Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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This issue is dedicated to Joe Gani (1924–2016)

- The Life and Work of John Wallis
- Van Aubel's Theorem using Complex Numbers
- Complex Numbers and Desargues' Theorem
- Computing Geographical Distances

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

David Sharpe, Editor

The Trustees of the Applied Probability Trust are indeed sad that this is the last issue of *Mathematical Spectrum* to appear with the Trust's support. This journal started in the flush of the Trust's life in 1969 when also the first volume of *Advances in Applied Probability* appeared. Soon after Joe Gani returned to Australia in 1974, editorial responsibility for the journal was entrusted to David Sharpe, a person with a goodly mix of interests and abilities in undergraduate and senior secondary school mathematics. After reading mathematics at St. John's College Oxford, David has spent most of his subsequent life at the University of Sheffield, as a research student under Professor D. G. Northcott and shortly thereafter as lecturer and sometimes Admissions Tutor. In this latter capacity he influenced many students who might otherwise have been destined for Oxbridge to come to Sheffield. His lectures here have been described, succinctly, as inspirational and idiosyncratic. These are qualities which readers of *Mathematical Spectrum* may have perceived in the abundant editorial and other pieces that he has contributed. In between he has written two books for Cambridge University Press (*Injective Modules*, 1972, and *Rings and Factorization*, 1987).

More recently, David has retired to his home county of Leicestershire where he has retained his active interest in music-making (he is a concert-goer and plays the organ), football-following and hiking, and continuing to travel more widely. His success as a teacher, editor and writer can be coupled with his deeper interest in people.

The Trustees have been most fortunate to have had the benefit of David's editorial interests for so long. It is with deep regret that they have felt obliged to bow to the financial realities of the electronic publishing era, at the same time acknowledging that were it not for David's efforts the journal would not have continued to 2016. We thank him, his editorial associates, referees, contributors and readers most sincerely for making the journal content what it became.

DARYL DALEY

April 2016 — AP Trustee

Our thanks go especially to Helen Parry for her exemplary work as Sub-Editor, preparing the manuscripts and compiling each issue. The high level of presentation of *Mathematical Spectrum* is entirely due to Helen's skill.

Photograph taken 2008

Joe Gani, 1924–2016

At the time of going to press for this final volume of Mathematical Spectrum, we have heard of the death of Professor Joe Gani. Joe was the founding father of Mathematical Spectrum. He has been enormously influential in the mathematical world, in particular in probability and statistics, throughout his working life. He was an innovator. He was professor at the University of Sheffield in UK before moving back to Australia to be Chief of what became known as the Division of Mathematics and Statistics within the Commonwealth Scientific and Industrial Research Organization (CSIRO). Following seven years there, he headed departments at the University of Kentucky and then the University of California at Santa Barbara. Subsequently, he had over twenty years in retirement as a Visiting Fellow at the Australian National University in Canberra, continuing to edit *The Mathematical Scientist* and write mathematical papers with a strong application flavour. It is fitting to dedicate this final issue to Joe Gani, our founder. Our sympathy goes to his family.

From the Editor

'I love numbers'

What do you say in a five-minute slot to a nonmathematical audience with ages ranging from 5 to 95? It was the beginning of January, so we could start by factorizing the year:

$$2016 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7.$$

A quick look back to the previous year gives

$$2015 = 5 \times 13 \times 31,$$

and a sneak look ahead gives 2017, which cannot be written as a product of smaller numbers; it is prime.

I was asked afterwards if 2019 is prime; the questioner realized that 2018 is divisible by 2. I was able to describe a test for a number to be divisible by 3, namely: is the sum of its digits divisible by 3? Since $2 + 0 + 1 + 9 = 12$, which is divisible by 3, then 2019 is also divisible by 3, so is not prime. The question was then asked: what is the next prime after 2017? I had to consult my list of primes up to 5000 to find out that the next two primes are 2027 and 2029, a 'prime pair', i.e. two primes only two apart, and was able to tell the questioner that, although there are infinitely many primes, it is not known whether there are infinitely many prime pairs.

Most of the audience knew that 2016 is a leap year, because 2016 is divisible by 4. Thus it has an extra day, or 24 hours, or 1440 minutes, or 86 400 seconds. What shall we do with all these extra seconds, we asked. Betty was born on 29 February, so although she is nearly 84 years old, she has only had 20 birthdays, a reminder of the paradox in Gilbert and Sullivan's *The Pirates of Penzance*. The children in the audience were full of sympathy for Betty's plight, but we cheered at the thought that Betty is about to have her 21st birthday.

Had there been time, we could have gone on to ask whether our year is the sum of two squares. We might have discovered that

$$2017 = 9^2 + 44^2$$

but that neither 2015 nor 2016 is a sum of two squares. In fact, a number is the sum of two squares if and only if all its prime factors of the form $4k + 3$ occur an even number of times in its prime factorization. Thus, the prime factors 7 of 2016 and 31 of 2015, both of the form $4k + 3$ and occurring only once in their prime factorizations (and 1 is odd), stop 2016 and 2015 being sums of two squares. On the other hand, 2017 has no prime factors of the form $4k + 3$ so, by default, 2017 satisfies the condition to be a sum of two squares.

Perhaps we should stop whilst we are ahead. After all, the aim was to stimulate a love of numbers, not to put the audience off.

Making Averages Whole

JONNY GRIFFITHS

The arithmetic, geometric, and harmonic means of two natural numbers may or may not be integers. When are all three averages integers together?

What exactly do we mean by the average of two numbers? There are several candidates for this title. The other day, I was trying to explain on paper how the arithmetic, geometric, and harmonic means are defined. As a reminder, the AM of two positive numbers x and y is $(x+y)/2$, the GM is $\sqrt{xy}$, and the HM is $2xy/(x+y)$. It is worth noting that $\text{AM}\times\text{HM}=\text{GM}^2$, and also that $\text{AM}\geq\text{GM}\geq\text{HM}$, with equality holding if and only if $x=y$ (a visual demonstration of this can be found in reference 1). I picked a pair of values without much thought to make the AM a natural number, and then noticed that my choice made the GM a natural number too. The HM, it turned out, was *not* a whole number here, but the seed of a question had been planted: which pairs of natural numbers (x, y) have an AM, a GM, and an HM that are all natural numbers?

It is not hard to find natural numbers that satisfy this condition; if $(x, y)=(45, 5)$, for example, then

$$\text{AM}=25,\qquad \text{GM}=15,\qquad \text{HM}=9.$$

Clearly, if x and y are the same natural number, then the three averages will all be this natural number. We are looking for a parametrisation for x and y that gives all possible solutions, rather in the way that the parametrisation for Pythagorean triples (PTs) gives all possible primitive PTs. (If $x^2+y^2=z^2$, where $\gcd(x, y, z)=1$ and where x is even, then we have $x=2mn$, $y=m^2-n^2$, and $z=m^2+n^2$, where m and n are natural numbers with $m>n$ of opposite parity, with no common factor. This parametrisation is derived in reference 2, p. 128.)

Let us start by saying $(x+y)/2=a$, $\sqrt{xy}=b$, and $2xy/(x+y)=c$, where a, b, and c are natural numbers, so $b^2=ac$. We will also say that $x>y$ without loss of generality, since we have dealt with the case $x=y$. Then $x+y=2a$ and $xy=b^2$, so $y=2a-x$ and $x(2a-x)=b^2$. This yields

$$x^2-2ax+b^2=0.$$

Solving for x we have $x=a+\sqrt{a^2-b^2}$, $y=a-\sqrt{a^2-b^2}$, and for these to be integers, $a^2-b^2=d^2$, for some natural number d. Substituting for b^2, we have $a^2-ac-d^2=0$, which gives $a=(c+\sqrt{c^2+4d^2})/2$. Once again, for this to be an integer, $c^2+4d^2=e^2$, for some natural number e. Let $\gcd(c, d)=f$. This yields $c=fp$, $d=fq$, $e=fr$, where $\gcd(p, q)=1$, and so $p^2+(2q)^2=r^2$. Maybe now we can use the parametrisation of PTs outlined above.

There are two cases to consider. Firstly, if p is odd (and so r is odd), we can write $p=u^2-v^2$, $q=uv$, $r=u^2+v^2$ (where u and v are coprime and of opposite parity), and so

$$c=f(u^2-v^2),\qquad d=fuv,\qquad e=f(u^2+v^2).$$

Substituting back, $a=fu^2$, $b=fu\sqrt{(u^2-v^2)}$, we obtain

$$x=fu^2+fuv=fu(u+v),\qquad y=fu(u-v).$$

For b to be an integer, $u^2 - v^2 = k^2$, for some integer k, or $u^2 = v^2 + k^2$ (note that k must be odd, since u and v are of opposite parity). Once again using our PT parametrisation, we can put

$$u = m^2 + n^2, \qquad k = m^2 - n^2, \qquad v = 2mn,$$

for coprime integers m, n that are of opposite parity. So we have the partial parametrisation

$$x = f(m^2 + n^2)(m + n)^2, \qquad y = f(m^2 + n^2)(m - n)^2.$$

Now for the second case, which is that p is even, which means q is odd (since $\gcd(p, q) = 1$), and r is even. Let us put $p = 2s$, $r = 2t$, which gives that $s^2 + q^2 = t^2$. Since q is odd, s is even, since if s and p are both odd, $t^2 \equiv 2 \pmod 4$, which is impossible. We can now say

$$s = 2uv, \qquad q = u^2 - v^2, \qquad t = u^2 + v^2,$$

where u and v are coprime integers of opposite parity. This gives us $c = 2fs$, $d = fq$, $e = 2ft$, and so $c = 4uvf$ and $d = f(u^2 - v^2)$. So

$$e = 2f(u^2 + v^2), \qquad a = f(u + v)^2, \qquad b = \sqrt{a^2 - d^2} = 2f(u + v)\sqrt{uv},$$
$$x = a + d = 2uf(u + v), \qquad y = 2vf(u + v).$$

Thus, for b to be an integer, we need $uv = k^2$. But we know that $\gcd(u, v) = 1$, and so for this to be true, we must have $u = m^2$, $v = n^2$, where m and n are coprime integers. Finally, we have the second half of the parametrisation, which is

$$x = 2f(m^2 + n^2)m^2, \qquad y = 2f(m^2 + n^2)n^2.$$

We have shown that any parametrisation that meets our conditions must be of these forms; it remains to check that our two formulations do in fact give three integer averages, and they do.

There are some maths problems that seem to set out to wilfully make life difficult for you, while others seem to fall over themselves to be obliging. I would say that this question is one of the second type – the way that repeatedly using the PT formula proves so profitable, and the way the facts concerning coprimeness work so neatly to our advantage, is pleasing. How many of these (x, y) pairs are there? A computer search reveals that there are 58 (x, y) pairs with $1000 \geq x > y$.

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Jonny Griffiths *taught mathematics at Paston Sixth Form College in Norfolk for over twenty years. He has studied mathematics and education at Cambridge University, the Open University, and the University of East Anglia. Possible claims to fame include being a member of 'Harvey and the Wallbangers', a popular band in the 1980s, and playing the character Stringfellow on the childrens' television programme 'Playdays'. He currently works for the Cambridge Mathematics Education Project.*

Variations of a Charming Putnam Problem

THOMAS KOSHY and ZHENGUANG GAO

We will study the Lucas, Pell, and Pell–Lucas counterparts of a charming Fibonacci problem that appeared in the 68th annual William Putnam Mathematical Competition in 2007.

Introduction

It is well known that *Fibonacci numbers* and *Lucas numbers* satisfy the recurrence

$$x_n = x_{n-1} + x_{n-2},$$

where $n \geq 3$. When $x_1 = 1 = x_2$, $x_n = \mathrm{F}_n$, the nth Fibonacci number; and when $x_1 = 1$ and $x_2 = 3$, $x_n = \mathrm{L}_n$, the nth Lucas number. Clearly, we can extend the recurrence to include the case $n = 0$ and hence nonpositive subscripts. For example, $\mathrm{F}_0 = 0$ and $\mathrm{L}_0 = 2$. Table 1 shows the first ten Fibonacci and Lucas numbers.

Pell numbers P_n and *Pell–Lucas numbers* Q_n satisfy the recurrence

$$x_n = 2x_{n-1} + x_{n-2},$$

where $n \geq 3$. When $x_1 = 1$ and $x_2 = 2$, $x_n = \mathrm{P}_n$; and when $x_1 = 1$ and $x_2 = 3$, $x_n = \mathrm{Q}_n$. Pell and Pell–Lucas numbers can also be extended to nonpositive subscripts. Table 1 also shows the first ten Pell and Pell–Lucas numbers.

Binet-like formulas

Fibonacci and Lucas numbers can also be defined by the *Binet-like formulas* as follows:

$$\mathrm{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{L}_n = \alpha^n + \beta^n,$$

Table 1 The first ten Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

n	F_n	L_n	P_n	Q_n
1	1	1	1	1
2	1	3	2	3
3	2	4	5	7
4	3	7	12	17
5	5	11	29	41
6	8	18	70	99
7	13	29	169	239
8	21	47	408	577
9	34	76	985	1 393
10	55	123	2 378	3 363

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are solutions of the equation

$$t^2 - t - 1 = 0$$

(see reference 1). Both can be confirmed by solving the recurrence or using induction.

