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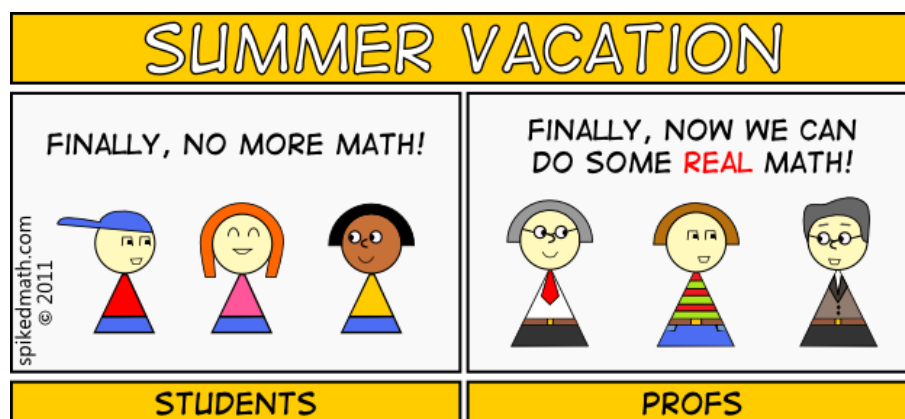
# EDITORIAL

Summer is here and at *Cruz* we are busy catching up to the new production schedule. You will notice more solutions appearing in this and the next few issues as we make our way through problems appearing in the previous Volume at double the pace. With the total of 45 solutions, this issue is jam-packed and I'd like to thank the amazing Editorial Board of *Cruz* for their hard work.

This is the last issue featuring *Contest Corner* solutions. This section is now replaced with a much more comprehensive and wide reaching *MathemAttic*. As this part of *Cruz* is aimed at secondary level students and teachers, we ask our more seasoned problem solvers to focus their attention and submit solutions to *Olympiad Corner* and *Problems* sections.

Enjoy the issue and let the summer begin!

Kseniya Garaschuk



# MATHEMATTIC

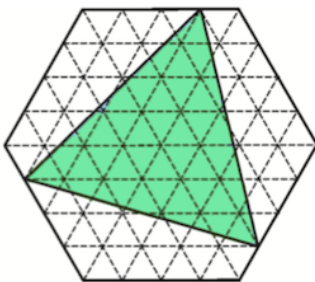
No. 5

*The problems in this section are intended for students at the secondary school level.*

*Click here to submit solutions, comments and generalizations to any problem in this section.*

*To facilitate their consideration, solutions should be received by **August 15, 2019**.*

**MA21.** An equilateral triangle is inscribed into a regular hexagon as shown below. If the area of the hexagon is 96, find the area of the inscribed triangle.



**MA22.** Do there exist three positive integers  $a$ ,  $b$  and  $c$  for which both  $a+b+c$  and  $abc$  are perfect squares? Justify your answer.

**MA23.** Integer numbers were placed in squares of a  $4 \times 4$  grid so that the sum in each column and the sum in each row are all equal. Seven of the sixteen numbers are known as shown below, while the rest are hidden.

1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Show that it is possible to determine at least one of the missing numbers. Is it possible to determine more than one of the missing numbers?

**MA24.** You have 5 cards with numbers 3, 4, 5, 6 and 7 written on their backs. How many five digit numbers are divisible by 55 with the digits 3, 4, 5, 6, and 7 each appearing once in the number (the card with 6 cannot be rotated to be used as a 9)?

**MA25.**

1. Find four consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, the third is divisible by 7 and the fourth is divisible by 9.
2. Can you find 100 consecutive natural numbers such that the first is divisible by 3, the second is divisible by 5, ... the 100th is divisible by 201?

.....

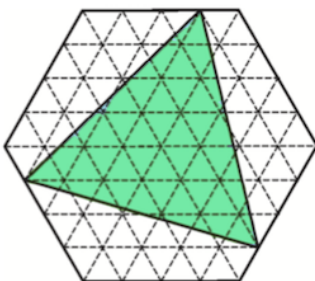
*Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.*

*Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2019**.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*

**MA21.** Un triangle équilatéral est inscrit dans un hexagone régulier tel qu'indiqué ci-bas. Si la surface de l'hexagone est 96, déterminer la surface du triangle inscrit.



**MA22.** Existe-t-il un ensemble de trois entiers positifs  $a$ ,  $b$  et  $c$  tels que  $a + b + c$  et  $abc$  sont des carrés parfaits? Justifier votre réponse.

**MA23.** Des nombres entiers sont placés dans les cases d'une grille  $4 \times 4$  de façon à ce que les sommes des entiers dans chaque colonne sont les mêmes et pareillement pour les rangées. Sept des seize nombres sont visibles et les autres ne le sont pas, comme indiqué ci-bas.

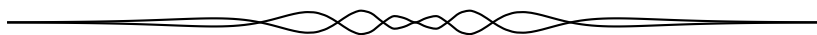
1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Prouver qu'il est possible de déterminer au moins un des nombres manquants. Est-ce possible d'en déterminer plus qu'un?

**MA24.** Vous disposez de cinq cartes, sur lesquelles sont inscrits les nombres 3, 4, 5, 6 et 7. Combien de nombres à cinq chiffres, divisibles par 55, peuvent être formés avec ces cartes, chacun des chiffres 3, 4, 5, 6 et 7 apparaissant une seule fois ? (Noter que la carte portant le chiffre 6 ne peut pas être réorientée pour lire 9.)

**MA25.**

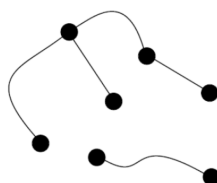
1. Déterminer quatre entiers naturels consécutifs tels que le premier est divisible par 3, le deuxième est divisible par 5, le troisième est divisible par 7 et le quatrième est divisible par 9.
2. Pouvez-vous déterminer 100 entiers naturels consécutifs tels que le premier est divisible par 3, le deuxième est divisible par 5, ... et le 100ième est divisible par 201?



# CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(9), p. 364–366; and 44(10), p. 406–407.

**CC341.** The graph below shows 7 vertices (the dots) and 5 edges (the lines) connecting them. An edge here is defined to be a line that connects 2 vertices together. In other words, an edge cannot loop back and connect to the same vertex. Edges are allowed to cross each other, but the crossing of 2 edges does not create a new vertex. What is the least number of edges that could be added to the graph, in addition to the 5 already present, so that each of the 7 vertices has the same number of edges?



Originally Problem 23 from the 2018 Indiana State Math Contest.

We received 4 solutions, out of which we present the one by Richard Hess.

There are currently 5 edges drawn and one vertex already has three edges. Suppose that each of the 7 vertices has  $n$  edges. The total number of edges is then  $7n/2$ , which requires that  $n$  be even. The smallest possible  $n$  is then  $n = 4$  and the total number of edges 14. Thus, 9 edges is the least number that must be added. This is achievable if we make the vertices those of a regular heptagon and connect each vertex to those that are one or two vertices away.

**CC342.** You are given the 5 points

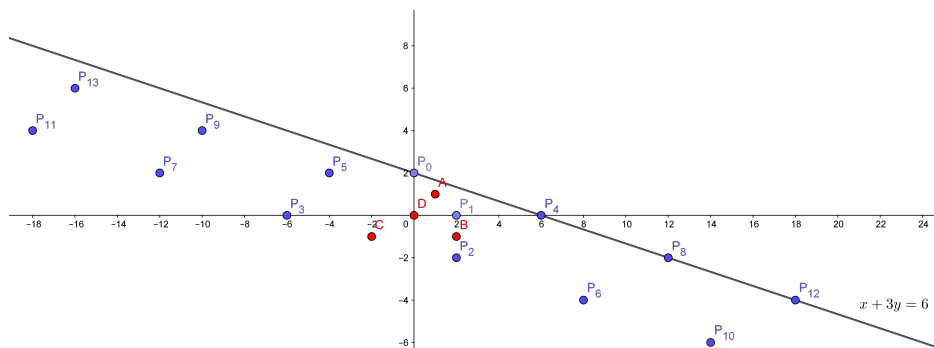
$$A = (1, 1), B = (2, -1), C = (-2, -1), D = (0, 0), P_0 = (0, 2).$$

$P_1$  equals the rotation of  $P_0$  around  $A$  by  $180^\circ$ ,  
 $P_2$  equals the rotation of  $P_1$  around  $B$  by  $180^\circ$ ,  
 $P_3$  equals the rotation of  $P_2$  around  $C$  by  $180^\circ$ ,  
 $P_4$  equals the rotation of  $P_3$  around  $D$  by  $180^\circ$ ,  
 $P_5$  equals the rotation of  $P_4$  around  $A$  by  $180^\circ$ , and so on repeating this pattern.  
 If  $P_{2016} = (a, b)$ , then what is the value of  $a + b$ ?

Originally Problem 13 from the 2016 Indiana State Math Contest.

We received 2 submissions, both correct and complete, and present both here.

*Solution 1, by Andrea Fanchini.*



$P_{2016}$  is the  $2016/4 = 504^{\text{th}}$  point on the line  $x + 3y = 6$  after  $P_0$ . It follows that

$$x = 504 \cdot 6 = 3024, \quad y = \frac{6 - x}{3} = -1006.$$

Thus

$$P_{2016} = (3024, -1006)$$

and  $a + b = 2018$ .

*Solution 2, by Ivko Dimitrić.*

Denote the reflections in the points  $A, B, C$  and  $D$  by  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$ , respectively. The rotation about a point by  $180^\circ$  is the reflection in that point. By the property of a mid-line in a triangle being half of the corresponding side in length, the composition  $\mathcal{B}\mathcal{A}$  of two point reflections is the translation by the vector  $2\overrightarrow{AB} = 2\langle 1, -2 \rangle$  and the composition  $\mathcal{D}\mathcal{C}$  is the translation by the vector  $2\overrightarrow{CD} = 2\langle 2, 1 \rangle$ . Hence, the composition of these two translations,  $\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A}$ , is the translation by the vector

$$2\overrightarrow{AB} + 2\overrightarrow{CD} = \langle 2, -4 \rangle + \langle 4, 2 \rangle = \langle 6, -2 \rangle,$$

so

$$(\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A})P = P + (6, -2)$$

and

$$(\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A})^n P = P + n(6, -2),$$

where addition of points and multiplication by a number is to be understood in vector sense, i. e. coordinate-wise. Then, since  $2016 = 4 \cdot 504$ , we have

$$P_{2016} = (\mathcal{D}\mathcal{C}\mathcal{B}\mathcal{A})^{504} P_0 = (0, 2) + 504(6, -2) = (6 \cdot 504, -2 \cdot 503).$$

Finally,

$$a + b = 6 \cdot 504 - 2 \cdot 503 = 2018.$$



**CC343.** In the following long division problem, most of the digits (26 in fact) are hidden by the symbol X. What is the sum of all of the 26 hidden digits?

$$\begin{array}{r}
 \text{X X} \overline{\text{X X X X X X X}} \\
 \underline{\text{X X X}} \\
 \text{X X} \\
 \underline{\text{X X}} \\
 \text{X X X} \\
 \underline{\text{X X X}} \\
 1
 \end{array}$$

*Originally Problem 17 from the 2017 Indiana State Math Contest.*

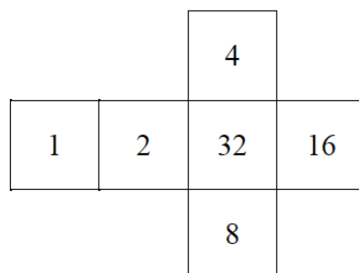
*We received 4 solutions, out of which we present the one by Kathleen E. Lewis.*

Since the divisor is a two digit number and 8 times the divisor is a two digit number, the divisor must be 10, 11, or 12. But some digit times the divisor gives a three digit number, so the divisor must be 12 and it must be multiplied by 9. From this, we can put together the entire division problem:

$$\begin{array}{r}
 90809 \\
 12 \overline{)1089709} \\
 \underline{108} \\
 97 \\
 \underline{96} \\
 109 \\
 \underline{108} \\
 1
 \end{array}$$

Therefore the sum of the hidden digits is 114.

**CC344.** Take the pattern below and form a cube. Then take three of these exact same cubes and stack them one on top of another on a table so that exactly 13 numbers are visible. What is the greatest possible sum of these 13 visible numbers?



*Originally Problem 16 from the 2015 Indiana State Math Contest.*

*We received 5 solutions, all of which were correct. We present the solution of Brad Meyer, modified by the editor.*

To maximize the total sum, we maximize the sum found on each cube.

The top cube has five visible faces and one hidden face at the bottom. To maximize the sum, we select the face with the smallest number to be “hidden” on the bottom; the 1 face. This leaves visible faces 32, 16, 8, 4, and 2. Resulting in a sum of 62.

The middle cube has 4 visible faces and 2 hidden faces on the top and bottom. The cubes opposite face pairs are  $[1, 32]$ ,  $[2, 16]$ , and  $[4, 8]$ . We select the pair of opposite faces with the smallest sum to be hidden on the top and bottom,  $[4, 8]$ . This leaves visible faces 32, 16, 2, and 1. Resulting in a sum of 51.

The same methodology used above for the middle cube holds for the bottom cube. Thus the total of the visible faces is  $62 + 51 + 51 = 164$ .

**CC345.** Your teacher asks you to write down five integers such that the median is one more than the mean, and the unique mode is one greater than the median. You then notice that the median is 10. What is the smallest possible integer that you could include in your list?

*Originally Problem 17 from the 2015 Indiana State Math Contest.*

*We received 5 submissions, all of which were correct and complete. We present the solutions by Kathleen Lewis, Richard Hess, and Charles Justin Shi (done independently), combined by the editor.*

Call the five integers  $a, b, c, d$ , and  $e$  in a non-decreasing order. Given that the median is 10,  $c = 10$  and our mean and mode are 9 and 11, respectively. Our mode is unique and therefore occurs at least twice. Given that the median is 10 and our mode is 11, our mode can occur at most twice - else our median would be 11. Thus  $d = e = 11$ . As our mean is 9, we have that

$$\frac{a + b + 10 + 11 + 11}{5} = 9 \Rightarrow a + b = 13,$$

where  $b \leq 9$ . The smallest value of  $a$  is 4.

**CC346.** An ant paces along the  $x$ -axis at a constant rate of one unit per second. He begins at  $x = 0$  and his path takes him one unit forward, then two back, then three forward, etc. How many times does the ant step on the point  $x = 10$  in the first five minutes of his walk?

*Originally Problem 10 of Game 3 from the 2015-16 Nova Scotia Math League.*

*We received 3 submissions to this problem, only one of which was correct. We present the solution by Ivko Dimitrić.*

By a single move we mean a stretch that the ant paces in the same direction before turning to move in the opposite direction. We use notation  $M_n^F(k)$  to denote the ant's  $n$ th move, which is a forward move ending up at number  $k$  on the  $x$ -axis. Similarly,  $M_n^B(-k)$  means that on the  $n$ th move the ant moved back (to the left

on the  $x$ -axis) and ended up at number  $-k$ . Since every subsequent move is one unit longer, the ant progresses 1 unit to the right on each move forward (an odd-numbered move) compared to the previous forward move and progresses 1 unit to the left on each even-numbered move:

$$M_1^F(1), M_3^F(2), M_5^F(3), M_7^F(4), \dots, M_{2k-1}^F(k), \dots$$

$$M_2^B(-1), M_4^B(-2), M_6^B(-3), M_8^B(-4), \dots, M_{2k}^B(-k), \dots$$

Therefore, after  $19 = 2 \cdot 10 - 1$  moves the ant finally reaches the point  $x = 10$  for the first time with  $M_{19}^F(10)$ , whereupon it turns back to make its next move  $M_{20}^B(-10)$ . The time to complete the first 20 moves is equal to the length traveled,

$$1 + 2 + 3 + \dots + 20 = \frac{20 \cdot 21}{2} = 210$$

seconds and during this time the ant stepped on  $x = 10$  only once. Then in each subsequent move the ant will step on the point 10 once each time as it moves forward or back. On the next four moves which last 21 seconds ( $F$ ), 22 seconds ( $B$ ), 23 seconds ( $F$ ) and 24 seconds ( $B$ ), the number 10 will be stepped on once during each move, thus four additional times. The total elapsed time to complete all 24 steps is

$$210 + 21 + 22 + 23 + 24 = 300 \text{ seconds} = 5 \text{ minutes},$$

during which time the ant stepped on the point  $x = 10$  exactly 5 times.

**CC347.** Find the sum of all fractions  $p/q$  between 0 and 1 that have denominator 100 when expressed in lowest terms.

*Originally Problem 9 of Game 1 from the 2017-18 Nova Scotia Math League.*

*We received 8 correct submissions. We present two of the solutions.*

*Solution 1, by Henry Ricardo.*

The number of positive integers relatively prime to 100 is

$$\phi(100) = 100(1 - 1/2)(1 - 1/5) = 40,$$

where  $\phi$  denotes Euler's totient function. Note that if a numerator  $k \in \{1, 2, \dots, 99\}$  is relatively prime to 100, then so is  $100 - k$ , thus we have  $\phi(100)/2 = 20$  pairs  $\{k, 100 - k\}$  of numerators, each pair adding to 100. Therefore the sum of all fractions  $p/q$  between 0 and 1 that have denominator 100 when expressed in lowest terms is  $(20 \cdot 100)/100 = 20$ .

*Solution 2, by Kathleen E. Lewis.*

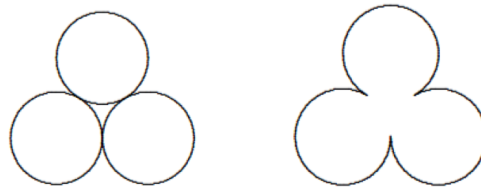
Since all of the fractions have the same denominator, we just need to add up the numerators and divide the sum by 100. The numerators we want are those that are relatively prime to 100, thus have no factor of 2 or 5. Using inclusion-exclusion,

we start with all the positive integers less than or equal to 100, then subtract the multiples of 2 and the multiples of 5. Finally we need to add the multiples of 10, as they have been subtracted twice. So we get

$$\begin{aligned} & \frac{100 \cdot 101}{2} - 2 \cdot \frac{50 \cdot 51}{2} - 5 \cdot \frac{20 \cdot 21}{2} + 10 \cdot \frac{10 \cdot 11}{2} \\ &= \frac{100}{2}(101 - 51 - 21 + 11) \\ &= 50 \cdot 40 = 2000. \end{aligned}$$

Dividing by 100, we obtain 20 for the sum of the fractions.

**CC348.** Three circles with the same radius  $r$  are mutually tangent as shown on the left figure. The arcs in the middle are removed, making a trefoil (right figure). Determine the exact length of the trefoil in terms of  $r$ .



*Originally from the 2017-18 Final Game of the Newfoundland and Labrador Teachers' Association Senior Math League.*

*We received 10 submissions, all correct. We present the solution provided by Richard Hess.*

Each of the three circles has a circumference of  $2\pi r$ , so the total length before removal is  $6\pi r$ . The removed segments add to  $\pi r$ , giving the arc length of the trefoil as  $5\pi r$ .

**CC349.** Let  $O$  be the centre of equilateral triangle  $ABC$  (i.e. the unique point equidistant from each vertex). Another point  $P$  is selected uniformly at random in the interior of  $\triangle ABC$ . Find the probability that  $P$  is closer to  $O$  than it is to any of  $A$ ,  $B$  or  $C$ .

*Originally Problem 10 of Game 3 from the 2017-18 Nova Scotia Math League.*

*We received 3 submissions, all correct. We present the solution provided by Richard Hess.*

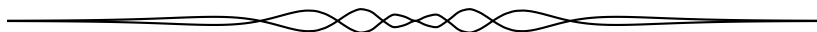
The perpendicular bisectors of  $OA$ ,  $OB$  and  $OC$  produce a regular hexagon inside the equilateral triangle. This hexagon has an area twice the sum of the three small triangular areas outside the hexagon but inside the large triangle. Thus  $P$  has a probability of  $2/3$  of being closer to  $O$  than any of  $A$ ,  $B$  or  $C$ .

**CC350.** A friend proposes the following guessing game: He chooses an integer between 1 and 100, inclusive, and you repeatedly try to guess his number. He tells you whether each incorrect guess is higher or lower than his chosen number, but you are allowed at most one high guess overall. You win the game when you guess his number correctly. You lose the game the instant you make a second high guess. What is the minimum number of guesses in which you can guarantee you will win the game?

*Originally Problem 10 of Game 3 from the 2011-12 Nova Scotia Math League.*

*We received 3 submissions of which only one was correct. We present the solution by Richard Hess.*

I can guess the number in 14 or fewer guesses if the number is 1 to 105 inclusive. My first guess is 14. If high, my next 13 guesses are 1 to 13. If low, my second guess is 27. If this is high, my next guesses are 15-26. If my second guess is low, my third guess is 39. If I continue to guess low, my next guesses are 50, 60, 69, 77, 84, 90, 95, 99, 102, 104, and 105. If any of these is a high guess, then the remaining guesses are enough to guess all possibilities from low to high in the gap between that guess and the prior low guess.



# PROBLEM SOLVING VIGNETTES

No.5

Shawn Godin

Introducing Induction

Mathematical induction is a very powerful technique used to show that a certain property holds for infinitely many cases. It is based on a nice piece of logic where we show that if the property holds for a case then it must hold for the “next” case. This induction step then can be used recursively to show that if the property holds for  $n = 1$ , then it must hold for  $n = 2$ , but then it must hold for  $n = 3$  and so on. We will examine induction in action by trying to prove that the sum of the first  $n$  positive integers is  $\frac{n(n+1)}{2}$ .

We may have come up with the expression  $\frac{n(n+1)}{2}$  by playing with a few cases, realizing it must be a quadratic because the second differences would be constant (why?). To show it is true for all positive integers  $n$ , we will first show it is true for a specific case:  $n = 1$ . In this case our “sum” is just 1, while our expression gives  $\frac{(1)(1+1)}{2} = 1$ , which shows the expression is good in this case.

Next we will do our induction step, we will *assume* that the statement works for some  $n = k$ . We can do this, because we know there is at least one case where the expression gives the correct result, that is  $n = 1$ . So if the statement is true for  $n = k$ , then we must have

$$\sum_{i=1}^k i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2} \quad (1)$$

so then

$$\begin{aligned} \sum_{i=1}^{k+1} i &= 1 + 2 + \cdots + k + (k+1) \\ &= \sum_{i=1}^k i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{by our assumption (1)} \\ &= (k+1) \left[ \frac{k}{2} + 1 \right] \\ &= \frac{(k+1)(k+2)}{2} = \frac{(\boxed{k+1})(\boxed{k+1}+1)}{2} \end{aligned} \quad (2)$$

which is just our formula (1) with  $k$  replaced by  $k + 1$ . This shows that if the formula holds for some positive integer  $n = k$ , then it holds for the next positive integer  $n = k + 1$ . When we couple this with our demonstration that the formula works for  $n = 1$ , it shows that the property is true for all positive integers  $n$ .  $\square$

Some people prefer to refer to the thing that we are to prove as a proposition,  $P(n)$ , since the proposition is being made for various positive integers,  $n$ . In our case we would define

$$P(n) : 1 + 2 + 3 + \cdots + n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

which is a *boolean* function. It is either *true* or *false* for any particular  $n$ .

Thus our proof goes by showing  $P(1)$  is true, that is  $1 = \frac{(1)(1+1)}{2}$  then assuming  $P(k)$  is true for some  $k$  (we can do this since we know it is true for at least  $k = 1$ ), which was (1). Then we try to show  $P(k) \Rightarrow P(k+1)$ , which we showed with (2).

