Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

EUREKA

Vol. 2, No. 10

December 1976

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education

Publié par le Collège Algonquin

EUREKA is published monthly (except July and August). The following yearly subscription rates are in Canadian dollars. Canada and USA: $6.00; elsewhere: $7.00. Bound copies of combined Volumes 1 and 2: $10.00. Back issues: $1.00 each.

All communications about the content of the magazine (articles, problems, solutions, book reviews, etc.) should be sent to the editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ont., K1S 1N3.

All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ont., K1S 0C5.

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CONTENTS

CHIPS FROM THE EDITOR'S WORKBENCH

The chips in the title are not all chips off the old block.[1] They are chips, hewn from several disparate blocks, that are now cluttering up the editor's desk, sundry bits of information from many sources that we[2] would like to impart to our readers before cleaning up our desk at year-end.[3]

1. *The Great October Flap.*

All Canadian readers receive their copies of EUREKA by second-class mail. But, from the beginning, all copies going outside Canada[6] had been sent by first-class mail, because of the distances involved and of the relatively short time span to submit solutions to proposed problems. Then, to conserve dwindling funds, Fred Maskell, whom I would like to call my circulation manager if it did not sound so pompous,[7] decided to send *all* copies of the October issue by second-class mail.

What happened after that lies in the lap of the gods who oversee the destinies of the Post Office. Before two weeks had gone by, I started receiving pleading letters and distraught telephone calls from as far away as California. I found out that some Americans, fearful that they might have been dropped from our mailing list, started calling their friends all over the United States to see if *they* had received their copy. One American wrote to me: "Where is my favorite (excuse me, favourite) magazine?". One caller said to me that reading EUREKA was habit-forming and that the withdrawal symptoms were severe when its arrival was delayed.[8] He sounded thoroughly convincing and I believed every word of it.[9]

Now penny-pinching Fred has relented. He has assured me that henceforth all copies going outside Canada will go by first-class mail, and damn the expense.[10] O.K., I will. No more footnotes.[11]

---

[1]Otherwise this article could have gone under the unappetizing title: *Editorial Dandruff.*

[2]Mark Twain once said that the only persons who use *we* instead of *I* are editors, kings, and people with tapeworms.

[3]A procedure our wife insisted upon after discovering on a remote corner of our desk a pile of letters that should have been mailed six months ago.[4]

[4]*Our wife*... I don't like the polyandrous sound of that. To hell with the editorial *we* and *our*.[5]

[5]What are you doing down here? Get back up to the text.

[6]Most of these go to Americans, but a few copies go to England, Sweden, India, Australia, and one copy goes to a penurious professor in some South American banana republic called Guayazuela.

[7]Fred had all pomposity pricked out of him a long time ago.

[8]Be the first EUREKA "pusher" in your neighbourhood.

[9]Don't be misled by the flippant tone of this article: I am not making anything up; these things really happened.

[10]The expense is... Look, chum, this footnote business is getting a bit tiresome. Why don't you lay off from now on?

[11]Thank you.

2. *How do you colour the empty set?*

One reader, who has been receiving EUREKA for over a year, wrote to me recently that he had observed that EUREKA now came with three holes punched along the left margin, from which he had brilliantly deduced that the editor's intent was that readers should remove the staple at the top left corner and file their copies away in a loose-leaf binder for ease of reference. He then congratulated me for having thought of this added convenience for readers.

Dear reader, the holes have been there all along, right from the Vol. 1, No. 1 issue. To make sure that all readers are aware of this convenience, I'm going to ask Fred to ask the printer if he can arrange to have the holes come in some highly visible colour.

3. *Meet the editorial conscience of EUREKA.*

Professor Leroy F. Meyers, The Ohio State University, has long been a faithful supporter of and contributor to EUREKA. He is at the same time Associate Problem Editor for *Mathematics Magazine*. There is no conflict of loyalties in this, since the mathematical level of *Mathematics Magazine* is somewhat higher than that of EUREKA. So when you begin to find the contents of EUREKA too trivial, your next step *up* should be to *Mathematics Magazine* and/or to *James Cook Mathematical Notes*, whose editor, Basil C. Rennie, is also one of our contributors. On the other hand, if you merely get *tired* of EUREKA, then you could switch *laterally* to *The Pi Mu Epsilon Journal* and/or to *The Pentagon*, both of whose Problem Editors (Leon Bankoff and Kenneth M. Wilke, respectively) are EUREKA contributors, and/or to *The Two-Year College Mathematics Journal*, one of whose associate editors (Peter Lindstrom) is a EUREKA subscriber.[12]

I had started out to tell you about Professor Meyers, whom I like to call the editorial conscience of EUREKA. In addition to giving me much valuable advice

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[12]Just one more footnote, OK? Here are the addresses to which readers should write for subscription information about the journals mentioned above.

For *Mathematics Magazine*, write to: A.B. Willcox, Executive Director, Mathematical Association of America, Suite 310, 1225 Connecticut Ave., N.W., Washington, D.C. 20036, USA.

For *James Cook Mathematical Notes*, write to its editor: Professor Basil C. Rennie, Mathematics Department, James Cook University of North Queensland, Post Office James Cook University, Q. 4811, Australia.

For *The Pi Mu Epsilon Journal*, write to: David C. Kay, Editor, The Pi Mu Epsilon Journal, 601 Elm, Room 423, The University of Oklahoma, Norman, Oklahoma 73069, USA.

For *The Pentagon*, write to: Wilbur J. Waggoner, Business Manager, Central Michigan University, Mount Pleasant, Michigan, USA.

For *The Two-Year College Mathematics Journal*, write to: TYCMJ Subscriptions Department, The Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036, USA.

about editorial matters, for which I would like to thank him publicly, he has been keeping a sharp eye out for errors, misprints, and other blemishes in the magazine. His latest letter to me contains the following information:

> In looking over the earlier issues of EUREKA, I find the following misprints.
>
> In Vol. 1 (1975), p. 81, it is stated that the smallest power of 2 whose decimal expansion begins with 7 is 2^{56}; it should be 2^{46}.
>
> In Vol. 2 (1976), p. 18, the impression is given that the divisor, $5 \cdot 2^{1947} + 1$, of F_{1945} has about 10^{585} digits. However, $5 \cdot 2^{1947} + 1$ has exactly 587 digits, whereas F_{1945} has approximately 10^{585} (closer: $9.6 \cdot 10^{584}$) digits.

For the first item, I plead guilty with extenuating circumstances, since the misprint occurs in the first printing of Professor Honsberger's book *Ingenuity in Mathematics*, from which I got the information. It may, of course, have been corrected in later printings of the book, which I have not seen. My mistake was in not checking the information. There are no extenuating circumstances for the second item. It's just a result of sloppy editing, for which I alone am responsible.

In any case, thanks and a tip of the editorial hat to Professor Meyers.

There is another piece of information that I have been guilty of publishing without first checking its accuracy. In Vol. 1, No. 2, p. 5, I said (quoting TIME Magazine) that the millionth decimal place in the expansion of π was a 5. Will Professor Meyers please check this, and thus make an honest editor out of me?

4. *Reflections of a Problem Editor.*

An important part of the work of the editor of a publication such as EUREKA lies in acting as a problem editor. I expected that the bulk of my remarks at this time would consist of recounting my experiences in this rôle and of finding ways to encourage more readers (especially Canadian readers) to submit problems and solutions. Since, however, my friend and mentor Leon Bankoff, problem editor of the *Pi Mu Epsilon Journal*, has recently published a remarkable article on just this subject, I decided to let him speak for me. His article is reprinted later in this issue.

And a retroactive Merry Christmas and Happy New Year to all of you.

* * *

ZIGZAG

Ziggery-zaggery,
Karl W.T. Weierstrass,[1]
Graphing some functions while
Working at night,

Found one: continuous,
Nondifferentiable.
Tangents? Too wiggly. *Mein*
Gott! what a sight!

L.F. MEYERS

---

[1]The initials are to be pronounced *à l'allemande*.

TWO HEXAHEDRONS

CHARLES W. TRIGG, Professor Emeritus, Los Angeles City College

There are two distinct hexahedrons with congruent regular polygons as faces. They are the cube with its six square faces, and the ditetrahedron (triangular dipyramid) with six equilateral triangle faces. When these two polyhedrons have equal edges they have an interesting relationship.

The Cube.

