Crux Mathematicorum

Editorial Board

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Crux Mathematicorum

Crux Mathematicorum
with Mathematical Mayhem

EDITORIAL

Sometimes school mathematics scares me. At the recent Math Matters seminar that I organize at my home university, one of our department members brought in his daughter's Grade 12 math binder. We saw pages and pages of exercises on completing the square, dozens of proofs of trigonometric identities, and quite literally hundreds of exponential and logarithmic equations (I am not actually exaggerating). There was some graphing and a couple of word problems sprinkled here and there. But what was more stunning than the amount of purely procedural work is how the binder was organized: every section consisted of problem sets with blank spaces for solutions. Pre-determined and pre-fixed blank spaces big enough to write *the* solution to the above problem. There was no room for anything but the one solution, no room for exploration, for mistakes and corrections, for drawing connections or drawing doodles. No room for creativity.

We expect painters to get paint all over their clothes and cooks to get flour all over the kitchen. Learning is messy. So if you're a teacher: give your students scrap paper and encourage them to explore. If you're a student, then don't let a careful planned lecture or presentation fool you: no one ever produces an immediately clean and polished mathematical product.

Be messy, be creative, write in circles and draw lots of pictures. Just make sure to hand in a clean, well-organized final copy.

Kseniya Garaschuk
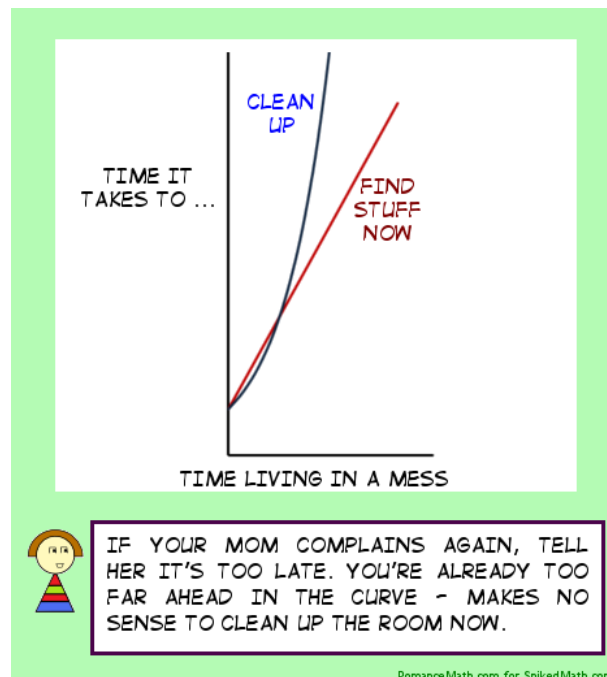
THE CONTEST CORNER
No. 63
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **August 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.
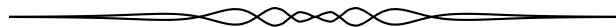
CC311. Suppose $1 \le a < b < c < d \le 100$ are four natural numbers. What is the minimum possible value of $\dfrac{a}{b} + \dfrac{c}{d}$?

CC312. Choose four points A, B, C and D on a circle uniformly at random. What is the probability that the lines AB and CD intersect outside the circle?

CC313. Consider a pyramid whose faces consist of a 60×60 square base $ABCD$ and four $60 - 50 - 50$ triangles that join at the apex E. If you are only allowed to move on the surfaces of the four triangles, what is the length of the shortest path between A and C?
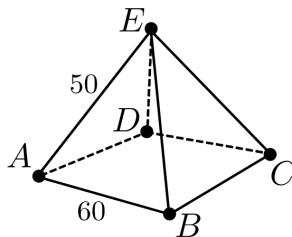
CC314. An infinite sequence $a_1, a_2, \ldots$ of 1's and 2's is uniquely defined by the following properties:

1. $a_1 = 1$ and $a_2 = 2$,

2. For every $n \ge 1$, the number of 1's between the nth 2 and the $(n+1)$st 2 is equal to a_{n+1}.

Is the sequence periodic from the beginning?

CC315. A square table is divided into a 3×3 grid with every cell having 3

coins. In every step of a game, Terry can take 2 coins from the table as long as they come from distinct but adjacent cells. (Here "adjacent" means the two cells share a common edge.) At most how many coins can Terry take?

. . . .

CC311.   Soit a, b, c et d quatre entiers tels que $1 \le a < b < c < d \le 100$. Quelle est la plus petite valeur possible de l'expression $\dfrac{a}{b} + \dfrac{c}{d}$?

CC312.   On choisit au hasard quatre points, A, B, C et D, sur un cercle. Quelle est la probabilité pour que les droites AB et CD se coupent à l'extérieur du cercle?

CC313.   On considère une pyramide d'apex E dont la base est un carré $ABCD$ mesurant 60×60 et dont les faces latérales sont des triangles $60 - 50 - 50$. Sachant qu'on peut se déplacer sur les faces latérales seulement, quelle est la longueur du chemin le plus court de A à C?
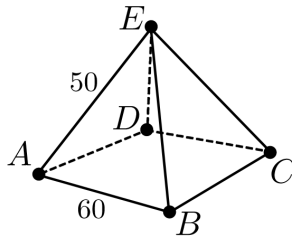
CC314.   On considère une suite $a_1, a_2, \ldots$ dont chaque terme est un 1 ou un 2. Elle est définie de façon non équivoque au moyen des deux propriétés suivantes:

1. $a_1 = 1$ et $a_2 = 2$,

2. Pour chaque n ($n \ge 1$), le nombre de 1 entre le $n^{\text{ième}}$ 2 et le $(n+1)^{\text{ième}}$ 2 est égal à a_{n+1}.

Cette suite est-elle périodique à partir du début?

CC315.   Une table carrée est divisée en un quadrillage 3×3. Chaque carreau du quadrillage contient 3 pièces de monnaie. À chaque étape d'un jeu, Terry peut prendre 2 pièces, à condition qu'elles proviennent de deux carreaux distincts adjacents. (Deux carreaux sont adjacents s'ils ont un côté commun.) Quel est le nombre maximal de pièces que Terry peut prendre en tout?
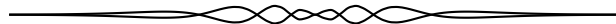
CONTEST CORNER
SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(3), p. 87–89.

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CC261.   Peter is walking through a train tunnel when he hears a train approaching. He knows that on this section of track trains travel at 60 mph. The tunnel has equally spaced marker posts, with post 0 at one end and post 12 at the other end. Peter is by post 7 when he hears the train. He quickly works out that whether he runs to the nearer end or the further end of the tunnel as fast as he can (at constant speed) he will just exit the tunnel before the train reaches him. How fast can Peter run?

Originally Problem 1 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.

We received 4 submissions of which three were correct. We present the solution by Digby Smith, modified by the editor.

We let

$$
\begin{aligned}
v &= \text{ constant running speed of Peter in miles per hour,} \\
x &= \text{ starting distance between the train and post 12 in miles,} \\
y &= \text{ distance between consecutive posts in miles,} \\
t_{12} &= \text{ time taken for Peter and the train to reach post 12, and} \\
t_0 &= \text{ time taken for Peter and the train to reach post 0.}
\end{aligned}
$$

Recall that, given a constant speed,

$$
\text{time } = \frac{\text{distance}}{\text{speed}}. \tag{1}
$$

Peter and the train will travel a distance of $5y$ and x, respectively, to reach post 12 in t_{12} units of time. Therefore, given (1) we have that

$$
t_{12} = \frac{5y}{v} = \frac{x}{60}. \tag{2}
$$

Peter and the train will travel a distance of $7y$ and $x + 12y$, respectively, to reach post 0 in t_0 units of time. Therefore, given (1) we have that

$$
t_0 = \frac{7y}{v} = \frac{x + 12y}{60}.
$$

Note that

$$
\frac{7y}{v} = \frac{x + 12y}{60} \quad \Longrightarrow \quad \frac{7y}{v} - \frac{y}{5} = \frac{x}{60}.
$$

By substituting the above into (2) we have that

$$\frac{7y}{v} - \frac{y}{5} = \frac{5y}{v},$$

which simplifies to

$$\frac{2}{v} = \frac{1}{5} \quad \implies \quad v = 10.$$

Therefore, the fastest running speed of Peter is 10 miles per hour.

CC262. In the diagram, the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.
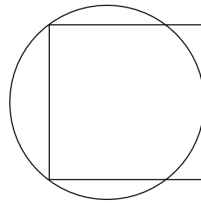
Originally Problem 4 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.

We received 13 solutions, all of which were correct. We present three different approaches to this problem.

Solution 1, by Andrea Fanchini, modified by the editor.

Let l be the side length of the square and let C be the center of the circle. We construct the following right angle triangle with vertices A, B, and C. By construction, AC has length 1, CB has length $l - 1$, and BA has length $\frac{l}{2}$.
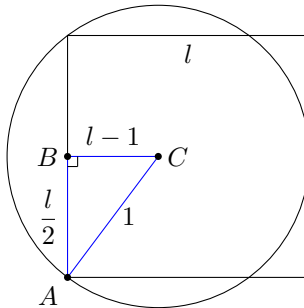
By the Pythagorean Theorem, we have that

$$(l-1)^2 + \left(\frac{l}{2}\right)^2 = 1 \quad \implies \quad l = \frac{8}{5}.$$

Therefore the area of the square is $l^2 = \frac{64}{25}.$

Solution 2, by Ángel Plaza, modified by the editor.

Let x be the side length of the square and let C be center of the circle. We construct the following two triangles and note the relationship between the angles subtending the same arc:
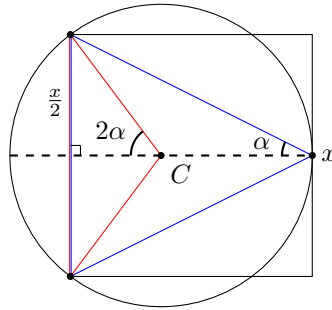
By construction, we have

$$\sin(\alpha) = \frac{\dfrac{x}{2}}{\sqrt{\left(\dfrac{x}{2}\right)^2 + x^2}} = \frac{1}{\sqrt{5}},$$

$$\cos(\alpha) = \frac{x}{\sqrt{\left(\dfrac{x}{2}\right)^2 + x^2}} = \frac{2}{\sqrt{5}},$$

$$\sin(2\alpha) = \frac{x}{2}.$$

Since $\sin(2\alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$ we have that $\frac{x}{2} = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}}$, so $x = \frac{8}{5}$. Therefore the area of the square is $x^2 = \dfrac{64}{25}$.

Solution 3, by Joel Schlosberg, modified by the editor.

