Editorial Board

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EDITORIAL

I must admit that putting together issue 1 of this Volume has been a bit of a challenge. In retrospect, it was the bigger than usual amount of administrative work, such as securing the name of the new section, re-shuffling the Editorial Board, setting new timelines and schedules. Obviously, my job generally involves a non-trivial administrative component; but I've quickly realized that for me to truly enjoy the work, it is the mathematical component of it that has to be far from trivial. The good news: with administrative heavy lifting out of the way, I've thoroughly enjoyed putting together this issue of **Crux** and hope you enjoy reading it as well.

I am particularly excited about *MathemAttic* coming together. With dedicated editors and a new target audience, there are inspiring ideas and pieces in the works. I encourage you to explore the section's contents and to encourage any secondary level students you know to submit solutions to MA problems.

As **Crux** is getting a new start, I've received numerous emails with words of encouragement and support; they were a real source of motivation for me, so thank you for those. I would also like to thank my home institution of University of the Fraser Valley for providing financial support for me to travel to Canadian Mathematical Society meetings on behalf of **Crux**.

As per usual, you can reach me at `crux.eic@gmail.com` : send your photos for *Snapshot* section and your comments about any and every section.

Without further ado, here is issue 2.

Kseniya Garaschuk

MathemAttic

No. 2

The problems in this section are intended for students at the secondary school level.

To facilitate their consideration, solutions should be received by **May 15, 2019***.*

MA6. A rectangular sheet of paper is labelled $ABCD$, with AB being one of the longer sides. The sheet is folded so that vertex A is placed exactly on top of the opposite vertex C. The fold line is XY, where X lies on AB and Y lies on CD. Prove that the triangle CXY is isosceles.

MA7. Sixteen counters, which are black on one side and white on the other, are arranged in a 4 by 4 square. Initially all the counters are facing black side up. In one 'move', you must choose a 2 by 2 square within the square and turn all four counters over once. Describe a sequence of 'moves' of minimum length that finishes with the visible colours of the counters of the 4 by 4 square alternating (as shown in the diagram).
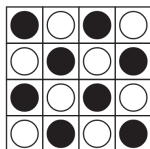
MA8. I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm. What is the smallest square that can be made with equal numbers of each type of tile?

MA9. The letters a, b, c, d, e and f represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e, \quad d + e = f.$$

Find all possible solutions for the values of a, b, c, d, e and f.

MA10. An arithmetic and a geometric sequence, both consisting of only positive integral terms, share the same first two terms. Show that each term of the geometric sequence is also a term of the arithmetic sequence.

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Les problémes dans cette section sont appropriés aux étudiants de l'école secondaire.

MA6. Une feuille de papier rectangulaire $ABCD$ est telle que AB est un de ses longs côtés. Cette feuille est pliée de façon à ce que le coin A se trouve directement par dessus le coin opposé C. La ligne de pli est dénotée XY, où X se trouve sur le côté AB et Y se trouve sur le côté CD. Démontrer que le triangle CXY est isocèle.

MA7. Seize jetons, noirs d'un côté et blancs de l'autre, sont disposés sur une grille 4 par 4. Au départ, chaque jeton montre sa face noire. À chaque "tour", on choisit un carré 2 par 2 dans le carré original et on y renverse tous ses jetons. Déterminer une suite de "tours" de longueur minimale telle que les couleurs visibles des jetons deviennent alternantes (voir schéma).
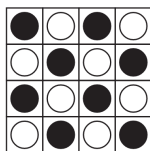
MA8. On dispose de deux tailles de tuiles carrées. La première sorte est de côtés 1 cm, tandis que la deuxième sorte a des côtés de 2 cm. Quel est le plus petit carré qui puisse être formé à l'aide de nombres égaux des deux sortes de tuile ?

MA9. Les lettres a, b, c, d, e et f dénotent des entiers à un chiffre, chaque lettre correspondant à un entier différent. De plus, ces lettres satisfont aux équations suivantes :
$$a + b = d, \quad b + c = e, \quad d + e = f.$$
Déterminer toutes les solutions possibles a, b, c, d, e et f.

MA10. Une suite arithmétique et une suite géométrique, consistant d'entiers positifs, partagent les mêmes deux premiers termes. Démontrer que chaque terme de la progression géométrique est aussi un des termes de la progression arithmétique.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2018: 44(3), p. 92–93; and 44(4), p. 133–136.

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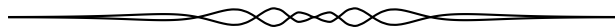
CC311. Suppose $1 \le a < b < c < d \le 100$ are four natural numbers. What is the minimum possible value of $\dfrac{a}{b} + \dfrac{c}{d}$?

Originally Problem 1 from Group round of 2016 Georgia Tech High School Mathematics Competition.

We received 14 submissions. We present two submissions.

Solution 1, by Ángel Plaza.

Since $1 \le a < b < c < d \le 100$, the minimum value is attained for $a = 1$ and $d = 100$ (otherwise the solution can be made smaller by making a smaller or d larger). Also since $b < c$, we may assume that $c = b + 1$ because otherwise if $c = b + 2$ or greater than that the result will be greater than for $c = b + 1$.

The problem is now an optimization problem with

$$f = \frac{1}{x} + \frac{x+1}{100}, \quad f'(x) = \frac{-1}{x^2} + \frac{1}{100} \quad \text{and} \quad f''(x) = \frac{2}{x^3} > 0$$

for $x > 0$. This implies that function $f(x)$ present a minimum value at $x = 10$ which is the root of $f'(x) = 0$, with $f(10) = 0.21$ and the problem is done.

Solution 2, by Digby Smith.

As above, we see that $a = 1$, $d = 100$, and $c = b + 1$. We will minimize

$$\frac{1}{b} + \frac{b+1}{100} = \left(\frac{1}{b} + \frac{b}{100} \right) + \frac{1}{100}.$$

Applying the AM-GM inequality to the bracketed terms, we get

$$\frac{1}{b} + \frac{b}{100} \ge 2\sqrt{\frac{1}{b} \cdot \frac{b}{100}} \ge \frac{2}{10}.$$

Then

$$\left(\frac{1}{b} + \frac{b}{100} \right) + \frac{1}{100} \ge \left(\frac{2}{10} \right) + \frac{1}{100} \ge \frac{21}{100}.$$

Equality holds for $b = 10$.

It then follows that the minimum value is $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{21}{100}$ when $a = 1$, $b = 10$, $c = 11$, and $d = 100$.

CC312.   Choose four points A, B, C and D on a circle uniformly at random. What is the probability that the lines AB and CD intersect outside the circle?

Originally Problem 8 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.

Since the points are chosen uniformly at random, any set of 4 points is as likely to be labelled in any one of the 24 ways as in any other way. But once A is labeled, there are four ways to label B, C and D for which the lines AB and CD intersect outside the circle and only two for which they intersect inside the circle. So the probability of intersecting outside is 2/3.

CC313.   Consider a pyramid whose faces consist of a 60×60 square base $ABCD$ and four $60 - 50 - 50$ triangles that join at the apex E. If you are only allowed to move on the surfaces of the four triangles, what is the length of the shortest path between A and C?
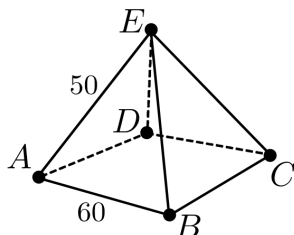
Originally Problem 10 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 11 submissions. We present the solution by Carlos Moreno and Ángel Plaza.

Imagine flattening two triangular faces of the pyramid as shown.
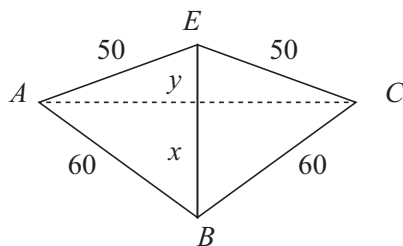
Since $EB = 50$, let $x + y = 50$ as shown. By the Pythagorean theorem, it is deduced $x = 36$, $y = 14$. Then the height over EB in triangle ABC is $h = 48$ and consequently the shortest path between A and C is $2 \times 48 = 96$.

CC314. An infinite sequence $a_1, a_2, \ldots$ of 1's and 2's is uniquely defined by the following properties:

1. $a_1 = 1$ and $a_2 = 2$,

2. For every $n \geq 1$, the number of 1's between the nth 2 and the $(n+1)^{\text{st}}$ 2 is equal to a_{n+1}.

Is the sequence periodic from the beginning?

Originally Problem 4 from Proof round of 2016 Georgia Tech High School Mathematics Competition.

We received 4 submissions of which 2 were both correct and complete. Both correct solutions used proof-by-contradiction strategy. We present the solution by Steven Chow.

Assume for the sake of contradiction that the sequence is periodic from the beginning. Let k be the fundamental period of the sequence. Let t be the number of 2's in $a_1, a_2, \ldots, a_k$. Due to periodicity, t is the number of 2's in any subsequence of length k.

We claim that for all integers $j \geq 2$, $a_j = a_{j+t}$. Indeed a_j is the number of 1's between the $(j-1)^{\text{st}}$ 2 and the jth 2, and a_{j+t} is the number of 1's between the $(j+t-1)^{\text{st}}$ 2 and the $(j+t)^{\text{th}}$ 2. Between the $(j-1)^{\text{st}}$ 2 and the $(j+t-1)^{\text{st}}$ 2, including the $(j-1)^{\text{st}}$ 2 and excluding the $(j+t-1)^{\text{st}}$ 2, there are exactly t 2's. The sequence is periodic with exactly t 2's in any subsequence of length equal to the fundamental period. Therefore the subsequence starting at the $(j+t-1)^{\text{st}}$ 2 must be a repeat of the subsequence starting at the $(j-1)^{\text{st}}$ 2. Therefore $a_j = a_{j+t}$.

Since $a_j = a_{j+t}$ for all integers $j \geq 2$, the sequence $a_2, a_3, \ldots$ is periodic with period t. However, the fundamental period is k, so $k \leq t$. From the definition of t, $t \leq k$. Therefore $t = k$ and the sequence consists of 2's only.

This is a contradiction. Therefore, the sequence is not periodic from the beginning.

Editor's Comment. The above solution can be slightly modified to obtain that the sequence is not periodic starting at n, for any $n \geq 1$.

CC315. A square table is divided into a 3×3 grid with every cell having 3 coins. In every step of a game, Terry can take 2 coins from the table as long as they come from distinct but adjacent cells. (Here "adjacent" means the two cells share a common edge.) At most how many coins can Terry take?

Originally Problem 9 from Fillins round of 2016 Georgia Tech High School Mathematics Competition.

We received 7 submissions. We present the solution by the Missouri State University Problem Solving Group.

The answer is 24 coins.

If we apply the usual checkerboard colouring with the central and corner cells coloured black and the side squares coloured white, there are 15 coins in black cells and 12 coins in white ones. But every time a pair of coins is chosen, one must be from a black cell and one from a white one. At best this will leave us with 3 coins remaining (i.e. 24 coins taken).

This result can actually be obtained by choosing pairs of coins from adjacent cells on the left-hand side of the first three rows and from two adjacent cells in the last column.

A similar argument shows that if k coins are placed in each cell of an $m \times n$ grid with m and n both odd, then at most $k(mn - 1)$ coins can be taken.

Editor's Comments. There was quite a variety of solutions here. Many did *one* of the following:

- Using a checkerboard argument, showed that it is impossible to take more than 24 coins.

- By construction, showed that it is possible to take exactly 24 coins.

Only a few showed both sides of the proof.

CC316. Positive integers (x, y, z) form a *Trenti-triple* if $3x = 5y = 2z$. Show that for every Trenti-triple (x, y, z) the product xyz must be divisible by 900.

Originally Question 3 from Hypatia 2011.

We received 13 submissions, all correct. We feature a composite solution based on essentially the same ones given by most of these solvers.

Note first that 3, 5, and 2 are mutually relatively co-prime.

Hence, from $3x = 5y = 2z$ we deduce that $5 \mid x$ and $2 \mid x$, so $10 \mid x$.

Similarly, from $5y = 2z = 3x$ we deduce that $2 \mid y$ and $3 \mid y$, so $6 \mid y$.

Finally, from $2z = 3x = 5y$ we deduce that $3 \mid z$ and $5 \mid z$, so $15 \mid z$.

It then follows that xyz is divisible by $10 \cdot 6 \cdot 15$ or 900.

$$\text{Compute} \quad \lim_{x \to k} \frac{s\, x^2 y \sin(k - x)}{k^2 - kx}.$$

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CC317. In the diagram, $ABCD$ is a trapezoid with AD parallel to BC and BC perpendicular to AB. Also, $AD = 6$, $AB = 20$, and $BC = 30$.
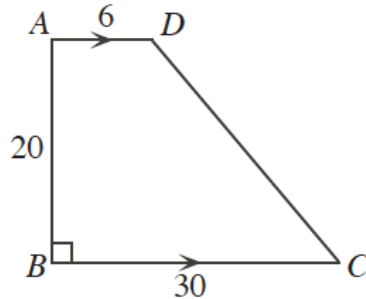
1. Determine the area of trapezoid $ABCD$.

