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E U R E K A

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All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, Rideau Campus, 200 *Lees Ave., Ottawa, Ont., K1S* 0*C*5.

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REPORT ON PRIMES II

VIKTORS LINIS, University of Ottawa

Goldbach's conjecture (stated in a letter to Euler, June 7, 1742) that every positive integer ≥ 7 is a sum of three primes still defies proof despite numerous and continual efforts of the best mathematicians. Nevertheless, considerable progress towards the solution has been made. A brief survey of the main results was recently given by W. Narkiewicz (*Jahresbericht d. Deutschen Mathem.*, Ver. Bd. 77, 2, 1975); here are some excerpts from that work.

The first important breakthrough was achieved by L. Schnirelman who proved in 1930, using density theorems and sieve methods, that every natural number greater than 1 can be represented as a sum of at most C primes. However, the value of C was enormous (of order $6\cdot 10^9$). On the other hand, Schnirelman's method shows that for sufficiently large natural numbers 12 primes are sufficient.

In 1937, I.M. Vinogradov proved Goldbach's conjecture for all natural numbers greater than some constant V. It turned out that this constant is of order $\exp\exp 16$, which defies direct verification even by the most powerful computers for the interval $(7,V)$.

A closely related conjecture is that every even number ≥ 6 is a sum of two odd primes. This conjecture implies Goldbach's conjecture, but not conversely. It turns out that this conjecture is even harder than Goldbach's, but the results are more varied and interesting.

The first important result was Viggo Brun's who proved in 1920 that every even number $2N$ can be written as a sum of two numbers each of which has at most 9 prime divisors. This can be expressed in the form $2N = P_9 + P_9$. Later this was improved to $P_4 + P_4$ (Tartakovskij and Buchštab) and to $P_1 + P_6$ (Esterman, using Riemann's hypothesis). A new powerful method was introduced by Rényi in 1948 with the so-called large sieve method (invented by Linnik); he proved Esterman's result unconditionally. The latest improvement is to $P_1 + P_2$ by J. Chen in 1966.

However, it appears unlikely that by any presently known method the desired result $P_1 + P_1$ can be achieved.

Other results concerning the "even number conjecture" deal with asymptotic evaluations. It has been proved that *almost* all even natural numbers are sums of two primes. The best recent result is by H.L. Montgomery and R.C. Vaughan (1975); they proved that the exceptional numbers have density of order $o(x^{1-a})$, $a > 0$.

* * *

LETTERS TO THE EDITOR

Dear editor:

Many thanks for all the beautiful material you have just sent me. I am delighted to have discovered the existence of EUREKA and am looking forward to the republication of Volume 1 in its entirety, as well as the current issues as they appear.

The enclosed material on the BUTTERFLY PROBLEM provides a further example of rediscovery in the field of Mathematics.

LEON BANKOFF,
Los Angeles, California.

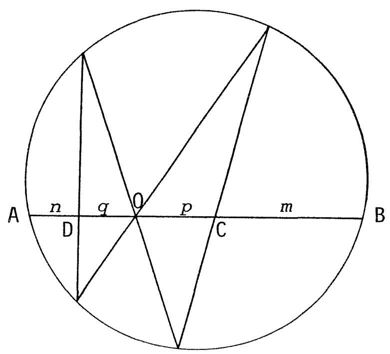
Editor's comment.

The enclosed material was a photocopy of the first part of an article by A.L. Candy, Lincoln, Nebraska, entitled *A general theorem relating to transversals, and its consequences* (*Annals of Mathematics*, 1896, pp. 175-190).

The principal result achieved, which was followed by a lengthy discussion of special cases and extensions, was the following: if (see figure) OA $= n$, OB $= m$, OC $= p$, OD $= q$, then

$$mn(p - q) = pq(m - n). \qquad (1)$$

If $m = n$, it follows that $p = q$, and we have the Butterfly Theorem.



If (1) is rewritten as

$$\frac{mn}{m-n} = \frac{pq}{p-q},$$

we see that we have a way of constructing two pairs of segments the products of whose lengths are proportional to their differences, an interesting bit of information.

The "rediscovery" mentioned by Bankoff alludes to the fact that the method used to prove (1) was essentially the same as that used by Conrad in his proof of the Butterfly Theorem [1976: 2], that is, by ratios of areas.

Dear editor:

Dan Eustice showed me your letter with the attached issue of EUREKA. The publication seems quite interesting. . .

I remember the Butterfly Problem from my time in high school. Boys High School (Brooklyn, New York) had in my time there a very active mathematics team, with two hours' practice per day. The Butterfly Problem, as we learned about it, had a rather complicated solution.

I very much like the solution to Problem 75 by Walter Bluger [1976: 10]. When I first heard of the problem, I found a solution, like yours [1976: 11], which involved the Butterfly Problem, but I had to look up the proof of the Butterfly Theorem, which I had forgotten! (I found it in Coxeter and Greitzer.) Apparently, several people heard of the problem at the same time.

I found several of the proposed problem in EUREKA interesting, and have written up solutions for them, which I am submitting. . .

I hope that your journal prospers!

LEROY F. MEYERS,
The Ohio State University,
Columbus, Ohio.

Dear editor:

It was very thoughtful of you to send me the copy of EUREKA containing your interesting article with a reference to one of my solutions. I thank you.

