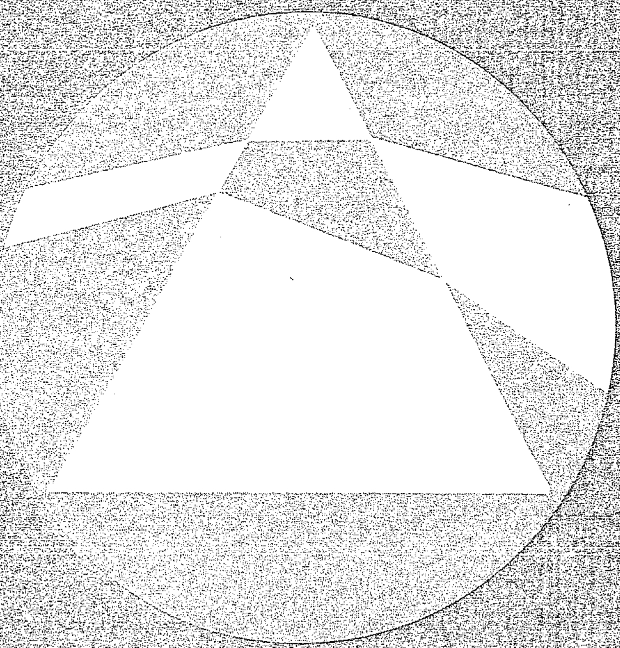
Mathematical Spectrum

1996/7 Volume 29 Number 1



- The 21-card trick
- The oddball problem
- What became of the Senior Wranglers?

A magazine for students and teachers of mathematics
in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year, consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column. The copyright of all published material is vested in the Applied Probability Trust.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions; prizes are available annually for students as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

All correspondence should be sent to

The Editor, Mathematical Spectrum,
Hicks Building, The University,
Sheffield S3 7RH, UK.

Subscriptions

Current volume

Volume 29, Issues 1, 2 and 3 (postage included) costs £9.00 or US$ 14.00 or $A 19.00

These prices apply to subscribers in all parts of the world. There is now only one set of sterling rates applicable to all countries.

A discount of 10% will be allowed on all orders for five or more copies of the current volume sent to the same address.

Details of reduced prices for two- and three-year subscriptions are available on request.

Back issues

Most back issues are available; information concerning prices and a list of the articles published may be obtained from the Editor.

* * *

Enquiries about rates, subscriptions and advertisements should be directed to

The Editor, Mathematical Spectrum,
Hicks Building, The University,
Sheffield S3 7RH, UK.

What Became of the Senior Wranglers?

D. O. FORFAR

Wrangler, a word that has something to do with American jeans? Not in this case!

During the 157 years (1753–1909) in which the results of the Cambridge Mathematical Tripos were published in order of merit and divided by class of degree into Wranglers (1st Class), Senior Optimes (2nd Class) and Junior Optimes (3rd Class), great prestige attached to those students who had come out in the top two or three places. The securing of the top position as Senior Wrangler was regarded at the time as the greatest intellectual achievement attainable in Britain, and the Senior Wrangler was feted well beyond Cambridge and accorded pre-eminent status among his peers. Indeed, years in Cambridge were often remembered in terms of who had been Senior Wrangler in that year. It is curious therefore that no systematic study has ever been made, in so far as the author is aware, of what became of these Senior Wranglers in later years after their triumph. This article may shed a little light on the matter.

Until 1850, mathematics in Cambridge was dominant over all other University subjects, so much so that it was obligatory, astonishing as it now seems, for students who were studying for honours in Classics first to have taken the Mathematical Tripos.

Because of the prestige attaching to the position of Senior Wrangler and the college from which the Senior Wrangler came, the students, especially the most promising, were subjected, like thoroughbred racehorses, to the most intense training for the Tripos race. The training was in the hands of private tutors and not the University professors, as often students attended very few lectures and, for example, Charles Babbage gave no lectures in the eleven years, 1828–39, during which he was Lucasian Professor. The best of the tutors, because of their reputation, were able to select the most able students, thus perpetuating their reputation for success.

The most famous private tutor was William Hopkins (1793–1866), who himself had been 7th Wrangler in 1827 and was a person of distinction outside his tutoring activities, being President of the Geological Society 1851–53 and President of the British Association 1853. In 1849 it was said of Hopkins that in the 22 years since his degree he had taught 17 Senior Wranglers, 27 Second or Third Wranglers and 200 Wranglers in total (reference 1). As William Hopkins continued to turn out Wranglers well after that date, his final tally must have been much higher. Hopkins' Wranglers included Clerk Maxwell, Cayley, Thomson (Lord Kelvin), Stokes and Tait. It can be seen with the benefit of hindsight that the greatest of Hopkins' pupils was Clerk Maxwell, but remarkably Hopkins recognised this even when Maxwell was an undergraduate saying 'he is unquestionably the most extraordinary man I have met with, in the whole range of my experience' (reference 2).

The Mathematical Tripos was a formidable examination taken by students after three years and one term at the University. The best students also sat the papers for the two Smith's prizes. For example, in 1854, the Tripos consisted of sixteen papers, two papers each day for eight days—a total of 44.5 hours in the examination room. The total number of questions set was 211. The best students then went on for a further three days' Smith's prize examinations consisting of 63 even more testing questions. The questions in the early papers contained bookwork in the first part of the question with riders based on that bookwork in the second part of the question. The questions became progressively more difficult in the later papers, particularly in the Smith's prize papers. To solve the more technically difficult problems within the short time available in the examination room, the students had to find the correct approach straight away. Sometimes the approach involved the use of subtle stratagems which the students could not have been expected to think up on the spur of the moment in the examination room. Hence constant practice at solving similar questions, as set in previous years, and familiarity with the right method of tackling the questions was all-important, and students wishing to perform well had to hone their technique, with the help of their tutor, to a fine pitch prior to the examination.

The actual marks were never published, but Sir Francis Galton in his book *Hereditary Genius* (reference 3) refers to having obtained marks in respect of three years (unspecified, but probably around the 1860s). In one of these years, out of a total possible mark of 17,000, the Senior Wrangler obtained 7634 marks, the second Wrangler obtained 4123 marks, the lowest Wrangler obtained around 1500 marks and the lowest candidate receiving an honours degree (Junior Optime) obtained 237 marks. In the second of these years the Senior obtained between 5500 and 6000 marks, the Second obtained between 5000 and 5500 and the lowest Junior Optime received 309 marks. In the third of these years when, according to Galton, the Senior was conspicuously eminent, he obtained 9422 marks and the Second 5642 marks. Galton makes considerable play of the large discrepancy between the marks obtained by the Senior and by the lowest Wrangler.

It can be seen that the Senior Wrangler would typically obtain less than 50% of the marks, the lowest Wrangler less than 10% and the lowest honours candidate less than 2%!

This seems to the author a rather curious result, and it is not clear what conclusions are to be drawn from it. It suggests that the candidates covered a very wide ability range, that the level of the lowest Wrangler and the lowest honours man was really rather poor by today's standards (perhaps university life was more relaxed, and the average student did not apply himself very hard!), and that the papers were too long and hard even for the best students. This contributed to the criticisms which, in the early 1900s, were levelled against the content and style of such a fierce examination and against publishing the results in order of merit which gave undue prominence to those occupying the top few places. The famous mathematician G. H. Hardy (reference 4) was particularly critical of the examination as we shall see later (reference 5).

A high position in the Tripos was very desirable as it gave a favoured entrée not only into academia and the actuarial profession, but also into professions such as the Law, the Church and even Medicine. The Senior Wrangler often entered these professions and not academic life.

To give examples of Senior Wranglers (SW) and Second Wranglers (2W) who attained eminence in various professions we may cite the following.

Legal

J. Rigby (1834–1903), 2W 1856, later Sir John Rigby. Solicitor General 1892–94, Attorney General 1894, Lord Justice of Appeal 1894. Privy Councillor 1901.

J. Stirling (1836–1916), SW 1860, later Sir James Stirling. Lord Justice of Appeal 1900–1906, Privy Councillor 1900.

R. Romer (1840–1918), SW 1863, later Sir Robert Romer. Lord Justice of Appeal 1899–1906, Privy Councillor 1899.

J. L. Moulton (1844–1931), SW 1868, later Lord Moulton of Bank. Lord Justice of Appeal 1906, Privy Councillor 1906.

It is interesting that the connection between mathematics and the law continues with the current Lord Chancellor, Lord Mackay of Clashfern, being a Wrangler in 1951.

Actuarial

T. B. Sprague (reference 6) (1830–1920), SW 1853, the only person to have been both President of the Faculty of Actuaries of Scotland and the Institute of Actuaries, and the dominant actuary of the second half of the 19th century both nationally and internationally.

It is interesting to note, in current times, Professor J. J. McCutcheon (Wrangler 1962) immediate Past-President of the Faculty of Actuaries and C. D. Daykin (Wrangler 1970) currently President of the Institute of Actuaries. Since actuarial mathematics is a branch of mathematics, it is natural that actuaries should have studied mathematics at Cambridge or elsewhere.

Church

H. Goodwin (1818–91), 2W 1840. Dean of Ely 1858–69, Bishop of Carlisle 1869–91.

C. F. Mackenzie (reference 7) (1845–62), 2W 1848. Archbishop of Natal 1855–59, First Bishop in Central Africa 1861–62.

J. M. Wilson (1836–1931), SW 1859. Headmaster of Clifton College 1879–90. Archdeacon of Manchester 1890–1905, Canon of Worcester 1905–26. He was also a great classical scholar.

Medicine

D. McAlister (reference 8) (1854–1934), later Sir Donald McAlister of Tarbet, SW 1877, President of the General Medical Council 1904–1931. He was also a very great linguist. His first language was Gaelic. His extraordinary linguistic ability is evidenced by the fact that he spoke well German, Norse, French, Italian, Dutch, Spanish, Portugese, Romansch, Czech, Basque, Turkish, Greek, Arabic, Swedish, Russian, Serbian, Africaans and Romany, and published translations of poems in one foreign language into another.

Political

L. H. Courtney (1832–1918), 2W 1855, later Baron Courtney of Penwith. Financial Secretary to the Treasury 1882–84, Deputy Speaker of the House of Commons 1886–92.

To hold a chair in Mathematics at Cambridge it seemed almost a necessary condition to have been Senior Wrangler as the following table shows, Babbage being the exception (to most things!).

Lucasian Chair		
1760–1798	Edward Waring	SW 1757
1798–1820	Isaac Milner	SW 1774
1820–1822	Robert Woodhouse	SW 1795
1822–1826	Thomas Turton	SW 1805
1826–1828	George Airy	SW 1823
1828–1839	Charles Babbage	did not sit the Tripos in 1813, his graduation year
1839–1849	Joshua King	SW 1819
1849–1903	George Stokes	SW 1841
1903–1932	Joseph Larmor	SW 1880

Sadleirian Chair		
1863–1895	A. Cayley	SW 1842
1895–1910	A. R. Forsyth	SW 1881
1910–1931	E. W. Hobson	SW 1878
1931–1942	G. H. Hardy	4W 1898
1945–1953	L. J. Mordell	3W 1909

Among the Wranglers are to be found those who, along with Michael Faraday (1791–1867), William Rowan Hamilton (1805–65) and James Prescott Joule (1818–89), secured for the UK world leadership in physics and mathematical physics in the second half of the 19th century, namely:

James Clerk Maxwell (reference 2) (1831–79), 2W 1854;

William Thomson (reference 9) (1824–1907), 2W 1845, later Lord Kelvin;

George Stokes (1819–1903), SW 1841, later Sir George Stokes;

John Couch Adams (1819–92), SW 1843, predicted theoretically the existence of the planet Neptune (also predicted independently by Le Verrier in France);

George Green (reference 10) (1793–1841), 4W 1837, first introduced the concept of potential in a paper of 1828;

Peter Guthrie Tait (reference 11) (1831–1901), SW 1852, author with Lord Kelvin of the epoch-making book *Treatise on Natural Philosophy*;

J. J. Thomson (1856–1940), 2W 1880, later Sir J. J. Thomson, discoverer of the electron in 1897.

University professorships throughout the UK and the British Empire were commonly held by Wranglers in the top two or three places. For example the Senior Wranglers of 1834, 1838, 1839, 1847, 1849, 1852, 1861, 1862, 1864, 1867, 1883, 1886 and 1889 were respectively professors at the following universities or colleges: Edinburgh, Royal Naval College (Portsmouth), Gresham College, Melbourne, Sydney, Edinburgh, Auckland, Manchester, Royal School of Naval Architecture, Aberdeen, Bangor, Belfast and Poona.

Given the great attention and prestige attaching to mathematics over the 157 years (1753–1909) we are considering, it is curious that the Tripos produced, in contrast to mathematical physics, only a few world class pure mathematicians—only Cayley, Sylvester, Hardy and Littlewood (reference 12). World leadership in pure mathematics in this period remained firmly in France and Germany, with each of these countries producing a plethora of world class mathematicians, e.g. Gauss, Bessel, Jacobi, Dirichlet, Kummer, Riemann, Dedekind, Kronecker, Cantor, Klein, Hilbert, Landau, Weyl in Germany and d'Alembert, Lagrange, Laplace, Legendre, Fourier, Poisson, Cauchy, Liouville, Galois, Hermite, Bertrand, Jordan, Poincaré, Hadamard, Cartan, Borel and Lebesgue (reference 13) in France.

