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MAXIMA AND MINIMA BY THE METHOD OF COINCIDENCE OF EQUAL VALUES WITH AN APPLICATION TO PHILO'S LINE

HOWARD EVES

A diagram picturing ordinary maxima and minima of a given function (see Figure 1) reveals that these extreme values occur at positions of coalescence of pairs of equal neighboring values of the function. This simple observation served, in precalculus days, as an approach to the calculation of maxima and minima of a given function. The procedure, aptly called *the method of coincidence of equal values*, was successfully exploited by geometers in the nineteenth century to obtain characterizations of a number of maximal and minimal situations that otherwise seemed to require special and individual tricks. The method deserves to be better known, and we here illustrate it by a set of four related problems. In each problem we will be given an angle RCS and a point P within the angle, and we shall be required to draw through P a line segment AB, with A on CR and B on CS, such that a certain quantity related to the figure becomes a minimum.

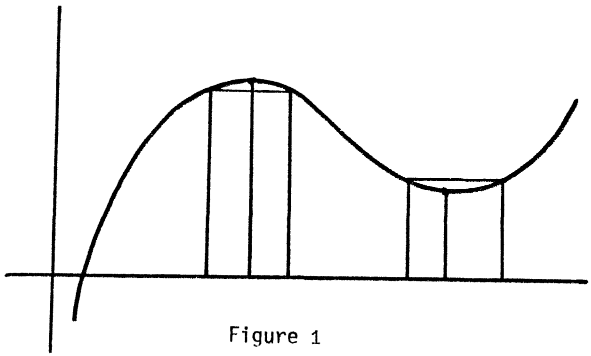

Figure 1

Problem 1. Draw APB *such that triangle* ABC *has a minimum area.*

Let A'PB' and A"PB" (see Figure 2) be two neighboring chords through P such that area A'CB' = area A"CB". Then area A'PA" = area B'PB", whence

$$PA' \cdot PA'' = PB' \cdot PB''.$$

Taking the limit as $\angle A'PA'' \to 0$ (that is, as A'B' and A"B" rotate about P into coincidence with the required segment AB), we find

$$PA^2 = PB^2 \quad \text{or} \quad PA = PB.$$

Thus AB is characterized by the fact that it must be bisected by P.

To construct the sought chord AB, draw the parallel through P to CS to cut CR in M, then on CR mark off $MA = CM$.

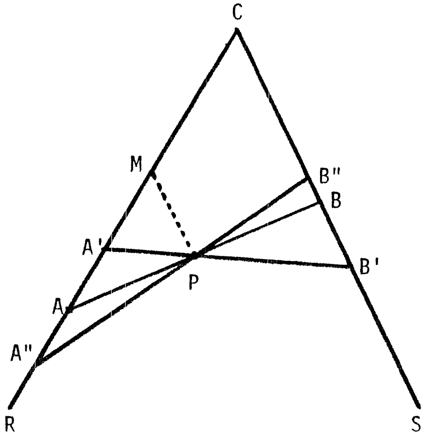

Figure 2

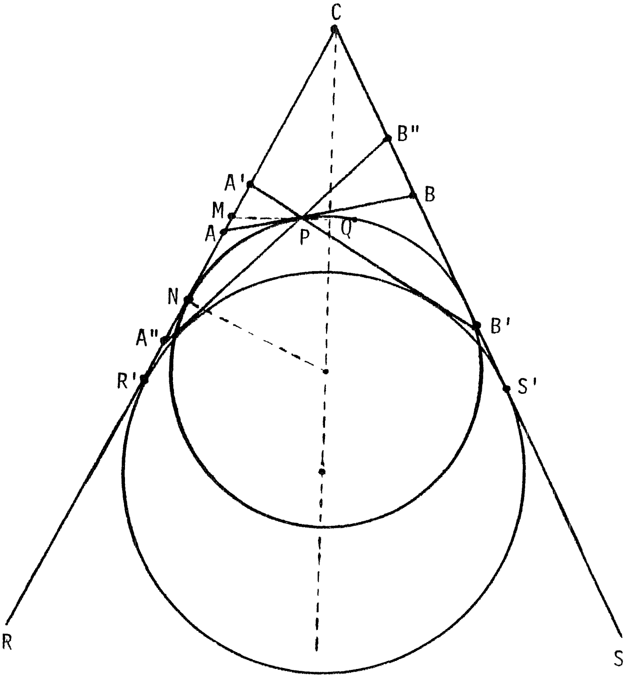

Figure 3

Problem 2. Draw APB *such that triangle* ABC *has a minimum perimeter.*

Let A'PB' and A"PB" (see Figure 3) be two neighboring chords through P such that triangles A'CB' and A"CB" have equal perimeters. Let the excircle of triangle A'CB' opposite vertex C touch CR and CS in R' and S', respectively. Then $CR' = CS'$ = half the perimeter of triangle A'CB'. It follows that triangle A"CB" possesses the same excircle opposite vertex C. Taking the limit as A'B' and A"B" coalesce with the sought line AB, we see that AB is the tangent to the circle through P and touching CR and CS.

The construction of AB is easy. First find Q, the reflection of P in the bisector of $\angle$RCS. Let QP produced cut CR in M. Now find N on CM produced such that MN is the mean proportional between MQ and MP. Line AB is then the tangent at P to the circle through P, Q, N; the center of this circle lies at the intersection of the bisector of $\angle$RCS and the perpendicular to CR at N.

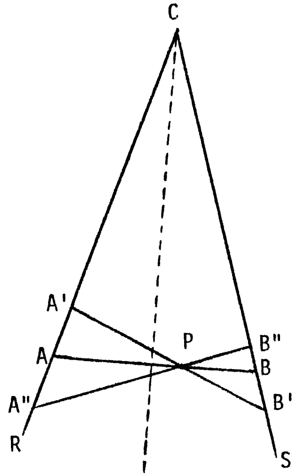

Figure 4

Problem 3. Draw APB *such that the product* AP • PB *is a minimum.*

Let A'PB' and A"PB" (see Figure 4) be two neighboring chords through P such that A'P • PB' = A"P • PB". Then A'P/A"P = PB"/PB' and triangles A'PA" and B"PB' are similar. It follows that $\angle$CA"P = $\angle$CB'P. Taking the limit as A'B' and A"B" coalesce with the sought line AB, we see that AB makes equal angles with CR and CS.

The segment AB can be drawn as the chord through P perpendicular to the bisector of $\angle$RCS.

Problem 4. Draw APB *such that* AB *is a minimum.*

Let A'B' and A"B" (see Figure 5) be equally lengthed neighboring segments passing through P. Take U on PA" such that PU = PA', and V on PB' such that PV = PB". Draw CT perpendicular to the bisector of $\angle$B'PB", cutting A'B' in Q' and A"B" in Q". Now, from similar triangles,

$$\frac{CQ'}{Q'B'} = \frac{B"V}{VB'}, \quad \frac{CQ"}{Q"A"} = \frac{A'U}{UA"}. \quad (1)$$

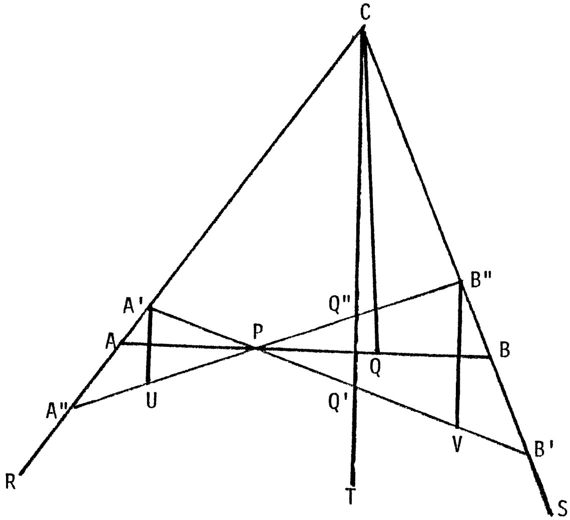

Figure 5

Also, since

$$A'P + PV + VB' = A"U + UP + PB"$$

and

$$A'P = UP, \quad PV = PB",$$

it follows that $A''U = VB'$. Therefore, from (1),

$$\frac{CQ'}{Q'B'} \cdot \frac{Q''A''}{CQ''} = \frac{B''V}{VB'} \cdot \frac{UA''}{A'U} = \frac{B''V}{A'U} = \frac{PB''}{PA'} :$$

Taking the limit as $\angle B'PB'' \to 0$ (that is, as $A'B'$ and $A''B''$ rotate about P into coincidence with the minimum segment AB, and Q' and Q'' approach Q, the foot of the perpendicular from C to AB), we find $QA/QB = PB/PA$. It follows that $PA = QB$, and AB may be characterized as that segment through P for which P and the foot Q of the perpendicular from C on AB are isotomic conjugates.

