Crux Mathematicorum

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Crux Mathematicorum

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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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MURRAY KLAMKIN WINS M.A.A. AWARD

Murray S. Klamkin, familiar to *Crux* readers as the former editor of the Olympiad Corner and as a prolific proposer and solver of *Crux* problems, has been presented the Mathematical Association of America's Award for Distinguished Service to Mathematics. The presentation took place on January 8th in Atlanta, Georgia at the annual meeting of the M.A.A.

To quote from the M.A.A.: "The award is being presented to Professor Klamkin for his outstanding service to mathematics which has significantly influenced the field of mathematics and mathematics education on a national scale. A life member of the M.A.A., Professor Klamkin is recognized internationally for his significant contributions in the area of problem solving and his tireless commitment to the U.S., Canadian and International Mathematical Olympiads."

Apart from his years of writing in *Crux*, Murray has been (often simultaneously) problem editor of the *SIAM Review*, *The American Mathematical Monthly*, *Mathematical Intelligencer*, *Pi Mu Epsilon Journal*, and as associate problem editor of *Mathematics Magazine*. More on Murray's many credentials for receiving this award, and some information on his background (including his birthday, March 5) can be obtained from the appreciative article by G.L. Alexanderson in the January *Monthly*.

The editor joins all other readers of *Crux* in a robust "three cheers" and hearty congratulations to Murray for this well–deserved honour.

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THE OLYMPIAD CORNER
No. 92
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month's column with two contest problem sets, from the 1987 Austrian Mathematics Olympiad and the 1987 Austrian–Polish Mathematics Competition. We thank Walther Janous of Innsbruck, Austria for forwarding them to us.

18th Austrian Math–Olympiad

2nd Round: May 5, 1987 (4 hours)

1. Show that $\sin x + \cos x \leq 0$ implies that $\sin^{1987} x + \cos^{1987} x \leq 0$, as well.

2. The solutions x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$ (a real, $a \neq 0$) also satisfy

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8.$$

Determine x_1, x_2, x_3.

3. Determine all sequences $x_0, x_1, \ldots$ of real numbers satisfying $0 < x_0 \leq 1$ and
$$0 < x_{n+1} \leq 2 - \frac{1}{x_n}, \; n \geq 0.$$

4. A triangle $A_1A_2A_3$ and its circumcircle κ are given in space. With κ as base a cylinder (of rotation) Z is constructed. Let P be a point on the boundary of Z not lying on κ. A point P^* is the point of intersection of the planes V_i through A_i and orthogonal to PA_i ($i = 1,2,3$). Determine the set of all "image–points" P^* as P varies over the boundary of Z minus κ.

18th Austrian Math-Olympiad

Final Round

1st Day: June 2, 1987 (4-1/2 hours)

1. The sides a, b and the angle bisector of the included angle γ of a triangle are given. Determine necessary and sufficient conditions for such triangles to be constructible, and show (with proof) how to reconstruct the triangle.

2. Determine the number of all sequences $(x_1,\ldots,x_n)$, with $x_i \in \{a,b,c\}$ for $1 \leq i \leq n$, that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $1 \leq i \leq n-1$.

3. Let $x_1,\ldots,x_n$ be positive real numbers. Prove that

$$\sum_{k=1}^{n} x_k + \sqrt{\sum_{k=1}^{n} x_k^2} \leq \frac{n+\sqrt{n}}{n^2}\left[\sum_{k=1}^{n}\frac{1}{x_k}\right]\left[\sum_{k=1}^{n} x_k^2\right].$$

2nd Day: June 3, 1987 (4-1/2 hours)

4. Determine all triples (x,y,z) of natural numbers satisfying
$$2xz = y^2 \quad \text{and} \quad x + z = 1987.$$

5. Let P be a point in the interior of a convex n–gon $A_1A_2\ldots A_n$, $n \geq 3$. Show that among the angles $\beta_{ij} = \angle A_iPA_j$, $1 \leq i < j \leq n$, there are at least

$n-1$ angles satisfying

$$90^\circ \leq \beta_{ij} \leq 180^\circ.$$

6. Determine all polynomials

$$P_n(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where the a_i are integers and where $P_n(x)$ has as its n zeros precisely the numbers $a_1,\ldots,a_n$ (counted in their respective multiplicities).

*

10th Austrian-Polish Mathematics Competition
Individual Competition

1st day: June 29, 1987 (4-1/2 hours)

1. Let P be a point in the interior of a sphere. Through P three pairwise orthogonal lines are drawn. Each line determines a chord on the great circle determined by the points of intersection of the line and the sphere. Show that the sum of the squares of the lengths of the three chords is independent of the directions of these lines.

2. Let n be the square of a natural number and suppose that if p is a prime dividing n then there are an even number of digits in the decimal representation of p. Furthermore let $P(x) = x^n - 1987x$. Show that if x and y are rational numbers with $P(x) = P(y)$, then $x = y$.

3. Let $\mathbb{R}$ be the set of real numbers and $f: \mathbb{R} \to \mathbb{R}$ a function with

$$f(x+1) = f(x) + 1, \quad x \in \mathbb{R}.$$

Let $a \in \mathbb{R}$ be given, and define a sequence $x_0, x_1, \ldots$ by $x_0 = a$ and

$$x_{n+1} = f(x_n), \qquad n \geq 0.$$

Assume that there is a positive natural number m such that the difference $x_m - x_0$ is a whole number k. Prove that the limit $\lim\limits_{n\to\infty} \frac{x_n}{n}$ exists, and evaluate it.

2nd Day: June 30, 1987 (4-1/2 hours)

4. Does the set $\{1,2,\ldots,3000\}$ contain a subset A of 2000 elements such that $x \in A$ implies $2x \notin A$?

5. Let space be partitioned into three non–empty (pairwise disjoint) sets A_1, A_2, A_3. Show that there is a set A_i such that for each positive real number d there is a pair of elements from A_i which are at distance d from each other.

6. Let a circle κ of radius 1 and a natural number n be given. Let A be a set of n points $P_1,\dots,P_n$ on the circumference of κ. For a diameter d of κ the nonnegative distance from P_i to d is denoted by $\delta(P_i,d)$. Define

$$D(A) = \max_d \min_i \delta(P_i,d).$$

Let $\mathcal{B}_n$ be the collection of all such sets A and let $D_n = \min_{A\in\mathcal{B}_n} D(A)$. Evaluate D_n, and determine all $A \in \mathcal{B}_n$ such that $D(A) = D_n$.

Team Competition

July 1, 1987 (4 hours)

7. Let $n = \sum_{i=0}^{k} a_i 10^i$, $0 \leq a_i \leq 9$, $a_k \neq 0$, be a natural number and define

$$p(n) = \prod_{i=0}^{k} a_i, \qquad s(n) = \sum_{i=0}^{k} a_i,$$

and

$$\overline{n} = \sum_{i=0}^{k} a_{k-i} 10^i.$$

We consider the set

$$P = \{n \in \mathbb{N} \colon n = \overline{n}, \tfrac{p(n)}{3} + 1 = s(n)\}$$

and the subset Q of P consisting of those n having all digits $a_i \geq 2$, $i = 0,\dots,k$.

(i) Show that P is an *infinite* set of numbers.

(ii) Show that Q is only a *finite* set.

(iii) Determine all members of Q.

8. A circle of circumference 1 is divided into four arcs B_1, B_2, B_3, B_4 of equal lengths. Let C be a closed plane curve formed from translates of the arcs B_i (where each B_i may appear arbitrarily often), such that adjacent arcs appear in either one of the ways shown in the figure ("half–circle" or "wave"). Show that the length of C is a whole number.

9. Consider the set M of all points in the plane whose coordinates (x,y) are both whole numbers that satisfy $1 \leq x \leq 12$, $1 \leq y \leq 13$.

(i) Show that every subset of M containing at least 49 points must contain the four vertices of a rectangle having its sides parallel to the coordinate axes.

(ii) Construct a counterexample to (i) if the subset is allowed to consist of only 48 elements.

* * *

We now turn again to solutions submitted to problems posed in this column in 1985.

7. [1985: 238] *1981 Leningrad High School Olympiad (Third Round).*

The sequence $\{a_i\}$ of natural numbers satisfies

$$1 < a_1 < a_2 < \dots < a_n < \dots$$

and

$$a_{p+a_p} = 2a_p$$

for each natural number p. Show that there is a natural number c such that $a_n = n + c$ for any n. (Grade 9)

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let $a_1 = 1 + c$, $c \in \mathbb{N}$. Letting $p = 1$ in the given equation gives $a_{c+2} = 2c + 2$. But since $a_i \in \mathbb{N}$ and $1 + c = a_1 < a_2 < \dots$, it follows that $a_j \geq j + c$ for $j = 1,2,3,\dots$. Also from this observation, $a_j = j + c$ implies that $a_k = k + c$ for $1 \leq k \leq j$. Since

$$a_{c+2} = 2c + 2 = (c + 2) + c$$

we have $a_k = k + c$, for $1 \leq k \leq c + 2$.

Now suppose $a_k = k + c$ for $1 \leq k \leq (2^m - 1)c + 2^m$. (Note this holds for $m = 1$.) Setting $p = (2^m - 1)c + 2^m$ gives

$$a_{(2^{m+1}-1)c+2^{m+1}} = a_{p+a_p} = 2a_p = 2[(2^m - 1)c + 2^m + c] = 2^{m+1}c + 2^{m+1}.$$

Thus for $j = (2^{m+1} - 1)c + 2^{m+1}$ we have $a_j = j + c$. It follows that for $1 \leq k \leq (2^{m+1} - 1)c + 2^{m+1}$, $a_k = k + c$. Thus by induction we have that for all m, $a_k = k + c$ whenever $1 \leq k \leq (2^m - 1)c + 2^m$. From this the result is immediate.