Induction works like a row of dominoes. The induction step  $P(k) \Rightarrow P(k+1)$ , says “if one domino falls over it will knock the next one over”. Then  $P(1)$  says “knock over the first domino” and away it goes! Domino #1 knocks over domino #2, which knocks over domino #3, .... The difference is with a mathematical proposition we have infinitely many dominoes!

Take a moment and try your hand at a proof by induction. The statement below is similar to the one we just did. Prove

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}. \quad (3)$$

The proof of this will appear at the end of the column.

We will now examine another proof by induction. The problem below is problem 10b from the 2019 Euclid contest put on by the CEMC. You can check out past CEMC contests on their website [www.cemc.uwaterloo.ca](http://www.cemc.uwaterloo.ca).

*Carlos has 14 coins, numbered 1 to 14. Each coin has exactly one face called “heads”. When flipped, coins 1, 2, 3, ..., 13, 14 land heads with probabilities  $h_1, h_2, h_3, \dots, h_{13}, h_{14}$ , respectively. When Carlos flips each of the 14 coins exactly once, the probability that an even number of coins land heads is exactly  $\frac{1}{2}$ . Must there be a  $k$  between 1 and 14, inclusive, for which  $h_k = \frac{1}{2}$ ? Prove your answer.*

We will actually prove the more general situation and use that to show that the specific case, with 14 coins, must then be true. Thus we will prove the following:

*If we have  $2n$  coins, with probabilities for flipping heads  $h_1, h_2, \dots, h_{2n}$  and the probability of flipping an even number of heads is  $\frac{1}{2}$  then there exists some  $k$ , with  $1 \leq k \leq 2n$ , such that  $h_k = \frac{1}{2}$ .*

Define  $E(2n)$  and  $O(2n)$  to be the probabilities of flipping an even number of heads and an odd number of heads, respectively, on a flip of  $2n$  coins with probabilities of heads for each coin  $h_1, h_2, \dots, h_{2n}$ . Then clearly  $E(2n) + O(2n) = 1$  implies

$$\text{if } E(2n) = \frac{1}{2}, \text{ then } O(2n) = \frac{1}{2}$$

for all  $n$  since the events are mutually exclusive and there is no other possibility.

Examining the case where  $n = 1$ , we have two coins with  $E(2) = O(2) = \frac{1}{2}$ . Hence, if we have an even number of heads we have both coins showing heads or neither coin showing heads. Thus

$$\begin{aligned} E(2) = \frac{1}{2} &\Rightarrow h_1 h_2 + (1 - h_1)(1 - h_2) = \frac{1}{2} \\ &\Rightarrow 2h_1 h_2 - h_1 - h_2 + 1 = \frac{1}{2} \\ &\Rightarrow 4h_1 h_2 - 2h_1 - 2h_2 + 2 = 1 \\ &\Rightarrow 4h_1 h_2 - 2h_1 - 2h_2 + 1 = 0 \\ &\Rightarrow (2h_1 - 1)(2h_2 - 1) = 0 \end{aligned}$$

hence  $h_1 = \frac{1}{2}$  or  $h_2 = \frac{1}{2}$  and our proposition is true for  $n = 1$ .

Now we will assume that the proposition is true for some  $n = k$ , that is we assume that if  $E(2k) = \frac{1}{2}$ , then at least one coin has probability of heads of  $\frac{1}{2}$ . It follows that if  $E(2(k+1)) = \frac{1}{2}$ , we will break our coins into two groups with probabilities  $p_1, p_2, \dots, p_{2k}$  and  $p_{2k+1}, p_{2k+2}$  and we will let  $E(2k)$  be the probability of flipping an even number of heads with the first  $2k$  coins.

To have an even number of coins showing heads, we have either both or neither coins  $2k+1$  and  $2k+2$  are heads and an even number of the first  $2k$  coins are heads; or exactly one of the coins  $2k+1$  and  $2k+2$  is heads and an odd number of the first  $2k$  coins are heads. Since  $E(2(k+1)) = \frac{1}{2}$ , we have

$$\begin{aligned} &(p_{2k+1}p_{2k+2} + (1 - p_{2k+1})(1 - p_{2k+2}))E(2k) \\ &\quad + (p_{2k+1}(1 - p_{2k+2}) + (1 - p_{2k+1})p_{2k+2})O(2k) = \frac{1}{2}, \\ &(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 2)E(2k) \\ &\quad + (2p_{2k+1} + 2p_{2k+2} - 4p_{2k+1}p_{2k+2})O(2k) = 1, \\ &(4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 + 1)E(2k) \\ &\quad - (4p_{2k+1}p_{2k+2} - 2p_{2k+1} - 2p_{2k+2} + 1 - 1)O(2k) = 1, \\ &(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) + E(2k) + O(2k) = 1, \end{aligned}$$



but since  $E(2k) + O(2k) = 1$  we have

$$(2p_{2k+1} - 1)(2p_{2k+2} - 1)(E(2k) - O(2k)) = 0$$

thus

$$2p_{2k+1} - 1 = 0 \Rightarrow p_{2k+1} = \frac{1}{2} \quad \text{or,}$$

$$2p_{2k+2} - 1 = 0 \Rightarrow p_{2k+2} = \frac{1}{2} \quad \text{or,}$$

$$E(2k) - O(2k) = 0 \Rightarrow E(2k) = O(2k) \Rightarrow E(2k) = O(2k) = \frac{1}{2}.$$

So either the probability of at least one of coins  $2k+1$  and  $2k+2$  have probability of  $\frac{1}{2}$  or the probability of there being an even number of heads in the first  $2k$  coins is  $\frac{1}{2}$ . But by our assumption if any set of  $2k$  coins has  $E(2k) = \frac{1}{2}$  then at least one of the coins has probability of  $\frac{1}{2}$ , which shows it works for  $n = k+1$  and hence the property is true for all positive integers  $n$  and hence must be true for  $n = 7$ , which is the original problem.  $\square$

Our proof showed that for any even number of coins, if the probability of flipping an even number of heads is  $\frac{1}{2}$  then at least one coin has probability of heads being  $\frac{1}{2}$ . It turns out that we were being too specific, and it can be shown that the property works when we have an odd number of coins as well. If you look at the official solution from the CEMC, they use a method of descent that is similar to induction. As an exercise, you may want to prove that the property holds no matter how many coins we have to start. The proof should be a bit easier than the one we presented.

Mathematical induction is a useful technique to add to your problem solving toolkit. Look for places that you can use it!

Ok, here is the proof of (3), namely that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since

$$\frac{(1)((1)+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1 = 1^2$$

the statement is true for  $n = 1$ .

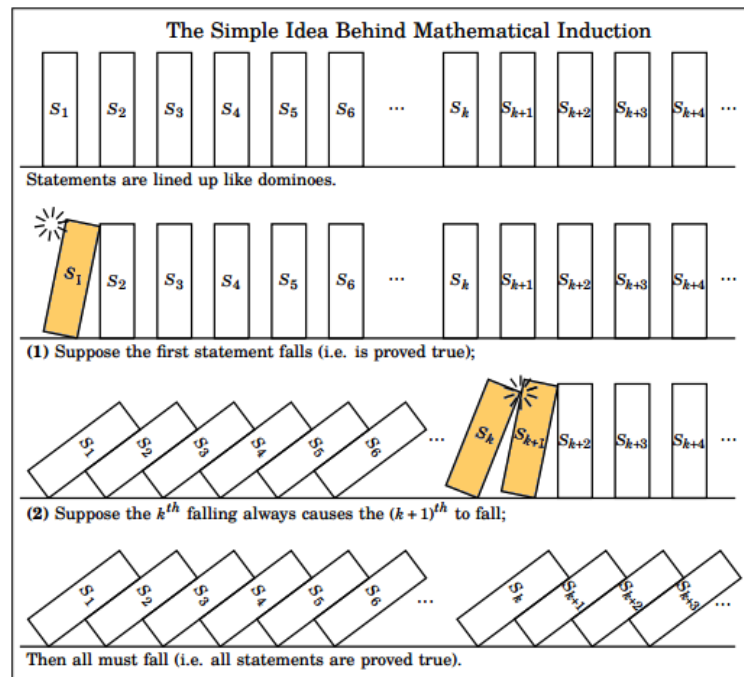
Let's assume the statement is true for  $n = k$ , that is

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}. \quad (4)$$

Then, we must have

$$\begin{aligned}
 \sum_{i=1}^{k+1} i^2 &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
 &= \sum_{i=1}^{k+1} i^2 + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by (4)} \\
 &= \frac{k+1}{6} (k(2k+1) + 6(k+1)) \\
 &= \frac{k+1}{6} (2k^2 + 7k + 6) \\
 &= \frac{k+1}{6} (k+2)(2k+3) \\
 &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
 \end{aligned}$$

which means the statement is true for  $n = k + 1$  and hence the statement is true for all positive integers by induction.  $\square$



# TEACHING PROBLEMS

No.2

John McLoughlin

## The Marching Band Problem



A marching band is attempting to get organized for a performance in a local parade. Unfortunately, the band is having difficulty getting lined up correctly. When the members line up in twos, there is one person left over. When they line up in threes, there are two people left over. When they line up in fours, there are three people left over. When they line up in fives, there are four people left over. When they line up in sixes, there are five people left over. Finally, they discover that when they line up in sevens, they line up neatly with nobody left over. What is the smallest possible number of people in the marching band?



Some readers may look at the Marching Band Problem and say that is not a problem. However, if this type of question is unfamiliar, it ought to represent a problem. If it is a problem, take some time to at least make sure you understand the problem. Play with it (or ideally solve it) before reading further along here. On the other hand, what if the form of this question is familiar? Then you are encouraged to outline how you would solve it before challenging yourself to solve the question in at least one other way.

This problem invites a variety of approaches. Indeed this is one of the merits of this problem. Further, there is a teasing element to this problem as one of these ideas can lead us so far without necessarily being straightforward to bring to conclusion. For instance, my experience is that students sense (correctly too) a connection with the idea of a lowest common multiple but struggle with actually applying that idea in the solution. Something is amiss in their efforts as the remainders are nonzero, thus, seemingly a bit out of step with their understanding of multiples.

This problem has many interesting features that add to its value for teaching. A brief discussion of these features here precedes the overriding quality of multiple approaches to be discussed subsequently with various forms of solution.

- Various mathematical concepts can be brought into play in the discussion and solution of the problem. Among these are multiples, divisibility, and modular arithmetic. Generally the problem promotes application and development of number sense.
- The problem is easy to understand. Accessibility is not a concern, thus, encouraging engagement with the problem at many levels. Brute force and/or trial and error have a place here in terms of both understanding the question and motivating insight to enable elegance in solution.

- The presence of redundant information is valuable. Not everything stated in the problem is offering new information. Recognition of such redundancies is an underappreciated skill in mathematical problem solving.
- The problem lends itself to engagement as people do not quickly see the solution and hence, there is time to delve into the problem at the various levels.
- Extensions or variations of this problem are relatively easy to develop. This allows for differentiating within a classroom setting, or even allowing students to create their own challenges for sharing with peers.

Let us turn our attention to some of the ways of solving this problem.

### The Last Digit Approach

The last digit of the number of band members must be 4 or 9, as there are four people left over when lined up in fives. However, the number of band members is odd as it is not divisible by 2. Hence, the final digit must be 9.

Brute force can result in people trying out all numbers ending in 9 until a result is found, and it will work. Rather, consider multiples of 7 that end in 9. Note that  $7 \cdot 7$  results in a product ending in 9. However, checking 49 we find that it does not meet the requirement when grouped in threes.

No other multiple of 7 less than 70 ends in 9, and it follows that the next number to check is  $17 \cdot 7 = 119$ . Checking we find that 119 satisfies all of the conditions. The smallest number of people in the band is 119.

Readers who are learning about congruences through recent issues of *MathemAttic* may wish to convince themselves that the numbers  $n = 0, 1, 2, 3, \dots, 9$  each produce a different remainder when considered as  $7n \pmod{10}$ . Of particular interest here is the fact that when  $n = 7$ , the result is  $9 \pmod{10}$ .

### A note on redundant information before proceeding further

Note that any number that leaves a remainder of 3 when divided by 4 must also leave a remainder of 1 when divided by 2. Further, any number that leaves a remainder of 5 when divided by 6 must leave a remainder of 2 when divided by 3. The initial two conditions stated in the problem can be removed as they are satisfied by default, so to speak.

This fact will be applied in each of the following methods of solution as a given, thus, making the problem one that is reduced to four conditions rather than six.

### The Lowest Common Multiple (LCM) Connection

Begin by noting that the addition of one band member would make the number of members divisible by 4, 5, and 6. So here is the leap: the number of people in the band must be 1 less than a number divisible by each of 4, 5, and 6.

The lowest common multiple of 2, 3, 4, 5, and 6 is 60. So the numbers to consider begin with 59 and there is only one question to answer, “Is this number divisible by 7?” The answer is no, so go up 60 and check 119 or  $2 \cdot 60 - 1$ . Aha! It works.

### Applying Congruences

A solution using congruences and modular arithmetic is offered here. Readers may wish to use tables to verify some of the results along the process.

We need to find a value  $n$  that satisfies four congruences:

$$n \equiv 3 \pmod{4}; n \equiv 4 \pmod{5}; n \equiv 5 \pmod{6}; n \equiv 0 \pmod{7}$$

Since  $n \equiv 0 \pmod{7}$ , we can represent  $n = 7t$  for some integer  $t$ .

Continuing we write  $7t \equiv 5 \pmod{6}$ . Removing  $6t$  will not change the remainder, and hence, we have  $t \equiv 5 \pmod{6}$ . Therefore,  $t = 6k + 5$  giving

$$n = 7t = 7(6k + 5) = 42k + 35.$$

Now it follows that  $42k + 35 \equiv 4 \pmod{5}$ . Simplifying gives  $2k \equiv 4 \pmod{5}$ . Therefore  $k \equiv 2 \pmod{5}$  and  $k = 5m + 2$ . Substituting, we get

$$n = 42k + 35 = 42(5m + 2) + 35 = 210m + 119.$$

Finally, we require that  $210m + 119 \equiv 3 \pmod{4}$  giving  $2m + 3 \equiv 3 \pmod{4}$  or  $2m \equiv 0 \pmod{4}$ .

This final congruence is actually more difficult in that 2 (the coefficient of  $m$ ) and the modulus of 4 share a common factor other than 1, and hence we have a situation that will not have a unique solution. Let us write a table here to see this fact.

$m$	0	1	2	3
$2m \pmod{4}$	0	2	0	2

This gives us two solutions, in that  $m \equiv 0$  or  $2 \pmod{4}$ . Prior to substituting these separately, we can make an observation that any number that leaves a remainder of 0 or 2 upon division by 4 is an even number. All even numbers are solutions, and they can be represented as being  $0 \pmod{2}$ . So we can write  $m = 2b$  and then substitute into  $n = 210m + 119$  giving  $n = 420b + 119$ .

The solution to the original set of 4 congruences is given by  $n \equiv 119 \pmod{420}$ . The smallest positive integer that satisfies the congruence is 119 or the smallest number of people in the band given the conditions.

Note that the number 420 is not appearing accidentally either as it represents the LCM of 4, 5, 6, and 7. So theoretically the same conditions would apply if we increased the band size by 420 or 840 or 1260 or any multiple of 420. For example, the second smallest possible band size would be 539 people.

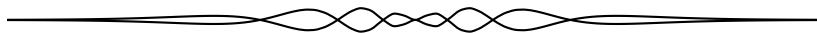
### Concluding Comments

Over the years this problem has been a rich example for me in work with teachers, as few are familiar with the problem but they can all solve it. Discussion of the problem or submissions of solutions have offered many correct answers with comments like “there must be an easier way” as partially (in)complete ideas have led to them checking every number that ends in 9, or perhaps all multiples of 7 that are odd. The appreciation of the insights shared above is enhanced through prior experience with the problem. At a secondary level (or with secondary teachers), the approach using congruences may be considered as a source of enrichment. Congruences have been featured in vignettes #3 and #4 in *MathemAttic*, and it is hoped that the inclusion of the application here will add to the growing appreciation of the value of modular arithmetic.

Prior to closing, a few suggestions are shared here. If this problem is too big for starters, it may be that reducing the number of conditions would be practical to consider. My first exposure to a band number problem involved having one person left over each time the groups were formed, but then the trivial band size of 1 had to be accounted for with a note mentioning there was more than one person in the band. Then it was thought that a requirement of exact groups for some number would take care of that. So it may be that there is one person left over when grouping in threes, fours, and fives, but no one left over when grouped in sixes. Of course, there is a problem as that final statement contradicts others as the number had to be a multiple of 6 but not a multiple of 3. Hence, going to a prime number like 7 as a factor makes it mathematically sound. Finally, it seems to be a richer problem when each group falls one short of being exact as in two left over in groups of three, three left over in groups of four and so on. There is not that immediate sense that a particular number will obviously work (like 1 if there is always one left over). The combination of these ideas exemplifies how a problem can be adapted to make another, and in this case, a better problem in my opinion.

This issue of *Teaching Problems* closes with a couple of variations that may be considered to ensure the concepts at hand are understood. Readers are encouraged to use a blend of methods in their solutions.

1. A marching band has 1 person left over when it lines up in twos, threes, fours, fives, or sixes. What is the smallest number of people in this band if it can line up with no people left over when arranged in rows of seven?
2. What is the smallest number that leaves a remainder of 1 when divided by 4, a remainder of 2 when divided by 5, a remainder of 4 when divided by 7, and no remainder when divided by 9?
3. The bandmaster claims that the band had one player left over when they tried to line up by twos, two when they tried to line up by threes, three when they tried to line up by fours, four when they tried to line up by fives, but successfully lined up by sixes. Why are you suspicious?



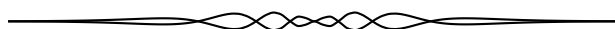
# OLYMPIAD CORNER

No. 373

*The problems featured in this section have appeared in a regional or national mathematical Olympiad.*

*Click here to submit solutions, comments and generalizations to any problem in this section*

*To facilitate their consideration, solutions should be received by **August 15, 2019**.*



**OC431.** All natural numbers greater than 1 are coloured with blue or red so that the sum of every two blue numbers (not necessarily distinct) is blue, and the product of every two red ones (not necessarily distinct) is red. It is known that the number 1024 is blue. What colour can the number 2017 be?

**OC432.** Find the smallest natural number that is a multiple of 80 such that you can rearrange two of its distinct digits and the resulting number will also be a multiple of 80.

**OC433.** Consider an isosceles trapezoid  $ABCD$  with bases  $AD$  and  $BC$ . A circle  $\omega$  passing through  $B$  and  $C$  intersects the side  $AB$  and the diagonal  $BD$  at points  $X$  and  $Y$ , respectively. The tangent to  $\omega$  at  $C$  intersects the line  $AD$  at  $Z$ . Prove that the points  $X$ ,  $Y$ , and  $Z$  are collinear.

**OC434.** The acute isosceles triangle  $ABC$  ( $AB = AC$ ) is inscribed in a circle with center  $O$ . The rays  $BO$  and  $CO$  intersect the sides  $AC$  and  $AB$  at the points  $B'$  and  $C'$ , respectively. A line  $l$  parallel to the line  $AC$  passes through point  $C'$ . Prove that the line  $l$  is tangent to the circumcircle  $\omega$  of the triangle  $B'OC$ .

**OC435.** There are  $n$  positive numbers  $a_1, a_2, \dots, a_n$  written on a blackboard. Under each number  $a_i$ , Vasya wants to write a number  $b_i \geq a_i$  so that for every pair of numbers chosen from  $b_1, b_2, \dots, b_n$ , the ratio of one of them to the other is an integer. Prove that Vasya can write out the required numbers so that

$$b_1 b_2 \cdots b_n \leq 2^{(n-1)/2} a_1 a_2 \cdots a_n.$$

.....

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

*Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.



**OC431.** Tous les nombres naturels supérieurs à 1 sont colorés en bleu ou en rouge de sorte que la somme de deux nombres bleus (pas nécessairement distincts) est bleue et le produit de deux nombres rouges (pas nécessairement distincts) est rouge. On sait que le nombre 1024 est bleu. Quelle est la couleur de 2017 ?

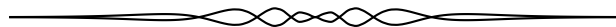
**OC432.** Trouvez le plus petit nombre naturel qui est un multiple de 80 tel que si deux de ses chiffres distincts sont réarrangés, le résultat est toujours un multiple de 80.

**OC433.** Considérez le trapèze isocèle  $ABCD$  dont les bases sont  $AD$  et  $BC$ . Un cercle  $\omega$  passant par  $B$  et  $C$  intercepte le côté  $AB$  et la diagonale  $BD$  aux points  $X$  et  $Y$ , respectivement. La tangente à  $\omega$  au point  $C$  intercepte le segment  $AD$  au point  $Z$ . Prouvez que  $X$ ,  $Y$ , et  $Z$  sont colinéaires.

**OC434.** Le triangle isocèle acutangle  $ABC$  ( $AB = AC$ ) est inscrit dans un cercle de centre  $O$ . Les rayons  $BO$  et  $CO$  interceptent les côtés  $AC$  et  $AB$  aux points  $B'$  et  $C'$ , respectivement. Un segment  $I$  parallèle au segment  $AC$  passe par le point  $C'$ . Prouvez que le segment  $I$  est tangent au cercle circonscrit  $\omega$  du triangle  $B'OC$ .

**OC435.** Il y a  $n$  nombres positifs  $a_1, a_2, \dots, a_n$  écrits sur un tableau. Sous chaque nombre  $a_i$ , Vasya veut écrire un nombre  $b_i \geq a_i$  tel que pour toute paire de nombres choisis parmi  $b_1, b_2, \dots, b_n$ , le quotient de un par l'autre est un entier. Prouvez que Vasya peut écrire ces nombres tels que

$$b_1 b_2 \cdots b_n \leq 2^{(n-1)/2} a_1 a_2 \cdots a_n.$$





# OLYMPIAD CORNER SOLUTIONS

*Statements of the problems in this section originally appear in 2018: 44(6), p. 243–244; and 44(7), p. 284–285.*



**OC386.** Find all monic polynomials  $P, Q$  which are non-constant, have real coefficients and satisfy

$$2P(x) = Q\left(\frac{(x+1)^2}{2}\right) - Q\left(\frac{(x-1)^2}{2}\right)$$

and  $P(1) = 1$  for all real  $x$ .

*Originally Problem 2 of the 2016 Greece National Olympiad.*

*We received 8 submissions, of which 6 were correct and complete. We present the solution by Oliver Geupel, slightly modified.*

It is straightforward to check that, for every real number  $c$ , the following pairs of polynomials  $P, Q$  are solutions to the problem:

$$P(x) = x, \quad Q(x) = x + c, \quad (1)$$

and

$$P(x) = x^3, \quad Q(x) = x^2 - x + c. \quad (2)$$

We prove that there are no other solutions.

Suppose polynomials  $P, Q$  satisfy the conditions of the problem where  $n = \deg Q$ . Since  $Q$  is monic, it has the form  $Q(x) = x^n + R(x)$  with a polynomial  $R$  of degree less than or equal to  $n - 1$ . Then,

$$P(x) = \frac{1}{2} \left( \frac{(x+1)^{2n}}{2^n} - \frac{(x-1)^{2n}}{2^n} + R\left(\frac{(x+1)^2}{2}\right) - R\left(\frac{(x-1)^2}{2}\right) \right).$$

Expanding the powers of the binomials and observing that  $\deg R \leq n - 1$ , we obtain that  $P$  has the degree  $2n - 1$  and leading coefficient  $2n/2^n$ . Since  $P$  is non-constant and monic, we obtain  $n > 0$  and  $2n = 2^n$ , that is,  $n \in \{1, 2\}$ .