Pell and Pell–Lucas numbers have their own Binet-like formulas:

$$\mathrm{P}_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad \mathrm{Q}_n = \frac{\gamma^n + \delta^n}{2},$$

where

$$\gamma = 1 + \sqrt{2} \quad \text{and} \quad \delta = 1 - \sqrt{2}$$

are solutions of the equation

$$t^2 - 2t - 1 = 0$$

(see reference 2). Again, both follow by solving the recurrence or using induction.

A Putnam delight

The following charming problem appeared in the 68th annual William Putnam Mathematical Competition in 2007.

Problem 1 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lfloor \sqrt{5}x_n \rfloor,$$

where $x_0 = 1$, $n \geq 0$, and $\lfloor x \rfloor$ denotes the *floor* of the real number x, the greatest integer $\leq x$.

The given recurrence, together with induction, can be used to show that

$$x_n = 2^{n-1}\mathrm{F}_{2n+3},$$

where $n \geq 0$ (see references 1 and 3). By virtue of the Binet formula for F_n, this gives an explicit formula for x_n. We encourage readers to construct a proof for themselves by adapting the proof of the Lucas counterpart given below.

The Lucas counterpart

We will now study the equally charming Lucas counterpart to problem 1.

Problem 2 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lceil \sqrt{5}x_n \rceil,$$

where $x_0 = 2$, $n \geq 0$, and $\lceil x \rceil$ denotes the *ceiling* of the real number x, the smallest integer $\geq x$.

The first few values of x_n exhibit a clear pattern:

$$\begin{aligned} x_0 &= 2 = 2^{-1} \cdot 4, \\ x_1 &= 11 = 2^0 \cdot 11, \\ x_2 &= 58 = 2^1 \cdot 29, \\ x_3 &= 304 = 2^2 \cdot 76, \\ x_4 &= 1592 = 2^3 \cdot 199, \end{aligned}$$

and so on, where the last number on each line is a Lucas number.

Conjecture 1 *We have*

$$x_n = 2^{n-1} \mathrm{L}_{2n+3},$$

where $n \geq 0$*; this provides an explicit formula for* x_n*.*

Proof We will establish this using induction. The statement is clearly true when $n = 0$. Now assume that it holds for an arbitrary integer $n \geq 0$. Then we have

$$\begin{aligned} x_{n+1} &= 3x_n + \lceil \sqrt{5} x_n \rceil \\ &= \lceil (3 + \sqrt{5}) x_n \rceil \\ &= \lceil 2\alpha^2 x_n \rceil \\ &= \lceil 2\alpha^2 \cdot 2^{n-1} \mathrm{L}_{2n+3} \rceil \\ &= \lceil 2^n \alpha^2 (\alpha^{2n+3} + \beta^{2n+3}) \rceil \\ &= \lceil 2^n (\alpha^{2n+5} + \beta^{2n+1}) \rceil \\ &= \lceil 2^n [(\alpha^{2n+5} + \beta^{2n+5}) - (\beta^{2n+5} - \beta^{2n+1})] \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil -2^n \beta^{2n+3} (\beta^2 - \alpha^2) \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil 2^n \beta^{2n+3} (\alpha^2 - \beta^2) \rceil \\ &= 2^n \mathrm{L}_{2n+5} + \lceil 2^n \beta^{2n+3} \sqrt{5} \rceil. \end{aligned}$$

Since

$$2\beta^2 = 3 - \sqrt{5},$$

we have

$$2^n \beta^{2n} = (3 - \sqrt{5})^n.$$

So

$$2^n \beta^{2n+2} = \frac{(3 - \sqrt{5})^{n+1}}{2};$$

hence,

$$2^n \beta^{2n+3} = \frac{1 - \sqrt{5}}{2} \cdot \frac{(3 - \sqrt{5})^{n+1}}{2}.$$

Then

$$2^n\beta^{2n+3}\sqrt{5} = \frac{\sqrt{5}-5}{4}(3-\sqrt{5})^{n+1}.$$

This implies that

$$-1 < 2^n\beta^{2n+3}\sqrt{5} < 0.$$

Consequently,

$$\lceil 2^n\beta^{2n+3}\sqrt{5}\rceil = 0.$$

Thus,

$$x_{n+1} = 2^n \mathrm{L}_{2n+5} + 0 = 2^n \mathrm{L}_{2n+5};$$

so the formula works for $n+1$ also. Hence, it works for all $n \geq 0$, as desired.

Interestingly, the Fibonacci and Lucas versions have Pell and Pell–Lucas counterparts.

The Pell counterpart

We now present the Pell counterpart to problem 1.

Problem 3 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lceil 2\sqrt{2}x_n\rceil,$$

$x_0 = 2$, and $n \geq 0$.

Conjecture 2 *Using the initial values* $x_0 = 2 = \mathrm{P}_2$, $x_1 = 12 = \mathrm{P}_4$, $x_2 = 70 = \mathrm{P}_6$, $x_3 = 408 = \mathrm{P}_8$, *and* $x_4 = 2\,378 = \mathrm{P}_{10}$, *we conjecture that* $x_n = \mathrm{P}_{2n+2}$, *where* $n \geq 0$.

Proof Clearly, the formula is true when $n = 0$. Now assume that it is true for an arbitrary integer $n \geq 0$. Then we have

$$\begin{aligned}
x_{n+1} &= 3x_n + \lceil 2\sqrt{2}x_n\rceil \\
&= \lceil (3+2\sqrt{2})x_n\rceil \\
&= \lceil \gamma^2 x_n\rceil \\
&= \lceil \gamma^2 \mathrm{P}_{2n+2}\rceil \\
&= \left\lceil \frac{\gamma^2}{\gamma-\delta}(\gamma^{2n+2}-\delta^{2n+2})\right\rceil \\
&= \left\lceil \frac{1}{\gamma-\delta}[(\gamma^{2n+4}-\delta^{2n+4})+(\delta^{2n+4}-\delta^{2n})]\right\rceil \\
&= \mathrm{P}_{2n+4} + \left\lceil \frac{\delta^{2n+2}}{\gamma-\delta}(\delta^2-\gamma^2)\right\rceil \\
&= \mathrm{P}_{2n+4} + \lceil -2\delta^{2n+2}\rceil.
\end{aligned}$$

Since

$$\delta^2 = 3-2\sqrt{2},$$

we have

$$\delta^{2n} = (3 - 2\sqrt{2})^n;$$

so

$$-2\delta^{2n+2} = -2(3 - 2\sqrt{2})^{n+1};$$

hence, $-1 < -2\delta^{2n+2} < 0$. Consequently,

$$\lceil -2\delta^{2n+2} \rceil = 0.$$

This implies that

$$x_{n+1} = \mathrm{P}_{2n+4} + 0 = \mathrm{P}_{2n+4};$$

so the formula works for $n + 1$ as well. Thus, the formula works for all $n \geq 0$, as desired.

The Pell–Lucas counterpart

We now present the Pell–Lucas counterpart to problem 1.

Problem 4 Find an explicit formula for x_n, where

$$x_{n+1} = 3x_n + \lfloor 2\sqrt{2}x_n \rfloor,$$

$x_0 = 3$, and $n \geq 0$.

Using similar steps, we can show that $x_n = \mathrm{Q}_{2n+2}$, where $n \geq 0$. Again, we encourage readers to confirm the formula for x_n by adapting the proof for the Pell counterpart.

Acknowledgement The authors would like to thank the Editor for his thoughtful comments and suggestions for improving the quality of the exposition of the original version.

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Thomas Koshy *is Professor Emeritus at Framingham State University. He has authored several books, including 'Catalan Numbers with Applications', 'Triangular Numbers with Applications', and 'Pell and Pell–Lucas Numbers with Applications'.*

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The Life and Work of John Wallis

SCOTT H. BROWN and LUKE A. SMITH

2016 marks the 400th anniversary of the birth of John Wallis

John Wallis was one of the leading mathematicians of the 17th century. He made significant achievements in mathematics and in other fields such as cryptography, science, speech and hearing, and theology. Highlights of his career included holding the post of Savillian Professor of Geometry at Oxford, deciphering difficult coded letters, and publishing his brilliant treatise *Arithmetica Infinitorum*. He made several important discoveries which would set the foundation for future mathematicians such as Isaac Newton to build upon. He was known to be very controversial because he often criticized the work of other mathematicians. However, his personal work and enthusiasm greatly inspired many of his contemporaries, as well as those in the future, to make significant achievements in several fields.

Wallis was born to John and Joanna Wallis on 23 November 1616 at Ashford in Kent, UK. His father was educated at Trinity College, Cambridge, and thereafter became a highly esteemed Minister of Ashford. Joanna was the daughter of the heiress of a wealthy merchant in London. Wallis had two older sisters and two younger brothers.

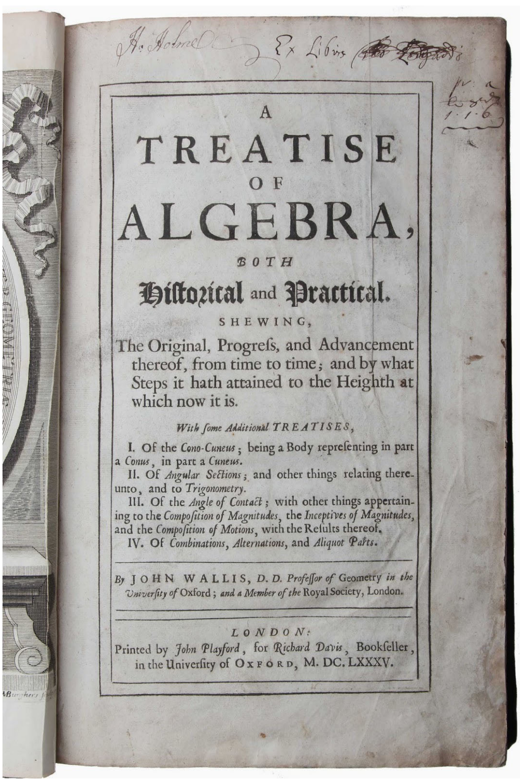
He lost his father when he was six years old. His mother took it upon herself to give him a 'pious and prudent education', which included English and some Latin. With the arrival of the black plague in Ashford, he was sent to Tenterden and later Felsted in Essex. Up to this

***John Wallis*, by Sir Godfrey Kneller**

point in his life, he had received very little education in mathematics. Then, prior to entering Emmanuel College, Cambridge, in 1632, he learned basic arithmetic from the textbooks of a younger brother.

During his undergraduate years, his studies included logic, physics, astronomy, and some parts of mathematics. He was selected early on at Emmanuel College as a *Fellow*. However, since there was already one from his County of Kent, Wallis was not given the honored position. He seemed not to be concerned with the matter, having made the remark, ‘I am already well-esteemed, and well beloved in the College…’. After receiving his Bachelor of Arts degree in 1637, he pursued a Masters of Arts, which he completed in 1640. During the same year, Wallis took Holy Orders in the Church of England and was ordained. He proceeded to serve as a Private Chaplain for a few years.

In 1642, he was given the opportunity to decipher a letter during the early stages of England’s Civil War. Wallis quickly deciphered the letter which was based on no more than a new alphabet with 23 characters. This achievement started his brilliant career in codebreaking. He was soon given another opportunity to decipher a letter sent from France by the Secretary of State to his son in England. This letter was written in a cipher consisting of over 700 numerical figures and numerous other characters intermixed. After several months, he was able to decipher the code. His success as a codebreaker was based on knowing that the ‘more difficult the cipher is, the more material is required’.


A
TREATISE
OF
ALGEBRA,
BOTH
Historical and Practical.
SHEWING,
The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With some Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body representing in part a *Conus*, in part a *Cuneus*.
II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.
III. Of the *Angle of Contact*; with other things appertaining to the *Composition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Composition of Motions*, with the Results thereof.
IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D. D. Professor of* Geometry *in the University of* Oxford; *and a Member of the* Royal Society, London.

LONDON:
Printed by *John Playford*, for *Richard Davis*, Bookseller, in the University of OXFORD, M. DC. LXXXV.

***Arithmetica Infinitorum*, by John Wallis**

In 1644, Wallis married Suzanne Glyde, from whom he had a son and two daughters. Being married meant he had to leave his position as Fellow at Queens' College, Cambridge, the next year. At about the same time he was selected as one of the secretaries to the assembly of Divinity at Westminister. During 1645, he began attending meetings in London with many highly esteemed individuals to discuss the progress of mathematics and natural philosophy.

Perhaps one of the most important events in Wallis' life was being appointed as the Savillian Professor of Geometry at Oxford in 1649. In this position, he was afforded the opportunity to conduct research. One of his initial works was the *Treatise of Speech*, published in 1653. In this paper, he developed a 'theoretical foundation' to teach deaf-mutes how to talk.