*

4. [1985: 239] *1984 Bulgarian Mathematical Olympiad.*

The numbers $a,b,a_2,a_3,\dots,a_{n-2}$ are all real and $ab \neq 0$. It is known that all the roots of the equation

$$ax^n - ax^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-2}x^2 - n^2bx + b = 0$$

are real and positive. Prove that the roots are all equal.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Denote the roots by r_i, $i = 1,2,\dots,n$, where each $r_i > 0$. The relations between the roots and coefficients of polynomials allow us to write

(i) $$\sum_{i=1}^{n} r_i = 1;$$

(ii) $$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} r_j = \left[\prod_{i=1}^{n} r_i\right]\left[\sum_{i=1}^{n} \frac{1}{r_i}\right] = (-1)^{n-1}\left[\frac{-n^2 b}{a}\right];$$

(iii) $$\prod_{i=1}^{n} r_i = (-1)^n \frac{b}{a}.$$

Thus

$$\sum \frac{1}{r_i} = \left[\sum \frac{1}{r_i}\right]\left[\sum r_i\right]$$

from (i), and

$$\sum \frac{1}{r_i} = n^2$$

from (ii) and (iii). Thus

$$\left[\sum \frac{1}{r_i}\right]\left[\sum r_i\right] = n^2.$$

But by the AM–HM inequality,

$$\frac{1}{n} \sum r_i \geq \frac{n}{\sum \frac{1}{r_i}}$$

or $\left[\sum \frac{1}{r_i}\right]\left[\sum r_i\right] \geq n^2$ with equality just in case $r_i = r_j$ for all i, j. Hence the roots are equal.

[*Editor's note*: The AM–HM inequality follows from the AM–GM inequality, which is that for $\alpha_i > 0$,

$$\left[\prod_{i=1}^{n} \alpha_i\right]^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$$

with equality just if $\alpha_i = \alpha_j$ for all i. From this, we have

$$\left[\prod r_i\right]^{1/n} \leq \frac{1}{n} \sum r_i \tag{1}$$

and

$$\left[\prod \frac{1}{r_i}\right]^{1/n} \leq \frac{1}{n} \sum \frac{1}{r_i} \tag{2}$$

and thus

$$1 \le \frac{1}{n^2}\left[\sum r_i\right]\left[\sum \frac{1}{r_i}\right]. \qquad (3)$$

If the inequality in either (1) or (2) is strict it must be strict in (3) as well.]

*

1. [1985: 239] *1983 Annual Greek High School Competition.*
The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = x^5 + x - 1.$$

(a) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ is a bijection.
(b) Show that $f(1001^{999}) < f(1000^{1000})$.
(c) Determine the roots of the equation $f(x) = f^{-1}(x)$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

(a) Since f is continuous, $\lim\limits_{x\to\infty} f(x) = \infty$, and $\lim\limits_{x\to-\infty} f(x) = -\infty$, f is onto. Since $f'(x) = 5x^4 + 1 > 0$, f is strictly increasing, so it is a bijection.

(b) Since f is increasing, it will suffice to show that $1001^{999} < 1000^{1000}$. This is equivalent to

$$\frac{\ln 1001}{1000} < \frac{\ln 1000}{999}. \qquad (1)$$

Now, the function $\dfrac{\ln(x+1)}{x}$ has derivative

$$\frac{x/(x+1) - \ln(x+1)}{x^2},$$

which is negative for $x \ge 2$, and so (1) holds.

(c) If $x > 1$ then $f(x) = (x^5 - 1) + x > x$, so $f(f(x)) > f(x) > x$ and $f(x) > f^{-1}(x)$ since f and f^{-1} are increasing.

Similarly, if $x < 1$, $f(x) < x$, $f(f(x)) < f(x) < x$ and $f(x) < f^{-1}(x)$. Hence the only possible solution is $x = 1$. This works since $f(1) = f^{-1}(1) = 1$.

3. [1985: 239] *1983 Annual Greek High School Competition.*
If the function $f\colon [0,+\infty) \to \mathbb{R}$ satisfies the relation

$$f(x)e^{f(x)} = x$$

for all x in its domain, prove:

(a) the function f is monotonic over its entire domain;
(b) $\lim\limits_{x\to+\infty} f(x) = +\infty$;
(c) $\lim\limits_{x\to+\infty} \dfrac{f(x)}{\ln x} = 1.$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

(a) Since $f(x)e^{f(x)} = x \ge 0$, we have $f(x) \ge 0$. The function ye^y is clearly increasing for $y \ge 0$ and increases from 0 to $+\infty$ on $[0,+\infty)$. Thus f is uniquely defined for each x. Now, $f(x) < f(x_1)$ implies

$$x = f(x)e^{f(x)} < f(x_1)e^{f(x_1)} = x_1.$$

Also, $f(x) = f(y)$ if and only if $x = y$. From this we conclude that f is a strictly increasing function.

(b) The inequality $e^x \geq x + 1$ holds for all $x \geq 0$. Thus

$$x = f(x)e^{f(x)} \leq (e^{f(x)} - 1)e^{f(x)} < e^{2f(x)}$$

and so

$$f(x) > 1/2 \ln x$$

for $x > 0$. Hence $\lim\limits_{x \to +\infty} f(x) = +\infty$.

(c) Note that

$$\frac{f(x)}{\ln x} = \frac{f(x)}{f(x) + \ln f(x)} = \frac{1}{1 + \dfrac{\ln f(x)}{f(x)}},$$

so to show $\lim\limits_{x \to +\infty} \dfrac{f(x)}{\ln x} = 1$ it suffices to show that $\lim\limits_{x \to +\infty} \dfrac{\ln f(x)}{f(x)} = 0$. But $f(x) \to +\infty$ as $x \to +\infty$ and $\lim\limits_{y \to +\infty} \dfrac{\ln y}{y} = 0$. Thus it follows that $f(x) \approx \ln x$.

*

1. [1985: 270] *1985 Bulgarian Winter Competition.*

Determine each term of the sequence $\{a_1, a_2, \ldots\}$ of positive numbers, given that $a_4 = 4$, $a_5 = 5$, and

$$\frac{1}{a_1a_2a_3} + \frac{1}{a_2a_3a_4} + \cdots + \frac{1}{a_na_{n+1}a_{n+2}} = \frac{(n+3)a_n}{4a_{n+1}a_{n+2}}$$

holds for $n = 1,2,3,\ldots$. (Grade 9)

Solution by George Evagelopoulos, Law student, Athens, Greece.

For $n = 1$, we get

$$\frac{1}{a_1a_2a_3} = \frac{4a_1}{4a_2a_3} \Leftrightarrow a_1^2 = 1.$$

Since $a_1 > 0$, we conclude that $a_1 = 1$.

For $n = 2$, we find that

$$\frac{1}{a_1a_2a_3} + \frac{1}{a_2a_3a_4} = \frac{5a_2}{4a_3a_4} \Leftrightarrow \frac{1}{a_2} + \frac{1}{4a_2} = \frac{5a_2}{16}$$
$$\Leftrightarrow a_2^2 = 4$$

and as before $a_2 = 2$.

For $n = 3$ we get, as above,

$$\frac{1}{1\cdot 2\cdot a_3} + \frac{1}{2\cdot a_3\cdot 4} + \frac{1}{a_3\cdot 4\cdot 5} = \frac{6a_3}{4\cdot 4\cdot 5} \Leftrightarrow a_3^2 = 9,$$

giving $a_3 = 3$. To proceed by induction we take as induction hypothesis that for $1 \leq j \leq n + 2$ we have $a_j = j$. We have just verified the base case $n = 1$. Applying the given equation with n and adding $\dfrac{1}{a_{n+1}a_{n+2}a_{n+3}}$ to both sides, and then using the given equation with $n+1$, yields

$$\frac{(n+3)a_n}{4a_{n+1}a_{n+2}} + \frac{1}{a_{n+1}a_{n+2}a_{n+3}} = \frac{(n+4)a_{n+1}}{4a_{n+2}a_{n+3}}.$$

Using that $a_j = j$ for $1 \le j \le n+2$ and multiplying by $4(n+1)(n+2)a_{n+3}$ we obtain

$$\begin{aligned} & n(n+3)a_{n+3} + 4 = (n+4)(n+1)^2 \\ \Leftrightarrow\quad & n(n+3)a_{n+3} = n^3 + 6n^2 + 9n + 4 - 4 \\ \Leftrightarrow\quad & n(n+3)a_{n+3} = n(n+3)^2 \\ \Leftrightarrow\quad & a_{n+3} = n+3. \end{aligned}$$

Thus $a_j = j$ for $j \le (n+1) + 2 = n+3$ and the induction is complete. The sequence is then $a_n = n$.

4. [1985: 270] *1985 Bulgarian Winter Competition.*
Determine the number of real solutions of the equation

$$x^2 + 2x\sin x - 3\cos x = 0. \qquad \text{(Grade 11)}$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Note that $x^2 + 2x\sin x - 3\cos x$ is an even function of x, so we restrict ourselves to nonnegative reals. Let x be a solution to the equation. Since

$$|a\sin x + b\cos x| = \sqrt{a^2+b^2}\cdot|\sin(x+\theta)| \le \sqrt{a^2+b^2},$$

where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad\text{and}\quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}},$$

we have

$$x^4 = |2x\sin x - 3\cos x|^2 \le (2x)^2 + 3^2,$$

so

$$(x^2-2)^2 \le 13 < 16$$

and

$$x^2 < 2 + 4 < 9.$$

We find that $|x| < 3 < \pi$. Now if $\pi/2 \le x < \pi$ then

$$x^2 + 2x\sin x - 3\cos x \ge \frac{\pi}{2} + 0 + 0 > 0,$$

so $0 < x < \pi/2$. Write

$$f(x) = x^2 + 2x\sin x - 3\cos x;$$

then

$$f'(x) = 2x + 5\sin x + 2x\cos x > 0$$

for $0 < x < \pi/2$, so f is strictly increasing on $(0,\pi/2)$. Since

$$f(0) = -3 < 0 < \frac{\pi^2}{4} + \pi = f(\tfrac{\pi}{2})$$

and f is continuous, we find that f has precisely one root in $(0,\pi/2)$. Therefore, f has exactly two roots, one in $(-\pi/2,0)$ and one in $(0,\pi/2)$.

