Mathematical Spectrum

2002/2003 Volume 35 Number 2

- **International Mathematical Olympiad 2002**
- **Going Round in Circles**
- **Oscillations Produced by a Cam**

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing. The copyright of all published material is vested in the Applied Probability Trust.

From the Editor

Looking for patterns

Abbas Rooholamine, a reader in Sirjan, Iran, has sent a number of items. He is intrigued by number patterns that he has come across. Of course, usually such patterns are coincidences. Occasionally they may point to something more, so they are always worth investigating. Here are some that Abbas has sent:

$$512 = (5+1+2)^3\,,$$

$$153 = 1^3 + 5^3 + 3^3\,,$$

$$\left.\begin{aligned} 370 &= 3^3 + 7^3 + 0^3\,, \\ 371 &= 3^3 + 7^3 + 1^3\,, \end{aligned}\right\}$$

$$5832 = (5+8+3+2)^3\,,$$

$$54748 = 5^5 + 4^5 + 7^5 + 4^5 + 8^5\,,$$

$$3^3+4^3+5^3+5^3+6^3+7^3 = (3+4+5+5+6+7)^2\,,$$

$$\sin^6 15 + \cos^6 15 + \sin^4 15 + \cos^4 15 = \tfrac{27}{16}\,,$$

$$\left.\begin{aligned} 8!+7!+1! &= 45\,361\,, \\ 4!+5!+3!+6!+1! &= 871\,, \end{aligned}\right\}$$

$${}_8\mathrm{C}_2 = 28\,,$$

$${}_{10}\mathrm{C}_5 + {}_{10}\mathrm{C}_0 + {}_{10}\mathrm{C}_5 = (505)_{10}\,.$$

Abbas has also sent intriguing ways of doing long division of polynomials, and also division of integers. For example, can you work out what is happening here? It is his way of dividing $3x^5 + 2x^3 - 4x^2 + 5$ by $x+1$:

$$\begin{array}{r|rrrrrr} & 3 & 0 & 2 & -4 & 0 & 5 \\ -1 & 3 & -3 & 5 & -9 & 9 & -4 \end{array}$$

The quotient is $3x^4 - 3x^3 + 5x^2 - 9x + 9$ and the remainder is -4. Intriguing!

Or how about dividing

$$x^7 + x^6 + 3x^5 + 8x^4 + 21x^3 + 55x^2 + 144x$$

by $x-1$:

$$\begin{array}{r|rrrrrrrr} & 1 & 1 & 3 & 8 & 21 & 55 & 144 & 0 \\ 1 & 1 & 2 & 5 & 13 & 34 & 89 & 233 & 233 \end{array}$$

The quotient is $x^6 + 2x^5 + 5x^4 + 13x^3 + 34x^2 + 89x + 233$ and the remainder is 233. Where have I seen these numbers before?

Or how about dividing $5x^4 + 3x^2 - 2x + 1$ by $x^2 - x - 3$?

$$\begin{array}{r|rrrrr} & 5 & 0 & 3 & -2 & 1 \\ \hline 1 & 0 & 5 & 5 & 23 & 0 \\ 3 & 0 & 0 & 15 & 15 & 69 \\ \hline & 5 & 5 & 23 & 36 & 70 \end{array}$$

The quotient is $5x^2 + 5x + 23$ and the remainder $36x + 70$.

What about dividing integers? Try dividing 517 438 by 11 $(= 10 + 1)$:

$$\begin{array}{r|rrrrrr} & 5 & 1 & 7 & 4 & 3 & 8 \\ -1 & 5 & -4 & 11 & -7 & 10 & -2 \end{array}$$

The remainder is $-2 + 11 = 9$, the quotient $47\,040 - 1 = 47\,039$.

Or try dividing 3 974 523 by 9 $(= 10 - 1)$:

$$\begin{array}{r|rrrrrrr} & 3 & 9 & 7 & 4 & 5 & 2 & 3 \\ 1 & 3 & 12 & 19 & 23 & 28 & 30 & 33 \end{array}$$

To get the remainder, divide 33 by 9:

$$\begin{array}{r|rr} & 3 & 3 \\ 1 & 3 & 6 \end{array}$$

Thus, the remainder is 6 and the quotient is

$$\begin{array}{r} 3 \\ 30 \\ 28 \\ 23 \\ 19 \\ 12 \\ \underline{3}\\ 441613 \end{array}$$

Or try dividing 745 213 964 by 99 $(= 10^2 - 1)$:

$$\begin{array}{r|rrrrr} & 7 & 45 & 21 & 39 & 64 \\ 1 & 7 & 52 & 73 & 112 & 176 \end{array}$$

Now divide 176 by 99:

$$\begin{array}{r|rr} & 1 & 76 \\ 1 & 1 & 77 \end{array}$$

***Mathematical Spectrum* Awards for Volume 34**

Prizes have been awarded to the following student readers for contributions in Volume 34:

Julia Moore for her article 'The problem of the pastilles' (with David K. Smith);

Daniel Lamy for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

The remainder is 77 and the quotient is

$$
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & 1 & 1 & 2 \\
 & & & 7 & 3 & & \\
 & 5 & 2 & & & & \\
7 & & & & & & \\
\hline
7 & 5 & 2 & 7 & 4 & 1 & 3
\end{array}
$$

Try dividing 39 426 537 by 89 $(= 10^2 - 10 - 1)$:

	3	9	4	2	6	5	3	7
1	0	3	12	19	33	58	96	0
1	0	0	3	12	19	33	58	96
	3	12	19	33	58	96	157	103

$$
\begin{array}{r}
157 \\
103 \\
\hline
1673
\end{array}
$$

	1	6	7	3
1	0	1	7	0
1	0	0	1	7
	1	7	15	10

$$
\begin{array}{r}
15 \\
10 \\
\hline
160
\end{array}
$$

	1	6	0
1	0	1	0
1	0	0	1
	1	7	1

The remainder is 71 and the quotient is

$$
\begin{array}{cccccc}
3 & & & & & \\
1 & 2 & & & & \\
 & 1 & 9 & & & \\
 & & 3 & 3 & & \\
 & & & 5 & 8 & \\
 & & & & 9 & 6 \\
 & & & & 1 & 7 \\
 & & & & & 1 \\
\hline
4 & 4 & 2 & 9 & 9 & 4
\end{array}
$$

I had better check these with my calculator!

Bernhard Neumann, AC, DSc, FAA, FRS
15 Ocober 1909–21 October 2002

The last issue of *Mathematical Spectrum* (Vol. 35, No. 1, pp. 2–3) contained a brief article by Professor Bernhard Neumann, reminiscing about his mathematical training, his teachers and his colleagues at the University of Berlin some 70 years ago. We are sad to report that Bernhard died on the morning of Monday 21 October; he had celebrated his 93rd birthday with his colleagues in the Mathematical Sciences Institute of the Australian National University, Canberra, only a few days before. He will be missed by mathematicians all over the world. We extend our deepest sympathy to his wife Dorothea, his five children and their families.

International Mathematical Olympiad 2002

PAUL JEFFERYS

The 43rd International Mathematical Olympiad was held in Glasgow, Scotland, on 19–31 July 2002. Over 80 countries each sent six pre-university mathematicians to compete in two gruelling four-and-a-half-hour papers. Each paper consisted of three questions, which were worth seven marks each, and were fiendishly difficult (thus, one's score is at most 'the answer to life, the universe and everything'). I was privileged to be able to represent the UK in this competition, which is one of the toughest challenges available to secondary school students of mathematics.

The selection process for the UK team was long. It was felt that we needed to have more people of a higher level of mathematical sophistication if we were to succeed at the IMO, so a training event was held in Bath during September 2001, to which 15 people were invited. Later, during the Christmas school break, we had another training event in Hungary with the Hungarian IMO squad. Such training is often mistakenly believed to be about learning theorems, but for the most part it is not. IMO mathematics is not about learning large numbers of theorems, but rather about finding creative ways to consider the problem, and to think 'outside the box'. Indeed, even the hardest IMO problem of this year could be solved using 'theorems' with which any A-level Further Maths student should be familiar.

The competitive process began with the Senior Maths challenge in October, open to all school students. The top scorers are invited to take part in BMO1 (British Mathematical Olympiad Round 1); however, anyone may enter who wishes to do so. The top 100 scorers from BMO1 are invited to sit BMO2 — this is a strictly invitation-only paper. From that paper, 20 people are chosen to go to a selection event held at Trinity College, Cambridge. On the final morning of this event a selection test is taken, known as the FST (first selection test), lasting four and a half hours, which traditionally is a mock-IMO paper, containing three questions of IMO standard. This year's paper in fact contained four questions. Following the FST, a squad of nine was selected, and they underwent the dreaded correspondence course, in which a sheet of eight questions was sent out every eight to ten days, and to which as many solutions as possible were to be found and sent off. In the first week of June the squad convened at Oundle School during their half term for more training, and on the last morning of that another selection test (appropriately dubbed the NST, or next selection test) was taken. Following that, a final team of six was selected. During the first two weeks of July the team underwent further training, taking a four-and-a-half-hour practice exam every day for two weeks, in exact IMO style, firstly at the national mathematics summer school, and later at Trinity College, Cambridge, and then had a week off before the start of the IMO. The team was Tim Austin, Nathan Bowler, Tom Coker, Jenny Gardner, Paul Jefferys and Gavin Johnstone. Our leader was Dr Geoff Smith, a lecturer from the University of Bath, and the deputy leader was Richard Atkins, a maths teacher from Oundle School.

On Monday 22 July, we arrived in Glasgow. As we only had to travel within the UK we travelled in separate groups, those being nearer to Glasgow taking the train, whilst the rest of us flew up using EasyJet. We met our guide, Hannah, a UK team member from last year, and looked around Strathclyde University. All team leaders were in the process of selecting the questions for this year's paper in a secret location — in fact, the jury, as the group of leaders was called, had already been considering questions for the paper for two days before we arrived in Glasgow.

Tuesday was the day of the opening ceremony, so we had to dress up for it in our IMO uniforms. (Luckily we didn't have to wear them when we actually sat the exam!) We got some press attention as we happened to be the team from the host country, and later we even found that Jenny got into *The Times*! The opening ceremony itself was surprisingly good, being shorter than expected.

Wednesday was the first day of the exams — we were bussed to where the exams were to be held, and arrived only five minutes before the start of the exam, so we had little time to settle down and relax. However, the exam then proceeded smoothly. Afterwards, we all thought that we had done reasonably well. Paul, Jenny, Tim and Tom had solved questions 1 and 2, Gavin had solved questions 1 and 3 with a small hole in, and Nathan had solved question 1, and, although he had not had time to finish question 2, he believed that the final steps were clear.

Therefore, we felt quite upbeat, although we realized that the paper was a lot easier than last year's — indeed, asking around people from other countries, there was uniform agreement that the paper was substantially easier. We later found out that Jenny had sent a redundant question to the jury regarding question 1, asking 'Am I missing something, or are all the points red?' Geoff pointed out to the jury that the answer should be 'yes' (indeed, the answer to such a question is always 'yes', regardless of whether or not the 'points' are in fact red), but requested politely that he be allowed to reply 'Read the question', to which the jury happily acquiesced.

Thursday, the second day of exams, began with less hurry, as we arrived at the exam hall moderately early. On Thursday's paper, Paul and Tim solved questions 4 and 5, Nathan solved question 5, Jenny and Tom solved question 4, and Gavin did most of question 5 other than the non-obvious case (which I won't say in case you want to do the question yourself). So we were expecting three silvers, a bronze and (if you believed Richard regarding the cut-offs) two golds or (if you believed Geoff) two silvers. (The gold cut-off score was anywhere between 28 and 31, depending on whom one asked.) It was generally recognized that the jury screwed up on question 1, as it should not have been in the paper at all, being too easy for an IMO question, meaning that the coordinators for the problem had to introduce an awful mark scheme, to try to penalize anyone for small mistakes.

Following the paper, we went to the nearby science centre, then back to the campus. Afterwards, we went to the Barony Hall, in which the opening ceremony had been held, to learn some Scottish dancing at a 'Ceilidh'. It was very enjoyable, especially after the stress of the exams. After that, some of us went to play cards with the Australians and South Africans — meeting people from other countries is a big part of the IMO once the papers have finished and there is no need to sleep much!

On Friday we went on an excursion to Edinburgh. (There were several options regarding excursions, and we chose all those with the shortest journey times.) It was reasonably interesting; however, as we already live in the United Kingdom, there is nothing that we haven't seen a variant of before.

In the evening we received some bad news regarding our scores. Paul's expected score was now 27, Tim's 26, Tom and Jenny were both low 20s, and Nathan and Gavin were below the expected bronze borderline. This was due in some part to 'solutions' of ours that were not rigorous. However, in Tim's case, it was due to a failure to explicitly check that $1 = 1$, and as a result he had lost four marks. There was some stupid argument as to why this should happen, but I believe that losing four marks for what is a trivial error is totally against the IMO ethos — it was a one-mark slip, as his solution was still a solution, and so should be in the 6–7 mark bracket (all questions are marked out of 7). We only knew our scores for question 5 so far, as that was the only question which had been 'coordinated'.

On Saturday we went to Strathclyde Park, which was nearby, meaning that we did not have to spend a long time travelling, and could get up reasonably late. We found that the park filled a gap in the tourist industry, and after sampling a few rides ended up playing a game of Frisbee with the Australians. Getting back that evening, we discovered our scores for questions 3 and 6. It seemed that, if I could get 7 on questions 1, 2 and 4, my total would be 29, which was the latest estimate for the gold medal boundary. In theory that should have been feasible, as I had paid lip-service to all trivial complications, so that there wouldn't be any minor slips. We weren't sure what the lower medal boundaries were going to be, but we knew that they would be higher than in previous years, as the earlier questions on the papers, that is to say questions 1, 2, 4 and 5 were somewhat easier than they have been recently.

