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Editor: Léo Sauvé, Architecture Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Gorettie C.F. Luk.

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CONTENTS

A GENERAL DIVISIBILITY TEST

CLAYTON W. DODGE

Divisibility tests have been "in the air" in recent years. As examples, within the last two or three years there have been: articles by Burton [1], De Leon [4], and Stinson [8]; others by Engle [5] and Yazbak [9] in the same issue of *Mathematics Teacher*, with a sequel by Smith [7]; and no doubt still others that have not come to the author's attention. This article was impelled by the prevailing winds, as well as by a letter to the editor [6] asking whether there is a general divisibility test for primes including 2 and 5. Such a test will be presented here. It was taught by the author for several years to his college freshman and other mathematics classes using his own texts [2, 3]. It is practical for testing divisibility by small numbers (composites as well as primes) and by certain larger numbers.

Definition 1. A set of *digit weights* is an ordered collection of integers

$$(b_0, b_1, b_2, \ldots, b_{k-1};; c_0, c_1, c_2, \ldots, c_{m-1})$$

divided into two subsets by a double semicolon (double for emphasis). Each subset is finite and the second subset is always nonempty. If the first subset is empty, the double semicolon is omitted.

Thus $(1,3;;2,4,1)$ is a set of digit weights in which $b_0 = 1$, $b_1 = 3$, $c_0 = 2$, $c_1 = 4$, and $c_2 = 1$. Also $(-3,0)$ is a set of digit weights having $c_0 = -3$ and $c_1 = 0$.

Digit weights are to be used for testing divisibility by means of weighted digit sums, which we next define.

Definition 2. Let S denote the set of digit weights of Definition 1, and let

$$q = a_n a_{n-1} \ldots a_1 a_0$$

be a base ten positive integer whose digits are a_n, a_{n-1}, ..., a_0. The *weighted digit sum* (w.d.s.) for the integer q using the set S is the sum

$$b_0 a_0 + b_1 a_1 + \ \ldots + b_{k-1} a_{k-1} + c_0 a_k + c_1 a_{k+1} + \ \ldots + c_{n-k} a_n,$$

in which $c_m = c_0$, $c_{m+1} = c_1$, and in general $c_{m+r} = c_r$ as needed. The weighted digit sum need not use all the c_r or all the b_r when q is small.

Thus the weighted digit sum is the sum of each digit of the integer q multiplied by a digit weight, using the b_r's only once each and running through the c_r's as many times as necessary. For example, the w.d.s. for 6312785 using the set $(3,-2;;1,4,0)$ is

$$3\cdot 5 + (-2)\cdot 8 + 1\cdot 7 + 4\cdot 2 + 0\cdot 1 + 1\cdot 3 + 4\cdot 6 = 41,$$

and the w.d.s. for 582 using the same set is

$$3\cdot 2 + (-2)\cdot 8 + 1\cdot 5 = -5.$$

Observe that the first digit weight is multiplied by the *units'* digit of q, the second by the tens' digit, and so forth. A convenient setup for calculating a w.d.s. is to write the integer q, spacing out its digits, then to write the digit weights, working from right to left, over the digits of q. Now we multiply down each column and then add the products to get the w.d.s. For the examples above we would write

$$\begin{array}{ccccccc} 4 & 1 & 0 & 4 & 1 & -2 & 3 \\ 6 & 3 & 1 & 2 & 7 & 8 & 5 \\ \hline \end{array}$$
$$24 + 3 + 0 + 8 + 7 - 16 + 15 = 41$$

and

$$\begin{array}{ccc} 1 & -2 & 3 \\ 5 & 8 & 2 \\ \hline \end{array}$$
$$5 - 16 + 6 = -5.$$

For the set (1,-3,2), the w.d.s. for 6312785 is given by

$$\begin{array}{ccccccc} 1 & 2 & -3 & 1 & 2 & -3 & 1 \\ 6 & 3 & 1 & 2 & 7 & 8 & 5 \\ \hline \end{array}$$
$$6 + 6 - 3 + 2 + 14 - 24 + 5 = 6$$

and that for 26 by

$$\begin{array}{cc} -3 & 1 \\ 2 & 6 \\ \hline \end{array}$$
$$-6 + 6 = 0.$$

We intend to show that an integer q can be tested for divisibility by an integer $d > 1$ by testing whether an appropriately chosen w.d.s. is divisible by d.

Definition 3. A set S of digit weights is called a *divisibility set (of digit weights) for (the divisor) d* if every positive integer q and its w.d.s. using S both leave the same remainder when divided by d.

We look at just one more preliminary item. If 100 is divided by 6, it is customary to take the quotient equal to 16 and the remainder 4, since

$$100 = 6\cdot 16 + 4.$$

Because 4 is greater than half the divisor, it will be convenient to increase the quotient by 1 and decrease the remainder accordingly. We take 100 divided by 6 to have quotient 17 and remainder -2 $(= 4 - 6)$ because

$$100 = 6\cdot 17 - 2.$$

Thus, when dividing by an integer $d > 1$, it is always possible to select the remainder

r so that

$$-\frac{d}{2} < r \le \frac{d}{2}.$$

Such a remainder is called the *least absolute remainder*. For the divisor 6, the least absolute remainders are 0, 1, 2, 3, -2, and -1 instead of the usual remainders 0, 1, 2, 3, 4, and 5, respectively.

THEOREM 1. If successive nonnegative integral powers of 10 (1, 10, 100, 1000, ...) *are divided by the positive integral divisor* $d > 1$, *then the least absolute remainders will eventually repeat with period no longer than* $d-1$ *digits.*

Proof. When dividing by d, there are only d different (least absolute) remainders, so when the division is carried through $d+1$ steps, the same remainder must appear at least twice. The sequence of remainders from one appearance to the next must repeat since the same division steps occur. If the remainder 0 ever appears, then all following remainders must also be 0. □

It is easy to calculate the remainders of Theorem 1. We illustrate by finding them for $d = 44$. By long division, divide a high power of 10 by 44, taking a higher power of 10 by appending more zeros, if needed, as the division progresses. Note the remainders. We have

```
     00227
 44)10000000000
     88
     120
      88
      320
      308
       12.
```

The division is stopped since the remainder 12 has appeared a second time, so the remainders 12 and 32 will repeat. Now 1 divided by 44 leaves remainder 1, 10 leaves 10, 100 leaves 12, and so forth. We list these remainders as a set of digit weights

$$(1,10;;12,32).$$

Notice that 1 and 10 do not repeat, so they appear before the double semicolon. For convenience, since $32 > 44/2$, we use $32 - 44 = -12$ in place of 32, obtaining the set

$$(1,10;;12,-12)$$

of least absolute remainders.

The remainders of Theorem 1 form a divisibility set for the divisor d, as we shall prove shortly. First we list some such sets. Observe that 1 is the first member of each set since 1 divided by any $d > 1$ always leaves the remainder 1.

d	divisibility set
2	(1;;0)
3	(1)
4	(1,2;;0)
5	(1;;0)
6	(1;;4)
7	(1,3,2,-1,-3,-2)
8	(1,2,4;;0)
9	(1)
10	(1;;0)
11	(1,-1)
12	(1,-2;;4)
13	(1,-3,-4,-1,3,4)
14	(1;;-4,2,6,4,-2,-6)
15	(1;;-5)
16	(1,-6,4,8;;0)
17	(1,-7,-2,-3,4,6,-8,5,-1,7,2,3,-4,-6,8,-5)
18	(1;;-8)
19	(1,-9,5,-7,6,3,-8,-4,-2,-1,9,-5,7,-6,-3,8,4,2)
20	(1,10;;0)

THEOREM 2. The set S of remainders of Theorem 1 forms a divisibility set for the divisor d.

Our proof, for convenience, uses modular arithmetic, but it is not difficult to construct a proof that does not rely on that tool.

Proof. Let the positive integer q have the decimal representation

$$q = a_n a_{n-1} \ldots a_1 a_0$$

and, for $k = 0, 1, 2, \ldots, n$, let r_k be the remainder when 10^k is divided by d, so that

$$10^k \equiv r_k \pmod{d}.$$

Each such congruence can be multiplied by the corresponding integer a_k, giving

$$a_k \cdot 10^k \equiv a_k r_k \pmod{d},$$

and then all of them can be added to yield

$$q = a_0 \cdot 1 + a_1 \cdot 10 + a_2 \cdot 10^2 + \ldots + a_{n-1} \cdot 10^{n-1} + a_n \cdot 10^n$$

$$\equiv a_0r_0 + a_1r_1 + a_2r_2 + \ldots + a_{n-1}r_{n-1} + a_nr_n \pmod{d}.$$

By definition of modular congruence, q and the latter sum, which is the w.d.s. for q using the remainders of Theorem 1, leave the same remainder when divided by d. In particular, q and its w.d.s. either both are, or both are not, divisible by d. □

Fermat's Theorem [3, p. 234] implies that, if a prime p does not divide 10, then

$$10^{p-1} \equiv 1 \pmod{p}.$$

Thus the remainder when 10^{p-1} is divided by p is 1. Since 1 is the first member of any divisibility set of Theorem 1, then we have shown that the divisibility set for a prime d different from 2 and 5 consists of a repetend part $(c_0, c_1, \ldots, c_{m-1})$ only. (Is it possible to find a rule for composite divisors d? What other patterns can be observed in the listed divisibility sets for primes? See especially those for 7, 13, 17, and 19.)