Strangely enough, a letter of mine to Professor Maskell inquiring about EUREKA must have crossed yours in the mails.

You are editing a delightful periodical. I look forward expectantly to future issues. Congratulations.

Enclosed are solutions of five problems that you may find acceptable.

CHARLES W. TRIGG,
San Diego, California.

Dear editor:

Thank you for the further copy of EUREKA, with the interesting details of the Steiner-Lehmus Theorem [1976: 19-24]. I like the proofs you give. . .

A high school girl recently wrote to the Library of Congress to ask for help in working it. Since she is in Minnesota, the Library of Congress suggested that she write to the Mathematics Library here, and since the librarian knows me, I was able to help her. . .

DAN PEDOE,
University of Minnesota,
Minneapolis, Minnesota.

Dear editor:

Your superb handling of the Steiner-Lehmus Theorem in the February 1976 issue of EUREKA [1976: 19-24] provides the interested reader with such an abundance of references that you can just about claim to have exhausted everything that can be said on the subject! That is, everything except one little item! This was discovered quite by accident.

In 1968 while browsing in Gibert's on Boulevard St-Michel in Paris, I came across a fascinating old geometry book. It was not dated, but judging by its white-on-black diagrams I estimated it to be about a hundred years old. As I flipped through the pages, my eye was attracted by a familiar diagram--one associated with such an unbelievably simple proof of the Steiner-Lehmus Theorem that my pulse raced with excitement. What a rare gem! At last--a proof to annihilate all rival proofs! The world-famous problem that had been responsible for universal frustration was here neatly and incredibly disposed of in two brief sentences!

I hurriedly paid for my newly-found treasure and could hardly wait to get back to my hotel to dash off letters to friends across the Altantic. Back at my room, I ran to the desk, opened my precious acquisition and re-read the proof. Slowly my jaw dropped with consternation; then suddenly, to the utter amazement of my wife, I burst out in hysterical laughter.

Enclosed is a photocopy of this astounding proof that I had thought was too good to be true but which turned out not true enough to be good. Later, in a private communication, Professor Coxeter aptly characterized the proof as "charmingly stupid."

Can your readers tell why--and, it is hoped, more quickly than I could?

LEON BANKOFF,
Los Angeles, California.

Editor's comment.

Bankoff found his serendipitous "proof" on p. 314 in *Exercices de Géométrie*, by M.Ph. André, 14th edition, Librairie classique de F.-E. André-Guédon, E. André fils, Successeur (no date).

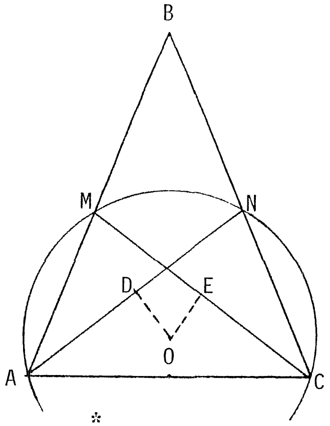
To find such a "proof" in a presumably reputable treatise on geometry is remarkable enough, but to find it in a *fourteenth* edition (and how many later ones?) is simply astounding. The similarity in names between author and publisher leads one to speculate whether the publication of this work might not have been more an act of filial piety rather than a desire to aid the cause of mathematical education. (Well, there's nothing wrong with nepotism, as long as it's kept in the family.)

Here, in its entirety, is the "proof" found by Bankoff. (An English translation is given in Problem 141 below.)

Lorsque dans un triangle deux bissectrices sont égales, le triangle est isocèle.

Sur le milieu des bissectrices, j'élève deux perpendiculaires qui se rencontrent en O; du point O comme centre, avec AO pour rayon, je décris une circonférence qui passera évidemment par les points A, M, N, C.

Or les angles MAN, MCN sont égaux comme ayant l'un et l'autre pour mesure $\frac{MN}{2}$; donc BAC = ACB, et le triangle ABC est isocèle.



* * *

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 89.

For the problems given below, solutions, if available, will appear in EUREKA Vol. 2, *No.* 8, *to be published around Oct.* 15, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Oct.* 1, 1976.

141. *Proposed by Leon Bankoff, Los Angeles, California.*

What is wrong with the following "proof" of the Steiner-Lehmus Theorem? (See figure above and p. 92 for the source.)

If in a triangle two angle bisectors are equal, then the triangle is isosceles.

At the midpoints of the angle bisectors, I erect two perpendiculars which meet in O; with O as center and AO as radius, I describe a circle which will evidently pass through the points A, M, N, C.

Now the angles MAN, MCN are equal since the measure of each is arc $\frac{MN}{2}$; hence BAC = ACB, and triangle ABC is isosceles.

142. *Proposed by André Bourbeau, École Secondaire Garneau.*

Find 40 consecutive positive integral values of x for which $f(x) = x^2 + x + 41$ will yield composite values only.

143. *Proposé par Léo Sauvé, Collège Algonquin.*

On donne

$$f(n) = x^n + y^n + z^n,$$

où (x,y,z) est un triplet de nombres complexes tels que $f(n) = n$ pour $n = 1,2,3$. Montrer que le triplet (x,y,z) n'est pas réel et calculer $f(4)$, $f(5)$, et $f(6)$.

