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THE OLYMPIAD CORNER: 75

M.S. KLAMKIN

All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

This month's problem set consists of the first and second rounds of the 1985 Spanish Mathematical Olympiad. I am grateful to Professor Francisco Bellot for transmitting these problems (in French) and to Andy Liu for their translation.

1985 SPANISH MATHEMATICAL OLYMPIAD - 1st ROUND

1. Consider the quadratic equations with complex coefficients,

$$x^2 - sx + p = 0 \qquad \text{and} \qquad x^2 - s'x + p' = 0.$$

Find conditions on the coefficients s, p, s', and p' such that the roots of each equation are different pairs of opposite vertices of the same square (in an Argand diagram).

2. Let n be a natural number. Prove that the expression

$$(n + 1)(n + 2) \ldots (2n - 1)(2n)$$

is divisible by 2^n.

3. Let a, b, and c be positive real numbers. Prove that

$$(b + c)(c + a)(a + b) \geq 8abc.$$

4. L and M are points on the sides AB and AC, respectively, of triangle ABC such that $\overline{AL} = 2\overline{AB}/5$ and $\overline{AM} = 3\overline{AC}/4$. If BM and CL intersect at P, and AP and BC intersect at N, determine $\overline{BN}/\overline{BC}$.

5. Determine all the real roots of $4x^4 + 16x^3 - 6x^2 - 40x + 25 = 0$.

6. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $(a_n - b_n)(b_n - c_n) \geq 0$ for all natural numbers n. If $\lim a_n = \lim c_n = 1$ for $n \longrightarrow \infty$, analyze the convergence of $\{b_n\}$.

7. Analyze the continuity and sketch the graph of the function

$$y = x^2 - 3x + 1 + |-x^2 + x + 2|.$$

8. A cone C with vertex V is inscribed in a sphere S with centre 0. Planes through V and 0 cut C in equilateral triangles. Furthermore the plane parallel to the base of the cone C and at distance 1 from the vertex

V cuts S and C in circles bounding an annulus of area k^2. Find the radius r of S (in terms of k).

*

1985 SPANISH MATHEMATICAL OLYMPIAD - 2nd ROUND

1. Let P be the set of all points in the plane and $f: P \longrightarrow P$ be a function satisfying the following three conditions:
   (i) f is a bijection,
   (ii) $f(r)$ is a line for every line r,
   (iii) $f(r)$ is parallel to r for every line r.

What possible transformations can f be?

2. Let $\mathbb{Z}$ be the set of integers and $\mathbb{Z} \times \mathbb{Z}$ be the set of ordered pairs of integers. On $\mathbb{Z} \times \mathbb{Z}$, define $(a,b) + (a',b') = (a + a',b + b')$ and $-(a,b) = (-a,-b)$. Determine if there exists a subset E of $\mathbb{Z} \times \mathbb{Z}$ satisfying:
   (i) Addition is closed in E,
   (ii) E contains $(0,0)$,
   (iii) For every $(a,b) \neq (0,0)$, E contains exactly one of (a,b) and $-(a,b)$.

3. Solve the equation
$$\tan^2 2x + 2 \tan 2x \cdot \tan 3x = 1.$$

4. Prove that for each positive integer k there exists a triple (a,b,c) of positive integers such that $abc = k(a + b + c)$. In all such cases prove that $a^3 + b^3 + c^3$ is not a prime.

5. Find the equation of the circle determined by the roots (in the Argand diagram) of the equation
$$z^3 + (-1 + i)z^2 + (1 - i)z + i = 0.$$

6. Let OX and OY be non-collinear rays. Through a point A on OX, draw two lines r_1 and r_2 that are antiparallel with respect to $\angle XOY$. Let r_1 cut OY at M and r_2 cut OY at N. (Thus, $\angle OAM = \angle ONA$.) The bisectors of $\angle AMY$ and $\angle ANY$ meet at P. Determine the location of P.

7. Determine the value of p such that the equation $x^5 - px - 1 = 0$ has two roots r and s which are the roots of an equation $x^2 - ax + b = 0$ where a and b are integers.

8. A square matrix is "sum-magic" if the sum of all the elements in each row, column, and major diagonal is constant. Similarly, a square matrix is "product-magic" if the product of all the elements in each

row, column, and major diagonal is constant. Determine if there exist 3×3 matrices of real numbers which are both "sum-magic" and "product-magic".

*   *   *

I now give solutions to some earlier problems.

7. [1981: 42] *1980 Oesterreichisch-Polnischer Mathematik Weltbewerb*

Find the greatest natural number n such that there exist natural numbers $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_{n-1}$ with $a_1 < a_2 < \ldots < a_{n-1}$ satisfying the following system of equations:

$$\begin{cases} x_1 x_2 \ldots x_n = 1980 \\ x_i + (1980/x_i) = a_i, \qquad i = 1,2,\ldots,n-1. \end{cases}$$

Solution.

We first claim that $x_1 > x_2 > \ldots > x_{n-1}$. The function $f(x) = x + 1980/x$ is decreasing for $x < \sqrt{1980}$ and increasing for $x > \sqrt{1980}$. Thus if all x_i's are less than $\sqrt{1980}$, to ensure that $a_1 < a_2 < \ldots < a_{n-1}$ we must have $x_1 > x_2 > \ldots > x_{n-1}$. On the other hand, if $x_i > \sqrt{1980}$ for some $i < n$, then for all $j \neq i$, $x_j \leq 1980/x_i < \sqrt{1980}$, and hence

$$a_j = f(x_j) \geq f(1980/x_i) = f(x_i) = a_i,$$

which implies $j > i$. Thus $i = 1$, and $x_1 > x_2 > \ldots > x_{n-1}$ again holds.

To maximize n, we would choose $x_n = 1$. Since $1980 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 11$, the maximum number of factors $x_1, x_2, \ldots, x_n$ satisfying the given equations is seven, given by

$$11 \cdot 6 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 1980.$$

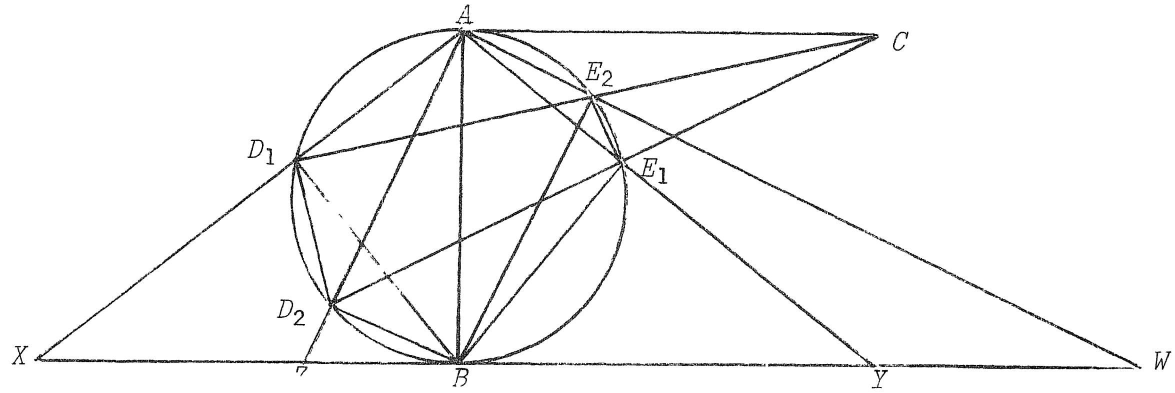
Thus, $n = 7$.

*

9. [1981: 42] *1980 Oesterreichisch-Polnischer Mathematik Weltbewerb*

Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let D_1D_2, E_1E_2 be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined by AE_1 and AE_2.

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.



Referring to the figure, we have:

$\Delta AE_2B \sim \Delta BE_2W$	and so	$AB/AE_2 = BW/BE_2$,	(1)
$\Delta AD_1B \sim \Delta BD_1X$	and so	$AB/AD_1 = BX/BD_1$,	(2)
$\Delta AE_1B \sim \Delta BE_1Y$	and so	$AB/AE_1 = BY/BE_1$,	(3)
$\Delta AD_2B \sim \Delta BD_2Z$	and so	$AB/AD_2 = BZ/BD_2$,	(4)
$\Delta E_2AC \sim \Delta AD_1C$	and so	$AE_2/AD_1 = AC/D_1C$,	(5)
$\Delta E_1AC \sim \Delta AD_2C$	and so	$AE_1/AD_2 = AC/D_2C$,	(6)
$\Delta E_2E_1C \sim \Delta D_2D_1C$	and so	$E_2C/D_2C = E_2E_1/D_2D_1$.	(7)

Then

$$\begin{aligned}\frac{AE_2\cdot AE_1}{AD_1\cdot AD_2} &= \frac{AC^2}{D_1C\cdot D_2C} && \text{by (5), (6)}\\ &= \frac{D_1C\cdot E_2C}{D_1C\cdot D_2C}\\ &= E_2C/D_2C\\ &= E_2E_1/D_2D_1 && \text{by (7),}\end{aligned}$$

and so

$$\begin{aligned}XZ &= BX - BZ\\ &= AB(BD_1/AD_1 - BD_2/AD_2) && \text{by (2), (4)}\\ &= \frac{AB(BD_1\cdot AD_2 - AD_1\cdot BD_2)}{AD_1\cdot AD_2}\\ &= \frac{AB^2\cdot D_1D_2}{AD_1\cdot AD_2} && \text{by Ptolemy's Theorem}\\ &= \frac{AB^2\cdot E_2E_1}{AE_2\cdot AE_1} && \text{from above}\\ &= \frac{AB(BE_2\cdot AE_1 - AE_2\cdot BE_1)}{AE_2\cdot AE_1} && \text{by Ptolemy's Theorem}\\ &= AB(BE_2/AE_2 - BE_1/AE_1)\\ &= BW - BY && \text{by (1), (3)}\\ &= WY.\end{aligned}$$

*

1. [1981: 44] *1980 Competition in Mariehamn, Finland*

In triangle ABC the perpendicular bisectors of AB and AC cut BC, produced if necessary, at X and Y, respectively.

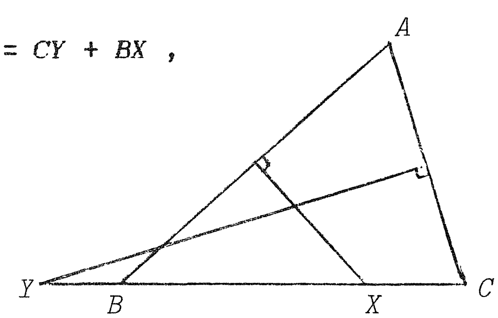
(a) Prove that a sufficient condition for $BC = XY$ is $\tan B \tan C = 3$.

