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CONTENTS

A PRIME-GENERATING TRINITY

CLAYTON W. DODGE
University of Maine at Orono

Steven R. Conrad's comment in [1] prompts me to indite some facts I have never, until recently,[1] seen in print, but which ought to be better known.

There are three commonly known "prime-generating" polynomials:

$$f(x) = x^2 - x + 41, \qquad \text{for } x = 0, 1, \ldots, 40;$$
$$g(x) = x^2 + x + 41, \qquad \text{for } x = 0, 1, \ldots, 39;$$

and, appearing to be vastly more powerful,

$$h(x) = x^2 - 79x + 1601, \text{ for } x = 0, 1, \ldots, 79.$$

Curiously, they are not independent of one another.

First notice that

$$f(1-x) = (1-x)^2 - (1-x) + 41 = f(x),$$

so we have

$$f(0) = f(1),\ f(-1) = f(2), \ldots,\ f(-39) = f(40).$$

Hence $f(x)$ actually generates 40 distinct primes for $x = -39, -38, \ldots, 40$, an interval of 80 consecutive integers.

Next we see that

$$f(x+1) = (x+1)^2 - (x+1) + 41 = x^2 + x + 41 = g(x),$$

so $g(x)$ generates the same 40 primes for the 80 consecutive integers $x = -40, -39, \ldots, 39$. Thus $f(x)$ and $g(x)$ are essentially the same formula.

Lastly, consider that

$$f(x-39) = (x-39)^2 - (x-39) + 41 = x^2 - 79x + 1601 = h(x).$$

Thus $h(x)$ is not a new, distinct formula, but merely a translation of $f(x)$, so $h(x)$ generates only the same 40 primes for $x = 0, 1, \ldots, 79$. Indeed, we do not have three different formulas $f(x)$, $g(x)$, and a vastly more powerful $h(x)$, but merely three forms of the same formula generating the same 40 primes for some sets of 80 consecutive integers.

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[1] In [2, p. 29], Honsberger discusses the function $f(x) = x^2 + x + 41$, then goes on to say: "Equivalently, the function $f(x-40) = x^2 - 79x + 1601 \ldots$." This paper expands on his word "equivalently."

It is easy to find polynomial formulas $p(x)$ that give prime values for as many consecutive integral values of x as desired. One technique is simply to require a polynomial of degree $n-1$ to take on n given prime values for $x = 1, 2, \ldots, n$. For example,

$$p(x) = (x-2)(x-3) - 3(x-1)(x-3) + \frac{5}{2}(x-1)(x-2)$$

is a quadratic having $p(1) = 2$, $p(2) = 3$, and $p(3) = 5$. Such a process is, of course, cumbersome and not especially interesting.

One would really like a formula that always gives primes. A rather trivial example is[2]

$$q(x) = 5.$$

Since you insist, quite reasonably, that it not always give the *same* prime, let us take

$$r(x) = 4 + (-1)^x.$$

An example that gives four distinct primes for integral x is

$$s(x) = 15 + 2\sin\frac{\pi x}{2} + 4\cos\frac{\pi x}{2}.$$

Other similar formulas are readily constructed.

Since you feel the above formulas are not really satisfactory, yet another one is offered. In [2, p. 33] Honsberger states:[3]

$$\text{"}f(x,y) = \frac{y-1}{2}[|B^2-1| - (B^2-1)] + 2,$$

where $B = x(y+1) - (y!+1)$, x and y natural numbers, generates only prime numbers, every prime number, and each odd prime number exactly once."

Because B is an integer, either $B^2 = 0$ or $B^2 \geq 1$. Now Wilson's Theorem states that a positive integer $p > 1$ is a prime if and only if p divides $(p-1)! + 1$. Thus $B = 0$ if and only if $y+1$ is prime and x is the quotient when $y!+1$ is divided by $y+1$, in which case

---

[2]In [2, p. 30] Honsberger states: "It is not difficult to show that no polynomial can yield a prime number for *every* $x = 0, 1, 2, \ldots$." One must insert "of positive degree" after "polynomial," as this example shows.

[3]Honsberger gives this formula without attribution. Queried about its source, he replied that he did not know it at the time his book was published, but that since then he has had reason to believe the formula may be due to Professor John D. Dixon, Carleton University, Ottawa. I have checked this matter with Professor Dixon personally. He confirms that he did indeed discover this formula which, to the best of his knowledge, appears in print for the first time in [2]. (Editor)

$$f(x,y) = \frac{y-1}{2}[|-1| - (-1)] + 2 = y+1.$$

In every other case, $B \neq 0$, so $B^2 - 1 \geq 0$ and we have

$$f(x,y) = \frac{y-1}{2}[0] + 2 = 2.$$

Thus Honsberger's claim is verified.

To test the practicality of Dixon's formula, you might wish to attempt to find out whether, say, 1007 is prime. Essentially, it is a more complicated form of Wilson's Theorem, nice, but not practical because of the large numbers involved.

Your disappointment is again noted. The prime-generating formula we seek, then, is a simple formula that will give at least one "new" prime (not any prime we already have), and the closest we seem to have been able to come to such a formula is Euclid's 2000-year-old

$$t(n) = p_1 p_2 \cdots p_n + 1,$$

where $p_1, p_2, \ldots, p_n$ are distinct primes. Then $t(n)$ is either a new prime or it contains only new prime factors, since clearly $t(n)$ is not divisible by any of p_1, $p_2, \ldots, p_n$. Taking $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$, we find that

$$t(1) = 3, \quad t(2) = 7, \quad t(3) = 31, \quad t(4) = 211, \quad t(5) = 2311$$

are all new primes, whereas

$$t(6) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 59 \cdot 509$$

is not prime, but gives the two new primes 59 and 509.

REFERENCES

1. Steven R. Conrad, Comment IV to Problem 142, EUREKA, Vol. 2 (1976), p. 177.

2. Ross Honsberger, *Mathematical Gems II*, The Mathematical Association of America, 1976, pp. 29-34.

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A SIMPLE PROOF OF THE REMAINDER THEOREM

$$\begin{array}{r@{}l} & \underline{\;f\qquad\qquad} \\ x-r) & f(x) \\ & \underline{f(x)-f(r)} \\ & \qquad\quad f(r) \end{array}$$

DAVID L. SILVERMAN

* * *

Il y a 260 *ans*

JEAN LE ROND D'ALEMBERT EST DÉCOUVERT SUR LES MARCHES D'UNE ÉGLISE[1]

MARIE-HÉLÈNE BOURQUIN

Dans la journée du 16 novembre 1717, vraisemblablement avant la tombée de la nuit, car on tenait à ce que la découverte en soit faite rapidement, un enfant nouveau-né, reposant dans une boîte de sapin, était déposé discrètement sur les marches de la petite église Saint Jean-le-Rond, accotée à la tour nord de Notre-Dame de Paris, dont elle avait été autrefois le baptistère. Ce lieu avait l'avantage, pour la responable de l'abandon, d'être situé à proximité immédiate de l'Hospice des Enfants trouvés, où l'enfant fut en effet porté, aussitôt découvert. Et c'est bien là la seule attention que cette étrange mère eut jamais pour son enfant.

Celle-ci, qui avait alors trente-cinq ans, était née à Grenoble, dans une famille estimée de la noblesse de robe du Dauphiné, les Guérin de Tencin. Cédant à l'autorité et aux menaces de son père, elle avait en 1698 prononcé ses vœux, qu'elle avait dès le lendemain rétractés devant notaire, ce qui lui permit cinq ans après la mort de ce père redouté, et à l'insu de sa famille, de faire parvenir une requête en bonne et due forme au pape Clément XI, qui en novembre 1712, estimant après enquête que son consentement avait été forcé, la relève de ses vœux.

Il apparaît toutefois que Claudine-Alexandrine de Tencin n'avait pas plus la vocation maternelle que religieuse puisqu'au lieu de garder près d'elle l'enfant du péché, ou tout au moins de le faire élever secrètement à la campagne, ce qui n'aurait en rien gêné sa nouvelle vie mondaine, elle préféra donner l'ordre de l'abandonner sur les marches d'une église.

