Crux Mathematicorum

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Editorial Board

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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YEAR-END FINALE

I am always a little intimidated when writing a year-end piece. It is somewhat reminiscent of a thesis defence: all the materials are finalized, submitted and even reviewed, yet it is this official culmination of the entire process that packs the most significance and emotions. You are supposed to summarize, for yourself and others, the results of all the hard work and do so in a very disproportional amount of time and space with respect to how long it actually took to arrive at said results: a 20-minute presentation for a 5-year degree. Or, in my current situation, a one-page year-end summary for a 450-page Volume 41 of ***Crux***. Intimidating. So I will be brief.

This Volume did not quite take a year; in fact, it took just over 7 months, which is great news considering our production backlog that has now been significantly reduced. This is a result of great efforts and even greater commitment on the part of my remarkable Editorial Board : my editors not only do not complain about the workload, but rather welcome the challenge while specifically upholding the high standards set for the publication. To everyone who has contributed to this Volume, my sincere thank you.

The Canadian Mathematical Society office has also been instrumental in allowing us to move forward at this pace : our Managing Editor Denise Charron and her staff have made sure the administrative side of things has always moved along smoothly and most efficiently. It is Denise and our developer Steve La Rocque that I have to thank for the most exciting recent addition to ***Crux*** : the implementation of our new online submission system!

Overall, it has been an eventful year, which is not surprising : ***Crux*** is a very dynamic publication as we draw the majority of the materials from our readers, so every issue is only superficially the same. Furthermore, different editors continuously provide different perspectives and altogether it moulds the publication into the unique problem-solving resource that it is. So, without further ado, here is the last issue of Volume 41 and I am looking forward to all your contributions to Volume 42.

Kseniya Garaschuk

THE CONTEST CORNER

No. 40

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

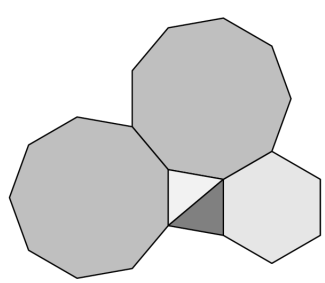
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC196. On dispose de neuf tuiles carrées dont les côtés ont des longueurs respectives de 1, 4, 7, 8, 9, 10, 14, 15 et 18 unités. Ces tuiles sont juxtaposées pour paver la surface d'un rectangle. Déterminer les longueurs des côtés du rectangle et montrer comment les tuiles doivent être placées.

CC197. On choisit au hasard deux entiers a et b, pas nécessairement distincts, parmi les entiers de 1 à 100. Quelle est la probabilité pour que le chiffre des unités du nombre $3^a + 7^b$ soit un 6 ?

CC198. La figure suivante est formée de cinq polygones, soit deux triangles, un hexagone régulier et deux ennéagones réguliers, placés de manière que certains polygones partagent un côté.



Démontrer que chacun des triangles est isocèle.

CC199. Étant donné un nombre réel u, soit $\{u\} = u - \lfloor u \rfloor$ où $\lfloor u \rfloor$ représente le plus grand entier inférieur ou égal à u. ($\{u\}$ est parfois appelé la partie fractionnaire de u.) Par exemple, $\{\pi\} = \pi - 3$ et $\{-2,4\} = -2,4 - (-3) = 0,6$. Déterminer tous les nombres réels x tels que $\{(x+1)^3\} = \{x^3\}$.

CC200. Déterminer tous les entiers positifs m et n tels que $m! + 76 = n^2$. (On rappelle que $m! = m \times (m-1) \times \cdots \times 2 \times 1$.)

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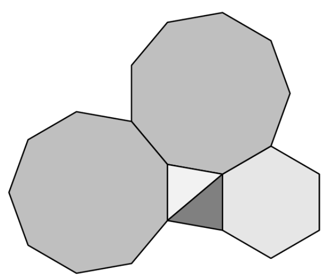
CC196. You are given nine square tiles, with sides of lengths 1, 4, 7, 8, 9, 10, 14, 15 and 18 units, respectively. They can be used to tile a rectangle without gaps or overlaps. Find the lengths of the sides of the rectangle, and show how to arrange the tiles.

CC197. Let a and b be two randomly chosen positive integers (not necessarily distinct) such that $a, b \leq 100$. What is the probability that the units digit of $3^a + 7^b$ is 6 ?

CC198. The diagram shows five polygons placed together edge to edge : two triangles, a regular hexagon and two regular nonagons.



Prove that each of the triangles is isosceles.

CC199. For any real number u, let $\{u\} = u - \lfloor u \rfloor$ denote the fractional part of u (here, $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u). For example, $\{\pi\} = \pi - 3$ and $\{-2.4\} = -2.4 - (-3) = 0.6$. Find all real numbers x such that $\{(x+1)^3\} = \{x^3\}$.

CC200. Find all positive integers m and n such that $m! + 76 = n^2$, where $m! = m \times (m-1) \times \cdots \times 2 \times 1$.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014 : 40(10), p. 413–414. All the problems in this section are from Savin's tournament "Mathematics 6-8", as printed in Kvant 2014 (1).

CC146. Determine the number of integer solutions (x, y) to the equation

$$xy = x + y + 999,999,999.$$

Originally problem #13 by Galperin.

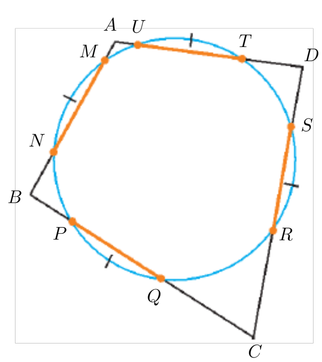
There were five correct solutions for this problem and two incorrect solutions. We present the solution by Joel Schlosberg.

The given equation is equivalent to

$$1,000,000,000 = xy - x - y + 1 = (x - 1)(y - 1),$$

so $x-1 \mid 10^9$ and $y = 10^9/(x-1)+1$. Conversely, if $d \mid 10^9$, $(x, y) = \big(d + 1, 10^9/d + 1\big)$ is a solution. Therefore, the number of integer solutions (x, y) equals the number of integer divisors, *not* necessarily positive, of $10^9 = (2\cdot 5)^9$. By unique factorization, those are of the form $\pm 2^a 5^b$ for $a, b \in \{0, 1, 2, \ldots, 9\}$, so they number $2 \cdot 10^2 = 200$.

CC147. A circle intersects every side of a quadrilateral in such a way that the sides of the quadrilateral cut away equal length arcs.



Show that you can inscribe a circle into this quadrilateral.

Originally problem #1 by Dvoryaninov.

We received three solutions, two of which were correct and complete. We present the solution by Titu Zvonaru.

We clearly have $MN = TU$. By the power of point A with respect to the circle, we have

$$\begin{aligned} AM \cdot AN &= AU \cdot AT \\ AM(AM + MN) &= AU(AU + AT) \\ 0 &= (AM - AU)(AM + AU) + MN(AM - AU) \\ 0 &= (AM - AU)(AM + AU + MN), \end{aligned}$$

and so $AM = AU$. Analogously, $DT = DS$, $CQ = CR$, $BN = BP$.

Therefore, we have

$$\begin{aligned} AB + CD &= MU + RS + AM + BN + DS + CR \\ &= TU + PQ + AU + DT + BQ + CQ \\ &= AD + BC \end{aligned}$$

and hence the quadrilateral has an incircle.

Editor's Comment. There is a simpler and more general solution that proceeds as follows. Let O be the center, R be the radius, and $2A$ the common length of the four chords. A chord of length $2A$ is tangent to a circle centered at O with radius $\sqrt{R^2 - A^2}$, so this is the incircle. This establishes concentricity and generalizes to polygons, simple or crossed, of any number of sides.

CC148. Using cubes of size $1 \times 1 \times 1$, Amanda puts together a rectangular brick of size $6 \times 10 \times 15$. How many little cubes does the main diagonal of the big brick cross ?

Originally problem #9, folklore.

We received no solutions for this problem.

CC149. Find all positive numbers x, y, z such that for any triangle with side lengths a, b, c there exists a triangle with sides ax, by, cz.

Originally problem #8 by Dvoryaninov.

We received only one correct solution. We present the solution by Digby Smith.

There exists triangle ABC with side lengths a, b, c if and only if the following "triangle" inequalities all hold, $a + b > c$, $b + c > a$ and $c + a > b$.

Suppose a, b, c, k are positive real numbers with $a+b > c$, $b+c > a$, and $c+a > b$. It follows that $ka + kb > kc$, $kb + kc > ka$ and $kc + ka > kb$. That is given a triangle with side lengths a, b, c there exists a triangle with side lengths ka, kb, kc for all positive real numbers k. So if $x = y = z$ we are satisfied.

Let x, y, z be real numbers and suppose without loss of generality that $x > y$. Since $x > y$ it then follows that $x - y > 0$ and there exists $n \in \mathbb{N}$ such that $n(x - y) > z$ with $nx > ny + z$. Let $a = n$, $b = n$ and $c = 1$. There exists a triangle with side lengths a, b, c but there does not exist a triangle with side lengths ax, by, cz.

It follows that only when $x = y = z$ does there exist a triangle with side lengths ax, by, cz for all triangles with side lengths a, b, c.

CC150. Shane writes down all numbers from 1 to 2015 in red and blue pen. The largest blue number is equal to the number of blue numbers; the smallest red number is equal to half the number of red numbers. How many red numbers did Shane write down?

Originally problem #6 by Dvoryaninov.

We received five correct solutions. We present the solution given by Fernando Ballesta Yagüe.

The largest blue number is equal to the number of blue numbers. Therefore, if b is the number of blue numbers, then b is the largest blue number, and every number smaller than it is also blue (because, if not, then there would not be b blue numbers with b being the largest one). Therefore, the rest of numbers, that is, from $b+1$ to 2015 are red.

$$\overbrace{1, 2, \ldots, b}^{b}, \overbrace{b+1, b+2, \ldots, 2015}^{r}$$

So the smallest red number is $b+1$. Call the number of red numbers $r = 2015 - b$. Then, as the smallest red number is equal to half the number of red numbers, we can deduce :

$$b + 1 = \tfrac{r}{2} \rightarrow b + 1 = \tfrac{2015-b}{2} \rightarrow 2b + 2 = 2015 - b \rightarrow$$
$$3b = 2013 \rightarrow b = 671 \rightarrow r = 2015 - 671 = 1344$$

There are 1344 red numbers.

ANTI-MAGIC SQUARES

Proposed by Zhang Zaiming, Yuxi Normal University, Yunnan, China.

An $n \times n$ square whose cells are filled with consecutive distinct numbers $1, 2, \ldots n^2$ is called *anti-magic* if the sums of its rows, its columns and all of its diagonals (including the broken diagonals that wrap around the edges) are all distinct numbers.

Here is one example of a 4×4 anti-magic square with the corresponding sums contained in shaded squares :

32		25	35	34	42
39	1	3	2	6	12
27	7	10	5	4	26
38	13	15	12	14	54
	16	8	11	9	44
	37	36	30	33	

Notice that in the above example, the sums contain eight consecutive numbers, namely 32 to 39 inclusive.

Question. Give an example of a 4×4 anti-magic square containing more than eight consecutive numbers in its sums. How many consecutive numbers can you get ?