2. There is a point K on AB such that the area of triangle KBC equals the area of quadrilateral $KADC$. Determine the length of BK.

3. There is a point M on DC such that the area of triangle MBC equals the area of quadrilateral $MBAD$. Determine the length of MC.

Originally Question 3 from Hypatia 2008.

We received 8 correct solutions. We present the solution by Ricard Peiró i Estruch.

1. The area of $ABCD$ is $\frac{30+6}{2} \cdot 20 = 360$.

2. Set $x = BK$. The area of triangle KBC is half the area of trapezoid $ABCD$, so

$$\frac{1}{2} \cdot 30x = \frac{1}{2} \cdot 360.$$

Solving gives $x = 12$.

3. Let D' be the projection of D onto line BC. Then $DD' = AB = 20$ and

$$D'C = BC - AD = 24.$$

Applying the Pythagorean Theorem to right triangle $DD'C$ gives

$$CD = \sqrt{24^2 + 20^2} = 4 \cdot \sqrt{61}.$$

Since the area of triangle MBC is half the area of trapezoid $ABCD$ and thus equal to the area of triangle KBC, and the two triangles share base $\overline{BC}$, they have the same height. Thus $\overline{KM}$ is parallel to both $\overline{BC}$ and $\overline{AD}$. It follows that

$$\frac{MC}{BK} = \frac{CD}{AB} \quad \text{or} \quad \frac{MC}{12} = \frac{4\sqrt{61}}{20}.$$

Solving gives

$$MC = \frac{12\sqrt{61}}{5}.$$

CC318. Right-angled triangle PQR is inscribed in the parabola with equation $y = x^2$, as shown. Points P, Q and R have coordinates $(p, p^2), (q, q^2)$ and (r, r^2), respectively. If p, q and r are integers, show that $2q + p + r = 0$.
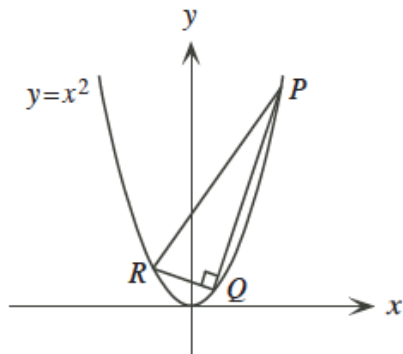
Originally Question 3c from Hypatia 2012.

We received 9 submissions of which all but one were correct and complete. We present the solution by Digby Smith.

Note from the diagram that $p > q > r$. Since $\angle RQP = 90°$ it follows that QR and QP are perpendicular. Moreover, the slope of line QP is

$$m_{QP} = (p^2 - q^2)/(p - q) = p + q,$$

the slope of line QR is

$$m_{QR} = (q^2 - r^2)/(q - r) = q + r,$$

and

$$m_{QP}m_{QR} = (p + q)(q + r) = -1.$$

Since p, q, and r are integers it then follows that either $p + q = 1$ and $q + r = -1$, or $p + q = -1$ and $q + r = 1$. In both cases, after adding the two equations, it follows that $2q + p + r = 0$.

CC319. In the diagram, square $ABCD$ has sides of length 4, and triangle ABE is equilateral. Line segments BE and AC intersect at P. Determine the exact area of triangle APE.
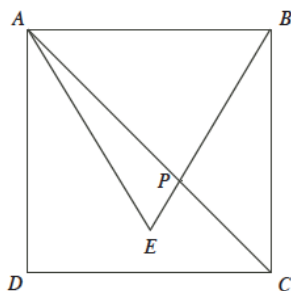
Originally Question 3 from Hypatia 2010.

We received 14 correct solutions from 11 solvers. The solver of one incorrect submission apparently misread the question thinking that the side length of the given square is 1. We present the solution by Richard Peiró i Estruch.

From P, draw line PH perpendicular to AB and let $\overline{AH} = x$.

Since $\angle BAP = 45°$, we have $\overline{PH} = \overline{AH} = x$.

Since
$$\overline{PH} = \overline{(BH)}\tan 60° = (4 - x)\sqrt{3},$$

we have
$$x = \sqrt{3}(4 - x).$$

Solving yields
$$x = \frac{4\sqrt{3}}{\sqrt{3} + 1} = 2\sqrt{3}(\sqrt{3} - 1) = 6 - 2\sqrt{3}.$$

Finally,
$$\begin{aligned}
|\triangle APE| &= |\triangle ABE| - |\triangle ABP| \\
&= \frac{1}{2}(4)(2\sqrt{3}) - \frac{1}{2}(4)(6 - 2\sqrt{3}) = 4\sqrt{3} - 4(3 - \sqrt{3}) \\
&= 8\sqrt{3} - 12.
\end{aligned}$$

CC320. A sequence of m P's and n Q's with $m > n$ is called *non-predictive* if there is some point in the sequence where the number of Q's counted from the left is greater than or equal to the number of P's counted from the left. For example, if $m = 5$ and $n = 2$, the sequences $PPQQPPP$ and $QPPPQPP$ are non-predictive.

1. If $m = 7$ and $n = 2$, determine the number of non-predictive sequences that begin with P.

2. Suppose that $n = 2$. Show that for every $m > 2$, the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.

3. Determine the number of non-predictive sequences with $m = 10$ and $n = 3$.

Originally Question 3 from Hypatia 2013.

We received five submissions. We present the solution by Kathleen E. Lewis.

1. The sequence must begin with PP or PQ. If it begins with PP and has only 2 Q's, then the only way that it can be non-predictive is to begin $PPQQ$. But then the rest of the terms must be P's, so the only such sequence is $PPQQPPPPP$. Now suppose that it begins with PQ. Then it's already non-predictive, so the

remaining terms can be in any order. Since there are 6 P's and one Q remaining, there are 7 possible arrangements. Therefore, there are a total of eight non-predictive sequences beginning with P.

2. Any sequence that begins with Q is automatically non-predictive. There are as many such sequences as there are arrangements of the other letters, one Q and m P's. Since there are $m+1$ places to put the Q, there are $m+1$ such sequences. The number of non-predictive sequences beginning with P can be analyzed as in part (1) above. There is only one such sequence beginning with PP, since it must start with $PPQQ$. Any sequence beginning with PQ is non-predictive, so we can arrange the other terms in any way. There are $m-1$ P's and only one Q remaining, so there are m ways to arrange them. That gives a total of $m+1$ non-predictive sequences beginning with P to match the $m+1$ beginning with Q.

3. Consider the following cases:

a) sequences starting with Q: Any sequence starting with Q is automatically non-predictive. The remaining 10 P's and 2 Q's can be arranged in any order, so there are $\binom{12}{2} = 66$ such sequences.

b) sequences starting with PQ: Again, all such sequences are non-predictive. There are 9 P's and 2 Q's still to arrange, so that gives $\binom{11}{2} = 55$ such sequences.

c) sequences starting with PPP: The only way such a sequence can be non-predictive is to have the three Q's immediately follow the first three P's, so the only non-predictive sequence of this form is $PPPQQQPPPPPPP$.

d) sequences starting with $PPQQ$: Any such sequence is non-predictive. Since there is one remaining Q to place among 8 P's, there are nine such sequences.

e) sequences starting with $PPQP$: The only way such a sequence can be non-predictive is to continue as $PPQPQQPPPPPPPP$. So there is only one sequence in this case.

Total: Hence, there is a total of $66+55+1+9+1 = 132$ non-predictive sequences with 10 P's and 3 Q's.
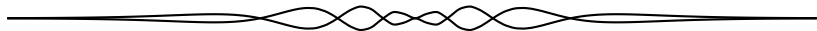
PROBLEM SOLVING VIGNETTES

No.2

Shawn Godin

Carefully Counting Canoeists

This month, we will look at problem **7a** from the 2018 *Euclid Contest*, hosted by the The Centre for Education in Mathematics and Computing at the University of Waterloo. You can check out the contest, and past contests on the CEMC website at `www.cemc.uwaterloo.ca`.

> Eight people, including triplets Barry, Carrie and Mary, are going for a trip in four canoes. Each canoe seats two people. The eight people are to be randomly assigned to the four canoes in pairs. What is the probability that no two of Barry, Carrie and Mary will be in the same canoe?

This problem can be tricky as it involves careful counting. Since it is a probability problem we may also interpret the process of putting people into canoes in different ways, as long as we are consistent throughout our solution. We need to compute the number of ways that the people can be seated, without restriction, and the number of ways that they can be seated with the triplets separated. We will look at the problem in several different ways.

Solution #1: Completely ordered.

We will impose an "order" on the canoes and on the seats in the canoes. That is, we will consider Alice and Barry in the red canoe as different from Alice and Barry in the green canoe. Similarly, we will consider Alice and Barry in the silver canoe with Alice in front different from Alice and Barry in the silver canoe with Barry in the front. Thus there are 8 positions into which we want to order our 8 people. We can do the ordering in 8! ways.

To ensure the triplets are in separate canoes, we will assign them separately. Barry can be placed in any of the 8 seats. When it comes to seating Carrie, she cannot be in the same seat as Barry or the other empty seat in his canoe. As such, there are 6 possible seats in which to seat Carrie. Similarly, there are 4 possible seat choices for Mary. Once the triplets have been seated, we can seat the remaining 5 people in any order, which can be done in 5! ways. Thus the total number of ways to seat the people, keeping the triplets separated is $8 \times 6 \times 4 \times 5!$.

Hence, the desired probability is

$$\frac{8 \times 6 \times 4 \times 5!}{8!} = \frac{8 \times 6! \times 4}{8 \times 6! \times 7} = \frac{4}{7}.$$

Solution #2: Partially ordered.

We will impose an order on the canoes, but not on the positions within the canoes. Thus we need to pick 2 people for the first canoe, which we can do in $\binom{8}{2}$ ways. Filling the other canoes in similar ways, we get the total number of ways to fill the canoes to be $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. We could have similarly assumed that the 8 people were given cards randomly that had either R, G, B, or S on it, indicating that the canoeist was assigned to the red, green, blue or silver canoe, respectively. We can then think of the process as the number of ways or ordering two each of R, G, B, and S. This can be done in $\frac{8!}{(2!)^4}$ ways. It is easy to show that

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{8!}{(2!)^4}$$

Next, since we want the triplets separated, we will deal with them first. First choose which canoes they will be placed into, which can be done in $\binom{4}{3}$ ways. Since we have "ordered" our canoes, the order the triplets are placed in the canoes is important. They can be placed in the chosen canoes in 3! ways. At this point we have one empty canoe, and three half filled canoes in which to place our remaining 5 canoeists. We will fill the empty canoe first, which can be done in $\binom{5}{2}$ ways. Then the remaining 3 canoeists have to be "ordered" into the three remaining canoes with the triplets, which can be done in 3! ways. Note we could have put the three canoeists with the triplets first and then filled the empty canoe with the remaining two. You may wish to show that this yields the same result. So the total number of ways to seat the people, keeping the triplets separated is $\binom{4}{3} \times (3!)^2 \times \binom{5}{2}$.

Once again, our desired probability is

$$\frac{\binom{4}{3} \times (3!)^2 \times \binom{5}{2}}{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}} = \frac{4}{7}.$$

Solution #3: Completely unordered.

In this solution we will not impose an order on either the canoes or the positions. From solution #2, the number of ways to place the canoeists in the boats where the boats are "ordered" but the seats are not is $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$. If we want the boats not to be ordered, we need to "remove" the order of the four boats by noticing that each possible set of groupings of pairs can be assigned to the "ordered" canoes in 4! ways. Thus the number of unordered assignments is $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}$.

To assign the pairs with the triplets separated, we can assume that each triplet, in turn, will select a person with whom they will be paired. This can be done in $5 \times 4 \times 3$ ways. The remaining two people form the final pair. Thus the desired probability is

$$\frac{5 \times 4 \times 3}{\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!}} = \frac{4}{7}$$

Solving a problem in different ways gives you different insights. Solution #3 has the most straightforward way of calculating the number of ways to seat the canoeists with the triplets separated. It also seems like the most "real world" way of thinking of the problem (i.e. canoeists probably were assigned a partner, not a particular canoe or seating order). On the other hand, determining the number of ways to seat the canoeists without any order, becomes a little trickier. Solution #2 seems like a natural way to think about the problem, and was the way I first thought of it. Even if we are not assigning groups to particular boats, if we pick the groups by picking names out of a hat, the *process* imposes an order on the groups, even if that order is removed later on. Solution #1, while seemingly unnatural, does provide the "cleanest" solution. When dealing with probability questions it is worth thinking about considering the problems as ordered, or not. In some cases, going against the "natural" way of thinking about the problem might provide you with a simplified solution. It is worth your time to check out the official solutions to the problem from the CEMC website. Their first solution is similar to our solution #2 with a few interesting variations. Their second solution is a clever way to determine the probability without counting the groups.

Careful counting is important when dealing with probability or counting problems. I suspect that many students who did the problem, but got it wrong, probably computed the two totals for the problem with different interpretations. I wouldn't be surprised if there were many solutions that gave

$$\frac{5 \times 4 \times 3}{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}},$$

where students were thinking that they were dealing with an unordered problem but, inadvertently, imposed an ordering to the canoes. We will revisit counting problems in future columns.

