Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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THE CONTEST CORNER

No. 49

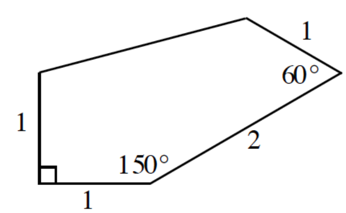
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2017**.

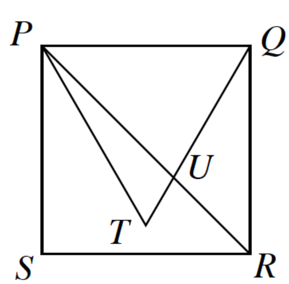
The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

CC241. Over many centuries, tilings have fascinated mathematicians and the society in general. Particularly interesting are tilings of the plane that use a single type of tile. You can tile the plane with some regular polygons (such as equilateral triangles, squares, regular hexagons). On the other hand, you cannot tile the plane using regular pentagons. Now, we know that some non-regular pentagons can be used to tile the plane, although not all of them are yet known (see page 397 of this issue for all known pentagonal tilings). It was therefore with great enthusiasm that in August 2015, the world welcomed the discovery of a new pentagonal tiling, illustrated below.



Use the lengths given and angle sizes to calculate the exact area of this pentagon.

CC242. The diagram below shows square $PQRS$ with sides of length 1 unit. Triangle PQT is equilateral. Show that the area of triangle UQR is $(\sqrt{3}-1)/4$ square units.



CC243. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands. Show that it must be possible to use these air services to fly between any pair of islands.

CC244. How many distinct solutions consisting of positive integers does this system of equations have?

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ y_1 + y_2 + y_3 &= 5, \\ z_1 + z_2 + z_3 &= 5, \\ x_1 + y_1 + z_1 &= 5, \\ x_2 + y_2 + z_2 &= 5, \\ x_3 + y_3 + z_3 &= 5. \end{aligned}$$

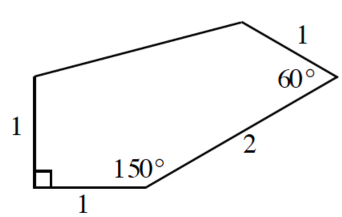
CC245. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that

$$V = \frac{1}{12} a^2 \sqrt{3b^2 - a^2}.$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Find the height of the pyramid in this position.

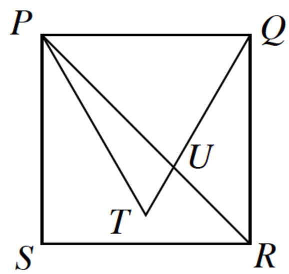
. . . .

CC241. Au fil des années, les pavages ont fasciné les mathématiciens et la société. Particulièrement utiles sont les pavages du plan à l'aide d'une seule tuile. Dans certains cas, des polygones réguliers (triangles équilatéraux, carrés, hexagones réguliers) jouent le rôle. Par contre, il est impossible de paver le plan à l'aide de pentagones réguliers. Or, on sait que certains pentagones non réguliers peuvent paver le plan, bien qu'on ne les connait pas tous (voir, à la page 397, tous les pavages pentagonaux connus). C'est donc avec beaucoup d'enthousiasme qu'en août 2015, le monde a accueilli la découverte d'un nouveau pavage pentagonal, illustré ci-bas.



Utiliser les longueurs et angles donnés afin de calculer la surface.

CC242. Le schéma ci-bas illustre un carré $PQRS$ avec côtés de longueur 1 unité. Le triangle PQT est équilatéral. Démontrer que la surface du triangle UQR est $(\sqrt{3}-1)/4$ unités carrées.



CC243. Huit îles sont chacune dotées d'au moins un service aérien. Un service aérien consiste de vols aller-retours entre deux îles ; deux services différents ne peuvent pas relier la même paire d'îles. Au total, on compte 17 services aériens. Démontrer qu'il est possible d'utiliser ces services aériens pour se déplacer d'une île à n'importe quelle autre.

CC244. Combien de solutions distinctes en entiers positifs le système suivant possède-t-il?

$$\begin{aligned} x_1 + x_2 + x_3 &= 5, \\ y_1 + y_2 + y_3 &= 5, \\ z_1 + z_2 + z_3 &= 5, \\ x_1 + y_1 + z_1 &= 5, \\ x_2 + y_2 + z_2 &= 5, \\ x_3 + y_3 + z_3 &= 5. \end{aligned}$$

CC245. Une pyramide se situe sur un terrain horizontal. Sa base est un triangle équilatéral avec côtés de longueur a, ses trois autres côtés sont de longueur b et son volume est V. Démontrer que

$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

La pyramide est maintenant placée de façon à ce qu'une des faces non équilatérales soit horizontale. Déterminer la hauteur de la pyramide dans cette nouvelle position.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(9), p. 373–374, unless otherwise specified.

CC123. Find how many pairs of integers (x, y) satisfy the inequality

$$2^{x^2} + 2^{y^2} < 2^{1976}.$$

Originally question 5 on the 1976 entrance exam to the All-republican Distance Education Moscow School.

*This problem originally appeared in **Crux** 40(5), p. 188–189, but received no correct submissions. We since received two correct solutions. We present the solution by Billy Jin and Zi-Xia Wang.*

Note first that a necessary condition for $2^{x^2} + 2^{y^2} < 2^{1976}$ to hold is $x^2 < 1976$ and $y^2 < 1976$, or $|x| < \sqrt{1976}$ and $|y| < \sqrt{1976}$.

Since x and y are integers and $\lfloor \sqrt{1976} \rfloor = 44$, it follows that

$$x, y \in \{-44, -43, \ldots, 0, 1, 2, \ldots, 44\}. \tag{1}$$

Conversely, if (1) holds, then we have $1976 - x^2 > 1$ and $1976 - y^2 > 1$, which together imply that

$$\frac{2^{x^2} + 2^{y^2}}{2^{1976}} = \frac{1}{2^{1976-x^2}} + \frac{1}{2^{1976} - y^2} < \frac{1}{2} + \frac{1}{2} < 1,$$

so $2^{x^2} + 2^{y^2} < 2^{1976}$.

Therefore, since each of x and y can take 89 possible values, the total number of pairs (x, y) of integers satisfying $2^{x^2} + 2^{y^2} < 2^{1976}$ is $89^2 = 7921$.

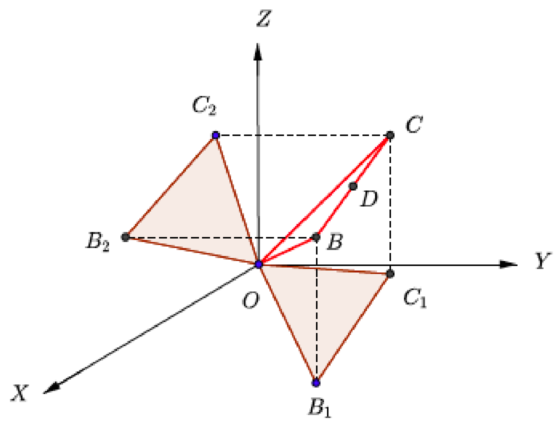
CC125. Orthogonal projections of a triangle ABC onto two perpendicular planes are equilateral triangles with side length 1. If the median AD of triangle ABC has length $\sqrt{\frac{9}{8}}$, find BC.

Originally question 4 on the 1969 entrance exam to the mathematical-mechanical department of Moscow State University.

*This problem originally appeared in **Crux** 40(5), p. 188–189, but received no correct submissions at that time. Here we present the solution by Somasundaram Muralidharan.*

Without loss of generality, we assume that the perpendicular planes on which the triangle is projected are the XOY and XOZ planes and that the vertex A of the triangle is at the origin. Let the other two vertices have coordinates $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. The projections of these on the XOY plane are $B_1(b_1, b_2, 0)$ and $C_1(c_1, c_2, 0)$. Since $OB_1 = OC_1 = 1$ and $\angle B_1OC_1 = 60°$, we have

$$b_1^2 + b_2^2 = 1, \quad c_1^2 + c_2^2 = 1, \quad b_1c_1 + b_2c_2 = \frac{1}{2}. \tag{2}$$



Similarly, the projections of B and C on the XOZ plane are $B_2(b_1, 0, b_3)$ and $C_2(c_1, 0, c_3)$. Using the fact that OB_2C_2 is an equilateral triangle with side equal to 1, we get

$$b_1^2 + b_3^2 = 1, \quad c_1^2 + c_3^2 = 1, \quad b_1c_1 + b_3c_3 = \frac{1}{2}. \tag{3}$$

Let D be the midpoint of BC. The length of the median $AD = \sqrt{\frac{9}{8}}$, so

$$\left(\frac{b_1 + c_1}{2}\right)^2 + \left(\frac{b_2 + c_2}{2}\right)^2 + \left(\frac{b_3 + c_3}{2}\right)^2 = \frac{9}{8} \tag{4}$$

$$(b_1^2 + b_2^2) + (c_1^2 + c_2^2) + 2(b_1c_1 + b_2c_2) + (b_3 + c_3)^2 = \frac{9}{2} \tag{5}$$

Using (2), we get $(b_3 + c_3)^2 = \frac{3}{2}$. Again, rewriting (4) as

$$(b_1^2 + b_3^2) + (c_1^2 + c_3^2) + 2(b_1c_1 + b_3c_3) + (b_2 + c_2)^2 = \frac{9}{2}$$

and using (3), we obtain $(b_2 + c_2)^2 = \frac{3}{2}$. Now from (4), we have $(b_1 + c_1)^2 = \frac{3}{2}$. Thus

$$b_1 + c_1 = \pm\sqrt{\frac{3}{2}}, \quad b_2 + c_2 = \pm\sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \pm\sqrt{\frac{3}{2}}$$

We need to discuss eight possibilities. We will consider two of these (the other cases are similar and can be mapped to one of these using rotation of axes).

Case 1. Suppose that

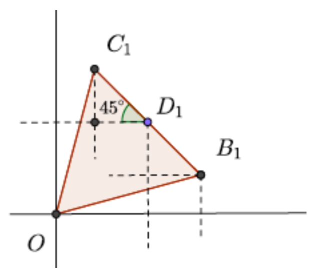
$$b_1 + c_1 = \sqrt{\frac{3}{2}}, \quad b_2 + c_2 = \sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \sqrt{\frac{3}{2}}.$$

Consider D_1 the projection of D on XOY plane. The coordinates of D_1 are

$$\left(\frac{b_1+c_1}{2}, \frac{b_2+c_2}{2}, 0\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}}, 0\right).$$

Thus D_1 lies on the line $y = x$ in the XOY plane and hence the coordinates of B_1 and C_1 are given by

$$(b_1, b_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right), (c_1, c_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).$$



If D_2 is the projection of D on the XOZ plane, then D_2 has coordinates

$$\left(\frac{b_1+c_1}{2}, 0, \frac{b_3+c_3}{2}\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, 0, \frac{1}{2}\sqrt{\frac{3}{2}}\right)$$

As before we find

$$(b_1, b_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right), (c_1, c_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).$$

Thus, in this case, we have

$$\begin{aligned}(b_1, b_2, b_3) &= \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right),\\ (c_1, c_2, c_3) &= \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).\end{aligned}$$

It readily follows that $BC = \sqrt{\frac{3}{2}}$.

