Mathematical Spectrum

2007/2008 Volume 40 Number 1

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Problem-Solving, e, and π

Mathematics is about solving problems – discuss! In recent years there has grown up a divide between those who see mathematics as about solving problems and those who see it as building structures. The latter view is exemplified by the Bourbaki school, which originated in France.

But how do you go about solving problems? Yes, ask the right questions in the right order.

1 What am I asked to prove or find?

2 How do I prove something like that?

3 What am I given?

Sometimes we don't make any headway with a problem because we don't ask the right questions. If only it were as easy as that!

Mathematical Spectrum is bombarded with books of problems, usually set in an Olympiad competition or suchlike. It's a bit like requiring us to perform in the Olympic Games when we haven't learned how to walk! Most of us need to start further back. Hence my pleasure in being sent a book of problems called *First Steps for Math Olympians* (see reference 1), published by The Mathematical Association of America. It uses questions set in American mathematics competitions, but starts sufficiently far back for most of us to get on the ladder. Each chapter deals with a particular topic, includes a bit of theory, some marked problems, and finishes with a good number of problems for readers to try and a choice of possible answers – there are full solutions at the back! But don't cheat and look up how to tackle a problem until you've given it a good go. Here are three as samples.

1 A spider has one sock and one shoe for each of its eight legs. On each leg the sock must be put on before the shoe. In how many different orders can the spider put on its socks and shoes? (Page 92.)

2 Mrs Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82 and 91. What was the last score Mrs Walter entered? (Page 139.)

3 For all integers n greater than 1, define

$$a_n = \frac{1}{\log_n 2002}.$$

Let

$$b = a_2 + a_3 + a_4 + a_5 \quad \text{and} \quad c = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}.$$

What is $b - c$? (Page 182.)

I have to confess that I cheated with the last question. As often the case, it's easy when you see the trick! That's what makes mathematicians get hooked on problem-solving!

Another book on the same theme has come our way, the second edition of *Solving Mathematical Problems: A Personal Perspective*, by Terence Tao, published by Oxford University Press (see reference 2). It does not start well. On page 9, of the five number theory problems given, one is wrong and another is either wrong or meaningless, probably both! And I gasped to turn over the page and read 'Basic number theory is a pleasant backwater of mathematics'. Also, researchers may not agree with the statement on page 35: 'Algebra is so important that most of its secrets have been discovered'. However, the author does discuss how to tackle some nice problems gleaned from various sources. Nor does it simply present the final solution, nicely honed, with no clue as to how it was arrived at and the head-scratching that led up to a solution. A useful next step to becoming an olympic problem-solver!

The great π/e debate

I wish I could be more enthusiastic about the DVD *The Great π/e Debate*, produced by The Mathematical Association of America (see reference 3). Which is the better number, π or e? That is the question asked in the debate held between two protagonists, one promoting π and descrying e, the other doing the reverse. I settled down for an enjoyable 40 minutes in front of the TV, but soon squirmed with embarrassment. To be fair, the audience seemed to find it 'hilariously funny', to quote the blurb. Maybe their sense of humour was more in tune with the whole thing than mine. I won't reveal how the audience voted, in case you are interested to see for yourself. As I watched, a formula which has been described as 'the most extraordinary formula in mathematics' came to mind:

$$e^{i\pi} = -1.$$

Doesn't that say it all?

References

1 J. Douglas Faires, *First Steps for Math Olympians: Using the American Mathematics Competitions* (Mathematical Association of America, Washington, DC, 2006).

2 Terence Tao, *Solving Mathematical Problems: A Personal Perspective* (Oxford University Press, 2006).

3 Colin Adams and Thomas Garrity, *The Great π/e Debate: Which is the Better Number?* (DVD-ROM, Mathematical Association of America, Washington, DC, 2007).

Complete the expression

$$(\cdots + \cdots)^2 = x^2 + 1 + \cdots .$$

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An Inequality Connected with a Special Point of a Triangle

ZHANG YUN

1. Introduction

Let F be an interior point of a triangle ABC. Then F is called the *Fermat point* of the triangle if

$$\angle AFB = \angle BFC = \angle CFA = 120^\circ.$$

Let Ω also be an interior point of a triangle ABC. Then Ω is called the *Brocard point* of the triangle if

$$\angle \Omega AB = \angle \Omega BC = \angle \Omega CA.$$

There are a good many articles about the Fermat point and the Brocard point of a triangle. I will give a new result on these.

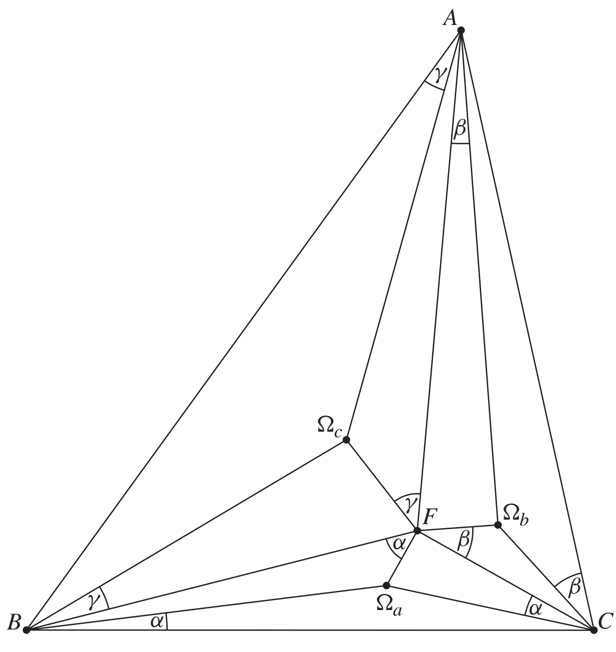

Figure 1

2. Main result

Theorem 1 *Let F be the Fermat point of a triangle ABC with $\angle ABC < 120°$, $\angle BCA < 120°$, and $\angle CAB < 120°$. Let Ω_a, Ω_b, and Ω_c denote the Brocard points of ΔFBC, ΔFCA, and ΔFAB respectively, and write*

$$\angle\Omega_a FB = \angle\Omega_a BC = \angle\Omega_a CF = \alpha,$$
$$\angle\Omega_b FC = \angle\Omega_b CA = \angle\Omega_b AF = \beta,$$
$$\angle\Omega_c AB = \angle\Omega_c BF = \angle\Omega_c FA = \gamma.$$

Then

$$\cot\alpha + \cot\beta + \cot\gamma \geq 5\sqrt{3}. \tag{1}$$

Lemma 1 *Let F be the Fermat point of a triangle ABC with*

$$\angle ABC < 120°, \qquad \angle BCA < 120°, \qquad \angle CAB < 120°,$$

denote the lengths of the sides BC, CA, and AB by a, b, and c respectively, and the area of ABC is Δ. Then we have

$$(AF + BF + CF)^2 = \tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\Delta.$$

Proof The cosine formula applied to the triangle BCF gives

$$a^2 = BF^2 + CF^2 - 2BF \times CF\cos\angle BFC = BF^2 + CF^2 + BF \times CF.$$

Similarly,

$$b^2 = CF^2 + AF^2 + CF \times AF, \qquad c^2 = AF^2 + BF^2 + AF \times BF,$$

so that

$$\begin{aligned} a^2 + b^2 + c^2 &= 2(AF^2 + BF^2 + CF^2) + AF \times BF + BF \times CF + CF \times AF \\ &= 2(AF + BF + CF)^2 - 3(AF \times BF + BF \times CF + CF \times AF). \end{aligned}$$

Hence,

$$(AF + BF + CF)^2 = \tfrac{1}{2}(a^2 + b^2 + c^2) + \tfrac{3}{2}(AF \times BF + BF \times CF + CF \times AF).$$

Since

$$\begin{aligned} \Delta &= \frac{1}{2}AF \times BF\sin\angle AFB + \frac{1}{2}BF \times CF\sin\angle BFC + \frac{1}{2}CF \times AF\sin\angle CFA \\ &= \frac{1}{2}AF \times BF\frac{\sqrt{3}}{2} + \frac{1}{2}BF \times CF\frac{\sqrt{3}}{2} + \frac{1}{2}CF \times AF\frac{\sqrt{3}}{2}, \end{aligned}$$

we have

$$AF \times BF + BF \times CF + CF \times AF = \frac{4\Delta}{\sqrt{3}},$$

so that

$$(AF + BF + CF)^2 = \tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\Delta.$$

Lemma 2 *Let Ω be the Brocard point of a triangle ABC, write*

$$\angle \Omega AB = \angle \Omega BC = \angle \Omega CA = \omega,$$

denote the lengths of the sides BC, CA, and AB by a, b, and c respectively, and denote the area of ABC by Δ. Then we have

$$\cot\omega = \frac{a^2 + b^2 + c^2}{4\Delta}.$$

Proof The sine formula applied to the triangle ΩAC gives

$$\frac{\Omega A}{\sin\omega} = \frac{b}{\sin(180^\circ - \omega - (A - \omega))} = \frac{b}{\sin A},$$

so

$$\Omega A = \frac{b}{\sin A}\sin\omega.$$

Similarly,

$$\Omega B = \frac{c}{\sin B}\sin\omega, \qquad \Omega C = \frac{a}{\sin C}\sin\omega.$$

We have

$$\begin{aligned}
\Delta &= \tfrac{1}{2}\Omega A \times \Omega B \sin\angle A\Omega B + \tfrac{1}{2}\Omega B \times \Omega C \sin\angle B\Omega C + \tfrac{1}{2}\Omega C \times \Omega A \sin\angle C\Omega A \\
&= \frac{1}{2}\frac{b}{\sin A}\sin\omega\frac{c}{\sin B}\sin\omega\sin B + \frac{1}{2}\frac{c}{\sin B}\sin\omega\frac{a}{\sin C}\sin\omega\sin C \\
&\quad + \frac{1}{2}\frac{a}{\sin C}\sin\omega\frac{b}{\sin A}\sin\omega\sin A \\
&= \frac{1}{2}\sin^2\omega\left(\frac{bc}{\sin A} + \frac{ca}{\sin B} + \frac{ab}{\sin C}\right).
\end{aligned}$$

Also,

$$\Delta = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C,$$

so that

$$\begin{aligned}
\Delta &= \frac{1}{2}\sin^2\omega\left(bc\frac{bc}{2\Delta} + ca\frac{ca}{2\Delta} + ab\frac{ab}{2\Delta}\right) \\
&= \frac{\sin^2\omega}{4\Delta}(a^2b^2 + b^2c^2 + c^2a^2).
\end{aligned}$$

Hence,

$$\sin^2\omega = \frac{4\Delta^2}{a^2b^2 + b^2c^2 + c^2a^2},$$

so that

$$\cos^2\omega = 1 - \frac{4\Delta^2}{a^2b^2 + b^2c^2 + c^2a^2} = \frac{a^2b^2 + b^2c^2 + c^2a^2 - 4\Delta^2}{a^2b^2 + b^2c^2 + c^2a^2}.$$

Hence,

$$\cot\omega = \frac{\sqrt{a^2b^2 + b^2c^2 + c^2a^2 - 4\Delta^2}}{2\Delta}.$$

Since

$$\Delta^2 = \left(\tfrac{1}{2}ab\sin C\right)^2 = \tfrac{1}{4}a^2b^2(1 - \cos^2 C),$$

we have

$$\begin{aligned}
4\Delta^2 &= a^2b^2\left(1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2\right) \\
&= a^2b^2\,\frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2} \\
&= -\tfrac{1}{4}(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2a^2c^2),
\end{aligned}$$

so that

$$\begin{aligned}
&a^2b^2 + b^2c^2 + c^2a^2 - 4\Delta^2 \\
&\quad = a^2b^2 + b^2c^2 + c^2a^2 + \tfrac{1}{4}(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2) \\
&\quad = \left(\tfrac{1}{2}(a^2 + b^2 + c^2)\right)^2.
\end{aligned}$$

Hence,

$$\cot\omega = \frac{a^2 + b^2 + c^2}{4\Delta}.$$

Lemma 3 *Let x_i be real numbers and let y_i be positive real numbers ($i = 1, 2, \ldots, n$, $n \ge 2$, $n \in \mathbb{N}^*$). Then we have*

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \ge \frac{(\sum_{i=1}^{n} x_i)^2}{\sum_{i=1}^{n} y_i}.$$

Proof We take $a_i = x_i/\sqrt{y_i}$ and $b_i = \sqrt{y_i}$, $i = 1, 2, \ldots, n$, in Cauchy's inequality,

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \ge \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

to give

$$\left(\sum_{i=1}^{n} \frac{x_i^2}{y_i}\right)\left(\sum_{i=1}^{n} y_i\right) \ge \left(\sum_{i=1}^{n} x_i\right)^2.$$

Hence,

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \ge \frac{(\sum_{i=1}^{n} x_i)^2}{\sum_{i=1}^{n} y_i}.$$

Proof of Theorem 1 We let Δ_a, Δ_b, and Δ_c denote the areas of ΔFBC, ΔFCA, and ΔFAB respectively. Since Ω_a is the Brocard point of ΔFBC, by Lemma 2 we have

$$\cot\alpha = \frac{1}{4\Delta_a}(FB^2 + BC^2 + FC^2).$$

Similarly,

$$\cot\beta = \frac{1}{4\Delta_b}(FC^2 + CA^2 + FA^2)$$

and

$$\cot\gamma = \frac{1}{4\Delta_c}(FA^2 + AB^2 + FB^2),$$

so that

$$\begin{aligned}&\cot\alpha + \cot\beta + \cot\gamma \\ &\quad = \frac{1}{4}\left(\frac{FB^2}{\Delta_a} + \frac{FC^2}{\Delta_b} + \frac{FA^2}{\Delta_c}\right) + \frac{1}{4}\left(\frac{FC^2}{\Delta_a} + \frac{FA^2}{\Delta_b} + \frac{FB^2}{\Delta_c}\right) + \frac{1}{4}\left(\frac{a^2}{\Delta_a} + \frac{b^2}{\Delta_b} + \frac{c^2}{\Delta_c}\right).\end{aligned}$$

