THE ACADEMY CORNER

No. 35

Bruce Shawyer

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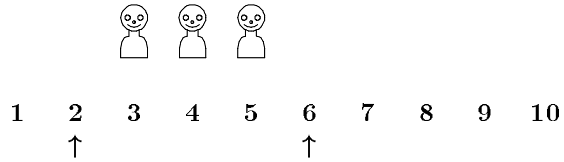
We present the last of the readers' solutions to the questions of the 1999 Atlantic Provinces Council on the Sciences Annual Mathematics Competition, which was held last year at Memorial University, St. John's, Newfoundland [1999 : 452].

2. The Memorial University Philosopher's Jockey Club has just received the bronze busts of the ten members of their hall of fame. Each will be placed in its designated place on a single shelf, above the gold plaque bearing the name of the member. The ten busts are drawn at random from the crate. What is the probability that at no time will there be an empty space between two busts already placed on the shelf?

Solution by Sarah Mathews, student, Memorial University of Newfoundland, St. John's, Newfoundland.

Let us index the places on the shelf from left to right with the numbers 1 through 10.

As each bust is pulled from the crate and put in its place on the shelf, the only way that no gap will be left between busts on the shelf is if the new bust belongs in the place to the immediate left, or the immediate right of those already on the shelf as illustrated below.



The new bust may go here or here.

Let us call the index of the place belonging to the first object pulled from the crate r. This leaves $r-1$ busts to be placed to the left of the first bust, labeled L_{r-1}, L_{r-2}, ..., L_1, and $10-r$ busts to be placed to the right of the first bust, labelled R_{r+1}, R_{r+2}, ..., R_{10}.

To avoid ever leaving gaps between busts already placed on the shelf, the 'left' busts must be pulled from the crate in order: L_{r-1}, L_{r-2}, ..., L_1, and the 'right' busts must be pulled from the crate in order: R_{r+1}, R_{r+2}, ..., R_{10}. Note that this has no effect on whether the next bust pulled from the crate must be a ‘left’ object or a ‘right’ object; it means only that if L_x is pulled from the crate, then the next ‘left’ bust must be L_{x-1} and if R_y is pulled from the crate, then the next ‘right’ bust must be R_{y+1}.

Since one of the 10 busts has been placed, pulling the remaining busts from the crate and placing them on the shelf will take 9 ‘moves’, where each move consists of placing either the next ‘left’ bust or the ‘right’ bust. Hence, we must find the number of permutations of the $r - 1$ ‘left’ and the $10 - r$ ‘right’ moves.

If, for example, the first bust placed was the bust at index 4, then there are $r - 1 = 4 - 1 = 3$ ‘left’ moves and $10 - r = 10 - 4 = 6$ ‘right’ moves, so some possible permutations of ‘left’ and ‘right’ moves are:

$$\text{P1:}\quad \begin{array}{ccccccccc} L_3 & R_5 & L_2 & R_6 & L_1 & R_7 & R_8 & R_9 & R_{10} \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$\text{P2:}\quad \begin{array}{ccccccccc} R_5 & R_6 & R_7 & L_3 & L_2 & R_8 & L_1 & R_9 & R_{10} \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$\text{P3:}\quad \begin{array}{ccccccccc} R_5 & L_3 & R_6 & R_7 & R_8 & L_2 & R_9 & L_1 & R_{10} \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

Recall that the order of all ‘left’ moves is fixed (L_{r-1}, L_{r-2}, ..., L_1) and the order of all ‘right’ moves is fixed (R_{r+1}, R_{r+2}, ..., R_{10}). Hence, once it is known that a move was a ‘left’ move, the exact ‘left’ bust placed in that move is determined as it must be the next ‘left’ bust in the above sequence. The same is true for all ‘right’ moves. Therefore, the indices of the ‘left’ and ‘right’ busts are not necessary. For example, the above permutations can be simplified to the following:

$$\text{P1:}\quad \begin{array}{ccccccccc} L & R & L & R & L & R & R & R & R \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$\text{P2:}\quad \begin{array}{ccccccccc} R & R & R & L & L & R & L & R & R \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$\text{P3:}\quad \begin{array}{ccccccccc} R & L & R & R & R & L & R & L & R \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

Thus, we are arranging 9 objects, $r - 1$ of one kind (L’s) and $10 - r$ of another (R’s). Hence, for each of the 10 choices for r, there are

$P(9; r-1, 10-r)$ ways to place the busts on the shelf without ever leaving a space between busts already on the shelf.

Alternatively, we could choose which $r-1$ of the 9 moves will be the 'left' moves, leaving the remaining $10-r$ moves to be the 'right' moves. For each of the 10 choices for r, this would give $\binom{9}{r-1}$ ways to place the busts on the shelf without ever leaving a space between busts already on the shelf. But

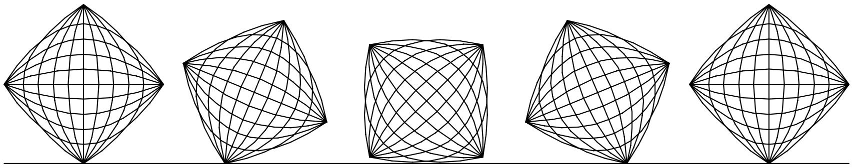
$$\begin{aligned} P(9; r-1, 10-r) &= \frac{9!}{(r-1)!(10-r)!} \\ &= \frac{9!}{(r-1)!(9-(r-1))!} \\ &= \binom{9}{r-1} \end{aligned}$$

Hence, the two approaches are equivalent.

It is clear that there are 10! ways in total to place the busts on the shelf (without regard as to whether or not gaps are left). Therefore, the probability that at no time will there be any space between busts already placed is:

$$\begin{aligned} \frac{\sum_{r=1}^{10} P(9; r-1, 10-r)}{10!} &= \frac{\sum_{r=1}^{10} \binom{9}{r-1}}{10!} \\ &= \frac{\sum_{r=0}^{9} \binom{9}{r}}{10!} \\ &= \frac{2^9}{10!} = \frac{2}{14175} \end{aligned}$$

A general solution, for n busts to be placed in n specific places on a shelf can be obtained by replacing 10 by n and following the same reasoning as was given for the case with 10 busts. Hence, a general solution for n busts is: $\frac{2^{n-1}}{n!}$.



THE OLYMPIAD CORNER

No. 208

R.E. Woodrow

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This number we start you off with the problems to the three selection tests of the 47th Latvian Olympiad, written 21 March, 3 May and 4 May, 1997. My thanks go to Richard Nowakowski, Canadian Team Leader at the IMO in Argentina, who collected them for the *Corner*.

47th LATVIAN MATHEMATICAL OLYMPIAD 1997

1st Selection Test

March 21, 1997 — Time: 4.5 hours

1. It is given that a and b are different natural numbers. Prove that there are infinitely many natural n's such that $a+n$ and $b+n$ are coprime.

2. Does there exist a function $f(x)$ which is defined for all reals and for which the identities

$$f(f(x)) = x \quad \text{and} \quad f(f(x)+1) = 1-x$$

hold?

3. The line t has no common points with a circle w centred at O. Point E lies on t; $OE \perp t$. Point M is another point on t; MA and MB are tangents to w, A and B being the points of tangency; AB intersects OE at X.

Prove that X does not depend on M.

4. For positive a and b and natural n prove

$$\frac{1}{a+b}+\frac{1}{a+2b}+\cdots+\frac{1}{a+nb} < \frac{n}{\sqrt{a(a+nb)}}\,.$$

5. There are 9 originally unacquainted persons. Find the least n with the property: no matter which n pairs of these persons we choose and no matter in which of these pairs the persons will become mutual friends and in which they will become enemies, it will always be possible to find three pairwise friends or three pairwise enemies.

2^{nd} Selection Test

May 3, 1997 — Time: 4.5 hours

1. There are $4n$ points on a circle; they are assigned numbers from 1 to $4n$ in an arbitrary manner. Prove that these points can be pairwise joined with $2n$ chords without common points so that the difference between the numbers of endpoints of any chord does not exceed $3n - 1$.

2. An equilateral triangle of side 1 is dissected into n triangles. Prove that the sum of squares of all sides of all triangles is at least 3.

3. Let us consider the sequence $\{x_n\}$ with $x_0 = a$ and $x_{n+1} = 3 \cdot (x_n + |x_n - 1| - |x_n + 1|)$. For how many real numbers a are $x_0, x_1, \ldots, x_{1996}$ all different but $x_{1997} = x_0$?

3^{rd} Selection Test

May 4, 1997 — Time: 4.5 hours

1. Let m and n be natural numbers such that $m^2 + n^2$ divides into $m \cdot n$. Prove that $m = n$.
[Ed: we know something is odd in this question. Can you clarify it for us?]

2. Let $ABCD$ be a parallelogram. The bisector of A cuts BC at M and cuts the extension of CD at N. The circumcentre of MCN is O. Prove that B, O, C, D are concyclic.

3. There are 100 coins with different masses and equal appearances. We can use a pan-balance without counterfeits; only one coin at the same time can rest on each pan. Find the least number of weighings which guarantees finding both the heaviest and the lightest coins.

Next we give the problems of the two days of the 1997 Mathematical Olympiad in Bosnia and Hercegovina. My thanks go to Richard Nowakowski, Canadian Team Leader at the International Mathematical Olympiad in Argentina for sending them for our use.

MATHEMATICAL OLYMPIAD IN BOSNIA AND HERCEGOVINA 1997

First Day

1. Solve the system of equations in $\mathbb{R}^3$:

$$\begin{aligned} 8(x^3 + y^3 + z^3) &= 73\,, \\ 2(x^2 + y^2 + z^2) &= 3(xy + yz + zx)\,, \\ xyz &= 1\,. \end{aligned}$$

2. In an isosceles triangle ABC with the base $\overline{AB}$, point M lies on the side $\overline{BC}$. Let O be the centre of its circumscribed circle, and S be the centre of the inscribed circle in the triangle ABC. Prove that:

$$SM \| AC \quad \Longleftrightarrow \quad OM \perp BS\,.$$

3. Let $f : A \to \mathbb{R}$, $(A \subseteq \mathbb{R})$ be a function with the following characteristic:

$$f(x+y) \;=\; f(x)\cdot f(y) - f(xy) + 1, \quad (\forall x, y \in A)\,.$$

(a) If $f : A \to \mathbb{R}$, $(\mathbb{N} \subset A \subseteq \mathbb{R})$ is such a function, prove that the following is true:

$$f(n) \;=\; \begin{cases} \frac{c^{n+1}-1}{c-1}\,, & \forall n \in \mathbb{N}\,,\ c \neq 1\,, \\ n+1\,, & \forall n \in \mathbb{N}\,,\ c = 1\,, \end{cases}$$

$(c = f(1) - 1)$.

(b) Solve the given functional equation for $A = \mathbb{N}$.

(c) If $A = \mathbb{Q}$, find all the functions f which satisfy the given equation and the condition $f(1997) \neq f(1998)$.

Second Day

1. (a) Let A_1, B_1, C_1 be the points of contact of the circle inscribed in the triangle ABC and the sides BC, CA, AB, respectively. Let B_1C_1, A_1C_1, B_1A_1 be the arcs which do not contain points A_1, B_1, C_1, respectively. Let I_1, I_2, I_3 be their respective arc lengths. Prove the following inequality:

$$\frac{a}{I_1} + \frac{b}{I_2} + \frac{c}{I_3} \;\geq\; 9\frac{\sqrt{3}}{\pi}$$

(where a, b, c denote the lengths of sides of the given triangle).

(b) Let $ABCD$ be a tetrahedron with:

$$\begin{array}{ccccc} AB & = & CD & = & a \\ BC & = & AD & = & b \\ AC & = & BD & = & c\,. \end{array}$$

Express the height of the tetrahedron in terms of the lengths a, b and c.

2. (a) Prove that for every positive integer n there exists a set M_n of positive integers which has n elements and possesses the property:

(i) the arithmetic mean of elements of an arbitrary non-empty subset of M_n is an integer

(ii) the geometric mean of elements of an arbitrary non-empty subset of M_n is an integer

(iii) both arithmetic and geometric mean of elements of an arbitrary non-empty subset of M_n are integral.

(b) Is there an infinite set M of natural numbers which has the property that the arithmetic mean of an arbitrary non-empty subset of M is an integer?

3. Let k, m, n be integers such that $1 < n \leq m - 1 \leq k$. Determine the maximum size of subsets S of the set $\{1, 2, 3, \ldots, k\}$ such that no sum of n distinct elements of S

(a) is equal to m,

(b) is bigger than m.

Next we give the final round of the 5th Japan Mathematical Olympiad 1995. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for them.

5th JAPAN MATHEMATICAL OLYMPIAD 1995

Final Round

February 11, 1995 — Time: 4.5 hours

1. Let n and r be positive integers such that $n \geq 2$ and $r \not\equiv 0 \pmod{n}$, and let g be the greatest common divisor of n and r. Prove that

$$\sum_{i=1}^{n-1} \left\langle \frac{ri}{n} \right\rangle = \frac{1}{2}(n-g),$$

where $\langle x \rangle = x - \lfloor x \rfloor$ is the fractional part of x.

2. Let $f(x)$ be a rational function (that is, a quotient of two polynomials) which is not a constant, and let a be a real number. Find all pairs of a and $f(x)$ which satisfy $f(x)^2 - a = f(x^2)$.

3. Let $ABCDE$ be a convex pentagon. Let S, R be the points of intersection of BE with AC, AD, respectively. Let T, P be the points of intersection of BD with CA, CE, respectively. Let Q be the point of intersection of CE and AD. Assume that the areas of triangles ASR, BTS, CPT, DQP, ERQ are all equal to 1.

(a) Determine the area of the pentagon $PQRST$.

(b) Determine the area of the pentagon $ABCDE$.

4. Define a sequence $\{a_i\}_{i \geq 1}$ by $a_{2n} = a_n$ and $a_{2n+1} = (-1)^n$. A point P moves on the coordinate plane as follows:

(a) Let P_0 be the origin. First P moves from P_0 to $(1, 0)$. Denote this point by P_1.

(b) After P has moved to P_i, it turns 90° to the left and moves forward 1 unit if $a_i = 1$, and turns 90° to the right and moves forward 1 unit if $a_i = -1$. Denote this point by P_{i+1}.

Can the point retrace an edge? That is, can $P_u P_{u+1} = P_w P_{w+1}$ for some u and w?

5. Let k and n be integers such that $1 \le k \le n$, and assume that a_1, $a_2, \ldots, a_k$ satisfy

$$\begin{aligned} a_1 + a_2 + \cdots + a_k &= n\,, \\ a_1^2 + a_2^2 + \cdots + a_k^2 &= n\,, \\ &\vdots \\ a_1^k + a_2^k \cdots + a_k^k &= n\,. \end{aligned}$$

Prove that

$$(x+a_1)(x+a_2)\cdots(x+a_k) = x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} + \cdots + \binom{n}{k}\,.$$

We next turn to solutions to problems of the Vietnamese Mathematical Olympiad 3/1996, Category A given in [1999 : 6].

1. Solve the system of equations:

$$\begin{cases} \sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2\,, \\ \sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}\,. \end{cases}$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution by Tsaoussoglou.

Note $x \ne 0$, $y \ne 0$ and $x, y > 0$. We have

$$1 + \frac{1}{x+y} = \frac{2}{\sqrt{3x}}\,, \qquad 1 - \frac{1}{x+y} = \frac{4\sqrt{2}}{\sqrt{7y}}\,.$$

Adding and subtracting, we get the equations

$$2 = \frac{2}{\sqrt{3x}} + \frac{4\sqrt{2}}{\sqrt{7y}}\,, \qquad \frac{2}{x+y} = \frac{2}{\sqrt{3x}} - \frac{4\sqrt{2}}{\sqrt{7y}}\,.$$

Dividing by 2 gives

$$1 = \frac{1}{\sqrt{3x}} + \frac{2\sqrt{2}}{\sqrt{7y}}\,, \qquad \frac{1}{x+y} = \frac{1}{\sqrt{3x}} - \frac{2\sqrt{2}}{\sqrt{7y}}\,.$$

Multiplying, we get

$$\frac{1}{x+y} = \frac{1}{3x} - \frac{8}{7y}.$$

Thus,

$$7y^2 - 38xy - 24x^2 = 0, \qquad (y-6x)(7y+4x) = 0,$$

giving $y = 6x$ (since $x, y > 0$).

Substituting, we get $1 = \frac{1}{\sqrt{3x}} + \frac{2\sqrt{2}}{\sqrt{6 \cdot 7x}}$, so that

$$\sqrt{3x} = 1 + \frac{2}{\sqrt{7}} = \frac{2+\sqrt{7}}{\sqrt{7}},$$

$$3x = \frac{4+7+4\sqrt{7}}{7}, \quad x = \frac{(2+\sqrt{7})^2}{21}, \quad y = \frac{2(2+\sqrt{7})^2}{7}.$$

2. Let $Sxyz$ be a trihedron (a figure determined by the intersection of three planes). A plane (P), not passing through S, cuts the rays Sx, Sy, Sz, respectively at A, B, C. In the plane (P), construct three triangles DAB, EBC, FCA such that each has no interior point of triangle ABC and $\triangle DAB = \triangle SAB$, $\triangle EBC = \triangle SBC$, $\triangle FCA = \triangle SCA$. Consider the sphere (T) satisfying simultaneously two conditions:

(i) (T) touches the planes (SAB), (SBC), (SCA), (ABC);

(ii) (T) is inside the trihedron $Sxyz$ and is outside the tetrahedron $SABC$.

Prove that the circumcentre of triangle DEF is the point where (T) touches (P).

Solution by Michel Bataille, Rouen, France.

