$\overline{BX_1} = m\overline{BF_1}\ ,\quad \overline{CX_1} = m\overline{CU_1}.$$

We apply Ptolemeus' theorem in $ABCX_1$ and get

$$a\cdot\overline{AF_1} - b\cdot\overline{BF_1} + c\cdot\overline{CU_1} = 0. \tag{1}$$

Let us put $\overline{AD} = x$, $\overline{BD} = y$, $\overline{CD} = z$, $\overline{F_1D} = p$, $\overline{F_2D} = q$. Then

$$\overline{AF_1} = x - p,\quad \overline{BF_1} = y + p,\quad \overline{CU_1} = z - \overline{DU_1} = z - p.$$

Hence (1) leads to

$$a(x-p) - b(y+p) + c(z-p) = 0$$

or

$$2ps = ax - by + cz \tag{2}$$

where $2s$ is the perimeter of ΔABC. The line through the incentre I perpendicular to CD meets CD at H and the incircle Γ at S_1 (in ΔACD) and S_2 (in ΔBCD). Putting $\overline{HS_1} = w$ we have

$$2\cdot\text{area}(ACD) = xr + br + z(w-r),$$
$$2\cdot\text{area}(BCD) = yr + ar - z(w-r).$$

Hence

$$xr + br + z(w-r) : yr + ar - z(w-r) = x : y.$$

It follows that

$$y(xr + br + zw - zr) = x(yr + ar - zw + zr),$$
$$(x + y)zw = r(xz + ax - by + yz),$$

and so, since $x + y = c$,

$$czw = r(ax - by + cz).$$

Therefore, in view of (2),

$$cwz = 2prs = 2p\cdot\text{area}(ABC) = pcz\sin\theta.$$

We conclude $w = p\sin\theta$, i.e. $\overline{HS_1} = \overline{DF_1}\sin\theta$. This means $F_1S_1 \perp HS_1$, or $F_1S_1 \| CD$. Analogously, $F_2S_2 \| CD$. As a consequence

$$\angle F_1DK_1 = \frac{1}{2}\angle F_1DC = \frac{1}{2}\angle EF_2S_2 = \angle EF_2I,$$
$$\angle F_2DK_2 = \angle EF_1I,$$

and so

$$\Delta F_1DK_1 \sim \Delta EF_2I, \quad \Delta F_2DK_2 \sim \Delta EF_1I.$$

Putting $\overline{F_1E} = x_1$, $\overline{F_2E} = x_2$ we get

$$r_1{:}p = r{:}x_2 \quad \text{and} \quad r_2{:}q = r{:}x_1,$$

or

$$r_1x_2 = pr \quad \text{and} \quad r_2x_1 = qr. \tag{3}$$

Let the line EI meet K_1K_2 at K. Then in the trapezium $F_1F_2K_1K_2$ we obviously have

$$\overline{EK} = \frac{r_1x_2 + r_2x_1}{x_1 + x_2},$$

or, using (3) and $x_1 + x_2 = \overline{F_1F_2} = p + q$,

$$\overline{EK} = r.$$

Hence K coincides with I, and K_1, K_2 and I are collinear. □

We find furthermore

$$\frac{x_1}{x_2} = \frac{r_1q}{pr_2} = \tan^2(\theta/2).$$

Hence

$$\frac{x_1}{x_1 + x_2} = \sin^2(\theta/2), \quad \frac{x_2}{x_1 + x_2} = \cos^2(\theta/2)$$

and therefore

$$r = \frac{r_1x_2 + r_2x_1}{x_1 + x_2} = r_1\cos^2(\theta/2) + r_2\sin^2(\theta/2).$$

An additional solution has since been received from JORDI DOU, Barcelona, Spain.

* * *

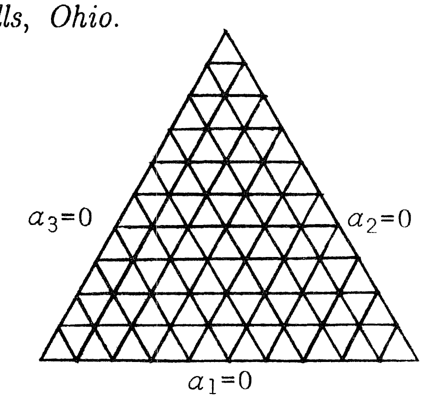
1299*. [1987: 321] *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

Three real numbers a_1, a_2, a_3 are chosen at random from the interval

[0,1] such that $\sum_{i=1}^{3} a_i = 1$. They are then rounded off to the nearest one-digit decimal to form $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$. What is the probability that $\sum_{i=1}^{3} \bar{a}_i = 1$?