(b) Prove that this condition is not necessary and find necessary and sufficient conditions for $BC = XY$.

Solution.

(a) Assuming $\tan B \tan C = 3$, both B and C must be acute, and we have the following implications:

$$\begin{aligned}\tan B \tan C = 3 &\Longrightarrow \sin B \sin C = 3 \cos B \cos C\\
&\Longrightarrow \cos(B - C) = 4 \cos B \cos C\\
&\Longrightarrow \sin(B + C)\cos(B - C) = 4 \cos B \cos C \sin(B + C)\\
&\Longrightarrow \sin 2B + \sin 2C = 8 \cos B \cos C \sin(B + C)\\
&\Longrightarrow \sin B \cos B + \sin C \cos C = 4 \cos B \cos C \sin(B + C)\\
&\Longrightarrow \frac{\sin B}{\cos C} + \frac{\sin C}{\cos B} = 4 \sin(B + C) = 4 \sin A\\
&\Longrightarrow 4a = \frac{a}{\sin A}\frac{\sin B}{\cos C} + \frac{a}{\sin A}\frac{\sin C}{\cos B}\\
&\qquad = \frac{b}{\sin B}\frac{\sin B}{\cos C} + \frac{c}{\sin C}\frac{\sin C}{\cos B} = \frac{b}{\cos C} + \frac{c}{\cos B}\\
&\Longrightarrow 2a = \frac{b}{2 \cos C} + \frac{c}{2 \cos B} = CY + BX,\end{aligned}$$

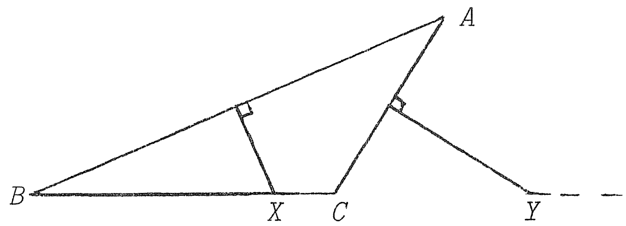


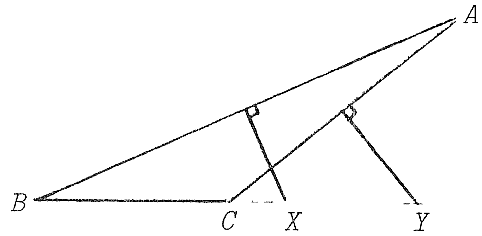
where X, Y are the intersections of the perpendicular bisectors of AB and AC, respectively, with BC. Assume without loss of generality that $CY \geq a$ and $BX \leq a$; then X and Y lie as in the diagram. Hence

$$2a = CY + BX = CB + XY = a + XY,$$

so $XY = a = BC$.

(b) If B and C are acute, then the above steps can all be reversed and thus $\tan B \tan C = 3 \Longleftrightarrow BC = XY$ in this case. However this equivalence does not hold in general. Assuming C is obtuse, we have one of the following diagrams (depending on whether X lies to the left or the right of C):





Then,

$$\begin{aligned} BC = XY &\Longrightarrow BX = CY \\ &\Longrightarrow \frac{c}{2\cos B} = \frac{b}{2\cos(\pi - C)} = \frac{-b}{2\cos C} \\ &\Longrightarrow \frac{a}{\sin A}\,\frac{\sin C}{\cos B} = \frac{c}{\sin C}\,\frac{\sin C}{\cos B} = \frac{c}{\cos B} = \frac{-b}{\cos C} \\ &\qquad\qquad = \frac{-b}{\sin B}\,\frac{\sin B}{\cos C} = \frac{-a}{\sin A}\,\frac{\sin B}{\cos C} \\ &\Longrightarrow \frac{\sin C}{\cos B} = -\frac{\sin B}{\cos C} \\ &\Longrightarrow \sin 2C = -\sin 2B \\ &\Longrightarrow \sin(C+B)\cos(C-B) = \frac{1}{2}(\sin 2C + \sin 2B) = 0 \\ &\Longrightarrow \cos(C-B) = 0 \\ &\Longrightarrow C - B = 90^\circ. \end{aligned}$$

Conversely, if $C - B = 90^\circ$, we can reverse the above steps and obtain $BC = XY$. Therefore the necessary and sufficient condition for $BC = XY$ is that

$$(C - B - 90^\circ)(B - C - 90^\circ)(\tan B \tan C - 3) = 0.$$

*

4. [1981: 44] *1980 Competition in Mariehamn, Finland*

A convex polygon with $2n$ sides is inscribed in a circle and $n - 1$ of its n pairs of opposite sides are parallel. For which values of n is it true that the remaining pair of opposite sides must be parallel?

(In the polygon $A_1A_2 \ldots A_{2n}$, A_1A_2 and $A_{n+1}A_{n+2}$, for example, are a pair of opposite sides.)

Solution.

We show that the result is valid only for odd values of $n \geq 3$. We use the easily proven lemma: two chords AB and CD of a circle are parallel if and only if minor arc AC = minor arc BD.

Case (1): $n = 2m + 1$ $(m \geq 1)$.

Let M_i denote the measure of minor arc A_iA_{i+1} for $i = 1,2,\ldots,4m+2$ (here

$A_{4m+3} \equiv A_1$). By hypothesis, all pairs of opposite sides of the polygon are parallel except for one pair, say $A_{4m+2}A_1$ and $A_{2m+1}A_{2m+2}$. Then applying the lemma for each pair of parallel sides, we obtain the following system of equations:

$$M_2 + M_3 + \dots + M_{2m+1} = M_{2m+3} + M_{2m+4} + \dots + M_{4m+2} \qquad (1)$$

$$M_3 + M_4 + \dots + M_{2m+2} = M_{2m+4} + M_{2m+5} + \dots + M_1 \qquad (2)$$

$$\vdots$$

$$M_{2m+1} + M_{2m+2} + \dots + M_{4m} = M_{4m+2} + M_1 + \dots + M_{2m-1} . \qquad (2m)$$

Subtracting (2) from (1), (4) from (3), ..., (2m) from (2m-1), yields

$$M_1 + M_2 = M_{2m+2} + M_{2m+3}$$

$$M_3 + M_4 = M_{2m+4} + M_{2m+5}$$

$$\vdots$$

$$M_{2m-1} + M_{2m} = M_{4m} + M_{4m+1} .$$

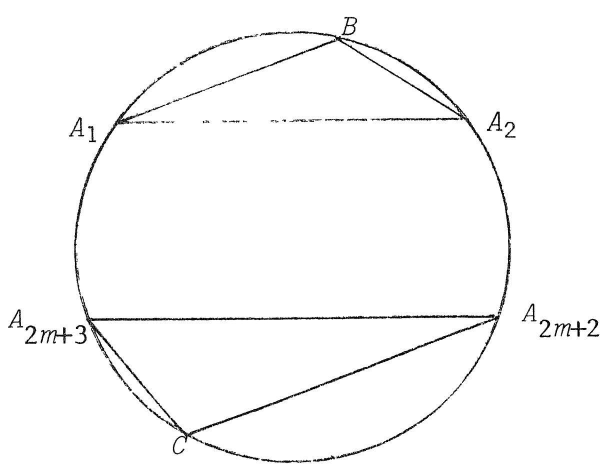
Adding these equations yields

$$M_1 + M_2 + \dots + M_{2m} = M_{2m+2} + M_{2m+3} + \dots + M_{4m+1}$$

and thus, by the lemma, $A_{4m+2}A_1 \| A_{2m+1}A_{2m+2}$.

Case (ii): $n = 2m + 2$ ($m \geq 0$).

The case $m = 0$ follows easily by considering an isosceles trapezoid. Suppose $m \geq 1$. It is easy to construct a cyclic $2(2m+1)$-gon in which all pairs of opposite sides are parallel and in which, say, $A_1A_2 \neq A_{2m+2}A_{2m+3}$. (In the figure we show only these two sides.) We can now always draw chords $A_1B \| A_{2m+2}C$ as shown. Since $A_1A_2 \neq A_{2m+2}A_{2m+3}$, $\angle A_1BA_2 \neq \angle A_{2m+2}CA_{2m+3}$, and so $A_2B \nparallel A_{2m+3}C$. Thus we have a configuration for $n = 2m + 2$ in which the result is not valid.



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5. [1981: 298] *Problems from Középiskolai Matematikai Lapok 1981*

If n is a given natural number, solve the equation

$$(2x - 1)^n + (1 - x)^n = x^n .$$

Solution by Noam Elkies, Harvard University.

Note that $x = 1/2$ is a solution for all n. We thus assume $x \neq 1/2$. Letting $t = \dfrac{1 - x}{2x - 1}$, the above equation becomes

$$(1 + t)^n = 1 + t^n.$$

All real solutions $\neq 1/2$ are given by the following three cases.

If $n = 1$, t and thus x is arbitrary.

If n is even, then: if $t < -1$, $0 < (1 + t)^n < t^n$ so there are no solutions; if $-1 \leq t < 0$, then $(1 + t)^n < 1 < 1 + t^n$ so there are no solutions; and if $t \geq 0$, expansion of $(1 + t)^n$ shows there are no solutions except $t = 0$, that is, $x = 1$.

If n is odd and ≥ 3, then: if $t < -1$, putting $t = -1 - s$ where $s > 0$ we have $(1 + t)^n = 1 + t^n \iff -s^n = 1 - (s + 1)^n \iff (s + 1)^n = s^n + 1$ which by expansion has no solutions; if $-1 \leq t \leq 0$, putting $t = -r$ where $r \geq 0$ we have $(1 + t)^n = 1 + t^n \iff (1 - r)^n = 1 - r^n \iff r^n + (1 - r)^n = 1$ which has solutions $r = 0,1$ only, that is $t = 0,-1$, that is $x = 1,0$; and if $t > 0$, then there are no solutions by expansion of $(1 + t)^n$.

Thus the only real solutions are $x = \frac{1}{2}$ and 1 for all n, $x = 0$ when n is odd, and x is arbitrary when $n = 1$. It is doubtful that there is any explicit representation for the complex roots.

*

1. [1982: 300] *West German Mathematical Olympiad 1982 - 2nd Round*

Max divides the positive integer p by the positive integer q, where $q \leq 100$. In the decimal expansion of the quotient p/q, Max finds somewhere after the decimal point the digit-block 1982. Prove that Max's division is wrong.

Solution.

As in problems #6 [1985: 141] and #4 [1986: 72] it suffices to show that

$$0.1982 \leq p/q \leq 0.1983 \tag{1}$$

is impossible in positive integers p, q with $q \leq 100$. We assume (1) is valid and obtain a contradiction. Since $p \leq 0.1983q$ and $q \leq 100$, $p \leq 19$. Also, from (1),

$$(5 + 90/1982)p \geq q \geq (5 + 85/1983)p. \tag{2}$$

Since $85p/1983 > 0$ and $90p/1982 < 1$, it follows from (2) that $5p \geq q \geq 5p + 1$, which is a contradiction.