Let Ω be the centre of (T) and O, O' be the points where (T) touches the planes (ABC) and (SBC), respectively. Since $\Omega O \perp (ABC)$ and $\Omega O' \perp (SBC)$, BC is orthogonal to ΩO and $\Omega O'$ so that BC is orthogonal to the plane $(\Omega O O')$. Hence, the (orthogonal) projection ω of Ω onto BC is the projection of O and of O' onto BC as well. Therefore,

$$O\omega \perp BC, \qquad O'\omega \perp BC. \tag{1}$$

Moreover, since $\Omega O = \Omega O'$ (equal to the radius R of (T)), we also have

$$O\omega = O'\omega, \tag{2}$$

by Pythagoras' Theorem. From (1) and (2), we easily deduce that the rotation with axis BC which transforms S into E, also transforms O' into O. Thus, $SO' = OE$.

Similarly, if (T) touches (SAB) at O'' and (SCA) at O''', we have $SO'' = OD$ and $SO''' = OF$. Since $SO' = SO'' = SO'''$ $(= \sqrt{S\Omega^2 - R^2})$, we obtain $OE = OD = OF$, so that O is actually the circumcentre of $\triangle DEF$.

3. Given two positive integers k and n, $0 < k \leq n$, find the number of k–arrangements $(a_1, a_2, \dots, a_k)$ of the first n positive integers $1, 2, \dots, n$ such that each k–arrangement $(a_1, a_2, \dots, a_k)$ satisfies at least one of the two conditions:

(i) there exist $s, t \in \{1, 2, \dots, k\}$ such that $s < t$ and $a_s > a_t$;

(ii) there exists $s \in \{1, 2, \dots, k\}$ such that $(a_s - s)$ is not divisible by 2.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's solution.

Let A be the set of k–arrangements $(a_1, a_2, \dots, a_k)$ satisfying (i) or (ii). Let $B = \{k\text{–arrangements } (a_1, a_2, \dots, a_k) \text{ of } (1, 2, \dots, n)\}$, and set $C = \{(a_1, a_2, \dots, a_k) \in B \mid a_i < a_{i+1},\ i = 1, 2, \dots, k-1$ and $a_i \equiv i \pmod 2$ for all $i = 1, 2, 3, \dots, k\}$.

Then we have

$$A = B \backslash C \qquad \text{and} \qquad |A| = |B| - |C| = \frac{n!}{(n-k)!} - |C|\,.$$

Set $D = \{(b_1, b_2, \dots, b_k) \mid b_i < b_{i+1}$ for all $i = 1, 2, \dots, k-1$, $b_i \in \{1, 2, \dots, n+k\}$ and $2|b_i$ for all $i = 1, \dots, k\}$, and consider the function

$$f : C \to D$$

$$\begin{aligned} (a_1, \dots, a_i, \dots, a_k) &\mapsto (b_1, \dots, b_i, \dots, b_k) \\ &= (a_1 + 1, \dots, a_i + i, \dots, a_k + k)\,. \end{aligned}$$

Note that f is a bijection, and then we have $|C| = |D| = \binom{\lfloor \frac{n+k}{2} \rfloor}{k}$. Hence,

$$|A| = \frac{n!}{(n-k)!} - \binom{\lfloor \frac{n+k}{2} \rfloor}{k}\,.$$

4. Determine all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying:

$$f(n) + f(n+1) = f(n+2)f(n+3) - 1996$$

for every $n \in \mathbb{N}^*$ ($\mathbb{N}^*$ is the set of positive integers).

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's answer.

From the equation we obtain

$$\begin{aligned} f(n+1) + f(n+2) &= f(n+3)f(n+4) - 1996\,, && (1) \\ f(n) + f(n+1) &= f(n+2)f(n+3) - 1996\,. && (2) \end{aligned}$$

Now, (1) minus (2) yields

$$f(n+2) - f(n) = f(n+3)(f(n+4) - f(n+2))\,.$$

Hence, $\forall n \in \mathbb{N}^*$,

$$f(3) - f(1) = f(4)f(6)\cdots f(2n+2)\left(f(2n+3) - f(2n+1)\right)\,, \tag{3}$$

$$f(4) - f(2) = f(5)f(7)\cdots f(2n+3)\left(f(2n+4) - f(2n+2)\right)\,. \tag{4}$$

From (3), and if $f(1) > f(3)$, we obtain an infinite decreasing sequence $f(1)$, $f(3)$, $\ldots$ of positive integers, a contradiction. Hence, $f(1) \leq f(3)$.

Case 1: $f(1) = f(3)$.

By (3) we have, $\forall n \in \mathbb{N}^*$,

$$f(2n+3) - f(2n+1) = f(3) - f(1) = 0\,,$$

and hence,

$$f(2n-1) = f(1)\,. \tag{5}$$

From (4) we get, $\forall n \in \mathbb{N}^*$,

$$f(4) - f(2) = (f(1))^n\left(f(2n+4) - f(2n+2)\right)\,. \tag{6}$$

If $f(1) = 1$, by (6), (1) and (5) we obtain $f(4) - f(2) = 1997$ and

$$f(n) = \begin{cases} 1 & \text{if } n \text{ odd,} \\ a + (\frac{n}{2} - 1) \cdot 1997 & \text{if } n \text{ even,} \end{cases}$$

where $a \in \mathbb{N}^*$.

If $f(1) > 1$, then by (6) we have, $\forall n \in \mathbb{N}^*$,

$$f(4) - f(2) = f(2n-4) - f(2n+2) = 0\,,$$

and then

$$f(2n) = f(2)\,. \tag{7}$$

Using (2), (5) and (7), we have $\{f(1), f(2)\} = \{2, 1998\}$, and hence,

$$f(n) = \begin{cases} 2 & \text{if } n \text{ odd,} \\ 1998 & \text{if } n \text{ even;} \end{cases} \quad \text{or} \quad f(n) = \begin{cases} 1998 & \text{if } n \text{ odd,} \\ 2 & \text{if } n \text{ even.} \end{cases}$$

Case 2: $f(3) > f(1)$.

By (3) we have $f(2n-1) < f(2n+1)$ for all $n \in \mathbb{N}^*$. Now, by (4) we have

$$f(4) - f(2) = f(2n+4) - f(2n+2) = 0\,, \quad \forall n \in \mathbb{N}^*\,.$$

Then

$$f(2n) = f(2) \quad \forall n \in \mathbb{N}^*\,, \tag{8}$$

and

$$f(3)-f(1) \;=\; (f(2))^n(f(2n+3)-f(2n+1)) \quad \forall n \in \mathbb{N}^*\,.$$

Because $f(2n+3)-f(2n+1)>0$, we see that $f(2)>1$ is impossible. Thus, $f(2)=1$ and $f(3)-f(1)=f(2n+3)-f(2n+1)$ for all positive n. Now, from (2), we obtain that $f(3)-f(1)=1997$, and f is given by

$$f(n) \;=\; \begin{cases} 1 & \text{if } n \text{ even}, \\ \left(a+\frac{n-1}{2}\right)1997 & \text{if } n \text{ odd}, \end{cases} \quad , \quad \text{where } a \in \mathbb{N}^*.$$

5. Consider the triangle ABC, the measure of side BC of which is 1 and the measure of angle BAC of which is a given number α $\left(\alpha > \frac{\pi}{3}\right)$. For triangle ABC, find the distance from the incentre to the centre of gravity of ABC which attains the least value. Calculate this least value in terms of α.

Let $f(\alpha)$ be the least value. When α varies in the interval $\left(\frac{\pi}{3},\pi\right)$, at which value of α does the function $f(\alpha)$ attain its greatest value?

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give Aassila's exposition.

The distance from the incentre to the centroid is always at least the corresponding distance for ABC isosceles. The incentre makes an isosceles triangle of vertex angle $\frac{\pi+\alpha}{2}$, so that its altitude is $\frac{1}{2}\cot(\frac{\alpha}{4}+\frac{\pi}{4})$. Meanwhile, the distance of the centroid to BC is $\frac{1}{6}\cot\frac{\alpha}{2}$. Hence, the desired distance is

$$f(\alpha) \;=\; \frac{1}{2}\cot\left(\frac{\alpha}{4}+\frac{\pi}{4}\right)-\frac{1}{6}\cot\frac{\alpha}{2}\,.$$

A simple computation gives

$$f'(\alpha) \;=\; \frac{1}{4}\frac{1}{\left(\sin\frac{\alpha}{2}+1\right)\sin^2\frac{\alpha}{2}}\left(\sin\frac{\alpha}{2}-\frac{1-\sqrt{13}}{6}\right)\left(\frac{1+\sqrt{13}}{6}-\sin\frac{\alpha}{2}\right)\,.$$

When α varies in the interval $(\frac{\pi}{3},\pi)$, the greatest value of $f(\alpha)$ is attained for $\alpha = 2\arcsin\left(\frac{1+\sqrt{13}}{6}\right)$.

6. We are given four non-negative real numbers a, b, c, d satisfying the condition:

$$2(ab+ac+ad+bc+bd+cd)+abc+abd+acd+bcd \;=\; 16\,.$$

Prove that:

$$a+b+c+d \;\geq\; \frac{2}{3}(ab+ac+ad+bc+bd+cd)\,.$$

When does equality occur?

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France;. We give Bornsztein's generalization of this problem and number B6 of the same contest. See page 336.

I will prove the following results: Let n be a positive integer, $n \geq 3$. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that:

$$(n-2)\sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k \;=\; \frac{2}{3}n(n-1)(n-2)\,.$$

Then

$$\sum_{i=1}^{n} x_i \;\geq\; \frac{2}{n-1}\sum_{i<j} x_i x_j\,. \tag{1}$$

And equality occurs if and only if:

(i) $n = 3$ and (x_1, x_2, x_3) is a permutation of $(0, 2, 2)$, or

(ii) $n \geq 3$ and $x_1 = x_2 = \cdots = x_n = 1$.

(i) First, we prove the statement for $n = 3$. Let x, y, z be non-negative real numbers such that

$$xy + yz + zx + xyz \;=\; 4\,. \tag{2}$$

With no loss of generality we can suppose that $x \leq y \leq z$. From (2), we deduce that $3x^2 + x^3 \leq 4$, and so, $x \leq 1$. In the same way $3z^2 + z^3 \geq 4$, and so, $z \geq 1$. Moreover

$$y \;=\; \frac{4 - xz}{x + z + xz}\,.$$

Thus, (1) is equivalent to:

$$\begin{aligned} f(x,z) \;&=\; x(x+z+xz) + 4 - xz + z(x+z+xz) - x(4-xz) \\ &\qquad -z(4-xz) - zx(x+z+xz) \\ &\geq\; 0\,. \end{aligned}$$

But it is easy to see that

$$f(x,z) \;=\; (x+z-2)^2 + xz(z-1)(1-x)\,.$$

The result, for $n = 3$, follows easily.

(ii) Secondly, we prove (1) for $n \geq 3$, by induction on $n \geq 3$. From the first step, we know that (1) is true for $n = 3$. Let $n \geq 3$ be a fixed integer. Suppose that (1) holds for n.

Let $x_1, \ldots, x_{n+1}$ be non-negative real numbers such that

$$(n-1)\sum_{i<j} x_i x_j + \sum_{1<j<k} x_i x_j x_k \;=\; \frac{2}{3}(n+1)n(n-1)\,.$$

Denote by

$$S_1 \;=\; \sum_{i=1}^{n+1} x_i\,, \qquad S_2 \;=\; \sum_{i<j} x_i x_j\,, \qquad S_3 \;=\; \sum_{i<j<k} x_i x_j x_k\,.$$

Then we know that

$$(n-1)S_2 + S_3 \;=\; \frac{2}{3}(n+1)n(n-1)\,, \tag{3}$$

and we want to prove that

$$S_1 \;\geq\; \frac{2}{n} S_2\,. \tag{4}$$

Let

$$P(x) \;=\; \prod_{i=1}^{n+1}(x - x_i) \;=\; x^{n+1} - S_1 x^n + S_2 x^{n-1} - S_3 x^{n-2} + \cdots\,.$$

Then

$$\begin{aligned} P'(x) \;&=\; (n+1)\left(x^n - \frac{nS_1}{n+1}x^{n-1} + \frac{(n-1)S_2}{n+1}x^{n-2}\right. \\ &\qquad \left. -\frac{(n-2)S_3}{n+1}x^{n-3} + \cdots\right)\,. \end{aligned}$$

Since all roots of P are non-negative real numbers, it follows from Rolle's Theorem that all the roots of P' are non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus, $P'(x) = (n+1)\prod_{i=1}^{n}(x - \alpha_i)$.

Denote by $S_1' = \sum_{i=1}^{n}\alpha_i$, $S_2' = \sum_{i<j}\alpha_i\alpha_j$, $S_3' = \sum_{i<j<k}\alpha_i\alpha_j\alpha_k$. Then $P'(x) = (n+1)(x^n - S_1'x^{n-1} + S_2'x^{n-2} - S_3'x^{n-3} + \cdots)$. Identifying the coefficients, we deduce that

$$S_1' \;=\; \tfrac{n}{n+1}S_1\,, \qquad S_2' \;=\; \tfrac{n-1}{n+1}S_2\,, \qquad S_3' \;=\; \tfrac{n-2}{n+1}S_3\,.$$

From (3) we have $(n-2)S_2' + S_3' = \frac{2}{3}n(n-1)(n-2)$. Then, by the induction hypothesis (for $\alpha_1, \ldots, \alpha_n$), we have

$$S_1' \;\geq\; \frac{2}{n-1}S_2'\,.$$

Thus, $\frac{n}{n+1}S_1 \geq \frac{2}{n-1}\cdot\frac{n-1}{n+1}S_2$, which is equivalent to (4). Then (1) holds for $n+1$, and the induction is complete.

(iii) Thirdly, the equality case in (1). Suppose $x_1 \leq x_2 \leq \cdots \leq x_n$. From the first step, we know that for $n = 3$, equality occurs if and only if $x_1 = 0$, $x_2 = x_3 = 2$, or $x_1 = x_2 = x_3 = 1$.

Suppose that equality occurs in (1) for $n = 4$. Then, with the notation that is used above, $S_1' = S_2'$. Thus, $\alpha_1 = 0$, and $\alpha_2 = \alpha_3 = 2$; or $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Case 1. $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = 2$.

Since 0 is a root of P' and is between two non-negative roots of P, then 0 is also a root of P. It follows that 0 is a root with order at least 2 of P. Thus, $x_1 = x_2 = 0$.

Now, (3) gives $x_3 x_4 = 8$, and $S_1 = \frac{2}{3} S_2$ gives $x_3 + x_4 = \frac{16}{3}$, but the discriminant of the resulting quadratic is negative. This is a contradiction.

Case 2. $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

From Rolle's Theorem, we deduce that P has not more than two distinct real roots: $P(x) = (x-a)^\alpha (x-b)^{4-\alpha}$ with $a \le b$ and $\alpha \in \{1,2,3\}$. Then

$$\begin{aligned} P'(x) &= (x-a)^{\alpha-1}(x-b)^{3-\alpha}(4x - (4a - a\alpha + b\alpha)) \\ &= 4(x-1)^3\,. \end{aligned}$$

If $\alpha \ge 2$, then $a = 1$, and $4a - a\alpha + b\alpha = 4$. Thus, $a = b = 1$.

If $\alpha = 1$, then $b = 1$, and $4a - a\alpha + b\alpha = 4$. Thus, $a = b = 1$.

Conversely, if $x_1 = x_2 = x_3 = x_4 = 1$, then (3) is satisfied and equality occurs in (1). Then, for $n = 4$, equality occurs if and only if $x_1 = x_2 = x_3 = x_4 = 1$.

(iv) For $n \ge 4$, it is an easy induction on $n \ge 4$, using the same reasoning as above. From the second step, if equality occurs in (1) for the value $n+1$, then it occurs for the value n.

It follows that equality occurs for $n \ge 4$ if and only if $x_1 = \cdots = x_n = 1$.

Now we give the solutions for Category B.

2. Let $ABCD$ be a tetrahedron with $AB = AC = AD$, inscribed in a sphere with centre O. Let G be the centre of gravity of triangle ACD, E be the mid-point of BG and F be the mid-point of AE. Prove that OF is perpendicular to BG if and only if OD is perpendicular to AC.

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France. We give the vector argument of Aassila.

[The vector from the origin to the point X is denoted by $\mathbf{X}$.] We have

$$\begin{aligned} &(\mathbf{O}-\mathbf{F})\cdot(\mathbf{B}-\mathbf{G}) \\ &\quad = \frac{1}{2}(\mathbf{A}+\mathbf{E})\cdot(\mathbf{B}-\mathbf{G}) \;=\; \frac{1}{4}\left((2\mathbf{A}+\mathbf{B}+\mathbf{G})\cdot(\mathbf{B}-\mathbf{G})\right) \\ &\quad = \frac{1}{36}\left(18\mathbf{A}\cdot\mathbf{B} - 6\mathbf{A}\cdot(\mathbf{A}+\mathbf{C}+\mathbf{D}) + 9\mathbf{B}^2 - (\mathbf{A}+\mathbf{C}+\mathbf{D})^2\right) \\ &\quad = \frac{1}{36}\left(2\mathbf{A}\cdot\mathbf{D} - 2C\cdot\mathbf{D}\right)\,. \end{aligned}$$

Hence, $OF \perp BG$ if and only if $OD \perp AC$.

3. Let us be given n ($n \geq 4$) numbers a_1, a_2, $\ldots$, a_n, distinct from one another. Determine the number of permutations of these n numbers such that in each permutation, no three of the four numbers a_1, a_2, a_3, a_4 lie in three consecutive positions.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Aassila's solution.

There are $n! - 24(n-2)! + 24(n-3)!$ such permutations. Indeed, from the $n!$ permutations in all we exclude $(n-2)!$ permutations for each arrangement of a_1, a_2, a_3, a_4 into an ordered triple and one remaining element, or also $24(n-2)!$ in all. But, we have twice excluded each of the $24(n-3)!$ permutations in which all four of a_1, a_2, a_3, a_4 are in a block.

