Mathematical Spectrum

1991/92
Volume 24
Number 2

A magazine for students and teachers of mathematics in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 24 of *Mathematical Spectrum* will consist of four issues, of which this is the second. The first was published in September 1991, and the third and fourth will appear in February and May 1992 respectively.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH

Mathematical Spectrum Awards for Volume 23

Prizes have been awarded to the following student readers for contributions published in Volume 23:

Oliver Johnson for his article 'Reduction of a Non-Linear Recurrence Relation to Linear Form' (pages 11–12);

Amites Sarkar for his article 'Binomial Identities by Leibniz's Theorem' (pages 56–58) and other contributions;

Dylan Gow for various contributions.

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Charles Babbage: The Man Behind the Machines

ROGER WEBSTER, *University of Sheffield*

The author lectures in pure mathematics, his main interests being in convexity and the history of mathematics. To commemorate the bicentenary of Babbage's birth, he has had a computer terminal installed in his office, but as yet has only mastered the art of switching it on!

Charles Babbage, the great English polymath, is celebrated today as the founding father of computing. His extraordinary mechanical engines, designed during the last century, incorporate many features of modern electronic computers. His Difference Engines were the first automatic calculating devices; his Analytical Engines were the first general-purpose programmable computing machines to be devised.

The breadth of Babbage's interests is astonishing by any standard. In addition to his revolutionary work on computing machines, he made distinguished contributions to a vast range of subjects. These include mathematics, philosophy, machine tool technology, chess, property tax, geology, cryptology and lock-picking. Babbage was a fascinating and complex figure, who has variously been described as 'the irascible genius', 'a plaintive snob' and 'an unfailingly charming companion'.

Thus began the formal announcement heralding a bicentennial conference on computing that was to have been held in London in July 1991 (but which in fact did not take place) to commemorate the 200th anniversary of Babbage's birth. As the thumbnail biography makes clear, Babbage was much more than a computer pioneer, as we hope to show by taking a glimpse into the colourful life of this nineteenth-century eccentric genius.

Charles Babbage was born on 26 December 1791 in London, just a short distance from the Elephant and Castle, a popular coaching inn of the times. Were he alive today, he *would* be surprised to witness the celebrations surrounding his bicentenary, for he thought that he was born in 1792! Indeed, many histories of mathematics, reference books and the *Times* newspaper have laboured under the same misapprehension. Confusion has also surrounded his birthplace, Totnes, Teignmouth and London all having been put forward. This double mystery concerning Babbage's date and place of birth was only solved in 1978 when his biographer, Anthony Hyman, discovered in the baptismal register of St Mary Newington, London, an entry for 6 January 1792 which read: Charles, son to Benjamin and Betty Plumleigh Babbage.

Babbage came from a highly respected Totnes family; his father, a wealthy banker, left Devon in middle age for a position in the City. From his earliest years, Charles showed an almost scientific curiosity in the working of his toys that often led to their destruction! Whilst still a child, he was so visibly excited by an exhibition of clockwork automata that the exhibitor invited him to inspect the workshop and see yet more wonderful machines. There he became captivated by a female dancing figure in silver, a foot high, on the forefinger of whose right hand perched a bird that wagged its tail, flapped its wings and opened its beak. Many years later he chanced once again on the silver lady, this time at an auction. He acquired her for his drawing room, dressed her in beautiful clothes, and showed her off in a glass case. She proved an irresistible attraction at Babbage's famous soirées, especially to those guests who could not appreciate the subtleties of the Difference Engine that was also on display.

Babbage's education was a somewhat interrupted one, for he was sent back and forth between various schools in London and Devon. He first became attracted to mathematics in his early teens when, as a boarder at a village school on the outskirts of London, he read *Ward's Young Mathematician's Guide*. This had a dramatic effect on the young Charles, for he decided that there were not enough hours in the day in which to learn and that he would have to work at night as well. He made a pact with a friend to rise each morning at three, slip downstairs, light a fire in the classroom and there study until past five. The plan worked well until some of the other boys came to know of it and asked to join in. All such requests were flatly refused. There was however one boy, Frederick

Charles Babbage 1791–1871
(Reproduced by courtesy of the Master and Fellows of Trinity College, Cambridge)

Marryat, later to achieve fame as author of *Children of the New Forest* and *Mr Midshipman Easy*, who would not take no for an answer and was determined to join the three o'clock club, come what may. So developed a battle of wits between them.

Marryat fired the first volley by positioning his bed so close to the door that he would be woken up as Babbage escaped from the bedroom in the middle of the night. Babbage countered by pushing the offending bed out of the way. Then Marryat tied one end of a length of thread to his foot and the other to the door, only to have it cut by a penknife. Next followed a chain, which succeeded in outwitting Babbage for one night, after which he procured a pair of pliers to unbend one of the links. Finally came an even stouter chain attached to a padlock, which proved too much for Babbage's instruments and expertise in picking locks, and he was forced to accept defeat, but not gracefully. He tied a string to the chain, tugging on it while Marryat slept. Up leapt Marryat, only to find no one near the door. Babbage repeated the prank, succeeding in annoying Marryat, but not in leaving the room unobserved. A truce was called and the future novelist became a member of the club. One by one, more boys joined, and frolics replaced study as its main activity. The episode came to a colourful conclusion when the revellers let off fireworks in the playground, and were of course found out.

In October 1810 Babbage entered Trinity College, Cambridge, looking forward to the congenial lifestyle that lay ahead. Once there, he plunged himself into a host of activities, reflecting his own wide interests and sense of humour. He played chess and whist, and joined several societies, including the Ghost Club and the Extractors. One rule of the latter was that, *if after a year a member fails to communicate his address to the secretary, then it will be assumed that he has been certified insane, and every effort, legal and illegal, shall be made to get him out of the madhouse.* Hence the club's name. Another rule was that *every candidate for admission shall produce six certificates, three that he is sane and three that he is insane.*

Babbage took great delight in sailing and kept a boat on the Cam. Occasionally he would take extended trips down the river, choosing his companions for these voyages more for their brawn than their brains, so that *they* could row when the wind was contrary. The expeditions were planned in some detail. First he sent his servant to the apothecary for a certificate stating that he was indisposed and that it would be injurious to his health to attend chapel, hall or lectures. This was forwarded to the authorities. The servant then went to the college chef to order a well-seasoned meat pie, a couple of chickens and other provisions, which were then packed, together with several bottles of wine and one of liqueur, in a hamper. After a week of open-air exercise, sailing, fishing and shooting, the crew returned to Cambridge in excellent health!

Academically, Babbage was better prepared than most freshmen, having spent his leisure time at school reading every mathematics book that came his way. So well versed was he in the calculus that he could 'work with equal facility in the dots of Newton, the d's of Leibniz and the dashes of Lagrange'. Whilst studying by himself, he had encountered many difficulties and looked forward to having them resolved on reaching university. A big surprise lay in wait. He quickly realized that mathematics at Cambridge was in a discreditable state, and that English mathematics in general had stagnated since its isolation (caused by the priority dispute over the invention of the calculus) from the exciting developments in the subject on the Continent. Always eager to take up a challenge, Babbage joined with fellow undergraduates George Peacock and John Herschel in founding the Analytical Society to promote Continental mathematics with its analytical style and Leibnizian notation—or, as Babbage wittily expressed it, 'to advocate the principles of the pure D-ism of the Continent in opposition to the Dot-age of the University'. Babbage played a leading part in the Society, helping to translate Lacroix's textbook *Sur le Calcul Différentiel et Intégral* and to produce two volumes of calculus examples. The Society provided the catalyst that Cambridge mathematics so desperately needed, and the ensuing revival in British mathematics is generally dated from 1812, the year of its formation.

Babbage found the dull routine of study at Cambridge distasteful and decided to boycott mathematics lectures, not to compete for honours and not to seek a fellowship, preferring instead to work on behalf of the Analytical Society and devour the original papers of the great mathematicians. Nonetheless, Babbage had early acquired the reputation as one of the leading men of his year, and duly graduated in 1814. In July of that year he married Georgiana Whitmore in Teignmouth, and shortly afterwards the couple set up home in London. The marriage produced seven sons and a daughter in thirteen years, ending with Georgiana's death at the early age of thirty-five. Of the eight children, only three survived to adulthood, the boys Herschel, Dugald and Henry.

As a gentleman of means, Babbage was free to spend his life pursuing those many and varied activities into which his restless, explosive mind drove him. Once settled in London, he soon made a name for himself in scientific circles, writing papers on mathematics and lecturing on astronomy at the Royal Institution. He was elected to the Royal Society in 1816. In 1828 he became Lucasian Professor of Mathematics at Cambridge, a position he held until 1838, although he never delivered a single lecture and continued to live in London. As part of his efforts to revitalize British scientific life, he helped found the Astronomical Society (1820), the British Society for the Advancement of Science (1831) and the Statistical Society of London (1834). His book *On the Economy of Machinery and Manufactures* (1832), a seminal contribution to what is now known as operational research, established his reputation as a political economist, and his analysis of the Post Office resulted in the introduction of the penny-post. He also published the first reliable actuarial life-tables. In his philosophical work *The Ninth Bridgewater Treatise* (1837), Babbage envisaged God as a type of scientific programmer.

During his life he fought many campaigns: he attempted reform of the Royal Society and the Greenwich Observatory, championed the causes of life peerages and decimal coinage, twice stood for Parliament, both times unsuccessfully, and tried to secure government funding for scientific research. As if that were not enough, in later years he waged a vigorous one-man vendetta against organ-grinders, maintaining in court that all his ideas vanished the moment they began to play.

Babbage was responsible for inventions of all kinds: the *cow-catcher* fitted to the fronts of trains for sweeping obstacles off the track; the *occulting light* used in lighthouses for signalling; *coloured lighting* for theatres to replace white gaslight—he even wrote a ballet to accompany one of his light shows; an *accident analysis recorder* for trains, forerunner of the black box used in aircraft, consisting of inked pens making dots on a rotating roll of paper; an *ophthalmoscope* for examining the back of the eye, one of the first ever devised. Other interests included statistical linguistics, meteorology, geophysics, military warfare and the use of tree rings as historic climatic records.