If  $n = 1$ , we have  $Q(x) = x + c$  and

$$P(x) = \frac{1}{2} \left( \frac{(x+1)^2}{2} - \frac{(x-1)^2}{2} \right) = \frac{1}{2} \cdot 2x = x,$$

which gives the solution (1).

If  $n = 2$ , the polynomial  $Q$  has the form  $Q(x) = x^2 + ax + c$ , with constants  $a$  and  $c$ . Hence,

$$\begin{aligned} P(x) &= \frac{1}{2} \left( \left( \frac{(x+1)^2}{2} \right)^2 - \left( \frac{(x-1)^2}{2} \right)^2 + a \left( \frac{(x+1)^2}{2} - \frac{(x-1)^2}{2} \right) \right) \\ &= x^3 + (a+1)x. \end{aligned}$$

The hypothesis  $P(1) = 1$  leads to  $a = -1$  and thus to the solution (2).

**OC387.** Let  $X_1, X_2, \dots, X_{100}$  be a sequence of mutually distinct nonempty subsets of a set  $S$ . Any two sets  $X_i$  and  $X_{i+1}$  are disjoint and their union is not the whole set  $S$ , that is,  $X_i \cap X_{i+1} = \emptyset$  and  $X_i \cup X_{i+1} \neq S$ , for all  $i \in \{1, \dots, 99\}$ . Find the smallest possible number of elements in  $S$ .

*Originally Problem 1, Day 1 of the 2016 USA Math Olympiad.*

*We received 3 submissions. We present the solution by Oliver Geupel.*

The smallest number of elements in  $S$  is 8.

We present a sequence  $X_1, X_2, \dots, X_{100}$  for  $S = \{1, 2, \dots, 8\}$ . Let  $A_0, \dots, A_9$  and  $B_0, \dots, B_{10}$  be the following subsets of  $\{1, 2, 3, 4\}$  and  $\{5, 6, 7, 8\}$ , respectively:

$$\begin{aligned} A_0 &= \emptyset, & A_1 &= \{2, 4\}, & A_2 &= \{1\}, & A_3 &= \{3, 4\}, & A_4 &= \{2\}, \\ A_5 &= \{1, 4\}, & A_6 &= \{3\}, & A_7 &= \{1, 2\}, & A_8 &= \{4\}, & A_9 &= \{2, 3\}, \end{aligned}$$

$$\begin{aligned} B_0 &= \emptyset, & B_1 &= \{5, 7\}, & B_2 &= \{6, 8\}, & B_3 &= \{5\}, & B_4 &= \{7, 8\}, \\ B_5 &= \{6\}, & B_6 &= \{5, 8\}, & B_7 &= \{7\}, & B_8 &= \{5, 6\}, & B_9 &= \{8\}, \\ B_{10} &= \{6, 7\}. \end{aligned}$$

For  $k \in \{1, 2, \dots, 100\}$ , let  $X_k = A_r \cup B_s$  if  $k \equiv r \pmod{10}$  and  $k \equiv s \pmod{11}$ .

By the Chinese Remainder Theorem, the  $X_k$  are mutually distinct. Since the simultaneous congruences  $k \equiv 0 \pmod{10}$  and  $k \equiv 0 \pmod{11}$  have no solution in the range  $k \in \{1, \dots, 99\}$ , the  $X_k$  are nonempty. Let denote  $A_{10} = A_0$  and  $B_{11} = B_0$ . For all  $i \in \{0, \dots, 9\}$ , the two sets  $A_i$  and  $A_{i+1}$  are disjoint, and for all  $j \in \{0, \dots, 10\}$ , the two sets  $B_j$  and  $B_{j+1}$  are disjoint. Hence, any two sets  $X_k$  and  $X_{k+1}$  are disjoint, for all  $k \in \{1, \dots, 99\}$ . For all  $i \in \{0, \dots, 9\}$ , the set  $A_i \cup A_{i+1}$  has at most 3 elements, and for all  $j \in \{0, \dots, 10\}$ , the set  $B_j \cup B_{j+1}$  has at most 4 elements. Thus,  $X_k \cup X_{k+1} \subsetneq S$  for all  $k$ . We have proven that  $X_1, \dots, X_{100}$  is a sequence with the required properties.

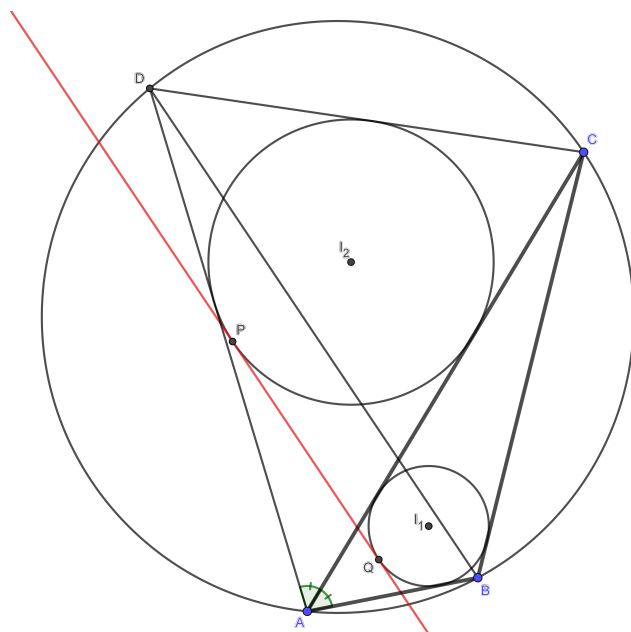
It remains to show that  $|S| \leq 7$  is impossible. The proof is by contradiction. Suppose  $X_1, X_2, \dots, X_{100}$  is a sequence with the desired properties for  $S$  with  $|S| \leq 7$ . We may in fact assume that  $|S| = 7$  (in fact, since  $2^{|S|} \geq 100$ , it must be  $|S| \geq 7$ ). The number of three-element sets  $X_i$  is at most  $\binom{7}{3} = 35$ . Every  $X_i$  with at least four elements is followed by  $X_{i+1}$  with one or two elements (except when  $i = 100$ ). The number of one- or two-element subsets of  $S$  is  $\binom{7}{1} + \binom{7}{2} = 28$ . It

follows that the number of sets  $X_i$  with four or more elements is not greater than 29. Hence, the number of distinct sets  $X_i$  in the sequence is at most  $35 + 28 + 29 = 92 < 100$ . This is the desired contradiction.

**OC388.** Let  $ABCD$  be a cyclic quadrilateral with  $\angle BAC = \angle DAC$ . Suppose  $I_1$  and  $I_2$  are the incircles of  $\triangle ABD$  and  $\triangle ADC$  respectively. Prove that one of the common external tangents of  $I_1$  and  $I_2$  is parallel to  $BD$ .

*Originally Problem 7, Day 2 of the 2016 China Western Mathematical Olympiad.*

*We received 2 submissions. We present the solution by Andrea Fanchini.*



We use barycentric coordinates with reference to the triangle  $ABC$ .

We denote with  $\Gamma$  the circumcircle of triangle  $ABC$ , then the point  $D$  is

$$D = AAD_{\infty} \cap \Gamma = (2a^2S_A : b^2(S_B - S_A) : 2S_A(S_A - S_B))$$

so the infinite point of line  $BD$  is

$$BD_{\infty}(a^2 : -b^2 : b^2 - a^2).$$

Centers and radii of the incircles  $I_1$  and  $I_2$  are

$$I_1 : (a : b : c), \quad \rho_1 = r,$$

$$I_2 : (a(c^2 - a^2 + ab) : b^2(a - b) : (b - a)(bc + S_A)), \quad \rho_2 = r \frac{s(b - a)}{c(s - a)}.$$

Then their common external tangent  $PQ$  is

$$PQ : b(S_B - S_A)x + a(S_A - S_B)y + ab(a + b)z = 0,$$

that has infinite point

$$PQ_\infty(a^2 : -b^2 : b^2 - a^2) \equiv BD_\infty.$$

**OC389.** Let  $n$  be a positive integer. In a kingdom there are  $2^n$  citizens and a king. In terms of currency, the kingdom uses paper bills with value  $2^n$  and coins with value  $2^a$  with  $a = 0, 1, \dots, n-1$ . Every citizen has infinitely many paper bills. Let the total number of coins in the kingdom be  $S$ . One fine day, the king decided to implement a policy which is to be carried out every night:

- each citizen must decide on a finite amount of money based on the coins that he/she currently has, and he/she must pass that amount to either another citizen or the king;
- each citizen must pass exactly 1 more than the amount he/she received from other citizens.

Find the minimum value of  $S$  which will guarantee that the king will be able to collect money every night eternally.

*Originally Problem 3 of the 2016 Japan Mathematical Olympiad Finals.*

*We received 1 incomplete submission.*

**OC390.** Let  $n \geq 2$  be an integer. Find the least value of  $\gamma$  which satisfies the inequality

$$x_1 x_2 \cdots x_n \leq \gamma (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)$$

for any positive real numbers  $x_1, x_2, \dots, x_n$  with  $x_1 + x_2 + \cdots + x_n = 1$  and any real numbers  $y_1, y_2, \dots, y_n$  with  $y_1 + y_2 + \cdots + y_n = 1$  and  $0 \leq y_1, y_2, \dots, y_n \leq \frac{1}{2}$ .

*Originally Problem 6, Day 2 of the 2016 Spain Mathematical Olympiad.*

*We received 2 submissions. We present the solution by the IISER Mohali Problem Solving Group.*

For  $n = 2$ , one can easily verify that  $\gamma = \frac{1}{2}$ . Henceforth, we shall assume  $n > 2$ . Without any loss of generality, we may assume that  $x_n \leq x_{n-1} \leq \cdots \leq x_1$ . Let  $S$  denote the set

$$\begin{aligned} \{(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \in \mathbb{R}^{2n} : \\ 0 < x_n \leq x_{n-1} \leq \cdots \leq x_1 < 1, x_1 + x_2 + \cdots + x_n = 1 \text{ and} \\ 0 \leq y_1, y_2, \dots, y_n \leq \frac{1}{2}, y_1 + y_2 + \cdots + y_n = 1\}. \end{aligned}$$

Now consider the following lemma:

*Lemma.* For all  $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \in S$ , we have

$$x_1 y_1 + x_2 y_2 + \dots + x_n y_n \geq \frac{1}{2}(x_n + x_{n-1}).$$

*Proof.* We will show that if  $y_n$  and  $y_{n-1}$  are less than  $\frac{1}{2}$ , then the value of the expression  $x_1 y_1 + x_2 y_2 + \dots + x_n y_n$  cannot be smaller than  $\frac{1}{2}(x_n + x_{n-1})$ . For real numbers  $s_1, s_2, \dots, s_{n-2}$  and  $t_1, t_2, \dots, t_{n-2}$  with  $0 \leq s_i, t_i \leq \frac{1}{2}$  for all  $i$ , we see that

$$\begin{aligned} & \left( \frac{1}{2} - \sum_{i=1}^{n-2} s_i \right) x_n + \left( \frac{1}{2} - \sum_{i=1}^{n-2} t_i \right) x_{n-1} + \sum_{i=1}^{n-2} (s_i + t_i) x_i \\ &= \frac{1}{2}(x_n + x_{n-1}) + \sum_{i=1}^{n-2} s_i (x_i - x_n) + \sum_{i=1}^{n-2} t_i (x_i - x_{n-1}) \\ &\geq \frac{1}{2}(x_n + x_{n-1}) \end{aligned} \quad (1)$$

where the last inequality follows from assumption  $x_n \leq x_{n-1} \leq \dots \leq x_1$ .  $\square$

Define a function  $F : S \rightarrow \mathbb{R}$  as

$$F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = \frac{x_1 x_2 \dots x_n}{x_1 y_1 + x_2 y_2 + \dots + x_n y_n}. \quad (2)$$

Then, in view of the above lemma, we have

$$F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \leq \frac{x_1 x_2 \dots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}}. \quad (3)$$

Notice that  $F$  is bounded. Indeed, from the AM-GM inequality we get

$$\frac{x_n x_{n-1}}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \leq \sqrt{x_n x_{n-1}}$$

so that combining this with inequality (2) gives

$$F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) \leq x_1 x_2 \dots x_{n-2} \sqrt{x_n x_{n-1}} < 1.$$

This shows that the supremum of  $F$  exists. Further, finding such a least number  $\gamma$  (as in the problem) amounts to finding the supremum of the function  $F$ .

Writing  $\alpha = x_n + x_{n-1}$ , we see that  $x_1 + x_2 + \dots + x_{n-2} = 1 - \alpha$ , and so from the AM-GM inequality, we have

$$x_n x_{n-1} \leq \frac{\alpha^2}{4} \quad \text{and} \quad x_1 x_2 \dots x_{n-2} \leq \left( \frac{1 - \alpha}{n - 2} \right)^{n-2}.$$

Therefore,

$$\frac{x_1 x_2 \dots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \leq \frac{\alpha}{2} \left( \frac{1 - \alpha}{n - 2} \right)^{n-2} \quad (4)$$

with equality if and only if

$$x_n = x_{n-1} = \frac{\alpha}{2} \quad \text{and} \quad x_1 = x_2 = \cdots = x_{n-2} = \frac{1-\alpha}{n-2}.$$

Since  $x_n \leq x_1$ , we must have  $\frac{\alpha}{2} \leq \frac{1-\alpha}{n-2}$  or  $\alpha \leq \frac{2}{n}$ . This means that  $\alpha \in (0, \frac{2}{n}]$ . Consider the function  $f : [0, 1] \rightarrow \mathbb{R}$  defined by

$$f(x) = \frac{x}{2} \left( \frac{1-x}{n-2} \right)^{n-2}.$$

Since  $f$  is a continuous function on a closed and bounded interval, it must assume its maximum value. Differentiating  $f$ , we find

$$f'(x) = \frac{1}{2} \left( \frac{1-x}{n-2} \right)^{n-2} - \frac{x}{2} \left( \frac{1-x}{n-2} \right)^{n-3}$$

so that  $f'(x) = 0$  only if  $x = 1$  or  $x = \frac{1}{n-1}$ . Moreover, we have

$$f(0) = f(1) = 0 < \frac{1}{2(n-1)^{n-1}} = f\left(\frac{1}{n-1}\right)$$

meaning that  $f$  attains its maximum value at  $x = \frac{1}{n-1}$ . But as  $0 < \frac{1}{n-1} < \frac{2}{n}$ , the maximum value of  $f$  on  $(0, \frac{2}{n}]$  is also  $\frac{1}{2(n-1)^{n-1}}$ . This in turn means that the greatest value of the quantity on the right hand side of (3) is  $\frac{1}{2(n-1)^{n-1}}$ . Combining all the inequalities, we get

$$\begin{aligned} F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) &\leq \frac{x_1 x_2 \cdots x_n}{\frac{1}{2}x_n + \frac{1}{2}x_{n-1}} \\ &\leq \frac{\alpha}{2} \left( \frac{1-\alpha}{n-2} \right)^{n-2} \\ &\leq \frac{1}{2(n-1)^{n-1}} \end{aligned}$$

with equalities when  $\alpha = \frac{1}{n-1}$ . In other words, equalities occur when

$$y_1 = y_2 = \cdots = y_{n-2} = 0, \quad y_{n-1} = y_n = \frac{1}{2}$$

and

$$x_1 = x_2 = \cdots = x_{n-2} = \frac{1}{n-1}, \quad x_{n-1} = x_n = \frac{1}{2(n-1)}.$$

Also note that for  $n = 2$ ,  $\gamma = \frac{1}{2} = \frac{1}{2(2-1)^{2-1}}$ . Therefore, the maximum value of  $F$  is  $\frac{1}{2(n-1)^{n-1}}$ , or equivalently, the smallest such value of  $\gamma$  is

$$\frac{1}{2(n-1)^{n-1}}$$

for all  $n \geq 2$ .

**OC391.** Let  $x_1, x_2, x_3, \dots$  be a sequence of positive integers such that for every pair of positive integers  $(m, n)$  we have  $x_{mn} \neq x_{m(n+1)}$ . Prove that there exists a positive integer  $i$  such that  $x_i \geq 2017$ .

*Originally Problem 6 of the 2017 Italy Mathematical Olympiad.*

*We received 1 correct submission by Oliver Geupel which is presented here.*

We prove that for every integer  $M$  there exists an index  $i$  such that  $x_i \geq M$ . The proof follows from two facts that we show below.

First, for every pair of integers  $i$  and  $j$ ,  $0 < i < j$ , the following are equivalent:

- (1) There are positive integers  $m$  and  $n$  such that  $i = mn$  and  $j = m(n+1)$ .
- (2) The difference  $j - i$  is a divisor of  $j$ .

Indeed, under (1), the number  $j - i = m$  is a divisor of  $j = m(n+1)$ , and (2) follows. Under (2), the difference  $j - i$  is a divisor of  $j - (j - i) = i$ . Therefore, we choose the positive integers  $m = j - i$  and  $n = i/(j - i)$ , such that (1) holds.

Secondly, we define a double sequence  $(a_{k,\ell})$  with  $k \in \mathbb{N}$  and  $1 \leq \ell \leq k$  as follows:

$$a_{k,\ell} = \begin{cases} 1 & \text{if } (k, \ell) = (1, 1) \\ a_{k-1,k-1}! & \text{if } 1 < k, \ell = 1 \\ a_{k-1,k-1}! + a_{k-1,\ell-1} & \text{if } 1 < k, 1 < \ell \leq k. \end{cases}$$

The first few terms of the double sequence are presented below:

$$\begin{pmatrix} a_{1,1} & & & \\ a_{2,1} & a_{2,2} & & \\ a_{3,1} & a_{3,2} & a_{3,3} & \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\ & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} 1 & & & \\ 1 & 2 & & \\ 2 & 3 & 4 & \\ 24 & 26 & 27 & 28 \\ & \dots & \dots & \dots \end{pmatrix}.$$

We establish the following property of the double sequence: for all positive integers  $k$ ,  $i$ , and  $j$ , with  $1 < k$  and  $i < j \leq k$ , the difference  $a_{k,j} - a_{k,i}$  is a divisor of  $a_{k,j}$ .

We use a proof by induction on  $k \geq 2$  to show this property. The base case  $k = 2$  is trivial, as  $a_{2,1} = 1$  and  $a_{2,2} = 2$ . Let  $k > 2$ . Notice that

$$a_{k-1,1} < a_{k-1,2} < \dots < a_{k-1,k-1}.$$

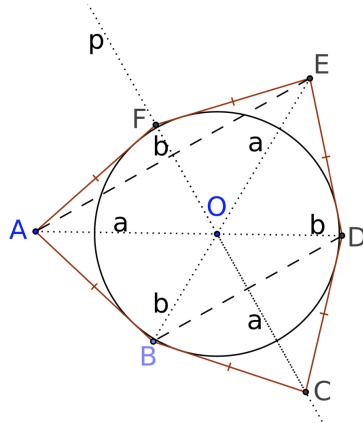
Hence, for  $1 < j \leq k$ , the difference  $a_{k,j} - a_{k,1} = a_{k-1,j-1}$  is a divisor of  $a_{k,j} = a_{k-1,k-1} + a_{k-1,j-1}$ . For  $1 < i < j \leq k$ , by induction hypothesis, the difference  $a_{k,j} - a_{k,i} = a_{k-1,j-1} - a_{k-1,i-1}$  divides  $a_{k-1,j-1}$ , which is a divisor of  $a_{k,j} = a_{k-1,k-1} + a_{k-1,j-1}$ . This completes the induction.

Using the two facts outlined above, we can proceed to solve the problem. Consider  $M$  members,  $x_i$ 's, of the given  $x$ -sequence with indices  $a_{M,1}, a_{M,2}, \dots, a_{M,M}$  that are specified by the double  $a$ -sequence. These  $M$  members are distinct positive integers. Consequently, there exists an index  $i$  such that  $x_i \geq M$ .

**OC392.** In a convex hexagon  $ABCDEF$  all sides are equal and also  $AD = BE = CF$ . Prove that a circle can be inscribed into this hexagon.

*Originally Problem 4 of Grade 8 of the 2017 Moscow Math Olympiad.*

*We received 4 correct submissions. We present the solution by Oliver Geupel.*



The triangles  $ABE$  and  $ADE$ , which have equal sides, are axially symmetric with respect to the perpendicular bisector  $p$  of the segment  $AE$ . Hence, the quadrilateral  $ABDE$  is an isosceles trapezoid with  $AE \parallel BD$ . Its diagonals intersect in a point  $O$  on the line  $p$ . Let  $OA = OE = a$  and  $OB = OD = b$ . Since  $BC = CD$  and  $EF = FA$ , the points  $C$  and  $F$  also lie on  $p$ . Thus, the diagonals  $AD$ ,  $BE$ , and  $CF$  are concurrent in  $O$ .

By an argument similar to that for  $ABDE$ , the quadrilateral  $BCEF$  is an isosceles trapezoid. Therefore,  $OC = a$  and  $OF = b$ . As a consequence, the triangles  $ABO$ ,  $CBO$ ,  $CDO$ ,  $EDO$ ,  $EFO$ , and  $AFO$  are congruent. This implies that the point  $O$  is equidistant from each of the six sides of the hexagon. Consequently, there is a circle centered at  $O$  that touches all the six sides.

**OC393.** The point  $O$  is the center of the circumcircle  $\Omega$  of the acute triangle  $ABC$ . The circumcircle  $\omega$  of the triangle  $AOC$  intersects the sides  $AB$  and  $BC$  again at the points  $E$  and  $F$ . Moreover, the line  $EF$  divides the area of the triangle  $ABC$  in half. Find  $\angle B$ .

*Originally Problem 3 of Grade 10 of the 2017 Moscow Math Olympiad.*

*We received 5 correct submissions. We present two solutions.*

*Solution 1, by Charles Justin Shi.*

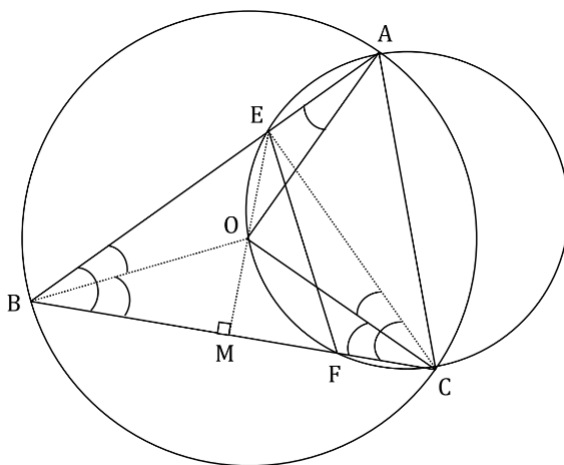
Since  $EF$  divides the area of triangle  $ABC$  in half, we have  $2[BFE] = [ABC]$ . Also, since  $ACFE$  is a cyclic quadrilateral, we have  $\angle BAC = 180^\circ - \angle CFE = \angle BFE$ , and  $\angle ACB = 180^\circ - \angle AEF = \angle BEF$ . Therefore, triangles  $BAC$  and



$BFE$  are similar, and

$$\frac{BC}{BE} = \sqrt{\frac{[ABC]}{[BFE]}} = \sqrt{2}.$$

Since  $O$  is the center of the circumcircle of the triangle  $ABC$ , it follows that  $OA$ ,  $OB$ , and  $OC$  are radii of the circumcircle, and  $OA = OB = OC$ . This implies that triangles  $OAB$  and  $OBC$  are isosceles triangles with  $\angle OBC = \angle OCB$  and  $\angle OAB = \angle OBA$ . In cyclic quadrilateral  $ACFE$ ,  $\angle OCE$  and  $\angle OAE$  subtend the same arc  $OE$ , hence,  $\angle OCE = \angle OAE$ . Then  $\angle OCE = \angle OAE = \angle OAB = \angle OBA$ , and  $\angle EBC = \angle ECB$ . This implies that the triangle  $EBC$  is isosceles with  $BE = BC$ .