Il est d'ailleurs curieux de noter que le procès-verbal d'admission du futur d'Alembert, à l'Hospice des Enfants-Trouvés, porte la signature du commissaire au Châtelet, Nicolas Delamare, qui écrivait alors un célèbre "Traité de la Police," encore fort apprécié et fort précieux de nos jours, et qui mourut en 1723, sans pouvoir deviner que cet enfant de cinq ans qui lui devait son état civil, allait devenir, mais sous un autre nom, un des personnages les plus illustres de son temps.

Ayant été baptisé sous le nom de Jean Le Rond, le bébé fut aussitôt envoyé à Crémery, petit village de la Somme, dans l'arrondissement de Montdidier, à une

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[1]Reproduit, avec la permission de *Historama*, de *Histoire Pour Tous,* No. 91 (novembre 1967), pp. 12-13.

centaine de kilomètres de Paris, ce qui, vu la saison et les moyens de transport alors accessibles à une pauvre nourrice picarde, dut mettre à rude épreuve la frêle santé de l'infortuné nouveau-né.

Cependant si jamais sa mère ne se soucia de lui, son père, Louis Camus-Destouches, lieutenant-général d'artillerie, plusieurs fois blessé au combat, dit encore Canon ou Petit-Canon, sans doute pour le distinguer de son frère Michel qui servait dans l'infanterie, de retour d'une mission aux Antilles qui l'avait tenu plusieurs mois éloigné de France, réclama de ses nouvelles. Informé de l'abandon, il fit en sorte de retrouver son fils qui selon les registres de l'Hospice, fut remis au médecin Molin, le 1er janvier 1718, pour le compte de sa famille. On peut alors vraisemblablement avancer que ce M. Molin, ou Dumoulin, "premier médecin de Paris pour la réputation," (il soignera tour à tour Louis XV et le Dauphin et mourra en 1755 à quatre-vingt-dix ans), ayant accouché la mère, et peut-être de ce fait complice malgré lui de l'abandon, fut heureux de s'entremettre pour la restitution.

Louis Destouches, qui pouvait s'honorer d'avoir été l'ami de Fénelon, lequel regrettait qu'il ne songe point à son salut, mais lui reconnaissait "beaucoup de grâce dans l'esprit et de noble dans le cœur," confia alors son fils à une certaine Madame Rousseau, femme d'un vitrier parisien, qui lui servira si bien de mère, que Jean Le Rond, alors même qu'il sera devenu d'Alembert, logera chez elle, en garçon, dans une petite chambre mansardée, jusqu'à l'âge de quarante-huit ans.

Malheureusement ce père, relativement attentionné, mourra le 11 mars 1726 à l'âge de cinquante-huit ans, ne léguant à son fils qu'une modeste rente d'un montant de 1,200 livres. Il est vrai aussi qu'il ne pouvait deviner que cet enfant de huit ans et demi, élevé par une pauvre vitrière, allait comme philosophe et littérateur lier son nom à l'Encyclopédie, entrer dans l'histoire des mathématiques par ses travaux sur les nombres imaginaires, la théorie des équations et l'analyse infinitésimale, enfin témoignerde l'intérêt pour l'acoustique, l'astronomie et la mécanique appliquée.

Cependant, grâce à la famille Destouches, fidèle au petit abandonné, il fit de bonnes études au Collège Mazarin ou des Quatre-Nations (aujourd'hui palais de l'Institut), bien que n'y fussent admis en principe que les gentilshommes pouvant prouver quatre degrés de noblesse. C'est sans doute pour cette raison qu'il fut inscrit sur les livres de l'école sous le nom de Daremberg, nom précédemment orthographié d'Arenbert dans le testament de son père, que lui-même modifiera en d'Alembert, et sur l'origine duquel personne à ce jour n'a encore pu avancer la moindre hypothèse valable.

Sortant de l'école, il fit son droit et fut reçu avocat en 1738; mais alors que l'exercice d'une profession tant soit peu lucrative aurait dû être le but de ce jeune homme sans nom et sans fortune, il préféra vivre dans une mansarde, et se livrer en toute liberté à des spéculations purement intellectuelles. Il eut sans doute raison, puisqu'en février 1746, il était admis à l'Académie des Sciences, avec le rang d'*associé géomètre* et 500 livres de pension. La même année ses *"Réflexions sur la cause générale des vents,"* lui ayant valu le prix proposé sur ce sujet par l'Académie de Berlin, il en adressait un exemplaire à Voltaire qui lui répondait le 13 décembre: "Monsieur, du temps de Voiture, on vous aurait dit que vous n'avez pas le vent contraire en allant à la gloire... "

C'est ainsi qu'en 1752, le vent de la gloire ayant continué de souffler, Frédéric II offrit à d'Alembert la présidence de l'Académie Royale de Berlin, à la place de Maupertuis alors en disgrâce. Mais d'Alembert déclina l'invitation, bien que sa fortune, dira-t-il, "soit au-dessous du médiocre." Bon prince, Frédéric, loin de s'offenser du refus, lui accordera en retour une pension annuelle de 1,200 livres.

C'est que ce vieux garçon, qui semble n'avoir eu pour les femmes que peu d'attirance physique, et qui ne rencontrera son unique amour qu'à trente-sept ans, goûtait avant tout le repos, la liberté et la tranquillité. Quand en 1759 l'Encyclopédie se voit supprimer son privilège, d'Alembert se retire prudemment de l'entreprise, préférant encore sa mansarde à quelque séjour à Vincennes ou à la Bastille. Et alors qu'en 1766, sa passion pour Mlle de Lespinasse le pousse à déménager pour la première fois de sa vie, afin de loger dans l'appartement voisin de celui de son amie, il écrit à Voltaire: "Mon Dieu! que deviendrais-je avec une femme et des enfants?"

Il tenait pour principe que le seul parti à prendre pour un philosophe, "est de ne dire que le quart de la vérité, s'il y a trop de danger à la dire tout entière," estimant avec beaucoup de bon sens, de sagesse peut-être même, "que ce quart sera toujours dit et fructifiera sans nuire à l'auteur." "Il est hardi, disait de lui Voltaire, il n'est point téméraire." Et c'est ainsi que Jean Le Rond d'Alembert, enfant trouvé, athée, pourfendeur des Jésuites, mourut le 29 octobre 1783, secrétaire perpétuel de l'Académie française, sous le règne du pieux roi Louis XVI.

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MAMA-THEMATICS II

8. Mother to Archimedes: "Mother didn't know it was lost, but now that you have found it why all the screaming?"

CHARLES W. TRIGG

REPORT ON THE EUREKA VALENTINE PROBLEM

In the February 1977 issue of EUREKA [1977: 60], Leon Bankoff offered a prize problem in the charming guise of a Valentine greeting. His calligraphic report on the solutions he received is given below. This is followed on the next page by Dr. Bankoff's choice, an article in which Charles W. Trigg gives ten (count 'em) solutions to the problem.

Excellent demonstrations of these relationships were offered by Clayton W. Dodge (Maine), Murray S. Klamkin (Edmonton, Canada), Dr. Sahib Ram Mandan (India), R. Robinson Rowe (California and Michigan) and Charles W. Trigg (California). Because of the incomparable elegance of all the solutions submitted, each participant will receive the magnificent prize of a ONE-YEAR SUBSCRIPTION TO EUREKA.

The second equation is merely a trivial and poetic way of expressing mutual and reciprocal dependence — a characteristic of so many comic Valentines. But the first equation arises from a sincere and serious devotion to mathematics by such giants as Archimedes, Apollonius, Pappus, Vieta, Descartes, Newton and Steiner, to name a few. It is one of the many ways of expressing the fundamental relation, $\rho = rr_1r_2/(r_1^2 + r_1r_2 + r_2^2)$. Consequently all the solvers focused their attention on various derivations of an expression for ρ.

The indefatigable Charles W. Trigg, top nominee for the Guiness Book of World Records because of his long and distinguished career as a problemist and mathematical author, could not resist submitting ten different solutions. These included methods used by the other solvers with the exception of Murray S. Klamkin, who based his demonstration on Descartes' Formula, as given in Coxeter's Introduction to Geometry, page 14, and on the further simplification by Soddy:

$$\frac{1}{\rho} + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} .$$

Leon Bankoff

HOW DO I LOVE THEE? LET ME COUNT THE WAYS.

