Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Sequences, questionnaires and hamburgers

Within the space of a few days I received requests for help in solving two problems which are curiously similar. The first was from Paul Hufton of Solihull Sixth Form College. He found the following in A. Gardiner's book *The Mathematical Olympiad Handbook* (OUP, 1997), p. 17:

> The sequence (u_n) is defined by
>
> $$u_1 = 1,$$
> $$u_n = (n-1)u_{n-1} + 1 \quad \text{for } n > 1,$$
>
> and the problem is to find those n for which n divides u_n.

I tried a few values till the numbers got too big for my brain, and found that this holds when $n = 1, 2, 4, 5, 10, 13$ — no pattern there that I could see, unless it is that these numbers have no prime factor of the form $4k+3$. I then tried a different tack and started with u_n and kept going, eventually reaching

$$\begin{aligned} u_n &= (n-1)(n-2)\cdots 2\cdot 1 + (n-1)(n-2)\cdots 2 \\ &\quad + (n-1)(n-2)\cdots 3 + \cdots + (n-1) + 1 \\ &= (n-1)!\Big(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}\Big) \\ &= (n-1)!\Big(\mathrm{e} - \frac{1}{n!} - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - \cdots\Big), \end{aligned}$$

so that, when $n > 1$,

$$\begin{aligned} (n-1)!\mathrm{e} - u_n &= \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots \\ &< \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \cdots \\ &= \frac{1}{n}\,\frac{1}{1-1/n} \\ &= \frac{1}{n-1} \\ &\le 1. \end{aligned}$$

Hence,

$$u_n = \big[(n-1)!\mathrm{e}\big] \quad \text{for } n > 1,$$

the integer part of $(n-1)!\mathrm{e}$. But that was no help, and I had to confess failure to Paul.

The second problem was presented to me by a former student of mine, another Paul, at church of all places!

> The sequence (u_n) is defined by
>
> $$u_0 = 3,\ u_1 = 0,\ u_2 = 2,$$
> $$u_n = u_{n-3} + u_{n-2} \quad \text{for } n > 2,$$
>
> and the problem again is to find those n for which n divides u_n.

I tried a few values again, and it soon looked as though it is the primes that we want. But how to prove it!

If any readers can throw any light on either of these problems, do write in with your thoughts.

These problems reminded me of a beautiful result which goes back to the nineteenth century French schoolteacher E. Lucas, which was improved by Lehmer in 1930. The problem is to find primes of the form $2^p - 1$, so-called Mersenne primes named after the seventeenth-century French monk Father Marin Mersenne. It is not difficult to see that, for $2^p - 1$ to be prime, p must also be prime. (Try factorizing $2^p - 1$ when $p = mn$.) But it doesn't work the other way round: $2^{11} - 1 = 23 \times 89$. So the problem is to test whether $2^p - 1$ is prime for a prime p. You could try testing whether it is divisible by a prime up to its square root, but you will soon give that up! The Lucas–Lehmer test is the following:

> The sequence (u_n) is defined by
>
> $$u_1 = 4,$$
> $$u_n = u_{n-1}^2 - 2 \quad \text{for } n > 1.$$
>
> Then, for $p > 2$, $2^p - 1$ is prime if and only if it divides u_{p-1}.

Thus $2^3 - 1 = 7$ divides $u_2 = 14$ and $2^5 - 1 = 31$ divides $u_4 = 37\,634$. The terms of the sequence can be reduced modulo $2^p - 1$ as you go along, which helps.

This isn't just a curiosity. Large prime numbers are of use in modern cryptography to design codes which cannot be broken. One such depends on the fact that today's computers are incapable of factorizing a 300-digit number which is the product of two unknown primes of 150 digits each (unknown, that is, to the person trying to break the code). So your security and the security of your financial assets (or overdraft) depend on prime numbers!

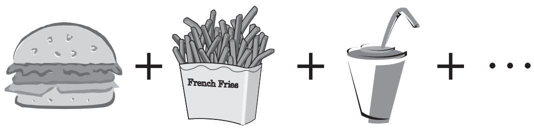
But to return to our two Pauls and their problems, I shall have to get the little grey cells working!

To change the subject completely, our thanks to all our readers who returned the questionnaire. We were surprised just how satisfied readers seem to be. Maybe the others were too cheesed off to reply! Inevitably, readers liked some items more than others, but often what reader A disliked, reader B liked! I suppose the analysis of questionnaires does not have the same precision as mathematics! A number of readers suggested that we review interesting websites. We invite readers to let us know what they have found of interest whilst surfing the web. You could, for a start, try searching for Mersenne primes. Another suggestion is to include descriptions of current research. We would love to do this if we could persuade researchers to write intelligibly for us. Maybe we are asking for the impossible! A third suggestion was to include ‘mathematical byways and eccentrivia’. Please, readers, do send in anything that you think might fit the bill.

Our Australian editor has sent us the following snippet of news. Apparently the Australian Mathematical Society has been lobbying for the creation of a National Sciences Institute along the lines of the Isaac Newton Institute in Cambridge, UK. The Institute would be to encourage mathematical research and disseminate news. Apparently, the Labor Party is to include this proposal in its next manifesto. Mathematics reaching into the political world. Will its influence never end!

McDunces?

Mathematics has even invaded the world of advertising. This headline in my local newspaper caught my eye. In their latest advertising campaign, McDonald’s® have been claiming that, from eight 99p items, customers have an amazing 40 312 meal combinations. Jason Baxter, a maths student at Sheffield Hallam University, says they have got it wrong, and the true figure is a mere 255. Where did they get these figures from, and who is right?



Statistics and History

JOE GANI

Statistical data are important in historical writing, but must be accurate and clearly presented. Statistical models may be used to support historical hypotheses.

1. Introduction

Most people would, at first glance, dismiss the notion that there is a serious relationship between statistics and history. After all, the dictionary defines statistics as the collection, organization and interpretation of numerical data, the recording and evaluation of population figures, or of imports and exports. On the other hand, history is defined as that branch of knowledge which records and analyses past events such as political changes, revolutions, or wars between nations.

But if one digs deeper, one begins to realize that such a relationship exists. For example, population pressure as recorded in official statistics, and aggravated by drought or disease, may trigger a famine, and famine may result in migration as in nineteenth century Ireland, or revolution as in twentieth century Russia. Thus, in a sense, history subsumes statistics. A careful scrutiny of bills and accounts preserved from the past has often led to a better understanding of important historical events. More recently, the lack of birth control measures (including records of births) in third world countries has contributed to their continuing poverty and political instability. Thus statistical data may be more closely interwoven with history than one might suspect.

In this article, I shall try to discuss

(a) the relevance of accurate numerical records to history,

(b) the optimal presentation of statistical data for historical purposes,

and

(c) the role of modelling in testing historical hypotheses.

2. Accurate numerical records

Australia is very fortunate to have accurate records of statistical data, such as the country’s population, as well as its production and business activities, compiled each year by the Commonwealth Statistician. The Australian Bureau of Statistics (ABS) is recognized internationally for its independence from the government, and its imaginative annual *Year Book of the Commonwealth of Australia*. In fact, when the UK accepted that its Central Statistical Office needed to be reorganized in the 1990s, it was to the ABS that it turned for help.

Let me give an example where accurate population records are relevant. A frequent topic of discussion among Australians is the composition of their present population. The 2001 *Year Book* (reference 1) lists the following countries of birth for Australians in 1999; see table 1.

Migrants from Europe and New Zealand appear to be over four times more numerous than those from Asia, despite the occasional misguided report that ‘one is surrounded by a sea of Asian faces’. This may well be true in those parts of a town where Asian restaurants abound, but it is an incorrect generalization about the Australian population, as the detailed figures of the *Year Book* indicate.

Table 1. Countries of birth for Australians (1999).

Country	Number (thousands)	
UK and Ireland	1 227.2	
New Zealand	361.6	
Italy	244.6	
Former Yugoslavia	208.4	
Greece	140.2	
Germany	123.5	
Netherlands	92.7	
Total		2 398.2
Vietnam	175.2	
China	156.8	
Hong Kong	62.0	
Philippines	116.9	
Total		510.9
Other		1 572.9
Australia		14 484.8
Grand total		18 966.8

Table 2. Males per 100 females, colonies and states 1861–1981.

Year	NSW	Vic	Qld	SA	Tas	WA	NT	ACT	Aust
1861	130	155	152	105	123	171			138
1871	121	121	149	106	114[a]	148[a]	593		121[a]
1881	121	110	142	112	112	135	3218		117
1891	118	109	132	106	112	149	1349		116
1901	110	101	126	102	108	158	558		110
1911	109	99	119	103	104	134	475	137	108
1921	104	97	112	101	102	114	270	156	103
1933	103	98	110	100	102	114	229	116	103
1947	100	97	105	98	101	106	211	116	100
1954	101	101	105	103	104	107	166	115	102
1961	101	101	104	102	103	104	149	110	102
1966	101	101	103	101	102	104	135	109	101
1971	101	100	102	100	101	106	129	104	101
1976	99	99	101	99	100	103	117	103	100
1981	99	98	101	98	99	102	113	99	99

[a] Figures for Tas and WA are from 1870 censuses.

In our present state of historical development, with government policies about migration still being debated, it is important for all Australians to know the exact facts of the situation. Accurate statistics are of importance in history, because they help to determine national policy.

3. Optimal presentation of historical statistics

Table 1 may not illustrate its message optimally. It is often more appropriate to present numerical data graphically, as I shall do in the following four examples (reproduced from reference 2).

Example 1. (*Masculinity of the Australian population, 1861–1981.*) Table 2 gives the number of males per 100 females from 1861 to 1981. The graphs in figure 1 provide information for 1861 and 1981 in a much clearer form: in 1861 there were on average 138 males per 100 females in Australia, while in 1981, there were on average slightly more females than males.

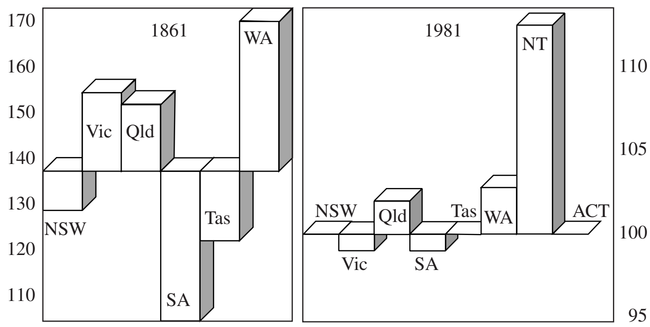


Figure 1. Masculinity of the Australian population.

Example 2. (*Farm output as a percentage of gross domestic product.*) Table 3 lists the amounts of farm and non-farm products in millions of dollars from 1950 to 1983. The graphs in figure 2 provide the percentage of gross domestic product for farm and non-farm products for the 4 years 1953, 1963, 1973 and 1983 in a much more striking manner.

Table 3. Gross domestic product, farm and non-farm, current prices, Australia 1950–1983.

Year 30 June	Farm $m	Non-farm $m	Total $m	Year 30 June	Farm $m	Non-farm $m	Total $m	Year 30 June	Farm $m	Non-farm $m	Total $m
1950	1116	3 989	5 105	1962	1611	13 357	14 968	1974	4199	47 167	51 366
1951	1831	4 950	6 781	1963	1792	14 390	16 182	1975	3724	58 049	61 773
1952	1225	6 052	7 277	1964	2192	15 779	17 971	1976	3765	69 061	72 826
1953	1542	6 714	8 256	1965	2170	17 599	19 769	1977	4191	78 974	83 165
1954	1530	7 495	9 025	1966	1891	18 879	20 770	1978	3933	86 407	90 340
1955	1423	8 190	9 613	1967	2326	20 542	22 868	1979	6489	95 674	102 163
1956	1488	8 928	10 416	1968	1769	22 642	24 411	1980	7448	107 307	114 755
1957	1664	9 681	11 345	1969	2355	25 207	27 562	1981	7091	123 722	130 813
1958	1326	10 212	11 598	1970	2137	28 408	30 545	1982	7257	140 685	147 942
1959	1556	10 907	12 463	1971	1943	31 794	33 737	1983	5611	155 195	160 806
1960	1616	12 130	13 746	1972	2217	35 463	37 680				
1961	1662	12 995	14 617	1973	3085	39 822	42 907				

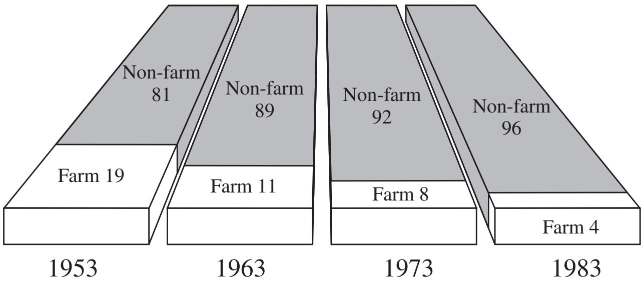


Figure 2. Farm output as a percentage of the gross domestic product for selected years.

