Mathematical Spectrum

Volume 13 1980/81 *Number 1*

A Magazine of *Contemporary Mathematics*

Published by the *Applied Probability Trust*

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 13 of *Mathematical Spectrum* will consist of three issues, the second of which will be published in January 1981, and the third in May 1981.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

The Editor, *Mathematical Spectrum*,
Hicks Building, The University, Sheffield S3 7RH.

Joseph Fourier—His Life and Work

R. J. WEBSTER
University of Sheffield

Roger Webster is a Lecturer in Pure Mathematics at the University of Sheffield. His main mathematical interest is convexity, but he is also very interested in the history of mathematics. Dr Webster tells us that he solved Fourier's problem, referred to at the end of his article, whilst listening to the first act of 'La Traviata' in the opera house, which is where he is usually to be found when not in the lecture theatre.

Introduction

Joseph Fourier, the hundred and fiftieth anniversary of whose death falls this year, was a French mathematical physicist whose researches into the transfer of heat had far-reaching consequences, completely transcending the particular problems to which they were directed. This telling of his story hopes to bring the name of Fourier before a much wider audience than that of research scientists. His life was certainly packed with incident: orphaned at nine, sponsored by a bishop, child prodigy, Benedictine monk, revolutionary, suspected terrorist under threat of death, gifted orator, university professor, friend of Napoleon, Egyptologist, prefect of a French department, drainer of swamps, planner of mountain roads, great-nephew of a saint, permanent secretary of the Académie des Sciences and an eccentric old man who, even on the hottest of days, would attire himself as befits a polar explorer. These are just some of the aspects of his colourful career.

Fourier the man

Jean Baptiste Joseph Fourier was born in the ancient French town of Auxerre on 21 March 1768. He was christened Joseph after his father, a master tailor of the town, and was the ninth child in a family of twelve; his father also had three children by a previous marriage. The future could not have looked at all bright for the young Fourier when in October 1777 his mother died, to be followed only a few months later by his father, who was grief-stricken at the loss of his wife. Thus Fourier was left an orphan at the age of nine. Happily for him, however, the cathedral organist of Auxerre showed interest in him, giving him lessons at the small preparatory school which he ran.

Fourier's geniality and natural abilities so impressed one of the ladies of Auxerre that she recommended him to the Bishop of Auxerre, who arranged for him (now aged twelve) to attend the local École Militaire run by the Benedictines. The military school at Auxerre was one of eleven such institutions in pre-revolutionary France noted for the teaching of science and mathematics—most famous of their pupils were Napoleon and Wellington. Fourier soon justified the confidence placed in him by the bishop. Within months of entering the school not only was he writing humorous verse to amuse his classmates, but he was also composing wonderful sermons for church dignitaries in Paris to deliver as if their own—quite an achievement for a twelve-year-old. At thirteen he had become completely fascinated by mathematics and this fascination was to exert its hold over him for the rest of his life. The hours prescribed for mathematical study could not satisfy his thirst for knowledge, so during the day he used to collect candle ends to enable him to pursue his researches deep into the night. Fourier had always wished to become a soldier, and on completing his studies at the military school he applied to join the artillery, but despite strong support from the school inspectors his application was rejected. Thereupon he resolved to join the Church.

After a further period of study, this time at the Benedictine Collège Montaigu in Paris, he returned to Auxerre to help in teaching mathematics at his old school. By 1787 he was ready to don the habit of a Benedictine monk and he entered the abbey of St Benoît-sur-Loire. While preparing for his vows, he instructed the other novices in mathematics. His stay at the abbey was not a particularly happy one; he was not sure that the Church was his true vocation and he felt mathematically isolated. His isolation, however, was not total, for he corresponded with his former mathematics teacher Bonard at Auxerre, exchanging items of mathematical interest. By 1789, when the Revolution broke out, he had not yet taken his vows and he seized this opportunity to go back to the military school at Auxerre as a professor of mathematics.

From its very outset Fourier was an enthusiastic supporter of the Revolution and he served on the local revolutionary committee. His gift for oratory manifested itself in a speech on conscription which he made to his local revolutionary assembly in 1793; so successful was his appeal for men to enlist in a national levy to defend the fatherland that the required local quota was met entirely from the ranks of volunteers. During the Reign of Terror (June 1793–July 1794), when 1400 heads rolled in Paris alone, he spoke out against the mindless brutality and did all within his power as a revolutionary committee member to protect innocent victims of the tyranny. On one occasion he discovered that his travelling companion was on a mission to arrest a citizen of the neighbouring town of Tonnerre and escort him to the Revolutionary Tribunal. On arrival in Tonnerre, Fourier invited his companion to lunch with him. During the meal, he made some pretext to leave the room, and on so doing he quietly locked the door behind him, thus giving him time to find the citizen and advise him to leave the town. On another occasion his unswerving support for victims of the Terror so incensed some revolutionary officials that an order was issued for his arrest and execution. He pleaded his case in person before Robespierre in Paris, but was unsuccessful and was imprisoned on his return to Auxerre. Fourier's life would have been in real danger at this time had he not been released under a general amnesty following the guillotining of Robespierre on 28 July 1794.

At the beginning of 1795 Fourier resigned his teaching position at Auxerre and went to Paris to become one of 1500 students enrolled at the newly founded École Normale to train as teachers. The school numbered amongst its professors such men as Lagrange and Laplace, foremost mathematicians of their generation. For the first time in his life he found himself at a centre of mathematical activity and he grasped the opportunity it afforded him with both hands. Not only did he receive inspiration through contact with great men, but he in turn impressed them with his own mathematical talents. Thus, when the École Normale closed after only a few months, due partly to the inadequate preparation of the students, Fourier was invited to join the staff of the newly formed (and enduring) École Polytechnique.

Joseph Fourier

Even now his life was in danger. In the middle of the night 7/8 June 1795 he was dragged from his bed, taken to prison and accused of acts of terror during the years 1793 and 1794. The concierge, after expressing the wish that he would soon see Fourier return, was told: 'Come and collect him yourself—in two pieces!' Fortunately he was soon released, after representations from his colleagues at the École Polytechnique. The next three years he passed in contented tranquillity, devoting himself to teaching and research. He earned a high reputation as a lecturer by the eloquent delivery of his well-researched lectures, which maintained a happy balance between abstraction and application, and which were filled with interesting historical detail.

The tranquillity of Fourier's life was shattered in March 1798 when he received a letter from the Minister of the Interior telling him to prepare himself for a top secret mission, and to be ready to leave at a moment's notice. On 19 May of the same year he found himself, together with an army of soldiers and a fleet of ships, under Napoleon's command, setting sail from Toulon for an undisclosed destination. He was no doubt soon informed that he had been chosen as a scientific member of Napoleon's Egyptian expedition. In the August following, Napoleon founded the

Institut d'Egypte, and Fourier was appointed its permanent secretary. Napoleon returned to France after only a year, leaving Fourier virtually in charge of the non-military side of the French occupation. Fourier soon proved his skill both as an administrator and as a negotiator, led an expedition to explore the antiquities of Upper Egypt, and still found time to read papers at the Institute on subjects ranging from the theory of equations and mechanics to researches into oases and wind-powered irrigation machines.

Fourier returned to Paris in 1801, hoping to resume his work at the École Polytechnique; but this was not to be, for Napoleon had spotted his talent as an administrator, and made him prefect of the department of Isère, a part of France bordering Italy and centred at Grenoble. The prefecture that Fourier took over was a difficult one; the region and its administration were in disarray, and his first task was to bring order out of the chaos; this he achieved. Initially he was viewed with suspicion in certain church circles, for it was thought that his theories on the age of Egyptian monuments were at variance with the Bible; the suspicion, however, was allayed when it was discovered that his great-uncle, the Blessed Pierre Fourier, was a saint! Fourier undertook several large-scale projects during his prefecture. His greatest achievement was the drainage of the Bourgoin marshes. He succeeded in persuading the 37 communities bordering the marshes to set aside their conflicting interests and agree to his drainage scheme, which was to turn disease-ridden swamps into prime farming land. The project which satisfied him most was the design of a road from Grenoble to Turin, passing through the French Alps. Throughout his time at Grenoble he somehow found time to continue his academic work on two fronts. He supervised the publishing of material amassed during the Egyptian expedition, which became the *Description de l'Egypte*, and to which he contributed a historical preface. Most importantly, it was at Grenoble that he began his epoch-making research into the diffusion of heat, which was destined to carry his name to every corner of the world where mathematical physics is studied.

When Napoleon abdicated in April 1814, Fourier was allowed to continue as prefect of Isère. This proved an embarrassment for Fourier when he learned that his former leader was to pass through Grenoble on his way to exile in Elba, and he hurriedly arranged a detour to Napoleon's route. No such detour was possible a year later when Napoleon's route again took him through Grenoble on his triumphant return to Paris, and Fourier fled the city. Napoleon, angered at Fourier's dereliction of duty, dismissed him as prefect of Isère. The two old friends did, however, meet at Bourgoin on 10 March 1815. Napoleon was in conciliatory mood, and appointed Fourier to the prefecture of the neighbouring department of the Rhône. Fourier, disgusted at the harshness of the new regime, resigned his office on 1 June and returned to the capital.

He was unemployed for a time, until one of his former students, now prefect of the department of the Seine, appointed him director of the department's Bureau of Statistics. His nomination to the Académie des Sciences in 1816 was vetoed by the king, but his renomination the following year was not opposed. He became one of the permanent secretaries of the Academy in 1822, was elected a member of the

Royal Society in 1823 and a representative of the Académie des Sciences on the Académie Française. He spent the final years of his life as no doubt he would have chosen to do, as a much-respected academic receiving international recognition for his pioneering work in the mathematics of heat transfer.

During the last five years of his life Fourier suffered from attacks of rheumatism, and to guard against such attacks he resorted to the most extreme measures. Even on the hottest days he would swathe himself like a mummy, and his friends likened his living quarters to the burning desert wind. In the final months, he used to sit in an enclosed chair with only his head and arms protruding, so that he could work. He died of a heart attack on 16 May 1830 aged sixty-two.

Fourier's mathematics

That is the story of his life, but what of the mathematics which made his name famous throughout the scientific centres of the world? It would not be appropriate here to go into the details of his researches, for this would demand a knowledge of sophisticated mathematics; on the other hand, it would be unforgivable not to mention, however briefly, some of the areas in which he worked. Two remarks should be made about his work at the outset: firstly, it is surprising that he succeeded in doing any worthwhile research in a life so full in other ways, and secondly, his claim to fame rests on a relatively small amount of work of the highest originality.