OLYMPIAD CORNER

No. 370

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **May 15, 2019**.
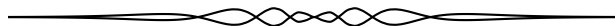
OC416. Given an acute nonisosceles triangle ABC with altitudes CD, AE, BF. Points E' and F' are symmetrical to E and F with respect to points A and B, respectively. Take a point C_1 on the ray $\overrightarrow{CD}$ such that $DC_1 = 3CD$. Prove that $\angle E'C_1F' = \angle ACB$.

OC417. Point M is the midpoint of side BC of a triangle ABC in which $AB = AC$. Point D is the orthogonal projection of M onto side AB. Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L, respectively. Lines tangent to ω which pass through M intersect line KL at X and Y, where points X, K, L and Y lie on KL in this order. Prove that points M, D, X and Y are concyclic.

OC418. Three sequences $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ of nonnegative real numbers are given such that for all $0 \le i, j \le n$ we have $a_i b_j \le (c_{i+j})^2$. Prove that

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \le \left(\sum_{k=0}^{2n} c_k \right)^2.$$

OC419. Prove that there exist infinitely many positive integers m such that there exist m consecutive perfect squares with sum m^3. Determine one solution with $m > 1$.

OC420. General Tilly and the Duke of Wallenstein play "Divide and rule!" (Divide et impera!). To this end, they arrange N tin soldiers in M companies and command them by turns. Both of them must give a command and execute it in their turn.

Only two commands are possible: The command "Divide!" chooses one company and divides it into two companies, where the commander is free to choose their size, the only condition being that both companies must contain at least one tin

soldier. On the other hand, the command "Rule!" removes exactly one tin soldier from each company.

The game is lost if in your turn you can't give a command without losing a company. Wallenstein starts to command.

(a) Can he force Tilly to lose if they start with 7 companies of 7 tin soldiers each?

(b) Who loses if they start with $M \geq 1$ companies consisting of $n_1 \geq 1$, $n_2 \geq 1$, ..., $n_M \geq 1$ $(n_1 + n_2 + \cdots + n_M = N)$ tin soldiers?

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

> ***Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

OC416. Soit un triangle acutangle non isocèle ABC dont les hauteurs sont CD, AE et BF. E' et F' sont symétriques à E et F par rapport aux points A et B respectivement. On prend un point C_1 sur le segment CD tel que $DC_1 = 3CD$. Montrer que $\angle E'C_1F' = \angle ACB$.

OC417. Le point M est le milieu du côté BC d'un triangle ABC où $AB = AC$. Le point D correspond à la projection orthogonale de M sur le côté AB. Le cercle ω est inscrit dans le triangle ACD et est tangent aux segments AD et AC aux points K et L respectivement. Les droites tangentes à ω qui passent par M interceptent le segment KL en X et Y, où les points X, K, L et Y sont sur le segment KL dans cet ordre. Montrer que les points M, D, X et Y sont concycliques.

OC418. Soit trois suites $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ de nombres réels non négatifs telles que pour toutes les suites $0 \leq i, j \leq n$ on a $a_i b_j \leq (c_{i+j})^2$. Montrer que

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \leq \left(\sum_{k=0}^{2n} c_k \right)^2.$$

OC419. Montrer qu'il existe une infinité d'entiers positifs m tels qu'il existe m carrés parfaits consécutifs dont la somme est m^3. Trouver une solution avec $m > 1$.

OC420. Le Général Tilly et le Duc de Wallenstein jouent à "diviser et régner!" Dans ce but, ils placent N soldats de plomb dans M groupes et leur donnent des commandes. Les deux doivent donner une commande et l'exécuter pendant leur tour. Seulement deux commandes sont possibles : la commande "Divise!" choisit un groupe et le divise en deux groupes où le commandant est libre de choisir la taille des groupes en autant que chaque groupe contienne au moins un soldat. La commande "Règne!" enlève exactement un soldat de plomb de chaque groupe. La partie est perdue si lors d'un tour, il n'est plus possible de donner un ordre sans perdre un groupe. C'est le Duc de Wallenstein qui débute les commandes.

(a) Peut-il faire perdre le Général Tilly s'ils commencent avec 7 groupes de 7 soldats de plomb chacun ?

(b) Qui perd s'ils commencent avec $M \geq 1$ groupes ayant $n_1 \geq 1$, $n_2 \geq 1$, ..., $n_M \geq 1$ ($n_1 + n_2 + \cdots + n_M = N$) soldats de plomb ?

OLYMPIAD CORNER
SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(10), p. 428–429 and 2018: 44(1), p. 13-14.
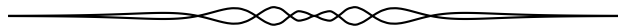
OC356. Suppose 2016 points of the circumference of a circle are colored red and the remaining points are colored blue. Given any natural number $n \geq 3$, prove that there is a regular n-sided polygon all of whose vertices are blue.

Originally Problem 5 of 2016 India National Olympiad.

We received 3 solutions. We present the solution by Ivko Dimitrić.

Without loss of generality, assume that the circle is unit and consider an arbitrary inscribed regular n-gon with at least one vertex red. Let $a > 0$ be the minimum arc-wise distance between distinct red points and let $b > 0$ the minimum arc-wise distance between blue vertices of the inscribed polygon (if any) and the red points. These minimum distances exist since the sets in question are finite. Let $\theta = \min\{a, b\}$. Then, rotate the selected n-gon through an angle $\theta/2$ (in radians). Then the vertices of the regular polygon would slide along the circumference the same arcwise distance equal to the angle of the rotation. Those vertices that were originally red will be rotated to blue points since $\theta/2 < a$, and none of the vertices that were originally blue can land at a red point under this rotation since $\theta/2 < b$, so they, too, will be rotated to blue points, and all the vertices of such rotated regular n-gon will be blue.

OC357. In $\triangle AEF$, let B and D be on segments AE and AF respectively, and let ED and FB intersect at C. Define K, L, M, N on segments AB, BC, CD, DA such that $\frac{AK}{KB} = \frac{AD}{BC}$ and its cyclic equivalents. Let the incircle of $\triangle AEF$ touch AE, AF at S, T respectively; let the incircle of $\triangle CEF$ touch CE, CF at U, V respectively. Prove that K, L, M, N concyclic implies S, T, U, V concyclic.

Originally Problem 2, Day 1 of 2016 China National Olympiad.

We received no solutions to this problem.

OC358. Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime of the form $4k + 1$, s is a positive integer of the form $4h + 1$, and $m \in \mathbb{Z}^+$, m is not divisible by p. Also, find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n - 1$ and $\frac{n(n+1)}{2}$ is a perfect number.

Originally Problem 3, Day 2 of Vietnam National Olympiad.

We received only 1 *solution. We present the solution by David Manes.*

Let $n = p_1^{s_1} p_2^{s_2} \ldots p_r^{s_r}$ be the prime factorization of the odd integer n and assume that n is perfect. Then $2n = \sigma(n) = \sigma(p_1^{s_1})\sigma(p_2^{s_2})\ldots\sigma(p_r^{s_r})$. Note that $2n \equiv 2$ (mod 4) since n is odd implies either $n \equiv 1$ (mod 4) or $n \equiv 3$ (mod 4). Therefore, $\sigma(n) = 2n$ is divisible by 2, but not by 4. Hence, exactly one of the $\sigma(p_i^{s_i})$, say $\sigma(p_1^{s_1})$, is an even integer (but $\sigma(p_1^{s_1})$ is not divisible by 4) and the remaining $\sigma(p_i^{s_i})$ are odd integers.

Given a prime p_i, either $p_i \equiv 1$ (mod 4) or $p_i \equiv 3$ (mod 4). If $p_i \equiv 3 \equiv -1$ (mod 4), then

$$\sigma(p_i^{s_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{s_i} \equiv 1 + (-1) + (-1)^2 + \cdots + (-1)^{s_i} \pmod 4.$$

Therefore,

$$\sigma(p_i^{s_i}) \equiv \begin{cases} 0 \ (\text{mod } 4), & \text{if } s_i \text{ is odd}, \\ 1 \ (\text{mod } 4). & \text{if } s_i \text{ is even}. \end{cases}$$

Thus, $p_1 \not\equiv 3$ (mod 4) since $\sigma(p_1^{s_1}) \equiv 2$ (mod 4). Hence, $p_1 = p \equiv 1$ (mod 4). Furthermore, $\sigma(p_i^{s_i}) \equiv 0$ (mod 4) $(i > 1)$ implies that 4 divides $\sigma(p_i^{s_i})$, a contradiction since $\sigma(n)$ is divisible by 2, but not by 4. Therefore, if $p_i \equiv 3$ (mod 4) for some $i > 1$, then its exponent s_i is an even integer. On the other hand, if the prime $p_i \equiv 1$ (mod 4), then

$$\begin{aligned} \sigma(p_i^{s_i}) &= 1 + p_i + p_i^2 + \cdots + p_i^{s_i} \\ &\equiv 1 + 1^1 + 1^2 + \cdots + 1^{s_i} \pmod 4 \\ &\equiv s_i + 1 \pmod 4. \end{aligned}$$

The statement $\sigma(p_1^{s_1}) \equiv 2$ (mod 4) implies $s_1 \equiv 1$ (mod 4). For the remaining values of i, we have $\sigma(p_i^{s_i}) \equiv 1$ or 3 (mod 4), and therefore $s_i \equiv 0$ or 2 (mod 4); in any case, s_i is an even integer. Hence, regardless of whether $p_i \equiv 1$ or 3 (mod 4), the exponent s_i is always even for $i \neq 1$. Therefore, an odd perfect number n can be expressed as

$$n = p^s p_2^{2h_2} \ldots p_r^{2h_r} = p^s (p_2^{h_2} \ldots p_r^{h_r})^2 = p^s m^2,$$

where $m = p_2^{h_2} \ldots p_r^{h_r}$, $p \equiv 1$ (mod 4) is a prime of the form $4k+1$, $s \equiv 1$ (mod 4) is an integer of the form $4h+1$, p is not a divisor of m by the prime factorization of n and $n \equiv 1$ (mod 4) since m is an odd integer implies $m^2 \equiv 1$ (mod 4).

For the last part of the problem, we will try to show that $n = 7$ is the only positive integer such that $n-1 = 6$ and $n(n+1)/2 = 28$ are perfect numbers. Consider the following cases that correspond to the four parity cases of $n - 1$ and $n(n+1)/2$.

Case 1: $n \equiv 0$ (mod 4). Then $n-1 \equiv -1 \equiv 3$ (mod 4) implies $n-1$ is not perfect by the solution to the first part of the problem.

Case 2: $n \equiv 1$ (mod 4). Then $n - 1$ is even and $n(n+1)/2$ is odd. Assume both integers are perfect. Then, by Euler's theorem, $n - 1 = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are odd primes. Then

$$n(n+1)/2 = (2^{2p-1} - 2^{p-1} + 1)(2^{2p-2} - 2^{p-2} + 1).$$

Both of these factors are relatively prime and the above solution requires that one of these terms is a square. Since p is odd,

$$2^{2p-1} - 2^{p-1} + 1 \equiv 2 \pmod 3$$

and so is not a square. The other factor lies between two consecutive squares; that is,

$$(2^{p-1} - 1)^2 < 2^{2p-2} - 2^{p-2} + 1 < (2^{p-1})^2.$$

Therefore, neither factor is a square so that $n(n+1)/2$ is not perfect.

Case 3: $n \equiv 2 \pmod 4$. Then $n + 1 \equiv 3 \pmod 4$ and therefore has a prime factor q congruent to 3 modulo 4 with an odd exponent in the prime factorization of $n + 1$. Since n and $n + 1$ are relatively prime, q has the same odd exponent in the prime factorization of $n(n + 1)/2$. By the above solution to the first part, $n(n + 1)/2$ cannot be perfect.

Case 4: $n \equiv 3 \pmod 4$. Then $n - 1$ is divisible by 2, but not by 4. The only even perfect number not divisible by 4 is 6. Therefore $n - 1$ is perfect only when $n = 7$. Also, $n(n + 1)/2 = 28$ is perfect.

OC359. Let a, b, c, d be positive numbers such that $a + b + c + d = 3$. Prove

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3 b^3 c^3 d^3}.$$

Originally Problem 7, Grade 11, Day 2 of 2016 AllRussian Olympiad.

We received 3 solutions. We present the solution by Oliver Geupel.

Without loss of generality assume $0 < a \leq b \leq c \leq d$. We successively obtain

$$a^3 b^3 c^3 \leq a^3 b^3 d^3 \leq a^2 b c^3 d^3 \leq a b^2 c^3 d^3,$$
$$a^3 b^3 d^3 + a^3 b^3 c^3 \leq 3 a^2 b c^3 d^3 + 3 a b^2 c^3 d^3,$$
$$b^3 c^3 d^3 + a^3 c^3 d^3 + a^3 b^3 d^3 + a^3 b^3 c^3 \leq (a^3 + 3a^2 b + 3ab^2 + b^3) c^3 d^3$$
$$= (a + b)^3 c^3 d^3. \qquad (1)$$

By the geometric mean - arithmetic mean inequality and the hypothesis, it holds

$$(a + b)^3 c^3 d^3 \leq \left(\frac{(a + b) + c + d}{3} \right)^9 = 1. \qquad (2)$$

Combining (1) and (2), we successively conclude

$$b^3 c^3 d^3 + a^3 c^3 d^3 + a^3 b^3 d^3 + a^3 b^3 c^3 \leq 1,$$

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} \leq \frac{1}{a^3 b^3 c^3 d^3}.$$

This completes the proof.

OC360. Let A, B and F be positive integers with $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

1. any two distinct intervals are disjoint and not adjacent;

2. there are at least F positive integers with no lava between any two intervals;

3. no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does, is $F = (n-1)A + B$, where n is the positive integer such that $\frac{A}{n+1} \leq B - A < \frac{A}{n}$.

Originally Problem 4 of 2016 Canadian Mathematical Olympiad.

We received no solutions to this problem.

OC361. Let $n \geq 2$ be a positive integer and define k to be the number of primes less than or equal to n. Let A be a subset of $S = \{2, \ldots, n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B of cardinality k such that $A \subseteq B \subseteq S$ and no two elements in B divide each other.

Originally Problem 4 of Day 2 of 2016 China National Olympiad.

We received only one submission. We present the solution by Mohammed Aassila.

We prove the result by backward induction on $|A|$.

For $|A| = k$, take $B = A$.

Now, suppose that the result is true for $|A| = m < k$ and we will prove it true for $|A| = m - 1$. Let $\{p_1, p_2, \ldots, p_k\}$ be the set of all primes less than or equal to n and for $i = 1, 2, \ldots, k$ define

$$A_i = \{x \in A : p_i \text{ is the largest prime divisor of } x\}.$$

Note that A is a disjoint union of the A_i's and $|A_i| < |A|$. If all the A_i's are nonempty, then

$$m - 1 = |A| = \sum_{i=1}^{k} |A_i| \geq k,$$

contradiction. So, A_j is empty for some $1 \leq j \leq k$. Let $p = p_j$ and let p^α be the largest power of $p \leq n$. We claim that A contains no divisors nor multiples of p^α. Indeed, if $p^\beta \in A$ for some $\beta \leq \alpha$, it would follow that $p^\beta \in A_j$, contradiction. Meanwhile, if $cp^\alpha \in A$ for some $c > 1$, then cp^α has a prime divisor $q > p$ because $cp^\alpha \notin A_j$. Hence, $q \mid c$, which implies that $cp^\alpha \geq qp^\alpha > p^{\alpha+1} > n$, contradiction. Thus, if $A' = A \cup \{p^\alpha\}$, then no two distinct elements of A' divide one another.

Since $|A'| = m$ and $A \subseteq A'$, we can then find the desired B by the inductive hypothesis applied to A'.

OC362. Given a positive prime number p, prove that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$ if and only if there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$.

Originally Problem 2 of Day 1 of 2016 Spain Mathematical Olympiad.

We received 4 submissions. We present 2 solutions.

Solution 1, by Oliver Geupel.

The assertion holds true for $p = 3$, because $\alpha = 3$ and $\beta = 2$ are integers with the required properties. In what follows we assume $p \neq 3$.

First, suppose that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$. We prove that there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$. Specifically, $\beta = 3\alpha - 1$ has the required property. In fact,

$$\beta(\beta - 1) + 25 = (3\alpha - 1)(3\alpha - 2) + 25 = 9\left(\alpha(\alpha - 1) + 3\right).$$

Next, suppose that there exists a positive integer β such that $p \mid \beta(\beta - 1) + 25$. We have to show that there exists a positive integer α such that $p \mid \alpha(\alpha - 1) + 3$. Observe that if an integer α has the property that $p \mid \alpha(\alpha - 1) + 3$, then every integer that is congruent to α modulo p, has the same property. As a consequence, it is enough to determine a (not neccessarily positive) integer α such that $p \mid \alpha(\alpha - 1) + 3$. Let $\omega = 3^{p-2}(2\beta - 1)$. We prove that $\alpha = \beta - \omega$ has the required property.

Applying Fermat's Little Theorem, we see that
$$\omega^2 \equiv 3^{2p-4}\left(4\left(\beta(\beta - 1) + 25\right) - 99\right)$$
$$\equiv 3^{2p-4} \cdot (-99)$$
$$\equiv \left(3^{p-1}\right)^2 \cdot (-11)$$
$$\equiv -11 \pmod{p}.$$

Also,
$$\omega(2\beta - 1) \equiv 3^{p-2}\left(4\left(\beta(\beta - 1) + 25\right) - 99\right)$$
$$\equiv 3^{p-2} \cdot (-99)$$
$$\equiv 3^{p-1} \cdot (-33)$$
$$\equiv -33 \pmod{p}.$$

Consequently,
$$\alpha(\alpha - 1) + 3 \equiv (\beta - \omega)(\beta - \omega - 1) + 3$$
$$\equiv \beta(\beta - 1) + \omega^2 + \omega(1 - 2\beta) + 3$$
$$\equiv -25 - 11 + 33 + 3$$
$$\equiv 0 \pmod{p}.$$

Hence α has the required property.

Solution 2, by David Manes.

Note that $p \neq 2$ since

$$\alpha(\alpha - 1) + 3 \equiv \beta(\beta - 1) + 25 \equiv 1 \pmod 2$$

for all positive integers α and β. Moreover, if $p = 11$, then $\alpha = \beta = 6$ and if $p = 3$, then $\alpha = 1$ and $\beta = 2$. Let $p \neq 3, 11$ be an odd prime and assume there exists a positive integer α such that $p | \alpha(\alpha - 1) + 3$. Then $\alpha^2 - \alpha + 3 \equiv 0 \pmod p$. Multiplying by 4 and simplifying, we obtain $(2\alpha - 1)^2 \equiv -11 \pmod p$. Therefore, -11 is a quadratic residue of p so that the Legendre symbol

$$\left(\frac{-11}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{11}{p}\right) = 1.$$

Consequently, -99 is also a quadratic residue of p since the Legendre symbol

$$\left(\frac{-99}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{11}{p}\right)\left(\frac{9}{p}\right) = 1.$$

Note that $(9/q) = 1$ for all odd primes $q \neq 3$. Thus, there exists a positive integer γ such that $\gamma^2 \equiv -99 \pmod p$. Solving the linear congruence $2x - 1 \equiv \gamma \pmod p$ for x a least residue of p produces the positive integer β. Therefore, $2\beta - 1 \equiv \gamma \pmod p \implies (2\beta - 1)^2 \equiv \gamma^2 \equiv -99 \pmod p \implies 4\beta^2 - 4\beta + 100 \equiv 0 \pmod p$. Since $p \neq 2$, it follows that 4 and p are relatively prime so that dividing the last congruence by 4, we obtain $\beta^2 - \beta + 25 \equiv 0 \pmod p$. Hence, there exists a positive integer β such that $p | \beta(\beta - 1) + 25$. The converse now follows by starting with the assumption for β and following the same steps that we used for α and using the fact that if -99 is a quadratic residue of the odd prime p, then so is -11.

Editor's Comments. The Missouri State University Problem Solving Group solved the generalization of the problem, i.e. the result holds for all p, whether prime or not. We give the idea of the proof, so that the reader can fill all the details. One implication is easy, because if there exists α such that $\alpha^2 - \alpha + 3 \equiv 0 \pmod p$, then taking $\beta = 3\alpha - 1$, we have

$$\beta^2 - \beta + 25 = 9(\alpha^2 - \alpha + 3) \equiv 0 \pmod p.$$

Conversely, if $p \not\equiv 0 \pmod 3$, then there exists γ such that $3\gamma \equiv 1 \pmod p$. Taking $\alpha = \gamma(\beta + 1)$, the reader can verify that $\alpha^2 - \alpha + 3 \equiv 0 \pmod p$. Now, one can prove by induction on $m > 0$ that the congruence $t^2 - t + 3 \equiv 0 \pmod{3^m}$ has a solution with $t \equiv 0 \pmod 3$ for all $m > 0$ (if $t = 3k$ is such that $t^2 - t + 3 = 3^m \ell$, letting $s = t + 3^m \ell$, it's easy to see that $s^2 - s + 3 \equiv 0 \pmod{3^{m+1}}$). Now, if $p \equiv 0 \pmod 3$, then $p = 3^m n$ with $m > 0$ and $n \not\equiv 0 \pmod 3$. If $\beta^2 - \beta + 25 \equiv 0 \pmod p$, then $\beta^2 - \beta + 25 \equiv 0 \pmod n$. Since $n \not\equiv 0 \pmod 3$, then there is a such that $a^2 - a + 3 \equiv 0 \pmod n$ and there is also t such that $t^2 - t + 3 \equiv 0 \pmod{3^m}$. Now it is sufficient to use Chinese Remainder Theorem to conclude that there is α such that $\alpha \equiv a \pmod n$ and $\alpha \equiv t \pmod{3^m}$, i.e. $\alpha^2 - \alpha + 3 \equiv 0 \pmod n$ and $\alpha^2 - \alpha + 3 \equiv 0 \pmod{3^m}$ and the conclusion follows.

OC363. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(y)f(x + f(y)) = f(x)f(xy)$$

for all positive real numbers x and y.

Originally Algebra Problem 3 of 3rd Round of 2016 Iranian Mathematical Olympiad.

We received only one submission. We present the solution by Mohammed Aassila.

Let $f(1) = a$ and let $P(x, y)$ be the assertion

$$f(y)f(x + f(y)) = f(x)f(xy).$$

Then,

$$P(x, 1) \implies f(x + a) = \frac{(f(x))^2}{a}$$

and

$$P(x + a, y) \implies \frac{f(y)(f(x + f(y)))^2}{a} = \frac{(f(x))^2 f(xy + ay)}{a}$$

$$\implies f(x + f(y)) = \frac{f(x)f(xy + ay)}{f(xy)}$$

$$\implies f(y)f(xy + ay) = (f(xy))^2.$$

Hence,

$$((f(2x))^2 = f(2)f(2x + 2a) = \frac{f(2)(f(2x + a))^2}{a} = \frac{f(2)(f(2x))^4}{a^3},$$

which gives

$$((f(2x))^2 = \frac{a^3}{f(2)},$$

which means that $f(x)$ is a constant function.

Hence, all solutions are $f(x) \equiv c$ for some constant $c > 0$ which clearly satisfies the functional equation.

OC364. Consider an acute triangle ABC. Suppose $AB < AC$, let I be the incenter, D the foot of perpendicular from I to BC, and suppose that altitude AH meets BI and CI at P and Q, respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L and suppose that the circumcircle of $\triangle AIL$ meets BC again at N. Prove that

$$\frac{BD}{CD} = \frac{BN}{CN}.$$

Originally Problem 7 of Day 2 of 2016 China Girls Mathematical Olympiad.

We received only one submission. We present the solution by Oliver Geupel.

Let E and F denote the feet of perpendiculars from I to CA and AB, respectively. Straightforward angle chasing gives us

$$\angle EDF = \angle PIQ = 90° - \angle A/2 \qquad \text{and} \qquad \angle FED = \angle QPI = 90° - \angle B/2.$$

Hence the triangles DEF and IPQ are similar.

Further, we can find that $\angle AIQ = \angle IPQ = 90° - \angle B/2$. Thus AI is tangent to the circumcircle of triangle IPQ. As a consequence, triangle PAI is similar to triangle EMD, where M is the point of intersection of BC and EF. By similarity, we have $\angle LMI = \angle DMI = \angle IAO = \angle IAL$. Hence, M is the point of intersection of BC and the circumcircle of triangle AIL, that is, $M = N$.

Suppose that the parallel to EM through B meets AC at point K. We have $\angle BKC = \angle FEC = \angle BFE = 180° - \angle MFB$. Let a, b, c, and s denote the lengths of sides and the semiperimeter, respectively, of triangle ABC. We obtain

$$\frac{BN}{s-b} = \frac{BM}{BF} = \frac{\sin \angle MFB}{\sin \angle BMF} = \frac{\sin \angle BKC}{\sin \angle CBK} = \frac{BC}{CK} = \frac{a}{b-c}.$$

Consequently,

$$BN = \frac{a(s-b)}{b-c},$$
$$CN = BC + BN = a + \frac{a(s-b)}{b-c} = \frac{a(s-c)}{b-c},$$
$$\frac{BD}{CD} = \frac{s-b}{s-c} = \frac{BN}{CN}.$$

The proof is complete.

OC365. A square $ABCD$ is divided into n^2 equal small (fundamental) squares by drawing lines parallel to its sides. The vertices of the fundamental squares are called vertices of the grid. A rhombus is called *nice* when:

1. it is not a square;

2. its vertices are points of the grid;

3. its diagonals are parallel to the sides of the square $ABCD$.

Find (as a function of n) the number of nice rhombuses (n is a positive integer greater than 2).

Originally Problem 4 of 2016 Greece National Olympiad Problem.

We received 2 correct submissions. We present the solution by the Missouri State University Problem Solving Group.