UNSOLVED CRUX PROBLEMS

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Chris Fisher published a list of unsolved problems from ***Crux*** *[2010 : 545, 547]. Below is a sample of two of these unsolved problems.*

283$\star$. [1977 : 250 ; 1978 : 115, 195–196]

Proposed by A. W. Goodman, University of South Florida.

The function
$$y = -\frac{2x \ln x}{1 - x^2}$$
is increasing for $0 < x < 1$ and, in fact, y runs from 0 to 1 in this interval. Therefore an inverse function $x = g(y)$ exists. Can this inverse function be expressed in closed form and if so what is it ? If it cannot be expressed in closed form, is there some nice series expression for $g(y)$? The series need not be a power series.

1581$\star$. [1990 : 266 ; 1991 : 308–309]

Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.

If T_1 and T_2 are two triangles with equal circumradii, it is easy to show that if the angles of T_2 *majorize* the angles of T_1, then the area and perimeter of T_2 is not greater than the area and perimeter, respectively, of T_1. (One uses concavity of $\sin x$ and $\log \sin x$ in $(0, \pi)$.) If T_1 and T_2 are two tetrahedra with equal circumradii, and the solid angles of T_2 majorize the solid angles of T_1, is it true that the volume, the surface area, and the total edge length of T_2 are not larger than the corresponding quantities for T_1 ?

Editor's comment. For any n-set $\{a_1, a_2, \ldots, a_n\}$ let the elements be sorted into order as $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$. A set $\{a_1, a_2, \cdots, a_n\}$ *majorizes* $\{b_1, b_2, \cdots, b_n\}$ if for all $m \in \{1, \ldots, n\}$ we have $\sum_{i=m}^{n} a(i) \geq \sum_{i=m}^{n} b(i)$.

THE OLYMPIAD CORNER

No. 338

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC256. Démontrer qu'il existe un nombre infini d'entiers positifs n tels que le plus grand diviseur premier de $n^4 + n^2 + 1$ est égal au plus grand diviseur premier de $(n + 1)^4 + (n + 1)^2 + 1$.

OC257. Soit $ABCD$ un trapèze (i.e. un quadrilatère ayant deux côtés opposés parallèles) tel que $AB < CD$. Supposons que AC et BD se rencontrent en E, puis que AD et BC se rencontrent en F. À partir de ça, construire les parallélogrammes $AEDK$ et $BECL$. Démontrer que EF passe par le mi point du segment KL.

OC258. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que pour tout x, $y \in \mathbb{R}$, la suivante tient

$$f(xf(y) - yf(x)) = f(xy) - xy.$$

OC259. À partir de polynômes $f(x)$ et $g(x)$ déjà inscrits dans une liste, on peut y ajouter les polynômes $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ et $cf(x)$ où c est une constante réelle arbitraire. Or les polynômes $x^3 - 3x^2 + 5$ et $x^2 - 4x$ se trouvent dans la liste. Est-il possible d'inscrire un polynôme non nul de la forme $x^n - 1$ dans un nombre fini d'étapes ?

OC260. Déterminer le maximum de

$$P = \frac{x^3y^4z^3}{(x^4 + y^4)(xy + z^2)^3} + \frac{y^3z^4x^3}{(y^4 + z^4)(yz + x^2)^3} + \frac{z^3x^4y^3}{(z^4 + x^4)(zx + y^2)^3}$$

où x, y et z sont des nombres réels positifs.

. . . .

OC256. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

OC257. Let $ABCD$ be a trapezoid (quadrilateral with one pair of parallel sides) such that $AB < CD$. Suppose that AC and BD meet at E and AD and BC meet at F. Construct the parallelograms $AEDK$ and $BECL$. Prove that EF passes through the midpoint of the segment KL.

OC258. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$

$$f(xf(y) - yf(x)) = f(xy) - xy.$$

OC259. If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of the form $x^n - 1$ after a finite number of steps?

OC260. Find the maximum of

$$P = \frac{x^3y^4z^3}{(x^4 + y^4)(xy + z^2)^3} + \frac{y^3z^4x^3}{(y^4 + z^4)(yz + x^2)^3} + \frac{z^3x^4y^3}{(z^4 + x^4)(zx + y^2)^3}$$

where x, y, z are positive real numbers.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(8), p. 326–327.

OC196. The number 5654 in base b is a power of a prime number. Find b if $b > 6$.

Originally problem 4 from the 2013 Italian Mathematical Olympiad.

We received five correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We must find b such that

$$5b^3 + 6b^2 + 5n + 4 = (b + 1)(5b^2 + b + 4) = p^n$$

where p is prime. By unique factorization, we must have that $b + 1 = p^\alpha$ and $5b^2 + b + 4 = p^\beta$, where $\alpha + \beta = n$. Solving for b in the first equation and substituting in the second gives

$$5(p^\alpha - 1)^2 + (p^\alpha - 1) + 4 = p^\beta$$

Now, $5b^2 + b + 4 > b + 1$, so $\beta > \alpha$. Reducing our equation modulo p^α yields $5 - 1 + 4 = 8 \equiv 0 \mod p^\alpha$. This forces $p^\alpha \mid 8$ and since $b > 6$, our only possibility is $p^\alpha = 8$ giving $b = 7$. In this case,

$$5 \cdot 7^3 + 6 \cdot 7^2 + 5 \cdot 7 + 4 = 2048 = 2^{11}$$

and hence $b = 7$ is the only solution.

OC197. A $n \times n \times n$ cube is constructed using $1 \times 1 \times 1$ cubes, some of them black and others white, such that in each $n \times 1 \times 1$, $1 \times n \times 1$, and $1 \times 1 \times n$ subprism there are exactly two black cubes, and they are separated by an even number of white cubes (possibly 0). Show it is possible to replace half of the black cubes with white cubes such that each $n \times 1 \times 1$, $1 \times n \times 1$ and $1 \times 1 \times n$ subprism contains exactly one black cube.

Originally problem 4 from the 2013 Mexico National Olympiad.

We received no solutions to this problem.

OC198. Determine all positive real M such that for any positive reals a, b, c, at least one of

$$a + \frac{M}{ab}, b + \frac{M}{bc}, c + \frac{M}{ca}$$

is greater than or equal to $1 + M$.

Originally problem 3 from day 1 of the 2013 Indonesia Mathematical Olympiad.

We received three correct submissions. We present the solution by Titu Zvonaru.

Let $0 < \epsilon < 1$ be an arbitrary real number. Let $a = b = c = 1 + \epsilon$. We obtain

$$\begin{aligned} 1 + \epsilon + \frac{M}{(1+\epsilon)^2} &\geq 1 + M \\ \epsilon &\geq \frac{M(\epsilon^2 + 2\epsilon)}{(1+\epsilon)^2} \\ \frac{(1+\epsilon)^2}{2+\epsilon} &\geq M \end{aligned}$$

Similarly, for $a = b = c = 1 - \epsilon$, it follows that

$$\frac{(1-\epsilon)^2}{2-\epsilon} \leq M.$$

As ϵ tends to 0, the above two inequalities imply that we must necessarily have $M = 1/2$ as the only valid solution.

In order to prove that $M = 1/2$ satisfies the statement of the problem, it suffices to show that

$$a + b + c + \frac{1}{2}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 3\left(1 + \frac{1}{2}\right)$$

which is equivalent to showing that

$$a + b + c \geq \frac{9abc}{2abc + 1}.$$

By the AM-GM inequality, we have $a + b + c \geq 3\sqrt[3]{abc}$. Thus, it suffices to show that

$$3\sqrt[3]{abc} \geq \frac{9abc}{2abc + 1}.$$

Isolating shows that

$$2(abc) - 3(abc)^{2/3} + 1 \geq 0$$

Factoring yields that the above is equivalent to

$$(\sqrt[3]{abc} - 1)^2(2\sqrt[3]{abc} + 1) \geq 0$$

which is true for any positive real values of a, b and c.

OC199. Determine all pairs of polynomials f and g with real coefficients such that

$$x^2 \cdot g(x) = f(g(x)).$$

Originally problem 4 from the 2013 South Africa National Olympiad.

We received four correct submissions and two incorrect submissions. We present the solution by Michel Bataille.

We show that the solutions are of three types :

(1) g is the zero polynomial and f is any polynomial divisible by polynomial x.

(2) $g(x) = ax + b$ and $f(x) = \frac{x(x-b)^2}{a^2}$ for some real numbers a, b with $a \neq 0$.

(3) $g(x) = ax^2 + b$ and $f(x) = \frac{x(x-b)}{a}$ for some real numbers a, b with $a \neq 0$.

It is readily checked that in cases $(1), (2), (3)$, the pair (f, g) is a solution.

Conversely, let (f, g) be a solution. If $g(x)$ is the zero polynomial, then $f(0) = 0$, hence f is divisible by x. Otherwise, f is not the zero polynomial either and we will denote by n and k the degrees of f and g, respectively. Since the respective degrees of $x^2 \cdot g(x)$ and $f(g(x))$ are $k+2$ and kn, we must have $k + 2 = kn$, that is, $(n-1)k = 2$. It follows that either $k = 1,\ n = 3$ or $k = n = 2$.

In the former case, let $g(x) = ax + b$ where $a \neq 0$. Then,

$$x^2 g(x) = x^2(ax+b) = f(ax+b) = f(b) + axf_1(b) + q(x)$$

where $q(x)$ is a polynomial divisible by x^2 and $f_1(x)$ is some polynomial. It follows that $f(b) = f_1(b) = 0$. Thus, $f(x)$ is divisible by $(x-b)^2$. Setting

$$f(x) = (x-b)^2(ux+v) \quad \text{and} \quad x^2 g(x) = f(ax+b)$$

now gives $ua^3 = a$ and $(bu+v)a^2 = b$ so that $u = \frac{1}{a^2}$ and $v = 0$ and we conclude that $f(x) = \frac{x(x-b)^2}{a^2}$.

In the latter case, substituting a root (possibly complex) of $g(x)$ for x in the equality $x^2 g(x) = f(g(x))$ we obtain $f(0) = 0$. As a result, $f(x)$ is divisible by x. Setting

$$f(x) = \alpha x^2 + \beta x \quad \text{and} \quad g(x) = ax^2 + bx + c$$

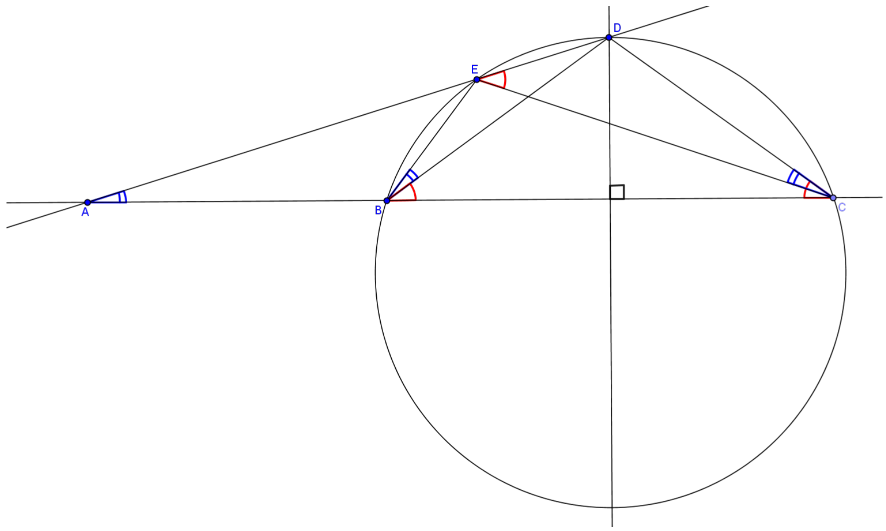
(with $a, \alpha \neq 0$), the equality $x^2 g(x) = f(g(x))$ yields $x^2 = \alpha(ax^2 + bx + c) + \beta$ and we deduce $\alpha = \frac{1}{a},\ b = 0,\ \beta = -\frac{c}{a}$. Therefore $f(x) = \frac{x(x-c)}{a}$ and $g(x) = ax^2 + c$, a pair of type (3).