Case 2. Suppose that

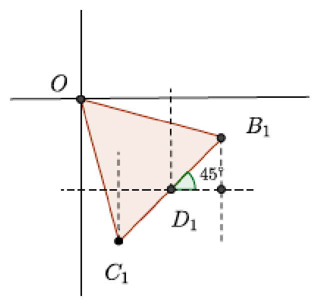
$$b_1 + c_1 = \sqrt{\frac{3}{2}}, \quad b_2 + c_2 = -\sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \sqrt{\frac{3}{2}}.$$

Consider D_1 the projection of D on XOY plane. The coordinates of D_1 are given by

$$\left(\frac{b_1+c_1}{2}, \frac{b_2+c_2}{2}, 0\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}}, 0\right).$$

In this case, D_1 lies on the line $y = -x$ and hence we find the coordinates of B_1, C_1 as (see the graphic)

$$(b_1, b_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right),$$
$$(c_1, c_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right).$$



As in **Case 1**, we find (b_3, c_3) as $b_3 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}$ and $c_3 = \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}$. Thus, in this case, we have

$$(b_1, b_2, b_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right)$$
$$(c_1, c_2, c_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right)$$

Hence in this case also, we have $BC = \sqrt{\frac{3}{2}}$. This completes the proof.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

Originally question 5 from 2013 Pan African Mathematics Olympiad.

This problem originally appeared in ***Crux*** *41(1), p. 4–5, but received no correct submissions. We since received one correct solution by Maria Aleksandrova, presented below.*

Suppose there is a 3×3 square with three corners all the same colour. In our figures, we denote black by X and white by O.

$$\begin{array}{ccc} X & - & - \\ - & - & - \\ X & - & X \end{array}$$

In order for no square to line of 3 in a row to have the same colour, the rest of the grid must be filled in as follows:

$$\begin{array}{ccc} X & X & O \\ O & O & X \\ X & O & X \end{array}$$

We observe that the square beneath the right hand column cannot be X since this would give 3 in a row vertically, and it cannot be O, since this would give 3 in a row diagonally. Thus, this configuration must be on a side of the grid. Simiarly, the square to the left of the top row cannot be filled. Thus, we see that the only place we can have 3 of the same colour in a 3×3 square is in a corner of the grid.

If we extend the 3×3 square up and to the right, we get:

$$\begin{array}{cccc} O & a & b & X \\ X & X & O & c \\ O & O & X & d \\ X & O & X & O \end{array}$$

Notice that c and d must both be different, since otherwise we would have 3 in a row vertically. If c is O and d is X then the square to the right of c could not be X or O, thus d is O and c is X. Similarly, a must be O and b must be X.

However, this gives a contradiction, as the diagonal containing a and d is monochromatic.

Thus, in any 3×3 square, two of the corners are each white and two are black.

CC191. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

We received one complete and correct solution, by Somasundaram Muralidharan. An edited version is presented here.

In the first round the players are divided into 16 pairs, each of which plays a match, resulting in 16 winners. These 16 winners are divided into 8 pairs, each of which plays a second-round match, resulting in 8 winners. Four third-round matches give us 4 winners, then two fourth-round matches give us 2 winners, and one final match returns the gold medal winner. Finding the gold medal winner thus requires $16 + 8 + 4 + 2 + 1 = 31$ gold-medal matches over five rounds.

The silver medal winner must be one of the players who lost only to the gold medal winner in the gold-medal matches. There will be 5 such players, say p_1, p_2, p_3, p_4, p_5, where p_i is the player who lost to the gold medal winner in the i^{th} round. The silver medal winner can be found among these 5 players over 4 silver-medal matches: p_1 plays p_2, the winner plays p_3, the winner of that match plays p_4, and the winner of that match plays p_5. Finding the gold and the silver medal winners is thus accomplished in a total of $31 + 4 = 35$ matches.

The bronze medal winner must lose to only the gold and silver medal winners, so they must be among the players who lost to either the silver or gold medal winner in the gold- and silver-medal rounds. We claim that there will be at most 5 such players. Among those who lost to the gold medal winner, let the silver medal winner be p_j, where j indicates that they lost to the gold winner in the j^{th} gold-medal round. This means that they won $j - 1$ matches in the gold-medal stage. In the silver-medal stage they won either $5 - j + 1$ matches (if $j > 1$) or $5 - j$ matches if $j = 1$. Thus, over both the gold- and silver-medal stages, the silver medal winner won a maximum of 5 matches (the exception is 4 matches if $j = 1$), and thus beat a maximum of 5 players. The bronze medal winner can be determined from amongst these 5 players in 4 matches. Thus we are guaranteed to find the top three ranked players by playing a total of $31 + 4 + 4 = 39$ matches in all three stages.

We note that if we start with n players, determining the gold, silver and bronze medal winners requires a minimum of

$$\lceil \log_2(n(n-1)) \rceil + n - 3$$

matches, where $\lceil x \rceil$ is the smallest integer $\geq x$. For a proof, see Serge Tabachnikov (Editor), *Kvant Selecta: Combinatorics I*, American Mathematical Society, 2000.

CC192. Let M be a 3×3 matrix with all entries drawn randomly (and with equal probability) from $\{0, 1\}$. What is the probability that $\det M$ will be odd?

We received three correct and complete solutions, of which we present the one by the Missouri State University Problem Solving Group.

If we consider the entries to be from $\mathbb{F}_2$, the field with two elements, the condition that $\det M$ be odd is equivalent to it being non-zero, which in turn is equivalent to M being invertible over $\mathbb{F}_2$. This is equivalent to the rows of M being linearly independent. There are $2^3 - 1$ choices for the first row (we cannot choose the zero vector), $2^3 - 2$ choices for the second row (we cannot choose a scalar multiple of the first row), and $2^3 - 2^2$ choices for the third row (we cannot choose an element in the span of the first two rows). This gives $7 \cdot 6 \cdot 4 = 168$ invertible matrices out of a total of $2^9 = 512$ matrices, so our probability is $\frac{168}{512} = \frac{21}{64}$.

CC193. Consider the set of numbers $\{1, 2, \ldots, 10\}$. Let $\{a_1, a_2, \ldots, a_{10}\}$ be some permutation of these numbers and compute

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

What is the maximum possible value of the above sum over all possible permutations and how many permutations give you this maximum value?

We received two correct solutions. We present the solution of the Missouri State University Problem Solving Group.

We will solve the analogous problem when 10 is replaced by $2k$. To determine the maximum values, we assume without loss of generality that $a_{2i-1} > a_{2i}$. We then wish to maximize

$$\sum_{i=1}^{k} a_{2i-1} - \sum_{i=1}^{k} a_{2i}.$$

This will clearly occur when $\sum_{i=1}^{k} a_{2i-1}$ is maximized and $\sum_{i=1}^{k} a_{2i}$ is minimized, namely when

$$\{a_{2i-1}\}_{i=1}^{k} = \{k+1, k+2, \cdots, 2k\}$$

and

$$\{a_{2i}\}_{i=1}^{k} = \{1, 2, \cdots, k\}.$$

This gives a maximum value of

$$\sum_{i=k+1}^{2k} i - \sum_{i=1}^{k} i = \sum_{i=1}^{k} (k+i) - \sum_{i=1}^{k} i = \sum_{i=1}^{k} k + \sum_{i=1}^{k} i - \sum_{i=1}^{k} i = k^2.$$

If a permutation achieves this maximum, then for each of the pairs $\{a_{2i-1}, a_{2i}\}$, one element must be sent to an element of $A = \{1, 2, \cdots, k\}$ and the other to an element of $B = \{k+1, k+2, \cdots, 2k\}$. There are $k!$ ways of assigning elements of A to each i and $k!$ ways of assigning elements of B to each i. There are also 2 choices for each i, whether a_{2i-1} is sent to A or B (and hence a_{2i} is sent to B or A respectively). This gives a total of $2k(k!)^2$ permutations. For the original problem, the maximum value is $5^2 = 25$ and the number of permutations is $25(5!)^2 = 460800$.

CC194. At a strange party, each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the party that they both knew. For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew. How many people were at the party?

We received one complete and correct solution, by the Missouri State University Problem Solving Group; it is presented here.

Consider K_n the complete graph on n vertices (one vertex for each person at the party). Colour an edge blue if the people corresponding to its end- points know one another and red if they don't. We will count the number of triangles with two blue sides and one red one in two different ways. First, each person knows exactly 22 others and none of these 22 know each other. Therefore, if we choose two blue edges emanating from one vertex, the third side of the resulting triangle must be red and every triangle with exactly two blue edges arises in this manner.This gives $\binom{22}{2}n$ triangles with exactly two blue sides. Second, consider the subgraph consisting only of blue edges. It is well known that the number of edges of a graph is half the sum of the degrees. In this case, since the degree of a vertex of a blue edge is 22, the number of blue edges is $22n/2 = 11n$. The total number of edges in K_n is $\binom{n}{2}$. Therefore the number of red edges is $\binom{n}{2} - 11n$. Each red edge is an edge in exactly 6 triangles with two blue edges, so the number of triangles with exactly two blue sides is $6\left(\binom{n}{2} - 11n\right)$. Therefore,

$$\binom{22}{2} n = 6\left(\binom{n}{2} - 11n\right).$$

Solving for n, we find $n = 0$, which we reject, or $n = 100$, so there are 100 people at the party.

CC195. A bisecting curve is one that divides a given region into two subregions of equal area. The shortest bisecting curve of a circle is clearly a diameter. What is the shortest bisecting curve of an equilateral triangle?

We received two submissions of which one was correct and complete. We present the solution by Somasundaram Muralidharan.

The shortest curve is an arc of a circle centred at one of the corners, with radius $\frac{3^{3/4}}{2\sqrt{\pi}}$ times the side length of the triangle.

Arrange six equilateral triangles to form a regular hexagon. Suppose that a curve dividing the area in half cuts two sides of the triangle. By piecing together six copies of the curve (see Figure 1), we obtain a curve that contains half the area of the hexagon. By the isoperimetric theorem, the shortest curve enclosing a given area is a circle. Thus, if we find the circle that encloses half the area of the hexagon, then the arc of that circle will also be the shortest bisecting curve for the equilateral triangle.

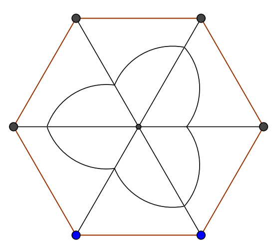


Figure 1: Curve bisecting the hexagon

Let the side of the equilateral triangle be 1. The area of the hexagon is $\frac{3\sqrt{3}}{2}$ and hence if r is the radius of the required circle, we have

$$\pi r^2 = \frac{3\sqrt{3}}{4} \Rightarrow r = \frac{3^{3/4}}{2\sqrt{\pi}}$$

Hence the length ℓ of the shortest arc bisecting the equilateral triangle is

$$\ell = \frac{2\pi r}{6} = \frac{\sqrt{\pi}}{2 \cdot 3^{1/4}}$$

Now suppose that there is a curve that bisects the area of the equilateral triangle and cuts only one of its sides. Piecing together two such curves, we obtain a curve enclosing an area equal to that of the triangle (See Figure 2). Again, we have a circle of radius r_1 which encloses the same area.

