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A magazine for students and teachers of mathematics in schools, colleges and universities

MATHEMATICAL SPECTRUM

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A Bird's Eye View of Spherical Triangles

A. TAN

1. Spherical trigonometry

The subject of spherical trigonometry has wide applications in navigation and aviation; in geography and astronomy; and in satellite tracking and space exploration. It is concerned with trigonometry on a spherical surface. The sphere under consideration is usually the surface of the Earth or the celestial sphere. Because of the curvature of the spherical surface, the rules of Euclidean geometry and plane trigonometry are no longer applicable. In fact there are more differences than similarities between triangles on a flat surface and those on a spherical surface. The most fundamental difference is that Pythagoras' theorem of Euclidean geometry is no longer valid on a spherical surface. Since plane trigonometry is based on Pythagoras' theorem, its equations are rendered useless on a spherical triangle.

In spite of its importance, spherical trigonometry is rarely a part of the curriculum these days and most of its books have become obsolete. In this article, we present a brief overview of spherical trigonometry as applied to spherical triangles. In a forthcoming article in the next issue of this magazine, we shall find applications of relevant formulas.

2. Great circles, lunes and spherical triangles

We begin with a few definitions. A *great circle* is a circle on a sphere whose centre is that of the sphere. The equator is a great circle on the surface of the Earth, and so are the meridians. Every great circle has two *poles*, which are points on the sphere farthest from the circle. For example, the north and south poles are the poles of the equator. There are no straight lines on a sphere. The 'shortest path' between two points (a *geodesic*) lies on a great circle.

A segment of the spherical surface bounded by two great circles is called a *lune*. A part of the surface bounded by three great circles is a *spherical triangle*. If one angle of the spherical triangle is a right angle, we have a *right spherical triangle*. Only one measure characterises a lune: the angle at either end on the surface. Two angles describe a right spherical triangle (the third being a right angle) whereas three angles are needed to specify a general (i.e. oblique) spherical triangle.

The lune has two sides, each of which is equal to half the circumference of the great circle. The spherical triangles have three sides. These sides also represent angles subtended at the centre of the sphere. If the radius of the sphere is taken as unity, the sides are equal to the respective angles in radian measure. In fact, the sides are treated as angles in spherical trigonometric relations. These angles can be converted to lengths by multiplying by the radius of the sphere.

A *diameter* is a chord passing through the centre of the sphere. The two points at the ends of a diameter are called *anti-podal points*. Thus the north and south poles are antipodal points of one another. Also recall that the Antipode Islands are diametrically opposite to Greenwich. An *antipodal triangle* is a spherical triangle whose vertices are antipodal points of a given spherical triangle.

It should be stressed that even though the surface of a sphere is curved, a small part of the surface is considered 'locally flat'. The sum of angles at any point on the surface still sums up to 2π radians. Also, Pythagoras' theorem holds for an infinitesimal triangle on the spherical surface.

3. Areas of lunes and spherical triangles

In Euclidean geometry, the sum of three angles of any triangle equals π radians. The angles have no bearing on the area of the triangle, which is determined by its sides. In spherical trigonometry, on the other hand, the area of a spherical triangle is solely determined by its angles. We first write down the expressions for the areas of a lune (of angle A), a right spherical triangle (of angles A and B) and a general spherical triangle (angles A, B and C) in terms of the radius of the sphere r:

$$\text{area of lune} = 2Ar^2, \tag{3.1}$$

$$\text{area of right spherical triangle} = (A + B - \tfrac{1}{2}\pi)r^2, \tag{3.2}$$

$$\text{area of oblique spherical triangle} = (A + B + C - \pi)r^2. \tag{3.3}$$

Equation (3.2) follows from (3.3) as a special case when $C = \frac{1}{2}\pi$. Equation (3.1) can also be derived from (3.3) by bisecting the lune into two equal right spherical triangles through its mid-section and by calculating and adding the two areas. In (3.3), the quantity $A + B + C - \pi$ is called the 'spherical excess'. For a 'flat' Euclidean triangle, the spherical excess is zero, which means that such a triangle is necessarily infinitesimal on a spherical surface. As a small triangle is made larger, it becomes curved; its angles increase and so do the spherical excess and the area.

Traditionally, (3.1) is derived first and then (3.3) from successive applications of (3.1) (references 1 and 2). Equation (3.2) then follows as a special case of (3.3). However, this derivation is rather difficult to visualise. In this article, we shall take a different route. We first derive (3.1) and then (3.2) and (3.3) as successive applications of the earlier results. This approach will be considerably simpler and more transparent to the reader.

Consider a lune of angle A. Next, consider the entire surface of the sphere as a lune of angle 2π. Then

$$\frac{\text{area of lune}}{\text{area of sphere}} = \frac{A}{2\pi}.$$

Substituting for the area of the sphere ($4\pi r^2$), we readily arrive at equation (3.1).

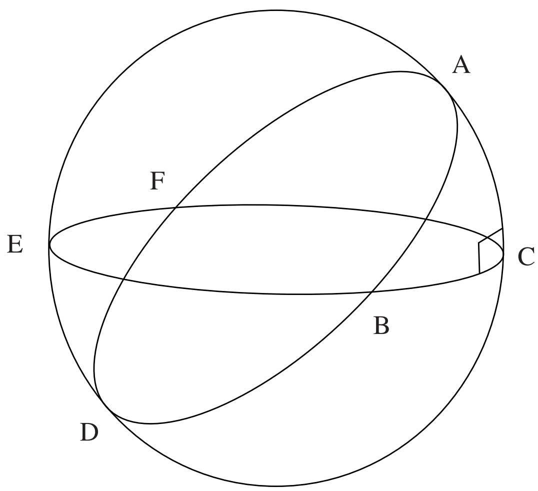

Figure 1. Right spherical triangle on a sphere.

In figure 1, ABC is a right spherical triangle where C is a right angle. D, F and E are respectively the anti-podal points of A, B and C. Also, ACF is the anti-podal triangle of BDE. Now, triangles ABC and ACF form a lune of angle B. Thus, by (3.1),

$$\triangle\text{ABC} + \triangle\text{ACF} = 2Br^2.$$

Since anti-podal triangles must have the same areas, we can replace $\triangle$ACF by $\triangle$BDE, getting

$$\triangle\text{ABC} + \triangle\text{BDE} = 2Br^2.$$

Next, triangles ABC and BDC form another lune of angle A, whence

$$\triangle\text{ABC} + \triangle\text{BDC} = 2Ar^2.$$

Adding the last two equations, we get

$$2\triangle\text{ABC} + \triangle\text{BDE} + \triangle\text{BDC} = 2(A+B)r^2.$$

Now, triangles BDE and BDC compose half of a hemisphere of area πr^2. The last equation finally yields

$$\triangle\text{ABC} = (A + B - \tfrac{1}{2}\pi)r^2,$$

which is equation (3.2).

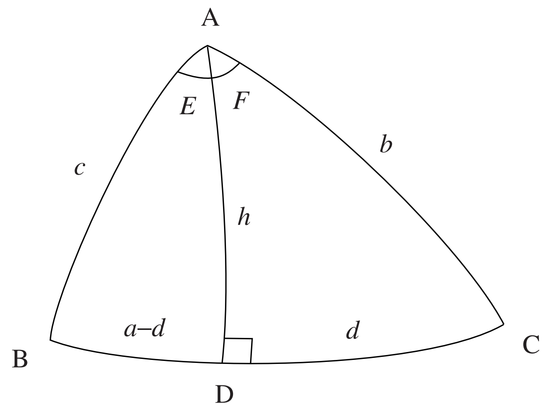

Figure 2. Oblique spherical triangle.

In figure 2, ABC is an oblique spherical triangle. Draw a great circular arc AD to intersect BC (if necessary produced) at right angles. We now have two right spherical triangles ABD and ADC. Applying (3.2) twice,

$$\triangle\text{ABD} = (E + B - \tfrac{1}{2}\pi)r^2,$$

$$\triangle\text{ADC} = (F + C - \tfrac{1}{2}\pi)r^2.$$

Adding and noting that angle A is the sum of the angles E and F, we get equation (3.3):

$$\triangle\text{ABC} = (A + B + C - \pi)r^2.$$

(This argument needs amending if D lies on BC produced; we leave this to the reader.)

4. Formulas relating to right spherical triangles

In figure 3, ABC is a right spherical triangle with $C = \frac{1}{2}\pi$; a, b and c are sides opposite A, B and C respectively. A right spherical triangle contains five elements: A, B, a, b and c (C is known). The number of combinations in which three elements can be chosen out of five is ${}^5\mathrm{C}_3 = 10$. There exist exactly 10 trigonometric relations connecting three elements of a right spherical triangle. In principle, any two elements define the right spherical triangle. From two given elements, any third element can be obtained from one of the 10 relations. By successive applications, all five elements can be determined.

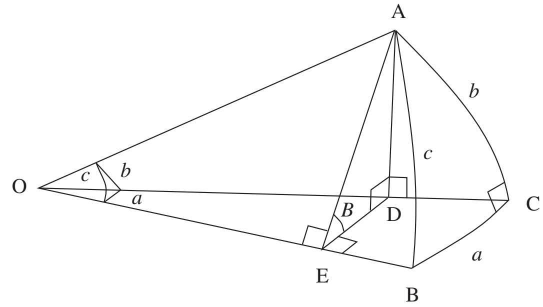

Figure 3. Right spherical triangle and its elements.

In figure 3, O is the centre of the sphere. Draw AD perpendicular to OC and DE perpendicular to OB. Join AE. Then angle AED equals angle B and angles ODA, ADE, OED and OEA are right angles. Thus, we have four right angled Euclidean triangles OAD, ODE, OAE and AED.

From triangles OAD, ADE and OAE:

$$\sin b = \frac{\mathrm{AD}}{\mathrm{OA}} = \frac{\mathrm{AD}}{\mathrm{AE}}\frac{\mathrm{AE}}{\mathrm{OA}} = \sin B \sin c.$$

From triangles ODE, ADE and OAE:

$$\tan a = \frac{\mathrm{DE}}{\mathrm{OE}} = \frac{\mathrm{DE}}{\mathrm{AE}}\frac{\mathrm{AE}}{\mathrm{OE}} = \cos B \tan c.$$

From triangles OAD, ADE and OED:

$$\tan b = \frac{\mathrm{AD}}{\mathrm{OD}} = \frac{\mathrm{AD}}{\mathrm{DE}}\frac{\mathrm{DE}}{\mathrm{OD}} = \tan B \sin a.$$

From triangles OAE, ODE and OAD:

$$\cos c = \frac{\mathrm{OE}}{\mathrm{OA}} = \frac{\mathrm{OE}}{\mathrm{OD}}\frac{\mathrm{OD}}{\mathrm{OA}} = \cos a \cos b.$$

Thus, we have four of the 10 relations:

$$\sin b = \sin B \sin c, \tag{4.1}$$
$$\tan a = \cos B \tan c, \tag{4.2}$$
$$\tan b = \tan B \sin a, \tag{4.3}$$

and

$$\cos c = \cos a \cos b. \tag{4.4}$$

Interchanging a and b and A and B in (4.1), (4.2) and (4.3), we have three more relations

$$\sin a = \sin A \sin c, \tag{4.5}$$
$$\tan b = \cos A \tan c, \tag{4.6}$$
$$\tan a = \tan A \sin b. \tag{4.7}$$

Equations (4.4), (4.5), (4.6) and (4.7) can also be obtained by dropping perpendiculars from B instead of from A.