144. *Proposed by Viktors Linis, University of Ottawa.*

In a triangle ABC, the medians AM and BN intersect at G. If the radii of the inscribed circles in triangles ANG and BMG are equal, show that ABC is an isosceles triangle.

(I found this problem in DELTA No. 1, 1976, a Polish journal for the popularization of physics and mathematics.)

145. *Proposed by Walter Bluger, Department of National Health and Welfare.*

A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 collinear points are equal.

Construct a magic pentagram with the smallest possible positive primes.

146. *Proposé par Jacques Marion, Université d'Ottawa.*

Montrer qu'il n'existe pas de fonction rationnelle $R(z)$ telle que $R(n) = n!$ pour tout nombre naturel n.

147. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

In square ABCD, $\overline{AC}$ and $\overline{BD}$ meet at E. Point F is in $\overline{CD}$ and $\angle CAF = \angle FAD$. If $\overline{AF}$ meets $\overline{ED}$ at G and if $EG = 24$, find CF.

(I wrote this question originally for the Bergen County, N.J. Math League.)

148. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

In $\triangle ABC$, $\angle C = 60°$ and $\angle A$ is greater than $\angle B$. The bisector of $\angle C$ meets $\overline{AB}$ in E. If CE is a mean proportional between AE and EB, find $\angle B$.

(I wrote this question originally for the New York City Senior Interscholastic Mathematics League.)

149. *Proposed by Kenneth S. Williams, Carleton University.*

Find the last two digits of 3^{1000}.

150. *Proposed by Kenneth S. Williams, Carleton University.*

If $[x]$ denotes the greatest integer $\le x$, it is trivially true that

$$\left[\left(\frac{3}{2}\right)^k\right] > \frac{3^k - 2^k}{2^k} \quad \text{for } k \ge 1,$$

and it seems to be a hard conjecture (see [1]) that

$$\left[\left(\frac{3}{2}\right)^k\right] \ge \frac{3^k - 2^k + 2}{2^k - 1} \quad \text{for } k \ge 4.$$

Can one find a function $f(k)$ such that

$$\left[\left(\frac{3}{2}\right)^k\right] \ge f(k)$$

with $f(k)$ between $\frac{3^k - 2^k}{2^k}$ and $\frac{3^k - 2^k + 2}{2^k - 1}$?

REFERENCE

1. G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford University Press 1960, p. 337, condition (f).

* * *

SOLUTIONS

111. *Proposed by H.G. Dworschak, Algonquin College.*

Prove that, for all distinct rational values of a, b, c, the expression

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}$$

is a perfect square.

I. *Solution by Léo Sauvé, Algonquin College.*

It is clear that, for non zero x, y, z,

$$x^2 + y^2 + z^2 = (x+y+z)^2 \iff yz + zx + xy = 0$$

$$\iff \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0. \qquad (1)$$

Since (1) is satisfied when

$$x = \frac{1}{b-c}, \quad y = \frac{1}{c-a}, \quad z = \frac{1}{a-b},$$

it follows that

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2} = \left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right)^2. \qquad (2)$$

II. *Comment by Charles W. Trigg, San Diego, California.*

If we clear the denominators in (2), we get

$$(c-a)^2(a-b)^2 + (a-b)^2(b-c)^2 + (b-c)^2(c-a)^2 = (a^2 + b^2 + c^2 - bc - ca - ab)^2.$$

Thus we have a three-parameter solution of the equation

$$x^2 + y^2 + z^2 = w^2,$$

namely $x = |(c-a)(a-b)|$, $y = |(a-b)(b-c)|$, $z = |(b-c)(c-a)|$, and the always positive $w = a^2 + b^2 + c^2 - bc - ca - ab$, where no two of a, b, c are equal. This is not the best possible solution of the Diophantine equation, since $(a+k,\ b+k,\ c+k)$ gives the same solution for all k. Also, the solution for $(a,\ a+d,\ a+2d)$ is $x = 2d^2$, $y = d^2$, $z = 2d^2$, $w = 3d^2$, which requires that $d = 1$ for a primitive solution. Furthermore, the two sets (a,b_1,c) and (a,b_2,c), with $a > b_i > c$, will give essentially the same solution if $a - b_1 = b_2 - c$.

Some typical distinct solutions are:

a	b	c	x	y	z	w
1	2	3	2	1	2	3
1	2	4	3	2	6	7
1	3	6	10	6	15	19
1	5	6	20	4	5	21
1	4	8	21	12	28	37

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; Charles W. Trigg, San Diego, California (solution as well); and the proposer.

112. *Proposed by H.G. Dworschak, Algonquin College.*

Let $k > 1$ and n be positive integers. Show that there exist n consecutive odd integers whose sum is n^k.

Solution by Jacques Sauvé, student, University of Ottawa.

Suppose the theorem is true, and let the n odd integers be

$$2a+1,\ 2a+3,\ \ldots,\ 2a+2n-1;$$

then

$$n^k = \sum_{i=1}^{n}(2a+2i-1) = 2an + \sum_{i=1}^{n}(2i-1) = 2an + n^2,$$

and so $2a = n^{k-1} - n$. This value of $2a$ is suitable, since it is even (for n^{k-1} and n are of the same parity) and nonnegative (for $k > 1$).

