Contributor Profiles:
Václav Konečný

Václav Konečný was born in Břest, Czechoslovakia in 1934. Both of his parents were elementary school teachers. He used to watch his mother teaching basic arithmetic and he became fascinated with mathematics at an early age.

After the end of the occupation of Czechoslovakia in 1945, his parents sent him and his brother Antonín to obtain their high school education at the Archbishop Seminary and Gymnasium in Kroměříž. There he excelled in Mathematics and Physics, but did poorly in Latin and Greek. The Communist government closed the school in 1950, and he completed his high school education in Gottwaldov in 1952, where he competed in his first Math Olympiad. After a year's work at a factory he studied physics at Masaryk University in Brno. He graduated in 1958, his thesis being on Lommel's problem in Optics (Prof. Albéric Boivin of Laval University verified Václav's correction of one of his formulas).

He was hired by the Physics Department of the Technical University of Brno on the strength of his thesis and his involvement in Physics Olympiads. He began by preparing physics labs, but soon became a Lecturer in Physics.

In 1959 he married and subsequently had two children.

The Vice Chancellor of the University of Khartoum hired Václav as a Lecturer in mathematics in 1964. There he began preparing students for the Cambridge exams until 1970. He also started publishing in the areas of Differential Equations, Combinatorics, Solid State Physics, Optics, Mechanics and problem solving. He received a Ph.D. in Mathematics in 1968 and an M.Sc. in Computer Science in 1984, all while he continued to work.

He was a Post-Doctoral Fellow in mathematics at the University of Saskatchewan for six months in 1970, then moved with his family to Hawkins, Texas, to teach mathematics and physics at Jarvis Christian College. His wife and children returned to Czechoslovakia in 1973. Václav followed, but three years later fled back to Hawkins. He was finally reunited with his family in Texas in 1980, the year he became a professor at Ferris State University.

He retired in 2001, but still promotes ***CRUX*** by presenting problems to faculty and students at Ferris nearly every year. When time permits he solves ***CRUX*** problems, especially the geometry problems. He likes symphony music, but he listens to the Beatles too. He enjoys reading books in foreign languages, especially Spanish, and *El conde de Montecristo* is one of his favourites.

Václav's sincerest hope is for world peace.

EDITORIAL

Václav Linek

The call goes out for nominations for ***CRUX*** editors, in particular the search is on for Problems Editors, Articles Editors, Olympiad Editors, and a new ***Mayhem*** Editor. Though such an extensive transition is a challenge, it is also an exciting time as new people come on board with fresh ideas.

Of course, a new Editor-in-Chief is needed as soon as possible! Nominations can be made either to the existing Editor-in-Chief, or to the Chair of the CMS Publications Committee.

The call for new editors comes with a strengthening of our preference for submissions in TEX or LATEX. We now strongly encourage our readers to submit their solutions in these languages, in order to save time and effort in the preparation of material for publication. As mentioned in this year's March editorial, the ideal submission consists of a LATEX file with a PDF file of the output.

For those not familiar with it, LATEX is a typesetting language ideal for preparing mathematical documents, and the language that we use to typeset the ***CRUX*** journal. For general information about LATEX, see the webpage `http://en.wikipedia.org/wiki/Latex_%28markup_language%29`

Readers not fluent in TEX or LATEX can begin learning it immediately, via the weblink `http://www.arachnoid.com/latex/index.html` to an online LATEX editor. (Note that this editor is set to "math mode", so material cut and pasted from it should be enclosed between two $ signs or two $$ signs.) For those who *are* fluent in LATEX, it is worth mentioning that such an editor is useful for latexing "on the go", for example, when one is visiting a far away, exotic country and only has internet access in a lobby, or if one is using an electronic device too small to support LATEX. LATEX packages are also available on the web for downloading, and one can install this software at no cost. Finally, there are some programs available for converting some common formats to LATEX.

In the meantime, we will continue to process other file formats (even the occasional solution written in lipstick on a napkin), but over time the preference for LATEX submissions will likely strengthen.

We are happy to present our third Contributor Profile of the year in this issue, and look forward to presenting profiles in future issues.

Finally, the call goes out for a variety of problem proposals, in particular send us your favourite problems in Geometry, Algebra, Number Theory, Game Theory, Logic, Calculus, and Probability. Our policy of allowing either new or well-established proposers up to two (2) expedited problem proposals remains in place until the end of 2010.

Václav (Vazz) Linek

SKOLIAD No. 126

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 December, 2010**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Final Round of the Swedish Junior High School Mathematics Contest 2009/2010 (3 hours are allowed for writing). Our thanks go to Paul Vaderlind, Stockholm University, Stockholm, Sweden for providing us with this contest and for permission to publish it.

We also thank Rolland Gaudet, University College of Saint Boniface, Winnipeg, MB, for translating this contest into French.

Swedish Junior High School Mathematics Contest Final Round, 2009/2010

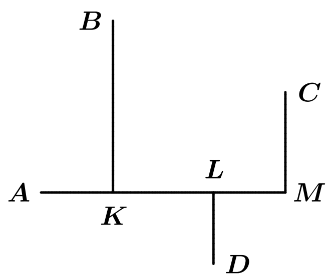
1. A 2009×2010 grid is filled with the numbers 1 and -1. For each row, calculate the product of the entries in that row. Do likewise for the columns. Show that the sum of all the row products and all the column products cannot be zero.

2. The square $ABCD$ has side length 6. The point P splits side AB such that $|AP| : |PB| = 2 : 1$. A point Q inside the square is chosen such that $|AQ| = |PQ| = |CQ|$. Find the area of $\triangle CPQ$.

3. The product of three positive integers is 140. Determine the sum of the three integers if the second integer is seven times the first one.

4. Five points are placed at the intersections of a rectangular grid. Then the midpoint of each pair of points is marked. Prove that at least one of these midpoints lands on an intersection point of the grid.

5. Points K and L on segment AM are placed such that $|AK| = |LM|$. Place the points B and C on one side of AM and point D on the other side of AM such that $|BK| = |KM|$, $|CM| = |KL|$, and $|DL| = |LM|$, and such that BK, CM, and DL are all perpendicular to AM. Prove that $ABCD$ is a square.



6. Let N be a positive integer. Ragnhild writes down all the divisors of N other than 1 and N. She then notes that the largest divisor is 45 times the smallest one. Which positive integers satisfy this condition?

Concours mathématique suédois
Niveau école intermédiaire
Ronde Finale, 2009/2010

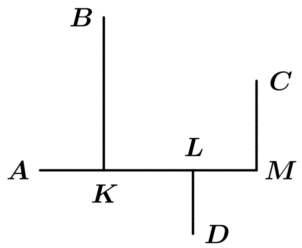
1. Un tableau 2009×2010 est rempli de nombres 1 et -1. Pour chaque rangée, on calcule le produit de tous les nombres dans la rangée. On fait de même pour chacune des colonnes. Démontrer que la somme de tous les produits en rangées et de tous les produits en colonne ne peut pas être nul.

2. Le carré $ABCD$ a côtés de longueur 6. Le point P sépare AB de façon à ce que $|AP| : |PB| = 2 : 1$. Un point Q à l'intérieur du carré est choisi de façon à ce que $|AQ| = |PQ| = |CQ|$. Déterminer la surface de $\triangle CPQ$.

3. Le produit de trois entiers positifs donne 140. Déterminer la somme des trois entiers si le second égale sept fois le premier.

4. Cinq points sont placés aux intersections d'un quadrillage rectangulaire. Ensuite, le point milieu est déterminé pour chaque paire de points parmi les cinq points. Démontrer qu'au moins un de ces points milieu se trouve à un point d'intersection du quadrillage.

5. Les points K et L sont placés sur le segment AM de façon à ce que $|AK| = |LM|$. Plaçons maintenant des points B et C d'un côté de AM et le point D de l'autre côté de AM de façon à ce que $|BK| = |KM|$, $|CM| = |KL|$, et $|DL| = |LM|$, avec en plus que BK, CM, et DL soient tous perpendiculaires à AM. Démontrer que $ABCD$ est un carré.



6. Soit N un entier positif. Ragnhild écrit alors la liste de tous les diviseurs de N sauf 1 et N. Elle note que le plus grand de ces diviseurs égale 45 fois le plus petit des ces diviseurs. Lesquels entiers satisfont ces conditions ?

Next follow solutions to the Maritime Mathematics Competition, 2009, given in Skoliad 120 at [2009 : 417–418].

1. Two cars leave city A at the same time. The first car drives to city B at 40 km/hr and then immediately returns to city A at the same speed. The second car drives to city B at 60 km/hr and then returns to city A at a constant speed, arriving at the same time as the first car. What was the second car's speed on its return trip?

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

Let x denote the distance (in km) between the two cities. Then it takes the first car $\frac{x}{40}$ hours to travel from city A to city B. The time to travel back is also $\frac{x}{40}$ hours, so the total time spent by the first car is $\frac{x}{20}$ hours.

Similarly, the second car takes $\frac{x}{60}$ hours to travel from city A to city B. If s denotes the return speed (in km/h) of the second car, then the return trip takes $\frac{x}{s}$ hours. Therefore the second car travels for a total of $\frac{x}{60} + \frac{x}{s}$ hours.

Since the two cars arrive simultaneously, $\frac{x}{60} + \frac{x}{s} = \frac{x}{20}$. Therefore $\frac{1}{60} + \frac{1}{s} = \frac{1}{20}$, so $\frac{1}{s} = \frac{1}{20} - \frac{1}{60} = \frac{2}{60} = \frac{1}{30}$. Thus $s = 30$, so the return speed of the second car is 30 km/h.

Also solved by IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; PAUL CHEN, student, Burnaby North Secondary School, Burnaby, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; ROWENA HO, student, École Banting Middle School, Coquitlam, BC; TIFFNEY HSIEH, student, Burnaby North Secondary School, Burnaby, BC; ETHAN LIN, student, Burnaby Mountain Secondary School, Burnaby, BC; and KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

2. The perimeter of a regular hexagon H is identical to that of an equilateral triangle T. Find the ratio of the area of H to the area of T.

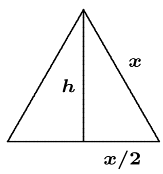
Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

The area of an equilateral triangle with side length x is $\frac{\sqrt{3}}{4}x^2$.

Let a be the side length of the hexagon. Then the side length of the equilateral triangle is $2a$, since H and T have the same perimeter. Therefore, the area of T is $\frac{\sqrt{3}}{4}(2a)^2 = \sqrt{3}a^2$. Cut H into six congruent equilateral triangles like this:  . Each of the six triangles has area $\frac{\sqrt{3}}{4}a^2$, so H has area $\frac{3\sqrt{3}}{2}a^2$. Therefore, the desired ratio is $\frac{3\sqrt{3}}{2}a^2 : \sqrt{3}a^2 = \frac{3}{2} : 1 = 3 : 2$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; TIFFNEY HSIEH, student, Burnaby North Secondary School, Burnaby, BC; and KENRICK TSE, student, Quilchena Elementary School, Vancouver, BC.

To prove our solver's formula for the area of an equilateral triangle, draw in the height, h, and use the Pythagorean Theorem: $h^2 + (\frac{x}{2})^2 = x^2$, so $h^2 = x^2 - \frac{x^2}{4} = \frac{3}{4}x^2$, so $h = \frac{\sqrt{3}}{2}x$. The area is then $\frac{1}{2}hx = \frac{\sqrt{3}}{4}x^2$.



However, one can also solve the problem without calculating the two areas: As our solver notes, you can cut H into triangles,  *. Note that the perimeter of H is six times the side length of the small triangles. You can also cut T into congruent equilateral triangles:*  *.*

Note that the perimeter of T is also six times the side length of the small triangles. Therefore the small triangles in H are congruent with the small triangles in T. Since H contains six triangles while T contains four, the ratio of their areas is $6 : 4 = 3 : 2$.

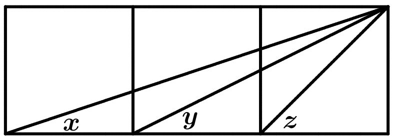
3. Some integers may be expressed as the sum of consecutive odd positive integers. For example, $64 = 13 + 15 + 17 + 19$. Is it possible to express 2009 as the sum of consecutive odd positive integers? If so, find all such expressions for 2009.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

Since 2009 is odd, but the sum of an even number of odd integers is even, the number of integers in the sum must be odd. Thus the sum has a central number, c. The neighbours of c are $c - 2$ and $c + 2$. Note that their sum is $2c$. The sum of their neighbours in turn is also $2c$, and so on. Thus, the sum is an odd multiple of c. That is, c is a divisor of 2009. Since $2009 = 7^2 \cdot 41$, the divisors are 1, 7, 41, 49, 287, and 2009. If $c = 2009$, the sum only has one term—a very silly case. If $c = 287$, the sum has seven terms: $281 + 283 + 285 + 287 + 289 + 291 + 293$. If $c = 49$, the sum has 41 terms: $9 + 11 + \cdots + 49 + \cdots + 87 + 89$. If $c = 41$, the sum has 49 terms: $-7 - 5 - \cdots + 41 + \cdots + 87 + 89$, but the terms were supposed to be positive. If $c = 7$ or $c = 1$, even more terms are negative. Hence, only three sums work out: $9 + \cdots + 89$, $281 + \cdots + 293$, and just 2009 by itself.

Several solvers found the sum $281 + \cdots + 293$ but not the other two. We can agree with our solver that a sum of a single term is silly. You can also consider the sum of zero terms; the value is zero. This is an example of the non-discriminatory nature of mathematics: borderline cases are accepted as long as a sensible definition is possible. The degenerate triangle with side lengths 1, 2, and 3 and that $0! = 1$ are other examples.

4. The diagram shows three squares and angles x, y, and z. Find the sum of the angles x, y, and z.



Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.

We'll use the trigonometric identity

$$\tan(\alpha + \beta) \;=\; \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\,.$$

Now, $\tan y = \dfrac{1}{2}$ and $\tan x = \dfrac{1}{3}$, so

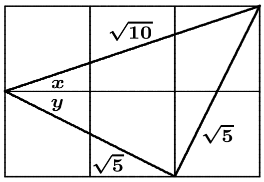
$$\tan(x + y) \;=\; \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} \;=\; \frac{\frac{5}{6}}{1 - \frac{1}{6}} \;=\; 1\,.$$

Therefore $x + y = 45^\circ$. Clearly $z = 45^\circ$, so $x + y + z = 90^\circ$.

The proof of our solver's trigonometric identity is too involved to present here. Fortunately, there is an easier solution: Several solvers noted that since $\tan y = \frac{1}{2}$ and $\tan x = \frac{1}{3}$, it follows that $y \approx 26.6^\circ$ and $x \approx 18.4^\circ$, whence $x + y$ is around

45°. This of course does not prove that $x + y$ is exactly *45°. However, the Pythagorean Theorem yields that the length of the "y-diagonal" is $\sqrt{5}$ and that the length of the "x-diagonal" is $\sqrt{10}$.*

Keeping these two lengths, the Pythagorean Theorem, and a suspected angle of 45° in your head at the same time may inspire you to draw the diagram on the right. By the Pythagorean Theorem, the isosceles triangle in the diagram is right angled. Thus $x + y = 45^\circ$, and $x + y + z = 90^\circ$.



5. Suppose that x_1, x_2, x_3, x_4, and x_5 are real numbers satisfying the following equations.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 4x_2 & + & 9x_3 & + & 16x_4 & + & 25x_5 & = & 1\,, \\ 4x_1 & + & 9x_2 & + & 16x_3 & + & 25x_4 & + & 36x_5 & = & 8\,, \\ 9x_1 & + & 16x_2 & + & 25x_3 & + & 36x_4 & + & 49x_5 & = & 23\,. \end{array}$$

Find the value of $x_1 + x_2 + x_3 + x_4 + x_5$.

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

First, note that the coefficients of each x_i in the three equations are consecutive squares, for example $4x_2$, $9x_2$, and $16x_2$. Second, note that consecutive squares have the form $(n-1)^2$, n^2, and $(n+1)^2$. Third, note that $(n-1)^2 + (n+1)^2 = n^2 - 2n + 1 + n^2 + 2n + 1 = 2n^2 + 2$, so that $(n-1)^2 + (n+1)^2 - 2n^2 = 2$. Therefore, if you add the first and the last equation, and then subtract twice the middle equation, then the coefficient of each x_i is 2, and

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \;=\; 1 + 23 - 2\cdot 8 \;=\; 8\,.$$

Therefore, $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

Also solved by PAUL CHEN, student, Burnaby North Secondary School, Burnaby, BC; and NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.

6. A math teacher writes the equation $x^2 - Ax + B = 0$ on the blackboard where A and B are positive integers and B has two digits. Suppose that a student erroneously copies the equation by transposing the two digits of B as well as the plus and minus signs. However, the student finds that her equation shares a root, r, with the original equation. Determine all possible values of A, B, and r.

Solution by the editors.

Since B has two digits, $B = 10b + c$, where b and c are digits. The teacher's polynomial is then $x^2 - Ax + (10b + c)$. You know that r is a root of this polynomial, so $r^2 - Ar + 10b + c = 0$.

If you reverse the digits of B, you get the number $10c+b$. The student's polynomial is then $x^2 + Ax - (10c + b)$. Again, you know that r is a root of this polynomial, so $r^2 + Ar - b - 10c = 0$.

If you subtract the equation $r^2 - Ar + 10b + c = 0$ from the equation $r^2 + Ar - b - 10c = 0$, you get that $2Ar - 11b - 11c = 0$. Since A, b, and c are positive integers, it follows that r is a positive fraction or an integer.

If you add $r^2 - Ar + 10b + c = 0$ and $r^2 + Ar - b - 10c = 0$ together, you get that $2r^2 + 9b - 9c = 0$, so

$$2r^2 = 9(c - b)\,.$$

Thus $2r^2$ is an integer. Since r is a fraction, r must actually be an integer. (If $c - b$ is even, then r is both a fraction and the square root of an integer, and in this case r must be an integer. If $c - b$ is odd, then there is no fraction r that solves the equation, and this can be shown by using the same argument that shows $\sqrt{2}$ is an irrational number.) It follows that $9(c - b)$ is twice a square. Since 9 is already a square, $c - b$ must be twice a square. Since b and c are digits, the possibilities are $c - b = 0$, $c - b = 2$, or $c - b = 8$.

If $c - b = 0$, then $r = 0$, and then from the original quadratic equation we have $B = 0$. However, B is given to be positive, so this cannot work.

If $c - b = 8$, then $c = 9$ and $b = 1$. Since $2r^2 = 9(c - b) = 72$, you have that $r = 6$. Since $2Ar - 11b - 11c = 0$, you have that $12A - 110 = 0$, contradicting the fact that A is an integer.

If $c - b = 2$, then $B = 10b + c$ is one of these values: 13, 24, 35, 46, 57, 68, or 79. Moreover, $2r^2 = 9 \cdot 2$, so $r = 3$. Since $2Ar - 11b - 11c = 0$, you have that $A = \dfrac{11(b + c)}{6}$. Only the values 24 and 57 yield an integer for A as required, namely 11 and 22, respectively. You can easily verify that both possibilities work out:

$$\begin{aligned}\text{teacher:}\quad x^2 - 11x + 24 &= (x - 3)(x - 8)\,,\\ \text{student:}\quad x^2 + 11x - 42 &= (x - 3)(x + 14)\,,\end{aligned}$$

or the other possible pair of polynomials

$$\begin{aligned}\text{teacher:}\quad x^2 - 22x + 57 &= (x - 3)(x - 19)\,,\\ \text{student:}\quad x^2 + 22x - 75 &= (x - 3)(x + 25)\,.\end{aligned}$$

The prize of a copy of ***CRUX with MAYHEM*** for the best solutions goes to Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

We invite readers to submit solutions to one or more of our problems.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

Mayhem Problems

*Please send your solutions to the problems in this edition by **1 December 2010**. Solutions received after this date will only be considered if there is time before publication of the solutions. The Mayhem Staff ask that each solution be submitted on a separate page and that the solver's name and contact information be included with each solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M445. *Proposed by the Mayhem Staff.*

The lines with equations $y = x + 1$, $y = mx - 1$, and $y = -4x + 2m$ pass through the same point. Determine all possible values for m.

M446. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Let a, b, and c be positive digits. Suppose that b equals the product of a, b, and c, and $\underline{ac} = a + b + c$. Determine a, b, and c. (Here $\underline{ab}$ is the two-digit positive integer with tens digit a and units digit b.)

M447. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let $ABCD$ be a parallelogram. The sides AB and AD are extended to points E and F (respectively) so that E, C, and F all lie on a straight line. Prove that $BE \cdot DF = AB \cdot AD$.

M448. *Proposed by the Mayhem Staff.*

A polyhedron with exactly $m + n$ faces has m faces that are quadrilaterals and n faces that are triangles. Exactly four faces meet at each vertex. Prove that $n = 8$.

M449. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $E(x) = \dfrac{4^x}{4^x + 2}$.

(a) Prove that $E(x) + E(1 - x) = 1$.

(b) Find the value of $E\Big(\dfrac{1}{2010}\Big) + E\Big(\dfrac{2}{2010}\Big) + \cdots + E\Big(\dfrac{2008}{2010}\Big) + E\Big(\dfrac{2009}{2010}\Big)$.

M450. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Prove that if n is an odd positive integer, then $n^{n+2} + (n + 2)^n$ is divisible by $2(n + 1)$.

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M445. *Proposé par l'Équipe de Mayhem.*

Les droites d'équations $y = x + 1$, $y = mx - 1$ et $y = -4x + 2m$ passent toutes par le même point. Trouver toutes les valeurs possibles de m.

M446. *Proposé par J. Walter Lynch, Athens, GA, É-U.*

On suppose que les chiffres positifs a, b et c sont tels que b égale le produit de a, b et c et que $\underline{ac} = a + b + c$. Déterminer a, b et c. (Ici $\underline{ab}$ dénote l'entier positif de deux chiffres, a étant celui des dizaines et b celui des unités.)

M447. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

On prolonge respectivement les côtés AB et AD du parallélogramme $ABCD$ jusqu'aux points E et F de sorte que E, C et F soient tous sur la même droite. Montrer que $BE \cdot DF = AB \cdot AD$.

M448. *Proposé par l'Équipe de Mayhem.*

Un polyèdre à exactement $m + n$ faces en possède m qui sont des quadrilatères et n qui sont des triangles. Chaque sommet est le point de rencontre d'exactement quatre faces. Montrer que $n = 8$.

M449. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit $E(x) = \dfrac{4^x}{4^x + 2}$.

(a) Montrer que $E(x) + E(1 - x) = 1$.

(b) Évaluer $E\Big(\dfrac{1}{2010}\Big) + E\Big(\dfrac{2}{2010}\Big) + \cdots + E\Big(\dfrac{2008}{2010}\Big) + E\Big(\dfrac{2009}{2010}\Big)$.

M450. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Montrer que si n est un entier positif impair, alors $n^{n+2} + (n+2)^n$ est divisible par $2(n+1)$.

Mayhem Solutions

M407. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not the square at right can be completed to form a 4×4 magic square using the integers from 1 to 16. (In a magic square, the sums of the numbers in each row, in each column, and in each of the two main diagonals are all equal.)

			12
	16	1	10
	2	15	8

Solution by Larry Rollins, student, Auburn University Montgomery, Montgomery, Alabama, USA, modified by the editor.

Since the magic square is to contain each of the integers from 1 to 16, then the sum of the entries would be $1+2+\cdots+15+16 = \frac{1}{2}(16)(17) = 136$. Since the sum of the entries in each row is the same, then this sum should be $\frac{1}{4}(136) = 34$. Also, the sum of the entries in each column and on each diagonal should be 34.

Suppose that we can complete the square to form a magic square. In this case, the missing entry in the fourth row should be $34 - 2 - 15 - 8 = 9$, the missing entry in the third row should be $34 - 16 - 1 - 10 = 7$, and the missing entry in the fourth column should be $34 - 12 - 10 - 8 = 4$. Also, the missing entry in the "northeast" diagonal should be $34 - 9 - 16 - 4 = 5$ and the missing entry in the third column should be $34 - 5 - 1 - 15 = 13$.

At this point, we would have the square on the right. The four integers unused are 3, 6, 11, and 14. To form a magic square, the missing entries in the second column would total $34 - 16 - 2 = 16$. No pair of the unused numbers totals 16. Therefore, the square cannot be completed to form a magic square.

		13	4
		5	12
7	16	1	10
9	2	15	8

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CARL LIBIS, Cumberland University, Lebanon, TN, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia. There were three incomplete solutions submitted.

M408. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Determine all three-digit positive integers $\underline{abc}$ that satisfy the equation $\underline{abc} = \underline{ab} + \underline{bc} + \underline{ca}$. (Here $\underline{abc}$ denotes the three-digit positive integer with hundreds digit a, tens digit b, and units digit c.)

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

Since $\underline{abc} = 100a + 10b + c$ and $\underline{ab} = 10a + b$ and $\underline{bc} = 10b + c$ and $\underline{ca} = 10c + a$, then from the given equation, we have $100a + 10b + c = (10a + b) + (10b + c) + (10c + a)$, or $89a = 10c + b$.

Since a, b, and c are digits, then $10c + b \leq 99$, so $89a$ is at most 89. Thus, $a = 1$ and so $10c + b = 89$. Since b and c are digits, then $b = 9$ and $c = 8$, so $\underline{abc} = 198$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; GEOFFREY A. KANDALL, Hamden, CT, USA; CARL LIBIS, Cumberland University, Lebanon, TN, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M409. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL and the Mayhem Staff.*

The three altitudes of a triangle lie along the lines $y = x$, $y = -2x + 3$, and $x = 1$. If one of the vertices of the triangle is at $(5, 5)$, determine the coordinates of the other two vertices.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

We label the vertices of the triangle as $A(5, 5)$, B, and C. We label the points D, E, and F on sides BC, AC, and AB, respectively, so that AD, BE, and CF are the altitudes of the triangle.

Since $A(5, 5)$ is on the line $y = x$ but not on the line $x = 1$ nor on the line $y = -2x + 3$, then it is the altitude from A that lies along the line $y = x$.

We will assume, without loss of generality, that the altitude from B lies along the line $y = -2x + 3$ and that the altitude from C lies along the line $x = 1$.

Since the altitude from B has a slope of -2 and BE is perpendicular to AC, then the slope of AC is $\frac{1}{2}$. Since $A(5, 5)$ also lies on segment AC, then the equation of the line through A and C is $y - 5 = \frac{1}{2}(x - 5)$ or $y = \frac{1}{2}x + \frac{5}{2}$. Since C lies on this line and on the line $x = 1$, then C has x-coordinate 1 and y-coordinate $y = \frac{1}{2} + \frac{5}{2} = 3$. Thus, C has coordinates $(1, 3)$.

Since the altitude from C is vertical and CF is perpendicular to AB, then AB is horizontal and passes through $A(5,5)$. Thus, the equation of the line through A and B is $y = 5$. Therefore, since B lies on the line $y = 5$ and on the line $y = -2x + 3$, then it is the point of intersection of these lines. Since the y-coordinate of B is 5, then its x-coordinate is the solution of $-2x + 3 = 5$ or $x = -1$. Thus, the coordinates of B are $(-1, 5)$.

Finally, the other two vertices have coordinates $(1, 3)$ and $(-1, 5)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, Western Canada High School, Calgary, AB; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; KONSTANTINOS AL. NAKOS, Agrinio, Greece; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M410. *Proposed by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.*

A cube with edge length a, a regular tetrahedron with edge length b, and a regular octahedron with edge length c all have the same surface area. Determine the value of $\dfrac{\sqrt{bc}}{a}$.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

We use the fact that the area of an equilateral triangle with side length s is $\dfrac{\sqrt{3}}{4}s^2$. (This can be derived by splitting the equilateral triangle into two congruent $30°$-$60°$-$90°$ triangles and using the known ratios of side lengths to calculate the area of the larger triangle.)

A cube has six square faces. Thus, the surface area of a cube with edge length a is $6a^2$.

A regular tetrahedron has four faces, each of which is an equilateral triangle. Thus, the surface area of a regular tetrahedron with edge length b is $4 \times \dfrac{\sqrt{3}}{4}b^2 = \sqrt{3}b^2$.

A regular octahedron has eight faces, each an equilateral triangle. Thus, the surface area of a regular octahedron of edge length c is $8\times\dfrac{\sqrt{3}}{4}c^2 = 2\sqrt{3}c^2$.

