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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE POWER MEAN AND THE HERON MEAN INEQUALITIES

Ji Chen and Zhen Wang

The Heron mean of two positive numbers a and b is defined by

$$\mathrm{He}(a,b) = \frac{a + \sqrt{ab} + b}{3}.$$

The power mean of two positive numbers a and b is defined by

$$M_p(a,b) = \left[\frac{a^p + b^p}{2}\right]^{1/p}$$

for any real number $p \neq 0$.

A result [1] about the ratios of the power means is that it is an increasing function of p, that is: *if*

$$b_1 \geq b_2 > 0, \quad \frac{a_1}{b_1} > \frac{a_2}{b_2} > 0, \quad \textit{and} \quad p < q,$$

then

$$\frac{M_p(a_1,a_2)}{M_p(b_1,b_2)} < \frac{M_q(a_1,a_2)}{M_q(b_1,b_2)}.$$

In this note, we determine the least value q and the greatest value p such that

$$\frac{M_p(a_1,a_2)}{M_p(b_1,b_2)} < \frac{\mathrm{He}(a_1,a_2)}{\mathrm{He}(b_1,b_2)} < \frac{M_q(a_1,a_2)}{M_q(b_1,b_2)} \tag{1}$$

is valid for all a_1, a_2, b_1, b_2 satisfying $b_1 \geq b_2 > 0$, $\frac{a_1}{b_1} > \frac{a_2}{b_2} > 0$.

Theorem 1. *If* $b_1 \geq b_2 > 0$ *and* $\frac{a_1}{b_1} > \frac{a_2}{b_2} > 0$ *then*

$$\frac{M_{1/2}(a_1,a_2)}{M_{1/2}(b_1,b_2)} < \frac{\mathrm{He}(a_1,a_2)}{\mathrm{He}(b_1,b_2)} < \frac{M_{2/3}(a_1,a_2)}{M_{2/3}(b_1,b_2)}.$$

Proof. We write the left hand inequality as

$$\frac{(\sqrt{a_1/a_2} + 1)^2}{a_1/a_2 + \sqrt{a_1/a_2} + 1} < \frac{(\sqrt{b_1/b_2} + 1)^2}{b_1/b_2 + \sqrt{b_1/b_2} + 1}. \tag{2}$$

Introducing the new variables

$$s = \sqrt{\frac{a_1}{a_2}}, \quad s' = \sqrt{\frac{b_1}{b_2}}$$

it follows that

$$s > s' \geq 1.$$

The function defined by

$$f(z) = \frac{(z+1)^2}{z^2 + z + 1}$$

is decreasing on $(1,\infty)$ since

$$f'(z) = \frac{1 - z^2}{(z^2 + z + 1)^2} < 0$$

for any $z > 1$. Since (2) is equivalent to $f(s) < f(s')$, it is valid.

Similarly, we write the right hand inequality as

$$\frac{(a_1/a_2 + \sqrt{a_1/a_2} + 1)^2}{((a_1/a_2)^{2/3} + 1)^3} < \frac{(b_1/b_2 + \sqrt{b_1/b_2} + 1)^2}{((b_1/b_2)^{2/3} + 1)^3}. \tag{3}$$

Introducing the new variables

$$t = \left[\frac{a_1}{a_2}\right]^{1/6}, \qquad t' = \left[\frac{b_1}{b_2}\right]^{1/6}$$

it follows that

$$t > t' \geq 1.$$

The function defined by

$$h(z) = \frac{(z^6 + z^3 + 1)^2}{(z^4 + 1)^3}$$

is decreasing on $(1,\infty)$ since

$$h'(z) = \frac{6z^2(1 + z^3 + z^6)(1 + z)(1 - z)^3}{(1 + z^4)^4} < 0$$

for $z > 1$. Since (3) is equivalent to $h(t) < h(t')$, it is also valid. □

The following theorem will show that the values 1/2 and 2/3 in Theorem 1 are indeed best possible.

Theorem 2. *If* $1/2 < r < 2/3$, *then*

$$\frac{M_r(a_1,a_2)}{M_r(b_1,b_2)} \quad \textit{and} \quad \frac{\mathrm{He}(a_1,a_2)}{\mathrm{He}(b_1,b_2)}$$

are not comparable.

Proof. For any $p > 1/2$, if we let

$$a_1 = (n + 1)^2, \quad a_2 = 1, \quad b_1 = n^2, \quad b_2 = 1,$$

then the left hand inequality of (1) becomes

$$\left[\frac{(n + 1)^{2p} + 1}{n^{2p} + 1}\right]^{1/p} < \frac{(n + 1)^2 + (n + 1) + 1}{n^2 + n + 1}. \tag{4}$$

We expand both sides of (4) in Maclaurin series in $1/n$, using $p > 1/2$ for the left side, and obtain

$$1 + \frac{2}{n} + \frac{1}{n^2} + o\left[\frac{1}{n^2}\right] < 1 + \frac{2}{n} + o\left[\frac{1}{n^2}\right],$$

which is not true. Thus

$$\frac{M_r(a_1,a_2)}{M_r(b_1,b_2)} \nleq \frac{\mathrm{He}(a_1,a_2)}{\mathrm{He}(b_1,b_2)}.$$

For the other direction, let

$$a_1 = 1 + x, \qquad a_2 = b_1 = b_2 = 1$$

for $x > 0$. Then the right hand inequality of (1) becomes

$$\mathrm{He}(1 + x,1) < M_q(1 + x,1). \tag{5}$$

Expand both sides of (5) as Maclaurin series in x. Then we have

$$1 + \frac{x}{2} - \frac{x^2}{24} + \cdots < 1 + \frac{x}{2} - \frac{1 - q}{8}x^2 + \cdots,$$

and so $q \geq 2/3$. Thus the theorem is proved. □

Reference.

[1] A.W. Marshall, I. Olkin, and F. Proschan, Monotonicity of ratios of means and other applications of majorization, in *Inequalities* (O. Shisha, ed.) 177–190, Academic Press, New York, 1967.

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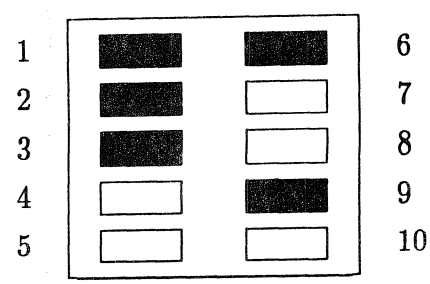
* * *

THE OLYMPIAD CORNER
No. 94
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first problems we present are from the American Invitational Mathematics Examination (A.I.M.E.) written March 22, 1988. The time allowed was three hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

1. One commercially available ten-button lock may be opened by depressing – in any order – the correct five buttons. The sample shown at right has {1,2,3,6,9} as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve



as combinations. How many additional combinations would this allow?

2. For any positive integer k, let $f_1(k)$ denote the square of the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

3. Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$.

4. Suppose that $|x_i| < 1$ for $i = 1,2,\ldots,n$. Suppose further that

$$|x_1| + |x_2| + \cdots + |x_n| = 19 + |x_1 + x_2 + \cdots + x_n|.$$

What is the smallest possible value of n?

5. Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find $m + n$.

6. It is possible to place positive integers into the twenty–one vacant squares of the 5×5 square shown on the right so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*).

			*	
	74			
				186
		103		
0				

7. In ΔABC, $\tan(\angle CAB) = 22/7$ and the altitude from A divides BC into segments of length 3 and 17. What is the area of ΔABC?

8. The function f, defined on the set of ordered pairs of positive integers, satisfies the following properties:

$$f(x,x) = x, \quad f(x,y) = f(y,x), \quad \text{and} \quad (x + y)f(x,y) = yf(x,x + y).$$

Calculate $f(14,52)$.

9. Find the smallest positive integer whose cube ends in 888.

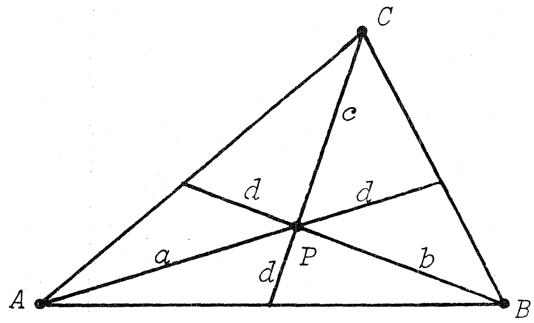
10. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face?

11. Let $w_1, w_2, \ldots, w_n$ be complex numbers. A line L in the complex plane is called a *mean line* for the points $w_1, w_2, \ldots, w_n$ if L contains points (complex numbers) $z_1, z_2, \ldots, z_n$ such that

$$\sum_{k=1}^{n} (z_k - w_k) = 0.$$

For the numbers $w_1 = 32 + 170i$, $w_2 = -7 + 64i$, $w_3 = -9 + 200i$, $w_4 = 1 + 27i$, and $w_5 = -14 + 43i$ there is a unique mean line with y–intercept 3. Find the slope of this mean line.

12. Let P be an interior point of ΔABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.



13. Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

14. Let C be the graph of $xy = 1$, and denote by C^* the reflection of C in the line $y = 2x$. Let the equation of C^* be written in the form

$$12x^2 + bxy + cy^2 + d = 0.$$

Find the product bc.

15. In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in–box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order 1, 2, 3, 4, 5, 6, 7, 8, 9.

While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after–lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.)

* * *

The other set of problems that we present this month is the *Tenth Annual Undergraduate Mathematics Competition* of the Atlantic Provinces Council on the Sciences, written Friday, October 30, 1987 (3 hours). Thanks go to Richard Nowakowski of Dalhousie University, Halifax, Nova Scotia for forwarding these problems. While some are not Olympiad level, I'm sure the readers will enjoy the challenge.

