Crux Mathematicorum

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Editorial Board

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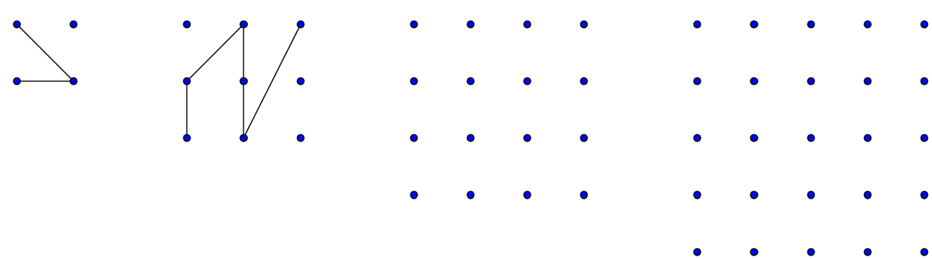
EDITORIAL

I have been involved with CMS Math Camps since 2004. That was the first year when Malgorzata Dubiel, Simon Fraser University CMS Math Camp coordinator, asked me if I wanted to help out and I was thrilled to be able to get involved. This was the beginning of my long-standing involvement with outreach. When I moved to the University of Victoria for my PhD, I discovered that the department there did not have regular outreach events. So I founded a CMS Math Camp there, making sure to involve graduate students in the running of the camp as I wanted to give them the same opportunity Malgorzata gave me years before. Being the main organizer, I saw all the presentations and helped develop many of them. The main lesson I took away in the process is that we should never stop playing mathematical puzzles, no matter what age or level of education, as they tend to either awaken or re-awaken our love for problem solving.

Every summer, with different presenters at my math camps, I discover different puzzles that allow my students (and me!) to discover mathematics in a non-standard way. This year, Shawn Desaulniers and I organized the CMS/PIMS/UBC Aboriginal Math Camp and he gave students the following activity which I thoroughly enjoyed doing myself. The puzzle is called Ariadne's String and it originally comes from Dr. Gordon Hamilton and his website `http://mathpickle.com`. (Note: this puzzle is not just for outreach. If you are a teacher, this activity fits well within school curriculum when you are covering the Pythagorean theorem). Suppose you have an $n \times n$ grid with vertices at lattice points. The rules of the game are as follows: draw a continuous zigzag line, where each line segment starts and ends at lattice points; line segments cannot touch (even at a vertex) and each subsequent line segment must be longer than the previous one. The goal: get as many line segments in as possible.

Here are some small maximal cases and some bigger ones for you to try!



Let me give you a little hint to make sure you're on the right track: you should be able to fit 9 line segments in the last grid. Have fun.

Kseniya Garaschuk

THE CONTEST CORNER

No. 28

Olga Zaitseva

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

CC136. Une toile d'araignée est de forme carrée 100×100, avec un noeud à chaque point d'intersection. L'araignée est assise à un des coins de la toile et 100 mouches se trouvent à certains noeuds de la toile, avec au plus une par noeud. Est-ce que l'araignée peut attraper toutes les mouches à l'aide d'au plus 2000 déplacements d'un noeud à un noeud voisin ?

CC137. Un empereur invite 2015 magiciens à un festival. Chaque magicien, mais pas l'empereur, sait exactement lesquels magiciens sont vertueux et lesquels sont vilains. Un magicien vertueux dit toujours la vérité, tandis qu'un magicien vilain peut dire la vérité ou mentir. Lors du festival, l'empereur remet à chaque magicien une carte comportant une seule question nécessitant une réponse oui-ou-non, ces questions pouvant être différentes pour différents magiciens ; l'empereur récolte toutes les réponses et expulse un seul magicien à travers une porte qui indique si le magicien expulsé était vertueux ou vilain. L'empereur prépare de nouvelles cartes et répète le processus avec les magiciens restants, aussi longtemps qu'il le désire. Démontrer que l'empereur peut expulser tous les magiciens vilains tout en expulsant au plus un magicien vertueux.

CC138. Démontrer que l'entier

$$\sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1} = 1^1 + 3^3 + 5^5 + \ldots + (2^n-1)^{2^n-1}$$

est un multiple de 2^n mais pas un multiple de 2^{n+1}.

CC139. Il est bien connu que si un quadrilatère possède un cercle inscrit et un cercle circonscrit avec le même centre, ce quadrilatère est obligatoirement un carré. Une affirmation similaire est-elle valide en 3 dimensions ? Notamment, si un cuboïde est inscrit dans une sphère et circonscrit autours d'une sphère et que les centres des sphères coïncident, ceci implique-t-il que le cuboïde est un cube ? (Un cuboïde est un polyèdre avec 6 faces quadrilatérales, où chaque sommet appartient à 3 arêtes.)

CC140. Soit $P(x)$ un polynôme à coefficients réels tel que l'équation $P(m) + P(n) = 0$ possède une infinité de couples de solutions entières (m, n). Démontrer que le graphique de $y = P(x)$ possède un centre de symétrie.

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CC136. A spiderweb is a square 100×100 grid with knots at each intersection. The spider sits at one corner of his spiderweb ; there are 100 flies caught in the web with at most one fly per knot. Can the spider get all the flies in no more than 2000 moves, if in one move it crawls to an adjacent knot ?

CC137. An Emperor invited 2015 wizards to a festival. Each wizard, but not the Emperor, knows which wizards are good and which ones are evil. A good wizard always tells the truth, while an evil wizard can either tell the truth or lie. At the festival, the Emperor gives every wizard a card with one "yes-or-no question (questions might be different for different wizards), learns all the answers and then expels one wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards and repeats the procedure with the remaining wizards until he wants to stop (with or without expelling a wizard). Prove that the Emperor can devise his questions so that all the evil wizards are expelled while expelling at most one good wizard.

CC138. Prove that the integer

$$\sum_{i=1}^{2^{n-1}} (2i-1)^{2i-1} = 1^1 + 3^3 + 5^5 + \ldots + (2^n - 1)^{2^n - 1}$$

is a multiple of 2^n but not a multiple of 2^{n+1}.

CC139. It is well-known that if in a quadrilateral the circumcircle and the incircle have the same centre, then the quadrilateral is a square. Is the similar statement true in 3 dimensions ? Namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of these spheres coincide, does it imply that the cuboid is a cube ? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

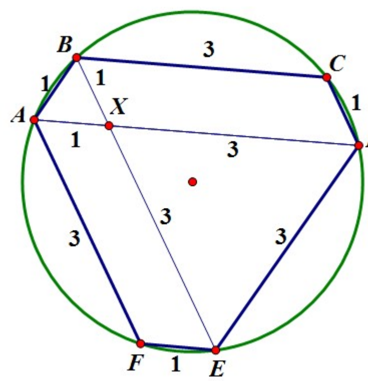
CC140. Let $P(x)$ be a polynomial with real coefficients so that the equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that the graph of $y = P(x)$ has a centre of symmetry.

CONTEST CORNER SOLUTIONS

CC86. A hexagon, H, is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Each side of length 1 is between two sides of length 3 and, similarly, each side of length 3 is between two sides of length 1. Find the area of H.

Originally 1998 W.J. Blundon Mathematics Contest, problem 10.

We received five correct submissions. We present the solution by John Heuver.



Consider a hexagon $H = ABCDEF$ with $AB = CD = EF = 1$ and $BC = DE = FA = 3$. Let the diagonals AD and BE intersect at X. Since $BE \| AF$ and $AD \| FE$, it follows that the quadrilateral $AFEX$ is a parallelogram with $AX = FE = 1$ and $AF = XE = 3$. Similarly, $BCDX$ is a parallelogram with $CD = BX = 1$ and $BC = XD = 3$.

Thus, both triangles ABX and DEX are equilateral with sides of length 1 and 3, respectively. Since their altitudes are correspondingly $\frac{\sqrt{2}}{2}$ and $\frac{3\sqrt{3}}{2}$, the area of the trapezoid $ABEF = \frac{1}{2} \cdot \frac{\sqrt{3}}{2}(4+3) = \frac{7}{4}\sqrt{3}$ and the area of the trapezoid $BCDE = \frac{1}{2} \cdot \frac{3\sqrt{3}}{2}(4+1) = \frac{15}{4}\sqrt{3}$.

This lets us conclude that the area of the hexagon H is $\frac{11}{2}\sqrt{3}$.

CC87. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is θ and the angle FEA is α, where F is the intersection of AC and BE. Find θ and $\cos\alpha$.

Originally 2004 W.J. Blundon Mathematics Contest, problem 10.

We received seven correct submissions. We present the solution by Matei Coiculescu, slightly modified by the editor.

Since the pentagon $ABCDE$ is regular, the internal angles all equal $\frac{3\cdot 180^\circ}{5} = 108^\circ$, and $1 = AB = EA$. Since $EA = AB$ the triangle ABE is isosceles, which implies that

$$\alpha = \angle FEA = \angle BEA = \frac{1}{2}(180^\circ - 108^\circ) = 36^\circ.$$

Similarly, triangle ABC is isosceles, so that $\angle BAC = \angle BAF = 36^\circ = \alpha$. Thus $\angle EFA = 2\alpha = 72^\circ$ (since it is the external angle of ΔABF at F). Since the triangles FAB and ABE are similar (having equal corresponding angles), we have $\frac{FB}{AB} = \frac{AE}{BE}$, or

$$FB = \frac{1}{\theta}.$$

Observe that $\angle FAE = \angle BAE - \angle BAF = 108^\circ - 36^\circ = 72^\circ = \angle AFE$. Thus, ΔEAF is isosceles, so that $1 = EA = EF$. Consequently, since $EB = EF + FB$,

$$\theta = 1 + \frac{1}{\theta}.$$

The positive solution of this equation is

$$\theta = \frac{1+\sqrt{5}}{2}.$$

Finally, the Law of Cosines applied to α in triangle ABE gives

$$\cos\alpha = \frac{1+\theta^2-1}{2\theta} = \frac{\theta}{2}.$$

In summary,

$$\theta = \frac{1+\sqrt{5}}{2} \text{ and } \cos\alpha = \frac{1+\sqrt{5}}{4}.$$

CC88. A cat is located at C, 60 metres directly west of a mouse located at M. The mouse is trying to escape by running at 7 m/s in a fixed direction. The cat, an expert in geometry, runs at 13 m/s in a suitable straight line path that will intercept the mouse as quickly as possible. Suppose that the mouse is intercepted after running a distance of d_1 metres in a particular direction. If the mouse had been intercepted after it had run a distance of d_2 metres in the opposite direction, show that $d_1 + d_2 \geq 14\sqrt{30}$.

Originally 2007 Canadian Open Mathematics Challenge, problem B4c).

We received two correct submissions. We present the solution by Titu Zvonaru and Neculai Stanciu.

Let A be the point where the cat catches the mouse after the mouse has run the distance d_1, and let B be the point where the cat catches the mouse after the mouse has run the distance d_2.

From M to A, the mouse runs for $\frac{d_1}{7}$ seconds, and from M to B, the mouse runs for $\frac{d_2}{7}$ seconds. It follows that the segment CA has length

$$|CA| = \frac{13d_1}{7} \tag{1}$$

and the segment CB has length

$$|CB| = \frac{13d_2}{7}. \tag{2}$$

By Stewart's theorem, we have that

$$|CA|^2 \cdot |BM| - |CM|^2 \cdot |AB| + |CB|^2 \cdot |AM| = |AM| \cdot |BM| \cdot |AB|. \tag{3}$$

Then by (1) and (2), and since $|CM| = 60$, $|AM| = d_1$, $|BM| = d_2$, and $|AB| = d_1 + d_2$, (3) becomes

$$\frac{169d_1^2 d_2}{49} - 3600(d_1 + d_2) + \frac{169d_1 d_2^2}{49} = d_1 \cdot d_2 \cdot (d_1 + d_2)$$

and we have

$$120d_1d_2(d_1 + d_2) = 49 \cdot 3600(d_1 + d_2).$$

Therefore,

$$d_1d_2 = 49 \cdot 30. \tag{4}$$

Now by the inequality of arithmetic and geometric means,

$$d_1 + d_2 \geq 2\sqrt{d_1d_2}$$

where $\sqrt{d_1d_2} = \sqrt{49 \cdot 30}$ by (4), so that

$$d_1 + d_2 \geq 14\sqrt{30}$$

as required.

CC89. Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function, and define $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ by $h(x, y) = \gcd(f(x), f(y))$. If $h(x, y)$ is a two-variable polynomial in x and y, prove that it must be constant.

Originally 2014 Sun Life Financial Repêchage Competition, problem 1.

No solutions to this problem were received.

CC90. For a given $k > 0$, $n \geq 2k > 0$, consider the square R in the plane consisting of all points (x, y) with $0 \leq x, y \leq n$. Color each point in R gray if $\frac{xy}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of n and k.