A cube can be viewed as an antiprism with two pyramidal caps (Trigg, 1975). It is dissected into these three pieces by planes through the three vertices adjacent to each of two opposite vertices of the cube.

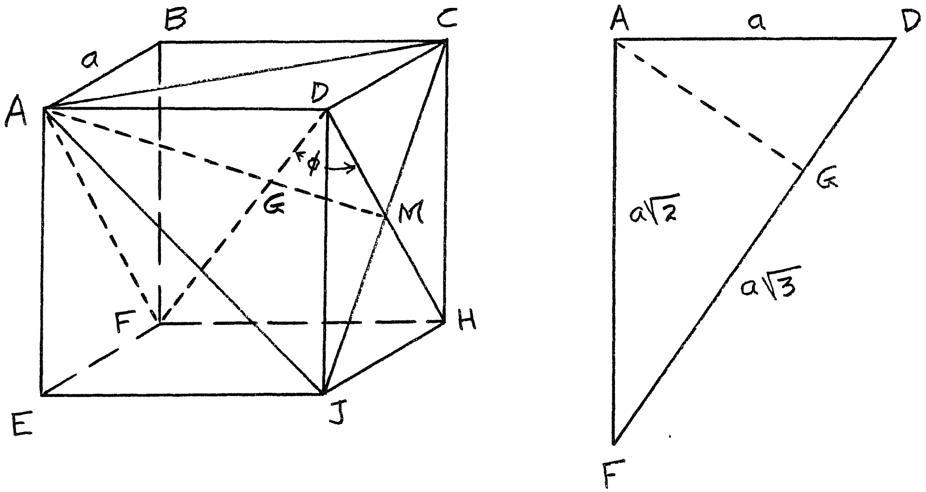

Figure 1

Consider the cube ABCD-EFHJ in Figure 1, with edges a. The intersections of the plane through A, C, and J with the faces form the equilateral triangle ACJ with sides $a\sqrt{2}$. The extremities of space diagonal DF are equidistant from A, C, and J. Therefore, DF is perpendicular to the plane ACJ at G, the centroid of triangle ACJ. It follows that DG and AG are perpendicular. Also, AD is perpendicular to the plane ABFE, so AD and AF are perpendicular. Then right triangles AFD and GAD are similar since they have a common acute angle. Therefore, DG/AD = AD/DF. It is well known that the space diagonal of a cube is $\sqrt{3}$ times an edge. Consequently,

$$DG = (AD)^2/DF = a^2/a\sqrt{3} = (a\sqrt{3})/3.$$

Thus G is a trisection point of the space diagonal DF. (This was proven with analytic geometry in Trigg-Buchman, 1969.)

The *smallest* angles between the space diagonal DF and any lines in the faces of the cube that pass through D are the angles made with the face diagonals. Such an angle, ϕ, can be determined from the right triangle DGM where

$$\phi = \angle GDM = \arccos DG/DM = \arccos (a\sqrt{3}/3)/(a\sqrt{2}/2) = \arccos \sqrt{2/3} .$$

The Ditetrahedron.

The ditetrahedron with edges a has a single diagonal which is twice the altitude of its constituent regular tetrahedrons with edge a. This diagonal PT in the ditetrahedron P-QRS-T of Figure 2 passes through the centroid N of triangle QRS, where NR = $(2/3)(a\sqrt{3}/2)$ or $a/\sqrt{3}$. Then from right triangle PNR,

$$PT = 2(PN) = 2\sqrt{(PR)^2 - (NR)^2} = 2\sqrt{a^2 - (a/\sqrt{3})^2} \quad \text{or} \quad 2a\sqrt{2/3} .$$

The *largest* angle between the space diagonal PT and any lines in the faces of the ditetrahedron that pass through P are the angles made with the edges. Such an angle, θ, can be determined from right triangle NPR where

$$\theta = \angle NPR = \arccos PN/PR = \arccos (a\sqrt{2/3})/a = \arccos \sqrt{2/3} .$$

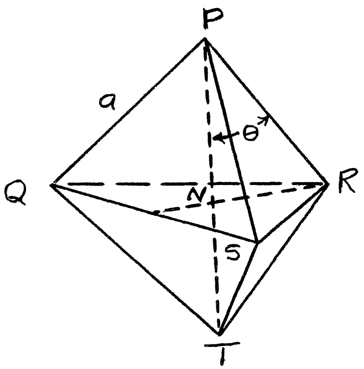

Figure 2

The Two Hexahedrons.

Since

$$PT = 2a\sqrt{2/3} < a\sqrt{3} = DF \quad \text{and} \quad \theta = \arccos \sqrt{2/3} = \phi,$$

the ditetrahedron with edge a can be put inside the cube with edge a with room to spare. (This was proven with analytic geometry in Trigg-Cormier, 1963.) Models of these two hexahedrons can be constructed easily to make an impressive demonstration.

Indeed, a double cone, with elements, x, equal to

$$a(\sqrt{3}/2\sqrt{2/3}) \quad \text{or} \quad 3a\sqrt{2}/4 \approx 1.061a$$

and axis coincident with the space diagonal of the cube, will touch the six faces of the cube. Consequently, it and its inscribed double tetrahedron with edges $x > a$ can be placed and rotated inside the cube of edge a.

Tetrahedrons in a Cube.

It is well known that the regular tetrahedron with edge $a\sqrt{2}$ and volume $a^3/3$ is the largest one that can be placed inside a cube with edge a. This is the tetrahedron determined by joining the extremities of non-parallel diagonals lying in opposite faces, as in Figure 3. Since each face contains two diagonals, two such tetrahedrons can be inscribed in the cube. The common volume of these interpenetrating tetrahedrons is $a^3/6$. Together they constitute Kepler's *stella octangula*, with a volume of $a^3/2$. (Trigg, 1971.)

As a by-product of the present discussion, the largest *pair* of congruent regular tetrahedrons that can be placed inside a cube of edge a is the pair with edges $3a\sqrt{2}/4$.

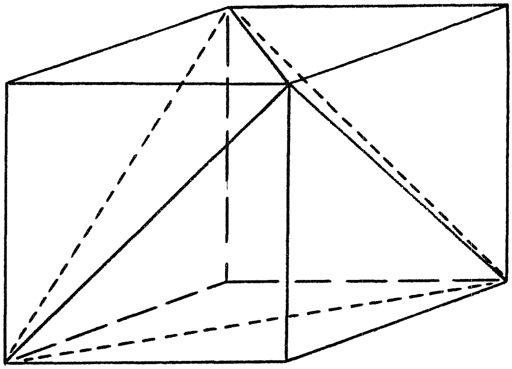

Figure 3

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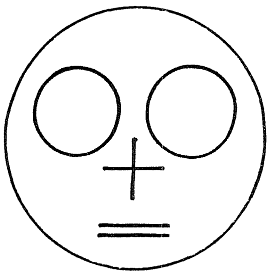
* * *

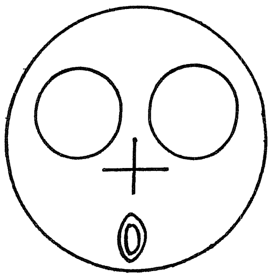
VARIATIONS ON A THEME BY BANKOFF I

The theme by Leon Bankoff



0 + 0 = 0

Variation No. 1



No-o-el — No-o-el

REFLECTIONS OF A PROBLEM EDITOR

LEON BANKOFF, Los Angeles, California

Introduction.

Telling is not teaching and listening is not learning. This terse truism summarizes the difficulties in communication so often encountered in mathematical education. Nevertheless properly directed telling and intelligently oriented listening are essential components of successful communication. The most effective way to measure the degree of such success is by appropriate testing of the student's problem solving ability.

Volumes can be and have been written on the importance of problem solving in the learning process and in the growth and development of mathematics. History is replete with instances where entire new branches of the art and science of mathematics have sprung up as a consequence of the search for the solution of some challenging problem. A noteworthy example is the successful attack on the brachistochrone problem by the Bernoulli brothers and the role played by this solution in the birth of the Calculus of Variations. Another familiar example is the emergence of the mathematical theory of probability as an offshoot of problems considered by Pacioli, Cardan and Tartaglia and the arousal of interest by the discussions between Pascal and Fermat. Even to this day, mathematicians continue to indulge in the age-old pleasurable activity of milking one another's brains through conversation or correspondence — exchanging ideas — collaborating on the solution of difficult and perplexing problems — hurling and accepting challenges emanating either from their own gnawing inquisitiveness or from the frustrated curiosity of others — building, forging, developing and inventing new tools and ingenious devices in the never-ending struggle for the establishment of mathematical order out of chaos.