Clearly not all of Babbage's life was spent closeted away in academic research, far from it; he was the most sociable of men and had a great sense of fun, as anyone leafing through his autobiographical *Passages from the Life of a Philosopher* will discover. On his frequent visits to the Continent he was fêted by members of the aristocracy and fellow scientists alike, meeting such mathematical luminaries as Laplace, Fourier and Poisson. Back at home he was in great demand, both as a party-goer and a party giver. In February 1842 he had at least thirteen invitations for every day of that month, Sundays included! His own Saturday soirées were highlights of the London season, each one being attended by two to three hundred of society's élite. He knew everyone who was anyone: the Prince Consort, the Duke of Wellington, John Stuart Mill, Browning, Dickens, Longfellow, Thackeray, Darwin, Faraday and Brunel, to name but a few.

It was while he was supervising the compilation of a set of scientific tables for the newly formed Astronomical Society, together with John Herschel, that an incident occurred which was to change the course of Babbage's life. During the tedium of checking results, they found numerous errors, causing Babbage to exclaim 'I wish to God these calculations had been executed by steam,' whereupon Herschel remarked, 'It is quite possible.' From this chance conversation, Babbage developed a lifelong obsession for constructing mechanical devices to compute and print tables of mathematical functions, an obsession that was to transform a carefree young man into a cantankerous old one.

Babbage immediately set about work on his first calculating machine, the so-called *Difference Engine*. The excitement of the whole enterprise made him ill, and his doctor advised him to rest. Nevertheless, by 1822 he had built a small pilot model which was capable of computing the values of a quadratic function and printing the results. So successfully did he promote his new toy that he secured government backing for the development of a much more ambitious engine to produce navigational and astronomical tables. Babbage expected the venture to take three years, but seriously underestimated the difficulties, theoretical, practical, mechanical, financial and personal, that he would meet in trying to carry through his plans. The project dragged on, limping from one crisis to another, until 1833 when it was dealt the final hammer-blow: Joseph Clement, Babbage's supervising engineer, walked out after a disagreement, taking with him his men, his specialized machine-tools and all the detailed technical drawings that had been prepared over the years. Although the Engine was never completed, a small working component of it can still be seen at London's Science Museum. In 1843, however, inspired by Babbage's ideas, two Swedish engineers built their own difference engine, a later version of which was purchased by the British Government in 1863.

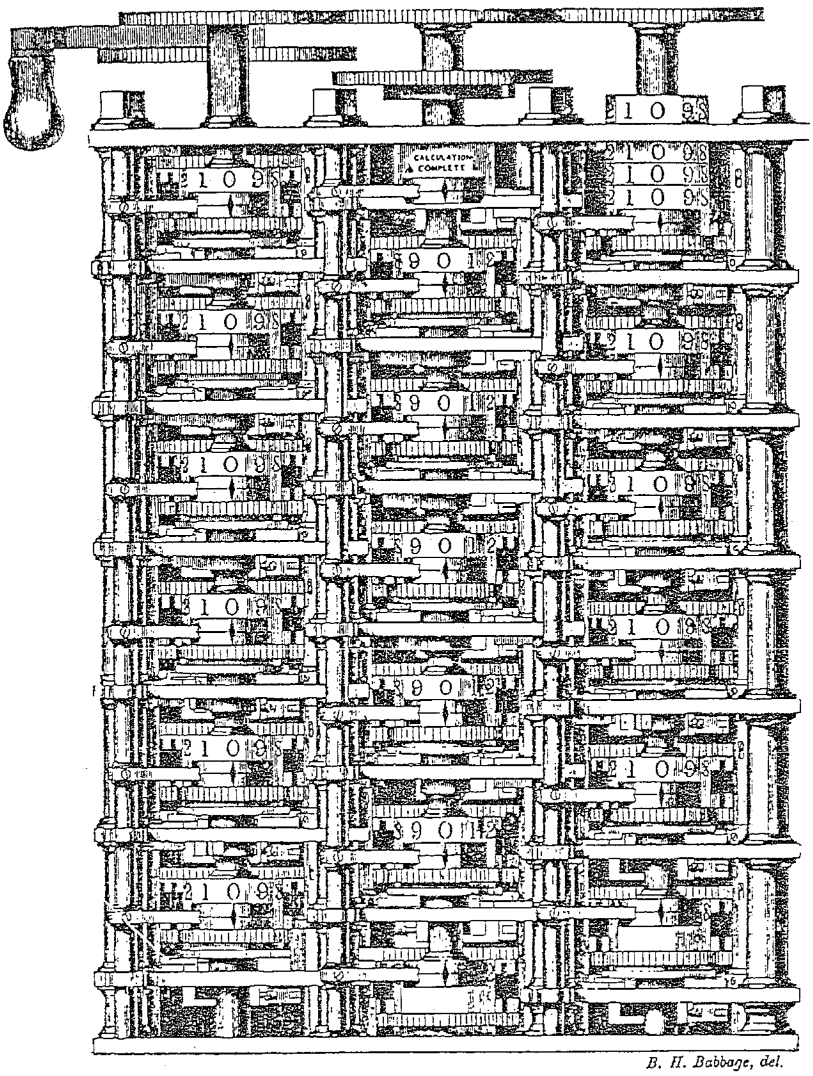


Figure 1. The Difference Engine

The Difference Engine was the mechanical realization in cog-wheels and cranks of the *principle of constant differences*. The principle is perhaps best illustrated by an example. Consider the values of x^2 at $1,2,3,4,\ldots$, i.e. the sequence $1,4,9,16,\ldots$. Form the differences between consecutive terms to generate the *first-differences* $3,5,7,9,\ldots$. Now form the differences between these first-differences to generate the *second-differences* $2,2,2,2,\ldots$, which are constant. By reversing the differencing procedure and knowing the constant second-difference 2, the first first-difference 3 and the first square 1, the sequence of squares can be recaptured by following a simple set pattern of additions. The principle extends to any polynomial: a cubic has constant third-differences, a quartic constant fourth-differences, and so on. Thus the values of a polynomial at $1,2,3,4,\ldots$ can be found by following a simple set pattern of additions, starting from an initial set of differences. Each addition can be performed physically by means of toothed wheels mounted on shafts, in a way similar to that used in a car's milometer, and it was this idea which lay at the heart of Babbage's machine.

In the autumn of 1834 Babbage had a truly grand vision, that of a completely automatic all-purpose calculating machine, the forerunner of today's electronic computer. The idea for the new machine, or *Analytical Engine* as he called it, developed out of his early work on the Difference Engine, when he became intrigued by the notion of the machine acting upon the results of its own calculations—'the engine eating its own tail' as he picturesquely expressed this. The Analytical Engine was to have an *input unit* to receive data and operating instructions, a *store* to hold numbers, a *mill* to perform arithmetical operations, a *control unit* to cause the machine to obey instructions, and an *output unit* to display results. Babbage regulated the sequence of operations needed in extended computations by means of a system of punched cards. Such a system had been devised in 1801 by J. M. Jacquard for use in his automatic looms, which could weave fabrics into intricate designs. Babbage himself owned a portrait of Jacquard, woven on a Jacquard loom and requiring no fewer than 24,000 cards to produce.

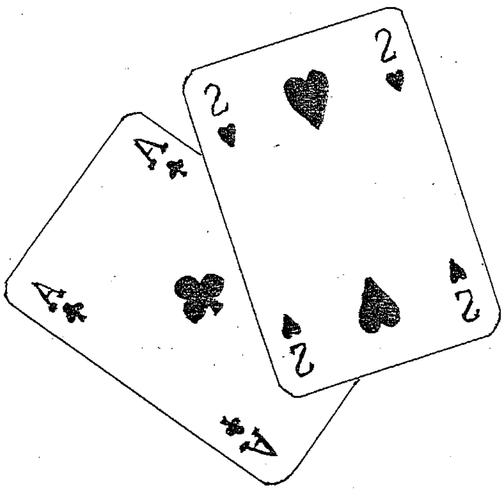
Babbage was too occupied developing his Analytical Engine to write any detailed account of it, although in 1840 he was invited to Turin to give a presentation of his work. In his audience was L. F. Menabrea, a young military engineer, later to become Prime Minister of Italy, who took detailed notes and published a summary of Babbage's ideas in a Geneva journal of 1842. This was translated into English in 1843 by Lord Byron's daughter Augusta Ada, the Countess of Lovelace, who added explanatory notes in collaboration with Babbage.

The Analytical Engine, like its predecessor, was never completed, even though Babbage continued work on it for the rest of his life, financing the project out of his own funds. His grand design was too ambitious: its execution was beyond the engineering expertise of the day, its very conception beyond the grasp of most of his contemporaries. As setback followed setback, he became more and more embittered by a sense of failure and isolation, believing that no one understood him. One can but sympathize with his plight, when he had to respond to such questions as 'Pray, Mr Babbage, if you put into the machine wrong figures, will the right answers come out?' asked of him by both a Member of Parliament and a Peer of the Realm! Babbage commented: 'I am not able rightly to apprehend the kind of confusion that could give rise to such a question.'

Babbage died on 18 October 1871, shortly before his eightieth birthday, leaving behind a mass of notebooks, blueprints, workshop instructions and components of uncompleted machines. His son, Major-General H. P. Babbage, spent years sifting through his father's papers and in 1899 published *Babbage's Calculating Engines*; he also helped build a small analytical mill. It is only in modern times that Babbage's contribution to computing science has been thoroughly evaluated, and his unique role as the great ancestral figure of the subject recognized. That being said, it must be mentioned that, to quote one present-day computer expert Maurice Wilkes: 'In writing of Babbage as a computer pioneer one must at once admit that his work, however brilliant and original, was without influence on the modern development of computers. The principles that Babbage elucidated, but regrettably failed to communicate, had to be rediscovered by the men who, a hundred years later, built the first automatic computers.'