Originally question 9 from the 2001 Stanford Math Tournament, Calculus.

We present the solution by Digby Smith.

Suppose $0 \leq y \leq k$. Then, $\frac{xy}{k} \leq \frac{xk}{k} = x \leq x+y$. That is, the portion of the square R where $0 \leq y \leq k$ is coloured gray.

Similarly, suppose $0 \leq x \leq k$. Then, $\frac{xy}{k} \leq \frac{ky}{k} = y = x + y$. That is the portion of the square R where $0 \leq x \leq k$ is coloured gray.

The next step is to determine the area of the blue region, A_b, contained in the portion of the square R where $k < x$ and $y \leq n$. To start with, let

$$a = \frac{kn}{n-k} \quad \text{and} \quad n = \frac{kx}{x-k}.$$

Solving for x, we have

$$kx = n(x-k) \implies kn = x(n-k) \implies x = \frac{kn}{n-k},$$

so that $x = a$. Similarly, let $\frac{ky}{y-k} = n$. Then solving for y, we have that $y = a$. The curve $y = \frac{kx}{x-k}$ intersects the line $y = n$ at the point $P = (a, n)$ and intersects the line $x = n$ at the point $Q = (n, a)$. We now make use of some basic properties. First, we have

$$(n-k)(n-a) = n(n-2k), \tag{5}$$

since

$$\begin{aligned}(n-k)(n-a) &= (n-k)\left(n - \left(\frac{kn}{n-k}\right)\right) \\ &= (n-k)n\left(\frac{n-k-k}{n-k}\right) \\ &= n(n-2k).\end{aligned}$$

Second, we have

$$\ln(n-k) - \ln(a-k) = 2\ln\left(\frac{n-k}{k}\right), \tag{6}$$

since

$$\begin{aligned}\ln(n-k) - \ln(a-k) &= \ln(n-k) - \ln\left(\frac{kn}{n-k} - k\right) \\ &= \ln(n-k) - \ln\left(\frac{kn - kn + k^2}{n-k}\right) \\ &= \ln(n-k) - 2\ln(k) + \ln(n-k) \\ &= 2\ln\left(\frac{n-k}{k}\right).\end{aligned}$$

Furthermore, the following basic inequality holds :

$$k < a \leq n. \tag{7}$$

Proof. Starting with $2k \leq n$, it follows that $2kn \leq n^2$ and $kn \leq n^2 - kn$, so that $kn \leq n(n-k)$. Since $n > k$, it follows that

$$\frac{kn}{n-k} \leq n$$

with $a \leq n$. Next, starting with $k^2 > 0$, we have $kn - k^2 < kn$ and $k(n-k) < kn$. Since $n > k$, it follows that

$$k < \frac{kn}{n-k}$$

with $k < a$. That is, $k < a \leq n$. $\square$

Now suppose that $k < x,\ y \leq n$. Then if $x + y < \frac{xy}{k}$, we have

$$kx < xy - ky = (x-k)y$$

making

$$\frac{kx}{x-k} < y \ \text{ and also } \ \frac{ky}{y-k} < x.$$

Applying (7), it follows that the points P and Q are contained in the portion of the square R where $k < x,\ y \leq n$. Thus the portion of the square R coloured blue is given by

$$\frac{kx}{x-k} \leq y \leq n$$

with $a \leq x \leq n$. If $a = n$ (when $n = 2k$), then $A_b = 0$ with the area of the gray region, A_g, being $A_g = n^2$. Otherwise, if $a \neq n$, then

$$\begin{aligned}
A_b &= \int_a^n n - \frac{kx}{x-k} dx \\
&= \int_a^n n - k\left(1 + \frac{k}{x-k}\right) dx \\
&= \int_a^n (n-k) - k^2\left(\frac{1}{x-k}\right) dx \\
&= (n-k)(n-a) - k^2(\ln(n-k) - \ln(a-k)).
\end{aligned}$$

so that

$$A_b = n(n-2k) - 2k^2 \ln\left(\frac{n-k}{k}\right)$$

by (5) and (6). It follows that

$$A_g = n^2 - A_b = 2nk + 2k^2 \ln\left(\frac{n-k}{k}\right),$$

so that the area of the area of the gray region in terms of n and k is

$$2nk + 2k^2 \ln\left(\frac{n-k}{k}\right).$$

THE OLYMPIAD CORNER

No. 326

Nicolae Strungaru and Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC196. Le nombre 5654 en base b est une puissance d'un nombre premier. Déterminer b si $b > 6$.

OC197. Un cube $n \times n \times n$ est construit à partir de cubes $1 \times 1 \times 1$, dont certains sont noirs et les autres blancs, de façon à ce que tout prisme $n \times 1 \times 1$, $1 \times n \times 1$ ou $1 \times 1 \times n$ contient exactement deux cubes noirs, séparés par un nombre pair de cubes blancs (possiblement 0). Démontrer qu'il est possible de remplacer la moitié des cubes noirs par des cubes blancs de façon à ce que tout prisme $n \times 1 \times 1$, $1 \times n \times 1$ ou $1 \times 1 \times n$ contient exactement un cube noir.

OC198. Déterminer tout nombre réel positif M tel que pour tous nombres réels a, b et c au moins un de $a + \dfrac{M}{ab}, b + \dfrac{M}{bc}, c + \dfrac{M}{ca}$ est plus grand ou égal à $1 + M$.

OC199. Déterminer toutes les paires de polynômes f et g à coefficients réels tels que

$$x^2 \cdot g(x) = f(g(x)).$$

OC200. Soient A, B et C trois points sur une ligne (dans cet ordre). Pour tout cercle k passant par les points B et C, soit D un point d'intersection de la bissectrice perpendiculaire de BC avec le cercle k. De plus, soit E le deuxième point d'intersection de la ligne AD et k. Démontrer que pour tout cercle k le ratio des longueurs $BE : CE$ est le même.

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OC196. The number 5654 in base b is a power of a prime number. Find b if $b > 6$.

OC197. A $n \times n \times n$ cube is constructed using $1 \times 1 \times 1$ cubes, some of them black and others white, such that in each $n \times 1 \times 1$, $1 \times n \times 1$, and $1 \times 1 \times n$ subprism there are exactly two black cubes, and they are separated by an even number of white cubes (possibly 0). Show it is possible to replace half of the black cubes with white cubes such that each $n \times 1 \times 1$, $1 \times n \times 1$ and $1 \times 1 \times n$ subprism contains exactly one black cube.

OC198. Determine all positive real M such that for any positive reals a, b, c, at least one of $a + \frac{M}{ab}, b + \frac{M}{bc}, c + \frac{M}{ca}$ is greater than or equal to $1 + M$.

OC199. Determine all pairs of polynomials f and g with real coefficients such that

$$x^2 \cdot g(x) = f(g(x)).$$

OC200. Let A, B and C be three points on a line (in this order). For each circle k through the points B and C, let D be one point of intersection of the perpendicular bisector of BC with the circle k. Further, let E be the second point of intersection of the line AD with k. Show that for each circle k, the ratio of lengths $BE : CE$ is the same.

Math Quotes

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work.

Richard Feynman in Nobel Lecture, 1966.

OLYMPIAD SOLUTIONS

OC136. *$ABCD$ is a quadrilateral inscribed in a circle with centre O. If $AB = \sqrt{2+\sqrt{2}}$ and $\angle AOB = 135^\circ$, find the maximum possible area of $ABCD$.*

Originally from the India National Olympiad 2012 Problem 1.

We received three correct submissions and one incorrect solution. We present the solution by Oliver Geupel.

We prove that the maximum possible area of quadrilateral $ABCD$ is

$$\frac{1}{8}\sqrt{2}(5+3\sqrt{3}). \tag{1}$$

Let r be the radius of the circle (O). Inspecting the triangle ABO, we find that

$$r^2 = \frac{AB^2}{4\cos^2\angle BAO} = \frac{AB^2}{2(1+\cos(180^\circ - \angle AOB))} = \frac{2+\sqrt{2}}{2(1+\cos 45^\circ)} = 1,$$

so that $r = 1$.

For the moment suppose that O is an interior point of the quadrilateral $ABCD$.

Let $\alpha = \angle BOC$, $\beta = \angle COD$, $\gamma = \angle DOA$. Then, $\alpha, \beta, \gamma \in (0,\pi)$, $\alpha+\beta+\gamma = 5\pi/4$, and $2[ABCD] = \sin 135^\circ + \sin\alpha + \sin\beta + \sin\gamma$. Since the sine function is concave on the interval $[0,\pi]$, we obtain by Jensen's inequality that

$$\begin{aligned}
2[ABCD] &\le \sin 45^\circ + 3\left(\frac{\alpha+\beta+\gamma}{3}\right) \\
&= \sin 45^\circ + 3\sin 75^\circ \\
&= \sin 45^\circ + 3(\cos 45^\circ \sin 30^\circ + \sin 45^\circ \cos 30^\circ) \\
&= \frac{1}{2}\sqrt{2} + 3\left(\frac{1}{2}\sqrt{2}\cdot\frac{1}{2} + \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{3}\right) \\
&= \frac{1}{4}\sqrt{2}(5+3\sqrt{3}).
\end{aligned}$$

The equality holds if and only if $\alpha = \beta = \gamma = 5\pi/12$. Consequently, the maximum area of quadrilaterals $ABCD$ with interior point O is as in (1).

Next suppose that O is not an interior point of $ABCD$. Then,

$$[ABCD] < \frac{\pi}{2} < \frac{1}{8}\sqrt{2}(5+3\sqrt{3}).$$

This completes the proof.

OC137. We denote by $S(k)$ the sum of the digits in the decimal representation of k. Prove that there are infinitely many positive integers n for which

$$S(2^n + n) < S(2^n)\,.$$

Originally from the Poland Math Olympiad 2012 Day 2 Problem 3.

There was one correct submission and one incorrect submission. We present the solution by Oliver Geupel.

For any integer $m \geq 2$ consider the number

$$n = 10^m - 2 = \underbrace{9\ldots\ldots9}_{m-1 \text{ digits}}8.$$

Let us write n in the form $n = 4q + 2$ with an integer q. Then,

$$2^n = 2^{4q+2} = 4 \cdot 16^q \equiv 4 \cdot 6 \equiv 4 \pmod{10}.$$

Therefore, the sum of carry digits in the addition of the integers 2^n and n according to the standard school method of adding column-by-column, right-to-left is at least m. Note that every single carry unit causes the sum of digits of the result of the addition to decrease by 9. Hence, the decrease in the sum of digits is at least $9m$. Thus,

$$S(2^n + n) \leq S(2^n) + S(n) - 9m = S(2^n) + 9(m-1) + 8 - 9m = S(2^n) - 1.$$

Consequently, for every $m \geq 2$, the integer n has the required property.

OC138. Find all positive integers $a, b, c, p \geq 1$ such that p is a prime and

$$a^p + b^p = p^c.$$

Originally from the France TST 2012, Day 1, Problem 3.

We received 2 correct submissions. We present the solution by Oliver Geupel.

It is straightforward to verify that the following quadruples (a, b, c, p) are solutions : $(2^\alpha, 2^\alpha, 2\alpha + 1, 2)$, $(3^\alpha, 2 \cdot 3^\alpha, 3\alpha + 2, 3)$, and $(2 \cdot 3^\alpha, 3^\alpha, 3\alpha + 2, 3)$ where α is a nonnegative integer. We prove that there are no other solutions.

Suppose that (a, b, c, p) is a solution.

First consider the case $p = 2$. We have $a = 2^\alpha a_1$, $b = 2^\beta b_1$ with nonnegative integers α, β, and odd a_1, b_1. Then, $2^{2\alpha}a_1^2 + 2^{2\beta}b_1^2 = 2^c$; whence $\alpha = \beta$ and $a_1^2 + b_1^2 = 2^{c-2\alpha}$. Taking this modulo 4, we see that $c - 2\alpha = 1$, $a_1 = b_1 = 1$. This completes the case $p = 2$.

It remains to consider $p \geq 3$.

Let $v_p(k)$ denote the exact exponent of the prime p in the factorization of the integer k into primes.

Lemma 1 (Lifting The Exponent (LTE) Lemma) *Let p be an odd prime. For any two different integers x, y with $p \nmid x$ and $x \equiv y \pmod p$ and any positive integer n, it holds $v_p(x^n - y^n) = v_p(x - y) + v_p(n)$.*

For a proof see T. Andreescu, ed., *Mathematical Reflections : The First Two Years*, XYZ Press, 2011, p.535.