In addition to the influence of private communication in the advancement of mathematical knowledge, it is important to recognize the tremendous impetus occasioned by the dissemination of provocative, non routine problems by way of mathematical journals. For the last three centuries, readers of periodicals that contained problem sections have been invited to submit solutions to proposed problems with the objective of competing with other solvers for the publication of what the editors later judged to be the "best" solution. First came the reader's pride in his successful bout with the challenging problem; then came his natural desire to display the results of his cerebration; and finally his curiosity as to how his solution

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stacked up against those submitted by other solvers. It has always been the function of the editor to solicit and select proposals suitable for the particular vehicle concerned and to use his best judgment in choosing solutions for publication. This often becomes a soul-searing problem for the editor, as will be discussed later.

One of the earliest periodicals to feature a section on problems was the *Ladies' Diary*, which first appeared in London in 1704. In 1841 the *Ladies' Diary* and the *Gentleman's Diary*, which made its debut in 1741, were united and published under the title of *The Lady's and Gentleman's Diary*, which came to an end in 1871. For some unaccountable reason, the title of the *Ladies' Diary* was changed to the singular form when it combined with the *Gentleman's Diary*. The treatment of proposals and their solutions in these and in several other British publications of that era became a model for the *Mathematical Questions from the Educational Times*, which had its inception in 1863 and continued uninterruptedly until 1918. The spirit of the problem departments of the British journals was picked up by various French publications such as *L'Enseignement Mathématique* and *Mathesis* (Belgium) and also by the early American journals, notably the *Mathematical Visitor*, which was launched at Erie, Pennsylvania in 1878.

In his introductory editorial to Volume I, Number 1 of the *Mathematical Visitor*, Artemas Martin, editor and publisher, had this to say:

> In England and Europe, periodical publications have contributed much to the diffusion of mathematical learning, and some of the greatest scientific characters of those countries commenced their mathematical career by solving the problems proposed in such works.
>
> It was stated nearly three-quarters of a century ago that the learned Dr. Hutton declared that the *Ladies' Diary* had produced more mathematicians in England than all the mathematical authors of that kingdom.
>
> Similar publications have produced like results in this country. Not a few of our ablest teachers and mathematicians were first inspired with a love of mathematical science by the problems and solutions published in the mathematical department of some unpretending periodical.

A world-renowned periodical that can certainly be considered "unpretending" despite its high level of sophistication is *The American Mathematical Monthly*, which was founded originally as a show case for proposed and solved problems. An exhaustive historical and statistical treatment of the problem departments of this journal from 1894 to 1954 appeared in the *Otto Dunkel Memorial Problem Book*, published by the *Mathematical Association of America* in August 1957 in commemoration of that Journal's fiftieth anniversary. The author of that survey, Mr. Charles W. Trigg, Dean Emeritus of Los Angeles City College, and one of the better known and most prolific problemists of our day, has put together a most informative, interesting and entertaining article well worth the attention of all mathematicians, whether active problemists or not.

One of the striking characteristics of most problem departments is the high incidence of participation by eminent mathematicians as well as by the "man on the street" lover of mental gymnastics. As one browses through the pages of the *Lady's and Gentleman's Diary*, the *Mathematical Questions from the Educational Times* or the *American Mathematical Monthly*, to name a few, one is impressed to discover what an attraction problems have held for so many who have achieved great prominence in mathematics. It comes as a surprise, for example, to learn that W.G. Horner, of Horner's Method fame, solved what is now known as the Butterfly Problem in the 1815 volume of the *Gentleman's Diary*. The list of problemists who participated in the problem department of the *Educational Times* reads like a veritable Who's Who in British Mathematics from 1863 to 1918. Among the active solvers may be found the names of Cayley, Cremona, Clifford, Sylvester, Whitworth, Todhunter, Hadamard, Hardy, Salmon, Beltrami and countless others far too numerous to list.

Currently the names of numerous prominent mathematicians may be found in the problem departments of the *American Mathematical Monthly*, the *Mathematics Magazine*, the *SIAM Review*, the *Pi Mu Epsilon Journal*, *Pentagon*, *School Science and Mathematics*, the *Journal of Recreational Mathematics*, the *Fibonacci Quarterly*, the *Technology Review*, the *Two-Year College Mathematics Journal*, *Elemente der Mathematik* (Switzerland), and the *Mathematics Student Journal*. It is hard to estimate how many high schools and two-year colleges publish "newsletters" primarily for their own students. Examples are the *Indiana School Mathematics Journal* and the *Oklahoma University Mathematics Letter*. Others are listed in a booklet issued by the National Council of Teachers of Mathematics, authored by William L. Schaaf and entitled "The High School Mathematics Library".

On a less formal basis, practically every issue of Martin Gardner's Mathematical Games Department in the *Scientific American* offers several intriguing problems for the entertainment and enlightenment of its readers, with solutions revealed in the following issue. Some of these problems have been known to generate heated controversy and discussion, all to the betterment of mathematical science.

In addition to its noteworthy expository articles, the *Mathematical Gazette*, while not containing a problem department, does nevertheless publish short provocative notes that frequently set off a chain-reaction of readership discussion and development. Furthermore, the *Gazette* maintains a Problem Bureau which offers assistance in the solution of problems whose sources are known. From those standpoints, the publication is a problemist's delight.

Of course, there are many specialized journals that do not maintain problem sections but most of the well-known ones do. It is hard to imagine the dismal change in character that would descend on a journal if its problem department were suddenly to be abandoned.

Problems of a Problem Editor.

After the foregoing prelude, let us now come home to our own *Pi Mu Epsilon Journal* and dwell a bit on what goes on behind the scenes in the conduct of the Problem Department. Let us also consider what can be done to improve the department and to provide more enthusiasm and enjoyment among our problem devotees and the readers in general.

Problems in a great variety of categories have appeared in the *Pi Mu Epsilon Journal* since the time of its first appearance in April 1949. The Fraternity, which started at the University of Syracuse in 1903 as a mathematics club, achieved the status of a full-fledged chartered organization shortly after the academic year 1914-15, but it was not until 1949 that the *Pi Mu Epsilon Journal* blossomed forth. In the first issue Editor Ruth W. Stokes got the problem department off to a good start by publishing eleven proposals, five of which were her own and the other six solicited from accommodating friends. With the exception of the Fall 1957 issue, the problem section has appeared regularly in each issue and it has been only on rare occasions that the editor was faced with a shortage of suitable proposals to the point where he was compelled to raid his own files to maintain an acceptable balance and variety in the proposal department.

Considering the relatively small circulation of the *Pi Mu Epsilon Journal* compared to some of the larger periodicals, the ratio of participants in the problem department is rather high. However, it is quite likely that many of the readers solve the problems, file them away and never get around to submitting the solutions. Readers are urged to try their hand at problem composition and to offer their solutions for possible publication. One never knows when the presence of an unusual gimmick or a clever solution device might in itself warrant the publication of the solution.

This could be interpreted as a cry for help. The most difficult task for the problem editor is not the selection of solutions for publication but rather the selection of proposals of a type that elicits reader response. By soliciting contributions from a wider cross-section of the membership and from other interested readers, the editor hopes to achieve a diversity of high-quality proposals in geometry, analysis, number theory, inequalities, mathematical logic, game theory, set theory, group theory, probability, paradoxes, fallacies and cryptarithms, to name a few. In general, problems should rise above the level of unimaginative text-book exercises and should strive to give solvers an opportunity to demonstrate ingenuity and inventiveness.

One of the essential attributes of a suitable proposal is the hard-to-define quality of elegance. This characteristic is usually associated more with solutions than with proposals but is nevertheless an important element in attracting the attention of would-be solvers. A beautiful example of an elegant proposal is the following one, due to W.J. Blundon, of the Memorial University of Newfoundland:

> Let I, O, H denote respectively the incenter, the circumcenter and the orthocenter of a triangle with sides a, b, c and the inradius r. Prove that the area K of the triangle IOH is given by
>
> $$K = |(a-b)(b-c)(c-a)|/8r.$$

This problem was proposed in the January 1967 issue of *Elemente der Mathematik* and a solution was published the following January. Opinions regarding beauty are often debatable but can anyone deny that the economy of expression in the displayed result constitutes a pure and austere elegance? One would hope that a proposal of such high artistic merit would elicit a solution of comparable elegance.