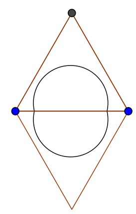


Figure 2: Circle bisecting the parallelogram

The radius r_1 is given by

$$\pi r_1^2 = \frac{\sqrt{3}}{4} \Rightarrow r_1 = \frac{3^{1/4}}{2\sqrt{\pi}}$$

The length ℓ_1 of the arc that bisects the equilateral triangle is given by

$$\ell_1 = \pi r_1 = \frac{\sqrt{\pi} \cdot 3^{1/4}}{2}$$

Clearly $\ell_1 > \ell$, and the proof is complete.

Editor's Note

Neculai Stanciu has brought it to our attention that the inequality discovered independently by A. Engel in 1998 and T. Andreescu in 2001 (see **Crux**,Volume 42 pages 216 and 342) was in fact known to H. Bergström as early as 1949.

THE OLYMPIAD CORNER

No. 347

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2017***.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC301. Solve the following Diophantine equation for integers x and y:

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that,

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

OC303. Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. Point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer bigger than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

. . . .

OC301. Résoudre l'équation diophantienne

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3,$$

x et y étant des entiers.

OC302. Soit x, y et z des nombres réels dont les sommes deux à deux ne sont pas égales à 1. Démontrer que

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2}\geq 2(x+y+z)-\frac{3}{4}$$

et déterminer les triplets (x,y,z) qui vérifient l'égalité.

OC303. Soit ABC un triangle, H son orthocentre et O le centre du cercle circonscrit au triangle. Soit K le milieu de AH et P le point sur AC tel que $\angle BKP = 90°$. Démontrer que OP est parallèle à BC.

OC304. Soit k un entier fixe strictement positif. Soit $F(n)$ le plus petit entier strictement positif supérieur à kn tel que $F(n)\cdot n$ est un carré parfait. Démontrer que si $F(n)=F(m)$, alors $m=n$.

OC305. Soit p un nombre premier pour lequel $\frac{p-1}{2}$ est aussi un nombre premier et soit a,b et c des entiers qui ne sont pas divisibles par p. Démontrer qu'il existe au plus $1+\sqrt{2p}$ entiers strictement positifs n tels que $n<p$ et que p soit un diviseur de $a^n+b^n+c^n$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(7), p. 288–289.

OC241. Let n be a natural number. For every positive real numbers $x_1,x_2,...,x_{n+1}$ such that $x_1x_2...x_{n+1}=1$ prove that:

$$\sqrt[x_1]{n}+\cdots+\sqrt[x_{n+1}]{n}\geq n^{\sqrt[n]{x_1}}+\cdots+n^{\sqrt[n]{x_{n+1}}}.$$

Originally problem 5 from day 2 of the 2014 Iran Team Selection Test.

We received 2 correct submissions. We present the solution by Michel Bataille.

We shall use the following inequality of means: if $x,a_1,\ldots,a_n>0$, then

$$\left(\frac{a_1^x+\cdots+a_n^x}{n}\right)^{1/x}\geq\sqrt[n]{a_1\cdots a_n}$$

which rewrites as

$$a_1^x+\cdots+a_n^x\geq n(a_1\cdots a_n)^{x/n}. \qquad (1)$$

Let

$$S = n^{1/x_1} + \cdots + n^{1/x_{n+1}},$$

$$p_j = \prod_{\substack{k=1\\k\neq j}}^{n} x_k = \frac{1}{x_{n+1}x_j},\ (j = 1, 2, \ldots, n),$$

$$S_{n+1} = S - n^{1/x_{n+1}} = n^{1/x_1} + \cdots + n^{1/x_n}.$$

Using (1), we obtain

$$S_{n+1} = \sum_{k=1}^{n} (n^{p_j})^{x_{n+1}} \geq n \left(n^{p_1+\cdots+p_n}\right)^{\frac{x_{n+1}}{n}} = n \left(n^{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)^{\frac{1}{n}}. \qquad (2)$$

Again by (1) with $x = 1$, that is, by AM-GM, we also have

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \geq n \left(\frac{1}{x_1 \cdots x_n}\right)^{\frac{1}{n}} = n(x_{n+1})^{\frac{1}{n}}.$$

It follows that $n^{\frac{1}{x_1}+\cdots+\frac{1}{x_n}} \geq n^{n(x_{n+1})^{\frac{1}{n}}}$ and (2) yields

$$S_{n+1} \geq n \cdot n^{x_{n+1}^{1/n}} = n \cdot n^{\sqrt[n]{x_{n+1}}}.$$

In the same way, if we set $S_j = S - n^{1/x_j}, \quad (j = 1, \ldots, n)$, we obtain

$$S_j \geq n \left(n^{n(x_j)^{\frac{1}{n}}}\right)^{\frac{1}{n}} = n \cdot n^{\sqrt[n]{x_j}}.$$

By addition,

$$S_1 + \cdots + S_{n+1} \geq n \left(n^{\sqrt[n]{x_1}} + \cdots + n^{\sqrt[n]{x_{n+1}}}\right)$$

and the result follows since $S_1 + \cdots + S_{n+1} = n(n^{1/x_1} + \cdots + n^{1/x_{n+1}})$.

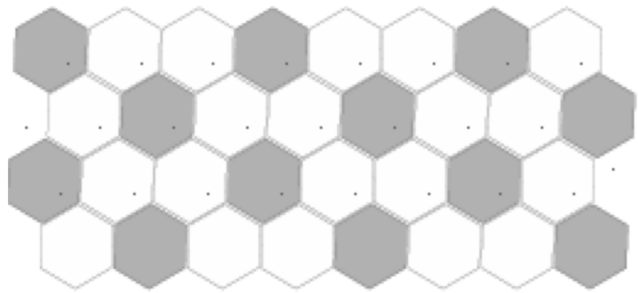
OC242. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

Originally problem 5 from day 2 of the 2014 USAJMO.

We present the solution by Oliver Geupel. There were no other submissions.

The answer is $k = 6$.

First we show that A cannot win when $k = 6$. Colour the cells white and grey according to the following pattern, continued to infinity:



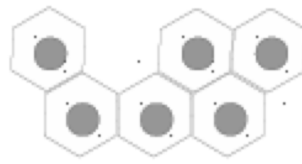
Player A can occupy only one grey cell per move. Player B's strategy is to remove the counter from the grey cell. Every arrangement of six cells in a line occupies at least two grey cells. By B's strategy, A cannot reach such a pattern. Hence A cannot win when $k = 6$.

In what follows we give a winning way for A when $k \leq 5$. Player A can reach an arrangement of 3 consecutive occupied cells in a line in two moves. In the next move A can create a pattern congruent to this one:

If B now would remove the counter from the lower cell, A could win immediately. Thus, B must remove a counter from the upper row. Then, A can reach an arrangement congruent to the following pattern P:

If B now would remove a counter from the lower row, A could win immediately. By reasons of symmetry, it is therefore enough to assume that B removes the upper-left or the upper-central counter. B should not remove the upper-left counter, since A could reach the pattern

which lets him win in the next move. Therefore, B must remove the upper-central counter in pattern P. Then, A moves to the following pattern Q:



To avoid immediate loss, B must remove a counter from the lower row in Q. If B removes the lower-left or the lower-central counter, A can reach to

which wins in the next move. Otherwise, B removes the lower-right counter in Q. Player A moves to

which ensures that A will win in the next move. We have proved that A has a winning strategy when $k \leq 5$.

OC243. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Originally problem 4 of the 2014 France Team Selection Test.

We received 4 correct submissions and 1 incorrect submission. We present the solution by David Manes.

Clearly, $f(x) = x$ for all $x \in \mathbb{Z}_{>0}$ satisfies the property since $m = n = x$ implies $m^2 + f(n) = mf(m) + n = x^2 + x$.

Assume the condition is satisfied and let $m = n = 2$. Then $4 + f(2) \mid 2f(2) + 2$ implies $(4+f(2))k = 2f(2)+2$ for some positive integer k. Therefore, $f(2) = \frac{4k-2}{2-k}$. Note that $k \neq 2$ since $f(2) \in \mathbb{Z}_{>0}$. Also, if $k > 2$, then $f(2) < 0$, a contradiction. Hence, $k = 1$ so that $f(2) = 2$. Let $m = 2$ and $n = 1$. Then $4 + f(1) \mid 5$ implies $f(1) = \frac{5-4k}{k}$ for some positive integer k. Again, the only value of k for which $f(k)$ is a positive integer is $k = 1$, whence $f(1) = 1$. Assume inductively that $v \geq 2$ is an integer and $f(v) = v$. Let $m = v$ and $n = v+1$. Then $v^2 + f(v+1) \mid v^2 + v + 1$ implies $(v^2 + f(v + 1))k = v^2 + v + 1$ for some positive integer k. Solving for

$f(v+1)$, one obtains

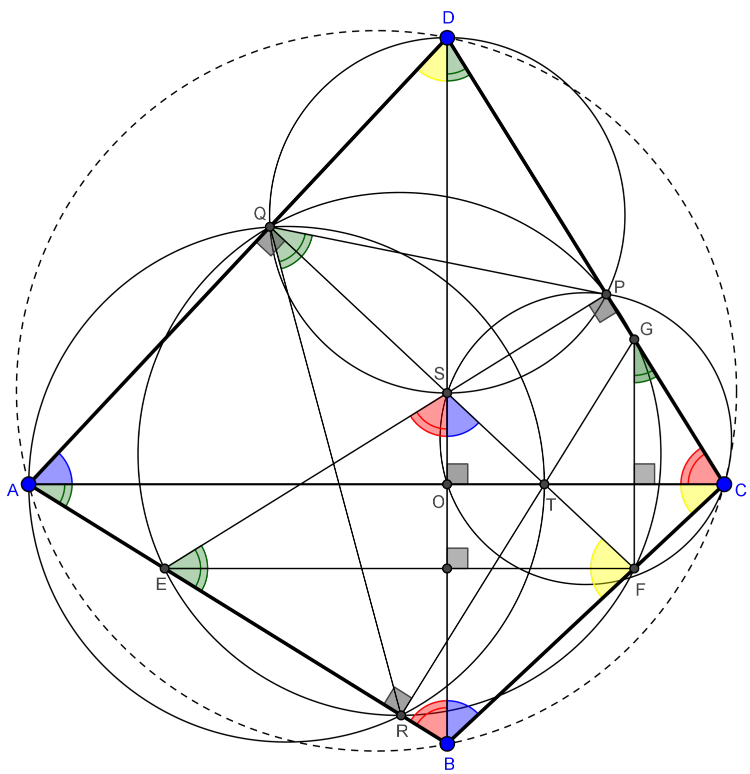
$$f(v+1) = \frac{v^2(1-k)+v+1}{k}$$

If $k > 1$, then $f(v+1) < 0$ for $v \geq 2$, a contradiction. Hence, $k = 1$ so that $f(v+1) = v+1$. Therefore, $f(x) = x$ for all positive integers x by induction.

