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CONTENTS

A NEW INEQUALITY FOR R, r, AND s

VEDULA N. MURTY

1. *Introduction.*

Given an ordered triple (R, r, s) of positive real numbers, does there always exist a triangle with circumradius R, inradius r, and semiperimeter s (which we will briefly call *an R-r-s triangle*)? If not, what is a necessary and sufficient condition for the existence of an *R-r-s* triangle? An inequality representing such a condition was given by Blundon [1]. Later, Bottema [2] derived anew Blundon's *fundamental inequality* and wrote that, because it is a necessary and sufficient condition, "it is the best condition available; it can not be improved and, in principle, all other inequalities for R, r and s are consequences of this one". (Bottema then went on to discuss several such inequalities.)

In this paper we will first, for completeness, give a brief outline of Bottema's derivation of Blundon's fundamental inequality and give a geometric interpretation. We will then derive an inequality for R, r and s which we believe to be new and, finally, use it to prove *Garfunkel's inequality* proposed earlier in this journal [4].

2. *Blundon's fundamental inequality.*

If a, b, c are positive real numbers, it is well known that there exists a triangle with sides a, b, c if and only if $u_i > 0$ $(i = 1, 2, 3)$, where

$$u_1 = b+c-a, \qquad u_2 = c+a-b, \qquad u_3 = a+b-c.$$

And if the $u_i > 0$ only are known, then the sides a, b, c are easily recaptured:

$$a = \tfrac{1}{2}(u_2+u_3), \qquad b = \tfrac{1}{2}(u_3+u_1), \qquad c = \tfrac{1}{2}(u_1+u_2).$$

It can be shown that the elementary symmetric functions of the u_i in terms of R, r, s are

$$u_1 + u_2 + u_3 = 2s,$$

$$u_2 u_3 + u_3 u_1 + u_1 u_2 = 4r(4R+r),$$

$$u_1 u_2 u_3 = 8r^2 s.$$

Hence the cubic equation with the roots u_i is

$$u^3 - 2su^2 + 4r(4R+r)u - 8r^2 s = 0 \qquad\qquad \text{. (1)}$$

and (assuming $R, r, s > 0$) an *R-r-s* triangle exists if and only if (1) has three

positive roots. It is clear from the signs of the coefficients that (1) has neither negative roots nor a zero root. Hence a necessary and sufficient condition for the existence of an R-r-s triangle is that (1) have *real* roots.

The substitution $u = 2v + 2s/3$ transforms (1) into

$$v^3 + pv + q = 0, \tag{2}$$

where

$$p = \frac{1}{3}(12Rr+3r^2-s^2), \qquad q = \frac{2}{27}s(18Rr-9r^2-s^2),$$

and the roots of (2) are real if and only if

$$4p^3 + 27q^2 \leq 0.$$

Hence the required condition is

$$(12Rr+3r^2-s^2)^3 + s^2(18Rr-9r^2-s^2)^2 \leq 0,$$

and this is equivalent to the following, which is Blundon's fundamental inequality:

$$(r^2+s^2)^2 + 12Rr^3 - 20Rrs^2 + 48R^2r^2 - 4R^2s^2 + 64R^3r \leq 0. \tag{3}$$

3. *Geometric interpretation.*

We observe that (3) is a homogeneous polynomial, so only the ratios of R, r, and s are of interest. Therefore we introduce variables $x > 0$ and $y > 0$ defined by

$$Rx = r \qquad \text{and} \qquad Ry = s. \tag{4}$$

This transforms (3) into

$$(x^2+y^2)^2 + 12x^3 - 20xy^2 + 48x^2 - 4y^2 + 64x \leq 0. \tag{5}$$

To each R-r-s triangle there corresponds a point (x,y) of the graph of (5) in (the first quadrant of) the Cartesian plane; and, conversely, to each point (x,y) of the graph of (5) there correspond infinitely many R-r-s triangles (one for each $R > 0$, for which r and s are then given by (4)). The graph of (5) is the shaded region in the figure on page 65.

Since the left member of (5) can be written

$$\{y^2 - (2+10x-x^2)\}^2 - 4(1-2x)^3,$$

arc OA_1 in the figure is the graph of

$$y = \sqrt{(2+10x-x^2) - 2(1-2x)^{3/2}}, \qquad 0 < x \leq \tfrac{1}{2},$$

and arc A_1D is the graph of

$$y = \sqrt{(2+10x-x^2) + 2(1-2x)^{3/2}}, \qquad 0 < x \leq \tfrac{1}{2}.$$

(Bottema [2] has shown that arcs OA_1 and A_1D are part of a hypocycloid of three cusps, or deltoid.) The points of arcs OA_1 and A_1D correspond to isosceles R-r-s triangles; the point A_1 corresponds to all equilateral R-r-s triangles; and the points on segment OD (which are not part of the graph of (5)) correspond to degenerate triangles. Arc OA_1 is concave downward, arc A_1D is concave upward, the line $y = \sqrt{3}(1+x)$ is tangent to both arcs at A_1, and the line $x = 0$ is tangent to arc OA_1 at O.

For every triangle ABC, we will henceforth assume without loss of generality that $A \leq B \leq C$. Every triangle falls into one of two types:

$$\text{type I: } B \geq \pi/3; \quad \text{type II: } B \leq \pi/3.$$

Bager [3] has shown that, with sums cyclic over A,B,C,

$$\text{for type I: } \sqrt{3}\Sigma \cos A \leq \Sigma \sin A;$$

$$\text{for type II: } \sqrt{3}\Sigma \cos A \geq \Sigma \sin A.$$

(Both of these inequalities are mistakenly reversed on page 15 of Bager's article.) Since

$$x = \frac{r}{R} = \Sigma \cos A - 1 \quad \text{and} \quad y = \frac{s}{R} = \Sigma \sin A,$$

we obtain

$$\text{for type I: } y \geq \sqrt{3}(1+x);$$

$$\text{for type II: } y \leq \sqrt{3}(1+x).$$

Hence the points in the region with vertical shading lines in the figure correspond to triangles of type I, and those in the region with horizontal shading lines correspond to triangles of type II.

4. *A new inequality for R, r, and s.*

We will prove the following

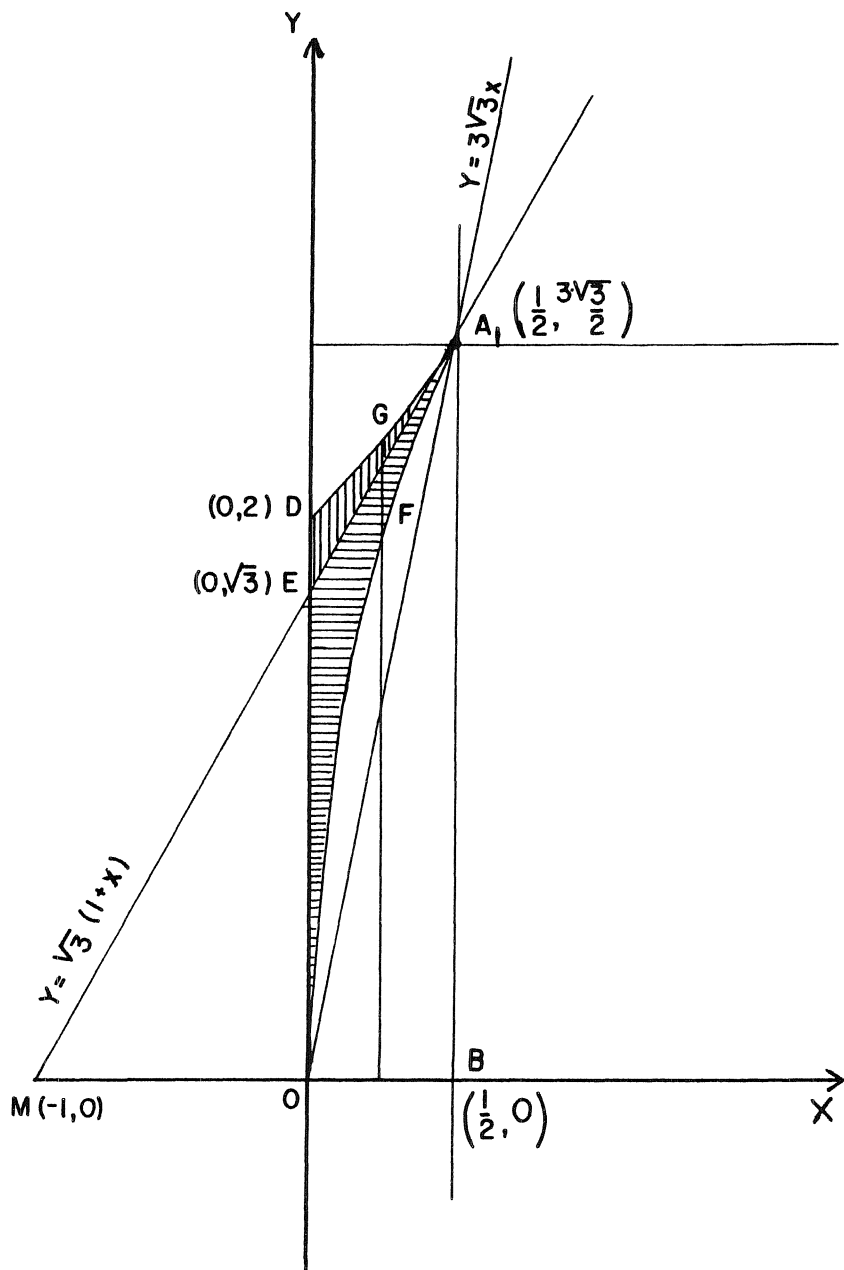*THEOREM. If an R-r-s triangle ABC is of type I, or is an acute-angled triangle of type II, then*