I. *Combined solutions of Richard I. Hess, Rancho Palos Verdes, California, and Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

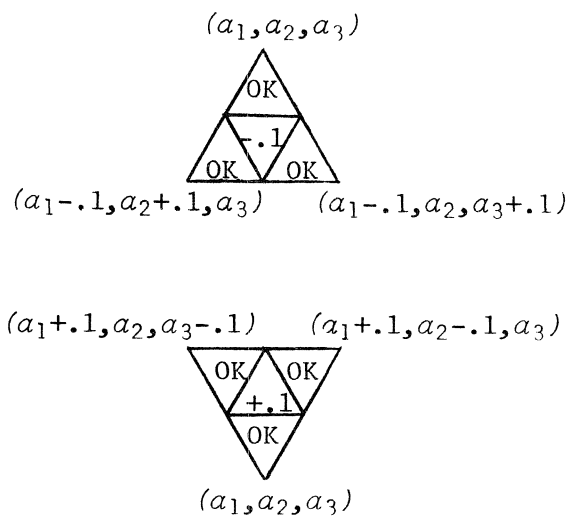
We show that the required probability is 3/4.



The figure is a representation of the portion of the plane $a_1 + a_2 + a_3 = 1$ enclosed by its intersections with the planes $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. Each point within the large triangle represents a triple (a_1,a_2,a_3) measured by the distances of the point from the three sides of the triangle. It is clear that the area of any region within the large triangle is proportional to the probability that a randomly chosen set (a_1,a_2,a_3) of positive real numbers adding to 1 lies within that region.

The lines parallel to the sides of the above triangle have been drawn at 0.1 intervals, so that each intersection point of these lines has coordinates of the form

$$(a_1,a_2,a_3) = (i/10,j/10,k/10),$$



where i, j, k are integers between 0 and 10. For such points $\Sigma\bar{a}_i = \Sigma a_i = 1$. If we further subdivide the above figure by parallel lines drawn at 0.05 intervals, each small triangle in the figure becomes split into four congruent subtriangles, as shown at the right. In the three corner subtriangles the coordinates of points differ by less than 0.05 from the corresponding coordinates of the corner point, and thus round to these corner coordinates, which add to 1. Furthermore, crossing into the interior subtriangle from any of the three corner subtriangles causes exactly one coordinate to differ from the corresponding corner coordinate by more than 0.05, and so the value of $\bar{a}_1 + \bar{a}_2 + \bar{a}_3$ for points in the interior subtriangle will increase or decrease by 0.1 depending upon the orientation of the triangle as shown. Thus the probability that $\bar{a}_1 + \bar{a}_2 + \bar{a}_3 = 1$ is 3/4.

Of the 100 interior subtriangles, 55 have $\Sigma\bar{a}_i = 0.9$ and 45 have $\Sigma\bar{a}_i = 1.1$. We are therefore led to the conclusion that the average value of $\bar{a}_1 + \bar{a}_2 + \bar{a}_3$ is less than one. This is an artifact of the finite size of the triangle.

II. *Editor's comments.*

It should have been made clearer what is meant by choosing a_1, a_2, a_3 "at random". The definition used in the above solution is that (a_1,a_2,a_3) is a point selected randomly and uniformly in the triangle $T \subset \mathbb{R}^3$ defined by

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x,y,z \le 1 \ , \ x + y + z = 1\}.$$

The answer to the problem then works out nicely to be 3/4. This was in fact the definition adopted by all solvers of this problem.

To realize this selection process (e.g., on a computer), G.P. HENDERSON, Campbellcroft, Ontario, first projects T onto the xy–plane, obtaining the isosceles right triangle T' with vertices (0,0), (1,0) and (0,1). He then selects two independent random variables x_1, x_2 uniformly in [0,1], and (if necessary) reflects the resulting point (x_1,x_2) of the unit square into T' by setting

$$(a_1,a_2) = \begin{cases} (x_1,x_2) & \text{if} \quad x_1 + x_2 \le 1 \ , \\ (1 - x_1, 1 - x_2) & \text{if} \quad x_1 + x_2 > 1 \ . \end{cases}$$

Of course, to generate such "random" points (a_1,a_2) of T' one could also simply ignore all choices of (x_1,x_2) lying outside T'.

