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THE CONTEST CORNER

No. 52

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

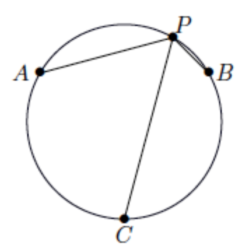
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er octobre 2017**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

CC256. Tous les sommets d'un polygone P se situent à des points à coordonnées entières dans le plan (c'est-à-dire, les deux coordonnées sont entières), et les côtés de P ont tous une longueur entière. Démontrer que le périmètre de P doit être pair.

CC257. On vous affirme qu'il est possible de déterminer S, un sous ensemble des entiers non négatifs, de façon à ce que tout entier non négatif peut être représenté uniquement sous la forme $x + 2y$ avec $x, y \in S$. Démontrer cette affirmation, ou démontrer le contraire.

CC258. Les trois points A, B et C au schéma ci-bas sont les sommets d'un triangle équilatéral. À partir d'un point P sur le cercle passant par A, B et C, considérer les trois distances AP, BP et CP. Démontrer que la somme des deux plus petites de ces distances égale la troisième.



CC259. Si on vous fournit la surface A et le périmètre P d'un rectangle, cette information suffit-elle pour déterminer les longueurs des côtés ?

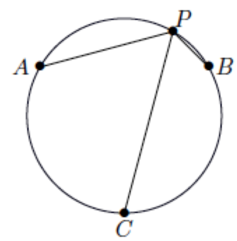
CC260. Supposons l'existence d'un dé à 9 faces, construit de façon à ce que, lorsqu'il est lancé, chacune des faces (numérotées de 1 à 9) intervient avec probabilité égale. Déterminer la probabilité qu'après n lancers du dé, le produit de tous les nombres obtenus est divisible par 14.

. . . .

CC256. All vertices of a polygon P lie at points with integer coordinates in the plane (that is to say, both their co-ordinates are integers), and all sides of P have integer lengths. Prove that the perimeter of P must be even.

CC257. It is asserted that one can find a subset S of the nonnegative integers such that every nonnegative integer can be written uniquely in the form $x+2y$ for $x, y \in S$. Prove or disprove the assertion.

CC258. The three points A, B and C in the diagram are vertices of an equilateral triangle. Given any point P on the circle containing A, B and C, consider the three distances AP, BP and CP. Prove that the sum of the two shorter distances gives the longer distance.



CC259. If you are told that a rectangle has area A and perimeter P, is that sufficient information to determine its side lengths?

CC260. Assume you have a 9-faced die, appropriately constructed so that when the die is thrown, each of the faces (which are numbered 1 to 9) occurs with equal probability. Determine the probability that after n throws of the die, the product of all the numbers thrown will be divisible by 14.

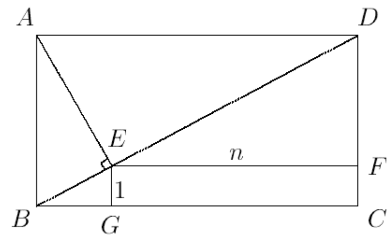
CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(2), p. 50–52.

CC206. A rectangle $ABCD$ has diagonal of length d. The line AE is drawn perpendicular to the diagonal BD. The sides of the rectangle $EFCG$ have lengths n and 1. Prove that $d^{2/3} = n^{2/3} + 1$.

Originally Question 2, Part B of the 1996 COMC paper (First Canadian Open Mathematics Challenge).

We received eleven correct solutions and present the solution by Ángel Plaza.



Let x be the length of BG. By the Pythagorean Theorem,

$$|BE| = \sqrt{1+x^2}, \quad |AE| = x\sqrt{1+x^2}, \quad |AB| = 1+x^2$$

since triangles ABE and BEG are similar. Therefore, $|DF| = x^2$ and $|EF| = n = x^3$ also because triangle ADF is similar to the previously considered triangles. By the Pythagorean Theorem,

$$|ED| = x^2\sqrt{1+x^2}, \quad \text{so} \quad d = |BE| + |ED| = (1+x^2)^{3/2}.$$

Then

$$d^{2/3} = 1 + x^2 = n^{2/3} + 1,$$

proving the claim.

CC207. Consider the ten numbers $ar, ar^2, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6, determine their product.

Originally Question 2, Part B of the 1997 COMC Paper.

We received ten correct solutions and one incorrect solution. We present the solution of Missouri State Problem Solving Group.

By the geometric series, the sum of the ten numbers is

$$\sum_{k=1}^{10} ar^k = ar\frac{1-r^{10}}{1-r}.$$

The sum of their reciprocals is also geometric, hence

$$\sum_{k=1}^{10} \frac{1}{ar^k} = \frac{1}{ar}\frac{1-r^{-10}}{1-r^{-1}} = \frac{1}{ar}\frac{r^{10}-1}{r^{10}-r^9} = \frac{1}{ar^{10}}\frac{1-r^{10}}{1-r}.$$

Therefore,

$$3 = 18 \div 6 = \left(\sum_{k=1}^{10} ar^k\right) \div \left(\sum_{k=1}^{10} \frac{1}{ar^k}\right) = ar\frac{1-r^{10}}{1-r}ar^{10}\frac{1-r}{1-r^{10}} = a^2r^{11}.$$

It then follows that the product of the ten numbers is

$$\prod_{k=1}^{10} ar^k = a^{10}r^{1+2+\cdots+10} = a^{10}r^{55} = \left(a^2r^{11}\right)^5 = 3^5 = 243.$$

CC208.

a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the *four* possible ordered pairs (A, B).

b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k.

   i) If a and b are integers and $10a + b = 7m$ for some integer m, prove that $a - 2b$ is a multiple of 7.

   ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

Originally Question 2, part B of the 2002 COMC Paper.

We received eight solutions. We present the solution by Titu Zvonaru.

a) Since $36 = 4 \cdot 9$ and the integer $2A5$ is odd, then $13B$ is divisible by 4, hence $B = 2$ or $B = 6$. If $B = 2$, then 132 is divisible by 3. We deduce that $2A5$ is divisible by 3, hence $A = 2, 5, 8$. If $B = 6$, then $2A5$ is divisible by 9, hence $A = 2$. It follows that $(A, B) \in \{(2, 2), (5, 2), (8, 2), (2, 6)\}$.

b) If $10a + b$ is a multiple of 7, then

$$5(10a + b) = 49a + 7b + a - 2b = 7(7a + b) + (a - 2b)$$

is a multiple of 7, hence $a - 2b$ is a multiple of 7.

If $5c + 4d$ is a multiple of 7, then

$$5(5c + 4d) = 21c + 21d + 4c - d = 7(3c + 3d) + (4c - d)$$

is a multiple of 7, hence $4c - d$ is a multiple of 7.

CC209.

a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.

b) Determine the *one* value of x such that $x - \sqrt{4x + 12} = 0$. Justify your answer.

c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

Originally Question 2, Part B of the 2004 COMC.

We received five correct solutions. We present the solution of Ángel Plaza.

a) $x = 6$ and $x = -2$ are the two solution of $x^2 - 4x - 12 = 0$. This can be seen through factoring or by the quadratic formula.

b) If $x - \sqrt{4x+12} = 0$ then $x = \sqrt{4x+12}$ and $x^2 = 4x+12$, or rather $x^2 - 4x - 12 = 0$. So either $x = 6$ or $x = -2$. Checking we find $x = 6$ is the *one* value of x such that $x - \sqrt{4x+12} = 0$.

c) Notice that if $P(x) = x^2 - 4x - c$, then the solutions of the equation $P(x) - \sqrt{8P(x)} = 0$ are between the solutions of the equation $(P(x))^2 = 8P(x)$, that is $P(x)\,(P(x) - 8) = 0$.

Let Δ the discriminant of $P(x)$, that is $\Delta = 16+4c$, and Δ^* the discriminant of $P(x)-8$, that is $\Delta^* = 16+4c+32 = 4(12+c)$. Then $P(x)$ has two different real roots if and only if $c > -4$, one double real root if $c = -4$ and two not real roots if $c < -4$. Analogously, $P(x) - 8$ has two different real roots if and only if $c > -12$, one double real root if $c = -12$ and two non-real roots if $c < -12$. Consequently, equation $P(x)\,(P(x) - 8) = 0$ will have two real roots if and only if $c \in (-12, -4)$.

CC210. There is a unique triplet of positive integers (a, b, c) such that $a \leq b \leq c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

Originally Question 2, Part B of the 2013 COMC paper.

We received 13 solutions, of which ten were correct and complete. We present the solution by Steven Chow slightly modified by the editor.

Since $a \leq b \leq c$, then

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \leq \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} \implies a \leq 4.$$

Furthermore,

$$\frac{25}{84} > \frac{1}{a} \implies a \geq 4.$$

It follows that $a = 4$. We have

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \implies \frac{25}{21} = 1 + \frac{1}{b} + \frac{1}{bc} \implies (4b - 21)c = 21.$$

It follows that $4b - 21 > 0$, so $b \geq 6$, which yields $c \geq 6$. Since c divides 21, an easy check gives $b = 6$ and $c = 7$. Therefore $a + b + c = 17$.

Editor's Comments. Konstantine Zelator proved also that without the condition $a \leq b \leq c$, the only positive integer solutions to the equation

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{25}{84}$$

are $(a, b, c) = (4, 6, 7)$ and $(a, b, c) = (7, 1, 12)$.

THE OLYMPIAD CORNER

No. 350

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er octobre 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC316. Soit ABC un triangle rectangle en B et soit BD la hauteur abaissée depuis B jusqu'à AC. Soit P, Q et I les centres des cercles inscrits dans les triangles respectifs ABD, CBD et ABC. Démontrer que le centre du cercle circonscrit au triangle PIQ est situé sur l'hypoténuse AC.

OC317. On a accueilli 110 équipes lors d'un tournoi de volleyball. Chaque équipe a rencontré chaque autre équipe une fois (il n'y a aucun match nul au volleyball). Or, dans n'importe quel ensemble de 55 équipes, il y a une équipe qui n'a pas perdu plus de 4 matchs contre les 54 autres équipes de l'ensemble. Démontrer que dans le tournoi, il y a une équipe qui n'a pas perdu plus de 4 des matchs contre les 109 autres équipes.

OC318. Soit n un entier strictement positif et k un entier de 1 à n. On considère un quadrillage blanc de dimensions $n \times n$ et on procède comme suit:

> On trace k rectangles délimités par les lignes du quadrillage de manière que chaque rectangle recouvre le coin supérieur droit 1×1 du quadrillage $n \times n$ et on peint les k rectangles en noir. De cette manière, il reste toujours de l'espace blanc sur le quadrillage.

Combien de figures blanches différentes peut-on former avec k rectangles de manière que ces figures ne puissent être formées avec moins de k rectangles?

OC319. Soit p un nombre premier supérieur à 30. Démontrer qu'un des nombres suivants,

$$p+1, 2p+1, 3p+1, \ldots, (p-3)p+1,$$

est la somme des carrés de deux entiers.

