Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
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Editorial Board

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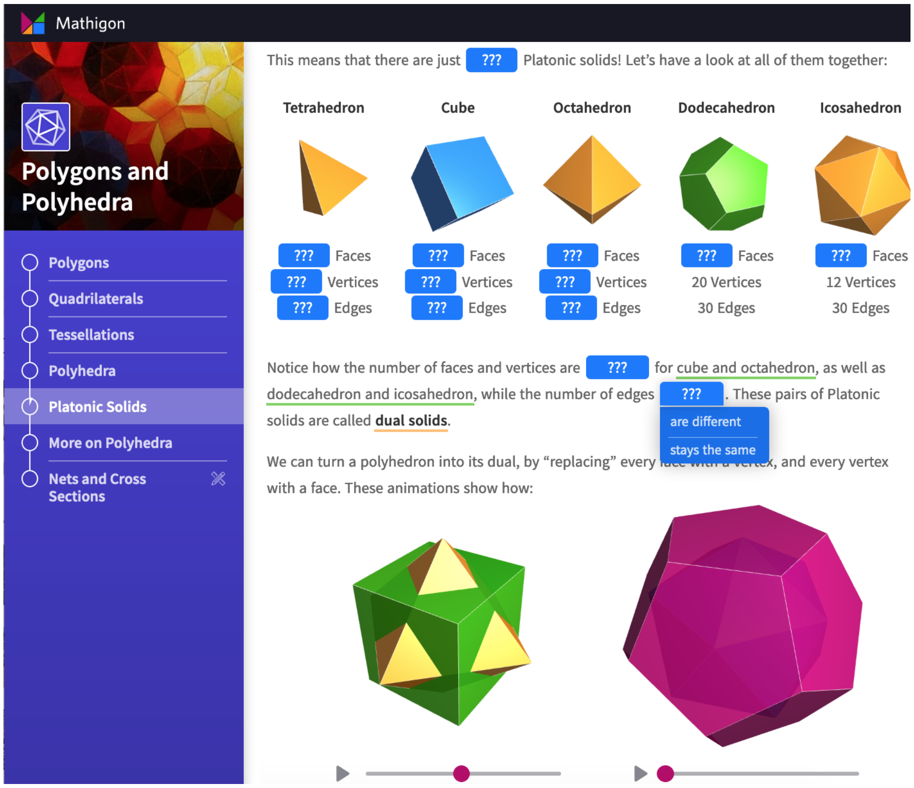
EDITORIAL

Finding good, educational and engaging mathematical resources can be quite a feat: there is a lot of information on the Internet. Typing "polygon" into Google yields over 28 million results. Just on the first page of the search, I get links to a dictionary definition, a gaming website and YouTube channel, a climate control organization and a trading network for jewellers. Oh, and, thankfully, a wikipedia page on plane figures. So when you find a mathematical resource that is worthwhile, you share it.

My latest discovery is Mathigon. This interactive textbook offers units for students and teachers of grades 6 and above on geometry, algebra, discrete math, probability, statistics, logic, sets, infinity, and many more. Some are still under construction, but many are available now. The website is easy to navigate, the content is excellent and the exposition is compelling. The layout and interactive features are conducive to active reading, experimenting and questioning. It is beautifully done. I encourage you to take a look. Meanwhile, I'll be busy thinking about the ways to incorporate it into my classroom and outreach activities this fall.

Kseniya Garaschuk



MathemAttic

No. 6

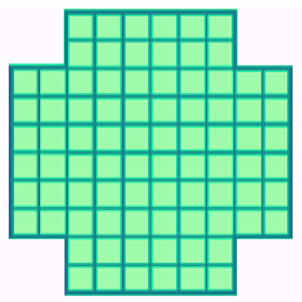
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 30, 2019**.

MA26. Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with?

MA27. You want to play Battleship on a 10×10 grid with 2×2 squares removed from each of its corners:



What is the maximum number of submarines (ships that occupy 3 consecutive squares arranged either horizontally or vertically) that you can position on your board if no two submarines are allowed to share any common side or corner?

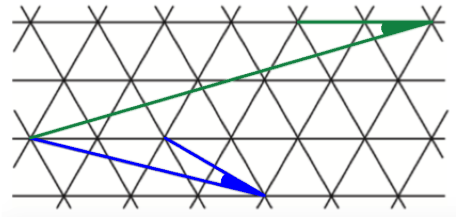
MA28. Prove that for all positive integers n, the number

$$\frac{1}{3}\left(4^{4n+1} + 4^{4n+3} + 1\right)$$

is not prime.

MA29. Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs so that if numbers in each pair are added and all the sums are multiplied together, the result is a perfect square.

MA30. Consider the two marked angles on a grid of equilateral triangles.



Prove that these angles are equal.

. . . .

Les problémes dans cette section sont appropriés aux étudiants de l'école secondaire.

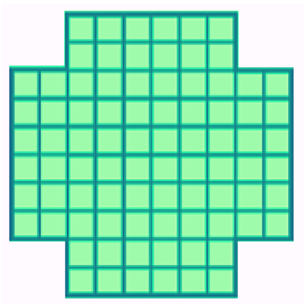
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA26. Neuf nombres à 9 chiffres sont formés, chacun se servant des chiffres de 1 à 9 une seule fois. Ces neuf nombres n'ont pas besoin d'être distincts. Quel est le nombre maximal de zéros pouvant se retrouver à la fin de la représentation décimale de la somme de ces neuf nombres?

MA27. Un jeu de bataille navale se tient sur un grillage 10×10, duquel on a enlevé les cases 2×2 de chacun des coins.



Étant donné qu'un sousmarin occupe 3 cases consécutives, horizontalement ou verticalement, quel est le nombre maximal de sousmarins qu'on puisse placer sur le grillage spécial de façon à ce que les sousmarins ne partagent jamais un coin ou un côté?

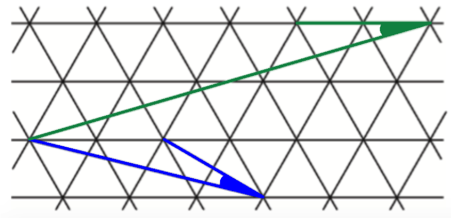
MA28. Démontrer que pour tout entièrs positifs n, le nombre

$$\frac{1}{3}\left(4^{4n+1}+4^{4n+3}+1\right)$$

n'est pas premier.

MA29. Déterminer tous les entièrs positifs n tels que si les nombres $1, 2, 3, \ldots, 2n$ sont regroupés en paires de façon bien choisie, que la somme de chaque paire est calculée et que ces sommes sont multipliées, le résultat est un carré parfait.

MA30. Considérer les deux angles situés sur un grillage de triangles équilatéraux.



Démontrer que ces angles sont égaux.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(1), p. 4–5.

MA1. How many two-digit numbers are there such that the difference of the number and the number with the digits reversed is a non-zero perfect square? Problem extension: What happens with three-digit numbers? four-digit numbers?

Originally Question 7 from the 1999 W.J. Blundon Contest.

We received 5 submissions of which one was correct and complete. We present the solution by Sophie Bekerman (Los Gatos High School), modified by the editor.

Two-digit numbers.

Let A be such a two digit number. We can express A as $10x + y$ where x is the first digit, y is the second digit, and $x, y \in [0, \ldots, 9]$. Let $\bar{A}$ be A with the digits reversed, note that $\bar{A} = 10y + x$. Given

$$A - \bar{A} = 10x + y - (10y + x) = 9x - 9y,$$

let $9x - 9y = a^2$ where $a \in \mathbb{N}$. It follows that

$$9x - 9y = a^2 \Leftrightarrow x - y = \left(\frac{a}{3}\right)^2$$

and $\left(\frac{a}{3}\right)^2 \in \mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{a}{3}\right)^2 \leq 9$ since $x - y \leq 9$. The only perfect squares that meet these conditions are 1, 4, and 9. Therefore, the differences of the digits of A are 1, 4, or 9. If $x - y = n$, their difference can be written as $(n + k) - k$ where $n + k = x$ and $k = y$. Since

$$n + k \leq 9 \Leftrightarrow k \leq 9 - n,$$

k can take any value from 0 to $9 - n$. In total, there are $10 - n$ ways to represent each difference. As $n \in [1, 4, 9]$, there are

$$(10 - 1) + (10 - 4) + (10 - 9) = 16$$

possible values of A.

Three-digit numbers.

Let B be such a three digit number. We can express B as $100x + 10y + z$, where x is the first digit, y is the second digit, z is the third digit, and $x, y, z \in [0, \ldots, 9]$. Let $\bar{B}$ be B with the digits reversed, note that $\bar{B} = 100z + 10y + x$. Given

$$B - \bar{B} = 100x + 10y + z - (100z + 10y + x) = 99x - 99z,$$

let $99\,(x-z) = b^2$ where $b \in \mathbb{N}$. It follows that

$$99\,(x-z) = b^2 \Leftrightarrow 11\,(x-z) = \left(\frac{b}{3}\right)^2$$

and $\left(\frac{b}{3}\right)^2 \in \mathbb{N}$ since it is the difference of two natural numbers. $\left(\frac{b}{3}\right)^2 \leq 9$ since $x - z \leq 9$. For $11\,(x-z)$ to be a perfect square, $(x-z)$ has to be a factor of 11. This is impossible since $x - z \leq 9$. Therefore, there are no possible forms of B.

Four-digit numbers.

Let C be such a four digit number. We can express C as $1000w + 100x + 10y + z$, where w is the first digit, x is the second digit, y is the third digit, z is the fourth digit, and $w, x, y, z \in [0, \ldots, 9]$. Let $\bar{C}$ be C with the digits reversed, note that $\bar{C} = 1000z + 100y + 10x + w$. Given

$$\begin{aligned} C - \bar{C} &= 1000w + 100x + 10y + z - (1000z + 100y + 10x + w) \\ &= 999w + 90x - 90y - 999z \end{aligned}$$

let $999\,(w-z) + 90\,(x-y) = c^2$ where $c \in \mathbb{N}$. It follows that

$$999\,(w-z) + 90\,(x-y) = c^2,$$

or, equivalently,

$$111\,(w-z) + 10\,(x-y) = \left(\frac{c}{3}\right)^2.$$

If $w = z \Leftrightarrow w - z = 0$ then that leaves $10\,(x-y) = \left(\frac{c}{3}\right)^2$. For $10\,(x-y)$ to be a perfect square, $(x-y)$ has to be a factor of 10, which is impossible since $x - y \leq 9$. Therefore, $w - z \neq 0$ and $111 \leq \left(\frac{c}{3}\right)^2 \leq 1089$.

The perfect squares between 111 and 1089 are 121, 484, 576, 625, 676, and 1089. These are found simply by searching through every perfect square in the range $[111, 1089]$ and seeing if the perfect square can be expressed in the form $111m + 10n$, where $m, n \in \mathbb{N}$. The only case where $x < y$ is $576 = 111 \cdot 6 - 10 \cdot 9$. For all the other possible squares, $x - y$ happens to be positive.

For 576, $w - z = 6$ and $x - y = -9$. There are $10 - 6 = 4$ possible pairs of w and z that yield a difference of 6. There is $10 - 9 = 1$ possible pair of x and y that yield a difference of 9. Therefore there are 4 possible combinations of w, x, y, and z that will yield 576.

This same methodology applies to the other possible perfect squares for a total of

$$\begin{aligned} &(10-1)\cdot(10-1) + (10-4)\cdot(10-4) + (10-6)\cdot(10-9) + \\ &\quad (10-5)\cdot(10-7) + (10-6)\cdot(10-1) + (10-9)\cdot(10-9) = 173 \end{aligned}$$

combinations of C.

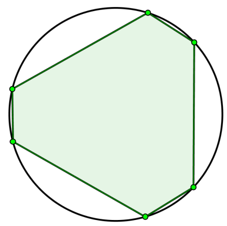
MA2. A sequence $t_1, t_2, \ldots$ beginning with any two positive numbers is defined such that for $n > 2$, $t_n = \dfrac{1 + t_{n-1}}{t_{n-2}}$. Show that such a sequence must repeat itself with a period of 5.

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received 5 solutions. We present the solution by Richard Hess.

Start with a and b. Then the next terms are $(1 + b)/a$, $(a + b + 1)/(ab)$, $(a + 1)/b$, a, b, $\ldots$. This sequence has a period of five since terms six and seven duplicate terms one and two.

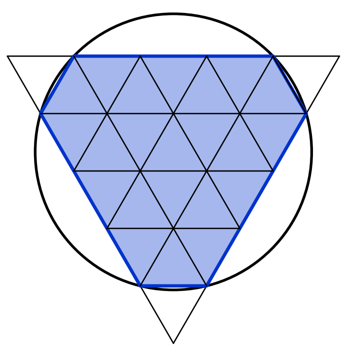
MA3. A hexagon H is inscribed in a circle, and consists of three segments of length 1 and three segments of length 3. Find the area of H.



Originally Question 10 from the 2000 W.J. Blundon Contest.

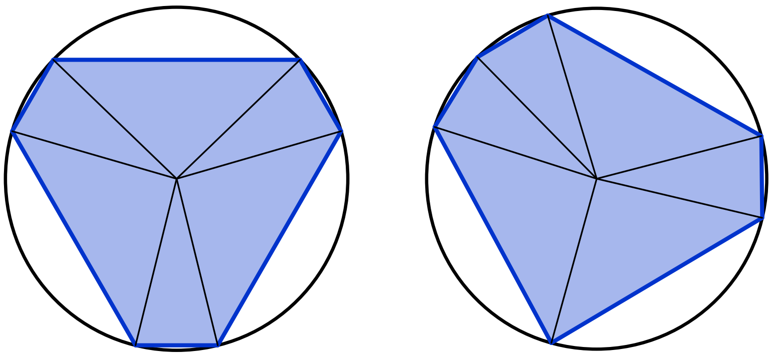
We received seven solutions, out of which we present the one by Valcho Milchev, lightly edited.

By symmetry, all the internal angles of the hexagon H are equal and thus 120°. This means that H may be tiled by equilateral triangles as shown in the figure:



H is composed of 22 equilateral triangles of side length 1, each of which has an area of $\dfrac{\sqrt{3}}{4}$. Therefore the area of H is $\dfrac{11\sqrt{3}}{2}$.

Editor's Comments. The statement of Problem MA3 did not specify in which order the segments appear in the hexagon, even though the picture suggested a specific arrangement. However, it turns out that all cyclic hexagons with three edges of length 1 and three edges of length 3 have the same area. This can be seen by drawing the radii from the centre of the circle to the six vertices of the hexagon (see figure below). This splits the hexagon into six isosceles triangles with leg lengths equal to the radius of the circles. Three of the isosceles triangles have base length 3 and three have base length 1, irrespective of the arrangement of the edges in the hexagon.



MA4. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

Originally Question 9 from the 2002 W.J. Blundon Contest.

We received seven submissions, all of which were correct and complete. We present the joint solution by Amit Kumar Basistha (Anundoram Borooah Academy High School) and Sophie Bekerman (Los Gatos High School), done independently, slightly modified by the editor.

$x + y = a$ is tangent to $x^2 + y^2 = b$ when the system

$$\begin{cases} x + y = a \\ x^2 + y^2 = b \end{cases}$$

has exactly one solution. Given that $x + y = a \Leftrightarrow y = a - x$, by substitution we see that

$$x^2 + y^2 = b \Leftrightarrow x^2 + (a - x)^2 = b \Leftrightarrow 2x^2 - 2ax + a^2 - b = 0$$

A quadratic equation has one solution if and only if the discriminant is equal to 0. By construction, our expression has only one solution, thus by setting the discriminant Δ of the above expression to 0 we see that

$$\Delta = 4a^2 - 4\left(2\left(a^2 - b\right)\right) = 0 \Leftrightarrow a^2 = 2b$$

When there is one solution for x, there is also only one solution for y since $y = a - x$. Hence, when $a^2 = 2b$, the line $x + y = a$ is tangent to the circle $x^2 + y^2 = b$.

MA5. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

Originally Question 8 from the 2002 W.J. Blundon Contest.

We received 5 submissions of which 3 were correct and complete. We present the solution by Vitthal Ingle and Konstantine Zelator, done independently.

Let θ_1, θ_2 be the angles made between the x-axis and the lines $y = 3x$ and $y = 2x$ respectively. Clearly, $\tan\theta_1 = 3$ and $\tan\theta_2 = 2$. Let $\alpha = \theta_1 - \theta_2$, the angle between the lines $y = 2x$ and $y = 3x$. By the angle subtraction identity for tangent:

$$\tan\alpha = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{3-2}{1+3\cdot 2} = \frac{1}{7}.$$

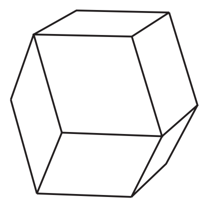
Let O be the origin. We have

$$\tan\alpha = \frac{1}{7} = \frac{\overline{PQ}}{\overline{OP}} = \frac{5}{\overline{OP}},$$

and so $\overline{OP} = 35$. Let P have coordinates $(a, 2a)$, and let M be the projection of P onto the x-axis. Now $\overline{OP}^2 = \overline{OM}^2 + \overline{MP}^2$, and so $35^2 = a^2 + 4a^2 = 5a^2$. It follows that $a = 7\sqrt{5}$ and so P has coordinates $(7\sqrt{5}, 14\sqrt{5})$. Note that the solution $a = -7\sqrt{5}$ gives a point in the third quadrant, and so can not be the answer.

CONTEST CORNER SOLUTIONS

CC339. A *rhombic dodecahedron* has twelve congruent rhombic faces; each vertex has either four small angles or three large angles meeting there. If the edge length is 1, find the volume in the form $\dfrac{p+\sqrt{q}}{r}$, where p, q, and r are natural numbers and r has no factor in common with p or q.

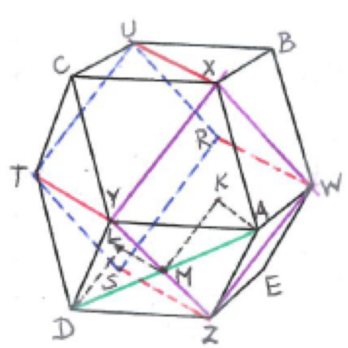


Originally from the 2018 Science Atlantic Math Competition.

The statement of this problem originally appeared in ***Crux*** *44(8). We received no solutions to this problem for the original publication. We have since received a solution by Ivko Dimitrić showing that the statement of the problem is not correct. We present the solution here.*

We have $f = 12$ rhombic faces, so the number of edges is $e = \frac{12\cdot 4}{2} = 24$. By Euler's formula, $v - e + f = 2$, we find the number of vertices to be $v = 14$. Moreover, if x is the number of vertices of degree 4 and y the number of those of degree 3, then from $x + y = 14$ and $e = \frac{4x+3y}{2} = 24$ we get $x = 6$ and $y = 8$.

It is a general fact that a rhombic dodecahedron is a semi-regular polyhedron with symmetries and transitive faces, which means that for each pair α, β of faces there is an isometry of the solid that takes α to β. As a consequence of the symmetries of this polyhedron, the arrangement of edges, the measures of angles in the same position and geometric picture about one vertex is exactly the same as the situation about another vertex of the same degree.



We use the labeling of vertices and points as shown on the diagram. Because of the symmetry, there exists an axis through a selected degree-4 vertex, which is equally inclined to each of the four edges incident to that vertex and there exists an angle so that the rotation through that angle about the axis will take each edge from that vertex to the next one in cyclic order and hence each vertex of the quadrilateral $WXYZ$ to its neighbouring vertex in cyclic order. Thus, these four vertices lie in the same plane perpendicular to the axis of rotation and in that plane each vertex is carried to its neighbour in the same sense by induced rotation in that plane through the same angle, so the quadrilateral is a square. Hence, each degree-4 vertex of the dodecahedron together with four neighboring vertices determines a pyramid with a square base formed by the four neighboring vertices, where the apex (a degree-4 vertex) is projected orthogonally to the center of the square base. Two square pyramids with apexes A and D are joined by the common edge YZ and the faces AYZ and DYZ that share that edge are two congruent triangular halves of rhomboid face $AZDY$ of the dodecahedron. Let a be the edge length of the polyhedron and h be the height of each of the six square pyramids with apex at a degree-4 vertex, such as pyramids $AWXYZ$ and $DTYZS$. The square bases of these six pyramids are congruent and their union forms the surface of a cube, since the dihedral angle at each edge is $90°$, for example $XY \perp XW$ and $XY \perp XU$, so XY is perpendicular to the square face $XURW$. We call this cube the core cube.

Let M be the midpoint of a shorter diagonal YZ of the rhombic face $AZDY$. Because of the congruence of the square pyramids with apexes A and D, the segments AM and DM are equally inclined to the corresponding bases of these pyramids and since the dihedral angle of the core cube at M is $90°$ and $\angle AMD = 180°$, it follows that $\angle KMA = \angle LMD = 45°$, i. e. $\triangle AKM$ and $\triangle DLM$ are two congruent isosceles right triangles and hence $s = h\sqrt{2}$, whereas the edge of the core cube is $2h$.