OC244. $ABCD$ is a cyclic quadrilateral, with diagonals AC, BD perpendicular to each other. Let point F be on side BC, the parallel line EF to AC intersect AB at point E, line FG parallel to BD intersect CD at G. Let the projection of E onto CD be P, projection of F onto DA be Q, projection of G onto AB be R. Prove that QF bisects $\angle PQR$.

Originally problem 1 from day 1 of the 2014 China Team Selection Test.

We received 4 correct submissions. We present the solution by Andrea Fanchini.



We first note that $PGRE$ and $PGFE$ are cyclic so $PGFRE$ is cyclic.

$ABCD$ is cyclic so $\angle BDC = \angle BAC$, then $AC \parallel EF$ implies that $\angle BAC = \angle BEF$ and $FG \parallel BD$ implies that $\angle FGC = \angle BDC$. Therefore, $\angle BEF = \angle FGC$.

$PGFE$ is cyclic so $\angle PEF + \angle PGF = 180°$, but $\angle PGF = 180° - \angle FGC$. Therefore, $\angle PEF = \angle FGC = \angle BEF$ and $PF = FR$.

Now, $SOCP$ is cyclic so $\angle BSE = \angle OCP$, but $ABCD$ is cyclic so

$$\angle ACD = \angle OCP = \angle ABD = \angle EBS \quad \Rightarrow \quad \angle BSE = \angle EBS.$$

Therefore $\triangle BES$ is isosceles and furthermore as seen $\angle PEF = \angle BEF$ so EF is the perpendicular bisector of BS and also $\triangle FBS$ is isosceles. Then

$$\angle SFE = \angle EFB = \angle ACB = \angle ADB.$$

Now, as shown in the picture, yellow and blue angles (single arch notation) are complementary and the sum of the red and green angles (double arch notation) is $90°$. So we have

$$\underbrace{\angle QAE}_{blue+green} + \underbrace{\angle AEF}_{180°-green} + \underbrace{\angle EFS}_{yellow} = 270°$$

therefore $FS \perp AD$ which implies S lies on FQ.

Let T be the intersection of AC with GR, we have similarly that also T lies on FQ. Note that $RTQA$ and $PDQS$ are cyclic. Then $\angle RQT = \angle RAT = \angle BAC$ and $\angle PQS = \angle PDS = \angle BDC$ but $\angle BAC = \angle BDC$ so $\angle RQT = \angle PQS$ and we are done.

OC245. Find all sets of 2014 not necessarily distinct rationals such that if we remove an arbitrary number in the set, we can divide the remaining 2013 numbers into three sets such that each set has exactly 671 elements and the product of all elements in each set is the same.

Originally problem 3 from day 2 of the 2014 Vietnam National Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

A multiset M of 2014 rationals satisfies the conditions of the problem if and only if:

1. M contains at least four occurrences of 0, or
2. all elements of M have the same absolute value and either all elements are equal or there are at least three negative and at least three positive ones.

First we show that every M with 1. or 2. satisfies the requirement.

If M satisfies condition 1. then after removing any element we can form three 671-element submultisets such that each of them contains an occurrence of 0. Hence M satisfies the requirement.

Now assume that M satisfies condition 2. If all elements are equal, everything is fine. After removing any element, we have m occurrences of a negative number $-q$, and $2013 - m$ occurrences of q, where $2 \le m \le 2011$. If m is an even number, then there are integers a, b, c such that $0 \le a, b, c \le 335$ and $m = 2(a + b + c)$.

Put $2a$, $2b$, and $2c$ negative elements in the first, second, and third submultiset, respectively. If m is odd, then there are integers a, b, c such that $0 \leq a, b, c \leq 335$ and $m = 2(a+b+c)+3$. Put $2a+1$, $2b+1$, and $2c+1$ negative elements in the first, second, and third submultiset, respectively. So M satisfies the condition of the problem.

It remains to prove that every multiset with the required property satisfies 1. or 2. Suppose M satisfies the condition of the problem with p as the common product value of the three multisets.

First consider the case $0 \in M$. If all elements of M are 0, then 1. is satisfied. Now assume that M contains an element $x \neq 0$. After removing x we have $p = 0$. Hence M contains at least three occurrences of 0. On the other hand, after removing 0 we have again $p = 0$. Therefore, M contains at least four occurrences of 0, i.e., M satisfies 1.

It remains to consider the case $0 \notin M$.

For every rational number q, the set $M_0 = \{qx \mid x \in M\}$ also has the required property. Let us choose q such that $1 \in M_0$. Removing 1 from M_0, the product of the remaining elements is p^3 for some rational number p. Removing any other element x from M_0, the product of the remaining elements is r^3 for some rational number r. Thus, $p^3 = xr^3$, that is, $x = (p/r)^3$, i.e., every element of M_0 is the 3rd power of a rational. Observe that the multiset $M_1 = \{\sqrt[3]{x} \mid x \in M_0\}$ also has the required property. Repeating the observation, we obtain an infinite sequence of multisets $M_0,\ M_1,\ M_2,\ \ldots$ such that $M_k = \{\sqrt[3^k]{x} \mid x \in M_0\}$ and M_k has the required property. Since the elements are rationals, this is possible only if all elements of M_0 are ± 1.

If there were exactly one -1 or two 1's, there were no appropriate partition after removing a 1. If there were exactly two -1's or one 1, there were no appropriate partition after removing a -1. Hence condition 2. is satisfied.

Editor's Note. Congratulations to Oliver Geupel who managed to solve all 5 OC problems in this edition! Well done!

FOCUS ON...

No. 24

Michel Bataille

Solutions to Exercises from Focus On... No. 17–21

From Focus On... No. 17

1. *Show that if k is a positive odd integer and $2^k + 3^k = a^n$ for some integers a, n with $n \geq 2$, then k is a multiple of 5.*

Since k is odd, $2^k + 3^k = (2+3) \cdot N = 5N$ where

$$N = 2^{k-1} - 3 \cdot 2^{k-2} + \cdots + 3^{k-1}.$$

It follows that 5 divides a^n, hence 5 divides a (since 5 is prime) and so 5^n divides $a^n = 2^k + 3^k = 5N$. Recalling that $n \geq 2$, we see that 5 divides N. However, since $2 \equiv -3 \pmod 5$ and k is odd, we have $2^{k-1} \equiv 3^{k-1} \pmod 5$ and more generally

$$(-1)^j 3^j \cdot 2^{k-j-1} \equiv 2^{k-1} \pmod 5.$$

Therefore $N \equiv k \cdot 2^{k-1} \pmod 5$ and so 5 divides $k \cdot 2^{k-1}$. Since 5 and 2^{k-1} are coprime, 5 divides k.

2. *Let m, n be integers such that $m > n \geq 1$ and suppose that $m(m+n) = k^2 + \ell^2$ and $n(m-n) = 2(k^2 - \ell^2)$ for some integers k, ℓ. Prove that m, n have the same parity.*

For the purpose of a contradiction, assume that m and n are of opposite parity. Then $m-n$ is odd and the second relation implies that n is even, say $n = 2r$. Thus, m is odd and the first relation then shows that k and ℓ are also of opposite parity, say $k = 2s$ and ℓ odd. We obtain $r(m-2r) = 4s^2 - \ell^2$, an odd integer, hence r must be odd and so $r+m$ is even, say $r+m = 2t$. Now, from $(2t-r)(2t+r) = 4s^2 + \ell^2$, we deduce $(2t)^2 - (2s)^2 = r^2 + \ell^2$, a contradiction since modulo 8, $r^2 + \ell^2 \equiv 2$ while $(2t)^2 - (2s)^2 \equiv 0$ or 4. The proof is similar if k is odd and ℓ is even.

3. *Find all odd positive integers a, b, c, d, n such that $a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n$.*

Suppose that (a, b, c, d, n) is a solution. The integers a^2, b^2, c^2, d^2 being congruent to 1 modulo 8, we have $a^2 = 1 + 8a_1, b^2 = 1 + 8b_1, c^2 = 1 + 8c_1, d^2 = 1 + 8d_1$ for some nonnegative integers a_1, b_1, c_1, d_1. Then $1 + 2(a_1 + b_1 + c_1 + d_1) = 7 \cdot 4^{n-1}$, which calls for $n = 1$ (since $7 \cdot 4^{n-1}$ must be odd). It follows that $a_1 + b_1 + c_1 + d_1 = 3$ and, up to permutations, $(a_1, b_1, c_1, d_1) = (0, 0, 0, 3)$ or $(0, 0, 1, 2)$ or $(0, 1, 1, 1)$. This leads to $(a^2, b^2, c^2, d^2) = (1, 1, 1, 25)$ or $(1, 1, 9, 17)$ or $(1, 9, 9, 9)$, up to permutations. Clearly, the second case cannot occur, and so $(a, b, c, d) = (1, 1, 1, 5)$ or $(1, 3, 3, 3)$ (up to permutations). Conversely, it is readily checked that taking $n = 1$, $(a, b, c, d) = (1, 1, 1, 5)$ or $(1, 3, 3, 3)$ or their permutations provides solutions.

From Focus On... No. 18

1. *Show that if p is an odd prime number, then $(p+1)(p+2)\cdots(2p-1) \equiv (p-1)!$ $\left(\bmod p^2\right)$.*

Let $L = (p+1)(p+2)\cdots(2p-1)$. It is readily checked that

$$2L = (p-1)!\binom{2p}{p}.$$

But it is well-known that

$$\binom{2p}{p} = \binom{p}{0}^2 + \binom{p}{1}^2 + \cdots + \binom{p}{p-1}^2 + \binom{p}{p}^2$$

(this is true for any positive integer p, even if p is not prime). Since $\binom{p}{j}$ is a multiple of p when $j = 1, 2, \ldots, p-1$, we have $\binom{2p}{p} \equiv 2 \left(\bmod p^2\right)$. It follows that $2L \equiv 2(p-1)! \left(\bmod p^2\right)$, so $L \equiv (p-1)! \left(\bmod p^2\right)$ (p is odd, 2 and p are coprime).

2. *Let a, b be positive integers and p be any prime. Show that $a^p - b^p$ is either coprime to p or divisible by p^2.*

If p and $a^p - b^p$ are not coprime, then p divides $a^p - b^p$. From Fermat Little Theorem, we have $a^p \equiv a \pmod p$ and $b^p \equiv b \pmod p$, hence

$$a^p - b^p \equiv a - b \pmod p$$

and so p divides $a - b$. But $a^p - b^p = (a-b)N$ where

$$N = a^{p-1} + a^{p-2}b + \cdots + ab^{p-2} + b^{p-1} \equiv pb^{p-1} \equiv 0 \pmod p$$

(since $a \equiv b \pmod p$). Thus, p divides each of the integers $a - b$ and N, hence p^2 divides $(a-b)N = a^p - b^p$.

3. *Let m be a positive integer such that $p = 1 + 4m$ is a prime. Show that the square of $(2m)!$ is congruent to -1 modulo p.*

We remark that modulo $p = 4m + 1$, we have

$$2 \equiv -(4m-1),\ 3 \equiv -(4m-2), \ldots, 2m \equiv -(2m+1).$$

It follows that $(2m)! \equiv -(2m+1)(2m+2)\cdots(4m-1)$ from which we obtain

$$((2m)!)^2 \equiv -[2\times 3\times\cdots\times(2m)]\cdot[(2m+1)(2m+2)\cdots(4m-2)(4m-1)] = -(p-2)!$$

Now, Wilson's theorem gives

$$-1 \equiv (p-1)! = (p-1)(p-2)! \equiv -(p-2)!$$

and we may conclude $((2m)!)^2 \equiv -1 \pmod p$.