It was this relative failure of British pure mathematics after the death of Colin Maclaurin in 1746 that so irked G. H. Hardy, and he put a large part of the blame on to the Tripos as is evident from his 1926 Address to the Mathematical Association (reference 5). Hardy's thesis was that the syllabus for the Tripos was out of date and far behind the times since it did not contain any of the important ideas which were dominating contemporary thought in pure mathematics at the time. It was therefore a poor training for a pure mathematician. Furthermore, the questions put too much stress on technique rather than ideas and were questions in which professional mathematicians had lost interest many years previously. While accepting these criticisms, it seems curious that those who became professional pure mathematicians apparently found difficulty in shaking off the legacy of the Tripos. After all, the Professors had spent only three years of their active lives on the Tripos during their undergraduate careers, and often took little interest in the Tripos thereafter, apart from setting some questions for the Smith's prizes. Given their small lecturing load, they had much free time for research, for familiarising themselves with the latest mathematical ideas and for trying to publish work matching the originality of the papers coming from continental pens. The Cambridge Mathematical Journal had been founded in 1837 by R. L. Ellis, SW 1840, and D. F. Gregory, 5W 1837 (reference 14). The relative failure of British pure mathematics during this period in comparison with France and Germany remains something of a paradox. A comparative study of the way mathematics was taught and research organised during this period at the Ecole Polytechnique and Ecole Normale Supérieure in Paris and at the Universities of Göttingen and Berlin, the centres of European pure mathematics, would be fascinating.

In contrast to pure mathematics, the British mathematical physicists, like Clerk Maxwell, were producing work of great originality by initiating wholly new ideas like the importation of statistics into the theory of gases and the development of a field theory of electrical and magnetic disturbances.

The Tripos was not without its amusing anecdotes. One concerns Lord Kelvin. He was undoubtedly the best and most original mathematician of his year and thought he was a 'dead cert' for Senior Wrangler. He said to one of the college servants on the day the Tripos results were published 'Oh, just run down to the Senate House, will you, and see who is Second Wrangler'. When the servant returned he said 'You, Sir'.

Lord Kelvin had been beaten by Stephen Parkinson, later President of St. John's College who, although not possessing great originality in mathematics, was highly intelligent and had schooled himself to perfection in the executional skills of solving Tripos problems at speed.

It perhaps rankled a little with Kelvin that he had not been Senior Wrangler. G. H. Hardy seems to have felt the same, although he took the Tripos a year earlier than was normal, as did James Jeans. The persons who beat Hardy were:

R. W. H. T. Hudson, SW 1898, who went into mathematics but was not able to fulfil his potential as he died in a climbing accident on Snowdon in 1904;

James Cameron, 2=W 1898, later Master of Caius College and Vice Chancellor of the University;

James Jeans, 2=W 1898, later Sir James Jeans, the famous mathematical physicist.

Family connections with the Tripos are interesting. For example W. E. Littlewood, 34=W 1854, E. T. Littlewood, 9W 1882, and J. E. Littlewood (the famous mathematician), SW 1905, are grandfather, father and son respectively. Women were first listed in the Tripos in 1882 but in a rather curious way. Only the men were ranked but the position of the women Wranglers was indicated by giving the placings between which the women fell (e.g. between the 6th and 7th Wrangler). A noteworthy result was achieved in 1890 when Philippa Fawcett (who the author believes was a cousin of Littlewood) was placed above G. T. Bennett, the 'official'

Senior Wrangler and therefore rightly should be considered the Senior Wrangler of 1890. Her father Henry Fawcett had been 7W of 1856. This is the only occasion that a woman was the Senior Wrangler.

Other interesting family connections are the four Niven brothers from Aberdeen who were respectively 3W, SW, 8W and 15W in 1866, 1867, 1874 and 1881; also the three Aldis brothers who were respectively SW, 2W and 6W in 1861, 1866 and 1863, and the three Phear brothers who were respectively 2W, 4W, 6W in 1849, 1852, 1847. A previous member of Clerk Maxwell's family on his mother's side had been ranked 2nd in the Tripos in 1752. Between 1748 and 1752 the candidates were listed in order of merit but the term Wrangler was not used. Also Clerk Maxwell's first cousin Charles Cay (reference 15) was 6W in 1864. Ability in both mathematics and classics is evinced by Cornelius Neale, SW 1812, who also won one of the two Chancellor's Classical Medals (the top awards for classics).

The Tripos had its share of sadness as well as triumph. Owing to the poorer medical facilities of the 19th century mortality was much higher and those who died relatively soon after taking the Tripos include J. Savage, SW 1855, Slesser, SW 1858, Purkiss, SW 1864, Hartog, SW 1869, R. W. H. T. Hudson, SW 1898 and J. E. Wright, SW 1900.

It is interesting to note that the philosopher Bertrand Russell was 7W 1893 and Lord Keynes 12W 1905. However, lack of excellence in mathematics was no hindrance to being Chancellor of the Exchequer as evidenced by H. C. E. Childers, 14 Senior Optime 1850, Chancellor 1882, and Sir William Harcourt, 27 Senior Optime 1851, Chancellor 1886 and 1892–95. Indeed it was clearly no hindrance to being Prime Minister—Sir Henry Campbell Bannerman, 22 Senior Optime 1858, Prime Minister 1905–08.

References

1. W. W. Rouse Ball, *A History of the Study of Mathematics at Cambridge* (1889).
2. L. Campbell and W. Garnett, *Life of James Clerk Maxwell* (1882).
3. F. Galton, *Hereditary Genius —An Enquiry into its Laws and Consequences* (1869).
4. C. P. Snow, *Variety of Men* (1969).
5. G. H. Hardy, The Case against the Mathematical Tripos — Presidential Address to the Mathematical Association (1926).
6. G. King, Memoir of Thomas Bond Sprague, *Journal of the Institute of Actuaries*, **52** (1921).
7. H. Goodwin, *Memoir of Bishop Mackenzie* (1864).
8. E. F. B. McAlister, *Sir Donald McAlister of Tarbet* (1935).
9. S. P. Thomson, *The Life of William Thomson —Baron Kelvin of Largs* (1910).
10. L. J. Challis, George Green — miller, mathematician and physicist, *Mathematical Spectrum* **20**, No. 2, pp. 45-53, (1988).
11. C. G. Knott, *Life and Scientific Work of Peter Guthrie Tait* (1911).
12. J. E. Littlewood, *Littlewood's Miscellany*, edited by Béla Bollobás, (1986).
13. Articles on Babbage, Littlewood, Fourier, d'Alembert, Lagrange, Cauchy and Kronecker in *Mathematical Spectrum*, **1**, **7**, **11**, **13**, **16**, **19**, **22** and **24**.
14. W. Walton, *The Mathematical Writings of Duncan Farquharson Gregory*, (1865).
15. D. O. Forfar, Origins of the Clerk (Maxwell) Genius, *Bulletin of the Institute of Mathematics and its Applications* **28**, Nos 1/2, (Jan/Feb 1992). □

__D. O. Forfar__ is an actuary. He is on the Council of the Faculty of Actuaries and Chairman of its Research Committee and has published twenty-five papers on actuarial/mathematical topics. He is also a Trustee of the Clerk Maxwell Foundation. He graduated in mathematics from Trinity College, Cambridge, and was himself a Wrangler!

Isaac Newton Institute for Mathematical Sciences: New Director

The Isaac Newton Institute for Mathematical Sciences is pleased to announce the appointment of Professor Keith Moffatt ScD, FRS as its new Director. Professor Moffatt will take up his five-year appointment on 1 October 1996. He takes over from the Institute's first Director Sir Michael Atiyah OM, FRS who together with Peter Goddard, the former Deputy Director, established it as an international centre for mathematical research and has overseen its development since it first came formally into existence on 1 October 1990.

Sir Michael said: 'I am delighted to be handing over to Keith Moffatt whose mathematical interests and experience should guarantee the continued success of the Newton Institute. In particular, I look forward to having his support as Director when I am organizing my own programme this autumn.'

Professor Moffatt is currently Professor of Mathematical Physics at the Department of Applied Mathematics and Theoretical Physics, and Fellow of Trinity College, Cambridge. His research activities cover all aspects of fluid dynamics. He already has links with the Newton Institute, having been Principal Organiser in 1992 of the Institute's research programme *Dynamo Theory*, which brought together specialists from a range of fields including fluid dynamics, geophysics and astrophysics. He was also a member of the Institute's Management Committee from 1990 to 1991.

Professor Moffatt said of his aims as Director: 'The purpose of the Isaac Newton Institute is to provide an interactive setting where talented scientists from all over the world can meet to promote cross-fertilisation of experience, techniques and ideas, and hence to make new discoveries in mathematics and in the mathematical sciences. I want to strengthen this image of the INI, and to ensure that it continues to provide a focus and melting pot for research at the highest level into the 21st century.'

Functional Equations in Cauchy's Book

TOSHIYA URABE and KAZUAKI KITAHARA

In this article we shall visit the origins of functional equations.

Do you know what 'functional equations' are? They are equations in which the functions are the unknowns. For example, in

$$f(xy) = f(x) + f(y), \tag{1}$$
$$f(x+y) = f(x)f(y), \tag{2}$$
$$f(x+y) = f(x) + f(y), \tag{3}$$
$$f(xy) = f(x)f(y) \tag{4}$$

and

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{5}$$

we search for functions f which satisfy the equations for all x and y. Functional equations orginated in the calculus and in the foundations of mechanics. The notion of function did not begin to take shape until the latter half of the 17th century, and, in fact, functional equations played an important role in clarifying mathematicians' thinking on this fundamental idea.

Cauchy (reference 3) treated the functional equations (1)–(5) in his book and obtained rigorously their continuous solutions. The functional equations (1)–(4) are called Cauchy's equations and (5) is D'Alembert's equation. In this article, we shall explain the origin of each equation except (4). We refer to references 1 and 6 for equation (1), to reference 2 for equation (5), and to references 1, 2, 4, 5 for equations (2) and (3). For the sake of clarity, modern notation is used throughout.

Equation (1) characterizes logarithms. It was Napier (1614) and Briggs (1624) who created logarithms. Napier's definition of logarithms is essentially as follows.

Let AB be a segment of length 1 and let A$'\infty$ be a half line. Suppose that

(a) moving points P, P$'$ make a simultaneous start with initial velocity v_0, travelling from A to B and from A$'$ to ∞, respectively,
(b) the velocity of the point P is in proportion to the length PB, and
(c) the point P$'$ moves with constant velocity v_0.

Then we define the length A$'$P$'$ as the logarithm of the length PB. In modern terminology this length is, in fact, minus the natural logarithm. To construct logarithms $\ell(x)$, $x > 0$, they used the property that

$$\ell(xy) = \ell(x) + \ell(y).$$

The discovery by de Saint-Vincent (1647) and his student de Sarasa (1649) was closely related to natural logarithms. Let $A_{a,b}$, $0 < a < b$, denote the area of the region

$$\{(x,y) \mid 0 \leqslant y \leqslant x^{-1},\ a \leqslant x \leqslant b\}$$

of R^2. De Saint-Vincent showed that $A_{a,b} = A_{\alpha a,\alpha b}$ for any positive number α. Furthermore, setting $L(x) = A_{1,x}\,(x \geqslant 1)$, and $L(x) = -A_{x,1}$ $(0 < x < 1)$, de Sarasa deduced from de Saint-Vincent's result that

$$L(xy) = L(x) + L(y)$$

for all positive numbers x, y. Readers will notice that $L(x)$ is the natural logarithm of x.

We next turn to equation (5) and readers will easily see that $f(x) = \cos x$ is a solution. This equation arose in connection with the determination of the composition, or sum, of forces in mechanics. D'Alembert (1769) argued as follows.

Let p, q be any unit vectors in the plane. Their sum $p+q$ acts along the bisector of the angle between them and the magnitude $|p+q|$ is a function h of the included angle $A(p,q)$; also set $f(x) = \frac{1}{2}h(2x)$.

Now let a, b, c, d be four unit vectors such that

$$A(a,b) = A(c,d) = 2y \quad \text{and} \quad A(a+b, c+d) = 2x.$$

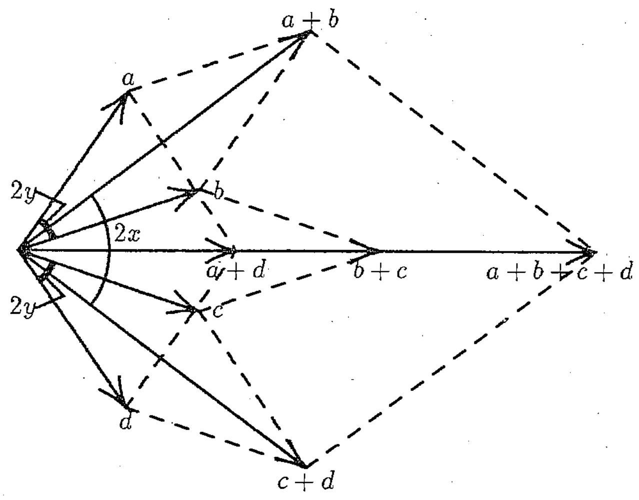


Figure 1

Then

$$A(a,d) = 2x + 2y \quad \text{and} \quad A(b,c) = 2x - 2y,$$

so that, by the definition of f,

$$|a+d| = 2f(x+y) \qquad \text{and} \qquad |b+c| = 2f(x-y).$$

Since clearly $A(a+d, b+c) = 0$,

$$\begin{aligned} \big|(a+d)+(b+c)\big| &= |a+d| + |b+c| \\ &= 2f(x+y) + 2f(x-y). \end{aligned} \tag{6}$$

On the other hand, if $|p| = |q| = \alpha$, then

$$|p+q| = \alpha \cdot 2f\left(\tfrac{1}{2}A(p,q)\right)$$

and so

$$\big|(a+b)+(c+d)\big| = 2f(y)\cdot 2f(x). \tag{7}$$

Equation (5) now follows at once from (6) and (7).