The one topic which Fourier worked on for the whole of his adult life was that of locating the roots of polynomial equations. At the age of fifteen he discovered a new (and better) proof of *Descartes' Rule of Signs*, a result which gives an upper bound for the number of positive roots of a real polynomial equation. The rule can be described as follows: *Consider the sequence of signs (plus or minus) of the coefficients of a real polynomial*

$$f(x) = x^n + ax^{n-1} + \cdots + bx + c.$$

Each pair of opposite signs in this sequence, i.e. either $+-$ *or* $-+$ *is called a* variation. *Then the number of positive roots of the equation* $f(x) = 0$ *cannot exceed the total number of variations of signs in the sequence.* A simple example will illustrate. The sequence of signs of the coefficients of the polynomial $x^5 - x^4 - x^3 - x^2 + x - 1$ is $+ - - - + -$, giving a total number of variations of 3; thus the equation $x^5 - x^4 - x^3 - x^2 + x - 1$ has at most 3 positive roots. Amongst many other results in this direction, Fourier generalized the above result to estimate the number of roots of the equation $f(x) = 0$ within any given interval.

His most original and significant mathematical work is that which is concerned with the transfer of heat, and it is contained in his classic book *Théorie analytique de la chaleur*, published in 1822. It was in this book that he introduced his notation for the definite integral $\int_a^b$ for the first time. So successful was this notation that it is now standard everywhere. The type of problem which interested Fourier in his study of heat is illustrated in the following example. Consider the flow of heat in a uniform bar which is perfectly insulated (so that no heat escapes from the side of the bar) and the ends of which are always maintained at zero temperature. Suppose that at some initial time the temperature $\varphi(x)$ at a distance x along the bar is known. The problem then is to find the temperature $\theta(x, t)$ at a distance x along the bar at any subsequent time t; so, in particular, $\theta(x, 0) = \varphi(x)$ and $\theta(0, t) = \theta(L, t)$, where L is the length of the bar. After performing many experiments and making several false starts, Fourier showed that the temperature function θ satisfied a certain equation, now known as the *diffusion equation*; for the record and because it symbolizes so much of Fourier's work, we exhibit the diffusion equation below:

$$\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2},$$

where k is a constant determined by the material of the bar. To deduce the equation was one thing, to solve it was another, and it was in the mathematical techniques he introduced to do this that his true genius showed itself. These techniques, now universally known as *Fourier analysis*, involved expressing the initial temperature function φ as an infinite combination of sines and cosines; such combinations are known now as *Fourier series*. As hinted at earlier, Fourier analysis today has manifold applications in fields far removed from heat transfer: in the vibration of strings, the bending of beams, electromagnetic theory and X-ray crystallography, to name but a few. Many of Fourier's ideas in this domain were not accepted immediately, because they were so different from previous ones. The clarification of his ideas led to an important contribution in pure mathematics, the modern concept of a *function*.

It was mentioned earlier that, during his isolation at the monastery, Fourier used to exchange items of a mathematical interest with his former teacher Bonard. The following is an extract from a letter from Fourier to Bonard:

Here is a little problem of rather a singular nature; it occurred to me in connection with certain propositions in Euclid we discussed on several occasions. Arrange 17 lines in the same plane so that they give 101 points of intersection.

Had you been Bonard, would you have been able to send the solution to the problem in your next letter to Fourier? If so, turn to p. 31.

From *Hudibras* by Samuel Butler (1612–1680)

In mathematics he was greater
Than Tycho Brahe, or Erra Pater:
For he by geometric scale
Could take the size of pots of ale;
Resolve by sines and tangents straight
If bread or butter wanted weight;
And wisely tell what hour o' the day
The clock doth strike, by algebra.

The Versatile Mathematician

R. C. FAUST
University of Sheffield

Trained as a physicist, Dr Faust has gained wide experience of research and development in several industrial concerns, and was a research manager before joining the University of Sheffield as a careers adviser. His interest in educational matters expresses itself in several ways; for example, membership of the Postgraduate Training Committee of the Science Research Council and tutor-counsellor with the Open University.

Introduction

In the days long before sex discrimination was an issue, Mathematics was described as the Queen of Sciences by mathematicians, or as the Handmaiden of the Sciences by scientists. Today we must describe it differently, not simply to take it out of the arena of sex, but to signify its spreading influence. Mathematics is indeed a sovereign subject. On the one hand it provides an intellectual challenge that its followers find irresistible; on the other it is the servant of many other disciplines, not all scientifically based, that would rapidly stultify without its aid.

For these reasons it is a very desirable subject to take at an advanced level in school as it represents a firm foundation for a very wide range of studies at universities and polytechnics. Mathematics is essential or highly desirable for courses in such subjects as mathematics, physics, astronomy, chemistry, engineering of all kinds, metallurgy, architecture and computer science. It is also useful for courses in agriculture, anatomy, physiology, pharmacology, pharmacy, zoology, biology, dentistry, medicine, ophthalmic optics, psychology, geology, geography, accounting, business and management studies, economics and computer studies (mainly concerned with commercial applications).

A word of explanation is required about computer science (in the first list) and computer studies (in the second list). Computer science has largely developed as an academic subject under the aegis of mathematics departments. Computer studies, on the other hand, has tended to be linked to departments of business studies or management science and thus the emphasis is less mathematical.

Within the compass of this article I must, however, restrict the discussion to the career prospects of graduate mathematicians, and here we will define the relevant degree courses as those in mathematics, mathematics with statistics or mathematics with computing science—although we should not forget that there are many degree courses in which mathematics is combined with another subject, such as economics, geography, music or philosophy. A helpful survey of the various courses in mathematics, mathematics and statistics and mathematics and computing science is given in *Mathematics and Statistics*, CRAC Degree Course Guide, Hobsons Press (Cambridge) Ltd; this guide is updated every two years.

The strands of mathematics

These courses cover, with varying emphasis, the main fields of pure mathematics, applied mathematics, statistics, operational research and computer science, but these divisions are not sharp.

In *pure mathematics* conclusions are drawn by logical steps from certain basic assumptions known as axioms or postulates. The geometry of Euclid, number theory, the algebra of sets and matrices are but a few examples. Although pure mathematics is not directed towards practical problems, this does not imply that it is without applications: for example Boolean algebra forms the basis of the logic circuits by means of which digital computers operate, and set theory impinges upon linguistics.

Applied mathematicians must accept external constraints in their attempts to describe the 'real world' in mathematical form. Typical of their work is the mathematical treatment of the flow of air over aircraft bodies or between cooling towers, the scattering of electromagnetic waves, lubrication, and the viscous behaviour of materials (such as thixotropic paints) under stress. The first step, often a very difficult one, is the formulation of the problem in mathematical terms. This will often take the form of differential equations with appropriate boundary conditions. These equations must then be solved either by established or new methods, and here the applied and the pure mathematician may well meet. But the problem might not be amenable to a formal solution; instead it might be necessary to utilise a computer to obtain numerical answers.

Traditionally, applied mathematics has concentrated on scientific and engineering problems, but of late it has found increasing use in the development of mathematical models to solve problems in the social sciences and in the world of business. Many of these applications fall under the umbrella of operational research, which is discussed after statistics.

Statistics and the associated subject of *probability* deal with the likelihood of certain events occurring and with the degree of reliability with which conclusions can be drawn from observations. How should the information be collected so as not to be misleading, how should it be analysed and then interpreted? The design of a census questionnaire and the associated computer programs for analysing the returns is one aspect of the statisticians' work. Another is the help the statistician can offer on the planning of experimental work and on the analysis of results.

Operational research is the application of a logical approach to the analysis of a complex system involving men, machines and other resources. The aim is to develop a model of the system so that the outcome of pursuing different strategies can be predicted and the most appropriate one be selected. Simulation, linear programming, network analysis, and queuing theory are a few of the techniques underlying OR. The scope is very wide and can range from a consideration of the various ways of disposing of rubbish collected by local authorities (e.g. tips, burning or recycling) to a determination of the optimum level of stocks that a manufacturing company should hold. Some of these problems inevitably involve the use of computers.

Computer science embraces the design of computer logic, the organisation of the computer store, how data is transferred between different parts of the computer or from one computer to another. It also covers the systems programs that control these operations, the applications programs that must be written for each specific application, and the special languages in which they are written. The use of computers for the solution of mathematical and statistical problems will be an essential part of the course.

The graduate's first move: a job or further study?

The openings for mathematicians are numerous, and the majority of mathematicians enter employment immediately after graduation. There are, however, two significant exceptions. Currently rather less than 10% of all mathematics graduates from British universities embark on a one-year postgraduate teacher-training course. The second group, about 15% of the graduates, stays on at university or polytechnic to work for a research degree (normally for three years leading to a doctorate) or to undertake a year's further study (lectures plus a small research project) to obtain a master's degree in some particular topic.

The research degree can prove advantageous to those wishing to make a career in research and development in industry or in government service and is virtually essential for those with an eye on the academic world. The one-year course can equip the graduate for jobs that might otherwise be hard to enter. The graduate can specialise in statistics, operational research, or numerical analysis if little of this was studied in the undergraduate course, or can learn about certain facets of engineering in preparation for a career in that direction.

Where do mathematicians start their careers?

Those graduates who take up posts after taking their first degree from a British university are distributed throughout the various employment sectors roughly as shown in Table 1. Table 2 gives us a broad picture of the work upon which they are engaged.

These figures include those who do a one-year teacher-training course after graduation. It has been assumed that they all enter teaching, an assumption that will slightly exaggerate the educational figure in the tables below.

TABLE 1

Industry (mainly engineering companies)	40%
Banks, insurance and accountancy practice	20%
Education	15%
Public service (central and local government, hospitals, HM Forces)	6%
Gas, water, electricity, mines and transport undertakings	4%
Others (mainly commerce)	15%

TABLE 2

Management services (computer work 38%, OR 2%, others 4%)	44%
Financial work (mainly chartered accountancy)	22%
Teaching/lecturing	14%
Scientific research, design, development	8%
Actuarial work	4%
Statistical work (other than scientific, actuarial or market research)	2%
Others (including market research)	6%

Mathematicians with higher degrees have a slightly different pattern of first employment. The three main employment sectors are education (36%), industry (30%), and public service (12%), and the most popular types of work are management services (40%), teaching/lecturing (25%) and scientific work (15%): financial work is as low at 6%.