Hence, by Lemma 3,

$$\begin{aligned}&\cot\alpha + \cot\beta + \cot\gamma \\ &\quad \geq \frac{1}{4}\frac{(FB + FC + FA)^2}{\Delta_a + \Delta_b + \Delta_c} + \frac{1}{4}\frac{(FC + FA + FB)^2}{\Delta_a + \Delta_b + \Delta_c} + \frac{1}{4}\frac{(a + b + c)^2}{\Delta_a + \Delta_b + \Delta_c}.\end{aligned}$$

By Lemma 1 and $\Delta = \Delta_a + \Delta_b + \Delta_c$, we have

$$\begin{aligned}\cot\alpha + \cot\beta + \cot\gamma &\geq \frac{1}{4}\frac{1}{\Delta}\left(\frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\Delta\right)2 + \frac{1}{4}\frac{1}{\Delta}(a + b + c)^2 \\ &= \frac{1}{4\Delta}(a^2 + b^2 + c^2) + \sqrt{3} + \frac{1}{4\Delta}(a + b + c)^2 \\ &= \frac{1}{4\Delta}(a^2 + b^2 + c^2) + \frac{1}{4\Delta}(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) + \sqrt{3}.\end{aligned}$$

Since

$$a^2 + b^2 \geq 2ab, \qquad b^2 + c^2 \geq 2bc, \qquad c^2 + a^2 \geq 2ca,$$

we have

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

We put $s = \frac{1}{2}(a + b + c)$, let R denote the radius of the circumcircle of ΔABC, and let r denote the radius of the inscribed circle of ΔABC. By reference 1, we obtain

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 16s^2r^2 = 16\Delta^2,$$

so that

$$(ab + bc + ca)^2 - 2abc(a + b + c) \geq 16\Delta^2$$

and

$$(ab + bc + ca)^2 \geq 2 \times 4Rrs \times 2s + 16\Delta^2 = 16Rrs^2 + 16\Delta^2.$$

Using Euler's inequality, $R \geq 2r$ (see reference 1), we have

$$(ab + bc + ca)^2 \geq 32r^2s^2 + 16\Delta^2 = 32\Delta^2 + 16\Delta^2 = 48\Delta^2,$$

so that

$$ab + bc + ca \geq 4\sqrt{3}\Delta$$

and

$$\begin{aligned}
\cot\alpha + \cot\beta + \cot\gamma &\geq \frac{1}{4\Delta}(a^2 + b^2 + c^2) + \frac{1}{4\Delta}(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) + \sqrt{3} \\
&\geq \frac{1}{4\Delta}(ab + bc + ca) + \frac{3}{4\Delta}(ab + bc + ca) + \sqrt{3} \\
&= \frac{1}{\Delta}(ab + bc + ca) + \sqrt{3} \\
&\geq 4\sqrt{3} + \sqrt{3} \\
&= 5\sqrt{3},
\end{aligned}$$

so the inequality (1) is proved.

Reference

1 Zhang Yun, An introduction to geometric inequalities, *Math. Spectrum* **32** (1999/2000), pp. 35–37.

Zhang Yun *is a senior teacher (associate professor) of mathematics at the high school attached to Xi'an Jiaotong University in China. He has published over 120 papers. His research interests include elementary number theory, algebraic inequalities, and geometric inequalities.*

For integers $x_1, \ldots, x_n$ with $n \geq 2$,

$$|x_1 - x_2| + |x_2 - x_3| + \cdots + |x_n - x_1|$$

is always even. Why?

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On the Divergence of *p*-like Series with *p* > 1

HOSSEIN BEHFOROOZ

Introduction

In calculus and real analysis, the theorem on 'p-series' states that, for a fixed real number p, $\sum(1/n^p)$ converges when $p > 1$ and diverges when $p \leq 1$. But what happens when p depends on n? Is it possible for $\sum(1/n^p)$ to be divergent when $p > 1$? In this article, we will present some interesting divergent series of the form $\sum(1/n^p)$ with $p > 1$. We will also use the techniques of reference 1 to delete and weed out some specific terms of these series to change them to convergent series.

Problem 1 For any positive real number α, show that $\lim_{x\to\infty} x^{1/x^\alpha} = 1$.

Solution 1 Suppose that $y = x^{1/x^\alpha}$. Then $\ln y = \ln x^{1/x^\alpha} = \ln x / x^\alpha$ and

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x^\alpha} = \lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0.$$

Hence, $\lim_{x\to\infty} x^{1/x^\alpha} = \mathrm{e}^0 = 1$.

Problem 2 If α is a positive real number, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n^\alpha}}$$

is a divergent series.

Solution 2 By using the limit comparison test and the result of Problem 1, we have

$$\lim_{n\to\infty} \frac{1/n}{1/n^{1+1/n^\alpha}} = \lim_{n\to\infty} \frac{n^{1+1/n^\alpha}}{n} = \lim_{n\to\infty} n^{1/n^\alpha} = 1.$$

Since the harmonic series $\sum_{n=1}^{\infty}(1/n)$ is divergent, $\sum_{n=1}^{\infty}(1/n^{1+1/n^\alpha})$ is also divergent.

Problem 2 shows that there are infinitely many divergent series $\sum_{n=1}^{\infty}(1/n^p)$, with $p = 1 + 1/n^\alpha > 1$, for different values of $\alpha > 0$.

Note 1 By using the solution to Problem 2, we can claim that any two similar subseries of $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^{1+1/n^\alpha})$ are either both convergent or both divergent. For example, since $\sum_{n=1}^{\infty}(1/n^2)$ is convergent, then $\sum_{n=1}^{\infty}(1/(n^2)^{1+1/n^{2\alpha}})$ is convergent too, and because $\sum_{n=1}^{\infty}(1/3n)$ is divergent then $\sum_{n=1}^{\infty}(1/(3n)^{1+1/(3n)^\alpha})$ is divergent too.

The following theorem is a generalization of Problem 2 (with $a_n = 1/n^\alpha$ and $\alpha > 0$). Also, the series in references 1 and 2 are two special cases of the following theorem.

Theorem 1 *For any sequence* $\{a_n\}$ *such that* $\lim_{n\to\infty} n^{a_n} = L > 0$, *the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+a_n}}$$

is divergent.

Proof By applying the limit comparison test, we have

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) \Big/ \left(\frac{1}{n^{1+a_n}}\right) = \lim_{n\to\infty} n^{a_n} = L > 0.$$

So, the series $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ behaves like the harmonic series and consequently it is divergent.

Note 2 In Theorem 1, it can be shown that $\lim_{n\to\infty} a_n = 0$. Since $\lim_{n\to\infty} n^{a_n} = L > 0$ then $\ln n^{a_n} \to \ln L$ or $a_n \ln n \to \ln L$, which implies that $a_n = a_n \ln n / \ln n \to 0$ as $n \to \infty$. If $\lim_{n\to\infty} a_n = k > 0$ then there exists a positive integer m such that, for all integers $n > m$, $a_n > k/2$. This implies that, for $n > m$, we have $1 + a_n > 1 + k/2$ and $1/n^{1+a_n} < 1/n^{1+k/2}$. Since $\sum_{n=1}^{\infty}(1/n^{1+k/2})$ is convergent, $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ will be convergent.

Note 3 The above limit comparison test implies that any two similar subseries of $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ are either both convergent or both divergent (with the condition $\lim_{n\to\infty} n^{a_n} = L > 0$).

Now, by using the weeding out techniques of reference 1 on harmonic series, we can eliminate some specific terms of $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ to change it to a convergent series. *In the rest of this article, the sequence* $\{a_n\}$ *in* $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ *satisfies the condition* $\lim_{n\to\infty} n^{a_n} = L > 0$.

Harmonic series

It is well known that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \tag{1}$$

is divergent and that its partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is very close to $\log n$. This shows that the harmonic series diverges very slowly. For example, it takes more than 1.5×10^{43} terms for its partial sum to reach 100.

Problem 3 Show that the harmonic series is divergent.

Solution 3 There are so many different approaches to proving this. The first attempt, by Nicole Oresme, dates back to 1350. Then it received very little attention for almost 400 years. The divergence of (1) was rediscovered in the late 17th century. One proof was by John Bernoulli (1667–1747).

The following proof (by contradiction) is perhaps the shortest proof; it was discovered quite recently by Leonard Gillman on a train ride (see reference 3). Suppose that (1) is convergent and its sum is equal to a positive real number S. Then,

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots \\ &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \\ &= S. \end{aligned}$$

This implies that $S > S$, which is impossible. Hence, the harmonic series is divergent.

Weeding out the harmonic series

We begin with the following interesting problems; see references 1 and 4 and the references therein.

Problem 4 Show that the following series is divergent:

$$\sum_{9 \in n} \frac{1}{n} = \frac{1}{9} + \frac{1}{19} + \frac{1}{29} + \cdots + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + \cdots + \frac{1}{99} + \frac{1}{109} + \cdots . \tag{2}$$

Here, $9 \in n$ means that n contains 9 or, in other words, n has at least one digit nine in it; $9 \notin n$ means that n does not contain any 9s. For example, $2 \in 256$, $5 \in 256$, and $7 \notin 256$. Also, $8 \in n \wedge 9 \in n$ means that n contains each of the digits 8 and 9 at least once, for example 281 099.

Solution 4 Missing out some terms, we have

$$\begin{aligned} \frac{1}{9} + \frac{1}{19} + \frac{1}{29} + \cdots + \frac{1}{10n+9} &> \frac{1}{10} + \frac{1}{20} + \cdots + \frac{1}{10(n+1)} \\ &= \frac{1}{10}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right) \\ &\to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Hence, the series (2) is divergent.

Problem 5 Show that the following 9-free series is convergent:

$$T_9 = \sum_{9 \notin n} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} + \frac{1}{10} + \cdots . \tag{3}$$

Solution 5 There are only 8×9^k different $(k+1)$-digit positive integers with their digits belonging to $\{0, 1, 2, \ldots, 8\}$. So, for any natural number k, there are only 8×9^k 9-free positive integers between 10^k and $10^{k+1} - 1$. We consider a partial sum of the infinite series (3) and arrange it as follows:

$$\begin{aligned}
&\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{10} + \cdots + \frac{1}{88}\right) \\
&\quad + \left(\frac{1}{100} + \cdots + \frac{1}{888}\right) + \cdots + \left(\frac{1}{10^k} + \cdots + \frac{1}{\overbrace{88\cdots8}^{(k+1)\text{ times}}}\right) \\
&< (\overbrace{1 + 1 + \cdots + 1}^{8\times 9^0 \text{ times}}) \left(\overbrace{\frac{1}{10} + \frac{1}{10} + \cdots + \frac{1}{10}}^{8\times 9^1 \text{ times}}\right) \\
&\quad + \left(\overbrace{\frac{1}{100} + \frac{1}{100} + \cdots + \frac{1}{100}}^{8\times 9^2 \text{ times}}\right) + \cdots + \left(\overbrace{\frac{1}{10^k} + \cdots + \frac{1}{10^k}}^{8\times 9^k \text{ times}}\right) \\
&= 8 \times 9^0 \times \frac{1}{10^0} + 8 \times 9^1 \times \frac{1}{10^1} + 8 \times 9^2 \times \frac{1}{10^2} + \cdots + 8 \times 9^k \times \frac{1}{10^k} \\
&= 8 \sum_{n=0}^{k} \left(\frac{9}{10}\right)^n \\
&< \frac{8}{1 - \frac{9}{10}} \\
&= 80.
\end{aligned}$$

This proves that the series (3) is convergent.

A similar argument shows that the series $T_0, T_1, \ldots, T_8$ (with obvious notation) is also convergent.

The series (3) converges very slowly and its exact value is approximately equal to 22.921. Baillie (see reference 5), has calculated (to 20 decimal places) the values of the ten individual convergent series $T_0, T_1, \ldots, T_9$. Table 1 gives the values of these sums to three decimal places.

Table 1

T_0	23.103	T_5	21.835
T_1	16.177	T_6	22.206
T_2	19.257	T_7	22.493
T_3	20.570	T_8	22.726
T_4	21.327	T_9	22.921

Table 2

k	N_k	M_k
1	8	1
2	80	19
3	728	271

Remark For any positive integer k, let N_k be the number of 9-free positive integers which are less than 10^k and M_k be the number of positive integers which are less than 10^k and contain at least one digit 9. For small ks, we have $N_k > M_k$; see table 2. However, when k is large enough, $M_k > N_k$ or, in probability notation,

$$\mathrm{P}(n \text{ contains } 9) > \mathrm{P}(n \text{ is } 9\text{-free}).$$

Problem 6 Show that

$$\lim_{k\to\infty} \frac{N_k}{M_k} = 0.$$

Solution 6 From the solution to Problem 5, we have

$$N_k = 8 \times 9^0 + 8 \times 9^1 + 8 \times 9^2 + \cdots + 8 \times 9^{k-1} = 9^k - 1$$

and

$$M_k = (10^k - 1) - (9^k - 1) = 10^k - 9^k.$$

So, for $k \geq 7$, we have $M_k > N_k$ and

$$\lim_{k\to\infty} \frac{N_k}{M_k} = \lim_{k\to\infty} \frac{9^k - 1}{10^k - 9^k} = \lim_{k\to\infty} \frac{1 - 1/9^k}{(10/9)^k - 1} = 0.$$

One surprising conclusion of Problem 6 is that, in a large range of natural numbers from 1 to 10^k, 'almost all' natural numbers in this range contain the digit 9. In other words, in a large scale random selection (the set of all natural numbers), the probability that a number chosen at random will not contain a 9 is zero.

It is obvious that, in the above discussion, the number 9 does not play a critical role, and similar results can be derived for any particular digit $r = 0, 1, \ldots, 8$. We have the following theorem.

Theorem 2 *For any number $r = 0, 1, 2, \ldots, 9$,*

(i) *the series $\sum_{r\in n}(1/n)$ is divergent,*

(ii) *the series $\sum_{r\notin n}(1/n)$ is convergent.*

In Note 3, we mentioned that $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ have the same behaviour and their similar corresponding subseries are either both convergent or both divergent. So, we have the following theorem for subseries of $\sum_{n=1}^{\infty}(1/n^{1+a_n})$.