It is now easily verified that the sum of n consecutive odd integers beginning with $2a+1 = n^{k-1} - n + 1$ is

$$\frac{n}{2}[2(n^{k-1} - n + 1) + 2(n-1)] = n^k .$$

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; Charles W. Trigg, San Diego, California; and the proposer.

Editor's comment.

Some solvers merely *assumed* that the theorem was true and deduced that the first odd integer had to be $n^{k-1} - n + 1$, but did not follow through with a verification as the above solver did.

The problem is not new. It can be found in Pólya-Szegö [2] with a solution different from the one given above. It has also appeared before in the *American Mathematical Monthly* (but then, *everything* has appeared before in the *Monthly*!). The problem was proposed there by W.C. Waterhouse [3] and a solution by Roger B. Eggleton was published in [1]. In his solution, Eggleton proved the following generalization:

THEOREM. If n be a positive integer and m be any integer with the same parity as n, the product mn is equal to the sum of n consecutive odd integers. These odd integers are all positive if and only if $m \geq n$.

The editor of the *Monthly* then gave the names of *one hundred fifty-six* other solvers. Among them could be found the names of Walter Bluger, who solved the problem once more for us; Kenneth S. Williams, Carleton University, also one of our contributors; and, lost in the crowd, the editor of EUREKA.

REFERENCES

1. Roger B. Eggleton, Solution to Problem E1641, *American Mathematical Monthly*, Vol. 71, 1964, p. 913.

2. G. Pólya, G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1976, Vol. II, pp. 153, 358.

3. W.C. Waterhouse, Problem E1641, *American Mathematical Monthly*, Vol. 70, 1963, p. 1099.

113. *Proposé par Léo Sauvé, Collège Algonquin.*

Si $\vec{u} = (b,c,a)$ et $\vec{v} = (c,a.b)$ sont deux vecteurs non nuls dans l'espace euclidien réel à trois dimensions, quelle est la valeur maximale de l'angle $(\vec{u},\vec{v})$ entre $\vec{u}$ et $\vec{v}$? Quand cette valeur maximale est-elle atteinte?

Solution d'André Ladouceur, École Secondaire De La Salle.

La relation bien connue $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}| \cos\theta$, où $\theta = (\vec{u},\vec{v})$ donne

$$bc + ca + ab = (a^2 + b^2 + c^2)\cos\theta.$$

Ainsi

$$\begin{aligned}(a+b+c)^2 &= a^2+b^2+c^2+2(bc+ca+ab)\\ &= a^2+b^2+c^2+2(a^2+b^2+c^2)\cos\theta\\ &= (a^2+b^2+c^2)(1+2\cos\theta).\end{aligned}$$

On doit donc avoir $1+2\cos\theta \geq 0$, d'où $\cos\theta \geq -\frac{1}{2}$ et $\theta \leq 120°$.

La valeur maximale de θ est donc $120°$, et elle est atteinte quand $1+2\cos\theta = 0$, c'est-à-dire quand $a+b+c = 0$.

Aussi résolu par F.G.B. Maskell, Collège Algonquin; et par le proposeur.

114. *Proposed by Léo Sauvé, Algonquin College.*

An arithmetic progression has the following property: for any even number of terms, the ratio of the sum of the first half of the terms to the sum of the second half is always equal to a constant k.

Show that k is uniquely determined by this property, and find all arithmetic progressions having this property.

Solution by the proposer.

If the given A.P. is

$$a,\ a+d,\ \ldots,\ a+(n-1)d,\ a+nd,\ \ldots,\ a+(2n-1)d,\ \ldots,$$

then the sums of the first n terms and of the next n terms are

$$\frac{n}{2}[2a+(n-1)d] \quad \text{and} \quad \frac{n}{2}[2a+(3n-1)d].$$

From the hypothesis, the ratio of these sums is

$$k = \frac{2a+(n-1)d}{2a+(3n-1)d} = 1 - \frac{2d}{3d+\dfrac{2a-d}{n}}.$$

Since this ratio is constant for all n, we must have either

$$d=0, \quad k=1, \quad \text{and the A.P. is the trivial } a,\ a,\ a,\ \ldots$$

or

$$d=2a, \quad k=\frac{1}{3}, \quad \text{and the A.P. is } a,\ 3a,\ 5a,\ \ldots$$

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; and Charles W. Trigg, San Diego, California.

115. *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality of Huygens:

$$2\sin\alpha+\tan\alpha \ge 3\alpha, \qquad 0 \le \alpha < \frac{\pi}{2}.$$

I. *Solution by Leon Bankoff, Los Angeles, California.*

According to Hobson [3, pp. 126, 127], we have

$$\sin\alpha \ge \alpha - \frac{\alpha^3}{6} \quad \text{and} \quad \tan\alpha \ge \alpha + \frac{\alpha^3}{3},$$

and the required result follows.

II. *Solution d'André Ladouceur, École Secondaire De La Salle.*

Posons

$$f(\alpha) = 2\sin\alpha+\tan\alpha-3\alpha, \qquad 0 \le \alpha < \frac{\pi}{2}.$$

Puisque $f(0)=0$ et que

$$f'(\alpha) = 2\cos\alpha+\sec^2\alpha-3 = \frac{(\cos\alpha-1)^2(2\cos\alpha+1)}{\cos^2\alpha} \ge 0,$$

la fonction f est croissante sur son domaine et $f(\alpha)\ge 0$, ce qu'il fallait démontrer.

