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All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor: Léo Sauvé, Math-Architecture, Algonquin College, Col. By Campus, 281 *Echo Drive, Ottawa, Ont., K1S* 1*N*4.

All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, Rideau Campus, 200 *Lees Ave., Ottawa, Ont., K1S* 0*C*5.

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THE STEINER - LEHMUS THEOREM

LÉO SAUVÉ, Algonquin College

1. *Introduction.*

It is a trivial exercise in Euclidean geometry to prove that in an isosceles triangle the internal bisectors (as well as the external bisectors) of the equal angles are equal. In each case the word *bisectors* refers to the segments from the vertices of the equal angles to the opposite sides (or sides produced).

Over the millenia since the infancy of geometry, many people must have wondered if the converse of this theorem is true, and some may even have found a satisfactory answer to the question; but if so history has no record of it until 1840. In that year Berlin professor C. L. Lehmus (1780 - 1863), whose name would otherwise have been long forgotten, asked the famous Swiss geometer Jacob Steiner (1796 - 1863) for a proof of the following theorem: *if two internal angle bisectors of a triangle are equal, then the triangle is isosceles.*

Lewin [20] writes: "Steiner soon found a proof but did not publish it until 1844 [1]. In 1850 Lehmus found a proof of his own [2]. However it was the French mathematician Rougevain who was the first to publish a proof in 1842." Coxeter and Greitzer [14] add: "Papers on the Steiner-Lehmus theorem appeared in various journals in 1842, 1844, 1848, almost every year from 1854 till 1864, and with a good deal of regularity during the next hundred years." From Lewin [20] again: "The seemingly never-ending stream culminated in a paper by Henderson [7] whose avowed aim it was 'to write an essay on the internal bisector problem to end all essays on the internal bisector problem.' To strengthen his point he supplied as many as ten different proofs." That should have ended the matter, but only six years later a centenary account by McBride [9] mentioned about sixty proofs! In the first edition of [15], published in 1961, Coxeter gave a simple proof which he attributed to H. G. Forder. In a review of Coxeter's book in [12], Martin Gardner referred to this proof and described the Steiner-Lehmus theorem in such an interesting manner that hundreds of readers sent him their own proofs. Gardner painstakingly went through all these proofs, selected the one he considered the best, and had it published in the *American Mathematical Monthly* [13]. It was by two English engineers, G. Gilbert and D. McDonnell. After publication it was discovered that the proof was essentially the same as the original 1850 proof of Lehmus!

And there, for the moment, the matter rests, except for the quarrel about direct vs. indirect proofs, which is still going on. This is described in section 3 below.

2. *Two simple proofs of the Steiner-Lehmus theorem:*

(a) I give first the gem mined by Gardner from the mountainous collection of proofs sent to him [13].

Let ABC be a triangle with equal internal angle bisectors BM and CN, as in Figure 1. If the angles B and C are not equal, one must be less, say $B < C$. Take L on BM so that $\angle LCN = \frac{1}{2}B$. Since this is equal to $\angle LBN$, the four points L, N, B, C are concyclic. Since

$$B < \tfrac{1}{2}(B+C) < \tfrac{1}{2}(A+B+C),$$

$\angle CBN < \angle LCB < 90°$. Since smaller chords of a circle subtend smaller acute angles, and $BL < CN$,

$$\angle LCB < \angle CBN.$$

We thus have a contradiction.

(b) The second proof I give, which is my own favorite, is due to a French engineer, M. Descube; it was published in [4] and I found it in [5]. (Note that both proofs given here were discovered by engineers and not by professional mathematicians, which proves something or other.) The proof is just as short and elegant as the first, and more elementary, since it makes no use of circles.

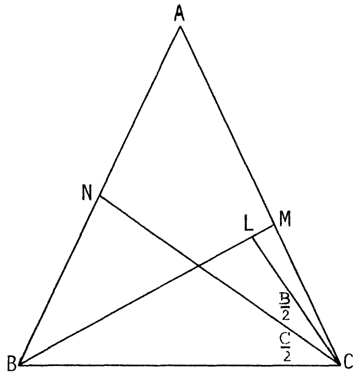

Figure 1

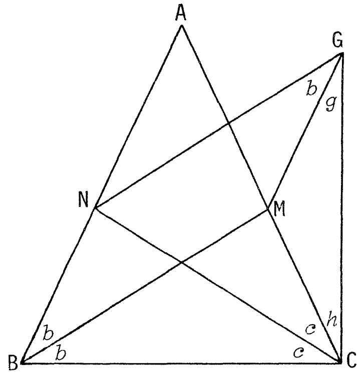

Figure 2

Let ABC be a triangle with equal internal angle bisectors BM and CN, as in Figure 2. Complete parallelogram NBMG, so that the three angles denoted by b are equal (as well as the two angles denoted by c). It will suffice to show that $b = c$. If $b \neq c$ then one is greater, say $b > c$. Since Δ's NBC and MBC have two corresponding sides equal and included angles b and c, we have

$$b > c \Rightarrow \mathrm{CM} > \mathrm{BN}.$$

In ΔCGN, we have CN = NG, so that $b + g = c + h$; hence

$$b > c \Rightarrow g < h \Rightarrow \mathrm{CM} < \mathrm{MG} \Rightarrow \mathrm{CM} < \mathrm{BN},$$

and we have a contradiction.