Not all proposals can aspire to a high level of elegance in their mere statement. Most problems are straightforward challenges to duplicate or improve upon results already found by the proposer, especially if the method of solution or the final result is significant, novel, generalized, instructive or entertaining. Ordinarily problem editors require solutions submitted along with proposals. The purpose of this is to assist the editor in the evaluation of the suitability of the proposal, the complexity of the solution or the expected readership response. On the other hand, conjectures and unsolved problems connected with related investigations or research projects are sometimes submitted with the hope that someone may successfully arrive at a satisfactory solution. When such proposals are published, the readers are alerted to the fact that solutions have not been provided.

Since the *Pi Mu Epsilon Journal* appears only twice a year, acceptable proposals are filed away for possible use some time in the future. This may entail long delays in publication, especially if other problems in like categories have priority. Unused or unusable proposals will be returned to the proposer upon request.

After an issue of the *Journal* comes off the presses and is sent to the subscribers, solutions begin to trickle in. In due course the contributions are acknowledged, the solutions are filed away and the envelopes in which they were mailed are discarded. That is why solvers who would like to receive credit for their labors should be sure to identify their solutions with their names and addresses. Solutions to more than one problem should be sent on separate sheets and, to facilitate filing, should not be commingled with extraneous correspondence. This saves the editor the inconvenience of photocopying portions for separate filing.

With the approach of deadlines for submitting the copy to the *Journal* Editor your problem editor examines all solutions received and is often confronted with difficult decisions as to which solution to publish. He is reminded of what motivates problemists to submit solutions in the first place. Why do they not simply solve it, file it and forget it? One incentive, of course, is the altruistic desire to share with others a well-thought-out and well-expressed solution; another is to gratify one's ego in a most acceptable way by seeing his creation appreciated and published. Some problemists are so well versed in so many diverse branches of mathematics that they breeze through most of the proposals with ease and take a delight in making a marathon game of their knack for prolixity. These are individuals who generally combine quality with quantity and in many cases are legitimate candidates for inclusion in the *Guinness Book of Records*. The frequency with which their solutions are published may lead other solvers to suspect favoritism on the part of the editor, but readers are hereby assured that every effort is made to select solutions objectively on the basis of merit.

When submitting a solution, the solver should try to present it in the format adopted by the problem department. This saves the editor time and trouble in retyping it for the printer. Most problem editors are their own secretaries — unsung heroes who make a labor of love out of serving as intermediaries between proposers and solvers. Consequently when they are confronted with a difficult choice between two otherwise excellent solutions they may just tip the scales in favor of the solution that permits them to follow the path of least resistance. On the other hand, neatness and good form cannot in themselves supersede content; while they are qualities that are greatly appreciated, editors are often grieved to have to turn down a solution despite the evidence of painstaking care in presentation.

On occasion, excellent solutions with widely separated approaches are found to be too good to be lost to posterity. In those cases a diligent editor will attempt to do justice by concocting an amalgam of the solutions or, if space permits, publishing multiple solutions. Here again, the best mathematical and literary expression is considered along with the quality of the solution.

It may be hard to believe, but your problem editor occasionally receives an answer to a problem instead of a solution. Participants in this arena are not really concerned with answers; their primary interest is in the *way* the solution was found — the train of thought that led to the solution, the transparency of the solver's heuristic approach to the problem, essentially, the solver's ability to take the reader by the hand and literally lead him over the various steps of the proof. One of the tests of elegance is finding a way of doing this adroitly without insulting the reader's intelligence by spelling out procedures that should be evident to him. At the same time, the solution should avoid the sins of omission — skipping steps that are necessary for a full understanding of the solution, proof or construction, as the case may be.

To achieve this ability, the solver should be familiar with the criteria for elegance — what we call the *ABCD*'s of Elegance. They are *A* for Accuracy, *B* for Brevity, *C* for Clarity, and *D* for the Display of Insight, Ingenuity, Imagination, Originality and, where possible, Generalization. It always helps to be able to instill a dramatic sense of awe, wonder and surprise. These are the intangible qualities that elevate mathematical creations to the realm of high art, whether they be proposals, solutions, short notes, expository essays or chapters in some impressive tome.

In conclusion, it is hoped that the enunciation of these high ideals will inspire readers to make efforts to achieve them without deterring them from their most welcome participation in the Problem Department of this *Journal*.

* * *

A Christmas Story

THE ANGEL ON THE TREE

CLAYTON W. DODGE, University of Maine at Orono

This is the legend telling why an angel always appears on the top of the Christmas tree.

Many years ago on December 24, it seems that Santa Claus awoke somewhat late and feeling rather poorly because he and the elves had partied the night before. When he was dressing in his Santa Claus suit, he discovered a rip in the pants. Now, last year after his annual flight, he had told Mrs. Claus about that rip and she had assured him she would take care of it, but she had forgotten to mend it. So he had to wait for repairs.

Next he went to get the sleigh ready, grumbling because he was running more than half an hour late, and there he found that the elves had not even started to pack the toys because they were still partying. After putting an end to the party in no uncertain terms, and now a full hour late, he went to the barn to get the reindeer. Alas, someone had let them out to pasture, so they had to be rounded up before they could be hitched to the sleigh.

Just as he had everything ready and was finally starting out on his deliveries, a little angel ran up to him and said, "Here is your Christmas tree, Santa. What shall I do with it?"

Editor's comment.

What is the mathematical relevance of this story? Professor Dodge does not say, but perhaps he is using the festive season as an excuse to remind us of the age-old problem about how many angels can dance on the point of a pin, a problem that medieval theologians were unable to solve, but which modern computers may perhaps tackle with success.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 205.

For the problems given below, solutions, if available, will appear in EUREKA *Vol. 3, No. 3, to be published around Mar. 15, 1977. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Mar. 1, 1977.*

191. *Proposed by R. Robinson Rowe, Sacramento, California.*

In the November 1976 *Scientific American*, p. 134, Martin Gardner recited an algorithm equivalent to

$$N_n N_{n-2} = N_{n-1} + 1$$

and demonstrated that, for any N_0 and N_1, the algorithm led to $N_5 = N_0$. Considering the more general relation

$$N_n N_{n-2} = N_{n-1} + e$$

(a) find sets of square integers N_0 and N_1 for which $N_5 = N_0$ when $e = 2$;

(b) find the general relation between N_0 and N_1 for any value of e.

192. *Proposed by Ross Honsberger, University of Waterloo.*

Let D, E, F denote the feet of the altitudes of ΔABC, and let (X_1, X_2), (Y_1, Y_2), (Z_1, Z_2) denote the feet of perpendiculars from D, E, F, respectively, upon the other two sides of the triangle. Prove that the six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 lie on a circle.

193. *Proposed by L.F. Meyers, The Ohio State University.*

A river with a steady current flows into a still-water lake at Q. A swimmer swims down the river from P to Q, and then across the lake to R, in a total of 3 hours. If the swimmer had gone from R to Q to P, the trip would have taken 6 hours. If there had been a current in the lake equal to that in the river, then the downstream trip PQR would have taken $2\frac{1}{4}$ hours. How long would the upstream trip RQP have taken under the same circumstances?

(This is a reconstruction of a problem that I could not solve while participating in a high school contest.)

194. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

A sequence $\{a_n\}$ is defined by

$$a_1 = X, \quad a_n = X^{a_{n-1}}, \quad n = 2, 3, \ldots$$

where $X = \left(\frac{4}{3}\right)^{3/4}$. Discuss the convergence of the sequence and find the value of the limit if any.

195. *Proposed by John Karam, Coop. Student, University of Waterloo.*

The following two problems are given together, since they both appear to be related to the celebrated Birthday Problem, which says that if 23 persons are in a room the odds are better than 50% that two persons in the room have the same birthday.

(a) How many persons would have to be in a room for the odds to be better than 50% that three persons in the room have the same birthday?

(b) In the Québec-based lottery *Loto Perfecta*, each entrant picks six distinct numbers from 1 to 36. If, at the draw, his six numbers come out in some order (*dans le désordre*) he wins a sum of money; if his numbers come out in order (*dans l'ordre*), he wins a larger sum of money. How many entries would there have to be for the odds to be better than 50% that two persons have picked the same numbers (i) *dans le désordre*, (ii) *dans l'ordre*?

196. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Montrer que, si $|a_i| < 2$ pour $1 \le i \le n$, alors l'équation

$$1 + a_1z + \ldots + a_nz^n = 0$$

n'a pas de racine à l'intérieur du disque $|z| = \frac{1}{3}$.

La réciproque est-elle vraie?

197. *Proposed by Charles W. Trigg, San Diego, California.*

In the octonary system, find a square number that has the form *aaabaaa*.

198. *Proposed by Gali Salvatore, Ottawa, Ont.*

Without using an acre of paper, find the coefficient of x^8 in the expansion of the polynomial

$$P = (1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^6.$$

199. *Proposed by H.G. Dworschak, Algonquin College, Ottawa, Ont.*

If a quadrilateral is circumscribed about a circle, prove that its diagonals and the two chords joining the points of contact of opposite sides are all concurrent.

200. *Proposed by the editor.*

(a) Prove that there exist triangles which cannot be dissected into two or three isosceles triangles.

(b) Prove or disprove that, for $n \ge 4$, every triangle can be dissected into n isosceles triangles.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

127. [1976: 41, 124, 140] *Proposed by Viktors Linis, University of Ottawa.*

A, B, C, D are four distinct points on a line. Construct a square by drawing two pairs of parallel lines through the four points.

IV. *Comment by Clayton W. Dodge, University of Maine at Orono.*

The following construction problem is an easy extension of my comment III [1976: 140]:

Given four points in the plane, construct a rectangle similar to a given rectangle so that the four points lie on the four sides of the rectangle (or the sides extended).

Note that it is possible for the points to coincide by twos.

Instead of drawing DC' the same length as AB, draw DC' of length AB times the ratio of the sides of the rectangle. The rest of the construction is unaltered.

133. [1976: 67, 144] *Proposed by Kenneth S. Williams, Carleton University.*

Let f be the operation which takes a positive integer n to $\frac{1}{2}n$ (if n even) and to $3n+1$ (if n odd). Prove or disprove that any positive integer can be reduced to 1 by successively applying f to it.

Example: $13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

(This problem was shown to me by one of my students.)

IV. *Comment by John L. Davison, Laurentian University, Sudbury, Ont.*

I have also worked on what C.W. Trigg calls the Collatz Algorithm, although I had known it referred to as the Syracuse Problem. My main results can be summarized as follows (I use $\frac{3n+1}{2}$ instead of $3n+1$ for the definition of f, as in (1) of the editor's comments [1976: 149]):

1. Let $V_a(m)$ denote the exponent of the highest power of a which divides m. Then, if $n \geq 3$ and odd, we have

$$n < f(n) < \ldots < f^k(n) \quad \text{and} \quad f^{k+1}(n) < f^k(n)$$

where $k = V_2(n+1)$.

2. We can thus compress the iteration as follows:

$$7 \overset{3}{\to} 26 \overset{1}{\to} 13 \overset{1}{\to} 20 \overset{2}{\to} 5 \overset{1}{\to} 8 \overset{3}{\to} 1.$$

The transitions T: $7 \to 13$, $13 \to 5$, $5 \to 1$ are called *circuits*.

3. We show that $T(n) = n$ is possible if and only if there exists a nontrivial solution to the diophantine equation

$$(2^{k+\ell} - 3^k)h = 2^\ell - 1.$$

134. [1976: 68, 151, 173] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

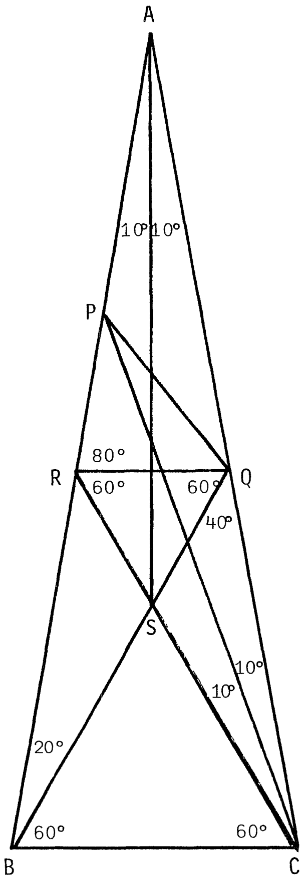
(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

V. *Comment by the proposer.*

In solution III [1976: 173], the proof is correct but the accompanying Figure 1 is incorrectly drawn. The lines QR, PC, and AS do *not* pass through the same point.

VI. *Solution by Don Baker, Presidio Junior High School, San Francisco, California.*

Draw QR $\parallel$ BC and draw RC, AS as shown in the adjoining figure. At this point, taking symmetry into account, the angles shown in the figure are known. Since $\triangle ACS \cong \triangle ACP$ (*ASA*), we have AS = CP; and AR = RC since $\triangle ARC$ is isosceles. Now $\triangle ARS \cong \triangle CRP$ (*SAS*), and so RS = RP. But $\triangle RSQ$ is equilateral, so RP = RQ. Hence $\angle RQP = 50°$ and $\angle PQA = 30°$.



Also solved geometrically by ANDREJS DUNKELS, *University of Luleå, Sweden.*

Editor's comment.

The figure submitted by the solver in solution III was correct in all respects, but the editor, who could not leave well enough alone, redrew it and ended up with the slight inaccuracy noted above by Williams. The figure accompanying solution VI on this page shows the correct relationship between segments QR, PC, and AS.

Some readers may consider that solution VI is more elementary than solution III, since it does not use theorems for angle bisectors.

In the last three months, four elementary geometric solutions to this problem have been received. Four months ago [1976: 152-153], the editor looked into his clouded crystal ball and predicted that a geometric solution would be hard to find.

135. [1976: 68, 153] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

How many 3×5 rectangular pieces of cardboard can be cut from a 17×22 rectangular piece of cardboard so that the amount of waste is a minimum?

IV. *Comment by Clayton W. Dodge, University of Maine at Orono.*

In my earlier comment [1976: 154] I stated that it appeared that 24 cards could not be cut using a paper cutter. A (tedious) proof follows. In cutting 3×5 cards from $m \times n$ sheets, the following observations become apparent:

1. A strip $1 \times n$ and a strip $2 \times n$ are all waste.

2. A strip $3 \times n$ has waste 0, 3, 6, 9, or 12 square inches according as n has remainder 0, 1, 2, 3, or 4 when divided by 5.

3. A strip $4 \times n$ produces no more cards than a $3 \times n$ strip. Hence its waste is n plus that for a $3 \times n$ strip.

4. A strip $5 \times n$ has waste 0, 5, or 10 sq in according as n has remainder 0, 1, or 2 when divided by 3.

5. A strip $6 \times n$ has waste 0, 6, 12, 3, or 9 according as n has remainder 0, 1, 2, 3, or 4 when divided by 5.

6. A strip $7 \times n$ produces no more cards than a $6 \times n$ strip. Hence its waste is n plus that for a $6 \times n$ strip.

7. A strip $8 \times n$ has the waste of a $3 \times n$ plus that of a $5 \times n$ strip, and it can always be so cut with minimum waste.

8. A strip $9 \times n$ has waste 0, 9, 18, 12, or 21 sq in according as n has remainder 0, 1, 2, 3, or 4 when divided by 5.

9. A strip $10 \times n$ has waste 0, 10, or 5 according as n has remainder 0, 1, or 2

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44
3	3	6	9	12	0	3	6	9	12	0	3	6	9	12	0	3	6	9	12	0	3	6
4	4	8	12	16	5	9	13	17	21	10	14	18	22	26	15	19	23	27	31	20	24	28
5	5	10	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10	0	5	10	0	5
6	6	12	3	9	0	6	12	3	9	0	6	12	3	9	0	6	12	3	9	0	6	12
7	7	14	6	13	5	12	19	11	18	10	17	24	16	23	15	22	29	21	28	20	27	34
8	8	16	9	17	10	3	11	19	12	5	13	6	14	22	0	8	16	9	17	10	3	11
9	9	18	12	21	0	9	18	12	21	0	9	18	12	21	0	9	18	12	21	0	9	18
10	10	20	0	10	5	0	10	5	0	10	5	0	10	5	0	10	5	0	10	5	0	10
11	11	22	3	14	10	6	17	13	9	5	16	12	8	19	0	11	22	3	14	10	6	17
12	12	24	6	18	0	12	24	6	18	0	12	24	6	18	0	12	24	6	18	0	12	24
13	13	26	9	22	5	3	16	14	12	10	8	6	19	17	0	13	11	9	22	5	3	16
14	14	28	12	26	10	9	23	22	21	5	19	18	17	31	0	14	28	12	26	10	9	23
15	15	30	0	15	0	0	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	16	32	3	19	5	6	22	8	9	10	11	12	13	14	0	16	17	3	19	5	6	22
17	17	34	6	23	10	12	29	16	18	5	22	24	11	28	0	17	34	6	23	10	12	29

Figure 1. Waste in square inches for an $m \times n$ sheet

when divided by 3, except a 10×2 card has waste 20 sq in.