Equations (2) and (3) are closely related to the binomial theorem, i.e.

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \cdots, \tag{8}$$

for $\alpha, x \in R$, with $|x| < 1$, where

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

for $k = 1, 2, \ldots$. Newton (1664) had investigated this theorem. Euler (1774) and Lacroix (1797) also took up the subject, but neither considered the question of the convergence of the series in (8).

Euler's approach involved, for a particular x, the functions

$$g(r) = (1+x)^r, \quad h(r) = \sum_{i=0}^{\infty}\binom{r}{i}x^i$$

of the rational variable r. He obtained the formula

$$h(r+s) = h(r)h(s)$$

(i.e. (2)) and, using the facts that $g(0) = h(0)\ (= 1)$ and $g(1) = h(1)\ (= 1+x)$ derived the identity $g(r) = h(r)$ for all rational r.

On the other hand, Lacroix showed that, in the series in (8), the nth ($n \geqslant 3$) coefficient can be obtained recursively from the second. Moreover, denoting by $f(\alpha)$ the second coefficient, he verified that f must satisfy

$$f(\alpha+\beta) = f(\alpha) + f(\beta)$$

(i.e. (3)) and, by using the power series of $\log(1+x)$, got $f(\alpha) = \alpha$ for any real number α.

The first rigorous proof of the binomial theorem was given by Abel in 1826.

Cauchy (1821) introduced basic concepts such as functions, limits, continuity and so on, and he obtained rigorously the continuous solutions of the functional equations (1)–(5). Below we find the continuous solutions of (1)–(4).

We first consider equation (3). By putting $x = y = 0$ we have $f(0) = 0$, and also, by putting $y = -x$, $f(-x) = -f(x)$. Next, it is easily seen that

$$f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n),$$

and, by putting $x_1 = x_2 = \cdots = x_n = x$, we have

$$f(nx) = nf(x).$$

Thus, if $x = (m/n)t$, where m, n are positive integers, then $nx = mt$, $f(nx) = f(mt)$, and so $nf(x) = mf(t)$, i.e.

$$f\left(\frac{m}{n}t\right) = \frac{m}{n}f(t).$$

By putting $t = 1$ and $f(1) = c$ we get $f(x) = cx$ for all positive rational x; and, since $f(-x) = -f(x)$, the identity holds for all rational x. Finally, assuming the continuity of $f(x)$, by taking limits at irrational points, we obtain

$$f(x) = cx$$

for all real x.

On the other hand, any function $f(x) = cx$ evidently satisfies equation (3).

Cauchy's equations (1) and (2) can be solved by reducing them to equation (3).

Consider equation (2). For any real x,

$$f(x) = \left[f\left(\tfrac{1}{2}x\right)\right]^2 \geqslant 0.$$

In fact, for a non-trivial solution we must have $f(x) > 0$ for all real x; for if there exists x_0 such that $f(x_0) = 0$, then, for any real x,

$$f(x) = f\big(x_0 + (x - x_0)\big) = f(x_0)f(x - x_0) = 0,$$

i.e. $f(x) \equiv 0$.

So suppose that f is continuous and $f(x) > 0$ for all real x. Defining the function g by $g(x) = \log f(x)$ we therefore have

$$\begin{aligned} g(x+y) &= \log f(x+y) \\ &= \log f(x)f(y) \\ &= \log f(x) + \log f(x) \\ &= g(x) + g(y). \end{aligned}$$

Thus the continuous function g satisfies (3) and therefore $g(x) = cx$. Hence

$$f(x) = \mathrm{e}^{cx}$$

for all real x.

Next we consider equation (1). If $f(0)$ is defined, then $f(0) = f(0) + f(0)$ and so $f(0) = 0$. But then, for any real x, $f(0) = f(x) + f(0)$, i.e. $f(x) = 0$. Thus we are led to the trivial solution $f(x) \equiv 0$.

Now suppose that $f(0)$ is not defined, but f is continuous on $(0, \infty)$ and on $(-\infty, 0)$. We define the function $g(t)$ for $-\infty < t < \infty$ by $g(t) = f(\mathrm{e}^t)$. Then, for any real s, t,

$$g(s) + g(t) = f(\mathrm{e}^s) + f(\mathrm{e}^t) = f(\mathrm{e}^{s+t}) = g(s+t),$$

i.e. the continuous function g satisfies (3). Hence $g(t) = ct$, i.e. $f(e^t) = ct$, or

$$f(x) = c \log x \quad (x > 0).$$

If $x < 0$, then

$$2f(x) = f(x^2) = f((-x)^2) = c\log(-x)^2 = 2c\log(-x),$$

i.e.

$$f(x) = c\log(-x).$$

Hence, for any real $x \neq 0$,

$$f(x) = c\log|x|.$$

Finally we solve equation (4). If $f(0)$ is defined, then $f(0) = f(0)f(0)$ and so $f(0) = 0$ or 1. If $f(0) = 1$, then, for all x, $f(0) = f(x)f(0)$, i.e. $f(x) = 1$, so that we have the trivial solution $f(x) \equiv 1$. Hence, for a non-trivial solution, $f(0) = 0$ or $f(0)$ is not defined.

Taking f to be continuous we define the function $g(t)$ for $-\infty < t < \infty$ by

$$g(t) = f(e^t).$$

Then, for any real s, t,

$$g(s)g(t) = f(e^s)f(e^t) = f(e^{s+t}) = g(s+t),$$

i.e. the continuous function g satisfies (2). Hence $g(t) = e^{ct}$ and so $f(e^t) = e^{ct} = (e^t)^c$. Thus, for any $x > 0$,

$$f(x) = x^c.$$

To obtain the values $f(x)$ for negative x, we first note that

$$[f(-1)]^2 = f((-1)^2) = f(1) = 1^c = 1,$$

so that $f(-1) = \pm 1$.

If $f(-1) = 1$, then, when $x < 0$, $f(x) = f(-1)f(-x) = (-x)^c$. Hence, for all $x \neq 0$,

$$f(x) = |x|^c. \tag{9}$$

If $f(-1) = -1$ then, when $x < 0$, $f(x) = f(-1)f(-x) = -(-x)^c$. Hence, for all $x \neq 0$,

$$f(x) = \begin{cases} x^c & \text{when } x > 0, \\ -|x|^c & \text{when } x < 0. \end{cases} \tag{10}$$

For a non-trivial continuous solution on $(-\infty, \infty)$ we require $f(0) = 0$ and $f(x) \to 0$ as $x \to 0$. The last condition is satisfied by both (9) and (10) provided that $c > 0$.

We simply state the non-trivial solutions of (5), namely

$$f(x) = \cos cx \quad \text{and} \quad f(x) = \cosh cx.$$

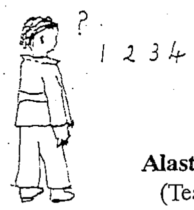
Since in D'Alembert's mechanics problem clearly $f(0) = 1$, $f(\frac{1}{2}\pi) = 0$ and $0 < f(x) < 1$ for $0 < x < \frac{1}{2}\pi$, it follows that $c = 1$, i.e. $f(x) = \cos x$.

References

1. H. Abe, *The History of the Differential and Integral Calculus*, (Morikita, Tokyo, 1989). (In Japanese.)
2. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, (Cambridge University Press, Cambridge, 1989).
3. A. L. Cauchy, *Cours d'analyse de l'École Polytechnique, Vol. I: Analyse algébrique*, (Debure, Paris, 1821).
4. I. Grattan-Guinness, Functional equations, *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, ed. I. Grattan-Guinness, (Routledge, London, 1994), pp. 557–562.
5. M. Pensivy, The binomial theorem, *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, ed. I. Grattan-Guinness, (Routledge, London, 1994), pp. 492–498.
6. R. Takekuma, *The History of Mathematics*, (Baihukan, Tokyo, 1959). (In Japanese.) □

***Toshiya Urabe** is a postgraduate at Saga University, reading mathematical education.*
***Kazuaki Kitahara** teaches mathematics at Kwansei Gakuin University.*

How many ways are there of choosing the same number of even and odd numbers from the first $2n$ natural numbers?



Alastair Summers
(Teacher, Stamford School)

Four people, Adam, Ben, Charles and Dale, were having a quarrel. Adam said to Ben, "You are a liar." Then Dale said to Adam, "No, you are a liar, Adam." Charles got angry and said, "Dale is a liar."

How many liars are there?

Junji Inaba
(Student, William Hulme's Grammar School, Manchester)

A Straightforward Method for Extracting Real Roots of Polynomial Equations of Arbitrary Order

R. A. DUNLAP

Introduction

Polynomial equations of the form

$$ax^N + bx^{N-1} + cx^{N-2} + \cdots = 0 \qquad (1)$$

appear in a wide variety of problems in classical and quantum physics. Some applications of the cubic equation in introductory thermodynamics have recently been discussed in reference 1. In the case of $N = 2$ (quadratic equation), $N = 3$ (cubic equation) and $N = 4$ (quartic equation) the roots may be expressed analytically (reference 2). For larger values of N it is customary to utilize an iterative numerical method; for a recent review, see reference 3. In the present article a method of obtaining the real roots of a polynomial equation of arbitrary order in terms of the limit of the ratio of terms in a corresponding sequence is described. Although this algorithm is straightforward and its implementation on a microcomputer is simple and reliable, its application appears not to be described in the literature. The present article describes the root-finding algorithm in terms of an example involving a quadratic equation. This is followed by a generalization of the appropriate equations to higher order and an example for a sixth-order polynomial equation.

A quadratic example

A quadratic equation may be expressed as

$$x^2 - a_1x - a_2 = 0, \qquad (2)$$

where the coefficients a_i are real numbers and may be either positive or negative. Trivial cases with a_1 or $a_2 = 0$ will not be considered. Equation (2) may be related to the form of equation (1) by dividing all terms by the leading coefficient.

Using the coefficients a_1 and a_2 in (2) we define the sequence (A_n) by means of the recursion relation

$$A_n = a_1A_{n-1} + a_2A_{n-2} \qquad (n \geqslant 2) \qquad (3)$$

and two seed values, i.e. the values of A_0 and A_1. If the coefficients of equation (3) satisfy the condition

$$a_1^2 \geqslant -4a_2, \qquad (4)$$

then the roots will be real and one of these roots will be given by the limit of the ratio of successive terms in the sequence formed by the recursion relation of equation (3):

$$x_1 = \lim_{n\to\infty} \frac{A_n}{A_{n-1}}. \qquad (5)$$

It follows from the explicit solution of the recursion relation (3) obtained in reference 4 that this limit exists and it is easy to show that, if it exists, then it is a root of the equation (2). First, multiplying (3) by A_n/A_{n-1}^2 yields

$$\left(\frac{A_n}{A_{n-1}}\right)^2 = a_1\left(\frac{A_n}{A_{n-1}}\right) + a_2\left(\frac{A_n}{A_{n-1}}\right)\left(\frac{A_{n-2}}{A_{n-1}}\right). \qquad (6)$$

If now the ratio A_n/A_{n-1} converges to the limit x_1 as $n \to \infty$, then

$$\left(\frac{A_n}{A_{n-1}}\right)\left(\frac{A_{n-2}}{A_{n-1}}\right) \to 1$$

and equation (6) gives $x_1^2 = a_1x_1 + a_2$. Hence x_1 is a root of equation (2).

A specific example will serve to illustrate the application of the above expressions. Consider the case where $a_1 = 3/2$ and $a_2 = -1/2$ and the seed values are $A_0 = 0$ and $A_1 = 1$. Actually the choice of seed values is immaterial: although different choices will yield different sequences (A_n), the limit of A_n/A_{n-1} as $n \to \infty$ is the same for all of them. The sequence generated by the given recursion relation together with the given seed values and the resulting ratios of successive terms are exhibited in table 1. It is evident that this ratio tends to the root $x_1 = 1$. The second real root, x_2, may be readily obtained from the relation

$$x_2 = a_1 - x_1. \qquad (7)$$

Table 1. Sequence terms and ratios for $a_1 = 3/2$ and $a_2 = -1/2$.

n	A_n	A_n/A_{n-1}
0	0	—
1	1.0000	—
2	1.5000	1.5000
3	1.7500	1.6667
4	1.8750	1.0714
5	1.9375	1.0333
6	1.9687	1.0161
7	1.9844	1.0079
8	1.9922	1.0039
9	1.9961	1.0020
10	1.9981	1.0010

The analytic solution of this quadratic equation yields the same roots. In all cases it is found that the root which is obtained from (5) is the root with the largest absolute value. For this example it is interesting to note that the values of

A_n obtained from the recursion relation form a convergent sequence. This is distinctly different from the Fibonacci sequence obtained from equation (3) with the coefficients $a_1 = 1$, $a_2 = 1$ and seed values $A_0 = 0$ and $A_1 = 1$. For a quadratic equation with imaginary roots A_n/A_{n-1} does not tend to a limit as $n \rightarrow \infty$. Application of this technique to some simple examples in which (4) is not satisfied will immediately show this to be the case.

Extensions to higher-order polynomials

The results presented above can be extended to cubic and higher-order equations. In general, a polynomial equation of arbitrary order (Nth order) may be written as

$$x^N - \sum_{i=1}^{N} a_i x^{N-i} = 0. \tag{8}$$

One real root (if it exists) of this polynomial equation is given by equation (5) for a sequence generated by the recursion relation

$$A_n = \sum_{i=1}^{N} a_i A_{n-i}. \tag{9}$$

The limit, if it exists, of the ratio of terms A_n/A_{n-1} as $n \rightarrow \infty$ can be seen to be a root of equation (8) by multiplying the terms in equation (9) by an appropriate factor to give $(A_n/A_{n-1})^N$ on the left-hand side. If the ratio A_n/A_{n-1} converges to a limit x_1 (say) as $n \rightarrow \infty$, then x_1 is a solution of equation (8).