Let $A = (0,0), B = (n,0), C = (n,n)$, and $D = (0,n)$, so the vertices of the small squares have integer coordinates. We claim that the number of nice rhombuses centered at the point (i, j) is

$$xy - \min(x, y), \text{ where } x = \min(i, n - i) \text{ and } y = \min(j, n - j)$$

(i.e. x and y are the distances from the point to sides of $ABCD$ nearest it). To see this, note that a rhombus satisfying conditions (2) and (3) and centered at (i, j) has vertices

$$(i + s, j), \ (i, j + t), \ (i - s, j) \text{ and } (i, j - t)$$

with $1 \le s \le x$ and $1 \le t \le y$. The fact that $x = \min(i, n - i)$ and $y = \min(j, n - j)$ guarantees that the vertices are points of the grid. Therefore the number of rhombuses satisfying conditions (2) and (3) and centered at (i, j) is xy. The rhombuses of this type that are squares have $s = t$ and there are $\min(x, y)$ of these. The claim follows.

We first deal with the case $n = 2k + 1$. Here we can sum over all $1 \le i, j \le k$ and multiply by 4 to obtain the final answer. Therefore, the number of nice rhombuses is

$$
\begin{aligned}
4 \left(\sum_{i=1}^{k} \sum_{j=1}^{k} (ij - \min(i, j)) \right) &= 4 \left(\sum_{i=1}^{k} \sum_{j=1}^{k} ij - \sum_{i=1}^{k} \sum_{j=1}^{k} \min(i, j) \right) \\
&= 4 \left(\sum_{i=1}^{k} i \sum_{j=1}^{k} j - \sum_{t=1}^{k} \sum_{i=t}^{k} \sum_{j=t}^{k} 1 \right) \\
&= 4 \left(\left(\frac{k(k+1)}{2} \right)^2 - \sum_{t=1}^{k} (k + 1 - t)^2 \right) \\
&= 4 \left(\frac{k^2(k+1)^2}{4} - \frac{k(k+1)(2k+1)}{6} \right) \\
&= \frac{(3k + 2)(k + 1)k(k - 1)}{3}.
\end{aligned}
$$

When $n = 2k$, summing over all $1 \leq i, j \leq k-1$ and over $i = k$, $1 \leq j \leq k-1$ multiplying this result by 4 and adding the value for $i = j = k$ gives the total number of nice rhombuses. In this case, we obtain

$$4 \left(\sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (ij - \min(i,j)) + \sum_{j=1}^{k-1} (kj - j) \right) + k^2 - k$$

$$= \frac{(3k-1)k(k-1)(k-2)}{3} + 4(k-1)\frac{k(k-1)}{2} + k^2 - k$$

$$= \frac{k(k-1)(3k^2 - k - 1)}{3}$$

as the number of nice rhombuses.
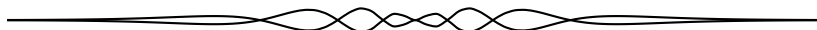
PROBLEM SOLVING TOOLBOX: Correspondence

Olga Zaitseva

Which set contains more elements?

There is a large class of problems asking to determine which of two sets contains more elements.

A direct approach is not always the best choice; sometimes it is not even applicable. However, there is a nice trick: one can try to establish a correspondence between elements of two sets (to pair elements of both sets). In kindergarten, not all children can count even a small number of objects. However, when they are placed in "girl-boy" pairs in a dance class, each can answer correctly if there are more boys than girls, more girls than boys or there are the same number of each.

Problem 1 (Lomonosov Academic Tournament 1996) *Among two sets of the numbers from 000 to 999, which set contains more elements, the set of numbers with the middle digit being strictly greater than the others or the set of numbers with the middle digit being strictly smaller than the others?*

Solution. For any number abc, let us place into correspondence with it the number $(9-a, 9-b, 9-c)$. Then all numbers from 000 to 999 are paired. It is clear that if a number belongs to any one of the two given sets, then the corresponding number belongs to the other set. Hence, the number of elements in both sets is the same. □

Problem 2 *One of n chosen points on a circle is coloured red. Consider all possible convex polygons with vertices at the chosen points. Which set contains more elements, the set of polygons that contain the red point or the set of polygons that does not contain the red point?*

Solution. Observe that any described polygon is uniquely defined by its vertices. Let R be the set of polygons that contain the red point and let X be the set of polygons that do not contain the red point. To any polygon in X, we can place into correspondence a polygon with the red point added. Consequently, to any polygon in R we can try to put into correspondence a polygon with the red point erased. However, there is no polygon in X corresponding to a triangle in R. Hence, set R contains more elements. □

Problem 3 (Tournament of Towns 2014, Fall Round, Seniors O-level)
Among 15 given integers, no two are the same. Pete wrote down sums of all seven element subsets while Basil wrote down sums of all eight element subsets of these numbers. Could it happen that Pete and Basil wrote down the same set of numbers? (To be considered the same, each integer must be repeated in Pete's set as many times as it is repeated in Basil's set).

Solution. Consider the set of numbers

$$S = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}.$$

Let P be the set of all seven element subsets of S and let B be the set of all eight element subsets of S. For each subset of S, let us place into correspondence the complement of set S with flipped signs. We can see that this is a one-to-one correspondence between P and B. One can check that the sum of elements in corresponding subsets is the same. This implies that Pete and Basil wrote down the same sets of numbers. □

Problem 4 (Tournament of Towns 2014, Fall Round, Seniors A-level)
Pete counted all possible words consisting of m letters, such that each letter can be only one of T, O, W, or N, and moreover that the number of letters T and O is the same. Basil counted all possible words consisting of $2m$ letters such that each letter is either T or O and the number of letters T and O is the same. Which of the boys obtained the greater number of words? (A word is an arbitrary finite sequence of letters.)

Hint. Establish a one-to-one correspondence between words on Pete's and Basil's lists. □

Correspondence in Games

In strategy games, a widely applied strategy is based on corresponding (symmetrical, reciprocal) moves of one of the players.

Problem 5 *Two players A and B take turns placing coins on a circular table without overlaps. The player who cannot make a move loses. Which of the players has a winning strategy, the first or the second?*

Solution. The first player has a winning strategy. His first move is to place a coin at the centre of the table. After that, he simply reciprocates the second player's move by placing a coin symmetrical about the centre. Thus, if the second player has a move, so does the first player. □

Note that any centrally symmetric table will work for this argument.

Problem 6 (Tournament of Towns 2002, Fall Round, Juniors O-level)
2002 cards with the numbers $1, 2, \ldots, 2002$ written on them are placed on a table face up. Two players take turns picking up one card at a time until all cards are gone. The winner is the player who gets the larger last digit from the sum of all numbers on their cards. Which of the players has a winning strategy the first or the second? How should this person play to win?

Solution. The first player has a winning strategy. Let us pair numbers up: $(k, 1000 + k)$ for $k = 1, \ldots, 1000$ and $(2001, 2002)$. Then in each pair except the last one the last digit of both numbers is the same. The first player starts and

picks up 2002. From this moment, his strategy is to pick up the other half of the pair chosen by the second player. So, eventually the second player is forced to pick up 2001. If not all cards are gone, then the first player takes any number leaving for the second player to pick up the other half of the pair. In the meantime, the first player echoes the second player's moves. Therefore, except for the last pair, the last digits of the sums of both players are the same (0 modulo 10). With the last pair, the first player has the last digit of the sum one greater than the second player. □

Problem 7 (Tournament of Towns, 2013 Fall, Juniors O-level) *On a table, there are eleven piles of ten stones each. Pete and Basil play a game. They take turns picking up 1, 2, or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses. Which of the players, Pete or Basil, has a winning strategy?*

Hint. Arrange stones into an 11×11 table so that piles correspond to columns. The diagonal from top left to bottom right is empty. Then Pete takes stones from one column while Basil takes stones from different columns. Find a reciprocal move for Basil. □

Practice Problems

Problem 8 (Tournament of Towns 2009, Fall Round, Juniors O-level) *There are forty weights: $1, 2, \ldots, 40$ grams. Ten even weights are placed on the left side of a balance while ten odd weights are placed on the right pan of the balance. It occur that the left and the right sides are in equilibrium. Prove that one of the sides contains two weights that differ by exactly 20 grams.*

Problem 9 (Olympiad Formula of Unity) *A convex 2015-gon and all its diagonals are drawn on a blackboard. Alex and Ben play the following game. In turns each boy erases either any number of adjacent sides from 1 to 10 or any number of diagonals from 1 to 9. The player who cannot make a move loses. Alex starts first. Which of the boys can win the game no matter how the other boy plays? What is the winning strategy?*

Problem 10 (Tournament of Towns, 2001, Spring, Seniors O-level) *Two players in turns place 1×2 tiles on a 3×100 board. The first player places tiles horizontally (along the longest side of the board), while the second player places tiles vertically. The player who can not make a move loses the game. Which of the players, the first or the second can always win (no matter how his opponent plays), and what is the winning strategy?*
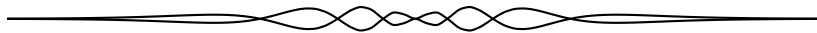
PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

*To facilitate their consideration, solutions should be received by **May 15, 2019**.*

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4411. *Proposed by Michel Bataille.*

Let n be a positive integer. Find the largest constant C_n such that

$$\frac{(xy)^n}{z^{n+1}} + \frac{(yz)^n}{x^{n+1}} + \frac{(zx)^n}{y^{n+1}} \geq C_n \left(\max(x,y,z)\right)^{n-1}$$

holds for all real numbers x, y, z satisfying $xyz > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$.

4412. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with incenter I. If I_a, I_b, I_c are the excenters of ABC, show that $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ if and only if ABC is equilateral.

4413. *Proposed by Miguel Ochoa Sanches and Leonard Giugiuc.*

Let ABC be a triangle with incenter I and circumcircle ω. The lines AI, BI, CI intersect ω a second time at M, N, P, respectively. Also suppose that NP intersects AB and AC at E and F, respectively. We define points G, H, J and D analogously (see the picture). Show that if $EF = GH = JD$, then triangle ABC is equilateral.

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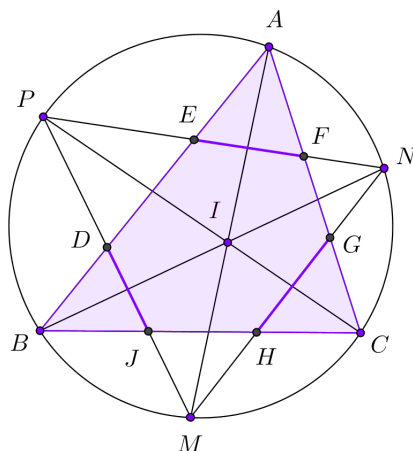
4414. *Proposed by Konstantin Knop.*

Let α and β be a pair of circles that intersect in points P and Q, and let the diameter AA' of α lie on the same line as the diameter BB' of β such that the end points lie in the order $AB'A'B$. Suppose that PB' intersects α again at the point C, that PA' intersects β again at D, and that the lines AD and BC intersect at R. Prove that the line QR intersects the segment AB at its midpoint.
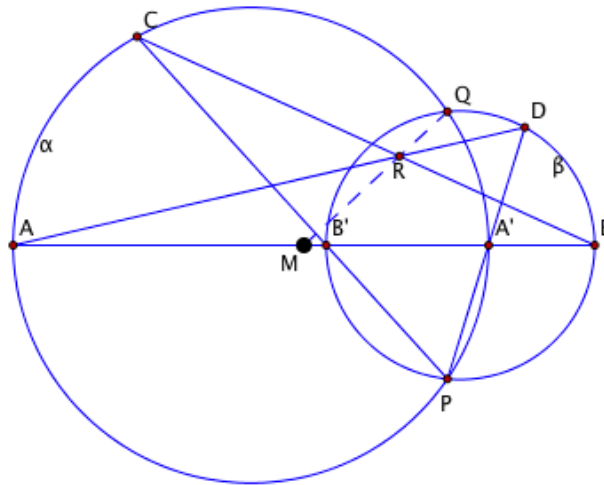
4415. *Proposed by Titu Zvonaru.*

Let ABC be an acute-angled triangle with $AB < AC$, where AD is the altitude from A, O is the circumcenter and M and N are the midpoints of the sides BC and AB, respectively. The line AO intersects the line MN at X. Prove that DX is parallel to OC.

4416. *Proposed by Nguyen Viet Hung.*

Let ABC be an acute triangle with orthocentre H. Denote by r_a, r_b, r_c the exradii opposite the vertices A, B, C, and by r_1, r_2, r_3 the inradii of triangles BHC, CHA, AHB, respectively. Prove that

$$r_1 + r_2 + r_3 + r_a + r_b + r_c = a + b + c.$$

4417. *Proposed by Dan Stefan Marinescu, Daniel Sitaru and Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $abc \geq 1$. Further, let x, y and z be real numbers such that $xy + yz + zz \geq 3$. Prove that

$$(y^2 + z^2)a + (z^2 + x^2)b + (x^2 + y^2)c \geq 6.$$

4418. *Proposed by Daniel Sitaru.*

Consider a convex cyclic quadrilateral with sides a, b, c, d and area S. Prove that

$$\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c} \geq 64S^2.$$

4419. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90°$. Let D on the hypotenuse BC produced beyond C be such that $CD = CB + BA$. The internal bisector of $\angle ABC$ intersects the line through the midpoints of AB and AC at T. Prove that $\angle TCA = \angle CDA$.

4420. *Proposed by Leonard Giugiuc and Marian Dinca.*

Let $A_0 A_1 \ldots A_{n-1}$, $n \geq 10$ be a regular polygon inscribed in a circle of radius r centered at O. Consider the closed disks $\omega(A_k), k = 0, \ldots, n-1$ centered at A_k of radius r. Prove that
$$\cap_{k=0}^{n-1} \omega(A_k) = \{O\}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2019**.*

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

4411. *Proposé par Michel Bataille.*
Soit n un entier positif. Trouver la plus grande constante possible C_n telle que

$$\frac{(xy)^n}{z^{n+1}} + \frac{(yz)^n}{x^{n+1}} + \frac{(zx)^n}{y^{n+1}} \geq C_n \left(\max(x, y, z)\right)^{n-1}$$

soit vraie pour tous les nombres réels x, y, z satisfaisant à $xyz > 0$ et $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$.

4412. *Proposé par Mihaela Berindeanu.*
Soit ABC un triangle acutangle de centre inscrit I. Si I_a, I_b, I_c sont les centres des cercles exinscrits de ABC, montrer que $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ si et seulement si ABC est un triangle équilatéral.

4413. *Proposé par Miguel Ochoa Sanches et Leonard Giugiuc.*
Soit ABC un triangle de centre inscrit I et le cercle circonscrit ω. Les segments AI, BI, CI interceptent ω une deuxième fois aux points M, N, P, respectivement. Supposons aussi que NP intercepte AB et AC aux points E et F respectivement. Similairement, les points G, H, J et D sont définis (voir l'image ci-dessous). Montrer que si $EF = GH = JD$, alors le triangle ABC est équilatéral.
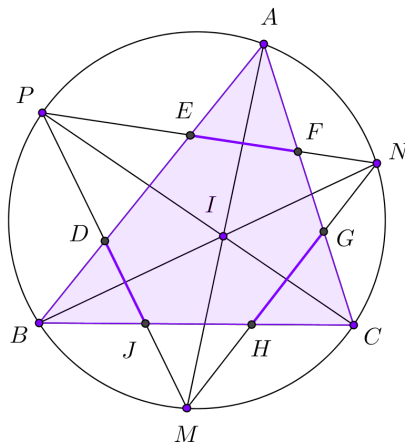
4414. *Proposé par Konstantin Knop.*

Soient α et β deux cercles, intersectant aux points P et Q. Supposer qu'un diamètre AA' de α se trouve sur la même ligne qu'un diamètre BB' de β, de façon à ce que les extrémités de ces diamètres se retrouvent dans l'ordre $AB'A'B$. De plus, supposer que PB' intersecte α de nouveau au point C, que PA' intersecte β de nouveau au point D, puis que les lignes AD et BC intersectent au point R. Démontrer que la ligne QR intersecte le segment AB à son mi point.
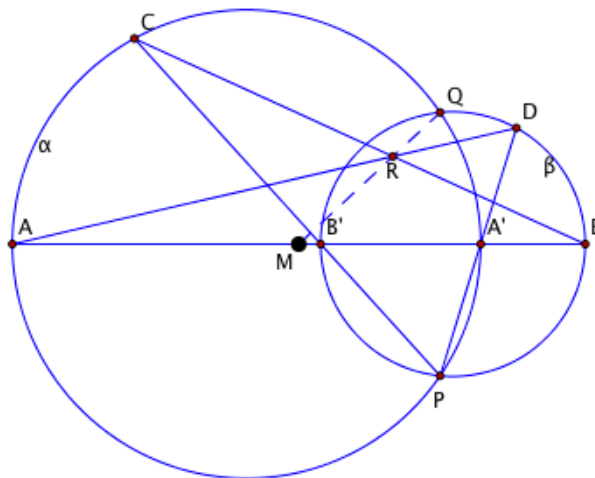
4415. *Proposé par Titu Zvonaru.*
Soit ABC un triangle acutangle tel que $AB < AC$ où AD est la hauteur issue de A, O est le centre circonscrit et M et N sont les points milieu des côtés BC et AB respectivement. Le segment AO intercepte le segment MN au point X. Montrer que DX est parallèle à OC.

4416. *Proposé par Nguyen Viet Hung.*
Soit ABC un triangle acutangle d'orthocentre H. Soit r_a, r_b, r_c les rayons des cercles exinscrits opposés aux sommets A, B, C, et soit r_1, r_2, r_3 les rayons des cercles inscrits des triangles BHC, CHA, AHB, respectivement. Montrer que

$$r_1 + r_2 + r_3 + r_a + r_b + r_c = a + b + c.$$

4417. *Proposé par Dan Stefan Marinescu, Daniel Sitaru et Leonard Giugiuc.*
Soit a, b et c des nombres réels positifs tels que $abc \geq 1$. De plus, soit x, y et z des nombres réels tels que $xy + yz + xz \geq 3$. Montrer que

$$(y^2 + z^2)a + (x^2 + z^2)b + (x^2 + y^2)c \geq 6.$$

4418. *Proposé par Daniel Sitaru.*
Soit un quadrilatère convexe cyclique de côtés a, b, c, d et d'aire S. Montrer que

$$\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c} \geq 64S^2.$$

4419. *Proposé par Michel Bataille.*
Soit ABC un triangle où $\angle BAC = 90°$. Soit le point D sur le prolongement de l'hypoténuse BC tel que $CD = CB + BA$. La bissectrice interne de $\angle ABC$ intercepte le segment par les points milieu de AB et AC au point T. Montrer que $\angle TCA = \angle CDA$.

4420. *Proposé par Leonard Giugiuc et Marian Dinca.*
Soit $A_0 A_1 \ldots A_{n-1}$, $n \geq 10$ un polygone régulier inscrit dans un cercle de rayon r centré en O. Considérons les disques fermés $\omega(A_k), k = 0, \ldots, n-1$ centrés en A_k de rayon r. Montrer que
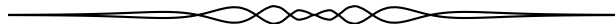$$\cap_{k=0}^{n-1} \omega(A_k) = \{O\}.$$
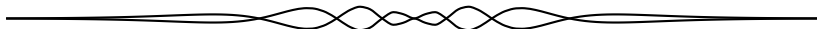
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2018: 44(2), p. 69–72.

$$\longrightarrow \diamond\!\!\diamond\!\!\diamond\!\!\diamond\!\!\diamond \longleftarrow$$

4300★. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers with $a + b + c = ab + bc + ca > 0$. Prove or disprove that

$$\sqrt{24ab + 25} + \sqrt{24bc + 25} + \sqrt{24ca + 25} \geq 21.$$

Arkady Alt pointed out some defects in the solution of 4300 published in 43(10). We present a complete solution here.

We received 3 submissions of which 2 were correct and complete. We feature the solution by the proposer modified by the editor.

Let $a + b + c = ab + bc + ac = k$. Since $(a + b + c)^2 \geq 3(ab + bc + ac)$, we have $k^2 \geq 3k$, and since $k > 0$ it follows that $k \geq 3$. Assume, without loss of generality, that $bc = \max\{ab, bc, ac\}$. Then $bc \geq 1$, and $a = \frac{b+c-bc}{b+c-1}$. Setting $b + c = 2s$ and $bc = p^2$, we know that $s \geq p$, $p \geq 1$, and $a = \frac{2s-p^2}{2s-1}$. We consider two cases.

Case 1: $1 \leq p \leq 2$. For any such p, let $f_p(t) = \frac{2s-p^2}{2t-1}$ on $[p, \infty)$. Since $p^2 \geq 1$, we know that f_p is increasing, which implies that $a = \frac{2s-p^2}{2s-1} \geq \frac{2p-p^2}{2p-1}$. If we apply the AM-GM inequality we have:

$$\sqrt{24ab + 25} + \sqrt{24ac + 25} \geq 2((24ab + 25)(24ac + 25))^{1/4}.$$

Then

$$2((24ab + 25)(24ac + 25))^{1/4}$$

$$= 2\left(576\left(\frac{2s - p^2}{2s - 1}\right)^2 p^2 + 1200\left(\frac{2s - p^2}{2s - 1}\right)s + 625\right)^{1/4}$$

$$\geq 2\left(576\left(\frac{2p - p^2}{2p - 1}\right)^2 p^2 + 1200\left(\frac{2p - p^2}{2p - 1}\right)p + 625\right)^{1/4}$$

$$= 2\left(\left[24\left(\frac{2 - p}{2p - 1}\right)p^2 + 25\right]^2\right)^{1/4}$$

$$= 2\sqrt{24\left(\frac{2 - p}{2p - 1}\right)p^2 + 25}.$$

So we have

$$\sqrt{24ab + 25} + \sqrt{24ac + 25} \geq 2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2 + 25},$$

and $\sqrt{24bc + 25} = \sqrt{24p^2 + 25}$. So it suffices to show that

$$\sqrt{24p^2 + 25} + 2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2 + 25} \geq 21.$$

Since $1 \leq p \leq 2$, it follows that

$$\sqrt{24p^2 + 25} + 2\sqrt{24\left(\frac{2-p}{2p-1}\right)p^2 + 25} \geq \sqrt{24p^2 + 25} + 2\sqrt{73 - 24p}$$

Using basic calculus, it follows that the function $f : [1, 2] \longrightarrow \mathbb{R}$,

$$f(p) = \sqrt{24p^2 + 25} + 2\sqrt{73 - 24p}$$

is strictly decreasing on $[1, 5/3]$ and it is strictly increasing on $[5/3, 2]$. Hence it has a minimum at $5/3$ and the minimum value is more than 21.

Case 2: $p > 2$. Then we have

$$\sqrt{24ab + 25} + \sqrt{24ac + 25} + \sqrt{24bc + 25} \geq 10 + \sqrt{24p^2 + 25} > 10 + 11 = 21.$$

4311. *Proposed by Mihaela Berindeanu.*

Let A and B be two matrices in $\mathfrak{M}_3(\mathbb{Z})$ with $AB = BA$ and $\det A = \det B = 1$. Find the possible values for $\det\left(A^2 + B^2\right)$ knowing that

$$\det\left(A^2 + 2AB + 4B^2\right) - \det\left(A^2 - 2AB + 4B^2\right) = -4.$$

We received 4 correct solutions and one incomplete submission. We will feature a solution by Leonard Giugiuc.

Since $\det(B) = 1$, then $\det(B^{-1}) = 1$ and $B^{-1} \in M_3(\mathbb{Z})$. This implies that $\det(AB^{-1}) = 1$ and $AB^{-1} \in M_3(\mathbb{Z})$. Since $AB = BA$, we set $C = AB^{-1}$ to get

$$\det(A^2 + 2AB + 4B^2) - \det(A^2 - 2AB + 4B^2) = \det(C^2 + 2C + 4I_3) - \det(C^2 - 2C + 4I_3),$$

and $\det(A^2 + B^2) = \det(C^2 + I_3)$.

Consider the polynomial $f(x) = \det(C - xI_3)$ over $\mathbb{C}$. Then $f(x) = 1 - kx + tx^2 - x^3$ for all $x \in \mathbb{C}$ with k and t integers.

Let $u = 1/2(-1 + i\sqrt{3})$. Then $C^2 + 2C + 4I_3 = (C - 2uI_3)(C - 2u^2I_3)$, which implies

$$\begin{aligned}\det(C^2 + 2C + 4I_3) &= \det(C - 2uI_3) \cdot \det(C - 2u^2I_3) \\ &= f(2u) \cdot f(2u^2) \\ &= 49 + 4k^2 + 16t^2 - 14k + 28t + 8kt.\end{aligned}$$

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Similarly,

$$\det(C^2 - 2C + 4I_3) = 81 + 4k^2 + 16t^2 - 18k - 36t - 8kt.$$

Thus

$$\det(C^2 + 2C + 4I_3) - \det(C^2 - 2C + 4I_3) = -4$$

if and only if $k + 16t + 4kt = 7$ or, equivalently, $4t(k + 4) = 7 - k$. Since clearly, $k \neq -4$, we have $4t = -1 + 11/(k + 4)$, which implies $11/(k + 4) \in \mathbb{Z}$. Hence $k + 4 \in \{\pm 1, \pm 11\}$. Since t is an integer as well, we obtain $k = -5$ and $t = 3$ or $k = 7$ and $t = 0$. Observe that

$$\det(C^2 + I_3) = \det(C - I_3) \cdot \det(C + I_3) = f(i) \cdot f(-i) = (k - 1)^2 + (t - 1)^2.$$

In conclusion, $\det(A^2 + B^2) \in \{37, 52\}$.

4312. *Proposed by William Bell.*

Prove that

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh\left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

Nine correct solutions were received. Six followed the strategy of the first solution and three the strategy of the second.

Solution 1.