Draw the diameter at the midpoint of tangency, and segments connecting the diameter's endpoints to one of the vertices on the circle. By Thales' theorem, this produces a right triangle.
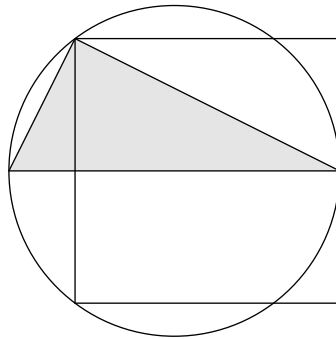
It is split into two smaller triangles by one of the square's sides; since the side is perpendicular to the diameter, the smaller triangles are also right triangles. By construction these smaller triangles are similar.

The right triangle inside the square has one leg twice the length of the other, and so the right triangle outside the square does also. Let s be the side length of the square. The triangle outside the square has a side length of $s/2$. This same length is twice the length of the the other leg which has length $2 - s$. Therefore

$$\frac{s}{2} = 2\,(2 - s).$$

Thus $s = 8/5$ and area of the square is $s^2 = 64/25$.

CC263. An old fashioned tram starts from the station with a certain number of men and women on board. At the first stop, a third of the women get out and their places are taken by men. At the next stop, a third of the men get out and their places are taken by women. There are now two more women than men and as many men as there originally were women. How many men and women were there on board at the start?

Originally Problem 2 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.

We received six solutions to this problem. We present the composite solution of Dan Daniel, Digby Smith, and Titu Zvonaru.

Let m_i and w_i be the number of men and women on board the train, respectively, at stop i. Note that m_0 and w_0 is the number of men and women on board the train, respectively, at the start of the trip.

At stop 1 we have that

$$m_1 = m_0 + \frac{w_0}{3} \quad \text{and} \quad w_1 = \frac{2w_0}{3}. \tag{1}$$

At stop 2 we have that

$$m_2 = \frac{2m_1}{3} \quad \text{and} \quad w_2 = w_1 + \frac{m_1}{3}.$$

Given (1), the above expands and simplifies to

$$m_2 = \frac{2m_0}{3} + \frac{2w_0}{9} \quad \text{and} \quad w_2 = \frac{7w_0}{9} + \frac{m_0}{3}. \tag{2}$$

Since $m_2 = w_0$ it follows from (2) that

$$w_0 = \frac{2m_0}{3} + \frac{2w_0}{9} \quad \Longrightarrow \quad m_0 = \frac{7w_0}{6}.$$

Since $m_2 + 2 = w_2$ it follows from (2) that

$$\frac{2m_0}{3} + \frac{2w_0}{9} + 2 = \frac{7w_0}{9} + \frac{m_0}{3} \quad \Longrightarrow \quad w_0 = \frac{3m_0 + 18}{5}. \tag{3}$$

By substituting $m_0 = \dfrac{7w_0}{6}$ into (3) and solving for w_0 we see that

$$w_0 = \frac{3\left(\dfrac{7w_0}{6}\right) + 18}{5} \qquad \Longrightarrow \qquad w_0 = 12.$$

Given $w_0 = 12$ and $m_0 = \frac{7w_0}{6}$ it follows that $m_0 = 14$. Therfore, there are 12 women and 14 men on board the train at the start of the trip.

CC264. Kirsty runs three times as fast as she walks. When going to school one day she walks for twice the time she runs and the journey takes 21 minutes. The next day she follows the same route but runs for twice the time she walks. How long does she take to get to school?

Originally Problem 3 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.

We received 7 submissions. We present the solution by Titu Zvonaru.

We denote by d the distance to school, r the speed when Kirsty runs, and w the speed when Kirsty walks. We have $r = 3w$. On the first day, Kirsty runs 7 minutes and she walks 14 minutes; this yields $d = 7r + 14w$. Solving with $r = 3w$,

$$d = 35w. \tag{1}$$

Let x be the number of minutes Kirsty walks on the second day. Then she runs $2x$ minutes. It follows that $d = 2xr + xw$, so

$$d = 7xw. \tag{2}$$

By equations 1 and 2 we obtain $x = 5$, hence the second day she takes 15 minutes to get to school.

CC265. In the diagram, ST is parallel to QR, UT is parallel to SR, $PU = 4$ and $US = 6$. Find the length of SQ.
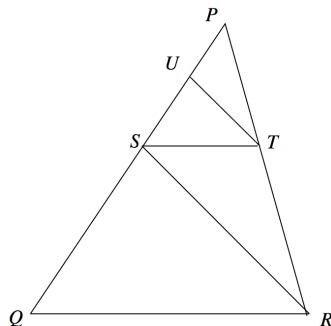
Originally Problem 5 of the 2016–2017 Scottish Mathematical Council Mathematical Challenge, Middle division.

We received 12 correct submissions. We present the solution by Šefket Arslanagić.

We have $\triangle PUT \sim \triangle PSR$ and from here:

$$\frac{PU}{PS} = \frac{PT}{PR} \quad \Longrightarrow \quad \frac{4}{10} = \frac{PT}{PR} \quad \Longrightarrow \quad \frac{PT}{PR} = \frac{2}{5} \tag{1}$$

We have too $\triangle PST \sim \triangle PQR$ and from here:

$$\frac{PS}{PQ} = \frac{PT}{PR}$$

and from here by equation 1 and $PS = 10$:

$$\frac{10}{PQ} = \frac{2}{5} \quad \Longrightarrow \quad PQ = 25.$$

Now, $SQ = PQ - PS = 25 - 10$, so

$$SQ = 15.$$
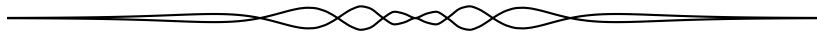
THE OLYMPIAD CORNER
No. 361

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **August 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.
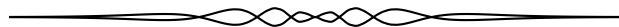
OC371. Let a, b and $c \in \mathbb{R}^+$ such that $abc = 1$. Prove that

$$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}.$$

OC372. In the circumcircle of a triangle ABC, let A_1 be the point diametrically opposite to the vertex A. Let A' the intersection point of A_1' and BC. The perpendicular to the line AA_1 from A' meets the sides AB and AC at M and N, respectively. Prove that the points A, M, A_1 and N lie on a circle whose center lies on the altitude from A of the triangle ABC.

OC373. Let a and b be positive integers. Denote by $f(a,b)$ the number of sequences $s_1, s_2, \ldots, s_a \in \mathbb{Z}$ such that $|s_1| + |s_2| + \ldots + |s_a| \leq b$. Show that $f(a,b) = f(b,a)$.

OC374. Let p be an odd prime and let $a_1, a_2, \ldots, a_p$ be integers. Prove that the following two conditions are equivalent:

1) There exists a polynomial $P(x)$ of degree less than or equal to $\frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.

2) For any natural number $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

where indices are taken modulo p.

OC375. Let $ABCD$ be a non-cyclic convex quadrilateral with no parallel sides. Suppose the lines AB and CD meet in E. Let $M \neq E$ be the intersection of circumcircles of ADE and BCE. Further, suppose that the internal angle bisectors of $ABCD$ form an convex cyclic quadrilateral with circumcenter I while

the external angle bisectors of $ABCD$ form an convex cyclic quadrilateral with circumcenter J. Show that I, J, M are collinear.

$\cdot\,.\,\cdot\,.$

OC371. Soit $a, b, c \in \mathbb{R}^+$ et $abc = 1$. Démontrer que

$$\frac{a+b}{(a+b+1)^2} + \frac{b+c}{(b+c+1)^2} + \frac{c+a}{(c+a+1)^2} \geq \frac{2}{a+b+c}.$$

OC372. On considère un triangle ABC et le cercle circonscrit au triangle. Soit A_1 le point diamétralement opposé au sommet A sur le cercle et A' le point d'intersection de AA_1 et de BC. La perpendiculaire à la droite AA_1 menée au point A' coupe les droites AB et AC aux points respectifs M et N. Démontrer que les points A, M, A_1 et N sont situés sur un cercle dont le centre est situé sur la hauteur du triangle ABC abaissée au sommet A.

OC373. Soit a, $b \in \mathbb{Z}^+$. $f(a,b)$ représente le nombre les suites numériques s_1, s_2, ..., s_a, $s_i \in \mathbb{Z}$ telles que $|s_1| + |s_2| + \ldots + |s_a| \leq b$. Démontrer que $f(a,b) = f(b,a)$.

OC374. Soit p un nombre premier impair et $a_1, a_2, \ldots, a_p$ des entiers. Démontrer que les deux énoncés suivants sont équivalents:

1) Il existe un polynôme $P(x)$ de degré inférieur ou égal à $\frac{p-1}{2}$ tel que $P(i) \equiv a_i \pmod{p}$ pour tous i $(1 \leq i \leq p)$.

2) Pour tout nombre naturel d inférieur ou égal à $\frac{p-1}{2}$, on a

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

les indices étant énumérés modulo p.

OC375. Soit $ABCD$ un quadrilatére convexe non inscriptible n'admettant pas de côtés parallèles. Les droites AB et CD se coupent en E. Soit M le deuxième point d'intersection des cercles circonscrits aux triangles ADE et BCE. Les bissectrices intérieures de $ABCD$ forment un quadrilatère convexe inscriptible dont le cercle circonscrit a pour centre I; les bissectrices extérieures de $ABCD$ forment un quadrilatère convexe inscriptible dont le cercle circonscrit a pour centre J. Démontrer que I, J et M sont alignés.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(1), p. 12–13.

OC311. Let $\triangle ABC$ be an acute-scalene triangle, and let N be the center of the circle which passes through the feet of the altitudes. Let D be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Prove that A, D and N are collinear if and only if $\angle BAC = 45°$.

Originally 2015 Brazil National Olympiad Problem 1 Day 1.

We received 8 submissions of which 7 were correct and complete. We present the solution by Ivko Dimitrić, modified by the editor.
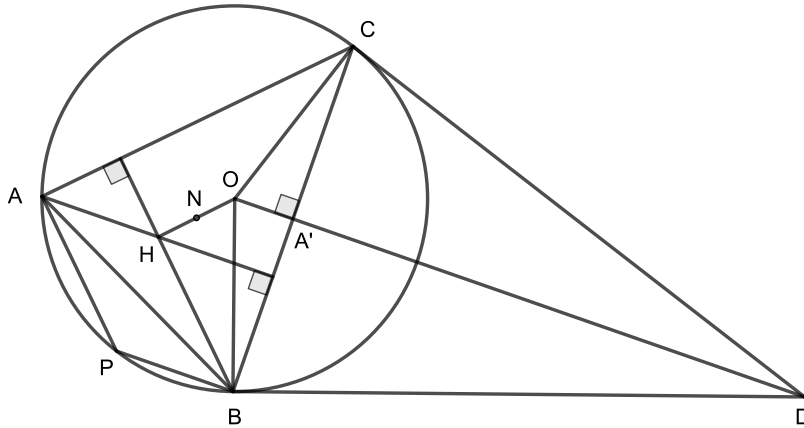
The circle that passes through the feet of the altitudes is the nine-point circle, whose center N is the midpoint of the segment OH joining the circumcenter O and the orthocenter H. Denote by A' the midpoint of the side BC.

We will first show that $OA' = \frac{1}{2}AH$. Let P be the point diametrically opposite C on the circumcircle of $\triangle ABC$. Then $\angle PBC = 90°$, so $PB \perp BC$ and thus $PB \| AH$. Also $\angle PAC = 90°$, so $PA \perp AC$ and $PA \| BH$. Hence $PBHA$ is a parallelogram, so $PB = AH$. In $\triangle CBP$, the segment OA' joins the midpoints of CP and CB, and hence $OA' = \frac{1}{2}PB$, allowing us to conclude $OA' = \frac{1}{2}AH$.

Since D is the point of intersection of the tangents to the circumcircle at B and C, $\angle OBD = \angle OCD = 90°$. Moreover, quadrilateral $OBDC$ is a deltoid in which $OB = OC$ and $DB = DC$. It follows that $OD \perp BC$, so O, A' and D are collinear and also $OD \| AH$ (since H is on the height from A to BC).

We have $\angle BAC = 45° \Leftrightarrow \angle BOC = 90°$ (since $\angle BOC = 2\angle BAC$) $\Leftrightarrow OBDC$ is a square $\Leftrightarrow OA' = A'D \Leftrightarrow OD = AH$ (using the fact that $OA' = \frac{1}{2}AH$ shown above) $\Leftrightarrow ODHA$ is a parallelogram $\Leftrightarrow$ the diagonal AD of $ODHA$ passes through the midpoint N of the diagonal $OH \Leftrightarrow A$, N and D are collinear.

OC312. Let a, b, c be nonnegative real numbers. Prove that

$$\frac{(a - bc)^2 + (b - ca)^2 + (c - ab)^2}{(a - b)^2 + (b - c)^2 + (c - a)^2} \geq \frac{1}{2}.$$

Originally 2015 China Second Round Olympiad Part B Problem 1.

We received 12 submissions, all correct. Two contained comments that are not true. We present the solution by Mohammed Aassila.

By expanding and simplifying, the inequality reduces to

$$a^2 b^2 + b^2 c^2 + c^2 a^2 + ab + bc + ca \geq 6abc,$$

which is true according to the AM-GM inequality.

OC313. Let $x_1, x_2, \cdots, x_n \in (0, 1)$, $n \geq 2$. Prove that

$$\frac{\sqrt{1 - x_1}}{x_1} + \frac{\sqrt{1 - x_2}}{x_2} + \cdots + \frac{\sqrt{1 - x_n}}{x_n} < \frac{\sqrt{n - 1}}{x_1 x_2 \cdots x_n}.$$

Originally 2015 China Girls Mathematics Olympiad Day 2 Problem 7.

We received seven submissions. Six of these are correct; the seventh is too long and not straight to the point. We present the solution by induction provided by Titu Zvonaru.

For $n = 2$ using Cauchy-Schwarz, we have

$$\begin{aligned}
\frac{\sqrt{1 - x_1}}{x_1} + \frac{\sqrt{1 - x_2}}{x_2} &= \frac{x_2 \sqrt{1 - x_1} + x_1 \sqrt{1 - x_2}}{x_1 x_2} \\
&< \frac{\sqrt{x_2} \sqrt{1 - x_1} + x_1 \sqrt{1 - x_2}}{x_1 x_2} \\
&\leq \frac{\sqrt{(x_1 + 1 - x_1)(x_2 + 1 - x_2)}}{x_1 x_2} = \frac{1}{x_1 x_2}.
\end{aligned}$$

Suppose that the inequality is true for n. We have to prove that

$$\frac{\sqrt{1 - x_1}}{x_1} + \frac{\sqrt{1 - x_2}}{x_2} + \cdots + \frac{\sqrt{1 - x_n}}{x_n} + \frac{\sqrt{1 - x_{n+1}}}{x_{n+1}} < \frac{\sqrt{n}}{x_1 x_2 \cdots x_n x_{n+1}}.$$

It suffices to show that

$$\frac{\sqrt{n - 1}}{x_1 x_2 \cdots x_n} + \frac{\sqrt{1 - x_{n+1}}}{x_{n+1}} < \frac{\sqrt{n}}{x_1 x_2 \cdots x_n x_{n+1}}. \tag{1}$$

Applying again the Cauchy-Schwarz Inequality, we obtain

$$
\begin{aligned}
\frac{\sqrt{n-1}}{x_1 x_2 \cdots x_n} + \frac{\sqrt{1-x_{n+1}}}{x_{n+1}} &= \frac{x_{n+1}\sqrt{n-1} + x_1 x_2 \cdots x_n \sqrt{1-x_{n+1}}}{x_1 x_2 \cdots x_n x_{n+1}} \\
&< \frac{\sqrt{x_{n+1}}\sqrt{n-1} + \sqrt{1-x_{n+1}}\sqrt{1}}{x_1 x_2 \cdots x_n x_{n+1}} \\
&\le \frac{\sqrt{(x_{n+1}+1-x_{n+1})(n-1+1)}}{x_1 x_2 \cdots x_n x_{n+1}} \\
&= \frac{\sqrt{n}}{x_1 x_2 \cdots x_n x_{n+1}}.
\end{aligned}
$$

It follows that the inequality (1) is true and the induction is completed.

OC314. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, z, we have

$$
(f(x) + 1)(f(y) + f(z)) = f(xy + z) + f(xz - y).
$$

Originally 2015 Korea National Olympiad Day 2 Problem 1.

We received three submissions, all correct. We present here an editor's amalgamation of all three solutions.

All three proved using the same techniques that there are two functions: $f(x) = 0$, and $f(x) = x^2$. To prove that one solution is $f(x) = x^2$, everyone used the following steps:

1. f is even.

2. $f(xy) = f(x)f(y)$, a functional equation of Cauchy type.

3. $f(x) = x^2$ for any natural number using induction.

4. $f(x) = x^2$ for any positive rational number x, using the results from the second and third steps.

5. f is an increasing function for all $x > 0$.

6. $\mathbb{Q}$ is dense in $\mathbb{R}$, therefore $f(x) = x^2$ for all real $x \ge 0$, using the results from the fourth and fifth steps.

7. Finally, using the fact that f is even (step 1), we have $f(x) = x^2$ for all real numbers.

OC315. Suppose that a is an integer and that $n! + a$ divides $(2n)!$ for infinitely many positive integers n. Prove that $a = 0$.

Originally 2015 South Africa National Olympiad Problem 6.

We received 2 correct submissions. We present the solution by Oliver Geupel.

Suppose $a \neq 0$. It is enough to prove that the condition is satisfied for not more than finitely many positive integers n.

Let n be such that $n > 2|a|$ and the number $n! + a$ divides $(2n)!$. Let $m = \dfrac{n! + a}{|a|}$. Then, the greatest common divisor of the numbers $n! + a$ and $n!$ is equal to $|a|$. Because $m \equiv \pm 1 \pmod{a}$, the numbers m and $n!$ are coprime. Since $n! + a$ divides $(2n)!$, all prime divisors of m are members of the interval $(n, 2n)$. Also every such prime divisor p has multiplicity 1, because $2p > 2n$.

We obtain

$$n! \cdot n^{n/2} \cdot \frac{n! + a}{|a|} \leq n! \left(\prod_{\substack{n < k \leq 2n \\ n \text{ even}}} k \right) \left(\prod_{\substack{n < k \leq 2n \\ n \text{ odd}}} k \right) = (2n)! \tag{1}$$

Taking logarithms and applying Stirling's formula $\log N! = N \log N + O(N)$, the left-hand side of the inequality (1) becomes $\dfrac{5}{2} n \log n + O(n)$, while the term on the right becomes $2n \log n + O(n)$. Thus, the inequality (1) can hold for only a finite number of positive integers n, which completes the proof.

FOCUS ON...

No. 30
Michel Bataille
Cauchy's Functional Equation

Introduction

This number is about functional equations, especially the most famous of them, Cauchy's equation

$$f(x + y) = f(x) + f(y), \tag{1}$$

where the unknown function $f : S \to S$ has to satisfy this equation for all elements x, y of S. Here, S can be any set in which an addition has been defined. A lot of solutions to functional equations proposed in problems eventually end up with an application of known results about this equation, so it seems of interest to review these results. As usual, a choice of past problems will illustrate the various properties under consideration.

Elementary results

Let f denote a solution to equation (1) and let x be an element of S. Taking $y = x$ in (1) immediately shows that $f(2x) = 2f(x)$ and an easy induction shows that

$$f(nx) = nf(x) \tag{2}$$

for all $n \in \mathbb{N}$. Simple as it may appear, this result can be applied to problems. An example is provided by Crux problem **4040** [2015 : 170, 172 ; 2016 : 189]:

Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2 \quad \forall a, b \in \mathbb{N}.$$

Three solutions of this interesting problem were featured, and we propose a fourth one. If f is a solution, we first observe that $f(b + f(b)) = b + f(b)$ for any $b \in \mathbb{N}$ ($a = b$ in the equation). Then, substituting $b + f(b)$ for b in the equation gives

$$(f(a) + b + f(b))f(a + b + f(b)) = (a + b + f(b))^2$$

while substituting $a + b$ for a yields

$$(f(a + b) + b)f(a + b + f(b)) = (a + b + f(b))^2.$$

The comparison leads to $f(a + b) = f(a) + f(b)$, which takes us back to equation (1). With $x = 1$ in (2), we see that $f(n) = nw$ for all $n \in \mathbb{N}$ where we set $f(1) = w$. To conclude, we remark that from the initial equation with $a = b = 1$ we obtain $f(1 + w) = 1 + w$, that is, $w(1 + w) = 1 + w$. Thus $w = 1$ and so $f(n) = n$ for all

$n \in \mathbb{N}$. Conversely, the identity function from $\mathbb{N}$ to itself is obviously a solution and so is the unique solution.

Now, suppose that S is an additive group with 0 as the neutral element. Then, taking $y = 0$ in (1), we obtain $f(0) = 0$. With $y = -x$, we obtain $f(-x) = -f(x)$ and it follows that (2) now holds for any $n \in \mathbb{Z}$. A good application is the following problem proposed at the 20th Austrian-Polish Mathematical Competition in 1997:

Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x + f(y)) = f(x) - y$$

for all integers x and y.