Since the surface area of each of these solids is the same, we then have $6a^2 = \sqrt{3}b^2 = 2\sqrt{3}c^2$. Thus, $(6a^2)^2 = (\sqrt{3}b^2)(2\sqrt{3}c^2)$ or $36a^4 = 6b^2c^2$. Therefore, $\dfrac{b^2c^2}{a^4} = 6$, so that $\left(\dfrac{b^2c^2}{a^4}\right)^{1/4} = \sqrt[4]{6}$ and $\dfrac{\sqrt{bc}}{a} = \sqrt[4]{6}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; G.C. GREUBEL, Newport News, VA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M411. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Triangle ABC has side lengths a, b, and c. If

$$2a+3b+4c \;=\; 4\sqrt{2a-2} \;+\; 6\sqrt{3b-3} \;+\; 8\sqrt{4c-4} \;-\; 20\,,$$

prove that triangle ABC is right-angled.

Solution by Pedro Henrique O. Pantoja, student, UFRN, Brazil.

From the given equation, we obtain the three equivalent equations

$$\begin{aligned} 2a+3b+4c-4\sqrt{2a-2}-6\sqrt{3b-3}-8\sqrt{4c-4}+20 &= 0\,, \\ (2a-2-4\sqrt{2a-2}+4)+(3b-3-6\sqrt{3b-3}+9) & \\ +\,(4c-4-8\sqrt{4c-4}+16) &= 0\,, \\ (\sqrt{2a-2}-2)^2+(\sqrt{3b-3}-3)^2+(\sqrt{4c-4}-4)^2 &= 0\,. \end{aligned}$$

Since x^2 is nonnegative for any real number x, then we can conclude that

$$\sqrt{2a-2}-2 \;=\; \sqrt{3b-3}-3 \;=\; \sqrt{4c-4}-4 \;=\; 0\,.$$

Therefore, $2a-2=2^2$ (which yields $a=3$) and $3b-3=3^2$ (which yields $b=4$) and $4c-4=4^2$ (which yields $c=5$).

Since $a=3$, $b=4$, and $c=5$, then $a^2+b^2=c^2$, which tells us that the triangle is right-angled.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GEOFFREY A. KANDALL, Hamden, CT, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA.

M412. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

For a real number x, let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x, and let $\{x\}=x-\lfloor x\rfloor$ denote the fractional part of x. Determine all real numbers x for which $\lfloor x\rfloor\cdot\{x\}=x$.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

If $x=0$, then $\lfloor x\rfloor\cdot\{x\}=\lfloor 0\rfloor\cdot\{0\}=0\cdot 0=0$, so $\lfloor x\rfloor\cdot\{x\}=x$.

If $x>0$, then $0\le\lfloor x\rfloor\le x$ and $0\le\{x\}<1$, so $\lfloor x\rfloor\cdot\{x\}<x\cdot 1=x$ and in this case $\lfloor x\rfloor\cdot\{x\}\ne x$.

If $x<0$, then there is an integer k with $k\ge 1$ and $-k\le x<-k+1$; that is, $\lfloor x\rfloor=-k$. Thus, $\{x\}=x-\lfloor x\rfloor=x-(-k)=x+k$. Therefore, the equation $\lfloor x\rfloor\cdot\{x\}=x$ is equivalent to $-k(x+k)=x$, or $(k+1)x=-k^2$, or $x=-\dfrac{k^2}{k+1}$.

Therefore, the solution set to the equation $\lfloor x \rfloor \cdot \{x\} = x$ is $x = 0$ or $x = -\dfrac{k^2}{k+1}$ for some positive integer k. These can be combined to give $x = -\dfrac{k^2}{k+1}$ for some nonnegative integer k.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; CARL LIBIS, Cumberland University, Lebanon, TN, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were two incorrect solutions submitted.

Problem of the Month

Ian VanderBurgh

Some of the best problems are ones that are simple to understand, do not require a whole lot of mathematical background, but that send you nicely down the "garden path".

Problem (2010 Pascal Contest) The product of N consecutive four-digit positive integers is divisible by 2010^2. What is the least possible value of N?

(A) 5 (B) 12 (C) 19 (D) 6 (E) 7

Since we want to find a product of consecutive integers divisible by 2010^2, it makes good sense to find the prime factors of 2010. (It's a good thing that we didn't ask this in 2011.) This isn't that difficult since 2010 is divisible by 10 (since its units digit is 0) and 3 (since the sum of its digits is 3 which is a multiple of 3). Therefore,

$$2010 = 10 \times 201 = 10 \times 3 \times 67 = 2 \times 3 \times 5 \times 67 .$$

But we want to find a product of consecutive integers divisible by 2010^2, so we'd better write out the prime factorization of 2010^2:

$$2010^2 = 2^2 \times 3^2 \times 5^2 \times 67^2 .$$

Now, let's try to solve the problem.

Solution 1. Since we are looking for a set of N consecutive four-digit positive integers whose product is divisible by 2010^2, then we need to look for integers that are divisible by the prime factors of 2010^2.

We start with the largest prime factor of 2010^2, namely 67. We want the product of the integers in our set to include two factors of 67, and so either two different integers in the set are multiples of 67 or one integer in the set has two factors of 67. In the first case, our set of consecutive integers would contain two different multiples of 67 which must be at least 67 apart from each other; this would mean that the set contains at least 68 positive integers. Since none of the available answers are anywhere close to 68, this must not be the case.

Since the integers in the set are four-digit positive integers, then our set should contain $67^2 = 4489$. We also need the product of the integers in the set to contain two multiples of 5. There is no multiple of 5^2 that is close to 4489, so we try expanding the set to include 4490 and 4485, the two closest multiples of 5 to 4489. (These are also the two multiples of 5 that we can include to minimize the total number of integers in the set thus far.)

Up to this point our set is

$$\{4485, 4486, 4487, 4488, 4489, 4490\} .$$

The product of these integers includes two factors of 67 (since $4489 = 67^2$), two factors of 5 (one in 4485 and one in 4490), two factors of 2 (one in 4486 and one in 4488), and also two factors of 3 (one in 4485 and one in 4488). We can check this last fact using a calculator (don't be tempted!) or by noting that the sum of the digits of 4485 is 21 and the sum of the digits of 4488 is 24; each of these sums is divisible by 3 so each of the integers is divisible by 3.

Therefore, the product of the four-digit integers in the set

$$\{4485, 4486, 4487, 4488, 4489, 4490\}$$

is divisible by 2010^2 as required. This set contains 6 integers, so the answer to the problem is (D). ■

This solution is not too difficult and makes good sense. In other words, it is a great solution except for one small problem. Solution 1 is wrong! Can you see what is wrong with it? Take a few minutes and read through it critically to see if you can spot the flaw. Don't be too alarmed if you can't find the flaw – a number of quite good mathematicians have missed this already!

The crucial mistake is in the sentence "Since the integers in the set are four-digit positive integers, then our set should contain $67^2 = 4489$." Can you see the flaw now? The sentence can be corrected be re-writing it as "Since the integers in the set are four-digit positive integers, then our set should contain a four-digit integer that is divisible by $67^2 = 4489$." Let's start Solution 2, picking up from Solution 1 in the third paragraph.

Solution 2. Since the integers in the set are four-digit positive integers, then our set should contain a four-digit integer that is divisible by $67^2 = 4489$. The four-digit multiples of 4489 are 4489 and $2 \times 4489 = 8978$. (Note that 3×4489 is too large, since it has five digits.)

In Solution 1, we saw that if the set includes 4489 and has the desired property, then the minimal size of the set is 6. So let's consider a set that includes 8978.

Let's look for multiples of 5 to include in the set. As in Solution 1, if we include two different multiples of 5, then the set includes at least 6 integers. (Can you see why?) Is it possible to include a multiple of 5^2 in our set in this case that is close to 8978? Yes! We can include 8975, which is divisible by 5^2.

So let's expand our set by including the intermediate integers to get

$$\{8975, 8976, 8977, 8978\} .$$

How are we doing so far with respect to the desired property? The product of these integers includes two factors of 67 (since 8978 is divisible by 67^2), two factors of 5 (since 8975 is divisible by 5^2), and two factors of 2 (at least one in each of 8976 and 8978). How about factors of 3? Since the sum of the digits of 8976 is 30, then 8976 is divisible by 3. However, 8976 is not divisible by 3^2 (since the sum of its digits is not divisible by 9) and none of the other three integers is divisible by 3. (Check the sum of the digits of each.)

Therefore, we need to expand the set to include a second multiple of 3. Is either 8974 or 8979 divisible by 3? Yes, 8979 is divisible by 3. Therefore, the set

$$\{8975, 8976, 8977, 8978, 8979\}$$

has the property that the product of its elements is divisible by 2010^2. Based on our reasoning, this set is the smallest set of four-digit integers with this property. Therefore, the answer to the problem is (A). ■

I really like this problem, because it really tests reasoning skills without requiring a whole lot of formal knowledge. There was quite a debate among the contest creation team as to whether to include the possible answer "(D) 6", given the trap that it sets. Leaving it in undoubtedly trapped a number of contestants, but really rewarded those who saw through this. On the other hand, leaving this answer out might have served as a good "teaching point" for students who got 6 as the answer and then found that their answer wasn't there and as a result persevered to find the real answer. Food for thought! In the end, in the context of the entire paper, the contest creation team decided to leave the answer (D) in.

THE OLYMPIAD CORNER

No. 287

R.E. Woodrow

We start this issue with problems from the Bulgarian National Olympiad, National Round, 2007. My thanks to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

2007 BULGARIAN NATIONAL OLYMPIAD
National Round
May 12–13, 2007

1. (Emil Kolev, Alexandar Ivanov) The quadrilateral $ABCD$ is such that $\angle BAD + \angle ADC > 180^\circ$ and is circumscribed around a circle of centre I. A line through I meets AB and CD at points X and Y, respectively. Prove that if $IX = IY$, then $AX \cdot DY = BX \cdot CY$.

2. (Alexandar Ivanov, Emil Kolev) Find the largest positive integer n such that one can choose 2007 distinct integers from the interval $[2 \cdot 10^{n-1}, 10^n)$ with the property that whenever $1 \le i \le j \le n$, then there exists a chosen number with decimal representation $a_1a_2 \ldots a_n$ and $a_j \ge a_i + 2$.

3. (Nikolai Nikolov, Oleg Mushkarov) Find the least positive integer n for which $\cos\frac{\pi}{n}$ cannot be expressed in the form $p + \sqrt{q} + \sqrt[3]{r}$, where p, q, r are rational numbers.

4. (Emil Kolev, Alexandar Ivanov) Let $k > 1$ be a fixed integer. A set of positive integers S is called *good* if all positive integers can be painted in k colours such that no element of S is a sum of two distinct numbers of the same colour. Find the largest positive integer t for which the set

$$S = \{a + 1, a + 2, a + 3, \ldots, a + t\}$$

is good for all positive integers a.

5. (Oleg Mushkarov, Nikolai Nikolov) Find the least number m for which any five equilateral triangles with combined area m can cover an equilateral triangle of area 1.

6. (Alexandar Ivanov, Emil Kolev) Let $f(x)$ be a monic polynomial of even degree with integer coefficients. Prove that if there are infinitely many integers x for which $f(x)$ is a perfect square, then there is a polynomial $g(x)$ with integer coefficients such that $f(x) = g^2(x)$.

Next we continue with problems from the IMO Team Selection Tests for the Bulgarian Team. Thanks again to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use.

48th IMO
Bulgarian Team First Selection Test
May 16-17, 2007

1. The sequence $\{a_i\}_{i=1}^{\infty}$ is such that $a_1 > 0$ and $a_{n+1} = \dfrac{a_n}{1+a_n^2}$ for $n \geq 1$.

(a) Prove that $a_n \leq \dfrac{1}{\sqrt{2n}}$ for $n \geq 2$.

(b) Prove that there exists an n such that $a_n > \dfrac{7}{10\sqrt{n}}$.

2. Let $A_1A_2A_3A_4A_5$ be a convex pentagon such that the triangles $A_1A_2A_3$, $A_2A_3A_4$, $A_3A_4A_5$, $A_4A_5A_1$, $A_5A_1A_2$ have the same area. Prove that there exists a point M such that the triangles A_1MA_2, A_2MA_3, A_3MA_4, A_4MA_5, A_5MA_1 have the same area.

3. Prove that there are no distinct positive integers x and y such that

$$x^{2007} + y! \;=\; y^{2007} + x!\,.$$

4. Given a point P on the side AB of a triangle ABC, consider all pairs of points (X, Y) such that $X \in BC$, $Y \in AC$ and such that $\angle PXB = \angle PYA$. Prove that the midpoints of the segments XY lie on a straight line.

5. The real numbers a_i, b_i for $1 \leq i \leq n$ are such that

$$\sum_{i=1}^{n} a_i^2 \;=\; \sum_{i=1}^{n} b_i^2 \;=\; 1 \qquad \text{and} \qquad \sum_{i=1}^{n} a_ib_i \;=\; 0\,.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \;\leq\; n\,.$$

6. For a finite set S denote by $\mathcal{P}(S)$ the set of all subsets of S. The function $f : \mathcal{P}(S) \to \mathbb{R}$ is such that

$$f(X \cap Y) \;=\; \min(f(X), f(Y))$$

for any two subsets X, $Y \in \mathcal{P}(S)$. Find the largest number of distinct values that f can take.

48^{th} IMO
Bulgarian Team Second Selection Test
May 26-27, 2007

1. is externally tangent to Γ_1 at Q and to Γ_2 at R. The lines PQ and PR meet Γ_3 again at points A and B, Two circles Γ_1 and Γ_2 with centres O_1 and O_2, ~~respectively are externally tangent at point P. A circle Γ_3~~ respectively. If AO_2 meets BO_1 at a point S, prove that

$$SP \perp O_1O_2 .$$

2. Find all positive integers m such that

$$\frac{2^m\alpha^m - (\alpha+\beta)^m - (\alpha-\beta)^m}{3\alpha^2+\beta^2}$$

is an integer for all integers α and β with $\alpha\beta \neq 0$.

3. Find all integers $n \geq 3$ such that for any two positive integers $m < n-1$ and $r < n-1$ there exist m distinct elements of the set $\{1, 2, \ldots, n-1\}$ whose sum is congruent to r modulo n.

4. Solve the system

$$\begin{aligned} x^2 + yu &= (x+u)^n , \\ x^2 + yz &= u^4 , \end{aligned}$$

where x, y, and z are prime numbers and u is a positive integer.

5. Find all pairs of functions f, $g : \mathbb{R} \to \mathbb{R}$ such that

(a) $f(xg(y+1)) + y = xf(y) + f(x+g(y))$ for any x, $y \in \mathbb{R}$, and

(b) $f(0) + g(0) = 0$.

6. Prove that $n = 11$ is the least positive integer such that for any colouring of the edges of a complete graph of n vertices with three colours there exists a monochromatic cycle of length 4.

Next we turn to the problems from Hellenic competitions and the problems of the Mediterranean Mathematical Competition 2007. Thanks again are due to Bill Sands for collecting them for our use.

10th MEDITERRANEAN MATHEMATICAL COMPETITION 2007

1. Let $x \le y \le z$ be real numbers satisfying the relation $xy + yz + zx = 1$. Prove that $xz < \frac{1}{2}$. Is it possible to improve the value of the constant $\frac{1}{2}$?

2. The quadrilateral $ABCD$ is convex and cyclic, and the diagonals AC and BD intersect at the point E. Given that $AB = 39$, $AE = 45$, $AD = 60$ and $BC = 56$, determine the length of CD.

3. In the triangle ABC the angle $\alpha = \angle A$ and the side $a = BC$ are given. It is known that $a = \sqrt{rR}$, where r is the inradius and R is the circumradius of ABC. Determine all such triangles, that is, compute the sides b and c of all such triangles.

4. Let $x > 1$ be a real number that is not an integer. Prove that

$$\left(\frac{x + \{x\}}{\lfloor x \rfloor} - \frac{\lfloor x \rfloor}{x + \{x\}}\right) + \left(\frac{x + \lfloor x \rfloor}{\{x\}} - \frac{\{x\}}{x + \lfloor x \rfloor}\right) > \frac{9}{2},$$

where $\lfloor x \rfloor$ and $\{x\}$ are the integer and the fractional part of x, respectively.

Continuing with the Hellenic theme we give the problems of the 24th Balkan Mathematical Olympiad written in Rhodes, Greece, April 2007. Thanks again go to Bill Sands for obtaining them for our use.

24th BALKAN MATHEMATICAL OLYMPIAD

April 26, 2007

1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \ne BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120°$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$,

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y .$$

3. Find all positive integers n for which there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ such that the number below is a rational number

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} .$$

[*Ed.: a permutation of the set* $\{1, 2, \ldots, n\}$ *is a one-to-one function of this set to itself.*]

4. For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane such that the sets $C_1 \cap C_2$, $C_2 \cap C_3$, $C_3 \cap C_1$ are finite. Find the maximum cardinality the set $C_1 \cap C_2 \cap C_3$ may have.

To complete the collection of problems for this number we give the Indian Team Selection Test problems, 2007. Thanks again go to Bill Sands for obtaining them for the *Corner*.

INDIAN TEAM SELECTION TEST 2007

1. Let ABC be a triangle with $AB = AC$, and let Γ be its circumcircle. The incircle γ of ABC moves (slides) on BC in the direction of B. Prove that when γ touches Γ internally, it also touches the altitude through A.

2. Consider the quadratic polynomial $p(x) = x^2 + ax + b$, where a, b are in the interval $[-2, 2]$. Determine the range of the real roots of $p(x)$ as a and b vary over $[-2, 2]$.

3. Let triangle ABC have side lengths a, b, c; circumradius R, and internal angle bisector lengths w_a, w_b, w_c. Prove that

$$\frac{b^2+c^2}{w_a} + \frac{c^2+a^2}{w_b} + \frac{a^2+b^2}{w_c} > 4R\,.$$

4. Let a_1, a_2, $\dots$, a_n be an ordering of the numbers 1, 2, $\dots$, n. Find

$$\min \sum_{j=1}^{n} |a_j - a_{j+1}| \quad \text{and} \quad \max \sum_{j=1}^{n} |a_j - a_{j+1}|\,,$$

where $a_{n+1} = a_1$ and the extrema are taken over all such possible orderings.

5. Show that in a non-equilateral triangle, the following are equivalent:

(a) The angles of the triangle are in arithmetic progression.

(b) The common tangent to the nine-point circle and the incircle is parallel to the Euler line.

6. Let X be the set of all bijective functions from $S = \{1, 2, 3, \dots, n\}$ to itself. Let $f^0(x) = x$ and $f^{(k)}(x) = f(f^{(k-1)}(x))$ for $k \geq 1$, and for each $f \in X$ define

$$T_f(j) = \begin{cases} 1, & \text{if } f^{(12)}(j) = j\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Determine

$$\sum_{f \in X} \sum_{j=1}^{n} T_f(j)\,.$$

7. Let a, b, and c be nonnegative real numbers such that $a + b \le c + 1$, $b + c \le a + 1$, and $c + a \le b + 1$. Prove that

$$a^2 + b^2 + c^2 \ \le\ 2abc + 1\,.$$

8. Given a finite string S of symbols a and b, we write $\triangle(S)$ for the number of a's in S minus the number of b's. (For example, $\triangle(abbabba) = -1$.) We call a string S *balanced* if every substring (of consecutive symbols) T of S has the property that $-1 \le \triangle(T) \le 2$. (Thus, $abbabba$ is not balanced, as it contains the substring $bbabb$ and $\triangle(bbabb) = -3$.) Find, with proof, the number of balanced strings of length n.

9. Define the functions f, g, h on $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ as follows:

$$\begin{aligned} f(x,y,z) &= (3x+2y+2z,\ 2x+2y+z,\ 2x+y+2z)\,,\\ g(x,y,z) &= (3x+2y-2z,\ 2x+2y-z,\ 2x+y-2z)\,,\\ h(x,y,z) &= (3x-2y+2z,\ 2x-y+2z,\ 2x-2y+z)\,. \end{aligned}$$

Given a primitive Pythagorean triple (x,y,z), with $x > y > z$, prove that (x,y,z) can be uniquely obtained by repeated application of f, g, h to the triple $(5,4,3)$. For example, $(697,528,455) = (f\circ h\circ g\circ h)(5,4,3)$.

10. (Short-List, IMO-2007) Circles Γ_1 and Γ_2, with centres O_1 and O_2 are externally tangent at the point D and internally tangent to a circle Γ at points E and F, respectively. Line ℓ is the common tangent to Γ_1 and Γ_2 at D. Let AB be the diameter of Γ perpendicular to ℓ so that A, E, O_1 are on the same side of the line ℓ. Prove that AO_1, BO_2 and EF are concurrent.

11. Find all pairs of integers (x,y) such that $y^2 = x^3 - p^2x$, where p is a prime such that $p \equiv 3 \pmod 4$.

12. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the equation

$$f(x+y) + f(x)f(y) \ =\ (1+y)f(x) + (1+x)f(y) + f(xy)\,,$$

for all x, $y \in \mathbb{R}$.

Next is an apology for having misfiled some solutions from Mohammed Aassila with a group of solutions to problems for a later number of the *Corner*. He should also appear as a solver for two problems discussed in the April number of the *Corner*. These are Problem 3, Thai Mathematical Olympiad [2010 : 155; 2009 : 22] and Problem 2, The Italian Mathematical Olympiad [2010 : 164; 2009 : 25–26].

Now we turn to solutions from our readers to problems of the Hungarian Mathematical Olympiad 2005–2006, National Olympiad, Grades 11–12, Second Round given at [2009 : 211].

1. Find the positive values of x that satisfy

$$x^{(2\sin x - \cos 2x)} \;<\; \frac{1}{x}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Sinc $\cos 2x = 1 - 2\sin^2 x$ and $x > 0$, the given inequality is equivalent to $x^{2\sin x(1+\sin x)} < 1$. Write $a = 2\sin x(1 + \sin x)$. For $x > 0$, the inequality $x^a < 1$ holds if and only if either

(1) $0 < x < 1$ and $a > 0$, or

(2) $x > 1$ and $a < 0$.

If $0 < x < 1$, then we have $\sin x > 0$; hence the condition (1) is satisfied.

If $x > 1$, then $a < 0$ holds if and only if $-1 < \sin x < 0$. Thus, (2) is equivalent to $(2k+1)\pi < x < \left(2k + \frac{3}{2}\right)\pi$ or $\left(2k + \frac{3}{2}\right)\pi < x < (2k+2)\pi$ for some nonnegative integer k.

Consequently, the solution set is the union of these open intervals:

$$(0,1) \;\cup\; \bigcup_{k=0}^{\infty} \left((2k+1)\pi,\; \left(2k + \frac{3}{2}\right)\pi\right) \;\cup\; \bigcup_{k=0}^{\infty} \left(\left(2k + \frac{3}{2}\right)\pi,\; (2k+2)\pi\right)\,.$$

2. For $f(x) = ax^2 - bx + c$ we know that $0 < |a| < 1$, $f(a) = -b$, and $f(b) = -a$. Prove that $|c| < 3$.

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania. We give Kandall's write-up.

First, suppose $a = b$. Then $f(x) = ax^2 - ax + c$ and $-a = f(a) = a^3 - a^2 + c$, that is, $c = a^2 - a^3 - a$. Then $|c| \le |a|^2 + |a|^3 + |a| < 3$.

Next, suppose $a \neq b$. Then

$$-b \;=\; f(a) \;=\; a^3 - ab + c\,, \qquad (1)$$

$$-a \;=\; f(b) \;=\; ab^2 - b^2 + c\,. \qquad (2)$$

Subtracting (2) from (1), we obtain successively

$$\begin{aligned} a - b &= a(a^2 - b^2) - b(a - b)\,,\\ 1 &= a(a+b) - b\,,\\ b(1-a) &= (a+1)(a-1)\,,\\ b &= -(a+1)\,. \end{aligned}$$

From (2) we now successively deduce

$$\begin{aligned} -a &= a(a+1)^2 - (a+1)^2 + c\,, \\ c &= 1 - a^2 - a^3\,, \\ |c| &\le 1 + |a|^2 + |a|^3 \;<\; 3\,, \end{aligned}$$

as desired.

Next we look at solutions to problems of the Final Round of the Hungarian Mathematical Olympiad 2005–2006 National Olympiad, Grades 11–12 given at [2009 : 211-212].

1. Define the function $t(n)$ on the nonnegative integers by $t(0) = t(1) = 0$, $t(2) = 1$, and for $n > 2$ let $t(n)$ be the smallest positive integer which does not divide n. Let $T(n) = t(t(t(n)))$. Find the value of S if

$$S = T(1) + T(2) + T(3) + \cdots + T(2006)\,.$$

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comǎneşti, Romania. We give Bataille's solution, modified by the editor.

We observe that $T(n) = 0$ if $n = 2$ or if n is an odd integer, hence $S = T(4) + T(6) + T(8) + \cdots + T(2006)$.

If $n \in \mathcal{P} = \{4,\,6,\,8,\,\ldots,\,2006\}$ and n is not a multiple of 3, then $t(n) = 3$ and $T(n) = 1$.

Let $\mathcal{P}_1 = \{6,\,18,\,30,\,\ldots,\,6\cdot 333\}$ be the set of elements of $\mathcal{P}$ of the form $6(2m-1)$, and let $\mathcal{P}_2 = \{12,\,24,\,36\ldots,\,6\cdot 334\}$ be the set of elements of $\mathcal{P}$ of the form $12m$.

For each $n \in \mathcal{P}_1$, we have $t(n) = 4$, hence $T(n) = 2$, and there are 167 numbers in $\mathcal{P}_1$.

If $n \in \mathcal{P}_2$ and n is not a multiple of 5 or n is not a multiple of 7, then $t(n) = 5$ or $t(n) = 7$, and then $T(n) = 1$. Otherwise $T(n)$ is one of the following: $T(420) = 2$, $T(840) = 1$, $T(1260) = 2$, or $T(1680) = 1$.

Thus, $T(n) = 1$ for each $n \in \mathcal{P}_2$ except for two numbers where T takes the value 2.

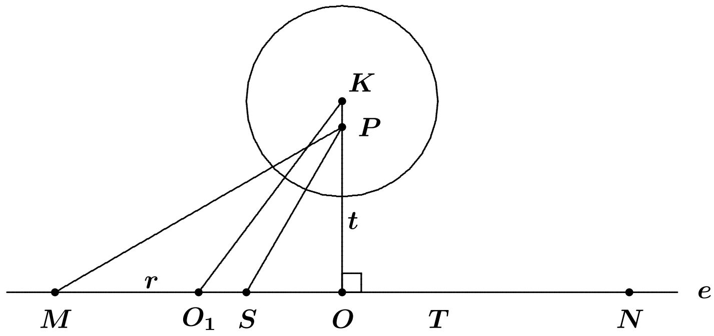
In summary, $T(n) = 2$ for $167{+}2 = 169$ numbers and $T(n) = 1$ for the $1002 - 169 = 833$ remaining numbers. Therefore, $S = 833 + 2\cdot 169 = 1171$.

3. A unit circle k with centre K and a line e are given in the plane. The perpendicular from K to e intersects e in point O and $KO = 2$. Let $\mathcal{H}$ be the set of all circles centred on e and externally tangent to K.

Prove that there is a point P in the plane and an angle $\alpha > 0$ such that $\angle APB = \alpha$ for any circle in $\mathcal{H}$ with diameter AB on e. Determine α and the location of P.

Solution by Titu Zvonaru, Comănești, Romania.

Let M and N be points on e such that $OM = ON$ and the circles with diameter OM and ON are in $\mathcal{H}$. Let ST be the diameter of the circle belonging to $\mathcal{H}$ and centred at O.



The $\triangle MPN$ is isosceles since $\angle MPO = \angle NPO = \alpha$ and O is the midpoint of MN, hence the point P lies on KO.

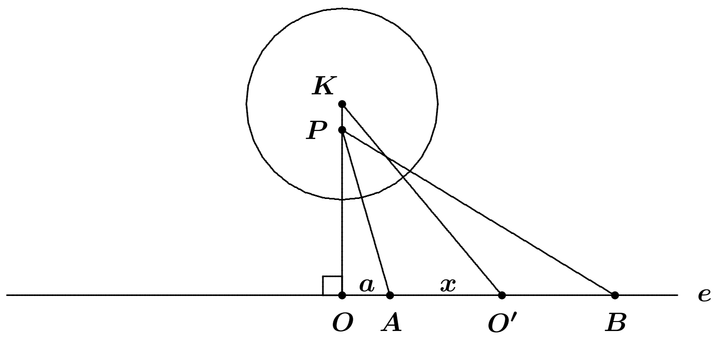
Let O_1 be the midpoint of OM, and let $r = MO_1 = O_1O$. In $\triangle KO_1O$ we have $O_1O^2 + OK^2 = O_1K^2$, or $r^2 + 4 = (r+1)^2$, and hence $r = \frac{3}{2}$.