On Sunday I woke up and discovered our scores. I had 28, dropping a mark on question 2, Tim had 26, Tom had 22, Jenny had 21, Gavin had 12 and Nathan had 7. The smart money was on the medal boundaries to be 29, 23 and 15. However, not wanting to believe this, I went down to the area in which the scores were posted to investigate for myself. I found that there were 39 people with scores greater than mine, which meant that the gold boundary would almost certainly be 29. I also found that the silver boundary was 23. This was devastating, as in previous years the silver boundary had meandered around between 19 and 21. Thus, Tom and Jenny were both very unlucky to end up with high-level bronzes. We spent the afternoon on the paddle steamer Waverley, and, talking to Geoff, I discovered that I had had a gold medal for approximately two seconds! In the coordination sessions, normally the team leader and deputy leader from the country would agree on scores for the scripts with mathematicians from the host country. However, as we were the host country, we had had to agree scores with the countries that proposed the question, with the coordinators from the United Kingdom in the background to ensure that the marks were approximately correct. Question 2 had been proposed by South Korea, so Richard and Geoff had had to agree a mark with them. So, on question 2, there was a condition on the size of an angle, and I pointed out in my script that the angle-chasing subsection of the solution was valid because the angle is within this range, thus paying lip-service to that concept. Or so I thought. The Koreans agreed, and were happy to give me a 7. However, the UK coordinators decided that 6 was the only fair score, and refused to allow a 7. So I ended up with six marks, and one mark off a gold medal.

Monday was the day of the closing ceremony, which was in the afternoon. In the morning, we went to the area in which the exams were held to see a competition being held between schools with children of about 12–13 years competing in teams, rather than as individuals, called 'Enterprising Maths in the UK'. Tim was recorded for a BBC Choice comedy show, explaining his 'favourite theorem'. His talk was hilarious, and will be screened on the Ralph Little Show at some point.

In the afternoon we had the closing ceremony which was short, sweet and to the point. The gold medals were presented by Princess Anne, and we were able to meet her before the ceremony, as she met several of the teams. We were given our medals, and then went back to our accommodation to change out of suits before the closing party, which was great fun. Thus ended the IMO.

The official report on this year's IMO by the UK team leader is at http://www.srcf.ucam.org/∼jsm28/imo-register/. A complete listing of IMO problems can be found at http://www.kalva.demon.co.uk/. The IMO has evolved over the years, and whilst the earlier problems may be a little easier than those from later years, attempting to solve any IMO problem is a rewarding experience and one that I would strongly recommend.

The 2002 IMO problems

First day

Question 1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

Question 2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

Question 3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

Second day

Question 4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n\,.$$

Define $D = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$.

(a) Prove that $D < n^2$.
(b) Determine all n for which D is a divisor of n^2.

Question 5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\big(f(x)+f(z)\big)\big(f(y)+f(t)\big) = f(xy-zt)+f(xt+yz)$$

for all $x,\ y,\ z,\ t$ in $\mathbb{R}$.

Question 6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $\mathrm{O}_1, \mathrm{O}_2, \ldots, \mathrm{O}_n$ respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{\mathrm{O}_i\mathrm{O}_j} \leq \frac{(n-1)\pi}{4}.$$

__Paul Jefferys__ was a member of the UK IMO teams in 2001 and 2002 and hopes to continue to represent the UK in 2003 and 2004. He has also been offered a place in the UK international physics olympiad team and has represented the UK at the international informatics olympiad. He is 16 and is doing AS-levels at Berkhamsted Collegiate School.

Suppose Snow White Agreed to Take Part as Well

YUNPENG LI and PAUL BELCHER

In the British Mathematical Olympiad, Round 1, in 2000 there was the following interesting question:

> The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy and Grumpy as a trio, and one of Bashful and Sneezy as a pair. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
>
> Suppose Snow White agreed to take part as well. In how many ways could the four teams then be formed?

The most direct way to solve the original question, about arranging dwarfs into teams, is to first list all the possible categories and then count the number of possible arrangements within all those categories. For seven dwarfs in four teams there are three categories describing the numbers in each team:

Category 1	1	1	1	4
Category 2	1	1	2	3
Category 3	1	2	2	2

In category 1 there are ${}_7\mathrm{C}_4 = 35$ ways of arranging the dwarfs; in category 2 there are $({}_7\mathrm{C}_3)({}_{7-3}\mathrm{C}_2) = 210$ ways; and in category 3 there are $({}_7\mathrm{C}_2)({}_{7-2}\mathrm{C}_2)({}_{7-2-2}\mathrm{C}_2)/{}_3\mathrm{P}_3 = 105$ ways. So the total number of ways is $35+210+105 = 350$.

If Snow White took part as well, the problem could be answered in a similar way. Let us look at the problem in general, letting ${}_n\mathrm{T}_r$ denote the number of ways of arranging n people into r teams (where there is no order to teams nor to the people within the teams) for $n, r \in \mathbb{N}$. Considering an extra person (Snow White) we can obtain a recurrence relationship for the ${}_n\mathrm{T}_r$.

Theorem 1. *For any $n, r \in \mathbb{N}$ with $n, r \geq 2$,*

$${}_n\mathrm{T}_r = r\,{}_{n-1}\mathrm{T}_r + {}_{n-1}\mathrm{T}_{r-1}.$$

Proof. Consider the last person S to join a team. There are two possibilities: either (i) S joins one of r existing teams, which can be done in $r\,{}_{n-1}\mathrm{T}_r$ ways, or (ii) S creates her own team, which leaves the others in $r-1$ teams in ${}_{n-1}\mathrm{T}_{r-1}$ ways. Hence ${}_n\mathrm{T}_r = r\,{}_{n-1}\mathrm{T}_r + {}_{n-1}\mathrm{T}_{r-1}$.

In the original question, we have ${}_8\mathrm{T}_4 = 4({}_7\mathrm{T}_4) + {}_7\mathrm{T}_3 = 1701$ choices with Snow White in one of the four teams. For $r = 1$, all the people have to go in one team so ${}_n\mathrm{T}_1 = 1$. Similarly, ${}_n\mathrm{T}_n = 1$ and ${}_n\mathrm{T}_r = 0$ if $r > n$. Table 1 gives the first few ${}_n\mathrm{T}_r$ numbers.

A straightforward induction proof using theorem 1 shows that ${}_n\mathrm{T}_2 = 2^{n-1} - 1$ for $n \geq 2$, $n \in \mathbb{N}$. Alternatively, a direct argument can easily be given. From looking at the columns in table 1, we make the conjecture that

$$\lim_{n\to\infty} \frac{{}_n\mathrm{T}_r}{{}_{n-1}\mathrm{T}_r} = r \quad \forall r \in \mathbb{N}^+.$$

We first prove a lemma.

Lemma 1. *When $r \geq 2$,*

$$\lim_{n\to\infty} \frac{{}_{n-1}\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r} = 0.$$

Table 1. Values of ${}_n\mathrm{T}_r$ for small n, r.

	r								
n	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	3	1						
4	1	7	6	1		(Triangle of ${}_n\mathrm{T}_r$)			
5	1	15	25	10	1				
6	1	31	90	65	15	1			
7	1	63	301	350	140	21	1		
8	1	127	966	1701	1050	266	28	1	
9	1	255	3025	7770	6951	2646	462	36	1

Proof. A method of arranging n people into r teams would be to arrange $n-1$ people into $r-1$ teams first and then to put the nth person (call him Q) into the rth team on his own. There are ${}_{n-1}\mathrm{T}_{r-1}$ ways of doing this. There is at least one team containing the largest number of people, call this team A, with k people in it. In this situation, if the person Q (who is alone in his team) exchanges his position with one of the people in team A, then we obtain another different arrangement. Therefore ${}_n\mathrm{T}_r \geq (k+1)\,{}_{n-1}\mathrm{T}_{r-1}$. Now $k \geq (n-1)/(r-1)$, thus

$$\frac{{}_{n-1}\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r} \leq \frac{1}{k+1} < \frac{1}{k} \leq \frac{r-1}{n-1}\,.$$

Hence, for r fixed, letting $n \to \infty$ gives the result.

Theorem 2. *For all $r \in \mathbb{N}^+$,*

$$\lim_{n\to\infty} \frac{{}_n\mathrm{T}_r}{{}_{n-1}\mathrm{T}_r} = r\,.$$

Proof. When $r = 1$,

$$\frac{{}_n\mathrm{T}_1}{{}_{n-1}\mathrm{T}_1} = \frac{1}{1} = r\,.$$

When $r \geq 2$, from theorem 1 we have ${}_n\mathrm{T}_r = r\,{}_{n-1}\mathrm{T}_r + {}_{n-1}\mathrm{T}_{r-1}$ for $n \geq 2$. So

$$1 = \frac{{}_n\mathrm{T}_r}{{}_n\mathrm{T}_r} = \frac{r\,{}_{n-1}\mathrm{T}_r}{{}_n\mathrm{T}_r} + \frac{{}_{n-1}\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r}\,.$$

From lemma 1, letting $n \to \infty$ gives

$$1 = r \lim_{n\to\infty} \frac{{}_{n-1}\mathrm{T}_r}{{}_n\mathrm{T}_r}\,,$$

and the result follows.

Corollary 1. *For all $r \in \mathbb{N}$, $r \geq 2$,*

$$\lim_{n\to\infty} \frac{{}_n\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r} = 0\,.$$

Proof. Using theorem 1,

$$\frac{{}_{n+1}\mathrm{T}_r}{{}_n\mathrm{T}_r} = \frac{r\,{}_n\mathrm{T}_r + {}_n\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r} = r\,\frac{{}_n\mathrm{T}_r}{{}_n\mathrm{T}_r} + \frac{{}_n\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r}\,.$$

Letting $n \to \infty$,

$$r = r + \lim_{n\to\infty} \frac{{}_n\mathrm{T}_{r-1}}{{}_n\mathrm{T}_r}\,,$$

from which the result follows.

To try to find a non-recursive formula for ${}_n\mathrm{T}_r$, we consider the generating function

$$F_r(x) = {}_r\mathrm{T}_r + {}_{r+1}\mathrm{T}_r x + {}_{r+2}\mathrm{T}_r x^2 + {}_{r+3}\mathrm{T}_r x^3 + \cdots\,.$$

To find the radius of convergence of this power series, we apply the ratio test having first taken the absolute value of the terms:

$$\lim_{m\to\infty} \frac{{}_{r+m+1}\mathrm{T}_r |x|^{m+1}}{{}_{r+m}\mathrm{T}_r |x|^m} = r|x|$$

using theorem 2. Therefore the series converges for $|x| < 1/r$ and diverges for $|x| > 1/r$ and the radius of convergence is hence $1/r$. When $|x| = 1/r$, the ratio test fails, so we shall investigate this case separately. For $|x| = 1/r$,

$$\begin{aligned} |{}_{r+m+1}\mathrm{T}_r x^{m+1}| &= [r\,{}_{r+m}\mathrm{T}_r + {}_{r+m}\mathrm{T}_{r-1}]|x|^{m+1} \\ &= |{}_{r+m}\mathrm{T}_r x^m| + {}_{r+m}\mathrm{T}_{r-1}|x|^{m+1} \\ &\geq |{}_{r+m}\mathrm{T}_r x^m|\,. \end{aligned}$$

Hence ${}_{r+m}\mathrm{T}_r x^m$ cannot tend to zero as m tends to infinity, showing that $F_r(x)$ diverges when $|x| = 1/r$. Therefore, $F_r(x)$ converges for $|x| < 1/r$ and diverges for $|x| \geq 1/r$.

Theorem 3. *For $r \in \mathbb{N}^+$ and $|x| < 1/r$,*

$$\begin{aligned} F_r(x) &= \sum_{m=0}^{\infty} {}_{r+m}\mathrm{T}_r x^m \\ &= \frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-rx)}\,. \end{aligned}$$

Proof. We proceed by induction on r. For $r = 1$, since ${}_n\mathrm{T}_1 = 1$,

$$F_1(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}\,, \qquad |x| < 1\,.$$

We now assume the result for k and attempt to prove it for $k+1$. We have

$$\begin{aligned} F_{k+1}(x) &= {}_{k+1}\mathrm{T}_{k+1} + {}_{k+2}\mathrm{T}_{k+1}\,x + {}_{k+3}\mathrm{T}_{k+1}\,x^2 + \cdots \\ &= {}_k\mathrm{T}_k + [{}_{k+1}\mathrm{T}_k + (k+1){}_{k+1}\mathrm{T}_{k+1}]x \\ &\quad + [{}_{k+2}\mathrm{T}_k + (k+1){}_{k+2}\mathrm{T}_{k+1}]x^2 + \cdots \\ &= [{}_k\mathrm{T}_k + {}_{k+1}\mathrm{T}_k x + {}_{k+2}\mathrm{T}_k x^2 + \cdots] \\ &\quad + (k+1)x[{}_{k+1}\mathrm{T}_{k+1} + {}_{k+2}\mathrm{T}_{k+1}x + \cdots] \\ &= F_k(x) + (k+1)xF_{k+1}(x)\,. \end{aligned}$$

Hence $[1-(k+1)x]F_{k+1}(x) = F_k(x)$, giving

$$\begin{aligned} F_{k+1}(x) &= \frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-kx)} \\ &\quad \times \frac{1}{(1-(k+1)x)}\,, \qquad |x| < \frac{1}{k+1}\,. \end{aligned}$$

Therefore by the principle of mathematical induction the proof is complete.