Theorem 3 *For any number $r = 0, 1, 2, \ldots, 9$,*

(i) *the series $\sum_{r\in n}(1/n^{1+a_n})$ is divergent,*

(ii) *the series $\sum_{r\notin n}(1/n^{1+a_n})$ is convergent.*

More surprising results

Now we extend this thinning idea and use the above results to obtain other theorems. We consider two special cases of Theorem 3, with $r = 9$ and $r = 8$. From Theorem 3, we have

(i) the series $\sum_{9 \in n}(1/n^{1+a_n})$ is divergent,

(ii) the series $\sum_{8 \notin n}(1/n^{1+a_n})$ is convergent.

The convergent series (ii) contains all terms of the divergent series (i) except those terms with n values containing both digits 8 and 9 (such as $n = 89$, 98, 189, and so on). So the divergence of (i) and also of the harmonic series and $\sum_{n=1}^{\infty}(1/n^{1+a_n})$ depends on these types of terms. This means that

$$\sum_{8 \in n \wedge 9 \in n} \frac{1}{n} = \frac{1}{89} + \frac{1}{98} + \frac{1}{189} + \cdots \qquad \text{and} \qquad \sum_{8 \in n \wedge 9 \in n} \frac{1}{n^{1+a_n}}$$

are divergent and their complements are convergent. In general, we have the following theorem.

Theorem 4 *For any two different integers $r, s = 0, 1, 2, \ldots, 9$,*

(i) *the series $\sum_{r \in n \wedge s \in n}(1/n^{1+a_n})$ is divergent,*

(ii) *the series $\sum_{r \notin n \vee s \notin n}(1/n^{1+a_n})$ is convergent.*

If we continue in this way, we obtain the following theorem.

Theorem 5 *Let D be the set of positive integers that contains each of the digits $0, 1, 2, \ldots, 9$ at least once. Then we have*

(i) *the series $\sum_{n \in D}(1/n^{1+a_n})$ is divergent,*

(ii) *the series $\sum_{n \notin D}(1/n^{1+a_n})$ is convergent.*

The first value of n to be included in the divergent series in Theorem 5(i) and so not included in the convergent series in Theorem 5(ii) is $n = 1\,023\,456\,789$. Now we state the following theorem which is similar to the result of Problem 6.

Theorem 6 *For any positive integer k, let N_k be the number of all positive integers n which are less than 10^k and $n \notin D$ and let M_k be the number of all positive integers m which are less than 10^k and $m \in D$. Then $\lim_{k \to \infty}(N_k/M_k) = 0$.*

Proof This theorem is a special case of Theorem 7, below; see also reference 1.

Theorem 7 *Suppose that C is a subset of positive integers such that $\sum_{n \in C}(1/n^{1+a_n})$ is convergent. For any positive integer k, let N_k be the number of elements in C that are less than or equal to 10^k and let $M_k = 10^k - N_k$. Then $\lim_{k \to \infty}(N_k/M_k) = 0$.*

Proof Since $\sum_{n\in C}(1/n^{1+a_n})$ is convergent and $\lim_{n\to\infty}((1/n)/(1/n^{1+a_n})) = L > 0$, therefore $\sum_{n\in C}(1/n)$ will be convergent too. Consider

$$\frac{N_k}{M_k} = \frac{N_k}{10^k - N_k} = \frac{(N_k/10^k)}{1-(N_k/10^k)}.$$

So, it suffices to prove that $\lim_{k\to\infty}(N_k/10^k) = 0$. But, $N_k - N_{k-1}$ is the number of elements of $C_k = \{n \in C \mid 10^{k-1} < n \le 10^k\}$ and

$$\sum_{n\in C_k}\frac{1}{n} \ge \sum_{n\in C_k}\frac{1}{10^k} = \frac{N_k - N_{k-1}}{10^k}.$$

So,

$$\sum_{k=1}^{\infty}\frac{N_k - N_{k-1}}{10^k} \le \sum_{n\in C}\frac{1}{n} < \infty.$$

Thus, $\lim_{k\to\infty}((N_k - N_{k-1})/10^k) = 0$. If we set $a_k = N_k/10^k$ then, for all k, we have $a_k \le 1$ and $\lim_{k\to\infty}(a_k - a_{k-1}/10) = 0$. Now we show that $\lim_{k\to\infty} a_k = 0$. Let $\varepsilon > 0$. There exists an integer n such that $a_k - a_{k-1}/10 < \varepsilon$ for all $k \ge n$. By induction on m, we show that, for any natural number m,

$$a_{n+m} \le \varepsilon\left(\sum_{k=0}^{m-1}\frac{1}{10^k}\right) + \frac{1}{10^m}. \tag{4}$$

For $m = 1$, we have

$$a_{n+1} = a_{n+1} - \frac{a_n}{10} + \frac{a_n}{10} < \varepsilon + \frac{a_n}{10} \le \varepsilon + \frac{1}{10}.$$

Suppose that (4) holds for $m \ge 1$, then we obtain

$$\begin{aligned} a_{n+m+1} &= a_{n+m+1} - \frac{a_{n+m}}{10} + \frac{a_{n+m}}{10} \\ &< \varepsilon + \frac{1}{10}\left[\varepsilon\sum_{k=0}^{m-1}\frac{1}{10^k} + \frac{1}{10^m}\right] \\ &= \varepsilon\sum_{k=0}^{m}\frac{1}{10^k} + \frac{1}{10^{m+1}}. \end{aligned}$$

Thus, (4) holds for all $m \ge 1$ and $a_{n+m} < \frac{10}{9}\varepsilon + 1/10^m < 2\varepsilon$ for all sufficiently large m. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{k\to\infty} a_k = 0$ or $\lim_{k\to\infty}(N_k/10^k) = 0$.

The case of *n* prime

Since the Euler series $\sum(1/n)$, over prime numbers n, is a divergent series, $\sum_{n \text{ prime}}(1/n^{1+a_n})$ is divergent too. Again, the convergent series in Theorem 5(ii) contains all the terms of $\sum_{n \text{ prime}}(1/n^{1+a_n})$ except for those prime numbers which belong to the set D of Theorem 5. Hence, the divergence of $\sum_{n \text{ prime}}(1/n^{1+a_n})$ is due to these types of prime numbers. For example, 10 123 457 689, 10 123 465 789, and 10 123 465 897 are the first three such primes. By removing all the terms of $\sum_{n \text{ prime}}(1/n^{1+a_n})$ corresponding to these types of primes, the remaining series will be convergent.

Theorem 8 *Let E be the set of all primes that contains each of digits $0, 1, 2, \ldots, 9$ at least once. Then we have*

(i) *the series $\sum_{n \in E}(1/n^{1+a_n})$ is divergent,*

(ii) *the series $\sum_{n \notin E}(1/n^{1+a_n})$ is convergent.*

Conclusions

Firstly, we introduced a family of divergent series of the form $\sum(1/n^p)$, with $p > 1$. Here, p was not a constant; the famous calculus theorem on $\sum(1/n^p)$ for constant p holds.

Secondly, if we change the base of the natural number system from 10 to any higher number b, then any integer L less than b can be considered as a single-digit number. Then we can generalize our discussion by changing 9 to L and obtain similar results. In this case, if we apply Problem 6 to this digit L, we get some surprising results. For example, with base $b = 10^7$, $L = 1729$ looks like a single-digit number. Thus, when k is large enough, for a randomly selected positive integer $n < b^k$, we have $\mathrm{P}(n \text{ contains } L) > \mathrm{P}(n \text{ is } L\text{-free})$. Intuitively, it would seem that this claim is not true. If we ask the entire population of the world to write down a positive integer, we would not expect to see that the number of written integers with block 1729 is more than the other written integers. But for large enough k we can expect this because in this case (with the notation of Problem 6), we have

$$\lim_{k \to \infty} \frac{N_k}{M_k} = \lim_{k \to \infty} \frac{(b-1)^k - 1}{b^k - (b-1)^k} = 0.$$

This is a good example of the Law of Large Numbers or the Empirical Concept of Probability. What number is really a large number? Sometimes ten trials is enough and it is a reasonable sample size, and sometimes 10^7 is not a big enough number. The reason is that in real life we deal with numbers which contain only a few digits not with millions and millions of digits.

Acknowledgment

I would like to thank the referees for their comments and suggestions that improved the contents of this article.

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The Picture-Perfect Numbers

JOSEPH L. PE

1. The elusive picture-perfect numbers

There is something very special about the number 10 311.

A number is *perfect* if it is equal to the sum of its proper divisors. The number 10 311 is not perfect; its proper divisors are 1, 3, 7, 21, 491, 1 473, and 3 437, which sum to 5 433. (Since $5\,433 < 10\,311$, 10 311 is said to be *deficient*.) Hence, the equation

$$10\,311 = 1 + 3 + 7 + 21 + 491 + 1\,473 + 3\,437$$

is *not* valid. However, reading this invalid equation *backwards* gives

$$7\,343 + 3\,741 + 194 + 12 + 7 + 3 + 1 = 11\,301,$$

which, amazingly, is valid!

A number n is called *picture-perfect* or *mirror-perfect* if the reverse of n is equal to the sum of the reverses of the proper divisors of n. (Leading zeros are ignored.) In other words, if n is placed on one side of an equality sign and the (unevaluated) sum of the proper divisors of n are placed on the other side, then the resulting equation read backwards is valid. This explains the use of the term 'picture-perfect', since a picture of an object is a mirror image (i.e. an orientation reversal) of that object.

The first picture-perfect number (PPN) is 6 (the first perfect number); but since 6 and its proper divisors are single digit, it is trivially picture-perfect. The first nontrivial PPN is 10 311.

If P is a PPN, then we call the expression $P \stackrel{\text{b}}{=} D$, where D is the (unevaluated) sum of the proper divisors of P, the *mirror equation* of P. The symbol '$\stackrel{\text{b}}{=}$' indicates that the mirror equation should be read backwards to be valid. For example, the mirror equation of 10 311 is

$$10\,311 \stackrel{\text{b}}{=} 1 + 3 + 7 + 21 + 491 + 1\,473 + 3\,437.$$

Here is the MATHEMATICA® code to generate PPNs not exceeding 10^{10}.

```
f[n_] := FromDigits[Reverse[IntegerDigits[n]]];
n = 2; (*Initial value of n*)
While[n<10^10, If[f[n]==
    Apply[Plus,Map[f, Drop[Divisors[n], -1]]], Print[n]]; n++]
```

When my computer search had not given any new PPNs less than 10^7, I was about to conjecture that 10 311 was the only nontrivial PPN. However, after several hours of MATHEMATICA running on my machine, I was rewarded with the third PPN, 21 661 371, which has mirror equation

$$21\,661\,371 \stackrel{\text{b}}{=} 1+3+9+27+443+1\,329+1\,811+3\,987+5\,433+11\,961 + 16\,299+48\,897+802\,273+2\,406\,819+7\,220\,457.$$

The discovery of this large PPN raised hopes that a fourth number would be found before long. Two problem enthusiasts, Daniel Dockery and Mark Ganson, soon joined me in the search. Our group has a discussion forum (reference 1) which we use to share ideas and results, as well as software customized for PPN search. For example, Ganson has provided a search utility he has written. The results and conjectures mentioned in this article first appeared in reference 1.

We focused our efforts on the interval from 10^9 to 10^{10}. Three weeks later, Dockery found the fourth PPN. It is 1 460 501 511, with mirror equation

$$1\,460\,501\,511 \stackrel{\text{b}}{=} 1+3+7+21+101+303+707+2\,121+688\,591+2\,065\,773 + 4\,820\,137+14\,460\,411+69\,547\,691+208\,643\,073+486\,833\,837.$$

After this find, our progress was slow, even though Ganson's C++® search program ran at least twice as fast as MATHEMATICA. Then Jens Kruse Andersen joined the team by announcing his discovery of a new PPN, 7 980 062 073, with mirror equation

$$7\,980\,062\,073 \stackrel{\text{b}}{=} 1+3+19+57+140\,001\,089+420\,003\,267+2\,660\,020\,691.$$

Devising an efficient algorithm that caches divisor information, Andersen quickly tested all numbers up to 10^{10}, and concluded that there are no more PPNs in this range other than the five already mentioned above. Within a month, our team exhaustively searched the interval from 10^{10} to 10^{12} using Andersen's program.

The sequence of PPNs

$$6,\ 10\,311,\ 21\,661\,371,\ 1\,460\,501\,511,\ 7\,980\,062\,073,\ 79\,862\,699\,373,\ 798\,006\,269\,373, \ldots$$

is sequence A069942 of reference 2, and also appears in its short index. 'Small' PPNs are rare pearls in the infinite ocean of numbers – there are only five PPNs below 10^{10}. The seven PPNs listed above are all the PPNs less than 10^{12}. At the time of writing, the value of the eighth PPN is unknown. Unexpectedly, this sparseness is somewhat relieved in the realm of large numbers, as the next section will show.

2. Andersen's theorem

The even perfect numbers can be generated by Euclid's expression $2^{n-1}(2n-1)$, where n is a prime such that $2^n - 1$ is also prime. Is there an expression yielding (not necessarily all) PPNs? Andersen discovered the following remarkable result.

Andersen's theorem *The decimal number $p = 140z10n89$ is prime if and only if the product $57\ p$ is picture-perfect (where z is any number (possibly none) of 0s and n is any number (possibly none) of 9s). In particular, if n has no 9s, then $57\ p$ has the form $798z62073$; if n has at least one 9, then $57\ p$ has the form $798z626m373$, where m has one 9 less than n.*

Table 1

Andersen prime	Andersen number
140 **00**1 089	7 98**0 0**62 073
1 401 0**99 9**89	79 862 6**99** 373
14 0**00** 10**9 9**89	798 **00**6 26**9** 373
140 **000** 10**9 999 9**89	7 98**0 00**6 26**9 999** 373

The proof of Andersen's theorem appears towards the end of this section. (In this article, a product such as $57\ p$ is written with a space between the multiplicands to distinguish it from concatenations of digits such as $140z10n89$.)

Call a prime p of the form $140z10n89$ an *Andersen prime*, and its corresponding PPN $57\ p$ an *Andersen number*. Examples of Andersen primes and corresponding Andersen numbers are provided in table 1. (The finite sequences z, n, and m (as defined above) appear in boldface.)