*

2. [1982: 300] *West German Mathematical Olympiad 1982 - 2^{nd} Round*

Decide whether every triangle ABC can be transformed by orthogonal projection on a certain plane into an equilateral triangle.

Solution.

This is a classical result; e.g., see Theorem 6.9.4, p. 276 in H. Eves, *A Survey of Geometry*, Allyn and Bacon, Boston, 1972. (Theorem 6.6.4 in the 1963 edition, Volume One.) We give two other solutions.

(i) Consider a circumscribed ellipse of the triangle whose centroid coincides with the centroid of the triangle. (Such an ellipse exists.) By an orthogonal projection, we can always transform this ellipse into a circle. Since centroids transform into centroids, the projected triangle inscribed in the circle can easily be shown to be equilateral. (Just consider the vectors from the center of the circle to the vertices of the triangle. Their sum must be zero and consequently the angle between any two of them is 120°.)

(ii) Consider an inscribed ellipse of maximum area. (Such an ellipse exists.) We now orthogonally project the ellipse into a circle and the triangle goes into a circumscribed triangle T of the circle. Since ratios of areas are preserved under this transformation, T must be a triangle of minimum area circumscribed about a circle. As is known, T is equilateral.

*

3. [1982: 300] *West German Mathematical Olympiad 1982 - 2^{nd} Round*

The nonnegative real numbers $a_1,a_2,\ldots,a_n$ satisfy $a_1 + a_2 + \ldots + a_n = 1$. Show that the sum

$$\frac{a_1}{1+a_2+a_3+\ldots+a_n} + \frac{a_2}{1+a_1+a_3+\ldots+a_n} + \ldots + \frac{a_n}{1+a_1+a_2+\ldots+a_{n-1}}$$

has a minimum and compute it.

Comment.

A generalization is established on [1985: 242].

*

4. [1982: 300] *West German Mathematical Olympiad 1982 - 2^{nd} Round*

If $4^n + 2^n + 1$ is a prime number for the positive integer n, then n is a power of 3.

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.

We first show that if t is an integer not divisible by 3, then $x^2 + x + 1$

divides $x^{2t} + x^t + 1$. Since

$$x^2 + x + 1 = \frac{x^3 - 1}{x - 1},$$

the roots of $x^2 + x + 1 = 0$ are ω and ω^2, the complex cube roots of unity, so that $\omega^3 = 1$. Thus, if $t = 3m + 1$ or $3m + 2$ and $x = \omega$ or ω^2, $x^t = \omega$ or ω^2; hence $x^{2t} + x^t + 1 = 0$ for $x = \omega$ or ω^2 *if* $t = 3m + 1$ or $3m + 2$, and so $x^2 + x + 1$ must divide into $x^{2t} + x^t + 1$.

Now let $n = st$ where $s \geq 1$ is a power of 3 and $3 \nmid t$. Using the above result with $x = 2^s$, we have that $4^n + 2^n + 1$ is divisible by $4^s + 2^s + 1$. Since $4^n + 2^n + 1$ is a prime, we must have $n = s$, so that n is a power of 3.

* * *

Here are the answers to the 4th AIME problems given in the previous corner [1986: 67].

1. 337	2. 104	3. 150	4. 181	5. 890
6. 033	7. 981	8. 141	9. 306	10. 358
11. 816	12. 061	13. 560	14. 750	15. 400.

Solution pamphlets for these problems can be obtained at a nominal cost from Professor W.E. Mientka, Mathematics Department, University of Nebraska, Lincoln, Nebraska 68588-0322.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1141. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

Disjoint spheres O_1 and O_2 are inside and tangent to a sphere O.

Four spheres S_1, S_2, S_3, S_4, each tangent to two of the others as well as to O_1, O_2, and O, are packed in a ring inside O and around O_1 and O_2. Show that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}$$

where r_i is the radius of S_i.

1142. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Suppose ABC is a triangle whose median point lies on its inscribed circle.

(a) Find an equation relating the sides a, b, c of ΔABC.

(b) Assume $a \geq b \geq c$. Find an upper bound for a/c.

(c) Give an example of a triangle with *integral* sides having the above property.

1143. *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

(a) Given integers $k \geq 0$ and $\ell \geq 1$, characterize all natural numbers n such that $\binom{n}{k+i-1}$ divides $\binom{n}{k+i}$ for $i = 1,2,\ldots,\ell$.

(b) Given a natural number n, determine the largest integer ℓ such that for some integer $k \geq 0$, $\binom{n}{k+i-1}$ divides $\binom{n}{k+i}$ for $i = 1,2,\ldots,\ell$.

1144. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and P an interior point at distances x_1, x_2, x_3 from the vertices A, B, C and distances p_1, p_2, p_3 from the sides BC, CA, AB, respectively. Show that

$$\frac{x_1x_2}{ab} + \frac{x_2x_3}{bc} + \frac{x_3x_1}{ca} \geq 4\left[\frac{p_1p_2}{ab} + \frac{p_2p_3}{bc} + \frac{p_3p_1}{ca}\right] .$$

1145. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given a plane convex figure and a straight line ℓ (in the same plane) which splits the figure into two parts whose areas are in the ratio $1:t$ $(t \geq 1)$. These parts are then projected orthogonally onto a straight line n perpendicular to ℓ. Determine, in terms of t, the maximum ratio of the lengths of the two projections.

1146. *Proposed by Jordi Dou, Barcelona, Spain.*

Let AD, BE, CF be cevians of ΔABC and V the foot of the bisector of $\angle A$. Prove that the conic through $DEFAV$ is perpendicular to AV at A.

1147. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is shown in Z.A. Melzak's *Companion to Concrete Mathematics* II (Wiley-Interscience, N.Y., 1976, p.81) that

$$\int_0^\infty \frac{2 - 2\cos u - u\sin u}{u^4}\,du = \frac{\pi}{12},$$

and it is noted that "this is quite simply obtained by residues and complex integration, but it is not quite so simple to obtain by real-variable methods alone". Obtain this result by real-variable methods alone.

1148. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find the triangle of smallest area that has integral sides and integral altitudes.

1149. *Proposed by Lanny Semenko, Erewhon, Alberta.*

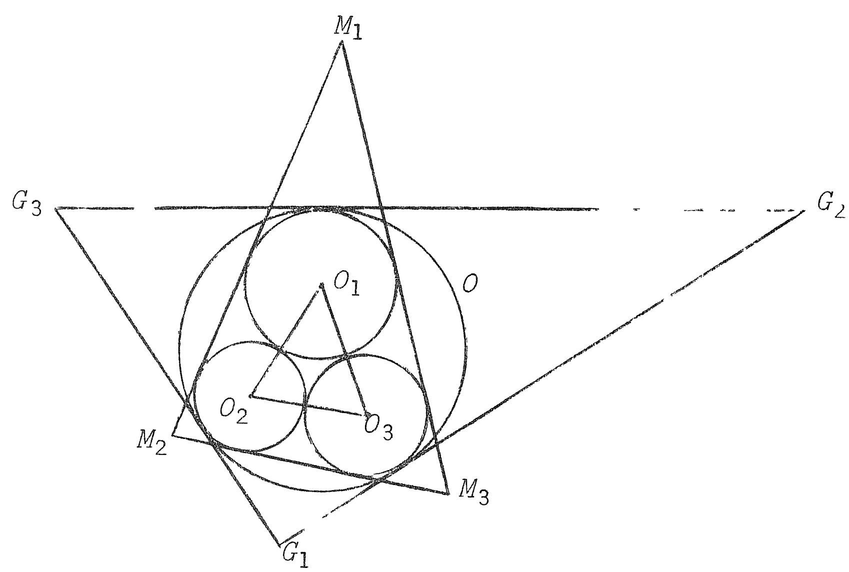
Solve the base ten alphametic

$$\text{HOT}\ ^\circ C = \text{COOL}\ ^\circ F.$$

(Do not, of course, replace C (Celsius) and F (Fahrenheit) by digits.)

1150*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

In the figure, $\Delta M_1M_2M_3$ and the three circles with centers O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles and the sides of triangle $G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that



$$P(\Delta G_1G_2G_3) \geq P(\Delta M_1M_2M_3) + P(\Delta O_1O_2O_3),$$

where P stands for perimeter. Equality is attained when $\Delta O_1O_2O_3$ is equilateral.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

994*. [1984: 318] *Proposed by Ernest J. Eckert, University of Wisconsin-Green Bay.*

Can two different primitive Pythagorean triangles with sides (a,b,c) and (r,s,t) be such that $abc = rst$?

Comment by Richard K. Guy, University of Calgary, Calgary, Alberta.

Since a Pythagorean triple is of the form $(u^2 - v^2, 2uv, u^2 + v^2)$, we are seeking non-trivial solutions to the Diophantine equation

$$uv(u^4 - v^4) = xy(x^4 - y^4).$$

I have consulted several experts (Andrew Bremner, John Leech, and D.H. & Emma Lehmer). The consensus is that it is unlikely that there are any non-trivial solutions, but that this is likely to be very difficult to prove. The above equation is a sextic in four unknowns, which may be thought of as representing a surface in three-dimensional projective space. Such equations are known to be very difficult. For example, no-one knows if there are non-trivial equal sums of two sixth powers,

$$u^6 + v^6 = x^6 + y^6 ,$$

nor can anyone prove their non-existence.

* * *

1000. [1984: 319] *Proposed by H.S.M. Coxeter, University of Toronto.*

In a tetrahelix (see R. Buckminster Fuller, *Synergetics*, Macmillan, New York, 1975, pp.518-524), points A_0, A_1, A_2,... are arranged so that every consecutive four are the vertices of a regular tetrahedron; in other words, there is an infinite sequence of regular tetrahedra $A_0A_1A_2A_3$, $A_1A_2A_3A_4$, $A_2A_3A_4A_5$,... . In terms of the edge length A_0A_1 as a unit of measurement, find the distance d and angle δ between the two skew lines A_0A_1 and A_nA_{n+1}.

Solution by the proposer.

