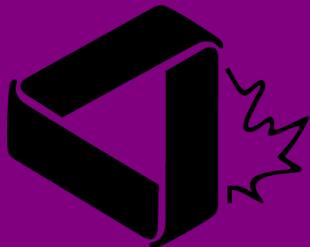


Mathematicorum

Crux

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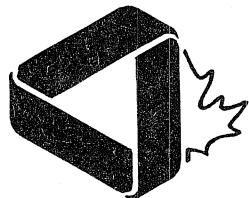
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Crux Mathematicorum

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UNCLE SAM AND THE U.S.A.M.O.
PERSONAL REMINISCENCES OF SAMUEL L. GREITZER
Murray S. Klamkin

The first time I met Sam, he, Wilhelm Magnus and I were the speakers at a high school mathematics teachers meeting in New York City back in the early 1950's. I was not particularly pleased to meet him due to his rather brash manner (actually his bark was much worse than his bite). My next intersection with him was an indirect one in 1966. I was on the editorial panel of the New Mathematical Library and contributed editorially to his excellent book with H.S.M. Coxeter, *Geometry Revisited*. I learned quite a bit of geometry from this work, as have many of the U.S.A.M.O. winners, since this book has frequently been awarded as prizes to them. Sam always railed against the poor teaching of geometry in U.S. schools, and I agree with him. This book at least helped the U.S. I.M.O. team members to prepare for geometry problems, one of that team's traditional weaknesses.

My next meeting with him was at a U.S.A.M.O. subcommittee meeting of which he was chairman, at Penn State University in 1971. At that time I was on the Board of Governors of the M.A.A. and wished to place the matter of starting a Math Olympiad on the agenda. Henry Alder informed me of the subcommittee which had been in existence for a number of years and put me on it. Nura Turner was on this committee and had tried for many years to get an Olympiad started without any success. Apparently, my vote on the committee tipped the balance in favor of having one. It was my impression that at that time Sam was not in favor of the Olympiad. But afterwards, at least, he was gung ho for it and he contributed immensely to it by being a very strong, forceful, and effective chairman of the committee for many years (up till 1982). He took charge of all the administrative work including all the letter writing, telephoning, grading of the papers, directing the training sessions, etc., with part of the costs coming out of his own pocket. At this time Nura Turner was the very effective chairman of the awards ceremonies subcommittee. Since she and Sam were both strong personalities, there were the inevitable conflicts between them (which I helped to resolve). However in later years these conflicts disappeared. As chairman of the Olympiad Examination committee for 14 years, I also had conflicts at the beginning with Sam, but with increasing mutual respect for each other over the years these also disappeared.

Sam was also instrumental in getting the U.S.A. invited to participate in the I.M.O. and in getting funds for the first two training sessions at Rutgers University where he was teaching at the time. Our first I.M.O. was held in East Berlin in 1974. I was supposed to assist Sam in the coaching, but unfortunately my employer (at that time not a university) would not permit me to attend. This led to my returning to academia the same year. The U.S.A. team did very well for a first-time entry, coming in second. This very strong showing

was no doubt due both to the quality of the team members and to the coaching of Sam. From 1975 to 1980, Sam and I shared the joys and burdens of coaching the U.S.A. team. He was called Uncle Sam (but not to his face!) by the students of the training sessions, since he always maintained a high standard of discipline which was necessary for these high-spirited youngsters.

In 1981, the I.M.O. was held in the U.S.A., the first time outside Europe. Again Sam was instrumental in this and in getting the necessary N.S.F. funding. Naturally, other people were highly involved in this too, e.g., Al Willcox, Executive Director of the M.A.A. Since at this Olympiad Sam was president of the international jury, he reluctantly had to give up being director of the training session and the coaching of the U.S. team. I took this over and had my colleague Andy Liu assist me for the next four years. I now fully realized all the work that Sam had put into directing the training sessions, arranging for travel abroad for the team and being the leader of the team, etc. In between, he also compiled the I.M.O. problems and solutions for 1959–1977 in an excellent book in the New Mathematical Library series for the M.A.A. This book also has been awarded many times to the U.S.A.M.O. winners.

Sam has contributed greatly to various South American Olympiads by visiting at his own expense and helping to make up the various competitions. He also helped in the start-up of the Australian Olympiad.

Samuel L. Greitzer passed away on February 22, 1988. This will be the first year since 1974 that he will not be attending the I.M.O. with the U.S.A. team. He will be sorely missed by his colleagues and former students here as well as by many of his counterparts in other countries. Although Uncle Sam is gone, he will be remembered for a long time.

University of Alberta
Edmonton, Alberta

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THE OLYMPIAD CORNER
No. 96
R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow,
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta,
Canada, T2N 1N4.*

We begin this column with the Canadian Mathematics Olympiad for 1988. In the

next number of the Corner we will discuss the exam and its outcome, and present the "official" solutions. Of course we remain open to other "nice" answers. Thanks go to Professor R.J. Nowakowski, Chairman of the Canadian Mathematical Association Olympiad Committee, for his cooperation in supplying information.

1988 CANADIAN MATHEMATICS OLYMPIAD

March 30, 1988

Time limit: 3 hours

1. For what values of b do the equations $1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?

2. A house is in the shape of a triangle, perimeter P metres and area A square metres. The garden consists of all the land within 5 metres of the house.

How much land do the garden and house together occupy?

3. Suppose that S is a finite set of points in the plane where some are coloured red, the others are coloured blue. No subset of three or more similarly coloured points is collinear. Show that there is a triangle

- (i) whose vertices are all the same colour; and such that
(ii) at least one side of the triangle does not contain a point of the opposite colour.

4. Let

$$x_{n+1} = 4x_n - x_{n-1}, x_0 = 0, x_1 = 1,$$

and

$$y_{n+1} = 4y_n - y_{n-1}, y_0 = 1, y_1 = 2.$$

Show for all $n \geq 0$ that $y_n^2 = 3x_n^2 + 1$.

5. Let $S = \{a_1, a_2, \dots, a_r\}$ denote a set of integers where r is greater than 1. For each non-empty subset A of S , we define $p(A)$ to be the product of all the integers contained in A . Let $m(S)$ be the arithmetic average of $p(A)$ over all non-empty subsets A . If $m(S) = 13$ and if $m(S \cup \{a_{r+1}\}) = 49$ for some positive integer a_{r+1} , determine the values of a_1, a_2, \dots, a_r and a_{r+1} .

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The next set of problems we give are from the *Seventeenth U.S.A. Mathematical Olympiad, 1988*. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director,

917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588-0322. As always we welcome your alternate elegant solutions for possible use in future columns. We would like to thank Leo J. Schneider of John Carroll University for forwarding the copy of the questions.

U.S.A. MATHEMATICAL OLYMPIAD

April 26, 1988

Time limit: 3 1/2 hours

1. By a *pure repeating decimal* (in base 10) we mean a decimal $0.\overline{a_1 \dots a_k}$ which repeats in blocks of k digits beginning at the decimal point. An example is $.243243243\dots = 9/37$. By a *mixed repeating decimal* we mean a decimal $0.b_1\dots b_m\overline{a_1 \dots a_k}$ which eventually repeats, but which cannot be reduced to a pure repeating decimal. An example is $.011363636\dots = 1/88$.

Prove that if a mixed repeating decimal is written as a fraction p/q in lowest terms, then the denominator q is divisible by 2 or 5 or both.

2. The cubic equation $x^3 + ax^2 + bx + c = 0$ has three real roots. Show that $a^2 - 3b \geq 0$, and that $\sqrt{a^2 - 3b}$ is less than or equal to the difference between the largest and smallest roots.

3. A function $f(S)$ assigns to each 9-element subset S of $\{1,2,3,\dots,20\}$ a whole number from 1 to 20. Prove that, regardless of how f is chosen, there will be a 10-element subset $T \subset \{1,2,3,\dots,20\}$ such that $f(T - \{k\}) \neq k$ for all $k \in T$.

4. Let I be the incenter of triangle ABC , and let A' , B' , and C' be the circumcenters of triangles IBC , ICA , and IAB , respectively. Prove that the circumcircles of triangles ABC and $A'B'C'$ are concentric.

5. A polynomial product of the form

$$(1-z)^{b_1}(1-z^2)^{b_2}(1-z^3)^{b_3}(1-z^4)^{b_4}(1-z^5)^{b_5}\dots(1-z^{32})^{b_{32}}$$

where the b_i are positive integers, has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, what is left is just $1 - 2z$. Determine, with proof, b_{32} . (The answer can be written as the difference of two powers of 2.)

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Next we present a selection of problems from the 1987 *KVANT*, the excellent U.S.S.R. journal for students. These problems can provide challenging "open ended" problem solving situations for enrichment classes at the Junior and Senior High School level. As usual we will consider publishing any really good solutions.

M1043. Can the set of all integers be partitioned into three subsets so that for any integer n the numbers n , $n - 50$, $n + 1987$ will belong to three different subsets? (S.V. Konyagin)

M1046. In the acute triangle ABC the angle A is 60° . Prove that one of the bisectors of the angle between the two altitudes drawn from the vertices B and C passes through the circumcircle of the triangle. (V. Progrebnyak, 10th form student, Vinnitsa)

M1047. In a one round chess tournament no less than $3/4$ of the games concluded at some moment were drawn. Prove that at this moment at least two players had the same number of points. (Miklos Bona, gymnasium student, Hungary)

M1051. The lower left hand corner of an 8×8 chessboard is occupied by nine pawns forming a 3×3 square. A pawn can jump over any other pawn landing symmetrically, if the corresponding square is empty. Using such moves is it possible to reassemble the 3×3 square

- (a) in the upper left hand corner?
- (b) in the upper right hand corner?

(Ya.E. Briskin)

M1056. In each little square of a 1987×1987 square table there is a number no greater than 1 in absolute value. In any 2×2 square of the table the sum of the numbers is 0. Prove that the sum of all the numbers in the table is no greater than 1987. (A.S. Merkuriev)

M1057. Two players in turn write natural numbers on a board. The rules forbid writing numbers greater than p and divisors of previously written numbers. The player who has no move loses.

(a) Determine which of the players has a winning strategy for $p = 10$ and describe this strategy.

(b) Determine which of the players has a winning strategy for $p = 1000$.

(D.V. Fomin)

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Next we turn to solutions to problems posed in the 1986 numbers of *Crux*.

3. [1986: 230] *17th Austrian Mathematical Olympiad 1986.*

Determine the set of all values of $x_0, x_1 \in \mathbb{R}$ such that the sequence defined by

$$x_{n+1} = \frac{x_{n-1}x_n}{3x_{n-1} - 2x_n}, \quad n \geq 1$$

contains infinitely many natural numbers.

Solution by Daniel Ropp, Washington University, St. Louis, MO.

We show that we must have $x_0 = x_1 =$ any positive whole number. We first prove by induction that

$$x_n = \frac{x_0 x_1}{(2^n - 2)(x_0 - x_1) + x_0}, \quad n = 0, 1, 2, \dots$$

This is clearly true for $n = 0$ and $n = 1$. Suppose it is true for all $n \leq k$ where $k \geq 1$. Then

$$\begin{aligned} x_{k+1} &= \frac{x_{k-1} x_k}{3x_{k-1} - 2x_k} = \frac{1}{3/x_k - 2/x_{k-1}} \\ &= \frac{x_0 x_1}{3(2^k - 2)(x_0 - x_1) + 3x_0 - 2(2^{k-1} - 2)(x_0 - x_1) - 2x_0} \\ &= \frac{x_0 x_1}{(2^{k+1} - 2)(x_0 - x_1) + x_0}. \end{aligned}$$

By induction, the formula holds for all n .

Since $x_0 = 0$ or $x_1 = 0$ eventually gives division by zero, we assume $x_0 \neq 0$ and $x_1 \neq 0$. Suppose $x_0 \neq x_1$; then

$$\lim_{n \rightarrow \infty} |(2^n - 2)(x_0 - x_1) + x_0| = +\infty,$$

and so

$$\lim_{n \rightarrow \infty} x_n = 0,$$

unless some denominator is zero, in which case the sequence breaks down. In any case at most finitely many x_n are natural numbers. Hence we must have $x_0 = x_1$. Then $x_n = x_0 = x_1$ and thus x_n will be a natural number for infinitely many n just in case $x_0 = x_1 \in \mathbb{N} - \{0\}$.

5. [1986: 231] *17th Austrian Mathematical Olympiad 1986.*

Show that for every convex n -gon ($n \geq 4$), the arithmetic mean of the lengths of all the sides is less than the arithmetic mean of the lengths of all the diagonals.

Solution by Daniel Ropp, Washington University, St. Louis, MO.

The number of diagonals in an n -gon equals

$$\left[\frac{n}{2} \right] - n = \frac{n(n-3)}{2},$$

so we wish to prove that

$$\left[\frac{n-3}{2} \right] \sum (\text{all sides}) < \sum (\text{all diagonals}).$$

Denote the vertices by P_i , $1 \leq i \leq n$, taken consecutively, and for convenience set $P_{i+n} = P_i$. Fix j with $2 \leq j \leq [n/2]$ (where $[x]$ is of course the greatest integer in x). Let Q_{ij} be the point of intersection of diagonals $P_i P_{i+j}$ and $P_{i+1} P_{i+j+1}$. By the triangle inequality

$$\begin{aligned} \overline{P_i P_{i+j}} + \overline{P_{i+1} P_{i+j+1}} &= (\overline{P_i Q_{ij}} + \overline{P_{i+1} Q_{ij}}) + (\overline{P_{i+j} Q_{ij}} + \overline{P_{i+j+1} Q_{ij}}) \\ &> \overline{P_i P_{i+1}} + \overline{P_{i+j} P_{i+j+1}}. \end{aligned}$$

We sum this from $i = 1$ to n obtaining

$$2 \sum_{i=1}^n \overline{P_i P_{i+j}} > 2 \sum_{i=1}^n \overline{P_i P_{i+1}}$$

or

$$\sum_{i=1}^n \overline{P_i P_{i+j}} > \sum_{i=1}^n \overline{P_i P_{i+1}}. \quad (*)$$

Every diagonal is of the form $P_i P_{i+j}$ for some i, j with $1 \leq i \leq n$, $2 \leq j \leq [n/2]$. If n is odd, these are all distinct. If n is even, they are distinct, except for each diagonal $P_i P_{i+n/2}$ being counted twice. Thus if n is odd, we sum $(*)$ from $j = 2$ to $j = [n/2] = (n-1)/2$ to obtain

$$\sum (\text{all diagonals}) > \left[\frac{n-3}{2} \right] \sum (\text{all sides}).$$

If n is even, we sum $(*)$ from $j = 2$ to $j = (n-2)/2$ and add the resulting inequality to the inequality

$$\frac{1}{2} \sum_{i=1}^n \overline{P_i P_{i+n/2}} > \frac{1}{2} \sum_{i=1}^n \overline{P_i P_{i+1}},$$

again obtaining

$$\sum (\text{all diagonals}) > \left[\frac{n-3}{2} \right] \sum (\text{all sides}).$$

6. [1986: 231] 17th Austrian Mathematical Olympiad 1986.

For n a given positive integer, determine all functions $F: \mathbb{N} \rightarrow \mathbb{R}$ such that $F(x+y) = F(xy-n)$ for all $x, y \in \mathbb{N}$ with $xy > n$.

Solution by Daniel Ropp, Washington University, St. Louis, MO.

For any $k \geq 1$, we may set $x = k+n$, $y = 1$ in the functional equation to find

$$F(k+n+1) = F(k). \quad (1)$$

Then

$$F(k+2(n+1)) = F((k+n+1)+n+1) = F(k+n+1) = F(k).$$

By an easy induction, we see that

$$F(k+j(n+1)) = F(k)$$

for $k \geq 1$, $j \geq 0$. Now, for any $k \geq 1$, we set $x = k$, $y = n+1$ in the functional equation to obtain

$$F(k+n+1) = F(k(n+1)-n) = F(1+(k-1)(n+1)) = F(1).$$

This with (1) implies that $F(k) = F(1)$ for $k \geq 1$. Thus, the only solutions are the constant functions.

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A reader has sent in the following observation.

Comment by G.R. Veldkamp, De Bilt, The Netherlands. In his very fine and stimulating Olympiad Corner 80, Professor Klamkin cites the following theorem of Hayashi [1986: 277]: *if a convex polygon inscribed in a circle be divided into triangles from one of its vertices, then the sum of the radii of the circles in these triangles is the same, whichever vertex is chosen.* He remarks that an inductive proof leads one to consider the case of the quadrilateral first. I should like to point out that the theorem may be seen as a direct consequence of a triangle property.

If O , R and r are the circumcentre, the circumradius and the inradius of a triangle, then $R + r$ equals the sum of the directed distances of O to the sides.

[e.g. see equation (2), item 2.16 of Bottema et al, *Geometric Inequalities* combined with 293(b), p.186 of Johnson, *Advanced Euclidean Geometry*.]

Applying this to each of the above mentioned $n - 2$ triangles of the n -gon (with circumcentre O and circumradius R) and adding the results, we find, thanks to the fact that distances to diagonals cancel, that the sum of the $n - 2$ inradii equals the sum of the sum of the distances of O to the sides minus $(n - 2)R$.

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These solutions and comments exhaust my backlog for 1986. This leaves quite a few problems for the readership to work on. Please send me your solutions. The problems from Volume 12 for which no solution has been discussed are given below.

[1986: 19, 20] 1985 Austria-Poland, 4, 7-9

[1986: 39, 40] 1982 Leningrad, 1-12 (see also [1988: 108])

[1986: 97] 1985 Spanish Mathematical Olympiad, 1st Round, 1, 4-8

[1986: 98] 1985 Spanish Mathematical Olympiad, 2nd Round, 1, 2, 7, 8

[1986: 202] 6 Proposals of 1986 U.S.A.M.O. Training Session

[1986: 230] 17th Austrian Mathematical Olympiad, 1 & 2

[1986: 231, 232] 9th Austrian-Polish Mathematical Competition 1986, 1-9

Lots of problems there. I hope that over the summer break you will have time to solve some of these and send the solutions in to me!

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We now turn to problems given in the 1987 numbers of *Crux*.

3. [1987: 3] 1985 Annual Greek High School Competition.

Let $S \subseteq \{(a,b): a, b \in \mathbb{R}\}$ be a set such that

- (i) at least one $(a,b) \in S$ has $a \neq 0$, $b \neq 0$ and $a \neq b$;
- (ii) if $(x_1, y_1) \in S$, $(x_2, y_2) \in S$ and $k \in \mathbb{R}$, then

$(x_1 + x_2, y_1 + y_2)$, (kx_1, ky_1) , and (x_1x_2, y_1y_2)
are all in S . Prove that $S = \mathbb{R}^2$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Let (x, y) be an arbitrary element of \mathbb{R}^2 . Then $x \in \mathbb{R}$ and $y \in \mathbb{R}$. To complete the solution it suffices to prove that $(x, y) \in S$.

By (i), fix an element (a, b) of S with $a \neq 0$, $b \neq 0$ and $a \neq b$. Then $-\frac{1}{a}, \frac{1}{b}, \frac{b}{b-a}$, and $\frac{a}{a-b}$ are all real numbers. Thus by (ii)

$$\left(-\frac{1}{a}a, -\frac{1}{a}b\right) = \left(-1, -\frac{b}{a}\right), \quad (1)$$

$$\left(\frac{1}{b}a, \frac{1}{b}b\right) = \left(\frac{a}{b}, 1\right), \quad (2)$$

and from (1)

$$\left((-1)(-1), (-1)\left(-\frac{b}{a}\right)\right) = \left(1, \frac{b}{a}\right), \quad (3)$$

are all elements of S . From (1) and (2), by (ii),

$$\left(-\frac{a}{b}, -\frac{b}{a}\right) \quad (4)$$

belongs to S . From (3) and (4)

$$\left(\frac{b-a}{b}, 0\right) \in S$$

and from (2) and (4)

$$\left(0, \frac{a-b}{a}\right) \in S,$$

both by (ii). Hence

$$\left(\frac{b}{b-a} \cdot \frac{b-a}{b}, \frac{b}{b-a} \cdot 0\right) = (1, 0) \in S$$

and similarly $(0, 1) \in S$. Finally $(x, 0)$ and $(0, y)$ belong to S , by (ii) applied to x and $(1, 0)$ and then to y and $(0, 1)$ respectively. This gives $(x, y) = (x, 0) + (0, y) \in S$, as required.

4. [1987: 3] 1985 Annual Greek High School Competition.

For each real k let $L(k)$ be the number of solutions of the equation

$$[x] = kx - 1985$$

($[x]$ is the greatest integer that is not greater than x). Prove that

- (i) if $k > 2$ then $1 \leq L(k) \leq 2$;
- (ii) if $0 < 1986k < 1$ then $L(k) = 0$;
- (iii) there is at least one k such that $L(k) = 1985$.

Solution by Daniel Ropp, student, Washington University, St. Louis, MO.

From $kx = [x] + 1985$, we see kx must be an integer, say $kx = n$. If $k \neq 0$, we conclude $x = n/k$, and the equation becomes

$$n = [n/k] + 1985.$$

For an integer n , this equation is equivalent to

$$n/k - 1 < n - 1985 \leq n/k,$$

or

$$1984 < n(1 - 1/k) \leq 1985,$$

or, for $k \neq 1$,

$$\frac{1984}{|1 - 1/k|} < |n| \leq \frac{1985}{|1 - 1/k|} \quad \text{and} \quad n(1 - 1/k) > 0.$$

Thus for k different from 0 and 1, $L(k)$ equals the number of integers contained in the half-open interval

$$\left[\frac{1984}{|1 - 1/k|}, \frac{1985}{|1 - 1/k|} \right]. \quad (*)$$

(i) The length of the interval (*) equals $\frac{1}{|1 - 1/k|}$. For $k > 2$, this length is $1 + \frac{1}{k-1}$. Since this is greater than 1, we see that for $k > 2$, $L(k) \geq 1$. If $L(k) \geq 3$ we would have $1 + \frac{1}{k-1} > 2$, or $k < 2$. Thus $1 \leq L(k) \leq 2$ for $k > 2$.

(ii) If $0 < 1986k < 1$, then $1/k > 1986$, so

$$|1 - 1/k| = 1/k - 1 > 1985$$

and so $1985/|1 - 1/k| < 1$. Hence (*) contains no integers in this case, so $L(k) = 0$.

(iii) For $k = 1985/1984$ we have

$$\frac{1}{|1 - 1/k|} = \frac{1}{1 - 1/k} = 1985,$$

so the interval (*) is the interval

$$(1984 \cdot 1985, 1985 \cdot 1985]$$

which contains precisely $1985 \cdot 1985 - 1984 \cdot 1985 = 1985$ integers; hence $L(1985/1984) = 1985$.

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Next we look at solutions for the 1987 Alberta High School Mathematics Scholarship Exam (A.H.S.M.S.E.).

1. [1987: 37] A.H.S.M.S.E. 1987.

[Fibonacci, AD 1225] If a and b are two positive integers that have no common factor and such that $a + b$ is even, show that the product $ab(a - b)(a + b)$ is divisible by 24.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Since a and b are two integers having no common factor with $a + b$ even, a and b are both odd. Thus $a^2 \equiv 1 \pmod{8}$ and $b^2 \equiv 1 \pmod{8}$, so $a^2 \equiv b^2 \pmod{8}$. Hence $8 | (a - b)(a + b)$.

Case (i): If $3 | a$ or $3 | b$ then $3 | ab$. It follows that $24 | ab(a - b)(a + b)$.

Case (ii): If $3 \nmid a$ and $3 \nmid b$ then $a^2 \equiv 1 \pmod{3}$ and $b^2 \equiv 1 \pmod{3}$, so $a^2 \equiv b^2 \pmod{3}$. Thus $3 | (a - b)(a + b)$. Because 3 and 8 are relatively prime, $24 | (a - b)(a + b)$. Hence $24 | ab(a - b)(a + b)$.

Solutions were also submitted by J.T. Groenman, Arnhem, The Netherlands and John Morvay, Dallas, Texas.

2. [1987: 38] A.H.S.M.S.E. 1987.

A (badly designed) table is constructed by nailing through its centre a circular disk of diameter two metres to a sphere of diameter one metre. The table tips over to bring the edge of the disc into contact with the floor. As the table rolls, the two points of contact with the floor trace out a pair of concentric circles. What are the radii of these circles?

Solution by the editors.

Instead of the table, we can consider a rolling cone whose base is the table top and in which the sphere is inscribed. As the cone rolls on the floor, its apex remains fixed. This may be intuitively obvious, but here is a proof. Consider an arbitrary circular cross-section S perpendicular to the axis of the cone. Nail the apex of the cone to the floor and roll the cone around its apex so that S traces out on the floor, *without slipping*, a circle centred at the apex. (Initially we don't worry whether the rest of the cone slips or not.) Since both the circumference of S , and its trace on the floor, have lengths which are proportional to the distance along the cone from S to the apex, the number of times the cone will turn during one rotation around its apex is independent of the position of S . Thus the entire cone rolls about its apex without slipping, as claimed.

Thus we need only calculate the distance r in the diagram. Here A is the apex of the cone, T_1T_2 is a diameter of the tabletop, and C is the centre of the sphere. We get, by similar triangles,

$$\frac{r}{1/2} = \frac{\sqrt{(r+1)^2 - 1}}{1}$$

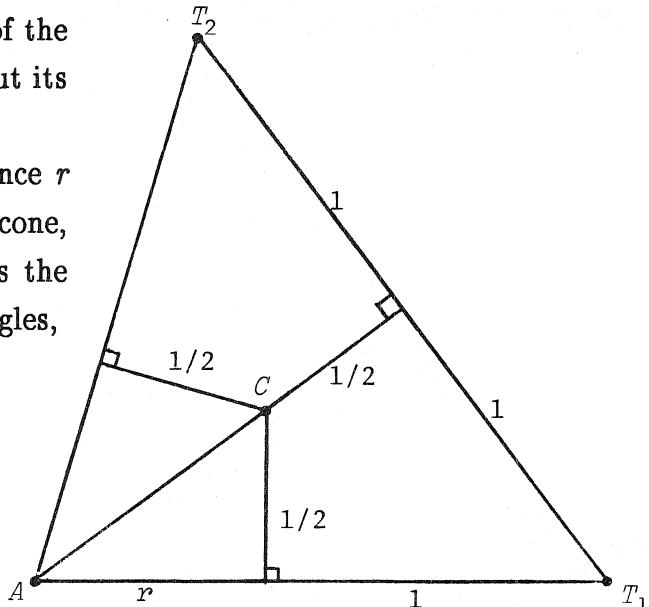
or

$$4r^2 = r^2 + 2r$$

or

$$3r = 2,$$

so $r = 2/3$. Thus the radii of the two circles are $2/3$ and $5/3$.



3. [1987: 38] A.H.S.M.S.E. 1987.

Find all pairs of real numbers (x,y) that are solutions to the simultaneous equations

$$x^{x+y} = y^3 \text{ and } y^{x+y} = x^6 y^3.$$

Solution by the editors.

Divide the exponents in the equations by 3 and $x + y$ respectively so

$$y = x^{(x+y)/3}$$

and

$$y = x^{(x+y+6)/(x+y)}.$$

Now unless $x = 0, \pm 1$ we deduce that

$$\frac{x+y}{3} = \frac{x+y+6}{x+y}.$$

The positive root of this quadratic equation in $x + y$ is 6. Clearly $y = x^2$, hence two solutions are $x = 2, y = 4$ and $x = -3, y = 9$.

The negative root gives $x + y = -3$. Now $y = 1/x$ and $x + y = -3$ gives two more solutions

$$x = \frac{-3 + \sqrt{5}}{2}, \quad y = \frac{-3 - \sqrt{5}}{2}$$

and

$$x = \frac{-3 - \sqrt{5}}{2}, \quad y = \frac{-3 + \sqrt{5}}{2}.$$

Returning to $x = 0, \pm 1$, we eliminate $x = 0$ since we would then get $y = 0$ and thus the indeterminate 0⁰. If $x = 1$ we see from $y = x^{(x+y)/3}$ that $y = 1$ as well, giving the added solution (1,1). For $x = -1$ we read from $y = x^{(x+y)/3}$ that $y = \pm 1$. Now (-1,1) is easily checked to be a solution, while (-1,-1) does not work. The solutions are thus (2,4), (-3,9), $\left[\frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}\right]$, $\left[\frac{-3 - \sqrt{5}}{2}, \frac{-3 + \sqrt{5}}{2}\right]$, (1,1) and (-1,1).

4. [1987: 38] *A.H.S.M.S.E. 1987.*

Adrian thinks of an integer A between 1 and 12 inclusive and Bernice tries to guess it. With each guess B that Bernice makes, Adrian makes one of the following replies:

- (a) "That's my number.", if $B = A$.
- (b) "You're close.", if $0 < |B - A| \leq 3$.
- (c) "You're not close.", if $|B - A| > 3$.

Show that Bernice can always force Adrian to say "That's my number" with four or fewer guesses.

Solution by John Morvay, Dallas, Texas.

The following procedure is sufficient, although not unique.

Step 1. $B = 6$. Answer: Close $\Rightarrow A \in \{3,4,5,7,8,9\}$, go to step 2a.

Not close $\Rightarrow A \in \{1,2,10,11,12\}$, go to step 2b.

Step 2a. $B = 4$. Answer: Close $\Rightarrow A \in \{3,5,7\}$.

Not close $\Rightarrow A \in \{8,9\}$.

In either case two more questions are sufficient. If $A \in \{3,5,7\}$, ask $B = 3$. Close implies $A = 5$, not close gives $A = 7$.

Step 2b. $B = 10$. Answer: Close $\Rightarrow A \in \{11,12\}$.
Not close $\Rightarrow A \in \{1,2\}$.

Again at most two questions are needed.

Comment by the editors.

There is an obvious general problem: with n instead of 12 what "closeness threshold" c (in place of 3), given in terms of n , allows the fewest number of guesses? Note that setting c to 0 and to " ∞ " both are useless. What is the best value?

5. [1987: 38] *A.H.S.M.S.E. 1987.*

$ABCD$ is a rectangular sheet of paper. E is the point on the side CD such that if the paper is folded along BE , C will coincide with a point F on the side AD . G is the point of intersection of the lines AB and EF . Prove that CG is perpendicular to BD .

Solution by J.T. Groenman, Arnhem, The Netherlands.

Let $\overline{AB} = a$, $\overline{BC} = b$, and let S be the intersection of CG and BD . Note that $\angle GFB$ is the supplement of $\angle EFB = \angle ECB = 90^\circ$ and hence is a right angle. Also FA is the altitude from F in the right triangle GFB . Thus triangles GAF , FAB and GFB are similar. Therefore $\overline{AB} : \overline{AF} = \overline{FA} : \overline{GA}$.

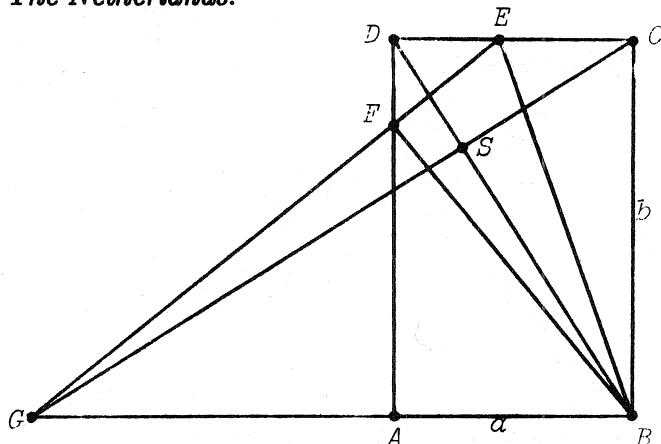
But $\overline{AF}^2 = b^2 - a^2 = \overline{GA} \cdot a$. Thus

$$\overline{GB} = \overline{GA} + \overline{AB} = \frac{b^2 - a^2}{a} + a = \frac{b^2}{a}$$

and $\tan \angle CGB = a/b$, $\tan \angle DBA = b/a$. Therefore

$$\angle SGB + \angle SBG = \angle CGB + \angle DBA = 90^\circ$$

and the angles at S are all 90° .



Please send in your contests and solutions.

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PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1989, although solutions received after that date will also be considered until the time when a solution is published.

1351. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find eleven consecutive positive integers, the sum of whose squares is the square of an integer.

1352. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine lower and upper bounds for

$$S_r = \cos^r A + \cos^r B + \cos^r C$$

where A, B, C are the angles of a non-obtuse triangle, and r is a positive real number, $r \neq 1, 2$. (The cases $r = 1$ and 2 are known; see items 2.16 and 2.21 of Bottema et al, *Geometric Inequalities*.)

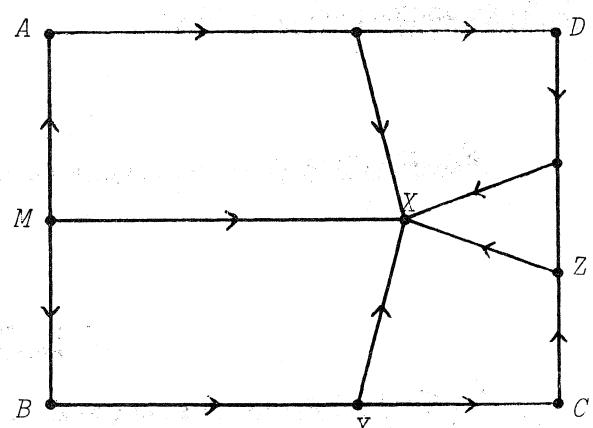
1353. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Find a linear recurrence with constant coefficients whose range is the set of all integers.

(b) Is there a linear recurrence with constant coefficients whose range is the set of all Gaussian integers (complex numbers $a + bi$ where a and b are integers)? *

1354. *Proposed by Jordi Dou, Barcelona, Spain.*

A girl stands at the midpoint M of one of the short sides of a rectangular swimming pool $ABCD$ 69 metres by 54.4 metres. A buoy is floating in the pool at a point X such that the girl can reach it (by a combination of swimming and walking) in the minimum time of 57.8 seconds; moreover she can achieve this minimum time in 5 different ways as shown in the diagram (that is, by swimming directly from M to X , by walking along MBY and swimming along YX , by walking along $MBYCZ$ and swimming along ZX , and symmetrically by two other paths). Find her swimming speed and her walking speed, assuming they are each constant.



1355. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and I its incenter. The perpendicular to AI at I intersects the line BC at the point A' . Analogously we define B' , C' . Prove that A' , B' , C' lie in a straight line.

1356* *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{n-1}}$$

for positive real numbers x_1, \dots, x_n ($n \geq 2$) satisfying $x_1 + \cdots + x_n = 1$.

1357* *Proposed by Jack Garfunkel, Flushing, N.Y.*

Isosceles right triangles $AA'B$, $BB'C$, $CC'A$ are constructed outwardly on the sides of a triangle ABC , with the right angles at A' , B' , C' , and triangle $A'B'C'$ is drawn. Prove or disprove that

$$\sin A' + \sin B' + \sin C' \geq \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2},$$

where A' , B' , C' are the angles of $\Delta A'B'C'$.

1358. *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia.*

Four different digits are chosen at random from the set $\{1, 2, 3, \dots, 9\}$. Denote by S the sum of all possible four-digit numbers formed by permuting these digits. What is the probability that S is square-free?

1359. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let PQR , PST , and PUV be congruent isosceles triangles with common apex P and having no vertex in common other than P . The sense $P \rightarrow Q \rightarrow R$, $P \rightarrow S \rightarrow T$, and $P \rightarrow U \rightarrow V$ is anticlockwise. We suppose moreover that VQ and RS meet in A , RS and TU in B , and TU and VQ in C . Prove that P is on the line joining the circumcentre of ΔABC to the symmedian point of ΔABC .

1360. *Proposed by Eric Holleman, student, Memorial University of Newfoundland.*

Find all increasing arithmetic progressions $a_1, a_2, \dots, a_{2\ell+1}$ of positive integers such that

$$a_{\ell+1}, \quad \sum_{i=1}^{2\ell+1} a_i, \quad \prod_{i=1}^{2\ell+1} a_i$$

is a geometric progression.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

- 1039.** [1985: 122; 1987: 152] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given are three collinear points O, P, H (in that order) such that $OH < 3OP$. Construct a triangle ABC with circumcentre O and orthocentre H and such that AP is the internal bisector of angle A . How many such triangles are possible?

II. Comment by G.R. Veldkamp, De Bilt, The Netherlands.

The accompanying figure has been constructed by following closely the prescription given in the published solution [1987: 152]. In this case however, AP is not the inner but the *outer* bisector of $\angle BAC$!

We have before us the case (iii)(β) of the solution, that is, the case for which the proof has not been worked out in detail but has been said to be similar to the proof for the case (iii)(α). What has gone wrong? Obviously the condition $OH < 3OP$ is not sufficient to guarantee that *any* point A with $\overline{AH}/\overline{AO} = \overline{PH}/\overline{PO}$ corresponds to a triangle meeting the requirements. Which extra condition has to be imposed on A ? The crucial situation will be where the order $B - A - S$ on K switches into $A - B - S$; that is, if A is the midpoint of HS . We have

$$\overline{HA} \cdot \overline{HS} = \overline{HO}^2 - R^2.$$

Let us assume $\overline{HP} = \lambda \overline{OP}$ ($0 < \lambda < 2$). Then $\overline{HA} = \lambda R$ and, putting $\angle HAO = \varphi$,

$$\overline{HO}^2 = R^2 + \lambda^2 R^2 - 2\lambda R^2 \cos \varphi.$$

Thus in the critical position $\overline{HA} = \overline{AS}$ and

$$2\lambda^2 R^2 = \overline{HA} \cdot \overline{HS} = \lambda^2 R^2 - 2\lambda R^2 \cos \varphi,$$

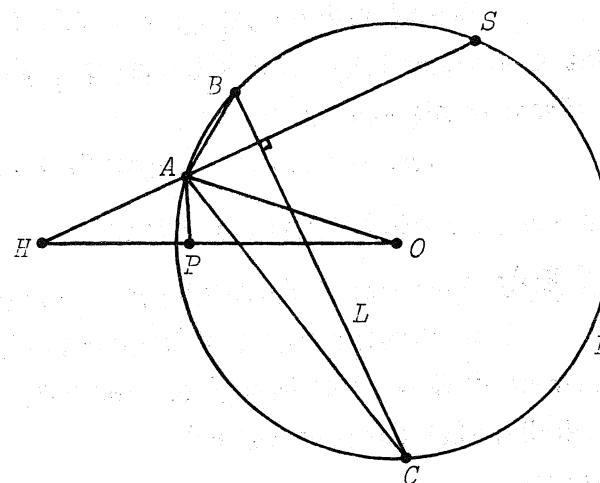
that is,

$$\cos \varphi = -\lambda/2.$$

We conclude that, given $\overline{OH} = 3\overline{OP}$, A has to be taken such that

$$\angle OAH < \arccos \left[\frac{-\overline{HP}}{2\overline{OP}} \right]$$

in order to get a triangle ABC meeting the requirements.



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1195* [1986: 282; 1988: 62] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle with medians m_a, m_b, m_c and circumcircle Γ . Let DEF be the triangle formed by the parallels to BC, CA, AB through A, B, C respectively, and let Γ' be the circumcircle of DEF . Let $A' B' C'$ be the triangle formed by the tangents to Γ at the points (other than A, B, C) where m_a, m_b, m_c meet Γ . Finally let A'', B'', C'' be the points (other than D, E, F) where m_a, m_b, m_c meet Γ' . Prove that lines $A'A'', B'B'', C'C''$ concur in a point on the Euler line of ABC .

Combination of solutions by R.C. Lyness, Southwold, Suffolk, England, and G.R. Veldkamp, De Bilt, The Netherlands.

We use homogeneous distance (trilinear) coordinates with respect to ΔABC . As usual the coordinates of A, B , and C are taken as $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ respectively, and the sides of ΔABC are a, b, c . Then the circumcircle of ΔABC is given by

$$\Gamma: ayz + bzx + cxy = 0, \quad (1)$$

and the median from A by

$$m_a: by - cz = 0.$$

The intersection, other than A , of m_a with Γ is then

$$K[-abc, c(b^2 + c^2), b(b^2 + c^2)].$$

The tangent $B' C'$ to Γ at the point K is

$$(b^2 + c^2)^2 x + ab^3 y + ac^3 z = 0. \quad (2)$$

By cyclic permutation, the tangent $C' A'$ to Γ will be given by

$$ba^3 x + (c^2 + a^2)^2 y + bc^3 z = 0. \quad (3)$$

From (2) and (3) we obtain the coordinates of C' as

$$C' [ac(c^2 + a^2 - b^2), bc(b^2 + c^2 - a^2), -2a^2b^2 - c^2(a^2 + b^2 + c^2)].$$

Therefore the equation of the line CC' can be taken as

$$\begin{aligned} b(b^2 + c^2 - a^2)x &= a(c^2 + a^2 - b^2)y, \\ (2b^2c \cos A)x &= (2a^2c \cos B)y, \\ (\sin^2 B \sec B)x &= (\sin^2 A \sec A)y, \end{aligned}$$

and finally

$$(\sec B - \cos B)x = (\sec A - \cos A)y. \quad (4)$$

Next, the sides of ΔDEF have equations

$$DE: ax + by = 0, \quad EF: by + cz = 0, \quad FD: cz + ax = 0,$$

so we obtain the coordinates

$$D(bc, -ca, -ab), \quad E(-bc, ca, -ab), \quad F(-bc, -ca, ab). \quad (5)$$

From (1), the equation of the circumcircle Γ' of ΔDEF can be written in the form

$$t(ayz + bzx + cxy) + (ux + vy + wz)(ax + by + cz) = 0, \quad (6)$$

where $ux + vy + wz = 0$ represents the radical axis of Γ and Γ' . Since D, E, F are on Γ' , (5) and (6) yield

$$t(a^3bc - ab^3c - abc^3) = abc(ubc - vca - wab), \text{ etc.}$$

Thus we can take

$$t = abc, \quad u = a^3, \quad v = b^3, \quad w = c^3,$$

so that the radical axis of Γ and Γ' is

$$a^3x + b^3y + c^3z = 0, \quad (7)$$

and

$$\Gamma': abc(ayz + bzx + cxy) + (a^3x + b^3y + c^3z)(ax + by + cz) = 0.$$

The coordinates of A'' , the intersection of Γ' with m_a : $by = cz$, will then satisfy

$$a^2b^2y + ab^3xy + abc^2xy + (a^3x + b^3y + bc^2y)(ax + 2by) = 0$$

which can be written

$$(ax + by)[a^2(ax + by) + 2by(b^2 + c^2)] = 0.$$

Putting $ax + by = 0$ yields the point D , so we want

$$a^2(ax + by) + 2by(b^2 + c^2) = 0.$$

This yields the coordinates

$$A''[bc(a^2 + 2b^2 + 2c^2), -ca^3, -ba^3]$$

and by cyclic permutation

$$B''[-cb^3, ca(b^2 + 2c^2 + 2a^2), -ab^3].$$

Now notice that from (3) and cyclic permutation we get

$$A'B': ca^3x + cb^3y + (a^2 + b^2)^2z = 0,$$

and so the intersection of $A'B'$ with the line AB : $z = 0$ will be the point

$$Z(b^3, -a^3, 0)$$

which from (7) lies on the radical axis of Γ and Γ' . Moreover Z, A'' , and B'' are collinear, since

$$\begin{vmatrix} b^3 & -a^3 & 0 \\ bc(a^2 + 2b^2 + 2c^2) & -ca^3 & -ba^3 \\ -cb^3 & ca(b^2 + 2c^2 + 2a^2) & -ab^3 \end{vmatrix} = 0.$$

Thus $AB, A'B'$, and $A''B''$ all concur on the radical axis, and so by symmetry the radical axis is an axis of perspective for each pair of the triangles $ABC, A'B'C'$, $A''B''C''$. By Desargues' theorem, each pair of these triangles will have a centre of perspective; in particular, $A'A'', B'B'', C'C''$ concur. Moreover these centres of perspective will be collinear. The centre of perspective for ΔABC and $\Delta A'B'C'$ will be the point

$$P(\sec A - \cos A, \sec B - \cos B, \sec C - \cos C)$$

by (4) and cyclic permutation. The centre of perspective for ΔABC and $\Delta A''B''C''$ will be the barycentre G of ΔABC . G lies on the Euler line of ΔABC , as does the orthocentre and circumcentre, and since these last two have respective coordinates

$(\sec A, \sec B, \sec C)$ and $(\cos A, \cos B, \cos C)$,

P lies on the Euler line as well. Thus the third centre of perspective, namely the intersection of $A'A''$, $B'B''$, and $C'C''$, will also lie on the Euler line.

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1198. [1986: 283; 1988: 85] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with incenter I , Gergonne point G , and Nagel point N , and let J be the isotomic conjugate of I . Prove that G , N , and J are collinear.

II. *Solution by G.R. Veldkamp, De Bilt, The Netherlands.*

We use homogeneous barycentric (areal) coordinates with respect to $\triangle ABC$.

Let $[XYZ]$ denote the area of $\triangle XYZ$. Then

$$BN_a = CG_a = s - c$$

and

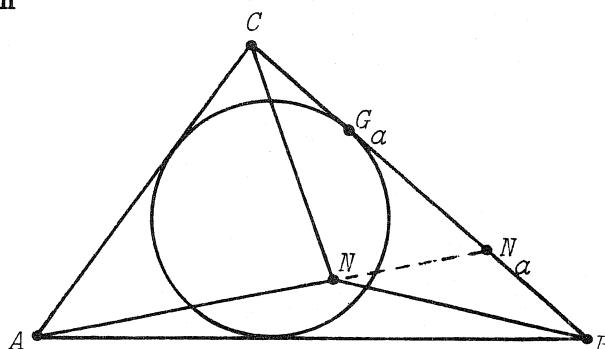
$$CN_a = s - b,$$

so

$$\frac{[ACN_a]}{[ABN_a]} = \frac{s - b}{s - c} = \frac{[NCN_a]}{[NBN_a]},$$

and thus

$$\frac{[ACM]}{[ABN]} = \frac{s - b}{s - c}.$$



It follows by symmetry that N is given by the coordinates $(s - a, s - b, s - c)$. G , being the isotomic conjugate of N , is therefore given by

$$((s - b)(s - c), (s - c)(s - a), (s - a)(s - b)).$$

Since I is obviously the point (a, b, c) , J is given by (bc, ca, ab) . Now we observe that

$$(s - b)(s - c) + s(s - a) = bc, \text{ etc.}$$

It follows that, in symbolic notation, $(G) + s(N) = (J)$, i.e. G , N , and J all lie on the line $x + sy - z = 0$, and so are collinear.

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1200. [1986: 283; 1988: 89] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

In a certain game, the first player secretly chooses an n -dimensional vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ all of whose components are integers. The second player is to determine \mathbf{a} by choosing any n -dimensional vectors \mathbf{x}_i , all of whose components are also integers. For each \mathbf{x}_i chosen, and before the next \mathbf{x}_i is chosen, the first player tells the second player the value of the dot product $\mathbf{x}_i \cdot \mathbf{a}$. What is the least number of vectors \mathbf{x}_i the second player has to choose in order to be able to determine \mathbf{a} ? [Warning: this is somewhat "tricky"!]

Editor's comment.

This office has received the following communication from R. ISRAEL of the

University of British Columbia, quote: "I must protest the published solution! How can the second player "choose" $\mathbf{x}_1 = \mathbf{a}$ without knowing what \mathbf{a} is? If you do allow "choices" that depend on the unknown \mathbf{a} , you can manage with just a single vector." He then went on to give an example. An easier example (supplied by a colleague of the editor) would be to let

$$\mathbf{x}_1 = (1, M, M^2, \dots, M^{n-1})$$

where $M = 2\sqrt{\mathbf{a} \cdot \mathbf{a}} + 1$, say. All this does is bypass the first vector in the published solution [1988: 89]. (Note the exponent $n - 1$ above, corrected from n on [1988: 89].)

The editor cannot find a nit of sufficient mass to pick with this or Israel's example, and so stands reproved (as does the problem).

Israel then went on to dispose of the conjecture, by the proposer and the published solver Hess, that if the components of each \mathbf{x}_i must be known then n such vectors are required. His short proof that this is so goes as follows. Given any vectors $\mathbf{x}_1, \dots, \mathbf{x}_{n-1}$ with integral or even rational coordinates, elementary linear algebra says there is a nonzero vector \mathbf{b} with rational coordinates which is orthogonal to every \mathbf{x}_i (\mathbf{b} lies in the kernel of the $(n-1) \times n$ matrix with rows the \mathbf{x}_i). Furthermore, by multiplying by a scalar we can take \mathbf{b} to have integer coordinates. Then $\mathbf{a} \cdot \mathbf{x}_i = (\mathbf{a} + \mathbf{b}) \cdot \mathbf{x}_i$ for all i , so the $n-1$ vectors \mathbf{x}_i will not enable the second player to distinguish between the integer vectors \mathbf{a} and $\mathbf{a} + \mathbf{b}$.

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1238. [1987: 119] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Let $A = a^4$ where a is a positive integer. Find all positive integers x such that

$$A^{15x+1} \equiv A \pmod{6814407600},$$

or prove that there are none.

Solution by David R. Stone, Georgia Southern College, Statesboro, Georgia.

Holy cow, who would have thought it — the given congruence is true for *all* a and x ! In fact, this big strange modulus is precisely the right one:

$$(a^4)^{15x+1} \equiv a^4 \pmod{N} \text{ for all } a, x \geq 1 \Leftrightarrow N | 6814407600.$$

Note first that

$$6814407600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$$

and that

$$\begin{aligned}\varphi(3^2) &= 6, & \varphi(5^2) &= 20, & \varphi(7) &= 6, & \varphi(11) &= 10, \\ \varphi(13) &= 12, & \varphi(31) &= 30, & \varphi(61) &= 60,\end{aligned}$$

each a divisor of 60, φ being Euler's phi function.

Let $N = 6814407600$. To show the given congruence is true for all $a, x \geq 1$, we can show

$$a^{60x} \cdot a^4 \equiv a^4 \pmod{n} \tag{1}$$

for $n = 2^4, 3^2, 5^2, 7, 11, 13, 31, and 61 (the prime power divisors of N).$

Begin with $n = 2^4$. For a even, a^4 is divisible by 16, so both sides of (1) are congruent to 0. For a odd, $a^4 \equiv 1 \pmod{16}$, so $a^{60x} \equiv 1$, so both sides of (1) are congruent to 1.

Now let n be any of the other possibilities. If a is relatively prime to n , then $a^{\varphi(n)} \equiv 1 \pmod{n}$, and recall that $\varphi(n) | 60$. Thus $a^{60x} \equiv 1$, so each side of (1) becomes a^4 . If $(a, n) > 1$ and n is prime, then $n | a$ so both sides of (1) become 0. If $(a, n) > 1$ and $n = 3^2$ (or 5^2), then 3 (or 5) divides a , so 3^4 (or 5^4) divides a^4 , and both sides of (1) are congruent to 0.

Thus

$$a^{60x} \cdot a^4 \equiv a^4 \pmod{6814407600},$$

and hence mod any divisor of 6814407600. Notice that we "got off easy" in each case: $a^{60x} \cdot a^4$ is congruent to a^4 either because a^{60x} is congruent to 1 or because a^4 is congruent to 0. Perhaps more work would be required for some yet untried modulus? But the converse assures us that no other modulus works.

For the proof, assume that

$$a^{60x} \cdot a^4 \equiv a^4 \pmod{N} \quad (2)$$

for all $a, x \geq 1$.

If $N = 2^i m$ with m odd, then $i \leq 4$; for if $i \geq 5$ then $32 | N$, so putting $a = 2, x = 1$ in (2) yields $2^{64} \equiv 2^4 \pmod{32}$, which is certainly not true. So N can have no more 2's than does 6814407600.

Let d be an odd divisor of N . Then by (2) with $a = 2, x = 1$ we have $2^{64} \equiv 2^4 \pmod{N}$, so $2^{60} \equiv 1 \pmod{d}$. Hence

$$d | 2^{60} - 1 = 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 41 \cdot 61 \cdot 151 \cdot 331 \cdot 1321.$$

In particular there is the possibility that d could be 41, say. But if it were, we would have (from (2) with $a = 3, x = 1$) $3^{64} \equiv 3^4 \pmod{N}$, so $3^{60} \equiv 1 \pmod{41}$. But actually

$$3^{60} = (3^4)^{15} \equiv (-1)^{15} = -1 \pmod{41}.$$

Similarly, d cannot be 151, 331, or 1321. Therefore d must divide $3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$, so N must be a divisor of 6814407600.

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; JURGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer. One partial solution was received.

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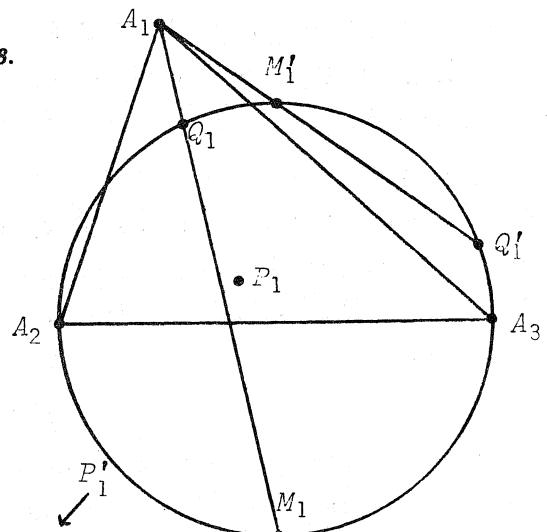
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1239. [1987: 120] Proposed by J.T. Groenman, Arnhem, The Netherlands.

Find all points whose pedal triangles with respect to a given triangle are isosceles and right-angled.

Solution by P. Penning, Delft, The Netherlands.

Let the given triangle be $A_1A_2A_3$. Draw a circle with (say) A_2A_3 as diameter, and denote with M_1 and M_1' the points on the circle midway between A_2 and A_3 . The straight lines M_1A_1 and $M_1'A_1$ intersect the circle again at Q_1 and Q_1' . Then two of the points we are looking for will be the points P_1 and P_1' isogonally associated with Q_1 and Q_1' with respect to $\Delta A_1A_2A_3$. Similarly we obtain points P_2, P_2', P_3, P_3' , for a total of six solutions.



To prove that P_1 and P_1' work, recall first that for any point P and its isogonally associated point Q (with respect to $\Delta A_1A_2A_3$), the lines A_1Q, A_2Q, A_3Q are perpendicular to the corresponding edges of the pedal triangle of P [e.g. Theorem 237, page 156 of R.A. Johnson, *Advanced Euclidean Geometry*]. Since we look for pedal triangles with angles $\pi/2, \pi/4, \pi/4$, the lines A_iQ must make angles of $\pi/2, \pi/4, \pi/4$ with each other. The construction gives the correct points Q_1, Q_1' ; for example, $\angle A_2Q_1A_3 = \pi/2$, and thus $\angle A_2Q_1M_1 = \angle A_3Q_1M_1 = \pi/4$ since arcs A_2M_1 and A_3M_1 are equal.

The same method can be used to find points whose pedal triangles have other angles.

Also solved by JORDI DOU, Barcelona, Spain; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

There were three essentially different solutions for this problem, yielding some varied and interesting properties of the six required points. The solutions of Smeenk and the proposer found the six points on the three circles of Apollonius of the triangle. Dou's solution concludes that the points come in pairs each of which are inverses with respect to the circumcircle.

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1240. [1987: 120] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find distinct positive integers a, b, c such that

$$a + b + c, \quad ab + bc + ca, \quad abc$$

forms an arithmetic progression.

Solution by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Taking symmetry into account, there are precisely two solutions

$$(3,6,27) \quad \text{and} \quad (3,7,16),$$

and if we drop the artificial requirement that the integers be distinct, then there is one more solution

$$(4,4,24).$$

Assume that (a,b,c) is a solution where, without loss of generality, $a \leq b \leq c$. Then

$$l = a + b + c + abc - 2(ab + bc + ca) = 0.$$

If $a \geq 6$, then

$$l > 6bc - 2(ab + bc + ca) = 2b(c-a) + 2c(b-a) \geq 0,$$

a contradiction. Thus $a \leq 5$ and there are five cases to be considered.

(i) If $a = 1$, then

$$l = 1 - bc - b - c < 0,$$

a contradiction.

(ii) If $a = 2$, then

$$l = 2 - 3b - 3c < 0,$$

a contradiction.

(iii) If $a = 3$, then

$$l = bc + 3 - 5b - 5c.$$

Thus $l = 0$ if and only if

$$(b-5)(c-5) = 22,$$

yielding $b-5 = 1, c-5 = 22$ or $b-5 = 2, c-5 = 11$. Hence we obtain the two solutions $(3,6,27)$ and $(3,7,16)$.

(iv) If $a = 4$, then

$$l = 2bc - 7b - 7c + 4.$$

Thus $l = 0$ if and only if

$$(2b-7)(2c-7) = 41,$$

yielding $2b-7 = 1, 2c-7 = 41$. This gives the triple $(4,4,24)$ which is a solution if we do not require the integers to be all distinct.

(v) If $a = 5$, then

$$l = 3bc - 9b - 9c + 5 = 0$$

which is clearly impossible since 3 does not divide 5.

Also solved by SEUNG-JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; JACK GARFUNKEL, Flushing, New York; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; MICHAEL SELBY, University of Windsor, Windsor, Ontario; ROBERT E. SHAFER, Berkeley, California; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Most solvers found both solutions and proved there were no others. Several solvers pointed out the nondistinct solution (4,4,24). Engelhaupt, McCallum, Selby, and Stone considered other orders of the terms $a + b + c$, $ab + bc + ca$, abc and thus obtained the additional solutions (1,3,7) and (1,1,3). Penning allowed a , b , c to be zero or negative as well as positive and so obtained several more solutions.

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1241. [1987: 149] *Proposed by Jordan B. Tabov, Sofia, Bulgaria. (Dedicated to Murray S. Klamkin.)*

A quadrilateral $ABCD$ and a triangle EFG are inscribed in a circle Γ . For an arbitrary point X of Γ , $s(X)$ denotes the sum of the distances from A , B , C , and D to the tangent to Γ at X . Prove that if $s(E) = s(F) = s(G)$, then $ABCD$ is a rectangle.

I. *Solution by Dan Sokolowsky, Williamsburg, Virginia.*

We assume A , B , C , D occur in that order around Γ . Let M_1 be the midpoint of AC , M_2 the midpoint of BD , and P the midpoint of M_1M_2 . Then if ℓ is the tangent line to Γ at E , and $d_\ell(X)$ denotes the distance from point X to ℓ , we get

$$d_\ell(M_1) = (d_\ell(A) + d_\ell(C))/2,$$

$$d_\ell(M_2) = (d_\ell(B) + d_\ell(D))/2,$$

and so

$$\begin{aligned} d_\ell(P) &= (d_\ell(M_1) + d_\ell(M_2))/2 \\ &= (d_\ell(A) + d_\ell(B) + d_\ell(C) + d_\ell(D))/4 \\ &= s(E)/4. \end{aligned}$$

Repeating this calculation for F and G , and since $s(E) = s(F) = s(G)$, we get that P is equidistant from three tangents to Γ . Thus P is the center of Γ .

Now note that if AC does not pass through P , then neither will BD (since P is the midpoint of M_1M_2); and in this case since $AC \perp PM_1$ and $BD \perp PM_2$ we get $AC \parallel BD$, impossible by the location of $ABCD$. Thus AC and BD must both be diameters of Γ , and it follows that $ABCD$ is a rectangle.

II. *Solution by Murray S. Klamkin, University of Alberta.*

We first derive a more general result from which the desired result will follow. Let A_0, A_1, \dots, A_n denote the vertices of an n -dimensional simplex and Γ its circumsphere. Let B_1, B_2, \dots, B_r be any set of $r > 1$ distinct points on Γ . For an arbitrary point X of Γ , $s(X)$ denotes the sum of the distances from B_1, \dots, B_r to the tangent hyperplane to Γ at X .

Theorem. If

$$s(A_0) = s(A_1) = \dots = s(A_n),$$

then the centroid of B_1, \dots, B_r is the center O of Γ .

Proof. For convenience we take Γ to have unit radius. Denote a vector from O to point P by \mathbf{P} . Since the distance from B_j to the tangent hyperplane to Γ at A_i is just the length of the projection of $\overrightarrow{B_j A_i}$ onto the unit vector \mathbf{A}_i , it follows that

$$\begin{aligned}s(A_i) &= \sum_{j=1}^r \overrightarrow{B_j A_i} \cdot \mathbf{A}_i = \sum_{j=1}^r (\mathbf{A}_i - \mathbf{B}_j) \cdot \mathbf{A}_i \\ &= r - \mathbf{A}_i \cdot \mathbf{B},\end{aligned}$$

where

$$\mathbf{B} = \sum_{j=1}^r \mathbf{B}_j.$$

Since

$$s(A_0) = s(A_1) = \dots = s(A_n),$$

we have

$$\mathbf{A}_0 \cdot \mathbf{B} = \mathbf{A}_1 \cdot \mathbf{B} = \dots = \mathbf{A}_n \cdot \mathbf{B}.$$

Thus if $\mathbf{B} \neq 0$, all the orthogonal projections of the \mathbf{A}_i 's onto \mathbf{B} would be the same. This would mean that the vertices A_i lie in a hyperplane orthogonal to \mathbf{B} , contradicting the assumption that they form a simplex. Therefore $\mathbf{B} = 0$, which is equivalent to the centroid of B_1, \dots, B_r being at the center O . \square

In the given problem we have $r = 4$ and $n = 2$, and the theorem implies

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0.$$

Then

$$\begin{aligned}(\mathbf{A} + \mathbf{B})^2 &= (\mathbf{C} + \mathbf{D})^2, \\ (\mathbf{A} + \mathbf{C})^2 &= (\mathbf{B} + \mathbf{D})^2, \\ (\mathbf{A} + \mathbf{D})^2 &= (\mathbf{B} + \mathbf{C})^2.\end{aligned}$$

Since $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = \mathbf{D}^2$, this implies $\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{D}$, etc., so we also have

$$\begin{aligned}(\mathbf{A} - \mathbf{B})^2 &= (\mathbf{C} - \mathbf{D})^2, \\ (\mathbf{A} - \mathbf{C})^2 &= (\mathbf{B} - \mathbf{D})^2, \\ (\mathbf{A} - \mathbf{D})^2 &= (\mathbf{B} - \mathbf{C})^2.\end{aligned}$$

Thus opposite sides of $ABCD$ are of equal length, and so are the diagonals, so $ABCD$ is a rectangle.

For the case $r = 3$, $n = 2$ we would get

$$\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 = 0$$

and thus $B_1 B_2 B_3$ must be an equilateral triangle. For $r = 4$, $n = 3$ we would get

$$\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 = 0$$

and thus, as above, $B_1 B_2 B_3 B_4$ must have opposite sides equal, i.e. it is an isosceles tetrahedron.

Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; DAN PEDOE, Minneapolis, Minnesota; and the proposer.

Pedoe pointed out Klamkin's theorem in the case $n = 2$.

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1242. [1987: 149] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

The following problem appears in a book on matrix analysis: "Show that

$\sum_{i,j=1}^n a_{ij}x_i x_j$ is positive definite if

$$\sum_i a_{ii}x_i^2 + \sum_{i \neq j} |a_{ij}|x_i x_j$$

is positive definite."

Give a counterexample!

Solution by Leroy F. Meyers, The Ohio State University.

Let $n = 3$ and

$$a_{ii} = 3, \quad a_{ij} = -2, \quad 1 \leq i, j \leq 3.$$

Then

$$\sum_i a_{ii}x_i^2 + \sum_{i \neq j} |a_{ij}|x_i x_j = \frac{7}{3}(x_1 + x_2 + x_3)^2 + \frac{1}{6}(x_1 + x_2 - 2x_3)^2 + \frac{1}{2}(x_1 - x_2)^2$$

is positive definite, whereas

$$\sum_i a_{ii}x_i^2 + \sum_{i \neq j} a_{ij}x_i x_j = -\frac{1}{3}(x_1 + x_2 + x_3)^2 + \frac{5}{6}(x_1 + x_2 - 2x_3)^2 + \frac{5}{2}(x_1 - x_2)^2$$

is negative for $x_1 = x_2 = x_3$, and so indefinite.

There is obviously no counterexample in the case of one indeterminate. There is also no counterexample in the case of two indeterminates, since the condition for positive definiteness, namely $a_{11}a_{22} > a_{12}a_{21}$ (assuming $a_{11}, a_{22} > 0$) is implied by $a_{11}a_{22} > |a_{12}| |a_{21}|$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.

The proposer notes that the author of the problem in question no doubt intended the two quadratic forms to be interchanged.

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1243. [1987: 149] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and M an interior point with barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$. The distances of M from the vertices A, B, C are x_1, x_2, x_3 and the circumradii of the triangles MBC, MCA, MAB, ABC are R_1, R_2, R_3, R . Show that

$$\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 \geq R \geq \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3.$$

Solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Murray S. Klamkin, University of Alberta.

We will use a more usual notation. The triangle will be denoted by $A_1A_2A_3$ and its sides by a_1, a_2, a_3 . The distances of M from the vertices and sides of $\Delta A_1A_2A_3$ will be R_1, R_2, R_3 and r_1, r_2, r_3 respectively. The area of ΔMA_2A_3 will be denoted $[MA_2A_3]$, etc. Then the circumradius of ΔMA_2A_3 (the proposer's R_1) becomes

$$\frac{a_1 R_2 R_3}{4[MA_2A_3]} = \frac{a_1 R_2 R_3}{2a_1 r_1} = \frac{R_2 R_3}{2r_1}, \text{ etc.}$$

Moreover

$$\lambda_1 = \frac{[MA_2A_3]}{F} = \frac{a_1 r_1}{2F}, \text{ etc.,}$$

where $F = [A_1A_2A_3]$. In terms of this new notation, the given inequalities reduce to

$$a_1 R_2 R_3 + a_2 R_3 R_1 + a_3 R_1 R_2 \geq 4RF = a_1 a_2 a_3 \geq 2(a_1 r_1 R_1 + a_2 r_2 R_2 + a_3 r_3 R_3). \quad (1)$$

The left-hand inequality is known and has even appeared recently in this journal [1987: 260].

For the right-hand inequality, from [1] and [2] comes the following duality of triangle inequalities: if

$$I(a_1, a_2, a_3, r_1, r_2, r_3, R_1, R_2, R_3) \geq 0$$

is a valid triangle inequality, then so is

$$I\left[\frac{a_1 r_1 R_1}{2R}, \frac{a_2 r_2 R_2}{2R}, \frac{a_3 r_3 R_3}{2R}, \frac{k}{R_1}, \frac{k}{R_2}, \frac{k}{R_3}, \frac{k}{r_1}, \frac{k}{r_2}, \frac{k}{r_3}\right] \geq 0,$$

and vice versa, where $k = r_1 r_2 r_3$. Applying this to the second inequality of (1), we have to show

$$\frac{a_1 a_2 a_3 k R_1 R_2 R_3}{8R^3} \geq \frac{k^2 (a_1 + a_2 + a_3)}{R}$$

or

$$\frac{R_1 R_2 R_3}{r_1 r_2 r_3} \geq \frac{4R}{r}.$$

Now item 12.26 of [3] is

$$\frac{R_1 R_2 R_3}{r_1 r_2 r_3} \geq \frac{1}{\sin(A_1/2)\sin(A_2/2)\sin(A_3/2)} = \frac{4R}{r}.$$

Done!

References:

- [1] A. Oppenheim, The Erdős inequality and other inequalities for a triangle, *Amer. Math. Monthly* 68 (1961) 226–230.
- [2] M.S. Klamkin, Triangle inequalities via transforms, *Notices A.M.S.*, January 1972, A–103, 104.
- [3] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.

Also solved by the proposer. The solutions of Janous and Klamkin were virtually identical, even to the point of altering the notation the same way.

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- 1244.** [1987: 149] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all triangles with integral sides a, b, c and area A such that a, b, c , and A form an arithmetic progression.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let the common difference of the progression be d (not necessarily > 0) and let $a = b - d$, $c = b + d$, $A = b + 2d$. Then $s = 3b/2$ and by Heron's formula,

$$(b + 2d)^2 = \frac{3b}{2}(\frac{3b}{2} - b + d)(\frac{3b}{2} - b - d) = \frac{3b^2}{4}(\frac{b^2}{4} - d^2). \quad (1)$$

Thus b must be even, say $b = 2B$ for some positive integer B . Then (1) reduces to

$$4(B + d) = 3B^2(B - d)$$

and finally

$$d = \frac{3B^2 - 4B}{3B^2 + 4} = B - \frac{8B}{3B^2 + 4}. \quad (2)$$

Now $3B^2 + 4 > 8B$ for $B > 2$, so the quotient in (2) will not be an integer. Checking $B = 1$ and $B = 2$, only $B = 2$ yields a solution, which is $d = 1$, $b = 4$, $a = 3$, $c = 5$, $A = 6$. Thus the unique solution is the 3–4–5 right triangle.

Also solved by SYDNEY BULMAN-FLEMING, Wilfrid Laurier University, Waterloo, Ontario; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; and the proposer.

Bulman-Fleming, Janous, and Wilke (see the above solution) allowed the arithmetic progression a, b, c, A to be decreasing as well as increasing. Janous in fact considered all orderings of a, b, c, A without finding any further solutions.

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1245. [1987: 150] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria. (Dedicated to Léo Sauv  .)*

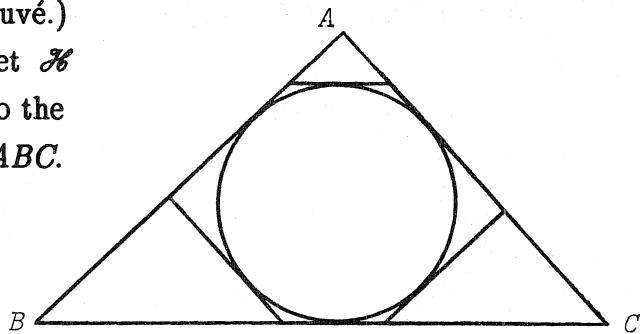
Let ABC be a triangle, and let \mathcal{H}

be the hexagon created by drawing tangents to the incircle of ABC parallel to the sides of ABC .

Prove that

$$\text{perimeter}(\mathcal{H}) \leq \frac{2}{3} \text{perimeter}(ABC).$$

When does equality occur?



Solution by Hans Engelhardt, Gundelsheim, Federal Republic of Germany.

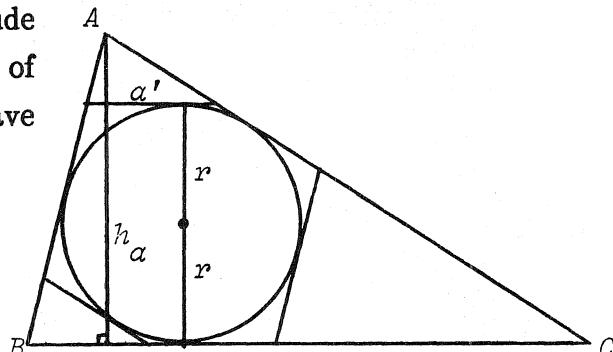
Letting r be the inradius, h_a the altitude from C , and a' , b' , c' the lengths of the sides of \mathcal{H} parallel to BC , CA , AB respectively, we have by similar triangles

$$\frac{a'}{a} = \frac{h_a - 2r}{h_a} = 1 - \frac{2r}{h_a} = 1 - \frac{a}{s}$$

where s is the semiperimeter. Thus

$$a' = a - a^2/s, \text{ etc.}$$

and since opposite sides of \mathcal{H} have equal lengths,



$$\begin{aligned} \frac{\text{perimeter}(\mathcal{H})}{\text{perimeter}(ABC)} &= \frac{a' + b' + c'}{s} = \frac{a + b + c - \frac{a^2 + b^2 + c^2}{s}}{s} \\ &= 2 - \frac{a^2 + b^2 + c^2}{s^2} = 2 - \frac{4(a^2 + b^2 + c^2)}{(a + b + c)^2}. \end{aligned} \quad (1)$$

Since

$$(a - b)^2 + (a - c)^2 + (b - c)^2 \geq 0,$$

we have

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2,$$

and thus from (1)

$$\frac{\text{perimeter}(\mathcal{H})}{\text{perimeter}(ABC)} \leq 2 - \frac{4}{3} = \frac{2}{3}.$$

Equality occurs for $a = b = c$.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1246. [1987: 150] *Proposed by Lanny Semenko, Erewhon, Alberta.*

Canadians may notice (in the summer only, of course) that 28°C and 82°F are almost, but not exactly, the same temperature, where 82 is 28 written in reverse order. Find positive integers M and N such that $M^{\circ}\text{C} = N^{\circ}\text{F}$, and M and N in decimal are reverses of each other, or prove that no such M and N exist.

Combined solutions of Hans Engelhaupt, Gundelsheim, Federal Republic of Germany; and the proposer.

We prove that there are no such integers M , N .

Suppose that

$$N = 1.8M + 32, \quad (1)$$

where the digits of M are a_1, a_2, \dots, a_n from left to right, so that

$$M = a_1 a_2 \dots a_n, \quad N = a_n \dots a_2 a_1.$$

We find necessary conditions on the a_i 's until there is a contradiction.

(i) First, from (1) 5 is a divisor of M , so $a_n = 5$. (Note $a_n = 0$ is impossible because then $N < M$.)

(ii) Since $N \equiv a_1 \pmod{10}$ and

$$\begin{aligned} 1.8M + 32 &\equiv 1.8(10a_{n-1} + 5) + 32 \pmod{10} \\ &\equiv 8a_{n-1} + 1 \pmod{10}, \end{aligned}$$

from (1) a_1 must be odd. By considering the first digits of N and M , from (1) $a_1 = 3$.

(iii) Thus from the above

$$8a_{n-1} + 1 \equiv 3 \pmod{10}$$

and so $a_{n-1} = 4$ or 9. If $a_{n-1} = 4$ then by considering the first two digits of N and M , from (1) $a_2 = 0$. But then $N \equiv 3 \pmod{100}$ while

$$\begin{aligned} 1.8M + 32 &\equiv 1.8(100a_{n-2} + 45) + 32 \pmod{100} \\ &\equiv 80a_{n-2} + 113 \pmod{100}, \end{aligned}$$

so that

$$80a_{n-2} + 110 \equiv 0 \pmod{100},$$

or

$$10 | 8a_{n-2} + 11,$$

a contradiction. Thus $a_{n-1} = 9$.

(iv) Now

$$N \equiv 10a_2 + 3 \pmod{100}$$

and

$$\begin{aligned} 1.8M + 32 &\equiv 1.8(100a_{n-2} + 95) + 32 \pmod{100} \\ &\equiv 80a_{n-2} + 203 \pmod{100}, \end{aligned}$$

so from (1)

$$a_2 \equiv 8a_{n-2} \pmod{10}. \quad (2)$$

Hence a_2 must be even. By considering the first two digits of N (i.e. 59) and M (i.e. $3a_2$), from (1) $a_2 = 2$.

(v) Finally we look at a_{n-2} . By considering the first two digits of M (i.e. 32) and the first three digits of N (i.e. $59a_{n-2}$), we see from (1) that $a_{n-2} \leq 3$. But from (2)

$$8a_{n-2} \equiv 2 \pmod{10},$$

and so $a_{n-2} = 4$ or 9, which is a contradiction.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

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1247. [1987: 150] *Proposed by Robert E. Shafer, Berkeley, California.*

Prove that for $0 \leq \phi < \theta \leq \pi/2$,

$$\begin{aligned} \cos^2 \frac{\phi}{2} \log \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \log \sin^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2} \log \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \log \sin^2 \frac{\theta}{2} \\ < \frac{3}{4} [\sin^{4/3} \theta - \sin^{4/3} \phi]. \end{aligned}$$

Solution by Kee-Wai Lau, Hong Kong.

Let $x = \cos \phi$, so that

$$\cos^2 \frac{\phi}{2} = \frac{1+x}{2}, \quad \sin^2 \frac{\phi}{2} = \frac{1-x}{2}, \quad \sin \phi = (1-x^2)^{1/2}.$$

Thus it suffices to show that the function

$$f(x) = \frac{1}{2}(1+x)\log\left[\frac{1+x}{2}\right] + \frac{1}{2}(1-x)\log\left[\frac{1-x}{2}\right] + \frac{3}{4}(1-x^2)^{2/3}$$

decreases for $0 < x < 1$. Now

$$\begin{aligned} f'(x) &= \frac{1}{2} \log\left[\frac{1+x}{1-x}\right] - x(1-x^2)^{-1/3} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} - \left[x + \sum_{n=1}^{\infty} \frac{1 \cdot 4 \cdot \dots \cdot (3n-2)x^{2n+1}}{3^n n!} \right] \\ &= \sum_{n=2}^{\infty} \left[\frac{1}{2n+1} - \frac{1 \cdot 4 \cdot \dots \cdot (3n-2)}{3^n n!} \right] x^{2n+1}. \end{aligned} \tag{1}$$

Consider

$$a_n = \frac{1 \cdot 4 \cdot \dots \cdot (3n-2)(2n+1)}{3^n n!}, \quad n = 2, 3, \dots$$

We have $a_2 = 10/9$ and

$$a_{n+1} = \frac{(3n+1)(2n+3)}{3(n+1)(2n+1)} a_n = \frac{6n^2 + 11n + 3}{6n^2 + 9n + 3} a_n > a_n.$$

It follows that $a_n > 1$ for $n = 2, 3, \dots$. Thus the coefficient of x^{2n+1} in (1) is negative, so $f'(x) < 0$ for $0 < x < 1$.

Also solved by VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and the proposer.

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1248. [1987: 150] Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.

Suppose that m and n are positive integers and that the decimal expansion of the rational number m/n has a repetend of 4356. Prove that n is divisible by 101.

Solution by Eddie Cheng, student, Memorial University of Newfoundland, St. John's.

Since we are only concerned with n , without loss of generality assume $m < n$. Then

$$\begin{aligned}\frac{m}{n} &= 0.a_1a_2\dots a_k\overline{43564356\dots} \\ &= \frac{a_1a_2\dots a_k \cdot 43564356\dots}{10^k} \\ &= \frac{A}{10^k} + \frac{0.\overline{43564356\dots}}{10^k} \quad \text{where } A = a_1a_2\dots a_k \\ &= \frac{A}{10^k} + \frac{4356}{9999 \cdot 10^k} \\ &= \frac{9999A + 4356}{9999 \cdot 10^k} = \frac{m'}{n'}.\end{aligned}$$

Now since $101|9999$, $101|n'$, so $101|mn'$. Since $mn' = m'n$, $101|m'n$. Now if 101 were to divide $m' = 9999A + 4356$, then since $101|9999$, 101 would divide 4356, which is not true. Thus 101 does not divide m' . Since 101 is a prime, 101 divides n .

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Shan and Wang point out more generally that the repetend 4356 could have been replaced by any four-digit number not of the form $abab$. Thus they remark: "It is a mystery why the proposer chose the particular number 4356 when the conclusion is true in general. Could it be the case that Professor Larson was born on May 6, 1943?"

Contacted about this, Larson replies that no, 4356 was chosen because it is divisible by 99, a restriction which is unnecessary but makes the solution a little neater. Could it be the case that Professor Larson is a Wayne Gretzky fan?

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