In 1656, he published the *De sectionibus conicis*. Wallis analyzed the conic sections in this treatise, which were named and defined by Apollonius, the Greek mathematician of antiquity. He redefined Apollonius' properties, in terms of equations, for the ellipse, parabola, and hyperbola as follows:

$$e^2 = ld - \frac{1}{t}d^2, \qquad p^2 = ld, \qquad h^2 = ld + \frac{1}{t}d^2.$$

For example, in the ellipse, the notation e stood for the ordinate of the point and d was the distance from the abscissa. The notation l was the *latus rectum* and t was the transverse diameter.

Wallis not only made improvements on Apollonius' properties of conic sections, but he was the first to introduce the symbol $1/\infty$ to represent an infinitely small quantity in the treatise. Unfortunately, his treatise on conics was not well received by the mathematical community. However, his next work, the *Arithmetica Infinitorum*, published in 1656, would be his finest achievement.

This work provided the foundation for the future development of integral calculus. As he did in his treatise on conic sections, he used an arithmetic approach to find the area under curved figures. Wallis had studied Bentura Cavalieri's geometric approach of using 'indivisibles', or lines, to find the area under the curve $y = x^n$. Cavalieri had used a lengthy procedure of pairing the geometric indivisibles in a parallelogram with those of one of the triangles formed by the diagonal of the parallelogram, from which he derived a result equivalent to the modern integral form:

$$\int_0^a x^n \,\mathrm{d}x = \frac{a^{n+1}}{n+1}.$$

Wallis, on the other hand, using his arithmetic method, gave a poof that

$$\int_0^1 x^m \,\mathrm{d}x = \frac{1}{m+1},$$

for positive integers m, by showing that

$$\frac{o^m + 1^m + 2^m + 3^m + \cdots + n^m}{n^m + n^m + n^m + n^m + \cdots + n^m} \quad \to \quad \frac{1}{m+1} \qquad \text{as } m \to \infty.$$

By using interpolation he also attempted to find the results for fractional powers.

Wallis' next contribution in the *Arithmetica Infinitorum* was deriving his famous infinite product formula. His goal was to find the ratio of the area of the quadrant of a circle to the square on the radius, which in this case was the determination of $\pi/4$.

He considered the semicircle with radius R and divided the radius into a equal subintervals, and erected perpendiculars equidistant from each other. Wallis proposed that the perpendiculars were the mean proportional of the segments of the diameter, which resulted in the series $(R^2 - p^2 \cdot 0a^2)^{1/2}$, $(R^2 - 1 \cdot a^2)^{1/2}$, $(R^2 - 4a^2)^{1/2}$, ... based on the general term $(R^2 - p^2 \cdot a^2)^{1/2}$ for $p = 0, 1, 2, 3, \ldots$.

In his first attempt to find the area of the quadrant of the circle, he considered the value between $(R^2 - p^2 \cdot a^2)^0$ and $(R^2 - p^2 \cdot a^2)^1$ believing that $(R^2 - p^2 \cdot a^2)^{1/2}$ was halfway between the two values. However, he found the result was obviously too large. Since Wallis was not familiar with the binomial theorem, he turned to a long and difficult process of discovering an infinite sequence in which interpolation of the term $(R^2 - p^2 \cdot a^2)^{1/2}$ would be more accurate.

The fruits of his labour using interpolation provided the desired value for the quadrant of the circle in terms of the infinite product

$$\frac{\pi}{4} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots},$$

from which one could find the product for π . Although there were previous discoveries of a formula for π , his result was the first to involve only rational operations. In the near future, his method of interpolation was to play an instrumental part in Sir Isaac Newton developing the binomial series.

Isaac Newton read the *Arithmetica Infinitorum* in the winter of 1664–1665 and was inspired to continue where Wallis had left off. Newton went beyond Wallis in introducing an algebraic variable. While Wallis used a simple numerical sum, Newton used an infinite power series. In doing so, he was able to find the partial area of a quadrant and express the result in terms of an infinite power series.

Newton continued to build upon a method of interpolation that was different from Wallis'. For example, Wallis generated the sequence $1, 3, 6, 10, 15, 21, \ldots$ by using a multiplier as follows:

$$1,\ 1 \times \tfrac{3}{1},\ 3 \times \tfrac{4}{2},\ 6 \times \tfrac{5}{3},\ 10 \times \tfrac{6}{4},\ 15 \times \tfrac{7}{5}, \ldots .$$

Newton generated the sequence by addition. He wrote it in the following form:

$$a,\ a + b,\ a + 2b + c,\ a + 3b + 3c,\ a + 4b + 6c,\ a + 5b + 10c, \ldots,$$

with $a = 1$, $b = 2$, and $c = 1$. The sequence had a pattern of constant second differences. In this case, taking the difference between the consecutive terms of the original sequence to create a new sequence $(b, b + c, b + 2c, b + 3c, b + 4c, \ldots)$, and then taking the difference again between the consecutive terms of this newly created sequence will create a common difference of c. Newton believed this pattern would hold for any intermediate terms and thus he would be able to interpolate terms between any two entries of a sequence. As a result, he could find the coefficients of the general binomial theorem.

Wallis continued to be productive during 1656 in his mathematical research. He wrote his treatise *Angle of Contact*. The focus of this work was determining the magnitude of the angle between the curved line and the tangent that touches the circumference of a circle. Euclid initially asked the question 'could a right line be drawn between the tangent and the circle'? Wallis concluded that the angle of contact did not have any magnitude.

In 1657, he published *Mathesis Universalis*, which was supposedly based on material from his lectures at Oxford. The first few chapters were dedicated to discussing the basic operations of arithmetic. He believed there are two pure mathematical disciplines, arithmetic and geometry.

According to Wallis, geometry was less pure and abstract than arithmetic. He first provided a history of the development of arithmetic notation from antiquity to the 16th century. The later chapters were devoted to giving an algebraic demonstration of the theorems from the second book of Euclid's *Elements*.

While working on *Mathesis Universalis* in 1657, Wallis was selected for the Custos Archivorum for Oxford University. In this position as keeper of the archives, he had considerable power and influence. He was able to defend the rights and privileges of the university in lawsuits. In a tribute to his work, it was noted, 'he put the records and other papers belonging to the university that were under his care, into such exact order'. He was also praised by his predecessor for developing a catalogue system, which would not be replaced at the university until the late 1900s.

Along with his many duties at Oxford University, Wallis continued his mathematical research. He followed up in 1659 with his two tracts: the first one was on the 'cycloid', and the second on the 'cissoids'. The driving force behind writing his first tract was Blaise Pascal's prize problem regarding the cycloid.

In 1658, Pascal also studied the geometry of the cycloid and succeeded in solving several problems connected to it. He posed two of those as prize problems to English and French mathematicians. The first was to determine 'the area of any segment of the cycloid, which is cut off by a right line parallel to the base' and the second was 'the content of the solid generated by the rotation of the same about the axis, and about the base of the segment'.

Wallis was one of two mathematicians to submit a solution to the commissioners, who determined his solution had numerous mistakes. As a result, no prize was given to Wallis or the other person. He was embittered by Pascal's demeaning remarks about his solution. However, Wallis went on to write his first tract on the cycloid, containing his solution, which was still not 'completely correct'.

In his second tract, Wallis used his method of interpolation and arithmetic quadrature procedure to determine the quadrature of a cissoid. He worked with the idea that the cissoid was associated with the circle $y^2 = x(1-x)$ with a diameter 1. He applied two proportion properties of the cissoid and obtained the equation $y = x^{3/2}(1-x)^{-1/2}$. Wallis had found in his *Arithmetic Infinitorum* an expression for the area $A = \pi$ of a circle with radius 1, so he was able to find the area of the generating semicircle of the cissoid, which was $A/8$. The generating semicircle, where $m = 1$, belonged to the family of curves $y = x^{m/2}(1-x)^{1/2}$. Wallis continued to find different quadratures on the interval [0, 1] using the methods in his *Arithmetica Infinitorum* for other integer values of m. He was able to create a recurrence relation from which he could show that for $m = 3$, the area was half of the area of the generating semicircle or $A/16$. Finally, Wallis turned to the family of curves $y = x^{3/2}(1-x)^{m/2}$ and found that for $m = -1$, the equation was for a cissoid. He continued to find the quadrature for various values of m. As a result, he made the generalization that 'the quadrature of a cissoid is three times the quadrature of the generating semicircle'.

When Wallis moved to Oxford, several of the individuals that he had met with in London to study mathematics and philosophy formed the *Oxonian Society*. By 1659, Oxford members would meet at Gresham College along with many new associates. In 1660, *The Royal Society, London*, was formed and Wallis was elected in 1663 as an original fellow. During this period, he attained a Doctor of Divinity and was appointed as a Royal Chaplain and a member of a committee to revise the Prayer Book.

During 1662, the Commissioner of the British Navy asked Wallis to assist with the design of a ship's hull which would have the least resistance. The Commissioner wanted to use a special geometric solid that would be used to shape a tool to build such ship hulls. The tool was known as the Shipwright's Circular Wedge, which made progressively curved strips of wood. Wallis would develop his mathematical principle to find the variation of curvature and to model the ship hulls. He would later publish an article regarding the results.

The Royal Society began publishing *Philosophical Transactions* in 1665, allowing Wallis to submit numerous articles on a wide range of subjects. In Volume 1, Wallis published an article on the hypothesis on the flux and the reflux of the sea. The focus of the article was on the high tides which occurred close to the vernal equinox and the autumnal equinox. Wallis believed the cause of these tides was the combination of the gravitational pull by the Moon and the Sun.

Following this article, Wallis turned his interest to the laws regarding the impact of bodies. Along with Sir Christopher Wren and Christian Huygens, Wallis was selected by the Royal Society in 1668 to examine René Descarte's *Laws of Motion*.

After studying Descarte's work, Wallis wrote 'A summary account of the general laws of motion' in 1668 and read his paper to the Royal Society in November of the same year. He discussed the collision of two hard bodies, where no distortion occurred and moved as one after impact. Based on the laws he established, Wallis showed that 'two bodies having equal momenta will upon impact, reduce each other to rest'.

His research concerning elastic bodies was included in his next paper 'Mechanics sive de motu', 1669–1671. This work was a significant contribution to mechanics and to the treatment of percussions. The brilliant mathematician David Gregory praised Wallis' work saying, 'The treatise was a full and complete mechanics in which he lays the true and solid foundation of the science, which had not been done before'.

Like many mathematicians, Wallis was also interested in music theory, and as a result, translated Ptolemy's *Harmonics* into Latin. In 1682, Wallis published his work, 'The harmonics of the ancients compared with today's'. The main focus of this work was the relationship between the ancient thinking on musical harmony by Ptolemy and the thinking on the subject during the 17th century in England.

By the early 1680s, Wallis had made several contributions in many fields. His contributions continued after being selected in 1682 as President of the Royal Society of Oxford. He turned his attention briefly to improving their relations with the Royal Society in London. Through his endeavours, the Royal Society of Oxford began to flourish.

After several years of hard work, Wallis published the 'Treatise of algebra, both historical and practical' in 1685. The book began with an extensive discussion of algebra from antiquity to the 17th century. He followed up with a fairly biased review of the algebraic works of his contemporaries. While he praised his fellow countrymen's publications on algebra, he criticized the work done by great mathematicians such as René Descartes and others from outside of England.

In the final portion of this work, Wallis was the first to contribute significantly to the 'geometric treatment of imaginary quantities'. He elaborated on the use of symbolical algebra, showing that it provided one with a brief and suitable way of notation. Some of his best material covered solutions of quadratic and cubic equations. Wallis also defended the method of induction by claiming induction by itself can be sufficient in demonstrating a general result.

Included with his treatise on algebra, Wallis published his *Treatise of Combinations, Alternations, and Aliquot Parts* and a few other works he had previously published. Despite the controversial problems, the treatise on algebra was very popular; and with regard to his work on equations, it provided a valuable source from which his contemporaries and algebraists drew much inspiration.

His next publication was entitled *Institutio Logicae*, completed when he turned seventy. According to Wallis, his purpose was not the same as those of other writings on logic. He believed that the business of logic is to teach how ‘to manage our reason to the best advantage, with strength of argument and in good order…’.

During the final decade of his life, he continued to decipher material. For his services, Queen Mary II offered him a deanery position at Hereford in 1692, but he did not accept it. Wallis also consolidated several of his books and articles into three volumes, which were entitled *Opera Mathematica*. The first volume was published in 1693 and the final volume in 1699.

He then published his theological material regarding the doctrine of The Holy Trinity in the early 1690s. In this work, he explained his doctrine using a geometrical model. He used a cube to relate the length, width, and height to the Father, Son, and Holy Ghost. His material on the Trinity was very popular throughout the various religious sects. He continued his theological work during his final years.

Wallis was fortunate to enjoy great health during his life. He was blessed with an incredible memory, which allowed him to perform large mathematical calculations in his head. Though known to have serious disputes with several mathematicians, he had many friends among his contemporaries. Wallis passed away on 28 October 1703 at the age of 86.