Although, as we have seen, the chord APB in each of the first three problems can be constructed with Euclidean tools, we will show that such is not the case for the chord APB of Problem 4. To accomplish this consider the rectangle CGPH shown in Figure 6, and let segment APB be such that $AP = QB$, where Q is the foot of the perpendicular from C on AB. Then C, G, P, Q, H are concyclic and

$$AG \cdot AC = AP \cdot AQ = BQ \cdot BP = BH \cdot BC,$$

whence

$$\frac{AC}{BC} = \frac{BH}{AG} .$$

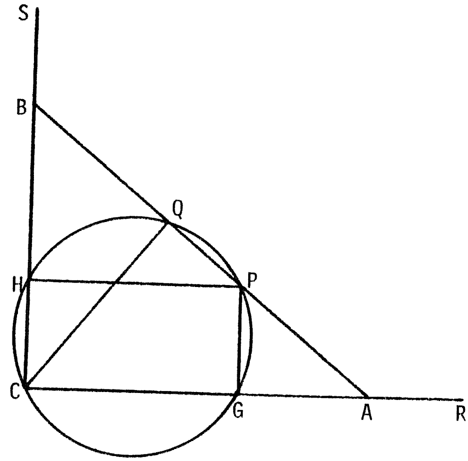

Figure 6

But, from similar triangles

$$\frac{AC}{BC} = \frac{PH}{BH} = \frac{AG}{PG} ,$$

and it follows that

$$\frac{PH}{BH} = \frac{BH}{AG} = \frac{AG}{PG}$$

or

$$\frac{GC}{BH} = \frac{BH}{AG} = \frac{AG}{HC} ,$$

which yields

$$GC \cdot AG = BH^2 \qquad \text{and} \qquad BH \cdot HC = AG^2.$$

Taking $GC = 2HC$ in the last two equations, and eliminating BH, we find

$$AG^3 = 2HC^3.$$

It follows that, if the segment APB can be drawn, we can solve the problem of the duplication of the cube. Since it is known that this latter problem is impossible of solution with Euclidean tools, it now follows, even for our very particular choice of angle RCS and position of P, that the minimum chord APB cannot be constructed with Euclidean tools.

The minimum chord APB in Problem 4 has become known as the *Philo line* of the point P for the given angle RCS. The line is named after Philo of Byzantium, an ancient Greek writer on mechanical devices, who flourished probably in the first or second century B.C., and who devised the interesting allied reduction of the famous problem of the duplication of a cube that we discussed above. Because of this connection with the duplication problem, and because of its own inherent attractions, Philo's line has excited interest over the ages, and many proofs showing the impossibility of the construction of the line with Euclidean tools have been devised. For some of these proofs, for constructions employing higher curves, for some interesting generalizations, and for important references, the reader may consult Howard Eves, "Philo's Line", *Scripta Mathematica*. vol. 26 (1959), pp. 141-148.

*

The interested reader might, employing the method of coincidence of equal values, care to try his hand at obtaining the following characterizations of certain extrema, which may lead to constructions of these extrema.

1. If a parallel to a given line MN cuts the semicircle on AB as diameter in points D and C, the quadrilateral ABCD has a maximum area when

$$2DC^2 + (AB\cos\alpha)\cdot DC - AB^2 = 0,$$

where α is the angle that MN makes with AB. (Note that when $\alpha = 0$ the maximum quadrilateral is half of a regular hexagon.)

2. If O is a given point on the prolongation of diameter BA of a given semicircle, and if ODC is a secant cutting the semicircle in D and C, then the quadrilateral ABCD has a maximum area when the projection of CD on AB is equal to the radius of the semicircle.

3. If a parallel to a given line MN cuts a circle in A and B, and if P is a fixed point on the circle, then triangle APB has a maximum area when

$$AB^2 = \frac{PA^2 + PB^2}{2}$$

or, equivalently, when

$$\cos APB = \frac{PA^2 + PB^2}{4(PA \cdot PB)} .$$

4. If a secant through a given point O on the tangent to a given circle at a given point P on the circle cuts the circle in A and B, then triangle APB has a maximum area when

$$AB^2 = \frac{PA^4 + PB^4}{PA^2 + PB^2}$$

or, equivalently, when

$$\cos APB = \frac{PA \cdot PB}{PA^2 + PB^2} .$$

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* * *

A GENERALIZATION OF A HANDSHAKE PROBLEM

RICHARD A. HOWLAND

The purpose of this note is to give a generalization of the following mathematical folklore problem:

Mr. and Mrs. Adams recently attended a party at which there were four other couples. Various handshakes took place. No one shook hands with his/her own spouse, no one shook hands with the same person twice, and, of course, no one shook his/her own hand.

After all the handshaking was finished, Mr. Adams asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Adams shake?

The basic ingredients of this problem are:

(a) The people.

(b) The couplings.

(c) The handshakes.

(d) The totals.

(e) The surprise.

We generalize as follows:

(a) Let X be a set of $2n$ distinct objects ($n \geq 1$), a particular one of which is distinguished by the name x_1.

(b) Let $*$ be a permutation of X such that

(i) $x^{**} = x$ for each $x \in X$,

(ii) $x^{*} \neq x$ for each $x \in X$.

(c) Let the function $h : X \times X \to \{0,1\}$ be such that

(i) $h(x,x) = 0$ for each $x \in X$,

(ii) $h(x,x^{*}) = 0$ for each $x \in X$,

(iii) $h(x,y) = h(y,x)$ for each $(x,y) \in X \times X$.

(d) Let $H(x) = \sum_{y \in X} h(x,y)$ for each $x \in X$.

(e) Assume that $H(x) = H(y)$ implies $x = y$ or $x = x_1$ or $y = x_1$.

Then we have the

THEOREM. Given (a)-(e) *above, we have* $H(x) + H(x^{*}) = 2n - 2$ *for each* $x \in X$, *and* $H(x_1) = n - 1$.

Proof. The proof is by induction on n. The theorem clearly holds if $n = 1$. For $n > 1$, we assume that the result holds for any set X' of $2(n-1)$ elements and deduce that it also holds for the set X of $2n$ elements.

By hypothesis, the set $\{H(x) | x \neq x_1\}$ has $2n - 1$ distinct elements and hence, since $0 \leq H(x) \leq 2n - 2$, there are elements $r,s \in X$ such that $H(r) = 2n - 2$ and $H(s) = 0$. Since $H(r) = 2n - 2$, we have $h(r,y) = 1$ for $y \neq r, r^{*}$. Thus $h(y,r) = 1$ for $y \neq r, r^{*}$ and so $H(y) \geq 1$ for $y \neq r^{*}$. Note that this implies that $s = r^{*}$. Furthermore, observe that if $r = x_1^{*}$ then $H(y) \geq 1$ for $y \neq x_1^{**} = x_1$, which contradicts the existence of s. Therefore $r \neq x_1, x_1^{*}$.

Now let $X' = X - \{r, r^{*}\}$, let h' be the restriction of h to X', and define H' by $H'(x) = \sum_{y \in X'} h'(x,y)$, so that

$$H'(x) = H(x) - h(x,r) - h(x,r^{*}) = H(x) - 1 - 0 = H(x) - 1.$$

Certainly X', h', and H' meet the induction hypothesis, since

$$H'(x) = H'(y) \Longrightarrow H(x) - 1 = H(y) - 1 \Longrightarrow H(x) = H(y) \Longrightarrow x = y \quad \text{or} \quad x = x_1 \quad \text{or} \quad y = x_1.$$

By the induction assumption we therefore have $H'(x) + H'(x^{*}) = 2(n-1) - 2$ for each $x \in X'$, so $H(x) + H(x^{*}) = 2n - 2$ for each $x \in X'$. But $H(r) + H(r^{*}) = 2n - 2 + 0 = 2n - 2$. Therefore $H(x) + H(x^{*}) = 2n - 2$ for each $x \in X$.

Finally, by the induction assumption we have $H'(x_1) = (n-1) - 1$, so $H(x_1) - 1 = n - 2$ and therefore $H(x_1) = n - 1$. □

Coming back to the original problem, we see that Mr. and Mrs. Adams shook four hands each and, in fact, each couple jointly shook eight hands.

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* * *

POSTSCRIPT TO "EQUAL CEVIANS"

One should not be surprised to learn that the results in my recent article "Equal Cevians" [1] are more simple than novel, and that the history of equal cevians reaches back to the 19th century. Not long after [1] appeared, the editor let me see a letter from O. Bottema of Delft that referred to a 1971 paper of his [2] and to his later discovery of papers by G. Brocard[1] [3] and J. Neuberg [4] published in 1896, all dealing successfully with the problem treated in [1]. I am grateful to Dr. Bottema and the editor for this opportunity to give a brief account of these earlier researches.

Bottema's paper begins with a very thorough treatment of the topic mentioned briefly at the end of [1], namely the locus Γ traced by the intersection point of *two* equal cevians of a given triangle. His methods make elegant use of various systems of homogeneous coordinates and of the circular points at infinity. With *three* equal cevians he finds that there are two real and two imaginary "normal" solutions, the common length l being expressed in terms of the sides as in (6) of [1]. He establishes, among other things, that the two real intersection points are isogonally conjugate with respect to the triangle whose sides are parallel to those of the given triangle and pass through its vertices (a known property of the Steiner foci — see below).

The papers of 1896 are somewhat less analytic, particularly that of Neuberg: such terms as isotomic conjugate, trilinear polar, barycentric coordinates, and Steiner angle reveal the rich geometrical soil of 19th-century France. Both authors obtain our equation (6) for l, and Neuberg states the equivalent result

$$\sqrt{c^2l^2 - 4\Delta^2} = \sqrt{a^2l^2 - 4\Delta^2} + \sqrt{b^2l^2 - 4\Delta^2}, \tag{9}$$

where Δ is the area of the triangle; this is implied by our equation (1), since (6) is unchanged when l is replaced by $\Delta\sqrt{3}/l$. (I can also derive algebraically the equivalent equation

$$\sqrt{l^2 - m_c^2} = \sqrt{l^2 - m_a^2} + \sqrt{l^2 - m_b^2}, \tag{10}$$

where m_a, m_b, m_c are the lengths of the medians.)

Most fascinating is the direct connection Brocard discovered with the so-called minimum circumscribed ellipse, or *Steiner ellipse*, of the given triangle, i.e., the ellipse passing through the vertices with tangents there parallel to the opposite sides. By regarding the triangle as obtained by parallel projection from an

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[1] Not to be confused with H. Brocard of "Brocard points" fame.

equilateral triangle, and noting that parallel segments undergo uniform magnification and change of direction, it is seen that the centre of the Steiner ellipse is the centroid of the triangle. A simple synthetic argument (communicated to Neuberg by a mysterious "M.A.C.") using the string and reflection properties of the ellipse now shows that its foci are precisely the points O_1, O_2 yielding equal cevians. (In the opposite direction, our equations (4) and (5) imply that $O_1A + AO_2 = O_1B + BO_2$, etc., and that the midpoint of O_1O_2 is the centroid. This yields the Steiner ellipse.)

Finally, the so-called Brocard angle ω of a given triangle ABC should be mentioned. It is known that circular arcs on AB, BC, CA touching sides BC, CA, AB, respectively, meet in a point K, and the angles KAB, KBC, KCA have a common value ω satisfying the relations

$$4\Delta\cot\omega = 4\Delta(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2.$$

It is easy to see that our equation (6) has positive roots whose *squares* are given by $\Delta(\cot\omega \pm \sqrt{\cot^2\omega - 3})$. The major and minor axes of the Steiner ellipse are 4/3 of these roots. Thus all triangles with given area and Brocard angle have the same Steiner ellipse and the same equal-cevian length l. (This observation leads immediately to the special results of [1] for isosceles triangles, among others.)

Direct geometrical proofs of equation (10) above and of the Fermat construction presented in [1] would probably represent the last word on the subject of equal cevians.

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1. J.R. Pounder, "Equal Cevians", this journal, 6 (1980) 98-104.

2. O. Bottema, "On Some Remarkable Points of a Triangle," *Nieuw Archief voor Wiskunde*, 19 (1971) 46-57.

3. Georges Brocard, "Centre de transversales angulaires égales," *Mathésis* (2), 6 (1896) 217-221.

4. J. Neuberg, "Note sur l'article précédent," *ibid.*, pp. 221-225.

J.R. POUNDER,
University of Alberta.

* * *

...BUT DON'T TELL YOUR STUDENTS

$$\sqrt{2\tfrac{2}{3}} = 2\sqrt{\tfrac{2}{3}}$$

W. KNIGHT, University
of New Brunswick.

THE OLYMPIAD CORNER: 18

MURRAY S. KLAMKIN

We start off this month with solutions to the problems asked at the Twelfth Canadian Mathematics Olympiad, which took place on 30 April 1980. The solutions we give were edited from those provided by the Canadian Mathematics Olympiad Committee (Chairman: J.H. Burry, Memorial University of Newfoundland).

TWELFTH CANADIAN MATHEMATICS OLYMPIAD (1980)

1. If $a679b$ is a five-digit number (in base 10) which is divisible by 72, determine a and b.

Solution.

An integer is divisible by 72 if and only if it is divisible by both 8 and 9. A satisfactory value of b exists if and only if $79b$ is divisible by 8, from which we get uniquely $b = 2$; and a satisfactory value of a exists if and only if $a+6+7+9+2$ is divisible by 9, from which we get uniquely $a = 3$.

2. The numbers from 1 to 50 are printed on cards. The cards are shuffled and then laid out face up in 5 rows of 10 cards each. The cards in each row are rearranged to make them increase from left to right. The cards in each column are then rearranged to make them increase from top to bottom. In the final arrangement, do the cards in the rows still increase from left to right?

Solution.

It is just as easy to consider mn cards laid out in an array of m rows and n columns. Our proof, which is indirect, shows that the answer to the question is YES. Let c_{ij} denote the number on the card in row i and column j in the final arrangement. Suppose there exist subscripts i,j,k, with $1 \le i \le m$ and $1 \le j < k \le n$, such that $c_{ij} > c_{ik}$. Since the columns are increasing from top to bottom, every one of the $m-i+1$ numbers in the set

$$A = \{c_{ij} \text{ and all the numbers below it in column } j\}$$

is greater than every one of the i numbers in the set

$$B = \{c_{ik} \text{ and all the numbers above it in column } k\}.$$

Now (after the rows were rearranged but) before the columns were rearranged, the

$m+1$ numbers in $A \cup B$ were distributed among the m rows of the array. By the pigeonhole principle, it follows that at least one number from A was in the same row as a number from B. This contradicts the fact that the rows were then increasing from left to right.

We have thus shown that $c_{ij} < c_{ik}$ whenever $1 \le i \le m$ and $1 \le j < k \le n$, that is to say, the rows are still increasing from left to right in the final arrangement. □

An equivalent problem appears in Martin Gardner's "Mathematical Games" column in the August 1980 issue of *Scientific American*. The following related problem is known from Game Theory. It also appeared, with a simple direct proof, earlier in this journal [1975: 13]:

A rectangular array of m rows and n columns contains mn distinct real numbers. For $i = 1,2,\ldots,m$, let s_i denote the smallest number of the ith row; and for $j = 1,2,\ldots,n$, let l_j denote the largest number of the jth column. If $A = \max\{s_i\}$ and $B = \min\{l_j\}$, then $A \le B$.

3. Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

Solution.

The perimeter P is given by

$$P = 2r\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right).$$

Since

$$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\sin(B+C)/2}{\sin(B/2)\sin(C/2)} = \frac{\cos(A/2)}{\sin(B/2)\sin(C/2)}, \tag{1}$$

$P_{\min}$ occurs when $\sin(B/2)\sin(C/2)$ is a maximum. Now

$$\sin\frac{B}{2}\sin\frac{C}{2} = \tfrac{1}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) = \tfrac{1}{2}\left(\cos\frac{B-C}{2} - \sin\frac{A}{2}\right),$$

and this is a maximum when $B = C$. Thus $P_{\min}$ occurs when the triangle is isosceles. □

If follows from (1) that, in any triangle ABC,

$$BC = a = \frac{r\cos(A/2)}{\sin(B/2)\sin(C/2)};$$

so we have also shown that, when A and r are fixed, side a is least when $B = C$.

4. A gambling student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring *exactly* n points is $\frac{1}{3}\{2 + (-\frac{1}{2})^n\}$.

Solution.

Let P_n denote the probability of scoring exactly n points. Since the only way of failing to score exactly n points is to score $n-1$ points and then throw a tail, we have

$$1 - P_n = \tfrac{1}{2}P_{n-1}. \tag{1}$$

Now P_n is uniquely determined from (1) and the obvious fact that $P_1 = 1/2$. To see that the suggested answer,

$$P_n = \frac{1}{3}\{2 + (-\frac{1}{2})^n\}, \tag{2}$$

is in fact the correct one, we need only verify that (2) holds for $n = 1$ and, by an easy induction, that it satisfies (1) for $n = 2,3,4,\ldots$. □

Alternatively, since the last toss yielding a score of n can be a head or a tail, we have

$$P_n = \tfrac{1}{2}P_{n-1} + \tfrac{1}{2}P_{n-2}, \tag{3}$$

and P_n is uniquely determined from (3) and the obvious results $P_1 = 1/2$ and $P_2 = (1/2)^2 + (1/2) = 3/4$. We then verify that (2) holds for $n = 1,2$, and another easy induction shows that it satisfies (3) for $n = 3,4,5,\ldots$. □

Finally, we give a proof in which the presumed answer (2) is not known in advance. We rewrite (1) in the form

$$P_n = 1 - \tfrac{1}{2}P_{n-1}. \tag{4}$$

From (4) and the knowledge that $P_1 = 1/2 = 1 - (1/2)$, the intuition being if necessary assisted by a few iterations,

$$P_2 = 1 - \tfrac{1}{2}P_1 = 1 - \tfrac{1}{2} + (\tfrac{1}{2})^2, \qquad P_3 = 1 - \tfrac{1}{2}P_2 = 1 - \tfrac{1}{2} + (\tfrac{1}{2})^2 - (\tfrac{1}{2})^3,$$

it is easy to conjecture that

$$P_n = \sum_{i=0}^{n} (-\frac{1}{2})^i, \qquad n = 1,2,3,\ldots . \tag{5}$$

Now (5) holds for $n = 1$, and an easy induction shows that it satisfies (4) for $n = 2,3,4,\ldots$. Finally formula (5), now known to be correct, is an $(n+1)$-term geometric series whose sum is given in (2).

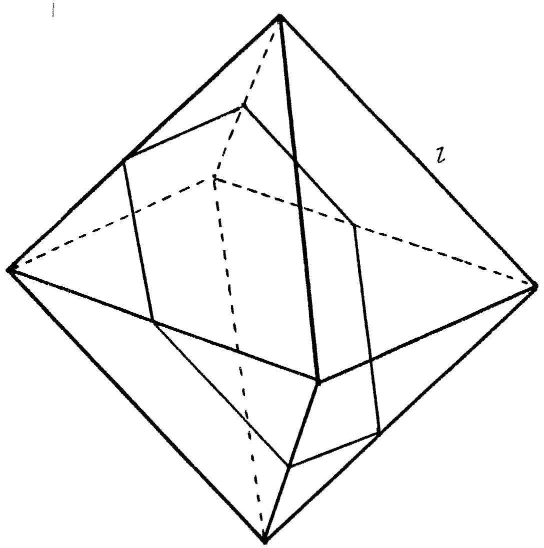
5. A parallelepiped has the property that all cross sections which are parallel to any fixed face F have the same perimeter as F. Determine whether or not any other polyhedron has this property.

Solution.

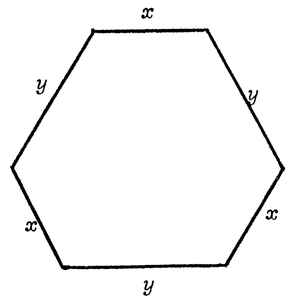
We cite the following exercise from Altshiller-Court [1]:

Any plane parallel to a face of a regular octahedron cuts the solid along a hexagon the length of whose perimeter is thrice the length of an edge of the octahedron.

This shows that the regular octahedron is one answer to our problem.



Octahedron



Cross section

The figures show why this is so. All cross sections parallel to a pair of opposite faces (with the exception of the faces themselves) are hexagons whose opposite sides are parallel. It follows easily that $x+y=l$ and that

$$\text{perimeter of cross section} = 3(x+y) = 3l = \text{perimeter of face.} \quad \square$$

As a rider, consider the same problem with "perimeter" replaced by "area".

REFERENCE

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Second Edition, Chelsea Publishing Co., Bronx, N.Y., p. 24, Ex. 24.

*

We now give solutions to the problems in Practice Set 14.

14-1. Consider the tetrahedra T_1 and T_2 with edge lengths a,b,c,d, as shown in Figures 1 and 2. Under what conditions (on a,b,c,d) is the volume of T_1 greater than that of T_2?

Solution.

We first show that the volume V of the general tetrahedron of Figure 3 is given by

$$V^2 = \frac{a^2b^2c^2}{36}(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma), \quad (1)$$

where $\alpha = \angle\,\mathrm{BPC}$, $\beta = \angle\,\mathrm{CPA}$, $\gamma = \angle\,\mathrm{APB}$. We choose an orthonormal basis $(\vec{\imath},\vec{\jmath},\vec{k})$ such that the unit vectors

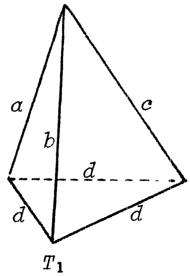


Figure 1

$$\vec{u} = \frac{\overrightarrow{\mathrm{PA}}}{a} = \vec{\imath}, \qquad \vec{v} = \frac{\overrightarrow{\mathrm{PB}}}{b} = v_1\vec{\imath}+v_2\vec{\jmath},$$
$$\vec{w} = \frac{\overrightarrow{\mathrm{PC}}}{c} = w_1\vec{\imath}+w_2\vec{\jmath}+w_3\vec{k}.$$

Now we have

$$V = \frac{abc}{6}|\vec{u}\cdot\vec{v}\times\vec{w}| = \frac{abc}{6}\left\|\begin{matrix}1 & 0 & 0\\ v_1 & v_2 & 0\\ w_1 & w_2 & w_3\end{matrix}\right\| = \frac{abc}{6}\cdot|v_2w_3|, \quad (2)$$

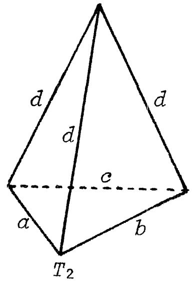


Figure 2

where the double bars indicate that the absolute value of the determinant is to be taken. Since $\vec{u}\cdot\vec{v} = v_1 = \cos\gamma$, we have $v_2 = \sin\gamma$. Also, from

$$\vec{u}\cdot\vec{w} = w_1 = \cos\beta \qquad \text{and} \qquad \vec{v}\cdot\vec{w} = v_1w_1+v_2w_2 = \cos\alpha,$$

we get $\cos\alpha = \cos\beta\cos\gamma + w_2\sin\gamma$. Hence, since $|\vec{w}| = 1$, we have

$$w_3^2 = 1-w_1^2-w_2^2 = 1-\cos^2\beta-\left(\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\gamma}\right)^2.$$

Now (2) is equivalent to

$$V^2 = \frac{a^2b^2c^2}{36}\cdot v_2^2w_3^2,$$

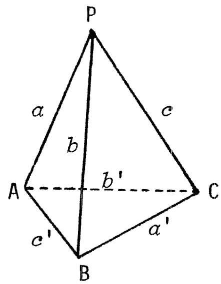


Figure 3

which reduces to (1). Since, from Figure 3,

$$\cos\alpha = \frac{b^2+c^2-a'^2}{2bc}, \quad \cos\beta = \frac{c^2+a^2-b'^2}{2ca}, \quad \cos\gamma = \frac{a^2+b^2-c'^2}{2ab}, \qquad (3)$$

whenever the edges of a tetrahedron are given its volume can conveniently be found by substituting (3) into (1).

The volume V_1 of T_1 can be found by setting $a' = b' = c' = d$ in (3) and substituting into (1). The result is

$$144V_1^2 = -d^6 + d^4\Sigma a^2 - d^2\Sigma(a^4 - b^2c^2),$$

where the sums (here and later) are cyclic. Similarly, the volume V_2 of T_2 can be found by setting $a = b = c = d$ in (1) and (3), and $a' = a$, $b' = b$, $c' = c$ in (3) and substituting into (1). Here the result is

$$144V_2^2 = -a^2b^2c^2 + d^2\Sigma(2b^2c^2 - a^4). \tag{4}$$

Thus we have

$$\begin{aligned} 144(V_1^2 - V_2^2) &= a^2b^2c^2 - d^2\Sigma b^2c^2 + d^4\Sigma a^2 - d^6 \\ &= (a^2 - d^2)(b^2 - d^2)(c^2 - d^2). \end{aligned}$$

Hence, if the edges are labeled so that $a \geq b \geq c$, we have $V_1 \geq V_2$ if and only if $a \geq d \geq b$ or $c \geq d$. □

Formula (4) for the volume of T_2 is well-known (see [1], [2]). It is usually written in the form

$$V_2 = \frac{1}{12}\sqrt{16p(p-a)(p-b)(p-c)d^2 - a^2b^2c^2},$$

where $2p = a + b + c$.

REFERENCES

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Second Edition, Chelsea Publishing Co., Bronx, N.Y., 1964, p. 122, Ex. 29.

2. F.G.-M., *Exercices de Géométrie*, Quatrième édition, Mame et Fils, Tours, 1907, p. 891.

14-2. Determine the maximum volume of a tetrahedron if it has exactly k edges ($1 \leq k \leq 3$) of length greater than 1. For the case $k = 3$, it is also assumed that the three longest edges are not concurrent, since otherwise the volume can be arbitrarily large.

Solution.

We consider three cases separately. Case 1 appeared in the Ninth International Olympiad (1967), but the proof we give here is different from that given in [1].

Case 1. $k = 1$. Let AD be the single edge that is greater than 1 (see Figure 1). If all the edges except AD are kept fixed, the volume will be a maximum when the altitude from A is greatest, that is, when the plane

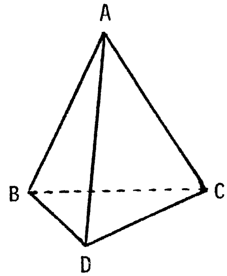


Figure 1

of ABC is perpendicular to that of BCD. We assume this is so and, to complete the problem, seek to maximize the area of BCD and the altitude from A (subject to the length constraints). By considering unit circular arcs struck from B and C, it follows easily that we must have

$$AB = AC = DB = DC = 1.$$

If AH is the altitude from A and $BC = 2a$, where $a \le \frac{1}{2}$ (see Figure 2), we have

$$AH = \sqrt{1-a^2} \quad \text{and} \quad \text{area } BCD = \text{area } ABC = a\sqrt{1-a^2},$$

so the volume is

$$V = \frac{a(1-a^2)}{3}, \quad 0 < a \le \tfrac{1}{2}.$$

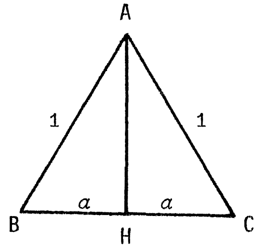


Figure 2

This function is strictly increasing over its interval of definition, with a value $V_{max} = 1/8$ when $a = \frac{1}{2}$ (and then $BC = 1$ and $AD = \sqrt{3/2} > 1$).

Case 2: $k = 2$. If opposite edges AD and BC are the only two edges that are greater than 1, then the analysis of Case 1 applies again, except that now we have $\frac{1}{2} < a < 1$. Thus the volume is given by

$$V = \frac{a(1-a^2)}{3}, \quad \tfrac{1}{2} < a < 1.$$

Here the maximum is $V = 2\sqrt{3}/27$ when $a = 1/\sqrt{3}$ (and then $AD = BC = 2/\sqrt{3} > 1$). We must compare this with the maximum volume when the two edges greater than 1 are concurrent. Say we have $AB > 1$ and $AD > 1$. As before, the maximum volume will occur when the plane of ABC is perpendicular to that of BCD. Now BCD has maximum area when $BC = CD = DB = 1$. The altitude from A will be a maximum when it is equal to $AC = 1$ and $AC \perp BC$. The volume is then $\sqrt{3}/12$ (and $AB = AD = \sqrt{2} > 1$). Since $\sqrt{3}/12 > 2\sqrt{3}/27$, we have $V_{max} = \sqrt{3}/12$.

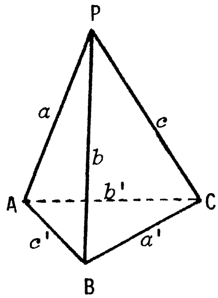
Case 3: $k = 3$. Here also there are two subcases to consider. If the three edges greater than 1 form a triangle, say ABC, it is clear that the maximum volume 1/6 occurs when DA, DB, and DC are each of unit length and mutually perpendicular (and then $BC = CA = AB = \sqrt{2} > 1$). The other subcase is when the three edges greater than 1 do not form a triangle (but are not all concurrent). If, say, AB, BC, and AD are greater than 1, then the volume will be a maximum when the area of BCD is a maximum and the altitude from A is a maximum. Now the area of BCD is a maximum when $BD = DC = 1$ and $BD \perp DC$. The maximum altitude from A occurs when it coincides with AC when AC $\perp$ BCD and AC $= 1$. Here AD $=$ BC $= \sqrt{2} > 1$, AB $= \sqrt{3} > 1$, and the volume is again $1/6$. Hence $V_{\max} = 1/6$.

REFERENCE

1. Samuel L. Greitzer, *International Mathematical Olympiads* 1959-1977, Mathematical Association of America, 1978, pp. 9, 100.

14-3. If tetrahedron P-ABC has edge lengths a,b,c,a',b',c' as shown in the figure, prove that

$$\frac{a'}{b+c} + \frac{b'}{c+a} > \frac{c'}{a+b}.$$



Solution.

Rather than attempting to establish the required inequality for an arbitrary order of edge lengths a,b,c, we will assume that $a \geq b \geq c$ and show that the quantities

$$\frac{a'}{b+c}, \quad \frac{b'}{c+a}, \quad \frac{c'}{a+b}$$

are the side lengths of some triangle.

We first prove that

$$\frac{a'}{b+c} + \frac{b'}{c+a} > \frac{c'}{a+b}. \tag{1}$$

Since $a' + b' > c'$, it suffices to show that

$$\frac{a'}{b+c} + \frac{b'}{c+a} \geq \frac{a'+b'}{a+b}.$$

But this follows easily from $a \geq b \geq c$.

Next we show that

$$\frac{b'}{c+a} + \frac{c'}{a+b} > \frac{a'}{b+c} \tag{2}$$

or, equivalently, that

$$\frac{ab'}{c+a} + \frac{ac'}{a+b} > \frac{aa'}{b+c}.$$

Since $bb' + cc' > aa'$ by the extension to 3 dimensions of the Ptolemaic inequality (see [1979: 131]), it suffices to prove that

$$\frac{ab'}{c+a} + \frac{ac'}{a+b} \ge \frac{bb' + cc'}{b+c} .$$

But this follows immediately from the inequalities

$$\frac{a}{c+a} \ge \frac{b}{b+c} \quad \text{and} \quad \frac{a}{a+b} \ge \frac{c}{b+c} ,$$

which are implied by $a \ge b \ge c$.

Finally, we show that

$$\frac{c'}{a+b} + \frac{a'}{b+c} > \frac{b'}{c+a} . \tag{3}$$

If

$$\frac{c'}{a+b} + \frac{a'}{b+c} \ge \frac{c' + a'}{c+a} ,$$

then (3) follows as in the proof of (1). So we assume that

$$\frac{c'}{a+b} + \frac{a'}{b+c} < \frac{c' + a'}{c+a}$$

or, equivalently, that

$$a'(a^2 - b^2) < c'(b^2 - c^2). \tag{4}$$

We establish the inequalities

$$b\left(\frac{c'}{a+b} + \frac{a'}{b+c}\right) \ge \frac{cc' + aa'}{c+a} > \frac{bb'}{c+a} , \tag{5}$$

from which (3) follows. The first inequality in (5) is equivalent to

$$ac'(b^2 - c^2) \ge ca'(a^2 - b^2),$$

and this follows from (4) and $a \ge c$; and the second inequality in (5) follows from the Ptolemaic inequality, as in the proof of (2). □

The inequality in this problem is due to D.J. Schattschneider [1]. Also, see the letter to and the reply by the editors in [2] and [3]. A. Meir and the author have extended this result and related ones in [4].

REFERENCES

1. D.J. Schattschneider, "A Multiplicative Metric", *Mathematics Magazine*, 49 (1976) 203-205.

2. Letter to the editors, *ibid.*, 50 (1977) 55-56.

3. Reply by the editors, *ibid.*, 51 (1978) 207-208.

4. M.S. Klamkin and A. Meir, "Ptolemy's Inequality, Chordal Metric, Multiplicative Metric," to appear in *The Pacific Journal of Mathematics*.

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Finally, one new Practice Set for which solutions will appear here next month.

PRACTICE SET 15

15-1. Determine an n-digit number (in base 10) such that the number formed by reversing the digits is nine times the original number. What other multiples besides nine are possible?

15-2. Solve the following system of equations:

$$\begin{array}{l} ax_1 + bx_2 + bx_3 + \ldots + bx_n = c_1, \\ bx_1 + ax_2 + bx_3 + \ldots + bx_n = c_2, \\ \vdots \\ bx_1 + bx_2 + bx_3 + \ldots + ax_n = c_n. \end{array}$$

(In the left member of each equation, all the coefficients except one are b's and the remaining one is a.)

15-3. Three circular arcs of fixed total length are constructed, each passing through two different vertices of a given triangle, so that they enclose the maximum area. Show that the three radii are equal.

Editor's note: All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

571. *Proposed by Sidney Kravitz, Dover, New Jersey.*

The party conventions are over and, come November, Americans will have to

$$\begin{array}{r} \text{ELECT} \\ \text{REAGAN} \\ \underline{\text{OR}} \\ \text{CARTER} \end{array}.$$

Members of the U.S. House of Representatives are also entitled to find the unique answer to this problem. Indeed, because of John Anderson, they may have to.

572.* *Proposed by Paul Erdős, Technion - I.I.T., Haifa, Israel.*

It was proved in Crux 458 [1980: 157] that, if ϕ is the Euler function and the integer $c > 1$, then each solution n of the equation

$$\phi(n) = n - c \qquad (1)$$

satisfies $c + 2 \le n \le c^2$.

Let $F(c)$ by the *number* of solutions of (1). Estimate $F(c)$ as well as you can from above and below.

573. *Proposed by Charles W. Trigg, San Diego, California.*

In Crux 430 [1980: 52], attention was called to the fact that the decimal digits of 8^3 sum to 8. Is there another power of 2, say $P > 8$, and a positive integer k such that the decimal digits of P^k sum to P?

574. *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Given five points A, B, C, D, E, construct a straight line l such that the three pairs of straight lines {AD,AE}, {BD,BE}, {CD,CE} intercept equal segments on l.

575. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

If $n \ge 2$ is an integer and square brackets denote the greatest integer function, evaluate

$$S = \sum_{r=1}^{[n/2]} \sum_{k=1}^{n+1-2r} \left[\frac{n-r}{k+r-1}\right].$$

576. *Proposed by Mats Röyter, student, Chalmers University of Technology, Sweden.*

Consider two n-digit numbers in base b, the digits of one being a permutation of the digits of the other. Prove that the difference of the numbers is divisible by $b-1$.

577. *Proposed by R.B. Killgrove, California State University at Los Angeles.*

Explain why, in calculus courses, we are never asked to find the exact value of the arc length for the general sine curve $y = a\sin bx$ from $x = 0$ to $x = c$, where $c \le 2\pi/b$.

578.* *Proposed by S.C. Chan, Singapore.*

An unbiased cubical die is thrown repeatedly until a 5 and a 6 have been obtained. The random variable X denotes the number of throws required. Calculate $E(X)$, the expectation of X and $V(X)$, the variance of X.

579. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

If n is a nonnegative integer, express in closed form the sum

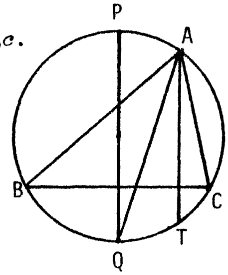
$$H_n = \sum_{r+s+t=n} a^r b^s c^t,$$

which is the sum of all homogeneous products of degree n in a,b,c.

580. *Proposed by Leon Bankoff, Los Angeles, California.*

In the figure, the diameter $PQ \perp BC$ and chord $AT \perp BC$. Show that

$$\frac{AQ}{PQ} = \frac{AB+AC}{PB+PC} = \frac{TB+TC}{QB+QC} .$$



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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

466. [1979: 200; 1980: 188] Late solution and comment from DIMITRIS VATHIS, Chalcis, Greece.