$$\sqrt{3}(r^2+s^2+2Rr-2R^2) - 8rs \geq 0. \tag{6}$$

Before proving the theorem, we establish a few results we will need. First, as we did with Blundon's inequality (3), we use (4) to obtain from (6) the equivalent inequality

$$x^2 - \frac{8}{\sqrt{3}}xy + y^2 + 2x - 2 \geq 0. \tag{7}$$

Next, observing that the left member of (7) is

$Y = 3\sqrt{3}x$

$A_1\left(\dfrac{1}{2}, \dfrac{3\sqrt{3}}{2}\right)$

G

(0,2) D

F

(0,√3) E

$Y = \sqrt{3}(1+x)$

M (-1,0)

O

B

$\left(\dfrac{1}{2}, 0\right)$

X

Y

$$(y - \frac{4x}{\sqrt{3}})^2 - \frac{1}{3}(13x^2 - 6x + 6),$$

we can write (7) in the form

$$\{(y - \frac{4x}{\sqrt{3}}) - \frac{1}{\sqrt{3}}\sqrt{13x^2 - 6x + 6}\}\{(y - \frac{4x}{\sqrt{3}}) + \frac{1}{\sqrt{3}}\sqrt{13x^2 - 6x + 6}\} \geq 0. \tag{8}$$

Now every point (x,y) in the shaded region of the figure lies above the line OA_1, whose equation is $y = 3\sqrt{3}x$. Hence, for *every* R-r-s triangle, the corresponding point (x,y) satisfies

$$y \geq 3\sqrt{3}x \geq \frac{4x}{\sqrt{3}},$$

and the second factor of (8) is always positive. Thus (8) holds if and only if its first factor is positive, and so (6) and (7) are both equivalent to

$$y \geq \frac{4x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sqrt{13x^2 - 6x + 6}. \tag{9}$$

Next, we observe that, for $0 < x \leq \frac{1}{2}$,

$$\sqrt{3}(1+x) \geq \frac{4x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sqrt{13x^2 - 6x + 6}, \tag{10}$$

since straightforward algebraic manipulations show that (10) is equivalent to $x^2 \leq \frac{1}{4}$. Finally, we will need the following

LEMMA. If an R-r-s triangle ABC *is acute-angled and of type II, then the corresponding* (x,y) *satisfies* $x \geq \frac{1}{4}$.

Proof. The graph of $\cos\theta$ is concave downward for $0 \leq \theta \leq \pi/2$, so we have the simple inequalities

$$\cos\theta \geq \begin{cases} 1 - \frac{3\theta}{2\pi}, & \text{for } 0 \leq \theta \leq \frac{\pi}{3}, \\ \frac{3}{\pi}(\frac{\pi}{2} - \theta), & \text{for } \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}. \end{cases}$$

Since $A \leq B \leq \frac{\pi}{3} \leq C \leq \frac{\pi}{2}$ in our triangle, we have

$$\cos A \geq 1 - \frac{3A}{2\pi}, \qquad \cos B \geq 1 - \frac{3B}{2\pi}, \qquad \cos C \geq \frac{3}{\pi}(\frac{\pi}{2} - C).$$

Hence

$$1 + x = \Sigma \cos A \geq 2 - \frac{3}{2\pi}(\pi - C) + \frac{3}{\pi}(\frac{\pi}{2} - C) = 2 - \frac{3C}{2\pi} \geq \frac{5}{4},$$

and so $x \geq \frac{1}{4}$.

Proof of the theorem.

If triangle ABC is of type I, then the corresponding point (x,y) lies above

the line MA_1 in the figure; hence

$$y \geq \sqrt{3}(1+x),$$

and then (9) follows from this and (10).

If triangle ABC is acute-angled and of type II, it follows from the lemma that the corresponding point (x,y) lies in the shaded portion of the figure to the right of line FG. Hence we have

$$y \geq \sqrt{(2+10x-x^2) - 2(1-2x)^{3/2}}, \qquad \tfrac{1}{4} \leq x \leq \tfrac{1}{2}. \qquad (11)$$

It is a simple exercise to verify that

$$\sqrt{(2+10x-x^2) - 2(1-2x)^{3/2}} \geq \frac{4x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sqrt{13x^2-6x+6}, \quad \tfrac{1}{4} \leq x \leq \tfrac{1}{2}, \quad (12)$$

with equality holding for $(x \approx 0.232 < \tfrac{1}{4}$ and$)$ $x = \tfrac{1}{2}$. Now (9) follows from (11) and (12), and the proof is complete. $\square$

Note that we have only established that condition (6) is necessary. It is not sufficient, as can be seen from the counterexample

$$A = 30°, \qquad B = 59°, \qquad C = 91°.$$

Here triangle ABC is of type II but not acute-angled, yet

$$x = \Sigma\cos A - 1 \approx 0.363611072$$

and

$$y = \Sigma\sin A \approx 2.357014996$$

satisfy (9).

5. *Proof of Garfunkel's inequality.*

Garfunkel's inequality, as given in [4], is that

$$\Sigma\cos\frac{B-C}{2} \geq \frac{2}{\sqrt{3}}\Sigma\sin A \qquad (13)$$

when $A+B+C = \pi$. We will show that (13) holds whenever A,B,C are the angles of a triangle. (It also holds for some, but not all, A,B,C which sum to π but are not the angles of a triangle. For example, it does not hold when $A = -\pi$ and $B = C = \pi$; but it does hold when $A = -\pi/18$ and $B = C = 19\pi/36$.)

Bager has shown [3, p. 10] that, if α,β,γ are the angles of an R-r-s triangle (with associated point (x,y)), then

$$\Sigma\cos\beta\cos\gamma = \frac{r^2+s^2-4R^2}{4R^2} = \frac{x^2}{4} + \frac{y^2}{4} - 1, \qquad (14)$$

$$\Sigma \sin \beta \sin \gamma = \frac{r^2+s^2+4Rr}{4R^2} = \frac{x^2}{4} + \frac{y^2}{4} + x, \tag{15}$$

$$\Sigma \sin 2\alpha = 4\Pi \sin \alpha = \frac{2rs}{R^2} = 2xy, \tag{16}$$

and adding (14) and (15) gives

$$\Sigma \cos(\beta-\gamma) = \tfrac{1}{2}(x^2+y^2+2x-2). \tag{17}$$

Now let ABC be any triangle. If it is of type I, so that

$$A \leq \frac{\pi}{3} \leq B \leq C,$$

then triangle A'B'C', where

$$A' = \frac{\pi}{2} - \frac{C}{2}, \qquad B' = \frac{\pi}{2} - \frac{B}{2}, \qquad C' = \frac{\pi}{2} - \frac{A}{2}, \tag{18}$$

is of type II and acute-angled. Hence, from (7), the point (x,y) associated with triangle A'B'C' satisfies

$$\tfrac{1}{2}(x^2+y^2+2x-2) \geq \frac{4}{\sqrt{3}} xy. \tag{19}$$

Now, from (17) and (16),

$$\Sigma \cos (B'-C') \geq \frac{2}{\sqrt{3}} \Sigma \sin 2A', \tag{20}$$

from which (13) follows.

If ABC is of type II, so that

$$A \leq B \leq \frac{\pi}{3} \leq C,$$

then triangle A'B'C' (again defined by (18)) is of type I and the point (x,y) associated with triangle A'B'C' again satisfies (19). So (20) holds and again (13) follows.

We conclude that (13) holds for all triangles ABC.

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3. Anders Bager, "A Family of Goniometric Inequalities", *ibid.*, pp. 5-25.

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AN INTERESTING RECURSIVE FUNCTION

RICHARD V. ANDREE

The recursive function of the title has been around for some time. I have used it with undergraduates for at least twenty years, but I have no recollection of seeing it discussed earlier in the literature (although it surely must have been). It is discussed, as far as I know for the first time in print, in Lesson 17 of *EXPLORE COMPUTING with TRS-80 and Computer Sense* (Prentice-Hall, 1982).