By considering the Maclaurin series for $F_r(x)$,

$$F_r(x) = \sum_{m=0}^{\infty} \frac{F_r^{(m)}(0)x^m}{m!}\,,$$

we see that ${}_{r+m}\mathrm{T}_r = F_r^{(m)}(0)/m!$, so

$${}_n\mathrm{T}_r = \frac{F_r^{(n-r)}(0)}{(n-r)!}\,.$$

In order to differentiate $F_r(x)$ it is easier to first convert it into partial fractions. Let

$$\begin{aligned} &\frac{1}{(1-x)(1-2x)(1-3x)\cdots(1-rx)} \\ &\quad = \frac{a_1}{1-x} + \frac{a_2}{1-2x} + \frac{a_3}{1-3x} + \cdots + \frac{a_r}{1-rx}\,. \end{aligned}$$

Then

$$\begin{aligned} 1 &= a_1(1-2x)\cdots(1-rx) \\ &\quad + (1-x)a_2(1-3x)\cdots(1-rx) \\ &\quad + \cdots + (1-x)\cdots(1-(r-1)x)a_r\,. \end{aligned}$$

Putting $x = 1/p$ for $p \in \mathbb{N},\ 1 \le p \le r$, gives

$$1 = a_p\Big(1-\frac{1}{p}\Big)\ldots\Big(1-\frac{p-1}{p}\Big)\Big(1-\frac{p+1}{p}\Big)\ldots\Big(1-\frac{r}{p}\Big)\,.$$

Hence

$$\begin{aligned} a_p &= p^{r-1}\frac{1}{[(p-1)(p-2)\cdots 2\cdot 1]} \\ &\quad \times \frac{1}{[(r-p)(r-p-1)\cdots 2\cdot 1\cdot(-1)^{r-p}]} \\ &= \frac{(-1)^{r-p}p^{r-1}}{(p-1)!\,(r-p)!}\,. \end{aligned}$$

Thus

$$F_r(x) = \sum_{p=1}^{r} \frac{a_p}{1-px}\,,$$

where

$$a_p = \frac{(-1)^{r-p}p^{r-1}}{(p-1)!\,(r-p)!}\,.$$

If we let $g_p(x) = a_p/(1-px)$, then

$$\begin{aligned} g_p^{(m)}(x) &= \frac{m!\,p^m a_p}{(1-px)^{m+1}} \\ &= \frac{(-1)^{r-p}p^{m+r-1}m!}{(p-1)!\,(r-p)!\,(1-px)^{m+1}}\,. \end{aligned}$$

Therefore

$$F_r^{(m)}(0) = \sum_{p=1}^{r} g_p^{(m)}(0) = \sum_{p=1}^{r} \frac{(-1)^{r-p}p^{m+r-1}m!}{(p-1)!\,(r-p)!}\,,$$

and

$$\begin{aligned} {}_n\mathrm{T}_r &= \frac{1}{(n-r)!}\sum_{p=1}^{r} \frac{(-1)^{r-p}p^{(n-r)+r-1}(n-r)!}{(p-1)!\,(r-p)!} \\ &= \sum_{p=1}^{r} \frac{(-1)^{r-p}p^n}{p!\,(r-p)!}\,. \end{aligned}$$

So we have proved the following theorem, which gives a non-recursive formula for ${}_n\mathrm{T}_r$.

Theorem 4. *For $n, r \in \mathbb{N}^+$,*

$${}_n\mathrm{T}_r = \sum_{p=1}^{r} \frac{(-1)^{r-p}p^n}{p!\,(r-p)!}\,.$$

This seems to be the type of formula that could be expected from the inclusion–exclusion principle. Let us then attempt a direct proof in this way. Number the teams: team 1, team 2, …, team r. If we did not mind having empty teams, there would be r^n ways of arranging n people into r ordered teams, since everyone has r free choices. For arrangements with no empty teams, we need to subtract the number of arrangements in which at least one team is empty. Let A_i be the set of ordered arrangements in which team i is empty. Then we require to calculate $|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_r|$ which, by the inclusion–exclusion principle, equals

$$\begin{aligned} &\sum_{i=1}^{r}|A_i| - \sum_{1\le i_1<i_2\le r}|A_{i_1}\cap A_{i_2}| \\ &\quad + \sum_{1\le i_1<i_2<i_3\le r}|A_{i_1}\cap A_{i_2}\cap A_{i_3}| \\ &\quad - \cdots + (-1)^{r-1}|A_1\cap A_2\cap A_3\cap\cdots\cap A_r|\,. \end{aligned}$$

Now $|A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap\cdots\cap A_{i_k}| = (r-k)^n$ since each person has $r-k$ choices. Therefore

$$\sum_{1\le i_1<i_2<i_3<\cdots<i_k\le r} |A_{i_1}\cap A_{i_2}\cap A_{i_3}\cap\cdots\cap A_{i_k}| = {}_r\mathrm{C}_k(r-k)^n\,.$$

Hence

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_r| = \sum_{k=1}^{r} (-1)^{k-1} {}_r\mathrm{C}_k (r-k)^n = \sum_{k=1}^{r-1} (-1)^{k-1} {}_r\mathrm{C}_k (r-k)^n$$

since $r - k = 0$ when $k = r$. Thus, the number of ways of arranging n people into r ordered teams is

$$r^n - \sum_{k=1}^{r-1} (-1)^{k-1} {}_r\mathrm{C}_k (r-k)^n = \sum_{k=0}^{r-1} (-1)^{k} {}_r\mathrm{C}_k (r-k)^n .$$

By letting $p = r - k$, this becomes

$$\sum_{p=1}^{r} (-1)^{r-p} {}_r\mathrm{C}_p p^n .$$

Now, for ${}_n\mathrm{T}_r$ the teams are not ordered, hence

$${}_n\mathrm{T}_r = \frac{\sum_{p=1}^{r} (-1)^{r-p} ({}_r\mathrm{C}_p) p^n}{r!} = \sum_{p=1}^{r} \frac{(-1)^{r-p} p^n}{p!\,(r-p)!} ,$$

agreeing with the earlier derivation.

The ${}_n\mathrm{T}_r$ numbers investigated here are called the Stirling set numbers after the Scottish mathematician James Stirling (1692–1770), best known for the approximation

$$n! \cong \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}}\right)^n .$$

An application of the ${}_n\mathrm{T}_r$ numbers is as follows. Let X be a set with n elements and Y be a set with r elements. Then ${}_n\mathrm{T}_r$ will be the number of surjections from set X to set Y.

References

1. J. Matousek and J. Nesetril, *Invitation to Discrete Mathematics* (Oxford University Press, 1998).

Yungpeng Li *is a 19-year-old student at Atlantic College who competed in the British Mathematical Olympiad in 2000 and 20001. He investigated this problem for his Extended Essay in the International Baccalaureate.*

Paul Belcher *is Head of Mathematics at Atlantic College and was Yunpeng's supervisor for his Extended Essay.*

Going Round in Circles

DEREK COLLINS

You've probably been to a theme park like Alton Towers, where there are lots of exciting rides to try. When you were there, did you realise how much mathematics and how much mechanics there are in these rides? Quite a few of them involve the rider going round in a circle. Mechanics tells us that for a particle of mass m to describe a circle of radius r at a constant angular speed ω there must be a force of magnitude $m\omega^2 r$, directed towards the centre O of the circle (figure 1). There must be some external agency, be it string or rod, which acts on the particle to provide this force.

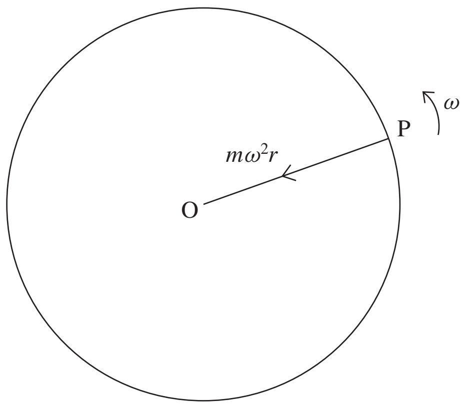


Figure 1.

The Rotor Ride is one in which the rider stands flat against the inside of a circular wall, feet resting on the floor of the ride. The wall and floor begin to rotate and, when they are rotating fast enough, the floor drops but the rider remains stationary against the wall. To model this we replace the rider by a ball. We can justify this since mechanics tells us the weight of the rider behaves as if it is all concentrated at the centre of gravity.

As the wall rotates, the ball B moves in a horizontal circle whose radius r is the wall's radius and whose centre O is where the horizontal through the ball meets the axis of rotation (figure 2). When the floor has descended, the wall rotates with angular speed ω and the forces on the ball are its weight mg vertically downwards together with the contact force of the wall on the ball. The component F upwards of

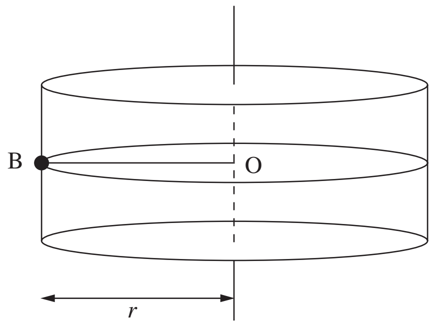


Figure 2.

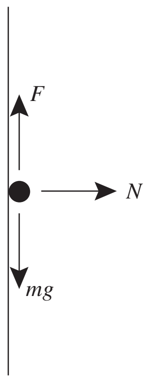


Figure 3.

this latter must balance the weight of the ball if the ball is not to drop, whilst the component N horizontally must provide the force towards the centre (figure 3), so that

$$F = mg\,, \qquad N = m\omega^2 r\,.$$

The vertical component is, however, the frictional force between the wall and the ball. A common assumption based on experiment is that the frictional force is less than or equal to a constant μ, the coefficient of friction, multiplied by the component of the contact force at right angles to the wall. For the real ride the coefficient of friction is that between the wall and the rider's clothes. Thus

$$F \le \mu N\,.$$

Hence, $mg \le \mu m\omega^2 r$, or, cancelling m which is positive,

$$g \le \mu\omega^2 r\,,$$

or

$$\omega^2 \ge \frac{g}{\mu r}\,.$$

This says that the rider remains stationary on the wall only if the wall is rotating at an angular speed no less than $(g/\mu r)^{1/2}$. In practice, of course, for safety reasons the floor is not dropped until the angular speed is well above this. In this ride the wall can be turning 30 times a minute, so the angular speed ω is $30 \times 2\pi$ radians per minute or π radians per second. A reasonable value for N is $2.6\,mg$, which gives $\mu \ge 0.38$, so that the coefficient of friction must be at least 0.38 which is also reasonable for friction between the wall and the rider's clothes.

Another ride based on circular motion is the Wave or Ug Swinger. In this the riders are in chairs at the lower ends of long bars, the upper ends of which are attached to a rotating circular disc (figure 4). In some rides the disc goes up and down or tilts, but we'll leave this out for simplicity. When the ride starts up, the chairs begin to swing outwards and, as the ride speeds up, so the riders swing out further and further. However, any rider swings out at the same angle to the vertical as any other rider. Can mechanics explain this?

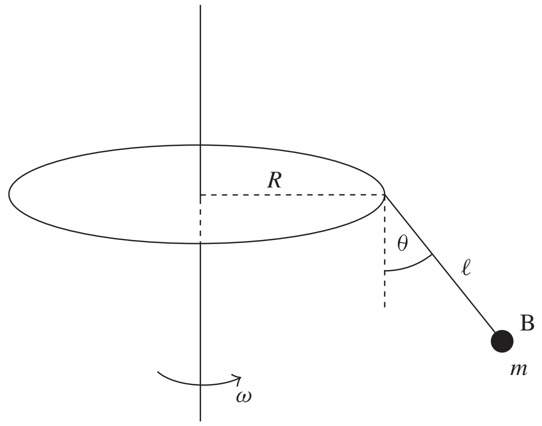


Figure 4.

Again it is reasonable to model a rider as a particle. What quantities are likely to be important in determining what is happening as the riders swing out? Five suggest themselves:

(i) the mass m of the particle/rider,
(ii) the length ℓ of the bar,
(iii) the radius R of the disc,
(iv) the angle θ that the rider's bar makes with the vertical,
(v) the angular speed ω.

Are they all equally important? Suppose that we ignore R, that is, make it zero. Then the ride becomes what is known in mechanics as a conical pendulum. The particle moves in a horizontal circle, radius r, centre at C where the horizontal plane through the particle meets the vertical axis about which the disc is rotating (figure 5).

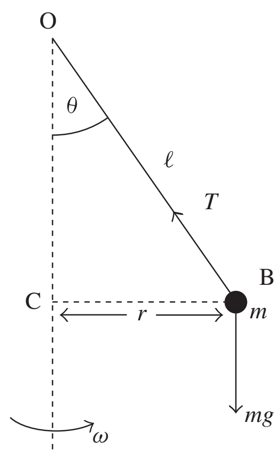


Figure 5.