Let $OP = t$; then in $\triangle PMO$ we have $\tan\alpha = \frac{3}{t}$, and in $\triangle SPO$ we have $\tan\frac{\alpha}{2} = \frac{1}{t}$. Using the formula $\tan\alpha = \frac{2\tan\alpha/2}{1-\tan^2\alpha/2}$ we obtain

$$\frac{3}{t} = \frac{\frac{2}{t}}{1-\frac{1}{t^2}} \iff \frac{3}{t} = \frac{2t}{t^2-1} \iff t^2 = 3\,,$$

so $OP = \sqrt{3}$ and $\alpha = 60^\circ$.

It remains to prove that the point P and the angle α have the property that $\angle APB = \alpha$ for any circle in $\mathcal{H}$ with diameter AB on e.



Suppose that A lies between O and B. Let $OA = a$, $AO' = x$, and let the midpoint of AB be O'.

In $\triangle KOO'$ we obtain

$$OO'^2 + OK^2 \; = \; O'K^2 \iff (a+x)^2 + 4 \; = \; (x+1)^2 \,,$$

hence, $x = \dfrac{a^2+3}{2(1-a)}$. It follows that $OB = a + 2 \cdot \dfrac{a^2+3}{2(1-a)} = \dfrac{a+3}{1-a}$.

Thus

$$\begin{aligned} \tan\angle APB &= \tan(\angle OPB - \angle OPA) \\ &= \frac{\tan\angle OPB - \tan\angle OPA}{1 + \tan\angle OPB \cdot \tan\angle OPA} \\ &= \frac{\dfrac{a+3}{\sqrt{3}(1-a)} - \dfrac{a}{\sqrt{3}}}{1 + \dfrac{a(a+3)}{3(1-a)}} \;=\; \frac{a^2+3}{\sqrt{3}(1-a)} \cdot \frac{3(1-a)}{3+a^2} \;=\; \sqrt{3}\,, \end{aligned}$$

hence $\angle APB = 60^\circ$.

For other locations of AB, the calculations are similar.

Next we move to the Hungarian Mathematical Olympiad 2005–2006, Specialized Mathematical Classes, First Round given at [2009 : 212].

1. Is it true that there are infinitely many palindromes in the arithmetic progression $7k+3$, $k = 0, 1, 2, \ldots$? (A number is a palindrome if reversing its digits yields the same number, for example, 12321 is a palindrome.)

Solution by Titu Zvonaru, Comănești, Romania.

The answer is YES.

To see this take $k = 10^n - 1$ (with n a positive integer), then we have $7k + 3 = 7 \cdot 10^n - 4 = 70\ldots00 - 4 = 699\ldots96$, which is a palindrome.

3. The interval $[0,1]$ is divided by 999 red points into 1000 equal parts and by 1110 blue points into 1111 equal parts. Find the minimum distance between a red point and a blue point. How many pairs of blue and red points achieve this minimum distance?

Solution by Titu Zvonaru, Comănești, Romania.

Let R_1, R_2, $\ldots$, R_{999} be the red points and B_1, B_2, $\ldots$, B_{1110} be the blue points. The distance between R_k and B_t is

$$d \;=\; \left|\frac{k}{1000} - \frac{t}{1111}\right| \;=\; \frac{|1111k - 1000t|}{1000 \cdot 1111}\,.$$

Since 1111 and 1000 are coprime, we deduce that $d \neq 0$ (the equation $1111k - 1000t = 0$ has the solution $k = 1000$ and $t = 1111$). So, we search for solutions to $|1111k - 1000t| = 1$ with $1 \le k \le 999$ and $1 \le t \le 1110$.

Case 1: $1111k - 1000t = 1$.
To solve this equation, let $k = \overline{abc}$, where a, b, c are the digits of k. Since $1111 \cdot \overline{abc} - 1 = 1000t$, we have $c = 1$ and successively

$$\begin{aligned} 1111(10 \cdot \overline{ab} + 1) - 1 &= 10000t\,, \\ 11110 \cdot \overline{ab} + 1110 &= 1000t\,, \\ 1111 \cdot \overline{ab} + 111 &= 100t\,. \end{aligned}$$

We deduce that $b = 9$, hence

$$\begin{aligned} 1111(10a + 9) + 111 &= 100t\,, \\ 11110 \cdot a + 10110 &= 100t\,, \\ 1111 \cdot a + 1011 &= 1001\,. \end{aligned}$$

We thus obtain the solution $k = 991$ and $t = 1001$.

Case 2: $1111k - 1000t = -1$.
Then $1111 \cdot \overline{abc} + 1 = 1000t$. We have $c = 9$, and successively

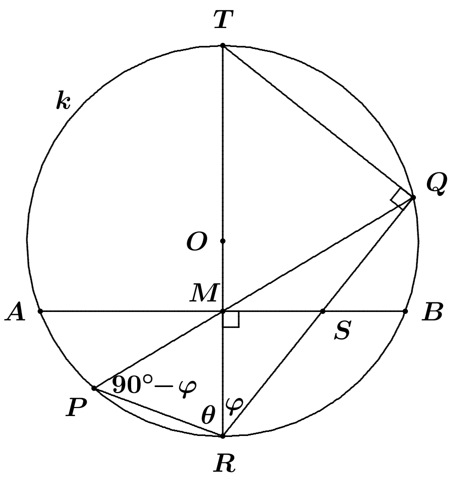
$$\begin{aligned} 1111(10\overline{ab} + 9) + 1 &= 1000t\,, \\ 11110 \cdot \overline{ab} + 10000 &= 1000t\,, \\ 1111 \cdot \overline{ab} + 1000 &= 100t\,. \end{aligned}$$

Hence, $\overline{ab} = 0$, and $k = 9$, $t = 10$.

The minimum distance between a red and a blue point is thus $\dfrac{1}{11110000}$, achieved for the pairs (R_{991}, B_{1001}) and (R_9, B_{10}) and no others.

5. Let k be a circle with centre O and let AB be a chord of k whose midpoint, M, is distinct from O. The ray from O through M meets k at R. Let P be a point on the minor arc AR of k, let PM meet k again at Q, and let AB meet QR at S. Which segment is longer, RS or PM?

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. Kandall's solution is presented.



Extend RO to meet k at T. Note that $\angle RMS$, $\angle RQT$ are right angles.

Let $\angle MRP = \theta$, $\angle MRS = \varphi$. Then $RM = RS\cos\varphi$ and $\angle RPQ = \angle RTQ = 90^\circ - \varphi$. Consequently,

$$\begin{aligned} \frac{PM}{\sin\theta} &= \frac{RM}{\sin(90 - \varphi)} \\ &= \frac{RM}{\cos\varphi} = RS\,, \end{aligned}$$

that is, $PM = RS\sin\theta < RS$.

We now give a solution to a problem of the 2005 Kürschák Competition, given at [2009 : 213].

2. Ann and Bob are playing tennis. The winner of a match is the player who is the first to win at least four games, being at least two games ahead of his or her opponent. Ann wins a game with probability $p \le \frac{1}{2}$ independently of the outcome of the previous games. Prove that Ann wins the match with probability at most $2p^2$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $p_{i,j}$ denote the probability that, after $i+j$ games, Ann has won exactly i games. Write $q = p - 1$ and $p_{i,j} = c_{i,j} \cdot p^i q^j$. For $1 \le i,\ j \le 3$ as well as for $i = j \ge 4$, it holds that $p_{i,j} = p_{i-1,j}p + p_{i,j-1}q$, hence

$$c_{i,j} \;=\; c_{i-1,j} + c_{i,j-1} \quad (1 \le i,j \le 3 \ \text{ or } \ i,j \ge 4)\,. \tag{1}$$

There are no intermediate scores such that one player won four games being two games ahead of his or her opponent. Therefore, for $j \ge 5$, we have $p_{j-2,j} = p_{j-2,j-1}q = p_{j-2,j-2}q^2$ and $p_{j,j-2} = p_{j-1,j-2}p = p_{j-2,j-2}p^2$; thus

$$c_{j-2,j} \;=\; c_{j-2,j-1} \;=\; c_{j-2,j-2} \;=\; c_{j-1,j-2} \;=\; c_{j,j-2} \qquad (j \ge 5)\,. \tag{2}$$

The constants $c_{i,j}$ for small indices i, j can now be directly computed from (1) and (2); some are computed at right.

$i \setminus j$	0	1	2	3	4	5	6	7	8
0		1	1	1	**1**				
1	1	2	3	4	**4**				
2	1	3	6	10	**10**				
3	1	4	10	20	20	**20**			
4	**1**	**4**	**10**	20	40	40	**40**		
5				**20**	40	80	80	**80**	
6					**40**	80	160	160	**160**
7						**80**	160	320	320
8							**160**	320	640

We guess from the examples that the common value of the constants in (2) is $2^{j-3} \cdot 5$. This can easily be proved by Mathematical Induction.

Consequently, Ann wins the match with probability

$$\begin{aligned}
&p_{4,0} + p_{4,1} + \sum_{i=4}^{\infty} p_{i,i-2} \\
&= \; p^4 + 4p^4q + 10p^4q^2 \sum_{i=0}^{\infty} (2pq)^i \;=\; p^4(1+4q) + \frac{10p^4q^2}{1-2pq} \\
&= \; \frac{15p^4 - 34p^5 + 28p^6 - 8p^7}{1 - 2p + 2p^2} \\
&= \; 2p^2 + 2p^2\left(\frac{1}{2} - p\right)(2 - 11p^2 + 12p^3 - 4p^4)\,.
\end{aligned}$$

It remains to prove that the function $f(p) = 2 - 11p^2 + 12p^3 - 4p^4$ is positive for $0 \le p \le \frac{1}{2}$. Since $f'(p) = -2p(11 - 18p + 8p^2)$ is negative, f is decreasing and $f(p) \ge f\left(\frac{1}{2}\right) = \frac{1}{2} > 0$, which completes the proof.

Now we move to solutions to the problems of the 8th Hong Kong (China) Mathematical Olympiad, given at [2009 : 213].

1. On a planet there are $3 \cdot 2005!$ aliens and 2005 languages. Each pair of aliens communicate with each other in exactly one language. Show that there are 3 aliens who communicate with each other in one common language.

Solution by Titu Zvonaru, Comănești, Romania.

We will prove by induction the statement of the problem for $3 \cdot n!$ aliens and n languages.

For $n = 2$, let A_1, A_2, A_3, A_4, A_5, and A_6 be the aliens and ℓ_1, ℓ_2 be the languages. By the Pigeonhole Principle, at least three pairs among (A_1, A_2), (A_1, A_3), (A_1, A_4), (A_1, A_5), (A_1, A_6) use the same language. We may assume that the pairs (A_1, A_2), (A_1, A_3) and (A_1, A_4) use the language ℓ_1.

If one of the pairs (A_2, A_3), (A_2, A_4) or (A_3, A_4) use the language ℓ_1, we are done. If not, then A_2, A_3, and A_4 communicate with each other in language ℓ_2.

Suppose that the statement is valid for $3 \cdot n!$ aliens and n languages.

Let $m = 3 \cdot (n+1)!$, $A_1, \dots, A_m$ be the aliens, and $\ell_1, \dots, \ell_{n+1}$ be the languages.

By the Pigeonhole Principle, at least $\left\lceil \frac{m-1}{n+1} \right\rceil = 3 \cdot n!$ pairs among (A_1, A_2), (A_1, A_3), $\dots$, (A_1, A_m) use the same language. We assume then that the pairs (A_1, A_2), (A_1, A_3), $\dots$, $(A_1, A_{3n!+1})$ use the language ℓ_1.

Now, if two aliens among A_2, A_3, $\dots$, $A_{3n!+1}$ use the language ℓ_1, then we are done. Otherwise, these $3 \cdot n!$ aliens use only the n languages ℓ_2, ℓ_3, $\dots$, ℓ_{n+1}, and then some three among them use a common language by the induction hypothesis.

This completes the proof.

2. Suppose that there are $4n$ line segments of unit length inside a circle of radius n. Given a straight line ℓ, prove that there exists a straight line ℓ' that is either parallel to or perpendicular to ℓ and such that ℓ' intersects at least two of the given line segments.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel's solution.

Let s_1, s_2, $\dots$, s_{4n} denote the $4n$ line segments inside the circle Γ; let x_i and y_i denote the projections of s_i onto ℓ and onto a fixed perpendicular

$\ell_\perp$ to ℓ, respectively. By the Triangle Inequality, we have

$$\sum_{i=1}^{4n}|x_i| + \sum_{i=1}^{4n}|y_i| \ \geq\ \sum_{i=1}^{4n}|s_i| \ =\ 4n\,.$$

We consider three cases. First, if $\sum\limits_{i=1}^{4n}|x_i| > 2n$, then there is a point P on ℓ that belongs to two of the x_i. In this case, the perpendicular to ℓ through P is a suitable choice for ℓ'. Second, if $\sum\limits_{i=1}^{4n}|y_i| > 2n$, then there is a point Q on $\ell_\perp$ that belongs to two of the y_i. The parallel line to ℓ through Q is a suitable choice for ℓ'. It remains to consider the case $\sum\limits_{i=1}^{4n}|x_i| = \sum\limits_{i=1}^{4n}|y_i| = 2n$. In this situation, the parallel line to ℓ through the midpoint of Γ is adequate for ℓ'.

3. Let a, b, c, and d be positive real numbers such that $a+b+c+d=1$. Prove that $6\big(a^3+b^3+c^3+d^3\big) \geq \big(a^2+b^2+c^2+d^2\big) + \frac{1}{8}$.

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Comănești, Romania. We give Alt's presentation.

By the Power Mean Inequality

$$\begin{aligned}\frac{a^3+b^3+c^3+d^3}{4} \ &\geq\ \left(\frac{a+b+c+d}{4}\right)^3\ ;\\ a^3+b^3+c^3+d^3 \ &\geq\ \frac{(a+b+c+d)^3}{16} \ =\ \frac{1}{16}\,,\end{aligned}$$

and by Chebychev's inequality

$$a^3+b^3+c^3+d^3 \ \geq\ \frac{a+b+c+d}{4}(a^2+b^2+c^2+d^2) \ =\ \frac{a^2+b^2+c^2+d^2}{4}\,.$$

This yields

$$\begin{aligned}&6(a^3+b^3+c^3+d^3)\\ &\quad=\ 4(a^3+b^3+c^3+d^3)+2(a^3+b^3+c^3+d^3)\\ &\quad\geq\ (a^2+b^2+c^2+d^2)+\frac{1}{8}\,,\end{aligned}$$

as desired

Next we move to solutions to problems of the Hong Kong Team Selection Test 1 given at [2009 : 214].

1. Find the integer solutions of the equation $7(x+y) = 3\big(x^2-xy+y^2\big)$.

Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's solution.

The pairs $(0,0), (4,5)$ and $(5,4)$ are clearly solutions for (x,y). We show that there are no other solutions.

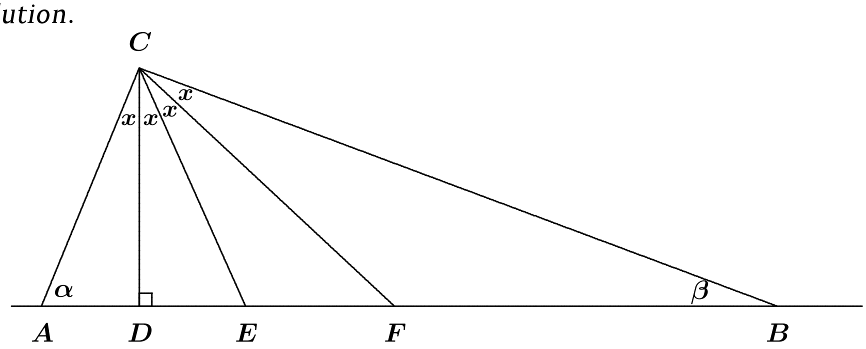
Let $(x,y) \neq (0,0)$ be a solution. Since $x^2 - xy + y^2 > 0$, we must have $x + y > 0$. Since 3 is coprime to 7, it divides $x + y$. Let $x + y = 3k$. Since $7(x+y) = 3((x+y)^2 - 3xy)$, we have $7k = 3(3k^2 - xy)$ so that 3 divides k as well and finally we see that $x + y$ is a multiple of 9.

Also, $xy \leq \dfrac{(x+y)^2}{4}$, hence $(x+y)^2 - 3xy \geq \dfrac{(x+y)^2}{4}$ and it follows that $7(x+y) \geq \dfrac{3(x+y)^2}{4}$. From this (and $x + y > 0$), we deduce that $x + y \leq \dfrac{28}{3}$.

Since $x + y$ is a multiple of 9 and $0 < x + y \leq \dfrac{28}{3}$, it follows that $x + y = 9$, so $7 \cdot 9 = 3(9^2 - 3xy)$ and $xy = 20$. Thus, $(x,y) = (4,5)$ or $(5,4)$, and we are done.

3. In triangle ABC, the altitude, angle bisector, and median from C divide $\angle C$ into four equal angles. Find $\angle B$.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give Smeenk's solution.



We let CD, CE, CF be the altitude, angle bisector, and median from C, respectively. It is given that $\angle ACD = \angle DCE = \angle ECF = \angle FCB = x$. Then we have $\alpha = 90^\circ - x$, $\beta = 90^\circ - 3x$.

It is well known that $\sin\angle ACF : \sin\angle FCB = \sin\alpha : \sin\beta$, hence

$$\begin{aligned}
\cos 3x : \sin x &= \sin(90^\circ - x) : \sin(90^\circ - 3x)\,, \\
\cos 3x : \sin x &= \cos x : \cos 3x\,, \\
\cos 6x &= \sin 2x\,.
\end{aligned}$$

This means that $6x + 2x = 180^\circ$, so $x = 22.5^\circ$ and $\angle B = \beta = 22.5^\circ$.

4. Let x, y, and z be positive real numbers such that $x+y+z=1$. For a positive integer n, let $S_n = x^n + y^n + z^n$. Also, let $P = S_2 S_{2005}$ and $Q = S_3 S_{2004}$.

(a) Find the smallest possible value of Q.

(b) If x, y, and z are distinct, determine which of P or Q is the larger.

Solution by Arkady Alt, San Jose, CA, USA.

(a) By the Power Mean inequality, for any positive integer k we have

$$\begin{aligned} \frac{x^k+y^k+z^k}{3} &\geq \left(\frac{x+y+z}{3}\right)^k = \frac{1}{3^k}, \\ S_k \geq x^k+y^k+z^k &\geq \frac{1}{3^{k-1}} \geq \frac{1}{3^{k-1}}, \end{aligned}$$

where equality occurs if and only if $x=y=z=\frac{1}{3}$. Thus the minimum value of Q is $\frac{1}{3^2}\cdot\frac{1}{3^{2003}} = \frac{1}{3^{2005}}$.

(b) We will prove that if x, y, z are distinct, then

$$\frac{S_{n+1}}{S_n} > \frac{S_n}{S_{n-1}}$$

holds for any positive integer n. Indeed,

$$\begin{aligned} &S_{n+1}S_{n-1} - S_n^2 \\ &= (x^{n+1}+y^{n+1}+z^{n+1})(x^{n-1}+y^{n-1}+z^{n-1}) - (x^n+y^n+z^n)^2 \\ &= x^{n+1}(y^{n-1}+z^{n-1}) + y^{n+1}(z^{n-1}+x^{n-1}) + z^{n+1}(x^{n-1}+y^{n-1}) \\ &\quad -2(x^ny^n+y^nz^n+z^nx^n) \\ &= \sum_{\text{cyclic}} (x^{n+1}y^{n-1}+x^{n-1}y^{n+1}-2x^ny^n) \\ &= \sum_{\text{cyclic}} x^{n-1}y^{n-1}(x-y)^2 \geq 0. \end{aligned}$$

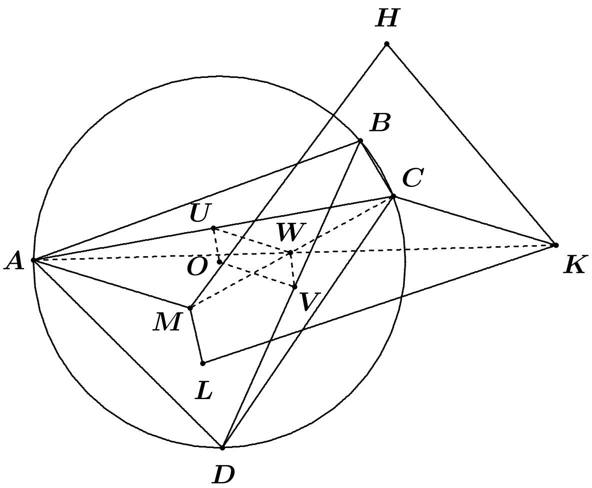
Therefore, $\frac{S_{n+1}}{S_n} > \frac{S_m}{S_{m-1}}$ for $n \geq m \geq 2$, or $S_{m-1}S_{n+1} > S_m S_n$. In particular for $m=3$, $n=2004$ we have $S_2S_{2005} > S_3S_{2004}$.

Next are solutions to the Hong Kong Team Selection Test 2, given at [2009 : 214–215].

1. Let $ABCD$ be a cyclic quadrilateral. Show that the orthocentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$ are the vertices of a quadrilateral congruent to $ABCD$ and show that the centroids of the same triangles are the vertices of a cyclic quadrilateral.

Solved by Michel Bataille, Rouen, France; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's solution.

Let H, K, L, and M be the orthocentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$, respectively and let O be the centre of the circle through A, B, C, D. Let U and V be the orthogonal projections of O onto AC and BD, respectively and let W be defined by $\overrightarrow{UW} = \overrightarrow{OV}$ (see the figure). From a well-known property in the triangle, we have $\overrightarrow{AM} = 2\overrightarrow{OV}$, and similarly $\overrightarrow{CK} = 2\overrightarrow{OV}$. Hence, $\overrightarrow{AM} = \overrightarrow{CK}$ and $AMKC$ is a parallelogram.



In addition, since U is the midpoint of AC and $\overrightarrow{UW} = \frac{1}{2}\overrightarrow{AM}$, W is the midpoint of CM that is, the centre of the parallelogram $AMKC$. In the same way, we obtain that $BHLD$ is a parallelogram with centre W and it follows that the quadrilateral $HKLM$ is the symmetric of $DABC$ about W. Thus, $HKLM$ is congruent to $DABC$.

In any triangle with circumcentre O, centroid G, and orthocentre H, we have $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OH}$. Here, the triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$ have the same circumcentre, namely O, and therefore the centroids of these triangles are the images of H, K, L, M under the homothety with centre O and factor $\frac{1}{3}$. Since H, K, L, M are concyclic (as A, B, C, D are concyclic), the four centroids are concyclic as well.

2. Let $ABCD$ be a cyclic quadrilateral with $BC = CD$. The diagonals AC and BD intersect at E. Let X, Y, Z, and W be the incentres of $\triangle ABE$, $\triangle ADE$, $\triangle ABC$, and $\triangle ADC$, respectively. Show that X, Y, Z, and W are concyclic if and only if $AB = AD$.

Solution by Titu Zvonaru, Comăneşti, Romania.

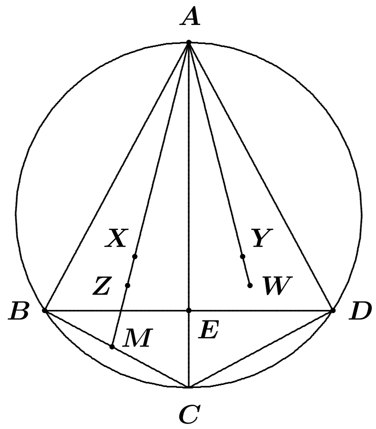
Let AZ meet BC at M. By the Bisector Theorem, $BM = \dfrac{BC \cdot AC}{AB + AC}$.

Since

$$AM = \frac{2AB \cdot AC \cdot \cos\frac{\angle BAC}{2}}{AB+AC},$$

applying the Bisector Theorem again in $\triangle ABM$ with the bisector BZ we obtain $\frac{AZ}{ZM} = \frac{AB}{BM}$, hence

$$AZ = \frac{2AB \cdot AC \cdot \cos\frac{\angle BAC}{2}}{AB+BC+AC} \qquad (1)$$



We let $a = BC = CD$, $x = AB$, $y = AD$, $p = BE$, $q = DE$, $t = AC$, and $m = AE$.

Since the quadrilateral $ABCD$ is cyclic we have $\angle BDC = \angle BAC$ and $\angle DBC = \angle CAD$, hence AE is a bisector in $\triangle ABD$. We have

$$\frac{BE}{ED} = \frac{AB}{AD} \Longrightarrow q = \frac{py}{x}. \qquad (2)$$

The quadrilateral $XZWY$ is cyclic if and only if $AX \cdot AZ = AY \cdot AW$. Using (1) and (2) we have successively

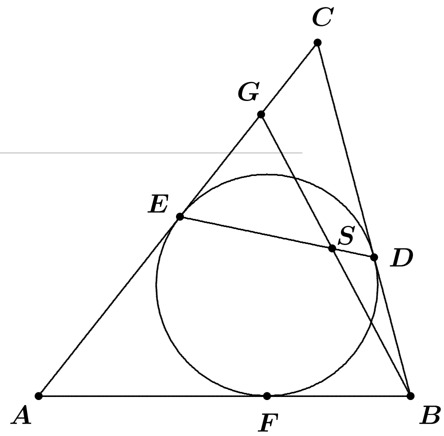
$$\begin{aligned}
&AX \cdot AZ = AY \cdot AW\,; \\
&\frac{2\cdot AB \cdot AE \cos\frac{\angle BAC}{2}}{AB+AE+BE} \cdot \frac{2\cdot AB\cdot AC\cdot\cos\frac{\angle BAC}{2}}{AB+AC+BC} \\
&\quad = \frac{2\cdot AD\cdot AE\cdot\cos\frac{\angle CAD}{2}}{AD+AE+ED} \cdot \frac{2\cdot AC\cdot AD\cos\frac{\angle CAD}{2}}{AD+AC+DC}\,; \\
&\frac{x^2}{(x+m+p)(x+t+a)} = \frac{y^2}{(y+m+q)(y+t+a)}\,; \\
&x^2\left[y^2 + y(m+q+t+a) + (m+q)(t+a)\right] \\
&\quad = y^2\left[x^2 + x(m+p+t+a) + (m+p)(t+a)\right]\,; \\
&(x-y)[xy(m+t+a) + m(t+a)(x+y) + py(t+a)] = 0 \\
&x - y = 0\,;
\end{aligned}$$

hence X, Y, Z, and W are concyclic if and only if $AB = AD$.

3. Points A and B lie in a plane and ℓ is a line in that plane passing through A but not through B. The point C moves from A toward infinity along a half-line of ℓ. The incircle of $\triangle ABC$ touches BC at D and AC at E. Show that the line DE passes though a fixed point.

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel's solution.

Let G be the point on the half line from A through C that satisfies $AB = AG$. We prove that the line DE passes through the midpoint of the line segment BG.



For the tangential segments CD and CE it holds

$$CD = CE. \qquad (1)$$

Let the incircle of $\triangle ABC$ touch the side AB at point F. Then, we have

$$\begin{aligned} BD &= BF = AB - AF \\ &= AG - AE = GE. \qquad (2) \end{aligned}$$

Let lines DE and BG intersect at the point S. By Menelaus' Theorem for $\triangle BCG$ and the line DE, and by the equations (1) and (2), we have

$$1 = \frac{BD}{CD} \cdot \frac{CE}{GE} \cdot \frac{GS}{BS} = \frac{GS}{BS}.$$

Consequently, S is the midpoint of BG, which completes the proof.

Next we look at solutions from our readers to problems of the 20th Nordic Mathematical Olympiad given at [2009 : 215].

1. Let B and C be points on two given rays from the same point A, such that $AB + AC$ is constant. Prove that there exists a point D distinct from point A such that the circumcircles of the triangles ABC pass through D for all choices of B and C subject to the given constraint.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's write-up.

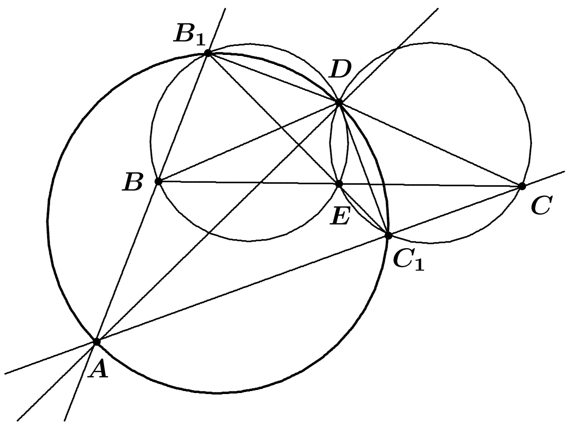
Let m be the given constant and let B_1, C_1 be on the two given rays with $AB_1 = AC_1 = \frac{m}{2}$. Let D be the point of intersection other than A of the circumcircle of $\triangle AB_1C_1$ with the bisector of $\angle B_1AC_1$ (see the figure on the next page).

Now, consider arbitrary points B, C satisfying all the constraints. Then $DB_1 = DC_1$, $BB_1 = CC_1$ and in addition, $\angle DB_1B = \angle DC_1C = 90^\circ$ (since AD is a diameter of the circumcircle of $\triangle AB_1C_1$).

It follows that the right-angled triangles $\triangle DB_1B$ and $\triangle DC_1C$ are congruent and so $\angle B_1DB = \angle C_1DC$. As a result,

$$\begin{aligned}\angle BDC &= \angle B_1DC_1\\ &= 180^\circ - \angle B_1AC_1\\ &= 180^\circ - \angle BAC\end{aligned}$$

and A, B, C, D are concyclic. In other words, the circumcircle of $\triangle ABC$ passes through D.



2. The real numbers x, y, and z are not all equal and satisfy

$$x + \frac{1}{y} \;=\; y + \frac{1}{z} \;=\; z + \frac{1}{x} \;=\; k\,.$$

Determine all possible values of k.

Solved by Arkady Alt, San Jose, CA, USA; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the solution of the latter.

We prove that $k = \pm 1$ and both values are attainable. We have $z = k - x^{-1} = \dfrac{kx-1}{x}$. Since $z \neq 0$, substituting for z in $y = k - z^{-1}$ yields $y = \dfrac{k^2x - k - x}{kx-1}$.

Since $y \neq 0$, we then have

$$x \;=\; k - y^{-1} \;=\; k - \left(\frac{kx-1}{k^2x-k-x}\right) \;=\; \frac{k^3x - k^2 - 2kx + 1}{k^2x - k - x}\,,$$

or $k^2x^2 - kx - x^2 = k^3x - k^2 - 2kx + 1$.

Simplifying, we obtain $(k^2-1)x^2 - k(k^2-1)x + k^2 - 1 = 0$, or $(k^2-1)(x^2 - kx + 1) = 0$.

If $x^2 - kx + 1 = 0$, then $k = x + x^{-1}$ and $k = x + y^{-1}$, hence $x = y$. From $x + y^{-1} = y + z^{-1}$ we deduce that $y = z$, so $x = y = z$, a contradiction.

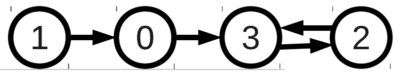
Thus, $k^2 = 1$ since $x^2 - kx + 1 \neq 0$.

For each value of k, infinitely many triples (x, y, z) satisfy the conditions. To see this, let $x \in \mathbb{R} - \{0, \pm 1\}$ and let $y = \dfrac{1}{1-x}$, $z = \dfrac{x-1}{x}$. Then clearly, $x + y^{-1} = y + z^{-1} = z + x^{-1} = 1$. On the other hand, if we let $y = \dfrac{-1}{1+x}$ and $z = -\dfrac{1+x}{x}$, then $x + y^{-1} = y + z^{-1} = z + x^{-1} = -1$.

3. The sequence $\{a_n\}$ of positive integers is defined by $a_0 = m$ and the recursion $a_{n+1} = a_n^5 + 487$ for all $n \geq 0$. Determine all values of m for which the sequence contains as many square numbers as possible.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The quadratic residues modulo 4 are 0 and 1. The table below shows the residues of the expression x^5+487 modulo 4, and the above transition diagram demonstrates that only the first and the second term of any $\{a_n\}$ can be square numbers.



$x \pmod 4$	0	1	2	3
$x^5 + 487 \pmod 4$	3	0	3	2

If $a_0 = a^2$ and $a_1 = b^2$ for positive integers a, b, then $b^2 = a^{10} + 487$; hence $(b-a^5)(b+a^5) = 487$. Since 487 is prime, $b-a^5 = 1$ and $b+a^5 = 487$; thus, $b = 244$, $a^5 = 243$, $a = 3$, $a_0 = 9$, and

$$a_1 \;=\; 9^5 + 487 \;=\; 59\,536 \;=\; 244^2\,.$$

Consequently, $m = 9$ is the unique solution.

Next are some solutions to problems of the XXXII Russian Mathematical Olympiad 2005–2006 Final Round 9th Form, given at [2009 : 216–217].

2. Show that there exist four integers a, b, c, and d whose absolute values are greater than $1\,000\,000$ and which satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \;=\; \frac{1}{abcd}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

We prove that for each positive integer N, there are integers a, b, c, d with absolute values greater than N which satisfy the equation. In fact, let $a = N+1$, $b = -N-2$, $p = -ab$, $c = 1-p$, and $d = p(p-1)-1$. Then,

$$\begin{aligned}
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} &= \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{p-1} + \frac{1}{p(p-1)-1} \\
&= \frac{1}{p} - \frac{1}{p-1} + \frac{1}{p(p-1)-1} \\
&= \frac{1}{p(p-1)[p(p-1)-1]} \;=\; \frac{1}{(-ab)(-c)d} \;=\; \frac{1}{abcd}\,,
\end{aligned}$$

which completes the proof.

5. Let $a_1, a_2, \ldots, a_{10}$ be positive integers such that $a_1 < a_2 < \cdots < a_{10}$. Let b_k be the greatest divisor of a_k such that $b_k < a_k$. If $b_1 > b_2 > \cdots > b_{10}$, prove that $a_{10} > 500$.

Solution by Titu Zvonaru, Cománeşti, Romania.

If some a_i with $1 \leq i \leq 9$ is a prime, then $b_i = 1$ and $1 > b_{10}$ a contradiction.

Let p_i be the smallest prime divisor of a_i. Then $a_i = p_i b_i$ for each i. Since $b_1 > b_2 > \cdots > b_9$ and $a_1 < a_2 < \cdots < a_9$, we deduce that $p_1 < p_2 < \cdots < p_9$.

The first 9 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, so it follows that $a_9 = p_9 \cdot b_9 \geq 23 \cdot 23 = 529$ (since $p_9 \leq b_9$) and $a_{10} \geq a_9 + 1 \geq 530$.

7. A 100×100 square board is cut into dominos (that is, into 2×1 rectangles). Two players play a game. At each turn, a player may glue together any two adjacent squares if there is a cut between them. A player loses if he or she reconnects the board (thus allowing the board to be lifted by a corner without it falling apart). Who has a winning strategy, the first player or the second player?

Solution by Oliver Geupel, Brühl, NRW, Germany.

We generalize to an $m \times n$ rectangular board such that $\frac{mn}{2} - m - n$ is an even integer and $mn + 12 \geq 8(m + n)$. We will prove that the second of two players, A and B, has a winning strategy.

Two squares are *adjacent* if they share an edge. A *cut* is a common line segment of length 1 between two adjacent unconnected squares. Glueing two squares together is equivalent to removing the cut between them.

A set C of squares is *connected* if, for any two squares s, $s' \in C$, there are squares $s = s_0, s_1, \ldots, s_n = s' \in C$ such that s_{k-1} is adjacent to s_k and for each k there is no cut between them. A *component* is a maximal connected set. At any moment the board is partitioned into components, in particular there are $\frac{mn}{2}$ components at the beginning. With each move (turn) the number of components decreases by at most one, and a player loses if he connects the last two components.

B's strategy is as follows.

(1) Remove all cuts having an endpoint on the border of the board. There are at most $2(m + n) - 4$ such cuts, requiring at most $4(m + n) - 8$ moves to be removed. Once this is achieved, then there are no less than

$$\frac{1}{2}mn - 4(m + n) + 8 \;=\; \frac{1}{2}\left(mn + 12 - 8(m + n)\right) + 2 \;\geq\; 2$$

components, one being an external component containing the border of the board, and at least one internal component not touching the border.

Call a set C of squares *contiguous* if, for any two s, $s' \in C$, there are squares $s = s_0, s_1, \ldots, s_n = s' \in C$ such that s_{k-1} is adjacent to s_k for

$k = 1, 2, \ldots, n$. A *cluster* is a contiguous union of internal components. It can easily be shown by Mathematical Induction on the number of squares of a cluster, that a cluster has an even number of cuts on its external border.

(2) In each move, remove a cut of a cluster which is not an external border edge of this cluster. This is possible because, before each move of B, the number of remaining cuts is odd. (Observe that the initial number of cuts is the even number $\frac{3mn}{2} - m - n$.) The number of clusters is not decreased in this way. Therefore, the number of clusters can only decrease when A moves. Consequently, player A will eventually break up the last cluster.

8. A quadratic polynomial $f(x) = x^2 + ax + b$ is given. Suppose that the equation $f\big(f(x)\big) = 0$ has four distinct real roots and that the sum of two of them is equal to -1. Prove that $b \le -\frac{1}{4}$.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Cománeşti, Romania. We give Bataille's solution.

We assume that $a, b \in \mathbb{R}$. Let x_1, x_2, x_3, x_4 be the distinct real roots of $f(f(x)) = 0$, so that the numbers $f(x_i)$ are real roots of $f(x) = 0$. Clearly $a^2 \ge 4b$, but if $a^2 = 4b$, then $f(x_i) = -\frac{a}{2}$ for each i and the x_i would be four distinct roots of $x^2 + ax + b + \frac{a}{2} = 0$, a contradiction. Thus, $a^2 > 4b$ and $f(x) = 0$ has two distinct real roots u_1, u_2. Note that $\{f(x_1), f(x_2), f(x_3), f(x_4)\} \subset \{u_1, u_2\}$. If three or more of $f(x_1), f(x_2), f(x_3), f(x_4)$ were equal, say to u_1, then $x^2 + ax + b - u_1 = 0$ for at least three distinct real values, a contradiction. It follows that we have (say)

$$f(x_1) = f(x_2) = u_1\,, \qquad f(x_3) = f(x_4) = u_2\,.$$

Now,

$$f(f(x)) \;=\; (x^2 + ax + b)^2 + a(x^2 + ax + b) + b \;=\; x^4 + 2ax^3 + \cdots\,,$$

so that $x_1 + x_2 + x_3 + x_4 = -2a$ with two of x_1, x_2, x_3, x_4 adding to -1 (by hypothesis). There are just two essential cases: $x_1 + x_2 = -1$ or $x_1 + x_3 = -1$. In the former case, we have $a = -1$ (since x_1, x_2 are the roots of $x^2 + ax + b - u_1 = 0$) and $x_3 + x_4 = 1 - 2a = 3$, contradicting the fact that x_3, x_4 are the roots of $x^2 + ax + b - u_2 = 0$. Thus, we must be in the latter case $x_1 + x_3 = -1$ with

$$x_1^2 + ax_1 + b - u_1 = 0\,, \qquad x_3^2 + ax_3 + b - u_2 = 0\,.$$

Recalling that $u_1 + u_2 = -a$, we deduce $x_1^2 + x_3^2 = -2b$. Since $2(x_1^2 + x_3^2) \ge (x_1 + x_3)^2 = 1$, we obtain $-4b \ge 1$ and the result follows.

Next are problems of the XXXII Russian Mathematical Olympiad 2005–2006, Final Round, 10th Form given at [2009 : 217].

2. Assume that the sum of the cubes of three consecutive positive integers is a cube of some positive integer. Prove that the middle number of these three numbers is divisible by 4.

Solved by Michel Bataille, Rouen, France; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Bataille's solution.

Let n be an integer such that $n \geq 2$ and

$$(n-1)^3 + n^3 + (n+1)^3 \;=\; m^3 \qquad (1)$$

for some positive integer m. We show that n is divisible by 4.

Since (1) is the same as $3n(n^2+2) = m^3$, we see that 3 divides m^3, hence 3 divides m. With $m = 3k$, equation (1) becomes

$$n(n^2+2) \;=\; 9k^3\,. \qquad (2)$$

First, we prove that n is even.

Assume on the contrary that n is odd. Then, n and n^2+2 are coprime (any common divisor must be odd and divide $2 = (n^2+2) - n(n)$). From (2), it follows that 9 divides n or n^2+2. In the former case, by comparing the standard prime factorization of each side of (2), we see that n^2+2 must be a cube. However, any cube is congruent to 0, 1, or 8 modulo 9. But $n^2+2 \equiv 2 \pmod 9$, since 9 divides n, a contradiction.

Similarly, if 9 divides n^2+2, then we see that $n \equiv 4$ or $5 \pmod 9$. Since n must be a cube, we again have a contradiction.

Thus, $n = 2q$ for a positive integer q and (2) yields $4q(2q^2+1) = 9k^3$. Then 2 divides k^3 and hence k, so $k = 2\ell$ and we obtain $q(2q^2+1) = 2\cdot(9\ell^3)$. Since $2q^2+1$ is odd, 2 must divide q, and finally n is divisible by 4.

8. A 3000×3000 square is divided into dominos (that is, into 2×1 rectangles). Prove that one can paint the dominos with three colours such that each colour is used equally often and each piece shares a side with no more than two pieces of the same colour.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The statement holds for each $N \times N$ board where $3 \mid N$. We use the Russian national colors white (W), blue (B), and red (R). First, we paint the 1×1 squares of the board as shown in Figure 1. Now, each domino covers two squares with distinct colors. Second, we paint each domino with the color that does not occur on the the two squares it covers. We will show that this coloring has the desired properties.

Let w, b, and r denote the number of white, blue, and red dominos. A domino is either white or blue, if and only if one of its parts covers one of the $\frac{2}{3}N^2$ red cells. Hence, $w + b = \frac{1}{3}N^2$. Similar, $w + r = \frac{1}{3}N^2$. Thus, $b = r$. By symmetry, we obtain $w = b = r$, hence the each color is used equally often.

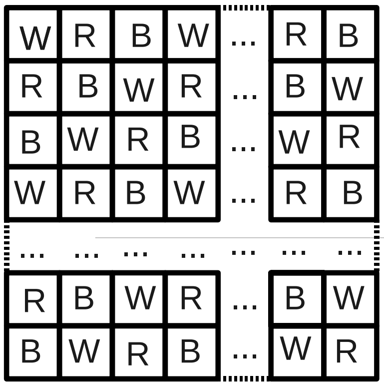


Figure 1

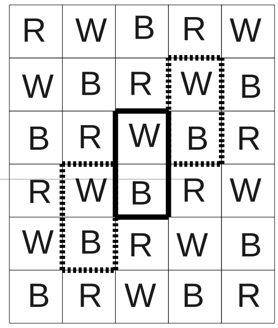


Figure 2

It remains to show that no domino shares a side with more than two other pieces of its color. By symmetry, it suffices to consider the case in Figure 2. There the central red domino has at most two red neighbors – the two dominos that cover a white and a blue square each.

This proof is complete.

Now we look at solutions to problems of the XXXII Russian Mathematical Olympiad 2005–2006, Final Round 11th Form, given at [2009 : 217–218].

1. Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ whenever $0 < x < \frac{\pi}{2}$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give Bataille's version.

We first prove the inequality for $0 < x \le 1$. Let $f(t) = \frac{\sin t}{t}$. Then $f'(t) = \frac{\cos t}{t^2}(t - \tan t)$ is negative on $(0, 1]$, so f decreases on this interval. Also $0 < f(x) < 1$ for x in this range, so $\sqrt{f(x)} > f(x) \ge f(\sqrt{x}\,)$, or

$$\sqrt{\frac{\sin x}{x}} > \frac{\sin x}{x} \ge \frac{\sin\sqrt{x}}{\sqrt{x}},$$

and we conclude that $\sqrt{\sin x} > \sin\sqrt{x}$.

Now, we suppose $1 < x < \frac{\pi}{2}$. From $0 < \sqrt{x} < \frac{x+1}{2} < \frac{\pi}{2}$, we obtain $\sin\sqrt{x} < \sin\left(\frac{x+1}{2}\right)$. Hence, it suffices to prove $\sin\left(\frac{x+1}{2}\right) \le \sqrt{\sin x}$. Since $\cos(x+1) = 1 - 2\sin^2\left(\frac{x+1}{2}\right)$, we will equivalently show that

$$g(x) = 2\sin x + \cos(x+1) \ge 1 \quad \text{for} \quad 1 \le x \le \frac{\pi}{2}. \tag{1}$$

We have

$$g'(x) \;=\; 2\cos x - \sin(x+1)\,,$$
$$g''(x) \;=\; -2\sin x - \cos(x+1)\,.$$

Since $2\sin x \geq 2\sin 1$ and $\cos(x+1) \geq \cos\left(1+\frac{\pi}{2}\right) = -\sin 1$, we have $g''(x) \leq -\sin 1 < 0$. Thus, g' decreases from $g'(1) = 2\cos 1(1 - \sin 1) > 0$ to $g'\left(\frac{\pi}{2}\right) = -\sin\left(1+\frac{\pi}{2}\right) = -\cos 1 < 0$. Thus, there exists $\alpha \in \left(1, \frac{\pi}{2}\right)$ such that $g'(x) > 0$ for $x \in (1, \alpha)$ and $g'(x) < 0$ for $x \in \left(\alpha, \frac{\pi}{2}\right)$. Therefore,

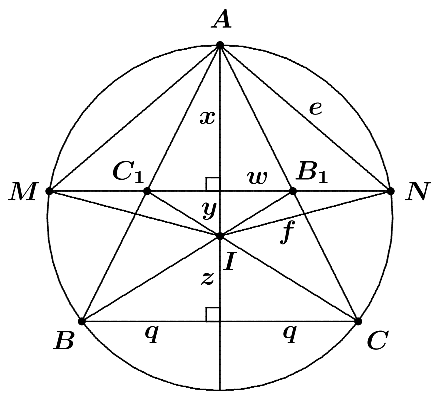
$$g(x) \;\geq\; \min\left\{g(1),\; g\left(\frac{\pi}{2}\right)\right\}\,. \tag{2}$$

Now, $g\left(\frac{\pi}{2}\right) = 2 - \sin 1 > 1$ and $g(1) = 1 + 2\sin 1(1 - \sin 1) > 1$, hence the minimum in (2) is greater than 1 and the inequality (1) follows immediately.

4. The angle bisectors BB_1 and CC_1 of $\triangle ABC$ (with B_1 on AC and C_1 on AB) meet at I. The line B_1C_1 meets the circumcircle of $\triangle ABC$ at M and N. Prove that the circumradius of $\triangle MIN$ is twice the circumradius of $\triangle ABC$.

Partial solution by Geoffrey A. Kandall, Hamden, CT, USA.

Here is a partial solution: the case where $\triangle ABC$ is isosceles with apex A.



Let R be the circumradius of $\triangle ABC$ and $\triangle AMN$, R^* be the circumradius of $\triangle MIN$, $p = AB = AC$, $2q = BC$, $e = AN$, $f = IN$, h be the altitude from A to BC, and x, y, z, w be as in the diagram.

Since the product of two sides of a triangle is equal to the product of its circumdiameter and the altitude to the third side, we have

$$R \;=\; \frac{e^2}{2x} \;=\; \frac{p^2}{2h} \quad\text{and}\quad R^* \;=\; \frac{f^2}{2y}\,.$$

In view of $\dfrac{AC_1}{C_1B} = \dfrac{AC}{CB} = \dfrac{p}{2q}$, we have $\dfrac{x}{h} = \dfrac{p}{p+2q}$, hence

$$x \;=\; \frac{ph}{p+2q} \quad\text{and}\quad e^2 \;=\; \frac{p^3}{p+2q}\,; \tag{1}$$

$$\frac{x}{y} \;=\; \frac{x}{y+z}\cdot\frac{y+z}{y} \;=\; \frac{p}{2q}\left(1+\frac{q}{w}\right) \;=\; \frac{p+q}{q}\,. \tag{2}$$

Thus,

$$y \;=\; \frac{q}{p+q}\cdot x \;=\; \frac{pqh}{(p+q)(p+2q)}\,.$$

Since $f^2 = e^2 - x^2 + y^2$ and $h^2 = p^2 - q^2$, we have

$$f^2 \;=\; \frac{p^3}{p+2q} - \frac{p^2(p^2-q^2)}{(p+2q)^2} + \frac{p^2q^2(p^2-q^2)}{(p+q)^2(p+2q)^2} \;=\; \frac{2p^3q}{(p+q)(p+2q)}\,.$$

Consequently,

$$R^* \;=\; \frac{1}{2}\cdot\frac{2p^3q}{(p+q)(p+2q)}\cdot\frac{(p+q)(p+2q)}{pqh} \;=\; \frac{p^2}{h} \;=\; 2R\,.$$

7. The polynomial $(x+1)^n - 1$ is divisible by a polynomial

$$P(x) \;=\; x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1x + c_0$$

of even degree k such that c_0, c_1, $\ldots$, c_{k-1} are odd integers. Prove that n is divisible by $k+1$.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

We drop the hypothesis that k is even.

The result is obvious if $k=0$. If $k=1$, then $(x+1)^n-1$ is divisible by $x+c_0$; that is $(-c_0+1)^n - 1 = 0$. Hence, $c_0 \in \{0, 2\}$, contradicting the fact that c_0 is odd. It remains to consider $k \geq 2$.

Write $n = (k+1)q + r$, where q, r are integers with $0 \leq r \leq k$. Considering the polynomials in the ring $\mathbb{Z}_2[x]$, we have

$$(x+1)^n - 1 \;=\; \left(x^k + x^{k-1} + \cdots + x + 1\right)Q(x)\,, \qquad (1)$$

$$(x+1)^n - x^n \;=\; x^{n-k}\left(x^k + x^{k-1} + \cdots + x + 1\right)Q\left(\frac{1}{x}\right)\,, \qquad (2)$$

where $Q(x)$ is a polynomial of degree $n-k$ and (2) follows from (1) upon replacing x by $1/x$ and multiplying by x^n. Writing $p_k(x) = x^k + \cdots + x + 1$, we deduce from (1) and (2) that

$$x^n - 1 \;=\; p_k(x)\left[Q(x) - x^{n-k}Q\left(\frac{1}{x}\right)\right] \;=\; p_k(x)S(x)\,,$$

where $S(x) \in \mathbb{Z}_2[x]$.

Thus, $p_k(x)$ divides $x^n - 1$ in $\mathbb{Z}_2[x]$. Moreover, $p_k(x)$ also divides $x^n - x^r$ in $\mathbb{Z}_2[x]$, since it divides $x^{k+1}-1$, which divides $x^{(k+1)q}-1$, which in turn divides $x^{(k+1)q+r} - x^r = x^n - x^r$.

Now, $p_k(x)$ is of degree k and it divides $x^r - 1 = (x^n-1) - (x^n - x^r)$, which is of degree r. But $0 \leq r \leq k$, hence $r = 0$, as desired.

That completes the material for this number of the *Corner*. Send me your nice solutions!

BOOK REVIEWS

Amar Sodhi

Mrs. Perkins's Electric Quilt: And Other Intriguing Stories of Mathematical Physics
By Paul J. Nahin, Princeton University Press, 2009
ISBN13: 978-0-691-13540-3, hardcover, 391+xxix pages, US$29.95
Reviewed by ***Nora Franzova****, Langara College, Vancouver, BC*

In the opening paragraph of chapter one the author states:

> *The central thesis of this book is that physics needs mathematics, but the converse is often true too.*

From that point on, Physics and Math play a sort of a game in which they pass a ball to each other, each time trying a more challenging shot to see if the other player can catch it and then reciprocate. Nahin supports each of the players with equal cheer. The game is not about who (Physics or Math) is better or more important; it is about playing a great game.

This game keeps physics enthusiasts and math lovers reading on. The strongest feature of the book is undoubtedly the enthusiasm of the author himself. His passion for the subjects is the key force behind all the interesting math problems that he suddenly turns into physics problems and vice versa. Immediately (in the preface of the book) Newton's second law becomes the main part of the proof of the oh so well-known limit, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. A convincing proof of this limit is rare in an introductory calculus course.

There are 19 chapters in the book — but the author calls them "discussions" and they definitely are that. The author invites you to sit down, grab a piece of paper and compute things first alone and then with him. Nahin spices up every step with a joke or a piece of history, personal comment or reminiscence, and in doing so one learns much about his jobs, and his personal path through math and physics. Each chapter ends with "Notes and References". These are worth reading in their own right as it is not just a traditional list of references, but something that leads to another discussion, involving equations and solutions, of the author with his audience.

Of course we cannot forget the Challenge Problems (the reviewer is still working on many of them, resisting the temptation to peek into the back of the book, where full solutions are included). These problems range from computing integrals, through modelling random walks. In the preface we are warned that the book is not the kind where an amazing problem is given a short succinct elegant two-line solution. The solutions are the realistic kind — one needs a good supply of paper, a pencil and sometimes even a computer facility like `MATLAB`. Nevertheless, the solutions do not send you to bed with the feeling that a question is done and finished with, but more with a need to further discuss or maybe just argue with the author a bit about things.

Even though the author stipulates that his "intended audience includes students of mathematical physics, starting with very bright high school seniors who have taken AP calculus and physics", the reviewer believes that some of the computational steps require a higher mathematical maturity to be fully enjoyed. Topics explore Newtonian gravity, lots of electrical circuits, ballistics, then more gravity, random walks, and even big noise.

Newtonian gravity starts with a non-traditional home experiment of measuring gravity with a stopwatch, a rubber ball, and a yardstick (and of course a formula). The reviewer repeated the author's very successful experiment in her kitchen (with a tiled floor) and got a result not as good as the author, but still pretty close. In another chapter about gravity and Newton we compute how much energy it would take to disassemble a planet of the Earth's size. It is definitely much more than movies want us to believe. Yes, the author discusses movies, poems, and songs throughout the book. In the discussion called Really Long Falls, we learn how long it would take for the Earth to fall into the Sun, which is immediately followed by a full analysis of the journey of fallen angels (from Heaven to Hell). Once we get this far into Heaven we get the challenge of calculating the mass of the Moon.

Extremely tempting integrals arise in the Zeta Function and Physics discussion, which in its references and notes section also includes lyrics of a mathematically spiced song. The discussion about Ballistics With and Without Air Drag naturally involves cannons, but also includes baseball studies.

Probability is taken on in three discussions with titles: Random Walks, Two More Random Walks, and One Last Random Walk. Topics start with mosquitoes and finish with electrical circuits. A Monte Carlo simulation is presented with its `MATLAB` code along with interesting computation. More probability takes us to an island of cannibal mathematicians in the Nearest Neighbors discussion. Reasons for finding an average distance to a nearest neighbor are easy to justify, the computation is elegant, but the result is quite surprising and convincingly supported by a Monte Carlo method.

Naturally, there has to be a discussion about an electric quilt. The story starts with a real square quilt cutting puzzle of Sam Loyd from 1907, but then the quest continues in finding the perfect squared square. Once the quilt turns electric, things are easier to see — mathematically and physically speaking.

There is much more to explore and capture in the book since pages are quilted with a fine thread of jokes and notes, personal comments that each reader needs to pick out on their own.

And as the ball is being tossed between Math and Physics throughout this book, we can only thank the author for helping us understand more of this game. Often we tell our students that to be able to do physics they have to learn "all this math", but this book suggests that the converse is also true: to be able to do "all this math", we need to be able to understand and use "all this physics".

A Taste of Mathematics Volume VIII, Problems for Mathematics Leagues III
By Peter I. Booth, John Grant McLoughlin, and Bruce L.R. Shawyer
Published by the Canadian Mathematical Society, 2008
ISBN: 0-919558-17-8, softcover, 55+vii pages, US$15
Reviewed by **Nancy Clarke**, *Acadia University, Wolfville, NS*

Problems for Mathematics Leagues III, the eighth volume in the ATOM series (A Taste of Mathematics) published by the Canadian Mathematical Society, is a collection of problems created for the 2003-2004 and 2004-2005 seasons of the Newfoundland and Labrador Senior Mathematics League. The league was founded by one of the authors, Bruce Shawyer, more than 20 years ago in St. John's and has since expanded throughout the province. The league was also the inspiration for a similar league which has been running in Nova Scotia in recent years.

The authors have considerable experience and skill with mathematical problem solving and, in particular, with helping to develop such skills in students. For example, Peter Booth has been a member of the CMS's Mathematical Olympiad Committee and Bruce Shawyer has served as a head coach of Canada's International Mathematical Olympiad team. Interestingly, it was during a competition of the Newfoundland Math League that the reviewer first met John Grant McLoughlin and began to develop her own "taste for mathematics".

The first four games in a season of the Newfoundland Math League are held regionally, with the highest ranked teams competing in a provincial championship game. The problems in this book are organized according to game. Each of the five games per season include ten "regular" problems, ordered in terms of increasing difficulty, followed by a special relay problem at the end. The relay problem has four parts and is constructed so that each part depends on the answer to the previous one.