The rest of the equations can be obtained from the above. From (4.4), (4.3) and (4.7), we have

$$\cos c = \cot A \cot B. \tag{4.8}$$

Similarly, from (4.2), (4.5) and (4.4),

$$\cos B = \cos b \sin A, \tag{4.9}$$

and, from (4.6), (4.1) and (4.4),

$$\cos A = \cos a \sin B. \tag{4.10}$$

5. Formulas relating to oblique spherical triangles

The oblique spherical triangle has six elements: three angles A, B and C; and their opposite sides a, b and c. In principle, any three elements determine the oblique spherical triangle. Trigonometric relations for the oblique spherical triangle contain four elements. If two angles and their opposite sides are given, we have the *law of sines*:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{5.1}$$

If three sides and one angle are given, we have the *law of cosines for sides*:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{5.2}$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \tag{5.3}$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{5.4}$$

When three angles and one side are given, we have the *law of cosines for angles*:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \tag{5.5}$$
$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \tag{5.6}$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{5.7}$$

Once again, it is easier to derive the formulas for the oblique spherical triangle from those of the right spherical triangles than to deduce them directly. Applying (4.1) to the two right spherical triangles ABD and ADC in figure 2, we have

$$\sin h = \sin B \sin c = \sin C \sin b,$$

from which

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Similarly, by dropping a great circular arc from B perpendicular to AC, we get

$$\frac{\sin c}{\sin C} = \frac{\sin a}{\sin A}.$$

The last two equations complete the *law of sines*.

Applying (4.4) to the right spherical triangle ABD, we have

$$\begin{aligned}\cos c &= \cos h \cos(a - d)\\ &= \cos h \cos a \cos d + \cos h \sin a \sin d.\end{aligned}$$

Applying (4.4), (4.7) and (4.5) to spherical triangle ADC, we get

$$\cos b = \cos h \cos d,$$
$$\sin d = \tan h \cot C,$$

and

$$\sin h = \sin b \sin C.$$

Substituting in the expression for $\cos c$, we get the law of cosines for the angle c (5.4); (5.2) and (5.3) can be obtained by cyclic permutation of letters.

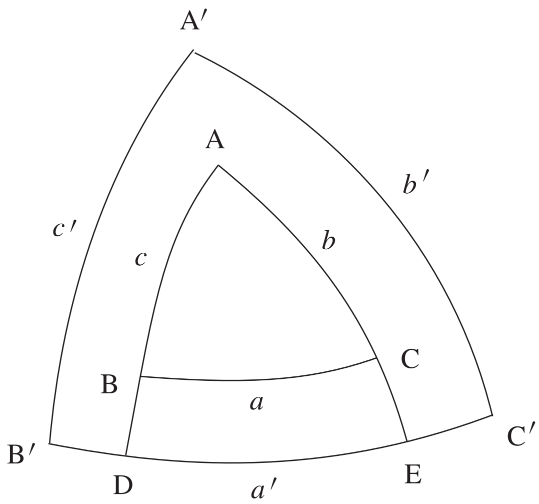

Figure 4. Polar triangle of oblique spherical triangle.

Equations (5.5), (5.6) and (5.7) may be derived with the help of 'polar triangles'. In figure 4, ABC is a spherical triangle. Draw great circular arcs B′C′, C′A′ and A′B′ with A, B and C as their respective poles. Then A′B′C′ is the *polar triangle* of ABC.

It follows that if A′B′C′ is a polar triangle of ABC, then ABC is also a polar triangle of A′B′C′, for B is the pole of A′C′ and C is the pole of A′B′, so A′ is a quadrant's distance ($\frac{1}{2}\pi$) from BC. Thus A′ is the pole of BC. Similarly, B′ and C′ are poles of A′C′ and A′B′ respectively. Hence ABC is a polar triangle of A′B′C′.

Next, we show that $a' = \pi - A$; $b' = \pi - B$ and $c' = \pi - C$. Extend AB and AC to intersect B′C′ at D and E respectively. Now, B′ is the pole of ACE; therefore

$$\mathrm{B'E} = \mathrm{B'D} + \mathrm{DE} = \tfrac{1}{2}\pi.$$

Similarly,

$$\mathrm{DC'} = \mathrm{DE} + \mathrm{EC'} = \tfrac{1}{2}\pi.$$

Adding and noting that $\mathrm{B'D} + \mathrm{DE} + \mathrm{EC'} = a'$, we get

$$\mathrm{DE} + a' = \pi.$$

Next, A is the pole of DE and ADE is just half of a lune through its midsection. Thus DE = A. From the previous equation, we get $a' = \pi - A$. Similarly, $b' = \pi - B$ and $c' = \pi - C$.

Since ABC and A′B′C′ are polar triangles of one another, we can interchange the dashed and undashed quantities to obtain $A' = \pi - a$; $B' = \pi - b$ and $C' = \pi - c$. Finally, applying (5.2) to the polar triangle A′B′C′ and setting $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$ and $A' = \pi - a$, we arrive at (5.5). Equations (5.6) and (5.7) follow in a similar manner.

Needless to say, all the equations relating to right spherical triangles follow as special cases of those relating to oblique spherical triangles.

6. Discussion

Evidently, the formulas of spherical trigonometry are far more numerous and complex than those of plane trigonometry. Furthermore, not much parallel can be found between the two with the possible exception of the law of sines.

In problems involving spherical trigonometry, the first priority is to seek the existence of right spherical triangles, whose equations are simpler and contain a smaller number of variables. In a forthcoming article in the next issue of this magazine, we shall find an illustration of this procedure.

References

1. I. Todhunter and J. G. Leathem, *Spherical Trigonometry* (MacMillan and Co., London, 1949).
2. P. Sperry, *Short Course in Spherical Trigonometry* (Johnson Publ. Co., Richmond, 1952).

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Solution to Braintwister 9
(Secret society)

Answer: 3
Solution: Let the society's secret number be the integer n. Then if everything were as it seemed member 1 would say $19 + n$, member 2 would say $19 + 2n, \ldots$ and member 19 would say $19 + 19n = 19(1 + n)$, which should therefore be divisible by 19. But we are told that the final answer is 67, which is **not** divisible by 19. So something is fishy!
If the society is secret how does the puzzle-teller know about it? He must be a member. So there are actually $19 - n$ members and the repetitive process will lead to a final answer of $(19 - n)(1 + n)$. This must equal $67 - n$. Hence $(19 - n)(1 + n) = 67 - n$ which tidies up to $n^2 - 19n + 48 = 0$ and n is 3 or 16. But if $n = 16$ there are only 3 members, contradicting the story-line.

VICTOR BRYANT

More about the convergence of the sequence $\{(1+(1/n))^n\}$

VIDAN GOVEDARICA AND MILAN JOVANOVIĆ

1. Introduction

The convergence of the sequence $\{x_n\}$, where

$$x_n = \left(1+\frac{1}{n}\right)^n ,$$

is usually established by showing that $\{x_n\}$ is bounded and monotonic. The standard proofs of these properties are

$$\begin{aligned} x_n &= 1+\sum_{k=1}^{n}\frac{\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)}{k!} \\ &< 1+\sum_{k=1}^{n}\frac{1}{2^{k-1}} = 3-\frac{1}{2^{n-1}} < 3\,, \end{aligned} \tag{B}$$

and

$$x_n < 1+\sum_{k=1}^{n+1}\frac{\left(1-\frac{1}{n+1}\right)\ldots\left(1-\frac{k-1}{n+1}\right)}{k!} = x_{n+1}\,. \tag{M}$$

However, in reference 1, Bernoulli's inequality

$$(1+x)^n \geqslant 1+nx \quad (x>-1, n\in\mathbb{N}) \tag{BI}$$

and the arithmetic-geometric mean inequality

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} = G_n(a) \leqslant A_n(a) = \frac{1}{n}\sum_{k=1}^{n} a_k\,, \tag{AGM}$$

where $a=(a_1,\ldots,a_n)$ and $a_k>0$ for $1\leqslant k\leqslant n$, are each used to establish both the boundedness and the monotonicity of $\{x_n\}$. We shall give several new proofs of these propositions. An additional inequality that we use is

$$\left(1+\frac{1}{l}\right)^k < 1+\frac{k}{l}+\frac{k^2}{l^2} \quad (l\in\mathbb{N}, k\in\{1,2,\ldots,l\}) \tag{I}$$

which is easily proved by induction on k (for fixed l).

2. Monotonicity of $\{x_n\}$

2.1. By (BI),

$$\frac{x_n}{x_{n-1}} = \left(1-\frac{1}{n^2}\right)^n\frac{n}{n-1} \geqslant \left(1-\frac{n}{n^2}\right)\frac{n}{n-1} = 1\,.$$

The same inequality is proved in reference 1 as follows:

$$\begin{aligned} \frac{x_n}{x_{n-1}} &= \left(1-\frac{1}{n^2}\right)^{n-1}\frac{n+1}{n} \geqslant \left(1-\frac{n-1}{n^2}\right)\frac{n+1}{n} \\ &= 1+\frac{1}{n^3} > 1\,. \end{aligned}$$

The inequality (M) can also be deduced from (BI) by noting that

$$\begin{aligned} \binom{n}{k}\frac{1}{n^k} \leqslant \binom{n+1}{k}\frac{1}{(n+1)^k} &\iff \left(1-\frac{1}{n+1}\right)^k \\ &\geqslant 1-\frac{k}{n+1}\,. \end{aligned}$$

2.2. In reference 1, (M) is obtained by applying (AGM) to the n-tuple

$$a = \left(\frac{n}{n-1},\ldots,\frac{n}{n-1},1\right)\,.$$

2.3. Let $k=n-1$, $l=n^2-1$ in (I). Then

$$\begin{aligned} \frac{x_{n-1}}{x_n} &= \left(1+\frac{1}{n^2-1}\right)^{n-1}\frac{n}{n+1} \\ &< \left(1+\frac{1}{n+1}+\frac{1}{(n+1)^2}\right)\frac{n}{n+1} \\ &< \left(1+\frac{1}{n}\right)\frac{n}{n+1} = 1\,. \end{aligned}$$

3. Boundedness of $\{x_n\}$

3.1. Let s be an integer $\geqslant 2$. Then, by (BI),

$$\begin{aligned} 1 &> \left(1-\frac{1}{s^2n^2}\right)^{sn} \\ &= \left[\left(1-\frac{1}{sn}\right)^n\right]^s x_{sn} \\ &\geqslant \left[1-\frac{n}{sn}\right]^s x_{sn} \\ &= \left(1-\frac{1}{s}\right)^s x_{sn}\,. \end{aligned}$$

When $s=2$, we have

$$x_{2n} < 4,$$

which yields $x_n<4$, as in reference 1; and when $s=6$,

$$x_{6n} < \left(\frac{6}{5}\right)^6 < 3,$$

so that $x_n < 3$, which is (B).

Alternatively, again by (BI),

$$\frac{x_{k-1}}{x_k} = \left(1 + \frac{1}{k^2-1}\right)^{k-1} \frac{k}{k+1}$$
$$> \left(1 + \frac{1}{k+1}\right) \frac{k}{k+1} = \frac{k(k+2)}{(k+1)^2}$$

and, for $n \geqslant k > 1$,

$$\frac{x_{k-1}}{x_n} = \prod_{i=k}^{n} \frac{x_{i-1}}{x_i} \geqslant \prod_{i=k}^{n} \frac{i(i+2)}{(i+1)^2}$$
$$= \frac{k}{n+1} \frac{n+2}{k+1} > \frac{k}{k+1}\,.$$

Hence $x_n < ((k+1)/k)x_{k-1}$, so that from $k = 2$ we obtain $x_n < 3$.

3.2. By use of (AGM), with $a = (\frac{1}{2}, \frac{1}{2}, 1, \ldots, 1)$, we have

$$\left(\frac{1}{4}\right)^{1/n} < \frac{n-1}{n} < \frac{n}{n+1}\,,$$

i.e. $1/4 < 1/x_n$, so that

$$x_n < 4.$$

On the other hand, choosing the n-tuple

$$a = (a_1, a_1, a_1, a_2, a_2, a_2, 1, \ldots, 1),$$

where

$$a_1 = \frac{5}{6} - \sqrt{\frac{25}{36} - \sqrt[3]{\frac{1}{3}}}\,, \quad a_2 = \frac{5}{6} + \sqrt{\frac{25}{36} - \sqrt[3]{\frac{1}{3}}}\,,$$

we get $x_n < 3$ when $n \geqslant 6$.

The simple inequality

$$G_{n+k+1}(a) < A_{n+k+1}(a)\,,$$

with

$$a = \left(\underbrace{\frac{k}{k+1}, \ldots, \frac{k}{k+1}}_{k+1}, \underbrace{\frac{n+1}{n}, \ldots, \frac{n+1}{n}}_{n},\right)$$

implies $x_n < ((k+1)/k)\,x_k$, which is weaker than the similar relation in section 3.1.

3.3. From the inequality (I), with $k = l = n$, $x_n < 3$ follows immediately.

Reference

1. Liang Shi-Lui, The convergence of the sequence $\{(1+(1/n))^n\}$, *Mathematical Spectrum* **22** (1989/90), pp. 17–18.

Milan Jovanović *received his Ph.D. degree in mathematics at Belgrade University, Yugoslavia. His main field of research covers convexity, inequalities and mathematical education. He works as Associate Professor at the Banja Luka University, Republic of Srpska, Bosnia and Herzegovina.*

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Reversing Digits Revisited

ROGER COOK

3320 − 0233 = 3032 (base 5).

Introduction

In a recent article (reference 1) we considered the problem of reversing digits. Starting with a 4-digit number $abcd$, where the digits are in descending order (e.g. 8742), we reverse the digits to give $dcba$ (2478) and subtract from the original number (to give 6264) and reorder the digits into descending order (6642). Iterating this process we produce a sequence of 4-digit integers and most numbers will end up at the fixed point 7641. One of the questions raised at the end of the article was to determine those bases B for which the analogous process produces a unique fixed point (other than the trivial fixed point 0000). In this article we give a partial answer to that question by determining the 4- and 5-digit fixed points.

4-digit numbers

We begin by determining the bases B (and the fixed points $abcd$) which satisfy

$$a \geqslant b \geqslant c \geqslant d > 0 \tag{1}$$

together with an initial assumption to simplify the argument

$$(a-d) > (b-c) > 0. \tag{2}$$

Because $b > c$, when we include the necessary 'borrows' the subtraction becomes

$$\begin{array}{ccccc} & a & b-1 & c-1+B & d+B \\ - & d & c & b & a \\ \hline & (a-d) & (b-c)-1 & B-1-(b-c) & B-(a-d). \\ \hline \end{array}$$

For a fixed point, these 4 digits must be a permutation of $abcd$. Since $d > 0$ we have

$$(b-c)-1 < (a-d) < a,$$

and from (2)

$$B-(a-d) \leqslant B-1-(b-c),$$

so

$$a = B-1-(b-c). \tag{3}$$

Also $B-(a-d) = d$ implies that $B = a$, which is not possible. Therefore

$$d = (b-c)-1. \tag{4}$$

Suppose now that $b = B-(a-d)$ and $c = (a-d)$ so that $b = B-c$. Also

$$c = a-d = B-1-b+c-d$$

so that

$$B = b+d+1 = (B-c)+(b-c-1)+1.$$

Hence $b = 2c$ and then

$$a = B-1-c < B-c = b$$

to give a contradiction. Therefore

$$b = (a-d) \tag{5}$$

and

$$c = B-(a-d). \tag{6}$$

Rewriting equations (3) to (6) we have

$$\begin{aligned} a+b-c \quad\;\; &= B-1, \\ b-c-d &= 1, \\ a-b \quad\;\; -d &= 0, \\ a \quad\;\; +c-d &= B. \end{aligned}$$

These equations simplify to give

$$\begin{aligned} a+b-c \quad\;\; &= B-1, \\ b-c-\;\; d &= 1, \\ c+3d &= B-3, \\ 5d &= B-5. \end{aligned}$$

Clearly the last equation has an integer solution d if and only if B is a multiple of 5; specifically $B = 5(d+1)$. We can then substitute back through the equations to find

$$c = 2d+2, \quad b = 3d+3, \quad a = 4d+3,$$

which can conveniently be expressed in vector notation

$$(a, b, c, d) = (4d, 3d, 2d, d) + (3, 3, 2, 0). \tag{7}$$

For example, in base 10 we get the fixed point 7641. It is straightforward to check that for any base $B > 5$ of the form $5(d+1)$ equation (7) does give a fixed point in base B.

Theorem 1. *There is a fixed point abcd satisfying $a \geqslant b \geqslant c \geqslant d > 0$ and $(a-d) > (b-c) > 0$ if and only if the base B is a multiple of 5 with $B > 5$. Writing $B = 5(d+1)$ the fixed point abcd is given by (7).*

The case $d = 0$

We have proved theorem 1 under the additional assumption $d > 0$ and (2), made in order to simplify our calculations. Now we have to see to what extent these assumptions can be removed. We begin by noting that (7) also gives us the fixed point 3320 in base 5, with $d = 0$. Suppose now that $d = 0$ and

$$a = (a-d) > (b-c) > 0. \tag{8}$$

Then the subtraction gives the 4 digits

$$a, \quad B-a, \quad (b-c)-1, \quad B-1-(b-c),$$

which must be a permutation of $a, b, c, 0$. Since $B > a$ we have

$$0 = d = b-c-1,$$

so that

$$b = c+1$$

and

$$B-1-(b-c) = B-2.$$

Our assumptions imply that $a > 1$ and we have the 4 digits $a, B-a, B-2, 0$, so $b = B-2$ and $c = B-a$. Then $c = b-1 = B-3$ and $a = B-c = 3$. Therefore the 4 digits are, in decreasing order, $3, B-2, B-3, 0$. Therefore

B can only be 4 or 5. Thus if $d = 0$ and (8) holds, the only possible fixed points are 3210 in base 4 and 3320 in base 5.

If $d = 0$ and $b = c$ then $abcd$ is $abb0$, which gives the 4 digits $(a-1)$, $(B-1)$, $(B-1)$, $(B-a)$ on subtraction. Since $B > 1$ and $B > a$ we have $a = 1$ and the only possible fixed point is 1110 in base 2. If $d = 0$ and $(a-d) = (b-c)$ then $abcd$ is $aa00$. We get just one fixed point, 1100 in base 2.

Theorem 2. *Non-trivial fixed points with $d = 0$ can only occur when B is 2, 4 or 5 and are 1100 or 1110 in base 2, 3210 in base 4 and 3320 in base 5.*

The exceptional cases

Theorem 2 deals completely with the case $d = 0$, so now we can suppose that $d > 0$ and consider what happens when the condition (2) fails.

When
$$(a-d) = (b-c) = 0$$
the number $abcd$ is $aaaa$ and we get the trivial fixed point 0000. When
$$(a-d) > (b-c) = 0$$
then $abcd$ is of the form $abbd$. On subtraction we get the 4 digits
$$(a-d)-1, \quad B-1, \quad B-1, \quad B-(a-d).$$
Therefore $a = b = c = B-1$ and $(a-d)-1 < a$ so $(a-d) = 1$. But this implies that $d = 0$, which contradicts the assumption $d > 0$. Finally we suppose that
$$(a-d) = (b-c) > 0$$
so that $abcd$ is of the form $aadd$. The 4 digits produced in the subtraction are
$$a-d > a-d-1, \quad B-(a-d) > B-1-(a-d).$$
These must be a, a, d, d in some order. Hence $a-d = a$ and $d = 0$, and again this contradicts the assumption that $d > 0$. Thus we can now state a stronger version of theorem 1:

Theorem 3. *There is a fixed point abcd with $d > 0$ if and only if the base B is a multiple of 5 and $B > 5$. Writing $B = 5(d+1)$ the fixed point is given by (7).*

5-digit numbers

The same ideas work with longer numbers. We consider a 5-digit non-trivial fixed point $abcde$ in base B, where
$$a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 0. \tag{9}$$
If we assume that $b > d$ then the subtraction (with the required 'borrows') becomes
$$\begin{array}{rccccc} & a & b-1 & c-1+B & d-1+B & e+B \\ - & e & d & c & b & a \\ \hline & (a-e) & (b-d)-1 & B-1 & B-1-(b-d) & B-(a-e) \end{array}$$

Clearly
$$a = B-1$$
and
$$(b-d)-1 < (a-e) \leqslant a.$$
If we assume that
$$(a-e) > (b-d) > 0, \tag{10}$$
then
$$B-(a-e) \leqslant B-1-(b-d).$$
Now $B-(a-e) = e$ implies that $a = B$, which is not possible, Therefore
$$e = (b-d)-1.$$
Since $a = B-1$ we have $B-(a-e) = e+1$, and this must be d. Then
$$b = e+d+1 = 2e+2.$$
The remaining two digits are
$$a-e = B-1-e$$
and
$$B-1-(b-d) = B-2-e.$$
Hence $c = B-2-e$ and
$$b = B-1-e = 2e+2,$$
so that
$$B = 3(e+1).$$
Thus the 5-digit fixed points which satisfy (10) occur only when B is a multiple of 3, say $B = 3(e+1)$, and are then given by
$$(a, b, c, d, e) = (3e, 2e, 2e, e, e) + (2, 2, 1, 1, 0). \tag{11}$$
For example, taking $e = 1$ we find that 54321 is a fixed point in base 6. It is easy to check that for any B of the form $3(e+1)$, (11) does indeed give a fixed point in base B.

Finally we have to eliminate the cases when (10) is not satisfied. If $b-d = 0$ then the fixed point is $abbbe$. The subtraction produces the 5 digits
$$a-e-1, \quad B-1, \quad B-1, \quad B-1, \quad B-(a-e).$$
The only such fixed point is 11110 in base 2. If $a-e = b-d$ then the fixed point is $aacee$. For a non-trivial fixed point we have $a > e$. Considering the digits generated by the subtraction we find that the only such case is 11100 in base 2.

Theorem 4. *Let abcde be a non-trivial fixed point in base B. Then either $B = 2$, and the fixed point is 11110 or 11100, or B is divisible by 3, $B = 3(e+1)$, and the fixed point is given by (11).*

One corollary of these results is that when the base B is such that a non-trivial 4- or 5-digit fixed point can exist it is the only fixed point, except for the case of base 2.

Even longer numbers

There is no difficulty in principle in extending these results to longer numbers, but the details become more cumbersome and some aspects of the problem change. For example, with 6-digit numbers in base 8 there are at least two non-trivial fixed points: 664320 and 774433. Some other 6-digit fixed points found by computer search are

Base	Number
6	443220
10	766431
13	a97541
17	ed9741

where in a base $B > 10$ we use the letters a to z to represent the numbers 10 to 35 respectively.

A short computer search of 7-digit numbers found an interesting sequence of *flip-flops*, i.e. pairs of numbers which alternate in the sequence:

Base	Numbers	
9	8665331	8755322
13	c997542	ca87533
17	gcc9753	gdb9744
21	kffb964	kgeb955
25	oiidb75	ojhdb66

As we cast our eyes down the columns of numbers the pattern becomes clear. The base B is of the form $4u + 5$, the first number is then

$$u(4, 3, 3, 2, 2, 1, 1) + (4, 3, 3, 3, 1, 2, 0)$$

and the second number is

$$u(4, 3, 3, 2, 2, 1, 1) + (4, 4, 2, 3, 1, 1, 1).$$

However there is a cheap trick that enables us to generate longer fixed points in some bases. The original fixed point 7641 in base 10 can be lengthened by inserting k copies of the *seed pair* (63) to give a fixed point $76(6\ldots6)4(3\ldots3)1$ with $2k + 4$ digits. More generally, in a base B of the form $3t + 1$ we can use $(2t, t)$ as a seed pair to lengthen fixed points; details are given in reference 1. Thus, when $B = 3t + 1$ and also $B = 5(d + 1)$ we have a 4-digit fixed point and a seed pair which can be used to lengthen it. The conditions on B are satisfied when

$$B = 15h + 10 = 5(3h + 2) = 3(5h + 3) + 1.$$

Then the fixed point in base $15h + 10$ is

$$\begin{aligned}(12h+4, 9h+3, 6h+2, 3h+1) &+ (3, 3, 2, 0)\\ &= (12h, 9h, 6h, 3h) + (7, 6, 4, 1)\end{aligned}$$

and the seed pair is

$$(10h + 6, 5h + 3).$$

The subtraction becomes

$$\begin{array}{rcccc} & 12h+7 & 10h+6\ldots10h+6 & 9h+6 \\ - & 3h+1 & 5h+3\ldots\ \ 5h+3 & 6h+4 \\ \text{with 'borrows'} & & & -1 \\ \hline & 9h+6 & 5h+3\ldots\ \ 5h+3 & 3h+1 \\ \hline \end{array}$$

$$\begin{array}{rcccc} & 6h+4 & 5h+3\ldots\ \ 5h+3 & 3h+\ \ 1 \\ - & 9h+6 & 10h+6\ldots10h+6 & 12h+\ \ 7 \\ \text{with 'borrows'} & 15h+9 & 15h+9\ldots15h+9 & 15h+10 \\ \hline & 12h+7 & 10h+6\ldots10h+6 & 6h+\ \ 4 \\ \hline \end{array}$$

Unfortunately the same trick does not work when we start with the 5-digit fixed points since the condition $B = 3t + 1$ for seed pairs is incompatible with the existence of fixed points. However, starting with the 6-digit fixed point $a97541$ in base 13 we can produce long fixed points in base 13 using the seed pair (8, 4).

Reference

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Roger Cook *is a member of the editorial board of* Mathematical Spectrum *and has recently been appointed recorder-elect of the mathematical section of the British Association for the Advancement of Science. If any of our readers have (sensible) suggestions about how to obtain more media coverage of mathematics he would be delighted to hear from you!*

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A ‘Three Digits’ Problem

P. RADMORE and G. STEPHENSON

A problem which has recently been circulating verbally, and which is a variant of the famous 4 digits, and 4 fours, problem (see Rouse Ball *Mathematical Recreations and Essays*, 1959, and other extensive literature) is that of forming a positive integer N from precisely three identical integers n (where n runs from 1 to 9) using a prescribed set of operations. These are $+$, $-$, $\div$, $\times$, $\sqrt{}$, !, !! (where, for example, $6!! = 6 \cdot 4 \cdot 2 = 48$), powers, the decimal point and $\cdot\dot{n}$, meaning $\cdot n$ recurring (for example, $\cdot\dot{4} = \cdot 44444\ldots = 4/9$). We take $\cdot\dot{9} = \cdot 9999\ldots$ as unity, but it is often possible to find separate expressions which avoid this. As illustrative examples, we give

$$(\cdot 5)^{-5} - 5 = 27, \quad \frac{6!}{6!!} - 6 = 9, \quad 6!! \times \cdot\dot{6} - 6 = 26,$$

$$2 + \sqrt{\frac{2}{\cdot\dot{2}}} = 5, \quad \frac{(\sqrt{9}!)!}{9} + 9 = 89.$$

For N in the range 1 to 24 (inclusive), we have found formulae for the special cases N =1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 15, 18, 20 and 24 as shown in the table, where n runs from 1 to 9. We have not found general formulae involving n for $N = 7$ and $N = 12$, although we have found separate expressions for $N = 7$ and $N = 12$ in terms of each of the integers 1 to 9. For example,

$$\sqrt{\frac{1}{\cdot\dot{1}}}! + 1 = 7, \qquad 3! + \frac{3}{3} = 7,$$

$$\sqrt{\frac{7!! + 7}{\cdot\dot{7}}} = 12, \qquad 8 + \sqrt{8 + 8} = 12.$$

We note that a formula for $N = 48$ can be written by taking the double factorial of that for $N = 6$ given above. For most other values of $N \leq 100$ we have found *at least* one expression (available on request) which produces a specified N for some n in the range 1 to 9. Exceptions are $N = 67, 69, 76$ and 85 for which no expressions have been found. We may conjecture whether 67 is the smallest integer for which an expression using three identical digits and the prescribed operations cannot be found.

N	Alternative expressions avoiding the use of $\cdot\dot{9}$
$n^{n-n} = 1$	
$\frac{(n+n)}{n} = 2$	
$\sqrt{\frac{\sqrt{n \times n}}{\cdot\dot{n}}} = 3$	$\sqrt{9+9-9} = 3$
$\frac{(\sqrt{n}+\sqrt{\cdot\dot{n}})}{\sqrt{\cdot\dot{n}}} = 4$	$\sqrt{9} + \frac{9}{9} = 4$
$\frac{n}{(\cdot n + \cdot n)} = 5$	
$\left(\sqrt{\frac{\sqrt{n \times n}}{\cdot\dot{n}}}\right)! = 6$	$(\sqrt{9+9-9})! = 6$
$\left(\frac{(\sqrt{n}+\sqrt{\cdot\dot{n}})}{\sqrt{\cdot\dot{n}}}\right)!! = 8$	$(\sqrt{9} + \frac{9}{9})!! = 8$
$\frac{(n - \cdot n)}{\cdot n} = 9$	
$\frac{\sqrt{n \times n}}{\cdot n} = 10$	
$\frac{(n + \cdot n)}{\cdot n} = 11$	
$\left(\frac{n}{(\cdot n + \cdot n)}\right)!! = 15$	
$\frac{(n+n)}{\cdot\dot{n}} = 18$	$(\sqrt{9} + \sqrt{9})\sqrt{9} = 18$
$\frac{(n+n)}{\cdot n} = 20$	
$\left(\frac{(\sqrt{n}+\sqrt{\cdot\dot{n}})}{\sqrt{\cdot\dot{n}}}\right)! = 24$	$\sqrt{9}^{\sqrt{9}} - \sqrt{9} = 24$

We are not aware of any publications relating to this problem, but if any reader or author knows of any such work we would be glad to hear from them.

We are grateful to Adrian and Philip Chapman for bringing this problem to our attention.

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In how many ways can n non-challenging rooks be placed on the white squares of an $n \times n$ chessboard?

An Introduction to Geometric Inequalities

ZHANG YUN

This article introduces inequalities which feature a number of geometric entities connected with a triangle.

Many inequalities hold for all values in a certain range of the variables concerned. For instance, if $a, b \geq 0$, then

$$\frac{a+b}{2} \geq (ab)^{1/2}. \tag{1}$$

The left-hand side of (1) is the arithmetic mean of a and b, while the right-hand side is the geometric mean. The inequality (1) is, in fact, a particular case of the famous arithmetic mean–geometric mean inequality

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n) \geq (a_1, a_2 \ldots a_n)^{1/n},$$

which holds whenever $a_1, a_2, \ldots, a_n$ are non-negative.

Another well-known inequality is

$$(x_1^2 + y_1^2)^{1/2} + (x_2^2 + y_2^2)^{1/2} \geq [(x_1 + x_2)^2 + (y_1 + y_2)^2]^{1/2}, \tag{2}$$

which is valid for all real numbers x_1, x_2, y_1, y_2. It has the geometrical interpretation

$$\mathrm{OP} + \mathrm{PR} \geq \mathrm{OR},$$

sketched in figure 1.

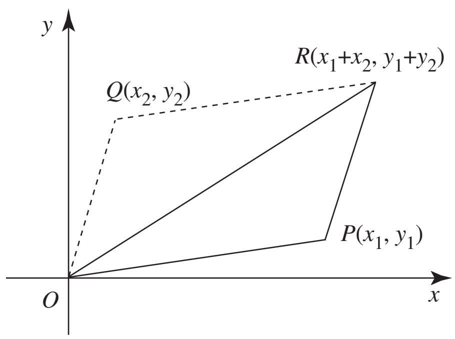

Figure 1.

The theme of this article is the derivation of some inequalities which involve various magnitudes relating to a given triangle ABC. We denote the lengths of the sides BC, CA, AB by a, b, c, respectively, and we put $s = \frac{1}{2}(a+b+c)$; also the radius of the circumcircle is R, the radius of the inscribed circle is r, and the area of ABC is $\triangle$. Moreover no confusion arises from the practice of denoting the angles of ABC at the vertices A, B, C, respectively, by the same letters A, B, C.

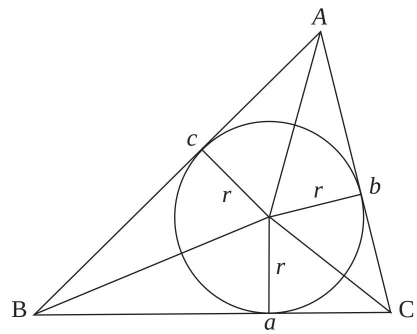

Figure 2.

A powerful tool for the investigation of inequalities is the notion of convexity explored in reference 1. The real function f on the interval I is called *convex* if, for all a, b in I,

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

For the geometrical meaning of the definition see figure 1 of reference 1. However, if f is twice differentiable on I, then the most useful criterion for convexity is that $f''(x) \geq 0$ in I.

Now, as is shown in reference 1, a notable characteristic of a convex function f is that not only does the defining property (3) hold, but, more generally,

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leq \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n} \tag{4}$$

for any real numbers $a_1, a_2, \ldots, a_n$ in I. For example, consider the function

$$f(x) = -\sin x$$

in the interval $[0, \pi]$. Since

$$f''(x) = \sin x \geq 0 \quad \text{for} \quad 0 \leq x \leq \pi,$$

f is convex in $[0, \pi]$ and so (4) holds, i.e.

$$-\sin\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leq \frac{-\sin a_1 - \sin a_2 - \cdots - \sin a_n}{n}$$

or

$$\sin\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geq \frac{\sin a_1 + \sin a_2 + \cdots + \sin a_n}{n}. \tag{5}$$

It is also of interest to note that the idea of convexity yields a particularly simple proof of the arithmetic mean–geometric mean inequality. (See corollary 2 of theorem 2 in reference 1.)

We are now ready for the proof of the first of our geometric inequalities.

Theorem 1. *(The Pólya–Szego inequality.)*

$$(abc)^{2/3} \geq \frac{4}{\sqrt{3}}\triangle\,.$$

Proof. The sine formula applied to the triangle ABC is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

and so

$$\triangle = \tfrac{1}{2}ab\sin C = \frac{abc}{4R}.$$

It is also easy to see that (see figure 2)

$$\triangle = \tfrac{1}{2}r(a+b+c) = sr \tag{6}$$

and therefore

$$abc = 4R\triangle = 2Rr(a+b+c). \tag{7}$$

Using the arithmetic mean–geometric mean inequality

$$\frac{1}{3}(a+b+c) \geq (abc)^{1/3},$$

we derive from (7) that

$$abc \geq 6Rr(abc)^{1/3},$$

and then from (6) that

$$(abc)^{2/3} \geq 6Rr = \frac{6R\triangle}{s}. \tag{8}$$

But, by (5),

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\Big(\frac{A+B+C}{3}\Big) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

or

$$\frac{a+b+c}{2R} \leq \frac{3\sqrt{3}}{2},$$

i.e.

$$s \leq \frac{3\sqrt{3}}{2}R.$$

Hence, by (8),

$$(abc)^{2/3} \geq 6R\triangle\frac{2}{3\sqrt{3}R} = \frac{4}{\sqrt{3}}\triangle,$$

which was to be proved.

Corollary. *(Goldstone's inequality.)*

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 16s^2r^2.$$

Proof. By the arithmetic mean–geometric mean inequality,

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 3(a^4b^4c^4)^{1/3} = 3(abc)^{4/3}.$$

Using also the Pólya–Szego inequality $(abc)^{2/3} \geq \frac{4}{\sqrt{3}}\triangle$, we have

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 3\Big(\frac{4}{\sqrt{3}}\triangle\Big)^2 = 16\triangle^2$$

and so, by (6),

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 16s^2r^2.$$

Theorem 2. *(Complement to the Pólya–Szego inequality.)*

$$(abc)^{2/3}\Big(\frac{2r}{R}\Big)^{1/2} \leq \frac{4}{\sqrt{3}}\triangle\,.$$

Proof. The identity (7) written

$$abc = 4R\triangle = 4Rsr$$

shows that

$$(abc)^{2/3} = \frac{4R\triangle}{(4Rsr)^{1/3}} = 2\Big(\frac{2R^2}{r}\Big)^{1/3}\frac{\triangle}{s^{1/3}}. \tag{9}$$

Also, by the arithmetic mean–geometric mean inequality,

$$s = \tfrac{1}{2}(a+b+c) \geq \tfrac{3}{2}(abc)^{1/3} = \tfrac{3}{2}(4R\triangle)^{1/3}$$

and so

$$s \geq \frac{3}{2}\Big(4\frac{R}{r}sr^2\Big)^{1/3}.$$

Hence

$$s^3 \geq \frac{27}{8}\frac{4R}{r}sr^2,$$

or

$$s^2 \geq 27\Big(\frac{R}{2r}\Big)r^2,$$

i.e.

$$s \geq 3\sqrt{3}\Big(\frac{R}{2r}\Big)^{1/2}r.$$

Thus, by use also of (9),

$$\begin{aligned}(abc)^{2/3} &\leq 2\Big(\frac{2R^2}{r}\Big)^{1/3}\triangle\frac{1}{\sqrt{3}}\Big(\frac{2r}{R}\Big)^{1/6}\frac{1}{r^{1/3}}\\ &= 2^{4/3}\Big(\frac{R}{r}\Big)^{2/3}\triangle\frac{1}{\sqrt{3}}\Big(\frac{2r}{R}\Big)^{1/6}\\ &= 2^2\Big(\frac{R}{2r}\Big)^{2/3}\triangle\frac{1}{\sqrt{3}}\Big(\frac{2r}{R}\Big)^{1/6}\\ &= \frac{4}{\sqrt{3}}\triangle\Big(\frac{R}{2r}\Big)^{1/2},\end{aligned}$$

and this is the desired inequality.

Combining theorems 1 and 2 we have

$$(abc)^{2/3}\left(\frac{2r}{R}\right)^{1/2} \le \frac{4}{\sqrt{3}}\triangle \le (abc)^{2/3}, \tag{10}$$

which implies that $(2r/R)^{1/2} \le 1$. Therefore finally we have obtained:

Theorem 3. *(Euler's inequality.)*

$$2r \le R.$$

The chain of inequalities (10) is of interest in its own right, but it should be noted that Euler's inequality can be proved without analytical tools. For instance theorem 23 of reference 2 provides a purely geometric proof of the identity

$$OI^2 = R^2 - 2Rr, \tag{11}$$

in which O and I are the circumcentre and incentre of the triangle ABC; and Euler's inequality is an immediate corollary of (11).

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Twin Primes

PANTELIS A. DAMIANOU

(3, 5), (5, 7), (11, 13), (17, 19) are twin primes. But how many are there?

1. Introduction

One of the most famous problems in number theory is the twin prime conjecture. Two primes p, q form a twin pair if $q - p = 2$. The twin prime conjecture is that the number of such pairs is infinite. This is a very simple statement but perhaps very difficult to prove. I do not know the exact origin of this problem. In 1849, A. de Polignac conjectured that for every even number $2n$ there are infinitely many pairs of consecutive primes which differ by $2n$ (reference 1). For $n = 1$ this statement reduces to the twin prime conjecture.

I will list some facts I know about twin primes. Most of them I saw in the excellent books by Burton and Ribenboim (references 2, 3).

1. (Clement, reference 4) If $n \ge 2$, the integers n and $n+2$ form a pair of twin primes if and only if

$$4\left[(n-1)!+1\right]+n \equiv 0 \pmod{n\,(n+2)}\,. \tag{1}$$

2. The twin prime conjecture is equivalent to the existence of infinitely many integers of the form $n^2 - 1$ with precisely four divisors in the positive integers. (Writing $p = n-1$, $q = n+1$ the divisors are just 1, p, q and pq.)

3. One can also prove that (p, q) form a pair of twin primes if and only if the number $n = pq$ satisfies

$$\phi(n)\sigma(n) = (n-3)(n+1)\,. \tag{2}$$

   We remind the reader that $\phi(n)$ is the number of positive integers less than n, and relatively prime to n, and $\sigma(n)$ equals the sum of positive divisors of n.

4. In reference 5, J. R. Chen has shown that there exists an infinite number of primes p such that $p+2$ has at most two factors. This was the final step in a sequence of results obtained using deep methods, known as 'sieve methods', each step reducing the number of prime factors of $p+2$. The next step is to show that $p+2$ is prime infinitely often.

   It is interesting to ask whether one of the two factors is always a twin prime. This fails for some values of p, for example $p = 1907$.

5. In 1919 Viggo Brun proved that the sum

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right) + \cdots + \left(\frac{1}{p}+\frac{1}{p+2}\right)+\cdots \tag{3}$$

   (where p, $p+2$ are twin primes) is either finite or convergent (reference 6). A more accessible reference is found in chapter 15 of the book by Rademacher (reference 7). The sum is called Brun's constant (i.e. 1.902 160 577 783 278 . . .). In the process of calculating this constant, T. Nicely discovered a bug in Intel's Pentium microprocessor.

6. In 1923 Hardy and Littlewood conjectured that there are about

$$C_2 \frac{2x}{(\log x)^2} \tag{4}$$

twin primes less than x. The constant C_2 is called the twin prime constant:

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.660\,16\ldots. \tag{5}$$

It is worth noting that the same constant appears also in conjunction with the Goldbach conjecture. Specifically, it is related to the number of representations of an even integer as the sum of two primes. The two problems can be regarded as duals. The methods Chen (reference 5) used can also be used to show that all large even integers can be expressed as a sum $p + p'$, where p is prime and p' has at most two prime factors.

In this article we propose an algorithm for generating twin prime pairs. The idea is a generalization of Euclid's method of proving that the number of primes is infinite. The proof, which is given in Book IX of Euclid, goes as follows: we form the number

$$p_1 p_2 \ldots p_n + 1\,, \tag{6}$$

where $p_1, p_2, \ldots, p_n$ are the first n primes. This number is either a prime or is divisible by a prime which is greater than $p_1, p_2, \ldots, p_n$. In either case we have produced a new prime from a list of known primes. It was conjectured by Dan Shanks that this process of Euclid generates all primes (references 8, 9). We propose a similar algorithm for generating a new pair of twin primes from a list of known ones.

2. An algorithm

Define

$$\begin{aligned} t_1 &= 3 \cdot 5 + 2\,, \\ t_2 &= 3 \cdot 5 \cdot 5 \cdot 7 + 2\,, \\ t_3 &= 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 2\,, \\ t_4 &= 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 + 2\,, \\ &\;\vdots \\ t_n &= D_1 \cdot D_2 \ldots D_n + 2\,, \end{aligned} \tag{7}$$

where $D_i = p(p+2)$ and $\Delta_i = (p, p+2)$ is the ith pair of twin primes. We use the notation Δ_i to denote the ith pair of twin primes, but sometimes it means the set consisting of the two twins. So $q \in \Delta_i$ means that q is one of the twin primes in the ith pair, i.e. $q = p$ or $q = p + 2$. We also use the notation $\Delta_k = \left(p^{(k)}, q^{(k)}\right)$, so $p^{(2)} = 5$ and $q^{(2)} = 7$. Therefore $D_k = p^{(k)} q^{(k)}$ and

$$t_n = \prod_{k=1}^{n} p^{(k)} q^{(k)} + 2\,. \tag{8}$$

The symbol Δ_* denotes a twin pair whose index is unknown. We propose the following algorithm: if $t_n \in \Delta_j$ for some j then we stop, since j is clearly bigger than n. Otherwise we factor

$$t_n = p_1 p_2 \ldots p_s \tag{9}$$

into a product of prime factors. We examine each of the factors p_k for $1 \le k \le s$. If p_k is less than or equal to $q^{(n)}$ we ignore it since it cannot produce a new pair. If $p_k > q^{(n)}$ then we form the numbers $p_k + 2$ and $p_k - 2$. If one of the numbers $p_k + 2$ or $p_k - 2$ is prime we stop and list the twin pair obtained. Otherwise we factor the two numbers $p_k + 2$ and $p_k - 2$ and repeat. (Equivalently, one can try another $p_j > p_k$ first.) This process clearly terminates after a finite number of steps.

We illustrate the algorithm with an example:

$$\begin{aligned} t_5 &= 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 31 + 2 \\ &= 218\,000\,532\,77, \end{aligned} \tag{10}$$

which turns out to be a prime p. We examine $p + 2$ and $p - 2$. Of course $p - 2$ is the product of (twin) primes we used to construct t_5. On the other hand, $p + 2$ factors as $p + 2 = rs$, a product of two primes with $r = 227 \in \Delta_{16}$ and $s = 960\,354\,77 \in \Delta_*$.

It turns out that t_5 and t_{10} are both primes. It is natural to ask whether t_n is prime for infinitely many values of n.

We make the following conjectures:

1. This algorithm generates a larger pair of twin primes each time. In other words, by applying the algorithm to t_n, we obtain a new pair Δ_j with $j > n$.

2. Given an arbitrary pair Δ_j with $j \ge 4$, this pair is obtained by applying the algorithm to some t_n, with $n < j$.

3. Experimental evidence

The table opposite gives some evidence for these conjectures. For each t_n we list some twin pairs obtained by applying the algorithm to t_n.

Acknowledgement

I would like to thank John Brillhart for showing me some factorization methods. Following his advice I was able to find some factors of t_n, for $n > 15$. This work was completed in the period 1997–1999 (another twin pair!).

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t_i	Some pairs obtained from t_i											
t_1	Δ_4											
t_2	Δ_4	Δ_5										
t_3	Δ_4	Δ_6	Δ_{14}									
t_4	Δ_{15}	Δ_{30}	Δ_{310}									
t_5	Δ_{16}	Δ_*										
t_6	Δ_7	Δ_{12}	Δ_*									
t_7	Δ_8	Δ_{10}	Δ_{18}	Δ_{55}	Δ_{60}	Δ_{68}	Δ_{1085}	Δ_*				
t_8	Δ_9	Δ_{11}	Δ_{17}	Δ_{19}	Δ_{21}	Δ_{23}	Δ_{25}	Δ_{39}	Δ_{40}	Δ_{65}	Δ_{299}	
t_9	Δ_{13}	Δ_{20}	Δ_{42}	Δ_{46}	Δ_{54}	Δ_{157}	Δ_{297}	Δ_{808}	Δ_*			
t_{10}	Δ_{12}	Δ_{13}	Δ_{25}	Δ_{61}	Δ_{421}	Δ_{1150}	Δ_*					
t_{11}	Δ_{13}	Δ_{14}	Δ_{15}	Δ_{16}	Δ_{17}	Δ_{98}	Δ_{128}	Δ_{482}	Δ_*			
t_{12}	Δ_{14}	Δ_{16}	Δ_{21}	Δ_{22}	Δ_{25}	Δ_{30}	Δ_{32}	Δ_{40}	Δ_{59}	Δ_{62}	Δ_{114}	Δ_{206}
t_{13}	Δ_{24}	Δ_{31}	Δ_{41}	Δ_{50}	Δ_{66}	Δ_{86}	Δ_{108}	Δ_{173}	Δ_{232}	Δ_{244}	Δ_{649}	Δ_{991}
t_{14}	Δ_{23}	Δ_{26}	Δ_{27}	Δ_{28}	Δ_{33}	Δ_{36}	Δ_{37}	Δ_{47}	Δ_{157}	Δ_{192}	Δ_{260}	Δ_{1077}
t_{15}	Δ_{17}	Δ_{18}	Δ_{27}	Δ_{28}	Δ_{32}	Δ_{42}	Δ_{67}	Δ_{132}	Δ_{155}	Δ_{260}	Δ_{333}	Δ_{833}

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***Pantelis Damianou** is an Associate Professor of Mathematics at the University of Cyprus. His only familiarity with twins until recently were his niece and nephew. His interest in twin primes began in 1997 when Christian Ballot, a number theorist from Université de Caen, visited Cyprus. Christian needed a vacation to calm down after finding out that his wife was expecting twins!*

Braintwister

10. Big fleas have little fleas . . .

A little flea was eight metres north of a big flea. The big flea set off eastwards at a steady speed and the little flea pursued it at three times that speed. Furthermore, throughout the pursuit, the little flea chose its direction so that it was aiming directly at the big flea.

After the big flea had walked a whole number of metres the little flea caught up with him.

How far had the big flea walked?

N.B. You can use some tricky calculus but it is **not** necessary.

VICTOR BRYANT

1999 puzzle

We received an interesting solution from Nick Grigg of Hampton School, Ashford, Middlesex, who was successful with all but two of the numbers between 1 and 100, which was better by three than the editor’s attempt. The three that Nick found which stumped the editor were

$$65 = -1 + \sqrt{\left[(\sqrt{9})!\left[((\sqrt{9})!)! + (\sqrt{9})!\right]\right]},$$

$$67 = 1 + \sqrt{\left[(\sqrt{9})!\left[((\sqrt{9})!)! + (\sqrt{9})!\right]\right]},$$

$$70 = -(1+9) + \left[((\sqrt{9})!)! \div 9\right].$$

So that only leaves 68 and 98.

The year 2000 certainly poses problems for our annual puzzle, as you will soon discover if you try it. Do not forget that you have to use each digit of the year once in order, and that you can use only $+$, $-$, $\times$, $\div$, ! and concatenation (e.g. putting 2 and 0 together to give 20).

Mathematics in the Classroom

Some aspects of logarithmic integration

The idea of logarithms was first conceived by a Scotsman, John Napier (1550–1617), who was also the first person to use the decimal point. He showed that *any* number can be expressed as a power of 10, or e (giving his name to these Naperian or natural logarithms). In this form, and using the rules of exponents, heavy computation can be achieved much more easily. Napier published his logarithmic calculations in convenient tables in 1614 and these have greatly aided scientists and engineers in their numerical work. With the advent of modern calculators, however, together with the slide rule (which was in effect a compact log table) these tables have fallen into disuse as far as A-level students are concerned. But logarithms remain an essential tool in calculus as the following article demonstrates.

The following cases show that sometimes a prescribed integral can be recast into the form:

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x,$$

which, apart from an arbitrary constant, is $\ln f(x)$.

The first case we consider is:

$$I = \int \frac{\mathrm{d}x}{x(k+x^n)} \quad (k = \text{constant}),$$

where n need not be an integer, so that partial fractions may not be an option. This may be written in alternative form:

$$I = \int \frac{x^{-n-1}}{kx^{-n}+1}\,\mathrm{d}x.$$

We see that, apart from an integration constant,

$$I = -\frac{1}{kn}\ln(1+kx^{-n}).$$

As another kind of example of logarithmic integration we consider the integral:

$$J = \int \frac{1+2x^2}{3x+4x^3}\,\mathrm{d}x.$$

We can rewrite J in the form:

$$J = \int \frac{x^r+2x^{r+2}}{3x^{r+1}+4x^{r+3}}\,\mathrm{d}x.$$

Can we now choose r so that J integrates to a logarithmic form? The derivative of the denominator of the integrand is:

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^{r+1}+4x^{r+3}) = 3(r+1)x^r + 4(r+3)x^{r+2}.$$

If we can choose r so that the last form is a constant times (x^r+2x^{r+2}) then J is expressed in a form suitable for logarithmic integration. Clearly, we require:

$$\frac{1}{3(r+1)} = \frac{2}{4(r+3)}, \qquad (*)$$

i.e.

$$r = 3.$$

With this value of r,

$$J = \int \frac{x^3+2x^5}{3x^4+4x^6}\,\mathrm{d}x = \frac{1}{12}\ln(3x^4+4x^6),$$

apart from an integration constant.

Frank Chorlton
Department of Computer Science
Aston University
Birmingham

Notes

(i) No difficulty would have arisen if $(*)$ had had a non-integral solution.
(ii) Of course J could have been evaluated by use of partial fractions, but the calculation would have been more tedious.

Another option is to note that $\mathrm{d}/\mathrm{d}x(3x+4x^3) = 3 + 12x^2$ and then to write the numerator $1+2x^2$ in the form $A + B(3+12x^2)$.

Computer Column

Things to do with your spare (CPU) time

It is not so long ago that computing power was precious enough for CPU time to be carefully scheduled and charged for by the millisecond. How things change! Nowadays, most CPUs on this planet spend their time idling while the monitors attached to them display flying toasters or some other screensaver. Just imagine what a powerful machine researchers would have access to if they could tap only a small fraction of the world's idle computational capacity! The recent phenomenal development of the internet means that researchers can do exactly this.

The idea behind large-scale distributed computing projects is for individuals to download a client program which — *when the individual's CPU would otherwise be idle* — works on a small chunk of a large problem. Once the small chunk of work is completed, the client connects to the coordinating site via the internet, returns the result to the server, and downloads another work unit to process.

In a previous column (*Mathematical Spectrum* **31**, Number 1) I discussed one such distributed computing project: GIMPS — the Great Internet Mersenne Prime Search. Since that article appeared, thousands more people have donated CPU time to the project; the combined power of all these processors is the equivalent of 17 of Cray's most powerful T932 supercomputers at peak power. With all this raw power at its disposal, GIMPS has recently found its fourth prime: $2^{6\,972\,593} - 1$ is the 38th known Mersenne prime.

If the search for prime numbers does not interest you, there are other worthy causes that would be happy to accept any CPU cycles you might wish to donate. Below, I outline three such projects.

Golomb rulers

Golomb rulers were first described by the American mathematician Solomon Golomb, the inventor of pentaminoes. A Golomb ruler is a set of integers $a_1 < \cdots < a_k$ such that all the differences $a_i - a_j$ $(i > j)$ are distinct; we may assume that $a_1 = 0$. If you think about this definition, it is clear that a Golomb ruler is a way to place marks along a line such that each pair of marks measures a unique linear distance. Here is a Golomb ruler with five marks:

The length of a Golomb ruler is simply a_k; the length of the ruler above, for instance, is 11. Mathematicians are interested in finding *optimal* Golomb rulers: the shortest ruler for a given number of marks, k. (This is not just a pure combinatorics problem; it has practical applications. For instance, by placing antennas on the marks of a Golomb ruler, astronomers can construct an interferometer that maximises the recovery of phase information from radio signals.)

You can check that a given ruler is Golomb by writing out a table of all the pairs of marks and their respective distances. For a large number of marks, however, proving the optimality of a Golomb ruler becomes computationally demanding. For example, in 1995 it took four CPU-years on 12 workstations simply to prove that an optimal 19-mark ruler must be larger than 246. This is where your spare CPU cycles can help! If the search for optimal Golomb rulers interests you, then you can download the relevant client from: `http://www.distributed.net/ogr/`

Cracking codes

Several popular encryption algorithms encode information in such a way that a 56-bit key (in other words, a binary number of 56 bits) is required to unlock the information. There are $2^{56} \approx 7 \times 10^{16}$ possible keys and, in general, codebreakers must try all possible keys until they find the correct one. (See *Mathematical Spectrum* **31**, Number 1, pp. 9–13 for a discussion of a particular type of encryption algorithm.)

Data security companies often post challenges; typically, they publish an encrypted message and, if you decode the message before a given date, you win a prize. These challenges increasingly demonstrate that a 56-bit key is vulnerable against a brute-force attack. (A brute-force attack simply means lots of computers trying out every possible decryption key until they find the right one — exactly the sort of attack provided by a distributed computing project.) The European company CS Communications & Systems recently issued a challenge that lasts until 17 March 2000. The prize for deciphering their encrypted message is 10 000 Euros. You are unlikely to win the money working alone: there are simply too many keys to work through for a single desktop machine. If you join a team, however, you can pool your efforts. If you want to join a team devoted to cracking the CS keys, try downloading the client from: `http://www.distributed.net/csc/`

Finding ET

By far the largest computing endeavour in history is the `SETI@home` project. More than one million users have downloaded the `SETI@home` client, and in its first full year of operation the project has accumulated well over 100 000 years of computing time! The project involves the search for extraterrestrial intelligence (SETI).

Astronomers at the Arecibo radio telescope study natural celestial phenomena; as they do so, they record *vast* amounts of data: about 35 Gb per day. There is a small — but intriguing — possibility that somewhere, hidden in all that data, is a signal from an alien intelligence. What would that signal look like? At what frequency would an alien civilisation transmit a signal? No one knows, of course; but `SETI@home` have decided that a good place to look for signals is in the 2.5 MHz band centred at 1420 MHz. Furthermore, if aliens wished to send a message using radio waves, there are good reasons to suppose that they would concentrate the power of the message into a very narrow frequency range. Finally, because Arecibo is fixed in position, the sky drifts past its focus: it takes 12 seconds for a target to move past the focus of the telescope. Terrestrial signals are more-or-less constant; an extraterrestrial signal should get louder and then softer, in Gaussian fashion, over a period of 12 seconds. Thus `SETI@home` combs the data looking for 12 s Gaussians.

The `SETI@home` project first splits the huge amount of telescope data into manageable work units, each of which is about 0.25 Mb in size. A work unit corresponds to 107 seconds of data of a 10 kHz piece of the spectrum. The work unit is sent over the internet to a project member, and the client then starts performing Fast Fourier Transforms on the data. A work unit typically requires over 1.75×10^{11} operations for a full analysis; this is heavy-duty number-crunching! Depending on the speed of the client machine, and the length of time it is idle, a typical home computer takes 10–50 hours to analyse one work unit. When the client has completed its task, it automatically sends the results back to `SETI@home` and requests another unit to process.

If you want to join the `SETI@home` project and — just maybe — discover ET, download the client from: `http://setiathome.ssl.berkeley.edu/`

Stephen Webb

Letters to the Editor

Dear Editor,

Peter Derlien's problem (Volume 31, Number 3, p. 54)

$$\begin{aligned}\sqrt[43]{\sqrt[7]{\sqrt[3]{\frac{1}{\sqrt{x}}}}} &= \left(\left((x^{-1/2})^{1/3}\right)^{1/7}\right)^{1/43} \\ &= x^{-1/1806} \\ &= x^{-1/2}\cdot x^{1/3}\cdot x^{1/7}\cdot x^{1/43} \\ &= \frac{1}{\sqrt{x}}\cdot\sqrt[3]{x}\cdot\sqrt[7]{x}\cdot\sqrt[43]{x}\end{aligned}$$

(since $-\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}=\frac{-1}{1806}$). Therefore the identity is correct.

It is noticed that successive denominators form a sequence in which each term (after the first) is one more than the product of the previous ones. Thus

$$3=2+1,\quad 7=(2\times 3)+1,\quad 43=(2\times 3\times 7)+1;$$

and the next term in this sequence is $2\cdot 3\cdot 7\cdot 43+1=1807$. A denominator can also be found as follows: to find the third denominator, write

$$\left((x^{-1/2})^{1/3}\right)^{1/n}=x^{-1/6n}=x^{-1/2}\cdot x^{1/3}\cdot x^{1/n},$$
$$\text{i.e.}\quad -\frac{1}{6n}=-\frac{1}{2}+\frac{1}{3}+\frac{1}{n},$$

whence $n=7$.

There seems to be a high incidence of primes in the sequence. The sixth term $(2\times 3\times 7\times 43\times 1807)+1=3263443$ is a prime, and only one of the first six, namely 1807, is compound (13×139). Is this a coincidence? Or have we discovered a prime-rich sequence?

Yours sincerely,
BOB BERTUELLO
(12 Pinewood Road
Midsomer Norton
Bath BA3 2RG)

Dear Editor,

Problem 31.7

I was interested in the very nice solution to Problem 31.7 in Volume 32, Number 1. As far as I can see, $\sqrt{3}$ can be replaced by any irrational number strictly between 1 and 2 and the proof still goes through. As the result is obvious when $\sqrt{3}$ is replaced by a rational number, it appears that the argument shows that, if $1\le r\le 2$, then the sequence $\{[nr]\}$ contains infinitely many perfect squares.

There is one small correction needed to the published proof; 'modulo 1' needs to be inserted in line 7 on page 21. The rest of the proof is not affected.

Yours sincerely,
KEITH AUSTIN
(University of Sheffield)

Dear Editor,

Unlucky 13

The following problem appeared on a worksheet for year 7 pupils (12 year olds).

Consider different sums of whole, positive numbers to make the number 13. You could choose, for example

$$4+9=13\quad\text{or}\quad 2+5+6=13.$$

Now multiply the numbers in your sum together. For example

$$4\times 9=36,\quad 2\times 5\times 6=60.$$

What set of numbers gives the highest product? What is the product?

Did you get the answer 3, 3, 3, 4 or 3, 3, 3, 2, 2 (which give the same product)? If so, you are right. If not, check that my answers give a better product. The question now is, then, why is the number 3 so 'good' in this?

I quickly realised that 1 is no good. And any number, x, above 4 would be no good, because you could partition it into 2 and $(x-2)$ to get a bigger product. We only want numbers where $2(x-2)\le x$, i.e. $x\le 4$. So I claim that the number 3 is the best number to use for all numbers (not just 13), i.e. use as many 3's as you can along with a 2 or a 4 if necessary.

If 2 is as good as 4, the curve of 'goodness' must peak somewhere in between these, making 3 good. But the best number must be pretty special and natural. I thought maybe 3 was good because it is close to $\mathrm{e}=2.71\ldots$.

So my next step was to allow decimal numbers. I tried 2.7, 2.7, 2.7, 2.7, 2.2 and this was better than my previous products. But annoyingly, 2.6, 2.6, 2.6, 2.6, 2.6 was even better.

This is, I claim, the best possible for 13. But 2.6 is just a freak result because it goes exactly into 13.

What about for other numbers, x, other than 13? I tried it on a spreadsheet and the 2.6 was a freak. For other numbers the best result jumped about, but stayed, I think, between 2.5 and 3.0. For larger numbers it looked like the best result was converging to 2.7 — maybe even 2.71.

Can you prove it?

Yours sincerely,
JIM WHITEMAN
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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

32.5 The point P lies inside the triangle ABC and AP, BP, CP meet the sides BC, CA, AB at D, E, F respectively. Evaluate the expression

$$\frac{\mathrm{AP}}{\mathrm{AD}}+\frac{\mathrm{BP}}{\mathrm{BE}}+\frac{\mathrm{CP}}{\mathrm{CF}}.$$

By taking P to be the orthocentre of the triangle, deduce that

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

where A, B, C are the angles of the triangle.

(Submitted by J. A. Scott, Chippenham)

32.6 Prove that

$$1^k+2^k+\cdots+n^k=\frac{n+1}{(k+1)!}\begin{vmatrix} \binom{2}{1} & 0 & \dots & \dots & \dots & 0 & (n+1)-1 \\ \binom{3}{2} & \binom{3}{1} & 0 & \dots & \dots & 0 & (n+1)^2-1 \\ \binom{4}{3} & \binom{4}{2} & \binom{4}{1} & 0 & \dots & 0 & (n+1)^3-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{k}{k-1} & \binom{k}{k-2} & \dots & \dots & \dots & \binom{k}{1} & (n+1)^{k-1}-1 \\ \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \dots & \dots & \binom{k+1}{2} & (n+1)^k-1 \end{vmatrix}$$

(Submitted by Seyyed Moosavi, Emam Hoseyn University, Iran)

32.7 Prove that, for integers $k>1$ and $n\geq 1$,

$$1+\frac{1}{k-1}\Big(\frac{1}{2^{k-1}}-\frac{1}{(n+1)^{k-1}}\Big) < \frac{1}{1^k}+\frac{1}{2^k}+\cdots+\frac{1}{n^k} < 1+\frac{1}{k-1}\Big(1-\frac{1}{n^{k-1}}\Big)$$

and hence find lower and upper bounds for $\sum_{k=1}^{\infty} 1/r^k$.

(Submitted by Seyyed Moosavi)

32.8 What is the largest number of ways of arranging p people in a row, where p is a prime number greater than 2, so that no two people are placed side by side more than once?

(Submitted by H. A. Shah Ali, Tehran, Iran)

Solutions to Problems in Volume 31 Number 3

31.9 A ladder 3 metres long is leaning against a wall, and there is a point on the ladder which is 1 metre from the wall and 1 metre from the ground. How far is the foot of the ladder from the wall?

Solution by Jeremy Young (Nottingham High School)

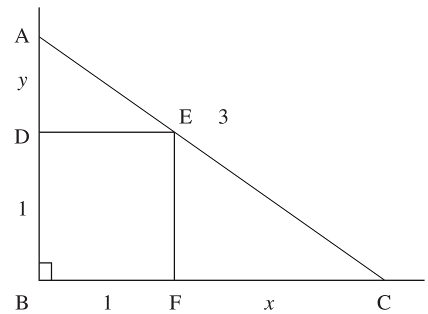

Figure 1.

Since triangles ADE, EFC are similar, AD/DE $=$ EF/FC, so $xy=1$. By Pythagoras' Theorem in $\triangle$ABC,

$$(x+1)^2+(y+1)^2=9,$$

so (since $xy=1$)

$$(x+y)^2+2(x+y)-9=0.$$

Hence (since $x+y>0$)

$$x+y=-1+\sqrt{10}.$$

Thus x, y are the roots of $z^2-(-1+\sqrt{10})z+1$, so

$$x,\,y=\tfrac{1}{2}\big\{-1+\sqrt{10}\pm\sqrt{(7-2\sqrt{10})}\big\}.$$

Hence

$$\mathrm{BC}=x+1=\tfrac{1}{2}\big\{1+\sqrt{10}\pm\sqrt{(7-2\sqrt{10})}\big\}.$$

Also solved by Inwook Kim, Ampleforth College.

31.10 What is the probability that, when arranged to form an non-decreasing sequence, the six numbers chosen in the UK national lottery (distinct numbers between 1 and 49) alternate odd–even or even–odd?

Solution by Jeremy Young, who proposed the problem

If $a_1<a_2<\cdots<a_6$ is such a sequence with a_1 odd (resp. even), then $50-a_6<50-a_5<\cdots<50-a_1$ is such a sequence with first term even (resp. odd). Thus there are equal numbers of such sequences beginning with odd and even terms. We count the number of sequences with an even first term.

Consider the more general problem of the number of ways of choosing $2k$ numbers from 1 to $2n+1$, with the first term even, to form such an 'alternating' sequence. Thus 1 is not chosen. The problem is then equivalent to the number of ways of choosing $2k$ balls in a row of $2n$ balls such that there are an even number (possibly zero) of unchosen balls between each pair of consecutive chosen balls and an even number of unchosen balls at each end.

Represent each chosen ball as red and each pair of consecutive unchosen balls as one single blue ball. We seek the number of ways of mixing in the $2k$ red balls with the $n-k$ blue balls, which is $\binom{n+k}{2k}$. The total number of all such sequences is thus $2\binom{n+k}{2k}$. For the national lottery, $k=3$ and $n=24$, giving a probability of

$$\frac{2\binom{27}{6}}{\binom{49}{6}} \approx 0.042\,.$$

31.11 (i) The sequence (p_n) satisfies the relation

$$p_{n+1} = (p_0 p_1 \dots p_n) + 2,$$

for $n = 0, 1, 2, \dots$, and $p_0 = 3$. Find p_n.
(ii) The sequence (q_n) satisfies the relation

$$q_{n+1} = (q_0 q_1 \dots q_n) + 4$$

for $n = 0, 1, 2, \dots$. What can be said about q_n for $n \geq 2$?

Solution by Jeremy Young

(i) The claim is that $p_n = 2^{2^n} + 1$ for all n. This is true for $n = 0, 1$ since $p_0 = 3$ and $p_1 = 5$. Also for $n \geq 1$,

$$\begin{aligned} p_{n+1} &= (p_0 p_1 \dots p_{n-1} p_n) + 2 \\ &= (p_n - 2) p_n + 2 \\ &= (p_n - 1)^2 + 1. \end{aligned}$$

The result follows from this by induction.
(ii) The claim is that q_n is a perfect square for all $n \geq 2$. For $n \geq 1$ we have

$$\begin{aligned} q_{n+1} &= (q_0 q_1 \dots q_{n-1} q_n) + 4 \\ &= (q_n - 4) q_n + 4 \\ &= (q_n - 2)^2, \end{aligned}$$

which proves the claim.

31.12 Prove that, for every positive integer n,

$$\sqrt[n]{n} < 1 + \frac{1}{\sqrt{n}}\,.$$

Solution by H. A. Shah Ali, who proposed the problem

The truth of the inequality can be verified directly for all values of n up to and including 16. For all natural numbers m, n, the geometric mean–arithmetic mean inequality gives that

$$\sqrt[n+1]{1 \times \left(1 + \frac{m}{n}\right) \times \cdots \left(1 + \frac{m}{n}\right)} < \frac{1 + n\left(1 + \frac{m}{n}\right)}{n+1},$$

whence

$$\left(1 + \frac{m}{n}\right)^n < \left(1 + \frac{m}{n+1}\right)^{n+1},$$

from which

$$\left(1 + \frac{m}{n}\right)^{n/m} < \left(1 + \frac{m}{n+1}\right)^{(n+1)/m}.$$

When $m = n$, the left-hand side is 2, from which it follows that, for all rational numbers $r > 1$,

$$\left(1 + \frac{1}{r}\right)^r > 2.$$

Now let α be a real number greater than 1. There is a sequence $\{r_i\}$ of rational numbers greater than 1 with limit α, and

$$\left(1 + \frac{1}{r_i}\right)^{r_i} > 2 \qquad \text{for every } i.$$

We now let $i \to \infty$ to give

$$\left(1 + \frac{1}{\alpha}\right)^\alpha \geq 2.$$

In particular,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} \geq 2,$$

so

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 2^{\sqrt{n}}.$$

We now consider the function $f(x) = (\log x)/x$ for $x > 0$. Then $f'(x) = (1 - \log x)/x^2$, and f attains its maximum value when $x = \mathrm{e}$. For $x > 4$,

$$\frac{\log x}{x} < \frac{\log 4}{4} = \frac{\log 2}{2}.$$

In particular, for $n > 16$,

$$\frac{\log \sqrt{n}}{\sqrt{n}} < \frac{\log 2}{2},$$

so $n < 2^{\sqrt{n}}$.

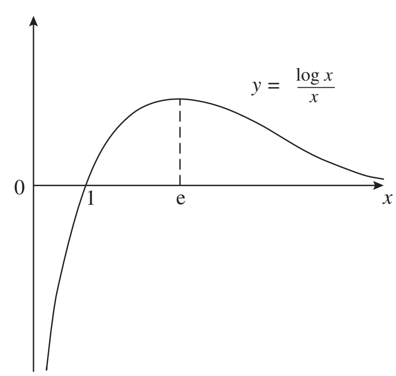

Figure 2.

It follows that, for $n > 16$,

$$n < 2^{\sqrt{n}} \leq \left(1 + \frac{1}{\sqrt{n}}\right)^n,$$

from which the result follows.

Reviews

Mathematically Speaking: A Dictionary of Quotations. By CARL C. GAITHER and ALMA E. CAVAZOS-GAITHER. IOP, Bristol, 1998. Pp. xiii+484. Paperback £19.95 (ISBN 0-7503-0503-7).

Love it or hate it, mathematics has always aroused strong feelings in all who encounter it. Some believe (like Simeon Poisson) that 'Life is good for only two things, discovering mathematics and teaching mathematics'. Others may rather agree that 'A talk in mathematics should be one of four things: beautiful, deep, surprising ... or short' (Michel Mendès). In its 360 pages of quotations, this substantial collection includes the thoughts of supporters of both viewpoints, particularly the admirers. Material is sorted by subject: for example, the first few topics are abstraction, addition, algebra, analogy, arithmetic and asymptotes. More importantly, indexes arranged by both subject and author are included. *Mathematically Speaking* is part of a series by the same authors, following equivalents in statistics (reviewed in Volume 29, Number 3) and physics.

The book probably works better as a reference work than as a source of entertainment for the general reader. Almost all the quotations are fairly serious, with many stretching to rather weighty paragraphs. Furthermore, the same basic ideas tend to be expressed repeatedly; the priority is to be comprehensive rather than readable. Sorting by keyword also has limitations, since the sections labelled simply 'mathematical', 'mathematician' and 'mathematics' between them contain well over a third of all the material. Yet we may excuse these minor criticisms (since even 'Mathematicians themselves are not infallible' — Henri Poincaré), and this book is certainly a useful reference.

Student, Nottingham High School JEREMY YOUNG

A History of Mathematics: An Introduction, 2nd edn. By VICTOR J. KATZ. Addison-Wesley, Harlow, 1998. Pp. xiv+879. Hardback £24.99 (ISBN 0-321-01618-1).

This book describes the development of mathematics, from its practical and mystical significance in ancient times to the modern applications of computers. It does this mainly by giving examples of results and methods taken from important textbooks of the time; such detail will be appreciated by those interested. Knowledge of basic calculus would be sufficient to understand the first two-thirds of the book, up to the end of the seventeenth century. However, after that the reader can still gain a general impression of new ideas.

The book is intended to be of particular use to teachers (although this should not deter others), and includes many exercises and discussion topics based on the material in each chapter. Several of the earlier chapters provide especially good coverage of non-Western mathematics. The subject is so vast that this can only claim to be an introduction, but fortunately references are not only given but critically discussed.

The greatest strength of this history is in its presentation and organization. A volume of this size and detail could easily become too daunting for the general reader, yet subjects are split into clearly-headed, manageable sections which encourage browsing. Within each chapter on a time period, material is arranged by topic, while the extensive index is also very useful. Each chapter begins with a scene-setting quotation and story, and ends with a summarizing chronology of the mathematicians described. A time line and map help to put everything into context, and there is even a pronunciation guide for mathematicians' names. Brief biographies in boxes beside the main text also contain some interesting tales. This book is an attractive and extremely thorough study, recommended for those with a serious interest in exploring their cultural history.

Student, Nottingham High School JEREMY YOUNG

The Golden Ratio and Fibonacci Numbers. By RICHARD A. DUNLAP. World Scientific Publishing, Singapore, 1997. Pp. vi+162. Hardback £23 (ISBN 981-02-3264-0).

The sequence of Fibonacci numbers is remarkable for its wealth of interesting properties. The ratio of consecutive terms of the Fibonacci sequence tends to the ubiquitous golden ratio, 1.61803 ..., known both to the ancient Greeks and to the ancient Egyptians. In this book Richard Dunlap, who is Professor of Physics at Dalhousie University, discusses some of the properties of the golden ratio and the Fibonacci numbers as well as their importance in geometry, plant growth, computer search algorithms and the crystallographic structure of certain solids.

The range of material covered in this book is enormous. As well as the standard material on the golden rectangle, the logarithmic spiral, the Platonic solids and Fibonacci's rabbit problem, there is the discovery of Fibonacci numbers in the field of optics, the periods of the Fibonacci and related Lucas sequence modulo a given number m, and the result that the fractional parts of iR for $i = 1, 2, 3, \dots, n$ are most uniformly distributed between 0 and 1 if R is the golden ratio. The latter leads on to the description of the Fibonacci search algorithm for approximating the minimum of a function. Full references are given if the reader is interested in the proofs of the mathematical results quoted. Tables and diagrams are used plentifully throughout to explain and illustrate ideas. The appendices include Euclid's construction of the regular pentagon and a list of 92 Fibonacci identities.

My favourite part of the book is about the Fibonacci rabbit sequence, which is relevant later to the chapters on Fibonacci lattices and Penrose's quasiperiodic tilings. This is because it helped me solve the following problem which has been bothering me for over two years:

> Does there exist an infinite 0–1 sequence such that no string of 0's and 1's appears four times consecutively in the sequence (e.g. 01010101 is forbidden because of the repeating string 01)?

We first describe the rabbit sequence. We start with a pair of adult rabbits and at each stage every pair of adults produces a pair of babies and every existing pair of babies becomes adults. Let us denote adult and baby pairs by A and b respectively and at each stage we shall put each b produced on the right of the A which are its parents. We thus have the following: $A \to Ab \to AbA \to AbAAb$ etc. This sequence of steps generates a unique sequence, the Fibonacci rabbit sequence:

$$AbAAbAbAAbAAbAbAAbAbA\ldots \tag{FRS}$$

It can be proved that the ratio of adults to babies in the limit is the golden ratio. Since the golden ratio is irrational, Dunlap derives from this that the sequence cannot be periodic, but it is quasiperiodic. He then considers the problem: given a string of A's and b's, how can we tell whether this string appears somewhere in (FRS)? We may notice that there are no occurrences of the pattern AAA, nor of bb, and this can be proved in general. We call these forbidden patterns. But this is not a sufficient condition. Dunlap now tells us of a wonderful test based on the deflation of the sequence. The deflation rules are as follows:

1. remove all isolated A's from the sequence;
2. replace all AA's by b's; and
3. replace all original b's by A's.

Perform this deflation repeatedly until either our string becomes empty, in which case it does appear in (FRS), or a forbidden pattern of elements occurs, in which case it does not.

Using this algorithm, we shall show that (FRS), with A's and b's signifying 0's and 1's, provides a solution to our problem in the affirmative. This is a very surprising result. Not only is (FRS) not periodic, it does not even have a string somewhere in the sequence repeating four times!

Suppose, for a contradiction, that such a repeating string exists. Choose such a string of minimum length. Let us denote the string by S and its length by k, so $SSSS$ occurs somewhere in (FRS). Note that $k > 1$, as $AAAA$ and $bbbb$ are both forbidden. It is easy to check that the deflation of a sequence of length greater than 1 must reduce the length of the sequence. Let X' denote the deflation of sequence X. Then, if $SSSS$ deflates to give $S'S'S'S'$, this would contradict the minimality of k. To prevent this from happening, S must begin and end with an A. Thus $S = AS_1A$, for some sequence S_1 and it is easy to see that S_1 cannot be empty.

Next note that if S_1 does not begin and end with a b, then

$$SSSS = AS_1AAS_1AAS_1AAS_1A$$

will have three consecutive A's which is impossible. Also, a quick check shows that S_1 cannot equal b, so that we can put $S = AbS_2bA$. We can also check that S_2 must have at least two elements. Thus S_2' cannot be empty and, after one operation, $SSSS$ deflates to include

$$AS_2'AbAS_2'AbAS_2'AbAS_2'A.$$

If S_2' begins and ends with a b, it will contain the forbidden string $bAbAb$. Therefore it must either begin or end with an A. Suppose without loss of generality that it begins with an A. Then, to avoid AAA, it must begin with Ab. Let $S_2' = AbS_3$. Then the above becomes

$$AAbS_3AbAAbS_3AbAAbS_3AbAAbS_3A.$$

If this appears somewhere in (FRS), it cannot be preceded by an A, so it must be preceded by a b. When we deflate this sequence, with the b included, we obtain:

$$AbA(S_3A)'AbA(S_3A)'AbA(S_3A)'AbA(S_3A)',$$

which is string $AbA(S_3A)'$ repeated four times. Now $AbA(S_3A)'$ has length less than or equal to:

$$\begin{aligned} 4 + (\text{length of } S_3) &= 2 + (\text{length of } S_2') \\ &\leq 2 + (\text{length of } S_2) \\ &= (\text{length of } S) - 2 \\ &= k - 2, \end{aligned}$$

contradicting the minimality of k. Hence the result.

This result is best possible in that (FRS) does contain strings repeating three times consecutively such as AbA. However, there are binary sequences with no string repeating three times and we challenge the reader to find one.

This is an excellent book. It is strongly recommended to all those who want to learn more about the Fibonacci numbers and the golden ratio, and furthermore, it may inspire you as it has inspired me.

Student, Trinity College, Cambridge MANSUR BOASE

Dissections: Plane and Fancy. By GREG N. FREDERICKSON. CUP, Cambridge, 1998. Pp. 336. Hardback £19.95 (ISBN 0521-571979).

In this book, Frederickson has tried to bring to the layman some of the beauty of the skill of geometric dissection — the art of cutting figures into as small a number of pieces as possible that will rearrange to form other figures.

Frederickson is somewhat of an expert in this field, and his enthusiasm for this fascinating branch of recreational mathematics is evident. His book contains a huge number of problems and their various solutions, covering everything from squares and triangles, to stars, Maltese crosses and tessellations, to how to cheat in solving puzzles. The main focus of the text, however, is not so much on the solutions as on how to arrive at them. Frederickson tries throughout to give the reader a feel for the various methods available, and introduces thorough mathematical descriptions of many dissections. The challenges posed become increasingly difficult as we progress through the chapters, moving in step with the reader's understanding of the techniques used, right up to some three-dimensional dissections towards the end.

The text is written in a conversational style that makes it easy to dip into at will. Diagrams abound, as indeed they must in such a book, and the text is also dotted with the histories of various puzzle-solvers, past and present. All in all, this is a very well written work, which is bound to interest anyone with a taste for recreational mathematics. It is, however, very much a coffee-table book, and is not really suitable for reading cover to cover — an excellent test of mental agility, but one to be taken in small doses only.

Student, Gresham's School, Holt — KIERAN GILLICK

Statistics in Sport. Edited by JAY BENNETT. Hodder Headline Plc, London, 1998. Pp. 288. Hardback £40.00 (ISBN 0-340-70072-6).

I approached *Statistics in Sport* reluctantly; I have limited knowledge of statistics and would not consider myself a great sportsman. The book is a compilation of essays by a variety of authors: nine chapters are dedicated to particular sports (American football, baseball, basketball, cricket, association football, golf, ice hockey, tennis and athletics); the other four chapters interrelate themes of sports statistics in general.

The early chapters begin interestingly with a brief description and a short history of the sport. Association football used to be an extremely violent affair having no decisive rules until 1863 (an Oxford student died in 1303 'whilst playing the ball in the High Street'). Those sports I was not familiar with were explained sufficiently for me to understand the statistics behind them, a glossary is provided listing sports terminology and readers who wish to study the subject in greater depth are provided with a substantial bibliography at the end of each chapter.

For the most part the statistics are dry, and sifting through the comprehensive tables, graphs and formulae does not make for exciting reading. One constantly hopes that workable strategies may be achieved from the detailed analysis, but the search tends to be fruitless and questions are left unanswered. Although this area of research only began in the 1970s when it was 'regarded with suspicion', I would have appreciated learning the opinion of the author given the *existing* data, albeit inconclusive, rather than being told that 'there is still much work to be done'. When discussing American football, for example, the author reaches the conclusion that 'more complete results about player evaluation and optimal strategy will require more data', but does decide that 'coaches should attempt fewer field goals' and 'take more fourth-down risks in pursuit of touchdowns'.

The chapter 'Developing Strategies in Tennis' does, however, reach probabilities of the server winning from different point situations. By focusing on the 1994 Wimbledon Men's Singles Final (Sampras v. Ivanisevic), John S. Croucher discusses service strategies, which makes more interesting reading, being specific to an identified match.

Some authors defend their view that sport 'merits academic study' as if they are subconsciously insecure about its value. In a book clearly aimed at the statistician, this justification seems superfluous. I would only recommend this book to those interested both in sports *and* in statistics and would certainly agree with the editor's view that the 'book's primary function is to provide a starting point for any serious researcher in sports statistics'.

Student, Nottingham High School — ANDREW HOLLAND

Introducing Pure Mathematics. By ROBERT SEDLEY AND GARRY WISEMAN. OUP, Oxford, 1998. Pp. 550. Paperback £20.00 (ISBN 0-19-914400-1).

This book covers all the content required for the current A-level syllabuses, plus additional topics in the new core, such as mathematical proof. The first two chapters are dedicated to algebra and geometry and are designed to prepare students, particularly those with a Grade C at GCSE, for the rigours of A-level mathematics. I feel that this book is clear and concise. The diagrams are easy to follow and the language is simple enough to get the theory across successfully. Hence it is useful when studying alone as a student or for revision purposes. There are lots of accessible exercises throughout the text and each chapter concludes with a selection of exam questions from many exam boards.

As a student on a modular course, I have appreciated the course content being distributed across six small books, one for each module. Although in some ways it is a good idea to have all the topics in one book such as this, it certainly makes the book heavy and cumbersome for such practicalities as carrying it to and from college. Hence I feel that it would be most useful as a reference book, to be kept either on the shelves at home or in the college library.

Student, Solihull Sixth Form College — KIM SKILLETT

Other books received

A Mathematical Mystery Tour: Discovering the Truth and Beauty of the Cosmos. By A. K. DEWDNEY. Wiley, Chichester, UK, 1999. Pp. 224. Hardback £17.99 (ISBN 0-471-23847-3).

Strength in Numbers : Discovering the Joy and Power of Mathematics in Everyday Life. By SHERMAN K. STEIN. Wiley, Chichester, UK, 1999. Pp. 288. Paperback £13.99 (ISBN 0-471-32974-6).

***Mathematical Spectrum* Awards for Volume 31**

Prizes have been awarded to the following student readers for contributions in Volume 31:

Mansur Boase
for his article 'Some prime results' (pages 25–28) and other contributions;

Michael Williams
for his article (with Linda J. S. Allen) 'The RSA algorithm: a public-key cryptosystem' (pages 9–13);

Jeremy Young
for problem 31.10 and solutions to various problems.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems and other items.

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