* * *

484. [1979: 265] *Proposed by Gali Salvatore, Perkins, Québec.*

Let A and B be two independent events in a sample space, and let χ_A, χ_B be their characteristic functions (so that, for example, $\chi_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$). If $F = \chi_A + \chi_B$, show that at least one of the three numbers

$$a = P(F{=}2), \quad b = P(F{=}1), \quad c = P(F{=}0)$$

is not less than 4/9.

Solution by M.S. Klamkin, University of Alberta and Léo Sauvé, Algonquin College, Ottawa (jointly).

We have $a \geq 0$, $b \geq 0$, $c \geq 0$, $a+b+c=1$, and we must show that $\max\{a,b,c\} \geq 4/9$. We will prove that

$$\min_{0 \leq x,y \leq 1} \max\{a,b,c\} = \frac{4}{9}, \tag{1}$$

where $x = P(A)$ and $y = P(B)$. From the given data, we have

$$a = xy, \quad b = x(1-y) + y(1-x), \quad c = (1-x)(1-y),$$

and it follows that

$$b^2 - 4ac = \{x(1-y) - y(1-x)\}^2 \geq 0.$$

If $b < 4/9$, then $ac \leq b^2/4 < 4/81$ and $a + c > 5/9$, so

$$a^2 + \frac{4}{81} > a^2 + ac = a(a+c) \geq \frac{5a}{9},$$

which implies that

$$(a - \tfrac{1}{9})(a - \tfrac{4}{9}) > 0.$$

If also $a < 4/9$, then $a < 1/9$ and $c > 4/9$. Hence $\max\{a,b,c\} \geq 4/9$. In particular, if $x = y = 2/3$, then $a = b = 4/9$, $c = 1/9$, and (1) is established. □

We now extend the problem by showing that

$$\max_{0 \leq x,y \leq 1} \min\{a,b,c\} = \frac{1}{4}. \tag{2}$$

If $a > 1/4$, then $b^2 \geq 4ac \geq c$ and $b + c < 3/4$, so

$$\sqrt{c} + c \leq b + c < \frac{3}{4},$$

which implies that

$$(\sqrt{c} + \tfrac{3}{2})(\sqrt{c} - \tfrac{1}{2}) < 0,$$

from which we get $\sqrt{c} < 1/2$ and $c < 1/4$. Hence $\min\{a,b,c\} \leq 1/4$. In particular, if $x = y = 1/2$, then $a = c = 1/4$, $b = 1/2$, and (2) is established.

We now use the familiar probabilistic model of dice to formulate problems which generalize (1) and (2). Suppose we have polyhedral dice D_i, $i = 1,2,\ldots,m$, each of which has on each of its faces one of the numbers $0,1,\ldots,n$. (The dice are not necessarily "fair" and they need not all have the same number of faces.) Let P_{ij} denote the probability that a face numbered j turns *down* on a throw of die D_i. (Note that we say "down" rather than "up" since there may not be any unique face turning "up", e.g., with a tetrahedral die.) The P_{ij}'s satisfy

$$0 \le P_{ij} \le 1 \quad \text{and} \quad \sum_{j=0}^{n} P_{ij} = 1, \qquad i = 1,2,\ldots,m,$$

but are otherwise arbitrary. Now let P_k be the probability that, in a random toss of the m dice, the sum of the down faces is k. Then we have the identity

$$P_0 + P_1 t + \ldots + P_{mn} t^{mn} = \prod_{i=1}^{m} (P_{i0} + P_{i1} t + \ldots + P_{in} t^n).$$

The problems we propose are to determine

$$A(m,n) \equiv \min_{0 \le P_{ij} \le 1} \max_k P_k \tag{3}$$

and

$$B(m,n) \equiv \max_{0 \le P_{ij} \le 1} \min_k P_k . \tag{4}$$

It is clear that

$$A(1,1) = \tfrac{1}{2}, \qquad B(1,1) = \tfrac{1}{2} ,$$

and we know from (1) and (2) that

$$A(2,1) = \frac{4}{9}, \qquad B(2,1) = \frac{1}{4} ;$$

but it appears difficult to determine (3) and (4) for arbitrary m and n.

It is easy enough to make a conjecture when $n = 1$. Here we have the identity

$$P_0 + P_1 t + \ldots + P_m t^m = (P_{10} + P_{11} t)(P_{20} + P_{21} t) \ldots (P_{m0} + P_{m1} t).$$

Our conjecture is that $A(m,1)$ for odd m, and $B(m,1)$ for all m, will both be attained when

$$P_{11} = P_{21} = \ldots = P_{m1} = \tfrac{1}{2}.$$

Then we have

$$P_0 + P_1 t + \ldots + P_m t^m = (\tfrac{1}{2} + \tfrac{1}{2} t)^m = \frac{(1+t)^m}{2^m} ,$$

from which

$$A(m,1) = \frac{\text{greatest coefficient in } (1+t)^m}{2^m} = \frac{\binom{m}{[m/2]}}{2^m}$$

and

$$B(m,1) = \frac{\text{smallest coefficient in } (1+t)^m}{2^m} = \frac{1}{2^m} .$$

Our conjecture is restricted to odd m for $A(m,1)$ because it does not give the correct value of $A(2,1)$.

No obvious conjecture suggests itself when $n>1$, but we can give a partial result for $A(2,n)$. For this problem, A. Meir (University of Alberta) recalled that the late Leo Moser once asked "how equal" the P_k's can be. It is a well-known problem by now that they cannot all be equal (see Crux 118 [1976: 101] for a special case). It is not difficult to get all of the P_k's except one equal. This gives us the bound

$$A(2,n) \le \frac{1}{2n},$$

achieved by letting

$$P_{1j} = \begin{cases} \frac{1}{n}, & \text{for } j \ne 0, \\ 0, & \text{for } j = 0, \end{cases} \qquad \text{and} \qquad P_{2j} = \begin{cases} 0, & \text{for } 0 < j < n, \\ \frac{1}{2}, & \text{for } j = 0, n. \end{cases}$$

In particular, these probabilities can be obtained as follows for $n=5$. (Here we will use $1,2,\ldots,6$ instead of $0,1,\ldots,5$ for the face numbers, as in the game of craps.) Let D_1 and D_2 be two "fair" regular icosahedral dice (20 faces), with each of the five numbers $2,3,4,5,6$ on four faces of D_1 and each of the two numbers $1,6$ on ten faces of D_2. Coincidentally, it turns out that with this set of icosahedral dice, which roll better than cubes, the game of craps is exactly fair. (Las Vegas will not be interested in this.)

Another measure for "closely equal" is to use

$$C(2,n) \equiv \min_{0 \le P_{ij} \le 1} \{\max_k P_k - \min_k P_k\}.$$

By choosing

$$P_{1j} = \begin{cases} \frac{1}{2n}, & \text{for } j = 0, n, \\ \frac{1}{n}, & \text{for } 0 < j < n, \end{cases} \qquad \text{and} \qquad P_{2j} = \begin{cases} 0, & \text{for } 0 < j < n, \\ \frac{1}{2}, & \text{for } j = 0, n, \end{cases}$$

we obtain

$$C(2,n) \le \frac{1}{4n}.$$

It would be of interest to determine other exact values or good bounds for the $A(m,n)$, $B(m,n)$, $C(m,n)$, and other related quantities, e.g., min (next to largest P_k). For two related dice problems, see Problem 3 in the Eighth U.S.A. Mathematical Olympiad [1979: 128] and Problem 80-5 in *SIAM Review* 22 (1980).

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; LEROY F. MEYERS, The Ohio State University; V.N. MURTY, Pennsylvania State University, Capitol Campus; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

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485. [1979: 265] *Proposed by M.S. Klamkin, University of Alberta.*

Given three concurrent cevians of a triangle ABC intersecting at a point P, we construct three new points A', B', C' such that $AA' = kAP$, $BB' = kBP$, $CC' = kCP$, where $k > 0$, $k \neq 1$, and the segments are directed. Show that A, B, C, A', B', C' lie on a conic if and only if $k = 2$.

This problem generalizes Crux 278 [1977: 227; 1978: 109].

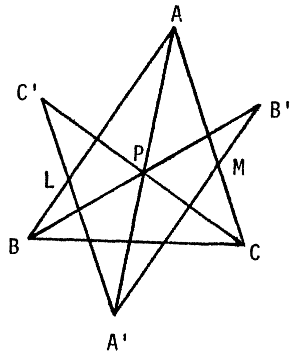
I. *Solution by Benji Fisher, New York, N.Y.*

We weaken the conditions on k to $k \neq 0,1$ (thus allowing negative values). The problem requires the additional hypothesis "P is not on any (extended) side of the triangle", for otherwise the six points A,B,C,A',B',C' (some of which may coincide) lie on a degenerate conic for every k (including even $k = 0,1$). Under these conditions the six points are all distinct.

Now A'B' and A'C' are the images of AB and AC, respectively, under the homothecy $H(P, 1-k)$. Hence, if

$$L = AB \cap A'C' \quad \text{and} \quad M = AC \cap A'B',$$

then ALA'M is a parallelogram (see figure).



Suppose the six points lie on a conic. Applying Pascal's Theorem to the hexagon ABB'A'C'C, we find that L, M, and P are collinear. Since the diagonals of parallelogram ALA'M bisect each other, their intersection P bisects AA', and so $k = 2$.