The tetrahelix is shifted one step along itself by the isometry $A_0A_1A_2A_3 \longrightarrow A_1A_2A_3A_4$, which is a twist: the product of rotation through a certain angle θ and translation through a certain distance c along the axis of rotation. Thus the points A_n lie on a cylinder, say $x^2 + y^2 = a^2$, and suitable Cartesian coordinates for A_n are

$$(a \cos n\theta\ ,\ a \sin n\theta\ ,\ nc).$$

In particular,

$$A_0 = (a,0,0).$$

Since for n = 1, 2, and 3

$$(A_0A_n)^2 = a^2(1 - \cos n\theta)^2 + (a \sin n\theta)^2 + (nc)^2$$

$$= 2a^2(1 - \cos n\theta) + n^2c^2 \tag{1}$$

are equal, we have

$$2a^2(1 - \cos \theta) + c^2 = 2a^2(1 - \cos 2\theta) + 4c^2$$

and

$$2a^2(1 - \cos \theta) + c^2 = 2a^2(1 - \cos 3\theta) + 9c^2$$

which simplify to

$$2a^2(\cos 2\theta - \cos \theta) = 3c^2$$

and

$$2a^2(\cos 3\theta - \cos \theta) = 8c^2\ ,$$

or

$$3(\cos 3\theta - \cos \theta) = 12(c/a)^2 = 8(\cos 2\theta - \cos \theta). \tag{2}$$

Putting $x = \cos \theta$ and using the identities

$$\cos 2\theta = 2 \cos^2\theta - 1$$

$$\cos 3\theta = 4 \cos^3\theta - 3 \cos\theta\ ,$$

(2) becomes

$$3(4x^3 - 4x) = 8(2x^2 - x - 1)$$

which simplifies to

$$(x - 1)^2(3x + 2) = 0.$$

We deduce that the angle of rotation θ is given by

$$\cos \theta = x = - 2/3\ , \tag{3}$$

and thus from (2) the translation-distance is given by

$$3(c/a)^2 = 2(2x^2 - x - 1) = 10/9\ ,$$

or

$$c = a \sqrt{\frac{10}{27}}\ .$$

Note that from (3) follows $\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{5}{6}}$ and $\cos \frac{\theta}{2} = \frac{1}{\sqrt{6}}$. Putting $(A_0A_1)^2 = 1$, from (1) we have

$$1 = 2a^2(1 - x) + c^2 = \frac{10}{3} a^2 + a^2 \cdot \frac{10}{27} = \frac{100}{27} a^2\ ,$$

and so

$$a = \frac{3\sqrt{3}}{10}$$

and

$$c = \frac{3\sqrt{3}}{10}\sqrt{\frac{10}{27}} = \frac{1}{\sqrt{10}} .$$

The vector $\overrightarrow{A_nA_{n+1}}$ is

$$[a(\cos(n+1)\theta - \cos n\theta) , a(\sin(n+1)\theta - \sin n\theta) , c]$$

which, using the identities

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$
$$\sin A - \sin B = 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2}$$

can be written as

$$\begin{aligned}\overrightarrow{A_nA_{n+1}} &= \left[-2a \sin \frac{\theta}{2} \sin(n + \tfrac{1}{2})\theta , 2a \sin \frac{\theta}{2} \cos(n + \tfrac{1}{2})\theta , c\right] \\ &= \left[-2\cdot\frac{3\sqrt{3}}{10}\cdot\frac{\sqrt{5}}{\sqrt{6}}\cdot\sin(n + \tfrac{1}{2})\theta , 2\cdot\frac{3\sqrt{3}}{10}\cdot\frac{\sqrt{5}}{\sqrt{6}}\cdot\cos(n + \tfrac{1}{2})\theta , \frac{1}{\sqrt{10}}\right] \\ &= \frac{1}{\sqrt{10}}\left[-3 \sin(n + \tfrac{1}{2})\theta , 3 \cos(n + \tfrac{1}{2})\theta , 1\right] . \qquad (4)\end{aligned}$$

In particular,

$$\overrightarrow{A_0A_1} = \frac{1}{\sqrt{10}}(-3 \sin \frac{\theta}{2} , 3 \cos \frac{\theta}{2} , 1), \qquad (5)$$

and hence the desired angle δ is given by

$$\begin{aligned}\cos \delta &= \frac{\overrightarrow{A_nA_{n+1}}\cdot\overrightarrow{A_0A_1}}{\|\overrightarrow{A_nA_{n+1}}\| \ \|\overrightarrow{A_0A_1}\|} \\ &= \frac{1}{10}\left[9 \sin(n + \tfrac{1}{2})\theta \sin \frac{\theta}{2} + 9 \cos(n + \tfrac{1}{2})\theta \cos \frac{\theta}{2} + 1\right] \\ &= \frac{1}{10}(9 \cos n\theta + 1).\end{aligned}$$

(Notice that $\delta = 0$ when $n = 0$, $\delta = 120^\circ$ when $n = 1$, and $\delta = 90^\circ$ when $n = 2$, as we should expect!)

From (4) and (5), the plane passing through A_nA_{n+1} and parallel to A_0A_1 is

$$\begin{vmatrix} x - a \cos n\theta & y - a \sin n\theta & z - nc \\ -3 \sin(n + \frac{1}{2})\theta & 3 \cos(n + \frac{1}{2})\theta & 1 \\ -3 \sin \frac{\theta}{2} & 3 \cos \frac{\theta}{2} & 1 \end{vmatrix} = 0 ,$$

which can be written as

$$-6(x - a \cos n\theta) \sin \frac{(n+1)\theta}{2} \sin \frac{n\theta}{2} + 6(y - a \sin n\theta) \sin \frac{n\theta}{2} \cos \frac{(n+1)\theta}{2}$$
$$-9(z - nc) \sin n\theta = 0 ,$$

or

$$-6(x - a\cos n\theta)\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2} + 6(y - a\sin n\theta)\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}$$
$$-9(z - nc)\,2\sin\frac{n\theta}{2}\cos\frac{n\theta}{2} = 0 ,$$

or

$$(x - a\cos n\theta)\sin\frac{(n+1)\theta}{2} - (y - a\sin n\theta)\cos\frac{(n+1)\theta}{2}$$
$$+ 3(z - nc)\cos\frac{n\theta}{2} = 0,$$

or finally

$$x\sin\frac{(n+1)\theta}{2} - y\cos\frac{(n+1)\theta}{2} + 3z\cos\frac{n\theta}{2} = -a\sin\frac{(n-1)\theta}{2} + 3nc\cos\frac{n\theta}{2} . \quad (6)$$

Similarly, the plane passing through A_0A_1 and parallel to A_nA_{n+1} is

$$\begin{vmatrix} x - a & y & z \\ 3\sin(n+\frac{1}{2})\theta & 3\cos(n+\frac{1}{2})\theta & 1 \\ -3\sin\frac{\theta}{2} & 3\cos\frac{\theta}{2} & 1 \end{vmatrix} = 0 ,$$

or

$$(x - a)\sin\frac{(n+1)\theta}{2} - y\cos\frac{(n+1)\theta}{2} + 3z\cos\frac{n\theta}{2} = 0 ,$$

or

$$x\sin\frac{(n+1)\theta}{2} - y\cos\frac{(n+1)\theta}{2} + 3z\cos\frac{n\theta}{2} = a\sin\frac{(n+1)\theta}{2} . \quad (7)$$

The required distance d is the distance between the parallel planes (6) and (7), that is,

$$d = \frac{\left|a\sin\frac{(n+1)\theta}{2} + a\sin\frac{(n-1)\theta}{2} - 3nc\cos\frac{n\theta}{2}\right|}{\sqrt{\sin^2\frac{(n+1)\theta}{2} + \cos^2\frac{(n+1)\theta}{2} + 9\cos^2\frac{n\theta}{2}}}$$

$$= \frac{\left|2a\cos\frac{\theta}{2}\sin\frac{n\theta}{2} - 3nc\cos\frac{n\theta}{2}\right|}{\sqrt{1 + 9\cos^2\frac{n\theta}{2}}}$$

$$= \frac{\left|2\cdot\frac{3\sqrt{3}}{10}\frac{1}{\sqrt{6}}\tan\frac{n\theta}{2} - 3n\cdot\frac{1}{\sqrt{10}}\right|}{\sqrt{\sec^2\frac{n\theta}{2} + 9}}$$

$$= \frac{\left|\frac{3}{5\sqrt{2}}\tan\frac{n\theta}{2} - \frac{3n}{\sqrt{10}}\right|}{\sqrt{\tan^2\frac{n\theta}{2} + 10}}$$

$$= \frac{3|n\sqrt{5} - t|}{5\sqrt{2(t^2 + 10)}}$$

where $t = \tan\frac{n\theta}{2}$. (Notice that $d = 0$ when $n = 0$ or 1, and $d = \frac{1}{\sqrt{2}}$ when $n = 2$, as we should expect!)

For a related paper, see the *Canadian Mathematical Bulletin*, Vol. 28, no. 4 (1985) 385-393.

Also solved by JORDI DOU, Barcelona, Spain.

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1010*. [1985: 17] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Are there integers $k \neq 1$ such that the sequence $\{3n^2k + 3nk^2 + k^3\}$, n an integer, contains infinitely many squares? If the answer is yes, determine all such k. (The case $k = 1$ is dealt with in *Crux* 873 [1984: 335].)

Partial solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let

$$p^2 = 3n^2k + 3nk^2 + k^3 \tag{1}$$

be a square member of such a sequence. Now multiply both sides of (1) by c^6, with c an integer > 1, and let $n_1 = nc^2$, $p_1 = pc^3$, and $k_1 = kc^2$. Then we obtain

$$p_1{}^2 = 3n_1{}^2k_1 + 3n_1k_1{}^2 + k_1{}^3,$$

which is of the same form as (1). Thus if k is an integer satisfying the question, kc^2 will also be such an integer for each integer c. In particular, since in *Crux* 873 it was proved there are infinitely many solutions for $k = 1$, there are solutions with $k = 4, 9, 16, \ldots$. In the ensuing discussion we will therefore assume that k is square-free.

Since the right-hand side of (1) contains k as a factor, the left-hand side must also. Since k is square-free, $k|p$, and we may set $p = kr$. Dividing both sides of (1) by k gives

$$kr^2 = 3n^2 + 3nk + k^2. \tag{2}$$

Since k appears in three of the four terms in (2), $k|3n^2$ and therefore $k|n$ or $3|k$. We will defer the latter possibility until later and set $n = km$ in (2). Dividing both sides by k gives

$$r^2 = k(3m^2 + 3m + 1).$$

We now see that $k|r$, so we set $r = ks$ and again divide by k, giving

$$ks^2 = 3m^2 + 3m + 1. \tag{3}$$

We note that the right-hand side is congruent to 1 modulo 6, so we must have $k \equiv 1 \pmod 6$ and $s \equiv \pm 1 \pmod 6$. Multiplying both sides of (3) by 12 gives, after some rearrangement,

$$[3(2m + 1)]^2 - 3k(2s)^2 = -3,$$

which is a Pell equation. It is well known that if a Pell equation has one solution, it has an infinite number of solutions. Substituting $s = 1$ in (3) gives

$$k = 3m^2 + 3m + 1,$$

so that if k is of this form there are an infinity of solutions. The first few such values of k are 1, 7, 19, 37, 61, 91, 127, 169 and 217. These numbers will be recognized as the first differences of the sequence of cubes, or the number of points in a hexagonal array. There may well be other values of k for which there are an infinity of solutions to (3).