Example 3. (*Medical practitioners per 100 000 population, 1960–1984.*) The graph (figure 3) shows clearly how the number of medical doctors increased steadily from 1960 to 1984, their number roughly doubling in 24 years.

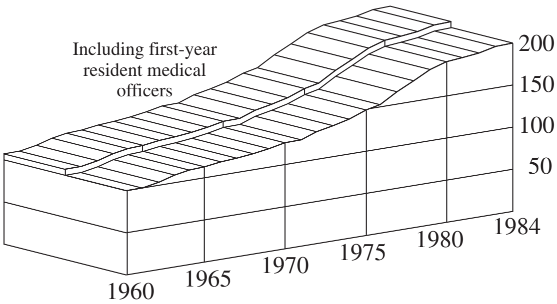


Figure 3. Medical practitioners, Australia 1960–1984. (Figures are per 100 000 population.)

Example 4. (*AIDS cases and deaths in Australia to January 1987.*) The rather complicated graph in figure 4 shows the number of AIDS cases and deaths (state by state) which have occurred in Australia up to 1987. Clearly, New South Wales (NSW) is the state which has been most affected by the disease.

A general conclusion is that, while tables may provide more data, graphs will be more effective in communicating information to the reader.

4. Modelling and historical hypotheses

Instead of presenting a theoretical discussion on modelling in the testing of historical hypotheses, let me work through a practical example on the settlement of Polynesia, due to my recently deceased colleague Richard Tweedie (reference 3).

While most Polynesian islands are inhabited by seaborne migrants assumed to have sailed in canoes from South East Asia, or inhabited islands further east, some uninhabited islands have been found which have yielded human skeletons. The question then arises as to why some island populations have survived and others not. To resolve this, it was assumed that canoes carrying 3, 5, 7 and 10 couples had set off from inhabited islands to search for new land. The average age of the men in the canoe was taken to be 22 and of women 19; reasonable birth and death rates and a set of mating rules were used in a computer micro-simulation to calculate the

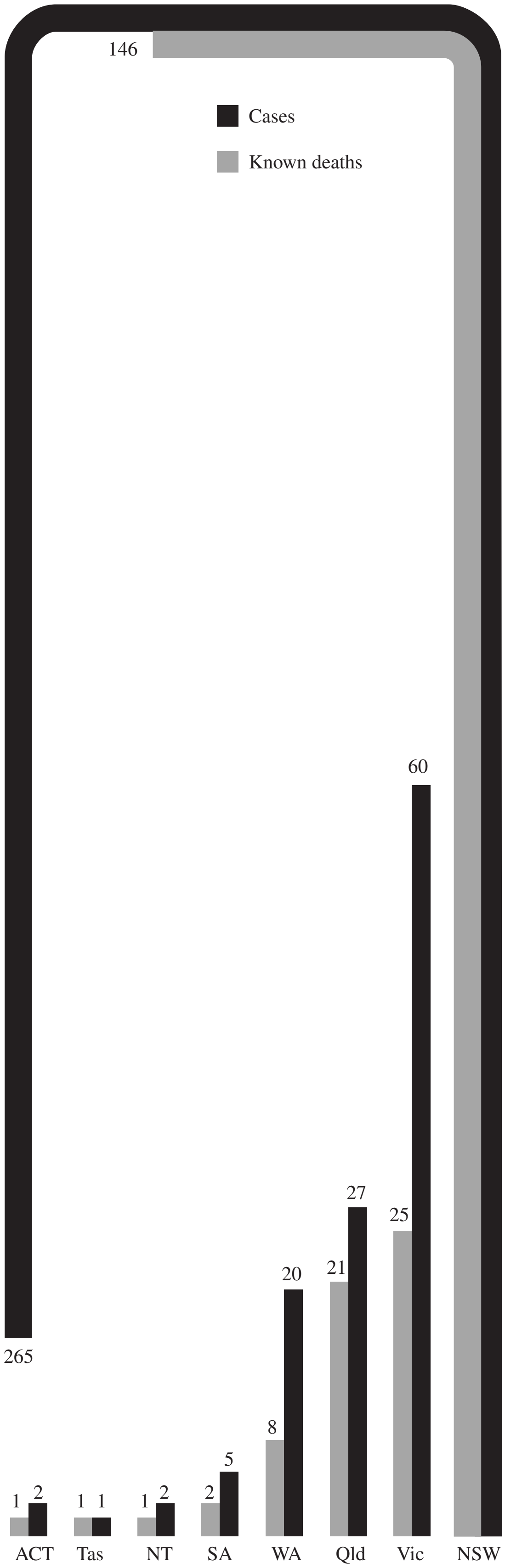


Figure 4. AIDS cases and known deaths in Australia to January 1987.

probability of survival of an island community. Survival was taken to occur if the total number of people who had ever lived on an island exceeded 500, or if, after 500 years, the population was over 30. Table 4 gives the probabilities of extinction calculated on the basis of this model.

Table 4. Probabilities of extinction of Polynesian islanders.

Initial size of group	With incest taboo	Without incest taboo
3 couples	0.65	0.70
5 couples	0.30	0.15
7 couples	0.00	0.15
10 couples	0.00	0.05

Note that the incest taboo does not appear to affect the viability of the population unduly; the larger the initial number of couples, the better the chance of survival. In Tweedie's words, 'the detailed model described here incorporates far more realistic conditions than could be considered in an analytic model, and throws considerable light on what could have happened in a "random" Polynesian population.' This statistical model bears out the hypothesis that the colonization of Polynesia was carried out by migrants braving the South Seas in their canoes.

5. Concluding remarks

Statistical data and their analysis have much to contribute to history. In this brief article, I have tried to illustrate these contributions by discussing the importance of accurate statistical records, the desirability of presenting historical statistics not only in tables, but more clearly in graphical form, and finally the role of modelling in testing historical hypotheses.

History has always subsumed some statistics. But with the advent of computers, the use of statistical records has become more widespread, and seems likely to play an increasingly important role in historical research.

Acknowledgement

We are grateful to Lansdowne Publishing Pty Ltd for permission to reproduce figures 1–4 and tables 2 and 3 from reference 2.

References

1. Australian Bureau of Statistics, *Year Book Australia 2001* (Australian Bureau of Statistics, Canberra, 2001).
2. W. Vamplew (ed.), *Australians — Historical Statistics* (Fairfax, Syme and Weldon, Broadway, NSW, 1967).
3. R. L. Tweedie, Computerised anthropology — finding and settling Polynesian islands, *Math. Spectrum* **11** (1978/79), pp. 75–81.

Joe Gani *is a retired statistician, one of whose hobbies is history. He is one of the three Associate Editors of the book* Statisticians of the Centuries *edited by C. C. Heyde and E. Seneta, and recently published by Springer.*

Squaring Up to Factorials

P. GLAISTER

The base of natural logarithms, e, can be defined in many ways, and has numerous interesting properties. One of the most appealing to me is the fact that e is the sum of the infinite series

$$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots, \tag{1}$$

whereas the corresponding alternating series

$$\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots$$

has the reciprocal of e, that is, $1/e$, as its sum. Thus,

$$\frac{1}{\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots}=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots,$$

which clearly raises the question as to whether there are other series $a_1+a_2+a_3+\cdots$ having this property, i.e.

$$\frac{1}{a_1+a_2+a_3+\cdots}=a_1-a_2+a_3-\cdots.$$

I have managed to generate some solutions using geometric series of the form $a+ar+ar^2+\cdots$, with first term a and common ratio r, and leave readers to verify that

$$\frac{1}{a+ar+ar^2+\cdots}=a-ar+ar^2-\cdots$$

provided $a^2+r^2=1$. Can you find other examples?

This is a digression, however, since the main purpose of this article is to consider the generalisation of the series in (1) to

$$P=\frac{1}{(0!)^2}+\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\frac{1}{(3!)^2}+\cdots \tag{2}$$

and its alternating counterpart

$$A=\frac{1}{(0!)^2}-\frac{1}{(1!)^2}+\frac{1}{(2!)^2}-\frac{1}{(3!)^2}+\cdots. \tag{3}$$

So that readers are not unduly disappointed, I should point out now that I am not about to show that $P = 1/A$. I shall evaluate P and A, however, using a well-known result in integral calculus. The series P and A appear in more advanced work on differential equations and I shall give brief details for interested readers.

The factorial $m!$ appears in many mathematical expressions, but the square of a factorial $(m!)^2$ is much less common. The one that immediately springs to mind occurs in a particular definite integral obtained using a simple recurrence relation. For the integral

$$I_n = \int_0^{\pi/2} \sin^n \theta \, \mathrm{d}\theta , \qquad n = 0, 1, 2, \dots ,$$

integration by parts twice yields the recurrence formula

$$I_n = \frac{n-1}{n} I_{n-2} , \qquad n = 2, 3, \dots ,$$

which we leave as an exercise. From this it is a straightforward matter to obtain

$$\begin{aligned} I_{2m} &= \frac{2m-1}{2m} I_{2m-2} = \frac{2m-1}{2m}\frac{2m-3}{2m-2} I_{2m-4} \\ &= \cdots = \frac{2m-1}{2m}\frac{2m-3}{2m-2}\cdots\frac{3}{4}\frac{1}{2} I_0 \\ &= \frac{2m(2m-1)(2m-2)\cdots 3\cdot 2\cdot 1}{\big(2m(2m-2)(2m-4)\cdots 4\cdot 2\big)^2} I_0 , \\ & \qquad\qquad\qquad\qquad m = 1, 2, \dots , \end{aligned} \tag{4}$$

by multiplying both the numerator and denominator by $2m(2m-2)\cdots 4\cdot 2$ to 'fill-in' the gaps in the numerator. The numerator on the right-hand side of (4) is clearly $(2m)!$, the denominator $(2^m m!)^2 = 2^{2m}(m!)^2$ and $I_0 = \int_0^{\pi/2} \sin^0\theta \, \mathrm{d}\theta = \frac{1}{2}\pi$, so that

$$I_{2m} = \int_0^{\pi/2} \sin^{2m}\theta \, \mathrm{d}\theta = \frac{(2m)!}{2^{2m}(m!)^2}\,\frac{1}{2}\pi , \quad m = 1, 2, \dots , \tag{5}$$

which is also valid for $m = 0$ by direct calculation, and so there it is, $(m!)^2$.

In order to use (5) to evaluate P and A we first rewrite it as

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{(2\sin\theta)^{2m}}{(2m)!} \mathrm{d}\theta = \frac{1}{(m!)^2} , \qquad m = 0, 1, 2, \dots . \tag{6}$$

From (6) we can form both series in (2) and (3), as follows. First we have

$$\begin{aligned} P &= \frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \cdots \\ &= \frac{2}{\pi}\int_0^{\pi/2} \Big(1 + \frac{(2\sin\theta)^2}{2!} + \frac{(2\sin\theta)^4}{4!} + \cdots\Big) \mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cosh(2\sin\theta)\, \mathrm{d}\theta \end{aligned} \tag{7}$$

using the series expansion $\cosh z = 1 + z^2/2! + z^4/4! + \cdots$, giving the sum P in terms of a definite integral. Secondly, we have

$$\begin{aligned} A &= \frac{1}{(0!)^2} - \frac{1}{(1!)^2} + \frac{1}{(2!)^2} - \cdots \\ &= \frac{2}{\pi}\int_0^{\pi/2} \Big(1 - \frac{(2\sin\theta)^2}{2!} + \frac{(2\sin\theta)^4}{4!} - \cdots\Big) \mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos(2\sin\theta)\, \mathrm{d}\theta \end{aligned} \tag{8}$$

using the series expansion $\cos z = 1 - z^2/2! + z^4/4! - \cdots$, giving the sum A also in terms of a definite integral.

Before we evaluate the sums P and A, we note that the expressions in (7) and (8) can be deduced from some advanced work on differential equations. The Bessel function of the first kind of order 0, denoted by $J_0(x)$, is a solution of the differential equation $xy'' + y' + xy = 0$, and a series expansion for $J_0(x)$ can be obtained from this differential equation as

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2} . \tag{9}$$

As a separate exercise, it can be shown that an integral representation of $J_0(x)$ is

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, \mathrm{d}\theta , \tag{10}$$

and by substituting $x = 2$ in (9) and (10) it is a simple matter to deduce the result in (8). The result in (7) can be obtained similarly using the modified Bessel function of the first kind of order 0, which is a solution of the differential equation $xy'' + y' - xy = 0$.