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Some remarkable continued fractions

$$1 = \cfrac{x+1}{x + \cfrac{x+2}{x+1+\cfrac{x+3}{x+2+\cfrac{x+4}{x+3+\cdots}}}} \qquad (x > 0)$$

$$\frac{x-a}{x+a} = \cfrac{x-a}{a+\cfrac{(x+a)^2-a^2}{a+\cfrac{(x+2a)^2-a^2}{a+\cfrac{(x+3a)^2-a^2}{a+\cdots}}}} \qquad (x \neq 0)$$

$$\frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n} = \cfrac{n}{1x+\cfrac{n^2-1}{3x+\cfrac{n^2-2^2}{5x+\cfrac{n^2-3^2}{7x+\cdots}}}} \qquad (x \notin [-1,1])$$

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Spiros P. Andriopoulos

Van Aubel's Theorem using Complex Numbers

GUIDO LASTERS and DAVID SHARPE

We show how complex numbers can be used to give a neat proof of van Aubel's theorem and an extension of it, and leave readers with a challenge.

Van Aubel (1830–1906) was a teacher of mathematics in the Royal Atheneum in Antwerp. The theorem which bears his name goes back to 1878 (see reference 1). Start with any quadrilateral in the plane and construct squares on its four edges outside the quadrilateral. Join the centres of opposite squares to give two line segments. Then these two line segments are equal in length and at right angles (see figure 1).

Readers will find various proofs of this delightful result on the web (e.g. reference 2). In a recent article in *Mathematical Spectrum*, Paul Glaister gave a proof using vectors (see reference 3). In this article we will use complex numbers to prove van Aubel's theorem and also to prove an extension of it.

We consider the quadrilateral in the Argand diagram and denote by a, b, c, d the complex numbers which represent its vertices (see figure 1). The centre of the square on the edge ab is found by moving from the centre of that edge at right angles to it a distance of half the length

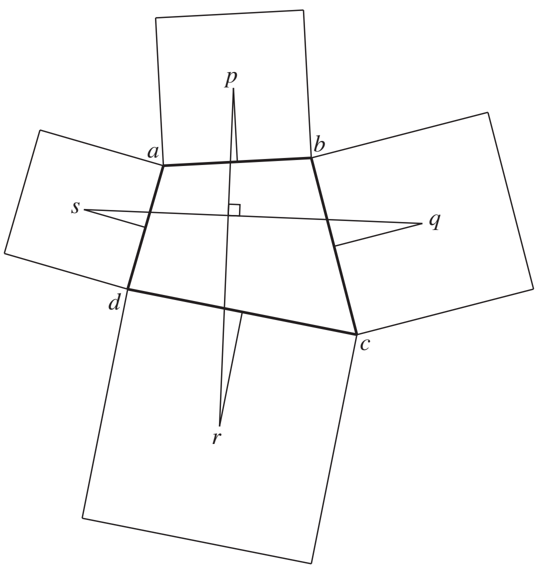

Figure 1

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of ab. Thus, it is represented by the complex number

$$p = \frac{a+b}{2} + \frac{1}{2}(b-a)\mathrm{i} = a\frac{1-\mathrm{i}}{2} + b\frac{1+\mathrm{i}}{2}.$$

The centres of the three other squares are represented by the complex numbers

$$q = b\frac{1-\mathrm{i}}{2} + c\frac{1+i}{2}, \qquad r = c\frac{1-\mathrm{i}}{2} + d\frac{1+i}{2}, \qquad s = d\frac{1-\mathrm{i}}{2} + a\frac{1+i}{2}.$$

Now,

$$r - p = (c-a)\frac{1-i}{2} + (d-b)\frac{1+\mathrm{i}}{2}.$$

We can turn the line segment pr anticlockwise through $90°$ by multiplying by i to give

$$\mathrm{i}(r-p) = (c-a)\frac{1+i}{2} + (b-d)\frac{1-\mathrm{i}}{2}.$$

But this is just $q - s$, which tells us that the line segments pr and qs are equal in length and at right angles, which is van Aubel's theorem.

We can prove something more if we start with a parallelogram rather than just a quadrilateral. Now we can say that the centres of the four squares form the vertices of a square (see figure 2). To see this, first note that

$$p - q = (a-b)\frac{1-i}{2} + (b-c)\frac{1+\mathrm{i}}{2}.$$

We can turn the line segment pq $90°$ anticlockwise about q by multiplying by i to give

$$\mathrm{i}(p-q) = (a-b)\frac{1+i}{2} + (c-b)\frac{1-\mathrm{i}}{2}.$$

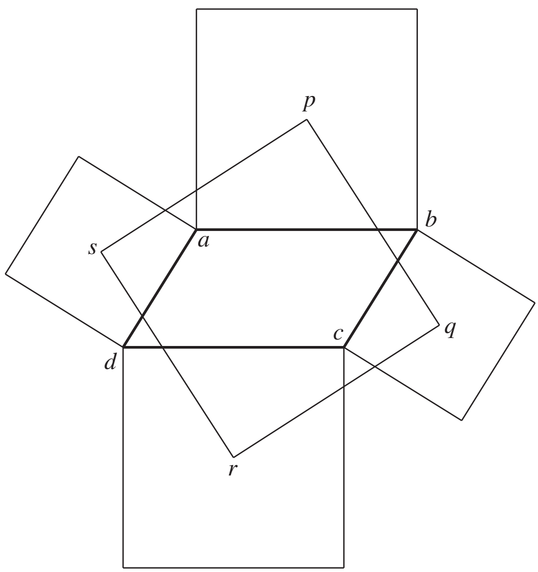

Figure 2

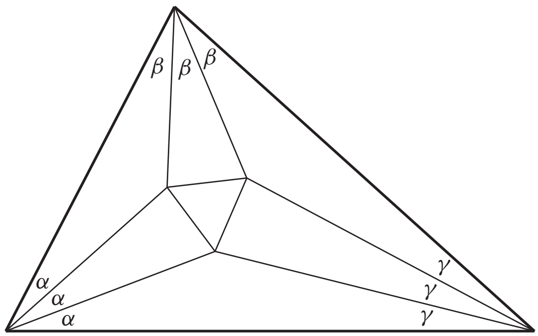

Figure 3

Now,

$$r - q = (c - b)\frac{1 - i}{2} + (d - c)\frac{1 + \mathrm{i}}{2}.$$

But opposite edges of the parallelogram are parallel and equal in length, so that

$$a - b = d - c,$$

which tells us that

$$\mathrm{i}(p - q) = r - q,$$

so that the line segments pq and qr are equal in length and at right angles. A cyclic rotation of the letters will give that qr and rs, rs and sp, and sp and pq are equal in length and at right angles, so that $pqrs$ is a square.

In case this all seems a bit too easy, consider Morley's theorem, which goes back to 1899. Start with any triangle in the plane and trisect the angles. The three points of intersection of adjacent trisectors form an equilateral triangle (see figure 3). The geometrical proofs of this theorem are not straightforward, and it seems as though it ought to be possible to prove it using complex numbers. But so far we have failed. Perhaps readers may be more successful. You may also be interested in the interactive GeoGebra links in references 4 and 5 to van Aubel's and Morley's theorems respectively.

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Four Functions and a Bijection

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By applying the floor and ceiling operations to real numbers, a real function can be discretized to give a new function on the integers. Much information will, of course, be lost in such a process, but perhaps some intrinsic property concerning order may be retained. Even for the integers, inequalities can be tricky, and the article may serve as a useful exercise in their manipulation.

1. Four integer functions from a real bijection

Let $\mathbb{U} = [1, \infty)$, the set of real numbers, $x \geq 1$, and $\phi\colon \mathbb{U} \to \mathbb{U}$ be an increasing bijection. The inverse ψ of ϕ is also such a bijection, and the two composite functions $\psi \circ \phi$ and $\phi \circ \psi$ are the identity function on $\mathbb{U}$, that is $\psi(\phi(x)) = x$ and $\phi(\psi(t)) = t$, for $x, t \in \mathbb{U}$.

Let $\mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. For $x \in \mathbb{U}$, the floor $\lfloor x \rfloor$ is the greatest integer $n \leq x$, and the ceiling $\lceil x \rceil$ is the least integer $n \geq x$, so that $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ with $\lfloor x \rfloor, \lceil x \rceil \in \mathbb{N}$; thus, $\lceil x \rceil = \lfloor x \rfloor + 1$, unless $x \in \mathbb{N}$, in which case $\lfloor x \rfloor = \lceil x \rceil$. Next, for a function $f : \mathbb{U} \to \mathbb{U}$, we define $f^+, f^- : \mathbb{N} \to \mathbb{N}$ by

$$f^+(n) = \lceil f(n) \rceil, \quad f^-(n) = \lfloor f(n) \rfloor, \qquad n \in \mathbb{N}.$$

Returning to ϕ and ψ, there are now four composite functions $f_i\colon \mathbb{N} \to \mathbb{N}$, $i = 1, 2, 3, 4$, given by

$$f_1(n) = \psi^-(\phi^+(n)), \quad f_2(n) = \psi^+(\phi^-(n)),$$
$$f_3(n) = \phi^-(\psi^+(n)), \quad f_4(n) = \phi^+(\psi^-(n)), \qquad n \in \mathbb{N},$$

so that each $f_i(1) = 1$. Since ϕ and ψ are strictly increasing in $\mathbb{U}$, each f_i is increasing in $\mathbb{N}$, although not necessarily strictly so. A natural question is: What mild conditions can be imposed on ϕ so that some f_i becomes the identity function on $\mathbb{N}$?

2. A condition on ϕ, and the sequences (h_n) and (k_n)

It turns out that if ϕ satisfies

$$\phi(n + 1) \geq \phi(n) + 1, \qquad n \in \mathbb{N}, \tag{1}$$

then f_1, f_2 are the identity function on $\mathbb{N}$, and f_3, f_4 are determined by two integer sequences (h_n) and (k_n), defined, for $n = 1, 2, \ldots$, by letting

$$\begin{aligned} h_n &= \text{the greatest } h \in \mathbb{N} \text{ such that } \psi(h) \leq n, \\ k_n &= \text{the least } k \in \mathbb{N} \text{ such that } \psi(k) \geq n. \end{aligned} \tag{2}$$

Both sequences (h_n) and (k_n) are strictly increasing: for suppose, if possible, that $h_n = h_{n+1}$ for some n, then the unit interval $n < x \leq n + 1$ would be free of images $\psi(h)$, so that there would be $h \in \mathbb{N}$ such that $\psi(h) \le n < n+1 < \psi(h+1)$; hence, $h \le \phi(n) < \phi(n+1) < h+1$, contradicting (1). Note that

$$\begin{aligned} n-1 < \psi(h_n) \le n \le \psi(k_n) < n+1, \\ \phi(n-1) < h_n \le \phi(n) \le k_n < \phi(n+1), \qquad n = 1, 2, \ldots, \end{aligned}$$

and that if $\psi(h_n) \in \mathbb{N}$ then $k_n = h_n$, otherwise $k_n = h_n + 1$. Thus, $h_1 = k_1 = 1$ and, writing $h_0 = 0$, the set $\mathbb{N}$ is partitioned into intervals of integers of the form $\{h_{n-1} < h \le h_n\}$, and another partitioning is $\{k_n \le k < k_{n+1}\}$, with $n = 1, 2, \ldots$.

3. Determination of f_i

Theorem 1 *Suppose that ϕ satisfies (1), and let h_n, k_n be defined by (2). Then f_1 and f_2 are the identity function on $\mathbb{N}$. Also, for $n = 1, 2, \ldots$, we have*

$$f_3(h) = h_n, \quad h_{n-1} < h \le h_n; \qquad f_4(k) = k_n, \quad k_n \le k < k_{n+1}.$$

Proof Let $n \in \mathbb{N}$, and set $k = \phi^+(n)$, so that $k-1 < \phi(n) \le k$. We also have $k < \phi(n+1)$, since otherwise we would have $k-1 < \phi(n) < \phi(n+1) \le k$, contradicting (1). From $\phi(n) \le k < \phi(n+1)$, and ψ being strictly increasing, we deduce that $n \le \psi(k) < n+1$; hence, $f_1(n) = \psi^-(\phi^+(n)) = \psi^-(k) = n$.

Similarly, on writing $h = \phi^-(n)$, so that $\phi(n-1) < h \le \phi(n) < h+1$, and hence $n-1 < \psi(h) \le n$, we deduce that $f_2(n) = \psi^+(\phi^-(n)) = \psi^+(h) = n$. Therefore, f_1 and f_2 are the identity function on $\mathbb{N}$.

Next, for $n = 1, 2, \ldots$, let $h_{n-1} < h \le h_n$, so that $n-1 < \psi(h) \le \psi(h_n) \le n < \psi(h_n+1)$. We deduce that

$$\psi^+(h) = \psi^+(h_n) = n \quad \text{and} \quad h_n \le \phi(n) < h_n + 1,$$

so that $f_3(h) = \phi^-(\psi^+(h)) = \phi^-(n) = h_n$.

