MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

Volume 22 1989/90 Number 2

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 22 of *Mathematical Spectrum* will consist of three issues, of which this is the second. The first was published in September 1989 and the third will appear in May 1990.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

EDITORIAL COMMITTEE

ADVISORY BOARD

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH

Mathematical Spectrum Awards for Volume 21

Prizes have been awarded to the following student readers for contributions published in Volume 21:

Robert Cannell for his article 'Highly Composite Numbers' (pages 13–17) and problems;

Amites Sarkar for his article 'A Fibonacci Sum' (pages 85–90) and other contributions.

The Editors are delighted to announce that a gift of £1000 has been received from a mathematics teacher from India who wishes to encourage student contributions by creating a prize fund. This generous donation has been invested and the interest on it will be used for future awards. Beginning with Volume 22 (1989–90) prizes of value up to £50 are available for articles and of up to £25 for letters, solutions to problems, or other items. In accordance with the wish of the donor, prizes will in future take the form of mathematical books or software, to be chosen by the recipients, rather than cash.

Pop Maths Roadshow

HILARY OCKENDON, *Somerville College, Oxford*

I've just visited the Pop Maths Roadshow in Leeds. It's an ambitious project to show mathematics as an exciting and dynamic subject and the programme at Leeds has something for everyone with the slightest interest in mathematics.

The core of the show, which will be touring the country for the next year, is a series of exhibitions; these include several 'hands-on' games and puzzles sections, computer games and simulations, and a series of stunning pictures of 'chaos' as well as much information on teaching and education. Each centre will add its own programme of lectures and discussions; in Leeds this included lectures designed for audiences from 7 upwards and subjects ranging from magic to education. There were also some exhibits from local industry and a beautiful bronze sculpture of knots and geometrical figures by the local artist John Robinson.

The puzzles sections will probably attract primary-school children most. Old chestnuts like optical illusions and rearranging squares to prove Pythagoras' theorem are there but there are some less familiar ones too. The games room provides equipment and instruction on 'simple' games from all over the world which, besides being challenging to play, show fascinating anthropological glimpses of how civilizations developed in parallel in different continents. The computer-generated pictures of fractals and chaos are dramatic and worth seeing, but I was disappointed that the opportunity to explain their generation using the software available on PCs had not been used to better effect. Generally, I'd have liked to see more attempts to relate 'fun' concepts seen in the exhibition with mathematics as done in the classroom. The show will certainly be a fruitful source of ideas for teachers but they too might appreciate more guidance on how it all fits in with the National Curriculum!

You should all visit this show and you should allow plenty of time to settle down and really tackle the projects while you are there. No doubt the details will change as the roadshow rolls along but what will really make it roll is active participation from lots of people, old and young.

If you have not already caught it, the roadshow will be stopping at the following places in 1990: Institute of Education, London (16 January–9 February); Southampton University (13–28 February); St Andrew's College of Education, Glasgow (7–16 March); Elgin Academy (20–29 March); Edinburgh Science Festival (1–17 April); Aberdeen University (18 April–3 May); Liverpool University (4–20 May); Gateshead (8–17 June); Sheffield City Polytechnic (18–28 June); Cambridge University (7–14 July); East of England Show, Peterborough (16–20 July); Science Museum, London (23 July–5 September); Stockport Local Education Authority (8–17 September); Birmingham City Council (20 September–2 October).

The unseen pages problem

Looking up a word in a dictionary is, let us stipulate, equivalent to selecting a page at random. How many years does it take until the average person has laid eyes on every page of the dictionary he regularly uses? More precisely, find the probability that, after n random references to an m-page book, exactly k pages have been referred to at least once.

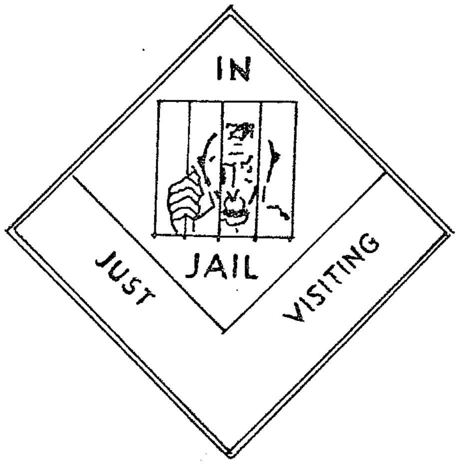
MICHAEL A. DE LA MAZA
10 Tumbleweed,
Irvine, CA 92715, USA.

Readers interested in this problem may like to refer to Feller's book *An Introduction to Probability Theory and its Applications* (third edition, 1968), John Wiley and Sons, New York, pages 101–106.—Editor.

Modelling Monopoly

NICK MACKINNON AND FORM 2Mr, *Winchester College*

Nick MacKinnon is a graduate of Oxford University. His main mathematical interest is in number theory. His other interests include mountaineering and pot-holing with his family.



1. Introduction

Which is the better square on which to put your first house in Monopoly,* Pentonville Road or Piccadilly?

As the costs of the houses are £50 and £150, respectively, and the rents received are £40 and £120, there seems to be little to choose as far as return on investment is concerned. But this argument pays no attention to the probabilities of landing on the squares. What is the probability of landing on a particular Monopoly square?

2. A Monte Carlo solution

When I put this question to my class, they quickly decided that the best thing to do was simulate Monopoly on a computer. The programming involved in this is an excellent exercise in abstracting from the rules of the game the essentials. These are:

(i) There are 40 squares numbered from 0 to 39.

(ii) Landing on square 30 (Go to jail) means moving directly to square 10.

(iii) Throwing three successive doubles means moving directly to square 10.

(iv) Landing on squares 7, 22 or 36 means drawing a chance card. There is then a 7/16 probability of having to move to some other square (e.g. Advance to Go).

*This is the British version of Monopoly, which uses London street names.

(v) Landing on squares 2, 17 or 33 means drawing a Community Chest card. There is then a 3/16 probability of having to move to some other square (e.g. Go back to Old Kent Road).

The program that simulates Monopoly is then easily written; it is given at the end of the article.

3. Results

As it stands, the program visits 100 000 squares before stopping. It is difficult to decide how many trials are needed before you can be reasonably certain of, say, three decimal places of accuracy in the results. We were pragmatic about this, and decided that, if there was not too much difference between the data for n trials and $10n$ trials, then the data ought to be not bad. There must be a criterion for deciding how much accuracy to expect but I don't know enough statistics to find one. The table below summarises the findings of the program. They agree with experience rather well, with the second most expensive property, Park Lane, being the least likely property visited and the Jail the most likely.

Table of Probabilities

Go	0.029	Free Parking	0.026
Old Kent Rd	0.025	The Strand	0.029
Community Chest	0.020	Chance	0.025
Whitechapel Rd	0.022	Fleet St	0.024
£200	0.022	Trafalgar Sq	0.030
Kings Cross Station	0.022	Fenchurch St Station	0.030
Angel Islington	0.022	Leicester Sq	0.025
Chance	0.022	Coventry St	0.025
Euston Rd	0.022	Waterworks	0.026
Pentonville Rd	0.022	Piccadilly	0.025
Jail	0.058	Go to Jail	0.024
Pall Mall	0.022	Regent St	0.024
Electric Company	0.022	Oxford St	0.025
Whitehall	0.023	Community Chest	0.026
Northumberland Ave	0.022	Bond St	0.023
Marylebone St Station	0.025	Liverpool St Station	0.022
Bow St	0.027	Chance	0.022
Community Chest	0.028	Park Lane	0.021
Marlborough St	0.026	£100	0.020
Vine St	0.028	Mayfair	0.025

4. Towards an analytic solution

Surprisingly for such a difficult seeming problem, an analytic solution is possible. Let us look at a simple two-square version of Monopoly, played with one die. Square A is a 'Chance' square. There are three chance cards, one marked 'Advance to B' and the other two marked 'Pay the Poll Tax, £200' and 'Your shares collapse, lose £100'. The square B just says Go to A on it. We can construct a transition matrix for this game, which stores the probabilities that you will be on a given square at the end of a move. If you are on A you will be on B next either if you draw the 'Advance to B' card (with probability $\frac{1}{3}$) or if you don't and you throw a 1, 3 or 5 (probability $\frac{2}{3}\times\frac{1}{2} = \frac{1}{3}$), a total probability of $\frac{2}{3}$. So you will stay on A with probability $\frac{1}{3}$. If on the other hand you start on B you will certainly be on A next. So the transition matrix is:

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc} A & B \end{array} \\ \text{To} & \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix}. \end{array}$$

If the vector

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

stores the probabilities that the counter is on square A and square B at the end of move n then

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$

If the starting square is A the probabilities that the counter will be on square A or B at some time in an infinite game is given by

$$\lim_{n\to\infty} \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The matrix has eigenvalues 1 and $-\frac{2}{3}$ with eigenvectors

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so we can rewrite the above as

$$\lim_{n\to\infty} \left(\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \right)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which is

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that the probabilities are

$$\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}.$$

The construction of the 40×40 transition matrix for real Monopoly is not particularly difficult. I should think that at least one reader out there has a BBC Archimedes and a program to do the diagonalising. So maybe someone can perform a service to Monopoly players everywhere.

```
  10 DIM a%(39)
  20 s%=0  [The starting square]
  30 dice1%=RND(6):dice2%=RND(6)
  40 IF dice1%=dice2% THEN d%=d%+1 ELSE d%=0  [d% checks for 3 doubles]
  50 IF d% = 3 THEN s%=10:d%=0: GOTO 70  [If 3 doubles then go to jail]
  60 s% = (s% + dice1% + dice2%) MOD 40  [Add throw on to s%]
  70 a%(s%) = a%(s%) + 1 [a%(s%) is the number of visits to square s%]
  80 visits = visits + 1  [Total no. of squares visited]
  90 IF visits = 100000 THEN GOTO 1000
 100 IF s% = 30 THEN s%=10:GOTO 70  [Go to jail]
 110 IF s% = 7 OR s% = 22 OR s% = 36 THEN GOTO 200  [Chance]
 120 IF s% = 2 OR s% = 17 OR s% = 33 THEN GOTO 300  [Community chest]
 130 GOTO 30
 200 ch% = RND(16)
 210 IF ch% > 7 THEN GOTO 30   [No movement]
 220 IF ch% = 1 THEN s% = 0    [Advance to Go]
 230 IF ch% = 2 THEN s% = 10   [Go to jail]
 240 IF ch% = 3 THEN s% = 25   [Take a trip to Marylebone Station]
 250 IF ch% = 4 THEN s% = 24   [Advance to Trafalgar Square]
 260 IF ch% = 5 THEN s% = 39   [Advance to Mayfair]
 270 IF ch% = 6 THEN s% = 21   [Advance to Pall Mall]
 280 IF ch% = 7 THEN s% = s%-3 [Go back 3 spaces]
 290 GOTO 70
 300 cc% = RND(16)
 310 IF cc% > 3 THEN GOTO 30 [No movement]
 320 IF cc% = 1 THEN s% = 0  [Advance to Go]
 330 IF cc% = 2 THEN s% = 10 [Go to jail]
 340 IF cc% = 3 THEN s% = 1  [Go back to Old Kent Road]
 350 GOTO 70
1000 FOR i% = 0 TO 39
1010 PRINT i%,a%(i%)/100000
1020 NEXT i%
```

A Locus in the Complex Plane

CHRISTOPHER ASH, *Fettes College, Edinburgh*

Christopher Ash is Head of Mathematics at Fettes College, Edinburgh. He read Engineering and Electrical Sciences at Cambridge and worked as a development engineer with British Telecom before becoming a teacher.