3. *Is a direct proof possible?*

It will be observed that both proofs given above are indirect. Sylvester [3], noting that all the proofs up to then had been by *reductio ad absurdum*, set about to prove the impossibility of a direct proof. He did not quite succeed in doing this but for a while his conclusion was accepted. Later "direct" proofs began to appear, and seven of the ten proofs given by Henderson in [7] were claimed to be direct. McBride [9] then dissected sixty different proofs, showing that each "direct" proof depended on indirectly proved lemmas. He then concluded by saying: "If it is held, as I hold, that Euc.I.14, Euc.I.29, Euc.I.32, and the Theorem of Pythagoras have no direct proof, then the Bisector Theorem has not been proved directly, nor is it likely to be." This was in 1943. In 1970, Malesevic published in [17] what he claimed to be a direct proof. His claim was demolished by Lewin in [20]. Finally, no less an authority than Howard Eves asks in 1972 [18, p.58]: *Give a direct proof of the Steiner-Lehmus Theorem*, and his solution is outlined at the back of the book on p. 390. I leave it to curious readers to determine if Eves's claim is justified.

Coxeter and Greitzer write in [14]: "A proof cannot properly claim to be direct if any one of [the] auxiliary theorems [used] has an indirect proof. Now, some of the simplest and most basic theorems have indirect proofs: consequently, if we insisted on complete directness, our store of theorems would be reduced to the merest trivialities. Is this observation any cause for sorrow?" They then quote the celebrated statement by G. H. Hardy [8]: "*Reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."

4. *The external angle bisector problem.*

After all that has been said about the internal angle bisector problem, one would naturally expect the external angle bisectors to have the same property, namely:

if two external angle bisectors of a triangle are equal, then the triangle is isosceles.

Surprisingly, this is not true: there are *pseudo-isosceles* triangles with equal external angle bisectors. Sastry [19] asked for a counterexample, and C. W. Trigg [22] and L. A. Ringenberg [21] (independently) answered his question, but N. A. Court [11] had already mentioned the fact, without however giving a proof. Trigg gives references to this problem for 1917, 1931, 1933, 1938, 1939, 1940. His solution in [22] is interesting. He begins by showing (the proof is easy) that triangle ABC, where

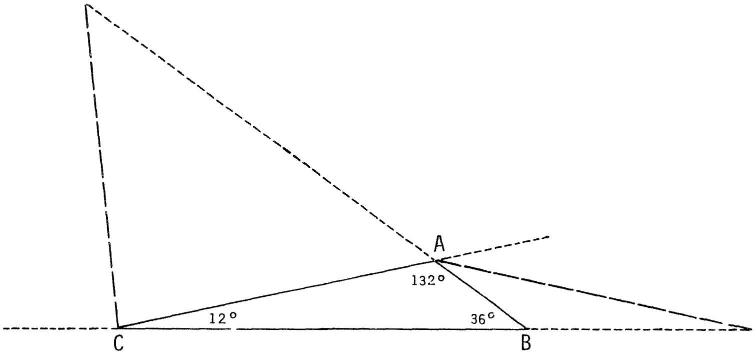

Figure 3

$$A = 132°, \quad B = 36°, \quad C = 12°, \tag{1}$$

has equal external angle bisectors at A and C (see Figure 3). He then goes on: "More generally, the square of an external bisector, T_A, of angle A of a triangle is equal to

$$bc\left[\frac{a^2}{(b-c)^2} - 1\right].$$

Thus, if $T_C = T_A$, then

$$ab\left[\frac{c^2}{(b-a)^2} - 1\right] = bc\left[\frac{a^2}{(b-c)^2} - 1\right].$$

This expression simplifies to

$$b(a - b + c)(c - a)[b^3 - (a + c)b^2 + 3acb - ac(a + c)] = 0.$$

The first two factors cannot be zero, so either $c - a = 0$ and the triangle is isosceles, or

$$b^3 - (a + c)b^2 + 3acb - ac(a + c) = 0. \tag{2}$$

This equation in b has one real and two complex roots. The real root is

$$b = M + \sqrt[3]{M^3 + \sqrt{N}} + \sqrt[3]{M^3 - \sqrt{N}},$$

where $M = \frac{a+c}{3}$ and the always positive $N = \frac{1}{27}ac[27a^2c^2 - 9ac(a + c)^2 + (a + c)^4]$.

When $a = c$, the corresponding triangle is equilateral. When $a \neq c$, the triangle is pseudo-isosceles.

After multiplication by $a+b+c=2s$, the equation (2) can be manipulated into the form

$$\left(\frac{s-b}{b}\right)^2 = \left(\frac{s-a}{a}\right)\left(\frac{s-c}{c}\right).$$

This expression, in turn, is equivalent to

$$\sin^2 \tfrac{1}{2}B = \sin \tfrac{1}{2}A \sin \tfrac{1}{2}C,$$

of which triangle (1) is a special case."

5. *Acknowledgments.*

I obtained references 1, 2 from [15]; references 3, 7, 9, 17 from [20]; reference 4 from [5]; reference 6 from [11]; reference 8 from [14]; reference 10 from [16]; reference 12 from [13]; all the remaining references are my own.

