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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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AN EXTENSION OF OPPENHEIM'S AREA INEQUALITY FOR TRIANGLES

Ji Chen

The area Δ of a triangle is a well known function of the lengths a, b, c of its sides:

$$\Delta = [s(s-a)(s-b)(s-c)]^{1/2}$$

(where $s = (a+b+c)/2$), or equivalently

$$16\Delta^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4. \tag{1}$$

In [3], A. Oppenheim proved that, if $0 \le p \le 1$, then

$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^p \le \frac{16}{3}\Delta^2(a^p,b^p,c^p) \tag{2}$$

(here $\Delta(x,y,z)$ denotes the area of the triangle with sides x, y, z), which from (1) can be written

$$3^{1-p}(16\Delta^2)^p \le 2b^{2p}c^{2p} + 2c^{2p}a^{2p} + 2a^{2p}b^{2p} - a^{4p} - b^{4p} - c^{4p}, \tag{3}$$

$0 \le p \le 1$. Equality holds (for $0 < p < 1$) if and only if $a = b = c$.

In this paper, we will prove that the *reverse* of inequality (3) holds for all other values of p.

Theorem. For $p < 0$ or $p > 1$,

$$3^{1-p}(16\Delta^2)^p \ge 2b^{2p}c^{2p} + 2c^{2p}a^{2p} + 2a^{2p}b^{2p} - a^{4p} - b^{4p} - c^{4p}; \tag{4}$$

equivalently, if a^p, b^p, c^p are the sides of a triangle,

$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^p \ge \frac{16}{3}\Delta^2(a^p,b^p,c^p). \tag{5}$$

Proof. If a^p, b^p, c^p are not the edges of a triangle, we have

$$(a^p + b^p + c^p)(b^p + c^p - a^p)(c^p + a^p - b^p)(a^p + b^p - c^p) \le 0,$$

and (4) is true obviously. Thus we suppose a^p, b^p, c^p are the edges of a triangle, and show (5). Also, equality holds in (5) when $a = b = c$, so we suppose that a, b, c are not all equal.

When $p > 1$, then $0 < 1/p < 1$. Using (2) on the triangle with edges a^p, b^p, c^p, we have

$$\left[\frac{16}{3}\Delta^2(a^p,b^p,c^p)\right]^{1/p} < \frac{16}{3}\Delta^2(a,b,c),$$

which is the same as (5).

Assume $p < 0$. We define a function f on $(-\infty,+\infty)$ by

$$f(x) = \left[\frac{b^{2x}c^{2x} + c^{2x}a^{2x} + a^{2x}b^{2x}}{3}\right]^{1/x} - \left[\frac{a^{4x} + b^{4x} + c^{4x} + 3(16\Delta^2/3)^x}{6}\right]^{1/x}, \quad x \ne 0,$$

$$f(0) = (abc)^{4/3} - (abc)^{2/3}\left[\frac{16\Delta^2}{3}\right]^{1/2}.$$

Then f is continuous at $x = 0$ (take logarithms and apply L'Hôpital's rule).

Since we are assuming that a, b, c are not all equal, from the well known inequality

$$\Delta < \frac{\sqrt{3}}{4}(abc)^{2/3}$$

(see item 4.14 of [2]) we know $f(0) > 0$. Thus we can find $q < 0$, $q > p$ such that $f(q) > 0$, i.e.

$$\left[\frac{b^{2q}c^{2q} + c^{2q}a^{2q} + a^{2q}b^{2q}}{3}\right]^{1/q} > \left[\frac{a^{4q} + b^{4q} + c^{4q} + 3(16\Delta^2/3)^q}{6}\right]^{1/q},$$

or

$$2b^{2q}c^{2q} + 2c^{2q}a^{2q} + 2a^{2q}b^{2q} < a^{4q} + b^{4q} + c^{4q} + 3(16\Delta^2/3)^q,$$

or, from (1),

$$\frac{16}{3}\Delta^2(a^q,b^q,c^q) < \left[\frac{16}{3}\Delta^2(a,b,c)\right]^q. \tag{6}$$

For $p < q < 0$, $0 < q/p < 1$. Using (2) on the triangle with edges a^p, b^p, c^p we obtain

$$\left[\frac{16}{3}\Delta^2(a^p,b^p,c^p)\right]^{q/p} < \frac{16}{3}\Delta^2(a^q,b^q,c^q). \tag{7}$$

From (6) and (7) we have

$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^q > \left[\frac{16}{3}\Delta^2(a^p,b^p,c^p)\right]^{q/p}$$

so that, since $q/p > 0$,

$$\left[\frac{16}{3}\Delta^2(a,b,c)\right]^p > \frac{16}{3}\Delta^2(a^p,b^p,c^p),$$

which is (5). □

As a special case, take $p = -1/2$ in (4); then we have the inequality

$$\frac{3\sqrt{3}}{4\Delta} \geq \frac{2}{bc} + \frac{2}{ca} + \frac{2}{ab} - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2}.$$

This could also be written

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{3\sqrt{3}}{4\Delta} + \left[\frac{1}{b} - \frac{1}{c}\right]^2 + \left[\frac{1}{c} - \frac{1}{a}\right]^2 + \left[\frac{1}{a} - \frac{1}{b}\right]^2.$$

For another application of the theorem, we extend the following inequality of S. Beatty [1]:

$$\Delta^2 \geq \frac{(K - H)(3K - 5H)}{12}, \tag{8}$$

where

$$H = (a^2 + b^2 + c^2)/2, \quad K = bc + ca + ab$$

(see also item 4.18 of [2]). Suppose that $p \geq 1$ or $p \leq 0$ and that the triples (a,b,c) and (a^p,b^p,c^p) are each the sides of a triangle. Then from (8) applied to the triangle with sides a^p, b^p, c^p,

$$\Delta^2(a^p,b^p,c^p) \geq (K_p - H_p)(3K_p - 5H_p),$$

where

$$H_p = \frac{a^{2p} + b^{2p} + c^{2p}}{2}, \quad K_p = b^pc^p + c^pa^p + a^pb^p.$$

By using (5) we thus get

$$\Delta^{2p} \geq 2^{2-4p}3^{p-2}(K_p - H_p)(3K_p - 5H_p).$$

References:

[1] S. Beatty, Upper and lower estimates for the area of a triangle, *Trans. Roy. Soc. Canada* III (3), 48 (1954), 1–5.

[2] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

[3] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, 496 (1974), 257–263

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THE OLYMPIAD CORNER
No. 101
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

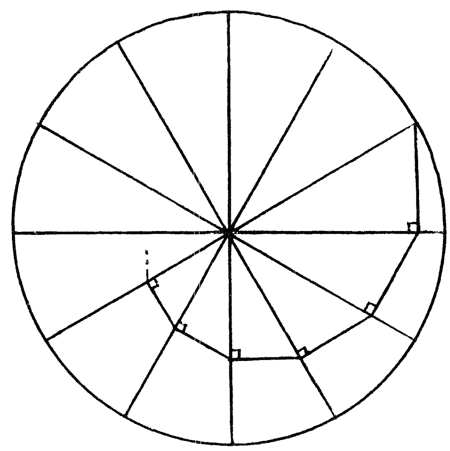
This column marks the second anniversary of my taking over the Corner. In spite of the odd error, the occasional misquoted problem, and misunderstanding of solutions, I hope that the resulting column provides a forum for discussion of the Olympiads, and is a source of good problems for those who coach others (or who are preparing themselves) for mathematics contests. For the rest of us the problems provide the pleasure of a good teaser. I'm grateful to all those who have sent in problem sets, comments, and solutions. Those whose contributions were used in Volume 14 include Beno Arbel, Francisco Bellot, Curtis Cooper, Ed Doolittle, George Evagelopoulos, Richard Gibbs, Douglass Grant, J.T. Groenman, Branko Grunbaum, R.K. Guy, F.D. Hammer, Walther Janous, Murray Klamkin, A.H. Lachlan, Andy Liu, Alan Mekler, John Morvay, V.N. Murty, Richard Nowakowski, Peter O'Halloran, Bob Prielipp, Josef Rifa i Coma, Daniel Ropp, Leo Schneider, M. Selby, Robert E. Shafer, Zun Shan, Bruce Shawyer, D.J. Smeenk, Dan Sokolowsky, D. Vathis, David Vaughan, G.R. Veldkamp, and Edward T.H. Wang. Keep up the contributions — without your help this column could not exist! (Looking over this list I find surnames beginning with each letter of the alphabet except F, I, Q, T, U, X, Y, Z. Perhaps this year we'll hit every letter.)

* * *

This month's Olympiad problems begin with the final rounds of the *Flanders Mathematics Olympiad*, 1985–86 and 1986–87. Thanks go to Bruce Shawyer who forwarded them to me. In each exam, students were allowed three hours to solve the problems. Some of these problems will be familiar to readers, and there are obvious possibilities for generalization. Please send in your "nice" solutions.

FLANDERS MATHEMATICS OLYMPIAD 1985–86 (final round)

1. A circle with radius R is divided into twelve equal parts. The twelve dividing points are connected with the centre of the circle, producing twelve rays. Starting from one of the dividing points a segment is drawn perpendicular to the next ray in the clockwise sense; from the foot of this perpendicular another perpendicular segment is drawn to the next ray, and the process is continued *ad infinitum*. What is the limit of the sum of these segments (in terms of R)?



2. Prove that, for every natural number n, we have

$$n! \le \left(\frac{n+1}{2}\right)^n .$$

3. A sequence of numbers $\{a_k\}$ is defined as follows:

$$a_0 = 0,$$
$$a_{k+1} = 3a_k + 1, \qquad k \ge 0.$$

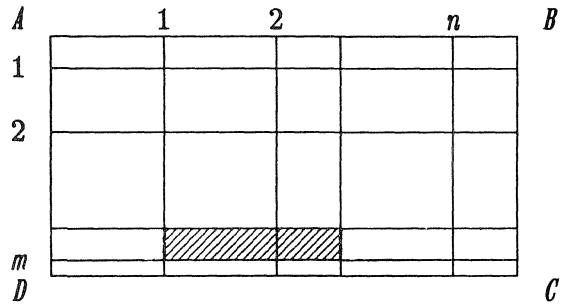
Show that a_{155} is divisible by 11.