CHARLES W. TRIGG
Professor Emeritus, Los Angeles City College

Leon Bankoff [1] has asked for elegant demonstrations of the verity of the equations at the end of the following problem, which is reproduced from EUREKA [1977: 60]:

EUREKA ~ We Love You

An Archimedean Valentine Greeting

Choose a point C anywhere on line AB and construct semicircles on the same side of diameters $AC = 2r_1$, $CB = 2r_2$, and $AB = 2r$. Reflect the outer semicircle in AB, as in the figure. We now have a shaded HEART-SHAPED area and an unshaded curvilinear triangle known as THE SHOEMAKER'S KNIFE OF ARCHIMEDES. Let ρ denote the radius of the circle inscribed in the KNIFE. then ~

$$\frac{\rho}{r} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2} = \frac{\text{Area of the SHOEMAKER'S KNIFE}}{\text{Area of the HEART}}$$

Let the centers of the circles with radii r, r_1, r_2, and ρ be O, D, F, and E, respectively. In Figure 1 on the next page, $\angle EDF = \phi$, $DE = r_1 + \rho$, $FE = r_2 + \rho$, $OE = r - \rho$, $DF = r_1 + r_2 = r$, $DO = r - r_1 = r_2$, and $OF = r - r_2 = r_1$. It follows that

$$2r_1r_2 = r^2 - r_1^2 - r_2^2.$$

In the following ten proofs, each of which has its own characteristics of elegance, selected expressions are substituted for their equivalents when appropriate.

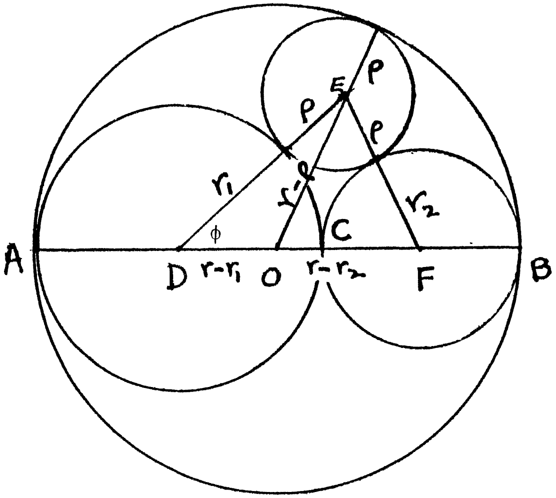

Figure 1

Method I. An established formula. It has been variously shown [2] that

$$\rho = \frac{r_1 r_2 (r_1 + r_2)}{r_1^2 + r_1 r_2 + r_2^2}.$$

Immediately,

$$\frac{\rho}{r} = \frac{2r_1 r_2}{2r_1^2 + 2r_1 r_2 + 2r_2^2} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2} = \frac{(\pi/2)(r^2 - r_1^2 - r_2^2)}{(\pi/2)(r^2 + r_1^2 + r_2^2)}$$

$$= \frac{\text{area of the SHOEMAKER'S KNIFE}}{\text{area of the HEART}}.$$

Thus the KNIFE goes into the HEART r/ρ times. Quite r/ρρing, old chappie!

Method II. Areas of triangles. In ΔDOE the semiperimeter $s_1 = r = s_2$, the semiperimeter of ΔFOE. Then by Heron's formula:

$$\text{area of } \Delta\text{DOE} = \sqrt{r(r-r_1-\rho)\rho r_1} = \sqrt{r(r_2-\rho)\rho r_1}$$

and

$$\text{area of } \Delta\text{FOE} = \sqrt{r(r-r_2-\rho)\rho r_2} = \sqrt{r(r_1-\rho)\rho r_2}.$$

Now Δs DOE and FOE have the same altitude, so their areas are to each other as their bases, that is,

$$\frac{\sqrt{r(r_2-\rho)\rho r_1}}{\sqrt{r(r_1-\rho)\rho r_2}} = \frac{r-r_1}{r-r_2} = \frac{r_2}{r_1}.$$

Squaring and simplifying:

$$(r_2-\rho)r_1^3 = (r_1-\rho)r_2^3,$$

$$r_1r_2(r_1^2 - r_2^2) = \rho(r_1^3 - r_2^3),$$

$$\rho(r_1^2 + r_1r_2 + r_2^2) = r_1r_2(r_1+r_2),$$

as before.

Method III. Equal areas. The semiperimeter of Δ DEF is $r+\rho$, so by Heron's formula its area is

$$\sqrt{(r+\rho)\rho(r-r_1)(r-r_2)} \quad \text{or} \quad \sqrt{r_1r_2\rho(r+\rho)}.$$

Since Δ DEF $= \Delta$ DEO $+ \Delta$ OEF, we get from method II

$$\sqrt{r_1r_2\rho(r+\rho)} = \sqrt{rr_1\rho(r_2-\rho)} + \sqrt{rr_2\rho(r_1-\rho)}.$$

Eliminating radicals and simplifying:

$$(r_1r_2-r^2)^2\rho^2 - 2(r_1r_2-r^2)r_1r_2r\rho - 3r_1^2r_2^2r^2 = 0,$$

$$[(r_1r_2-r^2)\rho - 3r_1r_2r][(r_1r_2-r^2)\rho + r_1r_2r] = 0.$$

Thus

$$\frac{\rho}{r} = \frac{3r_1r_2}{r_1r_2-r^2},$$

which is negative and extraneous, or

$$\frac{\rho}{r} = \frac{2r_1r_2}{2r^2-2r_1r_2} = \frac{r^2-r_1^2-r_2^2}{r^2+r_1^2+r_2^2},$$

as before.

Method IV. The Apollonian circles. It has been shown [3, 4] that if three circles with radii r_1, r_2, ρ are tangent externally, then the circle that touches and encompasses these three circles has the radius

$$r = \frac{r_1r_2\rho}{2\sqrt{r_1r_2\rho(r_1+r_2+\rho)} - (r_1r_2+r_1\rho+r_2\rho)}.$$

This situation exists when the semicircles in the valentine are reflected about AB (see Figure 1). It follows that

$$2r\sqrt{r_1r_2\rho(r+\rho)} = r_1r_2\rho + r(r_1r_2 + r\rho).$$

Squaring and collecting terms:

$$(r^2 - r_1r_2)^2\rho^2 - 2(r^2 - r_1r_2)r_1r_2r\rho + r_1^2r_2^2r^2 = 0,$$

$$[(r^2 - r_1r_2)\rho - r_1r_2r]^2 = 0,$$

whereupon

$$\frac{\rho}{r} = \frac{2r_1r_2}{2r^2 - 2r_1r_2},$$

as before.

Method V. Stewart's Theorem applied to Δ DEF and cevian OE gives

$$(r-\rho)^2r = (r_1+\rho)^2r_1 + (r_2+\rho)^2r_2 - r_1r_2r,$$

$$r^3 - 2r^2\rho + \rho^2r = r_1^3 + r_2^3 + 2r_1^2\rho + 2r_2^2\rho + \rho^2(r_1+r_2) - r_1r_2r,$$

$$r^3 - (r_1+r_2)(r_1^2 - r_1r_2 + r_2^2) + r_1r_2r = 2\rho(r^2 + r_1^2 + r_2^2),$$

$$r(r^2 - r_1^2 - r_2^2) + 2r_1r_2r = 2\rho(r^2 + r_1^2 + r_2^2),$$

$$2r(r^2 - r_1^2 - r_2^2) = 2\rho(r^2 + r_1^2 + r_2^2),$$

and finally

$$\frac{\rho}{r} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2},$$

as before.

Method VI. The law of cosines applied to Δs EDO and EDF, respectively, gives

$$(r-\rho)^2 = (r_1+\rho)^2 + (r-r_1)^2 - 2(r_1+\rho)(r-r_1)\cos\phi$$

and

$$(r_2+\rho)^2 = (r_1+\rho)^2 + r^2 - 2r(r_1+\rho)\cos\phi.$$

Eliminating $\cos\phi$:

$$r(r-\rho)^2 - (r-r_1)(r_2+\rho)^2 = r(r_1+\rho)^2 - (r-r_1)(r_1+\rho)^2 + r(r-r_1)^2 - r^2(r-r_1).$$

Expanding and collecting terms:

$$2\rho[r^2 + r_1^2 + r_2(r-r_1)] = r^3 - rr_1^2 - rr_2^2 + r^2r_1 + r_1(r_2^2 - r_1^2),$$

$$2\rho(r^2 + r_1^2 + r_2^2) = r(r^2 - r_1^2 - r_2^2) + r^2(r-r_2) + r_1(r_2+r_1)(r_2-r_1),$$

$$2\rho(r^2+r_1^2+r_2^2) = r(r^2-r_1^2-r_2^2)+r^3-rr_1^2-rr_2(r-r_1),$$

and finally

$$2\rho(r^2+r_1^2+r_2^2) = 2r(r^2-r_1^2-r_2^2),$$

as before.