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20. Mordechai Lewin, On the Steiner-Lehmus Theorem, *Mathematics Magazine*. Vol. 47, 1974, pp. 87 - 89.

21. Lawrence A. Ringenberg, Solution II to Problem 862, *Mathematics Magazine*, Vol. 47, 1974, p. 53.

22. Charles W. Trigg, Solution I to Problem 862, *ibid.*, pp. 52 - 53.

* * *

LETTER TO THE EDITOR

Dear editor:

Your remarks in the November issue on the mathematical illiteracy of many high school graduates invites comment. An increase in mathematical competence among high school teachers is not, I feel, an answer to the problem.

An improvement in quality of high school graduates will not occur as more teachers earn their type-A certificates. An incompetent teacher will still be incompetent after taking reams of courses in mathematics. This is not to belittle the esoteric, but it has little place in the mass production of graduates that is practiced in Canada in the name of public education.

Your point regarding the *doing* of mathematics was important. Any teacher who has had the satisfaction of working at such problems as are published in EUREKA should attempt to analyze that satisfaction and to realize that students need to have similar feelings.

Half the job is teaching mathematics; the other half is selling it. Aspects of salesmanship are as varied as the personalities of teachers themselves. One thing that cries out to be told (and sold) is that mathematics is beautiful, that it is creatively satisfying, that it can be fun. It should be possible to find joy in doing mathematics. Genuine enthusiasm, judicially communicated, does wonders for motivation.

And once the problem of motivation is solved, the teacher's hardest work is done. The teacher, textbooks, and the students' own minds are then simply sources of the knowledge sought;—if then truly sought, remembered.

SHEILA GRIBBLE,
Picton, Ont.

* * *

MATHEMATIQUES. Dessechent le coeur.

MECANIQUE. Partie inférieure des mathématiques.

ORTHOGRAPHE. Y croire comme aux mathématiques. N'est pas necessaire quand on a du style.

QUADRATURE DU CERCLE. On ne sait pas ce que c'est, mais il faut lever les epaules quand on en parle.

GUSTAVE FLAUBERT,
Dictionnaire des idées reçues.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 19.

*For the problems given below, solutions, if available, will appear in EUREKA Vol.*2, *No.*5, *to be published around May* 31, 1976. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than May* 20, 1976.

94. [1975; 97] *Correction.* In the statement of the problem, the word *sides* is inappropriate. Replace it by the word *edges*.

111. *Proposed by H.G. Dworschak, Algonquin College.*

Prove that, for all distinct rational values of a, b, c, the expression

$$\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}$$

is a perfect square.

112. *Proposed by H.G. Dworschak, Algonquin College.*

Let $k>1$ and n be positive integers. Show that there exist n consecutive odd integers whose sum is n^k.

113. *Proposé par Léo Sauvé, Collège Algonquin.*

Si $\vec{u}=(b,c,a)$ et $\vec{v}=(c,a,b)$ sont deux vecteurs non nuls dans l'espace euclidien réel à trois dimensions, quelle est la valeur maximale de l'angle $(\vec{u},\vec{v})$ entre $\vec{u}$ et $\vec{v}$? Quand cette valeur maximale est-elle atteinte?

114. *Proposed by Léo Sauvé, Algonquin College.*

An arithmetic progression has the following property: for any even number of terms, the ratio of the sum of the first half of the terms to the sum of the second half is always equal to a constant k.

Show that k is uniquely determined by this property, and find all arithmetic progressions having this property.

115. *Proposed by Viktors Linis, University of Ottawa.*

Prove the following inequality of Huygens:

$$2\sin\alpha + \tan\alpha \ge 3\alpha, \qquad 0 \le \alpha < \frac{\pi}{2}.$$

116. *Proposed by Viktors Linis, University of Ottawa.*

For which values of a, b, c does the equation

$$\sqrt{x+a\sqrt{x}+b} + \sqrt{x} = c$$

have infinitely many solutions?

117. *Proposé par Paul Khoury, Collège Algonquin.*

Le sultan dit à Ali Baba:

"Voici deux urnes, et a boules blanches et b boules noires. Répartis les boules dans les urnes, mais je rendrai ensuite les urnes indiscernables. Tu auras la vie sauve en tirant une boule blanche." Comment Ali Baba maximise-t-il ses chances?

118. *Proposé par Paul Khoury, Collège Algonquin.*

Peut-on piper deux dés de sorte que la somme des points soit uniformément répartie sur $2, 3, \ldots, 12$?

119. *Proposed by John A. Tierney, United States Naval Academy.*

A line through the first quadrant point (a,b) forms a right triangle with the positive coordinate axes. Find analytically the minimum perimeter of the triangle.

120. *Proposed by John A. Tierney, United States Naval Academy.*

Given a point P inside an arbitrary angle, give a Euclidean construction of the line through P that determines with the sides of the angle a triangle

(a) of minimum area;

(b) of minimum perimeter.

* * *

S O L U T I O N S

61. [1975; 56, 98] *Late solution: Walter Bluger, Department of National Health and Welfare.*

81. [1975; 84] *Proposed by H.G. Dworschak, Algonquin College.*

Which of the following are divisible by 6 for all positive integers n?

