MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

Volume 16 1983/84 Number 1

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 16 of *Mathematical Spectrum* will consist of three issues, of which this is the first. The second will be published in January 1984 and the third in May 1984.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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From Foundling to Philosopher

ROGER WEBSTER, *University of Sheffield*

The author is a Lecturer in Pure Mathematics at the University of Sheffield. His main mathematical interest is convexity, but he is also very interested in the history of mathematics. He shares with d'Alembert a love of the opera.

On 17 November 1717 a newly born baby boy with 'a head no bigger than an apple, hands like spindles and fingers as tiny as needles' was abandoned in a pine box on the steps of the church of St Jean le Rond near Notre Dame in Paris. The tiny infant was soon discovered and taken to the Foundlings' Home, where he was christened Jean le Rond after his first resting place. Such were the humble origins of Jean le Rond d'Alembert, who was to achieve fame throughout Europe as a mathematician, scientist, philosopher, man of letters and encyclopedist. To mark the bicentenary, on 29 October 1983, of his death and to introduce the name of d'Alembert to younger readers, we trace his life from the rejection by his mother to his acceptance into Parisian society as one of the most celebrated and influential figures of the French Enlightenment.

In reality, his origins were not as humble as they might at first appear. His mother was Madame de Tencin, a renegade nun, sister of a cardinal and the famous hostess of a literary salon. His father, who was abroad at the time of the birth, was the chevalier Destouches, an artillery officer in the French army. On returning to Paris, Destouches sought out his son and entrusted the child's upbringing to a glazier's wife named Madame Rousseau. D'Alembert always treated Madame Rousseau as a mother and lived with her until he was 47 when, for reasons of health, he moved to less cramped accommodation. Indeed, it was in the attic room of his foster-mother's house, from which he could see only 'three ells of sky', that he wrote the works which were to spread his fame throughout Europe.

Although Destouches never openly acknowledged his son, he did provide for his education. D'Alembert attended a private school from the age of four until the age of twelve when, through the influence of his father's family, he was admitted to the prestigious Collège des Quatre Nations. Initially he enrolled as Jean-Baptiste Daramberg, but later, for reasons not known, changed this to Jean d'Alembert. The choice of college was particularly fortunate in that it placed emphasis on mathematics and had a large mathematics library. He was a brilliant pupil, distinguishing himself in many subjects, and obtained his bachelor's degree in 1735. In spite of the advice of his teachers to follow a religious career, he decided to study law and graduated as an advocate in 1738.

Throughout his two years at law school d'Alembert maintained a lively interest in mathematics. Since he possessed only a few books and there was no one to whom he could turn when he was in difficulty, he made extensive use of the public libraries. Here he would skim through books, reading just enough of them to give him the main thread of the mathematical argument. Then he would return home to reconstruct the details for himself. In so doing, he often thought that he had discovered new results, only to be disappointed later (yet not without a certain satisfaction) at finding that they were already well known.

Not finding law to his liking, d'Alembert now turned to medicine. He took his new studies so seriously that he asked a friend to look after his mathematics books for him until after he had qualified—then they would be a recreation, not a distraction. The removal of the books, however, did not dim his interest in mathematics. From time to time, he would borrow a book back to consult it on a particular point. Before the year had passed he had taken back all his books! Realizing that resistance was useless, he forsook medicine and embraced mathematics, a decision that he was never to regret—the years ahead were the happiest of his life.

D'Alembert lost no time in introducing himself to the scientific community, presenting his first paper to the Academy of Sciences in July 1739. It was a critique of a calculus textbook and was well received. Over the next two years he submitted five papers on applied mathematics. On 29 May 1741, at the very young age of 23, he was elected to the Academy. In 1743 he published his most important scientific work, the *Treatise on dynamics*, containing the famous 'd'Alembert's principle' of classical mechanics. The following year he published a companion treatise, the *Treatise on fluids*, which dealt with the motion of fluids; it was, however, the earlier work that established d'Alembert's reputation as a mathematician.

During the next decade d'Alembert wrote a series of papers in which he made extensive use of partial differential equations, thus marking himself out as one of the founders of the subject and ensuring himself a permanent place in the history of mathematics. In 1746 he won the prize competition of the Berlin Academy for an essay on the cause of the winds and was admitted to the Academy. His paper of 1749 on the motion of a vibrating string is memorable for the first occurrence of the wave equation $\partial^2 u/\partial x^2 = \partial^2 u/\partial t^2$ which describes such motion. He published his explanation of the precession of the equinoxes and the nutation of the earth's axis in a paper of 1749. D'Alembert's essay on hydrodynamics is remarkable for its anticipation of the theory of complex variables and particularly for the derivation of the Cauchy–Riemann equations which play a central role in the theory. Between 1754 and 1756 he published a three-volume work on various topics in physical astronomy.

In 1746 d'Alembert's life took on a new dimension. After being introduced to the literary salon of Madame du Geoffrin, one of the most influential figures on the Parisian social scene, he began a hectic social life, soon acquiring a reputation as a witty conversationalist and a great mimic. It was at this time that he became associated with the *Encyclopédie*. The original plan of translating *Chambers' English Cyclopedia* of 1728 was abandoned in favour of a brand-new encyclopedia under the general editorship of Denis Diderot, with d'Alembert as the scientific editor. His contributions, however, were not confined to articles on science. To introduce the first volume of the *Encyclopédie* in 1751, he wrote his famous *Preliminary Discourse*, in which he attempted to summarize the state of human

Jean le Rond d'Alembert 1717–1783

knowledge. The *Discourse* was an instant success and launched d'Alembert on a new career, that of a philosopher; even today the work is regarded by many as the best introduction to the French Enlightenment. Its success was one of the reasons for d'Alembert's admission to the French Academy in 1754; later, in 1772, he became the permanent secretary of the Academy, a position which he held until his death.

During the latter half of his life, in addition to his scientific activities and his contributions to the *Encyclopédie*, d'Alembert wrote on a wide range of topics: literature, philosophy, religion, law, musical theory and biography. Even his scientific output reflected the broad scope of his interests: the study of achromatic lenses, acoustics, probability theory, classical mechanics and pure mathematics. His mathematical work was carried out at a time when there was a rapid development in the techniques of the calculus but little interest in its logical foundations. He was well aware of the shaky base on which the calculus was founded and he suggested, in an article for the *Encyclopédie*, one way in which this might be remedied. He regarded the key concept of the calculus as that of a limit, and he considered differentiation as the finding of a certain limit (i.e. dy/dx is the limit of $\delta y/\delta x$). In holding this point of view he was alone amongst his contemporaries, but time has vindicated him; when Cauchy placed the calculus on a firm foundation in 1822, it was d'Alembert's limit concept which played the central role. It was also his concern for rigour that led him to publish in volume five of his *Opuscules* of 1768 a test for convergence of infinite series which today bears his name: the infinite series $a_1 + a_2 + \cdots + a_n + \cdots$ of positive terms converges if the ratio a_{n+1}/a_n tend to a limit which is less than 1.

D'Alembert fell seriously ill in 1765 and went to stay with Julie de Lespinasse, a salonière with whom he had fallen in love. She nursed him back to health and he continued to live with her until her death in 1776. The remaining years of his life were spent in a small apartment at the Louvre, to which his position as secretary to the Academy entitled him. These final years were marked by a deterioration in both his health and his spirits. He died on 29 October 1783, refusing the services of a priest; he died as he had lived, a 'hardened unbeliever'.

When he realized that death was near, his thoughts turned to his foster-mother, and he made sure that she was provided for in his will. He must have remembered how she had taken him as a foundling, brought him up as her own son, and provided him with the home-base from which he was to go out and conquer the salons of Paris and the academies of Europe, to become friends with Voltaire and Rousseau, and to be fêted by Frederick the Great of Prussia and Catherine the Great of Russia. Maybe he recalled what she said to him as a young man at the outset of his career: 'You will never be anything but a philosopher, and what is a philosopher? A madman who torments himself during his lifetime in order that people may speak of him when he is dead.'

A piece of historical research

In which year was d'Alembert made a fellow of the Royal Society of London?

The Mean and Variance of the Roots of a Polynomial

NIGEL McCANN, *University of Hull*

The author wrote this article while he was a first-year undergraduate studying for an honours degree in mathematical statistics. He became interested in roots of polynomials whilst a sixth former at Immingham School, South Humberside, and has previously had a letter on this topic published in *Mathematical Spectrum* (Volume 14, Number 2). This article extends the results of his letter.

Consider the polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

with complex coefficients and of positive degree $n \geq 2$. It has n complex roots, which we denote by $\alpha_1, \ldots, \alpha_n$, and we can write

$$f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

In fact, we can expand this and equate coefficients of x^{n-1} and x^{n-2}. If we do this, we readily obtain the well-known relations between the roots and coefficients of a polynomial, namely

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_1}{a_0},$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \underset{(i<j)}{\alpha_i\alpha_j} + \cdots + \alpha_{n-1}\alpha_n = \frac{a_2}{a_0}.$$

We denote the mean of the roots of $f(x)$ by μ. Thus

$$\mu = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = -\frac{a_1}{na_0},$$

which shows how the mean of the roots of $f(x)$ may be obtained from the coefficients a_0, a_1 of $f(x)$.