The case of a sixth-order polynomial will be considered as an example. For the coefficients

$$a_1 = -2, \quad a_2 = -1, \quad a_3 = -1$$
$$a_4 = 5, \quad a_5 = -3, \quad a_6 = 3,$$

and the seed values for the sequence

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0,$$
$$A_3 = 0, \quad A_4 = 0, \quad A_5 = 1,$$

the ratio A_n/A_{n-1} is illustrated as a function of n in figure 1. This ratio is found to tend to the value

$$x_1 = 2.270\,391\,7764 \cdots.$$

It is straightforward to substitute this value into equation (8) with the appropriate coefficients and show that it is, in fact, a root of the sixth-order polynomial equation. Once one root of a polynomial equation of order N has been found, additional real roots, if any, may be obtained by factoring $(x - x_1)$ from the original equation and applying the same method to the resulting $(N-1)$th-order expression.

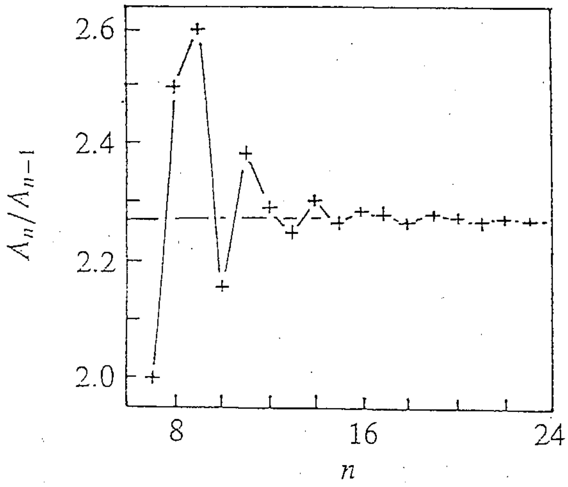


Figure 1. Ratio of terms A_n/A_{n-1} as a function of n as obtained from the recursion relation of equation (8) for $N = 6$ and values of the coefficients $a_1 = -2$, $a_2 = -1$, $a_3 = -1$, $a_4 = 5$, $a_5 = -3$ and $a_6 = 3$ and the seed values for the sequence $A_0 = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 0$, $A_4 = 0$ and $A_5 = 1$.

Conclusions

The above method is a straightforward technique for obtaining roots of higher-order equations. The computer code needed to implement this algorithm is simple and does not require the carefully designed iteration loops characteristic of the methods common in the literature (reference 3). This provides a useful computer technique for solving polynomial equations which appear in a variety of physics problems. It also provides a suitable coding problem to acquaint readers with simple computer methods for the numerical analysis and has the advantage that cpu and memory requirements are very undemanding. Sample programs may be obtained by writing to the author.

References

1. V. Thomsen, Solution of third-order polynomials, *Physics Teacher* 32, 24 (1994).
2. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972) p. 17.
3. W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, (Cambridge University Press, Cambridge, 1986) p. 259ff.
4. G. N. Thwaites, Difference and differential equations, *Mathematical Spectrum*, **27**, 38-40 (1994/5). □

***R. A. Dunlap** has been a professor of physics at Dalhousie University since 1980. In recent years his principal research interests have been in the field of experimental solid state physics. His work has concentrated on the structural, magnetic and electronic properties of metals and ceramics. As well, he has worked on a variety of problems involving the fundamental properties of crystallographic symmetry and Fibonacci lattices.*

The Dynamics of the 21-Card Trick

A. F. BEARDON

In the 21-card trick the trickster can tell the audience which card was chosen after three rounds; here we find the number of rounds needed for a pack of cards of any size.

The card trick

Consider the following card trick. The trickster has 21 cards in a pack; the audience chooses one card (leaving it in the pack) and the pack is shuffled. The trickster deals out the cards face up in a row of three, then in a second row of three, and so on, the cards finally making an array of seven rows of three cards each (or three columns of seven cards each). The audience states which column the chosen card is in, the trickster collects up the columns to reform the pack *ensuring that the chosen column is placed between the other two*. This process (which we call a *round*) is repeated twice more. The trickster can now tell the audience which card was chosen, for *it will be the card in the centre of the pack* with ten cards before it and ten cards after it. Of course, the trickster can add the usual embellishments to deflect the audience's attention from the crucial issues.

Why does this trick work? What will happen if we take a different size pack? We shall analyse the trick for a pack of any size, explain why it works, and show how many rounds are needed. Unexpectedly perhaps (but only at first), we shall see that if the trick were to be carried out with an array of, for example, 999×999 cards, the chosen card would find its way to the centre of the pack in just two rounds!

The convergence of the process

We consider a pack with PQ cards which will be arranged in an array of P columns, each of Q cards or, equivalently, Q rows, each of P cards. The case described above is with $P = 3$ and $Q = 7$. We shall only analyse the case when P and Q are odd and greater than 1 (leaving the other cases to the interested reader), and we write $P = 2p + 1$ and $Q = 2q + 1$, where $p \geqslant 1$ and $q \geqslant 1$. We are interested in the central card in the pack; this has exactly $\frac{1}{2}(PQ-1)$ cards above and below it and, labelling the cards $1, 2, 3, \ldots$ from the top of the pack, the central card is labelled C, where

$$C = \tfrac{1}{2}(PQ + 1) = pQ + q + 1 = qP + p + 1.$$

Suppose now that the chosen card lies in the X_nth position in the pack after n rounds have been played. During the play in the next round the first X_n cards will be spread out over the first few rows, and the chosen card will occur in the rth row, where r is the smallest integer greater than or equal to X_n/P. We write $\langle Y \rangle$ for the smallest integer m satisfying $m \geqslant Y$. It follows that, after reforming the pack by ensuring that the column containing the chosen card is the central column, the position X_{n+1} of the chosen card will satisfy the recurrence relation

$$X_{n+1} = pQ + \left\langle \frac{X_n}{P} \right\rangle.$$

This is a recurrence relation, but it is non-linear, and we do not know the initial value X_0 of the chosen card in the pack; all we know is that $1 \leqslant X_0 \leqslant PQ$. Nevertheless, this information is sufficient to give us a complete solution of our problem, and we shall see that the dynamics of the trick is reminiscent of the convergence of iterates of a function towards an attracting fixed point of the function.

We begin by noting that there is a symmetry in the process, so that we may confine our attention to those cases where the chosen card lies in the top half of the pack; thus we may assume that $1 \leqslant X_0 \leqslant C$. An induction argument now shows that, for all n, $X_n \leqslant C$. For suppose that $X_n \leqslant C$; then

$$\begin{aligned} X_{n+1} = pQ + \left\langle \frac{X_n}{P} \right\rangle &\leqslant pQ + \left\langle \frac{C}{P} \right\rangle \\ &= pQ + \left\langle \frac{qP + p + 1}{P} \right\rangle \\ &= pQ + q + 1 = C. \end{aligned}$$

This argument also shows that if $X_n = C$, then $X_{n+1} = C$.

Observe next that $X_1 > pQ$. We shall now show that there is some integer N with

$$\begin{aligned} pQ &< X_1 < X_2 < \cdots \\ &< X_{N-1} < X_N \\ &= C = X_{N+1} = X_{N+2} = \cdots . \end{aligned} \tag{1}$$

Indeed, if $pQ < X_n < C$, then $X_n = pQ + t$, where $1 \leqslant t \leqslant q$, and for this range of values of t we have $pQ + t > Pt$. This means that $pQ + t = Pt + \delta$, say, where $\delta > 0$; thus

$$X_{n+1} = pQ + \left\langle \frac{pQ + t}{P} \right\rangle = pQ + \left\langle \frac{Pt + \delta}{P} \right\rangle > pQ + t = X_n$$

and this establishes (1).

To summarise, given that P and Q are odd integers greater than 1, as each round is carried out the chosen card moves monotonically towards, and finally reaches, the central card in the pack where it remains thereafter. We now examine how many rounds are needed for the chosen card to reach the central position.

The number of rounds

In the case of the trick as first described, $P=3$, $Q=7$ and $C=11$. In this case,

$$X_1 \geqslant 8, \quad X_2 \geqslant 7+\langle 8/3\rangle = 10,$$
$$11 \geqslant X_3 = 7+\langle 10/3\rangle = 11.$$

Thus (as stated in the first section) the chosen card is in the central position after three rounds.

We turn now to discuss the number of rounds needed for the pack of general size. First, as $X_1 > pQ$ we deduce that

$$C \geqslant X_2 \geqslant pQ + \left\langle \frac{pQ+1}{P} \right\rangle.$$

Now $P \geqslant Q$ implies that $pQ \geqslant qP$ which, in turn, implies that

$$C \geqslant X_2 \geqslant pQ + \left\langle \frac{qP+1}{P} \right\rangle = pQ+q+1 = C.$$

Thus *if $P \geqslant Q$ (and, in particular, if $P = Q$) then the chosen card is already in the central position after the second round regardless of the size of P and Q.* A little thought should now show that, in retrospect, this is rather obviously so.

The situation when $Q > P$ is more complicated, and we shall show that the chosen card is always in the central position by the nth round provided that

$$n \geqslant 1 + \frac{\log Q}{\log P}. \tag{2}$$

We shall also give an example to show that in some cases this lower bound is indeed the smallest n for which the chosen card is centrally placed, so the inequality (2) cannot be improved upon. If $P \geqslant Q$, then (2) simply says that $X_n = C$ as soon as $n=2$, a fact we have already observed above. Note also that (2) shows that, for a fixed P, the number of rounds needed tends to $+\infty$ as Q does. We begin with the example to show that (2) cannot be improved upon.

Example Consider the trick with $P=3$, and $Q=3^N$, so that $C=\frac{1}{2}(3^{N+1}+1)$, and let the selected card be the first in the pack. Then

$$X_1 = pQ + \langle \tfrac{1}{3} \rangle = 3^N+1,$$
$$X_2 = 3^N + \langle \tfrac{1}{3}(3^N+1) \rangle = 3^N+3^{N-1}+1,$$

and a straightforward induction argument shows that, for $k=1,\dots,N$,

$$X_k = 3^N+3^{N-1}+\dots+3^{N-k+1}+1.$$

We deduce that

$$X_N = 3^N+3^{N-1}+\dots+3^2+3+1$$
$$= \tfrac{1}{2}(3^{N+1}-1) = C-1,$$

so that X_n first reaches C when

$$n = 1+N = 1+\frac{\log Q}{\log P}.$$

It remains to verify (2) for the general pack (with P and Q odd). We know that

$$X_{n+1} \geqslant pQ + \frac{X_n}{P}, \qquad X_0 \geqslant 1,$$

and it follows (by induction) that if we define the sequence V_n by

$$V_{n+1} = pQ + \frac{V_n}{P}, \qquad V_0 = 1,$$

then $X_n \geqslant V_n$ for all n. We can solve the linear recurrence relation for V_n and the solution is

$$V_n = \frac{PQ}{2} - \frac{PQ-2}{2P^n}.$$

This can be checked by induction. It follows that, if n satisfies

$$\frac{PQ}{2} - \frac{PQ-2}{2P^n} > C-1 = \frac{PQ-1}{2}, \tag{3}$$

then $X_n \geqslant V_n > C-1$ so that $X_n \geqslant C$ and hence $X_n = C$. The condition (3) is satisfied if $P^{n-1} \geqslant Q$, and this shows, as promised, that $X_n = C$ whenever (2) holds. □

***Alan Beardon** is a reader in pure mathematics at the University of Cambridge. His mathematical interests are geometry, complex analysis, dynamical systems and mathematical economics; his hobbies are woodwork and hill walking.*

Prove that the sum to $2n+1$ terms of the geometric series $x^n+x^{n-1}+x^{n-2}+\dots$, where x is positive, is at least $2n+1$.

A woman says: 'I have two children of whom at least one is a girl'. What is the probability that both are girls?

Twelve New Points on the Nine-Point Circle

JINGCHENG TONG and SIDNEY KUNG

The authors take another look at a famous circle.

Let $\triangle ABC$ be a given triangle, D, E, F be the midpoints of the sides BC, AC, AB, respectively. Let AG, BH, CI be the altitudes on the sides BC, AC, AB and G, H, I be the perpendicular feet. If O is the orthocentre and J, K, L are the midpoints of the segments AO, BO, CO, then the nine points D, E, F, G, H, I, J, K, L are on a circle.

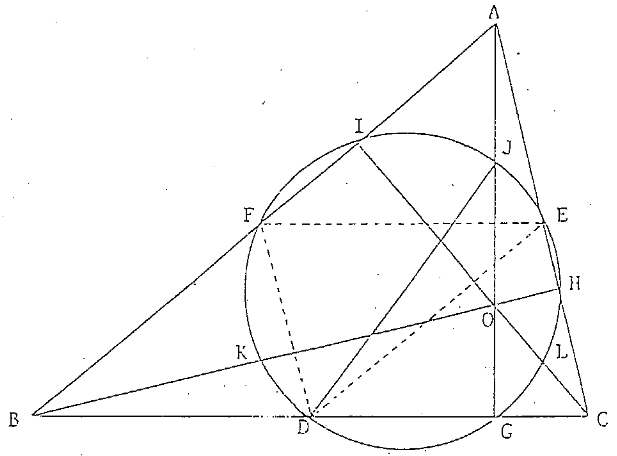


Figure 1

This nine-point circle theorem was discovered in the 19th century. The name was given by Poncelet (reference 1), and is commonly used in English-speaking countries. In France it is usually called Euler's circle, while in Germany it is called Feuerbach's circle (references 1 and 2).