6. Let x, y, z be three non-negative real numbers satisfying the condition: $xy + yz + zx + xyz = 4$.

Prove that: $x + y + z \geq xy + yz + zx$. When does equality occur?

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We gave Bornsztein's generalization to A6 and B6 above. See page 333. Here we give a different generalization by Klamkin to this problem. [Ed. In the next issue, readers will see the connection between this and Klamkin's solution to problem **2468**.]

More generally, we show that if

$$xy + yz + zx + xyz = 3a + a^3 \quad (a > 0), \tag{1}$$

then

$$x + y + z \geq xy + yz + zx, \tag{2}$$

provided also that $a \leq 1$. If $a > 1$, the inequality can go either way.

Case 1. $0 < a \leq 1$. We can assume that $x \geq y \geq z$. Then

$$3x^2 + x^3 \geq 3a + a^3 \geq 3z + z^3,$$

so that $x \geq a \geq z$. Solving for y in (1) and substituting in (2), we obtain

$$x^2(1 + z - z^2) + x(z^2 + z - p) + z^2 - pz + p \geq 0, \tag{3}$$

where $p = 3a + a^3$. Inequality (3) will be valid if the discriminant is less than or equal to zero; that is, if

$$(z^2 + z - p)^2 \leq 4(1 + z - z^2)(z^2 - pz + p). \tag{4}$$

Inequality (4) can be rewritten as

$$p(4 - p) + z(1 - z)\left(2p(1 - 2z) + z(5z + 3)\right) \geq 0.$$

Since $p \leq 4$ and $z \leq 1$, the inequality clearly holds for $z \leq 1/2$. For $z > 1/2$, we have

$$2p(1-2z)+z(5z+3) \;\geq\; 8(1-2z)+z(5z+3) \;=\; (1-z)(8-5z) \;\geq\; 0\,.$$

It now follows that the only time equality can occur is if $a = 1$ and $x = y = z = 1$, or $a = 1$ and $z = 0$, $x = y = 2$.

Case 2. Here $p > 4$. Let $y = z = \varepsilon$. Then $x \approx p/2\varepsilon$, so that (2) holds. Now let $z = 0$, $x = y$. Then $x = y = \sqrt{p}$, so that the inequality sign in (2) is reversed.

That completes the *Corner* for this issue. Send me your Olympiad Contest materials and your nice solutions to problems from the *Corner*.

Mathematical Poems

1. no solution

my mind is a matrix
that has been reduced
into row echelon form
and proven to be
- inconsistent

2. i'm tired of being a zero vector

i'm tired of being a zero vector
with no direction
no dimension
and no magnitude;
what i need is another element
- but that would be
a contradiction
of my definition

Eileen Tupaz, student, Ateneo University, Quezon City, Philippines.

BOOK REVIEWS

ALAN LAW

Mathematical Fallacies, Flaws, and Flimflam by Edward J. Barbeau,
published by The Mathematical Association of America, 1999,
ISBN # 0-88385-529-1, softcover, 162+ pages, $23.95 (US).
Reviewed by ***Catherine Shevlin****, Wallsend upon Tyne, England.*

This book is a delightful collection of mathematical "statements", problem "solutions", and howlers, along with assertions (erroneous or otherwise) which are often accompanied by analysis of a more searching nature. The author has collected material from the College Mathematics Journal's regular department devoted to fallacies for the last eleven years.

The volume presents 174 items of various levels of sophistication, grouped according to 11 classes: Numbers, Algebra and Trigonometry, Geometry, Finite Mathematics, Probability, Calculus: Limits and Derivatives, Calculus: Integration and Differential Equations, Calculus: Multivariate and Applications, Linear and Modern Algebra, Advanced Undergraduate Mathematics, and Parting Shots. Here are a few of the shorter items. (The black suits ♠ and ♣ indicate that there is accompanying analysis — a red suit ♡ that there is none).

- Using the well-known rule for multiplying numbers raised to powers:

  $$n^{\frac{a}{b}} \cdot m^{\frac{c}{d}} \;=\; (nm)^{\frac{a}{b}+\frac{c}{d}} \;=\; (nm)^{\frac{a+c}{c+d}}\,,$$

  a student recently evaluated $3^{2/3} \cdot 9^{7/6}$ as $(3 \cdot 9)^{9/9} = 27^1 = 27$. You don't like this? Then do it your way and see if you can get something better.

- **Problem**. Prove that $\sin(x+\pi) = -\sin x$.
  Solution.

  $$\sin(x+\pi) \;=\; \sin x + \sin\pi \;=\; \sin x - 1 \;=\; (-1)\sin x \;=\; -\sin x\,. \quad ♡$$

- **Problem**. Find all values of k for which the curves with equations

  $$y = x^2 + 3 \qquad \text{and} \qquad \frac{x^2}{4} + \frac{y^2}{k} = 1$$

  are tangent.
  Solution. Eliminating x yields the equation

  $$4y^2 + ky - 7k = 0$$

  for the ordinates of the intersection points of the two curves. If the curves are to be tangent, the quadratic equation should have a double root, so that its discriminant $k^2 + 112k$ vanishes. Since $k = 0$ is not admissible, k must be -112.

  With the aid of a sketch it is not hard to see that $k = 9$ also works. Why is it not turned up by this argument? ♣

- **Problem**. Determine the shortest distance from the point $(0, 5)$ to a parabola $16y = x^2$.

  Solution. We must minimize $f(y) = x^2 + (y - 5)^2 = 16y + (y - 5)^2$. Since $f'(y) = 2y + 6$, the only critical value of f is $y = -3$, which corresponds to an imaginary value of x. Hence, the minimum distance does not exist. ♠
- **Problem**. Expand about the origin $f(x) = (1 + x^2)/(1 - x^2)$.

  Solution. By the quotient rule, we find that

  $$f'(x) \;=\; 2\left[\frac{2x}{(1-x^2)^2}\right] \;=\; 2\left[\frac{1}{(1-x^2)}\right]' ,$$

  whence $f(x) = 2(1 - x^2)^{-1} = 2(1 + x^2 + x^4 + x^6 + \cdots)$ (so in particular, $f(0) = 2$). ♣
- Three non-zero vectors A, B, C in three-dimensional Euclidean space satisfy the following inequality

  $$||xA + yB + zC|| \geq ||xA|| + ||yB||$$

  for all real numbers x, y and z. Show that the three vectors are perpendicular to each other.

  (While the conclusion can certainly be established from the hypothesis, it turns out that there are no three non-zero vectors satisfying the hypothesis.)

The final item in the final section is a centennial tribute to Sam Loyd, the "prince of the puzzle makers". It includes his "Get Off the Earth" puzzle, which Martin Gardner called Loyd's greatest creation.

In his Foreword, Professor Barbeau expresses a hope to challenge and amuse readers, as well as to provide them with material suitable for teaching and student assignments. The book succeeds admirably.

Readings in Cooperative Learning for Undergraduate Mathematics, selected and edited by Ed Dubinsky, David Mathews, and Barbara E. Reynolds, published by the Mathematical Association of America, 1997,
ISBN 0-88385-153-9, softcover, 291+ pages, $34.95 (U.S.).
Reviewed by **John Grant McLoughlin**, *Memorial University of Newfoundland.*

The book features a collection of 17 papers chosen by the editors. The anthology is organized into three sections:

(1) Constructivism and the Teacher's Role;

(2) Research and Effectiveness; and

(3) Implementation Issues.

An annotated bibliography serves as an epilogue and an eighteenth paper.

Would I recommend this book to individuals? Likely not unless one is seeking a reference on the topic. Should the book find a place on the shelves of an institution? Sure. Math departments or individual faculty intent upon considering their own cooperative learning initiatives would benefit from reading pertinent chapters. Perhaps the greatest value of such readings may stem from the breadth of references cited. Whether one is looking to implement cooperative learning or learn more about its potential uses in calculus classes, for example, the book offers worthwhile discussion. Themes of readings range from subjects (for example, *Constructing Calculus Concepts: Cooperation in a Computer Laboratory*) to issues (for example, *Do Students Learn More in Heterogeneous or Homogeneous Groups?*) to contextual matters (for example, *Teaching Problem Solving Through Cooperative Grouping – Parts 1 and 2*). The quality of readings seemed to vary considerably from one to another.

While it is likely that individual interests will lead to selections of different favourites among the readings, I found "*A Framework for Research and Curriculum Development in Undergraduate Mathematics Education*" to be most informative. The authors, M. Asiala, A. Brown, D. DeVries, E. Dubinsky, D. Mathews, and W. Thomas, offer an insightful and thorough overview of the framework which they have been "using and refining". The three components of the framework are developed: an analysis of what it means to understand a concept and the potential construction of that understanding by the learner; design of an instructional treatment; and the gathering and analysis of data obtained through the implementation of instruction. The summary of the research along with an extensive bibliography lays a framework of its own for generating discussion in undergraduate mathematics education.

The enthusiasm for such an article does not extend unequivocally to the book as a whole. One curious feature of the book is the inclusion of a commentary (including discussion questions) preceding every reading. This model did not work for me. Rather it shaped views and perspectives that would have been preferable to develop fully on my own or at least after reading the article. Also, it seemed that reading the articles in order was important to follow the discussion questions and flow of perspective. This appears to reflect the fact that the editors have used this collection as a text in their own courses on cooperative learning. Unfortunately the sequential benefits put in place for a course marginally diminish the value of the book as a resource for those interested in reading select articles from the collection.

The merits of the book are bound to be determined individually, an appraisal of which may be aided by the editors' own introductory comments about the book's potential value (beyond that of a text): "We also hope that it is useful to the general reader who may want to use it in support of teaching with cooperative learning, or who just may be interested in a small snapshot of what the literature has to say about this innovative and challenging pedagogical strategy."

THE SKOLIAD CORNER

No. 48

R.E. Woodrow

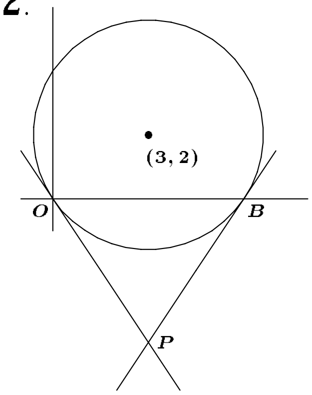
This issue we give the preliminary round of the Senior High School Mathematics Contest of the British Columbia Colleges written March 8, 2000. My thanks go to Jim Totten, The University College of the Cariboo, for sending them for use in the *Corner*.

BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. Antonino sets out on a bike ride of 40 km. After he has covered half the distance he finds that he has averaged 15 km/hr. He decides to speed up. The rate at which he must travel the rest of the trip in order to average 20 km/hr for the whole journey is:

(a) 25 km/hr (b) 30 km/hr (c) 35 km/hr (d) 36 km/hr (e) 40 km/hr

2.



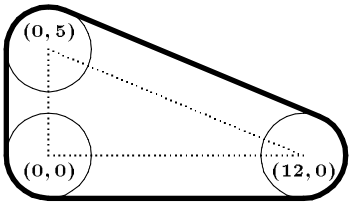
A circle with centre at $(3, 2)$ intersects the x-axis at the origin, O, and at the point B. The tangents to the circle at O and B intersect at the point P. The y-coordinate of P is:

(a) $-3\frac{1}{2}$ (b) -4 (c) $-4\frac{1}{2}$ (d) -5 (e) none of these

3. From five students whose ages are 6, 7, 8, 9, and 10, two are randomly chosen. The probability that the difference in their ages will be at least 2 years is:

(a) $\frac{1}{2}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{7}{10}$ (e) $\frac{3}{4}$

4. The centres of three circles of radius 2 units are located at the points $(0, 0)$, $(12, 0)$ and $(0, 5)$. If the circles represent pulleys, what is the length of the belt which goes around all 3 pulleys as shown in the diagram?

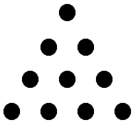


(a) $30 + \pi$ **(b)** $30 + 4\pi$ **(c)** $36 + \pi$ **(d)** $60 - 4\pi$ **(e)** none of these

5. If Mark gets 71 on his next quiz, his average will be 83. If he gets 99, his average will be 87. How many quizzes has Mark already taken?

(a) 4 **(b)** 5 **(c)** 6 **(d)** 7 **(e)** 8

6.



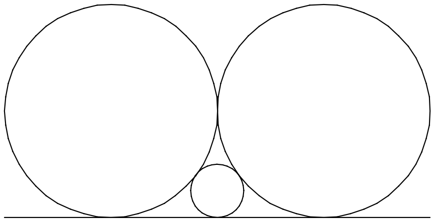
While 10 pin bowling (see diagram) Sam left 3 pins standing which formed the vertices of an equilateral triangle. How many such equilateral triangles are possible?

(a) 15 **(b)** 14 **(c)** 12 **(d)** 10 **(e)** none of these

7. If I place a 6 cm $\times 6$ cm square on a triangle, I can cover up to 60% of the triangle. If I place the triangle on the square, I can cover up to $\frac{2}{3}$ of the square. What is the area, in cm^2, of the triangle?

(a) $22\frac{4}{5}$ **(b)** 24 **(c)** 36 **(d)** 40 **(e)** 60

8. Two circles, each with radius 10 cm, are placed so they are tangent to each other and a straight line. A smaller circle is nestled between them so that it is tangent to the larger circles and the line. What is the radius, in centimetres, of the smaller circle?



(a) $\sqrt{10}$ **(b)** 2.5 **(c)** $\sqrt{2}$ **(d)** 1 **(e)** none of these

9. Arrange the following in ascending order:

$$2^{5555} \quad 3^{3333} \quad 6^{2222}$$

(a) 2^{5555} 3^{3333} 6^{2222} (b) 2^{5555} 6^{2222} 3^{3333}
(c) 6^{2222} 3^{3333} 2^{5555} (d) 3^{3333} 6^{2222} 2^{5555}
(e) 3^{3333} 2^{5555} 6^{2222}

[Editor's note: Astute readers will notice that this is the same question as Question 10 from the Junior Contest given last issue. The solution appears in this issue.]

10. Given that $0 < x < y < 20$, the number of integer solutions (x, y) to the equation $2x + 3y = 50$ is:

(a) 25 (b) 16 (c) 8 (d) 5 (e) 3

11. Suppose A, B, and C are positive integers such that

$$\frac{24}{5} = A + \cfrac{1}{B + \cfrac{1}{C+1}} .$$

The value of $A + 2B + 3C$ equals:

(a) 9 (b) 12 (c) 15 (d) 16 (e) 20

12. A box contains m white balls and n black balls. Two balls are removed randomly without replacement. The probability one ball of each colour is chosen is:

(a) $\frac{mn}{(m+n)(m+n-1)}$ (b) $\frac{mn}{(m+n)^2}$ (c) $\frac{2mn}{(m+n-1)(m+n-1)}$
(d) $\frac{2mn}{(m+n)(m+n-1)}$ (e) $\frac{m(m-1)}{(m+n)(m+n-1)}$

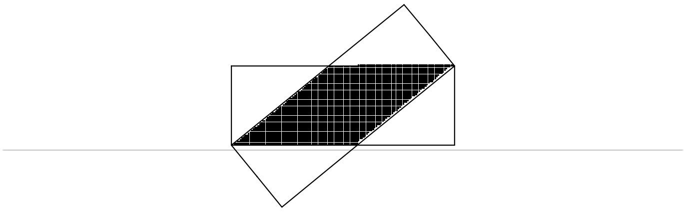
13. If it takes x builders y days to build z houses, how many days would it take q builders to build r houses? Assume these builders work at the same rate as the others.

(a) $\frac{qry}{xz}$ (b) $\frac{ryz}{qx}$ (c) $\frac{qz}{rxy}$ (d) $\frac{xyr}{qz}$ (e) $\frac{rz}{qxy}$

14. If $x^2 + xy + x = 14$ and $y^2 + xy + y = 28$, then a possible value for the sum of $x + y$ is:

(a) -7 (b) -6 (c) 0 (d) 1 (e) 3

15. Two congruent rectangles each measuring 3 cm ×7 cm are placed as in the figure. The area of overlap (shaded), in cm^2, is:



(a) $\frac{87}{7}$ (b) $\frac{29}{7}$ (c) $\frac{20}{7}$ (d) $\frac{21}{2}$ (e) none of these

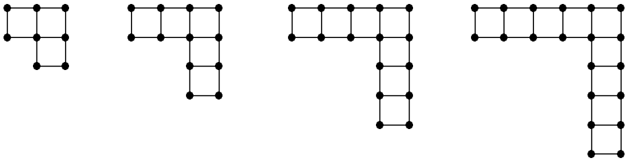
The problems given last issue were those of the preliminary round of the Junior High School Contest of the British Columbia Colleges. My thanks for these "official solutions" to Jim Totten, The University College of the Cariboo.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round — March 8, 2000

1. After 15 litres of gasoline was added to a partially filled fuel tank, the tank was 75% full. If the tank's capacity is 28 litres, then the number of litres in the tank before adding the gas was:

Answer. **(d)**. Let x be the number of litres of gasoline in the tank prior to filling. Then $x + 15 = \frac{3}{4} \cdot 28$, or $x = 6$.

2. The following figures are made from matchsticks.



If you had 500 matchsticks, the number of squares in the largest such figure you could build would be:

Answer. **(b)**. The first figure is composed of one square of side 1 (consisting of 4 matchsticks) plus 2 squares of side 1 each missing 1 matchstick, for a total of $4 + 2 \cdot 3 = 10$ matchsticks. Each subsequent figure consists of the previous figure plus 2 squares of side 1 each missing 1 matchstick. Thus, the n^{th} figure in the sequence contains $4 + 2 \cdot 3 \cdot n = 6n + 4$ matchsticks. The largest value n for which 500 matchsticks is sufficient is thus 82 (which uses up $6 \cdot 82 + 4 = 496$ matchsticks). Now the number of squares in the first figure is 3 and each subsequent figure contains 2 more squares than the previous one. Therefore the number of squares in the n^{th} figure is $2n + 1$. For $n = 82$ this means that 165 squares would be in the largest figure made with 500 matchsticks.