Editor's Comments. We received four solutions neglecting the zero polynomial case. They were considered correct though only the solution given below included the case when $g(x) = 0$.

OC200. Let A, B and C be three points on a line (in this order). For each circle k through the points B and C, let D be one point of intersection of the perpendicular bisector of BC with the circle k. Further, let E be the second point of intersection of the line AD with k. Show that for each circle k, the ratio of lengths $BE : CE$ is the same.

Originally problem 4 from part 1 of the 2013 Austrian Federal Competition For Advanced Students

We received five correct submissions. We present the solution by Andrea Fanchini.



We have that $\angle CBD = \angle CED$ because both are inscribed in the same arc of circle. Then we have that $BD = CD$ so $\angle BCD = \angle CBD$ so $\angle BCD = \angle CED$.

In the same way, also $\angle DBE = \angle DCE$ because both are inscribed in the same arc of circle.

Now, we note that $\triangle ACD$ and $\triangle CED$ have in common the $\angle ADC$, so $\angle CAD = \angle DCE$. Therefore, $\triangle ACD$ and $\triangle CED$ are similar. Also $\triangle ABD$ and $\triangle BED$ having three equal angles are similar.

So, we have

$$\frac{CE}{CD} = \frac{AC}{AD}, \qquad \frac{BE}{BD} = \frac{AB}{AD}$$

but $BD = CD$, then

$$\frac{CE}{AC} = \frac{BE}{AB}, \qquad \Rightarrow \qquad \frac{BE}{CE} = \frac{AB}{AC}$$

therefore the ratio $\frac{BE}{CE}$ is the same for each circle k.

A Mathematical Performance (II)

Hee-Joo Nam, Giavanna Valacco and Ling-Feng Zhu

*Part I of this article was published in **Crux** 41(9), p. 392–396. The three authors participated in 2010* International Mathematics Competition (IMC)*. Below, we present the contest rules as well as the 2010 individual and team contest papers. For further details, see their website* `http://www.imc-official.org/en_US/`

The twelve questions in the individual contest and the five questions in the team contest require answers only. The three problems in the individual contest and the five problems in the team contest require full solutions.

Two hours are allowed for the individual contest, and one hour for the team contest. In the first 10 minutes of the team contest, the four team members examine Part I together, with no writing allowed. During this time, they divide up the questions and problems among themselves in any way they wish, and then go their separate ways. In the next 35 minutes, they write down the answers to the questions and the solutions to the problems in their shares. In the final 15 minutes, the team members reconvene and work on Part II together.

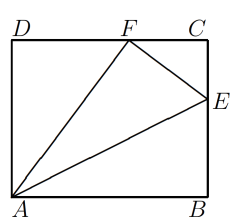
Individual Contest

Question 1.
Let $p,\ q$ and r be real numbers such that $p+q+r = 26$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 31$. What is the value of $\frac{p}{q}+\frac{q}{r}+\frac{r}{p}+\frac{q}{p}+\frac{r}{q}+\frac{p}{r}$?

Question 2.
At a charity dinner, each person consumed half a plate of rice, a third of a plate of vegetables and a quarter of a plate of meat. Overall, 65 plates of food were served. What was the number of people at the charity dinner ?

Question 3.
What is the number of triples (x, y, z) of positive integers which satisfy $xyz = 3^{2010}$ and $x \le y \le z < x + y$?

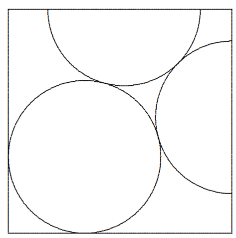
Question 4.
E is a point on the side BC of a rectangle $ABCD$ such that if a fold is made along AE, as shown in the diagram below, the vertex B coincides with a point F on the side CD. If $AD = 16$ cm and $BE = 10$ cm, what is the length, in cm, of AE ?



Question 5.
What is the smallest four-digit number which has exactly 14 positive divisors (including 1 and itself), such that the units digit of one of its prime divisors is 3 ?

Question 6.
Let $f(x)$ be a fourth degree polynomial. What is the remainder when $f(2010)$ is divided by 10 if $f(1) = f(2) = f(3) = 0$, $f(4) = 6$ and $f(5) = 72$?

Question 7.
A circle and two semicircles, all of radius 1 cm, touch one another inside a square, as shown in the diagram below. What is the area, in cm^2 of the square ?



Question 8.
Let $p \geq q$ be prime numbers such that $p^3 + q^3 + 1 = p^2q^2$. What is the maximum value of $p + q$?

Question 9.
The sum of n positive integers, not necessarily distinct, is 100. The sum of any 7 of them is less than 15. What is the minimum value of n ?

Question 10.
P is a point inside triangle ABC such that $\angle ABP = 20°$, $\angle PBC = 10°$, $\angle ACP = 20°$ and $\angle PCB = 30°$. What is the measure, in degrees, of $\angle CAP$?

Question 11.
A farmer puts 100 chickens and 100 pigs into four enclosures in two rows and two columns. There are 120 heads in the first row, 300 legs in the second row, 100 heads in the first column and 320 legs in the second column. How many different ways can this be done ?

Question 12.
An animal shelter consists of five cages in a row, labelled from left to right as shown in the diagram below. There is one animal in each cage.

Red Wolf	Silver Lion	Brown Fox	White Cow	Gray Horse

The five animals are indeed a wolf, a lion, a fox, a cow and a horse, and their colours are indeed red, silver, brown, white and gray. However, none of the labels matches any of the animals. Moreover, no animal is in or next to a cage whose label either matches its type or its colour. If the horse is not in the middle cage, what is the colour of the horse ?

Problem 13.
A segment divides a square into two polygons each of which has an incircle. One of the circles has radius 6 cm while the other one is larger. What is the difference, in cm, between twice the length of this segment and the side length of the square ?

Problem 14.
A small bag of candy contains 6 pieces. A medium bag of candy contains 9 pieces. A large bag of candy contains 20 pieces. If we buy candy in bags only, what is the largest number of pieces which we cannot obtain exactly ?

Problem 15.
For any positive integer n, S_n is the sum of the first n terms of a given sequence $a_1, a_2, a_3, \ldots$ where $a_1 = 2010$. If $S_n = n^2 a_n$ for every n, what is the value of a_{2010} ?

Team Contest

Part I.

Question 1.
Solve the following system of equations for real numbers $w,\ x,\ y$ and z :

$$\begin{aligned} w + 8x + 3y + 5z &= 20; \\ 4w + 7x + 2y + 3z &= -20; \\ 6w + 3x + 8y + 7z &= 20; \\ 7w + 2x + 7y + 3z &= -20. \end{aligned}$$

Problem 2.
In the convex quadrilateral $ABCD$, AB is the shortest side and CD is the longest. Prove that $\angle A > \angle C$ and $\angle B > \angle D$.

Question 3.
Let $m \geq n$ be integers such that $m^3 + n^3 + 1 = 4mn$. Determine the maximum value of $m - n$.

Problem 4.
Arranged in an 8×8 array are 64 dots. The distance between adjacent dots on the same row or column is 1 cm. Determine the number of rectangles of area 12 cm^2 having all four vertices among these 64 dots.

Question 5.
Determine the largest positive integer n such that there exists a unique positive integer k satisfying $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$.

Problem 6.
In each row and each column of a 9×9 table with 81 numbers, at most four different numbers appear. What is the maximum number of different numbers that can appear in this table ?

Question 7.
In the convex quadrilateral $ABCD$, we have $\angle ADB = 16°$, $\angle BDC = 48°$, $\angle ACD = 58°$ and $\angle BCA = 30°$. Determine the measure, in degrees, of $\angle ABD$.

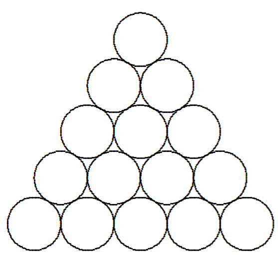
Problem 8.
Determine all ordered triples (x, y, z) of positive real numbers such that each of $x + \frac{1}{y}$, $y + \frac{1}{z}$ and $z + \frac{1}{x}$ is an integer.

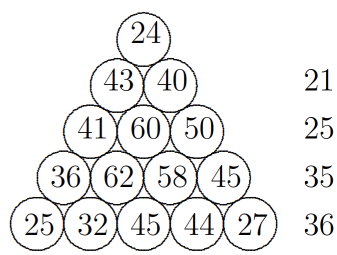
Part II.

Question 9.
Put each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15 into a different one of the fifteen circles in the diagram below on the left, so that

(1) for each circle, the sum of the numbers in it and in all circles touching it is as given by the diagram below on the right;

(2) for each row except the first, the sum of the numbers in the circles on it is as given by the diagram below on the right.





Problem 10.
The letters K, O, R, E, A, I, M and C are written in eight rows, with 1 K in the first row, 2 Os in the second row, and so on, up to 8 Cs in the last row. Starting with the lone K at the top, try to spell the words KOREAIMC by moving from row to row, going to the letter directly below or either of its neighbours, as illustrated by the path in boldface. It turns out that one of these 36 letters may not be used. As a result, the total number of ways of spelling KOREAIMC drops to 516. Determine the letter which may not be used.

K
O O
R **R** R
E **E** E E
A A **A** A A
I I I **I** I I
M M **M** M M M M
C C C **C** C C C C

Markov's Diophantine Equation

M. G. Krein

This article describes the story of one Diophantine equation (that is, a polynomial equation in which only integer variables are allowed). The solution uses only the most elementary properties of integers together with a familiar result for quadratic equations, making it accessible to younger audiences. This solution is based on a series of problems, which can be used to run a math circle.

In 1879 in St. Petersburg's University, a 23-year-old scholar defended his Master's thesis entitled "About Binary Quadratic Forms with Positive Determinant." In this thesis, he solved some of the hardest problems in number theory and this work paved the way for future developments of this mathematical area. That young man was Andrei Andreyevich Markov (1856–1922).

The basis for the thesis were two articles published by Markov in Germany in 1879 and 1880 in one of the most famous mathematical journals : *Mathematische Annalen*. Despite this, over 30 years passed before Markov's results were "discovered" by the western community. In 1913, German mathematician Ferdinand Georg Frobenius (1849–1917) published an article "On Markov's numbers." In its preface, he writes that despite Markov's research being "extremely remarkable and important", it is apparently not well known. Frobenius says it is due to the difficulty of exposition as Markov tended to use *continued fractions*, a technique unfashionable at the time.