Henderson then gave the editor a fright by introducing a *second* definition of randomness, moreover one which yielded a different answer to the problem! He merely selects three independent random variables x_1, x_2, x_3 uniformly in [0,1] and divides them by their sum to obtain the a_i's. With a lot of calculation he eventually obtains a probability of approximately 0.74772.

In the following remarks, the editor assumes that the *first* definition of randomness (or its natural extension) is to be used.

The problem can be generalized in at least three directions: (i) increase the number of round–off digits; (ii) increase the number of a_i's; (iii) do it for bases other than 10. PETER WINKLER, Emory University, has pointed out to the editor that the solution of the problem easily extends to handle (i), with the same answer of 3/4 no matter how many round–off digits are used. But now the following "paradox" arises. Let the number of round–off digits go to infinity; then the limit of the probabilities is of course 3/4, but the actual probability should be 1 since at the limit numbers aren't being rounded off at all! What is the explanation?

For (ii) and (iii) the editor has received no information. Extension (ii) seems to be difficult, even for the case of four real numbers a_1, a_2, a_3, a_4. Can anyone do this case, with one round–off digit? Some reader might like to approximate the probability by having a computer randomly (in the correct sense!) generate the a_i's. The editor would welcome all further results.

Also solved by RICHARD K. GUY, University of Calgary; G.P. HENDERSON, Campbellcroft, Ontario; P. PENNING, Delft, The Netherlands; and ROBERT E. SHAFER, Berkeley, California. One incorrect solution was received.

* * *

<u>1301</u>. [1988: 11] *Proposed by George Szekeres, University of New South Wales, Kensington, Australia.*

Given a positive rational number $q = a/b$ and an odd positive integer n, find a polynomial, with integer coefficients written in a simple closed form, that has $q^{1/n} + q^{-1/n}$ as a root. (See *Crux* 1187 [1988: 30].)

I. *Solution by P. Penning, Delft, The Netherlands.*

As on [1988: 31], write

$$q^{1/n} + q^{-1/n} = x$$

and solve for $q^{1/n}$; we get

$$q^{1/n} = \frac{x + \sqrt{x^2 - 4}}{2}$$

and so

$$q^{-1/n} = \frac{x - \sqrt{x^2 - 4}}{2} .$$

Thus

$$\begin{aligned} q + \frac{1}{q} &= [(x + \sqrt{x^2 - 4})^n + (x - \sqrt{x^2 - 4})^n]2^{-n} \\ &= 2^{1-n} \sum_{m=0}^{[n/2]} \binom{n}{2m} x^{n-2m}(x^2 - 4)^m \end{aligned}$$

which, using $q = a/b$, can be readily transformed into a polynomial with only integers as coefficients:

$$ab \sum_{m=0}^{[n/2]} \binom{n}{2m} x^{n-2m}(x^2 - 4)^m = 2^{n-1}(a^2 + b^2). \tag{1}$$

Working out the factor $(x^2 - 4)^m$ gives the polynomial of degree n that is asked for. One obtains

$$ab\sum_{t=0}^{[n/2]}(-1)^t x^{n-2t}\binom{n-t}{t}\frac{n}{n-t} = a^2 + b^2, \tag{2}$$

which is valid for any positive integer n.

The n roots of (2) are given by

$$x = (q^{1/n} + q^{-1/n})\cos\theta + i(q^{1/n} - q^{-1/n})\sin\theta,$$

where $\theta = 2\pi k/n$, $k = 0,1,\ldots,n-1$. The coefficients in (2) are related to the decomposition of 2 $\cosh(nx)$ in a power series of 2 $\cosh x$.