3. The perimeter of a rectangle is 56 metres. The ratio of its length to width is 4 : 3. The length, in metres, of a diagonal of the rectangle is:

Answer. **(b)**. Let ℓ and w be the length and width (in metres) of the rectangle in question. Since the perimeter is 56 metres, we have $2\ell + 2w = 56$, or $\ell + w = 28$. We are also told that $\ell : w = 4 : 3$, or $\ell = \frac{4}{3}w$. Using this in the first equation we get

$$\begin{aligned} \frac{4}{3}w + w &= 28 \\ \frac{7}{3}w &= 28 \\ w &= 12 \end{aligned}$$

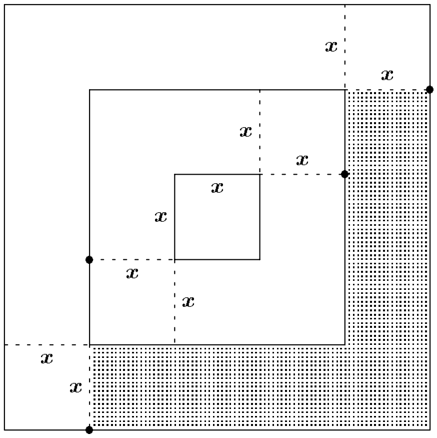
which implies that $\ell = 16$. By the Theorem of Pythagoras the length of the diagonal is

$$\sqrt{12^2 + 16^2} = \sqrt{400} = 20\,.$$

4. If April 23 falls on Tuesday, then March 23 of the same year was a:

Answer. **(a)**. Since there are 31 days in March, there are 31 days between March 23 and April 23. That is, the period in question is 4 weeks and 3 days. Since April 23 is a Tuesday, we must have March 23 a Saturday, namely 3 days earlier in the week.

5. Consider the dart board shown in the diagram. If a dart may hit any point on the board with equal probability, the probability it will land in the shaded area is:



Answer. **(d)**. The total area of the board is $25x^2$ square units. The area of the shaded region is $x \cdot 4x + x \cdot 3x = 7x^2$ square units. Therefore, the probability of hitting the shaded area is

$$\frac{7x^2}{25x^2} = \frac{7}{25} = 0.28 .$$

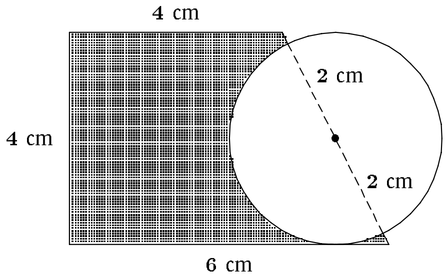
6. The proper divisors of a number are those numbers that are factors of the number other than the number itself. For example, the proper divisors of 12 are 1, 2, 3, 4 and 6. An *abundant* number is defined as a number for which the sum of its proper divisors is greater than the number itself. For example, 12 is an abundant number since $1 + 2 + 3 + 4 + 6 > 12$. Another example of an abundant number is:

Answer. **(c)**. Let us compute the sum of the proper divisors of each of the 5 possible answers in the list:

$$\begin{array}{ll}
13 : & 1 < 13 \\
16 : & 1 + 2 + 4 + 8 = 15 < 16 \\
30 : & 1 + 2 + 3 + 5 + 6 + 10 + 15 = 42 > 30 \\
44 : & 1 + 2 + 4 + 11 + 22 = 40 < 44 \\
50 : & 1 + 2 + 5 + 10 + 25 = 43 < 50
\end{array}$$

The only one of these which qualifies as an abundant number is 30.

7. The figure below is a right trapezoid with side lengths 4 cm, 4 cm, and 6 cm as labelled. The circle has radius 2 cm. The area, in cm^2, of the shaded region is:



Answer. **(d)**. The area in question is the area of a trapezoid less the area of a semicircle. The area of the semicircle is obviously $\frac{1}{2}\pi 2^2 = 2\pi$ cm^2. The area of the trapezoid is $\frac{1}{2}4(4 + 6) = 20$ cm^2. Thus, the shaded area is $20 - 2\pi$ cm^2.

8. Three vertices of parallelogram $PQRS$ were $P(-3,-2)$, $Q(1,-5)$, and $R(9,1)$ with P and R diagonally opposite. The sum of the coordinates of vertex S is:

Answer. (**e**). Let the coordinates of the point S be (x,y). Since $PS\|QR$, they must have the same slope:

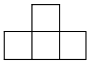
$$\begin{aligned} \frac{y+2}{x+3} &= \frac{-5-1}{1-9} = \frac{3}{4} \\ \text{or} \quad 4y-3x &= 1\,. \end{aligned}$$

Since $RS\|QP$, we also have (by the same argument):

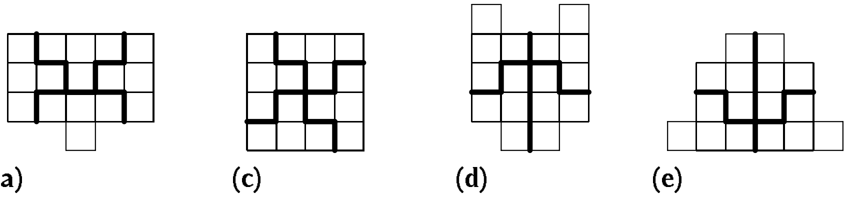
$$\begin{aligned} \frac{y-1}{x-9} &= \frac{-5+2}{1+3} = -\frac{3}{4} \\ \text{or} \quad 4y+3x &= 31\,. \end{aligned}$$

From these two equations in 2 unknowns we easily solve for $x = 5$ and $y = 4$. Thus, $x + y = 9$.

9. Which shape *cannot* be filled, without any overlapping, using copies of the tile shown on the right?



Answer. (**b**). The diagram below shows how figures (a), (c), (d), and (e) can be filled with copies of the "T" tile. No matter how one tries figure (b) cannot be filled with copies of it.



10. Arrange the following in ascending order:

$$2^{5555} \qquad 3^{3333} \qquad 6^{2222}$$

Answer. (**e**). Note first that

$$\begin{aligned} 2^{5555} &= (2^5)^{1111} = 32^{1111} \\ 3^{3333} &= (3^3)^{1111} = 27^{1111} \\ 6^{2222} &= (6^2)^{1111} = 36^{1111} \end{aligned}$$

Since $27 < 32 < 36$, we have $27^{1111} < 32^{1111} < 36^{1111}$, which means

$$3^{3333} < 2^{5555} < 6^{2222}\,.$$

11. 2000 days, 2000 hours, 2000 minutes, and 2000 seconds would be equivalent to N million seconds. Of the choices offered, the closest approximation of N is:

Answer. **(d)**. Let us first compute the number of seconds in 1 day, 1 hour, 1 minute, and 1 second, and then multiply by 2000. Now 1 day plus 1 hour is clearly 25 hours. Then 1 day, 1 hour, plus 1 minute is $25 \times 60 + 1 = 1501$ minutes. Expressed in seconds this is $1501 \times 60 = 90060$ seconds. Thus, 1 day, 1 hour, 1 minute, and 1 second is $90,061$ seconds. The answer to the problem is this figure multiplied by 2000; that is, $180,122,000$, which to the nearest million is $180,000,000$.

12. A three-digit decimal number abc may be expressed as $100a + 10b + c$ where each of the digits is multiplied by its respective place value and subsequently summed. If $a = b = c$ and $a > 0$, which of the following numbers must be a factor of the three-digit number abc?

Answer. **(e)**. If $a = b = c$, then $100a + 10b + c = 100a + 10a + a = 111a$. Since a can be any digit, in order for a number to be a factor of the three-digit number, it must be a factor of 111. The factors of 111 are 1, 3, 37, and 111. The only one of these appearing in the list is 37.

13. If $(x + y)^2 - (x - y)^2 > 0$, then

Answer. **(a)**.

$$\begin{aligned}(x + y)^2 - (x - y)^2 > 0 &\iff x^2 + 2xy + y^2 - x^2 + 2xy - y^2 > 0\\ &\iff 4xy > 0\\ &\iff xy > 0\,.\end{aligned}$$

The last condition clearly holds if and only if x and y have the same sign; that is, both are positive or both are negative.

14. Consider all non-congruent triangles with all sides having whole number lengths and a perimeter of 12 units. The following statements correspond to these triangles.

(i) There are only three such triangles.
(ii) The number of equilateral triangles equals the number of scalene triangles.
(iii) None of these triangles are right angled.
(iv) None of these triangles have a side of length 1 unit.

Of the four statements made, the number of true statements is:

Answer. **(d)**. Let a, b, c be the three sides of the triangle. Let us assume that $a \leq b \leq c$. Since the perimeter is 12, we have $a + b + c = 12$. Let us now list all possible sets of integers (a, b, c) satisfying the above conditions:

$$(1,1,10)\,,\quad (1,2,9)\,,\quad (1,3,8)\,,\quad (1,4,7)\,,\quad (1,5,6)\,,\quad (2,2,8)\,,$$

$$(2,3,7)\,,\quad (2,4,6)\,,\quad (2,5,5)\,,\quad (3,3,6)\,,\quad (3,4,5)\,,\quad (4,4,4)\,.$$

However, it is clear that some of these "triangles" do not actually exist, since in any triangle the sum of the lengths of the two shorter sides must be greater than the length of the longest side. With this additional condition we have only the following triangles (a, b, c):

$$(2,5,5)\,, \quad (3,4,5)\,, \quad (4,4,4)\,.$$

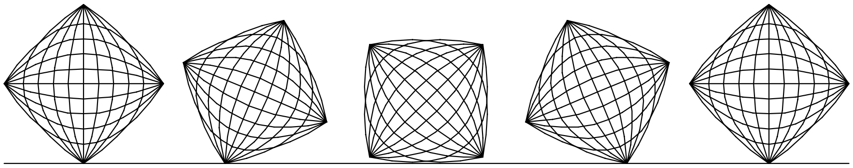
We can now examine the four statements and conclude that (i), (ii) and (iv) are clearly true. As for (iii), we see that triangle $(3,4,5)$ above is right-angled; hence, (iii) is false.

15. An altitude, h, of a triangle is increased by a length m. How much must be taken from the corresponding base, b, so that the area of the new triangle is one-half that of the original?

Answer. **(e)**. The area of the original triangle is $\frac{1}{2}bh$. The new triangle has altitude $h+m$ and base $b-x$. We need to find x such that the area of the new triangle is $\frac{1}{4}bh$. Clearly the area of the new triangle is $\frac{1}{2}(h+m)(b-x)$. Thus,

$$\begin{aligned}
\tfrac{1}{4}bh &= \tfrac{1}{2}(h+m)(b-x) \\
\frac{bh}{2(h+m)} &= b-x \\
x &= b-\frac{bh}{2(h+m)} = \frac{2bh+2bm-bh}{2(h+m)} = \frac{b(2m+h)}{2(h+m)}
\end{aligned}$$

That completes the *Skoliad Corner* for this issue. Send me your contest materials and any communications about the *Corner*.



MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M65 3G3. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Western Ontario). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University)

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 6 of 2001.

High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 `<ahchan@fas.harvard.edu>`

H257. Find all integers n such that $n^2 - 11n + 63$ is a perfect square.

Solution. Let $n^2 - 11n + 63 = k^2$, where k is a non-negative integer. Then

$$\begin{aligned} 4n^2 - 44n + 252 &= 4k^2 \\ \Longrightarrow (2n-11)^2 + 131 &= (2k)^2 \\ \Longrightarrow (2k)^2 - (2n-11)^2 &= 131 \\ \Longrightarrow (2k+2n-11)(2k-2n+11) &= 131\,. \end{aligned}$$

Notice that 131 is prime. And since $2k + 2n - 11$ and $2k - 2n + 11$ are both integers that add up to $4k \geq 0$, then there are only two possibilities:

Case 1: $2k + 2n - 11 = 131$, $2k - 2n + 11 = 1$.
Subtracting, $4n - 22 = 130$, which implies that $n = 38$.

Case 2: $2k + 2n - 11 = 1$, $2k - 2n + 11 = 131$.
Subtracting, $4n - 22 = -130$, which implies that $n = -27$.

Hence, only $n = 38$, -27 yields a perfect square for $n^2 - 11n + 63$.

Also solved by EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario.

H258. Solve in integers for x and y:

$$6(x! + 3) = y^2 + 5\,.$$

Solution. The given equation implies that $6x! + 13 = y^2$. Clearly $x \geq 0$. If $x \geq 5$, then $x! \equiv 0 \pmod 5$, implying that $y^2 \equiv 6x! + 13 \equiv 3 \pmod 5$. But the squares modulo 5 are 0, 1, 4, so $x \leq 4$.

If $x = 0$ or 1, then $y^2 = 19$, yielding no solutions in integers.

If $x = 2$, then $y^2 = 25$, yielding $(x, y) = (2, \pm 5)$.

If $x = 3$, then $y^2 = 49$, yielding $(x, y) = (3, \pm 7)$.

If $x = 4$, then $y^2 = 157$, yielding no solutions in integers.

Therefore, the four solutions are $(x, y) = (2, \pm 5)$ and $(3, \pm 7)$.

Also solved by EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario.

Note: Professor Wang had a similar solution opting to look at the question modulo 10.

H259. *Proposed by Alexandre Tritchtchenko, student, Carleton University.*

Solve for x:

$$2^{m-n} \sin(2^n x) \prod_{i=1}^{m-n} \cos(2^{m-i} x) \;=\; 1\,.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The given condition implies that $m - n$ is a positive integer. Iterating the formula $\sin 2x = 2 \sin x \cos x$ exactly $m - n$ times, we get

$$\begin{aligned}
\sin(2^m x) &= 2\sin(2^{m-1}x)\cos(2^{m-1}x) \\
&= 2^2 \sin(2^{m-2}x)\cos(2^{m-2}x)\cos(2^{m-1}x) \\
&\cdots \\
&= 2^{m-n}\sin(2^n x)\prod_{i=1}^{m-n}\cos(2^{m-i}x)\,.
\end{aligned}$$

Therefore, our original equation is equivalent to solving $\sin(2^m x) = 1$, which in turn is equivalent to $2^m x = (2k + 1/2)\pi$ for some integer k, so that $x = 2^{-m-1}(4k + 1)\pi$ where k is some integer.

H260. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\sum_{1\le i<j\le n} \cos(x_i - x_j) \ge -\frac{n}{2}\,.$$

Solution. Consider the following identities:

$$\left(\sum_{i=1}^{n} \cos x_i\right)^2 = \sum_{i=1}^{n} \cos^2 x_i + 2 \sum_{1\le i<j\le n} \cos x_i \cos x_j\,,$$

and

$$\left(\sum_{i=1}^{n} \sin x_i\right)^2 = \sum_{i=1}^{n} \sin^2 x_i + 2 \sum_{1\le i<j\le n} \sin x_i \sin x_j\,.$$

Summing together,

$$\begin{aligned} &\left(\sum_{i=1}^{n} \cos x_i\right)^2 + \left(\sum_{i=1}^{n} \sin x_i\right)^2 \\ &= \sum_{i=1}^{n} \cos^2 x_i + \sum_{i=1}^{n} \sin^2 x_i + 2\left(\sum_{1\le i<j\le n} \cos x_i \cos x_j + \sin x_i \sin x_j\right) \\ &= n + 2 \sum_{1\le i<j\le n} \cos(x_i - x_j) \\ &\ge 0 \end{aligned}$$

$$\Longrightarrow \sum_{1\le i<j\le n} \cos(x_i - x_j) \ \ge\ -n/2\,,$$ as desired.

Advanced Solutions

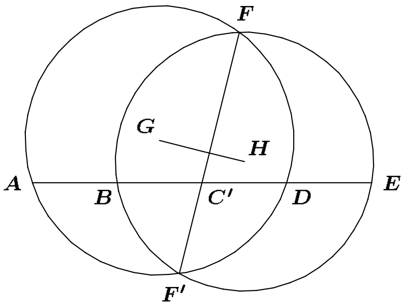
Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 `<dccheung@uwaterloo.ca>`

A232. Five distinct points, A, B, C, D, and E lie on a line (in this order) and $|AB| = |BC| = |CD| = |DE|$. The point F lies outside the line. Let G be the circumcentre of triangle ADF and H the circumcentre of triangle BEF. Show that the lines GH and FC are perpendicular.

(1997 Baltic Way)

Solution by Catherine Shevlin, Wallsend, England.

Suppose that the two circles intersect again at F'. Join FF' and suppose that FF' intersects $ABCDE$ at C'.



We have that $FF' \perp GH$. We shall show that $C = C'$.

By the Intersecting Chords Theorem, we have

$$AC' \cdot C'D = FC' \cdot C'F' = EC' \cdot C'B .$$

From this, it easily follows that $C = C'$.