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and Charles W. Trigg, San Diego, California.

Editor's comment.

The inequality of this problem is only the first half of the following double inequality

$$\frac{1}{3}(2\sin\alpha+\tan\alpha) > \alpha > \frac{3\sin\alpha}{2+\cos\alpha}, \quad 0<\alpha<\frac{\pi}{2} \tag{1}$$

which was used by Willebrord Snellius (1580-1626) and later by Christiaan Huygens (1629-1695) in their work on the rectification of the arc (approximating π). In his book *Cyclometricus* (1621), Snellius gave (but was unable to prove rigorously) two theorems equivalent to (1). Later, in *De circuli magnitude inventa* (1654), Huygens rigorously proved the two theorems as well as fourteen others which enabled him to find even better upper and lower bounds to the value of π. Huygens was then a young man of 25 who had only recently taken up serious study of mathematics and physics (he had been trained as a lawyer).

I don't know why Snellius and Huygens found it necessary to go to all that trouble to calculate π, since Albert von Sachsen (who died in 1390) had already written a long treatise called *De quadratura circuli* in which he said that, following the statement of many philosophers, the ratio of circumference to diameter is exactly $\frac{22}{7}$; of this, he says, there is proof, but a very difficult one!

Nearly all the information given so far in this comment can be found in Beckmann [1] and Hobson [2]. It is amusing to note that in Hobson [2] the double inequality (1) is given with the two inequality signs reversed; while in Hobson [3, p. 135] the right inequality is given as in (1). So take your pick: it is not often you get a Hobson's choice in mathematics!

As can be seen from the two solutions given above, an analytic proof of the inequality in this problem is fairly easy to come by, but since the problem is historically a geometrical one, it would be interesting if some reader could come up with a strictly geometrical proof, as Huygens himself did. Rouse Ball [4] says "almost all [Huygens'] demonstrations, like those of Newton, are rigidly geometrical, and he would seem to have made no use of the differential or fluxional calculus, though he admitted the validity of the methods used therein."

REFERENCES

1. Petr Beckmann, *A History of Pi*, The Golem Press, 1971, pp. 81-83, 106-116.

2. E.W. Hobson, Squaring the Circle, in *Squaring the Circle and Other Monographs*, Chelsea Pub. Co., 1953, pp. 27-31.

3. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, 1957.

4. W.W. Rouse Ball, *A Short Account of the History of Mathematics*, Dover, 1960, p. 305.

116. *Proposed by Viktors Linis, University of Ottawa.*

For which values of a, b, c does the equation

$$\sqrt{x+a\sqrt{x}+b} + \sqrt{x} = c$$

have infinitely many solutions?

Solution d'André Ladouceur, École Secondaire De La Salle.

Supposons que l'équation donnée ait une infinité de solutions. On ne peut évidemment avoir $c < 0$. Si $c = 0$, alors on a soit $b = 0$ et l'équation a la solution unique $x = 0$, soit $b \neq 0$ et l'équation n'a pas de solution. L'hypothèse exige donc $c > 0$.

Pour $0 \le x \le c^2$, l'équation équivaut à

$$x + a\sqrt{x} + b = (c - \sqrt{x})^2$$

qu'on peut réduire à

$$(a + 2c)\sqrt{x} = c^2 - b.$$

L'hypothèse exige maintenant $a + 2c = 0$ et $c^2 - b = 0$. Les conditions nécessaires pour qu'il y ait une infinité de solutions sont donc

$$c > 0, \quad a = -2c, \quad b = c^2.$$

Il est facile de vérifier que ces conditions sont aussi suffisantes, car lorsqu'elles sont satisfaites le membre gauche de l'équation donnée devient, pour tout $x \in [0, c^2]$,

$$\sqrt{x - 2c\sqrt{x} + c^2} + \sqrt{x} = \sqrt{(c - \sqrt{x})^2} + \sqrt{x} = c - \sqrt{x} + \sqrt{x} = c.$$

Also solved by G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and Charles W. Trigg, San Diego, California.

117. *Proposé par Paul Khoury, Collège Algonquin.*

Le sultan dit à Ali Baba:

"Voici deux urnes, et a boules blanches et b boules noires. Répartis les boules dans les urnes, mais je rendrai ensuite les urnes indiscernables. Tu auras la vie sauve en tirant une boule blanche." Comment Ali Baba maximise-t-il ses chances?

Solution by G.D. Kaye, Department of National Defence.

It is intuitively clear that the maximum chance occurs when one urn contains exactly one white ball and the other urn contains the rest of the white balls and all the black balls, giving a chance of winning of

$$P_{\max} = \tfrac{1}{2}\left(1 + \frac{a-1}{a+b-1}\right). \tag{1}$$

The problem is to prove it.

Let the urns be U_1 and U_2, the first being the one which contains at most half the balls. Suppose U_1 contains z balls, of which x are white.

If $z = \frac{1}{2}(a+b)$, then the chance of winning is $P = \dfrac{a}{a+b}$ for all possible ways of distributing the balls, and it is easy to show that this chance is less than (1) if $a > 1$. (If $a = 1$, the problem is trivial.)