4. To put a marble with radius 1 cm in a cube it is obvious that the cube must have an edge with at least a length of 2 cm. What is the minimum length of the edge of a cube which can contain two marbles of radius 1 cm? (Prove your answer.)

*

FLANDERS MATHEMATICS OLYMPIAD 1986–87 (final round)

1. A rectangle $ABCD$ is given. On the side AB, n different points are chosen strictly between A and B. Similarly, m different points are chosen on the side AD. Lines are drawn from the points parallel to the sides. How many rectangles are formed in this way? (One possibility is shown in the figure.)



2. Two parallel lines a and b meet two other lines c and d. Let A and A' be the points of intersection of a with c and d, respectively. Let B and B' be the points of intersection of b with c and d, respectively. If X is the midpoint of the line segment AA' and Y is the midpoint of the segment BB', prove that

$$|XY| \leq \frac{|AB| + |A'B'|}{2}.$$

($|XY|$ represents the length of the line segment XY.)

3. Determine all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that

$$(f(x))^3 = \frac{-x}{12}[x^2 + 7xf(x) + 16(f(x))^2].$$

4. Prove that, for every $r \in \mathbb{R}$ with $r > 1$

$$\lim_{n\to\infty}\left(\frac{1^r+2^r+\cdots+(n-1)^r+n^r+(n-1)^r+\cdots+2^r+1^r}{n^2}\right) = +\infty.$$

What is the value of the limit when $r = 1$?

* * *

Next we give four problems from the *Arany Daniel Competition 1987, Junior Level* (age 15), from Hungary. These were collected by Gy. Karolyi and J. Pataki, and forwarded to me by Bruce Shawyer.

1. The real numbers x, y, z satisfy the following equation:

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1.$$

Prove that two of the three fractions have the value 1.

2. The median lines of a convex quadrilateral divide it into four smaller ones. Prove that the sum of the areas of two of the quadrilaterals with no common side equals the sum of the areas of the other pair of such quadrilaterals.

3. Choose n points on a circle and label them with the numbers $1,2,\ldots,n$. Say that two non–neighbouring points A and B are *connectable* if the points on at least one of the two arcs containing A and B are all labelled with numbers that are less than

those for A and B. Prove that the number of connectable pairs of points is $n-3$.

4. Let four distinct points be given in space. Determine all the planes having the same distance from each of the points.

* * *

Now we return to problems from the 1987 I.M.O. (Havana) given in the October 1987 number of the Corner.

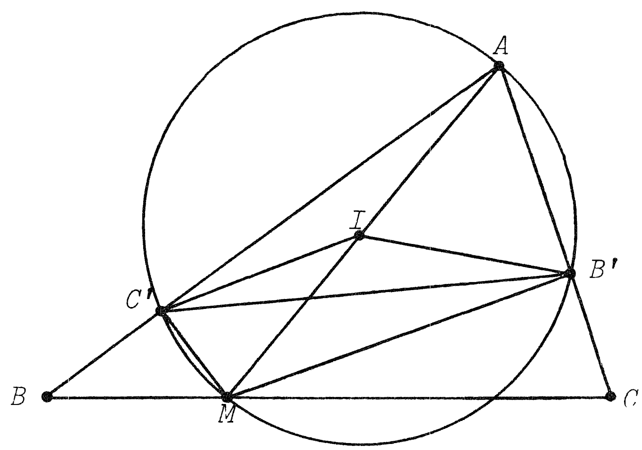
France 2. [1987: 246]

Let ABC be a triangle. For each point M of the segment BC denote by B' and C' the orthogonal projections of M on the lines AC and AB, respectively. Determine those points M for which the length of $B'C'$ is minimum.

Solutions independently by George Evagelopoulos, law student, Athens, Greece, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The four points A, C', M and B' lie on a circle with AM as a diameter. Using the law of sines it follows from this that

$$|B'C'| = |AM|\sin A.$$



[Let I be the center of the circle. In triangle $IC'B'$ we have

$$\angle C'IB' = 2A,$$

$$\angle IC'B' = \angle IB'C' = 90^\circ - A$$

and so

$$|B'C'| = \frac{|IB'|}{\sin(90^\circ - A)}\sin 2A = \frac{|AM|}{2}\cdot\frac{1}{\cos A}\cdot 2\sin A\cos A.]$$

But this means that the length of $B'C'$ is minimized where AM is as short as possible. This occurs when AM is an altitude, if angles B and C are both acute. If B (or C) is obtuse the point M should be chosen to be B (C, respectively).

Great Britain 1. [1987: 246]

Prove that if the equation

$$x^4 + ax^3 + bx + c = 0$$

has all its roots real then $ab \le 0$.

Solution I [*without the calculus*] *by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $p(x) = x^4 + ax^3 + bx + c = 0$ and let $V(p(x))$ denote the number of sign changes (variations) among the coefficients of $p(x)$. We use Descartes' rule of signs to prove the assertion by contradiction. We denote by $r^+(p)$, $r^0(p)$, and $r^-(p)$ the number of positive, zero, and negative real roots of $p(x)$, respectively. Suppose $ab > 0$. Then there are two cases:

Case (i). $a > 0$ and $b > 0$. There are three subcases.

If $c > 0$, then $V(p(x)) = 0$ and $V(p(-x)) = 2$. Thus

$$r^+(p) = 0,\ r^0(p) = 0 \text{ and } r^-(p) = 2 \text{ or } 0.$$

If $c = 0$, then $V(p(x)) = 0$, and $V(p(-x)) = 1$. Thus

$$r^+(p) = 0,\ r^0(p) = 1 \text{ and } r^-(p) = 1.$$

If $c < 0$, then $V(p(x)) = 1$ and $V(p(-x)) = 1$. Thus

$$r^+(p) = 1,\ r^0(p) = 0 \text{ and } r^-(p) = 1.$$

Case (ii). $a < 0$ and $b < 0$. We simply apply the conclusion of case (i) to

$$q(x) = p(-x) = x^4 + Ax^3 + Bx + C$$

where $A = -a$, $B = -b$, $C = c$.

In both cases $p(x)$ can have at most two real roots, a contradiction.

Solution II [*using Rolle's Theorem*] *by Murray S. Klamkin, University of Alberta.*

More generally, assume that the equation

$$x^n + a_1x^{n-1} + a_3x^{n-3} + a_4x^{n-4} + \cdots + a_n = 0 \tag{1}$$

has all of its roots real. We show that $a_1a_3 \leq 0$.

It follows from the above assumption that all the derivatives of the polynomial (1) have all real zeros. In particular, the equation

$$x^3 + a_1'x^2 + a_3' = 0,$$

where a_1' and a_3' are positive (real) multiples of a_1 and a_3, respectively, has real roots, r_1, r_2, r_3, say. Then

$$-a_1' = r_1 + r_2 + r_3$$

$$0 = r_2r_3 + r_3r_1 + r_1r_2$$

$$-a_3' = r_1r_2r_3.$$

If the three roots are 0, then trivially $a_1'a_3' = 0$. If not, there is some pair of roots whose sum is not zero. Suppose it is $r_2 + r_3$. Then

$$r_1 = \frac{-r_2r_3}{r_2 + r_3}.$$

Hence

$$a_1'a_3' = (r_1 + r_2 + r_3)r_1r_2r_3 = \left(r_2 + r_3 - \frac{r_2r_3}{r_2 + r_3}\right)\left(-\frac{(r_2r_3)^2}{r_2 + r_3}\right).$$

$$= -\left(\frac{r_2r_3}{r_2 + r_3}\right)^2(r_2^2 + r_2r_3 + r_3^2) \leq 0.$$

Hence $a_1a_3 \leq 0$.

[*Editor's note*: This problem was also solved by George Evagelopoulos, law student, Athens, Greece, and by M.A. Selby, Department of Mathematics, University of Windsor. They used methods similar to the above but did not generalize.]

<u>Great Britain 2</u>. [1987: 246]

Numbers $d(n,m)$, where n, m are integers and $0 \le m \le n$, are defined by

$$d(n,0) = d(n,n) = 1 \qquad \text{for all } \ n \ge 0$$

and

$$m \cdot d(n,m) = m \cdot d(n-1,m) + (2n-m) \cdot d(n-1,m-1)$$

for $0 < m < n$. Prove that all the $d(n,m)$ are integers.

Solutions by George Evagelopoulos, law student, Athens, Greece, R.K. Guy, Department of Mathematics and Statistics, The University of Calgary, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University.

The binomial coefficients $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ are integers, since they are the number of ways of choosing m things from n. Thus $\binom{n}{m}^2$ is an integer. We show by induction that

$$d(n,m) = \binom{n}{m}^2.$$

We assume $0 \le m \le n$, and recall that $0! = 1$. Now

$$d(1,0) = 1 = \binom{1}{0}^2 = \binom{1}{1}^2 = d(1,1).$$

So assume $0 < k$, and

$$d(k-1,m) = \binom{k-1}{m}^2$$

for $0 \le m \le k-1$. Then for $0 < m < k$,

$$\begin{aligned} m \cdot d(k,m) &= m\binom{k-1}{m}^2 + (2k-m)\binom{k-1}{m-1}^2 \\ &= \left\{\frac{(k-1)!}{m!(k-m)!}\right\}^2 [m(k-m)^2 + (2k-m)m^2] \\ &= m\binom{k}{m}^2. \end{aligned}$$

Thus

$$d(k,m) = \binom{k}{m}^2.$$

As

$$d(k,0) = 1 = \binom{k}{0}^2 = \binom{k}{k}^2 = d(k,k)$$

this completes the induction step and we conclude $d(n,m) = \binom{n}{m}^2$ for $0 \le m \le n$.

Editor's note. R.K. Guy asks "Is there a nicer combinatorial proof?"