Conversely, suppose $k = 2$. Then P bisects AA' and, since the diagonals of parallelogram ALA'M bisect each other, LM passes through P, that is, $P = LM \cap AA'$. As explained in Pedoe [1], the line AM intersects the conic through A,B,B',A',C' in $C'P \cap AM = C$. Thus the six points lie on a conic and the proof is complete.

II. *Solution by Jan van de Craats, Leiden University, The Netherlands.*

[As in solution I, we assume that $k \neq 0,1$ and that P does not lie on any extended side of the triangle.] We choose an affine coordinate system such that $P = (0,0)$, $A = (1,0)$, $B = (0,1)$, and $C = (c_1, c_2)$. Then, with $m = 1 - k \neq 1, 0$, we have $A' = (m,0)$, $B' = (0,m)$, and $C' = (mc_1, mc_2)$. Two distinct conics through A,B,A',B' are

$$K_1 \equiv xy = 0,$$

$$K_2 \equiv x^2 + y^2 - (1+m)(x+y) + m = 0,$$

and all conics through these four points are given by

$$K \equiv \lambda K_1 + \mu K_2 = 0, \qquad (\lambda,\mu) \neq (0,0).$$

The points C and C' are on the conic $K=0$ if and only if

$$\lambda c_1 c_2 + \mu\{c_1^2 + c_2^2 - (1+m)(c_1+c_2) + m\} = 0 \tag{1}$$

and

$$\lambda m^2 c_1 c_2 + \mu\{m^2(c_1^2+c_2^2) - m(1+m)(c_1+c_2) + m\} = 0. \tag{2}$$

If $m=-1$, then (1) and (2) are identical and the points C,C' (hence all six points) will lie on the conic $K=0$ when $\lambda = c_1^2+c_2^2-1$ and $\mu = -c_1c_2$. Here at least $\mu \neq 0$ since P is not on AC or BC.

Conversely, suppose c_1 and c_2 satisfy (1) and (2) for some $(\lambda,\mu)\neq(0,0)$. We have $c_1c_2 \neq 0$ since P is not on AC or BC, and therefore $\mu \neq 0$. Since $m \neq 0$, we may divide (2) by m^2 and subtract (1) to get an equation equivalent to

$$(m^2-1)(c_1+c_2-1) = 0.$$

Now $c_1+c_2 \neq 1$ since C is not on AB; hence $m^2=1$, and $m\neq 1$ implies $m=-1$.

We have thus shown that the six points lie on a conic if and only if $m=-1$, that is, if and only if $k=2$.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; G.C. GIRI, Midnapore College, West Bengal, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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1. Dan Pedoe, "Pascal Redivivus: I", this journal, 5 (November 1979) 254-258, esp. p. 255.

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486. [1979: 266] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

(a) Find all natural numbers N whose decimal representation

$$N = \overline{abcdefghi}$$

consists of nine distinct nonzero digits such that

$$2|a-b, \quad 3|a-b+c, \quad 4|a-b+c-d, \quad \ldots, \quad 9|a-b+c-d+e-f+g-h+i.$$

(b) Do the same for natural numbers $N = \overline{abcdefghij}$ consisting of ten distinct digits (leading zeros excluded) such that

$$2|a-b, \quad 3|a-b+c, \quad \ldots, \quad 10|a-b+c-d+e-f+g-h+i-j.$$

Solution by Charles W. Trigg, San Diego, California.

(a) Let $A = a + c + e + g + i$ and $B = b + d + f + h$. Then we have $A - B \equiv 0 \pmod 9$ and $A + B = 45 \equiv 0 \pmod 9$, from which $2A \equiv 0 \pmod 9$, so A and B are both multiples of 9. Since $10 \le B \le 30$, we must have either $B = 18$, $A = 27$ or $B = 27$, $A = 18$. Now from

$$2|a-b, \quad 4|(a-b)+(c-d), \quad 6|(a-b)+(c-d)+(e-f), \quad 8|(a-b)+(c-d)+(e-f)+(g-h),$$

it follows that each of the numbers

$$a-b, \quad c-d, \quad e-f, \quad g-h$$

is even. Thus, aside from i, A and B each contain two even and two odd addends. This implies that B is even, so we must have $B = 18$ and $A = 27$. Now $8|A-B-i = 9-i$, so $i = 1$ or 9. Since

$$7|(A-i) - (B-h) = 9 - i + h,$$

we have $h = 6$ if $i = 1$, and $h = 7$ if $i = 9$. We recapitulate what we have found out so far about the sets of addends in A and B:

(i) A contains two even and three odd addends, including 1 or 9, whose sum is 27; and B contains two even and two odd addends whose sum is 18.

(ii) If A contains 1 but not 9, B contains 6; if A contains 9 but not 1, B contains 7; and if A contains 1 and 9, B contains 6 or 7.

(iii) A satisfactory N must end in 61 or 79.

There are fourteen possible B sets consisting of four distinct digits with sums that are multiples of 9. But, when considered in conjunction with the corresponding possible A sets, only five satisfy conditions (i) and (ii). These are given in the adjoining table. Proceeding exhaustively by trial and error and using condition (iii) [the details are omitted (Editor)], the first three sets in the table are eliminated, and the last two yield exactly three answers to our problem:

A					B			
3	4	5	6	9	1	2	7	8
2	3	5	8	9	1	4	6	7
1	4	5	8	9	2	3	6	7
1	3	6	8	9	2	4	5	7
1	2	7	8	9	3	4	5	6

$$N_1 = 358264179,$$

$$N_2 = 823564179,$$

$$N_3 = 958473261.$$

(b) Let $A = a + c + e + g + i$ and $C = b + d + f + h + j$. Then $A - C = 10k$. But $A + C = 45$, so $2A = 5(2k + 9)$ with both factors odd. Thus A cannot be an integer and the problem has no solution.

Partial solutions were received from HERMAN NYON, Paramaribo, Surinam; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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487. [1979: 266] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

If a, b, c, and d are positive real numbers such that $c^2+d^2=(a^2+b^2)^3$, prove that

$$\frac{a^3}{c}+\frac{b^3}{d}\geq 1,$$

with equality if and only if $ad=bc$.

I. *Solution by M.S. Klamkin, University of Alberta.*

If $p>1$, $1/p+1/q=1$, and all $u_j, v_j>0$, Hölder's inequality gives

$$\left\{\sum_{j=1}^{n}u_j^r\right\}^{1/p}\left\{\sum_{j=1}^{n}v_j^{qr}/u_j^{qr/p}\right\}^{1/q}\geq\sum_{j=1}^{n}v_j^r,$$

with equality if and only if $u_j/v_j=$ constant for all j. The proposed problem corresponds to the special case

$$n=2,\quad u_1=c,\quad u_2=d,\quad v_1=a,\quad v_2=b,\quad r=2,\quad p=3,$$

which gives

$$(c^2+d^2)^{1/3}\left\{\frac{a^3}{c}+\frac{b^3}{d}\right\}^{2/3}\geq a^2+b^2,$$

with equality if and only if $c/a=d/b$. This is clearly equivalent to the proposed inequality.

II. *Solution by V.N. Murty, Pennsylvania State University, Capitol Campus.*

The Cauchy-Schwarz inequality gives

$$(x_1^2+x_2^2)(y_1^2+y_2^2)\geq(x_1y_1+x_2y_2)^2, \tag{1}$$

with equality if and only if $x_1y_2-x_2y_1=0$. With

$$x_1=a^{3/2}c^{-1/2},\quad x_2=b^{3/2}d^{-1/2},\quad y_1=a^{1/2}c^{1/2},\quad y_2=b^{1/2}d^{1/2},$$

(1) reduces to

$$\left(\frac{a^3}{c}+\frac{b^3}{d}\right)(ac+bd)\geq(a^2+b^2)^2, \tag{2}$$

with equality if and only if $ad=bc$; and with

$$x_1 = a, \quad x_2 = b, \quad y_1 = c, \quad y_2 = d,$$

(1) is equivalent to

$$(a^2+b^2)^{1/2}(c^2+d^2)^{1/2} \geq ac+bd, \tag{3}$$

with equality if and only if $ad=bc$. If we "cross-divide" (2) and (3), we obtain finally

$$\frac{a^3}{c}+\frac{b^3}{d} \geq \left\{\frac{(a^2+b^2)^3}{c^2+d^2}\right\}^{1/2} = 1,$$

with equality if and only if $ad=bc$.

Also solved by E.J. BARBEAU, University of Toronto; W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; ALLAN WM. JOHNSON JR., Washington, D.C.; VIKTORS LINIS, University of Ottawa; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (two solutions); JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

Editor's comment.

Five solvers used trigonometric substitutions and calculus in their solution. But it seems more esthetically satisfying to establish this algebraic inequality by purely algebraic means when this can be done conveniently, as in our featured solutions.

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488. [1979: 266] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given a point P within a given angle, construct a line through P such that the segment intercepted by the sides of the angle has minimum length.

I. *Part of the solution by M.S. Klamkin, University of Alberta.*

Let θ be the measure of the given angle AOB, and let a,b be the distances of P from OA,OB as shown in Figure 1. Our problem is equivalent to finding the length of the longest beam that can be carried horizontally around a corner where two corridors, of widths a and b, meet at an angle θ.