Then from (a half of) the isosceles $\triangle AYZ$ we get $(h\sqrt{2})^2 + h^2 = a^2$, which gives $h = a/\sqrt{3}$. The rhombic dodecahedron is constructed as the union of the cubical core of edge length $2h$ whose vertices are eight degree-3 vertices of the dodecahedron and six congruent square pyramids of height h, surmounting each of the six faces of the cube on the outside.Thus, its volume is

$$V = \left(\frac{2a}{\sqrt{3}}\right)^3 + 6 \cdot \frac{1}{3}\,(2h)^2 h = \frac{8a^3}{3\sqrt{3}} + 8\left(\frac{a}{\sqrt{3}}\right)^3 = \frac{16\sqrt{3}}{9}\,a^3.$$

When $a = 1$, the answer can be written as $\frac{0+\sqrt{768}}{9}$ which is the required expression, but it cannot be written in the form $\frac{p+\sqrt{q}}{r}$ where p, q, r are integers with r having no factor in common with p and q since 3 is the common factor of p, q, r.

Remark. Some nice pictures and additional information on rhombic dodecahedron can be found at `https://en.wikipedia.org/wiki/Rhombic.dodecahedron` and at `http://mathworld.wolfram.com/RhombicDodecahedron.html`.

PROBLEM SOLVING VIGNETTES

No.6

Shawn Godin

Repdigit Recreations

In this issue we will look at a couple of problems from the course C&O 380 that I took from Ross Honsberger that has been featured in previous columns. The next set of problems are:

#16. A is the integer $666\cdots66$, containing 666 sixes. B is the integer $333\cdots33$, containing 666 threes. State the value of AB.

#17. Show that a positive integer, with more than one digit, all of whose digits are the same, cannot be a perfect square.

#18. Show that the sum of the squares of 83 consecutive natural numbers is never a perfect square.

#19. Devise a method of trisecting a given line segment, using only straightedge and compasses, which does not involve parallel lines.

#20. Construct an equilateral triangle so that it has one vertex on each of three given parallel lines.

Problems #16 and #17 both deal with *repdigit* numbers. That is, numbers that are comprised of a single digit repeated a number of times. Don Rideout, in problem #3 of his vignette [2019: 45(3), p. 120], looked at *repunit* numbers, that is, repdigit numbers made up of only the digit 1. We will run into a couple of the properties of these numbers as we solve the two problems. We will look at #17 first.

How do we know if a number is a perfect square? Looking at the first few squares we start to see a pattern in the units digit.

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$9^2 = 361$	$20^2 = 400$

The unit digits follow the pattern

$$1, 4, 9, 6, 5, 6, 9, 4, 1, 0, 1, 4, 9, \ldots$$

If we use modular arithmetic, as in some recent columns, we would get the following:

$n \pmod{10}$	0	1	2	3	4	5	6	7	8	9
$n^2 \pmod{10}$	0	1	4	9	6	5	6	9	4	1

which tells us the same thing: the units digit of a perfect square is $0, 1, 4, 5, 6$, or 9. Hence the repdigit numbers $222\cdots22$, $333\cdots33$, $777\cdots77$, and $888\cdots88$ cannot be perfect squares.

We write the other repdigit numbers in the form

$$\begin{aligned}111\cdots11 &= 1 \times 111\cdots11\\ 444\cdots44 &= 4 \times 111\cdots11\\ 555\cdots55 &= 5 \times 111\cdots11\\ 666\cdots66 &= 6 \times 111\cdots11\\ 999\cdots99 &= 9 \times 111\cdots11.\end{aligned}$$

Clearly $555\cdots55$ is not a perfect square since $5 \mid 555\cdots55$, but $5 \nmid 111\cdots11$. Similarly, $666\cdots66$ is not a perfect square.

The remaining three candidates are written as a perfect square times $111\cdots11$. Thus if $111\cdots11$ is a perfect square, then so is $444\cdots44$ and $999\cdots99$. If $111\cdots11$ is not a perfect square then neither are $444\cdots44$ and $999\cdots99$.

To determine if $111\cdots11$ is a perfect square we will go back to modular arithmetic and look at numbers modulo 4.

$n \pmod 4$	0	1	2	3
$n^2 \pmod 4$	0	1	0	1

So if a number is a perfect square it must be congruent to 0 or 1 modulo 4. Taking into account that $4 \mid 100$ and hence $4 \mid 10^n$ when $n \geq 2$ (so $10^n \equiv 0 \pmod 4$) we get

$$111\cdots11 \equiv 11 \equiv 3 \pmod 4$$

and so $111\cdots11$ is not a perfect square and therefore no repdigit number, of more than one digit, is a perfect square.

Next, we will look at problem #16. A few computations suggest a pattern:

$$\begin{aligned}6 \times 3 &= 18 & 66 \times 33 &= 2\,178\\ 666 \times 333 &= 221\,778 & 6\,666 \times 3\,333 &= 22\,217\,778\\ 66\,666 \times 33\,333 &= 2\,222\,177\,778 & 666\,666 \times 333\,333 &= 222\,221\,777\,778\end{aligned}$$

that is,

$$\overbrace{666\cdots66}^{n\ \text{6s}} \times \overbrace{333\cdots33}^{n\ \text{3s}} = \overbrace{222\cdots22}^{n-1\ \text{2s}}1\overbrace{777\cdots77}^{n-1\ \text{7s}}8. \tag{1}$$

It is one thing to see a pattern and be certain it is true. It is another thing to *prove* that the pattern does indeed hold. The pattern seems to call out for mathematical induction like we saw in the last issue [2019: 45(5), p. 236-240].

To make our lives easier, we will introduce the sequence of repunit numbers

$$\{U_n\}_{n=1}^{\infty} = \{1, 11, 111, 1111, \ldots\}.$$

Our proposition that we would like to prove is

$$P_n : \quad (6U_n)(3U_n) = 10^n[2U_n - 1] + 7U_n + 1. \tag{2}$$

You may want to convince yourself that (2) is equivalent to (1).

To aid us in our proof we will need the following properties of the repunit numbers:

$$10 \times U_n + 1 = U_{n+1} \tag{3}$$
$$U_a + 10^a \times U_b = U_{a+b} \tag{4}$$

We leave the proofs of these as exercises. Now on to our proof by induction.

If we look at P_1, we get

$$(6U_1)(3U_1) = 6 \times 3 = 18$$

and

$$10^1[2U_1 - 1] + 7U_1 + 1 = 10 \times (2 - 1) + 7 + 1 = 18$$

so the proposition is true for $n = 1$.

Suppose P_n is true for some $n = k \in \mathbb{N}$, then

$$(6U_k)(3U_k) = 10^k[2U_k - 1] + 7U_k + 1. \tag{5}$$

So, using (3) we get

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= (6(10U_k + 1))(3(10U_k + 1)) \\ &= 100((6U_k)(3U_k)) + 360U_k + 18 \end{aligned} \tag{6}$$

Combining (5) with (6) yields

$$\begin{aligned}(6U_{k+1})(3U_{k+1}) &= 100(10^k[2U_k - 1] + 7U_k + 1) + 360U_k + 18 \\ &= 10^{k+2}[2U_k - 1] + 1060U_k + 118 \end{aligned} \tag{7}$$

Breaking the right side of (7) into two parts and using the properties (3) and (4), yields

$$\begin{aligned}10^{k+2}[2U_k - 1] &= 10^{k+1}[2(10U_k + 1 - 1) - 10] \\ &= 10^{k+1}[2(U_{k+1} - 1) - 10] \\ &= 10^{k+1}[2U_{k+1} - 12] \\ &= 10^{k+1}[2U_{k+1} - 1 - 11] \\ &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} \end{aligned} \tag{8}$$

and

$$\begin{aligned}
1060U_k + 118 &= 1000U_k + 60U_k + 118 \\
&= 1000(U_{k-2} + 10^{k-2}U_2) + 60(U_2 + 10^2U_{k-2}) + 118 \\
&= 7000U_{k-2} + 11 \times 10^{k+1} + 778 \\
&= 7000U_{k-2} + 777 + 1 + 11 \times 10^{k+1} \\
&= 7U_{k+1} + 1 + 11 \times 10^{k+1} \qquad (9)
\end{aligned}$$

Putting (8) and (9) back into (7) yields

$$\begin{aligned}
(6U_{k+1})(3U_{k+1}) &= 10^{k+1}[2U_{k+1} - 1] - 11 \times 10^{k+1} + 7U_{k+1} + 1 + 11 \times 10^{k+1} \\
&= 10^{k+1}[2U_{k+1} - 1] + 7U_{k+1} + 1
\end{aligned}$$

which shows that P_{k+1} is true and completes the induction. So for the problem at hand we have

$$\overbrace{666\cdots 66}^{666\ \text{6s}} \times \overbrace{333\cdots 33}^{666\ \text{3s}} = \overbrace{222\cdots 22}^{665\ \text{2s}}1\overbrace{777\cdots 77}^{665\ \text{7s}}8.$$

It is nice to have an opportunity to use a new tool, but it is always nice to find a slick solution such as

$$\begin{aligned}
\overbrace{666\cdots 66}^{666\ \text{6s}} \times \overbrace{333\cdots 33}^{666\ \text{3s}} &= 6 \times 3 \times \overbrace{111\cdots 11}^{666\ \text{1s}} \times \overbrace{111\cdots 11}^{666\ \text{1s}} \\
&= 9 \times 2 \times \overbrace{111\cdots 11}^{666\ \text{1s}} \times \overbrace{111\cdots 11}^{666\ \text{1s}} \\
&= \overbrace{999\cdots 99}^{666\ \text{9s}} \times \overbrace{222\cdots 22}^{666\ \text{2s}} \\
&= (10^{666} - 1) \times \overbrace{222\cdots 22}^{666\ \text{2s}} \\
&= \overbrace{222\cdots 22}^{666\ \text{2s}}\overbrace{000\cdots 00}^{666\ \text{0s}} - \overbrace{222\cdots 22}^{666\ \text{2s}} \\
&= \overbrace{222\cdots 22}^{665\ \text{2s}}1\overbrace{777\cdots 77}^{665\ \text{7s}}8.
\end{aligned}$$

The biggest hammer isn't always the best tool for the job.

A little manipulation tells us that the repdigit number $\overbrace{ddd\cdots dd}^{n\ \text{ds}}$, where $d \in \{1, 2, \ldots, 9\}$ can be written as

$$\overbrace{ddd\cdots dd}^{n\ \text{ds}} = \frac{d}{9}\left(\overbrace{999\cdots 99}^{n\ \text{9s}}\right) = \frac{d}{9}(10^n - 1)$$

which make sense as

$$\overbrace{ddd\cdots dd}^{n\text{ ds}} = d + 10d + 100d + \cdots + d \times 10^{n-1}$$

is a geometric series.

Enjoy the rest of the problems from the problem set, and you may enjoy the following problem from the vault:

> Find a quadratic polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated $2k$ times. (For example, $f(555) = 555555$.)

This was Mayhem problem M256 that appeared in [2006: 32(5), p. 265-266] and the solution is in [2007: 33(5), p. 271] which can be generalized to

> Find a degree d polynomial $f(x)$ such that, if n is a positive integer consisting of the digit 5 repeated k times, then $f(n)$ consists of the digit 5 repeated dk times.

TEACHING PROBLEMS

No. 3

John McLoughlin

Unique Teenage Factorization

Try it. Write down any two or three "teenages", not necessarily different, and multiply them together. Now break the product back down into its factors so that they can be rearranged into a product of the ages of teenagers. The result is unique.

For example, $14 \times 15 \times 17 = 3570$. If we try to identify factors, it may be evident that 35 is a factor and so we have 35×102. Breaking this down further we have 7×5 and $2 \times 3 \times 17$. These factors can be repackaged as $(2 \times 7) \times (3 \times 5) \times 17$. Note that it would not have mattered if we began by observing 2 or 5 or 10 was a factor instead, as ultimately the prime factorization is unique.

Let us consider the same idea in reverse. That is, given the product of the ages of a group of teenagers is 3570, find the ages of the teenagers. Indeed we could break 3570 down fully into prime factors and put them back together to make suitable ages. Alternatively, one can recognize properties like the divisibility by 10 (and hence, by 5) that necessitate the inclusion of age 15 among them. Likewise, the evident divisibility by 7 in this case ensures that there will be a 14 year old. The third age of 17 falls out through the division process.

You are encouraged to take a calculator and simply multiply a bunch of ages of teenagers together. Then take this product apart to find the individual ages. This will enhance appreciation of the process. Both students and teachers will realize how easy it becomes to generate different examples, thus enabling people to try their own problems at a suitable pace or engage peers with fresh challenges. Here is an example for you to try:

> The product of the ages of a group of teenagers is 10584000. Find the ages of the teenagers.

Another teaching point that can be offered here concerns the idea of lower and upper bounds. Informally these concepts can be considered through attention to a different matter. The focus can be placed on the number of teenagers in the group rather than the specific ages. Keep in mind that we require a value of n for which the product lies between 12^n and 20^n. In fact, using powers of 10 rather than 12 can provide a ballpark figure quite quickly. Reverting to our earlier example with 3570, we can readily see that $10^3 < 3570 < 20^3$. In fact, it can be verified that $n = 3$ when powers of 12 are used also. So in the problem with 10584000 or a little more than 10^7, it seems possible that there may be as many as seven teenagers. However, checking we find that 12^7 exceeds 35 million and there are only six teenagers.

Looking ahead...

The idea underlying *Teaching Problems* is to highlight problems that teachers have found to be particularly valuable. It may be that they illustrate features of mathematics. Some problems lend themselves to multiple solutions or approaches that vary widely. Submissions of your examples of teaching problems with accompanying commentary are welcomed. Send them along please.

Problem solvers enjoy solving problems. In anticipation of future issues of *Teaching Problems*, a trio of problems is offered here for your consideration. Discussion of them will appear in the coming months. Experience with trying these problems may enrich the reading experience in future, while adding to the discussion. Comments on the problems before or after that time are welcomed.

The Ruler Problem

An unmarked ruler is known to be exactly 6 cm in length. It is possible to exactly measure all integer lengths from 1 cm to 6 cm using only two marks, at 1 cm and 4 cm, since $2 = 6 - 4, 3 = 4 - 1$, and $5 = 6 - 1$.

Determine the smallest number of marks required on an unmarked ruler 30 cm in length to exactly measure all integer lengths from 1 cm to 30 cm.

A Geometry Problem inviting Multiple Approaches

Given square $ABCD$, with E the midpoint of CD and F the foot of the perpendicular from B to AE, show that $CF = CD$.

A Handshake Problem with a Twist

Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?

OLYMPIAD CORNER

No. 374

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **September 30, 2019**.

OC436. In a non-isosceles triangle ABC, let O and I be its circumcenter and incenter, respectively. Point B', which is symmetric to point B with respect to line OI, lies inside $\angle ABI$. Prove that the tangents to the circumcircle of the triangle $BB'I$ at points B' and I intersect on the line AC.

OC437. The magician and his helper have a deck of cards. The cards all have the same back, but their faces are coloured in one of 2017 colours (there are 1000000 cards of each colour). The magician and the helper are going to show the following trick. The magician leaves the room; volunteers from the audience place $n > 1$ cards in a row on a table, all face up. The helper looks at these cards, then he turns all but one card face down (without changing their order). The magician returns, looks at the cards, points to one of the face-down cards and states its colour. What is the minimum number n such that the magician and his helper can have a strategy to do the magic trick successfully?

OC438. A teacher gives the students a task of the following kind. He informs them that he thought of a monic polynomial $P(x)$ of degree 2017 with integer coefficients. Then he tells them k integers $n_1, n_2, \ldots, n_k$ and the value of the expression $P(n_1)P(n_2)\cdot \ldots \cdot P(n_k)$. According to these data, the students should then find teacher's polynomial. Find the smallest k for which the teacher can compose such a problem so that the polynomial found by the students must necessarily coincide with the one he thought of.

OC439. Let $(G, \cdot)$ be a group and let m and n be two nonzero natural numbers that are relatively prime. Prove that if the functions $f : G \to G$, $f(x) = x^{m+1}$ and $g : G \to G$, $g(x) = x^{n+1}$ are surjective endomorphisms, then the group G is abelian.

OC440. Let $f : [a, b] \to [a, b]$ be a differentiable function with continuous and positive first derivative. Prove that there exists $c \in (a, b)$ such that

$$f(f(b)) - f(f(a)) = (f'(c))^2(b - a).$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

OC436. Soit les points O et I, les centres des cercles circonscrit et inscrit du triangle non-isocèle ABC, respectivement. Le point B' qui est symétrique au point B par rapport à la droite OI est à l'intérieur de $\angle ABI$. Prouvez que les tangentes au cercle circonscrit du triangle $BB'I$ aux points B' et I s'interceptent sur le segment AC.

OC437. Un magicien et son assistant ont un jeu de cartes. Les cartes ont toutes la même face arrière, mais leur face avant sont colorées en une des 2017 couleurs (il y a 1000000 cartes de chaque couleur). Le magicien et l'assistant présentent le tour suivant. Le magicien quitte la pièce ; des volontaires de l'audience placent $n > 1$ cartes en une rangée sur une table, face avant sur le dessus. L'assistant regarde les cartes, puis retourne toutes les cartes sauf une (sans changer l'ordre). Le magicien revient, regarde les cartes, pointe une des cartes retournées et dit sa couleur. Quel est le nombre minimal n pour que le magicien et son assistant aient une stratégie pour réussir le tour ?

OC438. Un professeur donne à ses étudiants la tâche suivante. Il les informe qu'il pense à un polynôme unitaire $P(x)$ de degré 2017 dont les coefficients sont entiers. Il leur nomme ensuite k entiers $n_1, n_2, \ldots, n_k$ et la valeur d'expression $P(n_1)P(n_2) \cdot \ldots \cdot P(n_k)$. Selon ces informations, les étudiants devraient trouver le polynôme du professeur. Trouver la plus petite valeur de k pour laquelle le professeur peut composer un tel problème de sorte que le polynôme trouvé par les étudiants soit nécessairement le même que celui auquel il a pensé.

OC439. Soit $(G, \cdot)$ un groupe et soit m et n deux nombres naturels différents de zéro qui sont copremiers. Prouvez que si les fonctions $f : G \to G$, $f(x) = x^{m+1}$ et $g : G \to G$, $g(x) = x^{n+1}$ sont des endomorphismes surjectifs, alors le groupe G est abélien.

OC440. Soit $f : [a,b] \to [a,b]$ une fonction différentiable dont la dérivée première est continue et positive. Prouvez qu'il existe $c \in (a,b)$ tel que

$$f(f(b)) - f(f(a)) = (f'(c))^2(b-a).$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(8), p. 324–325; 44(9): 370-371; 44(10): 412-413.

OC396. Prove that there are infinitely many positive integers m such that the number of odd distinct prime factors of $m(m+3)$ is a multiple of 3.

Originally Problem 5 from the Final Round of 2017 Italy Math Olympiad.

We received no submissions for this problem.

OC397. In a triangle ABC with $\angle A = 45^\circ$, draw the median AM. The line b is symmetrical to the line AM with respect to the altitude BB_1 and the line c is symmetrical to AM with respect to the altitude CC_1. The lines b and c intersect at the point X. Prove that $AX = BC$.

Originally Problem 6 from Grade 9 competition of the 2017 Moscow Math Olympiad.

We received 3 correct submissions. We present two solutions.

Solution 1, by Oliver Geupel.

Put the triangle onto a complex plane such that the unit circle (O) is circumscribed about triangle ABC and identify each point with the corresponding complex number. By the hypothesis $\angle A = 45^\circ$, there is no loss of generality in putting $B = -(1+i)/\sqrt{2}$ and $C = (1-i)/\sqrt{2}$. Then,

$$M = \frac{B+C}{2} = \frac{-i}{\sqrt{2}}, \qquad \bar{M} = \frac{i}{\sqrt{2}}.$$

Let (O) intersect the lines AM, BB_1, and CC_1 for the second time at points D, E, and F, respectively. Since A and D are complex numbers of absolute value 1, the equation of the line AD is $Z = A + D - AD\bar{Z}$. We put $Z = M$ and solve for D, obtaining

$$D = \frac{M - A}{1 - A\bar{M}} = \frac{A\sqrt{2} + i}{Ai - \sqrt{2}}.$$

From $\angle EOA = 2\angle EBA = 90^\circ$ and the similar relation $\angle AOF = 90^\circ$, we deduce

$$E = A/i, \qquad F = Ai.$$

Let

$$Y = \frac{-Ai}{A\sqrt{2} + i}.$$

We shall show that $Y = X$ and that $AY = BC$. Indeed,

$$AY^2 = (A - Y)\overline{(A - Y)} = \frac{A^2\sqrt{2} + 2Ai}{A\sqrt{2} + i} \cdot \frac{\sqrt{2}/A^2 - 2i/A}{\sqrt{2}/A - i} = 2 = BC^2.$$

The orthogonal projection of Y onto the chord BE is $P = (B + E + Y - BE\bar{Y})/2$. The mirror image Q of Y under reflection in the axis BE satisfies $Q - P = P - Y$. Hence,

$$Q = 2P - Y = B + E - BE\bar{Y} = \frac{1 - i - A^2 i}{A + i\sqrt{2}}.$$

Similarly, the mirror image of Y under reflection in the axis CF is

$$S = C + F - CF\bar{Y} = \frac{A^2 + 1 - i}{\sqrt{2} - Ai}.$$

Since

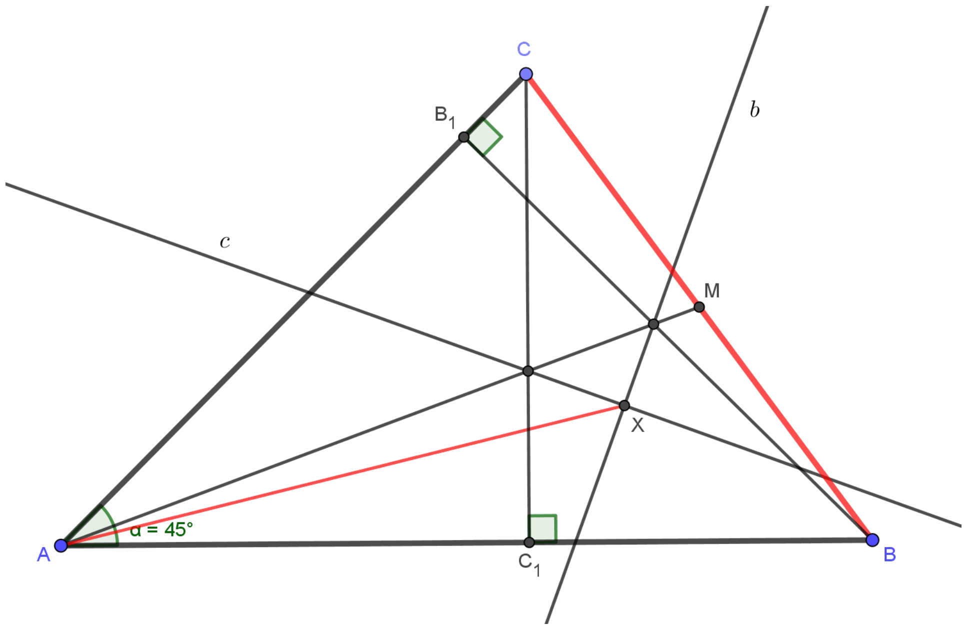
$$AD\bar{Q} = A \cdot \frac{A\sqrt{2} + i}{Ai - \sqrt{2}} \cdot \frac{1 + i + (i/A^2)}{(1/A) - i\sqrt{2}} = \frac{A^2 i - A^2 - 1}{Ai - \sqrt{2}} = A + D - Q,$$

the point Q belongs to the chord AD. Hence, the point Y lies on the line b. Analogously, $AD\bar{S} = A + D - S$, which implies that S belongs to AD, and Y lies on the line c. Consequently, $Y = X$ and $AX = AY = BC$.

Solution 2, by Andrea Fanchini.

We use barycentric coordinates with reference to the triangle ABC. Therefore, we have

$$AM : y - z = 0, \qquad BB_1 : S_A x - S_C z = 0, \qquad CC_1 : S_A x - S_B y = 0.$$



The line b, symmetrical to the line AM with respect to the altitude BB_1, is

$$b : 2S_Ax - (S_A + S_C)y + (S_A - S_C)z = 0$$

and the line c, symmetrical to AM with respect to the altitude CC_1, is

$$c : 2S_Ax + (S_A - S_B)y - (S_A + S_B)z = 0.$$

The lines b and c intersect at the point X

$$X = (a^2 : 2S_A + S_B - S_C : 2S_A - S_B + S_C),$$

and the distance between the points X and A is

$$\begin{aligned} AX^2 &= \frac{(S_A + S_B)(2S_A + S_B - S_C)^2 + 2S_A(2S_A + S_B - S_C)(2S_A - S_B + S_C)}{(4S_A + a^2)^2} \\ &+ \frac{(S_A + S_C)(2S_A - S_B + S_C)^2}{(4S_A + a^2)^2} \\ &= \frac{4S_A^2 + (S_B - S_C)^2}{4S_A + a^2}. \end{aligned}$$

Since $\angle A = 45°$, then $S_A = S$, and

$$AX^2 = \frac{4a^2S_A + (S_B + S_C)^2}{4S_A + a^2} = \frac{a^2(4S_A + a^2)}{4S_A + a^2} = a^2.$$

So $AX = BC = a$.

OC398. Detective Nero Wolfe is investigating a crime. There are 80 people involved in this case, among them one is the criminal and another is a witness of the crime (but it is not known who is who). Every day, the detective can invite one or more of these 80 people for an interview; if among the invited there is the witness, but there is no criminal, then the witness will tell who the criminal is. Can the detective solve the case in 12 days?

Originally Problem 3 of Grade 11 competition of the 2017 Moscow Math Olympiad.

We received 3 correct submissions. We present two solutions followed by a generalization of the question.

Solution 1, by Kathleen Lewis.

Yes, the detective can solve the case in 12 days.

Number the 80 people from 1 to 80, and convert all 80 numbers to base 3. All base 3 numbers have 4 or fewer digits. Add leading zeroes to the smaller numbers so that all of them have 4 digits. Then Detective Wolfe's strategy is as follows: the detective invites all people whose ones digit is 0 on day 1, all people whose ones digit is 1 on day 2, and all people whose ones digit is 2 on day 3. He continues using the threes, nines and 27s digits. This process will take him exactly $3 \times 4 = 12$ days. Since any two people must have numbers that differ in at least one digit, there must be at least one group that contains the witness but not the criminal.

Solution 2 and generalization by Oliver Geupel.

Yes, the detective can solve the case, even in 9 days. We propose a strategy that solves, for any natural number n, a case with a number $p \leq \binom{n}{\lfloor n/2 \rfloor}$ of people in n days. Since p is less than or equal to the number of subsets of size $\lfloor n/2 \rfloor$ of a set of size n, we can use these subsets to construct a matrix $A = (a_{ik})$ of size $p \times n$ with the following properties. All entries of A are either 1 or 0. Each row has exactly $\lfloor n/2 \rfloor$ entries that are equal to 1. Any two rows of A are not the same.

The matrix A is used to build our strategy. Identify the p people with the integers $1, 2, \ldots, p$. For each $1 \leq k \leq n$, the people that are invited on the k-th day are those people with numbers i such that $a_{ik} = 1$. By hypothesis, for every pair $i \neq j$ the rows i and j are not the same, and hence, there are column indices k and ℓ such that $a_{ik} = a_{j\ell} = 0$ and $a_{i\ell} = a_{jk} = 1$. Then, among those invited at day k, there is the person j but not the person i, and among those invited at day ℓ, there is the person i but not the person j. Thus, on some day, the witness but not the criminal is invited.

In our case we have 80 people and

$$80 < \binom{9}{\lfloor 9/2 \rfloor} = 126,$$

hence the detective has a strategy to find the criminal in 9 days.

OC399. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{P}$ if for any sequence of real numbers $(x_n)_{n\geq 1}$ such that the sequence $(f(x_n))_{n\geq 1}$ converges, then also the sequence $(x_n)_{n\geq 1}$ converges. Prove that a surjective function with property $\mathcal{P}$ is continuous.

Originally Problem 1 of Grade 11 competition of the 2017 Romania Math Olympiad.

We received 2 correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We start by establishing three facts.

Fact 1. Let $g : \mathbb{R} \to \mathbb{R}$. If g is continuous and injective, then it is strictly monotone.

We prove Fact 1 by contradiction. Assume g is continuous and injective, but not strictly monotone. Without loss of generality, assume there exists $a < b < c$ with $g(a) < g(c) < g(b)$. Let $k \in \mathbb{R}$ satisfy $g(a) < g(c) < k < g(b)$. Then by the Intermediate Value Theorem, there exists $a < x_1 < b < x_2 < c$ with $f(x_1) = f(x_2) = k$. This contradicts the assumption that g is injective. Therefore g must be strictly monotone. $\square$

Fact 2. Let $g : \mathbb{R} \to \mathbb{R}$. If g is monotone and surjective, then it is continuous.

We prove Fact 2 by contradiction. Assume g is monotone and surjective, but discontinuous at a. Without loss of generality, we can futher assume that g is increasing. Since g is monotone, the discontinuity must be a jump discontinuity and

$$\alpha = \sup\{g(x) : x < a\} < \inf\{g(x) : x > a\} = \beta.$$

Since $g : \mathbb{R} \to \mathbb{R}$ is surjective, there at least two distinct real numbers s and t such that $\alpha < g(s) < g(t) < \beta$. However, this contradicts that only $g(a)$ can possibly be between α and β. Therefore g must be continuous. $\square$

Fact 3. Given any two sequences (a_n) and (b_n), define the sequence

$$(a_n \sqcup b_n) := (a_1, b_1, a_2, b_2, a_3, b_3, \dots).$$

The sequence $(a_n \sqcup b_n)$ converges if and only if both sequences (a_n) and (b_n) converge to the same limit.

The proof of Fact 3 is trivial and is not included here.

Using these three facts, we proceed to prove the main question. Assume $f : \mathbb{R} \to \mathbb{R}$ is surjective and has property $\mathcal{P}$.

Let a and b be two real numbers such that $f(a) = f(b)$. Define the constant and convergent sequence

$$(f(a) \sqcup f(b)) = (f(a), f(b), f(a), f(b), \dots).$$

By property $\mathcal{P}$, the sequence $(a \sqcup b)$ also converges. Then $a = b$, and so f is injective. Now f is a bijection with inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$.

Assume the sequence (y_n) converges to y, and let $x_n = f^{-1}(y_n)$ and $x = f^{-1}(y)$. Then $(f(x_n) \sqcup f(x)) = (y_n \sqcup y)$ converges, and by property $\mathcal{P}$, $(x_n \sqcup x)$ converges. Using Fact 3, it follows that

$$\lim_{n\to\infty} f^{-1}(y_n) = \lim_{n\to\infty} x_n = x = f^{-1}(y),$$

and so f^{-1} is continuous. Now using Fact 1 for the continuous and injective function f^{-1} we get that f^{-1} is monotone. Then f must be monotone, as well. Lastly, using Fact 2 for the monotone and surjective function f we obtain that f is continuous, as required.

OC400. Let G be a finite group having the following property: for any automorphism f of G, there exists a natural number m such that $f(x) = x^m$ for all $x \in G$. Prove that G is abelian.

Originally Problem 3 of Grade 12 competition of the 2017 Romania Math Olympiad.

We received 1 correct submission by Oliver Geupel which is presented next.

Let n be the order of G. Let $o(g)$ denote the order of an element g of G. For any prime number p, let H_p be the set of all those elements of G, whose order is a divisor of p^n, that is, $H_p = \{g \in G : o(g) \mid p^n\}$.

Fact 1. Let $g, h \in G$ such that $o(g) \mid o(h)$. Then, g and h commute.

Proof. The conjugation by h, $x \mapsto hxh^{-1}$, is an automorphism of G. By hypothesis, there is a natural number m such that $hxh^{-1} = x^m$ for every $x \in G$. Putting $x = h$, we find that $h = h^m$; whence $o(h) \mid m-1$ and $o(g) \mid m-1$. Putting $x = g$, we obtain $hgh^{-1} = g^m = g$ and thus $hg = gh$. □

Fact 2. H_p constitutes an abelian subgroup of G for every prime number p.

Proof. The identity element e which has order 1, is in H_p. For every element g of H_p, its inverse g^{-1} is of the same order as g; whence $g^{-1} \in H_p$. For any two elements of H_p, the order of one element divides the order of the other one by the definition of H_p. Hence, by Fact 1, the two elements commute. Thus, for $g, h \in H_p$, we have $(gh)^{p^n} = g^{p^n} h^{p^n} = e$, which implies $gh \in H_p$. □

Fact 3. Let p and q be distinct primes and let $g \in H_p$ and $h \in H_q$. Then, g and h commute.

Proof. Since $(hgh^{-1})^{p^n} = e$, it holds $hgh^{-1} \in H_p$. Applying Fact 2, we obtain $g^{-1} \in H_p$ and $hgh^{-1}g^{-1} = (hgh^{-1})g^{-1} \in H_p$. Analogously, we deduce $hgh^{-1}g^{-1} \in H_q$. But $H_p \cap H_q = \{e\}$. It follows that $hgh^{-1}g^{-1} = e$. Consequently, $hg = gh$. □

We are now prepared for the proof that G is abelian. Let $n = \prod_{k=1}^{\ell} p_k^{a_k}$ be the canonical factorisation of n into primes. Then, the numbers $d_k = n/p_k^{a_k}$ are coprime for $k = 1, \ldots, \ell$. Hence, there exist integers $c_1, \ldots, c_\ell$ such that

$\sum_{k=1}^{\ell} c_k d_k = 1$. Let $g, h \in G$. It holds

$$g = \prod_{k=1}^{\ell} (g^{d_k})^{c_k}, \quad h = \prod_{k=1}^{\ell} (h^{d_k})^{c_k},$$

where $g^{d_k}, h^{d_k} \in H_{p_k}$, $k = 1, \ldots, \ell$. From Facts 2 and 3, we finally conclude $gh = hg$. This proves that G is an abelian group.

OC401. Determine all polynomials $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:

(a) $P(2017) = 2016$;

(b) $(P(x) + 1)^2 = P(x^2 + 1)$ for all real numbers x.

Originally Problem 1 from Final Round of the 2017 Austria Math Olympiad.

We received 6 *submissions. We present the solution by Oliver Geupel.*

A solution is $P(x) = x - 1$, and we prove that it is unique. Suppose $P(x)$ is a solution. Define a sequence $(x_n)_{n \in \mathbb{N}}$ by the recursion

$$x_1 = 2017, \qquad x_{n+1} = x_n^2 + 1 \quad (n \in \mathbb{N}).$$

We show by mathematical induction that $P(x_n) = x_n - 1$ holds for every positive integer n. The base case $n = 1$ is settled by condition (a). Assuming that for some specific n we have $P(x_n) = x_n - 1$, it follows by condition (b) that

$$P(x_{n+1}) = P(x_n^2 + 1) = (P(x_n) + 1)^2 = x_n^2 = x_{n+1} - 1,$$

which completes the induction. As a consequence, the infinitely many numbers $x_1, x_2, x_3, \ldots$ are roots of the polynomial $Q(x) = P(x) - x + 1$. Hence, Q is the null polynomial and $P(x) = x - 1$.

Editor's Comments. Walther Janous, who is also the author of this problem, investigated the case in which only condition (b) holds. We present his analysis.

With the substitution $Q(x) := P(x) + 1$, the functional equation given in (b) becomes

$$Q(x^2 + 1) = ((Q(x))^2 + 1, \qquad x \in \mathbb{R},$$

which we call (FG). From now on, we will look exclusively at (FG). Assuming that $Q(x) = a + bx + cx^2 + \cdots$, by a coefficient comparison, we get for small degrees of Q the following polynomials:

$$\begin{aligned} Q_0(x) &= x & &\text{if } \deg Q = 1 \\ Q_1(x) &= x^2 + 1 & &\text{if } \deg Q = 2 \\ Q_2(x) &= x^4 + 2x^2 + 2 & &\text{if } \deg Q = 4 \\ Q_3(x) &= x^8 + 4x^6 + 8x^4 + 8x^2 + 5 & &\text{if } \deg Q = 8 \end{aligned}$$

as solutions of the functional equation (FG).

There are no solutions Q with $\deg Q \in \{0,3,5,6,7\}$. In addition one recognizes that these polynomials satisfy the recurrence relation $Q_{n+1}(x) = (Q_n(x))^2+1$, $x \in \mathbb{R}$ for $n \in \{0,1,2\}$. Conversely, every polynomial $Q_n(x)$ satisfying this recurrence relation with $Q_0(x) = x$ is a solution of (FG). This is evident for $Q_0(x)$. Let $Q_n(x)$ be a solution of (FG). Then, in particular $Q_n(x^2+1) = (Q_n(x))^2+1$, so also $(Q_n(x^2+1))^2+1 = ((Q_n(x))^2+1)^2+1$, i.e. $Q_{n+1}(x^2+1) = (Q_{n+1}(x))^2+1$. Thus, $Q_{n+1}(x)$ is also a solution of (FG). For the general solution of the functional equation (FG) we need the following three lemmas.

Lemma. Let $P(x) \in \mathbb{R}[x]$ be a polynomial with $P(0) = 0$ and let f be a real-valued function with $f(x) > x$ for every $x \in \mathbb{R}$. Then, the functional equation $P(f(x)) = f(P(x))$, $x \in \mathbb{R}$, has the polynomial $P(x) = x$, $x \in \mathbb{R}$, as the only solution.

Proof. We define the sequence $(x_n)_{n\geq 0}$ recursively by $x_0 = 0$ and $x_{n+1} = f(x_n)$, $n \geq 0$. We prove by induction that $P(x_n) = x_n$, $n \geq 0$. By hypothesis, we have $P(0) = 0$, i.e $P(x_0) = x_0$. Let $P(x_k) = x_k$, where $k \geq 0$. Then, $P(x_{k+1}) = P(f(x_k)) = f(P(x_k)) = f(x_k) = x_{k+1}$. Moreover, $x_{k+1} = f(x_k) > x_k$, $k \geq 0$. This gives us a sequence of points $x_0 < x_1 < x_2 < \dots$ for which the polynomials $P(x)$ and $\mathrm{id}(x) = x$ coincide at all points of the sequence. So, it must be $P(x) = x$, $x \in \mathbb{R}$, and the conclusion follows. $\square$

Lemma. All polynomials Q that satisfy the functional equation (FG) are either even or odd.

Proof. Since $(Q(-x))^2 = Q((-x)^2+1) - 1 = Q(x^2+1) - 1 = ((Q(x))^2$, $x \in \mathbb{R}$, then for every $x \in \mathbb{R}$ we have

$$Q(-x) = Q(x) \qquad \text{or} \qquad Q(-x) = -Q(x).$$

That is, at least one of these two relations is fulfilled for an infinite number of $x \in \mathbb{R}$. Since Q is a polynomial, therefore, either $Q(-x) = Q(x)$, $x \in \mathbb{R}$ or $Q(-x) = -Q(x)$, $x \in \mathbb{R}$. So, the polynomial Q is either even or odd. $\square$

Lemma. If a polynomial Q with $Q(0) \neq 0$ is the solution of the functional equation (FG), then there exists a polynomial S, with $\deg S = \dfrac{1}{2}\deg Q$, which also satisfies (FG), where $Q(x) = S(x^2+1)$, $x \in \mathbb{R}$.

Proof. The second lemma and $Q(0) \neq 0$ show that Q must be even, i.e. $Q(x) = R(x^2)$, $x \in \mathbb{R}$, with $R \in \mathbb{R}[x]$. Therefore, the functional equation (FG) can be written in the form $R((x^2+1)^2) = (R(x^2))^2+1$, $x \in \mathbb{R}$. The variable substitution $\xi := x^2+1$ gives

$$R(\xi^2) = (R(\xi-1))^2+1 \implies R((\xi^2+1)-1) = (R(\xi-1))^2+1$$

for all $\xi \in [1,\infty)$. Since R is a polynomial, this relation holds even for every $\xi \in \mathbb{R}$. Therefore, with the function substitution $S(z) := R(z-1)$, $z \in \mathbb{R}$, we get $S(\xi^2+1) = (S(\xi))^2+1$ for $\xi \in \mathbb{R}$, so S is also a solution of (FG). $\square$

Let us go back to the solution of the functional equation (FG). We show by induction that for $n \geq 0$ there exists exactly one polynomial Q, with $2^n \leq \deg Q < 2^{n+1}$, which is a solution of (FG), namely Q_n. For $n = 0$, that is $\deg Q = 1$, the assertion is proved by setting $Q(x) = ax + b$ and by comparison of coefficients.

We assume that the statement is true for $n \geq 0$ and show that it then also holds for $n+1$. Let Q be a polynomial with $2^{n+1} \leq \deg Q < 2^{n+2}$ which is a solution of (FG). We have two cases.

(i) $Q(0) = 0$. Then, the first lemma applied to the function $f(x) = x^2 + 1$, $x \in \mathbb{R}$ that satisfies $f(x) > x$, $x \in \mathbb{R}$, implies that $Q(x) = x$, $x \in \mathbb{R}$. But this is not possible because $\deg Q \geq 2$.

(ii) $Q(0) \neq 0$. Then, by the second lemma Q must be even. By the third lemma, we have $Q(x) = S(x^2 + 1)$, where S satisfies the functional equation (FG), $\deg S = \dfrac{1}{2} \deg Q$ and $2^n \leq \deg S < 2^{n+1}$. By the induction hypothesis, $S = Q_n$ and thus $Q = Q_{n+1}$.

The general solutions to the functional equation considered in part (b) of the problem statement are therefore $P_n(x) = Q_n(x) - 1$. If $Q_0(2017) = 2017$, we obtain $Q_{n+1}(x) = (Q_n(x))^2 + 1 > (Q_n(x))^2$, $x \in \mathbb{R}$, $n \geq 0$, which gives immediately $Q_n(2017) > 2017$ for all $n \geq 1$. Therefore, $P(x) = x - 1$, $x \in \mathbb{R}$, is the only polynomial that satisfies the two conditions of the problem.

OC402. Find all natural numbers n that satisfy the following property: for each integer $k \geq n$ there is a multiple of n whose digits sum up to k.

Originally Problem 5 from Grade 10 competition of the 2017 Moscow Math Olympiad.