Now denote the variance of the roots of $f(x)$ by σ^2. Thus

$$\sigma^2 = \frac{\alpha_1^2 + \cdots + \alpha_n^2}{n} - \mu^2.$$

Now

$$(\alpha_1 + \cdots + \alpha_n)^2 = (\alpha_1^2 + \cdots + \alpha_n^2) + 2(\alpha_1\alpha_2 + \cdots + \alpha_i\alpha_j + \cdots + \alpha_{n-1}\alpha_n)$$

so that

$$n^2\mu^2 = n\sigma^2 + n\mu^2 + 2\left(\frac{a_2}{a_0}\right).$$

Hence

$$\sigma^2 = (n-1)\mu^2 - \frac{2a_2}{na_0} = \frac{(n-1)a_1^2}{n^2a_0^2} - \frac{2a_2}{na_0}.$$

Thus the variance of the roots of $f(x)$ may be obtained using this formula from the coefficients a_0, a_1, a_2 of $f(x)$. In fact.

$$\frac{a_1}{a_0} = -n\mu, \qquad \frac{a_2}{a_0} = \frac{n(n-1)}{2}\left(\mu^2 - \frac{\sigma^2}{n-1}\right)$$

so that the polynomial can be written as

$$f(x) = a_0\left\{x^n - n\mu x^{n-1} + \frac{n(n-1)}{2}\left(\mu^2 - \frac{\sigma^2}{n-1}\right)x^{n-2} + \cdots\right\}, \qquad (*)$$

where μ is the mean of its roots and σ^2 is their variance.

It is interesting to see what happens to the mean and variance of the roots of $f(x)$ when we form its successive derivatives. If we differentiate the expression (*) $n-1$ times, we obtain

$$f^{(n-1)}(x) = a_0\{n!x - n\mu(n-1)!\},$$

and this has the root μ, i.e. $f^{(n-1)}(x)$ has as its root the mean of the roots of $f(x)$. Since $f^{(n-1)}(x)$ can also be obtained by differentiating the rth derivative $f^{(r)}(x)$ $n-r-1$ times, where $1 \le r \le n-1$, it follows that $f^{(n-1)}(x)$ also has as its root the mean μ_r of the roots of $f^{(r)}(x)$. Thus

$$\mu = \mu_1 = \mu_2 = \cdots = \mu_{n-1},$$

i.e. *the means of the roots of $f(x)$ and its successive derivatives are all equal.*

Geometrically we can say that the centroid of the roots of $f(x)$ coincides with the centroids of all the successive derivatives of $f(x)$.†

For the variance, we look at the roots of the $(n-2)$th derivative of $f(x)$. We have

$$f^{(n-2)}(x) = a_0\left\{\frac{n!}{2}x^2 - n\mu(n-1)!x + \frac{n(n-1)}{2}\left(\mu^2 - \frac{\sigma^2}{n-1}\right)(n-2)!\right\}$$

$$= \frac{a_0 n!}{2}\left\{x^2 - 2\mu x + \left(\mu^2 - \frac{\sigma^2}{n-1}\right)\right\},$$

and this has roots

$$\mu \pm \frac{\sigma}{\sqrt{(n-1)}}.$$

Note again that the roots of $f^{(n-2)}(x)$ have mean μ. But, more importantly, they have variance σ^2_{n-2} given by

$$\sigma^2_{n-2} = \frac{\left(\mu + \dfrac{\sigma}{\sqrt{(n-1)}}\right)^2 + \left(\mu - \dfrac{\sigma}{\sqrt{(n-1)}}\right)^2}{2} - \mu^2$$

$$= \frac{\sigma^2}{n-1}.$$

If we now denote the variance of the roots of the rth derivative by σ_r^2, where $1 \le r \le n-2$, we obtain

$$\sigma^2_{n-2} = \frac{\sigma_r^2}{n-r-1}.$$

Thus the variances of the roots of the successive derivatives of $f(x)$ are related by

$$\frac{\sigma^2}{n-1} = \frac{\sigma_1^2}{n-2} = \frac{\sigma_2^2}{n-3} = \cdots = \sigma^2_{n-2}.$$

† We have also received a proof of this result from Liu Zhiqing, a student reader in China— see the letter in this issue.

COMPUTING COLUMN

The next issue of *Mathematical Spectrum* sees the start of a regular computing column. This will include programs that you can use on your computer at home or school, and suggestions for writing your own programs. We invite readers to send in programs of their own for inclusion, and any other items they think would interest fellow computer-buffs. In charge of the column will be Dr Michael Piff who, as well as being a Lecturer in Pure Mathematics in the University of Sheffield, is also something of a computerholic! The first column will contain a program for a fascinating game entitled 'Cannibals'. Don't miss it!

A Mathematical Christmas

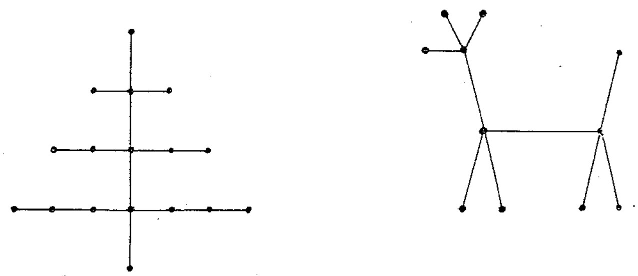
DAVID SHARPE, *University of Sheffield*

The author lectures in pure mathematics at the University of Sheffield, and is also the current editor of *Mathematical Spectrum*.

1. Trees

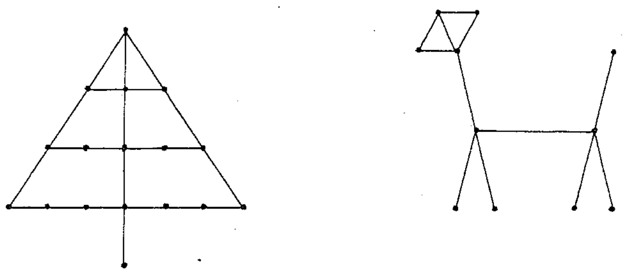
Most of us will have a tree in our homes over Christmas. To the mathematician, a tree is a special type of *graph*. In this context, a graph is simply a finite set of points in the plane, called *vertices*, with certain points joined to others by lines, called *edges*. So Figures 1 and 2 both give examples of graphs. Such configurations can arise, for example, with transport networks such as the London Underground system, where the vertices are stations and the edges are underground lines connecting various stations.

You may be able to spot a difference between the graphs in Figure 1 as opposed to those in Figure 2. In the graphs in Figure 2, it is possible to start at one of the vertices and travel along edges through distinct vertices so as to arrive back at your starting point (like travelling right round the Circle Line on the London Underground!). We say that the graphs in Figure 2 have *circuits*, whereas those in Figure 1 do not. The two graphs in Figure 1 are called *trees*, those in Figure 2 are not. Thus, mathematically at least, a tree is a graph with no circuits. (Even Figure 1(b) is a tree, 'well known for its bark', as Robin Wilson remarks in Reference 1!)



(a) (b)

Figure 1



(a) (b)

Figure 2

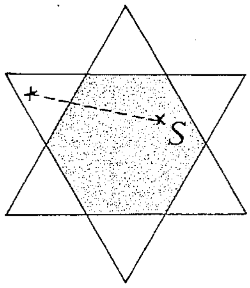


Figure 3

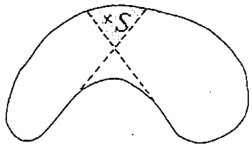


Figure 4

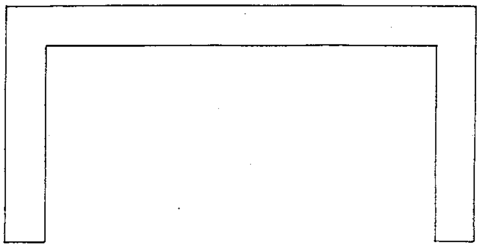


Figure 5

If we now count the numbers of vertices and edges of these various graphs, we see that Figure 1(a) has 17 vertices and 16 edges, Figure 1(b) has 11 vertices and 10 edges, Figure 2(a) has 17 vertices and 22 edges and Figure 2(b) has 11 vertices and 12 edges. Is it possible to tell whether a graph is a tree by comparing the number of its vertices with the number of its edges? If we assume that our graph is *connected* in the sense that it is possible to get from every vertex to every other by travelling along edges, a reasonable assumption in the case, say, of an underground system, then the answer is 'yes'. It is not difficult to convince yourself that the trees are the graphs which have one edge fewer than they have vertices. All the rest have at least as many edges as vertices.

2. Stars

Your Christmas tree will probably have either a fairy or a star on top. What is a star? Mathematically, it is a geometrical figure which contains a point S with the property that every other point of the figure can be joined to S by a straight line lying wholly within the figure. For example, any point in the inner hexagon of Figure 3 can be taken as S. Thus a star-shaped room is one which has a point from which every other point of the room is visible. No doubt trainee teachers are advised to stand at such a point in the classroom! Another example of a star-shaped figure is in Figure 4. This would look rather odd on your Christmas tree, but it nevertheless satisfies the requirements for a star. On the other hand, Figure 5 is not star-shaped. Such a shape would be most inadvisable for a classroom, since there is nowhere the teacher could stand so as to see every student.