Babbage's bicentenary is being celebrated in a variety of imaginative ways. The Royal Mail issued a commemorative stamp and authors William Gibson and Bruce Sterling have written a science fiction novel, *The Difference Engine*. This is set in the time of Lord(!) Babbage, when not only has the eponymous engine been built, but is the kingpin of Britain's continuing Industrial Revolution. At the Babbage–Faraday Symposium held in Cambridge in July 1991, Maurice Wilkes' play *Pray, Mr Babbage* was performed. The main event of the year is the Charles Babbage exhibition at the Science Museum, London, which began on 1 July 1991 and continues through until 1992, its centrepiece being the newly built Babbage's Difference Engine, designed to print out seventh-order polynomials to thirty decimal places. Mention must also be made of the recently published collected works of Babbage, in eleven volumes, and of the Babbage Room at the Totnes Museum, where a life-size model of the great man himself greets his visitors.

As old age beckoned, Babbage was once heard to remark that he would happily surrender the rest of his life to see the progress made in science five hundred years on. If he had hoped to see a society driven by the machines of his dreams, then he would not have had to wait quite so long—his time has come.



From a pack of playing cards, five cards are chosen, then replaced. Five cards are again chosen and replaced; five cards are again chosen and replaced. Finally one card is chosen. What is the probability that no diamonds are chosen, that the two of hearts is chosen three times, that the ace of clubs is chosen twice, and that all the other cards are different?

All that Glisters: An Old Story Retold as a Cautionary Tale

R. HAYDOCK

The author is now retired, but previously taught in secondary schools and at Matlock College of Education. He continues to write for mathematical publications, mainly at secondary school level.

In the Lower VI, Richard had the downy cheek and the dewy innocence of his kind. The mathematics he had met had been of the purest and consequently his eye was bright and his sleep untroubled.

Daphne Loveday was his appointed teacher for A-level mathematics. She soon saw that Richard had a special aptitude for the subject and tried in small ways to foster his enthusiasm and take him outside the narrow confines of the syllabus.

One day, late in the autumn term, when the caretaker had grudgingly allowed the central heating and the cosy glow of the old-fashioned light bulbs illuminated the small classroom, Daphne passed Richard's table and laid upon it a single sheet of paper (figure 1).

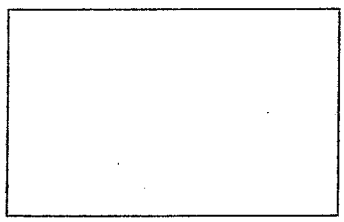


Figure 1

He stared at the drawing on its surface and stammered, 'M-m-miss Loveday,—b-b-but it's b-b-beautiful!'

She smiled, the special smile she reserved for pupils with mathematical promise.

'I see that you're favoured with the gift of aesthetic appreciation. That, Richard, is sometimes called the Golden Rectangle. Among all rectangles it is said to have the most pleasing proportions. The word "golden" was actually applied by mediaeval artists to the section which divided it into a square and a smaller similar rectangle (figure 2). Thus XY divides rectangle $ABCD$ into square $AXYD$ and rectangle $XBCY$, and the rectangles $ABCD$ and $XBCY$ are similar. The ratio AX:XB is called the Golden Section. It is approximately 13:8 but it cannot be given exactly using integers. You can find that ratio, using your knowledge of similar figures.'

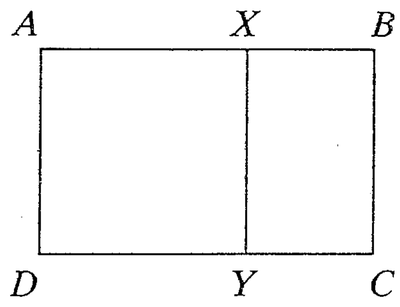


Figure 2

At home that evening Richard wasted no time in settling down to work on the Golden Section. He found little difficulty in its computation.

Let AX be x and XB be 1 unit. Then $AB = x+1$ and, since the two rectangles are similar,

$$\frac{AB}{BC} = \frac{BC}{XB};$$

$$\frac{x+1}{x} = \frac{x}{1};$$

$$x^2-x-1 = 0.$$

Thus, x, being positive, must be $\frac{1}{2}(1+\sqrt{5})$.

He remembered that if a is a positive integer, not a perfect square, then $\sqrt{a}$ is irrational and saw that indeed 13:8 or any other ratio of integers could be only an approximation to the mysterious section.

Now, just as primitive man having seen fire could not rest until he could make it, Richard resolved to construct a Golden Rectangle. He thought how wonderful it would be to lay the completed construction before Miss Loveday. Naturally he used only the instruments prescribed by the great Euclid. He took from the rosewood box his treasured compasses and reached for the boxwood ruler bequeathed him by his grandfather, automatically checking its edge for straightness. The expression $\frac{1}{2}(1+\sqrt{5})$ was deeply suggestive to his keen young mind. Adding 1 and halving were child's play using Euclidean methods. He also recalled how to form right angles and this also gave him the clue he needed for constructing $\sqrt{5}$; he would use the theorem of the mighty Pythagoras, relating to right-angled triangles. His first construction was a little clumsy but he found refinements and, sharpening his pencil, he produced a clean, uncluttered drawing.

Miss Loveday was pleased with his calculation and delighted by the drawing, which Richard insisted on presenting to her. As a reward she showed him how to construct an equiangular spiral, using repeated applications of the Golden Section (figure 3). The spiral could not really be drawn accurately, of course, but the lad went home that evening, his brain pleasantly occupied by thoughts of the spirals to be seen in sunflower

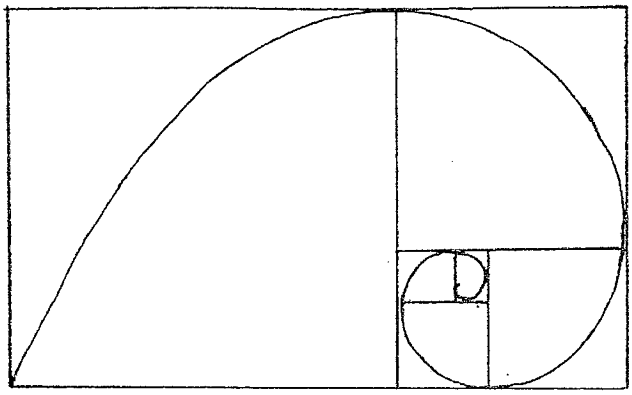


Figure 3. An equiangular spiral

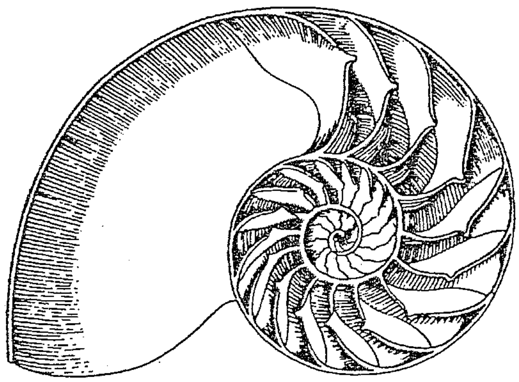


Figure 4. A nautilus shell

heads and nautilus shells (figure 4) and how they might all be part of a great over-arching plan.

Richard's thoughts were next brought back to the Golden Section at the end of the autumn term in his Upper VI year. As a pre-Christmas diversion, Miss Loveday had introduced the class to linear recurrence relations, showing that the solution of the relation

$$u_{n+2}+au_{n+1}+bu_n = 0$$

is

$$u_n = A\alpha^n+B\beta^n,$$

where α and β are the roots of the quadratic equation

$$x^2+ax+b = 0,$$

the constants A and B being determined by the values of u_1 and u_2. As a parting shot at the end of the last lesson of the term she threw out the comment, 'If you're familiar with the Golden Section you should now be able to relate this to the Fibonacci sequence.'

Richard responded to the challenge and set to work on the problem during the Christmas holidays. He recalled that the Fibonacci sequence $F_n = \{f_1, f_2, f_3, \ldots\}$ was generated by the relation

$$f_{n+2} = f_{n+1}+f_n,$$

so that the solution was

$$f_n = A\alpha^n+B\beta^n,$$

where α and β were the roots of

$$x^2-x-1 = 0.$$

When he saw the familiar equation his heart quickened. Hastily he completed the calculation:

$$\alpha = \tfrac{1}{2}(1+\sqrt{5}), \qquad \beta = \tfrac{1}{2}(1-\sqrt{5}).$$

If $f_1 = f_2 = 1$ then

$$\tfrac{1}{2}A(1+\sqrt{5}) + \tfrac{1}{2}B(1-\sqrt{5}) = 1,$$
$$\tfrac{1}{2}A(3+\sqrt{5}) + \tfrac{1}{2}B(3-\sqrt{5}) = 1,$$

whence

$$A = \frac{1}{\sqrt{5}}, \qquad B = -\frac{1}{\sqrt{5}}.$$

(He was pretty hot on solving simultaneous equations by now.) So

$$f_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

Some sort of connection with the Golden Section was obvious, but not yet clear. Richard racked his brain for a while to no great effect, then left the matter, confident that the problem would respond to 'sleeping on it'.

And so it was on Christmas Day he suddenly recalled Miss Loveday's introduction of the Golden Section. 'It is about 13:8 ... .' Surely 13 and 8 were terms of the Fibonacci sequence! He checked it: yes $f_7{:}f_6 = 13{:}8$. A calculation of f_8/f_7 showed that this was even closer to the Golden Number $\tfrac{1}{2}(1+\sqrt{5})$, which he had learned to call ϕ. That afternoon was spent happily with his Christmas present, a programmable calculator (reluctantly approved by Miss Loveday), in calculating members of the sequence (f_{n+1}/f_n). It was clear that as n increased the numbers drew closer to ϕ. Of course! The whole thing hinged on the fact that $|\tfrac{1}{2}(1-\sqrt{5})| < 1$, so that $\{\tfrac{1}{2}(1-\sqrt{5})\}^n \to 0$. This would mean that, for large n,

$$f_n \simeq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and

$$\frac{f_{n+1}}{f_n} \simeq \frac{1+\sqrt{5}}{2}.$$

Over the Christmas holidays an American uncle came to stay. Nick worked on Wall Street and came laden with expensive presents, but Richard found him sinister; he couldn't take to him. Nick didn't seem to mind this, though, and always treated Richard in a friendly fashion. A couple of days after his arrival he came upon Richard at work, proving rigorously that $f_{n+1}/f_n \to \phi$.