Since $\tanh u = 2 \coth 2u - \coth u$,

$$\sum_{r=1}^{n} \frac{1}{2^r} \tanh \frac{x}{2^r} = \sum_{r=1}^{n} \left(\frac{1}{2^{r-1}} \coth \frac{x}{2^{r-1}} - \frac{1}{2^r} \coth \frac{x}{2^r}\right) = \coth x - \frac{1}{2^n} \coth \frac{x}{2^n}.$$

Since, by l'Hôpital's Rule, for example, $\lim_{v \to 0} v \coth v = 1$,

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh \frac{x}{2^r} = \coth x - \frac{1}{x} \lim_{n \to \infty} \frac{x}{2^n} \coth \frac{x}{2^n} = \coth x - \frac{1}{x}.$$

Solution 2.

For $n \geq 1$, consider the product

$$P_n(x) = \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \ldots \cosh\left(\frac{x}{2^n}\right).$$

Since

$$P_n(x) \sinh\left(\frac{x}{2^n}\right) = \frac{1}{2} P_{n-1}(x) \sinh\left(\frac{x}{2^{n-1}}\right),$$

an induction argument leads to

$$P_n(x) = \frac{\sinh x}{2^n \sinh\left(\frac{x}{2^n}\right)}.$$

With D denoting differentiation, we have that

$$\sum_{r=1}^{n} \frac{1}{2^r} \tanh \frac{x}{2^r} = D\left(\sum_{r=1}^{n} \ln \cosh \frac{x}{2^r}\right) = D\ln\left(\prod_{r=1}^{n} \cosh \frac{x}{2^r}\right)$$
$$= D\ln P_n(x) - D(\ln \sinh x - n\ln 2 - \ln \sinh \frac{x}{2^n})$$
$$= \coth x - \frac{1}{2^n}\coth \frac{x}{2^n},$$

Let $n \to \infty$ to obtain

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh\left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

4313. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

Let I be the incenter of triangle ABC, and denote by H_a, H_b and H_c the orthocenters of triangles IBC, ICA and IAB, respectively. Prove that triangles ABC and $H_aH_bH_c$ have the same area.

We received six submissions, all correct, and feature the solution by Mohammed Aassila.

The result holds for any point P in the plane of triangle ABC that is not on a line joining two of its vertices; the argument is therefore not restricted to the special case of the problem, namely $P = I$. It is based on a familiar property of the mixed product $[\overrightarrow{p}, \overrightarrow{q}]$, which for vectors $\overrightarrow{p} = \langle p_1, p_2 \rangle$ and $\overrightarrow{q} = \langle q_1, q_2 \rangle$ in $\mathbf{R}^2$ is just the determinant,

$$[\overrightarrow{p}, \overrightarrow{q}] = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = p_1 q_2 - p_2 q_1.$$

Specifically, given a pair of triangles ABC and $A'B'C'$ in the plane, we have

$$[\overrightarrow{BA'}, \overrightarrow{AB'}] + [\overrightarrow{CB'}, \overrightarrow{BC'}] + [\overrightarrow{AC'}, \overrightarrow{CA'}] = [\overrightarrow{A'B'}, \overrightarrow{A'C'}] - [\overrightarrow{AB}, \overrightarrow{AC}].$$

This identity can be found as a straightforward exercise in texts that deal with the mixed product. Of course, when A', B', and C' are, respectively, the orthocentres of triangles PBC, PCA, and PAB, then each of the quantities on the left are zero (because the vectors $\overrightarrow{BA'}$ and $\overrightarrow{AB'}$ are both perpendicular to the line PC, etc.). We are therefore left with

$$[\overrightarrow{A'B'}, \overrightarrow{A'C'}] = [\overrightarrow{AB}, \overrightarrow{AC}],$$

which says (under the assumption that A', B', C' are the appropriate orthocentres) that the areas of triangles ABC and $A'B'C'$ are equal.

4314. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=1}^{n} k2^k \cdot \frac{\binom{n}{k}}{\binom{2n-1}{k}}.$$

We received three solutions, and we present two of them.

Solution 1, by Paolo Perfetti, slightly edited.

Denote the kth summand by s_k; that is,

$$s_k = k2^k \frac{\binom{n}{k}}{\binom{2n-1}{k}} = k2^k \frac{\binom{n}{k}k!(2n-k-1)!}{(2n-1)!}.$$

We will use the beta function, $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ defined for complex x, y such that $\mathfrak{Re}(x), \mathfrak{Re}(y) > 0$. One of the well known properties of the beta function (which follows from its relationship to the gamma function) is that $\beta(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$ for x, y positive integers. Hence we can write

$$s_k = k \cdot 2^k \cdot \binom{n}{k} \cdot 2n \cdot \beta(k+1, 2n-k),$$

and we have

$$\sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \left(k2^k \binom{n}{k} \cdot 2n \int_0^1 t^k(1-t)^{2n-k-1}\,dt \right)$$

$$= 2n \int_0^1 (1-t)^{2n-1} \sum_{k=1}^{n} k\binom{n}{k} \left(\frac{2t}{1-t} \right)^k dt. \tag{1}$$

Note that for any y we have the equality $\sum_{k=1}^{n} k\binom{n}{k}y^k = ny(y+1)^{n-1}$:

$$\sum_{k=1}^{n} k\binom{n}{k}y^k = n\sum_{k=1}^{n} \binom{n-1}{k-1}y^k = ny\sum_{k=0}^{n-1} \binom{n-1}{k}y^k = ny(1+y)^{n-1},$$

with the last equality following from the binomial theorem. Thus, continuing from (1) we get

$$\sum_{k=1}^{n} s_k = 2n \int_0^1 (1-t)^{2n-1} \left(\frac{2nt}{1-t} \right) \left(\frac{2t}{1-t} + 1 \right)^{n-1} dt$$

$$= 2n^2 \int_0^1 (1-t)^{n-1}(2t)(1+t)^{n-1}\,dt$$

$$= 2n^2 \int_0^1 2t(1-t^2)^{n-1}\,dt$$

$$= 2n^2 \cdot \left(-\frac{1}{n} \right) \cdot (1-t^2)^n \Big|_0^1 = 2n.$$

Solution 2, by the proposer.

Denote the kth summand by s_k as in the previous solution.

Let

$$u_k = -2^k \frac{\binom{n}{k}}{\binom{2n}{k}} = -2^k \frac{n!(2n-k)!}{(2n)!(n-k)!}$$

for $k = 1, 2, \ldots, n$ and $u_{n+1} = 0$. Note that for $k = 1, 2, \ldots, n-1$ we have

$$\begin{aligned}
u_{k+1} - u_k &= 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k-1)!} \cdot \left(-2 + \frac{2n-k}{n-k}\right) \\
&= 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k-1)!} \cdot \frac{k}{n-k} \\
&= k \cdot 2^k \cdot \frac{n!}{(2n)!} \cdot \frac{(2n-k-1)!}{(n-k)!} = \frac{s_k}{2n}.
\end{aligned}$$

The equality $u_{k+1} - u_k = \frac{s_k}{2n}$ holds trivially for $k = n$ since $u_{n+1} = 0$.

It follows that

$$\sum_{k=1}^{n} s_k = 2n \left(\sum_{k=1}^{n} u_{k+1} - u_k\right) = 2n(u_{n+1} - u_1) = 2n.$$

4315. *Proposed by Moshe Stupel, modified by the editors.*

Let H be the orthocenter of triangle ABC, and denote by R, r, and r' respectively the circumradius, inradius, and radius of the excircle that is opposite vertex A. Prove that $HA + r' = 2R + r$.

We received 11 submissions, all substantially correct. We present a combination of the solutions from Leonard Giugiuc and Cristóbal Sánchez-Rubio.

The statement of the problem is not quite correct. The identity we shall prove is

$$2R \cos A + r' = 2R + r. \tag{1}$$

Editor's comments. Giugiuc and just one other solver (Pranesachar) stated explicitly that $HA = 2R \cos A$ when $0° < \angle A \le 90°$; otherwise (when $\angle A$ is obtuse) $HA = -2R \cos A$. All other submissions tacitly took HA to be a directed length by setting it equal to $2R \cos A$. If you prefer to assume lengths to be nonnegative, then the correct identity in terms of HA is $r' \pm HA = 2R + r$, with the negative sign used when $\angle A$ is obtuse.

We shall make use of familiar formulas for the circumradius

$$R = \frac{abc}{4sr}, \tag{2}$$

and for the area,

$$rs = r'(s - a) = \sqrt{s(s-a)(s-b)(s-c)}. \tag{3}$$

Using expressions for R and r' (from (2) and (3)) reduces identity (1) (in the form $2R(\cos A - 1) = r - r'$) to

$$2\frac{abc}{4sr}(\cos A - 1) = r\left(1 - \frac{s}{s-a}\right) = -\frac{ra}{s-a},$$

so that the identity to be established is equivalent to

$$\frac{bc\,(1 - \cos A)}{2sr} = \frac{r}{s-a}. \tag{4}$$

But because

$$\begin{aligned}
1 - \cos A &= 1 + \frac{a^2 - b^2 - c^2}{2bc} \\
&= \frac{a^2 - (b-c)^2}{2bc} \\
&= \frac{(a+b-c)(a-b+c)}{2bc} \\
&= \frac{2(s-c)\cdot 2(s-b)}{2bc} \\
&= \frac{2(s-b)(s-c)}{bc},
\end{aligned}$$

equation (4) reduces to

$$(s-a)(s-b)(s-c) = sr^2,$$

which is the square of Heron's formula (see (3) above).

4316. *Proposed by Daniel Sitaru.*

Let $f : [0, 11] \to \mathbb{R}$ be an integrable and convex function. Prove that

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \leq \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

Ten correct solutions were received. Most followed the procedure of Solution 2.

Solution 1, by Roy Barbara.

Let $g(x) = ax + b$ be the linear function that satisfies $g(3) = f(3)$ and $g(8) = f(8)$.

Because $f(x)$ is convex, $f(x) \geq g(x)$ when $0 \leq x \leq 3$ or $8 \leq x \leq 10$, and $f(x) \leq g(x)$ when $3 \leq x \leq 8$. The left side does not exceed

$$\int_3^5 g(x)\,dx + \int_6^8 g(x)\,dx = 22a + 4b,$$

and the right side is not less than

$$\int_0^2 g(x)\,dx + \int_9^{11} g(x)\,dx = 22a + 4b.$$

The result follows.

Solution 2.

Since $f(x)$ is convex,

$$f(3+x) \leq \frac{2}{3}f(x) + \frac{1}{3}f(9+x) \quad \text{and} \quad f(6+x) \leq \frac{1}{3}f(x) + \frac{2}{3}f(9+x).$$

Therefore,

$$\begin{aligned}
\int_3^5 f(x)\,dx + \int_6^8 f(x)\,dx &= \int_0^2 [f(3+x) + f(6+x)]\,dx \\
&\leq \int_0^2 [f(x) + f(9+x)]\,dx \\
&= \int_0^2 f(x)\,dx + \int_9^{11} f(x)\,dx.
\end{aligned}$$

Solution 3, by Oliver Geupel.

Recall the Hermite-Hadamard Inequality for convex functions:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq \frac{1}{2}(b-a)(f(a)+f(b)).$$

Therefore

$$\begin{aligned}
\int_3^5 f(x)\ dx &\leq f(3) + f(5) \\
&\leq \left(\frac{7}{9}f(1) + \frac{2}{9}f(10)\right) + \left(\frac{5}{9}f(1) + \frac{4}{9}f(10)\right) \\
&= \frac{4}{3}f(1) + \frac{2}{3}f(10),
\end{aligned}$$

and, similarly,

$$\int_6^8 f(x)\,dx \leq f(6) + f(8) \leq \frac{2}{3}f(1) + \frac{4}{3}f(10).$$

Therefore

$$\int_3^5 f(x)\,dx + \int_6^8 f(x)\,dx \leq 2f(1) + 2f(10) \leq \int_0^2 f(x)\,dx + \int_9^{11} f(x)\,dx.$$

4317. *Proposed by Leonard Giugiuc.*

Solve the following system of equations over reals:

$$\begin{cases} a + b + c + d = 4, \\ abc + abd + acd + bcd = 2, \\ abcd = -\frac{1}{4}. \end{cases}$$

We received nine correct submissions. We present the solution by Oliver Geupel, modified and expanded by the editor.

Let $p = a + b, q = ab, r = c + d, s = cd$. Then $q \neq 0$ and $s = -\dfrac{1}{4q}$.

Since

$$2 = ab(c + d) + cd(a + b) = qr + ps = q(4 - p) - \frac{p}{4q},$$

we have

$$8q = 4q^2(4 - p) - p = 16q^2 - (4q^2 + 1)p$$

so

$$p = \frac{8q(2q - 1)}{4q^2 + 1} \qquad \text{and} \qquad r = 4 - p = \frac{4(2q + 1)}{4q^2 + 1}.$$

Since the quadratic $x^2 - px + q$ has two real roots a and b, its discriminant is non-negative.