There are two forces on the particle, its weight mg acting vertically downwards and the tension T in the bar. The vertical component $T\cos\theta$ of the tension must balance the weight, whilst the horizontal component $T\sin\theta$ must provide the force $m\omega^2 r$ towards the centre for the particle to move in a circle. Hence

$$T\cos\theta = mg\,,$$
$$T\sin\theta = m\omega^2 r\,.$$

Some simple trigonometry in the triangle OCB gives $r = \ell\sin\theta$, so

$$T\sin\theta = m\ell\omega^2\sin\theta\,. \qquad (1)$$

The solutions of this equation are $\sin\theta = 0$, which gives $\theta = 0$, and $T = m\ell\omega^2$. For the rider moving outwards, θ is not zero, so $T = m\ell\omega^2$ and hence

$$m\ell\omega^2\cos\theta = mg\,,$$

or, cancelling m,

$$\cos\theta = \frac{g}{\ell\omega^2}\,.$$

This tells us two things. Firstly, $\cos\theta$ decreases as ω increases, and, secondly, θ is independent of m. Thus, all the riders and even empty chairs swing out in the same way and do so further and further as ω increases, which is just what is observed. The equation tells us something else, though. Since $\cos\theta \le 1$,

$$\frac{g}{\ell\omega^2} \le 1 \implies \omega^2 \ge \frac{g}{\ell}\,.$$

This says that the rider only swings out when ω is greater than or equal to $(g/\ell)^{1/2}$. What happens when ω is less than $(g/\ell)^{1/2}$? This is where the other solution of (1) comes in. It says that $\theta = 0$, so that the bar to the rider remains vertical until ω reaches the value $(g/\ell)^{1/2}$ and the rider then swings out.

This is not what happens with the ride though. The riders start to swing out as soon as the ride starts up. Our model of the ride has broken down in this respect, even though it describes perfectly well what happens once the riders do swing out. When a model like this breaks down, we need to ask if we have ignored something which is actually important. One quantity we have left out is the radius R of the disc. What happens if we include it?

The forces on the particle are still the tension T and the weight mg, and just as before

$$T\cos\theta = mg\,,$$
$$T\sin\theta = m\omega^2 r\,.$$

The difference now is that the radius r of the rider's circle is

$$r = R + \ell\sin\theta$$

(see figure 6), so

$$T\sin\theta = m\omega^2(R + \ell\sin\theta)\,.$$

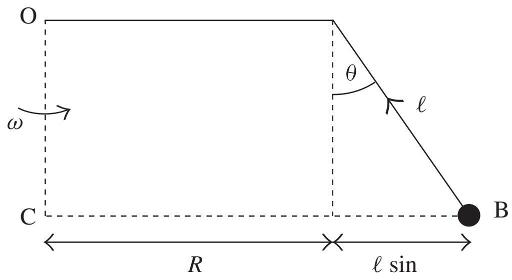


Figure 6.

Now $\sin\theta = 0$ is not a solution and eliminating T gives

$$\omega^2 = \frac{g\tan\theta}{R + \ell\sin\theta}\,,$$

so again θ is independent of m.

Some graphs of $\omega\sqrt{\ell/g}$ against θ for different σ, where $\sigma = R/\ell$, are plotted in figure 7. These tell us that again θ increases as ω increases but does so as soon as the ride starts up, that is, the chairs begin to swing out immediately on starting, provided that R is not zero. Our new model then satisfactorily describes all three features of the ride that we have observed.

The Rotor Ride and the Wave Swinger involve motion in only one circle. Several rides, however, for example, the Spider, Octopus and Turbostar Rides, involve motion in two circles in that they have a set of arms which rotate around a central hub. At the end of each of these arms is another set of arms which also rotate, although at a different rate and often in the opposite direction. The riders are at the free ends of this second set of arms. The designer of one of these rides can choose the rates at which the two sets of arms rotate as well as the lengths of these arms. How should they be chosen to give an exciting ride?

To look at this we can investigate the path of a rider and the speed he or she reaches. A simple version of the ride is a long arm and a short arm with the rider at the end of the short arm. We will let the arms rotate in a horizontal plane, so we are leaving out the up and down motion of some of these rides, but this simple model does have the main features of the ride.

We will use cartesian geometry to find coordinates of the rider. We take the x-axis to be along a position of the arms OB and BC when they are in a straight line with C furthest from O, the y-axis at right angles to this and take time $t = 0$ when the arms are in this position (figure 8). The arms OB and BC have lengths R_1 and R_2 with R_1 greater than R_2. At some subsequent time t, OB makes an angle A_1 with the x-axis and BC an angle A_2, the angles BOM and CBQ (figure 9), where BM and CN are perpendicular to the x-axis OMN and BQ is perpendicular to CN. Then the coordinates of the rider at C are

$$\begin{aligned} x &= \mathrm{OM} + \mathrm{MN} = \mathrm{OM} + \mathrm{BQ} \\ &= R_1\cos A_1 + R_2\cos A_2\,, \\ y &= \mathrm{CQ} + \mathrm{QN} = \mathrm{CQ} + \mathrm{BM} \\ &= R_1\sin A_1 + R_2\sin A_2\,. \end{aligned}$$

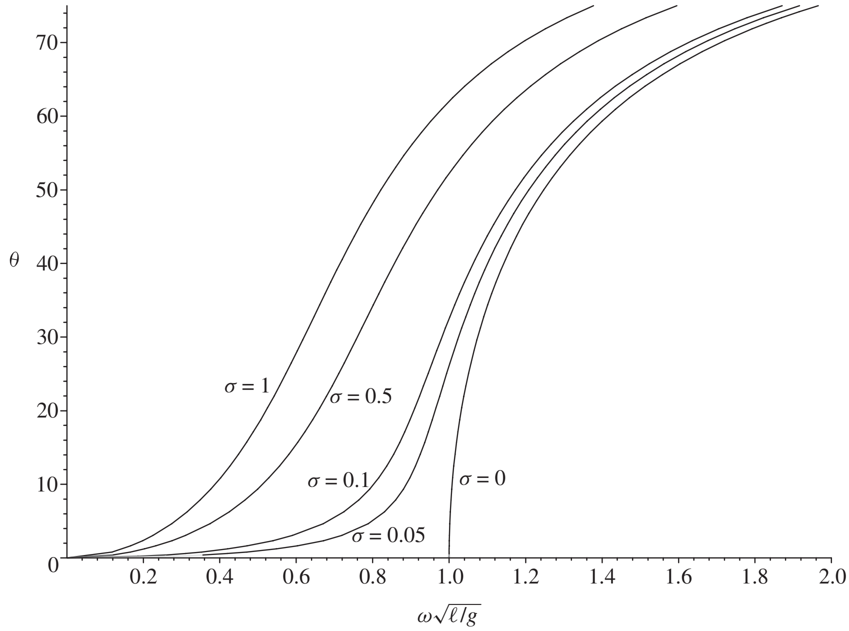


Figure 7.

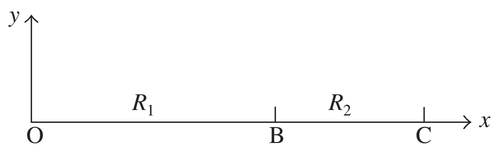


Figure 8. The arms OB and BC at rest.

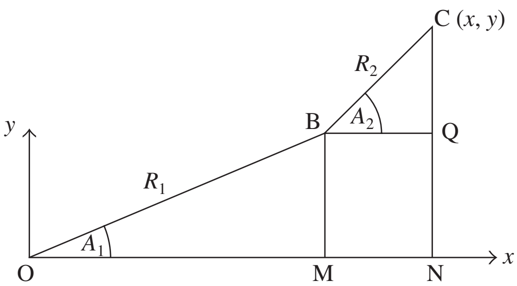


Figure 9. The arms OB and BC extended.

Suppose now that the long inner arm rotates with an angular speed Ω_1 anticlockwise. Then $A_1 = \Omega_1 t$. The inner arm rotates either anticlockwise, in which case $A_2 = \Omega_2 t$, or clockwise, in which case $A_2 = -\Omega_2 t$. Then

$$x = R_1 \cos \Omega_1 t + R_2 \cos \Omega_2 t\,,$$
$$y = R_1 \sin \Omega_1 t \pm R_2 \sin \Omega_2 t\,,$$

where in the expression for y the $+$ sign on the second term denotes the anticlockwise direction and the $-$ sign the clockwise one. The distance R of the rider from the centre O of the ride is given by

$$R^2 = x^2 + y^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos(\Omega_1 \mp \Omega_2)t\,,$$

where we have used the trigonometric addition formulae for $\cos(\theta \pm \phi)$.

The rider is furthest from the centre when R is greatest, which is when $(\Omega_1 \mp \Omega_2)t$ is 0 or 360°, so that

$$R^2 = R_1^2 + R_2^2 + 2R_1R_2\,,$$

that is, $R = R_1 + R_2$. Not surprisingly, the arms are then aligned, one beyond the other. The rider is nearest the centre when R is a minimum, which is when $(\Omega_1 \mp \Omega_2)t$ is 180°, so that

$$R^2 = R_1^2 + R_2^2 - 2R_1R_2\,,$$

that is, $R = R_1 - R_2$, so that the short arm lies alongside the long arm.

To find out when the speeds are greatest and least, we differentiate x and y with respect to t to give

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\Omega_1 R_1 \sin \Omega_1 t - \Omega_2 R_2 \sin \Omega_2 t\,,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Omega_1 R_1 \cos \Omega_1 t \pm \Omega_2 R_2 \cos \Omega_2 t\,.$$

The speed V is then given by

$$V^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \begin{cases} \Omega_1^2R_1^2 + \Omega_2^2R_2^2 + 2\Omega_1\Omega_2R_1R_2\cos(\Omega_1 - \Omega_2)t & \text{for BC anticlockwise} \\ \Omega_1^2R_1^2 + \Omega_2^2R_2^2 - 2\Omega_1\Omega_2R_1R_2\cos(\Omega_1 + \Omega_2)t & \text{for BC clockwise.} \end{cases}$$

For the short arm rotating anticlockwise, the maximum speed is when $(\Omega_1 - \Omega_2)t = 0,\ 360^\circ$, and is

$$V = \Omega_1 R_1 + \Omega_2 R_2\,,$$

and the minimum speed is when $(\Omega_1 + \Omega_2)t = 180^\circ$ and is

$$V = |\Omega_1 R_1 - \Omega_2 R_2|\,.$$

This means that, when both arms rotate in the same direction, the speed is a maximum when R is at its greatest, that is, when the rider is furthest from the centre, and a minimum when R is at its least, that is, when the rider is nearest the centre. When the arms rotate in opposite directions, the speed is a maximum when R is at its least, that is, when the rider is nearest the centre, and a minimum when R is at its greatest, that is, furthest from the centre. Since most rides have arms rotating in opposite directions, this suggests that designers of these rides think the most exciting ride is when you are travelling fastest nearest the centre, when your speed increases as you come crashing towards the centre and then decreases when you begin to pull out. Do you agree with them?

If the excitement of a ride depends on coming close to the centre with maximum speed, then it must also depend on how many times the riders come close to the centre in one revolution, say.

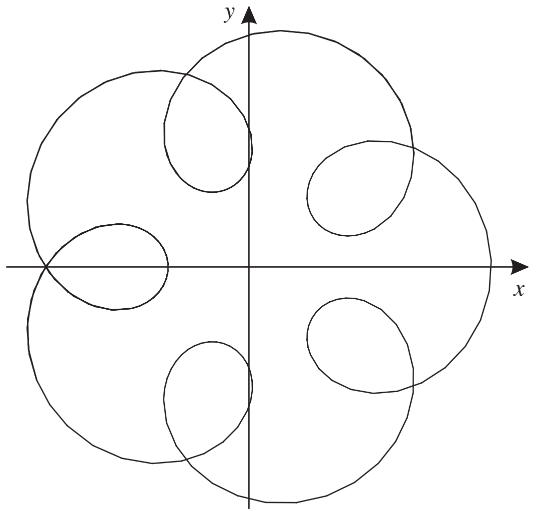


Figure 11. Arms in same direction.

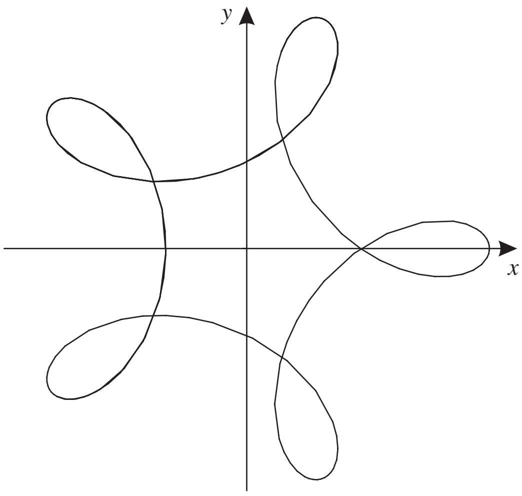


Figure 10. Arms in opposite directions.

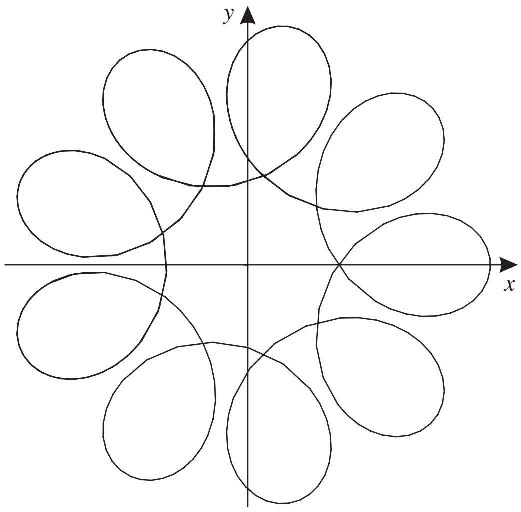


Figure 12. Arms going in opposite directions.