'Is that really necessary? It's pretty obvious, isn't it?'

'You know the Golden Section?'

'Indeed, yes. I had a great interest in mathematics until, for my sins, I took to accountancy and finally to stockbroking. Come to think of it, you may be interested in the connection between the Golden Section and the regular pentagon.'

At the beginning of the spring term, Richard was able to demonstrate this to Miss Loveday (figure 5).

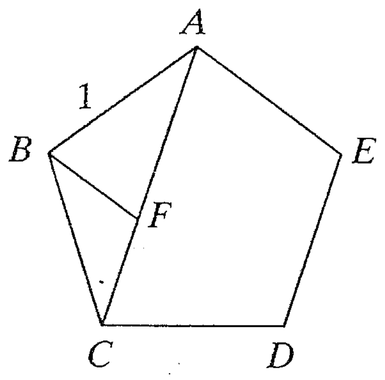


Figure 5

$ABCDE$ is a regular pentagon. Suppose that each side is 1 unit. Let the lengths of the diagonals be d. On the diagonal AC let F be the point such that $AF = 1$. Then it is easy to show that $\triangle BFC$ and $\triangle ABC$ are similar:

$$\frac{BC}{FC} = \frac{AC}{BC}$$

$$\frac{1}{d-1} = \frac{d}{1}$$

$$d^2 - d - 1 = 0.$$

Miss Loveday followed his argument closely.

'So d is ϕ; the ratio of the diagonal to the side is the Golden Section. That's splendid, Richard.'

Then she noticed that there were no construction lines on the drawing.

'Did you take the lengths from your Golden Section construction?'

'Oh, no; I just measured 1.62 times the side for the diagonal and it worked out pretty well.'

At the mention of measurement Miss Loveday paled, but she held her tongue. Though still inexperienced as a teacher she had already met such lapses from purity and knew that protestations from her could be counter-productive. The best she could do was to live in hope and to show by example the right way.

Her heart was heavy, though; the omens were not good. Like the others in the Upper VI, Richard had been allowed to call her Daphne. Until the present he had done so reluctantly and with a pretty hesitation. Now his conversation with her was sprinkled liberally with the use of her first name and he assumed an easy air of familiarity. He seemed to be using the computer a great deal and to be favouring applied mathematics over pure, though he tended to make such remarks as, 'Not much reality about that situation, Daphne. I ask you, all those weightless beams and perfectly smooth pegs!'

She finally realised that all was lost one day when he came into class from a chemistry lesson (which could have accounted for the smell of sulphur he brought with him) and demonstrated to her an extension of the regular pentagon to form a five-pointed star (figure 6).

'Isn't that the pentacle that magicians use, Daph?'

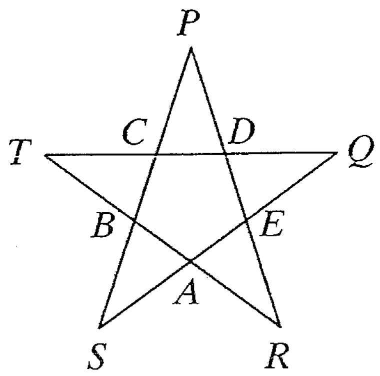
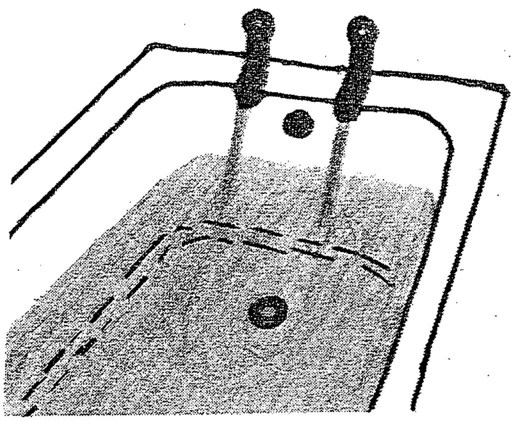


Figure 6. The number ϕ not only gives the ratio PC/AB but also PQ/PC. If we join $PQRST$ to form a second regular pentagon and then extend to form a further pentacle, and so on, we can form an infinite sequence of lengths in geometric progression with common ratio ϕ.

It was no surprise to Miss Loveday when Richard came up to her later in the term and said, 'I shan't be going on to university, Daphne: it's a mug's game. I've met someone who works in a management consultancy and she says that I could earn a bomb there. Should have a Porsche by the time I'm twenty-one. Beats teaching, eh!'



A bath takes 3 minutes to fill and 4 minutes to empty. How long does it take to fill the bath with the plug out?

Finite Fibonacci Sequences

I. M. RICHARDS, *Penwith Sixth Form College*

The author graduated from Exeter University with a B.A. in 1979 and a Ph.D. in 1983. He enjoys recreational mathematics, teaching, English literature, soccer, and, like many mathematicians, cricket.

Some time ago Dr Paul Strickland, now of Liverpool University, set me this puzzle:

> For any positive integer n, show that there are infinitely many numbers in the Fibonacci sequence which are divisible by n.

The puzzle is not difficult to solve, although I must admit I did not get to it immediately, but it opens up a whole range of interesting number theory. I have found it absorbing and have been surprised by how much the Fibonacci sequence can still yield in challenging problems even after 600 years of scrutiny.

To begin the discussion, I shall make my notation clear. The Fibonacci sequence traditionally begins 1, 1, 2, 3, 5, 8, ... and so I shall write $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$ and so on. Here f_i is to be understood as the ith term of the sequence. But the sequence extends backwards too. The generating relation, $f_{i+2} = f_i + f_{i+1}$, which produces the sequence from the initial values $f_1 = f_2 = 1$, may be reorganised to $f_i = f_{i+2} - f_{i+1}$. This allows us to work out from any two consecutive terms what the previous term must be. Applying this idea we see that the full Fibonacci sequence looks like this:

Index	...	f_{-4}	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3	f_4	...
Terms	...	-3	2	-1	1	0	1	1	2	3	...

The sequence alternates in sign to the left of f_0 but in magnitude the terms are same as those to the right. This can be expressed in a simple formula:

$$f_{-n} = (-1)^{n+1} f_n \quad \text{(for all integers } n\text{)}. \tag{1}$$

Returning now to the puzzle, I shall show that, if the Fibonacci sequence is reduced modulo n, that is, if the terms are divided by n and the remainders recorded, then the resulting sequence repeats itself cyclically.

Example

Fibonacci sequence	...	−1	1	0	1	1	2	3	5	8	13	21	34	55	89	144	...
Sequence reduced modulo 4	...	3	1	0	1	1	2	3	1	0	1	1	2	3	1	0	...

A cycle of length 6 appears. Why should this be? If the Fibonacci sequence is generated by $f_{i+2} = f_{i+1} + f_i$, then the reduced sequence is generated by $\bar{f}_{i+2} \equiv \bar{f}_{i+1} + \bar{f}_i \pmod{n}$, where $\bar{f}_j$ is f_j reduced modulo n. This means that the usual Fibonacci generating relation is being applied to produce the required sequence, the only difference being that the arithmetic is being performed modulo n. Notice now that, if we fix two consecutive remainders $\bar{f}_i$ and $\bar{f}_{i+1}$, then the relation $\bar{f}_{i+2} \equiv \bar{f}_{i+1} + \bar{f}_i \pmod{n}$ determines $\bar{f}_{i+2}$. We now know the values of the consecutive pair $\bar{f}_{i+1}$ and $\bar{f}_{i+2}$ and so can derive $\bar{f}_{i+3}$.

Repeatedly applying the generating relation, we see that the continuation of the reduced sequence is determined by any pair of consecutive residues. We next observe that the number of pairs of remainders modulo n is finite; it is n^2. This means that if we consider the pairs

$$(\bar{f}_1, \bar{f}_2), \qquad (\bar{f}_2, \bar{f}_3), \qquad \ldots, \qquad (\bar{f}_{n^2+1}, \bar{f}_{n^2+2}),$$

of which there are n^2+1, then, by the pigeonhole principle, there must be two the same. So there must be a pair of remainders that generates various pairs of consecutive remainders and then a copy of itself. Since this pair of residues, as remarked above, determines the continuation of the reduced sequence, then the same pattern repeats over and over. This accounts for the cyclic pattern.

The puzzle is finally solved by observing that $f_0 = 0$, when reduced modulo n, is 0, for any n. So the remainder 0 occurs within the repeating cycle of remainders, for any positive integer n. Each time the remainder 0 occurs we find a Fibonacci number divisible by n exactly. We conclude that there are infinitely many terms in the Fibonacci sequence that are divisible by n.

Cycle lengths. The arguments presented above show that the Fibonacci sequence, when reduced modulo n, always repeats cyclically. The question I shall ask is, 'How long is that cycle?'. In our example above, reduction modulo 4 produces a cycle of length 6. But what if we reduce by other moduli? The answer that I shall derive does not use high-level methods but provides a tour of number theory taking in several well-known results. To begin, I shall introduce some data. Table 1 gives the cycle length $C(n)$ for various moduli n. There seems little to say about $C(n)$ but the patterns begin to emerge when we examine prime moduli; p hereafter denotes a prime.

Table 1

Modulus n	2	3	4	5	6	7	8	9	10	11
Cycle length $C(n)$	3	8	6	20	24	16	12	24	60	10
Modulus n	12	13	14	15	16	17	18	19	29	47
Cycle length $C(n)$	24	28	48	40	24	36	24	18	14	32

Proposition 1. Let p be a prime. If $p \equiv 1$ or $4 \pmod 5$ then $C(p)$ divides $p-1$. If $p \equiv 2$ or $3 \pmod 5$ then $C(p)$ divides $2(p+1)$.