It now remains to add some substance to this framework in order to portray more clearly the work undertaken by mathematicians.

What posts do mathematicians fill?

The answers to this question are too numerous for adequate treatment in an article such as this. Instead only a few jobs can be depicted and then only in a superficial way. Those for which mathematical skills are essential for some of the practitioners—if not for all—are described first.

Computer work. It will be of little surprise to find mathematicians heavily involved in the rapid spread of computer applications. The principal starting point is as an applications programmer, not with computer manufacturers but with computer users, many of whom undertake a lot of non-technical computing. This is true not only of organisations such as local authorities, banks, retail stores and libraries but also of technologically-based industries, which must operate efficiently in a commercial sense, however well-designed their products. Many mathematicians are thus engaged in computer work for which non-mathematicians are equally suitable: the logical qualities needed in a programmer might as readily be found in an archaeologist or classical scholar.

There is, however, a very substantial proportion of computer applications in fields requiring mathematical, scientific or engineering backgrounds. Examples are the derivation of numerical solutions to technical problems; the computer-aided design of electrical circuits and of mechanical devices, the computer control of chemical processes and of electricity generating plant, the operation of military equipment, the processing of seismic data in oil exploration, and computerised medical instrumentation.

Computer users are especially interested in recruiting graduates—not necessarily mathematicians—to provide technical support in the marketing of their products: the support staff help to define the customer's problems, and specify the equipment and programs required to deal with it. Manufacturers also recruit staff, often mathematicians, to write the systems programs that make their computer systems 'tick'.

Software houses represent a third—and important—group of employers of computer personnel. These are specialist concerns which provide programs and other services for clients; the application could be commercial or technological.

Operational research. Like computer work, this activity is pursued in many different types of organisation. In the Ministry of Defence OR methods are applied to decisions affecting the choice of future equipment and to the best mix of weapons. The results have, paradoxically, been of value to our negotiators at international discussions on arms control and disarmament. Examples of other government projects are the development of models (1) to help in decisions on the number, location and mode of operation of vehicles engaged in winter maintenance of roadways, and (2) to simulate the effects of changes in the number and types of prison establishments and to relate these to the security classification of prisoners. Typical local government problems are transport policies, vehicle maintenance and replacement, housing planning, and forecasting the number and geographical distribution of school pupils. The National Coal Board employs OR for forecasts of domestic and industrial demand up to 15 years ahead, and for computer modelling of underground coal transport. In industry, OR applications are as diverse as the industries, and can range from determination of the optimum balance of products from an oil refinery to the examination of the way the profits of an international trading company will be affected by movements in the world exchange rates.

Scientific research, design, development. Mention has already been made of a few of the problems tackled by mathematicians in this general field of work. Further examples, drawn mainly from industry and government establishments, embrace theoretical work on the prediction of neutron behaviour within the complex configuration of a nuclear reactor, the development of radar navigational aids, the fluid dynamics involved in industrial-scale chemical processes, the dispersal of effluents discharged into rivers or into the air from chimney stacks, and the use of mathematical models to represent the diffusion of oxygen in the lungs into the blood stream. Of possible interest to the pure mathematician is the Government Communications HQ where the effort is directed to the analysis of practical communication problems; the emphasis is on pure mathematics, although statistics and computer programming are also utilised.

Actuarial work. Actuaries belong to a small professional group (about 1500 qualified practitioners in the UK) with an arduous training behind them. They ascertain, on the basis of past and present evidence, the likelihood of an event occurring; probability, compound interest and statistical considerations are all involved. For instance, the premiums payable on life assurance policies and general insurance policies (house, car, marine) and on pension schemes are fixed by actuarial calculations. The actuary will also advise on the best way for insurance companies and pension funds to invest the vast sums of money they receive; the likely level of interest earned by these investments will in turn influence the premiums payable or the benefits accruing to policy holders or pensioners. The handling of large amounts of data calls for the use of computers.

Most actuaries are employed by life assurance and general insurance companies, but actuaries are also found in consulting firms, on the Stock Exchange, where they provide investment advice, in government service, where they are primarily responsible for advice on social security schemes and pensions in the public sector, and in industry and commerce. The actuary is also well qualified to enter operational research and corporate planning.

Statistical work. This will be discussed in terms of general statistics, the design and analysis of experiments, and market research.

In the United Kingdom the Civil Service is the largest employer of statisticians, who are employed on family expenditure and price indices, earnings surveys, industrial productivity, balance of payments, educational statistics and other indicators of the economic and social well-being of the nation. Within the health service, medical statistics provide information that helps in the planning of preventive and other forms of medical care. Statisticians in industry work on pricing policies, production comparisons, manpower planning, the availability and depletion ratio of raw materials, and the reliability of engineering equipment. In commerce their work could cover the analysis and forecasting of property investment, the business activity of banks, the flow of goods through ports, and accidents to oil tankers.

Pharmaceutical firms are very interested in statisticians to engage in the design and analysis of the lengthy experiments required to develop and test new drugs, but careful design and planning is an essential aspect of much other experimental work, as for instance in field trials in agriculture. Another aspect is the correct choice of tests and sampling methods for the quality control of products.

Statisticians in market research are concerned with consumers' reactions to existing or proposed products and the progress a product is making once in the market place. Services as well as products are open to such examination; BBC audience research is but one example.

Teaching and lecturing. Despite the declining British school population, there is considerable concern over the shortage of graduate mathematicians amongst school teachers, and mathematicians interested in teaching are in a relatively strong position. They can enter teaching as their first appointment or after a stint elsewhere; it is desirable that they do the one-year teacher-training course before entry. Teaching posts in further and higher education are not numerous, and are more likely to go to those with higher degrees and relevant work experience.

The armed forces require mathematicians to teach their subject or engage in meteorology and allied activities. Various organisations, educational, industrial and commercial, accept mathematicians as trainers of computer staff.

Financial work. This is the major example of a class of work that has a strong attraction for mathematicians, although a mathematics degree is not required. The largest single group become trainees in chartered accountancy, a profession requiring numeracy and a systematic approach to problems, combined with the personal skills needed to deal with clients when auditing company accounts, commenting on the proposed merger of firms, or advising ordinary people on their financial affairs. Chartered accountants and accountants belonging to other professional bodies are also to be found in industry, commerce and the public sector. The number of mathematicians recruited to some form of accountancy far outweighs those entering other types of financial work, such as banking and insurance.

Other work. About a third of all jobs for fresh graduates do not call for any specific degree discipline—work study, production management, retail management, administration in central and local government, personnel work, social work, just to mention a few. Mathematicians are found in all these walks of life, their degree of representation being largely determined by the interest they exercise for mathematicians.

Is mathematics enough?

A degree in mathematics can lead to many posts for which that qualification is a necessary condition. However, it is not generally a sufficient condition; appropriate personal qualities, including the ability to work alongside graduates in other disciplines, are also sought by employers. Even greater importance is attached to these qualities whenever a mathematician seeks a post for which other types of graduate are suitable—and such posts account for nearly half of the jobs that mathematicians enter after graduation. These considerations of personal characteristics apply throughout one's working life; many mathematicians follow a career path that takes them to senior managerial positions within their organisations, but again this is rarely achieved solely on intellectual ability.

In this article I have sketched the position today. In five years' time the employment pattern could well be different, but the demand for mathematicians will still be there.

Further reading

There are two publications which give more detailed information about careers for mathematicians. One is *Mathematical, Statistical and Computer Work*. This is Career Booklet 109, prepared by the Careers and Occupational Information Centre and published by HMSO. It costs 50p and is available through good bookshops. The other is *Careers in Mathematics*, edited by C. Richards. This can be obtained from the Institute of Mathematics and its Applications, Maitland House, Warrior Square, Southend-on-Sea, Essex SS1 2JY. The cost to non-members of the IMA is £1·50.

The Problem of Buffon's Needle

J. GANI

CSIRO Division of Mathematics and Statistics, Canberra

Dr Gani is chief of the CSIRO Division of Mathematics and Statistics, whose headquarters are located in Canberra, Australia. He has taught in universities in the U.K., U.S.A. and Australia, and retains a strong interest in mathematical education. His other main interest is the application of probability to problems of biology.

1. Introduction

In 1977, a Bicentenary Symposium was held in Paris to celebrate Buffon's 'Essai d'Arithmétique Morale' which was first published in 1777. Buffon (1707–1788) was a French naturalist who had been interested since 1733 in the application of probability to geometrical problems; he formulated the famous game now referred to as 'Buffon's needle'. This he solved elegantly by using the integral calculus, though there was an error in his results which Laplace corrected 34 years later.

Buffon's main interests were biological and botanical: he was the creator and keeper of the Jardin des Plantes in Paris, and published a 36-volume 'Natural History'. His problem initiated the development of geometrical probability, an important subfield of probability theory in which concepts of randomness are applied to geometry.

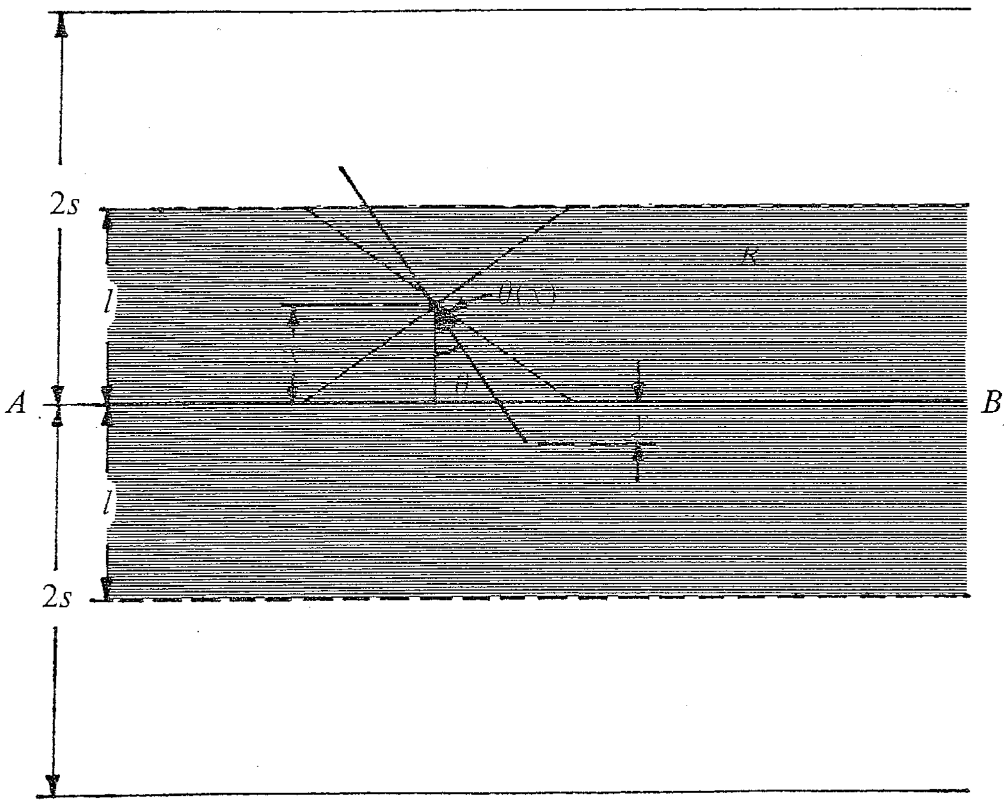


Figure 1. Buffon's needle on floorboards ($l \leq s$).