We now go back to (2) and consider the cases where $3|k$. Substituting $k = 3h$ into (2) and dividing by 3 gives

$$hr^2 = n^2 + 3nh + 3h^2.$$

It is clear that $h|n$ (k is square free), so we substitute $n = hm$ and divide by h, whence

$$r^2 = hm^2 + 3hm + 3h.$$

We now see that $h|r$, so we set $r = hs$ and obtain

$$hs^2 = m^2 + 3m + 3. \tag{4}$$

Multiplying by 4 and rearranging gives

$$(2m + 3)^2 - h(2s)^2 = -3$$

which is again a Pell equation (unless $h = 1$, when the only solution is $s = 1$, $m = -1$). Note that h, like k, is square-free, but in addition it cannot have 3 as a factor, since then $9|k$ and k is not square-free. Thus $m \equiv \pm 1 \pmod 3$. Setting $s = 1$ in (4) shows us that if h has the form

$$h = m^2 + 3m + 3,$$

so that k has the form

$$k = 3(m^2 + 3m + 3),$$

there will be an infinity of solutions. We may thus add to our set of values for k the sequence beginning 21, 39, 93, 129, 219. There may be other values of k which afford solutions to (4).

Combining the solutions to (3) and (4) with values obtained by multiplying each of these by a square gives the following list of values of $k < 250$ for which there are an infinity of solutions: 1, 4, 7, 9, 16, 19, 21, 25, 28, 36, 37, 39, 49, 61, 63, 64, 76, 81, 84, 91, 93, 100, 112, 121, 127, 129, 144, 148, 156, 169, 171, 175, 189, 196, 217, 219 and 225.

Some correct values of k were also found by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; and KENNETH M. WILKE, Topeka, Kansas. Hess conjectures that k is a solution to the problem if and only if it is of the form $k = p_1p_2 \cdots p_j\ell^2$ or $k = 3p_1p_2 \cdots p_{j+1}\ell^2$ where $j \geq 0$ and each p_i is a prime $\equiv 1 \bmod 6$.

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<u>1011</u>. [1985: 49] *Proposed by Charles W. Trigg, San Diego, California.*

In base six, find a nine-digit square of the form AAAAAAXYZ, given that it is the square of a number whose central triad is XYZ.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let n^2 be the nine-digit square and n its five-digit square root. Then clearly

$$r(111{,}111{,}000_6) \leq n^2 \leq r(111{,}111{,}000_6) + 555_6 \qquad (1)$$

for some integer r, $1 \leq r \leq 5$. Expressing (1) in decimal gives

$$2{,}015{,}496r \leq n^2 \leq 2{,}015{,}496r + 215.$$

Taking square roots gives

$$1419.6816\sqrt{r} < n < 1419.6817\sqrt{r} + 0.076/\sqrt{r}. \qquad (2)$$

Evaluating the left-hand term of (2) for the possible values of r gives the following values:

r	$1419.6816\sqrt{r}$
1	1419.6816
2	2007.7331
3	2458.9608
4	2839.3633
5	3174.5047

It is clear that $r = 3$ gives the only value that is sufficiently close to an integer. Thus

$$n = 2459 = 15215_6,$$

and a little exercise performing long multiplication in base 6 gives

$$n^2 = 333{,}333{,}521_6.$$

The fact that the middle three digits of n are the same as the last three digits of n^2 was not needed, but it serves as a convenient check.

Also solved by SAM BAETHGE, San Antonio, Texas; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Valencia Community College, Orlando, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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1012. [1985: 49] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

An amateur winemaker is siphoning wine from a carboy. To speed up the process, he tilts the carboy to raise the level of the wine. Naturally, he wants to maximize the height, H, of the surface of the liquid above the table on which the carboy rests. The carboy is actually a circular cylinder, but we will only assume that its base is the interior of a smooth closed convex curve, C, and that the generators are perpendicular to the base. P is a point on C, T is the line tangent to C at P, and the cylinder is rotated about T.

(a) Prove that H is a maximum when the centroid of the surface of the liquid is vertically above T.

(b) Let the volume of the wine be V and let the area inside C be A. Assume that $V \geq AW/2$, where W is the maximum width of C (i.e., the maximum distance between parallel tangents). Obtain an explicit formula for H_M, the maximum value of H. How should P be chosen to maximize H_M?

Solution by the proposer.

(a) We choose coordinate axes with the origin at P and T along the x-axis. The y-axis is in the base of the tilted cylinder with $y \geq 0$ for points of C. The z-axis is parallel to the generators of the cylinder and $z \geq 0$ for points in the liquid.

The equation of the surface of the wine has the form $y = u - mz$. We take u (≥ 0) to be the independent variable and calculate m so that the volume below the surface is V. Let R be the region bounded by C and let w be the width of C at P (w will be the largest y-coordinate of points on C). Then

$$V = \iint zdxdy = (1/m)\iint(u - y)dxdy \tag{1}$$

where, if $u \leq$ w, the region of integration is the part of R, call it R_u, between the lines $y = 0$ and $y = u$. If $u \geq$ w, the region of integration is R.

For $u \leq$ w, set

$$f(u) = \int_0^u (x_2 - x_1)dy$$

where x_1 and x_2 are the x-coordinates of the left and right sides of C for a given y. For $u \geq w$, set

$$f(u) = \int_0^w (x_2 - x_1)dy = A.$$

Similarly, set

$$y(u) = \int_0^{\min(u,w)} y(x_2 - x_1)dy.$$

If $u \geq w$, $g(u) = A\overline{y}$ where $\overline{y}$ is the y-coordinate of the centroid of R. If $u \leq w$, $g(u)/f(u)$ is the y-coordinate of the centroid of R_u. From (1),

$$m = (uf - g)/V. \tag{2}$$

This gives m (≥ 0) as a function of u for $u \geq 0$.

H is the perpendicular distance from the origin to the plane $y + mz = u$; that is,

$$H = \frac{u}{\sqrt{1 + m^2}} .$$

Differentiating with respect to u,

$$(1 + m^2)^{3/2} H' = 1 + m^2 - umm' .$$

From (2), if $u \leq w$

$$\begin{aligned} m' &= \frac{1}{V}(uf - g)' \\ &= \frac{1}{V}(f + uf' - g') \\ &= \frac{1}{V}(f + u(x_2 - x_1) - u(x_2 - x_1)) \\ &= f/V , \end{aligned}$$

and if $u \geq w$ we again have $m' = f/V$. Therefore

$$\begin{aligned} (1 + m^2)^{3/2} H' &= 1 + \frac{(uf - g)^2}{V^2} - u\cdot\frac{(uf - g)}{V}\cdot\frac{f}{V} \\ &= 1 + \frac{(uf - g)}{V^2} (uf - g - uf) \\ &= 1 - \frac{g(uf - g)}{V^2} , \end{aligned}$$

and we see that H' has the same sign as

$$\Phi(u) = V^2 - g(uf - g). \tag{4}$$

Since $(uf - g)' = f \geq 0$, Φ is a decreasing function. Also $\Phi(0) > 0$ and $\Phi < 0$ for u large. Hence there is a unique u, say u_0, for which $\Phi = 0$. Φ and thus H' changes sign from positive to negative at u_0 and therefore H is a maximum at u_0.

The surface of the liquid is obtained by projecting R (if $u \geq w$) or R_u (if $u \leq w$) onto the plane $y + mz = u$. Since centroids project onto centroids, by similar triangles the coordinates of the centroid of the surface are of the form

$$(x_0,\ g/f\ ,\ u/m - g/fm).$$

The line joining this point to the point $(x_0,0,0)$ on T has direction numbers

$$\left[0,g,\frac{uf - g}{m}\right] = (0,g,V)$$

by (2). The vertical direction (the normal to the surface) is $(0,1,m)$, which by (2) is

$$(0,V,uf - g).$$

When H is a maximum, $\phi(u) = 0$, so by (4) this direction is

$$(0,V,V^2/g)$$

or

$$(0,g,V),$$

and therefore H is a maximum if and only if the centroid of the surface is vertically above T.

(b) If $V \geq AW/2$, then

$$V \geq Aw/2 \geq A\sqrt{\bar{y}(w - \bar{y})}\ .$$

By (4),

$$\begin{aligned}\phi(w) &= V^2 - g(wf - g)\\ &= V^2 - f^2\cdot\frac{g}{f}(w - \frac{g}{f})\\ &= V^2 - A^2\cdot\bar{y}(w - \bar{y})\end{aligned}$$

and thus $\phi(w) \geq 0$. Hence $u_0 \geq w$, and thus $f(u_0) = A$ and $\frac{g(u_0)}{f(u_0)} = \bar{y}$. Since $\phi(u_0) = 0$, (4) implies

$$u_0 = \frac{V^2 + g^2}{fg} = \bar{y} + \frac{V^2}{A^2\bar{y}}\ ,$$

so at maximum H (2) implies

$$m = \frac{u_0 f - g}{V} = \frac{\bar{y}A + V^2/A\bar{y} - \bar{y}A}{V} = \frac{V}{A\bar{y}}\ .$$

Thus by (3),

$$H_M = \frac{u_0}{\sqrt{1 + m^2}} = \frac{\bar{y} + \dfrac{V^2}{A^2\bar{y}}}{\sqrt{1 + \dfrac{V^2}{A^2\bar{y}^2}}} = \frac{\bar{y}^2 + \dfrac{V^2}{A^2}}{\sqrt{\bar{y}^2 + \dfrac{V^2}{A^2}}} = \sqrt{\frac{V^2}{A^2} + \bar{y}^2}\ .$$

In particular, to maximize H_M, we should maximize $\bar{y}$. That is, we should choose P so as to maximize the perpendicular distance from the centroid G of R to the tangent to C at P. This is done by simply maximizing PG, so that P should be chosen as the point on C furthest from G. Then H_M will occur when the centroid of the surface of the liquid is vertically above P.

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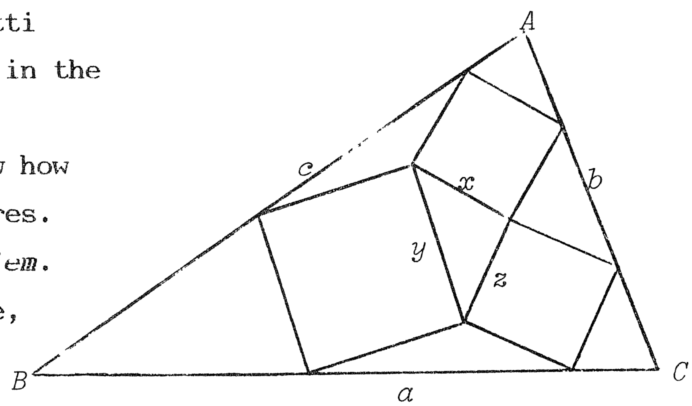
1013. [1985: 50] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai City, Aichi, Japan.*

This problem is about "Malfatti" squares, named by analogy with Malfatti circles. The concept is illustrated in the adjoining figure.