Here is a fascinating result about star-shaped regions in the plane. Suppose that, for every three points in the region, there is a point in the region from which all three points are visible. Then there is a point in the region from which *every* point is visible, i.e. the region is star-shaped. This is called *Krasnosellskii's Theorem*.

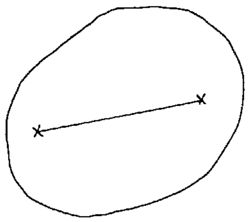


Figure 6

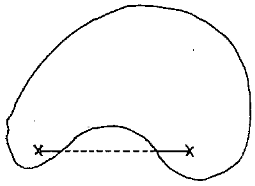


Figure 7

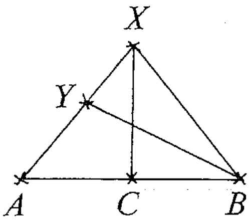


Figure 8

A *convex* figure is one such that every two points of it can be joined by a straight line lying wholly within the figure. Figure 6 is an example of a convex figure, whereas Figure 7 is not. A convex figure is star-shaped from every point of it.

In Figures 3 and 4, the points from which the figures are star-shaped are those in the shaded regions. We call these shaded regions the *kernels* of the figures. What property do these kernels possess? Why, they are convex. In 1912, a mathematician by the name of Brun proved that this is always the case, so that the kernel of a star-shaped figure is always convex.

We can quite easily give an argument to prove Brun's result. We refer to Figure 8. Let A, B be two points in the kernel of our star-shaped figure (whose boundary is now shown in Figure 8), and let C be any point on the line segment joining A, B. We must show that every point of the figure is visible from C, so consider any point X of the figure. We must show that the whole line segment CX lies in our figure. Now each point Y on the line segment AX lies in the figure (because A is in the kernel). Thus each line segment BY lies in the figure (because B is in the kernel). Since this is true for every Y on the line segment AX, the whole triangle ABX lies in the figure. In particular, the whole line segment CX lies in the figure, so C really is in the kernel, as we claimed.

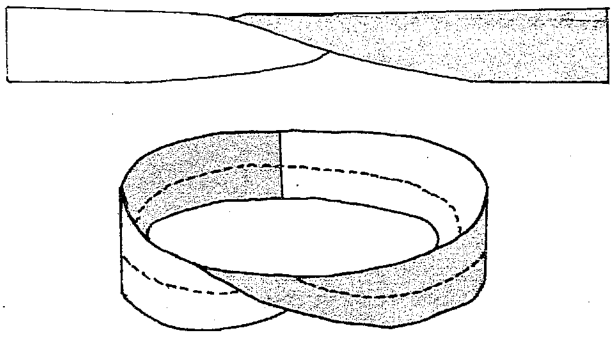


Figure 9

3. Decorations

We now describe a way of decorating your Christmas tree. Take a rectangular strip of paper, preferably brightly coloured, much longer than it is wide, and put a single twist in it. Now glue the two free ends together. We now have a *Möbius band*† (see Figure 9).

A Möbius band is a very curious mathematical object. For example, how many sides does it possess? Answer: only one. Try tracing round a side with a pencil as shown by the dotted line in Figure 9. You will eventually come back to your starting point. How many edges does the band possess? Answer: only one.

Now take a pair of scissors and cut your Möbius band down the middle. You will be amused at the result. Or you could cut it one-third of the way across. Your tree will end up festooned with exciting mathematical figures.

A mathematician confided
That a Möbius band is one-sided,
  And you'll get quite a laugh
  If you cut one in half,
For it stays in one piece when divided.‡

4. Fairy-tales

Christmas is a time for pantomimes and fairy-tales. Here is a mathematical fairy-tale. Take two Möbius strips. Each has only one edge. Bring the two edges together and sew the strips together. The result is a mathematical object called a *Klein bottle* after the German mathematician Felix Klein who invented it in 1882. We refer to a Klein bottle as a mathematical object because it cannot exist in three dimensions. In our drawing of it in Figure 10, it appears to pass through itself and therefore to have a circular hole in it (the dotted line in the figure). But for the mathematician this hole does not exist. The bottle has no rim and only one side. You would be ill-advised to try to use a Klein bottle to contain your Christmas drinks; nothing will stay in it because it has no inside or outside!

A mathematician named Klein
Thought the Möbius band was divine.
  Said he, 'If you glue
  The edges of two,
You'll get a weird bottle like mine.'‡

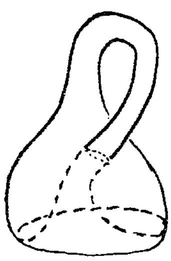


Figure 10

† A. F. Möbius was a German mathematician who in 1858 first considered such a band.
‡ See Reference 2.

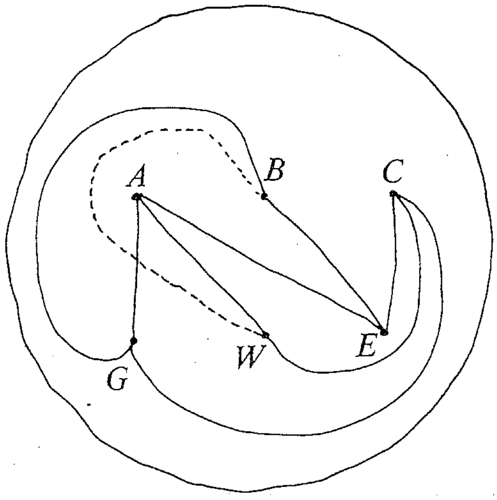


Figure 11

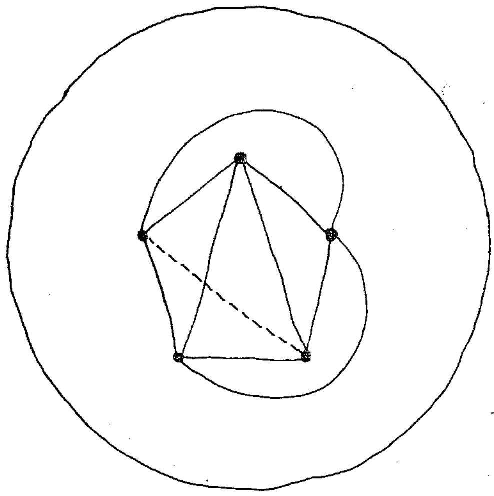


Figure 12

5. Balloons

No Christmas party is complete without balloons which, for our purposes, we may think of as spheres. We now describe a way of decorating a balloon.

The balloon represents the earth. Take a felt-tipped pen and mark three houses A, B, C. Depending on the size of the balloon these may have to be blobs. These three houses are to be connected to the gas, water and electricity services, denoted by three more blobs G, W, E. Thus each of A, B, C must be joined to each of G, W, E by a pipe or wire, which we draw with a line. The question is this: can this be done without these lines crossing? Try it on your balloon. (The problem is the same on a plain piece of paper, but less fun.) No matter how much you try, it cannot be done; two of the lines must cross (see Figure 11).

If we refer back to Section 1, we see that we have a graph with 6 vertices and 9 edges, and we say that this graph is *not planar*. Another example of a non-planar graph is obtained by marking five vertices on your balloon or piece of paper and joining every vertex to every other (Figure 12). We call this graph *the complete graph on 5 vertices*, and label it K_5.

Is there a strange planet upon which we might stumble, hitch-hiking around the galaxy, which might allow us to join our houses to the services without the supply lines crossing? If there is a doughnut- (or swimming-ring-) shaped planet, then the answer is yes. This we have shown in Figure 13, where the dotted lines go through the hole of the doughnut and the dashed line goes round the back. Mathematicians for some reason prefer to speak of a *torus* rather than a doughnut! It is also possible to draw the graph K_5 on a doughhnut/torus with no crossing edges (Figure 14).

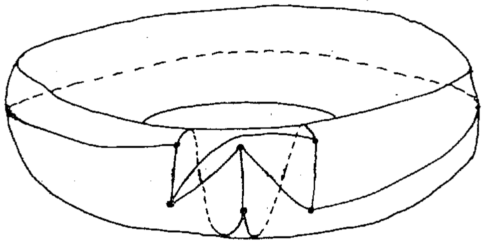


Figure 13

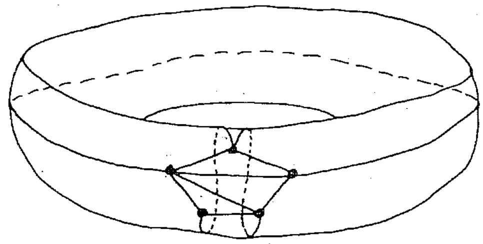


Figure 14

6. Food

This suggests an exciting addition to your Christmas party fare. Decorate doughnuts using piped icing with these graphs. Your friends will be fascinated! You can really go to town with this idea. You can also decorate your doughnut with K_6 and K_7. But try it with K_8 and it will not work. In fact, to draw K_8 without crossing edges you need a doughnut with two holes, shaped like a figure 8, called a *double torus* in the trade. And for K_9 you will need three holes! In general, for K_n you will need h holes, where h is the smallest integer greater than or equal to $\frac{1}{12}(n-3)(n-4)$. This was proved after a long and difficult struggle by two American mathematicians, Ringel and Youngs, in 1968. The corresponding result for the graph with m houses and n services is that you need p holes, where p is the smallest integer greater than or equal to $\frac{1}{4}(m-2)(n-2)$. Thus $K_{3,3}$ above needs one hole, as we saw, but $K_{4,5}$ needs two. (See reference 1, p. 69.)