2. The problem and its solution

Buffon described his problem as follows: 'Suppose that a thin rod is thrown in the air in a room whose floor consists of parallel boards. Of two players, one bets that the rod will not intersect any of the parallel floor joins, while the other bets the opposite, namely that the rod will intersect one of these joins. One may ask which of the two has the higher odds. This game can be played on a checker board with a sewing needle, or a headless pin.' His method of solution is simple and ingenious. Consider a needle of length $2l$ which falls on floorboards of width $2s$; we shall for simplicity begin by analysing the case where the length of the needle is less than or equal to the width of the boards, or $l \leq s$. Let us examine a particular join AB. Then, as Figure 1 indicates, the needle will not be able to intersect AB unless its centre lies within the shaded region R. Let the centre of the needle fall at a distance x $(-s \leq x \leq s)$ from the join, and an angle θ $(-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi)$ to the vertical. For any given x in R $(-l \leq x \leq l)$, the needle will intersect the join AB provided the angle θ lies between $-\theta(x)$ and $+\theta(x)$ where $\cos\theta(x) = x/l$.

When the needle is said to be thrown 'at random' on the floorboards, we take this to mean that x and θ are uniformly and independently distributed. Thus

$$\Pr\{\text{centre of needle falls in } (x, x+\delta x)\} = \frac{\delta x}{2s} \tag{2.1}$$

since for any join AB, the needle may fall uniformly in $-s \leq x \leq s$.

Likewise

$$\Pr\{\text{angle of needle falls in } (\theta, \theta+\delta\theta)\} = \frac{\delta\theta}{\pi} \tag{2.2}$$

since θ will lie uniformly in $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$. One should point out at this stage that the formulation and solution of the problem depends on how we define its sample space. Here we use (x, θ) to define it, and allow x and θ to be uniformly distributed. But we could equally well define the position of the needle by (x, y) where y is the distance of the needle tip (nearest the join) from the join. In this case, if x and y (rather than x and θ) are uniformly distributed, the probabilities in the problem will be different from those we have used, and its solution will therefore also differ. However, in the present case the probabilities (2.1) and (2.2) seem most natural.

For any given x lying in R $(-l \leq x \leq l)$, we see from (2.2) that

$$\Pr\{\text{needle intersects } AB, \text{ given } x\} = \int_{-\theta(x)}^{\theta(x)} \frac{d\theta}{\pi} = \frac{2\theta(x)}{\pi}. \tag{2.3}$$

Hence the probability that the needle will intersect AB is

$$\int_{x=-l}^{l} \Pr\{\text{needle intersects } AB, \text{ given } x\}$$

$$.\Pr\{\text{centre of needle falls in } (x, x+\delta x)\}$$

$$= \int_{-l}^{l} \frac{2\theta(x)}{\pi} \cdot \frac{dx}{2s} = 2\int_0^l \frac{\theta(x)\,dx}{\pi s} = 2\int_0^l \frac{\cos^{-1} x/l}{\pi s} dx. \tag{2.4}$$

We can integrate this inverse cosine function, but it is probably easier to carry out the transformation

$$\cos\theta(x) = x/l, \qquad -\sin\theta(x)\,d\theta(x) = dx/l,$$

so that the Pr{needle intersects AB} is

$$2\int_0^l \frac{\theta(x)}{\pi s}\,dx = -2\int_{\pi/2}^0 \frac{l\theta\sin\theta}{\pi s}\,d\theta$$

$$= \frac{-2l}{\pi s}[\sin\theta - \theta\cos\theta]_{\pi/2}^0 = \frac{2l}{\pi s}. \tag{2.5}$$

We see from this that the answer to Buffon's question about the player's odds is that

$$\Pr\{\text{needle intersects } AB\} > \tfrac{1}{2}$$

if and only if $2l/\pi s > \frac{1}{2}$ or $\frac{1}{2}l > \frac{1}{4}\pi > 0{\cdot}78539$. Thus, if the length of the needle were four-fifths of the floorboard width, the odds would be in favour of intersecting a join. If, however, the needle were only three-quarters the width of the floorboard, the odds would be against its intersecting the join.

3. A simplified proof and the case of $l > s$

Recently, a very simple method of proof for Buffon's problem was outlined by Dr B. R. Stonebridge (reference 2). He pointed out that, if the sample space (x, θ)

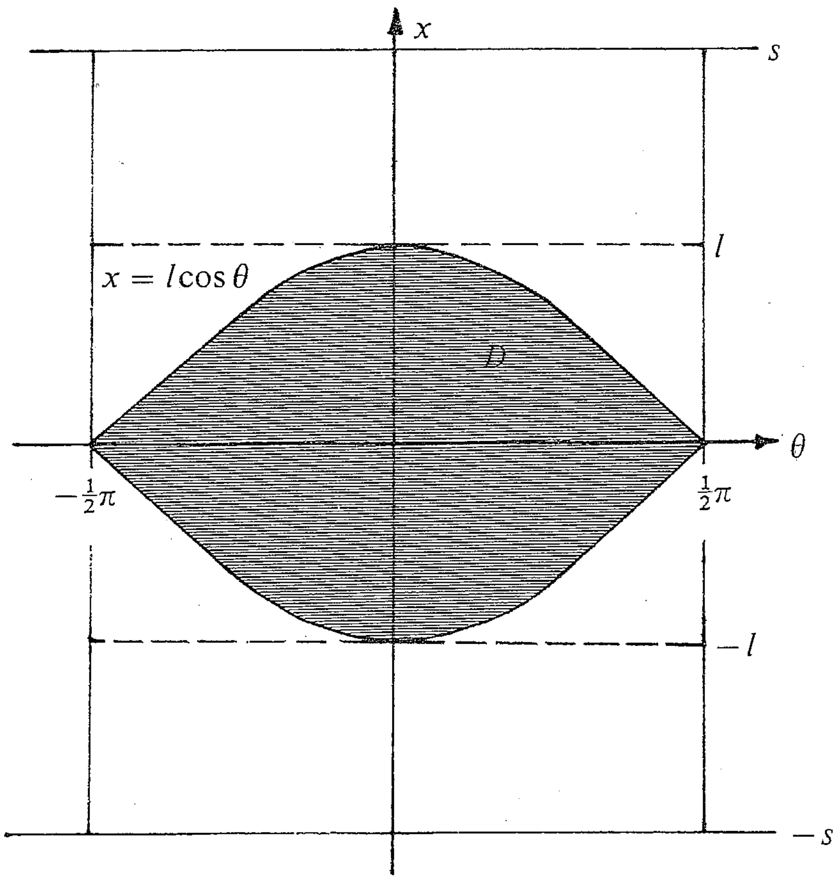


Figure 2. Sample space (x, θ) for Buffon's problem ($l \leq s$).

was mapped, then the set of values of x for which the needle would intersect a join is contained between the graphs $x = -l\cos\theta$ and $x = +l\cos\theta$ of Figure 2. Thus

$$\Pr\{\text{needle intersects join}\} = \frac{\text{shaded area } D}{\text{area of sample space}}$$

$$= 2\int_{-\pi/2}^{\pi/2} l\frac{\cos\theta}{2\pi s}\,d\theta = \frac{2l}{\pi s}. \tag{3.1}$$

This method lends itself readily to the proof of the slightly different result when the length of the needle is greater than the floorboard width, of $l > s$. In this case, as Figure 3 for the sample space (x, θ) indicates, the shaded area contained between the graphs

$$x = \max\{-l\cos\theta, -s\}, \qquad \text{and} \qquad x = \min\{l\cos\theta, s\},$$

gives the set of values of x for which the needle intersects a join. Hence

$$\Pr\{\text{needle intersects join}\} = \frac{\text{shaded area } E}{\text{area of sample space}}$$

$$= 2\int_{-\pi/2}^{\pi/2} \frac{\min\{l\cos\theta, s\}}{2\pi s}\,d\theta$$

$$= \frac{2}{\pi s}\{s\cos^{-1} s/l + l(1 - \sqrt{1 - s^2/l^2})\}$$

$$= \frac{2l}{\pi s} + \frac{2}{\pi}\cos^{-1} s/l - \frac{2l}{\pi s}\sqrt{1 - s^2/l^2}. \tag{3.2}$$

From this, it is clear that the previous result $2l/\pi s$ needs correction when $l > s$.