Although least absolute remainders are usually more convenient to use, other remainders can be used and may be better for some cases. For example, it seems easier to use (1;;10) in place of (1;;-5) when testing divisibility by 15 and the same (1;;10) in place of (1;;-8) for 18. Thus, to decide whether 638215236 is divisible by 18, we find its w.d.s. to be

$$\begin{array}{ccccccccc} 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 1 \\ 6 & 3 & 8 & 2 & 1 & 5 & 2 & 3 & 6 \\ \hline \end{array}$$
$$60 + 30 + 80 + 20 + 10 + 50 + 20 + 30 + 6 = 306.$$

It may not be clear whether 306 is divisible by 18 because 306 is a bit large for mental verification. In any case, 306 and 638215236 both are or both are not divisible by 18. Let us apply the digit weight test to 306 to decide the issue. We find its w.d.s. to be

$$\begin{array}{ccc} 10 & 10 & 1 \\ 3 & 0 & 6 \\ \hline \end{array}$$
$$30 + 0 + 6 = 36.$$

Since 36 is divisible by 18, then 306 is also divisible by 18, and finally 638215236 is divisible by 18.

Many of the common tests for divisibility are easily derived from the digit weight test. For example, to test for divisibility by 4, one commonly checks whether the number formed by the last two digits of the given number q is divisible by 4. Thus 63527186 is not divisible by 4 since 86 is not. According to the list above, a divisibility set for 4 is (1,2;;0). Since $2 \equiv 10 \pmod{4}$, we can replace the 2 by 10 to get the set (1,10;;0) for 4. Applying the latter set to 63527186, we get the w.d.s.

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 10 & 1 \\ 6 & 3 & 5 & 2 & 7 & 1 & 8 & 6 \\ \hline \end{array}$$
$$0+0+0+0+0+0+80+6=86,$$

the number formed by the last two digits of 63527186. Furthermore, by Theorem 2, since 86 leaves the remainder 2 when divided by 4, then we know that 63527186 will also leave remainder 2 when divided by 4.

Again, to test whether 63527186 is divisible by 11, we use the divisibility set (1,-1) and find the w.d.s.

$$\begin{array}{cccccccc} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 6 & 3 & 5 & 2 & 7 & 1 & 8 & 6 \\ \hline \end{array}$$
$$-6+3-5+2-7+1-8+6=-14.$$

Now -14 is not divisible by 11, so 63527186 is not divisible by 11. To find the remainder when 63527186 is divided by 11, we could divide -14 by 11, obtaining the quotient -2 and the remainder 8. It may be easier, however, to add a multiple of 11 to -14 and obtain a positive number. We take $-14+22=8$ and, of course, 8 divided by 11 leaves the remainder 8. Hence 63527186 also leaves the remainder 8. And this divisibility test also proves the common test for divisibility by 11 of alternately adding and subtracting the digits of the number q, always beginning with the units' digit, subtracting the tens' digit, adding the hundreds' digit, etc. (See also [5].)

The digit weight development is a convenient general method for arriving at divisibility tests for small primes and powers of primes. In general, it does not seem reasonable to require or to emphasize digit weights or divisibility sets as such. If one knows the common divisibility tests for 2, 3, 4, 5, 8, 9, 10, and 11, then there is little need to remember other tests. Who would ever find it convenient to use the divisibility set for 17, for example?

It is unfortunate that no simple set of digit weights exists for 7. But there are two worthwhile techniques for testing divisibility by 7. The first test, in fact, can be used for 11 and 13 as well.

THEOREM 3. (The 1001 test) *If the number formed by the last three digits of a decimal numberal*[1]*q is subtracted from the number formed by all the remaining digits of q, then the difference so formed is divisible by 7, by 11, or by 13 if and only if q is so divisible.*

Proof. The reader can easily verify that

$$(1,10,100,-1,-10,-100)$$

---

[1]The author spotted this typographical error when proofreading his article and decided to make good use of it. "Decimal numberal q" means, very appropriately here, "the number whose decimal numeral is q". (Editor)

is a divisibility set for 7, for 11, and for 13. □

To test 63527186, we repeatedly appy Theorem 3, obtaining

$$63527 - 186 = 63341$$

and

$$63 - 341 = -278.$$

We can conveniently do this arithmetic by using the form:

```
    63 527|186
        -186
    63|341
  -341
  -278.
```

Now 278 (and hence -278 also) is not divisible by 7, nor by 11, nor by 13. Therefore 63527186 is divisible by none of 7, 11, and 13.

This test is called "the 1001 test" because it is based on the fact that $7 \cdot 11 \cdot 13 = 1001$.

THEOREM 4. Remove the units' digit from a positive decimal integer q and subtract twice that digit from the number formed by the remaining digits. Then this difference and q both are or both are not divisible by 7.

Proof. We take $q = 10t + u$, where u is the units' digit of q and t is the number formed by all the other digits of q. Now q is divisible by 7 if and only if $2q$ is divisible by 7, and

$$2q = 20t + 2u = 21t - (t - 2u).$$

Since $21t$ is always divisible by 7, it follows that $2q$ is divisible by 7 if and only if $t - 2u$ is divisible by 7. □

For example, to test 63341 for divisibility by 7, we subtract $2 \cdot 1$ from 6334

```
6 3 3 4|1
     -2
6 3 3 2.
```

Now repeat the test by cutting off the units' digit 2 from 6332 and subtracting twice that digit from the remaining number 633:

```
6 3 3 4|1
     -2
6 3 3|2
   -4
6 2 9.
```

Since 629 is still rather large, we perform the operation once more:

$$\begin{array}{rl}
6\ 3\ 3\ 4 & \underline{|1} \\
-2 & \\
\hline
6\ 3\ 3 & \underline{|2} \\
-4 & \\
\hline
6\ 2 & \underline{|9} \\
-1\ 8 & \\
\hline
4\ 4. &
\end{array}$$

Since the difference 44 is clearly not divisible by 7, then neither are 629, 6332, nor 63341. If 63341 is divided by 7, the remainder is 5, and of course 44 leaves the remainder 2 when divided by 7. So we see that Theorem 4, and also Theorem 3, are used only to test for divisibility, not to calculate remainders. Theorem 3, however, can be altered to give the correct remainder.

The method of Theorem 4 extends to other prime divisors as well. For each prime p, different from 2 and 5, there is a multiplier m such that $q = 10t + u$ and $t + mu$ both are or both are not divisible by p. For $p = 7$, we have seen that $m = -2$. Values of m for a few primes appear below:

p	3	7	11	13	17	19	23	29
m	1	-2	-1	4	-5	2	7	3

To find each m and to prove such tests, find a multiple of p that differs by 1 from a multiple of 10. Thus, for $p = 17$, we have the multiples 17, 34, 51, ... Since 51 differs from 50, a multiple of 10, by 1, we examine $(50/10)q$. Then

$$5q = 50t + 5u = 51t - t + 5u = 51t - (t - 5u).$$

We see that $5q$ is divisible by 17 if and only if $t - 5u$ is divisible by 17, and $5q$ is divisible by 17 if and only if q is divisible by 17, since 5 is not divisible by 17.

So here we have a very practical test for divisibility by a prime. Unfortunately, it cannot be applied to the primes 2 and 5. But it can be applied to any positive integer relatively prime to 10, as discussed by Stinson [8].

REFERENCES

1. D.M. Burton, "Devising Divisibility Tests", *Journal of Recreational Mathematics*, 9 (1976-1977) 258-260.

2. Clayton W. Dodge, *Mathematics: Something to Think About*, 4th preliminary edition, mimeographed 1974, pp. 5-12.

3. ————, *Numbers and Mathematics*, Second Edition; Prindle, Weber and Schmidt, Boston, 1975, pp. 231-235.

4. M.J. De Leon, "A Simple Proof of an Old Divisibility Test", *Journal of Recreational Mathematics*, 11 (1978-1979) 186-189.

5. Jessie Ann Engle, "A Rediscovered Test for Divisibility by Eleven", *Mathematics Teacher*, 69 (December 1976) 669.

6. Charlene Oliver, Letter to the editor, *Mathematics Teacher*, 70 (May 1977) 387.

7. Lehi T. Smith, "A General Test of Divisibility", *Mathematics Teacher*, 71 (November 1978) 668-669.

8. Douglas Stinson, "A Rule for Divisibility", *Ontario Secondary School Mathematics Bulletin*, 13 (September 1977) 10-12.

9. Najib Yazbak, "Some Unusual Tests of Divisibility", *Mathematics Teacher*, 69 (December 1976) 667-668.

Mathematics Department, University of Maine, Orono, Maine 04469.