To evaluate the sums P and A, we use the trapezium rule for approximating integrals in the form

$$\begin{aligned} &\int_0^{\pi/2} f(\theta)\, \mathrm{d}\theta \\ &\approx \frac{\pi}{4N}\Big[f(0) + f\Big(\frac{\pi}{2}\Big) \\ &\quad + 2\Big(f\Big(1\frac{\pi}{2N}\Big) + f\Big(2\frac{\pi}{2N}\Big) + \cdots + f\Big((N-1)\frac{\pi}{2N}\Big)\Big)\Big], \end{aligned} \tag{11}$$

where N is the number of intervals. A bound on the error in the trapezium rule can be determined as

$$|\text{error}| \le \frac{\pi^3}{96N^2} \max_{0\le\theta\le\pi/2} \big|f''(\theta)\big|$$

(see reference 1 for example), so we can determine P and A as accurately as we please. In the case $f(\theta) = \cosh(2\sin\theta)$, the graph of $f''(\theta) = 4\cosh(2\sin\theta)\cos^2\theta - 2\sinh(2\sin\theta)\sin\theta$ shows that $|f''(\theta)|$ is greatest at $\theta = \frac{1}{2}\pi$, giving $|\text{error}| \le (\pi^3/48N^2)\sinh 2$. Taking $N = 70$

makes this error less than 0.0005, and the required approximation given by (11) yields

$$P = \frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \cdots = 2.280\,,$$

accurate to three decimal places, which we leave readers to check. For the other sum, with $f(\theta) = \cos(2\sin\theta)$, the graph of

$$f''(\theta) = -4\cos(2\sin\theta)\cos^2\theta + 2\sin(2\sin\theta)\sin\theta$$

shows that $|f''(\theta)|$ is greatest at $\theta = 0$, giving $|\text{error}| \le \pi^3/24N^2$ which is less than 0.0005 for $N = 52$, and hence the required approximation given by (11) which is also accurate to three decimal places is

$$A = \frac{1}{(0!)^2} - \frac{1}{(1!)^2} + \frac{1}{(2!)^2} - \cdots = 0.224\,.$$

Now, what about

$$\frac{1}{(0!)^3} + \frac{1}{(1!)^3} + \frac{1}{(2!)^3} + \cdots \quad \text{and} \quad \frac{1}{(0!)^3} - \frac{1}{(1!)^3} + \frac{1}{(2!)^3} - \cdots ?$$

Reference

1. Robert A. Adams, *Calculus: a Complete Course* (Addison-Wesley Longman, Don Mills, Ontario, 1999).

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, perturbation methods as well as mathematics and science education.

A Table of the Partition Function

ABDULKADIR HASSEN, THOMAS J. OSLER and TIRUPATHI R. CHANDRUPATLA

1. Introduction

A *partition* of a positive number n is a representation of this number as a sum of natural numbers, called parts or summands. The order of the summands is irrelevant. For example, $4+2+1$ is a partition of the number 7. Since order is irrelevant, $4+2+1$ is the same partition as $2+4+1$. The number of unrestricted partitions of the positive integer n is denoted by $p(n)$. For example, the partitions of 5 are $5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$. Thus, $p(5) = 7$. The reader can easily verify that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(6) = 11$ and $p(7) = 15$.

While it is simple to determine $p(n)$ for small numbers by actually counting all the partitions, this becomes difficult as the numbers grow. As George Andrews put it: 'Actual enumeration of the 3 972 999 029 388 partitions of 200 would certainly take more than a lifetime' (reference 1, p. 150).

The theory of partitions is a very active area of modern mathematical research. The authors searched ten online electronic journals available in JSTOR (Journal Storage on the world-wide web) and found 2213 papers which discussed partitions! However, most textbooks on number theory do not discuss partitions. Two of the exceptions are the excellent book of Andrews (reference 1) and the bible of number theory by Hardy and Wright (reference 2).

In reference 3, the first two authors used the following remarkable recursive algorithm to show how a table of $p(n)$ can be created with a BASIC program for large values of n. A heuristic explanation behind the proof of the formula

$$\begin{aligned} p(n) = {} & p(n-1) + p(n-2) - p(n-5) \\ & - p(n-7) + p(n-12) + p(n-15) \\ & - p(n-22) - p(n-26) + \cdots \end{aligned} \tag{1}$$

was also given. (The progression of numbers $1, 2, 5, 7, 12, 15, 22, 26, \ldots$ is related to the pentagonal numbers. Space does not permit us to give details here, but they can be found in reference 3.)

Most relations in the theory of partitions are not directly accessible. Euler initiated a beautiful theory of these in 1748 (reference 4) using generating functions. The interplay between the discrete and the continuous branches of mathematics contains a wonderland of amazing relations. For the reader unfamiliar with Euler's pentagonal number theorem or Jacobi's triple product, reference 1 has many delightful gems to sample.

It is the purpose of this article to show how $p(n)$ and certain related functions can be calculated. This is a story about building a table of numbers in the spirit of Pascal's triangle. We will use a recursion relation that is easy to understand and easy to implement, by hand or by computer. Dickson, in his extensive history of number theory (reference 5), seems to indicate that this method was known to Euler (reference 4).

In Section 2, we will look at some restricted partitions and some relations between them. In Section 3, we will show how to create a table of partitions by hand. Finally, we will give a BASIC program that generates a table of partitions.

2. Restricted partitions and a recursive relation

The partition function $p(n)$ is sometimes called the *unrestricted partition function* of n. Besides the partition function $p(n)$, we also consider partitions formed from numbers from some restricted set. For example, let $q(n)$ denote the number of partitions of the number n where all parts are distinct. The partitions of the number 7 into distinct parts are 7, $6+1$, $5+2$, $4+3$ and $4+2+1$. Thus $q(7) = 5$. Other examples of restricted partitions are the following:

$p(m, n) =$ the number of partitions of n in which no part is larger than m,

$q(m, n) =$ the number of partitions of n in which at most m parts appear,

$d(m, n) =$ the number of partitions of n into m distinct parts,

$D(m, n) =$ the number of partitions of n into distinct parts in which no part is greater than m,

$h(m, n) =$ the number of partitions of n whose least part is m,

$o(n) =$ the number of partitions of n in which all parts are odd,

$e(n) =$ the number of partitions of n in which all parts are even,

$p(S, n) =$ the number of partitions of n using summands in a set S.

Note that the above notation is not standard. For example, many authors use the notation $p_m(n)$ for our function $p(m, n)$. We challenge the reader to find the values of each of these functions for $n = 1, 2, 3, \dots, 6$ for different values of m. For relationships between these partition functions and similar ones, we advise the interested reader to read references 1 and 2.

The relation (1) is very good for calculating $p(n)$, but not easy to derive. We will use the restricted partition function $p(m, n)$ to compute $p(n)$. Let us consider an example. The partitions of the number 7 using parts not exceeding 2 are $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$ and $1+1+1+1+1+1+1$. Thus $p(2, 7) = 4$. The partition function $p(m, n)$ has a simple relation given by the following theorem.

Theorem 1. *We have*

$$p(m, n) = p(m-1, n) + p(m, n-m)\,. \qquad (2)$$

Proof. Imagine counting the partitions of n whose parts do not exceed m, namely $p(m, n)$. First we count all the partitions of n whose parts do not exceed $m-1$. This is $p(m-1, n)$, the first term on the right-hand side of (2). It remains to count all the partitions of n using the number m at least once. All of these partitions are of the form

$$m + x = n\,, \qquad (3)$$

where x is a sum of parts that do not exceed m. Looking only at x, we see that the number of partitions of the form (3) is $p(m, x)$. But (3) tells us that $x = n - m$, so $p(m, x) = p(m, n-m)$, which is the second term on the right-hand side of (2). This completes our proof.

To see how we obtain $p(n)$ from (2), we need only notice that

$$p(n) = p(n, n) = p(n+1, n) = p(n+2, n) = \cdots\,.$$

For example, the number of partitions of 5 using parts not exceeding 7, $p(7, 5)$, must be the same as the partitions of 5 using parts not exceeding 5, which is $p(5, 5) = p(5)$.

3. Creating a table of the partition function

We now show how to use (2) for hand calculation of the partition function. Here are the steps:

Step 1. Create a table, labelling the rows $n = 0, 1, 2, 3, \dots$ and the columns $m = 1, 2, 3, \dots$. (See table 1. Note that we use the *nonstandard* notation in which m counts the columns and n counts the rows.)

Step 2. Fill the first row with 1s. (We need to define $p(m, 0) = 1$.) You can also complete a few more rows easily using the definition of $p(m, n)$.

Step 3. Next we fill in the remaining rows one at a time using (2).

To see how this works, suppose we have completed all the rows down to $n = 9$ and are now working on row $n = 10$. We fill in the cells in this row from left to right. Relation (2) tells us that the number needed in the cell (m, n) is the number in the cell to the left plus the number directly above on the diagonal of bold numbers. For example, the number in the cell where $n = 10$ and $m = 6$, $p(6, 10) = 35$, is the sum of the number to its left, $p(6-1, 10) = 30$, and the number above it on the diagonal of bold numbers, $p(6, 10-6) = 5$.

Table 1. Calculation of $p(m, n)$ using (2).

	m										
n	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	**1**	1
1	1	1	1	1	1	1	1	1	**1**	1	1
2	1	2	2	2	2	2	2	**2**	2	2	2
3	1	2	3	3	3	3	**3**	3	3	3	3
4	1	3	4	5	5	**5**	5	5	5	5	5
5	1	3	5	6	**7**	7	7	7	7	7	7
6	1	4	7	**9**	10	11	11	11	11	11	11
7	1	4	**8**	11	13	14	15	15	15	15	15
8	1	**5**	10	15	18	20	21	22	22	22	22
9	**1**	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56

Notice that, on each row, from $n = m$ all the values of $p(m, n)$ are constant and equal $p(n)$. Thus the values in the last column, where $n = 11$, give the correct values of $p(n)$ for $n = 1, 2, \ldots, 11$. Having shown how to calculate a partition function by hand, we give a simple BASIC program to do the same thing in the next section.

4. A computer program to count partitions

The following computer program calculates the partition function using (2). Since we have given such a table in reference 3, we will not do so here. The reader familiar with BASIC will have no difficulty seeing how the program works.

Program: Counting partitions the easy way

```
100  CLS
110  DIM P(50,50)
120  MAXN = 20
200  FOR M = 0 TO MAXN
210    P(M,0) = 1
220    P(M,1) = 1
230  NEXT M
300  FOR N = 2 TO MAXN
310    FOR M = 1 TO N
320      P(M,N) = P(M-1,N) + P(M,N-M)
330    NEXT M
340    FOR M = N + 1 TO MAXN
350      P(M,N) = P(N,N)
360    NEXT M
370  PRINT N, P(N,N)
380  NEXT N
```

In line 120 the variable `MAXN` is the number of rows and columns in our table. In lines 200 to 230 we fill the first two rows of the table with the number 1 as we did in the hand-calculation in the previous section. Each time the loop from lines 300 to 380 is executed, it calculates one row of the table and prints out the partition function calculated on that row in line 370. In lines 310 to 330 we calculate the entries along row `N` for columns `M = 1, 2, 3, ..., N`, using (2) in line 320. In lines 340 to 360 we fill in the rest of the row with the constant value `P(N,N)`.

5. Concluding remarks

The method shown above works for other restricted partition functions as well. The reader might want to study the partition function $D(m, n)$, which counts the number of partitions of the number n into distinct summands that do not exceed the number m. Corresponding to (2), we have the relation

$$D(m, n) = D(m-1, n) + D(m-1, n-m). \qquad (4)$$

A table for hand calculation of $D(m, n)$, similar to our table for $p(m, n)$, can then be constructed using (4).

Another example which the reader might wish to examine is the function $h(m, n)$, which counts the number of partitions of n whose least part is m. For example, the partitions of 5 whose least part is 1 are $1+1+1+1+1$, $1+1+1+2$, $1+2+2$, $1+1+3$ and $1+4$. Thus $h(1, 5) = 5$. This is a particularly interesting partition with many nice properties. It satisfies the relation

$$h(m, n) = h(m-1, n-1) - h(m-1, n-m).$$

Finally, one has the deep results of the so-called Rogers–Ramanujan identities and the Ramanujan congruences. Andrews' book (reference 1) has an excellent treatment of these topics that should be accessible to most readers.

References

1. G. E. Andrews, *Number Theory* (Dover, New York, 1971).
2. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (5th edn, Clarendon Press, Oxford, 1979).
3. A. Hassen and T. J. Osler, Playing with partitions on the computer, *Math. Comput. Education* **35** (2001), pp. 5–17.
4. L. Euler, *Introduction to Analysis of the Infinite*, Vol. 1 (translation by J. D. Blanton, Springer, New York, 1988).
5. L. E. Dickson, *History of the Theory of Numbers*, Vol. II (Chelsea, New York, 1952).

Abdul Hassen *is Associate Professor of Mathematics at Rowan University. He obtained his BSc in 1984 and MSc in 1987 from Addis Ababa University in Ethiopia. He obtained his PhD in 1996 from Temple University, Philadelphia, under the directions of Professor Marvin Knopp. His research area is analytic number theory.*

Tom Osler *is Professor of Mathematics at Rowan University. He received his PhD from the Courant Institute at New York University in 1970 and is the author of 33 mathematical papers. In addition to teaching university mathematics for the past 40 years, Tom has a passion for long distance running. He has been competing for the past 47 consecutive years. Included in his over 1600 races are wins in three national championships in the late sixties at distances from 25 kilometres to 50 miles. He is the author of two running books.*

Tirupathi R. Chandrupatla *is Professor and Chair of Mechanical Engineering at Rowan University. His areas of interest include finite-element analysis, design, optimization, and manufacturing engineering. He is author of the books* Introduction to Finite Elements in Engineering *and* Optimization Concepts and Applications in Engineering, *both published by Prentice Hall.*

The Railway Station Problem

D. M. BURLEY and R. A. SMITH

Introduction

The second author was whiling away an hour on a railway station and was admiring the metalwork in the roof of the station. Being an engineer he was pondering how the roof was designed and constructed. A bit of trial and error and a good template would probably be quite sufficient for the manufacture. However, thoughts moved to a more mathematical formulation of the problem. Given the dimensions of the box and taking the curve to be an ellipse (see figure 1), where is the centre of the circle that touches the sides of the box and the ellipse?