Proof. We start from a well-known result which is a formula for f_i (see reference 2, pp. 96–98):

$$f_i = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right\}.$$

Let us assume that p is an odd prime and consider f_p and f_{p+1}:

$$f_p = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^p - \left(\frac{1-\sqrt{5}}{2}\right)^p\right\},$$

$$f_{p+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{p+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{p+1}\right\}.$$

If we expand these binomially and cancel, we obtain

$$\begin{aligned} 2^{p-1}f_p &= \binom{p}{1} + 5\binom{p}{3} + 5^2\binom{p}{5} + \ldots + 5^{\frac{1}{2}(p-1)}\binom{p}{p}, \\ 2^p f_{p+1} &= \binom{p+1}{1} + 5\binom{p+1}{3} + 5^2\binom{p+1}{5} + \ldots + 5^{\frac{1}{2}(p-1)}\binom{p+1}{p}. \end{aligned} \tag{2}$$

We now start to reduce modulo p. Consider the binomial coefficient

$$\binom{p}{r} = \frac{p!}{(p-r)!\,r!}.$$

The numerator contains a factor of p. The denominator will not contain a factor of p unless $r = 0$ or p. So $\binom{p}{r} \equiv 0 \pmod p$ except when $r = 0$ or p and in these cases $\binom{p}{r} = 1$. Similar considerations show that $\binom{p+1}{r} \equiv 0 \pmod p$ unless $r = 0, 1, p$ or $p+1$. In each of these four exceptional cases $\binom{p+1}{r} \equiv 1 \pmod p$. With these facts in mind the reduction of (2) is simple. Nearly all the binomial coefficients are congruent to 0 modulo p and the few remaining are congruent to 1. What remains is this:

$$2^{p-1}f_p \equiv 5^{\frac{1}{2}(p-1)} \pmod p, \qquad 2^p f_{p+1} \equiv 1 + 5^{\frac{1}{2}(p-1)} \pmod p. \tag{3}$$

The next well-known result to be called upon is Fermat's congruence (see reference 1, p. 87 or reference 2, p. 23).

Fermat's congruence. If p is a prime and z is an integer not congruent to 0 modulo p, then $z^{p-1} \equiv 1 \pmod{p}$. If z is any integer, $z^p \equiv z \pmod{p}$.

Fermat's congruence implies that, if we consider an odd prime p, $2^{p-1} \equiv 1 \pmod{p}$ and $2^p \equiv 2 \pmod{p}$. Then (3) reduces further to

$$f_p \equiv 5^{\frac{1}{2}(p-1)} \pmod{p}, \qquad 2f_{p+1} \equiv 1+5^{\frac{1}{2}(p-1)} \pmod{p}. \tag{4}$$

The right-hand side of (4) recalls a result of Euler (see reference 1, p. 111 or reference 2, p. 63 (Theorem 3.1)).

Euler's criterion. If z is not congruent to 0 modulo p and p is an odd prime, then

$$z^{\frac{1}{2}(p-1)} \equiv \begin{cases} 1 \pmod{p} & \text{(if } \bar{z} \text{ is a quadratic residue modulo } p\text{)}, \\ -1 \pmod{p} & \text{(if } \bar{z} \text{ is a quadratic non-residue modulo } p\text{)}. \end{cases}$$

To explain, if $z \not\equiv 0 \pmod{p}$, $\bar{z}$ is a quadratic residue if the congruence $x^2 \equiv \bar{z} \pmod{p}$ is soluble. If $x^2 \equiv \bar{z} \pmod{p}$ is not soluble then we say that $\bar{z}$ is a quadratic non-residue. In effect, to say that $\bar{z}$ is a quadratic residue means that 'z has a square root modulo p'. Applying Euler's criterion to (4) we obtain

$$\left.\begin{aligned} f_p &\equiv 1 \pmod{p} \\ f_{p+1} &\equiv 1 \pmod{p} \end{aligned}\right\} \begin{array}{l} \text{if } p\ (\neq 5) \text{ is an odd prime and} \\ x^2 \equiv 5 \pmod{p} \text{ is soluble,} \end{array}$$
$$\left.\begin{aligned} f_p &\equiv -1 \pmod{p} \\ f_{p+1} &\equiv 0 \pmod{p} \end{aligned}\right\} \begin{array}{l} \text{if } p\ (\neq 5) \text{ is an odd prime and} \\ x^2 \equiv 5 \pmod{p} \text{ is not soluble.} \end{array} \tag{5}$$

Now one of the most celebrated results in number theory plays its part. Gauss's law of quadratic reciprocity (see reference 1, p. 114 or reference 2, p. 67) solves the problem of when a residue is quadratic.

Gauss's law of quadratic reciprocity (special case). Let p be an odd prime ($p \neq 5$). Then $\bar{5}$ is a quadratic residue modulo p if and only if $\bar{p}$ is a quadratic residue modulo 5.

So when $p \neq 5$, $x^2 \equiv 5 \pmod{p}$ is soluble if and only if $x^2 \equiv \bar{p} \pmod{5}$ is soluble. But a simple inspection of the residues modulo 5 shows that only 1 and 4 are the squares of other residues. Therefore $x^2 \equiv 5 \pmod{p}$ is soluble if and only if $p \equiv 1$ or $4 \pmod{5}$. Formulae (5) now become simple:

$$f_p \equiv f_{p+1} \equiv 1 \pmod{p} \quad \text{if } p \text{ is a prime congruent to 1 or 4 modulo 5,} \tag{6}$$

$$f_p \equiv -1 \pmod{p} \text{ and } f_{p+1} \equiv 0 \pmod{p} \quad \text{if } p \text{ is an odd prime congruent to 2 or 3 modulo 5.} \tag{7}$$

We make some further deductions from (7) by use of (1).

$$\left.\begin{aligned} f_{-p} &= (-1)^{p+1}f_p \equiv -1 \pmod p \\ f_{-p-1} &= (-1)^{p+2}f_{p+1} \equiv 0 \pmod p \end{aligned}\right\}\text{ if } p \text{ is an odd prime congruent to 2 or 3 modulo 5.} \tag{8}$$

We now apply the Fibonacci generating relation $\bar{f}_{-p-2}+\bar{f}_{-p-1} \equiv \bar{f}_{-p} \pmod p$ to deduce the value of $\bar{f}_{-p-2}$:

$$\left.\begin{aligned} f_{-p-2} &\equiv -1 \pmod p \\ f_{-p-1} &\equiv 0 \pmod p \end{aligned}\right\}\text{ if } p \text{ is an odd prime congruent to 2 or 3 modulo 5.} \tag{9}$$

We next draw together statements (6), (7) and (9):

$$\left.\begin{aligned} f_1 &\equiv f_p \pmod p \\ f_2 &\equiv f_{p+1} \pmod p \end{aligned}\right\}\text{ if } p \text{ is a prime congruent to 1 or 4 modulo 5,} \tag{10}$$

$$\left.\begin{aligned} f_{-p-2} &\equiv f_p \pmod p \\ f_{-p-1} &\equiv f_{p+1} \pmod p \end{aligned}\right\}\text{ if } p \text{ is an odd prime congruent to 2 or 3 modulo 5.}$$

These statements are the aim of the proof. In each case we have a pair of residues that repeat themselves later in the reduced sequence. In the first case, where $p \equiv 1$ or $4 \pmod 5$, the remainders $\bar{f}_1$ and $\bar{f}_2$ recur as $\bar{f}_p$ and $\bar{f}_{p+1}$. This means that, if we think of a cycle as starting at $\bar{f}_1$, there follow a number of complete cycles ending with $\bar{f}_{p-1}$. Therefore the sequence $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{p-1}$ divides exactly into a whole number of cycles. Therefore, if p is a prime congruent to 1 or 4 modulo 5, $C(p)$ divides exactly into $p-1$. For similar reasons, if p is an odd prime congruent to 2 or 3 modulo 5, the sequence $\bar{f}_{-p-2}, \bar{f}_{-p-1}, \ldots, \bar{f}_{p-1}$ divides exactly into complete cycles. This list consists of $2(p+1)$ remainders. Therefore, if p is an odd prime congruent to 2 or 3 modulo 5, then $C(p)$ divides $2(p+1)$. We add in the special case when p is an even prime, that is, $p = 2$. Now $C(2) = 3$, so that in this case too $C(p)$ divides $2(p+1)$. Therefore, for all primes congruent to 2 or 3 modulo 5, $C(p)$ divides $2(p+1)$. That ends a circuitous but, I think, interesting proof.

So we now know that $C(p)$ divides exactly into $p-1$ or $2(p+1)$, unless $p = 5$. For small values of p, $C(p)$ often takes the value $p-1$ or $2(p+1)$, but this is not always so, as is shown by the values of $C(29)$ and $C(47)$ in table 1.

The next step is to consider powers of primes. I am able to prove the following result.

Proposition 2. Let p be prime. The $C(p^{k+1}) = C(p^k)$ or $pC(p^k)$, where k is any positive integer.

Now with a third proposition the prime powers are knitted together.

Proposition 3. If $\mathrm{hcf}(m, n) = 1$ then $C(mn) = \mathrm{lcm}(C(m), C(n))$.

The proof of Proposition 3 is a routine proof through elementary number theory which is omitted for the sake of brevity, although we give an illustration of the use of Proposition 3 below.

Example. To find $C(912)$. We factorise 912 into prime powers as $16\times3\times19$. From the data in table 1 we read that $C(16) = 24$, $C(3) = 8$ and $C(19) = 18$. Now, as $\mathrm{hcf}(16, 3) = 1$, by Proposition 3,

$$C(48) = C(16\times3) = \mathrm{lcm}(C(16), C(3)) = \mathrm{lcm}(24, 8) = 24.$$

Again, as $\mathrm{hcf}(48, 19) = 1$,

$$C(912) = C(48\times19) = \mathrm{lcm}(C(48), C(19)) = \mathrm{lcm}(24, 18) = 72.$$

We conclude that $C(912) = 72$, which readers may like to check.