If $z < \frac{1}{2}(a+b)$, the chance of winning is

$$P = \tfrac{1}{2}\left(\frac{x}{z} + \frac{a-x}{a+b-z}\right).$$

Now

$$\frac{\partial P}{\partial x} = \tfrac{1}{2}\left(\frac{1}{z} - \frac{1}{a+b-z}\right),$$

and this is positive since $z < a+b-z$. Thus, for fixed z, P is maximal when x has its largest possible value, namely, $x = z$ if $z \le a$ or $x = a$ if $z > a$. We must therefore look for $P_{\max}$ among those cases where x is maximal.

If $z > a$, the chance of winning can be increased by transferring $z - a$ black balls from U_1 to U_2, since this increases the chance for U_1 without affecting that for U_2. We need therefore only consider cases in which U_1 contains only z white balls, where $z \le a$. In these cases, $P_{\max}$ clearly occurs when $z = 1$, in which case the chance of selecting a white ball from U_1 is 1 and that for U_2 is greater than for any other value of z.

We can now conclude that $P_{\max}$ is given by (1).

Also solved by Walter Bluger, Department of National Health and Welfare; and by the proposer.

Editor's comment.

This problem, with a solution different from the one above, can be found in [1], where it is attributed to Anatole Joffe, University of Montreal.

REFERENCE

1. Gérard Letac, *Problèmes de Probabilité*, Presses Universitaires de France, 1970, pp. 10, 44.

118. *Proposé par Paul Khoury, Collège Algonquin.*

Peut-on piper deux dés de sorte que la somme des points soit uniformément répartie sur 2, 3, ..., 12?

Solution by Walter Bluger, Department of National Health and Welfare.

Let the probability that $i = 1, 2, \ldots, 6$ turns up be $q(i)$ for one loaded die and $r(i)$ for the other, and, for $j = 2, 3, \ldots, 12$, let $P(j)$ be the probability that the sum of the points on the two loaded dice be j. We have to determine if it is possible to load the dice so that

$$P(j) = k, \quad \text{for all } j = 2, 3, \ldots, 12, \tag{1}$$

where k is a constant (in fact $k = \frac{1}{11}$).

Suppose (1) holds; then

$$P(2) = q(1)\cdot r(1) = P(12) = q(6)\cdot r(6) = k,$$

and so

$$q(1)\cdot r(6) + q(6)\cdot r(1) = q(1)\cdot\frac{k}{q(6)} + q(6)\cdot\frac{k}{q(1)} = \left(\frac{q(1)}{q(6)} + \frac{q(6)}{q(1)}\right)k \ge 2k.$$

Now

$$P(7) > q(1)\cdot r(6) + q(6)\cdot r(1) \geq 2k > k = P(2),$$

which contradicts the hypothesis.

We have in fact proved the stronger result that it is impossible to load the dice so that $P(2) = P(7) = P(12)$. In fact, if $P(2) = P(12)$, then necessarily $P(7) > 2P(2)$.

Also solved by G.D. Kaye, Department of National Defence; and the proposer.

Editor's comment.

The problem can be found in [1], together with a solution which is far more complicated than the astonishingly simple one given above.

REFERENCE

1. Gérard Letac, *Problèmes de Probabilité*, Presses Universitaires de France, 1970, pp. 13, 51.

119. *Proposed by John A. Tierney, United States Naval Academy.*

A line through the first quadrant point (a,b) forms a right triangle with the positive coordinate axes. Find analytically the minimum perimeter of the triangle.

Solution by F.G.B. Maskell, Algonquin College.

Let the polar coordinates of (a,b) be (r,α); then the normal form of the equation of any line through (a,b) which meets both axes is

$$x\cos\theta + y\sin\theta = r\cos(\theta-\alpha), \qquad 0<\theta<\frac{\pi}{2},$$

and the perimeter of the triangle formed is

$$P = r\cos(\theta-\alpha)\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta} + \frac{1}{\cos\theta\sin\theta}\right)$$

which reduces to

$$P = r\left(1+\frac{1}{t}\right)\cos\alpha + \frac{2r\sin\alpha}{1-t}, \qquad t = \tan\frac{\theta}{2}.$$

The first derivative vanishes when

$$\frac{\cos\alpha}{t^2} = \frac{2\sin\alpha}{(1-t)^2} \quad \text{or} \quad \frac{a}{t^2} = \frac{2b}{(1-t)^2},$$

that is, when $t = \dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{2b}}$. The positive sign must be taken in the denominator, since $0<t<1$.

Since all terms of the second derivative are positive (or from geometrical considerations), this value of t gives the minimum perimeter, which is

$$P = a\left(1+\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{a}}\right) + 2b\left(\frac{\sqrt{a}+\sqrt{2b}}{\sqrt{2b}}\right) = 2(a+\sqrt{2ab}+b).$$

Also solved by G.D. Kaye, Department of National Defence; and the proposer.

Editor's comment.

A related problem asking for *the equation of the line through* (a,b) *which forms the triangle with minimum perimeter* was proposed by Steve Moore and Mike Chamberlain in *Mathematics Magazine*, Vol. 48, 1975, p. 238. Solutions to this problem have not yet been published.

120. *Proposed by John A. Tierney, United States Naval Academy.*

Given a point P inside an arbitrary angle, give a Euclidean construction of the line through P that determines with the sides of the angle a triangle

(a) of minimum area;

(b) of minimum perimeter.

Solution of part (a) by F.G.B. Maskell, Algonquin College.