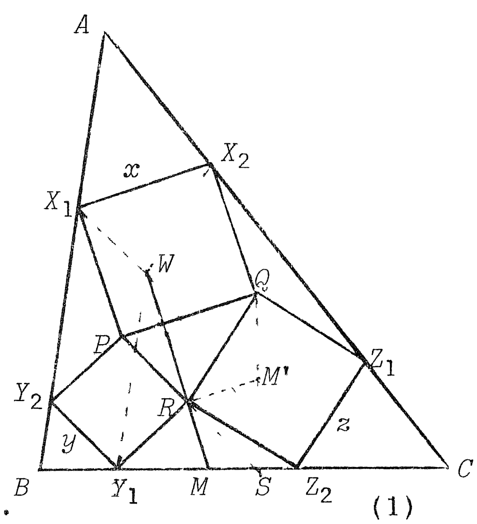
(a) Given a triangle ABC, show how to construct its three Malfatti squares.

(b) *The Malfatti squares problem.* Given the sides a, b, c of a triangle, calculate the sides x, y, z of its Malfatti squares.

(c) *The reverse Malfatti squares problem.* Given the sides x, y, z of the Malfatti squares of a triangle, calculate the sides a, b, c of the triangle.

Solution by Dan Sokolowsky, Brooklyn, N.Y.

(a) The figure shows a triangle ABC and its Malfatti squares $S_1 = QPX_1X_2$, $S_2 = PRY_1Y_2$, $S_3 = RQZ_1Z_2$. We provide an analysis of this figure which leads to a method for construction of these squares.



We first show that the medians m_a, m_b, m_c of ΔABC satisfy

$$m_a \perp PQ \ , \ m_b \perp PR \ , \ m_c \perp RQ. \qquad (1)$$

To do this, consider a rotation of ΔRY_1Z_2 through $90°$ about R, carrying Z_2 to Q, Y_1 to S, and the midpoint M of Y_1Z_2 to M'. Since $\angle PRY_1 = \angle Y_1RS = 90°$, the points P, R, and S are on a line. Then, since $PR = Y_1R = RS$ and $SM' = M'Q$, we have $RM = RM' = \frac{1}{2}PQ = x/2$ and $RM' \| PQ$, which implies that $RM \perp PQ$, or equivalently $RM \| X_1P$. Extend MR to W with $RW = X_1P = x$. Then WM is a median of ΔWY_1Z_2, and since $RW = x = 2RM$, R is in fact the centroid of ΔWY_1Z_2. Also, $WRPX_1$ is a parallelogram, so $X_1W = PR = Y_1Y_2$ and $X_1W \| PR \| Y_1Y_2$. Hence $WY_1Y_2X_1$ is a parallelogram, and

$$Y_1W = X_1Y_2 \ , \ Y_1W \| X_1Y_2 .$$

Similarly,

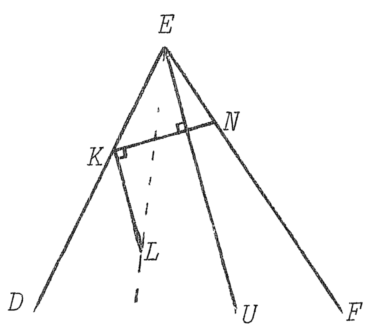
$$Z_2W = Z_1X_2 \ , \ Z_2W \| Z_1X_2 \ ,$$

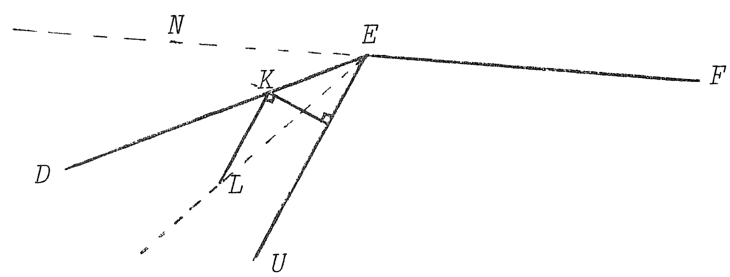
and thus

$$\Delta WY_1Z_2 \sim \Delta ABC .$$

Since the sides of ΔWY_1Z_2 are parallel to the corresponding sides of ΔABC, the same is true of their corresponding medians. Thus since $RM \perp PQ$ we have that $m_a \perp PQ$. Also, since R is the centroid of ΔWY_1Z_2, the median m_{Y_1} (resp. m_{Z_2}) of ΔWY_1Z_2 falls along line Y_1R (resp. Z_2R). Hence $m_b \| Y_1R$ and $m_c \| Z_2R$, and (1) follows.

Now consider the following general situation, consisting of an angle $\angle DEF < 180^\circ$ and a ray EU interior to $\angle DEF$. Let K be a point on ED, let N be on EF with $KN \perp EU$, and let L be interior to $\angle DEF$ such that $KL \| EU$ and $KL = KN$. The following diagrams show two typical configurations.



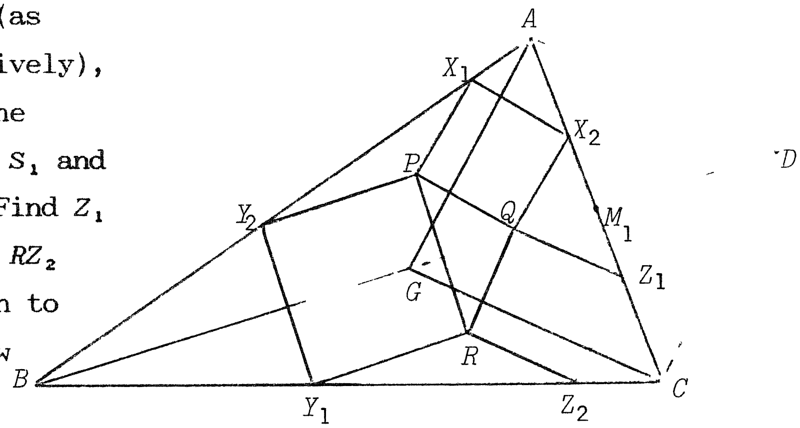


As K traverses ED the locus of L is a ray EV interior to $\angle DEU$ (hence to $\angle DEF$), as can easily be seen by similar triangles. This locus is easy to construct: one need only determine one point $L \neq E$ on it, and the ray EL is then the desired locus. Call this ray the locus relative to $\angle DEF$ and the ray EU.

Let G denote the centroid of ΔABC. By (1), P is the intersection of two of the above loci, L_1 and L_2, where L_1 is the locus relative to $\angle BAC$ and AG, and L_2 is the locus relative to $\angle ABC$ and BG. Note from the above that L_1 is interior to $\angle BAG$ and L_2 is interior to $\angle ABG$, so the intersection of L_1 and L_2 exists and is unique. Similar remarks for Q and R imply the uniqueness of the

Malfatti squares of ΔABC once their existence has been established. This we now do.

Having determined P as above, we can construct the lines PQ and PR (as perpendicular to AG and BG respectively), then the lines PX_1 and PY_2, then the points X_1 and Y_2, then the squares S_1 and S_2, and thus the points Q and R. Find Z_1 on AC and Z_2 on BC so that QZ_1 and RZ_2 are both perpendicular to QR. Then to complete the argument, we must show that $S_3 = RQZ_1Z_2$ is a square.



Since G is the centroid of ΔABC, BG meets AC at its midpoint M_1, and $BG = 2GM_1$. Extend GM_1 to D with $M_1D = GM_1$; then $GD = BG$, and moreover, since $AM_1 = M_1C$ and $GM_1 = M_1D$, $AGCD$ is a parallelogram, so $AD = CG$ and $AD \| CG$. Since $\Delta X_1PY_2 \sim \Delta AGB$,

$$\frac{PQ}{AG} = \frac{PX_1}{AG} = \frac{PY_2}{BG} = \frac{PR}{GD},$$

and since PQ and PR are perpendicular to AG and GD respectively, $\angle QPR = \angle AGD$. Hence $\Delta PQR \sim \Delta GAD$, and so

$$\frac{PR}{GD} = \frac{QR}{AD}$$

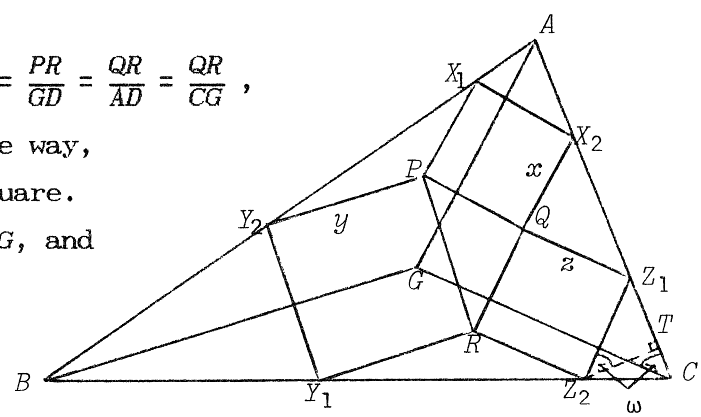
and also $\angle PQR = \angle GAD$. From this follows $QR \perp AD$, and so $QR \perp CG$. Thus $\Delta Y_1RZ_2 \sim \Delta BGC$, and hence

$$\frac{RZ_2}{CG} = \frac{Y_1R}{BG} = \frac{PR}{GD} = \frac{QR}{AD} = \frac{QR}{CG},$$

which implies $RZ_2 = QR$. In the same way, $QZ_1 = QR$. Therefore RQZ_1Z_2 is a square.

(b) Recall that $\Delta Y_1Z_2R \sim \Delta BCG$, and thus we may let

$$\frac{Y_1Z_2}{BC} = \frac{RY_1}{BG} = \frac{RZ_2}{CG} = \alpha,$$



that is,

$$\left.\begin{aligned} Y_1Z_2 &= a\alpha \\ y &= \frac{2}{3} m_b \alpha \\ z &= \frac{2}{3} m_c \alpha \end{aligned}\right\} \qquad (2)$$

since G is the centroid of ΔABC.

