

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of *Crux* as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol. 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

Mathematicorum

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 4, No. 5

May 1978

Sponsored by

Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton

A Chapter of the Ontario Association for Mathematics Education

Publié par le Collège Algonquin

(32 issues of this journal, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name EUREKA.)

CRUX MATHEMATICORUM is published monthly (except July and August). The yearly subscription rate for ten issues is \$8.00 in Canadian or U.S. dollars (\$1.50 extra for delivery by first-class mail). Back issues: \$1.00 each. Bound volumes: Vol. 1-2 (combined), \$10.00; Vol. 3, \$10.00. Cheques or money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, permission to reprint, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Nancy Makila.

*

*

*

CONTENTS

Théories Mathématiques de la Grande Pyramide	Roger Fischler	122
Harvard Was Old Hat	R. Robinson Rowe	129
Oops!		131
Supplementary List of References to the Morley Theorem	Charles W. Trigg	132
Problems - Problèmes		133
Solutions		135

For Vol. 4 (1978), the support of Algonquin College, the Samuel Beatty Fund, and Carleton University is gratefully acknowledged.

Peut-on vérifier une théorie avec des chiffres?

THÉORIES MATHÉMATIQUES DE LA GRANDE PYRAMIDE

ROGER FISCHLER

Nous avons deux buts: exposer les différentes théories qui ont été proposées pour expliquer la configuration de la Grande Pyramide de Gizeh et, en même temps, indiquer qu'il ne suffit pas de proposer une théorie et ensuite de la "vérifier" en montrant que la théorie donne des chiffres qui sont en accord avec les observations.

L'histoire de ces théories est assez embrouillée. Dans plusieurs cas l'origine en a été difficile à trouver, et dans un cas (Théorie V) on a universellement attribué la théorie à un plagiaire. Nous ne donnerons, pour chaque théorie, que le nom et la date d'origine, laissant de côté la plupart des détails bibliographiques, souvent obscurs, ainsi que l'histoire, pourtant intéressante, de chacune d'elles.

Pour de plus amples détails au sujet des pyramides, on pourra consulter [1], [7], et [10]; mais on se méfiera des livres de vulgarisation, comme [2] et [9].

La question qui nous intéresse ici est la valeur de l'angle α entre l'apothème et la base de la pyramide (Figure 1).

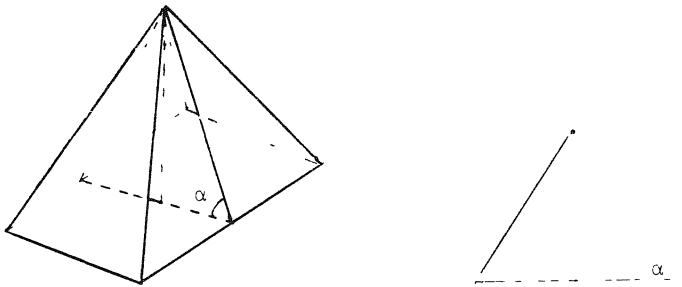


Figure 1

Commençons par la valeur observée. Petrie en 1883 a mesuré quelques morceaux du peu qui restait du revêtement de la pyramide. En suivant Borchardt (1922) nous allons adopter une des valeurs moyennes, 51.844° , comme valeur "observée".

Théorie I. "Seked".

La théorie qui est généralement acceptée dans le milieu égyptologue est celle du "seked". Cette façon de mesurer la pente d'une droite est donnée dans les problèmes

56-60 du Papyrus Rhind, qui date d'environ -1800 (la Grande Pyramide date d'environ -2800). Une analyse de ces problèmes se trouve dans Gillings [3. p. 185].

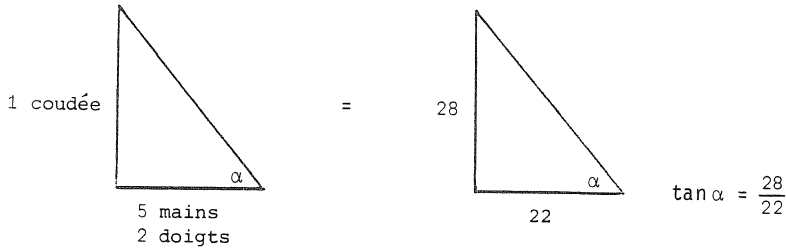


Figure 2

Le "seked" d'une pente est le nombre de coudées horizontales pour chaque coudée verticale. La coudée, qui valait à peu près 52.3 cm à l'époque, était divisée en 7 mains de 4 doigts. Petrie (1883) a trouvé qu'un "seked" de 5 mains et 2 doigts donnerait (Figure 2)

$$\tan \alpha = \frac{7 \cdot 4}{5 \cdot 4 + 2} = \frac{28}{22}.$$

Si nous remarquons que $28/22 = 4/(22/7)$, nous ne nous étonnerons pas quand le nombre π fera son apparition.

Cette hypothèse a été confirmée archéologiquement lors de la découverte par Petrie (1892) des "tracés régulateurs" au "mastaba"—tombeau en forme de pyramide tronquée—de Medun. Ce dernier a été construit juste avant la Grande Pyramide.

Théorie II. Base : hauteur = 8:5.

(a) Cette théorie remonte à Agnew (1838) qui la considère comme la vraie base théorique, et à Perring (1840) qui, apparemment, la croyait accidentelle.

Agnew la présente ainsi. On commence par le carré ABCD de côté 2 qui représente la base de la pyramide (Figure 3). Soit K le centre du cercle de rayon r qui passe

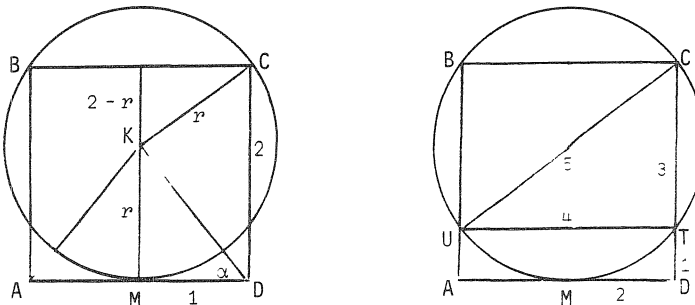


Figure 3

par B et C et qui est tangent à AD en son point de milieu M. Le théorème de Pythagore montre que $x = 5/4 = \tan \alpha$.

Mais Agnew ne s'arrête pas là. Il observe que UC est un diamètre du cercle, et que si on attribue la valeur 5 à UC, alors le cercle coupe le carré en segments de longueurs 1, 2, 3, 4. Les côtés du triangle UCT sont donc proportionnels à 3:4:5.

Or, remarque Agnew, ce dernier est "le plus beau des triangles" dont parle Plutarque dans son *Isis et Osiris* (56) (deux dieux égyptiens). Ce triangle et son rapport mythologique ont ensuite servi, parfois de façon très embrouillée, à d'autres auteurs comme justification du résultat $\tan \alpha = 5/4$.

(b) D'après cette théorie le rapport base : hauteur = 8:5 (Figure 4). Or le

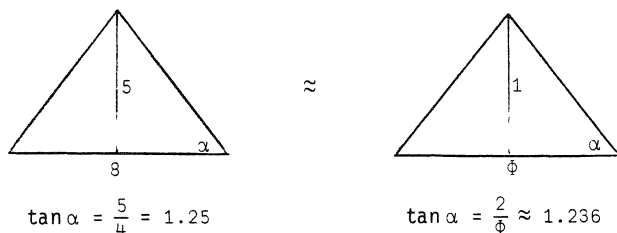


Figure 4

rapport $8/5 = 1.6$ est très proche du nombre d'or $\phi = (1 + \sqrt{5})/2 \approx 1.618$, qui va faire sa réapparition d'une façon plus spectaculaire un peu plus tard. Ce rapprochement a été remarqué pour la Grande Pyramide par Choisy (1899) mais, fait étrange, ne semble pas avoir été adopté comme théorie dans la littérature.

Théorie III. Apothème : base = 4:5.

Cette théorie (Figure 5) est due à Jomard (avant 1830), qui croyait que la présence des nombres 4 et 5 était due à leur utilisation à des fins astronomiques.

Théorie IV. Triangle de Kepler.

