

# Mathematicorum

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# CRUX

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## Mathematicorum

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**Problem proposals, solutions and short notes intended for publication should be sent to the Editor:**

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THE OLYMPIAD CORNER  
No. 108  
R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

We begin this month with eleven of the problems proposed to the jury, but not used, at the 30th I.M.O. at Braunschweig, West Germany. Thanks go to Professor Bruce Shawyer, Memorial University, who hunted out his copy and sent it to me by courier so that this month's column could appear! Send me your nice solutions.

1. *Proposed by Australia.*

Ali Baba the carpet merchant has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. If the two rooms have dimensions of 38 ft by 55 ft, and 50 ft by 55 ft, what are the carpet's dimensions?

2. *Proposed by Bulgaria.*

Prove that for every integer  $n > 1$ , the equation

$$\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + \frac{x^2}{2!} + \frac{x}{1!} + 1 = 0$$

has no rational roots.

3. *Proposed by Colombia.*

Find the roots  $r_i$  of the polynomial

$$p(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$$

satisfying  $(r_1)^{16} + (r_2)^{16} + \cdots + (r_n)^{16} = n$ .

4. *Proposed by Czechoslovakia.*

The circumcircle  $\kappa$  of triangle  $ABC$  has radius  $R$ . The bisectors of the angles of the triangle intersect the circle  $\kappa$  again in the points  $A'$ ,  $B'$ ,  $C'$ . If  $P$  (respectively  $Q$ ) is the area of triangle  $ABC$  (respectively  $A'B'C'$ ) prove the inequality

$$16Q^3 \geq 27R^4P.$$

5. *Proposed by Finland.*

Show that any two points lying inside a regular  $n$ -gon  $E$  can be joined by two circular arcs lying inside  $E$  and meeting at an angle of at least  $\left(1 - \frac{2}{n}\right)\pi$ .

6. *Proposed by Greece.*

Let  $g: \mathbb{C} \rightarrow \mathbb{C}$ ,  $\omega \in \mathbb{C}$ ,  $a \in \mathbb{C}$ , with  $\omega^3 = 1$  and  $\omega \neq 1$ . Show that there is one and only one function  $f: \mathbb{C} \rightarrow \mathbb{C}$  such that

$$f(z) + f(\omega z + a) = g(z), \quad z \in \mathbb{C}.$$

Find the function  $f$ .

7. *Proposed by Hungary.*

Define the sequence  $\{a_n\}_{n=1}^{\infty}$  of integers by

$$\sum_{d|n} a_d = 2^n.$$

Show that  $n|a_n$ . [Editor's note: Of course  $x|y$  means that  $x$  divides  $y$ .]

8. *Proposed by India.*

Let  $Q$  be a quadrilateral the incircle and circumcircle of which exist. Show that the centers of these two circles are collinear with the point of intersection of the diagonals.

9. *Proposed by Ireland.*

Let  $a, b, c, d, m, n$  be positive integers such that

$$a^2 + b^2 + c^2 + d^2 = 1989,$$

$$a + b + c + d = m^2,$$

and the largest of  $a, b, c, d$  is  $n^2$ . Determine, with proof, the values of  $m$  and  $n$ .

10. *Proposed by Israel.*

The set  $\{a_0, a_1, \dots, a_n\}$  of real numbers satisfies the following conditions:

(i)  $a_0 = a_n = 0$ ;

(ii) for  $1 \leq k \leq n - 1$ ,

$$a_k = c + \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1}).$$

Prove that  $c \leq \frac{1}{4n}$ .

11. *Proposed by Mongolia.*

Seven points are given in the plane. They are to be joined by a minimal number of segments such that at least two of any three points are joined. How many segments has such a figure? Give an example.

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We turn now to the solutions submitted for problems posed in the January 1988 number of the Corner. These problems were from the *Kürschak Competitions*, Hungary, 1982–1986.

### 1982.2 [1988: 1]

Prove that for any integer  $k > 2$ , there exist infinitely many positive integers  $n$  such that the least common multiple of

$$n, n+1, n+2, \dots, n+k-1$$

is greater than the least common multiple of

$$n+1, n+2, \dots, n+k.$$

*Solution by John Morvay, Dallas, Texas.*

Fix  $k > 2$ . Denote the least common multiple of the set  $\{n, n+1, \dots, n+k-1\}$  by  $M_n$ . Now  $M_n$  is clearly the product of the highest powers of primes dividing one of  $n, n+1, \dots, n+k-1$ . Set  $r_n$  to be the product of all prime powers of the form  $p^{\alpha-\beta}$ , where  $p^\alpha$  is the highest power of  $p$  in  $M_{n+1}$  and  $p^\beta$  is the highest power in  $M_n$ , and  $\alpha > \beta$ . Similarly, define  $l_n$  as the product of prime powers dividing  $M_n$  but not  $M_{n+1}$ . Then

$$M_{n+1} = M_n \cdot \frac{r_n}{l_n}.$$

Now let  $m$  be the largest nonnegative integer such that  $2^m$  divides  $k$  and let  $k = 2^m k'$ , where  $k'$  is odd. There are two cases.

*Case 1:*  $k' > 1$ . Let  $p$  be any prime number greater than  $k$ . Set  $n = 2^m p$ . Observe that  $l_n = p$ . Also  $n+k = 2^m(p+k')$ , so  $n+k$  is divisible by  $2^{m+1}$ . Also  $2^{m+1}$  divides  $2^m p + 2^m \cdot 1$ , and  $1 \leq 2^m \cdot 1 < k$ . Hence

$$r_n \leq \frac{n+k}{2^{m+1}} = \frac{p+k'}{2},$$

so

$$\frac{M_{n+1}}{M_n} = \frac{r_n}{l_n} \leq \frac{p+k'}{2p} < 1.$$

*Case 2:*  $k' = 1$ . Then  $k = 2^m$ , where  $m \geq 2$ . Now let  $l$  be an odd integer greater than 1. Set  $n = 2^{l+m}$ . Then  $n+k = 2^m(2^l+1)$  and it is easy to check that  $2^l$  divides  $l_n$ . Since  $l$  is odd,  $2^l+1$  is divisible by 3. As  $k \geq 4$  the list  $n, n+1, \dots, n+k-1$  contains at least one number divisible by 3. Also  $2^m$  divides  $n$ , and so

$$r_n \leq \frac{n+k}{3 \cdot 2^m} = \frac{2^l+1}{3}.$$

Thus

$$\frac{M_{n+1}}{M_n} = \frac{r_n}{l_n} \leq \frac{2^l + 1}{2^l \cdot 3} < 1.$$

In either case we see that there are infinitely many  $n$  for which  $M_{n+1} < M_n$ .

### 1982.3 [1988: 1]

The set of integers is coloured with 100 colours in such a way that all the colours are used and the following is true: for any choice of intervals  $[a,b]$  and  $[c,d]$  of equal length and with integral endpoints, if  $a$  and  $c$  have the same colour and  $b$  and  $d$  have the same colour, then the intervals  $[a,b]$  and  $[c,d]$  are identically coloured, in that for any integer  $x$ ,  $0 \leq x \leq b-a$ , the numbers  $a+x$  and  $c+x$  are of the same colour. Prove that -1982 and 1982 are of different colours.

*Solution by the editors.*

Let  $m$  be least such that two points  $a$  and  $b = a + m$ , at distance  $m$ , have the same colour, say  $c_1$ , and by renaming let  $w = c_2 \dots c_m$  be the sequence of colours between  $a$  and  $b$ , giving the pattern  $c_1 w c_1$ . We may also suppose  $a = 1$ ,  $b = m + 1$ .

Notice that for any interval  $[x,y]$  with  $x < y$  the number of distinct ways of colouring the two endpoints is  $100^2$ . Consider now  $100^2 + 1$  different intervals  $[x,y]$  of length  $l = 100^2 + m$ , each containing the interval  $[1,m + 1]$ , with  $x \leq 1$  and  $m + 1 \leq y$  (explicitly, the intervals  $[i,i + l]$  for  $-9999 \leq i \leq 1$ ). This means there are  $i, j$  with  $-9999 \leq i < j \leq 1$  for which the colours at  $i$  and  $j$  are the same, and the colours at  $i + l$  and  $j + l$  agree. These intervals overlap and both contain  $[1,m + 1]$ . This means that the leftmost interval contains another interval of length  $m + 1$  coloured  $c_1 w c_1$  to the left of  $[1,m + 1]$ , with no overlap except possibly that they have the endpoint 1 in common (because of the minimality of  $m$ ). Similarly the right interval has a block coloured  $c_1 w c_1$  to the right of  $[1,m + 1]$  with overlap possible only at the endpoint  $m + 1$ .

Next observe that if  $x + m < y$  and  $[x,x + m]$ ,  $[y,y + m]$  both have colour pattern  $c_1 w c_1$  then the interval between  $x + m$  and  $y$  must be filled by blocks of type  $c_1 w c_1$  too, since the condition may be applied to  $[x,y]$  and  $[x + m,y + m]$  to conclude that  $[x + m,x + 2m]$  and  $[y - m,y]$  each give blocks coloured  $c_1 w c_1$ , narrowing the gap by at least  $m$ .

Let  $(c_1 w)^n$  denote the colour pattern with repetition of blocks of type  $c_1 w$ , so that  $(c_1 w)^3$  is  $c_1 w c_1 w c_1 w$  for example. We have just showed that  $[1,m + 1]$  is part of a larger interval  $[\mu,\nu]$  with  $\mu < 1 < m + 1 < \nu$  such that the colour pattern of  $[\mu,\nu]$  is  $(c_1 w)^n c_1$  for some  $n$ .