1. Find all different right–angled triangles with all sides of integral length whose areas equal their perimeters.

2. Sketch the graph and find the measure of the area bounded by the relation

$$|X - 60| + |Y| = |X/4|\ .$$

3. Suppose $S(X)$ is given by

$$S(X) = X(1 + X^2(1 + X^3(1 + X^4(1 + \cdots\ .$$

Is $S(1/10)$ rational?

4. Given a function f and a constant $c \neq 0$ such that

(i) $f(x)$ is even,

(ii) $g(x) = f(x - c)$ is odd,

prove that f is periodic and determine its period.

5. Show that

$$\sqrt{3} \le \exp\left[\int_{\pi/6}^{\pi/2} \frac{\sin x}{x}\, dx\right] \le 3.$$

6. Three equal circles each pass through the centres of the other two. What is the area of their common intersection?

7. Prove that $\dfrac{(1987^2)!}{(1987!)^{1988}}$ is an integer.

8. Find the sum of the infinite series

$$1 + 1/2 + 1/3 + 1/4 + 1/6 + 1/8 + 1/9 + 1/12 + \cdots$$

where the terms are reciprocals of integers divisible only by the primes 2 or 3.

* * *

We now turn to solutions submitted to problems presented in Volume 12, 1986.

1. [1986: 3] *1985 Dutch Mathematical Olympiad — Second Round.*

For a certain real number p, the equation

$$x^3 + px^2 + 3x - 10 = 0$$

has three real roots a, b, c satisfying $c - b = b - a > 0$. Determine p, a, b, and c.

Solutions by Beno Arbel, School of Mathematics, Tel Aviv University, Israel; Edward Doolittle, The University of Toronto, Ontario, Canada; and Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

We have

$$a + b + c = -p, \tag{1}$$

$$ab + bc + ca = 3, \tag{2}$$

$$abc = 10, \tag{3}$$

and, from the given extra equation,

$$a + c = 2b. \tag{4}$$

Equations (1) and (4) together give $b = -p/3$, $a + c = -2p/3$. Then, by (3), $ac = 10/b = -30/p$. This with (2) gives·

$$3 = b(a + c) + ca = (-p/3)(-2p/3) - 30/p,$$

or

$$2p^3 - 27p - 270 = (p - 6)(2p^2 + 12p + 45) = 0.$$

Since p is real, we must have $p = 6$. Then $b = -p/3 = -2$. Also $a + c = -4$, and $ac = -5$. These and $b > a$ imply $a = -5$, $c = 1$.

2. [1986: 4] *1985 Dutch Mathematical Olympiad — Second Round.*

Among the numbers $11n + 10^{10}$, $1 \le n \le 10^{10}$, how many are squares?

Solutions by Curtis Cooper, Central Missouri State University, U.S.A.; Edward Doolittle, The University of Toronto, Ontario, Canada; and Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

If $11n + 10^{10}$ is a square we let

$$10^5 + m = \sqrt{11n + 10^{10}} \tag{1}$$

where $1 \le m \le [(2\sqrt{3} - 1)10^5] = 24610$ since $1 \le n \le 10^{10}$.

Now squaring both sides of (1) and simplifying we obtain

$$11n = m(m + 2 \cdot 10^5).$$

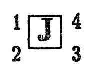
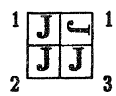
Thus $m \equiv 0$ or $2 \bmod 11$.

Therefore we have $22400 + 22401 = 44801$ numbers among $11n + 10^{10}$, $1 \le n \le 10^{10}$ which are squares, since the above steps are reversible.

3. [1986: 4] *1985 Dutch Mathematical Olympiad — Second Round.*

In a factory, square tables of size 40×40 cm^2 are tiled with four tiles of size 20×20 cm^2. All tiles are the same, and decorated in the same way with an asymmetric pattern such as the letter **J**. How many different types of tables can be produced in this way?

Solution by Edward Doolittle, The University of Toronto, Ontario, Canada.

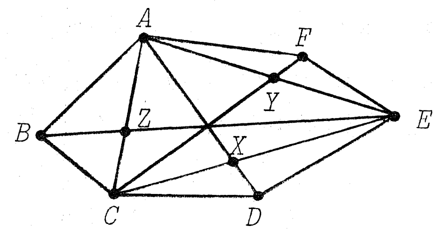
Since the tiles cannot be flipped, there are four possible orientations. Number the corners of each tile with the numbers 1, 2, 3, 4 (e.g. ). Now for each table, there is a 4–tuple consisting of the numbers of the tile corners at each of the corners in the table, listed counterclockwise starting at any corner. ( gives (1,2,3,1) or (1,1,2,3), etc.) Two tables are the same just in case one can be rotated to match the other, e.g. (1,1,2,3), (1,2,3,1), (2,3,1,1), (3,1,1,2) all give the same table.

Most configurations, like the above, admit 4 different labellings — in these the first pair of numbers can be chosen in 16 ways, and the second pair in 15 ways, giving $(16 \times 15)/4 = 60$ different tables. Other configurations have only two labellings, e.g. (1,3,1,3) and (3,1,3,1); in these the first pair can be chosen in 12 ways and the second pair is then determined, giving $12/2 = 6$ more tables. Finally we have (1,1,1,1), (2,2,2,2), (3,3,3,3) and (4,4,4,4). This gives a grand total of $60 + 6 + 4 = 70$ tables.

4. [1986: 4] *1985 Dutch Mathematical Olympiad – Second Round.*

A convex hexagon $ABCDEF$ is such that each of the diagonals AD, BE, and CF divides the hexagon into two parts of equal area. Prove that AD, BE, and CF are concurrent.

Solution by Edward Doolittle, The University of Toronto, Ontario, Canada.



Draw lines AC, CE and AE. Let AD intersect CE at X, let AE intersect CF at Y and let AC meet BE at Z. For a triangle MNP, let $[MNP]$ indicate the area, and let K be the area of the hexagon $ABCDEF$. Then because of equal altitudes we have

$$\frac{CX}{XE} = \frac{[ACX]}{[AXE]} = \frac{[CDX]}{[DEX]} = \frac{[ACX] + [CDX]}{[AXE] + [DEX]} = \frac{[ACD]}{[ADE]} = \frac{K/2 - [ABC]}{K/2 - [AEF]}.$$

Similarly we find

$$\frac{EY}{YA} = \frac{K/2 - [CDE]}{K/2 - [ABC]}, \qquad \frac{AZ}{ZC} = \frac{K/2 - [AEF]}{K/2 - [CDE]}.$$

Therefore

$$\frac{CX}{XE}\cdot\frac{EY}{YA}\cdot\frac{AZ}{ZC} = \frac{K/2 - [ABC]}{K/2 - [AEF]}\cdot\frac{K/2 - [CDE]}{K/2 - [ABC]}\cdot\frac{K/2 - [AEF]}{K/2 - [CDE]} = 1.$$

By Ceva's theorem the lines through AX, CY and EZ of triangle ACE are concurrent. Thus the diagonals of $ABCDEF$ meet in a point.

[*Editor's note*: For a proof of Ceva's theorem see, for example, pp.145–147 of *Advanced Euclidean Geometry* (*Modern Geometry*) by R.A. Johnson.]

*

1. [1986: 19] *1985 Austria–Poland Mathematical Competition.*

If a, b, c are distinct real numbers whose sum is zero, prove that

$$\left[\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right]\left[\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right] = 9.$$

Solutions by Curtis Cooper, Central Missouri State University, U.S.A.; Richard Gibbs, Fort Lewis College, Colorado, U.S.A.; Douglas Grant, University College of Cape Breton, N.S., Canada; and Bob Prielipp, University of Wisconsin–Oshkosh, U.S.A.

Clearly one must assume $abc(a-b)(a-c)(b-c) \neq 0$. It is not difficult to show that in general

$$\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z} = \frac{y^2z - yz^2 + xz^2 - x^2z + x^2y - xy^2}{xyz}$$
$$= \frac{-(x-y)(y-z)(z-x)}{xyz}.$$

Now for $a + b + c = 0$, letting $A = b - c$, $B = c - a$, $C = a - b$ we have

$$A - B = b - c - c + a = -3c,$$
$$B - C = -3a,$$
$$C - A = -3b.$$

Therefore

$$\left[\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right]\left[\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right]$$
$$= \left[-\frac{(b-c)(c-a)(a-b)}{abc}\right]\left[-\frac{1}{3}\right]\left[\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right]$$
$$= \left[-\frac{ABC}{a\,b\,c}\right]\left[\frac{1}{3}\frac{(B-C)(C-A)(A-B)}{ABC}\right]$$
$$= -\frac{1}{3}\frac{(-3a)(-3b)(-3c)}{abc}$$
$$= 9.$$

[*Editor's note*: while all four solutions were essentially the same, the presentation of Richard Gibbs, given above, was the most pleasant to read.]

2. [1986: 19] *1985 Austria–Poland Mathematical Competition.*

A given graph has $n \geq 8$ vertices. Is it possible for the vertices to have the respective valences $4,5,6,\ldots,n-4,n-3,n-2,n-2,n-2,n-1,n-1,n-1$?

Solution by Curtis Cooper, Central Missouri State University, U.S.A.

In this solution we will use a standard result from graph theory which can be found in many textbooks (for example, *Graph Theory* by F. Harary, Addison–Wesley, 1969, p.59).

