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- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

AN n-GONAL DELIGHT

K. R. S. Sastry

I have two consecutive integers. You place them side by side in their natural order to produce a longer integer. Do you want this longer integer to be a triangular number? A pentagonal number? $\cdots$ A 71-gonal number? $\cdots$

Granted, you are ambitious. You require that some of these numbers be initial links in infinite chains of n-gonal numbers. Want more? In some chain the longer numbers are all divisible by the larger of the corresponding consecutive pair. Okay, n-gonal numbers have the answers!

By placing the consecutive integers 49 and 50 side by side we obtain the four digit triangular number $4950 = \frac{1}{2}(99)(100)$. This is the initial link in the infinite chain of triangular numbers

$$4950, \quad 499500 = \frac{1}{2}(999)(1000), \quad 49995000 = \frac{1}{2}(9999)(10000), \cdots .$$

In the above infinite chain, it is easily seen that the (longer) numbers are all divisible by the larger members of the consecutive pair.

A *pentagonal number* is of the form $\frac{1}{2}r(3r-1)$. The consecutive integer pair 16, 17 generates an infinite chain of pentagonal numbers — sans the divisibility property:

$$1617 = \frac{1}{2}(33)(98), \quad 166167 = \frac{1}{2}(333)(998),$$

$$16661667 = \frac{1}{2}(3333)(9998), \quad \cdots .$$

An *n-gonal number* or a *polygonal number*, $n \geq 3$, is given by

$$P(n,r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2} = \frac{1}{2}r[(n-2)r - (n-4)], \quad r \geq 1, \tag{1}$$

see the references [1, 2] for details. We propose to outline a technique to produce pairs of consecutive integers that produce n-gonal numbers for various values of n as described in the preceding paragraphs. To this end we need the following result about n-gonal numbers.

THEOREM. If the integer N is n-gonal then $8(n-2)N + (n-4)^2$ is a perfect square.

Proof. The integer N is n-gonal implies that for some integers $n \geq 3$ and $r \geq 1$,

$$N = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2} .$$

Then

$$\begin{aligned} 8(n-2)N + (n-4)^2 &= 4(n-2)^2r^2 - 4(n-2)(n-4)r + (n-4)^2 \\ &= [2(n-2)r - (n-4)]^2 = M^2, \end{aligned} \tag{2}$$

a perfect square. □

It should be noted that the converse, that N is n-gonal for some $n \geq 3$ if $8(n-2)N + (n-4)^2 = M^2$, is true only when M has the form $2(n-2)r - (n-4)$ for some integer $r \geq 1$. This is a crucial point in our discussion because sometimes n may be integral but not r. We illustrate the technique by determining a pair of three-digit consecutive integers that becomes n-gonal. The technique is perfectly general: be it the determination of two-digit or two million-digit pairs of consecutive integers. As a referee points out, it can easily be modified to work in bases other than ten.

PROBLEM: Find a pair of consecutive three-digit integers such that when they are placed side by side in their natural order they form a six-digit n-gonal number for some $n \geq 3$ and $r > 2$.

SOLUTION: Let m, $m+1$ be consecutive integers such that $100 < m + 1 < 1000$. Then the six-digit integer

$$m\ m+1 = 1000(m) + m + 1 = 1001m + 1$$

is to be n-gonal. Then the necessary criterion (2) implies that

$$8(n-2)(1001m+1) + (n-4)^2 = M^2, \quad M = 2(n-2)r - (n-4). \tag{3}$$

A simplified restatement of the first equation above is

$$M^2 - n^2 = 8008(n-2)m. \tag{4}$$

The left hand side has the two-factor factorization $(M+n)(M-n)$. Furthermore, $M + n = (M-n) + 2n$. Hence both factors $M+n$ and $M-n$ are either odd or even. However, we can't have both $M+n$ and $M-n$ odd as the right hand side of (4) is even. So both factors $M+n$ and $M-n$ are even. Hence the following solution technique:

I. Consider a two-factor factorization of the right hand side expression in (4) such that both factors are even.

II. Equate one even factor to $M+n$ and the other to $M-n$. Then solve for M and m.

III. Choose a specific numerical value of $n \geq 3$ that makes m integral such that $100 < m + 1 < 1000$.

IV. Check that $M = 2(n-2)r - (n-4)$ yields an integral value of $r > 2$ for the value of n chosen in step III. If r is such an integer then we have a solution

$$P(n,r) = \frac{1}{2}(r)[(n-2)r - (n-4)].$$

If not, choose a different value of n and repeat steps III and IV.

(There will be only a finite number of such values n. When all such n are exhausted, we repeat the entire process by considering a different two-factor factorization. Of course the number of such factorizations is also finite.)

Here is a numerical illustration:

I. $(M+n)(M-n) = 8008(n-2)m = (4004)\cdot 2(n-2)m$, say. We may equate $M+n$ to either factor. For definiteness, let us solve the equations

II. $M+n = 4004$, $\quad M-n = 2(n-2)m$. So

$$M = 2(n-2)m + n \tag{5}$$

and

$$m = \frac{2002-n}{n-2} = \frac{2000}{n-2} - 1. \tag{6}$$

III. Suppose

$$n = 18. \text{ Then } m = 124. \tag{7}$$

IV. From (3), (5) and (7) we must have r integral. That is,

$$M = 2(n-2)r - (n-4) = 2(n-2)m + n,$$

or

$$2(18-2)r - (18-4) = 2(18-2)124 + 18,$$

which yields $r = 125$, an integer.

Hence, the consecutive integers 124, 125 yield the rth n-gonal number for $r = 125, n = 18$. In other words,

$$P(18,125) = 124125 = \frac{1}{2}(125)[(18-2)125 - (18-4)].$$

It is easily seen that the 18-gonal number 124125 is divisible by 125. Does it generate an infinite chain of such 18-gonal numbers? Yes, it does! Here it is:

$$12491250 = \frac{1}{2}(1250)(199986), \quad 1249912500 = \frac{1}{2}(12500)(1999986), \cdots.$$

REMARKS: 1. No doubt there are other solutions. For example, in (6) you could let $n = 7$. This produces the pair 399, 400 that yields the 400th heptagonal number 399400. The reader is encouraged to determine other such three-digit integer pairs. Hint: First note that

$$8008(n-2)m = 2^3 \cdot 7 \cdot 11 \cdot 13(n-2)m.$$

$P(25,175) = 346347$ and $P(71,58) = 114115$ are some examples.

2. In the statement of the problem $r > 2$ is necessary to make the problem non-trivial. If $r = 2$ then $P(n,2) = n$ for all $n \geq 3$. Thus any pair of consecutive integers — or for that matter any pair of positive integers in general — yields trivially a second n-gonal number. For instance, the pair 37, 38 yields the second 3738-gonal number, 3738 that is.

CONCLUSION: The reader can find (the easier case) two-digit pairs of consecutive integers that become four-digit n-gonal numbers, $n \geq 3$ and $r > 2$. For a greater challenge one can determine four, five, $\cdots$ digit pairs of consecutive integers that become n-gonal numbers non-trivially. Directions for further investigation include extensions of the basic problem we have discussed: Find triples of two, three, $\cdots$ digit consecutive integers that become n-gonal numbers. Find pairs of two, three, $\cdots$ digit integers differing by a specified integer (this specification cannot be done at will) that become n-gonal numbers, when placed side by side. Find triples of two, three, $\cdots$ digit integers in arithmetic progression that become n-gonal numbers when placed side by side in their natural order, $\cdots$.

References:

[1] K. R. S. Sastry, Cubes of natural numbers in arithmetic progression, *Crux Mathematicorum*, **18**(1992), pp. 161-164.

[2] K. R. S. Sastry, Pythagorean triangles of the polygonal numbers, *Mathematics and Computer Education Journal*, **27**(Spring 1993), pp. 135-143.

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GUIDELINES FOR ARTICLES

Articles for this section of *Crux* should satisfy the following:

- have length of two to four pages, ideally (we have allowed up to six pages in exceptional circumstances);
- be of interest to advanced high school and first or second year university students;
- contain some new material that leads to further interesting questions (for this level);
- be well referenced as to origin of the problem and related material;
- not contain long involved formulas or expressions, i.e. we like more elegant mathematics as opposed to that which involves tedious calculations and attention to detail. We really want to emphasize ideas and avoid many, many formulas. *Crux* does not want to be too technical in its appeal, rather we want to have wide general interest.

— Denis Hanson

THE SKOLIAD CORNER

No. 3

R. E. WOODROW

Last month we gave the problems of the junior contest of the University College of the Cariboo. This number we give the problems of the senior contest. The contest was forwarded to us by the organizer, Dr. J. Totten. The first twenty years of the contest with solutions are available in a book at \$14.95 plus handling through the University College of the Cariboo Bookstore, Box 3010, Kamloops, B.C. V2C 5N3 (tel: (604) 828-5141 for details). The collection was reviewed previously [1993: 139].

UNIVERSITY COLLEGE OF THE CARIBOO

Senior Preliminary 1994

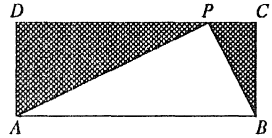
March 16, 1994

1. The sum of the two prime divisors of 1994 is:

(a) 1995 (b) 999 (c) 671 (d) 668 (e) 500.

2. If $g(x) = x(x-1)$, then $g(x+1) - g(x)$ equals:

(a) 0 (b) $g(1)$ (c) $2x^2$ (d) $2x$ (e) all of these.

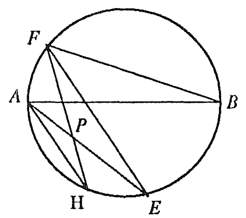


3. Point P is chosen on side DC of rectangle $ABCD$ so that $DP > PC$ and the two shaded triangles are similar. If $AB = 5$ and $AD = 2$, then DP equals:

(a) $2\sqrt{3}$ (b) 3 (c) $3\sqrt{2}$ (d) 4 (e) none of these.