Cette théorie est la plus intéressante du point de vue mathématique. C'est dans une lettre datée septembre 1597, de Kepler à son professeur Maestlin, que le triangle en question semble apparaître pour la première fois ([5], XIII, p. 141; voir [5]).

Le théorème de Kepler peut être présenté ainsi. Si un segment AB (Figure 6) est coupé "en extrême et moyenne raison" (Euc. VI, 30), c'est-à-dire si

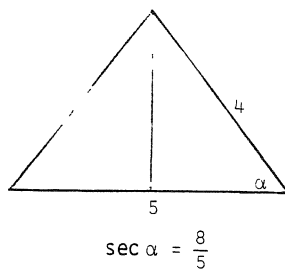


Figure 5

$$AB:AG = AG:GB = \phi,$$

alors que $CG \perp AB$ et $\angle ACB = 90^\circ$,
alors $CB = AG$. Voici une démon-
stration de ce résultat que
Kepler a, bien entendu, démontré
à la façon d'Euclide.

Soit $GB = 1$ et $AG = \phi$. Comme
 CG est moyenne géométrique entre
 AG et GB , on a $CG = \sqrt{1 \cdot \phi}$ et

$CB = \sqrt{1 + \phi}$. Or, comme ϕ est racine de l'équation $x^2 = x + 1$, on a

$$CB = \sqrt{\phi^2} = \phi = AG.$$

Il est à remarquer que les côtés du triangle de Kepler ($\triangle CGB$ dans la Figure 6)
forment une série géométrique. On démontre facilement que, à une constante multipli-
cative près, c'est le seul triangle ayant cette propriété.

Le triangle de Kepler en tant que base d'une théorie des pyramides paraît pour
la première fois chez Friedrich Röber (le Jeune) en 1855. Cependant le triangle
y est utilisé—sans connaissance de la lettre de Kepler—pour toutes les pyramides
à Gizeh sauf la Grande. Pour celle-ci, Röber a suivi Perring en adoptant la Théorie
II. Nous avons ici la toute première (mais non pas, hélas, la dernière) assertion
du genre "tel ou tel a utilisé le nombre d'or pour dessiner l'objet X".

Le triangle de Kepler reparaît, mais cette fois-ci de façon non explicite,
chez Taylor (1859). Sa théorie, que nous appellerons "théorie des surfaces", dit
que la pyramide a été construite telle qu'un carré, de côté égal à la hauteur de
la pyramide, a la même surface qu'une des faces de cette dernière. Nous avons donc
(Figure 7)

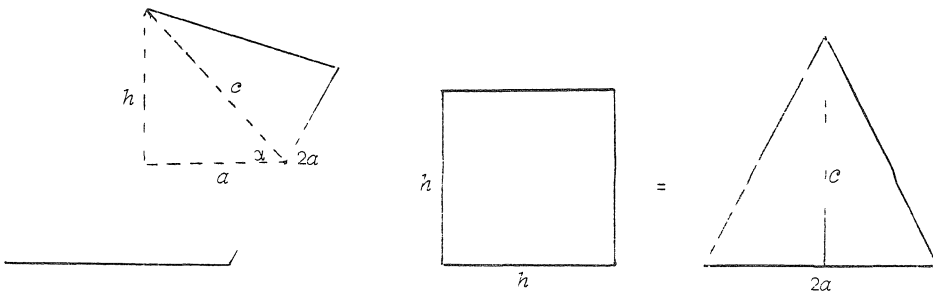
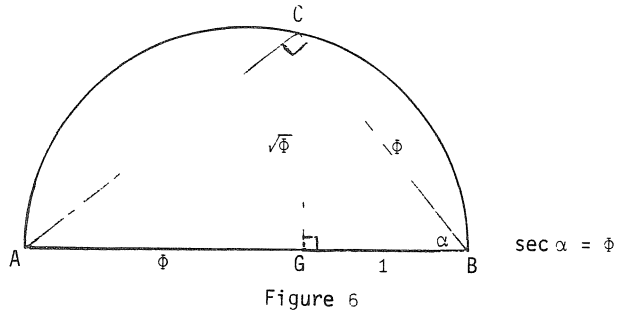


Figure 7



$$\tilde{h}^2 = \frac{1}{2}(2a\alpha) \quad \text{et aussi} \quad c^2 = \tilde{h}^2 + a^2;$$

par conséquent

$$c^2 = c\alpha + a^2 \quad \text{ou} \quad \left(\frac{c}{\alpha}\right)^2 - \frac{c}{\alpha} - 1 = 0.$$

La solution de cette équation est $c/\alpha = \sec \alpha = \phi$.

Taylor est arrivé à sa théorie de la façon suivante. Dans ses *Histoires* ([4], II, 124) Hérodote (-V^e siècle) écrit:

"Pour la construction de la pyramide même, le temps employé aurait été de vingt ans; elle a de tous les côtés un front de huit plèthres et une égale hauteur... ."

Un plèthre équivaut à 100 pieds grecs (à peu près 101 pieds anglais ou 30.8 mètres). Donc le texte d'Hérodote donne environ 246 mètres pour la hauteur et la base. Ceci correspond assez bien à la valeur arpentée (Cole 1925) de 230 mètres, mais fort peu à la hauteur de 146 mètres.

Donc, selon Taylor, Hérodote aurait ici une intention particulière. Par un tour de main il transforme la phrase d'Hérodote

(front 8 = hauteur 8 \Rightarrow une surface reliée en
quelque sorte à la hauteur
= la surface d'une face
 \Rightarrow la "théorie des surfaces")

et il manipule ensuite les chiffres d'une façon spectaculaire afin de démontrer que chaque surface égale 8 plèthres carrés.

Théorie V. Pi.

Cette théorie proclame que la pyramide a été construite pour montrer la connaissance du nombre π , de façon à ce qu'un cercle avec rayon égal à la hauteur \tilde{h} ait le même périmètre que la base de la pyramide (Figure 8). Cette hypothèse,

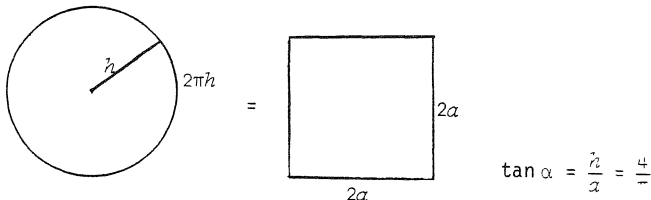
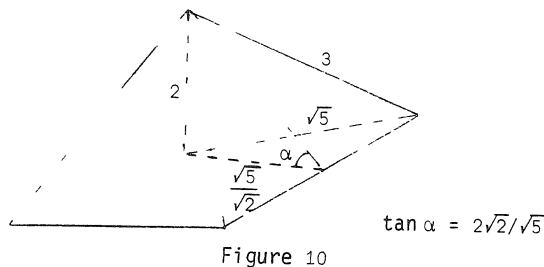
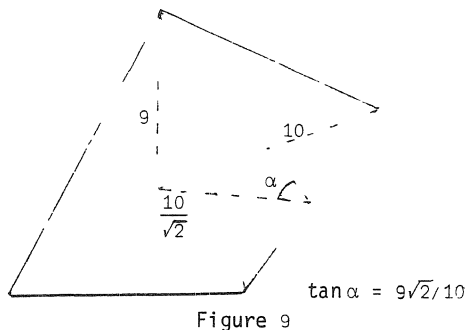


Figure 8

souvent attribuée par erreur à Taylor, fut appliquée à la Grande Pyramide par ce dernier (1859), et déjà à la troisième pyramide par Agnew (1833).

Théorie VI. Pente de l'arête = $\frac{9}{10}$ (Figure 9).

Certains auteurs ont cru que l'angle avait été mesuré sur l'arête. Selon un certain James (avant 1882), la pente était 9/10.



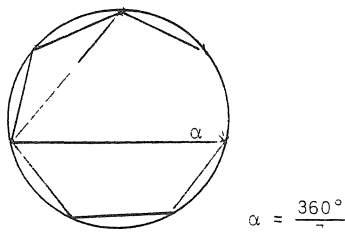
Théorie VII. 2 unités verticales pour 3 unités sur l'arête (Figure 10).

La source de cette théorie (avant 1883) reste inconnue.

Théorie VIII. Heptagone régulier.

On suppose, en suivant Texier (1934), que la Grande Pyramide peut être inscrite dans un heptagone régulier (Figure 11).