Since  $EBC$  is an isosceles triangle, a median from vertex  $E$  to  $BC$  is also the altitude of the triangle. Let  $M$  be the midpoint of  $BC$ . Then  $\angle EMB = 90^\circ$ , and triangle  $EMB$  is a right triangle. Using the previous result,  $BC = \sqrt{2}BE$ , it follows that

$$\cos(\angle ABC) = \frac{BM}{BE} = \frac{BC}{2BE} = \frac{\sqrt{2}}{2},$$

and  $\angle ABC = 45^\circ$ .

*Solution 2, by Andrea Fanchini.*

We use barycentric coordinates with reference to the triangle  $ABC$ . The circum-circle  $\omega$  of the triangle  $AOC$ , is described by

$$\omega : a^2yz + b^2zx + c^2xy - (x + y + z)S_By = 0.$$

Then the points  $E$  and  $F$  are

$$E = \omega \cap AB = (S_B : S_A : 0) \quad \text{and} \quad F = \omega \cap BC = (0 : S_C : S_B).$$

Therefore the area of triangle  $BEF$  is

$$[BEF] = \frac{[ABC]}{a^2 c^2} \times \begin{vmatrix} 0 & 1 & 0 \\ S_B & S_A & 0 \\ 0 & S_C & S_B \end{vmatrix} = [ABC] \times \frac{S_B^2}{a^2 c^2} = [ABC] \times \cos^2(\angle ABC).$$

However,  $[BEF] = \frac{1}{2}[ABC]$ , then  $\cos(\angle ABC) = \sqrt{2}/2$ , and  $\angle ABC = 45^\circ$ .

**OC394.** In Chicago, there are 36 criminal gangs, some of which are at war with each other. Each gangster belongs to several gangs and every pair of gangsters belongs to a different set of gangs. It is known that no gangster is a member of two gangs that are at war with each other. Furthermore, each gang that some gangster does not belong to is at war with some gang he does belong to. What is the largest possible number of gangsters in Chicago?

*Originally Problem 6 of Grade 10 of the 2017 Moscow Math Olympiad.*

*We received 1 correct submission by Oliver Geupel presented here.*

The answer is  $3^{12} = 531,441$ .

We establish this result using graph theory. A graph can be defined as follows. A node is assigned for each gang and two nodes are joined by an edge if the corresponding gangs are at war. The gangs that some gangster belongs to define a set of nodes. Such a set has the following two crucial properties. First, it is independent. Equivalently, there are no edges between any two nodes of the set, because a gangster does not belong to two gangs that are at war. Second, it is a maximal independent (MI) set, since it is not properly contained in any other independent set. The second property follows from the fact that each gang that some gangster does not belong to is at war with some gang he does belong to. The question asks for the maximum number of MI sets that are possible in a graph with 36 nodes.

We prove by induction over the number of nodes that the maximum number of MI sets that are possible in a graph with  $3k$  nodes is  $3^k$ .

The base case  $k = 1$  is obvious. The complete graph with 3 nodes, i.e. the triangle, has exactly 3 MI sets, specifically its 3 nodes. This graph is generated by 3 gangs that are at war. The maximum number of gangsters is 3, one gangster per gang.

Let  $G$  be a graph with  $3k \geq 6$  nodes, and let  $N$  be the set of its nodes. Assume that the maximum number of MI sets that are possible in a graph with  $3j$  nodes is  $3^j$ , for any  $1 \leq j \leq k-1$ . Split the set of nodes of  $G$  into a disjoint union of sets  $U$  and  $V$  with  $3(k-1)$  and 3 elements, respectively. Let  $[U]$  and  $[V]$  be the induced subgraphs formed from  $U$  and  $V$ , respectively.

For any MI set  $M$  of  $G$ , the sets  $M \cap U$  and  $M \cap V$  are MI sets of  $[U]$  and  $[V]$ , respectively. Let  $m(G)$  be the number of MI sets in the graph  $G$ . We have obtained that

$$m(G) \leq m([U])m([V]).$$

Hence, by induction,

$$m(G) \leq 3^{k-1} \cdot 3 = 3^k.$$

The equality holds when there are no edges between the nodes  $U$  and  $V$ ,  $V$  is the complete graph on 3 nodes, and  $U$  is a union of  $k-1$  disjoint complete graphs on 3 nodes, i.e.  $k-1$  disjoint triangles. This completes the proof and shows that the maximum number of gangsters in a city with 36 gangs is  $3^{12}$ .

This problem is discussed and solved on graphs in two articles:

- (1) J. W. Moon and L. Moser, On cliques in graphs, Israel J. Math. 3 (1965) 23-28
- (2) V. Vatter, Maximal independent sets and separating covers, Amer. Math. Monthly 118 (2011) 418-423.

Specifically, it is shown that the maximum number of possible MI sets in a graph with  $3k$  nodes is  $3^k$ , with  $3k+1$  nodes is  $4 \cdot 3^{k-1}$ , and with  $3k+2$  nodes is  $2 \cdot 3^k$ .

**OC395.** Let  $A_1, A_2, \dots, A_k \in \mathcal{M}_n(\mathbb{R})$  be symmetric matrices. Prove that the following statements are equivalent:

- (a)  $\det(A_1^2 + A_2^2 + \dots + A_k^2) = 0$ ;
- (b) for all matrices  $B_1, B_2, \dots, B_k \in \mathcal{M}_n(\mathbb{R})$  it holds

$$\det(A_1 B_1 + A_2 B_2 + \dots + A_k B_k) = 0.$$

*Originally Problem 2 of Grade 11 of the 2017 Romania Math Olympiad.*

*We received 1 correct submission by Oliver Geupel presented here.*

First we prove the implication "(a)  $\Rightarrow$  (b)". By the hypothesis (a), there exists a nonzero row vector  $v \in \mathbb{R}^{1 \times n}$  such that  $v(A_1^2 + A_2^2 + \dots + A_k^2) = o$ , where  $o = (0, \dots, 0) \in \mathbb{R}^{1 \times n}$ . Since the matrices  $A_1, A_2, \dots, A_k$  are symmetric, it follows

$$\begin{aligned} |vA_1|^2 + |vA_2|^2 + \dots + |vA_k|^2 &= vA_1 A_1^T v^T + vA_2 A_2^T v^T + \dots + vA_k A_k^T v^T \\ &= v(A_1^2 + A_2^2 + \dots + A_k^2) v^T = 0. \end{aligned}$$

Therefore,  $vA_1 = vA_2 = \dots = vA_k = o$  and, for all  $B_1, B_2, \dots, B_k \in \mathcal{M}_n(\mathbb{R})$

$$v(A_1 B_1 + A_2 B_2 + \dots + A_k B_k) = (vA_1)B_1 + (vA_2)B_2 + \dots + (vA_k)B_k = o.$$

The conclusion (b), follows.

Finally, for the implication "(b)  $\Rightarrow$  (a)", it is enough to put  $B_1 = A_1, B_2 = A_2, \dots, B_k = A_k$ .

# FOCUS ON...

No. 36

Michel Bataille

## Geometry with Complex Numbers (I)

### Introduction

Among the various angles of attack of a geometry problem, the use of complex numbers is sometimes chosen. The method, often very direct, involves clever calculations instead of auxiliary constructions or synthetic arguments and can produce elegant solutions. The purpose of this number and the next one is to show some examples of applications of this method. In this part I, we will focus on the frequent case when a triangle and its circumcircle are at the heart of the problem. In part II, we will consider and illustrate other applications.

For an introduction to the method with a selection of exercises, we refer the reader to [1]; for a more thorough treatise on the subject, a good reference is [2].

### Complex numbers and circumcircle: a direct approach

We will consider problems for which we can suppose that the circumcircle  $\Gamma$  of a given triangle  $ABC$  is the unit circle. We will denote by the lower-case letter  $m$  the affix of any point  $M$ . Thus, the affixes  $a, b, c$  of the vertices  $A, B, C$  satisfy  $a\bar{a} = b\bar{b} = c\bar{c} = 1$  while the affix of the circumcentre  $O$  is (conveniently) 0. We shall freely use the following lemma:

If  $A, B$  are two distinct points of  $\Gamma$  and  $U, V$  two distinct points of the plane, then

- (i)  $UV$  is parallel to  $AB$  if and only if  $v - u = -ab(\bar{v} - \bar{u})$ ;
- (ii)  $UV$  is perpendicular to  $AB$  if and only if  $v - u = ab(\bar{v} - \bar{u})$ .

*Proof.* (i) The line  $UV$  is parallel to  $AB$  if and only if  $\frac{v - u}{b - a}$  is a real number, that is, if and only if

$$\frac{v - u}{b - a} = \frac{\bar{v} - \bar{u}}{\bar{b} - \bar{a}}.$$

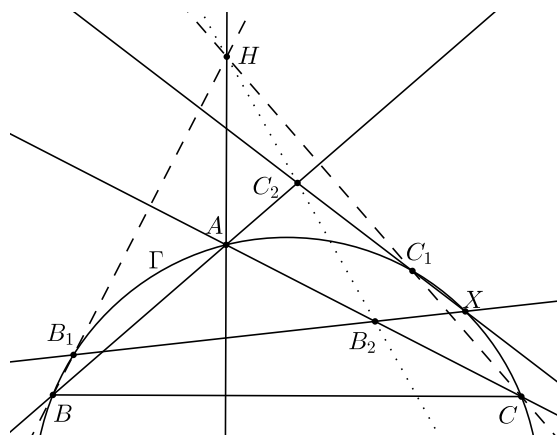
Since  $\bar{b} = \frac{1}{b}$  and  $\bar{a} = \frac{1}{a}$ , a simple calculation gives the condition  $v - u = -ab(\bar{v} - \bar{u})$ .

The proof of (ii) is similar, with  $UV$  being perpendicular to  $AB$  if and only if  $\frac{v - u}{b - a} = -\frac{\bar{v} - \bar{u}}{\bar{b} - \bar{a}}$ . □

With the help of this lemma, it is easy to obtain the equation of the line  $AB$ :  $z + ab\bar{z} = a + b$  and the equation of the tangent to  $\Gamma$  at  $A$ :  $z + a^2\bar{z} = 2a$ .

We are now ready to consider a first example, namely problem 11846 set in the *American Mathematical Monthly* in 2015. Here is the slightly modified statement:

Let  $ABC$  be a triangle with no right angle, and let  $B_1$  and  $C_1$  be the points where the altitudes from  $B$  and  $C$  intersect the circumcircle. Let  $X$  be a point of  $\Gamma$ , not diametrically opposite to  $B$  or  $C$ , and let  $B_2$  and  $C_2$  denote the intersections of  $XB_1$  with  $AC$  and  $XC_1$  with  $AB$ , respectively. Prove that the line  $B_2C_2$  contains the orthocenter of  $ABC$ .



Recall that the orthocentre  $H$  satisfies  $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ , hence its affix is  $h = a + b + c$ .

Since the line  $BB_1$ , whose equation is  $z + bb_1\bar{z} = b + b_1$ , passes through  $H$ , we have

$$(a + b + c) + bb_1\overline{(a + b + c)} = b + b_1$$

and so  $b_1 = -\frac{ca}{b}$ . It follows that the equation of  $XB_1$  is

$$z - \frac{cax}{b}\bar{z} = x - \frac{ca}{b}$$

(with the convention that  $XB_1$  is the tangent to  $\Gamma$  at  $B_1$  if  $X = B_1$ ). Since the equation of  $AC$  is  $z + ac\bar{z} = a + c$ , we readily find  $b_2 = \frac{hx - ca}{x + b}$ . Similarly,

$c_2 = \frac{hx - ab}{x + c}$  and we deduce

$$h - b_2 = \frac{hb + ca}{x + b} \quad \text{and} \quad h - c_2 = \frac{hc + ab}{x + c}.$$

Let  $\lambda = \frac{h - b_2}{h - c_2}$ . Using  $a\bar{a} = b\bar{b} = c\bar{c} = x\bar{x} = 1$ , a straightforward calculation gives  $\bar{\lambda} = \lambda$ , hence  $\lambda$  is a real number. The collinearity of the points  $H, B_2, C_2$  follows.

The efficiency and directness of the method are also noticeable in the following example proposed in the *Mathematical Gazette* in 2013:

Triangle  $A'B'C'$  is the image of a given triangle  $ABC$  after rotation through  $180^\circ$  about a given point  $P$  in its plane. Points  $A'', B''$  and  $C''$  are the reflections of  $A'$  in  $BC$ ,  $B'$  in  $CA$  and  $C'$  in  $AB$ , respectively. Prove that

- (i) the circumcentres of triangles  $ABC, A''B''C''$  coincide;
- (ii) triangles  $ABC, A''B''C''$  are similar;
- (iii) the orthocentre of triangle  $A'B'C'$  lies on the circle  $A''B''C''$ .

Let  $H$  and  $H'$  be the orthocentres of  $\triangle ABC$  and  $\triangle A'B'C'$ . Note that  $H'$  is the image of  $H$  under the rotation through  $180^\circ$  about  $P$ .

Because the midpoint of  $A'A''$  is on the line  $BC$ , we have

$$(a' + a'') + bc(\bar{a}' + \bar{a}'') = 2(b + c)$$

and because  $A'A''$  is perpendicular to  $BC$ , we have

$$a'' - a' = bc(\bar{a}'' - \bar{a}').$$

Also,  $a' = 2p - a$  expresses that  $P$  is the midpoint of  $AA'$ . From these relations, we readily obtain:

$$a'' = b + c + \bar{a}bc - 2bc\bar{p},$$

that is,  $a'' = \lambda\bar{a}$ , where

$$\lambda = ab + bc + ca - 2abc\bar{p} = abc(\bar{a} + \bar{b} + \bar{c} - 2\bar{p}) = abc(\overline{h - 2p}) = -ab\bar{c}h'.$$

From the symmetry of  $\lambda$  in  $a, b, c$ , we deduce  $b'' = \lambda\bar{b}$ ,  $c'' = \lambda\bar{c}$ . Now, assuming that  $h' \neq 0$ , that is,  $P$  is not the centre of the Euler circle of  $\triangle ABC$ , (i),(ii),(iii) result successively from

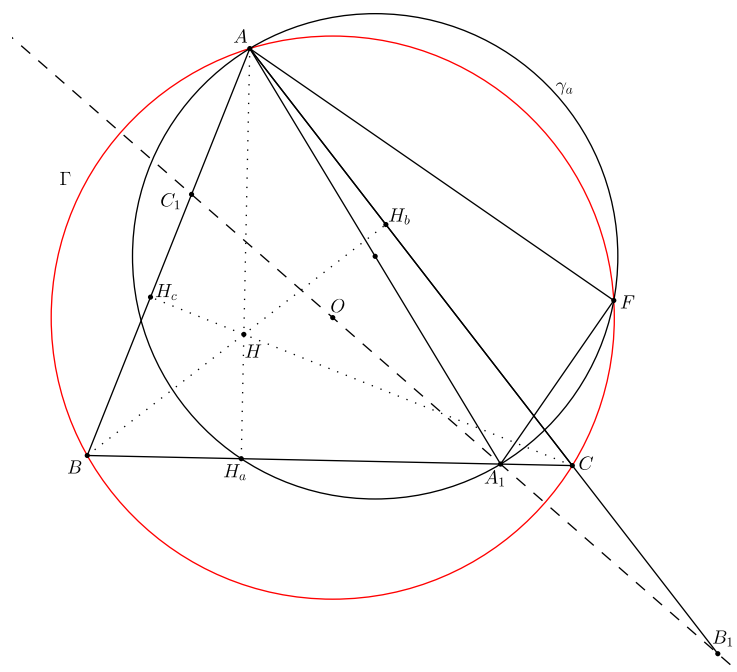
- $|a''| = |b''| = |c''| (= |\lambda|)$ ;
- $\frac{A''B''}{AB} = \frac{|\lambda||\bar{b} - \bar{a}|}{|b - a|} = |\lambda| = \frac{B''C''}{BC} = \frac{C''A''}{CA}$ ;
- $|h'| = |\bar{h}'| = |-ab\bar{c}h'| = |\lambda|$ .

(When  $P$  is the centre of the Euler circle of triangle  $ABC$ , we have that  $A'' = B'' = C'' = O = H'$ .)

We conclude the paragraph with a more difficult problem, problem **3585** [2013 : 414 ; 2014 : 399], of which two geometric solutions have been published.

Let  $ABC$  be a triangle and let  $F$  be a point that lies on the circum-circle of  $ABC$ . Further, let  $H_a, H_b$  and  $H_c$  denote projections of the

orthocenter onto sides  $BC, AC$  and  $AB$ , respectively. The three circles  $AH_aF, BH_bF$  and  $CH_cF$  meet the three sides  $BC, AC$  and  $AB$  at points  $A_1, B_1$  and  $C_1$ , respectively. Prove that  $A_1, B_1$  and  $C_1$  are collinear.



To ensure the existence of the circles  $\gamma_a = (AH_aF)$ ,  $\gamma_b = (BH_bF)$ ,  $\gamma_c = (CH_cF)$ , we assume that  $F$  is different from  $A, B, C$  and from the reflections of  $H$  in the sidelines of  $\triangle ABC$ . We note that the triangle  $AH_aA_1$  is right-angled at  $H_a$  and inscribed in  $\gamma_a$ , hence  $AA_1$  is a diameter of  $\gamma_a$ .

It follows that  $\angle AFA_1 = 90^\circ$  so that  $A_1$  is the point of intersection of  $BC$  and the perpendicular to  $AF$  at  $F$ . Since these two lines have respective equations

$$z + bc\bar{z} = b + c \quad \text{and} \quad z - af\bar{z} = f - a,$$

we obtain  $a_1 = \frac{k}{bc + af}$  where  $k = (ab + bc + ca)f - abc$ . By circular permutation,

we obtain  $b_1 = \frac{k}{ca + bf}$  and  $c_1 = \frac{k}{ab + cf}$ . Now,  $\frac{b_1}{a_1} = \frac{bc + af}{ca + bf}$  and,

$$\frac{\overline{b_1}}{a_1} = \frac{\frac{1}{bc} + \frac{1}{af}}{\frac{1}{ca} + \frac{1}{bf}} = \frac{b_1}{a_1}.$$

Thus,  $\frac{b_1}{a_1}$  is a real number and so  $B_1$  is on the line  $OA_1$ . Similarly,  $C_1$  is on  $OA_1$  and the conclusion follows with an additional result: the line through  $A_1, B_1, C_1$  passes through the centre  $O$  of the circumcircle of  $\triangle ABC$ .

### Complex numbers and circumcircle: when angle bisectors are involved

When the angle bisectors of  $\triangle ABC$  play a central role in the problem, the above approach is not quite suitable, as it leads to complicated calculations. It seems preferable to adopt the following way: keeping the circumcircle  $\Gamma$  as the unit circle, we suppose that the affixes of  $A, B, C$  are  $e^{i\alpha}, e^{i\beta}, e^{i\gamma}$ , respectively, with  $0 < \alpha < \beta < \gamma < 2\pi$ . Denoting the incentre by  $I$ , the bisectors  $IA, IB, IC$  then intersect  $\Gamma$  again at  $A', B', C'$  whose respective affixes are

$$e^{i(\beta+\gamma)/2}, -e^{i(\gamma+\alpha)/2}, e^{i(\alpha+\beta)/2}.$$

For sake of simplicity, we set

$$a = e^{i\alpha/2}, b = -e^{i\beta/2}, c = e^{i\gamma/2}$$

so that the affixes of  $A, B, C, A', B', C'$  are  $a^2, b^2, c^2, -bc, -ca, -ab$ , respectively. Note that  $A', B', C'$  are the respective circumcentres of the triangles  $IBC, ICA, IAB$ . From the equations  $z - a^2 b c \bar{z} = a^2 - bc$  and  $z - ab^2 c \bar{z} = b^2 - ca$  of the lines  $AA'$  and  $BB'$ , we easily obtain the affix  $-(ab + bc + ca)$  of  $I$ . In a similar way, the reader will obtain the respective affixes

$$ab + ca - bc, bc + ba - ca, ca + cb - ab$$

of the excenters  $I_a, I_b, I_c$ .

After these preliminaries, what about a starter and a main course? First consider problem **4268** [2017 : 303 ; 2018 : 312]:

Let  $I$  be the incenter of the acute triangle  $ABC$ , and let the triangle's internal angle bisectors intersect the circles  $IBC, ICA$ , and  $IAB$  again at  $A_1, B_1$ , and  $C_1$ , respectively. Show that  $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \vec{0}$  if and only if  $\triangle ABC$  is equilateral.

The solution is very short: the affixes of the vectors  $\overrightarrow{IA'}, \overrightarrow{IB'}, \overrightarrow{IC'}$  are  $ab + ac, bc + ab, ac + bc$ , respectively, hence the affix of

$$\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = 2(\overrightarrow{IA'} + \overrightarrow{IB'} + \overrightarrow{IC'})$$

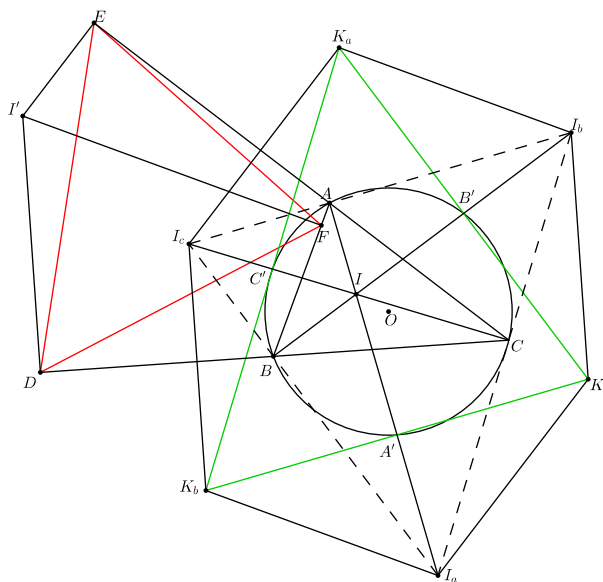
is  $4(ab + bc + ca)$ . Thus,  $\overrightarrow{IA_1} + \overrightarrow{IB_1} + \overrightarrow{IC_1} = \vec{0}$  if and only if  $ab + bc + ca = 0$ , if and only if  $I = O$ , that is, if and only if  $ABC$  is equilateral. Note that the hypothesis  $ABC$  acute is not needed.