Similarly, let $k_n \le k < k_{n+1}$, so that $\psi(k_n - 1) < n \le \psi(k_n) \le \psi(k) < n+1$. We deduce that

$$\psi^-(k) = \psi^-(k_n) = n \quad \text{and} \quad k_n - 1 < \phi(n) \le k_n,$$

so that $f_4(k) = \phi^+(\psi^-(k)) = \phi^+(n) = k_n$. The theorem is proved.

4. Two examples

For our first example, we let

$$\phi(x) = x^3, \quad x \ge 1, \qquad \text{and} \qquad \psi(t) = \sqrt[3]{t}, \quad t \ge 1.$$

Then ϕ and ψ form a pair of bijections on $\mathbb{U} = [1, \infty)$ which are inverse functions of each other, and $\phi \circ \psi$ and $\psi \circ \phi$ are the identity function on $\mathbb{U}$. It is clear that condition (1) is satisfied so that, by theorem 1, $f_1 = \psi^- \circ \phi^+$ and $f_2 = \psi^+ \circ \phi^-$ are the identity function on $\mathbb{N}$. To determine the composite functions $f_3 = \phi^- \circ \psi^+$ and $f_4 = \phi^+ \circ \psi^-$, we note that $\psi(n^3) = n$, so that $h_n = k_n = n^3$, that is

$$(k_n) = (h_n) = (1, 8, 27, 64, 125, \ldots, n^3, \ldots).$$

Table 1

n	$\phi(n)$	$\phi^+(n)$	$\phi^-(n)$	$\psi(\phi^+(n))$	$\psi(\phi^-(n))$	$f_1(n)$	$f_2(n)$
10	8 103.08 ...	8 104	8 103	10.000 11 ...	9.999 98 ...	10	10
12	59 874.14 ...	59 875	59 874	12.000 014 ...	11.999 997 ...	12	12
n	$\psi(n)$	$\psi^+(n)$	$\psi^-(n)$	$\phi(\psi^+(n))$	$\phi(\psi^-(n))$	$f_3(n)$	$f_4(n)$
402	6.996 4 ...	7	6	403.428 ...	148.413 ...	403	149
403	6.998 9 ...	7	6	403.428 ...	148.413 ...	403	149
404	7.001 4 ...	8	7	1 096.63 ...	403.428 ...	1 096	404
405	7.003 8 ...	8	7	1 096.63 ...	403.428 ...	1 096	404

It follows from theorem 1 that, for $n = 1, 2, \ldots,$

$$f_3(h) = \phi^-(\psi^+(h)) = n^3, \quad (n-1)^3 < h \le n^3;$$
$$f_4(k) = \phi^+(\psi^-(k)) = n^3, \quad n^3 \le k < (n+1)^3.$$

For another example, we take the exponential function exp and the logarithm function log, which form a pair of inverse functions, mapping between the intervals $0 \le x < \infty$ and $1 \le t < \infty$. If we now set

$$\phi(x) = \exp(x-1), \quad x \ge 1, \qquad \text{and} \qquad \psi(t) = 1 + \log t, \quad t \ge 1,$$

then ϕ and ψ form a pair of bijections on $\mathbb{U} = [1, \infty)$ which are inverse functions of each other, and $\phi \circ \psi$ and $\psi \circ \phi$ are again the identity function on $\mathbb{U}$. Since

$$\phi(n+1) - \phi(n) = \exp(n) - \exp(n-1) = \exp(n-1)(\mathrm{e}-1) \ge \mathrm{e} - 1 > 1, \qquad n = 1, 2, \ldots,$$

condition (1) is satisfied so that $f_1 = \psi^- \circ \phi^+$ and $f_2 = \psi^+ \circ \phi^-$ are again the identity function on $\mathbb{N}$.

For the composite functions $f_3 = \phi^- \circ \psi^+$ and $f_4 = \phi^+ \circ \psi^-$, we note that $\mathrm{e}^k \notin \mathbb{N}$ for $k \in \mathbb{N}$, a consequence of e being a transcendental number (see reference 1, theorem 204). It follows that $\log n \notin \mathbb{N}$ for $n > 1$, so that $k_n = h_n + 1$. Computation shows that

$$(h_n) = (1, 2, 7, 20, 54, 148, 403, 1\,096, 2\,980 \ldots),$$

so that

$$(k_n) = (1, 3, 8, 21, 55, 149, 404, 1\,097, 2\,981 \ldots),$$

and the values for $f_3(h) = \phi^-(\psi^+(h))$ and $f_4(k) = \psi^+(\psi^-(k))$ are delivered accordingly by theorem 1. See table 1 for some of the values involved in the definition of $f_i(n)$.

Reference

1 G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Clarendon Press, Oxford, 1960), 4th edn.

Peter Shiu *is a retired Reader in Pure Mathematics at Loughborough University. He has written many expository articles, several of which have appeared in Mathematical Spectrum. His research interest is in number theory.*

Dividing in base arithmetic

Example: divide 4071_8 by 6_8.

Conventional textbook method

$$
\begin{array}{r@{\,}l}
 & 5\;3\;6_8 \\
6_8 & \overline{)4\;0\;7\;1_8} \\
 & \underline{3\;6_8} \\
 & 2\;7_8 \\
 & \underline{2\;2_8} \\
 & 5\;1_8 \\
 & \underline{4\;4_8} \\
 & 5_8 \text{ rem}
\end{array}
$$

Base 8 table

$6_8 \times 1 = 6$
$6_8 \times 2 = 12 = 14_8$
$6_8 \times 3 = 18 = 22_8$
$6_8 \times 4 = 24 = 30_8$
$6_8 \times 5 = 30 = 36_8$
$6_8 \times 6 = 36 = 44_8$
$6_8 \times 7 = 42 = 52_8$

The answer is 536_8 rem 5.

New conversion method

$6_8\,)\,4\;0\;7\;1_8$	convert 40_8 to base 10
$\downarrow$	
$6\,)\,3\;2\,(\,5$	divide normally as in base 10
$-3\;0$	subtract normally as in base 10
$2\;7_8$	convert 27_8 to base 10
$\downarrow$	
$6\,)\,2\;3\,(\,3$	divide normally as in base 10
$-1\;8$	subtract normally as in base 10
$5\;1_8$	convert 51_8 to base 10
$\downarrow$	
$6\,)\,4\;1\,(\,6$	divide normally as in base 10
$-3\;6$	subtract normally as in base 10
5 rem	

The answer is 536_8 rem 5.

Note the following points for the new conversion method.

(i) At each step, before performing the normal division, you must first convert the base 8 numeral to base 10.

(ii) This method does not require construction of the base 8 table, or subtraction in base 8.

Delaware State University **Lyndon O. Barton**

Complex Numbers and Desargues' Theorem

GUIDO LASTERS and DAVID SHARPE

We show how Desargues' theorem in geometry can be proved using complex numbers and illustrate how a multiple use of Desargues' theorem can be used to prove a result in geometry.

In our companion article in this issue 'Van Aubel's theorem using complex numbers' we showed how complex numbers can be used to give a neat proof of Van Aubel's theorem in geometry. This spurred us on to try to do a similar thing with the better-known Desargues' theorem. According to Wikipedia, Desargues did not publish his theorem but it was in an appendix, entitled "Universal method of M. Desargues' for using perspectives", of a book on perspective by a pupil of his, Abraham Bosse, in 1648.

Desargues' theorem is illustrated in figure 1. Triangles ABC and $A'B'C'$ are such that AA', BB', CC' meet at a point P, say. We say that they are *in perspective* from P and call P their *point of perspectivity*. The sides BC and $B'C'$ are produced to meet at X, CA and $C'A'$ are produced to meet at Y, and AB and $A'B'$ are produced to meet at Z. Desargues' theorem says that X, Y, Z are collinear. We call X, Y, Z their *axis of perpectivity*. A beautiful theorem, we hope readers will agree.

We now show how this may be proved using complex numbers. We choose P as the origin of the Argand diagram, and denote by a, b, c the complex numbers representing A, B, C respectively, and by ka, lb, mc the complex numbers representing A', B', C' respectively, where k, l, m are real numbers. A point on BC will be represented by a complex number of the form $\lambda b + (1-\lambda)c$ for some real number λ, and a point on $B'C'$ by $\mu lb + (1-\mu)lc$ for

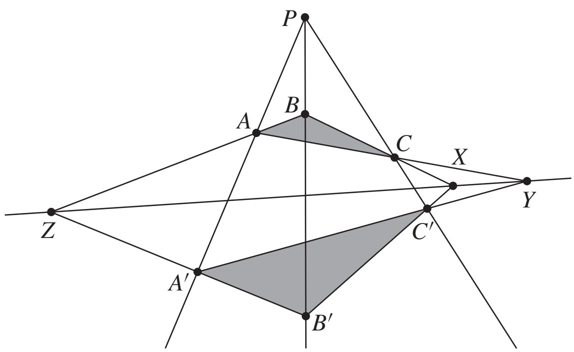

Figure 1

Accepted by C. Jordan, Pure Mathematics Editor.

some real number μ, so to obtain X we write

$$\lambda b + (1-\lambda)c = \mu l b + (1-\mu) m c.$$

If we equate coefficients of b and c we get

$$\lambda = \mu l, \qquad 1-\lambda = (1-\mu)m,$$

so that

$$\mu(l-m) = 1-m$$

and

$$\mu = \frac{1-m}{l-m}, \qquad \lambda = \frac{l(1-m)}{l-m}.$$

Hence, X will be represented by the complex number

$$\begin{aligned} x &= \lambda b + (1-\lambda)c \\ &= \frac{l(1-m)}{l-m} b + \left(1 - \frac{l(1-m)}{l-m}\right) c \\ &= \frac{l(1-m)b - m(1-l)c}{l-m}. \end{aligned}$$

We should point out that, if $l = m$, then BC and $B'C'$ will be parallel and so will not meet. Or we could say that BC and $B'C'$ will meet at infinity. This is a degenerate case of the theorem, which becomes trivial in this case. By cyclic rotation, CA and $C'A'$ meet at Y represented by

$$y = \frac{m(1-k)c - k(1-m)a}{m-k}$$

and AB and $A'B'$ meet at Z represented by

$$z = \frac{k(1-l)a - l(1-k)b}{k-l}.$$

In order that X, Y, Z are collinear, we need

$$z = x + \nu(y-x)$$

for some real number ν, i.e.

$$(1-\nu)x + \nu y - z = 0.$$

Or, symmetrically, we require real numbers α, β, γ not all zero such that

$$\alpha x + \beta y + \gamma z = 0,$$

with $\alpha + \beta + \gamma = 0$. This is the tricky bit. Put

$$\alpha = \frac{k-1}{(k-l)(k-m)}, \qquad \beta = \frac{l-1}{(l-k)(l-m)}, \qquad \gamma = \frac{m-1}{(m-k)(m-l)}.$$

We leave readers to check that

$$\alpha x + \beta y + \gamma z = 0 \quad \text{and} \quad \alpha + \beta + \gamma = 0.$$

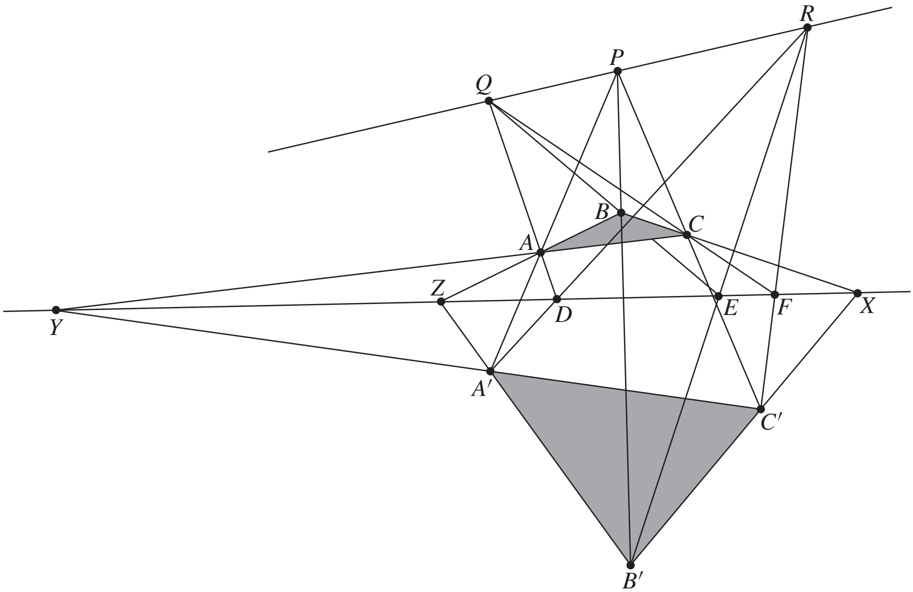

Figure 2

We finally point out that, if $\alpha = \beta = \gamma = 0$, then $k = l = m = 1$ and the two triangles coincide, in which case the theorem loses its point.