Lemma 2 *The sequence* $\left(n^{1/(n-1)}\right)_{n=3,4,5,\dots}$ *is decreasing.*

Proof. By the geometric - arithmetic mean inequality,

$$\left(\frac{n+1}{n}\right)^{n-1}\cdot\frac{1}{n}<\left(\frac{(n-1)\cdot\frac{n+1}{n}+\frac{1}{n}}{n}\right)^{n}=1.$$

Hence, $(n+1)^{n-1}<n^n$ and $(n+1)^{1/n}<n^{1/(n-1)}$. □

Returning to our problem, note that we have $a=p^{\alpha}a_1$ and $b=p^{\beta}b_1$ with non-negative integers α, β, and positive integers a_1 and b_1 which are not divisible by p. Thus, $\alpha=\beta$ and $a_1^p+b_1^p=p^{c-p\alpha}$. Hence, $a_1+b_1\equiv a_1^p+b_1^p\equiv 0 \pmod p$. By LTE, $v_p(a_1^p+b_1^p)=v_p(a_1+b_1)+1$. Therefore, $a_1^p+b_1^p=(a_1+b_1)p$. By the general means inequality, we have

$$(a_1+b_1)\cdot\left(\frac{a_1+b_1}{2}\right)^{p-1}=2\cdot\left(\frac{a_1+b_1}{2}\right)^{p}<a_1^p+b_1^p=(a_1+b_1)p.$$

Thus,

$$\frac{a_1+b_1}{2}<p^{1/(p-1)}.$$

In the case $p=3$, we obtain $a_1+b_1\le 3$, i.e. $(a_1,b_1)\in\{(1,2),\ (2,1)\}$.

In the case $p\ge 5$, Lemma 2 implies $a_1+b_1<2\sqrt[4]{5}$. Consequently, $a_1+b_1\le 2$, $a_1=b_1=1$, i.e. $a=b$, which is impossible.

OC139. The numbers $1,2,\dots,50$ are written on a blackboard. Each minute any two numbers are erased and their positive difference is written instead. At the end one number remains. Find all the values this number can take.

Originally from the Kyrgyzstan National Olympiad 2012, Problem 6.

We received three correct submissions. We present the solution by Oliver Geupel.

We show that the possible values of the last number are $1,3,5,\dots,47,49$.

Consider the general problem with numbers $1,2,\dots,n$ initially on the blackboard and let L_n denote the range of the values of the last number. The total sum of the numbers on the blackboard is decreased by the even number $a+b-|a-b|=2\min\{a,b\}$ when erasing a and b and writing their positive difference. Hence each subsequent sum has the same parity as the initial sum $1+2+\cdots+n=n(n+1)/2$. We prove by mathematical induction on $n\ge 1$ that

$$L_n \;=\; \left\{k:\quad 0\le k\le n,\quad k\equiv\frac{n(n+1)}{2}\pmod 2\right\}. \tag{1}$$

The base cases $n = 1$ and $n = 2$ are straightforward. For the induction step, assume that $n \geq 3$ is such that for every $m < n$ it holds

$$L_m = \left\{k : \quad 0 \leq k \leq m, \quad k \equiv \frac{m(m+1)}{2} \pmod 2\right\}.$$

We are to prove (1) for this number n. Let r denote the remainder of $n \pmod 4$. Consider the values $r = 0, 1, 2, 3$ in succession.

Case $r = 0$. By induction, $L_{n-1} = \{0, 2, 4, \ldots, n-2\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$ because we can start reducing the set $\{1, 2, \ldots, n-1\}$ up to the one remaining number k and finally process the numbers k and n. Hence $2, 4, 6, \ldots, n \in L_n$. By induction, $1 \in L_{n-2}$. Note that for every $k \in L_{n-2}$, we have $k - (n - (n-1)) \in L_n$. Thus, $0 \in L_n$.

Case $r = 1$. By induction, $L_{n-1} = \{1, 3, 5, \ldots, n-2\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Hence, $2, 4, 6, \ldots, n-1 \in L_n$. By induction, $1 \in L_{n-2}$. For every $k \in L_{n-2}$, we have $k - (n - (n-1)) \in L_n$. Thus, $0 \in L_n$.

Case $r = 2$. By induction, $L_{n-1} = \{1, 3, 5, \ldots, n-1\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Therefore, $1, 3, 5, \ldots, n-1 \in L_n$.

Case $r = 3$. By induction we have $L_{n-1} = \{0, 2, 4, \ldots, n-1\}$. For every $k \in L_{n-1}$, we have $n - k \in L_n$. Hence, $1, 3, 5, \ldots, n \in L_n$.

This completes the proof by induction.

OC140. Let ABC be an obtuse triangle with $\angle A > 90°$. Let circle O be the circumcircle of ABC. D is a point on the segment AB such that $AD = AC$. Let AK be the diameter of circle O, and let L be the point of intersection of AK and CD. A circle passing through D, K, L intersects the circle O at $P \neq K$. Given that $AK = 2, \angle BCD = \angle BAP = 10°$, prove that

$$DP = \sin\left(\frac{\angle A}{2}\right).$$

Originally from the Korea Mathematical Olympiad Day 1 Problem 1.

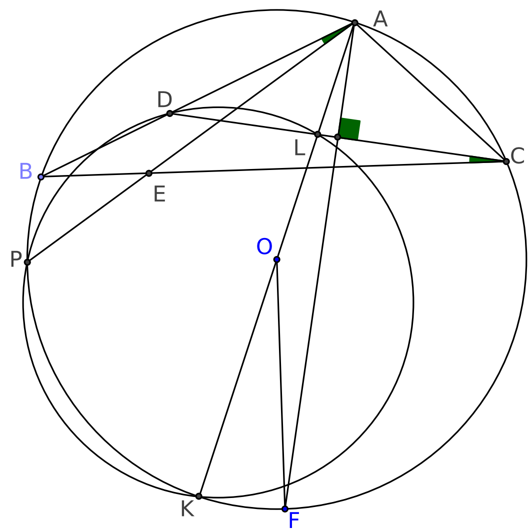
We present the solution by Oliver Geupel.

Let lines AP and BC intersect at point E. Let point F be the second intersection of the internal bisector of $\angle A$ with Γ. Let O be the centre of Γ.

We have

$$\angle C = \angle ACD + \angle DCB = \left(90° - \frac{\angle A}{2}\right) + 10° = 100° - \frac{\angle A}{2},$$

$$\angle B = 180° - \angle A - \angle C = 80° - \frac{\angle A}{2}.$$



Because $DC \perp AF$ and $CE \perp FO$, it holds

$$\angle KPF = \angle KAF = \angle AFO = \angle DCB = 10^\circ,$$

$$\angle DLK = \angle CLA = 90^\circ - \angle KAF = 80^\circ.$$

Since the quadrilateral $DPKL$ is cyclic, we obtain

$$\angle KPD = 180^\circ - \angle DLK = 100^\circ.$$

Hence,

$$\angle FPD = \angle KPD - \angle KPF = 100^\circ - 10^\circ = 90^\circ. \tag{1}$$

We have

$$\angle BCP = \angle BAP = 10^\circ,$$

$$\angle AFP = \angle ACP = \angle C + \angle BCP = \left(100^\circ - \frac{\angle A}{2}\right) + 10^\circ = 110^\circ - \frac{\angle A}{2},$$

$$\angle AFD = \angle CFA = \angle B = 80^\circ - \frac{\angle A}{2},$$

$$\angle DFP = \angle AFP - \angle AFD = 30^\circ. \tag{2}$$

From (1) and (2), we deduce that

$$DP = \frac{DF}{2} = \frac{CF}{2} = FO \cdot \sin\frac{\angle FOC}{2} = \frac{KA}{2} \cdot \sin\angle FAC = \sin\left(\frac{\angle A}{2}\right).$$

This completes the proof.

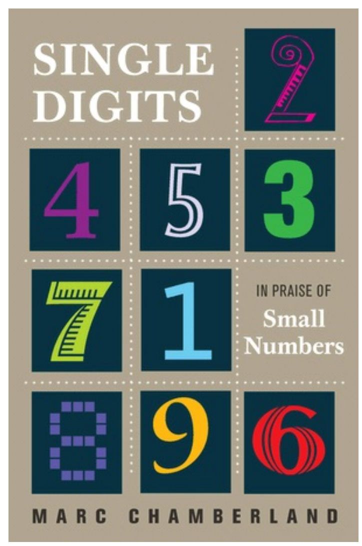
BOOK REVIEWS

Robert Bilinski

Single digits : in praise of small numbers by Marc Chamberland
ISBN 978-0-691-16114-3, 224 pages.
Published by Princeton University Press, 2015.

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

The author is a mathematics teacher at Grinnell College, Iowa. He specializes in experimental mathematics, the "new way" to do mathematics spearheaded by the Borwein brothers. His published research pertains to known problems rife with conjectures like "the $3x + 1$ problem", "the jacobian conjecture" and "the Ducci sequences". The book contains many results, some well-known and quite a few off the beaten path normally trodden in general interest math books. The originality of this book is the quantity of "obscure" lore (at least for me).



I have recently reviewed a few books which offer "general interest" mathematics. Even though the books might seem similar, I want to stress here that they differ in their aims and their targeted audiences. Marc Chamberland's book is definitely meant for a mathematically mature audience, that ideally has at least a few bachelor level analysis type courses, some knowledge of number theory, knows how to structure a proof, etc. That said, my most talented or math-interested high schoolers would still be able to follow, if they know what higher math is about or if they have a mentor to help them along.

The purported aim of the book is to show the wealth of mathematical lore related to our most seen and probably too overlooked digits. Yes! The book talks about the simple digits, the $1, 2, \ldots, 9$ we use to write numbers with. But here, they are the number of dimensions, the starting points of solution sets or sequences, the remainders of modular classes, the number of candidates in an election or the number of circles in a theorem, amongst many other properties. Each is like a candy in a sweets shop. This has probably been one of my slowest reads in a while, not because the book is badly written or hard to read, but because I wanted to take out a piece of paper and doodle a bit of math with a pen or look something up on the internet. To be clear, I thoroughly enjoyed this book. I learned a lot of facts and trivia, special sequences, new theorems and even a new mathematical tidbit on our Canadian loonie!

The book's structure makes it a bit intense to read since the topics covered for each digit keep changing as long as they are relevant to the digit itself. As an example, for the digit 2, the second chapter starts with geometry, computer science and chess (Jordan curves and parity arguments), then we consider aspect ratios (screens and paper), symmetry, Pythagoras and a few connected results, Beatty's sequence, Euler's formula, Golbach's conjecture, Squeezing primes, the twin prime conjecture, The ham sandwich theorem, power sets, the Hanoï towers, The Sylvester-Gallai theorem, The formulas for Pi, The Thue-Morse sequence, duality and graphs, circle packing, music and tuning, number theory, the AGM, positive polynomials, Newton's method, chaos, sequences, series, map representations and analysis. Naturally, the subjects are treated as flash cards more than as textbook material. On this point, one has to note that the author wrote it very well so that one doesn't feel overwhelmed by the rhythm. If you're like me, you haven't heard about a few of these subjects before. Probably, if I hadn't taken my exploratory breaks, I would have had to take breathers or I wouldn't have been able to remember everything. After all, there are 8 other chapters in the book!

With Chamberland's interest in experimental mathematics, one might expect to find a lot of "applied" mathematics used in computers, programming or symbolic calculators, but far from it! The book is rife with "pure" mathematics and deep questions, such as E8. It's like a compendium of mathematical gossip, like something that happens when two mathematicians from different fields meet and recount their adventures.

Happy reading!

Chebyshev polynomials and recursive relations (I)

N. Vasiliev and A. Zelevinskiy

People often imagine a mathematician to be a person who spends all of their time doing various arithmetic calculations or manipulating long and complicated formulas. ***Crux*** readers know very well that there is beautiful and important "formula-free" math, but this alternative view is not entirely false either. A large part of a mathematician's job consists of looking at formulas from an unexpected point of view, finding new formulas and explore connections between various formulas. In this two-part article, we will consider a cascade of curious formulas related to famous Chebyshev polynomials and also the general mathematical ideas hiding behind them.

1 Two curious sequences of polynomials

The polynomials that we shall consider play an important role within many problems in analysis, enumeration and algebra. They first appeared in 1854 in the works of a Russian mathematician Pafnutyi L'vovich Chebyshev in the following context. Consider all possible polynomials of degree n with leading coefficient equal to 1. Which polynomial is *closest to 0* on the interval $[-1,1]$; that is, for which polynomial $F_n(x) = x^n + \cdots$ is the value $c_n = \max_{[-1,1]} |F_n(x)|$ the smallest?