Soon after Andersen's announcement of his result, Ganson used Andersen's theorem to identify 204 (mostly huge) Andersen numbers, the largest with 177 digits. This is the gargantuan $798z626m373$, where z has 77 zeros and m has 91 nines. Eventually, Ganson and Andersen discovered thousands of Andersen primes. At the time of writing, the largest Andersen prime known is the 2461-digit $140 \times 10^{2458} + 1\,089$, verified as prime by Andersen using the program PRIMO by Marcel Martin. Its corresponding Andersen number is the 2 462-digit $798 \times 10^{2459} + 62\,073$. Andersen and Ganson have also found probable Andersen primes with more than 10^4 digits, but there currently seems to be no way to prove primality for numbers of this size.

The sequence of PPNs is an example of a sequence that appears at first to be extremely sparse – even finite – but yields many terms in the scale of the very large. Indeed, the sequence of PPNs has been compared to what first appears to be a faint star in the universe of numbers, but is then revealed to be an abundant galaxy by a computer-telescope. An open problem is whether PPNs such as 10 311 can generate other PPNs in the same way as Andersen's 7 980 062 073 does.

The proof of Andersen's theorem now follows. The proof is due to Andersen, except for the proof of Andersen's lemma, which I have provided. We denote by $R(n)$ the *reverse* of n, for example, $R(123) = 321$ and $R(120) = 21$.

Andersen's lemma *If p (not necessarily prime) is of the form $140z10n89$, then $R(57\ p) = 170 + R(p) + R(3\ p) + R(19\ p)$.*

Proof of Andersen's theorem Assuming for the meantime the validity of Andersen's lemma, suppose that p is prime. Then the proper divisors of $57\ p = 3 \times 19 \times p$ are

$$1,\ 3,\ 19,\ 57,\ p,\ 3\ p,\ 19\ p. \tag{1}$$

Adding the reverses of these proper divisors, we obtain

$$\begin{aligned} &R(1) + R(3) + R(19) + R(57) + R(p) + R(3\ p) + R(19\ p) \\ &\quad = 170 + R(p) + R(3\ p) + R(19\ p) \\ &\quad = R(57\ p), \end{aligned}$$

by Andersen's lemma. Hence, $57\ p$ is picture-perfect.

Conversely, if p is not prime, then $57\ p$ will have more divisors than (1), and the sum of the reversed divisors becomes larger than

$$170 + R(p) + R(3\ p) + R(19\ p) = R(57\ p).$$

Hence, $57\ p$ is not picture-perfect. This completes the proof of Andersen's theorem.

Proof of Andersen's lemma We consider two cases.

Case 1 (n has at least one 9, that is $p = 140z10n89$). In what follows, the finite sequences z, n, and m (as defined above) appear in boldface. To simplify notation (which can easily get cluttered with subscripts), z, n, and m appear as **00**, **999**, and **99** respectively. There is no loss of generality here: the boldfaced finite sequences can be replaced by arbitrary sequences in the appropriate manner without affecting the correctness of the proof. For example, everywhere in the proof **00** (i.e. z) and **999** (i.e. n) can be replaced by, say, **0000** and **9999** respectively, in which case **99** (i.e. m) must then be replaced by **999**.

We should convince ourselves of the generality of the argument by performing the arithmetical operations below by hand. In doing so, we gain a better understanding of the numerical patterns at work here.

With $p = 140\,\mathbf{00}1\,0\mathbf{99}\,\mathbf{9}89$, simple multiplication shows that

$$\begin{aligned}
57\ p &= 7\,98\mathbf{0}\,\mathbf{0}62\,6\mathbf{99}\,373, & R(57\ p) &= 3\,73\mathbf{9}\,\mathbf{9}62\,6\mathbf{00}\,897,\\
19\ p &= 2\,66\mathbf{0}\,\mathbf{0}20\,8\mathbf{99}\,791, & R(19\ p) &= 1\,97\mathbf{9}\,\mathbf{9}80\,2\mathbf{00}\,662,\\
3\ p &= 420\,\mathbf{00}3\,2\mathbf{99}\,\mathbf{9}67 & R(3\ p) &= 76\mathbf{9}\,\mathbf{99}2\,3\mathbf{00}\,024,\\
p &= 140\,\mathbf{00}1\,0\mathbf{99}\,\mathbf{9}89, & R(p) &= 98\mathbf{9}\,\mathbf{99}0\,1\mathbf{00}\,041.
\end{aligned}$$

Adding $R(19\ p)$, $R(3\ p)$, $R(p)$, and 170 as follows gives $R(57\ p)$, as required:

$$\begin{array}{r}
1\,97\mathbf{9}\,\mathbf{9}80\,2\mathbf{00}\,662\\
76\mathbf{9}\,\mathbf{99}2\,3\mathbf{00}\,024\\
98\mathbf{9}\,\mathbf{99}0\,1\mathbf{00}\,041\\
+ \qquad\qquad\qquad 170\\
\hline
3\,73\mathbf{9}\,\mathbf{9}62\,6\mathbf{00}\,897.\\
\hline
\end{array}$$

Case 2 (n has no 9s, that is $p = 140z1089$). The proof here is handled similarly to that in case 1. I only show the following final summation of $R(19\ p)$, $R(3\ p)$, $R(p)$, and 170 to $R(57\ p)$, using the same notation as case 1:

$$\begin{array}{r}
1\,960\,2\mathbf{00}\,662\\
762\,3\mathbf{00}\,024\\
980\,1\mathbf{00}\,041\\
+ \qquad\qquad 170\\
\hline
3\,702\,6\mathbf{00}\,897.\\
\hline
\end{array}$$

3. Conjectures and extensions

Recently, Andersen discovered an extension of his theorem.

Andersen's extended theorem *The decimal number* $p = 140\{\{0\}_{z(k)}10\{9\}_{n(k)}89\}_k$ *is prime if and only if* $57\ p$ *is picture-perfect, where* $\{0\}_{z(k)}$ *and* $\{9\}_{n(k)}$ *are finite sequences of* $z(k) \geq 0$ *zeros and* $n(k) \geq 0$ *nines respectively and* $\{\{0\}_{z(k)}10\{9\}_{n(k)}89\}_k$ *is a finite sequence of* $k \geq 0$ *finite sequences of the form* $\{0\}_{z(k)}10\{9\}_{n(k)}89$.

Andersen's extended theorem is proved similarly to Andersen's theorem.

Andersen conjectures that there are infinitely many Andersen primes (for both Andersen's theorem and Andersen's extended theorem) and, hence, infinitely many corresponding Andersen numbers.

Even before the third PPN 21 661 371 had been found, Ganson had conjectured that all PPNs are divisible by 3. Of course, the discoveries of 21 661 371, 1 460 501 511, and the many Andersen numbers (all multiples of 3), have further strengthened *Ganson's conjecture*.

Since the ancient Greeks first posed it, the conjecture that every perfect number is even has remained unanswered. Of course, in the 'mirror' things appear reversed. Ganson and I conjecture that every nontrivial PPN is odd. (The only trivial PPN, 6, is even.) We also believe that 6 is the only number that is both perfect and picture-perfect.

Ganson investigated PPNs in bases other than 10. In base 6, he found only four such numbers not exceeding 10^6, they are $28 = 44_6$, $145 = 401_6$, $901 = 4\,101_6$, and $1\,081 = 5\,001_6$. For example, $145 = 401_6$ has mirror equation

$$401_6 \stackrel{\text{b}}{=} 45_6 + 5_6 + 1_6,$$

where the sum on the right-hand side is a base-6 sum.

More intriguingly, Ganson found 41 base-5 PPNs less than 10^6. Continuing the computation, Dockery found 38 more such numbers less than 2.1×10^7. All but one of these 79 base-5 numbers are divisible by 3. The single exception is 5. However, Ganson noted that, for any prime p, p has only one proper divisor (i.e. 1) and has a base-p representation as 10; therefore p is trivially base-p picture-perfect. Also, 5 is trivially base-5 picture-perfect. Thus, Ganson conjectured that every nontrivial base-5 PPN is divisible by 3.

Observe that a perfect number n is (trivially) picture-perfect in any base b larger than n. This is because n and its proper divisors have one-digit base-b representations. For example, 6 and 28 are picture-perfect in base 29. A perfect number n is also picture-perfect in base $n-1$, where n is represented as the palindronomic 11 and all proper divisors of n are single-digit, so reversing changes nothing.

Here is the MATHEMATICA code Ganson used to investigate the situation for other bases.

```
f[n_] := FromDigits[Reverse[IntegerDigits[n, base]], base];
baseDivisors[n_, base_]
        := IntegerDigits[Drop[Divisors[n], -1], base];
Do[startFrom = 2;Do[If[f[n]
        == Apply[Plus, Map[f, Drop[Divisors[n], -1]]],
Print["base = ", base, ", n = ", n, ") ", IntegerDigits[n, base],
" divisors:  ", Drop[Divisors[n], -1], " base divisors:  ",
baseDivisors[n, base]]], {n, startFrom, 10000}], {base, 2, 10}]
```

To conclude this section, I mention two variations of the PPN sequence. A number n is called *tcefrep* ('perfect' backwards) if $R(n)$ is equal to the sum of the proper divisors of n; n is called *anti-perfect* if n is equal to the sum of the reverses of the proper divisors of n.

The first five tcefrep numbers are

$$6,\ 498\,906,\ 20\,671\,542,\ 41\,673\,714,\ 73\,687\,923,$$

which is sequence A072234 of reference 2. Andersen discovered that 4 158 499 614 is tcefrep, although it might not be the sixth term. Ganson conjectured that all tcefrep numbers are divisible by 3 (Nosnag's conjecture).

On the other hand, no obvious pattern applies to the first five anti-perfect numbers

$$6,\ 244,\ 285,\ 133\,857,\ 141\,635\,817,$$

which is sequence A072228 of reference 2. Andersen, who found the fifth term, checked that these are all the terms less than 10^{10}.

4. Picture-perfect semi-primes

Little is known about PPNs in general, especially those not generated by Andersen's theorem. In investigating PPNs, it is natural to start with numbers having few prime factors. Obviously, no prime can be a PPN. Hence, the simplest possible PPNs are the semi-prime PPNs, that is PPNs with exactly two prime factors. The smallest semi-prime PPN is $6 = 2 \times 3$; this is the only one currently known.

Ganson asked if a number of the form 3 p, with $p > 3$ prime, can be a PPN. I answered this question in the negative.

5. Picture-amicable pairs

Let $D(n)$ denote the sum of $R(d)$, where d ranges over all proper divisors of n. A pair a, b is called *picture-amicable* (or *mirror-amicable*) if $R(b) = D(a)$ and $R(a) = D(b)$.

I discovered the picture-amicable pair 2 320 000, 34 049. The mirror equations of a picture-amicable pair can be defined similarly as for PPNs. This pair has mirror equations

$$2\,320\,000 \overset{\text{b}}{=} 1 + 79 + 431, \tag{2}$$

$$\begin{aligned} 34\,049 \overset{\text{b}}{=} {} & 1 + 2 + 4 + 5 + 8 + 10 + 16 + 20 + 25 + 29 + 32 + 40 + 50 + 58 + 64 + 80 \\ & + 100 + 116 + 125 + 128 + 145 + 160 + 200 + 232 + 250 + 290 + 320 \\ & + 400 + 464 + 500 + 580 + 625 + 640 + 725 + 800 + 928 + 1\,000 + 1\,160 \\ & + 1\,250 + 1\,450 + 1\,600 + 1\,856 + 2\,000 + 2\,320 + 2\,500 + 2\,900 + 3\,200 \\ & + 3\,625 + 3\,712 + 4\,000 + 4\,640 + 5\,000 + 5\,800 + 7\,250 + 8\,000 + 9\,280 \\ & + 10\,000 + 11\,600 + 14\,500 + 16\,000 + 18\,125 + 18\,560 + 20\,000 + 23\,200 \\ & + 29\,000 + 36\,250 + 40\,000 + 46\,400 + 58\,000 + 72\,500 + 80\,000 + 92\,800 \\ & + 116\,000 + 145\,000 + 232\,000 + 290\,000 + 464\,000 + 580\,000 + 1\,160\,000. \end{aligned} \tag{3}$$

The sum of the proper divisors of 340 49 appears on the right-hand side of (2), and the sum of the proper divisors of 2 320 000 appears on the right-hand side of (3).

I used the following MATHEMATICA code to generate picture-amicable pairs $\{n, b\}$.

```
f[x_] := FromDigits[Reverse[IntegerDigits[x]]];
d[x_] := Apply[ Plus, Map[ f, Drop[Divisors[ x], -1] ] ];
Do[a = d[n]; b = f[a]; c = d[b]; u = Last[IntegerDigits[a]];
If[u != 0 && n != b && c == f[n], Print[{n, b}]], {n, 2, 10^8}]
```

For each n in the test range, the program computes $D(n)$. It must then find b such that $R(b) = D(n)$. The problem now is that there are infinitely many b which when reversed are equal to $D(n)$. For example, $321 = R(123) = R(1\,230) = R(12\,300)$, etc. The program makes the simplest choice of b, that is $b = R(D(n))$. However, to ensure that $R(b) = D(n)$, the last digit of $D(n)$ must not be 0 (which would be lost upon reversal); hence, the condition $u! = 0$ is necessary. Finally, the program checks that $R(n) = D(b)$ and that $n! = b$, since $n = b$ yields at most a PPN.

My computer search has reached as far as $n = 2 \times 10^7$. Using this code, I found another picture-amicable pair: 10 223 000, 1 790 947. As an exercise, the reader can find the mirror equations of this pair.

Readers may enjoy finding other picture-amicable pairs or even picture-sociable chains. Since the reverse function is not injective, my program does not perform an exhaustive search (not all possible values of b are tested). Perhaps a resourceful reader will write an exhaustive search program. This might appear to be an impossible task since infinitely many b have to be inspected for a particular n. However, some variation of Cantor's 'zigzag argument' for the countability of the rational numbers can probably be used to circumvent this difficulty.