__Great Britain 3__. [1987: 247]

Find, with proof, the smallest real number c with the following property: for every sequence $\{X_i\}$ of positive real numbers such that

$$X_1 + X_2 + \cdots + X_n \le X_{n+1} \qquad \text{for } n = 1,2,3,\ldots$$

we have

$$\sqrt{X_1} + \sqrt{X_2} + \cdots + \sqrt{X_n} \le c\sqrt{X_1 + X_2 + \cdots + X_n}$$

for $n = 1,2,3,\ldots$. [c is to be independent of the X_i and independent of n.]

Solutions by George Evagelopoulos, law student, Athens, Greece, by Murray S. Klamkin, Department of Mathematics, The University of Alberta, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The sequence $X_i = 2^{i-1}$ satisfies

$$X_1 + X_2 + \cdots + X_n \le X_{n+1} \tag{1}$$

since

$$1 + 2 + \cdots + 2^{n-1} = 2^n - 1.$$

Thus we must have

$$1 + 2^{1/2} + 2^{2/2} + \cdots + 2^{(n-1)/2} = \frac{2^{n/2} - 1}{2^{1/2} - 1} \le c(2^n - 1)^{1/2}.$$

This gives

$$c \ge \frac{1}{\sqrt{2} - 1}\cdot\frac{2^{n/2} - 1}{(2^n - 1)^{1/2}}$$

for all n. Since

$$\lim_{n\to\infty} \frac{2^{n/2} - 1}{(2^n - 1)^{1/2}} = \lim_{n\to\infty} \frac{1 - 2^{-n/2}}{\sqrt{1 - 2^{-n}}} = 1$$

we therefore require

$$c \ge \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1.$$

We show $c = \sqrt{2} + 1$ does have the required property by induction on n.

Let $\{X_i\}$ satisfy (1) for all n. Certainly $\sqrt{X_1} \le (\sqrt{2} + 1)\sqrt{X_1}$ holds, so the required inequality holds with $n = 1$. Now suppose

$$\sqrt{X_1} + \cdots + \sqrt{X_n} \le (\sqrt{2} + 1)\sqrt{X_1 + \cdots + X_n}\,.$$

Then

$$\sqrt{X_1} + \cdots + \sqrt{X_{n+1}} \le (\sqrt{2} + 1)\sqrt{Y_n} + \sqrt{X_{n+1}}\,,$$

where

$$Y_n = X_1 + \cdots + X_n.$$

We shall show

$$(\sqrt{2} + 1)\sqrt{Y_n} + \sqrt{X_{n+1}} \le (\sqrt{2} + 1)\sqrt{Y_n + X_{n+1}}\,,$$

by arguing that, more generally, if $0 < A \le B$ we have

$$(\sqrt{2}+1)\sqrt{A}+\sqrt{B} \le (\sqrt{2}+1)\sqrt{A+B}.$$

Squaring, we get the equivalent inequality

$$(3+2\sqrt{2})A + 2(\sqrt{2}+1)\sqrt{A}\sqrt{B} + B \le (3+2\sqrt{2})(A+B),$$

that is,

$$2(\sqrt{2}+1)\sqrt{A}\sqrt{B} \le 2(\sqrt{2}+1)B,$$

which is clear for $0 < A \le B$ (and strict unless $A = B$). This completes the induction step and the proof.

Greece 1. [1987: 247]

Consider the regular 1987–gon $A_1A_2...A_{1987}$ with center O. Show that the sum of vectors belonging to any proper subset of $M = \{OA_j\colon j = 1,2,...,1987\}$ is nonzero.

Solutions by George Evagelopoulos, law student, Athens, Greece, by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The number 1987 can be replaced by any prime p with the same result. We can represent the given set of vectors by the p complex numbers

$$1,\omega,\omega^2,...,\omega^{p-1} \tag{1}$$

where $\omega = e^{2\pi i/p}$ satisfies the equation

$$1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0. \tag{2}$$

It is also a known result (which follows by using Eisenstein's criterion) that the cyclotomic polynomial $f(x) = 1 + x + \cdots + x^{p-1}$ is irreducible over the reals, i.e. there is no real polynomial of degree less than $p-1$ which is satisfied by ω.

Our proof is indirect. Assume that there is a proper subset of the ω^j's whose sum is zero. It follows from (2) that the sum of the remaining ω^j's is also zero. This gives two real polynomial equations in ω (after dividing out the lowest power of ω in one of them)

$$1 + \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_r} = 0$$
$$1 + \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_s} = 0$$

where the α's and β's are natural numbers less than p. One of these equations must be of degree less than $p-1$ which contradicts the fact that f is irreducible. Thus there is no proper subset of the ω^j's whose sum is zero.

Editor's note. Murray Klamkin adds the following comment.

There is an extension of this result to n dimensions but it is less sophisticated. Consider the $n+1$ vectors from the centroid of a regular n-dimensional simplex to its vertices $A_0,...,A_n$. The sum of these $n+1$ vectors is zero. However, the sum of any subset of these vectors

$$\overrightarrow{GA_0} + \overrightarrow{GA_1} + \cdots + \overrightarrow{GA_{r-1}} = r\overrightarrow{GG_r} \neq 0 \qquad (r < n+1)$$

where G and G_r are the centroids of the given n-dimensional simplex and the $(r-1)$-

dimensional simplex with vertices $A_0, A_1, \ldots, A_{r-1}$, respectively.

Greece 2. [1987: 247]

Solve the equation

$$28^x = 19^y + 87^z$$

where x, y, z are integers.

Solution by Dave McDonald, Crimson Elk, Alberta.

Suppose (x,y,z) is a solution.

(i) None of x, y, z are negative, since otherwise there would be a prime (respectively 2, 19, 3) in the denominator of one term which could not be matched by another.

(ii) From (i), $y \geq 0$, $z \geq 0$ imply $x > 0$ and it is easy to see that $x \neq 1$, so $x \geq 2$.

(iii) From (ii), $0 \equiv 3^y + 7^z$ mod 16. Now $7^z \equiv 7$ or 1 mod 16 according as z is odd or even, respectively, while $3^y \equiv 1, 3, 9$ or 11 mod 16, according as $y \equiv 0, 1, 2$ or 3 mod 4. So z is odd and $y \equiv 2$ mod 4.

(iv) Working modulo 9, $1 \equiv 1 + (-3)^z$, so $z \geq 2$, and since z is odd, $z \geq 3$.

(v) By (iv), $1 \equiv (-8)^y$ mod 27, so 3 divides y, and by (iii), $y \equiv 6$ mod 12.

(vi) Working modulo 7, $0 \equiv (-2)^y + 3^z$. By (v), y is a multiple of 6, so $3^z \equiv -1$ mod 7. This gives $z \equiv 3$ mod 6.

(vii) Now, modulo 13, $2^x \equiv 6^y + (-4)^z$. By (v), y is a multiple of 6, so $6^y \equiv -1$ mod 13. By (vi), z is a multiple of 3, so $(-4)^z \equiv 1$ mod 13. This gives $2^x \equiv 0$ mod 13, a contradiction!

Thus there are *no* integer solutions of the equation.

Editor's note. A second correct, but somewhat more involved, solution was submitted by George Evagelopoulos, law student, Athens, Greece.

Hungary 1. [1987: 247]

Does there exist a set M in the usual Euclidean space such that for any plane σ, the intersection $M \cap \sigma$ is finite and non-empty?

Solution by George Evagelopoulos, law student, Athens, Greece, and also by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The answer is yes. Consider the curve

$$M = \{(t,t^3,t^5)\colon t \in \mathbb{R}\}.$$

If the plane σ has equation $Ax + By + Cz + D = 0$ (not all of A, B, C zero) then the points of intersection are given by the solutions of $At + Bt^3 + Ct^5 + D = 0$ which is a polynomial of odd degree. Thus there is at least one and there are only finitely many solutions so $M \cap \sigma$ is finite and non-empty.

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Next month we continue with solutions to these I.M.O. problems. The Olympiad season is fast approaching. Please remember to collect the Olympiads available to you and send them in to me for use in the Corner.

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PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

<u>1401</u>. *Proposed by P. Penning, Delft, The Netherlands.*

Given are a circle C and two straight lines l and m in the plane of C that intersect in a point S inside C. Find the tangent(s) to C intersecting l and m in points P and Q so that the perimeter of ΔSPQ is a minimum.

<u>1402</u>. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let M be an interior point of the triangle $A_1A_2A_3$ and B_1, B_2, B_3 the feet of the perpendiculars from M to sides A_2A_3, A_3A_1, A_1A_2 respectively. Put $r_i = B_iM$, $i = 1,2,3$. R' is the circumradius of $\Delta B_1B_2B_3$, and R, r the circumradius and inradius of $\Delta A_1A_2A_3$. Prove that

$$R'Rr \geq 2r_1r_2r_3.$$

<u>1403</u>*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $n \geq 2$, prove or disprove that

$$1 < \frac{x_1 + \cdots + x_n}{n} \leq 2$$

for all natural numbers $x_1,x_2,\ldots,x_n$ satisfying

$$x_1 + x_2 + \cdots + x_n = x_1\cdot x_2\cdot\ldots\cdot x_n .$$

1404. *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let ABC be a triangle with circumradius R and inradius ρ. A theorem of Poncelet states that there are an infinity of triangles having the same circumcircle and the same incircle as ΔABC.

(a) Show that the orthocenters of these triangles lie on a circle.

(b) If $R = 4\rho$, what can be said about the locus of the centers of the nine-point circles of these triangles?

1405. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

Two distinct congruent n–gons P and P' are inscribed in a noncircular ellipse E. Prove or disprove that if $n > 4$, P' must be obtainable from P by a reflection across the axes or center of E. (For the cases $n = 3$ and 4 see [1988: 131, 139].)

1406. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

If $0 < \theta < \pi$, prove without calculus that

$$\cot \theta/4 - \cot \theta > 2.$$

1407. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Given a rectangle $ABCD$ with $AB = CD > AD = BC$, construct points X, Y on CD between C and D such that $AX = XY = YB$.

1408. *Proposed by Jordi Dou, Barcelona, Spain.*

Given the equilateral triangle ABC, find all positive real numbers r for which there is a point $P(r)$ such that

$$\frac{PA}{1} = \frac{PB}{r} = \frac{PC}{r^2},$$

and describe the locus of $P(r)$.

1409. *Proposed by Shailesh Shirali, Rishi Valley School, India.*

Show that

$$\frac{n}{n+1} + \frac{2n(n-1)}{(n+1)(n+2)} + \frac{3n(n-1)(n-2)}{(n+1)(n+2)(n+3)} + \cdots = \frac{n}{2}$$

for all positive integers n. What if $n \geq 0$ is not an integer?

1410*. *Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.*

Given is a triangle with circumcentre O and circumradius R. Interior points P, P' are isogonal conjugates, and r_1, r_2, r_3 are the distances from P to the sides of the triangle. Prove that

$$(R^2 - \overline{OP}^2)^2(R^2 - \overline{OP'}^2) = 8r_1r_2r_3R^3.$$

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1199*. [1986: 283; 1988: 87] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

Prove that for acute triangles,

$$s^2 \le \frac{27R^2}{27R^2 - 8r^2}(2R + r)^2,$$

where s, r, R are the semiperimeter, inradius, and circumradius, respectively.

II. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let h_1, h_2, h_3 be the altitudes of an acute triangle and d_1, d_2, d_3 the distances from the circumcenter to the sides. Then since

$$\prod_{i=1}^{3} h_i = \frac{2s^2r^2}{R} \quad \text{and} \quad \prod_{i=1}^{3} d_i = \frac{R}{4}[s^2 - (2R + r)^2],$$

the above inequality has a remarkable interpretation, namely that

$$\prod_{i=1}^{3} \frac{h_i}{3} \ge \prod_{i=1}^{3} d_i ,$$

i.e.

$$\text{geometric mean}\left\{\frac{h_1}{3},\frac{h_2}{3},\frac{h_3}{3}\right\} \ge \text{geometric mean}\ \{d_1,d_2,d_3\}.$$

On the other hand it is known that

$$\text{arithmetic mean}\left\{\frac{h_1}{3},\frac{h_2}{3},\frac{h_3}{3}\right\} \le \text{arithmetic mean}\ \{d_1,d_2,d_3\}$$

(see e.g. item 12.3 of Bottema et al, *Geometric Inequalities*). Thus one could pose the problem: *find all exponents $t \neq 0$ such that*

$$M_t\left(\frac{h_1}{3},\frac{h_2}{3},\frac{h_3}{3}\right) \ge M_t(d_1,d_2,d_3)$$

holds for all acute triangles, where

$$M_t(u,v,w) = \left(\frac{u^t + v^t + w^t}{3}\right)^{1/t}.$$

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1292*. [1987: 320] *Proposed by Jack Garfunkel, Flushing, N.Y.*

It has been shown (see *Crux* 1083 [1987: 96]) that if A, B, C are the angles of a triangle,

$$\frac{2}{\sqrt{3}}\sum \sin A \le \sum \cos\left[\frac{B-C}{2}\right] \le \frac{2}{\sqrt{3}}\sum \cos\frac{A}{2},$$

where the sums are cyclic. Prove that

$$\sum \cos\left[\frac{B-C}{2}\right] \le \frac{1}{\sqrt{3}}\left[\sum \sin A + \sum \cos\frac{A}{2}\right],$$

which if true would imply the right hand inequality above.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Proceeding as in the solution of *Crux* 1083 [1987: 96] we put

$$\alpha = \frac{\pi - A}{2}, \text{ etc.}$$

so that α, β, γ are the angles of an acute triangle. Then, using the relations given in *Crux* [1982: 67–68],

$$\sum \cos\frac{B-C}{2} = \sum \cos(\beta-\gamma) = \frac{x^2+y^2+2x-2}{2},$$

$$\sum \cos\frac{A}{2} = \sum \sin\alpha = y,$$

$$\sum \sin A = \sum \sin 2\alpha = 2xy,$$

where $x = r/R$, $y = s/R$ (r, R, s being the inradius, circumradius, and semiperimeter of the $\alpha-\beta-\gamma$ triangle). The claimed inequality now reads

$$\frac{x^2+y^2+2x-2}{2} \le \frac{y+2xy}{\sqrt{3}},$$

i.e.

$$y^2 - \frac{2}{\sqrt{3}}y(1+2x) + x^2 + 2x - 2 \le 0, \tag{1}$$

where $0 < x \le 1/2$ and $0 < y \le 3\sqrt{3}/2$. In order to show (1) we have to prove that $y_1 \le y \le y_2$, where

$$y_1 = \frac{1}{\sqrt{3}}(1 + 2x - \sqrt{x^2 - 2x + 7})$$

and

$$y_2 = \frac{1}{\sqrt{3}}(1 + 2x + \sqrt{x^2 - 2x + 7}).$$

We first show $y_1 < 0$. Indeed,

$$(1+2x)^2 = 4x^2 + 4x + 1 < x^2 - 2x + 7$$

is equivalent to

$$x^2 + 2x < 2,$$

which is true for $0 < x \le 1/2$.

Hence we have to show $y \le y_2$, i.e.

$$y\sqrt{3} \le 1 + 2x + \sqrt{x^2 - 2x + 7}. \tag{2}$$

From item 5.4 of Bottema et al, *Geometric Inequalities* we take the inequality

$$y \leq 2 + (3\sqrt{3} - 4)x,$$

and thus inequality (2) will follow if we prove

$$2\sqrt{3} + (9 - 4\sqrt{3})x \leq 1 + 2x + \sqrt{x^2 - 2x + 7},$$

i.e.

$$\ell(x) := 2\sqrt{3} - 1 + (7 - 4\sqrt{3})x \leq \sqrt{x^2 - 2x + 7} =: r(x). \qquad (4)$$

Now $\ell(x)$ increases, whereas $r(x)$ decreases for $0 < x \leq 1/2$. As furthermore

$$\ell(1/2) = 5/2 = r(1/2),$$

inequality (4) follows.

* * *

<u>**1293**</u>. [1987: 320] *Proposed by Steve Maurer, Swarthmore College, Swarthmore, Pennsylvania and Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Solve the following "twin" problems (in both problems, O is the center of the circle and $OA \perp AB$).

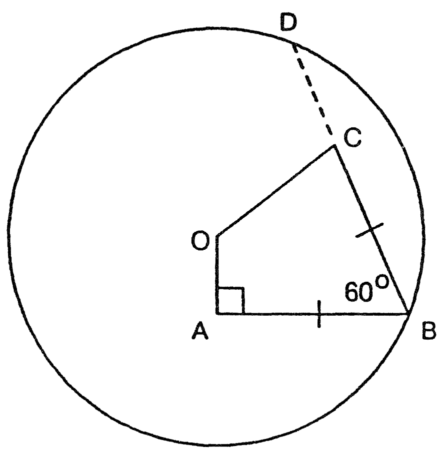

Figure (a)

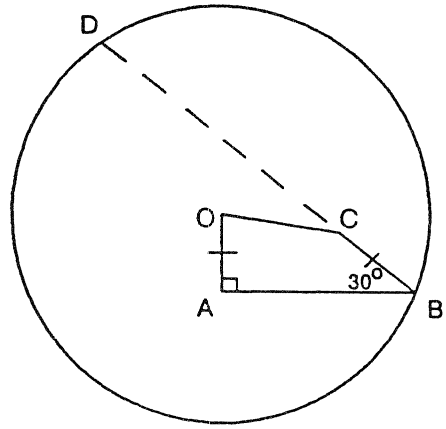

Figure (b)

(a) In Figure (a), $AB = BC$ and $\angle ABC = 60°$. Prove $CD = OA\sqrt{3}$.

(b) In Figure (b), $OA = BC$ and $\angle ABC = 30°$. Prove $CD = AB\sqrt{3}$.

Solution by Jordi Dou, Barcelona, Spain.

(a) Let E be on line AB such that ΔEBD is equilateral. Then $CD = EA$ and EO (extended) is the perpendicular bisector of BD. Thus $\angle OEA = 30°$, so $CD = EA = OA\sqrt{3}$.

(b) Let E be the other intersection of AB with the circle, and let F be on BD so that $EF \perp BD$. Then $EF = EB/2 = AB$. Since $\angle EBD = 30°$, we have $\angle EOD = 60°$ so that $ED = EO = OB$. Therefore $DF = OA = CB$, and thus $CD = BF = EF\sqrt{3} = AB\sqrt{3}$.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; S.C. CHAN, Singapore; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C.

FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; GEORGE TSINTSIFAS, Thessaloniki, Greece; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposers.

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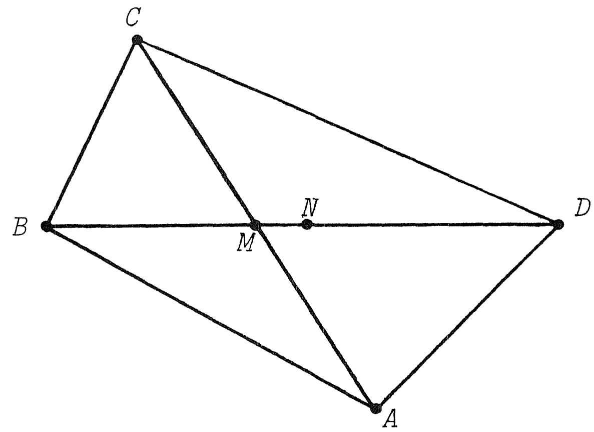
1294. [1987: 320] *Proposed by P. Penning, Delft, The Netherlands.*

Find a necessary and sufficient condition on a convex quadrangle $ABCD$ in order that there exist a point P (in the same plane as $ABCD$) such that the areas of the triangles PAB, PBC, PCD, PDA are equal.



Solution by Jordi Dou, Barcelona, Spain.

The simplest condition I have found is that *the midpoint of one diagonal lies on the other.* The point P will then be the midpoint of the other diagonal. Convexity is not necessary.