*

The remaining solutions in this number come from the 30 problems proposed for the 26th I.M.O. which appeared in Vol. 11, No. 10.

6. [1985: 305] *Proposed by Czechoslovakia.*

Let A be a set of positive integers such that $|x-y| \geq xy/25$ for any two elements x, y of A. Prove that A contains at most nine elements. Also, give an example of such a nine–element set.

Solution by F.D. Hammer, Cupertino, California, U.S.A.

Write $N = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, where

$$A_1 = \{1,2,3,4,5,6\},\quad A_2 = \{7,8,9\},\quad A_3 = \{10,11,12,13,14\},$$
$$A_4 = \{15,16,\dots,24\},\quad A_5 = \{25,26,\dots\}.$$

First note that $|A \cap A_i| \leq 1$ for $i = 2,\dots,5$. For A_5 this follows from

$$x > y \geq 25 \Rightarrow \frac{xy}{25} \geq x > |x-y|.$$

For A_2,A_3,A_4 it follows by noting that the *minimum* product divided by 25 is greater than the width (maximum difference) of the set, e.g. for A_4 we have

$$\frac{15\cdot 16}{25} > 9 = 24 - 15.$$

Thus if $|A| \geq 10$ it follows that $|A_1 \cap A| \geq 6$. Hence $5, 6 \in A_1 \cap A$. But $5\cdot 6/25 > 1$. An example with $|A| = 9$ is $\{1,2,3,4,5,7,10,17,54\}$.

[*Editor's note: Alternate solutions were submitted by Daniel Ropp, Washington University, St. Louis, MO, U.S.A. and by Ed Doolittle, University of Toronto, Canada.*]

8. [1985: 305] *Proposed by France.*

Determine eight positive integers $n_1,n_2,\dots,n_8$ with the following property: For every integer k such that $-1985 \leq k \leq 1985$, there are eight integers $\alpha_1,\alpha_2,\dots,\alpha_8$, each belonging to the set $\{-1,0,1\}$, such that

$$\alpha_1 n_1 + \alpha_2 n_2 + \cdots + \alpha_8 n_8 = k.$$

Solution by Ed Doolittle, University of Toronto, Ontario, Canada.

Every integer between 0 and $3^8 - 1 = 6560$ can be expressed (uniquely) in the ternary form

$$e_8 3^7 + e_7 3^6 + \cdots + e_1 3^0 \qquad (1)$$

with each $e_i \in \{0,1,2\}$. Subtracting $1\cdot 3^7 + 1\cdot 3^6 + 1\cdot 3^5 + \cdots + 1\cdot 3^0 = 3280$ from (1) gives the balanced ternary form (see reference)

$$\alpha_8 3^7 + \alpha_7 3^6 + \cdots + \alpha_1 3^0$$

in which each integer from –3280 to 3280 can be expressed (uniquely) with each $\alpha_i \in \{-1,0,1\}$. Thus one set of eight integers n_i is $\{3^0,3^1,\ldots,3^7\}$. (The set is not unique: e.g. 3^7 can be replaced by many smaller numbers.)

Reference: Kenneth H. Rosen, *Elementary Number Theory and its Applications*, Addison-Wesley, 1984, problem 8, p.30.

10. [1985: 305] *Proposed by Great Britain.*

A sequence of polynomials $\{P_m\}$, $m = 0,1,2,\ldots$, in x, y, and z is defined by $P_0(x,y,z) \equiv 1$ and, for $m > 0$, by

$$P_m(x,y,z) = (x+z)(y+z)P_{m-1}(x,y,z+1) - z^2P_{m-1}(x,y,z).$$

Prove that each $P_m(x,y,z)$ is symmetric in x, y, z.

Solution by Ed Doolittle, University of Toronto, Ontario, Canada.

First note that by an easy induction,

$$P_m(x,y,z) \equiv P_m(y,x,z) \tag{1}$$

for all m. Let $Q(m)$ be the statement: P_m and P_{m+1} are symmetric and

$$(x+y)P_m(x,y+1,z) - (x+z)P_m(x,y,z+1) + (z-y)P_m(x,y,z) \equiv 0.$$

As $P_1(x,y,z) = xy + yz + xz$, it is easy to check that $Q(0)$ is true. Suppose then that $Q(k)$ holds. Then

$$(x+y)P_k(x,y+1,z) - (x+z)P_k(x,y,z+1) + (z-y)P_k(x,y,z) \equiv 0.$$

Multiplying by $-z^2$ gives

$$-z^2(x+y)P_k(x,y+1,z) + z^2(x+z)P_k(x,y,z+1) - z^2(z-y)P_k(x,y,z) \equiv 0.$$

Adding and subtracting $y^2(x+z)P_k(x,y,z+1)$ yields

$$y^2(x+z)P_k(x,y,z+1) + (z^2-y^2)(x+z)P_k(x,y,z+1)$$
$$- z^2(x+y)P_k(x,y+1,z) - z^2(z-y)P_k(x,y,z) \equiv 0.$$

Adding and subtracting $(x+y)(x+z)(y+1+z)P_k(x,y+1,z+1)$, using the symmetry of P_k and factoring gives

$$(x+y)[(x+z)(y+1+z)P_k(x,y+1,z+1) - z^2P_k(x,y+1,z)]$$
$$- (x+z)[(x+y)(z+1+y)P_k(x,z+1,y+1) - y^2P_k(x,z+1,y)]$$
$$+ (z-y)[(x+z)(y+z)P_k(x,y,z+1) - z^2P_k(x,y,z)] \equiv 0.$$

Using the definition of P_{k+1}, and the assumed symmetry of P_{k+1}, this becomes

$$(x+y)P_{k+1}(x,y+1,z) - (x+z)P_{k+1}(x,y,z+1) + (z-y)P_{k+1}(x,y,z) \equiv 0. \tag{2}$$

Multiplying by $y+z$ we get

$$(x+y)(z+y)P_{k+1}(x,y+1,z) - (x+z)(y+z)P_{k+1}(x,y,z+1) + (z^2-y^2)P_{k+1}(x,y,z) \equiv 0.$$

Rewriting and using symmetry of P_{k+1} we have

$$(x+y)(z+y)P_{k+1}(x,z,y+1) - y^2P_{k+1}(x,z,y)$$
$$\equiv (x+z)(y+z)P_{k+1}(x,y,z+1) - z^2P_{k+1}(x,y,z).$$

In other words

$$P_{k+2}(x,z,y) \equiv P_{k+2}(x,y,z). \tag{3}$$

Combining (1) and (3), we see that P_{k+2} is symmetric. Now this, the symmetry of P_{k+1} and (2) give that $Q(k+1)$ is true. Thus P_m is symmetric for all m by induction.

* * *

Unfortunately limited space means that this month I won't be able to finish giving solutions to problems posed in 1985 or to publish the list of outstanding questions. Perhaps next month. Meanwhile keep sending in your solutions to problems posed since 1985!

* * *

PROBLEMS

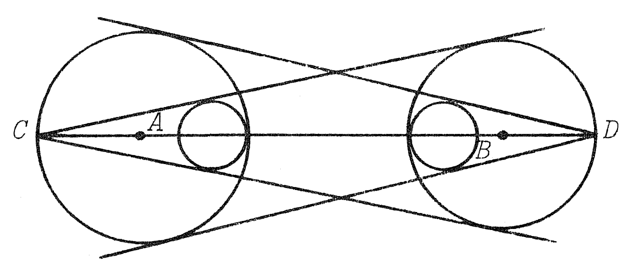
Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **September 1, 1988**, *although solutions received after that date will also be considered until the time when a solution is published.*

<u>1311</u>. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

In the figure, A and B are the centers of the large circles, and the lines from C and D are tangents. Show that the small inscribed circles have equal radii.



<u>1312</u>. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find all 27 solutions of the system of equations

$$y = 4x^3 - 3x$$
$$z = 4y^3 - 3y$$
$$x = 4z^3 - 3z.$$

1313. *Proposed by Wendel Semenko, Snowflake, Manitoba.*

Show that any triangular piece of paper of area 1 can be folded once so that when placed on a table it will cover an area of less than $\frac{\sqrt{5}-1}{2}$.

1314. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Solve the determinantal equation

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_2 & \cdots & a_n \\ \vdots & & & & \\ 1 & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ x & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = 0$$

for x.

1315. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with medians AD, BE, CF and median point G. We denote $\Delta AGF = \Delta_1$, $\Delta BGF = \Delta_2$, $\Delta BGD = \Delta_3$, $\Delta CGD = \Delta_4$, $\Delta CGE = \Delta_5$, $\Delta AGE = \Delta_6$, and let R_i and r_i denote the circumradius and inradius of Δ_i ($i = 1,2,\ldots,6$). Prove that

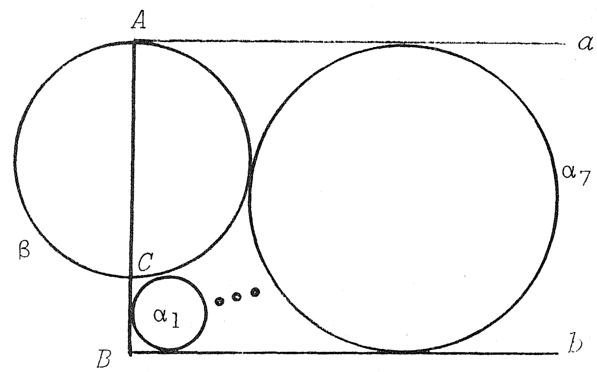
(i) $R_1R_3R_5 = R_2R_4R_6$;

(ii) $\frac{15}{2r} < \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6} < \frac{9}{r}$,

where r is the inradius of ΔABC.