In reference 3, three more points are added to the nine-point circle. They are the intersection points of the perpendicular bisectors of BC, AC, AB and the perpendicular bisectors of AO, BO, CO, respectively. Therefore, there are so far twelve special points on the nine-point circle.

It is easily seen that the altitudes of the triangle play a decisive role in the nine-point circle theorem. Six of the nine points are on the altitudes. For a triangle, medians and angle bisectors are just as important as altitudes. It is a little unreasonable that they have no contribution to this fantastic circle. In this note we prove that there are twelve new points on the nine-point circle by the help of medians and angle bisectors.

Before stating the result, we first point out a fact we are going to use. Referring to figure 1, DJ is a diameter of the nine-point circle because $\angle DGJ$ is a right angle. We assume that the radius of the nine-point circle is 1 and that $\angle C > \angle B$. Then

$$\angle DJG = \tfrac{1}{2}\text{arc}\, DG = \tfrac{1}{2}(\text{arc}\, DE - \text{arc}\, GE).$$

Also

$$\angle DFE = \tfrac{1}{2}\text{arc}\, DE \quad \text{and} \quad \angle EDC = \tfrac{1}{2}\text{arc}\, GE,$$

so that

$$\angle DJG = \angle DFE - \angle EDC.$$

However, since the triangles DEF, EDC, FBD are similar,

$$\angle DFE = \angle C \quad \text{and} \quad \angle EDC = \angle B.$$

Hence, finally,

$$\angle DJG = \angle C - \angle B.$$

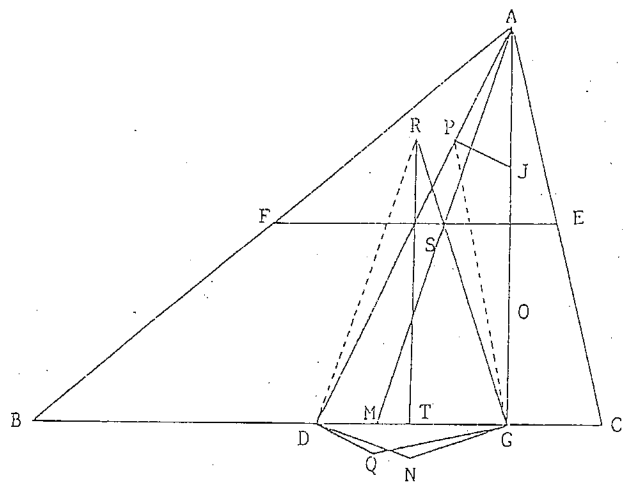


Figure 2

Theorem. Let $\triangle ABC$ be a triangle, D, E, F be the midpoints of the sides BC, AC, AB. Let AD, AG, be the median and the altitude on the side BC, and AM be the angle bisector of $\angle A$. If the orthocentre of $\triangle ABC$ is O and J is the midpoint of the segment AO, then the following statements are true.

(1) The intersection point P of the median AD with its perpendicular line passing through J is on the nine-point circle.

(2) The intersection point Q of the line passing through D and perpendicular to AD, and the line passing through G and perpendicular to PG, is on the nine-point circle.

(3) Let S be the intersection point of EF with the angle bisector AM. Then the intersection point R of the line GS with the perpendicular bisector of DG is on the nine-point circle.

(4) The intersection point N of the line passing through D and perpendicular to DR, and the line passing through G and perpendicular to GR, is on the nine-point circle.

Proof. (1) In figure 2, since $JP \perp AD$ and $AG \perp DG$, the four points P, J, G, D must be on a circle. Hence P is on the circle passing through J, G, D. This circle is the nine-point circle.

(2) Since $QP \perp PD$ and $QG \perp PG$, the four points P, D, Q, G must be on a circle. Hence Q is on the circle passing through P, D, G. By (1), we know that this circle is the nine-point circle.

(3) Let T be the midpoint of the segment DG. Since S is on the segment EF, which is the perpendicular bisector of AG, $AS = GS$ and $\angle SAG = \angle SGA$. Since AG is the altitude on BC and RT is the perpendicular bisector of DG, we have that TR is parallel to AG and $\angle TRG = \angle SGA$. Since R is on the perpendicular bisector of DG, we have $RG = RD$ and $\angle TRG = \angle TRD$. But $\angle SAG = \angle A/2 - \angle CAG = \angle A/2 - (\pi/2 - \angle C) = (\angle C - \angle B)/2$. Therefore $\angle DRG = \angle C - \angle B$. By the statement above the theorem, $\angle DJG = \angle C - \angle B$. Thus, $\angle DRG = \angle DGJ$ and the four points R, J, G, D are on a circle. The point R is thus on the circle passing through J, G, D. This circle is the nine-point circle.

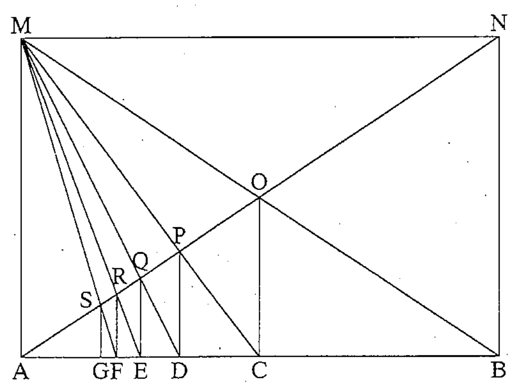
(4) Since $DN \perp DR$ and $NG \perp GR$, the four points R, D, N, G are on a circle. Thus N is on the circle passing through R, D, G. By (3), this circle is the nine-point circle.

Remark. If we discuss the sides AC, AB we can get eight more new points on the nine-point circle. Therefore we have twelve new points on the nine-point circle. Together with the twelve points already obtained, there are twenty-four special points constructed easily from $\triangle ABC$ by the help of altitudes, medians and angle bisectors. There could be more special points discovered in the future. Therefore, the name of the nine-point circle does not reflect the fact very well. It is high time to rename it as either Euler's circle, for short, or the Euler–Poncelet–Feuerbach circle to reflect the history of discovery.

References

1. N. Altshiller-Court, *College Geometry*, (Barnes and Noble, New York, 1952).
2. H. W. Eves, *Fundamentals of Geometry*, (Allyn and Bacon, Boston, 1969).
3. J. Tong and S. Kung, Proof without words: the nine-point circle is in fact a twelve-point circle, submitted. □

***Jingcheng Tong** received his B.S. from Guizhou University in China, and his Ph.D. from Wayne State University in the USA. He is now a professor at the University of North Florida. He has published more than sixty papers with topics from number theory, general topology, analysis and geometry. **Sidney Kung** received degrees in engineering and applied mathematics from the University of Rhode Island and from the University of Illinois. His interests are linear algebra and numerical methods. He has published about thirty articles in collegiate mathematics.*



ABNM is a rectangle with $AB = 1$, and OC, PD, QE etc. are perpendicular to AB. Then $AC = \frac{1}{2}$, $AD = \frac{1}{3}$, $AE = \frac{1}{4}$, $AF = \frac{1}{5}$, etc. The proof is simple but cumbersome, so it is not included here.

L. P. Knight
(Flat 1, 541 Babbacombe Road, Torquay, Devon TQ1 1HQ)

The International Mathematical Olympiad 1996

This was held in Bombay, India, 5–17 July. Teams from a record number of 75 countries took part. The top ten teams with their scores out of a maximum possible 252 points were:

Romania	187
USA	185
Hungary	167
Russia	162
United Kingdom	161
China	160
Vietnam	155
South Korea	151
Iran	143
Germany	137

The IMO is a rigorous competition composed of problems that would challenge most professional mathematicians. In addition to comprehensive mathematical knowledge, success in the IMO requires exceptional mathematical creativity and inventiveness.

The Oddball Problem

BRIAN D. BUNDAY

This problem has appeared in several collections of mathematical brain-teasers and from time to time gets resurrected to challenge yet another generation of puzzlers. The problem can be stated as follows. We are given twelve balls, identical in appearance but one of which is either heavier or lighter than the other eleven. We are allowed three weighings with a *balance*, to determine which is the odd ball and to find whether this ball is heavier or lighter than the others. What do we do?

We can suppose that we can, if it has not already been done, label the balls $1, 2, 3, \ldots, 10, 11, 12$ so that we can distinguish between and identify them using these labels. The solution usually given runs along the following lines.

Weigh $1, 2, 3, 4$ against $5, 6, 7, 8$

(1) They balance, so $9, 10, 11, 12$ contain the odd ball. Weigh $6, 7, 8$ against $9, 10, 11$.

   (a) They balance, therefore 12 is the odd ball and so weigh 12 against any other to discover whether it is heavy or light.

   (b) $9, 10, 11$ are heavy and so they contain an odd *heavy* ball. Weigh 9 against 10. If they balance, 11 is the odd heavy ball, otherwise the heavier of 9 and 10 is the odd ball.

   (c) If $9, 10, 11$ are light, we use the same procedure to reach the same conclusion for the odd light ball.

(2) $5, 6, 7, 8$ are heavy and so either they contain an odd heavy ball or $1, 2, 3, 4$ contain an odd light ball. Weigh $1, 2, 5$ against $3, 6, 10$.

   (a) They balance, so the odd ball is 4 (light) or 7 or 8 (heavy). Thus weigh 7 against 8. If they balance 4 is light, otherwise the heavier of 7 and 8 is the odd heavy ball.

   (b) $3, 6, 10$ are heavy, so the odd ball can be 6 (heavy) or 1 or 2 (light). Thus weigh 1 against 2. If they balance 6 is heavy, otherwise the lighter of 1 and 2 is the odd light ball.

   (c) $3, 6, 10$ are light, so the odd ball is 3 and light or 5 and heavy. We thus weigh 3 against 10. If they balance 5 is heavy, otherwise 3 is light.

(3) If $5, 6, 7, 8$ are light we use a similar procedure to that in 2.

This solution, which requires different courses of action depending on the outcomes of previous weighings, is not particularly elegant or easy to remember. We shall give a solution which involves a fixed course of action in all circumstances and which has the advantage of showing how this particular problem can be generalised.

In our method we weigh four specificed balls against four other specified balls in each of the three weighings and note the result. If we observe say the left-hand side of the balance, then for an individual weighing there are three possible alternatives: the left-hand side is heavy (H), light (L) or equal (E) as compared with the right-hand side of the balance.

Since three weighings are allowed, the number of different results that can be obtained is just the number of arrangements (with repetitions allowed) of the three symbols H, L, E, i.e. $3^3 = 27$. If we use all twelve balls in the three weighings, and ensure that no particular ball appears on the same side of the balance in all three weighings, the outcomes HHH, LLL, EEE are not possible. We thus have only 24 possible outcomes and we shall show that it is possible to set up a one–one correspondence between these 24 outcomes and the conclusion that a particular ball among the twelve is heavy or light.

The 24 outcomes can be divided into two groups of twelve in each group. If we call the *reverse* of an outcome the outcome obtained by replacing H by L, L by H and leaving E unchanged, one group of twelve will be the reverses of the other group and *vice versa*. We can thus write the 24 outcomes in the form of two arrays, each array having three rows (the three weighings) and twelve columns (the twelve balls), so that each row contains four Hs, four Ls and four Es. Thus we have, for example,

H	H	H	H	L	L	L	L	E	E	E	E
L	E	E	E	L	L	H	H	H	L	H	E
E	H	L	E	H	E	L	H	H	E	L	L
1	2	3	4	5	6	7	8	9	10	11	12
L	L	L	L	H	H	H	H	E	E	E	E
H	E	E	E	H	H	L	L	L	H	L	E
E	L	H	E	L	E	H	L	L	E	H	H

We consider just the top array, and for each weighing (row) place the balls corresponding to an H in the left, and the balls corresponding to an L in the right of the balance. Thus we would weigh $1, 2, 3, 4$ against $5, 6, 7, 8$; then $7, 8, 9, 11$ against $1, 5, 6, 10$; and finally $2, 5, 8, 9$ against $3, 7, 11, 12$. The results of these three weighings as observed on the left of the balance are noted. If the outcome is H, L, E we conclude that ball 1 is heavy; if H, E, H ball 2 is heavy, etc; if E, E, L ball 12 is heavy. If we obtain an outcome that appears in the lower array, we conclude that the corresponding ball is light. Thus for L, H, E ball 1 is light, etc; E, E, H means ball 12 is light.

This method of solution allows us to see how the problem can be generalised. If two weighings are permitted we would have $3^2 - 3 = 6$ different outcomes. These can be split into two groups of three, the one group being the reverses of the other. Thus with two weighings, weighing one ball against one other ball at each weighing allows us to find the odd ball among three.

The arrays are

H	L	E
E	H	L
1	2	3
L	H	E
E	L	H

Thus weigh 1 against 2 and then 2 against 3 and note the result. Of course this is too easy a problem to set our puzzlers no matter how we solve it.

But if four weighings are allowed, the problem is probably too difficult, except by this procedure, when it is equally easy, even though it takes a bit longer to enumerate the outcomes. With four weighings there are $3^4 - 3 = 78$ possible outcomes, and these can be divided into two groups with 39 in each group. In this case we weigh thirteen balls against thirteen others at each weighing and note the results. This will allow us to pick out the odd ball among 39 balls. If five weighings are allowed we can find the odd ball, and whether it is heavy or light, among $\frac{1}{2}(3^5 - 3) = 120$ balls.

In general, if n weighings are allowed, we can find the odd ball among $\frac{1}{2}(3^n - 3)$ apparently identical balls.