In this article, we will discuss a Diophantine equation that arose in Markov's research (an equation that now bears his name) :

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

On Diophantine equations

A *Diophantine equation* is an equation which can be written as a polynomial

$$P(x, y, \ldots, w) = 0$$

with integer unknowns $x, y, \ldots, w$ and with integer coefficients.

Diophantine equations often arise naturally. For example, suppose a country makes stamps of denomination 1, 2, 3 and 5 cents ; in how many ways can you make up a postage of n cents ? In solving this problem, you will arrive at the Diophantine equation

$$x + 2y + 3z + 5w = n.$$

Ancient Babylonian mathematicians were interested in constructing right-angle triangles with integer sides, which is equivalent to finding integer solutions to the equation

$$x^2 + y^2 = z^2.$$

The Pythagoreans found a method to construct all the solutions to this equation. And even though the equation might have been found even earlier in Babylon and India, its integer solutions (x, y, z) are called *Pythagorean triples.*

Finding all (integer) solutions to a Diophantine equation, even an easy-looking one, is generally a hard problem. Even for determining whether or not a Diophantine equation has any solutions, it is known that there is no universal method or algorithm. Nowadays, the field of algebraic geometry has developed various techniques to attack specific Diophantine equations. However, we are interested in Markov's Diophantine equation, which can be solved using elementary techniques, so let us come back to it.

Markov's tree

Consider an ordered triple of integers (a, b, c) that is a solution to the given Diophantine equation $P(x, y, z) = 0$. Integers a, b and c are called the *coordinates* of the solution. It is easy to see that if one of the coordinates of the solution of (1) is equal to zero, then all the coordinates are equal to zero ; for that reason, we will only consider solutions without zero coordinates.

The left-hand side of (1) is positive for any solution (a, b, c), so either all of a, b and c are positive or two of them are negative. In the former case, negating any of the two coordinates results in a new solution ; in the latter case, the triple $(|a|, |b|, |c|)$ is also a solution. Therefore, we will only consider solutions (a, b, c) with all positive coordinates.

The symmetry of equation (1) implies that we can form up to six solutions by permuting the coordinates of any given solution ; that is, if (a, b, c) is a solution, we have the six following solutions (not necessarily distinct) :

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Therefore, we will consider all 6 of the above permutations as one solution to Markov's Diophantine equation. As such, only the values of the solution's coordinates are important and not their order.

The equation has one obvious solution, namely $(1, 1, 1)$. We will now see how we can use one solution to find others. Suppose that (a, b, c) is a solution to the Markov's Diophantine equation ; then a is a root of the following quadratic equation :

$$\Phi_a(x) = x^2 + b^2 + c^2 - 3bcx = 0.$$

Then the well-known result for quadratic equations (sometimes also known as Viète's theorem) implies that there is another root $x = a'$ such that

$$a + a' = 3bc, \quad aa' = b^2 + c^2. \tag{2}$$

Clearly, $a' > 0$ and (a', b, c) is also a solution to (1). It is called the *adjacent solution in coordinate* a. Obviously, if (a', b, c) is an adjacent solution to (a, b, c), then (a, b, c) is an adjacent solution to (a', b, c). Similarly, we can define adjacent solutions in coordinates b and c.

Let us find an adjacent solution of $(1, 1, 1)$ in the first coordinate. For that, we need to solve the quadratic equation

$$x^2 + 1^1 + 1^1 - 3 \cdot 1 \cdot 1 \cdot x = 0.$$

This equation has two roots $x = 1$ and $x = 2$. So from the solution $(1, 1, 1)$, we obtain an adjacent solution $(2, 1, 1)$. Following Markov, we shall call these two solutions *singular*. The following important property distinguishes singular solutions from all other solutions.

Exercise 1. Prove that a solution (a, b, c) of Markov's Diophantine equation has two equal coordinates if and only if it is singular.

The first singular solution $(1, 1, 1)$ has only one adjacent solution. The second singular solution $(2, 1, 1)$ has two adjacent solutions : one is $(1, 1, 1)$ and the second one (adjacent in the second coordinate) comes from the equation

$$2^2 + y^2 + 1^2 - 3 \cdot 2 \cdot y \cdot 1 = 0.$$

and is equal to $(2, 5, 1)$. Furthermore, $(2, 5, 1)$ has three adjacent solutions : the previous $(2, 1, 1)$ and two new ones, $(13, 5, 1)$ and $(2, 5, 29)$. In general, any non-singular solution (a, b, c) gives rise to three adjacent solutions

$$(a', b, c), (a, b', c), (a, b, c'),$$

where (compare to (2))

$$a' = 3bc - a, \quad b' = 3ac - b, \quad c' = 3ab - c.$$

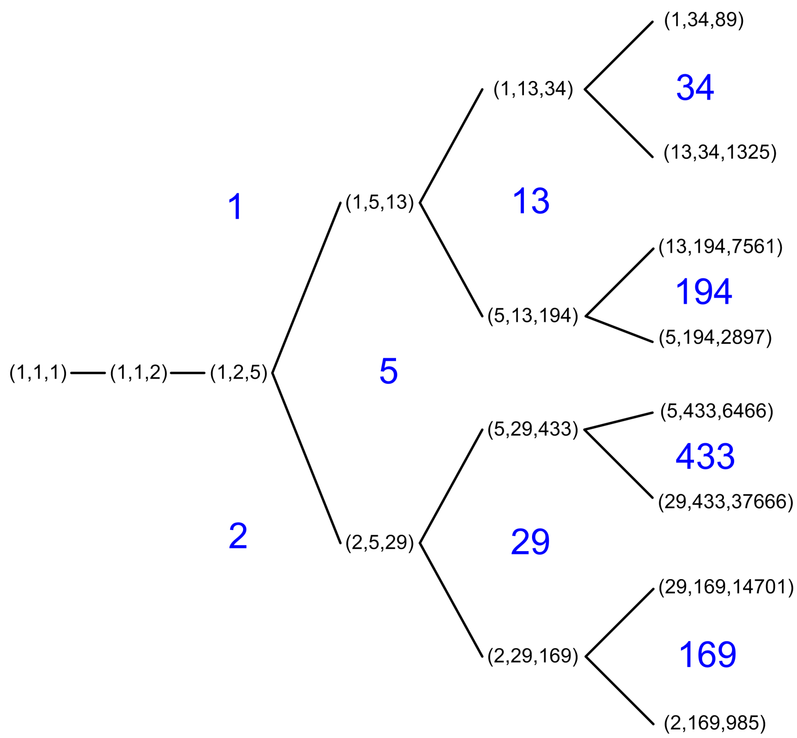
Exercise 2. Suppose that a solution (a, b, c) of Markov's Diophantine equation is non-singular. Prove that the largest coordinate of one of its adjacent solutions is larger than $\max(a, b, c)$ and the largest coordinate of another one of its adjacent solutions is smaller than $\max(a, b, c)$.

Markov's Theorem. Any solution to (1) is connected to the singular solution $(1, 1, 1)$ via a chain of adjacent solutions.

Proof sketch. Let (a, b, c) be a non-singular solution of (1). Use Exercise 2 to deduce that there is a (finite) sequence of adjacent solutions with decreasing maximum coordinate ; hence, we can arrive at a solution (a_n, b_n, c_n) with two equal coordinates. Use Exercise 1. □

Markov's Theorem implies that we can start at a singular solution $(1, 1, 1)$ and expand to adjacent solutions therefore constructing all possible solutions to Markov's Diophantine equation. This process results in *Markov's tree*, which also allows us for any given $N \geq 1$ to find all solutions to Markov's Diophantine equation whose coordinates are less than N. See the picture below indicating adjacent solutions and values of N.

Exercise 3. Prove that the coordinates of any solution to Markov's Diophantine equation are relatively prime.



Exceptionality of Markov's Diophantine equation

Consider the following question :

> If the sum of the squares of three natural numbers is divisible by their product, what can the quotient be ?

This question is equivalent to the following :

> For which natural numbers k does the Diophantine equation
>
> $$X^2 + Y^2 + Z^2 = kXYZ \tag{3}$$
>
> have a non-zero solution ?

For $k = 3$, (3) is actually Markov's Diophantine equation. It is easy to see that for $k = 1$ equation (3) has solutions, for example $(3, 3, 3)$. In fact, Hurwitz and Frobenius proved that equation (3) has solutions only for $k = 1$ and $k = 3$. This result can also be derived using elementary methods.

Let us start with the case $k = 1$. We shall see that solving (3) for $k = 1$ reduces to solving Markov's Diophantine equation.

Exercise 4. Let A, B and C be natural numbers and suppose that d of them are not divisible by 3 $(0 \le d \le 3)$. Prove that the division of $A^2 + B^2 + C^2$ by 3 leaves a remainder of d.

Exercise 5. Prove that all solutions (A, B, C) to the equation

$$X^2 + Y^2 + Z^2 = XYZ \tag{4}$$

are given by

$$A = 3a, \quad B = 3b, \quad C = 3c, \tag{5}$$

where (a, b, c) is a solution to

$$x^2 + y^2 + z^2 = 3xyz. \tag{6}$$

Let us now consider the case $k = 2$.

Exercise 6. Let A, B and C be natural numbers. Prove that the remainder of division of $A^2 + B^2 + C^2$ by 4 is equal to the number of odd integers among A, B and C.

Exercise 7. Prove that equation (3) has no solutions for $k = 2$.

Theorem. Equation (3) has non-zero solutions only for $k = 1$ and $k = 3$.

Proof sketch. Cases $k = 1$ and $k = 2$ are considered above in Exercises 5 and 7. So we have to consider the case $k > 3$. We will prove this case by contradiction; assume that equation (3) has a solution (a, b, c) for some $k > 3$. First, show that the coordinates of the solution are mutually distinct. Next, assume that $a > b > c$ and consider adjacent solutions (a, b, c) and (a', b, c). Using (3), conclude that $a > b > a'$ and hence the largest coordinate of (a', b, c) is less than the largest coordinate of (a, b, c). Proceed similarly to the proof of Markov's theorem by constructing adjacent solutions with smaller and smaller largest coordinates to arrive at a contradiction. □

Corollary. In any solution (a, b, c) to Markov's Diophantine equation all the coordinates are relatively prime.

Proof. Suppose a and b have a common divisor $d > 1$. Then by (1), d also divides c. Hence, we can find natural numbers X, Y and Z such that $a = dX, b = dY, c = dZ$ and $X^2 + Y^2 + Z^2 = 3dXYZ$, which contradicts the above theorem. □

The following equation is an immediate generalization of Markov's Diophantine equation to $n \geq 3$:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = nx_1x_2 \ldots x_n. \tag{7}$$

It is not hard to see that many of the properties of Markov's Diophantine equation proven above also apply to the general case. For example, there exists a singular solution $x_1 = x_2 = \ldots = x_n = 1$; for every solution, there exists an adjacent one; and so on. A general theory of equation (7) constitutes its own research topic.

This article appeared in Russian in Kvant, 1985(4), p. 13–16. It has been translated and adapted with permission. The images are courtesy of Wikipedia.