Let XOY be the given angle, and let P be the given point inside the angle (see Figure 1). The triangle OAB in which AB is bisected by P has minimum area. For let MN be any other segment through P, and let AC be parallel to OY. The congruency of Δs PNB, PCA is readily established; hence

$$\Delta\, PNB < \Delta\, PMA,$$

and adding quadrilateral OAPN to each side of this inequality gives

$$\Delta\, OAB < \Delta\, OMN.$$

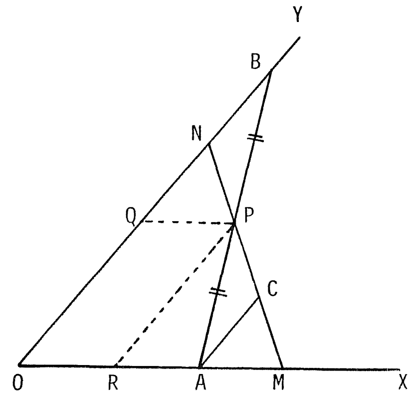


Figure 1

The figure shows the case when $OM > OA$, but the proof is similar when $OM < OA$, and it makes no difference if the angle at O is acute, right, or obtuse.

Having characterized geometrically the triangle of minimum area, we must now show how to construct it. This is easily done by drawing PQ parallel to OX, finding point B on OY such that $QB = OQ$, and joining BP to meet OX in A. The proof is obvious.

Part (a) was also solved by Walter Bluger, Department of National Health and Welfare; and by the proposer (who referred to [7]).

Editor's comment on part (a).

This interesting problem has been around quite a while. I found it in [1, p. 186], [2], [3], and [7]. All proofs are essentially the same as the one given above.

F.G.-M. makes the following interesting comment in [1, p. 186]: To every minimum problem corresponds a maximum problem, and conversely. In Figure 1, parallelogram OQPR, inscribed in Δ OAB, has maximum area when P is the midpoint of AB.

Solution of (b) by Léo Sauvé, Algonquin College.

The geometric characterization and the construction are more complicated in this problem, so we will creep up on them by easily understood stages.

THEOREM 1. *Let* AD *and* AE *be tangent to a circle, let* P *be a variable point on the minor arc, and let* BPC *be tangent to the same circle. Then the perimeter of* Δ ABC *is constant for all positions of* P *on the minor arc* (see Figure 2).

The conclusion easily follows from

$$\begin{aligned} AB + BC + CA &= AB + BP + AC + CP \\ &= AB + BD + AC + CE \\ &= AD + AE \\ &= 2AD, \end{aligned}$$

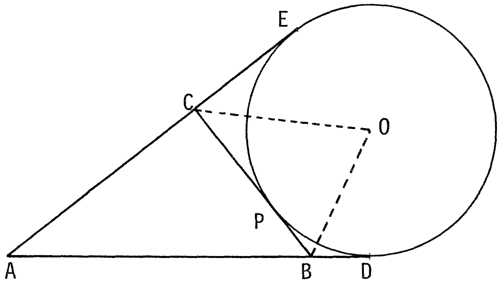


Figure 2

which is independent of the position of P.

I found Theorem 1 and its proof in [1, p. 315]; it can also be found in [5].

Although it has nothing to do with the problem under consideration, it is interesting to note in passing that the angle at the centre BOC in Figure 2 is also constant for all positions of P. This result is due to Poncelet [6].

THEOREM 2. *If, in Figure* 3, BC *is tangent to the circle at* P, *and* RS *is any other segment through* P, *then* Δ ABC *has a smaller perimeter than* Δ ARS.

For if we draw the tangent MN parallel to RS, then, from Theorem 1,

$$\begin{aligned} AB + BC + CA &= AM + MN + NA \\ &< AR + RS + SA. \end{aligned}$$

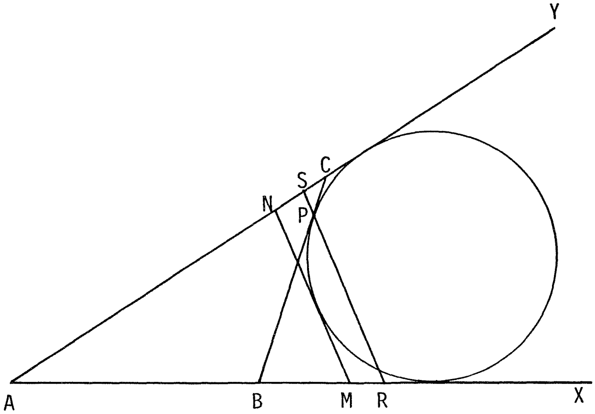


Figure 3

I found this theorem and its proof in [1, p. 437]; an equivalent form can also be found in [5].

Thus Theorem 2 gives us the required geometric characterization: *Given an angle* XAY *and a point* P *inside it, segment* BPC *determines the* ΔABC *of minimal perimeter when* BC *is tangent at* P *to the circle which passes through* P *and is tangent to the sides* AX, AY *of the given angle.*

Now we are left with the difficult question of *constructibility*, which Pólya completely ignores in [5]. Specifically, given an angle XAY and a point P inside it, we must, according to Theorem 2, describe by Euclidean means a circle which passes through P and is tangent to AX and AY (see Figure 4). Let AZ bisect ∠XAY. If P' is symmetric to P with respect to AZ, it is clear that if we can describe a circle which passes through P and P' and is tangent to AX, then that circle will also be tangent to AY, and will thus be the required circle.