A sequence of non–negative integers $D = (d_1,d_2,\ldots,d_n)$ with $d_1 \geq d_2 \geq \ldots \geq d_n$ is called *graphic* if there is a simple graph (with no loops or multiple edges) with degree ($\equiv$ valence) sequence D (in decreasing order).

Claim: if D is graphic then

(i) $d_1 + d_2 + \cdots + d_n$ is even, and

(ii) for $k = 1,2,\ldots,n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k,d_i).$$

Actually, Erdös and Gallai (1960) proved that (i) and (ii) are necessary and sufficient for D to be graphic. In this problem we shall only need the necessity stated in the claim. For completeness we reproduce a proof.

Assume $D = (d_1,d_2,\ldots,d_n)$ is graphic. Now $d_1 + d_2 + \cdots + d_n$ counts the number, e, of edges twice, since each edge is counted for each of its two endpoints. Hence

$$d_1 + d_2 + \cdots + d_n = 2e$$

is even, giving (i).

Now fix k with $1 \le k \le n$. Then

$$\sum_{i=1}^{k} d_i = d_1 + d_2 + \cdots + d_k$$

counts the number of edges incident with one of the vertices $v_1,\ldots,v_k$. These edges may be split into two sets, those for which both endpoints are in $\{v_1,\ldots,v_k\}$ and those connecting a vertex v_i, $1 \le i \le k$ with a vertex v_j, $k+1 \le j \le n$. Summing the contributions of the first sort we get at most $k(k-1)$ (remember these edges are counted twice!). For the second we count from the perspective of each v_j. The number of edges from v_j to $\{v_1,\ldots,v_k\}$ is at most k, and no more than the degree of v_j. Thus counting edges of this crossing type one has at most

$$\sum_{j=k+1}^{n} \min\{k,d_j\}\ .$$

Combining these two counts we have (ii).

Now to the problem at hand. Let $n \ge 8$ and

$$D = (n-1,n-1,n-1,n-2,n-2,n-2,n-3,n-4,\ldots,5,4).$$

For $n = 8$, $D = (7,7,7,6,6,6,5,4)$. A graph (in fact the only graph) realizing D is shown in Figure 1. In this and the next figure, all vertices inside the rectangle are joined to each other.

For $n = 9$, $D = (8,8,8,7,7,7,6,5,4)$. A graph realizing D is shown in Figure 2.

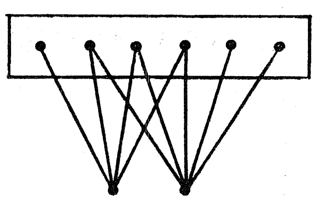


Figure 1

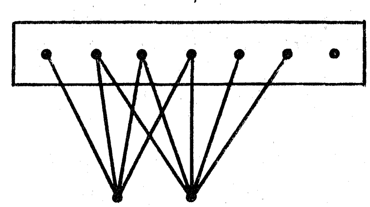


Figure 2

If $n \equiv 2 \bmod 4$ or $n \equiv 3 \bmod 4$ then D has an odd number of odd degrees. Hence $d_1 + d_2 + \cdots + d_n$ is odd and D is not graphic by (i).

Finally, suppose $n \equiv 0 \bmod 4$ or $n \equiv 1 \bmod 4$ and $n \ge 12$. Then, applying condition (ii) with $k = 7$, we get

$$\begin{aligned}\sum_{i=1}^{7} d_i &= (n-1)+(n-1)+(n-1)+(n-2)+(n-2)+(n-2)+(n-3)\\ &= 7n-12.\end{aligned}$$

But

$$7(7-1) + \sum_{i=8}^{n} \min(7,d_i) = 42 + 7(n-10) + 15$$

$$= 7n - 13$$

and we see that condition (ii) fails. Thus D is not graphic. Combining these observations we see that the degree sequence is possible for $n = 8$ and $n = 9$ and impossible for $n \geq 10$.

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3. [1986: 40] *1982 Leningrad High School Olympiad (Third Round).*

Write a sequence of digits, the first four of which are 1, 9, 8, 2 and such that each following digit is the last digit of the sum of the preceding four digits (in base 10). Does this sequence contain 3, 0, 4, 4 as consecutive digits? (Grade 8, 9.)

Solution by John Morvay, Dallas, Texas, U.S.A.

The solution is motivated by the one for problem 5 of the 1986 Canadian Mathematics Olympiad [1986: 176].

Group the sequence into blocks of 4 digits. Denote the first block $a_1 = 1982$. Then $a_2 = 0990$, $a_3 = 8637$, etc. Notice that not only does block a_i uniquely determine block a_{i+1}, but a_{i-1} is also uniquely determined. Indeed 3044 is the unique block of four that could precede 1982. (So "$a_0 = 3044$".) Next notice that as there are at most 10^4 possible blocks of length four, the sequence of blocks must repeat, i.e. $a_n = a_{n+k}$ for some n, k. Now, however, the fact that preceding blocks are repeated means that the cycle of blocks $a_n, a_{n+1}, \ldots, a_{n+k-1}$ must contain all possibilities, i.e. $a_1, a_2, \ldots, a_{n-1}$ also appear. Let $a_1 = a_i$ for some $i > 1$. Then $a_{i-1} = 3044$ and 3044 must appear.

5. [1986: 40] *1982 Leningrad High School Olympiad (Third Round).*

The cells in a 5×41 rectangular grid are two–colored. Prove that three rows and three columns can be selected so that the nine cells in their intersection have the same color. (Grade 8, 10.)

Solution by John Morvay, Dallas, Texas, U.S.A.

Any column of five cells has a "majority" color, that is, three or more cells in the column have the same color. One color must be the majority color for at least 21 columns. Now in each of these columns the rows containing the majority color could occur in at most $\binom{5}{3} = 10$ ways (always choosing the first three cells of the majority color, for definiteness). Thus, by the pigeonhole principle some selection of three rows must appear three or more times for the 21 columns. This gives the desired selection of rows and columns.

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We close this month's column with a reply to Murray Klamkin's appeal [1986: 39] for better formulations of problems 6 and 12 in the 1982 Leningrad High School Olympiad (Third Round). The reply is from Branko Grunbaum, The University of Washington, U.S.A. He says these are his "wild guesses" as to correct phrasings for 6 and 12 [1986: 40].

6. (*Rephrased*) The plane is divided into regions by $2n$ straight lines ($n > 1$) no two of which are parallel and no three of which pass through the same point. Prove that at most $2n - 1$ of these regions are "angles" (that is, bounded by just two of the lines).

12. (*Rephrased*) $4n$ points are marked on a circle and colored alternately red and blue. The points of each color are arbitrarily assigned to n pairs, and the points in each pair are connected by a line segment of the same color. The original points are chosen so that no three of the segments intersect at a point. Show that the red segments and the blue segments intersect in at least n points.

* * *

As always we solicit your problem sets and solutions. As this is the season for Olympiads I remind you to please send me a copy of your national Olympiad contest.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1988,** *although solutions received after that date will also be considered until the time when a solution is published.*

1331. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find the smallest positive integer a so that

$$13 \mid 11^{3n+1} + a\cdot 5^n \quad \text{and} \quad 31 \mid 23^{2n+1} + 2^{n+a}$$

both hold for all positive integers n.

1332. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is known that if A, B, C are the angles of a triangle,

$$\sin A/2 + \sin B/2 + \sin C/2 \geq 1,$$

with equality if and only if the triangle is degenerate with angles π, 0, 0. Establish the related non–comparable inequality

$$\sin A/2 + \sin B/2 + \sin C/2 \geq 5r/R - 1,$$

where r and R are the inradius and circumradius respectively.

1333. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

If a, b, c and a', b', c' are the sides of two triangles and F, F' are their areas, show that

$$\sum a[a' - (\sqrt{b'} - \sqrt{c'})^2] \geq 4\sqrt{3FF'},$$

where the sum is cyclic. (This improves *Crux* 1114 [1987: 185].)

1334. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Suppose Fibonacci had wanted to set up an annuity that would pay F_n lira on the nth year after the plan was established, for $n = 1,2,3,\ldots$ ($F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$). To fund such an annuity, Fibonacci had planned to invest a sum of money with the Bank of Pisa[1], which paid 70% interest per year, and instruct them to administer the trust. How much money did he have to invest so that the annuity could last in perpetuity?

(b) When he got to the bank, Fibonacci found that their interest rate was only 7% (he had misread their ads), not enough for his purposes. Despondently, he went looking for another bank with a higher interest rate. What rate must he seek in order to allow for a perpetual annuity?

1335. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Let A and B be points on the positive x and y axes, respectively, such that AB is tangent to the curve $xy = 1$. Let Q be on AB so that $OQ \perp AB$. Find the area in the first quadrant enclosed by the locus of Q.

1336. *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia.*

For each natural number $n \geq 2$, express the number

$$55\underbrace{99\ldots9}_{n-2}89\underbrace{33\ldots3}_{n-1}9$$

as a sum of squares of three natural numbers.

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[1][They'd held a lien on a tower he once owned.]

1337. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Consider integers a, b, c such that (*) the equation $ax^2 + by^2 + cz^2 = 0$ has a solution in integers x, y, z not all zero, and every solution $(x,y,z) \neq (0,0,0)$ satisfies

$$\gcd(x,y) > 1, \quad \gcd(y,z) > 1, \quad \gcd(z,x) > 1.$$

(i) Show that $a = 9$, $b = 25$, $c = -98$ satisfies (*).

(ii) Find infinitely many triples (a,b,c) satisfying (*).