As an illustration of the use of Desargues' theorem, we start with two triangles ABC, and $A'B'C'$ in perspective from the point P as before, with XYZ as their axis of perspectivity. Let Q be a point distinct from P. We also make it distinct from the vertices of the triangles to avoid complications. Draw the straight lines QA, QB, QC, produced as necessary, to meet the line of perspectivity at D, E, F respectively. Draw the straight lines $A'D$, $B'E$, $C'F$, produced as necessary. We claim that these three lines meet at a single point R which lies on the straight line through P and Q (see figure 2).

To prove this we note that triangles $BB'E$ and $CC'F$ are in perspective from X so, by Desargues' theorem, they have an axis of perspectivity, which will be the line through P and Q. Hence, $B'E$ and $C'F$ meet at a point R on PQ produced. Similarly, triangles $CC'F$ and $AA'D$ are in perspective at Y, so they have an axis of perspectivity, which will again be the line through P and Q. Thus, $A'D$ and $C'F$ meet at a point on PQ produced, which can only be R. This completes our proof. It is redundant but worth pointing out that triangles $AA'D$ and $BB'E$ are in perspective from Z.

Camilla Jordan has used GeoGebra to show Desargues' theorem and our application in action. Readers can move the vertices of the triangles or the points P and Q. Curiously, moving P and Q does not move the axis of perspectivity of the triangles. This can be found on the *Mathematical Spectrum* website; or see reference 1. Fun to play with!

Reference

1 http://www.camillaanddavidjordan.org.uk/desargues.html.

*The authors are retired teachers of mathematics, **Guido Lasters** at a school in Tienen, Belgium, and **David Sharpe** at the University of Sheffield, UK. The latter is Editor of Mathematical Spectrum.*

When the Coefficients are the Roots

JONNY GRIFFITHS

When are the coefficients of a polynomial equation equal to its roots?

The other day I ran into this question:

> if the roots of the quadratic equation $x^2 + ax + b = 0$ are a and b, what are the possible values for a and b?

The question is straightforwardly solved. We say $x^2 + ax + b = (x - a)(x - b) = x^2 - (a + b)x + ab$. This is true for all values of x, so we can equate the coefficients of each side, giving $-a - b = a$, and $ab = b$. Thus, on solving, the possibilities for (a, b) are $(0, 0)$ or $(1, -2)$.

So far, so good, but the question to my mind immediately invited generalisation. If we are given the equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$, then what possibilities for $(a_{n-1}, a_{n-2}, \ldots, a_0)$ are there if these values are also the roots of the equation?

We can note firstly that $(0, 0, \ldots, 0)$ will always be a solution (we will call this the *trivial solution*).

Secondly, if $(a_{n-1}, a_{n-2}, \ldots, a_0)$ is a solution for the degree-n equation, then by multiplying our degree-n equation by x, we see that $(a_{n-1}, a_{n-2}, \ldots, a_0, 0)$ will be a solution in the degree-$(n + 1)$ case. Thus, if we add 0s as we go, the number of solutions we find will be nondecreasing as n increases.

What happens with the degree-3 case, the cubic equation? We have $x^3 + ax^2 + bx + c = (x - a)(x - b)(x - c)$, and on expanding and comparing coefficients, we find

$$[2a + b + c, ab + ac + bc - b, c(ab + 1)] = [0, 0, 0].$$

Let me explain the notation here. The algebra rapidly becomes unfriendly with this problem, and the use of a computer algebra package becomes essential. I am using Derive 6, but there are many alternatives – if you do not currently use such a program, it will open up a wealth of mathematics investigations to you if you do. When it comes to algebra, Derive is infinitely quicker and more accurate than I can ever hope to be.

Here, $[x, y, z]$ denotes a three-dimensional vector, a format which makes the substitution of values within Derive easy. We effectively have three equations in three unknowns. Solving $2a + b + c = 0$ gives $c = -2a - b$ and, substituting in, we have $[0, -2a^2 - 2ab - b^2 - b, -(2a + b)(ab + 1)] = [0, 0, 0]$. Now, putting $(2a + b)(ab + 1) = 0$, we have $b = -2a$ or $b = -1/a$.

Taking the first of these options gives $[0, 2a(1-a), 0] = [0, 0, 0]$, and so $a = 0$ or 1. These give us the trivial solution $(0, 0, 0)$, and $(1, -2, 0)$. Taking $b = -1/a$ instead gives $[0, -2a^2 + 1/a - 1/a^2 + 2, 0]$, and solving for a here gives four solutions, two complex (we will ignore these), $a = 1$ and (curiously) $a = 0.565\,197\,717\,3\ldots$. Substituting back, we get the triplets for (a, b, c) of $(1, -1, -1)$ and $(0.565\,197\,717\,3\ldots, -1.769\,292\,354\ldots, 0.638\,896\,919\,3\ldots)$.

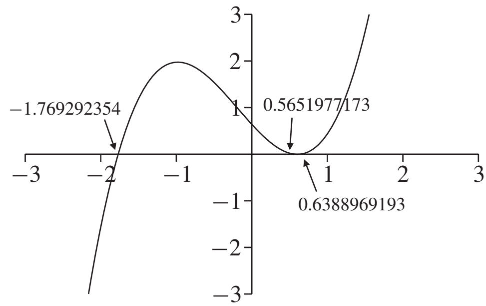

Figure 1 Plot of $y = x^3 + 0.565\,197\,717\,3x^2 - 1.769\,292\,354x + 0.638\,896\,919\,3$.

So we have four real solutions to our degree-3 problem. It is pleasing to ask a graphing program to plot this last possibility (see figure 1).

Can we make a conjecture now? Doing this on the basis of $n = 2$ and 3 alone seems rash. But it does seem appropriate to ask, will there always be one solution to the degree-n equation where all the coefficients are nonzero? (We have two such solutions here.) And what happens to these coefficients as n gets larger – will they perhaps form a sequence that tends to a limit?

On to the quartic. We have (with the added zeroes) four solutions already – is there a new one to be found, where none of the coefficients is zero? We have

$$x^4 + ax^3 + bx^2 + cx + d = (x - a)(x - b)(x - c)(x - d).$$

From this, Derive tells us that

$$\begin{aligned}&[2a + b + c + d, (a(b + c + d) + b(c + d - 1) + cd),\\ &(a(b(c + d) + cd) + c(bd + 1)), d(abc - 1)] = [0, 0, 0, 0].\end{aligned}$$

We have $d = -2a - b - c$, and so $[0, -2a^2 - a(2b + 2c) - b^2 - b(c + 1) - c^2, -2a^2(b + c) - a(b^2 + 3bc + c^2) - c(b^2 + bc - 1), (2a + b + c)(1 - abc)] = [0, 0, 0, 0]$. From the final element, we see that $c = -2a - b$, or $c = 1/ab$. Substituting in $c = -2a - b$ leads to solutions we have already with $d = 0$ added, that is $(0, 0, 0, 0)$, $(1, -2, 0, 0)$, $(1, -1, -1, 0)$, and $(0.565\,197\,717\,3\ldots, -1.769\,292\,354\ldots, 0.638\,896\,919\,3\ldots, 0)$. But $c = 1/ab$ is more interesting. This yields

$$\left[0, -\frac{f(a, b)}{a^2b^2}, -\frac{g(a, b)}{a^2b^2}, 0\right],$$

where f is a degree-6 polynomial, and g is degree-7. Derive is certainly earning its keep, but asking it to solve $f(a, b) = 0$ or $g(a, b) = 0$ for a or b, however, sees it down tools. We will need a different strategy. How about plotting the curves $f(a, b) = 0$ and $g(a, b) = 0$? Their intersection points will then give us what we need. This leads to (using Autograph) a closed curve for $f(a, b) = 0$, and a more widely-spread curve for $g(a, b) = 0$ (see figure 2).

The uppermost of the intersection points, $(1, -1)$, leads us to the solution $(1, -1, -1, 0)$, which we have already. But the lower point at $(1, -1.754\,877\,7)$ gives us a fresh solution, that is, $(1, -1.754\,877\,7\ldots, -0.569\,840\,28\ldots, 0.324\,717\,98\ldots)$; see figure 3.

So our conjecture has proven to be true for $n = 4$ at least, although this time we have only a single solution for $(a_{n-1}, a_{n-2}, \ldots, a_0)$ that is completely nonzero. Now on to the quintic,

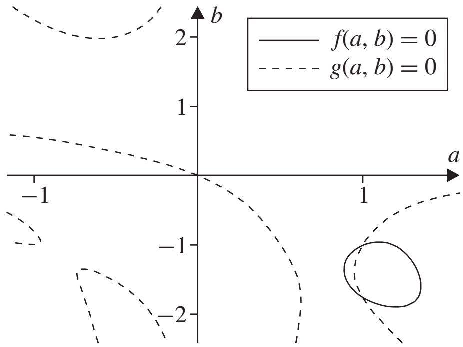

Figure 2

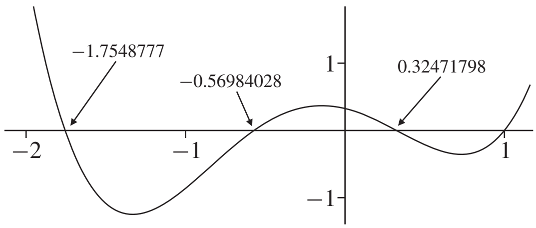

Figure 3 Plot of $x^4 + x^3 - 1.754\,877\,7x^2 - 0.569\,840\,28x + 0.324\,717\,98 = y$.

which initially solves much as before. We have $e = -2a - b - c - d$, and on substituting, we have $d = -2a - b - c$, which leads on to solutions we already have, or $d = -1/abc$, which gives us three polynomials in a, b, and c that must equate to zero. The hope is that we can now move into three dimensions, and plot three (fiendish-looking) surfaces; the points where they intersect should give us more possible values for a, b, and c. The power of Autograph is evident here – it plots the first two surfaces without a murmur. But try to plot the third, and – nothing. It is as if we are asking for a plot of $(x^2 + y^2 + z^2 + 1)(x^2y^2z^2 + 1) = 0$; there are simply no real points that work.

And so my conjecture fails and, not for the first time, the quintic proves to be a stumbling block. We are, in the degree-5 case it seems, asking for too much. Our new revised conjecture must be that for degrees higher than four, we can never find a solution for $(a_{n-1}, a_{n-2}, \ldots, a_0)$ where all the a_i are nonzero. Maybe someone out there is *au fait* with a package even more powerful than Derive – I would be delighted to hear of a counterexample if so.

Jonny Griffiths *taught mathematics at Paston Sixth Form College in Norfolk for over twenty years. He has studied mathematics and education at Cambridge University, the Open University, and the University of East Anglia. Possible claims to fame include being a member of 'Harvey and the Wallbangers', a popular band in the 1980s, and playing the character Stringfellow on the childrens' television programme 'Playdays'. He currently works for Underground Mathematics.*

Computing Geographical Distances; Derivation and Application

CHASE ASHBY and JOSH THOMPSON

We explain two existing methods for computing distances on Earth, one using great circles (orthodromes) and the other using great spirals (loxodromes). We derive formulas for these approaches, and use each of them to calculate the shoreline distances around each of the Great Lakes. The resulting calculations compare favourably with established statistics of the Great Lakes.

Introduction

In Euclidean geometry two points on a plane determine a unique straight line passing through them. In spherical geometry the notion of 'straight' can be interpreted in multiple ways. For example, a straight line connecting two points on a sphere could refer to the arc connecting them which is contained in a plane passing through the centre of the sphere. Alternatively, a straight line connecting these points could be defined as the path connecting the points which takes a constant bearing from one point to the other. These different approaches give different ways to measure distance on a sphere. We derive two methods for computing distance based on these ideas, apply them both to real data, and compare the results.

Loxodromes

A *loxodrome* is a path spiralling from the North Pole to South Pole that crosses each meridian (a great circle passing through the North Pole and South Pole) at the same angle. Two arbitrary geographic points are connected by many different loxodromic paths. There is, however, a unique shortest path corresponding to the path that spirals the least as it makes its way from point to point. In this method, we define the distance between two points on Earth to be the length of the shortest loxodromic path connecting the two points. To find this length we first use a Mercator projection to map the spiralling path into a flat plane. Then we take the standard Euclidean distance of the transformed path and define that to be the loxodromic distance between the two points on Earth. The Mercator map is a conformal cylindrical projection which preserves angles and maps lines of constant latitude to horizontal lines. Since angles are preserved under this mapping meridians are mapped to vertical lines. The Mercator projection assumes a spherical Earth and we make the same assumption.

We begin by writing parametric equations describing the line between two arbitrary locations on the Mercator projection. Because the projection is invertible, these equations will allow us to write parametric equations describing the loxodromic path between the same two points on Earth.

Let p denote a parallel of latitude (a great circle perpendicular to every meridian) north of the equator of radius r. Note that p makes an angle ϕ with the equator as shown in figure 1.