10. A strip $12 \times n$ has waste 0, 12, 24, 6, or 18 according as n has remainder 0, 1, 2, 3, or 4 when divided by 5.

11. A strip $15 \times n$ has 0 waste when $n > 7$.

12. A strip $m \times n$, not covered by the preceding observations, must be cut by the paper knife into two strips $k \times n$ and $(m-k) \times n$, or $m \times j$ and $m \times (n-j)$. The waste of the $m \times n$ strip is then the sum of the wastes of the two sub-strips. If one makes out a table of wastes row by row, then this sum can be minimized readily by looking at row m and at column n and forming all possible sums.

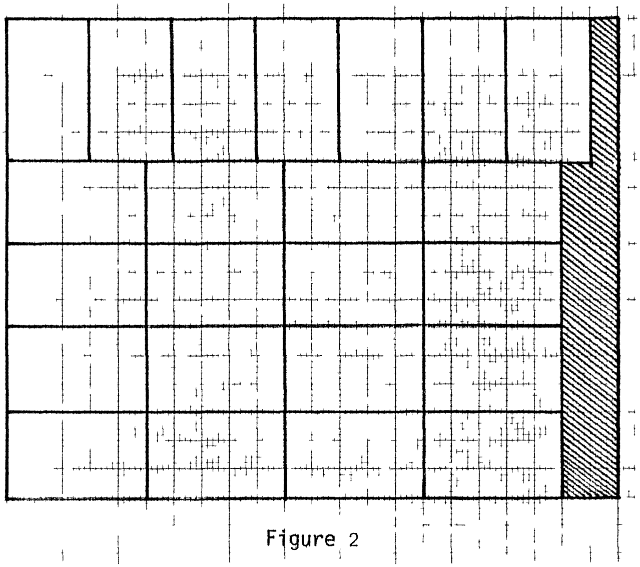


Figure 2

A table of wastes, as mentioned in Observation 12, appears in Figure 1. Its final entry shows that the minimum waste for a 17×22 sheet is 29 sq in, proving that only 23 cards can be cut with a paper knife.

By considering the various possible partitions of the 17×22 sheet, one observes that the 23 cards can all be cut from a 17×21 sheet, as shown in Figure 2.

Editor's comment.

Well, at least, now we *know*. And knowledge is power. We thank Professor Dodge for having laboured mightily to produce this minute but measurable addition to the accumulated mathematical knowledge of the ages.

Next question? How many cookies can be cut from a 23 by 38 layer of dough with a heart-shaped cookie cutter? Please! Save it for the next editor.

145. [1976: 94, 181] *Proposed by Walter Bluger, Department of National Health and Welfare, Ottawa, Ont.*

A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 collinear points are equal.

Construct a magic pentagram with the ten smallest possible positive primes.

Editor's comment.

Two readers have sent in comments on Langman's minimal solution with line sum 72. Each of them proved convincingly that any magic prime pentagram with line sum 72 must contain 1 as a "prime" entry. On this they are agreed. Then one went on to prove that the entries must be

1, 3, 5, 7, 11, 19, 23, 31, 37, 43

which can then be arranged as in Figure 1; and the other proved that the entries must be

1, 5, 7, 11, 13, 17, 19, 29, 37, 41

which can be arranged as in Figure 2.

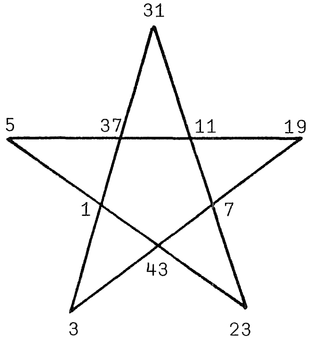


Figure 1

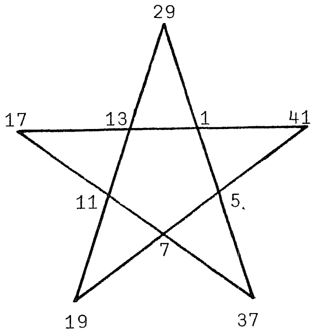


Figure 2

The editor will, upon request, send each of them the other's solution, with the suggestion that they meet at dawn, with primed pistols at 72 paces.

Meanwhile, the problem remains open: to find a magic prime pentagram with minimal line sum (1 not being considered a prime).

154. [1976: 110, 159, 197] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

1	2	3	4	5	6	7	8	9	10	11	12
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	3	5	9	7	15	11	27	25	21	13	45

We observe that $\ell = 17 = p_7$.

II. *Comment by the proposer.*

The proof in [1976: 197] is not correct as it stands, and so the problem remains unsolved. In the argument, the assumption that $\ell - 2$ is composite does not imply that $\ell - 2$ is the *largest* composite in M, nor that $\ell - 2 \le p_n - 2$, since M contains odd numbers other than those from 1 to p_n.

Editor's comment.

No solution having been received when publication time arrived, the editor, who should have stuck to his editing, made a quick attempt at a solution, and ended up with egg on his face.

This "full theorem" is hereby demoted to the rank of a conjecture, and solutions to it are earnestly sought.

161. [1976: 135] *Proposed by Viktors Linis, University of Ottawa.*

Evaluate

$$\int_0^{\pi/2} \frac{\sin^{25} t}{\cos^{25} t + \sin^{25} t}\, dt\,.$$

Solution de Hippolyte Charles, Waterloo, Québec.

Si $I_1 = \int_0^a f(t)\,dt$ et $I_2 = \int_0^a f(a-t)\,dt$, la relation suivante est presque triviale:

$$I_1 = I_2 = \tfrac{1}{2}(I_1 + I_2). \tag{1}$$

Or, pour $a = \frac{\pi}{2}$ et $f(t) = \dfrac{\sin^{25} t}{\cos^{25} t + \sin^{25} t}$, le membre droit de (1) devient

$$\tfrac{1}{2}\int_0^{\pi/2} dt = \frac{\pi}{4}\,.$$

Also solved by F.G.B. MASKELL, *Algonquin College, Ottawa;* B.C. RENNIE, *James Cook University of North Queensland, Australia;* R. ROBINSON ROWE, *Sacramento, California; and* KENNETH S. WILLIAMS, *Carleton University, Ottawa.*

162. [1976: 135] *Proposed by Viktors Linis, University of Ottawa.*

If $x_0 = 5$ and $x_{n+1} = x_n + \dfrac{1}{x_n}$, show that

$$45 < x_{1000} < 45.1\,.$$

This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

Adapted from the solutions submitted independently by John L. Davison, Laurentian University, Sudbury, Ont.; and Dan Eustice, The Ohio State University, Columbus, Ohio.

Since $x_{n+1} = x_n + \dfrac{1}{x_n}$, it follows that

$$x_{n+1}^2 = x_n^2 + 2 + \frac{1}{x_n^2}, \tag{1}$$

and so $x_{n+1}^2 > x_n^2 + 2$. Thus $x_1^2 > 27$, $x_2^2 > 29$, and an easy induction gives

$$x_n^2 > 2n + 25, \quad n = 1,2,\ldots \tag{2}$$

Setting $n = 1000$ in (2) gives $x_{1000} > 45$.

On the other hand, for $n \geq 1$ we get from (1) and (2)

$$x_{n+1}^2 < x_n^2 + 2 + \frac{1}{2n + 25}. \tag{3}$$

If we sum the n relations obtained by setting $n = 0$ in (1) and replacing n successively by $1,2,\ldots,n-1$ in (3), we get

$$x_n^2 < 2n + 25 + \sum_{k=0}^{n-1} \frac{1}{2k + 25}.$$

Since

$$\sum_{k=0}^{n-1} \frac{1}{2k + 25} < \int_{-1}^{n-1} \frac{dx}{2x + 25} = \tfrac{1}{2} \ln\left(\frac{2n + 23}{23}\right), \tag{4}$$

we obtain finally

$$x_n < \sqrt{2n + 25 + \tfrac{1}{2} \ln\left(\frac{2n + 23}{23}\right)}, \tag{5}$$

and setting $n = 1000$ gives $x_{1000} < 45.024865 < 45.1$, as required.