The recursive function in question generates, from an initial positive integer n_0, a sequence $\{n_0, n_1, n_2, \ldots\}$ defined as follows: if $n_i = ABC\ldots KL$ (in decimal notation), then $n_{i+1} = A'B'C'\ldots K'L'$, where

$$A' = |A-B|, \quad B' = |B-C|, \quad C' = |C-D|, \quad \ldots, \quad K' = |K-L|, \quad L' = |L-A|.$$

Here are a few examples (where $n_i \to n_{i+1}$):

$$n_0 = 2468 \to 2226 \to 0044 \to 0404 \to 4444 \to 0000 \to 0000 \to \ldots$$

$$n_0 = 3410 \to 1313 \to 2222 \to 0000 \to 0000 \to \ldots$$

$$n_0 = 4721 \to 3513 \to 2420 \to 2222 \to 0000 \to 0000 \to \ldots$$

$$n_0 = 7269 \to 5432 \to 1113 \to 0022 \to 0202 \to 2222 \to 0000 \to \ldots$$

A preliminary problem is to show that, if the initial number n_0 has four digits, then the sequence converges to 0000 and attains this limit in at most 8 steps. Formally,

$$1000 \le n_0 \le 9999 \implies n_i = 0000 \text{ for } i \ge 8.$$

Since there are not that many four-digit numbers, this is an ideal question that can be settled by a programmable calculator or small microcomputer.

The surprise comes when we take for the initial n_0 a number having more than four digits. When n_0 has five digits, for example, one rapidly concludes that the sequence does not always converge (or, even, seldom converges) to 00000. A similar phenomenon occurs if n_0 has six or seven digits. But it appears that all sequences with an eight-digit n_0 converge to 00000000.

Readers are invited to show that, if n_0 has any number of digits, the sequence will eventually enter a cycle containing $000\ldots0\,XX$, where XX represents a repeated digit not necessarily 00. It is conjectured (but this is only a wild conjecture with little evidence to support it) that one can guarantee that $XX = 00$ for all k-digit n_0 only if k is a power of 2.

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     ✲   ✲   ✲

THE OLYMPIAD CORNER: 33

M.S. KLAMKIN

A couple of months ago in this Corner [1982: 12], I was able to give, through the courtesy of Bernhard Leeb, the problems of the First and Second Rounds of the West German Mathematical Olympiad for 1981. This month, again thanks to Bernhard Leeb, I am able to give, soon after they were issued, the problems of the First Round of the West German Mathematical Olympiad for 1982. I hope to be able to publish the problems of the Second Round as soon as they are available. I understand that the problems are distributed by mail and that the students have 11 weeks to complete each round. As usual, I solicit elegant solutions from all readers for possible publication later in this column.

WEST GERMAN MATHEMATICAL OLYMPIAD 1982

First Round

1. Let S be the sum of the greatest odd divisors of each of the numbers $1,2,3,\ldots,2^n$. Prove that $3S = 4^n + 2$.

2. In a quadrilateral ABCD, the sides AB and CD are each divided into m equal parts, the points dividing AB being labeled (in order from A) $S_1, S_2, \ldots, S_{m-1}$, and those dividing CD being labeled (in order from D) $T_1, T_2, \ldots, T_{m-1}$. Similarly, the sides BC and AD are each divided into n equal parts by the points $U_1, U_2, \ldots, U_{n-1}$ (in order from B) and $V_1, V_2, \ldots, V_{n-1}$ (in order from A), respectively. Prove that each of the segments $S_i T_i (1 \le i \le m-1)$ is divided into n equal parts by the segments $U_j V_j (1 \le j \le n-1)$.

3. A 1982-gon is given in the plane. Let S be the set of all triangles whose vertices are also vertices of the 1982-gon. A point P lies on none of the sides of these triangles. Prove that the number of triangles in S that contain P is even.

4. A set of real numbers is called *simple* if it contains no elements x,y,z such that $x + y = z$. Find the maximum size of a simple subset of $\{1,2,\ldots,2n+1\}$.

*

Now follows, through the courtesy of Mark E. Saul, an English translation of the VI All-Russian Mathematical Olympiad 1979-80.

VI ALL-RUSSIAN MATHEMATICAL OLYMPIAD 1979-80

VIII Grade

1. A group of tourists decided to sit in a set of buses in such a way that each bus would contain the same number of tourists. At first they tried to sit 22 on each bus, but it turned out that one tourist was left over. But then one bus left empty, and the tourists were able to divide themselves equally among the remaining buses. If each bus holds fewer than 33 people, how many buses and how many tourists were there (originally)?

2. Along a segment AB, $2n$ points are chosen which are symmetric in pairs with respect to the midpoint of the segment. Any n of these points are colored blue, and the rest are colored red. Prove that the sum of the distances from the red points to A is equal to the sum of the distances from the blue points to B.

3. In a regular hexagon ABCDEF, points M and K are the midpoints of CD and DE, respectively, and L is the intersection of segments AM and BK. Prove that the area of triangle ABL is equal to that of quadrilateral MDKL. Also, find the measure of the angle between lines AM and BK.

4. If $\{x\}$ denotes the fractional part of x (e.g., $\{7/5\} = 2/5$), how many distinct numbers are there in the sequence

$$\{1^2/1980\}, \quad \{2^2/1980\}, \quad \{3^2/1980\}, \quad \ldots, \quad \{1980^2/1980\}?$$

5. From a point M on the circumcircle of a triangle ABC, perpendiculars MN and MK are drawn to lines AB and AC, respectively. For which point M will NK be longest?

IX Grade

1. Can the natural numbers from 1 to 30 be arranged in a 5×6 rectangular array in such a way that (a) all columns have the same sum and (b) all rows have the same sum?

2. For which natural numbers n is $2^8 + 2^{11} + 2^n$ a perfect square? [This problem also appeared in the 1981 Hungarian Mathematical Olympiad. See [1981: 267; 1982: 46].]

3. Each vertex of a convex $(2n+1)$-gon is colored with one of three different colors. No two adjacent vertices are colored the same. Prove that the polygon can be partitioned by nonintersecting diagonals into a set of triangles

each of which has its three vertices of different colours. [Diagonals which meet at a vertex are considered nonintersecting.]

4. In expressing the fraction m/n as a decimal, where m and n are natural numbers and $n \leq 100$, a student found, at a certain place after the decimal point, the sequence of digits 167. Show that the student must have made an error.

5. Equilateral triangles ABC and A'B'C' are drawn in a plane (both sets of vertices being labeled clockwise). The midpoints of segments BC and B'C' coincide. Find
   (a) the angle between the lines AA' and BB';
   (b) the ratio AA'/BB'.

X Grade

1. For each vertex of a tetrahedron, the point symmetric to that vertex with respect to the centroid of the opposite face is chosen. Find the ratio of the volume of the tetrahedron whose vertices are these new points to that of the original tetrahedron.

2. The map of a city has the shape of a convex polygon. Each diagonal of the polygon is a street, and the intersections of the diagonals are intersections of the streets (but the vertices of the polygon are *not* considered to be intersections of streets). Streetcar lines go through the city. Each line goes from one end of a street to the other end, and has stops at each intersection as well as at the endpoints. At each intersection only two streets cross, and a streetcar runs along at least one of them. Show that one can transfer from any intersection to any other, making no more than two transfers. (A transfer may be made whenever two streetcar lines have a common stop.)

3. Consider the $2k$ numbers
$$2^1 - 1, \quad 2^2 - 1, \quad \ldots, \quad 2^{2k} - 1,$$
where $k \geq 1$. Show that at least one of them is a multiple of $2k + 1$.

4. R being the set of all real numbers, find all functions $F: R \to R$ which satisfy
$$pF(a) + (1-p)F(b) \geq F(pa+(1-p)b)$$
for all $a, b, p \in R$.

5. The squares ABCD, $A_1B_1C_1D_1$, and $A_2B_2C_2D_2$ are coplanar (and their vertices

are labeled counterclockwise). Vertices A and A_1 coincide, and so do vertices C and C_2. Show that $D_1D_2 \perp BM$, where M is the midpoint of B_1B_2, and that $D_1D_2 = 2BM$.

<center>*</center>

I now present solutions to the problems, presented here last month, of the 1982 Alberta High School Prize Examination in Mathematics. The problems and solutions were prepared by a joint committee of the Departments of Mathematics of the University of Alberta and the University of Calgary, consisting of G. Butler, M.S. Klamkin, Andy Liu, J. Pounder, W. Sands, and R. Woodrow.

1. [1982: 40] A 9×12 rectangular piece of paper is folded so that a pair of diagonally opposite corners coincide. What is the length of the crease?

Solution.