Now repeat the argument of the last three paragraphs replacing intervals of length  $100^2 + m$  by those of length  $100^2 + \nu - \mu$  to increase the interval to an

interval  $[\mu', \nu']$  with  $\mu' < \mu$  and  $\nu < \nu'$  giving colour sequence  $(c_1 w)^N c_1$  where  $N > n$ . (The intervals increase in both directions but of course some more overlap may occur on intervals of colour type  $c_1 w c_1$ .)

Since this process does not stop we conclude that the colouring is cyclic, and indeed two integers receive the same colour just if the difference is divisible by  $m$ . As all 100 colours are used,  $m = 100$ . Now 1982 and -1982 obviously receive different colours.

[*Editor's note:* An idea sent in by John Morvay of Dallas, Texas inspired the above solution.]

### 1983.1 [1988: 2]

Let  $x$ ,  $y$ , and  $z$  be rational numbers satisfying

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0.$$

Prove that  $x = y = z = 0$ .

*Solutions by George Evangelopoulos, law student, Athens, Greece, and Zun Shan and Edward T.H. Wang, Wilfrid Laurier University.*

We may suppose that  $x$ ,  $y$  and  $z$  are integers, for we could replace them by  $lx$ ,  $ly$  and  $lz$  respectively where  $l$  is the least common multiple of the denominators. Assume that  $x_1$ ,  $y_1$ ,  $z_1$  give an integer solution with not all  $x_1$ ,  $y_1$ ,  $z_1$  zero. Set  $d$  to be the greatest common divisor of  $\{x_1, y_1, z_1\}$ . Then  $x = x_1/d$ ,  $y = y_1/d$ ,  $z = z_1/d$  give a nonzero integral solution for which the g.c.d. is 1.

With this solution fixed we see that from  $x^3 = 9xyz - 3y^3 - 9z^3$  we must have that  $x$  is divisible by 3. Let  $x = 3u$  and substitute to obtain

$$9u^3 + y^3 + 3z^3 - 9uyz = 0$$

from which  $y$  is clearly divisible by 3. Letting  $y = 3v$  and substituting we read

$$3u^3 + 9v^3 + z^3 - 9uvz = 0$$

and  $z$  is also divisible by 3, contradicting our assumption that  $\gcd(x, y, z) = 1$ . Therefore  $x = y = z = 0$ .

*Editor's note:* Alternate solutions were submitted by the late J.T. Groenman, Arnhem, The Netherlands; by H.N. Gupta, The University of Regina; and by M. Selby of the University of Windsor. Gupta and Selby both factored

$$a^3 + b^3 + c^3 - 3abc = 1/2(a + b + c)[(b - c)^2 + (c - a)^2 + (a - b)^2]$$

and used the irrationality of  $\sqrt[3]{3}$ . Gupta points out the following generalization: *if  $x^3 + ay^3 + a^2z^3 - 3axyz = 0$ , where  $a$  is a real number which is not the cube of a rational number, then the only rational solution is  $x = y = z = 0$ .*

### 1983.2 [1988: 2]

Prove that  $f(2) \geq 3^n$  where the polynomial

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1$$

has non-negative coefficients and  $n$  real roots.

*Solutions by George Evangelopoulos, law student, Athens, Greece, and by Zun Shan and Edward T.H. Wang of Wilfrid Laurier University.*

Since the coefficients are all non-negative the roots must all be negative. Denote them by  $-r_i$  with  $r_i > 0$ ,  $i = 1, 2, \dots, n$ . Then  $r_1 r_2 \cdots r_n = 1$ , and by the Arithmetic-Geometric Mean inequality we obtain

$$\begin{aligned} a_k &= \sum (r_{i_1} r_{i_2} \cdots r_{i_k}) \geq \binom{n}{k} \left( \prod r_{i_1} r_{i_2} \cdots r_{i_k} \right)^{1/\binom{n}{k}} \\ &= \binom{n}{k} (r_1 r_2 \cdots r_n)^{k/n} = \binom{n}{k}, \end{aligned}$$

where the sum and the product are taken over all the  $k$ -combinations of  $r_i$ 's. Consequently

$$f(2) = \sum_{k=0}^n a_k 2^{n-k} \geq \sum_{k=0}^n \binom{n}{k} 2^{n-k} = (2+1)^n = 3^n.$$

*Editor's note:* An alternate solution employing the Calculus was sent in by M.A. Selby, The University of Windsor, Ontario.

### 1986.2 [1988: 3]

Let  $n$  be an integer greater than 2. Find the maximum value for  $h$  and the minimum value for  $H$  such that for any positive numbers  $a_1, a_2, \dots, a_n$ ,

$$h < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \cdots + \frac{a_n}{a_n + a_1} < H.$$

*Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo.*

Let

$$S = \sum_{i=1}^n \frac{a_i}{a_i + a_{i+1}}$$

be the given cyclic sum (where  $a_{n+1} = a_1$ ). Then

$$S + 1 = \left( \sum_{i=1}^n \frac{a_i}{a_i + a_{i+1}} \right) + \frac{a_1 + a_2 + \cdots + a_n}{a_1 + a_2 + \cdots + a_n}$$

$$\begin{aligned}
 &= \sum_{i=1}^n \left( \frac{a_i}{a_i + a_{i+1}} + \frac{a_{i+1}}{a_1 + \cdots + a_n} \right) \\
 &< \sum_{i=1}^n \left( \frac{a_i}{a_i + a_{i+1}} + \frac{a_{i+1}}{a_i + a_{i+1}} \right) \\
 &= n .
 \end{aligned}$$

This implies that  $S < n - 1$ . Also

$$\begin{aligned}
 S - 1 &= \left( \sum_{i=1}^n \frac{a_i}{a_i + a_{i+1}} \right) - \frac{a_1 + a_2 + \cdots + a_n}{a_1 + a_2 + \cdots + a_n} \\
 &= \sum_{i=1}^n \left( \frac{a_i}{a_i + a_{i+1}} - \frac{a_i}{a_1 + a_2 + \cdots + a_n} \right) \\
 &> 0 .
 \end{aligned}$$

This implies  $S > 1$ .

If we set  $a_1 = 1$ ,  $a_2 = t$ ,  $a_3 = t^2$ , ...,  $a_n = t^{n-1}$  where  $t > 0$ , then it is easy to see that

$$S(t) = \frac{n-1}{1+t} + \frac{t^{n-1}}{t^{n-1}+1} .$$

Since  $\lim_{t \rightarrow 0^+} S(t) = n - 1$  and  $\lim_{t \rightarrow \infty} S(t) = 1$ , both bounds above are best possible.

Hence  $h = 1$  and  $H = n - 1$ .

Using the same arguments and example (with slightly more complicated notation) given in the above solution, one can show that if  $n$  and  $k$  are positive integers with  $n \geq 3$ ,  $1 < k < n$ , then the maximum value for  $h$  and the minimum value for  $H$  such that

$$h < \sum_{i=1}^n \frac{a_i}{a_i + a_{i+1} + \cdots + a_{i+k-1}} < H$$

for any positive numbers  $a_1, \dots, a_n$  are  $h = 1$  and  $H = n - k + 1$ .

\* \* \*

This exhausts the backlog of replies for these Kürschak problems. Send in your nice solutions and contests!

\* \* \*

## PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (\*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1990, although solutions received after that date will also be considered until the time when a solution is published.

**1471.** *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let  $A'B'C'$  be a triangle inscribed in a triangle  $ABC$ , so that  $A' \in BC$ ,  $B' \in CA$ ,  $C' \in AB$ , and so that  $A'B'C'$  and  $ABC$  are directly similar. If  $BA' = CB' = AC'$ , prove that the triangles are equilateral.

**1472.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For each integer  $n \geq 2$ , find the largest constant  $c_n$  such that

$$c_n \sum_{i=1}^n |a_i| \leq \sum_{i < j} |a_i - a_j|$$

for all real numbers  $a_1, \dots, a_n$  satisfying  $\sum_{i=1}^n a_i = 0$ .

**1473<sup>\*</sup>.** *Proposed by Murray S. Klamkin, University of Alberta.*

Given is a unit circle and an interior point  $P$ . Find the convex  $n$ -gon of largest area and/or perimeter which is inscribed in the circle and passes through  $P$ .

**1474.** *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let  $ABC$  be a triangle with sides  $a$ ,  $b$ ,  $c$  which we regard as being made of thin homogeneous material. The center of gravity of the perimeter of the triangle is denoted  $G$ . If  $G$  lies on the incircle of  $\Delta ABC$ , prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{10}{a+b+c}.$$

1475. Proposed by Richard Katz, California State University, Los Angeles; Raymond Killgrove, Indiana State University, Terre Haute; and Reginald Koo, University of South Carolina, Aiken.

Is there a function  $f$  from the reals onto the reals such that  $f(f(x)) = f(x)f(x)$  for all  $x$ ?

1476. Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.

A triangle is called *self-altitude* if it is similar to the triangle formed from its altitudes. Suppose  $\Delta ABC$  is self-altitude, with sides  $a \geq b \geq c$  and angle bisectors  $AP, BQ, CR$ . Prove that the lengths of  $CP, PB, BR, RA$  form a geometric progression.

1477. Proposed by Marcin E. Kuczma, Warszawa, Poland.

1024 tennis players, 128 professionals among them, are participating in a cup championship. All participants are numbered 1 through 1024. In the cup system, in the first round no. 1 plays against no. 2, no. 3 against no. 4, etc.; the winner in match 1 vs. 2 plays in the second round against the winner of 3 vs. 4, etc.; the final match constitutes the tenth round. Professionals bear numbers divisible by 8. In a match between a professional and an amateur, the former wins with probability 0.6. Is the cup winner more likely to be a professional or an amateur?