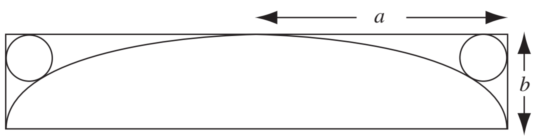


Figure 1.

Preliminary thoughts

A first thought is: 'has the problem got a sensible solution?' The circle must touch the sides of the box, so, with the axes shown in figure 2, the centre of the circle must have coordinates (R, R) and lie on the 45 degree line from the top corner. For small R the circle does not intersect the ellipse, but as R increases it will eventually intersect the ellipse in two points. At some value of R a circle with the correct radius will intersect the ellipse at a single point, that is, they will touch.

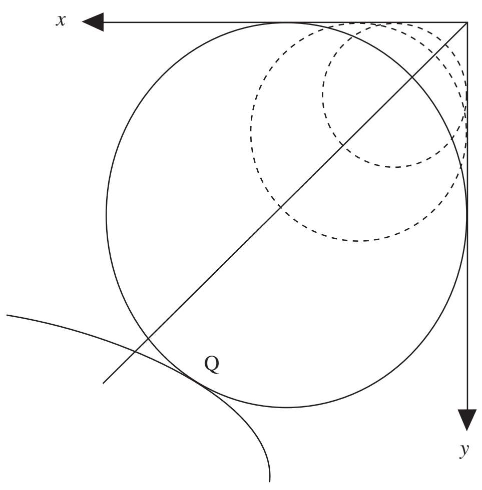


Figure 2.

For the simple case $a = b$, the ellipse becomes a circle and a little algebra shows that the two circles touch at the point

$$b\Big(1 - \frac{1}{\sqrt{2}},\ 1 - \frac{1}{\sqrt{2}}\Big)$$

and the radius of the circle is

$$R = b\Big(\frac{\sqrt{2}-1}{\sqrt{2}+1}\Big).$$

A second simple case is obtained by fixing b and letting a get very large. The ellipse is now very long and thin and in the limit as $a \to \infty$ the point Q where the circle and ellipse touch is $(\frac{1}{2}b, b)$ and the radius is $\frac{1}{2}b$.

There is an exact solution for the two limiting cases but is there a general solution for any a and b? If this is not possible, can the information already obtained be used to get an approximate solution to the problem posed?

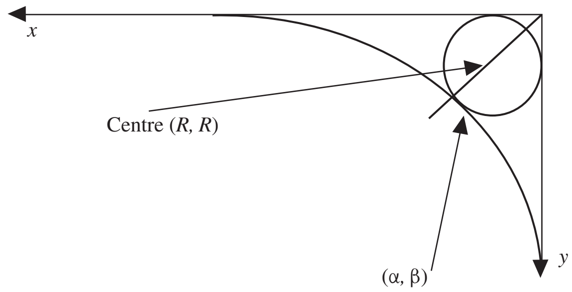


Figure 3.

Setting up the problem mathematically

Taking appropriate axes, the ellipse has equation

$$\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1 \qquad (1)$$

and the circle with radius R and centre (R, R) has equation

$$(x-R)^2 + (y-R)^2 = R^2. \qquad (2)$$

The mathematical question is to find the value of R for which the curves touch and to find the touching point (α, β) (as in figure 3). It sounds like a straightforward problem but, although many have been asked, no one has come up with a simple explicit formula for α, β and R.[1]

[1] However, see the note at the end of this article.

Method of solution

Method 1. In the section 'Preliminary thoughts' it was suggested that we find the points of intersection of the curves (1) and (2) and then use the condition that, to touch, these points must coincide. The method is perhaps the most demanding but it would be interesting to attempt it to find the appropriate conditions.

Method 2. A more geometrical approach is to compute the tangents of the two curves (1) and (2) and note that they must be identical if the two curves touch at the point (α, β). Any book on conics (see e.g. reference 1) gives the equation of the tangents as

$$\frac{(x-a)(\alpha-a)}{a^2} + \frac{(y-b)(\beta-b)}{b^2} = 1\,, \tag{3}$$

$$(x-R)(\alpha-R) + (y-R)(\beta-R) = R^2\,. \tag{4}$$

To simplify the interpretation, substitute α for x and β for y in (1) and (2) and subtract the resulting equations from, respectively, (3) and (4). These equations can then be arranged in standard form $y-\beta = m(x-\alpha)$ (that is, lines passing through (α, β) and having gradient m), as

$$y-\beta = \frac{b^2\alpha - a}{a^2\beta - b}(x-\alpha)\,,$$

$$y-\beta = \frac{\alpha - R}{\beta - R}(x-\alpha)\,.$$

Since the tangents are identical, the gradients must be equal, and thus

$$\frac{a^2(\alpha-R)}{\alpha-a} = \frac{b^2(\beta-R)}{\beta-b}\,. \tag{5}$$

The point (α, β) must lie on the two curves (1) and (2), so

$$\frac{(\alpha-a)^2}{a^2} + \frac{(\beta-b)^2}{b^2} = 1 \quad \text{and} \quad (\alpha-R)^2 + (\beta-R)^2 = R^2\,. \tag{6}$$

Solving the equations in (5) and (6) gives a precise mathematical formulation of the problem. Of course method 1 should give precisely the same or an equivalent set of equations.

Method 3. Another approach is to note that the shortest distance from a point to a curve is along a line that is perpendicular to the curve, that is, perpendicular to the tangent to the curve at the nearest point. The result is well-known for smooth curves; it is intuitively obvious but not so easy to prove.

Applying the result to the current problem, it may be seen that the shortest distance from the centre of the circle to the ellipse is required. The circle has centre (R, R) and the distance D to the ellipse is given by

$$D^2 = (R-x)^2 + (R-y)^2\,,$$

where the point (x, y) must lie on the ellipse. Mathematically the problem is to find, for given R,

$$\min_{x,y}\left[(R-x)^2 + (R-y)^2\right]$$

subject to

$$\frac{(x-a)^2}{a^2} + \frac{(y-b)^2}{b^2} = 1\,.$$

The problem is now a complicated two variable problem. Once this solution has been obtained, R must be obtained by checking that the resulting (x, y) lies on the circle.

Method 3′. To simplify, use the standard parameterisation of the ellipse

$$x-a = a\sin\theta \quad \text{and} \quad y-b = b\cos\theta\,,$$

and the minimisation is now the single variable problem of finding the minimum of the function

$$f(\theta) = (R-a-a\sin\theta)^2 + (R-b-b\cos\theta)^2\,.$$

Once the solution, for θ and hence (x, y), has been obtained, the correct value of R must be calculated by substituting into (2) to ensure that the point also lies on the circle.

This approach is not the most elegant but it should produce (5) and (6) after some manipulation — it is an interesting challenge to prove this to be true.

Table 1.

a	1	1.5	2	2.5	3
α	0.2929	0.3381	0.367	0.3871	0.4019
β	0.2929	0.3675	0.4226	0.4655	0.5
R	0.1716	0.2071	0.2327	0.2523	0.2679
a	4	5	6	8	10
α	0.4223	0.4356	0.4451	0.4575	0.4653
β	0.5528	0.5918	0.622	0.6667	0.6985
R	0.2918	0.3094	0.323	0.3431	0.3576
a	15	20	25	30	100
α	0.4763	0.482	0.4854	0.4878	0.4963
β	0.75	0.7818	0.8039	0.8204	0.9005
R	0.381	0.3957	0.4059	0.4136	0.4514

Numerical solution

None of the formulations has yielded a closed form solution but, of course, a numerical solution is always possible. Some are quite difficult to implement since the solution gives the point of contact as two coincident points. This means that the solution is close to giving solutions in complex numbers, and many numerical packages struggle. The program MAPLE® was found to give a solution quickly, provided sensible bounds for the unknowns could be given. Some experience of mathematical packages is required to progress quickly with this approach.

Taking $b = 1$, and there is no loss of generality in doing this, the solution for various values of a is given in table 1.

A plot of R against a is given in figure 4. In table 1, the values of $a = 3,\ 8,\ 15$ seem to give simple values of β,

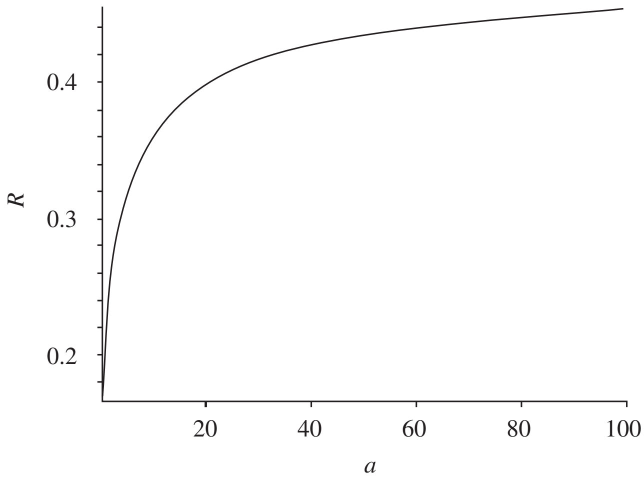


Figure 4.

namely $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$. Does this mean that explicit solutions can be computed for these cases?

When exact solutions cannot be obtained, can a simple approximate solution be computed? Often such approximations are quite sufficient for practical purposes. So it is left as an exercise to find a reasonably simple approximate formula that will fit the data to 5% accuracy over the range $1 < a < 30$. For instance, the formula, with $b = 1$,

$$R = \frac{\sqrt{2} - 1 + k(1-a)^n}{\sqrt{2} + 1 + 2k(1-a)^n}$$

fits the known solutions at $a = 1$ and $a \to \infty$ for any k and n. Can appropriate values of k and n be found that satisfy the accuracy criteria? A spreadsheet package is a very suitable method for obtaining these values.

Note

After this article was written, the referee discovered that there are simple explicit formulae for α, β and R. *If any student can find these formulae, there will be a small prize given for the first solution submitted.* The referee will publish this solution in a forthcoming issue.

Reference

1. C. J. Tranter and C. G. Lambe, *Advanced Level Mathematics* (Hodder and Stoughton, London, 1973).

Roderick Smith *is currently Head of Mechanical Engineering at Imperial College, London, having previously held appointments at Cambridge and Sheffield. His main research area is fatigue of metals, but he has published on topics as diverse as crowd control and railway engineering.*

David Burley *graduated in mathematics from King's College, London. He spent most of his working life at the University of Sheffield, latterly as Head of the Department of Applied Mathematics. He has now retired. His interests include applying mathematics to engineering problems, with a particular interest in the flow of molten glass in industrial processes.*

An interesting mistake

I was intrigued to see this solution of a problem on limits. Surprisingly, the student arrived at the right answer.

$$\begin{aligned}
\lim_{x\to\infty} \frac{(x+1)^{10} + (x+2)^{10} + (x+3)^{10}}{x^{10} + (x+3)^{10}} &= \lim_{x\to\infty} \frac{\log(x+1)^{10} + \log(x+2)^{10} + \log(x+3)^{10}}{\log x^{10} + \log(x+3)^{10}} \\
&= \lim_{x\to\infty} \frac{10\log(x+1) + 10\log(x+2) + 10\log(x+3)}{10\log x + 10\log(x+3)} \\
&= \lim_{x\to\infty} \frac{\log(x+1) + \log(x+2) + \log(x+3)}{\log x + \log(x+3)} \\
&= \lim_{x\to\infty} \frac{\log(3x+6)}{\log(2x+3)} = \lim_{x\to\infty} \frac{3x+6}{2x+3} = \frac{3}{2}\,.
\end{aligned}$$

ANAND KUMAR
Ramanujan School of Mathematics,
Chandpur Bela, Patna, India

[Readers may have come across other howlers. We would be pleased to hear from you, even if you were the perpetrator! — Ed.]

Heron Ladders

K. R. S. SASTRY

1. Introduction

The following ladder problem has surfaced in a number of places. The difference shows up only in the numerical values:

> Two old imperial ladders 30 ft and 20 ft long cross 10 ft above the ground when leaning against opposite walls in a passageway as shown in figure 1. What is the width of the passageway?

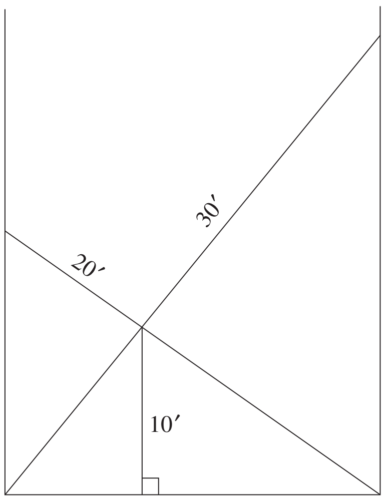


Figure 1. The ladder problem.