1338. *Proposed by Jean Doyen, Université Libre de Bruxelles, Brussels, Belgium; J. Chris Fisher, University of Regina, Regina, Saskatchewan; and Gunter Kist, Technische Universitat, Munich, Federal Republic of Germany.*

In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen from 1 to 49 (inclusive). A *t–prize* is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of winning a t–prize. Clearly $f(1) = 8$ and $f(6) = \binom{49}{6}$. Show that $f(2) \leq 19$. Can you do better?

1339. *Proposed by Weixuan Li, Changsha Railway Institute, Changsha, Hunan, China, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a, b, m, n denote positive real numbers such that $a \leq b$ and $m \leq n$. Show that

$$(b^m - a^m)^n \leq (b^n - a^n)^m$$

and determine all cases when equality holds.

1340. *Proposed by Jordi Dou, Barcelona, Spain.*

Given are lines a, b, m and a point P, all in the same plane. Find a line r through P such that the point $r \cap m$ is the midpoint of the segment joining $r \cap a$ and $r \cap b$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1207. [1987: 15] *Proposed by G.A. Chambers and M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If A and B are $m \times n$ and $n \times m$ matrices, respectively, with $m \geq n$, and AB is an identity matrix, prove that $m = n$. (A weaker form of this problem was proposed by the second proposer as a Quickie in *Mathematics Magazine* some years ago.)

Solution by Carles Romero, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain.

The product AB is a square $m \times m$ matrix and, since $AB = I$, we have rank$(AB) = m$. Moreover, rank$(A) \le \min\{m,n\}$ and rank$(B) \le \min\{m,n\}$. But always rank$(AB) \le$ rank(A) and rank$(AB) \le$ rank(B) and in both situations $m \le n$. This implies $m = n$.

Actually AB only needs to be a nonsingular matrix.

Also solved by SEUNG JIN BANG, Seoul, Korea; W. JOHN BRAUN, University of Calgary; J.L. BRENNER, Palo Alto, California; DUANE M. BROLINE, Eastern Illinois University, Charleston, Illinois; KEE–WAI LAU, Hong Kong; PETER ROSS, University of Santa Clara, Santa Clara, California; DANIEL B. SHAPIRO, The Ohio State University, Columbus, Ohio; and the proposers.

Shapiro points out that the result is well–known, and remains true if the entries of the matrices come from an arbitrary commutative ring. He gives the reference: P.M. Cohn, Some remarks on the invariant basis property, Topology *5 (1966) 215–228.*

* * *

1208. [1987: 15] *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Let A', B', C' be points on sides BC, CA, AB, respectively, of ΔABC such that

$$\frac{A'B}{BC} = \frac{B'C}{CA} = \frac{C'A}{AB} \quad (\ne 0\,,\,1/2\,,\,1)$$

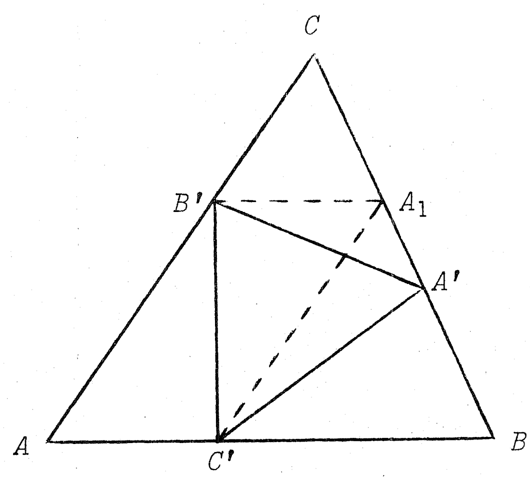
and so that some angle of ΔABC is equal to some angle of $\Delta A'B'C'$. Show that ΔABC and $\Delta A'B'C'$ are indirectly similar. In consequence, show that if they are directly similar then they are equilateral.

Solution by Jordi Dou, Barcelona, Spain.

Let A_1, B_1, C_1 be the points symmetric to A', B', C' with respect to the median points of BC, CA, AB, respectively. (Note $A_1 \ne A'$, B, or C, and similarly for B_1 and C_1.) Then

$$\frac{A'C}{BC} = \frac{A'B}{BC} = \frac{B'C}{CA},$$

so $B'A_1$ is parallel to AB. Similarly A_1C' is parallel to CA.



Suppose that $\angle B'C'A' = \angle B$. Then

$$\angle B'A_1A' + \angle B'C'A' = \angle B'A_1A' + \angle B = 180^\circ,$$

so $B'A_1A'C'$ is cyclic. Thus

$$\angle B'A'C' = \angle B'A_1C' = \angle A$$

and so triangles ABC and $A'B'C'$ are indirectly similar.

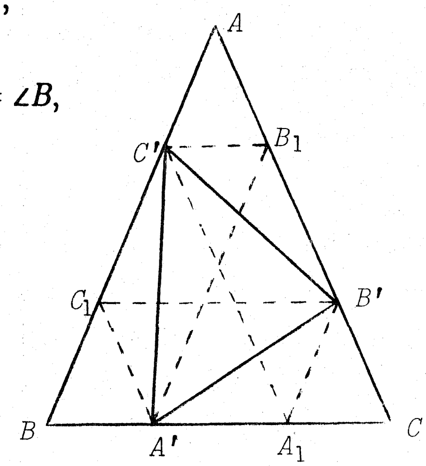
The only other case we need consider is that $\angle B'A'C' = \angle A$. Then

$$\angle B'A_1C' = \angle A = \angle B'A'C',$$

so again $B'A_1A'C'$ is cyclic. Thus

$$\angle B'C'A' = 180^\circ - \angle B'A_1A' = \angle B,$$

so triangles ABC and $A'B'C'$ are again indirectly similar.



If $\Delta A'B'C'$ and ΔABC are also directly similar then they are isosceles. We may suppose $\angle B = \angle C$. If $\angle B'C'A' = \angle A$ then, as above, A', B', C', B_1 all lie on a circle which, by symmetry, also passes through A_1 and C_1. Then

$$BA' \cdot BA_1 = BC_1 \cdot BC',$$

and so

$$1 = \frac{BA'}{BC_1} \cdot \frac{BA_1}{BC'} = \left[\frac{BC}{BA}\right]^2.$$

Therefore $BC = BA$ and ΔABC is equilateral. Analogously, if $\angle C'A'B' = \angle A$ then A', B', C', A_1 lie on a circle which also contains B_1 and C_1, and ΔABC is again equilateral.

Also solved (in a similar way) by the proposer. Two other readers sent in solutions which appear to be incomplete.

* * *

<u>1209</u>. [1987: 16] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Characterize all positive integers a and b such that

$$a + b + (a,b) \le [a,b],$$

and find when equality holds. Here (a,b) and $[a,b]$ denote respectively the g.c.d. and l.c.m. of a and b.

Solution by M.M. Parmenter, Memorial University of Newfoundland, St. John's.

We show that the inequality holds if and only if $a \neq (a,b)$ and $b \neq (a,b)$, that is, neither of a and b divide the other, and that equality holds if and only if $a = 2x$, $b = 3x$ or $a = 3x$, $b = 2x$ for some integer x.

If c is any positive integer, then a and b are a solution of the inequality if and only if ca and cb are. Hence we will find the "primitive" solutions, namely we will assume $(a,b) = 1$ and solve

$$a + b + 1 \le ab. \tag{1}$$

If $a = 1$ or $b = 1$, we see directly there is no solution.

Otherwise, (1) is equivalent to

$$1 + \frac{2}{a-1} \le b. \tag{2}$$

This can only fail when $a = 2$ and $b = 2$, but this would contradict $(a,b) = 1$.

The left side of (2) is an integer only when $a = 2$ or $a = 3$, so equality holds in (1) only when $a = 2$, $b = 3$ or $a = 3$, $b = 2$.

Also solved by SEUNG–JIN BANG, Seoul, Korea; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; DUANE M. BROLINE, Eastern Illinois University, Charleston, Illinois; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD K. GUY, University of Calgary; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; ROGER LEE, student, White Plains H.S., White Plains, N.Y.; BOB PRIELIPP, University of Wisconsin–Oshkosh; CARLES ROMERO, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; KENNETH M. WILKE, Topeka, Kansas; JURGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer.

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1210. [1987: 16] *Proposed by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

If A, B, C are the angles of an acute triangle, prove that

$$(\tan A + \tan B + \tan C)^2 \geq (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$

Solution by Duane M. Broline, Eastern Illinois University, Charleston, Illinois.

Since A, B, C are acute, all the standard trigonometric functions of these angles are positive.

Note that

$$\begin{aligned}(\cos A - \cos B)^2 &= \cos^2 A + \cos^2 B - 2\cos A\cos B \\ &= -\cos(B + C)\cos A - \cos(A + C)\cos B - 2\cos A\cos B \\ &= -\cos B\cos C\cos A + \sin B\sin C\cos A - \cos A\cos C\cos B \\ &\qquad + \sin A\sin C\cos B - 2\cos A\cos B \\ &= \cos A\cos B\cos C(\tan B\tan C + \tan A\tan C - 2\sec C - 2).\end{aligned}$$

Therefore, since the left side is nonnegative,

$$\tan B\tan C + \tan A\tan C \geq 2\sec C + 2.$$

Similarly,

$$\tan A\tan B + \tan A\tan C \geq 2\sec A + 2$$

and

$$\tan A\tan B + \tan B\tan C \geq 2\sec B + 2.$$

Adding, we get

$$\tan A\tan B + \tan A\tan C + \tan B\tan C \geq \sec A + \sec B + \sec C + 3. \qquad (1)$$

Hence

$$(\tan A + \tan B + \tan C)^2 = \tan^2 A + \tan^2 B + \tan^2 C$$
$$+ 2(\tan A \tan B + \tan A \tan C + \tan B \tan C)$$
$$\geq \tan^2 A + \tan^2 B + \tan^2 C + 2(\sec A + \sec B + \sec C) + 6$$
$$= \sec^2 A + \sec^2 B + \sec^2 C + 2(\sec A + \sec B + \sec C) + 3$$
$$= (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$

Also solved by S.J. BILCHEV, Technical University, Russe, Bulgaria; GEORGE EVAGELOPOULOS, Athens, Greece; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin–Whitewater; BOB PRIELIPP, University of Wisconsin–Oshkosh; ROBERT E. SHAFER, Berkeley, California; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Some solvers noted that inequality (1) *above appears as item* 2.63 *of Bottema et al,* Geometric Inequalities, *which incidentally should also have been restricted to acute triangles.*

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1211. [1987: 52] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Let $f(n)$ be the smallest integer greater than n such that $6^{f(n)}$ ends in the digits of 6^n. For example, $f(2) = 7$ since $6^7 = 279936$ is the next smallest power of 6 ending in $6^2 = 36$. Find a formula for $f(n)$.