(i) $n(n+1)(n+2)$

(ii) $n(n+1)(2n+1)$

(iii) $n(n^2+5)$

(iv) $(n+1)^{2k} - (n^{2k}+2n+1)$, k a positive integer.

Composite solution made up from the solutions submitted independently by R. Mahoney and Léo Sauvé, both from Algonquin College.

(i) The product of three consecutive integers is always divisible by $3! = 6$.

(ii) Since $n(n+1)(2n+1) = (n-1)n(n+1) + n(n+1)(n+2)$, divisibility by 6 follows from *(i)*.

(iii) Since $n(n^2+5) = (n-1)n(n+1) + 6n$, divisibility by 6 also follows from *(i)*.

(iv) The given expression, considered as a polynomial in n, vanishes for $n = 0, -1, -\frac{1}{2}$;

it is thus divisible by $n(n+1)(2n+1)$, and divisibility by 6 follows from *(ii)*.

Also solved by Walter Bluger, Department of National Health and Welfare; D.E. Fisher, Algonquin College; Sheila Gribble, Picton, Ont.; G.D. Kaye, Department of National Defence; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment.

Some solvers used congruences and some used induction to prove one or more parts of the problem. The proposer's proof of *(ii)* is interesting: $n(n+1)(2n+1) = 6\sum_{k=1}^{n} k^2$.

82. [1975; 84] *Proposé par Léo Sauvé, Collège Algonquin.*

Soit E un ensemble fini qui contient n éléments. Les faits suivants sont bien connus ou faciles à démontrer:

(a) Le nombre de sous-ensembles de E est 2^n.

(b) Le nombre de relations de la forme $A \subseteq B$, où $A \subseteq E$ et $B \subseteq E$, est $(2^n)^2 = 4^n$.

Combien des relations de (b) sont vraies?

I. *Solution d'André Ladouceur, Ecole Secondaire De La Salle.*

Je démontrerai par induction que le nombre de relations vraies est 3^n. Si $n = 0$, il y a $3^0 = 1$ relation vraie dans E, car la seule relation possible est $\phi \subseteq \phi$, qui est vraie.

Supposons maintenant que chaque ensemble de n éléments engendre 3^n relations vraies. Si E est un ensemble de $n+1$ éléments, donc non vide, et que $a \in E$, l'ensemble $E - \{a\}$ engendre 3^n relations vraies d'après l'hypothèse. Soit $A \subseteq B$ l'une quelconque de ces relations vraies. Celle-ci engendre seulement les trois relations vraies suivantes dans E:

$$A \subseteq B, \qquad A \subseteq B \cup \{a\}, \qquad A \cup \{a\} \subseteq B \cup \{a\}.$$

Le nombre des relations vraies dans E est donc $3 \cdot 3^n = 3^{n+1}$, et l'induction est complète.

II. *Solution by the proposer.*

For $k = 0,1,\ldots,n$, let A_k be any one of the $\binom{n}{k}$ k-element subsets of E. A relation $A_k \subseteq B$ is true if and only if $B = A_k \cup C_k$, where C_k is any one of the 2^{n-k} subsets of $E - A_k$. The number of true relations $A \subseteq B$ in which A is a k-element subset of E is therefore $\binom{n}{k}\cdot 2^{n-k}$, and the total number of true relations is

$$2^n + \binom{n}{1}\cdot 2^{n-1} + \ldots + \binom{n}{k}\cdot 2^{n-k} + \ldots + 1 = (2+1)^n = 3^n. \qquad (1)$$

Editor's comment.

This problem can be found, without answer or solution, in Simmons [2]. Equation (1) can be found in Burton [1], without, however, any indication of its set-theoretic significance.

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1. David M. Burton, *Elementary Number Theory*, Allyn and Bacon, 1976, p. 13.

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83. [1975; 84] *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que le produit de deux, trois ou quatre entiers positifs consécutifs n'est jamais un carré parfait.

Solution composée avec celles soumises par F.G.B. Maskell, Collège Algonquin; et par le proposeur.

(a) Puisque

$$n^2 < n(n+1) < (n+1)^2,$$

$n(n+1)$ n'est pas un carré parfait, étant situé entre deux carrés consécutifs.

(b) $n+1$ et $n(n+2)$ sont relativement premiers, puisque $n(n+2) = (n+1)^2 - 1$ et $(n+1)^2$ sont consécutifs. Donc $n(n+1)(n+2)$ ne peut être un carré que si $n+1$ et $n(n+2)$ sont tous deux des carrés, et $n(n+2) = (n+1)^2 - 1$ ne l'est pas.

(c) Puisque

$$\begin{aligned}(n^2+3n)^2 &< (n^2+3n)^2 + 2(n^2+3n)\\ &= n(n+1)(n+2)(n+3)\\ &= (n^2+3n+1)^2 - 1\\ &< (n^2+3n+1)^2,\end{aligned}$$

$n(n+1)(n+2)(n+3)$ n'est pas un carré parfait, étant situé entre deux carrés consécutifs.

Also solved by Walter Bluger, Department of National Health and Welfare; G. D. Kaye, Department of National Defence; and R. Mahoney, Algonquin College. One incorrect solution was received.

Editor's comment.