Also solved by L.F. MEYERS, *The Ohio State University, Columbus, Ohio;* B.C. RENNIE, *James Cook University of North Queensland, Australia; and* R. ROBINSON ROWE, *Sacramento, California.*

Editor's comments.

1. Meyers proved the following slight generalization, valid for $x_0 = a \geq 1$:

$$\sqrt{2n + a^2} < x_n < a + \sqrt{2n - 1 + a^2} - \sqrt{a^2 - 1}, \quad n \geq 1.$$

For $a = 5$ and $n = 1000$, this gives $45 < x_{1000} < 45.08991$. He also decided to break in his new SR-56 programmable calculator by calculating the table given below in which the first line gives the rank n of the term x_n which first exceeds the integer $[x_n]$ given in the second line.

n	6	12	20	28	38	48	60	72	85	100	300	600
$[x_n]$	6	7	8	9	10	11	12	13	14	15	25	35

n	999	1499	2099	4499	5499	9099	11999	20999	86098
$[x_n]$	45	55	65	95	105	135	155	205	415

2. Rennie's proof simply states that, by induction,

$$\sqrt{2n + 25} \leq x_n < \sqrt{2n + 25 + \tfrac{1}{2} \ln (n + 12)}.$$

It is not immediately clear to the editor whether the inductive proof of the second inequality is a straightforward one. (Being dense is not necessarily a fault in an editor, for what is clear to him will be clear to all.) Also, Rennie does not say by what (presumably obvious to him) line of reasoning he obtained the right member. This shows that the thought processes of those chaps in the antipodes (Rennie is from Australia) are much less sluggish than our own here in North America. Stands to reason: they're walking around upside down all the time, and more blood gets to the brain.

In any case, for $n = 1000$, Rennie's upper bound is $x_{1000} < 45.03843$, which is better than Meyers' but not as good as the one in our featured solution.

3. The upper bound in (5) can easily be improved by calculating the first few terms of the sum in (4). For example, if we calculate the terms for $k = 0,1,2$, we get

$$\sum_{k=0}^{n-1} \frac{1}{2k+25} < \frac{1}{25} + \frac{1}{27} + \frac{1}{29} + \int_2^{n-1} \frac{dx}{2x+25},$$

which leads to the following improved version of (5):

$$x_n < \sqrt{2n + 25 + \frac{1}{25} + \frac{1}{27} + \frac{1}{29} + \tfrac{1}{2}\ln\left(\frac{2n+23}{29}\right)}\ .$$

For $n = 1000$, this becomes $x_{1000} < 45.024816$. Since (5) gives $x_{1000} < 45.024865$, the improvement hardly seems worth the trouble, for the exact value, to the indicated accuracy, is $x_{1000} = 45.024524\ldots$. This was calculated by Eustice on a programmable hand calculator.

Incidentally, (2) and (5) give

$$414.992771 < x_{86097} < 414.998146,$$
$$414.995180 < x_{86098} < 415.000555,$$

which accords with the information given in Meyers' table, but also shows that lower bound (2) is not nearly as sharp as upper bound (5).

163. [1976: 135] *Proposed by Charles Stimler, Douglaston, N.Y.*

Find the value of the following infinite continued fraction:

$$\cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\ddots}}}}}$$

I. *Solution by John L. Davison, Laurentian University, Sudbury, Ont.*

If $\frac{p_n}{q_n}$ are the convergents to the continued fraction

$$\frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots ,$$

it is well known (see [1], for example) that

$$p_0 = 0,\ p_1 = b_1; \quad p_n = a_n p_{n-1} + b_n p_{n-2},\ n \geq 2,$$

$$q_0 = 1,\ q_1 = a_1; \quad q_n = a_n q_{n-1} + b_n q_{n-2},\ n \geq 2.$$

In this problem, we have $a_n = n$, $b_n = n+1$, and so

$$p_n = np_{n-1} + (n+1)p_{n-2}, \quad q_n = nq_{n-1} + (n+1)q_{n-2}.$$

An easy induction now shows that

$$q_n - p_n = (-1)^n, \quad n = 0,1,2\ldots$$

whence

$$\frac{p_n}{q_n} = 1 - \frac{(-1)^n}{q_n} \to 1,$$

since clearly $q_n \to \infty$. The value of the given continued fraction is 1.

II. *Comment by L.F. Meyers, The Ohio State University, Columbus, Ohio.*

An alternative notation for continued fractions, namely

$$b_0 + a_1/(b_1 + a_2/(b_2 + \ldots ,$$

contains one of the rare instances of unpaired parentheses in mathematics. For more on the sex life of brackets, see [2].

Also solved by L.F. MEYERS, *The Ohio State University, Columbus, Ohio (solution as well);* B.C. RENNIE, *James Cook University of North Queensland, Australia;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.;* R. ROBINSON ROWE, *Sacramento, California; and the proposer.*

REFERENCES

1. G. Chrystal, *Algebra*, Chelsea, 1952, Vol. II, p. 492.

2. C.E. Linderholm, *Mathematics made difficult*, London, Wolfe Publishing, 1971, pp. 156-161.

164. [1976: 135] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

In the five-digit decimal numeral $ABCDE$ $(A \neq 0)$, different letters do not necessarily represent different digits. If this numeral is the fourth power of an integer, and if $A + C + E = B + D$, find the digit C.

(This problem was originally written for the Fall 1975 Contest of the New York City Senior Interscholastic Mathematics League.)

I. *Essence of the solutions submitted independently by André Bourbeau, École Secondaire Garneau, Vanier, Ont.; R. Robinson Rowe, Sacramento, California; Charles W. Trigg, San Diego, California; Kenneth S. Williams, Carleton University, Ottawa, Ont.; and the proposer.*

The condition $A+C+E = B+D$ shows that the given number is a multiple of 11, and the only multiple of 11 whose fourth power has five digits is $11^4 = 14641$, so $C=6$.

II. *Essence of the solutions submitted independently by Clayton W. Dodge, University of Maine at Orono; G.D. Kaye, Department of National Defence, Ottawa, Ont.; and B.C. Rennie, James Cook University of North Queensland, Australia.*

The only integers having fourth powers of five digits are 10 through 17. Of their fourth powers only $11^4 = 14641$ is of the proper form. Hence $C=6$.

III. *Comment by Charles W. Trigg, San Diego, California.*

Having found $11^4 = 14641$, one could then digress into a discussion of Pascal's triangle and binomial coefficients, but that would be beating a well-worn rug.

Also solved by F.G.B. MASKELL, *Algonquin College, Ottawa, Ont.;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.; and* KENNETH M. WILKE, *Topeka, Kansas.*

Editor's comment.

Now why didn't *everybody* solve this easy problem? The editor put it in in the hope that many more readers would thereby be induced to become solvers, without the fear that their maiden efforts would be buried anonymously in a common grave bearing the epitaph "Three incorrect solutions were received."

The editor will try again, perhaps with questions like "Who wrote Gray's Elegy?" or "Who's buried in Grant's Tomb?"

165. [1976: 135] *Proposed by Dan Eustice, The Ohio State University, Columbus, Ohio.*

Prove that, for each choice of n points in the plane (at least two distinct), there exists a point on the unit circle such that the product of the distances from the point to the chosen points is greater than one.

I. *Solution by the proposer.*

Let the points be represented as complex numbers $z_1, z_2, \ldots, z_n$ and form

$$P(z) = (z-z_1)\ldots(z-z_n) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.$$

If we set $P^*(z) = z^nP(1/z)$, then $P^*(0) = 1$ and P^* is not constant since $P(z) \neq z^n$ when there are at least two distinct points. By the maximum modulus theorem, $P^*(z)$

has modulus greater than 1 at some point of $|z| = 1$. Since, on $|z| = 1$, $|P(z)| = |P^*(z)|$, the theorem follows.

