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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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IT'S ELEMENTARY (COMBINATORICS) I

William Moser

[*Editor's note.* This is the first of a three-part series of articles by Professor Moser. Parts II and III will appear in the next two issues of *Crux*.]

1. Introduction In this paper we solve in an elementary way a variety of old and new combinatorial counting problems and prove several old and new identities. The technique will be to construct and count appropriate displays of symbols in a row (or circle) which represent the required combinatorial structures.

We take without proof: for non-negative integers n and k the number $[n,k]$ of $k-$element subsets of an $n-$element set is $[n,k]=\binom{n}{k}$, where

$$\binom{n}{k}=\begin{cases} n!/k!(n-k)! & \text{if } 0\le k\le n,\\ 0 & \text{otherwise.}\end{cases}$$

A $k-$choice ($k-$element subset)

$$A=\{x_1,x_2,\dots,x_k\}\subseteq\{1,2,3,\dots,n\},\qquad 1\le x_1<x_2<x_3<\cdots<x_k\le n, \tag{1}$$

is nicely represented by its place-indicator sequence of k 1's and $n-k$ 0's :

$$\epsilon_1\ \epsilon_2\ \dots\ \epsilon_n \qquad \epsilon_i=\begin{cases}1 & \text{if } i=x_1,x_2,\dots,x_k,\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

For example,

$$\{2,\ 3,\ 6,\ 9,\ 10\}\subseteq\{1,\ 2,\ \dots,\ 10\}\quad\leftrightarrow\quad 0\,1\,1\,0\,0\,1\,0\,0\,1\,1.$$

Now consider the fact that a distribution of n like objects, say n 1's, into k unlike boxes $B_1,B_2,\dots,B_k$, with a_i 1's in B_i, $i=1,2,\dots,k$, corresponds to the ordered sum $a_1+a_2+\cdots+a_k=n$, $a_1,a_2,\dots,a_k\ge 0$, and is nicely represented by a linear display of n 1's and $k-1$ strokes:

$$\underbrace{1\,1\,\cdots\,1}_{a_1\,1\text{'s}}/\underbrace{1\,1\,\cdots\,1}_{a_2\,1\text{'s}}/\underbrace{1\,1\,\cdots\,1}_{a_3\,1\text{'s}}/\cdots/\underbrace{1\,1\,\cdots\,1}_{a_k\,1\text{'s}} \qquad \begin{array}{c} k-1 \text{ strokes}\\ n\ \ 1\text{'s}\end{array}$$

All such displays (of n 1's and $k-1$ strokes) are constructed by starting with $n+k-1$ 1's in a row, choosing $k-1$ of them, in $\binom{n+k-1}{k-1}$ ways, and replacing each chosen 1 by a stroke. Hence

Lemma 1. *n like objects can be distributed into k unlike boxes in $\binom{n+k-1}{k-1}$ ways.*

It may be convenient to remember this as:

$$\text{like objects into unlike boxes in}\quad\binom{\#\text{objects} + \#\text{boxes}-1}{\#\text{boxes}-1}\ \text{ways.} \tag{3}$$

Lemma 2. *The number of ways of distributing n like objects into k unlike boxes with at least w objects ($w \geq 0$) in each box is*

$$\binom{n-k(w-1)-1}{k-1}.$$

Proof. To see this, place $k-1$ strokes in a row (creating k boxes), put w 1's into each box, and distribute the remaining $n-kw$ 1's into the k boxes in (by (3))

$$\binom{(n-kw)+k-1}{k-1} = \binom{n-k(w-1)-1}{k-1}$$

ways. In the case $w=1$ we have

$$\text{like objects into unlike boxes, no box empty, in } \binom{\#\text{objects}-1}{\#\text{boxes}\ -1} \text{ ways.} \tag{4}$$

2. Linear displays

Problem 1.

(a) The number $[n,k\,|\,1]$ of k–choices (1) which satisfy: no two chosen integers are adjacent in the display $1,2,3,\ldots,n$ (i.e., $x_{i+1}-x_i-1 \geq 1,\ i=1,2,3,\ldots,k-1$) is

$$[n,k\,|\,1] = \binom{n-k+1}{k}.$$

Proof. The corresponding sequences (2) are constructed by starting with the display

$$\underbrace{\qquad}_{B_1} 1\ 0 \underbrace{\qquad}_{B_2} 1\ 0 \underbrace{\qquad}_{B_3} 1\ 0\ \cdots\ 1\ 0 \underbrace{\qquad}_{B_k} 1\ \underbrace{\qquad}_{B_{k+1}}$$

and distributing $n-k-(k-1)$ 0's into the $k+1$ boxes in (by (3))

$$\binom{(n-k-(k-1))+(k+1)-1}{(k+1)-1} = \binom{n-k+1}{k}$$

ways.

(b) More general is the count, for $w \geq 0$, of the number $[n,k\,|\,w]$ of k–choices (1) satisfying: any two chosen integers are separated by at least w non-chosen integers, i.e., $x_{i+1}-x_i-1 \geq w,\ i=1,2,3,\ldots,k-1$. The corresponding sequences (2) are now constructed by starting with the display

$$\underbrace{\qquad}_{B_1} 1 \underbrace{0\cdots0}_{w\ 0\text{'s}} \underbrace{\qquad}_{B_2} 1 \underbrace{0\cdots0}_{w\ 0\text{'s}} \underbrace{\qquad}_{B_3} \cdots 1 \underbrace{0\cdots0}_{w\ 0\text{'s}} \underbrace{\qquad}_{B_k} 1 \underbrace{\qquad}_{B_{k+1}}$$

and distributing $n-k-(k-1)w$ 0's into the $k+1$ boxes in

$$\binom{n-k-(k-1)w+(k+1)-1}{(k+1)-1}$$

ways. Hence

$$[n,k\,|\,w]=\binom{n-(k-1)w}{k}.$$

(c) Still more general is the count, for given non-negative integers $a_1,a_2,\ldots,a_{k-1}$, of the number of $k-$choices (1) satisfying: $x_{i+1}-x_i-1\geq a_i$, $i=1,2,\ldots,k-1$ (x_{i+1} and x_i are separated by a_i non-chosen integers). The corresponding sequences (2) are now constructed by starting with the display

$$\underbrace{\quad}_{B_1}1\underbrace{0\cdots0}_{a_1\ 0\text{'s}}\underbrace{\quad}_{B_2}1\underbrace{0\cdots0}_{a_2\ 0\text{'s}}\underbrace{\quad}_{B_3}\cdots1\underbrace{0\cdots0}_{a_{k-1}\ 0\text{'s}}\underbrace{\quad}_{B_k}1\underbrace{\quad}_{B_{k+1}}$$

and distributing $n-k-(a_1+a_2+\cdots+a_{k-1})$ 0's into the $k+1$ boxes in (by (3))

$$\binom{n-k-(a_1+\cdots+a_{k-1})+(k+1)-1}{(k+1)-1}=\binom{n-(a_1+\cdots+a_{k-1})}{k}$$

ways.

Problem 2.

(a) For given $0\leq k=2r\leq n$, how many of the $k-$choices (1) consist of r disjoint pairs of consecutive integers, i.e., satisfy

$$x_{2i-1},\ x_{2i}\quad\text{are consecutive integers},\quad i=1,2,\ldots,r?$$

Solution. Start with the display

$$\underbrace{\quad}_{B_1}1\,1\underbrace{\quad}_{B_2}1\,1\underbrace{\quad}_{B_3}1\,1\ \cdots\ 1\,1\underbrace{\quad}_{B_r}1\,1\underbrace{\quad}_{B_{r+1}}$$

and distribute $n-2r$ 0's into the $r+1$ boxes in

$$\binom{(n-2r)+(r+1)-1}{(r+1)-1}=\binom{n-r}{r}$$

ways.

(b) The obvious generalization is as follows. For given $0\leq k=rw\leq n$, $w\geq1$, how many $k-$choices (1) consist of r disjoint $w-$tuples of consecutive integers , i.e., satisfy, for $i=1,2,\ldots,r$,

$$x_{(i-1)w+1},\quad x_{(i-1)w+2},\quad x_{(i-1)w+3},\quad\ldots\quad,x_{iw}\qquad\text{are consecutive integers?}$$

Solution. Start with the display

$$\underbrace{}_{B_1}\underbrace{1\cdots1}_{w\,1\text{'s}}\underbrace{}_{B_2}\underbrace{1\cdots1}_{w\,1\text{'s}}\underbrace{}_{B_3}\underbrace{1\cdots1}_{w\,1\text{'s}}\cdots\underbrace{1\cdots1}_{w1\text{'s}}\underbrace{}_{B_r}\underbrace{1\cdots1}_{w\,1\text{'s}}\underbrace{}_{B_{r+1}}$$

and distribute $n-rw$ 0's into the $r+1$ boxes in (by (3))

$$\binom{(n-rw)+(r+1)-1}{(r+1)-1}=\binom{n-r(w-1)}{r}$$

ways.

(c) Still more general is that, for given integers $j_1, j_2, \ldots, j_r$ (all ≥ 1), the number of k-choices (1), $k=j_1+j_2+\cdots+j_r$, satisfying

$$\begin{array}{ll} x_1,\ x_2,\ \ldots,\ x_{j_1} & \text{are consecutive integers} \\ x_{j_1+1},\ x_{j_1+2},\ \ldots,\ x_{j_1+j_2} & \text{are consecutive integers} \\ \vdots & \vdots \\ x_{j_1+j_2+\cdots+j_{r-1}+1},\ x_{j_1+j_2+\cdots+j_{r-1}+2},\ \ldots,\ x_{j_1+j_2+\cdots+j_r} & \text{are consecutive integers} \end{array}$$

is

$$\binom{(n-(j_1+\cdots+j_r)+(r+1)-1)}{(r+1)-1}=\binom{n-(j_1+\cdots+j_r)+r}{r}.$$

(d) For given $0 \leq 2r = k \leq n$, how many k-choices (1) consist of r disjoint pairs of consecutive integers, but do not contain three consecutive integers?

Solution. Start with $n-k$ 0's in a row, choose r of the $n-k+1$ boxes they determine, and put a pair 1 1 into each chosen box. We have $\binom{n-k+1}{r}=\binom{n-2r+1}{r}$ displays, precisely those we want.

(e) For given $0 \leq 2r \leq k \leq n$, how many k-choices (1) *contain* exactly r disjoint pairs of consecutive integers but do not contain three consecutive integers?

Solution. Start with $n-k$ 0's in a row, choose r of the $n-k+1$ boxes they determine, put a pair 1 1 into each chosen box, choose $k-2r$ of the remaining $n-k+1-r$ boxes, put a single 1 into each of these. We now have

$$\binom{n-k+1}{r}\binom{n-k+1-r}{k-2r}=\binom{n-k+1}{k-r}\binom{k-r}{r}$$

desired displays.

(f) How many k-choices (1) contain no 3 consecutive integers?

Solution. From (e), the number is

$$\sum_{r=0}\binom{n-k+1}{k-r}\binom{k-r}{r}.$$

Problem 3. A sequence of n tosses of a coin is described by an $n-$sequence of H's and T's. How many of the 2^n sequences have precisely r occurrences of HT? (Sabharwal 1968)
Solution. These sequences can be constructed as follows. Start with a display of $r+1$ strokes and r pairs HT alternating in a row, creating $2r+2$ boxes:

$$\underbrace{\quad}_{B_1}/\underbrace{\quad}_{B_2}HT\underbrace{\quad}_{B_3}/\underbrace{\quad}_{B_4}HT\underbrace{\quad}_{B_5}/\cdots\underbrace{\quad}_{B_{2r-2}}HT\underbrace{\quad}_{B_{2r-1}}/\underbrace{\quad}_{B_{2r}}HT\underbrace{\quad}_{B_{2r+1}}/\underbrace{\quad}_{B_{2r+2}}.$$

Distribute $n-2r$ like symbols, say y's, into the $2r+2$ boxes, creating

$$\binom{(n-2r)+(2r+2)-1}{(2r+2)-1} = \binom{n+1}{2r+1}$$

displays which look like this:

$$\underbrace{y\cdots y}_{B_1}/\underbrace{y\cdots y}_{B_2}HT\underbrace{y\cdots y}_{B_3}/\underbrace{y\cdots y}_{B_4}HT\cdots HT\underbrace{y\ldots y}_{B_{2r-1}}/\underbrace{y\ldots y}_{B_{2r}}HT\underbrace{y\ldots y}_{B_{2r+1}}/\underbrace{y\ldots y}_{B_{2r+2}}.$$

Now replace each of the y's in $B_1, B_3, B_5, \ldots, B_{2r+1}$ by T, replace each of the y's in $B_2, B_4, B_6, \ldots, B_{2r+2}$ by H, and delete the strokes. The resulting sequences of H's and T's are the desired ones. Hence the answer is $\binom{n+1}{2r+1}$.

Problem 4. In how many ways can n be expressed as an ordered sum of positive integers? For example, 3 can be expressed in 4 ways: $3,\quad 1+2,\quad 2+1,\quad 1+1+1.$ (Moser 1951; Trigg 1967, Problem Q16).
Solution. Place n 1's in a row $\quad 1\ \ 1\ \ 1\ \ 1\ \ \cdots\ \ 1\ \ 1\quad$ creating $n+1$ boxes. Into each of the $n-1$ "in-between" boxes either put a single stroke or put no stroke. The resulting 2^{n-1} displays of n 1's and some strokes correspond to the desired sums. For example, the 4 sums for 3 correspond to

$$1\ 1\ 1, \qquad 1\,/\,1\ 1, \qquad 1\ 1\,/\,1, \qquad 1\,/\,1\,/\,1$$

Problem 5.

(a) How many $k-$choices (1) are there with x_i even if i is even and x_i odd if i is odd, $i = 1, 2, \ldots, k$, i.e., $x_i \equiv i \pmod 2$? This is known as Terquem's problem (Terquem 1839; Riordan 1958, p. 17, ex. 15).
Solution. The conditions on the x_i's are equivalent to

$$x_1 - 1 \equiv x_i - x_{i-1} - 1 \equiv 0 \pmod 2 \qquad i = 2, 3, 4, \ldots, k,$$

i.e., in the corresponding displays

$$\underbrace{\quad}_{B_1}1\underbrace{\quad}_{B_2}1\underbrace{\quad}_{B_3}1\ \cdots\ 1\underbrace{\quad}_{B_k}1\underbrace{\quad}_{B_{k+1}} \tag{5}$$

$|B_i| \equiv 0 \pmod 2$, $\quad i = 1, 2, \ldots, k$ ($|B|$ denotes the number of 0's in box B). This condition is met if we distribute the $n-k$ 0's in pairs. The $n-k$ 0's provide $\lfloor (n-k)/2 \rfloor$

pairs of 0's (there may be a single 0 left over), so we distribute these pairs into the $k+1$ boxes in

$$\binom{\left\lfloor\frac{n-k}{2}\right\rfloor + (k+1) - 1}{(k+1)-1} = \binom{\left\lfloor\frac{n+k}{2}\right\rfloor}{k}$$

ways; if there is a remaining 0 it must go into B_{k+1}.

(b) In Skolem's generalization of Terquem's problem (Netto 1927, pp. 313–314; Church and Gould 1967) the fixed modulus is $m \geq 2$, i.e., we seek the number of k–choices (1) for which $x_i \equiv i \pmod{m}$, or equivalently

$$x_1 - 1 \equiv x_i - x_{i-1} - 1 \equiv 0 \pmod{m}, \quad i = 2, 3, 4, \dots, k.$$

In the corresponding display (5), $|B_i| \equiv 0 \pmod{m}, i = 1, 2, 3, \dots k$. The $n-k$ 0's provide $\lfloor (n-k)/m \rfloor$ groups of m 0's each, and these groups can be distributed into the $k+1$ boxes in

$$\binom{\left\lfloor\frac{n-k}{m}\right\rfloor + (k+1) - 1}{(k+1)-1} = \binom{\left\lfloor\frac{n+(m-1)k}{m}\right\rfloor}{k}$$

ways; the remaining $n - k - m\lfloor (n-k)/m \rfloor$ 0's must go into B_{k+1}.

(c) The problem was further generalized by Abramson and Moser (1969) to: given $m \geq 2$ and $0 \leq r_1, r_2, \dots, r_k < m$, the number of k–choices (1) satisfying $x_1 - 1 \equiv r_1$, $x_i - x_{i-1} - 1 \equiv r_j \pmod{m}$, $i = 2, 3, 4, \dots k$, is

$$\binom{\left\lfloor\frac{n+(m-1)k-(r_1+r_2+\cdots+r_k)}{m}\right\rfloor}{k} .$$

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THE OLYMPIAD CORNER

No. 139

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first item this number is the 33rd I.M.O. which was held in Moscow, Russia, July 15–16, 1992. My sources this year are Professor Georg Gunther, the Canadian team leader; Barry Ferguson, University of Waterloo, observer for the Canadian team; Professor Walter Mientka of the MAA Committee on H.S. contests; and (via Andy Liu of the University of Alberta) Professors C.C. Yang and P.H. Cheung, who helped train the Hong Kong team. There were some discrepancies in the sources, and I hope that the results given below are the correct ones! Many thanks to those who sent me information. My apologies (in advance) for any errors or omissions.

This year a record total of 322 students from 56 countries participated officially, with another 28 students from eight other countries taking part unofficially. This is all the more remarkable because of the difficulties and uncertainties associated with hosting such a large event in Moscow at this time of political and economic change. It did mean, however, that the official invitations and the call for problems went out much later than is normal, and this may have contributed to the fact that this year's contest was very difficult, in both the questions and in the marking.

The contest is officially an individual competition and the six problems were assigned equal weights of seven points each (the same as the last 11 I.M.O.'s) for a maximum possible individual score of 42 (and a total possible of 252 for a national team of six students). For comparisons see the last 11 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], and [1991: 257].

This year there were only four perfect scores, three posted by students from China, and one from the C.I.S. To indicate the level of difficulty, there were six scores of 0, and the median score was 14, which was the cutoff for Bronze medals, of which 74 were awarded. The cutoff for Silver was 24, with 55 silver medals awarded. Finally, Gold medals were awarded to the 26 students who scored at least 32 points. Any student who did not receive a medal, but who got 7 on one problem was awarded Honourable Mention.

Congratulations to the gold medal winners:

Student	Country	Score
Dmitri A. Arinkin	CIS	42
Wei Luo	China	42
Kai Shen	China	42
Boazhong Yang	China	42
Yin Zhuang	China	41
Geffry Barad	Romania	40
Simai He	China	40

Student	Country	Score
Benjamin Burton	Australia	39
J.P. Grossman	Canada	36
Kiran Kedlaya	USA	36
Robert Kleinberg	USA	35
Hiroki Kodama	Japan	35
Mark Walters	United Kingdom	35
Eva Myers	United Kingdom	34
Sarban Nacu	Romania	34
Masmoudi Nader	Tunisia	34
Lenhard Lee Ng	USA	33
Nikolai V. Nikolov	Bulgaria	33
Jong Won Park	Republic of Korea	33
Nguyen Xuan Dao	Vietnam	33
Hong Zhou	China	33
Aleksey A. Chilikov	Russia	32
Pavel A. Kozgevnikov	Russia	32
Andray V. Malutin	CIS	32
Attila Por	Hungary	32
Patrick Popescu-Pampu	France	32

Here are the problems from this year's I.M.O. Competition. Solutions to these problems, along with those of the 1992 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1992*, which may be obtained for a small charge from:

Dr. W.E. Mientka
Executive Director
MAA Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, U.S.A. 68588.

33RD INTERNATIONAL MATHEMATICAL OLYMPIAD

Moscow, Russia

First Day — July 15, 1992 (4 1/2 hours)

1. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

2. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = y + (f(x))^2 \quad \text{for all } x,\ y \text{ in } \mathbb{R}.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured the set of coloured edges necessarily contains a triangle all of whose edges have the same colour.

Second Day — July 16, 1992 (4 1/2 hours)

4. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|$$

where $|A|$ denotes the number of elements in the finite set A. (Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

6. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.

(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

(b) Find an integer n such that $S(n) = n^2 - 14$.

(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

*

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. These totals (with possible errors!) are given in the following table. It is worth noting that with a 59 point spread between first place China, and the second place U.S.A., China would have won handily even if only its top five students' scores were used! Congratulations to the Chinese team.

	Country	Score	Gold	Silver	Bronze	Total Prizes
1	China	240	6	0	0	6
2	United States	181	3	3	0	6
3	Romania	177	2	2	2	6
4	Comm. of Indep. States	176	2	3	0	5
5	United Kingdom	168	2	2	2	6
6	Russia	158	2	2	2	6
7	Germany	149	0	4	2	6
8–9	Hungary	142	1	3	1	5
8–9	Japan	142	1	3	1	5
10–11	France	139	1	3	1	5
10–11	Vietnam	139	1	2	3	6
12	Yugoslavia	136	0	2	4	6
13	Czechoslovakia	134	0	2	3	5

	Country	Score	Gold	Silver	Bronze	Total Prizes
14	Iran	133	0	3	2	5
15	Bulgaria	127	1	1	3	5
16	D.P.R. of Korea	126	0	3	2	5
17	China Taipei	124	0	3	2	5
18	Republic of Korea	122	1	0	4	5
19	Australia	118	1	1	2	4
20	Israel	108	0	2	2	4
21	India	107	0	1	4	5
22	Canada	105	1	0	3	4
23	Belgium	100	0	1	2	3
24–25	Poland	90	0	1	3	4
24–25	Sweden	90	0	2	0	2
26–27	Hong Kong	89	0	1	2	3
26–27	Singapore	89	0	1	3	4
28	Italy	83	0	0	3	3
29	Norway	77	0	1	2	3
30	Netherlands	71	0	1	0	1
31	Austria	70	0	0	3	3
32	Argentina	67	0	1	1	2
33	Tunisia(team of 4)	64	1	0	1	2
34	Turkey	63	0	0	2	2
35	Colombia	55	0	0	1	1
36	Mongolia	51	0	0	0	0
37–38	Spain	50	0	0	1	1
37–38	Thailand	50	0	1	0	1
39	Brazil	48	0	0	1	1
40	Morocco	45	0	0	0	0
41–42	Denmark(team of 5)	42	0	0	0	0
41–42	Ireland	42	0	0	0	0
43	New Zealand	41	0	0	1	1
44	Philippines(team of 4)	40	0	0	1	1
45	Greece	37	0	0	0	0
46–47	Macau	35	0	0	0	0
46–47	Portugal	35	0	0	1	1
48	Cyprus	34	0	0	1	1
49	Finland	33	0	0	0	0
50	Mexico	32	0	0	0	0
51	Switzerland(team of 3)	30	0	0	0	0
52	Trinidad & Tobago	26	0	0	0	0
53	Indonesia	22	0	0	0	0
54	South Africa	21	0	0	0	0
55	Cuba(team of 3)	17	0	0	0	0
56	Iceland(team of 3)	16	0	0	0	0

*

This year the Canadian team slipped from 14th to 22nd place, but they put in a good performance. The team members and scores were

J.P. Grossman	36	Gold
Naoki Sato	21	Bronze
Alexander Nicholson	16	Bronze
Kevin Kwan	15	Bronze
Eric Lai	11	Honourable Mention
Kevin Cheung	6	

The Team Leaders were Dr. Georg Gunther, (Sir Wilfred Grenfell College, Corner Brook, Newfoundland) and Mr. Ravi Vakil (University of Toronto).

The I.M.O. venues for the next several years are as follows:

Turkey	1993
Hong Kong	1994
Canada	1995
India	1996

* * *

This issue we give an Olympiad from a country which has not been featured in the Corner for a while. Thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for collecting this contest and forwarding it to *Crux.*

1990 DUTCH MATHEMATICAL OLYMPIAD

Second Round — 9 September, 1990 (3 hours)

1. Prove for every integer $n > 1$:

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1) < n^n.$$

[*Editor's note.* The original problem as we received it had $n-1$ in place of $2n-1$.]

2. The numbers $a_1, a_2, a_3, a_4, \ldots$ are defined as follows:

$$a_1 = \frac{3}{2} \quad \text{and} \quad a_{n+1} = \frac{3a_n^2 + 4a_n - 3}{4a_n^2}.$$

(a) Prove that for all n holds: $a_n > 1$ and $a_{n+1} < a_n$.
(b) From (a) it follows that $\lim_{n\to\infty} a_n$ exists. Determine this limit.
(c) Determine $\lim_{n\to\infty} a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$.

3. Given is a function $f : x \to ax^4 + bx^3 + cx^2 + dx$ with the following properties:

- $a, b, c, d > 0$;
- $f(x)$ is an integer for all $x \in \{-2, -1, 0, 1, 2\}$;
- $f(1) = 1$ and $f(5) = 70$.

(a) Prove: $a = 1/24$, $b = 1/4$, $c = 11/24$, $d = 1/4$.
(b) Prove: $f(x)$ is an integer for every integer x.

4. Given is a regular 7-gon $ABCDEFG$. The sides have length 1. Prove for the diagonals AC and AD:

$$\frac{1}{AC} + \frac{1}{AD} = 1.$$

* * *

We now turn to comments and solutions to problems given in the past. The remainder of the column comes from correspondence received this summer. In some cases a solution has already appeared in the September or October numbers, which were prepared during the summer and before some of this mail was filed. I have tried to acknowledge the "extra" solutions.

The first block of generalizations come to problems for which we published solutions in earlier issues.

5. [1990: 3; 1992: 7] *1989 Indian Mathematical Olympiad.*

Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

must lie between the limits 3/2 and 2. Can equality hold at either limit?

Generalization by Murray S. Klamkin, University of Alberta.

Let $a_1, a_2, \ldots, a_n$ denote the sides of a polygon of perimeter p. Then

$$2 \geq \frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_n}{p-a_n} \geq \frac{n}{n-1} .$$

There is equality for the left hand inequality only for a degenerate polygon of sides $(a, a, 0, 0, \ldots, 0)$ and there is equality for the right hand inequality only if all the sides are equal. A proof for the left hand inequality is given in *Crux* 560 [1981: 280]. For the right hand inequality, we have by Cauchy's inequality that

$$\sum \frac{a_i}{p-a_i} = (n-1)^{-1}\left\{\sum(p-a_i)\cdot\sum\frac{1}{p-a_i}\right\} - n \geq \frac{n^2}{n-1} - n = \frac{n}{n-1} .$$

Incidentally, the latter inequality is still valid if the a_i are any real nonnegative numbers.

*

1. [1990: 290; 1992: 72] *20th Austrian Mathematical Olympiad, Final Round, Beginner's Level.*

Let a, b, c, d where $a \leq b \leq c \leq d$, be natural numbers such that $a+b+c+d = 30$. Find the maximum value of the product $P = a \cdot b \cdot c \cdot d$.

Comment by Murray S. Klamkin, University of Alberta.

We give a generalization of the previous generalization and with a "simple" proof. If $a_1, a_2, \ldots, a_n$ are natural numbers with $a_1 \le a_2 \le \cdots \le a_n$ and with given sum

$$a_1 + a_2 + \cdots + a_n = nk + m$$

where k, m are integers with $0 \le m < n$, then the vector $(nk+m-n-1, 1, 1, \ldots, 1)$ (where there are $n-1$ ones) majorizes the vector $(a_n, a_{n-1}, \ldots, a_1)$ and which majorizes the vector $(k+1, k+1, \ldots, k+1, k, k, \ldots, k)$ (where there are $m(k+1)$'s and $n-m$ k's). Consequently by the majorization inequality (see [1988: p. 129] or [1]),

$$\begin{aligned} mF(k+1) + (n-m)F(k) &\ge F(a_1) + F(a_2) + \cdots + F(a_n) \\ &\ge F(nk+m-n-1) + (n-1)F(1) \end{aligned}$$

for F a concave function over the range of the a_i's. Now letting $F(x) = \ln x$, the left hand inequality reduces to the previous generalization.

Reference:

[1] A.W. Marshall, I. Olkin, Inequalities: *Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

*

4. [1990: 290; 1992: 73] *20th Austrian Mathematical Olympiad, Final Round, Beginner's Level.*

Show that for any triangle each exradius is less than four times the circumradius.

Extensions and an alternate proof by Murray S. Klamkin, University of Alberta.

Here we will show that

$$\min(r_a, r_b, r_c) \le \frac{s}{\sqrt{3}} \le \frac{r_a + r_b + r_c}{3} \le \frac{3R}{2},$$

and the method of proof here will lead to another proof of the proposed problem. Incidentally, the latter inequalities provide an interpolation of the well known inequality $s/\sqrt{3} \le 3R/2$ which corresponds geometrically to the equilateral triangle being the maximum perimeter triangle inscribed in a given circle.

We can assume without loss of generality that $a \le b \le c$. Since

$$r_a = rs/(s-a), \quad r_b = rs/(s-b), \quad r_c = rs/(s-c),$$

$r_a \le r_b \le r_c$. Now $r_a \le s/\sqrt{3}$ reduces to

$$3r^2 \le (s-a)^2. \tag{1}$$

Our method of proof is to let $a = y+z$, $b = z+x$, $c = x+y$ where $x \ge y \ge z \ge 0$. This transformation is very useful since it gets rid of the inequality constraints on a, b, c (see [1984: p. 47]). Here, $x = s-a$, etc., and

$$r^2 = \frac{xyz}{x+y+z}, \quad R = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}.$$

Inequality (1) transforms into

$$3yz \leq x(x+y+z) \tag{1$'$}$$

and which is now obvious. There is equality if and only if $x = y = z$, i.e. the triangle is equilateral.

The second inequality

$$r_a + r_b + r_c = rs\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \geq s\sqrt{3} \tag{2}$$

transforms into

$$(xy+yz+zx)^2 \geq 3xyz(x+y+z) \tag{2$'$}$$

and which is well known. It is also equivalent to

$$x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2 \geq 0.$$

There is also equality here if and only if the triangle is equilateral.

The third inequality

$$\frac{9R}{2} \geq r_a + r_b + r_c = rs\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \tag{3}$$

transforms into

$$9(y+z)(z+x)(x+y) \geq 8(x+y+z)(yz+zx+xy) \tag{3$'$}$$

or

$$(x+y+z)(1/x+1/y+1/z) \geq 9.$$

This inequality is also well known and follows from Cauchy's inequality. Again there is equality if and only if the triangle is equilateral.

Finally, the proposed inequality

$$4R > r_c \tag{4}$$

transforms into

$$(y+z)(z+x)(x+y) \geq xy(x+y+z) \tag{4$'$}$$

or

$$z^2(x+y) + z(x^2+y^2) + xyz > 0.$$

Note that by allowing $z = 0$, the latter left hand expression becomes zero. This corresponds to a degenerate triangle of sides $(a, b, a+b)$. Hence, r_c/R can be arbitrarily close to 4.

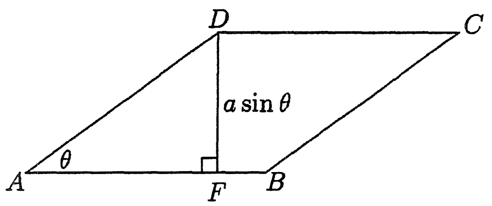
*

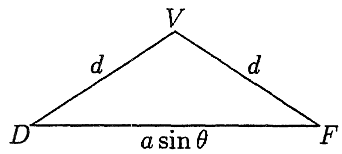
The next batch of solutions were sparked by the solutions given in the April Corner to problems from the Celebration of Chinese New Year Contests [1991: 1–2]. They all come from Leroy F. Meyers. He supplies us with his solutions to almost all the problems (some of which appear to be easier than the ones I used). We begin his solutions for problems which have not been given in the Corner. (See also [1992: 100–103].)

First the *1980 Celebration of Chinese New Year Contest* [1991: 1–2].

1. $ABCD$ is a rhombus of side length a, V is a point in space such that the distances from V to AB and CD are both d. Determine in terms of a and d the maximum volume of the pyramid $VABCD$.

Solution by Leroy F. Meyers, The Ohio State University.





Let the rhombus $ABCD$ be relabelled if necessary (by interchanging A with D and B with C) so that $\theta = \angle BAD$ is nonobtuse. Then the distance between sides AB and CD of rhombus $ABCD$ is $a \sin\theta$. The points V which are at distance d from sides AB and CD lie on the two lines which form the intersection of two circular cylindrical surfaces of radius d whose axes are the lines containing AB and CD. These cylinders intersect just when $a\sin\theta \le 2d$. Note that the foot F of the perpendicular from D to the *line* AB belongs to the *segment* AB, since AF is a leg of the right triangle ADF with hypotenuse $AD = a = AB$. The area of rhombus $ABCD$ (the base of pyramid $VABCD$) is $a^2 \sin\theta$, so that the volume of the pyramid is $(1/3)ha^2\sin\theta$, where h is the distance between V and the plane containing the rhombus, and is thus the altitude of the isosceles triangle VDF from vertex V to the midpoint of side DF. Hence

$$h = \sqrt{d^2 - (1/4)a^2\sin^2\theta}.$$

Thus the volume of the pyramid is

$$\frac{1}{3}(a^2\sin\theta)\cdot\frac{1}{2}\sqrt{4d^2 - a^2\sin^2\theta} = \frac{a}{6}\sqrt{(a^2\sin^2\theta)(4d^2 - a^2\sin^2\theta)}.$$

But by use of the arithmetic mean–geometric mean inequality $\sqrt{xy} \le (x+y)/2$ with $x = a^2\sin^2\theta$ and $y = 4d^2 - a^2\sin^2\theta$ we find that the volume is at most

$$\frac{a}{6}\cdot\frac{4d^2}{2} = \frac{ad^2}{3},$$

with equality just when $a^2\sin^2\theta = 4d^2 - a^2\sin^2\theta$, i.e., $\sin\theta = (\sqrt{2}\,d)/a$, which is always possible, since the requirement that the cylinders intersect is $0 \le \sin\theta \le (2d)/a$.

Note. The problem is somewhat ambiguously stated. What is intended is not that the rhombus $ABCD$ be fixed, but rather that the maximum volume of the pyramid is to be found among all rhombi with side length a.

3. A convex polygon is such that it cannot cover any triangle of area 1/4. Prove that it can be covered by some triangle of area 1.

Solution by Leroy F. Meyers, The Ohio State University.

Let P be any closed bounded convex set in the plane, such as a closed convex polygonal region, and let T be a closed triangular region of maximum area whose vertices belong to P. (There *is* a triangle of maximum area, since the area of a triangle depends continuously on the coordinates of its vertices, and the vertices belong to the compact set P.) Given any side AB of T, the third vertex C of the triangle must be as far as possible from AB. In other words, P must be a subset of the closed half-plane which contains T and whose boundary consists of the line through C parallel to AB. Similarly, P must be a subset of the closed half-planes which contain T and whose boundaries are, respectively, the line through B parallel to AC and the line through A parallel to BC. But the intersection of these three half-planes is a triangular region T' whose sides are twice as long as those of T, and whose area is thus 4 times that of T.

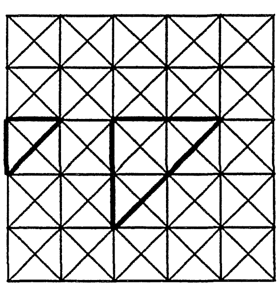
Since P cannot cover any triangle of area 1/4, the area of T must be less than 1/4. But T' covers P, and the area of T' is less than 1. An appropriate similarity with ratio greater than 1 will produce a triangle T^* of area exactly 1 which contains T' and hence contains P.

The requirement that P be convex is necessary, since for small positive ϵ and large positive M the quadrilateral with vertices at $(\epsilon, 0)$, $(M, 1)$, $(0, 0)$, and $(M, -1)$ has area ϵ, and so cannot cover any triangle of larger area, but any triangle covering it contains the three last-mentioned vertices and so has area at least M. In particular, let $\epsilon = 1/5$ and $M = 2$.

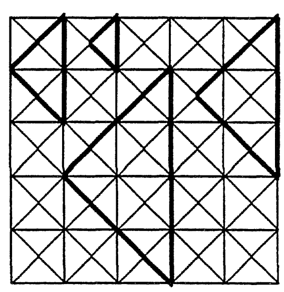
5. A square is divided into n^2 equal squares and the diagonals of each little square are drawn. Determine, in terms of n, the total number of isosceles right angled triangles of all sizes.

Solution by Leroy F. Meyers, The Ohio State University.

There are two types of isosceles right triangle in the figure: (1) those in which the hypotenuse is parallel to a diagonal of the square; and (2) those in which the hypotenuse is parallel to the side of a square.



Type (1)



Type (2)

For type (1), we consider those with right angle at the upper left corner. There are n^2 of these with leg length 1, $(n-1)^2$ with leg length $2, \dots, 1^2$ with leg length n. Hence the number of triangles of type 1 is

$$4(1^2+2^2+\cdots+n^2)=\frac{2n(n+1)(2n+1)}{3}\ ,$$

where all four orientations of triangles of type (1) are counted.

For type (2), we consider the triangles with right angle to the left of the hypotenuse. There are n^2 such triangles with hypotenuse length 1, $(n-1)n$ with hypotenuse length 2, $(n-2)(n-1)$ with hypotenuse length 3, $(n-3)(n-1)$ with hypotenuse length 4, and so on. If $n=2k$ for some positive integer k, then the number of type (2) triangles (with due regard to orientation) is

$$\begin{aligned}&4\cdot((2k)(2k)+(2k-1)(2k)+(2k-2)(2k-1)+\cdots+2\cdot(k+1)+1\cdot(k+1))\\&=4\cdot\sum_{i=1}^{k}(4i-1)(k+i)=4\cdot\sum_{i=1}^{k}(4ik+4i^2-k-i)=\frac{2k}{3}(20k^2+15k+1).\end{aligned}$$

If $n=2k-1$ for some positive integer k, then the number of type (2) triangles is

$$\begin{aligned}&4\cdot((2k-1)(2k-1)+(2k-2)(2k-1)+(2k-3)(2k-2)+\cdots+1\cdot k)\\&=4\cdot\sum_{i=1}^{k}(4i-3)(k+i-1)=4\cdot\sum_{i=1}^{k}(4ik+4i^2-3k-7i+3)=\frac{2k}{3}(20k^2-15k+1).\end{aligned}$$

Adding the numbers for types (1) and (2), we obtain

$$\frac{n}{2}(6n^2+9n+2)\quad \text{if } n \text{ is even,}$$

$$\frac{n+1}{2}(6n^2+3n-1)\quad \text{if } n \text{ is odd,}$$

as the number of isosceles right triangles in the figure.

*

That is all the space we have this month so we will continue next issue with the solutions sent me over the summer. Send me your problem sets and your nice solutions.

*　　*　　*　　*　　*

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1781. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $a > 0$ and $x_1, x_2, \ldots, x_n \in [0, a]$ $(n \geq 2)$ such that

$$x_1 x_2 \ldots x_n = (a - x_1)^2 (a - x_2)^2 \ldots (a - x_n)^2.$$

Determine the maximum possible value of the product $x_1 x_2 \ldots x_n$.

1782. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

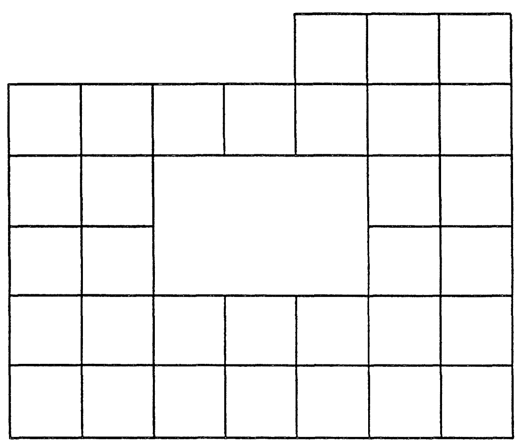
Triangle ABC, with angles α, β, γ, is inscribed in rectangle $APQR$ so that B lies on PQ and C lies on QR. Prove that

$$\cot\alpha \cdot [BCQ] = \cot\beta \cdot [ACR] + \cot\gamma \cdot [ABP],$$

where $[XYZ]$ denotes the area of triangle XYZ. (This problem is an extension of problem 2 of the 1987 Hungarian National Olympiad [1991: 68].)

1783. *Proposed by Andy Liu, University of Alberta, and Daniel van Vliet, student, Salisbury Composite H.S., Sherwood Park, Alberta.*

Dissect the figure into two congruent pieces.



1784. *Proposed by Murray S. Klamkin, University of Alberta, and Dale Varberg, Hamline University, St. Paul, Minnesota.*

A point in 3–space is at distances 9, 10, 11 and 12 from the vertices of a tetrahedron. Find the maximum and minimum possible values of the sum of the squares of the edges of the tetrahedron.

1785. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

(a) Show that the set of numbers of the form $2^r/5^n$, where r is a nonnegative rational number and n is a nonnegative integer, is dense in the set of nonnegative real numbers [that is, any nonnegative real number can be approximated arbitrarily closely by numbers of the form $2^r/5^n$].

(b)* What if r and n must both be nonnegative integers?

1786. *Proposed by R.P. Sealy, Mount Allison University, Sackville, New Brunswick.*

For which values of n can one construct a sequence of n consecutive positive integers so that the mean and variance are both integers?

1787. *Proposed by Isao Ashiba, Tokyo, Japan.*

ABC is an acute triangle and ℓ is a line in the same plane. Let ℓ_a be the line symmetric to ℓ with respect to line BC, and similarly define lines ℓ_b and ℓ_c. Let $A' = \ell_b \cap \ell_c$, $B' = \ell_c \cap \ell_a$, $C' = \ell_a \cap \ell_b$. Show that the incenter of $\Delta A'B'C'$ lies on the circumcircle of ΔABC.

1788. *Proposed by L.J. Bradley, Clifton College, Bristol, England.*

A pack of cards consists of m red cards and n black cards. The pack is thoroughly shuffled and the cards are then laid down in a row. The number of colour changes one observes in moving from left to right along the row is k. (For example, for $m = 5$ and $n = 4$ the row RRBRBBRBR exhibits $k = 6$.) Prove that k is more likely to be even than odd if and only if

$$|m - n| > \sqrt{m + n}.$$

1789*. *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Let a_1, a_2, a_3 be the sides of a triangle, w_1, w_2, w_3 the angle bisectors, F the area, and s the semiperimeter. Prove or disprove that

$$w_1^{a_1} w_2^{a_2} w_3^{a_3} \le (F\sqrt{3})^s.$$

1790. *Proposed by Neven Jurić, Zagreb, Croatia.*

$\mathcal{S}$ is the set of all finite sequences of 0's and 1's. For each $x \in \mathcal{S}$ let $\varphi(x)$ be the sequence obtained if each 1 in x is transformed into 01 and each 0 in x into 10. For example, $\varphi(01) = 1001$. Let $\varphi^2(x) = \varphi(\varphi(x))$ and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$, $n \ge 3$. How many pairs 00 are there in $\varphi^n(1)$?

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1684. [1991: 270] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let

$$f(x,y,z) = x^4 + x^3z + ax^2z^2 + bx^2y + cxyz + y^2.$$

Prove that for any real numbers b, c with $|b| > 2$, there is a real number a such that f can be written as the product of two polynomials of degree 2 with real coefficients; furthermore, if b and c are rational, a will also be rational.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Given b, c with $|b| > 2$, set

$$g(x,y,z) = x^2 + \frac{1}{2}xz + \frac{b}{2}y, \quad h(x,y,z) = \frac{\sqrt{b^2-4}}{2}\left(\frac{2c-b}{b^2-4}xz - y\right).$$

Then

$$g(x,y,z)^2 - h(x,y,z)^2 = x^4 + x^3z + ax^2z^2 + bx^2y + cxyz + y^2 = f(x,y,z),$$

with

$$a = \left(\frac{1}{2} + \frac{2c-b}{2\sqrt{b^2-4}}\right)\left(\frac{1}{2} - \frac{2c-b}{2\sqrt{b^2-4}}\right) = \frac{bc-c^2-1}{b^2-4}\,.$$

So, when $|b| > 2$, then $f = (g-h)(g+h)$, and of course a is rational, provided b, c are rational.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; JOSEPH M. LING, University of Calgary; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1685. [1991: 270] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

If equilateral triangles $A_2A_3P_1, A_3A_1P_2, A_1A_2P_3$ are erected externally on the sides of a triangle $A_1A_2A_3$, then A_1P_1, A_2P_2, A_3P_3 concur at a point R called the *isogonic center* (see p. 218 of R.A. Johnson, *Advanced Euclidean Geometry*). Prove that the line joining R and its isogonal conjugate is parallel to the Euler line of the triangle.

Summary of the partial solutions submitted by both Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain, and the proposer.

We use trilinear coordinates with the given triangle, relabelled ABC, as triangle of reference. The isogonic center R has coordinates

$$(\csc(A+\pi/3), \csc(B+\pi/3), \csc(C+\pi/3));$$

its isogonal conjugate is therefore

$$(\sin(A+\pi/3), \sin(B+\pi/3), \sin(C+\pi/3)),$$

and the line joining them has coefficients proportional to the cyclic permutations of $\sin^2(A+\pi/3)\sin(B-C)$. The Euler line, passing through the circumcenter ($\cos A, \cos B, \cos C$) and its orthogonal conjugate, the orthocenter ($\sec A, \sec B, \sec C$), has coefficients proportional to the cyclic permutations of $\sin 2A \sin(B-C)$. The two lines are parallel if and only if

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ \sin^2(A+\pi/3)\sin(B-C) & \sin^2(B+\pi/3)\sin(C-A) & \sin^2(C+\pi/3)\sin(A-B) \\ \sin 2A \sin(B-C) & \sin 2B \sin(C-A) & \sin 2C \sin(A-B) \end{vmatrix} = 0.$$

Editor's note. Neither Bellot Rosado nor the proposer completed the proof that this determinant is zero. Here is one way to do it, supplied by Chris Fisher (although he remarks that it may not be worth the effort). Since

$$\sin^2\left(A+\frac{\pi}{3}\right) = \left(\frac{1}{2}\sin A + \frac{\sqrt{3}}{2}\cos A\right)^2 = \frac{1}{4}(1+2\cos^2 A + \sqrt{3}\sin 2A), \quad \text{etc.},$$

by subtracting $\sqrt{3}/4$ times the third row from the second we see that the problem is to show

$$D = \begin{vmatrix} \sin A & \sin B & \sin C \\ (2+\cos 2A)\sin(B-C) & (2+\cos 2B)\sin(C-A) & (2+\cos 2C)\sin(A-B) \\ \sin 2A \sin(B-C) & \sin 2B \sin(C-A) & \sin 2C \sin(A-B) \end{vmatrix} = 0.$$

Expanding about the first row,

$$\begin{aligned} D &= \sum \sin A \sin(C-A)\sin(A-B) \begin{vmatrix} 2+\cos 2B & 2+\cos 2C \\ \sin 2B & \sin 2C \end{vmatrix} \\ &= \sum \sin A \sin(C-A)\sin(A-B)\sin(2C-2B)\cdot 2(\sin 2C - \sin 2B), \end{aligned}$$

where the sums are cyclic over A, B, C. Using the identities

$$\sin(2C-2B) = -2\sin(B-C)\cos(B-C),$$

$$\sin 2C - \sin 2B = -2\sin(B-C)\cos(B+C),$$

and $\cos(B+C) = -\cos A$, we find that

$$D = K\sum 4\sin A\cos A\sin(B-C)\cos(B-C) = K\sum \sin 2A\sin 2(B-C),$$

where $K = -2\sin(A-B)\sin(B-C)\sin(C-A)$. Finally, applying

$$\sin 2A\sin 2(B-C) = \frac{1}{2}[\cos(2A-2B+2C) - \cos(2A+2B-2C)]$$

we conclude that $D = 0$ as claimed.

Wanted: an informative explanation of why the two lines should be parallel. Does anybody have a suggestion? (For some related problems by the proposer, see the interesting article by David Gale in his Mathematical Entertainments column in the Spring 1992 *Mathematical Intelligencer.*)

The above claims about the coordinates can be deduced from an introduction to trilinear coordinates such as [2]. The other three references below were supplied by the solvers and apparently provide greater detail.

References:

[1] Casey, *Analytical Geometry*, Dublin U.P., Dublin, 1885.

[2] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, Math. Assoc. of America (New Math. Library #18), 1967.

[3] William Gallatly, *The Modern Geometry of the Triangle*, 2nd ed., Hodgson, London, 1913.

[4] Sommerville, *Analytical Conics*, Bell, London, 1951.

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1687. [1991: 270] *Proposed by Jisho Kotani, Akita, Japan.*

The octahedron $ABCDEF$ is such that the three space diagonals AF, BD, CE meet at right angles. Show that

$$[ABC]^2 + [ADE]^2 + [CDF]^2 + [BEF]^2 = [ACD]^2 + [ABE]^2 + [BCF]^2 + [DEF]^2,$$

where $[XYZ]$ is the area of triangle XYZ.

Solution by David Singmaster, South Bank Polytechnic, London, England.

Let O denote the intersection of the three space diagonals. By a theorem sometimes called de Gua's,

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2 \qquad (1)$$

(e.g., see H. Eves, *Great Moments in Mathematics (Before 1650)*, Dolciani Mathematical Exposition No. 5, MAA, Washington, 1980, pp. 26–42, esp. 37–38 and 41). Applying this to each of the eight faces of the octahedron, we see that both of the sums in the problem are equal to the sum of the squares of the areas of the twelve triangles determined by O and an edge of the octahedron.

More references to (1), and generalizations, are in my paper "The n-dimensional law of cosines" (submitted).

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta (two solutions); MARCIN E. KUCZMA, Warszawa, Poland; ALBERT KURZ, student, Council Rock H.S., Newtown, Pennsylvania; JOSEPH LING, University of Calgary; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; SHOBHIT SONAKIYA, Kanpur, India; and the proposer. One incorrect solution was sent in.

Several solvers knew and used the nice relation (1), *some giving a proof.*

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1688. [1991: 271] *Proposed by Leroy F. Meyers, The Ohio State University.*

Solve the equation

$$a \log b = \log(ab)$$

if a and b are required to be (i) positive integers, (ii) positive rational numbers.

Solution by David R. Stone, Georgia Southern University, Statesboro, and Tina H. Straley, Kennesaw State University, Marietta, Georgia.

The given equation can be rewritten as $b^a = ab$ or $b^{a-1} = a$ or (if $a \neq 1$)

$$b = a^{1/(a-1)}. \tag{1}$$

(i) One collection of solutions is $\{(1,b)|b \text{ any positive integer}\}$. If $a = 2$, then we see that $b = 2$ also. For a an integer greater than 2, the sequence $\{a^{1/(a-1)}\}$ decreases but never reaches its limit of 1, and thus never achieves any more integer values. So the solutions in this case are

$$\{(1,b)|b \text{ any positive integer}\} \quad \text{and} \quad (2,2).$$

(ii) Again we have the collection of solutions $\{(1,b)|b \text{ a positive rational}\}$. Note that the function $(a,b) \to (1/a, ab)$ establishes a one-to-one correspondence between those positive rational solutions with $a > 1$ and those with $a < 1$. Hence we need only find those with $a > 1$.

If we let $a = m/n$, with m and n relatively prime and $m > n > 1$, our condition (1) becomes

$$b = (m/n)^{n/(m-n)}. \tag{2}$$

We must find m and n which make b rational.

First suppose that $m = n+1$. Then $b = [(n+1)/n]^n$ is certainly rational. That is,

$$a = \frac{n+1}{n}, \quad b = \left(\frac{n+1}{n}\right)^n \tag{3}$$

is a solution.

We claim there are no other solutions (with $a > 1$). For this we need the easily proved result:

Lemma. *If m/n and u/v are reduced rationals and $(m/n)^{u/v}$ is rational, then m and n are both vth powers.*

Now suppose $m - n = c$ with $c > 1$; note that n and c are coprime. Then from (2)

$$b = \left(\frac{n+c}{n}\right)^{n/c}.$$

If we suppose b rational and apply the lemma, we see that $n + c = y^c$ and $n = x^c$, for some positive coprime integers x and y, with $y > x > 1$. Upon substitution,

$$c = y^c - x^c = (y-x)(y^{c-1} + y^{c-2}x + \cdots + yx^{c-2} + x^{c-1}).$$

But $y - x$ is positive and each of the c terms in the other factor is at least 2, forcing the product to be larger than c and yielding a contradiction.

In summary, all rational solutions are given by

$$a = 1, \quad b = \text{ any positive rational;}$$

$$a = \frac{n+1}{n}, \quad b = \left(\frac{n+1}{n}\right)^n, \quad n \geq 1;$$

$$a = \frac{n}{n+1}, \quad b = \left(\frac{n+1}{n}\right)^{n+1}, \quad n \geq 1.$$

Both parts also solved by HAYO AHLBURG, Benidorm, Spain; SAM BAETHGE, Science Academy, Austin, Texas; EUGENE A. HERMAN, Grinnell College, Grinnell, Iowa; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands (with only a trivial error); KENNETH M. WILKE, Topeka, Kansas (also a trivial error); and the proposer. Part (i), *but not all of part* (ii), *solved by H.L. ABBOTT, University of Alberta; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; DAVID HANKIN, Brooklyn, N.Y.; MARCIN E. KUCZMA, Warszawa, Poland; CORY PYE, student, Memorial University of Newfoundland, St. John's; and SHOBHIT SONAKIYA, Kanpur, India. One solution incorrect in both parts was sent in.*

Some solvers pointed out that the rational solutions (a, b) with $a \neq 1$ can all be expressed in the form (3) *if n is allowed to be any integer, negative or positive, except for* 0 *and* -1.

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1689. [1991: 271] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

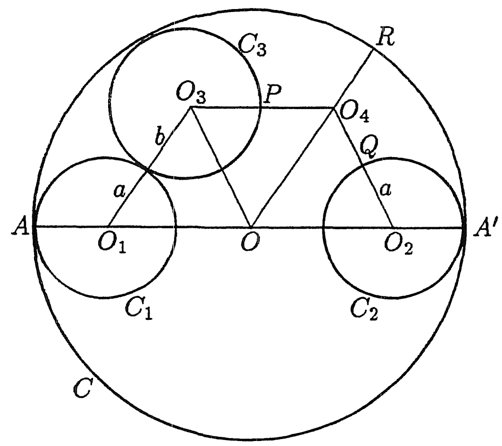
AA' is a diameter of circle $C = (O, r)$. Two congruent circles $C_1 = (O_1, a)$ and $C_2 = (O_2, a)$ $(a < r)$ are internally tangent to C at A and A' respectively. In one half of the circle C we draw two more circles (O_3, b) and (O_4, c) externally touching each other, both internally touching C, and also externally touching C_1 and C_2 respectively. Show that

$$\text{(i) } r = a + b + c\,; \qquad \text{(ii) } O_3O_4 \parallel AA'\,.$$

Solution by P. Penning, Delft, The Netherlands.

In the problem it is implicitly understood that there exists a unique solution for $C_4 = (O_4, c)$, once the other four circles are given. First I give the method to construct C_4.

To determine the centre O_4 of C_4 construct the parallelogram $OO_1O_3O_4$ with O_3O_4 parallel and equal to O_1O. Note that [since $O_1O = OO_2$] $OO_3O_4O_2$ is now a parallelogram also. That O_4 is the centre we look for follows from the fact that the distances to the circles $C_3 = (O_3, b)$, C_2 and C are equal:



$$O_3O_4 = O_1O = r - a, \qquad \text{so } O_4P = r - a - b;$$

$$O_2O_4 = OO_3 = r - b, \qquad \text{so } O_4Q = r - a - b;$$

$$OO_4 = O_1O_3 = a + b, \qquad \text{so } O_4R = r - a - b.$$

So (ii) O_3O_4 must be parallel to AA', and the radius of C_4 is $c = r - a - b$, giving (i).