OC320. Soit n un entier supérieur à 1. On écrit d'abord n ensembles au tableau, puis on leur fait subir une série de manoeuvres. Une *manoeuvre* se fait comme suit:

> On choisit deux ensembles A et B au tableau de manière que l'un ne soit pas un sous-ensemble de l'autre et on les remplace par $A \cap B$ et $A \cup B$.

Déterminer le nombre maximal de manoeuvres pour tous ensembles possibles présentés au départ.

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OC316. Let ABC be a right-angled triangle with $\angle B = 90°$. Let BD be the altitude from B. Let P, Q and I be the incenters of triangles ABD, CBD and ABC respectively. Show that the circumcenter of PIQ lies on the hypotenuse AC.

OC317. In a recent volleyball tournament, 110 teams participated. Every team has played every other team exactly once (there are no ties in volleyball). It turns out that in any set of 55 teams, there is one which has lost to no more than 4 of the remaining 54 teams. Prove that in the entire tournament, there is a team that has lost to no more than 4 of the remaining 109 teams.

OC318. Let n be a positive integer and let k be an integer between 1 and n inclusive. Given an $n \times n$ white board, we do the following process.

> We draw k rectangles with integer side lengths and sides parallel to the sides of the $n \times n$ board, and such that each rectangle covers the top-right corner of the $n \times n$ board. Then, the k rectangles are painted black. This process leaves a white figure in the board.

How many different white figures can be formed with k rectangles that cannot be formed with less than k rectangles?

OC319. Let $p > 30$ be a prime number. Prove that one of the following numbers

$$p + 1, 2p + 1, 3p + 1, \ldots, (p - 3)p + 1$$

is the sum of two integer squares $x^2 + y^2$ for integers x and y.

OC320. Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of *moves* as follows:

> choose two sets A and B on the blackboard such that neither of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$.

Find the maximum number of moves in a sequence for all possible initial sets.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(10), p. 425–426.

OC256. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

Originally problem 3 of the 2014 France Team Selection Test.

We present the solution by Steven Chow. There were no other submissions.

Let $F(x) = x^2 + x + 1$ for all x. Let $P(x)$ be the greatest prime divisor of x for all integers $x \geq 2$. Then, since

$$F(n^2) = n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1) = F(n)F(n-1)$$

we see that

$$P(F(n^2)) = P(F(n)F(n-1)) = \max\{P(F(n)), P(F(n-1))\}.$$

Therefore, $P(F(n^2)) = P(F((n+1)^2))$ if and only if

$$\max\{P(F(n)), P(F(n-1))\} = \max\{P(F(n+1)), P(F(n))\},$$

which is true if and only if either $P(F(n)) \geq \max\{P(F(n+1)), P(F(n-1))\}$ or $P(F(n-1)) = P(F(n+1)) \geq P(F(n))$.

Now, assume towards a contradiction that there exist finitely many integers $n \geq 1$ such that $P(F(n^2)) = P(F((n+1)^2))$. Then, the above implies that there exists an integer k such that for all integers $n \geq k$,

$$P(F(n)) < \max\{P(F(n+1)), P(F(n-1))\}$$

and

$$(P(F(n-1)) \neq P(F(n+1)) \text{ or } P(F(n-1)) = P(F(n+1)) < P(F(n))).$$

Rearranging the above implies that for all integers $n \geq k$, $F(n)$ is either an increasing function or a decreasing function.

If $F(n)$ is an increasing function, then for any integer $a \geq k+1$, $a < a^2$ and so

$$P(F(a)) < P(F(a^2)) = \max\{P(F(a)), P(F(a-1))\} = P(F(a))$$

which is a contradiction. Therefore, $F(n)$ is a decreasing function. However, since $P(F(n)) \geq 2$ for all positive integers n, we have a contradiction by infinite descent.

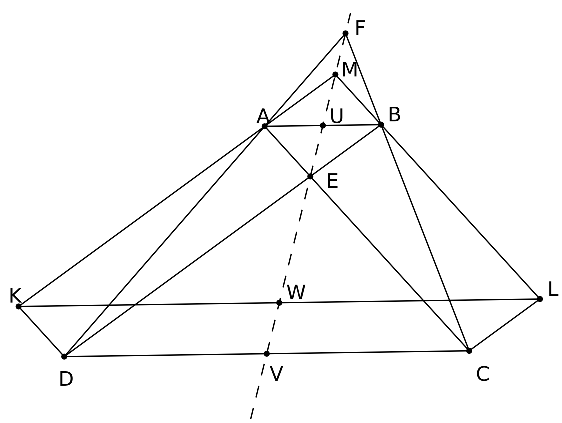
Thus, there exist infinitely many such n.

OC257. Let $ABCD$ be a trapezoid (quadrilateral with one pair of parallel sides) such that $AB < CD$. Suppose that AC and BD meet at E and AD and BC meet at F. Construct the parallelograms $AEDK$ and $BECL$. Prove that EF passes through the midpoint of the segment KL.

Originally problem 3 from day 1 of the 2014 Indonesian Mathematical Olympiad.

We received 5 correct submissions. We present the solution by Michel Bataille.

We assume that the parallel sides are AB and CD and we denote by U and V the midpoints of AB and CD, respectively. We first recall that U, V, E, F are collinear: indeed, the homothety h_F with centre F transforming A into D also transforms B into C, hence $\mathrm{h}_F(U) = V$ and U, V, F are collinear. Similarly, the homothety h_E with centre E such that $\mathrm{h}_E(B) = D$ and $\mathrm{h}_E(A) = C$ also satisfies $\mathrm{h}_E(U) = V$ so that U, V, E are collinear.



We have

$$\overrightarrow{KL} = \overrightarrow{KD} + \overrightarrow{DC} + \overrightarrow{CL} = \overrightarrow{AE} + \overrightarrow{DC} + \overrightarrow{EB} = \overrightarrow{AB} + \overrightarrow{DC}.$$

Thus, $\overrightarrow{KL}$ and $\overrightarrow{AB}$ are collinear and KL is parallel to AB (and CD).

Let M be the point of intersection of KA and BL. Then, $MA \parallel BE$ and $MB \parallel AE$ so that $AMBE$ is a parallelogram and U is also the midpoint of ME. It follows that the lines MU and UE coincide.

Now, consider the trapezoid $ABLK$ (with $AB \parallel KL$). From the property recalled above and applied to $ABLK$, the points M, U and the midpoint W of KL are collinear. As a result, W is on the line $MU = UE = EF$.

OC258. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$ hold:

$$f(xf(y) - yf(x)) = f(xy) - xy$$

Originally problem 1 of day 1 of the 2014 Serbian National Mathematical Olympiad.

We received 3 correct submissions and 1 incorrect submission. We present the solution by the Missouri State University Problem Solving Group.

First, we will show that $f(x) = 0$ if and only if $x = 0$. If $f(0) \neq 0$, then for any real z, if $x = z/f(0)$ and $y = 0$ in the given equation, then

$$f(z) = f(xf(0)) = f(0) - 0,$$

which implies that f is constant. This violates the given equation. Therefore, $f(0) = 0$. Now, if $x = y$, then the given equation becomes

$$f(x^2) - x^2 = f(xf(x) - xf(x)) = f(0) = 0$$

or equivalently, if $x = \sqrt{z}$, then $f(z) = z$ for all $z \geq 0$ (Equation 1).

Now, assume that $f(x) = 0$. Then using $y = 1$ and the fact that $f(1) = 1$ (seen by using $x = y = 1$ in the given equation), we see that

$$-x = f(x) - x = f(xf(1) - f(x)) = f(x) = 0.$$

Let $f(-1) = a$. Assume that $x \geq 0$. Then

$$f(f(-x) - ax) = f(-xf(-1) - (-1)f(-x)) = f(x) - x = 0.$$

Therefore, $f(-x) - ax = 0$ and hence, $f(-x) = ax$ for all $x \geq 0$ (Equation 2).

Now, if $x = 1$ and $y = -1$ or vice versa, then the given equation becomes

$$f(\pm(a+1)) = a + 1.$$

In particular, $a + 1 = f(|a+1|) = |a+1|$ which implies that $a + 1 \geq 0$. Then

$$a + 1 = f(-|a+1|) = a(a+1)$$

which implies that either $a = 1$ or $a = -1$. It follows from Equations 1 and 2 above that $f(x) = x$ or that $f(x) = |x|$. A quick check verifies that these two functions are indeed solutions.

OC259. If the polynomials $f(x)$ and $g(x)$ are written on a blackboard then we can also write down the polynomials $f(x) \pm g(x)$, $f(x)g(x)$, $f(g(x))$ and $cf(x)$, where c is an arbitrary real constant. The polynomials $x^3 - 3x^2 + 5$ and $x^2 - 4x$ are written on the blackboard. Can we write a nonzero polynomial of the form $x^n - 1$ after a finite number of steps?

Originally problem 3 from day 2 of the 2014 All Russian Mathematical Olympiad.

We present the solution by Arkady Alt. There were no other submissions.

Let

$$f_0(x) = x^3 - 3x^2 + 5, \quad g_0(x) = x^2 - 4x.$$

Noting that

$$f_0'(x) = 3x^2 - 6x = 3x(x-2) \quad \text{and} \quad g_0'(x) = 2x - 4 = 2(x-2),$$

we conclude that the derivative of the two initial polynomials have common root $x = 2$, that is $g_0'(2) = f_0'(2) = 0$.

Using basic derivative rules, we see that the derivative of any polynomial, obtained from polynomials $f_0(x)$ and $g_0(x)$, must be zero if $x = 2$.

Since $(x^n - 1)' = nx^{n-1}, n \in \mathbb{N}$ cannot be zero if $x = 2$, the answer is negative.

OC260. Find the maximum of

$$P = \frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} + \frac{y^3z^4x^3}{(y^4+z^4)(yz+x^2)^3} + \frac{z^3x^4y^3}{(z^4+x^4)(zx+y^2)^3}$$

where x, y, z are positive real numbers.

Originally problem 2 from day 2 of the 2014 Vietnam National Olympiad.

We received 2 correct submissions. We present the solution by Titu Zvonaru.

We will make use of the following inequalities:

$$\begin{aligned}
&(xy+z^2)^2 \geq 4xyz^2,\\
&\frac{1}{a+b} \leq \frac{1}{4} \cdot \left(\frac{1}{a} + \frac{1}{b}\right),\\
&x^2+y^2+z^2 \geq xy+yz+zx,\\
&x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z)
\end{aligned}$$

and lastly, $x^4 + y^4 \geq (2/3)xy(x^2 + y^2 + xy) \Leftrightarrow (x - y)^2(3x^2 + 4xy + 3y^2) \geq 0$. We have

$$\begin{aligned}
\frac{x^3y^4z^3}{(x^4+y^4)(xy+z^2)^3} &\leq \frac{3}{8} \cdot \frac{x^3y^4z^3}{xy(x^2+y^2+xy)xyz^2(xy+z^2)}\\
&= \frac{3}{8} \cdot \frac{xy^2z}{(x^2+y^2+xy)(xy+z^2)}\\
&= \frac{3}{8} \cdot \frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2+xy(x^2+y^2+z^2)}\\
&= \frac{3}{32} \cdot \left(\frac{xy^2z}{x^2y^2+y^2z^2+z^2x^2} + \frac{yz}{x^2+y^2+z^2}\right).
\end{aligned}$$

It follows that

$$P \leq \frac{3}{32}\left(\frac{xy^2z+xyz^2+x^2yz}{x^2y^2+y^2z^2+z^2x^2} + \frac{yz+zx+xy}{x^2+y^2+z^2}\right) \leq \frac{3}{32}(1+1) = \frac{3}{16}.$$

If $x = y = z > 0$, then $P = 3/16$ and hence the maximum value of P is $3/16$.

About the Side and Diagonals of the Regular Heptagon

Michel Bataille

In a regular heptagon $ABCDEFG$, we can distinguish two kinds of diagonals: a short one such as AC and a long one such as AD. In this article, we are interested in various relationships between the lengths $a = AB$ of the side of the heptagon and $b = AC, c = AD$ of the diagonals. Since $AC = BD$, a, b, c are the sidelengths of the triangle ABD (Figure 1). It is easily seen that the angles of this *heptagonal* triangle are $\angle ADB = \frac{\pi}{7}$, $\angle BAD = \frac{2\pi}{7}$ and $\angle ABD = \frac{4\pi}{7}$.

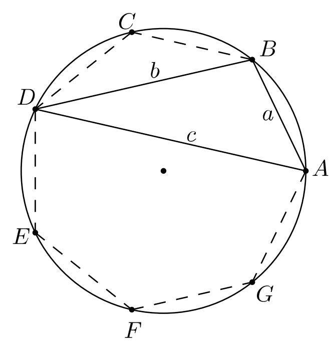


Figure 1

The heptagonal triangle was the subject of a celebrated 1973 article by Leon Bankoff and Jack Garfunkel ([**1**]). The two great problemists were gathering scattered results discovered by another great problemist, Victor Thébault. Here, guided by a closely related polynomial, we will re-obtain some of the numerous results of [**1**] and offer new relations between a, b, c.

Basic relationships

Let Γ be the circumscribing circle of the heptagon and R its radius. The Sine Law gives

$$a = 2R\sin\frac{\pi}{7}, \quad b = 2R\sin\frac{2\pi}{7}, \quad c = 2R\sin\frac{4\pi}{7}. \tag{1}$$

From $b = 4R\sin\frac{\pi}{7}\cos\frac{\pi}{7} = 2a\cos\frac{\pi}{7}$ and similar equalities, we deduce

$$\cos\frac{\pi}{7} = \frac{b}{2a}, \quad \cos\frac{2\pi}{7} = \frac{c}{2b}, \quad \cos\frac{4\pi}{7} = -\frac{a}{2c}. \tag{2}$$

Applying the Cosine Law, we readily obtain

$$a^2 = b^2 + c^2 - 2bc\cdot\frac{b}{2a} \quad \text{or} \quad a(b^2 + c^2) = a^3 + b^2c.$$

Similarly, we get

$$b(a^2+c^2) = b^3 + c^2 a \quad \text{and} \quad c(a^2+b^2) = c^3 - a^2 b.$$

However, simpler relations link the lengths a, b, c, e.g. the series

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}, \quad c^2 - a^2 = bc, \quad b^2 - a^2 = ca, \quad c^2 - b^2 = ab. \tag{3}$$

These four equalities result from judicious applications of Ptolemy's Theorem (see **[2]** if necessary), but elementary proofs may be of interest. For example, we can derive the first one from the following geometrical argument (Figure 2): The chords AB and CG are parallel and so are the chords BC and AD. If H is the point of intersection of AD and CG, it follows that $ABCH$ is a parallelogram, even a rhombus since $AB = BC$. But AC is parallel to DG, hence the triangle DHG is homothetic to the triangle AHC. Thus, $\frac{AB}{AC} = \frac{AH}{AC} = \frac{DH}{DG}$. Since $DG = DA$, we obtain

$$AB \cdot AD = AC \cdot DH = AC(AD - AH) = AC(AD - AB),$$

that is, $ac = b(c-a)$, the desired equality.

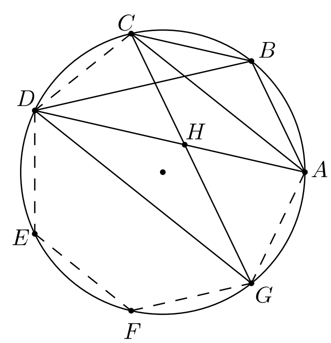


Figure 2

Let us prove the second relation by means of trigonometry. Note that using (1) and since $\sin\frac{4\pi}{7} = \sin\frac{3\pi}{7}$, we may rewrite the equality $c^2 - a^2 = bc$ as

$$\sin^2\frac{3\pi}{7} - \sin^2\frac{\pi}{7} = \sin\frac{2\pi}{7}\sin\frac{4\pi}{7},$$

which is directly obtained from the following calculation of the left-hand side:

$$\left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right)\left(\sin\frac{3\pi}{7} + \sin\frac{\pi}{7}\right) = 2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} \cdot 2\sin\frac{2\pi}{7}\cos\frac{\pi}{7} = \sin\frac{2\pi}{7}\sin\frac{4\pi}{7}.$$

Interestingly, we can remark that relations (3) lead to the proportions

$$a : b : c = b : a + c : b + c = c : b + c : a + b + c,$$

to be compared to the well-known golden proportion $a : b = b : a + b$ linking the side a and the diagonal b of a regular pentagon. Actually, these examples are particular cases of a remarkable, general pattern valid for all regular n-gons, the diagonal product formula, discovered and studied by P. Steinbach in [**3**].

To conclude this section, we propose the following exercise: using relations (1), (2) and (3), prove the beautiful formulas

$$a^2+b^2+c^2=7R^2, \quad a^4+b^4+c^4=21R^4 \quad \text{and} \quad a^2b^2+b^2c^2+c^2a^2=14R^4.$$

A useful polynomial

We introduce the polynomial

$$\begin{aligned} P(x) &= \left(x-\cos\frac{2\pi}{7}\right)\left(x-\cos\frac{4\pi}{7}\right)\left(x-\cos\frac{6\pi}{7}\right) \\ &= \left(x-\cos\frac{2\pi}{7}\right)\left(x-\cos\frac{4\pi}{7}\right)\left(x+\cos\frac{\pi}{7}\right). \end{aligned}$$

Let

$$\begin{aligned} s_1 &= \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}, \\ s_2 &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}, \\ s_3 &= \cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}. \end{aligned}$$

Using the formulas

$$\begin{aligned} 2\cos u\cos v &= \cos(u+v)+\cos(u-v), \\ 2\sin u\cos v &= \sin(u+v)+\sin(u-v), \end{aligned}$$

we successively obtain

$$2s_2=2s_1, \quad (2s_1)\sin\frac{\pi}{7}=-\sin\frac{\pi}{7} \quad \text{and} \quad (2s_3)\sin\frac{2\pi}{7}=\frac{1}{4}\sin\frac{2\pi}{7}.$$

Thus, the roots

$$x_1=\cos\frac{2\pi}{7}=\frac{c}{2b},\ x_2=\cos\frac{4\pi}{7}=-\frac{a}{2c},\ x_3=\cos\frac{6\pi}{7}=-\cos\frac{\pi}{7}=-\frac{b}{2a} \tag{4}$$

satisfy

$$x_1+x_2+x_3=x_1x_2+x_2x_3+x_3x_1=-\frac{1}{2}, \quad x_1x_2x_3=\frac{1}{8}.$$

We deduce that

$$P(x)=x^3+\frac{1}{2}x^2-\frac{1}{2}x-\frac{1}{8}$$

and two more formulas for a,b,c:

$$-\frac{1}{2}=\frac{c}{2b}-\frac{a}{2c}-\frac{b}{2a} \quad \text{and} \quad -\frac{1}{2}=-\frac{c}{2b}\cdot\frac{a}{2c}+\frac{a}{2c}\cdot\frac{b}{2a}-\frac{b}{2a}\cdot\frac{c}{2b},$$

that is,

$$\frac{a}{c}+\frac{b}{a}-\frac{c}{b}=1, \qquad \frac{a}{b}+\frac{c}{a}-\frac{b}{c}=2.$$

But we can go further! The fact that any symmetric polynomial in the roots x_1, x_2, x_3 can be expressed as a polynomial in s_1, s_2, s_3 opens the way to the discovery of a lot of relations between a, b, c. We give a few examples.

Consider first $x_1^2+x_2^2+x_3^2 = s_1^2 - 2s_2 = \frac{5}{4}$. Back to a, b, c, this yields a formula mentioned in [**1**]:

$$\frac{a^2}{c^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}=5. \tag{5}$$

Similarly, from the well-known

$$x_1^3+x_2^3+x_3^3 = 3x_1x_2x_3 + (x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2-(x_1x_2+x_2x_3+x_3x_1)),$$

we get

$$\frac{a^3}{c^3}+\frac{b^3}{a^3}-\frac{c^3}{b^3}=4$$

and from $x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2 = s_2^2 - 2s_3s_1$, we obtain

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=6, \tag{6}$$

a companion to (5). More complicated formulas can appear; for example, the identity

$$x_1x_2(x_1+x_2)+x_2x_3(x_2+x_3)+x_3x_1(x_3+x_1) = s_1s_2 - 3s_3$$

leads to

$$\frac{a^2}{bc}+\frac{bc}{a^2}-\left(\frac{b^2}{ca}+\frac{ca}{b^2}\right)-\left(\frac{c^2}{ab}+\frac{ab}{c^2}\right)=-1.$$

However, with the help of previous relations, this result simplifies into

$$b^3+c^3-a^3 = 4abc.$$

With the help of (5) and (6), one can obtain the very simple formula $abc = \sqrt{7}R^3$ (so that the area of our heptagonal triangle is $F = \frac{\sqrt{7}R^2}{4}$) (exercise!).

The reader is also encouraged to choose her/his favorite symmetric expressions in x_1, x_2, x_3 and discover a new formula for a, b, c!

Exploiting $P(x)$ in another way

The roots of $P(x)$ being simple roots, the decomposition of $\frac{1}{P(x)}$ into partial fractions is of the form

$$\frac{1}{P(x)} = \frac{\alpha_1}{x-x_1}+\frac{\alpha_2}{x-x_2}+\frac{\alpha_3}{x-x_3}. \tag{7}$$

By differentiation, we obtain another decomposition:

$$\frac{P'(x)}{(P(x))^2} = \frac{\alpha_1}{(x-x_1)^2} + \frac{\alpha_2}{(x-x_2)^2} + \frac{\alpha_3}{(x-x_3)^2}. \tag{8}$$

The values of the α_k $(k = 1, 2, 3)$ are easily obtained noticing that

$$\begin{aligned}\frac{1}{\alpha_1} = (x_1 - x_2)(x_1 - x_3) &= \left(\cos\frac{2\pi}{7} - \cos\frac{4\pi}{7}\right)\left(\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7}\right)\\ &= 4\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin^2\left(\frac{4\pi}{7}\right)\\ &= \frac{abc^2}{4R^4}.\end{aligned}$$

Similarly,

$$\frac{1}{\alpha_2} = -4\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = -\frac{a^2bc}{4R^4}$$

and

$$\frac{1}{\alpha_3} = 4\sin\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{ab^2c}{4R^4}.$$

The key idea is to specialize x in (7) and (8) and deduce formulas that can be transformed into relations between a, b, c by means of (1), (2) and (4). A few examples will clarify the method.

Making $x = 0$ in (7) and (8) first gives $8 = \sum_{k=1}^{3} \frac{\alpha_k}{x_k}$ and $-32 = \sum_{k=1}^{3} \frac{\alpha_k}{x_k^2}$ and then

$$8 = \frac{4R^4}{abc}\left(\frac{1}{cx_1} - \frac{1}{ax_2} + \frac{1}{bx_3}\right) \quad \text{and} \quad -32 = \frac{4R^4}{abc}\left(\frac{1}{cx_1^2} - \frac{1}{ax_2^2} + \frac{1}{bx_3^2}\right).$$

Finally, with the help of (4), we conclude that

$$\frac{1}{ac^3} + \frac{1}{ba^3} - \frac{1}{cb^3} = \frac{1}{R^4} \qquad \text{and} \qquad \frac{c}{ba^4} - \frac{a}{cb^4} - \frac{b}{ac^4} = \frac{2}{R^4}.$$

Because of the formulas $1 + \cos 2t = 2\cos^2 t$ and $1 - \cos 2t = 2\sin^2 t$, one can sensibly hope for some interesting results when taking $x = 1$ or -1 in (7) and (8). For example, we treat the case $x = -1$ in (7), leaving the similar case $x = 1$ to the reader. We are first led to

$$8 = \frac{4R^4}{abc}\left(\frac{1}{c(1+x_1)} - \frac{1}{a(1+x_2)} + \frac{1}{b(1+x_3)}\right)$$

with

$$1 + x_1 = 1 + \cos\frac{2\pi}{7} = 2\cos^2\frac{\pi}{7} = \frac{b^2}{2a^2}$$

and similarly

$$1 + x_2 = \frac{c^2}{2b^2}, \quad 1 + x_3 = \frac{a^2}{2c^2}.$$

This provides the relation

$$\frac{a}{b^3c^2} - \frac{b}{c^3a^2} + \frac{c}{a^3b^2} = \frac{1}{R^4}.$$

The calculations are slightly more complicated when applying (8); one can verify that the following formulas are obtained

$$\frac{1}{bc^4} + \frac{1}{ca^4} - \frac{1}{ab^4} = \frac{2abc}{7R^8}\left(= \frac{2\sqrt{7}}{7R^5}\right), \qquad \frac{a^3}{b^5c^2} - \frac{b^3}{c^5a^2} + \frac{c^3}{a^5b^2} = \frac{6}{R^4}.$$

Of course, the most courageous readers will be tempted to continue the pattern and first differentiate (8), getting

$$\frac{P''(x)}{(P(x))^2} - \frac{2(P'(x))^2}{(P(x))^3} = \sum_{k=1}^{3} \frac{-2\alpha_k}{(x-x_k)^3}.$$

Then, taking $x = 0$ leads to

$$\frac{b^3}{c^4} + \frac{c^3}{a^4} - \frac{a^3}{b^4} = \frac{5abc}{R^4}\left(= \frac{5\sqrt{7}}{R}\right).$$

Taking $x = 1$ and $x = -1$, with some additional effort, will produce two more amazing formulas:

$$\frac{1}{ca^6} - \frac{1}{ab^6} + \frac{1}{bc^6} = \frac{3abc}{7R^{10}}\left(= \frac{3\sqrt{7}}{7R^7}\right) \qquad \text{and} \qquad \frac{a^5}{b^7c^2} - \frac{b^5}{c^7a^2} + \frac{c^5}{a^7b^2} = \frac{31}{R^4}.$$

We conclude with a much easier exercise: obtain the relation

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}$$

(found in [**1**]) from the decomposition into partial fractions of $\dfrac{P'(x)}{P(x)}$.

References

[**1**] L. Bankoff, J. Garfunkel, The Heptagonal Triangle, *Math. Magazine*, Jan. Feb 1973, p. 7-19.

[**2**] A. Altintas, Some Collinearities in the Heptagonal Triangle, *Forum Geometricorum*, 2016, p. 249-256.

[**3**] P. Steinbach, Golden Fields: A Case for the Heptagon, *Math. Magazine*, Feb. 1997, p. 22-31.

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Two Famous Formulas (Part I)

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In this article, we will discuss *Pick's formula* for calculating the area of a lattice polygon (Part I) and *Euler's formula* for polyhedra (Part II), paying particular attention to the connection between these two formulas.

Before we proceed, let us define some terms. A *lattice* $\mathbb{Z}^2$ is the set of all points of the cartesian plane with integer coordinates. It is convenient to imagine a lattice as an infinite sheet of graph paper. A *lattice polygon* is a polygon with all its vertices on grid points. Unless otherwise specified, we consider only *simple polygons*, that is polygons that do not intersect themselves. Figure 1 shows examples of non-simple polygons.

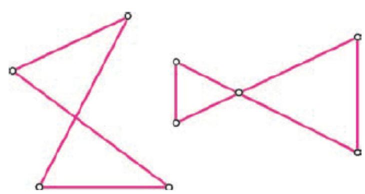


Figure 1

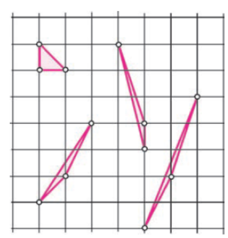


Figure 2

First of all, let us consider the smallest (and the most important) case. Suppose our lattice polygon is a triangle with no lattice points except its vertices or other lattice points on its perimeter. Such triangles are called *primitive* and example are depicted in Figure 2.

We will study their properties, beginning by showing that any triangle can be divided into primitive triangles. First, suppose that triangle ABC has no interior lattice points, but has some on at least one of its sides, say BC. Let us connect the vertex A with all lattice points on the side BC as in Figure 3. All the resulting triangles, except possibly ABP and AQC, are primitive. As for triangles ABP and AQC, they each have two sides that do not contain lattice points. Connecting points P and Q with lattice points on the sides AB and AC, we divide triangles ABP and AQC into primitive triangles.

Now suppose that the given triangle ABC has interior lattice points. Pick an arbitrary interior lattice point and connect it to the vertices A, B and C (see Figure 4). The three resulting triangles contain fewer interior lattice points than ABC. Since there are finitely many lattice points on the interior of ABC, by repeating this process, we will divide triangle ABC into triangles with no interior lattice points. To finish the decomposition into primitive triangles, we can apply the previously described process to eliminate lattice points on the sides of the resulting triangles.

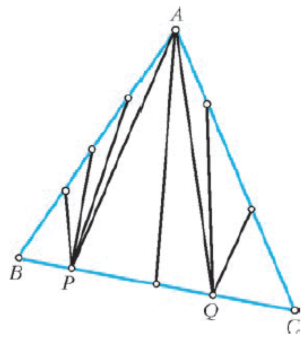


Figure 3

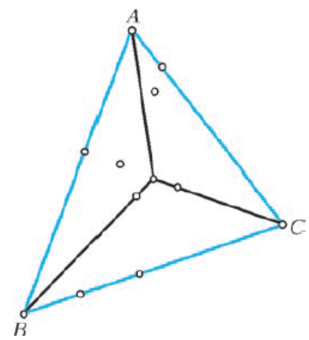


Figure 4

Theorem 1 *A triangle is primitive if and only if it has area of* $1/2$.

Proof. Let ABC be a primitive triangle. Consider the smallest lattice rectangle that contains ABC and has sides parallel to the coordinate axes. Because the rectangle is minimal, each of its four sides must pass through a vertex of the triangle, whence the pigeon-hole principle forces the rectangle to share at least one vertex with the triangle. Moreover, unless a side of the triangle is a diagonal of the rectangle, the triangle will necessarily contain a lattice point (as indicated by lattice point K in cases a) and b) of Figure 5, contrary to the assumption that it is primitive. We may therefore assume that AB is a diagonal of the rectangle $OAFB$ as shown in cases c) and d). Drop perpendiculars CD and CE to OA and OB, respectively (where C might coincide with D, E, or O).

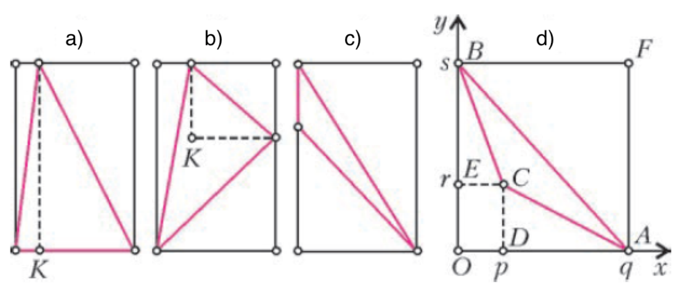


Figure 5

Suppose that the point O from Figure 5d is the origin and let $D = (p, 0), A = (q, 0), E = (0, r)$ and $B = (0, s)$. Let $I(P)$ denote the number of lattice points that lie inside a polygon P but not on its sides. Then

$$I(OAFB) = (q-1)(s-1).$$

Since AB does not contain lattice points other than A and B, we have

$$I(OAB) = I(OAFB)/2 = (q-1)(s-1)/2.$$

Similarly,

$$I(ACD) = (q-p-1)(r-1)/2 \quad \text{and} \quad I(CBE) = (s-r-1)(p-1)/2.$$

Since triangle ABC contains no interior lattice points, therefore

$$I(OAB) - I(ACD) - I(CBE) = pr,$$

the number of lattice points inside and on rectangle $ODCE$, excluding those on OD and OE. It follows that

$$(q-1)(s-1) - (q-p-1)(r-1) - (s-r-1)(p-1) = 2pr,$$

and so

$$qs - ps - qr = 1.$$

Letting square brackets denote the area of the region enclosed by the indicated polygon, it follows that

$$\begin{aligned}
[ABC] &= [OAB] - [ACD] - [CBE] - [ODCE] \\
&= \frac{sq}{2} - \frac{(p-q)r}{2} - \frac{(s-r)p}{2} - pr \\
&= \frac{qs - ps - qr}{2} \\
&= \frac{1}{2},
\end{aligned}$$

which proves one direction of the theorem.

Conversely, the area of a lattice triangle that is not primitive must exceed 1/2 because (as we have already seen) it can be divided into primitive triangles, each of which has an area of 1/2 by the first part of the theorem. □

Exercise 1. Prove that for any arbitrarily large number M, there exists a primitive lattice triangle such that each of its sides is larger than M.

Theorem 2 (G. Pick) *For any simple lattice polygon P, we have the following formula*

$$[P] = N_i + \frac{N_e}{2} - 1,$$

where N_i is the number of interior lattice points of P and N_e is the number of lattice points on the boundary of P.

For example, in Figure 6 we have $N_i = 9, N_e = 11$ and so $[P] = 9 + \frac{11}{2} - 1 = \frac{27}{2}$.

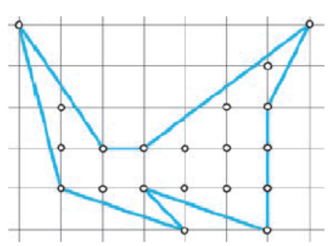


Figure 6

Proof. Assume that P is a simple polygon with k vertices. First of all, note that every simple polygon with four or more vertices has at least one diagonal that lies in its interior. This follows by definition for convex polygons; on the other hand, if the interior angle at some vertex is greater than $180°$, then a ray from that vertex sweeping the interior of the polygon must strike another vertex, and these two vertices determine the desired interior diagonal. From this, by induction, it follows that any simple polygon on k vertices can be split into $k-2$ triangles whose vertices are the vertices of the original polygon and are, in particular, lattice points. Therefore, the sum of all interior angles of a simple polygon on k vertices is equal $(k-2)\pi$.

Next, divide each of the resulting $k-2$ triangles into primitive triangles. Since the area of each primitive triangle is $1/2$, the number of primitive triangles in this case is $N = 2[P]$ and therefore does not depend on the way the polygon was divided into primitive triangles.

To finish the proof, we simply need to check that

$$N = 2N_i + N_e - 2.$$

We consider three cases as shown in Figure 7:

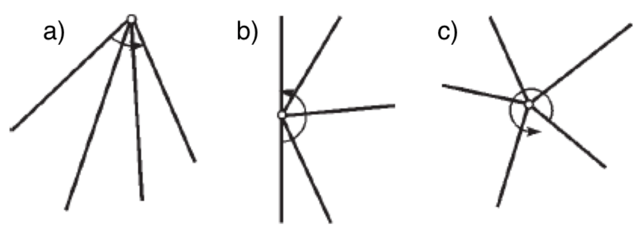


Figure 7

First of all, each vertex of P is also a vertex of one or more of our primitive triangles (see Figure 7a). The sum of the angles of all the triangles at these vertices equals to the sum of the interior angles of P and hence equals $180°(k-2)$.

Secondly, a lattice point which is not a vertex of P but lies on the boundary of P also serves as a vertex of our primitive triangles (see Figure 7b), and the sum of the angles at these vertices equals $180°(N_e - k)$.

Finally, we have to consider each of the N_i lattice points on the interior of P which also serve as vertices of our primitive triangles. The sum of the angles at these vertices is $360°$ (see Figure 7c). therefore, the sum of the angles of all primitive triangles with vertices on the interior lattice points equals $360°N_i$.

On the other hand, the sum of angles of all N of our primitive triangles is $180°N$, so we have

$$180°N = 360°N_i + 180°(N_e - k) + 180°(k-2).$$

Therefore, $N = 2N_i + N_e - 2$ and the proof is complete. □

Remark. Of course, we can replace the vertical lines of our lattice by any family of equally spaced parallel lines so that the square cells are replaced by congruent parallelograms. Pick's formula holds in this general case as well: For a lattice polygon P, we have

$$[P] = \left(N_i + \frac{N_e}{2} - 1\right) \cdot [a],$$

where $[a]$ is the area of each of the parallelograms. One can prove this claim with an argument similar to the one above; alternatively, one can apply a linear transformation to the cartesian lattice points by means of a 2×2 matrix whose determinant is $[a]$.

Let us summarize three proven statements:

$1°$. For any simple lattice polygon P, we have Pick's formula $[P] = N_i + \frac{N_e}{2} - 1$.

$2°$. The area of any primitive lattice triangle is $1/2$.

$3°$. For any decomposition of a simple polygon into N primitive triangles, we have that $N = 2N_i + N_e - 2$.

Let us consider logical connections between these statements and compare their relative strengths.

Statement $2°$ is an immediate consequence of $1°$, whereas $3°$ follows from $1°$ and $2°$ (see Figure 8a). Therefore, we could get all three statements immediately if we proved Pick's formula first and without using $2°$ and $3°$ (see Exercise 3.) However, we picked a different route: we proved $2°$ independently, then concluded $3°$ and then finally got $1°$ as a corollary of $2°$ and $3°$ (see Figure 8b).

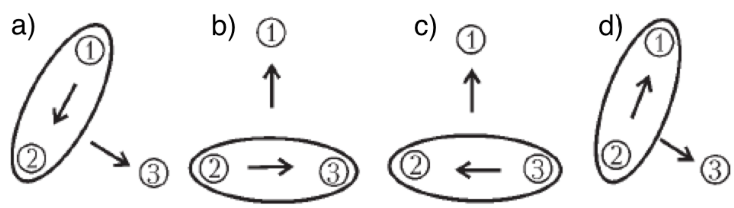


Figure 8

It is interesting to consider two other approaches, suggested by Figure 8c and 8d, to proving the three statements. Let us show how $2°$ follows from $3°$. First note that the area of any lattice triangle can be expressed as $n/2$ (for some integer n). (To see this, use coordinates for the vertices of the triangle, or see R.W. Gaskell, M.S. Klamkin, and P. Watson, "Triangulations and Pick's Theorem", *Mathematics Magazine*, **49**:1 (Jan. 1976) 35-37.) Now let T be a primitive triangle and P be a p by q lattice rectangle (whose sides lie along the lattice lines) that encloses T. Set $T_1 = T$ and use primitive triangles $T_j,\ j = 2, \ldots, N$, to triangulate the region (or regions) inside P lying outside of T (which are bounded by $OACB$ and AFB as in Figure 5d if the smallest rectangle is used for P). The number of primitive triangles required to cover P, according to $3°$, is

$$N = 2N_i + N_b - 2 = 2(p-1)(q-1) + 2p + 2q - 2 = 2pq.$$

Therefore, we have $2pq$ primitive triangles, each of area at least $1/2$, whose combined area equals

$$[P] = \sum_{j=1}^{2pq} [T_j] = pq.$$

Consequently, each of the $2pq$ triangles T_j must have area exactly $1/2$ so that their combined area does not exceed the area pq of the outer rectangle. Thus we see that $3°$ implies $2°$.

We will now prove that $2°$ implies $1°$. Consider the function

$$F(P) = N_i + \frac{N_e}{2} - 1,$$

defined on all simple lattice polygons. Split P into two lattice polygons P_1 and P_2 using a broken line passing through lattice points (see Figure 9); we write $P = P_1 + P_2$.

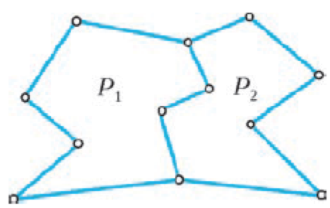


Figure 9

It is then easy to see that functions F and area are additive, that is

$$F(P_1 + P_2) = F(P_1) + F(P_2) \quad \text{and} \quad [P_1 + P_2] = [P_1] + [P_2].$$

Therefore, if Pick's formula holds for P_1 and P_2, then it also holds for $P = P_1 + P_2$. But since any simple polygon can be split into primitive triangles and by assumption Pick's formula holds for them, then it also holds for any given polygon.

To summarize, we have established that the three statements are equivalent even though a priori, $1°$ might appear to be the strongest of the three.

Exercise 2. Using the additive property of the function $F(P)$ and the proof of Theorem 1, find a proof of Pick's formula that does not require $2°$ and $3°$.

To be continued.

This article appeared in Russian in Kvant, 2008(2), p. 11–15. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er octobre 2017**.

Un astérisque ($\star$) *signale un problème proposé sans solution.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4211. *Proposé par Michel Bataille.*

Soient A et M deux matrices $n \times n$ avec valeurs complexes telles que A est inversible et M a rang égal à 1.

a) Evaluer $\mathrm{trace}(A^{-1}M)$ si $\det(A+M)=0$.

b) Déterminer $(A+M)^{-1}$ si $\det(A+M) \neq 0$.

4212. *Proposé par Florin Stanescu.*

Soient a, b et c les côtés d'un triangle, r le rayon du cercle inscrit et R le rayon du cercle circonscrit. Démontrer que

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{r}{R} \leq 2.$$

4213. *Proposé par Oai Thanh Dao et Leonard Giugiuc.*

Soit ABC un triangle dont aucun angle dépasse 120° et soit I le centre de son cercle inscrit. Considérer des points $D \in AI$, $E \in BI$ et $F \in CI$ tels que

$$\begin{aligned} AD &= \left(s - a - \frac{r}{\sqrt{3}}\right) \cos\frac{A}{3}, \\ BE &= \left(s - b - \frac{r}{\sqrt{3}}\right) \cos\frac{B}{3}, \\ CF &= \left(s - c - \frac{r}{\sqrt{3}}\right) \cos\frac{C}{3}, \end{aligned}$$

où a, b et c sont les côtés opposés aux angles A, B et C respectivement et où s dénote le demi périmètre et r dénote le rayon du cercle inscrit. Démontrer que le triangle DEF est équilatéral.

4214. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit ABC un triangle dont tous les angles dépassent $\frac{\pi}{6}$. Déterminer

$$\min(\cos A \cos B \cos C).$$

4215. *Proposé par Gheorghe Alexe et George-Florin Serban.*

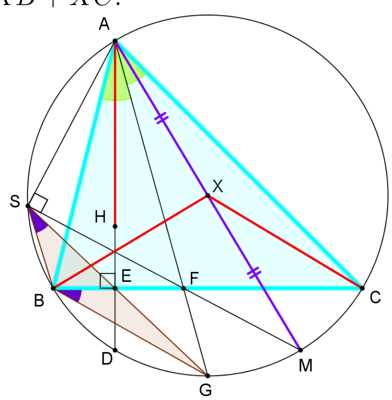
Déterminer des nombres naturels positifs a, b et c tels que

$$\frac{a+1}{b}, \quad \frac{b+1}{c} \quad \text{et} \quad \frac{c+1}{a}$$

sont tous des nombres naturels.

4216. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle avec orthocentre H et cercle circonscrit Γ. Or, $AH \cap BC = \{E\}$, $AH \cap \Gamma = \{A, D\}$, la bissectrice de l'angle A intersecte BC en F et Γ en G, $EG \cap \Gamma = \{G, S\}$, puis $SF \cap \Gamma = \{S, M\}$. Si X est le mi point de AM, démontrer que $\overrightarrow{AH} = \overrightarrow{XB} + \overrightarrow{XC}$.



4217. *Proposé par Dan Stefan Marinescu et Leonard Giugiuc.*

Soit $n \geq 3$ et soit $A_0 A_1 \ldots A_{n-1}$ un polygone convexe cyclique dont le centre O du cercle circonscrit coïncide avec le centre de gravité. Soient M et N deux points distincts tels que O se situe sur le segment MN et $ON = (n-1)OM$. Démontrer que

$$\sum_{k=0}^{n-1} MA_k \leq \sum_{k=0}^{n-1} NA_k.$$

4218. *Proposé par Daniel Sitaru.*

Démontrer que pour tout $a, b, c \in (0, \infty)$ et tout nombre naturel $n \geq 3$, l'inégalité suivante tient

$$\frac{1}{n}\sqrt[n]{a+b+c} \geq \frac{3\sqrt[3]{abc}}{(a+b+c)^{n-1}+n-1}.$$

4219. *Proposé par Nguyen Viet Hung.*

Soient a, b, c et d des entiers positifs distincts tels que

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a}$$

est un entier. Démontrer que $a+b+c+d$ n'est pas premier.

4220. *Proposé par Leonard Giugiuc, Daniel Sitaru et Hung Nguyen Viet.*

Soient s et r des nombres réels tels que $0 < r < s$ et soient $a, b, c \in [s-r, s+r]$ des nombres réels tels que $a+b+c = 3s$. Démontrer que

$$ab + bc + ca \geq 3s^2 - r^2 \quad \text{et} \quad abc \geq s^3 - sr^2.$$

. . . .

4211. *Proposed by Michel Bataille.*

Let A and M be two $n \times n$ matrices with complex entries such that A is invertible and M has rank 1.

a) Evaluate $\mathrm{trace}(A^{-1}M)$ if $\det(A+M) = 0$.

b) Find $(A+M)^{-1}$ if $\det(A+M) \neq 0$.

4212. *Proposed by Florin Stanescu.*

Let a, b and c be the sides of a triangle, r the inradius and R the circumradius. Show that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{r}{R} \leq 2.$$

4213. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let ABC be a triangle with no angle more than $120°$ and let I be its incentre. Consider points $D \in AI$, $E \in BI$ and $F \in CI$ such that

$$AD = \left(s - a - \frac{r}{\sqrt{3}}\right)\cos\frac{A}{3},$$
$$BE = \left(s - b - \frac{r}{\sqrt{3}}\right)\cos\frac{B}{3},$$
$$CF = \left(s - c - \frac{r}{\sqrt{3}}\right)\cos\frac{C}{3},$$

where a, b and c are sides opposite of angles A, B and C, respectively, s is the semiperimeter and r is the inradius of ABC. Prove that triangle DEF is equilateral.

4214. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle with every angle bigger than $\frac{\pi}{6}$. Find $\min(\cos A \cos B \cos C)$.

4215. *Proposed by Gheorghe Alexe and George-Florin Serban.*

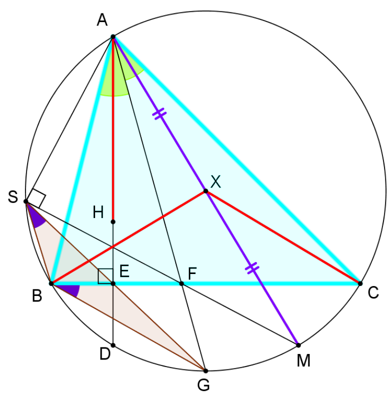
Find positive natural numbers a, b and c such that

$$\frac{a+1}{b}, \quad \frac{b+1}{c} \quad \text{and} \quad \frac{c+1}{a}$$

are all natural numbers.

4216. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let $AH \cap BC = \{E\}$, $AH \cap \Gamma = \{A, D\}$, the bisector of angle A cuts BC in F and Γ in G, $EG \cap \Gamma = \{G, S\}$, $SF \cap \Gamma = \{S, M\}$. If X is the midpoint of AM, prove that $\overrightarrow{AH} = \overrightarrow{XB} + \overrightarrow{XC}$.



4217. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Let $n \geq 3$ and consider a cyclic convex polygon $A_0A_1 \ldots A_{n-1}$ in which the circumcenter O coincides with the center of gravity. Let M and N be two distinct points such that O lies on the line segment MN and $ON = (n-1)OM$. Prove that

$$\sum_{k=0}^{n-1} MA_k \leq \sum_{k=0}^{n-1} NA_k.$$

4218. *Proposed by Daniel Sitaru.*

Prove that for all $a, b, c \in (0, \infty)$ and any natural number $n \geq 3$, we have

$$\frac{1}{n}\sqrt[n]{a+b+c} \geq \frac{3\sqrt[3]{abc}}{(a+b+c)^{n-1} + n - 1}.$$

4219. *Proposed by Nguyen Viet Hung.*

Let a, b, c and d be distinct positive integers such that

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}$$

is a integer. Prove that $a+b+c+d$ is not prime.

4220. *Proposed by Leonard Giugiuc, Daniel Sitaru and Hung Nguyen Viet.*

Let s and r be real numbers with $0 < r < s$ and let $a, b, c \in [s-r, s+r]$ be real numbers such that $a+b+c=3s$. Prove that

$$ab+bc+ca \geq 3s^2 - r^2 \quad \text{and} \quad abc \geq s^3 - sr^2.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(2), p. 75–78.

4111. *Proposed by Mihaela Berindeanu.*

The circumscribed circle of $\triangle ABC$ has circumcenter O and circumradius R. Let P be a point on the side BC. Calculate R given that the maximum value of the product $PA \cdot PB \cdot PC$ is 2016.

We received five solutions, all correct, and will feature two of them.

Solution 1, by Roy Barbara.

We provide some clarification:

- If 2016 is the maximum of the product $PA \cdot PB \cdot PC$ for a *given* triangle ABC with P on the side labeled BC, then all we can say is that $R \geq 3\sqrt[3]{63}$.
- If 2016 is the maximum of the product $PA \cdot PB \cdot PC$ among all triangles ABC that are inscribed in a circle whose radius is R, then $R = 3\sqrt[3]{63}$.

Indeed, let Γ be a circle of radius R. Then, according to problem 10282 (proposed by Paul Erdős) in the *American Mathematical Monthly* **102**:5 (May 1995) 468-469, if P is a point lying interior to, or on a side of a triangle ABC that is inscribed in Γ, then

$$PA \cdot PB \cdot PC \leq \frac{32}{27}R^3,$$

and this maximum can be reached only for a point that lies on a side of the triangle at distance $\frac{R}{3}$ from the center. Now if $\frac{32}{27}R^3 = 2016$, then $R = 3\sqrt[3]{63}$.

Solution 2, by C.R. Pranesachar.

Suppose that we are given two points B and C on a circle with center O and radius R. Let P be a point of the chord BC and let $y = OP$. Then by a theorem of Euclid, the product $PB \cdot PC$ depends only on y and R; specifically,

$$PB \cdot PC = R^2 - y^2.$$

Further, it is known (and easily shown) that as a point A moves about the circumference, the distance AP achieves its maximum value of $AP = R + y$ when A is the farthest point from P on the diameter through P. For this position of A, one has

$$PA \cdot PB \cdot PC = (R + y)(R^2 - y^2).$$

If we denote the product on the right by $f(y)$ and solve $f'(y) = 0$ for y, we get $y = \frac{R}{3}$. Because $f''\left(\frac{R}{3}\right) < 0$, we conclude that $f(y)$ reaches its maximum in the interval $[0, R]$ at $y = \frac{R}{3}$, and this maximum value is $f\left(\frac{R}{3}\right) = \frac{32}{27}R^3$. Equating this to $2016 = 32 \times 63$, one gets $R = 3\sqrt[3]{63}$. We can construct a triangle that achieves this maximum value by taking a point P at a distance of $\frac{R}{3}$ from O, A as described above, and BC any chord through P such that B and C are distinct from A.

Editor's Comments. A version of the problem appeared earlier in *Crux* as problem 1895 proposed by Ji Chen and Gang Yu [1993: 295; 1994: 263; 1995: 204]. Because our problem 1895 came out around the same time as Erdős' *Monthly* version mentioned in solution 1 above, *Crux* provided a reference to the two "short and attractive" solutions that the *Monthly* published, and published none of its own. Both sources reported that the problem also appears in *Bull. Math. (Wuhan)*, 1990, No. 3 (sum No. 224), p. 17, with solution in 1991, No. 10 (sum No. 243) p. 42. Our solution 2 is similar to one of the solutions published in the *Monthly.*

4112. *Proposed by Ardak Mirzakhmedov and Leonard Giugiuc.*

Let ABC be an acute triangle. Prove that

$$\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2 B + 25} + \sqrt{96\cos^2 C + 25} \geq 21.$$

We received two submissions both of which are correct. We present the solution by Sefket Arslanagić, modified and enhanced by the editor.

We shall prove the given inequality under the relaxed condition that $A, B, C \in [0, \frac{\pi}{2}]$ such that $A + B + C = \pi$.
We first consider the special case when two of the three angles are equal; $A = B$ say. Since $C = \pi - 2A$, it suffices to prove that

$$2\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2(\pi - 2A) + 25} \geq 21$$

$$\text{or} \quad \sqrt{96\cos^2 2A + 25} \geq 21 - 2\sqrt{96\cos^2 A + 25} \qquad (1)$$

Clearly (1) holds if its right side is negative. Hence, we may assume that

$$2\sqrt{96\cos^2 A + 25} \leq 21.$$

Squaring both sides of (1) and using $2\cos^2 A = 1 + \cos 2A$ we obtain the following equivalent equations:

$$96\cos^2 2A + 25 \geq 441 - 84\sqrt{96\cos^2 A + 25} + 4(25 + 96\cos^2 A)$$
$$96\cos^2 2A + 25 \geq 441 - 84\sqrt{48(1 + \cos 2A) + 25} + 4(25 + 48(1 + \cos 2A))$$
$$84\sqrt{73 + 48\cos 2A} \geq 708 + 192\cos 2A - 96\cos^2 2A$$
$$7\sqrt{73 + 48\cos 2A} \geq 59 + 16\cos 2A - 8\cos^2 2A. \qquad (2)$$

Let $t = 2\cos 2A$. Then $t \in [-2, 2]$ and (2) becomes

$$7\sqrt{73 + t} \geq 59 + 8t - 2t^2. \qquad (3)$$

Squaring (3) and simplifying, we obtain the following equations:

$$
\begin{aligned}
&49(73+t) \geq 3481+944t-172t^2-32t^3+4t^4 \\
&t^4-8t^3-43t^2-58t-24 \leq 0 \\
&(t+1)^2(t-12)(t+2) \leq 0. \qquad (4)
\end{aligned}
$$

Inequality (4) is true since $t \in [-2,2]$, and the proof of (1) is complete. Therefore, we have shown that the given inequality holds if two of the angles are equal. Note that equality holds in (4) if and only if $t=-1$ or $t=-2$.

Now, $t=-1 \Rightarrow \cos 2A = -\frac{1}{2} \Rightarrow 2A = \frac{2\pi}{3} \Rightarrow A = \frac{\pi}{3}$, so $A = B = \frac{\pi}{3}$.

Also, $t=-2 \Rightarrow \cos 2A = -1 \Rightarrow 2A = \pi \Rightarrow A = \frac{\pi}{2}$, so $A = B = \frac{\pi}{2}$ and $C=0$.

We now apply Lagrange's multipliers method to show that the minimum of

$$\sqrt{96\cos^2 A+25}+\sqrt{96\cos^2 B+25}+\sqrt{96\cos^2 C+25}$$

is attained on $[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$ when two of A,B, and C are equal.

Let $F=\sqrt{73+48\cos 2A}+\sqrt{73+48\cos 2B}+\sqrt{73+48\cos 2C}+2(A+B+C-\pi)$ where $A,B,C \in [0,\frac{\pi}{2}]$.