Assume that such a function f exists. Then for all integers x, y, we have

$$f(f(x)) = f(y + f(x + f(y))) = f(y) - (x + f(y))$$

hence $f(f(x)) = -x$. We deduce $f(-x) = f(f(f(x))) = -f(x)$ and so

$$f(y + x) = f(y + f(-f(x))) = f(y) + f(x).$$

Again with $w = f(1)$, we obtain $f(x) = wx$ for any integer x. However, substituting in the given equation leads to $-w^2 y = y$ for all y, which is clearly impossible.

It is worth noting that, in spite of this negative conclusion, functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying the same equation do exist. Indeed, the consequences $f(x + y) = f(x) + f(y)$ and $f(f(y)) = -y$ continue to hold when x, y are any real numbers. An appeal to the axiom of choice then allows one to construct a suitable function. We refer the interested reader to the solution of problem 1690 in *Mathematics Magazine*, Vol. 77, No 1, February 2005.

Suppose that S is a vector space over $\mathbb{Q}$ and let $p \in \mathbb{Z}, q \in \mathbb{N}, x \in S$. From (2), we get

$$pf(x) = f(px) = f\left(\left(q \cdot \frac{p}{q}\right)x\right) = qf\left(\frac{p}{q}x\right)$$

so that (2) is still valid if n is a rational number. As a result, the solutions to (1) in the case $S = \mathbb{Q}$ are the functions $x \mapsto wx$ where w is a rational number ($w = f(1)$). This is used in solving the following equation set at the 15th Irish Olympiad:

Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{Q}$.

Despite the very slight change with the previous equation, this equation does have solutions, for example, the functions $x \mapsto x$ and $x \mapsto -x$, as is readily checked. We show that there are no other solutions. Let f be an arbitrary solution and $a = f(0)$. The equation with $x = y = 0$ gives $f(a) = a$ and then, with $x = 0, y = a$, we obtain $a = 2a$. Thus $a = f(0) = 0$ and it immediately follows that $f(f(y)) = y$ for any rational number y. Replacing y by $f(y)$ in the equation finally shows that

(1) is satisfied for all $x, y \in \mathbb{Q}$. We first deduce that for some $w \in \mathbb{Q}$, $f(x) = wx$ and then, inserting in the equation, that $w = 1$ or $w = -1$, completing the proof.

The equation for $f : \mathbb{R} \to \mathbb{R}$

In view of the results obtained in the previous paragraph, we expect that, in the case $S = \mathbb{R}$, the solutions are the linear functions $x \mapsto wx$ where w is a real number. Clearly, these functions are solutions but the converse does not hold (again, the latter can be proved with the help of the axiom of choice). However, as we shall see, assuming some additional property of f, the solutions are the expected ones. Note that, $\mathbb{R}$ being a $\mathbb{Q}$-vector space, any solution $f : \mathbb{R} \to \mathbb{R}$ of (1) satisfies $f(rx) = rf(x)$ when $r \in \mathbb{Q}$ and $x \in \mathbb{R}$ (in particular $f(r) = rf(1)$).

Suppose now that a solution f is monotone, say nondecreasing, and let $x \in \mathbb{R}$. There exist sequences of rational numbers (r_n) and (s_n) such that $r_n \leq x \leq s_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} r_n = \lim_{n \to \infty} s_n = x$. Then $r_n f(1) \leq f(x) \leq s_n f(1)$ for all n and taking the limits as $n \to \infty$, we deduce $xf(1) \leq f(x) \leq xf(1)$ and so $f(x) = xf(1)$. Thus, the monotone solutions of Cauchy's equation with $S = \mathbb{R}$ are the linear functions.

Similarly, using an appropriate sequence of rational numbers, it is easy to obtain the same conclusion when monotone is replaced by continuous. Note also that a solution is continuous everywhere as soon as it is continuous at some point x_0 (since $f(x + h) - f(x) = f(x_0 + h) - f(x_0)$).

An interesting application is the determination of the morphisms of the field $\mathbb{R}$, that is, the mappings $f : \mathbb{R} \to \mathbb{R}$ such that the conditions $f(x + y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$ are verified. Clearly, the identity function $x \mapsto x$ is such a mapping. Let us show that it is the only one. We observe that if $x > 0$, then $f(x) = f(\sqrt{x}\sqrt{x}) = (f(\sqrt{x}))^2 \geq 0$ and deduce that $f(v) - f(u) = f(v - u) \geq 0$ if $v > u$. Thus f is nondecreasing on $\mathbb{R}$ and the conclusion follows.

A call to this property can prove useful, as exemplified by a problem posed to select the Indian IMO team in 2003 [2006 : 278 ; 2007 : 284]. Here, we propose a shortened version:

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f(x + y) + f(x)f(y) = f(x) + f(y) + f(xy)$$

for all x, y in $\mathbb{R}$.

A neat mix of the morphism conditions above!

Let f be any solution and let (E) denote the equation. Taking $x = y = 0$ in (E) yields $f(0) = 2$ or $f(0) = 0$. However, $f(0) = 2$ cannot hold, a contradiction with $f(1) = 1$ being reached if we take $x = 1, y = 0$ in (E). Thus, $f(0) = 0$. With $x = 1, y = -1$, the equation gives $f(-1) = -1$ and with arbitrary x and $y = 1$, we obtain $f(x+1) = 1+f(x)$. Substituting $x-1$ for x yields $f(x-1) = f(x)-1$.

Keeping x arbitrary and taking $y = -1$ leads to $f(x-1) = 2f(x) - 1 + f(-x)$ and comparing with the previous relation, we deduce that $f(-x) = -f(x)$, that is, f is odd. Changing y into $-y$ in (E) and adding to (E) itself shows that $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in \mathbb{R}$. It follows that $f(2x) = 2f(x)$ (with $y = x$) and then, changing x into $\frac{u+v}{2}$ and y into $\frac{u-v}{2}$, that $f(u+v) = f(u) + f(v)$ for all u, v. Returning to the equation, we deduce that $f(xy) = f(x)f(y)$ for all x, y. We can now conclude that f is a morphism of the field $\mathbb{R}$, hence is the identity function. Conversely, the identity function is obviously a solution, hence is the unique solution.

Exercises

1. (Problem 719 of the *College Mathematics Journal*, January 2002)
Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Prove that

$$f(x + y + f(y)) = f(x) + 2f(y)$$

for all real x and y if and only if

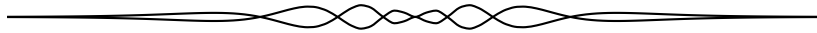$$f(x+y) = f(x) + f(y) \quad \text{and} \quad f(f(x)) = f(x)$$

for all real x and y.

2. (Problem 10854 of the *American Mathematical Monthly*, February 2001)
Find every function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at 0 and satisfies

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real numbers x and y.

The Method of Indirect Descent (Part I)
Adib Hasan

1   Introduction

The reader is probably familiar with the method of Infinite Descent [5] (or its child Vieta Jumping [3]). In its simplest form, we show that all natural numbers have some property by assuming a least exception and then, in contradiction, finding a still smaller one. In this note, we look at a powerful generalization of this technique, whereby the descent takes place not in the variable we are really interested in, but in its image under a suitably-chosen transformation.

Formally speaking, if $\mathcal{T}$ is the set of all possible values of the tuple $(x_1, x_2, \ldots, x_n)$ of variables of our interest, we assign each element of $\mathcal{T}$ a score $s : \mathcal{T} \to \mathbb{R}$, which is bounded either below or above. Then we describe a *transformation* $\chi : \mathcal{T} \to \mathcal{T}$ under which s monotonically heads towards its bound. Repeatedly applying χ on an element of $\mathcal{T}$ pushes s indefinitely in that direction, thereby upsetting its bound.

In practice, one should look for a suitable transformation first, and then set a score function to "fit" that transformation. This is because the former is often harder to find. To put all pieces together, let us solve the following dummy problem with indirect descent.

Example 1 *Show that $\sqrt{2}$ is irrational.*

Solution. (Mathematical Folklore.) Assume to the contrary that $\sqrt{2}$ is rational, ie, $\sqrt{2} = \frac{m}{n}$, $m, n \in \mathbb{N}$. Rewrite the equation as

$$m^2 = 2n^2. \tag{1}$$

Step 1. Suppose $\mathcal{T} \subseteq \mathbb{N} \times \mathbb{N}$ denotes the set of all ordered pairs of solutions to equation 1. We notice that $2|m$, and $(n, m/2) \in \mathcal{T}$. Hence, there is a transformation $\chi : (m, n) \to (n, m/2)$ for each $(m, n) \in \mathcal{T}$.

Step 2. Define a score function $s : \mathcal{T} \to \mathbb{R}$ as $s(m, n) = m + n \ \ \forall (m, n) \in \mathcal{T}$. Since there is a smallest solution to equation 1, s is bounded below.

Step 3. From 1, it is easy to see that $m > n > m/2$. Hence,

$$s(m, n) > s(\chi(m, n)) > \cdots > s(\chi^i(m, n)).$$

So, applying χ recursively on a fixed element of $\mathcal{T}$ will indefinitely decrease s, violating its lower bound. Consequently, $\mathcal{T}$ must be an empty set, and $\sqrt{2}$ has to be irrational.

$\square$

Now, we are ready to solve some 'real' problems from Mathematical Olympiads. However, let us save the more interesting problems from Number Theory and Combinatorics for part two, and limit ourselves only to Functional Equations and Functional Inequalities here.

2 Functional Equations & Functional Inequalities

Example 2 (IMO Shortlist 2010/A5) *Determine all functions* $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ *which satisfy the following equation for all* $x, y \in \mathbb{Q}^+$:

$$f\left(f(x)^2 y\right) = x^3 f(xy).$$

Solution. A little experimentation makes it apparent that only $f(x) = \frac{1}{x}$ satisfies the given equation. Alternatively, we are supposed to prove $xf(x) = 1 \ \forall x \in \mathbb{Q}^+$ from it. With that in mind, it is easy to stumble upon

$$[f(x)f(f(x))]^2 = (xf(x))^3. \tag{2}$$

Surprisingly, the LHS and the RHS of this equation are different powers. This observation lures us into more careful inspection of (2). So, we call a rational number r to be an nth *power* if $r = t^n$ for some $t \in \mathbb{Q}, n \in \mathbb{N}$.