With these suggestions, next Christmas could be the best ever!

References

1. Robin J. Wilson, *Introduction to Graph Theory*, 2nd edn. (Longman, London, 1979).
2. Albert W. Tucker and Herbert S. Bailey Jr, 'Topology', *Scientific American*, January 1950. Reprinted in *Mathematics in the Modern World* (W. H. Freeman, San Francisco, 1968).

Drilling a Cube

J. JABŁKOWSKI, *Warsaw*

Mr Jabłkowski is a glass technologist, now retired, with an active interest in mathematics. He is a graduate of Warsaw Polytechnic and also studied for a time at the University of Sheffield.

Back in 1973 *Mathematical Spectrum* published a letter from R. D. Kitchen who wrote that he had been told of the existence of a 'three or four line solution without calculus' to the following problem:

> A metal cube with 2 inch edges is drilled perpendicularly through the midpoint of one of its faces by a cylindrical drill of radius 1 inch. The drilling is then repeated along the other two perpendicular axes of the cube. In this way a circular cylinder of radius 1 inch is pierced along each of the three axes of the cube. What is the volume remaining?

Mr Kitchen wondered whether any reader might be able to supply the desired solution. Since no solution has appeared in *Mathematical Spectrum*, I now submit one that I have constructed. It takes up rather more than a mere 'three or four lines', so there is still scope for possible alternative approaches. But can there ever be a solution entirely 'without calculus'? Although no integral sign need appear in my solution, the very notions of area and volume involve integration. For we normally regard the area of a rectangle as a primitive concept and then define the area of a more general region by use of integration or of a limiting process akin to it. An analogous procedure leads from the volume of a rectangular box (which is taken as the starting point) to the volume of other solids. In my solution, I take for granted the area of a disc and the volume of a prism in terms of base area and height. In addition, I carry out what is essentially the integration of x^2 from first principles. There is, I believe, no escape from calculus, and all that can be expected is the avoidance of any sophisticated technique.

Solution

The volume of the drilled cube is denoted by V.

From Figure 1 it is evident that, after drilling, the remaining material is made up of 8 congruent cubes, each at a corner of the original cube, and 24 congruent pyramids, three attached to each small cube. One of these pyramids is $APQRS$.

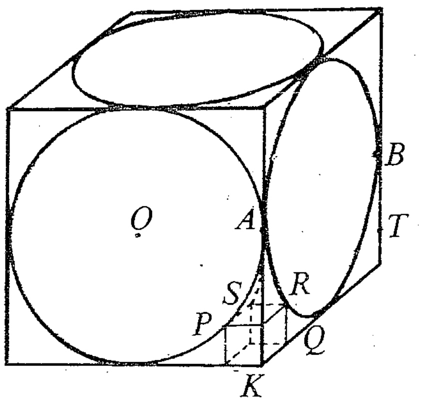


Figure 1

Denote the volumes of the cubes and the pyramids by V_1 and V_2, respectively, so that

$$V = 8V_1 + 24V_2.$$

The diagonal PK of a face of one of the small cubes is $OK - OP = \sqrt{2} - 1$. Hence the side of each cube is $(\sqrt{2} - 1)/\sqrt{2}$, and so

$$V_1 = \left(1 - \frac{1}{\sqrt{2}}\right)^3 = \frac{5}{2} - \frac{7}{2\sqrt{2}}.$$

To evaluate V_2 we first find V_3, the volume of the prism with opposite faces APQ, BUT (Figure 2). Now (Figure 3) the area of APQ (where AP is an arc) is

area (triangle OKA) – area (triangle PKQ) – area (sector OPA)

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}\frac{\pi}{4} = \frac{1}{2}\left(\sqrt{2} - \frac{1}{2} - \frac{\pi}{4}\right).$$

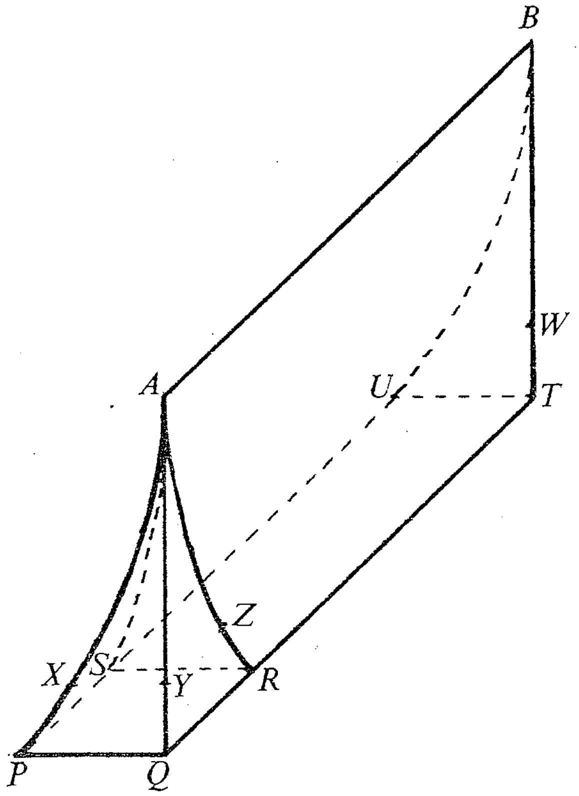


Figure 2

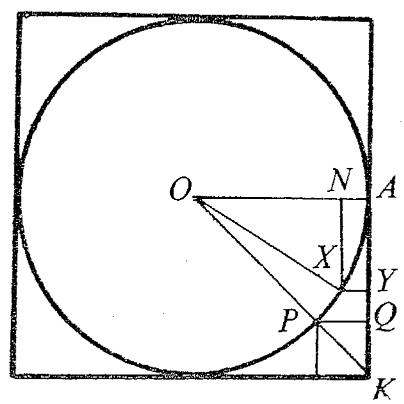


Figure 3

Since the length AB of the prism is 2,

$$V_3 = \sqrt{2} - \frac{1}{2} - \frac{\pi}{4}.$$

Next we obtain V_4, the volume of the solid left when the pyramid $APQRS$ is removed from the prism. Of course

$$V_3 = V_2 + V_4,$$

but it turns out that V_4 is easier to evaluate than V_2. We begin by calculating the area of the rectangular cross-section of the solid at a distance $y = AY$ from the top. The width of the rectangle (see Figure 3) is

$$XY = NA = OA - ON = OA - \sqrt{(OX^2 - XN^2)} = 1 - \sqrt{(1 - y^2)},$$

and its length (see Figure 4) is

$$ZW = YW - YZ = YW - XY = 2 - [1 - \sqrt{(1 - y^2)}] = 1 + \sqrt{(1 - y^2)}.$$

Thus the area of the rectangle is

$$XY \,.\, ZW = 1 - (1 - y^2) = y^2.$$

Hence the volume of a thin slice of thickness δ at a distance y from the top is $y^2\delta$.

If we are willing to use elementary calculus, we here note that y varies from 0 to $1/\sqrt{2}$, so that

$$V_4 = \int_0^{1/\sqrt{2}} y^2\, dy = \left[\frac{1}{3}y^3\right]_0^{1/\sqrt{2}} = \frac{\sqrt{2}}{12}.$$

The alternative is to divide the solid into n slices of equal thickness $AQ/n = 1/n\sqrt{2}$. The kth slice from the top is at a distance $y = k/n\sqrt{2}$ from the top and therefore its volume is

$$\frac{1}{n\sqrt{2}}\left(\frac{k}{n\sqrt{2}}\right)^2 = \frac{k^2}{(2\sqrt{2})n^3}.$$

So we have

$$V_4 = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{(2\sqrt{2})n^3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{(2\sqrt{2})n^3 \cdot 6} = \frac{1}{6\sqrt{2}} = \frac{\sqrt{2}}{12},$$

which is the result also obtained by integration.

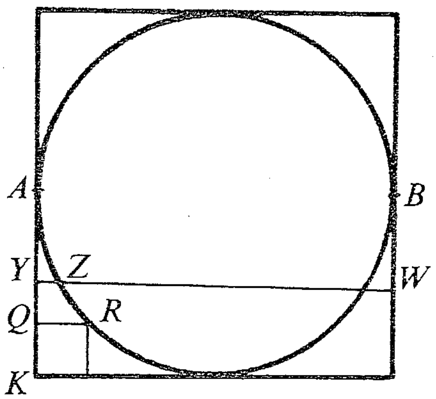


Figure 4

Finally,

$$V_2 = V_3 - V_4 = \frac{11\sqrt{2}}{12} - \frac{1}{2} - \frac{\pi}{4};$$

and

$$V = 8V_1 + 24V_2 = 8\left(\frac{5}{2} - \frac{7}{2\sqrt{2}}\right) + 24\left(\frac{11\sqrt{2}}{12} - \frac{1}{2} - \frac{\pi}{4}\right),$$

i.e.

$$V = 8 + 8\sqrt{2} - 6\pi.$$

Sums of Powers of Consecutive Integers

A. K. BAZLUL KARIM †

The author is a retired civil servant. He has written a number of articles on mathematics, as well as other literary sketches.