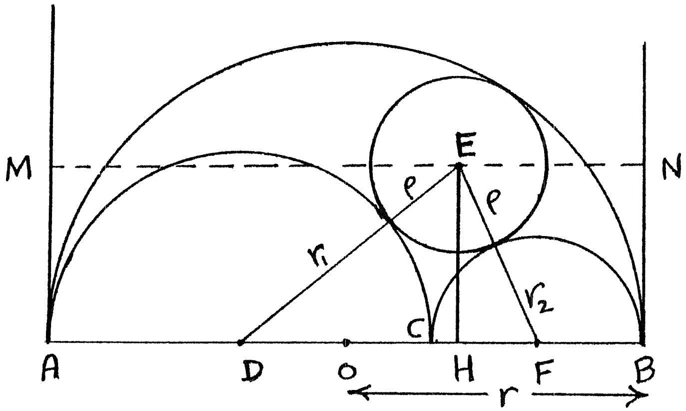

Figure 2

Method VII. A theorem of Pappus [5] states that in the arbelos of Figure 2 the perpendicular EH from E to AB equals 2ρ. Hence the area of Δ DEF is

$$\frac{2\rho r}{2} = \sqrt{r_1 r_2 \rho(r+\rho)}$$

from Heron's formula (see Method III). Squaring and collecting terms:

$$\rho(2r^2-2r_1r_2) = r(2r_1r_2)$$

and

$$\rho(r^2+r_1^2+r_2^2) = r(r^2-r_1^2-r_2^2),$$

as before.

Method VIII. A theorem of Casey [6] states that *if a variable circle touch two fixed circles, its radius has a constant ratio to the perpendicular from its centre on the radical axis.* In Figure 2, ME + EN = AB = $2r$.

Consider circles (O) and (D) to be fixed and circles (E) and (F) to be two positions of a variable circle; then

$$\frac{\rho}{\text{ME}} = \frac{r_2}{\text{AF}}, \qquad \text{so} \qquad \text{ME} = \frac{\rho(2r-r_2)}{r_2}.$$

Now consider circles (O) and (F) to be fixed and circles (D) and (E) to be two positions of a variable circle; then

$$\frac{\rho}{EN} = \frac{r_1}{DB}, \qquad \text{so} \qquad EN = \frac{\rho(2r - r_1)}{r_1}.$$

It follows that

$$ME + EN = 2r = \rho\left[\frac{2r - r_2}{r_2} + \frac{2r - r_1}{r_1}\right],$$

$$r(2r_1r_2) = \rho[2r(r_1 + r_2) - 2r_1r_2]$$

and

$$r(r^2 - r_1^2 - r_2^2) = \rho(r^2 + r_1^2 + r_2^2),$$

as before.

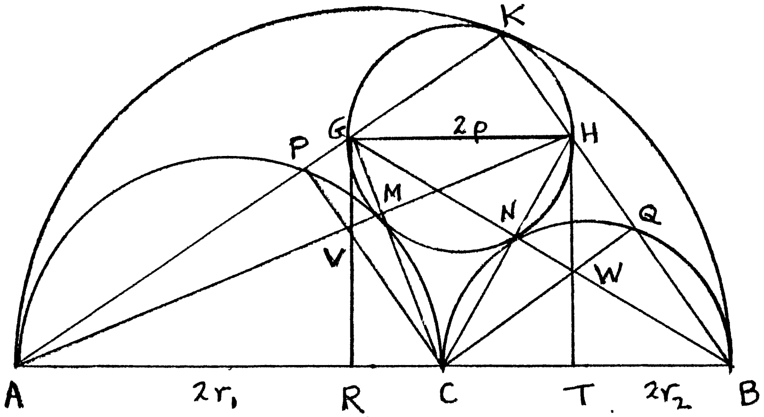

Figure 3

Method IX. Archimedes' treatment [7, pp. 307-308] of the configuration in Figure 3 is based upon his proposition [7, p. 301]: *If two circles touch at* A, *and if* BD *and* EF *be parallel diameters in them,* ADF *is a straight line.*

Since GH is parallel to AB, then AGK, KHB, HMA, GMC, GNB, and HNC are straight lines. Since they are inscribed in semicircles, angles AKB, APC, AMC, CNB, and CQB are right angles. Hence in Δs AGC and CHB the points V and W are orthocenters, so GR and HT are altitudes and are parallel. Thus RT = GH = 2ρ.

Furthermore, PC and KB are parallel, as are AK and CQ. It follows that

$$\frac{AR}{RT} = \frac{AV}{VH} = \frac{AC}{CB}, \qquad \text{so} \qquad AR = 2\rho \cdot \frac{2r_1}{2r_2}$$

and

$$\frac{TB}{RT} = \frac{WB}{GW} = \frac{CB}{AC}, \qquad \text{so} \qquad TB = 2\rho \cdot \frac{2r_2}{2r_1}.$$

Then

$$2r = \mathrm{AR} + \mathrm{RT} + \mathrm{TB} = 2\rho\left(\frac{r_1}{r_2} + 1 + \frac{r_2}{r_1}\right)$$

and

$$\frac{\rho}{r} = \frac{2r_1r_2}{2r_1^2 + 2r_1r_2 + 2r_2^2} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2},$$

as before.

Method X. Analytic geometry provides another procedure. In Figure 1, with AB as the x-axis and the origin at C, the centers of the circles with radii r, r_1, r_2, and ρ are

$$\mathrm{O}(r_2 - r_1,\ 0),\quad \mathrm{D}(-r_1,\ 0),\quad \mathrm{F}(r_2,\ 0),\quad \mathrm{E}(x,\ y),$$

respectively. Then by the distance formula:

$$|\mathrm{DE}|^2 = (r_1 + \rho)^2 = (x + r_1)^2 + y^2,$$

$$|\mathrm{EF}|^2 = (r_2 + \rho)^2 = (x - r_2)^2 + y^2,$$

$$|\mathrm{OE}|^2 = (r - \rho)^2 = (x + r_1 - r_2)^2 + y^2.$$

These three equations expand and simplify to give:

$$x^2 + y^2 + 2r_1x - 2r_1\rho - \rho^2 = 0, \tag{1}$$

$$x^2 + y^2 - 2r_2x - 2r_2\rho - \rho^2 = 0, \tag{2}$$

$$x^2 + y^2 + 2(r_1 - r_2)x - 4r_1r_2 + 2r\rho - \rho^2 = 0. \tag{3}$$

Subtracting (3) from (1):

$$2r_2x + 4r_1r_2 - 2(r + r_1)\rho = 0$$

and (2) from (3):

$$2r_1x - 4r_1r_2 + 2(r + r_2)\rho = 0.$$

Eliminating x:

$$4r_1r_2(r_1 + r_2) - 2(r_1r + r_1^2 + r_2r + r_2^2)\rho = 0,$$

whereupon

$$\frac{\rho}{r} = \frac{2r_1r_2}{r^2 + r_1^2 + r_2^2},$$

as before.

Note. When $r_1 = r_2 = \frac{r}{2}$, ρ attains its maximum value, $\frac{r}{3}$.

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1. Leon Bankoff, An Archimedean Valentine Greeting, EUREKA, 3 (1977), 60.

2. Howard Eves, A. Sisk, M.S. Klamkin, A.L. Epstein, and Leon Bankoff, Circles Inscribed in the Arbelos (Solutions of Problem 127), *Mathematics Magazine*, 26 (1952), 111-115.

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4. Charles W. Trigg, The Apollonius Problem, *The Fibonacci Quarterly*, 12 (1974), 326.