This is a wonderful book, with a variety of problems coming from areas of mathematics such as combinatorics, number theory and geometry. Complete solutions to the first ten problems per game are included. Answers, without solutions, are given for the relay problems. Multiple solutions to problems are sometimes included. This is especially instructive as it helps illustrate the connections that exist between the various mathematical ideas presented. This variety of analyses also shows students that different approaches to a problem can all lead to correct solutions.

Problems for Mathematics Leagues III is a book that makes problem solving accessible, and illustrates that it can be both rewarding and enjoyable. It is a valuable resource for teachers and a great adventure for keen students.

Ratio-Type Inequalities for Bisectors, Medians, Altitudes, and Sides of a Triangle

Mihály Bencze and Shan-He Wu

The inequalities relating angle-bisectors, medians, altitudes, and the sides of a triangle have attracted the interest of many geometers and have motivated a large number of research papers (for example, see [1], [3], [4], [6], and [9]. An excellent survey on these inequalities can be found in the well-known monograph [7], p. 200-229. In this paper, we present several interesting ratio-type inequalities. It is shown that the terms contained in the inequalities have a reciprocal relation to each other.

For a triangle ABC we let $A = 2\alpha$, $B = 2\beta$, $C = 2\gamma$ denote the angles at the respective vertex, a, b, c denote the lengths of the opposing sides, s, R, and r denote the semiperimeter, circumradius, and inradius, respectively. Let w_a, w_b, w_c; m_a, m_b, m_c; h_a, h_b, h_c denote respectively the angle-bisectors, medians, and altitudes emanating from A, B, C. All sums are cyclic sums unless otherwise indicated, so that (for instance)

$$\sum f(a) \;=\; f(a)+f(b)+f(c)\,, \qquad \sum f(a,b) \;=\; f(a,b)+f(b,c)+f(c,a)\,.$$

Our main result is the following.

Theorem For any triangle ABC the following inequalities hold:

$$\sum \frac{s-a}{w_a} \;\leq\; \frac{1}{3}\sum \frac{w_a}{s-a}\,, \tag{1}$$

$$\sum \frac{s-a}{m_a} \;\leq\; \frac{1}{3}\sum \frac{m_a}{s-a}\,, \tag{2}$$

$$\sum \frac{s-a}{h_a} \;\geq\; \frac{1}{3}\sum \frac{h_a}{s-a}\,, \tag{3}$$

$$\sum \frac{a}{w_a} \;\geq\; \frac{4}{3}\sum \frac{w_a}{a}\,, \tag{4}$$

$$\sum \frac{a}{h_a} \;\geq\; \frac{4}{3}\sum \frac{h_a}{a}\,, \tag{5}$$

and equality holds in each of these if and only if the triangle is equilateral.

Proof: Let I be the incentre of triangle ABC, then it is easy to see that $w_a \geq AI + r$. Thus, we obtain

$$\frac{w_a}{s-a} \;\geq\; \frac{AI}{s-a} + \frac{r}{s-a} \;=\; \frac{1+\sin\alpha}{\cos\alpha}\,. \tag{6}$$

By using the formula $w_a = \dfrac{2bc\cos\alpha}{b+c}$ for the bisector length, we have

$$\begin{aligned}
\frac{s-a}{w_a} &= \frac{(b+c)(s-a)}{2bc\cos\alpha} \\
&= \frac{(\sin B+\sin C)(\sin B+\sin C-\sin A)}{4\sin B\sin C\cos\alpha} \\
&= \frac{\sin B+\sin C}{4\cos\beta\cos\gamma} \;=\; \frac{1}{2}\left(\frac{\sin\beta}{\cos\gamma}+\frac{\sin\gamma}{\cos\beta}\right) . \qquad (7)
\end{aligned}$$

Summing cyclically yields

$$\begin{aligned}
\frac{1}{3}\sum\frac{w_a}{s-a}-\sum\frac{s-a}{w_a} &\geq \sum\left(\frac{1+\sin\alpha}{3\cos\alpha}-\frac{\sin\beta+\sin\gamma}{2\cos\alpha}\right) \\
&= \frac{1}{6}\sum\frac{1}{\cos\alpha}\Big(2+5\sin\alpha-3\sum\sin\alpha\Big). \qquad (8)
\end{aligned}$$

Using Čebyšev's inequality (see [8], p. 36) gives

$$\sum\frac{1}{\cos\alpha}\Big(2+5\sin\alpha-3\sum\sin\alpha\Big) \;\geq\; \frac{1}{3}\Big(\sum\frac{1}{\cos\alpha}\Big)\Big(6-4\sum\sin\alpha\Big) .$$

Now, by appealing to the known inequality $\sum\sin\alpha \leq \frac{3}{2}$, it follows that

$$\sum\frac{1}{\cos\alpha}\Big(2+5\sin\alpha-3\sum\sin\alpha\Big) \;\geq\; 0 .$$

From (8) and the above inequality, we obtain

$$\frac{1}{3}\sum\frac{w_a}{s-a} \;-\; \sum\frac{s-a}{w_a} \;\geq\; 0 ,$$

which yields the desired inequality (1).

Note the following known results (see [7], p. 223]):

$$m_a \;\geq\; w_a , \qquad m_b \;\geq\; w_b , \qquad m_c \;\geq\; w_c . \qquad (9)$$

By applying the above inequalities to the inequality (1), we obtain

$$\sum\frac{s-a}{m_a} \;\leq\; \sum\frac{s-a}{w_a} \;\leq\; \frac{1}{3}\sum\frac{w_a}{s-a} \;\leq\; \frac{1}{3}\sum\frac{m_a}{s-a} ,$$

which implies the inequality (2). In view of the two identities

$$h_a \;=\; 2R\sin B\sin C , \qquad s-a \;=\; 4R\sin\beta\sin\gamma\cos\alpha ,$$

we have

$$\begin{aligned}
&\sum \frac{s-a}{h_a} - \frac{1}{3}\sum \frac{h_a}{s-a} \\
&= 2\sum \frac{\sin\beta\sin\gamma\cos\alpha}{\sin B\sin C} - \frac{1}{6}\sum \frac{\sin B\sin C}{\sin\beta\sin\gamma\cos\alpha} \\
&= \frac{1}{2}\sum \frac{\cos^2\alpha}{\cos\alpha\cos\beta\cos\gamma} - \frac{2}{3}\sum \frac{\cos^2\beta\cos^2\gamma}{\cos\alpha\cos\beta\cos\gamma} \\
&= \frac{1}{2}\sum \frac{\cos^2\alpha}{\cos\alpha\cos\beta\cos\gamma} - \frac{1}{3}\sum \frac{\cos^2\alpha\left(\cos^2\beta+\cos^2\gamma\right)}{\cos\alpha\cos\beta\cos\gamma} \\
&= \frac{1}{6\cos\alpha\cos\beta\cos\gamma}\sum \cos^2\alpha\left(3-2\cos^2\beta-2\cos^2\gamma\right). \qquad (10)
\end{aligned}$$

Using Čebyšev's inequality we have

$$\sum \cos^2\alpha\left(3-2\cos^2\beta-2\cos^2\gamma\right) \ \geq\ \frac{1}{3}\left(\sum\cos^2\alpha\right)\left(9-4\sum\cos^2\alpha\right).$$

Thus, by Kooistra's inequality $\sum\cos^2\alpha \leq \frac{9}{4}$ (see [1], p. 26, item 2.29), we conclude that

$$\sum \cos^2\alpha\left(3-2\cos^2\beta-2\cos^2\gamma\right) \ \geq\ 0\,. \qquad (11)$$

Combining (10) and (11) yields

$$\sum \frac{s-a}{h_a} - \frac{1}{3}\sum\frac{h_a}{s-a} \ \geq\ 0\,,$$

which is the desired inequality (3).

The inequality (4) follows from the inequality (9) and an earlier result obtained by Liu and Chu (see [5], p. 146 and [2], p. 492):

$$\sum \frac{a}{w_a} \ \geq\ \frac{4}{3}\sum\frac{m_a}{a}\,. \qquad (12)$$

Further, from the inequality (4) and the well-known inequalities

$$w_a \ \geq\ h_a\,, \qquad w_b \ \geq\ h_b\,, \qquad w_c \ \geq\ h_c\,, \qquad (13)$$

we obtain

$$\sum\frac{a}{h_a} \ \geq\ \sum\frac{a}{w_a} \ \geq\ \frac{4}{3}\sum\frac{w_a}{a} \ \geq\ \frac{4}{3}\sum\frac{h_a}{a}\,,$$

which proves the inequality (5).

Next, we give a direct proof of the inequality (5).

From the identities

$$h_a \ =\ 2R\sin B\sin C\,, \qquad h_b \ =\ 2R\sin C\sin A\,, \qquad h_c \ =\ 2R\sin A\sin B\,,$$

we deduce that

$$\begin{aligned}
&\sum \frac{a}{h_a} - \frac{4}{3}\sum \frac{h_a}{a} \\
&\quad = \sum \frac{\sin A}{\sin B \sin C} - \frac{4}{3}\sum \frac{\sin B \sin C}{\sin A} \\
&\quad = \frac{1}{\sin A \sin B \sin C}\left(\sum \sin^2 A - \frac{4}{3}\sum \sin^2 B \sin^2 C\right) \\
&\quad = \frac{1}{\sin A \sin B \sin C}\left[\sum \sin^2 A - \frac{4}{9}\left(\sum \sin^2 A\right)^2\right. \\
&\qquad\qquad \left. + \frac{2}{9}\sum \left(\sin^2 B - \sin^2 C\right)^2\right] \\
&\quad = \frac{1}{\sin A \sin B \sin C}\left[\frac{4}{9}\left(\sum \sin^2 A\right)\left(\frac{9}{4} - \sum \sin^2 A\right)\right. \\
&\qquad\qquad \left. + \frac{2}{9}\sum \left(\sin^2 B - \sin^2 C\right)^2\right] .
\end{aligned}$$

Now, using Kooistra's inequality $\sum \sin^2 A \le \frac{9}{4}$ (see [1], p. 18, item 2.3), leads us to the inequality (5).

Finally, the condition for equality to hold in any of (1)-(5) can be deduced from the process of the proof stated above. This completes the proof of Theorem. ∎

We now make some remarks.

It is worth noting that the inequalities

$$\sum \frac{a}{m_a} \ \ge\ \frac{4}{3}\sum \frac{m_a}{a} \tag{14}$$

and

$$\sum \frac{a}{m_a} \ \le\ \frac{4}{3}\sum \frac{m_a}{a} \tag{15}$$

are not true in general.

Indeed, in a right-angled triangle with sides $a = 1$, $b = \sqrt{3}$, $c = 2$ we have $m_a = \frac{\sqrt{13}}{2}$, $m_b = \frac{\sqrt{7}}{2}$, $m_c = 1$; so that

$$\sum \frac{a}{m_a} - \frac{4}{3}\sum \frac{m_a}{a} \ = \ \frac{1092 + 52\sqrt{21} - 420\sqrt{13}}{819} \ = \ -0.224\ldots \ < \ 0$$

and inequality (14) fails.

Also, in a right-angled triangle with sides $a = 1$, $b = 1$, $c = \sqrt{2}$ we have $m_a = \frac{\sqrt{5}}{2}$, $m_b = \frac{\sqrt{5}}{2}$, $m_c = \frac{\sqrt{2}}{2}$; so that

$$\sum \frac{a}{m_a} - \frac{4}{3}\sum \frac{m_a}{a} \ = \ \frac{20 - 8\sqrt{5}}{15} \ = \ 0.140\ldots \ > \ 0$$

and inequality (15) fails.

Acknowledgments

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Mihály Bencze
Str. Harmanului 6
505600 Sacele-Négyfalu
Jud. Brasov
Romania
benczemihaly@yahoo.com

Shan-He Wu
Mathematics and Computer Science
Longyan University
Longyan, Fujian 364012
China
wushanhe@yahoo.com.cn

Polynomials Without Sign Changes

Gerhard J. Woeginger

Introduction

One of the simplest mathematical inequalities states that the square of a real number is nonnegative. Although this inequality is straightforward, it is quite powerful. For instance, it crunches the following problem:

Problem 1 Let a, b, c, d be four real numbers with $a + 2b + 3c + 4d \geq 30$. Prove that $a^2 + b^2 + c^2 + d^2 \geq 30$.

Indeed, the squares of the real numbers $a-1$, $b-2$, $c-3$, and $d-4$ are nonnegative. Thus, their sum is also nonnegative:

$$(a-1)^2 + (b-2)^2 + (c-3)^2 + (d-4)^2 \;\geq\; 0\,.$$

Rewriting this inequality and using the constraint then yields the result:

$$a^2 + b^2 + c^2 + d^2 \;\geq\; 2(a + 2b + 3c + 4d) - 30 \;\geq\; 30\,.$$

Here is a slightly different way of viewing (one quarter of) this solution: The solution actually exploits the fact that the polynomial $P(x) = (x-1)^2$ does not change its sign on the real numbers, and that it only takes nonnegative values on the entire set of real numbers.

Here is another problem whose solution exploits the more challenging observation that the polynomial $Q(x) = (x-1)(x-3)$ does not change sign on the real interval $[1, 3]$, and that this polynomial is nonpositive on this interval.

Problem 2 Determine all eight-tuples $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ of real numbers with entries in the closed interval $[1, 3]$ whose sum is 10 and whose squares add up to 16.

So let us consider eight real numbers x_i ($1 \leq i \leq 8$) from the closed interval $[1, 3]$. The polynomial $Q(x) = (x-1)(x-3)$ has its roots at $x = 1$ and $x = 3$, and hence does not change its sign on $[1, 3]$.

Thus $Q(x_i) = x_i^2 - 4x_i + 3$ is nonpositive for $1 \leq i \leq 8$, and adding up the corresponding eight inequalities yields

$$\begin{aligned} 0 \;\geq\; \sum_{i=1}^{8} Q(x_i) &= \sum_{i=1}^{8} x_i^2 - 4\sum_{i=1}^{8} x_i + 24 \\ &= 16 - 4 \cdot 10 + 24 \;=\; 0\,. \end{aligned}$$

We see that the inequality is in fact an equality, and this implies that equality holds in each inequality $Q(x_i) \leq 0$ for $1 \leq i \leq 8$. Hence $x_i \in \{1, 3\}$ for $1 \leq i \leq 8$. Now one easily derives that one of the x_i must be equal to 3, whereas the other seven numbers all take the value 1.

In this article we will discuss several related problems that all can be settled by the same type of approach: By investigating an appropriately chosen polynomial that does not change its sign on an appropriately chosen domain, and that hence only takes nonnegative (or only takes nonpositive) values on this domain. Of course choosing the right polynomial for such a problem remains an art.

A problem involving rational numbers

Let us analyze the following problem, which was discussed in the spring of 2006 in the German-language Usenet puzzle newsgroup `de.rec.denksport`:

Problem 3 Determine the smallest odd integer $n \geq 3$, for which there exist n rational numbers $x_1, x_2, \ldots, x_n$ with the following properties:

(a) $\sum\limits_{i=1}^{n} x_i = 0$ and $\sum\limits_{i=1}^{n} x_i^2 = 1$.

(b) $x_i x_j \geq -\dfrac{1}{n}$ for all i, j with $1 \leq i, j \leq n$.

Why does the problem statement only ask about odd integers n? Well, the answer is that the smallest even integer n would be easy to find: The case $n = 2$ does not work since $x_1 + x_2 = 0$ together with $x_1^2 + x_2^2 = 1$ yields irrational values. But $n = 4$ works with $x_1 = x_2 = -\dfrac{1}{2}$ and $x_3 = x_4 = \dfrac{1}{2}$. The even case is not interesting.

So let us turn to odd numbers n, and let us assume that the rational numbers $x_1, x_2, \ldots, x_n$ satisfy the conditions in this problem. Denote their minimum by α and denote their maximum by β, and note that this implies $\alpha\beta \geq -\dfrac{1}{n}$. Consider the polynomial $P(x) = (x - \alpha)(x - \beta)$ which does not change its sign on the interval $[\alpha, \beta]$. Hence, $P(x_i) \leq 0$ holds for $1 \leq i \leq n$, and adding up yields

$$\begin{aligned} 0 \;\geq\; \sum_{i=1}^{n} P(x_i) \;&=\; \sum_{i=1}^{n} x_i^2 - (\alpha + \beta) \sum_{i=1}^{n} x_i + n\alpha\beta \\ &=\; 1 + n\alpha\beta \;\geq\; 0\,. \end{aligned}$$

This implies that $P(x_i) = 0$, and hence $x_i \in \{\alpha, \beta\}$ for $1 \leq i \leq n$. Now let k denote the number of x_i's that are equal to α, and let ℓ denote the number of x_i's that are equal to β. We then have $1 \leq k, \ell \leq n-1$ and $n = k+\ell$. By rewriting the conditions of the problem we obtain

$$k\alpha + \ell\beta = 0 \qquad \text{and} \qquad k\alpha^2 + \ell\beta^2 = 1,$$

which leads to

$$\alpha^2 = \frac{\ell}{k(k+\ell)} \qquad \text{and} \qquad \beta^2 = \frac{k}{\ell(k+\ell)}.$$

Our goal is to find the smallest odd $n = k+\ell$ for which these values α and β are rational. If n is prime, then $\frac{\ell}{kn}$ cannot be a rational square (since the prime n only shows up once in the prime factorizations of numerator and denominator). This excludes the cases $n = 3, 5, 7, 11, 13, 17, 19, 23$. Some case analysis shows that the cases $n = 9, 15, 21$ are also impossible. This leaves $n = 25$ as the smallest odd candidate.

And indeed, $k = 9$ and $\ell = 16$ lead to the rational values $\alpha = -\frac{4}{15}$ and $\beta = \frac{3}{20}$. Hence, setting $x_i = -\frac{4}{15}$ for $1 \leq i \leq 9$ and $x_i = \frac{3}{20}$ for $10 \leq i \leq 25$ yields the desired rational solution for odd $n = 25$.

A problem involving real numbers

The following problem was devised by me and has appeared on the 2009 Mediterranean Mathematics Olympiad.

Problem 4 Determine all integers $n \geq 1$ for which there exist n real numbers $x_1, x_2, \ldots, x_n$ in the closed interval $[-4, 2]$ such that

the sum of these numbers is at least n,
the sum of their squares is at most $4n$,
the sum of their fourth powers is at least $34n$.

Since the conditions involve fourth powers, it is natural to look for a polynomial of degree 4. Furthermore, the boundaries of the interval $[-4, 2]$ suggest that two of the roots might be -4 and 2. Some experimentation eventually leads us to the polynomial $P(x) = (x+4)(x-2)(x-1)^2$. Then for x with $-4 \leq x \leq 2$ we have

$$0 \geq P(x) = x^4 - 11x^2 + 18x - 8.$$

Adding up the corresponding n inequalities for $x = x_1, x_2, \ldots, x_n$ and plugging in the appropriate bounds from the problem statement yields

$$\begin{aligned} 0 &\geq \sum_{i=1}^{n} x_i^4 - 11\sum_{i=1}^{n} x_i^2 + 18\sum_{i=1}^{n} x_i - 8n \\ &\geq 34n - 11\cdot 4n + 18n - 8n = 0. \end{aligned}$$

Hence $P(x_i) = 0$ for $1 \le i \le n$, which implies that $x_i \in \{-4, 1, 2\}$ for all i. Let a, b, c denote the number of x_i's that are equal to -4, 1, 2, respectively. Then $a+b+c = n$, and the bounds in the problem statement translate into

$$\begin{aligned} -4a + b + 2c &\ge a + b + c\,, \\ 16a + b + 4c &\le 4(a + b + c)\,, \\ 256a + b + 16c &\ge 34(a + b + c)\,. \end{aligned}$$

The first inequality boils down to $c \ge 5a$, and the second inequality boils down to $b \ge 4a$. From the third inequality we then find that in fact $b = 4a$ and $c = 5a$. Therefore, $n = 10a$ must hold.

In summary the answer to problem 4 is: Such real numbers $x_1, \ldots, x_n$ do exist if and only if n is a multiple of 10. For $n = 10a$, a feasible solution results by taking the number -4 a total of a times, the number 1 a total of $4a$ times, and the number 2 a total of $5a$ times.

Homework exercises

We encourage the reader to settle the following five problems along the lines indicated above. Titu Andreescu [1] has posed the following nice puzzle in the problem corner of *Mathematical Reflections*:

Problem 5 Determine all real solution pairs (x, y) for the system

$$\begin{aligned} x^4 + 2y^3 - x &= -\frac{1}{4} + 3\sqrt{3}\,, \\ y^4 + 2x^3 - y &= -\frac{1}{4} - 3\sqrt{3}\,. \end{aligned}$$

One possible approach to this problem involves adding the two given equations, and then recognizing the squares of two polynomials in their sum.

The following three problems can be attacked by considering polynomials of the form $P(x) = (x-a)(x-b)^2$ for appropriately chosen real numbers a and b.

Problem 6 Let $n \ge 2$ be an integer. Determine all $(n+2)$-tuples of real numbers such that $x_1, x_2, \ldots, x_{n+2} \ge 1$, and

the sum of these numbers is at most $n^2 + 2$,
the sum of their squares is at least $n^3 + 2$,
the sum of their third powers is at most $n^4 + 2$.

Problem 7 Let $n \ge 1$, and let $x_1, x_2, \ldots, x_n \ge -1$ be real numbers with $\sum_{i=1}^{n} x_i^3 = 0$. Prove that $\sum_{i=1}^{n} x_i \le \frac{n}{3}$.

Problem 8 Let $n \geq 1$, and let $x_1, x_2, \ldots, x_n \leq 2$ be real numbers with $\sum_{i=1}^{n} x_i = 0$. Prove that $\sum_{i=1}^{n} x_i^3 \leq 2n$.

Our last problem is taken from the final round of the 2002 Romanian National Olympiad. Svetoslav Savchev and Titu Andreescu (on page 16 of their book [2]) propose to attack this problem through clever trigonometric substitutions.

Problem 9 Find all real numbers $-2 \leq a, b, c, d, e \leq 2$ that satisfy the following system:

$$\begin{aligned} a+b+c+d+e &= 0\,, \\ a^3+b^3+c^3+d^3+e^3 &= 0\,, \\ a^5+b^5+c^5+d^5+e^5 &= 10\,. \end{aligned}$$

We propose that the reader try to find another solution approach that is based on the polynomial $P(x) = (x-2)(x^2+x-1)^2$. In fact, the answer to the problem does not change if one drops the upper bound constraint on a, b, c, d, e from the problem statement.

References

[1] T. Andreescu, Junior problem J1. Problem column of *Mathematical Reflections* 2006, Issue 1.

[2] S. Savchev and T. Andreescu. *Mathematical Miniatures*. Mathematical Association of America, 2002.

Gerhard J. Woeginger
Department of Mathematics and Computer Science
TU Eindhoven
P.O. Box 513, NL-5600 MB Eindhoven
The Netherlands
gwoegi@win.tue.nl

PROBLEMS

Solutions to problems in this issue should arrive no later than **1 March 2011**. *An asterisk (⋆) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3551. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $p \geq 2$ be an integer. Find the product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{\lfloor\sqrt[p]{n}\rfloor}\right)^{(-1)^{n-1}},$$

where $\lfloor a\rfloor$ is the greatest integer not exceeding a.

3552. *Proposed by N. Javier Buitrago Aza, Universidad Nacional de Colombia, Bogota, Colombia.*

Let θ be a real number. Prove that

$$\sum_{k=0}^{n-1}\frac{\sin\left(\frac{2k\pi}{n}-\theta\right)}{3+2\cos\left(\frac{2k\pi}{n}-\theta\right)} = \frac{(-1)^n n\sin(n\theta)}{5F_n^2+4(-1)^n\sin^2\left(\frac{n\theta}{2}\right)},$$

where F_n denotes the n^{th} Fibonacci number.

3553. *Proposed by Michel Bataille, Rouen, France.*

Let A, B, and C be the angles of a triangle. Prove that

$$\sum_{\text{cyclic}}\left(\sin A\cos\frac{B}{2}\cos\frac{C}{2}\right)^2 \leq \sum_{\text{cyclic}}\cos^6\frac{A}{2}.$$

3554. *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ca}}{c+a}+\frac{\sqrt{c^2+ab}}{a+b} \geq \sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}.$$

3555. *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let a and b be positive integers, $1 < a < b$, such that a does not divide b. Prove that there exists an integer x such that $1 < x \le a$ and both a and b divide $x^{\phi(b)+1} - x$, where ϕ is Euler's totient function.

3556. *Proposed by Arkady Alt, San Jose, CA, USA.*

For any acute triangle with side lengths a, b, and c, prove that

$$(a+b+c)\min\{a,b,c\} \;\le\; 2ab+2bc+2ca-a^2-b^2-c^2\,.$$

3557. *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers with $\sum\limits_{k=1}^{\infty} a_k = 1$ and $a_{k+1} \le \dfrac{a_k}{1-a_k}$. Let $S_n^{(p)} = \left(\sum\limits_{k=1}^{n} a_k^p\right)^{1/p}$, and for $p \ge 1$ prove that

$$\lim_{n\to\infty} \sum_{k=n+1}^{n} \frac{k}{n} \left(\prod_{j=1}^{n} \frac{j^{1/p} a_{k+j} a_j}{S_{k+j}^{(p)}} \right)^{1/n} \;=\; 0\,.$$

3558. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Given two distinct positive integers a and b, prove that there exists a positive integer n such that an and bn have different numbers of digits.

3559★. *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Let ABC be a triangle with side lengths a, b, c, inradius r, circumradius R, and semiperimeter s. Prove that

$$\frac{(b+c)^2}{4bc} \;\le\; \frac{s^2}{3r(4R+r)}\,.$$

3560. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2+y^2=1$. Find the maximum value of

$$f(x,y) \;=\; |x-y| + \left|x^3-y^3\right|\,.$$

3561. *Proposed by Mihály Bencze, Brasov, Romania.*

An n-sided polygon has perimeter k with $k^2 < 2n^2$. Prove that some three consecutive vertices along the polygon form a triangle with area less than 1 unit.

3562. *Proposed by Vahagn Aslanyan, Yerevan, Armenia.*

Let p be a prime number. Prove that there exists a prime number q such that $p \mid (q-1)$ and with the property that $q^k \mid \left(a^p - b^p\right)$ whenever $q^k \mid \left(a^{p^m} - b^{p^m}\right)$ for positive integers a, b, m, k with a and b not divisible by q.

3563. *Proposed by Mikhail Kochetov and Sergey Sadov, Memorial University of Newfoundland, St. John's, NL.*

A square $n \times n$ array of lamps is controlled by an $n \times n$ switchboard. Flipping a switch in position (i,j) changes the state of all lamps in row i and in column j.

(a) Prove that for even n it is possible to turn off all the lamps no matter what the initial state of the array is. Demonstrate how to do it with the minimum number of switches.

(b) Prove that for odd n it is possible to turn off all the lamps if and only if the initial state of the array has the following property: either the number of ON lamps in every row and every column is odd, or the number of ON lamps in every row and every column is even. If this property holds, provide an algorithm to turn off all the lamps.

. . . .

3551. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $p \geq 2$ un entier. Trouver le produit

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{\lfloor \sqrt[p]{n} \rfloor}\right)^{(-1)^{n-1}},$$

où $\lfloor a \rfloor$ est le plus grand entier n' excédant pas a.

3552. *Proposé par N. Javier Buitrago Aza, Université nationale de Colombie, Bogotá, Colombie.*

Soit θ un nombre réel. Montrer que

$$\sum_{k=0}^{n-1} \frac{\sin\left(\frac{2k\pi}{n} - \theta\right)}{3 + 2\cos\left(\frac{2k\pi}{n} - \theta\right)} = \frac{(-1)^n n \sin(n\theta)}{5F_n^2 + 4(-1)^n \sin^2\left(\frac{n\theta}{2}\right)},$$

où F_n désigne le $n^{\text{ième}}$ nombre de Fibonacci.

3553. *Proposé par Michel Bataille, Rouen, France.*

Soit A, B et C les angles d'un triangle. Montrer que

$$\sum_{\text{cyclique}} \left(\sin A \cos\frac{B}{2}\cos\frac{C}{2}\right)^2 \;\leq\; \sum_{\text{cyclique}} \cos^6\frac{A}{2}\,.$$

3554. *Proposé par Pham Huu Duc, Ballajura, Australie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\frac{\sqrt{a^2+bc}}{b+c}+\frac{\sqrt{b^2+ca}}{c+a}+\frac{\sqrt{c^2+ab}}{a+b} \;\geq\; \sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\,.$$

3555. *Proposé par Vahagn Aslanyan, Erevan, Arménie.*

Soit a et b deux entiers positifs, $1 < a < b$, tels que a n'est pas un diviseur de b. Montrer qu'il existe alors un entier x tel que $1 < x \leq a$ et que a et b sont tous les deux diviseur de $x^{\phi(b)+1} - x$, ϕ étant l'indicatrice d'Euler.