If M and N are the midpoints of AC and BD respectively, the locus of points P such that area ΔPAB = area ΔPBC is the line BM. This and analogous results show that if the areas of all four triangles PAB, PBC, PCD, PDA are to be equal, P must lie on lines BM and DM, and also on lines CN and AN. Thus either M lies on BD or N lies on AC.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. The conditions found appear to be more or less the same.

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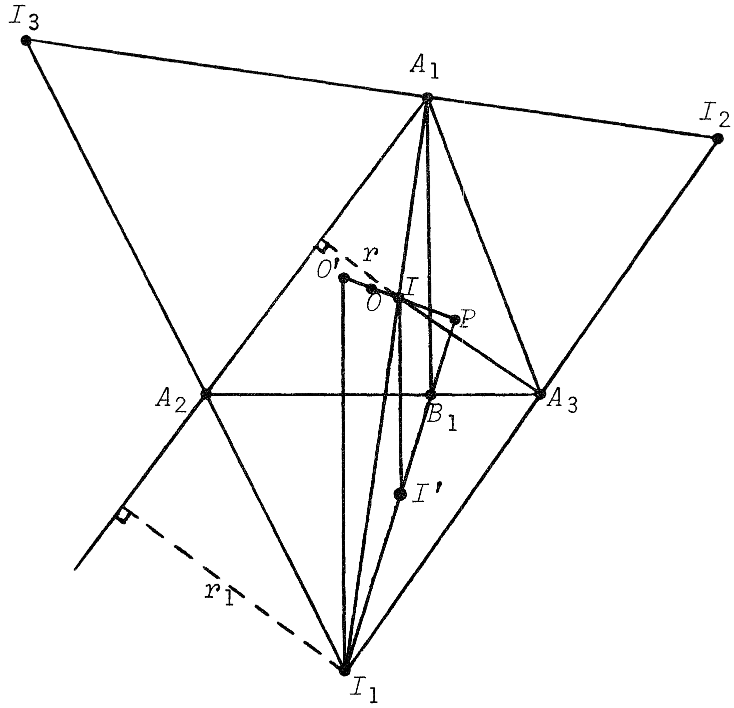
1295. [1987: 321] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with I_1, I_2, I_3 the excenters and B_1, B_2, B_3 the feet of the altitudes. Show that the lines I_1B_1, I_2B_2, I_3B_3 concur at a point collinear with the incenter and circumcenter of the triangle.

I. *Solution by Shiko Iwata, Gifu, Japan.*

(I,r), (O,R), and (O',R') are the centers and radii of the incircle and circumcircle of $\Delta A_1A_2A_3$ and the circumcircle of $\Delta I_1I_2I_3$, respectively. Then, since I is the orthocenter of $\Delta I_1I_2I_3$ and the circumcircle (O,R) of $\Delta A_1A_2A_3$ is the nine–point circle of $\Delta I_1I_2I_3$ ([1], page 197), O', O and I are collinear and $R' = 2R$. Also,



$$\begin{aligned}\angle O'I_1A_3 &= 90^\circ - \tfrac{1}{2}\angle I_1O'I_2 \\ &= 90^\circ - \angle I_3 \\ &= \angle A_2I_1I = \angle A_2A_3I \\ &= 90^\circ - \angle A_2A_3I_1,\end{aligned}$$

so $A_2A_3 \perp O'I_1$, i.e. $A_1B_1 \| O'I_1$. Let P be the meeting point of I_1B_1 and IO, and let I' be on PI_1 such that $II' \| O'I_1$. Then we have

$$PI:PO' = II':O'I_1. \tag{1}$$

On the other hand,

$$\frac{II'}{A_1B_1} = \frac{II_1}{A_1I_1} = \frac{r_1 - r}{r_1} = 1 - \frac{r}{r_1} = 1 - \frac{s - a_1}{s} = \frac{a_1}{s}$$

where s is the semiperimeter of $\Delta A_1A_2A_3$, so that

$$II' = \frac{a_1 \cdot A_1B_1}{s} = \frac{2\Delta}{s} = 2r,$$

where Δ is the area of $\Delta A_1A_2A_3$. Thus from (1)

$$PI:PO' = 2r:R' = r:R.$$

It follows that P is independent of I_1, so lies on I_2B_2 and I_3B_3 too.

Reference:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960.

II. *Generalization by the proposer.*

We prove more generally that *if P is any point in the plane of $\Delta A_1A_2A_3$, Q is its isogonal conjugate, and P_1 is the intersection of A_1P and A_2A_3 (with analogous definitions for P_2, P_3), then the lines P_1I_1, P_2I_2, P_3I_3 are concurrent at a point collinear with Q and the incenter I of the triangle.* The given result follows by letting P be the orthocenter so that Q is the circumcenter.

We use normal homogeneous triangular coordinates, with

$$A_1 = (1,0,0), \qquad A_2 = (0,1,0), \qquad A_3 = (0,0,1),$$
$$I = (1,1,1), \qquad I_3 = (1,1,-1), \text{ etc.}$$

Let P have coordinates (p_1,p_2,p_3). Then $P_3 = (p_1,p_2,0)$, so P_3I_3 has the equation

$$p_2x - p_1y + (p_2 - p_1)z = 0.$$

Analogous equations hold for P_1I_1 and P_2I_2. Thus to show these lines intersect we have to prove

$$\begin{vmatrix} p_2 & -p_1 & p_2 - p_1 \\ p_3 - p_2 & p_3 & -p_2 \\ -p_3 & p_1 - p_3 & p_1 \end{vmatrix} = 0.$$

This is easy as the sum of the three rows is zero. The coordinates of the intersection point S we get from the equations

$$p_2x - p_1y + (p_2 - p_1)z = 0$$
$$(p_3 - p_2)x + p_3y - p_2z = 0$$

from which comes

$$S = (p_1p_2 + p_3p_1 - p_2p_3\ ,\ p_1p_2 - p_3p_1 + p_2p_3\ ,\ -p_1p_2 + p_3p_1 + p_2p_3).$$

Next we can put

$$Q = (p_2p_3,p_3p_1,p_1p_2)$$

so to show that S, Q, and I are collinear we have to prove

$$\begin{vmatrix} p_2p_3 & p_3p_1 & p_1p_2 \\ p_1p_2 + p_3p_1 - p_2p_3 & p_1p_2 - p_3p_1 + p_2p_3 & -p_1p_2 + p_3p_1 + p_2p_3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} p_2p_3 & p_3p_1 & p_1p_2 \\ p_1p_2 + p_3p_1 + p_2p_3 & p_1p_2 + p_3p_1 + p_2p_3 & p_1p_2 + p_3p_1 + p_2p_3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

which is indeed true.

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; CLARK KIMBERLING, University of Evansville, Evansville, Indiana; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and JOSE YUSTY PITA, Madrid, Spain.

Kimberling notes that the proposal is a known result (John Casey, Analytic Geometry, *2nd ed., Hodges & Figgis, Dublin,* 1893, *p.*85).

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1296. [1987: 321] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let r_1, r_2, r_3 be the distances from an interior point of a triangle to its sides

a_1, a_2, a_3, respectively, and let R be the circumradius of the triangle. Prove that

$$a_1r_1^n + a_2r_2^n + a_3r_3^n \le (2R)^{n-2}a_1a_2a_3$$

for all $n \ge 1$, and determine when equality holds.

Solution by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.

Since $r_i < 2R$,

$$\begin{aligned} a_1r_1^n + a_2r_2^n + a_3r_3^n &\le (a_1r_1 + a_2r_2 + a_3r_3)(2R)^{n-1} \\ &= 2\cdot\text{Area}(A_1A_2A_3)\cdot(2R)^{n-1} \\ &= \frac{a_1a_2a_3}{2R}\cdot(2R)^{n-1} = a_1a_2a_3(2R)^{n-2}. \end{aligned}$$

Equality holds when $n = 1$.

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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__1297__. [1987: 321] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* (To the memory of Léo.)

(a) Let $C > 1$ be a real number. The sequence $z_1,z_2,\ldots$ of real numbers satisfies $1 < z_n$ and $z_1 + \cdots + z_n < Cz_{n+1}$ for $n \ge 1$. Prove the existence of a constant $a > 1$ such that $z_n > a^n$, $n \ge 1$.

(b)* Let conversely $z_1 < z_2 < \ldots$ be a strictly increasing sequence of positive real numbers satisfying $z_n \ge a^n$, $n \ge 1$, where $a > 1$ is a constant. Does there necessarily exist a constant C such that $z_1 + \cdots + z_n < Cz_{n+1}$ for all $n \ge 1$?

Solution by C. Wildhagen, Tilburg University, Tilburg, The Netherlands.

(a) Let $\theta = C^{-1}$ so that $0 < \theta < 1$. Then

$$z_n > \theta(z_1 + z_2 + \cdots + z_{n-1})$$

for each $n \ge 2$. This gives

$$\begin{aligned} z_n &> \theta[z_1 + z_2 + \cdots + z_{n-2} + \theta(z_1 + z_2 + \cdots + z_{n-2})] \\ &= \theta(1 + \theta)(z_1 + z_2 + \cdots + z_{n-2}), \end{aligned}$$

and using an inductive argument

$$z_n > \theta(1 + \theta)^{k-1}(z_1 + z_2 + \cdots + z_{n-k})$$

for each k, $1 \le k \le n - 1$. In particular when $k = n - 1$,

$$z_n > \theta z_1(1 + \theta)^{n-2} = D_n\cdot\lambda^n$$

where

$$D_n = \frac{\theta z_1}{(1 + \theta)^2}\left[\frac{1 + \theta}{1 + \theta/2}\right]^n, \quad \lambda = 1 + \theta/2.$$

Take an $N \in \mathbb{N}$ such that $D_n \ge 1$ for each $n > N$ and hence $z_n > \lambda^n$ for $n > N$. Since $z_n > 1$ for each $n \ge 1$, there exists some $\mu > 1$ such that $z^n > \mu^n$ for $1 \le n \le N$. Letting

$a = \min\{\lambda,\mu\}$, we have $a > 1$ and $z_n > a^n$ for all $n \geq 1$.

(b) The answer is "no"! Take $z_1 = 2$ and, for each integer $k \geq 2$, let

$$z_i = \left[k + \frac{i}{(k-1)k(k+1)}\right]^{k(k+1)}$$

for each integer i satisfying

$$k(k-1) \leq i < k(k+1).$$

Clearly $z_{i+1} > z_i \geq 2^i$ for each i. Let

$$S_n = \sum_{i=1}^{n} z_i$$

for each $n \in \mathbb{N}$. For $n = k(k+1)$ we have

$$\begin{aligned} S_{n-2} &> \sum_{i=k(k-1)}^{n-2} z_i \\ &> (2k-1)z_{k(k-1)} \\ &= (2k-1)\left[k + \frac{1}{k+1}\right]^{k(k+1)}, \end{aligned}$$

while

$$z_{n-1} < \left[k + \frac{1}{k-1}\right]^{k(k+1)}.$$

This implies

$$\begin{aligned} \frac{S_{n-2}}{z_{n-1}} &> (2k-1)\left(\frac{k + \frac{1}{k+1}}{k + \frac{1}{k-1}}\right)^{k(k+1)} \\ &= (2k-1)\left[1 - \frac{2}{(k-1)(k+1)\left[k + \frac{1}{k-1}\right]}\right]^{k(k+1)} \\ &\to \infty \quad (\text{as } k \to \infty). \end{aligned}$$

Hence the sequence (S_n/z_{n+1}) is unbounded.

Part (a) also solved by SEUNG–JIN BANG, Seoul, Korea; KEE–WAI LAU, Hong Kong; and the proposer.

* * *

<u>1298</u>. [1987: 321] *Proposed by Len Bos, University of Calgary, Calgary, Alberta.*

Let $A = (a_{ij})$ be an $n \times n$ matrix of positive integers such that $|\det A| = 1$, and suppose that $z_1,z_2,\ldots,z_n$ are complex numbers such that

$$z_1^{a_{i1}} z_2^{a_{i2}} \cdots z_n^{a_{in}} = 1$$

for each $i = 1,2,\ldots,n$. Show that $z_i = 1$ for each i.

I. *Solution by Seung–Jin Bang, Seoul, Korea.*

Note that

$$|z_1|^{a_{i1}}|z_2|^{a_{i2}} \cdots |z_n|^{a_{in}} = |z_1^{a_{i1}} \cdots z_n^{a_{in}}| = 1.$$

Taking logs, we have the system of linear equations

$$a_{i1}\log|z_1| + \cdots + a_{in}\log|z_n| = 0 \quad (i = 1,\ldots,n)$$

with coefficient matrix A. Since det $A \neq 0$, we have

$$\log|z_1| = \cdots = \log|z_n| = 0,$$

that is,

$$z_1 = e^{i\theta_1}, \ldots, z_n = e^{i\theta_n} \tag{1}$$

for some reals $\theta_1,\ldots,\theta_n$. From the original equation

$$z_1^{a_{i1}}z_2^{a_{i2}} \cdots z_n^{a_{in}} = 1$$

we see that

$$a_{i1}\theta_1 + \cdots + a_{in}\theta_n = 2k_i\pi, \quad i = 1,\ldots,n,$$

for integers k_i. Since $|\det A| = 1$, by Cramer's rule we have $\theta_i = 2\ell_i\pi$ for some integers ℓ_i, $i = 1,\ldots,n$. Hence by (1) we know that $z_1 = \cdots = z_n = 1$.

II. *Solution by C. Wildhagen, Tilburg University, Tilburg, The Netherlands.*

We slightly extend the problem by allowing A to have arbitrary integer entries (provided still that $|\det A| = 1$).

By applying a finite number of the following elementary row operations:

(i) multiplying a row by –1;

(ii) interchanging two rows;

(iii) adding an integral multiple of one row to another row,

A can be transformed to an upper triangular integral matrix $T = (t_{ij})$ with det $T = 1$ and $t_{ii} = 1$ for all i. Moreover the property (of A) that

$$z_1^{a_{i1}}z_2^{a_{i2}} \cdots z_n^{a_{in}} = 1 \quad (i = 1,2,\ldots,n)$$

is preserved by each of (i), (ii), (iii), and therefore

$$z_1^{t_{i1}}z_2^{t_{i2}} \cdots z_n^{t_{in}} = 1.$$

Now putting $i = n$ we have $t_{n1} = t_{n2} = \cdots = t_{n\,n-1} = 0$, $t_{nn} = 1$ so that $z_n = 1$; putting $i = n-1$ we have

$$z_1^0 \cdots z_{n-2}^0\, z_{n-1}^1\, 1^{t_{n-1\,n}} = 1$$

so that $z_{n-1} = 1$; and so on. Continuing in this way we find $z_i = 1$ for all i.

Also solved by CHRIS GODSIL, University of Waterloo, and EDWARD T.H. WANG, Wilfrid Laurier University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LEROY F. MEYERS, The Ohio State University; M.A. SELBY, University of Windsor; and the proposer. Two incorrect solutions were received.

Except for Wildhagen's solution II, all solutions received were similar to solution I, and go through if the entries of A are arbitrary integers. This extension was only pointed out *by Wildhagen, however.*

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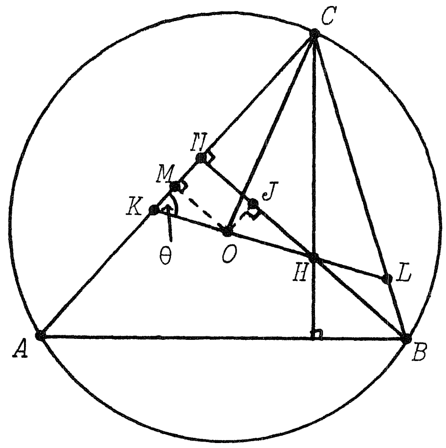
1300. [1987: 321] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle, not right angled, with circumcentre O and orthocentre H. The line OH intersects CA in K and CB in L, and $OK = HL$. Calculate angle C.

I. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

Let α, β, γ be the angles and a, b, c the sides of ΔABC, and let R be the circumradius. Label the figure as shown, and also put $\theta = \angle CKO$. Since



$$\angle KCO = \angle HCL (= 90^\circ - \beta),$$

we have

$$\frac{R}{\sin\theta} = \frac{CO}{\sin\theta} = \frac{KO}{\sin\angle KCO}$$
$$= \frac{HL}{\sin\angle HCL} = \frac{CH}{\sin\angle CLH}$$
$$= \frac{2R\cos\gamma}{\sin\angle CLH}$$

and thus

$$\sin\angle CLH = 2\sin\theta\cos\gamma$$
$$= \sin(\theta+\gamma) + \sin(\theta-\gamma).$$

But $\theta + \gamma + \angle CLH = 180^\circ$, so

$$\sin\angle CLH = \sin(\theta+\gamma),$$

and thus

$$\sin(\theta-\gamma) = 0,$$

i.e. $\theta = \gamma$. Now

$$\tan\gamma = \tan\theta = \tan\angle JOH$$
$$= \frac{HN - OM}{AN - AM} = \frac{2R\cos\alpha\cos\gamma - R\cos\beta}{c\cos\alpha - b/2}$$
$$= \frac{3R\cos\alpha\cos\gamma - R\sin\alpha\sin\gamma}{2R\sin\gamma\cos\alpha - R\sin\beta}$$
$$= \frac{3\cos\alpha\cos\gamma - \sin\alpha\sin\gamma}{\sin\gamma\cos\alpha - \sin\alpha\cos\gamma}$$
$$= \frac{3 - \tan\alpha\tan\gamma}{\tan\gamma - \tan\alpha}.$$

Thus $\tan^2\gamma = 3$, so that $\gamma = 60^\circ$ or 120°.

II. *Solution by P. Penning, Delft, The Netherlands.*

We use triangular coordinates based on the distances to the sides of ΔABC. Then

$$O(\cos A, \cos B, \cos C)$$

and

$$H(\cos B\cos C\,,\ \cos C\cos A\,,\ \cos A\cos B),$$

so the straight line OH has equation

$$\begin{vmatrix} x & y & z \\ \cos A & \cos B & \cos C \\ \cos B\cos C & \cos C\cos A & \cos A\cos B \end{vmatrix} = 0.$$

Thus the intersection of OH with $AC{:}\, y = 0$ is

$$K(\cos C(\cos^2 A - \cos^2 B)\,,\ 0\,,\ \cos A(\cos^2 C - \cos^2 B)). \qquad (1)$$

The actual (signed) distance of O from BC is $X_o = R\cos A$, of H from BC is $X_h = 2R\cos B\cos C$, and of K from BC is (from (1))

$$\frac{2R\sin A\sin B\sin C\cdot\cos C(\cos^2 A - \cos^2 B)}{\sin A\cdot\cos C(\cos^2 A - \cos^2 B) + \sin C\cdot\cos A(\cos^2 C - \cos^2 B)}, \qquad (2)$$

which, with quite some manipulation[1], equals

$$X_k = \frac{2R\cdot\cos C(\cos^2 A - \cos^2 B)}{2\cos A\cos C - \cos B}. \qquad (3)$$

The condition $OK = HL$ implies

$$X_k - X_o = X_h - X_\ell, \qquad (4)$$

which with (3) (and $X_\ell = 0$) becomes

$$\frac{2\cos C(\cos^2 A - \cos^2 B)}{2\cos A\cos C - \cos B} - \cos A = 2\cos B\cos C,$$

i.e.

$$2\cos C(\cos^2 A - \cos^2 B) = (2\cos A\cos C - \cos B)(2\cos B\cos C + \cos A).$$

This leads to

$$\cos A\cos B(4\cos^2 C - 1) = 0,$$

so $\cos C = \pm 1/2$, and thus $C = 60^\circ$ or 120°. (Note that $A = 90^\circ$ or $B = 90^\circ$ are possible solutions, but are excluded in the problem.)