Less easy is problem 123 in 2015 issue 1 of *Mathproblems*, which can be found at <http://www.mathproblems-ks.org>. Here is the statement, a little extended:

Let  $I_a, I_b, I_c$  be the excenters of a triangle  $ABC$  and let  $K_a$  be the point in which the perpendicular to  $AB$  through  $I_b$  meets the perpendicular to  $AC$  through  $I_c$ . Similarly define  $K_b$  and  $K_c$ . Prove that  $K_a, K_b, K_c$  are the respective reflections of  $I_a, I_b, I_c$  about the circumcentre  $O$  of  $\triangle ABC$  and that  $A', B', C'$  are the midpoints of  $K_b K_c, K_c K_a,$



$K_a K_b$ , respectively. Prove that the triangle  $K_a K_b K_c$  (known as the hexil triangle) is similar to the pedal triangle of the inverse of the incenter in the circumcircle.



For example, we show that  $K_a$  is the reflection of  $I_a$  about  $O$  and that  $C'$  is the midpoint of  $K_a K_b$ . The equations of the perpendiculars to  $AB$  through  $I_b$  and to  $AC$  through  $I_c$  are readily obtained:

$$z - a^2 b^2 \bar{z} = (b - a) \left( c + \frac{ab}{c} \right) \quad \text{and} \quad z - a^2 c^2 \bar{z} = (c - a) \left( b + \frac{ac}{b} \right).$$

Solving the system so formed shows that the affix of  $K_a$  is  $bc - ab - ca$ , clearly the opposite of the affix of  $I_a$ . The second assertion follows from

$$-ab = \frac{(bc - ab - ca) + (ca - bc - ab)}{2}.$$

Lastly, let  $p = -(ab + bc + ca)$  be the affix of  $I$ . The affix of the inverse  $I'$  of  $I$  in  $\Gamma$  is  $p' = \frac{1}{p}$ . The lines  $BC$ :  $z + b^2 c^2 \bar{z} = b^2 + c^2$  and the perpendicular to  $BC$  through  $I'$ :  $z - \frac{1}{p} = b^2 c^2 \left( \bar{z} - \frac{1}{p} \right)$  then yield the affix  $d$  of the projection  $D$  of  $I'$  onto  $BC$

$$d = \frac{1}{2} \left( b^2 + c^2 + \frac{1}{p} - \frac{b^2 c^2}{p} \right).$$

Cyclically, we obtain the affixes  $e$  and  $f$  of the projections  $E$  and  $F$  of  $I'$  onto  $CA$  and  $AB$

$$e = \frac{1}{2} \left( c^2 + a^2 + \frac{1}{p} - \frac{c^2 a^2}{p} \right) \quad \text{and} \quad f = \frac{1}{2} \left( a^2 + b^2 + \frac{1}{p} - \frac{a^2 b^2}{p} \right).$$

Simple calculations give

$$e - d = (a + b)(b + c)(c + a) \cdot \frac{b - a}{2p} \quad \text{and} \quad f - d = (a + b)(b + c)(c + a) \cdot \frac{c - a}{2p}$$

so that

$$\frac{e - d}{f - d} = \frac{b - a}{c - a}.$$

On the other hand, if  $p_a, p_b, p_c$  denote the affixes of  $I_a, I_b, I_c$ , then  $\frac{p_b - p_a}{p_c - p_a} = \frac{c(b - a)}{b(c - a)}$  and so

$$\frac{\overline{p_b} - \overline{p_a}}{\overline{p_c} - \overline{p_a}} = \frac{\frac{1}{c} \left( \frac{1}{b} - \frac{1}{a} \right)}{\frac{1}{b} \left( \frac{1}{c} - \frac{1}{a} \right)} = \frac{b - a}{c - a}$$

Thus,

$$\frac{e - d}{f - d} = \frac{\overline{p_b} - \overline{p_a}}{\overline{p_c} - \overline{p_a}}, \quad (*)$$

meaning that  $\triangle DEF$  and  $\triangle I_a I_b I_c$  are (inversely) similar and so are  $\triangle DEF$  and  $\triangle K_a K_b K_c$ . For the conclusion drawn from (\*), see part II or [2] p. 57-58.

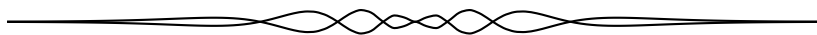
The reader is invited to solve the two following exercises with the help of complex numbers.

### Exercises

1. Given an acute triangle  $ABC$ , let  $O$  be its circumcenter, let  $M$  be the intersection of lines  $AO$  and  $BC$ , and let  $D$  be the other intersection of  $AO$  with the circumcircle of  $ABC$ . Let  $E$  be that point on  $AD$  such that  $M$  is the midpoint of  $ED$ . Let  $F$  be the point at which the perpendicular to  $AD$  at  $M$  meets  $AC$ . Prove that  $EF$  is perpendicular to  $AB$ . [Problem 11737 of the *American Mathematical Monthly*.]
2. Let  $H, I, O$  be the orthocenter, incentre, circumcentre of a triangle  $ABC$  and let  $J$  be the reflection of  $I$  about  $O$ . Prove that the line through the midpoints of  $JH$  and  $BC$  is parallel to  $AI$ .

### References

- [1] M. Bataille, Chapter 4 in *Géométrie plane, avec des nombres*, CMS/SMC Atom XV, 2015.
- [2] Liang-shin Hahn, *Complex numbers and Geometry*, MAA, 1994.



# PROBLEMS

*Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.*

To facilitate their consideration, solutions should be received by **August 15, 2019**.



**4441.** *Proposed by Mihaela Berindeanu.*

Let  $ABC$  be an acute triangle, with circumcenter  $O$  and orthocenter  $H$ . Let  $A', B'$  and  $C'$  be the intersection of  $AH, BH, CH$  with  $BC, AC, AB$ , respectively. Let  $A_1, B_1$  and  $C_1$  be the intersection of  $AO, BO, CO$  with  $BC, AC, AB$ , respectively. If  $A'', B''$  and  $C''$  are midpoints of  $AA_1, BB_1$  and  $CC_1$ , show that  $A'A'', B'B''$  and  $C'C''$  have a common intersection point.

**4442.** *Proposed by Nguyen Viet Hung.*

Find the following limit

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2n-1} + \sqrt{2n}} \right).$$

**4443.** *Proposed by Andrew Wu.*

Acute scalene  $\triangle ABC$  has circumcircle  $\Omega$  and altitudes  $\overline{BE}$  and  $\overline{CF}$ . Point  $N$  is the midpoint of  $\overline{EF}$  and line  $\overline{AN}$  meets  $\Omega$  again at  $Z$ . Let lines  $\overline{ZF}$  and  $\overline{ZE}$  meet  $\Omega$  again at  $V$  and  $U$ , respectively, and let lines  $\overline{CV}$  and  $\overline{BU}$  meet at  $P$ . Prove that  $\overline{UV}$  and  $\overline{BC}$  meet on the tangent from  $P$  to the circumcircle of  $\triangle APN$ .

**4444.** *Proposed by Michel Bataille.*

Let  $n$  be a positive integer. Evaluate in closed form

$$\sum_{k=0}^{n-1} \left( \tan^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) + \cot^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) \right).$$

**4445.** *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let  $ABC$  be a triangle with  $AC > BC > AB$ , incenter  $I$  and centroid  $G$ .

1. Prove that point  $A$  lies in one half-plane of the line  $GI$ , while points  $B$  and  $C$  lie in the other half-plane.
2. The line  $GI$  intersects the sides  $AB$  and  $AC$  at  $M$  and  $N$ , respectively. Prove that  $BM = CN$  if and only if  $\angle BAC = 60^\circ$ .

**4446.** *Proposed by Florin Stanescu.*

Let  $n$  be a prime number greater than 4 and let  $A \in M_{n-1}(\mathbb{Q})$  be such that  $A^n = I_{n-1}$ . Evaluate  $\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})$  in terms of  $n$ .

**4447.** *Proposed by Lorian Saceanu.*

Let  $ABC$  be a scalene triangle. Prove that

$$2 + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} \geq \sin^2 A + \sin^2 B + \sin^2 C.$$

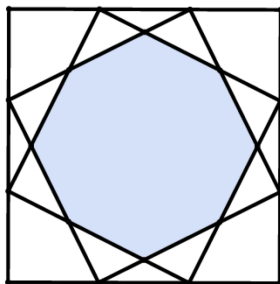
**4448.** *Proposed by Leonard Giugiuc.*

Let  $a, b, c$  and  $d$  be non-zero complex numbers such that  $|a| = |b| = |c| = |d|$  and  $\text{Arg}(a) < \text{Arg}(b) < \text{Arg}(c) < \text{Arg}(d)$ . Prove that

$$|(a-b)(c-d)| = |(a-d)(b-c)| \iff (a-b)(c-d) = (a-d)(b-c).$$

**4449.** *Proposed by Arsalan Wares.*

The figure shows two congruent overlapping squares inside a larger square. The vertices of the overlapping smaller squares divide each of the four sides of the largest square into three equal parts. If the area of the shaded region is 50, find the area of the largest square.



**4450.** *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let  $n \geq 3$  be an integer and consider positive real numbers  $a_1, a_1, \dots, a_n$  such that  $a_n \geq a_1 + a_2 + \cdots + a_{n-1}$ . Prove that

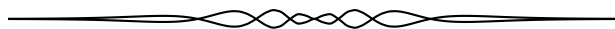
$$(a_1 + a_2 + \cdots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1).$$

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*Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le 15 août 2019.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*



**4441.** *Proposed by Mihaela Berindeanu.*

Soit  $ABC$  un triangle acutangle, dont le centre du cercle circonscrit est  $O$  et l'orthocentre est  $H$ . Soient  $A', B'$  et  $C'$  les points d'intersection de  $AH, BH$  et  $CH$  avec  $BC, AC$  et  $AB$ , respectivement. Soient  $A_1, B_1$  et  $C_1$  les points d'intersection de  $AO, BO$  et  $CO$  avec  $BC, AC$  et  $AB$ , respectivement. Si  $A'', B''$  et  $C''$  sont les mi points de  $AA_1, BB_1$  et  $CC_1$ , démontrer que  $A'A'', B'B''$  et  $C'C''$  ont un point d'intersection commun.

**4442.** *Proposed by Nguyen Viet Hung.*

Déterminer la limite suivante

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2n-1} + \sqrt{2n}} \right).$$

**4443.** *Proposed by Andrew Wu.*

Le triangle  $ABC$  est acutangle et scalène, avec cercle circonscrit  $\Omega$  et altitudes  $\overline{BE}$  puis  $\overline{CF}$ . Le point  $N$  est le mipoint de  $\overline{EF}$ ; la ligne  $\overline{AN}$  rencontre  $\Omega$  de nouveau en  $Z$ . Les lignes  $\overline{ZF}$  et  $\overline{ZE}$  rencontrent  $\Omega$  de nouveau en  $V$  et  $U$ , respectivement; les lignes  $\overline{CV}$  et  $\overline{BU}$  se rencontrent en  $P$ . Démontrer que  $\overline{UV}$  et  $\overline{BC}$  se rencontrent en un point se trouvant sur la tangente au cercle circonscrit de  $\triangle APN$  passant par le point  $P$ .

**4444.** *Proposed by Michel Bataille.*

Soit  $n$  un entier positif. Évaluer en forme close l'expression

$$\sum_{k=0}^{n-1} \left( \tan^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) + \cot^2 \left( \frac{2k+1}{2n+1} \cdot \frac{\pi}{4} \right) \right).$$

**4445.** *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Soit  $ABC$  un triangle tel que  $AC > BC > AB$ , le centre du cercle inscrit étant  $I$  et le centroïde étant  $G$ .

1. Démontrer que la ligne  $GI$  intersecte les intérieurs des segments  $AB$  et  $AC$ .
2. La ligne  $GI$  intersecte les côtés  $AB$  et  $AC$  en  $M$  et  $N$ , respectivement. Démontrer que  $BM = CN$  si et seulement si  $\angle BAC = 60^\circ$ .

**4446.** *Proposed by Florin Stănescu.*

Soit  $n$  un nombre premier supérieur à 4 et soit  $A \in M_{n-1}(\mathbb{Q})$  telle que  $A^n = I_{n-1}$ . Évaluer

$$\det(A^{n-2} + 2A^{n-3} + 3A^{n-4} + \cdots + (n-2)A + (n-1)I_{n-1})$$

en termes de  $n$ .

**4447.** *Proposed by Lorian Săceanu.*

Soit  $ABC$  un triangle scalène. Démontrer que

$$2 + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} \geq \sin^2 A + \sin^2 B + \sin^2 C.$$

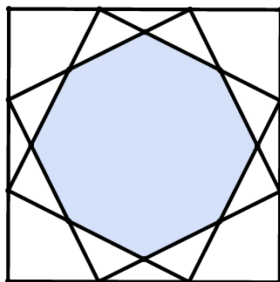
**4448.** *Proposed by Leonard Giugiuc.*

Soient  $a, b, c$  et  $d$  des nombres complexes non nuls tels que  $|a| = |b| = |c| = |d|$  et  $\text{Arg}(a) < \text{Arg}(b) < \text{Arg}(c) < \text{Arg}(d)$ . Démontrer que

$$|(a-b)(c-d)| = |(a-d)(b-c)| \iff (a-b)(c-d) = (a-d)(b-c).$$

**4449.** *Proposed by Arsalan Wares.*

La figure montre deux carrés congrus à l'intérieur d'un plus grand carré. Les sommets des deux petits carrés découpent chacun des côtés du grand carré en trois segments de mêmes longueurs. Si la surface colorée est de 50, déterminer la surface du grand carré.



**4450.** *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Soit  $n \geq 3$  un entier et soient des nombres positifs réels  $a_1, a_1, \dots, a_n$  tels que  $a_n \geq a_1 + a_2 + \dots + a_{n-1}$ . Démontrer que

$$(a_1 + a_2 + \dots + a_n) \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \geq 2((n-1)^2 + 1).$$



# SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(6), p. 256–259; 44(7), p. 302–305; 44(8), p. 340–343.

**4351.** Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.

Let  $ABC$  be a triangle and let  $\omega$  be its incircle. Let  $E$  and  $F$  be the tangency points of  $\omega$  and the sides  $AC$  and  $AB$ , respectively. Let  $G$  be the second intersection point of  $\omega$  and  $BE$  and, similarly, let  $D$  be the second intersection point of  $\omega$  and  $CF$ . Prove that

$$\frac{FE \cdot GD}{FG \cdot ED} = 3.$$

We received 6 submissions, all of which were correct, and will feature two of them.

*Solution 1, by Richard B. Eden.*

Let  $I$  be the incenter of  $\triangle ABC$  and  $K$  the point on  $FD$  such that  $\angle KGF = \angle DGE$ . Since  $\angle GFK = \angle GED$ , then

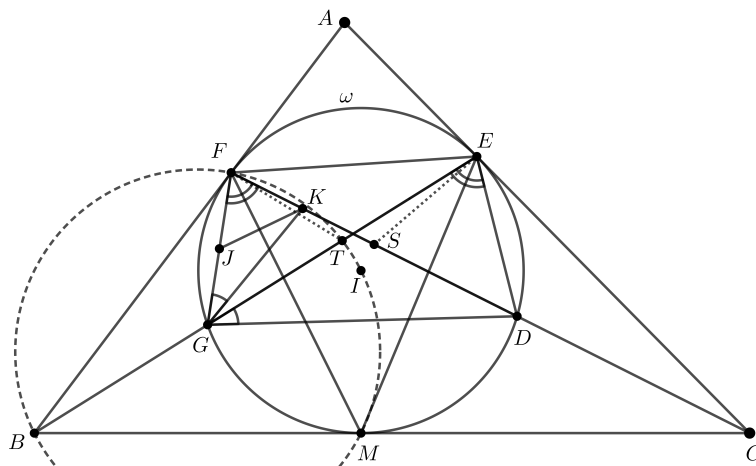
$$\triangle KGF \sim \triangle DGE.$$

Next, since  $\angle EGF = \angle EGK + \angle KGE = \angle EGK + \angle DGE = \angle DGK$  and  $\angle FEG = \angle KDG$ , then

$$\triangle KDG \sim \triangle FEG.$$

From the similarity of these triangles, we get  $\frac{FK}{FG} = \frac{ED}{EG}$  and  $\frac{DK}{DG} = \frac{EF}{EG}$ , so

$$\frac{FE \cdot GD}{FG \cdot ED} = \frac{DK \cdot EG}{FK \cdot EG} = \frac{DK}{FK}.$$





Our goal, therefore, is to prove that  $\frac{DK}{FK} = 3$ . Let  $S$  be the midpoint of  $DF$  so that the problem is reduced to showing that  $K$  is the midpoint of  $FS$ . Let  $T$  be the midpoint of  $EG$ , and  $M$  the point of contact of  $\omega$  and  $BC$ . First, we will show that  $\triangle TGF \sim \triangle EMF$ . We have  $\angle TGF = \angle EGF = \angle EMF$ . Since  $T$  is the midpoint of chord  $EG$  of  $\omega$ ,

$$\angle ITB = \angle ITG = 90^\circ = \angle IFB = \angle IMB,$$

so  $F, B, M$  and  $T$  are concyclic. Therefore,

$$\angle FTG = \angle FTB = \angle FMB = \angle FEM.$$

Therefore,  $\triangle TGF \sim \triangle EMF$ . Similarly,  $\triangle EDS \sim \triangle EMF$ . Therefore, we have  $\triangle TGF \sim \triangle EDS$ .

It follows that  $\frac{TG}{GF} = \frac{ED}{DS}$ , so  $\frac{ET}{FG} = \frac{ED}{FS}$ . Since

$$\angle GFS = \angle GFD = \angle GED = \angle TED,$$

it follows that  $\triangle GFS \sim \triangle TED$ . Now let  $J$  be the midpoint of  $FG$ . Since  $\triangle GFK \sim \triangle GED$  and the medians  $KJ$  and  $DT$  correspond to each other, then  $\triangle JFK \sim \triangle TED$ . This implies  $\triangle JFK \sim \triangle GFS$ , with vertices written in corresponding order. Since  $J$  is the midpoint of  $FG$ ,  $K$  is the midpoint of  $FS$ .

*Solution 2, by Michel Bataille.*

We shall use complex numbers, denoting by  $m$  (small letter) the complex affix of the point  $M$  (capital letter). Without loss of generality, we suppose that  $\omega$  is the unit circle. We observe that if  $M$  and  $N$  are points of  $\omega$ , then

$$MN^2 = |m - n|^2 = (m - n)(\bar{m} - \bar{n}) = (m - n) \left( \frac{1}{m} - \frac{1}{n} \right) = -\frac{(m - n)^2}{mn}.$$

It readily follows that

$$\frac{FE^2 \cdot GD^2}{FG^2 \cdot ED^2} = \frac{(e - f)^2(d - g)^2}{(g - f)^2(d - e)^2}$$

and therefore our problem amounts to showing that the ratio  $\rho = \frac{(e - f)(d - g)}{(g - f)(d - e)}$  equals 3 or  $-3$ .

Let  $W$  be the point of tangency of  $\omega$  and  $BC$ . From the respective equations  $z + w^2\bar{z} = 2w$  and  $z + e^2\bar{z} = 2e$  of the lines  $BC$  and  $CA$  (which are the tangents to  $\omega$  at  $W$  and  $E$ ), we obtain  $c = \frac{2ew}{e + w}$ . Since  $C$  is on the line  $DF$  whose equation is  $z + df\bar{z} = d + f$ , we have  $c + df\bar{c} = d + f$  and a short calculation yields

$$d = \frac{2we - wf - ef}{e + w - 2f} = \frac{w(e - f) + e(w - f)}{(e - f) + (w - f)} = \frac{w(e - f) + e(w - f)}{\alpha}$$

(with  $\alpha = (e - f) + (w - f)$ ). Exchanging  $e$  and  $f$  gives

$$g = \frac{w(f - e) + f(w - e)}{(f - e) + (w - e)} = \frac{w(f - e) + f(w - e)}{\beta}$$

(with  $\beta = (f - e) + (w - e)$ ). Then we easily calculate

$$\begin{aligned} g - f &= (\beta)^{-1}(f - e)(w - f), \\ d - e &= (\alpha)^{-1}(e - f)(w - e), \text{ and} \\ d - g &= (\alpha\beta)^{-1}(e - f)(3w^2 - 3we - 3wf + 3ef). \end{aligned}$$

This leads to

$$\rho = \frac{3(e - f)^2(w^2 - we - wf + ef)}{(f - e)(w - f)(e - f)(w - e)} = -3,$$

and the result follows.

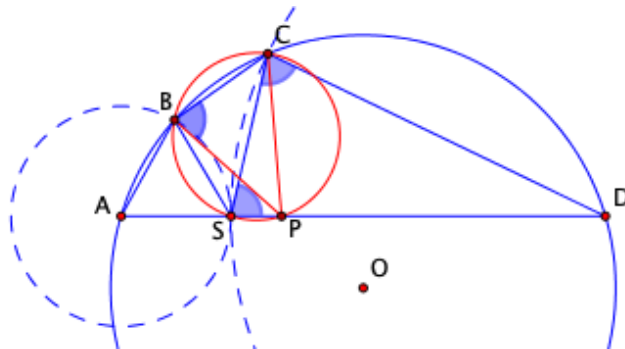
#### 4352. Proposed by Thanos Kalogerakis.

Explain how to locate six points  $A, B, C, D, E, F$  in that order about the circumference of a circle so that the resulting convex hexagon has an incircle, yet is not regular.

*The solutions from C.R. Pranesachar and the proposer were the only submissions we received. We present a composite of their work.*

A polygon that has both a circumcircle and an incircle is called *bicentric*. It is clear that for a convex hexagon to have an incircle, it is necessary that the angle bisectors at all six vertices be concurrent. (The point of concurrence will be equidistant from all six sides.) Our construction of a bicentric hexagon will be based on the following theorem, which is interesting in its own right.

*Theorem.* If  $ABCD$  is a cyclic quadrilateral, then the bisectors of the angles at  $B$  and at  $C$  meet at a point of the chord  $AD$  if and only if there exists a point  $S$  of  $AD$  for which  $AS = AB$  and  $DS = DC$ .



First, assume that for a point  $S$  of  $AD$  we have  $DS = DC$ . Assume further (without loss of generality) that the points have been labeled so that  $AB \leq CD$ .

Define  $P$  to be the second point where the circle  $BCS$  intersects the line  $AD$ . We show that  $BP$  bisects  $\angle ABC$ .

From  $BSPC$  cyclic, we have  $\angle PBC = \angle PSC$ .

From  $\triangle DSC$  isosceles, we have  $\angle PSC = \angle DSC = \angle DCS = \frac{180^\circ - \angle SDC}{2}$ .

From  $ABCD$  cyclic, we have  $\angle ABC = 180^\circ - \angle SDC$ .

It follows that  $\angle PBC = \frac{1}{2}\angle ABC$ ; that is,  $BP$  bisects  $\angle ABC$ , as claimed. Similarly,  $CP$  bisects  $\angle BCD$ .

For the converse, we assume that the bisectors of the angles at  $B$  and at  $C$  intersect at a point  $P$  of the chord  $AD$  and, moreover, that  $S$  is the point of  $AD$  for which  $AS = AB$ . We will show that  $DS = DC$ . We have proved above that the quadrilateral  $BSPC$  is cyclic. Similarly, for the point  $S'$  for which  $DC = DS'$  we have  $BS'PC$  is also cyclic. But there is just one point of  $AD$  other than  $P$  that can lie on the circle  $BCP$ , whence we conclude that  $S' = S$ , which finishes the proof.