More facts about cycle length. We have come a long way in understanding how the value of $C(n)$ is to be discovered, but there is still a good deal of uncertainty. Proposition 1 does not give the value of $C(p)$ exactly but restricts it to one of the divisors of $p-1$ and $2(p+1)$, unless $p = 5$. Now the situation may be improved by eliminating some of those divisors. I have proofs of the following results.

Proposition 4. If p is prime then $C(p)$ is even, unless $p = 2$.

Proposition 5. If p is an odd prime congruent to 2 or 3 modulo 5 then $C(p)$ is divisible by 4.

Uncertain conclusion. We have said a great deal about $C(n)$, yet we cannot pin it down precisely. There are further mysteries too if we consider a question such as 'Are there infinitely many primes for which $C(p) = p-1$?' The answer is 'yes' if a general conjecture of Emil Artin in number theory is true. But Artin's conjecture remains unproven. It is possible that there is some means of answering the question without recourse to Artin's conjecture, but I do not know of such means.

References

1. Calvin T. Long, *Elementary Introduction to Number Theory* (D. C. Heath, Boston, 1972).
2. I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers* (Wiley, New York, 1972).

Factorizing the Differential Operator

OLIVER JOHNSON, *King Edward's School, Birmingham*

The author was in the sixth form when he wrote this article.

It is well known that the homogeneous equation

$$p_0 x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_1 x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + p_n y = f(x)$$

is reduced to a linear equation with constant coefficients by means of the substitution $x = \mathrm{e}^t$. But I wonder how many readers have met the attractive formula

$$x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \left(\frac{\mathrm{d}}{\mathrm{d}t} - (n-1)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - (n-2)\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t}, \tag{1}$$

which is easily proved by induction.

First note that, if $x = \mathrm{e}^t$, then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t = x \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{x}.$$

Now (1) holds for $n = 1$, since

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Next, suppose that (1) holds for $n = k$, so that

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = \frac{1}{x^k}\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-2)\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Then

$$\begin{aligned}\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = {} & \frac{1}{x^k}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-2)\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t}\right\}\frac{\mathrm{d}t}{\mathrm{d}x} \\ & - \frac{k}{x^{k+1}}\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-2)\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t}\end{aligned}$$

and so

$$x^{k+1}\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \left(\frac{\mathrm{d}}{\mathrm{d}t} - k\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - (k-1)\right)\ldots\left(\frac{\mathrm{d}}{\mathrm{d}t} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}t},$$

i.e. (1) also holds for $n = k+1$.

Hence, by induction, (1) holds for all integers $n \geqslant 1$.

Home and Away

DAVID SHARPE, *University of Sheffield*

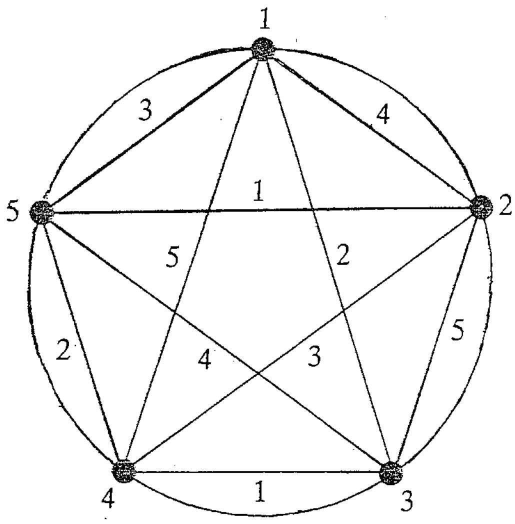
> The author is currently editor of *Mathematical Spectrum*, and wrote this whilst watching the rain through his window in the Lake District. He rejoices that Sheffield currently has two soccer teams in the first division, but laments that Leicester (his local team in formative years) languishes in the second.

There are currently 22 teams in the first division of the football league in England and Wales. If each team plays every other team twice in a season, home and away, and all matches take place on Saturdays, how many weeks are needed for the fixtures? What if there are 23 teams? Or n teams?

Denote each team by a 'vertex' and denote each match between two teams A and B by an 'edge' or line joining A and B.

BARCLAYS LEAGUE—Div I

		HOME					AWAY					
	P	W	D	L	F	A	W	D	L	F	A	Pts
Man U	11	4	2	0	12	1	4	1	0	6	2	27
Leeds	12	4	2	0	12	6	2	3	1	8	4	23
Arsenal	11	4	1	1	14	8	2	1	2	12	9	20
Sheff W	12	5	0	1	16	7	1	2	3	5	7	20
Coventry	12	4	1	2	10	3	2	1	2	5	7	20
Man City	12	3	0	3	8	8	3	1	2	7	7	19
Chelsea	12	2	3	1	12	8	2	2	2	9	10	17
Wimbledon	12	4	0	2	14	9	1	2	3	7	10	17
C Palace	11	3	1	2	11	12	2	1	2	9	11	17
Nottm For	11	3	1	1	12	9	2	0	4	9	8	16
Everton	12	3	3	0	12	5	1	1	4	7	11	16
Tottenham	9	1	0	2	4	5	4	1	1	13	9	16
Liverpool	10	3	2	0	8	4	1	2	2	3	4	16
Aston Villa	12	3	1	2	10	4	1	2	3	6	10	15
Norwich	12	2	3	1	7	6	1	3	2	8	11	15
Notts Co	12	2	1	3	7	12	2	2	2	8	7	15
Oldham	11	3	2	1	12	8	1	0	4	5	9	14
West Ham	12	1	2	3	5	7	1	3	2	7	9	11
QPR	12	0	4	2	5	9	1	2	3	5	10	9
Southamptn	12	1	1	4	4	13	1	2	3	6	8	9
Luton	12	2	2	1	5	4	0	1	6	1	23	9
Sheff U	12	1	2	3	4	8	0	1	5	10	18	6



First suppose that n is odd (and greater than 1!), and consider just half a season, in which each team plays every other team just once. We shall first show that these matches can be played in n weeks. Arrange the vertices denoting the teams to be equally spaced round a circle, and label them 1 to n; the case $n = 5$ is shown in the figure. Now label an edge i if it is parallel to the tangent to the circle at vertex i. This gives one of the labels 1 to n to each of the edges, and this label is the week on which that match is to be played. Thus the matches can be played in n weeks. The matches in the second half of the season can also be played in n weeks; two teams A and B play each other on A's ground (say) in the first half of the season and on B's ground in the second half. Thus the matches can be played in $2n$ weeks. But there can be at most $n-1$ teams in action on any given Saturday, and so at most $\frac{1}{2}(n-1)$ matches are being played. Since the total number of matches is $n(n-1)$, this requires at least $n(n-1)/\frac{1}{2}(n-1) = 2n$ weeks. Thus, when n is odd, the minimum number of weeks required in a season is $2n$.

We now consider the case when n is even, and again consider just the first half of the season. Take out one team, team X. We can arrange that the remaining $n-1$ teams play their matches in $n-1$ weeks in the manner already described. Each team has one week when it is not in action (consider the figure for $n = 5$, for example), and it is a different week for each team, so on that week this team can play team X. Hence n teams can play each other in $n-1$ weeks. The same is true for the second half of the season, each pair of teams again exchanging grounds, so that the season can be completed in $2(n-1)$ weeks. But each team plays $n-1$ other teams twice, so at least $2(n-1)$ weeks are needed and this is therefore the minimum number of weeks required.

Thus, with 22 teams, 40 weeks are needed, a fact well known to every football supporter; with 23 teams, 46 weeks are required.

Computer Column

MIKE PIFF

Shell sorting

In the last column, we investigated sorting a real array by means of insertion sorting. The main drawback to this method is that its running time is proportional to n^2, which means that it is impractical for large values of n.

However, just a slight modification brings the running time down to $n^{3/2}$, which extends its usefulness to values of n up to about 1000. Add the following lines to the body of `sort.mod`, and just the first declaration line to `sort.def` to make it visible to other modules. If you choose, the declaration of *InsertionSort* may be removed from `sort.def`, to make it invisible.

The idea behind Shell's method is first to move those items which need to move a long way, and gradually to reduce the distances moved down from roughly $\frac{1}{2}n$ to just one place.

```
PROCEDURE ShellSort(VAR a:ARRAY OF REAL);
VAR j,step:INTEGER;
BEGIN
  IF HIGH(a)>0 THEN
    step:=2;
    WHILE CARDINAL(step)<HIGH(a) DO
      step:=2×step;
    END;
    REPEAT
      step:=step DIV 2;
      FOR j:=0 TO (step-1) DO
        InsertionSort(a,j,HIGH(a),step)
      END;
    UNTIL step≤1;
  END;
END ShellSort;
```

Letters to the Editor

Dear Editor,

Summing powers of integers

In his article on summing powers of integers (*Mathematical Spectrum* Volume 23 Number 4) Oliver Anderson defined numbers A_k such that

$$S(k,n) = k\int_0^n S(k-1,x)\,\mathrm{d}x + A_k n,$$

where $S(k,n) = \sum_{i=1}^{n} i^k$ and $S(1,n) = \frac{1}{2}n^2+\frac{1}{2}n$. These are in fact the same as the Bernoulli numbers B_k defined by the formula

$$\frac{x}{\mathrm{e}^x-1} = \sum_{k=0}^{\infty} B_k\frac{x^k}{k!}$$

(except for $k=1$). Note that $B_k = 0$ for all odd $k>1$, $B_{4m}<0$ for all $m \geqslant 1$, $B_0 = 1$ and $B_1 = -\frac{1}{2}$. Also, although $|B_k|$ is initially small for small k, the general trend is exponential growth towards infinity. For example, B_{250} has an integer part of 294 digits in its decimal expansion. The first few values of B_k for even k, starting from B_2, are

$$\tfrac{1}{6},\quad -\tfrac{1}{30},\quad \tfrac{1}{42},\quad -\tfrac{1}{30},\quad \tfrac{5}{66},\quad -\tfrac{691}{2730},\quad \tfrac{7}{6},\quad -\tfrac{3617}{510},\quad \tfrac{43\,867}{798},\quad \ldots.$$

The only discrepancy is that, whereas $B_1 = -\frac{1}{2}$, $A_1 = +\frac{1}{2}$.