Nous voici à la fin des théories proposées par différents "chercheurs" (voir le tableau récapitulatif). Au lecteur maintenant de trouver encore d'autres théories pour la Grande Pyramide!



$$\alpha = \frac{360^\circ}{7}$$

BIBLIOGRAPHIE

1. Edwards, I., *The Pyramids of Egypt*, London, Parish, 1961.
2. Ghyka, M., *L'Esthétique des proportions dans la nature et dans les arts*, Paris, Gallimard, 1927.
3. Gillings, R., *Mathematics in the Time of the Pharaohs*, Cambridge, M.I.T. Press, 1972.
4. Hérodote, *Histoires* (trad. P. Legrand), Paris, Belles Lettres, 1972.
5. Hofmann, J., "Johannes Kepler als Mathematiker," *Praxis der Mathematik*, 13 (1971) 287-293, 318-324.
6. Kepler, J., *Gesammelte Werke* (M. Caspar et F. Hammer, rédacteurs), Munich, Beck, 1938-.

TABLEAU RÉCAPITULATIF

Nom	Détermination	α	Origine	Date
Valeur observée	51°50'40"±1'5"	51.844°	Petrie, Cole-Borchardt	1883, 1922, 1925
I "Seked"	$\tan \alpha = \frac{28}{22}$	51.843°	Papyrus Rhind, Petrie	-1800, 1883, 1892
II(a) base : hauteur = 8:5	$\tan \alpha = \frac{5}{4}$	51.340°	Agnew-Perring	1838, 1840
II(b) nombre d'or	$\tan \alpha = \frac{2}{\phi}$	51.027°	Choisy	1899
III apothème : base = 4:5	$\sec \alpha = \frac{8}{5}$	51.318°	Jomard	avant 1830
IV Triangle de Kepler	$\sec \alpha = \phi$	51.827°	Röber	1855
V Pi	$\tan \alpha = \frac{4}{\pi}$	51.854°	Agnew	1838
VI Pente 9/10 de l'arête	$\tan \alpha = \frac{9\sqrt{2}}{10}$	51.844°	James	avant 1882
VII 2 vertical pour 3 sur l'arête	$\tan \alpha = \frac{2\sqrt{2}}{\sqrt{5}}$	51.671°	Inconnue	avant 1883
VIII Heptagone	$\alpha = 360^\circ/7$	51.429°	Texier (?)	1934

$$\phi = \text{"nombre d'or"} = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

7. Lauer, J.-P., *Le problème des pyramides d'Égypte*, Paris, Payot, 1948.
8. Smeur, A., "On the Value Equivalent to π in Ancient Mathematical Texts, A New Interpretation," *Arch. Hist. Exact Sci.*, 6 (1969-1970) 249-270.
9. Tompkins, P., *Secrets of the Great Pyramid*, New York, Harper and Row, 1971.
10. Wheeler, N., "Pyramids and Their Purpose," *Antiquity*, 9 (1935) 5-21, 161-189, 292-304.

(Université Carleton et I.R.E.M. de Clermont-Ferrand)

Département de mathématiques, Université Carleton, Ottawa, Ontario K1S 5B6.

*

*

*

Reminiscences

HARVARD WAS OLD HAT

R. ROBINSON ROWE

When my uncle had given me the dress suit he had worn at Harvard (see [1978: 63]), I had been naïve enough to suppose he had worn it regularly to classes, so I took it along with me in September 1913—but it wouldn't be used until the Junior Prom. Harvard was "Old Hat." Headgear was a slouchy fedora or battered derby—or nothing at all. A three-piece suit, shirt and collar were standard and always clean, but indifferently pressed. Shoes, when and if polished, were dull, not shiny. The keynote was casualness; the media called it "Harvard indifference." But the real essence was "equality;" being cosmopolitan, rich and poor could look alike.

At the same time, my sister Ina had entered Lasell Seminary in nearby Auburndale. It was a "finishing school"—less academic than cultural—music, arts and the social graces. When she first invited me out to dinner, I was self-conscious as the only male in the large dining room. At each table for 10 or 12 students, a teacher sat at the head, enforcing strict rules of etiquette by noting demerits for any miscue. Sis had primed me on code words the girls used to warn others. A sotto voce to your neighbor using the word "gang" meant she should pass the word along for someone to "haul in the gang-plank," that is, someone's knife had its blade on the plate and handle on the cloth—a very plebeian habit. The word "balance" was to prompt someone on how to eat salad: instead of cutting the lettuce, one leaf at a time was to be lifted on the fork and nibbled, rabbit-wise. There was more, making me so nervous that I never used my knife nor ate the lettuce in my salad!

When Sis visited me at Harvard, she pointedly asked me to show her the glass

flowers in the Agassiz Museum—which I did. Coyly, she explained: "When I want to get out of the 'asylum' I'll ask permission to visit you; if they ask afterwards what we did, I can tell them about the wonderful glass flowers." So she had me as a dinner guest to prove she had a brother, but it made me feel very important—and the next time I used my knife and ate my lettuce.

Periodically there was a beer-cider-program soirée at the Harvard Union. A frequent visitor, often unannounced, was ex-President Teddy Roosevelt. He was one of the Overseers of the University and his two younger sons were undergraduates. Several times I joined the long line for the thrill of his gripping handshake, broad toothy smile, and his invariable "Dee-LIGHT-ed," spoken so very personably. Rough Rider, Bull Moose, and a really great American!

Although I would finish with the Class of 1916, I had started with that of 1917, and Archie was one of that class. Seats were assigned alphabetically, so we were often near—and next to each other in Astronomy. Unlike his father, he was quiet and reserved, but laconically polite and cordial. So it was surprising to the class, at the final lecture devoted to questions and answers preliminary to the final examination, when near the end Archie finally spoke: "Dr. Willson, I think now I understand how distances to the stars are measured and orbits of the planets computed, but you haven't told us how, by looking through a telescope, you read the names of Altair, Capella, Vega, and the other heavenly bodies." There was a stunned silence. Then Dr. Willson proved equal to the occasion. "An excellent question, Mr. Roosevelt, but it will not be on the examination. If there are no more questions, the class is dismissed."

On the other hand, Quentin, the youngest son, was as ebullient and extrovert as his father. He was in the Class of 1918, and I knew him only in seminars in Military Tactics for the Harvard Volunteer Advance Battalion early in 1916. Among other things we had to learn verbatim the "position of a soldier," which defines position for head, eyes, chin, shoulders, arms, and so on down, ending with "and weight of the body resting equally on the balls and soles of the feet." When Quentin was asked to recite it, he did so perfectly, enunciating it with a pause between phrases, ending with "and weight of the body.....resting equally on the balls.....and soles of the feet."

So far I have not mentioned the mathematics I had come to Harvard for. The undergraduate curriculum was a classical sequence familiar to most readers, beginning with analytic geometry and winding up with calculus of variations. Having time for one more course in my last semester, I opted for and was lucky enough to be one of

two seniors admitted to a graduate course, Mathematics 24A entitled "Introduction to Modern Algebra and Geometry."

That title was both vague and comprehensive. No textbook was assigned, so the scope was disclosed one item at a time as the teacher¹ lectured at the blackboard and we students kept copious notes to review before quizzes. Determinants, transformations, number theory, and invariants were followed by trilinear coordinates; and now the presentation at the board was so rapid that we could barely copy, without comprehension, before the board was erased for a new start. Our homework was done in small groups, where we compared notes to fill gaps and then tried to reconcile our interpretations.

When I asked the teacher for an outside-reading reference, he said that there wasn't any: the subject was too new. I mentioned this casually in my next letter to my mother²; and by return mail came *Conic Sections* by Charles Smith, Macmillan and Co., London (1889 reprint). In addition to her name on the first page was her note, "Excused from final by Prof Bohannon, EMR." Its 40-page Chapter XIII was entitled "Trilinear Co-ordinates." Delighted, and with tongue in cheek, I showed it to the teacher who, somewhat nettled, said only: "That's been out of print for years!" I still treasure this fine book.

However, it was not out of print³, though I am sure this very fine teacher thought it was. But there may be another explanation. In all my other courses, the assigned texts were by Harvard authors—Osgood, Byerly, Bocher, Peirce (pronounced PURSE, by the way), but none had yet written a text on trilinear coordinates.