A recent appearance is in reference 1. In most instances the width of the passageway has been an irrational number. It is not hard to set the above values rational to get the width also rational. However, when we do so, the sides of various right-angled triangles in figure 1 will have rational values. Moreover, if these rationals are not already integers, then by enlarging suitably we can make them integral. In other words, the various right-angled triangles of figure 1 become *Pythagorean triangles*. For this reason we call these ladders *Pythagorean ladders*. A general solution of the Pythagorean ladders problem follows.

In figure 2, e, f denote the lengths of the Pythagorean ladders; u, v denote the heights on the walls reached by the ladders; a the width of the passageway and h the height of the point above the ground at which the ladders cross. It is well known that, for relatively prime natural numbers m, n such that exactly one of them is even and $m > n$, we get all Pythagorean triangles with sides

$$\kappa(m^2+n^2), \quad \kappa(m^2-n^2), \quad \kappa(2mn), \qquad \kappa = 1, 2, 3, \ldots .$$

Applying the above to figure 2 twice, we see that, for appropriate values of κ_i, m_i, n_i, $i = 1, 2$, we could put

$$e = \kappa_1(m_1^2 + n_1^2), \quad a = \kappa_1(m_1^2 - n_1^2), \quad v = \kappa_1(2m_1n_1)$$

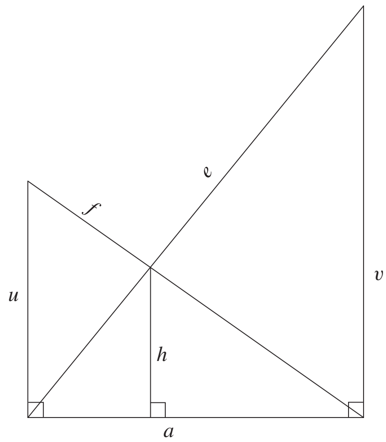


Figure 2. Pythagorean ladders.

and

$$f = \kappa_2(m_2^2 + n_2^2), \quad a = \kappa_2(m_2^2 - n_2^2), \quad u = \kappa_2(2m_2n_2)\,.$$

By equating the two values of a we could take $\kappa_1 = m_2^2 - n_2^2$, $\kappa_2 = m_1^2 - n_1^2$ for simplicity. Also, a consideration of similar triangles involving u, v, h gives the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{h}$$

from which we calculate h. We leave the details to the reader and obtain

$$\begin{aligned}
e &= (m_1^2 + n_1^2)(m_2^2 - n_2^2)\,, & f &= (m_1^2 - n_1^2)(m_2^2 + n_2^2)\,,\\
u &= 2m_2n_2(m_1^2 - n_1^2)\,, & v &= 2m_1n_1(m_2^2 - n_2^2)\,,\\
a &= (m_1^2 - n_1^2)(m_2^2 - n_2^2)\,, & h &= \frac{uv}{u+v}\,.
\end{aligned}$$

For example, $m_1 = 2$, $n_1 = 1$, $m_2 = 3$, $n_2 = 2$ yield the rational lengths $e = 25$, $f = 39$, $u = 36$, $v = 20$, $a = 15$ and $h = \frac{90}{7}$. We may now multiply by 7 to obtain all integral values. Of course, if all the above integral values have any common divisor greater than 4, then we may divide them by their gcd and get smaller values. The values $m_1 = 4$, $n_1 = 1$, $m_2 = 3$, $n_2 = 2$ illustrate this situation.

In the present study we ask the natural question: why should the walls be vertical? The answer is: they need not be as long as they are stable. Interestingly, in such a slanting-walls situation, the various triangles become *Heron triangles*, i.e. triangles whose sides and area are natural numbers. See reference 2 and the references therein for more on Heron triangles. To study the generalised situation it would be convenient to have the concept of a Heron angle with us. An angle A is called a *Heron angle* if both $\sin A$ and $\cos A$ are rational. Two examples of Heron angles are $\frac{1}{2}\pi$ and 0. The fundamental relation connecting them, $\sin^2 A + \cos^2 A = 1$, shows that we can take the parametric representation

$$\sin A = \frac{2uv}{u^2+v^2}\,, \qquad \cos A = \frac{u^2-v^2}{u^2+v^2} \tag{1}$$

for a Heron angle A, where u and v are relatively prime natural numbers with $u > v$. We also impose the constraint $0 < A < \frac{1}{2}\pi$. This is done to reduce the complexity of our study. Also, we have allowed the possibility that both u and v may be odd. This is done to obtain both the Heron angles A and $B = \frac{1}{2}\pi - A$ from (1). As an example of this, let $u = 2$, $v = 1$. Then $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$ yields the (acute) Heron angle A. If we put $u = 3$, $v = 1$ in (1), we obtain $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$.

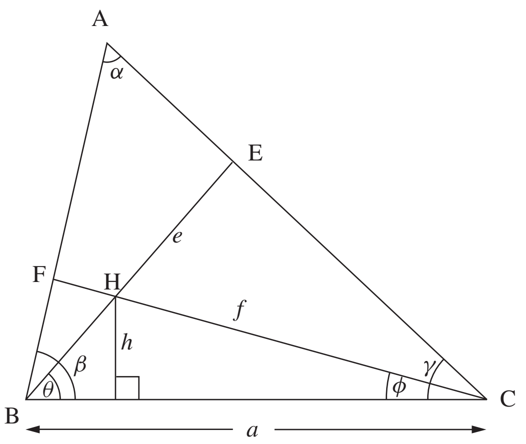


Figure 3. Heron ladders.

2. The ladder problem generalised: Heron ladders

Figure 3 shows the generalised situation. The walls BF and CE make angles β and γ with the passageway BC of width a. The lines BE, CF represent the ladders of lengths e, f respectively. They cross at H, height h above BC. Let $\angle\text{EBC} = \theta$, $\angle\text{FCB} = \phi$. We denote by A the point at which BF and CE extended meet and let $\alpha = \angle\text{BAC}$. This notation remains standard throughout our discussion. We determine the condition(s) under which e, f, h, a, BF and CE have rational lengths. As we shall see in the next section, this happens if and only if β, γ, θ and ϕ are Heron angles. For this reason we call these ladders *Heron ladders*.

3. Solution of the Heron ladders problem

We rephrase the Heron ladders problem in the following manner.

Theorem 1. *The lengths e, f, h, a,* BF *and* CE *of figure 3 have rational values if and only if β, γ, θ, and ϕ are Heron angles.*

Proof. Firstly assume that $\beta, \gamma, \theta, \phi$ are Heron angles. Therefore, $0 < \beta, \gamma, \theta, \phi < \frac{1}{2}\pi$ and, by (1),

$$\begin{aligned} \sin\beta &= \frac{2u_1v_1}{u_1^2+v_1^2}\,, \qquad \sin\gamma = \frac{2u_2v_2}{u_2^2+v_2^2}\,, \\ \sin\theta &= \frac{2m_1n_1}{m_1^2+n_1^2}\,, \qquad \sin\phi = \frac{2m_2n_2}{m_2^2+n_2^2} \end{aligned} \tag{2}$$

for appropriate pairs of natural numbers u_i, v_i and m_i, n_i, $i = 1, 2$. Moreover, by analogy with the classical ladders problem, we assume that $\theta \le \beta$, $\phi \le \gamma$. Angle $\alpha = \pi - (\beta + \gamma)$ so that

$$\begin{aligned} \sin\alpha &= \sin(\beta+\gamma) = \sin\beta\cos\gamma + \cos\beta\sin\gamma \\ &= \frac{2u_1v_1(u_2^2-v_2^2) + 2u_2v_2(u_1^2-v_1^2)}{(u_1^2+v_1^2)(u_2^2+v_2^2)}\,. \end{aligned} \tag{3}$$

The sine rule for the triangle ABC states that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}\,,$$

where $b = \text{AC}$ and $c = \text{AB}$. We substitute the expressions for $\sin\alpha$, $\sin\beta$, $\sin\gamma$ from (3) and (2) and take proportional values:

$$\begin{aligned} (a, b, c) = \big(&u_1v_1(u_2^2-v_2^2) + u_2v_2(u_1^2-v_1^2), \\ &u_1v_1(u_2^2+v_2^2), u_2v_2(u_1^2+v_1^2)\big)\,. \end{aligned} \tag{4}$$

Also, from the triangle BEC,

$$\frac{e}{\sin\gamma} = \frac{\text{EC}}{\sin\theta} = \frac{a}{\sin(\gamma+\theta)}\,,$$

so that

$$e = \frac{a\sin\gamma}{\sin(\gamma+\theta)}\,, \qquad \text{EC} = \frac{a\sin\theta}{\sin(\gamma+\theta)}\,. \tag{5}$$

Likewise, from the triangle BFC,

$$f = \frac{a\sin\beta}{\sin(\beta+\phi)}\,, \qquad \text{BF} = \frac{a\sin\phi}{\sin(\beta+\phi)}\,. \tag{6}$$

Finally, from the triangle BHC,

$$\cot\theta + \cot\phi = \frac{\text{BK} + \text{KC}}{\text{HK}} = \frac{a}{h}\,,$$

so that

$$h = \frac{a}{\cot\theta + \cot\phi}\,. \tag{7}$$

The expressions (5), (6), (7) show that when $\beta, \gamma, \theta, \phi$ are Heron angles, the designated lengths are indeed rational.

Before taking up the second part of the proof, let us consider a numerical illustration for a better grasp of the concept.

Example 1. Put $u_1 = 2$, $v_1 = 1$, $u_2 = 19$, $v_2 = 17$. Then $\sin\beta = \frac{4}{5}$, $\cos\beta = \frac{3}{5}$, $\sin\gamma = \frac{323}{325}$, $\cos\gamma = \frac{36}{325}$ and $(a, b, c) = (1113, 1300, 1615)$. Now put $m_1 = 3$, $n_1 = 1$, $m_2 = 3$, $n_2 = 2$. This gives $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\sin\phi = \frac{12}{13}$, $\cos\phi = \frac{5}{13}$. We observe that $\theta \le \beta$, $\phi \le \gamma$. Also $\sin(\beta+\phi) = \sin(\gamma+\theta) = \frac{56}{65}$. From (5), (6), (7) we calculate

$$\begin{aligned} e &= \tfrac{51\,357}{40}, & f &= \tfrac{2067}{2}, \\ h &= 636, & a &= 1113, \\ \mathrm{EC} &= \tfrac{6201}{8}, & \mathrm{BF} &= \tfrac{2385}{2}. \end{aligned}$$

Actually, this example gives us a *Brahmagupta quadrilateral*, a cyclic quadrilateral with integer sides, diagonals and area. To see this, note that $\cos(\beta+\phi) = \cos(\gamma+\theta) = -\frac{33}{65}$. Hence $\angle\mathrm{BFC} = \angle\mathrm{BEC}$ and BFEC is cyclic. To calculate EF, we apply Ptolemy's theorem (the sum of the products of the two pairs of opposite sides equals the product of the diagonals):

$$\mathrm{EF}\cdot\mathrm{BC} + \mathrm{BF}\cdot\mathrm{CE} = \mathrm{BE}\cdot\mathrm{CF}.$$

This gives $\mathrm{EF} = \frac{14\,469}{40}$. To get the integral values for all the lengths, we multiply each value by $\frac{40}{159}$: $e = 323$, $f = 260$, $h = 160$, $a = 280$, $\mathrm{EC} = 195$, $\mathrm{BF} = 300$. Furthermore,

$$\begin{aligned} \text{Area BFEC} &= \text{Area }\triangle\text{BFE} + \text{Area }\triangle\text{BEC} \\ &= \frac{e}{2}\big[\mathrm{BF}\sin(\beta-\theta) + \mathrm{BC}\sin\theta\big] \\ &= 40\,698. \end{aligned}$$

Hence BFEC of this example gives rise to a Brahmagupta quadrilateral. See reference 2 for more on this.

We now take up the second part of the proof. Here we assume that the lengths e, f, h, a, EC and BF of figure 3 are rational and show that $\beta, \gamma, \theta, \phi$ are Heron angles, i.e. the sines and the cosines of these angles are rational.

The cosine rule applied to the triangles BEC and BFC shows that $\cos\beta, \cos\gamma, \cos\theta, \cos\phi, \cos(\beta+\phi), \cos(\gamma+\theta)$ are rational. Let us expand the last ones. Since

$$\cos(\beta+\phi) = \cos\beta\cos\phi - \sin\beta\sin\phi$$

is rational, $\sin\beta\sin\phi$ (and likewise $\sin\gamma\sin\theta$) is rational too.