[Further information can be found in the following references:

1. B. L. Burrows and R. F. Talbot, Sums of powers of integers, *Amer. Math. Monthly* (1984), 394–403.

2. A. W. F. Edwards, Sums of powers of integers: a little of the history, *Math. Gazette* (1982), 22–28.

Editor.]

Yours sincerely,

JOSEPH MCLEAN
(9 Larch Road,
Glasgow G41 5DA)

Dear Editor,

Weed v. reed

Regarding David Singmaster's pondweed problem (Volume 23, Number 1, page 7), I feel obliged to disagree with the solution offered by Barry Christian (Volume 24, Number 1, page 23).

The precise wording of the problem is important:

> 'A water weed grows 3 feet on the first day, and its growth on each succeeding day is half that on the preceding day. A reed grows 1 foot on the first day, and its growth on each succeeding day is twice that on the preceding day. When are they of equal height?'

This means that the amount of *growth* on successive days is

$$3, \quad \tfrac{3}{2}, \quad \tfrac{3}{4}, \quad \ldots \qquad \text{for the weed}$$

and

$$1, \quad 2, \quad 4, \quad \ldots \qquad \text{for the reed.}$$

Let w_n and r_n denote the total amount of growth at the end of the nth day for the weed and the reed, respectively. Then, by summing the geometric progressions above one finds that

$$w_n = 6(1-2^{-n}) \quad \text{and} \quad r_n = 2^n-1. \tag{*}$$

We shall make two assumptions:
(1) the two plants have the same initial size, which we may take to be zero;
(2) the plants grow at such a rate during each day that the formulae (*) remain valid when n is not an integer.
(Instead of (2), we could assume, e.g., linear growth during each day.)

It follows that the plants are of equal size when $w_n = r_n$, i.e.

$$2^n-7+6\times2^{-n} = 0.$$

Multiplying by 2^n and factorising gives

$$(2^n-1)(2^n-6) = 0.$$

The two solutions are $n = 0$, which we assumed, and $n = \log_2 6$, and the size when the plants are equal is exactly 5 feet.

I find it striking that the equal size turns out to be an integer. Is this coincidence (or the work of David Singmaster) or was the answer known in 150 BC by Chiu Chang Suan Shu, to whom Singmaster attributes the problem?

Yours sincerely,
JEREMY BYGOTT
(Queens' College,
Cambridge CB3 9ET)

Father Christmas had 500 presents on his sleigh, costing a total of £6850. There were soft toys worth £6 each, cricket bats worth £50 each, and books at £11 each. He had more soft toys than books. What was the smallest number of cricket bats he could have been delivering?

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

24.4 (Submitted by Jeremy Bygott, Queens' College, Cambridge)
Prove that

$$\prod_{k=0}^{n} 2\cos 2^k\theta = 2\sum_{k=0}^{2^n-1} \cos(2k+1)\theta.$$

24.5 *Friends and strangers at a mathematics lecture.* (Submitted by Gregory Economides, University of Newcastle upon Tyne Medical School)
There are n students present at a mathematics lecture. Every two students are either friends of each other or strangers to each other. No two friends have a friend in common. Show that each student has the same number of friends at the lecture. (This situation can only occur for a limited number of values of n.)

24.6 *Mike the mountaineering mite.* (Submitted by John MacNeill, The Royal School, Wolverhampton)
Mike the mountaineering mite is at a point on the rim of the horizontal base of a right circular cone of radius r and height h. He wants to climb up to the vertex of the cone but unfortunately the direct route is too steep for him since he can only follow a path which never makes an angle greater than β with the horizontal. Find the length of Mike's shortest route to the vertex.

Later we see Mike at a point on the rim of the horizontal base of a hemisphere of radius r. He wants to climb up to the highest point of the hemisphere. The restriction on the steepness of his path still applies. Find the length of Mike's shortest route to the highest point of the hemisphere.

Solutions to Problems in Volume 23 Number 4

23.10 (a) Let $n = p^\lambda$, where p is a prime number and λ is a positive integer. Let r be an integer such that $1 \leqslant r \leqslant n-1$. Show that p divides the binomial coefficient ${}^nC_r$.

(b) Let r and n be integers such that $2 \leqslant r \leqslant n-2$. Show that ${}^nC_r$ is composite.

Solution by Amites Sarkar (Trinity College, Cambridge)

(a)
$${}^nC_r = \frac{n!}{r!\,(n-r)!}\,.$$

The highest power of p dividing the numerator is the $(p^{\lambda-1}+p^{\lambda-2}+\ldots+1)$th power. The highest power of p dividing the denominator is the

$$\left\{\left(\left[\frac{r}{p}\right]+\left[\frac{r}{p^2}\right]+\ldots+\left[\frac{r}{p^{\lambda-1}}\right]\right)+\left(\left[\frac{n-r}{p}\right]+\left[\frac{n-r}{p^2}\right]+\ldots+\left[\frac{n-r}{p^{\lambda-1}}\right]\right)\right\}\text{th}$$

power, where $[\alpha]$ denotes the integer part of α. We have the inequalities

$$\begin{aligned} p^{\lambda-1} &= \frac{n}{p} \geqslant \left[\frac{r}{p}\right]+\left[\frac{n-r}{p}\right], \\ p^{\lambda-2} &= \frac{n}{p^2} \geqslant \left[\frac{r}{p^2}\right]+\left[\frac{n-r}{p^2}\right], \\ \ldots &= \ldots \quad \ldots \quad \ldots \quad \ldots, \\ p &= \frac{n}{p^{\lambda-1}} \geqslant \left[\frac{r}{p^{\lambda-1}}\right]+\left[\frac{n-r}{p^{\lambda-1}}\right]. \end{aligned}$$

Adding these inequalities, we see that the power of p dividing the numerator of $n!/r!(n-r)!$ is greater than the power of p dividing its denominator, so that ${}^nC_r$ is divisible by p.

(b) Suppose that ${}^nC_r = p$, a prime. Then $p\,|\,n!$, so that $p \leqslant n$, i.e. ${}^nC_r \leqslant n$. But, for $2 \leqslant r \leqslant n-2$,

$${}^nC_r = \frac{n(n-1)\ldots(n-r-1)}{r(r-1)\ldots1}$$

and $n-1 > r$, $n-2 > r-1$, $\ldots$, $n-r+1 > 2$ because $n \geqslant r+2 > r+1$, and there are $r-1 \geqslant 1$ inequalities here, so that ${}^nC_r > n$. Hence ${}^nC_r$ must be composite.

23.11 Consider the infinite array of numbers

$$\begin{array}{ccccccc} 1 \\ 1 & 1 \\ 2 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

defined as follows. Write 1 in row 1. Read row 1 as 'one 1' and write this as 1 1 in row 2. Read row 2 as 'two 1s' and write 2 1 in row 3. Read row 3 as 'one 2 one 1' and write 1 2 1 1 in row 4, and so on. Prove that no term of the array exceeds 3 and that 3 3 3 can never occur in a row.

Solution by Andrew Thomson (The College, Winchester)

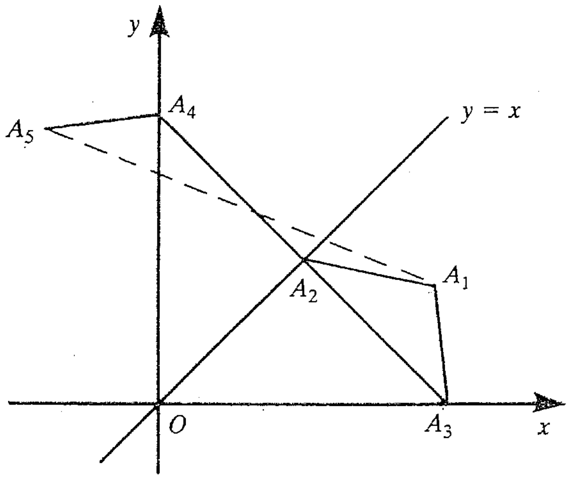
Suppose that there is a term greater than 3, and consider the first row in which this occurs. Then the previous row must have four equal consecutive terms $xxxx$. Each row is made up of pairs of numbers describing the previous row, so that $xxxx$ can be split up into xx xx or $*x$ xx $x*$. In either case x occurs at the end of two consecutive pairs. This is impossible, since a xs followed by b xs would appear in the next row as $(a+b)x$ not $axbx$.

For 3 3 3 to occur in a row, 3 3 3 must also occur in the previous row, as there is a 3 3 pair in the sequence, and so on back to the first row. But 3 3 3 does not occur in the first row. Hence 3 3 3 can never occur in a row.

Also solved by Amites Sarkar, Chris Warner (The King's School, Canterbury).

23.12 Let point A_1 have coordinates $(3,2,1)$. Find points A_2 and A_3 on the planes $y = x$ and $y = 0$, respectively, such that the circumference of triangle $A_1A_2A_3$ is a minimum.

Solution by Amites Sarkar



Clearly both the z-coordinates of points A_2 and A_3 are 1. We therefore restrict our attention to the plane $z = 1$. The point A_5 has coordinates $(-2,3,1)$ (so that OA_1 is rotated through 90° anticlockwise about the z-axis to OA_5) and A_4 is the reflection of A_3 in the plane $y = x$. The circumference of triangle $A_1A_2A_3$ is the same as the length of the path $A_1A_2A_4A_5$, and this will be a minimum when this path is a straight line. The straight line through A_1 and A_5 cuts $y = x$ at $(\frac{13}{6}, \frac{13}{6}, 1)$ and $x = 0$ at $(0, \frac{13}{5}, 1)$. Hence triangle $A_1A_2A_3$ has minimum circumference when $A_2 = (\frac{13}{6}, \frac{13}{6}, 1)$ and $A_3 = (\frac{13}{5}, 0, 1)$.