4. If the sum of two numbers is 15 and the sum of their squares is 135, then the product of the two numbers is:

(a) 45 (b) 50 (c) 54 (d) 56 (e) 90.



5. In the diagram AB is a diameter of the circle and FE is parallel to AH. If $\angle FBA = 20°$, then $\angle HPE$ equals:

(a) 20° (b) 25° (c) 30° (d) 35° (e) 40°.

6. A spherical fish bowl has a radius of 18 cm. If the greatest depth of water in the bowl is 26cm, then the area, in cm^2, of the circular surface of water is:

(a) 64π (b) 180π (c) 260π (d) 324π (e) 352π.

7. The smallest number that may be written both as the sum of 9 consecutive positive integers and 10 consecutive positive integers is:

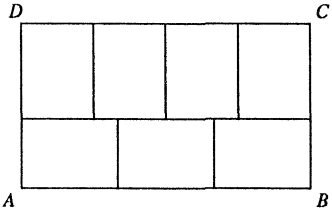
(a) 45 (b) 55 (c) 100 (d) 135 (e) 495.

8. The remainder on dividing $5^{22} + 7$ by eight is:

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4.

9. The rectangle $ABCD$ is composed of 7 congruent rectangles as in the figure at the right. If the area of $ABCD$ is 336 cm^2, then the perimeter of $ABCD$, in cm, is:

(a) 76 (b) 86 (c) 96 (d) 106 (e) none of these.



10. If $f(n+1) = f(n) + n$ for all integers $n \geq 0$, and if $f(0) = 1$, then $f(20)$ equals:

(a) 211 (b) 210 (c) 209 (d) 191 (e) 190.

11. Four boys and four girls are randomly seated in 8 adjacent seats at a theatre. If it is found that all the girls are seated in 4 adjacent seats, then the probability that the four boys are also in 4 adjacent seats is:

(a) 1 (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{5}$ (e) $\frac{1}{4}$.

12. Antonino, Bill, and Carol all work at the same rate, and working together could finish a job in 51 minutes. One day Bill and Carol arrive late to do the job. Bill arrived the same number of minutes after Antonino as Carol arrived after Bill. Once at work they all stayed until the job was finished. If Antonino worked twice as long as Carol, then the number of minutes it took to complete the job was:

(a) 58 (b) 60 (c) 64 (d) 68 (e) 70.

13. If $f(n+1) = \dfrac{n(n-1)f(n) - (n-2)f(n-1)}{n(n+1)}$, $f(0) = 1$, and $f(1) = 2$, then

$$\frac{f(1)}{f(2)} + \frac{f(2)}{f(3)} + \frac{f(3)}{f(4)} + \frac{f(4)}{f(5)} + \frac{f(5)}{f(6)} + \frac{f(6)}{f(7)}$$

equals:

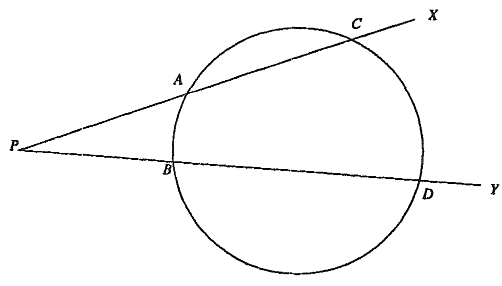
(a) $25\frac{1}{2}$ (b) 27 (c) 28 (d) 29 (e) none of these.

14. In a chess tournament among 6 contestants each contestant plays every other contestant exactly once. A loss counts 0 points, a tie 1 point, and a win 2 points. If one player got 8 points, another 6 points, and all the others had the same number of points, then the 4 contestants who had the same number of points each had n points, where n equals:

(a) 6 (b) 5 (c) 4 (d) 3 (e) none of these.

15. The rays PX and PY cut off arcs AB and CD of a circle with radius 4. If the length of arc CD is 2 times the length of arc AB and the length of arc CD is $\frac{4}{5}\pi$, then the number of degrees in angle APB is:

(a) 9 (b) 10 (c) 12 (d) 14 (e) 18.



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The answers for the Preliminary Round of the Junior Mathematics contest of the University College of the Cariboo are given below. Winners of the junior and senior contest are invited for a final round written at the College, followed by an afternoon of mathematical presentations and recreations, topped off by an awards banquet.

1. *e*	5. *d*	9. *e*	13. *d*
2. *c*	6. *b*	10. *d*	14. *e*
3. *d*	7. *b*	11. *d*	15. *a*
4. *d*	8. *e*	12. *a*	

* * *

That completes the Skoliad Corner for this issue. Send me your contest materials, comments, and student solutions.

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THE OLYMPIAD CORNER

No. 163

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this number with the 2nd Mathematical Olympiad of the Republic of China. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, who collected this contest (and several more!) when he was Canadian Team leader to the I.M.O. at Istanbul.

2nd MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA

April 17, 1993 — First Day

(Time: 4.5 hours)

1. A sequence $\{a_n\}$ of positive integers is defined by $a_n = [n + \sqrt{n} + (1/2)]$, $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Determine the positive integers which belong to the sequence.

2. Suppose that $ABCD$ is a parallelogram, E and F are two distinct points on the interior of the diagonal $\overline{AC}$. Prove that: if there is a circle passing through E, F, and tangent with two rays $\overrightarrow{BA}$, $\overrightarrow{BC}$, respectively, then there is another circle passing through E, F, and tangent with two rays $\overrightarrow{DA}$, $\overrightarrow{DC}$, respectively.

3. Determine all possible nonnegative integers x, y, z such that $7^x + 1 = 3^y + 5^z$.

April 20, 1993 — Second Day

(Time: 4.5 hours)

4. In the xy-coordinate plane, let C be a unit circle with center at origin O, $Q(x,y)$ be any point with $Q \neq 0$. Denote the intersection of the ray $\overrightarrow{OQ}$ with circle C by $\tilde{Q}$. Prove that: for any point P on the unit circle C and for any positive integer k, there exists a lattice point $Q(x,y)$ with $|x| = k$ or $|y| = k$ such that the length of the line segment $\overline{P\tilde{Q}}$ is less than $1/(2k)$.

5. Suppose that $A = \{a_1, a_2, \ldots, a_{11}, a_{12}\}$ is a set consisting of 12 positive integers $a_1 < a_2 < \cdots < a_{11} < a_{12}$, such that for each positive integer $n \leq 2500$, there is a subset S for which the sum of the elements of S is equal to n. What is the smallest possible value of a_{11}?

6. Suppose that $m \in \{1, 2\}$, $n \in \mathbb{N}$ and $n < 10799$. Determine all possible values of n satisfying the following equality:

$$\sum_{k=1}^{n} \frac{1}{\sin k \sin(k+1)} = m \times \frac{\sin n}{\sin^2 1}.$$

* * *

Murray S. Klamkin, Mathematics Department, The University of Alberta sent me a batch of Quickies which we will use this issue and in the next. Here are six Klamkin Quickies. Look for his solutions next number.

KLAMKIN QUICKIES

1. Are there any integral solutions (x, y, z) of the Diophantine equation

$$(x - y - z)^3 = 27xyz$$

other than $(-a, a, a)$ or such that $xyz = 0$?

2. Does the Diophantine equation

$$(x - y - z)(x - y + z)(x + y - z) = 8xyz$$

have an infinite number of relatively prime solutions?

3. It is an easy result using calculus that if a polynomial $P(x)$ is divisible by its derivative $P'(x)$, then $P(x)$ must be of the form $a(x - r)^n$. Starting from the known result that

$$\frac{P'(x)}{P(x)} = \sum \frac{1}{x - r_i}$$

where the sum is over all the zeros r_i of $P(x)$ counting multiplicities, give a non-calculus proof of the above result.

4. Solve the simultaneous equations

$$x^2(y + z) = 1, \quad y^2(z + x) = 8, \quad z^2(x + y) = 13.$$

5. Determine the area of a triangle of sides a, b, c and semiperimeter s if

$$(s - b)(s - c) = \frac{a}{h}, \quad (s - c)(s - a) = \frac{b}{k}, \quad (s - a)(s - b) = \frac{c}{l},$$

where h, k, l are consistent given constants.

6. Prove that

$$3(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x + y + z)^2$$

where $x, y, z \geq 0$.

* * *

Before moving on to readers' solutions to the next problem set from the December 1993 number of the Corner, I want to look at some comments, corrections, and solutions sent in by the readers about earlier material.

In the September number we gave the solution of K. R. S. Sastry to the following problem:

6. [1988: 35; 1994: 191-193] *18th Austrian Mathematical Olympiad, Final Round.*

Determine all polynomials

$$P_n(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where the a_i are integers and where $P_n(x)$ has as its n zeros precisely the numbers $a_1, \ldots, a_n$ (counted in their respective multiplicities).

In a comment about that solution Bill Sands raised the question whether there can be polynomials of degree larger than 4 with real coefficients whose roots are precisely the coefficients. Stan Wagon, Macalester College, St. Paul, Minnesota, writes pointing out an article of Paul R. Stein, "On polynomial equations with coefficients equal to their roots", M.A.A. Monthly 1966, pp. 272-274, in which it is shown that there can be none with *real* coefficients.

* * *

Next we give a different solution to an old problem of the 1st Nordic Mathematic Olympiad.

1. [1988: 289; 1990: 139-140]

In a group of nine mathematicians each speaks at most three languages and any two of them speak at least one common language. Show that at least five of them share a common language.

Solution by C.S. Yogananda, Indian Institute of Science, Bangalore, India.

If any one of them speaks less than three languages then by the pigeon-hole principle one of the languages he (or she!) speaks should be spoken by four others and we are done. Suppose now that each speaks three languages. Three cases arise:

(i) Two mathematicians have all the three languages in common. Here again we are through since one of the three languages should be spoken by at least three of the remaining seven mathematicians.