* * *

PASCAL REDIVIVUS: I

DAN PEDOE

Blaise Pascal (1623-1662) discovered what he called *l'Hexagramme Mystique* (called *la Pascale* by his contemporaries, and which we shall refer to as *Pascal's Theorem*) at the age of 15, four years before he invented his *machine d'arithmétique*. Pascal's Theorem says that if a hexagon ABCDEF is inscribed in a conic, then the intersections of pairs of opposite sides lie on a line. Thus, in Figure 1,

$$L = AB \cap DE, \quad M = BC \cap EF, \quad N = CD \cap FA$$

are collinear. (We have taken advantage of the fact that the hexagon is merely a set of 6 distinct points on the conic to obtain a concise figure.)

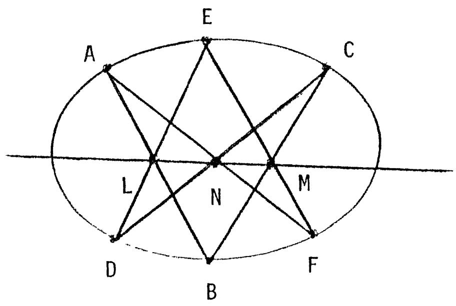


Figure 1

From this single theorem stemmed much if not most of Pascal's *Traité des Coniques*, "comprenant [as Pascal himself wrote] et les coniques d'Apollonius et d'innombrables autres résultats [obtenus] par une seule proposition ou presque". The adjective "innombrables" is clearly (and perhaps appropriately when discussing conics) hyperbolic. But it is understandably so for, according to Mersenne, Pascal deduced 400 propositions from "la Pascale". We shall never know what most of these propositions were because, although Leibniz, in a letter

(30 August 1676) to Étienne Périer, Pascal's nephew, had urged publication of the *Traité*, Périer never got around to it and the *Traité* is now lost, except for the two opuscules *Essai pour les Coniques* (which contains the Mystic Hexagram) and *Generatio Conisectionum*.

All the historical information given above can be found in [3].

The projective theory of conics came later (see [4] for the projective theory), and this tells us that a unique conic passes through 5 given distinct points A, B, C, D, and E. With the help of Pascal's Theorem, we can determine, using only an ungraduated straightedge, any number of points on this conic. We can find the point F (Figure 1) where any line through A intersects the conic again. For we can find $L = AB \cap DE$ and $N = CD \cap AF$, and we know that $M = BC \cap EF$ lies on LN, so we can find M as $BC \cap LN$, and then $F = AN \cap EM$.

The Pascal Theorem holds for special positions of the 6 points. In particular, we can find the tangent to the conic at A by considering the hexagon ABCDEA, with A replacing F, and interpreting AA as the tangent at A to the conic through the 5 points A, B, C, D, and E. This is shown in Figure 2.

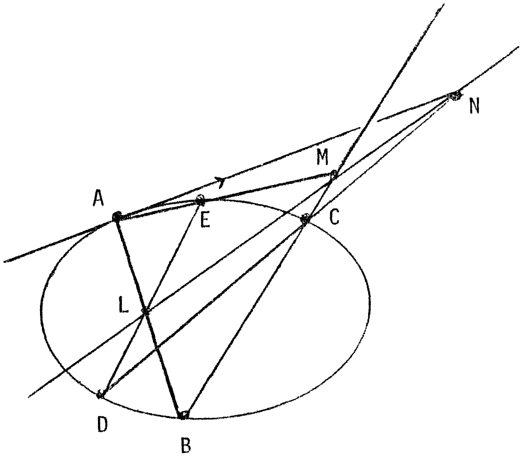


Figure 2

Pascal's 400 propositions may be lost, but the same fate should not be allowed to overtake Pascal's Theorem, which should be known to all geometers and especially to those, geometers or not, who write on geometry. But there is some disturbing evidence that this is not the case.

In a recent paper by Leon Henkin and William A. Leonard, the following problem is posed and solved: *Given a portion of a parabola, can we find the vertex using only a compass and straightedge?* According to the authors, this problem "seems to have passed around in meetings of the California Mathematics Council." But doubts remain with regard to the solution, and the title of the paper is "A Euclidean Construction?" [1].

The doubts arise from the foundational difficulties involved in a construction attempted with an infinite set of points which do not form a circle. An unknown referee gives a more "truly geometric" solution than that given by the authors, and the question is raised, for further study, whether solutions of this geometric nature can be found for the ellipse and hyperbola, "where we only have solutions based on analytical formulas."

In this note I shall show that the parabola problem can be solved with great ease by Euclidean methods, with no "foundational difficulties," *if we are furnished with a mere 5 points on the curve*, using Pascal's Theorem. In a further note[1], I propose to show how the centre, axes, foci, and so on of an ellipse and hyperbola can be found by Euclidean constructions, given 5 points on each curve. This involves some further explanation, since the methods make use of involution theory.

One can understand the lack of knowledge of the projective theory of conics in what seems to be a large circle of mathematicians, but the Pascal Theorem should surely be a part of every mathematician's alphabet. The notion that properties of the ellipse and hyperbola are known only from "analytical formulas" is a sad commentary on the holocaust of our geometrical heritage perpetrated by the "new math" from the beginning of the Sputnik era. That beautiful subject, "Geometrical Conics," which preceded Pascal in the works of Apollonius and Pappus, seems to have gone with the wind in geometrical education. Macaulay's fine textbook [2] happily survives to bring comfort to lovers of geometry.

To solve the parabola problem! The projective theory of conics defines a parabola as a conic which touches the "line at infinity." Let the point of contact be Ω (Figure 3). Then in the Euclidean plane all lines parallel to the axis of the parabola pass through Ω. If U is any point on the line at infinity, the polar of U passes through Ω, and the interpretation of harmonic ranges which have one point at infinity gives us the theorem: *The midpoints of parallel chords of a parabola lie on a line parallel to the axis.* If this line intersects the parabola at the finite point T, the tangent at T has the direction of the set of parallel chords.

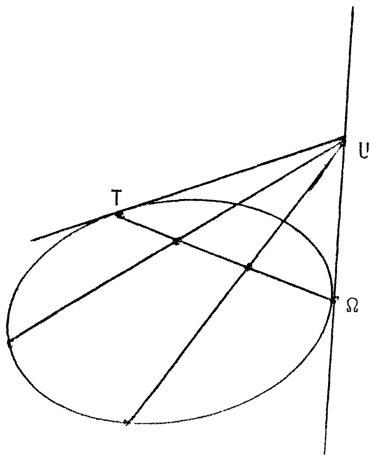


Figure 3

There is a point V on the parabola where the tangent is perpendicular to the

[1]To appear next month in this journal. (Editor)

axis. This is the vertex (Figure 4). The axis bisects all chords of the parabola which are perpendicular to the axis.

We are given 5 points A, B, C, D, and E on the parabola. If CD is parallel to AB, as in Figure 4, bisect both AB and CD and join their midpoints. This gives the direction of the axis. If CD is not parallel to AB, find the point on the parabola which is on the line through C parallel to AB, using the Pascal Theorem, and find the direction of the axis. Drop a perpendicular from A onto the direction of the axis, and find the point A' where this perpendicular intersects the parabola again, using the Pascal Theorem. Then the perpendicular bisector of AA' is the axis of the parabola.

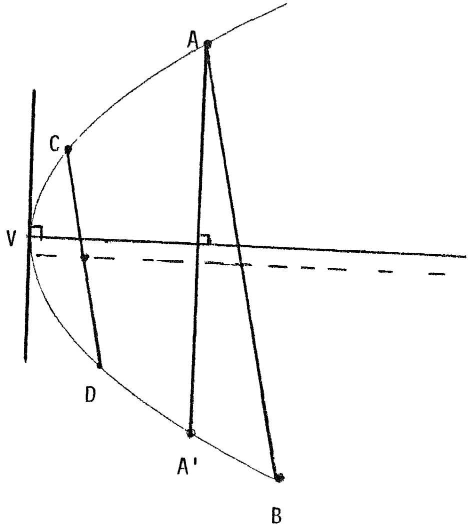


Figure 4

Let V be the unknown vertex on the axis of the parabola. We know that Ω is on the axis, and the join of any point P to Ω gives a line through P which is parallel to the axis.

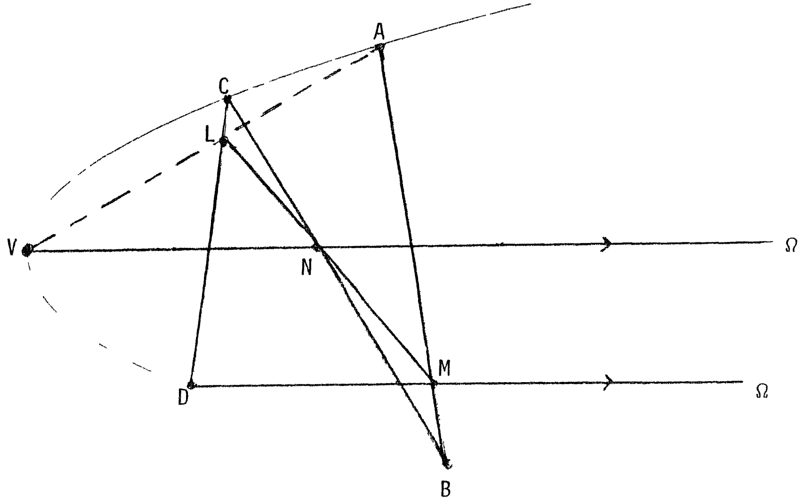


Figure 5

The Pascal Theorem holds for the inscribed hexagon VABCDΩ (Figure 5) and so the three points

$$L = VA \cap CD, \qquad M = AB \cap D\Omega, \qquad N = BC \cap \Omega V$$

are collinear. The point L lies on CD but is not known for the moment. The line DΩ through D is parallel to the axis, and so M is found. The line ΩV is the axis, so N is found. Now we have $L = CD \cap MN$, and LA intersects the axis in V.