Math Quotes

Our federal income tax law defines the tax y to be paid in terms of the income x; it does so in a clumsy enough way by pasting several linear functions together, each valid in another interval or bracket of income. An archeologist who, five thousand years from now, shall unearth some of our income tax returns together with relics of engineering works and mathematical books, will probably date them a couple of centuries earlier, certainly before Galileo and Vieta.

Hermann Weyl in "The Mathematical Way of Thinking", an address given at the Bicentennial Conference at the University of Pennsylvania, 1940.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er décembre 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

4091. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Déterminer le plus grand nombre strictement positif k de manière que

$$a+b+c+3k-3 \geq k\left(\sqrt[3]{\frac{b}{a}}+\sqrt[3]{\frac{c}{b}}+\sqrt[3]{\frac{a}{c}}\right)$$

pour tous nombres strictement positifs a, b et c tels que $abc = 1$.

4092. *Proposé par Mihaela Berindeanu.*

Démontrer que

$$\left[\frac{a^2+16a+80}{16\,(a+4)}+\frac{2}{\sqrt{2\,(b^2+16)}}\right]\left[\frac{b^2+16b+80}{16\,(b+4)}+\frac{2}{\sqrt{2\,(a^2+16)}}\right] \geq \frac{9}{4}$$

pour tous réels a et b strictement positifs. Quelles sont les conditions pour qu'il y ait égalité ?

4093. *Proposé par Dragolijub Milošević.*

Soit ABC un triangle quelconque. Soit r le rayon du cercle inscrit dans le triangle et R le rayon du cercle circonscrit au triangle. Soit m_a la longueur de la médiane du sommet A au côté BC et w_a la longueur de la bissectrice de l'angle A jusqu'au côté BC. Les longueurs m_b, m_c, w_b et w_c sont définies de la même façon. Démontrer que

$$\frac{a^2}{m_a w_a}+\frac{b^2}{m_b w_b}+\frac{c^2}{m_c w_c} \leq 4\left(\frac{R}{r}-1\right).$$

4094. *Proposé par Michel Bataille.*

Soit $x_1, x_2, \ldots, x_n$ des réels tels que $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Démontrer que

$$n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \leq \sum_{k=1}^{n}\cosh x_k \leq n-1+\cosh\left(\sum_{k=1}^{n}x_k\right).$$

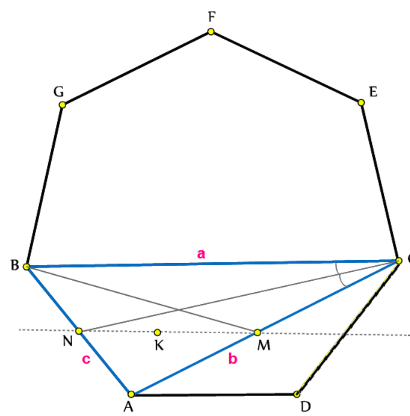
4095. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels positifs tels que $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Démontrer que

$$ab(a+b) + bc(b+c) + ac(a+c) \geq \frac{2}{3}(a^2 + b^2 + c^2) + 4abc.$$

4096. *Proposé par Abdilkadir Altintaş.*

Soit ABC un triangle heptagonal, $BC = a$, $AC = b$ et $AB = c$. Soit CN la bissectrice de l'angle BCA et BM la médiane issue du sommet B, N et M étant des points sur les côtés respectifs AB et AC. Soit K le point de Lemoine (point d'intersection des symédianes) du triangle ABC. Démontrer que les points N, K et M sont alignés.



4097. *Proposé par Leonard Giugiuc.*

Soit a_i des réels, $1 \leq i \leq 6$, tels que

$$\sum_{i=1}^{6} a_i = \frac{15}{2} \quad \text{et} \quad \sum_{i=1}^{6} a_i^2 = \frac{45}{4}.$$

Démontrer que $\prod_{i=1}^{6} a_i \leq \dfrac{5}{2}$.

4098. *Proposé par Ardak Mirzakhmedov.*

Soit α, β et γ des angles aigus tels que $\alpha + \beta = \gamma$. Démontrer que

$$\cos\alpha + \cos\beta + \cos\gamma - 1 \geq 2\sqrt{\cos\alpha \cdot \cos\beta \cdot \cos\gamma}.$$

4099. *Proposé par Lorian Saceanu.*

Soit ABC un triangle acutangle. Les bissectrices des angles A, B et C coupent les côtés du triangle ABC aux points respectifs A', B' et C' et elles coupent le cercle circonscrit au triangle ABC aux points respectifs L, M et N. Soit I le point d'intersection de ces bissectrices. Démontrer que :

a) $\dfrac{AI}{IL} = \dfrac{IA'}{A'L}$,

b) $\sqrt{\dfrac{AI}{IL}} + \sqrt{\dfrac{BI}{IM}} + \sqrt{\dfrac{CI}{IN}} \geq 3$.

4100. *Proposé par Daniel Sitaru et Leonard Giugiuc.*

Soit ABC un triangle quelconque avec $\angle A < 90°$. Soit S l'aire du triangle, $BC = a$, $AC = b$ et $AB = c$. Démontrer que

$$\frac{c\cos B}{ac + 2S} + \frac{b\cos C}{ab + 2S} < \frac{a}{2S}.$$

. . . .

4091. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the greatest positive number k such that

$$a + b + c + 3k - 3 \geq k\left(\sqrt[3]{\frac{b}{a}} + \sqrt[3]{\frac{c}{b}} + \sqrt[3]{\frac{a}{c}}\right)$$

for any positive numbers a, b and c with $abc = 1$.

4092. *Proposed by Mihaela Berindeanu.*

Show that

$$\left[\frac{a^2 + 16a + 80}{16\,(a + 4)} + \frac{2}{\sqrt{2\,(b^2 + 16)}}\right]\left[\frac{b^2 + 16b + 80}{16\,(b + 4)} + \frac{2}{\sqrt{2\,(a^2 + 16)}}\right] \geq \frac{9}{4}$$

for all $a, b > 0$. When does equality hold ?

4093. *Proposed by Dragolijub Milošević.*

Let ABC be an arbitrary triangle. Let r and R be the inradius and the circumradius of ABC, respectively. Let m_a be the length of the median from vertex A to side BC and let w_a be the length of the internal bisector of $\angle A$ to side BC. Define m_b, m_c, w_b and w_c similarly. Prove that

$$\frac{a^2}{m_a w_a} + \frac{b^2}{m_b w_b} + \frac{c^2}{m_c w_c} \leq 4\left(\frac{R}{r} - 1\right).$$

4094. *Proposed by Michel Bataille.*

Let $x_1, x_2, \dots, x_n$ be real numbers such that $0 \le x_1 \le x_2 \le \cdots \le x_n$. Prove that

$$n-1+\cosh\left(\sum_{k=1}^{n}(-1)^{k-1}x_k\right) \le \sum_{k=1}^{n}\cosh x_k \le n-1+\cosh\left(\sum_{k=1}^{n}x_k\right).$$

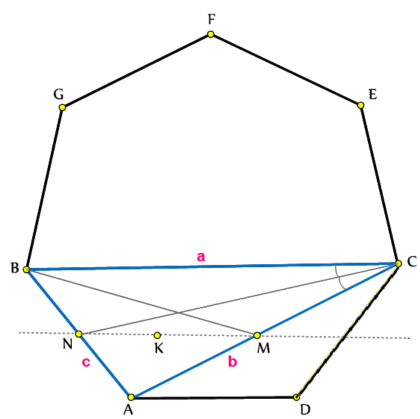
4095. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that

$$ab(a+b)+bc(b+c)+ac(a+c) \ge \frac{2}{3}(a^2+b^2+c^2)+4abc.$$

4096. *Proposed by Abdilkadir Altintaş.*

Let ABC be a heptagonal triangle with $BC=a$, $AC=b$ and $AB=c$. Suppose CN is the internal angle bisector of $\angle BCA$, BM is the median of triangle ABC and K is the symmedian point of ABC. Show that N, K and M are collinear.



4097. *Proposed by Leonard Giugiuc.*

Let $a_i, 1 \le i \le 6$ be real numbers such that

$$\sum_{i=1}^{6}a_i=\frac{15}{2} \quad \text{and} \quad \sum_{i=1}^{6}a_i^2=\frac{45}{4}.$$

Prove that $\prod_{i=1}^{6}a_i \le \dfrac{5}{2}$.

4098. *Proposed by Ardak Mirzakhmedov.*

Let α, β and γ be acute angles such that $\alpha+\beta=\gamma$. Show that

$$\cos\alpha+\cos\beta+\cos\gamma-1 \ge 2\sqrt{\cos\alpha\cdot\cos\beta\cdot\cos\gamma}.$$

4099. *Proposed by Lorian Saceanu.*

Let ABC be an acute angle triangle. Suppose the internal bisectors of angles A, B and C intersect the sides of ABC in points A', B' and C' and they intersect the circumcircle of ABC in points L, M and N respectively. Let I be the point of intersection of all internal bisectors. Show that :

a) $\dfrac{AI}{IL} = \dfrac{IA'}{A'L}$,

b) $\sqrt{\dfrac{AI}{IL}} + \sqrt{\dfrac{BI}{IM}} + \sqrt{\dfrac{CI}{IN}} \geq 3.$

4100. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let ABC be an arbitrary triangle with area S, $\angle A < 90^\circ$ and sides $BC = a$, $AC = b$ and $AB = c$. Show that

$$\frac{c\cos B}{ac+2S} + \frac{b\cos C}{ab+2S} < \frac{a}{2S}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014 : 40(10), p. 435–438.

3991. *Proposed by Michel Bataille.*

Let ABC be a triangle with $BC = a, CA = b, AB = c, \angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$ and let $m_a = AA', m_b = BB', m_c = CC'$ where A', B', C' are the midpoints of BC, CA, AB. Prove that

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma}.$$

We received eight correct solutions, from which two will be featured.

Solution 1, by Šefket Arslanagić and Dragoljub Milošević (done independently).

Without loss of generality we can suppose that $a \leq b \leq c$. This implies both

$$\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \quad \text{and} \quad m_a \geq m_b \geq m_c,$$

so that we can apply the Chebyshev Sum Inequality to get

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \;\geq\; \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(m_a + m_b + m_c). \tag{1}$$

We use the AM-HM inequality,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c},$$

together with the inequality

$$m_a + m_b + m_c \geq \frac{1}{2R}\left(a^2 + b^2 + c^2\right)$$

(where R is the circumradius of ΔABC), which can be found on p.13 of [1] or p. 213 of [2], to reduce inequality (1) to

$$m_a \cdot \frac{1}{a} + m_b \cdot \frac{1}{b} + m_c \cdot \frac{1}{c} \;\geq\; \frac{3(a^2 + b^2 + c^2)}{2R(a+b+c)}. \tag{2}$$

Finally, plug $a = 2R\sin\alpha,\ b = 2R\sin\beta,$ and $c = 2R\sin\gamma$ into the right-hand-side of (2) to finish the proof.

The equality holds if and only if $a = b = c$ (and ΔABC is equilateral).

Solution 2, a composite of the solutions of Arkady Alt and of Andrea Fanchini.