1316. *Proposed by Jordi Dou, Barcelona, Spain.*

AB is a segment of unit length and lines a, b are perpendicular to AB at A and B respectively. C is a point on the segment AB and β is the circle of diameter AC. Suppose that a chain of exactly seven circles $\alpha_1,\ldots,\alpha_7$ can be inscribed around β and between a and b as shown (i.e., α_1 is tangent to AB, β, and b; α_2 to α_1, β, and b ; ... ; α_7 to α_6, β, a, and b). Find a simple expression for the distance BC.



1317. *Proposed by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.*

Crux 1133 [1987: 225] suggests the following problem. In a triangle ABC the excircle touching side AB touches lines BC and AC at points D and E respectively. If $AD = BE$, must the triangle be isosceles?

1318. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Find, without calculus, the largest possible value of

$$\frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x}.$$

1319. *Proposed by Geng–zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.*

Let A, B, C be fixed points in the plane. An ant starts from a certain point P_0 of the plane and crawls in a straight line through A to a point P_1 such that $P_0A = AP_1$. It then crawls from P_1 through B to P_2 such that $P_1B = BP_2$, then through C to P_3, and so on. Altogether it repeats the same action 1986 times successively through the points $A,B,C,A,B,C,A,\ldots$, finally stopping in exhaustion at P_{1986}. Where is the ant now? (Compare with problem 2 of the 1986 I.M.O. [1986: 173].)

1320. *Proposed by Themistocles M. Rassias, Athens, Greece.*

Assume that $a_1,a_2,a_3,\ldots$ are real numbers satisfying the inequality

$$|a_{m+n} - a_m - a_n| \le C$$

for all m, $n \ge 1$ and for some constant C. Prove that there exists a constant k such that

$$|a_n - nk| \le C$$

for all $n \ge 1$.

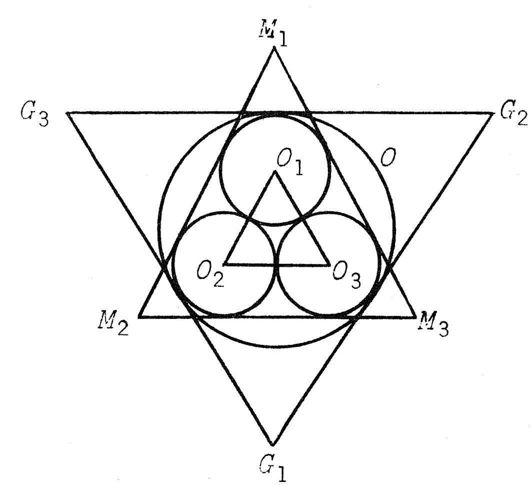
* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1150*. [1986: 108; 1987: 264] *Proposed by Jack Garfunkel, Flushing, N.Y.*

In the figure, $\Delta M_1M_2M_3$ and the three circles with centers O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles and the sides of triangle $G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that $P(\Delta G_1G_2G_3) \ge P(\Delta M_1M_2M_3) + P(\Delta O_1O_2O_3)$, where P stands for perimeter. Equality is attained when $\Delta O_1O_2O_3$ is equilateral.



Counterexample by G.P. Henderson, Campbellcroft, Ontario.

The proposed inequality is false. Let the radii of the Malfatti circles be $r_1 = r_2 = 1$ and r_3 slightly less than 2. Then $P(\Delta M_1M_2M_3)$ is arbitrarily large and the other two perimeters are of bounded size.

* * *

<u>1173</u>. [1986: 205] *Proposed by Jordi Dou, Barcelona, Spain.*

A knight is placed at random on a square of a chessboard. It then makes a sequence of legal moves, each chosen randomly from all possible legal moves. Find the average number of moves it has made when it first makes a move which is the reverse of its previous move.

Solution by the proposer.

To solve this problem we need the following ideas from graph theory. A *graph* consists of some points (called *vertices*) and lines (called *edges*) which connect some pairs of vertices. The *degree* of a vertex is the number of edges connected to it. Any sequence of vertices in which each pair of consecutive vertices is adjacent, i.e. connected by an edge, will be called a *path*. The *length* of a path is the number of edges in it. A *cycle* is a path in which the first and last vertices are the same, but which has no other repetition of vertices.

We first prove

Lemma. Let G be a graph in which every pair of vertices is contained in a cycle (G is "2–connected"). Let V_s be a vertex of G. Consider a path P in G satisfying

(i) P starts and ends at V_s;

(ii) V_s does not appear in P other than at an endpoint;

(iii) at each vertex V of P, the next vertex in P is chosen at random among all vertices adjacent to V in G other than the vertex immediately preceding V in P. (So P is not allowed to "backtrack".)

Then the expected length of P is given by

$$E(P) = \frac{2e}{e_s},$$

where e is the total number of edges in G and e_s is the degree of V_s.

Proof. For adjacent vertices V_r, V_h in G, where $r \neq s$, we let E_r^h denote the expected length of a path from V_r to V_s which satisfies (ii) and (iii) above and which does not have V_h as its second vertex. (Note that at least one such path exists, by the assumption on G.) We also define $E_s^h = 0$ for each vertex V_h adjacent to V_s. Then it follows easily that

$$E(P) = 1 + \frac{1}{e_s} \sum E_i^s ,$$

or

$$e_s E(P) = e_s + \sum E_i^s , \tag{1}$$

where the sum is over all vertices V_i adjacent to V_s. Also, for $r \neq s$ and V_h adjacent to V_r,

$$E_r^h = 1 + \frac{1}{e_r - 1} \sum E_i^r , \tag{2_{rh}}$$

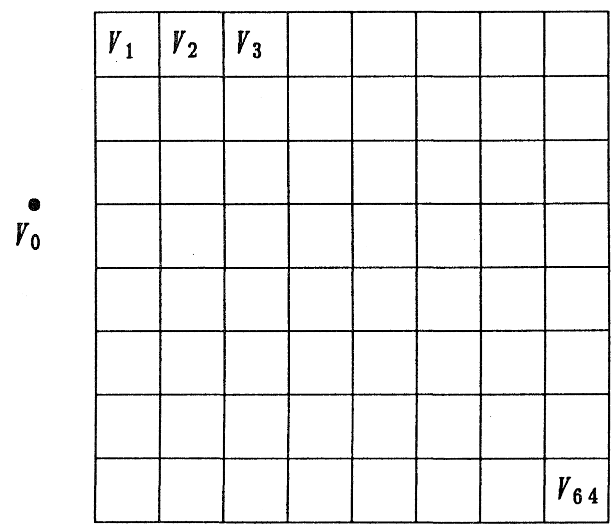
where e_r is the degree of V_r, and the sum is over all vertices V_i, except V_h, which are adjacent to V_r.

Now consider the result of adding (1) and all equations (2_{rh}) together. Each E_i^s, where V_i and V_s are adjacent, only occurs once on the right side of (1) and once on the left side of (2_{is}), and so will disappear in the sum of these equations. Each E_i^r, where V_i and V_r are adjacent and $r \neq s$, $i \neq s$, occurs once on the left side of (2_{ir}), and on the right side (with coefficient $(e_r - 1)^{-1}$) of each (2_{rh}) corresponding to a vertex V_h adjacent to V_r, $h \neq i$. Since there are $e_r - 1$ such vertices V_h, the sum of (1) and the equations (2_{rh}) will not contain E_i^r. There is an equation (2_{rh}) corresponding to each ordered pair (V_r, V_h) of adjacent vertices where $r \neq s$, for a total of $2e - e_s$ such equations. Thus the sum of (1) and all equations (2_{rh}) yields

$$e_s E(P) = e_s + (2e - e_s) = 2e,$$

and the lemma follows. □

To apply this lemma to the present problem, we consider the graph G with 65 vertices $V_1, \ldots, V_{64}$, V_0, where V_1 to V_{64} correspond to the 64 squares of the chessboard and V_0 is an extra vertex. The edges of G are all pairs $V_i V_j$ $(i,j \neq 0)$ such that $V_i \to V_j$ is a legal knight's move, along with *all* edges $V_0 V_i$, $i > 0$. The number of edges $V_i V_j$ $(i,j \neq 0)$ is



$$\frac{4 \cdot 2 \ + \ 8 \cdot 3 \ + \ 20 \cdot 4 \ + \ 16 \cdot 6 \ + \ 16 \cdot 8}{2} = 168 ,$$

which with the 64 edges $V_0 V_i$ make a total number of $e = 232$ edges.

Now note that each tour of a knight on a chessboard as in the problem can be associated in a unique way with a path in G as in the lemma, with $V_s = V_0$. The first edge of the path randomly selects the starting square of the knight. Thereafter the path corresponds to the knight's moves until it ends by reaching V_0 again, which corresponds to the knight having retraced its previous move. The length (number of moves) of the knight's tour is one less than the length of the corresponding path in G. By the lemma, the expected length of the path in G is

$$\frac{2e}{e_S} = \frac{2(232)}{64} = \frac{29}{4},$$

so the expected length of the knight's tour is $\frac{25}{4}$ moves.