Also solved by Chris Warner.

Reviews

Problem Solving Series (Booklets 1–5). By DEREK HOLTON. The Mathematical Association, 1990. 40 pages per booklet average. £1.25 each for members. (ISBN 0 906588 11 1/12 X/13 8/14 6/15 4).

The first five booklets in this series of 10 deal with 'how to', combinatorics, graph theory, number theory and geometry (respectively). Each contains 2–4 main topics with collections of exercises and solutions of varying degrees of completeness. Their aim is to extend able secondary mathematics students by considering areas of mathematics not usually covered in the school syllabus. In a series of this type the emphasis should be on providing 'interesting' problems to stimulate the mathematical ability of its intended audience; the author adopts a problem-solving approach to achieve this end. Whilst most of the problems discussed are of interest in themselves, a number of opportunities have been missed to illustrate some of the remarkable interconnections that exist within many parts of mathematics. The problems tackled in Booklet 1 ('How to') exemplify this:

(i) Problem 1, measuring out a certain amount of liquid given jugs of various sizes, is approached purely algebraically, although the ideas used are later connected (in Booklet 4) to the Euclidean division algorithm. The beautiful recasting of the problem in terms of the path of a particle undergoing repeated reflections within appropriately shaped containers is not mentioned. This omission is rather unfortunate in view of the author's highly commendable concern with geometrical ideas displayed in later booklets. In addition, the reflection idea leads, via some interesting parity ideas, to a geometrical discussion of highest common factors, a topic again treated purely algebraically in Booklet 4.

(ii) Problem 2, summing series, also relies in an essential way on parity considerations, but this theme is not really explored.

(iii) Problem 3, using a given set of stamps to produce higher-valued combinations, demonstrates an interesting connection with the jug problem—the latter is purely additive, whilst the former allows subtractive moves—but does not explore any of the consequences. Further, if we ask in how many ways a given value can be obtained from lower-denomination stamps, we are led to the powerful notion of generating functions. This latter concept may be explored in a later booklet; if so, a reference would be useful.

Of course, it may be possible that such considerations are excluded due to space limitations, but this aspect of mathematics is of such fundamental importance, and contributes in no small part to the endless fascination of the subject, that every opportunity should be taken to emphasize links even in introductory works.

I was pleased to see tessellations and Euclidean constructions appearing in the geometry booklet, although a historical perspective on the latter would be useful. The exercises on circle packings do not discuss the optimal (or otherwise) nature of the packings and a reference would be useful (Kravitz, *Math. Mag.* 1967 pp. 65–71, for example). In the graph theory booklet I would have liked an accessible reference to the Knight's tour (there are several) and some indication of the use of trees (in producing optimal schemes for power evaluation, for example). The discussion of the 4/5-colour theorem is very welcome, although it may prove too difficult for a student to follow unaided; further discussion of the triangulation approach would also be useful. The divisibility tests in Booklet 4 are very well presented, though I would have liked to see a second booklet on number theory (addressing aspects of factorization for example).

The booklets do contain a nice mixture of problems serving as a suitable introduction to various topics. The advice on problem solving in Booklet 1 is worth digesting, though the impression given that the teacher will always know the answer may also be a little misleading. There are a few misprints, which should cause no real problems. At a modest price the booklets are good value for the exercises alone. If you want to know whether regular hexagons are self-replicating (can be assembled to form a larger copy of themselves), or what the last two digits of $2^{222}-1$ are, take a look at these booklets. They deserve to be considered at sixth-form level and I hope they do well. I look forward to the publication of Booklets 6–10 and to seeing how the combinatorial and geometrical ideas are further developed.

Napier Polytechnic, Edinburgh LES SHORT

Core Maths for A-level. By L. BOSTOCK and S. CHANDLER. Stanley Thornes (Publishers) Ltd, Cheltenham, 1990. Pp. xvi+875. £11.99 (ISBN 0-7487-0067-6).

Many schools already use Bostock and Chandler's *Core Course for A-level* (ISBN 0-85950-308-2) as their main A-level textbook. The authors have produced this new book because they feel that, with GCSE courses, it can no longer be assumed that all students enter A-level courses with the algebraic skills and geometric knowledge that used to be expected. The first six (of 40) chapters cover basic algebra and geometry. The book covers most of the material in the *Core Course*, but not all: for example, induction is missing. The book is set out in a clear, pleasing and illuminating fashion. It is a pity that not all the errors have been spotted. For example, on page 3, 'remember that $(-3)\times(x) = +3x$'. One innovative idea is that there are symbols in the text indicating that a specific computer program would aid in the understanding of a topic.

It would be very expensive to replace existing sets of textbooks by this book. However, I would recommend schools to buy some copies for students who need a gentler introduction to sixth-form mathematics and more practice to attain confidence in basic algebra. If a new set of A-level books is required, this is certainly a strong contender.

United World College of the Atlantic
South Glamorgan

PAUL BELCHER

Examples in A-level Core Mathematics. By EWART SMITH. Stanley Thornes (Publishers) Ltd, Cheltenham, 1990. Pp. iii+181. £5.25 (ISBN 0-7487-0440-X).

This book consists of three parts. Part 1 states a few useful facts on a topic (33 topics in all) followed by numerous questions of varying difficulty on that topic. Part 2 consists of 10 revision papers, each containing 12 examination-standard questions on a variety of topics. Part 3 is similar, only it contains 15 papers with 10 questions each, with the questions being taken from recent A-level papers of various examining boards. Answers to all questions are given at the back; graphs and diagrams here are very clearly produced and on a sensible scale. In Part 1, the author had a problem deciding how many and which useful facts to include—some topics do not have any. For example in Topic 2 (quadratic equations) it gives $\alpha+\beta = -b/a$ and $\alpha\beta = c/a$, but not the quadratic formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

An erratum slip giving four corrections is enclosed with the book.

I cannot see schools buying classroom sets of this book, but individual students wishing to gain experience of tackling lots of problems might buy their own copy. For teachers it is a useful source of examples classified by topic. Worth having in the school library.

United World College of the Atlantic
South Glamorgan

PAUL BELCHER

A Guide to Numerical Analysis. By PETER R. TURNER. Macmillan, Basingstoke, 1989. Pp. x + 208. £8.95 (ISBN 0-333-44947-9).

This book serves as a good introduction to many numerical topics, although it occasionally lacks depth in those most useful to a first-year undergraduate. The book opens with a good description of the floating-point system used by most computers today for the storage of real numbers, outlining its pros and cons and the problems to which it gives rise. Chapters 2 and 3 discuss simple iterative methods for solving equations and give a readable but rigorous treatment of the convergence of sequences and series that will be useful for undergraduates new to ϵ, δ analysis. The remaining chapters introduce many more topics: Lagrange polynomials, Newton's divided difference formulae, splines, CORDIC algorithms, the standard algorithms for numerical calculus and the solution of ordinary differential equations and, finally, Gauss elimination and LU factorisation. I did feel that certain topics such as CORDIC algorithms and LU factorisation, although interesting, are of little use to a first-year undergraduate and should have been omitted to allow a deeper study of the standard algorithms. For example, regular singular points are not mentioned in the discussion of the Frobenius method, nor is Gauss–Jordan elimination in the solution of linear equations. There are many helpful examples and exercises with solutions and BASIC programs to illustrate the algorithms. A book worth reading.

Exeter College, Oxford — DOMINIC SYMES

Other books received

A Concise Course in A-level Statistics, second edition. By J. CRAWSHAW and J. CHAMBERS. Stanley Thornes, Cheltenham, 1990. (ISBN 0-7487-0455-8).

Collins Gem Basic Facts: Mathematics. Collins, 1991. £2.99.

Probabilistic Causality. By ELLERY EELLS. Cambridge University Press, 1991. Pp. xii+413. Hardback £30.00.

Problems in Mathematical Analysis. By PIOTR BILER and ALFRED WITKOWSKI. Marcel Dekker, New York, 1990. Pp. v+227. Hardback $49.75 US and Canada, $59.50 all other countries (ISBN 0-8247-8312-3).

Personal Mathematics and Computing: Tools for the Liberal Arts. Edited by FRANK WATTENBERG. MIT Press, Cambridge, MA, 1990. Pp. xiv+556. Hardback £26.95 (ISBN 0-262-23157-3).

What is Mathematical Logic? By JOHN CROSSLEY ET AL. Dover, New York, 1991. Pp. 77. Paperback £4.20 (ISBN 0-486-26404-1). This is a reprint of a book first published by OUP in 1972.

Statistics: Concepts and Controversies, third edition. By DAVID S. MOORE. W. H. Freeman, Oxford, 1991. Pp. xvii+439. Paperback £10.95 (ISBN 0-7167-2199-6).

A Primer in Probability, second edition. By KATHLEEN SUBRAMANIAN. Marcel Dekker, New York, 1990. Pp. xi+318. Hardback $45.00 (ISBN 0-8247-8348-4).

CONTENTS

ISSN 0025-5653

PRICES (*postage included*)

Prices for Volume 24 (Issues Nos. 1, 2, 3 and 4)

Subscribers in	*Price per volume*	
North, Central and South America	US$13.00 or £7.00	(Note: These overseas prices apply even if the order is placed by an agent in Britain.)
Australia	$A17.00 or £7.00	
Britain, Europe and all other countries	£6.00	

A discount of 10% will be allowed on all orders for five or more copies of Volume 24 sent to the same address.

Details of reduced prices for two- and three-year subscriptions available on request.

Back issues

All back issues except Volume 1 are available; information concerning prices and a list of the articles published may be obtained from the Editor.

Enquiries about rates, subscriptions and advertisements should be directed to:

Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, UK

Published by the Applied Probability Trust
Typeset by The Pi-squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK