MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH

Mathematical Spectrum Awards for Volume 17

This year the editors have decided to give three prizes to students. The prize for an article has been awarded to Liu Zhiqing for 'Pascal's Pyramid' on pages 1 to 3; prizes for solutions to problems and other contributions have been sent to Richard Dobbs and Ruth Lawrence.

In future, awards of up to £30 will be available for articles and of up to £15 for letters, solutions to problems, or other items. To qualify, contributors must be students at school, college or university.

Prime Beef

K. DEVLIN, *University of Lancaster*

Keith Devlin writes a regular mathematical column in the *Guardian*: this article is one which originally appeared there, and is reproduced here with permission. He also acted as consultant for the recent 'Horizon' programme on mathematics which appeared on BBC2 television.

Let me begin by quoting a formula. Given positive whole numbers M and N, calculate

$$K = M(N+1)-(N!+1),$$

and then calculate

$$P = \tfrac{1}{2}(N-1)[|K^2-1|-(K^2-1)]+2.$$

For any values of M and N, the value of P you obtain from the above formula will be a prime number. Moreover, every prime number will be a value of P for some values of M and N. So the formula generates all the primes, regardless of anything to the contrary that you may have read in either the popular press or elsewhere. Make no mistake about it, the primes can be generated by a simple formula.

For most values of M and N you find that $P = 2$. But if you are patient, the other primes do appear. For instance, for $M = 1$, $N = 2$ you get $P = 3$; for $M = 5$, $N = 4$ you get $P = 5$; for $M = 103$, $N = 6$ you get $P = 7$. To get $P = 11$, however, you need to be very patient indeed: this does not happen until you try $M = 329\,891$ and $N = 10$.

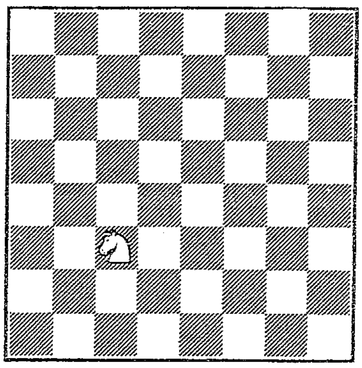
It is easy to write a computer program to calculate P using the above formula and check these results. But even without doing that, it seems clear that this is not an efficient way of generating primes. Most of the time you get the same output, 2, and only at rare intervals do you get any other output. (Question: where do you find $P = 13$?) The importance of the formula is that it demonstrates that the primes are capable of being generated to a formula, albeit inefficiently.

It is important to devise efficient methods of testing a given number to see if it is prime. The 'obvious' method of looking at all smaller numbers and checking that none of them divides it exactly (other than 1, which divides everything, of course), whilst perfectly adequate for computer use when the number concerned is not too large (say not larger than the word size of the computer), is hopelessly inefficient for very large numbers, say of 50 digits or more.

The fast primality tests being used are rather sophisticated mathematically, but they all start from the same idea, which goes back to the seventeenth-century French (amateur) mathematician Pierre de Fermat. He showed that if you take a number P which is prime, with $P > 2$, then the number $2^{P-1}-1$ is exactly divisible by P. For large numbers P, it is much quicker to calculate the number $2^{P-1}-1$ and see if P divides it than to look for factors of P. If P does not divide this number, then P cannot be prime. So this gives you a quick way to check that P is not prime. Unfortunately, you cannot conclude that if P does divide $2^{P-1}-1$ then P necessarily is prime. It probably is, but there are some numbers which are not prime and yet you do get divisibility here.

This provides a nice exercise to keep your micro busy. Find non-prime numbers P for which P divides $2^{P-1}-1$.

A Knight on a Chessboard



Is it possible for a knight to visit each square of a chessboard exactly once? Can it be done in such a way that, having reached the last square, one more move will take it back to its starting square? If the squares are labelled 1 to 64 in the order in which the knight visits them, can it be done in such a way that all sixteen row and column sums are the same?

A Problem Shared

K. AUSTIN, *University of Sheffield*

Keith Austin is a lecturer in pure mathematics at the University of Sheffield. He is interested in algorithms in mathematics and logic. This is a subject which has developed considerably since the Second World War, with the advent of the computer. Apart from his family and his work, his main interest is in his local methodist church.

Miss Maple was removing the greenfly from her rose trees when the inspector called.

'Good morning, inspector. Isn't it a lovely day?'

The lack of enthusiasm in his reply prompted a further comment on the weather.

'Come now, sunshine is not rationed; you don't need any coupons for a beautiful blue sky. Remember how we used to look forward to the days when the war clouds had cleared and we would see the sun again? Well, they have been gone this past year or two, but I suspect you have replaced them with some new clouds; you have a problem, haven't you?'

'Yes, Miss Maple, I have, and I have searched everywhere for a solution. To be honest, you are my last hope.'

'We shall go inside and you can tell me about it over a cup of tea.'

'There have been a number of unpleasant goings-on at Lower Woodney,' said the inspector as he took a piece of fruit cake. 'I need not worry you with the details, but the upshot is that we took a number of statements. Each witness listed three people and swore that they had seen at least one of them in the village whist drive at 8 o'clock on Wednesday last.

'Miss Potter, the schoolmistress, saw Mr Thetch or Mr Lamber or Mrs Meadow.

'Mr Copse, who runs the Post Office, saw Mr Dairman or Mrs Lamber or Mrs Meadow.

'Mrs Beech, the cook at Woodney Hall, saw Mrs Lamber or Mr Dairman or Mrs Dairman.

'Now each of the families mentioned has young children and so in each case the father or mother will have been at home. What I hoped to show was that it was not possible for all the witnesses' statements to be true and this would force one of them to confess to lying and allow us to get to the bottom of the matter. Unfortunately my plan has failed; as you can see, there are many possibilities which allow the statements to be true. For example, if Mr Thetch and Mr Dairman were at the whist drive then all three witnesses are telling the truth.'

'Will you have another cup of tea, inspector?'

'Thank you, Miss Maple, that would be very nice.'

'What do you plan to do now?'

'I was going to take another piece of fruit cake.'

'I was referring to Lower Woodney,' said Miss Maple in a rather frosty tone.

'The plan required a larger number of statements, and so my men have interviewed a lot more people who were at the whist drive. Each statement is similar to the earlier ones, that is, the witness saw at least one of the three people he or she names. Also each of the people named is the father or mother of young people.

'Thus I again have the possibility of showing that not all the witnesses are telling the truth. Unfortunately this is where my problem arises. The statements involve the parents of 30 families and with such a large number I don't know how to check if all the statements could be true.'

'There are two possibilities for each family, namely father or mother goes to the whist drive, which gives 2^{30} possibilities in all to check, inspector.'

'That is correct, Miss Maple. When I got to this point I decided to come to you for help.'

'Let me fill up your cup and pass you a biscuit, before we tackle your problem.

'The task of checking all the possibilities is straightforward and mechanical, which suggests that it should be done by a machine.'

'What sort of machine? A tractor or a bacon slicer?'

'Don't be silly. We want a calculating machine but one which does a little more than just arithmetic. Have you anything you can suggest?'

'Well, the police station at Brackford does have some sort of gadget that works by electricity and tells them where to position their men on the roads.'

'That sounds as if it is what we need. Can we go over and see it this afternoon?'

'Certainly, Miss Maple. Shall I call for you at 2 o'clock?'

'Constable Brown, will you explain to Miss Maple and the inspector what our machine does?'

'Certainly, sir. The machine was designed by a Mr Turing at the end of the war. He had been stationed up at Bletchley Hall for the duration, on some secret work. However, he was a keen runner and often when I was cycling round the lanes on my beat I would meet him and ride along with him, sort of pacing him.

'One day I told him of our security problem whenever the top brass came to visit the Hall. We were supposed to station men at various road junctions so that every stretch of road had a man at one end or the other. As our numbers were limited we were never sure we could do it.

'Mr Turing suggested we needed a machine to check through all the possible ways of stationing our men and, if it found a suitable way, to tell us.

'He drew up the plans for it and some of his colleagues at the Hall built it for us. We have been using it ever since.'

He pointed to a large array of valves and wires.

'If you want to station a certain number of men so as to cover certain roads then we first connect up the wires to indicate how many men and the layout of the roads. Then we switch on and wait. After an hour or so, the time depends on how many men and roads, either this bulb lights up, which means we cannot cover the roads with that number of men, or some of those bulbs light up, which tell us where to place the men.

'Mr Turing said it was more than a calculating machine as it had to check each possible placing in turn, and decide each time whether that placing covered the roads or not. He said it was computing and called it a computing machine.

'Are you wanting to use our machine?'

'No, I am afraid our problem is about statements, not about roads. Don't you agree, Miss Maple?'