For those who want a simple argument, assume, without loss of generality that $AB = 1$. Let $BC' = x$, so that $C'D = 2 - x$. Then $AC' = 1 + x$ and $EC' = 3 - x$. Hence, we have $(1 + x)(2 - x) = (3 - x)x$, or $2 + x - x^2 = 3x - x^2$, yielding $x = 1$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; ANDY LIU, University of Alberta, Edmonton, Alberta; DANIEL RIESZ, Université de Bourgogne, Dijon, France; D.J. SMEENK, Zaltbommel, the Netherlands; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

A233. *Proposed by Naoki Sato.*

In **C81**, we defined the following sequence: $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 4a_n - a_{n-1}$ for $n = 1, 2, \ldots$. This sequence exhibits the following curious property: For $n \geq 1$, if we set $(a, b, c) = (a_{n-1}, 2a_n, a_{n+1})$, then $ab + 1$, $ac + 1$, and $bc + 1$ are always perfect squares. For example, for $n = 3$, $(a, b, c) = (a_2, 2a_3, a_4) = (4, 30, 56)$, and indeed, $4 \cdot 30 + 1 = 11^2$,

$4{\cdot}56{+}1 = 15^2$, and $30{\cdot}56{+}1 = 41^2$. Show that this property holds. Generalize, using the sequence defined by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = Na_n - a_{n-1}$, and the triples $(a, b, c) = (a_{n-1}, (N-2)a_n, a_{n+1})$, where N is an arbitrary integer.

Solution by Masoud Kamgarpour, University of Waterloo, Waterloo, Ontario.

First, we show that $a_{n-1}a_{n+1}{+}1 = a_n^2$ for any positive integer n, using induction. Note that $a_2 = N$, so we know that $a_0a_2 + 1 = 0 \cdot N + 1 = 1^2$. Therefore, the proposition is true for the case $n = 1$.

Now, assume that the proposition is true for $n = k$; that is, $a_{k-1}a_{k+1} + 1 = a_k^2$. We then have

$$\begin{aligned} a_ka_{k+2} + 1 &= a_k(Na_{k+1} - a_k) + 1 \\ &= Na_ka_{k+1} - a_k^2 + 1 \\ &= Na_ka_{k+1} - (a_{k-1}a_{k+1} + 1) + 1 \\ &= a_{k+1}(Na_k - a_{k-1}) \;=\; a_{k+1}^2\,. \end{aligned}$$

Therefore, we have proved the proposition for the case $n = k + 1$. Hence, we have that $a_{n-1}a_{n+1} + 1 = a_n^2$ for all positive integers n.

Now, we proceed to prove that for any positive integer n, and integer N, we have that $(N-2)a_{n-1}a_n + 1 = (a_n - a_{n-1})^2$. From before, we have that

$$\begin{aligned} a_n^2 - (a_n^2 - 1) &= 1 \\ \Longrightarrow\quad a_n^2 - a_{n-1}(Na_n - a_{n-1}) &= 1 \\ \Longrightarrow\quad a_n^2 + a_{n-1}^2 - Na_na_{n-1} &= 1 \\ \Longrightarrow\quad a_n^2 + a_{n-1}^2 - 2a_na_{n-1} &= (N-2)a_na_{n-1} + 1\,. \end{aligned}$$

Therefore, we have proven that for any positive integer n, and integer N, $a_{n-1}a_{n+1} + 1 = a_n^2$ and $(N-2)a_{n-1}a_n + 1 = (a_n - a_{n-1})^2$. Hence, $ab + 1$, $ac + 1$, and $bc + 1$ are always perfect squares.

A234. In triangle ABC, AC^2 is the arithmetic mean of BC^2 and AB^2. Show that $\cot^2 B \geq \cot A \cot C$. (Note: $\cot\theta = \cos\theta/\sin\theta$.)

(1997 Baltic Way)

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $a = BC$, $b = AC$, and $c = AB$. Then we are given $a^2 + c^2 = 2b^2$ and we are to prove that

$$\cos^2 B \;\geq\; \sin^2 B \cot A \cot C\,. \tag{1}$$

From the Law of Cosines, we have

$$\cos B \;=\; \frac{a^2 + c^2 - b^2}{2ac} \;=\; \frac{b^2}{2ac}\,. \tag{2}$$

Let BH denote the altitude on side AC, $h = BH$, $b_1 = AH$, $b_2 = CH$, and let Δ denote the area of triangle ABC. Then from $\Delta = \frac{1}{2}ac\sin B$, we get

$$\sin B = \frac{2\Delta}{ac} = \frac{bh}{ac}. \tag{3}$$

Furthermore,

$$\cot A \cot C = \frac{b_1}{h}\cdot\frac{b_2}{h} = \frac{b_1b_2}{h^2}. \tag{4}$$

Using (2), (3), (4), we see that (1) is equivalent to

$$\left(\frac{b^2}{2ac}\right)^2 \geq \left(\frac{bh}{ac}\right)^2\left(\frac{b_1b_2}{h^2}\right) \quad \text{or} \quad b^2 \geq 4b_1b_2,$$

which is clearly true since

$$b^2 - 4b_1b_2 = (b_1+b_2)^2 - 4b_1b_2 = (b_1-b_2)^2 \geq 0.$$

Note that equality holds in the last inequality if and only if $b_1 = b_2$. Since it is easy to see that $b_1 = b_2$ implies $a = c = b$, we conclude that equality holds in the given inequality if and only if triangle ABC is equilateral.

A235. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

The convex polygon $A_1A_2\cdots A_n$ is inscribed in a circle of radius R. Let A be some point on this circumcircle, different from the vertices. Set $a_i = AA_i$, and let b_i denote the distance from A to the line A_iA_{i+1}, $i = 1, 2, \ldots, n$. Prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq 2nR.$$

Solution. Let $\theta_i = \angle AOA_i$, where O is the centre of the circumcircle. Since b_i is an altitude of triangle AA_iA_{i+1}, we have $b_i = a_i \sin\angle AA_iA_{i+1}$. Now, depending on whether O and A_i are on the same side of line $\overline{AA_{i+1}}$ or not, we either have $\angle AA_iA_{i+1} = \angle AOA_{i+1}/2$ or $\angle AA_iA_{i+1} = (2\pi - \angle AOA_{i+1})/2$. In both cases, $\sin\angle AA_iA_{i+1} = \sin(\theta_{i+1}/2)$, so we have $b_i = a_i\sin(\theta_{i+1}/2)$.

However, we also know that $a_i = 2R\sin(\theta_i/2)$, and so we get

$$\frac{a_i^2}{b_i} = \frac{a_i}{\sin(\theta_{i+1}/2)} = 2R\frac{a_i}{a_{i+1}}.$$

Using the AM–GM Inequality, we get

$$\frac{1}{n}\sum_{i=1}^{n}\frac{a_i}{a_{i+1}} \geq \sqrt[n]{\prod_{i=1}^{n}\frac{a_i}{a_{i+1}}} = 1,$$

giving us

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \;=\; 2R\sum_{i=1}^{n}\frac{a_i}{a_{i+1}} \;\geq\; 2nR\,,$$

as desired.

A236. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

For all positive integers n and positive reals x, prove the inequality

$$\frac{\binom{2n}{1}}{x+1}+\frac{\binom{2n}{3}}{x+3}+\cdots+\frac{\binom{2n}{2n-1}}{x+2n-1} \;<\; \frac{\binom{2n}{0}}{x}+\frac{\binom{2n}{2}}{x+2}+\cdots+\frac{\binom{2n}{2n}}{x+2n}\,.$$

Solution. We rearrange the inequality to get

$$\sum_{i=0}^{2n}\frac{(-1)^i\binom{2n}{i}}{x+i} \;>\; 0\,.$$

By summing in reverse, we get

$$\sum_{i=0}^{2n}\frac{(-1)^i\binom{2n}{i}}{x+i} \;=\; \sum_{i=0}^{2n}\frac{(-1)^{2n-i}\binom{2n}{2n-i}}{x+(2n-i)} \;=\; \sum_{i=0}^{2n}\frac{(-1)^i\binom{2n}{i}}{x+(2n-i)}\,.$$

Now, we have

$$2\sum_{i=0}^{2n}\frac{(-1)^i\binom{2n}{i}}{x+i} \;=\; \sum_{i=0}^{2n}\left(\frac{(-1)^i\binom{2n}{i}}{x+i}+\frac{(-1)^i\binom{2n}{i}}{x+(2n-i)}\right)\,.$$

However,

$$\begin{aligned}\frac{1}{x+i}+\frac{1}{x+2n-i} &= \frac{2x+2n}{(x+i)(x+2n-i)}\\ &= \frac{2x+2n}{(x+n-(n-i))(x+n+(n-i))}\\ &= \frac{2x+2n}{(x+n)^2-(n-i)^2}\\ &\geq \frac{2x+2n}{(x+n)^2} = \frac{2}{x+n}\,,\end{aligned}$$

with equality if and only if $i=n$. Since $n>0$, we have at least one term where $i\neq n$ so we get a strict inequality:

$$2\sum_{i=0}^{2n}\frac{(-1)^i\binom{2n}{i}}{x+i} \;>\; \sum_{i=0}^{2n}\frac{2(-1)^i\binom{2n}{i}}{x+n} \;=\; \frac{2}{x+n}\sum_{i=0}^{2n}(-1)^i\binom{2n}{i} = 0\,,$$

and we are done.

Challenge Board Solutions

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C87. *Proposed by Mark Krusemeyer, Carleton College.*

Find an example of three continuous functions $f(x)$, $g(x)$, and $h(x)$ from $\mathbb{R}$ to $\mathbb{R}$ with the property that exactly five of the six composite functions $f(g(h(x)))$, $f(h(g(x)))$, $g(f(h(x)))$, $g(h(f(x)))$, $h(f(g(x)))$, and $h(g(f(x)))$ are the same function and the sixth function is different.

Solution by the proposer. The idea is to try to use functions which are zero along most of $\mathbb{R}$ except for one small bump, and to design the bumps in such a way that there is only one order in which the functions can be composed to yield a non-zero function. Here is an example of three functions which interact as desired:

$$\begin{aligned}
f(x) &= \begin{cases} 1-|5-x| & \text{for } 4 \leq x \leq 6, \\ 0 & \text{otherwise,} \end{cases} \\
g(x) &= \begin{cases} 1-|3-x| & \text{for } 2 \leq x \leq 4, \\ 0 & \text{otherwise, and} \end{cases} \\
h(x) &= \begin{cases} 3-3|1-x| & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Since $f(x) \leq 1$ for all x, we get $g(f(x)) = 0$. Hence, the two compositions $h(g(f(x)))$ and $g(f(h(x)))$ are both identically 0. Similarly $g(x) \leq 1$ and $f(g(x)) = 0$, from which we find $h(f(g(x)))$ and $f(g(h(x)))$ to be identically 0. Moreover, $h(x) \leq 3$, so $f(h(x)) = 0$ and $f(h(g(x)))$ is identically 0. On the other hand,

$$g(h(f(5))) = g(h(1)) = g(3) = 1\,,$$

so unlike the other five compositions, $g(h(f(x)))$ is not identically 0.

C88.

(a) Let A be an $n \times n$ matrix whose entries are all either $+1$ or -1. Prove that $|\det A| \leq n^{n/2}$.

(b) It is conjectured that for there to exist an $n \times n$ matrix A whose entries are all either $+1$ or -1 and such that $|\det A| = n^{n/2}$, it is necessary and sufficient that $n = 1$, $n = 2$, or n is divisible by 4. Prove that this condition is necessary.

(c) Can you construct such a matrix, for n equal to a power of 2? For $n = 12$?

Solution. (a) Let $r_1, \ldots, r_n$ be the rows of $A = (a_{i,j})$. Since the determinant $|\det A|$ is equal to the volume of the parallelepiped

$R = \{\alpha_1 r_1 + \cdots + \alpha_n r_n \mid 0 \leq \alpha_i \leq 1 \text{ for } 1 \leq i \leq n\}$, we know that $|\det A| \leq |r_1| \cdots |r_n|$, where $|r_i|$ denotes the length of the vector r_i. Since the entries of A are all ± 1, each r_i is a vector of n coordinates, each ± 1, so that r_i has length exactly $n^{1/2}$. Hence, $|\det A| \leq (n^{1/2})^n = n^{n/2}$. A matrix A for which equality holds, above, is called a **Hadamard matrix**.

(b) From (a), we know that A is a Hadamard matrix if and only if $\text{volume}(R)$ is $|r_1| \cdots |r_n|$. Since the volume of a parallelepiped is equal to the product of the lengths of the defining edges if and only if these edges are perpendicular, we learn that $|\det A| = n^{n/2}$ if and only if the rows of A are mutually perpendicular; that is, if and only if the dot product $r_i \cdot r_j = 0$ for $i \neq j$.

Observe, without loss of generality, that we may assume that the first row of A consists entirely of 1's. Indeed, if we multiply any column (or columns) of A by -1 we obtain another Hadamard matrix, and so if we have a Hadamard matrix of size n, we may in this fashion produce a Hadamard matrix of size n whose first row is all 1's.

If $n > 1$, then write $r_1 = (1, \ldots, 1)$ and $r_2 = (a_{2,1}, \ldots, a_{2,n})$. The relation

$$r_1 \cdot r_2 = \sum_{j=1}^{n} a_{2,j} = 0$$

imposes the condition that $a_{2,j} = 1$ exactly as often as $a_{2,j} = -1$. Hence, n must be even.

If $n > 2$, we can additionally consider the third row r_3. As in the previous paragraph, the rows r_3 and r_1 must coincide in half the columns and differ in half, so r_3 consists of $\frac{n}{2}$ 1's and $\frac{n}{2}$ (-1)'s. Let N be the number of columns j in which $a_{2,j} = 1$ and $a_{3,j} = 1$. Then $n/2 - N$ is the number of columns in which $a_{2,j} = -1$ and $a_{3,j} = 1$ (since r_3 has $\frac{n}{2}$ 1's), and $n/2 - N$ is the number of columns in which $a_{2,j} = 1$ and $a_{3,j} = -1$ (since r_2 has $\frac{n}{2}$ 1's). Since the total number of columns is n, this leaves N columns in which $a_{2,j} = -1$ and $a_{3,j} = -1$. It follows that $r_2 \cdot r_3 = N - (n/2 - N) - (n/2 - N) + N = 0$; that is, $4N = n$ or $N = n/4$. Therefore n is divisible by 4.

(c) We begin by noting that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is a Hadamard matrix. In fact, if A is an $n \times n$ Hadamard matrix, then the matrix

$$\begin{pmatrix} A & A \\ A & -A \end{pmatrix}$$

is a $2n \times 2n$ Hadamard matrix, and in this fashion from our 2×2 Hadamard matrix we can obtain one of size 2^n. For example,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

is a Hadamard matrix of size 4. In fact, we can generalize this idea to create a Hadamard matrix of size mn from a Hadamard matrix of size m and another of size n. (How? Exercise!) Unfortunately, this construction does not help us build a Hadamard matrix of size 12. To do this, observe that the matrix

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 1 & -1 & 0 \end{pmatrix}$$

satisfies the identity $B^2 = 5I_6$, where I_n denotes the $n \times n$ identity matrix. Then

$$A = \begin{pmatrix} B + I_6 & B - I_6 \\ B - I_6 & -B - I_6 \end{pmatrix}$$

satisfies

$$A^2 = \begin{pmatrix} 12I_6 & 0 \\ 0 & 12I_6 \end{pmatrix} = 12I_{12}$$

and so is a Hadamard matrix.

Comments. Whether or not $n \times n$ Hadamard matrices exist for all $n > 2$ which are divisible by 4 is one of the major unsolved questions of coding theory; it is conjectured that they do always exist, and indeed if m is odd it is known that there exist Hadamard matrices of size $2^k m$ for k larger than an easily computable lower bound. However, while there are a number of families of constructions of Hadamard matrices, the construction of Hadamard matrices of size $2^k m$ with k small is in general quite difficult. A recent reference (though by now it may be out of date) lists 428 as the smallest size for which Hadamard matrices are conjectured to exist but are not yet known to exist.

As an example of a construction of a family of Hadamard matrices, it is an excellent exercise for the reader to attempt to use the arithmetic of the finite field $\mathbb{F}_q$ with q elements for $q \equiv 3 \pmod 4$ to build a Hadamard matrix of size $q + 1$. This constructs Hadamard matrices of all possible sizes less than 36. Using this construction, and using the fact that we can produce Hadamard matrices of size mn from ones of size m and ones of size n, what are the sizes up to 100 for which we have not yet shown Hadamard matrices to exist?

For interested readers, the state of the art for Hadamard matrices may be investigated starting at

http://mathword.wolfram.com/HadamardMatrix.html

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem.

Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1+a_2+\cdots+a_n < 1$. Prove that

$$\frac{a_1a_2\cdots a_n(1-a_1-a_2-\cdots-a_n)}{(1-a_1)(1-a_2)\cdots(1-a_n)(a_1+a_2+\cdots+a_n)} \le \frac{1}{n^{n+1}}.$$

Solution. Let $a_0 = 1 - a_1 - a_2 - \cdots - a_n > 0$.

The inequality is equivalent to

$$\frac{a_0a_1\cdots a_n}{(1-a_0)(1-a_1)\cdots(1-a_n)} \le \frac{1}{n^{n+1}}.$$

Now, by the AM–GM Inequality,

$$1 - a_j = \sum_{i=0}^{n} a_i - a_j = \sum_{\substack{0\le i\le n\\ i\ne j}} a_i \ge n \sqrt[n]{\prod_{\substack{0\le i\le n\\ i\ne j}} a_i}\,.$$

Thus, we have

$$\begin{aligned} &(1-a_0)(1-a_1)\cdots(1-a_n)\\ \ge\ & n^{n+1}\sqrt[n]{\prod_{\substack{0\le i\le n\\ i\ne 0}} a_i}\sqrt[n]{\prod_{\substack{0\le i\le n\\ i\ne 1}} a_i}\cdots\sqrt[n]{\prod_{\substack{0\le i\le n\\ i\ne n}} a_i}\\ =\ & n^{n+1}a_0a_1\cdots a_n\,, \end{aligned}$$

since each of the a_i's occurs n times in the product. The result follows.

Ellipses in Polygons

Naoki Sato

In this article, we look into the intriguing problem of inscribing an ellipse in a polygon. This problem once caught the eye of Isaac Newton [D], and the solutions are pleasantly elegant and unexpected. Along the way, we touch on many diverse geometric principles as well. We begin by stating a pivotal theorem, especially for the purposes of this article.