1478<sup>\*</sup>. Proposed by D.M. Milosevic, Pranjani, Yugoslavia.

A circle of radius  $R$  is circumscribed about a regular  $n$ -gon. A point on the circle is at distances  $a_1, a_2, \dots, a_n$  from the vertices of the  $n$ -gon. Prove that

$$\sum_{i=1}^n a_i^3 \geq 2R^3 n\sqrt{2}.$$

1479. Proposed by Vedula N. Murty, Pennsylvania State University at Harrisburg.

Given  $x > 0, y > 0$  satisfying  $x^2 + y^2 = 1$ , show without calculus that  $x^3 + y^3 \geq \sqrt{2}xy$ .

1480. Proposed by Juan Bosco Romero Márquez, Valladolid, Spain.

$ABC$  and  $A'B'C'$  are triangles connected by a dilatation ( $BC \parallel B'C'$ ,  $CA \parallel C'A'$ ,  $AB \parallel A'B'$ ), and  $A'' = BC' \cap B'C$ ,  $B'' = AC' \cap A'C$ ,  $C'' = AB' \cap A'B$ . Show that  $\Delta A''B''C''$  is connected to either of the two given triangles by a dilatation, and that the centroids of the three triangles are collinear.

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## SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1114.** [1986: 26; 1987: 185] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let  $ABC, A'B'C'$  be two triangles with sides  $a, b, c, a', b', c'$  and areas  $F, F'$  respectively. Show that

$$aa' + bb' + cc' \geq 4\sqrt{3}FF'.$$

III. *Further comment by Murray S. Klamkin, University of Alberta.*

We give a wider generalization than that given by Janous ((4) on [1987: 186]). Let  $a_i, b_i, c_i, F_i$ , and  $R_i$  be the sides, area, and circumradius of triangle  $T_i$ , and let

$$G_i(t) = \exp(H_i(\ln t))$$

where  $H$  is a convex nondecreasing function, all for  $i = 1, 2, \dots, n$ . From the A.M.-G.M. inequality and the properties of  $H$ ,

$$\begin{aligned} \prod_{i=1}^n G_i(a_i) + \prod_{i=1}^n G_i(b_i) + \prod_{i=1}^n G_i(c_i) &\geq 3 \prod_{i=1}^n \sqrt[3]{G_i(a_i)G_i(b_i)G_i(c_i)} \\ &= 3 \exp \left( \sum_{i=1}^n \frac{H_i(\ln a_i) + H_i(\ln b_i) + H_i(\ln c_i)}{3} \right) \\ &\geq 3 \exp \left( \sum_{i=1}^n H_i \left( \ln \frac{a_i b_i c_i}{3} \right) \right) \\ &= 3 \exp \left( \sum_{i=1}^n H_i \left( \ln \frac{4R_i F_i}{3} \right) \right) \\ &\geq 3 \exp \left( \sum_{i=1}^n H_i \left( \ln 2\sqrt{F_i/\sqrt{3}} \right) \right), \end{aligned}$$

where for the last inequality we have used the known one

$$2\sqrt{F} \leq 3^{3/4}R.$$

Janous' generalization (4) corresponds to the special case  $H_i(x) = t_i x$ . Also, like Janous we can involve the medians for any number of the triangles since for any triangle of sides  $a, b, c$  and area  $F$  we also have a triangle of sides  $m_a, m_b, m_c$  and

area  $3F/4$ . Similarly, we can use other duality relations to obtain other mixed inequalities in a general setting.

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**1294.** [1987: 320; 1989: 17] *Proposed by P. Penning, Delft, The Netherlands.*

Find a necessary and sufficient condition on a convex quadrangle  $ABCD$  in order that there exists a point  $P$  (in the same plane as  $ABCD$ ) such that the areas of the triangles  $PAB$ ,  $PBC$ ,  $PCD$ ,  $PDA$  are equal.

II. *Comment by Toshio Seimiya, Kawasaki, Japan.*

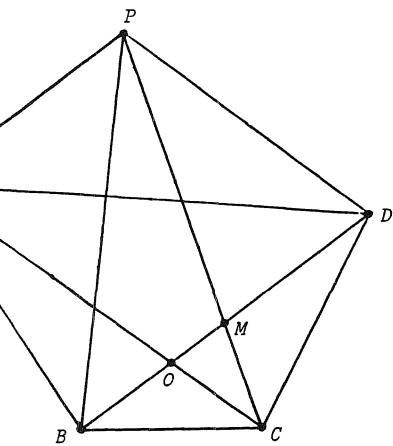
The solution given on [1989: 17] is not satisfactory. There remains another case.

Let  $O$  be the point of intersection of the two diagonals  $AC$  and  $BD$ . We assume that  $OA > OC$ ,  $OD > OB$ , and

$$\frac{OA}{OC} = \frac{OD + OB}{OD - OB}. \quad (1)$$

Let  $P$  be the point that satisfies  $AP \parallel BD$  and  $DP \parallel CA$ , and let  $M$  be the midpoint of  $BD$ . Then from (1)

$$\frac{PD}{OC} = \frac{AO}{OC} = \frac{BD}{OD - OB} = \frac{2DM}{2MO} = \frac{DM}{MO}.$$



Therefore  $P$ ,  $M$ ,  $C$  are collinear, and we can easily deduce that

$$\text{area } PAB = \text{area } PAD = \text{area } PCD = \text{area } PBC. \quad (2)$$

It can also be proved that in the case  $OA > OC$  and  $OD > OB$ , (1) is a necessary condition for (2) to hold.

[*Editor's comment:* It must be pointed out that the proposer's original problem spoke of the *oriented* areas of the triangles being equal, in which case the point  $P$  must be inside the quadrangle and the published proof [1989: 17] is correct. This condition was dropped by the editor, though, and he and all other solvers missed the above additional solution. Condition (1) is not easy to visualize. Can any reader find a simpler way to state it?]

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**1300.** [1987: 321; 1989: 23] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

$ABC$  is a triangle, not right angled, with circumcentre  $O$  and orthocentre  $H$ . The line  $OH$  intersects  $CA$  in  $K$  and  $CB$  in  $L$ , and  $OK = HL$ . Calculate angle  $C$ .

IV. *Further comments by P. Penning, Delft, The Netherlands.*

In the editor's comment [1989: 24] a new category of solutions is indicated and a particular solution is given. The order of points on the straight line is no longer *KOHL* or *LOHK*, but *OHKL* (underlining indicates points outside the triangle). I note that for  $C = 120^\circ$  points outside the triangle were already considered with the solutions *OLKH* and *OKLH*.

The condition  $OK = HL$  may imply either  $HK = OL$  or  $OH = KL$ . All solvers chose the first and the editor suggests the second. This solution is obtained by giving the right-hand side of equation (4) on [1989: 24] the other sign. It is not difficult to arrive at the equation

$$2 \cos B = \cos A \left( 2 \cos C + \frac{1}{2 \cos C} \right).$$

From this result follows that all angles must be acute, because if one cosine is negative, another must be negative also, which is impossible in a triangle. The equation can be solved for  $A$ :

$$\tan A = \frac{5 + 4 \cos 2C}{2 \sin 2C}.$$

All values  $C$  from 0 to  $90^\circ$  are allowed. The solution given by the editor is for

$$\cos 2C = -0.8, \quad \tan A = 1.5,$$

which corresponds to the minimum value of  $A$  ( $\approx 56.3^\circ$ ).

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**1344.** [1988: 140] *Proposed by Florentin Smarandache, Craiova, Romania.*

There are given  $mn + 1$  points such that among any  $m + 1$  of them there are two within distance 1 from each other. Prove that there exists a sphere of radius 1 containing at least  $n + 1$  of the points.

*Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Let  $S$  be the set of  $mn + 1$  points. Suppose, by way of contradiction, that every sphere of radius 1 contains at most  $n$  points of  $S$ . Choose an arbitrary point  $P_1$  of  $S$ . The unit sphere with center at  $P_1$  "covers" at most  $n$  points of  $S$ . Choose any "uncovered" point  $P_2$  of  $S$ . The unit sphere with center at  $P_2$  covers at most  $n$  previously uncovered points of  $S$ . Continue to select an uncovered point as the center of the next unit sphere. Since the union of any  $k$  of these spheres covers at most  $kn$  points of  $S$ , it requires at least  $m + 1$  of them to cover  $S$ . We obtain our desired contradiction upon observing that the centers of these spheres comprise a set of (at least)  $m + 1$  points of  $S$ , no two of which are within distance 1 of each other.

*Also solved by C. WILDHAGEN, Breda, The Netherlands; and the proposer.*

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**1353.** [1988: 174] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Find a linear recurrence with constant coefficients whose range is the set of all integers.

(b)\* Is there a linear recurrence with constant coefficients whose range is the set of all Gaussian integers (complex numbers  $a + bi$  where  $a$  and  $b$  are integers)?