Let $P_n = m(m+1)\cdots(m+n-1)$. The present problem shows that P_2, P_3, and P_4 are not squares. The following additional information about P_n can be found in Dickson [2; pp. 679 - 680]:

(a) P_n is not a square or higher power if at least one factor $m,\ldots,m+n-1$ is a prime, or if $n > m-5$ (Liouville, 1857). In particular, $n!$ is not an exact power if $n > 1$.

(b) P_3 is not an exact power (Mlle A.D., 1857).

(c) P_5, P_6, and P_7 are not squares (Gerono, 1860).

(d) P_5 is not a cube (Lebesgue, 1860).

(e) P_n is not a square if $8 \le n \le 17$, and P_6 and P_9 are not cubes (Guibert, 1862).

(f) P_4 is not a cube (Aubry, 1913).

(g) P_2 is not a cube (Hayashi, 1916).

(h) P_n is not a square if $1 \le n \le 203$ (Narumi, 1917).

Information about the character of the related number $Q_n = n! + 1$ can also be found in Dickson [2, p. 681]: H. Brocard asked in 1876 for values of n making Q_n a square.

He later (1887) conjectured that only Q_4, Q_5, and Q_7 are squares. Exactly one hundred years after the question was raised, Burton [1, p. 9] reports that it is still unknown if Q_n is ever a square for any $n > 7$.

REFERENCES

1. David M. Burton, *Elementary Number Theory*, Allyn and Bacon, 1976.

2. L. E. Dickson, *History of the Theory of Numbers*, Chelsea, 1952, Vol. II.

84. [1975; 84] *Proposed by Viktors Linis, University of Ottawa.*

Prove that for any positive integer n

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

Solution by D.E. Fisher, Algonquin College.

The inequality clearly holds for $n = 1$. For $n \geq 2$, we have

$$\begin{aligned}\left(1+\sqrt{\frac{2}{n}}\right)^n &= 1 + n\sqrt{\frac{2}{n}} + \frac{n(n-1)}{2}\cdot\frac{2}{n} + \ldots \\ &= 1 + \sqrt{2n} + (n-1) + \ldots \\ &= n + \sqrt{2n} + \ldots \\ &> n,\end{aligned}$$

and the desired inequality follows on taking nth roots.

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; R. Mahoney, Algonquin College; F.G.B. Maskell, Algonquin College; Léo Sauvé, Collège Algonquin; and the proposer.

Editor's comment.

All solvers used approximately the same method. However, in addition to the above solver, only the proposer observed that independent verification of the inequality was necessary for $n = 1$.

85. [1975; 84] *Proposed by Viktors Linis, University of Ottawa.*

Find n natural numbers such that the sum of any number of them is never a square.

I. *Solution by F.G.B. Maskell, Algonquin College.*

The sum of the numbers in every subset of $\{10, 10^3, 10^5, \ldots, 10^{2n-1}\}$ ends in an odd number of zeros, so that it is the product of an odd power of 10 and a number which is not a power of 10. Hence it cannot be a perfect square.

The same argument may be extended to the number base k, provided k itself is not a perfect square, and the set $\{k, k^3, k^5, \ldots, k^{2n-1}\}$ has the desired property.

II. *Solution by G.D. Kaye, Department of National Defence.*

A natural number ending in 2, 3, 7, or 8 cannot be a perfect square; neither can one ending in 20, 30, 70, or 80. Thus the set

$$\{2, 20, 200, \ldots, 2\cdot 10^{n-1}\} \qquad (1)$$

has the required property since the sum of the numbers in any subset is a number ending in

$$2\cdot 10^{2r} \quad \text{or} \quad 20\cdot 10^{2r},$$

where r is a nonnegative integer, and hence is not a perfect square.

The sets obtained by replacing 2 by 3, 7, or 8 in (1) also have the required property.

III. *Solution d'André Ladouceur, École Secondaire De La Salle.*

On définit d'abord par récurrence les deux suites

$$A = \{a_1, a_2, \ldots, a_n\} \qquad \text{et} \qquad S = \{s_1, s_2, \ldots, s_{n-1}\}.$$

On pose $a_1 = 2$; pour $k \geq 1$, lorsque $a_1, \ldots, a_k$ ont été définis, on définit ensuite

$$s_k = a_1 + \ldots + a_k \qquad \text{si cette somme est impaire,}$$
$$s_k = a_1 + \ldots + a_k + 1 \qquad \text{dans le cas contraire;}$$

puis, si s_k (qui est toujours impair) est égal à $2m+1$, on définit $a_{k+1} = (m+1)^2 + 1$.

On peut maintenant démontrer que la suite A a la propriété désirée. Soit E un sous-ensemble non vide de A, et notons ΣE la somme de ses éléments. Si le plus grand (et donc le seul) élément de E est $a_1 = 2$, alors ΣE n'est pas un carré. Si le plus grand élément de E est a_{k+1}, $k \geq 1$, et que $s_k = 2m+1$, alors

$$\begin{aligned}(m+1)^2 &< a_{k+1}\\ &\leq \Sigma E \leq a_{k+1} + s_k = (m+1)^2 + 1 + 2m + 1 = m^2 + 4m + 3\\ &< (m+2)^2.\end{aligned}$$

La somme ΣE, étant située entre deux carrés consécutifs, ne peut elle-même être un carré.