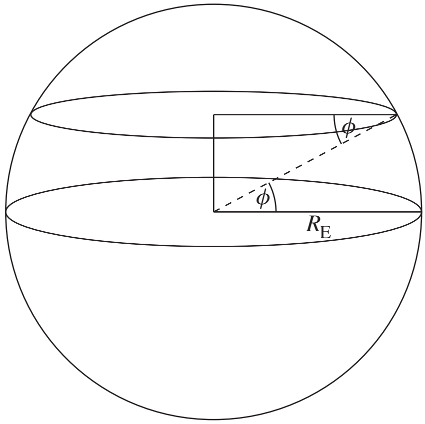

Figure 1 Parallels of latitude.

This means that ϕ can be thought of as a measure of latitude. If R_E denotes the radius of the Earth then it follows from right-angle trigonometry that $r = R_E \cos(\phi)$. Let s denote the ratio of the length of the equator to the length of p. It follows easily that

$$s = \frac{2\pi R_E}{2\pi R_E \cos(\phi)} = \sec(\phi).$$

Thus, the length of p is equal to the length of the equator multiplied by $\cos(\phi)$. In the Mercator projection all parallels are drawn with the same length, so each must be scaled by $1/\cos(\phi) = \sec(\phi)$. Indeed, $\sec(\phi)$ serves as a scaling factor to compute distances in the Mercator projection.

We define (x, y) axes in the Mercator projection so that x parametrizes longitude and y parametrizes latitude. In this axis the image $\mathrm{proj}(p)$ of the parallel p is a horizontal line that stretches across the entire width of the map. Observe that $\mathrm{proj}(p)$ appears as long as the image of the equator. It has been stretched by a factor of $\sec(\phi)$. This means that to compute the length of $\mathrm{proj}(p)$ we multiply the length of p by $\sec(\phi)$.

Since the Mercator projection is conformal it must preserve angles. To do this it must distort distances equally in both the x and the y directions. We have just seen that the projection distorts horizontal distances by a factor of $\sec(\phi)$, it must do the same for vertical distances. Since y is a function of latitude, the conformality of the projection implies that $\mathrm{d}y/\mathrm{d}\phi = \sec(\phi)$. The integral of $\sec(x)$ from 0 to ϕ defines a function

$$y(\phi) = \int_0^{\phi} \sec(x)\,\mathrm{d}x = \ln|\sec(\phi) + \tan(\phi)|, \tag{1}$$

which gives the y-coordinate of a point on the Mercator projection corresponding to the latitude ϕ.

Consider two points on the Mercator projection $A(x_1, y_1)$ and $B(x_2, y_2)$. Note that, since the vertical edges of the map are identical, there are many different straight lines connecting A and B. This can be seen by gluing multiple copies of the map side by side and connecting points on different copies of the map. Four different loxodromes connecting the same points

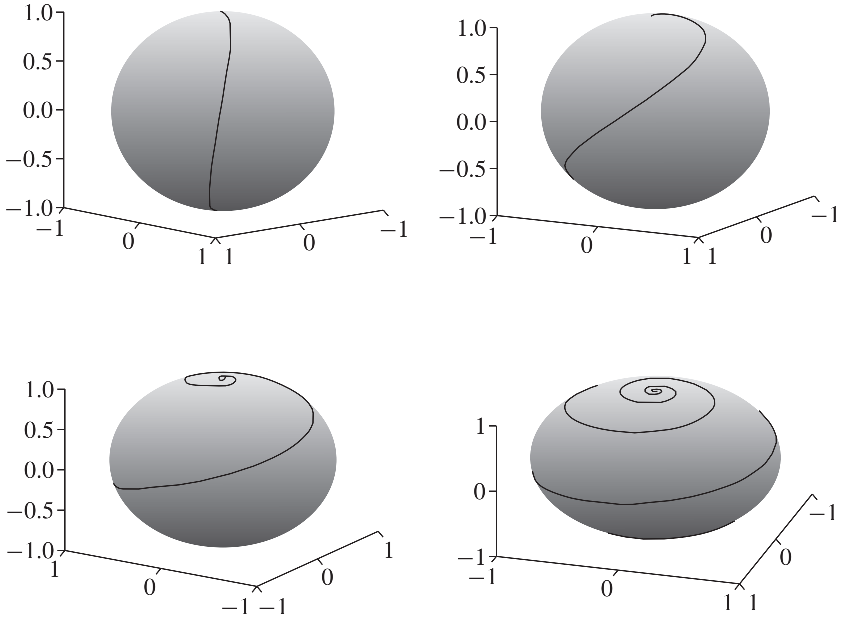

Figure 2 Four different loxodromes connecting the same two points.

are shown in figure 2. A straight line segment joining A and B may be parametrized by

$$X(t) = x_1 + t(x_2 - x_1), \qquad Y(t) = y_1 + t(y_2 - y_1),$$

for $t \in [0, 1]$. We find a parametric equation for the latitude of the corresponding loxodrome between these points on Earth by evaluating the inverse of $y(t)$ at each point of $Y(t)$. Exponentiating both sides of (1) gives $\mathrm{e}^{y(\phi)} = \sec(\phi) + \tan(\phi)$.

The trigonometric identity

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \sec(\phi) + \tan(\phi)$$

implies $\mathrm{e}^{y(\phi)} = \tan(\pi/4 + \phi/2)$. Solving for ϕ gives the latitude as a function of the y-coordinate in the Mercator projection:

$$\phi(y) = 2\arctan\left(\mathrm{e}^{y}\right) - \frac{\pi}{2}.$$

Evaluating this function at $Y(t)$ gives a parametric equation for the latitude of the loxodrome connecting A and B on Earth:

$$\lambda(t) = \phi(Y(t)) = 2\arctan\left(\mathrm{e}^{Y(t)}\right) - \frac{\pi}{2}.$$

The parametric equation for longitude of this loxodrome is

$$L(t) = x_1 + t(x_2 - x_1).$$

Now we have parametric equations defining a loxodrome connecting A and B. Next we show how to compute the length of this loxodrome on Earth.

The standard Euclidean metric in spherical coordinates $(r, L, \pi/2 - \lambda)$ is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\lambda^2 + r^2 \sin^2\left(\frac{\pi}{2} - \lambda\right)\mathrm{d}L^2,$$

so restricting to $r = R_\mathrm{E}$ for Earth gives $\mathrm{d}s^2 = R_\mathrm{E}^2(\mathrm{d}\lambda^2 + \cos^2(\lambda)\,\mathrm{d}L^2)$. Thus, the length of this loxodrome on Earth is

$$\ell_{\mathrm{lox}}(A, B) = R_\mathrm{E}\int_0^1 \sqrt{\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 + ((x_2 - x_1)\cos(\lambda))^2}\,\mathrm{d}t.$$

Orthodromes

An *orthodrome* is the shortest path between two points on Earth in the standard spherical metric and is commonly referred to as a *great circle*. Finding the arc length α between points A and C (see figure 3) is quite simple. Notice that α is proportional to the circumference as the subtended angle is proportional to 360°, meaning $\alpha/2\pi R_\mathrm{E} = \theta/360$. Therefore,

$$\alpha = 2\pi R_\mathrm{E}\frac{\theta}{360}.$$

To find the angle between two points A and C on Earth we treat them as vectors of length R_E with the origin identified as the centre of the Earth. We assume that A lies in the xy-plane. Therefore, the z-coordinate of A is zero and it is determined by an angle ν_A. The Cartesian coordinates of A are

$$A = (R_\mathrm{E}\cos(\nu_A), R_\mathrm{E}\sin(\nu_A), 0).$$

The vector describing point C can be described by two angles and the scalar R_E. The angle ν_C is an angle in the xy-plane and ϕ is the latitude of C. The Cartesian coordinates of C are

$$C = (R_\mathrm{E}\cos(\nu_C)\cos(\phi), R_\mathrm{E}\sin(\nu_C)\cos(\phi), R_\mathrm{E}\sin(\phi)).$$

We define the angle between the two points as the angle between their position vectors. This is computed as the inverse cosine of the dot product of the position vectors which can be seen as

$$\theta = \cos^{-1}[\cos(\nu_A)\cos(\nu_C)\cos(\phi) + \sin(\nu_A)\sin(\nu_C)\cos(\phi)].$$

We now define the orthodrome-distance between any abitrary pair of points as

$$\ell_{\mathrm{orth}} = \frac{2\pi}{360}R_\mathrm{E}\theta.$$

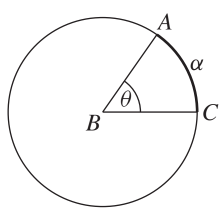

Figure 3

Application

We apply both of these methods to calculate the shoreline lengths of the Great Lakes in two different ways. In each method we use a freely available GPS dataset from the National Geophysical Data Laboratory (see reference 5). The data is an ordered list of GPS coordinates approximating the shorelines. Given an ordered set $N = (\nu_n, \phi_n)$ of (longitude, latitude) coordinates along the shoreline, we approximate $\mathrm{d}\phi/\mathrm{d}t$ at ϕ_n as $(\phi_n - \phi_{n-1})$. This allows us to compute the loxodrome and orthodrome distances between each consecutive pair of data points. These intermediate distances are summed to give an approximation to the length of the shorelines. In summary, we have developed two ways to calculate distances on Earth. Firstly, via loxodromes,

$$d_{\mathrm{lox}}(A, B) = R_{\mathrm{E}} \sum_{n=1}^{N} \int_0^1 \sqrt{(\phi_n - \phi_{n-1})^2 + ((\nu_n - \nu_{n-1}) \cos(\phi_n))^2}\, \mathrm{d}t,$$

and, secondly, via orthodromes,

$$d_{\mathrm{orth}}(A, B) = \frac{2\pi R_{\mathrm{E}}}{360} S,$$

where

$$S = \sum_{n=1}^{N} \cos^{-1}[\cos(\nu_n)\cos(\nu_{n-1})\cos(\phi_n) + \sin(\nu_n)\sin(\nu_{n-1})\cos(\phi_n)].$$

Both methods are implemented in MATLAB®. When calculating distances using orthodromes and loxodromes, there is no difference when compared to the built in *distance* command which allows the user to find either the loxodromic or great circle distance between two points. Using shoreline data for all five Great Lakes from NOAA (see reference 5), we tested our methods against published records (see table 1).

Since the distances between the data set points are small, we receive the same results from both methods. The loxodrome method returns a greater distance than the orthodrome method when we measure data points separated by 200 miles or more. The comparisons are made with published shoreline lengths retrieved from NOAA-GLERL (see reference 4).

Table 1 Shoreline miles excluding islands.

	Cartesian	Loxodrome	Orthodrome	NOAA-GLERL
Lake Superior	1625.3	1629.1	1629.1	1628.3
Lake Michigan	1333.4	1336.6	1336.6	1336.2
Lake Huron	1794.1	1798.6	1798.3	1827.2
Lake Ontario	929.9	932.1	932.1	932.3
Lake Erie	813.4	815.3	815.3	820.4

Testing our results

We verify our results by computing Euclidean distances between the Cartesian coordinates of the data. The data is given in terms of longitude ν and latitude ϕ and we follow the standard conversion procedure. First the (ν, ϕ) coordinates are converted to spherical coordinates $(R_{\mathrm{E}}, \nu, \phi)$. In these coordinates, the latitude is expressed as its defect from the North Pole. The conversion from spherical coordinates to Cartesian coordinates is shown here:

$$x = \cos(\phi)\cos(\nu), \qquad y = \cos(\phi)\sin(\nu), \qquad z = \sin(\phi).$$

We now calculate the distance between adjacent points using the standard Euclidean definition of distance

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Instead of following a path along the surface from point to point, this measures the length of the line connecting the points as it burrows through the crust of the Earth. This method gives an approximation of the distance around each lake. If the distances we find are close to our original findings, it is safe for us to dismiss the differences in our results as differences in data sets. In table 1 we compare results from applying all three different distances measured to the NOAA data.

Conclusion

The Great Lakes are the defining geographical feature for many people who live near them. They affect the weather, provide shipping routes, and provide countless means of recreation. They also serve as a natural backdrop for applying mathematics. Motivated by a desire to measure distances around the Great Lakes we develop and compare methods for computing geographical distances on Earth. We see both methods deliver similar correct results on data whose points were less than 200 miles apart. Using freely available data (see reference 5), the Mapping Toolbox in MATLAB, and some basic mathematics we apply these tools to understand better the world around us.

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7 http://www.mathworks.com/products/mapping.

Chase Ashby *is a mathematics major at Colorado State University, USA. He is currently doing undergraduate research in Electrical Impedance Tomography with Dr. Jennifer Mueller. Before transferring to CSU he was a student at Northern Michigan University. It was there that he combined his love of mathematics and Lake Superior while learning about loxodromes with Dr. Josh Thompson.*

Josh Thompson *is an Assistant Professor of Mathematics at Northern Michigan University in Marquette, MI, USA. He enjoys skiing, surfing, and doing mathematics on Lake Superior.*

Letters to the Editor

Dear Editor,

Represeting integers

I have observed that every integer can be expressed in the form $4A^2 + 2B^2 + C^3 + D^3$, where A, B, C, and D are integers. This can be derived from the following identities:

$$4(a)^2 + 2(2a-2)^2 + (2a-2)^3 + (-2a+1)^3 = 2a+1,$$
$$4(a)^2 + 2(a-2)^2 + (-a)^3 + (a-2)^3 = 2(2a),$$
$$4(a+1)^2 + 2(a)^2 + (a-1)^3 + (-a-1)^3 = 2(4a+1),$$
$$4(a)^2 + 2(a-4)^2 + (-a+1)^3 + (a-3)^3 = 2(4a+3).$$

Yours sincerely,
Rafael Jakimczuk
(División Matemática
Universidad Nacional de Luján
Buenos Aires
Argentina)

Dear Editor,

The sum of the digits of prime numbers

We will prove that the sum of the digits of the sequence of primes cannot be bounded by any number.