Let ABCD be the rectangle, with AD = 9 and AB = 12, and suppose that vertices A and C coincide after the fold, which produces the crease EF, as shown in the figure. The Pythagorean Theorem applied to triangle ADC gives AC = 15. It is clear that triangle FCG is right-angled and similar to triangle ACD, so
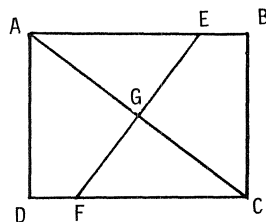
$$FG = \frac{AD \cdot CG}{DC} = \frac{45}{8} \quad \text{and} \quad EF = 2FG = \frac{45}{4}.$$

2. [1982: 40] Let $a = \sin A$, $b = \sin B$, and $c = \sin (A+B)$. Determine $\cos (A+B)$ as a quotient of two polynomials in a, b, c with integral coefficients.

Solution.

We have

$$\cos (A+B) = \cos A \cos B - ab = abx,$$

where $x = \cot A \cot B - 1$. Since

$$\cot A + \cot B = \frac{\cos A}{a} + \frac{\cos B}{b} = \frac{a \cos B + b \cos A}{ab} = \frac{c}{ab},$$

it follows that

$$\frac{c^2}{a^2b^2} = (\cot A + \cot B)^2 = (\cot^2 A + 1) + (\cot^2 B + 1) + 2x$$

$$= \frac{1}{a^2} + \frac{1}{b^2} + 2x,$$

so $c^2 = a^2 + b^2 + 2a^2b^2x$ and

$$\cos (A+B) = abx = \frac{c^2 - a^2 - b^2}{2ab}.$$

3. [1982: 40] A cylindrical tank with diameter 4 feet and open top is partially
filled with water. A cone 2 feet in diameter and 3 feet in height is
suspended (vertex up) above the water so that the bottom of the cone just touches
the surface of the water. The cone is then lowered at a constant rate of 10 feet
per minute into the water. How long does it take until the cone is completely
submerged, given that the water does not overflow?

Solution.

The volume of the cone is $(1/3)\cdot\pi\cdot1^2\cdot3 = \pi$ cubic feet. A cylinder of diame-
ter 4 feet and equal volume would have a height of $\pi/(\pi\cdot2^2) = \frac{1}{4}$ foot. Hence the
cone is lowered through a distance of $3 - \frac{1}{4} = 11/4$ feet. This process will take
$(11/4)/10 = 11/40$ minute.

4. [1982: 40] John added the squares of two positive integers and found that
his answer was the square of an integer. He subtracted the squares of the
same two positive integers and again found that his answer was the square of a
positive integer. Show that John must have made an error in his calculation.

Solution.

Suppose there exist positive integer solutions of the system

$$a^2 + b^2 = c^2 \tag{1}$$

$$a^2 - b^2 = d^2. \tag{2}$$

We consider in particular the solution (a,b,c,d) in which c is minimal. It
follows that a,b,c,d are pairwise relatively prime. Moreover, a,c,d are odd and
b is even. Adding (1) and (2), we obtain

$$a^2 = \left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2. \tag{3}$$

Both $(c+d)/2$ and $(c-d)/2$ are integers, one odd and one even. Let them be denoted
by p and q, with p odd and q even. Note that a,p,q are pairwise relatively
prime. From (3), we have

$$q^2 = (a+p)(a-p). \tag{4}$$

Now both $a+p$ and $a-p$ are even, so we may write $a+p = 2h$, $a-p = 2k$, and (4) becomes

$$q^2 = 4hk. \tag{5}$$

Since $a = h+k$ and $p = h-k$, it follows that h and k are relatively prime; hence,
from (5), each is a square, say $h = m^2$ and $k = n^2$. Thus

$$2p = 2(m^2-n^2) \tag{6}$$

and, from (5),

$$q = 2mn. \tag{7}$$

Now from (1), (2), (6), and (7),

$$b^2 = \tfrac{1}{2}(c^2 - d^2) = 2pq = 4mn(m+n)(m-n).$$

Since m, n, $m+n$, and $m-n$ are pairwise relatively prime, each is a square, say

$$m = w^2, \qquad n = x^2, \qquad m+n = y^2, \qquad m-n = z^2.$$

Thus (w,x,y,z) is also a solution of the original system (1)-(2). Furthermore,

$$y^2 = w^2 + x^2 = m+n < 2mn = q \le \frac{c+d}{2} < c < c^2,$$

so that $y < c$, a contradiction.

5. [1982: 40] Twenty-five Knights gather at the Round Table for a jolly evening. They belong to various Orders, with every two Orders having at least one common member. Members of the same Order occupy consecutive seats at the Round Table.

(a) If each Order has at most nine members, prove that there is a Knight who belongs to no Order and there is a Knight who belongs to every Order.

(b) Without the restriction on the sizes of the Orders, prove that there are two Knights such that between them they hold membership to every Order.

Solution.

Both statements are vacuously true if there are no Orders. Let the Knights be labeled consecutively from 1 to 25 around the Round Table. For each Order, the member with the lowest label will be designated its leader.

(a) We may assume that the largest Order K consists of 9, 10, ..., k, where $k \le 17$. It is easy to see that no Order can contain both 1 and 25, as otherwise it cannot intersect K. Suppose 1 belongs to some Order and 25 to another. These two Orders cannot intersect. Hence either 1 or 25 belongs to no Order. Let n be the leader with the highest label and let N be an Order of which he is the leader. Suppose there is an order X which does not contain n. Now its leader must have a label less than n, and it follows *that every member of X has a label* less than n. This is a contradiction, for then X and N do not intersect as required by the problem. Hence n belongs to every Order.

(b) Again let n be the leader with the highest label and N an Order of which he is the leader. If $n = 1$, then he belongs to every order and the condition

of the problem is trivially satisfied. We now assume that $n > 1$. Suppose there is an Order X which contains neither 1 nor n. Again, its leader must have a label less than n, and so has every member of X. This is a contradiction, for then X and N cannot intersect as required. Hence 1 and n between themselves hold membership to every Order.

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

<div align="center">

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THE PUZZLE CORNER
</div>

Puzzle No. 11: Rebus (7)

$$L: y = x^2$$

In this WHOLE way
Descartes, René,
A curve would say.

Puzzle No. 12: Rebus (5 4 3 12)

$$|x| < \delta, \quad \delta \to 0$$

So said the Indians on the strand:
(An ethnic slur, you understand—
Columbus was about to land.).

<div align="right">ALAN WAYNE, Holiday, Florida</div>

Puzzle No. 13: Alphametic

SQUARE is a cube and CUBE is a square
And THREE is not prime... Stop pulling your hair.
I'll wager some money the answer you'll get
But never again such a perfect bet!

Puzzle No. 14: Alphametic

The road to solutions is not always clear;
From left to right you may often veer.
My own advice is: Stay close to the center;
Approaching a "One-Way",   DO
                            NOT.
                          ENTER

<div align="right">HANS HAVERMANN, Weston, Ontario</div>

Answer to Puzzle No. 10 ⌈1982: 35⌉: (a) 10; (b) no.

PROBLEMS--PROBLÈMES

*Problem proposals and solutions should be sent to the editor, whose address
appears on the front page of this issue. Proposals should, whenever possible, be
accompanied by a solution, references, and other insights which are likely to be
of help to the editor. An asterisk (*) after a number indicates a problem submit-
ted without a solution.*

*Original problems are particularly sought. But other interesting problems
may also be acceptable provided they are not too well known and references are
given as to their provenance. Ordinarily, if the originator of a problem can be
located, it should not be submitted by somebody else without his permission.*

*To facilitate their consideration, your solutions, typewritten or neatly hand-
written on signed, separate sheets, should preferably be mailed to the editor before
October 1, 1982, although solutions received after that date will also be con-
sidered until the time when a solution is published.*

721. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port
Richey, Florida.*

A propos of the editor's comment following Crux 611 [1982: 30], verify that,
with decimal integers,

(a) uniquely, TRIGG is three times WRONG;

(b) independently, but also uniquely, WAYNE is seven times RIGHT.

722. *Proposed by Paul R. Beesack, Carleton University, Ottawa, Ontario.*

A very large prison has 10000 cells numbered from 1 to 10000 (each
occupied by one prisoner), an eccentric warden, and an ingenious electronic device
for opening and closing cell doors. Early one day the warden announces that all
those prisoners whose cell doors are left open at the end of the day will be free
to leave the prison. During the day he presses buttons which open or close cell
doors as follows. First he opens all cell doors beginning with cell 1. Next,
beginning with cell 2, he operates on every 2nd cell door, closing those that are
open and opening those that are closed. This operation is repeated throughout
the day so that at the nth step ($n = 1,2,\ldots,10000$), every nth cell beginning with
cell n is closed if it was open, or opened if it was closed. How many prisoners
are freed at the end of the day?

723. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let G be the centroid of a triangle ABC, and suppose that AG,BG,CG
meet the circumcircle of the triangle again in A',B',C', respectively. Prove that

(a) GA' + GB' + GC' ≥ AG + BG + CG;

(b) AG/GA' + BG/GB' + CG/GC' = 3;

(c) GA'·GB'·GC' ≥ AG·BG·CG.

724, *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Let ABC be a triangle (with sides a,b,c) in which the angles satisfy $C + A = 2B$ (that is, the angles are in arithmetic progression). Such a triangle has many interesting properties. Establish the following (and possibly others):

(a) $\sin(A-B) = \sin A - \sin C$.

(b) $a^2 - b^2 = c(a - c)$.

(c) Vertices A and C, circumcentre O, incentre I, orthocentre H, and excentre I_b all lie on a circle of radius R, where R is the circumradius of the triangle. Furthermore, if this circle meets the lines AB and BC again in A' and C', respectively, then AA' = CC' = $|c-a|$.

725, *Proposed by H. Kestelman, University College, London, England.*

An $n \times n$ matrix is called *simple* if its eigenvectors span C^n and is called *deficient* if they do not. If $\underline{A}$ and $\underline{B}$ are simple, can $A + B$ and AB both be deficient? If $\underline{A}$ is simple, show that adj A is simple; is the converse true?

(The (r,s) element of adj A is the (s,r) cofactor of $\underline{A}$.)

726, *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Show that, in a regular n-simplex,

(a) the dihedral angle is Arccos $(1/n)$;

(b) the product of the altitude and the circumdiameter is half the square of an edge (independently of n).

727, *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let t_b and t_c be the symmedians issued from vertices B and C of triangle ABC and terminating in the opposite sides b and c, respectively. Prove that $t_b = t_c$ if and only if $b = c$.

728, *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let $E(P,Q,R)$ denote the ellipse with foci P and Q which passes through R. If A,B,C are distinct points in the plane, prove that no two of $E(B,C,A)$, $E(C,A,B)$, and $E(A,B,C)$ can be tangent.

729, *Proposed jointly by Dick Katz and Dan Sokolowsky, California State University at Los Angeles.*

Given a unit square, let K be the area of a triangle which covers the square. Prove that $K \geq 2$.

730, *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Prove that $D = 0$ for all real θ if $D = |a_{ij}|$ is a determinant with

$$a_{ij} = \cos(i+j)\theta, \quad i,j = 0,1,2,\ldots,n; \quad n \geq 2.$$

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

492. [1979: 291; 1980: 291; 1981: 50, 117, 277] *Proposed by Dan Pedoe, University of Minnesota.*

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}AB$ are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{2}AB$?

IV. *Further comment by the proposer.*

Part (b) has been solved when the entire segment AB is given, but no complete solution has been forthcoming when only the points A and B are given. I am now happy to communicate a beautiful solution by the professors L. Yang and J. Zhang of the China University of Science and Technology, Hefei, Anhui, People's Republic of China. A translation was sent to me by Geng-zhe Chang, Visiting Scholar at Brown University, Providence, R.I.

I have been corresponding on geometrical matters with Drs. Yang and Zhang for some time. They have expressed their pleasure at their solution being published in this journal. I submitted to them proofs of the theorems used in their construction and a simplification of one part of their proof, and they have agreed to both proofs and modification.

Without loss of generality, we assume that the fixed radius of the compass opening is $r = 1$, so that AB > 2. We shall simply say "draw a circle" instead of "draw a circle of radius 1". We use the notation of Stanley Rabinowitz [1981: 277], according to which X $\triangledown$ Y = Z means that XYZ is an equilateral triangle with vertices labelled X,Y,Z in counterclockwise order.

LEMMA 1. If A,B,P *are distinct points, and the points* P',B',C *are defined by* A $\triangledown$ P = P', $\overrightarrow{B'B} = \overrightarrow{P'P}$, *and* P' $\triangledown$ B' = C, *respectively, then* A $\triangledown$ B = C.

The proof given below, by transformation geometry, is not easily replaced by a direct Euclidean proof (see [1]). We will denote by $(X)T$ the effect of a transformation T on a point X, and by $R(Y,\theta)$ a rotation with centre Y through an angle θ.

Proof. A given translation can be composed of two opposite rotations. We have (see Figure 1)

$$(P')R(P',\pi/3) = P',$$

$$(P')R(A,-\pi/3) = P,$$

and we also have

$(B')R(P',\pi/3) = C.$

Since $\vec{B'B} = \vec{P'P}$, we must therefore have

$(C)R(A,-\pi/3) = B.$ □

We now consider a rhombus PQ_1BQ_2 with sides $PQ_1 = Q_1B = BQ_2 = Q_2P$ of unit length, with $PB \leq 1$, and show:

LEMMA 2. (a) We can rotate this unit rhombus about the vertex P through an angle $\pi/3$, resulting in a rhombus $PQ_1'B'Q_2'$.

(b) We can translate the original unit rhombus to a unit rhombus $P'Q_1''B''Q_2''$, where $PP' = 1$.

Figure 1

Proof. (a) For $i = 1,2$, draw circles with centres at P and Q_i, denoting their intersections by Q_i' (chosen so that $P \triangledown Q_i = Q_i'$). Then draw circles with centres at Q_i', taking their intersection other than P as B'. The required unit rhombus $PQ_1'B'Q_2'$ results. (Note that this construction is essentially the five-circle construction given earlier ⌈1980: 291⌉ for an equilateral triangle PBB' given P, B, and a rusty compass of radius greater than $\frac{1}{2}PB$.)

(b) We are given that $PP' = 1$. For $i = 1,2$, draw circles with centres at P' and Q_i, taking the intersections, other than P, of these two circles as Q_i''. Then draw circles with centres at Q_i'', taking their intersection, other than P', as B''. The required rhombus $P'Q_1''B''Q_2''$ results. □

These unit rhombuses are a tool for rotating PB through $\pi/3$ or translating it.

Our final construction is of a lattice of points covering the plane, using our rusty compass to draw equilateral triangles, starting from an arbitrary equi-lateral triangle ARS such that $A \triangledown R = S$ (see Figure 2).

Assume that $\vec{a} = \vec{AR}$ and $\vec{b} = \vec{AS}$. Then in the affine system $\{A; \vec{a},\vec{b}\}$ we can construct any lattice point (m,n) by counting.

LEMMA 3. There is a lattice point P such that $PB \leq 1$.

Proof. The system of closed circular disks with centres at the lattice points covers the entire plane, so at least one of these disks covers the point B. If P is the centre of this disk, then $PB \leq 1$.
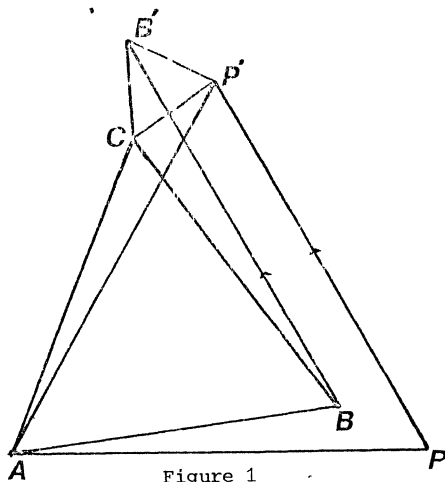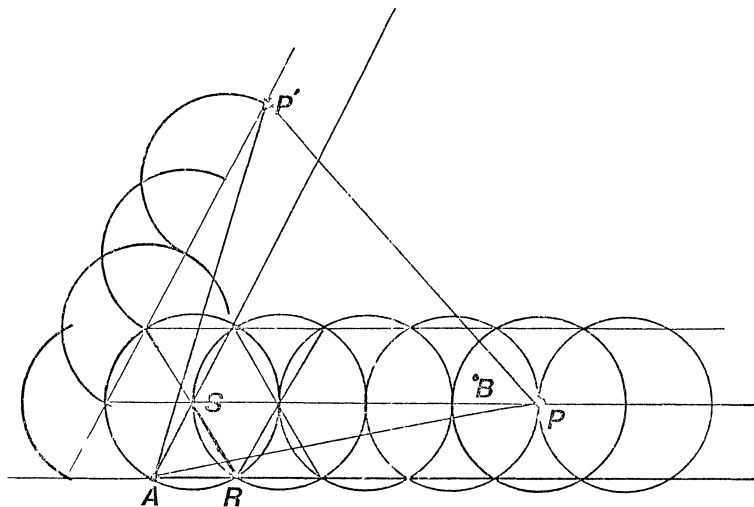
Figure 2

LEMMA 4. There is a lattice point P' *such that* A ∇ P = P'. *If* P = (k,l), *then* P' *is either* $(-l, k+l)$ *or* $(k+l, -k)$.