II. *Solution by B.C. Rennie, James Cook University of North Queensland, Australia.*

The plane potential function due to a unit point source is the logarithm of the radius. Now using circular symmetry, the mean on the unit circle of the potential due to a unit point source is the potential at any one point of the unit circle due to a distributed unit source on a concentric circle; so that it is zero from a source inside and positive from a source outside. Now consider the sum of the logarithms of the radial distances to the n points; its mean on the unit circle is non-negative, also non-constant, so that somewhere it must be strictly positive.

III. *Comment by L.F. Meyers, The Ohio State University, Columbus, Ohio.*

This problem is somewhat more general than, but can easily be reduced to, the following which appears, with a solution, in Pólya-Szegö [1]:

We assume that the given points $P_1, P_2, \ldots, P_n$ *are all inside a circle of radius* R *and* P *is moving along this circle. Then*

$$\sqrt[n]{\overline{PP}_1 \cdot \overline{PP}_2 \cdots \overline{PP}_n}$$

(the geometric mean of the n *distances* $\overline{PP}_\nu$*) attains a maximum* $> R$ *and a minimum* $< R$ *unless all the* P_ν*'s coincide with the center of the circle.*

REFERENCE

1. G. Pólya-G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1972, Vol. I, pp. 132, 327, Problem 139.

166. [1976: 136] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Find a simple proof for the following problem, which is not new:

Prove that for all real x and positive integers k

$$\sum_{i=0}^{k-1} \left[x + \frac{i}{k}\right] = [kx],$$

where brackets denote the greatest integer function.

I. *Solution by B.C. Rennie, James Cook University of North Queensland, Australia.*

Let $kx = km + n + u$, where m and n are integers, $0 \le n < k$ and $0 \le u < 1$. Since $\left[\frac{y}{k}\right] = \left[\frac{[y]}{k}\right]$, the integral part of $x + \frac{i}{k}$ is $m + \left[\frac{n+i}{k}\right]$, and

$$\sum_{i=0}^{k-1}\left[\frac{n+i}{k}\right] = \sum_{i=k-n}^{k-1} 1 = n$$

implies

$$\sum_{i=0}^{k-1}\left[x+\frac{i}{k}\right] = km+n = [kx].$$

II. *Comment by Gali Salvatore, Ottawa, Ont.*

The simplest proof I know is that given in Pólya-Szegö [5], which I paraphrase as follows:

It is sufficient to consider the case $0 \le x < 1$, since $[y+n] = [y]+n$ shows that changing x by an integer produces the same change on both sides of the given equation.

If m is a positive integer such that

$$x + \frac{m-1}{k} < 1 \le x + \frac{m}{k},$$

then

$$k-m \le kx < k-m+1$$

and

$$\sum_{i=0}^{k-1}\left[x+\frac{i}{k}\right] = \sum_{i=m}^{k-1}\left[x+\frac{i}{k}\right] = k-m = [kx].$$

Also solved by W.D. BURGESS, *University of Ottawa;* G.D. KAYE, *Department of National Defence, Ottawa;* R. ROBINSON ROWE, *Sacramento, California;* KENNETH M. WILKE, *Topeka, Kansas;* KENNETH S. WILLIAMS, *Carleton University, Ottawa; and the proposer.*

Editor's comment.

This problem is certainly not new, since Pólya-Szegö [5] credit it to Charles Hermite [1].

One reader writes: "I am curious to know what is the more complicated proof whose existence is implied by the wording of the question." I don't know whether the proposer had any particular "complicated" proof in mind when he formulated his problem, although reference [2] (supplied by him) contains the problem with a hint that would seem to lead to a solution that is slightly more involved than the solution the proposer himself supplied. The problem appears in many books on number theory (for example [3], [4]), usually as an exercise without solution, so it will be helpful to readers to have one or two simple solutions available.

There is probably little essential difference among the various known proofs of this theorem, although some may be expressed more felicitously than others. The two given above are the most esthetically satisfying that the editor has come across.

If someone can devise a more elegant way to fry an egg, the world, which hungers after beauty, is thereby enriched.

REFERENCES

1. Charles Hermite, *Acta Math.* 5, 1884, p. 315.

2. Calvin T. Long, *Elementary Introduction to Number Theory*, D.C. Heath and Co., 1967, p. 122.

3. Trygve Nagell, *Introduction to Number Theory*, Chelsea, 1964, p. 41.

4. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers*, Third Edition, Wiley, 1972, p. 83.

5. G. Pólya-G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1976, Vol. II, pp. 112, 303.

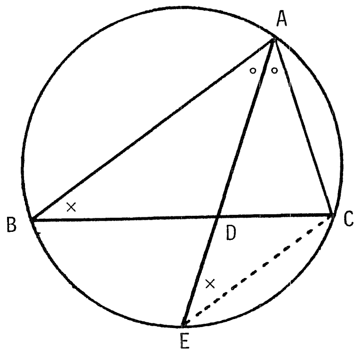
168. [1976: 136] *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y.*

If a, b, c are the sides of a triangle ABC, t_a, t_b, t_c are the angle bisectors, and T_a, T_b, T_c are the angle bisectors extended until they are chords of the circle circumscribing the triangle ABC, prove that

$$abc = \sqrt{T_aT_bT_ct_at_bt_c}\ .$$

Essence of the solutions submitted independently by Leon Bankoff, Los Angeles, California; André Bourbeau, École Secondaire Garneau, Vanier, Ont.; Clayton W. Dodge, University of Maine at Orono; and Kenneth M. Wilke, Topeka, Kansas.

If the bisector of $\angle$A meets the opposite side at D and the circumcircle at E (see figure), then Δ's ABD and AEC are similar, and so $bc = T_at_a$. Similarly $ca = T_bt_b$, $ab = T_ct_c$, and the stated result follows immediately.



Also solved by G.D. KAYE, *Department of National Defence, Ottawa;* VIKTORS LINIS, *University of Ottawa;* B.C. RENNIE, *James Cook University of North Queensland, Australia; and the proposer.*

169. [1976: 136] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

Prove that

$$\sqrt{5} + \sqrt{22 + 2\sqrt{5}} = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}.$$

(This identity is due to Daniel Shanks, Naval Ship Research and Development Center, Bethesda, Maryland.)

Essence of the solutions submitted independently by Clayton W. Dodge, University of Maine at Orono; Viktors Linis, University of Ottawa; F.G.B. Maskell, Algonquin College, Ottawa; and R. Robinson Rowe, Sacramento, California.

The given identity follows immediately upon adding corresponding members of the identities

$$\sqrt{5}+\sqrt{11-2\sqrt{29}} = \sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}}\;,$$

$$\sqrt{22+2\sqrt{5}} = \sqrt{11+2\sqrt{29}}+\sqrt{11-2\sqrt{29}}\;,$$

each of which can be verified by inspection by squaring both sides.

Also solved by ANDRÉ BOURBEAU, *École Secondaire Garneau, Vanier, Ont.; the* COMA *Council at its meeting of October 19, 1976;* STEVEN R. CONRAD, *Benjamin N. Cardozo H.S., Bayside, N.Y.;* G.D. KAYE, *Department of National Defence, Ottawa;* B.C. RENNIE, *James Cook University of North Queensland, Australia; and the proposer.*

Editor's comment.

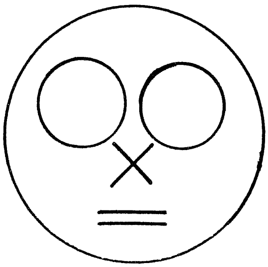
Conrad proved in a straightforward manner that, for suitable x, y, and z,

$$\sqrt{z}+\sqrt{x+2\sqrt{y}} = \sqrt{\frac{x}{2}+2\sqrt{\frac{x^2-4y}{16}}} + \sqrt{z+\frac{x}{2}-2\sqrt{\frac{x^2-4y}{16}}+2\sqrt{\frac{zx}{2}-2z\sqrt{\frac{x^2-4y}{16}}}}\;,$$

which reduces to our own identity for $x = 22$, $y = 5$, $z = 5$, and from which more monstrous identities can be manufactured at will.

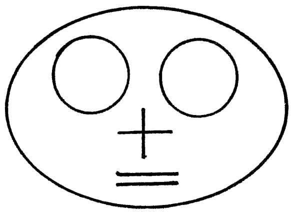
* * *

Variation No. 2



$0 \times 0 = 0$
or
Where is that smell coming from?

Variation No. 3



Yes, it hurts on both sides, Dr. Bankoff.[1]

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[1] Dr. Leon Bankoff, D.D.S., is by profession a dentist of, he says, Russian extraction.

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