Having found V, the line through V perpendicular to the axis is the tangent to the parabola at V. If we find the tangent at A, using the Pascal Theorem, and assume the theorem [2, p. 38] that the feet of the perpendiculars from the focus of a parabola onto the tangents lie on the tangent at the vertex, we can find the focus S. If the tangent at A intersects the tangent at V in T, then the perpendicular at T to the tangent at A intersects the axis in the focus S.

There is no doubt that approaching the conic sections via their algebraic equations is a dismal path, the foci being introduced quite arbitrarily as a rule. If the conic sections are considered as envelopes of their tangents, paper-folding very quickly produces the curves themselves, and all the focal properties are easily demonstrated, as I show in my recent book [5].

REFERENCES

1. Leon Henkin and William A. Leonard, "A Euclidean Construction?", *Mathematics Magazine*, 51 (1978) 294-298.

2. F.S. Macaulay, *Geometrical Conics*, Cambridge University Press, 1921.

3. Blaise Pascal, *Oeuvres Complètes*, Aux Éditions du Seuil, Paris, 1963.

4. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, New York, 1970.

5. ————, *Geometry and the Liberal Arts*, St. Martin's Press, New York, 1978, in North America; Penguin Books, London, 1976, elsewhere.

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

* * *

MATHEMATICAL SWIFTIES

Tom Swifties were all the rage some years ago and many of a mathematical nature appeared in various journals. The fad, long thought to be dead, was merely quiescent. In the Book Review section of the *New York Times* for November 4, 1979 appeared a brief announcement to the effect that the *Tom Swift* series of books for retarded adolescents will be reissued in 1980. A new rash of Tom Swifties is sure to follow. CRUX readers will later recall where the fad was first revived. Right here:

"There are solutions to $F(x) = x$," Tom maintained fixedly.

MURRAY S. KLAMKIN

THE OLYMPIAD CORNER: 9

MURRAY S. KLAMKIN

We present this month one new Practice Set, No. 7, and the solutions, promised last month, for the 15th British Mathematical Olympiad (1979).

PRACTICE SET 7 (3 hours)

7-1. (a) Determine $F(x)$ if, for all real x and y,

$$F(x)F(y) - F(xy) = x + y.$$

(b) Generalize.

7-2. $A_1A_2A_3A_4$ denotes a kite (i.e., $A_1A_2 = A_1A_4$ and $A_3A_2 = A_3A_4$) inscribed in a circle. Show that the incenters I_1, I_2, I_3, I_4 of the respective triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$ are the vertices of a square. (Jan van de Craats, The Netherlands.)

7-3. Show that the polynomial equation with real coefficients

$$P(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_{n-3}x^3 + x^2 + x + 1 = 0$$

cannot have all real roots.

*

15TH BRITISH MATHEMATICAL OLYMPIAD (1979)

1. Find all triangles ABC for which

$$AB + AC = 2 \text{ cm} \quad \text{and} \quad AD + BC = \sqrt{5} \text{ cm},$$

where AD is the altitude through A, meeting BC at right angles in D.

Solution.

In such a triangle ABC, let $x = BD$, $y = DC$, and $z = AD$, where the distances are directed so that $x + y > 0$ and $z > 0$. (There is no *a priori* reason to expect that the altitude AD will fall inside the triangle, so we must allow for the possibility $x < 0$ or $y < 0$, subject to $BC = x + y > 0$.) The hypothesis then yields

$$\sqrt{x^2 + z^2} + \sqrt{y^2 + z^2} = 2 \tag{1}$$

and

$$x + y + z = \sqrt{5}. \tag{2}$$

Equations (1) and (2) hold only if

$$\sqrt{x^2+z^2}+\sqrt{y^2+z^2} = \frac{2}{\sqrt{5}}(x+y+z). \tag{3}$$

Now

$$\begin{aligned}(x-2z)^2 \ge 0 \implies x^2+z^2 &\ge \frac{(2x+z)^2}{5}\\ \implies \sqrt{x^2+z^2} &\ge \frac{|2x+z|}{\sqrt{5}} \ge \frac{2x+z}{\sqrt{5}}\,,\end{aligned} \tag{4}$$

with equality if and only if $x = 2z$; and similarly

$$\sqrt{y^2+z^2} \ge \frac{2y+z}{\sqrt{5}}\,, \tag{5}$$

with equality if and only if $y = 2z$. From (4) and (5), we now get

$$\sqrt{x^2+z^2}+\sqrt{y^2+z^2} \ge \frac{2}{\sqrt{5}}(x+y+z),$$

with equality if and only if $x = y = 2z$. Thus (3) requires $x = y = 2z$ and then, from (1) or (2),

$$x = y = \frac{2}{\sqrt{5}} \qquad \text{and} \qquad z = \frac{1}{\sqrt{5}};$$

so the only triangle that satisfies the problem has sides

$$\mathrm{BC} = \frac{4}{\sqrt{5}}\ \mathrm{cm}, \qquad \mathrm{CA} = \mathrm{AB} = 1\ \mathrm{cm}.$$

We leave as a follow-up exercise an analogous 3-dimensional problem: *Determine all tetrahedra* P-ABC *such that*

$$\mathrm{PA} + \mathrm{PB} + \mathrm{PC} = 3 \qquad \textit{and} \qquad \mathrm{QA} + \mathrm{QB} + \mathrm{QC} + \mathrm{QP} = \sqrt{10},$$

where PQ *is an altitude of* P-ABC.

2. From a point O in 3-dimensional space, three given rays OA, OB, OC emerge, the angles BOC, COA, AOB being α, β, γ, respectively ($0 < \alpha,\beta,\gamma < \pi$).

Prove that, given $2s > 0$, there are unique points X, Y, Z on OA, OB, OC, respectively, such that the triangles YOZ, ZOX, and XOY have the same perimeter $2s$, and express OX in terms of s and $\sin\frac{1}{2}\alpha$, $\sin\frac{1}{2}\beta$, $\sin\frac{1}{2}\gamma$.

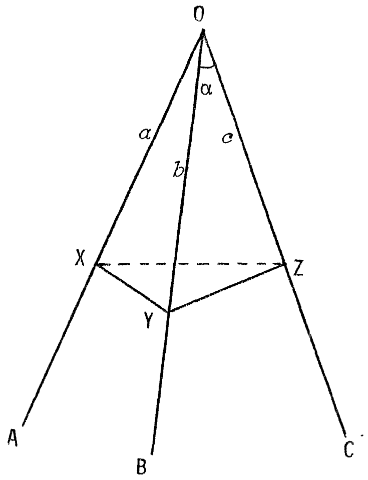
Solution.

Let Y and Z be points on OB and OC, respectively, such that $\mathrm{OY} = b > 0$ and $\mathrm{OZ} = c > 0$ (see figure). We will show that triangle YOZ has perimeter $2s$ if and only if

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}\,, \tag{1}$$

which is equivalent to

$$(s/b-1)(s/c-1)=\sin^2\frac{\alpha}{2}\,. \qquad (2)$$



The *only if* part is the usual derivation of the half-angle formula (1), which can be found in all trigonometry texts. For the converse (which is *not* proved in most texts), suppose (2) holds for some s greater than both b and c. Then

$$\begin{aligned}YZ^2 &= b^2+c^2-2bc\cos\alpha\\ &= b^2+c^2-2bc(1-2\sin^2\tfrac{\alpha}{2})\\ &= (b-c)^2+4(s-b)(s-c)\\ &= (b+c)^2+4s^2-4s(b+c)\\ &= [2s-(b+c)]^2;\end{aligned}$$

so $YZ = 2s-(b+c)$ and $YZ+b+c=2s$, as required.

It follows from the above that, if X is a point on OA such that $OX = a > 0$, then triangle ZOX and XOY also have the same perimeter $2s$ if and only if

$$(s/c-1)(s/a-1)=\sin^2\frac{\beta}{2} \quad\text{and}\quad (s/a-1)(s/b-1)=\sin^2\frac{\gamma}{2}. \qquad (3)$$

Now (2) and (3) imply

$$(s/a-1)(s/b-1)(s/c-1)=\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \qquad (4)$$

and then, from (4) and (2),

$$s/a-1 = \frac{\sin(\beta/2)\sin(\gamma/2)}{\sin(\alpha/2)},$$

from which the unique value of OX is

$$OX = a = \frac{s\sin(\alpha/2)}{\sin(\alpha/2)+\sin(\beta/2)\sin(\gamma/2)}\,. \qquad (5)$$

The unique values of OY and OZ are not required by the problem; but they are easily found from (5) by a circular permutation of α, β, γ.