Acknowledgments

I would like to thank Jens Kruse Andersen, Daniel Dockery, and Mark Ganson for their valuable contributions to and interest in picture-perfect numbers; and Mohammad Khadivi for his suggestions to improve this article.

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The author received a PhD in Mathematics at the University of Illinois, Urbana-Champaign, in 1991. His recent interests and publications deal with fractal geometry, number theory, and recreational mathematics. Attracted by the 'bubble' of the late 1990s, he left his job in academia in 1998 to work in industry. He is now a senior engineer involved in software and statistical work at Motorola in Arlington Heights, Illinois.

Sums of Consecutive Numbers Modulo *n*

PHILIP MAYNARD

Problems involving sums of consecutive positive integers have a history dating back to J. J. Sylvester (1814–1897); see reference 1. It is well known that a positive integer, m, can be expressed as the sum of at least two consecutive positive integers if and only if m is not a power of two; see references 2 and 3. More generally, for $n \in \mathbb{N}$, we say that $t \in \mathbb{N}$ is a *cs-number modulo* n if it can be expressed as the sum of at least two consecutive positive integers (from the set $\{1, 2, 3, \ldots, n-1\}$) modulo n. Obviously, any positive integer m, not a power of two, is a *cs-number modulo* n for $m < n$. The purpose of this short article is to show that, if $n > 3$, then every positive integer m, $m < n$, is a *cs-number modulo* n.

Theorem 1 *Let $n \in \mathbb{N}$ with $n > 3$. Then every positive integer t less than n is a cs-number modulo n.*

Proof We deal with the cases when n is odd or even in turn. Thus, suppose that $n = 2m + 1$ for some $m \geq 2$. First we show that any even number is a cs-number modulo n. For any $1 \leq l \leq m$, consider the following consecutive sum:

$$\begin{aligned}\sum_{i=m-l+2}^{m+l+1} i &= \sum_{i=1}^{l}\{(m+2-i)+(m+1+i)\}\\ &\equiv \sum_{i=1}^{l} 2 \pmod{n}\\ &\equiv 2l \pmod{n}.\end{aligned}$$

This completes the case when t is even. Now if t is odd and at least 3, so that $t = 2u + 1$ for $u \geq 1$, then we simply have $t = u + (u + 1)$. It remains to show that 1 can be expressed as the sum of consecutive numbers modulo n. Note that $\sum_{i=1}^{n-1} i \equiv 0 \pmod{n}$. Hence, $\sum_{i=1}^{n-2} i \equiv 1 \pmod{n}$.

Next, assume that n is even, say $n = 2m$ for $m \geq 2$. We first show that if $t \in \{1, 2, \ldots, m\}$ then t can be expressed as the sum of consecutive numbers modulo n. For any $t \leq m$ we consider the following consecutive sum:

$$\begin{aligned}\sum_{i=m-t+1}^{m+t} i &= \sum_{i=1}^{t}\{(m+1-i)+(m+i)\}\\ &\equiv \sum_{i=1}^{t} 1 \pmod{n}\\ &\equiv t \pmod{n}.\end{aligned}$$

For any $l \in \{0, 1, 2, \ldots, m-2\}$ we consider the following consecutive sum (modulo n):

$$\begin{aligned}\sum_{i=m-l-1}^{m+l} i &= (m-l-1) + m + \sum_{i=m-l}^{m-1} i + \sum_{i=m+1}^{m+l} i \\ &= 2m - l - 1 + \sum_{i=1}^{l} \{(m+i) + (m-i)\} \\ &\equiv 2m - l - 1 \pmod{n}.\end{aligned}$$

In this way we can express every $t \in \{m+1, \ldots, 2m-1\}$ as the sum of consecutive numbers modulo n. This completes the proof.

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Sudoku

Is this the simplest 9×9 Sudoku solution?

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

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Bath BA3 2RG, UK

Bob Bertuello

Oblique-Angled Diameters and Conic Sections

EDWARD GREVE and THOMAS J. OSLER

1. Introduction

Recently the authors were translating a paper by Leonhard Euler (see reference 1) from French into English. Euler wrote of *oblique-angled diameters* and *orthogonal diameters* without explanation. Clearly these ideas were familiar to mathematicians in his day, but have been ignored in the education of modern mathematicians. It is the purpose of this article to bring these simple ideas to the attention of today's readers.

Definition 1 Given a curve ACTDB shown in figure 1, the line ET, which intersects the curve at T, is called an *oblique-angled diameter* if it bisects all chords (such as AB and CD) that are parallel to the tangent line at T.

In other words,

$$\mathrm{AM} = \mathrm{MB} \quad \text{and} \quad \mathrm{CN} = \mathrm{ND}$$

since the chords AB and CD are parallel to the tangent line at T where the *oblique-angled diameter* ET intersects the curve ACTDB.

Many curves possess an oblique-angled diameter. For example, any curve that is symmetric about an axis has this property. In figure 2 we have a curve that is symmetric about the line OY. Clearly this line bisects all horizontal chords. In this special case we call OY an *orthogonal diameter*. It is also clear that every diameter of a circle is an orthogonal diameter.

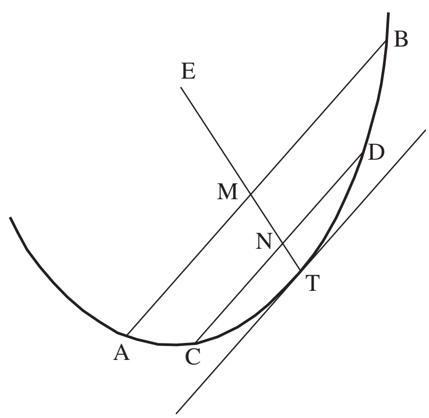

Figure 1 ET is an oblique-angled diameter.

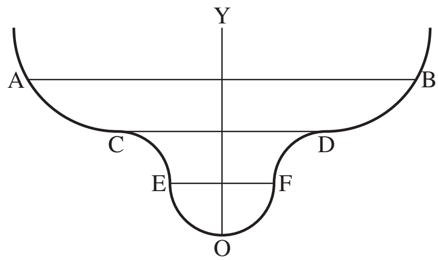

Figure 2 An orthogonal diameter OY.

While many curves possess one oblique-angled diameter, in the case of the conic sections *all* appropriately defined diameters have this property. For the parabola, we define a *diameter* as any line parallel to the axis of the parabola. In this case, all diameters are oblique-angled diameters. We will prove this in Section 2. For the ellipse and the hyperbola, we define a *diameter* as any line that passes through the centre of the curve. In Section 3 we will prove that *all* such diameters are oblique-angled diameters.

2. The parabola

Theorem 1 *Let* ATB *be a parabola and let* ET *be any line parallel to the axis of the parabola (see figure 3). Let* AB *be any chord parallel to the tangent line at* T. *Then* ET *bisects the chord* AB *at the point of intersection* M.

Proof Since all parabolas are similar, without loss of generality we can call the equation of the parabola $y = x^2$. Let the coordinates of point T be $(x_{\mathrm{T}}, y_{\mathrm{T}})$. The slope at this point is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = m = 2x_{\mathrm{T}}.$$

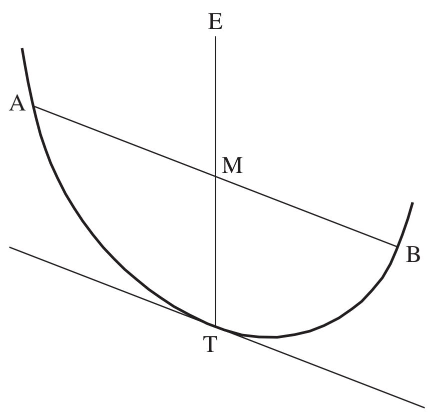

Figure 3 The parabola.

The line AB has the equation $y = mx + b$, and intersects the parabola at the points where $x^2 - mx - b = 0$. The solutions of this equation are

$$\frac{m}{2} \pm \frac{\sqrt{m^2 + 4b}}{2} = x_T \pm \sqrt{x_T^2 + b}.$$

Thus, the x-coordinates of points A and B, respectively, are

$$x_T - \sqrt{x_T^2 + b} \quad \text{and} \quad x_T + \sqrt{x_T^2 + b}.$$

Since point M has x-coordinate x_T it is clear from this result that point M bisects the line segment AB. Thus, any line parallel to the axis of a parabola is an oblique-angled diameter.

3. The ellipse and the hyperbola

Theorem 2 *Every diameter of an ellipse is an oblique-angled diameter.*

Proof Recall that a diameter is any line that passes through the centre of the ellipse. In figure 4 we see a section of the ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

The line OE is, by definition, a *diameter* of the ellipse since it passes through the centre of the ellipse. This diameter intersects the ellipse at point T which has coordinates (x_T, y_T). Differentiating (1), we obtain

$$y' = -\frac{b^2 x}{a^2 y}.$$

Thus, the slope of the tangent line at point T is given by

$$m = -\frac{b^2 x_T}{a^2 y_T}. \tag{2}$$

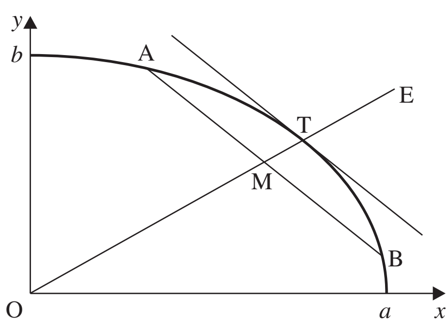

Figure 4 The ellipse.

The equation of the chord AB, which is parallel to the tangent at T, is

$$y = mx + c, \tag{3}$$

for some number c. The coordinates of points A and B are found by solving (1) and (3) simultaneously. We are led to consider

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1,$$

which simplifies to the quadratic equation

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0.$$

The roots of this quadratic are

$$x = -\frac{a^2cm}{b^2 + a^2m^2} \pm \Delta,$$

where Δ is an expression involving a, b, c, and m whose exact value will not concern us. The x-coordinates of points A and B, respectively, are thus given by

$$x_{\mathrm{A}} = -\frac{a^2cm}{b^2 + a^2m^2} - \Delta$$

and

$$x_{\mathrm{B}} = -\frac{a^2cm}{b^2 + a^2m^2} + \Delta.$$

Thus, it is clear that the x-coordinate of the midpoint M of AB is

$$x_{\mathrm{M}} = -\frac{a^2cm}{b^2 + a^2m^2}. \tag{4}$$

We must show that the diameter, which passes through T, also bisects the chord AB and thus passes through point M. This diameter has the equation

$$y = \frac{y_{\mathrm{T}}}{x_{\mathrm{T}}}x. \tag{5}$$

Thus, we must solve (3) and (5) simultaneously and show that the solution is identical to (4). Therefore, we have $(y_{\mathrm{T}}/x_{\mathrm{T}})x = mx + c$ and, solving for x, we obtain

$$x = \frac{cx_{\mathrm{T}}}{y_{\mathrm{T}} - mx_{\mathrm{T}}}. \tag{6}$$

Using (2) to eliminate m, after some manipulation, we see that (4) and (6) both become

$$x = \frac{a^2cx_{\mathrm{T}}y_{\mathrm{T}}}{a^2y_{\mathrm{T}}^2 + b^2x_{\mathrm{T}}^2},$$

and are thus identical; the theorem is therefore proved.

Theorem 3 *Every diameter of a hyperbola is an oblique-angled diameter.*

Proof The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Repeating the above argument with this equation produces terms which differ only by an occasional sign. We leave it to the reader to complete the details.

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Tom Osler *is professor of mathematics at Rowan University and is the author of 95 mathematical papers. In addition to teaching university mathematics for the past 46 years, Tom has a passion for long-distance running. He has been competing for the past 53 consecutive years. Included in his over 1950 races are wins in three national championships in the late 1960s at distances from 25 kilometres to 50 miles. He is the author of two books on running.*

What is

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}\ ?$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Picture This: From Inscribed Circle to Pythagorean Proposition

D. G. ROGERS

The Editor of *Mathematical Spectrum*, in one of his occasional capsules (see reference 1), recently confronted readers with the following two expressions for the radius r of the inscribed circle of a right-angled triangle with legs a and b and hypotenuse c:

$$r = \frac{a+b-c}{2}, \qquad r = \frac{ab}{a+b+c}.$$

A natural response is to try setting these two expressions equal, whence a little algebra reveals that they are indeed equivalent, subject to the following even more celebrated Pythagorean relation between a, b, and c:

$$a^2 + b^2 = c^2. \tag{1}$$

This algebra is reversible to the extent that, given (1), each of the expressions for r implies the other. But it might seem rather artificial to go in these directions. However, as we shall *see*, the connection between the inscribed circle and the Pythagorean proposition is closer yet.

The two expressions for the radius of the inscribed circle of a right-angled triangle have a long history, and already, some 17 centuries ago, a Chinese mathematician, Liu Hui (220–280), gave a neat dissection argument that makes both transparent – an early instance of a *proof without words*. First of all, Liu Hui dismembered the right-angled triangle as shown in figure 1(a). Then, as in figure 1(b), he reassembled the pieces from four copies of the

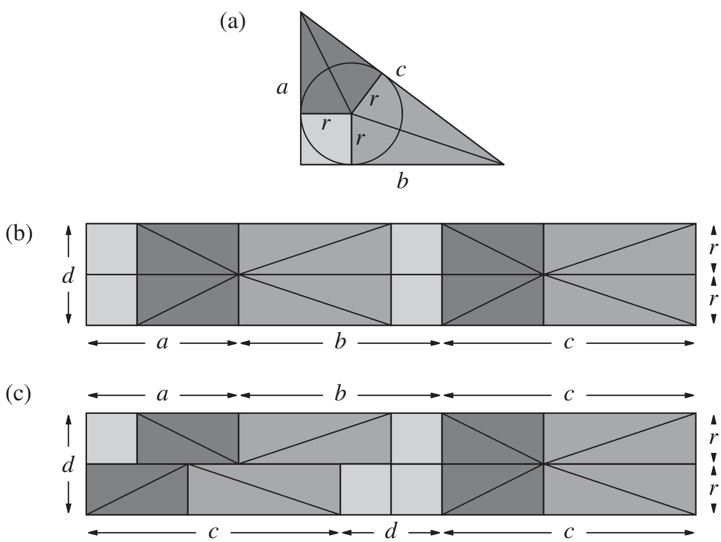

Figure 1 Liu Hui's dissection.

right-angled triangle into a long rectangle whose sides we recognize to be the perimeter of the right-angled triangle and the diameter d of the inscribed circle, i.e. $a + b + c$ and $d = 2r$. From the alternative way we have placed these pieces in figure 1(c) or, more directly, from the dissection in figure 1(a), we see that

$$a + b = c + d; \tag{2}$$

and, since the area of the four right-angled triangles is conserved in the long rectangle,

$$2ab = (a + b + c)d. \tag{3}$$

Of course, these are just the equations from reference 1 rewritten in terms of the diameter, rather than the radius.