(ii) Some two mathematicians have two languages in common, say, M_1 and M_2 have L_1 and L_2 in common. Let L_3 and L_4 be the third languages of M_1 and M_2 respectively. If there is one more mathematician having L_1 and L_2 as two of his languages then we are done as follows: Suppose M_3 speaks (L_1, L_2, L_5). Of the remaining six mathematicians if there are more than two who speak L_1 or L_2 then we are through; if not, there are four mathematicians who do not speak L_1 or L_2. But then these four are forced to speak L_3, L_4, L_5 to be able to converse with M_1, M_2, M_3 and so there will be five mathematicians who speak L_3, L_4 and L_5. So suppose only two have L_1 and L_2 common. Consider the pair (L_3, L_4). If some three mathematicians speak both these languages we are done as before; if not, there are five who do not speak L_3 and L_4 simultaneously. But then these five should speak either L_1 or L_2 to be able to converse with both M_1 and M_2 which in turn implies that either L_1 or L_2 is spoken by at least five mathematicians.

(iii) Any two of the nine mathematicians have exactly one language in common. Let M_1 speak (L_1, L_2, L_3). The remaining eight have to speak either L_1 or L_2 or L_3 and so one of these three languages has to be spoken by at least three more mathematicians, say L_1 is spoken by M_2, M_3, M_4 also. We will show that L_1 is spoken by all the nine mathematicians. Suppose not; say M_5 has L_2 in common with M_1. Now M_5 has to speak to each of M_2, M_3, M_4 in a different language other than L_2 (since any two mathematicians have only one common language) which is not possible as no mathematician speaks more than three languages and we are done.

* * *

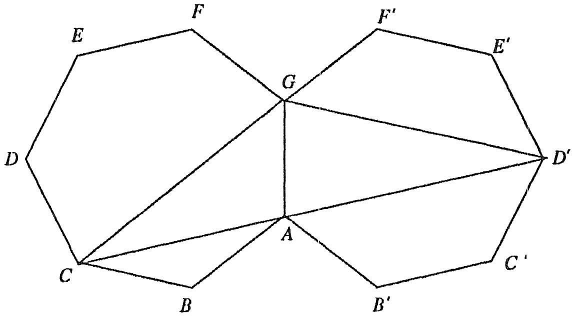
Before Christmas I received a large envelope from Federico Ardila M. It contained many very nice solutions, which, on a cursory glance, appeared to be solutions to problems that had been missed in earlier numbers of the Corner, or to be proposed improvements on solutions I had discussed. On coming to use them I realize that toward the end were several solutions to problems that I did not discuss until the December and January numbers of the Corner. My apologies to Federico for not listing him as a solver to problems 1-8 of the 28th Spanish Mathematical Olympiad [1993: 131; 1994: 278-283] and problems 1.2, 1.5 and 2.1 of the Fourth Irish Mathematical Olympiad [1993: 192-193; 1995: 11-14]. We now give some of his solutions to problems from the "archives".

4. [1992: 267; 1994: 13] 1990 *Dutch Mathematical Olympiad.*

Given is a regular 7-gon $ABCDEFG$. The sides have length 1. Prove for the diagonals AC and AD:

$$\frac{1}{AC} + \frac{1}{AD} = 1.$$

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.



Reflect the 7-gon with AG as an axis to obtain another 7-gon $AB'C'D'E'F'G$. Since

$$\angle GAC + \angle GAD' = \frac{4\pi}{7} + \frac{3\pi}{7} = \pi,$$

C, A and D' are collinear. Besides,

$$\angle GCA = \angle GD'A = \frac{\pi}{7} = \angle CAB = \angle ACB.$$

Therefore $\Delta GCD' \sim \Delta BAC$ giving

$$\frac{AC}{AB} = \frac{CD'}{CG} = \frac{AC + AD'}{AD} = \frac{AC + AD}{AD}\,.$$

But $AB = 1$ so

$$\frac{1}{AC} + \frac{1}{AD} = \frac{AC + AD}{AC \cdot AD} = \frac{1}{AB} = 1,$$

as required.

4. [1993: 4; 1994: 67,98] *AHSMC Part II.*

Suppose x, y and z are real numbers which satisfy the equation $ax + by + cz = 0$, where a, b and c are given positive numbers.

(a) Prove that $x^2 + y^2 + z^2 \geq 2xy + 2yz + 2xz$.

(b) Determine when equality holds in (a).

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

The fact that $ax + by + cz = 0$, $a, b, c > 0$ only tells us that two of x, y, z have the same sign and the other one has the opposite sign (considering 0 to be both positive and negative). Thus we may assume without loss of generality that $xy \geq 0$, $xz \leq 0$ and $yz \leq 0$. But then we have

$$x^2 + y^2 + z^2 \geq x^2 + y^2 \geq 2xy \geq 2xy + 2yz + 2xz.$$

Equality holds if $z = 0$, and, as $xy \geq 0$ and $ax + by = 0$, necessarily $x = y = z = 0$.

2. [1993: 100; 1994: 153-154] *XXV Soviet Mathematical Olympiad.*

On the blackboard are written n numbers. One may erase any pair of them, say a and b, and write down the number $(a + b)/4$ to replace them. After this procedure is repeated $n - 1$ times, only one number remains on the blackboard. Prove that given that the n numbers at the beginning are all 1, the last number will not be less than $1/n$. (B. Berlov)

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

By induction on n. For $n = 1, 2$ the result is obvious. Assume the result for $n = 1, 2, \ldots, m - 1$ and consider the procedure with m numbers. At the second last step, we have two numbers a and b. Now a comes from some k of the ones, so $a \geq 1/k$. Similarly $b \geq 1/(n - k)$. Then the final result is

$$\frac{a + b}{4} \geq \frac{\frac{1}{k} + \frac{1}{n-k}}{4} \geq \frac{1}{n}$$

by the Arithmetic Mean-Harmonic Mean inequality. The bound is sharp for $n = 2^t$, $t \in \mathbb{Z}$. To find the exact bound $f(n)$ we notice that

$$f(n) = \min_{1 \leq k \leq n} \frac{f(k) + f(n - k)}{4}\,.$$

An easy induction appears to prove

$$f(2n) = \frac{f(n)}{2},$$

$$f(2n+1) = \frac{(f(n) + f(n+1))}{4}.$$

5. [1993: 105; 1994: 219–220] *Final Round of the 22nd Austrian Mathematical Olympiad.*

Show that for all natural numbers $n > 1$ the inequality

$$\left(\frac{1 + (n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$$

is valid.

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

By the power means inequality we have

$$\sqrt[n]{\frac{1 + (n+1)^{n+1}}{n+2}} = \sqrt[n]{\frac{1 + (n+1)^n + \cdots + (n+1)^{n+1}}{n+2}}$$
$$> \sqrt[n-1]{\frac{1 + (n+1)^{n-1} + \cdots + (n+1)^{n-1}}{n+2}}.$$

If we take away one of the $(n+1)^{n-1}$ terms in this average, the average will obviously decrease, so

$$\sqrt[n]{\frac{1 + (n+1)^{n+1}}{n+2}} > \sqrt[n-1]{\frac{1 + (n+1)^{n-1} + \cdots + (n+1)^{n-1}}{n+1}}$$
$$> \sqrt[n-1]{\frac{1 + n^{n-1} + \cdots + n^{n-1}}{n+1}}$$
$$= \sqrt[n-1]{\frac{1 + n \cdot n^{n-1}}{n+1}} = \sqrt[n-1]{\frac{1+n^n}{n+1}}$$

and the proof is complete.

1.1. [1993: 192] *Fourth Irish Mathematical Olympiad.*

Three points X, Y and Z are given which are respectively, the circumcentre of triangle ABC, the midpoint of BC, and the foot of the altitude from B on AC. Show how to reconstruct the triangle ABC.

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

In the usual notation, M_a is the center of the circle through B, C, H_b, H_c, so $M_aH_b = M_aB = M_aC$. We are given M_a, H_b, and O. Construct the perpendicular to OM_a through M_a, and find the points of intersection of this line with the circle with center M_a and radius M_aH_b. These are B and C. Now we can construct the circumcircle of ABC and we can draw $\overrightarrow{CH_b}=\overrightarrow{CA}$, so we can determine the position of A.

2.5. [1993: 193] *Fourth Irish Mathematical Olympiad.*

Let $\mathbb{Q}$ denote the set of rational numbers. A nonempty subset S of $\mathbb{Q}$ has the following properties:

(i) 0 is not in S;

(ii) for each s_1, s_2 in S, s_1/s_2 is in S also;

(iii) there exists a nonzero rational number q which is not in S and which has the property that every nonzero rational number not in S is of the form qs for some s in S.

Prove that if x is in S, then there exist elements y, z in S such that $x = y + z$.

Solution by Federico Ardila, student, MIT, Cambridge, Massachusetts.

For any $s \in S$, $1 = s/s \in S$. We prove $x^2 \in S$ for $x \in Q\setminus\{0\}$.

(i) If $x \in S$, $1/x \in S$, so $x/(1/x) = x^2 \in S$.

(ii) If $x \notin S$, $1/x \notin S$, so by (iii) there is q with $x/q \in S$, $(1/x)/q = 1/(xq) \in S$. Then $(x/q)/(1/xq) = x^2 \in S$.

Now consider any $t \in S$. As $25/9 \in S$ and $25/16 \in S$, $t/(25/9) = (9t)/25$ and $t/(25/16) = (16t/25)$ are in S. So $t = (9t)/25 + (16t)/25$, where $(9t)/25, (16t)/25 \in S$, as required.