Outline of proof. If we rotate the whole lattice around the point A through π/3, all the lattice points fall on lattice points, so that P falls on a lattice point P'.

The second part of the lemma shows that we can determine P' by counting. To prove that P' is either $(-l, k+l)$ or $(k+l, -k)$, we can use the Bankoff result stated as a Lemma by Rabinowitz [1981: 277], using only equilateral triangles, with vertices P,Q,R,S at lattice points. This gives an elegant proof. An alternative is to work out the transformation of coordinates formulae for rotations of π/3 in oblique coordinates. □

We can now give the construction for the point C such that A ∇ B = C. Draw a unit rhombus PO_1BQ_2 and translate it step by step, with P going from lattice point to lattice point until it reaches P', where A ∇ P = P'. Then we also have $\overrightarrow{B'B} = \overrightarrow{P'P}$, defining B', and rotating the unit rhombus through π/3 about P', we have the situation shown in Figure 1, and B' falls on C, the desired point making ABC equilateral.

<div align="center">REFERENCE</div>

1. Daniel Pedoe, "A Key Theorem in Transformation Geometry", *The Mathematics Teacher*, 67 (December 1974) 716-718.

617. [1981: 80] *Proposed by Charles W. Trigg, San Diego, California.*

The sum of two positive integers is 5432 and their least common multiple is 223020. Find the numbers.

Solution by W.J. Blundon, Memorial University of Newfoundland.

Suppose the required integers, if they exist, are ad and bd, where a,b,d are positive integers and $(a,b) = 1$. It easily follows that also $(ab,a+b) = 1$. Since

$$\frac{a+b}{ab} = \frac{ad+bd}{abd} = \frac{5432}{223020} = \frac{28 \cdot 194}{28 \cdot 7965} = \frac{194}{7965}, \tag{1}$$

and the first and last fractions in (1) are both in lowest terms, we must have $a + b = 194$ and $ab = 7965$. Thus a and b are the roots of the quadratic

$$x^2 - 194x + 7965 = 0,$$

that is, $\{a,b\} = \{59, 135\}$. It follows from (1) that $d = 28$, from which

$$\{ad, bd\} = \{1652, 3780\}.$$

These numbers, which are satisfactory in all respects, constitute the only solution.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; JAYANTA BHATTACHARYA, Midnapur, West Bengal, India (2 solutions); JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; N. ESWARAN, student, Indian Institute of Technology, Kharagpur, India; BIKASH K. GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; FRED A. MILLER, Elkins, West Virginia; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; RAM REKHA TIWARI, Radhaur, Bihar, India (2 solutions); ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; ALAN WAYNE, Holiday, Florida; KENNETH M. WILKE, Topeka, Kansas; DAVID ZAGORSKI, student, Stuyvesant H.S., New York, N.Y.; and the proposer.

Editor's comment.

Meyers reported that this problem appears, with answer but without solution, in Uspensky and Heaslet [1].

REFERENCE

1. J.V. Uspensky and M.A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York and London, 1939, p.41, problem 9.

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618. [1981: 80] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

For $i = 1,2,3$, let I_i be the centres and r_i the radii of the three

Malfatti circles of a triangle ABC, as shown in the figure below. Calculate the sides a = BC, b = CA, and c = AB of the triangle in terms of the r_i.

[The notation in the problem has been revised to conform with the solution which follows.]

Solution by George Tsintsifas, Thessaloniki, Greece.
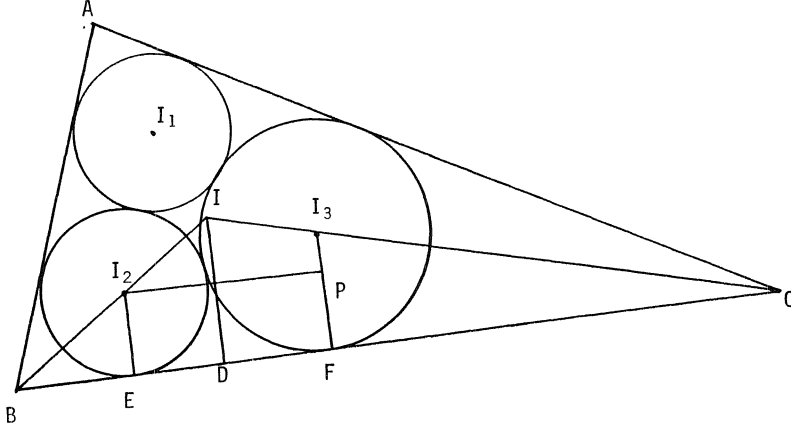
We adopt the usual notations I, r, and s for the incentre, inradius, and semiperimeter of the triangle. From the similar triangle pairs BEI_2, BDI and CFI_3, CDI (see figure), we have

$$ED = \frac{(r-r_2)(s-b)}{r} \quad \text{and} \quad DF = \frac{(r-r_3)(s-c)}{r}; \tag{1}$$

and from right triangle PI_2I_3 arises

$$ED + DF = EF = 2\sqrt{r_2r_3} . \tag{2}$$

Now, from (1) and (2) and similar relations, we obtain the system of equations

$$(r-r_2)(s-b) + (r-r_3)(s-c) = 2r\sqrt{r_2r_3}$$

$$(r-r_3)(s-c) + (r-r_1)(s-a) = 2r\sqrt{r_3r_1}$$

$$(r-r_1)(s-a) + (r-r_2)(s-b) = 2r\sqrt{r_1r_2}$$

from which we get

$$\begin{cases} s-a = \dfrac{r}{r-r_1}(-\sqrt{r_2r_3} + \sqrt{r_3r_1} + \sqrt{r_1r_2}) \\[2mm] s-b = \dfrac{r}{r-r_2}(\sqrt{r_2r_3} - \sqrt{r_3r_1} + \sqrt{r_1r_2}) \\[2mm] s-c = \dfrac{r}{r-r_3}(\sqrt{r_2r_3} + \sqrt{r_3r_1} - \sqrt{r_1r_2}). \end{cases} \tag{3}$$

The well-known formula for r,

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s},$$

is equivalent to

$$r^2\{(s-a)+(s-b)+(s-c)\} = (s-a)(s-b)(s-c),$$

and substituting (3) into this yields an equation equivalent to

$$\left(\frac{1}{\sqrt{r_1}}+\frac{1}{\sqrt{r_2}}+\frac{1}{\sqrt{r_3}}\right)r^2 - 2(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3})r + 2\sqrt{r_1 r_2 r_3} = 0. \tag{4}$$

With this remarkable equation, we can solve our problem. The appropriate root of (4) is

$$r = \frac{(\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}) + \sqrt{r_1+r_2+r_3}}{\sqrt{r_2 r_3}+\sqrt{r_3 r_1}+\sqrt{r_1 r_2}} \cdot \sqrt{r_1 r_2 r_3}. \tag{5}$$

Since $a = (s-b)+(s-c)$, we obtain from (3)

$$a = \frac{r}{r-r_2}(\sqrt{r_2 r_3}-\sqrt{r_3 r_1}+\sqrt{r_1 r_2}) + \frac{r}{r-r_3}(\sqrt{r_2 r_3}+\sqrt{r_3 r_1}-\sqrt{r_1 r_2}), \tag{6}$$

with r given by (5). The values of b and c can be obtained from (6) by cyclic permutations of r_1, r_2, r_3.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. In addition, two incorrect solutions and one unhelpful comment were received.

Editor's comment.

Our featured solution was the only really satisfactory one received. The other two solutions credited above were technically correct, but they did not, as the problem seems to require, give explicit formulas for the sides in terms of the Malfatti radii. Instead, they gave the sides in terms of other functions in which the Malfatti radii were implicit and very well concealed. One of the incorrect solutions was a two-line production giving the sides in terms of the Malfatti radii and the angles of the triangle, the last of which are not known. This deserves a booby prize of some sort.

The *Malfatti problem*, which dates from 1803 and has been extensively treated in the literature, runs as follows: given a triangle (or its sides), construct the (Malfatti) circles (or calculate their radii). Our problem is just the opposite: given the Malfatti radii of a triangle, calculate its sides. It seems improbable that this *inverse Malfatti problem* has never appeared in the literature since 1803. If it has, the editor would appreciate receiving a reference.

Our second incorrect solution (incorrect only because it solved the wrong

problem) gave one of the known solutions of the Malfatti problem (and references to several others), and our unhelpful commentator decided to throw the editor a bone by giving him one reference to the Malfatti problem. Thank you very much, but that is not what our problem is about.