Liu Hui's demonstration comes from an illustrated commentary on a then famous mathematical compilation, the *Jiu Zhang Suan Shu* (conventionally translated as *Nine Chapters on the Mathematical Arts*) from perhaps a century and a half earlier. Among the many problems contained in this work is, to give but one further instance, an early, exemplary version of Brahmagupta's problem of the broken bamboo, familiar to readers of *Mathematical Spectrum* from reference 2. Unfortunately, in regard to the problem on the inscribed circle of a right-angled triangle, so far from actually having a proof without words, the illustrative diagrams have disappeared, although what survives of the text indicates that they were coloured – for example, yellow for the little square of side r in figure 1(a) and crimson and indigo for the pairs of triangles. The reconstruction shown in figure 1(b) is standard, with the four small (yellow) squares grouped in pairs (sometimes with all four together), whereas in figure 1(c) they are placed so as to make (2) more apparent (with due deference to reference 3, p. 104).

But, now we have figure 1(b), we are free to play as we please with the set of 20 pieces that go to make up the rectangle, and to explore what other shapes can be obtained from them, much as in the game of *Tangrams*. Thus, we have in figure 2 two further rearrangements of this set of pieces that bring the Pythagorean relation (1) into view, so to say, *in silhouette*. We see that,

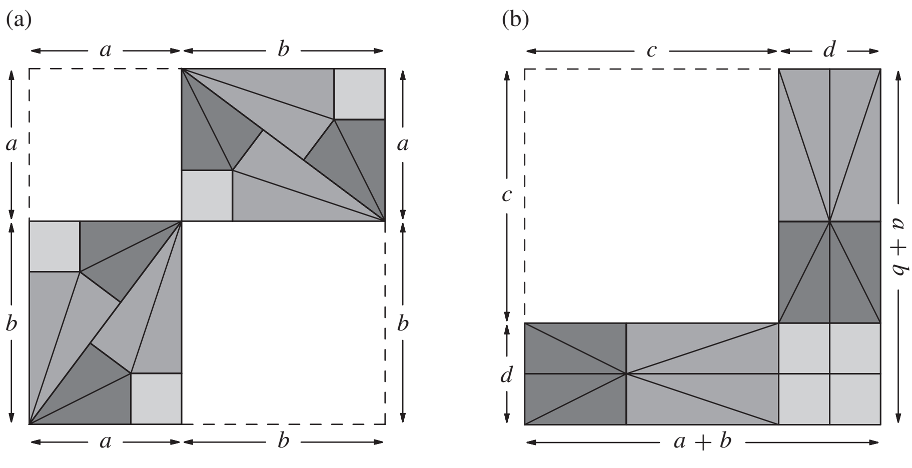

Figure 2 $a^2 + b^2 = c^2$.

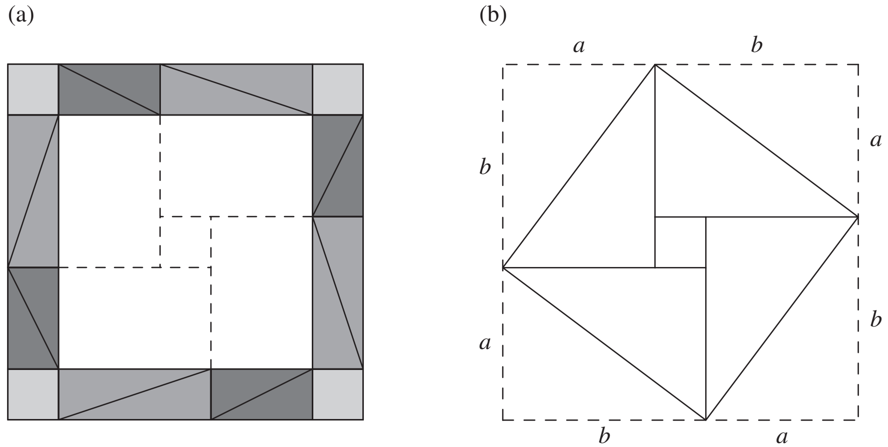

Figure 3

by (2), the containing rectangles in figures 2(a) and (b) have equal area. So, the complements of our set of 20 pieces in these rectangles have equal area, i.e. the unshaded squares of sides a and b in figure 2(a) and the unshaded square of side c in figure 2(b).

Now, making the rearrangements of the pieces shown in figure 2 is only a matter of mathematical *play*, without suggestion that this has any *historical* basis. In fact, purely as part of this play, we observe that the selected rearrangements are such that the pieces of figure 1(b) can be slid with rotations into the positions in figures 2(a) or (b) *without turning them over*, as though in a board game. But still we might wonder whether it was within the *scope* of Liu Hui, knowing both that he favoured dissection arguments and that he seemed to have accepted, in combination with them, demonstrations that turn on complementary figures? Liu Hui discussed the Pythagorean relation (1), but the passage, as it has come down to us, is obscure, and possibly corrupt, so that it has been a matter of debate what he intended, beyond some kind of dissection. Several candidates are mentioned in the references below, each with its own champions.

In a further rearrangement of the 20 pieces in figure 1(b), we can form a frame inside a square of side $a + b$, so as to leave a square of side c aligned with it inside the frame, as in figure 3(a). It is instructive to juxtapose this rearrangement of the pieces with a much more familiar diagram associated with the Pythagorean proposition, seen recently in *Mathematical Spectrum* (see figure 4 of reference 4), and shown again here in figure 3(b) (a version of this diagram appeared in the logo for the International Congress of Mathematicians (ICM) held in Peking in 2002 and, indeed, there has been some presumption that it featured in a commentary from the same century as Liu Hui on another Chinese mathematical classic, the *Zhou Bi*). We see that figures 3(a) and (b) share the same *underlying* rotational symmetry, as suggested by the dotted lines in figure 3(a); full rotational symmetry can be obtained in figure 3(a) if we break the convention of not turning pieces over. Moreover, in view of figure 1(a), each outer triangle in figure 3(b) has the same area as the corresponding L-shaped section of the frame, namely half the area of a rectangle with sides a and b. By sweeping the area of the four outer triangles into the frame, as it were, the inner square of side c is brought into alignment with the outer square of side $a + b$.

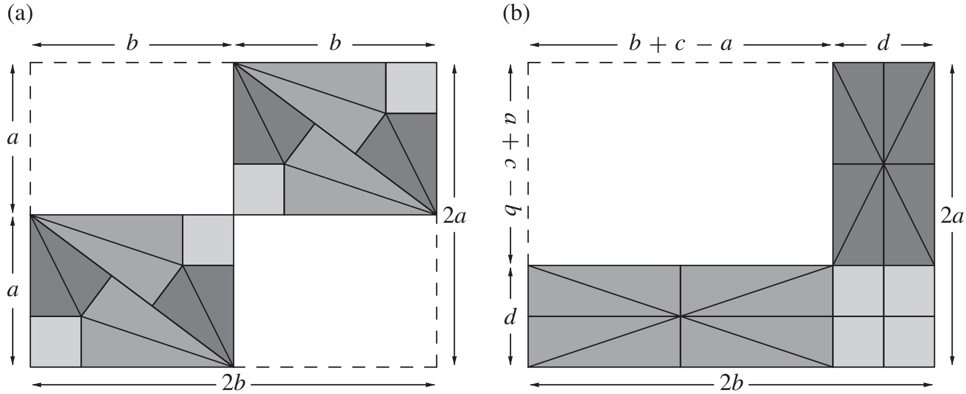

Figure 4 $2ab = (a + c - b)(b + c - a)$.

Again, this is only a mathematical observation, without any claim to historical foundation. But, curiously enough, a novel rendition of a passage from the *Zhou Bi* (see reference 5, p. 786) does speak of forming a ring of L-shaped trysquares, in conscious departure from the hitherto generally accepted translation (compare with reference 6, p. 134, n. 37).

The 20 pieces in figure 1(b) can be slid on their imagined board into two more configurations, as shown juxtaposed in figure 4, where now the pieces are held in a common containing rectangle of sides $2a$ and $2b$. Reasoning as earlier with figure 2, we see that the unshaded portion of this rectangle in figure 4(a) has the same area as the unshaded portion in figure 4(b). Hence,

$$2ab = (2a - d)(2b - d) = (a + c - b)(b + c - a),$$

where we have made use of (2) for the last equality. While this may seem no more than a minor variant on (3), both take on greater significance on recalling that, for $x = a, b, c$, the diameter d_x of the escribed circle of the right-angled triangle with legs a and b and hypotenuse c touching the side x externally and the other two sides produced is given by

$$d_a = a + c - b, \qquad d_b = b + c - a, \qquad d_c = a + b + c.$$

Thus, the same set of 20 tiles can be used to provide dissection demonstrations of four related results, i.e.

$$\begin{aligned} 2ab &= (a + b + c)d, & 2ab &= (a + b - c)d_c, \\ 2ab &= (b + c - a)d_a, & 2ab &= (a + c - b)d_b. \end{aligned} \tag{4}$$

On the other hand, it seems more difficult to establish directly by dissection that the shaded and unshaded portions of figure 4(b), like those of figure 4(a), have equal area or, equivalently, that

$$dd_c = d_a d_b,$$

although, as we see in figures 2 and 4, both sides represent areas equal to $2ab$. (It may be of interest to note that if a triangle with sides a, b, and c has inscribed and escribed circles of diameters d, d_a, d_b, and d_c, then any of the conditions

$$dd_c = 2ab, \qquad d_a d_b = 2ab, \qquad dd_c = d_a d_b$$

is sufficient to ensure that the triangle is a right-angled triangle with legs a and b and hypotenuse c.)

Once again, these are entirely mathematical observations. But it so happens that all four results in (4) appear in *Ce Yuan Hai Jing* (conventionally translated as *Sea Mirror of Circle Measurements*), a work by Li Ye (1192–1279) that was completed in 1248, so roughly a millennium after Liu Hui (see reference 7, pp. 43–149, especially figure 11.1). Indeed, reference 7 presents some 170 problems based on a single diagram in which, in effect, a circle is inscribed in and escribed to four similar right-angled triangles. Commentators have often been struck by an apparent duality between problems in this collection. However, tackled by means of dissections of the sort used by Liu Hui, as with (4), Li Ye's set of problems loses some of this mystique: the problems are less difficult than commonly supposed and it is possible to move between solutions to different problems quite easily. Liu Hui's commentary on the *Jiu Zhang Suan Shu* had been studied intensively, including for official examinations, in the intervening centuries. But the historical problem is the regard in which Li Ye and his contemporaries may have viewed these older dissection methods compared with the algebraic ones described in *Ce Yuan Hai Jing* and for which it is most noted.

The lack of documentation for figures 2, 3(a), and 4 is not just a matter of the historical record, since they seem to be missing from more recent mathematical and teaching discussions too. If truly absent, it would seem strange that such a versatile set of shapes has attracted so little comment.

Mathematical Spectrum has already carried a general synopsis (see reference 4) of mathematics from Chinese antiquity. Some popular account of the work of Liu Hui is given in references 5, 8, 9, and 10 (reference 10 goes so far as to reproduce a version of figure 1 in colours approximating Liu Hui's own choice). Two works of reference to note are references 3 and 7, while a more definitive account (see reference 11) of the *Jiu Zhang Suan Shu* has only recently appeared in French, supplementing a similar exercise (see reference 12) in English. But none of these works includes mention of figures 2, 3(a), or 4. A comment of Liu Hui regarding the way certain algorithms emerge from 'the same transformation of one particular figure' has recently been taken up at length in reference 6. That might sound somewhat akin to the various deductions made here merely by rearranging one set of 20 tiles in different ways. However, the figure on which reference 6 focuses is not one of these arrangements, being instead more closely related to figure 3(b).

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In the thirty years or so since graduation, ***Douglas Rogers*** *has travelled widely with his sums. Consequently, he has become tolerably well used to picking up the pieces, and reassembling them.*

Mathematics in the Classroom

Archimedes and parabolic segments

Twenty two centuries ago, when mathematics was in a state of infancy, Archimedes was able to make several significant physical and mathematical contributions, which were much ahead of his time. For example, his analysis of the quadrature of the parabola is really amazing. In one result, he states that the area S of a parabolic segment is one third of the area of the bounding triangle T which is formed by a chord of the segment, a tangent to the parabola at one extremity of the chord, and a line parallel to the axis of the parabola passing through the other extremity of the chord (see reference 1).

Archimedes exhibited an amazing understanding of the basic concept of limits by demonstrating that S cannot be less than $\frac{1}{3}T$ and also that S cannot be greater than $\frac{1}{3}T$. Therefore, he deduced that $S = \frac{1}{3}T$.

Archimedes' original proof of this proposition is rather long; the interested reader may find it in reference 1. Here we establish this proposition by using calculus. Consider the parabola with the equation $y = ax^2$ (see figure 1).