Also solved by Walter Bluger, Department of National Health and Welfare; H.G. Dworschak, Algonquin College; R. Mahoney, Algonquin College; and Léo Sauvé, Collège Algonquin.

Editor's comment.

One of the above solvers misunderstood the problem and proved (correctly) a related problem.

86. [1975; 84] *Proposed by Viktors Linis, University of Ottawa.*

Find all rational Pythagorean triples (a,b,c) such that

$$a^2 + b^2 = c^2 \qquad \text{and} \qquad a + b = c^2.$$

Solution by Léo Sauvé, Algonquin College.

Our task is to find all the rational triples (a,b,c) which satisfy the system

$$a^2 + b^2 = c^2, \tag{1}$$
$$a + b = c^2. \tag{2}$$

We will accomplish this in two stages: (a) we will find all rational triples which satisfy (1), and then (b) we will determine which of these solutions of (1) satisfy (2).

(a) It is well-known (see [1], for example) that the general solution in rationals of (1) is

$$a = 2tr, \qquad b = (t^2 - 1)r, \qquad c = (t^2 + 1)r, \tag{3}$$

where t and r are arbitrary rationals. To adequately describe the complete solution of (1), two important facts, which are implicit in (3), must be brought out.

First, observe that since a, b, c are all squared in (1), a specific choice of t and r in (3) yields, not one solution (a,b,c), but eight solutions $(\pm a, \pm b, \pm c)$; in other words, for each separate solution of (1), we must be free to choose the signs of a, b, c independently. Since we do not have that freedom in (2), where $a+b$ must be positive, we must try to describe the solutions of (1) in a manner that is compatible with (2). The sign of a in (3) can be chosen independently of b and c by choosing the sign of t. The sign of b can be chosen by specifying the sign of r. This does not affect the sign of a (because of the freedom of t), but it does affect the sign of c. We can disregard this provided we agree that whenever (a,b,c) is a solution, so is $(a,b,-c)$.

Second, we note that a and b enter symmetrically in (1), so that whenever (a,b,c) is a solution, so is (b,a,c).

To resume, the complete solution of (1) is given by (3) and the condition

$$(a,b,c) \text{ is a solution} \Rightarrow (a,b,-c) \text{ and } (b,a,\pm c) \text{ are solutions.} \tag{4}$$

Note that condition (4) is thoroughly compatible with (2).

(b) If we substitute the values of a, b, c from (3) into (2), we obtain

$$(t^2 + 2t - 1)r = (t^2 + 1)^2 r^2. \tag{5}$$

For $r = 0$, (3) gives the trivial solution $(0,0,0)$. If $r \neq 0$, then (5) yields

$$r = \frac{t^2 + 2t - 1}{(t^2 + 1)^2},$$

and substitution in (3) gives

$$a = \frac{2t(t^2 + 2t - 1)}{(t^2 + 1)^2}, \qquad b = \frac{(t^2 - 1)(t^2 + 2t - 1)}{(t^2 + 1)^2}, \qquad c = \frac{t^2 + 2t - 1}{t^2 + 1}.$$

To wrap it all up, the complete solution of the system (1)-(2) consists of

1) the trivial solution $(0,0,0)$;

2) the nontrivial solutions

$$(a,b,c) = \left(\frac{2t(t^2 + 2t - 1)}{(t^2 + 1)^2}, \frac{(t^2 - 1)(t^2 + 2t - 1)}{(t^2 + 1)^2}, \frac{t^2 + 2t - 1}{t^2 + 1} \right),$$

where t is an arbitrary rational;

3) the additional nontrivial solutions generated by the condition

$$(a,b,c) \text{ is a solution} \Rightarrow (a,b,-c) \text{ and } (b,a,\pm c) \text{ are solutions.}$$

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment.

Not all the solutions received were correct in all respects, and one was quite fragmentary.

REFERENCE

1. Oystein Ore, *Number Theory and its History*, McGraw-Hill, 1948, p. 169.

87. [1975; 84] *Proposed by H.G. Dworschak, Algonquin College.*

(a) If $u_n = x^{2n} + x^n + 1$, for which positive integers n is u_n divisible by u_1?

(b) For which positive integers n does $x + \frac{1}{x} = 1$ imply $x^n + \frac{1}{x^n} = 1$?

Solution of (a) by the proposer.

The zeros of $u_1(x)$ are ω and ω^2, the imaginary cube roots of unity; hence $u_n(x)$ is divisible by $u_1(x)$ if and only if $u_n(\omega)$ and $u_n(\omega^2)$ both vanish. Since

$$u_{3k}(\omega) = 3 \neq 0,$$
$$u_{3k+1}(\omega) = \omega^2 + \omega + 1 = 0, \qquad u_{3k+1}(\omega^2) = \omega + \omega^2 + 1 = 0,$$
$$u_{3k+2}(\omega) = \omega + \omega^2 + 1 = 0, \qquad u_{3k+2}(\omega^2) = \omega^2 + \omega + 1 = 0,$$

we conclude that u_n is divisible by u_1 for all $n = 3k + 1$ and $n = 3k + 2$, where k is a nonnegative integer.