Solution by Kee–wai Lau, Hong Kong.

We show that for $n = 1,2,\ldots$,

$$f(n) = 5^{[n \log 6]} + n, \tag{1}$$

where $[x]$ is the greatest integer less than or equal to x, and the logarithm is of base 10.

Since $6^{f(n)}$ ends in the digits of 6^n, the integer

$$6^{f(n)} - 6^n = 6^n(6^{f(n)-n} - 1)$$

ends with at least $[n \log 6] + 1$ zeros. As

$$(6^n, 6^{f(n)-n} - 1) = 1$$

we must have

$$2^{[n \log 6]+1} \mid 6^n \tag{2}$$

and

$$5^{[n \log 6]+1} \mid 6^{f(n)-n} - 1. \tag{3}$$

Note that (2) is trivial since it can be checked easily that for $n = 1,2,\ldots$, $n \geq [n \log 6] + 1$. Therefore, in view of (3), to prove (1) it suffices to show that the least positive solution of the congruence equation

$$6^x \equiv 1 \bmod 5^k \tag{4}$$

is $x = 5^{k-1}$ $(k = 1,2,\ldots)$.

We first prove by induction that $x = 5^{k-1}$ is a solution of (4). The assertion is clearly true for $k = 1$. If the assertion is true for $k = t$, then there is an integer m such that

$$\begin{aligned} 6^{5^t} &= \left(6^{5^{t-1}}\right)^5 = (m\cdot 5^t + 1)^5 \\ &= m^5 5^{5t} + 5m^4 5^{4t} + 10m^3 5^{3t} + 10m^2 5^{2t} + m5^{t+1} + 1 \\ &\equiv 1 \bmod 5^{t+1}. \end{aligned}$$

Thus the assertion is also true for $k = t + 1$.

It follows that the least positive solution of (4) is a factor of 5^{k-1}. Hence to show that the least positive solution is 5^{k-1}, it remains to show that

$$6^{5^{k-2}} \not\equiv 1 \bmod 5^k$$

for $k = 2,3\dots$. In fact,

$$6^{5^{k-2}} \equiv 5^{k-1} + 1 \bmod 5^k,$$

which can be readily shown by induction.

Also solved by RICHARD K. GUY, University of Calgary; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Altogether, five readers submitted incorrect answers. Four of them, ***including the proposer****, claimed that $f(n) = 5^n + n$, due to an assumption that 6^n has $n + 1$ digits. Such an argument is, of course, off base.*

Janous proved, in a very similar fashion to Lau, the following generalization: if p is an odd prime, and $N = p^a\cdot m$ where $a \geq 1$ and $m < p$ divides $p + 1$, then the minimal exponent $f(n) > n$ such that $(p + 1)^{f(n)}$ in base N ends in the base N digits of $(p + 1)^n$ is

$$f(n) = n + p^{a[n \log_N(p+1)]+a-1}.$$

The given problem corresponds to $p = 5$, $a = 1$, $m = 2$, $N = 10$.

* * *

<u>1212</u>. [1987: 52] *Proposed by Svetoslav Bilchev, Technical University, and Emilia Velikova, Mathematikalgymnasium, Russe, Bulgaria.*

Prove that

$$\frac{u}{v+w}\cdot\frac{bc}{s-a} + \frac{v}{w+u}\cdot\frac{ca}{s-b} + \frac{w}{u+v}\cdot\frac{ab}{s-c} \geq a + b + c$$

where a, b, c are the sides of a triangle and s is its semiperimeter, and u, v, w are arbitrary positive real numbers.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We give in fact a joint solution of this problem and *Crux* 1221 [1987: 85], which says that if a, b, c, u, v, w are as above, $0 < t \leq 2$, and F is the area of the triangle, then

$$\frac{u}{v+w}(bc)^t + \frac{v}{w+u}(ca)^t + \frac{w}{u+v}(ab)^t \geq \frac{3}{2}\left[\frac{4F}{\sqrt{3}}\right]^t.$$

In [1986: 253] the inequality

$$\sum \frac{u}{v + w}(bc)^2 \geq 8F^2, \tag{1}$$

where the sum is cyclic, is reported. (For a proof, see [1].) We also employ the following result of Oppenheim (cf. [3]): for $0 < p \leq 1$, a^p, b^p, c^p are the sides of a triangle of area

$$F_p \geq \left[\frac{\sqrt{3}}{4}\right]^{1-p} F^p. \tag{2}$$

Then (1) and (2) yield

$$\sum \frac{u}{v + w}(bc)^{2p} \geq 8F_p^2 \geq \frac{3}{2}\left[\frac{4F}{\sqrt{3}}\right]^{2p}. \tag{3}$$

Upon letting $2p = t$, *Crux* 1221 follows.

For the present problem, it is known (cf. [2], p.51, problem 20) that, if a, b, c form a triangle, then so do

$$\sqrt{a(s - a)}, \quad \sqrt{b(s - b)}, \quad \sqrt{c(s - c)},$$

and that the area of the new triangle is half the area of the old one. Replacing a by $\sqrt{a(s - a)}$, etc. in (3), we get for $0 < p \leq 1$

$$\sum \frac{u}{v + w}[bc(s - b)(s - c)]^p \geq \frac{3}{2}\left[\frac{2F}{\sqrt{3}}\right]^{2p}.$$

As

$$(s - b)(s - c) = \frac{F^2}{s(s - a)}, \qquad \text{etc.,}$$

this inequality becomes

$$\sum \frac{u}{v + w}\left[\frac{bc}{s - a}\right]^p \geq \frac{3}{2}\left[\frac{2F\sqrt{s}}{F\sqrt{3}}\right]^{2p} = \frac{3}{2}\left[\frac{4s}{3}\right]^p.$$

The special case $p = 1$ gives the required result.

References:

[1] M.S. Klamkin, Two non–negative quadratic forms, *Elem. der Mathematik*, 28 (1973) 141–146.

[2] G. Chrystal, *Algebra*, Chelsea, New York, 1964.

[3] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Univ. Belgrade Publ. Fak. Ser. Mat. Fiz.*, No.461–497 (1974) 257–263.

Also solved by MURRAY S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposers.

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<u>1213</u>*. [1987: 52] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

In *Math. Gazette* 68 (1984) 222, P. Stanbury noted the two close approximations $e^6 \approx \pi^5 + \pi^4$ and $\pi^9/e^8 \approx 10$. Can one show without a calculator that (i) $e^6 > \pi^5 + \pi^4$ and (ii) $\pi^9/e^8 < 10$?

Solution by Kee–wai Lau, Hong Kong.

(i) The number $(e-1)/2$ is greater than the seventh convergent of its continued fraction

$$\cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cdots}}},$$

i.e.

$$\frac{e-1}{2} > \frac{342762}{398959}$$

or

$$e > \frac{1084483}{398959}.$$

On the other hand, π is less than the sixth convergent of its continued fraction

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cdots}}},$$

i.e.

$$\pi < \frac{104348}{33215}.$$

Thus to prove (i) it suffices to show that

$$\left[\frac{1084483}{398959}\right]^6 > \left[\frac{104348}{33215}\right]^5 + \left[\frac{104348}{33215}\right]^4,$$

or

$$1084483^6 \cdot 33215^5 > 398959^6 \cdot 104348^4 \cdot 137563.$$

The last inequality can be verified by direct multiplication although the process is very tedious. In fact the left hand side is

$$6576696114902547737143476392266091928625707490557145333 4375$$

and the right hand side is

$$65766958301005110425229294448683820887085925791566542019328.$$

(ii) π is less than its fourth convergent $355/113$ and $(e-1)/2$ is greater than its fifth convergent $860/1001$, i.e.

$$e > \frac{2721}{1001}.$$

Thus it suffices to show that

$$\left[\frac{355}{113}\right]^9 < 10\left[\frac{2721}{1001}\right]^8,$$

or

$$226 \cdot 307473^8 > 71 \cdot 355355^8.$$

In fact the left hand side is

18053621007711264160103955243621587965885364706

and the right hand side is

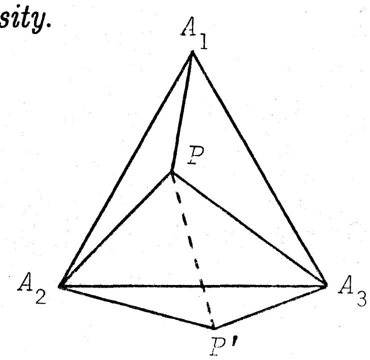
18053349630056540587683693471311397928527734375.