2701 Third Avenue, Sacramento, California 95818.

*

*

*

Oops!

The following classified advertisement appeared in *Saturday Review*, April 1, 1978, p. 51:

Due to computer error, we have 100,000 party balloons on hand with openings at both ends. Best offer. SR Box K.J.S.

¹Julian Lowell Coolidge, Harvard Professor of Mathematics; President, Mathematical Association of America; and the author of several well-known books.

²Eckka Mazola Robinson Rowe, The Ohio State University, BS 1892.

³Fred Maskell tells me he has a copy of the 1927 reprint which bears on the Copyright page:

First Edition printed 1882. Second Edition printed 1883. Reprinted with slight corrections 1384, with slight corrections 1885, 1887, 1888, 1889, January and November 1890, 1892, 1893, 1894, 1896, 1898, 1899, 1901, 1902, 1904, 1905, 1906, 1908, 1909. Revised and enlarged 1910. Reprinted 1912, 1914, 1916, 1918, 1919, 1920, 1921, 1924, 1927.

SUPPLEMENTARY LIST OF REFERENCES
TO THE MORLEY THEOREM¹

CHARLES W. TRIGG

149. Leon Bankoff, The beauty and truth of the Morley Theorem, *Crux Math.*, 3 (December 1977) 294-296.
150. E.J. Ebdon, Problem proposal 1655, *Mathesis*, 28 (1908) 32.
151. ———, Question 16381, *Mathematical Questions from the Educational Times*, New Series, 15 (1909) 22, 46, 110.
152. E. Ehrhart, Sur les trisectrices des angles d'un triangle, *Mathesis*, LXII (1953) 154-155.
153. D.J. Kleven, Morley's Theorem and a converse, *Amer. Math. Monthly*, 85 (February 1978) 100-105.
154. Judah Milgram and Anders Bager, Solution to Problem E 1030, *Amer. Math. Monthly*, 82 (December 1975) 1010-1011. (This is the same as [98], where it is credited to Salkind, who proposed but did not solve the problem.)
155. Frank Morley, On the intersections of the trisectors of the angles of a triangle, *Crux Math.*, 3 (December 1977) 273-275. (A reprint of [81].)
156. C.O. Oakley, A list of references to the Morley Theorem, *Crux Math.*, 3 (December 1977) 282-289.
157. Dan Pedoe, Notes on Morley's proof of his theorem on angle bisectors, *Crux Math.*, 3 (December 1977) 276-279.
158. A. Robson, Proof of Morley's Theorem, *Crux Math.*, 3 (December 1977) 280-281. (A reprint of [96].)
159. John Satterly, Equilateral and other triangles associated with triangles, *Sch. Sci. and Math.*, 56 (January 1956) 11-20.
160. Dan Sokolowsky, An elementary geometric proof of the Morley Theorem, *Crux Math.*, 3 (December 1977) 291-294.
161. C.E. Springer and Roscoe Woods, Area of the Morley triangle of a triangle (Solution of Problem 4477), *Amer. Math. Monthly*, 60 (May 1953) 337-338.
162. Charles W. Trigg, Supplementary list of references to the Morley Theorem, *Crux Math.*, 3 (December 1977) 289-290.
- 2404 Loring St., San Diego, California 92109.

¹This list extends the one ending in [1977: 290]. Note that *Crux Mathematicorum* was still called EUREKA at the time references [149], [155]-[158], [160], and [162] were published.

PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

341. *Proposé par Herman Nyon, Paramaribo, Surinam.*

L'addition décimale suivante est doublement vraie, et SEPT est divisible par 7. Reconstituer ses chiffres.

TROIS
TROIS
SEPT
SEPT
VINGT

342* *Proposed by James Gary Propp, Great Neck, N.Y.*

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets S_1, S_2, \dots, S_n as follows: for each positive integer m , we have $m \in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m \in S_n$ for all sufficiently large m . (If $n = 2$, this is essentially equivalent to Problem 226 [1977: 205].)

343* *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

It was proved in Problem 166 [1976: 231] that the greatest integer function satisfies the functional equation

$$f(nx) = \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

for all real x and positive integers n . Are there other functions which satisfy this equation? Find as many as possible.

344. *Proposed by Viktors Linis, University of Ottawa.*

Given is a set S of n positive real numbers. With each nonempty subset

P of S , we associate the number

$$\sigma(P) = \text{sum of all its elements.}$$

Show that the set $\{\sigma(P) \mid P \subseteq S\}$ can be partitioned into n subsets such that in each subset the ratio of the largest element to the smallest is at most 2.

345, *Proposed by Charles W. Trigg, San Diego, California.*

It has been shown [*Pi Mu Epsilon Journal*, 5 (Spring 1971) 209] that when the nine nonzero digits are distributed in a square array so that no column, row, or unbroken diagonal has its digits in order of magnitude, the central digit must always be odd.

(a) Can such a distribution be made for every odd central digit?

(b) Do any such distributions exist in which odd and even digits alternate around the perimeter of the array?

346, *Proposed by Leroy F. Meyers, The Ohio State University.*

It has been conjectured by Erdős that every rational number of the form $4/n$, where n is an integer greater than 1, can be expressed as the sum of three or fewer unit fractions (reciprocals of positive integers, also called Egyptian fractions), not necessarily distinct. As a partial verification of the conjecture, show that at least 23/24 of such numbers have the required expansions.

347, *Proposed by M.S. Klamkin, University of Alberta.*

Determine the maximum value of

$$\sqrt[3]{4 - 3x + \sqrt{16 - 24x + ax^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + ax^2 - x^3}}$$

in the interval $-1 \leq x \leq 1$.

348, *Proposed by Gilbert W. Kessler, Canarsie High School, Brooklyn, N.Y.*

I launched a missile, airward bound;

Velocity—the speed of sound;

Its angle—30. Can you tell

How far from here that missile fell?

349, *Proposed by R. Robinson Rowe, Sacramento, California.*

Solve in positive integers a and b the continued fraction equation

$$2 \left\{ \frac{1}{a+} \frac{1}{x+} \frac{1}{a+} \dots \right\} - \left\{ \frac{1}{b+} \frac{1}{b+} \frac{1}{b+} \dots \right\} = 1.$$

350. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

What regular n -gons can be constructed by paper folding? (Scissors and paste are assumed to be available if needed.)

*

*

*

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

282. [1977: 250; 1978: 114] Late solution: CHARLES W. TRIGG, San Diego, California.

287. [1977: 251] *Proposed by M.S. Klamkin, University of Alberta.*

Determine a real value of x satisfying

$$\sqrt{2ab + 2ax + 2bx - a^2 - b^2 - x^2} = \sqrt{ax - a^2} + \sqrt{bx - b^2}$$

if $x > a, b > 0$.

Composite of the solutions received from Gali Salvatore, Ottawa, Ontario; and the proposer.

The restriction $0 < a, b < x$ is equivalent to

$$0 < \sqrt{a}, \sqrt{b} < \sqrt{x}. \quad (1)$$

The radicand on the left of the given equation can be factored by inspection, giving

$$\sqrt{(\sqrt{a} + \sqrt{b} + \sqrt{x})(\sqrt{b} + \sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b} - \sqrt{x})} = \sqrt{ax - a^2} + \sqrt{bx - b^2}. \quad (2)$$

Now (1) ensures that the right side of (2) and the first three factors of the radicand on the left are all positive; hence for a solution to exist the fourth factor must also be positive, and we require

$$\sqrt{x} < \sqrt{a} + \sqrt{b}. \quad (3)$$

Thus with (1) and (3) the given equation is equivalent to the one obtained by squaring both sides. This procedure gives, after simplification,

$$x(a+b-x) = 2\sqrt{ab} \{ \sqrt{(x-a)(x-b)} - \sqrt{ab} \} = \frac{2x\sqrt{ab}(x-a-b)}{\sqrt{(x-a)(x-b)} + \sqrt{ab}}. \quad (4)$$

One solution of (4) is clearly $x = a+b$; and it is the only one since $x \neq a+b$ makes one side of (4) positive and the other side negative.

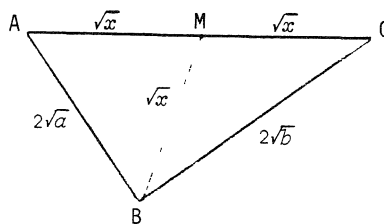
Since $x = a+b$ also satisfies both (1) and (3), it is the unique solution to the given equation.

Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin (two solutions); CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, City University of N.Y.; SAMUEL L. GREITZER, Rutgers University; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; F.G.B. MASKELL, Collège Algonquin, Ottawa; DAN PEDOE, University of Minnesota, Minneapolis; SIDNEY PENNER, Bronx Community College, N.Y.; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; KENNETH M. WILKE, Washburn University, Topeka, Kansas (two solutions); KENNETH S. WILLIAMS, Carleton University, Ottawa; and JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico. One incorrect solution was received.

Editor's comment.

Several solvers obtained the correct answer by squaring away to their heart's content, seemingly unaware that they were leaving themselves wide open to insidious attacks by extraneous roots. And of those who conscientiously tried to establish the existence and uniqueness of the solution, not all were successful. A couple of solvers apparently found the problem so obvious that they submitted *only* an answer: they are just about ready to graduate and go on to a more advanced journal.

The restrictions (1) and (3) ensure that \sqrt{a} , \sqrt{b} , \sqrt{x} (or any constant multiple thereof) are the lengths of the sides of a triangle. In fact, the proposer pointed out that the given equation can be interpreted geometrically by the following area relationship (see figure):



$$|ABC| = |ABM| + |MBC|.$$

This is most easily verified from (2) by Heron's formula.

Thus the given equation and its solution could be used to give us another proof, if one were needed, of the following theorem:

If the midpoint of the longest side of a triangle is equidistant from the three vertices, then the triangle is right-angled.

Conversely, this theorem itself can be used to provide an unexpected solution to the given equation.

*

*

*

288. [1977: 251] Proposed by W.J. Blundon, Memorial University of Newfoundland.

Show how to construct (with compass and straightedge) a triangle given the circumcenter, the incenter and one vertex.

Solution by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.

Let O, I, A be the three distinct given points which are destined to serve as the circumcenter, incenter, and one vertex of the required triangle (if it exists).

Construction.

Draw circle $(O)OA$ (circle with center O and radius OA). We assume that I is an interior point of this circle (see Figure 1). (Otherwise there is no solution and we are done; for if a solution triangle exists, then I must be one of its interior points, and hence an interior point of its circumcircle $(O)OA$.) Draw Line AI to meet the circle again in T . Draw the circle $(T)TI$ to meet circle $(O)OA$ in B and C . Finally, draw BC, CA, AB . Then ABC is the required triangle.

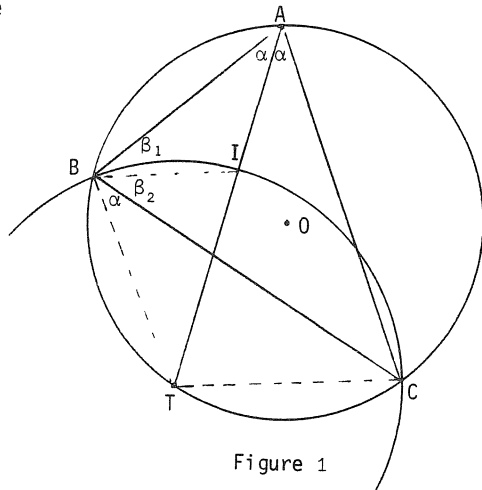


Figure 1

Proof. (a) We prove that the triangle ABC we have constructed is a solution to our problem by showing that I is its incenter.

Join TB, TC, BI . Since $TB = TC$, AI bisects $\angle A$, and the angles marked α in Figure 1 are all equal. Let the angles at B be denoted by β_1, β_2 as shown in Figure 1; then

$$\alpha + \beta_1 = \angle TIB = \angle TBI = \alpha + \beta_2,$$

so $\beta_1 = \beta_2$, BI bisects $\angle B$, and I is the incenter of triangle ABC .

(b) We prove that our solution is the only one by assuming ABC to be a solution triangle and showing that B and C are the intersections of circles $(O)OA$ and $(T)TI$, that is, that ABC must be the triangle we have in fact constructed. For this, it suffices to show that $TB = TI = TC$.

We have immediately $TB = TC$ since T lies on the bisector of $\angle A$. Furthermore, we have by assumption $\beta_1 = \beta_2$ so

$$\angle TIB = \alpha + \beta_1 = \alpha + \beta_2 = \angle TBI;$$

hence $TB = TI$ and the proof is complete.

Also solved by LEON BANKOFF, Los Angeles, California; CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.;

SAMUEL L. GREITZER, Rutgers University; M.S. KLAMKIN, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; R. ROBINSON ROWE, Sacramento, California; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

Editor's comment.

The other solutions submitted were about evenly divided between two types of constructions: that given in our featured solution and one based on a formula of Euler. But the *proofs* of several were incomplete (in some cases nonexistent).

I describe briefly the second type of construction, based on Euler's formula

$$OI^2 = R^2 - 2Rr,$$

from which r can be constructed given OI and $R = OA$. Draw the circumcircle, construct r , and then draw the incircle (Figure 2). Now from A draw tangents to the incircle to meet the circumcircle in B and C . Finally, join BC , and ABC is the required triangle.

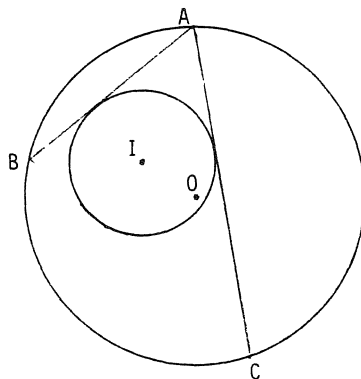


Figure 2

Although its proof is more complicated, the construction based on Euler's formula seems conceptually simpler than the one in our featured solution. So how should one weigh the worth of one against that of the other?

Constructions form a part of practical geometry, and the worth of a construction depends, not upon the brevity and elegance of its proof, but upon the accuracy with which the desired result can in practice be achieved. This accuracy, in turn, depends upon the number and type of Euclidean operations performed (setting a compass, striking an arc, drawing a line). In general, the fewer the operations, the greater the accuracy.

The French, to whom we owe so many of the good things in life, have found a way of assessing the worth of a construction. The procedure is thus described in Eves [1]:

The problem of finding the "best" Euclidean solution to a required construction has also been considered, and a science of *geometrography* was developed in 1907 by Émile Lemoine for quantitatively comparing one construction with another. To this end, Lemoine considered the following five operations:

- S_1 : to make the straightedge pass through one given point,
- S_2 : to rule a straight line,
- C_1 : to make one compass leg coincide with a given point,

C_2 : to make one compass leg coincide with an arbitrary point of a given locus,

C_3 : to describe a circle.

If the above operations are performed m_1, m_2, n_1, n_2, n_3 times in a construction, then

$$m_1S_1 + m_2S_2 + n_1C_1 + n_2C_2 + n_3C_3$$

is regarded as the *symbol* of the construction. The total number of operations, $m_1 + m_2 + n_1 + n_2 + n_3$, is called the *simplicity* of the construction, and the total number of coincidences, $m_1 + n_1 + n_2$, is called the *exactitude* of the construction. The total number of loci drawn is $m_2 + n_3$, the difference between the simplicity and the exactitude of the construction.

As simple examples, the symbol for drawing the straight line through A and B is $2S_1 + S_2$, and that for drawing the circle with center C and radius AB is $3C_1 + C_3$.

I have calculated that the symbol of our featured construction is

$$8S_1 + 4S_2 + 4C_1 + 0C_2 + 2C_3,$$

so its simplicity is 18 and its exactitude 12. But the most economical symbol I've been able to calculate for the construction based on Euler's formula is

$$18S_1 + 9S_2 + 22C_1 + 1C_2 + 11C_3$$

so its simplicity is 61 and its exactitude 41.

So judge for yourself. In the second construction, it is easy to say "construct r from Euler's formula;" but actually doing so costs a lot of Brownie points.

REFERENCE

1. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972, pp. 181-182.

*

*

*

289. [1977: 251] *Proposed by L.F. Meyers, The Ohio State University.*

Derive the laws of reflection and refraction from the principle of least time without use of calculus or its equivalent. Specifically, let L be a straight line, and let A and B be points not on L . Let the speed of light on the side of L on which A lies be c_1 , and let the speed of light on the other side of L be c_2 . Characterize the points C on L for which the time taken for the route ACB is smallest, if

- (a) A and B are on the same side of L (reflection);
- (b) A and B are on opposite sides of L (refraction).