We received no submissions to this problem.

OC403. Let S be the point of tangency of the incircle of a triangle ABC with the side AC. Let Q be a point such that the midpoints of the segments AQ and QC lie on the incircle. Prove that QS is the angle bisector of $\angle AQC$.

Originally Problem 2 from Grade 11 competition of the 2017 Moscow Math Olympiad.

We received no submissions to this problem.

OC404. Let $(A, +, \cdot)$ be a ring simultaneously satisfying the conditions:

(i) A is not a division ring;

(ii) $x^2 = x$ for every invertible element $x \in A$.

Prove that:

(a) $a + x$ is not invertible for any $a, x \in A$, where a invertible and $x \neq 0$ is not invertible;

(b) $x^2 = x$ for all $x \in A$.

Originally Problem 4 from Grade 12 competition of the District Round of the 2017 Romania Math Olympiad.

We received 1 *submission. We present the solution by the Missouri State University Problem Solving Group.*

We claim that condition (a) holds regardless of whether A is a division ring and condition (b) does *not* follow from conditions (i) and (ii). We replace condition (i) with one that will (along with condition (ii)) imply condition (b).

If x is invertible and $x^2 = x$, then $x = 1$. Therefore the only invertible element of A is 1. If a is invertible, then $a = 1$ and if $x \neq 0$ (whether invertible or not), $a + x = 1 + x \neq 1$ and hence $a + x$ is not invertible, so condition (a) is satisfied.

Let $A = \mathbb{F}_2[t]$. Clearly A is not a division ring and the only invertible element of A is the only invertible element in $\mathbb{F}_2$, namely 1, which satisfies $1^2 = 1$. However, $t^2 \neq t$.

If the additional criterion that A is finite is added, then condition (b) follows from condition (ii) alone.

The characteristic subring of A must be isomorphic to $\mathbb{F}_2$, otherwise -1 would be an invertible element distinct from 1. Clearly $0^2 = 0$ and $1^2 = 1$. Choose any element $x \in A, x \neq 0, 1$. The subring of A generated by x and 1 is isomorphic to $\mathbb{F}_2[t]/(f(t))$ (via the map sending t to x), where $f(t)$ is a polynomial of degree greater than 1 (we need the fact that A is finite to guarantee that $f(t)$ is not the zero polynomial). Let

$$f(t) = \prod_{i=1}^{k} p_i(t)^{m_i}$$

be the factorization of $f(t)$, where $p_i(t) \neq p_j(t)$ for $i \neq j$. By the Chinese Remainder Theorem,

$$\mathbb{F}_2[t]/(f(t)) \cong \prod_{i=1}^{k} \mathbb{F}_2[t]/(p_i(t)^{m_i}).$$

If $m_i > 1$ for any i, then $1 - p_i(t)$ is a non-trivial unit in $\mathbb{F}_2[t]/(p_i(t)^{m_i})$, which gives a non-trivial unit in $\mathbb{F}_2[t]/(f(t))$, so this cannot occur and $m_i = 1$ for all i. If $\deg(p_i) > 1$ for any i, then

$$\mathbb{F}_2[t]/(p_i(t)^{m_i}) = \mathbb{F}_2[t]/(p_i(t))$$

is a field of order greater than 2 and hence has non-trivial units. Therefore, $f(t)$ must be a product of distinct linear factors. But there are only two linear polynomials over $\mathbb{F}_2$, so $f(t) = t(t-1)$ and hence $x^2 = x$.

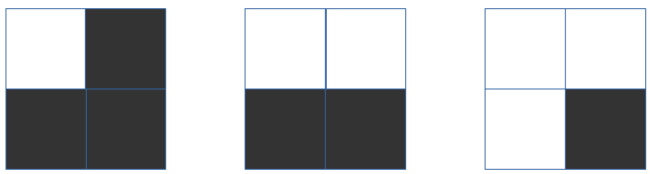
We end by classifying all finite rings A satisfying condition (ii). Since $x^2 = x$ for all $x \in A$, $a + b = (a + b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$. Therefore $ab = -ba = ba$ (since the characteristic of A is 2) and A is a commutative ring. All finite commutative rings are products of local rings. If any of these local rings is not isomorphic to $\mathbb{F}_2$, we will have a non-trivial unit leading to a contradiction. Hence $A \cong \mathbb{F}_2^k$.

OC405. Each cell of a 100×100 table is painted either black or white and all the cells adjacent to the border of the table are black. It is known that in every 2×2 square there are cells of both colours. Prove that in the table there is a 2×2 square that is coloured in the chessboard manner.

Originally Problem 8 from Grade 9 competition of the 2017 Russia Math Olympiad.

We received 1 *submission. We present the solution by Oliver Geupel.*

We prove the property for an $m \times n$ table where m and n are even numbers greater than 2. The proof is by contradiction. Assume there is no 2×2 chessboard. Then, up to rotation about the centre, there are only the following three types of 2×2 squares:



Consider sides of cells that separate a black cell from a white one. Those sides can be arranged in disjoint closed nonintersecting lattice paths. Traverse any such lattice path, starting at a lattice point and arriving at the same point. Write L, R, U, D every time you pass a single side to the left, right, up and down, respectively. For a closed lattice path, the number of L's is the same as the number of R's, and the number of U's is equal to the number of D's. Hence, the lattice path consists of an even number of sides. Therefore, every such lattice path connects an even number of lattice points. As a consequence, the total number of lattice points that is part of any such lattice path is even.

Since in every 2×2 square there are cells of both colours, every interior lattice point of the table belongs to exactly one of our lattice paths. But the number of interior lattice points is $(m-1) \times (n-1)$, which is an odd number. This is the desired contradiction, which completes the proof.

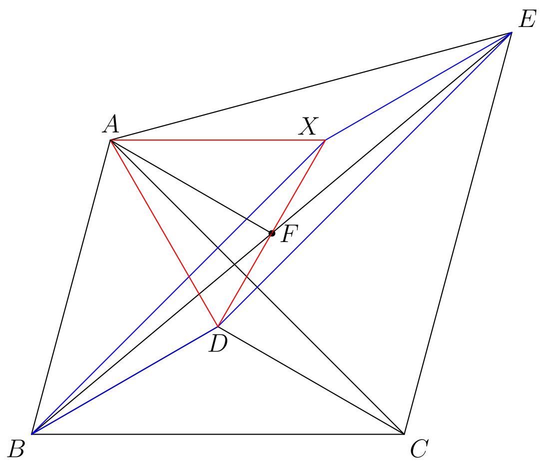
OC406. Let D be a point inside the triangle ABC such that $BD = CD$ and $\angle BDC = 120^\circ$. Let E be a point outside the triangle ABC such that $AE = CE$, $\angle AEC = 60^\circ$ and points B and E are in different half-planes with respect to AC. Prove that $\angle AFD = 90^\circ$, where F is the midpoint of the segment BE.

Originally Problem 2 from Grade 11 competition of the 2017 Moscow Math Olympiad.

We received 5 *submissions. We present two solutions.*

Solution 1, by Sushanth Sathish Kumar.

Let X be a point such that $DBXE$ is a parallelogram. Then, F is the midpoint of XD. Therefore, it is enough to show that A lies on the perpendicular bisector of XD, or that $AX = AD$.



Note that $AE = AC$, and $XE = BD = DC$, by construction. We claim that Ψ, the $60°$ rotation about A, maps D to X. Note that this will imply $AX = AD$, and we will be done. Clearly, under Ψ, C maps to E, since $\angle CAE = 60°$. To show D maps to X, note that the angle formed by lines CD and EX is the same as the angle formed by lines CD and DB, which is $180° - \angle BDC = 60°$. Hence, the claim is proven, and we are done.

Remark. Note that the above proof additionally shows that $\angle DAX = 60°$, which implies that triangle AFD is actually a 30-60-90 triangle, with the $30°$ angle at A.

Solution 2, by Ivko Dimitrić

We use a combination of vectors and complex numbers. Consider the given triangle ABC in the plane of complex numbers with vertices labeled counterclockwise and point D at the origin. Represent each point by a complex number denoted by the same capital letter. Each vector $\overrightarrow{XY}$ is represented by a complex number $Y - X$. In particular, $\overrightarrow{AC} = C - A$. Since $\overrightarrow{DC}$ is the rotation of $\overrightarrow{DB}$ through the angle $2\pi/3$, we have $C = B\,e^{i(2\pi/3)}$. Since $\triangle ACE$ is isosceles $(EA = EC)$ with vertex angle of $60°$ at E, it is, in fact, equilateral and $\angle EAC = 60°$. Hence, $\overrightarrow{AE}$ is the result of rotation of $\overrightarrow{AC}$ through $\pi/3$. Hence,

$$E - A = (C - A)\,e^{i(\pi/3)} \quad \Longrightarrow \quad E = A + (C - A)\,e^{i(\pi/3)}.$$

Then,

$$\begin{aligned} 2\overrightarrow{DF} &= \overrightarrow{DB} + \overrightarrow{DE} \\ &= B + E \\ &= B + A + (C - A)e^{i(\pi/3)} \\ &= B + A + (Be^{i(2\pi/3)} - A)\,e^{i(\pi/3)} \\ &= A(1 - e^{i(\pi/3)}) \end{aligned}$$

Similarly,

$$\begin{aligned}2\overrightarrow{AF} &= \overrightarrow{AB} + \overrightarrow{AE}\\ &= (B-A)+(E-A)\\ &= B-A+(C-A)e^{i(\pi/3)}\\ &= B-A+\left(Be^{i(2\pi/3)}-A\right)e^{i(\pi/3)}\\ &= -A(1+e^{i(\pi/3)}).\end{aligned}$$

The dot product of two vectors, represented by complex numbers $v = 2\overrightarrow{DF}$ and $w = 2\overrightarrow{AF}$ is $\langle v, w\rangle = \frac{1}{2}(v\bar{w}+\bar{v}w)$, which is zero if and only if the two vectors are perpendicular. In our case,

$$\begin{aligned}\langle v, w\rangle &= \frac{1}{2}\Big[A(1-e^{\frac{\pi}{3}i})(-\bar{A})(1+e^{-\frac{\pi}{3}i}) + \bar{A}(1-e^{-\frac{\pi}{3}i})(-A)(1+e^{\frac{\pi}{3}i})\Big]\\ &= -\frac{1}{2}\,A\bar{A}\left(1+e^{-\frac{\pi}{3}i}-e^{\frac{\pi}{3}i}-e^{0}+1+e^{\frac{\pi}{3}i}-e^{-\frac{\pi}{3}i}-e^{0}\right)\\ &= -\frac{1}{2}\,A\bar{A}\cdot 0\\ &= 0.\end{aligned}$$

Hence, vectors $\overrightarrow{AF}$ and $\overrightarrow{DF}$ are perpendicular i. e. $\angle AFD = 90^\circ$.

OC407. The acute isosceles triangle ABC ($AB = AC$) is inscribed in a circle with center O. The rays BO and CO intersect the sides AC and AB in the points B' and C', respectively. A line l parallel to the line AC passes through point C'. Prove that the line l is tangent to the circumcircle ω of the triangle $B'OC$.

Originally Problem 3 from Grade 10 competition of the 2017 Russia Math Olympiad.

We received 5 *submissions. We present the solution by Sushanth Sathish Kumar.*

Let X be the reflection of A over $B'C'$. Given concyclic points P_1,P_2,...,P_n, let $(P_1P_2...P_n)$ denote the circle passing through them.

By construction A, O, X are collinear and X lies on l. We claim that l is tangent to $(B'OC)$ at X. Note that O is the circumcenter of triangle $XB'C'$. Indeed, the homothety at O taking BC to $B'C'$ also takes X to A, since $XC'||AC$ and $XB'||AB$. The same argument shows triangles OXC' and OAC are similar. Invoking symmetry,

$$\angle OXC' = \angle OAC = \angle OCA = \angle OBA = \angle OBC',$$

which shows that X lies on $(B'OC)$. To show tangency, note that

$$\angle B'XO = \angle OXC' = \angle XC'O = \angle XBO = \angle XBB',$$

and we are done.

OC408. Does there exist an infinite increasing sequence $a_1, a_2, a_3, \ldots$ of positive integers such that the sum of any two distinct terms of the sequence is coprime with the sum of any three distinct terms of the sequence?

Originally Problem 4 from Grade 9 competition of the 2017 Moscow Math Olympiad.

We received 1 *submission. We present the solution by Oliver Geupel.*

The answer is yes. We are going to prove that the sequence with $a_1 = 7$ and $a_n = (3a_{n-1})! + 1$ for $n > 1$ has the desired property. Note that all members of the sequence are odd. Also, for every $n \geq 2$ and integers u, v with the property $1 \leq v \leq 3a_{n-1}$, it holds that $a_n \equiv 1 \pmod{v}$; whence

$$(u,\ v) = (u + 1 - a_n,\ v).$$

We have to show that, for any indices i, j, k, ℓ, m with the property $i < j < k$ and $\ell < m$, it holds that $(a_i + a_j + a_k,\ a_\ell + a_m) = 1$.

First, consider the case where $|\{i, j, k, \ell, m\}| = 3$. Let a, b, c be members of (a_n) such that $a < b < c$. Then,

$$\begin{aligned}
(a+b+c,\ a+b) &= (a+b+1,\ a+b) = 1,\\
(a+b+c,\ a+c) &= (b,\ a+c) = (b,\ a+1) = (1,\ a+1) = 1,\\
(a+b+c,\ b+c) &= (a,\ b+c) = (a,\ b+1) = (a,\ 2) = 1,
\end{aligned}$$

which completes the case $|\{i, j, k, \ell, m\}| = 3$.

Next, consider the case where $|\{i, j, k, \ell, m\}| = 4$. Let a, b, c, d be members of (a_n) such that $a < b < c < d$. Then,

$$\begin{aligned}
&(a+b+c,\ a+d) = (a+b+c,\ a+1) = (a+2,\ a+1) = 1,\\
&(a+b+c,\ b+d) = (a+b+c,\ b+1) = (a+b+1,\ b+1) = (a,\ b+1) = 1,\\
&(a+b+c,\ c+d) = (a+b+c,\ c+1) = (a+b-1,\ c+1) = (a+b-1,\ 2) = 1,\\
&(a+b+d,\ a+c) = (a+b+1,\ a+c) = (a+2,\ a+1) = 1,\\
&(a+b+d,\ b+c) = (a+b+1,\ b+c) = (a+b+1,\ b+1) = 1,\\
&(a+b+d,\ c+d) = (a+b+d,\ c-a-b) = (a+b+1.\ 1-a-b) = 1,\\
&(a+c+d,\ a+b) = (a+2,\ a+1) = 1,\\
&(a+c+d,\ b+c) = (a+c+1,\ b+c) = (a+c+1,\ b-a-1) = (a+2,\ a) = 1,\\
&(a+c+d,\ b+d) = (a-b+1,\ b+1) = (a+2,\ b+1) = (a+2,\ 2) = 1,\\
&(b+c+d,\ a+b) = (b+2,\ a+b) = (b+2,\ a-2) = (3,\ a-2) = 1,\\
&(b+c+d,\ a+c) = (b+c+1,\ a+c) = (b+1-a,\ a+1) = (b+2,\ a+1) = 1,\\
&(b+c+d,\ a+d) = (b+c-a,\ a+1) = (2-a,\ a+1) = 1,
\end{aligned}$$

which completes the case $|\{i, j, k, \ell, m\}| = 4$.

Finally, consider the case where $|\{i, j, k, \ell, m\}| = 5$. Let a, b, c, d, e be members of (a_n) such that $a < b < c < d < e$. For every arrangement of the numbers

a, b, c, d, e as a sum of three and a sum of two numbers, we can successively reduce the numbers e, d, c, b, a in the gcd to 1, obtaining $(3, 2) = 1$, for example:

$$(a + d + e,\ b + c) = (a + d + 1,\ b + c) = (a + 2,\ b + c) = (a + 2,\ 2) = (3, 2) = 1.$$

This completes the proof that (a_n) has the required property.

OC409.

(a) Give an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1$$

and $f(x)/x$ has no limit as $x \to \infty$.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be an increasing function such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1.$$

Prove that $f(x)/x$ has a limit as $x \to \infty$ and determine this limit.

Originally Problem 4 from Grade 12 competition of the 2017 Romania Math Olympiad.

We received 1 *submission. We present the solution by Omran Kouba.*

(a) First, consider $f(x) = x \sin x + 2x$. Clearly $f(x)/x$ has no limit as $x \to \infty$, while

$$\frac{1}{x^2} \int_0^x f(t)\, dt = 1 + \frac{\sin x - x \cos x}{x^2}$$

and this tends to 1 as $x \to \infty$.

(b) Let

$$G(x) = \frac{1}{x^2} \int_0^x f(t) dt.$$

For $0 < u < v$, and because f is increasing we have

$$(v - u)f(u) \le \int_u^v f(t) dt \le (v - u)f(v),$$

or

$$f(u) \le \frac{v^2 G(v) - u^2 G^2(u)}{v - u} \le f(v),$$

Now, consider $\lambda > 1$, $x > 0$ and apply the previous inequality with $(u, v) = (x, \lambda x)$ and $(u, v) = (x/\lambda, x)$ we conclude that

$$f(x) \le x \frac{\lambda^2 G(\lambda x) - G^2(x)}{\lambda - 1}, \quad \text{and} \quad x \frac{G(x) - G^2(x/\lambda)/\lambda^2}{1 - 1/\lambda} \le f(x)$$

Equivalently

$$\frac{\lambda^2 G(x) - G^2(x/\lambda)}{\lambda(\lambda-1)} \leq \frac{f(x)}{x} \leq \frac{\lambda^2 G(\lambda x) - G^2(x)}{\lambda - 1}.$$

Recalling that $\lim_{x\to\infty} G(x) = 1$ we conclude that, for all $\lambda > 1$ we have

$$1 + \frac{1}{\lambda} \leq \liminf_{x\to\infty} \frac{f(x)}{x} \leq \limsup_{x\to\infty} \frac{f(x)}{x} \leq 1 + \lambda$$

But $\lambda > 1$ is arbitrary. So,

$$2 = \liminf_{x\to\infty} \frac{f(x)}{x} \leq \limsup_{x\to\infty} \frac{f(x)}{x} = 2,$$

i.e.

$$\lim_{x\to\infty} \frac{f(x)}{x} = 2$$

which is the desired conclusion.

OC410. Let $a_0, a_1, \ldots, a_{10}$ be integers such that $a_0 + a_1 + \cdots + a_{10} = 11$. Find the maximum number of distinct integer solutions to the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$

Originally Problem 1 of Category 3 of the 2017 Hungary Math Olympiad.

We received no submissions to this problem.

The Orthocentric Distances

Sushanth Sathish Kumar

In this article, I introduce some distances related to the orthocenter, and their applications in the resolutions of challenging geometry problems. The most curious application of the distances is an alternate proof of the existence of the nine point circle in a triangle. If you are unfamiliar with the classic proof of this theorem, trying to prove all nine points are concyclic might seem like a nightmare. In Section 3, I give an intuitive proof of it using the results developed in Section 2.

1 What are the distances?

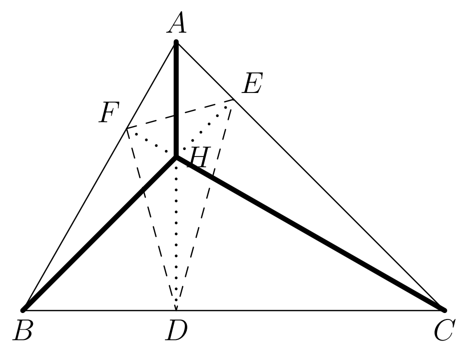
Theorem 1.1 (Orthocentric Distances, [3]) *Let ABC be any triangle with orthocenter H. Denote by D, E, F the feet of the A-, B-, C- altitudes, respectively. If we invoke the notion of directed lengths, then the following hold:*

a) $HA = 2R\cos A$,

b) $HD = 2R\cos B\cos C$,

c) $EF = a\cos A$

We invite the reader to prove the theorem. Note that invoking the notion of directed lengths does not restrict the distances to just acute triangles. Pictorially, the lemma gives these lengths,



It is worth noting a few facts that follow immediately.

Corollary 1.2 *In $\triangle ABC$ with orthocenter H and feet of altitudes D, E, F, we have $(HA)(HD) = (HB)(HE) = (HC)(HF)$.*

Corollary 1.3 *The power of orthocenter H with respect to the circumcircle of $\triangle ABC$ is $8R^2\cos A\cos B\cos C$.*

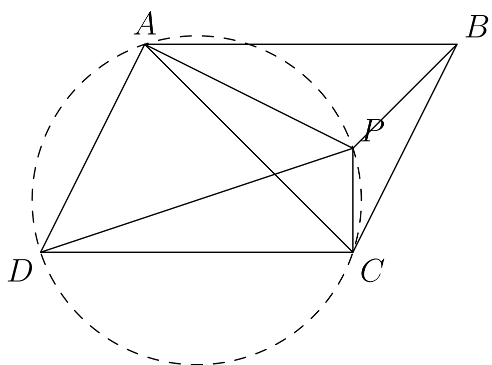
Corollary 1.3 follows from noting that the reflection of H over the sides of $\triangle ABC$ lie on its circumcircle. Looking back at the diagram, we see that A is the orthocenter of HBC, B is the orthocenter of HCA and that C is the orthocenter of HAB. For this reason, the points A, B, C, H are said to form an *orthocentric tetrad*.