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   619. [1981: 80] *Proposed by Robert A. Stump, Hopewell, Virginia.*
     If k is a positive integer, find the value of

$$\sum_{i=1}^{\infty} \frac{1}{i(i+k)}.$$

  I. *Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*
For $n > k$, we have

$$S_n \equiv \sum_{i=1}^{n} \frac{1}{i(i+k)} = \frac{1}{k}\sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+k}\right) = \frac{1}{k}\left(\sum_{i=1}^{n}\frac{1}{i} - \sum_{i=1}^{n}\frac{1}{i+k}\right)$$

$$= \frac{1}{k}\left(\sum_{i=1}^{n}\frac{1}{i} - \sum_{i=k+1}^{n+k}\frac{1}{i}\right) = \frac{1}{k}\left(\sum_{i=1}^{k}\frac{1}{i} - \sum_{i=n+1}^{n+k}\frac{1}{i}\right)$$

$$= \frac{1}{k}\sum_{i=1}^{k}\frac{1}{i} - R_n,$$

where

$$R_n = \frac{1}{k}\sum_{i=n+1}^{n+k}\frac{1}{i} < \frac{1}{k}\sum_{i=n+1}^{n+k}\frac{1}{n} = \frac{1}{k}\cdot\frac{k}{n} = \frac{1}{n}.$$

Since $R_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{i} = \frac{1}{k}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}\right).$$

  II. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*
[Just as in solution I], we can show that, more generally, for suitable a, b, and positive integer k,

$$\sum_{i=1}^{\infty} \frac{1}{(a+bi)\{a+b(i+k)\}} = \frac{1}{bk}\sum_{i=1}^{k}\frac{1}{a+bi}, \tag{1}$$

from which our problem follows when $a = 0$ and $b = 1$. The more general form (1), with appropriate values for a, b, and k, will give many of the results tabulated by Jolley [1], in particular his relations numbered (204), (205), (208), (221), (224), (231), and (233).

Also solved by PAUL R. BEESACK, Carleton University, Ottawa, Ontario; JAMES BOWE, Erskine College, Due West, South Carolina; S.C. CHAN, Singapore; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; BIKASH K. GHOSH, Bombay, India; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; F.G.B. MASKELL, Algonquin College, Ottawa, Ontario; BOB PRIELIPP, University of Wisconsin-Oshkosh; LAWRENCE SOMER, Washington, D.C.; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; ALAN WAYNE, Holiday, Florida; KENNETH M. WILKE, Topeka, Kansas; DAVID ZAGORSKI, student, Stuyvesant H.S., New York, N.Y.; and the proposer. Two incorrect solutions were received.

Editor's comment.

Two solutions were labeled incorrect for the good and sufficient reason that they arrived at the wrong answer. A couple of other solvers figuratively skated over a patch of nonexistent ice when they wrote

$$\sum_{i=1}^{\infty} \frac{1}{i(i+k)} = \frac{1}{k}\left(\sum_{i=1}^{\infty} \frac{1}{i} - \sum_{i=1}^{\infty} \frac{1}{i+k}\right),$$

but they managed to land on their feet and arrive at the correct answer. For this singular feat of equilibrium they were credited with a correct solution.

REFERENCE

1. L.B.W. Jolley, *Summation of Series*, Second Revised Edition, Dover, New York, 1961.

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620. [1981: 80] *Proposed by Fred A. Miller, Elkins, West Virginia.*

Using the digits 0 and 1, express each of the following in the nega-binary system of notation (base -2):

$$\frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{5}, \ \frac{1}{6}, \ -\frac{1}{2}, \ -\frac{1}{3}, \ -\frac{1}{4}, \ -\frac{1}{5}, \ -\frac{1}{6}.$$

Solution de Robert Tranquille, Collège de Maisonneuve, Montréal, Québec.

Dans ce qui suit, seules les représentations qui contiennent un "point décimal" sont des représentations négabinimales et, comme d'habitude, un groupe de chiffres surlignés indique une représentation négabinimale infinie périodique. Par exemple,

$$11.\overline{101} = (-2)^1 + (-2)^0 + \{(-2)^{-1} + (-2)^{-3}\} + \{(-2)^{-4} + (-2)^{-6}\} + \ldots = -\frac{14}{9}.$$

On trouve immédiatement

$$\frac{1}{2} = (-2)^0 + (-2)^{-1} = 1.1, \qquad -\frac{1}{2} = (-2)^{-1} = 0.1,$$

$$\frac{1}{4} = (-2)^{-2} = 0.01, \qquad -\frac{1}{4} = (-2)^{-1} + (-2)^{-2} = 0.11.$$

Tous les autres nombres proposés ont des représentations négabinimales infinies périodiques. Ces représentations correspondent donc à des progressions géométriques infinies ayant comme raison r un des nombres $(-2)^{-1}$, $(-2)^{-2}$, $(-2)^{-3}$, ... Si n est un de ces nombres et si g est le premier terme de la progression géométrique correspondante, on doit donc avoir

$$n = \frac{g}{1-r}.$$

Trouvons d'abord les nombres n pour lesquels $r = (-2)^{-1}$. On a alors $n = 2g/3$. On obtient $n = -1/3$ pour $g = (-2)^{-1}$ et $n = 1/6$ pour $g = (-2)^{-2}$; donc

$$-\frac{1}{3} = 0.\overline{1} \qquad \text{et} \qquad \frac{1}{6} = 0.0\overline{1}.$$

Pour $r = (-2)^{-2}$, il vient $n = 4g/3$; alors $n = 1/3$ correspond à $g = (-2)^{-2}$ et $n = -1/6$ correspond à $g = (-2)^{-3}$, d'où

$$\frac{1}{3} = 0.\overline{01} \qquad \text{et} \qquad -\frac{1}{6} = 0.0\overline{01}.$$

Ces deux dernières représentations ne sont pas uniques. En effet, puisque

$$\frac{1}{3} = (-2)^0 + (-2)(1/3) \qquad \text{et} \qquad -\frac{1}{6} = (-2)^{-1} + (1/3),$$

on obtient des représentations précédentes

$$\frac{1}{3} = 1.\overline{10} \qquad \text{et} \qquad -\frac{1}{6} = 0.1\overline{10}.$$

Enfin, pour $r = (-2)^{-4}$, il vient $n = 16g/15$. On obtient alors

$$n = 1/5 \quad \text{pour} \quad g = 3/16 = (-2)^{-2} + (-2)^{-3} + (-2)^{-4} = 0.0111$$

et

$$n = -1/5 \quad \text{pour} \quad g = -3/16 = (-2)^{-1} + (-2)^{-2} + (-2)^{-4} = 0.1101;$$

d'où

$$\frac{1}{5} = 0.\overline{0111} \qquad \text{et} \qquad -\frac{1}{5} = 0.\overline{1101}.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; LEROY F. MEYERS, The Ohio State University; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

According to Gilbert and Green [1], for an integer $s > 1$, the numbers with two different representations in base $-s$ are those of the form

$$(-s)^k \left(a + \frac{1}{s+1} \right),$$

where a and k are integers.   Readers who wish to pursue the subject further are invited to look also in [2] and [3].

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1.   William J. Gilbert and R. James Green, "Negative Based Number Systems", *Mathematics Magazine*, 52 (September 1979) 240-244.

2.   Wacław Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964, Chapter VII: Representation of Numbers by Decimals in a Given Scale.

3.   David Hale and Peter Wells, "Base Negative Two", *Mathematics Teaching*, No. 60 (September 1972), pp. 32-33.

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621.   [1981: 80] *Proposed by Herman Nyon, Paramaribo, Surinam.*

For the adjoining alphametic, there is unfortunately no solution in which SQUARE is a square, but there is one in which the digital sum of SQUARE is, very appropriately, the square

```
  THREE
  THREE
  THREE
  EIGHT
  EIGHT
 SQUARE
```

$$3 + 3 + 3 + 8 + 8 = 25.$$

Find this solution.

Solutions were received from J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; CHARLES W. TRIGG, San Diego, California; DAVID ZAGORSKI, student, Stuyvesant H.S., New York, N.Y.; and the proposer.

Editor's comment.