The long arm OB describes one revolution, that is, 360°, in time T, where $\Omega_1 T = 360$. Suppose we want to design the ride so that the riders approach the centre n times. This means that, as the long arm describes one revolution, the short one must describe n revolutions, that is, 360°. For it to do this in time T, we must have $\Omega_2 T = 360\,n$. But this gives

$$\Omega_2 T = n\Omega_1 T\,,$$

so that $\Omega_2 = n\Omega_1$. In other words, the short arm must rotate n times as fast as the big arm. In one of the Alton Towers rides $n = 5$, so $\Omega_2 = 5\Omega_1$. It is not too difficult to write a computer program which plots out the path of a rider. Figures 10 and 11 show this path for rotation of the small arm in the opposite and in the same direction as the long arm respectively. Here $R_1 = 6$, $R_2 = 3$, $\Omega_1 = 15$, $\Omega_2 = \mp 75$.

By increasing n we can increase the number of loops, as shown in figure 12 for $n = 9$ with the arms going in opposite directions. Here $R_1 = 6$, $R_2 = 3$, $\Omega_1 = 15$, $\Omega_2 = -135$. Ride designers do not seem to take n as large as this, perhaps for technical reasons to do with the stresses on the ride's framework or perhaps because there are limits to how much excitement the riders can take!

Next time you go to a theme park why not pick a ride and see how much mathematics and mechanics you can find in it.

Derek Collins *is Emeritus Professor of Applied Mathematics at the University of Sheffield. His research interests include non-linear oscillations of electric motors and contact problems for solid bodies. He is a member of the Sheffield Mechanics in Action team and a co-author of* Mechanics *in the AEB Mathematics for AS- and A-level series.*

Oscillations Produced by a Cam

DAVID BURLEY

1. Introduction

Many mechanical devices from the first steam engine, the Newcomen engine, to modern sixteen-valve car engines involve the opening and closing of valves. The opening and closing has to be done with great precision since the appropriate machine operation has to start and stop at exactly the right moment. What is the best method of controlling these actions? One of the traditional methods is with a cam which consists of a metal disc, of roughly elliptical shape, that rotates about an axis. The cam lifts the valve to an open position and then the valve returns to its closed position by gravity or spring action. A schematic diagram shows the rotating cam lifting the valve, which is represented by a plate in figure 1.

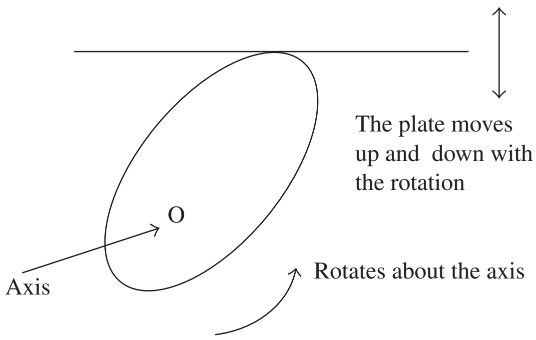


Figure 1. A schematic diagram of the action of a cam.

Given the shape of the cam, what is the motion of the plate? Alternatively, if a specific movement of the plate is required, what is the shape of the cam that will produce the movement? It is assumed that the cam rotates with a constant angular velocity and that the cam is smooth and convex. Corners and indentations in the shape of the cam would not produce a movement of the plate that is practical.

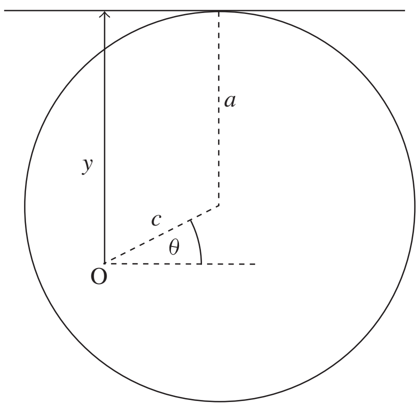


Figure 2. A circular cam rotating about the axis O.

2. A simple case

The simplest curve that can be used to produce the cam effect is a circle with the axis of rotation off-centre. The height of the plate, y, above the fixed axis of rotation, O, can be calculated by some simple trigonometry (see figure 2) to be

$$y = a + c\sin\theta\,.$$

Thus the cam produces simple harmonic motion. For the axis to be inside the circle we require $c < a$. It is sensible to use non-dimensional distances by putting $Z = y/a$ and $C = c/a$ so that

$$Z = 1 + C\sin\theta \quad \text{with } C < 1\,,$$

and the graph is illustrated in figure 3.

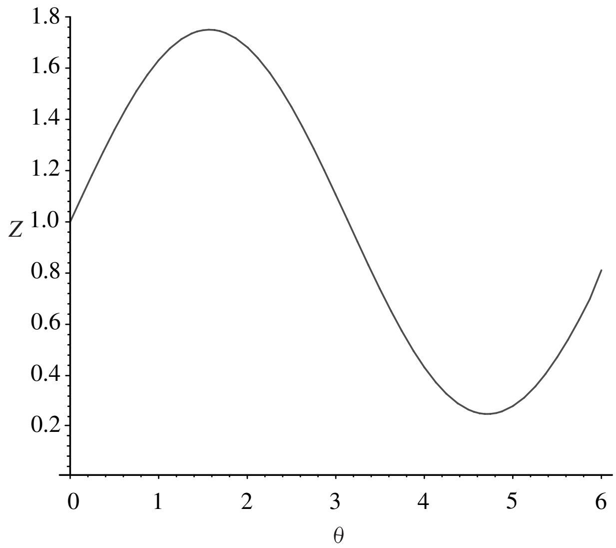


Figure 3. The displacement, Z, of the plate for a cam of circular shape with the axis of rotation off-centre, $C = 0.75$.

3. An elliptic cam

Replacing the circular, off-centred cam with an ellipse rules out a simple trigonometric solution and a more formal approach is required. It is convenient to represent the curve in parametric form as

$$X = f(t) \quad \text{and} \quad Y = g(t)\,,$$

and to proceed generally at this stage; the exact representation of the ellipse will be inserted later.

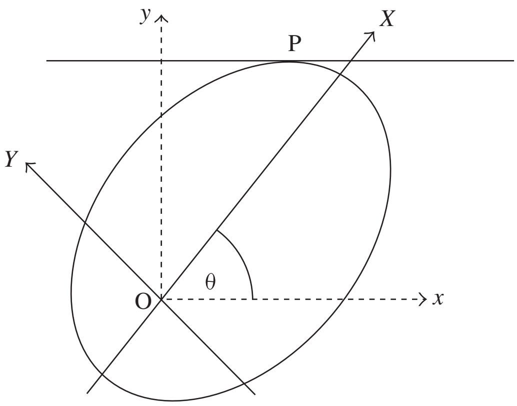


Figure 4. An elliptical cam rotating about the origin, O.

The coordinates of the two sets of axes shown in figure 4 are related by

$$x = X\cos\theta - Y\sin\theta$$

and

$$y = X\sin\theta + Y\cos\theta\,,$$

the relations being obtained by standard trigonometry. To find the point P for a fixed θ requires that the curve has zero gradient. Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{\dot{f}\sin\theta + \dot{g}\cos\theta}{\dot{f}\cos\theta - \dot{g}\sin\theta} = 0\,,$$

and hence t must be obtained from the solution of

$$\tan\theta = -\frac{\dot{g}}{\dot{f}} \qquad (1)$$

and the height of the plate is then calculated from

$$y = f(t)\sin\theta + g(t)\cos\theta\,. \qquad (2)$$

Now apply the result to the ellipse with centre at $(c, 0)$ in the X, Y coordinate system

$$\frac{(X-c)^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad \text{with } c < a\,.$$

The parametric form is

$$X = f(t) = c + a\sin t\,,$$
$$Y = g(t) = b\cos t\,,$$

and hence (1) gives

$$\tan t = \frac{a}{b}\tan\theta\,.$$

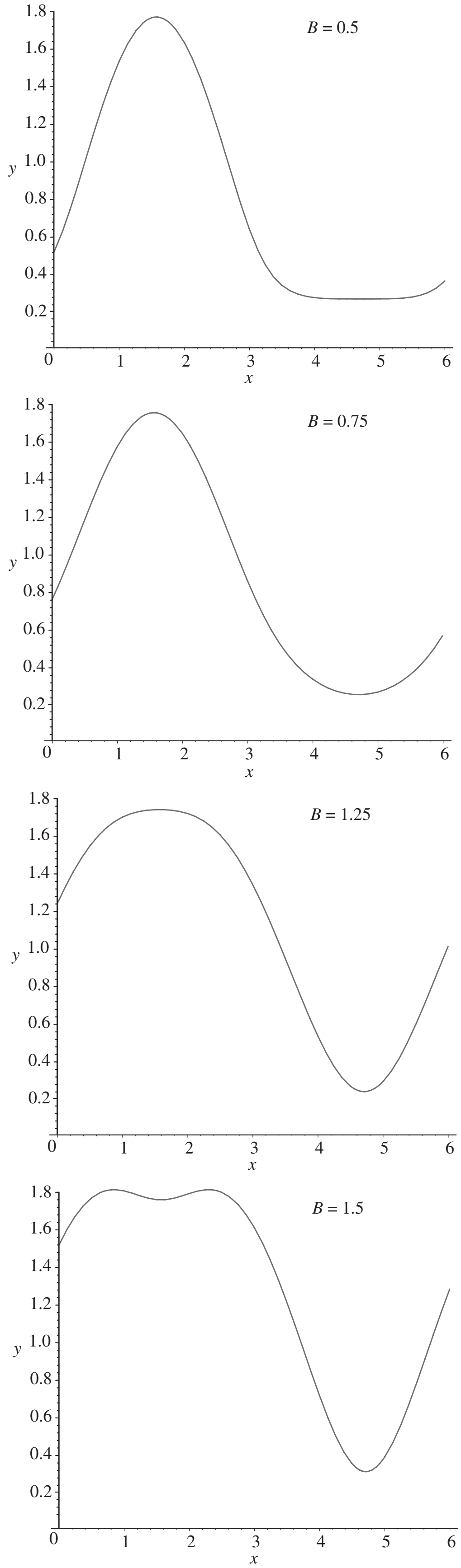


Figure 5. Displacement, Z, of the plate for a cam of elliptical shape, $C = 0.75$, and for various B.

Using standard trigonometric identities, it is easily shown that

$$\cos t = \frac{b\cos\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}$$

and

$$\sin t = \frac{a\sin\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}\,.$$

The form of y is now obtained from (2) as

$$y = c\sin\theta + (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}\,.$$

When $a = b$, the ellipse becomes a circle and the result agrees with the previous section. Again it is sensible to use non-dimensional distances in the same manner as section 2 by putting

$$Z = \frac{y}{a}, \qquad C = \frac{c}{a} \quad \text{and} \quad B = \frac{b}{a}.$$

The height then becomes

$$Z = C\sin\theta + (\sin^2\theta + B^2\cos^2\theta)^{1/2}\,.$$

The graphs of the function are plotted in figure 5 for $C = 0.75$ and various values of B. The case $B = 1$ corresponds to the circular cam and gives the simple harmonic motion shown in figure 3.

The graph shows that for B close to 1, that is when the cam is close to the circular shape, the behaviour of the plate is very similar to that produced by the circular cam in figure 3. The peaks are a little more flattened and the drops are a little steeper. However, for large and small values of B, for instance 0.5 and 1.5 in figure 5, there are additional bumps or flattening at either the peak or the trough. The ellipses in these cases are longer and thinner, more like a cigar, and it is interesting to look at the geometry of such an ellipse and try to explain why these bumps occur. From a practical point of view, elliptical cams, and presumably cams with more complicated shapes, do not always give an expected rise and fall without bumps or other curious behaviour. Clearly, any old shape will not do and some serious design has to be undertaken to produce the exact effects required by any piece of machinery.

4. More complicated shapes

In the previous section, the procedure for a cam of general shape was derived. The parametric equation of the cam shape is required but it then appears unlikely that (1) can be solved for t as a function of θ. As an example, the Wankel engine requires a near-triangular shaped rotor that lifts valves with the appropriate timing. A curve that has roughly the right shape and a comparatively simple representation is

$$X = 1 + 2.5\cos t + 0.6\cos 2t\,,$$
$$Y = 2\sin t\,.$$

It is illustrated in figure 6 with the cam rotating about the origin.

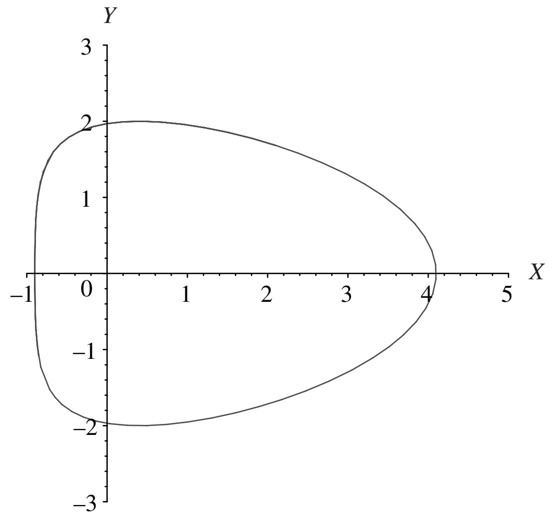


Figure 6. A cam with a roughly triangular shape.