Remark *The formula $AD = bc/2R$ is much more useful than the "areal" length of AD for problems using these distances.*

2 Examples

Here we will present instructive problems that succumb to the formulas described.

Example 2.1 (CMIMC 2016) In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ while $PD = 821$ what is AC?



Solution. Perhaps the only difficulty in solving this problem is finding the right triangle to apply our lemma on. A natural candidate is $\triangle ABC$ since two of its altitudes are already drawn in. Letting R be the circumradius, our formulas give

$$PB = 2R\cos B, PC = 2R\cos C, PA = 2R\cos A.$$

The only length we need now is PD. But we can easily get this from Ptolemy's Theorem on $PADC$! That gives

$$(PD)(AC) = (PA)(CD) + (PC)(AD),$$

and plugging in the distances, we see the right hand side is just $2Rb$. Hence $2R = 821$. Also,

$$PB = 2R\cos B = 700,$$

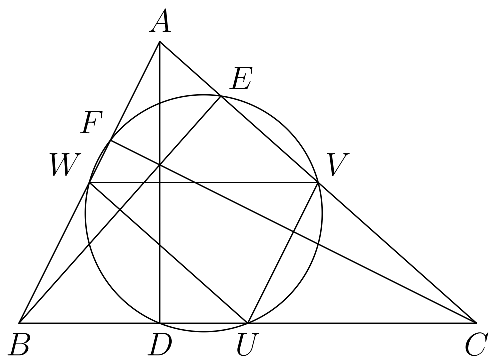
meaning $\cos B = 700/821$. Hence $\sin B = 429/821$. Applying the law of sines to $\triangle ABC$ gives

$$\frac{AC}{\sin B} = 2R = 821.$$

Finally, $AC = 429$. □

Now we give the proof I promised of the existence of the nine-point circle.

Theorem 2.2 (9-pt Circle). *In $\triangle ABC$, U, V, W are the midpoints of segments BC, CA, AB. Let D, E, F be the feet of the $A-$, $B-$, $C-$ altitudes respectively. Denote by K, L, M the midpoints of HA, HB, HC where H is the orthocenter of $\triangle ABC$. Prove U, V, W, D, E, F, K, L, M are concyclic.*



Solution. The first 6 points were due to Brianchon in 1821. The points K, L, and M were added by Terquem (see [4]). First, draw in the circumcircle of triangle UVW. By using power of a point and radical axis we will show that the other points lie on this circle as well.

To prove $UDWF$ is cyclic it suffices to show $(BD)(BU) = (BW)(BF)$. Note that $BD = c\cos B$ and $BU = a/2$. Furthermore, $BW = a\cos B$ and $BF = c/2$. Thus, $(BD)(BU) = (BW)(BF)$, establishing that $UDWF$ is cyclic.

By symmetry, $WFEV$ and $EVUD$ are also cyclic. Assume by way of contradiction that the circumcircles of $UDWF$, $WFEV$ and $EVUD$ do not coincide. By the radical axis theorem on those three circles, WF, EV and UD must concur. However this is impossible! Thus, the circumcircles coincide, and we have established that $WFEVUD$ is cyclic.

It remains to prove that K, L, M lie on this circle. By power of a point it suffices to show

$$(AK)(AD) = (AF)(AW).$$

Plugging $AK = R\cos A$, and $AD = bc/2R$ gives

$$(AK)(AD) = bc\cos A/2 = (c/2)b\cos A = (AF)(AW),$$

as needed. By symmetry, L and M lie on the circle as well. □

The good news is we do not have to repeat this for an obtuse triangle because we have used the notion of directed distances. An experienced reader would have noted that points K, L, M lie on the nine-point circle trivially by the orthocentric tetrad. Indeed, after we proved that the outer 6 points were concyclic, we could apply that result to triangles HAB and HBC to show that the midpoints of the segments joining the orthocenter to the vertices also lay on the circle.

Example 2.2 (Turkey 1999). In acute triangle ABC with circumradius R, altitudes AD, BE, CF have lengths h_1, h_2 and h_3 respectively. If t_1, t_2 and t_3 are the lengths of the tangents from A, B and C to the circumcircle of triangle DEF, prove that

$$\sum_{i=1}^{3} \left(\frac{t_i}{\sqrt{h_i}}\right)^2 \leq \frac{3}{2}R.$$

Solution. Write $\displaystyle\sum_{i=1}^{3} \left(\frac{t_i}{\sqrt{h_i}}\right)^2$ as $\displaystyle\sum_{i=1}^{3} \left(\frac{t_i^2}{h_i}\right)$. Actually, does the t_i^2 term look familiar? It's just the power of points A, B and C with respect to the nine point circle. Specifically, we have $t_1^2 = bc\cos A/2$, and similar formulas for the others. What about h_i? Well, $h_1 = bc/2R$ from the remark in Section 2, so we have

$$\sum_{i=1}^{3} \left(\frac{t_i}{\sqrt{h_i}}\right)^2 = R\cos A + R\cos B + R\cos C.$$

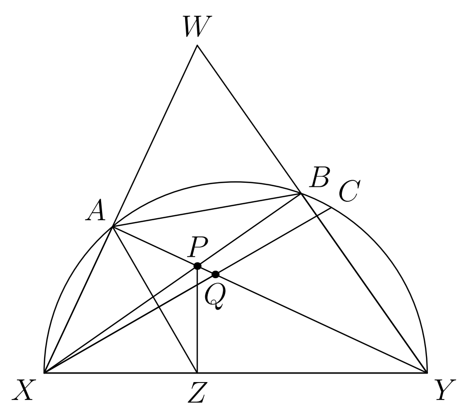
Thus, it suffices to show $\cos A + \cos B + \cos C \leq \frac{3}{2}$. But this follows by Jensen's inequality, and we are done. □

3 A Non-Trivial Example

Our last example is from the USAJMO in 2013. It illustrates the use of the orthocentric distances with law of sines.

Example 3.1 (USAJMO 2013) Quadrilateral $XABY$ is inscribed in semicircle ω with diameter XY. Let $P = \overline{AY} \cap \overline{BX}$. Point Z is the foot of the perpendicular from P to XY. Point C is on ω such that XC is perpendicular to AZ. Let $Q = \overline{AY} \cap \overline{XC}$. Prove that

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$



Solution. It does not look as if we have an orthocenter to apply our distances to. Or do we? Letting $W = \overline{AX} \cap \overline{BY}$, we see YA is perpendicular to WX and XB is perpendicular to WY. Since $P = \overline{AY} \cap \overline{BX}$, it is the orthocenter of triangle WXY. For brevity, I will write $\cos X$ for $\cos\angle WXY$, and $\cos Y$ for $\cos\angle WYX$.

Let $XY = w$, $WX = y$, and $YW = x$. Then,

$$\frac{AY}{AX} = \frac{wx/2R}{w\cos X} = \frac{x}{2R\cos X}$$

and

$$\frac{BY}{XP} = \frac{w\cos Y}{2R\cos X}.$$

Now we just need CY/XQ. Applying the law of sines to $\triangle CXY$ gives

$$\frac{CY}{\sin\angle CXY} = \frac{XY}{\sin 90^\circ} = XY = w.$$

Actually because $\triangle XAZ \sim \triangle XYW$, we have $\angle CXY = \angle WXB = 90 - \angle W$. Thus $CY = w\cos W$. Now we calculate XQ. The law of sines on $\triangle QAX$ gives

$$\frac{XQ}{\sin 90^\circ} = \frac{w\cos X}{\sin\angle AQX}.$$

Remark that $\angle AQX = 90^\circ - \angle AXQ = 90 - \angle YXB = \angle Y$, where the second-to-last step follows from $\triangle XAZ \sim \triangle XYW$. Hence, $XQ = w\cos X/\sin Y$. Therefore,

$$\begin{aligned}\frac{BY}{XP} + \frac{CY}{XQ} &= \frac{w\cos Y}{2R\cos X} + \frac{\cos W\sin Y}{\cos X}\\ &= \frac{w\cos Y}{2R\cos X} + \frac{y\cos W}{2R\cos X}\\ &= \frac{x}{2R\cos X}\\ &= \frac{AY}{AX},\end{aligned}$$

as wanted. □

4 Practice Problems

We end this article with several practice problems.

Problem 4.1 (TKMT, David Altizio) Let ABC be a triangle with $AB = 3$ and $AC = 4$. Points O and H denote the circumcenter and orthocenter of $\triangle ABC$ respectively. If $OH||BC$, what is $\cos A$?

Problem 4.2 (HMMT November 2016) Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let DF intersect the circumcircle of AHF again at X. Find the length of EX.

Problem 4.3 (HMMT November 2010) Triangle ABC is given with $AB = 13, BC = 14, CA = 15$. Let E and F be the feet of the altitudes from B and C respectively. Let G be the foot of the altitude from A in triangle AFE. Find AG.

Problem 4.4 (APMO 2013) Let ABC be an acute triangle with altitudes $\overline{AD}, \overline{BE}, \overline{CF}$, and let O be the circumcenter of $\triangle ABC$. Show segments OA, OF, OB, OD, OC, and OE dissect triangle ABC into three pairs of triangles with equal area.

Problem 4.5 (APMO 2004) Let O and H be the circumcenter and orthocenter of an acute $\triangle ABC$ respectively. Prove that the area of one of triangles AOH, BOH, and COH is equal to the sum of the areas of the other two.

Problem 4.6 (IMO 2008) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Problem 4.7 (Tuymaada 2002) The points D and E on the circumcircle of an acute triangle ABC are such that $AD = AE = BC$. Let H be the common point of the altitudes of triangle ABC. Given that $AH^2 = BH^2 + CH^2$, prove that H lies on the segment DE.

Problem 4.8 (Tuymaada 2010) Let ABC be an acute triangle, H its orthocenter, D a point on the side $\overline{BC}$, and P a point such that $ADPH$ is a parallelogram. Show that $\angle BPC > \angle BAC$.

Finally, I will finish with a few nice inequalities.

Problem 4.9 (Britain 2011) Let triangle ABC be acute. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine when equality holds.

Problem 4.10 (Canada 2015) Let triangle ABC have altitudes AD, BE and CF. Denote by H the orthocenter of triangle ABC. Prove that

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

Problem 4.11 In $\triangle ABC$ let D, E and F be the feet of the altitudes from A, B and C, respectively. Let K be the intersection of AO with BC, where O is the circumcenter of triangle ABC. Prove that

$$\frac{DE}{DF} = \frac{KB}{KC}.$$

Some hints and answers to the problems are presented in the next section.

5 Hints

Problem 4.1 If $OH||BC$, the distance from H and O to BC must be the same. The answer is

$$\frac{25-\sqrt{113}}{32}.$$

Problem 4.2 First things first: Power of a point. Prove $DX = DE$. Why does DA bisect $\angle XDE$? DA is the perpendicular bisector of XE. The answer is 190/49.

Problem 4.3 Similarity: $\triangle AFE \sim \triangle ACB$. Add in the foot of the altitude from A to BC. The answer is 396/65.

Problem 4.4 Prove $[AOF] = [COD]$. Invoke symmetry.

Problem 4.5 Sine areas:

$$[AOH] = \frac{(AO)(AH)}{2} \sin \angle HAO.$$

Problem 4.6 Prove $(AB_1)(AB_2) = (AC_1)(AC_2)$. To do this let N be the midpoint of AC. Find HN by using Stewart's.

Problem 4.7 Draw a circle at A with radius BC. Consider radical axis. Use Corollary 2.3.

Problem 4.8 You need $\cot \angle BPC < \cot \angle BAC$. Use the cotangent angle sum formula on the appropriate triangles.

Problem 4.9 The condition is $AB = AC$. Expand the $\cos B$ and $\cos C$, and bash away.

Problem 4.10 Again, bash away. You should be left with

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Problem 4.11 You will need to use the fact that AO and AH are isogonal. The exact result is

$$\frac{BD}{DC} \cdot \frac{BK}{KC} = \frac{c^2}{b^2}.$$

Acknowledgements

Finally, a big thanks to Dr. Bogdan Suceava of California State University Fullerton who took the time to validate all of this material.

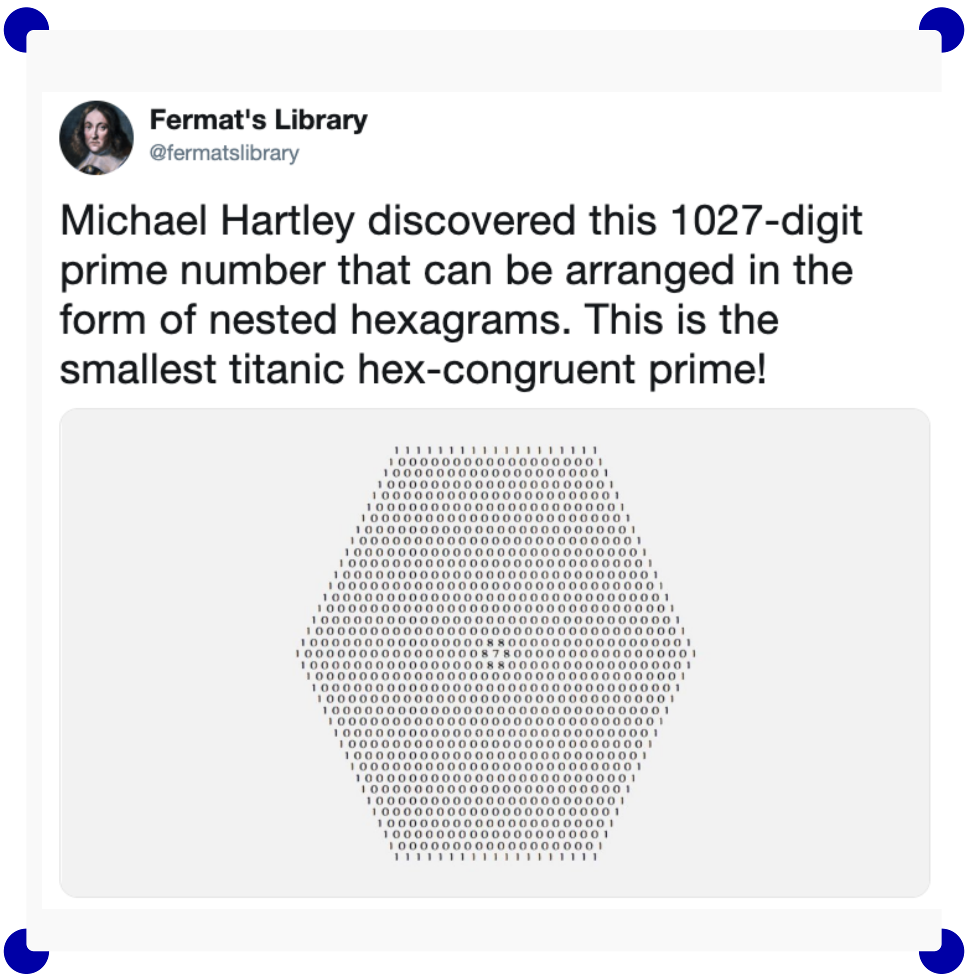
6 References

[1] E. Chen, Chapter 3 in *Euclidean Geometry in Mathematical Olympiads*, MAA, 2016.

[2] AoPS Forum. https://artofproblemsolving.com/community

[3] A. D. Gardiner and C. J. Bradley, *Plane Euclidean Geometry: Theory and Problems*, UKMT, 2012, pp. 86–89.

[4] J. S. Mackay, History of the Nine-point Circle, *Proceedings of the Edinburgh Mathematical Society*, **11**, 19–57 (1892).



PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 30, 2019**.

4451. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$ with $n \geq 2$ and $0 < a < b < 1$, let

$$I(a,b) = \int_a^b \frac{(x+1)((2n-3)x^{n+1} - (2n-1)x^n + 3x - 1)}{x^2(x-1)^2}\, dx.$$

Find

$$\lim_{a \to 0^+} \left(\frac{1}{a} + \lim_{b \to 1^-} I(a,b) \right).$$

4452. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with orthocenter H. If A', B', C' are the circumcenters of $\triangle\ HBC, \triangle\ HAC$ and $\triangle\ HAB$, respectively, and $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}$, show that ABC is an equilateral triangle.

4453. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be a triangle with no angle larger than $\frac{2\pi}{3}$ and let T be its Fermat-Torricelli point, that is the point such that the total distance from the three vertices of ABC to T is minimum possible. Suppose BT intersects AC at D and CT intersects AB at E. Prove that if $AB + AC = 4DE$, then ABC is equilateral.

4454. *Proposed by Nguyen Viet Hung.*

Prove the identity

$$\binom{4n}{0} - \binom{4n}{2} + \cdots + (-1)^n \binom{4n}{2n} = \frac{(-4)^n + (-1)^n \binom{4n}{2n}}{2}.$$

4455. *Proposed by Marian Maciocha.*

Find all integer solutions (if any) for the equation

$$(A + 3B)(5B + 7C)(9C + 11A) = 1357911.$$

4456. *Proposed by Leonard Giugiuc.*

Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$(a+b+c)(ab+bc+ac)+3 \geq 4(a+b+c).$$

4457. *Proposed by Hung Nguyen Viet.*

Prove that for all $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$, $x \neq -y$, we have that

$$\tan^2 x + \tan^2 y + \cot^2(x+y) \geq 1.$$

4458. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let a, b, c, d be the sides of a cyclic quadrilateral with circumradius R and lengths of diagonals d_1 and d_2. Prove that

$$\sum_{cyclic} \frac{a}{b+c+d-a} \geq \frac{4R}{\sqrt{d_1 d_2}}.$$

4459. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be an isosceles triangle with $AB = AC$. For a point P on side AB let Q be a point of the extension of AC beyond C for which the midpoint N of PQ lies on the segment BC; similarly, for a point R on side AC let S be a point of the extension of AB beyond B for which the midpoint M of RS lies on the segment BC. Prove that

$$\frac{PQ}{RS} = \frac{\cos \angle RMN}{\cos \angle PNM}.$$

4460. *Proposed by Gantumur Choijilsuren and Leonard Giugiuc.*

Let $(x_n)_{n \geq 1}$ be a sequence of real numbers such that $(3x_{n+1} - 2x_n)_{n \geq 1}$ is convergent. Show that $(x_n)_{n \geq 1}$ is convergent.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 septembre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4451. *Proposed by Michel Bataille.*

Pour $n \in \mathbb{N}$ tel que $n \geq 2$ and $0 < a < b < 1$, soit

$$I(a,b) = \int_a^b \frac{(x+1)((2n-3)x^{n+1} - (2n-1)x^n + 3x - 1)}{x^2(x-1)^2}\,dx.$$

Déterminer

$$\lim_{a\to 0^+}\left(\frac{1}{a} + \lim_{b\to 1^-} I(a,b)\right).$$

4452. *Proposed by Mihaela Berindeanu.*

Soit ABC un triangle avec orthocentre H. Si A', B', C' sont les centres des cercles circonscrits de $\triangle\, HBC, \triangle\, HAC$ et $\triangle\, HAB$ respectivement et si

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0},$$

démontrer que ABC est un triangle équilatéral.

4453. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Soit ABC un triangle dont aucun angle est supérieur à $\frac{2\pi}{3}$ et soit T son point Fermat-Torricelli, c'est-à-dire le point tel que la distance totale de T vers les sommets de ABC est minimale. Supposer que BT intersecte AC en D et que CT intersecte AB en E. Démontrer que si $AB + AC = 4DE$, alors ABC est équilatéral.

4454. *Proposed by Nguyen Viet Hung.*

Démontrer l'identité

$$\binom{4n}{0} - \binom{4n}{2} + \cdots + (-1)^n\binom{4n}{2n} = \frac{(-4)^n + (-1)^n\binom{4n}{2n}}{2}.$$

4455. *Proposed by Marian Maciocha.*

Déterminer toute solution entière, s'il y en a, à l'équation

$$(A+3B)(5B+7C)(9C+11A) = 1357911.$$

4456. *Proposed by Leonard Giugiuc.*

Soient a, b, c des nombres réels tels que $abc = 1$. Démontrer que

$$(a+b+c)(ab+bc+ac)+3 \geq 4(a+b+c).$$

4457. *Proposed by Hung Nguyen Viet.*

Démontrer que pour tout x et y tels que $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$ et $x \neq -y$, l'inégalité suivante tient:

$$\tan^2 x + \tan^2 y + \cot^2 (x+y) \geq 1.$$

4458. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Soient a, b, c, d les côtés d'un quadrilatère cyclique de diagonales d_1 et d_2 et dont le rayon du cercle associé est R. Démontrer que

$$\sum_{cyclic} \frac{a}{b+c+d-a} \geq \frac{4R}{\sqrt{d_1 d_2}}.$$

4459. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Soit ABC un triangle isocèle tel que $AB = AC$. Pour un point P sur le côté AB, soit Q un point sur le prolongement de AC au delà de C pour lequel le mi point N de PQ se situe sur le segment BC; de façon similaire, pour un point R sur le côté AC, soit S un point sur le prolongement de AB au delà de B pour lequel le mi point M de RS se trouve sur le segment BC. Démontrer que

$$\frac{PQ}{RS} = \frac{\cos \angle RMN}{\cos \angle PNM}.$$

4460. *Proposed by Gantumur Choijilsuren and Leonard Giugiuc.*

Soit $(x_n)_{n\geq 1}$ une suite de nombres réels telle que $(3x_{n+1} - 2x_n)_{n\geq 1}$ est convergente. Démontrer que $(x_n)_{n\geq 1}$ est convergente.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(8), p. 340–343; 44(9), p.387–390; 44(10): 423–426.