In the course of an investigation into the summation of consecutive integers raised to a given power, I came across certain identities which readers may find interesting.

If we expand the expression

$$(p+r)^k - (p-r)^k$$

by the binomial theorem, we obtain

$$2\binom{k}{1}p^{k-1}r + 2\binom{k}{3}p^{k-3}r^3 + 2\binom{k}{5}p^{k-5}r^5 + \cdots;$$

the last term in this sum involves pr^{k-1} or r^k according to whether k is even or odd, and $\binom{k}{1}$, $\binom{k}{3}$, ... are the familiar binomial coefficients. If we now sum r from 1 to n and separate the first term on the right-hand side from the others, we obtain

$$\sum_{r=1}^{n} (p+r)^k - \sum_{r=1}^{n} (p-r)^k = 2kp^{k-1}\sum_{r=1}^{n} r + R,$$

where

$$R = 2\sum_{r=1}^{n}\left\{\binom{k}{3}p^{k-3}r^3 + \binom{k}{5}p^{k-5}r^5 + \cdots\right\}.$$

Now $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, so that we have

$$\sum_{r=1}^{n} (p+r)^k - \sum_{r=1}^{n} (p-r)^k = kn(n+1)p^{k-1} + R.$$

We now choose $p = kn(n+1)$ and rearrange this identity to give

$$\sum_{r=0}^{n} (p-r)^k = \sum_{r=1}^{n} (p+r)^k - R.$$

Note that $R = 0$ when $k < 3$.

We shall be able to understand better what this identity says if we try certain values for k and n. When $k = 1$ and $n = 1, 2, 3$ successively, then p takes the successive values 2, 6, 12, and we obtain

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15.$$

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When $k=2$ and $n=1,2,3$ successively, we have

$$3^2+4^2=5^2$$

$$10^2+11^2+12^2=13^2+14^2.$$

$$21^2+22^2+23^2+24^2=25^2+26^2+27^2.$$

When $k>2$, we no longer obtain identities which involve consecutive powers alone, because R is no longer 0. Thus, when $k=3$ and $n=1,2,3$ successively, we have

$$5^3+6^3=7^3-2$$

$$16^3+17^3+18^3=19^3+20^3-18$$

$$33^3+34^3+35^3+36^3=37^3+38^3+39^3-72;$$

when $k=4$ and $n=1,2,3$ successively, we have

$$7^4+8^4=9^4-64$$

$$22^4+23^4+24^4=25^4+26^4-1728$$

$$45^4+46^4+47^4+48^4=49^4+50^4+51^4-13{,}824.$$

The general identity is

$$(p-n)^k+\cdots+(p-1)^k+p^k=(p+1)^k+\cdots+(p+n)^k-R,$$

where $p=kn(n+1)$ and R is given above.

Obituary—Sir Maurice Kendall

Sir Maurice Kendall, FBA, for many years a member of the Advisory Board of this magazine, died on 23 March 1983 at the age of 75. Sir Maurice was well known as the author of *The Advanced Theory of Statistics*, a two-volume treatise on the subject published during and shortly after World War II; this was later enlarged and revised with other colleagues. He became Professor of Statistics at the London School of Economics in 1949, moved on to the company now called SCICON in 1961 and retired in 1972. He was then appointed Director of the World Fertility Survey, a post which he held until 1980; he was awarded the UN Peace Medal for his work on world population problems.

Sir Maurice will be sadly missed by the Editors and members of the Advisory Board of *Mathematical Spectrum*, as well as by his many friends in the statistical profession throughout the world.

15 June 1983

J. GANI
University of Kentucky

Beyond the Calculator

J. W. HILLE AND E. McPHERSON, *Chisholm Institute of Technology*

John Hille and Ewen McPherson are both Lecturers in Mathematics at the Chisholm Institute of Technology at Frankston.

John's main interests include applied probability and statistics, number theory and the history of mathematics. He is completing research for his M.Sc. in the area of mental test theory. He wrote the article 'A Bayesian look at the jury system' which appeared in Volume 11, Number 2 of *Mathematical Spectrum*.

Ewen's main interests include modern algebra and number theory and he is currently involved in extending his knowledge of computing; he holds a B.Sc. degree in Pure Mathematics and Mathematical Statistics.

When we are asked to evaluate numbers such as $3{\cdot}165^{100}$, the use of logarithms or calculators necessarily focuses attention upon the order of magnitude and a relatively small number of digits. If one requires a particular digit in a particular place outside the range of places used in the calculator most techniques appear quite inadequate or impossibly tedious for the task. Recently a colleague mentioned one such problem where both the units and tenths digits in the decimal expansion of $(\sqrt{2}+\sqrt{3})^{1980}$ are required. Initial attempts using binomial expansions, surd manipulation, approximations, etc. all proved disappointing. What seemed to be required was an algorithm which at least kept us informed about the units digit in successive expansions of $(\sqrt{2}+\sqrt{3})^n$, and hopefully only a minor computation would be needed for the first few decimal places.

An approach which suggested itself was the use of continued-fraction notation, since this gives both the integral part and a readily interpretable expression for the fractional part of the number being represented. Further, every quadratic irrational $a \pm \sqrt{b}$ can be expressed as a continued fraction. For example,

$$\begin{aligned}
1+\sqrt{2} &= 2+(\sqrt{2}-1)\\
&= 2+\cfrac{1}{1+\sqrt{2}}\\
&= 2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}\\
&= 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}.
\end{aligned}$$

Continuing in this way, we obtain the continued fraction† expansion of $1+\sqrt{2}$ as

$$1+\sqrt{2}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}.$$

Since $(\sqrt{2}+\sqrt{3})^2=5+2\sqrt{6}$, a quadratic irrational, this approach seemed reasonable and was investigated. Now $5+2\sqrt{6}$ is the larger root of $x^2-10x+1=0$ or

$$\begin{aligned} x &= 10-\frac{1}{x} \\ &= 10-\cfrac{1}{10-\cfrac{1}{10-\ddots}}. \end{aligned}$$

This 'negative' continued fraction form differs from the usual form in that the numerators of each partial fraction are -1 rather than $+1$. Using $[u]$ to denote the 'negative' continued fraction

$$u-\cfrac{1}{u-\cfrac{1}{u-\ddots}},$$

we can write $5+2\sqrt{6}=[10]$, and

$$5+2\sqrt{6}=10-\frac{1}{[10]}\approx 10-\frac{1}{10}=9{\cdot}9.$$

Repeating the above procedure for $(5+2\sqrt{6})^2=49+20\sqrt{6}$, we are led to the equation $x^2-98x+1=0$, so

$$\begin{aligned} x &= 98-\frac{1}{x} \\ &= 98-\cfrac{1}{98-\cfrac{1}{98-\ddots}} \\ &= [98]. \end{aligned}$$

† For a more complete explanation of continued fractions, see the article by K. E. Hirst, in *Mathematical Spectrum* Volume 15, Number 2.

Again,

$$49+20\sqrt{6}=98-\frac{1}{[98]}\approx 98-\frac{1}{98}\approx 97{\cdot}99.$$

A further squaring gives $(5+2\sqrt{6})^4=4801+1960\sqrt{6}$, which is the larger solution of $x^2-9602x+1=0$, so

$$\begin{aligned}x&=9602-\frac{1}{x}\\&=9602-\cfrac{1}{9602-\cfrac{1}{9602-\ddots}}\\&=[9602].\end{aligned}$$

Again,

$$4801+1960\sqrt{6}=9602-\frac{1}{[9602]}\approx 9601{\cdot}999.$$

At this point the beginning of a pattern was evident, since $98=10^2-2$ and $9602=98^2-2$, and since further computations were becoming laborious we decided to investigate the properties of these 'negative' continued fractions. In particular, we sought answers to the following two questions:

Q.1. Does $x^n=(5+2\sqrt{6})^n$, where n is a positive integer, always produce solutions of the form

$$x^n=[u_n]=u_n-\cfrac{1}{u_n-\cfrac{1}{u_n-\ddots}}?$$

We show in Appendix 1 that this is indeed so.

Q.2. In multiplying x^m and x^n, that is, $[u_m]$ and $[u_n]$, does a general rule involving the numbers u_m and u_n exist?

This question resulted in our deriving a general rule for the product $[u_m][u_n]$ which takes the form

$$[u_m][u_n]=[u_mu_n-u_{m-n}]$$

for $m\geq n$. The proof of this is in Appendix 2.