Next, the sine rule applied to the triangles BEC and BFC yields the equations

$$\frac{\mathrm{BE}}{\mathrm{CE}} = \frac{\sin\gamma}{\sin\theta}, \qquad \frac{\mathrm{CF}}{\mathrm{BF}} = \frac{\sin\beta}{\sin\phi},$$

which shows that

$$\frac{\sin\gamma}{\sin\theta} \quad\text{and}\quad \frac{\sin\beta}{\sin\phi}$$

are rational. It follows that $\sin^2\gamma$, $\sin^2\beta$ are rational. Hence, there are rational numbers $\lambda_1, \lambda_2, r_1, r_2$ such that

$$\begin{aligned} \sin\gamma &= \sqrt{r_1}, & \sin\theta &= \frac{\lambda_1}{\sqrt{r_1}}, \\ \sin\beta &= \sqrt{r_2}, & \sin\phi &= \frac{\lambda_2}{\sqrt{r_2}}. \end{aligned}$$

Let $\cos\theta = \lambda_3$ and $\cos\phi = \lambda_4$, both rational numbers. Now $\cot\theta + \cot\phi = h/a$ must be rational, so that

$$\Big(\frac{\lambda_3}{\lambda_1}\Big)\sqrt{r_1} + \Big(\frac{\lambda_4}{\lambda_2}\Big)\sqrt{r_2} = \frac{h}{a}$$

must be rational. However, this equation is impossible unless $\sqrt{r_1}$ and $\sqrt{r_2}$ are themselves rational. This shows that β, γ, θ and ϕ must be Heron angles and completes the proof.

4. Special cases

We deduce two special cases from theorem 1. The first of these solves the classical ladders problem in yet another way.

4.1. Vertical walls

Theorem 1 reduces to the classical ladders problem when $\beta = \gamma = \frac{1}{2}\pi$. The expression (4) does not yield the value of a. However, (5), (6), (7) yield

$$a = e\cos\theta = f\cos\phi = h(\cot\theta + \cot\phi).$$

If θ and ϕ are Heron angles, then a (along with e, f, h) will be rational which leads to the Pythagorean ladders case when the values are scaled up suitably. In any case, this provides yet another approach to solving the ladders problem. The problem with which we began the discussion has $e = 30$, $f = 20$, $h = 10$, so

$$a = 30\cos\theta = 20\cos\phi = 10(\cot\theta + \cot\phi).$$

We challenge the reader to determine $\cos\theta$ (or $\cos\phi$) and hence the value of a.

4.2. Walls equally inclined to the pathway

In this case we put $\gamma = \beta$, i.e. $u_1 = u_2 = u$ (say), $v_1 = v_2 = v$ (say) and in general $\theta \ne \phi$. Then the expressions (4), (5), (6) and (7) give

$$(a, b, c) = \big(2(u^2 - v^2), u^2 + v^2, u^2 + v^2\big)$$

after removing the gcd. (Incidentally, these describe isosceles Heron triangles completely.) Now

$$\begin{aligned} a &= 2(u^2 - v^2), & e &= \frac{a\sin\beta}{\sin(\beta+\theta)}, \\ f &= \frac{a\sin\beta}{\sin(\beta+\phi)}, & h &= \frac{a}{\cot\theta + \cot\phi}. \end{aligned}$$

For example, $u = 2$, $v = 1$, $m_1 = 3$, $n_1 = 1$, $m_2 = 5$, $n_2 = 1$ result in $a = 6$, $e = \frac{24}{5}$, $f = \frac{104}{21}$, $h = \frac{45}{28}$.

5. Conclusion

This has been yet another demonstration of the fact that the study of mathematics is full of pleasant surprises. We invite the reader to generalise the ladder problem in some other way. We pose the following challenges to the reader.

1. The expressions (4) describe the complete set of Heron triangles up to similarity. Use (4) to solve *Hoppe's problem*: to determine Heron triangles whose sides are in arithmetic progression.
2. In the vertical walls problem, assuming that e, f, h, a are rational lengths, prove that u, v are also rational.
3. Given u_1, v_1, u_2, v_2, determine their combinations for m_1, n_1, m_2, n_2 that make BFEC (of figure 3) a Brahmagupta quadrilateral.
4. Can we strengthen the second part of the proof of theorem 1, i.e. assuming only the rationality of e, f, h, a, can we deduce that $\beta, \gamma, \theta, \phi$ are Heron angles?
5. In our discussion, we assumed that the Heron angles $\beta, \gamma, \theta, \phi$ are all acute and that $0 \leq \beta,\ \phi \leq \gamma$ to reduce the complexity of the study. However, this need not be the case. Discuss the various cases that arise when one or more of these angles are non-acute (including the cases $\theta > \beta$, $\phi > \gamma$).

References

1. Problem 33.2, The ladder problem, *Math. Spectrum*, **33** (2000/2001), pp. 67–68.
2. K. R. S. Sastry, A family of Heron triangles, *Math. Spectrum*, **33** (2000/2001), pp. 49–52.

K. R. S. Sastry *taught mathematics in India and then in Ethiopia. He now spends time contributing problems and articles to mathematics journals.*

Mathematics in the Classroom

Innumeracy

Calling at the garage for petrol on the way home from college the other day, there was a frustratingly long queue to pay. The majority of people in the queue were not lining up to pay for petrol, rather they were there to purchase their lottery tickets for that evening's draw. Whilst I could not help feeling pleased at this enthusiasm for paying what I perceive to be a voluntary tax, I despaired at their lack of understanding of their infinitesimally low chance of actually winning a sizeable amount of money. Most will quote you the chance of one in fourteen million if you enquire whether or not they know the likelihood of winning, but actually they seem blissfully unaware of just how remote this possibility is.

Even students studying for A-level maths can fail to have a real feel for numbers. Earlier in the same day, when checking through a project which was attempting to ascertain the fitness levels of students in college by studying body–mass indices constructed from data on the weights and heights of a sample of students, I was surprised to observe that, in a purportedly representative sample of 30 female students, 3 of them had heights between $5'$ and $5'6''$ but weights no more than 5 st. 5 lb. When querying the likelihood of this, the student concerned said that he had accepted the data from the students without question, and no, it didn't seem strange to him!

Royal Statistical Society News regularly reports strange mathematical perceptions that appear in various documents in a column aptly named 'Forsooth!'. Two that were reported recently (references 1 and 2) were:

- Candidate to be top scorer in Euro 2000:

Vieri (Italy),	matches:	17,
	goals:	10,
	mean (goals per match):	0.17.

(*Marca* (Spain), 26 April 2000.)

Wherever did that mean come from?

- A rail route takes up four times less land than a motorway.
  (Railtrack advert at Euston Station, December 2001.)

Four times more I can understand, but what is four times *less*?

Perhaps, therefore, it is not surprising that Application of Number is to have a higher profile in the 16–19 curriculum in September, when all students on an advanced course without a Level 2 qualification in AON will be required to undertake such a course alongside their other advanced studies. There is evidently a need for a greater understanding of all things numerical amongst our students.

Reference 3 is a wonderful book which highlights many interesting and varied applications of number across numerous settings, and identifies some widespread misconceptions that we all regularly encounter. In the hope of whetting the reader's appetite, I will give you a taste of what it contains, some problems it raises that have been of interest to my students, and refer you to the text for the answers.

A scientific consultant from MIT has used the following problem to sort out prospective employees during job interviews. He asks how long it would take to remove Mount Fujiyama (Japan's highest peak at 12 395 ft) if dump trucks were available to cart away the rock and soil. Assumptions to be made are that trucks arrive every 15 minutes, 24 hours a day, are instantaneously filled with the debris that constitutes the mountain, and depart without getting in each other's way. Some rough estimates need to be made, but reasonable answers are surprisingly large.

Another large number arises in a problem in an area that A-level students frequently have difficulty with, namely permutations and combinations. Consider a G8 conference where the 8 leaders stand in line for the inevitable photograph. If this is done in a random fashion, then we are familiar with the fact that this can be done in $8! = 40\,320$ ways. What, therefore, is the probability that Tony Blair will find himself positioned next to George W. Bush? In the same area, we are invited to calculate how many matches it would take if a baseball manager wants to play every possible combination of a team of 9 chosen from his squad of 25 players. One innumerate sportswriter actually suggested this!

Courts of law are places where estimates of probabilities sometimes need to be made in order to ascertain the likelihood of some event that has been known to happen. But gauging what is the right probability to calculate can cause difficulty, and inappropriate choice can alter the outcome of a trial. One such case happened in Los Angeles where a blond woman with a ponytail snatched a purse from another woman. The thief fled on foot but was later spotted entering a yellow car driven by a black man with a beard and moustache. Subsequently, the police located a blond woman with a ponytail who regularly associated with a bearded and moustached black man who owned a yellow car. There was no other evidence to link the couple with the crime and no witnesses able to identify either party.

The prosecutor assigned the following probabilities to the characteristics in question:

$$\begin{aligned}
\text{P(yellow car)} &= \tfrac{1}{10},\\
\text{P(man with moustache)} &= \tfrac{1}{4},\\
\text{P(woman with ponytail)} &= \tfrac{1}{10},\\
\text{P(woman has blond hair)} &= \tfrac{1}{3},\\
\text{P(black man with beard)} &= \tfrac{1}{10},\\
\text{P(interracial couple in a car)} &= \tfrac{1}{1000}.
\end{aligned}$$

He argued that these characteristics were independent, so it is straightforward to show that the probability of a randomly selected couple having all of them is $\frac{1}{12\,000\,000}$, which is so small that the couple must be the guilty pair. They were convicted by the jury on the basis of this calculation.

However, an appeal went forward to the California Supreme Court, where the defence attorney argued that this was not an appropriate probability on which to make a guilty/not guilty verdict. Los Angeles had a population of approximately 2 000 000 couples. So what is the chance that more than 1 couple exist in the city with the given characteristics? You can find this probability using the binomial distribution and the probability calculated at the first trial. You will see that it is big enough (i.e. larger than 5%) to be a reasonable possibility, and led the court to overturn the original verdict.

Paulos considers lots of games and advocates the calculation of the expected winnings as a good way of deciding whether or not to play. Returning to the lottery problem above, and assuming that a prize of £1 000 000 is available for the purchase of a £1 ticket, can you show that the expected loss is about 93 p? On the other hand, consider another game where a coin is tossed until a tail appears for the first time. If you pay £100 to play and win £1 billion if this first tail occurs on the 20th or later toss, what is your expected gain? Surprisingly large, but would it tempt you to play? Probably not. But what if you could play lots of times and didn't have to settle up until you had finished playing? Perhaps a different response? (This is a variation of the St Petersburg paradox.)

I commend Paulos's book to you as an easy and very enjoyable read.

Carol Nixon

References

1. *R. Statist. Soc. News* **29** (2002), No. 5, p. 9.
2. *R. Statist. Soc. News* **29** (2002), No. 6, p. 10.
3. J. A. Paulos, *Innumeracy: Mathematical Illiteracy and Its Consequences* (Penguin Books, London, 1990).

Computer Column

Functional programming

Functional programming is a method of programming which uses functions. (What else?) That in itself isn't unusual: most modern programming languages allow you to define and use things that look a bit like functions, but appearances can be deceptive. Appendix A lists a section of code, written in a language called C, for sorting a list of numbers into numerical order using an algorithm known as quicksort. (Don't worry too much about trying to understand the details!) It defines a function called `qsort`, depending on `a`, `lo` and `hi`, which can then be used in the rest of the program. However, it's not really a function: writing `b=qsort(a,lo,hi)` doesn't make any sense, because `qsort(a,lo,hi)` can't be thought of as having a value. This is because, unlike in mathematics, program variables can change as the program runs; this code uses the fact to rewrite the list `a` in numerical order, destroying the original list. When we solve a mathematical problem, on the other hand, the 'variables' are not really variable at all; they're just unknown to begin with. Thus, we could denote an unsorted list of numbers by a, but the sorted version would need to be called something else.

Functional programming languages, by contrast, are much more like normal mathematics: variables, once defined, never change their value. In fact, the entire notion of 'commands' goes out of the window, since commands are generally for changing the values of variables, which isn't allowed. Essentially, a program is thought of as one big function, depending on the input data, whose value is the result of the program. Look now at appendix B, which is the same sorting algorithm as that used in appendix A but written in a functional language known as Haskell. (The language is named after Haskell Brooks Curry, whose work in mathematical logic serves as the foundation for functional languages.)

Two things are immediately evident: the functional version is a lot closer to normal mathematical notation, and hence easier to understand, and it is also a lot shorter! In this version, even without knowing much about Haskell, we get a pretty good idea that the program says the following.

- The result of sorting an empty list is an empty list.
- To sort a list whose first element is x and the rest of which is called xs, sort all the elements of xs that are less than x (call them `elts_lt_x`), sort all the elements that are greater than or equal to x (call them `elts_greq_x`) and put the results together, with x sandwiched in the middle.

It takes a bit of thought to see that this does work, but that's due to the nature of the algorithm, not the program. The aim of functional languages is to make translating a problem from mathematical form to computer code as simple and transparent as possible. In most programs written in languages like C, this step often takes quite a lot of work and is often the source of errors in the final program.