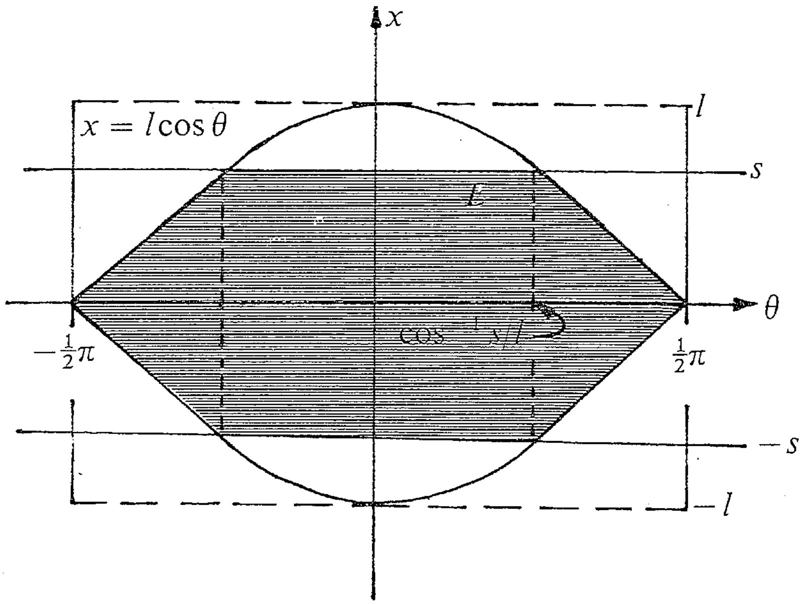


Figure 3. Sample space (x, θ) for $l > s$.

4. Estimates of π

The formula (3.1) can be used to estimate the value of π by the 'Monte Carlo' method of carrying out the Buffon needle experiment in practice. In 1873 Asaph Hall reported in his note 'On an experimental determination of π' that 'In 1864, my friend Captain O. C. Fox was unable to do active duty on account of a severe wound, and I proposed that he should make some experiments for determining the ratio n/m.' Here m indicates the number of throws of the needle and n the number of times it intersects a join. Hence n/m would be the 'Monte Carlo' estimate of the probability that the needle intersects a join. For the case $l \leq s$, this means that

$$\frac{n}{m}\text{ estimates }\frac{2l}{\pi s}, \qquad \text{or } \frac{2lm}{ns}\text{ estimates } \pi. \tag{4.1}$$

Hall reports that 'Captain Fox had made a plane wooden surface ruled with equidistant parallel lines, and on this he threw at random a fine steel wire.' In order to avoid any systematic error arising from his manner of throwing the wire, the surface was sometimes rotated slightly before the wire was thrown. The values below are extracted from Captain Fox's table of results:

	m	n	l	s	Condition of surface
Trial 1	500	236	3 inches	4 inches	surface stationary
Trial 2	530	253	3 inches	4 inches	surface revolved

From (4.1) we see that estimates of π from trials 1 and 2 respectively are:

$$\hat{\pi}_1 = \frac{2 \times 500 \times 3}{4 \times 236} = 3{\cdot}1780 \qquad \hat{\pi}_2 = \frac{2 \times 530 \times 3}{4 \times 253} = 3{\cdot}1423$$

which are reasonably close to 3·1416.

5. Further developments

The field of geometrical probability has grown very rapidly over the past 200 years. Those interested in the more recent developments outlined at the Bicentenary Symposium may consult the volume of its proceedings (reference 1).

Note that the method outlined in Section 4 can equally well be used to estimate l or s, as Professor G. S. Watson of Princeton University has pointed out. If the width of a floorboard is known to be $s = 4$ inches, and the value of π is assumed known, then by repeatedly throwing a small needle ($l \leq s$) m times and counting the number n of times it intersects a join, one can obtain an estimate

$$\hat{l} = \frac{\pi s n}{2m}$$

of the needle length l. If you have the patience, why not try it?

References

1. R. E. Miles and J. Serra (editors) *Geometrical Probability and Biological Structures: Buffon's 200th Anniversary* (Springer-Verlag, Berlin, 1978).
2. B. R. Stonebridge, Proof of Buffon's result, *Bull. IMA* **15**, (1979) 237.

Configurations

S. VAJDA
University of Sussex

Steven Vajda obtained his first degree and his doctorate from the University of Vienna. Thereafter he spent a number of years in actuarial work on the Continent and in England, but in 1944 he joined the Royal Naval Scientific Service as a statistician. His career took yet another turn in 1965 when he became Professor of Operational Research at the University of Birmingham. After his official retirement in 1967 he held various research fellowships in Birmingham and Sussex; at present he is Visiting Professor at both these universities.

In agricultural field trials one wishes to compare the yields of a number of varieties, or of treatments of the same variety. The yield may depend on the fertility of the plot or block where the variety is planted, and therefore the experimenter uses a number of blocks, and he plants the same variety in more than one block. Statistical analysis eliminates the effects of differential block fertility, and assesses the significance of differences in the yields.

As an example, we quote from reference 5 a design where the varieties, denoted by numbers 1 to 6, are arranged in blocks as follows:

$$\begin{array}{ccccc} 123 & 124 & 135 & 146 & 156 \\ 236 & 245 & 256 & 345 & 346. \end{array}$$

Each of the ten blocks contains the same number of varieties (3), and each of the six varieties appears the same number of times (5).

Now consider the following case:

Out of seven workers three are chosen every day of the week to perform some task, and the choice is to be such that no two workers work together on more than one single day.

Denoting the workers by numbers 1 to 7, this can be done by choosing the following triples on successive days:

$$\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \\ 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3. \end{array}$$

The two examples of arrangements have formal features in common. In either case we are dealing with objects of two types (varieties and blocks, or workers and triples), with an 'incidence relation' between them (blocks contain varieties, triples are made up of workers). Each object of the second type contains the same number of objects of the first type, and each object of the first type is contained in the same number of objects of the second type.

Such an arrangement is called a *configuration.*

It is usual to denote the number of objects of the first type by v (for varieties) and those of the second type by b (for blocks). The number of repetitions is denoted by r, and the size of a 'block' by k.

In our two examples we have, respectively,

$$v = 6, \qquad b = 10, \qquad r = 5, \qquad k = 3$$

and

$$v = 7, \qquad b = 7, \qquad r = 3, \qquad k = 3.$$

The first example has one important feature: any pair of varieties appears in two blocks. For instance, the pair (1, 2) appears in the first two blocks, and the pair (1, 3) appears in the first and in the third block. Such a configuration is called a *balanced incomplete block design*, and the construction and analysis of these designs is one of the fundamental tasks of mathematical statistics.

But configurations do not only have practical importance; part of their attraction is, as we shall see, their interaction with other branches of mathematics, particularly geometry.

A configuration can be described by using points and lines in a plane as the two objects with an incidence relation. In this geometrical representation a configuration is an arrangement of v points on b lines (which need not be straight) so that there are k points on each line and r lines through each point. Not every configuration has a significant geometrical representation, but our second example can be depicted as in Figure 1. The lines are the sides and the perpendicular heights of the equilateral triangle, and also the in-circle. The intersections of the circle with the heights are not counted as points.

Another example is the complete quadrilateral with all its points of intersection (see Figure 2), where $v = 6$, $b = 4$, $r = 2$, $k = 3$.

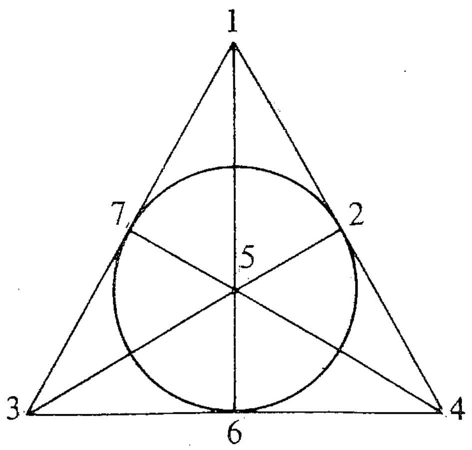


Figure 1

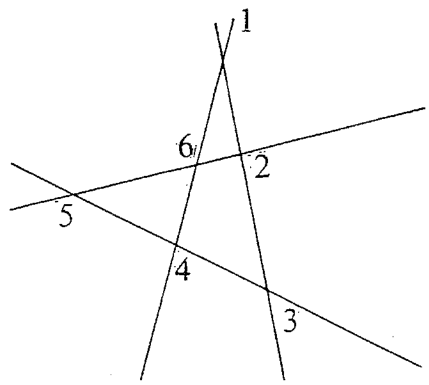


Figure 2

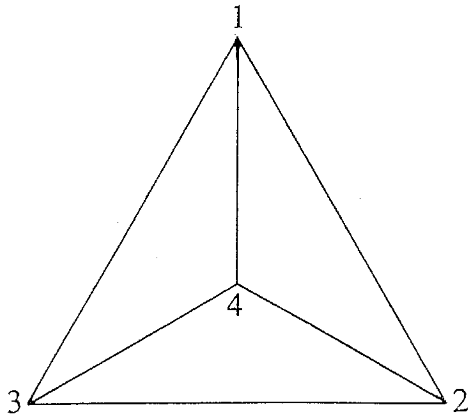


Figure 3

Yet another example is the (projection of a) tetrahedron, with $v = 4$, $b = 6$, $r = 3$, $k = 2$. (See Figure 3.)

In all examples we find that $bk = vr$. This must be so, because on each of the b lines there are k points; so there are altogether bk points, if we count each point as often as there are lines through it, viz. r times. On the other hand, each of the v points lies on r lines, which gives vr points, again each point counted r times.

The proof applies, of course, also to balanced incomplete block designs (our first example), provided that 'line' is replaced by 'block' and 'point' by 'variety'.

In Figure 1 we have $b = v$, and (hence) $r = k$. Such a configuration is called symmetric; we denote it by

$$(v_r).$$

A (v_2) is simply a polygon of v vertices and v sides.

There exists only one (7_3), shown in Figure 1, and only one (8_3), which can be represented as in Figure 4. To get the intersections right, in particular to get the straight lines 278 and 475 to intersect on the circle 176, the distances must be as indicated. (This is easy to check.)

There are three essentially different (9_3). One of them is the diagram illustrating a theorem of the 4th century BC Greek geometer Pappus.

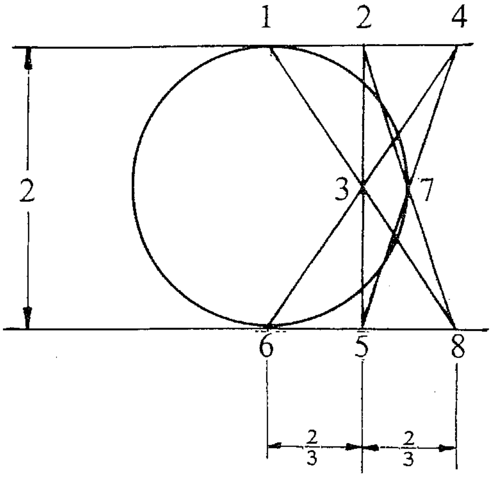


Figure 4

Pappus' Theorem. Let $l, \bar{l}$ be any two straight lines. If A, B, C are any three points on l and $\bar{A}, \bar{B}, \bar{C}$ are any three points on $\bar{l}$, then the three intersections of the pairs of lines $B\bar{C}$ and $\bar{B}C$, $C\bar{A}$ and $\bar{C}A$, $A\bar{B}$ and $\bar{A}B$ are collinear. (We assume here that none of the three pairs of lines is a pair of parallel lines.)