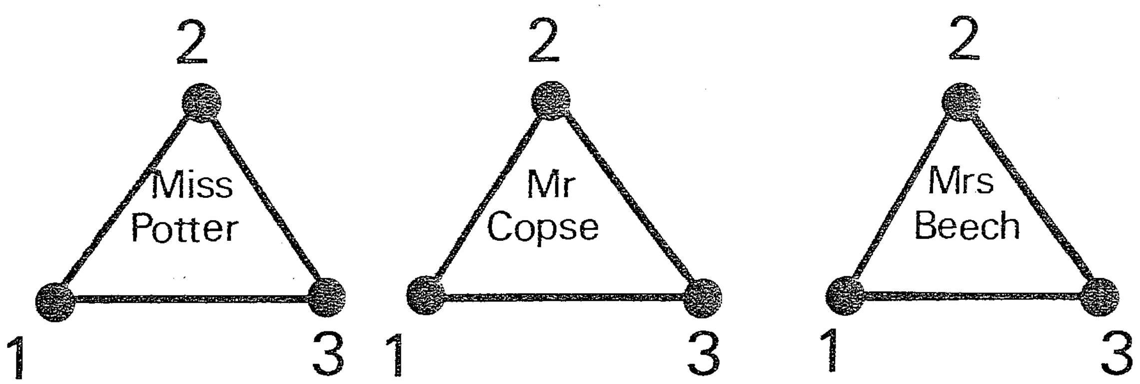
'I am not so sure. We want a machine to check a lot of cases, and that is just what this does. Perhaps we can formulate our statement problem as a road problem. Let me have a few minutes to think about it. Can we meet here again in one hour?'

'I think I have a solution,' said Miss Maple as the three of them came together again. 'Let me show you how it works with the three statements you brought me this morning. I will use them to draw a map as follows.

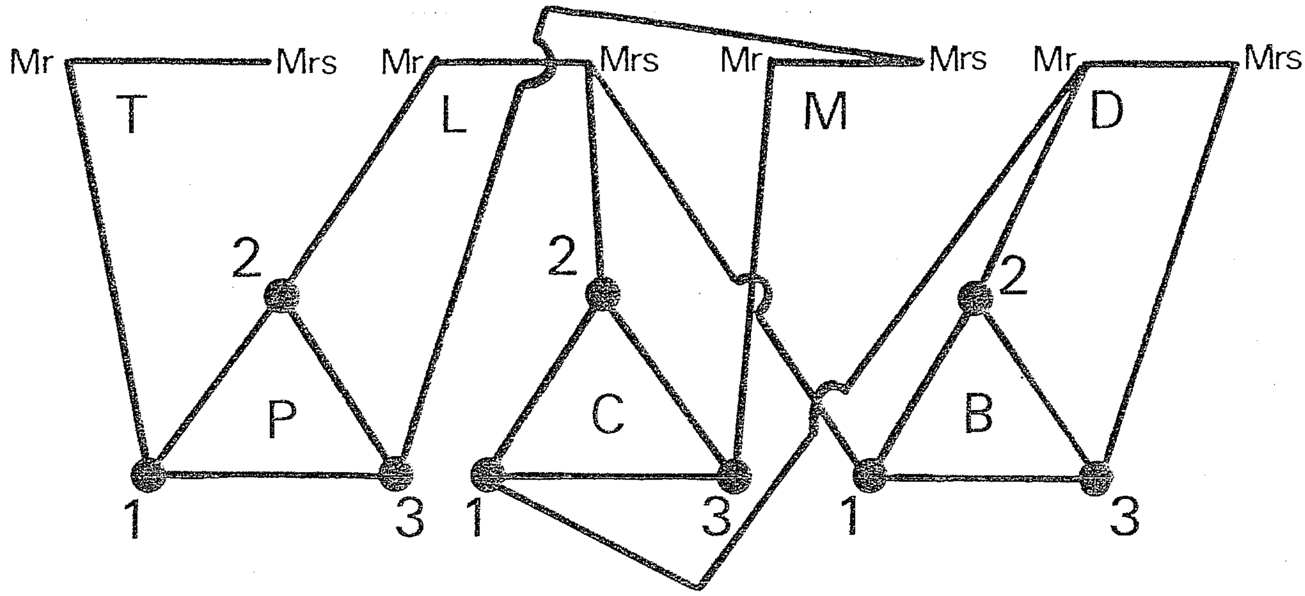
'There are four families, Thetch, Lamber, Meadow and Dairman, so I draw a road for each and mark one end as Mr and the other as Mrs:

Mr $\underset{\text{Thetch}}{\text{———}}$ Mrs   Mr $\underset{\text{Lamber}}{\text{———}}$ Mrs   Mr $\underset{\text{Meadow}}{\text{———}}$ Mrs   Mr $\underset{\text{Dairman}}{\text{———}}$ Mrs

'There are three statements so I draw a triangle of roads for each and number the three junctions 1, 2 and 3 denoting the first, second and third person mentioned in the statement, respectively.



'Finally I join the two parts together as follows. Miss Potter's first name is Mr Thetch, so I draw a road from 1 on Miss Potter's triangle to Mr Thetch. Similarly for all the others.'



'Miss Maple, you have your roads, but what about your policemen?'

'I add the number of families to twice the number of statements to get $4+6 = 10$ policemen. Now, Constable Brown, could you plug in your wires so that the machine solves that road problem?'

'Certainly, Miss Maple.'

After connecting up the wires, the constable switched on and the machine began to hum.

'This may take some time. Would you and the inspector care to come this way for a cup of tea, Miss Maple?'

When they returned the humming had stopped and a number of bulbs were lit. The constable looked carefully at them.

'One possible solution is to put the police at Mr T, Mr L, Mrs M, Mr D, P1, P2, C2, C3, B1, B3.'

'Then one possible solution for our statement problem is that Mr T, Mr L, Mrs M and Mr D went to the whist drive,' said Miss Maple.

'That is because there must be one policeman on each family road and two on each statement triangle,' agreed the inspector. 'However, suppose there had been no solution to the road problem, could we then say there was no solution to the statement problem?'

'Yes, for a statement solution readily converts into a road solution; the process is more or less the reverse of the above conversion,' she replied.

'So, Miss Maple, I can take my 30-family problem and solve it by using the road machine.

'There is just one thing that I don't understand. Surely when a machine is built to do one job, it is curious that we should be able to use it for another.'

'When Mr Turing was designing his computing machine he talked about building a machine which was designed with no particular job in mind. He called it a general-purpose machine. The idea is that, whenever we have a problem, we simply connect the wires in the appropriate way and then it works away and produces a solution. Mr Turing and his colleagues believed that such a machine would be built within the year.'

'It all sounds very exciting, constable. It really is a historic moment, the creation of a machine of universal adaptability.'

'How can a machine adapt to anything we ask? Tell me that, Miss Maple.'

'You said, 'we ask', inspector. That is the point. Just as we have to connect the wires carefully to specify our map, so we will have to set up the general-purpose machine precisely to answer the question we have in mind.

'However, I do have one worry. Most machines can be used for good or ill. For example, the telephone can be used for contacting distant relatives or for summoning help from the fire service or your men, inspector. On the other hand, it is often used in the village for a purpose that its inventors never intended, the spreading of malicious gossip.

'Now that Mr Turing has built adaptability into his machines, his next task is to incorporate a sense of right and wrong into them. What do you think, inspector?'

'I would say that is a very difficult problem, Miss Maple.'

'Oh dear, inspector. You have come full circle since this morning and you are still faced with a problem. When someone has as many problems as you, there are only two possible answers. One is to get a general-purpose machine, and the other is to take up gardening to take your mind off your problems.'

'Is that why you were tackling the greenfly?'

'Oh, thank you for reminding me. I must be getting back to them. Can we be going, inspector?'

As they were getting into the inspector's car, Miss Maple remarked, wistfully, 'I wonder what the general-purpose machine will recommend for greenfly?'

Further Reading

More examples of the technique of connecting different problems together are given in *Computers and Intractability: A Guide to the Theory of NP-completeness* by Michael Garey and David Johnson, W. H. Freeman and Co., 1979.

Ampleforth's Answers

The following problems were posed in Ampleforth College's *Mathematics and Science Society Bulletin*, and are reproduced here by the kind permission of the editor of the *Bulletin*. See how you get on. We should be pleased to hear of any other such magazines produced by schools.

1. A perfectly analogue clock explodes when the minute hand is exactly on top of the hour hand. After the explosion only the second hand remains in place and it reads between 5 and 6 seconds past the minute. What was the time when it exploded?

2. Each letter stands for a different digit:

$$\begin{array}{r} \mathsf{LPBRR}\ + \\ \mathsf{TLSRR}\ + \\ \underline{\mathsf{SBRRR}} \\ \underline{\underline{\mathsf{RLMPL}}} \end{array}$$

What is the actual sum?

3. I am trying to cover a square of side a with a number of smaller squares of a different size, without any overlapping. What is the least number of such squares and how is it done?

4.

$$[3(230+t)]^2 = 492a04.$$

What are the integers t and a ?