For example

$$x^6x^4=(5+2\sqrt{6})^6(5+2\sqrt{6})^4=[u_6][u_4]=[u_6u_4-u_2].$$

For the case $m = n$,

$$x^m x^m = (5 + 2\sqrt{6})^{2m} = [u_m][u_m] = [u_m^2 - u_0].$$

Now

$$[u_0] = u_0 - \frac{1}{[u_0]} \qquad \text{and} \qquad [u_0] = x^0 = 1, \qquad \text{so } u_0 = 2.$$

Hence

$$[u_m]^2 = [u_m^2 - 2].$$

We return to our original question involving $(\sqrt{2} + \sqrt{3})^{1980} = (5 + 2\sqrt{6})^{990}$, and note that 990 has binary representation 1111011110. Let $x = [10]$. Then, using the result $[u_n]^2 = [u_n^2 - 2]$, we have

$$\begin{aligned} x^2 &= [98], \\ x^4 &= [9602], \\ x^8 &= [\ldots 02], \\ x^{16} &= [\ldots 02], \\ &\vdots \\ x^{512} &= [\ldots 02], \end{aligned}$$

where $[\ldots 02]$ indicates a u_k value whose tens and units digits are 0 and 2 respectively.

Now

$$\begin{aligned} x^6 = x^4 x^2 &= [u_4 u_2 - u_2] \\ &= [(\ldots 02)98 - 98] \\ &= [\ldots 98] \\ x^{14} = x^8 x^6 &= [u_8 u_6 - u_2] \\ &= [(\ldots 02)(\ldots 98) - 98] \\ &= [\ldots 98] \\ x^{30} = x^{16} x^{14} &= [u_{16} u_{14} - u_2] \\ &= [(\ldots 02)(\ldots 98) - 98] \\ &= [\ldots 98], \\ x^{94} = x^{64} x^{30} &= [u_{64} u_{30} - u_{34}] \\ &= [u_{64} u_{30} - (u_{32} u_2 - u_{30})] \\ &= [(\ldots 02)(\ldots 98) - \{(\ldots 02)98 - (\ldots 98)\}] \\ &= [\ldots 98], \end{aligned}$$

and similarly we can show that x^{222}, x^{478} and finally x^{990} are of the form $[\ldots 98]$ where $[\ldots 98]$ indicates a u_k value whose tens and units digits are 9 and 8 respectively.

Thus

$$(\sqrt{2}+\sqrt{3})^{1980} = [\ldots 98] = [N], \text{ say}$$
$$= N - \cfrac{1}{N - \cfrac{1}{N - \cfrac{1}{\ddots}}},$$

and the magnitude of N ensures that

$$(\sqrt{2}+\sqrt{3})^{1980} = N - \varepsilon, \qquad \text{where } \varepsilon < 10^{-2},$$
$$= (\ldots 97.9\ldots).$$

Thus, the units and tenths digits are 7 and 9 respectively.

This exercise in problem-solving raised several interesting questions which some readers might like to pursue. For example, those readers involved in computer programming activities might like to verify (or otherwise) the result obtained above. In particular, it would be invaluable to know whether a program could be constructed to generate successive digits to a large number of places (as some have done for π) for numbers like $(\sqrt{2}+\sqrt{3})^{1980}$ and we should be pleased to hear from anybody who could do this.

In this article attention has focused on the class of numbers $[u_n]$ derived from $(5+2\sqrt{6})^n$ and interest has centred around the product $[u_m][u_n]$. Readers may be interested in investigating $[u_m] \pm [u_n]$ and other numbers based on $(5+2\sqrt{6})$.

We saw earlier that

$$1 = [2] = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \ddots}},$$

and integer representation investigations may lead to some interesting discoveries. For example,

$$x^2 - 2ax + a^2 = 0$$
$$\Rightarrow x = 2a - \frac{a^2}{x}$$
$$= 2a - \cfrac{a^2}{2a - \cfrac{a^2}{2a - \ddots}}.$$

A convenient notation for this expression would be $[2a; -a^2]$, and since $(x-a)^2 = 0$ has solution $x = a$ only, we are led to conclude that integer a may be expressed in this form. Some examples would be

$$1 = [2 \cdot 1; -(1^2)] = [2; -1],$$

$$2 = [2 \cdot 2; -(2^2)] = [4; -4],$$

$$5 = [2 \cdot 5; -(5^2)] = [10; -25].$$

The numbers $[u_n]$ used in the early part of this article could have been more fully expressed as $[u_n; -1]$, and whereas $(5 + 2\sqrt{6})^n$ leads us to consider the class of numbers of this form, other possibilities obviously exist. For example, what class of numbers of the form $(a + b\sqrt{q})^n$ are represented by numbers of the type $[u_n; -2]$? What properties does this class possess? Does the multiplication rule deduced for $[u_n; -1]$ still apply? Is there a general multiplicative property for the numbers of the type $[u_n; -k]$? Under what conditions are $[u_n; -k]$ and $[v_m; -j]$ conformable for multiplication?

Another very interesting area for investigation involves the complex numbers, since these naturally arise in the solution of many quadratic equations.

Thus, if $x = a + ib$,

$$(x-a)^2 = -b^2,$$

$$x = 2a - \frac{(a^2+b^2)}{x}$$

$$= [2a; -(a^2+b^2)].$$

For example, if $a = 2$, $b = 1$, the result suggests

$$2 + i = [4; -5]$$

$$= 4 - \cfrac{5}{4 - \cfrac{5}{4 - \ddots}},$$

or

$$i = 2 - \cfrac{5}{4 - \cfrac{5}{4 - \ddots}}.$$

Other apparently real expressions for the purely imaginary number i can be developed using other values for a and b, and some readers may like to explore the nature of such infinite 'negative' continued fractions with the objective of shedding light on new interpretations of the complex numbers. A certain consistency is evident in the above since if $b = 0$, then $a + ib = a = [2a; -a^2]$, as found earlier for the integer a.

However, if $a=0$ and $b=1$, then $a+ib=i=[0;-1]$, and i is seen to be a member of our original class of numbers $[u_n;-1]$ with $u_n=0$.

The 'negative' continued fraction in this case is given by

$$i = 0 - \cfrac{1}{0 - \cfrac{1}{0 - \cfrac{1}{0 - \ddots}}},$$

a form which indicates the probable complexity to be found in interpreting complex numbers with functions of real numbers.

While many other avenues of exploration are possible, we hope these brief comments indicate some of the ways in which possibly rewarding investigations may be undertaken by interested readers. It is also likely that other quite different approaches to the original problem will yield the required solution (hopefully in agreement with ours!).

The exercise we have undertaken is probably typical of those initiated by frustration in failing to find a solution to what appears to be a simple enough problem. We hope some of the questions raised here will cause some readers to be similarly motivated, or at least encouraged to pursue solutions to problems which initially appear intractable.

Appendix 1

If $x=5+2\sqrt{6}$, then $x^n=u_n-1/x^n$ for some positive integer n.

Proof.

$$\begin{aligned}
x^n &= (5+2\sqrt{6})^n \\
&= \left\{5^n + \binom{n}{2} 5^{n-2}.2^2.6 + \binom{n}{4} 5^{n-4}.2^4.6^2 + \cdots\right\} \\
&\quad + \left\{\binom{n}{1} 5^{n-1}.2 + \binom{n}{3} 5^{n-3}.2^3.6 + \binom{n}{5} 5^{n-5}.2^5.6^2 + \cdots\right\}\sqrt{6} \\
&= u_n - \left(\left\{5^n + \binom{n}{2} 5^{n-2}.2^2.6 + \binom{n}{4} 5^{n-4}.2^4.6^2 + \cdots\right\}\right. \\
&\quad \left. - \left\{\binom{n}{1} 5^{n-1}.2 + \binom{n}{3} 5^{n-3}.2^3.6 + \binom{n}{5} 5^{n-5}.2^5.6^2 + \cdots\right\}\sqrt{6}\right),
\end{aligned}$$

where

$$u_n = 2\left(5^n + \binom{n}{2} 5^{n-2}.2^2.6 + \binom{n}{4}.5^{n-4}.2^4.6^2 + \cdots\right),$$

which is a positive integer. Hence,

$$\begin{aligned} x^n &= u_n - (5 - 2\sqrt{6})^n \\ &= u_n - 1/(5 + 2\sqrt{6})^n \\ &= u_n - 1/x^n. \end{aligned}$$

The result in Question 1 follows from this.

Appendix 2

If $x^m = [u_m]$, $x^n = [u_n]$, $x^{m-n} = [u_{m-n}]$ for $m \geq n$, then $x^{m+n} = [u_m u_n - u_{m-n}] = [u_m][u_n]$.

Proof.

$$\begin{aligned} x^m = [u_m] &\Rightarrow x^m = u_m - 1/x^m \\ &\Rightarrow x^{2m} = u_m x^m - 1, \\ x^n = [u_n] &\Rightarrow x^n = u_n - 1/x^n \\ &\Rightarrow x^{2n} = u_n x^n - 1, \\ x^{m-n} = [u_{m-n}] &\Rightarrow x^{m-n} = u_{m-n} - 1/x^{m-n} \\ &\Rightarrow u_{m-n} = x^m/x^n + x^n/x^m. \end{aligned}$$

Hence

$$\begin{aligned} x^{2(m+n)} &= (u_m x^m - 1)(u_n x^n - 1) \\ &= u_m u_n x^{m+n} - u_m x^m - u_n x^n + 1 \\ &= u_m u_n x^{m+n} - (x^{2m} + 1) - (x^{2n} + 1) + 1 \\ &= (u_m u_n - (x^m/x^n + x^n/x^m))x^{m+n} - 1 \\ &= (u_m u_n - u_{m-n})x^{m+n} - 1 \\ \Rightarrow x^{m+n} &= (u_m u_n - u_{m-n}) - 1/x^{m+n} \\ \Rightarrow [u_m][u_n] &= [u_m u_n - u_{m-n}], \end{aligned}$$

as required.