Comment. The above lemma, which I originally used for the proof of my proposal *Crux* 499, is remarkably similar to a result of Rowe [1]. The only difference is that in Rowe's result (which applies to *all* connected graphs) the interior vertices of the path P are chosen at random from among *all* adjacent vertices, i.e. the path is allowed to backtrack. *The formula for* $E(P)$ *is the same.* Even the proofs are very similar.

As an aside, this means that the published proof of *Crux* 499 [1980: 327] is incorrect, in that it should have used the lemma above rather than Rowe's. Thus it happens that by a curious and amusing irony of chance, a false solution leads to a correct result!

Reference:

[1] R. Robinson Rowe, Roundtripping a chessboard, *Journal of Recreational Mathematics* 4 (1971) 265–267.

* * *

<u>1185</u>*. [1986: 242] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the set M of all integers $k \geq 2$ such that there exists a positive real number u_k satisfying $[u_k^n] \equiv n \pmod{k}$ for all natural numbers n, where $[x]$ denotes the greatest integer $\leq x$.

(Problem A–5 in the 1983 Putnam Competition is equivalent to showing that $2 \in M$.)

Solution by Richard K. Guy, University of Calgary.

The set M contains all positive integers k, including (trivially) $k = 1$, since we may choose u_1 to be any positive real number.

In general we can choose u_k in the interval $(k + 1, k + 2)$ so that u_k^2 is in

$$(k^2 + 2k + 1\ ,\ k^2 + 4k + 4).$$

We next ensure $[u_k^2] \equiv 2 \bmod k$, and obtain a closer approximation to u_k, by choosing u_k^2 to be in the unit interval

$$(k^2 + 3k + 2\ ,\ k^2 + 3k + 3),$$

for example. u_k^2 now lies in

$$((k^2 + 3k + 2)^{3/2}, (k^2 + 3k + 3)^{3/2}).$$

This is an interval of length more than $k + 1$, since

$$((k + 1)^2 + 1)^{3/2} > (k + 1)((k + 1)^2 + 1)$$

and thus

$$(k^2 + 3k + 3)^{3/2} - (k^2 + 3k + 2)^{3/2} > ((k + 1)^2 + 1)^{3/2} - ((k + 1)^2)^{3/2} > k + 1,$$

and so contains at least one unit interval, with integer endpoints, in each residue class mod k. This process continues indefinitely, providing closer and closer upper and lower bounds on u_k.

We illustrate with $k = 2$. Choose $u_2 \in (3,4)$, so that

$$[u_2] \equiv 1 \bmod 2.$$

Then $u_2^2 \in (9,16)$. Next choose $u_2^2 \in (12,13)$ (we could have chosen $(10,11)$ or $(14,15)$) so that

$$[u_2^2] \equiv 2 \bmod 2,$$

and thus

$$u_2 \in (3.464, 3.605),$$
$$u_2^3 \in (41.569, 46.872).$$

Since $43 \equiv 3 \bmod 2$, we may choose u_2^3 more precisely to be in $(43,44)$. The next few refinements might yield

n	$u_2 \in$	$u_2^n \in$	so choose $u_2^n \in$
4	(3.5034,3.5303)	(150.646,155.335)	(152,153) where $152 \equiv 4 \bmod 2$
5	(3.5112,3.5170)	(533.7089,538.1016)	(535,536) where $535 \equiv 5 \bmod 2$
6	(3.5129,3.5142)	(1879.423,1883.639)	(1880,1881) where $1880 \equiv 6 \bmod 2$

and so on.

Similarly, for $k = 3$, choose $u_3 \in (4,5)$ so that $[u_3] \equiv 1 \bmod 3$. Then $u_3^2 \in (16,25)$. Next choose $u_3^2 \in (20,21)$ so that $[u_3^2] \equiv 2 \bmod 3$. The process can then continue

n	$u_3 \in$	$u_3^n \in$	so choose $u_3^n \in$
3	(4.472,4.583)	(89.443,96.234)	(90,91) where $90 \equiv 3 \bmod 3$
4	(4.4814,4.4979)	(403.326,409.312)	(406,407) where $406 \equiv 4 \bmod 3$
5	(4.4888,4.4915)	(1822.458,1828.07)	(1823,1824) where $1823 \equiv 5 \bmod 3$

and so on.

For each k there are uncountably many choices for u_k. For example, start with

$u_k \in (2k + 1, 2k + 2)$, so that

$$u_k^2 \in (4k^2 + 4k + 1\ ,\ 4k^2 + 8k + 4).$$

Then we may choose u_k^2 in any of

$$(4k^2 + 4k + 2\ ,\ 4k^2 + 4k + 3),$$
$$(4k^2 + 5k + 2\ ,\ 4k^2 + 5k + 3),$$
$$(4k^2 + 6k + 2\ ,\ 4k^2 + 6k + 3),$$
$$(4k^2 + 7k + 2\ ,\ 4k^2 + 7k + 3),$$

or

$$(4k^2 + 8k + 2\ ,\ 4k^2 + 8k + 3).$$

For each choice we will have u_k^2 in an interval of length greater than $2k + 1$, so that there will be, at this stage and at each succeeding stage, at least *two* unit intervals in the required residue class.

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<u>1189</u>*. [1986: 242] *Proposed by Kee–wai Lau, Hong Kong.*

Find the surface area and volume of the solid formed by the intersection of eight unit spheres whose centres are located at the vertices of a unit cube.

Solution by Rex Westbrook, University of Calgary.

We assign coordinates so that the origin is one corner of the cube and (1,1,1) is the opposite corner.

Split the required region into eight congruent parts by the three planes $x = 1/2$, $y = 1/2$, $z = 1/2$. That part in the octant $x \geq 1/2$, $y \geq 1/2$, $z \geq 1/2$ has boundaries

$$x^2 + y^2 + z^2 = 1, \qquad x = 1/2, \qquad y = 1/2, \qquad z = 1/2,$$

with volume V and surface area S given by

$$V = \int_{1/2}^{1/\sqrt{2}} dx \int_{1/2}^{\sqrt{3/4-x^2}} (\sqrt{1 - x^2 - y^2} - \tfrac{1}{2})\,dy\ ,$$

$$S = \int_{1/2}^{1/\sqrt{2}} dx \int_{1/2}^{\sqrt{3/4-x^2}} \frac{dy}{\sqrt{1 - x^2 - y^2}}\ . \tag{1}$$

Changing to polar coordinates in the xy–plane and using symmetry, we get

$$V = 2\int_{\pi/4}^{\tan^{-1}\sqrt{2}} d\theta \int_{\frac{1}{2}\sec\theta}^{\sqrt{3}/2} ((1 - r^2)^{1/2} - \tfrac{1}{2})\,r\,dr$$

$$= 2\int_{\pi/4}^{\tan^{-1}\sqrt{2}} \left(\tfrac{1}{24}(4-\sec^2\theta)^{3/2} - \tfrac{11}{48} + \tfrac{1}{16}\sec^2\theta\right)d\theta$$

$$= \tfrac{1}{3}\int_{\pi/4}^{\beta} (4-\sec^2\theta)^{1/2}d\theta \;-\; \tfrac{1}{12}\int_{\pi/4}^{\beta} \sec^2\theta(4-\sec^2\theta)^{1/2}d\theta$$
$$- \tfrac{11}{24}(\beta - \tfrac{\pi}{4}) + \tfrac{1}{8}(\sqrt{2}-1)$$

where we have put $\beta = \tan^{-1}\sqrt{2}$, and

$$S = 2\int_{\pi/4}^{\beta} d\theta \int_{\frac{1}{2}\sec\theta}^{\sqrt{3}/2} (1-r^2)^{-1/2}r\,dr$$

$$= 2\int_{\pi/4}^{\beta} \left(\tfrac{1}{2}(4-\sec^2\theta)^{1/2} - \tfrac{1}{2}\right)d\theta$$

$$= \int_{\pi/4}^{\beta} (4-\sec^2\theta)^{1/2}d\theta \;-\; \beta \;+\; \tfrac{\pi}{4}.$$

Now, using the substitution $t = \tan\,\theta$,

$$\int_{\pi/4}^{\beta} \sec^2\theta(4-\sec^2\theta)^{1/2}d\theta = \int_{\pi/4}^{\beta} \sec^2\theta(3-\tan^2\theta)^{1/2}d\theta$$

$$= \int_{1}^{\sqrt{2}} (3-t^2)^{1/2}dt$$

$$= \tfrac{3}{2}(\sin^{-1}\sqrt{2/3} - \sin^{-1}(1/\sqrt{3}))$$

$$= \tfrac{3}{2}(\beta - (\tfrac{\pi}{2} - \beta)) = 3\beta - \frac{3\pi}{4}.$$

Similarly, putting $t = \tan\,\theta$ and then $t = \sqrt{3}\,\sin\,\phi$, we get

$$\int_{\pi/4}^{\beta} (4-\sec^2\theta)^{1/2}d\theta = \int_{1}^{\sqrt{2}} \frac{(3-\,t^2)^{1/2}}{1+\,t^2}\,dt$$

$$= \int_{\frac{\pi}{2}-\beta} \frac{3\;\cos^2\phi}{1+3\;\sin^2\phi}\,d\phi$$

$$= \int_{\frac{\pi}{2}-\beta}^{\beta} \left[\frac{4}{1+3\sin^2\phi} - 1\right] d\phi$$

$$= 4\int_{\frac{\pi}{2}-\beta}^{\beta} \frac{\sec^2\phi}{1+4\tan^2\phi}\, d\phi - 2\beta + \frac{\pi}{2}$$

$$= 2\tan^{-1}(2\tan\phi)\Big|_{\frac{\pi}{2}-\beta}^{\beta} - 2\beta + \frac{\pi}{2}$$

$$= \frac{5\pi}{2} - 8\beta,$$

where we have used

$$\tan^{-1}(2\tan\beta) = \tan^{-1}\left[\frac{-2\tan\beta}{1-\tan^2\beta}\right] = \tan^{-1}(-\tan 2\beta) = \pi - 2\beta$$