4376. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let A and B be two matrices in $M_n(\mathbb{C})$ such that $AB = -BA$. Prove that

$$\det(A^4 + A^2B^2 + 2A^2 + I_n) \geq 0.$$

We received 6 solutions and will feature just one of them here. We present the solution by Missouri State University Problem Solving Group.

The result is false as stated. Let

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where $\alpha = (1+i)/\sqrt{2}$. Then $AB = -BA$, but

$$\det\left(A^4 + A^2B^2 + 2A^2 + I_n\right) = \left(1+\alpha^2\right)^4 = -4.$$

We assume that the intended hypothesis was $A, B \in M_n(\mathbb{R})$. Note that

$$\begin{aligned}
&\left(I_n + A^2 + AB\right)\left(I_n + A^2 + BA\right) \\
&= I_n + A^2 + BA + A^2 + A^4 + A^2BA + AB + ABA^2 + ABBA \\
&= I_n + A^2 - AB + A^2 + A^4 + A^2BA + AB - A^2BA - ABAB \\
&= I_n + A^2 - AB + A^2 + A^4 + A^2BA + AB - A^2BA + A^2B^2 \\
&= A^4 + A^2B^2 + 2A^2 + I_n.
\end{aligned}$$

Now

$$\begin{aligned}
\det\left(I_n + A^2 + AB\right) &= \det\left(I_n + A\left(A+B\right)\right) \\
&= \det\left(I_n + \left(A+B\right)A\right) \\
&= \det\left(I_n + A^2 + BA\right),
\end{aligned}$$

using Sylvester's Theorem that $\det(I_n + XY) = \det(I_n + YX)$. Finally, we have

$$\begin{aligned}
\det\left(A^4 + A^2B^2 + 2A^2 + I_n\right) &= \det\left(\left(I_n + A^2 + AB\right)\left(I_n + A^2 + BA\right)\right) \\
&= \det\left(I_n + A^2 + AB\right)\det\left(I_n + A^2 + BA\right) \\
&= \left(\det\left(I_n + A^2 + AB\right)\right)^2 \\
&\geq 0.
\end{aligned}$$

4377. *Proposed by Tidor Vlad Pricopie and Leonard Giugiuc.*

Let $x \geq y \geq z > 0$ such that $x + y + z + xy + xz + yz = 1 + xyz$. Find $\min x$.

We received 7 correct solutions and 1 incorrect submission. We feature the solution based on the approach of Ramanujan Srihari and the collaboration between Leonard Giugiuc and Tidor Vlad Pricopie, done independently.

We first show that $xy+xz+yz \neq 1$. Otherwise, we would have that $x+y+z = xyz$ and

$$z = z(xy + xz + yz) = x + y + z + z^2(x + y),$$

leading to

$$0 = (x + y)(1 + z^2),$$

which contradicts the hypotheses.

Let $x = \tan u$, $y = \tan v$, $z = \tan w$ where $\pi/2 > u \geq v \geq w > 0$. Then

$$\tan(u + v + w) = \frac{x + y + z - xyz}{1 - xy - xz - yz} = 1.$$

Therefore

$$u + v + w = \pi/4 \quad \text{or} \quad u + v + w = 5\pi/4.$$

In the latter case, we would have $u > 5\pi/12$ and $x > 1$.

When $u + v + w = \pi/4$, then $u \geq \pi/12$. From the formula for $\tan 3\theta$, we see that $\tan \pi/12$ satisfies the equation

$$0 = t^3 - 3t^2 - 3t + 1 = (t + 1)(t^2 - 4t + 1),$$

so that $x \geq 2 - \sqrt{3}$. However, it is possible that $u = v = w = \pi/12$, so that the equation is satisfied by $x = y = z = 2 - \sqrt{3}$.

Therefore, for all positive solutions of the equation, $x \geq 2 - \sqrt{3}$, and the lower bound is attained.

4378. *Proposed by Tarit Goswami.*

Find all k such that the following limit exists

$$\lim_{n\to\infty} \{k \cdot F_{n+1} - \sum_{i=0}^{n} \tau^i\},$$

where F_n is the n^{th} Fibonacci number and τ is the golden ratio.

Five correct solutions and three incorrect or incomplete solutions were received to the problem when the brackets are used to simply display the summand. The first solution by Sushanth Sathish Kumar solves this problem. However, the Problem Solving Group at Missouri State University took the brackets to refer to the fractional part. Theirs is the second solution below.

Solution 1, by Sushanth Sathish Kumar.

Using Binet's formula for F_{n+1} and the relation $1-\tau = -1/\tau$, we have that

$$\begin{aligned}
kF_{n+1} - \sum_{i=0}^{n} \tau^i &= \frac{k}{\sqrt{5}}(\tau^{n+1} - (1-\tau)^{n+1}) - \frac{\tau^{n+1}-1}{\tau-1} \\
&= \frac{k}{\sqrt{5}}(\tau^{n+1} - (1-\tau)^{n+1}) - \tau^{n+2} + \tau \\
&= \tau + \left(\frac{k}{\sqrt{5}} - \tau\right)\tau^{n+1} - \frac{k}{\sqrt{5}}(1-\tau)^{n+1}.
\end{aligned}$$

Since $\lim_{n\to\infty} \tau^{n+1} = \infty$ and $\lim_{n\to\infty}(1-\tau)^{n+1} = 0$, the limit can exist if and only if $k = \sqrt{5}\tau = (5+\sqrt{5})/2$, in which case the limit is τ.

Solution 2, by Missouri State University Problem Solving Group.

Let

$$x_n = kF_{n+1} - \sum_{i=0}^{n} \tau^i$$

and let $\{x_n\} = x_n - \lfloor x_n \rfloor$ be the fractional part of x_n. We prove that the limit of $\{x_n\}$ exists if and only if $k = u + v\tau$ for some integers u and v.

Suppose first that $k = u + v\tau$. Since $\tau^n = \tau F_n + F_{n-1}$ for each positive integer n,

$$x_n = kF_{n+1} - (\tau^{n+2} - \tau) = (k-1)F_{n+1} - \tau F_{n+2} + \tau. \tag{1}$$

By Binet's formula, $\lim_{n\to\infty} \tau F_n - F_{n+1} = 0$, so that

$$\begin{aligned}
x_n &= (k-1)F_{n+1} - \tau F_{n+2} + \tau \\
&= (u-1)F_{n+1} + v\tau F_{n+1} - \tau F_{n+2} + \tau \\
&= (u-1)F_{n+1} + vF_{n+2} - F_{n+3} + 1 + (\tau - 1 + \epsilon_n)
\end{aligned}$$

where $\lim_{n\to\infty} \epsilon_n = 0$. Hence, for sufficiently large n, $\{x_n\} = \tau - 1 + \epsilon_n$ and so $\lim_{n\to\infty}\{x_n\} = \tau - 1$.

Now, suppose that $\lim\{x_n\} = L$ and let

$$y_n = \lfloor x_n \rfloor - 1 = x_n - \{x_n\} - 1.$$

Then, for $n \geq 3$, from (1), we have that

$$\begin{aligned}
y_n - y_{n-1} - y_{n-2} &= (x_n - x_{n-1} - x_{n-2}) - (\{x_n\} - \{x_{n-1}\}) - \{x_{n-2}\}) + 1 \\
&= -\tau + (\{x_{n-1}\} + \{x_{n-2}\} - \{x_n\}) + 1.
\end{aligned}$$

Since $\{y_n - y_{n-1} - y_{n-2}\}$ is a convergent sequence of integers with limit $L - (\tau - 1)$, it is eventually constant. Since $0 \leq L \leq 1$ and $0 < \tau - 1 < 0$, the limit must be 0 and so $y_n = y_{n-1} + y_{n-2}$ for $n \geq N$. Therefore, for $n \geq N$,

$$y_n = aF_{n+1-N} + bF_{n-N}$$

for $n \geq N$, where $a = y_N$ and $b = y_{N-1}$.

From (1),

$$\begin{aligned} k - 1 &= \frac{x_n + \tau F_{n+2} - \tau}{F_{n+1}} \\ &= \frac{y_n + \{x_n\} + 1 + \tau F_{n+2} - \tau}{F_{n+1}} \\ &= a\frac{F_{n+1-N}}{F_{n+1}} + b\frac{F_{n-N}}{F_{n+1}} + \tau\frac{F_{n+2}}{F_{n+1}} + \frac{\{x_n\} - (\tau - 1)}{F_{n+1}}. \end{aligned}$$

Let n tend to infinity. Then

$$\begin{aligned} k - 1 &= a\tau^{-N} + b\tau^{-(N+1)} + \tau^2 \\ &= a(\tau - 1)^N + b(\tau - 1)^{N+1} + \tau^2, \end{aligned}$$

so that $k = u + v\tau$ for some integers u and v.

4379. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let triangle ABC share its vertices with three vertices of a regular heptagon; in particular, let B coincide with vertex 1, C with vertex 2, and A with vertex 4. Let I be the incenter and let G be the centroid of ABC, respectively. Suppose BI intersects AC in D and CI intersects AB in E. Show that the points D, G and E are collinear.

We received 5 solutions. Presented is the one by Andrea Fanchini, lightly edited.

We use barycentric coordinates with reference to the triangle ABC, with side lengths a, b, and c. The vertices of the triangle have coordinates $A(a : 0 : 0)$, $B(0 : b : 0)$ and $C(0 : 0 : c)$, the incentre has coordinates $I(a : b : c)$ and the centroid $G(1 : 1 : 1)$. Using that D is on both lines BI and AC, we obtain the coordinates $D(a : 0 : c)$. Similarly we get $E(a : b : 0)$. The points G, D and E are collinear if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ a & 0 & c \\ a & b & 0 \end{vmatrix} = ac - bc + ab = 0.$$

Now we know that the side lengths of the heptagonal triangle satisfy the relations

$$bc = c^2 - a^2, \quad ac = b^2 - a^2, \quad ab = c^2 - b^2.$$

After substituting these relations, we are done.

4380. *Proposed by George Apostolopoulos.*

Let a, b and c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$a^2 \tan\frac{A}{2} + b^2 \tan\frac{B}{2} + c^2 \tan\frac{C}{2} \le \frac{3\sqrt{3}R^3(R-r)}{2r^2}.$$

We received 12 submissions, including the one from the proposer, all correct. We present two solutions, the second one of which gives a sharper inequality.

Solution 1, by Kee-Wai Lau.

Let S denote the semiperimeter of triangle ABC. The following identities and inequalities are all well known:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{1}$$

$$\sin A\cos A + \sin B\cos B + \sin C\cos C = \frac{rS}{R^2} \tag{2}$$

$$R \ge 2r \quad \textit{(Euler's Inequality)} \tag{3}$$

$$s \le \frac{3\sqrt{3}R}{2} \tag{4}$$

By (1) we have

$$\begin{aligned} a^2 \tan\frac{A}{2} &= 4R^2(\sin^2 A)(\tan\frac{A}{2}) \\ &= 4R^2(\sin A)(2\sin\frac{A}{2}\cos\frac{A}{2})(\tan\frac{A}{2}) \\ &= 8R^2(\sin A)(\sin^2\frac{A}{2}) \\ &= 4R^2(\sin A)(1-\cos A). \end{aligned}$$

Similarly, $b^2\tan\frac{B}{2} = 4R^2(\sin B)(1-\cos B)$ and $c^2\tan\frac{C}{2} = 4R^2(\sin C)(1-\cos C)$. Hence, by (1), (2), (3), and (4) we have

$$\begin{aligned} \sum_{cyc} a^2\tan\frac{A}{2} = 4R^2\Big(\sum_{cyc}\sin A - \sum_{cyc}\sin A\cos A\Big) &= 4R^2\Big(\frac{a+b+c}{2R} - \frac{rs}{R^2}\Big) \\ &= 4s(R-r) \\ &\le 6\sqrt{3}R(R-r) \\ &= \frac{3\sqrt{3}R(2r)^2(R-r)}{2r^2} \\ &\le \frac{3\sqrt{3}R^3(R-r)}{2r^2} \end{aligned}$$

and we are done.

Solution 2, by Arkady Alt.

We prove the inequality that

$$\sum_{cyc} a^2 \tan\frac{A}{2} \le 6\sqrt{3}R(R-r)$$

which is sharper than the proposed result since

$$6\sqrt{3}R(R-r) \le \frac{3\sqrt{3}R^3(R-r)}{2r^2} \iff 2r \ \le R$$

which is Euler's Inequality.

Using the known results that

$$\tan\frac{A}{2} = \frac{r}{s-a}, \quad \sum_{cyc}\frac{a}{s-a} = \frac{4R-2r}{r} \quad \text{and} \quad s \le \frac{3\sqrt{3}R}{2},$$

we obtain

$$\begin{aligned}
&\sum_{cyc} a^2 \tan\frac{A}{2} \le 6\sqrt{3}R(R-r) \iff \\
&\sum_{cyc}\frac{a^2}{s-a} \le \frac{6\sqrt{3}R(R-r)}{r} \iff \\
&\sum_{cyc}\left(\frac{a^2}{s-a}+a\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&\sum_{cyc}\left(\frac{sa}{s-a}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s\cdot\left(\frac{4R-2r}{r}\right) \le \frac{6\sqrt{3}R(R-r)}{r} + 2s \iff \\
&s(2R-r) \le 3\sqrt{3}R(R-r) + sr \iff \\
&2s(R-r) \le 3\sqrt{3}R(R-r) \iff \\
&s \le \frac{3\sqrt{3}R}{2}
\end{aligned}$$

and the proof is complete.

4381. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle Γ_1 and circumcenter O. Suppose the open ray AO intersects Γ_1 at point D and E is the middle point of BC. The perpendicular bisector of BE intersects BD in P and the perpendicular bisector of EC intersects CD in Q. Finally suppose that circle Γ_2 with center P and radius PE intersects the circle Γ_3 with center Q and radius QE in X. Prove that AX is a symmedian in $\triangle\, ABC$.

We received 6 submissions, all of which were correct, and feature the solution by K.V. Sudharshan.

Define X' to be the intersection of the A-symmedian with Γ_1. We shall prove that $X' = X$. Let AE intersect Γ_1 at a point Y. Observe that since AX', AY are isogonal (by definition), we have $\angle BAX' = \angle YAC$, and so $BX' = CY$. Since E is the midpoint of BC, we see that $BX'E$ and CYE are congruent triangles (by side-angle-side). Consequently,

$$\angle EX'B = \angle CYE = \angle CYA = \angle CBA.$$

Also, since AD is a diameter of Γ_1 and $P \in BD$, we have $PB \perp AB$, so that $\angle CBA = 90^\circ - \angle PBE$. Thus (because P is on the perpendicular bisector of the segment BE),

$$\angle EPB = 180^\circ - 2 \cdot \angle PBE = 2 \cdot \angle CBA = 2 \cdot \angle EX'B.$$

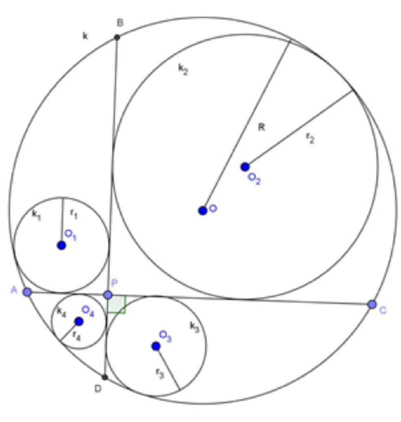
But $\angle EPB$ is the angle at the center of Γ_2 that is subtended by the chord EB, which is twice any angle on the circumference subtended by EB; consequently, X' is a point of Γ_2 as well as lying on Γ_1. Similarly, X' is a point where Γ_3 intersects Γ_1, whence X' is the point other than E where Γ_2 intersects Γ_3. That is, $X' = X$, so that AX must be the A-symmedian, as desired.

Editor's comment. If directed angles are used, then the featured argument does not require the given triangle to be acute: the result holds for an arbitrary ΔABC.

4382. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

Let $ABCD$ be an orthogonal cyclic quadrilateral with $AC \perp BD$. Let O and R be the circumcenter and the circumradius of $ABCD$ respectively and let P be the intersection of AC and BD. Denote by r_1, r_2, r_3 and r_4 the inradii of the minor circular sectors PAB, PBC, PCD and PDA respectively. Prove that

$$r_1 + r_2 + r_3 + r_4 + 8R = (R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right).$$



We received 7 submissions of which 6 were correct and complete. We feature two of them.

Solution 1, by Sushanth Sathish Kumar.

The proof is by coordinates. Set P to be the origin, and let $O = (x, y)$. Assume the figure is oriented such that B and C lie on the positive y and x-axis, as in the accompanying figure. Let the center of the incircle of sector PBC be O_2.

Note that $O_2 = (r_2, r_2)$, and $OO_2 = R - r_2$. Thus, by the distance formula

$$\begin{aligned}\sqrt{(x-r_2)^2+(y-r_2)^2} = R - r_2 \quad &\iff \quad x^2+y^2-2r_2(x+y)+r_2^2 = R^2-2Rr_2\\ &\iff \quad 2Rr_2-2r_2(x+y)+r_2^2 = R^2-OP^2,\end{aligned}$$

where the last step follows from $OP^2 = x^2+y^2$. Therefore, we have

$$\frac{R^2-OP^2}{r_2} = 2R + r_2 - 2(x+y).$$

Similarly, we may compute

$$\begin{aligned}\frac{R^2-OP^2}{r_1} &= 2R + r_1 + 2(x-y),\\ \frac{R^2-OP^2}{r_3} &= 2R + r_3 + 2(y-x)\\ \frac{R^2-OP^2}{r_4} &= 2R + r_4 + 2(x+y).\end{aligned}$$

Upon adding, we arrive at

$$(R^2-OP^2)\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right) = r_1+r_2+r_3+r_4+8R,$$

which is precisely the desired result.

Solution 2, by Ioannis D. Sfikas.

Let O_i be the center of the incircle whose radius is r_i, $i = 1, \ldots, 4$. Assuming labelling as in the figure, we have

$$O_1O = R - r_1, \quad O_3O = R - r_3, \quad O_1P = \sqrt{2}r_1, \quad \text{and} \quad O_3P = \sqrt{2}r_3;$$

furthermore, P lies on the line segment O_1O_3. By Stewart's theorem applied to the cevian OP of ΔOO_1O_3 we have

$$OP^2 = (R-r_3)^2\frac{r_1}{r_1+r_3} \; + \; (R-r_1)^2\frac{r_3}{r_1+r_3} \; - \; 2r_1r_3,$$

so that

$$\frac{R^2-OP^2}{r_1r_3} = 1 + \frac{4R}{r_1+r_3},$$

and, finally,

$$\left(R^2 - OP^2\right)\left(\frac{1}{r_1} + \frac{1}{r_3}\right) = r_1 + r_3 + 4R.$$

Similarly,

$$\left(R^2 - OP^2\right)\left(\frac{1}{r_2} + \frac{1}{r_4}\right) = r_2 + r_4 + 4R.$$

Adding these last two equations gives us the desired result, namely

$$(R^2 - OP^2)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}\right) = r_1 + r_2 + r_3 + r_4 + 8R.$$

Editor's comments. It is interesting to compare our problem 4382 with Problem 1.4.6 on pages 9 and 85 of *Japanese Temple Geometry Problems: San Gaku* (edited by Hidetosi Fukagawa and Dan Pedoe and published in 1989 by the Charles Babbage Research Centre). The Japanese problem applies to essentially the same figure, except they use a small circle inside the region bounded by arcs of the given circles and externally tangent to those four given circles (rather than our large circumcircle that encloses the four given circles and is internally tangent to them). This requires replacing our radius R by $-R$. They find a relation among the four radii that involves neither R nor OP, namely

$$r_1r_3(r_2+r_4)^2 + r_2r_4(r_1+r_3)^2 = (r_1r_3-r_2r_4)^2 + (r_1+r_3)(r_2+r_4)(r_1r_3+r_2r_4).$$

4383. *Proposed by Michel Bataille.*

Evaluate the integral

$$I = \int_0^1 (\ln x) \cdot \sqrt{\frac{x}{1-x}}\, dx.$$

We received 14 submissions, all correct. We present the solution by Kee-Wai Lau.

We show that the given integral equals

$$I = \frac{(1 - 2\ln 2)\pi}{2}.$$

By substitution $x = \sin^2\theta$ and using half angle formula $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$, we obtain

$$\begin{aligned} I &= 4\int_0^{\pi/2} \ln(\sin\theta)\sin^2\theta\, d\theta \\ &= 2\left(\int_0^{\pi/2} \ln(\sin\theta)\, d\theta - \int_0^{\pi/2} \ln(\sin\theta)\cos(2\theta)\, d\theta\right). \end{aligned} \tag{1}$$

The first integral in (1) is well know:

$$\int_0^{\pi/2} \ln(\sin\theta)\,d\theta = \frac{1}{2}\int_0^{\pi} \ln(\sin\theta)\,d\theta = -\frac{\pi\ln 2}{2}$$

(see, for example, p.246 of G. Boros and V. Moll *"Irresistible Integrals"*, Cambridge University Press, 2004.)