Finally, for an example of a bicentric hexagon  $ABCDEF$  that is not regular, denote its circumcircle by  $\gamma$ , let  $AD$  be a diameter, and choose a point  $S$  of  $AD$  different from its midpoint. Define  $B$  and  $F$  to be the points where  $\gamma$  intersects the circle  $(A, AS)$  (with center  $A$  and radius  $AS$ ); define  $C$  and  $E$  to be the points where  $\gamma$  intersects the circle  $(D, DS)$  labeled so that  $B$  and  $C$  are on the same side of  $AD$ . By our theorem, the bisectors of the angles at  $B$  and  $C$  meet in a point of  $AD$  as do the bisectors of the angles at  $E$  and  $F$ . By symmetry all four of those angle bisectors must meet in the same point of  $AD$  while  $AD$  bisects the angles at both endpoints. In other words, all six angle bisectors are concurrent in the center of the incircle of  $ABCDEF$ . This concludes the required construction.

*Further comments.* Once we have constructed one bicentric hexagon, we can construct infinitely many of them according to the Great Poncelet Theorem for Circles: If there is an  $n$ -sided polygon inscribed in a circle  $\alpha$  and circumscribed about a circle  $\beta$ , then for any point  $A$  of  $\alpha$  there exists an  $n$ -sided polygon, also inscribed in  $\alpha$  and circumscribed about  $\beta$ , which has  $A$  as one of its vertices. In fact, the theorem applies more generally to families of conics, but the proof, even in the simplest case of a pair of bicentric polygons, is not easy. Note that Poncelet's theorem combined with our result provides a recipe for the construction of all bicentric hexagons — given any bicentric hexagon with circumcircle  $\alpha$  and incircle  $\beta$ , there exists a bicentric hexagon  $ABCDEF$  inscribed in  $\alpha$  and circumscribed about  $\beta$  having  $AD$  a diameter of  $\alpha$ . Simply start the Poncelet construction with  $A$  on the line containing the two centers.

*Editor's comments.* For the “simplest” proof of Poncelet's theorem known to this editor, see the article “A Simple Proof of Poncelet's Theorem” by Lorenz Halbeisen and Norbert Hungerbühler, *The American Mathematical Monthly* **122**:6 (January 2014) 537-551.

**4353.** *Proposed by Michel Bataille.*

Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{j=1}^n \frac{1}{k \binom{j+k-1}{j}}.$$

We received four solutions. We present the one by Bao Do, lightly edited.

Let

$$s(j) = \sum_{k=1}^{\infty} \frac{1}{k \binom{j+k-1}{j}} = \sum_{k=1}^{\infty} \frac{j!}{(k+j-1) \cdots (k+1)k^2}.$$

Note that the sum in the definition converges, since

$$\sum_{k=1}^{\infty} \frac{1}{k \binom{j+k-1}{j}} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

We need to evaluate  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n s(j)$ . Consider

$$\begin{aligned} \frac{s(j)}{j} &= \sum_{k=1}^{\infty} \frac{(j-1)!}{(k+j-1) \cdots (k+1)k^2} \\ &= (j-2)! \sum_{k=1}^{\infty} \left( \frac{1}{(k+j-2) \cdots (k+1)k^2} - \frac{1}{(k+j-1) \cdots (k+1)k} \right) \\ &= \frac{1}{j-1} \sum_{k=1}^{\infty} \frac{(j-1)!}{(k+j-2) \cdots (k+1)k^2} - (j-2)! \sum_{k=1}^{\infty} \frac{1}{(k+j-1) \cdots k} \\ &= \frac{s(j-1)}{j-1} - \frac{(j-2)!}{(j-1)} \sum_{k=1}^{\infty} \left( \frac{1}{(k+j-2) \cdots k} - \frac{1}{(k+j-1) \cdots (k+1)} \right) \\ &= \frac{s(j-1)}{j-1} - \frac{1}{(j-1)^2}. \end{aligned}$$

We rewrite this as

$$\frac{s(j)}{j} - \frac{s(j-1)}{j-1} = -\frac{1}{(j-1)^2}.$$

We take the sum from  $j = 2$  to  $n$  on both sides to obtain

$$\begin{aligned} \sum_{j=2}^n \left( \frac{s(j)}{j} - \frac{s(j-1)}{j-1} \right) &= - \sum_{j=2}^n \frac{1}{(j-1)^2} \\ \Rightarrow \quad \frac{s(n)}{n} - \frac{s(1)}{1} &= - \sum_{j=1}^{n-1} \frac{1}{j^2} \\ \Rightarrow \quad s(n) &= n \left( s(1) - \sum_{j=1}^{n-1} \frac{1}{j^2} \right) = n \left( \frac{\pi^2}{6} - \sum_{j=1}^{n-1} \frac{1}{j^2} \right). \end{aligned}$$

Finally, we apply the Stolz-Cesàro theorem twice to calculate the limit [Ed.: The editor found that most statements of Stolz-Cesàro require the sequence used for the denominators to be strictly monotone and divergent. However, the case when both denominator and numerator sequences converge to 0 and the denominator sequence is strictly monotone holds as well.]:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n s(j) &= \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^{n+1} s(j) - \sum_{j=1}^n s(j)}{(n+1) - n} = \lim_{n \rightarrow \infty} s(n+1) \\
 &= \lim_{n \rightarrow \infty} (n+1) \left( \frac{\pi^2}{6} - \sum_{j=1}^n \frac{1}{j^2} \right) = \lim_{n \rightarrow \infty} \frac{\frac{\pi^2}{6} - \sum_{j=1}^n \frac{1}{j^2}}{\frac{1}{n+1}} \\
 &= \lim_{n \rightarrow \infty} \frac{-\sum_{j=1}^{n+1} \frac{1}{j^2} + \sum_{j=1}^n \frac{1}{j^2}}{\frac{1}{n+1} - \frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{-\frac{1}{(n+1)^2}}{-\frac{1}{n(n+1)}} \\
 &= \lim_{n \rightarrow \infty} \frac{n(n+1)}{(n+1)^2} = 1.
 \end{aligned}$$

**4354.** Proposed by Ruben Dario Auqui and Leonard Giugiuc.

Let  $ABCD$  be a square with side length 1. Consider points  $M \in AB$ ,  $N \in BC$  and  $P \in CA$  such that the triangles  $BMN$  and  $PMN$  are congruent. Prove that

$$\frac{1}{MB} + \frac{1}{BN} = 2 + \frac{2}{MB + BN}.$$

Thirteen correct solutions were received, along with one incorrect one. We provide a sample of the variety of approaches used.

*Solution 1.*

The triangles are congruent through a reflection in the axis  $MN$ . Assign coordinates with  $A \sim (0, 1)$ ,  $B \sim (0, 0)$ ,  $C \sim (1, 0)$ ,  $M \sim (0, a)$ ,  $N \sim (c, 0)$ . The line  $MN$  has equation  $(x/c) + (y/a) = 1$  and slope  $-a/c$ . The line  $BP$  is perpendicular to  $MN$  and has slope  $c/a$ . Hence  $P \sim (a(a+c)^{-1}, c(a+c)^{-1})$ . Since  $BP$  and  $MN$  intersect at the midpoint of  $BP$ , we have

$$\begin{aligned}
 \frac{a}{2c(a+c)} + \frac{c}{2a(a+c)} &= 1 \\
 \iff a^2 + c^2 &= 2ac(a+c) \\
 \iff (a+c)^2 &= 2ac(a+c+1) \\
 \iff \frac{1}{a} + \frac{1}{c} &= \frac{a+c}{ac} = 2 \left( 1 + \frac{1}{a+c} \right)
 \end{aligned}$$

as desired.

*Solution 2, by Ivko Dimitrić.*

Using the notation and initial results from Solution 1, along with the formula for the distance of a point to the line  $MN$  with equation  $ax + cy - ac = 0$ , we find that

$$\text{dist}(B, MN) = \frac{ac}{\sqrt{a^2 + c^2}}$$

and

$$\text{dist}(P, MN) = \frac{|a^2(a+c)^{-1} + c^2(a+c)^{-1} - ac|}{\sqrt{a^2 + c^2}} = \frac{1}{\sqrt{a^2 + c^2}} \left( \frac{a^2 + c^2}{a+c} - ac \right),$$

since  $a^2 + c^2 - ac(a+c) = a^2(1-c) + c^2(1-a) > 0$ . (The first distance can also be found by taking the area of triangle  $MBN$  in two ways.) Since the two distances are equal,

$$\frac{a^2 + c^2}{a+c} = 2ac \Leftrightarrow (a+c)^2 = 2ac(a+c) + 2ac,$$

from which the result follows.

*Solution 3, by I.J.L. Garces.*

Let  $a$  and  $c$  be the respective lengths of  $BM$  and  $BN$ , and let  $\angle BMN = \theta$ , so that  $c = a \tan \theta$ .

$$\frac{1}{a} + \frac{1}{c} - \frac{2}{a+c} = \frac{1}{a} \left( 1 + \frac{\cos \theta}{\sin \theta} - \frac{2 \cos \theta}{\sin \theta + \cos \theta} \right) = \frac{1}{a \sin \theta (\cos \theta + \sin \theta)}.$$

Observe that  $MP = MB$  and  $\angle APM = 2\theta - 45^\circ$ . Use the Sine Law on triangle  $AMP$  to obtain

$$\frac{1}{a\sqrt{2}} = \frac{\sin(2\theta - 45^\circ)}{1-a},$$

whence

$$\frac{1}{a} = \sin 2\theta - \cos 2\theta + 1 = 2 \sin \theta (\cos \theta + \sin \theta).$$

Hence

$$\frac{1}{a} + \frac{1}{c} - \frac{2}{a+c} = 2,$$

as desired.

*Solution 4, by Cristóbal Sánchez-Rubio.*

Let  $Q$  be the foot of the perpendicular from  $P$  to  $AB$ , and let  $a, c, d$  be the respective lengths of  $BM, BN, PQ$ . The altitudes of triangles  $PAM$  and  $PNC$  have respective lengths  $d$  and  $1-d$ . With  $[\dots]$  denoting area, we have that

$$\begin{aligned} \frac{1}{2} &= [ABC] = [PAM] + [PMBN] + [PNC] \\ &= [PAM] + 2[MBN] + [PNC] \\ &= \frac{1}{2}(1-a)d + ac + \frac{1}{2}(1-c)(1-d) \end{aligned}$$

Hence  $d(a - c) = c(2a - 1)$ .

Since  $QP \parallel BN$  and  $BP \perp MN$ ,

$$\angle QPB = \angle PBN = 90^\circ - \angle MBP = \angle BMN.$$

It follows that the right triangles  $PQB$  and  $MBN$  are similar, so  $d/(1 - d) = a/c$  and  $d = a/(a + c)$ . Plugging this into the previous equation yields that  $a(a - c) = c(a + c)(2a - 1)$  which can be unravelled to the desired result

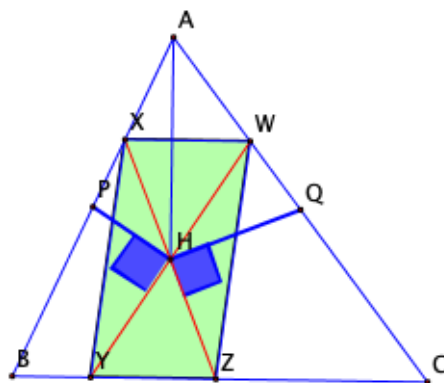
$$\frac{1}{a} + \frac{1}{c} = 2 + \frac{2}{a + c}.$$

*Editor's comments.* For the situation to be viable,  $P$  must be distant no more than 1 from each of the vertices  $A$  and  $C$ . If the triangles  $BMN$  and  $PMN$  are congruent with each of  $M$  and  $N$  corresponding to the other, then  $BNPM$  is a rectangle and  $MB + BN = 1$ . In this case, the equation holds only when  $MB = BN$  and  $P$  is the midpoint of  $AC$ .

#### 4355. *Proposed by Mihaela Berindeanu.*

Let  $H$  be the orthocenter of triangle  $ABC$  with  $P$  the midpoint of  $AB$  and  $Q$  the midpoint of  $AC$ . If  $WY$  is the line perpendicular to  $HP$  at  $H$  with  $W \in AC$  and  $Y \in BC$ , while  $XZ$  is the perpendicular to  $HQ$  at  $H$  with  $X \in AB$  and  $Z \in BC$ , prove that the quadrilateral  $WXYZ$  is a parallelogram.

*In rewording the proposer's problem, the editors mistakenly omitted the necessary condition that  $WY$  and  $XZ$  both had to pass through  $H$ . All four submissions corrected that error. We present the solution by Leonard Giugiuc.*



To prove that  $WXYZ$  is a parallelogram it is sufficient to prove that  $H$  is the midpoint of both diagonals. To that end, we introduce Cartesian coordinates, choosing (without loss of generality)

$$A(0, 1), \quad B(-u, 0), \quad \text{and} \quad C(v, 0),$$

where  $u = \cot B$  and  $v = \cot C$ . We therefore have

$$H(0, uv), \quad P\left(-\frac{u}{2}, \frac{1}{2}\right), \quad \text{and} \quad Q\left(\frac{v}{2}, \frac{1}{2}\right).$$

The slope of the line  $PH$  is  $\frac{2uv-1}{u}$ , so the equation of  $WY$  is  $(1-2uv)(y-uv) = ux$ . Also,  $BC$  is  $y = 0$  and  $AC$  is  $x + vy = v$ . Thus, from

$$(1-2uv)(y-uv) = ux \quad \text{and} \quad x + vy = v$$

we get  $W(v(1-2uv), 2uv)$ , while from

$$(1-2uv)(y-uv) = ux \quad \text{and} \quad y = 0$$

we get  $Y(v(2uv-1), 0)$ . We deduce that  $H$  is the midpoint of the segment  $WY$ . Similarly (by interchanging the roles of  $-u$  and  $v$ ), we find  $X(u(2uv-1), 2uv)$  and  $Z(u(1-2uv), 0)$ , so that  $H$  is the midpoint of  $XZ$ , which concludes the proof.

### 4356. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Solve the following system over reals:

$$\begin{cases} a + b + c + d = 6, \\ a^2 + b^2 + c^2 + d^2 = 12, \\ abc + abd + acd + bcd = 8 + abcd. \end{cases}$$

*We received 14 submissions including that from the proposers. All are correct and we present the solution by Ramanujan Srihari, modified slightly by the editor.*

From the first two equations we readily obtain

$$ab + ac + ad + bc + bd + cd = 12.$$

Let  $abcd = k$  and let

$$f(x) = x^4 - 6x^3 + 12x^2 - (k+8)x + k \tag{1}$$

be the polynomial with  $a, b, c, d$  as its roots.

Then

$$f'(x) = 4x^3 - 18x^2 + 24x - (k+8). \tag{2}$$

Suppose  $f'(r) = 0$  where  $r \in \mathbb{R}$ . Then the slope of  $y = f(x)$  at  $x = r$  is 0.

Now, by straightforward computations we have

$$\begin{aligned} f(r) &= f(r) + (1-r)f'(r) \\ &= r^4 - 6r^3 + 12r^2 - (6+8)r + k + (1-r)(4r^3 - 18r^2 + 24r - (k+8)) \\ &= -3r^4 + 16r^3 - 30r^2 + 24r - 8 \\ &= -(r^2 - 4r + 4)(3r^2 - 4r + 2) \\ &= -(r-2)^2(3r^2 - 4r + 2). \end{aligned}$$



Since  $3r^2 - 4r + 2 = 3\left(r - \frac{2}{3}\right)^2 + \frac{2}{9} > 0$  we see that  $f(r) \leq 0$  with  $f(r) = 0$  if and only if  $r = 2$ .

Assume that  $f'(2) \neq 0$ . Then for all  $t$  with  $f'(t) = 0$  we have  $f(t) < 0$ . This implies that  $f$  has only two simple roots which is a contradiction since  $f(x)$  has four roots,  $a, b, c$ , and  $d$ .

Thus,  $f'(2) = 0$  and  $t = 2$  is the only multiple root of  $f$ . From (2),  $f'(2) = 0 \implies k = 0$ .

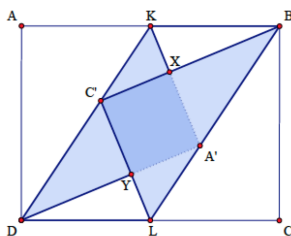
Hence we get from (1) that  $f(x) = x^4 - 6x^3 + 12x^2 - 8x = x(x-2)^3$  and so

$$(0, 2, 2, 2), \quad (2, 0, 2, 2), \quad (2, 2, 0, 2), \quad \text{and} \quad (2, 2, 2, 0)$$

are the only solutions.

### 4357. *Proposed by Arsalan Wares.*

Suppose  $ABCD$  represents a 9 by 12 rectangular sheet. Points  $K$  and  $L$  are midpoints of sides  $AB$  and  $DC$ , respectively. First, edge  $AD$ , of the rectangular sheet  $ABCD$ , is folded over by making a crease along  $DK$ . Then edge  $BC$  is folded over by making a crease along  $BL$ . Folded corners of the sheet overlap over a polygonal region  $C'XA'Y$  as shown.



Find the area of the overlapping polygon  $C'XA'Y$ .

*We received 14 correct solutions and 4 incorrect submissions. We present the solution by Skidmore College Problem Group.*

We claim that the area of  $C'XA'Y$  is  $\frac{2160}{169}$  square units.

Note first that

$$\angle CBL = \angle C'BL = \angle ADK = \angle KDA'.$$

Thus

$$\angle ABL = 90^\circ - \angle CBL = \angle AKD = 90^\circ - \angle ADK.$$

But  $\angle ABL = \angle AKD$ , implying that  $DK$  and  $BL$  are parallel. Thus, by alternating interior angles,  $\angle KC'B = \angle C'BL$ . Now

$$\angle C'XK = 180^\circ - \angle KCB' - \angle C'KX = 90^\circ$$

and  $\angle XC'L = \angle KA'D = 90^\circ$ , so  $C'B$  and  $DA'$  are parallel, implying that  $C'XA'Y$  is a rectangle.

We know that  $AB = DC = 12$ ,  $BC = AD = 9$ , and  $CL = 6$ . Set  $a = A'Y = C'X$  and  $b = XA' = C'Y$ . Then  $DY^2 + YL^2 = DL^2$ , so

$$(9 - a)^2 + (6 - b)^2 = 36,$$

i.e.,

$$a^2 - 18a + b^2 - 12b + 81 = 0.$$

But  $\triangle C'DY$  is similar to  $\triangle KDA'$ , so

$$\frac{9 - a}{9} = \frac{b}{6},$$

giving  $a = 9 - \frac{3}{2}b$ . Thus

$$\left(9 - \frac{3}{2}b\right)^2 - 18\left(9 - \frac{3}{2}b\right) + b^2 - 12b + 81 = 0,$$

which, after simplifying, is  $\frac{13}{4}b^2 - 12b = 0$ , implying that  $b = \frac{48}{13}$  and thus  $a = \frac{45}{13}$ .

This gives the area of  $\frac{2160}{169}$ .

### 4358. *Proposed by George Stoica.*

Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a non-increasing function such that

$$\int_0^\infty f(x) \sin(2\pi x) dx = 0.$$

Prove that  $f$  is constant on each of the intervals  $(n, n+1)$ ,  $n \in \mathbb{N}$ .

*We received 13 submissions, including the one from the proposer. Among them 12 are correct. The other solver apparently misread the question and assumed that the function is non-decreasing instead, but arrived at the same conclusion! We present the solution by Kee-Wai Lau.*

For any  $k \in \mathbb{N}$ , let  $a_k = \int_0^k f(x) \sin(2\pi x) dx$ . Then

$$\begin{aligned} a_k &= \sum_{n=0}^{k-1} \left( \int_n^{n+1/2} f(x) \sin(2\pi x) dx + \int_{n+1/2}^{n+1} f(x) \sin(2\pi x) dx \right) \\ &= \sum_{n=0}^{k-1} (J_n + K_n) = \sum_{n=0}^{k-1} I_n, \end{aligned}$$

where  $I_n = J_n + K_n$ ,  $J_n = \int_n^{n+1/2} f(x) \sin(2\pi x) dx$ , and  $K_n = \int_{n+1/2}^{n+1} f(x) \sin(2\pi x) dx$ . Substituting  $x = n + y$  and  $x = n + y + 1/2$  into  $J_n$  and  $K_n$ , respectively, we then obtain

$$I_n = \int_0^{1/2} (f(n + y) - f(n + y + 1/2)) \sin(2\pi y) dy. \quad (1)$$

Since  $f$  is non-increasing, we have  $I_n \geq 0$  and since  $\lim_{k \rightarrow \infty} a_k = 0$ , we obtain  $I_n = 0$  for any  $n \in \mathbb{N} \cup \{0\}$ . It follows from (1) that except for a set of measure zero,

$$f(n+y) - f(n+y+1/2) = 0 \text{ for } y \in (0, 1/2).$$

Let  $\epsilon \in (0, 1/2)$ . Then there exist

$$t_0 \in (n+1/2, n+1/2+\epsilon) \quad \text{and} \quad t_1 \in (n+1-\epsilon, n+1)$$

such that  $f(t_0) = f(t_0 - 1/2)$  and  $f(t_1) = f(t_1 - 1/2)$ . Since  $f$  is non-increasing,

$$\begin{aligned} f(y) &= f(n+1/2) \text{ for } y \in [n+\epsilon, n+1/2] \cup [n+1/2, n+1-\epsilon] \\ &= [n+\epsilon, n+1-\epsilon]. \end{aligned}$$

Letting  $\epsilon \rightarrow 0^+$  we then obtain  $f(y) = f(n+1/2)$  for all  $y \in (n, n+1)$ . Thus,  $f$  is constant on each of the intervals  $(n, n+1)$ ,  $n \in \mathbb{N}$ .

### 4359. Proposed by Daniel Sitaru.

Let  $a, b$  and  $c$  be positive real numbers. Prove that

$$3 \ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27.$$

*We received 10 submissions, including the one from the proposer, all of which are correct. We present a composite based on the solutions given by Richard B. Eden and Ramanujan Srihari, which are similar to all the other solutions submitted.*

Let  $f(x) = \ln x$ ,  $x > 0$ . Then  $f''(x) = -\frac{1}{x^2} < 0$  so  $f$  is concave. By Jensen's Inequality we then have

$$3 \ln \left( \frac{a^b + b^c + c^a}{3} \right) \geq b \ln a + c \ln b + a \ln c \quad (1)$$

with equality if and only if  $a = b = c$ .

Next, consider  $g(x) = \ln x + \frac{1}{x} - 1$ ,  $x > 0$ . Then  $g'(x) = \frac{x-1}{x^2}$  which implies  $g(x) \geq g(1) = 0$  so  $\ln x \geq 1 - \frac{1}{x}$  for all  $x > 0$ . Hence,

$$b \ln a \geq b(1 - \frac{1}{a}), \quad c \ln b \geq c(1 - \frac{1}{b}), \quad a \ln c \geq a(1 - \frac{1}{c}). \quad (2)$$

From (1) and (2), we then obtain

$$3 \ln(a^b + b^c + c^a) \geq b(1 - \frac{1}{a}) + c(1 - \frac{1}{b}) + a(1 - \frac{1}{c}) + \ln 27$$

so

$$3 \ln(a^b + b^c + c^a) + \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq a + b + c + \ln 27$$

follows, completing the proof.

**4360.** *Proposed by H. A. ShahAli.*

Let  $a, b, c$  be non-negative real numbers such that  $a + b + c = 1$ . Find the minimum and maximum values of the expression

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac}.$$

When do those extreme values occur?