5. Roger A. Johnson, *Modern Geometry*, Houghton Mifflin, 1929, p. 117 (165h).

6. John Casey, *A Sequel to the Elements of Euclid*, Longman's, Green & Co., London, 1884, p. 118.

7. T.L. Heath, *The Works of Archimedes*, Dover, New York, circa 1950.

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LETTERS TO THE EDITOR

Dear Editor:

Undoubtedly the limerick to which you refer in [1977: 169] is:

There was a young girl of Topeka
Who went out with a bookkeepah.
He said, "Here, look.
I can't balance this book."
So she tried it and shouted, "Eureka!"

After all, what else could it be?

On another topic, it is easy to see that Edith Orr's blank mind, which produces only blank verse [1977: 129], is also a one-track mind (see [1977: 151]). It is reported that Ms. Orr is not generally rated in millihelens (see [1977: 129]), but rather in *milliclocks*, according to those who have seen her, a fact which probably accounts for the tone of her thoughts. (Women can also be rated from 1 to 1000 milliclocks, the idea being that a face rating 1000 milliclocks would be enough to stop Big Ben.)

CLAYTON W. DODGE,
University of Maine at Orono.

Dear Editor:

It seems to me that you could have shown a bit more discrimination in the letters you publish. Since you did not, I hope you will allow me space to respond to Professor Dodge's snide letter.

I will not comment on his "limerick," other than to say that he does not seem to know an amphibrach from a pibroch.

After a reassuring glance in the mirror, which confirmed anew that clocks have nothing to fear from me, I was moved to pen the following:

FROM BLANK TO BLANKETY-BLANK

This snotty professor from Cow College,[1]
His breath he should save for his porridge
    To cool;
Not for venting his spleen in his letters
    From school
About those he perceives are his betters.
A wish that he will make a find of
    Rocks in his socks
    And shatterproof clocks
(Which he'll need) is what comes to the _____ mind of

EDITH ORR

Editor's comment.

Hell hath no fury... . It is now abundantly clear that Ms. Orr, if sufficiently provoked, is indeed capable of writing rhyming verse, and she does it with an acid-dipped pen.

Let us now prudently retreat to a safe distance and see if anything of mathematical significance can be salvaged from this encounter. It is clear that *milliclocks* are nothing but negative *millihelens*; hence the closed interval [-1000, 1000], scaled in millihelens, can serve as the maximum range of all female beauty functions. This could form the basis of a detailed mathematical analysis of female beauty, a field hitherto reserved for hucksters of depilatory devices and vanishing creams.

Dodge's poorly constructed *Topeka* limerick will not induce me to publish *in extenso* in these pages the X-rated one I referred to in [1977: 169]. He can look it up himself in the reference I gave or, if he prefers, I can send him the full text upon receipt of proof of age (see footnote in [1977: 120]) and of a stamped, self-addressed plain brown envelope.

*　　　*　　　*

Our Own Mother Goose I

    Little Miss Muffet
    Sat on a tuffet
While computing surds her own way.
Along there came an immy one:
"Go 'way, square root of -1,
You scare me more than spiders do!"

CLAYTON W. DODGE

[1]*Cow College* is a name familiarly given to the University of Maine at Orono. Whether this name is bestowed affectionately or not, I am unable to say; but it at least indicates a heightened awareness of something or other in the ambient atmosphere of the place. (Edith Orr)

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

271*. *Proposed by Shmuel Avital, I.I.T. Technion, Haifa, Israel.*

Find all possible triangles ABC which have the property that one can draw a line AD, outside the triangular region, on the same side of AC as AB, which meets CB (extended) in D so that triangles ABD and ACD will be isosceles.

272. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Perhaps by coincidence, the following problem occurs in three different books (to be revealed when a solution is published here):

Solve the system

$$z^x = y^{2x}$$

$$2^z = 2(4)^x$$

$$x + y + z = 16.$$

Perhaps also by coincidence (?), the same incomplete answer is given in all three sources.

Give a complete solution of the system.

273. *Proposed by M.S. Klamkin, University of Alberta.*

Prove that

$$\lim_{n\to 0}\int_c^\infty \frac{(x+a)^{n-1}}{(x+b)^{n+1}}\,dx = \int_c^\infty \frac{(x+a)^{-1}}{x+b}\,dx, \qquad (a,b,c > 0),$$

without interchanging the limit with the integral.

274. *Proposed by Charles W. Trigg, San Diego, California.*

Find triangular numbers of the form $abcdef$ such that

$$abc = 2def.$$

275*. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Given are the points $P(a,b)$ and $Q(c,d)$, where a, b, c, d are all rational. Find a formula for the number of lattice points (integral coordinates) on segment PQ.

276. *Proposed by Sidney Penner, Bronx Community College, Bronx, N.Y.*

How many unit squares must be deleted from a 17×22 checkerboard so that it is impossible to place a 3×5 polyomino on the remaining portion of the board? (A 3×5 polyomino covers exactly 15 squares of the board.)

277. *Proposed by R. Robinson Rowe, Naubinway, Michigan.*

Literally, EUREKA is multipowered; find its roots

$\sqrt{\text{EUREKA}} = \text{UEA}$ (UEA are alternates of eUrEkA),

$\sqrt[3]{\text{EUREKA}} = \text{RT}$ (RT is the cube RT),

$\sqrt[4]{\text{EUREKA}} = ?$

278. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

If each of the medians of a triangle is extended beyond the sides of the triangle to $4/3$ its length, show that the three new points formed and the vertices of the triangle all lie on an ellipse.

279. *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa, Ont.*

On donne sur une droite trois points distincts A, O, B tels que O est entre A et B, et $AO \neq OB$. Montrer que les trois coniques ayant deux foyers et un sommet aux trois points donnés sont concourantes en deux points.

280. *Proposed by L.F. Meyers, The Ohio State University.*

A jukebox has N buttons.

(a) If the set of N buttons is subdivided into disjoint subsets, and a customer is required to press exactly one button from each subset in order to make a selection, what is the distribution of buttons which gives the maximum possible number of different selections?

(b) What choice of n will allow the greatest number of selections if a customer, in making a selection, may press any n distinct buttons out of the N? How many selections are possible then?

(Many jukeboxes have 30 buttons, subdivided into 20 and 10. The answer to part (a) would then be 200 selections.)

* * *

9. Mrs. Agnesi to Maria Gaetana: "A pious girl like you should not be messing around with a witch."

Mrs. IDA RHODES, Washington, D.C.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

200. [1976: 220; 1977: 134] *Proposed by the editor.*

(a) Prove that there exist triangles which cannot be dissected into two or three isosceles triangles.

(b) Prove or disprove that, for $n \geq 4$, every triangle can be dissected into n isosceles triangles.

II. *Comment by Shmuel Avital, I.I.T. Technion, Haifa, Israel.*

This problem deals, *inter alia*, with the possibility of dissecting an oblique triangle into two isosceles triangles. One of the solutions given in [1977: 134] is: "All triangles in which one angle is twice another." To the best of my knowledge, the quantifier "all" is *not* justified in this case. From Figure 1 on page 134, the solution assumes $180° - 3a \geq a$ (at least in the Euclidean plane); hence $a \leq 45°$ and the solution fails for $a > 45°$. (For $a = 45°$, we have again the right-angled case.) This raises a new question (see Problem 271 in this issue).

Editor's comment.

Avital's point is well taken. But the original solver is not to blame for the mistake. He had written "all triangles in which one *acute* angle is twice another," which is correct; but the crucial adjective "acute" unaccountably disappeared on its way to our typewriter.

Our thanks to sharp-eyed reader Avital.

* * *

227. [1977: 66] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

It is well-known that

$$\sqrt{a^2+1} = \langle a, \overline{2a} \rangle = a + \frac{1}{2a+}\,\frac{1}{2a+}\,\frac{1}{2a+}\cdots$$

for all positive integers a. Solve completely in positive integers each of the equations

$$\sqrt{a^2+y} = \langle a, \overline{x, 2a} \rangle \quad \text{and} \quad \sqrt{a^2+y} = \langle a, \overline{x, x, 2a} \rangle,$$

where in both cases $x \neq 2a$.

Adapted from the solutions of Clayton W. Dodge, University of Maine at Orono; and the proposer.