It is now straightforward to follow the ideas of the previous section to obtain a displacement curve for the plate. Equation (1) now becomes

$$\tan\theta = \frac{2\cos t}{2.5\sin t + 1.2\sin 2t}\,.$$

It is not possible to obtain an exact solution as in the elliptical case, but a numerical solution is straightforward. Given t, the value of θ can be calculated easily, although care must be taken to choose the correct branch of the inverse tangent, and then the displacement of the plate can be computed from

$$y = (1 + 2.5\cos t + 0.6\cos 2t)\sin\theta + 2\sin t\cos\theta\,.$$

Most computer packages will then plot y against θ as shown in figure 7. Note the very sharp peak and the flat trough corresponding to the sharp right-hand end of the cam and the flat left-hand end respectively.

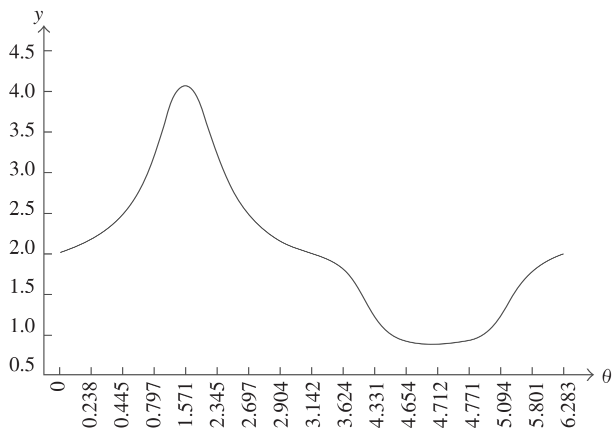


Figure 7. Plate displacement for the cam illustrated in figure 6.

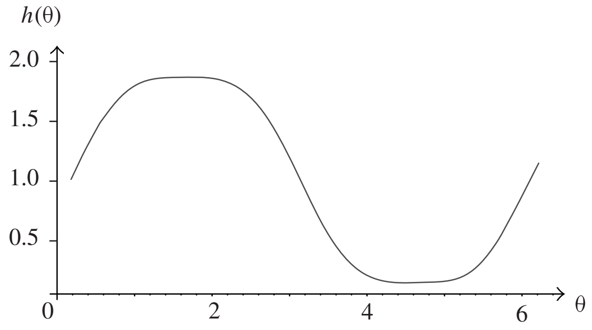


Figure 8. A graph of the given plate displacement.

5. Prescribing the valve movement

The question now posed is, given a prescribed movement for the valve or plate, can an appropriate cam be constructed? Return to the basic equations (1) and (2) with $y = h(\theta)$ as a given function and the functions $f(t)$ and $g(t)$ unknown. Equation (2) is now written as

$$h(\theta) = f(t)\sin\theta + g(t)\cos\theta\,,$$

and (1) is rearranged to give

$$0 = \dot{f}(t)\sin\theta + \dot{g}(t)\cos\theta\,.$$

Differentiating the first of these equations with respect to θ gives

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} = [\dot{f}(t)\sin\theta + \dot{g}(t)\cos\theta]\frac{\mathrm{d}t}{\mathrm{d}\theta} + f(t)\cos\theta - g(t)\sin\theta\,.$$

The first term is zero so $f(t)$ and $g(t)$ can be calculated from

$$h(\theta) = f(t)\sin\theta + g(t)\cos\theta\,,$$

$$\frac{\mathrm{d}h}{\mathrm{d}\theta} = f(t)\cos\theta - g(t)\sin\theta\,,$$

giving

$$f = \frac{\mathrm{d}h}{\mathrm{d}\theta}\cos\theta + h\sin\theta\,,$$
$$g = -\frac{\mathrm{d}h}{\mathrm{d}\theta}\sin\theta + h\cos\theta\,.$$

The equations give f and g in parametric form with parameter θ. It can be checked that

$$h(\theta) = a + c\sin\theta$$

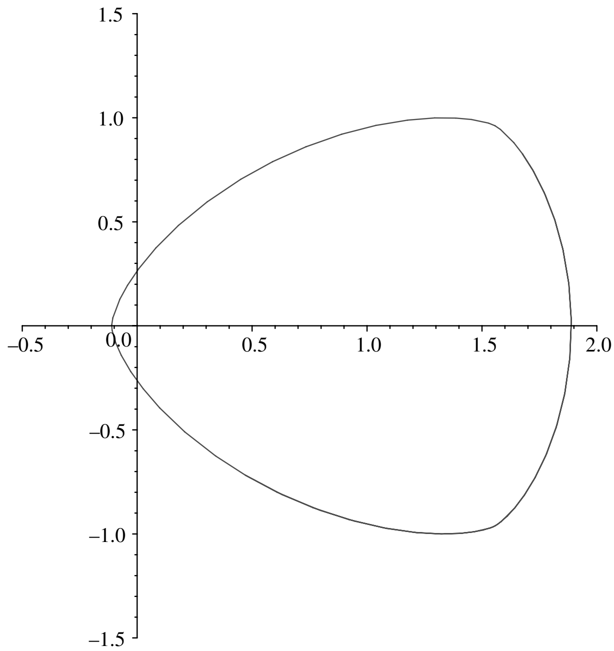


Figure 9. The cam shape required to produce the displacement of figure 8.

and

$$h(\theta) = c\sin\theta + (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2},$$

deduced in sections 2 and 3, recover the circle and the ellipse respectively.

For a more complicated situation, the plate movement

$$h(\theta) = 1 + \sin\theta + \tfrac{1}{9}\sin 3\theta\,,$$

is plotted in figure 8. The resulting cam shape can be derived as

$$f = 1 + \sin\theta + \tfrac{1}{3}\cos\theta\cos 3\theta + \tfrac{1}{9}\sin\theta\sin 3\theta\,,$$
$$g = \cos\theta - \tfrac{1}{3}\sin\theta\cos 3\theta + \tfrac{1}{9}\cos\theta\sin 3\theta\,,$$

as illustrated in figure 9. Although the method that has been described works in the cases given, it can produce very curious cam shapes. There is nothing in the method that guarantees that the cam will be convex. More complicated cases produce cams with spikes on them and shapes that fold in on themselves. There is clearly a lot of work to be done to sort out a foolproof approach that produces either a correct cam or an error message saying that it is not possible — at least using the present approach.

David Burley *graduated in mathematics from King's College, University of London. He spent most of his working life at the University of Sheffield, latterly as Head of the Department of Applied Mathematics. He has now retired. His interests include applying mathematics to engineering problems and a particular interest is the flow of molten glass in industrial processes.*

Mathematics in the Classroom

Birthdays, other goals, randomness and the exponential distribution

Birthdays

Birthday data are always interesting and usually surprising to students. Whenever I ask them to guess how many people you need to have in order that the probability that at least two share a birthday exceeds 0.5, I invariably get an answer which exceeds 100. It is left to the reader to demonstrate that it only takes 23 people to achieve this. As my teaching sets are frequently at least this size, we invariably give it a try, and if it does not contain at least two with the same birthday, then enlisting the help of the set in the room next door, we always have a success with the number of students involved being well below 100.

This year's teaching set was no exception. Their birthdays are shown in table 1.

Table 1.

Month	Day
January	16
February	25
March	1, 21
April	3, 9
May	22
June	22
July	19, 29, 29, 30
August	1, 8, 12, 14, 31
September	4, 21
October	15, 27
November	2, 21
December	17

So we do have two students who share a birthday. These data can be transferred to table 2 showing the distribution of the number of birthdays per month.

Table 2.

Number of birthdays per month	Number of months
0	0
1	5
2	5
3	0
4	1
5	1

But are these birthdays really random? Or, put another way, do these data follow a Poisson distribution? Clearly the mean is 2. The reader is left to show that fitting a Poisson model

$$P(X = x) = \frac{e^{-2}2^x}{x!}, \qquad x = 0, 1, \ldots, 5,$$

and testing for goodness of fit using a χ^2 test leaves our hypothesis of randomness in some doubt at a 5% significance level although acceptable at a 1% level. But our sample size is small which may well account for this uncertainty.

Michael Bedwell (reference 1) suggests that such a data set could also be used for demonstrating the exponential distribution. To do this we need to return to the data in table 1 and express it in inter-birthday intervals, i.e. the number of days between the consecutive birthdays. This gives rise to the distribution in table 3.

Table 3.

Number of days between birthdays	Frequency
0–9	10
10–19	6
20–29	3
30–39	2
40–49	2
50–59	0

Using the original rather than the grouped data, the mean of this distribution is 19.0, so in fitting an exponential distribution we should use $\lambda = \frac{1}{19}$ in the model

$$P(X \leq x) = 1 - e^{-\lambda x}.$$

The reader is left to show that these data are well modelled by an exponential distribution.

Goals

Football goals usually work even better than birthdays in demonstrating the Poisson and exponential distributions. Taking the data for the FA Premiership and the Nationwide Division 1 for Saturday 26 October 2002, we have the distribution for the number of goals scored per team given in table 4.

Table 4.

Number of goals per team	Frequency
0	11
1	12
2	10
3	2
4	2
5	1

This distribution has a mean of 1.34 goals and a variance of 1.58 goals^2 (the closeness of these two values giving support to the hypothesis that these data can be modelled by a Poisson distribution). The reader is left to fit the appropriate Poisson distribution and test the goodness of fit ($\chi^2 = 0.48$, 2 d.f. giving a good fit).

If we now consider the times between the goals being scored (reported in most newspapers to the nearest minute), then we have the distribution for that particular day in table 5.

Table 5.

Number of minutes between goals	Frequency
1–10	11
11–20	11
21–30	12
31–40	10
41–50	4
51–60	0
61–70	1
71–80	2

The immediate impression is that it does not look too much like an exponential distribution. Working with the original data, rather than the grouped version above, the mean is 24.78 minutes. The reader is left to fit the appropriate exponential distribution and see that this hypothesis is indeed in some doubt at the 5% significance level ($\chi^2 = 11.54$, 3 d.f. CV = 11.345 (1% level)).

Ah well — that's the nice thing about the unpredictability of real data. Next week's goal scorers will probably do it properly!

Carol Nixon

Reference

1. M. Bedwell, Many happy returns to the birthday problem, *Teaching Statist.* **24** (2002), pp. 43–45.

Computer Column

The games computers play

It made the headlines recently when the chess program Deep Fritz managed a 4–4 draw against the current world champion, Vladimir Kramnik. In 1997, IBM's Deep Blue went one better against the then champion, Garry Kasparov, winning 3.5–2.5. Chess, however, is just the latest in a series of games where computers have been able to match, or even beat, human players. Computers can now play Othello® (also called reversi) far better than any human, and a program called Chinook won the world draughts title in 1994. However, the Japanese game go is still very much a human preserve, with no systems yet developed which can challenge even amateur players (despite a prize of $1.4 million being offered for the first program to defeat a top-level player). Why should this be?

Virtually all game-playing programs are based around the same basic method, usually called the minimax algorithm, which was first proposed by John von Neumann in 1928. Imagine a game being played by two players, Max and Min, where the possible outcomes are either a win for Max (scoring $+1$), a draw (scoring 0) or a win for Min (scoring -1). Max is therefore trying to maximize the final score, while Min is trying to minimize it. If the computer takes the role of Max, and is due to play next, how should it move? If Min is a very good player, who will always make the best possible move, then it should try to limit the damage that Min can do; in other words, it should select the move that leads to the best guaranteed outcome. To do this, it considers all the moves it is allowed to make, all the moves Min can make in response, all the moves it can make in response to Min's move, and so on. This is usually represented as a tree, as illustrated in figure 1.

The tree in figure 1 is part of the tree for a very simple game, where each player always has three possible moves

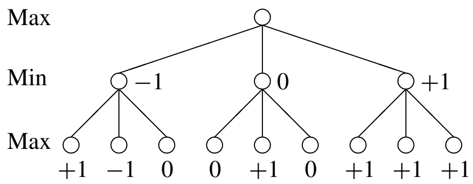


Figure 1. Game tree.

available. At the top of the tree is the current board position, with Max due to play next. For each of the three possible moves, Min has three counter-moves, and we leave it just as Max is considering responses to Min's counter-moves. The tree would continue in this way, with each node branching into three nodes on the next level down unless it represented a position where the game had already been won or drawn. For the moment, let's imagine that the game must be over after two moves, so that figure 1 is the complete tree. Since all the nodes on the bottom line represent end game positions, we can assign them scores of ± 1 or 0, according to how the game worked out.

For this tree, if Max decided to make the left-most move, then Min's three options would lead to a win for Max, a win for Min or a draw. If we assume that Min, being a perfect player, will choose the move that leads to a win, we can assign the left-most move a score of -1. Similarly, the middle move will lead to a draw, and the right-most move will lead to a win for Max. Max's choice is now clear: make the right-most move and prepare to celebrate!

The most basic version of the method works just like this: work out the complete tree of possible moves, then start at the bottom and work back up the tree, assigning each node a score that represents the worst possible outcome from that position, from the point of view of the player who would be due to play. At each level, we assume that Max will always select the maximum available score, while Min will always select the minimum available score. It's not too hard to see that, if the computer had enough power to find a move this way before Min got bored waiting, Max would always play perfectly. The problem, of course, is that in practice it's not currently possible to play any game much more complicated than noughts and crosses this way, so we need to compromise.

The way that we get round this problem is to only look ahead a certain number of moves, rather than all the way to the end of the game, and then find a way to assign a score to the unfinished games at the bottom of the tree. The score should somehow represent Max's chances of winning from that position, on a scale from $+1$ (sure to win) to -1 (sure to lose), but finding a good way of estimating this score is not easy. The most common methods consider things like the number of pieces of each type that each side has on the board, and features such as 'good pawn structure' and 'king safety', but it is more of an art than a science, usually involving a good deal of input from good human players. There is also a trade-off here: some programs use a very crude — but fast — way of assigning a score, looking many moves ahead, while others use a more accurate — but slower — one and consider fewer moves.