* * *

1214. [1987: 52] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be an equilateral triangle and let P be an interior point. Show that there is a triangle with side lengths PA_1, PA_2, PA_3.

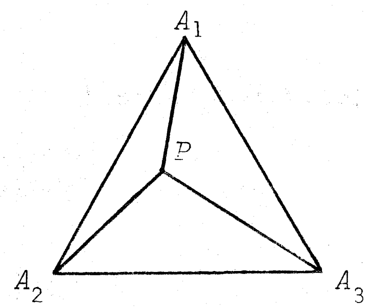
I. *Solution by Leroy F. Meyers, The Ohio State University.*



Rotate ΔA_1A_2P through an angle of 60° about A_2 so that it becomes the triangle A_3A_2P'. Then $P'A_2 = PA_2$ and $P'A_3 = PA_1$. Since $\angle PA_2P' = 60^\circ$, $\Delta A_2PP'$ is equilateral, and so $PP' = PA_2$. But then $\Delta PA_3P'$ has sides of lengths PA_1, PA_2, and PA_3.

Note that if P is outside the triangle, then the same argument can be used, although it is possible to obtain a degenerate triangle PA_3P'.

II. *Solution by several readers.*



Let

$$PA_1 \leq PA_2 \leq PA_3$$

without loss of generality. Then

$$PA_3 < A_2A_3 = A_1A_2 < PA_1 + PA_2,$$

so the result follows.

III. *Editor's comments.*

Several of the solvers of this problem (all of whom are listed below) pointed out that stronger results are already known. Konhauser mentions the theorem of Pompeiu: *the distances from the vertices of an equilateral triangle to a point in its plane are the sides of a triangle except if the point is on the circumcircle of the equilateral triangle* (*in which case the triangle formed is degenerate*). Klamkin gives the even more general result: if $A_1A_2A_3$ is an arbitrary triangle and P is any point in or out of its plane, then

$$A_2A_3 \cdot PA_1, \quad A_3A_1 \cdot PA_2, \quad \text{and} \quad A_1A_2 \cdot PA_3$$

satisfy the triangle inequality. There is equality (i.e. the triangle formed is degenerate) if and only if P lies on the circumcircle of $\Delta A_1A_2A_3$. The given problem corresponds to the case $A_1A_2 = A_2A_3 = A_3A_1$. Both of the above results are listed under item 15.5 of Bottema et al, *Geometric Inequalities.*

Solution II of the given problem is also contained in solution II of the closely related problem *Crux* 39 [1975: 65].

Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; JOSEPH D.E. KONHAUSER, Macalester College, St. Paul, Minnesota; KEE–WAI LAU, Hong Kong; DAN PEDOE, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

<u>1215</u>. [1987: 53] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a, b, c be nonnegative real numbers with $a + b + c = 1$. Show that

$$ab + bc + ca \le a^3 + b^3 + c^3 + 6abc \le a^2 + b^2 + c^2 \le 2(a^3 + b^3 + c^3) + 3abc,$$

and for each inequality determine all cases when equality holds.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany.

First we prove two inequalities which hold for all nonnegative real numbers a, b, c.

(i) $a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \le a^3 + b^3 + c^3 + 3abc.$

Because the inequality is symmetrical, let $a \le b \le c$. Then from

$$a(b - a)(c - a) \ge 0$$

we get

$$a^2b + a^2c \le a^3 + abc, \tag{1}$$

and from

$$(c - b)^2(c + b - a) \ge 0$$

we get

$$b^2a + b^2c + c^2a + c^2b \le b^3 + c^3 + 2abc. \tag{2}$$

(1) and (2) give (i). Equality holds in (i) if and only if $a = 0$, b, or c (from (1)) and $b = c$ or $b + c = a$ (from (2)), that is, if and only if either $a = b = c$ or one of a, b, c is 0 and the other two are equal.

(ii) $6abc \le a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$

By the AM–GM inequality,

$$\frac{a^2b + b^2c + c^2a}{3} \ge abc$$

and

$$\frac{ab^2 + bc^2 + ca^2}{3} \ge abc,$$

and (ii) follows. Equality holds in (ii) if and only if at least two of a, b, c are 0 or if $a^2b = b^2c = c^2a$, that is, $a = b = c$.

Now to the three inequalities of the problem.

The first is

$$(ab + bc + ca)(a + b + c) \leq a^3 + b^3 + c^3 + 6abc$$

which is just (i). Equality holds for $a = b = c = 1/3$ and for $a = 0$, $b = c = 1/2$ etc.

The second is

$$a^3 + b^3 + c^3 + 6abc \leq (a^2 + b^2 + c^2)(a + b + c)$$

which is just (ii). Equality holds for $a = b = c = 1/3$ and for $a = b = 0$, $c = 1$ etc.

The third is

$$(a^2 + b^2 + c^2)(a + b + c) \leq 2(a^3 + b^3 + c^3) + 3abc$$

which is again (i). Equality holds for $a = b = c = 1/3$ and for $a = 0$, $b = c = 1/2$ etc.

Also solved by S.J. BILCHEV, Technical University, Russe, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; KEE–WAI LAU, Hong Kong; BOB PRIELIPP, University of Wisconsin–Oshkosh; and the proposer. Partial solutions (missing only degenerate cases) were obtained by GEORGE EVAGELOPOULOS, Athens, Greece; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; and VEDULA N. MURTY, Penn State University at Harrisburg. A further solution was received which was correct only for the second inequality.

Prielipp, whose solution also used inequality (i) *above, notes that it has appeared in* Crux *as problem* 14 *of the* 1974 *U.S.S.R. National Olympiad, with solution on* [1985: 310].

* * *

<u>1216</u>*. [1987: 53] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \leq \frac{9\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

Solution by Murray S. Klamkin, University of Alberta.

First we show that the function

$$y = x^{-1}\sin x$$

is concave in $[0,\pi/2]$. (We assume that the removable singularity at $x = 0$ has been removed, so that the function has value 1 at $x = 0$; and similarly for some other functions defined subsequently.) We have

$$y'(x) = x^{-1}\cos x - x^{-2}\sin x$$

and

$$y''(x) = -x^{-1}\sin x - 2x^{-2}\cos x + 2x^{-3}\sin x.$$

Then putting

$$F(x) = x^3y''(x) = -x^2\sin x - 2x\cos x + 2\sin x,$$

we have $F(0) = 0$ and

$$F'(x) = -x^2\cos x \le 0$$

for $0 \le x \le \pi/2$. Thus $F(x) \le 0$ and $y''(x) \le 0$ on $[0,\pi/2]$, and so y is concave.

Let

$$G(A,B,C) = \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}.$$

From the above, G is concave. We first consider the set of non–obtuse triangles (including degenerate ones), and use the majorization inequality. Since for all such triangles, with $A \ge B \ge C$, we have

$$(\tfrac{\pi}{2},\tfrac{\pi}{2},0) \succ (A,B,C) \succ (\tfrac{\pi}{3},\tfrac{\pi}{3},\tfrac{\pi}{3})$$

[this means that $\pi/2 \ge A \ge \pi/3$ and $\pi/2 + \pi/2 \ge A + B \ge \pi/3 + \pi/3$], it follows that

$$G(\tfrac{\pi}{2},\tfrac{\pi}{2},0) \le G(A,B,C) \le G(\tfrac{\pi}{3},\tfrac{\pi}{3},\tfrac{\pi}{3}),$$

or

$$1 + \frac{4}{\pi} \le G(A,B,C) \le \frac{9\sqrt{3}}{2\pi}, \tag{1}$$

equality holding in the left inequality for the degenerate $(\pi/2,\pi/2,0)$ triangle and in the right inequality for the equilateral triangle.

Now we consider the set of non–acute triangles (again including degenerate ones). We assume $A \ge B \ge C$, so that $B + C = x$ where $0 \le x \le \pi/2$. Again by the majorization inequality, from

$$(B,C) \prec (x,0)$$

we have

$$\frac{\sin B}{B} + \frac{\sin C}{C} \ge 1 + \frac{\sin x}{x}$$

so that

$$G(A,B,C) \ge 1 + \frac{\sin x}{x} + \frac{\sin(\pi - x)}{\pi - x}. \tag{2}$$

By *Crux* 519 [1981: 64], the minimum value of

$$\frac{\sin \pi t}{\pi t(1-t)} = \frac{\sin \pi t}{\pi t} + \frac{\sin \pi t}{\pi - \pi t}, \qquad 0 \le t \le \frac{1}{2},$$

occurs at $t = 0$. Thus, putting $x = \pi t$ in (2),

$$G(A,B,C) \ge 1 + 1 + 0 = 2, \tag{3}$$

equality holding for the degenerate triangle $(0,0,\pi)$.