* * *

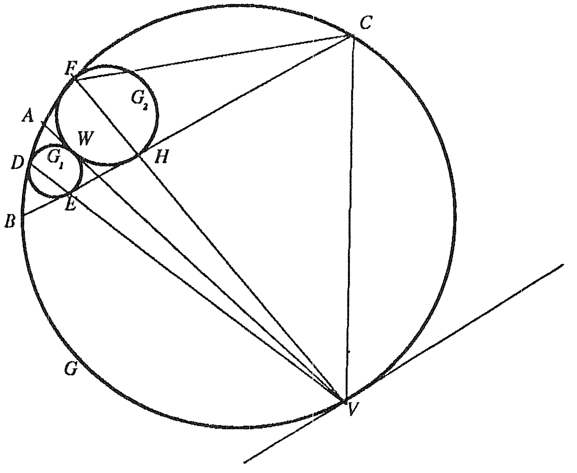
Next we give a solution to one of the problems proposed to the jury but not used for the 33rd International Mathematical Olympiad in Moscow.

8. [1993: 255] *Proposed by India.*

Circles G, G_1, G_2 are three circles related to each other as follows: circles G_1 and G_2 are externally tangent to one another at a point W and both these circles are internally tangent to the circle G. Points A, B, C are located on the circle G as follows: line BC is a direct common tangent to the pair of circles G_1 and G_2, and line WA is the transverse common tangent at W to G_1 and G_2, with W and A lying on the same side of the line BC.

Problem: Prove that W is the incenter of the triangle ABC.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.



Let l be a line parallel to BC and tangent to the circle G at V as shown in the figure (thus V is the midpoint of the arc BC). Line BC is tangent to G_1 and G_2 at E and H respectively; G_1 and G_2 are tangent to G at D and F respectively. Since l and BC are parallel, the homothety with center D which transforms G_1 on G, transforms E to V. Thus the points D, E, V are collinear. Similarly, F, H, V are collinear. Since $\angle BCV = \angle VFC$, triangle VHC and VCF are similar, which gives $(VC)^2 = VH \cdot VF$. Similarly, $(VB)^2 = VE \cdot VD$. Now we show that the line VW is the common transverse tangent of G_1 and G_2.

Suppose that line VW is not tangent to the both circles, and let distinct points W_1 and W_2 be the second intersections of the line VW with the circles G_1 and G_2 respectively. Then

$$VW \cdot VW_1 = VE \cdot VD = (VB)^2 = (VC)^2 = VH \cdot VF = VW \cdot VW_2,$$

which means that $W_1 = W_2$, a contradiction. Therefore A, W, V are collinear, which implies that W lies on the angle bisector of $\angle BAC$. Moreover

$$(VB)^2 = VE \cdot VD = (VW)^2,$$

which, according to Lemma 1 in the solution of *Crux* 1871 [1994: 199], means that W must be the incenter of the triangle ABC.

* * *

That is all the space we have for this number. The olympiad season is upon us. Send me your national and regional olympiads, as well as your nice solutions.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Game Theory and Strategy, by Philip D. Straffin. Published by The Mathematical Association of America as The New Mathematical Library no. 36, 1993. Softcover, 244 pages, ISBN 0-88385-637-9, US$ 27.50. *Reviewed by Mogens Esrom Larsen, Copenhagen.*

This is a textbook on the classical theory of games founded by von Neumann and Morgenstern. It is not about the games we usually play, or about the mathematical games such as those treated in *Winning Ways* by Berlekemp, Conway and Guy. The emphasis here is on analyzing decision making in general as games against competitors or nature. Such games usually have very few alternative strategies, in contrast to more "natural" games. However, this limitation is not uncommon in most practical situations.

The book contains a lot of thought-provoking examples. In the competition between companies about production planning, the author discusses how the strategies change with the level of information. So, it is not obvious whether we should just choose the best strategy based on information currently available, or wait until some investment is made in further research.

In games against nature, the interesting aspect is that nature does not play to win: it just plays! This leaves us with two options. We could choose the best strategy against nature on the understanding that it has no interest in the outcome of the game. Although this seems reasonable, it may not be wise. Even if nature in the long run assigns more or less fixed probabilities to each of its alternatives, the frequencies in the short run may allow for a better counter-strategy. This leaves us with a different approach, to play as though nature played against us.

In the case where no winning strategy is available and the game is played repeatedly, we should choose a mixed strategy which randomizes the alternatives with fixed probabilities. These can be chosen so that the mixed strategy produces the best possible result for us. The practical difficulty here is selling this seemingly unpredictable approach to society in general or to some political body consisting of less sophisticated people. Politicians usually want a "policy" with a single optimal solution. The funny thing is that some societies might use the following way of convincing people to adopt a mixed strategy. They ask some witch-doctor to announce the decision obtained by following the mixed strategy as a divine revelation. Paradoxically. a society of believers might be better off than a society of scientists or atheists in the Darwinian struggle for survival. The moral is that one should never refuse a second thought!

I used this book as the text for a class about mathematical modeling. My students found the exercises both interesting and manageable, and enjoyed working on them. I recommend this book most sincerely!

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1995**, *although solutions received after that date will also be considered until the time when a solution is published.*

2021. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is a variable interior point of triangle ABC, and AP, BP, CP meet BC, CA, AB at D, E, F respectively. Find the locus of P so that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

2022. *Proposed by K. R. S. Sastry, Dodballapur, India.*

Find the smallest integer of the form

$$\frac{A * B}{B},$$

where A and B are three-digit positive integers and $A * B$ denotes the six-digit integer formed by placing A and B side by side.

2023. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let a, b, c, d, e be positive numbers with $abcde = 1$.

(a) Prove that

$$\frac{a + abc}{1 + ab + abcd} + \frac{b + bcd}{1 + bc + bcde} + \frac{c + cde}{1 + cd + cdea} + \frac{d + dea}{1 + de + deab} + \frac{e + eab}{1 + ea + eabc} \ge \frac{10}{3}.$$

(b) Find a generalization!

2024. *Proposed by Murray S. Klamkin, University of Alberta.*

It is a known result that if P is any point on the circumcircle of a given triangle ABC with orthocenter H, then $(PA)^2 + (PB)^2 + (PC)^2 - (PH)^2$ is a constant. Generalize this result to an n-dimensional simplex.

2025. *Proposed by Federico Ardila, student, MIT, Cambridge, Massachusetts.*

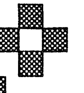
(a) An equilateral triangle ABC is drawn on a sheet of paper. Prove that you can repeatedly fold the paper along the lines containing the sides of the triangle, so that the entire sheet of paper has been folded into a wad with the original triangle as its boundary. More precisely, let f_a be the function from the plane of the sheet of paper to itself defined by

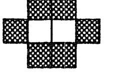
$$f_a(x) = \begin{cases} x & \text{if } x \text{ is on the same side of } BC \text{ as } A \text{ is} \\ \text{the reflection of } x \text{ about line } BC & \text{otherwise} \end{cases}$$

(f_a describes the result of folding the paper along line BC), and analogously define f_b and f_c. Prove that there is a finite sequence $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$, with each $f_{i_j} = f_a, f_b$ or f_c, such that $f_{i_n}(\ldots(f_{i_2}(f_{i_1}(x)))\ldots)$ lies in or on the triangle for every point x on the paper.

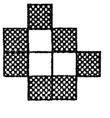
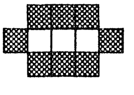
(b)* Is the result true for arbitrary triangles ABC?

2026. *Proposed by Hiroshi Kotera, Nara City, Japan.*

One white square is surrounded by four black squares: 

Two white squares are surrounded by six black squares: 

Three white squares are surrounded by seven or eight black squares:

What is the largest possible number of white squares surrounded by n black squares? [According to the proposer, this problem was on the entrance examination of the junior high school where he teaches!]

2027. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Quadrilateral $ABCD$ is inscribed in a circle Γ, and has an incircle as well. EF is a diameter of Γ with $EF \perp BD$. BD intersects EF in M and AC in S. Show that $AS : SC = EM : MF$.

2028. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

If $n \geq m \geq k \geq 0$ are integers such that $n + m - k + 1$ is a power of 2, prove that the sum $\binom{n}{k} + \binom{m}{k}$ is even.

2029*. *Proposed by Jun-hua Huang, The Middle School Attached To Hunan Normal University, Changsha, China.*

ABC is a triangle with area F and internal angle bisectors w_a, w_b, w_c. Prove or disprove that

$$w_b w_c + w_c w_a + w_a w_b \geq 3\sqrt{3}F.$$

2030. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

For which complex numbers s does the polynomial $z^3 - sz^2 + \bar{s}z - 1$ possess exactly three distinct zeros having modulus 1?

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1928. [1994: 75] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the tetrahedron $A_1A_2A_3A_4$, not necessarily regular, let a_i be the triangular face opposite vertex A_i ($i = 1, 2, 3, 4$). Let Q be any point in the interior of a_1 and P a point on the segment A_1Q. For $i = 2, 3, 4$ let B_i be the point where the plane through P parallel to a_i meets the edge A_1A_i. Prove that

$$\frac{A_1B_2}{A_1A_2} + \frac{A_1B_3}{A_1A_3} + \frac{A_1B_4}{A_1A_4} = \frac{A_1P}{A_1Q}.$$

I. *Solution by P. Penning, Delft, The Netherlands.*

Let A_1 be the origin and $\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ be the vectors to the corresponding vertices. An arbitrary vector $\mathbf{X}$ uniquely corresponds to the triple (a, b, c) with $\mathbf{X} = a\mathbf{A}_2 + b\mathbf{A}_3 + c\mathbf{A}_4$. If $\mathbf{Q} = u\mathbf{A}_2 + v\mathbf{A}_3 + w\mathbf{A}_4$, then $u + v + w = 1$ because Q lies in a_1. Let $\mathbf{P} = r\mathbf{Q}$. A vector $\mathbf{Y}$ in the plane a_2 is given by $\mathbf{Y} = b\mathbf{A}_3 + c\mathbf{A}_4$, so that a parallel shift of this plane to make it pass through P gives $\mathbf{Y} = r\mathbf{Q} + b\mathbf{A}_3 + c\mathbf{A}_4$. For $\mathbf{Y} = \mathbf{B}_2 = x\mathbf{A}_2$ (say),

$$\mathbf{B}_2 = r\mathbf{Q} + b\mathbf{A}_3 + c\mathbf{A}_4 = r(u\mathbf{A}_2 + v\mathbf{A}_3 + w\mathbf{A}_4) + b\mathbf{A}_3 + c\mathbf{A}_4 = x\mathbf{A}_2;$$

consequently, $x = ru = A_1B_2/A_1A_2$. Similarly, $A_1B_3/A_1A_3 = rv$ and $A_1B_4/A_1A_4 = rw$. Thus

$$\frac{A_1B_2}{A_1A_2} + \frac{A_1B_3}{A_1A_3} + \frac{A_1B_4}{A_1A_4} = r(u + v + w) = r = \frac{A_1P}{A_1Q}.$$

II. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.*

We denote the area of a tetrahedron by its name enclosed in parentheses. Since the plane through B_2 is parallel to a_2, we have

$$\frac{A_1B_2}{A_1A_2} = \frac{(B_2A_4A_3A_1)}{(A_1A_2A_3A_4)} = \frac{(PA_4A_3A_1)}{(A_1A_2A_3A_4)} .$$

Similarly,

$$\frac{A_1B_3}{A_1A_3} = \frac{(PA_1A_2A_4)}{(A_1A_2A_3A_4)} \quad \text{and} \quad \frac{A_1B_4}{A_1A_4} = \frac{(PA_3A_2A_1)}{(A_1A_2A_3A_4)} .$$

By summing these quantities we have

$$\begin{aligned}\frac{A_1B_2}{A_1A_2} + \frac{A_1B_3}{A_1A_3} + \frac{A_1B_4}{A_1A_4} &= \frac{(A_1A_2A_3A_4) - (PA_2A_3A_4)}{(A_1A_2A_3A_4)} \\ &= 1 - \frac{PQ}{A_1Q} = \frac{A_1Q - PQ}{A_1Q} = \frac{A_1P}{A_1Q} ,\end{aligned}$$

which was to be shown.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MURRAY S. KLAMKIN, University of Alberta; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

As several solvers remarked, both solutions given above can be extended in a natural way to n-dimensional simplexes. Klamkin adds that moreover, Q can be any point in the plane of a_1.

[Editor's comment by Chris Fisher. The proposer says that the result describes a theorem in economic production theory, namely

> *for a continuously differentiable production function, the sum of the factor input elasticities is equal to the degree of returns to scale exhibited by the production function.*

The rendering of this result in proper undecipherable jargon was aided by University of Regina Economics professor Chris Nicol. Please note that immediately following 1928 came the famous stock-market crash.]

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1931. [1994: 107] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

M is the midpoint of side BC of a triangle ABC, and Γ is the circle with diameter AM. D and E are the other intersections of Γ with AB and AC respectively. Let P be the point such that PD and PE are tangent to Γ. Prove that $PB = PC$.

I. *Solution by Waldemar Pompe, student, University of Warsaw, Poland.*

The conditions should add $\angle BAC \neq 90°$, otherwise P is undefined.

[See Figure 1.] Let Γ_1 and Γ_2 be the circumcircles of ΔBMD and ΔMCE respectively. Since ΔBMD and ΔMCE are right-angled and $BM = MC$, Γ_1 and Γ_2 are congruent and externally tangent at the point M. Let K and L be the other points of intersection of PD and PE with Γ_1 and Γ_2, respectively. Since $\angle KDM = \angle DAM$ and $\angle DKM = \angle ABM$, triangles ABM and DKM are similar. Analogously, triangles ACM and ELM are similar. Hence

$$\angle KMD + \angle LME = \angle BMA + \angle CMA = 180^\circ .$$

Therefore, since Γ_1 and Γ_2 are congruent, $DK = LE$. Moreover $PD = PE$, which gives $PK = PL$. Thus

$$PK \cdot PD = PL \cdot PE .$$

This means that P lies on the radical axis of Γ_1 and Γ_2, which in this case is the line passing through M and perpendicular to BC. Hence $PB = PC$.

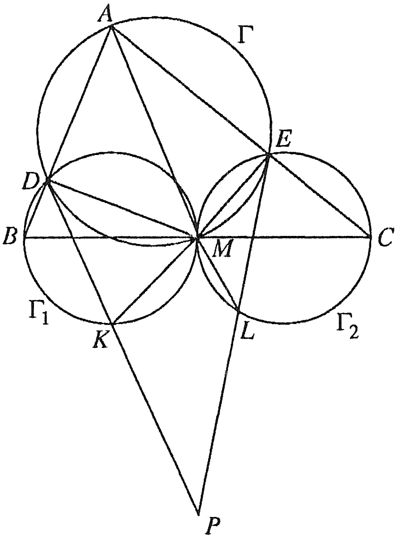


Figure 1

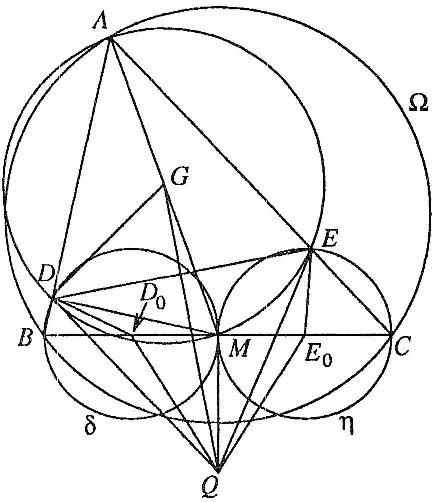


Figure 2

II. *Solution by Jordi Dou, Barcelona, Spain.*

[See Figure 2.] Let Ω be the circumcircle of ΔABC. When A varies along Ω [with B, C and Ω fixed], D varies along the circle δ of diameter BM and centre D_0, since $\angle MDB = \angle MDA = 90^\circ$; similarly, E varies along the circle η of diameter MC and centre E_0. The radii D_0D, E_0E form a constant angle α, where

$$\alpha = \angle MD_0D - \angle CE_0E = 2B - 2(90^\circ - C) = 2(90^\circ - A). \qquad (1)$$

Let Q be the intersection of the perpendicular bisectors of DE and BC. Then triangles QD_0D and QE_0E are congruent [by SSS, since circles δ and η have the same radius]. Thus $\angle DQE = \angle D_0QE_0 = \alpha$. Since D_0 and E_0 are fixed, Q is fixed because of $D_0Q = E_0Q$ and $\angle D_0QE_0 = \alpha$. Let G be the midpoint of

MA. Then $\angle QGD = A$ and $\angle GQD = \alpha/2$ [because GQ is the perpendicular bisector of DE]. Therefore $\angle QDG = 90°$, because $A + \alpha/2 = 90°$ by (1). Thus QD, and similarly QE, are tangent to Γ, and $P = Q$ follows. It is clear that not only does $PB = PC$, but $PB = PC$ is constant (as A varies on Ω), since P is in fact fixed. [If A lies on the lower arc BC of Ω, the proof goes through with only minor changes.—*Ed.*]

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * * *

1932*. [1994: 107] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Trivially, if $N = 1, 10, 100, \ldots$ then in each case the initial digits of N^3 are the digits of N. A nontrivial example is $N = 32$, since $32^3 = 32768$. Find another positive integer N with this property.

Solution by Peter Hurthig, Columbia College, Burnaby, B.C.

We claim that: *if the $(p+1)$st digit to the right of the decimal in $\sqrt{10}$ is greater than or equal to 6, then $N = [10^p\sqrt{10}] + 1$ has the desired property.* Since $\sqrt{10}$ begins 3.16227766... we have the solutions:

p	N	N^3
1	32	32768
4	31623	31623446801367
5	316228	31622846796684352
6	3162278	31622786796633508952
7	31622777	31622777796632428411433

For N to have the desired property, $N^3 = 10^m N + a$ where $0 \le a < 10^m$. Therefore

$$0 \le N^3 - 10^m N < 10^m. \tag{1}$$

If $m = 2p$, then $N(N^2 - 10^{2p}) \ge 0$ implies $N \ge 10^p$ where equality yields the trivial solutions. However if $N = 10^p + k$ for some positive integer k then

$$N^3 - 10^m N = N(N^2 - 10^{2p}) = (10^p + k)(2k \cdot 10^p + k^2) > 10^m,$$

contradicting (1). Therefore the non-trivial solutions can occur only if m is an odd number, $m = 2p+1$. Since $N^3 - 10^{2p+1}$ must be positive, $N > 10^p\sqrt{10}$. So if there are any solutions of (1) for $m = 2p + 1$, then $N = [10^p\sqrt{10}] + 1$, the smallest integer which is greater than $10^p\sqrt{10}$, must be one of them.

If we let $q = 10^p\sqrt{10}$ and $\delta = N - q$ then $N(N^2 - 10^m) < 10^m$ becomes

$$(q + \delta)(2q\delta + \delta^2) < q^2$$

or

$$(1 - 2\delta)q^2 - 3\delta^2 q - \delta^3 > 0.$$

If $\delta \geq 1/2$ then the left side is negative and there are no solutions. [In particular, $N = [10^p\sqrt{10}] + 1$ is the only possible choice for N for each value of p, because any larger choice would mean $\delta \geq 1$.—*Ed.*] If $\delta < 1/2$ then the inequality is satisfied if

$$q > \frac{3\delta^2 + \delta\sqrt{\delta^2 + 4\delta}}{2 - 4\delta}.$$

In particular, if $\delta \leq 2/5$ then

$$\frac{3\delta^2 + \delta\sqrt{\delta^2 + 4\delta}}{2 - 4\delta} \leq \frac{6 + 2\sqrt{11}}{5} < 3 < 10^p\sqrt{10} \quad \text{for all } p.$$

Therefore [since $\delta = N - q$ is the difference between $10^p\sqrt{10}$ and the next highest integer] if the $(p + 1)$st digit after the decimal in the expansion of $\sqrt{10}$ is greater than or equal to 6, then

$$10^p\sqrt{10} - [10^p\sqrt{10}] = 1 - \delta \geq 0.6,$$

and $[10^p\sqrt{10}] + 1$ will have the desired property.

