Crux Mathematicorum

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EDITORIAL

I do a lot of outreach presentations. I love sharing with younger audiences the mathematics that they do not see in school. More importantly, I enjoy showing the students that mathematics can be exciting, beautiful, social and approachable. But I always make it my goal not to trivialize the material I'm presenting. It's an easy trap to fall into: between all the fun and games, you might forget that you also wanted students to learn something mathematical.

Who doesn't like origami? It allows one to make beautiful three dimensional objects out of two dimensional paper. Origami's minimalistic requirements and impressive outcomes are also what make it a popular outreach topic: if the students are old enough to follow simple instructions, they will leave the presentation with a cool piece of origami. But while the presenters focus on providing clear folding instructions and ensuring everyone is engaged, they tend to forget to mention any math. So students leave with origami and better paper-folding skills, but without their critical thinking being challenged and without any understanding about how origami is related to math.

Another paper activity that is popular with outreach audiences is construction of hexaflexagons, that is hexagons that can be "flexed" or turned inside out to reveal different faces. Generally, you are given a template of pre-cut strip of paper that has been pre-marked with 9 equilateral triangles that you now have to fold. But why? Why do presenters choose to deprive their audiences of the ability to discover how to take a rectangular piece of paper and divide it into 9 equilateral triangles. Can you do it using a protractor? Can you do it using a straightedge and compass? Can you do it with just your bare hands? Can you do it in more than one way? (Thanks to my friend Vanessa Radzimski who made us do it with no tools.)

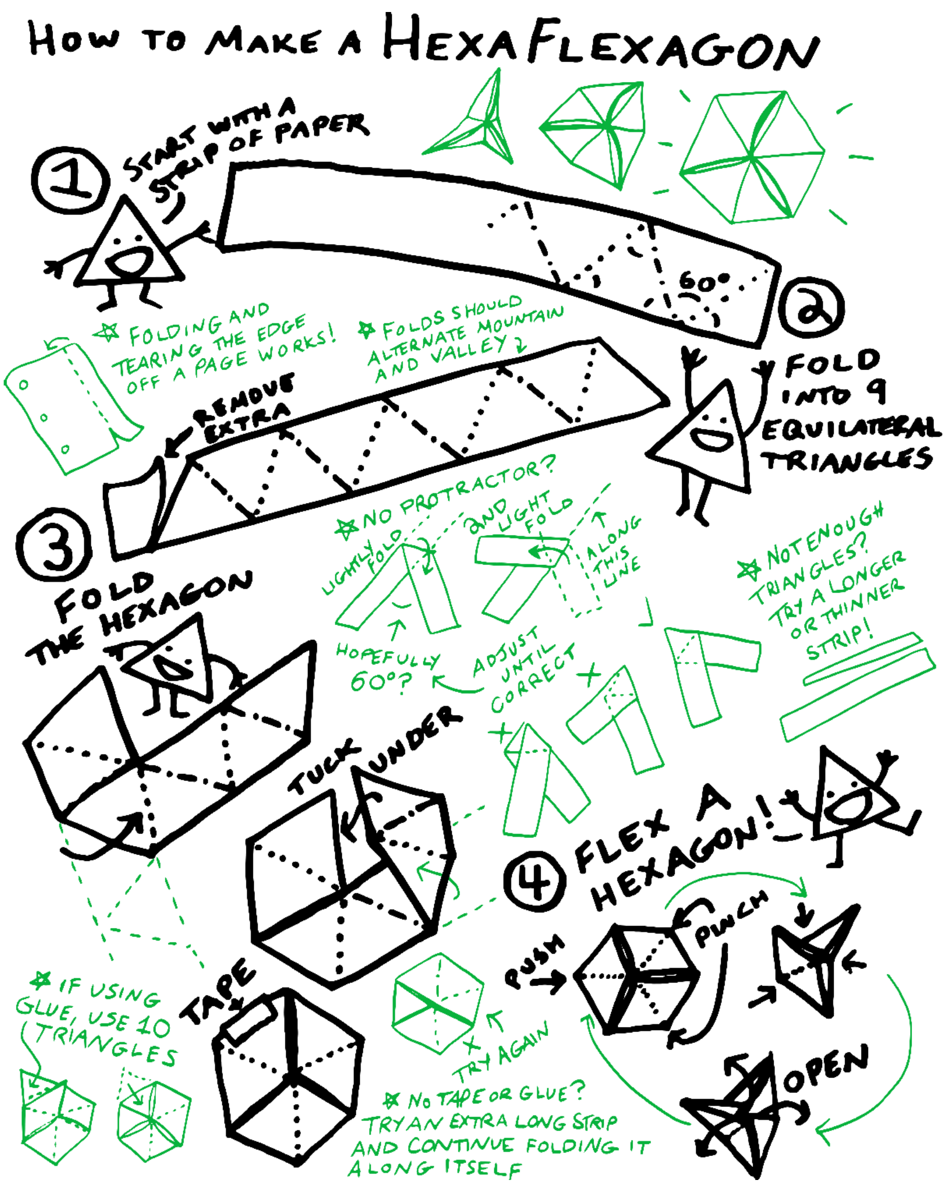
It is therefore with appreciation for math behind beauty that I welcome an origami article in this issue.

In this issue, we also welcome a new Olympiad Corner editor Alessandro Ventullo.

For those of you who like sharing mathematics in the form of various hands-on activities, I urge you to not skip over the math parts. After all, they are the best parts.

P. S. The fascinating theory of how hexaflexagons were discovered and studied by Stone, Tuckerman, Feynman and Tukey can be found in this article by Gardner: `www.maa.org/sites/default/files/pdf/pubs/focus/Gardner_Hexaflexagons12_1956.pdf`

Kseniya Garaschuk



Graphics are by Vi Hart, `http://vihart.com/hexaflexagons/`

THE CONTEST CORNER

No. 65

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **October 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC321. Six boxes are numbered $1, 2, 3, 4, 5$ and 6. Suppose that there are N balls distributed among these six boxes. Find the least N for which it is guaranteed that for at least one k, box number k contains at least k^2 balls.

CC322. Suppose that the vertices of a polygon all lie on a rectangular lattice of points where adjacent points on the lattice are at distance 1 apart. Then the area of the polygon can be found using Pick's Formula: $I + \frac{B}{2} - 1$, where I is the number of lattice points inside the polygon, and B is the number of lattice points on the boundary of the polygon. Pat applied Pick's Formula to find the area of a polygon but mistakenly interchanged the values of I and B. As a result, Pat's calculation of the area was too small by 35. Using the correct values for I and B, the ratio $n = \frac{I}{B}$ is an integer. Find the greatest possible value of n. (Ed.: For more information on Pick's formula, take a look at the article *Two Famous Formulas (Part I)*, ***Crux*** 43 (2), p. 61–66.)

CC323. Evaluate

$$\frac{\log_{10}(20^2) \cdot \log_{20}(30^2) \cdot \log_{30}(40^2) \cdots \log_{990}(1000^2)}{\log_{10}(11^2) \cdot \log_{11}(12^2) \cdot \log_{12}(13^2) \cdots \log_{99}(100^2)}$$

CC324. On the inside of a square with side length 60, construct four congruent isosceles triangles each with base 60 and height 50, and each having one side coinciding with a different side of the square. Find the area of the octagonal region common to the interiors of all four triangles.

CC325. Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

. . . .

CC321. Six boîtes sont numérotées $1, 2, 3, 4, 5$ et 6. On suppose que N balles sont distribuées dans ces six boîtes. Déterminer la plus petite valeur de N pour laquelle il est certain que pour au moins une valeur de k, la boîte numéro k contient au moins k^2 balles.

CC322. On considère un quadrillage avec une distance de 1 unité entre les lignes verticales et entre les lignes horizontales. Un polygone est tracé sur le quadrillage et ses sommets sont des points de treillis. On peut alors calculer l'aire du polygone à l'aide de la formule de Pick : $A = i + \frac{b}{2} - 1$, i étant le nombre de points de treillis à l'intérieur du polygone et b étant le nombre de points de treillis sur le bord du polygone. Pat a utilisé la formule de Pick pour calculer l'aire d'un polygone, mais il a changé l'une pour l'autre les valeurs de i et de b avec comme résultat que sa réponse était 35 de moins que la bonne réponse. Si on utilise les bonnes valeurs de i et de b, le rapport $n = \frac{i}{b}$ est un entier. Déterminer la plus grande valeur possible de n. (N.D.L.R. Pour en connaître davantage sur la formule de Pick, voir l'article *Two Famous Formulas (Part I)*, ***Crux*** 43 (2), p. 61–66.)

CC323. Évaluer

$$\frac{\log_{10}(20^2) \cdot \log_{20}(30^2) \cdot \log_{30}(40^2) \cdots \log_{990}(1000^2)}{\log_{10}(11^2) \cdot \log_{11}(12^2) \cdot \log_{12}(13^2) \cdots \log_{99}(100^2)}.$$

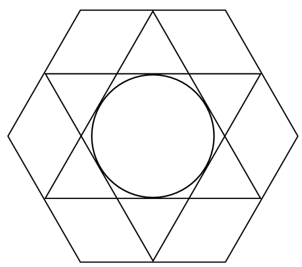
CC324. À l'intérieur d'un carré de 60 de côté, on construit quatre triangles isocèles, chacun avec une base de 60 et une hauteur de 50, chaque triangle ayant un côté qui coïncide avec un côté du carré. Déterminer l'aire de la région octogonale commune aux quatre triangles.

CC325. Sept personnes d'âges différents assistent à une réunion. Les sept personnes quittent la réunion une par une dans un ordre aléatoire. Sachant que la personne la plus jeune quitte avant que la personne la plus âgée ne quitte, la probabilité que les troisième, quatrième et cinquième personnes quittent la réunion dans l'ordre de leurs âges (de la plus jeune à la plus âgée) est égale à $\frac{m}{n}$, m et n étant des entiers premiers entre eux. Déterminer $m + n$.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(5), p. 184–189.

CC271. Warren's lampshade has an interesting design. Within a regular hexagon (six sides) are two intersecting equilateral triangles, and within them is a circle which just touches the sides of the triangles. (See the diagram.) The points of the triangles are at the midpoints of the sides of the hexagons.



If each side of the hexagon is 20 cm long, find:

a) the area of the hexagon;

b) the area of each large equilateral triangle;

c) the area of the circle.

Originally Question 5 from the 2011 University of Otago Junior Mathematics Competition.

We received two submissions to this problem, both of which were correct. We present the solution by David Manes, modified by the editor.

a) The formula for the area A_H of a regular hexagon is $A_H = \dfrac{3\sqrt{3}}{2}s^2$, where s is the side length. Since $s = 20$ cm, we have

$$A_H = \frac{3\sqrt{3}}{2}(20)^2 = 600\sqrt{3}\ \text{cm}^2.$$

b) Note that the two large equilateral triangles are congruent, the six smaller equilateral triangles are all congruent and the six rhombi are also congruent. Since the vertices of the equilateral triangles occur at the midpoints of the sides of the hexagon, it follows that the side length of each rhombus and each smaller equilateral triangle is 10 cm. Thus the side length for the two larger equilateral triangles is $a = 30$ cm. Therefore, the area A_T for each of the larger equilateral triangles is

$$A_T = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(30)^2 = 225\sqrt{3} \text{ cm}^2.$$

c) The circle is the inscribed circle for each of the two equilateral triangles with side length $a = 30$ cm. Therefore, its radius r is given by $r = \frac{\sqrt{3}}{6}a = 5\sqrt{3}$ cm. Hence, the area A_C of the circle is

$$A_C = \pi r^2 = \pi(5\sqrt{3})^2 = 75\pi \text{ cm}^2.$$

CC272. A *sum-palindrome number* (SPN) is a number that, when there are an even number of digits, the first half of the digits sums to the same total as the second half of the digits, and when odd, the digits to the left of the central digit sum to the same total as the digits to the right of the central digit. A *product-palindrome number* (PPN) is like a sum-palindrome, except the products of the digits are involved, not the sums.

a) How many three-digit SPNs are there?

b) The two SPNs 1203 and 4022 sum to 5225, which is itself a SPN. Is it true that, for any two four-digit SPNs less than 5000, their sum is also a SPN?

c) How many four-digit non-zero PPNs are there?

Originally Question 2 from the 2011 University of Otago Junior Mathematics Competition.

We received 2 solutions, one of which was correct and complete. We present the solution by Ivko Dimitrić.

a) A three-digit SPN is of the form aba where b, the digit of tens, can be any digit 0 to 9 (thus ten choices) and digit a of hundreds cannot be zero but can be any number 1 to 9, thus 9 choices. Then there are $10 \cdot 9 = 90$ ways to choose a and b independently, and hence there are 90 three-digit SPNs.

b) The statement is not true, in particular in some situations that involve "carrying over" of units to a higher-value place, such as in examples

$$3544+4316 = 7860,\ 2222+1964 = 4186,\ 4114+2727 = 6841,\ 1991+1991 = 3982.$$

c) First, we determine the number of four-digit PPNs in which 0 does not appear among the digits. We do the counting according to the number of distinct digits used in the representation of a PPN.

(1) There are nine PPNs if all the digits are equal.

(2) If exactly two distinct digits a and b are used, then the same pair is used in the first half of the number as in the second half for the following four

possibilities for each choice of a and b : $abab$, $abba$, $baab$, $baba$. Since a pair of distinct digits can be chosen from the set of nine in $\binom{9}{2} = 36$ ways, each choice producing four PPNs, the total number produced this way is $36 \cdot 4 = 144$.

(3) If exactly three digits a, b and c are used, for such a number to be a PPN the product of two of them equals the square of the third, e. g. $a \cdot b = c^2$, whereas a pair a, b appears in one half of the number and digit c, repeated twice, in the other half. This happens for the following four products of pairs

$$1 \cdot 4 = 2 \cdot 2, \quad 1 \cdot 9 = 3 \cdot 3, \quad 2 \cdot 8 = 4 \cdot 4, \quad 4 \cdot 9 = 6 \cdot 6,$$

each of the cases producing four PPNs, $abcc$, $bacc$, $ccab$, $ccba$, therefore there are sixteen PPNs obtained this way.