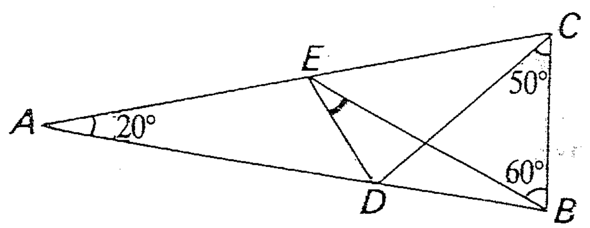
5. A group of explorers lost on an otherwise uninhabited planet similar to our own sent this last message:

'Last week we left base camp, we drove 1000 miles south, 1000 miles east and 1000 miles north and we found ourselves back at base camp.'

The rescue team sent to look for them thoroughly searched the North Pole for the camp but was not successful. Where could it be?

6. What is the last digit of 3^{55} ?

7. Triangle ABC is isosceles with $AB = AC$ and D and E are as in the diagram. Find angle DEB.



The Distributive Law in Geometry

GUIDO LASTERS, *Secondary Modern School, Sint Truiden, Belgium*

The author is a teacher of mathematics.

We are all familiar with the so-called distributive law in algebra for working out brackets, namely

$$x.(y+z) = (x.y)+(x.z).$$

A similar law makes an appearance in elementary geometry in the following way. Given points X and Y in the plane, we denote by $X+Y$ and $X.Y$, respectively, the points U and V on the line segment joining X and Y such that the ratios $XU:UY$ and $XV:VY$ are given fixed positive quantities. Figure 1 now illustrates the distributive law in this geometrical situation. Elementary properties of similar triangles give that the point T shown is both $X.(Y+Z)$ and $(X.Y)+(X.Z)$.

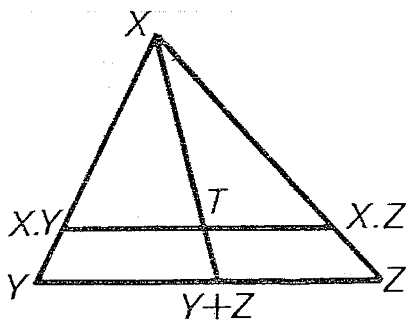


Figure 1

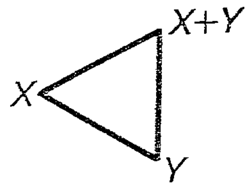


Figure 2

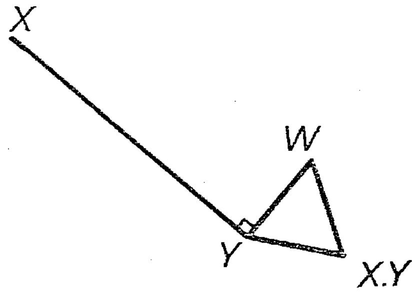


Figure 3

As a second illustration, the point $X+Y$ is now defined to be that point in the plane such that X, Y and $X+Y$ form an equilateral triangle with X, Y, $X+Y$ in anticlockwise order (see figure 2). The point $X.Y$ is defined by first constructing the point W such that the lines XY and YW are at right angles with $YW = \frac{1}{3}XY$ and with X, Y, W in anticlockwise order; then $X.Y = W+Y$ (see figure 3). Then it is quite easy to convince yourself that, for three points X, Y and Z,

$$X.(Y+Z) = (X.Y)+(X.Z).$$

Readers may like to prove this also by means of complex numbers in an Argand diagram. The particular factor $\frac{1}{3}$ is immaterial; it can be altered at will.

On A Sequence Of Trees With Fibonacci Weights

JOHN C. TURNER, *University of Waikato*

John Turner lectures in statistics and operations research. He completed his bachelor's degree (1954), and master's degree (1966) in reliability theory, at Leeds University, England. His recent researches have been in knot theory (D.Phil., Waikato University, Hamilton, New Zealand, 1984), and in random processes on graphs. He complements his mathematical work with strong interests in music-making, on a variety of instruments. He is past-president of both the N.Z. Mathematical Society and the N.Z. Federation of Classical Guitar Societies.

1. Introduction

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ has been studied by mathematicians for over 750 years. It takes its name from the famous Italian mathematician known as Leonardo of Pisa (Fibonacci is an abbreviation of *filius Bonacci*, meaning 'son of Bonacci'). He wrote a voluminous mathematical work in 1202, in which one problem studied was introduced by the question: 'How many pairs of rabbits are born of one pair in a year?' After properly formulating the question, so that mathematical reasoning could be applied, the answer he obtained was that the numbers of pairs at the end of each month follow the number sequence given above. Note that the sequence is continued according to the rule

$$f_n = f_{n-1} + f_{n-2}.$$

That is, any number in the sequence (after the first two) is the sum of the previous two.

Since the 13th century, very many remarkable properties of Fibonacci numbers have been discovered. There is a current journal, *The Fibonacci Quarterly*, in which new results about them and about a great variety of related number sequences and functions are published four times a year.

In this paper we describe the construction of a sequence of weighted tree graphs, denoted by $F_1, F_2, \ldots, F_n, \ldots$. We shall show that these trees have many properties related to the Fibonacci sequence.

Using these trees we have discovered many more properties of Fibonacci numbers than we can describe here. Student readers are invited to try to discover new properties for themselves, and they can begin by supplying proofs of the results quoted below. We have omitted proofs, again for reasons of space; but they may all be constructed using elementary number and graph theorems, or inductive reasoning on the manner of construction of the trees.

Before describing the trees themselves, we define the concept of *convolution of two sequences*, and give the example which we shall need later.

2. Convolutions

Consider the two infinite sequences of integers

$$u = \{u_1, u_2, u_3, \ldots\} \quad \text{and} \quad v = \{v_1, v_2, v_3, \ldots\}.$$

The *convolution* of u and v is written $u*v$, and it is the sequence of integers

$$u*v = \{u_1v_1, u_1v_2+u_2v_1, u_1v_3+u_2v_2+u_3v_1, \ldots\},$$

with general term

$$\{u*v\}_n = u_1v_n+u_2v_{n-1}+\ldots+u_nv_1.$$

Our example considers the case where both u and v are the Fibonacci sequence. Then

$$\begin{aligned} f*f &= \{1,1,2,3,5,\ldots\}*\{1,1,2,3,5,\ldots\} \\ &= \{1,2,5,10,20,\ldots\}. \end{aligned}$$

We shall denote the nth term of this sequence by either c_n or $(f*f)_n$. Thus, for example, from $\{u*v\}_n$ above we see that

$$c_4 = (f*f)_4 = 1\times3+1\times2+2\times1+3\times1 = 10.$$

The reader is invited to prove that terms of the convolution sequence satisfy the recurrence relation $c_n = c_{n-1}+c_{n-2}+f_n$, with $c_1 = 1$ and $c_2 = 2$.

3. Fibonacci convolution trees

We begin by mentioning a few properties of graphs and trees. First, a graph is a set of *nodes* (otherwise called points, or vertices), and *edges* (otherwise called lines, joins, or arcs) which join certain pairs of the nodes. If it is possible to get from any node in the graph to any other node by stepping along edges in the graph, then the graph is said to be *connected*. The *valency* of a node is the number of edges joined to it.

A *tree* is a connected graph with no circuits; or equivalently, it is a connected graph which has n nodes and $n-1$ edges for some n. If we single out one of the nodes of a tree, and call it the *root* (perhaps putting a ring around that node to mark it), then we have a *rooted tree*. In a tree there is always a set of nodes which have valency 1. These nodes (other than root, if that is in the set) are called *leaf-nodes*. If numbers, or letters, are attached to the nodes, we obtain a labelled, or *weighted*, tree. Figure 1 gives an example of such a tree; its root is the node weighted 0.

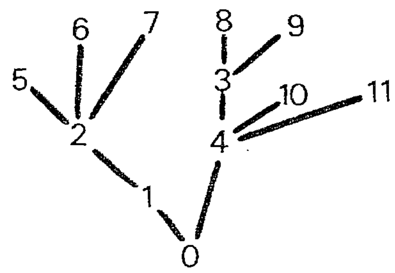


Figure 1. A tree with 12 nodes, 11 edges, 7 leaf-nodes and weights $0, 1, 2, \ldots, 11$

Finally we can describe our so-called Fibonacci convolution trees. The first five in the sequence $F_1, F_2, F_3, \ldots$ are shown in figure 2. The lowest nodes are roots; notice that their weights are the Fibonacci numbers.