The following problem was set in the magazine *Acorn User*, and is reproduced here with permission. Why not try it?

Use the digits 1 to 9 once and once only to make a number divisible by 9 such that the first eight digits give a number divisible by 8, the first seven give a number divisible by 7, etc. How many such numbers are there?

Letter to the Editor

Dear Editor,

Roots of polynomials

In Volume 14 Number 2, Nigel McCann showed an interesting phenomenon. I was inspired by his letter and obtained a more general result.

Consider a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with real coefficients, where $n \geq 1$ and $a_n \neq 0$. If we differentiate $f(x)$ k times ($1 \leq k \leq n-1$), we obtain

$$\frac{d^k f(x)}{dx^k} = a_n \frac{n!}{(n-k)!} x^{n-k} + a_{n-1} \frac{(n-1)!}{(n-1-k)!} x^{n-1-k} + \cdots + a_{k+1} \frac{(k+1)!}{1!} x + a_k \frac{k!}{0!}.$$

Now $f(x)$ has n complete roots. If we denote these by $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$a_{n-1} = -a_n(\alpha_1 + \alpha_2 + \cdots + \alpha_n). \tag{1}$$

If we denote the roots of $d^k f(x)/dx^k$ by $\beta_1, \beta_2, \ldots, \beta_{n-k}$, we have

$$a_{n-1} \frac{(n-1)!}{(n-1-k)!} = -a_n \frac{n!}{(n-k)!} (\beta_1 + \beta_2 + \cdots + \beta_{n-k}). \tag{2}$$

From (1) and (2) we obtain

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = \frac{\beta_1 + \beta_2 + \cdots + \beta_{n-k}}{n-k}.$$

In other words, the arithmetic mean of the roots of $f(x)$ is equal to the arithmetic mean of the roots of the kth derivative of $f(x)$.

Further to Nigel McCann's letter, consider a quadratic polynomial

$$y = x^2 + ax + b$$

with real coefficients *and any kind of roots*. The root of the derivative will be $-a/2 = (x_1 + x_2)/2$, where x_1, x_2 are the roots of the quadratic polynomial. Now

$$y = (x - x_1)(x - x_2)$$

so, when $x = (x_1 + x_2)/2$, then $y = -((x_1 - x_2)/2)^2$ and the vertex of the parabola $y = x^2 + ax + b$ is the point $((x_1 + x_2)/2,\ -((x_1 - x_2)/2)^2)$.

Yours sincerely,

LIU ZHIQING
(No. 4 Middle School, Xishiku Street,
Beijing, People's Republic of China)

Editor: We congratulate Liu Zhiqing on his excellent English.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

16.1. Evaluate $(9+4\sqrt{5})^{1/3}+(9-4\sqrt{5})^{1/3}$.

16.2. A lady enters a supermarket with her calculator, which she uses to add up her bill. She buys four items, but unfortunately she presses the multiplication button instead of the addition button. When she gets to the checkout, the cashier tells her that her bill is £7·11. 'Yes, that is what I got', says the customer. How much did each item cost?

16.3. The trump suit in a game of bridge is given as shown, and South leads. How can North and South guarantee to take at least three tricks no matter how the other five trumps are distributed among the other two hands?

	North	
	K J 3 2	
West		East
	A 9 5 4	
	South	

Solutions to Problems in Volume 15, Number 2

15.4. Given positive real numbers a, b and positive integers m, n with $m > n$, prove that

$$(a^m + b^m)^n < (a^n + b^n)^m.$$

Solution (by Richard Hilditch of Sydney Sussex College, Cambridge)

Let $u = a^n/(a^n + b^n)$ and $p = m/n > 1$. Then $0 < u < 1$, so that $u^p < u$ and $(1-u)^p < 1-u$. If we add these we obtain $u^p + (1-u)^p < 1$. We now multiply both sides by $(a^n + b^n)^p$ to give $(a^n)^p + (b^n)^p < (a^n + b^n)^p$. If we now take the nth power of both sides, we obtain the required result.

Also solved by Ruth Lawrence (Huddersfield), Lloyd Taylor (University of Nottingham), Ian Wright (Winchester College), Nishad Gumaste (Northfleet Grammar School) and Datta Gumaste (Open University).

15.5. Determine all real numbers α which satisfy the double inequality

$$\tfrac{1}{3}\tan\alpha \le \tan 3\alpha \le 3\tan\alpha.$$

Solution (by Ruth Lawrence of Huddersfield)
We have

$$\tan 3\alpha = \frac{3t - t^3}{1 - 3t^2}, \qquad \text{where } t = \tan\alpha.$$

We can therefore rewrite the double inequality as

$$\frac{\beta t}{3}(1 - 3t^2) \le \beta(3t - t^3) \le 3\beta t(1 - 3t^2),$$

where $\beta = 1$ or -1 depending on whether $1 - 3t^2 > 0$ or $1 - 3t^2 < 0$. (Note that $t = \pm 1/\sqrt{3}$ would mean that $\tan 3\alpha = \infty$ and the doubly inequality cannot be satisfied.) Thus

$$\beta t - 3\beta t^3 \le 9\beta t - 3\beta t^3 \le 9\beta t - 27\beta t^3,$$

or

$$8\beta t \ge 0 \qquad \text{and} \qquad 24\beta t^3 \le 0.$$

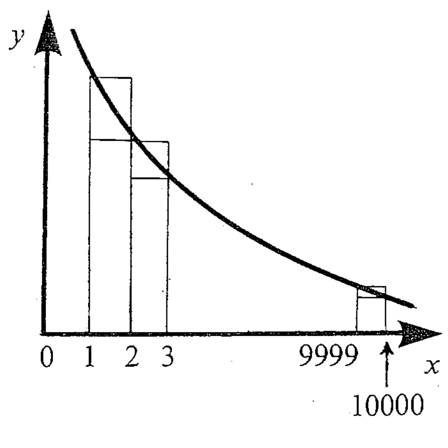
Since t and t^3 take the same signs, the only possibility is $t = 0$. Certainly $t = 0$ satisfies the double inequality. Therefore the real numbers α which satisfy the double inequality are those for which $\tan\alpha = 0$, i.e. all integer multiples of π.

Also solved by Richard Hilditch (Sydney Sussex College, Cambridge), Ian Wright (Winchester College), Don Jurries and the St Olaf problem-solving group, St Olaf College, Northfield, Minnesota.

15.6. What is the integer part of the number

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{10{,}000}}?$$

We include two methods of solution of this problem. The former is the one used by the majority of readers who sent in solutions, and involves the integration of $1/\sqrt{x}$ between 1 and 10,000. The latter cleverly avoids integration.



Solution 1 (by Richard Hilditch of Sydney Sussex College, Cambridge)
The area under the curve $y = 1/\sqrt{x}$ between $x = 1$ and $x = 10{,}000$ is

$$\int_1^{10{,}000} (1/\sqrt{x})\, dx = [2\sqrt{x}]_1^{10{,}000} = 198.$$

This area is bounded by 'lower' and 'upper' sums. If S denotes the sum in question, we have

$$S-1<198<S-\frac{1}{\sqrt{10{,}000}}.$$

Thus $198{\cdot}01<S<199$, and the integer part of S is 198.

Also solved using essentially this method by Benny Cheung (Ipswich School), Ruth Lawrence (Huddersfield), Ian Wright (Winchester College).

Solution 2 (by Ede Vásquez of Escuela de Ingeniería de Antioquia, Medellín, Colombia)

We first prove that

$$2\sqrt{(n+1)}-2\sqrt{n}<\frac{1}{\sqrt{n}}<2\sqrt{n}-2\sqrt{(n-1)}$$

for all positive integers n. In fact,

$$\begin{aligned}2\sqrt{(n+1)}-2\sqrt{n}&=\frac{2(\sqrt{(n+1)}-\sqrt{n})(\sqrt{(n+1)}+\sqrt{n})}{\sqrt{(n+1)}+\sqrt{n}}\\&=\frac{2}{\sqrt{(n+1)}+\sqrt{n}}<\frac{2}{\sqrt{n}+\sqrt{n}}=\frac{1}{\sqrt{n}},\end{aligned}$$

and

$$\begin{aligned}2\sqrt{n}-2\sqrt{(n-1)}&=\frac{2(\sqrt{n}-\sqrt{(n-1)})(\sqrt{n}+\sqrt{(n-1)})}{\sqrt{n}+\sqrt{(n-1)}}\\&=\frac{2}{\sqrt{n}+\sqrt{(n-1)}}>\frac{2}{\sqrt{n}+\sqrt{n}}=\frac{1}{\sqrt{n}}.\end{aligned}$$

Now we have

$$\begin{aligned}&\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{10{,}000}}\\&\quad>2[(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{10{,}001}-\sqrt{10{,}000})]\\&\quad=2\sqrt{10{,}001}-2\\&\quad>198.\end{aligned}$$

Also,

$$\begin{aligned}&\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{10{,}000}}\\&\quad<1+2[(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{10{,}000}-\sqrt{9999})]\\&\quad=1+2(100-1)\\&\quad=199.\end{aligned}$$

It follows that the integer part of the given sum is 198.