The inequality continues to hold when the medians m_x are replaced by the altitudes h_x ; more precisely, we shall prove that

$$\frac{m_a}{a}+\frac{m_b}{b}+\frac{m_c}{c}\ge\frac{h_a}{a}+\frac{h_b}{b}+\frac{h_c}{c}\ge\frac{3(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)}{\sin\alpha+\sin\beta+\sin\gamma},$$

with equality if and only if $a=b=c$.

The left inequality is clear because $m_x\ge h_x$ for each side x. For the right inequality, in terms of the area K of ΔABC we know that

$$\sin\alpha=\frac{2K}{bc},\quad \sin\beta=\frac{2K}{ac},\quad \sin\gamma=\frac{2K}{ab},$$

so that

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{4K^2(a^2+b^2+c^2)}{a^2b^2c^2},\quad \sin\alpha+\sin\beta+\sin\gamma=\frac{2K(a+b+c)}{abc}.$$

Since the altitudes satisfy

$$h_a=\frac{2K}{a},\quad h_b=\frac{2K}{b},\quad h_c=\frac{2K}{c},$$

the inequality on the right becomes

$$\frac{2K}{a^2}+\frac{2K}{b^2}+\frac{2K}{c^2}\ge\frac{6K(a^2+b^2+c^2)}{abc(a+b+c)},$$

which reduces to

$$(a+b+c)(a^2b^2+b^2c^2+c^2a^2)\ge 3abc(a^2+b^2+c^2). \tag{3}$$

Warning! Inequality (3) is guaranteed to hold only when a,b,c are the sides of a triangle. It might not hold for an arbitrary triple of positive real numbers; for example, when $b=c=1$ the inequality fails for a sufficiently large.

Because a,b,c are the sides of a triangle, we can set

$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$

and (3) expands to

$$x^5+y^5+z^5+x^2y^2z+x^2yz^2+xy^2z^2\ge x^3y^2+x^3z^2+x^2y^3+y^3z^2+y^2z^3+x^2z^3.$$

Let us write $[k,\ell,m]=\sum p^kq^\ell r^m$, the sum being taken over the six permutations (p,q,r) of (x,y,z). In this notation our inequality becomes

$$[5,0,0]+[2,2,1]\ge 2[3,2,0],$$

which holds by Muirhead's Theorem (or, if you prefer, by an inequality of Schur). By the equality condition of Muirhead's Theorem, the equality holds if and only if ΔABC is equilateral.

Editor's Comments. The proposer observed that his inequality resembles a known one :

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3\sqrt{3}}{2}$$

(see, for example, p. 211 of [2]). However, neither is a refinement of the other ; for example,

$$\text{with } \alpha = \tfrac{\pi}{2},\ \beta = \gamma = \tfrac{\pi}{4},\text{ then } \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma} < \frac{3\sqrt{3}}{2},$$

$$\text{while if } \alpha = \tfrac{\pi}{2},\ \beta = \tfrac{\pi}{12},\ \gamma = \tfrac{5\pi}{12},\text{ then } \frac{3(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)}{\sin\alpha + \sin\beta + \sin\gamma} > \frac{3\sqrt{3}}{2}.$$

References :

[**1**] Marin Chirciu, *Inegalităti Geometriće.* Editura Paralela 45, Pitesti Romania, 2015.

[**2**] D.S. Mitrinović et al. *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

3992. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let α, a, b, c be positive real numbers such that $a + b + c + 3 = 6abc$. Find the maximum value of the expression

$$\frac{1}{a^\alpha + b^\alpha + 1} + \frac{1}{b^\alpha + c^\alpha + 1} + \frac{1}{c^\alpha + a^\alpha + 1}.$$

Eight correct solutions were received. There was one incomplete solution. The solution presented draws on ideas from several solvers.

By the AM-GM inequality, we have that

$$6abc = a + b + c + 1 + 1 + 1 \geq 6(abc)^{1/6},$$

whence $abc \geq 1$. Let $a^\alpha = x^3$, $b^\alpha = y^3$ and $c^\alpha = z^3$, so that $xyz \geq 1$.

Observe that, by the AM-GM inequality,

$$x^3 + y^3 + 1 = \frac{1}{3}(2x^3 + y^3) + \frac{1}{3}(x^3 + 2y^3) + 1 \geq xy(x + y) + 1 \geq \frac{x + y + z}{z}.$$

Apply this and analogous inequalities for $y^3 + z^3 + 1$ and $z^3 + x^3 + 1$ to obtain

$$\frac{1}{x^3 + y^3 + 1} + \frac{1}{y^3 + z^3 + 1} + \frac{1}{z^3 + x^3 + 1} \leq \frac{z + y + x}{x + y + z} = 1,$$

with equality if and only if $x = y = z = 1$.

3993. *Proposed by Dragoljub Milošević.*

Let h_a, h_b and h_c be the altitudes and r the inradius of a triangle. Prove that

$$\frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} \geq \frac{3}{5}.$$

We received 21 correct solutions. We present the solution by Kee-Wai Lau.

Let s be the semiperimeter of the triangle. The area equals

$$\frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2} = rs.$$

Hence

$$\frac{h_a - 2r}{h_a + 2r} = \frac{2s}{s+a} - 1, \quad \frac{h_b - 2r}{h_b + 2r} = \frac{2s}{s+b} - 1 \text{ and } \frac{h_c - 2r}{h_c + 2r} = \frac{2s}{s+c} - 1,$$

so that the inequality of the problem is equivalent to

$$\frac{1}{s+a} + \frac{1}{s+b} + \frac{1}{s+c} \geq \frac{9}{5s}.$$

For $x > 0$, the function $\frac{1}{x}$ is convex. Hence

$$\frac{1}{s+a} + \frac{1}{s+b} + \frac{1}{s+c} \geq 3\left(\frac{1}{\frac{(s+a)+(s+b)+(s+c)}{3}}\right) = \frac{9}{5s},$$

and this completes the solution.

3994. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$a^4 + b^4 + c^4 \geq abc.$$

We received 32 *correct solutions. We present three of the shorter solutions.*

Solution 1, by AN-anduud Problem Solving Group.

Using the AM-GM inequality we have

$$\begin{aligned} a^4 + b^4 + c^4 &= \frac{a^4 + a^4 + b^4 + c^4}{4} + \frac{a^4 + b^4 + b^4 + c^4}{4} + \frac{a^4 + b^4 + c^4 + c^4}{4} \\ &\geq \sqrt[4]{a^4a^4b^4c^4} + \sqrt[4]{a^4b^4b^4c^4} + \sqrt[4]{a^4b^4c^4c^4} \\ &= a^2bc + ab^2c + abc^2 \\ &= abc(a + b + c) = abc. \end{aligned}$$

Equality holds if and only if $a = b = c = \frac{1}{3}$.

Solution 2, by Cao Minh Quang.

For all positive x, y, z, by the rearrangement inequality we have

$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

Hence,

$$\begin{aligned} a^4 + b^4 + c^4 &\geq (ab)^2 + (bc)^2 + (ca)^2 \\ &\geq abbc + bcca + caab = abc(a + b + c) = abc. \end{aligned}$$

Solution 3, by Joel Schlosberg.

By Muirhead's inequality,

$$\sum_{\text{symmetric}} a^4 b^0 c^0 \geq \sum_{\text{symmetric}} a^2 b^1 c^1,$$

as $(4, 0, 0)$ majorizes $(2, 1, 1)$. That is,

$$2(a^4 + b^4 + c^4) \geq 2(a^2bc + b^2ac + c^2ab) = 2abc(a + b + c) = 2abc,$$

which via cancellation of the factor 2 yields the inequality.

3995. *Proposed by Michel Bataille.*

For positive x and y, let $\mathcal{M}_0(x, y) = \sqrt{xy}$ and $\mathcal{M}_\alpha(x, y) = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{\frac{1}{\alpha}}$ if α is a nonzero real number. Given an equilateral triangle ABC, determine for which values of α the following property holds : $\mathcal{M}_\alpha(PB, PC) \leq PA$ for every point P distinct from B and C on the line BC.

Of the 4 submissions we received, only the one from Joel Schlosberg was complete and correct. We present his solution.

The desired property holds if and only if $\alpha \leq 4$. Set Cartesian coordinates so that the vertices of the equilateral triangle are

$$A = (0, \sqrt{3}), \quad B = (-1, 0), \quad C = (1, 0).$$

Any point P on line BC, the x-axis, has coordinates $(x, 0)$ for some $x \in \mathbb{R}$.

For $P = (x, 0)$, $PB = |x + 1|$ and $PC = |x - 1|$, so

$$\mathcal{M}_\alpha(PB, PC) = \begin{cases} \left(\frac{|x+1|^\alpha + |x-1|^\alpha}{2}\right)^{1/\alpha} & \text{if } \alpha \neq 0, \\ \sqrt{|x + 1| \cdot |x - 1|} & \text{if } \alpha = 0. \end{cases}$$

moreover, by the Pythagorean Theorem,

$$PA = \sqrt{x^2 + 3}.$$

Then because $2(x^2+3)^2 - (x+1)^4 - (x-1)^4 = 16 > 0$, we have

$$\mathcal{M}_4(PB, PC) = \left(\frac{(x+1)^4 + (x-1)^4}{2}\right)^{1/4} < \sqrt{x^2+3} = PA.$$

If $\alpha \le 4$, by the power mean inequality,

$$\mathcal{M}_\alpha(PB, PC) \le \mathcal{M}_4(PB, PC) < PA.$$

This includes $\alpha = 0$, since the power mean inequality includes the case of one of the power means being the zeroth power, defined by convention as the geometric mean. But the inequality is also easily proved directly : Because

$$-(x^2+3) \;\le\; x^2 - 1 \;\le\; x^2+3,$$

we deduce that

$$\mathcal{M}_0(PB, PC) = \sqrt{|x^2-1|} \le \sqrt{x^2+3} = PA.$$

For the converse, consider the function

$$f : z \mapsto \left(\frac{(1+z)^\alpha + (1-z)^\alpha}{2}\right)^{2/\alpha}.$$

By straightforward, if sometimes tedious calculations, $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 2(\alpha - 1)$. By Taylor's theorem,

$$f(z) = f(0) + f'(0)z + h(z)z^2 = 1 + h(z)z^2,$$

with $\lim_{z\to 0} h(z) = \frac{1}{2}f''(0) = \alpha - 1$.

Suppose that the desired inequality holds for $\alpha \ne 0$. Then for all $x > 1$, we have

$$\left(\frac{(x+1)^\alpha + (x-1)^\alpha}{2}\right)^{1/\alpha} = \mathcal{M}_\alpha(PB, PC) \le PA = \sqrt{x^2+3},$$

$$\left(\frac{x^\alpha[(1+x^{-1})^\alpha + (1-x^{-1})^\alpha]}{2}\right)^{2/\alpha} \le x^2+3,$$

$$\left(\frac{(1+x^{-1})^\alpha + (1-x^{-1})^\alpha}{2}\right)^{2/\alpha} \le 1 + 3x^{-2}.$$

So for $z = x^{-1} \in (0,1)$, we have $f(z) = 1 + h(z)z^2 \le 1 + 3z^2$, whence $h(z) \le 3$. Taking $z \to 0^+$,

$$\alpha - 1 = \lim_{z\to 0^+} h(z) \le 3.$$

Therefore $\alpha \le 4$, as claimed.