Solution and comment by the proposer.

- (a) Let B' be the point symmetric to B with respect to line L (see Figure 1).

The path ACB takes minimum time just when it is shortest, since the speed of light is the same on the entire path. But, for any position of C on L,

$$AC + CB = AC + CB',$$

and the length of the path ACB' is smallest only if ACB' is a straight segment, that is, only if the angle of incidence ACD is equal to angle B'CE and hence to the angle of reflection BCD, where DE is the line through C perpendicular to L.

It is easy to show, conversely, that if we assume the law of reflection (angle of incidence i = angle of reflection r), then ACB' is a straight segment, so the length (and the time) of path ACB is minimal.

The law of reflection was enunciated by Heron of Alexandria in his *Catoptrica* (ca. 60 A.D.; see, for example, [3] or [7]).

(b) The Editor was right when he wrote [1976: 96] that "everything has appeared before in the *Monthly*!" Shortly after this proposal appeared (starred), I received a copy of [1]; and while browsing through it I came across *three* noncalculus derivations of Snell's law of refraction, in two papers [4, 8]. One of the derivations in [4] is essentially algebraic, using the Cauchy inequality; the other is due to Huygens [5, 6]. The derivation in [8], by a well-known contributor to *Cruce Mathematicorum*, uses an inequality which is an extension of Ptolemy's Theorem.

Solutions and/or comments were also received from M.S. KLAMKIN, University of Alberta; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; HERMAN NYON, Paramaribo, Surinam; and DAN PEDOE, University of Minnesota.

Editor's comment.

A noncalculus (indeed, almost a *precalculus*) proof of Snell's law of refraction was given by Huygens around 1676 (see [5] and [6]). In 1961 Beckenbach and Bellman [2] implied that a noncalculus proof would be hard to find. A response from the mathematical community was not long in coming: in 1964 Golomb [4] and Pedoe [8] published their noncalculus proofs simultaneously. No response from Huygens has been recorded so far. All three of the known noncalculus proofs are easily accessible from [1], [4], and [8], and Pedoe's proof can also be found in [9]; so interested readers can look them up and select their own favourite.

The proposer's proof of part (a) and the references for part (b) assume that \angle

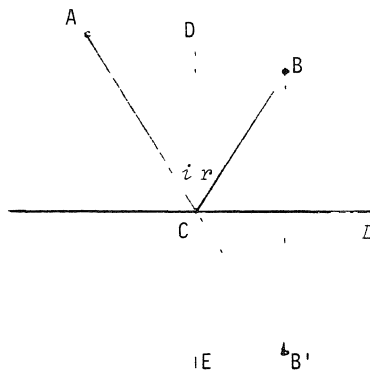


Figure 1

is a straight line (the mirror is flat), as indeed the proposal specified. Furthermore, the proof of part (a) shows that the law of reflection is equivalent to Heron's postulate of *minimal path* and to Fermat's principle of *least time* (1662). But, as Klamkin pointed out, this need not hold if the mirror is not flat. In

Figure 2, for example, L is a smooth curve, the law of reflection still holds ($i = r$, $i' = r'$), and the path ACB is maximal while the path $A'C'B'$ is minimal. All that can be said is that for smooth curves the law of reflection holds if and only if the path-length function has a stationary value. But there is not much hope that anyone can give a non-calculus proof of *that*! An elementary discussion of these matters can be found in [7].

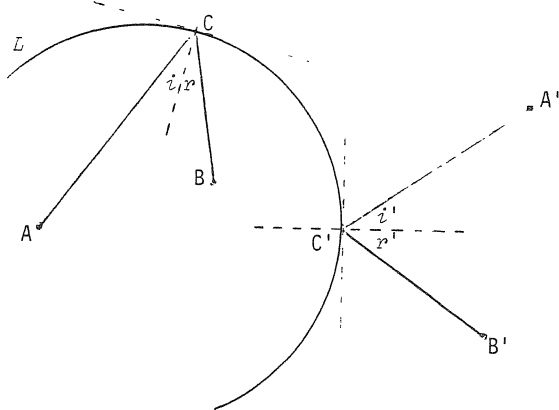


Figure 2

REFERENCES

1. Tom M. Apostol, Gulbank D. Chakerian, Geraldine C. Darden, and John D. Neff, *Selected papers on precalculus*, Volume 1 of The Raymond W. Brink Selected Mathematical Papers. Mathematical Association of America, 1977.
2. Edwin Beckenbach and Richard Bellman, *An Introduction to Inequalities*, Random House, The L.W. Singer Company, 1961, pp. 83-86.
3. R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, New York, 1943, p. 331.
4. Michael Golomb, Elementary proofs for the equivalence of Fermat's Principle and Snell's Law, *American Mathematical Monthly*, 71 (1964) 541-543. Reprinted in [1], pp. 204-206.
5. Christiaan Huygens, *TRAITE DE LA LUMIERE*, Où sont expliquées Les causes de ce qui luy arrive Dans la REFLEXION, & dans la REFRACTION. Et particulièrement Dans l'etrange REFRACTION DU CRISTAL D'ISLANDE. Leide, 1690, pp. 39-41. Reprinted in t. 19 of *OEuvres complètes de Christiaan Huygens*, La Haye, 1937, pp. 487-488. An earlier version in manuscript, dating from 1676-1677, appears on pp. 416-417.
6. ———, *TREATISE ON LIGHT*, In which are explained The causes of that which occurs In REFLEXION, & in REFRACTION. And particularly In the strange

REFRACTION OF ICELAND CRYSTAL, pp. 42-45. Translation of [5] by Silvanus P. Thompson. London, 1912; reprint, New York, Dover, 1962.

7. Edna E. Kramer, *The Nature and Growth of Modern Mathematics*, Hawthorn Books, New York, 1970, pp. 552-554.

8. Daniel Pedoe, A geometric proof of the equivalence of Fermat's Principle and Snell's Law, *American Mathematical Monthly*, 71 (1964) 543-544. Reprinted on pp. 206-207 of [1].

9. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, London and New York, 1970, pp. 91-93.

*

*

*

290. [1977: 251] *Proposed by R. Robinson Rowe, Sacramento, California.*

Find a 9-digit integer A representing the area of a triangle of which the three sides are consecutive integers.

Editor's comment.

In *Mathematics Magazine*, 50 (September 1977) 211, Problem 1023*, proposed by Steven R. Conrad, asked if there are infinitely many Heronian triangles whose sides are consecutive integers. Since a Heronian triangle is one with integral sides and integral area, Problem 1023* is closely related to our own. The solution given below was submitted by its author both to this journal and, in slightly different form, to *Mathematics Magazine*. Since the deadline for submitting solutions to the *Magazine* was April 1, 1978, all solutions are now presumed to be in, and there should be no objection to the publication of our own solution in advance of theirs.

Solution by Clayton W. Dodge, University of Maine at Orono.

Let a be a positive integer such that the triangle with sides $a-1$, a , and $a+1$ has integral area A . Since the semiperimeter is $3a/2$, Heron's formula gives

$$A = \sqrt{\frac{3a}{2} \cdot \frac{a}{2} \left(\frac{a}{2} - 1 \right) \left(\frac{a}{2} + 1 \right)} = \frac{a}{2} \sqrt{3 \left(\frac{a^2}{4} - 1 \right)}, \quad (1)$$

so a must be even, say $a = 2x$, and the value of the radical on the right must be a multiple of 3, say $3y$. Then (1) becomes

$$A = 3xy, \quad (2)$$

where (x, y) is a solution of the Pell equation

$$x^2 - 3y^2 = 1. \quad (3)$$

Equation (3) has infinitely many solutions (x_n, y_n) which can be found by standard

Pell equation techniques. The fundamental solution $(x_1, y_1) = (2, 1)$ is easily found by inspection, and all solutions are given by

$$x_n + y_n\sqrt{3} = (x_1 + y_1\sqrt{3})^n = (2 + \sqrt{3})^n, \quad n = 1, 2, \dots \quad (4)$$

From (4) and its conjugate $x_n - y_n\sqrt{3} = (2 - \sqrt{3})^n$, we obtain

$$x_n = \frac{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}{2} \quad (5)$$

and

$$y_n = \frac{(2 + \sqrt{3})^n - (2 - \sqrt{3})^n}{2\sqrt{3}}. \quad (6)$$

Thus we must have, for some positive integer n ,

$$\alpha = a_n = 2x_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \quad (7)$$

and, from (2), (5), and (6),

$$A = A_n = 3x_n y_n = \frac{3\{(2 + \sqrt{3})^{2n} - (2 - \sqrt{3})^{2n}\}}{4\sqrt{3}}. \quad (8)$$

Conversely, suppose that, for some positive integer n , $\alpha = a_n$ is defined by (7). Then α is an integer greater than 2 and, since

$$(\alpha - 1) + \alpha > \alpha + 1,$$

there is a triangle having integral sides $\alpha - 1$, α , $\alpha + 1$. Its area A is given by (1), and it is easy to verify that the definition of α leads to an integral value for A .