3556. *Proposé par Arkady Alt, San José, CA, É-U.*

Montrer que pour tout triangle acutangle de côtés a, b et c, on a

$$(a+b+c)\min\{a,b,c\} \;\leq\; 2ab+2bc+2ca-a^2-b^2-c^2\,.$$

3557. *Proposé par Paolo Perfetti, Département de Mathématiques, Université de Rome, "Tor Vergata", Rome, Italie.*

Soit $\{a_k\}_{k=1}^{\infty}$ une suite de nombres réels positifs tels que $\sum\limits_{k=1}^{\infty} a_k = 1$ et $a_{k+1} \leq \dfrac{a_k}{1-a_k}$. Soit $S_n^{(p)} = \left(\sum\limits_{k=1}^{n} a_k^p\right)^{1/p}$ et, pour $p \geq 1$, montrer que

$$\lim_{n\to\infty} \sum_{k=n+1}^{n} \frac{k}{n}\left(\prod_{j=1}^{n}\frac{j^{1/p}a_{k+j}a_j}{S_{k+j}^{(p)}}\right)^{1/n} \;=\; 0\,.$$

3558. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

On donne deux entiers positifs distincts a et b . Montrer qu'il existe un entier positif n tel que an et bn ont un nombre de chiffres différents.

3559★. *Proposé par Thanos Magkos, 3ième -Collège de Kozanie, Kozani, Grèce.*

Soit a, b et c les côtés d'un triangle ABC, r le rayon de son cercle inscrit, R celui de son cercle circonscrit, et s son demi-périmètre. Montrer que

$$\frac{(b+c)^2}{4bc} \leq \frac{s^2}{3r(4R+r)} .$$

3560. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Soit x et y deux nombres réels tels que $x^2 + y^2 = 1$. Trouver le maximum de

$$f(x,y) = |x-y| + \left|x^3 - y^3\right| .$$

3561. *Proposé par Mihály Bencze, Brasov, Roumanie.*

On donne un polygone à n côtés de périmètre k avec $k^2 < 2n^2$. Montrer qu'il possède trois sommets consécutifs formant un triangle dont l'aire est plus petite que 1.

3562. *Proposé par Vahagn Aslanyan, Erevan, Arménie.*

Soit p un nombre premier. Montrer qu'il existe un nombre premier q tel que $p \mid (q-1)$, avec la propriété que $q^k \mid \left(a^p - b^p\right)$ chaque fois que $q^k \mid \left(a^{p^m} - b^{p^m}\right)$, a, b, m et k étant des entiers positifs avec a et b non divisibles par q.

3563. *Proposé par Mikhail Kochetov et Sergey Sadov, Université Memorial de Terre-Neuve, St. John's, NL.*

Un tableau carré formé de $n \times n$ lampes est contrôlé par un tableau de distribution $n \times n$. Actionner un bouton en position (i,j) change l'état de toutes les lampes de la ligne i et de la colonne j.

(a) Montrer que pour n pair, il est possible d'éteindre toutes les lampes quel que soit le statut initial du tableau. Indiquer comment procéder en actionnant un minimum de boutons.

(b) Montrer que pour n impair, il est possible d'éteindre toutes les lampes si et seulement si l'état initial du tableau répond à la condition suivante : le nombre de lampes allumées dans chaque ligne et chaque colonne est soit impair, soit pair. Dans le cas où cette condition est satisfaite, proposer un algorithme pour éteindre toutes les lampes.

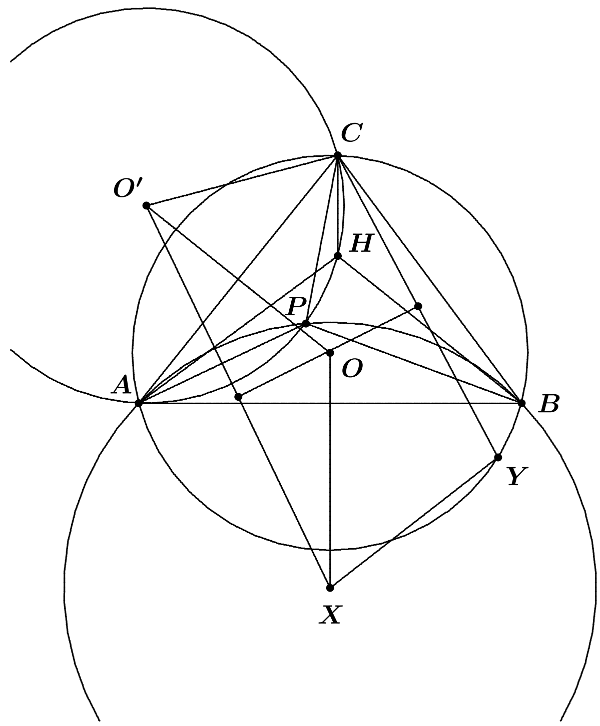
TOTTEN SOLUTIONS

These are the solutions to the special section of problems appearing in the September 2009 issue and dedicated to the memory of Jim Totten.

TOTTEN–01. [2009 : 320, 322] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let H be the orthocentre of triangle ABC and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of $\angle BAC$. If X is the circumcentre of triangle APB and if Y is the orthocentre of triangle APC, prove that the length of XY is equal to the circumradius of triangle ABC.

Solution by John G. Heuver, Grande Prairie, AB.



The points A, B, C, H form an orthocentric set, so the circumcircles of the four triangles formed have congruent radii. In particular, triangles ABC and AHC have equal circumradii with AC as axis of symmetry. Now A, P, C, Y also form an orthocentric set with the circumcircles of triangles APC and ACY having AC as axis of symmetry, and P lies on the circumcircle of triangle AHC, hence Y lies on the circumcircle of triangle ABC.

Let O, O' be the circumcentres of triangles ABC, AHC. Line segments YC and $O'X$ are parallel as both are perpendicular to AP. Consider the perpendicular bisector of YC passing through O and making a right angle with $O'X$. The rays forming $\angle OO'X$ make right angles with the rays of $\angle PAC$, hence $\angle OO'X = \frac{1}{2}\angle A$, and similarly $\angle OXO' = \frac{1}{2}\angle A$. It follows that the perpendicular bisector of YC also bisects $O'X$, since triangle

OXO' is isosceles. Thus, $O'C$ and XY are symmetric in the perpendicular bisector of YC, and hence $O'C = XY$, which solves the problem since triangles ABC and APC have the same circumradius.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

TOTTEN–02. [2009 : 320, 322] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k \geq 2$ be an integer and let $f : [0, \infty) \to \mathbb{R}$ be a bounded continuous function. If x is a positive real number, find the value of

$$\lim_{n\to\infty} \sqrt[k]{n} \int_0^x \frac{f(t)}{\left(1+t^k\right)^n}\, dt\,.$$

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The substitution $u = t \cdot \sqrt[k]{n}$ transforms

$$I(n,k,x) \;=\; \sqrt[k]{n} \int_0^x \frac{f(t)}{\left(1+t^k\right)^n}\, dt$$

into

$$I(n,k,x) \;=\; \int_0^{x\cdot\sqrt[k]{n}} \frac{f\left(\frac{u}{\sqrt[k]{n}}\right)}{\left(1+\frac{u^k}{n}\right)^n}\, du \;=\; \int_0^{\infty} \frac{f\left(\frac{u}{\sqrt[k]{n}}\right)}{\left(1+\frac{u^k}{n}\right)^n} \cdot \chi(u)\, du\,,$$

where $\chi(u) = 1$ if $u \in \left[0, x \cdot \sqrt[k]{n}\right]$ and $\chi(u) = 0$ otherwise. Define

$$g_n(u) \;=\; \frac{f\left(\frac{u}{\sqrt[k]{n}}\right)}{\left(1+\frac{u^k}{n}\right)^n} \cdot \chi(u)\,.$$

For fixed $u \geq 0$ and k, the denominator is an increasing function of n, so that with $|f| \leq M$ we have

$$|g_n(u)| \;\leq\; \frac{M}{1+u^k}\,,$$

that is, an integrable function dominates $g_n(u)$. By Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty} I(n,k,x) \;&=\; \int_0^{\infty} \left[\lim_{n\to\infty} g_n(u)\right]\, du \;=\; \int_0^{\infty} \frac{f(0)}{\exp(u^k)}\, du \\ &=\; f(0)\int_0^{\infty} e^{-u^k}\, du \;=\; \frac{f(0)}{k}\Gamma\left(\frac{1}{k}\right)\,.\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incorrect solution and one incomplete solution submitted.

TOTTEN–03. [2009 : 320, 323] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{n}{\dfrac{1}{x_1}+\cdots+\dfrac{1}{x_n}}\right) dx_1 \ldots dx_n\,.$$

Solution by the proposer, modified by the editor.

Let k be a given positive integer. For any positive integer n, let

$$I_n \;=\; \int_0^1 \cdots \int_0^1 \left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)^k dx_1 \cdots dx_n\,.$$

Put $C_k = \int_0^\infty t^{k-1}e^{-t}\,dt < \infty$. We will first show that

$$\lim_{n\to\infty} I_n = 0\,. \tag{1}$$

Making the substitutions $y_i = \dfrac{1}{x_i}$, $i = 1, \ldots, n$ in I_n, we obtain

$$I_n \;=\; n^k \int_1^\infty \cdots \int_1^\infty \frac{dy_1 \cdots dy_n}{(y_1+\cdots+y_n)^k\, y_1^2 \cdots y_n^2}$$

and also we have

$$\begin{aligned}
&\frac{1}{C_k}\int_0^\infty e^{-t(y_1+\cdots+y_n)}t^{k-1}\,dt \\
&\quad= \frac{1}{C_k}\int_0^\infty e^{-u}\left(\frac{u}{y_1+\cdots+y_n}\right)^{k-1}\frac{du}{y_1+\cdots+y_n} \\
&\quad= \frac{1}{C_k}\cdot\frac{1}{(y_1+\cdots+y_n)^k}\int_0^\infty e^{-u}u^{k-1}\,du \\
&\quad= \frac{1}{(y_1+\cdots+y_n)^k}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_n &= n^k \int_1^\infty \cdots \int_1^\infty \left(\frac{1}{C_k}\int_0^\infty e^{-t(y_1+\cdots+y_n)}\,dt\right)\frac{dy_1\cdots dy_n}{y_1^2\cdots y_n^2} \\
&= \frac{n^k}{C_k}\int_0^\infty t^{k-1}\left(\int_1^\infty \frac{e^{-ty}}{y^2}\,dy\right)^n dt \\
&= \frac{1}{C_k}\int_0^\infty s^{k-1}\left(\int_1^\infty \frac{e^{-sy/n}}{y^2}\right)^n ds\,, \quad \text{where } t = \frac{s}{n}\,.
\end{aligned}$$

Let

$$f_n(s) \;=\; s^{k-1}\left(\int_1^\infty \frac{e^{-sy/n}}{y^2}\,dy\right)^n$$

so that

$$I_n \;=\; \frac{1}{C_k}\int_0^\infty f_n(s)\,ds\,. \tag{2}$$

Now, for all $n \geq 1$,

$$f_n(s) \;\leq\; s^{k-1}\left(e^{-\frac{s}{n}}\int_1^\infty \frac{1}{y^2}\,dy\right)^n \;=\; s^{k-1}e^{-s}\,,$$

and thus, f_n is integrable for all $n \geq 1$. So, if we can show $\lim\limits_{n\to\infty} f_n(s) = 0$, then (1) would follow from the Lebesgue Dominated Convergence Theorem and (2). To show that $\lim\limits_{n\to\infty} f_n(s) = 0$, put

$$X_n \;=\; \int_1^\infty \frac{e^{-sy/n}}{y^2}\,dy\,.$$

Then $0 < X_n < 1$ for all n and by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} X_n \;=\; \int_1^\infty \left(\lim_{n\to\infty}\frac{e^{-sy/n}}{y^2}\right)dy \;=\; \int_1^\infty \frac{1}{y^2}\,dy \;=\; 1\,.$$

A power series expansion for $\ln t$ is

$$\ln t = (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \frac{(t-1)^4}{4} + \cdots\,, \quad 0 < t < 2\,,$$

so that

$$|n\ln X_n - n(X_n-1)| = n(X_n-1)^2\left|\tfrac{1}{2} - \tfrac{1}{3}(X_n-1) + \tfrac{1}{4}(X_n-1)^2 - \cdots\right|.$$

Since $\lim\limits_{n\to\infty} X_n = 1$, the sum inside the absolute signs in the above equation approaches $\frac{1}{2}$ in the limit as $n \to \infty$. Also,

$$\begin{aligned}
X_n - 1 \;&=\; \int_1^n \frac{e^{-sy/n}}{y^2}\,dy + \int_n^\infty \frac{e^{-sy/n}}{y^2}\,dy - 1\\
&=\; \left(\frac{-e^{-sy/n}}{y}\right)_1^n - 1 - \frac{s}{n}\int_1^n \frac{e^{-sy/n}}{y}\,dy + \int_n^\infty \frac{e^{-sy/n}}{y^2}\,dy\\
&=\; \frac{-e^{-s}}{n} + \left(e^{-s/n} - 1\right) - \frac{s}{n}\int_1^n \frac{e^{-sy/n}}{y}\,dy + \int_n^\infty \frac{e^{-sy/n}}{y^2}\,dy\,.
\end{aligned}$$

Because

$$\int_1^n \frac{e^{-sy/n}}{y}\,dy \;<\; \int_1^n \frac{1}{y}\,dy \;=\; \ln n$$

and
$$\int_n^\infty \frac{e^{-sy/n}}{y^2}\,dy \;\le\; \int_n^\infty \frac{1}{y^2}\,dy \;=\; \frac{1}{n}\,,$$
it follows that there is some constant C such that for sufficiently large n
$$X_n - 1 \;\le\; C\left(\frac{\ln n}{n}\right)\,.$$
Thus, $\lim_{n\to\infty} n(X_n-1)^2 = 0$ and
$$\lim_{n\to\infty} n\ln X_n \;=\; \lim_{n\to\infty} n(X_n-1)\,,$$
in the sense that both limits equal the same extended real number or both limits do not exist. In particular
$$\lim_{n\to\infty} X_n^n \;=\; \lim_{n\to\infty} e^{n\ln X_n} \;=\; \lim_{n\to\infty} e^{n(X_n-1)}\,. \tag{3}$$
Recalling that s is positive, we have the estimate
$$\begin{aligned} n(X_n-1) \;&=\; n\int_1^\infty \frac{e^{-sy/n}-1}{y^2}\,dy \;\le\; n\int_1^n \frac{e^{-sy/n}-1}{y^2}\,dy \\ &=\; n\left(\frac{1-e^{-sy/n}}{y}\right)_1^n - s\int_1^n \frac{e^{-sy/n}}{y}\,dy \\ &\le\; 1 - s\int_1^n \frac{e^{-sy/n}}{y}\,dy \\ &\le\; 1 - s\int_1^n \frac{e^{-s}}{y}\,dy \;=\; 1 - se^{-s}\ln n\,, \end{aligned}$$
hence, $\lim_{n\to\infty} n(X_n-1) = -\infty$. Therefore, from (3), $\lim_{n\to\infty} X_n^n = 0$, and so
$$\lim_{n\to\infty} f_n(s) \;=\; s^{k-1}\lim_{n\to\infty} X_n^n \;=\; 0\,,$$
which establishes (1). In particular, this implies that for any polynomial $p(t) = a_0 + a_1t + \cdots + a_mt^m$, we have
$$\lim_{n\to\infty}\int_0^1\cdots\int_0^1 p\left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)\,dx_1\cdots dx_n \;=\; a_0 \;=\; p(0)\,.$$
Finally, let f be any continuous function on $[0,1]$ and let $\epsilon > 0$ be arbitrary. By the Weierstrass Approximation Theorem, there is a polynomial p_ϵ such that $|f(x) - p_\epsilon(x)| \le \epsilon$ for all $x \in [0,1]$. Hence,
$$\begin{aligned} L \;&=\; \lim_{n\to\infty}\int_0^1\cdots\int_0^1 (f-p_\epsilon)\left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)\,dx_1\cdots dx_n \\ &\quad+ \lim_{n\to\infty}\int_0^1\cdots\int_0^1 p_\epsilon\left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)\,dx_1\cdots dx_n \\ &=\; \lim_{n\to\infty}\left(\int_0^1\cdots\int_0^1 (f-p_\epsilon)\left(\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)\,dx_1\cdots dx_n\right) + p_\epsilon(0)\,. \end{aligned}$$

As $0 < n\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^{-1} \leq 1$ for all $x_1, \ldots, x_n$ in $(0, 1]$, we have

$$-\epsilon \;\leq\; (f - p_\epsilon)\left(\frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}\right) \;\leq\; \epsilon$$

for all $x_1, \ldots, x_n$ in $(0, 1]$. It follows that $|L - p_\epsilon(0)| \leq \epsilon$, which together with the fact that $p_\epsilon(0) \to f(0)$ as $\epsilon \to 0^+$ implies that $L = f(0)$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and ALBERT STADLER, Herrliberg, Switzerland;

TOTTEN–04. [2009 : 320, 323] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Suppose that $0 < a < b$, $m_0 = \sqrt{ab}$, and $m_1 = \frac{a+b}{2}$. If $x \geq 0$, prove that

$$\frac{x}{m_1(x + m_1)} \;\leq\; \frac{1}{b-a}\log\frac{b(x+a)}{a(x+b)} \;\leq\; \frac{x}{m_0(x+m_0)}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

For $x \geq 0$, let

$$\begin{aligned} f(x) &= \frac{x}{m_1(x+m_1)}\,, \\ g(x) &= \frac{1}{b-a}\log\frac{b(x+a)}{a(x+b)}\,, \\ h(x) &= \frac{x}{m_0(x+m_0)}\,. \end{aligned}$$

Since f, g, and h are differentiable for $x \geq 0$ and $f(0) = g(0) = h(0)$, it suffices to show that

$$f'(x) \;\leq\; g'(x) \;\leq\; h'(x) \tag{1}$$

for all $x \geq 0$. Since $m_0 \leq m_1$, we have

$$2m_0x + m_0^2 \;\leq\; (a+b)x + ab \;\leq\; 2m_1x + m_1^2\,.$$

Hence,

$$(x+m_0)^2 \;\leq\; (x+a)(x+b) \;\leq\; (x+m_1)^2\,,$$

so that

$$\frac{1}{(x+m_1)^2} \;\leq\; \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right) \;\leq\; \frac{1}{(x+m_0)^2}\,,$$

and (1) is established.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There were two incomplete solutions submitted.

TOTTEN–05. [2009 : 320, 323] *Proposed by Michel Bataille, Rouen, France.*

Let I be the incentre of triangle ABC. Let the point A' be such that $\overrightarrow{AA'} = (\cos A)\overrightarrow{AI}$, and let points B' and C' be defined similarly. Find the radius of the circle passing through A', B', and C' and locate its centre.

I. *Solution by J. Chris Fisher, University of Regina, Regina, SK.*

We shall see that the circumradius of $\triangle A'B'C'$ equals the inradius of the original triangle, while its circumcentre is the orthocentre of the triangle whose vertices are the points where the incircle of $\triangle ABC$ touches its sides.

Denote by I_a, I_b, and I_c the images of the incentre I of $\triangle ABC$ under reflections in the sides BC, CA, and AB, respectively. Because AI bisects $\angle A$, we have $\angle IAI_c = \angle IAI_b = \angle A$. Consequently, setting $X = I_bI_c \cap AI$ we see that $\angle I_cXA = 90^\circ$ implies that

$$\cos A = \frac{AX}{AI_c} = \frac{AX}{AI},$$

whence $X = A'$. Similarly,

$$B' = I_cI_a \cap BI, \quad \text{and} \quad C' = I_aI_b \cap CI.$$

Because A', B', and C' are the midpoints of the sides of $\triangle I_aI_bI_c$,

> *the sides of $\triangle A'B'C'$ are parallel to and half the length of the sides of $\triangle I_aI_bI_c$.*

By the definition of reflection, the midpoints D of II_a, E of II_b, and F of II_c are the feet of the perpendiculars from I to the sides of the given triangle $\triangle ABC$; in other words, the circumcircle of $\triangle DEF$ is the incircle of $\triangle ABC$. Moreover, the dilatation with centre I and ratio $1/2$ takes $\triangle I_aI_bI_c$ to $\triangle DEF$; whence,

> *the sides of $\triangle DEF$ are parallel to and half the length of the sides of $\triangle I_aI_bI_c$.*

We conclude that the triangles $A'B'C'$ and DEF are congruent, so that the circumradius of $\triangle A'B'C'$ equals the inradius of $\triangle ABC$, as claimed. We now use the parallel corresponding sides of these two triangles to locate the circumcentre of $\triangle A'B'C'$, call it M. Since D is the midpoint of II_a and $\angle IB'I_a = IC'I_a = 90°$, $DB' = DC'$. That is, the perpendicular bisector of $B'C'$ passes through D as well as through M. But we saw that $B'C' \| EF$, so that DM must be perpendicular also to EF. Similarly, $EM \perp FD$, whence M is the orthocentre of $\triangle DEF$.

II. *Solution by the proposer.*

Let $X(\sigma)$ be the circle with centre X and radius σ. Since $\sin\frac{A}{2} = \frac{r}{IA}$, where r is the inradius of ABC, and $\overrightarrow{IA'} = (1-\cos A)\overrightarrow{IA} = (2\sin^2\frac{A}{2})\overrightarrow{IA}$, we see that $IA \cdot IA' = 2r^2$. Thus, A', B', and C' are the inverses of A, B, and C in the circle $I(\sqrt{2}r)$, and the circumcircle of $\triangle A'B'C'$, say $M(\rho)$, is the inverse of the circumcircle $O(R)$ of $\triangle ABC$. It follows that $M(\rho)$ is the image of $O(R)$ under the homothety with centre I and factor $\frac{2r^2}{p}$, where p is the power of I with respect to $O(R)$. [*Ed.: Details can be found in references dealing with inversive geometry, such as H.S.M. Coxeter,* Introduction to Geometry, *Section 6.3.*] Since $p = IO^2 - R^2 = -2rR$, this factor is $-\frac{r}{R}$, whence M is defined by $\overrightarrow{MI} = \frac{r}{R}\overrightarrow{IO}$ and $\rho = \frac{r}{R}\cdot R = r$.

Comment. Using classical expressions for R and r, one easily obtains trilinear coordinates of M relative to ABC: $M(\cos B + \cos C,\ \cos C + \cos A,\ \cos A + \cos B)$. Thus, M is point X_{65} of Clark Kimberling's *Encyclopedia of Triangle Centers.* (See *Math. Magazine*, **67**:3 (June 1994), p. 179, or the web page, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`). In particular, M is also on the line through the Nagel and Gergonne points of the original triangle.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

Bataille further commented that under the inversion of Solution II, the inverses of the sides of triangle ABC are the circles $(IB'C')$, $(IC'A')$, and $(IA'B')$. Since the distance to each side from I is r, we see that each of these circles has radius r and contains I. We recognize this configuration of three congruent circles containing a common point from earlier problems 2455 [1999 : 307; 2000 : 314] and 3337 [2008 : 173, 175; 2009 : 191-192], where further references are provided. From results established there, we see that I is the orthocentre of $\triangle A'B'C'$; because that triangle is interchanged with $\triangle DEF$ by a halfturn about the common midpoint of A' and D, B' and E, C' and F, and M and I (according to Solution I above), we again see that $\rho = r$ and M is the orthocentre of $\triangle DEF$.

TOTTEN–06. [2009 : 321, 323] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Jim and three of his buddies played a round of golf. As usual, Jim won the game. In fact, he beat every two of his three buddies, in the following sense. Let his three buddies be A, B, and C and let a_i be A's score on hole i, for all $1 \le i \le 18$, and similarly define b_i and c_i. Set $S_{ab} = \sum\limits_{i=1}^{18} \min(a_i, b_i)$, and similarly define S_{ac} and S_{bc}. Then Jim's total score was less than S_{ab}, S_{ac}, and S_{bc}. However, Jim's score was more than $S_{abc} = \sum_{i=1}^{18} \min(a_i, b_i, c_i)$. Jim's score was 72. What was the minimum possible score of any of his buddies?

Solution by Oliver Geupel, Brühl, NRW, Germany.

The following example shows that Jim's buddies can have score 75 each:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_{18} \\ b_1 & b_2 & \dots & b_{18} \\ c_1 & c_2 & \dots & c_{18} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 & 5 & 5 & 4 & 4 & \dots & 4 \\ 5 & 3 & 5 & 5 & 5 & 4 & 4 & \dots & 4 \\ 5 & 5 & 3 & 5 & 5 & 4 & 4 & \dots & 4 \end{bmatrix} .$$

We will prove that 75 is the minimum possible score of any of Jim's buddies.

Note that $S_{abc} \leq 71$, whereas each of S_{ab}, S_{bc}, and S_{ac} is at least 73. Let $S_a = \sum_{i=1}^{18} a_i$. We show that $S_{ab} < S_a$, the proof being by contradiction.

Assume that $S_a = S_{ab}$. Then for each $1 \leq i \leq 18$ we have that $a_i = \min(a_i, b_i) \leq b_i$; hence $73 \leq S_{ac} = S_{abc} \leq 71$, a contradiction.

Thus, $74 \leq S_{ab} + 1 \leq S_a$. It remains to show that $S_a = 74$ is impossible. The proof is again by contradiction.

Assume that $S_a = 74$. Let us define the quantities

$$m_i = \min(a_i, b_i, c_i) ,$$

$$a'_i = a_i - m_i , \qquad S_{a'} = S_a - S_{abc} ,$$

$$b'_i = b_i - m_i , \qquad S_{a'b'} = S_{ab} - S_{abc} = \sum_{i=1}^{18} \min(a'_i, b'_i) ,$$

$$c'_i = c_i - m_i , \qquad S_{a'c'} = S_{ac} - S_{abc} = \sum_{i=1}^{18} \min(a'_i, c'_i) .$$

For each $1 \leq i \leq 18$, at least one of the numbers $\min(a'_i, b'_i)$ or $\min(a'_i, c'_i)$ is zero. Hence, $\min(a'_i, b'_i) + \min(a'_i, c'_i) \leq a'_i$. Therefore,

$$\begin{aligned} S_{a'b'} + S_{a'c'} \leq S_{a'} &= S_a - S_{abc} \\ &= S_{ab} + 1 - S_{abc} \\ &= S_{a'b'} + 1 , \end{aligned}$$

which implies that $S_{a'c'} \leq 1$. On the other hand, $S_{a'c'} = S_{ac} - S_{abc} \geq 73 - 71 = 2$, a contradiction.

This completes the proof.

Also solved by TOM LEONG, The University of Scranton, Scranton, PA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Note that if $S_a = 74$, then $S_a = S_{ab} + 1$, hence $a_i = \min(a_i, b_i) \leq b_i$ fails for exactly one index $i = j$ and $b_j = a_j - 1$. The contradiction $S_{abc} \geq S_{ac} - 1 \geq 73 - 1 = 72$ then arises by an argument similar to the third paragraph of our featured solution.

TOTTEN–07. [2009 : 321, 323] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, and c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove or disprove that

(a) $1 \le \dfrac{a}{1-ab}+\dfrac{b}{1-bc}+\dfrac{c}{1-ca} \le \dfrac{3\sqrt{3}}{2}$,

(b) $1 \le \dfrac{a}{1+ab}+\dfrac{b}{1+bc}+\dfrac{c}{1+ca} \le \dfrac{3\sqrt{3}}{4}$.

Composite of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and the second proposer.

We show that all four inequalities hold. Both inequalities are cyclic symmetric, so without loss of generality we can assume that either $a \le b \le c$ or $a \le c \le b$.

(a) The left hand side inequality follows from (b).