A potential problem with only looking a fixed number of moves ahead is that, often, strong-looking board positions can turn out to have a fatal weakness, such as an easy path to checkmate for Min. This could lead to the program selecting a move which, as far as it could see, looked good but which, had it looked a few more moves ahead, it would have realised was terrible. To avoid this, many programs check to see whether the positions they reach are 'quiescent' (i.e. there are not likely to be any wild swings in the near future); if not, then they keep looking ahead until they do reach a quiescent position. Deep Blue, for example, usually looked 14 moves ahead, but would look more than 40 moves ahead in some cases.

This leads us on to the reasons why computers can play some games very well, but not others. The effort involved in calculating the complete tree of moves depends on the likely number of moves remaining until the end of the game, m, and the average 'branching factor', b (the average number of possible moves). The number of board positions to consider is of the order of b^m, so computers can approach this most closely (and therefore have the best chance of winning) in short games with relatively low branching factors. For example, chess games typically last about 100 moves, with a branching factor of 35, so a complete tree would consist of 35^{100} nodes (despite the fact that there are 'only' about 10^{40} legal board positions). By contrast, Othello has a branching factor of 5–15, while the branching factor for go can approach 360.

The branching factor problem becomes even worse in the case of games that include a random element, such as backgammon. For these, the program must consider all the moves that would be possible for every possible dice score. The minimax score for each node is then an average of the different possible scores, weighted according to how likely the associated dice score is. (For example, in games that use a single die, this would just be an ordinary average; with two or more dice, some scores are more likely than others.)

Of course, the minimax method is not the only one that can be used: for example, programs which attempt to play go are unlikely to use the minimax algorithm (or any other search method) in the near future, and rely instead on large databases of hand-coded rules for deciding how to play. Some methods also use so-called 'neural networks', which allow programs to learn their own set of rules through repeated trial and error against a human opponent; however, the training periods that would be required for complex games such as go would probably be prohibitively long.

As computer power increases, the standard of game-playing programs will steadily approach the perfect standard that would be possible if they could evaluate the complete game tree for every move. In particular, it will only take a few more years at most before chess joins draughts and reversi in succumbing to the strength of computer play. Ultimately, it will largely be a victory of sheer brute force over intelligence and intuition, but it just goes to show that very different approaches to a problem can yield very similar results!

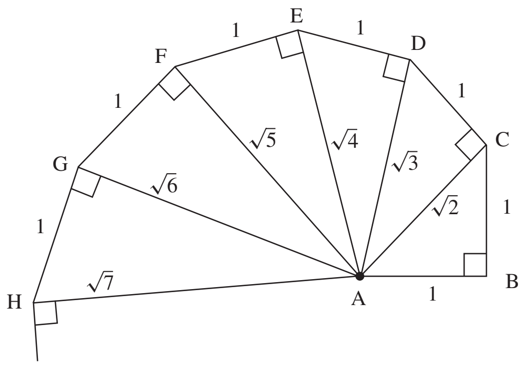
Peter Mattsson

Websites

1. Deep Fritz: http://www.brainsinbahrain.com
2. Deep Blue: http://www.research.ibm.com/deepblue/
3. Chinook: http://www.cs.ualberta.ca/~chinook/
4. Go: http://www.aaai.org/AITopics/html/go.html

Constructing $\sqrt{n}$

The diagram shows at a glance how to construct straight lines of lengths of $\sqrt{2}$, $\sqrt{3}$, ... successively with a straight edge and compass, given a unit of length. By trisecting a line of length $\sqrt{n}$ for $n = 1, 2, \ldots, 9$, we can construct straight lines of length $\sqrt{0.nnnn\ldots}$ for $n = 1, 2, \ldots, 8$.



BABLU CHANDRA DEY
12/15 Kamardanga Rly Quarter,
Kolkata 700 046, India.

Letters to the Editor

Dear Editor,

The integration of x^n

The integrals

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c\,, \qquad n \neq -1\,,$$

and

$$\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$$

are well-known results. A common question is why, when $1/x$ is simply x^n with $n = -1$, does the integral of $1/x$ with respect to x not follow the rule that holds for $n \neq -1$. Here is the secret.

Instead of the usual form, the result for $\int x^n \,\mathrm{d}x$ can be written as $(x^{n+1} - 1)/(n+1) + c$. In fact, the particular integral may be chosen as $(x^{n+1} - 1)/(n+1)$ instead of $x^{n+1}/(n+1)$. But

$$\lim_{n\to -1} \frac{x^{n+1} - 1}{n+1} = \lim_{m\to 0} \frac{x^m - 1}{m} = \lim_{m\to 0} \frac{x^m \ln x}{1} = \ln x\,,$$

using l'Hôpital's rule. So $\ln x$ is within the general rule, not to be seen at $n = -1$ but as $n \to -1$. So we can write

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1} - 1}{n+1} + c\,, \qquad n \neq -1\,,$$

and, by way of continuity,

$$\int x^{-1} \,\mathrm{d}x = \lim_{n\to -1} \frac{x^{n+1} - 1}{n+1} + c = \ln x + c\,.$$

Yours sincerely,

CHAN WEI MIN
(404 Pandan Gardens,
Singapore 600404.)

Dear Editor,

A Tale of Two Series — A Dickens of an Integral

In my letter in Volume 34, Number 2, pp. 45–46, following from P. Glaister's article in Volume 33, Number 2, pp. 25–27, I asked whether there are functions other than x^{-x} which are differentiable for $x > 0$ and satisfy the condition

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{x=1}^{\infty} f(x)\,.$$

There is a simple way to construct such functions. Let $g(x)$ be any differentiable even function on $\mathbb{R}$, so that $g(x) = g(-x)$ for all x. Then $g(x - \frac{1}{2})$ is symmetrical about $x = \frac{1}{2}$. Since $\sin(2\pi x)$ is odd about $x = \frac{1}{2}$, this means that $g(x - \frac{1}{2}) \sin 2\pi x$ is also odd about $x = \frac{1}{2}$, so that

$$\int_0^1 g(x - \tfrac{1}{2}) \sin 2\pi x \,\mathrm{d}x = 0 = \sum_{x=1}^{\infty} g(x - \tfrac{1}{2}) \sin 2\pi x\,.$$

Yours sincerely,

MILTON CHOWDHURY
(16 Caledonian Avenue,
Layton,
Blackpool FY3 8RB.)

Dear Editor,

A Tale of Two Series — A Dickens of an Integral

In his letter in Volume 34, Number 2, Milton Chowdhury gives some answers to the question I posed in my letter in Volume 33, Number 3, p. 65, as to whether there are functions f, other than x^{-x}, which satisfy

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{x=1}^{\infty} f(x)\,. \tag{1}$$

He asks at the end of his letter for differentiable functions which satisfy (1), giving $\sin(2n\pi x)$ for n a positive integer, as an example. Actually we are spoiled for choice! May I suggest four other lines of attack to this problem.

(i) Take virtually any function for which the sum and integral exist, and modify it in the neighbourhood of, say, $x = 2$ to fit the criterion (1). Differentiability to any finite degree can be achieved in this way.

(ii) Take any differentiable function depending on a single parameter k and find k to fit (1). This will often be possible, for example $f(x) = \mathrm{e}^{-kx}$ fits if $k = 2(\cosh k - 1)$, $k > 0$, giving $k \approx 0.9308212$.

(iii) Let $f(x) = g(x) + kh(x)$, where $h(x) = 0$ for positive integers x, and solve for k with $g(x)$, $h(x)$ integrable as necessary. For example,

$$f(x) = \mathrm{e}^{-x} + k \sin^2 \pi x$$

leads to

$$1 - \mathrm{e}^{-1} + \tfrac{1}{2}k = \frac{\mathrm{e}^{-1}}{1 - \mathrm{e}^{-1}}\,,$$

i.e.

$$k = \frac{-2(\mathrm{e}^2 - 3\mathrm{e} + 1)}{\mathrm{e}(\mathrm{e} - 1)}\,,$$

so (1) is satisfied.

(iv) Let $h(x)$ be a function such that

$$\int_0^1 h(x)\,\mathrm{d}x = b > 0\,,$$

$$\sum_{x=1}^{\infty} h(x) = a > 0$$

and $b < a$. Let $a = cb$, so $c > 1$. For simplicity, suppose that $h(x) > 0$ for all x. Let $f(x) = h(x)\mathrm{e}^{-kx}$ $(k > 0)$, and let

$$\int_0^p h(x)\,\mathrm{d}x = qb$$

where $0 < p < 1$ and, clearly, $0 < q < 1$, so $c/q > 1$. Then

$$\begin{aligned}\int_0^p f(x)\,\mathrm{d}x &= \int_0^p -\mathrm{e}^{-kx}h(x)\,\mathrm{d}x \\ &\geq \mathrm{e}^{-pk}\int_0^p h(x)\,\mathrm{d}x = \mathrm{e}^{-pk}qb\,.\end{aligned}$$

Now

$$\int_0^1 f(x)\,\mathrm{d}x > \mathrm{e}^{-pk}qb > \mathrm{e}^{-k}cb$$

provided that $\mathrm{e}^{(1-p)^k}q > c$, i.e. $k > (1/(1-p))\ln(c/q) > 0$. In this case

$$\begin{aligned}\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} h(x)\mathrm{e}^{-kx} &< \mathrm{e}^{-k}\sum_{x=1}^{\infty} h(x) \\ &= \mathrm{e}^{-k}a = \mathrm{e}^{-k}cb\,.\end{aligned}$$

So, for suitable $k > 0$,

$$F(k) = \sum_{x=1}^{\infty} f(x) - \int_0^1 f(x)\,\mathrm{d}x < 0\,.$$

But

$$F(0) = \sum_{x=1}^{\infty} h(x) - \int_0^1 h(x)\,\mathrm{d}x = a - b > 0\,.$$

Clearly $F(k)$ is a continuous function of k, and so for a suitable choice of k, $F(k) = 0$, and so $f(x)$ satisfies (1).

Yours sincerely,

Alastair Summers
(57 Conduit Road,
Stamford,
Lincs PE9 1QL.)

Dear Editor,

A Tale of Two Series — A Dickens of an Integral

In his letter in Volume 33, Number 3, A. G. Summers asked whether there exist any functions f for which

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} f(n)\,, \tag{1}$$

other than the example in P. Glaister's article of Volume 33, Number 2, i.e.

$$f(x) = x^{-x}\,.$$

M. Chowdury gave examples of other suitable non-differentiable functions in his letter in Volume 34, Number 2, but asked whether there exist suitable differentiable functions. In fact, examples can easily be formed as follows. Take two differentiable functions g and h (for which the sum and the integral in (1) converge), ensuring that they are not proportional to each other. Then the following function satisfies (1):

$$\begin{aligned}f(x) = &\left[\sum_{n=1}^{\infty} g(n) - \int_0^1 g(x)\,\mathrm{d}x\right]h(x) \\ &- \left[\sum_{n=1}^{\infty} h(n) - \int_0^1 h(x)\,\mathrm{d}x\right]g(x)\,.\end{aligned}$$

The fact that g and h are not proportional ensures that f is not identically zero. This method is interesting as it shows that what at first sight appears to be a tricky calculus problem may be solved from the point of view of linear algebra. The only bit of calculus needed is the fact that both integration and the taking of (convergent) infinite sums are linear machines acting on functions, i.e.

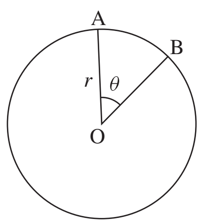
$$\begin{aligned}\int_0^1 \alpha_1 f_1(x) + \alpha_2 f_2(x)\,\mathrm{d}x &= \alpha_1\int_0^1 f_1(x)\,\mathrm{d}x \\ &\quad + \alpha_2\int_0^1 f_2(x)\,\mathrm{d}x\,, \\ \sum_{n=1}^{\infty}\alpha_1 f_1(n) + \alpha_2 f_2(n) &= \alpha_1\sum_{n=1}^{\infty} f_1(n) + \alpha_2\sum_{n=1}^{\infty} f_2(n)\,.\end{aligned}$$

Of course, this method extends immediately to give examples of infinitely differentiable functions satisfying (1).

Yours sincerely,

Will Donovan
(Student, Queens' College,
Cambridge, CB3 9ET.)

At the fair



A particle of mass m moves in a circle of radius r from A to B, where OA is vertical and $\angle\mathrm{AOB} = \theta$. (It could be a ride at a theme park!) Its speed at A is u and at B is v. What is the work done against friction between A and B?

Joe Collis
Student, Netherthorpe School,
Staveley, Derbyshire.

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

35.5 The real numbers $\alpha_1, \ldots, \alpha_n$ are such that

$$m\lfloor \alpha_1 + \cdots + \alpha_n \rfloor = \lfloor m\alpha_1 \rfloor + \cdots + \lfloor m\alpha_n \rfloor$$

holds for infinitely many values of the integer m, where $\lfloor x \rfloor$ denotes the integer part of the real number x. Prove that $\alpha_1, \ldots, \alpha_n$ are rational.

(Submitted by Hassan Shah Ali, Tehran)

35.6 The triangle ABC has angles $\mathrm{A} = 90^\circ$, $\mathrm{B} = 60^\circ$, $\mathrm{C} = 30^\circ$. Outward equilateral triangles BCD, CAE, ABF are attached to the triangle. Show that AD, BE and CF are concurrent and that their point P of intersection lies on the circle with diameter A′B, where A′ is the midpoint of BC. If inward instead of outward equilateral triangles are attached to the triangle, which point corresponds to P? (These two points are called the *first and second Fermat points* of the triangle.)