Since $\lim_{x\to 0^+} x\ln x = 0$, using a substitution for the second integral of (1), we get

$$\int_0^{\pi/2} \ln(\sin\theta)\cos(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta)\,d(\sin(2\theta)) = -\int_0^{\pi/2} \cos^2\theta\,d\theta = -\frac{\pi}{4}.$$

Combining the integrals, we get that $I = \dfrac{(1-2\ln 2)\pi}{2}$ as claimed.

4384. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$. Find all real numbers x such that

$$\sum_{0\leq i<j\leq n-1} \left\lfloor x+\frac{i}{n}\right\rfloor \cdot \left\lfloor x+\frac{j}{n}\right\rfloor = 0.$$

We received 4 correct solutions. We present the composite solution.

For any real number x and integers i, j with $0 \leq i < j \leq n-1$,

$$\lfloor x\rfloor \leq \left\lfloor x+\frac{i}{n}\right\rfloor \leq \left\lfloor x+\frac{j}{n}\right\rfloor < \lfloor x+1\rfloor = \lfloor x\rfloor + 1.$$

Therefore, each term of the sum assumes exactly one of the values a^2, $a(a+1)$ and $(a+1)^2$ for $a = \lfloor x\rfloor$, and so is non-negative. Therefore the sum vanishes if and only if each term $\lfloor x+i/n\rfloor \cdot \lfloor x+j/n\rfloor$ vanishes.

Suppose that $-1/n \leq x < 2/n$. Then

$$0 = \lfloor x+1/n\rfloor = \lfloor x+(n-2)/n\rfloor$$

so that each summand has at least one factor equal to 0 and the equation is satisfied. On the other hand, when $x < -1/n$, then

$$\lfloor x\rfloor \left\lfloor x+\frac{1}{n}\right\rfloor = (-1)^2 > 0$$

and when $x \geq 2/n$, then

$$\left\lfloor x+\frac{n-2}{n}\right\rfloor \cdot \left\lfloor x+\frac{n-1}{n}\right\rfloor = 1 > 0.$$

Therefore the equation is satisfied if and only if $-1/n \leq x < 2/n$.

4385. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with circumcircle ω and $AB < AC$. The tangent at A to ω intersects the line BC at P. The internal bisector of $\angle APB$ intersects the sides AB and AC at E and F, respectively. Show that

$$\frac{PE}{PF} = \sqrt{\frac{EB}{FC}}.$$

We received 14 correct submissions. We present the solution by K. V. Sudharshan.

Let AD be the internal angle bisector of $\angle BAC$, with $D \in BC$. We can see that

$$\angle PAD = \angle PAB + \angle BAD = \angle ACB + \angle CAD = \angle PDA.$$

Thus $PA = PD$. This implies that EF is the perpendicular bisector of AD.

Since AD bisects $\angle EAF$, we see that $EAFD$ is a rhombus, and so $AE = AF$.

Now note that $\triangle PEA \sim \triangle PFC$, so

$$\frac{PE}{PF} = \frac{AE}{FC}.$$

Also, $\triangle PEB \sim \triangle PFA$ and so

$$\frac{PE}{PF} = \frac{BE}{AF}.$$

Combining with $AE = AF$ and the previous result, we have $AE^2 = BE \cdot CF$. Thus,

$$\frac{PE}{PF} = \frac{AE}{FC} = \sqrt{\frac{BE}{CF}}.$$

4386. *Proposed by Thanos Kalogerakis.*

Let $ABCD$ be a cyclic quadrilateral with $AD > BC$, where $X = AB \cap CD$ and $Y = BC \cap AD$. The bisectors of angles X and Y intersect BC and CD at P and S, respectively. Finally, let Q and T be points on the sides AD and AB such that $PQ \perp AD$ and $ST \perp AB$. Prove that $ABCD$ is bicentric if and only if $PQ = ST$.

We received four submissions, but one was withdrawn. We feature the proposer's solution, modified by the editor. The proposer clearly intended for the given quadrilateral to have no parallel sides (rather than demanding inequalities among the edges).

Because a cyclic quadrilateral $ABCD$ has an inscribed circle if and only if the sums of opposite sides are equal, we are to prove that $AD + BC = AB + CD$ if and only if $PQ = ST$. We first show that the area of $ABCD$ is $\frac{1}{2}PQ(AD + BC)$.

From $ABCD$ cyclic we have

$$\angle BAD = \angle BCX. \tag{1}$$

Denoting by $B'C'$ the reflection of the segment BC in the mirror XP, the triangles PBC' and $PB'C$ are congruent (since P is fixed by the reflection) and, therefore, have the same area. It follows that the quadrangles $ABCD$ and $AC'B'D$ have the same area. Furthermore we have

$$\angle BCX = \angle B'C'X \quad \text{and} \quad BC = B'C'. \tag{2}$$

From (1) and (2) we have $\angle BAD = \angle BCX = \angle B'C'X$, so that the lines AD and $B'C'$ are parallel and, therefore, $AC'B'D$ is a trapezoid with altitude PQ and area

$$\frac{1}{2}PQ(AD + B'C').$$

Because this is also the area of our given quadrilateral and (by (2)) $BC = B'C'$, the area of $ABCD$ is $\frac{1}{2}PQ(AD + BC)$, as claimed.

Similarly, using the same argument with X, XP, PQ, AD, BC replaced by Y, YS, ST, AB, CD, we deduce that the area of $ABCD$ is also given by $\frac{1}{2}ST(AB+CD)$. That is,

$$PQ(AD + BC) = ST(AB + CD),$$

from which we conclude that $AD + BC = AB + CD$ if and only if $PQ = ST$, as desired.

Editor's Comment. If $ABCD$ has an incircle, its diameter has length $PQ = ST$. This follows from the observations that the bisector of $\angle X$ must pass through the incenter, which implies that the reflection in this line fixes the incircle and therefore takes its tangent BC to another tangent $B'C'$; the length of the common perpendicular PQ is the distance between parallel tangents AD and $B'C'$ which, of course, is the diameter of the incircle.

4387. *Proposed by Nguyen Viet Hung.*

Let

$$a_n = \sum_{k=1}^{n} \sqrt[k]{1 + \frac{k^2}{(k+1)!}}, \quad n = 1, 2, 3, \ldots$$

Determine $\lfloor a_n \rfloor$ and evaluate $\lim_{n\to\infty} \dfrac{a_n}{n}$.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Sushanth Sathish Kumar.

We claim that $\lfloor a_n \rfloor = n$. Clearly, $a_n \geq n$, since each term in the sum is at least 1. By the generalized Bernoulli inequality, $(1 + x)^{1/k} \leq 1 + \frac{x}{k}$ for any $k \in \mathbb{N}$ and

$x \geq -1$, so in particular we have

$$a_n = \sum_{k=1}^{n} \left(1 + \frac{k^2}{(k+1)!}\right)^{\frac{1}{k}} \leq \sum_{k=1}^{n} \left(1 + \frac{k}{(k+1)!}\right)$$
$$= \sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right)$$
$$= n + \left(1 - \frac{1}{(n+1)!}\right).$$

Thus, $n \leq a_n < n+1$, which implies $\lfloor a_n \rfloor = n$. By the squeeze theorem,

$$1 = \lim_{n\to\infty} \frac{n}{n} \leq \lim_{n\to\infty} \frac{a_n}{n} \leq \lim_{n\to\infty} \frac{n+1}{n} = 1;$$

that is, $\lim\limits_{n\to\infty} \frac{a_n}{n} = 1$.

4388. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

For positive real numbers a, b and c, prove

$$8abc(a^2 + 2ac + bc)(b^2 + 2ab + ac)(c^2 + 2bc + ab) \leq [(a+b)(b+c)(c+a)]^3.$$

We received 6 submissions, including the one from the proposers. One of the given solutions used Maple outputs. We present the proof by Daniel Văcaru.

The given inequality is equivalent to

$$\frac{(a^2 + 2ac + bc)(b^2 + 2ba + ca)(c^2 + 2cb + ab)}{(a+b)^2(b+c)^2(c+a)^2} \leq \frac{(a+b)(b+c)(c+a)}{8abc} \tag{1}$$

Let E denote the LHS of (1). Then by direct computations we have

$$E = \left(\frac{a}{a+b} + \frac{c}{c+a}\right)\left(\frac{b}{b+c} + \frac{a}{a+b}\right)\left(\frac{c}{c+a} + \frac{b}{b+c}\right). \tag{2}$$

Since $a+b \geq 2\sqrt{ab}$, $b+c \geq 2+\sqrt{bc}$, $c+a \geq 2\sqrt{ca}$, we get from (2) that

$$E \leq \frac{1}{8}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{c}{a}}\right)\left(\sqrt{\frac{b}{c}} + \sqrt{\frac{a}{b}}\right)\left(\frac{c}{c+a} + \frac{b}{b+c}\right)$$
$$= \frac{1}{8abc}(a+\sqrt{bc})(b+\sqrt{ca})(c+\sqrt{ab})$$
$$= \frac{1}{8abc}\left(2abc + ab\sqrt{ab} + bc\sqrt{bc} + ca\sqrt{ca} + a^2\sqrt{bc} + b^2\sqrt{ca} + c^2\sqrt{ab}\right). \tag{3}$$

Using AM-GM Inequality again we then obtain from (3) that

$$\begin{aligned} E &\leq \frac{1}{8abc}\Big(2abc + ab\Big(\frac{a+b}{2}\Big) + bc\Big(\frac{b+c}{2}\Big) + ca\Big(\frac{c+a}{2}\Big) \\ &\qquad + a^2\Big(\frac{b+c}{2}\Big) + b^2\Big(\frac{c+a}{2}\Big) + c^2\Big(\frac{a+b}{2}\Big)\Big) \\ &= \frac{1}{8abc}\Big(2abc + a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2\Big) \\ &= \frac{1}{8abc}(a+b)(b+c)(c+a), \end{aligned}$$

so (1) holds and the proof is complete.

4389. *Proposed by Daniel Sitaru.*

Consider the real numbers a, b, c and d. Prove that

$$a(c+d) - b(c-d) \leq \sqrt{2(a^2+b^2)(c^2+d^2)}.$$

We received 21 solutions, all correct, and will feature the solution by Michel Bataille.

The inequality certainly holds if $a(c+d) - b(c-d) < 0$ and otherwise is equivalent to

$$(ac + ad - bc + bd)^2 \leq 2(a^2+b^2)(c^2+d^2).$$

Now, a simple calculation shows that

$$2(a^2+b^2)(c^2+d^2) - (ac+ad-bc+bd)^2 = (ac+bd-ad+bc)^2 \geq 0$$

so we are done.

4390. *Proposed by Marius Drăgan and Neculai Stanciu.*

Let x, y and z be positive real numbers with $x + y + z = m$. Find the minimum value of the expression

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}.$$

We received 5 submissions of which 2 were correct and complete. We present the solution by the proposers, with minor edits.

We use the following result.

Let $s > 0$ and let $F(x_1, x_2, \ldots, x_n)$ be a symmetrical continuous function on the compact set in $\mathbb{R}^n$

$$S = \{(x_1, x_2, \ldots, x_n)\colon x_1 + x_2 + \cdots + x_n = s, x_1 \geq 0, \ldots, x_n \geq 0\}.$$

If

$$F(x_1, x_2, \ldots, x_n) \geq \min\Big\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), \\ F(0, x_1+x_2, x_3, \ldots, x_n)\Big\} \qquad (1)$$

for all $(x_1, \ldots, x_n) \in S$ with $x_1 > x_2 > 0$, then

$$F(x_1, \ldots, x_n) \geq \min_{1 \leq k \leq n} F\left(\frac{s}{k}, \ldots, \frac{s}{k}, 0, \ldots, 0\right),$$

for all $(x_1, \ldots, x_n) \in S$.

A proof of this can be found in *Algebraic inequalities, Old and New Methods*, V. Cârtoaje, Gil Publishing house, 2006, 267-269.

Let $F(x,y,z) = \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2}$, and let x, y, z be positive reals such that $x+y+z = m$ and $x > y > 0$.

We verify (1) by proving that

$$\text{if } F(x,y,z) < F\left(\tfrac{x+y}{2}, \tfrac{x+y}{2}, z\right), \text{ then } F(x,y,z) \geq F(0, x+y, z).$$

Put $t = \frac{x+y}{2}$, and $p = xy$. Rearranging the inequality $F(x,y,z) < F\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right)$, gives

$$\frac{(t^2-p)(4t^2+2p-2)}{(1+x^2)(1+y^2)(1+t^2)} < 0,$$

and since $t^2 - p > 0$, we have $4t^2 < 2-2p$. It follows that $4t^2p < 2p-2p^2 \leq 2-2p$, and so $2-4t^2p-2p > 0$. The inequality $F(x,y,z) \geq F(0, x+y, z)$ can be rearranged to obtain the equivalent inequality

$$\frac{xy(2-4t^2p-2p)}{(1+x^2)(1+y^2)(1+t^2)} \geq 0,$$

which follows from the above. Thus we can apply the cited result, which gives

$$\begin{aligned} F(x,y,z) &\geq \min\left\{F(m,0,0), F\left(\frac{m}{2}, \frac{m}{2}, 0\right), F\left(\frac{m}{3}, \frac{m}{3}, \frac{m}{3}\right)\right\} \\ &= \min\left\{\frac{2m^2+3}{m^2+1}, \frac{m^2+12}{m^2+4}, \frac{27}{m^2+9}\right\} \\ &= \begin{cases} \dfrac{2m^2+3}{m^2+1}, & m \in (0, \sqrt{2}] \\[2ex] \dfrac{m^2+12}{m^2+4}, & m \in (\sqrt{2}, \sqrt{6}] \\[2ex] \dfrac{27}{m^2+9}, & m \in (\sqrt{6}, \infty). \end{cases} \end{aligned}$$

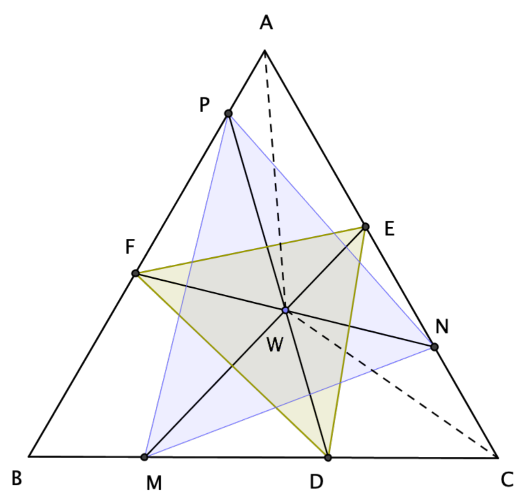
4391. *Proposed by Leonard Giugiuc and Oai Thanh Dao.*

Let ABC be an equilateral triangle and let W be a point inside ABC. A line l_1 through W intersects the segments BC and AB in D and P, respectively. Similarly, a line l_2 through W intersects AC and BC in E and M, and a line l_3 through W intersects AB and AC in F and N. If

$$\angle DWE = \angle EWF = \angle FWD = 120^\circ,$$

show that the triangles DEF and MNP are similar.

We received 6 solutions. We present the solution by C. R. Pranesachar.



It is easy to see that each of the six angles made by l_1, l_2 and l_3 at W is 60°. Thus $MWNC$, $DWEC$ and $EWFA$ each have opposite angles which add up to 180°, and are hence cyclic quadrilaterals. So

$$\angle WNM = \angle WCM = \angle WCD = \angle WED, \text{ and}$$
$$\angle WNP = \angle WAP = \angle WAF = \angle WEF.$$

Adding, we get $\angle MNP = \angle DEF$. Similarly, $\angle NPM = \angle EFD$ and $\angle PMN = \angle FDE$. Thus triangles DEF and MNP are similar.

Editor's Comments. As noted by J. Chris Fisher, this problem is a special consequence of Miquel's theorem. Namely, in any triangle ABC, if we choose points D on BC, E on AC and F on AB, the circumcircles of the three triangles DCE, EAF and FBD intersect at one point, called the Miquel point for DEF. If MNP is constructed analogously and has the same Miquel point as DEF then MNP is similar to DEF. For more details, see Roger A. Johnson's "Advanced Euclidean Geometry", paragraphs 183-188, and problem 1992: 176, 1993: 152-153 proposed by J. Chris Fisher, Dan Pedoe and Robert E. Jamison.

4392. *Proposed by Leonard Giugiuc and Kadir Altintas.*

Let M be an interior point of a triangle ABC with sides $BC = a$, $CA = b$ and $AB = c$. If $MA = x, MB = y$ and $MC = z$, then prove that if

$$\sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} + \sqrt{(c+x-y)(c-x+y)} = \sqrt{3}(x+y+z),$$

then ABC is equilateral.

We received 5 solutions, 4 of which were correct. We present the solution by Sushanth Sathish Kumar.

By Cauchy-Schwarz, we may estimate

$$\begin{aligned} a+b+c &= \sqrt{[(a+y-z)+(b+z-x)+(c+x-y)]} \\ &\quad \cdot \sqrt{[(a-y+z)+(b-z+x)+(c-x+y)]} \\ &\geq \sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} \\ &\quad + \sqrt{(c+x-y)(c-x+y)} \\ &= \sqrt{3}(x+y+z) \end{aligned}$$

But on the other hand, the estimate

$$\sqrt{(a+y-z)(a-y+z)} = \sqrt{a^2-(y-z)^2} \geq a,$$

gives that

$$\sqrt{3}(x+y+z) \geq a+b+c.$$

Thus, it follows that

$$a+b+c = \sqrt{3}(x+y+z).$$

Additionally, we have $x = y = z$, since otherwise equality cannot occur in the second estimate.

Since $x = y = z$, M must be the circumcenter of ABC. Hence, $x = y = z = R$, where R is the circumradius of the triangle. So the equation

$$a+b+c = \sqrt{3}(x+y+z)$$

reduces to

$$\sin A + \sin B + \sin C = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} = \frac{3\sqrt{3}}{2}.$$

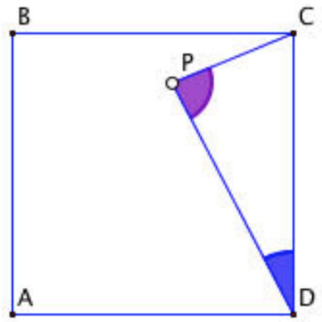
But Jensen's inequality implies that

$$\sin A + \sin B + \sin C \leq 3 \sin\left(\frac{A+B+C}{3}\right) = 3\sqrt{3}/2,$$

with equality holding if and only if $A = B = C = \pi/3$. Consequently, equality must occur in the above equation, which gives $A = B = C = \pi/3$, or that ABC is equilateral.

4393. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square. Find the locus of points P inside $ABCD$ such that $\cot\angle CPD + \cot\angle CDP = 2$.



We received 13 solutions, all correct, We feature 4 of them here.

Solution 1, by Paul Bracken and Sushanth Sathish Kumar (done independently).

Assign coordinates $A \sim (0,0)$, $B \sim (0,1)$, $C \sim (1,1)$, $D \sim (1,0)$, $P \sim (x,y)$, and let γ, δ and θ be the respective angles at C, D and P in triangle CDP. Then

$$\cot\gamma = \frac{1-y}{1-x}, \quad \cot\delta = \frac{y}{1-x},$$

and

$$\cot\theta = \frac{1-\cot\gamma\cot\delta}{\cot\gamma+\cot\delta} = \frac{(1-x)^2+(y^2-y)}{1-x}.$$

Therefore

$$2(1-x) = (1-x)(\cot\theta+\cot\delta) = (1-2x+x^2)+y^2,$$

whence $x^2+y^2 = 1$. The locus of P is a quarter-circle centred at A passing through the vertices B and D.

Solution 2, by Michel Bataille.

Let c, d, p, r be the respective lengths of DP, PC, CD, PA; let h and k be the respective distances from P to CD and AD; let γ, δ, θ be the respective angles at C, D, P in triangle CPD, and let S be the area of triangle CPD.

Noting that

$$2S = pd\sin\gamma = cd\sin\theta = cp\sin\delta = ph,$$

we have that

$$2 = \frac{\cos\theta}{\sin\theta} + \frac{\cos\delta}{\sin\delta} = \frac{\sin(\theta+\delta)}{\sin\theta\sin\delta} = \frac{\sin\gamma}{\sin\theta\sin\delta} = \frac{c^2(pd\sin\gamma)}{(cd\sin\theta)(cp\sin\delta)} = \frac{c^2}{2S},$$

whence $c^2 = 4S = 2ph$.

Then

$$r^2 = k^2 + (p-h)^2 = (c^2 - h^2) + (p^2 - 2ph + h^2) = p^2.$$

Therefore, for every position of P, its distance from A is equal to p, so that the locus of P is a quarter-circle with centre A passing through B and D.

Solution 3, by C.R. Pranesachar.