It works out to be $\hat{T}_n = \frac{1}{2^{n-1}} T_n(x)$, which is the polynomial from the first column of Table 1, divided by its leading coefficient. Among quadratics, it is $x^2 - \frac{1}{2}$ whose closeness to 0, denoted by c_2, is $\frac{1}{2}$; among cubics, it is $x^3 - \frac{3}{4}x$ for which $c_3 = \frac{1}{4}$. In general, closeness to 0 of the polynomial $\hat{T}_n$ is $c_n = \frac{1}{2^{n-1}}$ and it is smaller than for any other polynomial $F_n(x) = x^n + \cdots$ on the interval $[-1,1]$ (the proof is left as an exercise for the reader).

n	T_n	U_n
0	1	1
1	x	$2x$
2	$2x^2 - 1$	$4x^2 - 1$
3	$4x^3 - 3x$	$8x^3 - 4x$
4	$8x^4 - 8x^2 + 1$	$16x^4 - 12x^2 + 1$
5	$16x^5 - 20x^3 + 5x$	$32x^5 - 32x^3 + 6x$
6	$32x^6 - 8x^4 + 18x^2 - 1$	$64x^5 - 80x^4 + 24x^2 - 1$
	$\cdots$	$\cdots$

Table 1: Chebyshev polynomials of the first and second type. Multiply any one of them by $2x$ and subtract the previous one to get the next one.

What if we decide to measure closeness to 0 differently ? For example, we can replace c_n by $l_n = \int_{-1}^{1} |F_n(x)|dx$. In this case, the polynomial of degree n with leading coefficient 1 that is closest to 0 is $\hat{U}_n = \frac{1}{2^n}U_n(x)$, where $U_n(x)$ is the polynomial from the second column of Table 1. For the polynomial $U_n(x)$, the value of l_n (equal to the grey shaded area in Figure 1) is equal to 2 ; therefore, for $\hat{U}_n$ this area is equal to $\frac{1}{2^{n-1}}$. For any other polynomial $F_n(x) = x^n + \cdots$ this area is bigger (this is a theorem of Korkin and Zolotarev).

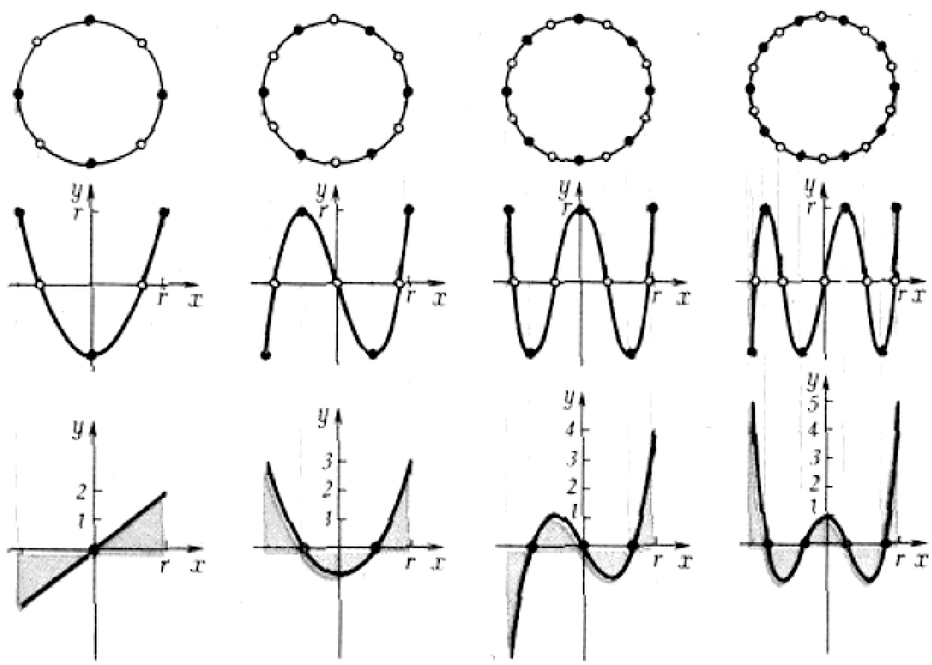


FIGURE 1: Take a transparent piece of paper with the graph of $y = r\cos nx$ $(0 \le x \le 2\pi r,\ -r \le y \le r)$ drawn on it and roll it into a cylinder of diameter and height of $2r$. Now look at it from the side so that the graphs on the front and back side coincide, you will see the graph of the nth Chebyshev polynomial of the first type : these graphs for $n = 2, 3, 4, 5$ are pictured in the middle row above. For $r = 1$, we get graphs of $y = T_n(x)$; for $r = 2$, we get graphs of $y = Q_n(x)$ (see exercise 3). In the bottom row, underneath the graph of the nth polynomial, we have its derivative divided by n : this is the $(n-1)$st Chebyshev polynomial of the second type ; the grey shaded area in this graph is equal to 2.

These facts imply some characteristic properties of the Chebyshev polynomials.

1. The values of the polynomial $T_n(x)$ at every extremum point as well as at the endpoints of the interval $[-1, 1]$ are equal in absolute value. The area of each of the $n+1$ pieces whose boundaries are formed by the lines $x = \pm 1$, the x-axis and the graph of $U_n(x)$ is the same (see Figure 1).

2. $T_n(\cos\phi) = \cos n\phi$ and $\sin\phi \cdot U_{n-1}(\cos\phi) = \sin n\phi$.

3. On top of the trigonometric formulas of 2, which determine the values of $T_n(x)$ and $U_n(x)$ for $|x| \le 1$, for $|x| \ge 1$ we have the following very different type of relations :

$$\begin{aligned} T_n(x) &= \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}, \\ U_n(x) &= \frac{(x+\sqrt{x^2-1})^{n+1} + (x-\sqrt{x^2-1})^{n+1}}{\sqrt{x^2-1}}. \end{aligned}$$

4. The roots of $T_n(x)$ and $U_n(x)$ can also be seen from the following :

$$\begin{aligned} T_n(x) &= 2^{n-1}\left(x-\cos\frac{\pi}{2n}\right)\left(x-\cos\frac{3\pi}{2n}\right)\dots\left(x-\cos\frac{(2n-1)\pi}{2n}\right), \\ T_n(x) &= 2^n\left(x-\cos\frac{\pi}{n+1}\right)\left(x-\cos\frac{2\pi}{n+1}\right)\dots\left(x-\cos\frac{n\pi}{n+1}\right). \end{aligned}$$

We could take any of the above properties as the defining property of Chebyshev polynomials, but it is more convenient for our purposes to define them using the recurrence relation stated in the caption of Table 1.

In what follows, we will use the polynomials derived from T_n and U_n by scaling : $P_n(x) = U_n(x/2)$ and $Q_n(x) = 2T_n(x/2)$. So instead of the interval $[-1, 1]$ for T_n and U_n, we will consider the interval $[-2, 2]$. The new polynomials are convenient because they have integer coefficients with leading coefficient equal to 1. As a rule, when we prove something for P_n, we will propose an analogous property for Q_n as an exercise to the reader. So we are asking you, the reader, to arm yourself with pen and paper and follow along.

2 Recurrence relations and induction

Let $P_0(x) = 1$, $P_1(x) = x$ and

$$P_{n+1}(x) = x \cdot P_n(x) - P_{n-1}(x). \tag{1}$$

Using this recursion, we then have :

$$\begin{aligned} P_2(x) &= x^2 - 1, \\ P_3(x) &= x(x^2-1) - x = x^3 - 2x, \\ P_4(x) &= x(x^3-2x) - (x^2-1) = x^4 - 3x^2 + 1 \end{aligned}$$

and so on.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1
7	1	7	21	35	35	21	7
8	1	8	28	56	70	56	28
9	1	9	36	84	126	126	84
10	1	10	45	120	210	252	210
...	...	...	...	...	...	...	...

TABLE 2: Pascale's Triangle. The binomial coefficients presented in this table have many properties. Yet another one is the fact that numbers located in the nth diagonal taken with alternating signs are coefficients of $P_n(x)$; exercises 6a and 8a (in part II of this article) talk about their sums.

Polynomials $P_n(x)$ arise in various scenarios. For example, consider the fractions

$$R_1(x) = x,\ R_2(x) = x - \frac{1}{x},\ R_3(x) = x - \frac{1}{x - \frac{1}{x}},\ R_4(x) = x - \frac{1}{x - \frac{1}{x - \frac{1}{x}}},\ \dots$$

These multilevel, so-called *continued* fractions, serve as a useful tool in problems concerning approximations of various functions, a topic also studied by Chebyshev. After easy transformations, we get (check this!) :

$$R_2 = \frac{x^2-1}{x},\ R_3(x) = \frac{x^3-2x}{x^2-1},\ R_4(x) = \frac{x^4-3x^2+1}{x^3-2x},\ \dots$$

Clearly, the numerators and denominators of these fractions are exactly the polynomials $P_n(x)$.

As another example, consider the function $\sin n\phi$ and express it in terms of $\sin\phi$ and $\cos\phi$ (check this!) :

$$\begin{aligned} \sin 2\phi &= 2\sin\phi\cos\phi, \\ \sin 3\phi &= \sin\phi(4\cos^2\phi - 1), \\ \sin 4\phi &= \sin\phi(8\cos^3\phi - 4\cos\phi), \dots \end{aligned}$$

It appears that $\sin n\phi = \sin\phi \cdot P_{n-1}(2\cos\phi)$ for all $n \geq 1$. In other words, for $\sin\phi \neq 0$, we have

$$P_n(2\cos\phi) = \frac{\sin{(n+1)\phi}}{\sin\phi}. \tag{2}$$

The two relations for $R_n(x)$ above and equation (2) are also easily derived from (1) using mathematical induction, which we leave as an exercise for the reader.

Exercises.

1. Using induction and recurrence (1), show that for $|x| > 2$, we have :

$$P_n(x) = \frac{(x+\sqrt{x^2-4})^{n+1} - (x-\sqrt{x^2-4})^{n+1}}{2^{n+1}\sqrt{x^2-4}}. \tag{3}$$

2. Show that

a) $P_n(2) = n+1$,

b) $P_n(-2) = (-1)^n(n+1)$.

Do this in three different ways : using (1) ; using limits in (2) as $\phi \to 0$ and $\phi \to \pi$ and in (3) with $x \to \pm 2$.

3. Consider the sequence of polynomials $Q_0(x), Q_1(x), Q_2(x), \ldots$ that satisfy (1) with initial conditions $Q_0(x) = 2, Q_1(x) = x$. Write out the first 6 polynomials. Prove the following :

a) $Q_n(x)/Q_{n-1}(x) = x - \cfrac{1}{x - \cfrac{1}{\ddots - \frac{1}{x - \frac{2}{x}}}}$,

b) $2\cos n\phi = Q_n(2\cos\phi)$, $\quad (2')$

c) for $|x| > 2$,

$$Q_n(x) = \frac{(x+\sqrt{x^2-4})^n - (x-\sqrt{x^2-4})^n}{2^n}.$$

4. Prove that any sequence of polynomials $R_0(x), R_1(x), R_2(x), \ldots$ satisfying (1), can be written in terms of polynomials $P_n(x)$ as follows :

$$R_n(x) = R_1(x)P_{n-1}(x) - R_0(x)P_{n-2}(x).$$

In particular, $Q_n(x) = xP_{n-1}(x) - 2P_{n-2}(x) = P_n(x) - P_{n-2}(x)$. From here, derive all equations from Exercise 3.

This article originally appeared in Kvant, 1982 (1). It has been translated and adapted with permission. Part II of the article is to appear in Volume 40, n. 10.

Variations on a theme : The sum of equal powers of natural numbers (I)

Arkady Alt

This is the first of a series of notes, mostly dedicated to the following problem :

Find, in closed form, the sum

$$S_p\left(n\right) := \sum_{k=1}^{n} k^p = 1^p + 2^p + ... + n^p \text{ where } p, n \in \mathbb{N}.$$

These notes can be seen as a continuation of the article "Sums of equal powers of natural numbers" by V. S. Abramovich, published in ***Crux*** Vol. 40 (6).

Let us first consider the simplest special cases of $S_p\left(n\right)$ for $p = 1, 2, 3$, testing various approaches to find the most suitable way for consideration of the general case.

1 Finding $S_1(n) := 1 + 2 + 3 + \cdots + n$

We reproduce the way that young K.F. Gauss apocryphally solved the problem at the age of 10, by grouping terms. Since

$$S_1\left(n\right) = \sum_{k=1}^{n} k = \sum_{k=1}^{n} \left(n - k + 1\right)$$

we have

$$2S_1\left(n\right) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} \left(n - k + 1\right) = \sum_{k=1}^{n} \left(k + \left(n - k + 1\right)\right) = \sum_{k=1}^{n} \left(n + 1\right) = n\left(n + 1\right)$$

and thus

$$S_1\left(n\right) = \frac{n\left(n + 1\right)}{2}. \tag{1}$$

We can also solve it by the method popularly known as *telescoping*, or the *difference method of summation.* Suppose that it is possible to find a sequence $b_1, b_2, ..., b_k, ...$ such that $a_k = b_{k+1} - b_k,\ k = 1, 2, ...,$. Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(b_{k+1} - b_k\right) = b_{n+1} - b_1\ . \tag{2}$$

This is justified in Appendix 1.