II. *Editor's comment.*

M.S. KLAMKIN, University of Alberta, notes that equation (1) above is just

$$2abT_n(x/2) = a^2 + b^2, \tag{3}$$

where $T_n(x)$ is the nth degree Chebyshev polynomial of the first kind. For instance (1) can be rewritten as

$$ab\,\mathcal{Re}\,(x + i\sqrt{4-x^2})^n = 2^{n-1}(a^2 + b^2),$$

or, putting $x = 2\cos\theta$,

$$2^n\cdot ab\,\mathcal{Re}\,(\cos\theta + i\sin\theta)^n = 2^{n-1}(a^2 + b^2),$$

$$2ab\,\mathcal{Re}\,(e^{in\theta}) = a^2 + b^2,$$

and so

$$2ab\cos n\theta = a^2 + b^2,$$

from which (3) follows. The coefficients in (2) are then well known and appear in many books (e.g. see p.365 of P.J. Davis, *Interpolation and Approximation*, Dover, 1975).

Also solved by the proposer. Four other readers gave polynomials whose coefficients were ***not*** *"written in a simple closed form".*

* * *

<u>1306</u>. [1988: 12] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Ellipses

$$\frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} = 1, \qquad i = 1,2,\ldots,n,$$

all satisfy the condition

$$\frac{1}{a_i^2} + \frac{1}{b_i^2} = 3.$$

Prove that the ellipses all pass through the same point.

Solution by Sam Maltby, student, Calgary.

Dividing each side of the given condition by three gives

$$\frac{1/3}{a_i^2} + \frac{1/3}{b_i^2} = 1.$$

This implies that whenever $x^2 = y^2 = 1/3$, the point (x,y) will be on the ellipses. Therefore, all ellipses satisfying the condition pass through the points

$$(\pm\sqrt{3}/3, \pm\sqrt{3}/3).$$

Also solved by SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; ROBERT E. SHAFER, Berkeley, California; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; W.R. UTZ, University of Missouri, Columbia; G.R. VELDKAMP, De Bilt, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

1307. [1988: 13] *Proposed by Jordi Dou, Barcelona, Spain.*

Let A', B', C' be the intersections of the bisectors of triangle ABC with the opposite sides, and let A'', B'', C'' be the midpoints of $B'C'$, $C'A'$, $A'B'$ respectively. Prove that AA'', BB'', CC'' are concurrent.

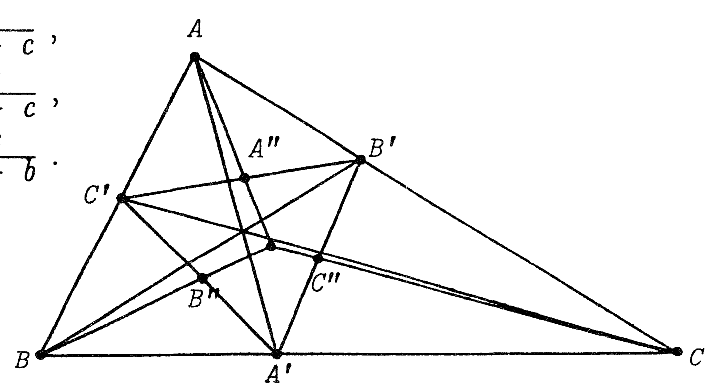
I. *Solution by C. Festraets–Hamoir, Brussels, Belgium.*

Letting a, b, c denote the lengths of the sides of ΔABC, we know that the interior bisectors determine on the sides of ΔABC segments of lengths

$$BA' = \frac{ac}{b+c}, \qquad CA' = \frac{ab}{b+c},$$

$$CB' = \frac{ab}{a+c}, \qquad AB' = \frac{bc}{a+c},$$

$$AC' = \frac{bc}{a+b}, \qquad BC' = \frac{ac}{a+b}.$$



In $\Delta AA''B'$,

$$\frac{A''B'}{AB'} = \frac{\sin \angle A''AB'}{\sin \angle AA''B'},$$

and in $\Delta AA''C'$,

$$\frac{A''C'}{AC'} = \frac{\sin \angle A''AC'}{\sin \angle AA''C'} .$$

Also

$$A''B' = A''C', \qquad \sin \angle AA''B' = \sin \angle AA''C' ,$$

from which we get

$$\frac{\sin \angle A''AB'}{\sin \angle A''AC'} = \frac{AC'}{AB'} = \frac{a+c}{a+b} .$$

Similarly,

$$\frac{\sin \angle B''BC'}{\sin \angle B''BA'} = \frac{b+a}{b+c} ,$$

$$\frac{\sin \angle C''CA'}{\sin \angle C''CB'} = \frac{c+b}{c+a} .$$

Thus we have

$$\frac{\sin \angle A''AB'}{\sin \angle A''AC'} \cdot \frac{\sin \angle B''BC'}{\sin \angle B''BA'} \cdot \frac{\sin \angle C''CA'}{\sin \angle C''CB'} = 1,$$

which is a necessary and sufficient condition (by Ceva's theorem) in order for the lines AA'', BB'', CC'' to be concurrent.