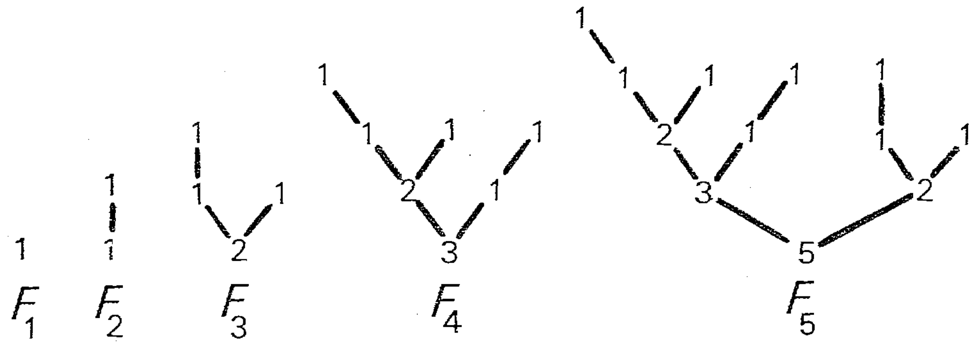


Figure 2. The first five convolution trees

The construction rules for convolution trees are simple. Leaving aside the attachment of weights for the moment, examination of the structure of the trees reveals that each one (after F_2) is obtained by combining the previous two and placing them together on a new stem and root. Thus, in general, to obtain F_n we must take the previous trees F_{n-1} and F_{n-2}, and combine them as in figure 3. Note how similar this procedure is to that used to obtain the Fibonacci numbers, namely to compute $f_n = f_{n-1} + f_{n-2}$.

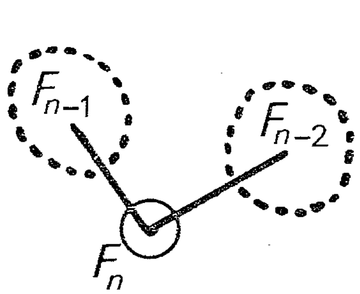


Figure 3. Combination of F_{n-1} and F_{n-2} to form F_n

We now observe that the weights of the nodes are attached in almost exactly the same way, beginning with the weights shown for F_1 and F_2. The only extra thing to do is that each time a new tree is formed (say F_n), one new weight must be introduced: the number f_n is attached to the new root-node.

4. Some properties of the convolution trees

Table 1 gives the formulae which may be obtained for the parameters listed in the left-hand column. Many of the parameters have values expressible very simply in terms of the Fibonacci numbers.

TABLE 1. Parameter values for F_n

PARAMETER		VALUE FOR TREE F_n
Number of nodes		$f_{n+2}-1 = \sum_{i=1}^{n} f_i$
Number of edges		$f_{n+2}-2$
Number of nodes	$d = 1$	f_n
of valency d	$d = 2$	$f_{n-1}+1$
$(n > 2)$:	$d = 3$	f_n-2
Number of leaf-nodes		f_n
Sum of all node weights		$(f*f)_n$

Proofs of the formulae given in table 1 are easily obtained from elementary theorems on graphs and/or from the recursive manner of construction of the trees. The last proof follows easily from the formula $c_n = c_{n-1}+c_{n-2}+f_n$ that we gave above for convolutions. Student readers are invited to prove them for themselves.

5. Projects

The following work would set an interested student on the way to making further (possibly new) discoveries about the integers and Fibonacci numbers.

(i) Construct the next three convolution trees, F_6, F_7 and F_8.

(ii) Check for these three trees that all the formulae given in the table 'work'.

(iii) Take any tree (say F_5) and 'prune' it; that is, cut off all its leaf-nodes and their joining edges. Study the effects of this operation and try to generalise your findings to the case F_n.

(iv) Again take tree F_5 and do the following. Start at the right-most leaf-node, then trace a path along the edges to the root. Add up all the weights on the nodes you visit. How many leaf-to-root paths are there? What do you discover about the sums of path weights? Generalise your discoveries to all trees in the sequence.

(v) Over to you! There are many more things you can try to do with convolution trees, to unlock the secrets of number sequences.

I welcome correspondence on discoveries made about these trees and shall be pleased to supply copies of papers on my own findings.

Squares which use the first nine natural numbers

$139\,854\,276 = 11\,826^2$	$549\,386\,721 = 23\,439^2$
$152\,843\,769 = 12\,363^2$	$597\,362\,481 = 24\,441^2$
$215\,384\,976 = 14\,676^2$	$627\,953\,481 = 25\,059^2$
$326\,597\,184 = 18\,072^2$	$735\,982\,641 = 27\,129^2$
$361\,874\,529 = 19\,023^2$	$842\,973\,156 = 29\,034^2$
$412\,739\,856 = 20\,316^2$	$923\,187\,456 = 30\,384^2$
$523\,814\,769 = 22\,887^2$	

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Keshabpur,
Jessore,
Bangladesh.

1986

In the previous two years, we have challenged readers to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations +, −, ×, ÷, √, ! and brackets. For example, for 1985,

$$2 = [1\times(\sqrt{9})!]\div(8-5), \qquad 92 = -1+98-5.$$

How about 1986? Readers who manage more than 90 of the numbers are invited to send in their solutions. (The editor failed on 10 of them!)

The Planning of Regression Studies

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1. Introduction

It is a sad but true reflection on the way that statistics is taught, both at school and university, that the subject and people who practise it are misunderstood. This attitude seems to stem from a belief that it is something of a cookbook subject in which techniques are set out to permit analysis of data in the 'best' possible way, and that these techniques were given to us by statisticians who, like magicians, draw white rabbits from top hats! I intend to take a specific area of statistics, regression, and try to indicate that there is much more to being a statistician than blind application of formulae. I shall suggest that a statistician can have a strategic role to play in the collection of data to make efficient use of available experimental resources.

What is regression, and why do we study it at all? The fact is that in all walks of life we encounter problems where variables that we measure are known to be related. In physics we know that the period of a pendulum is related to its length; in agriculture the yield of barley is known to be dependent upon the amount of fertiliser added to the soil; in medicine it is known that the value of our systolic blood pressure is related to our age. The *extent* of these relationships could be determined by collecting some data and calculating the correlation coefficient, but this statistic is of limited use in practice. What is required is a means of *relating* the variable which is of interest, for example the systolic blood pressure, to the value of the quantity which helped to determine it, namely the age of the patient. The form of this relationship can be found by collecting data in an appropriate way and then by *regressing* the response variable, y, on the stimulus variable, x, or appropriate functions of it. In the simplest case, that of simple linear regression, we consider the estimation of a relationship $y = \alpha + \beta x$, where α is the intercept, the value of y when $x = 0$, and β is the slope of the line. It is true that, given a set of data, the estimates a and b of α and β can be obtained by blind application of formulae. However, there are two additional facets to this problem which must be emphasised and understood.

(1) Why have we estimated this relationship at all? The answer may well be 'for the purposes of prediction'. For example, an individual aged 40 who presents himself at a doctor's surgery and has his blood pressure taken must have confidence that the doctor knows whether the value obtained is normal or abnormal for a person of his age. From the fitted relationship obtained

above the prediction of the normal value would be $\hat{y} = a+40b$. The assessment of whether or not the measured value was abnormal would be made taking account of the uncertainty present in the predicted value, which arises because of uncertainty in the values of the estimates of a and b. These estimates have been obtained using data from selected people, and we know that, even for people of the same age, the values of their blood pressure will differ.

(2) Imagine that you are a scientist and have been requested to estimate the relationship between blood pressure and age, which is believed to be a straight line. The results of your work are to be used as a basis for future prediction on people between the ages of 16 and 64. You are told that you are able to use a maximum of twelve normal individuals to determine this relationship and that you have the freedom to select the ages of the people that you will use. How would you carry out this selection process in order to estimate this straight line as accurately as possible? Indeed, does it matter *what* the ages are of the people you select? Some people might be surprised to learn that this is as much a statistical problem as is the fitting of the relationship.

In the remainder of this article I shall review the ideas underlying simple linear regression and give an example. I shall then discuss principles which provide objective means of choosing experimental points and compare the relative efficiencies of several choices of design. The results are drawn together and some further examples are provided for the reader to consider.

2. Review of simple linear regression

A set of data is collected consisting of n observations of the values of a stimulus variable x_i and the corresponding values y_i of the response variable. To estimate the parameters α and β of the straight line $y = \alpha+\beta x$ a standard procedure is to assign values a and b to α and β so that the sum of the squared vertical distances between the observations and the fitted line is as small as possible. When this procedure, which is called *least squares*, is carried out, we find that β is estimated by

$$b = \frac{\sum_{i=1}^{n} (x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n} (x_i-\bar{x})^2}, \qquad \text{where} \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

and α is estimated by $a = \bar{y}-b\bar{x}$. Thus the fitted line is given by $\hat{y} = a+bx$, and this can be used for prediction. To discuss the inherent precision of this estimated response for a given value of x we must recognise that the values of the observed response y for a number of observations with the same value x

will not all be the same. This variation is referred to as *random variation*. We denote its magnitude by σ^2 and estimate it by

$$s^2 = \frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2.$$

This is essentially an average value of the squared discrepancies between the observations and the fitted line. The use of $n-2$ on the bottom line can be justified in a commonsense way by saying that we have n observations which have already been used to estimate two quantities α and β. We say that we have used up two degrees of freedom in the data. The resultant formula is justified on statistical grounds because it yields an unbiased estimate of the true variance between individuals.