Editor's Comments. Two of the incomplete solutions restricted P to the interior of the edge BC, in which case the segment PA is longer than PB and PC

(because the longer side is opposite the larger angle) and, therefore, greater than $\mathcal{M}_\alpha(PB, PC)$ for all α. It is only when P is outside ΔABC on the line BC that the problem becomes interesting.

3996. *Proposed by Marcel Chiriţă.*

Let $a \in (1, \infty)$ and $b, c \in \mathbb{R}$. For each $\lambda \in (0, \infty)$, find all differentiable functions $f : [1, \infty) \to \mathbb{R}$ such that

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$

for all $x \in [1, \infty)$.

We received three submissions, all of which were correct and complete. We present the solution by Michel Bataille.

Since $a^x \geq 1$ is equivalent to $x \geq 0$, I suppose that "$x \in [1, \infty)$" at the end of the statement is to be replaced by "$x \in [0, \infty)$". This said, here are two solutions. The first one answers the problem as set. The second one answers the problem modified so as to make it more interesting (and probably closer to the one intended).

1. $a > 1$ and b, c given, suppose the differentiable function $f : [1, \infty) \to \mathbb{R}$ satisfies

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c \tag{1}$$

for all $\lambda \in (0, \infty)$ and $x \in [0, \infty)$.

Taking $\lambda = 1$, we see that $4f(a^x) = bx + c$ for all $x \geq 0$ and it follows that f defined by $f(y) = \frac{1}{4}(c + b\log_a(y))$ is the only possible solution.

Conversely, let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(y) = \frac{1}{4}(c + b\log_a(y))$ for $y \geq 1$. Then f is differentiable and a simple calculation gives

$$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = \frac{bx(\lambda + 1)^2}{4} + c.$$

It readily follows that for (1) to be verified for any $\lambda > 0$ and $x \geq 0$, we must have $b = 0$.

In conclusion, there is no solution if $b \neq 0$ and one solution if $b = 0$, namely the constant function f defined by $f(y) = \frac{c}{4}$ for all $y \geq 1$.

2. We modify the problem as follows :

> Let $a \in (1, \infty)$, $\lambda \in (0, \infty)$ and $b, c \in \mathbb{R}$. Find all differentiable functions $f : [1, \infty) \to \mathbb{R}$ such that
>
> $$f(a^{\lambda^2 x}) + 2f(a^{\lambda x}) + f(a^x) = bx + c$$
>
> for all $x \in [0, \infty)$.

We prove that the unique solution is the function f defined by

$$f(y) = \frac{b\log_a(y)}{(\lambda+1)^2} + \frac{c}{4} \tag{2}$$

for all $y \in [1,\infty)$. It is easy to check that this function satisfies all the requirements.

Conversely, let f be differentiable on $[1,\infty)$ such that (1) holds for all $x \in [0,\infty)$. If $\lambda = 1$, the calculations of part 1 show that f must be defined by $f(y) = \frac{1}{4}(c + b\log_a(y))$, in accordance with (2).

Let $g(x) = f(a^x)$ and $h(x) = g(x) + g(\lambda x) = f(a^x) + f(a^{\lambda x})$ $(x \geq 0)$. Note that h satisfies

$$h(x) + h(\lambda x) = bx + c \tag{3}$$

for all $x \in [0,\infty)$.

Suppose now that $\lambda \in (0,1)$. Let x be an arbitrary nonnegative real number. For all nonnegative integers k, we have $h(\lambda^k x) + h(\lambda^{k+1}x) = b\lambda^k x + c$ so that

$$h(x) - h(\lambda^{2n}x) = \sum_{k=0}^{2n-1} (-1)^k \left(h(\lambda^k x) + h(\lambda^{k+1}x)\right) = bx\sum_{k=0}^{2n-1}(-\lambda)^k + c\sum_{k=0}^{2n-1}(-1)^k$$

for any $n \in \mathbb{N}$. Thus,

$$h(\lambda^{2n}x) + \frac{bx(1-(-\lambda)^{2n})}{1+\lambda} = h(x).$$

Letting n go to ∞ and because h is continuous at 0 with $h(0) = \frac{c}{2}$ ((3) with $x = 0$), we obtain $\frac{c}{2} + \frac{bx}{1+\lambda} = h(x)$.

If $\lambda > 1$, then from (3) we obtain $h(x) + h(\mu x) = b\mu x + c$ where $\mu = \frac{1}{\lambda} < 1$. A similar method then gives $h(x) = \frac{c}{2} + \frac{b\mu x}{1+\mu}$, that is, $h(x) = \frac{c}{2} + \frac{bx}{1+\lambda}$ again. We deduce that in the case $\lambda \neq 1$, we have

$$g(x) + g(\lambda x) = \frac{bx}{\lambda+1} + \frac{c}{2}.$$

Applying the above method once more easily leads to

$$g(x) = \frac{bx}{(\lambda+1)^2} + \frac{c}{4}.$$

Since x is arbitrary and $g(x) = f(a^x)$ we may conclude that f is defined by

$$f(y) = \frac{b\log_a(y)}{(\lambda+1)^2} + \frac{c}{4},$$

which completes the proof.

Note : the hypothesis “f differentiable” can be replaced by “f continuous at 1”.

3997. *Proposed by Mihaela Berindeanu.*

Let a, b, c be positive numbers with product 8. Prove that

$$\frac{a^4+b^4}{c^3}+\frac{a^4+c^4}{b^3}+\frac{b^4+c^4}{a^3}\geq 64\left(\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\right)+6.$$

We received 20 *correct and two incorrect solutions. We present the solution by the AN-anduud Problem Solving Group.*

Using the AM-GM inequality, we have

$$\begin{aligned}\sum_{cyc}\frac{a^4+b^4}{c^3}\geq\sum_{cyc}\frac{2\sqrt{a^4b^4}}{c^3}&=\sum_{cyc}\frac{a^2b^2}{c^3}+\sum_{cyc}\frac{a^2b^2}{c^3}\\&\geq\sum_{cyc}\frac{(\frac{8}{c})^2}{c^3}+3\sqrt[3]{\frac{a^2b^2}{c^3}\cdot\frac{b^2c^2}{a^3}\cdot\frac{c^2a^2}{b^3}}\\&=64\left(\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\right)+6.\end{aligned}$$

Equality holds if and only if $a=b=c=2$.

3998. *Proposed by George Apostolopoulos.*

Let a_i, $i=1,2,\ldots,n$ be positive real numbers such that $\sum_{i=1}^n a_i=n$. Prove that

$$\sum_{i=1}^n\left(\frac{a_i^3+1}{a_i^2+1}\right)^4\geq n.$$

We received 17 submissions, all of which were correct and complete. We present two solutions.

Solution 1, by Joel Schlosberg.

For $x\in[0,\infty)$, define

$$f: x\mapsto\left(\frac{x^3+1}{x^2+1}\right)^4$$

and

$$g: x\mapsto 2x^2-2x+1.$$

For all $x\geq 0$, clearly $(x^2+1)^4>0$ and

$$\begin{aligned}&(x^3+1)^4-(x^2+1)^4(2x^2-2x+1)=\\&x(x-1)^2(x^9+2x^8+x^7+6x^6+2x^5+6x^4+5x^2+1+(x-1)^2)\geq 0\end{aligned}$$

so $f(x)\geq g(x)$. Since $g''(x)=4>0$, g is convex.

By Jensen's inequality,

$$\sum_{i=1}^{n}\left(\frac{a_i^3+1}{a_i^2+1}\right)^4=\sum_{i=1}^{n}f(a_i)\geq\sum_{i=1}^{n}g(a_i)\geq n\cdot g\left(\frac{\sum_{i=1}^{n}a_i}{n}\right)=n\cdot g(1)=n.$$

Note that the proof works if real $a_1,\ldots,a_n$ summing to n are nonnegative, rather than the stricter assumption that they are positive.

Solution 2, by Titu Zvonaru.

First, we will prove that for all positive real x we have the inequality

$$\frac{x^3+1}{x^2+1}\geq\sqrt{x}. \tag{1}$$

Denoting $t=\sqrt{x}$, we obtain

$$\begin{aligned}\frac{t^6+1}{t^4+1}\geq t &\iff t^6-t^5-t+1\geq 0\\ &\iff (t-1)(t^5-1)\geq 0\\ &\iff (t-1)^2(t^4+t^3+t^2+t+1)\geq 0,\end{aligned}$$

which is true. Using (1) yields

$$\sum_{i=1}^{n}\left(\frac{a_i^3+1}{a_i^2+1}\right)^4\geq\sum_{i=1}^{n}a_i^2. \tag{2}$$

Applying Cauchy-Schwarz inequality we deduce that

$$\sum_{i=1}^{n}1\sum_{i=1}^{n}a_i^2\geq\left(\sum_{i=1}^{n}a_i\right)^2\implies n\sum_{i=1}^{n}a_i^2\geq n^2,$$

so that

$$\sum_{i=1}^{n}a_i^2\geq n. \tag{3}$$

By (2) and (3) the desired inequality follows.

3999. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Consider real numbers a,b,c such that $a\geq 1\geq b\geq c>-3$ and

$$ab+bc+ca=3.$$

Prove that $a+b+c\geq 3$.

We received 11 submissions, of which 10 were correct and complete. We present two solutions.

Solution 1, by Michel Bataille, slightly modified by the editor.

By expanding and rearranging $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$, we get that $x^2+y^2+z^2 \geq xy+yz+zx$ for all real numbers x, y, z. Hence we have

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca) \geq 3(ab+bc+ca) = 9. \qquad (1)$$

Thus, it is sufficient to prove that $b+c \geq -1$, since then $a+b+c \geq 0$ (as $a \geq 1$), which combined with (1) gives $a+b+c \geq 3$, as desired.

Towards contradiction, assume that $b+c < -1$. We will recursively construct a sequence $\{\alpha_n\}$ such that $c < \alpha_n$ for all n. To start with, let $\alpha_0 = -\frac{1}{2}$. Since $b+c < -1$ and $b \geq c$, we must have $c < -\frac{1}{2} = \alpha_0$; also note that $b+c < 2\alpha_0$.

Now, suppose that $b+c < 2\alpha_n$ for some α_n a negative real number ($n \geq 0$). Then

$$a(b+c) < 2a\alpha_n \leq 2\alpha_n,$$

hence

$$3 = bc + a(b+c) < bc + 2\alpha_n.$$

Since $bc > 0$ and $\frac{1}{c} < -\frac{1}{3}$, we deduce that

$$b = (bc)\left(\frac{1}{c}\right) < -\frac{1}{3}(bc) < -\frac{1}{3}(3-2\alpha_n) = \alpha_{n+1}$$

where α_{n+1} is the negative real number defined by $\alpha_{n+1} = \frac{2\alpha_n}{3} - 1$. We conclude that $b, c \leq \alpha_n$ for all positive integers n, where the sequence $\{\alpha_n\}$ is defined by $\alpha_0 = -\frac{1}{2}$ and the recursion $\alpha_{n+1} = \frac{2\alpha_n}{3} - 1$ for $n \geq 0$.