II. *Solution by Clark Kimberling, University of Evansville.*

Triangle $A''B''C''$ is inscribed in and perspective with triangle $A'B'C'$, which is inscribed in and perspective with triangle ABC. It follows by a known result (e.g. p.165 of N. Altshiller–Court, *College Geometry* or p.159–160 of R.A. Johnson, *Advanced Euclidean Geometry*) that triangle $A''B''C''$ is perspective with triangle ABC.

Also solved by HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; J.T. GROENMAN, Arnhem, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; JORDAN B. TABOV, Sofia, Bulgaria; GEORGE TSINTSIFAS, Thessaloniki, Greece; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer. Three other readers made minor errors in their solutions.

Klamkin, Tabov, and Tsintsifas also pointed out the more general result in II.

* * *

1308. [1988: 13] *Proposed by Seung–Jin Bang, Seoul, Korea.*

Find $f(x,y)$ and $g(x,y)$ such that

(i) $\displaystyle\frac{\partial f}{\partial x} = \frac{1}{g}\cdot\frac{\partial g}{\partial y}$, (ii) $\displaystyle\frac{\partial f}{\partial y} = -\frac{1}{g}\cdot\frac{\partial g}{\partial x}$,

and

(iii) $g(x,y)\sin f(x,y) = x$

all hold.

Solution by G.R. Veldkamp, De Bilt, The Netherlands.

It follows from (iii) that

$$g_x \sin f + g f_x \cos f = 1,$$
$$g_y \sin f + g f_y \cos f = 0,$$

or, using (i) and (ii),

$$g_x \sin f + g_y \cos f = 1,$$
$$-g_x \cos f + g_y \sin f = 0.$$

Hence

$$g_x = \sin f, \qquad g_y = \cos f, \tag{1}$$

and (i)–(iii) may be rewritten as

$$g f_x = \cos f, \qquad g f_y = -\sin f, \qquad g g_x = x.$$

Hence

$$g^2 = x^2 + 2u(y)$$

and therefore

$$g g_y = u'(y).$$

So we get in view of (1) that

$$x^2 + 2u(y) = g^2 = g^2(g_x^2 + g_y^2) = x^2 + (u'(y))^2.$$

Therefore

$$(u'(y))^2 = 2u(y),$$

whence we obtain

$$u(y) = (y + c)^2/2,$$

c an arbitrary constant. Thus

$$g^2 = x^2 + (y + c)^2$$

so that

$$g g_y = y + c.$$

Now (i) and (ii) provide us with

$$f_x = \frac{y + c}{x^2 + (y + c)^2}, \qquad f_y = \frac{-x}{x^2 + (y + c)^2}.$$

This leads to

$$f(x,y) = \arctan\left(\frac{x}{y + c}\right) + v(y)$$

and so

$$f_y = \frac{-x}{x^2 + (y + c)^2} + v'(y).$$

Hence $v'(y) = 0$, i.e. $v(y) = a$, a constant. It follows from (iii) that, identically with respect to y, we must have

$$g(0,y) \sin f(0,y) = 0,$$

or

$$\pm(y + c)\sin a = 0.$$

Thus $a = k\pi$, k an integer, and the functions answering the question are

$$g(x,y) = \pm\sqrt{x^2 + (y + c)^2},$$

$$f(x,y) = \arctan\left(\frac{x}{y + c}\right) + k\pi.$$

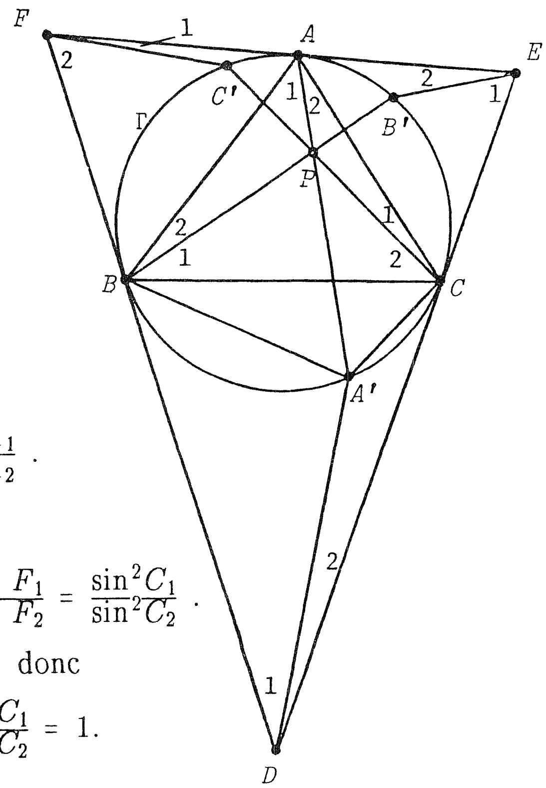
Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

* * *

1309. [1988: 13] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle with circumcircle Γ, and let DEF be the triangle formed by the lines tangent to Γ at A, B, C. Call a triangle $A'B'C'$ a *circumcevian triangle* if for some point P, A' is the point other than A where the line AP meets Γ, and similarly for B' and C'. Prove that DEF is perspective with every circumcevian triangle.

Solution par C. Festraets–Hamoir, Brussels, Belgium.



Dans le triangle DBA',

$$\frac{DA'}{BA'} = \frac{\sin \angle DBA'}{\sin D_1} = \frac{\sin A_1}{\sin D_1}.$$

Dans le triangle DCA',

$$\frac{DA'}{CA'} = \frac{\sin \angle DCA'}{\sin D_2} = \frac{\sin A_2}{\sin D_2}.$$

De plus,

$$\frac{BA'}{CA'} = \frac{\sin A_1}{\sin A_2},$$

d'où

$$\frac{\sin D_1}{\sin D_2} = \frac{\sin^2 A_1}{\sin^2 A_2}.$$

De manière analogue,

$$\frac{\sin E_1}{\sin E_2} = \frac{\sin^2 B_1}{\sin^2 B_2}, \qquad \frac{\sin F_1}{\sin F_2} = \frac{\sin^2 C_1}{\sin^2 C_2}.$$

AA', BB', CC' sont des céviennes concourantes, donc

$$\frac{\sin A_1}{\sin A_2}\cdot\frac{\sin B_1}{\sin B_2}\cdot\frac{\sin C_1}{\sin C_2} = 1.$$

De là

$$\frac{\sin D_1}{\sin D_2}\cdot\frac{\sin E_1}{\sin E_2}\cdot\frac{\sin F_1}{\sin F_2} = 1,$$

ce qui entraîne que DA', EB' et FC' sont concourantes et que, donc les triangles DEF et $A'B'C'$ sont perspectifs.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.

* * *

1310. [1988: 13] *Proposed by Robert E. Shafer, Berkeley, California.*

Let

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{N_n}{D_n}$$

where N_n and D_n are positive integers having no common divisor. Find all primes $p \geq 5$ such that $p | N_{p-4}$.

Solution by Colin Springer, student, University of Waterloo.

We show 11 is the only such prime.

For $p = 5$ and $p = 7$, p does not divide N_{p-4}, so we assume $p \geq 11$. Then

$$\frac{N_{p-4}}{D_{p-4}} = 1 + \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{p-4}\right) + \left(\frac{1}{5} + \frac{1}{p-5}\right) + \cdots + \left(\frac{1}{(p-1)/2} + \frac{1}{(p+1)/2}\right)$$

$$= \frac{11}{6} + p\left(\frac{1}{4(p-4)} + \frac{1}{5(p-5)} + \cdots + \frac{1}{(p^2-1)/4}\right).$$

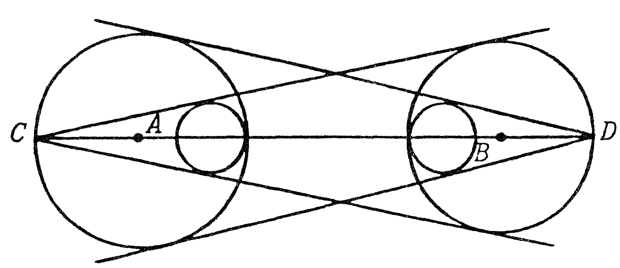
Clearly $p | N_{p-4}$ if and only if $p | 11$, i.e. if and only if $p = 11$.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; M.M. PARMENTER, Memorial University of Newfoundland; G.R. VELDKAMP, De Bilt, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