From the fitted line we can calculate that, on average, observations with a stimulus value of X have a predicted response of $\hat{y} = a+bX$, and the uncertainty associated with this prediction is

$$\operatorname{var}(\hat{y}) = s^2 \left\{ \frac{1}{n} + \frac{(X-\bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right\}.$$

Equivalently, this uncertainty is sometimes expressed in the form of the standard error, which is the square root of the $\operatorname{var}(\hat{y})$ given above.

3. Example

Table 1 gives the values of systolic blood pressure (y) and age (x) for a sample of 12 normal individuals. From this set of data we can compute $b = 0.98$ and $a = 102.1$ and hence the fitted line $\hat{y} = 102.1+0.98x$. The data and the fitted line are shown in figure 1. The estimate of the random

Table 1
Data on systolic blood pressure and age of twelve individuals

Person Number	Systolic blood pressure (y)	Age (x)	Person Number	Systolic blood pressure (y)	Age (x)
1	114	17	7	160	44
2	124	19	8	142	50
3	125	25	9	158	53
4	130	29	10	148	56
5	136	36	11	162	65
6	144	39	12	175	69

variation present in these data is given by $s^2 = 49.15$. Thus for individuals aged 40 the prediction of the mean normal systolic blood pressure is $\hat{y} = 141.3$ and the uncertainty present in this prediction is $\mathrm{var}(\hat{y}) = 4.15$.

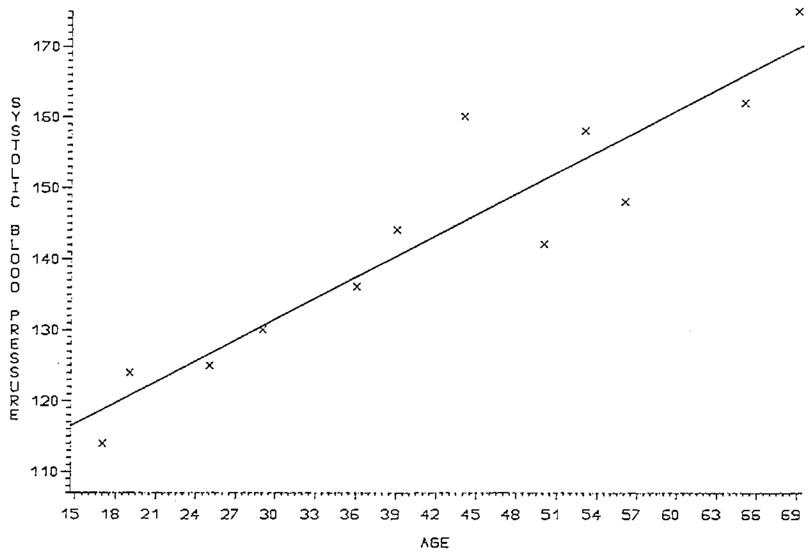

Figure 1. Relationship between systolic blood pressure and age

From these results we could obtain a 95% confidence interval for the mean systolic blood pressure of normal individuals aged 40 as $141.3 \pm 2.23\sqrt{4.15}$, that is between 137.2 and 145.4. For individuals aged 60 the mean prediction is 161.0, with a 95% confidence interval from 155.0 to 167.0. The question which remains to be answered is whether, by careful selection of the ages of these twelve individuals, we could have obtained a more precise prediction of $\hat{y}$.

4. Design principles

There are two fundamental questions to be answered in the problem posed above.

(a) Do I believe the relationship is a straight line, and if so, how should I choose my experimental points?

(b) If I have some doubt about whether the relationship is a straight line, what should I do?

In case (a), where we believe the relationship between systolic blood pressure and age is well represented by a straight line, our objective is to obtain predicted values which are as accurate as possible for people between the ages of 16 and 64. We have to choose 12 people of known age from whom to estimate the relationship. How should this choice be made? Before you read on, answer the question yourself, ask your friends, your teacher (or your class) and record their answers.

It is likely that the most popular answer will be, 'If I could use thirteen people I would choose them with ages equally spaced between 16 and 64, namely 16, 20, 24, …, 64.' Let us look briefly at the theoretical result and compare the results with the above suggestion omitting the observation at age 40, which I call *design D1*. You can make the same comparison with any other designs you have obtained. If prediction is to be accurate, the variance of the predicted value should be small. Thus we need to minimise

$$\operatorname{var}(\hat{y}) = s^2 \left\{ \frac{1}{n} + \frac{(X-\bar{x})^2}{\sum_{i=1}^{n} (x_i-\bar{x})^2} \right\} = ks^2.$$

The control we have is over the values of $x_1, \ldots, x_n$, and, if through our choice we are to minimise the value of $\operatorname{var}(\hat{y})$, thereby producing an 'optimal' design D_{opt}, we must maximise the value of

$$\sum_{i=1}^{n} (x_i-\bar{x})^2.$$

(Note in passing that the value of s^2 is independent of the choice of the x's.) You can show by algebraic expansion or some simple calculus that this maximisation occurs when $\bar{x}$ is in the middle of the range of interest (here it is 16–64 years) and when half of the observations are chosen at one end of the range and half at the other end! If this is the case, then

$$\sum_{i=1}^{12} (x_i-\bar{x})^2 = 6912 \quad \text{and} \quad k_{\text{opt}} = \frac{1}{12} + \frac{(X-40)^2}{6912}.$$

By contrast, twelve observations taken at intervals of four years from 16 to 64, omitting the age of 40, give a value of

$$\sum_{i=1}^{12} (x_i-\bar{x})^2 = 2912.$$

Note that, for each of these choices of observations, $\bar{x} = 40$, and for prediction of results for people of age 40 they are indistinguishable, since $k = \frac{1}{12}$ in each case. However, if prediction is required for people of age other than 40, the design with six observations at each end of the age range is better as the age of the people moves away from the mean chosen value of 40. This can be seen in table 2, where we define the efficiency of any design, D, relative to the optimal design as

$$\text{relative efficiency} = \frac{k_{\text{opt}}}{k_D}.$$

Table 2
Relative efficiency of design $D1$ to the 'optimal' design

Age	16	20	24	28	32	36	40
Relative efficiency	0.59	0.64	0.70	0.78	0.88	0.96	1.00
Age	44	48	52	56	60	64	
Relative efficiency	0.96	0.88	0.78	0.70	0.64	0.59	

This, of course, is not the whole story. Case (b) above poses the problem of what to do if we are in any doubt about the straight-line relationship. It will be obvious that the 'optimal' design is useless if the true relationship is curved, because it is quite unable to provide any information about the form of the curvature. We therefore need to take observations at more than two points, whilst at the same time not losing too much efficiency in prediction if the relationship is indeed straight. Many designs could be chosen, but two simple extensions are:

$D2$: Three observations at each of 16, 32, 48, 64, for which

$$\sum_{i=1}^{12} (x_i - \bar{x})^2 = 3840.$$

$D3$: Four observations at each of 16, 40, 64, for which $\sum_{i=1}^{12} (x_i - \bar{x})^2 = 4608$.

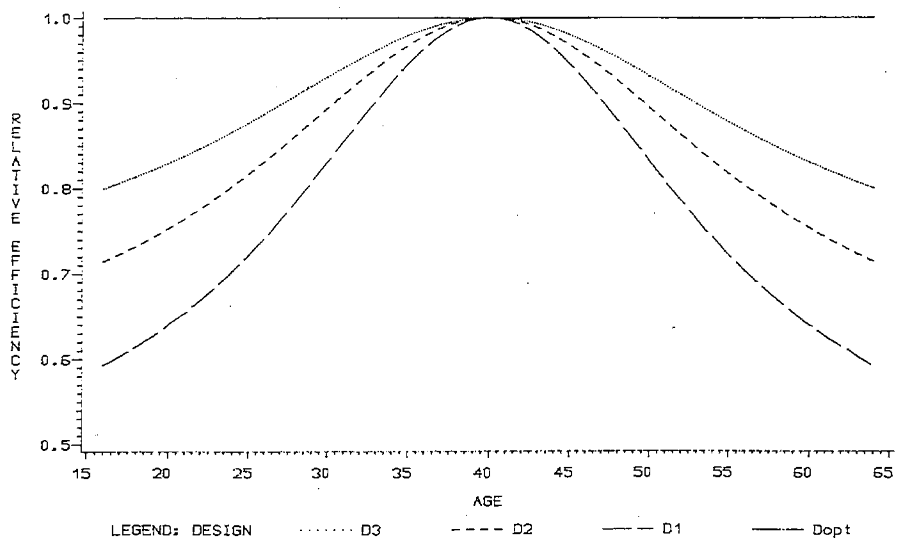

Figure 2. Relationship between relative efficiency and age

Curves of the relative efficiencies of designs $D1$, $D2$, $D3$, to D_{opt} are shown in figure 2. From this it can be seen that the loss of efficiency from the use of three or four distinct observation points is comparatively small. However, splitting into more distinct points causes the efficiency to drop off much more rapidly.