Complex numbers may be expressed either in the form $z = x + \mathrm{i}y$ (Cartesian form) or in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$ (polar form), where x is the real part of z (written $\operatorname{Re} z$), y is the imaginary part of z (written $\operatorname{Im} z$), $r = \sqrt{x^2+y^2}$ is the modulus of z (written $|z|$) and θ is the argument of z (written $\arg z$). The argument of zero is not defined. In particular we note that $\theta = \arg z$ is measured anticlockwise from the positive real axis and that we work modulo 2π. These relationships are illustrated in figure 1.

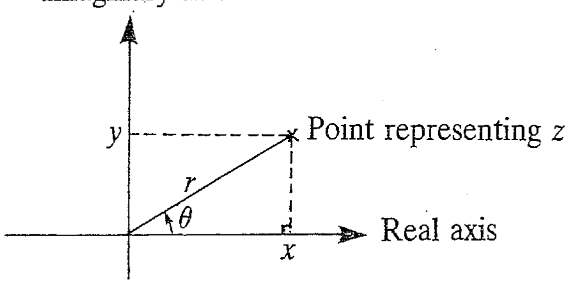
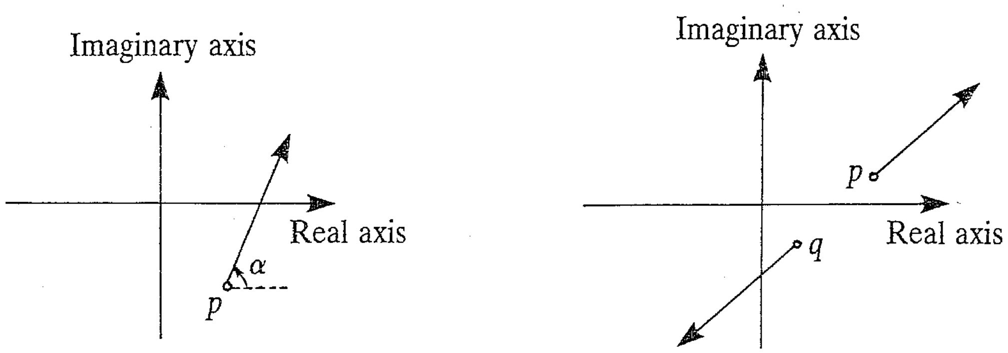


Figure 1

Various geometrical loci may conveniently be described in terms of complex numbers. Some of these appear in school textbooks (see the references). In particular, $\arg(z-p) = \alpha$ ($p \in \mathbb{C}, \alpha \in \mathbb{R}$) gives a half-line and $\arg(z-p) - \arg(z-q) = 0$ ($p, q \in \mathbb{C}, p \neq q$) gives a pair of half-lines, excluding the end points in each case. These are illustrated in figure 2.



(*a*) $\arg(z-p) = \alpha$ (*b*) $\arg(z-p) - \arg(z-q) = 0$

Figure 2

The line segment joining the points representing p and q in figure 2 (b) is produced by $\arg(z-p)-\arg(z-q) = \pi$. (Again, the end points are excluded.)

In this article we consider the locus

$$\arg(z-p)+\arg(z-q) = 0,$$

obtained by replacing the subtraction by an addition in the equation of figure 2 (b). This locus is not nearly as simple after this alteration. It is helpful to consider first the simpler problem

$$\arg(z-p)+\arg(z+p) = 0,$$

choosing $q = -p$, so that the fixed points p and q are symmetrically placed about the origin. We note first that $\arg z_1+\arg z_2 = \arg z_1 z_2$ when arguments are defined modulo 2π as we have done; also that

$$\arg z = 0 \Leftrightarrow \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z = 0.$$

In order to obtain a Cartesian equation for the locus we write $p = a+ib$ and $z = x+iy$. We take $a \neq 0$ and $b \neq 0$, and consider the special cases at the end. Our working then proceeds as follows:

$$\begin{aligned}
& \arg(z-p)+\arg(z+p) = 0 \\
\Leftrightarrow\ & \arg\{(z-p)(z+p)\} = 0 \\
\Leftrightarrow\ & \arg(z^2-p^2) = 0 \\
\Leftrightarrow\ & \arg\{(x^2-y^2+2ixy)-(a^2-b^2+2iab)\} = 0 \\
\Leftrightarrow\ & \arg\{(x^2-a^2)-(y^2-b^2)+2i(xy-ab)\} = 0 \\
\Leftrightarrow\ & x^2-a^2 > y^2-b^2 \text{ and } xy = ab.
\end{aligned}$$

Substitution of $x = ab/y$ in the inequality leads to

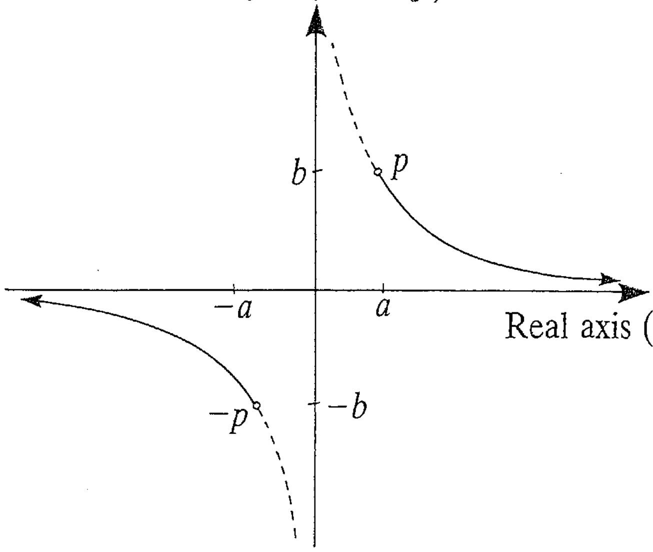
$$\frac{(b^2-y^2)(a^2+y^2)}{y^2} > 0 \Leftrightarrow |y| < |b|.$$

The resulting locus is illustrated in figure 3 (drawn for $a, b > 0$). We note that by a similar argument the remaining parts of the hyperbola (shown dotted but again excluding the points p and $-p$) are obtained from

$$\arg(z-p)+\arg(z+p) = \pi.$$

The special cases $a = 0$ or $b = 0$ are illustrated in figure 4. These are limiting forms of the hyperbola.

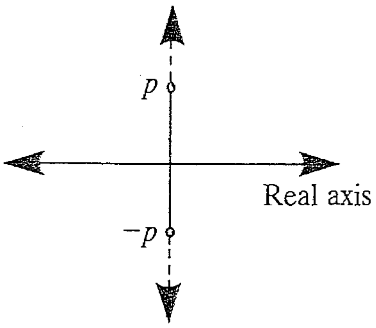
Before returning to the original problem it is worth noting that we have stumbled on an interesting geometrical property of the rectangular



$\arg(z-p)+\arg(z+p) = 0$
(full lines)

$\arg(z-p)+\arg(z+p) = \pi$
(dotted lines)

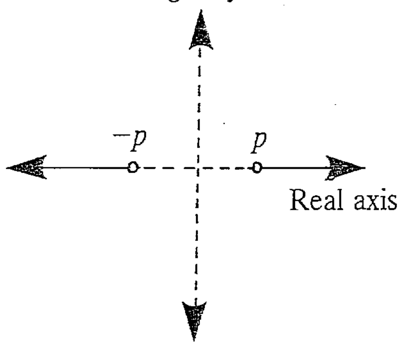
Figure 3



(a) $a = 0$

$\arg(z-p)+\arg(z+p) = 0$
(full line)

$\arg(z-p)+\arg(z+p) = \pi$
(dotted line)



(b) $b = 0$

$\arg(z-p)+\arg(z+p) = 0$
(full line)

$\arg(z-p)+\arg(z+p) = \pi$
(dotted line)

Figure 4

hyperbola. If P and Q are any two points on the hyperbola symmetrically placed about its centre of symmetry, and R is a distant point on the hyperbola, then the bisectors of $\angle PRQ$ are parallel to the asymptotes of the hyperbola. This is illustrated in figure 5. The result follows from our equation because $\arg(z-p)$ corresponds to $-\alpha$ and $\arg(z+p)$ corresponds to $+\beta$ in the figure.

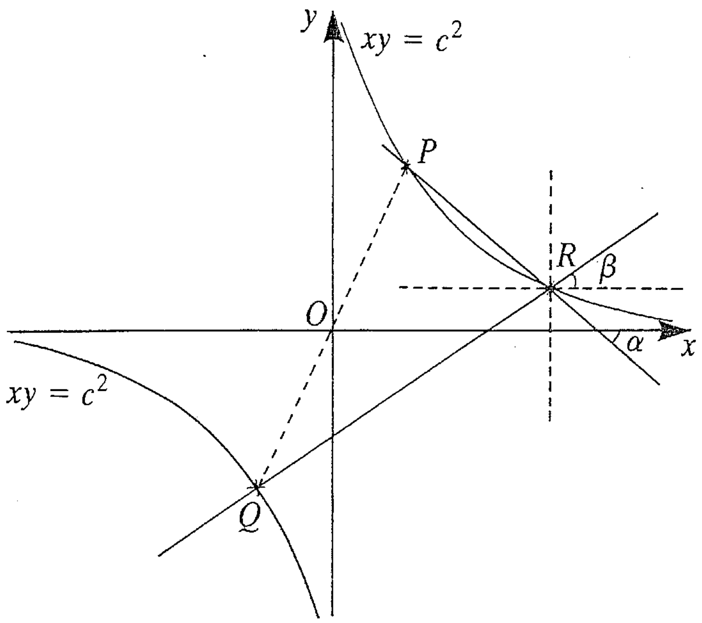


Figure 5

This property may be proved more simply by using the usual parametric representation

$$P = \left(pc, \frac{c}{p}\right), \qquad Q = \left(-pc, -\frac{c}{p}\right), \qquad R = \left(rc, \frac{c}{r}\right).$$

Hence

$$\text{gradient of } PR = \frac{c/r - c/p}{cr - cp} = -\frac{1}{pr}$$

and

$$\text{gradient of } QR = \frac{c/r + c/p}{cr + cp} = \frac{1}{pr}$$

so that PR and QR are equally inclined (in opposite senses) to the x-axis, and the result follows. This provides an interesting link between two topics (complex numbers and conic sections) which are often thought of in a compartmentalised way.

Returning to the original problem, we note that since our geometrical property is clearly not affected by a translation, we may proceed as follows. The points p and q in the general loci

$$\arg(z-p) + \arg(z-q) = 0, \pi$$

are symmetrically placed about their midpoint $\frac{1}{2}(p+q)$. Thus if we rewrite the loci in the form

$$\arg[\{z - \tfrac{1}{2}(p+q)\} - \tfrac{1}{2}(p-q)] + \arg[\{z - \tfrac{1}{2}(p+q)\} + \tfrac{1}{2}(p-q)] = 0, \pi$$

(equivalent to translating the complex plane appropriately), then we can solve this for $z-\frac{1}{2}(p+q)$ and hence for z. If we take $p = a+ib$, $q = c+id$, the resulting locus is

$$\{x-\tfrac{1}{2}(a+c)\}\{y-\tfrac{1}{2}(b+d)\} = \tfrac{1}{4}(a-c)(b-d)$$

with y restricted to lie inside or outside the interval $b < y < d$ (taking $b < d$ without loss of generality) as appropriate. The special cases $a = c$ (where p and q lie on the same vertical line) and $b = d$ (where p and q lie on the same horizontal line) should be considered separately, along the same lines as figure 4.

The family of pairs of loci

$$\arg(z-p)+\arg(z-q) = \alpha, \alpha+\pi \quad (\alpha \neq 0)$$

is also worth considering.

Further reading

1. T. J. Heard and D. R. Martin, *Extending Mathematics* 2, Chapter 5 (Oxford University Press).
2. S. L. Parsonson, *Pure Mathematics* Volume 2, Chapter 17 (Cambridge University Press).
3. *SMP Revised Advanced Mathematics*, Book 3, Chapter 32 (Cambridge University Press).
4. Turner, Knighton and Budden, *Advanced Mathematics* 2, Section 12.5 (Longman, Harlow).
5. D. Griffiths, *Pure Mathematics* Volume 2, Sections 31.9, 31.10 (Harrap, London).

The biggest known prime number is now

$$391\,581 \times 2^{216\,193}-1,$$

found by John Brown and co-workers at Amdahl Corporation, Sunnyvale, California, USA.

AMITES SARKAR
Student at
Winchester College,
Winchester SO23 9NA.

An Approximate and Simple Method of Trisecting an Acute Angle

RADHA PERUMAL RAMASAMY, *Vanavani Matriculation Higher Secondary School*

The author is a seventh standard student in Madras, India.

'Can you trisect an angle?' If you put this question to any mathematician, the answer is that there is no way in which this can be done for an arbitrary angle.

I now present an approximate method of trisecting an acute angle. My method of construction is given below.

Let BOA be the angle to be trisected, with $OA = OB$. Let C be the midpoint of OB. With A as centre and AC as radius, draw an arc of a circle. With B as centre and BC as radius, draw another arc. Let the two arcs intersect at the point D. Join OD. It is found by measurement that angle BOD is approximately a third of angle BOA. See figure 1.

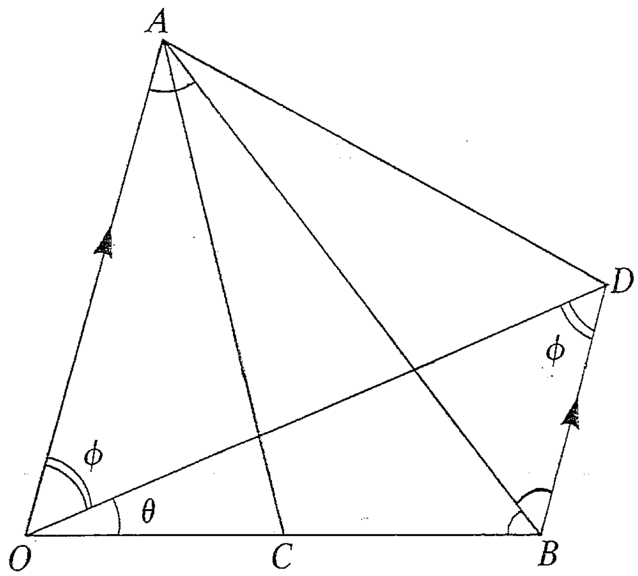


Figure 1

I tried the above method of trisecting angles lying between 10° and 90°. Practically every angle was trisected by my method.

Recently, I was given an opportunity to present my method at the Monthly Meeting of the Tamilnadu Academy of Sciences held at the Department of Nuclear Physics, University of Madras. It was theoretically proved at that time by Dr P. R. Subramanian that my construction is only approximate, but it works very well for small angles. Since I am only a seventh standard student, I do not understand the proof. For the sake of completeness, the proof and theoretical error in trisection are given in the appendix.

To conclude, I would like to state that this is a very simple but fairly accurate method of trisecting an acute angle.

Appendix: Theoretical error estimate
by P. R. SUBRAMANIAN

Triangles ABD and ABC are congruent since $AD = AC$, $BD = BC$ and AB is common. Hence angle DBA = angle CBA. Since $OA = OB$, angle OBA = angle OAB. Hence angle OAB = angle ABD. Therefore, OA and BD are parallel, implying that angle AOD = angle $ODB = \phi$, say. From triangle ODB,

$$\frac{BD}{\sin\theta} = \frac{OB}{\sin\phi}.$$

Since $OB = 2CB = 2BD$,

$$\sin\phi = 2\sin\theta.$$

For small values of θ and ϕ, $\sin\theta \approx \theta$ and $\sin\phi \approx \phi$, so that $\phi \approx 2\theta$. Hence Ramasamy's construction works well for small angles. The theoretical error in trisection can be obtained, and is given in table 1.

Table 1

Error in trisecting an angle by Ramasamy's technique

x° (angle BOA)	θ°	Percentage error $100(x-3\theta)/x$
6	1.999	0.0406
12	3.993	0.1630
18	5.977	0.3685
24	7.947	0.6594
30	9.896	1.0390
36	11.818	1.5117
42	13.708	2.0831
48	15.558	2.7601
54	17.360	3.5513
60	19.106	4.4669

Divisibility by 7

Y. L. CHEUNG, *University of Hong Kong*

Y. L. Cheung has taught mathematics in several schools and a university in Malaysia. He now lectures on mathematics/mathematical education at the University of Hong Kong. His major interests are mathematics and computing education, discrete mathematics and number theory.

In Volume 20, Number 1, pages 21–22, two contributions were published on divisibility by 7. This is explored further in the present article.

We begin with a traditional test for divisibility by 7, which may be stated as follows:

> 'Multiply the unit digit of the given number by 2 and subtract the product from the number with the unit digit removed. Repeat until the result is a one-digit or two-digit number. If the final result is divisible by 7, then the given number is also divisible by 7, and conversely.'

For example, to test 1995 for divisibility by 7, the rule gives $199-2\times5 = 189$ and then $18-2\times9 = 0$. Since 0 is divisible by 7, therefore 1995 is also divisible by 7.

For large numbers, the test is inefficient; many steps are required, as shown in table 1. This shows that the number 987 654 321 234 is divisible by 7. The test takes 11 steps, and may be proved as follows.

A number can be represented by Nj where j denotes the unit digit and N is the number left by removing the unit digit. Since

$$Nj = 10N+j \quad \text{and} \quad (10N+j)-3(N-2j) = 7(N+j),$$

the numbers $10N+j$ and $N-2j$ are either both divisible by 7 or both not divisible by 7. Hence the rule is proved.

We now ask whether we can produce a more efficient algorithm for divisibility by 7. If we replace $N-2j$ by $N-kj$, it is possible to discover values of the multiplier k to obtain a more efficient and general rule for divisibility by 7 (see table 2). The explanation is simple. For example, if Nj denotes the number where j represents the rightmost four-digit number and N represents the original number with the last four digits removed, the rule is $N-5j$. It works because

$$(10\,000N+j)-4(N-5j) = 21(476N+j).$$

Consider a numerical example (see table 3). The rule $N-5j$ takes only one step to show that the number 14 460 285 is divisible by 7 as, compared with six steps using the traditional rule $N-2j$ (see table 4). Since 10^6-1 is

Table 1

```
9 8 7 6 5 4 3 2 1 2 3|4
                    8|
9 8 7 6 5 4 3 2 1 1|5
                1 0|
9 8 7 6 5 4 3 2 0|1
                2|
9 8 7 6 5 4 3 1|8
            1 6|
9 8 7 6 5 4 1|5
          1 0|
9 8 7 6 5 3|1
          2|
9 8 7 6 5|1
        2|
9 8 7 6|3
      6|
9 8 7|0
    0|
9 8|7
1 4|
8|4
8|
0
```

Table 2

Number of digits in j	Rule
1	$N-2j$
2	$N-3j$
3	$N-\ j$
4	$N-5j$
5	$N-4j$
6	$N-6j$
7	$N-2j$
8	$N-3j$
9	$N-\ j$
10	$N-5j$
11	$N-4j$
12	$N-6j$

Table 3

```
  1 4 4 6|0 2 8 5
 -1 4 2 5|             (k = 5)
     2 1
```

Table 4

```
1 4 4 6 0 2 8|5
          1 0|
1 4 4 6 0 1|8
        1 6|
1 4 4 5 8|5
      1 0|
1 4 4 4|8
    1 6|
1 4 2|8
  1 6|
1 2|6
1 2|
0
```

divisible by 7, it follows that the value of k in the generalised rule of divisibility repeats after every six digits in j, as shown in table 2.

It is interesting to note two particular cases: $N-j$ and $N-6j$, which give rise to two simplified tests as follows.

1. For a six-digit number, the *difference* between the two trios is divisible by 7 if and only if the number is divisible by 7. For example, 192 164 is divisible by 7 because $192-164 = 28$ and 28 is divisible by 7.

2. For a twelve-digit number, the *sum* of the two six-digit numbers is divisible by 7 if and only if the original number is divisible by 7.

($k = 6 \equiv -1 \pmod 7$). For example, the number 412 384 251 258 is tested as follows:

$$\begin{array}{r|l}
412384 & 251258 \\
+251258 & \\
\hline
663642 &
\end{array}$$

Now 663 642 is divisible by 7 because $663-642 = 21$ and 21 is divisible by 7. Therefore, the number 412 384 251 258 is divisible by 7.

A corresponding set of multipliers k can be found for addition instead of subtraction in the rule of divisibility by 7 (see table 5). In comparing the addition test with the subtraction test, there seems to be no advantage of one over the other. Although addition may be easier than subtraction,

Table 5

Number of digits in j	Subtraction	Addition
1	$N-2j$	$N+5j$
2	$N-3j$	$N+4j$
3	$N-\ j$	$N+6j$
4	$N-5j$	$N+2j$
5	$N-4j$	$N+3j$
6	$N-6j$	$N+\ j$

Table 6

Subtraction

$$\begin{array}{r|ll}
24576 & 3189 & \\
-15945 & & (k = 5) \\
\hline
86 & 31 & \\
-93 & & (k = 3) \\
\hline
-7 & &
\end{array}$$

$$\begin{array}{r|ll}
63189 & 2457 & \\
-12285 & & (k = 5) \\
\hline
509 & 04 & \\
-12 & & (k = 3) \\
\hline
49 & 7 & \\
-14 & & (k = 2) \\
\hline
35 & &
\end{array}$$

Addition

$$\begin{array}{r|ll}
24576 & 3189 & \\
+\ 6378 & & (k = 2) \\
\hline
309 & 54 & \\
+216 & & (k = 4) \\
\hline
52 & 5 & \\
+25 & & (k = 5) \\
\hline
77 & &
\end{array}$$

$$\begin{array}{r|ll}
63189 & 2457 & \\
+\ 4914 & & (k = 2) \\
\hline
681 & 03 & \\
+\ 12 & & (k = 4) \\
\hline
69 & 3 & \\
+15 & & (k = 5) \\
\hline
84 & &
\end{array}$$

Table 7

$$
\begin{array}{rrl}
 & 987654\,|\,321234 & \\
+ & \underline{321234\,|} & \qquad (k = 1) \\
 & 1308\,|\,888 & \\
- & \underline{888\,|} & \qquad (k = 1) \\
 & 420 &
\end{array}
$$

the computation depends also on the size of the multiplier k and on the numbers to be tested. Comparisons are given in table 6. In fact, a combination of addition and subtraction can speed up computation, as illustrated in table 7. Since 420 is divisible by 7, so is 987 654 321 234, which has already been shown to be divisible by 7 using only the $N-2j$ rule (table 1).

The generalised rule of divisibility by 7, which is efficient for testing large numbers, can be extended to other prime numbers such as 11, 13, etc. The determination of the multipliers and the proofs for the rules are left to the readers.

Diophantus rules (OK?)

Here are three more problems that one of our readers, Arthur Pounder, set at his Maths Club at St. Peter's Grammar School, Prestwich, Manchester. Why not have a go at them?

1. *Off to the seaside.* A men's club decides to run a trip to the seaside and (unusually) allows women and children to take part as well. They decide that each man will pay £10, each woman 50p and each child $12\frac{1}{2}$p. (This was before the abolition of the $\frac{1}{2}$p.—Ed.) Altogether, 100 people went on the trip and £100 was taken. How many men, women and children took part?

2. *Unlucky 17.* Can you prove that there are only two rectangles with integer sides such that their perimeters are numerically equal to their areas. Hence explain why the Pythagoreans considered the number 17 as very unlucky!

3. *Whoops!* A owes B a sum of money in pounds and shillings (shades of yore here!) A writes B a cheque and then discovers that he has placed the pounds number in the shillings column and vice versa. The amount he has written is exactly double what he owed. What did A write on the cheque? (Recall that £1 = 20 shillings.)

Expectation in a Sudden-Death Situation

D. J. COLWELL, J. R. GILLETT AND B. C. JONES,
North Staffordshire Polytechnic

The authors are lecturers in mathematics at the North Staffordshire Polytechnic. They are involved with teaching mathematics and statistics to students on engineering and science degree courses.

In sports such as soccer, when a game between two teams ends in a draw after extra time, the game is awarded to the winner of a penalty competition. Such a competition usually takes the form of a fixed number of penalties being taken in turn by different members of each competing team, with the game being awarded to the team converting (or scoring) most penalties. However, if the number of penalties converted at the end of this stage of the competition is the same for both teams, then the competition continues on a 'sudden-death' basis. (That is, the teams take penalties in turn until the team which converts a penalty, when the other does not, is declared the winner.)

Of course, such a penalty competition usually decides the winner quite quickly, but it is amusing to envisage a situation where the competition drags on and on. After all, when replays are used instead of penalty competitions, it is not unusual for several matches to be needed before the winner is found. For instance, in 1955, in the English Football Association Cup, Stoke City were involved in five matches and 9 hours 22 minutes of play (one match had to be abandoned) before they managed to beat Bury! Is there much chance of an equivalent situation occurring in a penalty competition?

In a mathematician's language this question could be posed as 'what is the expected number of rounds in such a penalty competition?' More colloquially, how long on average does a penalty competition last?

To answer a question of this kind, we need to construct a mathematical model of a penalty competition and, as is usually the case with such models, there are several awkward questions to be faced before detailed construction can begin. For example, do we assume that all members of a team convert a penalty with the same probability? If so, do we assume that each team converts with the same probability? In trying to answer questions such as these, a balance usually has to be struck between what seems realistic and what leads to a mathematical model which can be solved.

Facing up to the question on penalty performance of individual team members, it is highly unlikely that any such data are available for every member of each team. Hence, it does not seem unreasonable to assume that all the members of a team convert penalties with the same probability, although deciding on the value of this probability poses a problem.

Addressing the question of the efficiency of each team at taking penalties, there seems to be no obvious reason to suppose that they should be the same, although they presumably would not differ by much.

Denoting the teams by A and B, we shall base the construction of our model on the following assumptions.

1. A and B take penalties in turn.
2. A and B convert penalties independently of one another and independently of the round of the competition, with probabilities p $(0 < p < 1)$ and q $(0 < q < 1)$, respectively.
3. Initially the competition consists of each team taking up to m penalties, with the competition finishing if one team takes a winning lead (that is, a lead that cannot be overtaken). However, if the teams are level after m penalties, a 'sudden-death' competition ensues.

We shall first consider the situation before the possibility of a sudden-death competition occurs. Suppose X and Y are the independent random variables denoting the number of successful penalty conversions by each of A and B, respectively, when both have taken r $(\leqslant m)$ penalties. Then X and Y have binomial distributions $\mathrm{Bi}(r,p)$ and $\mathrm{Bi}(r,q)$, respectively.

Thus, for $x = 0, 1, \ldots, r$ and $y = 0, 1, \ldots, r$,

$$P(X = x) = {}^rC_x p^x(1-p)^{r-x}; \qquad P(Y = y) = {}^rC_y q^y(1-q)^{r-y}.$$

The probability of A taking an unassailable lead in any round up to, and including, the rth round of penalties is

$$G(r) = \sum_{x=m-r+1}^{r} \sum_{y=0}^{x-(m-r+1)} P(X = x)\,P(Y = y),$$

since A must have a lead of at least $m-r+1$. It should be borne in mind here that the only possible values r can take are

$$[\tfrac{1}{2}m]+1, \qquad [\tfrac{1}{2}m]+2, \qquad \ldots, \qquad m,$$

where $[\frac{1}{2}m]$ is the integral part of $\frac{1}{2}m$. Thus, for example, if $m = 5$, $[\frac{1}{2}m] = 2$ and r could be 3, 4 or 5.

Similarly, the probability of B taking an unassailable lead in any round up to, and including, the rth round of penalties is

$$H(r) = \sum_{y=m-r+1}^{r} \sum_{x=0}^{y-(m-r+1)} P(Y = y)\,P(X = x).$$

Hence the probability that one of the teams takes an unassailable lead in any round up to, and including, the rth round of penalties is

$$F(r) = G(r)+H(r),$$

where $r = [\frac{1}{2}m]+1, [\frac{1}{2}m]+2, \ldots, m$.

If $0 \leqslant r \leqslant [\frac{1}{2}m]$ then, of course, $F(r) = 0$.

The probability that a sudden-death competition ensues is $1-F(m)$. Also the probability that exactly one of the teams scores in a penalty round is

$$\begin{aligned}\beta &= P(A \text{ scores})\,P(B \text{ does not score}) + P(A \text{ does not score})\,P(B \text{ scores})\\ &= p(1-q)+q(1-p)\\ &= p+q-2pq \quad (0<\beta<1).\end{aligned}$$

Hence the probability that the two teams have to move into a sudden-death situation and that each team is involved in a further n rounds of penalties before one team goes into the lead, and is thus declared the winner, is

$$[1-F(m)](1-\beta)^{n-1}\beta \quad (n = 1,2,\ldots).$$

Let R be the random variable denoting the number of rounds of penalties which have to be taken before one of the teams wins. Then, drawing on both the first stage and the sudden-death stage of the competition, we see that the expected value of R, as shown in the appendix, is

$$m - \sum_{r=t+1}^{m} F(r) + \frac{1-F(m)}{p+q-2pq}, \quad \text{where } t = [\tfrac{1}{2}m].$$

Further, the expected length of the sudden-death competition alone, if it takes place, is

$$\frac{1}{p+q-2pq}.$$

A commonly used value for m in competitions is $m = 5$. For this case table 1 gives the values of $E(R)$ for a range of values of p and q. It should be noted that the value of $E(R)$ is unchanged if p and q are interchanged or p and q are replaced by $1-p$ and $1-q$, respectively.

Table 1

				p			
	$E(R)$	0.05	0.1	0.2	0.3	0.4	0.5
	0.05	11.721					
	0.1	8.698	7.493				
	0.2	6.320	6.164	5.820			
	0.3	5.332	5.421	5.468	5.388		
	0.4	4.779	4.924	5.128	5.223	5.219	
q	0.5	4.404	4.549	4.808	5.005	5.130	5.172
	0.6	4.112	4.245	4.510	4.757	4.970	
	0.7	3.860	3.981	4.234	4.496		
	0.8	3.626	3.740	3.979			
	0.9	3.393	3.508				
	0.95	3.271					

The results in table 1 suggest that, on average, the chances of a high number of rounds in a penalty competition are slight, unless both p and q are very small (for example $p = q = 0.05$) or both p and q are very large (for example $p = q = 0.95$).

Appendix

Setting $[\frac{1}{2}m] = t$, we have

$$
\begin{aligned}
E(R) &= \sum_{r=t+1}^{m} rP(R=r) + \sum_{r=m+1}^{\infty} rP(R=r) \\
&= \sum_{r=t+1}^{m} r\{F(r)-F(r-1)\} + \{1-F(m)\} \sum_{n=1}^{\infty} (m+n)(1-\beta)^{n-1}\beta \\
&= mF(m) - \sum_{r=t+1}^{m-1} F(r) + \{1-F(m)\} m\beta \sum_{n=1}^{\infty} (1-\beta)^{n-1} \\
&\qquad + \{1-F(m)\}\beta \sum_{n=1}^{\infty} n(1-\beta)^{n-1}, \quad \text{since } F(t) = 0, \\
&= m - \sum_{r=t+1}^{m-1} F(r) - \beta\{1-F(m)\}\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{\beta}\right), \quad \text{since } 0 < \beta < 1, \\
&= m - \sum_{r=t+1}^{m-1} F(r) + \frac{1-F(m)}{p+q-2pq}, \quad \text{where } \beta = p+q-2pq.
\end{aligned}
$$

The expected length of the sudden-death competition, if it takes place, is

$$
\sum_{n=1}^{\infty} n(1-\beta)^{n-1}\beta = \beta\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{-1}{1-(1-\beta)}\right) = \frac{1}{\beta} = \frac{1}{p+q-2pq}.
$$

To solve the equation $0 \times x = 1$

Multiply both sides of the equation by x.

$$0 \times x^2 = x$$

Use the result that $0 \times a = 0$ for all a to reduce the left hand side to 0.

$$0 = x$$

The required answer.

A. K. AUSTIN
University of Sheffield.

Self-Median Triangles

K. R. S. SASTRY, *Box 21862, Addis Ababa, Ethiopia*

> The author thinks that he is teaching mathematics if ten per cent of his students are genuinely interested in the subject. To inspire a sense of creativity in the minds of this minority, occasionally he tries to do an interesting piece such as the material of this article. The other ninety per cent is very quick to expose his ignorance by shouting: Sir, this is not found in the prescribed textbook!

A *median* of a triangle is the line segment between a vertex and the mid point of its opposite side. Suppose AD is the median through the vertex A of triangle ABD shown in figure 1.

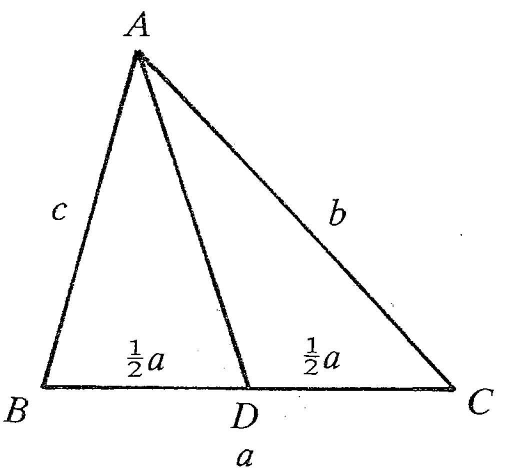


Figure 1

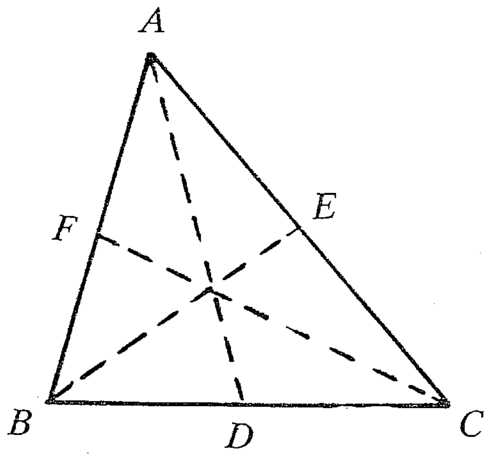


Figure 2

With the usual notation a, b, c for the lengths of the sides BC, CA, AB, respectively, we can express the length of AD in terms of a, b and c. (Just consider the triangles ABD and ADC and use the cosine rule to write down the expressions for AB^2 and AC^2.) In fact, if AD, BE and CF are the three medians of triangle ABC, shown in figure 2, then we have

$$\begin{aligned} 4AD^2 &= 2b^2+2c^2-a^2, \\ 4BE^2 &= 2c^2+2a^2-b^2, \\ 4CF^2 &= 2a^2+2b^2-c^2. \end{aligned} \tag{1}$$

Now take the triangle $(a,b,c) = (17,23,7)$, use the expressions (1) and calculate the lengths of the medians AD, BE and CF. Surprise surprise! The lengths of these medians, viz. $\frac{17}{2}\sqrt{3}$, $\frac{7}{2}\sqrt{3}$ and $\frac{23}{2}\sqrt{3}$ are proportional to the lengths of the sides of the triangle! We shall call such triangles, in which the medians are proportional to the sides, *self-median* triangles. We are going to reveal a method of generating such triangles. In the meantime, try to discover a relationship between the numbers 17, 23 and 7. (*Hint*: square them.) Before obtaining a characterisation of self-median triangles, we establish two useful results concerning the medians of a triangle.

Theorem 1. In any triangle, four times the sum of the squares of the medians equals three times the sum of the squares of the sides.

Proof. Addition of the three expressions in (1) yields

$$4(AD^2+BE^2+CF^2) = 3(a^2+b^2+c^2). \tag{2}$$

Let us impose the ordering $c \leqslant a \leqslant b$ on the sides of triangle ABC and stick to it throughout our discussion. The following theorem shows that this has induced an ordering on the medians as well.

Theorem 2. Let ABC be a triangle in which $c \leqslant a \leqslant b$. Then the medians AD, BE and CF satisfy the relations

$$BE \leqslant AD \leqslant CF. \tag{3}$$

Proof. From (1) and the hypothesis of theorem 2, we see that

$$4(AD^2-BE^2) = 3(b^2-a^2) \geqslant 0.$$

This shows that $BE \leqslant AD$. Similarly, $AD \leqslant CF$.

Theorem 3. A triangle is a self-median triangle if and only if the squares of its sides are in arithmetic progression.

Proof. Consider triangle ABC with $c \leqslant a \leqslant b$, in which AD, BE and CF are the medians. Firstly, suppose that the triangle is self-median. Therefore by theorem 2, we have

$$\frac{AD}{a} = \frac{BE}{c} = \frac{CF}{b} = K \quad \text{(proportionality constant)}.$$

Substituting for AD etc. in the equation (2), we obtain $K = \frac{1}{2}\sqrt{3}$. Now, using (1), we readily obtain the equation

$$b^2+c^2 = 2a^2,$$

that is, $b^2-a^2 = a^2-c^2$, which shows that the squares of the sides of the triangle are in arithmetic progression.

Conversely, if the squares of the sides of the triangle are in arithmetic progression, then $b^2+c^2 = 2a^2$ and, from (1), we have

$$AD = \tfrac{1}{2}\sqrt{3}a, \qquad BE = \tfrac{1}{2}\sqrt{3}c, \qquad CF = \tfrac{1}{2}\sqrt{3}b.$$

Hence the triangle is self-median.

Theorem 3 at once shows that a *non-equilateral isosceles triangle cannot be a self-median triangle.* An inquisitive mind might wonder about a right-angled triangle being self-median. For interested readers here is that problem:

Prove that any self-median, right-angled triangle is similar to the triangle $(1, \sqrt{2}, \sqrt{3})$.

How to determine self-median triangles

Theorem 3 has shown that the problem of determining self-median triangles is equivalent to the problem of determining three squares c^2, a^2, b^2 in arithmetic progression—and, of course, the triplet (c, a, b) should yield a triangle. This has been an interesting problem since ancient times. A variety of solutions to this problem since Diophantus of Alexandria in the third century AD can be found in reference 1. One solution is

$$a = m^2+n^2, \qquad b = m^2-n^2+2mn, \qquad c = |m^2-n^2-2mn|$$

(yielding the triangle opening this discussion for $m = 4$ and $n = 1$). Dickson also remarks that 'J. Neuberg and J. Déprez investigated "automedian" triangles viz. those whose medians are proportional to the sides a, b, c' and inspired the present investigation—and another companion one which will appear in a succeeding issue. In the meantime, keep guessing and making your own investigations about it.

Reference

1. L. E. Dickson, *History of the Theory of Numbers*, Volume II, Chapter XIV (Chelsea, New York).

'Curiouser and curiouser', said Alice

Malcolm Smithers has previously sent us the curious number

$$3435 = 3^3+4^4+3^3+5^5.$$

He has since written to tell us that this is the only number known of this type, apart from $1^1 = 1$. Curiously, the binary representation of 3435 is

$$\underbrace{110101}_{53}\underbrace{101011}_{43}$$

Computer Column

MIKE PIFF

Modula2

Instead of the usual unintelligible BBC BASIC program, I shall be looking in this column at one of the modern generation of programming languages which is widely available on micros, and can be used for a wide variety of tasks, such as number-crunching, writing operating systems, and so on. It does not suffer from the archaisms of C or the baroque monstrosities of Ada.

The Modula2 'program' is called a *module*, and looks like this:

```
MODULE hcf;
FROM InOut IMPORT ReadInt,WriteInt,WriteString,WriteLn;
VAR m,n,temp:INTEGER;
BEGIN
   WriteString('m=');ReadInt(m);
   WriteString('n=');ReadInt(n);
   IF (m<=0) OR (n<=0) THEN
      WriteString('One of the numbers is zero or negative.');WriteLn;
   ELSE
      REPEAT
         temp:=m MOD n;
         m:=n;
         n:=temp
      UNTIL temp=0;
      WriteString('The HCF is');WriteInt(m,10);WriteLn;
   END;
END hcf.
```

and the idea is that you put your calculations between the `BEGIN` and `END`. Certain facilities are only available if you `IMPORT` them from another module, but the compiler takes care of that for us.

Any variables we use must be declared, and the compiler will check for inconsistent uses of a variable name. The last parameter to `WriteInt` simply gives a fieldwidth in which to right-justify the answer. The rest is self-explanatory.

There is little new about what we have met so far; the structure is similar to Pascal, except for the `END` matching the `IF` keyword. However, in future programs we shall be exploring some of the more advanced features of this language. For now, see if you can modify the above to produce the LCM of two numbers. If you don't have Modula2, either (*a*) buy it straight away—there are some excellent implementations for around £50—or (*b*) translate into any other language that you have.

Letters to the Editor

Dear Editor,

π from Pascal's triangle

I was reminded by the letter in *Mathematical Spectrum*, Volume 21, Number 3, pages 97–98 from I. M. Richards, of a connection between π and Pascal's triangle. It involves the coefficients of the expansion of $(1-4x)^{-1/2}$ which can be read off from Pascal's triangle. In the triangle, the coefficients are indicated in bold,

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & \mathbf{2} & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & \mathbf{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \mathbf{20} & & 15 & & 6 & & 1
\end{array}$$

Following I. M. Richards and giving the sequence 1, 2, 6, 20, ... the labels $A_1, A_2, A_3, A_4, \ldots$, we have that

$$\frac{2\pi}{3\sqrt{3}} = \sum_{r=1}^{\infty} \frac{1}{(2r-1)A_r}. \qquad (*)$$

To prove this, we note that

$$A_n = \binom{2(n-1)}{n-1}$$

(the binomial coefficient), so that the right-hand side of (*) is

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)!}.$$

By repeated integration by parts, we see that

$$\int_0^1 x^n(1-x)^n \, dx = \frac{(n!)^2}{(2n+1)!},$$

so that

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{(2r-1)A_r} &= \sum_{n=0}^{\infty} \int_0^1 x^n(1-x)^n \, dx \\
&= \int_0^1 \sum_{n=0}^{\infty} x^n(1-x)^n \, dx
\end{aligned}$$

$$= \int_0^1 \frac{1}{1-x(1-x)}\,dx$$

and this integral can be evaluated to give the result.

Yours sincerely,
ALAN FEARNEHOUGH
(Portsmouth Sixth Form College)

Dear Editor,

Matrices and Fibonacci Numbers

If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

then it is easily seen (by induction) that

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \tag{1}$$

where

$$F_0 = 0, \qquad F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \quad (n = 2, 3, \ldots).$$

From this many results follow, such as that

$$\begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}^2,$$

so that, by multiplying out,

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \qquad F_{2n} = F_n(F_{n+1} + F_{n-1}).$$

Now it is easily verified that $A^2 = A + I$ (which will not surprise those who know the Cayley–Hamilton theorem), so that

$$\begin{aligned} A^{2n} &= (A+I)^n \\ &= I + \binom{n}{1}A + \binom{n}{2}A^2 + \ldots + \binom{n}{n}A^n \end{aligned}$$

by the binomial theorem. By substituting for each power of A its expression in terms of the sequence $F_0, F_1, F_2, \ldots$ (as in (1)) and looking at the first row, second column element, we see that

$$F_{2n} = \binom{n}{1}F_1 + \binom{n}{2}F_2 + \ldots + \binom{n}{n}F_n,$$

as in Amites Sarkar's article 'A Fibonacci sum'.

Your sincerely,
NORMAN ROUTLEDGE
(Eton College)

Dear Editor,

Ramanujan's third problem

In an article in *Mathematical Spectrum* (Volume 21 Number 3) Dermot Roaf asked for another elegant solution of Ramanujan's third problem:

'Show that it is possible to solve the equations:

$$\begin{aligned} x+\ y+\ z &= a \quad (1) & p^3x+q^3y+r^3z &= d \quad (4)\\ px+\ qy+\ rz &= b \quad (2) & p^4x+q^4y+r^4z &= e \quad (5)\\ p^2x+q^2y+r^2z &= c \quad (3) & p^5x+q^5y+r^5z &= f \quad (6)\end{aligned}$$

where x, y, z, p, q and r are unknowns. Solve the above when $a = 2$, $b = 3$, $c = 4$, $d = 6$, $e = 12$ and $f = 32$.'

Consider the sequence $4, -1, 7, 5, 19, \ldots$ which is governed (from the third term onwards) by the rule (the 'recurrence relation')

$$u_n = u_{n-1}+2u_{n-2},$$

where the sequence is $u_0, u_1, u_2, u_3, \ldots$.

The recipe to find u_n *in terms of* n is as follows:

1. Try $u_n = t^n$, where t is to be found.
2. Substitute this in the recurrence relation, obtaining the 'auxiliary equation' $t^2 = t+2$.
3. Solve for t, obtaining $t = 2$ or -1.
4. Put $u_n = x(2^n)+y(-1)^n$, where x and y are constants to be evaluated.
5. Obtain two equations in x and y by putting first $n = 0$, $u_0 = 4$ and then $n = 1$, $u_1 = -1$ in the last equation (here we see why it is convenient to regard the first term as u_0, the second term as u_1 and so on).
6. Solve the simultaneous equations for x and y, obtaining $x = 1$ and $y = 3$.
7. Put in these values for x and y to obtain $u_n = 2^n+3(-1)^n$.

This recipe works whenever the recurrence relation gives u_n as a linear combination of previous terms, provided of course we can solve the auxiliary equation, except that some modification is needed when the auxiliary equation has a repeated root. The recipe resembles trying $y = e^{kx}$ to solve a linear differential equation.

Returning to Ramanujan's problem, we see that the left sides of equations (1)–(6) are of the form $xp^n+yq^n+zr^n$. So the sequence a, b, c, d, e, f is governed by a recurrence relation, the roots of the auxiliary equation are p, q and r and, since the auxiliary equation is cubic, the recurrence relation gives u_n as a linear combination of the three previous terms:

$$u_n = Au_{n-1}+Bu_{n-2}+Cu_{n-3}.$$

Now the actual work starts.

1. From the recurrence relation,

$$d = Ac+Bb+Ca, \quad e = Ad+Bc+Cb, \quad f = Ae+Bd+Cc.$$

2. Solve the simultaneous equations for A, B and C.
3. Put the values of A, B and C into the auxiliary equation $t^3 = At^2+Bt+C$.
4. Solve the auxiliary equation for t; the roots are the values of p, q and r.

5. Using these values for p, q and r, solve equations (1)–(3) for x, y and z. And that's that.

Notice that once A, B and C are found it is possible to calculate $p^k x + q^k y + r^k z$ for $k = 6, 7, 8, \ldots$ (if desired) by using the recurrence relation and without knowing p, q, r, x, y or z.

Yours sincerely,
JOHN MACNEILL
(The Royal Wolverhampton School)

Editor's Note: Norman Routledge of Eton College has also written to draw readers' attention to this solution of Ramanujan's third problem.

Dear Editor,

Perfect boxes

K. R. S. Sastry's article 'Perfect boxes' (reference 1) is interesting as number theory, but as geometry it is misconceived. To be definite, take Sastry's definition (i) of a perfect box, that the volume is equal to the sum of the edges, so that

$$4(l+m+n) = lmn, \tag{1}$$

where l, m and n are positive integers. This formula equates quantities of different dimensions: the left side has dimension of length and the right side has dimension of $(\text{length})^3$. It follows that a box may be perfect with one unit of length, and imperfect with a different unit. Thus, for example, the box with edges 10, 3 and 2 metres is perfect if the unit chosen is the metre, but not if it is the centimetre; for in the latter case the edges are 1000, 300 and 200 which do not satisfy (1).

This illustrates how important it is in geometry (and physics) to ensure that each term in an equation has the same dimensions.

1. K. R. S. Sastry, Perfect boxes, *Mathematical Spectrum*, **21**, 90 (1988/1989).

Yours sincerely,
W. B. BONNOR
Queen Mary College,
London E1 4NS

Dear Editor,

Cyclic numbers

In Volume 17 Number 2 page 58, you suggested that we should try multiplying 142 857 successively by 2, 3, 4, 5 and 6 to see what happens. And in Volume 17, Number 3, page 76 Richard Dobbs suggested that the number 0588 235 294 117 647 be multiplied by 2, 3, 4 up to 16. A similar thing happens. I think that there are many numbers with the same property. Note that

$$142\,857 = \frac{10^6-1}{7}, \qquad 0588\,235\,294\,117\,647 = \frac{10^{16}-1}{17}.$$

Let p be a prime number. Then Fermat's 'little theorem' says that

$$10^{p-1} \equiv 1 \pmod{p}.$$

It may be that there is a positive integer n smaller than $p-1$ such that $10^n \equiv 1 \pmod{p}$, although it will always be true that n divides $p-1$. For example, $10^6 \equiv 1 \pmod{13}$. This means that, when $\frac{1}{13}$ is expanded as an infinite decimal, it has six terms in its recurring part; in fact $\frac{1}{13} = 0.\dot{0}7692\dot{3}$. But, when $p = 7$ or 17, then $p-1$ is the smallest value of n for which $10^n \equiv 1 \pmod{p}$, and

$$\tfrac{1}{7} = 0.\dot{1}4285\dot{7}, \qquad \tfrac{1}{17} = 0.\dot{0}58\,823\,529\,411\,764\dot{7},$$

with recurring parts having six and sixteen terms, respectively. The ten smallest prime numbers with this property are 7, 17, 19, 23, 29, 47, 59, 61, 97, 109. Let p be such a prime number, and write the positive integer $(10^{p-1}-1)/p$ in terms of $p-1$ digits. (For all except $p = 7$, we shall need to write some 0's at the beginning, as with $p = 17$.) Then, if we multiply this number by $2, 3, \ldots, p-1$, the numbers obtained are just the cyclic rearrangements of the original number. And it is easy to see why if you examine how the decimal expansion of $1/p$ is obtained; look, for example, at $p = 7$.

Reference: M. R. Schroeder *Number Theory in Science and Communication*, Second Enlarged Edition (Springer-Verlag, 1986).

Yours sincerely,

ESMAEEL BABAKI
191 Mofatteh St,
Gonbad-Kavous,
Mazanderan, Iran

Dear Editor,

Hidden powers

I was fascinated by the two articles in Volume 22 Number 1 about the sequence of Problem 22.1. About a year ago, when the problem first appeared, I programmed it on a BBC micro. I had learned of the series through a different newspaper article from Dr Devlin's, one which did not mention Dr Mallows' answer of 3173 375 556. Nevertheless, I was quite surprised when I found that 1490 appeared to be the actual solution. I was even more surprised when powers of 2 started turning up. I soon found that, if $n = 2^k$ $(k > 0)$, $s_n = 2^{k-1}$. Also $s_{2^k-(m-1)} = 2^{k-1}$ for $1 \leqslant m \leqslant k$. Perhaps this is what 'Hidden powers' means in the title of Dr Devlin's article. I am working on a proof of this observation.

Yours sincerely,

AMITES SARKAR
Student at Winchester College,
Winchester SO23 9NA.

Amites Sarkar has sent a computer printout of the first 2000 terms of the sequence. Editor.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

22.5 (Submitted by Gregory Economides, Sixth Form, Royal Grammar School, Newcastle upon Tyne)
The methane molecule is based on a regular tetrahedron with the carbon atom at the centroid and the four hydrogen atoms at the vertices. What is the angle between pairs of carbon valency links—all to hydrogen?

22.6 (Submitted by Malcolm Smithers, a student of the Open University)
How many non-similar integer-angled triangles are there (the angles being measured in degrees)?

22.7 (Submitted by Dylan Gow, Sixth Form, Oakham School)
Show that

$$\frac{\sum\limits_{r=0}^{n-1} \sin(rd+a)}{\sum\limits_{r=0}^{n-1} \cos(rd+a)} = \tan\{\tfrac{1}{2}(n-1)d+a\}.$$

22.8 (Submitted by Amites Sarkar, a student at Winchester College)
The sequence (A_r) is defined as follows:

$$A_1 = A_2 = A_3 = 1, \qquad A_{r+1} = A_{r-1}+A_{r-2} \quad (r > 2).$$

Prove the identity $A_{2n} = 2A_nA_{n-1}+A_{n-2}^2$ for $n > 2$.

Solutions to Problems in Volume 21 Number 3

21.9. The speed of projection of a particle projected to clear a wall of height H when the point of projection is at a distance D from the base of the wall is given by

$$u^2 = \frac{gD^2}{D\sin 2\theta - H - H\cos 2\theta},$$

where θ is the angle of projection. (Since $\tan\theta > H/D$, $u^2 > 0$.) Find, *without the use of calculus*, the minimum speed of projection.

Solution by Aimée Davidson (Form 3C, Hutcheson's Grammar School, Glasgow)

The speed of projection will be a minimum when $D\sin 2\theta - H - H\cos 2\theta$ is a maximum. We can write

$$D\sin 2\theta - H\cos 2\theta = R\sin(2\theta-\phi)$$
$$= R\sin 2\theta\cos\phi - R\cos 2\theta\sin\phi,$$

where $D = R\cos\phi$ and $H = R\sin\phi$, so that

$$R^2 = D^2+H^2, \qquad \sin\phi = H/R, \qquad \cos\phi = D/R.$$

The maximum value of $D\sin 2\theta - H - H\cos 2\theta$ is thus $R-H = \sqrt{D^2+H^2}-H$, so that

$$u_{\text{min}}^2 = \frac{gD^2}{\sqrt{D^2+H^2}-H}.$$

Also solved by Amites Sarkar, Dylan Gow, Nigel Mottram (Sandbach) and Thomas Hadfield (King Edward VI School, Southampton).

21.10. Let E be a finite non-empty set of natural numbers and, for each non-empty subset X of E, denote by $\|X\|$ the sum of the elements of X. What is $\sum_X \|X\|$, where the sum is taken over all non-empty subsets of E?

Solution by Dylan Gow

Consider an element e of E. For each subset containing e, the other $n-1$ elements are either in the subset or not, so that e belongs to exactly 2^{n-1} subsets of E. Thus, when evaluating $\sum_X \|X\|$, each element of E is counted exactly 2^{n-1} times, so that

$$\sum_X \|X\| = 2^{n-1}\|E\|.$$

Also solved by Gregory Economides, Amites Sarkar, Nigel Mottram and Daniel Brandow (Lake Superior State University, U.S.A.).

21.11. Prove that

$$\frac{1}{1}\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\binom{n}{n} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n},$$

where $\binom{n}{r}$ denotes the binomial coefficient.

Solution 1. Essentially this solution was received from Gregory Economides, Dylan Gow and Thomas Hadfield.

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \ldots + (-1)^n\binom{n}{n}x^n$$

$$\Rightarrow \qquad \frac{1-(1-x)^n}{x} = \binom{n}{1} - \binom{n}{2}x + \binom{n}{3}x^2 - \ldots + (-1)^{n-1}\binom{n}{n}x^{n-1}$$

$$\Rightarrow \quad \int_0^1 \frac{1-(1-x)^n}{x}\,dx = \left[\binom{n}{1}x - \binom{n}{2}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} - \ldots + (-1)^{n-1}\binom{n}{n}\frac{x^n}{n}\right]_0^1$$
$$= \frac{1}{1}\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}.$$

In the integral, we substitute $u = 1-x$ to give

$$\int_0^1 \frac{1-u^n}{1-u}\,du = \int_0^1 (1+u+u^2+\ldots+u^{n-1})\,du$$

$$= \left[u+\frac{u^2}{2}+\frac{u^3}{3}+\ldots+\frac{u^n}{n}\right]_0^1$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

The result follows.

Solution 2 by Amites Sarkar

Let

$$F(n) = \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r}\binom{n}{r}.$$

We prove by induction that

$$F(n) = \sum_{r=1}^{n} \frac{1}{r}.$$

The identity is true when $n = 1$, so we assume the result when $n = k$. Now

$$F(k+1)-F(k) = \sum_{r=1}^{k+1} \frac{(-1)^{r-1}}{r}\binom{k+1}{r} - \sum_{r=1}^{k} \frac{(-1)^{r-1}}{r}\binom{k}{r}$$

$$= \sum_{r=1}^{k} \frac{(-1)^{r-1}}{r}\left[\binom{k+1}{r}-\binom{k}{r}\right] + \frac{(-1)^k}{k+1}$$

$$= \sum_{r=1}^{k} \frac{(-1)^{r-1}}{r}\binom{k}{r-1} + \frac{(-1)^k}{k+1}$$

$$= \sum_{r=1}^{k} (-1)^{r-1}\frac{k!}{r!(k-r+1)!} + \frac{(-1)^k}{k+1}$$

$$= \frac{1}{k+1}\sum_{r=1}^{k} (-1)^{r-1}\binom{k+1}{r} + \frac{(-1)^k}{k+1}$$

$$= \frac{1}{k+1}[-(1-1)^{k+1}+1]$$

$$= \frac{1}{k+1}.$$

Hence

$$F(k+1) = F(k)+\frac{1}{k+1} = \sum_{r=1}^{k+1} \frac{1}{r}.$$

This completes the inductive step.

Solution 3 by Amites Sarkar

Let $w = x^n \ln x$. It is easily shown by induction that

$$\frac{d^r w}{dx^r} = \frac{n!\,x^{n-r}}{(n-r)!} \sum_{s=n-r+1}^{n} \frac{1}{s} + \frac{n!\,x^{n-r}}{(n-r)!} \ln x \quad (1 \leqslant r \leqslant n).$$

Therefore

$$\frac{d^n w}{dx^n} = n! \sum_{r=1}^{n} \frac{1}{r} + n! \ln x.$$

However, by Leibnitz's theorem with $y = x^n$ and $z = \ln x$,

$$\begin{aligned}\frac{d^n w}{dx^n} &= \sum_{r=0}^{n} \binom{n}{r} \frac{d^r y}{dx^r} \frac{d^{n-r} z}{dx^{n-r}} \\ &= \sum_{r=0}^{n-1} \binom{n}{r} \frac{n!\,x^{n-r}}{(n-r)!} \frac{(-1)^{n-r-1}(n-r-1)!}{x^{n-r}} + n! \ln x \\ &= n! \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \binom{n}{r} + n! \ln x.\end{aligned}$$

Comparing answers, we obtain the result.

21.12. Show that, for all positive integers i and j with $i < j$, the sum

$$\frac{1}{i} + \frac{1}{i+1} + \ldots + \frac{1}{j} \qquad (*)$$

is not an integer.

Solution 1 by Gregory Economides

Let the least common multiple of $i, i+1, \ldots, j$ be $2^n m$, where m is odd, so that 2^n is the highest power of 2 to divide $i, i+1, \ldots, j$. Note that $n > 0$. Now 2^n can divide only one of these integers. (For suppose that it divides k and l for some $k < l$, say $k = 2^n p$ and $l = 2^n q$, where $p < q$. Then $2^n(p+1)$ lies between i and j, and $p+1$ is even, so that $2^{n+1} | 2^n(p+1)$.) Thus, when we take $2^n m$ as a common denominator in (*), every term in the sum in the numerator is even except one, which is odd. Hence (*) is of the form $r/2^n m$, where r is odd and $n > 0$, so that (*) is not an integer.

Note that this solution does not use Bertrand's postulate, which was given as a hint when the problem was originally posed. We now give a second solution which uses Bertrand's postulate.

Solution 2 by Amites Sarkar

If (*) is an integer, then

$$1 \leqslant (*) < \frac{1}{i} + \ldots + \frac{1}{i} = \frac{j-i+1}{i},$$

so that $j-i+1 > i$ and $j > 2i-1$, whence $j \geqslant 2i$. Assume that there is no prime number p such that $[\frac{1}{2}j] < p \leqslant j$ (where $[\alpha]$ denotes the integer part of α). Let p_r be the largest prime number such that $p_r \leqslant [\frac{1}{2}j]$. (We may suppose that $j \geqslant 4$, since the cases $j = 2,3$ are easy to see.) Then the next prime p_{r+1} satisfies $p_{r+1} > j \geqslant 2[\frac{1}{2}j] \geqslant 2p_r$, which contradicts Bertrand's postulate that $p_{r+1} < 2p_r$. Hence there is a prime number p such that $[\frac{1}{2}j] < p \leqslant j$. Also $i \leqslant [\frac{1}{2}j]$, so that $i < p \leqslant j$. Let p be the largest prime number such that $i < p \leqslant j$. Then $2p > j$, otherwise there is a prime number q such that $p < q < 2p \leqslant j$, by Bertrand's postulate again, and this contradicts the maximality of p. Hence p does not divide $p+1, p+2, \ldots, j$. Now

$$\frac{1}{i} + \frac{1}{i+1} + \ldots + \frac{1}{j} = \frac{\ldots + i(i+1)\ldots(p-1)(p+1)\ldots j + \ldots}{i(i+1)\ldots p \ldots j}.$$

Every term but one in the sum in the numerator is divisible by p, so that the numerator is not divisible by p, whereas the denominator is divisible by p. Hence the expression cannot be an integer.

Also solved by Dylan Gow.

Correction to the article 'Recurring Decimals' by Oliver D. Anderson in Volume 22 Number 1

We apologise for the mistakes which occurred in the printing of this article. On page 9, line 8 should read

$$29 \times 3448\,275\,862\,068 = \frac{99\,999\,975 \qquad\qquad}{\dfrac{24\,999\,972}{99\,999\,999\,999\,972}}$$

The formula at the bottom of page 9 should be labelled (2) rather than the one so labelled on page 10, and line 6 on page 10 should include (2) before $\Leftrightarrow$.

The 1990 Puzzle

Our annual puzzle is not quite so interesting this year. The aim is to construct as many of the numbers 1 to 100 as possible from the digits of the year in order, using only $+$, $-$, $\times$, $\div$, $\surd$, ! and concatenation (i.e. constructing the number 19 from 1 and 9, for example). To start you off, $1 = 1+9-9+0$.

Reviews

An Olympiad Down Under. Edited by W. P. GALVIN, D. C. HUNT and P. J. O'HALLORAN. Australian Mathematics Competition Publications, Canberra, 1988. Pp. xv+256. £11·00. (ISBN 0-7316-5118-9).

This book gives a description of the 29th International Mathematical Olympiad (IMO for short), held from 9 to 21 July 1988 in Canberra, Australia.

In the contest itself, teams of six school-aged students from 49 countries tried to solve six extremely hard maths problems, set in two batches of three on successive days and with $4\frac{1}{2}$ hours for each batch. Awards were presented to the most successful individuals.

The book is a fascinating document covering all aspects of the IMO—mathematical, organisational and social. It contains over 90 problems which were submitted to the IMO, together with their solutions, and it concludes with a comprehensive set of statistics about the results of individuals and teams. Two of the statistics I found interesting were the sad lack of girls among the contestants and Vietnam coming fifth and beating the USA.

Incidentally, team selection for this country is run by the Mathematical Association through the National Maths Contest, British Mathematical Olympiad and further courses and tests. Details and past papers are available from the Mathematical Association.

This book is strongly recommended to all sixth-form mathematics students and to anyone interested in and concerned about mathematics, both for its general fascination and as a source of difficult problems.

Oakham School G. N. THWAITES

Fractal Report. Reeves Telecommunications Laboratories Ltd, West Towan House, Porthtowan, Cornwall TR4 8AX. £10 for 6 issues (including postage and packing).

Fractal enthusiast John de Rivaz has had the bright idea of starting this magazine as a forum for exchanging ideas about fractals and how they may be generated and displayed on various microcomputers. Issue 0 got his scheme off the ground with articles ranging from random fractals to Mandelbrot set sketching on specific machines. Issue 1, under review here, is, in the reviewer's opinion, a substantial advance on its predecessor. It features an article by Dietmar Saupe on the new method of drawing the Mandelbrot set based on the Milnor–Thurston formula for the distance of a given point from that set. Saupe is one of the editors of *The Science of Fractal Images* (Springer, 1988), which is a wonderful source of ideas on all aspects of fractals on computers, and his article here gives the essential algorithms and some subsidiary hints and tips for building a program to take full advantage of the new method. He does not, however, give specific code: that is left to the reader.

Most of the remaining articles do give code. Nick Day gives a BBC BASIC listing for a 'Fractal Turtle Graphics' package: turtle graphics with a drunken turtle!

The precise degree of drunkenness is controllable, of course. W. E. Thompson gives a PASCAL program to display the fractal patterns resulting from applying Newton's method to solve $z^n = 1$ in the complex plane. The idea is to colour a point z according to which root you find if you adopt z as your starting value. The different coloured regions all have the same boundary! Ed Hersom presents the results of a little trip through the Mandelbrot set drawing the iteration $z := z^2+c$ at a sequence of values of c. Finally, John Marriage presents an intricate C++ program to explore interesting parts of a Mandelbrot set picture without wasting time on the boring bits, and Bruce Milne gives a few practical tips for obtaining pleasing Mandelbrot pictures on the R. M. Nimbus.

One small grouse I have is about the price: for £10 one gets a postal subscription for 6 issues and issue 1 is 20 pages long. I find this price rather high. However, if the standard of issue 1 can be maintained, this should prove a valuable magazine, bringing together the many computer buffs who have become fascinated by the blend of the aesthetic and the mathematical which fractal pictures offer. I wish John de Rivaz and his production staff, Karen, all the best with their new venture.

University of Sheffield PETER DIXON

The Puzzling Adventures of Dr Ecco. By DENNIS SHASHA. W. H. Freeman and Company, New York, 1988. Pp. xiii + 183. Hardback £12·95 (ISBN 0-7167-1958-4), paperback £7·95 (ISBN 0-7167-1978-9).

The forty puzzles in this book largely arise from aspects of graphs and networks and other aspects of mathematics relevant to computer design. Although some of the puzzles might give a sixth-former a preview of some aspects of mathematics which he or she had not met before, in general it is not a very thrilling collection. In addition, the whole thing is dressed up in a long and elaborate (and generally rather silly) tale about the mysterious Dr Ecco.

University of Sheffield VICTOR BRYANT

Whys & Wherefores. By MARTIN GARDNER. The University of Chicago Press, 1989. Pp. ix + 261. £15·95. (ISBN 0-226-28245-7).

The Editor said 'if you cannot review it, please return it'!

Who can even put down a Martin Gardner Book? This is a collection of short articles and reviews ranging widely around his interests in literary criticism—a longish and theological piece on 'The Ancient Mariner' and critical approaches to it is one of the best—3D illusions, polywater, chaos, 'Physics: End of the Road?' the puzzles in *Ulysses*, information theory, the fantasies of G. K. Chesterton and Lord Dunsany and many others, not least a pseudonymous review of his own *Whys of a Philosophical Scrivener*! The last title is a very good indicator of this lively and thoughtful man's scope; there is something for everyone, as usual.

The Perse School, Cambridge JONATHAN PINHEY

Guide to Linear Algebra. By DAVID A. TOWERS. Macmillan Education, Basingstoke, 1988. Pp. x+210. Paperback £8·95. (ISBN 0-333-43627-X).

This is a textbook I can really recommend, not least because of a layout which is visually pleasing and easy on the eye, and enhanced by clear and helpful diagrams. The book is aimed at first-year undergraduates, but would also be suitable for sixth formers studying Further Mathematics at A level, and especially those who wish to proceed to a mathematics course at university.

The author begins by assuming that the reader is familiar only with the notation and ideas of basic set theory, and in the first half of the book makes steady progress through vectors, matrix algebra and determinants before going on in the second half to introduce the more abstract ideas of modern linear algebra in chapters on vector spaces, linear transformations and eigenvectors. The final chapter on 'Orthogonal reduction of symmetric matrices' is concerned with the application of linear algebra to the classification of conic and quadric curves, which, as the author states, emphasizes the underlying geometry.

The book is one of a series whose aim is to prepare the student with an A-level background for the 'more rigorous and abstract approach' required at university, and, in my view, it succeeds in this admirably, leading the reader progressively and, as painlessly as possible, from the simple and well-understood ideas to the more complex and challenging. Its value is greatly enhanced by the exercises containing problems of varying difficulty which are incorporated into each chapter and which allow the student to proceed by independent and self-paced study. These exercises are accompanied by virtually full solutions; there are also numerous worked examples. A useful extra feature is an index of notation, in addition to the general index.

De la Salle Sixth Form College, Salford PETER ROWLANDS

Guide to Analysis. By MARY HART. Macmillan Education, Basingstoke, 1988. Pp. xi+202. Paperback £8·95. (ISBN 0-333-43788-8).

In the life of a mathematics student, the highest hurdles to be climbed are those at the start of a course—get through the early days and you'll be all right. The first year in higher education has often proved to be a crucial time for students; the intuitive, informal and sometimes even invalid mathematics of school gives way to a logically thorough approach, more so in the field of analysis than elsewhere.

This book is one of a series aimed at easing this transition. It does not pretend to be a classical treatise on nor an encyclopaedia of analysis. It is an attempt to explain clearly and simply the central ideas which one would expect to meet in any course in elementary analysis. The path is a familiar one, from the elementary properties of numbers through the convergence of infinite sequences and series and thence to continuity and differentiability, culminating in Taylor's theorem and its applications. Throughout, the text shows an admirable clarity both of phrase and, very importantly, type and layout.

I thoroughly enjoyed revisiting this material and found it as easy to read as a novel. It should be considered that someone meeting these ideas for the first time might not read it in the same way. This is true, but there is a good supply of examples and well-chosen exercises to help. (There could be more, but this criticism applies to all elementary texts.) I cannot think where these basic results are treated more clearly for the student meeting them for the first time.

I was pleased, too, to see that the text was interspersed with historical background. The majority of school pupils are told little of the men and women who have created mathematics (or is it discovered?). The lesson that mathematics is advanced through personal insight and imagination is not always taught. It is an important lesson if the next generation is to take on the responsibility for the further advancement of mathematics.

If I can find one fault it is that although the book is clearly the work of an experienced teacher, and it is thoroughly refined in its presentation, I think that the astonishing nature of some of the results has been lost through familiarity. The fact that the real numbers cannot be listed, and so there exist different 'infinite numbers', should startle on its first meeting. That conditionally convergent series can be rearranged to *any* sum should cause wonderment. These facts are treated rather quietly in Dr Hart's book. I think that mathematics should celebrate itself, and much more could be made of these, which are among its great ornaments. This apart, the book is, within its own aims, excellent and highly recommended.

Penwith Sixth Form College — I. M. RICHARDS

Guide to Abstract Algebra. By CAROL WHITEHEAD. Macmillan Education, Basingstoke, 1988. Pp. x+257. Paperback £8·95. (ISBN 0-333-42657-6).

This textbook is one of the first of a series intended to guide undergraduate students through the first year of their mathematics course. The contents include chapters on sets, relations, mappings, the integers, binary operations, and groups, so one could be forgiven for assuming that here is yet another dry-as-dust, highly rigorous approach guaranteed to engender feelings of despair in all but the fearless. Even a cursory glance through the pages shows this to be untrue, and a closer look reveals the work of an author who has ordinary humans in mind and has a deep understanding of first-year students' needs, particularly those who may well be encountering this type of mathematical treatment for the first time at this level.

The development is deceptively gradual, the pace being diluted with a fair sprinkling of textual comments and numerous worked examples plus a selection of graded problems (degree of difficulty not indicated) accompanied for the most part by expanded answers. Apart from a few lapses, the readability is considerably enhanced by an uncommonly good page layout for this type of material and the inclusion in boxes of main theorems and results.

This is a 'why wasn't this book around when I was a student' type of book and any student progressing to second year with most of this material under his belt should have no difficulties. Sixth formers should find a good deal to interest them in the early parts of the chapters but only the dedicated (and perhaps more senior) will penetrate the later, more exacting, depths—but it is well worth the effort.

Formerly of the Universities of Edinburgh and Keele — VICTOR CLARKE

Mathematics for the Biosciences. By ANN C. MENELL and MICHAEL J. BAZIN. Ellis Horwood Ltd, Chichester, 1988. Pp. 231. £16·95 (ISBN 0-7458-0492-6).

The aim of this book is to introduce the mathematics needed for the biological sciences at undergraduate level. Many of the topics covered are illustrated by biological examples. Thus the book will be useful to students since, although the mathematics covered can be found in many texts, it is rare to see it motivated by biological applications. The mathematics is elementary, starting with an introduction to logarithms and trigonometry and moving on to basic calculus. Statistics has been deliberately omitted, but there is a chapter on linear regression. I find it a pity that no linear algebra has been included.

There is always a problem at this level of mathematics in finding applications which do not require more techniques than have been covered. Towards the end of the book, in particular in the chapter on differential equations, the mathematics needed is on a different level from that at the beginning, and would present real difficulties for students with only the background given.

On the whole, I welcome this book because it does fill a gap for this type of student, particularly as mathematics is being used increasingly in the biological sciences.

University of Sheffield CAMILLA JORDAN

Advanced Level Mathematics. By R. C. SOLOMON. DP Publications Ltd, London, 1988. Pp. 583. £6·50 (ISBN 0-905435-99-0).

This book, the author claims, is designed as support for formal tuition, rather than as a replacement of it. Relevant theorems are quoted but are not proved. Formulae are given but are not derived. The main body of the text consists of 46 chapters divided into: (*a*) The Common Core at A-level, (*b*) Extra Topics in Pure Mathematics, (*c*) Mechanics, (*d*) Statistics. Each chapter includes worked examples and routine exercises and concludes with a list of common errors and questions from past examination papers. Candidates for AS-level Mathematics, it is claimed, should find that the questions from Additional Mathematics papers are at a suitable level of difficulty.

Unfortunately, I find that the typography is in many places bad and that the book contains many printing errors, inaccurate statements and many serious errors. For example on page 60: the codomain is confused with the range; on page 98: the definition of a vector is incomplete; on page 109: the condition stated $\mathrm{d}^2y/\mathrm{d}x^2 = 0$ is not sufficient to ensure a point of inflection (e.g. $y = x^4$ at the origin); on page 117: we find the inexcusable statement: 'e is the number 2.718 281 828'; on page 293: the solution of Example 1 is wrong; on page 322: the solution of Example 1 is partly wrong. My list of errors extends to 12 pages of A4 paper. My overall impression of this book is that it is a disservice to A- and AS-level Mathematics. I cannot recommend it, either as support for a course or for revision purposes.

Sixth Form, Royal Grammar School, GREGORY D. ECONOMIDES
Newcastle Upon Tyne

In our last issue we published an announcement on behalf of a reader in Iran who wished to find a penfriend in Britain. One of our student readers responded and has been given the address of the young Iranian.

We have now received a request from an Iranian girl who would like to correspond with a girl in Britain. We shall be glad to assist by passing on the name and address of any reader who may be interested. Please contact:

The Editor - Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH.

CONTENTS

ISSN 0025-5653

PRICES (*postage included*)

Prices for Volume 22 (Issues Nos. 1, 2 and 3)

Subscribers in	*Price per volume*	
North, Central and South America	US\$12.00 or £6.50	(Note: These overseas prices apply even if the order is placed by an agent in Britain.)
Australia	\$A14.00 or £6.50	
Britain, Europe and all other countries	£5.25	

A discount of 10% will be allowed on all orders for five or more copies of Volume 22 sent to the same address.

Back issues

All back issues except Volume 1 are available; information concerning prices and a list of the articles published may be obtained from the Editor.

Enquiries about rates, subscriptions and advertisements should be directed to:

Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Published by the Applied Probability Trust
Typeset by the University of Nottingham
Printed in England by Galliard (Printers) Ltd, Great Yarmouth