Thus, we have the following derivatives:

$$
\begin{aligned}
\frac{\partial F}{\partial A} &= \frac{-48\sin 2A}{\sqrt{73+48\cos 2A}}+\lambda, & \frac{\partial F}{\partial B} &= \frac{-48\sin 2B}{\sqrt{73+48\cos 2B}}+\lambda, \\
\frac{\partial F}{\partial C} &= \frac{-48\sin 2C}{\sqrt{73+48\cos 2C}}+\lambda, & \frac{\partial F}{\partial \lambda} &= A+B+C-\pi.
\end{aligned} \qquad (5)
$$

Setting $\frac{\partial F}{\partial A}=\frac{\partial F}{\partial B}=\frac{\partial F}{\partial C}=\frac{\partial F}{\partial \lambda}=0$, we see that $\lambda \geq 0$ since $A,B,C \in [0,\frac{\pi}{2}]$.

Now, if $\lambda=0$, then $\sin 2A=\sin 2B=\sin 2C=0$ which together with $A+B+C=\pi$ imply that $(A,B,C)=(0,\frac{\pi}{2},\frac{\pi}{2}),(\frac{\pi}{2},0,\frac{\pi}{2})$, or $(\frac{\pi}{2},\frac{\pi}{2},0)$.

Thus, we now assume that $\lambda>0$. Then from (5) we have

$$\left(\frac{48}{\lambda}\right)^2=\frac{73+48\cos 2A}{\sin^2 2A}=\frac{73+48\cos 2A}{1-\cos^2 2A}=\frac{73+48\cos 2B}{1-\cos^2 2B}=\frac{73+48\cos 2C}{1-\cos^2 2C}.$$

Let $f(x)=\frac{73+48x}{1-x^2}$ where $x \in (-1,1)$. Then

$$f'(x)=\frac{48(1-x^2)+2t(73+48x)}{(1-x^2)^2}=\frac{48x^2+146x+48}{(1-x^2)^2}=\frac{2(8x+3)(3x+8)}{(1-x^2)^2}.$$

Hence, $f'(x)<0$ on $I_1=\left(-1,-\frac{3}{8}\right)$ and $f'(x)>0$ on $I_2=\left(-\frac{3}{8},1\right)$. Note that two of $\cos 2A$, $\cos 2B$, and $\cos 2C$ must belong to the same I_n. So, assume $\cos 2A, \cos 2B \in I_1$. Then since $f(\cos 2A)=f(\cos 2B)=f(\cos 2C)$ and since

f is injective on each of I_1 and I_2, we then have $\cos 2A = \cos 2B$. Now since $2A, 2B \in [0, \pi]$, it follows that $2A = 2B$ and thus $A = B$.

Using this together with the result established in the first half of the solution, the proof is complete.

4113. *Proposed by Dragoljub Milošević.*

Let m_a, m_b and m_c be the lengths of medians, w_a, w_b and w_c be the lengths of the angle bisectors, r and R be the inradius and the circumradius, respectively, of a triangle. Prove that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \frac{1}{2} + \frac{r}{R} + \frac{R}{r}.$$

We received three correct solutions and present the solution by Šefket Arslanagić.

We have by the Cauchy-Buniakowski-Schwarz inequality

$$\begin{aligned} \frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} &= (m_a\sqrt{a})\left(\frac{1}{w_a\sqrt{a}}\right) + (m_b\sqrt{b})\left(\frac{1}{w_b\sqrt{b}}\right) + (m_c\sqrt{c})\left(\frac{1}{w_c\sqrt{c}}\right) \\ &\le \sqrt{(am_a^2 + bm_b^2 + cm_c^2)\left(\frac{1}{aw_a^2} + \frac{1}{bw_b^2} + \frac{1}{cw_c^2}\right)} \\ &= \sqrt{(am_a^2 + bm_b^2 + cm_c^2) \cdot \frac{1}{2F}\left(\frac{h_a}{w_a^2} + \frac{h_b}{w_b^2} + \frac{h_c}{w_c^2}\right)}. \end{aligned} \qquad (1)$$

From p. 211, equation 10.5 of [2],

$$am_a^2 + bm_b^2 + cm_c^2 = \frac{s}{2}(s^2 + 2Rr + 5r^2) \qquad (2)$$

and from p. 57, equation (5) of [1],

$$\frac{h_a}{w_a}^{2} + \frac{h_b}{w_b^2} + \frac{h_c}{w_c^2} = \frac{R + 2r}{2Rr}. \qquad (3)$$

It follows now from (1), (2), and (3) that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \sqrt{\frac{s}{2}(s^2 + 2Rr + 5r^2) \cdot \frac{1}{2rs} \cdot \frac{R + 2r}{2Rr}},$$

i.e.,

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \sqrt{\frac{1}{8}(s^2 + 2Rr + 5r^2) \cdot \frac{R + 2r}{Rr^2}}.$$

We will prove that

$$\sqrt{\frac{1}{8}(s^2 + 2Rr + 5r^2) \cdot \frac{R + 2r}{Rr^2}} \le \frac{1}{2} + \frac{r}{R} + \frac{R}{r}. \qquad (4)$$

This is successively equivalent to

$$\begin{aligned}
&\frac{1}{8}(s^2+2Rr+5r^2)\cdot\frac{R+2r}{Rr^2}\le\left(\frac{1}{2}+\frac{r}{R}+\frac{R}{r}\right)^2,\\
&\frac{R+2r}{8Rr^2}(s^2+2Rr+5r^2)\le\frac{1}{4}+\frac{r^2}{R^2}+\frac{R^2}{r^2}+\frac{r}{R}+\frac{R}{r}+2,\\
&R(R+2r)(s^2+2Rr+5r^2)\le 2R^2r^2+8r^4+8R^4+8Rr^3+8R^3r+16R^2r^2,\\
&R(R+2r)s^2+R(R+2r)(2Rr+5r^2)\le 8R^4+8r^4+8R^3r+8Rr^3+18R^2r^2,\\
&R(R+2r)s^2\le 8R^4+6R^3r+9R^2r^2-2Rr^3+8r^4,\\
&s^2\le\frac{8R^4+6R^3+9R^2r^2-2Rr^3+8r^4}{R^2+2Rr}.
\end{aligned}$$

We will use inequality 5.8 on p. 50 of [3]:

$$s^2\le 4R^2+4Rr+3r^2, \tag{5}$$

and we will prove that

$$4R^2+4Rr+3r^2\le\frac{8R^4+6R^3+9R^2r^2-2Rr^3+8r^4}{R^2+2Rr}. \tag{6}$$

This is successively equivalent to

$$\begin{aligned}
&(R^2+2Rr)(4R^2+4Rr+3r^2)\le 8R^4+6R^3+9R^2r^2-2Rr^3+8r^4,\\
&2R^4-3R^3r-R^2r^2-4Rr^3+4r^4\ge 0,\\
&2\left(\frac{R}{r}\right)^4-3\left(\frac{R}{r}\right)^3-\left(\frac{R}{r}\right)^2-4\left(\frac{R}{r}\right)+4\ge 0.
\end{aligned}$$

Using the substitution $t=\frac{R}{r}$, this becomes successively

$$\begin{gathered}
2t^4-3t^3-t^2-4t+4\ge 0,\\
(t-2)(2t^3+t^2+t-2)\ge 0,\\
(t-2)[2(t^3-1)+t^2+t]\ge 0,
\end{gathered}$$

which is clearly true by Euler's inequality $R\ge 2r$. From (5) and (6), inequality (4) holds, completing the proof of the claimed inequality. Equality holds for $t=2$, i.e, for equilateral triangles.

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4114. *Proposed by Michel Bataille.*

In the plane, let BC be a given line segment. Find the locus of A such that the centroid G of the triangle ABC satisfies $\angle GAB = \angle GBC$ and $\angle GAC = \angle GCB$.

We received six submissions, all correct. We present the solution by C.R. Pranesachar.

Let AG meet BC in the midpoint D of BC. We first observe that only one of the problem's angle requirements is needed; specifically,

$$\angle GAB = \angle GBC \quad \text{if and only if} \quad \angle GAC = \angle GCB.$$

For, if $\angle GAB = \angle GBC = \angle GBD$, for instance, then triangles GBD and BAD are similar, whence one has $DB^2 = DG \cdot DA$. Since $DB = DC$, it follows that $DC^2 = DG \cdot DA$, which implies that triangles GCD and CAD are similar, and one obtains $\angle GCB = \angle GCD = \angle DAC = \angle GAC$, as desired. It is now easily seen that the locus of A is a circle minus two of its points: From the statement $DB^2 = DG \cdot DA$ one has $\frac{1}{4}BC^2 = \frac{1}{3}DA^2$, whence

$$DA = \frac{\sqrt{3}}{2}BC.$$

Thus, for a fixed segment BC, the vertex A of triangles satisfying $\angle GAB = \angle GBC$ lies on a circle with D as centre and $\frac{\sqrt{3}}{2}BC$ as radius. Conversely, since all claims are reversible, if A is any point of this circle not on BC (so that ABC is a proper triangle), we have $\angle GAB = \angle GBC$, which completes the proof.

Comment. It may be noted that $\frac{1}{4}BC^2 = \frac{1}{3}DA^2$ is equivalent to the relation $2a^2 = b^2 + c^2$ (since by Stewart's theorem, $4AD^2 = 2(b^2 + c^2) - a^2$, where $a = BC$, etc.).

Editor's Comments. Bataille observed that any triangle ABC obtained from a point A of the locus is what is variously called a root-mean-square, or self-median, or quasi-isosceles triangle (see J. Chris Fisher, "Recurring ***Crux*** Configurations", [2011: 304-307]). The above solution shows that the angle condition $\angle GAB = \angle GBC$ characterizes such triangles.

4115. *Proposed by Daniel Sitaru.*

Prove that for all natural numbers $n \geq 2$, we have

$$n^{\ln 2} \leq \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}.$$

We received 6 solutions and 2 incomplete submissions. We present the solution by the Missouri State University Problem Solving Group.

Since $3\ln 2 = \ln 8 < \ln 9 = 2\ln 3$, we have $\frac{\ln 2}{2} < \frac{\ln 3}{3}$. If $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1-\ln x}{x^2} < 0$ for all $x > e$, so $f(x)$ is decreasing for $x > e$. Together these two observations imply

$$\frac{\ln 3}{3} \geq \frac{\ln n}{n} \text{ for all natural } n \geq 2.$$

As an upper Riemann sum,

$$\sum_{k=n}^{2n} \frac{1}{k} > \int_n^{2n+1} \frac{1}{x}\,dx = \ln \frac{2n+1}{n} = \ln\left(2 + \frac{1}{n}\right) > \ln 2.$$

Using these two inequalities we get

$$\begin{aligned}
\ln\left(\sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}\right) &= \frac{\ln 3}{3} + \frac{\ln n}{n+1} + \frac{\ln n}{n+2} + \ldots + \frac{\ln n}{2n} \\
&\geq \frac{\ln n}{n} + \frac{\ln n}{n+1} + \frac{\ln n}{n+2} + \ldots + \frac{\ln n}{2n} \\
&= \ln n \cdot \sum_{k=n}^{2n} \frac{1}{k} \\
&> \ln n \ln 2 \\
&= \ln n^{\ln 2}.
\end{aligned}$$

Exponentiation yields the desired result.