(since $2\beta > \pi/2$) and

$$\tan^{-1}(2\tan(\tfrac{\pi}{2}-\beta)) = \tan^{-1}(\frac{2}{\tan\beta}) = \tan^{-1}(\frac{2}{\sqrt{2}}) = \beta.$$

Thus

$$V = \frac{1}{3}(\frac{5\pi}{2} - 8\beta) - \frac{1}{12}(3\beta - \frac{3\pi}{4}) - \frac{11}{24}(\beta - \frac{\pi}{4}) + \frac{1}{8}(\sqrt{2}-1)$$

$$= \frac{97}{96}\pi - \frac{27}{8}\beta + \frac{1}{8}(\sqrt{2}-1)$$

and

$$S = \frac{5\pi}{2} - 8\beta - \beta + \frac{\pi}{4} = \frac{11\pi}{4} - 9\beta.$$

Multiplying by 8, we obtain the required volume

$$\frac{97}{12}\pi - 27\tan^{-1}\sqrt{2} + \sqrt{2} - 1 \approx 0.0152$$

and surface area

$$22\pi - 72\tan^{-1}\sqrt{2} \approx 0.33224.$$

Also solved by HANAFI FARAHAT, University of Calgary; RICHARD K. GUY, University of Calgary; and JONATHAN SCHAER, University of Calgary. There was one incorrect solution submitted.

Interestingly, MACSYMA failed to evaluate the above integrals (1) *for* V *and* S.

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<u>1190</u>*. [1986: 242] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Prove or disprove that

$$\lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{(-1)^k(n-k)^k e^{n-k}}{k!} - 2n\right] = \frac{2}{3}.$$

Solution by G.P. Henderson, Campbellcroft, Ontario.

Set

$$a_n = \sum_{k=0}^{n}\frac{(-1)^k(n-k)^k e^{n-k}}{k!}, \qquad n = 1,2,\dots .$$

Consider the function

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

$$= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(n-k)^k e^{n-k} z^n}{k!}$$

$$= 1 + \sum_{n=1}^{\infty} e^n z^n + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^k(n-k)^k e^{n-k} z^n}{k!}.$$

Setting $n = r + k$ in the double sum,

$$F(z) = 1 + \sum_{n=1}^{\infty} e^n z^n + \sum_{r=0}^{\infty} e^r z^r \sum_{k=1}^{\infty}\frac{(-1)^k r^k z^k}{k!}$$

$$= \sum_{n=0}^{\infty} e^n z^n + \sum_{r=0}^{\infty} e^r z^r (e^{-rz} - 1)$$

$$= \sum_{r=0}^{\infty} e^r z^r e^{-rz} = \frac{1}{1 - ze^{1-z}}.$$

F is analytic except for the poles which occur at the zeros of $1 - ze^{1-z}$. There is one at $z = 1$ and the others are outside the circle $|z| = 1$. To see this, set $z = 1 + x + iy$ where x and y are real. The zeros of $1 - ze^{1-z}$ are determined by solving the system

$$1 + x = e^x \cos y,$$

$$y = e^x \sin y.$$

The second equation implies that $e^x \geq 1$, that is, $x \geq 0$. If $x = 0$ we must have $y = 0$; this gives us the pole at $z = 1$. If $x > 0$ then

$$|z|^2 = (x+1)^2 + y^2 = e^{2x} > 1,$$

that is, the other poles of F are outside $|z| = 1$.

To find the Laurent expansion of F at $z = 1$, set $z = 1 - t$. Then

$$\begin{aligned} F(z) &= [1-(1-t)e^t]^{-1} \\ &= [1-(1-t)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots)]^{-1} \\ &= (\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{8}+\cdots)^{-1} \\ &= \frac{2}{t^2}(1+\frac{2}{3}t+\cdots)^{-1} \\ &= \frac{2}{t^2}(1-\frac{2}{3}t+\cdots). \end{aligned}$$

Thus the pole at $z = 1$ is of order 2, the principal part being

$$\begin{aligned} G(z) &= \frac{2}{(1-z)^2} - \frac{4}{3(1-z)} \\ &= \sum_{n=0}^{\infty} 2(n+1)z^n - \frac{4}{3}\sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} (2n+2/3)z^n. \end{aligned}$$

The function

$$\begin{aligned} (F-G)(z) &= 1 + \sum_{n=1}^{\infty} a_n z^n - \sum_{n=0}^{\infty}(2n+2/3)z^n \\ &= 1/3 + \sum_{n=1}^{\infty}(a_n - 2n - 2/3)z^n \end{aligned}$$

is analytic for $|z| \le 1$. Therefore the above series converges for $z = 1$, and in particular its terms approach 0. Hence

$$\lim_{n\to\infty}(a_n - 2n) = 2/3.$$

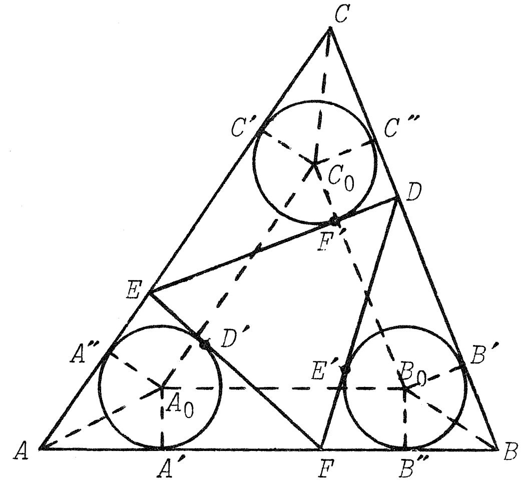
*　　　　*　　　　*

1191. [1986: 281] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

Let ABC be a triangle, and let points D, E, F be on sides BC, CA, AB respectively such that triangles AEF, BFD, and CDE all have the same inradius r. Let r_1 and r_2 denote the inradii of DEF and ABC respectively. Show that $r + r_1 = r_2$.

Solution by Jordi Dou, Barcelona, Spain.

Following the notation of the figure, we have that



$DE + EF + FD$

$= DF' + F'E + ED' + D'F + FE' + E'D$

$= DC'' + C'E + EA'' + A'F + FB'' + B'D$

$= C'A'' + A'B'' + B'C''$

$= C_0A_0 + A_0B_0 + B_0C_0.$

Also, letting $[X]$ denote the area of figure X, it is well known that for any triangle T,

$[T]$ = inradius $(T)\cdot$semiperimeter(T).

Thus

$$\begin{aligned}[DEF] &= [ABC] - ([DCE] + [EAF] + [FBD]) \\ &= [ABC] - \tfrac{r}{2}(AB + BC + CA + DE + EF + FD) \\ &= [ABC] - \tfrac{r}{2}(AB + BC + CA + C_0A_0 + A_0B_0 + B_0C_0) \\ &= [ABC] - \frac{r(AB + A_0B_0)}{2} - \frac{r(BC + B_0C_0)}{2} - \frac{r(CA + C_0A_0)}{2} \\ &= [A_0B_0C_0].\end{aligned}$$

Therefore the triangles DEF and $A_0B_0C_0$ have the same perimeter and the same area, and so their inradii are equal. Since the inradius of $\Delta A_0B_0C_0$ is obviously $r_2 - r$, we are done.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. Groenman's proof was almost identical to Dou's.

The problem was taken from the 1810 Japanese mathematics book Sanpo Tensho Ho Sinan.