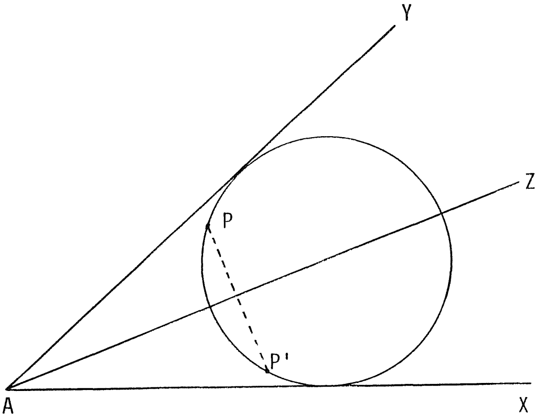


Figure 4

The problem of constructing a circle to pass through two given points P and P' and to be tangent to a given line AX was solved by Lemoine in [4]. His construction, as given in [1, p. 397], runs as follows:

Let P and P' be the given points, and AX the given line. Join PP' to meet AX in E, and let P" be symmetric to P' with respect to AX (see Figure 5). On PP" describe a circular segment containing an angle equal to ∠PEA, and let this segment (not shown in the figure) meet AX in D. Then the circle through P, P', and D is the required circle. For since

$$\angle PDP'' = \angle PEA = \angle PDE + \angle EPD,$$

it follows that ∠EDP" = ∠P'PD, and finally ∠P'DE = ∠P'PD, so that the circle drawn through P and P' is tangent to AX at D.

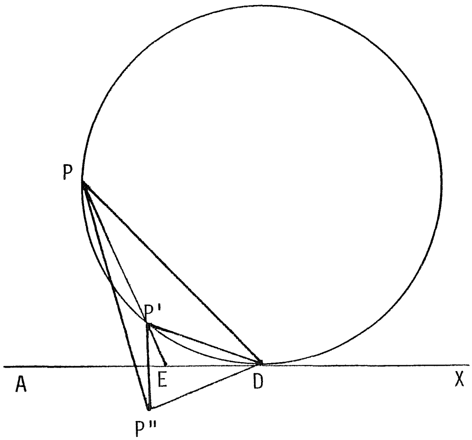


Figure 5

To resume: given a point P inside an angle XAY, the following construction steps will yield a segment BC through P such that ΔABC has minimum perimeter (see Figure 6).

1. Bisect ∠XAY by line AZ.
2. Find the point P' symmetric to P with respect to AZ.

3. Using the Lemoine construction (or otherwise), describe a circle with centre O which passes through P and P' and is tangent to AX.

4. Draw tangent BPC ⊥ OP.

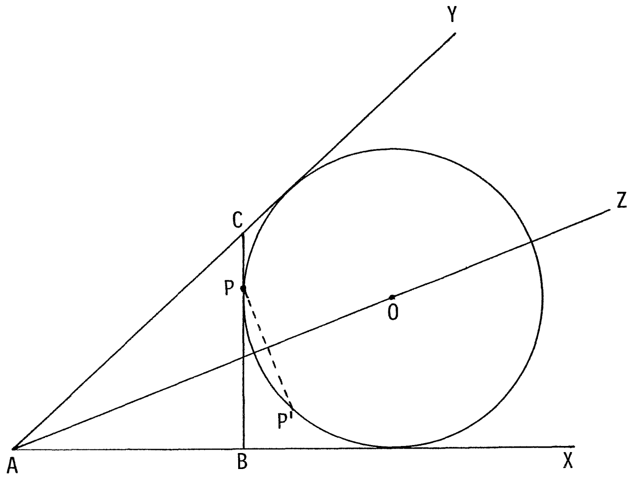


Figure 6

Part (b) was also solved by Walter Bluger, Department of National Health and Welfare; F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment on part (b).

Some of these solvers merely gave a geometric characterization of the minimum perimeter triangle, but did not show how to construct it. Lange [3] briefly discusses both parts (a) and (b), giving a direct construction for part (a) but not for part (b). He then goes on with an interesting discussion of three-dimensional analogues of these problems.

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2. Nicholas D. Kazarinoff, *Geometric Inequalities*, The Mathematical Association of America, 1961, pp. 89, 122.

3. L.H. Lange, Cutting Certain Minimum Corners, *American Mathematical Monthly*, Vol. 83, 1976, pp. 361-365.

4. Emile Lemoine, *Journal de Mathématiques Élémentaires*, 1879, p. 21; quoted in F.G.-M. [1, pp. 397-398].

5. G. Pólya, *Mathematics and Plausible Reasoning*, Revised Edition, Princeton University Press, 1968, Vol. 2, pp. 205-207.

6. Jean-Victor Poncelet, *Traité des propriétés projectives des figures*, Nos. 462, 463; quoted in F.G.-M. [1, p. 315].

7. John A. Tierney, Elementary Techniques in Maxima and Minima, *The Mathematics Teacher*, Vol. XLVI, 1953, pp. 484-486.

* * *

All language is treacherous and beguiling especially the language of mathematics.

J. NEYMAN in *The Heritage of Copernicus*, MIT Press, 1974, p. 265.