4116. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of the expression

$$a^2(a^2 - a + 1) + b^2(b^2 - b + 1) + c^2(c^2 - c + 1).$$

There were 23 correct solutions, with two from one submitter. Most of the solvers used the approach of one of the following two solutions, with 11 relying on Jensen's Inequality. One solver pointed out that the result holds even when a, b, c are not all positive. Two used Lagrange Multipliers, with Kayla Cowan and Amanda Fox, students at the Southeast Missouri State University at Cape Girardeau, MI giving a careful and comprehensive treatment. We present two composite solutions.

Solution 1.

The minimum value is 3. Using $x^2 - x + 1 \geq x$ for $x \geq 0$ and a power means inequality, we get that

$$\begin{aligned}
a^2(a^2 - a + 1) + b^2(b^2 - b + 1) + c^2(c^2 - c + 1) &\geq a^3 + b^3 + c^3 \\
&\geq 3\left(\frac{a+b+c}{3}\right)^3 = 3.
\end{aligned}$$

Equality occurs iff $a = b = c = 3$.

Solution 2.

Let $f(x) = x^2(x^2 - x + 1)$. Because $f''(x) = (3x - 1)^2 + (3x^2 + 2) > 0$, $f(x)$ is a convex function, and we can apply Jensen's Inequality. Thus,

$$\begin{aligned}&\frac{1}{3}\left[a^2(a^2-a+1)+b^2(b^2-b+1)+c^2(c^2-c+1)\right]\\&=\frac{1}{3}[f(a)+f(b)+f(c)]\geq f\left(\frac{a+b+c}{3}\right)=f(1)\geq 1,\end{aligned}$$

with equality iff $a=b=c=1$. Thus the minimum value of $f(a)+f(b)+f(c)$ for real a,b,c is 3.

Editor's Comments. The inequalities required in Solution 1 were obtained in some interesting ways. We will note the key ideas and leave the reader to complete the details.

$$\begin{aligned}&\sum a^4+\sum a^2\geq 2\sqrt{\sum a^4\cdot\sum a^2}\geq 2\sum a^3;\\&3\sum a^2(a^2-a+1)\geq\sum a^2(a^2+a+1);\\&\sum a^3=3\sum\left(\frac{a^3+1+1}{3}-\frac{2}{3}\right)\geq 3\sum\left(a-\frac{2}{3}\right)=3;\end{aligned}$$

$$\begin{aligned}a^3+b^3+c^3&=a^2\cdot a+b^2\cdot b+c^2\cdot c\geq\frac{1}{3}(a^2+b^2+c^2)(a+b+c)\\&=a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2=3.\end{aligned}$$

4117. *Proposed by Martin Lukarevski.*

The sequence (x_n) is given recursively by $x_0=0, x_1=1$,

$$x_{n+1}=x_n\sqrt{x_{n-1}^2+1}+x_{n-1}\sqrt{x_n^2+1},\quad n\geq 1.$$

Find x_n.

We received 15 solutions. We present the solution by Prithwijit De.

Obviously $x_n\geq 0$ for all $n\geq 0$. The function $f:[0,\infty)\to[0,\infty)$ defined by

$$f(x)=\sinh(x)=\frac{e^x-e^{-x}}{2}$$

is a bijection. Thus for every nonnegative integer n there exists a $\theta_n\in[0,\infty)$ such that $x_n=\sinh(\theta_n)$. Using the given values for x_0 and x_1 we calculate that $\theta_0=0$ and $\theta_1=\ln(1+\sqrt{2})$.

Substitute $x_n=\sinh(\theta_n)$ in the given recursion, and use the hyperbolic sine formulas

$$\begin{aligned}&\cosh^2(x)=1+\sinh^2(x),\text{ and}\\&\sinh(x+y)=\sinh(x)\cosh(y)+\cosh(x)\sinh(y)\end{aligned}$$

to obtain

$$\sinh(\theta_{n+1}) = \sinh(\theta_n)\cosh(\theta_{n-1}) + \sinh(\theta_{n-1})\cosh(\theta_n) = \sinh(\theta_n + \theta_{n-1}).$$

As sinh is one-one on $[0, \infty)$ it follows that for $n \geq 1$ we have $\theta_{n+1} = \theta_n + \theta_{n-1}$, and hence $\theta_n = \theta_1 F_n$ where F_n is the n^{th} Fibonacci number (recall that the Fibonacci numbers are defined by the recursion $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).

Thus for $n \geq 0$, we have

$$x_n = \sinh(\ln(1+\sqrt{2})F_n) = \frac{e^{\ln(1+\sqrt{2})F_n} - e^{-\ln(1+\sqrt{2})F_n}}{2} = \frac{(\sqrt{2}+1)^{F_n} - (\sqrt{2}-1)^{F_n}}{2}.$$

4118. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $a \in (0, \frac{\pi}{2}], b \in [\frac{\pi}{2}, \pi)$ with $a + b = \pi$. Calculate $\int_a^b \frac{x}{\sin x} dx$.

We received 11 correct solutions and will feature the one by Šefket Arslanagić here.

Substitute $x = \pi - y$, and using the fact that $a + b = \pi$, we have

$$I = \int_a^b \frac{x}{\sin x} dx = \int_a^b \frac{\pi - x}{\sin x} dx.$$

Then

$$\begin{aligned} 2I &= \int_a^b \frac{x}{\sin x} dx + \int_a^b \frac{\pi - x}{\sin x} dx \\ &= \pi \int_a^b \frac{dx}{\sin x} = \pi \ln(\tan(x/2))|_a^{\pi - a} \\ &= \pi \ln\left(\tan\left(\frac{\pi - a}{2}\right) / \tan\left(\frac{a}{2}\right)\right). \end{aligned}$$

Hence

$$I = \frac{\pi}{2} \ln\left(\tan\left(\frac{\pi - a}{2}\right) / \tan\left(\frac{a}{2}\right)\right).$$

4119. *Proposed by Ovidiu Furdui.*

Let $m, n, p \in \mathbb{N}$, $m \neq n$, and let A and B be 2×2 matrices with complex entries for which $mAB - nBA = pI_2$. Prove that

$$(AB - BA)^2 = O_2.$$

We received four submissions, three of which were correct and complete. We present two of the solutions.

Solution 1, by Michel Bataille.

Let $U = AB - BA$. Since $\mathrm{tr}(U) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$, the characteristic polynomial of U is $\chi_U(x) = x^2 + \det(U)$. From the Cayley-Hamilton Theorem, we have

$$U^2 + (\det(U))I_2 = 0,$$

so

$$U^2 = -(\det(U))I_2 = O_2$$

if $\det(U) = 0$. Thus, it suffices to show that $\det(U) = 0$ necessarily holds.

For the purpose of a contradiction, assume that $\det(U) \neq 0$. From the characteristic polynomial of U, we can conclude that U has two distinct eigenvalues, namely the square roots $\lambda_0,\ -\lambda_0$ of the nonzero complex number $-\det(U)$. It follows that there exist two independent column vectors X_1, X_2 such that $UX_1 = \lambda_0 X_1,\ UX_2 = -\lambda_0 X_2$.

Now, from $AB = \frac{n}{m}BA + \frac{p}{m}I_2$, we deduce

$$\lambda_0 X_1 = UX_1 = ABX_1 - BAX_1 = \frac{n-m}{m}\, BAX_1 + \frac{p}{m}\, X_1$$

so that

$$BAX_1 = \frac{p - \lambda_0 m}{m-n}\, X_1.$$

Similarly, we obtain

$$BAX_2 = \frac{p + \lambda_0 m + p}{m-n}\, X_2.$$

It follows that $\frac{p-\lambda_0 m}{m-n}$ and $\frac{p+\lambda_0 m}{m-n}$ are the eigenvalues of BA.

Using the same argument with $BA = \frac{m}{n}AB - \frac{p}{n}I_2$ we also have

$$ABX_1 = \frac{p-\lambda_0 n}{m-n}\, X_1 \quad \text{and} \quad ABX_2 = \frac{p+\lambda_0 n}{m-n}\, X_2,$$

so that $\frac{p-\lambda_0 n}{m-n}$ and $\frac{p+\lambda_0 n}{m-n}$ are the eigenvalues of AB.

Since AB and BA have the same characteristic polynomial, the product of the respective eigenvalues must be the same, that is

$$\frac{\lambda_0 m - p}{n-m} \cdot \frac{\lambda_0 m + p}{m-n} = \frac{\lambda_0 n - p}{n-m} \cdot \frac{\lambda_0 n + p}{m-n}.$$

It follows that $\lambda_0^2 m^2 - p^2 = \lambda_0^2 n^2 - p^2$, implying $m = n$, in contradiction with the hypothesis.

Solution 2, by the proposer.

We have $m(AB - BA) = pI_2 + (n-m)BA$ and thus, by passing to determinants,

$$\begin{aligned} m^2 \det(AB - BA) &= det(pI_2 + (n-m)BA) \\ &= (n-m)^2 \det\left(\frac{p}{n-m} I_2 + BA\right) \\ &= p^2 + p(n-m)\mathrm{tr}(BA) + (n-m)^2 \det(BA). \end{aligned}$$

Repeating the argument with $n(AB - BA) = pI_2 + (n-m)BA$, we also get

$$n^2 \det(AB - BA) = p^2 + p(n-m)\mathrm{tr}(AB) + (n-m)^2 \det(AB).$$

Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\det(AB) = \det(BA)$, we obtain, by subtracting the previous equalities,

$$(m^2 - n^2)\det(AB - BA) = 0,$$

implying $\det(AB - BA) = 0$. Using the Cayley-Hamilton Theorem, we conclude $(AB - BA)^2 = O_2$, finishing the proof.

4120. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the minimum value of the function $f : [1,2] \mapsto \mathbb{R}$, where

$$f(x) = \sqrt{\frac{8-3x}{x}} + 2\sqrt{4x+1} - \sqrt{4x^2 - 8x + 49}.$$

We received nine correct solutions and present the solution by AN-anduud Problem Solving Group.

We have

$$\begin{aligned} f(x) \geq 0 &\Leftrightarrow \sqrt{\frac{8-3x}{x}} + 2\sqrt{4x+1} \geq \sqrt{4x^2 - 8x + 49} \\ &\Leftrightarrow \sqrt{8-3x} + 2\sqrt{4x^2 + x} \geq \sqrt{4x^3 - 8x^2 + 49x} \\ &\Leftrightarrow (x-1)^2(2-x)(x^3 - 8x^2 + 23x - 2) \geq 0 \\ &\Leftrightarrow (x-1)^2(2-x)[x(x-4)^2 + 5x + 2(x-1)] \geq 0. \end{aligned}$$

Since $f(1) = f(2) = 0$, the minimum value of $f(x)$ is 0.

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