For the first equation let

$$\sqrt{a^2+y} = \langle a, \theta \rangle = a + \frac{1}{\theta} = a + \cfrac{1}{x + \cfrac{1}{a + \sqrt{a^2+y}}}, \tag{1}$$

where

$$\theta = \langle \overline{x, 2a} \rangle = \langle x, 2a, \theta \rangle.$$

Then

$$\theta = x + \cfrac{1}{2a + \cfrac{1}{\theta}} = \frac{x(2a\theta+1)+\theta}{2a\theta+1},$$

whence

$$\frac{2a}{x} = \frac{2a}{\theta} + \frac{1}{\theta^2} = y.$$

Thus the possible solutions consist of all choices of a, x, y such that $2a = xy$ with $y > 1$. Conversely, it is easily verified that (1) is satisfied whenever $x = 2a/y$.

Proceeding similarly with the second equation, we have

$$\sqrt{a^2+y} = \langle a, \phi \rangle = a + \frac{1}{\phi} = a + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{a + \sqrt{a^2+y}}}}, \tag{2}$$

where

$$\phi = \langle \overline{x, x, 2a} \rangle = \langle x, x, 2a, \phi \rangle.$$

Then

$$\phi = x + \cfrac{1}{x + \cfrac{1}{2a + \cfrac{1}{\phi}}} = \frac{(x^2+1)(2a\phi+1)+x\phi}{\phi(2ax+1)+x},$$

whence

$$\frac{2ax+1}{x^2+1} = \frac{2a}{\phi} + \frac{1}{\phi^2} = y.$$

Since $2ax+1$ is odd, x^2+1 must also be odd. Thus x is even and y is odd, and the condition $x \neq 2a$ gives $y > 1$. Let $x = 2m$ and $y = 2k+1$ where m and k are positive integers; then $(4m^2+1)(2k+1) = 4am+1$ and so

$$a = 2mk + m + \frac{k}{2m}.$$

Putting $k/2m = n$ gives all possible solutions in parametric form:

$$x = 2m, \qquad y = 4mn+1, \qquad a = 4m^2n + m + n. \tag{3}$$

Finally, substitution in (2) and a great deal of tedious algebra show that (3) does indeed generate all solutions.

Also solved by KENNETH M. WILKE, Washburn University, Topeka, Kansas; and R. ROBINSON ROWE, Naubinway, Michigan.

Editor's comment.

Much of the tediousness in the verification of the solutions to the second equation can be avoided by proceeding as did Sierpiński in [1], a reference sent to me by Wilke. In an elegant solution involving judicious use of inequalities and little algebraic manipulation, Sierpiński shows that if a and y are as given in (3) then

$$\sqrt{a^2+y} = \langle a, \overline{x, x, 2a} \rangle \tag{4}$$

with $x = 2m$ as in (3).

Sierpiński proves elsewhere [2] the following stronger result: If D is a positive integer for which the expansion of $\sqrt{D}$ into arithmetic continued fraction has a three-term period, then $D = a^2 + y$, with a and y as given in (3), and so (4) must hold with x as in (3).

REFERENCES

1. W. Sierpiński, *250 Problems in Elementary Number Theory*, American Elsevier, New York, 1970, pp. 22, 119-120 (Problem 247).

2. __________, O liczbach naturalnych D, dla których okres rozwinięcia $\sqrt{D}$ na ułamek łańcuchowy arytmetyczny ma trzy wyrazy, (On positive integers D for which the period of expansion of $\sqrt{D}$ into an arithmetic continued fraction has three terms; in Polish) *Roczniki Pol. Tow. Matem., ser. II: Wiadomości Matem.*, 5 (1962), 53-55.

* * *

228. [1977: 66] *Proposed by Charles W. Trigg, San Diego, California.*

(a) Find four consecutive primes having digit sums that, in some order, are consecutive primes.

(b) Find five consecutive primes having digit sums that are distinct primes.

List of the answers submitted, some by several of the solvers identified below.

(a)

Four consecutive primes				*Consecutive prime digit sums*			
2	3	5	7	2	3	5	7
3	5	7	11	3	5	7	2
191	193	197	199	11	13	17	19
821	823	827	829	11	13	17	19
1321	1327	1361	1367	7	13	11	17
2081	2083	2087	2089	11	13	17	19
3251	3253	3257	3259	11	13	17	19

(b)	*Five consecutive primes*				*Distinct prime digit sums*				
1291	1297	1301	1303	1307	13	19	5	7	11
3257	3259	3271	3299	3301	17	19	13	23	7
402131	402133	402137	402139	402197	11	13	17	19	23
1102313	1102333	1102337	1102393	1102397	11	13	17	19	23
4775231	4775233	4775293	4775297	4775299	29	31	37	41	43

Solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montreal, Quebec; HARRY L. NELSON, Livermore, California; R. ROBINSON ROWE, Naubinway, Michigan; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

It was not to be expected that any reader would come up with a reasoned process that would unfailingly churn out solutions, and none did. Most answers were obtained by an attentive examination of a table of primes or by consulting a friendly neighbourhood computer.

Note that the last three answers in part (b) consist of five consecutive primes with consecutive prime digit sums.

In addition to several of the above answers, the proposer also submitted a few near misses, but in this game a miss is as good as a mile.

Several solvers conjectured (conjectures are cheap) that the problem has infinitely many solutions, but don't hold your breath for a proof.

* * *

229. [1977: 66] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

On an examination, one question asked for the largest angle of the triangle with sides 21, 41, and 50. A student obtained the correct answer as follows: Let C denote the desired angle; then $\sin C = 50/41 = 1 + 9/41$. But $\sin 90° = 1$ and $9/41 = \sin 12° 40' 49''$. Thus

$$C = 90° + 12° 40' 49'' = 102° 40' 49'',$$

which is correct. Find the triangle of least area having integral sides and possessing this property.

Adapted from the proposer's solution.

Let ABC be one such triangle, with $a < b < c$. We have

$$\sin C = \frac{c}{b} = 1 + \frac{c-b}{b};$$

hence $C = 90° + \theta$ where $\sin\theta = \dfrac{c-b}{b}$ and so

$$\cos C = \cos(90^\circ + \theta) = -\sin\theta = \frac{b-c}{b}.$$

Now applying the law of cosines to $\angle C$ we get

$$\frac{a^2+b^2-c^2}{2ab} = \frac{b-c}{b},$$

which is equivalent to

$$a^2 + (b-a)^2 = (c-a)^2,$$

the familiar Pythagorean relationship. Hence we must have either

$$\text{(I)} \qquad a = m^2 - n^2, \qquad b - a = 2mn, \qquad c - a = m^2 + n^2$$

or

$$\text{(II)} \qquad a = 2mn, \qquad b - a = m^2 - n^2, \quad c - a = m^2 + n^2,$$

where m, n are integers of opposite parity, $m > n$ and $(m,n) = 1$, since for minimal area in our problem we need only consider primitive Pythagorean triangles. Thus we have two families of possible solution triangles,

$$\text{(I)} \quad \begin{cases} a = m^2 - n^2 \\ b = m^2 + 2mn - n^2 \\ c = 2m^2 \end{cases}, \qquad \text{(II)} \quad \begin{cases} a = 2mn \\ b = m^2 + 2mn - n^2 \\ c = (m+n)^2 \end{cases}.$$

Conversely, it is easily verified that all triangles in the two families have the desired property.

By Heron's formula, the areas of the solution triangles are

$$\text{(I)} \quad m(m+n)(m-n)\sqrt{n(2m-n)},$$

$$\text{(II)} \quad mn(m+n)\sqrt{(m+3n)(m-n)}.$$

It is clear that in each family minimal area occurs when $m = 2$ and $n = 1$, giving

$$\text{(I)} \quad a = 3,\ b = 7,\ c = 8,\ \text{area } 6\sqrt{3},$$

$$\text{(II)} \quad a = 4,\ b = 7,\ c = 9,\ \text{area } 6\sqrt{5}.$$

The first of these is thus the triangle sought. The triangle in the proposal belongs to family (I) with $m = 5$, $n = 2$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; and R. ROBINSON ROWE, Naubinway, Michigan.

Editor's comment.

The proposer pointed out that this problem is a variation of a fallacy in E.A. Maxwell's *Fallacies in Mathematics*, Cambridge University Press.

Rowe expressed surprise that such a whimsical solution was applicable to such a variety of shapes, with angle C ranging from 91.85° to 125.50° in the dozen or so examples he computed.