5. Conclusions and further thoughts

The main conclusions I hope are clear.

(i) It is important, where possible, to consider the choice of experimental design points when performing a regression study.

(ii) The 'optimal' choice of these levels can be made analytically by assuming a particular form for the response relationship, e.g. a straight line.

(iii) It is then necessary to 'trade-off' the gain in efficiency from using the optimal design against the consequent inability to estimate any more complex relationship.

Where do we go from here? Any article like this is of limited value if it does not pose additional problems or suggest practical exercises. The results presented here are true, but are theoretical in nature. They can be studied in many practical ways by collecting data, on variables which are believed to be linearly related, using a variety of designs. The relationship can then be fitted using the data from the separate designs, and the results compared.

Another area for further study is to consider how to design experiments to estimate other relationships. The possibilities are infinite but I shall mention two examples briefly.

(i) It is believed that the period of a pendulum (t) is related to its length (l) through the equation

$$t \propto \sqrt{\frac{l}{g}},$$

where g is the acceleration due to gravity. Choose the values of l in an experiment in order to estimate g as accurately as possible. Notice the new concept here—that the interest lies not in prediction but in the estimation of the value of an unknown parameter. Try to discover whether this change in objective leads to a change in the form of the best experimental design.

(ii) The simplest form of curved relationship between a response variable (y) and a stimulus variable (x) is the quadratic

$$y = a+bx+cx^2.$$

Choose the values of x in order to estimate the parameters a, b and c as accurately as possible. This is a more difficult problem to solve analytically, but provides an ideal opportunity to try out a range of designs, evaluate the variances of the parameter estimates and then compare the designs with each other.

I conclude by returning to the estimation of the straight line, and by posing two final questions. An experimenter wishes to fit a straight line in a given range, but wishes to use the fitted line mainly for prediction at the lower end of the range. Is the 'optimal' design with half of the observations at each end of the range really optimal for this situation? If not, how should it be modified?

Computer Column

MIKE PIFF

Take a couple of functions of four variables, such as

$$y1 = x1{*}x1 - x2{*}x2 + c1, \qquad y2 = 2{*}x1{*}x2 + c2.$$

For this example, the functions each individually only involve three of the variables, but that doesn't matter. We can regard $c1$ and $c2$ as constants, and think of these functions as moving the point $(x1, x2)$ in the plane to $(y1, y2)$. (If you are familiar with complex numbers, the function can be written $w = z{*}z + c$.) We illustrate with the case $c1 = 1$, $c2 = 1$, and $x1 = 0$, $x2 = 0$, so that $y1 = 1$ and $y2 = 1$, and $(0,0)$ is moved to $(1,1) = (c1, c2)$.

We now ask, what happens on applying the same transformation to $x1 = 1$, $x2 = 1$? This time $y1 = 1$ and $y2 = 3$. We can repeat or 'iterate' this transformation indefinitely and ask what happens to the point $(y1, y2)$, in the limit. Well, this is most easily programmed by the segment:

```
10 c1 = 1
20 c2 = 1
30 x1 = 0
40 x2 = 0
50 for i = 1 to k
60     temp = x1*x1-x2*x2+c1
70     x2 = 2*x1*x2+c2
80     x1 = temp
90 next i
```

It is known that if $x1*x1+x2*x2$ exceeds 4 at any stage, then $(x1,x2)$ moves off towards infinity with successive iterations, and the converse is, of course, obvious. The point $(c1,c2)$ is said to be in the Mandelbrot set if $(x1,x2)$ always remains close to the origin.

Thus, we iterate k times and see if

$$x1*x1+x2*x2 > 4 \qquad (*)$$

after the iterations. If so, $(c1,c2)$ is not in the Mandelbrot set. But we may have chosen $k = 10$, say, for a point $(c1,c2)$ where it would have taken 12 iterations to satisfy $(*)$; in other words, some points $(c1,c2)$ we can be certain are not in the Mandelbrot set, but for others we are never absolutely sure. We can only test whether they are by doing more iterations.

But this is where graphics come to our aid. We run lines 10 to 90 with $k = 10$ and check condition $(*)$. If it is satisfied, then $(c1,c2)$ is coloured black and we go on to examine another point $(c1,c2)$. If not, we branch back to run 50 to 90 again, and again check $(*)$. If it is now satisfied, we colour $(c1,c2)$ yellow say, otherwise go back to 50 and try again. This way, we can use a sequence of colours (however many our graphics facilities allow) to delimit the Mandelbrot set. After a predetermined number of repetitions of 50 to 90, we colour $(c1,c2)$ red, say, if it still hasn't satisfied condition $(*)$, and then it probably is in the Mandelbrot set.

The next fragment examines one point (h,v) and decides which of four colours to assign to it:

```
 90 rem $$ colour (h,v) with colour 4
100 x = 0
110 y = 0
120 for c = 1 to 3
130    for i = 1 to 10
140       temp = x*x-y*y+h
150       y = 2*x*y+v
160       x = temp
170    next i
180    if x*x+y*y <= 4 goto 210
190    rem $$ colour (h,v) with colour c
200    goto 220
210 next c
220 rem examine next point (h,v)
```

Lines 90 and 190 have to be replaced with lines putting a blob of the appropriate colour at (h,v). All we need is a containing loop letting h and v range over the values between -2 and $+2$, say, and our program is complete. If our screen is 300 wide and 200 deep we can use

```
10 for w = 1 to 300
20     h = -2+4*w/300
30     for d = 1 to 200
40         v = 2-4*d/200
           ........
250    next d
260 next w
```

to achieve this. Note that it is position (w, h) on the screen that is coloured at lines 90 and 190 now. Thus apart from a couple of minor alterations our program is complete.

For those of you with an experimental bent, let us see how we can modify the above program to produce an even greater wealth of graphics. First, the functions $x*x-y*y+h$ and $2*x*y+v$ at lines 140 and 150 can be replaced by any functions of x, y, h and v you like, for example, change the second to $2*x*\mathrm{abs}(y)$ and see what effect this has on the pictures produced. Also, both functions can involve all four variables if you like. The mathematics of when the iterations tend to infinity will not carry through, of course, but let us dispense with that interpretation and just ask how rapidly the iterations approach the region $x*x+y*y>2$. Having now made that alteration in our interpretation, why not substitute some other region, say $\mathrm{abs}(x)*y-x*\sin(xy)>5$? The possibilities are endless.

ANNOUNCEMENT

Mathematics for those aged between 14 and 17: do they really need it?

This will be the theme for the 38th conference of the International Commission for the study and improvement of mathematics teaching, the CIEAEM. It will be held at Southampton, England, from 24 to 30 July 1986.

The objectives are to develop recommendations on the theme and sub-themes. These will be summarised at the final plenary sessions and will be published together with the papers that are initially contributed.

Further information may be obtained from

Peter Bowie,
Shalbourne,
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Letters to the Editor

Dear Editor,

On sums of consecutive integers

On p. 15 of *Mathematical Spectrum* Volume 17 Number 1, Mr L. B. Dutta expresses the powers of 3, from the zeroth to the eighth, in terms of sums of consecutive positive integers and asks for explanation and generalisation. Clearly we seek a representation of the form $3^n = T_k - T_m$, where T_x denotes the triangular number $\frac{1}{2}x(x+1)$. In this way the eighth-power representation becomes $3^8 = T_{121} - T_{40}$. A very simple process of iteration enabled me to obtain the analogous results for 3^9 and 3^{10}:

$$3^9 = T_{202} - T_{40}, \qquad 3^{10} = T_{364} - T_{121}.$$

I then considered the differences in the sequences $k_1, k_2, \ldots$ and $m_1, m_2, \ldots$, where $3^n = T_{k_n} - T_{m_n}$, and found the following formula (applying a bit of common sense):

$$k_n = 2 + 2\sum_{i=0}^{[(n-2)/2]} 3^i + \sum_{i=1}^{[(n-1)/2]} 3^i, \qquad m_n = \sum_{i=0}^{[(n-2)/2]} 3^i.$$

(Here, square brackets denote the integer part.) Thus, for example,

$$3^{20} = T_{88573} - T_{29524}.$$

The same technique produced similar formulae when 3 was replaced by 5 and 7, and enabled me to obtain a general formula, as follows.

Let $x^n = T_{k(x,n)} - T_{m(x,n)}$ where x is odd, $x \geqslant 3$ and $n \geqslant 0$. Then

$$k(x,n) = \tfrac{1}{2}(x+1) + (x-1)\sum_{i=0}^{[(n-2)/2]} x^i + \tfrac{1}{2}(x-1)\sum_{i=1}^{[(n-1)/2]} x^i$$

and

$$m(x,n) = \sum_{i=0}^{[(n-2)/2]} x^i + \tfrac{1}{2}(x-3)\sum_{i=0}^{[(n-1)/2]} x^i.$$

In this way, for example, we have $11^7 = T_{8651} - T_{5989}$.