To prove the right hand side inequality, we homogenize it and prove more generally that for all nonnegative real numbers a, b, c we have:

$$\sum_{\text{cyclic}} \frac{a}{a^2+b^2+c^2-ab} \le \frac{3\sqrt{3}}{2}\frac{1}{\sqrt{a^2+b^2+c^2}},$$

or

$$\left(\frac{3\sqrt{3}}{2\sqrt{a^2+b^2+c^2}}\right)^2 \ge \left(\sum_{\text{cyclic}} \frac{a}{a^2+b^2+c^2-ab}\right)^2.$$

This is equivalent to

$$\begin{aligned}
&\frac{1}{4(a^2+b^2+c^2)(a^2+b^2+c^2-ab)^2(a^2+b^2+c^2-bc)^2(a^2+b^2+c^2-ca)^2}\cdot\\
&[23a^{12}-2a^{11}(31b+27c)+a^{10}(173b^2+78bc+169c^2)\\
&-2a^9(151b^3+104b^2c+128bc^2+139c^3)\\
&+a^8(481b^4+164b^3c+752b^2c^2+196bc^3+473c^4)\\
&-2a^7(294b^5+107b^4c+418b^3c^2+370b^2c^3+159bc^4+286c^5)\\
&+a^6(658b^6-38b^5c+1399b^4c^2+98b^3c^3+1395b^2c^4+58bc^5+658c^6)\\
&-2a^5(286b^7-29b^6c+556b^5c^2+264b^4c^3+316b^3c^4+556b^2c^5+19bc^6\\
&+294c^7)+a^4(473b^8-318b^7c+1395b^6c^2-632b^5c^3+1863b^4c^4\\
&-528b^3c^5+1399b^2c^6-214bc^7+481c^8)-2a^3(b^2+c^2)(139b^7-98b^6c\\
&+231b^5c^2+49b^4c^3+33b^3c^4+267b^2c^5-82bc^6+151c^7)\\
&+a^2(b^2+c^2)(169b^8-256b^7c+583b^6c^2-580b^5c^3+816b^4c^4-532b^3c^5\\
&+579b^2c^6-208bc^7+173c^8)-2a(b^2+c^2)^2(27b^7-39b^6c+50b^5c^2-4b^4c^3\\
&-20b^3c^4+66b^2c^5-39bc^6+31c^7)+(b^2+c^2)^2(23b^8-62b^7c+127b^6c^2\\
&-178b^5c^3+204b^4c^4-170b^3c^5+123b^2c^6-54bc^7+23c^8)] \ge 0\,.
\end{aligned}$$

We denote by $g(a,b,c)$ the numerator of the previous fraction. We need to prove that $g(a,b,c) \geq 0$.

In the first case we suppose that $a \leq b \leq c$. Then $b = a + s$ and $c = a + s + t$, where s and t are nonnegative real numbers. Substituting these into the expression for g yields

$$\begin{aligned}
&g(a,a+s,a+s+t) = 576a^{10}(s^2+st+t^2) + 16a^9(224s^3+363s^2t+411st^2\\
&+136t^3) + 16a^8(691s^4+1544s^3t+2136s^2t^2+1283st^3+295t^4)\\
&+4a^7(5428s^5+15433s^4t+25730s^3t^2+21866s^2t^3+9557st^4+1708t^5)\\
&+a^6(29876s^6+102904s^5t+202027s^4t^2+220046s^3t^3+139651s^2t^4\\
&+48604st^5+7364t^6) + 2a^5(14998s^7+60523s^6t+136935s^5t^2\\
&+181046s^4t^3+149499s^3t^4+76506s^2t^5+22655st^6+3006t^7)\\
&+a^4(22279s^8+102756s^7t+263080s^6t^2+407706s^5t^3+412688s^4t^4\\
&+277136s^3t^5+121173s^2t^6+31622st^7+3815t^8) + 2a^3(6060s^9+31334s^8t\\
&+89386s^7t^2+158387s^6t^3+189197s^5t^4+156661s^4t^5+90153s^3t^6+34826s^2t^7\\
&+8248st^8+918t^9) + a^2(4656s^{10}+26560s^9t+83312s^8t^2+165648s^7t^3\\
&+227450s^6t^4+223346s^5t^5+158989s^4t^6+81084s^3t^7+28487s^2t^8+6256st^9\\
&+662t^{10}) + 2a(576s^{11}+3576s^{10}t+12192s^9t^2+26796s^8t^3+41472s^7t^4\\
&+47006s^6t^5+39796s^5t^6+25167s^4t^7+11688s^3t^8+3810s^2t^9+791st^{10}\\
&+80t^{11}) + (4s^4+8s^3t+8s^2t^2+4st^3+t^4)(36s^8+168s^7t+472s^6t^2\\
&+796s^5t^3+919s^4t^4+722s^3t^5+389s^2t^6+130st^7+23t^8) \geq 0\,.
\end{aligned}$$

In the second case we have $a \leq c \leq b$, so that $b = a + s + t$ and $c = a + s$, where s and t are nonnegative real numbers. As in the first case, substituting these into the expression for g and simplifying (with the help of a computer algebra system) yieds a polynomial in a, s, and t all of whose coefficients are nonnegative, which completes the proof of part (a).

(b) The left end side inequality is equivalent to

$$\begin{aligned}
&(1+ab)(1+bc)(1+ca)\\
&\quad\leq\quad a(1+bc)(1+ca) + b(1+ab)(1+ca) + c(1+ab)(1+bc)\,,
\end{aligned}$$

which upon expanding becomes

$$\begin{aligned}
&1 + ab + ac + bc + abc(a+b+c) + a^2b^2c^2\\
&\quad\leq a + b + c + a^2c + ab^2 + bc^2 + abc(ab+ac+bc) + 3abc\,.
\end{aligned}$$

The above inquality follows by adding the three inequalities below, so it remains to prove each of these:

$$\begin{aligned}
abc(a+b+c) &\leq 3abc\,, &(1)\\
a^2b^2c^2 &\leq abc(ab+bc+ca)\,, &(2)\\
1+ab+bc+ca &\leq a+b+c+ab^2+bc^2+ca^2\,. &(3)
\end{aligned}$$

The inequality (1) follows from

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2) = 3\,.$$

Also, since $0 \leq c \leq 1$, we have $abc \leq ab \leq ab+bc+ca$, from which inequality (2) follows.

Lastly, we have

$$a(1-a)(1-c)+b(1-a)(1-b)+c(1-b)(1-c) \geq 0\,,$$

which yields (3).

To prove the right hand inequality, we homogenize again and prove more generally that:

$$\sum_{\text{cyclic}} \frac{a}{a^2+b^2+c^2+ab} \leq \frac{3\sqrt{3}}{4}\frac{1}{\sqrt{a^2+b^2+c^2}}\,,$$

or

$$\left(\frac{3\sqrt{3}}{4\sqrt{a^2+b^2+c^2}}\right)^2 \geq \left(\sum_{\text{cyclic}} \frac{a}{a^2+b^2+c^2+ab}\right)^2.$$

This is equivalent to

$$\frac{1}{16(a^2+b^2+c^2)(a^2+b^2+c^2+ab)^2(a^2+b^2+c^2+bc)^2(a^2+b^2+c^2+ca)^2}\cdot$$
$$\begin{aligned}
&[11a^{12}+2a^{11}(11b-5c)+a^{10}(61b^2+2bc+45c^2)\\
&+2a^9(39b^3-24b^2c+8bc^2-9c^3)\\
&+a^8(129b^4-44b^3c+208b^2c^2-44bc^3+97c^4)\\
&+2a^7(46b^5-57b^4c-10b^3c^2-74b^2c^3-41bc^4+14c^5)\\
&+a^6(142b^6-122b^5c+351b^4c^2-242b^3c^3+335b^2c^4-122bc^5+142c^6)\\
&+2a^5(14b^7-61b^6c-68b^5c^2-164b^4c^3-148b^3c^4-68b^2c^5-61bc^6\\
&+46c^7)+a^4(97b^8-82b^7c+335b^6c^2-296b^5c^3+471b^4c^4\\
&-328b^3c^5+351b^2c^6-114bc^7+129c^8)-2a^3(b^2+c^2)(9b^7+22b^6c\\
&+65b^5c^2+99b^4c^3+99b^3c^4+49b^2c^5+22bc^6-39c^7)\\
&+a^2(b^2+c^2)(45b^8+16b^7c+163b^6c^2-36b^5c^3+188b^4c^4-100b^3c^5\\
&+147b^2c^6-48bc^7+61c^8)-2a(b^2+c^2)^2(5b^7-b^6c+14b^5c^2+24b^4c^3\\
&+24b^3c^4+14b^2c^5-bc^6-11c^7)+(b^2+c^2)^2(11b^8+22b^7c+39b^6c^2\\
&+34b^5c^3+40b^4c^4+2b^3c^5+23b^2c^6-10bc^7+11c^8)] \geq 0\,.
\end{aligned}$$

We denote by $f(a,b,c)$ the numerator of the previous fraction. We need to prove that $f(a,b,c) \geq 0$.

In the first case, $a \leq b \leq c$. Then $b = a+s$ and $c = a+s+t$, where s and t are nonnegative real numbers. Substituting these into f we obtain

$$
\begin{aligned}
&f(a, a+s, a+s+t) = 9216a^{10}(s^2+st+t^2) + 128a^9(478s^3+663s^2t+669st^2\\
&+242t^3) + 128a^8(1484s^4+2644s^3t+2931s^2t^2+1771st^3+422t^4)\\
&+8a^7(45222s^5+98637s^4t+123266s^3t^2+96630s^2t^3+42063st^4+7642t^5)\\
&+a^6(467076s^6+1207100s^5t+1705863s^4t^2+1593386s^3t^3+956463s^2t^4\\
&+328916st^5+49204t^6) + 2a^5(213278s^7+637841s^6t+1014189s^5t^2\\
&+1085926s^4t^3+805617s^3t^4+393396s^2t^5+113243st^6+14582t^7)\\
&+a^4(278867s^8+948076s^7t+1683432s^6t^2+2022590s^5t^3+1753596s^4t^4\\
&+1082132s^3t^5+449425s^2t^6+112522st^7+12883t^8) + 2a^3(64428s^9+245574s^8t\\
&+483174s^7t^2+642585s^6t^3+630183s^5t^4+461513s^4t^5+245631s^3t^6+89646s^2t^7\\
&+20090st^8+2094t^9) + a^2(40296s^{10}+170312s^9t+368644s^8t^2+537504s^7t^3\\
&+584030s^6t^4+488290s^5t^5+311941s^4t^6+147356s^3t^7+48511s^2t^8+9960st^9\\
&+966t^{10}) + 2a(3856s^{11}+17912s^{10}t+42384s^9t^2+67268s^8t^3+79860s^7t^4\\
&+74270s^6t^5+54608s^5t^6+31241s^4t^7+13420s^3t^8+4076s^2t^9+783st^{10}\\
&+72t^{11}) + (4s^4+8s^3t+8s^2t^2+4st^3+t^4)(172s^8+528s^7t+844s^6t^2\\
&+940s^5t^3+815s^4t^4+546s^3t^5+261s^2t^6+78st^7+11t^8) \geq 0\,.
\end{aligned}
$$

In the second case we have $a \leq c \leq b$, so that $b = a+s+t$ and $c = a+s$, where s and t are nonnegative real numbers. Again, substituting these into f and simplifying (with the help of a computer) yields a polynomial in a, s, and t all of whose coefficients are nonnegative, which completes the proof of part (b).

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA.

Bracken's solution also used a computer, and involved expressions even longer than those of our featured solution. The editor has no other solutions to offer, and would be interested in receiving a more "human" solution to this problem.

TOTTEN–08. [2009 : 321, 324] *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

In triangle ABC suppose that $AB < AC$. Let D and M be the points on side BC for which AD is the angle bisector and AM is the median. Let F be on side AC so that AD is perpendicular to DF. Finally, let E be the intersection of AM and DF. Prove that $AB \cdot DE + AB \cdot DF = AC \cdot EF$.

Solution by Edmund Swylan, Riga, Latvia.

Define the points C', F', G' on AB and B', G on AC so that BB', CC', FF', and GG' are all perpendicular to AD, with $M \in GG'$. Because

$$AB \cdot DE + AB \cdot DF = AB(DE+DF) = AB(F'D+DE) = AB \cdot F'E\,,$$

the problem reduces to proving that

$$\frac{F'E}{EF} = \frac{AC}{AB}. \qquad (1)$$

Because the dilatation with centre A that takes E to M takes $F'E$ to $G'M$ and EF to MG, the left-hand side of (1) satisfies

$$\frac{F'E}{EF} = \frac{G'M}{MG}.$$

Because ACC' and ABB' are similar isosceles triangles, the right-hand side of (1) satisfies

$$\frac{AC}{AB} = \frac{CC'}{BB'}.$$

The proof concludes by noting that in triangles $BC'C$ and $CB'B$, the midline $G'M$ is half the base CC' while the midline MG is half BB', so that

$$\frac{CC'}{BB'} = \frac{G'M}{MG}.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect submission.

TOTTEN–09. [2009 : 321, 324] *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

Let n and k be integers with $n \geq 2$ and $k \geq 0$. Consider n dinner guests sitting around a circular table. Let $g_n(k)$ be the number of ways that k of these n guests can be chosen so that no two chosen guests are sitting next to one another. To illustrate, $g_6(0) = 1$, $g_6(1) = 6$, $g_6(2) = 9$, $g_6(3) = 2$, and $g_6(k) = 0$ for all $k \geq 4$. For each $n \geq 2$, let

$$f_n(x) = \sum_{k \geq 0} g_n(k)x^k.$$

For example, $f_6(x) = 1+6x+9x^2+2x^3 = (1+2x)\left(1+4x+x^2\right)$. Determine all n for which $(1 + 2x)$ is a factor of $f_n(x)$.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The number of strings consisting of k ones and $n-k$ zeros such that no two ones are adjacent is $\binom{n-k-1}{k}$. Treating the k chosen guests as ones and the others as zeros, and arbitrarily assigning a guest as the start of a string of length n, we have

$$g_n(k) = \binom{n-k+1}{k} - \binom{n-k-1}{k-2},$$

because we must eliminate strings in which the first and last elements are both ones. Thus,

$$\begin{aligned} g_n(k) &= \frac{(n-k-1)!}{k!(n-2k+1)!}\big[(n-k)(n-k+1)-k(k-1)\big] \\ &= \frac{n(n-k-1)!}{k!(n-2k)!}, \end{aligned}$$

so that

$$\begin{aligned} f_n(x) &= \sum_{k\geq 0} \frac{n(n-k-1)!}{k!(n-2k)!}x^k \\ &= n\sum_{k\geq 0} \frac{1}{n-k}\binom{n-k}{k}x^k. \end{aligned}$$

By equation (5.75) of *Concrete Mathematics* (Graham, Knuth, and Patashnik, 2nd ed.),

$$f_n(x) = \left(\frac{1+\sqrt{1+4x}}{2}\right)^n + \left(\frac{1-\sqrt{1+4x}}{2}\right)^n.$$

Noting that $1+2x$ is a factor of $f_n(x)$ if and only if $f_n\left(-\frac{1}{2}\right)=0$, we calculate

$$\begin{aligned} f_n\left(-\frac{1}{2}\right) &= \left(\frac{1+i}{2}\right)^n + \left(\frac{1-i}{2}\right)^n \\ &= \left(\frac{\sqrt{2}}{2}e^{\frac{i\pi}{4}}\right)^n + \left(\frac{\sqrt{2}}{2}e^{\frac{-i\pi}{4}}\right)^n \\ &= 2\left(\frac{\sqrt{2}}{2}\right)^n \cos\frac{n\pi}{4}. \end{aligned}$$

Hence, $1+2x$ is a factor of $f_n(x)$ if and only if $\cos\frac{n\pi}{4}=0$, that is, if and only if $n=2(2\ell+1)$, where ℓ is a nonnegative integer.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Geupel and the proposer both solved the problem by using the recursive relation $f_n(x)=f_{n-1}(x)+xf_{n-2}(x)$.

TOTTEN–10. [2009 : 322, 324] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Determine all triangles ABC whose side lengths are positive integers and such that $\cos C = \frac{4}{5}$.

Solution by Oliver Geupel, Brühl, NRW, Germany, modified by the editor.

Let ABC be a triangle with $a = BC$, $b = CA$, $c = AB$ and $\cos C = \frac{4}{5}$. By the Law of Cosines, we have

$$c^2 \;=\; a^2 + b^2 - \frac{8}{5}ab\,. \tag{1}$$

Conversely, if a, b, c are positive integers satisfying (1), then

$$(a-b)^2 \;<\; c^2 \;<\; (a+b)^2\,,$$

hence a, b, c are the side lengths of a triangle, and again by the Law of Cosines, $\cos C = \frac{4}{5}$. Thus, we seek all positive integer solutions of (1).

Either a or b is divisible by 5. Observe that if (a, b, c) is a solution, then so is (b, a, c), hence it suffices to find the solutions for which 5 divides a. Let $a = 5d$.

Then (1) becomes

$$c^2 \;=\; 25d^2 + b^2 - 8bd\,,$$

or

$$c^2 \;=\; (b-4d)^2 + (3d)^2\,.$$

This is the well-known Pythagorean Equation. We distinguish two cases:

Case 1: $b - 4d = 2lmn$, $3d = l(m^2 - n^2)$, $c = l(m^2 + n^2)$.
In this case we solve for a, b, and c to obtain

$$a = \frac{5}{3}l(m^2 - n^2)\,, \qquad b = \frac{4}{3}l(m^2 - n^2) + 2lmn\,, \qquad c = l(m^2 + n^2)\,.$$

Here a, b, c are positive integers if and only if $3|l(m-n)(m+n)$, $l > 0$, and either $m > n \geq 0$ or $m > -2n \geq 0$.

We distinguish three subcases:

Case 1a: $3|l$ and $l = 3k$. Then

$$a = 5k(m^2 - n^2)\,, \qquad b = 4k(m^2 - n^2) + 6kmn\,, \qquad c = 3k(m^2 + n^2)\,,$$

where $k > 0$ and either $m > n \geq 0$ or $m > -2n \geq 0$.

Case 1b: $3|(m-n)$ with $m - n = 3k$. Then $m^2 - n^2 = 3k(2n + 3k)$ and thus

$$a = 5lk(2n+3k)\,, \quad b = 4lk(2n+3k) + 2ln(n+3k)\,, \quad c = l(2n^2+6nk+9k^2)\,,$$

where $l > 0$ and either $k > 0$ and $n \geq 0$, or $k > -n \geq 0$.

Case 1c: $3|(m+n)$ with $m+n = 3k$. Then $m^2 - n^2 = 3k(3k-2n)$, and thus

$$a = 5lk(3k-2n)\,, \qquad b = 4lk(3k-2n)+2ln(3k-n)\,, \qquad c = l(2n^2-6nk+9k^2)\,,$$

where $l > 0$ and either $3k > 2n \geq 0$ or $3k > -n \geq 0$.

This completes the first case.

Case 2: $b - 4d = l(m^2 - n^2)$, $3d = 2lmn$, $c = l(m^2 + n^2)$.

In this case we obtain

$$a = l(m^2 - n^2) + \frac{8}{3}lmn\,, \qquad b = \frac{10}{3}lmn\,, \qquad c = l(m^2 + n^2)\,.$$

Now a, b, c are positive integers if and only if $3|lmn$, $l > 0$, and $3m > n \geq 0$. In this case three subcases also arise:

Case 2a: $3|l$ with $l = 3k$. Then

$$a = 3k(m^2 - n^2) + 8kmn\,, \qquad b = 10kmn\,, \qquad c = 3k(m^2 + n^2)\,,$$

where $k > 0$ and $3m > n \geq 0$.

Case 2b: $3|m$ with $m = 3k$. Then

$$a = l(9k^2 - n^2) + 8lkn\,, \qquad b = 10lkn\,, \qquad c = l(9k^2 + n^2)\,,$$

where $l > 0$ and $9k > n \geq 0$.

Case 2c: $3|n$ with $n = 3k$. Then

$$a = l(m^2 - 9k^2) + 8lmk\,, \qquad b = 10lmk\,, \qquad c = l(m^2 + 9k^2)\,,$$

where $l > 0$ and $m > n \geq 0$.

This completes the second and last case.

The above parametrizations explicitly yield all required triples (a, b, c).

Also solved by ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Two incomplete solutions were submitted.

Janous gave the reference `http://mathworld.wolfram.com/PythagoreanTriple.html` *for the well-known fact that all Pythagorean triangles have a leg divisible by* 3*, and he used this to express his solution in terms of Pythagorean triples.*

TOTTEN–11. [2009 : 322, 324] *Proposed by Walther Janous, Ursulinen-gymnasium, Innsbruck, Austria.*

(a) Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{8\sqrt{3}}{9} \le \left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)\left(\frac{1}{\sqrt{z}}-\sqrt{z}\right).$$

(b) ★. Let $n \ge 2$ and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1+x_2+\cdots+x_n=1$. Prove or disprove that

$$\left(\frac{n-1}{\sqrt{n}}\right)^n \le \prod_{k=1}^{n}\left(\frac{1}{\sqrt{x_k}}-\sqrt{x_k}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland.

(a) This is a special case of (b) when $n=3$.

(b) The inequality fails for $n=2$. [*Ed.: Both Bataille and Geupel provided the counterexample* $x_1=\frac{1}{4}$, $x_2=\frac{3}{4}$.]

We assume $n \ge 3$ and apply the following inequality proved in the Right-Left Convex Function Theorem (RLCF-Theorem) [*Ed.: See V. Cîrtoaje,* Algebraic Inequalities - Old and New Methods, *GIL Publishing House, Romania, 2006.*]

RLCF Theorem Let $f(u)$ be a function defined on an interval I. Suppose f is convex for either $u \le s$ or $u \ge s$ for some $s \in I$ and for some fixed positive integer n, $f(x)+(n-1)f(y) \ge n f(s)$ for all x, $y \in I$ such that $x+(n-1)y=ns$. Then $\sum\limits_{k=1}^{n} f(x_k) \ge nf(s)$ for all $x_1, x_2, \ldots, x_n \in I$ satisfying $\sum\limits_{k=1}^{n} x_k = ns$. ∎

If we consider the function $f(x)=\ln\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)$ on $I=(0,1)$, and let $s=\frac{1}{n}$, then the given inequality is the same as $\sum\limits_{k=1}^{n} f(x_k) \ge nf(s)$.

We have that $f'(x)=\dfrac{x+1}{2x(x-1)}$ and $f''(x)=\dfrac{1-2x-x^2}{2x^2(1-x)^2}$. It is easily verified that for $x \in (0,1)$, $1-2x-x^2 \ge 0$ if and only if $x \le \sqrt{2}-1$, so $f''(x) \ge 0$ for $x \le \sqrt{2}-1$. Hence, to apply the RLCF Theorem, it suffices to prove that

$$f(x)+(n-1)f(y) \ \ge\ nf\left(\frac{1}{n}\right) \tag{1}$$

for all x, $y \in (0,1)$ such that $x+(n-1)y=1$.

Since $y = \dfrac{1-x}{n-1}$, the inequality (1) is successively equivalent to

$$
\begin{aligned}
\ln\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right) + (n-1)\ln\left(\frac{1}{\sqrt{\frac{1-x}{n-1}}} - \sqrt{\frac{1-x}{n-1}}\right) &\geq n\ln\left(\sqrt{n} - \frac{1}{\sqrt{n}}\right); \\
\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{\frac{1-x}{n-1}}} - \sqrt{\frac{1-x}{n-1}}\right)^{n-1} &\geq \left(\frac{n-1}{\sqrt{n}}\right)^{n}; \\
\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{n-1}{1-x}\right)^{\frac{(n-1)}{2}}\left(1 - \frac{1-x}{n-1}\right)^{n-1} &\geq \left(\frac{n-1}{\sqrt{n}}\right)^{n}; \\
(1-x)^{1-\frac{(n-1)}{2}}(n-2+x)^{n-1} &\geq n^{\frac{-n}{2}}(n-1)^{2n-1-\frac{(n-1)}{2}}\sqrt{x}; \\
(1-x)^{\frac{(3-n)}{2}}(n-2+x)^{n-1} &\geq n^{\frac{-n}{2}}(n-1)^{\frac{(3n-1)}{2}}\sqrt{x}; \\
n^n(n-2+x)^{2n-2} &\geq (n-1)^{3n-1}(1-x)^{n-3}x\,.
\end{aligned}
$$

Let $f_n(x) = n^n(n-2+x)^{2n-2} - (n-1)^{3n-1}(1-x)^{n-3}x$. We will show that $f_n(x) \geq 0$ for all $x \in (0,1)$. We have

$$
\begin{aligned}
f_n'(x) &= n^n(2n-2)(n-2+x)^{2n-3} - (n-1)^{3n-1}(1-x)^{n-3} \\
&\quad + (n-3)(n-1)^{3n-1}(1-x)^{n-4}x \\
f_n''(x) &= n^n(2n-2)(2n-3)(n-2+x)^{2n-4} \\
&\quad + 2(n-3)(n-1)^{3n-1}(1-x)^{n-4} \\
&\quad - (n-3)(n-4)(n-1)^{3n-1}(1-x)^{n-5}x \\
&= 2n^n(n-1)(2n-3)(n-2+x)^{2n-4} \\
&\quad + (n-3)(n-1)^{3n-1}(1-x)^{n-5}[2-(n-2)x] \qquad (2)
\end{aligned}
$$

Note that

$$
\begin{aligned}
f_n\left(\frac{1}{n}\right) &= n^n\left(n-2+\frac{1}{n}\right)^{2n-2} - (n-1)^{3n-1}\left(1-\frac{1}{n}\right)^{n-3}\left(\frac{1}{n}\right) \\
&= n^{2-n}(n-1)^{4n-4} - n^{2-n}(n-1)^{4n-4} = 0 \qquad (3)
\end{aligned}
$$

and

$$
\begin{aligned}
f_n'\left(\frac{1}{n}\right) &= n^n(2n-2)\left(n-2+\frac{1}{n}\right)^{2n-3} - (n-1)^{3n-1}\left(1-\frac{1}{n}\right)^{n-3} \\
&\quad + (n-3)(n-1)^{3n-1}\left(1-\frac{1}{n}\right)^{n-4}\left(\frac{1}{n}\right)
\end{aligned}
$$

$$\begin{aligned} &= 2n^{3-n}(n-1)^{4n-5} - n^{3-n}(n-1)^{4n-4} \\ &\quad + n^{3-n}(n-3)(n-1)^{4n-5} \\ &= n^{3-n}(n-1)^{4n-5}[2-(n-1)+(n-3)] = 0\,. \end{aligned} \tag{4}$$

Also, we see from (2) that $f_n''(x) \geq 0$ for $0 < x \leq \dfrac{2}{n-2}$. Combining this with (3) and (4) we conclude that

$$f_n(x) \geq 0 \quad \text{for} \quad 0 < x \leq \frac{2}{n-2}\,. \tag{5}$$

Hence, (1) is true for $n = 3$ and $n = 4$. We now assume that $n \geq 5$.

Let $g(x) = x(1-x)^{n-3}$. We find that g decreases on $\left[\frac{1}{n-2}, 1\right)$, hence f_n increases on $\left[\frac{1}{n-2}, 1\right)$ since $n^n(n-2+x)^{2n-2}$ increases on $(0,1)$. Now, $f_n\left(\frac{1}{n-2}\right) \geq 0$ by (5), so we conclude that $f_n(x) \geq 0$ for all $x \in (0,1)$.

This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and ROY BARBARA, Lebanese University, Fanar, Lebanon. Part (a) only was solved by ARKADY ALT, San Jose, CA, USA (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; and the proposer. Two incorrect solutions were submitted.

TOTTEN–12. [2009 : 322, 324] *Proposed by Mihály Bencze, Brasov, Romania.*

Let w, x, y, and z be positive real numbers with $w+x+y+z = wxyz$, and let

$$f(x) = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{x^3}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{x^3}}}\,.$$

Prove that $\sqrt[3]{wxy} + \sqrt[3]{xyz} + \sqrt[3]{yzw} + \sqrt[3]{zwx} \geq f(w) + f(x) + f(y) + f(z)$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

By the AM-GM Inequality, we have

$$\begin{aligned} wxy &= \frac{w}{z} + \frac{x}{z} + \frac{y}{z} + 1 \\ &\geq 4\left(\frac{wxy}{z^3}\right)^{\frac{1}{4}}, \end{aligned}$$

and by symmetry the cyclic variants of this inequality also hold.

Using the above inequality and the AM–GM Inequality once more, we deduce that

$$\begin{aligned}\sum_{\text{cyclic}} \sqrt[3]{wxy} &\geq 4^{\frac{1}{3}} \sum_{\text{cyclic}} \frac{(wxy)^{\frac{1}{12}}}{z^{\frac{1}{4}}} \\ &\geq 4^{\frac{1}{3}} \cdot 4\,. \end{aligned} \tag{1}$$

The function $y = g(x) = x^{\frac{1}{3}}$ is concave for $x > 0$. Hence, by Jensen's inequality, we have

$$\begin{aligned} f(w) &= g\biggl(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{w^3}}\,\biggr) + g\biggl(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{w^3}}\,\biggr) \\ &\leq 2\,g\Bigl(\frac{1}{2}\Bigr) = 2\left(\frac{1}{2}\right)^{\frac{1}{3}} = 4^{\frac{1}{3}}\,. \end{aligned}$$

Similarly, $f(x) \leq 4^{\frac{1}{3}}$, $f(y) \leq 4^{\frac{1}{3}}$, and $f(z) \leq 4^{\frac{1}{3}}$; and hence

$$f(w) + f(x) + f(y) + f(z) \leq 4 \cdot 4^{\frac{1}{3}}\,. \tag{2}$$

The desired inequality follows immediately from (1) and (2).