***Brian Bunday** is Head of the Department of Mathematics at the University of Bradford. His research interests include stochastic processes, mathematical programming and using computers in the teaching of mathematics. He finds relaxation in gardening (besides an acre of garden he grows his own vegetables and fruit on an allotment), chess and music.*

Smarandache's Periodic Sequences

1. Start with a positive integer N with not all its digits the same, and let N' be its digital reverse. Put $N_1 = |N - N'|$ and let N_1' be the digital reverse of N_1. Put $N_2 = |N_1 - N_1'|$ and let N_2' be the digital reverse of N_2. And so on. By Dirichlet's box principle, we eventually obtain a repetition. For example,

$$42, 18, \mathbf{63}, 27, 45, 09, 81, \mathbf{63}, 27, \ldots.$$

Starting with a two-digit number, it seems that we always reach the cycle 63, 27, 45, 09, 81 or a cyclic permutation of it, and that a repetition occurs after between five and seven steps. An example of a three-digit number sequence is

$$321, 198, \mathbf{693}, 297, 495, 099, 891, \mathbf{693}, \ldots,$$

which is the same cycle as the two-digit sequence with a 9 inserted between each digit.

2. Let c be a fixed positive integer. Start with a positive integer N and let N' be its digital reverse. Put $N_1 = |N' - c|$ and let N_1' be its digital reverse. Put $N_2 = |N_1' - c|$ and let N_2' be its digital reverse. And so on. We shall eventually obtain a repetition. For example, with $c = 1$ and $N = 52$ we obtain the sequence

$$\mathbf{52}, 24, 41, 13, 30, 02, 19, 90, 08, 79, 96, 68, 85, 57, 74, 46, 63, 35, \mathbf{52}, \ldots.$$

Here a repetition occurs after 18 steps and the length of the repeating cycle is 18.

3. Let $c > 1$ be a fixed integer. Start with a positive integer N, multiply each digit x of N by c and replace that digit by the last digit of cx to give N_1. And so on. We shall eventually obtain a repetition. For example, with $c = 7$ and $N = 68$ we obtain the sequence

$$\mathbf{68}, 26, 42, 84, \mathbf{68}, \ldots.$$

4. Let N be a two-digit number. Add the digits, and add them again if the sum is greater than 10. Also take the absolute value of their difference. These are the first and second digits of N_1. Now repeat this. For example, with $N = 75$ we obtain the sequence

$$\mathbf{75}, 32, 51, 64, 12, 31, 42, 62, 84, 34, 71, 86, 52, 73, 14, 53, 82, 16, \mathbf{75}, \ldots.$$

There will always be a periodic sequence whenever we have a function $f : S \to S$, where S is a finite set of positive integers and we repeat the function f. Readers may like to investigate the above examples and produce examples of their own.

Reference

F. Smarandache, Sequences of Numbers, *University of Craiova Conferences* (1975).

M. R. Popov
(Student at Chandler College, Arizona)

Elementary Solution of a Second-Order Differential Equation

FRANK CHORLTON

This article shows how to solve any second-order differential equation with constant coefficients using a minimum of theory.

The object of this article is to solve for x the differential equation

$$\ddot{x} + 2a\dot{x} + bx = f(t), \tag{1}$$

where x is an unknown function of t (with $\dot{x}$ and $\ddot{x}$ denoting $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}^2x/\mathrm{d}t^2$ respectively), a and b are given constants and f is a prescribed function of t. The equation has not only intrinsic interest, but it also arises in many physical situations such as the swing of a pendulum, motion controlled by an elastic string and certain electrical circuits where the current varies with time.

We start by reducing (1) to the simpler form

$$\ddot{y} + (b - a^2)y = \mathrm{e}^{at} f(t). \tag{2}$$

This is achieved simply by putting

$$x(t) = \mathrm{e}^{-at} y(t) \tag{3}$$

into (1) and simplifying. We then consider separately the three cases $b - a^2 = 0$, $b - a^2 > 0$, $b - a^2 < 0$.

(a) When $b - a^2 = 0$, equation (2) reduces to

$$\ddot{y} = \mathrm{e}^{at} f(t). \tag{4}$$

Then at once

$$\dot{y} = \int \mathrm{e}^{at} f(t)\,\mathrm{d}t + A, \tag{5}$$

where A is an integration constant. Integrating (5) now gives the solution

$$y(t) = \iint \mathrm{e}^{at} f(t)\,\mathrm{d}t\,\mathrm{d}t + At + B \tag{6}$$

where B is another integration constant. Substitution of this form for y into (3) then gives the required solution $x(t)$, involving two integration constants. It may be noted that the integrations in (5) and (6) can certainly be carried out if $f(t)$ is a so-called admissible function, i.e. a sum of products of polynomials and exponential and trigonometric functions.

(b) When $b - a^2 = \omega^2 > 0$, (2) reduces to the form

$$\ddot{y} + \omega^2 y = g(t),$$

where $g(t) = \mathrm{e}^{at} f(t)$. Again $g(t)$ can be any admissible function of t, but for simplicity let us take the important special case $g(t) = k \sin \omega t$ which will illustrate the technique sufficiently well. So the differential equation for solution is now specifically

$$\ddot{y} + \omega^2 y = k \sin \omega t. \tag{7}$$

Multiplying (7) through by $\sin \omega t$ we have

$$\ddot{y} \sin \omega t + \omega^2 y \sin \omega t = \tfrac{1}{2}k(1 - \cos 2\omega t).$$

It is easily seen that the left-hand side of this is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{y} \sin \omega t - \omega y \cos \omega t).$$

Thus (7) may be integrated to give

$$\dot{y} \sin \omega t - \omega y \cos \omega t = \tfrac{1}{2}k\left(t - \frac{1}{2\omega} \sin 2\omega t\right) + A, \tag{8}$$

where A is an arbitrary constant. Similarly, multiplying (7) through by $\cos \omega t$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{y} \cos \omega t + \omega y \sin \omega t) = \tfrac{1}{2}k \sin 2\omega t,$$

so that integration of this gives

$$\dot{y} \cos \omega t + \omega y \sin \omega t = -\frac{k}{4\omega} \cos 2\omega t + B, \tag{9}$$

where B is an arbitrary constant. If we now eliminate $\dot{y}$ between the equations (8) and (9) we obtain the general solution of (7) in the form

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t - \frac{kt}{2\omega} \cos \omega t,$$

where $C_1 = (k + 4\omega B)/4\omega^2$ and $C_2 = -A/\omega$. Thus C_1 and C_2 are the integration constants.

(c) When $b - a^2 < 0$, (2) is of the form

$$\ddot{y} - n^2 y = g(t). \tag{10}$$

If we multiply (10) through by e^{nt}, it is easy to show that it becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{nt}\dot{y} - n\mathrm{e}^{nt} y) = \mathrm{e}^{nt} g(t),$$

which leads to the integral

$$e^{nt}(\dot{y} - ny) = \int_0^t e^{nu} g(u)\,du + A$$

or

$$\dot{y} - ny = e^{-nt} \int_0^t e^{nu} g(u)\,du + Ae^{-nt}. \qquad (11)$$

A similar result is obtained by multiplying (10) through by e^{-nt}. In fact the corresponding result may be derived from (11) by replacing n by $-n$ and changing A to B to give

$$\dot{y} + ny = e^{nt} \int_0^t e^{-nu} g(u)\,du + Be^{nt}. \qquad (12)$$

On eliminating $\dot{y}$ between (11) and (12) and simplifying, we obtain the general solution of (10) as

$$y(t) = C_1 e^{nt} + C_2 e^{-nt} + \frac{1}{2n}\left(e^{nt} \int_0^t e^{-nu} g(u)\,du - e^{-nt} \int_0^t e^{nu} g(u)\,du\right),$$

where $C_1 = B/2n$, and $C_2 = -A/2n$. □

In 1982 **Frank Chorlton** *took early retirement from his position of Senior Lecturer in Mathematics at Aston University. He has published many books and papers on mathematics and still does a modicum of teaching at Aston. He has contributed to* Mathematical Spectrum *and other similar journals. Apart from mathematics, his other main interest is in music, especially that of Bach.*

Computer Column

Julian dates

The previous computer column discussed how to calculate the times of sunrise and sunset in northern latitudes. Astronomers often need to calculate the dates and times of other celestial events. For instance, they may wish to know where to point their telescope to observe Venus at 2:15 a.m. on 14 March 1996, or they may wish to know the position of a comet at midnight on 25 December 4 BC. Computer programs can solve these problems. To handle such dates in a computer program, however, we have to convert the date and time into a single unique number. How can we do this?

A common method is simply to count the days that have elapsed since a particular agreed-upon event. First, then, we have to agree that some particular instant will be our starting point; astronomers call this instant the *initial epoch*. In many countries of the world it is customary to use the birth of Christ as the starting point for the calendar. Astronomers instead adopt noon as measured at Greenwich on 1 January 4713 BC as their starting point.

Any given calendar date can thus be converted into the number of days that have elapsed since the initial epoch; this number is called the *Julian day number* or the *Julian date*.

You may think that calculating the Julian date is simple: just count the days from 1 January 4713 BC — remembering to count the correct number of days for each month (this could be 28, 29, 30 or 31). However, there are complications!

Before 5 October 1582, the Julian calender was used throughout Europe. (To make matters even more confusing here, the Julian calendar and the Julian date are named after different people, and there is no connection between the two!) In the Julian calendar there were 365 days in the year, unless the year was divisible by 4 in which case there were 366 days. The average year was thus 365.25 days long. The leap year was to allow for the fact that the tropical year is 365.2422 days long; without a leap year the calendar would soon be out of phase with the seasons.

After 128 years the Julian calendar begins almost one day too late. So although the Julian calendar worked quite well for several centuries, by 1582 the accumulated error between the Julian year and the tropical year was noticeable. In order to bring the calendar back into line, Pope Gregory XIII abolished the dates 5–14 October 1582 inclusive (causing consternation among lots of people), and decreed that years ending in 00 (1600, 1700 and so on) would be leap years only if they were divisible by 400. This made the average year to be 365.2425 days long, which is a better approximation to the tropical year. (Even the Gregorian calendar is not perfect. In about another three thousand years the differences from the tropical year will have accumulated to one full day.) Many countries now use the Gregorian calendar, so when we convert a calendar date into a Julian date we have to take account of these complications.

There are several methods for calculating the Julian date. The following formula can be programmed on a PC or even into a calculator; it gives the Julian date (JD) from a day–month–year date for all dates after 15 October 1582.

Suppose y is the year with all four digits, m is the month and d is the day. Furthermore, suppose

$$f = \begin{cases} y & m \geqslant 3 \\ y-1 & m = 1 \text{ or } 2 \end{cases}$$

$$g = \begin{cases} m & m \geqslant 3 \\ m+12 & m = 1 \text{ or } 2 \end{cases}$$

and

$$A = 2 - \mathrm{INT}(f/100) + \mathrm{INT}(f/400).$$

The Julian date at 0 hours GMT is then

$$\begin{aligned} \mathrm{JD} = {} & \mathrm{INT}(365.25f) + \mathrm{INT}(30.6001(g+1)) \\ & + d + A + 1\,720\,994.5 \end{aligned}$$

where INT is the truncated-integer value. The .5 in the final term in the above formula arises because the Julian day changes at noon.

We can include time by converting it to a fraction of a day and adding it to the date. Thus 10 a.m. is equal to $10/24 = 0.4167$ days. To pick a time at random, 10 a.m. on 2 February 1989 has a Julian date of 2 447 559.9167.

For dates on or before 15 October 1582 (but after the initial epoch) the formula needs to be modified slightly. Readers may wish to find the correct formula to convert calendar dates between 1 January 4713 BC and 15 October 1582 into Julian dates.

Stephen Webb

Mathematics in the Classroom

Number theory

Recently, a television programme enthralled us with the eight-year struggle of one mathematician in deriving a proof of a long-unsolved theorem known as Fermat's Last Theorem. Pierre de Fermat lived more than three hundred years ago; his conjecture was that, although we can easily find two perfect squares that sum to a perfect square ($9 + 16 = 25$, for example), we cannot find two perfect cubes summing to a perfect cube, nor can we for any powers higher than cubes. In other words, the equation $x^n + y^n = z^n$ has no integer solutions for $n \geqslant 3$. Despite much mathematical effort, this theorem was only finally proved in 1994 by Andrew Wiles who needed to develop a whole new area of mathematics in order to access a proof.

In class the day after the programme, we admired the persistence of mathematicians so fascinated by these unsolved mysteries that they were willing to devote much of their lives in pursuit of something that had no obvious usefulness. (It was, however, very reassuring to read Ian Stewart's description of the up-and-coming applications of number theory (TES, 24 May 1996), the area of mathematics to which this theorem belongs. Here he tells of the important part played by number theory in the error-free transmission of messages on the Internet as well as in underlying the structure of secure codes which enable purchases by credit card to be made over the Internet.)

It seemed to us that numbers, which have so many amazing properties, were being remarkably intractable in the case of Fermat's Last Theorem. To reassure ourselves that this was far from being the norm, we decided to look at another situation where squares and cubes are more obliging, that is, we investigated some constellations (cf. reference 1).

Constellations

A constellation of six numbers $x_1, x_2, x_3, y_1, y_2, y_3$ has the property that

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

and

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2.$$

As an example, 4, 5, 9, 3, 7, 8 form a constellation as $4+5+9 = 3+7+8$ and $4^2+5^2+9^2 = 3^2+7^2+8^2$.

Can you find another?