For the special case where $\theta = 90^\circ$, the problem appears in many calculus texts (see, e.g., [1] and [2]). The expected solution is, naturally, by calculus. We show that the answer can easily be found by purely algebraic means. As can be seen from Figure 2, we have to minimize $L=\sqrt{h^2+k^2}$ subject to constraint $a/h+b/k=1$, where a and b are given positive numbers. By Hölder's inequality, we have

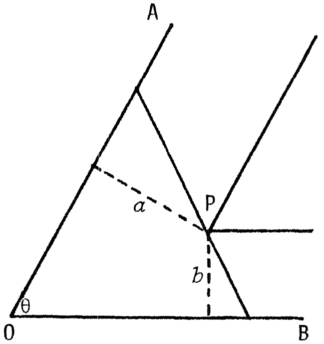

Figure 1

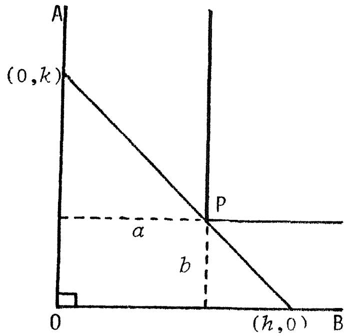

Figure 2

$$(h^2+k^2)^{1/3}\left(\frac{a}{h}+\frac{b}{k}\right)^{2/3} \geq a^{2/3}+b^{2/3}$$

or

$$L = \sqrt{h^2+k^2} \geq (a^{2/3}+b^{2/3})^{3/2},$$

with equality if and only if $h^3/a = k^3/b$ or

$$h = a^{1/3}(a^{2/3}+b^{2/3}), \qquad k = b^{1/3}(a^{2/3}+b^{2/3}).$$

With these values of h and k, we get the required answer:

$$L_{\min} = (a^{2/3}+b^{2/3})^{3/2}.$$

II. *Solution by Jan van de Craats, Leiden University, The Netherlands (revised by the editor).*

We choose a rectangular coordinate system with origin at the vertex of the given angle AOB and such that the given point P = (0,1). We assume that the sides of the angle are labeled so that the ray OA has the greater slope a and the ray OB the lesser slope b, as shown in Figure 3. Then any line l through P which meets OA, OB in distinct points is of the form

$$l\colon\ y = mx+1, \qquad b < m < a,$$

and it meets OA and OB, respectively, in the points

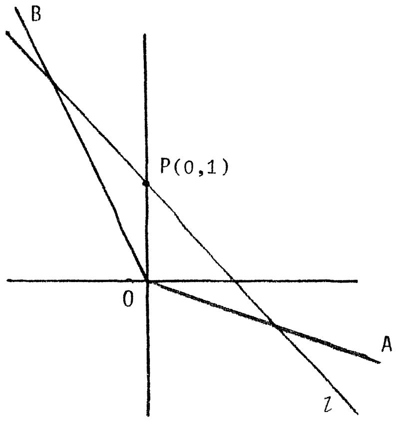

Figure 3

$$\left(\frac{1}{a-m}, \frac{a}{a-m}\right) \quad \text{and} \quad \left(\frac{1}{b-m}, \frac{b}{b-m}\right).$$

The length of the intercepted segment, which is the quantity to be minimized, is now easily found to be

$$f(m) = \frac{(a-b)\sqrt{1+m^2}}{(a-m)(m-b)}, \qquad b < m < a,$$

and differentiation leads to

$$\operatorname{sgn} f'(m) = \operatorname{sgn}\left(m^3 + (2-ab)m - a - b\right). \tag{1}$$

It is geometrically obvious that the function f has a unique critical point and hence that the equation

$$g(m) \equiv m^3 + (2-ab)m - a - b = 0$$

has only one real root m_0 (which can be found exactly, if desired, by Cardan's formula). Nevertheless, we give an analytic proof of this fact. This is clear if $2-ab \geq 0$, so we assume $2-ab < 0$. The desired result then follows from the fact that, if m_1 and m_2 are the roots of $g'(m) = 0$, then

$$g(m_1)g(m_2) = (a+b)^2 - \frac{4}{27}(2-ab)^3 > 0.$$

Finally, it is clear from (1) that $f'(m)$ changes sign from - to + as m increases through m_0, so $f(m_0)$ is the required minimum length of the intercepted segment.

It is clear that the required line l can be constructed by Euclidean means if and only if a segment of length $|m_0|$ is so constructible. Since m_0 is a root of a cubic equation, the construction is not in general possible. In particular, since a unit segment OP is given, the construction is never possible when a and b are rational and $g(m) = 0$ has no rational root (see the theorem cited in the solution of Crux 415 [1979: 306]).

III. *Solution by Basil Rennie, James Cook University of North Queensland, Australia.*

The proposal does not specify if a Euclidean construction is desired. We give a counterexample to show that such a construction is not in general possible.

Suppose the given angle is a right angle whose sides are the positive semiaxes of a rectangular coordinate system, and let P be the point (1,2). It is an easy calculus problem to show that the sides of the angle intercept a segment of minimal length on a line through P of slope m if and only if $m = -\sqrt[3]{2}$. So a Euclidean construction for our problem implies a Euclidean construction for the Delian problem (duplication of the cube), a construction long known to be impossible (see [3]).

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; VIKTORS LINIS, University of Ottawa; HERMAN NYON, Paramaribo, Surinam; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer (partial solution).

Editor's comment.

The segment of minimal length through P is part of a line that is well known (by few people) as the *Philo Line* of the angle with respect to point P. For more information, see the article by Howard Eves which, not coincidentally, appears in this issue (pages 232-237).

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1. W.E. Granville, P.F. Smith, and W.R. Longley, *Elements of the Differential and Integral Calculus*, Ginn and Co., Boston, 1934, p. 61, Problem 25.

2. John A. Tierney, *Calculus and Analytic Geometry*, Third Edition, Allyn and Bacon, Boston, 1975, p. 157, Problem 27.

3. Felix Klein, *Famous Problems of Elementary Geometry*, Dover, New York, 1956, Chapter II.

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489. [1979: 266] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Find all real numbers x, y, z such that

$$(1-x)^2+(x-y)^2+(y-z)^2+z^2 = \frac{1}{4}.$$

Solution by M.S. Klamkin, University of Alberta.

More generally, we use the power-mean inequality to show that, if $r<0$ or $r>1$ and

$$|a-t_1|^r+|t_1-t_2|^r+\ldots+|t_{n-1}-t_n|^r+|t_n-b|^r = \frac{|a-b|^r}{(n+1)^{r-1}}$$

(with $a\neq b$ if $r<0$), then

$$t_i = \frac{a(n-i+1)+ib}{n+1}, \qquad i=1,2,\ldots,n. \tag{1}$$

We have

$$\frac{|a-b|}{n+1} = \frac{|(a-t_1)+(t_1-t_2)+\ldots+(t_{n-1}-t_n)+(t_n-b)|}{n+1} \tag{2}$$

$$\leq \frac{|a-t_1|+|t_1-t_2|+\ldots+|t_{n-1}-t_n|+|t_n-b|}{n+1} \tag{3}$$

$$\leq \left\{\frac{|a-t_1|^r+|t_1-t_2|^r+\ldots+|t_{n-1}-t_n|^r+|t_n-b|^r}{n+1}\right\}^{1/r} \tag{4}$$

$$= \frac{|a-b|}{n+1},$$

so equality holds throughout. The equality (3) = (4) shows that the numbers

$$a-t_1, \quad t_1-t_2, \quad \ldots, \quad t_{n-1}-t_n, \quad t_n-b \tag{5}$$

have equal absolute values, and the equality (2) = (3) shows that they all have the same sign. Hence the $n+1$ numbers (5) are all equal, each being equal to $(a-b)/(n+1)$, and (1) follows by induction.

In the present problem, we have $n = 3$, $r = 2$, $a = 1$, $b = 0$, and so

$$x = t_1 = \frac{3}{4}, \quad y = t_2 = \frac{1}{2}, \quad z = t_3 = \frac{1}{4}.$$

Also solved by E.J. BARBEAU, University of Toronto (3 solutions); W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; BENJI FISHER, New York, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y. (2 solutions); BOB PRIELIPP, University of Wisconsin-Oshkosh; SANJIB KUMAR ROY, Research scholar, Indian Institute of Technology, Kharagpur, India; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

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QUOTATIONS

And when it is said in the Gospel that *Christ made the Blind to see,* and *the Deaf to hear,* and *the Lame to walk,* we ought not to infer here that *Christ performed Contradictions.*

Quoted from

I. WATTS, Logic, or the Right Use of Reason in the Enquiry after Truth, with a Variety of Rules to Guard Against Error, in the Affairs of Religion and Human Life, as well as in the Sciences; London, 1775;

in John Woods and Douglas Walton, "Composition and division", p. 404 of pp. 381-406 of *Studia Logica*, vol. 36, no. 4: On Leśniewski's Systems: Proceedings of XXII Conference on the history of logic.

But the limit of desired knowledge was unattainable, nor could I ever foretell the approximate point after which I might imagine myself satiated, because of course the denominator of every fraction of knowledge was potentially as infinite as the number of intervals between the fractions themselves.

VLADIMIR NABOKOV in "That in Aleppo once ...", from *Nabokov's Dozen*, Doubleday, 1958.

Do not worry about your difficulties in mathematics; I can assure you that mine are still greater.

EINSTEIN

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