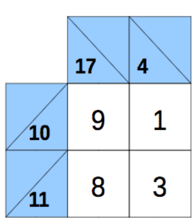
(4) If four distinct digits are used for a PPN with two different sets of pairs a, b and c, d in each half, we must have $ab = cd$. Since all four digits are non-zero and different, it is easily seen that $ac \neq bd$ and $ad \neq bc$, which means that the numbers from different pairs cannot be swapped and combined in the same half of the number and would have to stay within the original pair. This situation happens for the following five products of pairs:

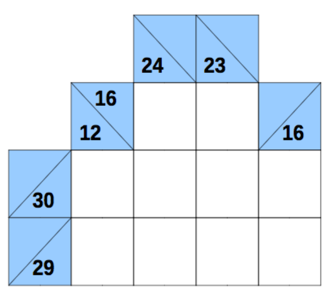
$$1 \cdot 6 = 2 \cdot 3, \quad 1 \cdot 8 = 2 \cdot 4, \quad 2 \cdot 6 = 3 \cdot 4, \quad 2 \cdot 9 = 3 \cdot 6, \quad 3 \cdot 8 = 4 \cdot 6.$$

Each of them gives rise to eight PPNs: $abcd$, $abdc$, $bacd$, $badc$, $cdab$, $cdba$, $dcab$, $dcba$. Thus, there are $8 \cdot 5 = 40$ PPNs obtained this way.

Adding up the numbers for all the possibilities we get $9 + 144 + 16 + 40 = 209$ PPNs in which the product of digits in each half of the number is the same and 0 is not one of the digits.

CC273. Kakuro is the name of a number puzzle where you place numbers from 1 to 9 into empty boxes. There are three rules in a Kakuro puzzle: only numbers from 1 to 9 may be used, no number is allowed in any line (across or down) more than once, the numbers must add up to the totals shown at the top and the left. The left figure in the diagram shows a small finished Kakuro puzzle. Solve the Kakuro puzzle on the right in the diagram. Is your solution unique?





Originally Question 1 from the 2008 University of Otago Junior Mathematics Competition.

We received one submission. We present the solution by Ivko Dimitrić.

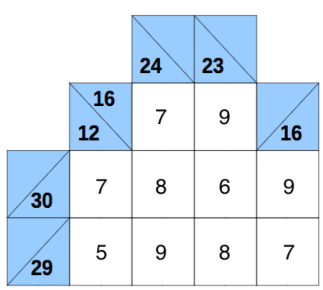
The only way (up to the order) to write 16 as a sum of two distinct positive integers which are less than ten is $16 = 7 + 9 = 9 + 7$. Thus, these are the choices for the Sum-16 row and Sum-16 column.

Assume the Sum-16 row (the first row) is $[\,9 \mid 7\,]$. The Sum-23 column is then $[\,7 \mid a \mid b\,]^T$, where subscript T denotes transpose, i.e. the triple of numbers should be seen in a vertical column. Then $a+b = 16$, where a, b are distinct integers from 1 to 9, so one of them must be 7 again, which cannot happen, since we already have one 7 in the top box of that column. Hence, our assumption cannot stand and the first row is then $[\,7 \mid 9\,]$, whereupon the Sum-23 column takes the form $[\,9 \mid a \mid b\,]^T$, with $a + b = 14$.

Let the Sum-24 column be $[\,7 \mid c \mid d\,]^T$ where $c+d = 17$ so that one of the numbers c, d is 9 the other one 8, in particular c is either 8 or 9. Since 9 already appears in the Sum-23 column, according to the rules, $a + b = 14$ is possible only if it is the sum of 8 and 6. Assume first that the Sum-23 column is $[\,9 \mid 8 \mid 6\,]^T$. We show that this is not possible.

If the Sum-16 column (the last column) is $[\,7 \mid 9\,]^T$, then the Sum-29 row has the form $[\,f \mid d \mid 6 \mid 9\,]$, with $f + d = 14 = 9 + 5 = 8 + 6$, so one term in the sum would have to be 6 or 9, clashing with the same number already in the last row. If the Sum-16 column is $[\,9 \mid 7\,]^T$, then the Sum-30 row would appear as $[\,e \mid c \mid 8 \mid 9\,]$, so the number c (which, from the above, is either 8 or 9) would clash with one of the last two numbers. Thus, both possibilities for the last column contradict the choice of the Sum-23 column as $[\,9 \mid 8 \mid 6\,]^T$, which means that that column would have to be changed to $[\,9 \mid 6 \mid 8\,]^T$ instead. Then, since $c + d = 17 = 9 + 8$ in the Sum-24 column, we have $c = 8$ and $d = 9$ and that column is completed as $[\,7 \mid 8 \mid 9\,]^T$ so that the last column could be only $[\,9 \mid 7\,]^T$, to avoid having two 9s in the last row. Then the remaining cell in the Sum-30 row is filled with $e = 7$ and the remaining cell in the Sum-29 row is filled with $f = 5$ to make the required sums, including the remaining Sum-12 column.

Since every choice at every step is forced upon by previous choices or contradictions resulting from alternative possibilities, every number is uniquely determined, i. e. there is only one solution to the puzzle, namely



CC274. In his office, Shaquille had nine ping pong balls which he used for therapeutic recovery by throwing them into the waste basket at slack times. Each time he threw the nine balls, some of them would land in the basket, with the rest of them landing on the floor.

a) If the balls are identical, how many different results could there be?

b) Suppose now that the balls are numbered 1 to 9. How many different results could there be now? (For example one possible result is for balls 1 to 4 to land in the basket, with 5 to 9 on the floor.)

c) Suppose instead that the balls are not numbered, but five are coloured yellow and four blue. Now how many different results could there be? (For example one possible result is for two yellow balls and three blue balls to land in the basket, and the rest to land on the floor.)

d) One day another basket appeared in the office. So now Shaquille had a choice of baskets to aim at. How did this change the answers to (a), (b), and (c)?

e) Now suppose that every time he threw the balls at the two baskets, each basket received at least two balls. How would this change the answers to (a), (b), and (c)?

Originally Question 5 from the 2008 University of Otago Junior Mathematics Competition.

We received one solution to this problem. We present the solution by Ivko Dimitrić, modified by the editor.

In each case, the number and the selection of the balls that land on the floor is uniquely determined by the number and the selection of the balls that land in the basket(s), so it suffices to count the number of balls that land in the basket(s) to get the number of different results.

a) Any number $k = 0, 1, \ldots, 9$ of balls can land in the basket. Therefore there are 10 different results.

b) A numbered ball is either in or out of the basket. Therefore each numbered ball has 2 possible states. As there are 9 balls, there are $2^9 = 512$ different results.

c) Since any number $y = 0, 1, \ldots 5$ of yellow balls and any number $b = 0, 1, \ldots 4$ of blue balls can independently land in the basket we have $6 \cdot 5 = 30$ different results.

d) Let B_L and B_R refer to the "left basket" and "right basket," respectively.

To answer a), for any number $k = 0, 1, \ldots, 9$ of balls that land in B_L there are $10-k$ choices of balls that may land in B_R. Therefore there are $\sum_{k=0}^{9}(10-k) = 55$ different results.

To answer b), a numbered ball is either in B_L, in B_R, or not in either. Therefore each numbered ball has 3 possible states. As there are 9 balls, there are $3^9 = 19{,}683$ different results.

To answer c), for any number $y = 0, 1, \ldots, 5$ of yellow balls that land in B_L there are $6 - y$ choices of yellow balls that may land in B_R. Therefore there are $\sum_{y=0}^{5}(6-y) = 21$ ways to distribute the 5 yellow balls. Likewise, for any number $b = 0, 1, \ldots, 4$ of blue balls that land in B_L there are $5 - b$ choices of blue balls that may land in B_R. Therefore there are $\sum_{b=0}^{4}(5-b) = 15$ ways to distribute the 4 blue balls. Therefore there are $21 \cdot 15 = 315$ different results.

e) To answer a) for any number $k = 2, 3, \ldots, 7$ of balls that land in B_L there are $8 - k$ choices of balls that may land in B_R. Therefore there are $\sum_{k=2}^{7}(8-k) = 21$ different results.

To answer (b), choose $n \geq 4$ balls out of nine in $\binom{9}{n}$ ways to be distributed in baskets B_L and B_R. Given the n chosen balls, k land in basket B_L and $n - k$ land in basket B_R. Therefore $2 \leq k \leq n - 2$. Thus, the number of results is

$$\begin{aligned}
\sum_{n=4}^{9}\sum_{k=2}^{n-2}\binom{9}{n}\binom{n}{k} &= \sum_{n=4}^{9}\binom{9}{n}\sum_{k=2}^{n-2}\binom{n}{k} \\
&= \sum_{n=4}^{9}\binom{9}{n}\Big[\sum_{k=0}^{n}\binom{n}{k} - \binom{n}{0} - \binom{n}{1} - \binom{n}{n-1} - \binom{n}{n}\Big] \\
&= \sum_{n=4}^{9}\binom{9}{n}[2^n - 2 - 2n].
\end{aligned}$$

We complete the sum so that it starts from $n = 0$ by adding and subtracting terms corresponding to $n = 0, 1, 2, 3$, which are equal to $-1, -18, -72, 0$, respectively. The above sum becomes

$$\begin{aligned}
\sum_{n=0}^{9}\binom{9}{n}[2^n - 2 - 2n] + 91 &= \sum_{n=0}^{9}\binom{9}{n}2^n - 2\sum_{n=0}^{9}\binom{9}{n} - 2\sum_{n=0}^{9}\binom{9}{n}n + 91 \\
&= (2+1)^9 - 2(1+1)^9 - 2 \cdot 9 \cdot 2^8 + 91 \\
&= 3^9 - 11 \cdot 2^9 + 91 = 14{,}142.
\end{aligned}$$

Therefore there are 14,142 different results.

To answer c), we count the number of results in which either or both of the baskets end up with less than two balls and subtract that number from 315 (the number of results we found in part (d) without any restrictions).

Let the sets (L_Y, L_B) and (R_Y, R_B) correspond to the number of yellow and blue balls that land in B_L and B_R, respectively. If B_L ends up with less than 2 balls then the set $(L_Y, L_B) \in \{(0,0), (1,0), (0,1)\}$. We consider these three cases:

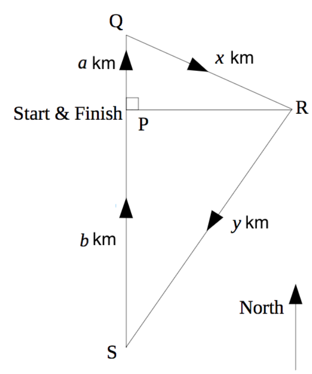
1. If $(L_Y, L_B) = (0,0)$, then there are 6 choices of yellow balls and 5 choices of blue balls landing in B_R. Totalling $6 \cdot 5 = 30$ different results.

2. If $(L_y, L_b) = (1,0)$, then there are 5 choices of yellow balls and 5 choices of blue balls landing in B_r. Totalling $5 \cdot 5 = 25$ different results.

3. If $(L_y, L_b) = (0,1)$, then there are 6 choices of yellow balls and 4 choices of blue balls landing in B_r. Totalling $6 \cdot 4 = 24$ different results.

A total of 79 results. If the situation is reversed and the basket B_R receives less than two balls, we will also have 79 results. In their sum the number of results where both baskets got less than two balls is counted twice. That number is $3 \cdot 3 = 9$, since there are three possibilities for each basket as listed above. The total number of results in which at least one basket has less than two balls is $2 \cdot 79 - 9 = 149$. The number of results in which both baskets received at least two balls is therefore $315 - 149 = 166$.

CC275. The local sailing club is planning its annual race. By tradition the boats always start at P, sailing due North for a distance of a km until they reach Q. They then turn and sail a distance of x km to R (which is due East of P). Next they turn and sail a distance of y km to S (which is South of P) before finally sailing due North for a distance of b km until the finish line, which is back at the starting point P (see the diagram). Bernie, the Club Commander, makes four extra rules for this year's race:



- $a + b = 40$,
- $x + y = 50$,
- $a < b$,
- the four lengths (a, b, x, y) must each be a whole number of kilometres. (Bernie doesn't like decimals.)

a) Find four numbers (a, b, x, y) which satisfy Bernie's four rules.

b) Are there four different numbers (a, b, x, y), which also satisfy Bernie's four rules apart from the four numbers you found in part (a)? Explain.

Originally Question 3 from the 2008 University of Otago Junior Mathematics Competition.

We received four solutions to this problem, all of which were correct. We present the composite solution of Konstantine Zelator, Digby Smith, and Ivko Dimitrić, modified by the editor.

We solve a) and b) by determining all solutions to Bernie's problem. From the right angle triangles PQR and PRS we have that

$$(PR)^2 = x^2 - a^2 = y^2 - b^2$$

Since $a + b = 40 \Rightarrow b = 40 - a$. Since $x + y = 50 \Rightarrow y = 50 - x$. Through substitution we see that

$$x^2 - a^2 = (50 - x)^2 - (40 - a)^2 ,$$

which simplifies to $x = 9 + \frac{4a}{5}$. Since x is a whole number it follows that a is divisible by 5. Note that, since $a < b$, $a < \frac{a+b}{2} = 20$. Therefore a is a whole number that is less than 20 and divisible by 5. Assuming $a > 0$ the possible values of a are 5, 10, and 15. Below we list these 3 cases of a:

$$\begin{array}{lcl} a = 5 & \Rightarrow & b = 35, x = 13, y = 37. \\ a = 10 & \Rightarrow & b = 30, x = 17, y = 33. \\ a = 15 & \Rightarrow & b = 25, x = 21, y = 29. \end{array}$$

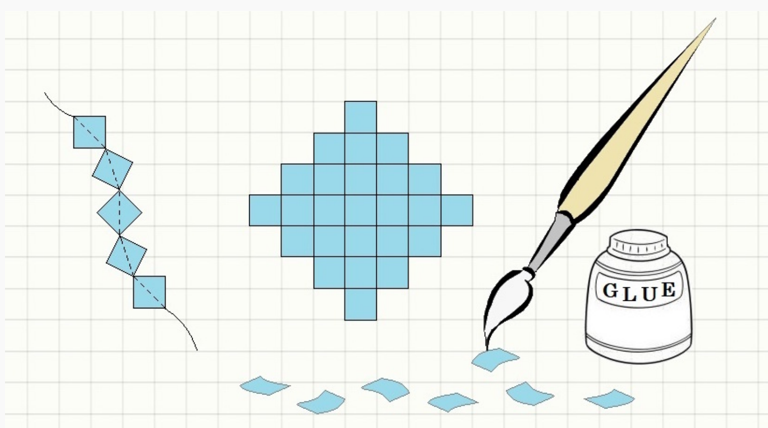
The three quadruples that solve Bernie's problem are therefore

$$(5, 35, 13, 37), \quad (10, 30, 17, 33) \quad \text{and} \quad (15, 25, 21, 29).$$

Note that if $a \geq 0$ then (0,40,9,41) is also a solution to Bernie's problem.

Square Garlands

Using construction paper, a string and some glue, make 5 garlands consisting of 5 squares each (for a sturdy garland, glue squares to both sides of the string).



Using these garlands, can you completely cover the figure shown above?

Puzzle by Nikolai Avilov.

THE OLYMPIAD CORNER

No. 363

Alessandro Ventullo

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **October 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC381. The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with one copy of their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

a) Prove that there is a sequence of replacements that will make the final number equal to 2.

b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

OC382. There are $n > 1$ cities in a country and some pairs of cities are connected by two-way non-stop flights. Moreover, every two cities are connected by a unique route (possibly with stopovers). A mayor of every city X counted the number of labelings of the cities from 1 to n so that every route beginning with X has the rest of the cities on that route occurring in ascending order. Every mayor, except one, noticed that the resulting number of their labelling is a multiple of 2016. Prove that last mayor's number of labelings is also a multiple of 2016.

OC383. Let ABC be a triangle. Let r and s be the angle bisectors of $\angle ABC$ and $\angle BCA$, respectively. The points E in r and D in s are such that $AD \| BE$ and $AE \| CD$. The lines BD and CE cut each other at F. The point I is the incenter of ABC. Show that if A,F and I are collinear, then $AB = AC$.

OC384. Solve the equation $xyz + yzt + xzt + xyt = xyzt + 3$ over the set of natural numbers.

OC385. A subset $S \subset \{0, 1, 2, \cdots, 2000\}$ satisfies $|S| = 401$. Prove that there exists a positive integer n such that there are at least 70 positive integers x such that $x, x + n \in S$.

. . . .

OC381. On a écrit les entiers $1, 2, 3, \ldots, 2016$ au tableau. Une action consiste à enlever n'importe quels deux nombres du tableau et ajouter leur moyenne à la liste, ce qui a pour effet de raccourcir la liste d'un terme. Par exemple, vous pouvez enlever les nombres 1 et 2 et ajouter le nombre 1,5 ou vous pouvez enlever les nombres 1 et 3 et ajouter le nombre 2. Après 2015 actions, il n'y aura plus qu'un seul nombre dans la liste.

a) Démontrer qu'il existe une suite d'actions telle qu'à la fin, le dernier nombre soit 2.

b) Démontrer qu'il existe une suite d'actions telle qu'à la fin, le dernier nombre soit 1000.

OC382. Il y a n ($n > 1$) villes dans un pays et certaines d'entre elles sont reliées par des vols directs aller-retour. Il y a exactement un lien aérien entre chaque paire de villes, possiblement avec escales et changements d'avion. Le maire de chaque ville X a compté le nombre d'étiquetages des villes de 1 à n de manière que sur chaque route à partir de X, les autres villes de cette route soient placées en ordre ascendant. Chaque maire, sauf un, remarque que le nombre d'étiquetages comptés est un multiple de 2016. Démontrer que le nombre d'étiquetages de ce dernier maire est aussi un multiple de 2016.

OC383. Soit un triangle ABC. Soit r et s les bissectrices des angles respectifs ABC et BCA. On considère les points E sur r et D sur s tels que $AD \| BE$ et $AE \| CD$. Soit F le point d'intersection des droites BD et CE. Soit I le centre du cercle inscrit dans le triangle ABC. Démontrer que si A, F et I sont alignés, alors $AB = AC$.

OC384. Résoudre l'équation $xyz + yzt + xzt + xyt = xyzt + 3$ dans l'ensemble des nombres naturels.

OC385. Un sous-ensemble S de $\{0, 1, 2, \cdots, 2000\}$ vérifie $|S| = 401$. Démontrer qu'il existe un entier strictement positif n de manière qu'il existe au moins 70 entiers strictement positifs x tels que $x, x + n \in S$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(3), p. 94–95.

OC321. Solve in positive integers

$$x^y y^x = (x+y)^z.$$

Originally 2015 Kazakhstan National Olympiad.

We received 3 submissions of which 1 was correct and complete. We present the solution by Steven Chow.

If $x = 1$, then $x^y y^x = (x+y)^z \implies y = (y+1)^z \geq y+1$ which is a contradiction.

Therefore $x \geq 2$ and similarly, $y \geq 2$.

If p is a prime such that $p \mid x$, then $p \mid x^y y^x = (x+y)^z$, so $p \mid x+y$ and $p \mid y$. If p is a prime such that $p \mid y$, then similarly, $p \mid x+y$ and $p \mid x$.

Therefore x, y, and $x+y$ have the same primes in their prime-power factorizations.

Let $\prod_{j=1}^{k} p_j^{\alpha_j} = x$ and $\prod_{j=1}^{k} p_j^{\beta_j} = y$ be the prime-power factorizations of x and y.

Since $x^y y^x = (x+y)^z$, therefore $z \mid \alpha_j y + \beta_j x$ for all j and

$$\prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j} = x + y = (x^y y^x)^{\frac{1}{z}} = \prod_{j=1}^{k} p_j^{\frac{\alpha_j y + \beta_j x}{z}}.$$

If there exists i such that $\alpha_i \neq \beta_i$, then WLOG, assume that $\alpha_i < \beta_i$, so

$$p_i^{\alpha_i} \mid \prod_{j=1}^{k} p_j^{\alpha_j} + \prod_{j=1}^{k} p_j^{\beta_j},$$

so

$$\alpha_i = \frac{\alpha_i y + \beta_i x}{z} > \frac{\alpha_i (x+y)}{z} \implies z > x + y \implies (x+y)^z > x^y y^x$$

from the Binomial Theorem, which is a contradiction.

Therefore for all j, $\alpha_j = \beta_j$.

Therefore $x = y$, so $x^y y^x = (x+y)^z \iff x^{2x} = 2^z x^z$.

If there exists a prime $p \neq 2$ such that $p \mid x$, then since $x^{2x} = 2^z x^z$, $2x = z$, so $1 = 2^{2x} \implies x = 0$, which is a contradiction.

Let $x_1 \geq 1$ be the integer such that $2^{x_1} = x$.

Therefore

$$\begin{aligned}x^{2x} = 2^z x^z \iff \left(2^{x_1}\right)^{2(2^{x_1})} &= 2^z \left(2^{x_1}\right)^z \\ \iff x_1 2^{x_1+1} = (x_1+1)\, z \iff z &= \frac{x_1 2^{x_1+1}}{x_1+1}.\end{aligned}$$

Since $\gcd(x_1, x_1+1) = 1$, the last number is a positive integer if and only if $x_1 = 2^{x_2} - 1$ for some integer $x_2 \geq 1$. Then $z = (2^{x_2} - 1)\, 2^{2^{x_2} - x_2}$.

Hence, the solution is $x = y = 2^{2^j - 1}$ and $z = \left(2^j - 1\right) 2^{2^j - j}$ for any integer $j \geq 1$.

Editor's Comments. The other 2 solvers misinterpreted the problem and found the positive integer solutions to the equation

$$x^y y^x = (x+y)^x .$$

This is an easier problem and we will leave it as an exercise to the reader.

OC322. Let $a,b,c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$a^2 b + b^2 c + c^2 a \geq \sqrt{(a+b+c)(ab+bc+ca)}.$$

Originally 2015 Macedonia National Olympiad Problem 2.

We received 6 solutions. We present the solution by Titu Zvonaru.

Using the AM-GM Inequality and $abc = 1$, we have

$$\begin{aligned}a^2b + a^2b + b^2c &\geq 3\sqrt[3]{a^4b^4c} = 3ab \\ b^2c + b^2c + c^2a &\geq 3\sqrt[3]{b^4c^4a} = 3bc \\ c^2a + c^2a + a^2b &\geq 3\sqrt[3]{c^4a^4b} = 3ca.\end{aligned}$$

Adding these three inequalities, we get

$$a^2b + b^2c + c^2a \geq ab + bc + ca. \tag{1}$$

Similarly,

$$\begin{aligned}a^2b + b^2c + b^2c &\geq 3\sqrt[3]{a^2b^5c^2} = 3b \\ b^2c + c^2a + c^2a &\geq 3\sqrt[3]{b^2c^5a^2} = 3c \\ c^2a + a^2b + a^2b &\geq 3\sqrt[3]{c^2a^5b^2} = 3a,\end{aligned}$$

hence

$$a^2b + b^2c + c^2a \geq a + b + c. \tag{2}$$

By (1) and (2), we get the desired inequality. The equality holds if and only if $a = b = c = 1$.

Editor's comments. Dan Daniel used the same approach, but using the clever substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, where $x,y,z > 0$. Indeed, then the given inequality becomes

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} \geq \sqrt{\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)},$$

which is equivalent to

$$(x^3 + y^3 + z^3)^2 \geq (x^2y + y^2z + z^2x)(x^2z + y^2x + z^2y).$$

Now, it is sufficient to prove that $x^3+y^3+z^3 \geq x^2y+y^2z+z^2x$ and $x^3+y^3+z^3 \geq x^2z + y^2x + z^2y$. For that, the approach is the same as the one used by Titu Zvonaru in the solution above.

OC323. Let ABC be a triangle. M, and N points on BC, such that $BM = CN$, with M in the interior of BN. Let P and Q be points in AN and AM respectively such that $\angle PMC = \angle MAB$, and $\angle QNB = \angle NAC$. Prove that $\angle QBC = \angle PCB$.

Originally 2015 Spain Mathematical Olympiad Day 2 Problem 3.

We received 3 correct solutions and will present 2 solutions.

Solution 1, by Mohammed Aassila.

Let A' and P' be the reflections of A and P with respect to the perpendicular bisector of BC, respectively. Let $\{X\} = NQ \cap A'B$ and $\{Y\} = NP' \cap AB$. Then it is easy (from symmetry) to see that $P' \in A'M$.

Since $AA'MN$ is an isosceles trapezoid, then A, A', M, N are concyclic. Since $\angle XNM = \angle NAC = \angle XA'M$, then A', X, M, N are concyclic. Since $\angle YNM = \angle NMP = \angle YAM$, then A, Y, N, M are concyclic.

Therefore, A, A', M, N, X, Y are concyclic, so from Pascal's theorem (applied to $AYNXA'M$), we conclude that

$$P' \in BQ \implies \angle QBC = \angle P'BC = \angle PCB.$$

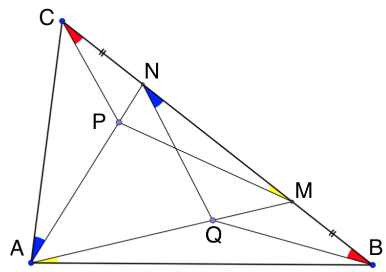
Solution 2, by Andrea Fanchini.

We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC.

Points M and N have coordinates

$$M(0 : a - t : t), \qquad N(0 : t : a - t)$$

where t is a parameter.



We now calculate the coordinates of point P. Recall that the oriented angle θ $(0 \leq \theta \leq \pi)$ of the oriented lines $d_i \equiv p_i x + q_i y + r_i z = 0 (i = 1{,}2)$, is given from

$$S_\theta = S \cot \theta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

so $\angle MAB$ between the two lines $AB : z = 0$ and $AM : ty + (t - a)z = 0$ is

$$S_{\angle MAB} = \frac{ac^2 - S_B t}{t}.$$

Now the side $BC : x = 0$ forms an angle $\angle PMC = \angle MAB$ with the line PM. Therefore, the point at infinity is

$$PM_\infty \left(a^2 t : (S_B - S_C)t - ac^2 : ac^2 - 2S_B t\right).$$

Then the line that passes from M and has PM_∞ as the point at infinity is

$$MPM_\infty : (at^2 - 2S_B t + ac^2)x + at^2 y + at(t - a)z = 0.$$

The line AN is give by $(t - a)y + tz = 0$. Thus, the point P has coordinates

$$P = AN \cap MPM_\infty = \left(a^2 t(2t - a) : t(2S_B t - at^2 - ac^2) : (t - a)(at^2 - 2S_B t + ac^2)\right)$$

Let us now calculate the coordinates of point Q. The $\angle NAC$ between the two lines $AN : (t - a)y + tz = 0$ and $AC : y = 0$ is

$$S_{\angle NAC} = \frac{ab^2 - S_C t}{t}.$$

Now the the side $BC : x = 0$ forms an oriented angle $\pi - \angle QNB = \pi - \angle NAC$ with the line QN, but $S_{\pi - \angle QNB} = -S_{\angle NAC}$. Therefore the point at infinity of this line is

$$QN_\infty \left(a^2 t : ab^2 - 2S_C t : (S_C - S_B)t - ab^2\right).$$

Then the line that passes from N and has the infinite point QN_∞ is

$$NQN_\infty : (2S_C t - at^2 - ab^2)x + at(a - t)y - at^2 z = 0.$$

Therefore the point Q has coordinates

$$Q = AM \cap NQN_\infty = \left(a^2 t(2t-a) : (t-a)(at^2 - 2S_C t + ab^2) : t(2S_C t - at^2 - ab^2)\right)$$

Finally, we will show that $\angle QBC = \angle PCB$. The $\angle QBC$ between the two lines $BC : x = 0$ and $BQ : (2S_C t - at^2 - ab^2)x + a^2(a-2t)z = 0$ is

$$S_{\angle QBC} = \frac{a\left(t^2 - 2at + S_A + S_B + S_C\right)}{a - 2t}$$

The $\angle PCB$ between the two lines $PC : (at^2 - 2S_B t + ac^2)x + a^2(2t-a)y = 0$ and $BC : x = 0$ is

$$S_{\angle PCB} = \frac{a\left(t^2 - 2at + S_A + S_B + S_C\right)}{a - 2t}$$

and we are done.

OC324. Given an integer $n > 1$ and its prime factorization $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, define a function

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} ... \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many integers n such that $f(n) = f(n-1) + 1$.

Originally 2015 Brazil National Olympiad Problem 3 Day 1.

No solutions received.

OC325. Let $S = \{1,2,\ldots,n\}$, where $n \geq 1$. Each of the 2^n subsets of S is to be coloured red or blue. (The subset itself is assigned a colour and not its individual elements.) For any set $T \subseteq S$, we then write $f(T)$ for the number of subsets of T that are blue.

Determine the number of colourings that satisfy the following condition: for any subsets T_1 and T_2 of S,

$$f(T_1)f(T_2) = f(T_1 \cup T_2)f(T_1 \cap T_2).$$

Originally 2015 USAMO Day 1 Problem 3.

No solutions received.

FOCUS ON...

No. 31

Michel Bataille

Mean Value and Rolle's Theorems

Introduction

Recall the statement of the Mean Value Theorem:

> If a real function f is continuous on $[a,b]$ and differentiable on (a,b), then $f(b) - f(a) = (b-a)f'(c)$ for some $c \in (a,b)$.

This theorem establishes a link between a function f and its derivative. As such, one of its main applications is to prove the familiar results connecting the sign of the derivative f' and the monotony of the function f on intervals. Here we will show it at work in various problems, sometimes rather unexpectedly. Frequently, it will intervene as Rolle's Theorem, which is nothing else than the particular case when $f(a) = f(b)$ even if generally proved before the Mean Value Theorem.

Warm-up exercises

We start with a series of simple exercises where our theorems lead to a short solution.

Exercise 1 *Let $T > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic, differentiable function. Show that f' has at least two zeros in $[0,T)$.*

Of course, since $f(0) = f(T)$, Rolle's Theorem provides a number $c \in (0,T)$ such that $f'(c) = 0$. But we need another zero of f' between 0 and T! Since $f(c) = f(c+T)$, Rolle's Theorem again gives d such that $f'(d) = 0$. However $d \in (c,c+T)$ and is suitable only if $d \in (c,T)$. But otherwise, $d - T \in [0,c)$ and $f'(d-T) = f'(d) = 0$ so that $d - T$ is another zero of f' in $[0,T)$. [f' is also T-periodic since $f(x+T) = f(x)$ for all x implies $f'(x+T) = f'(x)$ for all x.]

Exercise 2 *Find* $\lim\limits_{x\to\infty} \left(x^{\frac{x+1}{x}} - (x-1)^{\frac{x}{x-1}} \right)$.

A very short solution is obtained by remarking that we are asked the limit of $f(x) - f(x-1)$ if we set $f(x) = x^{\frac{x+1}{x}} = xe^{\frac{\ln x}{x}}$. For $x > 0$, the derivative of f is easily calculated:

$$f'(x) = e^{\frac{\ln x}{x}} \left(1 + \frac{1}{x} - \frac{\ln x}{x} \right)$$

and since $\lim\limits_{x\to\infty} \frac{\ln x}{x} = 0$, we see that $\lim\limits_{x\to\infty} f'(x) = 1$. Now, by the Mean Value Theorem, $f(x) - f(x-1) = f'(c)$ with $c \in (x-1,x)$. This number c depends on x, but is certainly large when x is. It readily follows that

$$\lim_{x\to\infty} \left(x^{\frac{x+1}{x}} - (x-1)^{\frac{x}{x-1}} \right) = 1.$$

Exercise 3 *Let $f : I \to \mathbb{R}$ be differentiable on the interval I and let $a,b \in I$. Prove that if $f'(a) < f'(b)$ and $\gamma \in (f'(a),f'(b))$, then $\gamma = f'(c)$ for some $c \in (a,b)$.*

This is called Darboux's property for a derivative. The result is obvious if f' is continuous, but this is not assumed here. Various proofs are known but the following one is particularly straightforward (see for example **[1]** p. 112 for a different one). For x in I, let $g(x) = f(x) - \gamma x$.

The function g is differentiable and $g'(x) = f'(x) - \gamma$; since $g'(a) < 0$ and $g'(b) > 0$ the function g is not strictly monotone on $[a,b]$. But an injective, continuous function on an interval must be strictly monotone. Therefore g is not injective and α,β in $[a,b]$ exist such that $\alpha < \beta$ and $g(\alpha) = g(\beta)$. Applying Rolle's Theorem to g gives $c \in (\alpha,\beta)$ such that $g'(c) = 0$, that is, $f'(c) = \gamma$.

We conclude our series by an exercise involving integrals.

Exercise 4 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function whose derivative is continuous on the interval $[a,b]$. Prove that*

$$\left| \int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx \right| \le \frac{(b-a)^2}{4} \sup_{x\in[a,b]} |f'(x)|.$$

This is problem 927 proposed in 2010 in *The College Mathematics Journal* and we present a variant of solution. The substitutions $x = \frac{u+a}{2}$ in the first integral and $x = \frac{u+b}{2}$ in the second one show that the left-hand side $\mathcal{L}$ of the inequality satisfies

$$\mathcal{L} = \frac{1}{2}\left| \int_a^b \left(f\left(\frac{u+a}{2}\right) - f\left(\frac{u+b}{2}\right)\right) du \right| \le \frac{1}{2}\int_a^b \left| f\left(\frac{u+a}{2}\right) - f\left(\frac{u+b}{2}\right)\right| du.$$

Now, if $u \in [a,b]$, from the Mean Value Theorem we obtain

$$\left| f\left(\frac{u+a}{2}\right) - f\left(\frac{u+b}{2}\right)\right| = \left| \frac{a-b}{2} f'(c) \right| \le \frac{b-a}{2} \cdot M$$

where c is in (a,b) and $M = \sup_{x\in[a,b]} |f'(x)|$. We deduce that

$$\mathcal{L} \le \frac{1}{2}\int_a^b \frac{b-a}{2} \cdot M\,du = \frac{(b-a)^2}{4} \cdot M.$$

A selection of problems

For a polynomial $P(x)$ with real coefficients, Rolle's Theorem implies that between two roots of $P(x)$, there is a root of its derivative $P'(x)$. More precisely, taking multiple roots into account, if $P(x)$ vanishes at k distinct real values among which m have multiplicity ≥ 2, then its derivative $P'(x)$ vanishes at $k - 1 + m$ distinct values. The solution we offer to the following problem set in a 1997 Croatian competition ([2001 : 90 ; 2003 : 163]) illustrates this result.

> Let a,b,c,d be real numbers such that at least one is different from zero. Prove that all roots of the polynomial $P(x) = x^6 + ax^3 + bx^2 + cx + d$ cannot be real.

For the purpose of a contradiction, let us assume that all the roots of $P(x)$ are real and form a set of k distinct real numbers among which m_i have multiplicity i ($i = 1,2,\ldots,6$). Then we have

$$\sum_{i=1}^{6} m_i = k \qquad \text{and} \qquad \sum_{i=1}^{6} i m_i = 6 \tag{*}$$

We have the following:

$P'(x)$ vanishes at $k' = k - 1 + m_2 + m_3 + m_4 + m_5 + m_6$ distinct real values,
$P''(x)$ vanishes at $k'' = k' - 1 + m_3 + m_4 + m_5 + m_6$ distinct real values, and
$P'''(x)$ vanishes at $k''' = k'' - 1 + m_4 + m_5 + m_6$ distinct real values.

Now, $k''' = 3 - m_5 - 2m_6$ (using $(*)$) while $P'''(x) = 120x^3 + 6a$ has exactly one simple root. Thus, we must have $m_6 = 1$ (note that a consequence of $(*)$ is $m_5, m_6 \in \{0,1\}$). However, this yields $P(x) = x^6$, that is, $a = b = c = d = 0$, in contradiction with the hypothesis.

We continue with problem **2989** [2004 : 502, 506 ; 2005 : 556] which proposes an inequality elementarily proved with the Mean Value Theorem (the featured solution is shorter but resorts to Karamata's Majorization Inequality).

> Prove that if $0 < a < b < d < \pi$ and $a < c < d$ satisfy $a + d = b + c$, then
> $$\frac{\cos(a-d) - \cos(b+c)}{\cos(b-c) - \cos(a+d)} < \frac{ad}{bc}.$$

Using the hypothesis $a + d = b + c$, we obtain

$$\frac{\cos(a-d) - \cos(b+c)}{\cos(b-c) - \cos(a+d)} = \frac{\sin a \sin d}{\sin b \sin c}$$

and the required inequality is equivalent to: $f(b) + f(c) - f(a) - f(d) > 0$ where the function f is given by $f(x) = \ln\left(\frac{\sin x}{x}\right)$. We now distinguish two cases.

Case 1: $a < b \leq c < d$. From the Mean Value Theorem, we have

$$f(b) - f(a) + f(c) - f(d) = (b-a)f'(\lambda_1) + (c-d)f'(\lambda_2)$$

for some $\lambda_1 \in (a,b)$ and $\lambda_2 \in (c,d)$. Now, $b - a = d - c, \lambda_1 < \lambda_2$ and f' is decreasing on $(0,\pi)$ (since $f''(x) = \frac{1}{x^2} - \frac{1}{\sin^2 x} < 0$ for $0 < x < \pi$), hence

$$f(b) + f(c) - f(a) - f(d) = (b-a)(f'(\lambda_1) - f'(\lambda_2)) > 0.$$

Case 2: $a < c < b < d$. Similarly,

$$f(c) - f(a) + f(b) - f(d) = (c-a)f'(\mu_1) + (b-d)f'(\mu_2),$$

where $\mu_1 \in (a,c)$ and $\mu_2 \in (b,d)$ so that

$$f(b) + f(c) - f(a) - f(d) = (c-a)(f'(\mu_1) - f'(\mu_2)) > 0.$$

Before passing to a problem involving integrals, we prove a lemma that is interesting in its own.

> Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f'(a) = f'(b)$. Show that $f'(c) = \frac{f(c)-f(a)}{c-a}$ for some $c \in (a,b)$.

Geometrically, the conclusion means that for some point $C(c,f(c))$ of the graph, the chord AC (where $A(a,f(a))$ coincides with the tangent at C. With the help of a quickly drawn diagram, the reader will observe that at such a c, the slope of the chord AM (where $M(x,f(x))$ seems to be locally extremal. This motivates the following proof.

Define the function g by

$$g(a) = f'(a) \quad \text{and} \quad g(x) = \frac{f(x) - f(a)}{x - a}$$

for $x \in (a,b]$. Then g is continuous on $[a,b]$, differentiable on $(a,b]$ and its derivative satisfies $f'(x) = g(x) + (x-a)g'(x)$ for all $x \in (a,b]$. We are looking for c in (a,b) such that $g'(c) = 0$.

If $g(b) = f'(b)$, then $g(b) = g(a)$ and Rolle's Theorem gives the sought number c.

If $g(b) \neq f'(b)$, we introduce $d \in (a,b)$ such that $g(b) - g(a) = (b-a)g'(d)$ and remark that $f'(b) - g(b)$ is equal to $(b-a)g'(b)$ as well as to $-(b-a)g'(d)$ (note that $g(a) = f'(a) = f'(b)$). As a result, we obtain $g'(b) \cdot g'(d) < 0$ and from Darboux's property (seen in exercise (3) above) $g'(c) = 0$ for some $c \in (d,b)$.

As our last problem, we propose problem 1824 set in June 2009 in *Mathematics Magazine*.

> Let f be a continuous real-valued function defined on $[0,1]$ and satisfying
> $$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx.$$
> Prove that there exists a real number c, $0 < c < 1$, such that $cf(c) = \int_0^c xf(x)\,dx$.

For $x \in [0,1]$, let $F(x) = \int_0^x f(t)\,dt$, $G(x) = \int_0^x F(t)\,dt$, $H(x) = \int_0^x tf(t)\,dt$. Integrating by parts, we obtain

$$H(x) = [tF(t)]_0^x - \int_0^x F(t)\,dt = xF(x) - G(x). \qquad (**)$$

Since $H(1) = F(1)$, $(**)$ gives $G(1) = 0$. But we also have $G(0) = 0$, hence there exists $a \in (0,1)$ such that $G'(a) = 0$ (by Rolle's Theorem).

Now, $G'(0) = G'(a)$ and by the lemma above, there exists $b \in (0,a)$ such that

$$G'(b) = \frac{G(b) - G(0)}{b - 0},$$

that is $bF(b) = G(b)$. From $(**)$, we deduce that $H(b) = 0 = H(0)$ and by Rolle's Theorem applied to the function $x \mapsto e^{-x} \cdot H(x)$, there exists $c \in (0,b)$ such that $e^{-c}(H'(c) - H(c)) = 0$, that is, $cf(c) = \int_0^c xf(x)\,dx$.

Exercises

1. (Exercise 5-21 in **[1]**) Assume that f is nonnegative and has a finite third derivative f''' in the open interval (0,1). If $f(x) = 0$ for at least two values of x in (0,1), prove that $f'''(c) = 0$ for some c in (0,1).

2. (Problème 500-1, *Bulletin de l'APMEP*, 2012) Let $f : [0,1] \to \mathbb{R}$ be continuous and such that $\displaystyle\int_0^1 f(t)\,dt = 0$. Prove that there exists $c \in (0,1)$ such that $\displaystyle\int_0^c tf(t)\,dt = 0$.

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Construction of a regular hendecagon by two-fold origami

Jorge C. Lucero

1 Introduction

Single-fold origami refers to geometric constructions on a sheet of paper by performing a sequence of single folds, one at a time [1]. Each folding operation achieves a minimal set of specific incidences (alignments) between given points and lines by folding along a straight line, and there is a total of eight possible operations [2]. The set of single-fold operations allows for the geometric solution of arbitrary cubic equations [3, 4]. As a consequence, the operations may be applied to construct regular polygons with a number of sides n of the form $n = 2^r 3^s p_1 p_2 \ldots p_k$, where r, s, k are nonnegative integers and $p_1, p_2, \ldots, p_k$ are distinct Pierpont primes of the form $2^m 3^n + 1$, where m, n are nonnegative integers [5]. For example, previous articles in *Crux Mathematicorum* have shown the construction of the regular heptagon [6] and nonagon [7]. Let us note that this family of regular polygons is the same that can be constructed by straightedge, compass and angle trisector.

Number 11 is the smallest integer not of the above form (the next are 22, 23, 25, 29, 31,...); therefore, the hendecagon is the polygon with the smallest number of sides that can not be constructed by single-fold origami. In fact, its construction requires the solution of a quintic equation [5], which can not be obtained by single folds. It has been shown that any polynomial equation of degree n with real solutions may be geometrically solved by performing $n-2$ simultaneous folds [1]. Hence, a quintic equation could be solved by performing three simultaneous folds. However, a recent paper presented an algorithm for solving arbitrary quintic equations with only two simultaneous folds [8]. In this article, the algorithm will be applied to solve the quintic equation associated to the regular hendecagon, and full folding instructions for its geometric construction will be given.

Let us note that another problem related to a quintic equation, the quintisection of an arbitrary angle, has also been solved by two-fold origami [9].

2 Quintic equation for the regular hendecagon and its solution

2.1 Equation

A similar approach to that used for the heptagon [6] is followed.

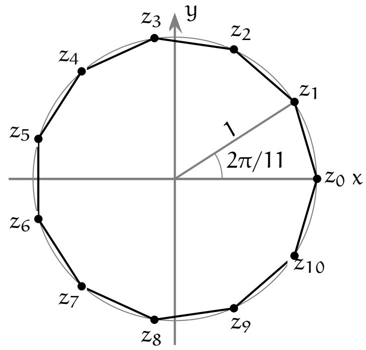


Figure 1: The regular hendecagon.

Consider an hendecagon in the complex plane, inscribed in a circle of unitary radius (Fig. 1). Its vertices are solutions of the equation

$$z^{11} - 1 = 0 \tag{1}$$

One solution is $z = z_0 = 1$, and the others are solutions of

$$\frac{z^{11} - 1}{z - 1} = z^{10} + z^9 + z^8 + \cdots + z + 1 = 0 \tag{2}$$

Assuming $z \neq 0$ and dividing both sides by z^5 produces

$$z^5 + z^4 + z^3 + \cdots + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} = 0 \tag{3}$$

Next, let $t = z + \overline{z} = 2\operatorname{Re} z$. Since $|z| = |\overline{z}| = 1$, then $\overline{z} = z^{-1}$ and $t = z + z^{-1}$. Expressing Eq. (3) in terms of t, the equation reduces to

$$t^5 + t^4 - 4t^3 - 3t^2 + 3t + 1 = 0 \tag{4}$$

Solutions of Eq. (4) have the form $t_k = 2\operatorname{Re} z_k = 2\cos(2k\pi/11)$, where $k = 1, 2, \ldots, 5$ and z_k are vertices as indicated in Fig. 1. Note that, due to symmetry on the x-axis, vertices except z_0 appear as complex conjugate pairs with a common real part.

2.2 Solution

Any quintic equation may be solved by applying the following two-fold operation (see Fig. 2): given two points P and Q and three lines ℓ, m, n, simultaneously fold along a line γ to place P onto m, and along a line δ to place Q onto n and to align ℓ and γ [8].

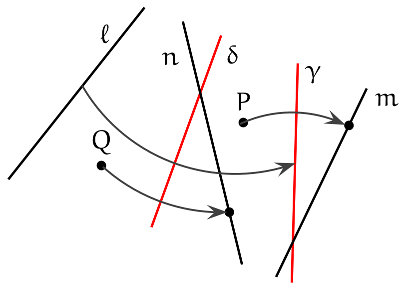


Figure 2: A two-fold operation. Red lines γ and δ are the fold lines.

The coordinates of points P and Q and equations of lines ℓ, m, and n are computed from the coefficients of the quintic equation to be solved. In the case of Eq. (4), we have $P(-\frac{5}{2},-3)$, $Q(0,1)$, $\ell : x = 0$, $m : x = -\frac{3}{2}$, and $n : y = -1$ (Fig. 3). Their calculation is omitted here for brevity; complete details of the algorithm may be found in Ref. [8].

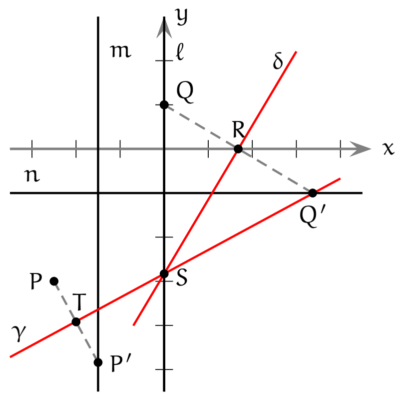


Figure 3: Geometric solution of Eq. (4), for $t = 2\cos(2\pi/11)$.

Let us demonstrate that the folds solve Eq. (4). Folding along line δ reflects point Q onto $Q' \in n$. Assume that point Q' is located at $(2t,-1)$, where t is a parameter. Then, the slope of segment $\overline{QQ'}$ is $-1/t$, and its midpoint R is at $(t, 0)$. The fold line δ is perpendicular to $\overline{QQ'}$ and passes through R; therefore, it has an equation

$$y = t(x - t) \tag{5}$$

Folding along line γ reflects point P on $P' \in m$. Assume that point P' is located at $(-\frac{3}{2}, 2s)$, where s is another parameter. Then, the slope of segment $\overline{PP'}$ is $2s + 3$, and its midpoint T is at $(-2, s - \frac{3}{2})$. The fold line γ is perpendicular to

$\overline{PP'}$ and passes through T; therefore, it has an equation

$$\begin{aligned} y &= -\frac{x+2}{2s+3} + s - \frac{3}{2} \\ &= -\frac{x}{2s+3} + \frac{2s^2 - \frac{13}{2}}{2s+3} \end{aligned} \tag{6}$$

Now, the same fold along δ reflects line ℓ over γ. Let S be the point of intersection of δ and ℓ. The y-intercept may be obtained by letting $x = 0$ in Eq. (5), which produces $y = -t^2$. Then, the slope of segment $\overline{SQ'}$ is $(t^2 - 1)/(2t)$. Line γ must pass through both Q' and S, and therefore it has an equation

$$y = \frac{t^2 - 1}{2t} x - t^2 \tag{7}$$

Since Eqs. (6) and (7) describe the same line, then their respective coefficients must be equal:

$$\frac{-1}{2s+3} = \frac{t^2 - 1}{2t} \tag{8}$$

$$\frac{2s^2 - \frac{13}{2}}{2s+3} = -t^2 \tag{9}$$

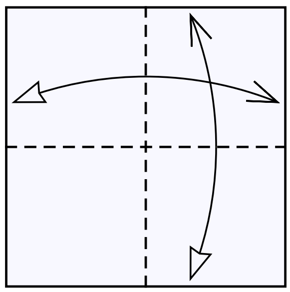
Solving Eq. (8) for s and replacing into Eq. (9), we obtain Eq. (4). Therefore, the x-intercept of δ (i.e., t at point R) is a solution of Eq. (4). Note that the equation has five possible solutions, and Fig. 3 shows the case of $t = 2\cos(2\pi/11) \approx 1.6825\ldots$.

3 Folding instructions

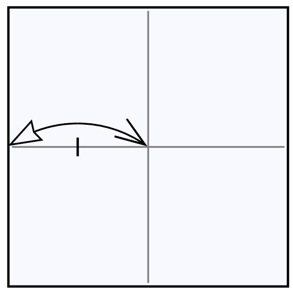
The following diagrams present full instructions for folding the regular hendecagon on a square sheet of paper.

Steps (1) to (7) produce points P and Q and lines ℓ, m, n of Fig. 3. The center of the paper is assumed to have coordinates $(0, -1)$, and each side has length 8. In step (1), the vertical and horizontal folds define lines ℓ and n, respectively. In step (3), the intersection of the fold line with the vertical crease is point Q. In step (5), the small crease at the bottom will mark the position of line m, after folding the paper backwards in the next step. Finally, step (7) defines point P.

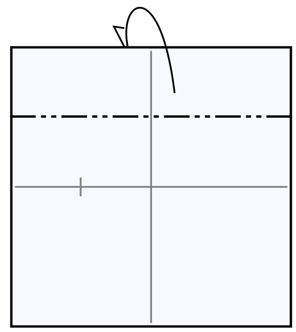
Next, steps (8) and (9) produce the fold lines γ and δ, respectively, of Fig. 3. As a result, point Q' in step (11) is at a distance of $4\cos(2\pi/11)$ from the center of the paper. In the same figure, point A is adopted as one vertex (z_0 in Fig. 1) of an hendecagon with radius of 4 units. The fold in step (11) produces a vertical line through Q', and steps (12) and (13) rotate a 4-unit length (from the paper center to point A) so as to find the next vertex of the hendecagon (point D in step 14). The remaining steps, (14) to (20) are used to find the other vertices and folding the sides.



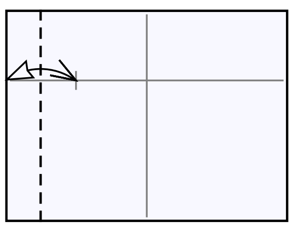
(1) Fold and unfold vertically and horizontally.



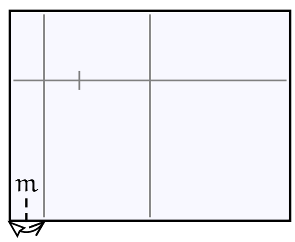
(2) Make a small crease at the midpoint.



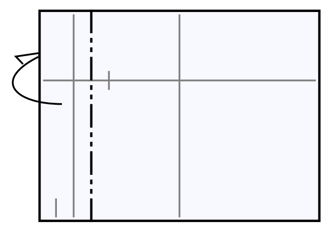
(3) Fold backwards the top edge to the horizontal center line.



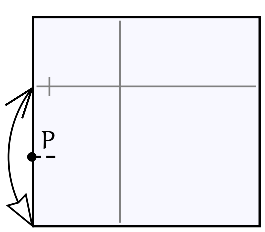
(4) Fold and unfold.



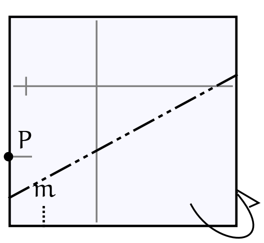
(5) Make a small crease at the bottom edge, and call this crease m.



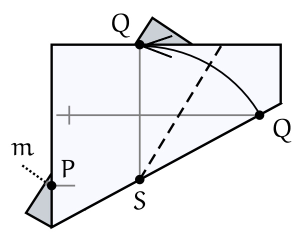
(6) Fold backwards to align the vertical crease made in step (4) with the small crease at the right.



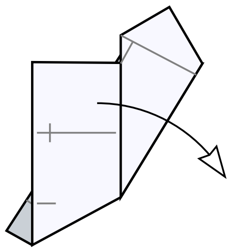
(7) Make a small crease at the left edge, and call this point P.



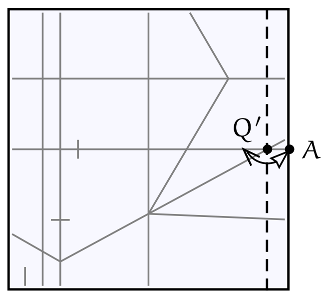
(8) Fold backwards to place P onto crease m made in step (5), so that...



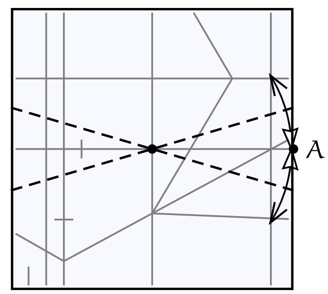
(9) ...a fold through point S places Q' onto Q.



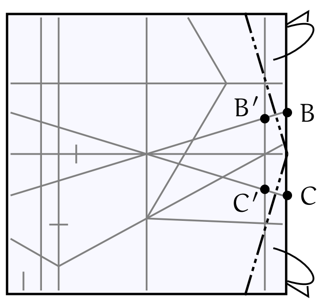
(10) Unfold.



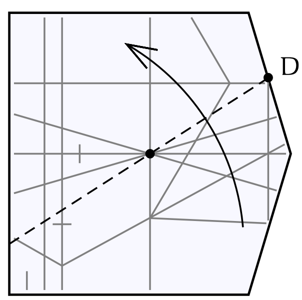
(11) Fold along a vertical line through point Q′ to place point A onto the horizontal crease and unfold.



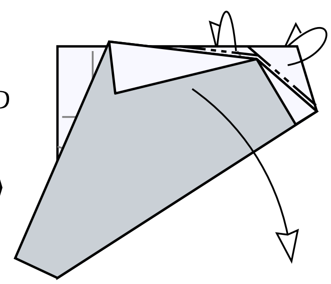
(12) Fold through the center of the paper to place point A onto the vertical crease made in step (11), and unfold.



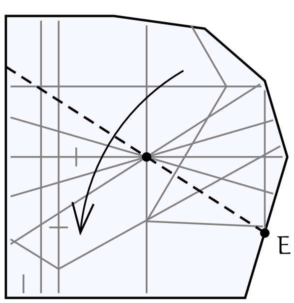
(13) Fold backwards to place point B onto point B′, and point C onto point C′.



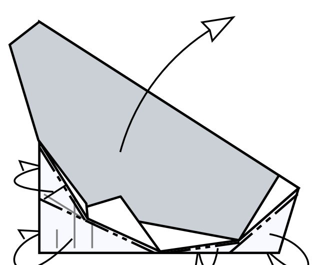
(14) Fold through the center of the paper and point D, at the upper intersection of the crease made in step (11) with the right edge.



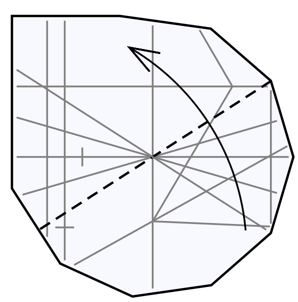
(15) Fold backwards the lower layer using the edges of the upper layer as guidelines, and next unfold the upper layer.



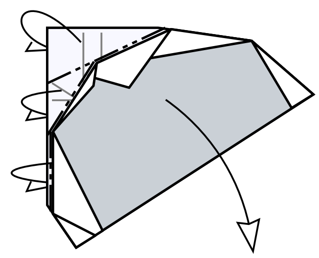
(16) Fold through the center of the paper and point E, at the lower intersection of the crease made in step (11) with the right edge.



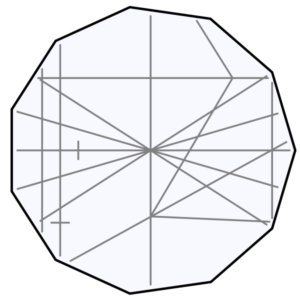
(17) Fold backwards the lower layer using the edges of the upper layer as guidelines, and next unfold the upper layer.



(18) Repeat the fold in step (14).



(19) Fold backwards the lower layer using the edges of the upper layer as guidelines, and next unfold the upper layer.



(20) Finished hendecagon.

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PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **October 1, 2018**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

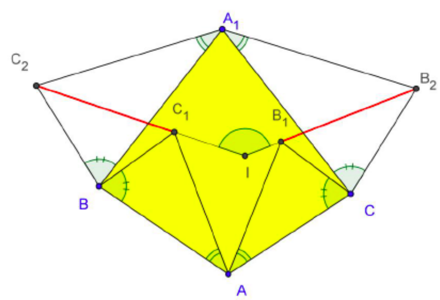
4341. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let ABC be an arbitrary triangle. Show that

$$\sum_{\text{cyc}} \sin A(|\cos A| - |\cos B \cos C|) = \sin A \sin B \sin C.$$

4342. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

In a convex quadrilateral ABA_1C, construct four similar triangles ABC_1, A_1BC_2, ACB_1 and A_1CB_2 as shown in the figure.



Show that $C_1C_2 = B_1B_2$ and that the directed angles satisfy $\angle(C_1C_2, B_1B_2) = 2\angle C_1BA_1$.

4343. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and E be the center of the excircle tangent at F and G to the extended sides AB and AC, respectively. If $GF \cap BE = \{B_1\}$, $FG \cap CE = \{C_1\}$ and B' and C' are feet of the altitudes from B, respectively C, show that $B_1C_1B'C'$ is a cyclic quadrilateral.

4344. *Proposed by Michel Bataille.*

Let n be a positive integer. Find all polynomials $p(x)$ with complex coefficients and degree less than n such that $x^{2n} + x^n + p(x)$ has no simple root.

4345. *Proposed by Mihai Miculiţa and Titu Zvonaru.*

Let ABC be a triangle with $AB < BC$ and incenter I. Let F be the midpoint of AC. Suppose that the C-excircle is tangent to AB at E. Prove that the points E, I and F are collinear if and only if $\angle BAC = 90°$.

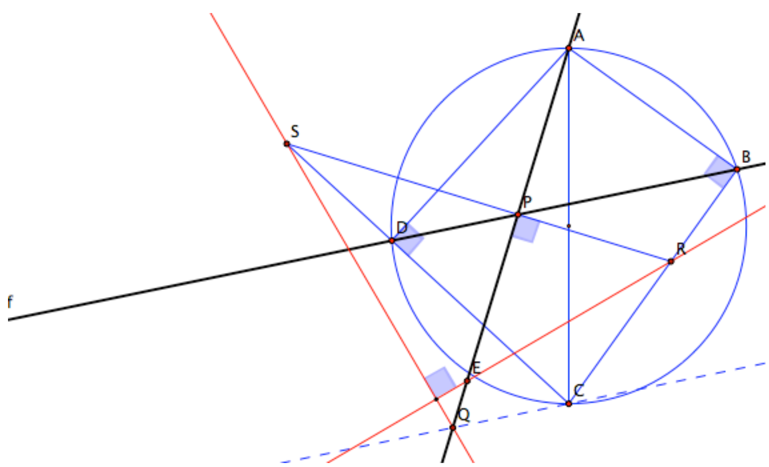
4346. *Proposed by Daniel Sitaru.*

Find all x,y,$z \in (0,\infty)$ such that

$$\begin{cases} 64(x+y+z)^2 = 27(x^2+1)(y^2+1)(z^2+1), \\ x+y+z = xyz. \end{cases}$$

4347. *Proposed by J. Chris Fisher.*

Given a cyclic quadrilateral $ABCD$ with diameter AC (and, therefore, right angles at B and D), let P be an arbitrary point on the line BD and Q a point on AP. Let the line perpendicular to AP at P intersect CB at R and CD at S. Finally, let E be the point where the line from R perpendicular to SQ meets AP. Prove that P is the midpoint of AE if and only if CQ is parallel to BD.



Comment by the proposer: This is a slightly generalized restatement of OC266 [2017:137-138], Problem 5 on the 2014 India National Olympiad. The solution featured recently in *Crux* was a long and uninformative algebraic verification; in particular, it failed to explain why the triangle had to be acute (it probably didn't), and it hid the true nature of the problem.

4348. *Proposed by Marius Drăgan.*

Let $p \in [0,1]$. Then for each $n > 1$, prove that

$$(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n.$$

4349. *Proposed by Hoang Le Nhat Tung.*

Let x, y and z be positive real numbers such that $x+y+z=3$. Find the minimum value of
$$\frac{x^3}{y\sqrt{x^3+8}}+\frac{y^3}{z\sqrt{y^3+8}}+\frac{z^3}{x\sqrt{z^3+8}}.$$

4350. *Proposed by Leonard Giugiuc.*

Let $f:[0,1]\mapsto\mathbb{R}$ be a decreasing, differentiable and concave function. Prove that
$$f(a)+f(b)+f(c)+f(d)\leq 3f(0)+f(d-c+b-a),$$
for any real numbers a,b,c,d such that $0\leq a\leq b\leq c\leq d\leq 1$.

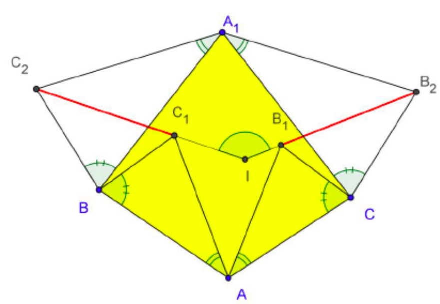
. . . .

4341. *Proposé par Daniel Sitaru et Leonard Giugiuc.*

Soit ABC un triangle quelconque. Démontrer que
$$\sum_{\text{cyc}}\sin A(|\cos A|-|\cos B\cos C|)=\sin A\sin B\sin C.$$

4342. *Proposé par Oai Thanh Dao et Leonard Giugiuc.*

À partir d'un quadrilatère convexe ABA_1C, construire quatre triangles semblables ABC_1, A_1BC_2, ACB_1 et A_1CB_2, tels qu'illustrés ci-bas.



Démontrer que $C_1C_2=B_1B_2$ et que les angles orientés vérifient $\angle(C_1C_2,B_1B_2)=2\angle C_1BA_1$.

4343. *Proposé par Mihaela Berindeanu.*

Soit $\triangle\ ABC$ un triangle acutangle, E le centre du cercle exinscrit tangent à F et G, situés sur les côtés prolongés AB et AC respectivement. Si $GF\cap BE=\{B_1\}$,

$FG \cap CE = \{C_1\}$ et si B', C' sont les pieds des hauteurs issues de B et de C respectivement, démontrer que $B_1C_1B'C'$ est un quadrilatère inscriptible.

4344. *Proposé par Michel Bataille.*

Soit n un entier positif. Déterminer tous les polynômes $p(x)$ à coefficients complexes et de degré inférieur à n, tels que $x^{2n} + x^n + p(x)$ n'a aucune racine simple.

4345. *Proposé par Mihai Miculiţa et Titu Zvonaru.*

Soit ABC un triangle tel que $AB < BC$ et soit I le centre de son cercle inscrit. Soit F le milieu de AC. Supposer que le cercle exinscrit opposé à C est tangent à AB en E. Démontrer que les points E, I et F sont aligneés si et seulement si $\angle BAC = 90°$.

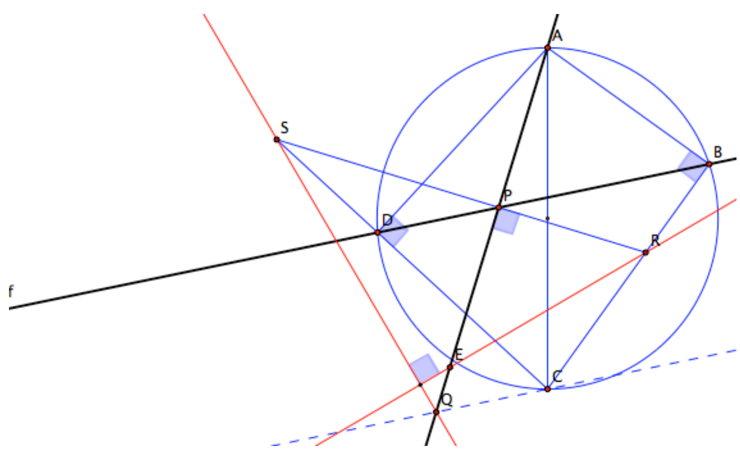
4346. *Proposé par Daniel Sitaru.*

Déterminer toutes valeurs de $x,y,z \in (0,\infty)$ telles que

$$\begin{cases} 64(x+y+z)^2 = 27(x^2+1)(y^2+1)(z^2+1), \\ x+y+z = xyz. \end{cases}$$

4347. *Proposé par J. Chris Fisher.*

Soit un quadrilatère inscriptible $ABCD$ avec diamètre AC (et donc ayant des angles droits à B et D) et soient P un point quelconque sur la droite BD, puis Q un point sur AP. La ligne passant par P et perpendiculaire à AP coupe CB en R, puis CD en S. Enfin, soit E le point où la ligne passant par R et perpendiculaire à SQ rencontre AP. Démontrer que P est le milieu de AE si et seulement si CQ est parallèle à BD.



Commentaire de l'auteur : Il s'agit d'une version légèrement généralisée de OC266 [2017 :137-138] Problème 5 de l'Olympiade Nationale en Inde 2014. La solution présentée récemment dans *Crux* n'était qu'une longue vérification algébrique offrant aucune révélation ; en particulier, elle n'expliquait pas pourquoi le triangle avait besoin d'être acutangle (ce qui n'est probablement pas nécessaire) et ne révélait pas la vraie nature du problème.

4348. *Proposé par Marius Drăgan.*

Soit $p \in [0,1]$. Pour chaque $n > 1$, démontrer

$$(1-p)^n + p^n \geq (2p^2 - 2p + 1)^n + (2p - 2p^2)^n.$$

4349. *Proposé par Hoang Le Nhat Tung.*

Soient x, y et z des nombres réels positifs tels que $x + y + z = 3$. Déterminer la valeur minimale de

$$\frac{x^3}{y\sqrt{x^3+8}} + \frac{y^3}{z\sqrt{y^3+8}} + \frac{z^3}{x\sqrt{z^3+8}}.$$

4350. *Proposé par Leonard Giugiuc.*

Soit $f : [0,1] \mapsto \mathbb{R}$ une fonction décroissante, différentiable et concave. Démontrer que

$$f(a) + f(b) + f(c) + f(d) \leq 3f(0) + f(d - c + b - a),$$

pour tous nombres réels a, b, c, d tels que $0 \leq a \leq b \leq c \leq d \leq 1$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017 : 43(5), p. 215–218.

4241. *Proposed by Margarita Maksakova.*

Place the numbers $1, 2, \ldots, 11$ and some real number r on the edges of a cube so that at every vertex the sum of the numbers on the incident edges is the same. What is the smallest value of r for which this is possible?

We received ten solutions, and all ten were correct. We present the solution by Roy Barbara together with the derivation of an upper bound for r that was included by Brian Beasley.

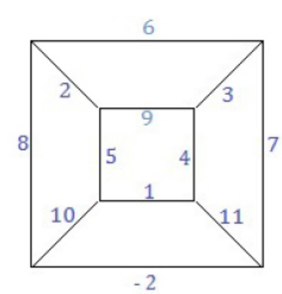
We define some terminologies to be used later. If a number x is placed on an edge that is incident to a vertex V, let us say that x belongs to V and that V contains x. Thus, every number belongs precisely to 2 vertices and every vertex contains exactly 3 numbers.

We show that $r = -2$ is the smallest suitable value. Consider a solution with some real r. Let s denote the common sum at each vertex. Pick a vertex not containing r, concluding that s is an integer, then pick a vertex containing r, concluding that r is an integer. The total sum corresponding to the 8 vertices is $8s$. Since each number belongs to 2 vertices, each number is counted twice in this total sum. Hence, $8s = 2(1 + 2 + \cdots + 11 + r)$. Therefore, $s = (66 + r)/4$. In order for s to be an integer, we must have $r \equiv -66 \pmod 4 \equiv 2 \pmod 4$, that is, $r \in \{\ldots, -10, -6, -2, 2, 4, 10, \ldots\}$.

Let V and V' be the two vertices containing r. Adding two sums at V and V', we get $2s$. But, $2s = (66 + r)/2$. But clearly, $2s \leq r + r + 11 + 10 + 9 + 8 = 2r + 38$. Hence, $(66 + r)/2 \leq 2r + 38$ yielding $r \geq -10/3 > -4$. This and $r \equiv 2 \pmod 4$ yield $r \geq -2$.

To obtain the upper bound on r, we use that $2s \geq r + r + 1 + 2 + 3 + 4 = 2r + 10$. Hence, $(66+r)/2 \geq 2r+10$ yielding $r \leq 46/3$. Again, $r \equiv 2 \pmod 4$ yields $r \leq 14$.

The lower and upper bounds on r indicate that there are 5 possible values for r, namely $-2, 2, 6, 10,$ and 14. A solution for $r = -2$ is shown on the attached planar graph. For the propose of graphically presenting the solution the 3-dimensional structure of the cube is irrelevant, and hence the cube was modified into a planar graph.



Editor's comments. All authors presented a number configuration including $r = -2$ on the cube edges, however no author included explanations on how to construct the configuration. Steven Chow presented 3 nonequivalent configurations including $r = -2$. Two authors, Brian Beasley and Digby Smith showed that there are only 5 possible values for r. In fact for each of $r \in \{-2, 2, 6, 10, 14\}$ a configuration can be constructed.

4242. *Proposed by Mihály Bencze.*

Let $x_1 = 4$ and $x_{n+1} = [\sqrt[3]{2}x_n]$ for all $n \geq 1$, where $[\cdot]$ denotes the integer part function. Determine the largest positive $n \in \mathbb{N}$ for which x_n, x_{n+1}, x_{n+2} form an arithmetic progression.

We received seven submissions, all of which were correct. We present the solution by Joseph DiMuro.

We prove that the largest such n is $n^* = 7$. By direct computations, we find that $x_7 = 12$, $x_8 = 15$, and $x_9 = 18$, which form an A.P. Hence, $n^* \geq 7$. Since $x_{10} = 22$, we see that $n^* \neq 8$.

Now, for any $n \geq 9$ we have $x_n \geq 18$ since x_n is an increasing sequence. Note that

$$x_{n+1} = \left[\sqrt[3]{2}x_n\right] = x_n + \left[(\sqrt[3]{2} - 1)x_n\right] = x_n + y_n,$$

where $y_n \geq [0.25x_n] \geq 4$. Since $4(\sqrt[3]{2} - 1) > 1$ we then have :

$$\begin{aligned} x_{n+2} = \left[\sqrt[3]{2}(x_{n+1})\right] &= x_{n+1} + \left[(\sqrt[3]{2} - 1)x_{n+1}\right] = x_{n+1} + \left[(\sqrt[3]{2} - 1)(x_n + y_n)\right] \\ &\geq x_{n+1} + \left[(\sqrt[3]{2} - 1)(x_n + 4)\right] \geq x_{n+1} + \left[(\sqrt[3]{2} - 1)x_n + 1\right] \\ &= x_{n+1} + \left[(\sqrt[3]{2} - 1)x_n\right] + 1 = x_{n+1} + y_n + 1. \end{aligned}$$

Hence, $x_{n+2} - x_{n+1} \geq y_n + 1 = [(\sqrt[3]{2} - 1)x_n] + 1 = [\sqrt[3]{2}x_n] - x_n + 1 > x_{n+1} - x_n$ so x_n, x_{n+1}, and x_{n+2} do not form an A.P., from which $n^* = 7$ follows.

4243. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Hung Nguyen Viet.*

Let ABC be a triangle. Let I, r and R be the incenter, the inradius and the circumradius of ABC, respectively. Let D be the point of intersection of the line AI and the circumcircle of ABC. Similarly, define points E and F. Prove that

$$AD \cdot BE \cdot CF \geq 16rR^2.$$

We received 9 correct solutions. We present the solution by Oliver Geupel.

Let $t_A = AA'$, t_B, and t_C denote the internal angle bisectors of the triangle. Let a, b, c, s, and K denote its sides, semiperimeter, and area, respectively. Remember the standard identities

$$K = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R}, \qquad t_A = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}.$$

We deduce $t_A \le \sqrt{s(s-a)}$, with analogous relations for t_B and t_C.

From similar triangles $AA'B$ and ACD, we obtain $AD = bc/t_A$, with analogous identites for BE and DF. Therefore,

$$\begin{aligned} AD \cdot BE \cdot CF &= \frac{a^2b^2c^2}{t_At_Bt_C} \ge \frac{a^2b^2c^2}{s\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \frac{a^2b^2c^2}{sK} = 16 \cdot \frac{K}{s} \cdot \left(\frac{abc}{4K}\right)^2 = 16rR^2. \end{aligned}$$

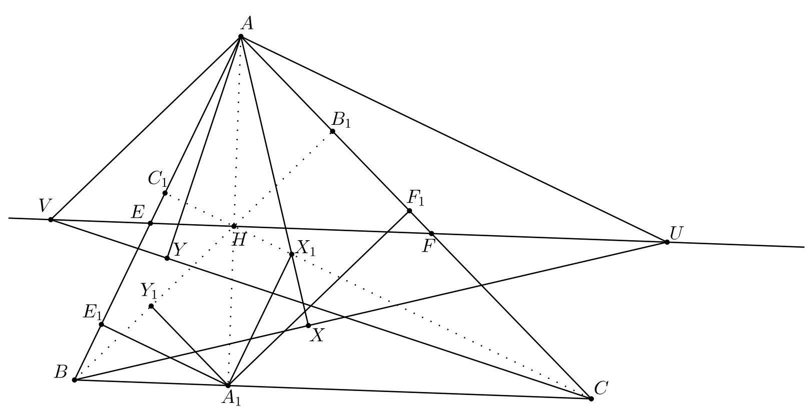
The equality holds if and only if the triangle is equilateral.

4244. *Proposed by Michel Bataille.*

Let ABC be a triangle with no right angle and let H be its orthocenter. The parallel to BC through H intersects AB and AC at E and F and the perpendiculars through A to AB and AC at U and V, respectively. Let X and Y be the orthogonal projections of A onto BU and CV, respectively. Prove that E,F,X,Y are concyclic.

We will feature two of the six (all correct and complete) solutions that we received.

Solution 1, by the proposer.



Let $\mathcal{C}_b$, $\mathcal{C}_c$, $\mathcal{C}_u$, $\mathcal{C}_v$ denote the circles with diameters AB,AC,AU,AV, respectively, and A_1,B_1,C_1 the feet of the altitudes from A,B,C, respectively. Note that H, X are on $\mathcal{C}_u$, H, Y on $\mathcal{C}_v$, A_1, B_1, X on $\mathcal{C}_b$ and A_1, C_1, Y on $\mathcal{C}_c$).

We shall use the inversion $\mathbf{I}$ with pole at A that exchanges H and A_1. Since $\mathbf{I}(\mathcal{C}_b)$ is a line through H perpendicular to AB, we have $\mathbf{I}(\mathcal{C}_b) = CC_1$ and so $\mathbf{I}(B) = C_1$; similarly, $\mathbf{I}(C) = B_1$. Now, the line EF inverts into the circle with diameter AA_1, hence $\mathbf{I}(E) = E_1$, the orthogonal projection of A_1 onto AB. Similarly, $\mathbf{I}(F) = F_1$, the orthogonal projection of A_1 onto AC.

Since $\mathcal{C}_u$,$\mathcal{C}_b$ are orthogonal circles passing through A, they invert into perpendicular lines through $\mathbf{I}(X) = X_1$, the orthogonal projection of A_1 onto CC_1 (because $\mathbf{I}(\mathcal{C}_u)$ passes through $A_1 = \mathbf{I}(H)$ and $\mathbf{I}(\mathcal{C}_b) = CC_1$). Similarly, $\mathbf{I}(Y) = Y_1$, the orthogonal projection of A_1 onto BB_1 ; it follows that E_1,Y_1, F_1 are the orthogonal projections of A_1 onto the sides of ΔBAB_1 and as such are collinear on the Simson line of A_1 (note that A_1 is on $\mathcal{C}_b$, the circumcircle of ΔBAB_1). In the same way, E_1,X_1, F_1 are collinear and consequently, E_1,X_1, Y_1,F_1 are collinear. As a result, their inverses E,X,Y,F are on a circle (passing through A).

Solution 2, by Mohammed Aassila.

Soit Z le point d'intersection des droits BU et CV. D'après le théorèm "triangle inscriptible dans un demi-cercle", les points X et Y sont sur le cercle de diamètre AZ. Notons $\mathcal{C}$ ce cercle.

Les triangles ΔAVU et ΔHBC sont homothétiques. D'apres le théorèm de Desargues, les droites AH, VB et UC sont concourantes. Notons W ce point de concours.

- Les triangles ΔEAV et ΔEBH étant homothétiques, donc $\frac{EA}{EB} = \frac{AV}{BH}$.
- Les triangles ΔWAV et ΔWHB étant homothétiques, donc $\frac{AV}{HB} = \frac{WV}{WB}$.
- Les triangles ΔWVU et ΔWBC étant homothétiques, donc $\frac{WV}{WB} = \frac{VU}{BC}$.
- Les triangles ΔZVU et ΔZCB étant homothétiques, donc $\frac{VU}{CB} = \frac{ZU}{ZB}$.

On conclut que $\frac{EA}{EB} = \frac{ZU}{ZB}$. D'après le théorèm de Thalès [Thales Intercept Theorem] appliqué dans le triangle BUA on déduit que $ZE||UA$. Par hypothès, $UA \perp AB$, par conséquent $ZE \perp AB$, ou encore $ZE \perp AE$. Ainsi, d'après le théorème "triangle inscriptible dans un demi-cercle", on déduit que $E \in \mathcal{C}$; de même, nous voyons que $F \in \mathcal{C}$. En conclusion, les points E,X,Y,Z,F et A sont concycliques.

4245. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle Γ, and let t be its tangent at A. Define T and E to be the points where the circle with centre B and radius BA again intersects t and AC, respectively, while T' and F are the points where the circle with centre C and radius CA intersects t and AB, respectively. If X is the point where TE and $T'F$ intersect, and Y is the second point where the line AX intersects Γ, prove that BC is the perpendicular bisector of the line segment XY.

We received six submissions, all of which were correct; we feature the solution by Mihai Miculiţa and Titu Zvonaru.

Denote by S the second point where the circles (B, BA) and (C, CA) intersect; since the common chord is perpendicular to the common diameter, we have

$$AS \perp BC. \tag{1}$$

Comparing angles inscribed and at the centre of (C, CA), and the angles inscribed in Γ and between a tangent and chord, we have

$$\angle AT'S = \frac{\angle ACS}{2} = \angle ACB = \angle TAB.$$

From this we conclude that $ST' || AB$, whence the quadrilateral $T'AFS$ (which is inscribed in the circle (C, CA)) is an isosceles trapezoid. Denoting by H the intersection of its diagonals FT' and AS, we see that the trapezoid is symmetric about the diameter CH; in particular, $CH \perp AF$. Together with (1) this implies that H is the orthocentre of ΔABC. Similarly, one shows that the line ET passes through H. But we are given that $ET \cap FT' = X$, so that $X = H$ and it follows that the point X is the orthocentre of ΔABC. Because the reflection of the orthocentre of a triangle in a side BC lies on the circumcircle, we conclude that BC is the perpendicular bisector of the line segment XY.

Editor's comments. Steven Chow observed that the result holds for *all* triangles : there was no need to require the angles of ΔABC to be acute. Indeed, with the use of directed angles modulo π, our featured solution is valid for all triangles.

4246. *Proposed by Leonard Giugiuc.*

Find the best lower bound for $abc + abd + acd + bcd$ over all positive a,b,c and d satisfying

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

We received six solutions, all complete and correct, and will feature here the one by Paolo Perfetti.

Suppose $b = c = d$. The condition $a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ yields

$$a = \frac{3 - 3b^2 + \sqrt{9 - 14b^2 + 9b^4}}{2b} \sim \frac{3}{2b}$$

as $b \to 0$. It then follows that

$$abc + bcd + cda + dab \sim \frac{3}{2}b + b^3 + \frac{3}{2}b + \frac{3}{2}b = \frac{9}{2}b + b^3$$

so the infimum (not minimum) is zero.

4247. *Proposed by Missouri State University Problem Solving Group.*

Let B and C be two fixed points on a circle centered at O that are not diametrically opposite. Let A be a variable point on the circle distinct from B and C and not belonging to the perpendicular bisector of BC. Let M and N be the midpoints of the segments BC and AO, respectively. The line AM intersects the circle again at D, and finally, NM and OD intersect at P. Determine the locus of points P as A moves around the circle.

We received six solutions, all of which used cartesian coordinates to find the equation of the locus. We feature the solution by Michel Bataille.

We show that the required locus is a hyperbola $\mathcal{H}$ minus its vertices. The foci of $\mathcal{H}$ are O and the inverse M' of M in the given circle, and its asymptotes are parallel to OB and OC.

To prove these results, we choose axes such that $O(0{,}0), M(m{,}0)$ with $0 < m < r$, where r is the radius of the circle. Let $A(u{,}v)$ with $u^2 + v^2 = r^2$, $u \neq m$, $v \neq 0$. The equation of the line AM is $x = m - \frac{m-u}{v}\,y$, hence the ordinate y_D of D is the solution distinct from v of the quadratic equation $\left(m - \frac{m-u}{v}\,y\right)^2 + y^2 = r^2$. A short calculation [observing that the product of the roots equals the constant term divided by the coefficient of y^2] gives y_D and then x_D :

$$y_D = \frac{v(m^2 - r^2)}{v^2 + (m-u)^2} = \frac{v(m^2 - r^2)}{r^2 + m^2 - 2mu},$$
$$x_D = m + \frac{(m-u)(r^2 - m^2)}{r^2 + m^2 - 2mu} = \frac{2mr^2 - ur^2 - um^2}{r^2 + m^2 - 2mu}.$$

Solving the system of the equations $y_D x - x_D y = 0$, $vx - y(u - 2m) = vm$ of the lines OD and MN provides the coordinates $(p{,}q)$ of the point P :

$$p = \frac{2mr^2 - ur^2 - um^2}{2m(m-u)}, \qquad q = \frac{v(m^2 - r^2)}{2m(m-u)} \neq 0.$$

We next eliminate $u{,}v$ using the relation $u^2 + v^2 = r^2$, successively obtaining

$$u = \frac{2mr^2 - 2pm^2}{r^2 + m^2 - 2mp}, \qquad v = \frac{2qm^2}{r^2 + m^2 - 2mp},$$

and the equation of the locus of P :

$$\frac{(2mr^2 - 2pm^2)^2}{(r^2 + m^2 - 2mp)^2} + \frac{4m^4q^2}{(r^2 + m^2 - 2mp)^2} = r^2, \quad q \neq 0.$$

This equation can be arranged into

$$\left(p - \frac{r^2}{2m}\right)^2 - \frac{m^2}{r^2 - m^2}\,q^2 = \frac{r^2}{4},$$

showing that P describes the hyperbola $\mathcal{H}$ whose centre is $\Omega(\frac{r^2}{2m}{,}0)$, the midpoint of the segment OM' where M' is the inverse of M in the given circle (since $M'(\frac{r^2}{m}{,}0)$). Since $q \neq 0$, the vertices of $\mathcal{H}$ are excluded.

Moving the origin of the axes from O to Ω, the equation of $\mathcal{H}$ takes the classical form

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad \text{with} \quad a = \frac{r}{2},\ b = \frac{r\sqrt{r^2 - m^2}}{2m}.$$

It follows that $\sqrt{a^2 + b^2} = \frac{r^2}{2m}$ and so the foci of $\mathcal{H}$ are O and M'. The eccentricity is $e = \dfrac{\sqrt{a^2 + b^2}}{a} = \frac{r}{m}$ ($e > 1$, as expected), while asymptotes $Y = \pm\dfrac{\sqrt{r^2 - m^2}}{m}\,X$ are the parallels to OB and OC through the centre Ω of $\mathcal{H}$. Note that their directions were expected since when A is at one of B or C, then D is at the other, and MN is parallel to OD so that P is at infinity.

4248. *Proposed by Michel Bataille.*

Let n be a positive integer and let $p(x) = 1 + p_1(x) + p_2(x) + \cdots + p_n(x)$ where the polynomials $p_k(x)$ are defined by $p_0(x) = 2$, $p_1(x) = x^2 + 2$ and the recursion

$$p_{k+1}(x) = (x^2 + 2)p_k(x) - p_{k-1}(x)$$

for $k \in \mathbb{N}$. Find all the complex roots of $p(x)$.

There were five correct solutions. Two of them followed the strategy of Solution 1, while the others worked from the solution of the recursion.

Solution 1, by Arkady Alt.

For $k \geq 0$, let $q_k(t) = p_k(t - 1/t)$, and let $q(t) = p(t - 1/t)$. Then $q_0(t) = 2$ and an induction argument reveals that

$$q_k(t) = t^{2k} + \frac{1}{t^{2k}}$$

for $k \geq 1$. Hence

$$q(t) = \sum_{k=-n}^{n} t^{2k} = t^{-2n} \sum_{k=0}^{2n} t^{2k} = \frac{t^{4n+2} - 1}{t^{2n}(t^2 - 1)}.$$

For $1 \leq k \leq 2n$, let

$$t_k = \cos\left(\frac{k\pi}{2n+1}\right) + i \sin\left(\frac{k\pi}{2n+1}\right), \quad x_k = t_k - \frac{1}{t_k} = 2i \sin\left(\frac{k\pi}{2n+1}\right).$$

Then $p(x_k) = q(t_k) = 0$.

For $k \geq 0$, the degree of $p_k(x)$ is $2k$ so that the degree of $p(x)$ is $2n$. We have identified $2n$ distinct roots, x_k, of $p(x)$. Thus

$$\left\{2i \sin\left(\frac{k\pi}{2n+1}\right) : k = 1, 2, \ldots, 2n\right\}$$

is the set of roots of $p(x)$.

Solution 2, by Ivko Dimitrić.

Solving the recursion for $p_k(x)$ yields $p_k(x) = u^k + v^k$ for $k \geq 0$ where

$$u = \frac{x^2 + 2 + x\sqrt{x^2 + 4}}{2} \qquad \text{and} \qquad v = \frac{1}{u} = \frac{x^2 + 2 - x\sqrt{x^2 + 4}}{2}.$$

Therefore,when $x \neq 0$,

$$\begin{aligned} p(x) &= 1 + u\left(\frac{u^n - 1}{u - 1}\right) + v\left(\frac{v^n - 1}{v - 1}\right) \\ &= \left[\frac{u^{n+1}}{u - 1} + \frac{v^{n+1}}{v - 1}\right] + \left[1 - \frac{u(v - 1) + v(u - 1)}{(u - 1)(v - 1)}\right] \\ &= \frac{u^{n+1}}{u - 1} + \frac{v^{n+1}}{v - 1}, \end{aligned}$$

while $p(0) = 2n+1$. Thus, when $x \neq 0$ and $1 \leq k \leq 2n$, we have

$$\begin{aligned} p(x) = 0 &\Longleftrightarrow u^{2n+2} = \left(\frac{u}{v}\right)^{n+1} = \frac{u-1}{1-v} = u \\ &\Longleftrightarrow u^{2n+1} = 1 \\ &\Longleftrightarrow u = \cos\left(\frac{2k\pi}{2n+1}\right) + i\sin\left(\frac{2k\pi}{2n+1}\right). \end{aligned}$$

For each such value of u, the corresponding value of x is given by $x\sqrt{x^2+4} = 2(u-1) - x^2$. Squaring leads to

$$\begin{aligned} x^2 &= \frac{(u-1)^2}{u} = u + \overline{u} - 2 = -2(1 - \mathrm{Re}(u)) \\ &= -2\left(1 - \cos\left(\frac{2k\pi}{2n+1}\right)\right) = -4\sin^2\left(\frac{k\pi}{2n+1}\right), \end{aligned}$$

from which we obtain the set of $2n$ roots of the polynomial $p(x)$:

$$\left\{\pm 2i\sin\left(\frac{k\pi}{2n+1}\right) : k = 1, 2, \ldots, n\right\}.$$

We can deduce an interesting equation from the factorization of $p(x)$. Since

$$p(x) = \prod_{k=1}^{n}\left(x^2 + 4\sin^2\left(\frac{k\pi}{2n+1}\right)\right),$$

we can set $x = 0$ to get

$$\prod_{k=1}^{n}\sin\left(\frac{k\pi}{2n+1}\right) = \frac{\sqrt{1+2n}}{2^n}.$$

Solution 3, by Paul Bracken.

We first show that

$$p(x) = \frac{p_{n+1}(x) - p_n(x)}{x^2}.$$

Using the fact that $p_1(x) = x^2+2$ and $p_2(x) = x^4+4x^2+2$, we see that this holds for $n = 1$. For $n \geq 2$, we have that

$$\begin{aligned} 0 &= \sum_{k=3}^{n+1} p_k(x) - (x^2+2)\sum_{k=2}^{n} p_k(x) + \sum_{k=1}^{n-1} p_k(x) \\ &= [p(x) + p_{n+1}(x) - 1 - p_1(x) - p_2(x)] - (x^2+2)[p(x) - 1 - p_1(x)] \\ &\quad + [p(x) - p_n(x) - 1] \\ &= -x^2p(x) + p_{n+1}(x) - p_n(x) + x^2 + (x^2+1)(x^2+2) - (x^4+4x^2+2) \end{aligned}$$

from which the representation of $p(x)$ follows.

With u and v defined as in the foregoing solution, we have that x is a root of $p(x)$ if and only if $p_{n+1}(x) - p_n(x) = 0$, equivalent to

$$\begin{aligned} 0 &= u^{n+1} + v^{n+1} - u^n - v^n = v^{n+1}(u^{2n+2} + 1 - u^{2n+1} - u) \\ &= v^{n+1}(u^{2n+1} - 1)(u - 1). \end{aligned}$$

Since $u \neq 0, 1$, then u is a nontrivial $(2n+1)^{\text{th}}$ root of unity and we can finish off as in Solution 2.

Editor's comments. The proposer found that $(t - 1/t)q(t) = (t^{2n+1} - t^{-(2n+1)})$, noted that $q(t) = 0$ whenever $t^{2n+1} = 1$, and obtained the set of roots of $p(x)$ in the form

$$\left\{ 2i \sin\left(\frac{2k\pi}{2n+1}\right) : k = 1, 2, \ldots, 2n \right\}.$$

4249. *Proposed by Daniel Sitaru.*

Let a,b,c be real numbers with at most one of them equal to zero. Prove that

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2 + b^2c^2 + c^2a^2} \leq 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

There were twelve correct solutions. Five of them applied a standard inequality; the first solution gives a sample. Four solvers used symmetric functions; while most used a discriminant condition on either a quadratic or cubic to obtain the inequality, the solver of our third solution kept the complications to a minimum. An additional solver used Maple.

Solution 1, by Mihai Bunget; Dionne Bailey, Elsie Campbell and Charles Diminnie; Oliver Geupel; and Kevin Soto Palacios (independently).

The inequality is trivial if one of a, b, c vanishes. Otherwise, observe that

$$(a-b)(b-c)(c-a) = ab(b-a) + bc(c-b) + ca(a-c)$$

and that

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (b-a)^2 + (c-b)^2 + (a-c)^2.$$

The result follows directly from the Cauchy-Schwarz Inequality applied to the vectors (ab, bc, ca) and $(b-a, c-b, a-c)$. Equality occurs if and only if $a = b = c$.

Solution 2, by Titu Zvonaru.

With the convention that each sum is cyclic in a, b, c over three terms, we have

$$\begin{aligned}
&2(a^2+b^2+c^2-ab-bc-ca)(a^2b^2+b^2c^2+c^2a^2)-(a-b)^2(b-c)^2(c-a)^2 \\
&\quad = [2\sum a^4b^2+2\sum a^2b^4+6a^2b^2c^2-2\sum a^3b^3-2\sum ab^2c^3-2\sum a^3b^2c] \\
&\qquad\quad -[\sum a^4b^2+\sum a^2b^4+2\sum ab^2c^3+2\sum a^3b^2c \\
&\qquad\quad -2\sum a^3b^3-2\sum a^4bc-6a^2b^2c^2] \\
&\quad = \sum a^4b^2+\sum a^2b^4+12a^2b^2c^2+2\sum a^4bc-4\sum a^3b^2c-4\sum ab^2c^3 \\
&\quad = \sum a^2(ab+ac-2bc)^2 \geq 0,
\end{aligned}$$

from which the inequality follows, with equality when $a=b=c\neq 0$.

Solution 3, by Arkady Alt.

The inequality holds when one of a, b, c vanishes. Assume $abc \neq 0$. Since the inequality is homogeneous, we may assume that $a+b+c=1$. Let $u=ab+bc+ca$ and $v=abc$. Then

$$a^2+b^2+c^2-ab-bc-ca=1-3u,$$

$$a^2b^2+b^2c^2+c^2a^2=u^2-2v,$$

and

$$(a-b)^2(b-c)^2(c-a)^2=u^2-4u^3+18uv-4v-27v^2.$$

Hence

$$\begin{aligned}
&2(a^2+b^2+c^2-ab-bc-ca)(a^2b^2+b^2c^2+c^2a^2)-(a-b)^2(b-c)^2(c-a)^2 \\
&\quad = 2(1-3u)(u^2-2v)-(u^2-4u^3+18uv-4v-27v^2) \\
&\quad = u^2-2u^3-6uv+27v^2 \\
&\quad = \frac{(9v-u)^2}{3}+\frac{2u^2(1-3u)}{3}.
\end{aligned}$$

Since

$$1-3u=(a+b+c)^2-3(ab+bc+ca)=(1/2)[(a-b)^2+(b-c)^2+(c-a)^2]\geq 0,$$

the right side of the equation is nonnegative, as desired.

Editor's comments. The proposer took an entirely different approach. The matrix $A=\begin{pmatrix} B \\ C \end{pmatrix}$ with

$$B=\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \end{pmatrix} \quad \text{and} \quad C=\begin{pmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{pmatrix}$$

is a 3×3 matrix. We note that

$$\det(B\cdot B^T)=2(a^2+b^2+c^2-ab-bc-ca),$$

$$\det(C \cdot C^T) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

and

$$\det A = \frac{(b-a)(c-a)(c-b)}{abc}.$$

Then the problem amounts to showing that

$$|\det A|^2 \leq \det(B \cdot B^T) \det(C \cdot C^T),$$

which requires a complicated evaluation of determinants and an application of the Cauchy-Schwarz Inequality.

4250. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Let ABC be an acute angle triangle such that $\sin A = \sin B \sin C$. Prove that

$$\tan A \tan B \tan C \geq \frac{16}{3}.$$

We received 15 submissions, all of which were correct. We present the solution by Steven Chow.

From the given condition, we have

$$\sin B \sin C = \sin A = \sin(B+C) = \sin B \cos C + \cos B \sin C,$$

so $\tan B \tan C = \tan B + \tan C = x$.

Since $\tan B > 0$, $\tan C > 0$, we have by the A.M.-G.M. Inequality that $\frac{x}{2} \geq \sqrt{x}$ so $x \geq 4$.

Then in sequence we have

$$(3x-4)(x-4) \geq 0 \implies 3x^2 \geq 16x - 16 \geq 0 \implies 16(x-1) \leq 3x^2 \implies$$
$$\frac{16}{3} \leq \frac{x^2}{x-1} = \frac{\tan B + \tan C}{\tan B \tan C - 1} \cdot x = -\tan(B+C)\, x = \tan A \tan B \tan C,$$

completing the proof.

Editor's comments. From the proof above, it is easy to see that equality holds if and only if $\tan B = \tan C = 2$, or equivalently, $\tan B \tan C = 4$. These were pointed out, in various forms, by many solvers.