From Focus On... No. 20

1. *Prove the identity*

$$vw(v-w)+wu(w-u)+uv(u-v)+(v-w)(w-u)(u-v)=0$$

where u, v, w are complex numbers and deduce another proof of Hayashi's inequality.

The identity is readily checked and supposing that u, v, w are distinct, we have

$$\frac{vw}{(w-u)(u-v)}+\frac{wu}{(v-w)(u-v)}+\frac{uv}{(v-w)(w-u)}=-1.$$

Using the triangle inequality, we obtain

$$\frac{|v||w|}{|w-u||u-v|}+\frac{|w||u|}{|v-w||u-v|}+\frac{|u||v|}{|v-w||w-u|}\geq 1. \qquad (1)$$

Taking P as the origin and u, v, w as the respective complex affixes of A, B, C, we have $|u|=PA$, $|v|=PB$, $|w|=PC$ and $|u-v|=AB$, $|v-w|=BC$, $|w-u|=AC$ and (1) then yields Hayashi's inequality:

$$\frac{PA\cdot PB}{CA\cdot CB}+\frac{PB\cdot PC}{AB\cdot AC}+\frac{PC\cdot PA}{BC\cdot BA}\geq 1.$$

2. *Using complex numbers, prove the identity*

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=(a^2b+b^2c+c^2a-abc)^2+(ab^2+bc^2+ca^2-abc)^2$$

for real numbers a, b, c. Deduce that if a, b, c are the sidelengths of a triangle, then

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)>(a^3+b^3+c^3)^2.$$

Clearly, we have $(b^2+c^2)(c^2+a^2)(a^2+b^2)=|(b+ic)(c+ia)(a+ib)|^2$. Expanding the product $p=(b+ic)(c+ia)(a+ib)$ gives

$$p=abc-ab^2-bc^2-ca^2+i(a^2b+b^2c+c^2a-abc)$$

and the identity is obtained from $(b^2+c^2)(c^2+a^2)(a^2+b^2)=|p|^2$.

Using the well-known $2(X^2+Y^2)\geq (X+Y)^2$, this gives

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)\geq (a^2b+b^2c+c^2a+ab^2+bc^2+ca^2-2abc)^2$$

or

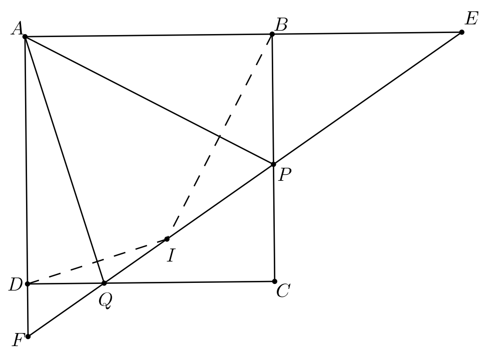
$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)\geq (a^3+b^3+c^3+(a+b-c)(b+c-a)(c+a-b))^2.$$

Since a, b, c are the side lengths of a triangle, the product $(a+b-c)(b+c-a)(c+a-b)$ is positive and the desired inequality follows.

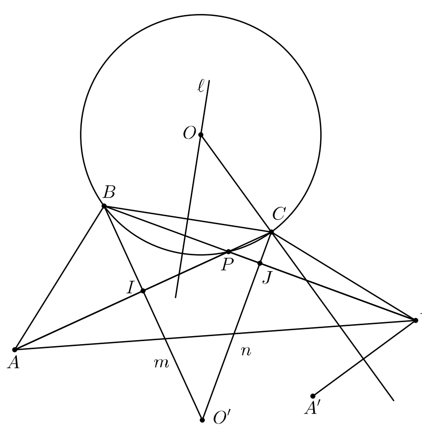
From Focus On... No. 21

1. *Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD, respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD, respectively. Prove that $AE + AF \geq 2\sqrt{2}a$.*

The transformation $\mathbf{R}_{AQ} \circ \mathbf{R}_{AP}$ is the right angle rotation $\boldsymbol{\rho}_A$ with centre A such that $\boldsymbol{\rho}_A(B) = D$. Let $I = \boldsymbol{R}_{AP}(B)$. Since $AB \perp BP$, we have $AI \perp IP$. In addition, $\boldsymbol{R}_{AQ}(I) = \boldsymbol{R}_{AQ} \circ \boldsymbol{R}_{AP}(B) = \boldsymbol{\rho}_A(B) = D$, hence $I = \boldsymbol{R}_{AQ}(D)$ and so $AI \perp IQ$ as well. As a result, I is on the line PQ and I is the foot of the altitude from A in the right-angled triangle AEF. Since the hypotenuse EF is twice the median from A, we see that $EF \geq 2AI = 2a$ and so $AI \cdot EF \geq 2a^2$. Observing that $AI \cdot EF = AE \cdot AF$, the arithmetic-geometric mean inequality then gives $AE + AF \geq 2\sqrt{AE \cdot AF} = 2\sqrt{2}a$, as required.



2. *Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$ and such that AD and BC are not parallel. Let P be the intersection of the diagonals AC and BD. If $AP : BD = DP : AC$, prove that $AB \perp CD$.*



Following the given hints, we introduce the perpendicular bisectors ℓ, m, n of BC, CA, BD, respectively, and the circumcentre O of ΔBPC. Note that A

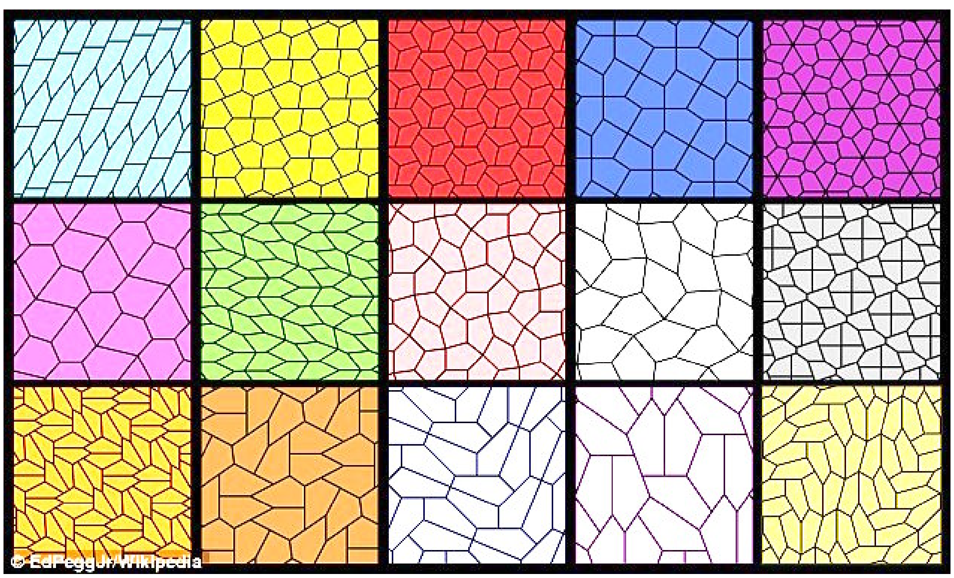
and D have the same power with respect to the circumcircle of ΔBPC (since $AP \cdot AC = DP \cdot DB$) and therefore $OA = OD$.

Since ℓ passes through O, $\boldsymbol{R}_{OC} \circ \boldsymbol{R}_\ell$ is a rotation $\boldsymbol{\rho}_O$ with centre O satisfying $\boldsymbol{\rho}_O(B) = \boldsymbol{R}_{OC}(C) = C$.

Let $A' = \boldsymbol{R}_\ell(A)$. Then $CA' = BA = CD$ and $OA' = OA = OD$ so that OC is the perpendicular bisector of DA'. It follows that $D = \boldsymbol{R}_{OC}(A') = \boldsymbol{\rho}_O(A)$. Since $\boldsymbol{\rho}_O(A) = D$ and $\boldsymbol{\rho}_O(B) = C$, the angle θ of $\boldsymbol{\rho}_O$ is $\theta = \angle(\overrightarrow{AB}, \overrightarrow{DC})$.

Similarly, since AC and BD are not parallel, m and n intersect and $\boldsymbol{R}_n \circ \boldsymbol{R}_m$ is a rotation $\boldsymbol{\rho}_{O'}$ with centre at the point O' of intersection of m and n. We have $\boldsymbol{\rho}_{O'}(A) = \boldsymbol{R}_n(C) = C$ and $\boldsymbol{\rho}_{O'}(B) = \boldsymbol{R}_n(B) = D$ so that the angle of $\boldsymbol{\rho}_{O'}$ is $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = \theta + \pi$.

Now, the midpoints I and J of AC and BD are on the circle with diameter $O'P$, hence $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = 2\angle(m, n) = 2\angle(O'I, O'J) = 2\angle(PI, PJ) = 2\angle(PC, PB) = \angle(\overrightarrow{OC}, \overrightarrow{OB}) = -\theta$. Thus, $\theta + \pi = -\theta$ so that θ is a right angle and $AB \perp DC$.



There are now 15 known convex pentagons that can tile the plane. See problem CC241 in this issue to learn more about the latest discovery in the world of tiling.

Selected Problems from the Early Years of the Moscow Mathematical Olympiad

Zhi Kin Loke

I am a trainer of the Malaysian National Team which competes in the International Mathematical Olympiad. I was a team member myself from 2006 to 2009, and was the first Malaysian to win a silver medal. I attended the 2016 IMO in Hong Kong as a Leader Observer.

In assembling training material, I turned to a place with a rich mathematical tradition, the former Soviet Union. I came across a file containing problems from this contest between 1935 and 1941 inclusive. The papers were incomplete as parts had been lost due to passage of time, so I treated them as isolated problems. The content, style and level of difficulty were certainly quite different from the IMO problems nowadays. Nevertheless, I found some of them useful for my purpose.

Below is a sample of ten Moscow Mathematical Olympiad problems. The solutions will be given in the next issue.

1. Solve the system of equations $x^2 + y^2 - 2z^2 = 2a^2,\ x + y + 2z = 4(a^2 + 1)$ and $z^2 - xy = a^2$ where a is a real constant. (1935)

2. Which is larger, 300! or 100^{300}? (1940)

3. (a) Find all possible values of a such that for all x and some integers b and c, $(x - a)(x - 10) + 1 = (x - b)(x - c)$.

   (b) Find all possible triples (a, b, c) of distinct non-zero integers such that $x(x-a)(x-b)(x-c)+1$ is the product of two non-constant polynomials with integer coefficients. (1941)

4. How many planes are equidistant from four given points not all in a plane? (1938)

5. Given a line and a circle, construct a unit circle tangent to both. How many solutions are there? (1940)

6. O is the circumcentre of triangle ABC. P, Q and R are the points symmetric to O about BC, CA and AB respectively. Construct ABC given only the points P, Q and R. (1940)

7. Construct a triangle given the points of intersection of its circumcircle with the extensions of the altitude, angle bisector and median from the same vertex. (1935)

8. Given two points A and B not on a line ℓ, construct a point P on ℓ such that $AP + BP = 1$. (1937)

9. Given three non-collinear points, construct three circles, each passing through two of them, such that every two circles intersect, and the tangents to the circles at each point of intersection are perpendicular to each other. (1937)

10. When an infinite circular cone is cut along a line through its vertex, its surface opens up into a circular sector. A straight line ℓ is drawn perpendicular to the bisector of the central angle which has measure θ. When the cone is reconstructed, determine the number of points of self-intersection of ℓ in terms of θ. (1940)

Historical Remarks on the first Moscow Mathematical Olympiad

In the spring of 1935, the Board of the Moscow Mathematical Society, following the example of Leningrad, decided to organize the first Moscow Mathematical Olympiad. The organizing committee included all professors of mathematics from Moscow University and was headed by P. S. Alexandrov, who was then the President of the Moscow Mathematical Society. The purpose of the Olympiad was to find the most talented students, to attract attention of the young people at large to some of the most important problems and methods of modern mathematics, and to show the students, at least partly, what Soviet mathematicians are working on, what progress they have made and what challenges they have.

In the preliminary round, 314 high school students participated, and 120 of them made it into the final round. Three students were awarded first prizes, five got second prizes and, in addition, 44 students received honorable mentions. A place at the top of the Olympiad determined for many their future scientific career. By the initiative of the famous mathematician A. N. Kolmogorov, the problems in the final round focused on the following three different mathematical abilities:

geometric,

computational and algorithmic,

combinatorial and logical.

The Moscow Mathematical Olympiad has been held every year since 1935 except for the war years 1942 to 1944.

Zhi Kin Loke
Selangor
Malaysia

PROBLEMS

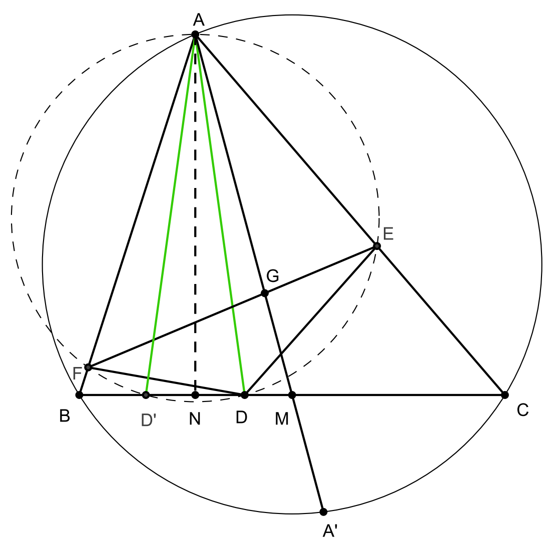
Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2017**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4181. *Proposed by Marius Stănean.*

Let $D \in BC$ be the foot of the A-symmedian of triangle ABC with centroid G (where the A-symmedian is the reflection of the median at A in the bisector of angle A). The circle passing through A, D and tangent to the line parallel to BC passing through A intersects sides AB and AC at E and F, respectively. If $3AD^2 = AB^2 + AC^2$, prove that G lies on EF.



4182. *Proposed by Michel Bataille.*

Let F_m denote the mth Fibonacci number (defined by $F_0 = 0, F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all integers $m \geq 0$) and let n be a positive integer. For $k = 1, 2, \ldots, n$, let

$$U_k = \frac{k}{F_{n+1-k}F_{n+3-k}} + (-1)^{k+1}\frac{2F_k}{F_{k+2}}.$$

Prove that $|U_1 + U_2 + \cdots + U_n - n|$ is the quotient of two Fibonacci numbers.

4183. *Proposed by Lorian Saceanu and Leonard Giugiuc.*

Let ABC be a non obtuse triangle with orthocenter H and circumradius R. Prove that
$$(3\sqrt{3}-4)\cdot AH\cdot BH\cdot CH \geq abc - 4R^3$$
and determine when the equality holds.

4184. *Proposed by Mihaela Berindeanu.*

Evaluate the following integral
$$\int_{32}^{63} \frac{\ln 2016x}{x^2+2016}\mathrm{d}x.$$

4185. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

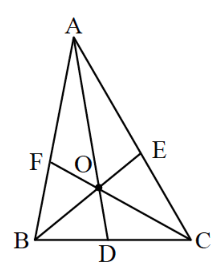
Prove that for any positive real numbers a, b, c and k, we have
$$\left[\frac{a^{k-1}(a^2+bc)}{(b+c)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{b^{k-1}(b^2+ca)}{(c+a)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{c^{k-1}(c^2+ab)}{(a+b)^{k+1}}\right]^{\frac{1}{k}} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

4186. *Proposed by Florin Stanescu.*

Let $f, g : [0,1] \to [0,\infty)$, $f(0) = g(0) = 0$ be two continuous functions such that f is convex and g is concave. If $h : [0,1] \to \mathbb{R}$ is an increasing function, show that
$$\int_0^1 g(x)h(x)dx \cdot \int_0^1 f(x)dx \leq \int_0^1 g(x)dx \cdot \int_0^1 h(x)f(x)dx.$$

4187. *Proposed by Avi Sigler and Moshe Stupel.*

A point P inside triangle ABC divides the three cevians AD, BE, CF through P into segments whose harmonic means are
$$K_A = \frac{2AP\cdot PD}{AP+PD},\quad K_B = \frac{2BP\cdot PE}{BP+PE},\quad K_C = \frac{2CP\cdot PF}{CP+PF}.$$
Prove that these three harmonic means, each associated with a cevian, are proportional to the sines of the angles $\angle CPE, \angle EPA, \angle APF$ formed between the other two cevians.



4188. *Proposed by Daniel Sitaru.*

Let $0 < x < y < z < \frac{\pi}{2}$. Prove that

$$(x + y)\sin z + (x - z)\sin y < (y + z)\sin x.$$

4189. *Proposed by Mihaela Berindeanu.*

Prove that the equation

$$3y^2 = -2x^2 - 2z^2 + 5xy + 5yz - 4xz + 1$$

has infinitely many solutions in integers.

4190. *Proposed by Leonard Giugiuc.*

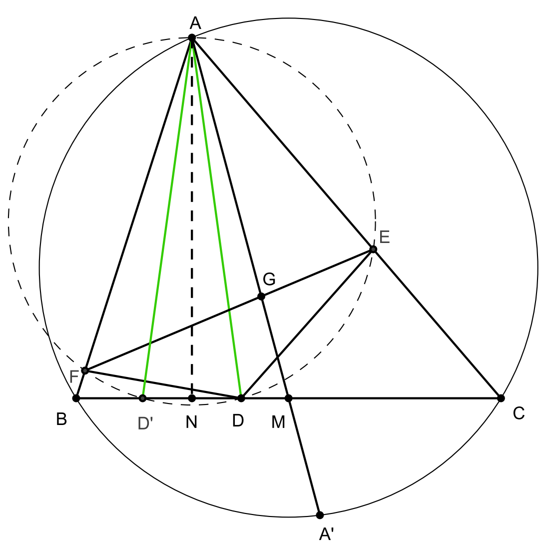
Let a, b, c, d and e be real numbers such that $a + b + c + d + e = 20$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 100$. Prove that

$$625 \leq abcd + abce + abde + acde + bcde \leq 945.$$

. . . .

4181. *Proposé par Marius Stănean.*

Soit $D \in BC$ le pied de la symédiane-A du triangle ABC avec centroïde G (où cette symédiane-A est la réflexion de la médiane à A avec la bissectrice de l'angle A). Le cercle, passant par A et D, puis tangent à la ligne parallèle à BC et passant par A, intersecte les côtés AB et AC en E et F, respectivement. Si $3AD^2 = AB^2 + AC^2$, démontrer que G se situe sur EF.



4182. *Proposé par Michel Bataille.*

Soit F_m le $m^{\text{ième}}$ nombre de Fibonacci (définis par $F_0 = 0, F_1 = 1$ et $F_{m+2} = F_{m+1} + F_m$ pour tout entier $m \geq 0$) et soit n un entier positif. Pour $k = 1, 2, \ldots, n$, soit

$$U_k = \frac{k}{F_{n+1-k}F_{n+3-k}} + (-1)^{k+1}\frac{2F_k}{F_{k+2}}.$$

Démontrer que $|U_1 + U_2 + \cdots + U_n - n|$ est le quotient de deux nombres de Fibonacci.

4183. *Proposé par Lorian Saceanu et Leonard Giugiuc.*

Soit ABC un triangle non obtus avec orthocentre H, où R dénote le rayon du cercle circonscrit. Démontrer que

$$(3\sqrt{3} - 4) \cdot AH \cdot BH \cdot CH \geq abc - 4R^3$$

et déterminer quand l'égalité tient.

4184. *Proposé par Mihaela Berindeanu.*

Évaluer l'intégrale

$$\int_{32}^{63} \frac{\ln 2016x}{x^2 + 2016} \mathrm{d}x.$$

4185. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Démontrer que pour tous nombres réels positifs a, b, c et k, l'inégalité suivante tient

$$\left[\frac{a^{k-1}(a^2+bc)}{(b+c)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{b^{k-1}(b^2+ca)}{(c+a)^{k+1}}\right]^{\frac{1}{k}} + \left[\frac{c^{k-1}(c^2+ab)}{(a+b)^{k+1}}\right]^{\frac{1}{k}} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

4186. *Proposé par Florin Stanescu.*

Soient $f, g : [0,1] \to [0, \infty)$, $f(0) = g(0) = 0$ deux fonctions continues telles que f est convexe et g est concave. Si $h : [0,1] \to \mathbb{R}$ est une fonction croissante, démontrer que

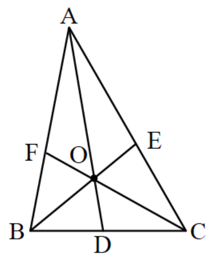
$$\int_0^1 g(x)h(x)dx \cdot \int_0^1 f(x)dx \leq \int_0^1 g(x)dx \cdot \int_0^1 h(x)f(x)dx.$$

4187. *Proposé par Avi Sigler et Moshe Stupel.*

Un point P à l'intérieur d'un triangle ABC divise les trois céviennes passant par P, AD, BE et CF, en segments dont les moyennes harmoniques sont

$$K_A = \frac{2AP \cdot PD}{AP + PD}, \quad K_B = \frac{2BP \cdot PE}{BP + PE}, \quad K_C = \frac{2CP \cdot PF}{CP + PF}.$$

Démontrer que ces trois moyennes harmoniques, chacune associé à une cévienne, sont proportionnelles aux sinus des angles $\angle CPE, \angle EPA, \angle APF$, formés par les deux autres céviennes.



4188. *Proposé par Daniel Sitaru.*

Soit $0 < x < y < z < \frac{\pi}{2}$. Démontrer que

$$(x+y)\sin z + (x-z)\sin y < (y+z)\sin x.$$

4189. *Proposé par Mihaela Berindeanu.*

Démontrer que l'équation

$$3y^2 = -2x^2 - 2z^2 + 5xy + 5yz - 4xz + 1$$

possède un nombre infini de solutions entières.