Using the fact that this recursion rewrites as

$$\alpha_{n+1} + 3 = \frac{2}{3}(\alpha_n + 3),$$

we readily see that

$$\alpha_n = \frac{5}{2} \cdot \left(\frac{2}{3}\right)^n - 3$$

for $n \geq 0$. Since $\lim_{n\to\infty} \alpha_n = -3$ and $c < \alpha_n$ for any n, we would have $c \leq -3$, in contradiction with the hypothesis.

Solution 2, by Digby Smith.

Using $(x+y+z)^2 \geq 3(xy+yz+zx)$, which holds for all real numbers x, y, z, we get that

$$(a+b+c)^2 \geq 3(xy+yz+zx) = 9,$$

which means that either $a+b+c \geq 3$ or $a+b+c \leq -3$. Use proof by contradiction to show that $a+b+c \geq 3$ is the only condition consistent with the problem.

Assume $a+b+c \leq -3$. The given condition $a \geq 1$ implies $b+c \leq -4$, whence $-3 < c \leq b < -1$. Using these inequalities, as well as $b+c<0$, we have

$$ab+bc+ca = a(b+c)+bc \leq b+c+bc = (b+1)(c+1)-1 < (-2)(-2)-1 = 3.$$

Thus $a+b+c \leq -3$ implies $ab+bc+ca < 3$, contradicting $ab+bc+ca = 3$. It follows that $a+b+c \geq 3$, with equality if and only if $a=b=c=1$.

4000. *Proposed by Marcel Chiriţă.*

Let $x_1, x_2, \dots x_n$ with $x_1 > x_2 > \dots > x_n > 0$, $x_1 x_2 \dots x_n = 1$ and $n \geq 3$. Show that

$$\frac{x_1^2+x_2^2}{x_1-x_2} \cdot \frac{x_2^2+x_3^2}{x_2-x_3} \cdots \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} > 2^{3/n}.$$

While the inequality holds as written, the right hand side was supposed to be the much stronger $2^{3n/2}$; we apologize for the error. As a consequence however, the solutions we received (11 in total) varied greatly in their approach. The strength of the proven results also varied, with the right hand side being variously replaced by $2^{n/3}$, 2^{n-1}, 2^n or $(2+2\sqrt{2})^{n-1}$, as well as the intended $2^{3n/2}$.

We present two of the solutions.

Solution 1, by Arkady Alt, modified by the editor. This solution had the best bound.

For $1 \leq k \leq n-1$, let $t_k := \frac{x_k}{x_{k+1}}$. Note that $t_k > 1$, and

$$\frac{x_k^2+x_{k+1}^2}{x_k-x_{k+1}} = \frac{x_{k+1}^2\left(\frac{x_k^2}{x_{k+1}^2}+1\right)}{x_{k+1}\left(\frac{x_k}{x_{k+1}}-1\right)} = x_{k+1} \cdot \frac{\left(\frac{x_k}{x_{k+1}}\right)^2+1}{\frac{x_k}{x_{k+1}}-1} = x_{k+1} \cdot \frac{t_k^2+1}{t_k-1}.$$

Let $a := \frac{x_n}{x_1}$ and note $0<a<1$; then

$$\frac{x_1^2+x_n^2}{x_1-x_n} = \frac{x_1^2(1+a^2)}{x_1(1-a)} = x_1 \cdot \frac{1+a^2}{1-a}.$$

Since $x_1 x_2 \dots x_n = 1$, it follows that

$$\prod_{k=1}^{n-1} \frac{x_k^2+x_{k+1}^2}{x_k-x_{k+1}} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} = \prod_{k=1}^{n-1} \frac{t_k^2+1}{t_k-1} \cdot \frac{a^2+1}{1-a}. \qquad (1)$$

For any $t>1$ we claim that $\frac{t^2+1}{t-1} \geq 2\sqrt{2}+2$.
To see this, note that

$$\begin{aligned} t^2+1-(2\sqrt{2}+2)(t-1) &= t^2+1-2\sqrt{2}t-2t+2\sqrt{2}+2 \\ &= (t-\sqrt{2}-1)^2 \geq 0, \end{aligned}$$

whence, dividing by $t-1$, we can obtain the claimed inequality.

On the other hand, for $0<a<1$, it is easy to check that $\frac{a^2+1}{1-a} > 1$.

In (1), we had $t_k > 1$ and $a < 1$, so using the last two observations, we get

$$\prod_{k=1}^{n-1} \frac{x_k^2 + x_{k+1}^2}{x_k - x_{k+1}} \cdot \frac{x_1^2 + x_n^2}{x_1 - x_n} > (2\sqrt{2}+2)^{n-1}.$$

Solution 2, by Joel Schlosberg, slightly modified by the editor. This solution addressed the question of when the minimum is achieved.

Suppose $a, b \in \mathbb{R}$ with $0 < a < b$. Consider the function $g(x) = \frac{(b^2+x^2)(x^2+a^2)}{x(b-x)(x-a)}$. We claim that on the interval (a, b), $g(x)$ has a unique minimum at $x = \sqrt{ab}$.

Multiplying,

$$g(x) = \frac{b^2x^2 + b^2a^2 + x^4 + x^2a^2}{x(bx + xa - (ba + x^2))}.$$

For $x > 0$ we can use the arithmetic mean-geometric mean inequality. In the numerator, we have

$$b^2a^2 + x^4 \geq 2\sqrt{b^2a^2 \cdot x^4} = 2bax^2$$

(with equality if and only if $x = \sqrt{ab}$), and in the denominator

$$ba + x^2 \geq 2\sqrt{bax^2} = 2x\sqrt{ba}$$

(once again, with equality if and only if $x = \sqrt{ab}$). Moreover, for $x \in (a, b)$ we have that the denominator

$$x(b - x)(x - a) = x(bx + xa - (ba + x^2)) > 0$$

and hence

$$g(x) \geq \frac{b^2x^2 + x^2a^2 + 2bax^2}{x(bx + xa - 2x\sqrt{ba})} = \frac{b^2 + a^2 + 2ba}{b + a - 2\sqrt{ba}}.$$

Note that the right-hand side does not depend on x, and equality holds if and only if $x = \sqrt{ab}$. This concludes the proof that, for $x \in (a, b)$, $g(x)$ has a unique minimum at $x = \sqrt{ab}$. Note that g is differentiable on (a, b), and so $x = \sqrt{ab}$ must also be the unique critical point of g on (a, b).

Now suppose $z_1, z_n \in \mathbb{R}$ are fixed, with $z_1 > z_n > 0$, and define $f : \mathbb{R}^{n-2} \to \mathbb{R}$ by

$$f : (z_2, \ldots, z_{n-1}) \to \frac{1}{z_1 \cdot \ldots \cdot z_n} \cdot \frac{z_1^2 + z_2^2}{z_1 - z_2} \cdot \ldots \cdot \frac{z_{n-1}^2 + z_n^2}{z_{n-1} - z_n}.$$

Consider the domain

$$D := \{(t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-2} : z_1 > t_2 > \ldots > t_{n-1} > z_n\}.$$

As $(z_2, \ldots, z_{n-1})$ approaches the boundary of D, some $z_{i-1} - z_i \to 0$, and so $f \to +\infty$. Since f is real-valued and continuous on the open domain D, an absolute minimum value of f must be attained at a critical point. The partial derivatives of f look like the derivatives of the function g discussed at the beginning of the proof. Hence a critical point of f must satisfy $z_i = \sqrt{z_{i-1}z_{i+1}}$ for $i = 2, \ldots, n-1$; that is, the sequence $\{z_1, \ldots, z_n\}$ is a geometric progression.

Now suppose $x_1, \ldots, x_n$ are as given in the question, with $x_1 > \cdots > x_n > 0$ and $x_1 \cdot \ldots \cdot x_n = 1$. Let $y_1, \ldots, y_n$ be the geometric progression with $y_1 = x_1$, $y_n = x_n$. Denote the common ratio $\sqrt[n-1]{\frac{x_1}{x_n}}$ by r; that is, $r > 1$ and $r = y_{i-1}/y_i$ for $i = 2, \ldots, n$. Let $f : \mathbb{R}^{n-2} \to \mathbb{R}$ be as described above, with $z_1 = x_1$ and $z_n = x_n$. Then, since $x_1 \ldots x_n = 1$,

$$\begin{aligned}\frac{x_1^2+x_2^2}{x_1-x_2} \cdot \ldots \cdot \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} &= \frac{1}{x_1 \cdot \ldots \cdot x_n} \cdot \frac{x_1^2+x_2^2}{x_1-x_2} \cdot \ldots \cdot \frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n} \cdot \frac{x_1^2+x_n^2}{x_1-x_n} \\ &= f(x_2, \ldots, x_{n-1}) \\ &\geq f(y_2, \ldots, y_{n-1}).\end{aligned}$$

Using the fact that $y_{i-1} = ry_i$ for $i = 2, \ldots, n$, and $x_1 = y_1$, $x_n = y_n$, we have

$$\begin{aligned}f(y_2, \ldots, y_{n-1}) &= \frac{1}{y_1 \cdots y_n} \cdot \frac{y_1^2 + \frac{y_1^2}{r^2}}{y_1 - \frac{y_1}{r}} \cdot \ldots \cdot \frac{y_{n-1}^2 + \frac{y_{n-1}^2}{r^2}}{y_{n-1} - \frac{y_{n-1}}{r}} \cdot \frac{(r^{n-1}y_n)^2 - y_n^2}{r^{n-1}y_n - y_n} \\ &= \frac{1}{y_1 \cdots y_n} \cdot \frac{y_1\left(1+\frac{1}{r^2}\right)}{1-\frac{1}{r}} \cdot \ldots \cdot \frac{y_{n-1}\left(1+\frac{1}{r^2}\right)}{1-\frac{1}{r}} \cdot \frac{y_n(r^{2(n-1)}-1)}{r^{n-1}-1} \\ &= \left(\frac{r^2+1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}+1}{r^{n-1}-1}.\end{aligned}$$

We now find a lower bound for this formula in r; this bound will not be sharp. Replace r^2+1 by r^2-1, and use Mahler's inequality (which is that for positive $u_1, \ldots, u_n$, $v_1, \ldots, v_n$, $\sqrt[n]{\prod_{i=1}^n (u_i+v_i)} \geq \sqrt[n]{\prod_{i=1}^n u_i} + \sqrt[n]{\prod_{i=1}^n v_i}$). That is,

$$\begin{aligned}\left(\frac{r^2+1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}+1}{r^{n-1}-1} &> \left(\frac{r^2-1}{r(r-1)}\right)^{n-1} \cdot \frac{r^{2(n-1)}-1}{r^{n-1}-1} \\ &= \underbrace{\left(1+r^{-1}\right) \ldots \left(1+r^{-1}\right)}_{n-1 \text{ factors}}(1+r^{n-1}) \\ &\geq \left(\sqrt[n]{1^{n-1}1} + \sqrt[n]{(r^{-1})^{n-1}r^{n-1}}\right)^n = 2^n > 2^{3/n}.\end{aligned}$$

This proves the inequality with the sharper right-hand side bound 2^n.

Editor's comments. Instead of approximating as shown, the second proof could be finished similarly to the first, obtaining the improved bound $(2+2\sqrt{2})^{n-1}$.

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