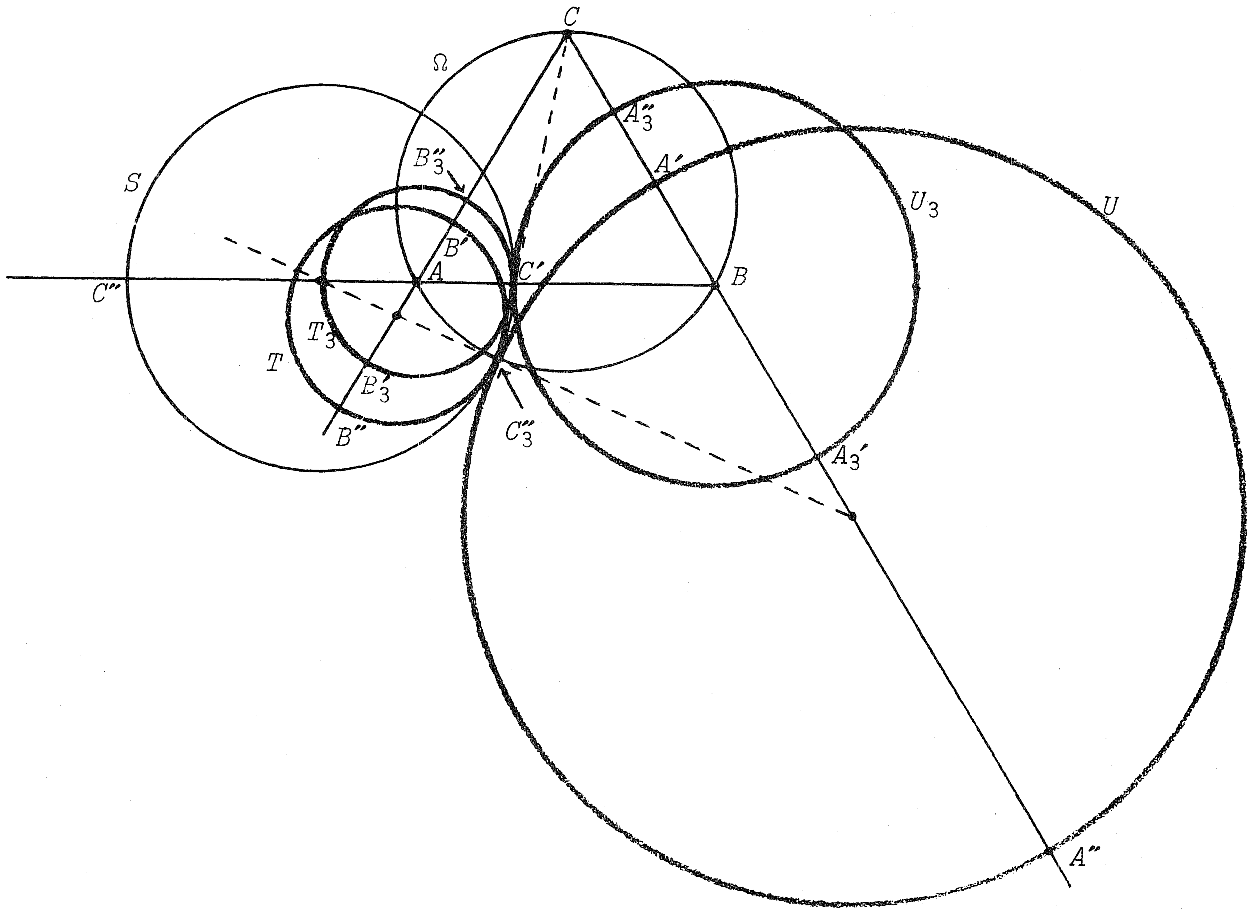
* * *

1192. [1986: 281] *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*

Let ABC be an equilateral triangle and v, w be arbitrary positive real numbers. S (resp. T, U) is the Apollonius circle which is the locus of points whose distances from A and B (resp. A and C, B and C) are in the ratio $v:w$ (resp. $v:v+w$, $w:v+w$). Prove that S, T, U have just one point in common, and that it lies on the circumcircle of ΔABC.

Solution by Jordi Dou, Barcelona, Spain.

Suppose $AB = BC = CA = 1$ and $v < w$. Let $A'A''$ be the diameter of U which lies on the line BC, where A' is interior to side BC. Define $B'B''$ and $C'C''$ similarly as diameters of T and S respectively.



Let I_3 be the inversion with centre C and power 1, and denote $I_3(X) = X_3$ for any point or figure X. Then $U_3 = I_3(U)$ is a circle with centre B (since

$$CA'_3 + CA''_3 = \frac{1}{CA'} + \frac{1}{CA''} = 2$$

follows from $CBA'A''$ harmonic), and similarly the centre of $T_3 = I_3(T)$ is A. The radius of U_3 is

$$u_3 = CA'_3 - 1 = \frac{1}{CA'} - 1 = \frac{1 - CA'}{CA'} = \frac{BA'}{CA'} = \frac{w}{v + w}\,.$$

Analogously the radius of T_3 is

$$t_3 = \frac{v}{v + w}\,.$$

Since $u_3 + t_3 = 1 = AB$, the circles U_3 and T_3 are tangent, in fact at C' since

$$\frac{C'A}{C'B} = \frac{v}{w} = \frac{t_3}{u_3}\,.$$

Thus U and T are tangent at $I_3(C') = C'_3$, and since $I_3(AB)$ is the circumcircle Ω of ΔABC, C'_3 lies on Ω. C'_3 belongs also to S, since

$$\frac{C_3'A}{C_3'B} = \frac{C_3'A/C_3'C}{C_3'B/C_3'C} = \frac{v/(v+w)}{w/(v+w)} = \frac{v}{w}.$$

Since S, T, U are orthogonal to Ω, the line through their centres is tangent to Ω at C_3'.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

* * *

1193*. [1986: 282] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Is there a Heronian triangle (sides and area rational) with one side twice another?

I. *Solution by Hayo Ahlburg, Benidorm, Alicante, Spain.*

The answer is no, already more or less implied by the asterisk, i.e. the fact that Mr. Rabinowitz's computer had not supplied a solution. We consider the more general question: what values $\lambda = a/b$ are possible in a Heronian triangle?

We assume the triangle ABC has sides a, b, c and area Δ all integers. This is no loss of generality since any rational triangle will give integers simply by using a suitable unit of measurement.

Now we consider

$$\lambda = \frac{a}{b} = \frac{\sin A}{\sin B} = \frac{2\tan A/2}{1+\tan^2 A/2} \div \frac{2\tan B/2}{1+\tan^2 B/2}$$

$$= \frac{\cot A/2\,(\cot^2 B/2 + 1)}{\cot B/2\,(\cot^2 A/2 + 1)}.$$

This gives

$$\cot\frac{B}{2} = \frac{\lambda(\cot^2 A/2 + 1) \pm \sqrt{\lambda^2\cot^4 A/2 + (2\lambda^2 - 4)\cot^2 A/2 + \lambda^2}}{2\cot A/2}. \tag{1}$$

In a Heronian triangle, the inradius $\rho = \Delta/s$ is rational (s is the semiperimeter), and with it also

$$\cot\frac{A}{2} = \frac{s-a}{\rho}, \quad \cot\frac{B}{2} = \frac{s-b}{\rho}.$$

This is only possible if the expression under the root sign in (1) is the square of a rational number. If we now write x/y for $\cot A/2$, and replace λ by a/b, this leads to

$$a^2x^4 + 2(a^2 - 2b^2)x^2y^2 + a^2y^4 = z^2, \tag{2}$$

a, b, x, y, z all positive integers, as a necessary (and sufficient) condition.

For $\lambda = 2$ ($a = 2b$), (2) becomes

$$4b^2(x^4 + x^2y^2 + y^4) = z^2,$$

which has no solution in positive integers. (This result was already known to Euler. See pp. 19–20 of [2] for a proof.) So $\lambda = 2$ is out.

For $\lambda = 1$ ($a = b$), (2) becomes

$$[a(x^2 - y^2)]^2 = z^2$$

with an infinity of solutions in positive integers a, x, y, z.

For $\lambda = 6$ ($a = 6b$), (2) becomes

$$4b^2(9x^4 + 17x^2y^2 + 9y^4) = z^2,$$

where a solution is $x = 8$, $y = 9$, $z = 858b$.

For $\lambda = 13$ ($a = 13b$), (2) becomes

$$169b^2x^4 + 334b^2x^2y^2 + 169b^2y^4 = z^2,$$

where a solution is $x = 15$, $y = 26$, $z = 11687b$.

While there are many solutions in integers for (2), there are also many values of λ for which there is none. For examples of both types see the literature quoted on pp. 634–639 of Dickson [1].

Finally, (2) can be written

$$[a(x^2 + y^2)]^2 = (2bxy)^2 + z^2,$$

so the general solution (Pythagoras) in integers is

$$a(x^2 + y^2) = t\,(r^2 + s^2)$$
$$2bxy = t\cdot 2rs$$
$$z = t\,(r^2 - s^2)$$

with r, s, t integers and $(r,s) = 1$. So all possible "Heronian" values of λ are of the form

$$\lambda = \frac{a}{b} = \frac{xy(r^2 + s^2)}{rs(x^2 + y^2)}, \tag{3}$$

where x, y, r, s are positive integers. (3) was already known to Euler (see p. 193 of [1]), but his derivation was lost.

References:

[1] L.E. Dickson, *History of the Theory of Numbers, Vol.* II, 1920, reprint by Chelsea, New York, 1952.

[2] L.J. Mordell, *Diophantine Equations*, Academic Press, 1969.

II. *"Solution" by N. Withheld.*

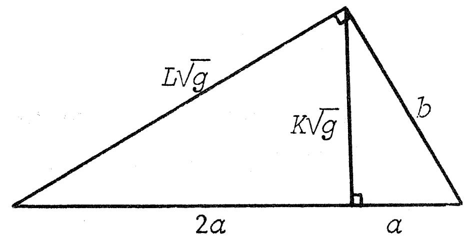
Without loss of generality, assume that the sides are integers a, b, and $2a$. Then

$$\text{Area} = \sqrt{\frac{3a+b}{2}\cdot\frac{3a-b}{2}\cdot\frac{b+a}{2}\cdot\frac{b-a}{2}} = \frac{1}{4}\sqrt{(9a^2 - b^2)(b^2 - a^2)}\,.$$

We may assume that

$$b^2 - a^2 = K^2g, \quad 9a^2 - b^2 = L^2g$$

and that we may construct right triangles $(b,a,K\sqrt{g})$ and $(3a,b,L\sqrt{g})$. However, by similar triangles $a/b = b/3a$ or $a\sqrt{3} = b$, contrary to the initial assumption that a and b are integers.



[*Editor's note*: There were altogether *three* incorrect solutions submitted for this problem. Two involved simple arithmetical errors, but the one above is more subtle. Readers might enjoy spotting the flaw.]

* * *

1194. [1986: 282] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

My uncle's ritual for dressing each morning except Sunday includes a trip to the sock drawer where he (1) picks out three socks at random, (2) wears any matching pair and returns the third sock to the drawer, (3) returns the three socks to the drawer if he has no matching pair and repeats steps (1) and (3) until he completes step (2). The drawer starts with 16 socks each Monday morning (8 blue, 6 black, 2 brown) and ends up with 4 socks each Saturday evening.

(a) On which day of the week does he average the longest time at the sock drawer?

(b) On which day of the week is he least likely to get a matching pair from the first three socks chosen?

Solution by the proposer.

We show that he averages the longest time at the sock drawer on Saturday, and that he is least likely to get a first match on Friday.