Also solved using this method by Roy Crole (Maidstone School for Boys), Nishad Gumaste (Northfleet Grammar School).

Book Reviews

Surface Topology. By P. A. FIRBY and C. F. GARDINER. Ellis Horwood, Chichester, 1982. Pp. 216. £19·50 hardback, £6·90 ($13·85) paperback.

What is the smallest number of colours that you would need to colour a map so that no two countries with a common boundary are coloured with the same colour? Answer: Four. But what if you draw a map on the surface of a torus—a rubber ring, for the uninitiated? This book will give you the answer. It is full of exciting things like Klein bottles, with no inside or outside, projective planes and hairy balls.

The authors have attempted the difficult task of giving the flavour of this exciting branch of mathematics to students without losing them in a forest of technicalities. In this they have largely succeeded. Certain results are not proved; in others, difficult points are rather glossed over. But the book provides a good introduction to the subject for a final-year undergraduate class which is not made up of budding research topologists, i.e. every final-year undergraduate class! It may well inspire some students to investigate the basis of the subject further.

The book is generally well written with clear, interesting diagrams. One small point which amused and then irritated your reviewer: the authors seem to think that every sentence should begin a new paragraph. Page 14 has no fewer than 16 paragraphs. Is this a record!

Recommended for final-year undergraduates and others for whom Alice's words 'Curiouser and curiouser' strike a sympathetic chord.

University of Sheffield D. W. SHARPE

aha! Gotcha: Paradoxes to Puzzle and Delight. By MARTIN GARDNER. W. H. Freeman and Company, San Francisco, 1982. Pp. vii + 164. Hardback £11·20. Paperback £5·60.
Mathematical Fallacies and Paradoxes. By BRYAN H. BUNCH. Van Nostrand Reinhold Company Ltd, London, 1982. Pp. xi + 216. £14·40.

In *aha! Gotcha*, Martin Gardner takes a paradox to be a result so contrary to common sense and intuition that it produces an immediate reaction of surprise. Thus the author is able to use each paradox to stimulate the reader.

First, there is the stimulus to laughter and enjoyment. The format of the book is, generally, to let each double-page spread cover one topic; this makes it ideal for dipping into. The left-hand page consists of a series of humorous cartoons depicting a story which ends with a paradox.

For the second stimulus, the paradox acts like a magician's trick which astonishes everyone so that they instantly want to know how it is done. On the right-hand page opposite the presentation of the paradox, the author does his best to explain in non-technical language, and as briefly as possible, why each paradox is paradoxical. Also he shows, again briefly, how these paradoxes have influenced the development of mathematics through the ages. It is hoped that this will stimulate the reader to pursue some of the topics further and the book includes a list of easily accessible readings.

The six chapters deal with paradoxes involving logic, number, geometry, probability, statistics and time. The appropriate age for the different topics varies from the first year of secondary school onwards.

aha! Gotcha is similar in style and format to Martin Gardner's earlier book *aha! Insight*, which considers problems rather than paradoxes. It is possible to obtain filmstrips to go with the two books; however, the drawings are sufficiently simple for a teacher to draw them as transparencies for an overhead projector.

As a sample of the book's goodies, the following makes an excellent introduction to the harmonic series $1 + 1/2 + 1/3 + 1/4 + \cdots$.

'A worm is at one end of a rubber rope, one kilometre long. It crawls steadily along the rope at one centimetre per second. After the first second the rope stretches to have length two kilometres, after the next second it stretches to three kilometres, and so on. Intuition says the worm will never reach the end of the rope. However, a careful calculation produces a surprise.'

Thus I can thoroughly recommend *aha! Gotcha*, either for the library or for classroom use. It should first attract pupils and then challenge them: 'This does not make sense! Are you going just to walk away, beaten by it, or are you going to do something about it?'

While Martin Gardner limits himself to an explanation that can be given briefly, Bryan Bunch, in *Mathematical Fallacies and Paradoxes*, tries to give as complete an explanation as possible without going into the technical details. The result is not as successful as *aha! Gotcha*, although many paradoxes are common to both books. The style is not particularly attractive. For example, in the proofs, all the straightforward steps are often given, and these tend to be rather dull. Also there are quite a number of errors, which may have been caused by the author's attempt to avoid the more difficult ideas.

Thus my choice is *aha! Gotcha*.

University of Sheffield — KEITH AUSTIN

Magic Cubes: New Recreations. By WILLIAM H. BENSON and OSWALD JACOBY. Dover Publications, New York, 1981. Pp. 142. £3·00.

Admirers of the same authors' *New Recreations with Magic Squares* should enjoy this extension of different combinations of 'cyclical' and 'interchange' systems to constructing magic cubes. A *perfect* magic cube—one with *all* the lines of N numbers (including the diagonals) adding to the same constant—is impossible, sadly, for N less than 7. But the authors show in detail how to get the best achievable results for all N. If N is odd or a multiple of 8, fairly elementary (though quite laborious) routines suffice. For N singly or doubly even, there are further stages of tricky trial and error.

STEPHEN AINLEY

Solving Real Problems with Mathematics, Volumes 1–3. By THE SPODE GROUP. Cranfield Press, Bedford, 1982. Pp. 134. £7·50.

Mathematical modelling as the basis of applied mathematics is clearly an idea in which there is a good deal of current interest. These volumes from the Spode Group at Cranfield should help both pupils and teachers by providing real (or pseudo-real) problems with examples of possible mathematical models that can be used in their solution. Most of the suggested solutions in Volumes 1 and 2 require only a good knowledge of O-level mathematics, though some use elementary ideas of calculus and trigonometry more usually introduced in the sixth form. Volume 3 concentrates on problems which can be solved using only CSE mathematics. Examples of areas of application include transfer fees in football, time to allow in hill walking, placing wine storage racks (with an error in the solution), arranging furniture, school dinners, planning a nursing timetable, proportional representation, using lifts efficiently, easing the traffic jam, car-park layouts and X-rays. Several problems, such as population prediction and the retail price index, use statistical ideas.

The approach to mathematics by modelling requires several skills which are not acquired by standard mathematics teaching. These skills are particularly in the areas of identifying basic structures within the problem, quantifying such vague ideas as the 'best solution', checking whether the mathematical solution is sufficiently useful in practice and developing a more realistic model if it is not. Each of the problems consists of a statement of the problem, teaching notes and a possible solution. Some also include an extension to related problems or to more sophisticated models.

The authors suggest two ways in which the problems might be used. The first is to photocopy the problem statement so that students together may develop appropriate models for a solution. They also suggest that particular examples might be chosen to motivate particular mathematical topics. The 'rugby goal-kicking' problem clearly leads in to some sixth-form-level trigonometry. A third way of using them would be for the students to look through some of the proposed solutions, to criticise them constructively and to improve them. This might give more confidence for tackling such problems from scratch.

The books provide a useful source of ideas and are well worth a place in the school library and as a recommended study source for sixth-form mathematics students.

University of Sheffield PETER HOLMES

Great Moments in Mathematics (After 1650). By HOWARD EVES. The Mathematical Association of America (distributed by John Wiley and Sons Ltd., Chichester), 1981. Pp. xii + 263. £15·75.

This book gives the same treatment to the development of mathematics after 1650 as the author's previous volume (reviewed in *Mathematical Spectrum* Volume 15 Number 2) does to the earlier period. It is an interesting and exciting, though somewhat anachronistic, account of certain topics in the later history of the subject. During this period the rate of mathematical advance increased, and the discoveries made were frequently at a high mathematical level. The author has therefore had to be doubly selective, and on the whole only the greater and more accessible of the great moments are the ones which are analysed.

The book begins with the birth of mathematical probability and ends with the use of the computer to resolve the four-colour conjecture. In between, the reader is treated to the development of the calculus, the non-Euclidean approaches to geometry and the evolution of mathematical structures. A not-very-historical chapter on the group concept is followed by a crisp examination of Klein's Erlanger Program. The remainder of the book is mainly concerned with the foundations of the subject, with all the usual heroes receiving a mention; there is a particularly interesting discussion of Gödel's discoveries.

This book has the distinct flavour of an elementary college text (more so than the previous volume), but the author's ability to express concepts simply and to give brief but quintessential character sketches of the great mathematicians will make this an absorbing book for anyone wishing to survey the development of mathematical ideas.

F. GARETH ASHURST

Soma World, the Complete Soma Cube. By SIVY FARHI. Canberra Publishing and Printing Co., 1982. Obtainable from the author, Box 476, Belconnen, ACT 2616, Australia. £2·20 surface mail; £3·00 airmail.

A tiny book containing over 2000 Soma-cube configurations with solutions. (You will find information about the Soma cube in Martin Gardner, *More Mathematical Puzzles and Diversions*, Penguin Books Ltd, London, 1966.)

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ISSN 0025-5653

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Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Published by the Applied Probability Trust
Printed in England by Galliard (Printers) Ltd, Great Yarmouth