Solution of (b) by F.G.B. Maskell, Algonquin College.

If we set $v_n(x) = x^{2n} - x^n + 1$, the given problem is equivalent to the following: *for which positive integers n is v_n divisible by v_1?*

Since $v_n(x) = u_n(-x)$ whenever n is odd, we conclude that the solutions of (b) consist of the odd solutions of (a), namely, when $k = 2m$,

$$n = 3k + 1 = 6m + 1,$$

and when $k = 2m + 1$,

$$n = 3k + 2 = 6m + 5,$$

where m is a nonnegative integer.

Also solved by Walter Bluger, Department of National Health and Welfare; D.E. Fisher, Algonquin College; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College (also part (a)); Léo Sauvé, Collège Algonquin; and the proposer (also part (b)).

Editor's comment.

Not all these solutions were correct in all respects. Two were wrong in part (a) and one was wrong in part (b).

One solver persists in sending in solutions which consist of little more than the answer, with hardly any indication of how the answer was arrived at.

One solver proposed the following generalization of (a): *if* $u_n = 1 + x^n + \ldots + x^{(k-1)n}$, *then* u_n *is divisible by* u_1 *whenever* $n \not\equiv 0 \pmod{k}$. Unfortunately it is false. A counterexample is not hard to find.

88. [1975; 85] *Proposé par F.G.B. Maskell, Collège Algonquin.*

Evaluer l'intégrale indéfinie

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}}$$

Solution by H.G. Dworschak, Algonquin College.

The integrand being defined and continuous for all $x \neq -1$, the required primitive must be differentiable for all $x \neq -1$, and all steps in the process of arriving at this primitive must be meaningful and true for all $x \neq -1$. This requires that the substitution function used to bring the integral into a more tractable form be defined, differentiable, and monotonic for all x in each of the intervals $(-\infty,-1)$ and $(-1,\infty)$. The function

$$u = \frac{x}{\sqrt[3]{1+x^3}}, \qquad x \neq -1,\ u \neq 1 \tag{1}$$

has all these desirable characteristics.

We easily obtain from (1)

$$du = \frac{dx}{(1+x^3)^{\frac{4}{3}}} \quad \text{and} \quad \frac{1}{1-u^3} = 1+x^3,$$

and so

$$I = \int \frac{dx}{\sqrt[3]{1+x^3}} = \int \frac{(1+x^3)dx}{(1+x^3)^{\frac{4}{3}}} = \int \frac{du}{(1-u)(1+u+u^2)}.$$

The use of partial fractions and other standard integration techniques now yields

$$I = -\frac{1}{3}\ln|u-1| + \frac{1}{6}\ln(u^2+u+1) + \frac{1}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}} + C, \qquad u = \frac{x}{\sqrt[3]{1+x^3}}.$$

Also solved by Léo Sauvé, Algonquin College; and the proposer. One incorrect solution was received.

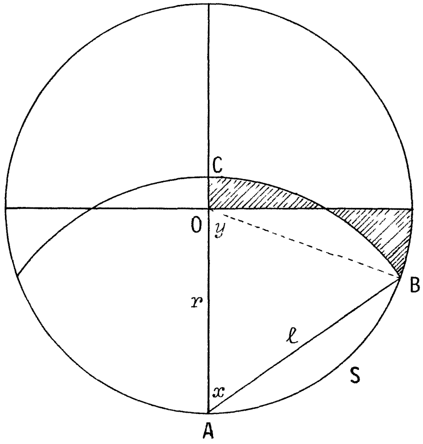
Editor's comment.

The trap to be avoided in this problem was that certain other substitution functions, such as $u = (1+x^{-3})^{\frac{1}{3}}$, made the integration process slightly less painful, but at the cost of validity of the result, since the function mentioned (and the resulting primitive) is not defined at $x = 0$. This is the trap that our lone incorrect solver fell into.

89. [1975; 85] *Proposed by Vince Bradley, Algonquin College and, independently, by Christine Robertson, Canterbury High School.*

A goat is tethered to a point on the circumference of a circular field of radius r by a rope of length ℓ. For what value of ℓ will it be able to graze over exactly half of the field?

I. *Solution adapted from that submitted by G.D. Kaye, Department of National Defence.*



It is clear from the figure that

$$\ell = 2r\cos x,$$

that the area of sector ABC is

$$\tfrac{1}{2}\ell^2 x = r^2 x(1+\cos 2x),$$

and that the area of segment ASB is

$$\tfrac{1}{2}r^2 y - \tfrac{1}{2}\ell r\sin x = \tfrac{1}{2}r^2(\pi - 2x) - \tfrac{1}{2}r^2\sin 2x.$$

Adding sector ABC to segment ASB gives half the grazing area. The total grazing area is therefore

$$2r^2 x(1+\cos 2x) + r^2(\pi - 2x) - r^2\sin 2x = \frac{\pi r^2}{2},$$

from which it follows that acute angle x is a solution of the equation

$$f(x) = 2x\cos 2x - \sin 2x + \frac{\pi}{2} = 0.$$

At this point it is important to get a really good first approximation to the solution. If the figure is drawn carefully to make the two shaded areas appear approximately equal, and angle x is then measured, it turns out to be about $54.5° \approx 0.95$ radian. A second approximation can be found by equating to zero the expansion of $f(0.95+\varepsilon)$ as far as the first degree terms in ε. This yields

$$\varepsilon \approx \frac{1.9\cos 1.9 - \sin 1.9 + \frac{1}{2}\pi}{3.8\sin 1.9} \approx 0.0028,$$

so that $x \approx 0.9528$ and $\ell = 2r\cos x \approx 1.159r$.