Since $k^2 - (k-1)^2 = 2k - 1$, then

$$\begin{aligned}
& \sum_{k=1}^{n} \left(k^2 - (k-1)^2\right) = \sum_{k=1}^{n} (2k-1) \\
\Rightarrow \quad & n^2 - (1-1)^2 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \quad \text{(telescoping the left hand side)} \\
\Rightarrow \quad & n^2 = 2\sum_{k=1}^{n} k - n \\
\Rightarrow \quad & \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \; .
\end{aligned}$$

Exercise 1 *Prove the same identity starting with the observation that*

$$k^2 - (k-1)^2 + 1 = 2k.$$

2 Finding $S_2(n) := 1^2 + 2^2 + 3^2 + \cdots + n^2$

We can find this by analogy with the previous example, by telescoping sums of cubes and making use of our knowledge of S_1 and S_0. Since $k^3 - (k-1)^3 = 3k^2 - 3k + 1$, we have

$$\begin{aligned}
& \sum_{k=1}^{n} \left(k^3 - (k-1)^3\right) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
\Rightarrow \quad & n^3 = 3S_2(n) - 3S_1(n) + n \\
\Rightarrow \quad & 3S_2(n) = n^3 - n + 3S_1(n) \\
\Rightarrow \quad & 3S_2(n) = n^3 - n + \frac{3n(n+1)}{2} = \frac{n(n+1)(2n+1)}{2} \\
\Rightarrow \quad & S_2(n) = \frac{n(n+1)(2n+1)}{6} \; . \qquad (3)
\end{aligned}$$

Noting that

$$(k+2)(k+1)k - (k+1)k(k-1) = 3k(k+1),$$

we get

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1) &= \frac{1}{3}\sum_{k=1}^{n} \left((k+2)(k+1)k - (k+1)k(k-1)\right) \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

and, using the identity $k^2 = k(k+1) - k$, we obtain

$$S_2(n) = \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6} .$$

This method may appear non-obvious, but the trick is one that is often used in combinatorics and probability theory, and well worth learning. The heart of it is that the *falling factorials*

$$(k)_n := k(k-1)\cdots(k-n+1)$$

and *rising factorials*

$$k^{(n)} := k(k+1)\cdots(k+n-1)$$

play better with their neighbors than powers do. In particular,

$$(k)_n - (k-1)_n = n \cdot (k-1)_{n-1} \text{ and } k^{(n)} - (k-1)^{(n)} = n \cdot k^{(n-1)} . \qquad (4)$$

The case $n = 3$ of the first identity is what we started with.

Exercise 2 *Prove* (4)*.*

Exercise 3 *Use* (4) *and* (2) *to show that*

$$\sum_{k=1}^{n} k^{(m)} = \frac{n^{(m+1)}}{m+1}.$$

3 Finding $S_3(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$

Exercise 4 *Note that* $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$ *. Use this, and the methods of subsection* 2*, to evaluate* $S_3(n)$*.*

This time we will use rising factorials. First, we will find a representation of k^3 in the form

$$k^3 = a + bk + ck(k+1) + k(k+1)(k+2) .$$

This can be done by expanding and equating like terms, but (as when finding partial fractions expansions without repeated roots) it is easier to evaluate by plugging in values that make one or more summands vanish. By substituting $k = 0, -1, -2$ in the equation above, we obtain $a = 0, a - b = -1, a - 2b + 2c = -8$, whence $a = 0, b = 1, c = -3$. Since

$$k^3 = k - 3k(k+1) + k(k+1)(k+2) ,$$

we have

$$S_3(n) = \sum_{k=1}^{n} k - 3\sum_{k=1}^{n} k(k+1) + \sum_{k=1}^{n} k(k+1)(k+2) .$$

We already know the closed forms of $\sum_{k=1}^{n} k\,(k+1)$ and $\sum_{k=1}^{n} k$. By Exercise 3, we have

$$\sum_{k=1}^{n} k\,(k+1)\,(k+2) = \frac{n\,(n+1)\,(n+2)\,(n+3)}{4}$$

and, therefore,

$$\begin{aligned} S_3\,(n) &= \frac{n\,(n+1)}{2} - n\,(n+1)\,(n+2) + \frac{n\,(n+1)\,(n+2)\,(n+3)}{4} \\ &= \frac{n^2\,(n+1)^2}{4}\,. \end{aligned} \tag{5}$$

Exercise 5 *Do the same using falling factorials : that is, find $S_3\,(n)$ by setting*

$$k^3 = a + bk + ck\,(k-1) + k\,(k-1)\,(k-2)$$

and finding the appropriate sums.

Exercise 6 *Prove the identity*

$$\frac{k^2\,(k+1)^2}{4} - \frac{(k-1)^2\,k^2}{4} = k^3$$

and use it together with (2) *to obtain the closed form for $S_3\,(n)$.*

4 Recurrences For The General Case

Modifying the ideas of sections 1 and 2, we obtain

$$\begin{aligned} & (k+1)^{p+1} - k^p = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i} \\ \Rightarrow\quad & \sum_{k=1}^{n} \left((k+1)^{p+1} - k^p\right) = \sum_{k=1}^{n}\sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i} \\ \Rightarrow\quad & (n+1)^{p+1} - 1^p = \sum_{i=1}^{p+1} \binom{p+1}{i} \sum_{k=1}^{n} k^{p+1-i} \\ \Rightarrow\quad & (n+1)^{p+1} - 1 = \sum_{i=1}^{p+1} \binom{p+1}{i} S_{p+1-i}\,(n) \\ \Rightarrow\quad & (n+1)^{p+1} - 1 = (p+1)\,S_p\,(n) + \sum_{i=2}^{p} \binom{p+1}{i} S_{p+1-i}\,(n) + S_0\,(n) \\ \Rightarrow\quad & S_p\,(n) = \frac{(n+1)^{p+1} - 1 - \sum_{k=0}^{p-1} \binom{p+1}{k} S_k\,(n)}{p+1} \end{aligned} \tag{6}$$

for all $p \in \mathbb{N}$, letting $S_0(n) := n$.

Exercise 7 *For any $p \in \mathbb{N}$, show that*

$$k^{p+1} - (k-1)^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i}.$$

Now, sum left and right sides for $k = 1, \ldots, n$ and thus show that

$$S_p(n) = \frac{n^{p+1} + \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p+1}{k} S_k(n)}{p+1} \tag{7}$$

for $p \in \mathbb{N}$.

So we have recursive representations of $S_p(n)$ in terms of $S_{p-1}(n), \ldots, S_1(n)$. An easy corollary of this result is the fact that $S_p(n)$ is polynomial of degree $p+1$ in n (use mathematical induction on (6).)

5 Appendix

Why does the difference method work ? Suppose that $a_k = b_{k+1} - b_k,\ k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_{k+1} - b_k) = \sum_{k=1}^{n} b_{k+1} - \sum_{k=1}^{n} b_k = \sum_{k=2}^{n+1} b_k - \sum_{k=1}^{n} b_k = b_{n+1} - b_1.$$

Informally,

$$a_1 + a_2 + a_3 + \cdots + a_n = (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) + \cdots + (b_{n+1} - b_n)$$

and all terms except $-b_1$ and b_{n+1} cancel.

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Un astérisque ($\star$) *signale un problème proposé sans solution.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

3971. *Proposé par Marcel Chiriţă.*

Soit x, y et z des réels strictement positifs tels que $4(x+y+z)^2 = 9(x^2+y^2+z^2)$. Démontrer que

$$4-\sqrt{15} \le \frac{x}{z} \le 4+\sqrt{15}.$$

3972. *Proposé par Michel Bataille.*

Déterminer toutes les fonctions $f : (0,\infty) \to (0,\infty)$ telles que

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$$

pour tous les réels x et y.

3973. *Proposé par Dragoljub Milošević.*

Soit m_a, m_b et m_c les médianes d'un triangle, r le rayon du cercle inscrit dans le triangle, r_a, r_b et r_c les rayons des cercles exinscrits du triangle et R le rayon du cercle circonscrit au triangle. Démontrer que

$$\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c} \le \frac{2R}{r}-1.$$

3974. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels strictement positifs tels que $a+b+c=3$. Démontrer que

$$\sqrt{\frac{a}{b}+\frac{1}{a}}+\sqrt{\frac{b}{c}+\frac{1}{b}}+\sqrt{\frac{c}{a}+\frac{1}{c}} \ge 3\sqrt{2}.$$

3975. *Proposé par Ovidiu Furdui.*

Soit k un entier strictement positif. Calculer

$$\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx.$$

3976. *Proposé par Cristinel Mortici et Leonard Giugiuc.*

Déterminer les triplets d'entiers strictement positifs qui vérifient l'équation

$$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{x}{y-z}.$$

3977. *Proposé par Dragoljub Milošević.*

Démontrer que dans un triangle ABC, on a

$$\frac{1}{3-2\cos A}+\frac{1}{3-2\cos B}+\frac{1}{3-2\cos C}\geq\frac{3}{2}.$$

3978. *Proposé par Billy Jin et Edward T.H. Wang.*

Soit n un entier supérieur à 2. Une permutation $\sigma=(a_1,a_2,\ldots,a_n)$ de $S(n)=\{1,2,\ldots,n\}$ est une *permutation zigzag* si les termes a_i changent de parité en alternance et s'ils augmentent et diminuent en alternance. Par exemple, $(5,6,3,4,1,2)$ est une permutation zigzag de $S(6)$. Déterminer le nombre de permutations zigzags de $S(n)$.

3979. *Proposé par George Apostolopoulos.*

Dans le triangle ABC, les bissectrices en A, B et C coupent les côtés opposés aux points respectifs D, E et F. Démontrer que

$$\frac{[DEF]}{[ABC]}\leq\frac{R_1}{4r},$$

R_1 étant le rayon du cercle circonscrit au triangle DEF, r étant le rayon du cercle inscrit dans le triangle ABC et $[ABC]$ et $[DEF]$ étant les aires des triangles.

3980. *Proposé par S. Viswanathan.*

Soit a, b et c des réels distincts. Démontrer que

$$\left(\sum_{cyc}\frac{a+b}{a-b}\right)\left(\prod_{cyc}\frac{a+b}{a-b}\right)<\frac{1}{3}.$$

. . . .

3971. *Proposed by Marcel Chiriţă.*

Let x, y, z be positive real numbers such that $4(x+y+z)^2 = 9(x^2+y^2+z^2)$. Show that

$$4-\sqrt{15} \leq \frac{x}{z} \leq 4+\sqrt{15}.$$

3972. *Proposed by Michel Bataille.*

Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right)$$

for all positive real numbers x, y.

3973. *Proposed by Dragoljub Milošević.*

Let m_a, m_b, m_c be the medians, r_a, r_b, r_c the exradii, R the circumradius and r the inradius of a triangle. Prove that

$$\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c} \leq \frac{2R}{r}-1.$$

3974. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c = 3$. Prove that

$$\sqrt{\frac{a}{b}+\frac{1}{a}}+\sqrt{\frac{b}{c}+\frac{1}{b}}+\sqrt{\frac{c}{a}+\frac{1}{c}} \geq 3\sqrt{2}.$$

3975. *Proposed by Ovidiu Furdui.*

Let $k \geq 1$ be an integer. Calculate

$$\int_0^\infty \frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx.$$

3976. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

Find positive integer solutions for the following equation:

$$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{x}{y-z}.$$

3977. *Proposed by Dragoljub Milošević.*

Given a triangle ABC, prove that

$$\frac{1}{3-2\cos A}+\frac{1}{3-2\cos B}+\frac{1}{3-2\cos C}\geq \frac{3}{2}.$$

3978. *Proposed by Billy Jin and Edward T.H. Wang.*

Let $n>2$ be a positive integer. A permutation $\sigma=(a_1,a_2,\ldots,a_n)$ of $S(n)=\{1,2,\ldots,n\}$ is called a *zigzag permutation* if, when reading from left to right, the a_i's alternately change their parity and increase/decrease in magnitudes. For example, $(5,6,3,4,1,2)$ is a zigzag permutation of $S(6)$. Determine the number of zigzag permutations of $S(n)$.

3979. *Proposed by George Apostolopoulos.*

Let AD, BE and CF be the internal bisectors of the triangle ABC. Prove that

$$\frac{[DEF]}{[ABC]}\leq \frac{R_1}{4r},$$

where R_1 denotes the circumradius of DEF, r denotes the inradius of ABC and $[\cdot]$ represents the area of the corresponding triangle.

3980. *Proposed by S. Viswanathan.*

Let a, b, c be distinct real numbers. Prove that

$$\left(\sum_{cyc}\frac{a+b}{a-b}\right)\left(\prod_{cyc}\frac{a+b}{a-b}\right)<\frac{1}{3}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3871. *Proposed by Cristinel Mortici.*

Solve in positive real numbers:

$$(x^2y-1)\ln x+(xy^2-1)\ln y=0\,.$$

We received ten correct submissions and two incorrect solutions. We present two solutions.