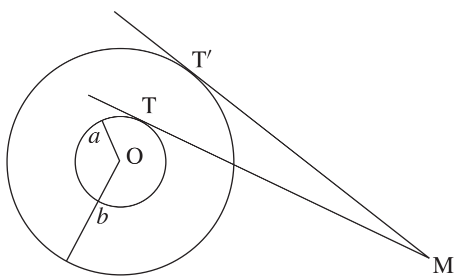
We do pay a price for all this, of course: the functional version is shorter and clearer because we have left it up to the computer to decide how to go about evaluating the function, whereas the C version gives precise, step-by-step instructions. While this is not usually a problem, it does mean that functional programs are often relatively slow; they probably wouldn't be the best choice for writing arcade games! (That said, however, there is an ongoing project to rewrite id Software's Quake — one of the most popular 3D action games of the last few years — in Haskell.)

Another benefit of the functional program is that it makes no explicit reference to the type of object it is sorting, and would in fact work just as well for sorting names or anything else where $y < x$ has a meaning. The C program, on the other hand, works for integer numbers only. In fact, it would be very easy to adapt the program to sort almost anything at all by replacing the references to $y < x$ and $y >= x$ by a function of x and y which determined which order x and y should be in. Functional languages make it very easy to re-use and adapt parts of other programs.

Functional programming languages often also have another feature, called *lazy evaluation*. To see what this means, imagine a word-search function that picks out all the places a given word occurs in a piece of text. Now imagine that you just want to find the first one, not all of them. Writing in C, say, you would either have to modify the function or add another function to take the results of the original function and output the first one — an option that no sensible programmer would consider, because the original function would have to finish its work before the results could be processed. Thus, finding the first occurrence of a word would take as long as finding all of them! In a functional language with lazy evaluation, on the other hand, each occurrence of the word would be passed to the new function as soon as it was found. If the new function was designed to output the first word and stop, then work on the original function would stop too. This again makes adapting existing programs to new purposes much easier.

To sum up, I hope that I've given you a taste of what functional languages have to offer, and not just to mathematicians. A number of large companies, such as the mobile-phone giant Ericsson, are starting to use functional languages; maybe you should, too!

Concentric circles



Given concentric circles with centre O and radii a, b, with $a < b$, is there a point M such that $\mathrm{MT} \cdot \mathrm{MT}' = ab$, where MT and MT′ are tangents to the circles, touching them at T and T′? Is there such a point with $\mathrm{MT} \cdot \mathrm{MT}' = ab$ replaced by $\mathrm{MT} \cdot \mathrm{MT}' = \mathrm{OM}^2$?

Seyamack Jafari
Razi Petrochemical Complex,
Bandar Imam, Khozestan, Iran

Appendix A. Quicksort in C

```
qsort( a, lo, hi ) int a[], hi, lo;
{
  int h, l, p, t;

  if (lo < hi) {
    l = lo;
    h = hi;
    p = a[hi];

    do {
      while ((l < h) && (a[l] <= p))
          l = l+1;
      while ((h > l) && (a[h] >= p))
          h = h-1;
      if (l < h) {
          t = a[l];
          a[l] = a[h];
          a[h] = t;
      }
    } while (l < h);

    t = a[l];
    a[l] = a[hi];
    a[hi] = t;

    qsort( a, lo, l-1 );
    qsort( a, l+1, hi );
  }
}
```

Appendix B. Quicksort in Haskell

```
qsort []     = []
qsort (x:xs) = qsort elts_lt_x ++ [x]
                    ++ qsort elts_greq_x
   where
     elts_lt_x = [y | y <- xs, y < x]
     elts_greq_x = [y | y <- xs, y >= x]
```

Peter Mattsson

Websites

Haskell home page, from which the above examples of program code were taken: http://www.haskell.org/
Frequently asked questions about functional programming: http://www.cs.nott.ac.uk/∼gmh/faq.html

Show that every positive integer has a multiple of the form $11\ldots100\ldots0$.

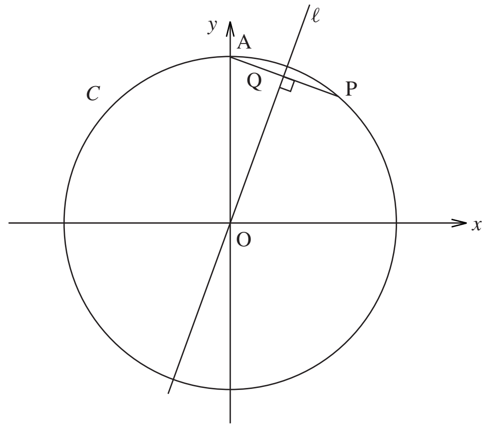
Letters to the Editor

Dear Editor,

Pythagorean triples

In Volume 34, No. 1, pp. 10–12, we described a geometrical way of obtaining the well-known formulae for primitive Pythagorean triples. Readers may be interested in another geometrical argument which does the same thing. As explained in the article, the problem amounts to finding all points P on the unit circle C in the first quadrant with rational coordinates, not including those on the coordinate axes. Such points may be found as reflections of the point $\mathrm{A} = (0, 1)$ in a straight line ℓ through the origin with rational slope greater than 1, say $y = (s/t)x$ with s, t coprime natural numbers such that $s > t$. The coordinates of P are easily found; the line AP has slope $-t/s$, so a point on this line has coordinates of the form

$$(0, 1) + \lambda\Big(1, -\frac{t}{s}\Big).$$



This line crosses ℓ at the point Q given by

$$1 - \lambda\frac{t}{s} = \frac{s}{t}\lambda,$$

so that, for Q,

$$\lambda = \frac{st}{s^2 + t^2}.$$

Now, P is given by

$$\lambda = \frac{2st}{s^2 + t^2},$$

i.e. P has the rational coordinates

$$(0, 1) + \frac{2st}{s^2 + t^2}\Big(1, -\frac{t}{s}\Big) = \Big(\frac{2st}{s^2 + t^2}, \frac{s^2 - t^2}{s^2 + t^2}\Big),$$

as in the article.

The rest of the discussion in the article gives us the standard formulae for a primitive Pythagorean triple x, y, z with x even, namely

$$(x, y, z) = (2st, s^2 - t^2, s^2 + t^2),$$

where s, t are coprime positive integers with $s > t$, one even and the other odd.

Your sincerely,

GUIDO LASTERS (Ganzendries 245, 3300 Tienen/Oplinter, Belgium.)
and
DAVID SHARPE (School of Mathematics and Statistics, University of Sheffield, Sheffield S3 7RH, UK.)

Dear Editor,

Pythagorean triples

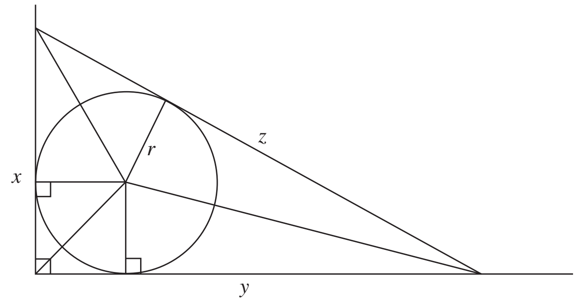
There are well-known formulae involving two parameters for positive integers x, y, z to form a primitive (i.e. the highest common factor is 1) Pythagorean triple (i.e. $x^2 + y^2 = z^2$) with x even, namely

$$x = 2st, \qquad y = s^2 - t^2, \qquad z = s^2 + t^2,$$

where s, t are positive coprime integers with $s > t$, one even and the other odd.

Two consequences may interest readers:

1. The radius r of the inscribed circle of a right-angled triangle with sides of integral length is an integer. To prove this, consider the area of the triangle in two ways.



2. If we restrict ourselves to primitive Pythagorean triples with x even and $z - y = 2$, we obtain formulae which involve a single parameter r which can take any positive integral value:

$$x = 4r, \qquad y = 4r^2 - 1, \qquad z = 4r^2 + 1.$$

In this case, $z + y = 8r^2$, which is twice an even perfect square.

It seems not to be possible to describe all primitive Pythagorean triples using a single parameter.

Yours sincerely,

MUNEER JEBREEL
(SS-Math-Hebron UNRWA, Field Education Officer, PO Box 19149, Jerusalem, Israel.)

Dear Editor,

Integrating by differentiating

You are confronted by the integral

$$I = \int e^{ax} \cos bx \, dx \,,$$

with a, b not both zero, and have forgotten how to integrate by parts. What do you do? You remember how to differentiate, including how to differentiate a product, so you write

$$y = e^{ax} \cos bx \,,$$

whence

$$y' = a e^{ax} \cos bx - b e^{ax} \sin bx \,,$$

so

$$y' - ay = -b e^{ax} \sin bx \,.$$

Now differentiate again to give

$$y'' - ay' = -ab e^{ax} \sin bx - b^2 e^{ax} \cos bx \,,$$

so

$$y'' - ay' = a(y' - ay) - b^2 y \,,$$

and so

$$y = \frac{2ay' - y''}{a^2 + b^2} \,.$$

You remember that integration is a linear process and that it is the reverse of differentiation (which means that you could have worked out how to integrate by parts from the rule for differentiating a product, but you did not spot this!), so

$$I = \int y \, dx = \frac{2ay - y'}{a^2 + b^2} + c \,,$$

where c is a constant. Hence

$$\begin{aligned} I &= \frac{1}{a^2 + b^2}(2ay - ay + b e^{ax} \sin bx) + c \\ &= \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c \,. \end{aligned}$$

Yours sincerely,

SEYAMACK JAFARI
(PO Box 161,
Razi Petrochemical Complex,
Bandar Imam,
Khozestan, Iran.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

34.9 Find all natural numbers $n > 2$ for which there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ such that $\{a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + a_1\}$ forms a set of n consecutive natural numbers.

(Submitted by Hassan Shah Ali, Tehran)

34.10 For $a, b > 0$, evaluate

$$\int_0^\infty \frac{1}{x} \left\{ \frac{1}{1 + ax} - \frac{1}{1 + bx^2} \right\} dx \,.$$

(Submitted by J. A. Scott, Chippenham)

34.11 For non-negative real numbers $x_1, \ldots, x_n$, prove that

$$\prod_{i=1}^{n}(1-x_i)+\Bigl(1+\frac{1}{n}\sum_{i=1}^{n} x_i\Bigr)^n \geq \prod_{i=1}^{n}(1+x_i)+\Bigl(1-\frac{1}{n}\sum_{i=1}^{n} x_i\Bigr)^n \,.$$

(Submitted by Hassan Shah Ali, Tehran)

34.12 For $x > 0$, let

$$f(x) = \sum_{n=0}^{\infty} \frac{\log(x + n)}{2^n} \,.$$

Evaluate

$$\lim_{x \to \infty} \frac{f(x)}{\log x} \,.$$

(Submitted by J. A. Scott, Chippenham)

Solutions to Problems in Volume 34 Number 1

34.1 Given positive integers n and k, what is the smallest positive integer s such that, whenever $x_1, \ldots, x_n$ are non-negative real numbers such that $x_1^k + \cdots + x_n^k = 1$, we can always choose s of the x_i s whose sum is at least 1.

Solution by Hassan Shah Ali, who proposed the problem

Choose $x_1 = \cdots = x_n = (n)^{-1/k}$. Then $x_1^k + \cdots + x_n^k = 1$. To get a sum of at least 1, we need at least $\lceil n^{1/k} \rceil$ of these x_i s, where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α. To show that this is sufficient, let $s = \lceil n^{1/k} \rceil$. We may suppose that $x_1 \geq \cdots \geq x_n$. Then

$$\begin{aligned}(x_1 + \cdots + x_s)^k &= x_1^k + \cdots + x_s^k + (s^k - s \text{ terms})\\ &= 1 - x_{s+1}^k - \cdots - x_n^k + (s^k - s \text{ terms}).\end{aligned}$$

Each of the $s^k - s$ terms is of the form $x_1^{i_1} \cdots x_s^{i_s}$, where $i_1 + \cdots + i_s = k$, and so is greater than or equal to $x_{s+1}^k, \ldots, x_n^k$. Since $s \geq n^{1/k}$, $s^k \geq n$, and so $s^k - s \geq n - s$. It follows that $(x_1 + \cdots + x_s)^k \geq 1$, so that $x_1 + \cdots + x_s \geq 1$. Thus, the required number is $\lceil n^{1/k} \rceil$.

34.2 Simplify the expression

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C},$$

where A, B, C are the angles of a triangle.

Solution by J. A. Scott, who proposed the problem

$$\begin{aligned}&(\sin A + \sin B + \sin C)(\cos A + \cos B + \cos C)\\ &\quad = \tfrac{1}{2}(\sin 2A + \sin 2B + \sin 2C)\\ &\qquad + \sin(B + C) + \sin(C + A) + \sin(A + B)\\ &\quad = \tfrac{1}{2}(\sin 2A + \sin 2B + \sin 2C) + \sin A + \sin B + \sin C.\end{aligned}$$

Hence, the given expression is

$$2(\cos A + \cos B + \cos C - 1).$$

Also solved by Daniel Lamy (Trinity College, Cambridge).