For the proof of the theorem, consider Figure 5.

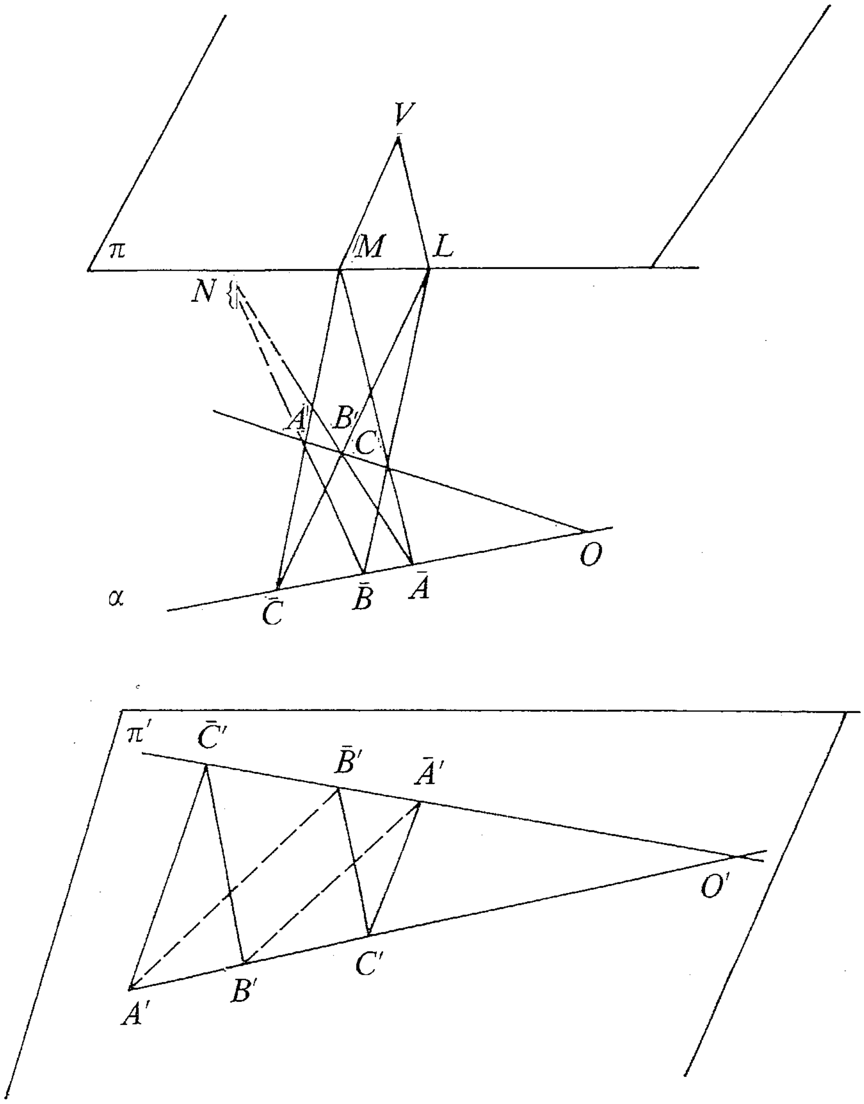


Figure 5

The lines $l = ABC$ and $\bar{l} = \bar{A}\bar{B}\bar{C}$ are in the central part of the figure, on a plane α. $\bar{B}C$ and $B\bar{C}$ meet in L, while $A\bar{C}$ and $\bar{A}C$ meet in M. We have also drawn $\bar{A}B$ and $A\bar{B}$. These two lines meet in N, and we have to prove that N is on the straight line through L and M.

Take a plane π through LM, different from α, and a point V in π. Project the lines and points on α from V onto a plane π', parallel to π. We note that pairs of lines on π' are parallel if and only if they are projections from V of pairs of lines on α which meet on LM or which are parallel to LM.

The lines $MA\bar{C}$ and $M\bar{A}C$ of α are projected into parallel lines $A'\bar{C}'$ and $\bar{A}'C'$ on π'. Also, the lines $LB\bar{C}$ and $L\bar{B}C$ are projected into parallel lines $B'\bar{C}'$ and $\bar{B}'C'$ on π'.

If the projections on π' of l and $\bar{l}$ meet at O', then, as a consequence of the parallelism of the two pairs of lines mentioned, we have

$$\frac{O'B'}{O'C'} = \frac{O'\bar{C}'}{O'\bar{B}'} \qquad \text{and} \qquad \frac{O'C'}{O'A'} = \frac{O'\bar{A}'}{O'\bar{C}'}.$$

Hence

$$\frac{O'B'}{O'A'} = \frac{O'\bar{A}'}{O'\bar{B}'},$$

which means that $\bar{A}'B'$ and $A'\bar{B}'$ are also parallel. Since, by hypothesis, $\bar{A}B$ and $A\bar{B}$ are not parallel, their point of intersection N lies on LM.

If the projections of l and $\bar{l}$ are parallel, then the triangles $A'B'\bar{C}'$ and $\bar{A}'\bar{B}'C'$ are congruent. Hence $B'C' = \bar{B}'\bar{C}'$, the triangles $A'\bar{B}'\bar{C}'$ and $\bar{A}'B'C'$ are also congruent, and therefore $\bar{A}'B'$ and $A'\bar{B}'$ are parallel. So again, L, M, N are collinear.

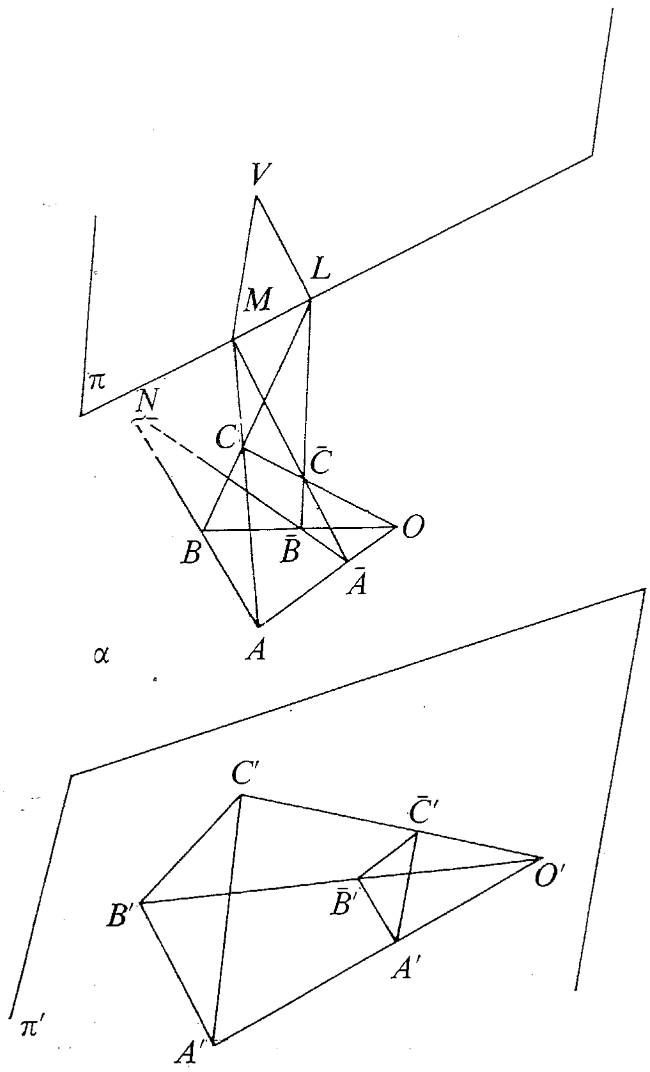


Figure 6

There exist ten different (10_3) and again one of them is a configuration relating to a classical theorem, that of Desargues (1591–1661).

Desargues' Theorem. If two triangles, ABC and $\bar{A}\bar{B}\bar{C}$ not degenerating into straight lines, and without common vertices, are such that $A\bar{A}$, $B\bar{B}$, $C\bar{C}$ pass through the same point O, then the intersections of the pairs of lines BC and $\bar{B}\bar{C}$, CA and $\bar{C}\bar{A}$, AB and $\bar{A}\bar{B}$ are collinear.

The proof can proceed on similar lines to that of the theorem of Pappus. This time the projection $B'C'$ will be parallel to the projection $\bar{B}'\bar{C}'$, and $C'A'$ will be parallel to $\bar{C}'\bar{A}'$. Hence

$$\frac{O'B'}{O'C'} = \frac{O'\bar{B}'}{O'\bar{C}'} \qquad \text{and} \qquad \frac{O'C'}{O'A'} = \frac{O'\bar{C}'}{O'\bar{A}'},$$

so that

$$\frac{O'B'}{O'A'} = \frac{O'\bar{B}'}{O'\bar{A}'}.$$

Therefore $A'B'$, $\bar{A}'\bar{B}'$ are parallel. Going back to plane α, it follows that AB and $\bar{A}\bar{B}$ meet on the line LM.

Desargues' theorem holds also if the lines $A\bar{A}$, $B\bar{B}$, $C\bar{C}$ are parallel.

These proofs of the theorems of Pappus and of Desargues by projection are taken from reference 2, Chapter 6.

If we describe a configuration by writing the symbols of the points on the same line in a row, then these symbols can be permuted without changing the meaning of the representation.

When the configuration is symmetric, and we write the rows so that they form a rectangle, as we have done in our second example, then we shall see that the symbols can be arranged in such a way that the first column of the rectangle contains every symbol once, and so do all the other columns. Of course, no symbol will appear more than once in the same row. Such an arrangement is called a *Latin Rectangle*.