III. *Editor's comments.*

After much manipulation indeed, the editor came up with the following argument to show (2) equals (3). We need to show

$$\sin A\sin B\sin C(2\cos A\cos C - \cos B) = \sin A\cos C(\cos^2 A - \cos^2 B) + \sin C\cos A(\cos^2 C - \cos^2 B)$$

which can be written

$$\cos A\cos C[\sin A(\sin B\sin C - \cos A) + \sin C(\sin A\sin B - \cos C)] + \cos^2 B(\sin A\cos C + \sin C\cos A) = \sin A\sin B\sin C\cos B. \qquad (5)$$

[1]See III, *Editor's comments.*

Using

$$\cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C, \quad \text{etc.}$$

(5) becomes

$$\cos A \cos C(\sin A \cos B \cos C + \sin C \cos A \cos B)$$
$$+ \cos^2 B(\sin A \cos C + \sin C \cos A) = \sin A \sin B \sin C \cos B$$

or

$$\cos B(\cos A \cos C + \cos B)(\sin A \cos C + \cos A \sin C) = \sin A \sin B \sin C \cos B.$$

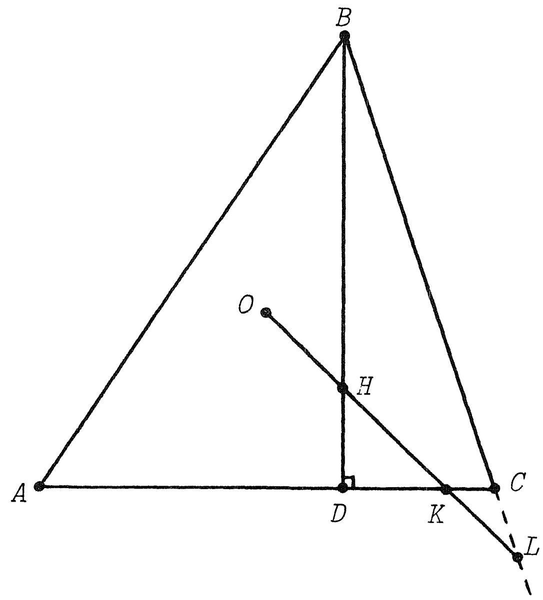
This last equation now follows from

$$\cos B = \sin A \sin C - \cos A \cos C$$

and

$$\sin A \cos C + \cos A \sin C = \sin(A + C) = \sin B.$$

None of the solutions received for this problem were completely satisfactory, in that they don't appear to work in all cases. Groenman's proof, for instance, depends on the given diagram, while Penning's, which seems to be the most general, uses (at (4)) that point K is farther from line BC than the circumcentre O is. In fact, if (as is reasonable) one allows the points K and L to lie on the *extended* lines CA and CB respectively, the conclusion of the problem may not hold! An interesting counterexample is the triangle ABC illustrated at the right, where D is the foot of the perpendicular from B, and $DC = 1$, $AD = 2$, $BD = 3$. The reader can check that $OK = HL$, while of course $\angle C \neq 60^\circ$.



Also solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS–HAMOIR, Brussels, Belgium; GEORGE TSINTSIFAS, Thessaloniki, Greece; JOSE YUSTY PITA, Madrid, Spain; and the proposer.

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1302. [1988: 12] *Proposed by Mihaly Bencze, Brasov, Romania.*

Suppose $\alpha_k > 0$ for $k = 1,2,\ldots,n$ and $\sum_{k=1}^{n} \tanh^2\alpha_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sinh \alpha_k} \geq n \sum_{k=1}^{n} \frac{\sinh \alpha_k}{\cosh^2\alpha_k}.$$

Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg.

We may without loss of generality assume that $0 < \alpha_1 \leq \cdots \leq \alpha_n$. Let

$$x_k = \frac{1}{\sinh\ \alpha_k}, \qquad w_k = \tanh^2\alpha_k, \qquad k = 1,2,\ldots,n.$$

Then it is easily seen that

$$x_1 \geq x_2 \geq \cdots \geq x_n$$

and

$$w_1 \leq w_2 \leq \cdots \leq w_n,$$

i.e. the correlation coefficient between the x's and w's is ≤ 0. Therefore by Chebyshev's inequality

$$\sum_{k=1}^{n} x_k w_k \leq \frac{1}{n}\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} w_k\right).$$

Since

$$\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} \tanh^2\alpha_k = 1$$

is given, and

$$\frac{\tanh^2\alpha_k}{\sinh\ \alpha_k} = \frac{\sinh\ \alpha_k}{\cosh^2\alpha_k},$$

the required inequality follows.

Also solved by SEUNG–JIN BANG, Seoul, Korea; C. FESTRAETS–HAMOIR, Brussels, Belgium; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

As in Crux 1288 [1988: 312], *about half the solvers simply applied Chebyshev's inequality.*

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<u>1303</u>. [1988: 12] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC and $A_1B_1C_1$ be two triangles with sides a, b, c and a_1, b_1, c_1 and inradii r and r_1, and let P be an interior point of ΔABC. Set $AP = x$, $BP = y$, $CP = z$. Prove that

$$\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq 4rr_1.$$

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

First proof. Let F and F_1 be the areas of triangles ABC and $A_1B_1C_1$, respectively. The given inequality can then be written as

$$a_1x^2 + b_1y^2 + c_1z^2 \geq 8Fr_1. \tag{1}$$

In [1] (see also item 12.56 of [3]) is shown the inequality

$$(a_1x + b_1y + c_1z)^2 \geq \frac{M}{2} + 8FF_1, \tag{2}$$

where

$$M = \sum a_1^2(b^2 + c^2 - a^2),$$

the sum being cyclic. Moreover, in [2] inequality (2) is extended to *all* points P of the space. Applying the Cauchy–Schwarz inequality to (2) we get

$$\left(\sum a_1x\right)^2 \leq \left(\sum a_1\right)\left(\sum a_1x^2\right) = 2s_1 \sum a_1x^2. \tag{3}$$

Furthermore, we take from item 10.8 of [3] the "Neuberg–Pedoe" inequality

$$M \geq 16FF_1. \tag{4}$$

Finally (2), (3) and (4) lead to the better estimation

$$\sum a_1x^2 \geq \frac{M}{4s_1} + 4Fr_1 \geq 8Fr_1,$$

the inequality holding for all points of the space.

Second proof. The polar moment of inertia inequality [4] states: if $u,v,w \geq 0$ then

$$(u + v + w)(ux^2 + vy^2 + wz^2) \geq a^2vw + b^2wu + c^2uv. \tag{5}$$

Furthermore, from *Crux* 1181(a) [1988: 25] the inequality

$$a^2vw + b^2wu + c^2uv \geq 4F\sqrt{uvw(u + v + w)} \tag{6}$$

is known. (5) and (6) yield

$$ux^2 + vy^2 + wz^2 \geq 4F\sqrt{\frac{uvw}{u + v + w}}. \tag{7}$$

Putting in (7) $u = a_1$, $v = b_1$, $w = c_1$ and noting $a_1b_1c_1 = 4F_1R_1$ we get

$$a_1x^2 + b_1y^2 + c_1z^2 \geq 4F\sqrt{\frac{4F_1R_1}{2s_1}} = 4F\sqrt{2R_1r_1}. \tag{8}$$

As $R_1 \geq 2r_1$ we obtain from (8) the desired inequality (1). Note that (8) gives another interpolation of (1). Again (as can be seen from [4]) there are no restrictions on the position of the point P.

References:

[1] O. Bottema and M.S. Klamkin, Joint triangle inequalities, *Simon Stevin* 48, I–II (1974) 3–8.

[2] D.S. Mitrinovic and J.E. Pecaric, Note on O. Bottema's inequality for two triangles, *C.R. Math. Rep. Acad. Sci. Canada* 8, No. 2 (1986) 141–144.

[3] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.

[4] M.S. Klamkin, Geometric inequalities via the polar moment of inertia, *Math. Magazine* 48 (1975) 44–46.

II. *Solution by Murray S. Klamkin, University of Alberta.*

First proof. The given result will follow by successive use of some known stronger inequalities. First we use the polar moment of inertia inequality [1]

$$(w_1 + w_2 + w_3)(w_1x^2 + w_2y^2 + w_3z^2) \geq w_2w_3a^2 + w_3w_1b^2 + w_1w_2c^2$$

where w_1, w_2, w_3 are arbitrary real numbers. (A simple proof follows by expanding out

$$(w_1\mathbf{X} + w_2\mathbf{Y} + w_3\mathbf{Z})^2 \geq 0,$$

where **X**, **Y**, **Z** are vectors from P to A, B, C, respectively.) Letting $(w_1,w_2,w_3) = (a_1,b_1,c_1)$, we get

$$\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq \frac{b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2}{(a+b+c)(a_1+b_1+c_1)}$$

$$= \frac{(b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2)rr_1}{4FF_1}.$$

It thus suffices to show that

$$b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2 \geq 16FF_1, \tag{9}$$

or equivalently,

$$\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{4F}{R_1}. \tag{10}$$

To prove (10), we use the known stronger inequality ([2], eq. 41)

$$\frac{a^2}{a_2^2} + \frac{b^2}{b_2^2} + \frac{c^2}{c_2^2} \geq \frac{8F\sqrt{F_2\sqrt[4]{3}}}{a_2b_2c_2}, \tag{11}$$

where a_2, b_2, c_2, F_2 are the sides and area of a third triangle. In (11), let

$$(a_2,b_2,c_2) = (\sqrt{a_1},\sqrt{b_1},\sqrt{c_1})$$

to give

$$\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{8F\sqrt{F_2\sqrt[4]{3}}}{\sqrt{a_1b_1c_1}}.$$

It now remains to show that

$$\frac{2\sqrt{F_2\sqrt[4]{3}}}{\sqrt{a_1b_1c_1}} \geq \frac{1}{R_1},$$

or

$$4R_1^2F_2\sqrt{3} \geq a_1b_1c_1 = 4F_1R_1,$$

or

$$R_1F_2\sqrt{3} \geq F_1. \tag{12}$$

But this follows from the Finsler-Hadwiger inequality ([3], item 10.3)

$$4F_2^2 \geq F_1\sqrt{3},$$

and the known inequality ([3], item 4.14)

$$3R_1^2\sqrt{3} \geq 4F_1,$$

which is equivalent to the fact that the largest triangle (in area) that can be inscribed in a circle is the equilateral one.

An inequality similar to (9), and due to the proposer, appears as problem E3154 of the *Amer. Math. Monthly* (solution in Vol. 95 (1988), pp.659–660). Here one was to show that

$$b_1c_1a^2 + c_1a_1b^2 + a_1b_1c^2 \geq 4F^2$$

where the triangle $A_1B_1C_1$ is inscribed in ΔABC.

References:

[1] M.S. Klamkin, Geometric inequalities via the polar moment of inertia, *Math. Magazine* 48 (1975) 44–46.

[2] M.S. Klamkin, Asymmetric inequalities, *Publ. Electrotehn. Fur. Ser. Mat. Fiz. Univ. Beograd* No. 357–380 (1971) 33–44.

[3] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968.

[*Editor's note.* Klamkin gave a second solution, somewhat like Janous' first proof above. References to lines in this proof have been added by the editor.]

Second proof. Let $r \geq 1$. We start out with the power mean inequality

$$\frac{a_1x^r + b_1y^r + c_1z^r}{a_1 + b_1 + c_1} \geq \left(\frac{a_1x + b_1y + c_1z}{a_1 + b_1 + c_1}\right)^r .$$

Then using (2) and (4),

$$\frac{a_1x^r + b_1y^r + c_1z^r}{a + b + c} \geq \frac{(a_1x + b_1y + c_1z)^r}{2s(2s_1)^{r-1}}$$

$$\geq \frac{(4\sqrt{FF_1})^r}{ss_1^{r-1}2^r} = \frac{2^r(FF_1)^{r/2}}{ss_1^{r-1}} .$$

The given inequality corresponds to the case $r = 2$.

Also solved by SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; COLIN SPRINGER, student, University of Waterloo; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.

Bilchev and Velikova also obtained the stronger inequality (8) in Janous' second solution.

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1304. [1988: 12] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If p, q, r are the real roots of

$$x^3 - 6x^2 + 3x + 1 = 0,$$

determine the possible values of

$$p^2q + q^2r + r^2p$$

and write them in a simple form.

Solution by Sam Baethge, Science Academy, Austin, Texas.

Let

$$A = p^2q + q^2r + r^2p\ ,\ \ B = p^2r + q^2p + r^2q,$$

the only two possible values of expressions of the given type. We also have

$$\begin{aligned} p + q + r &= 6, \\ pq + qr + rp &= 3, \\ pqr &= -1. \end{aligned}$$

In the equations that follow, all summations are symmetric over p, q and r.

(i) $$18 = (p + q + r)(pq + qr + rp) = \sum p^2q\ + 3pqr = \sum p^2q\ - 3$$

or

$$A + B = \sum p^2q = 21.$$

(ii) $$216 = (p + q + r)^3 = \sum p^3 + 3\sum p^2q\ + 6pqr$$

or

$$\sum p^3 = 216 - 3(21) - 6(-1) = 159.$$

(iii) $$27 = (pq + qr + rp)^3 = \sum p^3q^3 + 3\sum p^3q^2r\ + 6p^2q^2r^2$$

or

$$\sum p^3q^3 = 27 - 6(1) - 3pqr\sum p^2q = 21 - 3(-1)(21) = 84.$$

(iv) $$\begin{aligned} AB &= \sum p^4qr\ + \sum p^3q^3 + 3p^2q^2r^2 = pqr\sum p^3 + 84 + 3 \\ &= (-1)(159) + 87 = -72. \end{aligned}$$

Using (i) and (iv), A and B are the roots of

$$y^2 - 21y - 72 = 0,$$

so the possible values are 24 and –3.

Also solved by FRANCISCO BELLOT ROSADO, Emilio Ferrari High School, and MARIA ASCENSION LOPEZ CHAMORRO, Leopoldo Cano High School, Valladolid, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SIDNEY KRAVITZ, Dover, New Jersey; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; G.R. VELDKAMP, De Bilt, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer. There were two partial solutions submitted.

Some solvers, including the proposer, answered the problem for an arbitrary cubic.

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1305. [1988: 12] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

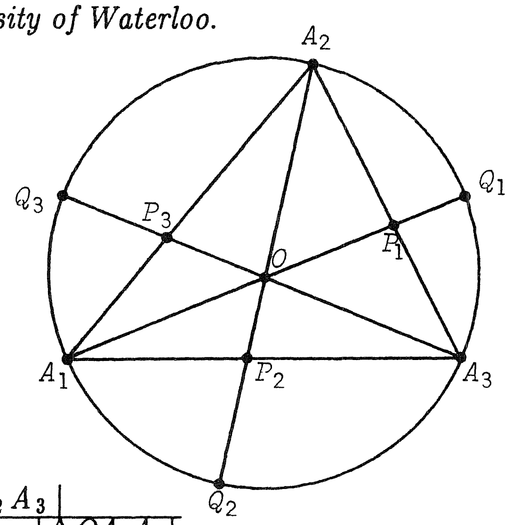
Let $A_1A_2A_3$ be an acute triangle with circumcenter O. Let P_1, Q_1 ($Q_1 \neq A_1$) denote the intersection of A_1O with A_2A_3 and with the circumcircle, respectively, and define P_2, Q_2, P_3, Q_3 analogously. Prove that

(a) $$\frac{\overline{OP_1}\cdot\overline{OP_2}\cdot\overline{OP_3}}{\overline{P_1Q_1}\cdot\overline{P_2Q_2}\cdot\overline{P_3Q_3}} \geq 1;$$

(b) $$\frac{\overline{OP_1}}{\overline{P_1Q_1}} + \frac{\overline{OP_2}}{\overline{P_2Q_2}} + \frac{\overline{OP_3}}{\overline{P_3Q_3}} \geq 3;$$

(c) $$\frac{\overline{A_1P_1}\cdot\overline{A_2P_2}\cdot\overline{A_3P_3}}{\overline{P_1Q_1}\cdot\overline{P_2Q_2}\cdot\overline{P_3Q_3}} \geq 27.$$

Solution by Colin Springer, student, University of Waterloo.



We write $|OP_1|$ for $\overline{OP_1}$ etc. and also write $|T|$ for the area of the triangle T.

Let

$$\alpha_1 = \angle A_2OA_3,$$
$$\alpha_2 = \angle A_3OA_1,$$
$$\alpha_3 = \angle A_1OA_2,$$

and, without loss of generality,

$$|OA_1| = |OA_2| = |OA_3| = 1.$$

Then, since $\Delta A_1A_2A_3$ is acute,

$$|OP_1| = \frac{|OP_1|}{|OA_1|} = \frac{|\Delta OA_2A_3|}{|\Delta OA_1A_2| + |\Delta OA_1A_3|}$$
$$= \frac{\sin\ \alpha_1}{\sin\ \alpha_3 + \sin\ \alpha_2},$$

with similar expressions for $|OP_2|$ and $|OP_3|$. Thus

$$|P_1Q_1| = 1 - |OP_1| = \frac{-\sin\ \alpha_1 + \sin\ \alpha_2 + \sin\ \alpha_3}{\sin\ \alpha_2 + \sin\ \alpha_3}, \text{ etc.}$$

Let

$$x = -\sin\ \alpha_1 + \sin\ \alpha_2 + \sin\ \alpha_3,$$
$$y = \sin\ \alpha_1 - \sin\ \alpha_2 + \sin\ \alpha_3,$$
$$z = \sin\ \alpha_1 + \sin\ \alpha_2 - \sin\ \alpha_3;$$

then

$$\frac{|OP_1|\cdot|OP_2|\cdot|OP_3|}{|P_1Q_1|\cdot|P_2Q_2|\cdot|P_3Q_3|}$$

$$= \frac{\sin\alpha_1\sin\alpha_2\sin\alpha_3}{(-\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1-\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1+\sin\alpha_2-\sin\alpha_3)}$$

$$= \frac{(y+z)(z+x)(x+y)}{8xyz}$$

$$= \left(\frac{y+z}{2\sqrt{yz}}\right)\left(\frac{z+x}{2\sqrt{zx}}\right)\left(\frac{x+y}{2\sqrt{xy}}\right) \geq 1$$

by the A.M.–G.M. inequality. This is (a).

For (b),

$$\frac{|OP_1|}{|P_1Q_1|} + \frac{|OP_2|}{|P_2Q_2|} + \frac{|OP_3|}{|P_3Q_3|} \geq 3\sqrt[3]{\frac{|OP_1|\cdot|OP_2|\cdot|OP_3|}{|P_1Q_1|\cdot|P_2Q_2|\cdot|P_3Q_3|}} \geq 3$$

by the A.M.–G.M. inequality and part (a).

Finally, since

$$|A_1P_1| = 1 + |OP_1| = \frac{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3}{\sin\alpha_2 + \sin\alpha_3}, \text{ etc.,}$$

we have

$$\frac{|A_1P_1|\cdot|A_2P_2|\cdot|A_3P_3|}{|P_1Q_1|\cdot|P_2Q_2|\cdot|P_3Q_3|}$$

$$= \frac{(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)^3}{(-\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1-\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1+\sin\alpha_2-\sin\alpha_3)}$$

$$\geq \frac{27\sin\alpha_1\cdot\sin\alpha_2\cdot\sin\alpha_3}{(-\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1-\sin\alpha_2+\sin\alpha_3)(\sin\alpha_1+\sin\alpha_2-\sin\alpha_3)}$$

$$\geq 27,$$

again by the A.M.–G.M. inequality and part (a).

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; JORG HARTERICH, Winnenden, Federal Republic of Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; D.J. SMEENK, Zaltbommel, The Netherlands; G.R. VELDKAMP, De Bilt, The Netherlands; and the proposer.

Janous' solution to part (c) reduced it to Crux *1199 [1988: 87].*

*　　　　　　*　　　　　　*

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