It would of course be very gratifying to show that $\sqrt{10}$ is a *normal* number (i.e., all the digits occur in the expected frequency) so that there would be infinitely many such N.

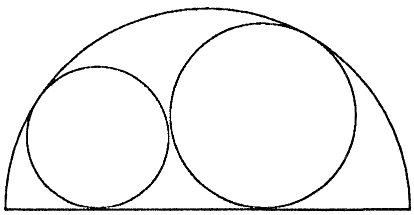
Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; THE BOOKERY SOLVERS GROUP, Walla Walla, Washington; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; TIM CROSS, Wolverley High School, Kidderminster, U. K.; ROBERT J. M. DAWSON, Saint Mary's University, Halifax, Nova Scotia; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; HANS HAVERMANN, Weston, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; DAVID E. MANES, State University of New York, Oneonta; J. A. MCCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University, Columbus; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; CORY PYE, student, Memorial University of Newfoundland, St. John's; GREG TSENG, student, Thomas Jefferson High School for Science and Technology, Alexandria, Virginia; and the proposer. One other reader apparently misunderstood the problem.

Several solvers (Dawson, Godin, Havermann, Hess, Konečný, Pompe, Pye, Tseng) also observed the connection between solutions to the problem and the *size of the digits of $\sqrt{10}$. Some add that a further solution "almost certainly" exists whenever a 5 occurs in the expansion of $\sqrt{10}$; can anyone remove all lingering doubt? There is never a solution associated with a digit in $\sqrt{10}$ which is 4 or less, as is contained in the above proof.*

As Hurthig implies, it is **not known** *whether $\sqrt{10}$ contains infinitely many digits greater than 5. In fact, almost nothing is known for sure about the digits of any number $\sqrt{N}$ where N is not a square; perhaps (though noone believes this of course) all digits of $\sqrt{10}$ beyond a certain point are 0's and 1's! If any reader could prove that such a thing can't happen, he or she would achieve instant fame.*

* * * * *

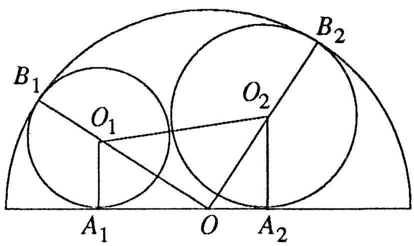
1933. [1994: 107] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*



Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$$R_1 + R_2 \le 2(\sqrt{2} - 1).$$

I. *Solution by N. T. Tin, Hong Kong.*



Let O_1, O_2, and O denote the centres of the circles, and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the semicircle, as shown in the diagram. Then $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$, and $O_2A_2 = R_2$, so

$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2} = \sqrt{(R_1+R_2)^2 - (R_1-R_2)^2} = 2\sqrt{R_1R_2}.$$

Also $OB_1 = 1$, $OB_2 = 1$, $O_1B_1 = R_1$ and $O_2B_2 = R_2$, so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$. Therefore

$$\begin{aligned} A_1A_2 &= OA_1 + OA_2 = \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2} \\ &= \sqrt{(1-R_1)^2 - R_1^2} + \sqrt{(1-R_2)^2 - R_2^2} = \sqrt{1-2R_1} + \sqrt{1-2R_2}. \end{aligned}$$

Thus

$$\sqrt{1-2R_1} + \sqrt{1-2R_2} = 2\sqrt{R_1R_2}.$$

Squaring, then dividing each term by 2 and rearranging the terms, we get

$$\sqrt{(1-2R_1)(1-2R_2)} = 2R_1R_2 + R_1 + R_2 - 1.$$

Square both sides and simplify:

$$8R_1R_2 = (2R_1R_2 + R_1 + R_2)^2, \tag{1}$$

so

$$2\sqrt{2R_1R_2} = 2R_1R_2 + R_1 + R_2.$$

Thus

$$\begin{aligned} R_1 + R_2 &= 2\sqrt{R_1R_2}(\sqrt{2} - \sqrt{R_1R_2}) \qquad (2) \\ &\le (R_1 + R_2)(\sqrt{2} - \sqrt{R_1R_2}), \end{aligned}$$

and therefore $\sqrt{R_1R_2} \le \sqrt{2} - 1$.

Now consider the function $f(x) = 2x(\sqrt{2} - x)$. $f(x)$ is increasing on the interval $(0, 1/\sqrt{2})$ since $f'(x) = 2\sqrt{2} - 4x > 0$ for x in the interval. Since $0 < \sqrt{R_1R_2} \le \sqrt{2} - 1 < 1/\sqrt{2}$ and $R_1 + R_2 = f(\sqrt{R_1R_2})$ from (2), $R_1 + R_2$ attains its maximum when $\sqrt{R_1R_2} = \sqrt{2} - 1$. Hence

$$R_1 + R_2 \le 2(\sqrt{2} - 1)[\sqrt{2} - (\sqrt{2} - 1)] = 2(\sqrt{2} - 1).$$

Equality holds when $R_1 = R_2$.

II. *Solution by Albert W. Walker, Toronto, Ontario.*

[*Editor's note.* Walker first derived the relation

$$2R_1R_2 = \sqrt{8R_1R_2} - (R_1 + R_2),$$

which is equivalent to (2) in Tin's solution, and then went on as follows.]

$$\begin{aligned} &2R_1R_2[(\sqrt{8} - 2) - (R_1 + R_2)] \\ &\quad= 2R_1R_2\sqrt{8} - 4R_1R_2 - (R_1 + R_2)\sqrt{8R_1R_2} + (R_1 + R_2)^2 \\ &\quad= \left(R_1 + R_2 - 2\sqrt{R_1R_2}\right)\left(R_1 + R_2 + (2 - \sqrt{8})\sqrt{R_1R_2}\right) \\ &\quad= \left(\sqrt{R_1} - \sqrt{R_2}\right)^2\left[\left(\sqrt{R_1} - \sqrt{R_2}\right)^2 + (4 - \sqrt{8})\sqrt{R_1R_2}\right] \ge 0. \end{aligned}$$

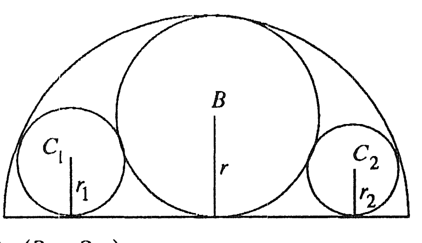
Thus

$$R_1 + R_2 \le \sqrt{8} - 2 = 2(\sqrt{2} - 1),$$

with equality if and only if $R_1 = R_2$.

III. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

As any given circle (B, r) inscribed in the semicircle has two "neighbours" (C_1, r_1) and (C_2, r_2), as shown in the diagram, we also should estimate $r + r_1 + r_2$. First we have



$$r_1 + r_2 = \frac{2r(3 - 2r)}{(2r + 1)^2}. \tag{3}$$

[*Editor's note.* In fact this can be seen by rewriting Tin's equation (1) using r, x instead of R_1, R_2:

$$8rx = (2rx + r + x)^2 = ((2r + 1)x + r)^2,$$

from which we obtain

$$(2r + 1)^2 x^2 + 2r(2r - 3)x + r^2 = 0.$$

Now the roots of this quadratic are just the radii of the neighbours of the circle of radius r, which implies that their sum is given by (3).]

Thus it holds that

$$r + r_1 + r_2 = r + \frac{2r(3 - 2r)}{(2r + 1)^2} = \frac{4r^3 + 7r}{(2r + 1)^2} =: f(r), \qquad 0 \le r \le 1/2.$$

Since

$$f'(r) = \frac{8r^3 + 12r^2 - 14r + 7}{(2r + 1)^3} > 0, \qquad 0 \le r \le 1/2,$$

we immediately get $f(r) \le f(1/2) = 1$, i.e.,

$$r + r_1 + r_2 \le 1.$$

This provokes the question: What about a "chain" of n circles inscribed in the semicircle? What is the best upper bound for the sum $r_1 + r_2 + \cdots + r_n$ of their radii?

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; NIELS BEJLEGAARD, Stavanger, Norway; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castello, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. There was one incorrect and one incomplete solution sent in.

Konečný notes the similar problem Crux 1627 [1992: 95], *also proposed by Tsintsifas, and the following result contained in the published proof by Bejlegaard* [1992: 96]*: if two perpendicular chords of a unit circle divide the circle into four parts, and circles of radii r_1 and r_3 are inscribed into two diagonally opposite parts, then $r_1 + r_3 \le 2(\sqrt{2} - 1)$. Is there some relationship between these two problems, perhaps a common proof or a common generalization?*

* * * * *

1934. [1994: 107] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

A cone of radius 1 metre and height h metres is lowered point first at a constant rate of 1 metre per second into a tall cylinder of radius R (> 1) metres which is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

Solution by Leroy F. Meyers, The Ohio State University, Columbus.

To make the formulas dimensionally homogeneous, let the radius of the cone be r metres, where $r < R$. Assume that the axes of the cone and the cylinder are kept vertical. Let the initial height of the water in the cylinder be H metres. When the cone has been moving for t seconds, its tip is t metres below the initial level of the top of the water. If at this time, the water has risen x metres, then the bottom $x + t$ metres of cone have been submerged. The total volume of water and submerged cone then satisfies the equation

$$\pi R^2(H + x) = \pi R^2 H + \left(\frac{x + t}{h}\right)^3 \frac{\pi r^2 h}{3},$$

which reduces to

$$3R^2h^2x = r^2(x + t)^3.$$

Differentiating this implicitly with respect to t gives

$$\frac{dx}{dt} = \frac{r^2(x + t)^2}{R^2h^2 - r^2(x + t)^2}$$

as the rate that the water is rising after t seconds. At the instant when the cone is completely submerged, we have $x + t = h$, so that

$$\frac{dx}{dt} = \frac{r^2}{R^2 - r^2}.$$

Note that the result is independent of h. Putting $r = 1$, we obtain

$$\frac{1}{R^2 - 1}$$

as the answer to the given problem.