We go on to find the upper bound of $G(A,B,C)$ in the non–acute case. Once more by the majorization inequality, letting $t = x/2$,

$$(B,C) \succ (t,t)$$

so that

$$\frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{2\sin t}{t}$$

and thus

$$G(A,B,C) \leq \frac{2\ \sin\ t}{t} + \frac{\sin\ 2t}{\pi -\ 2t} \equiv 2H(t).$$

We show that $H(t)$ is nondecreasing on $[0,\pi/4]$, so that

$$G(A,B,C) \leq G(\tfrac{\pi}{4},\tfrac{\pi}{4},\tfrac{\pi}{2}) = \frac{2\ +\ 4\sqrt{2}}{\pi}. \tag{4}$$

Here

$$\begin{aligned} H'(t) &= \frac{\sin\ 2t +\ (\pi\ -\ 2t)\cos\ 2t}{(\pi\ \ -\ 2t)^2} - \frac{\sin\ t\ -t\ \cos\ t}{t^2} \\ &= \phi(\pi - 2t) - \phi(t) \end{aligned}$$

where

$$\phi(t) = \frac{\sin\ t\ -t\ \cos\ t}{t^2},$$

so it suffices to show that

$$\phi(\pi - 2t) \geq \phi(t), \qquad 0 \leq t \leq \pi/4. \tag{5}$$

Since

$$t^3\phi'(t) = 2t\cos t - 2\sin t + t^2\sin t$$

and

$$(t^3\phi'(t))' = t^2\cos t,$$

it follows that $t^3\phi'(t)$ increases on $[0,\pi/2]$ and decreases on $[\pi/2,\pi]$. Also,

$$t^3\phi'(t) = \begin{cases} 0 & \text{when } t = 0 \\ -2\pi & \text{when } t = \pi, \end{cases}$$

so $\phi(t)$ increases from 0 to some point between $\pi/2$ and π and then decreases. Since we calculate that

$$\phi(\pi/2) > \phi(\pi) > \phi(\pi/4),$$

(5) follows. Now (1), (3) and (4) give the required result.

Finally, we give some related inequalities which are proved in a similar but easier fashion using the concavity of $\log\left[\frac{\sin\ x}{x}\right]$ in $[0,\pi]$ (see [1]). For non–obtuse triangles,

$$\left(\frac{2}{\pi}\right)^2 \leq \frac{\sin\ A}{A}\cdot\frac{\sin\ B}{B}\cdot\frac{\sin\ C}{C} \leq \left[\frac{3\sqrt{3}}{2\pi}\right]^3,$$

and for non–acute triangles,

$$0 \leq \frac{\sin\ A}{A}\cdot\frac{\sin\ B}{B}\cdot\frac{\sin\ C}{C} \leq \left[\frac{2}{\pi}\right]^2.$$

Reference:

[1] M.S. Klamkin, On Yff's inequality for the Brocard angle of a triangle, *Elemente der Mathematik* 32 (1977) 118.

Also solved by KEE–WAI LAU, Hong Kong; and GEORGE TSINTSIFAS, Thessaloniki, Greece. Tsintsifas's proof was similar to Klamkin's.

* * *

<u>1217</u>. [1987: 53] *Proposed by Niels Bejlegaard, Stavanger, Norway.*

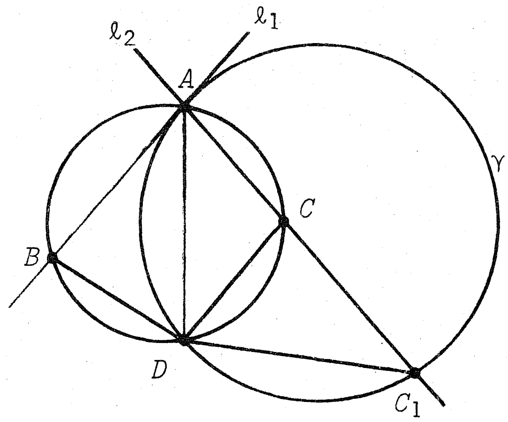
Given are two lines l_1 and l_2 intersecting at A, and a point P in the same plane, where P does not lie on either angle bisector at A. Also given is a positive real number r.

(a) Construct a line through P, intersecting l_1 and l_2 at B and C respectively, such that $AB + AC = r$.

(b) Construct a line through P, intersecting l_1 and l_2 at B and C respectively, such that $|AB - AC| = r$.

I. *Solution by George Tsintsifas, Thessaloniki, Greece.*

We need a preliminary result.



Case (i)

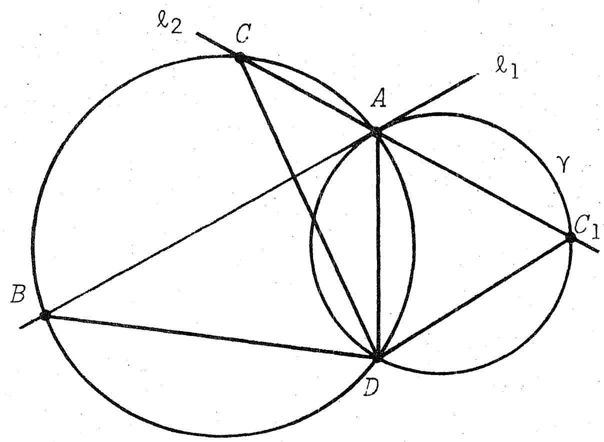
Lemma. Let C_1 on l_2 be such that $AC_1 = r$, and let γ be the circle through C_1 and tangent to l_1 at A. Let D be the intersection of γ with the angle bisector at A. Consider a circle through A and D, intersecting l_1 at B and l_2 at C.

(i) If B is on the same side of l_2 as D and C is on the same side of l_1 as D, then

$$AB + AC = r.$$

(ii) If B is on the same side of l_2 as D and C is on the opposite side of l_1 as D, then

$$AB - AC = r.$$



Case (ii)

Proof. In both cases we must show

$$AB = CC_1,$$

and we consider triangles ABD and C_1CD. Then

$$\angle ABD = \angle C_1CD$$

(since $ABCD$ is cyclic), and

$$\angle C_1AD = \angle BAD = \angle AC_1D = \angle CC_1D$$

(since DA bisects $\angle BAC_1$ and γ is tangent to BA at A), and thus

$$AD = DC_1.$$

Hence

$$\Delta ABD \equiv \Delta C_1CD,$$

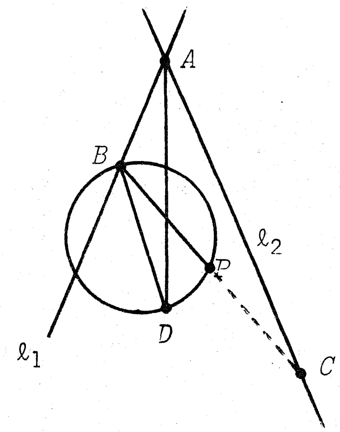
so $AB = CC_1$ as claimed. □

Conversely, if B on l_1 and C on l_2 satisfy (i) or (ii), then the circle through A, B, and C will also pass through D.

Now the problem is easy. We are given the point P and the number $r > 0$, and we can find the point D as above. It remains to find a circle through A with intersection points B on l_1 and C on l_2 (suitably located) so that BC passes through P. But, since $ABCD$ is cyclic, it must be true that

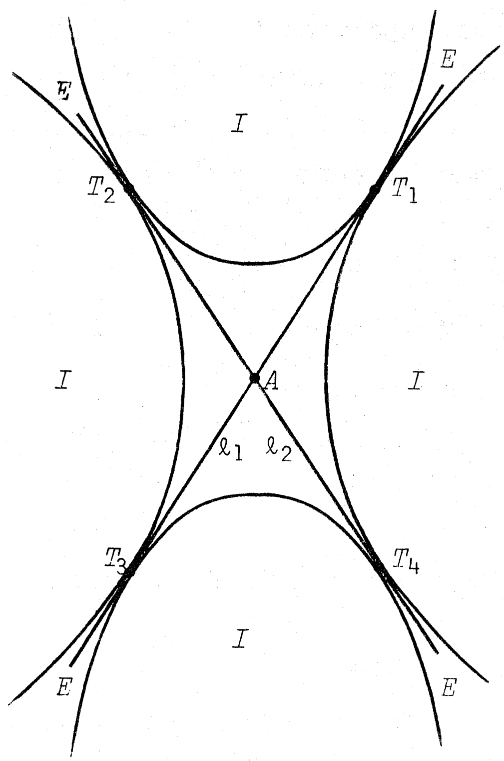
$$\angle PBD = \angle CBD = \angle CAD,$$

a known angle, so that B will be the intersection of l_1 with a known circle through P and D. (We can arrange that this circle will intersect l_1 and that C lies on the correct half of l_2 by choosing the appropriate bisector at A on which to locate D.) We see that several solutions are possible.



II. *Solution by Jordi Dou, Barcelona, Spain.*

Let T_1, T_3 on l_1 and T_2, T_4 on l_2 be such that $AT_i = r$ for each i, and consider the four parabolas tangent to l_1 and l_2 at T_1 and T_2, T_3 and T_2, T_3 and T_4, T_1 and T_4, respectively. Then any line tangent to one of the arcs T_1T_2, T_2T_3, T_3T_4, T_4T_1 of these parabolas will intersect l_1 and l_2 at points X_1, X_2 such that $AX_1 + AX_2 = r$. To solve (a), we simply find such a tangent which also passes through P. Similarly, a line tangent to one of these parabolas at a point outside the above-mentioned arcs will intersect l_1 and l_2 at points X_1, X_2 such that $|AX_1 - AX_2| = r$.



The number of solutions depends on the location of P. If P is in the region enclosed by the above four arcs T_iT_{i+1}, then there are 4 solutions of (a) and 4 solutions of (b). If P is in one of the four infinite regions E outside the parabolas, then there are 2 solutions of (a) and 6 of (b). Finally if P is in the interior I of one of the parabolas, then there are 2 solutions of (a) and 4 of (b).