34.3 For a positive integer n, $S'(n)$ denotes the smallest positive integer m such that n divides $m!!$, where

$$m!! = \begin{cases} m(m-2)(m-4)\cdots 4 \cdot 2 & \text{if } m \text{ is even,}\\ m(m-2)(m-4)\cdots 3 \cdot 1 & \text{if } m \text{ is odd.}\end{cases}$$

($m!!$ is called m *double factorial* and S' is called the *Smarandache double factorial function*.) For a prime number p, determine the number of positive integers x such that $S'(x) = p$.

Solution by Daniel Lamy

First take $p > 2$. If $S(q) = p$, then $q \mid p!!$ but $q \nmid (p-2)!!$. Since p is prime this means that $p \mid q$. In fact $S(q) = p$ if and only if $q \mid p!!$ and $p \mid q$, or $p \mid q$ and $q/p \mid (p-2)!!$. So the number of solutions of the equation $S'(x) = p$ is the number of positive divisors of $(p-2)!!$. This also holds when $p = 2$, as $S'(q) = 2$ only when $q = 2$ and $0!! = 1$.

Also solved by Karthik Reddy (Madras). L. Kuciuk of the University of New Mexico points out that this can be extended to k-factorials.

34.4 For real numbers a, b, c, determine when

$$(b-c)^2 + (c-a)^2 + (a-b)^2 \quad (= X)$$

and

$$(2a-b-c)^2 + (2b-c-a)^2 + (2c-a-b)^2 \quad (= Y)$$

are equal, and generalize.

Solution by Daniel Lamy

$$\begin{aligned}Y &= \big((a-b) + (a-c)\big)^2 + \big((b-a) + (b-c)\big)^2\\ &\quad + \big((c-a) + (c-b)\big)^2\\ &= 2\big[(a-b)^2 + (a-c)^2 + (b-c)^2 + (a-b)(a-c)\\ &\quad + (b-a)(b-c) + (c-a)(c-b)\big]\\ &= 2X + (a-b)(a-c-b+c)\\ &\quad + (a-c)(a-b-c+b)\\ &\quad + (b-c)(b-a-c+a)\\ &= 3X,\end{aligned}$$

so $Y = X$ if and only if $X = 0$, i.e. if and only if $a = b = c$.

J. MacNeill, who proposed the problem, generalized it in two ways. The first was to replace Y by

$$\big[(h+k)a - hb - kc\big]^2 + \big[(h+k)b - hc - ka\big]^2 + \big[(h+k)c - ha - kb\big]^2,$$

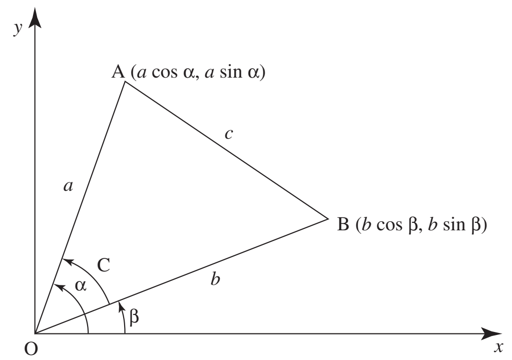
where $h^2 + hk + k^2 \neq 1$ and make the same deduction. The second was to replace a, b, c by $a_1, a_2, \ldots, a_n$ $(n \geq 2)$, put

$$X = \sum_{i<j} (a_i - a_j)^2,$$

$$Y = \sum_{i=1}^{n} \Big((n-1)a_i - \sum_{j \neq i} a_j\Big)^2$$

and prove that $Y = X$ if and only if $a_1 = \cdots = a_n$.

Prove the cosine formula



Use the diagram to prove the cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

(You will need the cosine formula in Volume 34, No. 1, p. 22.)

Guido Lasters
Tienen, Belgium

Reviews

A Mathematical Mystery Tour. By A. K. DEWDNEY. John Wiley, New York, 2001. Pp. 224. Paperback $15.95 (ISBN 0-471-40734-8).

In *A Mathematical Mystery Tour*, Dewdney describes his meetings with four different mathematicians in Greece, Jordan, Italy and England as part of his quest to find out more about the roots of mathematics. At the outset he asks two questions: firstly, 'why is mathematics so incredibly useful in the natural sciences?', and secondly, 'is mathematics discovered, or is it created?'

The four mathematicians each look at different discoveries within maths, as well as looking at the influence that individual cultures have had on mathematical discoveries. The first speaker, Pygonopolis, takes a great interest in Pythagoras, who reappears at various intervals throughout the book. The reader is shown several proofs of Pythagoras' theorem, as well as Euler's solution to the Königsberg bridge problem and a description of the proof. However, the most interesting sections look at the historical background, where proofs would be traced out in the sand, and the fact that the ancient Greeks did not use algebra to express the proofs. The religious aspects also seem to be in sharp contrast to reaction to scientific discoveries today, especially Pythagoras' Brethren of Purity, which has a distinctly spiritual nature hard to reconcile with the way maths is viewed today.

The second speaker, an astronomer named al-Flayli, looks at the history of mathematical systems that I had previously taken for granted, for example the Indian system of expressing numbers as tens and units, and how basic addition came to be. The theme of Pythagoras is continued and we see its use in the realms of astronomy, where Dewdney succeeds in providing understandable explanations to some apparently complex discoveries. Again there is a mention of a more modern figure when looking at Kepler's theories of planetary motion, which does initially seem a little out of context, as was the mention of Euler in the previous chapter. However, the thing that I found most interesting, once more, was the impact that different cultures have had on the discoveries that al-Flayli describes.

Calzoni, the third speaker, takes the example of Balmer and Ångström's work on hydrogen gas wavelengths to show how mathematics describes the physical world. Although the subject matter itself is initially less accessible, she does succeed in showing how the process of discovery appears to continue regardless of the background of the discoverer. Another example is that of Newton and Leibniz both developing calculus.

Finally, Dewdney talks to Sir John Brainard, whose examples I found much less accessible. He talks at length about 'horping zooks' to illustrate the major points of group theory, but because all the points are contained within the same example, I found that it became increasingly hard to follow. He also considers that mathematics could contain internal contradictions, which was much more interesting and also comprehensible. The very last speaker (linked to Brainard) is David Gridbourne, who runs a computer program that he believes may be evolving independently of any human intervention. This last example seems at first unlikely, but Dewdney approaches it with an open mind, and thus encourages the same approach for the reader.

Each of the speakers discusses a different aspect of mathematics, and for the most part their accounts are engaging and accessible. However, at times the characters themselves do not come alive to the reader as much as the subject matter — this seemed especially to be the case when describing al-Flayli and his family. Also Dewdney does not always stick rigidly to his quest, in trying to provide answers to the two questions mentioned earlier. Although the examples of uses of mathematics in the natural sciences are extremely interesting, at the end of the book I did not feel that he had offered any explanation as to why this might be so. The same cannot be said for the second question. There are many passages dealing explicitly with this, all discussed clearly and in a thought-provoking manner. However, none of the speakers took the view that mathematics is created; the arguments for discovery appear compelling and are carefully discussed, but the other side of the question is not subjected to the same scrutiny. It is also left until the final chapter of the book for the nature of existence itself to be considered.

However, I did enjoy this book. The insights into mathematics in a historical and cultural context were especially illuminating and it does give a much broader view of mathematics than that given in textbooks. I would recommend it to anyone with an interest in the subject, with some basic mathematical knowledge.

Student, Stamford High School — LOUISE EGGETT

Mathematical Chestnuts from Around the World. By ROSS HONSBERGER. MAA, Washington, DC, 2001. Pp. 320. Paperback $32.95 (ISBN 0-88385-330-2).

Mathematical Chestnuts, as the name suggests, is a collection of problems with elegant solutions. The most immediately striking feature of this book is the presentation: with a smart front cover, and a clear layout throughout. This is a quality which makes the mathematical arguments easier to follow, as do the numerous diagrams and concise algebra. In fact there is very little clutter or unnecessary detail, so a fairly advanced knowledge of algebra, geometry, number and binomial theory is assumed, although preliminary work is shown where relevant to the solution. The downside to this is that sometimes no explanation is given to justify a step where it is not intuitively obvious. Also, the problems are ordered by where they were taken from (e.g. IMO 1995), and not grouped into categories such as combinatorics, calculus, etc. This makes the book feel disjointed, as adjacent questions are often set on diverse topics. However, at the back of the book is an index of the problems with a brief description of each, placed in categorical order and with a page reference.

The problems come from many sources, mainly Olympiads, so they are of similar style with a large number of geometric and combinatorial questions. However, the solutions vary greatly in complexity, with numerous techniques and theorems employed, sometimes for tackling the same problem in different ways. There are also a small number of exercises for the reader to attempt, but it is probably worthwhile trying some of the problems before reading through the solutions. The problems themselves are challenging, yet accessible, and often they can be cracked with a little inspiration using only basic concepts.

This book is perfect preparation for Olympiads, or even for enthusiasts of problem solving. Readers of all levels will find something new amongst the 150 or so diverse challenges posed, which include proofs of the nine-point circle, and Coolidge's, Cantor's, and Napoleon's theorems. *Mathematical Chestnuts from Around the World* is a recreational read, designed for leisure as opposed to rigorous mathematical examination. I enjoyed a selective study of the material, especially as most of the solutions are concise and aided with clear diagrams. I recommend this book to anyone with a keen interest in maths, and who would appreciate the subtleties and insights contained within.

Student, Nottingham High School — DANIEL LAMY

Cryptological Mathematics. By ROBERT LEWAND. MAA, Washington, DC, 2001. Pp. 220. Paperback $29.95 (ISBN 0-88385-719-7).

This book covers many mathematical topics which play a role in enciphering and deciphering secret messages, ranging from discrete maths to pure maths and including probability and statistics. Some of the concepts I found hard to understand as they involved some complicated mathematics, but they were introduced in a thorough and gentle manner.

I found this book to be extremely factual, yet fascinating. It is clear that the author has made a big effort to keep the reader interested and active. The antics of the book's three main characters Beth, Stephanie and Molly help to hold the reader's attention. There are also numerous diagrams and examples throughout, including exam-style questions for the reader to try.

I would highly recommend this book to anyone who is interested in mathematics and computing. After reading it, I certainly felt that I had learned a great deal about cryptological mathematics.

Student, Solihull Sixth Form College — RICHARD BANKS

A Primer on Number Sequences. By SHAILESH SHIRALI. Universities Press (India) Ltd, Hyderabad, 2001. Pp. 240. Paperback Rs 290 (ISBN 81-7371-369-3).

This book surprised me with a variety of unusual and unexpected number sequences, building on ideas that I have already met in mathematics, and in developing a logical and methodical way of thinking. The text, which includes incredible examples of patterns found in shapes and space itself, successfully held my attention. Readers are encouraged to make their own discoveries by applying techniques carefully explained in the book. Whilst the proofs are complex in parts, the concepts are within one's grasp, and this book will leave you thinking that mathematics is no set of coincidences. Definitely an interesting read.

Student, Solihull Sixth Form College — SAARAH BADR

First Steps in Number Theory — A Primer on Divisibility. By SHAILESH SHIRALI. Universities Press (India) Ltd, Hyderabad, 2001. Pp. 240. Paperback Rs 215 (ISBN 81-7371-368-5).

This book is an interesting read for any mathematician. It explains the more complex ideas behind number theory that many students have not seen or worked with before, and then sets examples to test understanding. Because of the author's remarkably illuminating way of explaining these ideas and problems, students like myself (just starting the second year of an A-level Further Mathematics course) can begin to understand them and use them in their own mathematics. Powerful mathematics made not only accessible but riveting as well, this book comes recommended.

Student, Solihull Sixth Form College — DANIELLE SMITH

Other books received

Useful Mathematical and Physical Formulae. By MATTHEW WATKINS. Wooden Books, Presteigne, Powys, 2001. Pp. 58. Paperback (ISBN 1-902418-33-6).

An attractively produced, pocket-sized volume, well illustrated with its own friendly wizard. A handy reference.

Triangle of Thoughts. By ALAIN CONNES, ANDRÉ LICHNEROWICZ AND MARCEL PAUL SCHÜTZENBERGER. American Mathematical Society, Providence, RI, 2001. Pp. 180. Hardback £18.00 (ISBN 0-8218-2614-X).

This is a conversation among three distinguished mathematicians, Alain Connes, André Lichnerowicz (who died in 1998) and Marcel Paul Schützenberger (who died in 1996). They cover an enormous amount of ground, from logic to cosmology, quantum mechanics and time. It has to be said that this is not the sort of conversation that you would normally overhear on the bus. It is at the same time perplexing and intriguing. For most of our readers, the former of these reactions is likely to predominate. Do mathematicians really talk to each other like this?

Essays in the History of Lie Groups and Algebraic Groups. By ARMAND BOREL. American Mathematical Society, Providence, RI, 2001. Pp. 184. Hardback £23.50 (ISBN 0-8218-0288-7).

11–16 Numeracy Starters Pack. By PETER JOHNSON AND ABIGAIL TWYMAN. Oxford University Press, 2001. Pp. Paperback £40.00 (ISBN 0-19-914832-5).

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