By the conditions for equality to hold in the AM–GM Inequality and Jensen's inequality, we see that equality holds in the given inequality if and only if $w = x = y = z = 4^{\frac{1}{3}}$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The statement of the problem should have included the extra conditions that $x^3 \geq 4$, $y^3 \geq 4$, $z^3 \geq 4$, and $w^3 \geq 4$. Otherwise, the function f may not be real valued; for instance if $x = y = 1$, $z = 2$, and $w = 4$, then $w + x + y + z = wxyz = 8$ but $f(x)$ and $f(y)$ are not real numbers.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Our apologies to Albert Stadler, Herrliberg, Switzerland, for misfiling his correct solution to #3449 (which was the only other solution submitted for that problem other than the proposer's).

3451. [2009 : 325, 327] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $(X, <\cdot,\cdot>)$ be a real or complex inner product space and let x, y, and z be nonzero vectors in X. Prove that

$$\sum_{\text{cyclic}} \left| \frac{<z,x><x,y>}{\|x\|} \right|^{1/2} \leq \sum_{\text{cyclic}} \left(\frac{\|x\|}{\|y\|\,\|z\|} \right)^{1/2} |<y,z>| \,.$$

Solution by Albert Stadler, Herrliberg, Switzerland.

Put

$$\begin{aligned} a &= \sqrt{\|x\| \cdot |\langle y,z\rangle|}\,, \\ b &= \sqrt{\|y\| \cdot |\langle z,x\rangle|}\,, \\ c &= \sqrt{\|z\| \cdot |\langle x,y\rangle|}\,. \end{aligned}$$

By the Cauchy–Schwartz Inequality,

$$\begin{aligned} |ab+bc+ca| &\leq \sqrt{a^2+b^2+c^2}\sqrt{b^2+c^2+a^2} \\ &= a^2+b^2+c^2\,, \end{aligned}$$

or

$$\sum_{\text{cyclic}} \sqrt{\|x\| \cdot \|y\| \cdot |\langle y,z\rangle| \cdot |\langle z,x\rangle|} \leq \sum_{\text{cyclic}} \|x\| \cdot |\langle y,z\rangle|\,.$$

Hence, by dividing this last inequality by $\sqrt{\|x\| \cdot \|y\| \cdot \|z\|}$, we obtain the required inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3452. [2009 : 325, 327] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Prove the following and generalize these results.

(a) $\tan^2 36^\circ + \tan^2 72^\circ = 10$,

(b) $\tan^4 36^\circ + \tan^4 72^\circ = 90$,

(c) $\tan^6 36^\circ + \tan^6 72^\circ = 850$,

(d) $\tan^8 36^\circ + \tan^8 72^\circ = 8050$.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Set $a = \tan^2 36^\circ$, $b = \tan^2 72^\circ$ and $T_n = a^n + b^n$ for $n \geq 1$. It is well-known that

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4} \quad \text{and} \quad \cos^2 72^\circ = 2\left(\frac{\sqrt{5}+1}{4}\right)^2 - 1 = \frac{\sqrt{5}-1}{4}\,.$$

Since $\tan^2 x = \dfrac{1}{\cos^2 x} - 1$ and $\dfrac{4}{\sqrt{5}+1} = \sqrt{5} - 1$, we have

$$\begin{aligned} a &= (\sqrt{5}-1)^2 - 1 = 5 - 2\sqrt{5}\,, \\ b &= (\sqrt{5}+1)^2 - 1 = 5 + 2\sqrt{5}\,, \end{aligned}$$

so that $T_n = (5 - 2\sqrt{5})^n + (5 + 2\sqrt{5})^n$.

To verify equations (a)-(d), we give a way to calculate the values of T_n inductively. From $(a+b)(a^{n+1}+b^{n+1}) = (a^{n+2}+b^{n+2}) + ab(a^n+b^n)$ and $a+b = 10$, $ab = 5$, we obtain the recurrence $T_{n+2} = 10T_{n+1} - 5T_n$ for $n \geq 1$. Since $T_1 = 10$ and $T_2 = (5-2\sqrt{5})^2 + (5+2\sqrt{5})^2 = 90$, we easily obtain $T_3 = 10 \cdot 90 - 5 \cdot 10 = 850$, $T_4 = 10 \cdot 850 - 5 \cdot 90 = 8050$, $T_5 = 10 \cdot 8050 - 5 \cdot 850 = 76250$, and so forth.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHELE ARNOLD, Southeast Missouri State University, Cape Girardeau, MO, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; JESI BAYLESS, Southeast Missouri State University, Cape Girardeau, MO, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DOUGLASS L. GRANT, Cape Breton University, Sydney, NS; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; JOSHUA LONG, Southeast Missouri State University, Cape Girardeau, MO, USA; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; JOHN POSTL, St. Bonaventure University, St. Bonaventure, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; VASILE TEODOROVICI, Toronto, ON; PANOS E. TSAOUSSOGLOU, Athens, Greece; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

Bataille remarked that there exists a general result for the sum $\sum_{k=1}^{(m-1)/2} \tan^{2n}\left(\frac{k\pi}{m}\right)$, *where* m *is an odd positive integer and refers readers to his Problem 11044 in the* American Math Monthly *with solution in Vol. 112, No. 7, 2005; pp. 657-9. Grant pointed out that the formula* $\tan^n(36^\circ) + \tan^n(72^\circ) = (5-2\sqrt{5})^{n/2} + (5+2\sqrt{5})^{n/2}$ *holds for any positive integer* n.

3453. [2009 : 325, 328] *Proposed by Scott Brown, Auburn University, Montgomery, AL, USA.*

Triangle ABC has side lengths $a = BC$, $b = AC$, $c = AB$; and altitudes h_a, h_b, h_c from the vertices A, B, C, respectively. Prove that

$$8\left(\sum_{\text{cyclic}} h_a^2(h_b+h_c)\right) + 16h_ah_bh_c \;\le\; 3\sqrt{3}\left(\sum_{\text{cyclic}} a^2(b+c)\right) + 6\sqrt{3}abc\,.$$

I. *Solution by Joe Howard, Portales, NM, USA.*

It suffices to show that

$$8(h_a+h_b)(h_b+h_c)(h_c+h_a) \;\le\; 3\sqrt{3}(a+b)(b+c)(c+a)\,.$$

Since $h_a = c\sin B = b\sin C$ etc., then $h_a + h_b = (a+b)\sin C$ etc. Multiplying yields

$$\prod_{\text{cyclic}}(h_a+h_b) \;=\; \left(\prod_{\text{cyclic}}(a+b)\right)\left(\prod_{\text{cyclic}}\sin A\right)\,.$$

The result now follows from

$$\prod_{\text{cyclic}}\sin A \;\le\; \frac{3\sqrt{3}}{8}\,,$$

which is item 2.7, page 19 of O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

II. *Solution by George Apostolopoulos, Messolonghi, Greece.*

Let F denote the area of triangle ABC. It is known that $h_a = \dfrac{2F}{a}$ etc. Now,

$$\begin{aligned}
&8\sum_{\text{cyclic}} h_a^2(h_b+h_c) \;+\; 16h_ah_bh_c \\
&\qquad=\; 8\prod_{\text{cyclic}}(h_a+h_b) \;=\; 8(2F)^3\prod_{\text{cyclic}}\left(\frac{1}{a}+\frac{1}{b}\right) \\
&\qquad=\; \frac{64F^3}{a^2b^2c^2}\prod_{\text{cyclic}}(a+b)\,.
\end{aligned}$$

Also
$$3\sqrt{3}\sum_{\text{cyclic}} a^2(b+c) \;+\; 6\sqrt{3}abc \;=\; 3\sqrt{3}\prod_{\text{cyclic}}(a+b)\,,$$

so the conclusion is equivalent to

$$F^3 \;\leq\; \frac{3\sqrt{3}}{64}a^2b^2c^2\,.$$

[*Ed.: Apostolopoulos then gave a proof of this last inequality. However, it is equivalent to item 4.14, page 46 of O. Bottema et al.,* Geometric Inequalities, *Groningen, 1969, as some other solvers pointed out.*]

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Most solvers used one of the above two methods.

3454. [2009 : 326, 328] *Proposed by Richard Hoshino, Government of Canada, Ottawa, ON.*

Let a, b, c, and d be positive real numbers such that $a+b+c+d=1$. Prove that
$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b} \;\geq\; \frac{1}{8}\,.$$

I. *Similar solutions by George Apostolopoulos, Messolonghi, Greece and Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

By the AM–GM Inequality we have

$$\frac{a^3}{b+c}+\frac{b+c}{16}+\frac{1}{32} \;\geq\; 3\sqrt[3]{\frac{a^3(b+c)}{(b+c)\cdot 16\cdot 32}} \;=\; \frac{3a}{8}\,.$$

Similarly,

$$\begin{aligned}
\frac{b^3}{c+d}+\frac{c+d}{16}+\frac{1}{32} &\;\geq\; \frac{3b}{8}\,;\\
\frac{c^3}{d+a}+\frac{d+a}{16}+\frac{1}{32} &\;\geq\; \frac{3c}{8}\,;\\
\frac{d^3}{a+b}+\frac{a+b}{16}+\frac{1}{32} &\;\geq\; \frac{3d}{8}\,.
\end{aligned}$$

Adding the four inequalities, and using $a+b+c+d=1$, we have

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}+\frac{1}{8}+\frac{1}{8} \;\geq\; \frac{3}{8}\,,$$

and the inequality follows.

II. *Similar solutions by Tom Leong, The University of Scranton, Scranton, PA, USA and Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

By the Generalized Hölder Inequality, we have

$$\begin{aligned}&\left(\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\right)\\ &\qquad\cdot\Big[(b+c)+(c+d)+(d+a)+(a+b)\Big]\cdot(1+1+1+1)\\ &\geq\; (a+b+c+d)^3\,.\end{aligned}$$

Hence,

$$8\left(\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}\right) \;\geq\; 1\,,$$

and the inequality follows.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2^{nd} solution); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALVATORE INGALA, student, Scuola Superiore di Catania, University of Catania, Catania, Italy; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3^{rd} High School of Kozani, Kozani, Greece; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

3455. [2009 : 326, 328] *Proposed by Michel Bataille, Rouen, France.*

Find the minimum value of $x^2+y^2+z^2$ over all triples (x,y,z) of real numbers such that

$$13x^2+40y^2+45z^2-36xy+12yz+24xz \;\geq\; 2009$$

and characterize all the triples at which the minimum is attained.

Solution by Kee-Wai Lau, Hong Kong, China.

Since

$$\begin{aligned}&x^2+y^2+z^2\\ &\quad=\;\frac{(6x+3y-2z)^2+13x^2+40y^2+45z^2-36xy+12yz+24xz}{49}\\ &\quad\geq\;\frac{2009}{49}\;=\;41\,,\end{aligned}$$

the required minimum is 41.

The minimum is attained if and only if

$$\begin{aligned} 6x + 3y - 2z &= 0\,, \\ 13x^2 + 40y^2 + 45z^2 - 36xy + 12yz + 24xz &= 2009\,. \end{aligned}$$

Eliminating z from the last equation yields $40x^2 + 36xy + 13y^2 = 164$, which is an ellipse, E, in the xy-plane. Hence, the triples at which the minimum is attained are given by $(x, y, z) = \left(s, t, \dfrac{6s + 3t}{2}\right)$, where (s, t) is any point on E.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution and five incomplete solutions were submitted.

Curtis' solution was similar to the featured solution. All five incomplete solutions used Lagrange Multipliers and determined the minimum value of the objective function to be 41*, but did not determine the complete set of triples at which the minimum is attained.*

3456. [2009 : 326, 328] *Proposed by Michel Bataille, Rouen, France.*

Given a triangle ABC with circumcircle Γ, let circle Γ' centred on the line BC intersect Γ at D and D'. Denote by Q and Q' the projections of D and D' on the line AB, and by R and R' their projections on AC; assume that none of these projections coincide with a vertex of the triangle.

Show that if Γ' is orthogonal to Γ, then $\dfrac{BQ}{BQ'} = \dfrac{CR}{CR'}$. Does the converse hold?

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

We shall see that the converse does not hold. Let T be the point where the tangents to Γ at D and D' meet. We prove that the given ratios are equal when T lies on the line BC.

Because $ABCD$ is cyclic, the directed angle from line BA to BD equals the directed angle from CA to CD; thus, because Q is on AB and R is on AC, the acute angles $\angle QBD$ and $\angle RCD$ must be equal, whence the right triangles BQD and CRD are similar. Likewise, $\triangle BQ'D' \sim \triangle CR'D'$. From these two pairs of similar triangles we deduce that

$$\frac{BQ}{CR} = \frac{DB}{DC} \quad \text{and} \quad \frac{BQ'}{CR'} = \frac{D'B}{D'C}\,. \tag{1}$$

If we assume that $T \in BC$ and that the given triangle has been labeled so that B is between T and C, then we have $\angle TDB = \angle DCB = \angle DCT$ (the angle between a chord and tangent equals the angle subtended by the chord). We deduce that $\triangle TCD \sim \triangle TDB$. Likewise, $\triangle TCD' \sim \triangle TD'B$, and these two pairs of similar triangles give us

$$\frac{DB}{CD} = \frac{TD}{TC} \quad \text{and} \quad \frac{D'B}{CD'} = \frac{TD'}{TC}\,. \tag{2}$$

But $TD = TD'$, so all four quotients in (2) are equal, and (1) implies that $BQ/CR = BQ'/CR'$, or

$$\frac{BQ}{BQ'} = \frac{CR}{CR'}\,. \qquad (3)$$

When Γ' is orthogonal to Γ, T becomes the centre of Γ' and lies on BC by assumption. Thus, the orthogonality of the two circles implies that (3) holds, as desired. But when BC is the diameter of Γ, T lies on BC for all circles Γ' whose centre lies on BC. For each choice of D on Γ, only one of these circles will be orthogonal to Γ; the rest serve as counterexamples to the converse.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

Both Bataille and Geupel proved that the converse holds exactly when $\angle BAC$ is not a right angle. In other words, equation (3) holds if and only if (a) Γ' is orthogonal to Γ, or (b) $\angle BAC = 90^\circ$, which is the case (in the notation of our featured solution) if and only if T lies on BC.

3457. [2009 : 326, 328] *Proposed by Michel Bataille, Rouen, France.*

Let $A_n = \sum\limits_{j=0}^{\lfloor n/2\rfloor} \dfrac{(-3)^j}{2j+1} a_j$, where $a_j = \sum\limits_{k=2j}^{n} \dbinom{k}{2j} \dfrac{k+1}{2^k}$ and n is a positive integer. Prove that $A_1 + A_2 + \cdots + A_n \geq n$ with equality for infinitely many n.

Solution by George Apostolopoulos, Messolonghi, Greece, modified by the editor.

By using the Binomial Theorem, the identity $\dbinom{k}{2j}\dfrac{k+1}{2j+1} = \dbinom{k+1}{2j+1}$, and the fact that $\dbinom{k+1}{2j+1} = 0$ if $2j > k$, we obtain

$$\begin{aligned}
A_n &= \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(-3)^j}{2j+1} \sum_{k=2j}^{n} \binom{k}{2j} \frac{k+1}{2^k} \;=\; \sum_{j=0}^{\lfloor n/2\rfloor} \sum_{k=0}^{n} \frac{(-3)^j}{2^k} \binom{k+1}{2j+1} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \frac{(-3)^j}{2^k} \binom{k+1}{2j+1} \;=\; \frac{1}{\sqrt{3}i} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{n} \binom{k+1}{2j+1} (\sqrt{3}i)^{2j+1} \\
&= \frac{1}{\sqrt{3}i} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{k+1} \binom{k+1}{j} \left(\frac{(\sqrt{3}i)^j - (-\sqrt{3}i)^j}{2} \right) \\
&= \frac{1}{\sqrt{3}i} \sum_{k=0}^{n} \frac{1}{2^{k+1}} \left[(1+\sqrt{3}i)^{k+1} - (1-\sqrt{3}i)^{k+1} \right] \\
&= \frac{1}{\sqrt{3}i} \sum_{k=0}^{n} \left[\left(\frac{1+\sqrt{3}i}{2} \right)^{k+1} - \left(\frac{1-\sqrt{3}i}{2} \right)^{k+1} \right] ,
\end{aligned}$$

where $i^2 = -1$. Let $\alpha = \frac{1+\sqrt{3}i}{2}$ and $\beta = \frac{1-\sqrt{3}i}{2}$, then $\alpha + \beta = \alpha\beta = 1$ and $\alpha - \beta = \sqrt{3}i$. Set $S_\ell = \frac{\alpha^\ell - \beta^\ell}{\alpha - \beta}$; then $S_0 = 0$, $S_1 = 1$, and for each integer ℓ we have $S_{\ell+2} = S_{\ell+1} - S_\ell$. It follows that $S_{6k+1} = S_{6k+2} = 1$, $S_{6k+4} = S_{6k+5} = -1$, and $S_{3k} = 0$ for all integers k.

This allows us to write $A_n = \sum\limits_{k=0}^{n} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \sum\limits_{k=1}^{n+1} S_k$, and we deduce that $A_{n+6} = A_n$ for all n. The initial values are $A_1 = 2$, $A_2 = 2$, $A_3 = 1$, $A_4 = 0$, $A_5 = 0$, and $A_6 = 1$. Thus, $A_1 + A_2 + \cdots + A_n \geq n$ with equality if and only if n is congruent to 0 or 5 modulo 6.

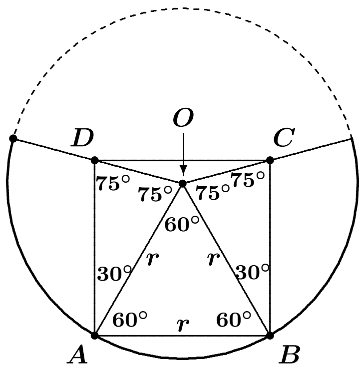
Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, The University of Scranton, Scranton, PA, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3458. [2009 : 326, 329] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Determine the central angle of a sector, such that the square drawn with one vertex on each radius of the sector and two vertices on the circumference, has area equal to the square of the radius of the sector.

Similar solutions by Richard I. Hess, Rancho Palos Verdes, CA, USA; Tom Leong, The University of Scranton, Scranton, PA, USA; and Albert Stadler, Herrliberg, Switzerland.

Call the square $ABCD$ where A and B lie on the the given circle with centre O. The given conditions imply that the side length of the square equals the radius of the sector; hence, $\triangle ABO$ is equilateral and $\angle AOB = 60^\circ$. Since $\angle OAD = \angle BAD - \angle BAO = 90^\circ - 60^\circ = 30^\circ$, and $\triangle OAD$ is isosceles, we find that $\angle DOA = 75^\circ$. Similarly, $\angle BOC = 75^\circ$. Thus, the central angle of the sector is



$$\begin{aligned} \angle DOC &= \angle DOA + \angle AOB + \angle BOC \\ &= 75^\circ + 60^\circ + 75^\circ \;=\; 210^\circ\,. \end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; EDMUND SWYLAN, Riga, Latvia; and the proposer.

3459. [2009 : 326, 329] *Proposed by Zafar Ahmed, BARC, Mumbai, India.*

Let a, b, c and p, q, r be positive real numbers. Prove that if $q^2 \leq pr$ and $r^2 \leq pq$, then

$$\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \leq \frac{3}{p+q+r}\,.$$

When does equality hold?

Solution by Arkady Alt, San Jose, CA, USA, expanded by the editor.

Consider the system of linear equations below:

$$\begin{aligned} pa+qb+rc &= x\,, \\ ra+pb+qc &= y\,, \\ qa+rb+pc &= z\,, \end{aligned}$$

The coeffficent matrix has determinant $\Delta = p^3+q^3+r^3-3pqr$. If $\Delta = 0$, then by the AM–GM Inequality $p=q=r$, in which case equality holds in the required inequality.

Otherwise, by Cramer's Rule, we obtain that $a = \frac{1}{\Delta}(x\alpha + y\gamma + z\beta)$, $b = \frac{1}{\Delta}(x\beta + y\alpha + z\gamma)$, and $c = \frac{1}{\Delta}(x\gamma + y\beta + z\alpha)$, where $\alpha = p^2 - qr$, $\beta = q^2 - rp$, and $\gamma = r^2 - pq$.

Thus,

$$\begin{aligned} &\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \\ &= \frac{1}{\Delta}\left(\frac{x\alpha+y\gamma+z\beta}{x} + \frac{x\beta+y\alpha+z\gamma}{y} + \frac{x\gamma+y\beta+z\alpha}{z}\right) \\ &= \frac{3\alpha}{\Delta} + \frac{\beta}{\Delta}\left(\frac{z}{x}+\frac{x}{y}+\frac{y}{z}\right) + \frac{\gamma}{\Delta}\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\,. \end{aligned} \tag{1}$$

Since x, y, and z are positive, the AM–GM Inequality yields

$$\frac{z}{x}+\frac{x}{y}+\frac{y}{z} \geq 3 \quad\text{and}\quad \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \geq 3\,. \tag{2}$$

Since $\beta \leq 0$ and $\gamma \leq 0$ by assumption, we obtain from (1) that

$$\begin{aligned} &\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \\ &\leq \frac{3\alpha}{\Delta} + \frac{3\beta}{\Delta} + \frac{3\gamma}{\Delta} = \frac{3(\alpha+\beta+\gamma)}{\Delta} = \frac{3}{p+q+r}\,. \end{aligned}$$

Equality holds if $p=q=r$ or if $\Delta \neq 0$ and $x=y=z$ (necessary for equality to hold in (2)), and in the latter case $a=b=c=\frac{x}{\Delta}(\alpha+\beta+\gamma)$.

Conversely, if $a = b = c$ then equality holds; hence, equality holds if and only if $p = q = r$ or $a = b = c$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

3460. [2009 : 327, 329] *Proposed by Tran Quang Hung, student, Hanoi National University, Vietnam.*

The triangle ABC has circumcentre O, orthocentre H, and circumradius R. Prove that

$$3R - 2OH \;\leq\; HA + HB + HC \;\leq\; 3R + OH\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

We need two known results:

$$\overrightarrow{OH} \;=\; \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\,, \tag{1}$$

and, for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in Euclidean space,

$$|\vec{b}+\vec{c}| + |\vec{c}+\vec{a}| + |\vec{a}+\vec{b}| \;\leq\; |\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{a}+\vec{b}+\vec{c}|\,. \tag{2}$$

The first is the observation (in vector notation) that the segment joining the orthocentre of a triangle to any vertex (represented, for example, by the vector $\overrightarrow{HA} = \overrightarrow{HO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OH}$) is parallel to and twice as long as the segment joining the midpoint of the opposite side to the circumcentre (namely $-\frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)$). The second, follows from the triangle inequality applied to Hlawka's identity (see D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, page 171 item 2.25.2, for the proof and for further references).

For the rightmost inequality use (1) together with $HA = |\overrightarrow{OB} + \overrightarrow{OC}|$, $R = |\overrightarrow{OA}|$, and analogous expressions for the other two vertices to rewrite the inequality $HA + HB + HC \leq 3R + OH$ as

$$\begin{aligned} &|\overrightarrow{OB} + \overrightarrow{OC}| + |\overrightarrow{OC} + \overrightarrow{OA}| + |\overrightarrow{OA} + \overrightarrow{OB}| \\ &\qquad\leq\; |\overrightarrow{OA}| + |\overrightarrow{OB}| + |\overrightarrow{OC}| + |\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}|\,, \end{aligned}$$

which holds by (2).

For the inequality on the left, without loss of generality we can assume that $\angle A \leq 60^\circ$. Then $|\overrightarrow{HA}| = 2R\cos A \geq 2R\cos 60^\circ = R$, while

$$|\overrightarrow{HB}| + |\overrightarrow{OH}| \;\geq\; |\overrightarrow{OB}| \;=\; R\,, \quad \text{and} \quad |\overrightarrow{HC}| + |\overrightarrow{OH}| \;\geq\; |\overrightarrow{OC}| \;=\; R\,.$$

Add these three inequalities to obtain

$$3R - 2|\overrightarrow{OH}| \ \leq \ |\overrightarrow{HA}| + |\overrightarrow{HB}| + |\overrightarrow{HC}|\,,$$

and the solution is complete.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; EDMUND SWYLAN, Riga, Latvia; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer

In addition to Apostolopoulos and Tran, only Geupel and Swylan established both inequalities correctly for all triangles; the other submissions relied on an identity that holds only for those triangles having no obtuse angles, namely $HA + HB + HC = 2R + 2r$. This formula can be found in O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969, page 103 item 12.2, where unfortunately the authors have omitted the necessary condition on the angles; should the triangle have $\angle A > 90^\circ$, for example, the identity would become $-HA + HB + HC = 2R + 2r$. The proposer observed that because $R \geq 2r$, the identity implies that for acute triangles his inequality on the right can be strengthened to $HA + HB + HC \leq 3R$, an inequality that fails for obtuse triangles.

3461. [2009 : 327, 329] *Proposed by Tran Quang Hung, student, Hanoi National University, Vietnam.*

Let I be the incentre of triangle ABC and let A', B', and C' be the intersections of the rays AI, BI, and CI with the respective sides of the triangle. Prove that

$$IA + IB + IC \ \geq \ 2(IA' + IB' + IC')\,.$$

Comment by Oliver Geupel, Brühl, NRW, Germany.

The problem is well-known: see
`http://www.mathlinks.ro/viewtopic.php?t=144776` (dated April 2007).

[*Editor's note. Geupel then gave a solution from this website, which he attributes to C. Pohoaţă. As all of the solutions received have some similarities to this one, no solution will be presented here.*

Readers are reminded once again that **CRUX** *is not interested in receiving problems that have appeared recently elsewhere, and certainly no problems taken from another source should ever be submitted to* **CRUX** *unless that source is identified.*]

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incorrect solution was received.

3462. [2009 : 327, 329] *Proposed by Sotiris Louridas, Aegaleo, Greece.*

Let x, y, and z be positive real numbers such that

$$\left(x^3+z^3-y^3\right)\left(y^3+x^3-z^3\right)\left(z^3+y^3-x^3\right) > 0 .$$

Prove that

$$\left(x^3+y^3+z^3+3xyz\right)\prod_{\text{cyclic}}\left(x^3+y^3-z^3+xyz\right) \le 3\prod_{\text{cyclic}}\sqrt[3]{x^4\left(x^2+yz\right)^4} .$$

Composite of similar solutions by Arkady Alt, San Jose, CA, USA, and Thanos Magkos, 3rd High School of Kozani, Kozani, Greece, modified by the editor.

First note that the hypotheses imply that each of the terms $x^3+y^3-z^3$, $y^3+z^3-x^3$, and $z^3+x^3-y^3$ is positive, since if two of them are negative, say $x^3+y^3-z^3<0$ and $y^3+z^3-x^3<0$, then we would have $2y^3<0$, or $y<0$, a contradiction. Hence, if we set $a=x^3+3xyz$, $b=y^3+3xyz$, and $c=z^3+3xyz$, then $a+b-c=x^3+y^3-z^3+3xyz>0$, which implies that $a+b>c$. Similarly, $b+c>a$ and $c+a>b$. Therefore, since a, b, and c are positive, they are the side lengths of a triangle ABC. In this context, the inequality to be proved is now rewritten as

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) \le 3\sqrt[3]{a^4b^4c^4} . \qquad (1)$$

Let s, R, and F denote the semiperimeter, the circumradius, and the area of triangle ABC. The following formulas are well known:

$$\begin{aligned} F &= \sqrt{s(s-a)(s-b)(s-c)} ; \\ \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R ; \\ abc &= 4RF . \end{aligned}$$

Hence, inequality (1) is equivalent to each of the following:

$$\begin{aligned} 16s(s-a)(s-b)(s-c) &\le 3\sqrt[3]{(4RF)^4} , \\ 16^3F^6 &\le 3^3\cdot 4^4\cdot R^4\cdot F^4 , \\ abc = 4RF &\le 3\sqrt{3}R^3 , \\ 8(\sin A\sin B\sin C)R^3 &\le 3\sqrt{3}R^3 , \\ \sin A\sin B\sin C &\le \frac{3\sqrt{3}}{8} , \end{aligned}$$

and it is well known that the last inequality is true [*Ed: c.f. Formula 2.8 on p. 20 of O. Bottema etal., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969.*]

Thus, inequality (1) is established, and the problem is solved.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was one incorrect solution submitted.