Let T be on AC so that $Z_2T \perp AC$. Then $\angle Z_1Z_2T = \angle ACG = \omega$, say. Thus

$$\cos \omega = \frac{AC^2 + CG^2 - AG^2}{2AC \cdot CG}$$

$$= \frac{b^2 + \frac{4}{9} m_c{}^2 - \frac{4}{9} m_a{}^2}{\frac{4}{3} m_c b}$$

$$= \frac{9b^2 + 4m_c{}^2 - 4m_a{}^2}{12m_c b} . \tag{3}$$

Using the known formulas

$$\left.\begin{aligned} m_a{}^2 &= \frac{2b^2 + 2c^2 - a^2}{4} \\ m_b{}^2 &= \frac{2a^2 + 2c^2 - b^2}{4} \\ m_c{}^2 &= \frac{2a^2 + 2b^2 - c^2}{4} \end{aligned}\right\} \tag{4}$$

(3) becomes

$$\cos \omega = \frac{9b^2 + (2a^2 + 2b^2 - c^2) - (2b^2 + 2c^2 - a^2)}{12m_c b}$$

$$= \frac{3a^2 + 9b^2 - 3c^2}{12m_c b}$$

$$= \frac{a^2 + 3b^2 - c^2}{4m_c b} .$$

Hence from (2),

$$Z_2T = z \cos \omega = \frac{2}{3} m_c \alpha \cdot \frac{a^2 + 3b^2 - c^2}{4m_c b}$$

$$= \frac{a^2 + 3b^2 - c^2}{2b} \cdot \frac{\alpha}{3}$$

$$= \left[a \cdot \frac{a^2 + b^2 - c^2}{2ab} + b\right]\frac{\alpha}{3}$$

$$= (a \cos C + b)\frac{\alpha}{3}$$

and so

$$Z_2C = \frac{Z_2T}{\sin C} = \frac{\alpha}{3}\left[\frac{a \cos C + b}{\sin C}\right] .$$

Similarly,

$$Y_1B = \frac{\alpha}{3}\left[\frac{a \cos B + c}{\sin B}\right] ,$$

and thus

$$Y_1B + Z_2C = \frac{\alpha}{3}\left[\frac{a\cos B + c}{\sin B} + \frac{a\cos C + b}{\sin C}\right]$$

$$= \frac{\alpha}{3}\left[\frac{a(\sin B\cos C + \cos B\sin C) + b\sin B + c\sin C}{\sin B\sin C}\right]$$

$$= \frac{\alpha}{3}\left[\frac{a\sin(B + C) + b\sin B + c\sin C}{\sin B\sin C}\right]$$

$$= \frac{\alpha}{3}\left[\frac{a\sin A + b\sin B + c\sin C}{\sin B\sin C}\right]. \qquad (5)$$

Letting Δ = area of ΔABC and using

$$\sin A = \frac{2\Delta}{bc}$$

$$\sin B = \frac{2\Delta}{ac}$$

$$\sin C = \frac{2\Delta}{ab}$$

(5) becomes

$$Y_1B + Z_2C = \frac{\alpha\cdot 2\Delta}{3}\left[\frac{\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}}{\frac{4\Delta^2}{a^2bc}}\right]$$

$$= \frac{\alpha}{6\Delta}\cdot a^2bc\left[\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right]$$

$$= \frac{\alpha a}{6\Delta}(a^2 + b^2 + c^2).$$

Thus from (2),

$$a = Y_1B + Z_2C + Y_1Z_2$$

$$= \frac{\alpha a}{6\Delta}(a^2 + b^2 + c^2) + \alpha a$$

whence

$$\alpha = \frac{6\Delta}{6\Delta + (a^2 + b^2 + c^2)},$$

which is symmetric in a, b, and c. Therefore from (2) we have

$$\left.\begin{aligned} x &= \frac{2}{3}m_a\alpha \\ y &= \frac{2}{3}m_b\alpha \\ z &= \frac{2}{3}m_c\alpha, \end{aligned}\right\} \qquad (6)$$

and thus from (4),

$$x = \frac{\alpha}{3}(2b^2 + 2c^2 - a^2)^{1/2}$$

$$y = \frac{\alpha}{3}(2a^2 + 2c^2 - b^2)^{1/2}$$

$$z = \frac{\alpha}{3}(2a^2 + 2b^2 - c^2)^{1/2} .$$

(c) Inverting (4) yields the formulas

$$a^2 = \frac{4}{9}(2m_b^2 + 2m_c^2 - m_a^2)$$
$$b^2 = \frac{4}{9}(2m_a^2 + 2m_c^2 - m_b^2)$$
$$c^2 = \frac{4}{9}(2m_a^2 + 2m_b^2 - m_c^2) ,$$

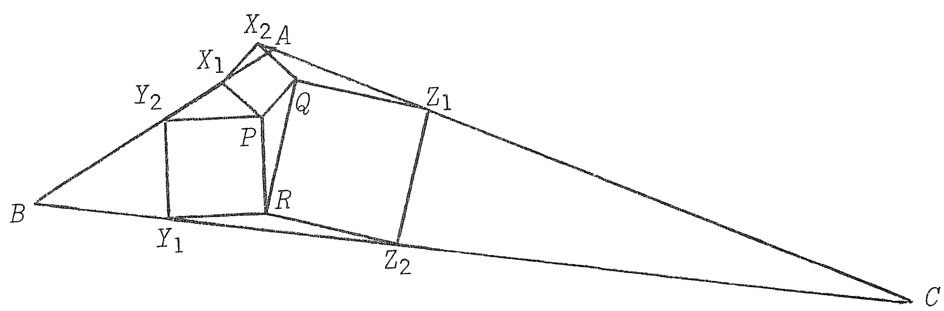
and from (6) we obtain

$$a^2 = \frac{4}{9}\left[2\cdot\frac{9}{4\alpha^2}y^2 + 2\cdot\frac{9}{4\alpha^2}z^2 - \frac{9}{4\alpha^2}x^2\right]$$
$$= \frac{1}{\alpha^2}(2y^2 + 2z^2 - x^2)$$

or

$$a = \frac{1}{\alpha}(2y^2 + 2z^2 - x^2)^{1/2}$$
$$b = \frac{1}{\alpha}(2x^2 + 2z^2 - y^2)^{1/2}$$
$$c = \frac{1}{\alpha}(2x^2 + 2y^2 - z^2)^{1/2} .$$

Finally, note that the Malfatti squares of a triangle need not be contained in the triangle. For instance, let $S_2 = PRY_1Y_2$ be an arbitrary square, and PX_1Y_2 an arbitrary triangle with $\angle PX_1Y_2 > 90^\circ$. Construct squares $S_1 = PX_1X_2Q$ and $S_3 = QZ_1Z_2R$ as shown. Intersect the lines X_1Y_2, Y_1Z_2, and Z_1X_2 to form ΔABC. Then S_1, S_2, S_3 are the Malfatti squares for ΔABC, but S_1 is not contained in ΔABC.

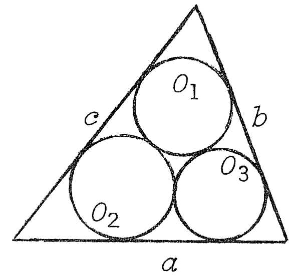


Comment by the proposer.

Here is a brief account of the famous Malfatti problem in the old Japanese mathematics, where it was named *Sansya Sanen* (problem of three circles in a given triangle).

- Chokuen Ajima (1732-1798), one of the great Japanese mathematicians of the time, gave the following problem and its solution in his manuscript of 1771-1773, about 30 years prior to Malfatti's 1803 paper (Memorie di Mate. e di Fisica della Società Italiana delle Scienze, Mondens, 10):

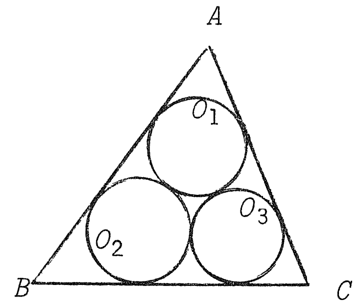
Problem: Given the sides a, b, and c of a triangle, calculate the radii of circles O_1, O_2, and O_3, as shown in the figure.



He gave the solution in general and the following example: if $a = 507$, $b = 375$, and $c = 252$, then $2r_1 = 128$, $2r_2 = 112.5$, and $2r_3 = 72$. (Japanese mathematicians of the time always considered the diameter instead of the radius, and tried to give examples where all numbers are integral or at least rational.)

- Teisi Fujita (1734-1807), a friend of Ajima, proposed an inverse problem, in a sense, as problem 42 in his 1781 book *Seiyo Sanpo* (Mathematics Detailed) Vol. 3:

Problem: Given three circles $O_1(r_1)$, $O_2(r_2)$ and $O_3(r_3)$ touching each other, construct the triangle ABC, as shown in the figure, and find the inradius of ΔABC.



He solved this problem, arriving at the same answer as G. Tsintsifas in this journal [1982: 83], although the methods differ.

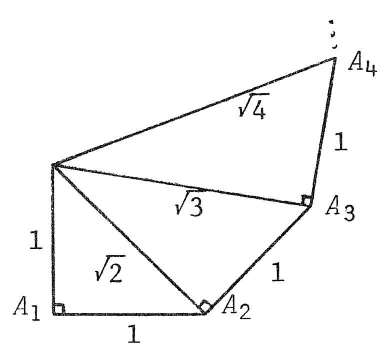
It was also in *Seiyo Sanpo* that Fujita proposed the Malfatti squares problem (not his terminology of course), which was part (b) of my proposed problem. Part (c), the inverse problem, was proposed by Gazen Yamamoto in 1841.

Also solved by the proposer.

* * *

1014*. [1985: 50] *Proposed by Shmuel Avital, Technion - Israel Institute of Technology, Haifa, Israel.*

The points $A_1, A_2, A_3, \ldots$ are chosen, by the familiar construction illustrated in the figure, in such a way that $OA_n = \sqrt{n}$, $n = 1,2,3,\ldots$.



(a) What is the nature of the smooth spiral that passes through $A_1, A_2, A_3, \ldots$?

(b) Find, in terms of n, an explicit formula for the measure of the rotation that ray OA_1 must undergo to bring it into coincidence with ray OA_n.

Comment by the Editor.

No solutions for this problem have been received. Hayo Ahlburg points out a related problem of Hans Havermann (#789 in *J. Recr. Math.* 11 (1978-79) 301), namely: do two of the lines OA_i, OA_j, $i \neq j$, ever coincide? Sadly, no answers were forthcoming for that problem either (*J. Recr. Math.* 12 (1979-80) 310). Ahlburg also notes the related problem #13 in Hugo Steinhaus' *One Hundred Problems in Elementary Mathematics* (reprinted by Dover, 1979). Murray Klamkin located a reference to the configuration of the problem in Arthur Engel, *Elementary Mathematics from an Algorithmic Standpoint* (English translation by F.R. Watson), Keele Mathematical Education Publications, University of Keele, Staffordshire, U.K., pp.79, 245. Here it is noted that OA_n increases by about π with each revolution.