By labourious computations, we find that

$$p^2 - 4q = \frac{64q^2(2q - 1)^2}{(4q^2 + 1)^2} - 4q = \frac{4q}{(4q^2 + 1)^2} D$$

where

$$\begin{aligned} D &= 16q(4q^2 - 4q + 1) - (16q^4 + 8q^2 + 1) \\ &= -16q^4 + 64q^3 - 72q^2 + 16q - 1 \\ &= -16(q^4 - 4q^3 + \tfrac{9}{2}q^2 - q + \tfrac{1}{16}) \\ &= -16(q^2 - 2q + \tfrac{1}{4})^2 \\ &= -16((q - 1 + \tfrac{\sqrt{3}}{2})(q - 1 - \tfrac{\sqrt{3}}{2}))^2. \end{aligned}$$

Since $p^2 - 4q^2 \geq 0$ and $q \neq 0$, it follows that

$$\text{either} \quad q < 0 \quad \text{or} \quad q = 1 \pm \frac{\sqrt{3}}{2} \tag{1}$$

Similarly, since the quadratic $x^2 - rx + s$ has two real roots c and d, its discriminant is non-negative.

By labourious computations, we find that

$$
\begin{aligned}
r^2 - 4s^2 &= \frac{16(2q+1)^2}{(4q^2+1)^2} + \frac{1}{q} \\
&= \frac{16q(2q+1)^2 + (4q^2+1)^2}{q(4q^2+1)^2} \\
&= \frac{16}{q} \cdot \frac{q^4 + 4q^3 + \frac{9}{2}q^2 + q + \frac{1}{16}}{(4q^2+1)^2} \\
&= \frac{16}{q} \left(\frac{q^2 + 2q + \frac{1}{4}}{4q^2+1} \right)^2 \\
&= \frac{16}{q} \left(\frac{(q+1+\frac{\sqrt{3}}{2})(q+1-\frac{\sqrt{3}}{2})}{4q^2+1} \right)^2.
\end{aligned}
$$

Since $r^2 - 4s^2 \geq 0$ it follows that

$$
\text{either} \quad q > 0 \quad \text{or} \quad q = -1 \pm \frac{\sqrt{3}}{2}. \tag{2}
$$

From (1) and (2) it is readily seen that there are four possible values of q given by $q = \pm 1 \pm \frac{\sqrt{3}}{2}$.

For the first case $q = 1 + \frac{\sqrt{3}}{2}$, we have

$$
\begin{aligned}
p &= \frac{8q(2q-1)}{4q^2+1} = 1 + \sqrt{3}, \\
r &= 4 - p = 3 - \sqrt{3}, \\
s &= \frac{-1}{4q} = \frac{-1}{4+2\sqrt{3}} = -1 + \frac{\sqrt{3}}{2}.
\end{aligned}
$$

Solving the equations

$$
a + b = p, \ ab = q, \ c + d = r, \ cd = s,
$$

we find by tedious calculations that

$$
(a, b, c, d) = \left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{5-3\sqrt{3}}{2} \right). \tag{3}
$$

Similarly, the second case $q = 1 - \frac{\sqrt{3}}{2}$ yields

$$
p = 1 - \sqrt{3}, \ r = 3 + \sqrt{3}, \ s = 1 - \frac{\sqrt{3}}{2},
$$

which would lead to the solution

$$
(a, b, c, d) = \left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{5+3\sqrt{3}}{2} \right). \tag{4}
$$

The third case $q = -1 + \frac{\sqrt{3}}{2}$ is symmetric to the first case with p and q interchanged with r and s, respectively, leading to a permutation of the solution in (3). Finally, the fourth case $q = -1 - \frac{\sqrt{3}}{3}$ would eventually lead to a permutation of the solution in (4).

In conclusion, all the solutions are given by the two ordered quadruples together with all of their permutations.

4318. *Proposed by Thanos Kalogerakis.*

Given a pair of intersecting circles (just their circumferences, not their centres), let AB be the common diameter with one end on each circle and neither end inside either circle. Show how to construct the midpoint of AB using only a straightedge and prove that your construction is correct.
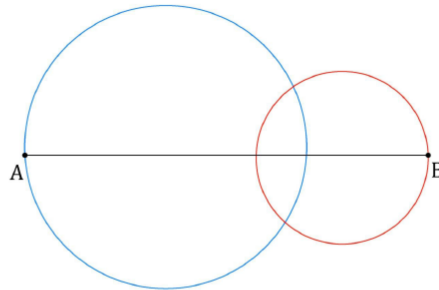
We received four submissions, but one was incomplete. We present the solution sent in by the Missouri State University Problem-Solving Group, modified by the editor.

Denote the circle containing A by α, the circle containing B by β, one of the points where α and β intersect by X, and the point other than B where AB meets β by B'. Our preliminary step is to construct the center of α. We note that once we have the center of α, by the Poncelet-Steiner Theorem we can construct any point constructible with compass and straightedge using straightedge alone.
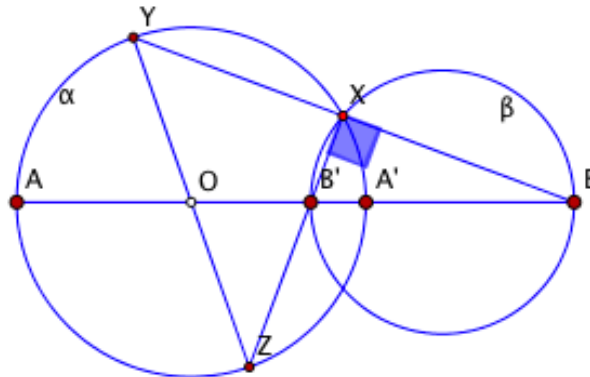
Since $\angle B'XB$ is inscribed in a semi-circle, XB' is perpendicular to XB. If XB is not tangent to α, it meets α in another point Y. Let Z denote the second point where XB' meets α. Since $\angle YXZ$ is a right angle, YZ is a diameter of α and the intersection of YZ and AB gives the center of α, which we denote by O. If XB is tangent to α, then XB' contains a diameter of α so that $B' = O$.

We now construct the midpoint M of AB given that O is the midpoint of the diameter AA'. Choose P to be a point between A and Y on the segment AY, and define

$$Q = PA' \cap YO, \quad R = QA \cap YA', \quad \text{and} \quad S = YO \cap PR,$$
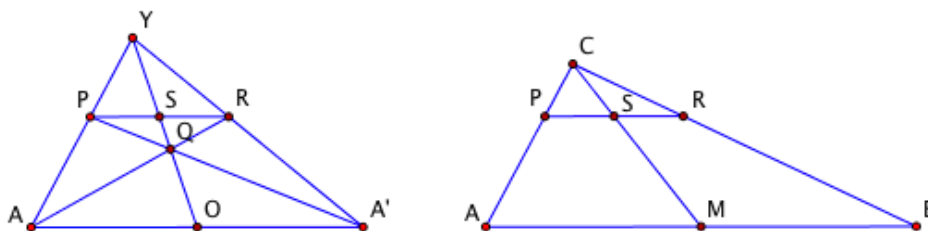
as in the figure on the left. It is a known property of the quadrangle $AA'RP$ that PR is parallel to AA' and, therefore, that S is the midpoint of PR. [For a proof featuring harmonic conjugates see the *Crux* article *"Problem Solver's Toolkit No. 5: Harmonic Sets Part 2"*, 2013: 174-177, especially Ex. 3, page 176.]

Finally, define $C = BR \cap AP$ and $M = CS \cap AB$, as in the figure on the right. Because S is the midpoint of the segment PR, M must be the midpoint of the parallel segment AB, as required.

Comment. The problem as stated dealt with intersecting circles. We can still construct the center of α by straightedge when the given circles α and β are tangent (so that X, A', and B' coincide in a single point). Choose any point P different from A and X on α such that PB intersects β in a second point Q, and α in a second point S. Denote the second point where XQ meets α by R. Now QR is perpendicular to PB since $\angle XQB$ is inscribed in a semicircle. Similarly, QR is perpendicular to AR since $\angle ARX$ is inscribed in a semicircle. Therefore PB is parallel to AR. The convex quadrilateral with vertices A, P, S, R is an isosceles trapezoid. Let T be the intersection of PR and AS and let U be the intersection of AP and RS. Then TU contains a diameter of α, so the intersection of TU and AB gives the center of α.

Editor's comments. The featured solution requires drawing 10 lines, but the incomplete submission found M drawing only 5 lines. Unfortunately that solution came as a diagram with no accompanying justification. Perhaps the author just wanted to challenge his fellow readers, so we pass along that challenge as one of this month's problems, namely number 4414.

4319. *Proposed by Marius Drăgan.*

Let $x_1, x_2, \ldots, x_n \in (0, +\infty)$, $n \geq 2$, $\alpha \geq \frac{3}{2}$ such that $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha = n$. Prove the following inequality:

$$\prod_{i=1}^{n}(1 + x_i + x_i^{\alpha+1}) \leq 3^n.$$

The original proposal contained a typo corrected here.

We received 1 correct solution, by the proposer, and we present it here.

Setting $a = \dfrac{1}{\alpha} \in \left[0, \dfrac{2}{3}\right]$, we consider the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \ln(1 + x^a + x^{a+1}).$$

We have

$$f'(x) = \frac{ax^{a-1} + (a+1)x^a}{1 + x^a + x^{a+1}},$$

$$f''(x) = \frac{-(a+1)x^{2a} - 2ax^{2a-1} - ax^{2a-2} + (a^2+a)x^{a-1} + (a^2-a)x^{a+2}}{(1 + x^a + x^{a+1})^2}.$$

We show that $f''(x) < 0$, for all $x \in (0, \infty)$. It will be sufficient to prove that for all $x \in (0, \infty)$,

$$-(a+1)x^{a+2} - 2ax^{a+1} - ax^a + (a^2+a)x + a^2 - a \leq 0$$

or

$$(a^2 + a)x < a - a^2 + ax^a + 2ax^{a+1} + (a+1)x^{a+2}.$$

For this it will be enough to prove that

$$(a^2 + a)x < a - a^2 + 2ax^{a+1}$$

or

$$x < \frac{1-a}{a+1} + \frac{2}{a+1}x^{a+1}.$$

From the power mean inequality, we obtain

$$\frac{a}{1+a} \cdot \frac{1-a}{a} + \frac{1}{1+a} \cdot 2x^{a+1} \geq \left(\frac{1-a}{a}\right)^{\frac{a}{a+1}} (2x^{a+1})^{\frac{1}{1+a}} = \left(\frac{1-a}{a}\right)^{\frac{a}{a+1}} 2^{\frac{1}{1+a}}x.$$

It will thus be sufficient to prove that $\left(\dfrac{a}{1-a}\right)^a \leq 2$. But since $a \in \left[0, \dfrac{2}{3}\right]$, we have $1 \leq \dfrac{2}{1-a} \leq 2$ or $\left(\dfrac{a}{1-a}\right)^a \leq 2^a$. But since $2^a < 2$, the claimed inequality holds. Thus f is a concave function.

From Jensen's inequality we have

$$\sum_{i=1}^{n} f(x_i^{\alpha}) \leq n f \left(\frac{\sum_{i=1}^{n} x_i^{\alpha}}{n} \right)$$

or $\sum_{i=1}^{n} f(x_i^{\alpha}) \leq n f(1)$ or $\ln \prod_{i=1}^{n} (1 + x_i + x_i^{\alpha+1}) \leq n \ln 3.$

4320. *Proposed by Abhay Chandra.*

For positive real numbers a, b, c, d, prove that

$$(a + b)(a + c)(a + d)(b + c)(b + d)(c + d) \geq 16 (a + b + c + d) \sqrt[4]{a^5 b^5 c^5 d^5}.$$

We received 8 submissions, all correct. We present the proof by Šefket Arslanagić.

We first establish the following lemma.

Lemma   If a, b, c, d are positive reals, then

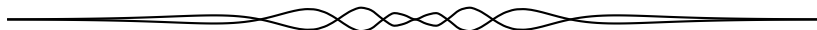$$(a + b)(b + c)(c + d)(d + a) \geq (a + b + c + d)(abc + bcd + cda + dab).$$

Proof   We have $(a + b)(b + c)(c + d)(d + a)$

$$= (ac + bd + ad + bc)(ac + bd + ab + cd)$$

$$= (ac + bd)^2 + \sum_{cyc} a^2 (bc + cd + db)$$

$$\geq 4abcd + \sum_{cyc} a^2 (bc + cd + db)$$

$$= (a + b + c + d)(abc + bcd + cda + dab), \text{ as claimed.}$$

By the lemma and the *AM-GM Inequality*, we have

$$(a + b)(a + c)(a + d)(b + c)(b + d)(c + d)$$

$$\geq (a + b + c + d)(abc + bcd + cda + dab) \cdot 2\sqrt{ac} \cdot 2\sqrt{bd}$$

$$\geq (a + b + c + d) 4 \sqrt[4]{a^3 b^3 c^3 d^3} \cdot 4 \sqrt{abcd}$$

$$= 16(a + b + c + d) \sqrt[4]{a^5 b^5 c^5 d^5}, \text{ completing the proof.}$$

Equality holds if and only if $a = b = c = d$.

SNAPSHOT

Enjoy the pictures from Grong, Trondelag in Norway with greetings from Lorian Saceanu.