However, a cursory inspection of similar equalities for even x proved ineffective, although the even powers *can* be written in the desired form; for example $6^5 = T_{136} - T_{55}$. Perhaps other readers could tackle this problem.

Yours sincerely,

Joseph McLean
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Dear Editor,

Curious numbers

Whilst considering the 'Peking Pyramid' and the 'Bangladeshi Powers of 3' in *Mathematical Spectrum* Volume 17 Number 1, pp. 1 and 15, I came across the following curious relationships between numbers. For a Biblical reference, see *Revelation* 13, v. 18 (note that $13\times18 = 234$).

$$3^5-5! = 123, \qquad 6!-2\times3^5 = 234, \qquad 7!-2\times3^7 = 666,$$

$$666-4! = 2\times321, \qquad 666-5! = 546, \qquad 6!-666 = 54,$$

$$\frac{2\times666-4!}{2} = 654, \qquad \frac{3\times666-4!}{2} = 987, \qquad \frac{2\times3^{10}-8!}{2\times3^2} = 4321 = \frac{2\times3^{12}-9!}{2\times3^4},$$

$$10!+3^7-5\times6\times10^5-3\times210\times10^3 = 987, \qquad \frac{11!+2\times3^8}{6}-3\times4\times5\times10^5 = 654\,987,$$

$$1+2+3+\ldots+666 = 222\,111.$$

Yours sincerely,

MALCOLM SMITHERS
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Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

18.4. (Submitted by Richard Dobbs, Magdalen College, Oxford)
Expressed in Roman digits, the number IV ($= 4$) increases by $\frac{1}{2}$ when the right-hand digit is moved to the front i.e. giving VI ($= 6$). Find a number which does this when expressed in the so-called Arabic numerals (i.e. $1, 2, 3, \ldots$).

18.5. (Submitted by K. R. S. Sastry, Addis Ababa)
Determine the lengths of the diagonals of a cyclic quadrilateral in terms of the lengths of its sides.

18.6. (From an article by Keith Devlin in the *Guardian*)
Determine the probability that two whole numbers chosen at random have highest common factor 1. (You may need to use the fact that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}.)$$

Solutions to Problems in Volume 17 Number 3

17.7. Let n be a positive integer and let m be the number of odd positive divisors of n. Show that there are exactly m different ways of expressing n as a sum of consecutive positive integers.

Solution by Guy Willard (The Haberdashers' Aske's School, Elstree)

Let d be an odd positive divisor of n. Then

$$n = \left(\frac{n}{d} - \frac{d-1}{2}\right) + \ldots + \left(\frac{n}{d} - 1\right) + \left(\frac{n}{d}\right) + \left(\frac{n}{d} + 1\right) + \ldots + \left(\frac{n}{d} + \frac{d-1}{2}\right).$$

Any negative terms in this sum will cancel with the corresponding positive terms (and 0 can be omitted if it occurs), to express n as a sum of consecutive positive integers. We claim that these expressions are all different. For suppose that the sums arising from the distinct odd positive divisors d_1 and d_2 of n are the same. In particular, their last terms are the same, so

$$\frac{n}{d_1} + \frac{d_1 - 1}{2} = \frac{n}{d_2} + \frac{d_2 - 1}{2}$$

$$\Rightarrow \frac{n}{d_1 d_2}(d_2 - d_1) = \tfrac{1}{2}(d_2 - d_1)$$

$$\Rightarrow n = \tfrac{1}{2} d_1 d_2.$$

But $\frac{1}{2}d_1 d_2$ is not an integer, so this is impossible.

We have to show that we have now produced all possible expressions of n as a sum of consecutive positive integers. Suppose first that n is the sum of an odd number of consecutive positive integers, say

$$n = (a-k) + \ldots + (a-1) + a + (a+1) + \ldots + (a+k)$$
$$= (2k+1)a.$$

Put $d = 2k+1$. Then d is an odd positive divisor of n and

$$n = \left(\frac{n}{d} - \frac{d-1}{2}\right) + \ldots + \left(\frac{n}{d} - 1\right) + \left(\frac{n}{d}\right) + \left(\frac{n}{d} + 1\right) + \ldots + \left(\frac{n}{d} + \frac{d-1}{2}\right).$$

Now suppose that n is the sum of an even number of consecutive positive integers, say $n = (a+1) + \ldots + (a+2k)$. We can write

$$n = (-a) + (-a+1) + \ldots + 0 + 1 + \ldots + a + (a+1) + \ldots + (a+2k).$$

There are now $2k+2a+1$ terms in this sum and it can be written as

$$n = [k-(a+k)] + \ldots + k + (k+1) + \ldots + [k+(a+k)]$$
$$= k(2a+2k+1).$$

Put $d = 2a+2k+1$. Then d is an odd positive divisor of n and the sum becomes

$$n = \left(\frac{n}{d} - \frac{d-1}{2}\right) + \ldots + \left(\frac{n}{d} - 1\right) + \left(\frac{n}{d}\right) + \left(\frac{n}{d} + 1\right) + \ldots + \left(\frac{n}{d} + \frac{d-1}{2}\right)$$

as before.

Please note that the penultimate word 'positive' was omitted in error from the question as posed in Volume 17 Number 3.

Also solved by Ruth Lawrence (St. Hugh's College, Oxford) and Michael McQuillan (University of Glasgow).

17.8. In a series of independent games between players, each player has probability $\frac{1}{2}$ of winning each game. The series terminates as soon as either player has won three consecutive games. Find the probability that the series terminates just after the nth game.

Solution

Denote by $Q(n)$ the probability that the series terminates just after the nth game, A having won the first game, and denote by $R(n)$ the probability that the series terminates just after the nth game, B having won the first game, respectively. Then the probability $P(n)$ that the series terminates just after the nth game is given by

$$P(n) = Q(n)+R(n).$$

Suppose that A wins the first game and that the series terminates just after the nth game with $n \geqslant 4$. Then A and B win the second game each with probability $\frac{1}{2}$. If A wins the second game, then B must win the third game because $n \geqslant 4$. It follows that

$$Q(n) = \frac{1}{2^2}R(n-2)+\frac{1}{2}R(n-1).$$

Similarly,

$$R(n) = \frac{1}{2^2}Q(n-2)+\frac{1}{2}Q(n-1).$$

Thus, for $n \geqslant 4$,

$$P(n) = \frac{1}{2^2}P(n-2)+\frac{1}{2}P(n-1)$$

with $P(2) = 0$, $P(3) = \frac{1}{4}$. Hence, for $n \geqslant 2$,

$$P(n) = a\left(\frac{1+\sqrt{5}}{4}\right)^n+b\left(\frac{1-\sqrt{5}}{4}\right)^n$$

for some constants a and b. If we substitute $n = 2,3$ we obtain

$$P(n) = \frac{(1+\sqrt{5})^{n-2}-(1-\sqrt{5})^{n-2}}{\sqrt{5}(2^{2n-3})}.$$

Also solved by Guy Willard (The Haberdashers' Aske's School), Richard Dobbs (Magdalene College, Oxford) and Michael McQuillan (University of Glasgow).

17.9. (i) Let f be a function defined for $0 < x < 1$ such that $f(x) > 0$ for all such x and such that

$$\frac{f(x)}{f(y)}+\frac{f(1-x)}{f(1-y)} \leqslant 2 \qquad (*)$$

for all x and y $(0 < x, y < 1)$. Show that f must be a constant function.

(ii) Show that there exist infinitely many non-constant functions f defined for $0 < x < 1$ such that $f(x) \neq 0$ for all such x and which satisfy condition (*) for all x and y $(0 < x, y < 1)$.

(iii) Determine all functions f defined for $0 < x < 1$ such that $f(x) > 0$ for all such x and such that

$$\frac{f(x)}{f(1-x)}+\frac{f(y)}{f(1-y)} \leqslant 2$$

for all x and y $(0 < x, y < 1)$.

Solution by Michael McQuillan (University of Glasgow)

(i) From (*) we have

$$f(x)f(1-y)+f(1-x)f(y) \leqslant 2f(y)f(1-y)$$

since $f(y), f(1-y) > 0$ for all y $(0 < y < 1)$. If we put $x = 1-y$, then

$$f^2(x)-2f(x)f(1-x)+f^2(1-x) \leqslant 0,$$

so that $\{f(x)-f(1-x)\}^2 \leqslant 0$, and $f(x) = f(1-x)$. Hence we may rewrite (*) as $2f(x)/f(y) \leqslant 2$, so that $f(x) \leqslant f(y)$. This is true for all x and y $(0 < x, y < 1)$; so we can reverse x and y to give $f(y) \leqslant f(x)$. Hence $f(x) = f(y)$ for all x and y and f is a constant function.