We conclude that there are infinitely many Heronian triangles whose sides are consecutive integers. They consist of all the triangles whose middle side and area are given by (7) and (8). The first few are given in the table on the following page. With modern calculators, these are easily calculated from (7) and (8). But I describe below a simpler method, which can be very useful when calculator batteries run down, or for calculations beyond the capacity of the machine.

We have from (4)

$$x_{n+1} + y_{n+1}\sqrt{3} = (2 + \sqrt{3})(x_n + y_n\sqrt{3}) = (2x_n + 3y_n) + (x_n + 2y_n)\sqrt{3},$$

and hence

$$x_{n+1} = 2x_n + 3y_n, \quad y_{n+1} = x_n + 2y_n,$$

from which we obtain the recurrence relations

$$x_1 = 2, \quad x_2 = 7, \quad x_{n+2} = 4x_{n+1} - x_n, \quad n = 1, 2, \dots; \quad (9)$$

$$y_1 = 1, \quad y_2 = 4, \quad y_{n+2} = 4y_{n+1} - y_n, \quad n = 1, 2, \dots. \quad (10)$$

It is now easy to calculate successive values of x_n and y_n from (9) and (10), and then we have $a_n = 2x_n$ and $A_n = 3x_n y_n$.

n	a_n	A_n
1	4	6
2	14	84
3	52	1170
4	194	16296
5	724	226974
6	2702	3161340
7	10084	44031786
8	37634	613283664
9	140452	8541939510
10	524174	118973869476*
11	1956244	1657092233154
12	7300802	23080317394680
13	27246964	321467351292366
14	101687054	4477462600698444

*[The last five lines of the table were taken from the solution submitted by Winterink.]

A glance at the table above shows that the number required by our proposal is $A = 613283664$. But it can also be calculated directly with little trouble. Since $(2 - \sqrt{3})^{2n}$ is quite small, we have from (8)

$$A_n = \left[\frac{\sqrt{3}}{4} (2 + \sqrt{3})^{2n} \right], \quad (11)$$

where the brackets denote the greatest integer function; so we need a value of n such that

$$10^8 < \frac{\sqrt{3}}{4} (2 + \sqrt{3})^{2n} < 10^9.$$

Flicking a few calculator keys soon produces $7.31 < n < 8.19$; so $n=8$ and (11) gives $A = 613283664$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; ROBERT S. JOHNSON, Montréal, Québec; HARRY L. NELSON, Livermore, California; HARRY D. RUDERMAN, Hunter College, New York; KENNETH M. WILKE, Washburn University, Topeka, Kansas; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

Editor's comment.

The literature on Heronian triangles (both rational and integral) is very

extensive. Dickson [1] devotes to it 11 closely packed pages. From this vast compilation I select just a few items for special mention:

(a) Euler noted [1, p. 193] that in any triangle with rational sides a, b, c and rational area,

$$a:b:c = \frac{(ps \pm qr)(pr \mp qs)}{pqrs} : \frac{p^2 + q^2}{pq} : \frac{r^2 + s^2}{rs} . \quad (12)$$

Dickson adds that the portion of Euler's paper containing his derivation of (12) is missing. A recent derivation of this formula can be found in Sands [4].

(b) G. Heppel noted [1, p. 198] that there are 219 triangles with integral sides ≤ 100 and integral areas. (He actually said there were 220, but one triangle was repeated.) Our table shows that only 3 of them have sides that are consecutive integers.

(c) R. Müller (1887) [1, p. 198] and L. Aubry (1911) [1, p. 200] considered Heronian triangles whose sides are consecutive integers. Their proofs, as far as can be judged from the scanty details given, merely approximate the one given here.

(d) E.N. Barisien [1, p. 200] gave complicated formulas for the integral sides of a triangle, with integral values for the altitudes, area, radius of circumscribed circle, radii of tritangent circles, segments of the sides made by the altitudes, and segments of the altitudes made by the orthocenter!

(e) E.T. Bell stated and W. Hoover proved incompletely (1917) [1, p. 201] that if $u_0 = 2, u_1 = 4, \dots, u_{n+2} = 4u_{n+1} - u_n$, then $u_n - 1, u_n, u_n + 1$ are the consecutive sides of a triangle with integral area, and all such triangles are given by this method. A complete proof follows from our own solution.

Garfunkel [2] showed, among other things, how to find all Heronian triangles with at least one integral altitude (this always occurs, in particular, when the sides are consecutive integers).

The *Heronian problem* is to find a set of expressions for the sides which will yield *all* (integral) Heronian triangles. A complete solution of the Heronian problem does not exist as yet. A recent attempt at a solution was made by Sastry [5]. He proved that all *right-angled* Heronian triangles have sides of the form

$$(m^2 - n^2)q, \quad 2mnq, \quad (m^2 + n^2)q,$$

where m, n, q are positive integers and $m > n$. He added that, in the general case, he knew of only one set of expressions which yields infinitely many Heronian triangles.

He then gave and derived the expressions

$$\begin{cases} a = 2(p^2 + q^2)(r^2 + s^2), \\ b = \frac{1}{2}[(p+r)^2 + (q+s)^2][(p-r)^2 + (q-s)^2], \\ c = \frac{1}{2}[(p+s)^2 + (q-r)^2][(p-s)^2 + (q+r)^2], \end{cases} \quad (13)$$

where p, q, r , and s are integers such that no term vanishes in the Lagrange identity

$$(p^2 + q^2 + r^2 + s^2)^2 = (p^2 + q^2 - r^2 - s^2)^2 + (2pr + 2qs)^2 + (2ps - 2qr)^2.$$

These expressions yield a Heronian triangle of sides ka, kb, kc for all even k and some odd k . He ends by saying: "It seems unlikely that all Heronian triangles are obtainable from [(13)] in this way. However I have been unable to demonstrate this by exhibiting a Heronian triangle whose sides are not multiples of the numbers a, b, c generated by [(13)]." But Tagg [7] found a counterexample in the Heronian triangle with sides 13, 14, 15, since none of these numbers can be written in the form $2(p^2 + q^2)(r^2 + s^2)$ or $4(p^2 + q^2)(r^2 + s^2)$.

Pargeter [3] also recently found a partial solution to the Heronian problem. He proved that

(i) if x, y, z are positive integers, the triangle with sides

$$(x+y)|xy - z^2|, \quad x(y^2 + z^2), \quad y(z^2 + x^2) \quad (14)$$

is Heronian;

(ii) every Heronian triangle is similar to one with sides given by (14).

Strange [6] found more Heronian triangles not obtainable from (13). He also derived, for Heronian triangles whose sides are consecutive integers, the following interesting formulas, which are, believe it or not, equivalent to (7) and (8):

$$a_n = 2 \cosh(n\alpha), \quad A_n = \frac{\sqrt{3}}{2} \sinh(2n\alpha), \quad n = 1, 2, \dots,$$

where $\alpha = \operatorname{argsinh} \sqrt{3}$. He ends by mentioning two Heronian problems still awaiting solution: given a natural number n , determine

(i) every Heronian triangle whose semiperimeter is n ;

(ii) every Heronian triangle whose area is n .

REFERENCES

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. II, pp. 191-201.
2. Jack Garfunkel, More about integral triangles, *Pi 'n' Epsilon Journal*, 6 (Spring 1975) 51-54.