More generally, let a body of height h and horizontal cross section area $a(y)$ at a distance y above the bottom be gradually immersed in a tub containing water initially of depth H, where the horizontal cross section area of the tub is $A(y)$ at depth y below the initial level. At the time when the body has been lowered so that its bottom is at a distance s below the original water level, and the water in the tub has risen a distance x, the volume of water and submerged body satisfies the relation

$$\int_{-x}^{H} A(y)\,dy = \int_0^H A(y)\,dy + \int_0^{x+s} a(y)\,dy.$$

If the body is moved downwards keeping the same orientation, then differentiation of the above relation with respect to time t yields

$$A(-x)\frac{dx}{dt} = 0 + a(x+s)\left(\frac{dx}{dt} + \frac{ds}{dt}\right),$$

i.e.

$$\frac{dx}{dt} = \frac{a(x+s)}{A(-x) - a(x+s)}\frac{ds}{dt}.$$

Thus the rate at which the water level is rising depends on only the areas of the cross sections and the rate of lowering of the body at the instant of measurement. For the circular cone of radius r metres and height h metres, lowered point first at a constant rate of 1 metre per second into a circular cylindrical tub of radius R metres containing water, the rate at which the water is rising at the instant of total submersion is $r^2/(R^2 - r^2)$ metres per second, as derived earlier.

This agrees with the intuitive idea that at each instant, the tub and the body are approximately cylinders of cross section areas $A(-x)$ and $a(x+s)$, respectively, so that the rate at which the water is rising is equal to the rate at which the body is being lowered multiplied by the ratio of the area of the newly immersed cross section to that of the water surface. At the instant that the body is completely submerged, we have $x + s = h$.

Also solved by JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; PAUL PENNING, Delft, The Netherlands; SKIDMORE COLLEGE PROBLEM GROUP, Skidmore College, Saratoga Springs, New York; and the proposer. There were five incorrect solutions sent in, all making the same mistake of not allowing for the rising of the water level as the cone descends.

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1936. [1994: 108] *Proposed by Bolian Liu, South China Normal University, Guangzhou, China.*

Ten players participate in a ping-pong tournament, in which every two players play against each other exactly once. If player i beats player j, player j beats player k, and player k beats player i, then the set $\{i, j, k\}$ is called a *triangle*. Let w_i and l_i denote the number of games won and lost, respectively, by the ith player. Suppose that, whenever i beats j, then $l_i + w_j \geq 8$. Prove that there are exactly 40 triangles in the tournament.

Solution by Kee-Wai Lau, Hong Kong (slightly revised and expanded by the editors, in this case Bill Sands and Edward Wang).

Let P_i denote the ith player, $i = 1, 2, \ldots, 10$. Without loss of generality, assume that $w_1 \leq w_2 \leq \cdots \leq w_{10}$. Let $M = \max_{1\leq i\leq 10} w_i = w_{10}$ and $m = \min_{1\leq i\leq 10} w_i = w_1$. Since $w_i + l_i = 9$ for all i, the condition $l_i + w_j \geq 8$, whenever P_i beats P_j, is equivalent to

$$w_i \leq w_j + 1. \tag{1}$$

Clearly

$$\sum_{i=1}^{10} w_i = \binom{10}{2} = 45. \tag{2}$$

We first show that

$$(w_1, w_2, \ldots, w_{10}) = (4,4,4,4,4,5,5,5,5,5).$$

Since P_1 beats m players and loses to $9 - m$ others, we get from (1) and (2) that

$$45 \le m + Mm + (9 - m)(m + 1)$$

or

$$36 \le (M - m + 9)m. \tag{3}$$

Similarly, since P_{10} beats M players and loses to $9 - M$ others, we have

$$45 \ge M + M(M - 1) + (9 - M)m = M^2 + (9 - M)m. \tag{4}$$

Since $M \le 9$, from (3) we obtain $36 \le (18 - m)m$, which implies $m \ge 3$. From (4) we get $M^2 \le 45$, i.e. $M \le 6$. If $M = 6$, then from (4), $3m \le 9$ or $m \le 3$ and so $m = 3$. Similarly, if $m = 3$ then from (3) $M \ge 6$ and so $M = 6$. Since $10 \nmid 45$, $M \ne m$. Therefore, the only possibilities are $(M, m) = (6, 3)$ and $(5, 4)$.

[*Editor's note.* At this point, Lau listed all 10-tuples $(w_1, w_2, \ldots, w_{10})$ satisfying $(M, m) = (6, 3)$ and (2), and then showed that none were possible. But the case $(M, m) = (6, 3)$ can be eliminated as follows. If $m = w_1 = 3$, then six players beat P_1, so by (1) at least seven of the w_i's (including w_1) are ≤ 4. If $M = w_{10} = 6$, then P_{10} beats six players, so by (1) at least seven of the w_i's (including w_{10}) are ≥ 5. But since there are only ten w_i's this is impossible.]

Thus $(M, m) = (5, 4)$, and $(4,4,4,4,4,5,5,5,5,5)$ is the only possible solution. By exhibiting the tournament graphically, one can see that this is indeed a solution.

Now, there is a theorem in graph theory which states that the number of transitive triples [3 players P_i, P_j and P_k such that P_i beats P_j, P_j beats P_k, and P_i beats P_k —*Ed.*] in a tournament with n vertices and score sequence $(w_1, w_2, \ldots, w_n)$ is $\sum_{i=1}^{n} w_i(w_i - 1)/2$ (for instance, see Theorem 16.13 on p. 207 of Harary's *Graph Theory*, Addison-Wesley, 1969). [*Editor's note.* This result is true because the number of such triples is just the number of unordered pairs $\{P_j, P_k\}$ of (distinct) players beaten by P_i, which is $\binom{w_i}{2}$, added up over all P_i.] It follows that, with $n = 10$, the number of triangles (cyclic triples) is

$$\binom{10}{3} - \frac{1}{2}\sum_{i=1}^{10} w_i(w_i - 1) = 120 - \frac{1}{2}(5)(5)(5-1) - \frac{1}{2}(5)(4)(4-1) = 40.$$

This completes the proof.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; and the proposer.

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1937. [1994: 108] *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcenter O, orthocenter H, and altitudes AD, BE and CF (with D on BC, etc.). Suppose $OH \parallel AC$.

(a) Show that EF, FD and DE are in arithmetic progression.

(b) Determine the possible values of angle B.

Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.

(a) If OH is parallel to AC then the perpendicular distances from O and H to AC are equal. These are $R\cos B$ and $2R\cos A\cos C$ respectively, both well-known expressions. Hence

$$2\cos A\cos C = \cos B = -\cos(A+C) = \sin A\sin C - \cos A\cos C$$

which implies

$$3\cos A\cos C = \sin A\sin C, \tag{1}$$

$$\cos(A-C) = -2\cos(A+C) = 2\cos B, \tag{2}$$

and finally

$$\sin 2A + \sin 2C = 2\sin(A+C)\cos(A-C) = 4\sin B\cos B = 2\sin 2B. \tag{3}$$

Now the angles of the pedal triangle DEF are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$ provided ABC is acute-angled. This is the case, for from (1)

$$\tan A\tan C = 3,$$

and since not both A and C can be obtuse, then both must be acute. And then (2) shows $\cos B$ positive and hence B is acute. So

$$DE : EF : FD = \sin 2A : \sin 2B : \sin 2C$$

and we have EF, FD, DE in arithmetic progression by (3).

(b) For a start, (2) implies $0 \le \cos B \le 1/2$, so $\pi/3 \le B \le \pi/2$. If $B = \pi/3$ then [$\cos(A-C) = 1$ and thus] $A = C = \pi/3$, and OH does not exist as a line. The case $B = \pi/2$ is not allowed for then the triangle degenerates. The conclusion is

$$\frac{\pi}{3} < B < \frac{\pi}{2}.$$

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Bellot gives the reference Mathesis, 1888, *page* 245 *for some properties of the triangles of this problem. He also mentions the related problems* Crux 1235 [1988: 156] *and* #1232 *of* Mathematics Magazine, *solution on page* 43 *of the* 1987 *issue, both of which were also proposed (or co-proposed) by Dr. Smeenk.*

*   *   *   *   *

1938. [1994: 108] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Find the exact value of

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cdots\cos\left(\frac{16\pi}{17}\right).$$

I. *Solution by Bill Correll, Jr., student, Denison University, Granville, Ohio.*

Let

$$I=\cos\frac{2\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\cdots\cos\frac{16\pi}{17};$$

then

$$\begin{aligned}
I\sin\frac{2\pi}{17} &= \frac{1}{2}\sin\frac{4\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\cdots\cos\frac{16\pi}{17}\\
&= \frac{1}{4}\sin\frac{8\pi}{17}\cos\frac{8\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}\cos\frac{16\pi}{17}\\
&= \frac{1}{8}\sin\frac{16\pi}{17}\cos\frac{16\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}\\
&= -\frac{1}{16}\sin\frac{2\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}.
\end{aligned}$$

Thus

$$\begin{aligned}
I\sin\frac{6\pi}{17} &= -\frac{1}{16}\sin\frac{6\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}\\
&= -\frac{1}{32}\sin\frac{12\pi}{17}\cos\frac{12\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17}\\
&= \frac{1}{64}\sin\frac{10\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17} = -\frac{1}{128}\sin\frac{14\pi}{17}\cos\frac{14\pi}{17}\\
&= -\frac{1}{256}\sin\frac{28\pi}{17} = \frac{1}{256}\sin\frac{6\pi}{17}.
\end{aligned}$$