I. *Solution to (a) by C. Wildhagen, Breda, The Netherlands.*

Let  $(c_n)_{n=1}^{\infty}$  be the sequence of integers given by

$$c_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Clearly the range of this sequence is the set of all integers. Moreover we have

$$c_n = \frac{1 + (2n-1)(-1)^n}{4}.$$

From this expression for  $c_n$ , one easily finds that the sequence  $(c_n)$  satisfies the linear recurrence (with constant coefficients)

$$c_{n+2} + 2c_{n+1} + c_n = 1.$$

II. *Solution to (a) by Murray S. Klamkin, University of Alberta.*

One solution is obtained by simply mixing the two sequences  $\{-n\}$  and  $\{n\}$  alternately, i.e.

$$\{a_0, a_1, a_2, \dots\} = \{0, 1, -1, 2, -2, \dots\}.$$

Since here  $a_{2n} = -n$  and  $a_{2n-1} = n$ , it follows by induction that

$$a_{n+3} + a_{n+2} - a_{n+1} - a_n = 0$$

where  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = -1$ .

More generally, if we have two given sequences  $\{a_n\}$  and  $\{b_n\}$  which are defined by linear difference equations with constant coefficients (and which can be assumed to be homogeneous without loss of generality), we can always find a linear difference equation with constant coefficients satisfied by the mixed sequence

$$\{u_n\} = a_1, b_1, a_2, b_2, a_3, b_3, \dots.$$

Assume for example that both  $\{a_n\}$  and  $\{b_n\}$  satisfy second order equations whose characteristic roots are  $r_1, r_2$  and  $r_3, r_4$  respectively. Then

$$a_n = \alpha r_1^n + \beta r_2^n, \quad b_n = \gamma r_3^n + \delta r_4^n$$

for suitable constants  $\alpha, \beta, \gamma, \delta$ . Since  $u_n$  can be expressed in the form

$$u_n = \frac{1}{2}(1 + (-1)^{n+1})a_{(n+1)/2} + \frac{1}{2}(1 + (-1)^n)b_{n/2},$$

the characteristic roots for the equation for  $u_n$  will be

$$\pm\sqrt{r_1}, \pm\sqrt{r_2}, \pm\sqrt{r_3}, \pm\sqrt{r_4}.$$

Then if the 8th degree polynomial equation whose roots are these is

$$c_0 + c_1x + c_2x^2 + \cdots + c_8x^8 = 0,$$

the difference equation for  $u_n$  is

$$c_0u_n + c_1u_{n+1} + \cdots + c_8u_{n+8} = 0.$$

If we were to mix three second-order sequences, we would end up with a 12th degree recurrence. For, corresponding to each root  $r$  of the given recurrence equations, we would have to consider  $r^{1/3}$ ,  $\omega r^{1/3}$ , and  $\omega^2 r^{1/3}$  where  $\omega$  is a cube root of unity. In a similar fashion, but with more work, we can mix any finite number of given sequences.

In view of the above procedure, it appears that one would have to mix a countable number of simple sequences to get all the Gaussian integers, and so it is reasonable to conjecture that it cannot be done.

*Part (a) also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University, and the proposer.*

*Part (b) remains open, although Klamkin, Meyers, and the proposer all believe the answer is "no". Can anyone find a rigorous proof? For another question of the same type, what if the range is required to be the set of all rational numbers?*

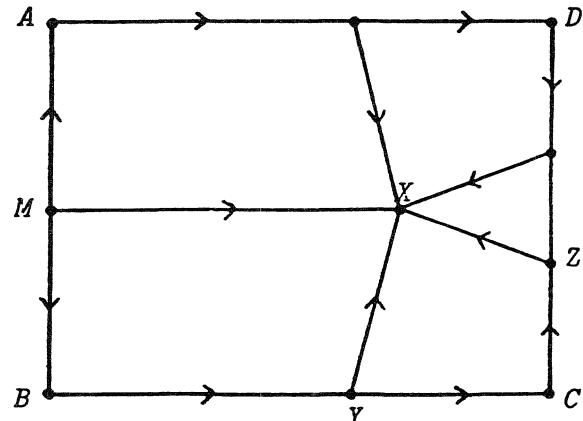
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### 1354. [1988: 174] *Proposed by Jordi Dou, Barcelona, Spain.*

A girl stands at the midpoint  $M$  of one of the short sides of a rectangular swimming pool  $ABCD$  69 metres by 54.4 metres. A buoy is floating in the pool at a point  $X$  such that the girl can reach it (by a combination of swimming and walking) in the minimum time of 57.8 seconds; moreover she can achieve this minimum time in 5 different ways as shown in the diagram (that is, by swimming directly from  $M$  to  $X$ , by walking along  $MBY$  and swimming along  $YX$ , by walking along  $MBYCZ$  and swimming along  $ZX$ , and symmetrically by two other paths). Find her swimming speed and her walking speed, assuming they are each constant.



*Solution by Richard I. Hess, Rancho Palos Verdes, California.*

Let  $v_s$  be the girl's swimming speed (in m/s),  $v_w$  her walking speed,  $t_m = 57.8$  sec. the minimum time, and  $l = 69m$ ,  $2w = 54.4m$  the dimensions of the pool. Also put  $x = MX = v_s t_m$ . Then from the refraction condition for minimum time,  $\angle XYC = \angle XZD = \theta$  and

$$v_s = v_w \cos \theta \quad (1)$$

so

$$\sin \theta = \sqrt{1 - \frac{v_s^2}{v_w^2}}. \quad (2)$$

Thus, from (1) and (2), the path  $MBYX$  yields

$$\begin{aligned} t_m &= \frac{w + x - w \cot \theta}{v_w} + \frac{w}{v_s \sin \theta} = \frac{w + x}{v_w} + \frac{w}{v_s \sin \theta} \left(1 - \frac{v_s^2}{v_w^2}\right) \\ &= \left(\frac{w + v_s t_m}{v_s}\right) \cos \theta + \left(\frac{w}{v_s}\right) \sin \theta. \end{aligned} \quad (3)$$

Similarly, the path  $MBYCZX$  yields

$$\begin{aligned} t_m &= \frac{2w + l - (l - x) \cot \theta}{v_w} + \frac{l - x}{v_s \sin \theta} \\ &= \frac{2w + l}{v_w} + \frac{l - x}{v_s \sin \theta} \left(1 - \frac{v_s^2}{v_w^2}\right) \\ &= \left(\frac{2w + l}{v_s}\right) \cos \theta + \left(\frac{l - v_s t_m}{v_s}\right) \sin \theta. \end{aligned} \quad (4)$$

From (1), (3) and (4),

$$w \cos \theta + w \sin \theta = v_s t_m (1 - \cos \theta)$$

and

$$(2w + l) \cos \theta + l \sin \theta = v_s t_m (1 + \sin \theta),$$

so

$$\frac{(2w + l) \cos \theta}{1 + \sin \theta} + \frac{l \sin \theta}{1 + \sin \theta} = \frac{w(\cos \theta + \sin \theta)}{1 - \cos \theta}.$$

Hence

$$\frac{69}{27.2} = \frac{l}{w} = \frac{1 + \sin \theta}{1 - \cos \theta} - \frac{2 \cos \theta}{\cos \theta + \sin \theta},$$

from which follows

$$\tan \theta = 2.4.$$

Thus, from (3) and then (1),

$$v_s = 1, \quad v_w = 2.6.$$

*Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; P. PENNING, Delft, The Netherlands; and the proposer.*

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**1355.** [1988: 175] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let  $ABC$  be a triangle and  $I$  its incenter. The perpendicular to  $AI$  at  $I$  intersects the line  $BC$  at the point  $A'$ . Analogously we define  $B'$ ,  $C'$ . Prove that  $A'$ ,  $B'$ ,  $C'$  lie in a straight line.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We show more generally:

Let  $ABC$  be a triangle and  $P$  be a point lying in its plane such that  $A'$ ,  $B'$ ,  $C'$ , defined as follows, exist. The perpendicular to  $PA$  at  $P$  intersects the line  $BC$  at the point  $A'$ , and analogously we get  $B'$  and  $C'$ . Then  $A'$ ,  $B'$  and  $C'$  are collinear.

*Proof.* Suppose that  $P$  is such that, say,  $PB \perp PC$ . Then  $B = C'$  and  $C = B'$  and we're done.

Therefore we assume that  $PA$ ,  $PB$  and  $PC$  are mutually non-perpendicular. Clearly

$$\overrightarrow{PA} \cdot \overrightarrow{PA'} = \overrightarrow{PB} \cdot \overrightarrow{PB'} = \overrightarrow{PC} \cdot \overrightarrow{PC'} = 0. \quad (1)$$

Furthermore, since  $A'$  is on  $BC$  etc., there exist  $\lambda, \mu, \nu \in \mathbb{R} \setminus \{0,1\}$  such that

$$\overrightarrow{PA'} = \lambda \overrightarrow{PB} + (1 - \lambda) \overrightarrow{PC},$$

$$\overrightarrow{PB'} = \mu \overrightarrow{PC} + (1 - \mu) \overrightarrow{PA},$$

$$\overrightarrow{PC'} = \nu \overrightarrow{PA} + (1 - \nu) \overrightarrow{PB}.$$

Thus

$$\overrightarrow{PA} \cdot \overrightarrow{PA'} = \lambda \overrightarrow{PA} \cdot \overrightarrow{PB} + (1 - \lambda) \overrightarrow{PA} \cdot \overrightarrow{PC}, \text{ etc.} \quad (2)$$

Via (1) we get from (2)

$$(\lambda - 1) \overrightarrow{PA} \cdot \overrightarrow{PC} = \lambda \overrightarrow{PA} \cdot \overrightarrow{PB},$$

$$(\mu - 1) \overrightarrow{PA} \cdot \overrightarrow{PB} = \mu \overrightarrow{PB} \cdot \overrightarrow{PC},$$

$$(\nu - 1) \overrightarrow{PB} \cdot \overrightarrow{PC} = \nu \overrightarrow{PA} \cdot \overrightarrow{PC}.$$

Since  $\overrightarrow{PA} \cdot \overrightarrow{PC}$  etc. are nonzero, by multiplication we get

$$(\lambda - 1)(\mu - 1)(\nu - 1) = \lambda \mu \nu.$$

Thus

$$\frac{A'C}{A'B} \cdot \frac{B'A}{B'C} \cdot \frac{C'B}{C'A} = \frac{\lambda}{1 - \lambda} \cdot \frac{\mu}{1 - \mu} \cdot \frac{\nu}{1 - \nu} = -1$$

and we're done by Menelaus' theorem.

*Also solved by BENO ARBEL, Tel Aviv University, Tel Aviv, Israel; FRANCISCO BELLOT ROSADO and MARIA ASCENSIÓN LOPEZ CHAMORRO, Valladolid, Spain; JORDI DOU, Barcelona, Spain; R.H. EDDY, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel,*

*The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.*

*Hut also proved the above generalization.*

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**1356\*** [1988: 175] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{n-1}}$$

for positive real numbers  $x_1, \dots, x_n$  ( $n \geq 2$ ) satisfying  $x_1 + \cdots + x_n = 1$ .

I. *Solution by Jeff Vanderkam, student, North Carolina School of Science and Mathematics, Durham.*

Since

$$f(x) = \frac{x}{\sqrt{1-x}}$$

is convex for all  $x < 1$ , we may use Jensen's inequality for convex functions to find that

$$\frac{1}{n} \sum_{i=1}^n f(x_i) \geq f\left(\sum_{i=1}^n \frac{x_i}{n}\right) = f\left(\frac{1}{n}\right),$$

or, equivalently,

$$\sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \sqrt{\frac{n}{n-1}}. \quad (1)$$

We also know, by the power-mean inequality (in this case  $P_1 \geq P_{1,2}$ ) that

$$\frac{1}{n} = \frac{1}{n} \sum_{i=1}^n x_i \geq \left( \sum_{i=1}^n \frac{\sqrt{x_i}}{n} \right)^2,$$

or, by taking square roots and multiplying by  $n$ ,

$$\sqrt{n} \geq \sum_{i=1}^n \sqrt{x_i},$$

or

$$\sqrt{\frac{n}{n-1}} \geq \frac{\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n}}{\sqrt{n-1}}. \quad (2)$$

Combining (1) and (2) gives the desired result.

II. *Solution by Kee-Wai Lau, Hong Kong.*

For  $0 < x < 1$  let

$$f(x) = \frac{x}{\sqrt{1-x}} - \sqrt{\frac{x}{n-1}}$$

so that

$$f''(x) = \frac{1}{4} \left( \frac{4-x}{(1-x)^{5/2}} + \frac{1}{\sqrt{n-1} \cdot x^{3/2}} \right) > 0.$$

Hence  $f(x)$  is convex and

$$\begin{aligned} f(x_1) + \cdots + f(x_n) &\geq nf\left(\frac{x_1 + \cdots + x_n}{n}\right) \\ &= nf\left(\frac{1}{n}\right) = 0. \end{aligned}$$

The inequality of the problem follows.

III. *Solution by the proposer.*

As the given inequality is symmetric we may and do assume

$$x_1 \leq x_2 \leq \cdots \leq x_n. \quad (3)$$

Consequently,

$$\frac{1}{\sqrt{1-x_1}} \leq \frac{1}{\sqrt{1-x_2}} \leq \cdots \leq \frac{1}{\sqrt{1-x_n}}. \quad (4)$$

(3) and (4) are two sequences ordered in the same direction. Thus Chebyshev's inequality applies and yields

$$\sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \frac{1}{n} \sum_{i=1}^n x_i \sum_{i=1}^n \frac{1}{\sqrt{1-x_i}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{1-x_i}}.$$

Noting that  $M_1 \geq M_{-2}$  where

$$M_t = \left( \frac{a_1^t + \cdots + a_n^t}{n} \right)^{1/t},$$

we have, with  $a_i = 1/\sqrt{1-x_i}$ ,

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{1-x_i}} \geq \sqrt{\frac{n}{\sum_{i=1}^n (1-x_i)}} = \sqrt{\frac{n}{n-1}}.$$

Thus

$$\sum_{i=1}^n \frac{x_i}{\sqrt{1-x_i}} \geq \sqrt{\frac{n}{n-1}} \geq \frac{1}{\sqrt{n-1}} \sum_{i=1}^n \sqrt{x_i}$$

[as in (2)].

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y. and CLIFFORD GARDNER, Austin, Texas; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; DAVID VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; ZENG ZHENBING, Institute of Mathematical Sciences, Chengdu, People's Republic of China; and the proposer (a second solution).

Festraet-Hamoir's solution was like solution III. Most of the others were like solution I.

Klamkin and the proposer gave generalizations (provable as above), Klamkin's a bit more broad, i.e.

$$\sum_{i=1}^n F(x_i) \geq n^s F\left(\frac{1}{n}\right) \sum_{i=1}^n x_i^s$$

where  $F$  is a convex function and  $s \leq 1$ . In particular when

$$F(x) = \frac{x^p}{(1-x^q)^r},$$

where  $p \geq 1$ ,  $q > 0$ ,  $r > 0$ , this yields

$$\sum_{i=1}^n \frac{x_i^p}{(1-x_i^q)^r} \geq \frac{n^{s-p+qr}}{(n^q-1)^r} \sum_{i=1}^n x_i^s.$$

The given inequality corresponds to  $s = 1/2$ ,  $p = 1$ ,  $q = 1$ ,  $r = 1/2$ .

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**1357\*** [1988: 175] Proposed by Jack Garfunkel, Flushing, N.Y.

Isosceles right triangles  $AA'B$ ,  $BB'C$ ,  $CC'A$  are constructed outwardly on the sides of a triangle  $ABC$ , with the right angles at  $A'$ ,  $B'$ ,  $C'$ , and triangle  $A'B'C'$  is drawn. Prove or disprove that

$$\sin A' + \sin B' + \sin C' \geq \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2},$$

where  $A'$ ,  $B'$ ,  $C'$  are the angles of  $\Delta A'B'C'$ .

*Editor's note:* No solutions to this problem have been received.

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**1358.** [1988: 175] Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia.

Four different digits are chosen at random from the set  $\{1, 2, 3, \dots, 9\}$ . Denote by  $S$  the sum of all possible four-digit numbers formed by permuting these

digits. What is the probability that  $S$  is square-free?

*Solution by P. Penning, Delft, The Netherlands.*

Let the four digits be  $a, b, c, d$ . Then

$$10 \leq a + b + c + d \leq 30.$$

The number of permutations is  $4 \cdot 3 \cdot 2 = 24$ . In the 24 four-digit numbers all digits appear 6 times at all positions. So

$$\begin{aligned} S &= 6(a + b + c + d) \cdot 1111 \\ &= 2 \cdot 3 \cdot 11 \cdot 101(a + b + c + d). \end{aligned}$$

$S$  is square-free (i.e. does not contain a prime factor more than once) for

$$a + b + c + d = 13, 17, 19, 23, 29.$$

Counting yields the number of possibilities for

$$\begin{aligned} a + b + c + d = 13 &\text{ to be } 3 && (\text{namely } 1237, 1246, 1345), \\ a + b + c + d = 17 &\text{ to be } 9, \\ a + b + c + d = 19 &\text{ to be } 11, \\ a + b + c + d = 23 &\text{ to be } 9, \\ a + b + c + d = 29 &\text{ to be } 1, \end{aligned}$$

for a total of

33.

The total number of different choices for  $\{a, b, c, d\}$  is

$$\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 126.$$

So the probability that  $S$  is square-free is equal to

$$\frac{33}{126} = \frac{11}{42}.$$

*Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD K. GUY, University of Calgary; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; SAM MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; and C. WILDHAGEN, Breda, The Netherlands. Four other readers, and the proposer, submitted incorrect solutions.*

*Janous suggests generalizing the problem to sets of  $j$  distinct "digits" taken from  $\{1, 2, \dots, b-1\}$ , where the  $j$ -digit numbers formed are considered to be in base  $b$ . He also suggests choosing  $n$  digits from  $\{1, 2, \dots, 9\}$ , repetitions allowed, and investigating the probability that the analogously defined  $S$  is square-free as  $n \rightarrow \infty$ .*

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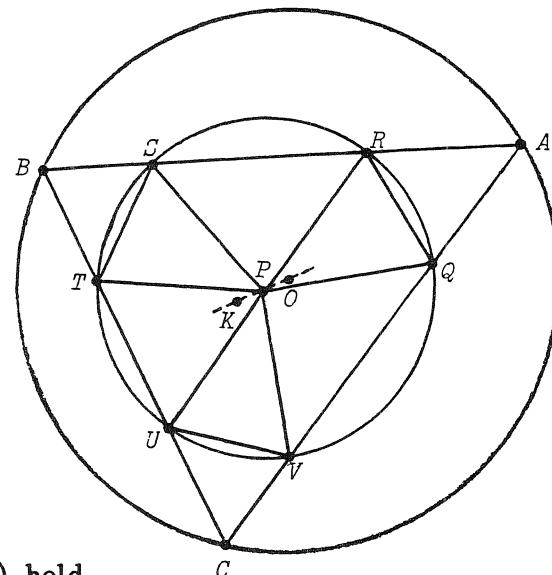
**1359.** [1988: 175] *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let  $PQR$ ,  $PST$ , and  $PUV$  be congruent isosceles triangles with common

apex  $P$  and having no vertex in common other than  $P$ . The sense  $P \rightarrow Q \rightarrow R$ ,  $P \rightarrow S \rightarrow T$ , and  $P \rightarrow U \rightarrow V$  is anticlockwise. We suppose moreover that  $VQ$  and  $RS$  meet in  $A$ ,  $RS$  and  $TU$  in  $B$ , and  $TU$  and  $VQ$  in  $C$ . Prove that  $P$  is on the line joining the circumcentre of  $\triangle ABC$  to the symmedian point of  $\triangle ABC$ .

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

Let  $O$  be the circumcenter and  $K$  the symmedian point of triangle  $ABC$ . The line  $RQ$  is antiparallel to  $SV$  with respect to  $AB$  and  $AC$  (because  $RSVQ$  is a cyclic quadrilateral). The arcs  $ST$  and  $UV$  are equal and of the same sense in the circle of center  $P$  which contains  $R, S, T, U, V, Q$ ; therefore  $SV$  is parallel to  $TU$ . But this signifies that  $RQ$  is antiparallel to  $TU$  (i.e.  $BC$ ) with respect to  $AB$  and  $AC$ ; that is,  $RQ$  is parallel to the tangent to the circumcircle of  $ABC$  at  $A$ . Analogous results for  $ST$  (with respect to  $B$ ) and  $UV$  (with respect to  $C$ ) hold.



It is well known (perhaps by few people) that, under these conditions, the circle through  $R, S, T, U, V, Q$  has its center  $P$  collinear with  $O$  and  $K$  (see pp. 172–173 of [2]); in fact, it is one of the Tucker circles associated with the triangle  $ABC$  (see also Theorem 456, p. 274 of [1]).

*References:*

- [1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.
- [2] Y. & R. Sortais, *La Géométrie du Triangle – Exercices Résolus*, Hermann, Paris, 1987.

II. *Solution by R.H. Eddy, Memorial University of Newfoundland.*

We denote, as usual, the measures of the angles of triangle  $ABC$  by  $\alpha$ ,  $\beta$ ,  $\gamma$ . If we next put

$$\lambda_1 = \angle BTS = \angle VUC,$$

$$\lambda_2 = \angle RQA = \angle CVU,$$

$$\lambda_3 = \angle TSB = \angle ARQ,$$

it becomes an elementary exercise to show that  $\lambda_1 = \alpha$ ,  $\lambda_2 = \beta$ ,  $\lambda_3 = \gamma$ . Now with the measure of the base angles of the isosceles triangles denoted by  $\theta$ , the trilinear coordinates of  $P$  are easily seen to be

$(x,y,z) = (\sin(\alpha + \theta), \sin(\beta + \theta), \sin(\gamma + \theta))$   
 [since  $\angle PTU = 180^\circ - \alpha - \theta$ , etc.], which implies

$$\frac{x}{\sin(\alpha + \theta)} = \frac{y}{\sin(\beta + \theta)} = \frac{z}{\sin(\gamma + \theta)}.$$

It now follows that

$$(x \sin \beta - y \sin \alpha) \cos \theta + (x \cos \beta - y \cos \alpha) \sin \theta = 0$$

and

$$(x \sin \gamma - z \sin \alpha) \cos \theta + (x \cos \gamma - z \cos \alpha) \sin \theta = 0,$$

which is a system of homogeneous linear equations in  $\cos \theta$  and  $\sin \theta$ . Since the solution  $(\cos \theta, \sin \theta)$  is nontrivial for each triple  $(x,y,z)$ , we must have

$$(x \sin \beta - y \sin \alpha)(x \cos \gamma - z \cos \alpha) = (x \cos \beta - y \cos \alpha)(x \sin \gamma - z \sin \alpha).$$

Simplifying, we obtain the line

$$x \sin(\beta - \gamma) + y \sin(\gamma - \alpha) + z \sin(\alpha - \beta) = 0,$$

which clearly contains the symmedian point

$$K(\sin \alpha, \sin \beta, \sin \gamma)$$

and the circumcentre

$$O(\cos \alpha, \cos \beta, \cos \gamma).$$

This gives the required result.

The line  $OK$  is the Brocard axis of the triangle  $ABC$ , which is the radical axis of the circles of Apollonius. See p. 295 of R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

*Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.*

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**1360.** [1988: 175] *Proposed by Eric Holleman, student, Memorial University of Newfoundland.*

Find all increasing arithmetic progressions  $a_1, a_2, \dots, a_{2l+1}$  of positive integers such that

$$a_{l+1}, \sum_{i=1}^{2l+1} a_i, \prod_{i=1}^{2l+1} a_i$$

is a geometric progression.

*Solution by M.A. Selby, University of Windsor.*

Let  $d \geq 1$  be the difference between consecutive terms of the progression. Then the progression can be represented as

$$a - ld, a - (l-1)d, \dots, a - d, a, a + d, \dots, a + ld.$$

Hence

$$\sum_{i=1}^{2l+1} a_i = (2l + 1)a .$$

If

$$a_{l+1} = a , \quad \sum_{i=1}^{2l+1} a_i , \quad \text{and} \quad \prod_{i=1}^{2l+1} a_i$$

are in geometric progression,

$$\frac{\prod_{i=1}^{2l+1} a_i}{(2l+1)a} = \frac{(2l+1)a}{a}$$

or

$$(2l+1)^2 = (a-ld)(a-(l-1)d)\cdots(a-d)(a+d)\cdots(a+ld). \quad (1)$$

Since  $a-ld \geq 1$ ,  $a^2 \geq (ld+1)^2$  and thus

$$(a-ld)(a+ld) = a^2 - l^2d^2 \geq 2ld + 1.$$

If  $l > 1$ , then in the right-hand side of (1) we have the terms  $a^2 - l^2d^2$  and  $a^2 - d^2$ , and

$$a^2 - d^2 > a^2 - l^2d^2 \geq 2ld + 1 \geq 2l + 1.$$

Hence the right-hand side of (1) is greater than  $(2l+1)^2$ , so (1) does not hold.

This gives that  $l = 0$  or  $l = 1$ .  $l = 0$  gives the trivial solution of one term  $a > 0$ . The other possibility  $l = 1$  implies, from (1),

$$(a+d)(a-d) = 9.$$

Hence  $a-d = 1$ ,  $a+d = 9$ , so  $a = 5$ ,  $d = 4$ . Thus the only possible arithmetic progression is

$$1, 5, 9,$$

which produces

$$a_{l+1} = 5 , \quad \sum_{i=1}^{2l+1} a_i = 15 , \quad \prod_{i=1}^{2l+1} a_i = 45 .$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; ROBERT E. SHAFER, Berkeley, California; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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**1361.** [1988: 201] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let  $ABC$  be a triangle with sides  $a, b, c$  and angles  $\alpha, \beta, \gamma$ , and let its circumcenter lie on the escribed circle to the side  $a$ .

- (i) Prove that  $-\cos \alpha + \cos \beta + \cos \gamma = \sqrt{2}$ .
- (ii) Find the range of  $\alpha$ .

*Solution by Murray S. Klamkin, University of Alberta.*

(i) We use the following known results concerning the escribed radius  $r_1$  to side  $a$  and the center  $I_1$  of the corresponding escribed circle:

$$OI_1^2 = R^2 + 2Rr_1 , \quad (1)$$

$$r_1 = 4R \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \sin \frac{\alpha}{2} , \quad (2)$$

where  $R$  is the circumradius. [See for example pp. 187 and 189 of R.A. Johnson, *Advanced Euclidean Geometry*.] By hypothesis,  $OI_1 = r_1$ , so that from (1),

$$R = r_1(\sqrt{2} - 1).$$

Then using (2) we get

$$\begin{aligned} \sqrt{2} + 1 &= 4 \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \sin \frac{\alpha}{2} = 4 \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cos \frac{\beta + \gamma}{2} \\ &= 4 \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - 4 \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \\ &= (1 + \cos \beta)(1 + \cos \gamma) - \sin \beta \sin \gamma \\ &= 1 + \cos \beta + \cos \gamma + \cos(\beta + \gamma) , \end{aligned}$$

or

$$-\cos \alpha + \cos \beta + \cos \gamma = \sqrt{2} .$$

(ii) Clearly  $\alpha \geq \pi/2$  since the circumcenter cannot lie inside the triangle. For the limiting case  $\alpha = \pi/2$ , we must have  $\beta = \gamma$  and  $O$  coincides with the midpoint of side  $a$ . We now consider  $\alpha > \pi/2$  and let  $\beta$  be the smallest angle, so that

$$\beta \leq \frac{\pi - \alpha}{2} .$$

Replacing  $\gamma$  in (i) by  $\pi - \alpha - \beta$ , we get

$$\begin{aligned} \cos \alpha + \sqrt{2} &= \cos \beta - \cos(\alpha + \beta) \\ &= (1 - \cos \alpha)\cos \beta + \sin \alpha \sin \beta . \end{aligned}$$

Then dividing both sides of this equation by the square root of

$$\sin^2 \alpha + (1 - \cos \alpha)^2 = 2 - 2 \cos \alpha = 4 \sin^2 \frac{\alpha}{2}$$

and simplifying, we get

$$\begin{aligned} \frac{\cos \alpha + \sqrt{2}}{\sqrt{2}(1 - \cos \alpha)} &= \frac{2 \sin^2 \frac{\alpha}{2} \cos \beta + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \beta}{2 \sin \frac{\alpha}{2}} \\ &= \sin(\beta + \frac{\alpha}{2}) . \end{aligned} \quad (3)$$

First we must satisfy

$$\frac{\cos \alpha + \sqrt{2}}{\sqrt{2}(1 - \cos \alpha)} \leq 1 ,$$

or

$$(\cos \alpha + \sqrt{2})^2 \leq 2 - 2 \cos \alpha ,$$

and this implies that  $\alpha \geq \pi/2$ . Then since

$$\frac{\pi - \alpha}{2} \geq \beta > 0 ,$$

we have  $\alpha/2 < \beta + \alpha/2 \leq \pi/2$ , so

$$\sin \frac{\alpha}{2} < \sin(\beta + \frac{\alpha}{2}).$$

From (3),

$$\sqrt{\frac{1 - \cos \alpha}{2}} < \frac{\cos \alpha + \sqrt{2}}{\sqrt{2}(1 - \cos \alpha)} ,$$

$$1 - \cos \alpha < \cos \alpha + \sqrt{2} ,$$

or

$$\cos \alpha > \frac{1 - \sqrt{2}}{2} .$$

Finally, the range of  $\alpha$  is the interval

$$\left[ 90^\circ, \cos^{-1}\left(\frac{1 - \sqrt{2}}{2}\right) \right] \approx \left[ 90^\circ, 101.9529^\circ \right] .$$

*Also solved by EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. Solutions to (i), with incorrect limits for  $\alpha$  in (ii), were found by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and D.J. SMEENK, Zaltbommel, The Netherlands. A proof of (ii) only was sent in by JORDI DOU, Barcelona, Spain.*

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**1362.** [1988: 202] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine the sum

$$\sum_{j=0}^n \sum_{k=0}^n \binom{n}{j+k} \binom{n}{j} \binom{n}{k} \omega^{-j-2k}$$

where  $\omega$  is a primitive cube root of unity.

*Solution by G.P. Henderson, Campbellcroft, Ontario.*

The coefficient of  $t^n$  in

$$(1 + xt)^a (1 + yt)^b (1 + zt)^c$$

is

$$\sum_{j+k \leq n} \binom{a}{j} \binom{b}{k} \binom{c}{n-j-k} x^j y^k z^{n-j-k}.$$

Setting  $a = b = c = n$ ,  $x = \omega^{-1}$ ,  $y = \omega^{-2}$ ,  $z = 1$ , we see that the required sum is the coefficient of  $t^n$  in

$$\left[ (1+t) \left( 1 + \frac{t}{\omega} \right) \left( 1 + \frac{t}{\omega^2} \right) \right]^n = [(1+t)(\omega+t)(\omega^2+t)]^n \\ = (1+t^3)^n.$$

Therefore the sum is  $\binom{n}{n/3}$  or 0 according as  $n$  is or is not divisible by 3.

*Also solved by EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; and (with a small error) by the proposer.*

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**1363\*** [1988: 202] *Proposed by P. Erdos, Hungarian Academy of Sciences.*

Let there be given  $n$  points in the plane, no three on a line and no four on a circle. Is it true that these points must determine at least  $n$  distinct distances, if  $n$  is large enough? I offer \$25 U.S. for the first proof of this.

*Editor's comment.*

No solutions have been received for this problem.

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**1364.** [1988: 202] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Let  $a$  and  $b$  be integers. Find a polynomial with integer coefficients that has  $\sqrt[3]{a} + \sqrt[3]{b}$  as a root.

*Solution by Leroy F. Meyers, The Ohio State University.*

If  $x = \sqrt[3]{a} + \sqrt[3]{b}$ , then

$$(x - \sqrt[3]{a})^3 = b, \\ x^3 - a - b = 3x \sqrt[3]{a}(x - \sqrt[3]{a}), \\ (x^3 - a - b)^3 = 27x^3a(x - \sqrt[3]{a})^3 = 27x^3ab,$$

so that the required polynomial is given by

$$p(x) = (x^3 - a - b)^3 - 27abx^3 \\ = x^9 - 3(a + b)x^6 + 3(a^2 - 7ab + b^2)x^3 - (a + b)^3.$$

Of course, this is equal to

$$\prod_{j=0}^2 \prod_{k=0}^2 (x - \omega^j \sqrt[3]{a} - \omega^k \sqrt[3]{b}),$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2}.$$

In certain cases, there is a lower-degree polynomial.

- (i) If  $\sqrt[3]{a}$  and  $\sqrt[3]{b}$  are the integers  $c$  and  $d$ , then we may set

$$p(x) = x - c - d.$$

- (ii) If  $\sqrt[3]{a}$  is the integer  $c$  or  $\sqrt[3]{b}$  the integer  $d$ , then we may set

$$p(x) = (x - c)^3 - b \quad \text{or} \quad p(x) = (x - d)^3 - a,$$

respectively.

- (iii) If there are integers  $m$ ,  $n$ ,  $c$  such that  $\sqrt[3]{a} = m\sqrt[3]{c}$  and  $\sqrt[3]{b} = n\sqrt[3]{c}$  [equivalently,  $\sqrt[3]{a/b}$  is rational], then we may set

$$p(x) = x^3 - (m + n)^3 c.$$

*Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; BRIAN CALVERT, Brock University, St. Catharines, Ontario; EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; JACK GARFUNKEL, Flushing, N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; SAM MALTBY, student, University of Waterloo; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. There was one incorrect solution sent in.*

Wang mentioned the further case (iv)  $a = b^2$  (or  $b = a^2$ ) when a lower-degree polynomial solution can be found. Ahlborg gave a polynomial with integer coefficients and a root  $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$ , where  $a$ ,  $b$ ,  $c$  are integers. Bellot mentioned the similar Crux problems 1187 [1988: 30] and 1301 [1989: 56], as well as a problem in Mathematical Intelligencer 8 (1986) 31–33.

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**1365.** [1988: 202] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}$$

where  $A, B, C$  are the angles (in radians) of an acute triangle.

I. *Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg.*

We first note the following inequality (which is not that well known): if  $x, y, z$  are positive numbers, then

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}. \quad (1)$$

For, from the Arithmetic-Harmonic mean inequality, we obtain

$$2(x+y+z)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right) \geq 9,$$

and (1) follows by multiplying out.

The function  $y = \sin x$  ( $0 \leq x \leq \pi/2$ ) is concave down and hence the line segment connecting the points  $(0,0)$  and  $(\pi/2,1)$  is below the curve. This gives the inequality

$$\sin x \geq \frac{2}{\pi}x, \quad 0 \leq x \leq \frac{\pi}{2},$$

where strict inequality holds for  $0 < x < \pi/2$ . Hence

$$\sin A > \frac{2}{\pi}A, \quad \sin B > \frac{2}{\pi}B, \quad \sin C > \frac{2}{\pi}C,$$

and thus

$$\begin{aligned} \frac{\sin A}{\pi-A} + \frac{\sin B}{\pi-B} + \frac{\sin C}{\pi-C} &= \frac{\sin A}{B+C} + \frac{\sin B}{C+A} + \frac{\sin C}{A+B} \\ &> \frac{2}{\pi}\left(\frac{A}{B+C} + \frac{B}{C+A} + \frac{C}{A+B}\right) \\ &\geq \frac{3}{\pi} \end{aligned}$$

by (1). This completes the proof of the left side of the proposed inequality.

From  $A, B, C < \pi/2$  follows

$$\frac{1}{\pi-A}, \quad \frac{1}{\pi-B}, \quad \frac{1}{\pi-C} < \frac{2}{\pi}.$$

Hence

$$\sum \frac{\sin A}{\pi-A} < \frac{2}{\pi} \sum \sin A \leq \frac{2}{\pi} \cdot \frac{3\sqrt{3}}{2} = \frac{3\sqrt{3}}{\pi}.$$

This completes the proof of the proposed inequality.

II. *Solution by G.P. Henderson, Campbellcroft, Ontario.*

Set

$$F(A, B, C) = \sum \frac{\sin A}{\pi-A}, \quad 0 \leq A, B, C \leq \frac{\pi}{2}, \quad \sum A = \pi.$$

We will prove that  $F$  takes on its minimum value at  $(\pi/4, \pi/4, \pi/2)$ , its value there being

$$\frac{2 + 4\sqrt{2}/3}{\pi}, \quad (1)$$

and that the maximum value of  $F$  is

$$F\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{4}{\pi}.$$

First we establish the above maximum for  $F$ . The line  $y = \sqrt{2}A/\pi$  is the chord through  $(0,0)$  and  $(\pi/2, \sqrt{2}/2)$  of the curve  $y = \sin(A/2)$ . Therefore, for  $0 \leq A \leq \pi/2$ ,

$$\begin{aligned} \sin \frac{A}{2} &\geq \frac{\sqrt{2}A}{\pi}, \\ \cos A &= 1 - 2 \sin^2 \frac{A}{2} \leq 1 - \frac{4A^2}{\pi^2}, \\ \sin A &= \cos\left(\frac{\pi}{2} - A\right) \leq 1 - \frac{4(\pi/2 - A)^2}{\pi^2} = \frac{4A(\pi - A)}{\pi^2}, \end{aligned}$$

and finally

$$F \leq \frac{4(A + B + C)}{\pi^2} = \frac{4}{\pi}.$$

To prove that (1) is the minimum value of  $F$ , we use an inequality of the form

$$\sin A \geq -p(\pi - A)^2 + q(\pi - A) + r, \quad 0 \leq A \leq \frac{\pi}{2}, \quad (2)$$

where  $p$ ,  $q$  and  $r$  are constants to be chosen. Using this for the first two terms of  $F$ ,

$$\begin{aligned} F &\geq -p(2\pi - A - B) + 2q + r\left(\frac{1}{\pi - A} + \frac{1}{\pi - B}\right) + \frac{\sin C}{\pi - C} \\ &= -p(\pi + C) + 2q + \frac{r(\pi + C)}{(\pi - A)(\pi - B)} + \frac{\sin C}{\pi - C}. \end{aligned} \quad (3)$$

With no loss in generality we assume  $A, B \leq C$ . Then

$$\frac{\pi}{3} \leq C \leq \frac{\pi}{2} \quad \text{and} \quad \pi - 2C \leq A, B \leq C.$$

Keeping  $C$  fixed and assuming  $r \geq 0$ , the minimum value of the right side of (3) occurs when  $A = B = (\pi - C)/2$ . Hence

$$F \geq -p(\pi + C) + 2q + \frac{4r}{\pi + C} + \frac{\sin C}{\pi - C},$$

and we want this to be greater than or equal to (1). Or, multiplying by  $\pi - C$ , we want

$$g(C) = \sin C + (\pi - C)\left[-p(\pi + C) + 2q + \frac{4r}{\pi + C} - \frac{2 + 4\sqrt{2}/3}{\pi}\right] \geq 0 \quad (4)$$

for  $\pi/3 \leq C \leq \pi/2$ .

We now get some necessary conditions on  $p$ ,  $q$  and  $r$ . Setting  $A = \pi/4$  in (2) yields

$$-9\pi^2p + 12\pi q + 16r \leq 8\sqrt{2},$$

while setting  $C = \pi/2$  in (4) yields

$$-9\pi^2p + 12\pi q + 16r \geq 8\sqrt{2}.$$

Hence

$$-9\pi^2p + 12\pi q + 16r = 8\sqrt{2}. \quad (5)$$

This means that the parabola in (2) coincides with  $\sin A$  at  $\pi/4$ . Since we do not want the curves to cross at this point, we will make them touch. This gives

$$\cos A = 2p(\pi - A) - q$$

at  $A = \pi/4$ , or

$$3\pi p - 2q = \sqrt{2}. \quad (6)$$

Setting  $A = \pi/2$  in (2), we get

$$-\pi^2p + 2\pi q + 4r \leq 4.$$

Using (5) and (6) to eliminate  $q$  and  $r$ , this becomes

$$p \geq \frac{-16 + 8\sqrt{2} + 2\pi\sqrt{2}}{\pi^2}.$$

It turns out that this lower bound is a suitable value for  $p$ . Using (5) and (6), we set

$$p = \frac{-16 + 8\sqrt{2} + 2\pi\sqrt{2}}{\pi^2} \approx 0.4255,$$

$$q = \frac{-48 + 24\sqrt{2} + 5\pi\sqrt{2}}{2\pi} \approx 1.298,$$

$$r = \frac{36 - 16\sqrt{2} - 3\pi\sqrt{2}}{4} \approx 0.01098 > 0,$$

and need only verify (2) and (4).

To prove (2) we set

$$f(A) = \sin A + p(\pi - A)^2 - q(\pi - A) - r$$

for  $0 \leq A \leq \pi/2$ . Then

$$f'(A) = \cos A - 2p(\pi - A) + q,$$

$$f''(A) = -\sin A + 2p.$$

Thus  $f''$  is a decreasing function that is positive at  $\pi/4$  and negative at  $\pi/2$ .

$$f'(0) = 1 - 2\pi p + q = 1 + \frac{16 - 8\sqrt{2} - 3\pi\sqrt{2}}{2\pi} < 0,$$

$$f'(\pi/4) = 0,$$

$$f'(\pi/2) = -\pi p + q = \frac{-16 + 8\sqrt{2} + \pi\sqrt{2}}{2\pi} < 0.$$

Thus  $f'(A)$  increases for  $0 \leq A \leq \pi/4$ , crosses the  $A$ -axis at  $\pi/4$ , increases to a maximum in  $\pi/4 < A < \pi/2$  and decreases to a negative value at  $\pi/2$ .

$$f(0) = \pi^2 p - \pi q - r = \frac{-4 + \pi\sqrt{2}}{4} > 0 ,$$

$$f(\pi/4) = f(\pi/2) = 0.$$

Thus  $f(A)$  is positive at 0, decreases for  $0 \leq A \leq \pi/4$ , touches the  $A$ -axis at  $\pi/4$ , increases to a maximum in  $\pi/4 < A < \pi/2$  and decreases to 0 at  $\pi/2$ . Therefore  $f(A) \geq 0$  for  $0 \leq A \leq \pi/2$ .

(4) can be proved in the same way. We have

$$g'(C) = \cos C + 2pC - 2q - \frac{8\pi r}{(\pi + C)^2} + \frac{2 + 4\sqrt{2}/3}{\pi} ,$$

$$g''(C) = -\sin C + 2p + \frac{16\pi r}{(\pi + C)^3} .$$

Since  $r > 0$ ,  $g''$  is a decreasing function.

$$g''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} + 2p + \frac{27r}{4\pi^2} = -\frac{\sqrt{3}}{2} + \frac{460 - 176\sqrt{2} - 17\pi\sqrt{2}}{16\pi^2} < 0 .$$

Therefore  $g''(C) < 0$  for  $\pi/3 \leq C \leq \pi/2$ .

$$g'\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{2\pi p}{3} - 2q - \frac{9r}{2\pi} + \frac{2 + 4\sqrt{2}/3}{\pi}$$

$$= \frac{1}{2} + \frac{-28 + 16\sqrt{2} - 7\pi\sqrt{2}}{24\pi} > 0 ,$$

$$g'\left(\frac{\pi}{2}\right) = \pi p - 2q - \frac{32r}{9\pi} + \frac{2 + 4\sqrt{2}/3}{\pi}$$

$$= \frac{18 - 4\sqrt{2} - 3\pi\sqrt{2}}{9\pi} < 0 .$$

Thus  $g'$  decreases from a positive value at  $\pi/3$  to a negative value at  $\pi/2$ .

$$g\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{8\pi^2 p}{9} + \frac{4\pi q}{3} + 2r - \frac{4}{3} - \frac{8\sqrt{2}}{9}$$

$$= \frac{\sqrt{3}}{2} + \frac{-20 + \pi\sqrt{2}}{18} > 0 ,$$

$$g\left(\frac{\pi}{2}\right) = 0 .$$

Thus, starting with a positive value at  $\pi/3$ ,  $g$  increases to a maximum and then decreases to 0 at  $\pi/2$ . Therefore  $g(C) \geq 0$  for  $\pi/3 \leq C \leq \pi/2$ .

*Also solved by EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; COLIN SPRINGER, student, University of Waterloo; and the proposer. The right-hand inequality only was solved by J.T. GROENMAN, Arnhem, The Netherlands and by D.J. SMEENK, Zaltbommel, The Netherlands.*

The sharp lower and upper bounds established by Henderson were also given, without proof, by Hess.

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**1368.** [1988: 203] Proposed by Florentin Smarandache, Craiova, Romania.

Let  $ABCD$  be a tetrahedron and  $A_1 \in CD$ ,  $A_2 \in CB$ ,  $C_1 \in AD$ ,  $C_2 \in AB$  be four coplanar points. Let  $E = BC_1 \cap DC_2$  and  $F = BA_1 \cap DA_2$ . Prove that the lines  $AE$  and  $CF$  intersect.

*Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

As  $A_1$ ,  $A_2$ ,  $C_1$  and  $C_2$  are coplanar,  $A_1A_2$  and  $C_1C_2$  intersect in a point  $S$  of  $BD$ . Let  $AE$  intersect  $BD$  in  $T$ . Then  $B$ ,  $D$ ,  $S$  and  $T$  form a set of four harmonic points:

$$\frac{TD}{TB} : \frac{SD}{SB} = -1$$

[e.g. see Theorem 220, p. 149 of R.A. Johnson, *Advanced Euclidean Geometry*].

Similarly, if  $CF$  intersects  $BD$  in  $T'$ , we have

$$\frac{T'D}{T'B} : \frac{SD}{SB} = -1 .$$

Thus  $T = T'$ .

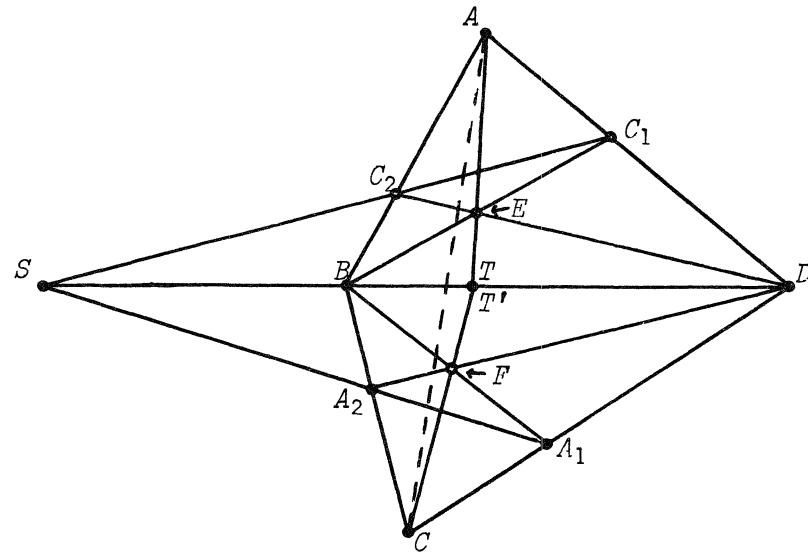
Also solved by JORDI DOU, Barcelona, Spain; EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. There was one incorrect solution submitted.

Dou's solution was the same as Smeenk's.

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