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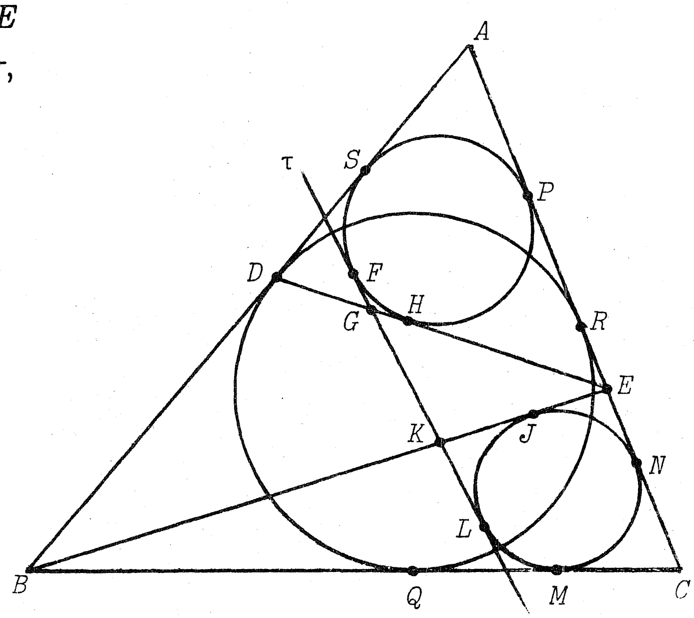
1219. [1987: 53] *Proposed by Herta Freitag, Roanoke, Virginia, and Dan Sokolowsky, Williamsburg, Virginia.* (Dedicated to Léo Sauvé.)

Let the incircle of ΔABC touch AB at D, and let E be a point of side AC.

Prove that the incircles of triangles ADE, BCE, and BDE have a common tangent.

Solution by Alfred Aeppli, University of Minnesota, Minneapolis.

Draw the incircles of triangles ADE and BCE, and their common tangent τ, with points labelled as shown. Then



$$\begin{aligned} BD + FL &= BD + NP \\ &= BQ + NR + RP \\ &= BQ + QM + RP \\ &= BM + RP \\ &= BJ + RP, \end{aligned}$$

so

$$\begin{aligned} BD + GK &= BD + FL - FG - KL \\ &= BJ + RP - FG - KL \\ &= BJ + DS - GH - KJ \\ &= (BJ - KJ) + (DH - GH) \\ &= BK + DG. \end{aligned}$$

Thus the four lines AB, BE, ED, τ touch a common circle, and we are done.

Also solved by the proposers.

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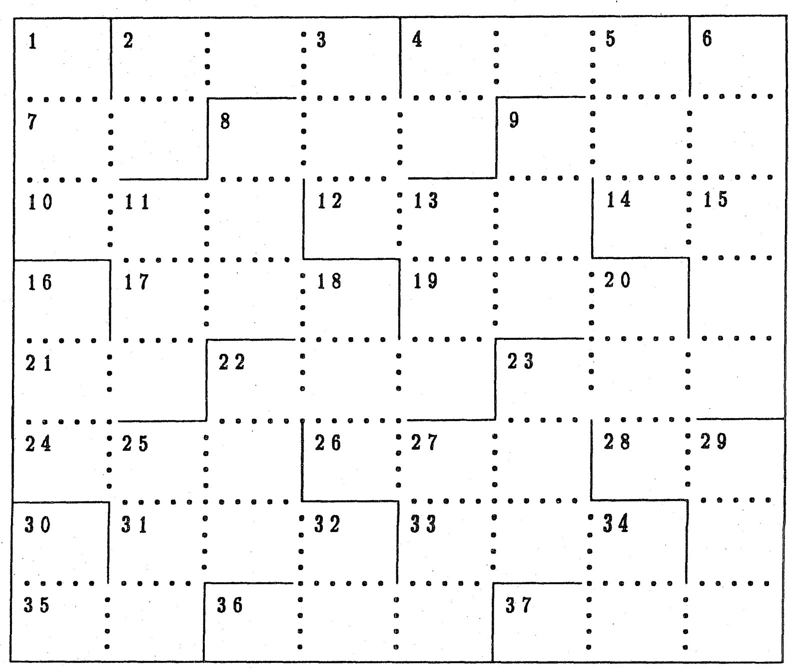
1220. [1987: 54] *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*



31B, 1D, 30D	14A, 28A, 6U	27D, 36B, 23B	24A, 12A, 32D
27D, 2B, 6D	18U, 12B, 7A	31A, 33A, 9B	8U, 17A, 20D
8U, 29D, 3D	15D, 8A, 28A	19B, 22D, 13U	22B, 18D, 16U
4U, 35A, 2D	31B, 33B, 9A	14A, 34U, 21A	19B, 37B, 25D
5D, 4A, 11D	31A, 10A, 26A	15U, 3D, 34U	9U, 23U, 20D

The twenty clues are triples (a,b,c) of two– and three–digit numbers, which form the sides of primitive integer triangles ABC with angle B twice the size of angle A.

A = across, B = back, D = down, U = up. For example, 31B has its hundreds and units digits in the squares labelled 32 and 31 respectively; 4U has its units digit in the square labelled 4.

Solution.

1 4	2 2	3	3 1	4 5	9	5 4	6 2
7 6	4	8 9	9	2	9 5	8	3
10 2	11 2	8	12 5	13 5	2	14 4	15 9
16 7	17 4	2	18 5	19 1	6	20 3	6
21 1	5	22 4	2	3	23 5	3	1
24 5	25 2	9	26 2	27 1	7	28 6	29 3
30 4	31 1	4	32 4	33 2	7	34 6	7
35 3	5	36 6	7	1	37 6	5	4

Solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; JURGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer.

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1221*. [1987: 85] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

Let u, v, w be non–negative numbers and let $0 < t \le 2$. If a, b, c are the sides of a triangle and if F is its area, prove that

$$\frac{u}{v+w}(bc)^t + \frac{v}{w+u}(ca)^t + \frac{w}{u+v}(ab)^t \geq \frac{3}{2}\left[\frac{4F}{\sqrt{3}}\right]^t.$$

[See Solution II of *Crux* 1051 [1986: 252].]

Editor's comment.

This has already been proved in this issue, by Walther Janous in his proof of *Crux* 1212 [1988: 115].

Also solved by MURRAY S. KLAMKIN, University of Alberta.

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<u>1223</u>. [1987: 86] *Proposed by H.S.M. Coxeter, University of Toronto, Toronto, Ontario.*

(a) Show that, if all four faces of a tetrahedron in Euclidean space are right-angled triangles, there must be two vertices at each of which two right angles occur. In other words, the tetrahedron must be an orthoscheme (Coxeter, *Introduction to Geometry*, Wiley, New York, 1969, p.156).

(b) Show that this also holds in hyperbolic space but not in elliptic space.

I. *Solution by the proposer.*

Let the tetrahedron have three pairs of opposite edges aa', bb', cc', such that a, b, c form a triangle while a', b', c' form a trihedron. Consider the possible distributions of the four hypotenuses. If they are all distinct, they could form a skew quadrilateral such as $aba'b'$, or a triangle with a tail, such as $abca'$. If just two coincide, they could be $abba'$ or $aaba'$[1]. If there are two coinciding pairs, they could be $aaa'a'$.

Since the hypotenuse of a right–angled triangle is its longest side, the lengths of the edges of the skew quadrilateral $aba'b'$ must satisfy either

$$a < b < a' < b' < a$$

or

$$a > b > a' > b' > a,$$

which is a contradiction. Similarly for $abca'$, either $a' < b < c < a'$ or $a' > b > c > a'$, another contradiction.

The case of $abba'$ is the orthoscheme.

The case of $aaba'$ is impossible in all three geometries. For Euclidean geometry, Pythagoras yields

$$b'^2 + c'^2 = a^2 = b^2 + c^2 = a'^2 + c'^2 + c^2 = b'^2 + c^2 + c'^2 + c^2$$

whence $c = 0$. Similarly, for elliptic geometry,

[1][Or $aabc$? See the editor's comment at the end.]

$$\cos b' \cos c' = \cos a = \cos b \cos c = \cos a' \cos c' \cos c$$
$$= \cos b' \cos c \cos c' \cos c$$

whence $\cos^2 c = 1$; and for hyperbolic geometry we have the same with "cos" changed to "cosh".

Finally, for the case of $aaa'a'$, let (ab) denote the angle between a and b, so that

$$(bc) = (cb') = (b'c') = (c'b) = \pi/2.$$

Then the trihedra at the four vertices yield

$$(ac) + (ab') > \pi/2, \qquad (a'b) + (a'c) > \pi/2,$$
$$(ab) + (ac') > \pi/2, \qquad (a'b') + (a'c') > \pi/2,$$

while the four right–angled faces yield

$$(ab) + (ac) \leq \pi/2, \qquad (a'b) + (a'c') \leq \pi/2,$$
$$(a'b') + (a'c) \leq \pi/2, \qquad (ab') + (ac') \leq \pi/2.$$

Adding each set of four inequalities, we again obtain a contradiction.

An elliptic counterexample is the tetragonal disphenoid in which the two "double" hypotenuses have length a while the remaining four edges all have length b, where $\cos a = \cos^2 b$; for instance, $a = \pi/3$, $b = \pi/4$.

II. *Editor's comment.*

Two readers, J.T. Groenman and Walther Janous, sent in solutions to part (a) which missed the final case $aaa'a'$ above. On the other hand, the proposer's solution seems to have left out the possibility $aabc$ (see the footnote), corresponding to one vertex having three right angles, which both Groenman and Janous considered. Here is Janous' simple proof of the impossibility of this case: if at one vertex there are three right angles, then the fourth face cannot be right–angled, as its normal projection onto each of the other faces is already right–angled.

Well, as long as we are all admitting our mistakes, the editor must confess that he thought he had a counterexample: just take two isosceles right triangles ABC and ABD hinged along a common hypotenuse AB, and fold them together until angles CAD and CBD become $90°$. The puncturing, or should we say flattening, of this "counterexample" is left to the readers.

* * *

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