Therefore

$$I=\frac{1}{256}.$$

II. *Solution by Himadri Choudhury, student, Hunter High School, New York.*

By De Moivre's theorem we have

$$\cos ma + i\sin ma = (\cos a + i\sin a)^m.$$

Expanding the right and equating imaginary parts gives

$$\sin ma = \binom{m}{1} \cos^{m-1} a \sin a - \binom{m}{3} \cos^{m-3} a \sin^3 a + \cdots .$$

Let $m = 2n + 1$ and we get

$$\begin{aligned} \sin(2n+1)a &= \binom{2n+1}{1} \cos^{2n} a \sin a - \binom{2n+1}{3} \cos^{2(n-1)} a \sin^3 a + \cdots \\ &= \sin a \left[\binom{2n+1}{1} \cos^{2n} a - \binom{2n+1}{3} \cos^{2(n-1)} a (1 - \cos^2 a) + \cdots \right. \\ &\qquad \left. + (-1)^n (1 - \cos^2 a)^n \right] . \end{aligned}$$

Note that

$$\sin(2n+1)a = 0 \quad \text{when} \quad a = \frac{\pi}{2n+1}, \quad \frac{2\pi}{2n+1}, \quad \ldots, \quad \frac{n\pi}{2n+1},$$

but $\sin a \neq 0$ for any of these values of a. So the expression in the brackets must be equal to zero for all these n values of a. Therefore $\cos^2 a$ for these values of a are roots of

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1}(1-x) + \cdots + (-1)^n (1-x)^n.$$

Note that this is a polynomial of the nth degree with the roots

$$\cos^2 \frac{\pi}{2n+1}, \quad \cos^2 \frac{2\pi}{2n+1}, \quad \ldots, \quad \cos^2 \frac{n\pi}{2n+1} .$$

The constant term of the polynomial is $(-1)^n$. The coefficient of x^n is

$$\binom{2n+1}{1} + \binom{2n+1}{3} + \binom{2n+1}{5} + \cdots + \binom{2n+1}{2n+1}$$

which we know is equal to 2^{2n} by a well known theorem of binomial coefficients. So the product of the roots of the polynomial is

$$\left(\cos \frac{\pi}{2n+1} \cos \frac{2\pi}{2n+1} \cdots \cos \frac{n\pi}{2n+1} \right)^2 = (-1)^n \cdot \frac{(-1)^n}{2^{2n}},$$

so

$$\cos \frac{\pi}{2n+1} \cos \frac{2\pi}{2n+1} \cdots \cos \frac{n\pi}{2n+1} = \frac{1}{2^n} . \tag{1}$$

Now note that

$$\cos \frac{\pi}{2n+1} = -\cos \frac{2n\pi}{2n+1}, \quad \cos \frac{3\pi}{2n+1} = -\cos \frac{(2n-2)\pi}{2n+1}, \quad \text{etc.},$$

and considering the cases when n is even and odd separately we get

$$\begin{aligned}\cos\frac{2\pi}{2n+1}\cos\frac{4\pi}{2n+1}\cdots\cos\frac{2n\pi}{2n+1} &\\ &= (-1)^{[(n+1)/2]}\cos\frac{\pi}{2n+1}\cos\frac{2\pi}{2n+1}\cdots\cos\frac{n\pi}{2n+1}\\ &= \frac{(-1)^{[(n+1)/2]}}{2^n}\,.\end{aligned}$$

For the particular problem we have $n = 8$, so the answer is $1/256$.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MURRAY S. KLAMKIN, University of Alberta; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; JISHO KOTANI, Akita, Japan; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; DAVID E. MANES, State University of New York, Oneonta; BENGT MÅNSSON, Partille, Sweden; BEATRIZ MARGOLIS, Paris, France; J. A. MCCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University, Columbus; VEDULA N. MURTY, Maharanipeta, India; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; CORY PYE, student, Memorial University of Newfoundland, St. John's; D. J. SMEENK, Zaltbommel, The Netherlands; VADLA MANI SWAPNA, student, Maharanipeta, India; N. T. TIN, Hong Kong; MICHAEL TING, student, Lisgar Collegiate, and KENNETH S. WILLIAMS, Carleton University, Ottawa; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer.

Several solvers gave references for this or similar results, for example, Crux 222 [1977: 200] *and* 234 [1977: 257]; *in fact equation* (1) *appears in an editor's comment on* [1977: 201], *with further references listed there. Also see S. Galovich, Products of sines and cosines,* Mathematics Magazine 60 (1987) 105–113 *(the present problem is contained in Theorem 2 of that paper).*

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1939. [1994: 108] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

Let ABC be an acute-angled triangle with circumcentre O, incentre I and orthocentre H. Let AI, BI, CI meet BC, CA, AB respectively in U, V, W, and let AH, BH, CH meet BC, CA, AB respectively in D, E, F. Prove that O is an interior point of triangle UVW if and only if I is an interior point of triangle DEF.

Solution by the proposer.

Lemma. With respect to triangle ABC as triangle of reference, suppose the internal point J has areal coordinates (k, m, n) so that $k, m, n > 0$. Let AJ, BJ, CJ meet BC, CA, AB respectively in K, M, N. Then the point P with areal coordinates (p, q, r) is internal to triangle KMN if and only if

$$q/m + r/n > p/k,$$

$$r/n + p/k > q/m,$$

$$p/k + q/m > r/n.$$

Proof. [*Editor's note by Cathy Baker.* The areas of the triangles BCJ, ACJ and ABJ are proportional to the areal coordinates of J. For a discussion of areal coordinates, see, for example, p. 218 of H. S. M. Coxeter, *Introduction to Geometry*, Wiley (1961).]

The equation of MN is $-mnx + nky + kmz = 0$. Since m and n are positive, the point P lies on the opposite side of MN to A if and only if $-mnp + nkq + kmr > 0$, which is the first of the above conditions.

Now P is internal to triangle KMN if and only if it is on the opposite side of MN to A, on the opposite side of NK to B and on the opposite side of KM to C. So the three inequalities are necessary and sufficient. □

In the given problem, we have

the areal coordinates of I proportional to $(\sin A, \sin B, \sin C)$,

the areal coordinates of H proportional to $(\tan A, \tan B, \tan C)$,

and

the areal coordinates of O proportional to

$$(\sin A \cos A, \sin B \cos B, \sin C \cos C).$$

Using the lemma with $J \equiv I$, $P \equiv O$ and again with $J \equiv H$ and $P \equiv I$ gives in both cases the three inequalities

$$\cos B + \cos C > \cos A,$$

$$\cos C + \cos A > \cos B,$$

$$\cos A + \cos B > \cos C.$$

The required result follows.

No other solutions were received.

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1940. [1994: 108] *Proposed by Ji Chen, Ningbo University, China.*
Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.
Writing $y + z = a$, $z + x = b$ and $x + y = c$, we have

$$x = \frac{b+c-a}{2}, \qquad \text{etc.,}$$

thus

$$yz = \frac{a^2 - (b-c)^2}{4} = \frac{a^2 - b^2 - c^2 + 2bc}{4}, \qquad \text{etc.,}$$

and hence, in cyclic sum notation,

$$\sum yz = \frac{1}{4}\sum(2bc - a^2).$$

Now assume $a \geq b \geq c$ without loss of generality; then

$$2bc - a^2 \leq 2ca - b^2 \leq 2ab - c^2$$

and therefore, by Chebyshev's inequality and the A.M.-G.M. inequality,

$$\begin{aligned}
\frac{4}{9}\left(\sum yz\right)\left(\sum \frac{1}{(y+z)^2}\right) &= \frac{1}{3}\sum(2bc - a^2) \cdot \frac{1}{3}\sum\frac{1}{a^2} \\
&\geq \frac{1}{3}\sum\left((2bc - a^2)\cdot\frac{1}{a^2}\right) \\
&= \frac{1}{3}\left(\sum\frac{2bc}{a^2}\right) - 1 \\
&\geq \left(\prod\frac{2bc}{a^2}\right)^{1/3} - 1 = 2 - 1 = 1.
\end{aligned}$$

The inequality follows.

Also solved (but not as neatly!) by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; VEDULA N. MURTY, Maharanipeta, India; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There were two incorrect solutions sent in.

Both Janous and the proposer note that the given inequality implies that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{9}{4r(4R+r)}, \tag{1}$$

where a, b, c are the sides of a triangle of inradius r and circumradius R. They put $x = s - a$, etc., where s is the semiperimeter of the triangle, which is the same as Kuczma's substitution! Then $y + z = a$, etc., and also

$$\sum yz = 4Rr + r^2$$

(e.g. item 15, *page* 54 *of Mitrinović, Pečarić and Volenec,* Recent Advances in Geometric Inequalities*), and* (1) *follows. This is an improvement of part of item* 5.9, *page* 173 *of* Recent Advances. *Actually,* (1) *was the proposer's original problem, which also included the inequality*

$$\frac{r_a^2}{a^2} + \frac{r_b^2}{b^2} + \frac{r_c^2}{c^2} \geq \frac{9}{4}, \tag{2}$$

where r_a, r_b, r_c are the exradii of the triangle; this follows immediately from the inequality of the problem via the known relation

$$a = \frac{r_a(r_b + r_c)}{\sqrt{r_b r_c + r_c r_a + r_a r_b}}, \quad \textit{etc.}$$

(2) *improves the case $p = -2$ of an inequality of Janous given on page* 195 *of* Recent Advances.

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BOOKS WANTED

To follow up his suggestion on [1994: 240], Kenneth Williams asks if any *Crux* reader can help him obtain copies of (one or more of) the following books:

- A. Wintner, *Eratosthenian Averages*, Waverly Press, Baltimore, 1943;
- Allan Cunningham, *Quadratic Partitions*, Francis Hodgson, London, 1904;
- Richard Dedekind, *Collected Works*, Vieweg, 1931.

He will pay a reasonable price, or possibly arrange a trade. Readers with information can write to

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