From the Cosine Law, we obtain for any triangle ABC,

$$\cot A = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{b^2+c^2-a^2}{4[ABC]},$$

where $[ABC]$ is the area of the triangle. Applying this to triangle CPD and using the notation of the previous solution, we have that

$$2 = \cot\theta + \cot\delta = \frac{(c^2+d^2-p^2)+(c^2+p^2-d^2)}{4[CPD]} = \frac{2c^2}{2ph}.$$

Thus $c^2 = 2ph$ and we can complete the argument as in Solution 2.

Solution 4, by Václav Konečný.

Adopt the notation of Solution 2. Let Q and R be the respective feet of the perpendiculars from A and C to the line DP. Then $\angle QAD = \delta$ and so the length of QD is equal to $p\sin\delta$.

The length p of PD is equal to the sum of the lengths of PR and RD when $\theta \le 90^\circ$ and to the difference of these lengths when $\theta > 90^\circ$. The length of RD is equal to $p\cos\delta$. The length of PR is equal to $p\sin\delta\cot\theta$ when $\theta \le 90^\circ$ and $-p\sin\delta\cot\theta$ when $\theta > 90^\circ$. In any case, we find that the length of PD is equal to

$$p\cos\delta + p\sin\delta\cot\theta = p\cos\delta + p\sin\delta(2-\cot\delta) = 2p\sin\delta,$$

i.e. twice the length of QD. It follows that the triangle PAD is isosceles and so the lengths of AP and AD are both equal to p.

Therefore P lies on the quarter-circle with centre A through B and C.

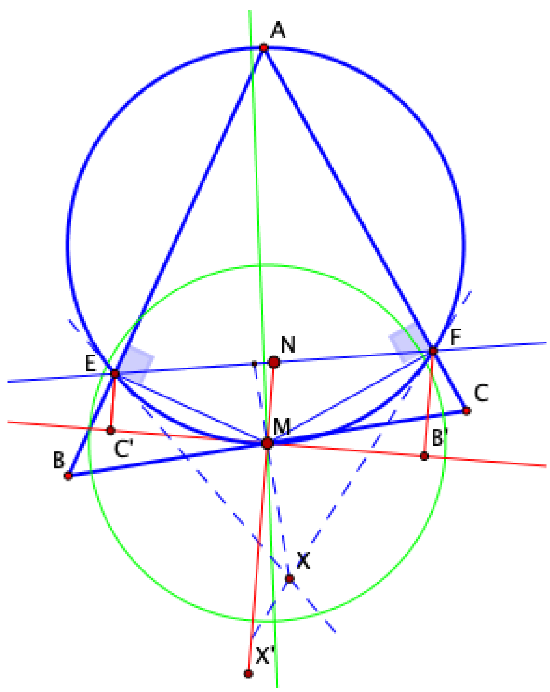
4394. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and $M \in BC$, $BM \equiv MC$, $E \in AB$, $F \in AC$, $\measuredangle BEM \equiv \measuredangle CFM = 90^\circ$. The two tangents at the points E and F to the circumcircle of $\triangle MEF$ intersect at the point X. If $XM \cap EF = \{Y\}$, show that $YB = YC$.

We received 4 submissions, all correct, and feature two of them.

Solution 1, by Shuborno Das.

To show $YB = YC$, it suffices to show that $YM \perp BC$. But since X, M, and Y are collinear, we just need to prove that $XM \perp BC$. As X is the intersection of the tangents to the circle MEF at E and F, MX is the M-symmedian in ΔMEF. Let's consider a mapping Ψ that is the product of the inversion in the circle centered at M with radius $\sqrt{ME \cdot MF}$ followed by the reflection in the bisector of $\angle EMF$. Note that E and F are interchanged by Ψ. As in the figure, we shall use a prime to denote the image of a point under Ψ.

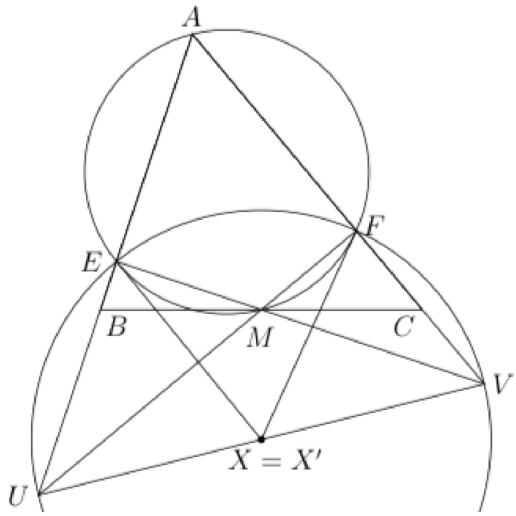


Because $\angle MEB = 90°$, we have $\angle MB'E' = \angle MB'F = 90°$, and similarly, $\angle MC'E = 90°$. Since $MB = MC$ and $M \in BC$, it follows that $MB' = MC'$ and $M \in B'C'$. Moreover, because MX is the M-symmedian in ΔMEF, MX' is the M-median in ΔMEF. Because lines through M are sent by Ψ to their reflection in the bisector of $\angle EMF$, our problem is reduced to showing that $MX' \perp B'C'$. Let MX' meet EF at N; then $NE = NF$. But $MB' = MC'$ and $EC' || FB'$ (because both lines are perpendicular to $B'C'$), while MN is the line that joins the midpoints of $B'C'$ and FE, whence $NM \perp B'C'$, and we are done.

Solution 2, by Sushanth Sathish Kumar.

As in Solution 1, we just need to prove that $XM \perp BC$.

Define $U = FM \cap AE$ and $V = EM \cap AF$. Let X' be the midpoint of UV. We claim that $X = X'$: note that $\angle UEV = \angle BEM = 90^\circ = \angle MFC = \angle UFV$, so X' is the circumcenter of cyclic quadrilateral $UEFV$. As such, triangles $X'UE$, $X'EF$, and $X'FV$ are all isosceles with vertex X'.



Abbreviating $\angle AUV = U$, $\angle AVU = V$ and angle chasing yields

$$\begin{aligned}\angle EX'F &= 180^\circ - \angle UX'E - \angle FX'V \\ &= 180^\circ - (180^\circ - 2U) - (180^\circ - 2V) \\ &= 2U + 2V - 180^\circ.\end{aligned}$$

Thus, from triangle $X'EF$

$$\angle X'EF = \angle X'FE = 180^\circ - U - V = \angle UAV = \angle EAF,$$

which implies that $X'E$ and $X'F$ are tangent to $(AEMF)$. Moreover, this implies X is the circumcenter of $(UEFV)$. Let rays MB, and MC meet $(UEFV)$ at points P, and Q, respectively. By the converse to the butterfly theorem, M is the midpoint of the chord PQ, and (recalling that X is the center of the circle $(UEFV)$) we see this implies that $XM \perp PQ$. Consequently, $XM \perp BC$, as desired.

We end with a projective proof of the converse of the butterfly theorem. Perspectivities from points E and F of the circle $(UPEFUV)$ give

$$(P, M; B, Q) \stackrel{E}{=} (P, V; U, Q) \stackrel{F}{=} (P, C; M, Q),$$

where $(P, M; B, Q)$, is the cross-ratio. Hence (using the hypothesis $BM = MC$),

$$\frac{BP}{BM} \cdot \frac{QM}{QP} = \frac{MP}{MC} \cdot \frac{QC}{QP} \Longrightarrow (BP)(QM) = (MP)(QC).$$

Taking into account that $BP = MP - MB$, we get

$$(MP - MB)(QM) = (MP)(QC) \implies MB(QM) = MP(QM - QC) = MP(MC),$$

which gives $MP = MQ$; thus, M is the midpoint of chord PQ and we are done.

Editor's comments. This editor was unable to find an explicit statement of the butterfly theorem's converse. Many of the vast variety of published proofs of the theorem are reversible, and thus the converse has been tacitly, yet firmly, established. See, for example the first of the editor's proofs in the second volume of ***Crux*** (when the journal was called *Eureka*) [1976: 2-3], or about half of the many proofs presented by Leon Bankoff in "The Metamorphosis of the Butterfly Problem" [*Mathematics Magazine*, **60**:4 (Oct. 1987) 195-210], including a version of the argument above in Solution 2.

4395. *Proposed by Michel Bataille.*

Let $ABCD$ be a tetrahedron and let $A_1, B_1, C_1, A_2, B_2, C_2$ be the midpoints of BC, CA, AB, DA, DB, DC, respectively. Prove that

$$(\overrightarrow{DA} \cdot \overrightarrow{BC})A_1A_2^2 + (\overrightarrow{DB} \cdot \overrightarrow{CA})B_1B_2^2 + (\overrightarrow{DC} \cdot \overrightarrow{AB})C_1C_2^2 = 0,$$

where $\overrightarrow{X} \cdot \overrightarrow{Y}$ denotes the dot product of the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$.

We received 7 solutions, all of which were correct. We present the solution by Oliver Geupel.

Consider location vectors relative to the origin at point D. We have

$$2\overrightarrow{A_1} = \overrightarrow{B} + \overrightarrow{C},\ \ 2\overrightarrow{B_1} = \overrightarrow{C} + \overrightarrow{A},\ \ 2\overrightarrow{C_1} = \overrightarrow{A} + \overrightarrow{B},\ \ 2\overrightarrow{A_2} = \overrightarrow{A},\ \ 2\overrightarrow{B_2} = \overrightarrow{B},\ \ 2\overrightarrow{C_2} = \overrightarrow{C}.$$

We use the alternative notation $\left\langle \overrightarrow{X}, \overrightarrow{Y} \right\rangle$ for the inner product of vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ for reasons of readability. Let

$$a = \left\langle \overrightarrow{B}, \overrightarrow{C} \right\rangle, \qquad b = \left\langle \overrightarrow{C}, \overrightarrow{A} \right\rangle, \qquad c = \left\langle \overrightarrow{A}, \overrightarrow{B} \right\rangle.$$

It follows that

$$\begin{aligned} 4\left\langle \overrightarrow{DA}, \overrightarrow{BC} \right\rangle A_1A_2^2 &= \left\langle \overrightarrow{A}, \overrightarrow{C} - \overrightarrow{B} \right\rangle \left(\overrightarrow{A} - \overrightarrow{B} - \overrightarrow{C} \right)^2 \\ &= (b - c)\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2 + 2(a - b - c) \right) \\ &= (b - c)\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2 \right) + 2\left(c^2 - b^2 + ab - ac \right). \end{aligned}$$

With similar identities for the other two terms, we obtain

$$\begin{aligned}
&4\left(\left\langle \overrightarrow{DA}, \overrightarrow{BC}\right\rangle A_1A_2^2 + \left\langle \overrightarrow{DB}, \overrightarrow{CA}\right\rangle B_1B_2^2 + \left\langle \overrightarrow{DC}, \overrightarrow{AB}\right\rangle C_1C_2^2\right)\\
&= ((b-c)+(c-a)+(a-b))\left(\overrightarrow{A}^2 + \overrightarrow{B}^2 + \overrightarrow{C}^2\right)\\
&\quad + 2\left(c^2-b^2+ab-ca\right) + 2\left(a^2-c^2+bc-ab\right) + 2\left(b^2-c^2+ca-bc\right)\\
&= 0,
\end{aligned}$$

which proves the desired identity.

4396. *Proposed by David Lowry-Duda.*

Show that there is a bijection $f : \mathbb{N} \mapsto \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} \dfrac{1}{n+f(n)}$ converges or show that no such bijection exists.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Oliver Geupel.

Such a bijection exists. For $n \in \mathbb{N}$, let $a_n = n^2$ be the sequence of all perfect squares in ascending order and let

$$b_1 = 2, b_2 = 3, b_3 = 5, b_4 = 6, b_5 = 7, b_6 = 8, b_7 = 10, \ldots$$

be the sequence of all positive non-squares in ascending order. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(a_n) = b_n$ and $f(b_n) = a_n$ for all $n \in \mathbb{N}$. This defines a bijection because $\mathbb{N}$ is the disjoint union of $\{a_n : n \in \mathbb{N}\}$ and $\{b_n : n \in \mathbb{N}\}$. For every $k \in \mathbb{N}$ each pair $\{a_k, b_k\}$ appears as $\{n, f(n)\}$ for exactly two distinct values of $n \in \mathbb{N}$. We get

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)} = 2\sum_{n=1}^{\infty} \frac{1}{a_n+b_n} < 2\sum_{n=1}^{\infty} \frac{1}{a_n} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}.$$

Consequently, the series is convergent.

4397. *Proposed by George Stoica.*

Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n\}$. Show that there exists $k' \in \{0, 1, \ldots, 2^{n+1}\}$ such that

$$\left|\sin \frac{k'\pi}{2^{n+2}} - \frac{k}{2^n}\right| \le \frac{1}{2^n}.$$

We received 2 solutions, both of which were correct. We present the solution by Omran Kouba.

Let

$$a_m = \sin\left(\frac{\pi m}{2^{n+2}}\right) \text{ for } m \in \{0, 1, \ldots, 2^{n+1}\}.$$

Clearly,

$$0 = a_0 < a_1 < \cdots < a_m < a_{m+1} < \cdots < a_{2^n} = 1.$$

Further, for $0 \leq x \leq y \leq \pi/2$, we have

$$0 \leq \sin y - \sin x = \int_x^y \cos t \, dt \leq \int_x^y dt \leq y - x.$$

So, since $\pi < 4$, we have

$$a_{m+1} - a_m < \frac{\pi}{2^{n+2}} < \frac{1}{2^n}, \quad \text{for } m = 0, 1, \ldots, 2^{n+1} - 1.$$

Now, given $k \in \{0, 1, \ldots, 2^n\}$ we consider

$$\mathcal{N}_k = \left\{ m \in \{0, 1, \ldots, 2^{n+1}\} : a_m \leq \frac{k}{2^n} \right\}.$$

Clearly, $\mathcal{N}_k \neq \emptyset$ because it contains 0 and it has 2^{n+1} as an upper bound. So we may define $k' = \max \mathcal{N}_k$.

- If $k' = 2^{n+1}$ then $a_{k'} = 1$; this corresponds to the case $k = 2^n$ and the desired inequality holds trivially in this case.
- If $k' < 2^{m+1}$ then by definition of k' we have $a_{k'} \leq k2^{-n} < a_{k'+1}$ so

  $$0 < \frac{k}{2^n} - a_{k'} < a_{k'+1} - a_{k'} < \frac{1}{2^n}.$$

  and the desired inequality follows.

4398. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}^*$, we have

$$\frac{1}{2n-1} + \int_0^1 \sin^2(x^n) dx \geq \frac{2}{n}(1 - \cos 1).$$

We received 10 submissions, including the one from the proposer, all of which are correct. We present the nearly identical solution by Michel Bataille, Leonard Giugiuc, Digby Smith, and Daniel Văcaru.

Since $a^2 + b^2 \geq 2ab$ for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{2n-1} + \int_0^1 \sin^2(x^n) dx &= \int_0^1 (x^{2n-2} + sin^2(x^n)) dx \\
&\geq 2 \int_0^1 x^{n-1} \sin(x^n) dx \\
&= -\frac{2}{n} \cos(x^n) \Big|_0^1 \\
&= -\frac{2}{n}(\cos 1 - \cos 0) \\
&= \frac{2}{n}(1 - \cos 1).
\end{aligned}$$

4399. *Proposed by Lacin Can Atis.*

Let $ABCDE$ be a pentagon. Prove that

$$|AB||EC||ED| + |BC||EA||ED| + |CD||EA||EB| \geq |AD||EB||EC|.$$

When does equality hold?

We received 7 solutions. We present the one by Michel Bataille.

We shall use the notation XY instead of $|XY|$. We observe that the left-hand side $\mathcal{L}$ of the inequality rewrites as

$$\begin{aligned}\mathcal{L} &= ED \cdot (AB \cdot EC + EA \cdot BC - EB \cdot AC) \\ &\quad + EB \cdot (ED \cdot AC + EA \cdot CD - AD \cdot EC) + AD \cdot EB \cdot EC.\end{aligned}$$

From Ptolemy's inequality, we have

$$AB \cdot EC + EA \cdot BC - EB \cdot AC \geq 0$$

with equality if and only if A, B, C, E lie, in this order, on a circle and

$$ED \cdot AC + EA \cdot CD - AD \cdot EC \geq 0$$

with equality if and only if A, C, D, E lie, in this order, on a circle.

It follows that $\mathcal{L} \geq AD \cdot EB \cdot EC$, the desired inequality, and that equality holds if and only if $ABCDE$ is a cyclic pentagon.

4400. *Proposed by Daniel Sitaru.*

Prove that in any triangle ABC, the following relationship holds:

$$\sum_{cyc} \frac{\sin(\frac{\pi}{3} - \frac{A}{2}) \sin(\frac{\pi}{3} - \frac{B}{2})}{\cos(\frac{C-A}{2}) \cos(\frac{C-B}{2})} < 1.$$

We received 6 submissions, including the one from the proposer. As it turned out, the proposed inequality is false, and four of the five solvers give various counterexamples. We will feature below some of the given solutions and comments.

Counterexample 1, given by Leonard Giugiuc, enhanced by the editor.

Let E denote the LHS of the given inequality. Set $A = B \in (0, \frac{\pi}{2})$ so $C = \pi - 2A$. Then

$$\frac{\sin(\frac{\pi}{3} - \frac{A}{2}) \sin(\frac{\pi}{3} - \frac{B}{2})}{\cos(\frac{C-A}{2}) \cos(\frac{C-B}{2})} = \frac{\sin^2(\frac{\pi}{3} - \frac{A}{2})}{\cos^2(\frac{\pi-3A}{2})}, \tag{1}$$

$$\begin{aligned}\frac{\sin(\frac{\pi}{3} - \frac{B}{2}) \sin(\frac{\pi}{3} - \frac{C}{2})}{\cos(\frac{A-B}{2}) \cos(\frac{A-C}{2})} &= \frac{\sin(\frac{\pi}{3} - \frac{A}{2}) \sin(A - \frac{\pi}{6})}{\cos(\frac{3A-\pi}{2})} \\ &= \frac{\sin(\frac{\pi}{3} - \frac{A}{2}) \sin(A - \frac{\pi}{6}) \cos(\frac{\pi-3A}{2})}{\cos^2(\frac{\pi-3A}{2})}, \qquad (2)\end{aligned}$$

and

$$\frac{\sin(\frac{\pi}{3}-\frac{C}{2})\sin(\frac{\pi}{3}-\frac{A}{2})}{\cos(\frac{B-C}{2})\cos(\frac{B-A}{2})} = \frac{\sin(A-\frac{\pi}{6})\sin(\frac{\pi}{3}-\frac{A}{2})}{\cos(\frac{3A-\pi}{2})}$$
$$= \frac{\sin(A-\frac{\pi}{6})\sin(\frac{\pi}{3}-\frac{A}{2})\cos(\frac{3A-\pi}{2})}{\cos^2(\frac{\pi-3A}{2})}. \qquad (3)$$

From (1)+(2)+(3) we then obtain

$$E = \frac{1}{\cos^2(\frac{\pi-3A}{2})}\left(\sin^2\left(\frac{\pi}{3}-\frac{A}{2}\right)+2\sin\left(\frac{\pi}{3}-\frac{A}{2}\right)\sin\left(A-\frac{\pi}{6}\right)\cos\left(\frac{3A-\pi}{2}\right)\right).$$

Now, let $A \to 0^+$. Then

$$\cos^2\left(\frac{\pi-3A}{2}\right)\to 0^+, \sin\left(\frac{\pi}{3}-\frac{A}{2}\right)\to\frac{\sqrt{3}}{2}, \text{ and } \sin\left(A-\frac{\pi}{6}\right)\to -\frac{1}{2}$$

so

$$\lim_{A\to 0^+} E = \infty.$$

Counterexample 2, by Alexandru Daniel Pîrvuceanu, with all the details supplied by the editor.

Let $A = 150°, B = C = 15°$. Then with calculations carried to 4 decimal places, we have:

$$\frac{\sin(\frac{\pi}{3}-\frac{A}{2})\sin(\frac{\pi}{3}-\frac{B}{2})}{\cos(\frac{C-A}{2})\cos(\frac{C-B}{2})} = \frac{-(\sin 15°)(\sin 52.5°)}{\cos(67.5°)\cos 0°} = -0.5366$$
$$\frac{\sin(\frac{\pi}{3}-\frac{B}{2})\sin(\frac{\pi}{3}-\frac{C}{2})}{\cos(\frac{A-B}{2})\cos(\frac{A-C}{2})} = \frac{\sin^2(52.5°)}{\cos^2(67.5°)} = 4.2980$$
$$\frac{\sin(\frac{\pi}{3}-\frac{C}{2})\sin(\frac{\pi}{3}-\frac{A}{2})}{\cos(\frac{B-C}{2})\cos(\frac{B-A}{2})} = \frac{\sin(52.5°)(-\sin 15°)}{\cos(67.5°)} = -0.5366$$

Hence, $E = 4.2980 - 2(0.5366) = 3.2248 > 1$.

Editor's note. Digby Smith remarked that the given inequality holds if each of $A, B, C > \frac{\pi}{6}$, and Pranesachar gave examples to show that $E < 1, E = 1$, and $E > 1$ are all possible.