*We received 9 correct solutions and 3 incorrect submissions. We present the solution by Paolo Perfetti, modified by the editor.*

We note first that

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \leq a+b+b+c+c+a = 2,$$

with equality for  $a = 1, b = c = 0$  and its permutations, so the maximum is 2.

The minimum is  $9/5$  and it attained for  $a = b = c = 1/3$  or  $a = b = 1/2, c = 0$  and its permutations. To prove this we show that

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \geq \left[ \frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \right] \Big|_{\frac{a+b}{2}, \frac{a+b}{2}, c=0}.$$

Equivalently, we need to prove that

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \geq \frac{a+b}{1+\frac{(a+b)^2}{4}} + \frac{a+b}{2} + \frac{a+b}{2} = \frac{a+b}{1+\frac{(a+b)^2}{4}} + 1.$$

To do so, it suffices to show that

$$\frac{a+b}{1+ab} \geq \frac{a+b}{1+\frac{(a+b)^2}{4}} \quad \text{and} \quad \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \geq 1.$$

The first of these follows from the AM-GM inequality since  $ab \leq \frac{(a+b)^2}{4}$ . The second can be successively rewritten as

$$\begin{aligned} (b+c)(1+ac) + (a+c)(1+bc) &\geq (1+bc)(1+ac) \\ (a+b) + 2c + 2abc + (a+b)c^2 - 1 - c(a+b) - abc^2 &\geq 0 \\ 1 + c + 2abc + (a+b)c^2 - 1 - c(a+b) - abc^2 &\geq 0 \\ c[1 + ab(2-c) + c(1-c) - (1-c)] &\geq 0 \\ c[ab(2-c) + c(2-c)] &\geq 0 \\ c(2-c)(ab+c) &\geq 0, \end{aligned}$$

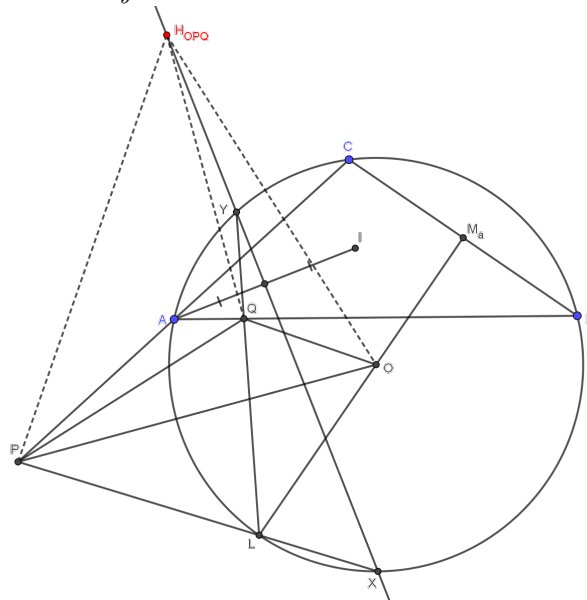
which is true. It follows that

$$\frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \geq \left[ \frac{a+b}{1+ab} + \frac{b+c}{1+bc} + \frac{a+c}{1+ac} \right] \Big|_{\frac{1}{2}, \frac{1}{2}, 0} = \frac{9}{5}.$$

**4361.** *Proposed by Andrew Wu.*

Let  $ABC$  be a scalene triangle with circumcircle  $\Gamma$ , circumcenter  $O$  and incenter  $I$ . Suppose that  $L$  is the midpoint of the arc  $BAC$  of  $\Gamma$ . The perpendicular bisector of  $AI$  meets at  $X$  the arc  $AC$  that contains  $B$ , and at  $Y$  the arc  $AB$  that contains  $C$ . Let  $XL$  and  $AC$  meet at  $P$ ; let  $YL$  and  $AB$  meet at  $Q$ . Show that the orthocenter of triangle  $OPQ$  lies on  $XY$ .

*Three of the four submissions that we received were correct, and one was flawed. We feature the solution by Andrea Fanchini.*



We use barycentric coordinates with respect to triangle  $ABC$ , where its circumcircle  $\Gamma$  is described by the equation  $a^2yz + b^2zx + c^2xy = 0$ . It will turn out that the orthocenter of  $\triangle OPQ$ , call it  $H'$ , is the intersection of  $XY$  and  $BC$ .

The midpoint  $L$  of the arc  $BAC$  of  $\Gamma$  is

$$L = OM_a \cap \Gamma = (a^2 : -b(b-c) : c(b-c)).$$

The perpendicular bisector of  $AI$  is

$$XY : bcx - c(a+c)y - b(a+b)z = 0,$$

whence the points  $X$  and  $Y$  are

$$X = XY \cap \Gamma = (a(a+b) : b(a+b) : -c^2),$$

$$Y = XY \cap \Gamma = (a(a+c) : -b^2 : c(a+c)).$$

Therefore, the lines  $XL$  and  $YL$  are

$$XL : c(c-b)x + acy + a(a+b)z = 0,$$

$$YL : b(b-c)x + a(a+c)y + abz = 0,$$

so that the points  $P$  and  $Q$  are

$$\begin{aligned}P &= XL \cap AC = (a(a+b) : 0 : c(b-c)), \\Q &= YL \cap AB = (a(a+c) : b(c-b) : 0).\end{aligned}$$

Finally, two altitudes of  $\triangle OPQ$  are given by the polars with respect to  $\Gamma$  of  $Q$  and of  $P$ , namely

$$\begin{aligned}POQ_{\infty\perp} &: bc(b-c)x - ac(a+c)y - ab(a+b)z = 0, \quad \text{and} \\QOP_{\infty\perp} &: bc(b-c)x + ac(a+c)y + ab(a+b)z = 0.\end{aligned}$$

Their intersection is the orthocenter of triangle  $OPQ$ , namely

$$H' (0 : -b(a+b) : c(a+c)).$$

One easily checks that it lies on  $XY$  (obtained above), as desired. As a bonus, we see that  $H'$  also lies on the line  $BC$ .

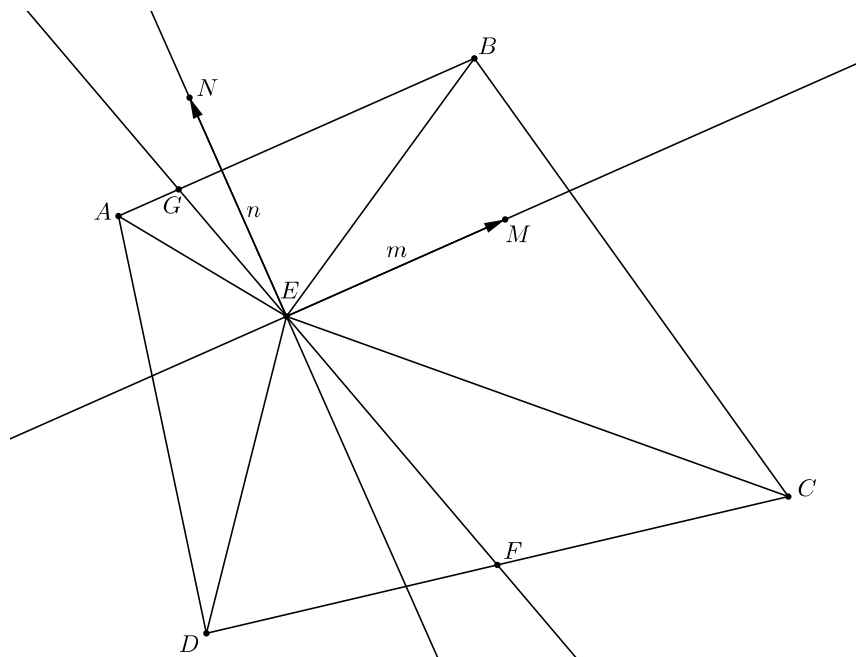
*Editor's comments.* Note the relationship between our Problem 4361 and Brocard's theorem that the center of a circle is the orthocenter of triangles that are self-polar with respect to the circle (meaning each side of the triangle is the pole of the opposite vertex):  $\triangle PQH'$  is an appropriate self-polar triangle with respect to  $\Gamma$ , therefore its orthocenter is  $O$  (which implies, of course, that  $H'$  is the orthocenter of  $\triangle OPQ$ ). Bataille observed that the polar of  $H'$ , namely the line  $PQ$ , contains the incenter  $I$ . That can easily be confirmed using the coordinates of our featured solution: subtract the coordinates of  $Q$  from those of  $P$  and you get the coordinates of  $I(a : b : c)$ .

### 4362. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let  $ABCD$  be a convex quadrilateral and let  $F$  be the midpoint of  $CD$ . Consider a point  $E$  inside  $ABCD$  such that  $AE \cdot CE = BE \cdot DE$ . The lines  $EF$  and  $AB$  intersect at  $G$ . If  $\angle AED + \angle CEB = 180^\circ$ , prove that  $\angle AED = \angle AGE$ .

*We received 4 solutions and will feature just one of them here, by Michel Bataille.*

Let lines  $m$  and  $n$  pass through  $E$  and be parallel and perpendicular to  $AB$ , respectively. Let  $M$  on  $m$  and  $N$  on  $n$  be such that  $EM = EN = 1$ . Consider  $m$  as the  $x$ -axis and  $n$  as the  $y$ -axis of a system of axes with origin at  $E$ . Then to each point is assigned a complex affix. If  $y_0$  denotes the distance from  $E$  to the line  $AB$ , the affixes  $a$  and  $b$  of  $A$  and  $B$  are  $a = x_1 + iy_0$  and  $b = x_2 + iy_0$  for some real numbers  $x_1, x_2$ . Let  $\alpha = \angle AED$ . Then  $\alpha \in (0, \pi)$  and we may suppose that  $M$  and  $N$  are chosen such that the oriented angle  $\angle(\overrightarrow{EA}, \overrightarrow{ED})$  equals  $\alpha$ , and then, by assumption  $\angle(\overrightarrow{EC}, \overrightarrow{EB}) = \pi - \alpha$  (see the figure on the next page).



Let  $c$  and  $d$  denote the affixes of  $C$  and  $D$ . Since  $AE \cdot CE = BE \cdot DE$ , we have  $\left| \frac{d}{a} \right| = \left| \frac{c}{b} \right| = \rho$ , say, so that

$$\frac{d}{a} = \rho e^{i\alpha} \quad \text{and} \quad \frac{b}{c} = \frac{1}{\rho} e^{i(\pi-\alpha)}.$$

We deduce

$$\frac{c}{b} = -\rho e^{i\alpha}$$

and the affix  $f$  of  $F$  is given by

$$f = \frac{c+d}{2} = \rho e^{i\alpha}(a-b) = \rho e^{i\alpha}(x_1 - x_2).$$

From the equation

$$e^{-i\alpha}z - e^{i\alpha}\bar{z} = 0$$

of  $EF$  and  $z - \bar{z} = 2iy_0$  of  $AB$ , we readily obtain the affix  $g$  of  $G$ :

$$g = \frac{y_0 e^{i\alpha}}{\sin \alpha}.$$

Observing that  $y_0 > 0$  and  $\sin \alpha > 0$ , we see that  $\alpha$  is an argument of  $\vec{g}$ , which means that  $\angle(\overrightarrow{EM}, \overrightarrow{EG}) = \alpha$ . Since  $AG \parallel EM$ , it follows that  $\angle(\overrightarrow{GA}, \overrightarrow{GE}) = \alpha$  and so  $\angle AGE = \angle ADE$ , as required.

**4363.** *Proposed by Michel Bataille.*

Let  $(a_n)_{n \geq 0}$  be the sequence defined by  $a_0 > 0$  and the recursion

$$a_{n+1} = \frac{a_n}{1 + (n+1)a_n^2}.$$

Prove that the series  $\sum_{n=0}^{\infty} a_n^2$  is convergent and find  $\lim_{n \rightarrow \infty} \left( n \cdot \sum_{k=n}^{\infty} a_k^2 \right)$ .

*We received 7 solutions. We present the solution by Oliver Geupel, lightly edited.*

The sequence  $b_n = (n+1)a_n$ , where  $n \geq 0$ , satisfies the recursion

$$b_{n+1} = (n+2)b_n / (n+1+b_n^2).$$

Note that  $b_n > 0$  for all  $n$ . We show by induction that  $b_n \leq 1$  for  $n \geq 1$ . First, we have  $b_1 = 2b_0/(1+b_0^2) \leq 1$ , which is the base case. Assuming the assertion holds for some index  $n$ , we obtain

$$b_{n+1} = \frac{nb_n + 2b_n}{n+1+b_n^2} \leq \frac{n+(1+b_n^2)}{n+1+b_n^2} = 1,$$

and the induction is complete. Hence  $a_n \leq \frac{1}{n+1}$  for  $n \geq 1$ , and we obtain

$$\sum_{n=0}^{\infty} a_n^2 \leq a_0^2 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = a_0^2 + \frac{\pi^2}{6} - 1;$$

that is,  $\sum_{n=0}^{\infty} a_n^2$  converges.

For  $n \geq 1$ , note that

$$b_{n+1} = \frac{(n+2)b_n}{n+1+b_n^2} \geq \frac{(n+2)b_n}{n+2} = b_n.$$

The sequence  $(b_n)_{n \geq 1}$  is thus increasing and bounded above by 1, which implies that it has a limit  $0 < L \leq 1$ . From the recursion formula for  $b_{n+1}$  (taking the limit as  $n \rightarrow \infty$  of both sides) we get  $L = \frac{(n+2)L}{n+1+L^2}$ . The only positive root of this equation is  $L = 1$ , which must be the desired limit.

Note that

$$n \cdot \sum_{k=n}^{\infty} a_k^2 = n \cdot \sum_{k=n}^{\infty} \frac{b_k^2}{(k+1)^2}.$$

For every  $\varepsilon > 0$  there is an index  $n_0$  such that, whenever  $n > n_0$ , we have  $1 - \varepsilon < b_n \leq 1$ . Moreover, for  $n > 0$ ,

$$\frac{1}{n+1} = \int_{n+1}^{\infty} x^{-2} dx < \sum_{k=n}^{\infty} \frac{1}{(k+1)^2} < \int_n^{\infty} x^{-2} dx = \frac{1}{n}.$$



Hence for  $n > n_0$ ,

$$\frac{n}{n+1} \cdot (1-\varepsilon)^2 < n \cdot \sum_{k=n}^{\infty} a_k^2 \leq 1,$$

and so  $\lim_{n \rightarrow \infty} \left( n \cdot \sum_{k=n}^{\infty} a_k^2 \right) = 1$ .

**4364.** *Proposed by George Stoica.*

Let  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be a binary operation, and define  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by

$$g(a, b) = \begin{cases} a & \text{if } b = 1 \\ f(g(a, b-1), a) & \text{if } b \geq 2. \end{cases}$$

If  $f$  is associative and  $g$  is commutative, prove that  $f(a, b) = a + b$  and  $g(a, b) = ab$ .

*We received 6 solutions. We present the solution by the Missouri State University Problem Solving Group.*

Note that, since  $g$  is commutative, from the definition of  $g$  it follows that  $g(1, a) = a$  for all  $a \in \mathbb{N}$ .

Let  $n \in \mathbb{N}$ . Then

$$n + 1 = g(1, n + 1) = f(g(1, n), 1) = f(n, 1).$$

That is,

$$f(n, 1) = n + 1 \tag{1}$$

for every  $n \in \mathbb{N}$ .

We now show that  $f(n, m) = n + m$ . Fix  $n$  and use induction on  $m \geq 1$ .

$$\begin{aligned} f(n, m + 1) &= f(n, f(m, 1)) && \text{(by (1), the base case)} \\ &= f(f(n, m), 1) && \text{(by associativity of } f) \\ &= f(n, m) + 1 && \text{(by (1))} \\ &= n + m + 1 && \text{(by the induction hypothesis).} \end{aligned}$$

Next we show  $g(n, m) = nm$ . Once again fix  $n$  and use induction on  $m \geq 1$ . Note that  $g(n, 1) = n \cdot 1$  by the definition of  $g$ .

$$\begin{aligned} g(n, m + 1) &= f(g(n, m), n) && \text{(by the definition of } g) \\ &= f(nm, n) && \text{(by the induction hypothesis)} \\ &= nm + n && \text{(by the formula for } f \text{ proved above)} \\ &= n(m + 1). \end{aligned}$$

This completes the proof.

**4365.** *Proposed by Marius Drăgan and Neculai Stanciu.*

Let  $a$  and  $b$  be real numbers such that  $a + b, a^4$  and  $b^4$  are rational numbers and  $a + b \neq 0$ . Prove that  $a$  and  $b$  are rational numbers.

*We received 7 solutions and will feature the solution by Madhav Modak.*

Let  $a + b = r$ ,  $a^4 = c$  and  $b^4 = d$ . Then by assumption,  $r, c, d$  are rational numbers and  $r \neq 0$ . Eliminating  $b$  from  $a + b = r$  and  $b^4 = d$ , we get  $(r - a)^4 = d$  or

$$a^4 - 4ra^3 + 6r^2a^2 - 4r^3a + r^4 = d.$$

Hence

$$-4ar(a^2 + r^2) = t - 6a^2r^2, \quad \text{where } t = d - r^4 - c. \quad (1)$$

Squaring (1), we get:

$$\begin{aligned} 16a^2r^2(c + r^4 + 2a^2r^2) &= t^2 + 36cr^4 - 12a^2r^2t, \\ a^2(16cr^2 + 16r^6) + 32cr^4 &= t^2 + 36cr^4 - 12a^2r^2t, \\ a^2(16cr^2 + 16r^6 + 12r^2t) &= t^2 + 4cr^4, \\ a^2(16cr^2 + 16r^6 + 12r^2d - 12r^6 - 12cr^2) &= t^2 + 4cr^4, \\ a^2(4cr^2 + 4r^6 + 12r^2d) &= t^2 + 4cr^4. \end{aligned} \quad (2)$$

Now  $r \neq 0$  so that  $r^6 > 0$  and so  $4cr^2 + 4r^6 + 12r^2d > 0$  as  $c, d \geq 0$ . Hence by (2),

$$a^2 = \frac{t^2 + 4cr^4}{4cr^2 + 4r^6 + 12r^2d},$$

so that  $a^2$  is rational. Then by (1),

$$a = (6a^2r^2 - t)/4r(a^2 + r^2),$$

as  $4r(a^2 + r^2) \neq 0$ , we see that  $a$  is rational, hence  $b = r - a$  is rational, as was to be shown.

**4366.** *Proposed by Daniel Sitaru.*

Let  $x_n$  be the base angle of a right triangle with base  $n$  and altitude 1. Find

$$\sum_{k=1}^{\infty} x_{k^2+k+1}.$$

*There were 15 correct solutions, all variants of the following two solutions.*

*Solution 1.*

The arms of the right triangle have lengths 1 and  $n$ , and  $x_n$  is the angle adjacent to the latter arm. Thus,  $x_n = \arctan \frac{1}{n}$ . Observe that

$$\tan(x_k - x_{k+1}) = \frac{\frac{1}{k} - \frac{1}{k+1}}{1 + \frac{1}{k(k+1)}} = \frac{1}{k^2 + k + 1} = \tan(x_{k^2+k+1}).$$

Therefore

$$\begin{aligned}\sum_{k=1}^n x_{k^2+k+1} &= \sum_{k=1}^n (x_k - x_{k+1}) \\ &= x_1 - x_{n+1} \\ &= \frac{\pi}{4} - \arctan \frac{1}{n+1},\end{aligned}$$

so that

$$\sum_{k=1}^{\infty} x_{k^2+k+1} = \frac{\pi}{4}.$$

*Solution 2.*

Let

$$u_n = \tan \left( \sum_{k=1}^n x_{k^2+k+1} \right).$$

Checking the values of  $u_n$  for small values of  $n$ , we are led to the conjecture that  $u_n = n/(n+2)$ . Suppose that this holds for  $n = m-1$ . Then

$$\begin{aligned}u_n &= \tan \left( x_{m^2+m+1} + \sum_{k=1}^{m-1} x_{k^2+k+1} \right) \\ &= \frac{\frac{1}{m^2+m+1} + \frac{m-1}{m+1}}{1 - \frac{m-1}{(m+1)(m^2+m+1)}} = \frac{m(m^2+1)}{(m+2)(m^2+1)} = \frac{m}{m+2}.\end{aligned}$$

Thus, an induction argument, along with  $u_1 = 1/3$ , establishes that  $u_n = n/(n+2)$  for each positive integer  $n$ . Since the limit as  $n$  tends to infinity of  $u_n$  is 1, the sum of the given series is  $\frac{\pi}{4}$ .

**4367.** *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let  $a, b$  and  $c$  be distinct complex numbers such that  $|a| = |b| = |c| = 1$  and  $|a+b+c| \leq 1$ . Prove that

$$\left| \frac{(a+b)(b+c)}{(a-b)(b-c)} \right| + \left| \frac{(b+c)(c+a)}{(b-c)(c-a)} \right| + \left| \frac{(c+a)(a+b)}{(c-a)(a-b)} \right| = 1.$$

*We received 3 correct solutions. We present the solution by Michel Bataille.*

$$\text{Let } u = \frac{(c+a)(a+b)}{(c-a)(a-b)}, \quad v = \frac{(a+b)(b+c)}{(a-b)(b-c)}, \quad w = \frac{(b+c)(c+a)}{(b-c)(c-a)}.$$

If for example  $b+c=0$ , then  $v=w=0$  and  $u=-1$ , hence  $|u|+|v|+|w|=1$  is satisfied.

In what follows, we suppose that  $b + c, c + a, a + b$  are nonzero complex numbers.

Since the conjugate of  $a, b, c$  are  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ , respectively, an easy calculation gives  $\bar{u} = u, \bar{v} = v, \bar{w} = w$ , meaning that  $u, v, w$  are real numbers.

Similarly,  $z = \frac{(a+b)(b+c)(c+a)}{(a-b)(b-c)(c-a)}$  is a nonzero purely imaginary number and so  $uvw = z^2$  is a negative real number.

The three following identities are readily checked:

$$(a+b)(b+c)(c-a) + (b+c)(c+a)(a-b) + (c+a)(a+b)(b-c) = -(a-b)(b-c)(c-a) \quad (1)$$

$$(a+b)(b-c)(c-a) + (b+c)(c-a)(a-b) + (c+a)(a-b)(b-c) = 8abc - (a+b)(b+c)(c+a) \quad (2)$$

$$(a+b+c)(ab+bc+ca) = (a+b)(b+c)(c+a) + abc. \quad (3)$$

Identity (1) shows that  $u + v + w = -1$  while (2) easily gives

$$uv + vw + wu = z^2 \left( \frac{8abc}{(a+b)(b+c)(c+a)} - 1 \right). \quad (4)$$

But the condition  $|a + b + c| \leq 1$  writes as

$$(a+b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq 1,$$

that is,

$$\frac{(a+b)(b+c)(c+a) + abc}{abc} \leq 1$$

(using (3)) or finally

$$\frac{(a+b)(b+c)(c+a)}{abc} \leq 0.$$

From this and (4), we deduce that  $uv + vw + wu > 0$ .

Now, let

$$p(x) = (x-u)(x-v)(x-w) = x^3 + x^2 + (uv + vw + wu)x - uvw.$$

The derivative

$$p'(x) = 3x^2 + 2x + (uv + vw + wu)$$

is positive on  $[0, \infty)$  (since  $uv + vw + wu > 0$ ), hence  $p$  is an increasing function on  $[0, \infty)$  and therefore

$$p(x) > p(0) = -uvw > 0$$

for  $x \in [0, \infty)$ . Thus, the roots  $u, v, w$  of  $p(x)$  are negative real numbers and so

$$|u| + |v| + |w| = -u - v - w = -(u + v + w) = -(-1) = 1,$$

as desired.