Step 1. Consider the set
$$\mathcal{T} = \{n \mid xf(x) \text{ is an } n^{\text{th}} \text{ power} \ \ \forall x \in \mathbb{Q}^+\}.$$

Now, $n \in \mathcal{T}$ implies both $xf(x)$ and $f(x)f(f(x))$ are n^{th} powers. Therefore, LHS of 2 is a $2n^{\text{th}}$ power while RHS of it is a $3n^{\text{th}}$ power. Hence, to be equal, both sides have to be at least $6n^{\text{th}}$ powers. However, it implies $xf(x)$ itself is a $\frac{6n}{3} = 2n^{\text{th}}$ power. Consequently, $2n \in \mathcal{T}$ and there exists a transformation $\chi : n \to 2n$ among the elements of $\mathcal{T}$.

Step 2. Set $s(n) = \frac{1}{n} \ \ \forall n \in \mathcal{T}$. We know that no rational number except 1 can be n^{th} powers for infinitely many n's. Hence, s has to be bounded below unless $xf(x) = 1 \ \ \forall x$.

Step 3. By repeatedly applying χ on a fixed $n \in \mathcal{T}$, we can decrease s below any positive real number. Hence, $xf(x) = 1$ and $f(x) = \dfrac{1}{x} \ \forall x \in \mathbb{Q}^+$.

$\square$

Example 3 (IMO 2013/5) *Let* $f : \mathbb{Q}^+ \to \mathbb{R}$ *be a function satisfying the following three conditions:*

   i) For all $x, y \in \mathbb{Q}^+$, *we have* $f(x)f(y) \geq f(xy)$;

   ii) For all $x, y \in \mathbb{Q}^+$, *we have* $f(x + y) \geq f(x) + f(y)$;

   iii) There exists a rational number $a > 1$ *such that* $f(a) = a$.

Prove that $f(x) = x$ *for all* $x \in \mathbb{Q}^+$.

Solution. First prove that,

1. f is strictly increasing and $f(n) \geq n \; \forall n \in \mathbb{N}$.

2. $a^k \geq f(a^k) \;\; \forall k \in \mathbb{N}$.

3. For $p, q \in \mathbb{N}$, $\frac{f(p)}{q} \geq f\left(\frac{p}{q}\right) \geq \frac{f(p)}{f(q)}$. Hence, if somehow we manage to prove $f(n) = n$ for all $n \in \mathbb{N}$, then it is also true for all $n \in \mathbb{Q}^+$.

The first two properties imply $f(x) \leq x$ for some x's and $f(x) \geq x$ for other x's. This makes us wonder how $f(x) - x$ changes across the whole domain.

Step 1. Consider the set

$$\mathcal{T} = \{(n, \varepsilon) \mid n \in \mathbb{N}, f(n) = n + \varepsilon\}.$$

Set $x = y = n$ in (ii) to get

$$f(2n) \geq 2f(n) = 2n + 2\varepsilon \implies f(2n) - 2n \geq 2\varepsilon.$$

Assuming $f(2n) - 2n = 2\varepsilon + \delta$, with $\delta \geq 0$, we discover a transformation $\chi : (n, \varepsilon) \to (2n, 2\varepsilon + \delta)$.

Step 2. Set $s(n, \varepsilon) = f(n) - n = \varepsilon$. Suppose k is a very large integer so that $\lfloor a^k \rfloor > n$. (Such a k exists because $a > 1$.) We know that $a^k \geq f(a^k) \geq f\left(\lfloor a^k \rfloor\right)$ since f is strictly increasing. So,

$$\begin{aligned} a^k \geq f\left(\lfloor a^k \rfloor\right) &= f\left(\lfloor a^k \rfloor - n + n\right) \\ &\geq f\left(\lfloor a^k \rfloor - n\right) + f(n) \\ &\geq \lfloor a^k \rfloor - n + n + \varepsilon \\ &= \lfloor a^k \rfloor + \varepsilon. \end{aligned}$$

Hence $s(n, \varepsilon)$ must be bounded above. In particular, for each $n \in \mathbb{N}$, we must have

$$s(n, \varepsilon) = \varepsilon \leq a^k - \lfloor a^k \rfloor < 1.$$

Step 3. We can repeatedly apply χ on any $T \in \mathcal{T}$ and double s infinitely many times. This will upset the upper bound of s unless $s(T) = f(n) - n = 0$. So, $f(n) = n \; \forall n \in \mathbb{N}$. Now, we also have $f(\mathbb{Q}^+) = \mathbb{Q}^+$ from the third property.

$\square$

Example 4 (IMO Shortlist 2013/N6) *Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying*

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{N}$.

Solution. First prove that either f is a constant function, or, it has the following properties:

- $f(n) = n \quad \forall n \in \mathbb{Z}$, and $f(x + a) = f(x) + a, \ \forall a \in \mathbb{Z}$ and $\forall x \in \mathbb{Q}$.

- $f\left(\frac{1}{2}\right) \in \{0, 1\}$. Now, assume $f\left(\frac{1}{2}\right) = 0$. (The other case can be proven analogously.) Now inductively prove for $k \in \mathbb{N}, 0 \le i < 2^k$, $f\left(\frac{i}{2^k}\right) = 0$.

We shall prove $f\left(\frac{p}{q}\right) = 0$ for $p, q \in \mathbb{N}$ with $p < q$. Suppose $f\left(\frac{p}{q}\right) = \delta$. We want to prove $\delta = 0$.

Step 1. Consider the set

$$\mathcal{T} = \left\{ f\left(\frac{p + n\delta}{q + n}\right) \quad \forall n \in \mathbb{N}_0 \right\}.$$

Set $x = \frac{p}{q}, a = p, b = q + 1$ in the main equation to get

$$f\left(\frac{p}{q}\right) = f\left(\frac{p + \delta}{q + 1}\right) = \delta. \tag{3}$$

Hence, there is a transformation

$$\chi : f\left(\frac{r}{s}\right) \to f\left(\frac{r + \delta}{s + 1}\right) \quad \forall f\left(\frac{r}{s}\right) \in \mathcal{T}.$$

Step 2. Define the score function as $s\left(f\left(\frac{r}{s}\right)\right) = f\left(\frac{p - q\delta}{s}\right)$. We know that $r = p + n\delta, s = q + n$ and $\delta \in \mathbb{Z}$. Therefore,

$$s\left(f\left(\frac{r}{s}\right)\right) = f\left(\frac{p - q\delta}{q + n}\right) = f\left(\frac{p + n\delta}{q + n} - \delta\right) = f\left(\frac{r}{s} - \delta\right) = f\left(\frac{r}{s}\right) - \delta = 0.$$

So, $s\left(f\left(\frac{r}{s}\right)\right) = 0$ for each $f\left(\frac{r}{s}\right) \in \mathcal{T}$.

Step 3. Note that $p - q\delta \ne 0$ (otherwise $\delta = \frac{p}{q}$ =non-integer). Suppose $T = f\left(\frac{p}{q}\right)$. If $\delta > 1$, then for $n = q\delta - p - q > 0$,

$$s\left(\chi^n(T)\right) = f\left(\frac{p - q\delta}{q\delta - p}\right) = f(-1) = -1.$$

A contradiction. Similarly, if $\delta < 0$, then for $n = p - q\delta - q > 0$,

$$s\left(\chi^n(T)\right) = f\left(\frac{p - q\delta}{p - q\delta}\right) = f(1) = 1.$$

This is also a contradiction. So $\delta \in \{0, 1\}$. However, if $\delta = 1$, then (3) asserts that there exists a $T \in \mathcal{T}$ so that $s = 2^k$, $r < s$ and

$$0 = f\left(\frac{r}{2^k}\right) = f\left(\frac{r}{s}\right) = \delta = 1.$$

This is another contradiction. Consequently, $f\left(\frac{p}{q}\right) = \delta = 0$. So, $f(x) = \lfloor x \rfloor$ for $0 \le x \le 1$. Since $f(a + \frac{p}{q}) = a + f(\frac{p}{q}) \ \forall a \in \mathbb{Z}$, we get $f(x) = \lfloor x \rfloor \ \forall x \in \mathbb{Q}$.

Similarly, by assuming $f\left(\frac{1}{2}\right) = 1$, one can prove $f(x) = \lceil x \rceil \ \forall x \in \mathbb{Q}$.

$\square$

3   Selected Problems

1. (IMOMath [4]) Let $\mathbb{R}^* = [1, \infty)$. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ that satisfy:

   i) $f(x) \leq 2(1 + x)$ $\forall x \in \mathbb{R}^*$,

   ii) $xf(x + 1) = f(x)^2 - 1$ $\forall x \in \mathbb{R}^*$.

2. (AoPS [1]) Determine all polynomials $P(x) \in \mathbb{Z}[x]$ such that $P(x)$ is bijective over $\mathbb{R}$ and
$$P\left(P(x)\right) = P\left(x^2\right) - 2P(x) + a$$
   for a constant $a \in \mathbb{R}$ and for all $x \in \mathbb{R}$.

3. (Vietnam NMO 2012/7 [2]) Find all surjective, strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x)) = f(x) + 12x.$$

4. (IMO 2011/3) Let $f : \mathbb{R} \to \mathbb{R}$ follows
$$f(x + y) \leq yf(x) + f(f(x))$$
   for all $x, y \in \mathbb{R}$. Prove $f(x) = 0$ for all $x \leq 0$.

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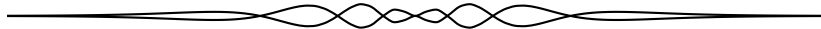
PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **August 1, 2018**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

4321. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Find the greatest positive real number k such that

$$(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq k(a^4 + b^4 + c^4 + d^4 + e^4)$$

for all real numbers a, b, c, d and e satisfying $a + b + c + d + e = 0$.

4322. *Proposed by Marius Drăgan.*

Let a, b, c be the side lengths of a triangle, x, y, z be positive numbers and let $u = bz - cy$, $v = ay - bx$, $w = cx - az$. Prove that $uv + vw + wu \leq 0$.

4323. *Proposed by Kadir Altintas.*

Let ABC be a triangle with $\angle C = 60°$. Let H denote the orthocenter, G the centroid, N the nine-point circle center and O the circumcenter of ABC. Let Q be the midpoint of NO. Prove that the parabola with vertex at Q and focus at G is tangent to the perpendicular bisector of both AC and BC.
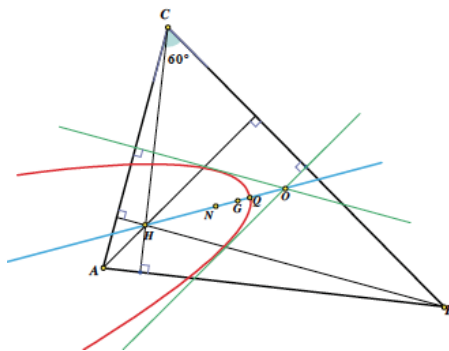
4324. *Proposed by Michel Bataille.*

Let f be a continuous, positive function on $[0, 1]$ such that $\mathcal{S} = \left\{ \int_0^1 (f(x))^n \, dx : n \in \mathbb{N} \right\}$ is bounded above. Find the value of $\sup \mathcal{S}$.

4325. *Proposed by Alessandro Ventullo.*

Solve in real numbers the system of equations:

$$\begin{aligned} x^4 - 2y^3 - x^2 + 2y &= -1 + 2\sqrt{5} \\ y^4 - 2x^3 - y^2 + 2x &= -1 - 2\sqrt{5}. \end{aligned}$$

4326. *Proposed by Tran Quang Hung.*

Let ABC be a triangle inscribed in circle (O). Suppose S is the midpoint of arc BC containing A, T is a point on arc BC not containing A, M is on (O) such that $SM \parallel OT$, P is a point on SM. Let points E and F lie on CA and AB, respectively, such that $PE \parallel MC$ and $PF \parallel MB$. Finally, let Q be on (O) such that AT is bisector of $\angle PAQ$. Prove that $QE = QF$.
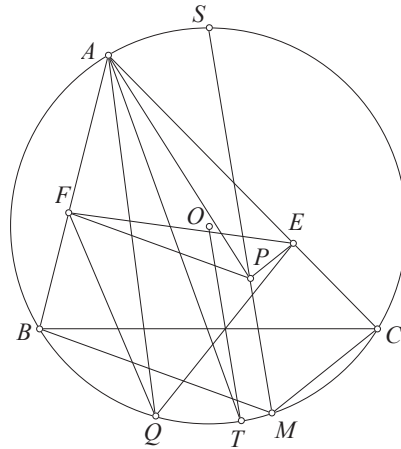
4327. *Proposed by Daniel Sitaru.*

Prove the following inequality for all $x > 0$:

$$\arctan(x)\arctan\left(\frac{1}{x}\right) < \frac{\pi}{2(x^2 + 1)}.$$

4328. *Proposed by Van Khea and Leonard Giugiuc.*

A circle I is inscribed in a triangle ABC and the points of tangency on the sides BC, CA and AB are D, E and F, respectively. The rays AD, BE and CF cut the circle I in points X, Y, Z, respectively. Prove that

$$\frac{1}{\frac{AX}{XD} + \frac{1}{4}} + \frac{1}{\frac{BY}{YE} + \frac{1}{4}} + \frac{1}{\frac{CZ}{ZF} + \frac{1}{4}} = 4.$$

4329. *Proposed by Mihaela Berindeanu.*

For $x, y, z \geq 1$, show that

$$\frac{\log_2 xy}{(\log_2 2z)^2} + \frac{\log_2 yz}{(\log_2 2x)^2} + \frac{\log_2 xz}{(\log_2 2y)^2} \geq \frac{\log_2 xyz}{1 + \left(\log_2 \sqrt[3]{xyz}\right)^2}.$$

4330★. *Proposed by Mohammed Aassila.*

Let a and b be integers such that $a^2 - 20b + 24 = 0$. Find the complete set of solutions of the following equation over integers:

$$5x^2 + axy + by^2 = 11.$$

. . . .

4321. *Proposé par Leonard Giugiuc and Diana Trailescu.*

Déterminer k, la plus grande valeur réelle positive telle que

$$(a^2 + b^2 + c^2 + d^2 + e^2)^2 \geq k(a^4 + b^4 + c^4 + d^4 + e^4)$$

pour tous les nombres réels a, b, c et d tels que $a + b + c + d + e = 0$.

4322. *Proposé par Marius Drăgan.*

Soient a, b, c les longueurs des côtés d'un triangle, puis x, y, z des nombres positifs. Poser $u = bz - cy$, $v = ay - bx$ et $w = cx - az$. Démontrer que $uv + vw + wu \leq 0$.

4323. *Proposé par Kadir Altintas.*

Soit ABC un triangle tel que $\angle C = 60°$. Soient H l'orthocentre, G le centroïde, N le cercle des neuf points et O le centre du cercle circonscrit de ABC. Soit Q le mi point de NO. Démontrer que la parabole avec sommet Q et foyer G est tangente à la bissectrice de AC puis celle de BC.
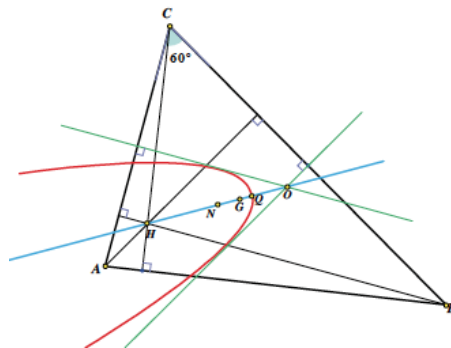
4324. *Proposé par Michel Bataille.*

Soit f une fonction continue et positive sur $[0,1]$ telle que $\mathcal{S} = \left\{ \int_0^1 (f(x))^n \, dx : n \in \mathbb{N} \right\}$ est bornée supérieurement. Déterminer la valeur de $\sup \mathcal{S}$.

4325. *Proposé par Alessandro Ventullo.*

Déterminer les solutions réelles au système d'équations suivant:

$$\begin{array}{rcl} x^4 - 2y^3 - x^2 + 2y & = & -1 + 2\sqrt{5} \\ y^4 - 2x^3 - y^2 + 2x & = & -1 - 2\sqrt{5}. \end{array}$$

4326. *Proposé par Tran Quang Hung.*

Soit ABC un triangle inscrit dans le cercle (O). Supposons que S est le mi point de l'arc BC contenant A, que T est un point sur l'arc BC ne contenant pas A, que M se situe sur (O) de façon à ce que $SM \parallel OT$, puis que P est un point sur SM. Les points E et F se situent sur CA et AB, respectivement, tels que $PE \parallel MC$ et $PF \parallel MB$. Enfin, soit Q sur (O) de façon à ce que AT bissecte $\angle PAQ$. Démontrer que $QE = QF$.
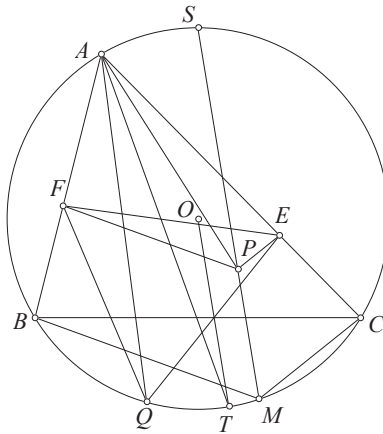
4327. *Proposé par Daniel Sitaru.*

Démontrer l'inégalité pour tout $x > 0$:

$$\arctan(x) \arctan\left(\frac{1}{x}\right) < \frac{\pi}{2(x^2 + 1)}.$$

4328. *Proposé par Van Khea and Leonard Giugiuc.*

Un cercle I est inscrit dans un triangle ABC; les points de tangence avec BC, CA et AB sont dénotés D, E et F respectivement. Les segments AD, BE et CF

intersectent le cercle aux points X, Y et Z, respectivement. Démontrer que

$$\frac{1}{\frac{AX}{XD}+\frac{1}{4}} + \frac{1}{\frac{BY}{YE}+\frac{1}{4}} + \frac{1}{\frac{CZ}{ZF}+\frac{1}{4}} = 4.$$

4329. *Proposé par Mihaela Berindeanu.*

Démontrer l'inégalité suivante, pour tout $x, y, z \geq 1$:

$$\frac{\log_2 xy}{(\log_2 2z)^2} + \frac{\log_2 yz}{(\log_2 2x)^2} + \frac{\log_2 xz}{(\log_2 2y)^2} \geq \frac{\log_2 xyz}{1 + \left(\log_2 \sqrt[3]{xyz}\right)^2}.$$

4330★. *Proposé par Mohammed Aassila.*

Soient a et b des entiers tels que $a^2 - 20b + 24 = 0$. Déterminer toutes les solutions entières de l'équation

$$5x^2 + axy + by^2 = 11.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(3), p. 101–104.

<center>⬦⬦⬦⬦⬦</center>

4221. *Proposed by Nguyen Viet Hung.*

Let a, b, c, p, q be distinct positive real numbers satisfying

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = p,$$

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2} = q.$$

Evaluate

$$\frac{a}{(b-c)^2} + \frac{b}{(c-a)^2} + \frac{c}{(a-b)^2}$$

in terms of p and q.

We received 13 submissions, 12 of which were correct, and we present the same solutions by Michel Bataille and Prithwijit De.

Let $r = \dfrac{a}{(b-c)^2} + \dfrac{b}{(c-a)^2} + \dfrac{c}{(a-b)^2}$. We show that $r = \sqrt{q(p-2)}$.

Let

$$s = \frac{1}{(c-a)(a-b)} + \frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)},$$

$$t = \frac{a}{(c-a)(a-b)} + \frac{b}{(a-b)(b-c)} + \frac{c}{(b-c)(c-a)},$$

$$u = \frac{bc}{(c-a)(a-b)} + \frac{ca}{(a-b)(b-c)} + \frac{ab}{(b-c)(c-a)}.$$

Then

$$s = \frac{(b-c) + (c-a) + (a-b)}{(a-b)(b-c)(c-a)} = 0,$$

$$t = \frac{a(b-c) + b(c-a) + c(a-b)}{(a-b)(b-c)(c-a)} = 0,$$

$$u = \frac{bc(b-c) + ca(c-a) + ab(a-b)}{(a-b)(b-c)(c-a)} = -1.$$

Hence,

$$\left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right)^2 = q + 2s = q$$

and $\quad \left(\dfrac{a}{b-c} + \dfrac{b}{c-a} + \dfrac{c}{a-b} \right)^2 = p + 2u = p - 2.$

Thus,

$$
\begin{aligned}
q(p-2) &= \left(\left(\dfrac{a}{b-c} + \dfrac{b}{c-a} + \dfrac{c}{a-b} \right) \left(\dfrac{1}{b-c} + \dfrac{1}{c-a} + \dfrac{1}{a-b} \right) \right)^2 \\
&= \left(r + \dfrac{b+c}{(c-a)(a-b)} + \dfrac{c+a}{(a-b)(b-c)} + \dfrac{a+b}{(b-c)(c-a)} \right)^2 \\
&= (r + (a+b+c)s - t)^2 \\
&= r^2.
\end{aligned}
$$

Since $p > 2$ and $r > 0$, $r = \sqrt{q(p-2)}$ follows.