The fact that such an arrangement is always possible (which is not at all obvious) can be proved in a variety of ways. Our proof here relies on an important theorem established by P. Hall in 1935. The theorem can be illustrated by the following situation. Let us suppose that in a school an athletics council is to be set up and it is to be composed of one representative from each of the school's first teams (in football, cricket, tennis, etc.). If the various teams have a non-overlapping membership, there is no difficulty about choosing a representative from each team. But what happens if some teams do overlap (as is likely to be the case in practice)? Hall's theorem gives a necessary and sufficient condition for the existence of a council of the desired kind: If there are n teams, then any m teams ($m = 1, 2, \ldots, n$) must between them have at least m different members.

We can now state the theorem formally.

Hall's Theorem. Let the sets $S_1, S_2, \ldots, S_n$ be given, the set S_i $(i = 1, \ldots, n)$ consisting of k_i elements. It is possible to choose n different elements $x_1, x_2, \ldots, x_n$, where x_i belongs to S_i $(i = 1, \ldots, n)$ and is called the 'representative of S_i', if and only if any m of the sets $(m = 1, 2, \ldots, n)$ contain between them at least m different elements.

It is not assumed in the theorem that all k_i are equal, but we shall apply it to a case where this is so.

In symbols, Hall's condition can be written as follows:

$$\left|\bigcup_{i\in M} S_i\right| \geq |M|$$

for any subset M of $\{1, 2, \ldots, n\}$. Here $|S|$ denotes the number of elements in a set S; and

$$\bigcup_{i\in M} S_i$$

is the set consisting of all elements which appear in any set S_i for which i belongs to M. For instance, if $M = \{2, 3, 5, 8\}$, then

$$\bigcup_{i\in M} S_i = S_2 \cup S_3 \cup S_5 \cup S_8.$$

Note that the necessity of the condition is obvious. We shall now show that it is also sufficient, by a proof due to R. Rado.

We do this by showing that, if Hall's condition holds, then it is possible to remove one element from any set S_i for which $|S_i| = k_i \geq 2$, and the condition will still hold. The element to be removed must be correctly chosen, though.

If, in this manner, we can reduce all sets until each contains just one single item, and the condition still holds, then clearly the remaining items form a system of distinct representatives (also called a 'transversal') of the S_i.

The proof is indirect.

We look at a set, say S_1, which contains the elements x and y (and possibly others as well) and assume that, if x is removed from S_1, then Hall's condition is violated, and that it is also violated when y is removed.

This will lead to a contradiction, which will show that at least one of x, y can be removed from S_1 without invalidating the condition.

Let us then assume that there is a subset A of $\{2, \ldots, n\}$ such that the union of $\bigcup_{i\in A} S_i$ and of $S_1 - \{x\}$, the set S_1 without x, contains fewer than $|A| + 1$ items, i.e. that

$$\left|\left(\bigcup_{i\in A} S_i\right) \cup (S_1 - \{x\})\right| \leq |A|.$$

Let us also assume that, if we remove y from S_1, then there is a subset B of $\{2, \ldots, n\}$ such that

$$\left|\left(\bigcup_{i\in B} S_i\right) \cup (S_1 - \{y\})\right| \leq |B|.$$

Adding, we have

$$\left|\left(\bigcup_{i\in A} S_i\right)\cup(S_1-\{x\})\right|+\left|\left(\bigcup_{i\in B} S_i\right)\cup(S_1-\{y\})\right|\le |A|+|B|. \tag{1}$$

Now generally, if P and Q are two sets, then

$$|P|+|Q|=|P\cup Q|+|P\cap Q|,$$

where $|P\cap Q|$ is the number of elements which appear in both P and Q.

Applying this to the left-hand side of (1), we have

$$\left|\left[\left(\bigcup_{i\in A} S_i\right)\cup(S_1-\{x\})\right]\cup\left[\left(\bigcup_{i\in B} S_i\right)\cup(S_1-\{y\})\right]\right|$$
$$+\left|\left[\left(\bigcup_{i\in A} S_i\right)\cup(S_1-\{x\})\right]\cap\left[\left(\bigcup_{i\in B} S_i\right)\cup(S_1-\{y\})\right]\right|. \tag{2}$$

But

$$(S_1-\{x\})\cup(S_1-\{y\})=S_1,$$

because both x and y will still be present in the union. Therefore the first term of (2) is greater than or equal to $|(\bigcup_{i\in A\cup B} S_i)\cup S_1|$. The second term of (2) is greater than or equal to $|\bigcup_{i\in A\cap B} S_1|$.

Thus the left-hand side of (1) is greater than or equal to

$$\left|\left(\bigcup_{i\in A\cup B} S_i\right)\cup S_1\right|+\left|\bigcup_{i\in A\cap B} S_i\right|. \tag{3}$$

By the condition of the theorem, the first term of (3) is greater than or equal to $|A\cup B|+1$, and the second term is greater than or equal to $|A\cap B|$.

Adding these upper bounds we reach the conclusion that the left-hand side of (1) is greater than or equal to

$$|A\cup B|+1+|A\cap B|=|A|+|B|+1.$$

But according to (1), its left-hand side is less than or equal to $|A|+|B|$, and we have reached a contradiction.

We shall now use this theorem to prove the possibility of writing a symmetric configuration as a Latin Rectangle.

First, we establish the fact that any m rows contain between them at least m different elements.

Altogether they contain mk elements, but some may, of course, be repeated. However, none of them can be repeated more than $r=k$ times in the m sets, so that each element appears at least $mk/r=m$ times.

According to Hall's theorem we can now choose a different element from each row to put into the first column. Then, when this element is omitted from its row, we are still left with a symmetric configuration, and the argument can be repeated for the remaining columns.

If we write, as continuation of each row, those of the v elements which do not appear in that row, then the added elements can again be ordered into a Latin Rectangle, and the two Latin Rectangles together form a *Latin Square*. No two equal elements appear in any row or in any column.

L. Euler (1707–1783), who studied such patterns, used roman numerals where we used numbers, and this accounts for the name of the square.

We have already written (7_3) as a Latin Rectangle. It might be of interest to see this also for the other symmetric configurations we have been dealing with.

(8_3)	(9_3) (Pappus)	(10_3) (Desargues)
124	186	012
235	293	195
346	351	269
457	462	304
568	528	427
671	637	583
782	745	650
813	879	731
	914	846
		978

In the last two configurations we have replaced our previous symbols by numbers, as follows:

O	A	$\bar{A}$	B	$\bar{B}$	C	$\bar{C}$	L	M	N
0	1	2	3	4	5	6	7	8	9

We finally turn to configurations which have the following property: each pair of varieties appears precisely in one block (not in two blocks as in a balanced incomplete block design). In particular, we take k, the number of varieties in a block, to be 3. These configurations are called *Steiner Triple Systems*, after J. Steiner (1796–1863) who considered them in a study of the bi-tangents of plane quartic curves. (A bi-tangent of a curve touches the curve at two points.)

It has been shown that such systems exist if and only if $v = 3$, $6t + 1$, or $6t + 3$, where t is a positive integer. The number of triples in the system is then $t(6t + 1)$ when $v = 6t + 1$, and $(2t + 1)(3t + 1)$ when $v = 6t + 3$. In the former case $r = 3t$, and in the latter case $r = 3t + 1$.

Except for $v = 7$, a Steiner Triple System is not symmetric.

For $v = 3$, 7 and 9 only one such system exists. There are two for $v = 13$, and 80 for $v = 15$.

The case $v = 7$ is the same as (7_3). The reader might amuse himself by constructing the system for $v = 9$, when $b = 12$, $r = 4$.

We quote here one of the systems for $v = 15$:

1	2	3	2	4	6	3	4	7	3	5	6
4	8	12	3	9	13	6	10	14	7	11	12
7	9	14	5	11	14	1	8	11	2	8	13
6	11	13	7	8	10	5	12	13	1	9	10
5	10	15	1	12	15	2	9	15	4	14	15

1	4	5	1	6	7	2	5	7
2	10	11	5	8	9	1	13	14
6	9	12	4	10	13	3	10	12
3	8	14	2	12	14	4	9	11
7	13	15	3	11	15	6	8	15

This particular design has the additional property of being 'resolvable': it is possible to group the blocks into sets, as we have done, so that the blocks of each set contain all elements once between them. Resolvable Steiner Triple Systems exist if $v = 6t + 3$.

The system which we have quoted solved the schoolgirl problem of Kirkman (see reference 1 or reference 3, Chapter X):

Fifteen girls go for a walk every day of the week, three abreast. How can these walks be arranged so that during the seven days of the week every pair of girls walks once in the same line?

For further information on problems of the type we have dealt with here the reader could consult reference 4.

References

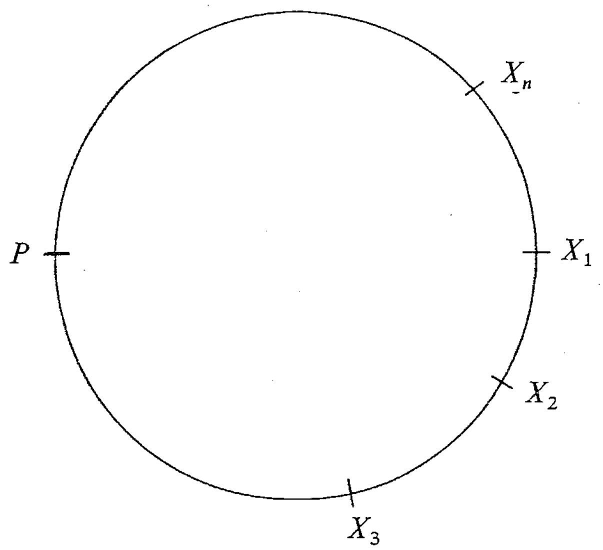
1. N. Biggs, The schoolgirls problem. *Math. Spectrum* **12** (1979), 7–11.
2. Hazel Perfect, *Topics in Geometry* (Pergamon Press, Oxford, 1963).
3. W. W. Rouse Ball, revised by H. S. M. Coxeter, *Mathematical Recreations and Essays* (Macmillan, London, 1942).
4. S. Vajda, *Patterns and Configurations in Finite Spaces* (Griffin, London, 1967).
5. F. Yates, Complex experiments. *J.R. Statist. Soc. Suppl.* **2** (1935), 181–247.

Dear Editor,

Minimizing delivery costs

I can hardly believe that the following problem is new, but I have not been able to find it in the obvious places: $X_1, X_2, \ldots, X_n$ are n shops occurring in this order round a circular lake (or on a ring road round a central business district where traffic is not allowed). Each shop has to receive one daily delivery from a warehouse P. Where should P be situated in order to minimize the total travelling distance of the delivery van? In other words we must find P to minimize $\sum_{i=1}^{n} |PX_i|$, where $|PX_i|$ is the length of the shorter of the two arcs PX_i, X_iP. For convenience, we number the shops in the clockwise sense.

