Indian IMO Team Selection Tests 2002

First Practice Test

1. Points A, B, C are given on a line in that order. The semicircles $\Gamma_1, \Gamma_2, \Gamma_3$ with diameters AC, AB, BC respectively are constructed on the same side of the line. Let l be the line through B perpendicular to AC and let Γ be the circle tangent to l, to Γ_1 internally, and to Γ_3 externally at D. The diameter of Γ through D meets l in G. Prove that $DG = AB$.

2. Show that there exists a block of 2002 consecutive positive integers containing exactly 150 primes. (Note that there are 168 primes less than 1000.)

3. Consider the set $X = \{2^m 3^n \mid 0 \le m, n \le 9\}$. Find the number of quadratic equations $ax^2 + 2bx + c = 0$ with equal roots, where a, b, c are distinct elements of X.

Second Practice Test

1. In an acute triangle ABC, H is the orthocenter and O the circumcenter. Show that there are points D, E, F on BC, CA, AB respectively such that AD, BE and CF concur and $DO + DH = EO + EH = FO + FH$.

2. If a, b, c are positive numbers with $a^2 + b^2 + c^2 = 3abc$, prove that

$$\frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} \ge \frac{9}{a+b+c}.$$

3. Find the number of n-tuples of integers $(x_1, \ldots, x_n)$ such that

$$|x_i| \le 10 \quad \text{and} \quad |x_i - x_j| \le 10 \quad \text{for } 1 \le i, j \le 10.$$

First Test

1. Given two distinct circles touching each other internally, show how to construct a triangle with the inner circle as its incircle and the outer circle as its nine-point circle.

2. For a natural number n denote by $\sigma(n)$ the sum of the positive divisors of n.

   (a) Show that $\sigma(mn) = \sigma(m)\sigma(n)$ whenever $\gcd(m, n) = 1$.

   (b) Find the natural numbers n for which $\sigma(n)$ is a power of 2.

3. During their tour of West Indies, Sourav and Srinath have either an apple or an orange along with breakfast in the following sequence:

   Sourav has oranges for the first m days, apples for the next m days followed by oranges for m days and so on, while Srinath has oranges for the first n days, apples for the next n days and so on.

   If $\gcd(m,n) = 1$ and the tour lasted for mn days, how many days did they have the same kind of fruit?

Second Test

1. Let T denote the set of all ordered triples (p,q,r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ such that

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \dfrac{1}{6} \quad \begin{aligned} &(f(p+1,q-1,r) + f(p-1,q+1,r) \\ &+ f(p-1,q,r+1) + f(p+1,q,r-1) \\ &+ f(p,q+1,r-1) + f(p,q-1,r+1)) \end{aligned} & \text{otherwise.} \end{cases}$$

2. Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

3. Let $a < b$ be two positive integers. A set of three nonnegative integers $\{x,y,z\}$ with $x < y < z$ is called *olympic* if $\{z-y, y-x\} = \{a,b\}$. Show that the set of all nonnegative integers can be written as the union of disjoint olympic sets.

Third Test

1. Two triangles ABC and PQR have the following properties:

   (i) P is the midpoint of BC and A is the midpoint of QR;

   (ii) QR bisects $\angle BAC$ and BC bisects $\angle QPR$.

   Prove that $AB + AC = PQ + PR$.

2. Let p be an odd prime and a be an integer not divisible by p. Show that the number of triples (x,y,z) with $0 \le x,y,z < p$ satisfying $(x+y+z)^2 \equiv axyz \pmod{p}$ equals $p^2 + 1$.

3. Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Fourth Test

1. Is it possible to find 100 positive integers $a_1, \ldots, a_{100}$ not exceeding 25,000 such that all pairwise sums of them are different?

2. For an integer $n \geq 2$, let $(1+iT)^2 = f(T) + ig(T)$, where $i^2 = -1$ and f, g are polynomials with real coefficients. Show that for any real number k the equation $f(T) + kg(T)$ has only real roots.

3. Consider the square grid with the opposite vertices $A(0,0)$ and $C(n,n)$. A path is composed of one unit up and one unit right steps. Let C_n be the number of paths from A to C which stay on or below diagonal AC. (C_n are the *Catalan numbers*.) Show that the number of paths from A to C which go above the diagonal AC at most twice is equal to $C_{n+2} - 2C_{n+1} + c_n$.

Fifth Test

1. Given an acute triangle PQR, construct triangles SRP, TPQ, UQR exterior to $\triangle PQR$ with $SP = SR$, $TP = TQ$, $UQ = UR$, and $\angle PSR = 2\angle QPR$, $\angle QTP = 2\angle RQP$, $\angle RUQ = 2\angle PRQ$. Lines SQ and TU meet in S', TR and US in T', and UP and ST in U'. Evaluate $\dfrac{SQ}{SS'} + \dfrac{TR}{TT'} + \dfrac{UP}{UU'}$.

2. If a, b, c are arbitrary positive numbers, prove the inequality
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

3. Given a prime p, show that there is a positive integer n such that the decimal representation of p^n has a block of 2002 zeros.