Editor's comment. Steven Chow pointed out that the solution to this problem includes the proof of problem 1 of the 2017 Canadian Mathematical Olympiad which asks to show that $p > 2$ and is, in essence, a sub-problem of the current problem.

4222. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a quadrilateral inscribed in a circle and X a mobile point on the small arc CD. If E, F, G, H are the orthogonal projections of X on the lines AD, BC, AC, BD show that the angle between EH and GF is always constant, regardless of the position of X on the arc.

Soit $ABCD$ un quadrilatère inscrit dans un cercle et soit X un point situé sur le petit arc CD. Si E, F, G et H sont les projections orthogonales de X vers AD, BC, AC et BD, démontrer que l'angle entre EH et GF est constant, quel que soit le point X sur l'arc.
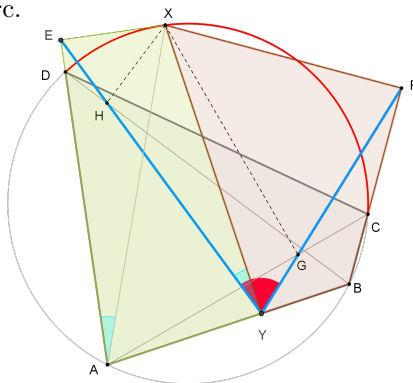
We received 12 submissions, all correct; most of the solutions were quite similar to our featured solution by Jean-Claude Andrieux, which was singled out to remind our readers that solutions can be submitted in either of our two official languages.

Rappelons le théorème de *la droite de Simson*: Soit ABC un triangle quelconque et M un point du plan. On note P, Q et R les projetés orthogonaux de M respec-

tivement sur (AB), (BC) et (CA); alors, P, Q et R sont alignés si et seulement si M appartient au cercle circonscrit à ABC.

Dans le problème posé, notons Y le projeté orthogonal de X sur (AB).

Considérons le triangle ABC: X appartient au cercle circonscrit au triangle ABC donc les points F, G et Y projetés orthogonaux de X respectivement sur (BC), (CA) et (AB) sont alignés.

Considérons le triangle ABD: X appartient au cercle circonscrit au triangle ABD donc les points E, H et Y projetés orthogonaux de X respectivement sur (AD), (BD) et (AB) sont alignés.

Les droites (FG) et (EH) se coupent donc en Y. Il faut montrer que l'angle $\widehat{EYF}$ est indépendant de la position de X sur l'arc CD.

Les triangles XEA, XGA et XYA sont des triangles rectangles de même hypoténuse $[XA]$. Les points X, E, A, G et Y sont donc cocycliques.

On a alors:
$$\widehat{EYF} = \widehat{EYG} = \widehat{EAG} = \widehat{DAC}.$$
L'angle $\widehat{EYF}$ est donc constant et on a
$$\widehat{EYF} = \widehat{DAC} = \widehat{DBC}.$$

Editor's comments. Steven Chow observed that if directed angles are used (modulo π), the point X need not be restricted to the small arc CD: the angle between EH and GF remains constant (and the featured proof remains valid) for all positions of X on the circle. Somasundaram Muralidharan observed, similarly, that the final line of our argument shows that while C and D must be fixed points, A and B are free to move about the circle without changing the angle between EH and GF. Chow also suggested that the editors perhaps should not have included the proposer's diagram with the statement of his problem since it essentially provides the solution. Maybe that explains the similarity of so many of the submissions. Bataille's solution, however, was based on the spiral similarity with fixed point X that takes C to D (and therefore G to E and F to H). He added to his solution the observation that the lines EG, FH, and CD are concurrent in a point common to the circles on diameter XC and XD. For more information about arguments that exploit intersecting circles and spiral similarities, see his article "Focus On... No. 12" [**40**:5 (May 2014) 203-206].

4223. *Proposed by Leonard Giugiuc and Dorin Marghidanu.*

Let a, b and c be positive real numbers such that $a + b + c \leq 1$. Prove that

$$\sqrt[3]{(1 - a^3)(1 - b^3)(1 - c^3)} \geq 26abc.$$

We received 15 correct solutions and we present a very succinct proof by Titu Zvonaru.

Since $1 \geq (a+b+c)^3$, we have by the AM-GM inequality that

$$
\begin{aligned}
1 - a^3 &\geq (a+b+c)^3 - a^3 \\
&= b^3 + c^3 + 3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2) + 6abc \\
&\geq 26 \sqrt[26]{a^{24}b^{27}c^{27}},
\end{aligned}
$$

with equality if and only if $a+b+c = 1$ and $a = b = c$ or two of a, b, and c are 0.

Multiplying by the other two inequalities obtained by considering $1 - b^3$ and $1 - c^3$ we then obtain

$$
(1 - a^3)(1 - b^3)(1 - c^3) \geq (26)^3 \sqrt[26]{a^{78}b^{78}c^{78}},
$$

from which $\sqrt[3]{(1 - a^3)(1 - b^3)(1 - c^3)} \geq 26abc$ follows.

The equality holds if and only if $(a, b, c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$.

Editor's comments. Geupel remarked that the given inequality can be generalized to the following result:

> if n is a natural number and $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n \leq 1$, then
>
> $$
> \sqrt[n]{(1 - a_1^n)(1 - a_2^n) \cdots (1 - a_n^n)} \geq (n^n - 1)a_1 a_2 \cdots a_n.
> $$

4224. *Proposed by Michel Bataille.*

Find the complex roots of the polynomial

$$
16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1.
$$

We received 16 solutions. We present 2 solutions.

Solution 1, by Prithwijit De.

Observe that

$$
16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1 = (2x^2)^3 + (2x^2)^3 + (2x - 1)^3 - 3(2x^2)(2x^2)(2x - 1).
$$

Using the identity

$$
a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca),
$$

we get

$$
(2x^2)^3 + (2x^2)^3 + (2x - 1)^3 - 3(2x^2)(2x^2)(2x - 1) = (4x^2 + 2x - 1)(2x^2 - 2x + 1)^2.
$$

Thus the roots (ignoring multiplicity) are $x = \dfrac{-1 \pm \sqrt{5}}{4}, \dfrac{1 \pm i}{2}$.

Solution 2, by Somasundaram Muralidharan.

Let $f(x) = 16x^6 - 24x^5 + 12x^4 + 8x^3 - 12x^2 + 6x - 1$. Let us first check whether f has repeated roots. Such repeated roots, if any, will be roots of $\gcd(f(x), f'(x))$, where $f'(x)$ is the derivative of $f(x)$. In this case, it is easy to see that $\gcd(f(x), f'(x)) = 2x^2 - 2x + 1$ and hence the roots of this gcd, namely $\dfrac{1 \pm i}{2}$, are double roots of $f(x) = 0$. Thus we have found four of the roots of f. We now find the remaining two roots of f. We have

$$f(x) = (2x^2 - 2x + 1)^2(4x^2 + 2x - 1)$$

and hence the remaining roots are roots of $4x^2 + 2x - 1 = 0$. These are $\dfrac{-1 \pm \sqrt{5}}{4}$. So, the complex roots of f are

$$\frac{-1 + \sqrt{5}}{4}, \ \frac{-1 - \sqrt{5}}{4}, \ \frac{1 + i}{2}, \ \frac{1 + i}{2}, \ \frac{1 - i}{2}, \ \frac{1 - i}{2}.$$

4225. *Proposed by Leonard Giugiuc, Daniel Dan and Daniel Sitaru.*

Prove that in any triangle ABC we have:

$$3(\cos^2 A + \cos^2 B + \cos^2 C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3.$$

We received 13 correct solutions. We present the solution by Arkady Alt.

Since $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ in any triangle ABC, the original inequality is successively equivalent to

$$3(1 - 2\cos A \cos B \cos C) + \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 3$$
$$\iff \cos A \cos B + \cos A \cos C + \cos B \cos C \geq 6 \cos A \cos B \cos C.$$

Setting $t = \sqrt[3]{\cos A \cos B \cos C}$ and using the AM-GM inequality, we obtain

$$\begin{aligned}
1 - 2t^3 &= 1 - 2\cos A \cos B \cos C \\
&= \cos^2 A + \cos^2 B + \cos^2 C \\
&\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \\
&= 3t^2.
\end{aligned}$$

Therefore $2t^3 + 3t^2 - 1 \leq 0$, implying successively that $(2t - 1)(t + 1)^2 \leq 0$ and then $t \leq \frac{1}{2}$. Hence, $2 \cdot \sqrt[3]{\cos A \cos B \cos C} \leq 1$, and again by the AM-GM inequality we have

$$\begin{aligned}
\cos A \cos B + \cos A \cos C + \cos B \cos C &\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \\
&\geq 3 \cdot \sqrt[3]{\cos^2 A \cdot \cos^2 B \cdot \cos^2 C} \cdot \sqrt[3]{\cos A \cos B \cos C} \\
&= 6 \cos A \cos B \cos C.
\end{aligned}$$

4226. *Proposed by Daniel Sitaru.*

Prove that if $0 < a < b$ then:

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x} dx \right)^2 > (b-a)^2 + \ln^2 \left(\frac{b}{a} \right).$$

We received nine submissions, eight of which are correct and the other is incorrect. We present a composite of virtually the same solutions by Arkady Alt; Michel Bataille; M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal (jointly); and Digby Smith.

Note first that

$$\left(\int_a^b \frac{\sqrt{1+x^2}}{x} dx \right)^2 > (b-a)^2 + \ln^2 \frac{b}{a}$$

$$\iff \left(\int_a^b \frac{\sqrt{1+x^2}}{x} dx \right)^2 - \left(\int_a^b \frac{1}{x} dx \right)^2 > (b-a)^2$$

$$\iff \int_a^b \frac{\sqrt{1+x^2}+1}{x} dx \cdot \int_a^b \frac{\sqrt{1+x^2}-1}{x} dx > (b-a)^2. \qquad (1)$$

Let $f(x) = \dfrac{\sqrt{1+x^2}+1}{x}$, $x \in [a,b]$. Then $f(x) > 0$ and $\dfrac{1}{f(x)} = \dfrac{\sqrt{1+x^2}-1}{x}$. By the integral form of the Cauchy-Schwarz Inequality, we have

$$\left(\int_a^b f(x)\, dx \right) \left(\int_a^b \frac{1}{f(x)} dx \right) = \left(\int_a^b \left(\sqrt{f(x)} \right)^2 dx \right) \left(\int_a^b \left(\sqrt{\frac{1}{f(x)}} \right)^2 dx \right)$$

$$\geq \left(\int_a^b 1 dx \right)^2$$

$$= (b-a)^2. \qquad (2)$$

But equality cannot hold in (2) as f is not a constant on $[a,b]$. Hence, from (1) and (2) the result follows.