3. S is a set of distinct positive odd integers $\{a_i\}$, $i=1, \ldots, n$. No two differences $|a_i-a_j|$ are equal, $1 \le i < j \le n$. Prove that

$$\sum_{i=1}^{n} a_i \ge \frac{1}{3}n(n^2+2).$$

Solution.

We can assume, without loss of generality, that $a_1 < a_2 < \ldots < a_n$. Let r be an integer such that $2 \le r \le n$. For $1 \le i,j \le r$, there are $\frac{1}{2}r(r-1)$ distinct differences $|a_i - a_j|$, and these are all even numbers. Thus

$$a_r - a_1 \ge 2 \cdot \tfrac{1}{2}r(r-1) = r(r-1)$$

and so

$$a_r \ge a_1 + r(r-1) \ge 1 + r(r-1). \qquad (1)$$

Observing that (1) holds also for $r = 1$, we get

$$\sum_{r=1}^{n} a_r \ge \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} r(r-1) = n + \frac{(n-1)n(n+1)}{3} = \frac{1}{3}n(n^2+2).$$

4. The function f is defined on the rational numbers and takes only rational values. Prove that f is constant if, for all rational x and y,

$$f(x + f(y)) = f(x)f(y). \qquad (1)$$

Solution.

The constant functions taking the values 0 and 1 are clearly solutions of the functional equation (1), and they are the only constant function solutions. For if $f(x) = c$ for all (rational) x, then, from (1), $c = c^2$ and $c = 0$ or 1. We show there are no other solutions of any kind.

If $f(y) = 0$ for some y, then $f(x) = 0$ for all x by (1). Consequently, we now assume that $f(y) \ne 0$ for all y. We first show that there is a rational r such that $f(r) = 1$. Let y be an arbitrary rational number. For $x = y - f(y)$, (1) becomes

$$f(y) = f(y - f(y))f(y);$$

hence $f(y - f(y)) = 1$ since $f(y) \ne 0$, and we can take $r = y - f(y)$. Now, for any x,

$$f(x + f(r)) = f(x)f(r) \qquad \text{or} \qquad f(x+1) = f(x),$$

and it follows inductively that

$$f(x+n) = f(x) \qquad (2)$$

holds for all integers n and all (rational) x.

With y still arbitrary, suppose that $f(y) = p/q$, where p and q are integers and $q > 0$. It is easy to show from (1) by induction that, for positive integer n,

$$f(nf(y)) = f(0)\{f(y)\}^n.$$

Hence

$$f(p) = f(qf(y)) = f(0)\{f(y)\}^q,$$

and so $f(y) = \pm 1$ since $f(p) = f(0) \ne 0$ by (2). But if $f(y) = -1$, then $f(x-1) = -f(x)$

by (1) and $f(x)=0$ by (2), contrary to our assumption about f. Thus $f(y)=1$ for all y and the proof is complete.

5. For n a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4)=5$, because there are 5 different sums, namely,

$$1+1+1+1, \quad 1+1+2, \quad 1+3, \quad 2+2, \quad 4.$$

Prove that, for $n>1$,

$$p(n+1)-2p(n)+p(n-1)\geq 0.$$

Solution

The example given for $n=4$ shows that $p(n)$ is the number of ways of expressing n as a sum of one or more positive integers, *irrespective of the order of the terms*.

By adjoining an extra 1 to any of the $p(n)$ expressions for n, we obtain an expression for $n+1$ which contains the summand 1 at least once. In this way, we get different sums for $n+1$ from different sums for n. Furthermore, every sum for $n+1$ containing a 1 is obtained in this way. Consequently,

$$q(n+1) \equiv p(n+1)-p(n)$$

is the number of ways of expressing $n+1$ as a sum of positive integers none of which is a 1, and the required inequality is equivalent to

$$q(n+1)\geq q(n).$$

Consider any one of the $q(n)$ expressions for n (i.e. one with terms at least 2) and add 1 to the largest term. This gives an expression for $n+1$ with terms at least 2. Also, in this way we get different q-sums for $n+1$ from different q-sums for n. Consequently, $q(n+1)\geq q(n)$.

6. Prove that in the infinite sequence of integers

$$10001, \quad 100010001, \quad 1000100010001, \ldots$$

there is no prime number.

Note that each integer after the first (ten thousand and one) is obtained by adjoining 0001 to the digits of the previous integer.

Solution.

The sequence

$$1, \quad 10001, \quad 100010001, \quad 1000100010001, \ldots$$

can be written

$$1, \quad 1+10^4, \quad 1+10^4+10^8, \quad 1+10^4+10^8+10^{12}, \ \ldots,$$

where the nth term a_n is the sum of n terms of a geometric progression. Thus

$$a_n = \frac{10^{4n}-1}{10^4-1}, \qquad n=1,2,3,\ldots,$$

and we must show that a_n is composite for $n \geq 2$.

According to the parity of n, we have

$$a_{2k} = \frac{10^{8k}-1}{10^4-1} = \frac{10^{8k}-1}{10^8-1} \cdot \frac{10^8-1}{10^4-1}, \qquad k=2,3,\ldots$$

and

$$a_{2k+1} = \frac{10^{4(2k+1)}-1}{10^4-1} = \frac{10^{2(2k+1)}-1}{10^2-1} \cdot \frac{10^{2(2k+1)}+1}{10^2+1}, \qquad k=1,2,\ldots$$

In each case, a_n is the product of two integers both greater than 1. Hence a_n is composite for all $n > 2$. Finally, $a_2 = 10001 = 73 \cdot 137$.

As noted earlier in this journal [1979: 203], this problem appeared in a 1975 Federal Republic of Germany Olympiad. See Crux 389 [1979: 202] for a related but harder problem.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

481. *Proposé par Herman Nyon, Paramaribo, Surinam.*

Résoudre la cryptarithmie doublement vraie suivante dans laquelle, pour l'unicité, nous exigeons raisonnablement que DEUX soit divisible par 2 et, moins raisonnablement, que DOUZE soit divisible par 8:

$$\begin{array}{r}\text{DEUX}\\ \text{DEUX}\\ \text{DEUX}\\ \text{TROIS}\\ \text{TROIS}\\ \hline \text{DOUZE}\end{array}$$

482. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In the prime year 1979, we celebrate the 190th anniversary of the French Revolution, which occurred in the prime year 1789. *Allons, enfants de la Patrie,* let us commemorate *le jour de gloire* by constructing a fourth-order magic square composed of distinct two-digit primes, four of which are situated as shown:

	17	89	
	19	79	

483. *Proposed by Stanley Collings, The Open University, Milton Keynes, England.*

Let ABCD be a convex quadrilateral; let $AB \cap DC = F$ and $AD \cap BC = G$; and let I_A, I_B, I_C, I_D, be the incentres of triangles BCD, CDA, DAB, ABC, respectively. Prove that:

(a) ABCD is a cyclic quadrilateral if and only if the internal bisectors of the angles at F and G are perpendicular.

(b) If ABCD is cyclic, then $I_AI_BI_CI_D$ is a rectangle. (*) Is the converse true?

484. *Proposed by Gali Salvatore, Perkins, Québec.*

Let A and B be two independent events in a sample space, and let χ_A, χ_B be their characteristic functions (so that, for example, $\chi_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$). If $F = \chi_A + \chi_B$, show that at least one of the three numbers

$$a = P(F{=}2), \quad b = P(F{=}1), \quad c = P(F{=}0)$$

is not less than 4/9.

485. *Proposed by M.S. Klamkin, University of Alberta.*

Given three concurrent cevians of a triangle ABC intersecting at a point P, we construct three new points A', B', C' such that AA' = kAP, BB' = kBP, CC' = kCP, where $k > 0$, $k \neq 1$, and the segments are directed. Show that A, B, C, A', B', C' lie on a conic if and only if $k = 2$.

This problem generalizes Crux 278 [1977: 227; 1978: 109].

486. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

(a) Find all natural numbers N whose decimal representation

$$N = \overline{abcdefghi}$$

consists of nine distinct nonzero digits such that

$$2|a-b, \quad 3|a-b+c, \quad 4|a-b+c-d, \quad \ldots, \quad 9|a-b+c-d+e-f+g-h+i.$$

(b) Do the same for natural numbers $N = \overline{abcdefghij}$ consisting of ten distinct digits (leading zeros excluded) such that

$$2|a-b, \quad 3|a-b+c|, \quad \ldots, \quad 10|a-b+c-d+e-f+g-h+i-j.$$

487. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

If a, b, c, and d are positive real numbers such that $c^2+d^2 = (a^2+b^2)^3$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1,$$

with equality if and only if $ad = bc$.

488.* *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given a point P within a given angle, construct a line through P such that the segment intercepted by the sides of the angle has minimum length.

489. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Find all real numbers x, y, z such that

$$(1-x)^2 + (x-y)^2 + (y-z)^2 + z^2 = \frac{1}{4}.$$

490.* *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Are there infinitely many palindromic primes (e.g., 131, 70207)?

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

401. [1979: 14] *Proposed by Herman Nyon, Paramaribo, Surinam.*

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 HAPPY
   NEW
  YEAR
 -----
*1979
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In this decimal alphametic, replace the eight distinct letters and the asterisk by nine distinct *nonzero* digits. Since years come in cycles of seven (seven lean years, seven fat years), YEAR should be divisible by 7.

I hope this will turn out to be a doubly-true alphametic for all readers of this journal.

Solution by Charles W. Trigg, San Diego, California.

Of course, H + 1 = *. Then, determining possibilities and eliminating contradictions, the values of the letters and the asterisk are determined in the order Y, W, R; P, E, A; and N, H, *, with the divisibility by 7 fixing the values of the otherwise interchangeable W and R, and P and E. The unique result of the slugfest:

$$\begin{array}{r} 72119 \\ 536 \\ \underline{9324} = 7\cdot 1332. \\ 81979 \end{array}$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ALLAN Wm. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Johnson made a computer search and found 28 unrestricted solutions, of which only five have YEAR divisible by 7, among which just one (the one given above) is composed of nonzero digits.

* * *

402. [1979: 14] *Proposed by the late R. Robinson Rowe, Sacramento, California.*

An army with an initial strength of A men is exactly decimated each day of a 5-day battle and reinforced each night with R men from the reserve pool of P men, winding up on the morning of the 6th day with 60% of its initial strength. At least how large must the initial strength have been if

(a) R was a constant number each day;

(b) R was exactly half the men available in the dwindling pool?

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

(a) The positive integers A and R satisfy the problem if and only if

$$(0.9)^5A + R\sum_{i=0}^{4}(0.9)^i = 0.6A, \qquad (1)$$

that is,

$$0.59049A + 4.0951R = 0.6A,$$

which is equivalent to

$$409510R = 951A.$$

Since 409510 and 951 are relatively prime, the least acceptable value of A is 409510, and then $R = 951$.

(b) Here we merely replace the second term in (1) by

$$P\sum_{i=0}^{4}(0.9)^i \times (0.5)^{5-i} = Q\sum_{i=0}^{4}(1.8)^i = 22.3696Q,$$

where $Q = P/32$ is an integer, and then obtain the equivalent equation

$$2236960Q = 951A.$$

Again, 2236960 and 951 are relatively prime, so the least acceptable value of A is 2236960, and then $Q = 951$ and $P = 32 \times 951 = 30432$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ALLAN Wm. JOHNSON Jr., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; HERMAN NYON, Paramaribo, Surinam; CHARLES W. TRIGG, San Diego, California; and the proposer.

Editor's comment.

Some solvers arrived at a different answer by assuming that no reinforcements came during the fifth night, since the battle was won by the end of the fifth day. But the proposal clearly implies that the process is carried on until the morning of the sixth day, so it must be assumed that the enemies only acknowledged their defeat by folding their tents and silently stealing away during the fifth night, leaving the victors with a clear field the next morning. In any case, such was the proposer's intent, for his answer agrees with the one given above.

The proposer, our late friend "Bob" Rowe, wrote that this was one of the problems he created to while away the long hours when he was lying flat on his back, during what turned out to be his terminal illness.

* * *

403. [1979: 15] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $Z^+ = \{0,1,2,...\}$ and set

$$A_1 = \{3m^2 + 6mn + 3n^2 + 2m + 3n + 1 \ : \ m,n \in Z^+\},$$
$$A_2 = \{3m^2 + 6mn + 3n^2 + 4m + 5n + 2 \ : \ m,n \in Z^+\},$$
$$A_3 = \{3m^2 + 6mn + 3n^2 + 5m + 6n + 3 \ : \ m,n \in Z^+\},$$
$$A_4 = \{3m^2 + 6mn + 3n^2 + 6m + 7n + 4 \ : \ m,n \in Z^+\},$$
$$A_5 = \{3m^2 + 6mn + 3n^2 + 7m + 8n + 5 \ : \ m,n \in Z^+\},$$
$$A_6 = \{3m^2 + 6mn + 3n^2 + 9m + 10n + 8 \ : \ m,n \in Z^+\},$$

so that

$$A_1 = \{1, 6, 7, 17, 18, 19, 34, 35, 36, 37, 57, 58, 59, 60, 61, \ldots\},$$
$$A_2 = \{2, 9, 10, 22, 23, 24, 41, 42, 43, 44, \ldots\},$$
$$A_3 = \{3, 11, 12, 25, 26, 27, 45, 46, 47, 48, \ldots\},$$
$$A_4 = \{4, 13, 14, 28, 29, 30, 49, 50, 51, 52, \ldots\},$$
$$A_5 = \{5, 15, 16, 31, 32, 33, 53, 54, 55, 56, \ldots\},$$
$$A_6 = \{8, 20, 21, 38, 39, 40, 62, 63, 64, 65, \ldots\}.$$

Prove or disprove that

(a) the elements of A_i are all distinct for $1 \le i \le 6$;

(b) $A_i \cap A_j = \emptyset$ for $1 \le i < j \le 6$;

(c) $\{0\} \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = Z^+$.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Instead of by letting m and n range independently over Z^+, the numbers in each of the sets $A_1, \ldots, A_5$ can be generated by letting $a = m+n = 0, 1, 2, \ldots$ in succession and then, for each a, letting $n = 0, 1, 2, \ldots, a$. But for A_6 we let $a = m+n+1 = 1, 2, 3, \ldots$ in succession and then, for each a, we let $n = 0, 1, 2, \ldots, a-1$. Thus we have

$$A_1 = \{3a^2 + 2a + n + 1 \colon 0 \le a,\ 0 \le n \le a\},$$
$$A_6 = \{3a^2 + 3a + n + 2 \colon 1 \le a,\ 0 \le n \le a-1\},$$
$$A_2 = \{3a^2 + 4a + n + 2 \colon 0 \le a,\ 0 \le n \le a\},$$
$$A_3 = \{3a^2 + 5a + n + 3 \colon 0 \le a,\ 0 \le n \le a\},$$
$$A_4 = \{3a^2 + 6a + n + 4 \colon 0 \le a,\ 0 \le n \le a\},$$
$$A_5 = \{3a^2 + 7a + n + 5 \colon 0 \le a,\ 0 \le n \le a\}.$$

For fixed $a \ge 0$, the six sets contain the following disjoint subsets of consecutive numbers (note that the subset of A_6 is empty when $a = 0$):

in A_1: $3a^2 + 2a + 1$ through $3a^2 + 3a + 1$,
in A_6: $3a^2 + 3a + 2$ through $3a^2 + 4a + 1$,
in A_2: $3a^2 + 4a + 2$ through $3a^2 + 5a + 2$,
in A_3: $3a^2 + 5a + 3$ through $3a^2 + 6a + 3$,
in A_4: $3a^2 + 6a + 4$ through $3a^2 + 7a + 4$,
in A_5: $3a^2 + 7a + 5$ through $3a^2 + 8a + 5$,

and the union of these subsets is the following set of consecutive positive integers:

$$S_a \equiv \{3a^2 + 2a + 1,\ 3a^2 + 2a + 2,\ \ldots,\ 3a^2 + 8a + 5\}.$$

Observe now that the first integer in S_{a+1} is

$$3(a+1)^2 + 2(a+1) + 1 = 3a^2 + 8a + 6, \qquad (1)$$

which is the next integer following the last element of S_a. It follows from (1) that in each A_i the elements are all distinct, which proves part (a); and part (b) follows from (1) and the disjointness of the six subsets forming S_a. Now the relation

$$\bigcup_{i=0}^{a} S_i = \{1, 2, 3, \ldots, 3a^2+8a+5\}$$

follows from (1) and the fact that $S_0 = \{1,2,3,4,5\}$. Finally, we have

$$\bigcup_{i=1}^{6} A_i = \bigcup_{a=0}^{\infty} S_a = Z^+ - \{0\},$$

which proves part (c).

Also solved by E.J. BARBEAU, University of Toronto; ALLAN Wm. JOHNSON Jr., Washington, D.C.; KELLY NOEL, Ridgemont H. S., Ottawa; JEREMY D. PRIMER, student, Columbia H. S., Maplewood, N.J.; and the proposer.

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404. [1979: 15] *Proposed by A. Liu, University of Alberta.*

Let A be a set of n distinct positive numbers. Prove that

(a) the number of distinct sums of subsets of A is at least $\frac{1}{2}n(n+1)+1$;

(b) the number of distinct subsets of A with sum equal to half the sum of A is at most $2^n/(n+1)$.

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

(a) We induct. The result is clearly true for $n=1$ since then there are exactly 2 distinct sums (one of them being 0 for the empty subset of A). Assume it is true for $n=k\geq 1$ and consider the set

$$A = \{a_1, a_2, \ldots, a_{k+1}\}, \qquad (1)$$

where we assume, without loss of generality, that the elements are listed in increasing order. Now consider the $k+1$ sums

$$\sum_{i=1}^{k+1} a_i \quad \text{and} \quad \sum_{\substack{i=1\\ i\neq j}}^{k+1} a_i, \qquad j=1,\ldots,k.$$

These sums are all distinct and, since $a_{k+1} > a_j$ for $j=1,\ldots,k$, each is greater than any of the (at least) $\frac{1}{2}k(k+1)+1$ distinct sums determined by $\{a_1, \ldots, a_k\}$. Thus (1) determines at least

$$\tfrac{1}{2}k(k+1) + 1 + (k+1) = \tfrac{1}{2}(k+1)(k+2) + 1$$

distinct sums, and the induction is complete.

(b) We prove, more generally, that *no* sum can occur more than $2^n/(n+1)$ times.

For $n \geq 1$, let $A = \{a_1, \ldots, a_n\}$ be a set of n distinct positive numbers with sum S (listed in no particular order), and let B be a subset of A having sum T, so that $0 \leq T \leq S$. The sums $T = 0$ and $T = S$ each occur only once, and $1 \leq 2^n/(n+1)$; so we now assume that $0 < T < S$ (note that this implies $n \geq 2$). Without loss of generality, we may assume that

$$B = \{a_1, \ldots, a_k\}, \quad \text{where} \quad 1 \leq k < n.$$

Consider the n subsets

$$B_i = B - \{a_i\}, \qquad i = 1, \ldots, k$$

and

$$B_i = B \cup \{a_i\}, \qquad i = k+1, \ldots, n.$$

Note that all n subsets B_i are distinct and have sums distinct from T. Now suppose that subset

$$B' = \{a'_1, \ldots, a'_m\} \neq B$$

also has sum T and that n subsets B'_j are defined similarly. We claim that any B_i is distinct from any B'_j. For suppose $B_i = B'_j$ for some i and j. If $i \leq k$ and $j \leq m$, then we must have $a_i = a'_j$ since B and B' have the same sum; but this implies $B = B'$, a contradiction. A similar contradiction follows if $i > k$ and $j > m$. Moreover, we cannot have either $i \leq k$ and $j > m$, or $i > k$ and $j \leq m$, since then one of B_i or B'_j would have a sum greater than T and the other a sum less than T, contradicting the assumption $B_i = B'_j$.

Thus, to each subset with sum T we can associate a collection of n distinct subsets with sums distinct from T, and furthermore any two such collections are disjoint. So, if there are K subsets with sum T, we have established a collection of $K(n+1)$ distinct subsets of A. Since there are 2^n subsets of A, we have $K(n+1) \leq 2^n$, and the required result follows.

Also solved by MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio (part (a) only); LEROY F. MEYERS, The Ohio State University; MICHAEL M. PARMENTER, Memorial University of Newfoundland; JEREMY D. PRIMER, student, Columbia H. S., Maplewood, N.J., KENNETH M. WILKE, Topeka, Kansas (part (a) only); and the proposer.

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405. [1979: 15] *Proposed by Viktors Linis, University of Ottawa.*

A circle of radius 16 contains 650 points. Prove that there exists an annulus of inner radius 2 and outer radius 3 which contains at least 10 of the given points.

Editor's comment.

Our proposer found this problem in the Russian journal *Kvant* (No. 8, 1977, p. 46, Problem M 419), where it appeared with a solution by I. Klimova. The original proposer, if not Klimova, was not identified and the earlier issue where the problem was first proposed was not available. Since no solution was received from our readers, we give the Russian solution below. It was edited from a translation supplied by V. Linis.

Solution by I. Klimova.

We shall need the following

LEMMA. *Suppose a region S of area σ contains regions $S_1, S_2, \ldots, S_n$ of areas $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that each point of S is covered by no more than k regions S_i. Then $\sum\sigma_i \le k\sigma$.*

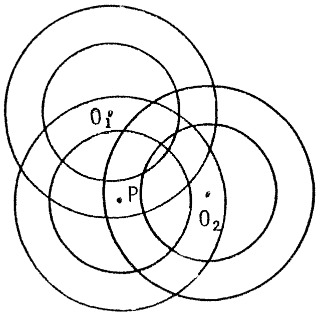
Proof. Consider the disjoint regions T_j, of areas t_j, formed by the intersections of the regions S_i with one another and with S. On the one hand, we have $\sum t_j \le \sigma$ and, on the other, $\sum\sigma_i \le k\sum t_j$. Hence $\sum\sigma_i \le k\sigma$. □

Now, to solve our problem, with each of the given 650 points as centre, we draw two concentric circles of radii 2 and 3, making in all 650 annuli. The union of all these annuli is contained in a circle of radius $16+3=19$. We show that at least one point in this circle is covered by at least 10 annuli. Suppose, on the contrary, that every point is covered by at most 9 annuli. The circle of radius 19 has area $\sigma = 361\pi$ and each of the 650 annuli has area $\sigma_i = 5\pi$. By our assumption and the lemma, we have

$$\sum\sigma_i = 3250\pi \le 9\sigma = 3249\pi,$$

which is the required contradiction.

Let P be such a point that is covered by (at least) the 10 annuli with centres $O_1, \ldots, O_{10}$ (see the figure, where only 2 of the 10 annuli are shown). The distances ρ_i of P from the O_i all satisfy $2 \le \rho_i \le 3$. Hence an annulus with centre P and radii 2 and 3 will contain (at least) the 10 given points $O_1, \ldots, O_{10}$.



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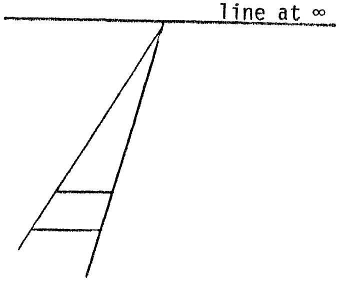
406. [1979: 16] *Proposed by W.A. McWorter Jr., The Ohio State University.*

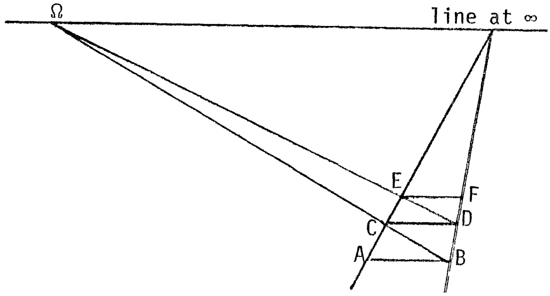
The figure shows an unfinished perspective drawing of a railroad track with two ties drawn parallel to the line at ∞. Can the remaining ties be drawn, assuming that the actual track has equally spaced ties?



Solution by E.J. Barbeau, University of Toronto.

It suffices to show that one more tie can be drawn.

Consecutive ties form with the track congruent rectangles whose corresponding diagonals are parallel and hence are concurrent in the line at infinity. If AB, CD are the two given ties (see figure), let BC intersect the line at infinity in Ω. If ΩD intersects AC in E, then EF, drawn parallel to CD, is the required third tie.



Also solved by ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; JEREMY D. PRIMER, student, Columbia H. S., Maplewood, N.J.; and the proposer.

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407. [1979: 16] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

There are decimal integers whose representation in some number base $B = 2, 3, 4, \ldots$ consists of three nonzero digits whose cubes sum to the integer. For example,

$$43_{10} = 223_4 = 2^3 + 2^3 + 3^3,$$
$$134_{10} = 251_7 = 2^3 + 5^3 + 1^3,$$
$$433_{10} = 661_8 = 6^3 + 6^3 + 1^3.$$

Prove that infinitely many such integers exist.

Solution and comment by the proposer.

The problem asks for an infinite set of solutions of the Diophantine equation

$$x^3 + y^3 + z^3 = xB^2 + yB + z, \qquad (1)$$

subject to the constraint $1 \le x,y,z \le B-1$. There is little hope of finding the *complete* solution of (1), but we will find several infinite sets of solutions by imposing additional constraints on the variables.

(a) Equation (1) is equivalent to

$$x(B+x)(B-x) = y(y^2-B) + z(z+1)(z-1). \tag{2}$$

With the additional constraint $y^2 = B$, this becomes

$$x(B+x)(B-x) = z(z+1)(z-1).$$

If we further require $B-x = z-1$, we obtain

$$x(B+x) = (B-x+1)(B-x+2),$$

from which $(B+1)(B-3x+2) = 0$ and $x = (B+2)/3$. Thus

$$x = \frac{1}{3}(k^2+2), \qquad y = k, \qquad z = \frac{1}{3}(2k^2+1), \qquad B = k^2, \qquad k = 2,4,5,7,\ldots$$

is a solution of (1) for any $k > 1$ that is not a multiple of 3. In particular, for $k = 2$ we get

$$43_{10} = 223_4 = 2^3 + 2^3 + 3^3.$$

(b) Coming back to (2), we set $z = 1$ and obtain

$$x(B+x)(B-x) = y(y^2-B). \tag{3}$$

With the additional constraint $y = B-x$, we get

$$x(B+x) = (B-x)^2 - B,$$

from which $x = (B-1)/3$, which is a positive integer if and only if $B = 3k+1$. We have thus found the following infinite solution set of (1):

$$x = k, \qquad y = 2k+1, \qquad z = 1, \qquad B = 3k+1, \qquad k = 1,2,3,\ldots$$

In particular, for $k = 2$ we get

$$134_{10} = 251_7 = 2^3 + 5^3 + 1^3.$$

(c) Now back to (3). With the alternative constraint $x = y$, we obtain $x^2 = B(B+1)/2$, and we are faced with the well-known problem of finding triangular numbers that are squares. According to Dickson [1], Euler proved in 1778 that $B(B+1)/2 = x^2$ only if $B = B_n$ and $x = x_n$, where

$$B_n = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2}{4}, \qquad x_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}, \qquad n = 0,1,2,\ldots$$

Hence

$$x = y = x_n, \quad z = 1, \quad B = B_n, \quad n = 2,3,4,\ldots$$

is also an infinite solution set of (1). As mentioned in [1], the values of B_n and x_n are conveniently obtained from the recurrence relations

$$B_0 = 0, \quad B_1 = 1, \quad B_n = 6B_{n-1} - B_{n-2} + 2, \quad n = 2,3,4,\ldots$$

and

$$x_0 = 0, \quad x_1 = 1, \quad x_n = 6x_{n-1} - x_{n-2}, \quad n = 2,3,4,\ldots$$

In particular, for $n = 2$ we get

$$433_{10} = 661_8 = 6^3 + 6^3 + 1^3.$$

(d) Many other solution sets can be found by applying various constraints to the variables in (1). We will find just one more, an obvious one. We set

$$x^3 = z, \quad y^3 = yB, \quad z^3 = xB^2$$

in (1) and obtain

$$x = k, \quad y = k^2, \quad z = k^3, \quad B = k^4, \quad k = 2,3,4,\ldots$$

(e) We end with a comment on various extensions of the problem.

The restriction $1 \le x,y,z$ was imposed in (1) mainly to encourage readers to find relatively nontrivial solution sets. But it is clear that if $\overline{xy1}_B$ satisfies the problem, then so does $\overline{xy0}_B$. This yields two additional infinite solution sets, obtained by setting $z = 0$ in parts (b) and (c). On the other hand, if $x = 0$ is allowed in (1), solution sets for the equation

$$y^3 + z^3 = yB + z, \quad 1 \le y,z \le B - 1$$

are not hard to find; for example,

$$y = k, \quad z = k + 1, \quad B = 2k^2 + 3k + 2, \quad k = 1,2,3,\ldots$$

and

$$y = k + 1, \quad z = k, \quad B = 2k^2 + k + 1, \quad k = 1,2,3,\ldots$$

Also, if $y = 0$ and $B - x = z - 1$ in (2), we get the solution set

$$x = k + 1, \quad y = 0, \quad z = 2k + 1, \quad B = 3k + 1, \quad k = 2,3,4,\ldots$$

The equation

$$w^3 + x^3 + y^3 + z^3 = wB^3 + xB^2 + yB + z,$$

where $0 \le w,x,y,z \le B - 1$ has solutions even when $w \ne 0$; for example,

$$(w,x,y,z,B) = (1,0,5,5,6),\ (1,3,8,8,9),\ (1,1,10,10,12),\ (1,5,14,14,16);$$

but we have not been able to isolate an infinite solution set. Finally, for $n \geq 5$, the Diophantine equation

$$\sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i B^{n-i}, \qquad 1 \leq x_i \leq B-1$$

has no solution because the left side is at most $n(B-1)^3$ while the right side is at least B^{n-1}, and $B^{n-1} > n(B-1)^3$ for $n \geq 5$.

One related question about which little seems to be known is the following: for a given base B, how many numbers greater than unity are there which are the sums of the cubes of their digits? For base 10, it is known that there are exactly four such numbers. One of them is 407, *the number of this problem* (which proves something or other about the editor). The others will be mentioned at the end of this comment. A proof that there are no others can be found in Barbeau et al. [2].

The numbers discussed in this problem belong to the category of *narcissistic numbers*, which are defined in Madachy [3] as "those that are representable, in some way, by mathematically manipulating the digits of the numbers themselves." Furthermore, our numbers belong to the sub-category of *perfect digital invariants* (PDI's). Madachy [3] defines an nth-order PDI as an integer greater than unity that is equal to the sum of the nth powers of its digits. Finally, the numbers in our problem belong to the sub-sub-category of *Armstrong numbers*, which Spencer [4] defines as n-digit PDI's of order n. In base 10, the smallest Armstrong number is 153, and the largest we have been able to find in the literature is 4679307774:

$$4679307774 = 4^{10} + 6^{10} + 7^{10} + 9^{10} + 3^{10} + 0^{10} + 7^{10} + 7^{10} + 7^{10} + 4^{10}.$$

This extraordinary number was discovered in 1963 by Harry L. Nelson, according to Madachy [3] who, however, adds that "recently the work has been extended to the seventeenth order."

Our often-quoted Madachy [3] gives the following excerpt from G.H. Hardy's *A Mathematician's Apology*: "There are just four numbers, after unity, which are the sums of the cubes of their [decimal] digits:

$$153 = 1^3 + 5^3 + 3^3, \quad 370 = 3^3 + 7^3 + 0^3, \quad 371 = 3^3 + 7^3 + 1^3, \quad \text{and} \quad 407 = 4^3 + 0^3 + 7^3.$$

These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician." Ouch!

Also solved by E.J. BARBEAU, University of Toronto; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; JEREMY D. PRIMER, student, Columbia H. S., Maplewood, N.J.; and KENNETH M. WILKE, Topeka, Kansas.

REFERENCES

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1966, Vol. II, p. 16.

2. E. Barbeau, M. Klamkin, W. Moser, *1001 Problems in High School Mathematics*, Canadian Mathematical Society, 1978, Book 3, pp. 8, 52-53.

3. Joseph S. Madachy, *Mathematics on Vacation*, Charles Scribner's Sons, New York, 1966, pp. 163-164.

4. Donald D. Spencer, *Game Playing with BASIC*, Hayden, Rochelle Park, N.J., 1977, p. 77.

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408. [1979: 16] A solution of this problem will appear next month.

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409. [1979: 16] *Proposed by L.F. Meyers, The Ohio State University.*

In a certain bingo game for children, each move consists in rolling (actually, "popping") two dice. One of the dice is marked with the symbols B, I, N, G, O, and *, and the other die is marked with 1, 2, 3, 4, 5, and 6. A disadvantage of this form of bingo, in comparison with the adult form of the game, is that a combination (such as B3) may appear repeatedly. What is the expected number of the move at which the first repetition occurs in each of these cases:

(a) all 36 combinations (B1 through *6) are considered to be different (and equally likely)?

(b) all 36 combinations (B1 through *6) are considered to be equally likely, but the 6 combinations containing * are considered to be the same?

Solution by the proposer.

This problem is related to the classical birthday problem and can be solved by a similar approach. (See Crux 195 [1976: 220; 1977: 83, 195] for another related problem and for references to the birthday problem.)

The first repetition occurs at move n just when the first $n-1$ moves are all different and move n produces a combination which is the same as one of the first $n-1$ moves.

(a) For $2 \le n \le 37$, the probability that the first repetition occurs at move n is

$$p(n) = \frac{36 \cdot 35 \cdot \ldots \cdot (38-n)}{36^{n-1}} \cdot \frac{n-1}{36}.$$

Hence the expected (or mean) number of the move at which the first repetition occurs is

$$\sum_{r=2}^{37} np(n) \approx 8.203.$$

(b) For $2 \le n \le 32$, the first repetition can occur at move n in either of two ways: (1) none of the first n moves results in *; or (2) exactly one of the first $n-1$ moves results in * (and move n may result in * or not). Let $q(n)$ be the probability that the first repetition occurs at move n. Then

$$q(2) = \frac{30}{36} \cdot \frac{1}{36} + \frac{6}{36} \cdot \frac{6}{36} = \frac{11}{216}$$

and, for $3 \le n \le 32$,

$$\begin{aligned} q(n) &= \frac{30 \cdot 29 \cdot \ldots \cdot (32-n)}{36^{n-1}} \cdot \frac{n-1}{36} + (n-1) \cdot \frac{6}{36} \cdot \frac{30 \cdot 29 \cdot \ldots \cdot (33-n)}{36^{n-2}} \cdot \frac{6+(n-2)}{36}, \\ &= \frac{30 \cdot 29 \cdot \ldots \cdot (33-n)}{36^{n-2}} \cdot \frac{n-1}{36} \cdot \frac{5n+56}{36}. \end{aligned}$$

The expected number of the move at which the first repetition occurs is then

$$\sum_{n=2}^{32} nq(n) \approx 6.704.$$

The fact that the expected number of moves is not an integer in (a) and (b) should cause no difficulty. Of course, all that is meant is that if the experiment should be carried out many times, then the average of the numbers of the move at which repetition first occurs is expected to be close to 8.203 or 6.704, respectively.

Editor's comment.

The game mentioned in the proposal is called Pop-o-matic Bingo (a Registered Trademark). Buy one for your children for fast-approaching Christmas, and ask them to guess the number of the move at which a repetition first occurs. With the answers given above you can then bet accordingly, and in no time at all their weekly allowance will be back in your pocket.

* * *

MAMA-THEMATICS

Mrs. Cantor to son Georg: "I'm very glad that you gave up singing for mathematics."

CHARLES W. TRIGG