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAC KONEČNÝ, Ferris State University, Big Rapids, Michigan; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; SHOBHIT SONAKIYA, Kanpur, India; and the proposer.

The problem was taken from the 1877 Japanese mathematics book Sanpo Kigen Syu.

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1690. [1991: 271] *Proposed by Charlton Wang, student, Waterloo Collegiate Institute, and David Vaughan and Edward T.H. Wang, Wilfrid Laurier University.*

When working on a calculus problem, a student misinterprets "the average rate of change of $f(x)$ from a to b" to mean "the average of the rates of change of $f(x)$ at a and b", but obtains the correct answer. Determine all infinitely differentiable functions $f(x)$ for which this occurs, i.e., for which

$$\frac{f(b) - f(a)}{b - a} = \frac{f'(b) + f'(a)}{2}$$

for all $a \neq b$.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.

For fixed a and b, the given condition implies

$$2f(x) - 2f(a) = (f'(x) + f'(a))(x - a)$$

and

$$2f(x) - 2f(b) = (f'(x) + f'(b))(x - b)$$

for all real x. Assuming $a \neq b$, these equations yield

$$f'(x) = \frac{f'(b) - f'(a)}{b - a}\,x + \frac{2(f(b) - f(a)) - (bf'(b) - af'(a))}{b - a}\ .$$

Thus f takes the general form

$$f(x) = a_2x^2 + a_1x + a_0,$$

for some constants a_0, a_1, a_2. It is easy to check that this satisfies the given equation for arbitrary a_0, a_1, a_2.

Note. We assumed only that f was continuously differentiable.

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D. HNIDEI, student, University of Windsor; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOSEPH M. LING, University of Calgary; BEATRIZ MARGOLIS, Paris, France; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; CORY PYE, student, Memorial University of Newfoundland, St. John's; SHOBHIT SONAKIYA, Kanpur, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposers.

Many solvers pointed out that it was not necessary to assume f was infinitely differentiable.

Prielipp spotted this result as part of the solution of Crux 374 [1979: 141–142]*! He also points out that it follows from the note "A property of quadratic polynomials" by S. Haruki, in the* American Mathematical Monthly (1979) 577–579.

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1691*. [1991: 301] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$. Determine the best upper bound of

$$\frac{x_1}{x_2x_3\ldots x_n + 1} + \frac{x_2}{x_1x_3\ldots x_n + 1} + \cdots + \frac{x_n}{x_1x_2\ldots x_{n-1} + 1}\ ,$$

over all $x_1, \ldots, x_n$ with $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$.

Solution by Manuel Benito Muñoz, I.B. Sagasta, Logroño, Spain (slightly altered by the editor).

First we prove

LEMMA. For $n \geq 2$ and $0 \leq x_i \leq 1$ $(1 \leq i \leq n)$,

$$f(x_1, x_2, \ldots, x_n) := x_1 + x_2 + \cdots + x_n - x_1 x_2 \ldots x_n - (n-1) \leq 0.$$

Proof. We use induction over n. For $n = 2$, we have

$$f(x_1, x_2) = x_1 + x_2 - x_1 x_2 - 1 = (x_1 - 1)(1 - x_2) \leq 0.$$

Suppose now that the inequality holds for $n = k - 1$. Then

$$\begin{aligned} f(x_1, x_2, \ldots, x_k) &= (1 - x_2 \ldots x_k)x_1 + x_2 + \cdots + x_k - (k-1) \\ &\leq 1 - x_2 \ldots x_k + x_2 + \cdots + x_k - (k-1) \\ &= x_2 + \cdots + x_k - x_2 \ldots x_k - (k-2) \\ &\leq 0 \end{aligned}$$

by the induction hypothesis. Thus the inequality is true for $n = k$. □

After this lemma, the solution to the problem:

$$\begin{aligned} &\frac{x_1}{x_2 x_3 \ldots x_n + 1} + \frac{x_2}{x_1 x_3 \ldots x_n + 1} + \cdots + \frac{x_n}{x_1 x_2 \ldots x_{n-1} + 1} \\ &\qquad \leq \frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \ldots x_n + 1} \\ &\qquad = \frac{x_1 + x_2 + \cdots + x_n - (n-1)(x_1 x_2 \ldots x_n + 1)}{x_1 x_2 \ldots x_n + 1} + (n-1) \\ &\qquad \leq \frac{x_1 + x_2 + \cdots + x_n - x_1 x_2 \ldots x_n - (n-1)}{x_1 x_2 \ldots x_n + 1} + (n-1) \\ &\qquad \leq n - 1 \end{aligned}$$

by the lemma. Putting $x_1 = 0$, $x_2 = \cdots = x_n = 1$, we see that $n - 1$ is the best upper bound.

Also solved by GENE ARNOLD, Ferris State University, Big Rapids, Michigan; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; and the proposer. One incorrect solution was sent in.

Klamkin and the proposer both offered generalizations, to

$$\frac{x_1}{x_2 x_3 \ldots x_n + k} + \frac{x_2}{x_1 x_3 \ldots x_n + k} + \cdots + \frac{x_n}{x_1 x_2 \ldots x_{n-1} + k}$$

$(k > 0)$ *for example, for which the upper bound is*

$$\begin{cases} (n-1)/k & \text{if } n \geq k+1, \\ n/(k+1) & \text{if } n \leq k+1. \end{cases}$$

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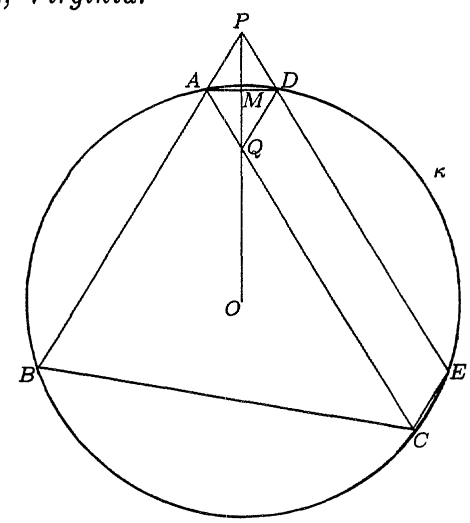
1692. [1991: 301] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcenter O and with $\overline{AB} < \overline{AC}$. Let P be a point on BA produced beyond A such that $\overline{BP} = \overline{AC}$, and let Q be a point on side AC such that $\overline{CQ} = \overline{AB}$. Suppose that P, Q, O are collinear. Find $\angle BAC$.

I. *Solution by Dan Sokolowsky, Williamsburg, Virginia.*

Since $BP = AC = b$ and $CQ = AB = c$,

$$AP = b - c = AQ. \qquad (1)$$



Let κ be the circumcircle of ΔABC. Draw $AD \perp OP$ (D on κ) and let AD and OP meet at M. Then $AM = MD$, while (1) implies $PM = MQ$, so $AQDP$ is a rhombus. Hence $PD = AP$. Extend PD to meet κ again at E. Then $AP = PD$ implies $DE = AB = CQ$. Hence, since $PD \| AQ$, $CEDQ$ is a parallelogram, so $EC = DQ$, while $DE \| AC$ implies $EC = AD$. Then $AD = DQ = AP = PD$, so ΔAPD is equilateral. Hence $\angle BAC = \angle APD = 60°$.

Conversely we can show that $\angle BAC = 60°$ implies O, P and Q are collinear. To this end draw $CE' \| AB$ (E' on κ), then draw $E'P' \| AC$ (P' on BA). Let OP' meet AC at Q'. It will then suffice to show $AP' = AQ' = b - c$. Let $E'P'$ meet κ at D'. By construction $ACE'P'$ is a parallelogram, so $CE' = AP'$, also $\angle AP'D' = \angle BAC = 60°$. $E'P' \| AC$ implies $CE' = AD'$, so $AP' = AD'$. Since $\angle AP'D' = 60°$, $\Delta AP'D'$ is equilateral, so $AP' = D'P'$. This implies $BP' = E'P' = AC = b$, so $AP' = BP' - AB = b - c$. By obvious symmetry OP' bisects $\angle AP'D'$, so $\angle AP'Q' = 30°$. Since $\angle P'AQ' = 180° - \angle BAC = 120°$, $\angle AQ'P' = 30° = \angle AP'Q'$, so $AQ' = AP' = b - c$.

II. *Solution by José Yusty Pita, Madrid, Spain.*

Let AD be the internal bisector of $\angle A$; then [because $AP = AQ$] PQ is parallel to AD. Let O' be the projection of O on AD, and let A' be the projection of A on the line PQO; then $AA' = OO'$. Since

$$\angle OAO' = \angle CAD - \angle CAO = \frac{A}{2} - (90° - B) = \frac{B-C}{2} ,$$

we have

$$R \sin\frac{B-C}{2} = OO' = AA' = AP \sin\frac{A}{2} = (b-c)\sin\frac{A}{2}$$
$$= 2R(\sin B - \sin C)\sin\frac{A}{2}$$

$$= 4R\cos\frac{B+C}{2}\sin\frac{B-C}{2}\sin\frac{A}{2}$$
$$= 4R\sin\frac{B-C}{2}\sin^2\frac{A}{2},$$

that is, $4\sin^2(A/2) = 1$, or $A = 60°$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Dou's solution was similar to Solution I.