4227. *Proposed by Dan Marinescu and Leonard Giugiuc.*

Let P be a point in the interior of an equilateral triangle ABC whose sides have length 1, and let R' and r' be the circumradius and inradius of the triangle whose sides are congruent to PA, PB and PC (which exists by Pompeiu's theorem). Prove that

$$3R' \geq 1 \geq 6r'.$$

Among the four submissions, three were complete and correct; in the fourth, Michel Bataille simply provided a reference where the proof can be found: Proposition 7 in Jószef Sándor's "On the Geometry of Equilateral Triangles", Forum Geometricorum, vol. 5 (2005) 107-117. Here we present the solution by Roy Barbara.

Let square brackets denote area and let $T = \Delta A'B'C'$ denote the given Pompeiu triangle with sides $a = PA, b = PB, c = PC$ (with a opposite A', etc.).

Lemma.

$$r' = \frac{\sqrt{3}}{6}\left(\frac{2 - (a^2 + b^2 + c^2)}{a + b + c}\right).$$

Proof of Lemma. Let X, Y, and Z be the reflections of P through BC, CA, and AB, respectively. Triangle AZY satisfies $AY = AZ = a$ and $\angle YAZ = 120°$; hence,

$$[AZY] = \frac{\sqrt{3}}{4}a^2 \quad \text{and, similarly,} \quad [BXZ] = \frac{\sqrt{3}}{4}b^2 \quad \text{and} \quad [CYX] = \frac{\sqrt{3}}{4}c^2. \quad (1)$$

Triangle XYZ, having sides $a\sqrt{3}, b\sqrt{3}, c\sqrt{3}$ is similar to T. Clearly, the area of the hexagon $AZBXCY$ is twice $[ABC]$ so,

$$[AZBXCY] = \frac{\sqrt{3}}{2}. \quad (2)$$

But also, $[AZBXCY] = [XYZ] + [AZY] + [BXZ] + [CYX]$. From this, (1), and (2) we get

$$[XYZ] = \frac{\sqrt{3}}{4}\left(2 - (a^2 + b^2 + c^2)\right). \quad (3)$$

The inradius r'' of ΔXYZ is $\frac{[XYZ]}{s''}$, where $s'' = \frac{\sqrt{3}}{2}(a+b+c)$ is the semiperimeter of ΔXYZ. From this and (3) we get

$$r'' = \frac{1}{2}\left(\frac{2 - (a^2 + b^2 + c^2)}{a + b + c}\right).$$

Finally, since T is similar to ΔXYZ with ratio $\frac{1}{\sqrt{3}}$, we obtain $r' = \frac{1}{\sqrt{3}}r''$, and the lemma follows.

Proof that $R' \geq \frac{1}{3}$. Since ΔABC is equilateral, its Fermat point (that minimizes $a + b + c$) is its centroid. Hence,

$$a + b + c \geq \sqrt{3}. \quad (4)$$

By the Law of Sines we have

$$2R' = \frac{a}{\sin A'} = \frac{b}{\sin B'} = \frac{c}{\sin C'} = \frac{a + b + c}{\sin A' + \sin B' + \sin C'}. \quad (5)$$

Further, any triangle $A'B'C'$ satisfies $\sin A' + \sin B' + \sin C' \leq \frac{3\sqrt{3}}{2}$. This together with (5) and (4) yields

$$2R' = \frac{a + b + c}{\sin A' + \sin B' + \sin C'} \geq \frac{\sqrt{3}}{\left(\frac{3\sqrt{3}}{2}\right)} = \frac{2}{3};$$

that is, $R' \geq \frac{1}{3}$, as desired. Equality holds if and only if P is the centroid of ΔABC (in which case $\Delta A'B'C'$ is also equilateral).

Proof that $r' \leq \frac{1}{6}$. Set $S = a + b + c$. By the lemma, $r' = \frac{\sqrt{3}}{6}\left(\frac{2-(a^2+b^2+c^2)}{a+b+c}\right)$; moreover, $a^2 + b^2 + c^2 \geq \frac{S^2}{3}$ is a basic inequality. Consequently,

$$r' \leq \frac{\sqrt{3}}{18}\left(\frac{6 - S^2}{S}\right). \tag{6}$$

By (4), $S \geq \sqrt{3}$. Furthermore, $\frac{6-S^2}{S}$ is decreasing with respect to S, whence

$$\frac{6 - S^2}{S} \leq \frac{6 - (\sqrt{3})^2}{\sqrt{3}} = \sqrt{3}.$$

The desired result follows immediately from this together with (6); also here, equality holds if and only if $\Delta A'B'C'$ is equilateral.

4228. *Proposed by Mihály Bencze.*

Let $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$ such that $\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} z_k^2 = 0$. Prove that

$$n\sum_{k=1}^{n} |z_k|^2 \leq (n-2)\left(\sum_{k=1}^{n} |z_k|\right)^2.$$

We received three correct solutions and one incorrect solution. We present two solutions here.

Solution 1, by the proposer.

For each k with $1 \leq k \leq n$, we have that

$$2z_k^2 = z_k^2 + (z_1 + z_2 + \cdots \hat{z_k} + \cdots + z_n)^2$$

$$= \sum_{k=1}^{n} z_k^2 + 2\sum\{z_i z_j : 1 \leq i < j \leq n; i, j \neq k\}$$

$$= 2\sum\{z_i z_j : 1 \leq i < j \leq n; i, j \neq k\}$$

from which

$$2|z_k|^2 \leq 2\sum\{|z_i z_j| : 1 \leq i < j \leq n; i, j \neq k\}.$$

Adding all these n inequalities leads to

$$2\sum_{k=1}^{n} |z_k|^2 \leq 2(n-2)\sum_{1 \leq i < j \leq n} |z_i z_j|.$$

Adding $(n-2)\sum_{k=1}^{n}|z_k|^2$ to each side yields

$$n\sum_{k=1}^{n}|z_k|^2 \le (n-2)\left(\sum_{k=1}^{n}|z_k|^2 + 2\sum_{1\le i<j\le n}|z_i z_j|\right)$$
$$= (n-2)\left(\sum_{k=1}^{n}|k|\right)^2.$$

Solution 2, by Michel Bataille.

When $n = 1, 2, 3$, we have equality on both sides. For $n = 1$, $z_1 = 0$. For $n = 2$, $z_1 + z_2 = z_1 z_2 = 0$ so that $z_1 = z_2 = 0$. For $n = 3$ and $i \neq j$,

$$2(z_i^3 - z_j^3) = (z_i - z_j)[z_i^2 + z_j^2 + (z_i + z_j)^2)] = 0,$$

so that $z_1^3 = z_2^3 = z_3^3$, whence $|z_1| = |z_2| = |z_3|$.

Let $n \geq 4$. Then

$$0 = |z_1 + \cdots + z_n|^2 = (z_1 + \cdots + z_n)(\overline{z_1} + \cdots + \overline{z_n}) = \sum_{k=1}^{n} |z_k|^2 + 2 \sum_{1 \leq i < j \leq n} \text{Re } (z_i \overline{z_j}).$$

Therefore

$$\sum_{k=1}^{n} |z_k|^2 = -2 \sum_{1 \leq i < j \leq n} \text{Re } (z_i \overline{z_k}) \leq 2 \sum_{1 \leq i < j \leq n} |\text{Re } (z_i \overline{z_k})|$$

$$\leq 2 \sum_{1 \leq i < j \leq n} |z_i \overline{z_j}| = 2 \sum_{1 \leq i < j \leq n} |z_i||z_j|$$

$$= \left(\sum_{k=1}^{n} |z_k| \right)^2 - \sum_{k=1}^{n} |z_k|^2.$$

Rearranging like terms and multiplying by $n/2$ yields the inequality

$$n \sum_{k=1}^{n} |z_k|^2 \leq \frac{n}{2} \left(\sum_{k=1}^{n} |z_k| \right)^2,$$

which is the required inequality when $n = 4$ and is stronger for larger n.

4229. *Proposed by Leonard Giugiuc.*

Let n be an integer with $n \geq 2$ and let p be a prime number with $p > n$. Consider an $n \times n$ matrix X over $\mathbb{Z}_p$ with $X^p = I_n$. Prove that $(X - I_n)^n = O_n$.

There were 5 correct solutions. We present the solution obtained independently by Roy Barbara and Trey Smith.

Since $(X - I_n)^p = X^p - I_n^p = O_n$, $X - I_n$ is nilpotent. But this implies that $(X - I_n)^n = O_n$.

Editor's comment. For the linear algebra result invoked, let D be nilpotent and m the minimum exponent for which $D^m = O_n$. If $m \leq n$, then $D^n = O_n$. If $m > n$, suppose, if possible, (by Cayley's theorem) that $O_n = D^n + c_{n-1} D^{n-1} + \cdots + c_k D^k$ with $c_k \neq 0$ for some $0 \leq k \leq n - 1$. Multiply the equation by D^{m-k-1} to get a contradiction.

4230. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle in which $\angle B = 2\angle C$ and let M be the midpoint of BC. The internal bisector of $\angle ACB$ intersects AM in D. Prove that $\angle CDM \leq 45°$ and find $\angle C$ for which the equality holds.

Miguel Amengual Covas observed that this problem appeared as problem **1562** of **Crux** [1990:204], posed by Toshio Seimiya and that three solutions were given in [1991 : 252-254]. He observes further that it appears in *The Olympiad Corner No. 161* [1995 : 9-10], with solution by Covas, distinct from those published in 1991 given in [1996 : 265-267].

References to further properties of triangles whose angles satisfy $\angle B = 2\angle C$ can be found in J. Chris Fisher's "Recurring Crux Configurations 7", [2012 : 238-240].
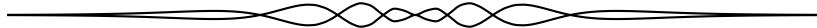
Bouncing Balls

A pair of balls A and B of negligible radius (so you can treat them as points) lie on a perfectly flat surface with ball B lying between ball A and a wall. Ball A has mass 100^n and ball B has mass 1. Ball A is pushed towards ball B and, as the balls interact, we count the number of collisions.

When $n = 0$, ball A strikes ball B and stops. Ball B bounces off the wall and returns to strike ball A. Ball B then stops and ball A rolls away into the distance. A total of 3 collisions occurred.

When $n = 1$, there are a total of 31 collisions.

When $n = 5$, there are a total of 314159 collisions.

See a pattern? Can you prove it?