All solvers are agreed that

```
  45066
  45066
  45066
  68354
  68354
 271906
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is the only solution with 25 as the digital sum of SQUARE, and Hunter noted that in this solution, interestingly enough, the digital root of THREE is 3 and that of EIGHT is 8.

Apparently no solver was able to get a good mathematical "handle" on the problem and thereby avoid extensive use of brute force. Two sent in only the answer, and the other three carefully swept most of their extensive brute force calculations out of sight. If in fact no such "handle" exists, some may think that this is the mark of a good alphametic, but, as far as we are concerned, it makes this a problem for computers or a low-grade amusement for children and other people who have nothing better to do with their time, about on a par with reconstructing a 500-piece jigsaw puzzle. It is not mathematics.

622. [1981: 115] *Proposed by ,r.T. Groenman, Arnhem, The Netherlands.*

The Diophantine equation

$$8x^3 - 21x^2y + 35xy^2 - 83y^3 = z^3$$

has the obvious solutions $(x,y,z) = (k,0,2k)$, where k is any integer. Find at least two solutions for which $y \neq 0$.

I. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

If we subtract $(2x-3y)^3$ from both sides of the given equation, we get

$$15x^2y - 19xy^2 - 56y^3 = z^3 - (2x-3y)^3,$$

or

$$y(3x-8y)(5x+7y) = z^3 - (2x-3y)^3.$$

For $z = 2x-3y$ one of the factors on the left must vanish. The trivial solution set given in the proposal results if we set $y = 0$. If $3x-8y = 0$, then

$$\frac{x}{8} = \frac{y}{3} = \frac{2x-3y}{7}$$

and we have the solution set

$$(x, y, z) = (8k, 3k, 7k); \tag{1}$$

and $5x+7y = 0$ gives

$$\frac{x}{7} = \frac{y}{-5} = \frac{2x-3y}{29}$$

and the solution set

$$(x, y, z) = (7k, -5k, 29k). \tag{2}$$

A computer search reveals no other solution (x,y,z) with $|x|,|y|,|z| \leq 100$.

II. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

[Solution sets (1) and (2) having been found], we will obtain other solutions (x,y,z) with $y \neq 0$ from rational solutions (u,v) of the equation

$$8u^3 - 21u^2 + 35u - 83 = v^3, \tag{3}$$

where $u = x/y$ and $v = z/y$. We describe a technique of Fermat [1, p. 566] which we will use to find rational solutions of (3). Consider the equation

$$Au^3 + Bu^2 + Cu + D = v^3. \tag{4}$$

If $A = a^3$, set $v = au + B/3a^2$; if $D = d^3$, set $v = d + Cu/3d^2$; if both $A = a^3$ and $D = d^3$, set $v = au+d$. Each of these substitutions, when appropriate, will reduce (4) to a linear equation in u.

Only the Fermat substitution $v = 2u-(7/4)$ is appropriate for (3), and it

yields $u = 4969/1064$, from which we get $v = 8076/1064$ and the solution set

$$(x, y, z) = (4969k, 1064k, 8076k). \tag{5}$$

To find additional rational solutions of (3), we can use the "tangent method" of Lagrange, which is described in [1, p. 595] and in [2], [3]. This involves starting from a known solution (u_0, v_0) of (3), substituting $u = t+u_0$ in (3), and applying the Fermat technique to the resulting equation.

With $u_0 = 8/3$ from (1), for example, we substitute $u = t+(8/3)$ in (3) and obtain

$$8t^3 + 43t^2 + \frac{281t}{3} + \frac{343}{27} = v^3.$$

Now three Fermat substitutions are possible for this equation:

$$v = 2t + \frac{7}{3}, \qquad v = 2t + \frac{43}{12}, \qquad v = \frac{7}{3} + \frac{281t}{49}.$$

It can be verified that the first and second substitutions lead to solution sets (2) and (5), respectively. But the third substitution yields

$$t = -\frac{7341572}{7082283}, \qquad u = \frac{3848172}{2360761}, \qquad v = \frac{-8525447}{2360761},$$

and we have the new solution set

$$(x, y, z) = (3848172k, 2360761k, -8525447k). \tag{6}$$

Similarly, starting with $u_0 = -7/5$ from (2), the tangent method yields solution sets (1), (5), and the new solution set

$$(x, y, z) = (8555030953k, 3393217903k, -1111978813k). \tag{7}$$

Readers who love to crunch numbers will enjoy using the tangent method of Lagrange to find still more solution sets from (5), (6), and (7).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; BENGT MÅNSSON, Lund, Sweden; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

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1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. II.

2. L.J. Mordell, *Diophantine Equations*, Academic Press, London and New York, 1969, pp. 66-70.

3. H.D. Grossman, "Investigation of Lagrange's Tangent Method on Diophantine Binary Cubics", *American Mathematical Monthly*, 47 (1940) 305-306.

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623. [1981: 116] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If PQR is the equilateral triangle of smallest area inscribed in a given triangle ABC, with P on BC, Q on CA, and R on AB, prove or disprove that AP, BQ, and CR are concurrent.

Solution by J.T. Groenman, Arnhem, The Netherlands.

We disprove the theorem and show that AP, BO, and CR are concurrent if and only if triangle ABC is isosceles.

It is well known that the three Apollonius circles of a triangle have two common points, J and J', called the *isodynamic points* of the triangle; that the distances from either isodynamic point to the vertices are inversely proportional to the sides of the triangle; and that the pedal triangle of either isodynamic point is equilateral. (See Johnson [1].)

The pedal triangle of one of the isodynamic points, say J, is inscribed in triangle ABC. Let this (equilateral) triangle be PQR (with P on BC, etc.). We show that PQR is the inscribed equilateral triangle of smallest area. Let P'Q'R' be *any* inscribed equilateral triangle (with P' on BC, etc.), and let θ be the angle of smallest absolute value through which PQR must be rotated around J to make its sides parallel to those of P'Q'R'. This rotation makes PQR homothetic to P'Q'R', with J the homothetic center, and we have the homothetic ratio

$$\frac{QR}{Q'R'} = \frac{JP}{JP'} = \cos\theta \; ;$$

hence, with brackets denoting area, we have

$$[PQR] = [P'Q'R']\cos^2\theta \le [P'Q'R'],$$

with equality if and only if $\theta = 0$, and so PQR is the inscribed equilateral triangle of smallest area.

As noted above, we have

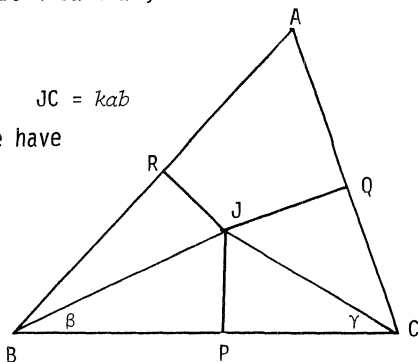$$JA : JB : JC = \frac{1}{a} : \frac{1}{b} : \frac{1}{c} = bc : ca : ab,$$

where a = BC, etc., and so

$$JA = kbc, \qquad JB = kca, \qquad JC = kab$$

for some $k \ne 0$. Referring to the figure, we have

$$a = BP + PC$$
$$= JB\cos\beta + JC\cos\gamma$$
$$= kca\cos\beta + kab\cos\gamma,$$

and so

$$1 = kc \cos \beta + kb \cos \gamma. \tag{1}$$

From $(kb \cos \gamma)^2 = (1 - kc \cos \beta)^2$, we obtain

$$k^2 b^2 - k^2 b^2 \sin^2 \gamma = 1 - 2kc \cos \beta + k^2 c^2 - k^2 c^2 \sin^2 \beta. \tag{2}$$

Now, observing that

$$JP = JB \sin \beta = JC \sin \gamma$$
$$= kca \sin \beta = kab \sin \gamma,$$

so that $c \sin \beta = b \sin \gamma$, we see that (2) reduces to

$$k^2 b^2 = 1 - 2kc \cos \beta + k^2 c^2;$$

hence

$$kc \cos \beta = \tfrac{1}{2}\{1 - k^2 (b^2 - c^2)\},$$

and

$$kb \cos \gamma = \tfrac{1}{2}\{1 + k^2 (b^2 - c^2)\}$$

then follows from (1). Finally, we have

$$BP = JB \cos \beta = kca \cos \beta = \tfrac{1}{2}a\{1 - k^2 (b^2 - c^2)\},$$
$$PC = JC \cos \gamma = kab \cos \gamma = \tfrac{1}{2}a\{1 + k^2 (b^2 - c^2)\},$$

and similar expressions can be found for CQ, QA and AR, RB.

Now by Ceva's Theorem and its converse, AP, BQ, and CR are concurrent if and only if

$$BP \cdot CQ \cdot AR = PC \cdot QA \cdot RB.$$

With the expressions found above, the necessary and sufficient condition is equivalent to

$$\{1 - k^2 (b^2 - c^2)\}\{1 - k^2 (c^2 - a^2)\}\{1 - k^2 (a^2 - b^2)\}$$
$$= \{1 + k^2 (b^2 - c^2)\}\{1 + k^2 (c^2 - a^2)\}\{1 + k^2 (a^2 - b^2)\},$$

which is itself equivalent to

$$(b^2 - c^2)(c^2 - a^2)(a^2 - b^2) = 0.$$

We conclude that AP, BQ, and CR are concurrent if and only if $b = c$ or $c = a$ or $a = b$, that is, if and only if triangle ABC is isosceles.

Also disproved by W.J. BLUNDON, Memorial University of Newfoundland; O. BOTTEMA, Delft, The Netherlands; BIKASH K. GHOSH, Bombay, India; DMITRY P. MAVLO, Moscow, U.S.S.R; VADIM V. MUZYCHENKO, Moscow, U.S.S.R.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

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