**4368.** *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=2}^{\infty} [2^n (\zeta(n) - 1) - 1],$$

where  $\zeta$  denotes the Riemann zeta function defined as  $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ .

*We received 9 submissions, all correct, although one used WolframAlpha software. We present the solution by Paolo Perfetti, which is short, simple, and similar to the other seven non-WolframAlpha solutions.*

$$\begin{aligned} \sum_{n=2}^{\infty} [2^n (\zeta(n) - 1) - 1] &= \sum_{n=2}^{\infty} \left[ 2^n \sum_{k=2}^{\infty} \frac{1}{k^n} - 1 \right] \\ &= \sum_{n=2}^{\infty} \sum_{k=3}^{\infty} \frac{2^n}{k^n} \\ &= \sum_{k=3}^{\infty} \sum_{n=2}^{\infty} \frac{2^n}{k^n} \\ &= \sum_{k=3}^{\infty} \frac{4}{k^2} \frac{1}{1 - \frac{2}{k}} \\ &= 2 \sum_{k=3}^{\infty} \left[ \frac{1}{k-2} - \frac{1}{k-1} + \frac{1}{k-1} - \frac{1}{k} \right] \\ &\quad \underbrace{\hspace{1.5cm}}_{\text{telescoping}} 2\left(1 + \frac{1}{2}\right) = 3. \end{aligned}$$

**4369.** *Proposed by Mihaela Berindeanu.*

On the sides of triangle  $ABC$ , take points

$$A_1, A_2 \in (BC), B_1, B_2 \in (AC), C_1, C_2 \in (AB),$$

so that

$$BA_1 = A_2C, CB_1 = B_2A, AC_1 = C_2B.$$

On  $B_2C_1$ ,  $A_1C_2$ ,  $A_2B_1$  take  $A_3$ ,  $B_3$ ,  $C_3$  so that

$$\frac{C_1A_3}{A_3B_2} = \frac{A_1B_3}{B_3C_2} = \frac{B_1C_3}{C_3A_2} = k.$$

Find all values of  $k$  for which  $AA_3$ ,  $BB_3$ ,  $CC_3$  are concurrent lines.

*We received 5 correct submissions. We present the solution by Michel Bataille.*

Let  $\alpha > 0$  be such that  $\overrightarrow{BA_1} = \alpha \overrightarrow{BC} = \overrightarrow{A_2C}$ . Then,

$$A_1 = (1 - \alpha)B + \alpha C \quad \text{and} \quad A_2 = \alpha B + (1 - \alpha)C.$$

In a similar way, there exist  $\beta, \gamma > 0$  such that

$$\begin{aligned} B_1 &= \beta A + (1 - \beta)C, \\ B_2 &= (1 - \beta)A + \beta C, \\ C_1 &= (1 - \gamma)A + \gamma B, \\ C_2 &= \gamma A + (1 - \gamma)B. \end{aligned}$$

Expressing that  $\overrightarrow{C_1A_3} = k\overrightarrow{A_3B_2}$ , we obtain  $(1 + k)A_3 = kB_2 + C_1$ , that is,

$$(1 + k)A_3 = \rho A + \gamma B + k\beta C,$$

where  $\rho = 1 + k - \gamma - k\beta$ . Similarly,

$$\begin{aligned} (1 + k)B_3 &= k\gamma A + \sigma B + \alpha C, \\ (1 + k)C_3 &= \beta A + k\alpha B + \tau C, \end{aligned}$$

where  $\sigma, \tau \in \mathbb{R}$ . It readily follows that the equations of the lines  $AA_3$ ,  $BB_3$ ,  $CC_3$  respectively are

$$k\beta y - \gamma z = 0, \quad \alpha x - k\gamma z = 0, \quad k\alpha x - \beta y = 0.$$

Now, the common point of  $BB_3$  and  $CC_3$  is  $J = (k\beta\gamma : k^2\alpha\gamma : \alpha\beta)$  (not at infinity since  $k\beta\gamma + k^2\alpha\gamma + \alpha\beta$  is positive, hence not zero), and this point  $J$  is on  $AA_3$  if and only if  $k\beta(k^2\alpha\gamma) - \gamma(\alpha\beta) = 0$ , that is, if and only if  $k^3 = 1$ .

We conclude that  $AA_3$ ,  $BB_3$ ,  $CC_3$  are concurrent if and only if  $k = 1$  (which means that  $A_3$ ,  $B_3$ ,  $C_3$  are the midpoints of  $B_2C_1$ ,  $C_2A_1$ ,  $A_2B_1$ , respectively).

**4370.** *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Solve the following system of equations:

$$\begin{cases} a + b + c + d = 4, \\ a^2 + b^2 + c^2 + d^2 = 7, \\ abc + abd + acd + bcd - abcd = \frac{15}{16}. \end{cases}$$

*We received 10 submissions, including the one from the proposers, 9 of which are correct, and the other one, incomplete. We present the same solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Trey Smith (jointly); and Ramanujan Srihari.*

Note first that

$$\begin{aligned} 16 &= (a + b + c + d)^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd) \\ &= 7 + 2(ab + ac + ad + bc + bd + cd) \end{aligned}$$

so

$$ab + ac + ad + bc + bd + cd = \frac{9}{2}.$$

Letting  $w = a - 1$ ,  $x = b - 1$ ,  $y = c - 1$ , and  $z = d - 1$ , we then have

$$w + x + y + z = 0$$

and

$$\begin{aligned} wxyz &= (a-1)(b-1)(c-1)(d-1) \\ &= (abcd) - (abc + abd + acd + bcd) \\ &\quad + (ab + ac + ad + bc + bd + cd) - (a + b + c + d) + 1 \\ &= -\frac{15}{16} + \frac{9}{2} - 4 + 1 \\ &= \frac{9}{16}. \end{aligned}$$

Also,

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= (a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 \\ &= a^2 + b^2 + c^2 + d^2 - 2(a + b + c + d) + 4 \\ &= 7 - 8 + 4 \\ &= 3. \end{aligned}$$

By the AM-GM Inequality, we have

$$3 = w^2 + x^2 + y^2 + z^2 \geq 4\sqrt[4]{w^2x^2y^2z^2} = 4\sqrt{\frac{9}{16}} = 3$$

with equality if and only if

$$w^2 = x^2 = y^2 = z^2 = \frac{3}{4}.$$

Hence,

$$(w, x, y, z) = \left( \pm \frac{\sqrt{3}}{2}, \pm \frac{\sqrt{3}}{2}, \pm \frac{\sqrt{3}}{2}, \pm \frac{\sqrt{3}}{2} \right).$$

Since  $w + x + y + z = 0$ , we then easily see that exactly two of  $w, x, y$ , and  $z$  must be  $\frac{\sqrt{3}}{2}$ , and the other two equal  $-\frac{\sqrt{3}}{2}$ . Therefore, there are six solutions given by

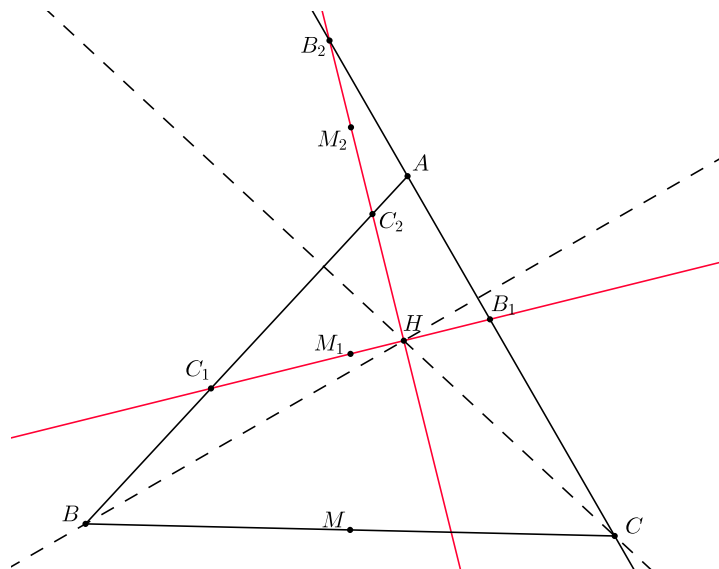
$$(a, b, c, d) = \left( 1 + \frac{\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2} \right)$$

together with its five permutations.

**4371.** *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let two perpendicular lines pass through the orthocenter  $H$  of a triangle  $ABC$ . Suppose they meet the sides  $AB$  and  $AC$  in  $C_1, B_1$  and  $C_2, B_2$ , respectively. Define  $M_i$  as the midpoint of  $B_iC_i$  ( $i = 1, 2$ ) and  $M$  as the midpoint of  $BC$ . Prove that  $M, M_1$  and  $M_2$  are collinear.

*We received 6 submissions, of which 5 were complete and correct. We present the solution by Michel Bataille.*



The perpendicular lines through  $H$  define a system of orthonormal axes with origin at  $H$ . We assign the coordinates

$$B_1(2b_1, 0), C_1(2c_1, 0), B_2(0, 2b_2), C_2(0, 2c_2)$$

for some real numbers  $b_1, b_2, c_1, c_2$ . Then,  $M_1(b_1 + c_1, 0)$ ,  $M_2(0, b_2 + c_2)$  and the equation of the line  $M_1M_2$  is

$$(b_2 + c_2)x + (b_1 + c_1)y = (b_1 + c_1)(b_2 + c_2). \quad (1)$$

The equation of the line  $AB = C_1C_2$  is  $c_2x + c_1y = 2c_1c_2$ . The altitude from  $B$  is perpendicular to  $AC = B_1B_2$  and passes through the origin  $H$ , so its equation is  $b_1x - b_2y = 0$ . The lines  $AB$  and  $BH$  intersect in  $B$ , so we calculate

$$B \left( \frac{2c_1c_2b_2}{b_1c_1 + b_2c_2}, \frac{2c_1c_2b_1}{b_1c_1 + b_2c_2} \right).$$

Similarly (*mutatis mutandis*), we have

$$C \left( \frac{2b_1b_2c_2}{b_1c_1 + b_2c_2}, \frac{2b_1b_2c_1}{b_1c_1 + b_2c_2} \right).$$



Thus the midpoint  $M$  of  $BC$  has coordinates

$$x_M = \frac{b_2 c_2 (b_1 + c_1)}{b_1 c_1 + b_2 c_2} \quad \text{and} \quad y_M = \frac{b_1 c_1 (b_2 + c_2)}{b_1 c_1 + b_2 c_2};$$

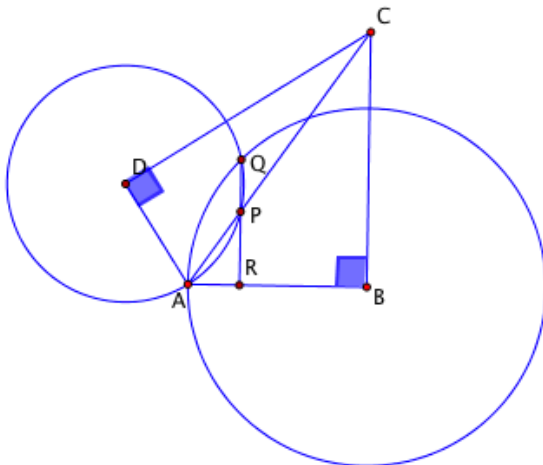
and

$$\begin{aligned} (b_2 + c_2)x_M + (b_1 + c_1)y_M &= \frac{b_2 c_2 (b_1 + c_1)(b_2 + c_2)}{b_1 c_1 + b_2 c_2} + \frac{b_1 c_1 (b_1 + c_1)(b_2 + c_2)}{b_1 c_1 + b_2 c_2} \\ &= (b_1 + c_1)(b_2 + c_2) \end{aligned}$$

and from (1) we obtain that  $M$  is on the line  $M_1 M_2$ , as desired.

**4372.** *Proposed by J. Chris Fisher.*

Given a quadrangle  $ABCD$  with right angles at  $B$  and  $D$ , let the circle  $(D, DA)$  (with center  $D$  and radius  $DA$ ) intersect the line  $AC$  at  $P$  and the circle  $(B, BA)$  at  $Q$ . Prove that  $PQ$  is perpendicular to  $AB$ .



*We received 7 submissions, of which 5 were correct and complete. We present a solution based on the submission by Sushanth Sathish Kumar, completed and corrected by the editor.*

Since  $AQ$  is the radical axis of circles  $(B, BA)$  and  $(D, DA)$ , the line  $DB$  bisects  $\angle QDA$ . By the inscribed angle theorem,

$$\angle QPA = \frac{360^\circ - \angle QDA}{2} = 180^\circ - \angle BDA.$$

Since  $\angle QPC$  and  $\angle QPA$  are supplementary, it follows that  $\angle QPC = \angle BDA$ . Quadrilateral  $ABCD$  is cyclic, so also  $\angle BDA = \angle ACB$ . Therefore, we have  $\angle QPC = \angle PCB$  (since  $\angle PCB$  and  $\angle ACB$  are the same angle), which implies  $\overline{PQ} \parallel \overline{CB}$ . From this we can conclude  $PQ \perp AB$ , as desired.

**4373.** *Proposed by Michel Bataille.*

Let  $p$  be an odd prime. Let  $q$  and  $r$  be the quotient and the remainder in the division of the positive integer  $n$  by  $p$  and let  $S_n = \sum_{k=r}^n (-1)^{k-r} \binom{k}{r}$ . Show that  $S_n \equiv 0 \pmod{p}$  if and only if  $q$  is odd and  $2^r \equiv 1 \pmod{p}$ .

*We received three submissions, including the one from the proposer. We present the solution by Modhav R. Modak, modified slightly by the editor.*

By assumption, we have  $n = qp + r$ . For  $i = 0, 1, 2, \dots, q-1$ , let

$$A_i = \sum_{k=0}^{p-1} (-1)^k \binom{r+ip+k}{r}.$$

Since  $p$  is odd,  $(-1)^{ip} = 1$  or  $-1$  depending on whether  $i$  is even or odd. Hence we can write

$$\begin{aligned} S_n &= \sum_{k=0}^{p-1} (-1)^k \binom{r+k}{r} + \sum_{k=p}^{2p-1} (-1)^k \binom{r+k}{r} + \cdots + \sum_{k=q}^{qp-1} (-1)^k \binom{r+k}{r} \\ &= A_0 - A_1 + \cdots + (-1)^{qp-1} A_{q-1} + (-1)^{qp} \binom{r+qp}{r}. \end{aligned} \quad (1)$$

Since  $p > 2$  and  $r < p$ , we have for  $r \leq k$ ,

$$\begin{aligned} (k+p)(k+p-1) \cdots (k+p-r+1) &\equiv (k)(k-1) \cdots (k-r+1) \pmod{p} \\ \text{so that } \binom{k+p}{r} &\equiv \binom{k}{r} \pmod{p} \quad \text{since } p \nmid r!. \end{aligned} \quad (2)$$

By (2) we then have, for  $i = 0, 1, 2, \dots, q-1$ , that

$$\begin{aligned} A_i &\equiv \binom{r}{r} - \binom{r+1}{r} + \cdots + \binom{r+p-1}{r} \pmod{p} \\ \text{i.e. } A_i &\equiv A_0 \pmod{p}. \end{aligned}$$

Hence from (1) we get, modulo  $p$ ,

$$S_n \equiv \begin{cases} 1, & \text{if } q \text{ is even,} \\ A_0 - 1 & \text{if } q \text{ is odd.} \end{cases} \quad (3)$$

We next establish the following identity:

$$\sum_{k=0}^m (-1)^k \binom{r+k}{r} = \frac{1}{2^{r+1}} \left[ 1 + (-1)^m \left\{ \sum_{k=0}^r 2^k \binom{m+k}{k} \right\} \right], \quad (4)$$

To prove (4), note that

$$\sum_{k=0}^{\infty} (-1)^k \binom{m+k}{k} x^k = \frac{1}{(1+x)^{m+1}}.$$

Hence

$$\sum_{k=0}^m (-1)^k \binom{r+k}{r} = \text{coefficient of } x^m \text{ in } f(x) = \frac{1}{(1-x)(1+x)^{m+1}}.$$

Now  $f(x)$  can be decomposed into partial fractions as

$$f(x) = \frac{1}{2^{r+1}} \left[ \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{(1+x)^2} + \frac{2^2}{(1+x)^3} + \cdots + \frac{2^r}{(1+x)^{r+1}} \right] \quad (5)$$

since the right hand side of (5) can be written as

$$\begin{aligned} &= \frac{1}{2^{r+1}} \left[ \frac{1}{1-x} + \frac{1}{2} \sum_{k=1}^{r+1} \frac{2^k}{(1+x)^k} \right] \\ &= \frac{1}{2^{r+1}} \left[ \frac{1}{1-x} + \frac{1}{2} \cdot \frac{2}{1+x} \cdot \frac{(2/(1+x))^{r+1} - 1}{(2/(1+x)) - 1} \right] \\ &= \frac{1}{2^{r+1}} \left[ \frac{1}{1-x} + \frac{2^{r+1} - (1+x)^{r+1}}{(1-x)(1+x)^{r+1}} \right] = f(x). \end{aligned}$$

Hence (4) follows by taking the coefficient of  $x^m$  in (5).

Letting  $m = p-1$  in (4), we then have

$$\begin{aligned} A_0 &= \sum_{k=0}^{p-1} (-1)^k \binom{r+k}{r} = \frac{1}{2^{r+1}} \left( 1 + (-1)^{p-1} \left( \sum_{k=0}^{p-1} 2^k \binom{p-1+k}{k} \right) \right) \\ &= \frac{1}{2^{r+1}} \left( 1 + 1 + 2 \binom{p}{1} + 2^2 \binom{p+1}{2} + \cdots + 2^{p-1} \binom{2p-2}{p-1} \right), \end{aligned} \quad (6)$$

so

$$A_0 \equiv \frac{1}{2^r} \pmod{p},$$

since

$$\binom{p}{1}, \binom{p+1}{2}, \dots, \binom{2p-2}{p-1}$$

are all congruent to zero modulo  $p$ .

Hence, (3) becomes

$$S_n \equiv \begin{cases} 1, & \text{if } q \text{ is even,} \\ \frac{1}{2^r} \cdot (1 - 2^r), & \text{if } q \text{ is odd.} \end{cases} \quad (7)$$

Thus if  $q$  is odd and  $2^r \equiv 1 \pmod{p}$ , then (6) shows that  $S_n \equiv 0 \pmod{p}$ . Further, if  $q$  is even or if  $2^r \not\equiv 1 \pmod{p}$ , then  $S_n \not\equiv 0 \pmod{p}$ , completing the proof.

**4374.** *Proposed by Šefket Arslanagić.*

For a fixed positive integer  $n$ , solve the equation

$$|1 - |2 - |3 - \cdots - |n - x \underbrace{|\cdots\cdots|}_{n \text{ vertical bars}}| = 1.$$

*There were 6 correct solutions. We present the standard approach.*

Let  $E_n$  denote the given equation,  $S_n$  the set of its solutions, and  $T_n = \frac{1}{2}n(n+1)$ , the sum of the first  $n$  positive integers. It is readily checked that  $S_1 = \{0, 2\}$ ,  $S_2 = \{0, 2, 4\}$ ,  $S_3 = \{-1, 1, 3, 5, 7\}$ . We prove by induction that

$$S_n = \{-T_{n-2}, -T_{n-2} + 2, -T_{n-2} + 4, \dots, -T_{n-2} + 2(T_{n-1} + 1) = T_n + 1\}$$

for  $n \geq 3$ .

Suppose that this holds for  $n = m$ . Then  $x$  is a solution of  $E_{m+1}$  if and only if  $|m+1-x|$  is a nonnegative solution of  $E_m$ , if and only if

$$|m+1-x| = -T_{m-2} + 2k$$

for  $\frac{1}{2}T_{m-2} \leq k \leq T_{m-1} + 1$ .

The smallest value assumed by  $|m+1-x|$  for a solution  $x$  is equal to 0 when  $m \equiv 1, 2 \pmod{4}$  and 1 when  $m \equiv 0, 3 \pmod{4}$ . It follows that the numbers in  $S_{m+1}$  are all consecutively even when  $m \equiv 0, 1 \pmod{4}$  and all consecutively odd when  $m \equiv 2, 3 \pmod{4}$ .

The solution  $x$  assumes its smallest value when

$$x = (m+1) - (T_m + 1) = -(T_m - m) = -T_{m-1}$$

and its largest value when

$$x = (m+1) + (T_m + 1) = T_{m+1} + 1.$$

Thus,  $S_{m+1}$  consists of all integers between  $-T_{m-1}$  and  $T_{m+1} + 1$  inclusive that share the same parity.

**4375.** *Proposed by George Stoica.*

Consider two sequences  $\{x_n\}$  and  $\{y_n\}$  such that  $\{x_n\} \cup \{y_n\} = \mathbb{N}$ . Let  $l > 1$  be given. Prove that  $\liminf x_n/n \geq l$  if and only if  $\limsup y_n/n \leq l/(l-1)$ .

*The problem is incorrect as it stands. A counterexample was offered by Oliver Geupel: let  $x_n = n^2$ ,  $y_{2n-1} = n^2$  and  $y_{2n} = n$ . If you want one where the two sequences are exhaustive but not overlapping take  $x_n = 3n$  and  $y_{2^m} = 2^{m+1}$  with the other  $y_n$  picking up the missing numbers. In this case, take  $l = 3$  and  $l/(l-1) = 3/2$ .*

The following solution is essentially that of the proposer, with additional hypotheses to make it work.

We assume that both  $\{x_n\}$  and  $\{y_n\}$  are increasing, nonoverlapping and jointly exhaustive sequences of positive integers. Let  $t$  be a positive integer and let  $N(t)$  be the number of indices  $n$  for which  $y_n \leq t$  and  $M(t)$  the number of indices  $n$  for which  $x_n \leq t$ . Then  $M(t) + N(t) = t$ .

Suppose  $x_m \leq t < x_{m+1}$ . Then  $M(t) = m$  and

$$\frac{x_m}{m} \leq \frac{t}{M(t)} < \left( \frac{x_{m+1}}{m+1} \right) \left( \frac{m+1}{m} \right)$$

so that

$$\liminf_{t \rightarrow \infty} \frac{t}{M(t)} = \liminf_{m \rightarrow \infty} \frac{x_m}{m} = a \geq 1.$$

Similarly

$$\limsup_{t \rightarrow \infty} \frac{t}{N(t)} = \limsup_{m \rightarrow \infty} \frac{y_m}{m} = b \geq 1.$$

Since

$$\frac{M(t)}{t} + \frac{N(t)}{t} = 1$$

and  $\limsup M(t)/t = 1 - \liminf N(t)/t$ , we find that  $\frac{1}{a} + \frac{1}{b} = 1$  and  $b = \frac{a}{a-1}$ .

Let  $l$  be as in the hypothesis. Then,

$$\begin{aligned} \liminf x_n/n &= \liminf t/M(t) \geq l \Leftrightarrow \limsup M(t)/t \leq 1/l \\ &\Leftrightarrow \liminf N(t)/t \geq 1 - (1/l) = (l-1)/l \\ &\Leftrightarrow \limsup t/N(t) = \limsup y_n/n \leq l/(l-1). \end{aligned}$$