There are many sets of equations which will generate constellations. One of these is given by

$$\begin{array}{lll} x_1 = a + c, & x_2 = b + c, & x_3 = 2a + 2b + c \\ y_1 = c, & y_2 = 2a + b + c, & y_3 = a + 2b + c, \end{array}$$

where a, b and c are integers of your choice. Can you find another set of such equations which will generate constellations, this time involving four unknowns, a, b, c and d? (See reference 1 if you have any difficulties.)

More complex constellations

It is possible to have constellations with more than six numbers, and to extend the property so that the sum of the cubes of the numbers on one side of the equation is the same as the sum of the cubes on the other side. For example, consider the eight number constellation 1, 2, 9, 10, 4, 7, 11, 0. Then the following equations are all true:

$$\begin{aligned} 1 + 2 + 9 + 10 &= 4 + 7 + 11 + 0 \\ 1^2 + 2^2 + 9^2 + 10^2 &= 4^2 + 7^2 + 11^2 + 0^2 \\ 1^3 + 2^3 + 9^3 + 10^3 &= 4^3 + 7^3 + 11^3 + 0^3. \end{aligned}$$

Again, there is a set of equations for the eight numbers in the constellation. Can you find it?

Higher-order constellations

Consider the twelve-number constellation 2, 6, 11, 19, 24, 28, 3, 4, 14, 16, 26, 27 and show that this works for powers up to and including the fifth. It would seem that the sky is the limit for those persistent mathematicians amongst us!

Carol Nixon

Reference

1. B. A. Kordemsky, *The Moscow Puzzles*, ed. M. Gardner, (Penguin, London, 1975).

Letters to the Editor

Dear Editor,

The domino problem

I have had some further thoughts on the domino problem inspired by George Jellis' letter in Volume 28, Number 2, page 44. First of all there is a straightforward way of indicating the formula for $T_{4,n}$ given by Mr Jellis on similar lines to the one I indicated in my letter in the same issue.

For $m > 4$ I do not think any such simple approach to finding $T_{m,n}$ is possible. I attempted to find a recurrence relation to calculate $V_{5,n}$ but soon realised it was not much harder to find a recurrence relation for $T_{5,n}$ directly. Further investigation shows that the order of complexity increases quickly as m increases. For example,

$$T_{5,n} - 14T_{5,n-2} + 17(T_{5,n-4} + T_{5,n-6}) - 14T_{5,n-8} + T_{5,n-10} = 0!$$

And so far I have found the eliminations required to find a single recurrence formula for $T_{6,n}$ too difficult to complete by hand!

However, may I suggest the following general approach which could be translated into a computer program. Perhaps one of your readers would like to do this and let us know the exact number of ways of covering the chess board!

Code the dominoes for any column by stating the *rows* in which the left-hand end of a horizontal domino intersects that column. For example, the arrangement shown in figure 1 would be coded 14, 23, 14, 0 for the four columns. The 0 indicates that there is *no* qualifying row. The *last* column will always be coded 0.

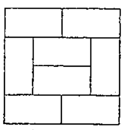


Figure 1

Now I consider the $4 \times n$ array in detail, explained in such a way that it would generalise immediately for a $2m \times n$ array. A slight modification is needed for the $(2m-1) \times n$ array.

(1) Write down every possible selection of none, two and four integers from $1, 2, 3, 4$ which contain equal numbers of odd and even integers.

(2) Asterisk those selections which contain odd and even alternating when the integers are placed in order of size *beginning with an odd*. (See (4) below.)

(3) Create *linking* selections as follows. Selections *link* if and only if they *unite* to give an asterisked selection *and* they have *no rows in common*. 0 links with the asterisked selections only, which includes itself for a $2m \times n$ array. For example, $\{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}$ which is asterisked since it 'goes' odd, even, odd, even in order. Hence $\{1, 4\}$ and $\{2, 3\}$ link. This means in practice that they can follow each other as consecutive columns of the domino array as shown in the diagram above.

(4) The six selections are: 0*, 12*, 14*, 23, 34*, 1234*. The network shown in figure 2 illustrates the linking. For example, the fact that 14 and 23 are linked implies that one may follow the other as the arrangement is built up in a $4 \times n$ array.

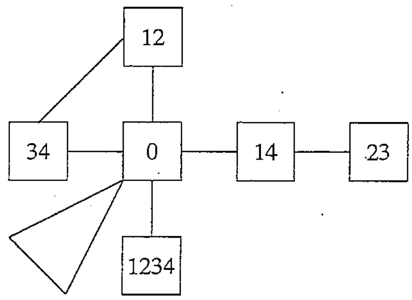


Figure 2

(5) Let a_r, b_r, c_r d_r, e_r, f_r be the number of arrangements which arrive at 0, 12, 14, 23, 34, 1234 respectively in the rth column. We may then deduce the simultaneous recurrence relations:

$$\begin{aligned} a_r &= a_{r-1} + b_{r-1} + c_{r-1} + e_{r-1} + f_{r-1} \\ b_r &= a_{r-1} + e_{r-1} \\ c_r &= a_{r-1} + d_{r-1} \\ d_r &= c_{r-1} \\ e_r &= a_{r-1} + b_{r-1} \\ f_r &= a_{r-1}. \end{aligned}$$

(These results follow because, for example, to get to 14 in the rth column, we must have reached either 0 or 23 in the $(r-1)$th column. There are, by definition, a_{r-1} ways of reaching 0 and d_{r-1} ways of reaching 23 in the $(r-1)$th column and so $c_r = a_{r-1} + d_{r-1}$.)

(6) $T_{4,n} = a_n$. It is worth noting that evaluating $T_{4,n}$ is equivalent to counting the number of routes of length n through the network in figure 2 starting and ending at 0.

(7) The initial values are $a_1 = 1$, $b_1 = 1$, $c_1 = 1$, $d_1 = 0$, $e_1 = 1$, $f_1 = 1$. Note that the 1s occur here where the corresponding codes are asterisked and 0s otherwise. (See (4) above.)
The results for $n = 1$ to 8 are tabulated below, confirming the results in Mr Jellis' letter.

n	a_n	b_n	c_n	d_n	e_n	f_n
1	1	1	1	0	1	1
2	5	2	1	1	2	1
3	11	7	6	1	7	5
4	36	18	12	6	18	11
5	95	54	42	12	54	36
6	281	149	107	42	149	95
7	781	430	323	107	430	281
8	2245					

(8) I have applied the same technique to evaluate $T_{6,n}$. This requires ${}^{6}C_{3} = 20$ simultaneous recurrence relations (see page 7) which I managed to program into my graphics calculator using 28 memory stores. It gave 2 720 246 633 arrangements for a 6×14 array!

Yours sincerely,
ALASTAIR SUMMERS
(Teacher, Stamford School, Lincs.)

Dear Editor,

The geometrical problem of Junji Inaba
Volume 28, Number 1, Page 18

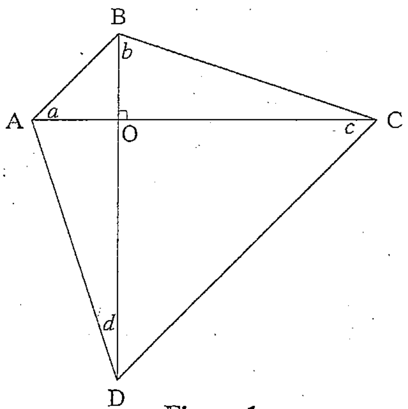


Figure 1

Similar problems can be constructed if we observe that

$$(\tan a)(\tan b)(\tan c)(\tan d) = \frac{\mathrm{OB}}{\mathrm{OA}}\frac{\mathrm{OC}}{\mathrm{OB}}\frac{\mathrm{OD}}{\mathrm{OC}}\frac{\mathrm{OA}}{\mathrm{OD}} = 1 \qquad (1)$$

in figure 1. Now

$$\begin{aligned}\tan(60-x)\tan(60+x)\tan x &= \frac{\tan 60 - \tan x}{1+\tan 60 \tan x} \\ &\quad \times \frac{\tan 60 + \tan x}{1 - \tan 60 \tan x}\tan x \\ &= \tan x\left(\frac{\tan^2 60 - \tan^2 x}{1-\tan^2 60 \tan^2 x}\right) \\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\end{aligned}$$

and

$$\begin{aligned}\tan 3x = \tan(2x+x) &= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \\ &= \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x}\tan x} \\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x},\end{aligned}$$

so

$$\tan(60-x)\tan(60+x)(\tan x)(\tan(90-3x)) = 1. \qquad (2)$$

In Junji Inaba's problem, $a = 60$, $b = 50$, $d = 70$ (say), so, from (1),

$$(\tan 50)(\tan 70)(\tan c)(\tan 60) = 1.$$

If we put $x = 10$ in (2), we obtain

$$(\tan 50)(\tan 70)(\tan 10)(\tan 60) = 1.$$

Hence $c = 10$ and the required angle is $(10 + 40)° = 50°$.

This solution uses trigonometry, of course. A solution not using trigonometry was given in Volume 28, Number 2, page 43. There are five similar problems corresponding to the other five cyclic arrangements of the four angles 50°, 70°, 10°, 60°. Readers may like to solve these without using trigonometry.

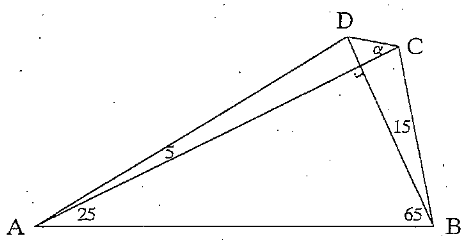


Figure 2

We can give x any value up to 30 and obtain a similar problem; if we put $x = 25$, we obtain $\alpha = 35$ in figure 2.

Yours sincerely,
A. VAHDATI
(40 Orchard Road, Street, BA16 0BT)

Dear Editor,

The Collatz problem

I write in response to the appeal by Junji Inaba, *Mathematical Spectrum* 1995/96, Volume 28, Number 2, page 33, to add some substantive references to this fascinating problem. It is called the Collatz problem, having been (first?) posed by L. Collatz as a student. R. K. Guy, in the first edition of his book *Unsolved Problems in Number Theory*, problem E16, cites verification of the tree-like structure (apart from the cycle $4, 2, 1, 4, 2, 1 \cdots$) up to 10^9 by D. H. and Emma Lehmer and J. L. Selfridge, and up to 7×10^{11} by others. The related problem using $3n - 1$ (n odd) seems likely to conclude with one of the cycles (1, 2) or (5, 14, 7, 20, 10) or $(17, 50, \cdots, 34)$. This has been tested up to at least 10^8.

David C. Kay, in *Pi Mu Epsilon Journal* 1972, Volume 5, page 338, defines a more general problem using $a_{n+1} = a_n/p$ if p divides a_n whilst $a_{n+1} = a_n q + r$ otherwise, and asks if there are any numbers (p, q, r) for which the problem can be settled.

Further to this, R. E. Crandall, in *Mathematics of Computation* 1978, Volume 32, pages 1281–1292 'On the '$3x+1$' problem', looks at the $qx+r$ problem and shows cyclic trajectories for $q=5$, 181 and 1093. He concludes with the sobering thought that it is unknown whether even a single x in a single $qx+1$ problem gives rise to an unbounded trajectory even though $x=3$ and $7x+1$ reaches 10^{2000} 'and beyond, with no apparent tendency to return'.

Yours sincerely,
MICHAEL R. MUDGE
(22 Gors Fach,
Carmarthen,
Dyfed, SA33 4AQ)

Dear Editor,

Brahmagupta

I think there is an error in the recent article on the mathematics of Brahmagupta (Volume 28, Number 3, pages 49–51). It is claimed that Brahmagupta proved that:

> If the elements of two Pythagorean triples (a, b, c) and (A, B, C) are combined to form products aC, cB, bC and cA representing the sides of a quadrilateral then that quadrilateral is cyclic.

This cannot be true because a quadrilateral is not determined only by its four sides. In fact there exist infinitely many quadrilaterals with the same sides in the same order. Only one of them is cyclic and this cyclic quadrilateral has the greatest area of all.

Yours sincerely,
HANS ENGELHAUPT
(Lärchenweg 11,
96163 Gundelsheim,
Germany)

The author replies

Contrary to the assertion in the article, Brahmagupta showed only that if the quadrilateral is generated from Pythagorean triples then its diagonals are perpendicular. This indeed falls short of always producing cyclic quadrilaterals. It is a necessary but not sufficient condition of cyclicity. Brahmagupta also observed that the area is rational but not that it is a maximum.

In the article reference is made to the quadrilateral with sides 39, 60, 52 and 25, generated by the triples (3, 4, 5) and (5, 12, 13). This quadrilateral can, indeed, have diagonals of length 63 and 56, but it need not, and in addition to his letter Hans Engelhaupt furnished a drawing of such a quadrilateral with a diagonal of length 40.

Chris Pritchard

Dear Editor,

Primes of the form $x^y + y^x$

I read Peter Castini's letter (Volume 28, Number 3, page 68), and I want to add more comments on the Smarandache general expression $x_1^{x_2} + x_2^{x_3} + \cdots + x_n^{x_1}$, where $n > 1$, $x_1, x_2, \ldots, x_n > 1$ and have greatest common divisor 1.

There are infinitely many composite numbers of this form. For example, choose the n numbers $x_1, x_2, \ldots, x_n$ so that an even number of them are odd, while the others are even, and keeping the above conditions. Then the result is even and therefore a composite number.

I am not sure if there are an infinite or finite number of primes among them. What do your readers think?