The shaded area S in figure 1 is given by

$$\begin{aligned}
S &= \text{area of trapezium AMNB} - \text{area bounded by AM, MN, NB, and the parabolic arc} \\
&= \tfrac{1}{2}\mathrm{MN}(\mathrm{AM} + \mathrm{BN}) - \int_{x_1}^{x_2} ax^2\,\mathrm{d}x \\
&= \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2) - \tfrac{1}{3}a(x_2^3 - x_1^3) \\
&= \tfrac{1}{6}a(x_2 - x_1)(3(x_1^2 + x_2^2) - 2(x_2^2 + x_2x_1 + x_1^2)) \\
&= \tfrac{1}{6}a(x_2 - x_1)(x_2 - x_1)^2 \\
&= \tfrac{1}{6}a(x_2 - x_1)^3.
\end{aligned}$$

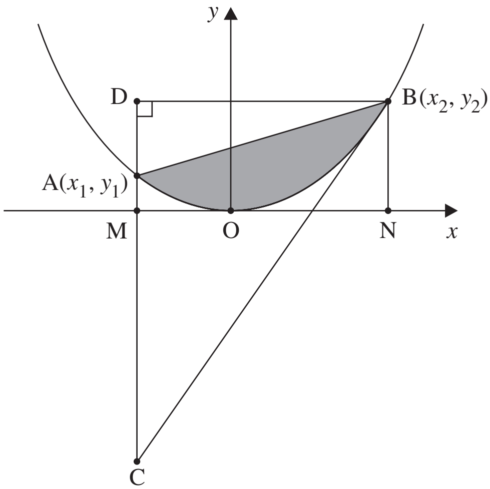

Figure 1 Shaded segment S of the parabola $y = ax^2$ cut off by the chord AB and the bounding triangle ABC.

Next,

$$\begin{aligned}\text{area } \Delta\text{ABC} &= \text{area } \Delta\text{BCD} - \text{area } \Delta\text{ABD}\\ &= \tfrac{1}{2}(\text{CD})(\text{BD}) - \tfrac{1}{2}(\text{AD})(\text{BD})\\ &= \tfrac{1}{2}(\text{AC})(\text{BD}).\end{aligned}$$

The tangent to the parabola at B has equation $y - y_2 = 2ax_2(x - x_2)$, so C, which is the point of intersection of the tangent BC and the line AC with the equation $x = x_1$, has y-coordinate

$$y_2 + 2ax_2(x_1 - x_2) = 2ax_2x_1 - ax_2^2.$$

Hence,

$$\begin{aligned}\text{area } \Delta\text{ABC} &= \tfrac{1}{2}(\text{AC})(\text{BD})\\ &= \tfrac{1}{2}(ax_1^2 - 2ax_2x_1 + ax_2^2)(x_2 - x_1)\\ &= \tfrac{1}{2}a(x_2 - x_1)^3\\ &= 3S, \quad \text{as required.}\end{aligned}$$

Reference

1 http://www.math.ubc.ca/～cass/archimedes/parabola/.

Lucknow, India **M. A. Khan**

Letters to the Editor

Dear Editor,

Curious powers

In previous volumes, attention has been drawn to 'curious cubes' such as 512, i.e.

$$512 = (5 + 1 + 2)^3;$$

see Volume 37, Number 3, p. 111 and Volume 38, Number 2, p. 85. Are there curious higher powers apart from the obvious number 1?

Yours sincerely,

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Dear Editor,

Stability analysis on SARS epidemics

This letter provides an alternative explanation for SARS epidemics to that given by J. Gani in *Modelling SARS* (*Math. Spectrum*, Volume 39, Number 1). The original article proposed a simple deterministic model for SARS epidemics. This study proposes another elucidation for the SARS epidemic based on stability analysis.

If we select the SIR model to describe the phenomenon of a SARS epidemic, three categories in the total population may be classified at the time $t \geq 0$ to characterize the epidemic, namely the numbers X, Y, and U of susceptibles, infectives, and removals respectively. These are governed by the differential equations

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= -bXY + rY = f(X, Y), && X(0) = X_0,\\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= bXY - (r + g)Y = g(X, Y), && Y(0) = Y_0,\\
\frac{\mathrm{d}U}{\mathrm{d}t} &= gY, && U(0) = 0,
\end{aligned} \tag{1}$$

where b, r, and g are the infection, recovery, and removal rates respectively. Suppose that the stability of a SARS-free system is the prime concern for analysis, i.e. we may simply exclude (1). In addition, if the epidemic will die out in the population, the system should be stable at the point $(X_0, Y_0) = (X_0, 0)$. Note that the point $\hat{X}_0 = (X_0, Y_0) = (X_0, 0)$ is also a critical point of the system. First, the system equations can be linearized at (X_0, Y_0) as follows:

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= f(X, Y) \cong \left.\frac{\partial f}{\partial X}\right|_{(X_0, Y_0)} (X - X_0) + \left.\frac{\partial f}{\partial Y}\right|_{(X_0, Y_0)} (Y - Y_0),\\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= g(X, Y) \cong \left.\frac{\partial g}{\partial X}\right|_{(X_0, Y_0)} (X - X_0) + \left.\frac{\partial g}{\partial Y}\right|_{(X_0, Y_0)} (Y - Y_0).
\end{aligned}$$

Define $\delta X_0 = X - X_0$ and $\delta Y_0 = Y - Y_0$. The system equations may then be approximated in a neighbourhood of the critical point $\hat{X}_0$ by the matrix

$$\begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}\delta X_0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\delta Y_0 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \delta X_0 \\ \delta Y_0 \end{bmatrix},$$

where the matrix

$$\boldsymbol{A} = \begin{bmatrix} \left.\dfrac{\partial f}{\partial X}\right|_{(X_0,Y_0)} & \left.\dfrac{\partial f}{\partial Y}\right|_{(X_0,Y_0)} \\ \left.\dfrac{\partial g}{\partial X}\right|_{(X_0,Y_0)} & \left.\dfrac{\partial g}{\partial Y}\right|_{(X_0,Y_0)} \end{bmatrix} = \begin{bmatrix} 0 & -bX_0 + r \\ 0 & bX_0 - (r+g) \end{bmatrix}$$

is the *Jacobian matrix* at $\hat{X}_0$. Thus, the eigenvalues λ_1 and λ_2 of $\boldsymbol{A}$ can be determined via the characteristic equation $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$, i.e. $\lambda_1 = 0$ and $\lambda_2 = bX_0 - (r+g)$. To have a locally stable system at $\hat{X}_0$, we must have a *negative* real part of the eigenvalue λ_2. That is, the initial population X_0 should not exceed the 'threshold' $X_{\text{th}} = (r+g)/b$ to resist SARS epidemics in the population, i.e. $\hat{X}_0$ should be locally stable or $Y \cong Y_0 = 0$. Thus, if the initial population X_0 is greater than the threshold, i.e. $X_0 > X_{\text{th}}$, then the epidemic will unstably spread away from $\hat{X}_0$. In contrast, provided that $X_0 < X_{\text{th}}$, the epidemic will stabilize at $\hat{X}_0$.

Yours sincerely,

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Dear Editor,

Rational approximation to square roots

In M. A. Khan's article 'Rational approximation to square roots of integers' (*Math. Spectrum*, Volume 39, Number 2), Khan proposed a recursive relation to find rational approximations to square roots of integers using the Diophantine equation

$$y^2 = nx^2 + 1.$$

This proposed approximation is not only applicable to integers but also to rational numbers. Let (x_0, y_0) satisfy

$$y^2 = \frac{n}{m}x^2 + 1, \tag{1}$$

so that

$$y_0^2 - \frac{n}{m}x_0^2 = 1 \quad \text{or} \quad \left(y_0 + \sqrt{\frac{n}{m}}x_0\right)\left(y_0 - \sqrt{\frac{n}{m}}x_0\right) = 1.$$

If (x, y) is also a solution to (1), we obtain

$$\left(y_0 + \sqrt{\frac{n}{m}}x_0\right)\left(y + \sqrt{\frac{n}{m}}x\right)\left(y_0 - \sqrt{\frac{n}{m}}x_0\right)\left(y - \sqrt{\frac{n}{m}}x\right) = 1$$

or

$$\left(\left(y_0 y + \frac{n}{m} x_0 x\right) + \sqrt{\frac{n}{m}}(y_0 x + x_0 y)\right)\left(\left(y_0 y + \frac{n}{m} x_0 x\right) - \sqrt{\frac{n}{m}}(y_0 x + x_0 y)\right) = 1,$$

so that

$$\left(y_0 y + \frac{n}{m} x_0 x\right)^2 - \frac{n}{m}(y_0 x + x_0 y)^2 = 1.$$

Thus, $(y_0 x + x_0 y,\, y_0 y + (n/m) x_0 x)$ is also a solution to (1). We may denote this solution in matrix form as follows:

$$\begin{bmatrix} y_0 & (n/m)x_0 \\ x_0 & y_0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} y \\ x \end{bmatrix}.$$

Thus, we can obtain a sequence of solutions to (1), namely

$$\begin{bmatrix} y_r \\ x_r \end{bmatrix} = \boldsymbol{A}^r \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}, \quad \text{for } r = 0, 1, 2, \dots,$$

and y_r/x_r gives the rth rational approximation to $\sqrt{n/m}$. For example, to find $\sqrt{7/3}$, a first guess is chosen to be $(x_0, y_0) = (2, 3)$. Then we obtain

$$\begin{bmatrix} y_r \\ x_r \end{bmatrix} = \begin{bmatrix} 3 & 14/3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} y_{r-1} \\ x_{r-1} \end{bmatrix},$$

and successive approximations of $\sqrt{7/3}$ for $r = 0, 1, 2, \dots$ are

$$\sqrt{\frac{7}{3}} \cong \frac{3}{2}, \frac{55}{36}, \frac{333}{218}, \frac{6\,049}{3\,960}, \frac{36\,627}{23\,978}, \dots.$$

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Dear Editor,

Why is $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ *?*

Abbas Roohol Amini gave a geometrical explanation of this in Volume 38, Number 3, p. 107. Here is another explanation. Write $1 + 2 + \cdots + r = T_r$, so that $T_r = \frac{1}{2}r(r+1)$. Then

$$T_r^2 - T_{r-1}^2 = \tfrac{1}{4}r^2(r+1)^2 - \tfrac{1}{4}r^2(r-1)^2 = r^3,$$

so that

$$1^3 + 2^3 + \cdots + n^3 = (T_1^2 - T_0^2) + (T_2^2 - T_1^2) + \cdots + (T_n^2 - T_{n-1}^2) = T_n^2.$$

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

40.1 Prove that

$$(\sqrt{3}+1)\sin\left\{\frac{5\pi}{6}\right\} - (\sqrt{3}-1)\cos\left\{\frac{5\pi}{6}\right\} = 2(\sin 2 + \cos 2),$$

where $\{\cdot\}$ denotes the fractional part.

(Submitted by Mihály Bencze, Brasov, Romania)

40.2 A student chooses between two exam strategies. In the first, he answers three questions. If at least two answers are correct, then he passes, otherwise he fails. In the second, he answers an unlimited number of questions until he answers two consecutive questions correctly, in which case he passes, or two consecutive questions incorrectly, in which case he fails. If the probability of his answering each question correctly is the same and these are independent, which strategy is preferable?

(Submitted by Czesław Stępniak, University of Rzeszów, Poland)

40.3 Let a, b, c be the length of the sides of a triangle ABC and let r_A, r_B, r_C, be the radii of its escribed circle. Prove that

$$\frac{a^2}{\mathrm{r}_B\mathrm{r}_C} + \frac{b^2}{\mathrm{r}_C\mathrm{r}_A} + \frac{c^2}{\mathrm{r}_A\mathrm{r}_B} \geq 4.$$

(Submitted by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain)

40.4 You are given a straight line with equation $ax + by + c = 0$ and a point (p, q) not on the line. The line is then 'altered' to go through (p, q) by adjusting either a alone, b alone, or c alone. If s is the percentage change required in a if we alter a alone, t is the percentage change required in b if we alter b alone, and u is the percentage change required in c if we alter c alone, show that

$$\frac{1}{s} + \frac{1}{t} + \frac{1}{u} + \frac{1}{100} = 0.$$

(Submitted by Jonny Griffiths, Paston College, Norfolk, UK)

Solutions to Problems in Volume 39 Number 2

39.5 The squares $ABCD$ and $A_1B_1C_1D_1$ in the plane have equal sides. Prove that

$$AA_1^2 + CC_1^2 = BB_1^2 + DD_1^2.$$

Solution by Bor-Yann Chen, University of California, Irvine, USA

Choose axes so that A, B, C, D have respective coordinates $(0,0)$, $(a,0)$, (a,a), $(0,a)$, denote the coordinates of A_1 by (x_1, y_1), and let $\overrightarrow{A_1B_1}$ make an angle α with the positive directions of the x-axis. Then B_1, C_1, D_1 have respective coordinates

$$\begin{aligned}
&(x_1 + a\cos\alpha,\ y_1 + a\sin\alpha),\\
&\left(x_1 + a\sqrt{2}\cos\left(\alpha + \frac{\pi}{4}\right),\ y_1 + a\sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)\right)\\
&\qquad = (x_1 + a\cos\alpha - a\sin\alpha,\ y_1 + a\sin\alpha + a\cos\alpha),\\
&\left(x_1 + a\cos\left(\frac{\pi}{2} + \alpha\right),\ y_1 + a\sin\left(\frac{\pi}{2} + \alpha\right)\right)\\
&\qquad = (x_1 - a\sin\alpha,\ y_1 + a\cos\alpha).
\end{aligned}$$

Now,

$$\begin{aligned}
&AA_1^2 + CC_1^2\\
&\quad = x_1^2 + y_1^2 + (x_1 + a\cos\alpha - a\sin\alpha - a)^2 + (y_1 + a\sin\alpha - a\cos\alpha - a)^2\\
&\quad = 2(x_1^2 + y_1^2) + 2a(\cos\alpha - \sin\alpha - 1)x_1 + 2a(\sin\alpha - \cos\alpha - 1)y_1 + 4a^2 - 4a^2\cos\alpha,\\
&BB_1^2 + DD_1^2\\
&\quad = (x_1 + a\cos\alpha - a)^2 + (y_1 + a\sin\alpha)^2 + (x_1 - a\sin\alpha)^2 + (y_1 + a\cos\alpha - a)^2,
\end{aligned}$$

which is seen to be equal to the same expression.

39.6 A set S with n elements, where n is a positive integer, has m distinct subsets $A_1, \ldots, A_m$ such that $A_i \cup A_j = S$ whenever $i \neq j$. Show that $m \leq n + 1$ and that if $m = n + 1$ then $A_1, \ldots, A_m$ are uniquely determined up to order.

Solution by H. A. Shah Ali, who proposed the problem

Put $B_i = S \setminus A_i$ for $i = 1, \ldots, m$. For $i \neq j$,

$$B_i \cap B_j = (S \setminus A_i) \cap (S \setminus A_j) = S \setminus (A_i \cup A_j) = \varnothing.$$

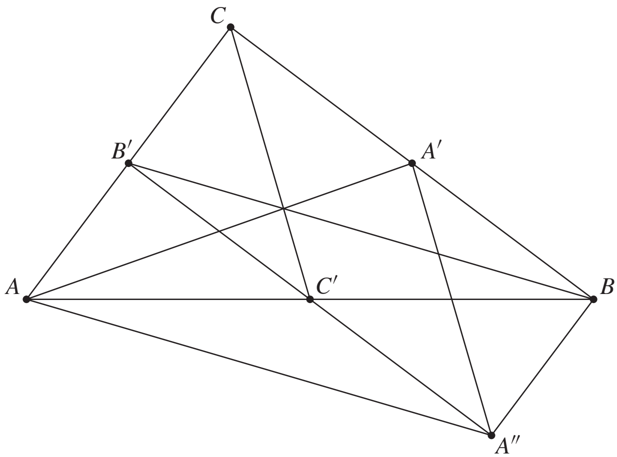
Also, $B_1, \ldots, B_m$ are distinct. Hence, at most one of them is the empty set and $m - 1 \leq n$, so that $m \leq n + 1$. If $m = n + 1$, then the sets $B_1, \ldots, B_m$ must be $\{x_1\}, \ldots, \{x_n\}, \varnothing$ in some order, so that $A_1, \ldots, A_m$ must be $S \setminus \{x_1\}, \ldots, S \setminus \{x_n\}, S$ in some order.

39.7 Denote the lengths of the sides of a triangle ABC by a, b, and c and the lengths of its medians by d, e, and f. Express d, e, and f in terms of a, b, and c.

Show how to construct a triangle $A'B'C'$ from ABC with sides d, e, and f, and deduce that a similar construction applied to triangle $A'B'C'$ will give a triangle similar to ABC.

Solution by J. A. Scott, who proposed the problem

Produce $B'C'$ to A'', where $B'C' = C'A''$. Triangles $AB'C'$ and ACB are similar, so that $B'A''$ is parallel and equal to CB, so that $B'A''BC$ is a parallelogram. Since A' and C' are the midpoints of BC and $A''B'$ respectively, $A''A' = CC' = f$. Also, $A''B = B'C = AB'$ and is parallel to AB', so that $AA''BB'$ is a parallelogram and $AA'' = BB' = e$. Since $AA' = d$, $\Delta A'AA''$ has sides d, e, and f.



Denote $\angle AA'B$ by θ. Using the cosine rule in triangles ABA' and ACA', we have

$$c^2 = d^2 + \frac{a^2}{4} - 2d\frac{a}{2}\cos\theta,$$
$$b^2 = d^2 + \frac{a^2}{4} - 2d\frac{a}{2}\cos(\pi - \theta),$$

so that

$$b^2 + c^2 = 2d^2 + \frac{a^2}{2}$$

and

$$d^2 = -\frac{a^2}{4} + \frac{1}{2}b^2 + \frac{1}{2}c^2,$$

with similar expressions for e^2 and f^2. Thus,

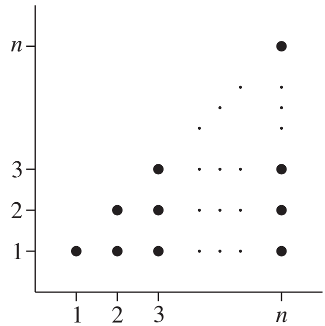
$$\begin{bmatrix} d^2 \\ e^2 \\ f^2 \end{bmatrix} = M \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}, \quad \text{where } M = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

If we carry out a similar construction on $\Delta A'AA''$, we obtain a triangle with sides g, h, and j, where

$$\begin{bmatrix} g^2 \\ h^2 \\ j^2 \end{bmatrix} = M \begin{bmatrix} d^2 \\ e^2 \\ f^2 \end{bmatrix} = M^2 \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix} = \begin{bmatrix} \frac{9}{16} & 0 & 0 \\ 0 & \frac{9}{16} & 0 \\ 0 & 0 & \frac{9}{16} \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix},$$

so that the resulting triangle is similar to ΔABC, with sides three-quarters those of ΔABC.

39.8 Two points P and Q are chosen at random from the array of $\frac{1}{2}n(n+1)$ points as shown below.



Determine the probabilities that the line PQ

(a) is parallel to,
(b) is perpendicular to,
(c) makes an acute angle with,
(d) makes an obtuse angle with

the positive direction of the x-axis.

Solution by M. A. Khan, who proposed the problem

The number of pairs of points is the binomial coefficient

$$\binom{\frac{1}{2}n(n+1)}{2} = \frac{\frac{1}{2}n(n+1)(\frac{1}{2}n(n+1)-1)}{2}$$
$$= \tfrac{1}{8}n(n+1)(n-1)(n+2).$$

(a) The number of pairs PQ in which PQ is parallel to the positive direction of the x-axis is

$$\sum_{i=2}^{n}\binom{i}{2} = \sum_{i=1}^{n}\tfrac{1}{2}i(i-1)$$
$$= \tfrac{1}{2}\sum_{i=1}^{n}i^2 - \tfrac{1}{2}\sum_{i=1}^{n}i$$
$$= \tfrac{1}{2}\tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}\tfrac{1}{2}n(n+1)$$
$$= \tfrac{1}{12}n(n+1)(2n-2),$$

so the probability is

$$\frac{\frac{1}{12}n(n+1)(2n-2)}{\frac{1}{8}n(n+1)(n-1)(n+2)} = \frac{4}{3(n+2)}.$$

(b) By symmetry, this is the same as (a).

(c) Suppose that P is in the (i, j) position. Then the number of possibilities for Q is

$$\tfrac{1}{2}n(n+1) - \tfrac{1}{2}i(i+1) - (n-i)j,$$

so the number of pairs PQ which make an acute angle with the positive direction of the x-axis is

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{i}\left[\tfrac{1}{2}n(n+1)-\tfrac{1}{2}i(i+1)-(n-i)j\right] \\
&= \tfrac{1}{2}n(n+1)\sum_{i=1}^{n} i-\tfrac{1}{2}\sum_{i=1}^{n} i^2(i+1)-\sum_{i=1}^{n}(n-i)\tfrac{1}{2}i(i+1) \\
&= \tfrac{1}{2}n(n+1)\tfrac{1}{2}n(n+1)-\tfrac{1}{2}n\sum_{i=1}^{n} i^2-\tfrac{1}{2}n\sum_{i=1}^{n} i \\
&= \tfrac{1}{4}n^2(n+1)^2-\tfrac{1}{2}n\tfrac{1}{6}n(n+1)(2n+1)-\tfrac{1}{2}n\tfrac{1}{2}n(n+1) \\
&= \tfrac{1}{12}n^2(n+1)[3(n+1)-(2n+1)-3] \\
&= \tfrac{1}{12}n^2(n+1)(n-1).
\end{aligned}$$

Hence, the required probability is

$$\frac{\tfrac{1}{12}n^2(n+1)(n-1)}{\tfrac{1}{8}n(n+1)(n-1)(n+2)}=\frac{2n}{3(n+2)}.$$

(d) Since the four probabilities add to one, this probability is

$$1-\frac{4}{3(n+2)}-\frac{4}{3(n+2)}-\frac{2n}{3(n+2)}=\frac{n-2}{3(n+2)}.$$

Reviews

Bayes Linear Statistics. By Michael Goldstein and David Wooff. John Wiley, Chichester, 2007. Hardback, 508 pages, £80.00 (ISBN 0-470-01562-9).

Publisher's note: '…essential reading for all statisticians concerned with the theory and practice of Bayesian methods. There is an accompanying website hosting free software and guides to the calculations within the book.'

The R Book. By Michael J. Crawley. John Wiley, Chichester, 2007. Hardback, 950 pages, £55.00 (ISBN 0-470-51024-7).

The high-level language of R is recognized as one of the most powerful and flexible statistical software environments, and is rapidly becoming the standard setting for quantitative analysis, statistics, and graphics. R provides free access to unrivalled coverage and cutting-edge applications, enabling the user to apply numerous statistical methods ranging from simple regression to time series or multivariate analysis.

American Regions Math League & ARML Power Contests 1995–2003. By Donald Barry and Thomas Kilkelly. ARML, Andover, MA, 2004. Paperback, 406 pages, \$35.00.

The American Regions Math League is a competition between regions in America, who each supply a team of 15 talented High School mathematicians. These teams meet up at a University campus. This book gives the problems and the solutions for the years indicated. A typical year has the following types of questions: Team Round, Power Question, Individual Round, Relay Round, Tiebreakers, and Super Relay. Here is what each involves.

Team Round The 15 students have 20 minutes to solve 10 problems. They decide how to spread their efforts and calculators are allowed. Each question has a single numerical answer that is required.

Example The Luzors played y games, winning some and losing the rest. They then won three in a row and improved their winning percentage by exactly 10%. Compute the least value of y.

Power Round The team is given 60 minutes to solve a series of in-depth questions and/or prove a number of theorems on one topic. The teams' papers are then graded by a group of teachers.

Example A triangle T has sides of a, b, and c. Definitions are then given for a triangle T^2 with sides of a^2, b^2, and c^2 and for a triangle $\sqrt{T}$ with sides of $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. Ten questions are then asked exploring these ideas.

Individual Round Here they all attempt to solve the same eight questions individually, with no collaboration or calculators. The questions are given in pairs with 10 minutes for each pair. Just the correct numerical answer is required.

Example If

$$\frac{4}{2001} < \frac{a}{a+b} < \frac{5}{2001},$$

compute the number of integral values that b/a can take on.

Relay Round The team is divided into five groups of three. The first person in each group has the same problem. They pass the answer to their problem to their second person, where this number is needed to solve the second problem. Similarly, the answer to the second problem is needed to find the answer to the third problem, this final answer is the only one that is scored.

Example Let $T = \text{TNYWR}$. Compute the number of lattice points that lie strictly inside the triangle formed by $x = 0$, $y = 0$, and $x + y = T$. (I was able to translate 'TNYWR' as American for 'the number you will receive'.)

Tiebreaker Round In the case of teams being equal at the end of the previous rounds they are presented with a tiebreak question where the winners are determined by who has the correct answer most quickly.

Example Let $f(x) = (x+3)^2 + \frac{9}{4}$ for $x \geq 3$. Compute the shortest possible distance between a point on f and a point on f^{-1}.

Super Relay This is just for fun after the main competition and is similar to the relay round but involves all 15 team members in solving 15 different problems and passing the answer on to be used in the next problem.

Example Let $T = \text{TNYWR}$. If $|a + bi| = T$ and $|a + 2bi| = \sqrt{T^2 + 96}$, compute $|a|$.

The book also contains questions and answers to the Power Contest. The idea is similar to the Power Round and is designed to simulate mathematical research activity. The questions are sent to the teams, rather than the teams travelling all to be together, and then the scripts posted back to be marked. There is no limit to the size of the team, calculators are allowed, and there is a time limit of 45 minutes.

Example A 'totally repeating decimal' is a number such as $\frac{1}{7} = 0.\overline{142\,857}$. For every totally repeating decimal $0.\overline{d_1 d_2 \cdots d_n}$, a rotating function is defined by

$$r(0.\overline{d_1 d_2 \cdots d_n}) = 0.\overline{d_n d_1 d_2 \cdots d_{n-1}}.$$

Questions are then asked investigating these concepts.

It is interesting to compare this competition with the British Maths Contest and the British Mathematical Olympiad for the same age group. The first is comprised of 25 multiple-choice questions to be answered in 90 minutes with no calculator. The second involves answering six questions, in 3.5 hours with no calculator, which often require ingenuity and where an emphasis is placed on rigorous proof. The contests are taken in the students' own schools.

In this book I particularly liked the Power Questions and the Power Round. There are possibilities here for teachers to use and amend these ideas, if they have to provide guided coursework for their students to do.

For any student taking this competition, this book is a must. The best revision is to do questions in the same style as the questions that are going to appear. It would also be interesting for anyone involved in mathematics competitions to compare the different styles and ideas used in America to those that they are used to.

Atlantic College **Paul Belcher**

The Edge of the Universe: Celebrating Ten Years of Math Horizons. Edited by Deanna Haunsperger and Stephen Kennedy. MAA, Washington, DC, 2006. Hardback, 320 pages, $57.50 (ISBN 0-88385-550-0).

I can do little better than endorse what is written on the back cover. '*Math Horizons* celebrates the people and ideas that are mathematics. Containing the editors' selection from the first ten years of the magazine's existence, this volume features exquisite expositions of mathematics accessible at the level of an undergraduate or advanced high school student. Broad and appealing, the coverage also includes fiction with mathematical themes; literary, theatrical, and cinematic criticism; humour; history; and social history. Mathematics is shown as a human endeavour through biographies and interviews of mathematicians and users of mathematics including artists, writers, and scientists. The puzzles, games, and activities throughout make it a valuable resource for student math clubs.'

Anyone with an interest in mathematics will delight in engaging with this volume. Its subject matter is wide ranging. I enjoyed the book, learned some new mathematics, but above all I have had my horizons widened regarding the breadth of subjects which have appealed to mathematicians in the last 100 years or so. Amongst the 75 articles this fascinating book contains are ones on magic, mathematical sculpture, the arts in general, mathematical genius, number theory, plenty on geometry, space filling curves, decision mathematics, discrete mathematics, wallpaper patterns, the hyperbolic plane, knots, tilings, chess tasks, women in mathematics, and many potted biographies of mathematicians.

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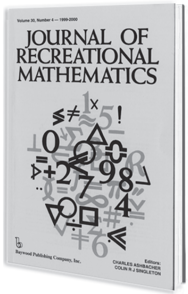

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Mathematical Spectrum

2007/2008 Volume 40 Number 1

ISSN 0025-5653

Published by the Applied Probability Trust

Printed by MFK Pear Tree Press Ltd, Stevenage, Hertfordshire, UK