In number theory Dirichlet's theorem states that for any two coprime integers a and b there are infinitely many primes in the arithmetic progression consisting of terms $ax+b$ where x is a natural number. Dirichlet first proved this theorem using analytic number theory and calculus. Now we are going to use it to prove our theorem.

Theorem 1 *Given any natural number N, there exists a prime whose sum of the digits is greater than N.*

Proof Take the number $b = 11111111\ldots11$ (N times) and take the number $a = 10^N$. Now we consider the arithmetic progression $ax + b$ where $x = 0, 1, 2, 3, \ldots$. We notice that the numbers a and b are coprime. Hence, by Dirichlet's theorem the arithmetic progression defined above contains infinitely many primes. Also we see that the sum of the digits of each number in the arithmetic progression above is greater than or equal to $N + 1$ since the last N digits of each of the terms of the arithmetic progression are all 1s and the leading digit of each term (which is actually the leading digit of x) is at least 1, thus making the sum of the digits of each term at least $N + 1$. Hence, we find infinitely many primes, each of whose sum of the digits is greater than N. This completes the proof.

Yours sincerely,
Arpan Sadhukhan
(2nd year undergraduate
Indian Statistical Institute
Kolkata
India)

Problems and Solutions

Problems

48.9 Solve the simultaneous equations

$$x^2 - 9y + 22 = 2\sqrt{z-4},$$
$$y^2 - 9z + 22 = 2\sqrt{x-4},$$
$$z^2 - 9x + 22 = 2\sqrt{y-4}.$$

(Submitted by Mihàly Bencze, Brasov, Romania)

48.10 Let a, b, c be positive integers such that $a^2 + b^2 = c^2$. Prove that, for all positive integers n,

$$a^{2n+1} + b^{2n+1} + c^{2n+1} \quad \text{and} \quad (b+c)(c+a)(a+b)$$

are divisible by $a + b + c$.

(Submitted by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece)

48.11 Two straight lines meet at a point inside the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and their inclinations with their positive x-axis are α and $180° - \alpha$. Show that the four points where these straight lines meet the ellipse lie on a circle.

(Submitted by Zhang Yun, Sunshine High School of Xi An Jiaotong University, China)

48.12 The distinct points A, B, C, with x-cooridnates a, b, c respectively, lie on the rectangular hyperbola $xy = 1$. Show that the orthocentre of $\triangle ABC$ also lies on the rectangular hyperbola.

Suppose that $\triangle ABC$ is equilateral and let h be the x-coordinate of its orthocentre. Show that a, b, c are the roots of the cubic equation

$$x^3 - 3hx^2 - \frac{3}{h^2}x + \frac{1}{h} = 0$$

and that the radius of the circumcircle of $\triangle ABC$ is $2\sqrt{h^2 + 1/h^2}$. Deduce the minimum area of the triangle.

(Submitted by J. A. Scott, Great Somerford, Chippenham, UK)

Solutions to Problems in Volume 48 Number 1

48.1 Express x and y in terms of a, b, and c (see figure 1).

Solution by Billy Suandito, University of Music Charitas, Palembang, Indonesia

We have $b^2 - x^2 = c^2 - y^2$ and $x = a - y$, so that

$$(a - y)^2 - y^2 = b^2 - c^2;$$

whence,

$$y = \frac{a^2 - b^2 + c^2}{2a}$$

and

$$x = a - y = \frac{a^2 + b^2 - c^2}{2a}.$$

Also solved by Henry Ricardo, New York Math Circle.

48.2 A sequence is formed by adding to a given number A the terms of an arithmetic progression (e.g. 11, 13, 17, 23, 31, ...). Find the sum of the first n terms of such a sequence.

Solution

Denote the initial term of the arithmetic progression by a and its common difference by d. Then the rth term ($r > 1$) of the given sequence is

$$A + a + (a + d) + (a + 2d) + \cdots + (a + (r - 2)d) = A + (r - 1)a + \tfrac{1}{2}(r - 2)(r - 1)d.$$

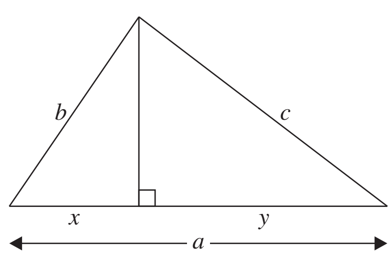

Figure 1

This is also true when $r = 1$. Hence, the sum of the first n terms ($n > 2$) is

$$\begin{aligned}
&nA + (1 + 2 + \cdots + (n-1))a + \left(\sum_{r=1}^{n} \tfrac{1}{2}(r-2)(r-1)\right)d \\
&\quad = nA + \tfrac{1}{2}(n-1)na + \sum_{r=1}^{n-2} \tfrac{1}{2}r(r+1)d \\
&\quad = nA + \tfrac{1}{2}(n-1)na + \tfrac{1}{2}\left(\sum_{r=1}^{n-2} r^2 + \sum_{r=1}^{n-2} r\right)d \\
&\quad = nA + \tfrac{1}{2}(n-1)na + \tfrac{1}{2}\big(\tfrac{1}{6}(n-2)(n-1)(2n-3) + \tfrac{1}{2}(n-2)(n-1)\big)d \\
&\quad = nA + \tfrac{1}{2}(n-1)na + \tfrac{1}{12}(n-2)(n-1)(2n-3+3)d \\
&\quad = nA + \tfrac{1}{2}(n-1)na + \tfrac{1}{6}(n-2)(n-1)nd.
\end{aligned}$$

This is also true when $n = 1, 2, \ldots$. For example, the sum of the first nine terms of the sequence $11, 13, 17, 23, 31, \ldots$ is

$$9 \times 11 + \tfrac{1}{2} \times 8 \times 9 \times 2 + \tfrac{1}{6} \times 7 \times 8 \times 9 \times 2 = 339.$$

Also solved by Billy Suandito.

48.3 In figure 2, $ABCD$ is a rectangle with $AB = 2BC$, angles PAB and PBA are $15°$, and angles DEP and CFP are right angles. What is the area of the quadrilateral $DEFC$?

Solution by Billy Suandito

Put $BC = a$. Then

$$\begin{aligned}
\frac{EF}{AB} &= \frac{EP}{AP} \\
&= \frac{AP - AE}{AP} \\
&= 1 - \frac{AE}{AP}
\end{aligned}$$

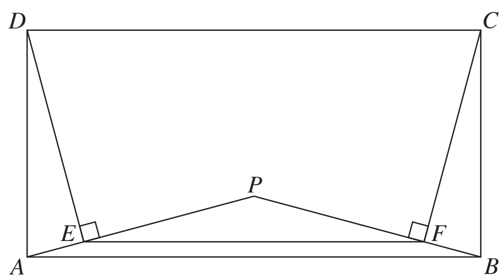

Figure 2

$$
\begin{aligned}
&= 1 - \frac{a\cos 75^\circ}{a\sec 15^\circ}\\
&= 1 - \cos 75^\circ \cos 15^\circ\\
&= 1 - \tfrac{1}{2}(\cos 90^\circ + \cos 60^\circ)\\
&= 1 - \tfrac{1}{4}\\
&= \tfrac{3}{4},
\end{aligned}
$$

so that $EF = \frac{3}{4}AB = \frac{3}{2}a$. Now,

$$
\begin{aligned}
\text{area } DEFC &= \tfrac{1}{2}(EF + DC)DE\sin 75^\circ\\
&= \tfrac{1}{2}\big(\tfrac{3}{2}a + 2a\big)a\sin^2 75^\circ\\
&= \tfrac{7}{4}a^2\sin^2(45^\circ + 30^\circ)\\
&= \tfrac{7}{4}a^2(\sin 45^\circ\cos 30^\circ + \cos 45^\circ \sin 30^\circ)^2\\
&= \tfrac{7}{4}a^2\tfrac{1}{2}\big(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\big)^2\\
&= \tfrac{7}{16}a^2(2 + \sqrt{3}).
\end{aligned}
$$

48.4 Prove that

$$\frac{2\tan 54^\circ}{\sqrt{(\tan 54^\circ)^2 + 1}}$$

is equal to the golden ratio.

Solution

The given expression is equal to $2\sin 54^\circ$. Now

$$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5.$$

If we equate real parts, we obtain

$$
\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\
&= \cos\theta((1 - \sin^2\theta)^2 - 10(1 - \sin^2\theta)\sin^2\theta + 5\sin^4\theta)\\
&= \cos\theta(16\sin^4\theta - 12\sin^2\theta + 1).
\end{aligned}
$$

Put $\theta = 54^\circ$. Then $\cos 5\theta = \cos 270^\circ = 0$, so that

$$16\sin^4\theta - 12\sin^2\theta + 1 = 0;$$

whence,

$$\sin^2\theta = \frac{6 + \sqrt{36 - 16}}{16} = \frac{3 + \sqrt{5}}{8}.$$

But $(1 + \sqrt{5})^2 = 6 + 2\sqrt{5}$ so that

$$\sin 54^\circ = \frac{1 + \sqrt{5}}{4},$$

and the given expression is equal to $(1 + \sqrt{5})/2$, the golden ratio.

Also solved by Billy Suandito and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Reviews

L.A. Math: Romance, Crime, and Mathematics in the City of Angels. By James D. Stein. Princeton University Press, 2016. Hardback, 256 pages, £18.95 (ISBN 9780691168289).

James Stein's *L.A. Math* is a collection of short stories featuring a private detective in Los Angeles, who must use a range of mathematical ideas to solve problems in his professional and personal life.

Each short story focuses on a different area of mathematics, and across the collection a wide variety of concepts, from compound interest to set theory, is covered. This presentation allows someone reading for leisure to dip in and out as they feel interested, and would also makes the book an ideal companion for teaching, particularly as the contents clearly list what mathematical topics are covered in each chapter, so identifying relevant stories is easy! There are no in-depth calculations in the stories themselves; however, around half the book consists of appendices in which the full calculations that are discussed in the stories, along with further mathematical discussion and examples, are presented. This allows the reader to appreciate the mathematical concepts in the context of the stories without becoming bogged down in detail, while also providing a clear explanation of all techniques discussed.

L.A. Math offers a lighthearted answer to the question of 'Why is this maths useful in real life?'. It is ideal for those looking to motivate mathematics teaching, and equally to those interested in the importance of numeracy in our daily lives.

The University of Sheffield **Caitlin McAuley**

Leonhard Euler: Mathematical Genius in the Enlightenment. By Ronald S. Calinger. Princeton University Press, 2016. Hardback, 696 pages, £40.95 (ISBN 9780691119274).

The book consists of 15 chapters, each focusing on a particular era of Euler's life, in chronological order, ranging from ten to fifty pages in length. Each chapter gives a detailed account of Euler's progress and mathematical interest at the time. Few mathematical formulas are written explicitly in the book. The author often describes mathematical concepts with words, yet a mathematical background beyond high-school level is desirable if one wishes to appreciate the beauty of the maths. What is remarkable in this book, is that the author gives a systematic account of Euler's articles; when and where they were written, who was interested in them, who either agreed or disagreed with their contents, along with information about their publication.

I found the book at times a challenging read; there are a substantial number of facts and references to indulge in. The author presents the scientific, and sometimes the political, currents of that era. Not only does he present Euler's life and achievements, but at each stage, puts it in perspective with the rest of the European academic world. Also, when mentioning other important members of the then scientific society, he does not restrict himself to a passive reference, rather he gives a small exposition of their accomplishments using formal terms from disciplines such as Philosophy, Physics, and so on. It is worth mentioning that passages portraying Euler's relations with friends and relatives are touching, and the author frequently illustrates this world-renowned genius as simple and accessible to all.

Overall, I feel this book provides an inspiring and vivid taste of Euler's everlasting brilliance and energy, and will provide great insight into Euler's life to all of his fans.

The University of Sheffield **Magdalini Flari**

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Articles

From the Editor

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Consecutive numbers divisible by squares of consecutive Fibonacci numbers

$$(\mathrm{F}_4, \mathrm{F}_5, \mathrm{F}_6, \mathrm{F}_7) = (3, 5, 8, 13),$$
$$1\,323\,774 = 9 \times 147\,086,$$
$$1\,323\,775 = 25 \times 52\,951,$$
$$1\,323\,776 = 64 \times 20\,684,$$
$$1\,323\,777 = 169 \times 7\,833.$$

Is it possible to find more than four numbers with this property?

Lucknow, India **M. A. Khan**

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