Brianchon's Theorem[Co]. Let $ABCDEF$ be a hexagon, such that each side is tangent to a conic section (circle, ellipse, parabola, or hyperbola). Then the diagonals AD, BE, and CF are concurrent (or possibly parallel) (see Figure 1).

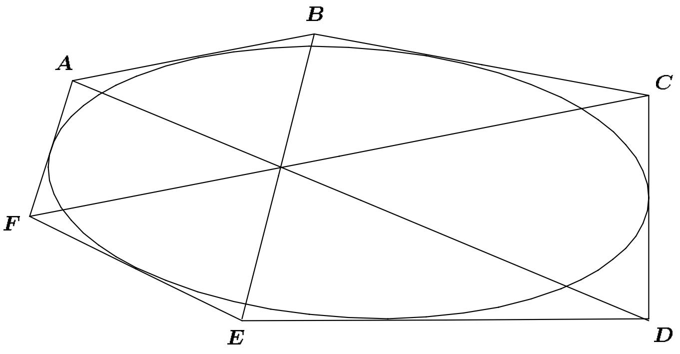

Figure 1.

Furthermore, the converse is also true: in hexagon $ABCDEF$, if the diagonals AD, BE and CF are concurrent, then there exists a conic section to which each side of hexagon $ABCDEF$ is tangent.

We give an example configuration in Figure 1, but as stated above, the theorem holds for any conic section, and the hexagon $ABCDEF$ need not be convex or simple.

Inscribing a curve in a polygon entails finding a curve that is tangent to each side of that polygon. We look first at the case of a pentagon, and for this case, we require a result about conic sections passing through each vertex of a pentagon.

Theorem. Five points in general position uniquely determine a conic section; that is, given five points in general position, there is a unique conic section that passes through each of these points.

Here, "in general position" means that there is nothing special about the positions of the five points, such as all being collinear.

We will not give a rigorous proof here, but we do give an idea of why this is so. The general equation of a conic section, in Cartesian coordinates, is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0\,. \tag{5}$$

For five points then, whose coordinates we substitute, we obtain five equations in six unknowns, which is an undetermined system and does not have a unique solution. However, we can multiply (5) by any non-zero constant without changing the solution, namely the underlying conic section. Thus, we can reduce the number of variables by dividing one of them say a, out of (5), giving us

$$x^2 + b'xy + c'y^2 + d'x + e'y + f' \;=\; 0\,, \tag{6}$$

which then gives us a system of five equations and five unknowns, which we can then solve. This determines the conic section. If a turns out to be 0 in our original system, then we can divide out by another coefficient.

We can now prove the analogous result for lines, instead of points.

Theorem. Five lines in general position uniquely determine a conic section; that is, given five lines in general position, there is a unique conic section that is tangent to each of these lines.

Proof. We can prove this using a system of coordinates called "dual", or "line" coordinates, but here, we will take another approach using Brianchon's Theorem. Let $ABCDE$ be a pentagon formed by the lines, and let ω be the conic section. Note that ω may only be tangent to the sides when they are suitably extended.

First we show that such a conic section ω must be unique. Let ω be tangent to side AB at P. Then we can legitimately view $APBCDE$ as a hexagon, in which each side is tangent to ω; sides AP and PB happen to be tangent at an end-point (see Figure 2). By Brianchon's Theorem, diagonals AC, BE, and DP are concurrent.

Turning this argument around, let Q be the intersection of AC and BE. Then P is the intersection of AB and DQ. Thus, P, the point of tangency of side AB to ω, is determined uniquely by the vertices A, B, C, D, and E.

In a similar manner, points of tangency to all sides of pentagon $ABCDE$ to ω are uniquely determined. The conic section ω must pass through all five of these points, and as we know, five points uniquely determine a conic section, which proves the first part.

We must now prove that such a conic section ω exists. Construct points P and Q as above. By the converse to Brianchon's Theorem, since AC, BE, and DP concur, each side of hexagon $APBCDE$ is tangent to some conic section ω.

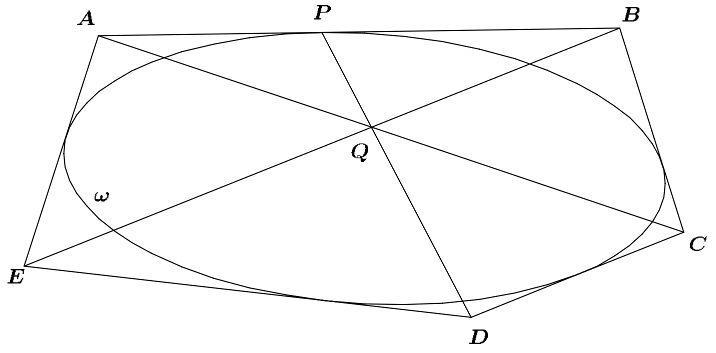

Figure 2.

Hence, sides BC, CD, DE, and EA are tangent to ω. Since P lies on AB, we conclude that side AB is tangent to ω at precisely the point P. Therefore, such an ω exists, which concludes the proof. ■

This result shows why we began looking at the case of the pentagon – it has the "right number" of sides to accommodate a unique conic section. Only certain hexagons will have a conic section tangent to each side. Furthermore, it is clear that the conic section is an ellipse if the pentagon is convex.

Our next case is the quadrilateral. Let us focus on the case of an ellipse inscribed in a convex quadrilateral. Since we have freed up one condition (in that we have one less side to worry about), we can speculate that there is more than one ellipse tangent to all sides of the quadrilateral, and in fact, that the family of such ellipses is described by one parameter. This does turn out to be the case, as seen in our next result.

Theorem[Ch,D]. Let ω be an ellipse inscribed in a convex quadrilateral $ABCD$. Then the locus of the centre of ω is the line segment joining the mid-points of diagonals AC and BD.

Proof. In this proof we use the concept of an orthogonal projection. In layman's terms, imagine drawing an image on a sheet of paper, and then viewing the sheet at an angle. The image becomes distorted, but perspective aside, many important geometric properties are preserved (see Figure 3).

For example, distances and angles are not preserved. However, lines that are concurrent and points that are collinear remain so. Furthermore, ratios of line segments and areas are also preserved. If one figure has four times the area of another, then this is still so. The technical name for this distortion is an **orthogonal projection**. It is an example of an affine transformation, all of which have these properties.

The usefulness of projections and other transformations becomes apparent when we use them to transform general geometric figures into ones with nice properties. This is very much like the principle of making an as-

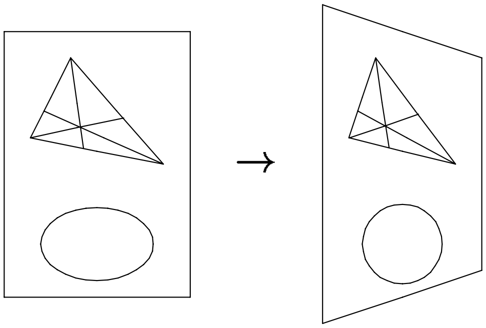

Figure 3.

sumption without losing generality. For example, any triangle can be projected to become an equilateral triangle. Similarly, any ellipse can be projected to become a circle. We can then take advantage of the properties of circles, which we do here.

Thus, apply a projection so that ω becomes a circle. Let O and r denote the centre and radius of this circle, respectively. Also, let M and N be the mid-points of the diagonals AC and BD, respectively (see Figure 4). We must show that O lies on line segment MN. Since collinearity is preserved under projection, this will show that O lies on MN in the original configuration.

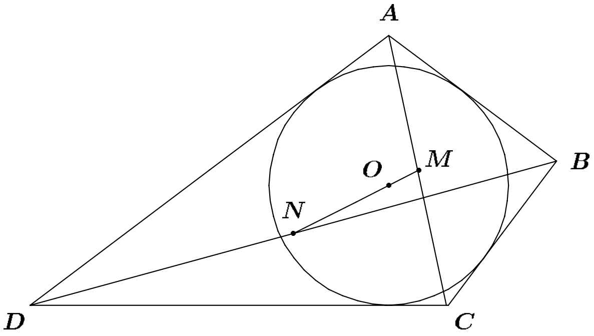

Figure 4.

Let $[X]$ denote the area of polygon X, and so, for a point P, let f be the function

$$f(P) \;=\; [PAB] - [PBC] + [PCD] - [PDA]\,.$$

Letting $P = (x, y)$, one can show that f is a linear function in x and y, which implies that the set of points P such that $f(P) = 0$ is a line.

Since M is the mid-point of AC, $[MAB] = [MBC]$ and $[MCD] = [MDA]$, giving $f(M) = 0$. Similarly, $[NBC] = [NCD]$ and $[NDA] = [NAB]$, giving $f(N) = 0$. Therefore, $f(P) = 0$ if and only if P lies on MN. Thus, the problem has been reduced to showing that $f(O) = 0$.

We have that

$$\begin{aligned}[OAB] &= r \cdot AB/2\,, \\ [OBC] &= r \cdot BC/2\,, \\ [OCD] &= r \cdot CD/2\,, \text{ and} \\ [ODA] &= r \cdot DA/2\,.\end{aligned}$$

Let Q, R, S, and T be the points of tangency of ω with the sides AB, BC, CD, and DA, respectively. Then

$$\begin{aligned}AB + CD &= AQ + QB + CS + SD \;=\; AT + RB + CR + TD \\ &= AT + TD + CR + RB \;=\; DA + BC\,.\end{aligned}$$

Thus,

$$\begin{aligned}f(O) &= [OAB] - [OBC] + [OCD] - [ODA] \\ &= r(AB - BC + CD - DA)/2 \;=\; 0\,,\end{aligned}$$

as required. ∎

We finally consider the case of an ellipse inscribed in a triangle. We have even one less side to worry about. Thus, we expect a two-parameter solution family.

Theorem[Ch]. Let ω be an ellipse inscribed in a triangle. Then the locus of the centre of ω is the medial triangle; that is, the triangle formed by the mid-points of the sides. Furthermore, for any point P in the medial triangle, there exists a unique ellipse inscribed in the triangle that is centred at P.

We do not prove this theorem here, and reluctantly leave the result to the reader. We will say that projection plays a large role in the proof, as in our next result, for which we will also have to define **areal coordinates**.

Let P be a point inside triangle ABC. Then the areal coordinates of point P are (α, β, γ), where

$$\alpha \;=\; \frac{[PBC]}{[ABC]}\,, \quad \beta \;=\; \frac{[APC]}{[ABC]}\,, \quad \text{and} \quad \gamma \;=\; \frac{[ABP]}{[ABC]}\,.$$

For example, the vertex A itself has coordinates $(1, 0, 0)$, and the centroid G has coordinates $(1/3, 1/3, 1/3)$. Note that $\alpha + \beta + \gamma = 1$ for all points P. We can now state our next result.

Theorem[Ch]. Let K denote the area of triangle ABC, and let K' denote the area of an ellipse ω inscribed in triangle ABC. Let P be the centre of ω, with areal coordinates (α, β, γ). Then

$$K' \;=\; \pi K \sqrt{(1 - 2\alpha)(1 - 2\beta)(1 - 2\gamma)}\,.$$

Proof. Apply a projection to triangle ABC, so that ω becomes a circle, namely the incircle of triangle ABC, and P becomes the incentre I. Let s and r denote the semi-perimeter and inradius of triangle ABC, respectively. Under a projection, the ratio K'/K remains the same, as well as the coordinates α, β, and γ.

Recall that $K = rs$. Furthermore, the distance from I to side BC is r, so that $[IBC] = ra/2$, and

$$\alpha = \frac{[IBC]}{[ABC]} = \frac{ra/2}{rs} = \frac{a}{2s}.$$

Similarly, $\beta = b/2s$ and $\gamma = c/2s$. Hence,

$$\begin{aligned} \frac{K'}{K} &= \frac{\pi r^2}{K} = \frac{\pi(\frac{K}{s})^2}{K} = \frac{\pi K}{s^2} \\ &= \frac{\pi\sqrt{s(s-a)(s-b)(s-c)}}{s^2} \\ &= \pi\sqrt{\left(1-\frac{a}{s}\right)\left(1-\frac{b}{s}\right)\left(1-\frac{c}{s}\right)} \\ &= \pi\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}\,, \end{aligned}$$

as desired. ■

Using this formula, one can show that the ellipse of maximal area inscribed in a triangle is the one centred at the centroid G of triangle ABC. This ellipse is called the **mid-point ellipse**, because it has the additional property that it is tangent to each of the sides at their mid-points.

We now present a remarkable result about the mid-point ellipse. The plane can be identified with the complex plane, by assigning the complex number $x + yi$ to the point (x, y). Hence, if P has coordinates $(3, 4)$, we will write $P = (3, 4) = 3 + 4i$. With this in mind, we have the following theorem.

Theorem. Let ABC be a triangle in the plane, and let z_1, z_2, and z_3 be the complex numbers corresponding to the vertices A, B, and C, respectively. Let

$$p(z) = (z - z_1)(z - z_2)(z - z_3)\,.$$

Then the foci of the mid-point ellipse ω are the roots of the equation $p'(z) = 0$, and the centre of ω is the root of the equation $p''(z) = 0$.

Proof. We can translate triangle ABC in the plane so that its centroid G coincides with the origin $(0, 0)$. In the complex plane, this corresponds to subtracting a constant from each of z_1, z_2, and z_3. The foci and centre are also translated by this same number, and a translation leaves p, and its derivatives, invariant, so that the equations are still satisfied.

Assume without loss of generality that the centroid G of ABC does coincide with the origin. The complex number corresponding to G is $(z_1 + z_2 + z_3)/3$, so that $z_1 + z_2 + z_3 = 0$. For $n = 1$, 2, and 3, let $z_n = (x_n, y_n) = x_n + y_n i$, so that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0$.

Let F_1 and F_2 denote the foci of ω, and let $F_1 = (p, q) = p + qi$. We know that the mid-point ellipse ω is centred at the centroid G, which by assumption is the origin. Hence, ω is symmetric around the origin, and $F_2 = -F_1 = (-p, -q) = -p - qi$. By definition, every point P on the ellipse ω satisfies $PF_1 + PF_2 = 2d$ for some positive constant d. Therefore, the equation of ω is given by

$$\sqrt{(x-p)^2 + (y-q)^2} + \sqrt{(x+p)^2 + (y+q)^2} \;=\; 2d$$

$$\begin{aligned}
\Longrightarrow\quad \sqrt{(x+p)^2 + (y+q)^2} &= 2d - \sqrt{(x-p)^2 + (y-q)^2} \\
\Longrightarrow\quad (x+p)^2 + (y+q)^2 &= 4d^2 - 4d\sqrt{(x-p)^2 + (y-q)^2} \\
&\quad\; +(x-p)^2 + (y-q)^2 \\
\Longrightarrow\quad 4px + 4qy &= 4d^2 - 4d\sqrt{(x-p)^2 + (y-q)^2} \\
\Longrightarrow\quad d\sqrt{(x-p)^2 + (y-q)^2} &= d^2 - px - qy \\
\Longrightarrow\quad d^2[(x-p)^2 + (y-q)^2] &= d^2x^2 - 2d^2px + d^2p^2 + d^2y^2 \\
&\quad\; -2d^2qy + d^2q^2 \\
&= d^4 + p^2x^2 + q^2y^2 \\
&\quad\; -2d^2px - 2d^2qy + 2pqxy\,,
\end{aligned}$$

which simplifies to

$$(d^2 - p^2)x^2 - 2pqxy + (d^2 - q^2)y^2 \;=\; d^2(d^2 - p^2 - q^2)\,,$$

or

$$\frac{d^2 - p^2}{d^2(d^2 - p^2 - q^2)}\,x^2 - \frac{2pq}{d^2(d^2 - p^2 - q^2)}\,xy + \frac{d^2 - q^2}{d^2(d^2 - p^2 - q^2)}\,y^2 \;=\; 1\,.$$

Let a, b, and c be the coefficients, so that

$$a \;=\; \frac{d^2 - p^2}{d^2(d^2 - p^2 - q^2)}\,, \qquad b \;=\; -\frac{2pq}{d^2(d^2 - p^2 - q^2)}\,, \qquad \text{and}$$

$$c \;=\; \frac{d^2 - q^2}{d^2(d^2 - p^2 - q^2)}\,.$$

Thus, the equation of the ellipse becomes $ax^2 + bxy + cy^2 = 1$. Now we must try to link d, p, and q to z_1, z_2, and z_3 through a, b, and c.

Let M_1, M_2, and M_3 be the mid-points of the sides BC, AC, and AB, respectively. We know that ω passes through these three points. Let N_1 be the mid-point of AG. Since the centroid G divides the median AM_1 into the ratio $2:1$, N_1 is the reflection of M_1 through G, which is also the origin, so that N_1 also lies on ω. This similarly holds for N_2 and N_3, the mid-points of BG and CG, respectively (see Figure 5).

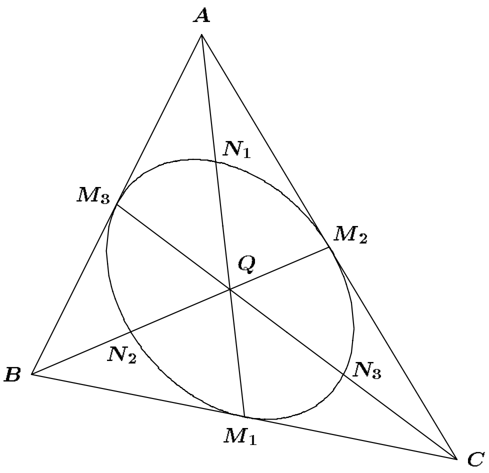

Figure 5.