(ii) For a real number $\lambda \neq 0$, define the function f_λ by

$$f_\lambda(x) = \begin{cases} \lambda & (\text{if } 0 < x \leqslant \frac{1}{2}), \\ -\lambda & (\text{if } \frac{1}{2} < x < 1). \end{cases}$$

For every real number $\lambda \neq 0$, f_λ satisfies (*).

(iii) From the given condition we obtain

$$f(x)f(1-y)+f(y)f(1-x) \leqslant 2f(1-x)f(1-y).$$

If we put $x = 1-y$, this gives

$$f^2(x)-2f(x)f(1-x)+f^2(1-x) \leqslant 0,$$

so that $\{f(x)-f(1-x)\}^2 \leqslant 0$ and $f(x) = f(1-x)$, and this must be true for all x $(0 < x < 1)$. But this is also a sufficient condition, so it describes all these functions, i.e. they are the functions symmetrical about $x = \frac{1}{2}$.

Also solved by Ruth Lawrence (St. Hugh's College, Oxford) and Philip Wadey (University of Exeter); a partial solution received from Jeremy Rosten (The Haberdashers' Aske's School).

Miss Fatland 1985

This problem was posed on p. 67. Richard Dobbs (Magdalen College, Oxford) writes:

'I would suggest a flatness factor given by

$$\frac{360\pi(\text{volume})^2}{(\text{surface area})^3},$$

where 'ten' would be their Bo Derick! This is independent of size and dependent on shape only. It will be maximum for a sphere as a sphere has maximum volume for a given surface area.'

Sums of five cubes

On p. 71 we asked if readers could prove that every integer is the sum of five cubes. Guy Willard (The Haberdashers' Aske's School) has sent us the following proof. Firstly, n^3-n is divisible by 6 for all integers n, so, given an integer n, we can write

$$\begin{aligned} n &= n^3-6k \\ &= n^3+(-k-1)^3+(-k+1)^3+k^3+k^3 \end{aligned}$$

for some integer k. This gives, for example,

$$7 = 7^3+(-57)^3+(-55)^3+56^3+56^3.$$

Book Reviews

The Biographical Dictionary of Scientists—Mathematicians. Edited by D. ABBOTT. Blond Educational Publishers, London, 1985. Pp. 175. £12·95.

The main part of this book consists of 180 articles on mathematicians, each being a short biography and an account of the subject's work in mathematics. There is then a glossary of mathematical terms and an index. The book is very readable and attractively printed.

As with anthologies, the editor has been faced with the problem of selection. One wonders why he has chosen to include textbook writers such as Cocker, and writers on mathematics such as Bronowski and J. R. Newman, while omitting such mathematicians as Einstein, Erdös, Landau, Matijasevich, Mordell, Tchebychef and Vinogradov, to name a few who come to mind. (Some of these are mentioned in articles on others, or in the glossary but that is hardly sufficient.) Space could have been saved by reducing the glossary. (For example, the trigonometric functions are defined at length as in school textbooks, though the cotangent is, curiously, omitted.) The index, though incomplete, is needlessly repetitive and use of initials instead of all forenames would have saved further space in it.

Proof-reading has been careless, but there are errors that go beyond that. The dates of Beltrami, Cantor, Fermat and Hermite are not those that other reference books give. J. E. Littlewood is described as 'in what Hardy would have called the "second eleven" among mathematicians'; in fact Hardy (*A Mathematician's Apology*, p. 148) regarded Littlewood as at least his equal.

Although the book was published in 1985 it bears signs of having been written long before. Hall, said to be still alive, died in 1982; and a solution of the four-colour problem (mentioned under Ore) was announced by Appel and Haken in 1976. Even though this proof was initially suspect it should have been mentioned.

Much has happened in mathematics since the standard histories were written, and there is a need to bring the story up to date. The present book does not fill that need.

Royal Holloway College, University of London H. J. GODWIN

Charles Babbage—On the Principles and Development of the Calculator and Other Seminal Writings by Charles Babbage and Others. Edited by PHILIP MORRISON and EMILY MORRISON. Reprinted by Constable & Co., London, 1985. Pp. xxxviii + 400. £7·95.

This book is a fascinating mixture. It contains mostly writings of Charles Babbage himself, but there are some descriptive pieces from other people.

To start with, it is biographical. Babbage describes much of his early life, through school and Cambridge, and indicates how his interest in calculating machines developed. Then there are details of his career and the problems in trying to acquire money to develop his ideas. Finally there are some of his more technical papers, which describe his machines in more detail.

In between all this there are excursions into life in the nineteenth century. He gives us his recollections of prominent contemporaries. These are given in a very personal style—as is most of the writing—which makes it very interesting and readable. He leaves us in no doubt about his disaffection with Sir Humphrey Davy!

He describes a meeting with the celebrated thief-taker Vidocq, who could 'alter his height by an inch and a half', but was no help in the art of picking locks, something that Babbage was particularly interested in. In fact he called deciphering 'one of the most fascinating arts', and his patterns for locks had much the same basis as his calculating machines. There are also descriptions of his other interests, including railways, tunnels and fire!

The main impression that comes over is that Babbage was always an inventor. There is an amusing episode at the seaside, when he sees the possibility of walking on the water by using a pair of 'folding feet' made of wood. This invention was not very successful!

But of course his main concern was the building of the big machines. There was only one real success, in that the Difference Engine was the only one to be completed. This was mainly because, after working at his engine for so long he would get new and better ideas. So he would give up on the current design and start again! But he did leave diagrams of these other machines, and it is interesting to see how these ideas have now been utilised in modern computers. One of his main contributions is showing how the punch-card system—which had been developed in the silk-weaving industry—could be used in calculating machines.

This book is well worth reading, even for dipping into, and it gives a graphic and detailed picture of the man—and inventor—Charles Babbage.

University College of Swansea G.V. Wood

Micro-Maths. Mathematical Problems and Theorems to Consider and Solve on a Computer. By Keith Devlin. Macmillan, London. Pp. xiii + 103. £5·95.

This book, from its title and subtitle, appears to be aiming at one target but in fact hits another. The apparent intention is to provide users of microcomputers with problems of mathematical interest, but only about 10 per cent of the book contains problems, and many of these are not connected with the text. There is no indication of how the problems are to be tackled by computer except for some hints on how to implement multilength routines to deal with large numbers.

What most of the book does, and does well, is to give an account of various branches of number theory, up to such recent results as Gross and Zagier's work on the class-number of real quadratic fields and the 26-variable polynomial of Jones et al., whose positive values are the primes. In this it should appeal both to the mathematically educated layman and to the specialist, though the latter may be irritated by an almost complete lack of references.

The book is based on a series of articles that appeared in the *Guardian*: this accounts for its readability, but also for some repetitiveness. It is attractively produced, and there seem to be few misprints or errors, though I did notice the incorrect expansion of $\frac{22}{7}$ on page 11, and that Wunderlich's stay at Queen Mary College (page 82) should have been described as some weeks rather than some months.

Three things that I should like to have seen included (and for which room could have been made by omitting the rather feeble drawings) are an index, more detail on the public-key method, with a worked example, and Pollard's 'rho' method (see, e.g., *Mathematical Spectrum*, **11** (1978/9), 21–22). The 'rho' method is simple in principle, easy to implement on a computer, and a powerful tool for factorization.

My scoring is, therefore, target aimed at, a miss; target not aimed at, an inner.

Royal Holloway College, University of London H.J. GODWIN

Basic Numerical Methods. By R. E. SCRATON. Edward Arnold (Publishers) Ltd., London, 1985. Pp. viii+92. £4·95 paperback.

It is probably realistic of the author to suggest, as he does in the preface, that this book as a whole is more suitable for first-year undergraduates than for sixth-form mathematics students. It would, however, be a useful acquisition to a school departmental library for the use of specific chapters by interested students (or their teachers). Thus the initial chapter on computer accuracy contains many basic points of which the student needs to be aware when writing any program—not just those illustrated in the remainder of the book. The section on error analysis should be followed carefully, as this is utilised in discussions of the accuracy of subsequent processes. The treatment of iterative techniques, solution of equations and numerical integration are all clearly-written chapters involving material common to most A-level courses. Thus the provision of a resource to the A-level (or even O-level) teacher which only requires a demonstration program to be typed in once and then saved will be most acceptable. As the author indicates, however, the operator needs to be fully aware of the changes in syntax which need to be made if one is not using a PET. Otherwise the programs may not run as written.

I particularly appreciate the way in which the chapters on differential equations include the development of programs by the insertion of new lines without the necessity of complete renumbering. By this means a sequence of programs can be produced—the reader choosing either the one which makes most sense to him or that which will solve the majority of the problems he is likely to formulate. This approach suggests that great care and thought went into their production, and that, as is admitted in Chapter 4, the 'author [took] a long time to get this program right'. The paragraph in which this appears can at first sight seem to be slightly patronising, but on second reading it is seen to be an attempt to show that this book on computing has been written by a human being and not by another computer.

Rowlinson School, Sheffield J. ASHMORE

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