Case $n = 3$. Suppose arc $X_1X_2 = \max(\text{arc}\, X_1X_2,\ \text{arc}\, X_2X_3,\ \text{arc}\, X_3X_1)$, all arcs being measured in the clockwise sense. Then it is obvious that P must be at X_3, i.e. at the vertex not on the largest arc.



Case $n = 4$. Suppose $X_1X_2 = \max(X_1X_2, X_2X_3, X_3X_4, X_4X_1)$, all arcs again being taken in the clockwise sense. It can then be shown that (i) at most one of $X_4X_1X_2$ and $X_1X_2X_3$ can be less than a semicircle; (ii) if $X_4X_1X_2$ is less than a semicircle, then $\sum|PX_i|$ is a minimum for P anywhere on arc X_4X_1, and (iii) if $X_1X_2X_3$ is less than a semicircle, then $\sum|PX_i|$ is a minimum for P anywhere on arc X_2X_3 [so that in (ii) and (iii), P is on the shortest of $X_1X_2, X_2X_3, X_3X_4, X_4X_1$]; (iv) if both $X_4X_1X_2$ and $X_1X_2X_3$ are larger than a semicircle, then $\sum|PX_i|$ is a minimum for P anywhere on X_3X_4 [i.e. P is on the arc opposite the largest of $X_1X_2, X_2X_3, X_3X_4, X_4X_1$]. A proof is based on inequalities and elementary geometry. For $n > 4$ the problem seems more difficult and I do not know the answer.

Yours sincerely,

A. V. Boyd
(University of the Witwatersrand, Johannesburg 2001, South Africa)

Dear Editor,

First and last digits of large numbers

Thank you for a most interesting issue of *Mathematical Spectrum* (Volume 12, Number 3). On page 70, the author pointed out that

$$(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17) + 1 = 19 \times 26869,$$

and so is not prime. The smaller number

$$(2 \times 3 \times 5 \times 7 \times 11 \times 13) + 1 = 59 \times 509,$$

and so is not prime either. Also, on page 71, $2^{11937} - 1$ is given as the largest prime number known in 1971. This should be $2^{19937} - 1$. In fact, $2^{11937} - 1$ is not prime because 11937 is divisible by 3, and the author pointed out that, if $2^n - 1$ is prime, then n must be prime. Also, the initial digits in the decimal expansion of $2^{21701} - 1$ are 448, not 488 as asserted on page 90.

Readers may be interested in the following method of finding the numbers of decimal digits and the starting and ending digits for large numbers such as $2^{21701} - 1$, using a calculator which displays 10 digits. Let $N = 2^{21701}$. Then

$$\log N = 21701 \log 2 = 6532{\cdot}651936.$$

Hence the number of decimal digits of N is 6533. Also, N starts with the digits of antilog 0·651936, i.e. 448679 .... Finding the last five digits depends on the fact that the last five digits of the product of two numbers are the same as for the product of their last five digits; for example, the last five digits of

$$123456789 \times 987654321$$

are the same as the last five digits of

$$56789 \times 54321,$$

i.e. 35269. Thus $2^{32} = 4294967296$, so we have the following table:

number	last five digits
2^{23}	67296
2^{64}	51616
2^{128}	11456
2^{256}	39936
2^{512}	84096
2^{1024}	37216
2^{2048}	30656
2^{4096}	90336
2^{8192}	92896
2^{16384}	66816

Now

$$21701 = 16384 + 4096 + 1024 + 128 + 64 + 5,$$

so that the last five digits of 2^{21701} are the same as the last five digits of

$$66816 \times 90336 \times 37216 \times 11456 \times 51616 \times 32,$$

which gives 82752. Hence

$$2^{21701} - 1 = 448679 \ldots 82751.$$

Yours sincerely,

BOB BERTUELLO (Open University)
(12 Pinewood Road, Midsomer Norton, Bath, Avon)

(Ed: We are grateful to Bob Bertuello for pointing out slips in Volume 12 Number 3.)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

13.1. (Submitted by L. Mirsky, University of Sheffield.) Let n be a natural number. Show that every set of n natural numbers contains a non-empty subset the sum of whose elements is divisible by n.

13.2. How may 17 straight lines be drawn in the plane so that they meet in exactly 101 points? (A problem first proposed by Fourier—see the article by R. J. Webster in this issue.)

13.3. Solve the matrix equation $X = Y + A \times B$ for X when $A^k = 0$ for some positive integer k.

Solutions to Problems in Volume 12, Number 2

12.4. Find the number of digits required to print simultaneously all the integers from zero to a million inclusive in (a) decimal form (base ten), (b) binary form (base two), (c) duodecimal form (base twelve).

Solution

To base b, the numbers from 0 to $b-1$ require b digits, those from b to b^2-1 require $2(b^2-b)$, those from b^2 to b^3-1 require $3(b^3-b^2)$, and so on. Consider the natural number n, and suppose that the integer k is such that $b^k \leq n < b^{k+1}$. The integers from b^k to n require $(k+1)(n-b^k+1)$ digits. The total number of digits required to print the integers from 0 to n inclusive to base b is thus

$$\begin{aligned} b + 2(b^2-b) + 3(b^3-b^2) + \cdots + k(b^k - b^{k-1}) + (k+1)(n-b^k+1) \\ = (k+1)(n+1) - (b + b^2 + \cdots + b^k) \\ = (k+1)(n+1) - \frac{b(b^k-1)}{b-1}. \end{aligned}$$

This gives the three answers (in decimal form)

(a) 5888897, (b) 18951446, (c) 5728554.

12.5. (i) In a sequence of real numbers, the sum of every N consecutive terms is negative, whereas the sum of every M consecutive terms is positive. Show that the sequence must have fewer than $M+N-D$ terms, where D is the highest common factor of M and N. (ii) In a sequence of positive real numbers, the product of every N consecutive terms is less than 1, whereas the product of every M consecutive terms is greater than 1. Show that, again, the sequence must have fewer than $M+N-D$ terms.

Solution

(i) Put $M = mD$, $N = nD$, and consider a sequence of real numbers possessing at least $M + N - D = (m + n - 1)D$ terms. Denote the sum of the first d of the terms of the sequence by s_1, the sum of the next d by s_2, and so on to s_{m+n-1}. By hypothesis,

$$\left.\begin{array}{l} s_1 + s_2 + \cdots + s_m < 0, \\ s_2 + s_3 + \cdots + s_{m+1} < 0, \\ \text{-----------------} \\ s_n + s_{n+1} + \cdots + s_{m+n-1} < 0, \end{array}\right\} \quad (1)$$

whereas

$$\left.\begin{array}{l} s_1 + s_2 + \cdots + s_n > 0, \\ s_2 + s_3 + \cdots + s_{n+1} > 0, \\ \text{-----------------} \\ s_m + s_{m+1} + \cdots + s_{m+n-1} > 0. \end{array}\right\} \quad (2)$$

But the sum of the left-hand sides of the inequalities (1) is the same as that of the left-hand sides of (2), which is impossible.

(ii) This follows from (i) if we take the log of each term of the sequence.

12.6. There are six events, and in each year from AD 1 to 1979 exactly one of these events occurs. Show that there exist years x, y, z, with $x = y + z$, in which the same event occurs.

Solution

Consider six sets, set i consisting of the years from AD 1 to 1979 in which event i occurs. Since $1979/6 > 329$, one of the sets, say set 1, must contain at least 330 years, say $x_1, x_2, \ldots, x_{330}$, labelled in increasing order. Consider the 329 years

$$x_2 - x_1, x_3 - x_1, \ldots, x_{330} - x_1.$$

If one of these occurs in set 1, we can find x, y, z with the required property, so we suppose that none of these occurs in set 1. Since $329/5 > 65$, one of the sets 2 to 6, say 2, contains at least 66 of these, say $y_1, y_2, \ldots, y_{66}$, labelled in increasing order. Consider the 65 years

$$y_2 - y_1, y_3 - y_1, \ldots, y_{66} - y_1.$$

These are differences of two x's as well as differences of two y's, so that, if any of these occurs in either set 1 or set 2, we can find x, y, z with the required property, so suppose they do not occur in sets 1, 2. Then one of sets 3 to 6, say set 3, must contain at least 17 of these years, say $z_1, z_2, \ldots, z_{17}$, labelled in increasing order. Consider the 16 years

$$z_2 - z_1, z_3 - z_1, \ldots, z_{17} - z_1.$$

As before, if any of these occurs in years 1, 2, 3, we can find suitable x, y, z, so suppose they do not so occur. Then we can suppose that set 4 contains at least 6 of these years, say $t_1, t_2, \ldots, t_6$, labelled in increasing order. Consider the 5 years

$$t_2 - t_1, t_3 - t_1, \ldots, t_6 - t_1.$$

As before, we can suppose that these all occur in sets 5, 6, so one of these, say 5, must contain at least 3 of them, say v_1, v_2, v_3, labelled in increasing order. Consider finally the three years

$$v_2 - v_1, v_3 - v_1, v_3 - v_2.$$

If any of these occurs in one of the sets 1 to 5, we have found suitable x, y, z. If not, then they all occur in set 6, and now

$$v_3 - v_1 = (v_3 - v_2) + (v_2 - v_1).$$

Contents

R. J. WEBSTER 1 Joseph Fourier—his life and work

R. C. FAUST 7 The versatile mathematician

J. GANI 14 The problem of Buffon's needle

S. VAJDA 19 Configurations

29 Letters to the Editor

31 Problems and Solutions

ISSN 0025–5653

PRICES (*postage included*)

Prices for Volume 13 (*Issues Nos.* 1, 2, *and* 3):

Subscribers in Britain and Europe: £2·00
Subscribers overseas: £4·00 (US\$9.00; \$A8.00)
(These prices apply even if the order is placed by an agent in Britain.)
A discount of 10% is allowed on all orders for five or more copies.

Back issues:

All back issues are available; information concerning prices may be obtained from the Editor.

Enquiries about rates, subscriptions and advertisements should be directed to:

Editor—*Mathematical Spectrum*,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Printed in England by Galliard (Printers) Ltd, Great Yarmouth