3. A.R. Pargeter, Letter to the Editor, *Mathematical Spectrum*, 9 (1976-1977) 58-59.
4. A.D. Sands, Letter to the Editor, *Mathematical Spectrum*, 10 (1977-1978) 30-31.
5. K.R.S. Sastry, Heronian Triangles, *Mathematical Spectrum*, 8 (1975-1976) 77-80.
6. John Strange, More on Heronian Triangles, *Mathematical Spectrum*, 10 (1977-1978) 15-24.
7. Donovan Tagg, Letter to the Editor, *Mathematical Spectrum*, 9 (1976-1977) 58.

*

*

*

291. [1977: 297] Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.

Using soap, on a mirror, please trace
The apparent outline of your face;
Now explain (if you're wise)
Why it turns out "half size",
Using geometry as your base.

I. Solution by R. Robinson Rowe, Sacramento, California.

Two points on your face, C and D,
Reflect mirror-wise at A, B

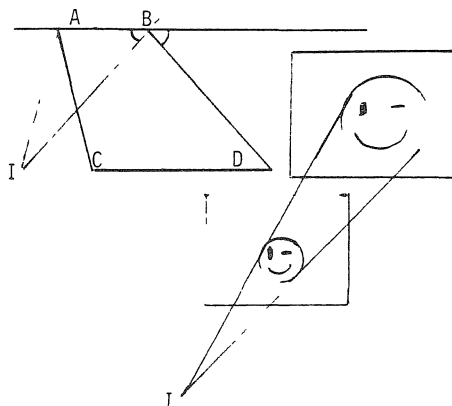
F-----G

To your eye

At point I

Which projects the image
Of points on your visage
Twice as far as AB at FG;
Whence angularity
And similarity
Prove FG just equal to CD
And its half is equal to AB,
Though maybe a bit wide
If you're looking two-eyed.

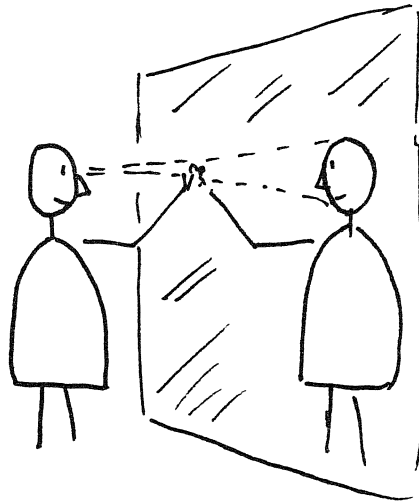
You are amply repaid
For whenever you shave
With a virtual blade
The time that you will save



Is three-fourths, as the trace
Is one-quarter your face.

II. *Solution by Leon Bankoff, Los Angeles, California.*

The virtual image I see
Is twice the distance from me
To the tracing in soap;
So I fervently hope
That their sizes run similarly.



Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, City University of N.Y.; ROBERT S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; MARK E. SAUL, Bronx High School of Science, N.Y.; and the proposer.

Editor's comment.

This problem awoke the poetic impulses of several other solvers, as it was meant to do. But in some cases the result was *de la prose où les vers se sont mis*. A rough translation of this would be: *if Shakespeare were alive today, he'd turn over in his grave.*

*

*

*

292. [1977: 297] *Proposed by Charles W. Trigg, San Diego, California.*

Fold a square piece of paper to form four creases that determine angles with tangents of 1, 2, and 3.

I. *Solution by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Three creases suffice, as shown in Figure 1, where it is clear that $\tan \alpha = 1$, $\tan \beta = 2$, and then $\tan \gamma = 3$ follows from the well-known relation

$$\text{Arctan } 1 + \text{Arctan } 2 + \text{Arctan } 3 = \pi.$$

(1)

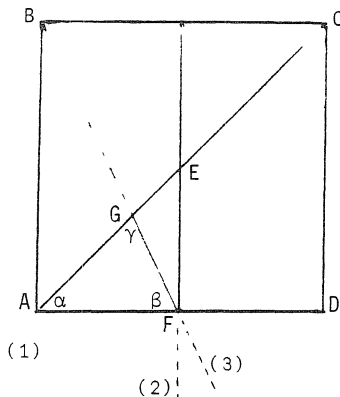


Figure 1

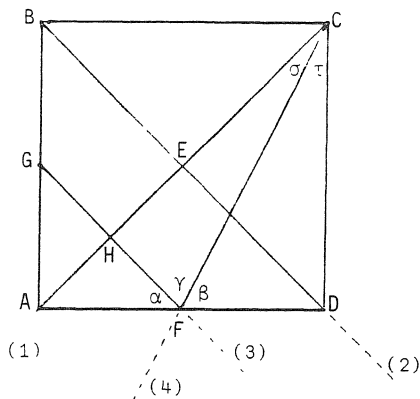


Figure 2

II. Solution by R. Robinson Rowe, Sacramento, California.

The four creases required are shown and numbered in Figure 2. (The crease FG is obtained by folding corner A to center E.) If a side of the square is assumed to be 4 units long, then

$$\tan \alpha = \frac{AG}{AF} = \frac{2}{2} = 1, \quad \tan \beta = \frac{DC}{DF} = \frac{4}{2} = 2, \quad \tan \gamma = \frac{HC}{HF} = \frac{3\sqrt{2}}{\sqrt{2}} = 3.$$

Also solved by LEON BANKOFF, Los Angeles, California; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CECILE M. COHEN, John F. Kennedy H.S., New York; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, City University of N.Y. (two solutions); ROBERT S. JOHNSON, Montréal, Québec; HERMAN NYON, Paramaribo, Surinam; BASIL RENNIE, James Cook University of North Queensland, Townsville, Australia; MALCOLM A. SMITH, Georgia Southern College, Statesboro, Georgia; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer (two solutions).

Editor's comment.

After the proposal appeared, the proposer wrote that the wording of the problem did not quite convey its intent that the creases (with the sides of the square) not only form the sides of the angles, but also that the lengths of the segments formed by the creases determine the tangent ratios. Solution II is in this spirit, and it would seem to require four creases.

The proposer pointed out that, in Figure 2, $\sigma + \tau = \pi/4$, where $\tan \sigma = \frac{1}{3}$ and $\tan \tau = \frac{1}{2}$, so solution II provides a geometric proof of the equality

$$\text{Arctan } 1 = \text{Arctan } \frac{1}{2} + \text{Arctan } \frac{1}{3}. \quad (2)$$

A different geometric proof of (2) was recently given in [1].

One could add that this same solution II provides a geometric proof of (1). But Bankoff noted that (1) can also be proved from Figure 1 as follows. Draw (do not crease) diagonal BD. Since AE and BF are medians of $\triangle ABD$, we have $GE = \frac{1}{3}AE = \frac{1}{3}BE$ and $\tan \gamma = \tan(\angle BGE) = 3$, and it has already been noted in solution I that $\tan \alpha = 1$ and $\tan \beta = 2$.

REFERENCE

1. D.M. Hallows, " $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 45^\circ$ ", *Mathematical Gazette*, 62 (March 1978) 53-54.

*

*

*

293. [1977: 297] *Proposed by David R. Stone, University of Kentucky, Lexington.*

For which b is the exponential function $y = b^x$ tangent to the given line $y = mx$? Conversely, given $y = b^x$, for which m is $y = mx$ tangent to $y = b^x$?

Solution by Cecile M. Cohen, John F. Kennedy H.S., New York.

We seek a relation between numbers $m \neq 0$ and b , $0 < b \neq 1$, that will hold if and only if the curves $y = mx$ and $y = b^x$ are tangent at some point.

Suppose they are tangent at the point (x_0, y_0) ; then $x_0 \neq 0$ and

$$y_0 = mx_0 = b^{x_0}, \quad y'_0 = m = b^{x_0} \ln b.$$

Thus $1/x_0 = \ln b$ and $m = b^{1/\ln b} \ln b$. But $b^{1/\ln b} = e$, since the natural logarithm of each side is 1; hence we have the equivalent necessary conditions

$$m = e \ln b \quad \text{and} \quad b = e^{m/e}, \tag{1}$$

from which either one of m or b can be found if the other is given.

Conversely, if either of conditions (1) holds, it is easy to verify that the two curves are tangent at the point

$$\left(\frac{1}{\ln b}, e \right) = \left(\frac{e}{m}, e \right),$$

and so the equivalent conditions (1) are also sufficient.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, City University of N.Y.; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; BERNARD VANBRUGGHE, Université de Moncton; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

*

*

*