If there are ℓ blue pairs, m black pairs, and n brown pairs, then the probability of not pairing when drawing three socks is

$$q(\ell,m,n) = \frac{2\ell\cdot 2m\cdot 2n}{\binom{2(\ell+m+n)}{3}} = \frac{48\ell mn}{t(t-1)(t-2)},$$

where $t = 2(\ell + m + n)$. The probability of pairing with blue on the first draw is

$$P_1 = \frac{\binom{2\ell}{2}(2m+2n) + \binom{2\ell}{3}}{\binom{2(\ell+m+n)}{3}} = \frac{2\ell(2\ell-1)(3t-4\ell-2)}{t(t-1)(t-2)},$$

and thus the probability that the socks eventually chosen are blue is

$$P_1(1+q+q^2+\cdots) = \frac{P_1}{1-q} = \frac{2\ell(2\ell-1)(3t-4\ell-2)}{t(t-1)(t-2)-48\ell mn},$$

with analogous probabilities for black and brown.

Using the above relations and the initial state

$$(\ell,m,n) = (4,3,1)$$

on Monday, one can compute the probabilities of being in all possible states on Tuesday through Saturday. On a given morning, if the state probabilities are p_i then the probability of not getting a pair is

$$B = \sum p_i q_i,$$

where the q_i are computed for each state according to the formula for $q(\ell,m,n)$ above. On a given morning the expected time at the sock drawer is proportional to

$$A = \sum p_i d_i \ ,$$

where d_i is the expected delay for state i. If a transaction at the sock drawer (choose 3, put back either 1 or 3) takes unit time then the expected delay is determined from

$$d_i = (1 - q_i) + q_i(d_i + 1) = 1 + q_i d_i \ ,$$

so

$$A = \sum \frac{p_i}{1 - q_i} .$$

The following table gives the required numerical values, from which the stated result is clear.

Day	States	p_i	q_i	B	A
Monday	{4,3,1}	1	.1714	.1714	1.207
Tuesday	{3,3,1}	.6034	.1978	.1838	1.227
	{4,2,1}	.3664	.1758		
	{4,3,0}	.0302	0		
Wednesday	{3,2,1}	.8522	.2182	.1973	1.251
	{4,1,1}	.0782	.1455		
	{3,3,0}	–	0		
	{4,2,0}	–	0		
Thursday	{2,2,1}	.5450	.2667	.2109	1.280
	{3,1,1}	.3275	.2000		
	{3,2,0}	–	0		
	{4,1,0}	–	0		
Friday	{2,1,1}	.7684	.2857	.2195	1.307
	{2,2,0}	–	0		
	{3,1,0}	–	0		
	{4,0,0}	–	0		
Saturday	{1,1,1}	.5379	.4000	.2151	1.359
	{2,1,0}	–	0		
	{3,0,0}	–	0		

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1195*. [1986: 282] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle with medians m_a, m_b, m_c and circumcircle Γ. Let DEF be the triangle formed by the parallels to BC, CA, AB through A, B, C respectively, and let Γ' be the circumcircle of DEF. Let $A'B'C'$ be the triangle formed by the tangents to Γ at the points (other than A, B, C) where m_a, m_b, m_c meet Γ. Finally let A'', B'', C'' be the points (other than D, E, F) where m_a, m_b, m_c meet Γ'. Prove that lines $A'A''$, $B'B''$, $C'C''$ concur in a point on the Euler line of ABC.

Editor's comment.

There have been no solutions submitted for this problem.

* * *

1196. [1986: 282] *Proposed by Jordi Dou, Barcelona, Spain.*

Let I be the incentre and O the circumcentre of ΔABC. Let D on AC and E on BC be such that $AD = BE = AB$. Prove that DE is perpendicular to OI.

Editor's comment.

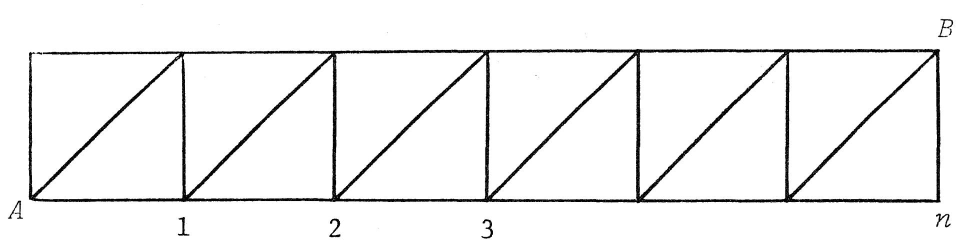
This problem has already been solved in this journal, by J.T. Groenman in his solution of *Crux* 1100 [1987: 160]. See [1987: 258] for the editor's lame excuse for publishing both problems.

Solved (this time) by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

* * *

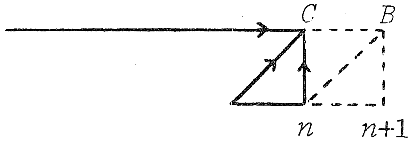
1197. [1986: 283] *Proposed by Peter Andrews and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

In the network illustrated by the figure below, where there are n adjacent squares, what is the number of paths (not necessarily shortest) from A to B which do not pass through any intersection twice?

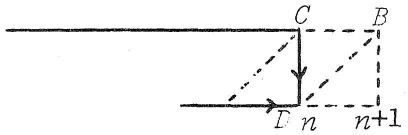


Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

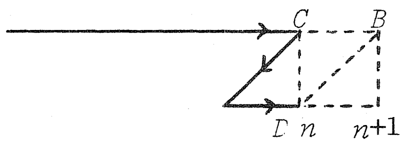
In order to get a recursion for a_n, the desired number of paths from A to B, we introduce three subtypes of paths.

(i) 

The paths from A to C which do not pass through B. There are clearly a_n of them.

(ii) 

The paths from A to D which either go directly from C to D or do not pass through C at all. We denote their number by b_n.

(iii) 

The zigzag paths from A through C to D. Their number is c_n.

For the total number we have

$$a_{n+1} = a_n + 2b_n + 2c_n. \tag{1}$$

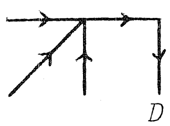
We now claim that

$$b_n = a_{n-1} + 2b_{n-1} + 2c_{n-1} \tag{2}$$

and

$$c_n = a_{n-2} + b_{n-2} + c_{n-2} \tag{3}$$

for $n \geq 2$. Since $a_0 = b_0 = 1$, $a_1 = b_1 = 3$, $a_2 = b_2 = 9$, $c_0 = c_1 = 0$, and $c_2 = 2$, (2) and (3) hold for $n = 2$. Assume $n \geq 3$. Then paths of type (ii) can arise as follows:



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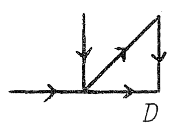

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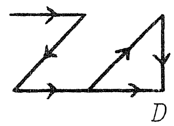
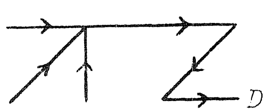


This gives (2). And the paths of type (iii) arise in the following ways:



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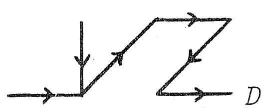
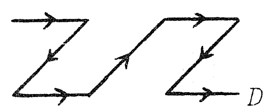


or



This gives (3). Comparing (1) and (2) we conclude that $a_n = b_n$. Therefore (1) and (3) become

$$a_{n+1} = 3a_n + 2c_n \tag{4}$$

and

$$c_n = 2a_{n-2} + c_{n-2}. \tag{5}$$

From (4) we get

$$a_{n-1} = 3a_{n-2} + 2c_{n-2}. \tag{6}$$

Subtracting (4) and (6) yields in view of (5)

$$\begin{aligned} a_{n+1} - a_{n-1} &= 3a_n - 3a_{n-2} + 2(c_n - c_{n-2}) \\ &= 3a_n - 3a_{n-2} + 4a_{n-2}, \end{aligned}$$

so

$$a_{n+1} = 3a_n + a_{n-1} + a_{n-2}, \qquad n \geq 2. \tag{7}$$

Using $a_0 = 1$, $a_1 = 3$, $a_2 = 9$, recursion (7) defines a_n for all $n \geq 3$. For example, $a_3 = 31$, $a_4 = 105$, etc.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

Three other readers, ***and the proposers****, all arrived at the incorrect answer of* $a_n = 3^n$, *due to not considering the zigzag paths in* (iii) *above. If only* Crux *were a democracy... .*

*   *   *

WORDS OF MILD ALARM FROM THE EDITOR

Despite its evident popularity and a recent modest increase in subscription rates, *Crux Mathematicorum* is currently losing money. This situation cannot continue indefinitely because our publisher, the Canadian Mathematical Society, requires that all of its publications break even. I urge you to help us find more subscribers. High school teachers can join the Canadian Mathematical Society for $20, receive the *Notes of the CMS* as a privilege of membership, and then subscribe to *Crux* for just another $15. A subscription without CMS membership costs $30, surely a remarkably low price for a publication of such wide interest. A cheque to the Managing Editor, whose address appears on the inside front cover of this issue, will start the flow of *Crux* at once. In the back of future issues we will include a form, which can be conveniently removed and duplicated if necessary, and which your friends and colleagues can use to obtain a subscription to *Crux* with or without CMS membership. Please do what you can to ensure the future of *Crux Mathematicorum.*

And while I have your attention, remember that *Crux* is always, and especially lately, in need of nice accessible problems, especially in areas such as number theory, combinatorics, or high school mathematics. *Crux* has a small, but dependable, group of reader-proposers whose membership can be determined by anyone who regularly scans the problem section, and whom the editor excepts with gratitude from the next sentence. All other readers of *Crux* should resolve to contribute at least one problem sometime this year.

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