Since N_1 is the mid-point of AG, $N_1 = x_1/2 + y_{1/2}i$, etc. Therefore,

$$a\,\frac{x_1^2}{4} + b\,\frac{x_1y_1}{4} + c\,\frac{y_1^2}{4} \;=\; 1\,, \qquad a\,\frac{x_2^2}{4} + b\,\frac{x_2y_2}{4} + c\,\frac{y_2^2}{4} \;=\; 1\,,$$

$$\text{and} \qquad a\,\frac{x_3^2}{4} + b\,\frac{x_3y_3}{4} + c\,\frac{y_3^2}{4} \;=\; 1\,,$$

or

$$ax_1^2 + bx_1y_1 + cy_1^2 \;=\; 4\,, \qquad ax_2^2 + bx_2y_2 + cy_2^2 \;=\; 4\,,$$

$$\text{and} \qquad ax_3^2 + bx_3y_3 + cy_3^2 \;=\; 4\,.$$

Let D be the determinant

$$\begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 \\ x_2^2 & x_2y_2 & y_2^2 \\ x_3^2 & x_3y_3 & y_3^2 \end{vmatrix}\,,$$

and for $i = 1$, 2, and 3, let D_i be the determinant with the i^{th} column of D replaced by 4s. Then by Cramer's Rule, $a = D_1/D$, $b = D_2/D$, and $c = D_3/D$.

We can simplify the determinants by noticing that D factors as

$$D \;=\; (x_1y_3 - x_3y_1)(x_2y_1 - x_1y_2)(x_3y_2 - x_2y_3)\,,$$

and that

$$\begin{aligned} x_1y_3 - x_3y_1 &= x_1(-y_1 - y_2) - (-x_1 - x_2)y_1 \\ &= -x_1y_1 - x_1y_2 + x_1y_1 + x_2y_1 \\ &= x_2y_1 - x_1y_2\,, \end{aligned}$$

which can also be shown to be equal to $x_3y_2 - x_2y_3$. Thus, the three factors of D are equal. Let k denote this common value, so that $D = k^3$.

Similarly, we can show that

$$\begin{aligned} D_1 &= 2k(y_1^2 + y_2^2 + y_3^2)\,, \\ D_2 &= -4k(x_1y_1 + x_2y_2 + x_3y_3)\,, \\ D_3 &= 2k(x_1^2 + x_2^2 + x_3^2)\,. \end{aligned}$$

We are now ready to pull all our equations together. Expanding $p(z)$, keeping in mind that $z_1 + z_2 + z_3 = 0$,

$$\begin{aligned} p(z) &= (z - z_1)(z - z_2)(z - z_3) \\ &= z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_1z_3 + z_2z_3)z - z_1z_2z_3 \\ &= z^3 + (z_1z_2 + z_1z_3 + z_2z_3)z - z_1z_2z_3\,. \end{aligned}$$

Thus,

$$p'(z) = 3z^2 + (z_1z_2 + z_1z_3 + z_2z_3)\,.$$

Since $z_1 + z_2 + z_3 = 0$, $z_1^2 + z_2^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 + 2z_2z_3 = 0$, so that

$$\begin{aligned} p'(z) &= 3z^2 - (z_1^2 + z_2^2 + z_3^2)/2 \\ &= 3z^2 + ((-x_1^2 - x_2^2 - x_3^2 + y_1^2 + y_2^2 + y_3^2)/2 \\ &\qquad +(-x_1y_1 - x_2y_2 - x_3y_3)i)\,. \end{aligned}$$

To prove that $F_1 = p + qi$ and $F_2 = -p - qi$ are the roots of this polynomial, we must show that $F_1 + F_2 = 0$, which we already know, and that

$$\begin{aligned} F_1F_2 &= (p + qi)(-p - qi) \;=\; -p^2 + q^2 - 2pqi \\ &= (-x_1^2 - x_2^2 - x_3^2 + y_1^2 + y_2^2 + y_3^2)/6 \\ &\qquad + i\left((-x_1y_1 - x_2y_2 - x_3y_3)/3\right)\,. \end{aligned}$$

Thus, we do not need to solve for p and q, but only for $-p^2 + q^2$ and $-2pq$. From our expressions for a, b, and c,

$$\begin{aligned} b^2 - 4ac &= \frac{4p^2q^2 - 4(d^2 - p^2)(d^2 - q^2)}{d^4(d^2 - p^2 - q^2)^2} \\ &= -\frac{4d^2(d^2 - p^2 - q^2)}{d^4(d^2 - p^2 - q^2)^2} \;=\; -\frac{4}{d^2(d^2 - p^2 - q^2)}\,. \end{aligned}$$

Thus,

$$\begin{aligned} d^2 - p^2 &= ad^2(d^2 - p^2 - q^2) \;=\; -\frac{4a}{b^2 - 4ac}\,, \\ -2pq &= -\frac{4b}{b^2 - 4ac}\,, \quad \text{and} \\ d^2 - q^2 &= -\frac{4c}{b^2 - 4ac}\,, \end{aligned}$$

which implies that

$$\begin{aligned}
&-p^2+q^2\\
&= \frac{4(c-a)}{b^2-4ac} = \frac{4D(D_3-D_1)}{D_2^2-4D_1D_3}\\
&= \frac{4k^3\cdot 2k(x_1^2+x_2^2+x_3^2-y_1^2-y_2^2-y_3^2)}{16k^2(x_1y_1+x_2y_2+x_3y_3)^2-16k^2(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)}\\
&= -\frac{8k^4(x_1^2+x_2^2+x_3^2-y_1^2-y_2^2-y_3^2)}{16k^2[(x_1y_3-x_3y_1)^2+(x_2y_1-x_1y_2)^2+(x_3y_2-x_2y_3)^2]}\\
&= -\frac{8k^4(x_1^2+x_2^2+x_3^2-y_1^2-y_2^2-y_3^2)}{16k^2\cdot 3k^2}\\
&= \tfrac{1}{6}\left(-x_1^2-x_2^2-x_3^2+y_1^2+y_2^2+y_3^2\right),
\end{aligned}$$

and

$$\begin{aligned}
-2pq &= -\frac{4b}{b^2-4ac} = -\frac{4DD_2}{D_2^2-4D_1D_3}\\
&= -\frac{4k^3\cdot 4k(x_1y_1+x_2y_2+x_3y_3)}{48k^4}\\
&= \tfrac{1}{3}\left(-x_1y_1-x_2y_2-x_3y_3\right),
\end{aligned}$$

as required. Finally, $p''(z)=6z$, whose root is 0, and which is indeed the centre of ω. ■

As one can see, much of this proof is simply an algebraic tour de force. Can anyone supply a shorter, or more insightful proof?

Problems

1. Quadrilateral $ABCD$ is circumscribed about an ellipse, which is tangent to the sides AB, BC, CD, and DA at points P, Q, R, and S, respectively. Show that lines AC, BD, PR, and QS are concurrent.
2. (a) Prove the theorem concerning the locus of the centre of an ellipse inscribed in a given triangle.

   (b) Prove that the ellipse of maximal area inscribed in a given triangle is centred at the centroid of the triangle, and is tangent to each of the sides at their mid-points.

   (c) Triangle ABC is circumscribed about an ellipse, which is tangent to the sides AB, BC, and CA at points P, Q, and R, respectively. Show that lines AP, BQ, and CR are concurrent.

   Hint: This may be reminiscent of Ceva's Theorem, but it is a special case of Brianchon's Theorem.

3. Quadrilateral $ABCD$ is circumscribed about a given circle S, which touches sides AB, BC, CD, and DA at E, F, G, and H, respectively. Let I be the intersection of AC and BD. Let $a = AE = AH$ and $c = CF = CG$. Prove that

$$\frac{AI}{IC} = \frac{a}{c} .$$

(R. Honsberger, *More Mathematical Morsels*, Math. Assoc. Amer., Washington, DC, USA, 1991.)

4. The vertices of a convex pentagon are $(3,5)$, $(4,3)$, $(2,0)$, $(0,1)$, and $(1,4)$. Show that the equation of the unique ellipse inscribed in this pentagon is

$$f(x,y) := 3529x^2 - 2016xy + 2016y^2 - 9004x - 6192y + 9076 = 0 .$$

Jean Collins observed that when you evaluate f at any of the vertices, you always get a perfect square. For example, $f(3,5) = 3025 = 55^2$. Is there a good reason for this?

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1. [Ch] Chakerian G.D., "A Distorted View of Geometry", *Mathematical Plums*, Math. Assoc. Amer., Washington, DC, USA, 1979.

2. [Co] Coxeter H.S.M. and S.L. Greitzer, *Geometry Revisited*, Math. Assoc. Amer., Washington, DC, USA, 1967.

3. [D] Dorrie H., *100 Great Problems of Elementary Mathematics: Their History and Solutions*, Dover, New York, USA, 1965.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2539. Correction. [The editor missed a subtle error in the proposer's proof. Thanks to Peter Y. Woo, Biola University, La Mirada, CA, USA for pointing out that the result could not be true.] *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea, adapted by the editor.*

Let $ABCD$ be a convex quadrilateral with vertices oriented in the clockwise sense. Let X and Y be interior points on AD and BC, respectively. Suppose that P is a point between X and Y such that $\angle AXP = \angle BYP = \angle APB = \theta$ and $\angle CPD = \pi - \theta$ for some θ.

(a) Prove that $AD \cdot BC \geq 4\,PX \cdot PY$.

(b)$\star$ Find the case(s) of equality.

2563. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

You are given that angle x satisfies the equation $a\sin x + b\cos x = c$.

(a) If a, b and c are real numbers, calculate angle x.

(b) Considering a, b and c as line segments, find a straight edge and compass construction for angle x.

2564. *Proposed by Darko Veljan, University of Zagreb, Zagreb, Croatia.*

(a) Find all integer solutions (a, b, c) of the equation $\binom{a}{2} + \binom{b}{2} = \binom{c}{2}$, such that $2 \leq a \leq b \leq c$.

(b) For each integer $n \geq 1$, find at least one integer solution (a, b, c) $(n \leq a \leq b \leq c)$ of the equation $\binom{a}{n} + \binom{b}{n} = \binom{c}{n}$.

(c) For $n = 3$, find at least one further solution for (b).

2565. *Proposed by K.R.S. Sastry, Dodballapur, India.*

A *Heron* triangle has integer sides and integer area. Show that there are exactly three pairs of Heron Triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that $B_1C_1 = B_2C_2$, $A_1C_1 = A_2C_2$, $\angle A_1B_1C_1 = \angle A_2B_2C_2$ and $A_2B_2 - A_1B_1 = 10$.

2566. *Proposed by K.R.S. Sastry, Dodballapur, India.*

Suppose that each of the three quadratics ax^2+bx+c, $ax^2+bx+(c+d)$ and $ax^2 + bx + (c + 2d)$ factors over the integers. Let $S = ad > 0$. Show that S represents the area of some Pythagorean triangle (integer sided right triangle).

2567. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In triangle ABC, points P and Q are on the line segment BC such that AP and AQ are trisectors of $\angle BAC$ and $BQ = QC$. If $AC = \sqrt{2}\,AQ$, find the measure of $\angle BAC$.

2568. *Proposed by K.R.S. Sastry, Dodballapur, India.*

The sides a, b and c of a non-degenerate triangle ABC satisfy the relations $b^2 = ca + a^2$ and $c^2 = ab + b^2$. Find the measures of the angles of triangle ABC.

2569. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Suppose that a, b, c and d, e, f are real numbers satisfying

1. the pairwise sums of a, b, c are (in some order) d, e and f; and
2. the pairwise products of d, e, f are (in some order) a, b and c.

Find all possible values of $a + b + c$.

2570. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let C be a conic with focus F and directrix d. Let A and B be the points of intersection of the conic with a line through the focus F. Let I, J and K be the feet of the perpendiculars from A, F and B to d, respectively.

Prove that the length of FJ is the harmonic mean of the lengths of AI and BK.

2571. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Suppose that a, b and c are the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \geq \frac{3(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a+b+c}\,.$$

2572. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\leq\; \left(\frac{a+b+c}{3}\right)^{a+b+c}\,.$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

2573. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let H be the orthocentre of triangle ABC. For a point P not on the circumcircle of triangle ABC, denote by X, Y, Z the reflections of P in the sides BC, CA, and AB, respectively. Show that the areas of triangles HYZ, HZX, and HXY are in constant proportions.

2574. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Let P be a point in the interior of triangle ABC, whose centroid is G. Extend AP to a point X such that PX is bisected by the line BC. Similarly, extend BP to Y and CP to Z such that PY and PZ are each bisected by CA and AB, respectively. Show that the 6 points A, B, C, X, Y, Z, lie on a conic, and that the centre of the conic is the point Q dividing PG externally in the ratio $PQ : QG = 3 : -1$.

2575. *Proposed by H. Fukagawa, Kani, Gifu, Japan.*

Suppose that $\triangle ABC$ has a right angle at C. The circle, centre A and radius AC meets the hypotenuse AB at D. In the region bounded by the arc DC and the line segments BC and BD, draw a square $EFGH$ of side y, where E lies on arc DC, F lies on DB and G and H lie on BC. Assume that BC is constant and that $AC = x$ is variable.

Find $\max y$ and the corresponding value of x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2460. [1999 : 308] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Let $y(x) = \sqrt[3]{x+\sqrt{x^2-1}} + \sqrt[3]{x-\sqrt{x^2-1}}$ for $0 \le x \le 1$.

(a) Show that $y(x)$ is real-valued.

(b) Find an infinite sequence $\{x_n\}_{n=0}^{\infty}$ such that $y(x_n)$ can be expressed in terms of square roots only.

I. *Solution by David Doster, Choate Rosemary Hall, Wallingford, CT, USA.*

(a) We write $\sqrt{x^2-1} = i\sqrt{1-x^2}$ and assume that $z \mapsto \sqrt[3]{z}$ defines the branch of the cube root function with $-\pi < \arg z \le \pi$. Let $\theta = \cos^{-1} x$. Then $\sqrt{1-x^2} = \sin\theta$, so that

$$x + i\sqrt{1-x^2} = e^{i\theta}, \qquad x - i\sqrt{1-x^2} = e^{-i\theta}.$$

Hence,

$$y(x) = e^{i\theta/3} + e^{-i\theta/3} = 2\cos\frac{\theta}{3}.$$

This shows that $y(x)$ is real.

(b) Let $x_n = \cos\frac{3\pi}{2^{n+3}}$, $n = 0, 1, 2, \ldots$. (The largest angle, $3\pi/8$, is chosen to be less than $\pi/2$ as required.) Then $y(x_n) = 2\cos\frac{\pi}{2^{n+3}}$. From the identity $\cos 2\theta = 2\cos^2\theta - 1$, it follows that $y(x_{n+1}) = \sqrt{2 + y(x_n)}$, and we know that $y(x_0) = 2\cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$. Thus, $y(x_n)$ can be expressed in terms of square roots only.

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Editor's comment. Woo's solution to part (a) is the same as Doster's, so we give only his solution to part (b). Although his solution does not involve square roots, it appears to honour the spirit of the problem.

We shall give an infinite sequence of solutions where both x_n and y_n are rational numbers. For each $m = 0, 1, 2, \ldots$, let $\mu = m + 1$. Then $\{m + \mu, 2m\mu, m^2 + \mu^2\}$ forms a Pythagorean triple.

Let $\sin\theta_m = (m+\mu)/(m^2+\mu^2)$. Then $\cos\theta_m = 2m\mu/(m^2+\mu^2)$. If we let

$$x_m = \cos 3\theta_m = \frac{2m\mu(4m^2\mu^2 - 3(m+\mu)^2)}{(m^2+\mu^2)^3},$$

then

$$y(x_m) \; = \; 2\cos\theta_m \; = \; \frac{4m\mu}{m^2+\mu^2}\,.$$

For $m = 2$, $x_2 = 828/2197$, and $y(x_2) = 24/13$.

For $m = 3$, $x_3 = 10296/15625$, and $y(x_3) = 48/25$.

This shows that $x_2, x_3, \ldots$ is a sequence of (rational) numbers between 0 and 1, converging to 1, for which $y(x_n)$ is rational and converging to 2.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer. PANOS E. TSAOUSSOGLOU, Athens, Greece, had a solution for part (a) *only. There was one incorrect solution.*

2461. [1999 : 308] *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Suppose that x_0, x_1, ..., x_n are integers which satisfy $x_0 > x_1 > \ldots > x_n$. Let

$$F(x) \; = \; \sum_{k=0}^{n} a_k x^{n-k}\,, \qquad a_k \in \mathbb{R}\,,\, a_0 = 1\,.$$

Prove that at least one of the numbers $|F(x_k)|$, $(k = 0, 1, \ldots, n)$ is greater than $\frac{n!}{2^n}$.

Combination of solutions by Michel Bataille, Rouen, France; and Kee-Wai Lau, Hong Kong, China.

Let

$$P(x) \; = \; (x-x_0)(x-x_1)\cdots(x-x_n)\,.$$

The decomposition of the rational function F/P into partial fractions has the form:

$$\frac{F(x)}{P(x)} \; = \; \frac{b_0}{x-x_0} + \frac{b_1}{x-x_1} + \cdots + \frac{b_n}{x-x_n}\,.$$

Coefficient b_k is found by multiplying both sides by $x - x_k$, simplifying the left-hand side, and letting $x = x_k$. This provides

$$b_k \; = \; \frac{F(x_k)}{\prod\limits_{j\neq k}(x_k - x_j)}\,,$$

and thus we have $\quad F(x) \; = \; \sum\limits_{k=0}^{n} F(x_k) \prod\limits_{j\neq k} \dfrac{x-x_j}{x_k-x_j}\,.$

[This result is known as Lagrange Interpolation.]