4190. *Proposé par Leonard Giugiuc.*

Soient a, b, c, d et e des nombres réels tels que $a+b+c+d+e = 20$ et $a^2+b^2+c^2+d^2+e^2 = 100$. Démontrer que

$$625 \leq abcd + abce + abde + acde + bcde \leq 945.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(9), p. 397–400.

4081. *Proposed by Daniel Sitaru.*

Determine all $A, B \in M_2(\mathbb{R})$ such that:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix}, \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix}. \end{cases}$$

We received 17 correct solutions and will feature the solution by Joseph DiMuro.

Summing the two equations, we obtain:

$$(A+B)^2 = A^2 + AB + BA + B^2 = \begin{pmatrix} 32 & 64 \\ 16 & 32 \end{pmatrix}.$$

We can diagonalize this matrix in order to find its square roots:

$$(A+B)^2 = PDP^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix},$$

$$A+B = PD^{1/2}P^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 8 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix} = \pm \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}.$$

We can also subtract the original two equations to obtain:

$$(A-B)^2 = A^2 - AB - BA + B^2 = \begin{pmatrix} 12 & 24 \\ 12 & 24 \end{pmatrix}.$$

As before, we diagonalize this matrix:

$$(A-B)^2 = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix},$$

$$A-B = PD^{1/2}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} = \pm \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

Now we have the two equations

$$A+B=\pm\begin{pmatrix}4 & 8\\ 2 & 4\end{pmatrix}, A-B=\pm\begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix},$$

which can easily be solved to produce four possible pairs of matrices for A and B. One solution is

$$A=\begin{pmatrix}3 & 6\\ 2 & 4\end{pmatrix}, B=\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}.$$

The other solutions may be obtained by interchanging A and B, and/or replacing A and B with their negatives.

4082. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a right-angle triangle with $\angle A = 90^\circ$ and $BC = a$, $AC = b$ and $AB = c$. Consider the Fibonacci sequence F_n with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all non-negative integers n. Prove that

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq\frac{3}{2a^2}$$

for all non-negative integers m, n, p.

We received 8 correct solutions and present the solution by Adnan Ali.

From the Cauchy-Schwarz Inequality,
$(b^2+c^2)(F_k^2+F_\ell^2) = a^2(F_k^2+F_\ell^2) \geq (bF_k+cF_\ell)^2$, for all $k, \ell \geq 0$. Thus,

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq$$
$$\frac{F_m^2}{a^2(F_n^2+F_p^2)}+\frac{F_n^2}{a^2(F_p^2+F_m^2)}+\frac{F_p^2}{a^2(F_m+F_n^2)}\geq\frac{3}{2a^2},$$

where the last inequality follows from Nesbitt's Inequality. Equality holds iff $F_m = F_n = F_p$ and $b = c$.

Editor's Comments. As solvers pointed out, the fact that the F_n's were Fibonacci numbers was irrelevant; it was only necessary that they were nonnegative.

4083. *Proposed by Ovidiu Furdui.*

Calculate

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\int_0^n\frac{x}{1+n\cos^2 x}\mathrm{d}x.$$

We received 10 solutions, of which 6 were correct and complete. We present the solution by Michel Bataille.

We show that the required limit is $\dfrac{1}{2}$.

Let $f_n(x) = \frac{1}{1+n\cos^2 x}$.

The π-periodicity of f_n and the change of variables $x = \tan^{-1}(t),\ dx = \frac{dt}{1+t^2}$ easily yield

$$\int_{(2k-1)\pi/2}^{(2k+1)\pi/2} f_n(x)\,dx = \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \int_{-\infty}^{\infty} \frac{dt}{n+1+t^2} = \frac{\pi}{\sqrt{n+1}}$$

for any $k, n \in \mathbb{N}$.

This said, for every $n \in \mathbb{N}$ with $n \geq 2$, let $p_n = \left\lfloor \frac{n}{\pi} + \frac{1}{2} \right\rfloor$ and $I_n = \int_0^n x f_n(x)\,dx$. Then, $(2p_n - 1)\frac{\pi}{2} \leq n < (2p_n + 1)\frac{\pi}{2}$ and

$$I_n = \int_0^{\pi/2} x f_n(x)\,dx + \sum_{k=1}^{p_n-1} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx + \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx.$$

Clearly,

$$0 \leq \int_0^{\pi/2} x f_n(x)\,dx \leq \frac{\pi}{2} \cdot \frac{1}{2} \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \frac{\pi^2}{4\sqrt{n+1}}$$

and for $k \in \{1, 2, \ldots, p_n - 1\}$,

$$(2k-1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}} \leq \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx \leq (2k+1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}}.$$

Similarly,

$$0 \leq \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx \leq n \int_{(2p_n-1)\frac{\pi}{2}}^{n} f_n(x)\,dx \leq \frac{n\pi}{\sqrt{n+1}}.$$

Thus,

$$\frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k-1) \leq I_n \leq \frac{\pi^2}{4\sqrt{n+1}} + \frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k+1) + \frac{n\pi}{\sqrt{n+1}}$$

so that

$$\frac{\pi^2(p_n-1)^2}{2\sqrt{n+1}} \leq I_n \leq \frac{\pi}{\sqrt{n+1}} \left(\frac{\pi}{4} + \frac{\pi}{2} \cdot p_n^2 + n \right) = \frac{\pi p_n^2}{\sqrt{n+1}} \left(\frac{\pi}{2} + \frac{\pi}{4p_n^2} + \frac{n}{p_n^2} \right).$$

Since $p_n \sim \frac{n}{\pi}$ as $n \to \infty$, we obtain

$$I_n \sim \frac{\pi^2 p_n^2}{2\sqrt{n+1}} \sim \frac{n\sqrt{n}}{2}$$

as $n \to \infty$. The result follows.

4084. *Proposed by Michel Bataille.*

In the plane, let Γ be a circle and A, B be two points not on Γ. Determine when $\frac{MA}{MB}$ is not independent of M on Γ and, in these cases, construct with ruler and compass I and S on Γ such that

$$\frac{IA}{IB} = \inf\left\{\frac{MA}{MB} : M \in \Gamma\right\} \quad \text{and} \quad \frac{SA}{SB} = \sup\left\{\frac{MA}{MB} : M \in \Gamma\right\}.$$

We feature the proposer's solution; we received no others.

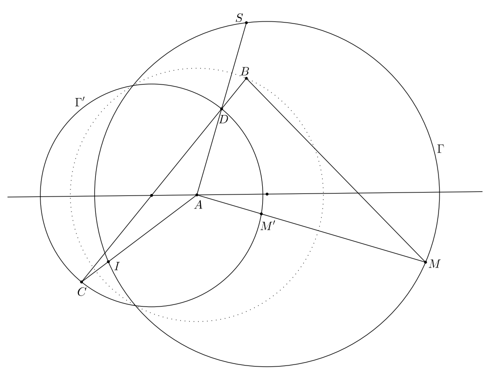
Because A is not on Γ, inversion in the circle with centre A and radius AB takes Γ to another circle, call it Γ'. For any point M on Γ, this inversion takes the pair of points M, B to another pair M', B, whose distances satisfy

$$MB = \frac{AB^2 \cdot M'B}{AM' \cdot AB} = \frac{AM \cdot AM' \cdot M'B}{AM' \cdot AB} = MA\frac{M'B}{AB};$$

consequently,

$$\frac{MA}{MB} = \frac{AB}{BM'}. \tag{1}$$

From (1), $\frac{MA}{MB}$ is independent of M on Γ if and only if BM' is constant. This occurs if and only if B is the centre of Γ'; that is, if and only if A and B are an inverse pair with respect to Γ. Otherwise, let the diameter of Γ' through B intersect Γ' at C and D with $BC > BD$. Then $\frac{MA}{MB}$ is minimal when BM' is maximal; that is, when $M' = C$; $\frac{MA}{MB}$ is maximal when BM' is minimal, in which case $M' = D$. Thus, I coincides with C' (the image of C under our inversion), and S coincides with D'. The construction of I and S is immediate once Γ' has been drawn. The circle Γ' can be readily constructed from the inverses of three points of Γ (as in the figure):



Editor's Comments. For any two fixed points A and B, the locus of points M for which $\frac{MA}{MB}$ is constant is called *the circle of Apollonius*; inversion in that circle interchanges A and B. See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (Mathematical Association of America, 1967), exercise 5.4.1, pages 114 and 172. Also, Theorem 5.41 there provides the distance formula used above to obtain (1).

4085. *Proposed by José Luis Díaz-Barrero. Correction.*

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

We received eight submissions, six of which are correct. We present the solution by Titu Zvonaru.

It is well known that $\cos A+\cos B+\cos C\leq\frac{3}{2}$ [Item 2.16 on p.22 of the book *Geometric Inequalities* by O. Bottema et al; Groningen, 1969]. Using this, together with the facts that $\sin x<x$ for $0<x<\frac{\pi}{2}$, $xy+yz+zx\leq x^2+y^2+z^2$, and $(x+y+z)^2\leq 3(x^2+y^2+z^2)$ we then have

$$\begin{aligned}\sum_{\text{cyc}}\sqrt[4]{\sin(\cos A)\cdot\cos B}&<\sum_{\text{cyc}}\sqrt[4]{\cos A\cdot\cos B}\leq\sum_{\text{cyc}}\sqrt{\cos A}\\&\leq\sqrt{3(\cos A+\cos B+\cos C)}\leq\sqrt{3(\frac{3}{2})}=\frac{3\sqrt{3}}{2}.\end{aligned}$$

Editor's comments. Arkady Alt proved the stronger result that the given upper bound could be replaced by $3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$ which is less than $\frac{3\sqrt{3}}{2}$ since $\sin\frac{1}{2}<\frac{1}{2}$. This new upper bound is attained if and only if the triangle is equilateral. His proof used the Cauchy-Schwarz Inequality, concavity of the functions $\sqrt{\sin x}$ and $\sqrt{\cos x}$, Jensen's Inequality as well as the fact that $\sum\cos A=1+\frac{r}{R}$ and the Euler's Inequality $2r\leq R$.

4086. *Proposed by Daniel Sitaru.*

Let be $f:[0,1]\to\mathbb{R}$; f twice differentiable on $[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$. Prove that

$$25\int_{\frac{1}{5}}^{1}f(x)dx\geq 16\int_{0}^{1}f(x)dx+4f(1).$$

We received seven solutions and present two of them.

Solution 1, by AN-anduud Problem Solving Group.

From the given conditions, f is concave on $[0,1]$. Using Hermite-Hadamard's inequality we get

$$\begin{aligned}16\int_{\frac{1}{5}}^{1}f(x)dx+9\int_{\frac{1}{5}}^{1}f(x)dx&\geq 16\cdot\int_{\frac{1}{5}}^{1}f(x)dx+9\cdot\frac{1-\frac{1}{5}}{2}\cdot\left(f(1)+f\left(\frac{1}{5}\right)\right)\\&=16\int_{\frac{1}{5}}^{1}f(x)dx+\frac{18}{5}f(1)+\frac{18}{5}f\left(\frac{1}{5}\right).\end{aligned}$$

On the other hand, we have

$$f\left(\frac{1}{5}\right)=f\left(\frac{1}{9}\cdot 1+\frac{8}{9}\cdot\frac{1}{10}\right)\geq\frac{1}{9}f(1)+\frac{8}{9}\cdot f\left(\frac{1}{10}\right),$$

so
$$\frac{18}{5} f\left(\frac{1}{5}\right) \geq \frac{2}{5} f(1) + \frac{16}{5} f\left(\frac{1}{10}\right).$$

From here, we get
$$25 \int_{\frac{1}{5}}^{1} f(x)dx \geq 16 \int_{\frac{1}{5}}^{1} f(x)dx + 4f(1) + \frac{16}{5} f\left(\frac{1}{10}\right).$$

Using Hermite-Hadamard's inequality, we get
$$f\left(\frac{1}{10}\right) = f\left(\frac{\frac{1}{5}+0}{2}\right) \geq \frac{1}{\frac{1}{5}-0} \int_0^{\frac{1}{5}} f(x)dx \iff \frac{1}{5} f\left(\frac{1}{10}\right) \geq \int_0^{\frac{1}{5}} f(x)dx.$$

Hence, we get
$$25 \int_{\frac{1}{5}}^{1} f(x)dx \geq 16 \int_0^{1} f(x)dx + 4f(1).$$

Solution 2, by Leonard Giugiuc.

In $\displaystyle\int_{\frac{1}{5}}^{1} f(x)\ dx$, we make the substitution $x \to \dfrac{5x-1}{4}$ and clear fractions to get
$$25 \int_{\frac{1}{5}}^{1} f(x)\ dx = 20 \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx.$$

We need to prove
$$\begin{aligned} 20 \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx \geq 16 \int_0^1 f(x)\ dx + 4f(1) &\iff \\ \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx \geq \frac{4}{5} \int_0^1 f(x)\ dx + \frac{1}{5} f(1) &\iff \\ \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx \geq \int_0^1 \left[\frac{4}{5} f(x) + \frac{1}{5} f(1)\right]\ dx.& \end{aligned}$$

But $f''(x) < 0\ \forall x \in [0,1]$, so f is concave on $[0,1]$ and from here
$$f\left(\frac{4x+1}{5}\right) \geq \frac{4}{5} f(x) + \frac{1}{5} f(1).$$

Integrating, we conclude that
$$\int_0^1 f\left(\frac{4x+1}{5}\right)\ dx \geq \int_0^1 \left[\frac{4}{5} f(x) + \frac{1}{5} f(1)\right]\ dx.$$

Editor's Comments. Henry Ricardo observed that this problem appears as problem MA 110 (with solution) in the Daniel Sitaru's book *Math Phenomenon*, published in English by the Romanian publisher Editura Paralela 45 in 2016.

4087. *Proposed by Lorian Saceanu.*

If S is the area of triangle ABC, prove that

$$m_a(b+c)+2m_a^2 \geq 4S\sin A,$$

where b and c are the lengths of sides that meet in vertex A, and m_a is the length of the median from that vertex; furthermore, equality holds if and only if $b=c$ and $\angle A = 120^\circ$.

We received seven correct submissions and present the solution by Leonard Giugiuc.

Let A' be the reflection of A in the midpoint M of BC. Because $\Delta AMC \cong \Delta A'MB$, we have

$$AA' = 2m_a,\ A'B = b,\ \angle A'BA = \pi - A, \text{ and } [A'AB] = [ABC] = S$$

(where the square brackets denote area). Let $m = 2m_a$ and denote by r', R', and s' $(= \frac{m+b+c}{2})$ the inradius, circumradius, and semiperimeter, respectively, of $\Delta A'AB$. We need to prove that

$$m(b+c)+m^2 \geq 8S\sin\angle A'BA,$$

which is equivalent, in turn, to

$$\begin{aligned} m(m+b+c) &\geq 8S\sin(\pi - A) \\ \frac{2mS}{r'} &\geq 8S\sin A \\ \frac{m}{\sin A} &\geq 4r' \\ R' &\geq 2r'. \end{aligned}$$

But the final line is Euler's inequality applied to $\Delta A'AB$, which completes the proof. Equality holds for Euler's inequality if and only if $\Delta A'AB$ is equilateral, which implies that $b=c$ and $\angle A = 120^\circ$, as desired.

4088. *Proposed by Ardak Mirzakhmedov.*

Let a, b and c be positive real numbers such that $a^2b+b^2c+c^2a+a^2b^2c^2 = 4$. Prove that

$$a^2+b^2+c^2+abc(a+b+c) \geq 2(ab+bc+ca).$$

We received four submissions all of which are correct. We present the solution by the proposer, expanded by the editor with some details.

We first show that the given condition implies

$$\frac{a}{2a+bc^2}+\frac{b}{2b+ca^2}+\frac{c}{2c+ab^2} = 1 \tag{1}$$

or

$$\begin{aligned} a(2b+ca^2)(2c+ab^2)+b(2c+ab^2)(2a+bc^2)+c(2a+bc^2)(2b+ca^2) \\ = (2a+bc^2)(2b+ca^2)(2c+ab^2). \end{aligned} \tag{2}$$

Let S and P denote the left side and the right side of (2), respectively. Then by straightforward computations, we find

$$\begin{aligned} S &= \sum_{\text{cyc}} a(4bc + 2ab^3 + 2c^2a^2 + a^3b^2c) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(a^3b + b^3c + c^3a) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(4 - a^2b^2c^2) \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3 \end{aligned}$$

and

$$\begin{aligned} P &= (4ab + 2ca^3 + 2b^2c^2 + a^2bc^3)(2c + ab^2) \\ &= 8abc + 4(c^2a^3 + a^2b^3 + b^2c^3) + 2abc(a^3b + b^3c + c^3a) + a^3b^3c^3 \\ &= 8abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + 2abc(4 - a^2b^2c^2) + a^3b^3c^3 \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3. \end{aligned}$$

Hence, $S = P$ which establishes (2).

Now, for all $u, v, w > 0$, we have by the Cauchy-Schwarz Inequality that

$$(u + v + w)\left(\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w}\right) \geq (a + b + c)^2. \tag{3}$$

Setting

$$u = 2a^2 + bc^2a, \quad v = 2b^2 + ca^2b \quad \text{and} \quad w = 2c^2 + ab^2c,$$

we then have by (1) and (3) that

$$1 = \frac{a^2}{2a^2 + bc^2a} + \frac{b^2}{2b^2 + ca^2b} + \frac{c^2}{2c^2 + ab^2c} \geq \frac{(a + b + c)^2}{u + v + w},$$

so

$$(2a^2 + bc^2a) + (2b^2 + ca^2b) + (2c^2 + ab^2c) = u + v + w \geq (a + b + c)^2$$

from which it follows that

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq 2(ab + bc + ca).$$

4089. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a, b, c and d be real numbers with $0 < a < b < c < d$. Prove that

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > 3 + \ln\frac{d}{a}.$$

There were 14 correct solutions. We present four of them here. Most of the solvers approached the problem along the lines of one of the first two solutions.

Solution 1.

Since $x > 1 + \ln x$ for $x \neq 1$,

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > \left(1 + \ln\frac{b}{a}\right) + \left(1 + \ln\frac{c}{b}\right) + \left(1 + \ln\frac{d}{c}\right) = 3 + \ln\frac{d}{a}$$

as desired.

Solution 2.

Applying the AM-GM Inequality, we find that the left side is not less than

$$3\sqrt[3]{\frac{d}{a}} > 3\left(1 + \ln\sqrt[3]{\frac{d}{a}}\right) = 3 + \ln\frac{d}{a}.$$

Solution 3, by Kee-Wai Lau.

For $0 < a < b < c < d$, let

$$\begin{aligned} f(a,b,c,d) &= \frac{b}{a} + \frac{c}{b} + \frac{d}{c} - \ln\frac{d}{a}, \\ g(a,b,c) &= \frac{b}{a} + \frac{c}{b} + 1 - \ln\frac{c}{a}, \\ h(a,b) &= \frac{b}{a} + 2 - \ln\frac{b}{a}. \end{aligned}$$

An analysis of the partial derivatives reveals that each of its functions strictly increases in its final variable, so that

$$f(a,b,c,d) > f(a,b,c,c) = g(a,b,c) > g(a,b,b) = h(a,b) > h(a,a) = 3,$$

which yields the desired result.

Solution 4, by the proposers.

Let $f(x) = 1/x$. A diagram shows that

$$(b-a)f(a) + (c-b)f(b) + (d-c)f(c) > \int_a^d \frac{dx}{x},$$

whence

$$\frac{b}{a} - 1 + \frac{c}{b} - 1 + \frac{d}{c} - 1 > \ln d - \ln a,$$

as desired.

Editor's Comments. Two solvers provided a straightforward generalization for an increasing sequence $\{a_k\}$ of $n+1$ positive reals:

$$\sum_{k=1}^{n} \frac{a_{k+1}}{a_k} > n + \ln\frac{a_{n+1}}{a_1}.$$

4090. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a, b and c be non-negative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{3}{8}.$$

We received two correct solutions. We present the solution of the proposers, slightly modified by the editor.

Using Jensen's inequality for $f(x)=\frac{1}{x}$ (which is convex on $(0,\infty)$), we have

$$\frac{a}{a+b+c}\cdot\frac{1}{3b^2+6c-bc}+\frac{b}{a+b+c}\cdot\frac{1}{3c^2+6a-ca}+\frac{c}{a+b+c}\cdot\frac{1}{3a^2+6b-ab}$$
$$\geq\left(\frac{a(3b^2+6c-bc)}{a+b+c}+\frac{b(3c^2+6a-ca)}{a+b+c}+\frac{c(3a^2+6b-ab)}{a+b+c}\right)^{-1},$$

which we can rearrange to

$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}.$$

In order to prove the inequality given in the question, it thus suffices to show

$$\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}\geq\frac{3}{8},$$

which holds (by cross multiplying and rearranging) if and only if

$$8(a+b+c)^2\geq 9(ab^2+bc^2+ca^2)+18(ab+bc+ca)-9abc\iff$$
$$8(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)+2(ab+bc+ca)-9abc.$$

By the Cauchy-Schwarz inequality, $ab+bc+ca\leq a^2+b^2+c^2$. Note for later that equality holds if and only if $a=b=c=1$. Hence, it suffices to show that

$$6(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)-9abc.$$

Finally, since $a^2+b^2+c^2=3$, this reduces to

$$2\geq(ab^2+bc^2+ca^2)-abc. \tag{1}$$

Assume that $a\geq b\geq c$. Then $(a-b)(b-c)\geq 0$, equivalent to $ab+bc\geq b^2+ac$. Multiply both sides by $a>0$ and rearrange to get $abc\geq ab^2+a^2c-a^2b$. Note that the cubic $g(b)=3b-b^3$ has a local maximum at $b=1$, and in fact for all $b\geq 0$ we have $3b-b^3\leq g(1)=2$. Hence

$$abc+2\geq ab^2+a^2c-a^2b+3b-b^3,$$

which is equivalent to

$$abc+2\geq ab^2+a^2c+c^2b-b(a^2+b^2+c^2)+3b.$$

Since $a^2+b^2+c^2=3$, this shows that $abc+2\geq ab^2+a^2c+c^2b$, which is equivalent to (1), concluding the proof.

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