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1693. [1991: 301] *Proposed by Murray S. Klamkin, University of Alberta.*

If **A**,**B**,**C**,**D** are four distinct unit vectors in space such that

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{C} = \mathbf{C}\cdot\mathbf{A} = \mathbf{D}\cdot\mathbf{B} = \mathbf{D}\cdot\mathbf{C} = 1/2,$$

determine $\mathbf{A}\cdot\mathbf{D}$.

Solution by P. Penning, Delft, The Netherlands.

Let O be the origin and A, B, C, D the endpoints of the unit vectors. Then $OABC$ is a regular tetrahedron; and D must be the mirror image of A in the plane OBC (because $\mathbf{B}$ and $\mathbf{C}$ make equal angles with $\mathbf{D}$). The projection P of A on OBC is the centre of the equilateral triangle OBC, so the vector $\mathbf{P} = \overrightarrow{OP} = (\mathbf{B}+\mathbf{C})/3$. So

$$\mathbf{D} = \mathbf{A} - 2\overrightarrow{PA} = \mathbf{A} - 2(\mathbf{A}-\mathbf{P}) = -\mathbf{A} + 2(\mathbf{B}+\mathbf{C})/3$$

and

$$\mathbf{A}\cdot\mathbf{D} = -\mathbf{A}\cdot\mathbf{A} + \frac{2}{3}(\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}) = -\frac{1}{3}.$$

In case the scalar products, given to be $1/2$, are equal to t, a similar treatment leads to the result $\mathbf{P} = (\mathbf{B}+\mathbf{C})t/(1+t)$ and

$$\mathbf{A}\cdot\mathbf{D} = -1 + \frac{4t^2}{t+1}.$$

Also solved by SAM BAETHGE, Science Academy, Austin, Texas (two solutions); SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, *Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOSEPH M. LING, University of Calgary; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; and the proposer. There was one incorrect solution sent in.*

Several solutions were like the one above. The proposer's original problem was in fact the generalization given by Penning, but the editor printed only a special case.

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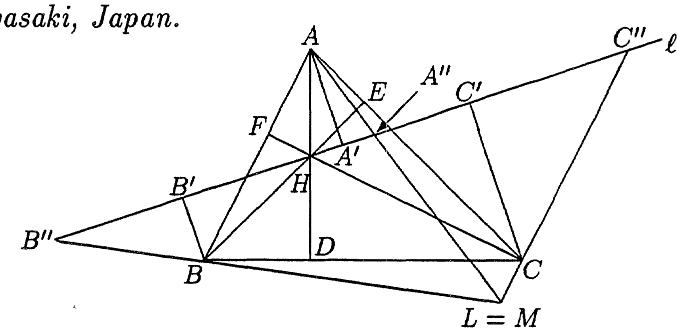
1694. [1991: 301] *Proposed by Jordi Dou, Barcelona, Spain.*

Let ℓ be a line through the orthocentre H of ΔABC. Let A', B', C' be the feet of the perpendiculars to ℓ from A, B, C, and let A'', B'', C'' be chosen on ℓ so that A', B', C' are the midpoints of HA'', HB'', HC'', respectively. Prove that AA'', BB'', CC'' meet in a point L, and determine the locus of L as ℓ rotates about H.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let D, E, F be the feet of the perpendiculars from A, B, C to BC, CA, AB, respectively. Then AD, BE, CF concur at H. [Assume ℓ cuts the segments AB and AC.] Let the intersections of BB'' and CC'' with AA'' be L and M respectively. Because $AA' \perp HA''$ and $HA' = A'A''$, we have

$$\angle AHA' = \angle AA''A'.$$

Similarly we get

$$\angle BHB' = \angle BB''B'.$$

Therefore

$$\begin{aligned}\angle ALB &= \angle AA''B'' - \angle A''B''L = \angle AHA' - \angle BHB' \\ &= \angle AHA' - \angle EHA' = \angle AHE = \angle ACD.\end{aligned}$$

Thus A, B, L, C are concyclic, and L is the intersection of AA'' with the circumcircle of ΔABC other than A. Similarly we can prove that M is the intersection of AA'' with the circumcircle of ΔABC other than A. Therefore M coincides with L. Hence AA'', BB'', CC'' meet in a point L, and L moves on the circumcircle of ΔABC as ℓ rotates about H. Therefore the locus of L is the circumcircle of ΔABC.

[*Editor's note.* A similar argument holds if ΔABC is obtuse with $\angle A > 90°$.]

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; CHRIS FISHER, Clemson University, Clemson, South Carolina; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

* * * * *

1695. [1991: 301] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ with $a_0 > 0$ and

$$a_0 + \frac{a_0 + a_2}{3} + \frac{a_2 + a_4}{5} + \frac{a_4}{7} < 0.$$

Prove that there exists at least one zero of $p(x)$ in the interval $(-1, 1)$.

I. *Solution by Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain.*

Note that

$$\int_{-1}^{1} p(x)\,dx = 2\left(a_0 + \frac{a_2}{3} + \frac{a_4}{5}\right) \quad \text{and} \quad \int_{-1}^{1} x^2 p(x)\,dx = 2\left(\frac{a_0}{3} + \frac{a_2}{5} + \frac{a_4}{7}\right),$$

so that

$$\frac{1}{2}\int_{-1}^{1}(1 + x^2)p(x)\,dx = a_0 + \frac{a_0 + a_2}{3} + \frac{a_2 + a_4}{5} + \frac{a_4}{7} < 0.$$

But $1 + x^2 > 0$, so that $p(x)$ will be strictly negative at some point of $(-1, 1)$. As $p(0) = a_0 > 0$, we have that there exists at least one zero of $p(x)$ in $(-1, 1)$.

II. *Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

We show that there is a zero of $p(x)$ in the interval $(-3/\sqrt{14}, 3/\sqrt{14})$.

The given condition is equivalent to $14a_2 + 9a_4 < -35a_0$. Using this inequality we obtain

$$\begin{aligned} p\left(\frac{3}{\sqrt{14}}\right) + p\left(-\frac{3}{\sqrt{14}}\right) &= 2\left(a_0 + \frac{9}{14}a_2 + \frac{81}{196}a_4\right) = \frac{9}{98}\left(\frac{196}{9}a_0 + 14a_2 + 9a_4\right) \\ &< \frac{9}{98}\left(\frac{196}{9}a_0 - 35a_0\right) < 0, \end{aligned}$$

which means that at least one of the values $p(3/\sqrt{14})$, $p(-3/\sqrt{14})$ is negative. Since $p(0) = a_0 > 0$ we conclude that at least one of the intervals $(-3/\sqrt{14}, 0)$, $(0, 3/\sqrt{14})$ contains at least one zero of $p(x)$, which completes the proof.

Also solved by GENE ARNOLD and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; FRANK FLANIGAN, San Jose State University, San Jose, California; EUGENE A. HERMAN, Grinnell College, Grinnell, Iowa; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, *Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; and the proposer.*

Several solutions, including the proposer's, were similar to solution I *(in fact, Flanigan and Janous gave more general results). The improved interval* $(-3/\sqrt{14}, 3/\sqrt{14})$ *given by Bluskov in Solution* II *was also found by Hess. Can this be further shortened?*

*   *   *   *   *

1697. [1991: 302] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Suppose that P, Q, R are lattice points (i.e., points with integer coordinates) in the plane such that Q and R lie on the parabola $y = x^2$ and PQ, PR are tangents to the parabola. Let n be the total number of lattice points on the sides of triangle PQR, including the vertices. Find the area of ΔPQR.

Solution by Albert Kurz, student, Council Rock H.S., Newtown, Pennsylvania.

Let $Q = (a, a^2)$, $R = (b, b^2)$, with $a < b$ and both integers of course. Then PQ and PR have slopes $2a$ and $2b$ respectively. We see that all integral x-values on the graph of segment QP correspond to integral y-values, since Q is a lattice point and the slope of QP is integral. Likewise on PR, y is an integer whenever x is. Including the endpoints, there are a total then of $b + 1 - a$ lattice points on the "V" formed by PQ and PR. The slope $(a^2 - b^2)/(a - b) = a + b$ of QR is also integral, and so on the graph of QR we again see that y is an integer whenever x is, giving us $b - 1 - a$ additional lattice points, this time not including Q and R, for a grand total of $n = 2(b - a)$ lattice points on the sides of ΔPQR.

We next determine the coordinates of P. Given the slope of PQ along with Q, we can determine the equation of PQ. We do likewise for PR, and find the point of intersection to be $P = ((a + b)/2, ab)$.

Using the well-known formula for area in terms of coordinates, we get

$$\text{area}(\Delta PQR) = \frac{1}{2}\left(\begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} + \begin{vmatrix} b & b^2 \\ (a+b)/2 & ab \end{vmatrix} + \begin{vmatrix} (a+b)/2 & ab \\ a & a^2 \end{vmatrix}\right) = \frac{(a-b)^3}{4} = \frac{n^3}{32}\,.$$

Also solved by H.L. ABBOTT, University of Alberta; GENE ARNOLD, Ferris State University, Big Rapids, Michigan; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; MERCHE SANCHEZ BENITO, Madrid, Spain; DAN SOKOLOWSKY, Williamsburg, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Three incorrect solutions were sent in.

Some solvers pointed out that (as is seen by the above solution) it is not necessary to assume P *is a lattice point. With that assumption, the proposer notes that the minimum possible area of triangle* PQR *is* 2, *occurring when* $a - b = 2$.

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