A close rational approximation would be $\ell = \dfrac{7r}{6}$.

II. *Comment by H.G. Dworschak, Algonquin College.*

This problem can be found, complete with a solution (different from solution I), on p. 460 of Barnard and Child's *Higher Algebra*, Macmillan, 1949.

Also solved by Leo Sauvé, Algonquin College. Two incorrect solutions were received.

90. [1975; 85] *Proposed by Léo Sauvé, Algonquin College.*

(a) Determine, as a function of the positive integer n, the number of odd binomial coefficients in the expansion of $(a+b)^n$.

(b) Do the same for the number of odd multinomial coefficients in the expansion of $(a_1+a_2+\ldots+a_r)^n$.

Solution by the proposer.

The proof will require the following

LEMMA. If the natural number n is represented in the scale of p by

$$n = a_0p^h + a_1p^{h-1} + \ldots + a_h,$$

where p is prime and

$$a_0 \neq 0, \qquad 0 \le a_i < p, \qquad i = 0,1,\ldots,h,$$

then the index of the highest power of p contained in n! is

$$\frac{n-(a_0+a_1+\ldots+a_h)}{p-1}.$$

The lemma is due to Legendre [1]. More accessible references are [4,9,10,12].

(a) Now we can prove

THEOREM 1. *For any natural number n, the number of odd coefficients in the expansion of $(a+b)^n$ is 2^k, where k is the sum of the digits in the binary representation of n.*

Proof. A typical binomial coefficient is

$$\binom{n}{r}=\frac{n!}{r!s!}, \tag{1}$$

where $0\le r\le n$ and $s=n-r$. Let

$$n=\sum_{i=0}^{h} n_i 2^{h-i}, \qquad n_i=0 \text{ or } 1,\ n_0=1,$$

$$r=\sum_{i=0}^{h} r_i 2^{h-i}, \qquad r_i=0 \text{ or } 1,$$

$$s=\sum_{i=0}^{h} s_i 2^{h-i}, \qquad s_i=0 \text{ or } 1.$$

From the lemma, the index of the highest power of 2 contained in $n!$, $r!$, and $s!$, respectively, is

$$n-\Sigma n_i, \qquad r-\Sigma r_i, \qquad s-\Sigma s_i.$$

Since (1) is an integer, we must have

$$n-\Sigma n_i \ge (r-\Sigma r_i)+(s-\Sigma s_i), \tag{2}$$

and (1) will be odd if and only if equality is attained in (2), that is, if and only if

$$\Sigma n_i = \Sigma r_i + \Sigma s_i. \tag{3}$$

Now, given n, the number of ways in which r can be chosen to satisfy (3) is easily determined by observing that, for $i=0,1,\ldots,h$,

(i) if $n_i=0$, r_i can be chosen in only one way, since we must have $r_i=0$ (and $s_i=0$);

(ii) if $n_i=1$, r_i can be chosen in two ways, since we can have $r_i=0$ or 1 (and $s_i=1$ or 0).

The required number of ways is thus 2^k, where k is the number of 1's (or the sum of the digits) in the binary representation of n.

(b) Since a typical multinomial coefficient is

$$\frac{n!}{s!t!\cdots w!},$$

where $s+t+\ldots+w=n$, a similar approach would show that the necessary and sufficient condition for oddness is

$$\Sigma n_i = \Sigma s_i + \Sigma t_i + \ldots + \Sigma w_i.$$

Continuing the proof in a manner analogous to (a), we soon arrive at the following generalization of Theorem 1:

THEOREM 2. *For all natural numbers r and n, the number of odd coefficients in the expansion of* $(a_1 + a_2 + \ldots + a_r)^n$ *is* r^k, *where k is the sum of the digits in the binary representation of n.*

Comment by the proposer.

Nearly two years ago, I read the following problem in Klambauer [11]: *Show that the number of odd binomial coefficients in any finite expansion is a power of* 2. The problem was followed by an outline of a proof by induction. I wrote to Dr. Klambauer, sending him the proof given above and asking for information about the history of the problem. He replied:

> I encountered the result first when I participated in the W.L. Putnam Mathematical Competition in 1956 and the solution you see sketched in my book is the one I found during the examination session. The *Amer. Math. Monthly* reported on the competition in question a year later and solutions to the problems have also been published in the said journal. The famous German algebraist Ernst Witt considered the problem and devised a sophisticated solution in the early sixties while he was a visiting professor at McMaster University; he also obtained a number of nontrivial extensions, but unfortunately did not seem too impressed with them and never published his findings.

The reference mentioned by Dr. Klambauer is given below as [5]. Other references to this problem or extensions of it are [2,3,6 - 8].

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3. N.J. Fine, Binomial coefficients modulo a prime, *American Mathematical Monthly*, Vol. 54, 1947, pp. 589 -592, especially Theorem 2.

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