(Submitted by J. A. Scott, Chippenham)

35.7 O_1, O_2 and A_0 are three distinct points in the plane. The sequence of points $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \ldots$ is constructed by the following rule: given A_{2n}, the point A_{2n+1} lies on $\mathrm{O}_1\mathrm{A}_{2n}$ produced such that A_{2n} is the midpoint of $\mathrm{O}_1\mathrm{A}_{2n+1}$ and A_{2n+2} lies on $\mathrm{O}_2\mathrm{A}_{2n+1}$ produced such that A_{2n+1} is the midpoint of $\mathrm{O}_2\mathrm{A}_{2n+2}$. Show that $\mathrm{A}_0, \mathrm{A}_2, \mathrm{A}_4, \ldots$ are collinear, that $\mathrm{A}_1, \mathrm{A}_3, \mathrm{A}_5, \ldots$ are collinear, and that these straight lines are parallel.

(Submitted by Guido Lasters, Tienen, Belgium)

35.8 Show that the set of natural numbers is the union of two disjoint subsets neither of which contains an infinite arithmetic progression.

(Submitted by Farshid Arjomandi, University of California, San Diego)

Solutions to Problems in Volume 34 Number 3

34.9 Find all natural numbers $n > 2$ for which there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ such that $\{a_1+a_2, a_2+a_3, \ldots, a_{n-1}+a_n, a_n+a_1\}$ forms a set of n consecutive natural numbers.

Solution by Hassan Shah Ali, who proposed the problem

Suppose that there exists such a permutation for some n, and write

$$\{a_1+a_2, a_2+a_3, \ldots, a_n+a_1\} = \{k+1, \ldots, k+n\}.$$

Then

$$\begin{aligned}(a_1+a_2)+(a_2+a_3)+\cdots+(a_n+a_1)\\ = (k+1)+(k+2)+\cdots+(k+n),\end{aligned}$$

so

$$2(a_1+\cdots+a_n) = nk + \frac{n(n+1)}{2}.$$

Now

$$2(a_1+\cdots+a_n) = n(n+1),$$

so

$$n+1 = 2k$$

and n must be odd.

Conversely, let n be odd, say $n = 2m+1$. Then the permutation

$$m+1, 1, m+2, 2, m+3, 3, \ldots, 2m, m, \ 2m+1$$

satisfies the requirements. Hence the required natural numbers are the odd ones.

34.10 For $a, b > 0$, evaluate

$$\int_0^\infty \frac{1}{x}\left\{\frac{1}{1+ax} - \frac{1}{1+bx^2}\right\} \mathrm{d}x.$$

Solution by Milton Chowdhury

With $X, Y > 0$,

$$\begin{aligned}\int_X^Y \frac{1}{x}\left\{\frac{1}{1+ax} - \frac{1}{1+bx^2}\right\}\mathrm{d}x\\ &= \int_X^Y \left(\frac{bx}{1+bx^2} - \frac{a}{1+ax}\right)\mathrm{d}x\\ &= \left[\tfrac{1}{2}\log(1+bx^2) - \log(1+ax)\right]_X^Y\\ &= \left[\log\frac{\sqrt{1+bx^2}}{1+ax}\right]_X^Y\\ &= \log\frac{\sqrt{1+bY^2}}{1+aY} - \log\frac{\sqrt{1+bX^2}}{1+aX}\\ &= \log\frac{\sqrt{b+1/Y^2}}{a+1/Y} - \log\frac{\sqrt{1+bX^2}}{1+aX}\\ &\to \log\frac{\sqrt{b}}{a}\end{aligned}$$

as $Y \to \infty$ and $X \to 0$.

34.11 For non-negative real numbers $x_1, \dots, x_n$, prove that

$$\prod_{i=1}^{n}(1-x_i)+\Big(1+\frac{1}{n}\sum_{i=1}^{n}x_i\Big)^n \geq \prod_{i=1}^{n}(1+x_i)+\Big(1-\frac{1}{n}\sum_{i=1}^{n}x_i\Big)^n .$$

Solution by Hassan Shah Ali, who proposed the problem

There is equality for $n = 1, 2$, so we suppose that $n \geq 3$. Consider some integers m, k, $1 \leq m \leq n$, $2 \leq k \leq n$, and write

$$x_1 + \cdots + x_k = ka ,$$

so that a is the mean of $x_1, \dots, x_k$. By reordering the x_i if necessary, we can suppose that $x_{k-1} \leq a \leq x_k$. Now

$$\begin{aligned}(1+a)^{m-1}&(1+x_1)\cdots(1+x_{k-2})\\ &\geq (1-a)^{m-1}(1-x_1)\cdots(1-x_{k-2}) ,\end{aligned}$$

and

$$(a-x_{k-1})(x_k-a) \geq 0 .$$

Since

$$\begin{aligned}&(a-x_{k-1})(x_k-a)\\ &\quad = (1-a)(1-x_{k-1}-x_k+a)-(1-x_{k-1})(1-x_k) ,\\ &\quad = (1+a)(1+x_{k-1}+x_k-a)-(1+x_{k-1})(1+x_k) ,\end{aligned}$$

this gives

$$\begin{aligned}&(1+a)^{m-1}\big[(1+a)(1+x_{k-1}+x_k-a)\\ &\qquad -(1+x_{k-1})(1+x_k)\big]\\ &\quad\times(1+x_1)\cdots(1+x_{k-2})\\ &\qquad \geq (1-a)^{m-1}\big[(1-a)(1-x_{k-1}-x_k+a)\\ &\qquad\qquad -(1-x_{k-1})(1-x_k)\big]\\ &\qquad\quad \times(1-x_1)\cdots(1-x_{k-2}) .\end{aligned}$$

Put $x'_{k-1} = x_{k-1}+x_k-a$. Then $x'_{k-1} \geq 0$ and

$$\begin{aligned}(1+a)^{m-1}&\prod_{i=1}^{k}(1+x_i)-(1-a)^{m-1}\prod_{i=1}^{k}(1-x_i)\\ &\leq (1+a)^m(1+x'_{k-1})\prod_{i=1}^{k-2}(1+x_i)\\ &\qquad -(1-a)^m(1-x'_{k-1})\prod_{i=1}^{k-2}(1-x_i) .\end{aligned}$$

Also

$$x_1+\cdots+x_{k-2}+x'_{k-1} = x_1+\cdots+x_k-a = (k-1)a .$$

We can now repeat this procedure, eventually reaching

$$\begin{aligned}(1+a)^{m-1}&\prod_{i=1}^{k}(1+x_i)-(1-a)^{m-1}\prod_{i=1}^{k}(1-x_i)\\ &\leq (1+a)^{m-1+k}-(1-a)^{m-1+k} .\end{aligned}$$

If we now take $m = 1$ and $k = n$, we obtain

$$\prod_{i=1}^{n}(1+x_i)-\prod_{i=1}^{n}(1-x_i) \leq (1+a)^n-(1-a)^n ,$$

which gives the result.

34.12 For $x > 0$, let

$$f(x) = \sum_{n=0}^{\infty}\frac{\log(x+n)}{2^n} .$$

Evaluate

$$\lim_{x\to\infty}\frac{f(x)}{\log x} .$$

Solution by J. A. Scott, who proposed the problem

For $x > 0$,

$$f(x) = \sum_{n=0}^{\infty}\frac{\log(x+n)}{2^n} \geq \sum_{n=0}^{\infty}\frac{\log x}{2^n} = 2\log x .$$

Also,

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty}\frac{\log x+\log(1+n/x)}{2^n}\\ &= \log x\sum_{n=0}^{\infty}\frac{1}{2^n}+\sum_{n=0}^{\infty}\frac{\log(1+n/x)}{2^n}\\ &\leq 2\log x+\sum_{n=0}^{\infty}\frac{n}{2^n x}\\ &= 2\log x+\frac{1}{x}\Big(\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots\Big)\\ &= 2\log x+\frac{1}{2x}\frac{1}{(1-\frac{1}{2})^2}\\ &= 2\log x+\frac{2}{x} .\end{aligned}$$

Hence,

$$2 \leq \frac{f(x)}{\log x} \leq 2+\frac{2}{x\log x} .$$

It follows that

$$\lim_{x\to\infty}\frac{f(x)}{\log x} = 2 .$$

Also solved by Farshid Arjomandi (University of California, San Diego).

Where is the error?

For any fixed $k \neq 0$, we have

$$\lim_{x\to 0}(\sin x - kx) = 0 ,$$

so

$$\lim_{x\to 0}\frac{\sin x}{x} = k .$$

But I thought that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 !$$

SEYAMACK JAFARI
Razi Petrochemical Complex,
Bandar Imam, Khozestan, Iran.

Reviews

Studying Mathematics and Its Applications. By Peter Kahn. Palgrave Macmillan, Basingstoke, 2001. Pp. 208. Paperback £11.99 (ISBN 0-33-392279-4).

This book is fairly unusual as a study guide because it does not tell the reader *what* to learn, rather *how* to learn. For this reason the book can be useful to anyone studying subjects that contain concepts that are often too difficult to comprehend at first glance.

I found the book very reassuring because it told me I was studying maths in the right way and for the right reasons. When I asked friends who had not enjoyed studying maths how they saw the subject, it was clear they had not learned it in a way suited to its enjoyment.

Peter Kahn writes very clearly, and he avoids dictating to the reader the right and wrong way to learn. Instead, he takes the reader through the problem, indicating the way that he finds best, which is usually best for the reader too. At times, it was possible for the point being made to become lost and diluted in writing, although in most cases this served to make the problem easier to understand. By splitting up the ideas, the reader can also understand how they were originally put together to create new topics.

I do not think I can really fully appreciate this book because most of the material it contains is post A-level. However, if and when I do a degree in maths, this book will be useful to help me understand what I am studying, because it also contains ideas for preparing to study the subject — pay attention to what you are being taught. (Surely no-one's mind wanders during lectures!)

Student, Stamford High School for Girls — Jo Faux

Teaching Mathematics in Colleges and Universities: Case Studies for Today's Classroom. By Solomon Friedberg *et al.* American Mathematical Society, Providence, RI, 2001. Pp. 158. Paperback $29.00 (ISBN 0-8218-2875-4).

This volume is designed to help in the training of TAs (Teaching Assistants) in US universities, many of whom are graduate students (UK: postgraduate students). The context is American, with many parallel 'math classes', usually calculus. Fourteen case studies are presented, ranging from students transferring from one class to a parallel one with a more sympathetic teacher, setting exams and quizzes, students not understanding and resenting the introduction of background physical material. Teachers will recognize the problems even if they cannot provide solutions! In the training program (UK: programme) used in Boston, TAs (remember what they are?) look at the case studies in interactive groups. This book is somewhat repetitive, and does not provide ready-made answers to the problems that it poses.

Although the usefulness of this book is likely to be limited to teachers from other cultures, the problems faced by students and teachers of math (UK: mathematics) are universal, and it is useful to have them set out in this novel way. At the very least, teachers can use some of the examination questions posed, for example:

- solve $5^{2x-1} = 0.2$,
- solve $4^x - 2^{x+1} + 35 = 0$ (*sic*; maybe students have to correct the question before solving it),
- find $\int 2^\alpha \cos(3\alpha)\,\mathrm{d}\alpha$.

University of Sheffield — David Sharpe

Modern Statistics for the Life Sciences. By Alan Grafen and Rosie Hails. Oxford University Press, 2002. Pp. xv+351. Paperback £22.99 (ISBN 0-19-925231-9).

This book is designed to meet the needs of biologists and other scientists who wish to understand the principles of modern statistics, while relying mostly on computer packages such as MINITAB, SAS, SPSS, GENSTAT, BDMP, GLIM and S-PLUS for their statistical computations. The authors, who have taught the contents of their book to first- and second-year biology students at Oxford University for about ten years, state that their aim has been to choose 'a different conceptual plane on which to explain statistics .... Some of the ideas ... are introduced, using geometrical pictures, to explain a bit about how GLM (General Linear Model) works.'

The authors have succeeded in producing a very readable account of modern statistical methods in the following 13 chapters: 1. An introduction to analysis of variance; 2. Regression; 3. Models, parameters and GLMs; 4. Using more than one explanatory variable; 5. Designing experiments — keeping it simple; 6. Combining continuous and categorical variables; 7. Interactions — getting more complex; 8. Checking the models I: independence; 9. Checking the models II: the other three assumptions; 10. Model selection I: principles of model choice and designed experiments; 11. Model selection II: datasets with several explanatory variables; 12. Random effects; 13. Categorical data. Each chapter is followed by a summary of its contents and some exercises. The book ends with two further chapters, a revision section and three appendices: 14. What lies beyond; 15. Answers to exercises; Revision section: The basics; Appendix 1: The meaning of p-values and confidence intervals; Appendix 2: Analytical results about variances of sample means; Appendix 3: Probability distributions. In conclusion, there is a bibliography of useful texts, and an index.

This book will be a boon to all those who wish to use statistical methods intelligently, without necessarily following all the mathematics required to prove each result. The authors have done an excellent job of providing clear explanations of complex statistical procedures; they are to be congratulated on their achievement. I would recommend every library to have a copy of the book on its shelves.

The Australian National University — Joe Gani

Mathematical Spectrum
2002/2003 Volume 35 Number 2

ISSN 0025-5653

Published by the Applied Probability Trust
Printed by Pear Tree Press Ltd, Stevenage, Herts, UK