Solution 1, by Václav Konečný.

There is only one solution, namely $x=1$, $y=1$. Put

$$F(x,y)=(x^2y-1)\ln x+(xy^2-1)\ln y.$$

One solution of $F(x,y)=0$ is $x=1$, $y=1$. We show that it is the only solution.

Using the substitutions $x=e^X$, $y=e^Y$ (X and Y are real) we get

$$G(X,Y)=(e^{2X+Y}-1)X+(e^{X+2Y}-1)Y=0.$$

Using the inequality $e^z\geq 1+z$, where z is real, with equality when $z=0$, we get

$$G(X,Y)\geq(2X+Y)X+(X+2Y)Y=g(X,Y)$$

with equality when $2X+Y=0$ and $X+2Y=0$. Thus $X=0$ and $Y=0$. Furthermore,

$$g(X,Y)=X^2+Y^2+(X+Y)^2\geq 0$$

with equality only if $X=0$ and $Y=0$. From the substitutions above, $x=1$ and $y=1$ is the only solution of $F(x,y)=0$.

Solution 2, by Nermin Hodžić and Salem Malikić.

First, we prove the following inequalities:

1. If $x,y>0$ then

$$x\ln x+y\ln y\geq(x+y)\ln\frac{x+y}{2}. \tag{1}$$

   This inequality easily follows from *Jensen's inequality* applied on the function $f(a)=a\ln a$.

2. If $x > 0$ then
$$x^x \geq x. \tag{2}$$
This inequality is equivalent to $x \ln x \geq \ln x$ that is further equivalent to $(x-1)\ln x \geq 0$ and can be easily proved by discussing cases $x \geq 1$ and $x < 1$. Equality is achieved for $x = 1$.

3. If $x, y > 0$ then
$$\frac{x+y}{2} \geq \sqrt{xy}. \tag{3}$$
This well known inequality follows directly from the inequality of arithmetic and geometric means.

Now we turn back to our problem. It is equivalent to

$$xy(x \ln x + y \ln y) = \ln(xy).$$

We have

$$\begin{aligned} xy(x \ln x + y \ln y) &\overset{(1)}{\geq} xy(x+y) \ln \frac{x+y}{2} = 2xy \ln \left(\frac{x+y}{2}\right)^{\frac{x+y}{2}} \\ &\overset{(2)}{\geq} 2xy \ln \frac{x+y}{2} \\ &\overset{(3)}{\geq} 2xy \ln \sqrt{xy} = \ln(xy)^{xy} \geq \ln(xy), \end{aligned}$$

hence, the equality sign must hold in all of the inequalities used above. From (3) it follows that $x = y$ and (2) implies $\frac{x+y}{2} = 1$, so we get $x = y = 1$ as the only candidate solution.

It is easy to verify that this is indeed a solution of the given equation.

3872. *Proposed by Farrukh Rakhimjanovich Ataev. Correction.*

Let x, y, z be the distances from the vertices of a triangle to its incircle and let r be the inradius of the triangle. Show that the area of the triangle is given by

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r}.$$

We received 18 correct submissions. We provide a solution similar to the one given by all of the solvers.

Let x, y, z be the respective distances of A, B, C from the incircle. Thus, the respective distances of A, B, C from the centre of the incircle are $x + r$, $y + r$, $z + r$. The lengths of the tangents to the incircle from A, B, C are, respectively, $s - a$, $s - b$, $s - c$. By Pythagoras' theorem, $(s-a)^2 = (x+r)^2 - r^2 = x(x+2r)$, $(s-b)^2 = y(y+2r)$ and $(s-c)^2 = z(z+2r)$. Therefore

$$\frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r} = \frac{s(s-a)(s-b)(s-c)}{rs} = \frac{A^2}{A} = A.$$

Editor's comments. Unfortunately, there were four solvers who interpreted the problem to mean that x, y, z are the distances from the vertices to the centre rather than the nearest point on the incircle and provided counterexamples.

3873. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that the following inequality holds

$$\frac{a}{8b^3+8c^3+11bc}+\frac{b}{8c^3+8a^3+11ca}+\frac{c}{8a^3+8b^3+11ab}\geq\frac{1}{9}\,.$$

We received seven correct submissions. We present a solution that uses ideas from all of the solvers and computer algebra to avoid lengthy computations.

Since by Cauchy's inequality

$$\sum_{\text{cyc}}a^3(8b^3+8c^3+11bc)\cdot\sum_{\text{cyc}}\frac{a}{8b^3+8c^3+11bc}\geq(a^2+b^2+c^2)^2,$$

it suffices to show that

$$9\left(\sum_{\text{cyc}}a^2\right)^2-16\sum_{\text{cyc}}a^3b^3-11abc\sum_{\text{cyc}}a^2\geq 0.$$

Using the constraint $a+b+c=3$, we rewrite this as

$$3(a+b+c)^2\left(\sum_{\text{cyc}}a^2\right)^2-48\sum_{\text{cyc}}a^3b^3-11abc(a+b+c)\sum_{\text{cyc}}a^2\geq 0.$$

Call the left-hand side of this last inequality $G(a,b,c)$. Since G is symmetric, we may assume that $a\leq b\leq c$. We may thus write $b=a+s$ and $c=a+s+t$, where s and t are nonnegative. Then $G(a,b,c)=G(a,a+s,a+s+t)$. Using computer algebra to expand this last expression gives a polynomial in a, s, and t, all of whose coefficients are nonnegative.

Editor's Comments. Arslanagić and the proposers each use an approach referred to by the proposers as the "UVW method", using the substitutions $a+b+c=3u$, $ab+bc+ca=3v^2$, $abc=w^3$, thereby transforming the claimed inequality into one of the form $f(u,v,w)\geq 0$, where f is a symmetric polynomial homogeneous of degree 6. They verify that, as a function of w^3, f is concave and conclude from this that the inequality $f(u,v,w)\geq$ holds if it holds when $a=b$ and when $c=0$. Verification in these two cases is straightforward. The other solvers either had laborious algebraic calculations or used computer assistance, or both.

3874. *Proposed by Panagiote Ligouras.*

In triangle ABC, the sides satisfy $AB+AC=2BC$. Let I and O be the incentre and circumcentre of triangle ABC respectively, and let M be the intersection of

AI with the circumcircle of triangle ABC. Let N be a point in the plane of triangle ABC and let P be the point on the line AI such that $IC^2+IN^2 = 2IP^2$, $\angle NIP = \angle CIP$ and $INPC$ is a cyclic quadrilateral. Let Q be the intersection of AP and CN, and let R be the intersection of OI and CN. Prove that $IQ = IR$.

We received two correct submissions. We present the solution by Oliver Geupel.

We have $\angle BAM = \angle CAM = \angle BAC/2$, so that $\angle BAM$ and $\angle CAM$ are subtended by equal-length arcs; thus triangle BMC is isosceles with $BM = CM$, and $\angle MBC = \angle BAM/2$. Therefore, $\angle MBI = \angle MBC + \angle ABC/2 = (\angle CAB + \angle ABC)/2 = \angle BIM$, using the fact that I is the incentre of $\triangle ABC$. Thus, $CM = BM = IM$. By Ptolemy's theorem, we obtain

$$AM \cdot BC = AB \cdot CM + AC \cdot BM = (AB + AC) \cdot IM = 2BC \cdot IM.$$

Hence, I is the midpoint of the chord AM of circle circumscribing $\triangle ABC$. We conclude that AM is perpendicular to IO, that is, $\angle RIQ = 90^\circ$.

By hypothesis $\angle CIP = \angle NIP$, so these angles are subtended by equal-length arcs, and thus $NP = CP$. Applying the Law of Cosines to $\triangle ICP$ and $\triangle INP$, we obtain

$$\begin{aligned} IC^2 + CP^2 - 2IC \cdot CP \cos\angle ICP =& IN^2 + NP^2 - 2IN \cdot NP \cos\angle PNI \\ (IC^2 - IN^2) + -2IC \cdot CP \cos\angle ICP =& -2IN \cdot CP \cos(180^\circ - \angle ICP) \\ (IC - IN)(IC + IN) =& (IC + IN)2CP \cos\angle ICP, \end{aligned} \qquad (1)$$

where $180^\circ - \angle ICP = \angle PNI$ because these are opposite angles in a cyclic quadrilateral.

By applying hypothesis $IC^2+IN^2 = 2IP^2$, we obtain another identity for $\cos\angle ICP$ from

$$\begin{aligned} 2IP^2 &= IC^2 + IN^2 \\ &= (IC^2 + CP^2 - 2IC \cdot CP \cos\angle ICP) + (IN^2 + NP^2 - 2IN \cdot NP \cos\angle PNI) \\ &= IC^2 + IN^2 + 2CP^2 + 2CP(IN - IC) \cos\angle ICP. \end{aligned} \qquad (2)$$

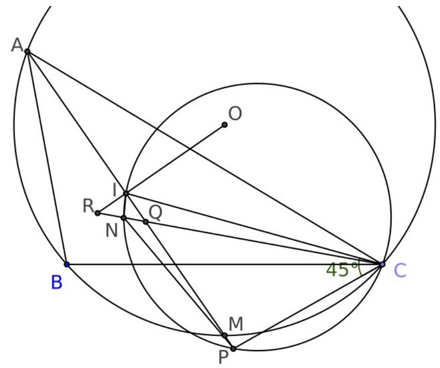
Equation (1) yields $\cos\angle ICP = \frac{IC-IN}{2CP}$ and Equation (2) yields $\cos\angle ICP = \frac{CP}{IC-IN}$. Combining these equations, $2\cos^2\angle ICP = 1$. Therefore the size of $\angle ICP$ is either 45° or 135°. We shall consider each case $\angle ICP = 45^\circ$ and $\angle ICP = 135^\circ$ separately after deriving two more properties, as follows.

Considering inscribed angles, we see that $\angle NIP = \angle NCP$, and by hypothesis $\angle CIP = \angle NIP$. Thus

$$\angle QIC = \angle CIP = \angle NIP = \angle NCP = \angle QCP. \qquad (3)$$

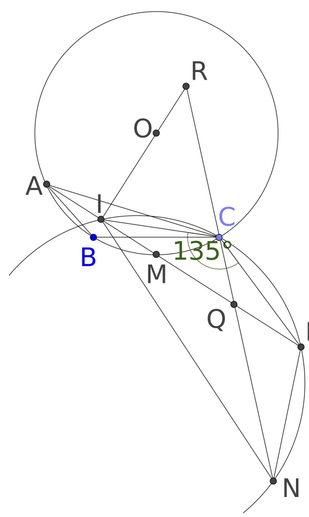
Furthermore, either (i) $\angle IQR = \angle IQC$ or (ii) $\angle IQR = 180^\circ - \angle IQC$, depending on the location of R on the line CN.

First consider the case $\angle ICP = 45^\circ$:



We have $\angle ICQ = \angle ICP - \angle QCP = 45° - \angle QCP = 45° - \angle QIC$ by (3), so that $\angle IQC = 180° - \angle ICQ - \angle QIC = 135°$. Thus, (i) is not possible, since $\angle RIQ = 90°$, so we have $\angle IQR = 180° - \angle IQC = 45°$, and thus $\angle QRI = 45°$. Consequently, $\triangle IRQ$ is isosceles with $IQ = IR$, which is the desired result.

It remains to consider the case $\angle ICP = 135°$:



We have $\angle ICQ = \angle ICP - \angle QCP = 135° - \angle QCP = 135° - \angle QIC$, so that $\angle IQC = 180° - \angle ICQ - \angle QIC = 45°$. Thus, (ii) is not possible, so $\angle IQR = \angle IQC = 45°$ and $\angle QRI = 45°$. Consequently, $\triangle IRQ$ is isosceles with $IQ = IR$, which is the desired result and completes the proof.

3875. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(A, +, \cdot)$ be a ring with unity, $1 \neq 0$, such that if $xy = 1$ then $yx = 1$. If $a, b \in A$, and there is a $u \in A$, where u is invertible, with $ua = au$, $ub = bu$, such that $uab + a + b = 0$, then prove that $ab = ba$.

We received six correct submissions. We present the solution by Roy Barbara *although all the solutions were similar.*

Let $x = ua + 1$ and $y = ub + 1$. Note that

$$(ua)(ub) = u(au)b = u(ua)b = u^2ab.$$

Similarly, $(ub)(ua) = u^2ba$. Hence,

$$xy = (ua + 1)(ub + 1) = u^2ab + ua + ub + 1 = u(uab + a + b) + 1 = 0 + 1 = 1.$$

Similarly, $yx = u^2ba + ub + ua + 1$. Since $xy = 1$, we have $yx = 1 = xy$ by one of the given hypotheses.

We have

$$u^2ab + ua + ub + 1 = u^2ba + ub + ua + 1,$$

so $u^2ab = u^2ba$. Since u^2 is invertible, $ab = ba$ follows.