* * *

230. [1977: 66] *Proposed by R. Robinson Rowe, Sacramento, California.*

Find the least integer N which satisfies

$$N = a^{ma+nb} = b^{mb+na}$$

with m and n positive and $1 < a < b$. (This generalizes Problem 219.)

I. *Solution by Gali Salvatore, Ottawa, Ontario.*

We have $b = ra$ and $n = sm$ with $r > 1$ and $s \neq 1$ (since $m = n$ implies $a = b$). The given relation holds if and only if

$$a = r^{\frac{r+s}{(r-1)(s-1)}} = r^u, \text{ say,} \tag{1}$$

and then

$$N = \left(a^{a(1+rs)}\right)^m. \tag{2}$$

Since $a > 1$ and $r > 1$, it follows from (1) that $s > 1$. The value of N in (2) depends on three independent parameters r, s, and m; hence, if we now assume that m is a positive integer, minimality for N requires $m = 1$, so that $s = n$. On the assumption that the proposer's intent was, as in Problem 219 [1977: 173], to consider this a Diophantine problem, with N, m, n, a, b all positive integers, we see that we must have r, s, and u all positive integers. So we must find integral values of $r \geq 2$ and $s \geq 2$ which make u an integer and minimize

$$N = a^{a(1+rs)}.$$

We need only consider values of r and s which are of the same parity, since otherwise u is not an integer.

Now, for fixed r, consider

$$u = u(s) = \frac{r+s}{(r-1)(s-1)} = \frac{1}{r-1}\left(1 + \frac{r+1}{s-1}\right).$$

For even r, the two largest values of $u(s)$ are

$$u(2) = 1 + \frac{3}{r-1} \qquad \text{and} \qquad u(4) = \frac{1}{3}\left(1 + \frac{5}{r-1}\right).$$

Since when $r > 4$ we have

$$1 < u(2) < 2 \qquad \text{and} \qquad u(4) \leq \frac{2}{3},$$

it follows that $u(s)$ is never an integer for any even $r > 4$. For odd r, the largest value of $u(s)$ is

$$u(3) = \frac{1}{2}\left(1 + \frac{4}{r-1}\right).$$

Since $u(3) < 1$ when $r > 5$, it follows that $u(s)$ is never an integer for odd $r > 5$.

The table which follows therefore lists all the possibilities that need be investigated to determine the minimal N.

Table 1

r	s	u	a	b	N	*Digits in N*
2	2	4	16	32	2^{320}	97
2	4	2	4	8	2^{72}	22
3	5	1	3	9	3^{48}	23
4	2	2	16	64	2^{576}	174
5	3	1	5	25	5^{80}	56

The least solution N has 22 digits. It is

$$N = 4^{1\cdot 4 + 4\cdot 8} = 8^{1\cdot 8 + 4\cdot 4} = 2^{72}$$

$$= 4{,}722{,}366{,}482{,}869{,}645{,}213{,}696.$$

II. *Comment by David Stone, Georgia Southern College, Statesboro, Georgia.*

Since it was not ruled out by the proposal, I will assume here that N, m, n are positive integers and a, b positive reals.

We have $b = ka$ with $k > 1$. The given relation holds if and only if

$$a = k^{\frac{mk+n}{(n-m)(k-1)}} \tag{3}$$

and then

$$N = N(k) = a^{a(m+nk)}, \qquad k > 0,\ k \neq 1.$$

(Note that I have dispensed with the imposition $a < b$ or, equivalently, $k > 1$, since $N(k) = N(1/k)$.)

I have found by L'Hôpital's rule that

$$\lim_{k\to 1} N(k) = e^{\frac{(n+m)^2}{n-m}\cdot e^{\frac{n+m}{n-m}}} = e^{v}, \text{ say.}$$

So $N(k)$ is a continuous function after removing the discontinuity at $k = 1$. Moreover,

$$\lim_{k\to 1} N'(k) = 0.$$

I conjecture, but cannot quite prove, that $N(k)$ has an absolute minimum at $k=1$ for any choice of m and n. So, for fixed m and n, the answer to our problem would be the first integer above e^v, that is,

$$N = [e^v] + 1. \qquad (4)$$

I did not determine which specific values of m and n would yield the least value of N in (4).

III. *Comment by Edith Orr, Ottawa, Ontario.*

If considered as Diophantine,
This problem is elephantine;
And I'd hate to tell you how it feels
When it's considered on the reals.

Also solved by the proposer. One incorrect solution was received.

Editor's comment.

Shut up, Edith. This is serious work.

Without describing his calculations in complete detail, the proposer gave the following list of the first 12 least solutions for the Diophantine case, in increasing order. In Table 2, $k = b/a$ and w is the exponent in (3).

Table 2

m	n	k	w	a	b	N	*Digits in N*
1	4	2	2	4	8	2^{72}	22
1	5	3	1	3	9	3^{48}	23
2	8	2	2	4	8	2^{144}	44
2	10	3	1	3	9	3^{96}	46
1	3	5	1	5	25	5^{80}	56
3	12	2	2	4	8	2^{216}	66
3	15	3	1	3	9	3^{144}	69
2	7	4	1	4	16	2^{240}	73
2	5	2	3	8	16	2^{288}	87
4	16	2	2	4	8	2^{288}	87
4	20	3	1	3	9	3^{192}	92
1	2	2	4	16	32	2^{320}	97

Note that the *largest* of these, 2^{320}, is the *least* solution for Problem 219.

If we test (4) with the values of m and n used in Table 2, the smallest two values of N obtained are

$$(m,n) = (1,4) \implies N = 1.4501016\ldots \times 10^{19}, \quad \text{a 20-digit number}$$

and

$$(m,n) = (1,5) \implies N = 3.2912092\ldots \times 10^{17}, \quad \text{an 18-digit number.}$$

So the minimal values of N do not occur for the same m and n in the Diophantine and real cases.

* * *

231. [1977: 104] *Proposed by Viktors Linis, University of Ottawa.*

Find the period P of the Easter dates based on the Gaussian algorithm (see pp. 102 - 103 in the April issue), that is, the smallest positive integer P satisfying the conditions:

$$D(Y+P) = D(Y) \quad \text{and} \quad M(Y+P) = M(Y)$$

for all Y, where D and M are the day and month functions of year number Y.

Solution and comment by R. Robinson Rowe, Naubinway, Michigan; with an assist by Carl-Eric Fröberg [1].

Let P' be a number such that, for all Y,

$$D(Y+P') = D(Y) \quad \text{and} \quad M(Y+P') = M(Y);$$

then the required period P is a divisor of P'. With $k, p, \ldots$ for the year Y as in [2], we will use $k', p', \ldots$ for the year $Y+P'$. Thus

$$k' = [(Y+P')/100] = k + P'/100; \tag{1}$$

$$p' = [(13+8k')/25] = p + 2P'/625; \tag{2}$$

$$q' = [k'/4] = q + P'/400; \tag{3}$$

$$a' \equiv Y+P' \pmod{19} = a \text{ when } P' = 19u; \tag{4}$$

$$b' \equiv Y+P' \pmod{4} = b \text{ when } P' = 4v; \tag{5}$$

$$c' \equiv Y+P' \pmod{7} = c \text{ when } P' = 7w. \tag{6}$$

To satisfy (1) to (6), P' must have the factors 2^4, 5^4, 7, 19, so we can set

$$P' = 2^4 \cdot 5^4 \cdot 7 \cdot 19 \cdot Q = 1{,}330{,}000Q. \tag{7}$$

Continuing,

$$m' \equiv 15 - p' + k' - q' \equiv m + 5719Q \equiv m \pmod{30}$$

provided we set $Q = 30R$, and then (7) becomes

$$P' = 1{,}330{,}000Q = 39{,}900{,}000R; \tag{8}$$

$$n' \equiv 4 + k' - q' \equiv n + 9975Q \equiv n \pmod{7},$$

since $7|9975$;

$$d' \equiv 19a' + m' \equiv 19a + m + 171570R \equiv d \pmod{30},$$

since $30|171570$; and

$$e' \equiv 2b' + 4c' + 6a' + n' \equiv 2b + 4c + 6a + n \equiv e \pmod{7}.$$

So P' is given by (8) for any R. Taking $R = 1$, we get

$$P' = 39{,}900{,}000 = 2^5 \cdot 3 \cdot 5^5 \cdot 7 \cdot 19.$$

Thus the required period P is the smallest divisor of P' which satisfies all the conditions.