Since the leading coefficient a_0 of $F(x)$ is 1,

$$\sum_{k=0}^{n} b_k = \sum_{k=0}^{n} \frac{F(x_k)}{\prod\limits_{j\neq k}(x_k - x_j)} = 1\,.$$

From the hypothesis $x_0 > x_1 > \ldots > x_n$, any difference $x_k - x_j$ satisfies $|x_k - x_j| \geq |k - j|$, so that

$$\begin{aligned}\prod_{j\neq k} |x_k - x_j| &= \prod_{j=0}^{k-1}(x_j - x_k)\cdot \prod_{j=k+1}^{n} (x_k - x_j) \\ &\geq \prod_{j=0}^{k-1}(k-j)\cdot \prod_{j=k+1}^{n} (j-k) = k!(n-k)! = \frac{n!}{\binom{n}{k}}\,.\end{aligned}$$

This yields $$|b_k| \leq \frac{|F(x_k)|}{n!}\binom{n}{k}\,.$$

Thus, we have

$$1 = \sum_{k=0}^{n} b_k \leq \sum_{k=0}^{n} |b_k| \leq \frac{\max|F(x_k)|}{n!}\sum_{k=0}^{n}\binom{n}{k} = \frac{2^n}{n!}\max|F(x_k)|\,.$$

Therefore, $\max|F(x_k)| \geq \dfrac{n!}{2^n}$, which implies that at least one of the numbers $|F(x_k)|$ is $\geq n!/2^n$.

Remark. Consider the integers $x_0 = n$, $x_1 = n-1$, $\ldots$, $x_n = 0$, and the polynomial

$$F(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\prod_{j\neq k}(x - x_j)\,.$$

Then F satisfies the hypothesis and $|F(x_k)| = n!/2^n$ for $k = 0, 1, \ldots, n$. Thus, in the statement of the problem, "greater than $n!/2^n$" must be understood as "$\geq n!/2^n$", not as "$> n!/2^n$".

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer, all with the same method.

2462. [1999 : 308] *Proposed by Vedula N. Murty, Visakhapatnam, India.*

If the angles, A, B, C of $\triangle ABC$ satisfy

$$\cos A \sin\frac{A}{2} = \sin\frac{B}{2}\sin\frac{C}{2}\,,$$

prove that $\triangle ABC$ is isosceles.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Replace C by $\pi - (A + B)$ to obtain

$$\cos A \sin \frac{A}{2} = \sin \frac{B}{2} \sin \left(\frac{\pi}{2} - \frac{A+B}{2} \right)$$

or

$$\cos A \sin \frac{A}{2} = \sin \frac{B}{2} \cos \left(\frac{A+B}{2} \right) .$$

The last equality can be rewritten as

$$\sin \left(\frac{A}{2} + A \right) + \sin \left(\frac{A}{2} - A \right) = \sin \left(B + \frac{A}{2} \right) + \sin \left(- \frac{A}{2} \right)$$

or

$$\sin \frac{3A}{2} = \sin \left(B + \frac{A}{2} \right) .$$

Hence,

$$\frac{3A}{2} = B + \frac{A}{2} \quad \text{or} \quad \frac{3A}{2} + B + \frac{A}{2} = \pi .$$

The first case leads to $A = B$ and the second to $A = C$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; IAN GARCES, Ateneo de Manila University, Manila, Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria (two solutions); MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; GERRY LEVERSHA, St. Paul's School, London, England; MICHAEL LIAW, student, Biola University, La Mirada, CA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; AMRITPREET SINGH, student, Angelo State University, San Angelo, TX, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; ARAM TANGBOONDOUANGJIT, Carnegie Mellon University, Pittsburgh, PA, USA; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

2463*. [1999 : 366] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Suppose that $k \in \mathbb{Z}$. Prove or disprove that

$$\left(\tan\left(\frac{3n\pi}{11}\right) + 4\sin\left(\frac{2n\pi}{11}\right)\right)^2 = \left(\tan\left(\frac{5n\pi}{11}\right) - 4\sin\left(\frac{4n\pi}{11}\right)\right)^2 = \begin{cases} 11 & \text{for} \quad n \neq 11k\,, \\ \\ 0 & \text{for} \quad n = 11k\,. \end{cases}$$

Solution by Charles Diminnie and Larry White, San Angelo, TX, USA.

The statement is easily verified when $n = 11k$. Assume that $n \neq 11k$ and let

$$R = \cos\left(\frac{2n\pi}{11}\right) + i\sin\left(\frac{2n\pi}{11}\right)\,.$$

Since $R^{11} = 1$, we have

$$\sum_{j=0}^{10} R^j = \frac{R^{11}-1}{R-1} = 0\,,$$

and hence,

$$\sum_{j=1}^{10} R^j = -1\,.$$

Using DeMoivre's Theorem and the fact that

$$\cos\left(\frac{2(11-j)n\pi}{11}\right) = \cos\left(\frac{2jn\pi}{11}\right)$$

for all $j \geq 0$, we get

$$\begin{aligned} -1 &= \mathfrak{Re}\left(\sum_{j=1}^{10} R^j\right) = \sum_{j=1}^{10} \cos\left(\frac{2jn\pi}{11}\right) \\ &= \sum_{j=1}^{5} \cos\left(\frac{2jn\pi}{11}\right) + \sum_{j=6}^{10} \cos\left(\frac{2(11-j)n\pi}{11}\right) \\ &= \sum_{j=1}^{5} \cos\left(\frac{2jn\pi}{11}\right) + \sum_{j=1}^{5} \cos\left(\frac{2jn\pi}{11}\right) = 2\sum_{j=1}^{5} \cos\left(\frac{2jn\pi}{11}\right)\,. \end{aligned}$$

We will also use the identities:
$\sin^2(x) = \frac{1}{2}\left(1-\cos(2x)\right)$, $\cos^2(x) = \frac{1}{2}\left(1+\cos(2x)\right)$,
$\sin(2x) = 2\sin(x)\cos(x)$, $\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y)+\cos(x-y)\right)$,
$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y)-\cos(x+y)\right)$.

Then
$$\begin{aligned}
&\left[\sin\left(\frac{3n\pi}{11}\right) + 4\sin\left(\frac{2n\pi}{11}\right)\cos\left(\frac{3n\pi}{11}\right)\right]^2 \\
= \;& \sin^2\left(\frac{3n\pi}{11}\right) + 8\sin\left(\frac{2n\pi}{11}\right)\cos\left(\frac{3n\pi}{11}\right)\sin\left(\frac{3n\pi}{11}\right) \\
& \qquad +16\sin^2\left(\frac{2n\pi}{11}\right)\cos^2\left(\frac{3n\pi}{11}\right) \\
= \;& \frac{1-\cos\left(\frac{6n\pi}{11}\right)}{2} + 4\sin\left(\frac{2n\pi}{11}\right)\sin\left(\frac{6n\pi}{11}\right) \\
& \qquad +4\left[1-\cos\left(\frac{4n\pi}{11}\right)\right]\left[1+\cos\left(\frac{6n\pi}{11}\right)\right] \\
= \;& \frac{9}{2} + \frac{7}{2}\cos\left(\frac{6n\pi}{11}\right) - 2\cos\left(\frac{4n\pi}{11}\right) - 2\cos\left(\frac{8n\pi}{11}\right) \\
& \qquad -4\cos\left(\frac{4n\pi}{11}\right)\cos\left(\frac{6n\pi}{11}\right) \\
= \;& \frac{9}{2} + \frac{7}{2}\cos\left(\frac{6n\pi}{11}\right) - 2\cos\left(\frac{4n\pi}{11}\right) - 2\cos\left(\frac{8n\pi}{11}\right) \\
& \qquad -2\cos\left(\frac{10n\pi}{11}\right) - 2\cos\left(\frac{2n\pi}{11}\right) \\
= \;& \frac{11}{2}\left(1+\cos\left(\frac{6n\pi}{11}\right)\right) - 2\sum_{j=1}^{5}\cos\left(\frac{2jn\pi}{11}\right) - 1 \\
= \;& 11\cos^2\left(\frac{3n\pi}{11}\right)\,,
\end{aligned}$$

and the first identity is proved.

Editor's Comment. Diminnie and White provided a similar argument to verify the second identity. Instead, one could rename the parameter in the first identity (call it m), then use $\tan(\pi - x) = -\tan x$ and let $-m = 2n$. Specifically, for $n \neq 11k$,

$$\begin{aligned}
\left(\tan\left(\frac{5n\pi}{11}\right) - 4\sin\left(\frac{4n\pi}{11}\right)\right)^2 &= \left(-\tan\left(\frac{6n\pi}{11}\right) - 4\sin\left(\frac{4n\pi}{11}\right)\right)^2 \\
&= \left(\tan\left(\frac{3m\pi}{11}\right) + 4\sin\left(\frac{2m\pi}{11}\right)\right)^2 \;=\; 11\,,
\end{aligned}$$

and the proof is complete.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions).

Alternative approaches are suggested by the two featured solutions to **2378** [1999 : 513] *and the references provided there.*

2465*. [1999 : 367] *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, NY, USA.*

For $n \geq 1$, prove that

$$\sum_{i=0}^{n-1} \binom{n}{i} \sum_{j=0}^{n-1-i} \binom{n-1}{j} \;=\; 4^{n-1}\,.$$

I. *Solution by Choongyup Sung, Pusan High School, Pusan, Korea.*

Let S_n denote the given summation. Then

$$\begin{aligned} S_n &= \sum_{\substack{i+j\leq n-1\\ i,j\geq 0}} \binom{n}{i}\binom{n-1}{j} = \sum_{k=0}^{n-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} \binom{n}{i}\binom{n-1}{j} \\ &= \sum_{k=0}^{n-1} \left(\text{the coefficient of } x^k \text{ in } (x+1)^n(x+1)^{n-1}\right) \\ &= \sum_{k=0}^{n-1} \left(\text{the coefficient of } x^k \text{ in } (x+1)^{2n-1}\right) \\ &= \sum_{k=0}^{n-1} \binom{2n-1}{k} = \frac{1}{2}\left(\sum_{k=0}^{n-1}\binom{2n-1}{k} + \sum_{k=0}^{n-1}\binom{2n-1}{2n-1-k}\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{n-1}\binom{2n-1}{k} + \sum_{k=n}^{2n-1}\binom{2n-1}{k}\right) \\ &= \frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k} = \frac{1}{2}\cdot 2^{2n-1} = 4^{n-1}\,. \end{aligned}$$

II. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA (slightly modified by the editor).*

Let S denote the sum on the left side of the identity. Let b_i denote $\binom{n}{i}$ and c_j denote $\binom{n-1}{j}$ where $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n-1$. Consider the $(n+1)\times n$ matrix $M = (a_{ij})$ where $a_{ij} = b_i c_j$. Note that each summand in S is some a_{ij} with $i+j \leq n-1$. Since $a_{ij} = a_{n-i,n-1-j}$ and $n-i+n-1-j = 2n-1-(i+j) \geq n$ we see that the $n(n+1)/2$ summands in S constitute exactly half of all the entries of M and furthermore, S equals half of the sum of all the entries of M, which is

$$\frac{1}{2}(b_0+b_1+\cdots+b_n)(c_0+c_1+\cdots+c_{n-1}) = \frac{1}{2}\left(2^n\right)\left(2^{n-1}\right) \;=\; 4^{n-1}\,.$$

Also solved by The ANGELO STATE PROBLEM GROUP, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ÓSCAR CIAURRI, Departamento de Mathemáticas y Computación, Universidad de La Rioja, Logroño, Spain, CON AMORE PROBLEM GROUP, Royal Danish School *of Educational Studies, Copenhagen, Denmark; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; (2 solutions), KEE-WAI LAU, Hong Kong, China; JOSÉ H. NIETO, Universidad del Zulia, Maracáibo, Venezuela, HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; KENNETH M. WILKE, Topeka, KS, USA; and JEREMY YOUNG, student, University of Cambridge, Cambridge, UK.*

Most of the 23 submitted solutions are variants of either I *or* II *above. Several of them quoted the well-known and easy-to-prove Vandermonde Identity* $\sum\limits_{k=0}^{l} \binom{m}{k}\binom{n}{l-k} = \binom{m+n}{l}$. *A proof of this is actually contained in Solution* I *above.*

2466. [1999 : 367] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given a circle (but not its centre) and two of its arcs, AB and CD, and their mid-points M and N (which do not coincide and are not the end points of a diameter), prove that all the unmarked straightedge and compass constructions that can be carried out in the plane of the circle can also be done with an unmarked straightedge alone.

Solution by Toshio Seimiya, Kawasaki, Japan.

Let the circle be Γ. If we can find the centre of Γ using only an unmarked straightedge, then we can solve all the problems of Euclidean construction using only an unmarked straightedge by the Poncelet-Steiner Construction Theorem. (See H. Eves, A Survey of Geometry, Vol 1, p. 209.) Thus our aim is to find the centre of Γ.

The construction is as follows. Let E and F be the intersections of AN, BM and AM, BN, respectively. Also, let E' and F' be the intersections of CN, DM and CM, DN, respectively. Next, let T be the intersection of EF and $E'F'$. (See Figure 1.) For K a point on the line MT, let X and Y be the intersections of AM, BK and AK, BM, respectively. Call M' the intersection of XY with AB. For H a point on the line TN, let X' and Y' be the intersections of CN, DH and CH, DN, respectively. Let N' be the intersection of $X'Y'$ with CD, and O the intersection of MM' and NN'. (See Figure 2.) Then O is the centre of Γ.

Proof. Let P and Q be the intersections of MN with AB and CD, respectively. Since MN is not a diameter, P and Q are not the centre of Γ. We assume that P and Q are different points. Since E and F are intersections of AN, BM and AM, BN, respectively, and P is the intersection of AB and MN, then EF is the polar of P. Similarly, $E'F'$ is the polar of Q. As T is the intersection of EF and $E'F'$, so T is the pole of PQ. That is, T is the pole of MN. Consequently, MN is the polar of T. It follows that TM and TN are tangents to Γ. Since M and N are mid-points of arc AB and arc CD, then

$$MT \parallel AB \text{ and } NT \parallel CD\,.$$

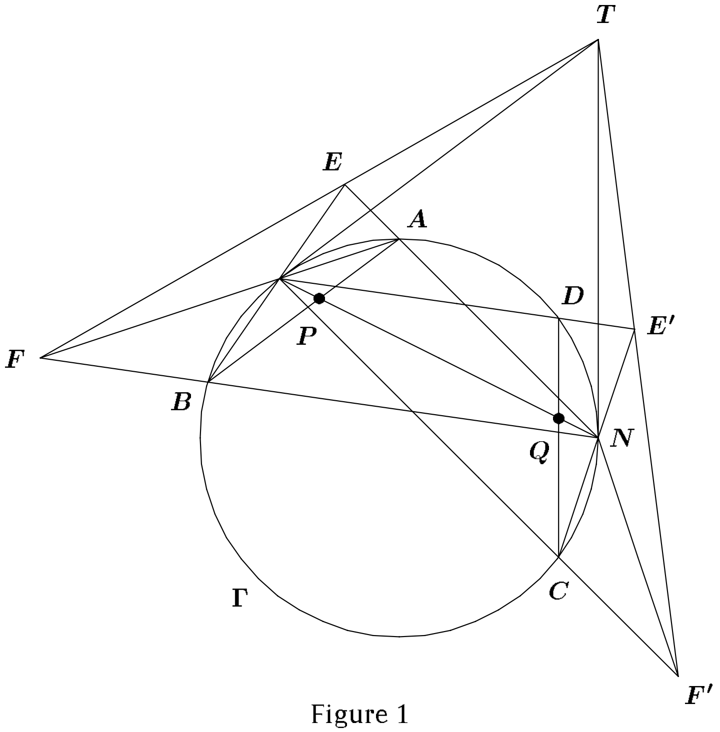

Figure 1

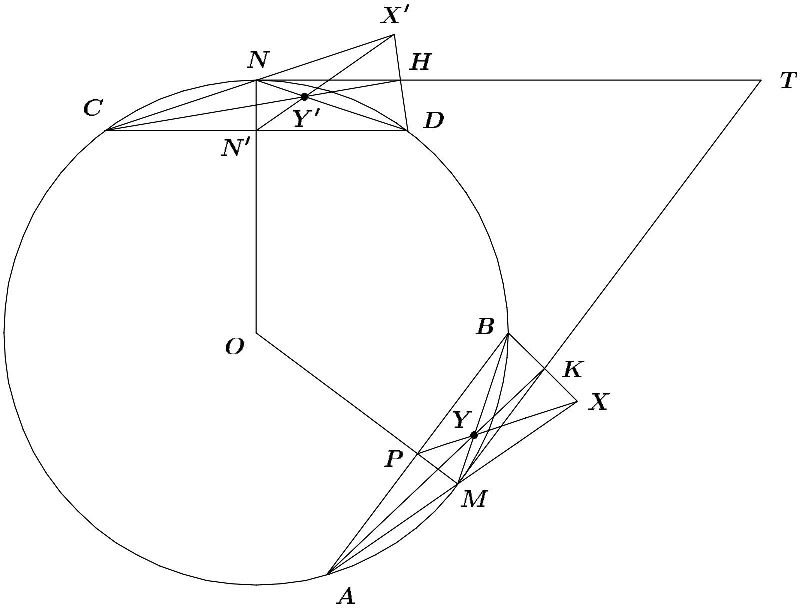

Figure 2

Since $MK \parallel AB$, we have $\dfrac{XM}{MA} = \dfrac{XK}{KB}$.

By Ceva's Theorem, we have $\dfrac{AM'}{M'B} \cdot \dfrac{BK}{KX} \cdot \dfrac{XM}{MA} = 1$,

from which we get $\dfrac{AM'}{M'B} = 1$; that is, $AM' = M'B$.

Hence, MM' is the perpendicular bisector of AB. Similarly, NN' is the perpendicular bisector of CD. Since O is the intersection of MM' and NN', O is the centre of Γ, as claimed.

Also solved by MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer.

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