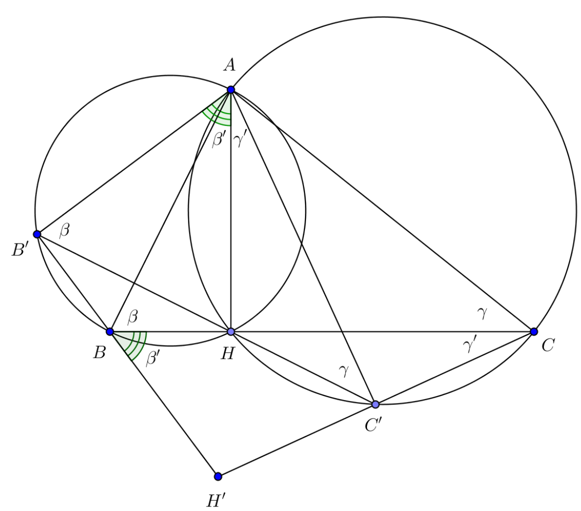
3876. *Proposed by Michel Bataille.*

Let ABC be a triangle, H the foot of the altitude from A and γ_b, γ_c the circles with diameters AB, AC, respectively. A line ℓ passing through H intersects γ_b, γ_c at H and B', C', respectively. If BB' and CC' intersect at H', show that

$$(H'C)(HC') = k(H'B)(HB')$$

for some constant k independent of the chosen line ℓ.

We received six correct submissions and one faulty submission. We present the solution of Peter Woo with details supplied by the editor.



We shall prove a more general result. Let H be any point on the side BC of ΔABC, and replace the circles from the statement of the problem (on diameters

AB and AC) by the circumcircles of triangles ABH and AHC. Then

$$\frac{(H'C)(HC')}{(H'B)(HB')} = \frac{\sin\angle CBA}{\sin\angle ACB}\,,$$

which is independent of the choice of H and of the chosen line through it.

In circle $AB'BH$, we have

$$\begin{aligned}\sin\angle HB'A &= \sin\angle HBA = \sin\angle CBA = \sin\beta,\\ \sin\angle B'AH &= \sin\angle B'BH = \sin\angle H'BC = \sin\beta'.\end{aligned}$$

In circle $AHC'C$, we have

$$\begin{aligned}\sin\angle HAC' &= \sin\angle HCC' = \sin\angle BCH' = \sin\gamma',\\ \sin\angle AC'H &= \sin\angle ACH = \sin\angle ACB = \sin\gamma.\end{aligned}$$

We apply the sine law four times. In $\Delta BH'C$,

$$\frac{H'C}{BC} = \frac{\sin\angle H'BC}{\sin\angle CH'B} = \frac{\sin\beta'}{\sin\angle CH'B}$$

and

$$\frac{H'B}{BC} = \frac{\sin\angle BCH'}{\sin\angle CH'B} = \frac{\sin\gamma'}{\sin\angle CH'B};$$

in $\Delta AHC'$,

$$\frac{HC'}{AH} = \frac{\sin\angle HAC'}{\sin\angle AC'H} = \frac{\sin\gamma'}{\sin\gamma}\,;$$

and in $\Delta AB'H$,

$$\frac{HB'}{AH} = \frac{\sin\angle B'AH'}{\sin\angle HB'A} = \frac{\sin\beta'}{\sin\beta}.$$

It follows that

$$(H'C)(HC') = (BC)(AH)\frac{\sin\beta'}{\sin\angle CH'B}\cdot\frac{\sin\gamma'}{\sin\gamma}\,,$$

and

$$(H'B)(HB') = (BC)(AH)\frac{\sin\gamma'}{\sin\angle CH'B}\cdot\frac{\sin\beta'}{\sin\beta}.$$

The quotient of the last two equations gives us the claimed result.

Editor's comments. Most solvers showed that the desired quotient is the constant $k = \frac{AC}{AB}$ (which equals $\frac{\sin\angle CBA}{\sin\angle ACB}$ by way of the sine law applied to ΔABC). A few solvers incorrectly claimed that B' and C' must lie on opposite sides of the line BC; to the contrary, as B' moves around its circle ABH, each of H, B', and C' lies between the other two for a portion of the journey. Although this turns pairs of equal angles into supplementary angles, this error is not critical because the sine of an angle equals the sine of its supplement.

Unlike the featured solution, all others made use of the right angle at H, which was an unnecessary part of the problem as originally stated. Other properties they observed, however, hold for all positions of H and lead to alternative approaches, namely *the triangles ABC and $AB'C'$ are similar, and H' lies on the circumcircles of both triangles.*

3877. *Proposed by Marian Dincă.*

Let $x_1, x_2, \ldots, x_m$ be real numbers such that $\prod_{k=1}^{m} x_k = 1$. Prove that

$$\sum_{k=1}^{m} \frac{x_k^2}{x_k^2 - 2x_k \cos\frac{2\pi}{m} + 1} \geq 1.$$

We received four correct submissions. We present a slightly expanded version of the solution by Oliver Geupel.

The inequality does not hold when $m < 3$. The denominator is 0 for $m = 1$, and so the inequality does not even make sense. For $m = 2$, we know that $x_2 = \frac{1}{x_1}$, and the inequality simplifies, upon substitution, to

$$\frac{x_1^2 + 1}{(x_1 + 1)^2} \geq 1,$$

which only holds when $x_1 < 0$. For example, take $x_1 = \frac{1}{2}$ and $x_2 = 2$ to see a counterexample. Thus, we add the hypothesis $m \geq 3$, and give proofs for the three cases $m = 3$, $m = 4$, and $m \geq 5$ in succession.

First suppose $m = 3$. Since $0 < x_k^2 + 2x_k \cos\frac{2\pi}{3} + 1 \leq x_k^2 + |x_k| + 1$, it suffices to prove that the inequality $\sum\limits_{k=1}^{3} \frac{x_k^2}{x_k^2+|x_k|+1} \geq 1$ holds. We make the substitution $(|x_1|, |x_2|, |x_3|) = (a^3, b^3, c^3)$. Then, a, b, c are positive numbers such that $abc = 1$. We homogenize the inequality as

$$\sum_{\text{cyc}} \frac{a^6}{a^6 + a^4bc + a^2b^2c^2} \geq 1,$$

using the fact that $abc = 1$. Clearing denominators and simplifying (which is an exercise in patience), we obtain

$$\sum_{\text{sym}} a^8b^8c^2 \geq \sum_{\text{sym}} a^8b^5c^5,$$

which holds by Muirhead's inequality (as $8 \leq 8$, $8 + 5 \leq 8 + 8$, and $8 + 5 + 5 = 8 + 8 + 2$). This proves the inequality in the case $m = 3$.

In the case $m=4$, we use a substitution $(|x_1|,|x_2|,|x_3|,|x_4|)=\left(\frac{a}{b},\frac{b}{c},\frac{c}{d},\frac{d}{a}\right)$ with positive numbers a,b,c,d. We are readily done with the observation that

$$\sum_{k=1}^{4}\frac{x_k^2}{x_k^2-2x_k\cos\frac{2\pi}{4}+1}=\sum_{\text{cyc}}\frac{a^2}{a^2+b^2}>\sum_{\text{cyc}}\frac{a^2}{a^2+b^2+c^2+d^2}=1.$$

It remains to consider $m\geq 5$. We have

$$\sum_{k=1}^{m}\frac{x_k^2}{x_k^2-2x_k\cos\frac{2\pi}{m}+1}\geq\sum_{k=1}^{m}\frac{x_k^2}{(|x_k|+1)^2}=\sum_{k=1}^{m}f(\log|x_k|).$$

where $f(y)=\dfrac{e^{2y}}{(e^y+1)^2}$. It is clear that f is positive, and the first derivative is positive, hence f is increasing. Checking the second derivative, we see that $f(y)$ is convex for $y\leq\log 2$ and is concave for $y\geq\log 2$.

Let $y_1,\ldots,y_m$ be the numbers $\log|x_1|,\ldots,\log|x_m|$ arranged in increasing order. Note that since

$$\prod_{k=1}^{m}|x_k|=\left|\prod_{k=1}^{m}x_k\right|=1,$$

we have

$$\sum_{k=1}^{m}y_k=\log(1)=0.$$

We consider the cases $y_m\leq\log 2$ and $y_m>\log 2$ in succession.

If $y_m\leq\log(2)$, then Jensen's inequality yields:

$$\sum_{k=1}^{m}f(y_k)\geq mf\left(\frac{1}{m}\sum_{k=1}^{m}y_k\right)=mf(0)=\frac{m}{4}\geq 1,$$

since $m\geq 5$. So this case is done.

If $y_m>\log(2)$, then we have more work to do. First, we will show that the minimum of the sum $\sum\limits_{k=1}^{m}f(y_k)$ occurs when the first $m-1$ of the y_k are equal. Since the sum of the y_k is 0, there is some positive integer n such that $y_n\leq\log 2\leq y_{n+1}$. By Jensen's inequality applied to $y_1,\ldots,y_n$ and $m-(n+1)$ copies of $\log(2)$, we obtain:

$$\sum_{k=1}^{n}f(y_k)+(m-n-1)f(\log 2)\geq(m-1)f\left(\frac{1}{m-1}\left(\sum_{k=1}^{n}y_k+(m-n-1)\log 2\right)\right).$$

Moreover, the finite sequence $y_m,y_{m-1},\ldots,y_{n+1}$ is majorized by

$$\sum_{k=n+1}^{m}y_k-(m-(n+1))\log(2),\log(2),\ldots,\log(2),$$

and f is concave on this region, so by the Majorization inequality (also known as Karamata's inequality), we have

$$\sum_{k=n+1}^{m} f(y_k) \geq (m-n-1)f(\log 2) + f\left(\sum_{k=n+1}^{m} y_k - (m-n-1)\log 2\right).$$

Thus, adding the two inequalities together gives us:

$$\sum_{k=1}^{m} f(y_k) \geq (m-1)f\left(\frac{1}{m-1}\left(\sum_{k=1}^{n} y_k + (m-n-1)\log 2\right)\right) \\ +f\left(\sum_{k=n+1}^{m} y_k - (m-n-1)\log 2\right).$$

As well, since f is strictly concave on this region, we may apply the equality case of Jensen's inequality to show that the only time the minimum is achieved in this case is when $y_1 = \cdots = y_{m-1}$. Since $\log(x)$ is one-to-one, we may set $y_1 = \cdots = y_{m-1} = \log(x)$, for some $x > 0$. Then we have $y_m = -(m-1)\log(x)$, and so:

$$\sum_{k=1}^{m} f(y_k) = \frac{(m-1)e^{2\log(x)}}{(e^{\log(x)}+1)^2} + \frac{e^{-2(m-1)\log(x)}}{(e^{-(m-1)\log(x)}+1)^2} = \frac{(m-1)x^2}{(x+1)^2} + \frac{1}{(x^{m-1}+1)^2},$$

which we may write as $g(x)$.

It remains to prove that $g(x) \geq 1$. After clearing denominators, this rewrites as $P(x) \geq 0$ where

$$P(x) = (m-2)x^{2m-2} - 2x^{2m-3} - x^{2m-4} + (2m-4)x^{m-1} - 4x^{m-2} - 2x^{m-3} + m - 1.$$

Using some clever addition of 0 and algebraic manipulation, we write P in the form

$$P(x) = (x-1)x^{m-3}(3x^m + x^{m-1} + 6x + 2) + (m-5)x^{m-1}(x^{m-1}+2) + (m-1),$$

and see that $P(x) > 0$ for $x \geq 1$. For $0 < x < 1$, we write P as

$$P(x) = 2(1-x^{m-3}) + (1 + (m-2)x^{m-1} - 4x^{m-2}) \\ +(1 + (m-2)x^{2m-2} - x^{2m-4}) + ((m-2)x^{m-1} - 2x^{2m-3}) + m - 5.$$

Observing that

$$\frac{1+(m-2)x^{m-1}}{4} \geq \frac{1+(m-2)x^{m-1}}{m-1} \geq x^{m-2}$$

and similarly

$$1 + (m-2)x^{2m-2} > \frac{1+(m-2)x^{2m-2}}{m-1} \geq x^{2m-4},$$

by the AM-GM Inequality, and rewriting

$$(m-2)x^{m-1} - 2x^{2m-3} = x^{m-1}((m-2) - 2x^{m-2}),$$

we verify that $P(x) > 0$ for $0 < x < 1$. Therefore, $g(x) \geq 1$, and we are done.

Editor's comments. The denominator of the fraction is $|x - e^{\frac{2\pi i}{n}}|^2$, the square of the complex absolute value; this was put to good use via Cauchy-Schwarz in an attempted solution which treated the situation where all of the x_i were positive, but which had a fatal flaw afterwards. The proposer's solution uses the so-called 'Lagrange inequality', and some geometry of convex polygons embedded in the complex plane. Perhaps the appearance of the complex absolute value was a clue to take a more geometric approach to the problem.

3878. *Proposed by Dao Thanh Oai.*

Let H be the orthocentre of triangle ABC and let D be any point in the plane different from A, B, C, H. Prove that if A', B', and C' are the points where the perpendiculars from H to DA, DB, and DC meet the lines BC, CA and AB, respectively, then they are collinear. Conversely, if a line (not through a vertex or orthocentre) intersects BC in A', CA in B' and AB in C', then prove that the lines through A, B, C that are perpendicular to HA', HB', HC', respectively, are concurrent.

We received four correct submissions and two incomplete solutions. We present the solution by Michel Bataille.

We shall use barycentric coordinates with reference to the triangle $\triangle ABC$, with side-lengths denoted a, b, c, where $a = BC$, $b = CA$, $c = AB$. We recall some basic facts and conventions. A point P is denoted by $(p_1 : p_2 : p_3)$ and, if we insist on normalized barycentric coordinates (also called *areal coordinates*) we have $p_1 + p_2 + p_3 = 1$; otherwise P is identical to $(kp_1 : kp_2 : kp_3)$, for any non-zero k. A line ℓ is given by the equation $ux + vy + wz = 0$, where u, v, and w are real constants, and the line at infinity is $x + y + z = 0$. A line through points $P = (p_1 : p_2 : p_3)$ and $Q = (q_1 : q_2 : q_3)$ may be denoted PQ and is given by the determinant

$$\begin{vmatrix} x & p_1 & q_1 \\ y & p_2 & q_2 \\ z & p_3 & q_3 \end{vmatrix} = 0. \tag{1}$$

The intersection point of the lines $ux + vy + wz = 0$ and $u'x + v'y + w'z = 0$ is given by

$$\left(\begin{vmatrix} v & w \\ v' & w' \end{vmatrix} : - \begin{vmatrix} u & w \\ u' & w' \end{vmatrix} : \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \right). \tag{2}$$

In our dealings with the orthocentre and various perpendiculars we shall make use of the special cases of Conway's notation: $S_A = (b^2 + c^2 - a^2)/2$; $S_B =$

$(c^2+a^2-b^2)/2$; and $S_C = (a^2+b^2-c^2)/2$, with which we can write the orthocentre as $H = (S_BS_C : S_CS_A : S_AS_B)$. In addition, if $(f : g : h)$ is the point at infinity of the line ℓ (*i.e.*, the intersection of ℓ with the line $x + y + z = 0$), then $(gS_B - hS_C : hS_C - fS_A : fS_A - gS_B)$ is the point at infinity of the lines perpendicular to ℓ.

Let $D = (\alpha : \beta : \gamma)$ where $\alpha+\beta+\gamma = 1$. Equation (1) gives the equation of the line DA as $\gamma y-\beta z = 0$, and Equation (2) yields its point at infinity, $(-(\beta+\gamma) : \beta : \gamma)$. Hence the point at infinity of the line HA' is $(\beta S_B-\gamma S_C : \gamma b^2+\beta S_A : -\beta c^2-\gamma S_A)$. Since $H = (S_BS_C : S_CS_A : S_AS_B)$, and again from (1), the equation of HA' is

$$\begin{vmatrix} x & S_BS_C & \beta S_B - \gamma S_C \\ y & S_CS_A & \gamma b^2 + \beta S_A \\ z & S_AS_B & -\beta c^2 - \gamma S_A \end{vmatrix} = 0.$$

The point $A' = (x_1 : y_1 : z_1)$ is on the intersection of the line $BC : x = 0$ and HA', above, so by Equation (2), $x_1 = 0$,

$$\begin{aligned} y_1 =& S_CS_A(\beta S_B - \gamma S_C) - S_BS_C(\gamma b^2 + \beta S_A), \text{ and} \\ z_1 =& S_BS_C(\beta c^2 + \gamma S_A) + S_AS_B(\beta S_B - \gamma S_C). \end{aligned}$$

We need a few identities to simplify A'. Let S denote twice the area of the reference triangle $\triangle ABC$. We have $S^2 = S_AS_B + S_BS_C + S_CS_A$, and notice that $S_A + S_B = c^2$; $S_B + S_C = a^2$; and, $S_C + S_A = b^2$. From this we derive $c^2S_C + S_AS_B = b^2S_B + S_CS_A = S^2$, and we now readily obtain $A' = (0 : -\gamma S_C : \beta S_B)$. In a similar way, $B' = (\gamma S_C : 0 : -\alpha S_A)$ and $C' = (-\beta S_B : \alpha S_A : 0)$. In order to finish this part of the solution, we show that the triangle determined by A', B', and C' has zero area by computing the determinant

$$\begin{vmatrix} 0 & \gamma S_C & -\beta S_B \\ -\gamma S_C & 0 & \alpha S_A \\ \beta S_B & -\alpha S_A & 0 \end{vmatrix} = 0.$$

Thus, the points A', B', C' are collinear completing the proof of the first part.

Conversely, suppose that A', B', and C' occur on the line $\ell : ux + vy + wz = 0$ (at the intersections with BC, CA, and AB, respectively), such that ℓ does not pass through A, B, C, or H. Then, $A' = (0 : -w : v)$; $B' = (w : 0 : -u)$; and, $C' = (-v : u : 0)$ (i.e., by Equation (2)) and the equation of the line HA' is $x(vS_CS_A + wS_AS_B) - yvS_BS_C - zwS_BS_C = 0$. Its point at infinity is $\big((v - w)S_BS_C : vS_AS_C + wb^2S_B : -wS_AS_B - vc^2S_C\big)$.

It follows that the point at infinity of the perpendiculars to HA' is $(\delta : -vS_C(c^2S_C+S_AS_B) : -wS_B(S_AS_C + b^2S_B))$ for some real number δ; we have again simplified this point using the aforementioned identities on S_A, S_B, and S_C, and we need not simplify δ because we shall not use it. We readily deduce that the equation of the perpendicular ℓ_a to HA' through A is $y(wS_B) - z(vS_C) = 0$ by employing Equation (1) with A and the above-derived point at infinity. Similarly, we find the equations of ℓ_b, perpendicular to HB' through B and ℓ_c, perpendicular to HC' through C: $x(wS_A) - z(uS_C) = 0$ and $x(vS_A) - y(uS_B) = 0$, respectively. It is easily verified that

$$\begin{vmatrix} 0 & wS_A & vS_A \\ wS_B & 0 & -uS_B \\ -vS_C & -uS_C & 0 \end{vmatrix} = 0,$$

and with Equation (2) we interpret this as showing that the intersection point of ℓ_b and ℓ_c satisfies the equation of ℓ_a. As such, ℓ_a, ℓ_b, and ℓ_c are concurrent or parallel. Their common point being $(uS_BS_C : vS_AS_C : wS_AS_B)$, they are actually concurrent because $uS_BS_C + vS_AS_C + wS_AS_B \neq 0$, since H is not on ℓ.

Editor's comments. We featured Bataille's solution because (normalized) barycentric coordinates are particularly well-suited for a large class of (Olympiad) triangle-geometry problems. There are several IMO-flavoured introductions to barycentric coordinates available online, for example at
`http://www.mit.edu/~evanchen/handouts/bary/bary-full.pdf` and
`http://math.fau.edu/Yiu/GeometryNotes020402.pdf`.

3879. *Proposed by Ovidiu Furdui.*

Calculate

$$\sum_{n=2}^{\infty} \left(n \ln\left(1 - \frac{1}{n}\right) + 1 + \frac{1}{2n} \right).$$

We received ten correct submissions. We present the solution of Joel Schlosberg.

Taking the logarithm of both sides of Stirling's formula $N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$ and rearranging, gives

$$\lim_{N\to\infty} \left(\ln N! - \left(N + \frac{1}{2}\right) \ln N + N \right) - \frac{1}{2}\ln(2\pi).$$

The Euler-Mascheroni constant γ is given by $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right)$. Denoting by S_N the Nth partial sum for $N \geq 2$, we have

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \left(n \ln\left(1 - \frac{1}{n}\right) + 1 + \frac{1}{2n} \right) \\ &= \sum_{n=2}^{N} \left[(n-1)\ln(n-1) - n\ln n\right] + \sum_{n=2}^{N} \ln(n-1) + N + \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \frac{3}{2} \\ &= -N\ln N + \ln N! - \ln N + N + \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n} - \frac{3}{2} \\ &= \left(\ln N! - \left(N + \frac{1}{2}\right)\ln N + N \right) + \frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{n} - \ln N \right) - \frac{3}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(n\ln\left(1-\frac{1}{n}\right)+1+\frac{1}{2n}\right)\\
&=\left[\left(\ln N!-\left(N+\frac{1}{2}\right)\ln N+N\right)+\frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln N\right)-\frac{3}{2}\right]\\
&=\frac{\ln(2\pi)+\gamma-3}{2}
\end{aligned}$$

Editor's Comments. All of the submitted solutions were similar, invoking Stirling's formula to estimate the partial sums. Bataille remarked both that "the result can readily be found using Wolfram Alpha" and that "the problem is equivalent to solved Problem 3.23 in the proposer's book: Ovidiu Furdui, *Limits Series, and Fractional Part Integrals*, Spring 2013."

3880. *Proposed by Iliya Bluskov.*

Let $B=[b_{ij}]$ be an $n\times k$ matrix with entries in the set of residues modulo v, such that the k entries in each row of B are pairwise distinct. Form the $n\times[k(k-1)]$ matrix of differences $D=[d_{ip}]$ where $d_{ip}=b_{ij}-b_{is} \pmod v$, $j\neq s$, $1\leq j,s\leq k$ and $1\leq i\leq n$. Let O_q, $q=1,2,\ldots,v-1$ be the number of occurrences of the residue q in the matrix D. Show that the sum

$$\sum_{q=1}^{v-1} O_q$$

does not depend on v.

We received three correct submissions. We present the solution by Oliver Geupel.

Consider the sets

$$A_q=\{(i,p):\ 1\leq i\leq n,\ 1\leq p\leq (k-1)k,\ d_{ip}=q\},\qquad 1\leq q\leq v-1.$$

The sets A_q are pairwise disjoint and $|A_q|=O_q$. Hence,

$$\sum_{q=1}^{v-1} O_q=\sum_{q=1}^{v-1}|A_q|=\left|\bigcup_{q=1}^{v-1}A_q\right|=n(k-1)k,$$

which does not depend on the modulus v.

Editor's comments. Note that the matrix of differences is not unique, and that this does not matter for the solution. The solution works because the matrix of differences has no entries of residue 0 (mod v), and so the union of the sets A_q is just the collection of indices of the matrix of differences.

SOLUTIONS

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

George Apostolopoulos, Messolonghi, Greece: 3954, 3959
Michel Bataille, Rouen, France: 3952
Marcel Chiriţă, Bucharest, Romainia : 3951
Ovidiu Furdui, Campia Turzii, Cluj, Romania: 3955
Billy Jin, University of Waterloo, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON : 3958
Dragoljub Milošević, Gornji Milanovac, Serbia: 3953, 3957
Cristinel Mortici and Leonard Giugiuc, Romania : 3956
S. Viswanathan, Central India Research Institute Maharashtra, India : 3960

Solvers - individuals

Arkady Alt, San Jose, CA, USA: 3872, 3873, 3879
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : 3872
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina: CC86, CC87, 3872, 3873, 3876, 3878
Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan : 3872
Michel Bataille, Rouen, France: 3871, 3872, 3875, 3876, **3878**, 3879
Roy Barbara, Lebanese University, Fanar, Lebanon: **3875**
Ricardo Barroso Campos, University of Seville, Seville, Spain: 3872
Iliya Bluskov, University of Northern British Columbia, BC: 3880
Matei Coiculescu, East Lyme High School, East Lyme, CT, USA: **CC87**
Roberto de la Cruz, Centre de Recerca Matemàtica, Barcelona : 3871
Marian Dincă, Bucharest, Romania: 3877
Ovidiu Furdui, Campia Turzii, Cluj, Romania: 3879
Hannes Geupel, Max-Ernst-Gymnasium, Brühl, Germany : CC87 (original version)
Oliver Geupel, Brühl, NRW, Germany: **OC136, OC137, OC138, OC139, OC140**, 3872, **3874**, 3876, **3877**, 3879, **3880**
John G. Heuver, Grande Prairie, AB: **CC86**, CC87, 3872
Václav Konečný, Big Rapids, MI, USA: **3871**, 3872
Dimitrios Koukakis, Kilkis, Greece: 3879
Kee-Wai Lau, Hong Kong, China: 3871, 3872, 3879
Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy: 3874
Salem Malikić, student, Simon Fraser University, Burnaby, BC: 3872
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA: CC86, CC87, 3871
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Solvers - collaborations