Fröberg reports in [1] (a reference supplied by the proposer) that of the numbers

$$P'/2, \quad P'/3, \quad P'/5, \quad P'/7, \quad P'/19,$$

only $P'/7$ satisfies all the conditions, a claim that we are all at liberty to verify for ourselves. Hence

$$P = P'/7 = 5{,}700{,}000 \text{ years}.$$

Comment. This problem is intended to be academic and apparently fantastic. I can show that it is also very erroneous. The Gaussian algorithm was predicated on the Gregorian calendar which has 97 leap years in 400, that is, the mean Gregorian year is 365.2425 days. Consulting an ephemeris, I find that a year is postulated to be

$$365.24219879 - 0.00000614T \text{ days}, \tag{9}$$

where T is in centuries since 1900. Thus, after P years, a year would only be 364.89222 days long, averaging 365.06721 days over the period P and departing from the Gregorian year by an average of 0.17529 days per year, with a total discrepancy of 999,160 days, or 2,736 years. Astronomers may be expected to revise and/or refine (9), but it indicates that, long before the cycle P has run its course, there will have to be a revision of the Gregorian calendar and a consequent disturbance of any cyclic character of Easter dates.

REFERENCES

1. Carl-Eric Fröberg, "Om påskmatematik," *Elementa*, (Uppsala), årg. 59, 1 (1976).
2. Viktors Linis, "Gauss and Easter Dates," EUREKA, 3 (1977), 102.

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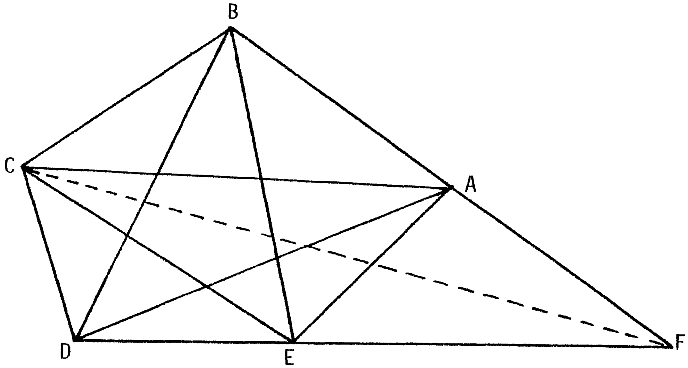
232. [1977: 104] *Proposed by Viktors Linis, University of Ottawa.*

Given are five points A, B, C, D, E in the plane, together with the segments joining all pairs of distinct points. The areas of the five triangles BCD, EAB, ABC, CDE, DEA being known, find the area of the pentagon ABCDE.

The above problem with a solution by Gauss was reported by Schumacher [*Astronomische Nachrichten*, Nr. 42, November 1823]. The problem was given by Möbius in his book (p. 61) on the Observatory of Leipzig, and Gauss wrote his solution in the margins of the book.

I. *Solution by L.F. Meyers, The Ohio State University.*

We will assume that A, B, C, D, E are the successive vertices of a *convex* pentagon. (In other cases, a suitable definition of area will give a corresponding result.) At least one pair of nonadjacent sides of the pentagon must be nonparallel. Suppose BA and DE produced meet at F, and draw CF (see figure).



Since the areas of two triangles on the same base are to each other as their altitudes[1], we have

$$\frac{BCF}{BDF} = \frac{ABC}{ABD} \quad \text{and} \quad \frac{CDF}{BDF} = \frac{CDE}{BDE},$$

so that

$$\frac{BCD}{BDF} = \frac{BCF + CDF - BDF}{BDF} = \frac{ABC}{ABD} + \frac{CDE}{BDE} - 1. \qquad (1)$$

Also

$$\frac{AEF}{BEF} = \frac{ADE}{BDE};$$

hence

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[1]Or, musically speaking, the arias of two trios on the same bass are to each other as their altos. (L.F.M.)

$$\text{BDF} = \text{ABDE} + \text{AEF} = \text{ABDE} + \text{BEF} \cdot \frac{\text{ADE}}{\text{BDE}} = \text{ABDE} + (\text{BDF} - \text{BDE}) \cdot \frac{\text{ADE}}{\text{BDE}},$$

so that

$$\text{BDF} \cdot (\text{BDE} - \text{ADE}) = \text{ABDE} \cdot \text{BDE} - \text{ADE} \cdot \text{BDE} = \text{ABD} \cdot \text{BDE}. \tag{2}$$

Multiplying (1) and (2) gives

$$\text{BCD} \cdot (\text{BDE} - \text{ADE}) = \text{ABC} \cdot \text{BDE} + \text{CDE} \cdot \text{ABD} - \text{ABD} \cdot \text{BDE}. \tag{3}$$

It will be convenient to denote the areas of

	pentagon ABCDE and triangles	EAB,	ABC,	BCD,	CDE,	DEA
by	p,	a,	b,	c,	d,	e.

Then

$$\text{BDE} = p - a - c, \qquad \text{ABD} = p - c - e,$$

and (3) becomes

$$c(p - a - c - e) = b(p - a - c) + d(p - c - e) - (p - c - e)(p - a - c),$$

which simplifies to

$$p^2 - (a + b + c + d + e)p + (ab + bc + cd + de + ea) = 0.$$

Thus the required area p of the pentagon, if it exists, is one of the roots of the equation

$$x^2 - ux + v = 0, \tag{4}$$

where

$$u = a + b + c + d + e \quad \text{and} \quad v = ab + bc + cd + de + ea,$$

a necessary condition for existence being $u^2 - 4v \geq 0$.

But—which root? Since the sum of the roots of (4) is u, the other root must be $u - p$. Now

$$p = i + t + \omega,$$

where i is the area of the "inside" pentagon enclosed by the diagonals of ABCDE, t is the sum of the areas of the "teeth" of the inscribed pentagram (star), and ω is the area outside the star. But $u = 2\omega + t$; hence

$$u - p = 2\omega + t - (i + t + \omega) = \omega - i.$$

Since $\omega - i \leq \omega \leq p$ (in the convex case), we can see that p must be the larger root of (4):

$$p = \frac{u + \sqrt{u^2 - 4v}}{2}.$$

II. *Comment by Murray S. Klamkin, University of Alberta.*

There is a unique area p for the pentagon, when one exists, but the pentagon itself is not uniquely determined: there is a whole family of pentagons which are area-preserving affine transforms of one another.

The special case corresponding to the five given triangles being all of unit area was set as a problem in the 1st USA Mathematical Olympiad in 1972. The top student, James Saxe, showed that all solutions were area-preserving affine transforms of a regular pentagon.

III. *Solution by Carl Friedrich Gauss (as translated by the proposer).*

Denote the five given points by 1, 2, 3, 4, 5 and the angles 213, 214, 215 by p, q, r, respectively. Also denote the sides 12, 13, 14, 15 by t, u, v, w, respectively and the areas of triangles 123, 234, 345, 451, 512, 124, 134, 135 by a, b, c, d, e, x, y, z, respectively. Finally, let the area of pentagon 12345 be ω.

We have the following relations:

$$tu\sin p = 2a, \qquad tv\sin q = 2x, \qquad tw\sin r = 2e,$$

$$vw\sin(r-q) = 2d, \quad uw\sin(r-p) = 2z, \quad uv\sin(q-p) = 2y.$$

Since $ad - xz + ey = 0$ and

$$b+d+x = a+d+y = a+c+z = \omega,$$

elimination of x, y, z yields

$$ad - (\omega - b - d)(\omega - a - c) + e(\omega - a - d) = 0,$$

which simplifies to

$$\omega^2 - (a+b+c+d+e)\omega + (ab+bc+cd+de+ea) = 0.$$

Also solved by MURRAY S. KLAMKIN, University of Alberta (solution as well).

Editor's comment.

Gauss's solution shows the touch of the master. But it leaves many things unsaid, to be filled in by lesser minds, about convexity, existence of solutions, and discrimination between the roots of the quadratic.

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Ars longa, vita brevis

EPITAPH IN A COUNTRY CHURCHYARD

John Longbottom III

1875 - 1877

A proud addition to II, too soon cancelled

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