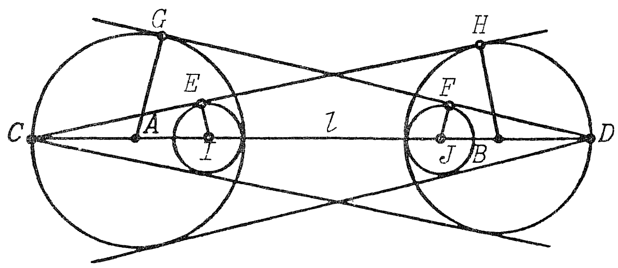
* * *

1311. [1988: 44] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

In the figure, A and B are the centers of the large circles, and the lines from C and D are tangents. Show that the small inscribed circles have equal radii.



I. *Solution by Eric Holleman, student, Memorial University of Newfoundland.*



Let R_a and R_b be the radii of the large circles centered at A and B, respectively. Let r_a and r_b be the radii of the small circles near A and B, respectively. Then since

$$\Delta CEI \sim \Delta CHB$$

we have

$$\frac{R_b}{r_a} = \frac{2R_a + R_b + \ell}{2R_a - r_a},$$

so

$$r_a = \frac{2R_aR_b}{2R_a + 2R_b + \ell}.$$

Since

$$\Delta DFJ \sim \Delta DGA,$$

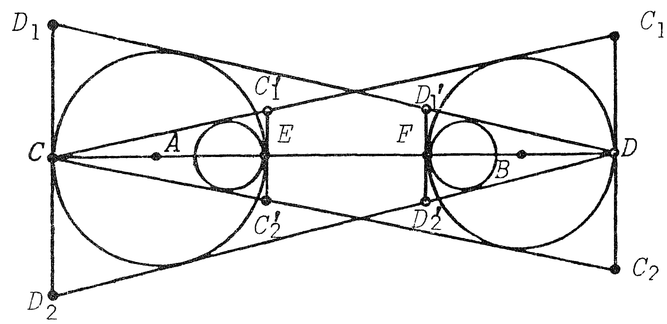
we similarly have

$$r_b = \frac{2R_aR_b}{2R_a + 2R_b + \ell},$$

thus the small inscribed circles have equal radii.

II. *Solution by Jordi Dou, Barcelona, Spain.*

[Dou's solution is so short and simple that it will be presented in the original Spanish! He first denotes the radii of the four circles by R_a, R_b, r_a, r_b as in solution I above. Then ...]



En los triangulos semejantes CC_1C_2, $CC_1'C_2'$ los inradios son proporcionales a las alturas:

$$R_b/r_a = CD/CE,$$

ó

$$r_a = \frac{CE \cdot R_b}{CD} = \frac{2R_aR_b}{CD}.$$

Analogamente en los triangulos DD_1D_2, $DD_1'D_2'$,

$$r_b = \frac{DF \cdot R_a}{CD} = \frac{2R_bR_a}{CD}.$$

Luego $r_a = r_b$.

Es un problema muy bello.

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; SIDNEY KRAVITZ, Dover, New Jersey; M.M PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

The solution by Festraets–Hamoir was similar to Dow's. The other solutions were much like solution I, *except that Janous also gave a generalization.*

The problem was quoted from a lost 1843 sangaku.

*　　　*　　　*

REMINDERS FROM THE EDITOR

Crux Mathematicorum depends in several ways on the contributions of its readers. Here are some of these ways. How can you help?

– *Crux* is now, and always, in need of interesting, varied, and accessible problem proposals. Problems on the topics of number theory, combinatorics, and high school mathematics are usually in short supply. Problems from students are especially encouraged!

– *Crux* would like to print more short articles. See [1988: 160] for some guidelines.

– *Crux* could use more subscriptions. Do you have a friend who would enjoy receiving *Crux*, but hasn't heard of it? Do you know of a school or university which doesn't subscribe to *Crux*, but ought to?

– What better clearing–house for the world's mathematics contests is there than the Olympiad Corner? If your country's (or region's) contests do not appear in *Crux*, perhaps you can do something about it.

– What do you think of *Crux*; its readability, the level of its problems, anything to do with its content? Write to the editor.

See you next month!

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