* * *

1015. [1985: 50] *Proposed by Yang Lu, China University of Science and Technology, Hefei, Anhui, People's Republic of China.*

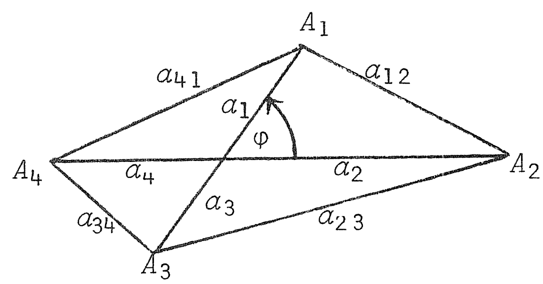
Let $A_1A_2A_3A_4$ be a convex quadrilateral, let a_{ij} denote the length of segment A_iA_j $(i,j = 1,2,3,4)$, and let R_1, R_2, R_3, R_4 be the circumradii of triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Prove that

$$(R_1R_2 + R_3R_4)a_{12}a_{34} + (R_1R_4 + R_2R_3)a_{14}a_{23} = (R_1R_3 + R_2R_4)a_{13}a_{24}\ . \qquad (1)$$

(This is an extension of Ptolemy's Theorem, for if $A_1A_2A_3A_4$ is cyclic, then $R_1 = R_2 = R_3 = R_4$, and (1) is equivalent to $a_{12}a_{34} + a_{14}a_{23} = a_{13}a_{24}$.)

Solution by N. Chavdarov and J. Tabov, Sofia, Bulgaria.

We use the following notation: O is the point of intersection of the diagonals of the quadrilateral, S_1 is the area of $\Delta A_2A_3A_4$, a_i is the length of the segment OA_i, and $\varphi = \angle A_1OA_2$.



First we will show that

$$a_{12}^2a_3a_4 + a_{23}^2a_4a_1 + a_{34}^2a_1a_2 + a_{41}^2a_2a_3 = a_{13}a_{24}(a_1a_3 + a_2a_4). \qquad (2)$$

Indeed, by the law of cosines,

$$a_{12}^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos \varphi$$
$$a_{23}^2 = a_2^2 + a_3^2 + 2a_2a_3 \cos \varphi$$
$$a_{34}^2 = a_3^2 + a_4^2 - 2a_3a_4 \cos \varphi$$
$$a_{41}^2 = a_4^2 + a_1^2 + 2a_4a_1 \cos \varphi$$

and obviously

$$a_{13} = a_1 + a_3$$

$$a_{24} = a_2 + a_4 \ .$$

Thus

$$\begin{aligned} & a_{12}^2a_3a_4 + a_{23}^2a_4a_1 + a_{34}^2a_1a_2 + a_{41}^2a_2a_3 \\ &\quad = (a_1^2 + a_2^2)a_3a_4 + (a_2^2 + a_3^2)a_4a_1 + (a_3^2 + a_4^2)a_1a_2 + (a_4^2 + a_1^2)a_2a_3 \\ &\quad = a_1a_3(a_1a_4 + a_3a_4 + a_2a_3 + a_1a_2) + a_2a_4(a_2a_3 + a_2a_1 + a_4a_1 + a_4a_3) \\ &\quad = (a_1a_3 + a_2a_4)(a_1 + a_3)(a_2 + a_4) \\ &\quad = a_{13}a_{24}(a_1a_3 + a_2a_4) \end{aligned}$$

so (2) is true.

Now since

$$S_1 = \frac{1}{2} a_{24}a_3 \sin \varphi$$

we have

$$\begin{aligned} R_1 &= \frac{a_{23}a_{34}a_{24}}{4S_1} \\ &= \frac{a_{23}a_{34}}{2a_3 \sin \varphi} \ . \end{aligned}$$

Similarly,

$$R_2 = \frac{a_{34}a_{41}}{2a_4 \sin \varphi} \ , \quad R_3 = \frac{a_{41}a_{12}}{2a_1 \sin \varphi} \ , \quad R_4 = \frac{a_{12}a_{23}}{2a_2 \sin \varphi} \ .$$

Putting

$$\lambda = \frac{a_{12}a_{23}a_{34}a_{41}}{4a_1a_2a_3a_4 \sin^2\varphi} \ ,$$

we obtain

$$\begin{aligned} a_{12}a_{34}R_1R_2 &= a_{12}a_{34} \cdot \frac{a_{23}a_{34}}{2a_3 \sin \varphi} \cdot \frac{a_{34}a_{41}}{2a_4 \sin \varphi} \\ &= \lambda a_{34}^2 a_1a_2 \end{aligned}$$

and

$$\begin{aligned} a_{13}a_{24}R_1R_3 &= a_{13}a_{24} \cdot \frac{a_{23}a_{34}}{2a_3 \sin \varphi} \cdot \frac{a_{41}a_{12}}{2a_1 \sin \varphi} \\ &= \lambda a_{13}a_{24}a_2a_4 \ , \end{aligned}$$

and similarly

$$\begin{aligned} a_{23}a_{41}R_2R_3 &= \lambda a_{41}^2 a_2a_3 \\ a_{34}a_{12}R_3R_4 &= \lambda a_{12}^2 a_3a_4 \\ a_{41}a_{23}R_4R_1 &= \lambda a_{23}^2 a_4a_1 \\ a_{24}a_{13}R_2R_4 &= \lambda a_{24}a_{13}a_1a_3 . \end{aligned}$$

Thus from (2),

$$\begin{aligned} & (R_1R_2 + R_3R_4)a_{12}a_{34} + (R_1R_4 + R_2R_3)a_{14}a_{23} \\ &\quad = \lambda a_{34}^2a_1a_2 + \lambda a_{12}^2a_3a_4 + \lambda a_{23}^2a_4a_1 + \lambda a_{41}^2a_2a_3 \\ &\quad = \lambda(a_{12}^2a_3a_4 + a_{23}^2a_4a_1 + a_{34}^2a_1a_2 + a_{41}^2a_2a_3) \end{aligned}$$

$$= \lambda a_{13}a_{24}(a_1a_3 + a_2a_4)$$
$$= a_{24}a_{13}R_2R_4 + a_{13}a_{24}R_1R_3$$
$$= a_{13}a_{24}(R_1R_3 + R_2R_4).$$

We remark that the following extension of Ptolemy's Theorem has recently appeared (E. Kraemer, Zobecneni vety Ptolemaiovy, *Rozhledy Mathematicko-fyzikalni* (Czechoslovakia) 63, no. 8, 345-349):

$$a_{13}^2a_{24}^2 = a_{12}^2a_{34}^2 + a_{23}^2a_{41}^2 - 2a_{12}a_{23}a_{34}a_{41}\cos(A_1 + A_3).$$

Other related ideas and results for quadrilaterals may be found in a paper of J.L. Coolidge, *Amer. Math. Monthly* 46 (1939) 345-347 (reprinted in the book *Selected Papers on Precalculus*, MAA, 1977).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

*   *   *

1016. [1985: 50] *Proposed by Andrew P. Guinand, Trent University, Peterborough, Ontario.*

(a) Show that, for the triangle with angles 120°, 30°, 30°, the nine-point centre lies on the circumcircle.

(b) Characterize all the triangles for which the nine-point centre lies on the circumcircle.

I. *Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

(a) Let ΔABC have $\alpha = \beta = 30^\circ$ and $\gamma = 120^\circ$. Let the circumcircle have centre O and radius R and let H be the orthocentre. Then it is easy to verify that ΔABH is equilateral, and thus $CH = CO = R$. Hence C is the midpoint of OH, so C is the centre of the nine-point circle.

(b) Such triangles are characterized by $OH = 2R$, that is, $(OH)^2 = 4R^2$. Using the known relation

$$(OH)^2 = 9R^2 - (a^2 + b^2 + c^2)$$

we have the characterization

$$a^2 + b^2 + c^2 = 5R^2.$$

II. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

A necessary and sufficient condition for the nine-point (Feuerbach) center to lie on the circumcircle is that $R = \frac{1}{2}OH$, or $R^2 = \frac{1}{4}(OH)^2$. Using

$$(OH)^2 = R^2(1 - 8\cos A\cos B\cos C)$$

(see, for example, (17) page 199 of E.W. Hobson's *A Treatise on Plane Trigonometry*), we have the condition

$$\cos A \cos B \cos C = -\frac{3}{8}. \tag{1}$$

Furthermore, using the known relation

$$2 \cos A \cos B \cos C = 1 - (\cos^2 A + \cos^2 B + \cos^2 C),$$

we have alternatively

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{7}{4}. \tag{2}$$

In particular, when $A = 120^\circ$, $B = C = 30^\circ$, (1) and (2) hold.

III. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We solve the problem in the complex plane. Choose the origin to be the circumcenter of ΔABC and let $A = e^{iu}$, $B = e^{iv}$, $C = e^{iw}$. Then

$$H = e^{iu} + e^{iv} + e^{iw}$$

and so the nine-point center F is

$$F = \frac{e^{iu} + e^{iv} + e^{iw}}{2}.$$

F lies on the circumcircle if and only if

$$1 = \left|\frac{e^{iu} + e^{iv} + e^{iw}}{2}\right|;$$

that is,

$$\begin{aligned}
4 &= |e^{iu} + e^{iv} + e^{iw}|^2 \\
&= (e^{iu} + e^{iv} + e^{iw})\overline{(e^{iu} + e^{iv} + e^{iw})} \\
&= (e^{iu} + e^{iv} + e^{iw})(e^{-iu} + e^{-iv} + e^{-iw}) \\
&= 3 + e^{i(u-v)} + e^{-i(u-v)} + e^{i(v-w)} + e^{-i(v-w)} + e^{i(w-u)} + e^{-i(w-u)} \\
&= 3 + 2\cos(u - v) + 2\cos(v - w) + 2\cos(w - u)
\end{aligned}$$

and thus

$$\cos(u - v) + \cos(v - w) + \cos(w - u) = \frac{1}{2}.$$

Since

$$\begin{aligned}
\cos(u - v) &= \cos \angle AOB = \cos(360^\circ - 2C) = \cos 2C \\
\cos(v - w) &= \cos \angle BOC = \cos 2A \\
\cos(w - u) &= \cos \angle AOC = \cos 2B
\end{aligned}$$

the required characterization is

$$\cos 2A + \cos 2B + \cos 2C = \frac{1}{2}.$$

The 30°, 30°, 120° triangle clearly satisfies this condition.

Also solved by R.H. EDDY, Memorial University of Newfoundland, St. John's, Newfoundland; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAN SOKOLOWSKY, Brooklyn, N.Y.; and the proposer.

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