Yours sincerely,
JOSE CASTILLO
(Box 722, Vail,
AZ 85671, USA)

Reference

1. F. Smarandache, Properties of the Numbers (II), *University of Craiova Conferences* (1975).

Dear Editor,

Another look at the Euler constant

The Euler (Mascheroni) constant γ is the limit of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

(where here, and below, $\log x$ means $\log_e x$). Convergence is normally established by showing, with the help of integration, that γ_n increases with n and is bounded. However, readers of *Mathematical Spectrum* may be interested to see an approach without integration that uses no more than the exponential series. Instead of working with γ_n we consider

$$\delta_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n+1).$$

This makes no essential difference since

$$\gamma_n - \delta_n = \log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right) \to 0 \qquad (1)$$

as $n \to \infty$.

The inequality

$$\log(1+x) < x < -\log(1-x) \qquad (0 < x < 1) \qquad (2)$$

will be used. To prove it we note that (2) is equivalent to

$$1 + x < e^x < \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

for $0 < x < 1$, which is true since, for all x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Next we use induction to prove that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}<\log 3n \qquad (3)$$

for all $n \geqslant 1$. Since $\mathrm{e}<3$, $1=\log \mathrm{e}<\log 3$, i.e. (3) holds for $n=1$. But, if (3) holds for $n=k$, then

$$\begin{aligned}1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1} &< \log 3k+\frac{1}{k+1}\\ &= \log 3(k+1)+\frac{1}{k+1}\\ &\quad +\log\left(1-\frac{1}{k+1}\right)\\ &< \log 3(k+1),\end{aligned}$$

by the right-hand side of (2) with $x=1/(k+1)$. Thus (3) holds for $n=k+1$ and so for all $n \geqslant 1$.

It now follows from (3) that

$$\delta_n<\log 3n-\log(n+1)=\log\left(\frac{3n}{n+1}\right)<\log 3.$$

Also, by the left-hand side of (2) with $x=1/(n+1)$,

$$\begin{aligned}\delta_{n+1}-\delta_n &= \frac{1}{n+1}-\log(n+2)+\log(n+1)\\ &= \frac{1}{n+1}-\log\left(1+\frac{1}{n+1}\right)\\ &> 0.\end{aligned}$$

We have now shown that the sequence δ_n is bounded and increasing and therefore converges. Hence, by (1), γ_n converges to the same limit.

Yours sincerely,

J. A. SCOTT
(1 Shiptons Lane,
Great Somerford,
Chippenham,
Wilts SN15 5EJ)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

29.1 Prove that

$$\sin A \sin B(2\cos C-1)+\sin B \sin C(2\cos A-1) + \sin C \sin A(2\cos B-1) \geqslant 0,$$

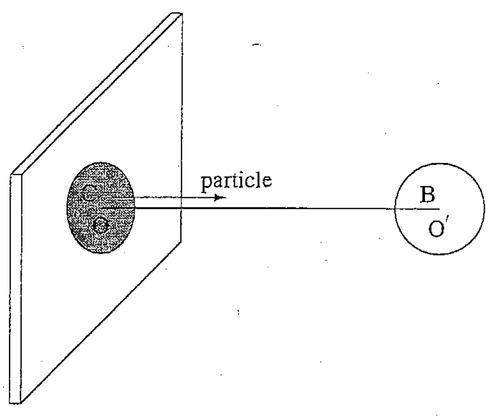
where A, B and C are the angles of a triangle.

(Submitted by Mansur Boase, student at St Paul's School, London)

29.2 There are n sheep in a field, numbered 1 to n and some integer $m>1$ is given such that $m^2 \leqslant n$. It is required to separate the sheep into two groups such that (1) no sheep has number m times the number of a sheep in the same group, and (2) no sheep has number the sum of the numbers of two sheep in its group. For which values of m, n is this possible?

(Submitted by Junji Inaba, student at William Hulme's Grammar School, Manchester)

29.3 A very small spherical particle is fired horizontally in the direction OO′ from the plane circular firing area C of radius 1 — see the diagram. The line OO′ is at right angles to C and O is the centre of C. The particle is fired at a fixed sphere B, with centre O′, also of radius 1, and rebounds from B in a perfectly elastic manner. The particle is equally likely to be fired from anywhere in C. Show that all directions of rebound from B are equally likely. (This was proved by James Clerk Maxwell in his paper 'Illustration of the Dynamical Theory of Gases', 1859).



(Submitted by D. Forfar — see his article 'What became of the Senior Wranglers?' in this issue.)

29.4 Let a, b, c be the lengths of the sides of a non-degenerate triangle. Prove that

$$\frac{3}{2} \leqslant \frac{\tanh a}{\tanh b + \tanh c} + \frac{\tanh b}{\tanh c + \tanh a} + \frac{\tanh c}{\tanh a + \tanh b} < 2.$$

(Submitted by Toby Gee, The John of Gaunt School, Trowbridge)

Solutions to Problems in Volume 28 Number 2

28.5 Prove that there are no natural numbers n which make $3^n + n^3$ a cube.

Solution by Junji Inaba

We prove a more general result, namely that there are no natural numbers k, m, n such that $3^k + n^3 = m^3$. We prove this by contradiction using the 'method of infinite descent'. Suppose that such k, m, n do exist. Then

$$3^k = (m-n)(m^2 + mn + n^2).$$

Since $m^2 + mn + n^2 > 1$, this means that

$$m^2 + mn + n^2 \equiv 0 \pmod 3.$$

But

$$m^2 + mn + n^2 = (m-n)^2 + 3mn,$$

so $(m-n)^2 \equiv 0 \pmod 3$ so $m \equiv n \pmod 3$. Write $m - n = 3r$, where r is a natural number. Then

$$3^k = 3r(9r^2 + 3mn)$$

so

$$3^{k-2} = r(3r^2 + mn).$$

Since $3r^2 + mn > 1$, it follows that

$$3r^2 + mn \equiv 0 \pmod 3$$

so

$$mn \equiv 0 \pmod 3$$

so either $m \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$. Since $m \equiv n \pmod 3$, this means that both m and n are divisible by 3, say $m = 3m'$ and $n = 3n'$ for some natural numbers m' and n'. If we now substitute back in the original equation, we obtain

$$3^{k-3} + n'^3 = m'^3,$$

and a similar argument will give that m', n' are divisible by 3. And so on. Thus m, n are infinitely divisible by 3, which is impossible.

Also solved by Can A. Minh (University of California, Berkeley) and Lan Nguyen (University of California, Berkeley).

28.6 A triangle has angles α, β and γ which are whole numbers of degrees, and $\alpha^2 + \beta^2 = \gamma^2$. Find all possibilities for α, β and γ.

Solution by Can A. Minh

From $\alpha^2 + \beta^2 = \gamma^2$ and $\alpha + \beta + \gamma = 180$, we have

$$\begin{aligned}\alpha^2 + \beta^2 &= (180 - \alpha - \beta)^2 \\ &= 180^2 + \alpha^2 + \beta^2 - 360\alpha - 360\beta + 2\alpha\beta,\end{aligned}$$

which simplifies to

$$\alpha = 180 - \frac{90 \times 180}{180 - \beta}.$$

Since $0 < \alpha < 180$, we require that $0 < \beta < 90$. Also, $180 - \beta$ must divide $90 \times 180 = 2^3 \times 3^4 \times 5^2$ and is greater than 90, so the possibilities for $180 - \beta$ are 2×3^4, $2^2 \times 3^3$, $3^3 \times 5$, $2 \times 3 \times 5^2$, $2^3 \times 3 \times 5$, $2^2 \times 5^2$. This gives $\beta = 18$, 72, 45, 30, 60, 80. Thus the possible triplets (α, β, γ) are $(18, 80, 82)$, $(30, 72, 78)$, $(45, 60, 75)$ and those where α and β are interchanged.

Also solved by Junji Inaba, Borr-Yann Chen (University of California, Irvine) and Noah Rosenberg (Rice University, Houston).

28.7 Evaluate

$$\int \sqrt{\sec^2 x + A}\,\mathrm{d}x,$$

where $A \geqslant 0$ is a constant.

Solution by Konstantin Ardakov (Dr Challoner's Grammar School, Amersham)

Denote the integral by I and make the substitution $\tan x = \sqrt{A+1}\sinh\theta$. Then

$$\sec^2 x\,\mathrm{d}x = \sqrt{A+1}\cosh\theta\,\mathrm{d}\theta$$

so

$$\mathrm{d}x = \frac{\sqrt{A+1}\cosh\theta}{1 + (A+1)\sinh^2\theta}\,\mathrm{d}\theta.$$

Thus

$$\begin{aligned}I &= \sqrt{A+1}\int\sqrt{1 + A + (A+1)\sinh^2\theta} \\ &\qquad \times \frac{\cosh\theta\,\mathrm{d}\theta}{1 + (A+1)\sinh^2\theta} \\ &= (A+1)\int\frac{\cosh^2\theta\,\mathrm{d}\theta}{(A+1)\cosh^2\theta - A} \\ &= \theta + A\int\frac{\mathrm{d}\theta}{(A+1)\cosh^2\theta - A}.\end{aligned}$$

In this integral, substitute $\tanh\theta = t$. Then $\mathrm{d}t = \mathrm{sech}^2\theta\,\mathrm{d}\theta$ so $\mathrm{d}\theta = \mathrm{d}t/(1-t^2)$ and

$$\begin{aligned}I &= \theta + A\int\frac{\mathrm{d}t}{(A+1) - A(1-t^2)} \\ &= \theta + A\int\frac{\mathrm{d}t}{1 + At^2} \\ &= \theta + \sqrt{A}\tan^{-1}(t\sqrt{A}) \\ &= \theta + \sqrt{A}\tan^{-1}(\sqrt{A}\tanh\theta).\end{aligned}$$

Now, $\sinh\theta = \tan x/\sqrt{A+1}$ so

$$\cosh^2\theta = 1 + \frac{\tan^2 x}{A+1} = \frac{A+1+\tan^2 x}{A+1}$$

and

$$\operatorname{sech}^2\theta = \frac{A+1}{A+1+\tan^2 x}.$$

Hence

$$\tanh\theta = \frac{\tan x}{\sqrt{A+1+\tan^2 x}} = \frac{\tan x}{\sqrt{A+\sec^2 x}}.$$

Hence

$$I = \sinh^{-1}\left(\frac{\tan x}{\sqrt{A+1}}\right) + \sqrt{A}\tan^{-1}\left(\frac{\sqrt{A}\tan x}{\sqrt{A+\sec^2 x}}\right) + \text{constant}.$$

28.8 Prove that

$$\prod_{n=1}^{p-1}(1+pn^{-1}) \equiv 1 \pmod{p}.$$

Solution by Can A. Minh

$$\prod_{n=1}^{p-1}(1+pn^{-1}) = \frac{(p+1)(p+2)\ldots(2p-1)}{1\times 2\times\cdots\times(p-1)} = \binom{2p-1}{p-1},$$

the binomial coefficient, and so is an integer. Now

$$(p-1)!\binom{2p-1}{p-1} = (p+1)(p+2)\ldots(2p-1)$$
$$\equiv 1\times 2\times\ldots(p-1) \pmod{p}.$$

Since $\gcd\big(p, (p-1)!\big) = 1$, we deduce that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p}.$$

Also solved by Junji Inaba and Mansur Boase.

Reviews

Vector Calculus. 4th edn. By J. E. MARSDEN AND A. J. TROMBA. W. H. Freeman, New York, 1996. Pp 627. Hardback £29.95. (ISBN 0716724324).

The fourth edition of this standard undergraduate text on the subject. The layout is attractive, there are many graded exercises, applications of the theory are emphasized and there are historical notes. What a pity that George Green, unsung hero of the subject, gets little mention apart from his name being attached to his theorem and functions. You will need to read the article in *Mathematical Spectrum* Volume 20, Number 2, pages 45–52 to find out about the man behind the mathematics.

University of Sheffield — DAVID SHARPE

Statistical Tests: An Introduction with MINITAB Commentary. By G. P. BEAUMONT AND J. D. KNOWLES. Prentice Hall, London, 1996. Pp. ix + 285. Paperback £16.95. (ISBN 0-13-842576-0).

This book is intended as a second undergraduate course in statistics which concentrates on statistical tests, but omits regression and analysis of variance. Numerical aspects are dealt with through the statistical package MINITAB, though the material can be followed independently of it. Sadly, the first author Geoffrey Beaumont died while the book was in production.

The book begins with four introductory chapters on probability and distributions, data and its exploration, point and interval estimation, and the testing of hypotheses, in particular those related to the normal distribution. They are followed by three chapters concerned with tests related to the binomial distribution, the hypergeometric distribution and Spearman's and Kendall's rank correlation coefficients.

The authors conclude with a chapter on distribution-free tests for many samples, and another on the theory of hypothesis testing. There are appendices on sampling distributions, sampling from finite populations, and tables of various distribution functions. A three-page index at the end of the book makes it easy to trace specific topics.

The book is written clearly, and contains a wealth of examples, problems and solutions which students should find very helpful. The χ^2 distribution is mentioned on p. 15, but surprisingly not the χ^2 test. Undergraduates and sixth-formers interested in understanding statistical tests and practising their use will find this book a helpful supplement to their usual texts.

Australian National University, Canberra, Australia — JOE GANI

Oxford Concise Dictionary of Mathematics. By CHRISTOPHER CLAPHAM. Oxford University Press, Oxford, 1996. Pp. 313. Paperback £5.99 (ISBN 0-19-280041-8).

This is a new, expanded edition covering pure and applied mathematics and statistics, including short biographies of leading mathematicians, tables of areas, volumes, derivatives, integrals, trigonometric formulae, standard series and standard symbols. It is surprising to see how much information is given on each item. Yet the style is relaxed, with no sense of compression. An ideal present to grace the bookshelf of a student mathematician.

University of Sheffield — DAVID SHARPE

Mathematical Spectrum

1996/7 Volume 29 Number 1

ISSN 0025-5653

Published by the Applied Probability Trust
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK