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CONTENTS

AN EXTENSION OF AN IDENTITY OF FELLER

V.N. MURTY

The following identity involving binomial coefficients is given by William Feller in his classical text *An Introduction to Probability Theory and Its Applications*, Volume 1, Third Edition, John Wiley & Sons, Inc., New York, 1968, page 65, Problem 18:

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1}(r+1)^{-1} = \sum_{i=1}^{n} \frac{1}{i} . \tag{1}$$

Feller does not give a detailed proof, but he indicates that (1) can be established by mathematical induction (using the binomial theorem) or by integrating the following identity (2) from 0 to 1:

$$\sum_{\nu=0}^{n-1} (1-t)^{\nu} = \{1-(1-t)^n\}\, t^{-1}. \tag{2}$$

The object of this note is to obtain a recurrence relation that will enable us to evaluate

$$\mu_k \equiv \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1}(r+1)^{-k} \tag{3}$$

for integral values of $k \geq 2$.

Let

$$\nu_k = \sum_{i=1}^{n} \frac{1}{i^k}. \tag{4}$$

From (3) and the binomial theorem, we have $\mu_0 = 1 - (1-1)^n = 1$, and $\mu_1 = \nu_1$ follows from (1) and (4).

To evaluate μ_k for $k \geq 2$, we consider the rational functions

$$M(t) \equiv \frac{n!}{\prod_{i=1}^{n} (i-t)} \tag{5}$$

and

$$\Psi(t) \equiv \sum_{i=1}^{n} \frac{1}{i-t} . \tag{6}$$

If we express (5) in partial fractions as

$$M(t) = \sum_{r=0}^{n-1} \frac{A_r}{1+r-t},$$

then we have

$$A_r = \lim_{t \to 1+r} \{M(t)(1+r-t)\}$$

$$= \left[\frac{n!}{\prod\limits_{\substack{i=1 \\ i \neq 1+r}}^{n} (i-t)} \right]_{t = 1+r}$$

$$= \frac{n!}{\left\{ \prod\limits_{i=1}^{r} (i-1-r) \right\} \left\{ \prod\limits_{i=r+2}^{n} (i-1-r) \right\}}$$

$$= \frac{n!}{(-1)^r r!(n-r-1)!}$$

$$= (-1)^r \ (r+1) \binom{n}{r+1}.$$

Thus (5) can be written as

$$M(t) = \sum_{r=0}^{n-1} \frac{(-1)^r \binom{n}{r+1} (r+1)}{1+r-t}.$$

Noting that, for $|t/(r+1)| < 1$,

$$\frac{r+1}{1+r-t} = \frac{1}{1 - \frac{t}{r+1}} = \sum_{k=0}^{\infty} t^k (r+1)^{-k},$$

we can restrict the domains of the functions (5) and (6) to the unit interval (0,1) and obtain

$$M(t) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} \sum_{k=0}^{\infty} t^k (r+1)^{-k}$$

$$= \sum_{k=0}^{\infty} t^k \sum_{r=0}^{n-1} (-1)^r \binom{n}{r+1} (r+1)^{-k}$$

$$= \sum_{k=0}^{\infty} t^k \mu_k .$$

Thus μ_k is the coefficient of t^k in the Maclaurin expansion of $M(t)$, and hence

$$k!\mu_k = M^{(k)}(0) = \left[\frac{d^k}{dt^k} M(t)\right]_{t=0}. \tag{7}$$

To evaluate (7), we start by differentiating both sides of the equation

$$\log M(t) = \log n! - \sum_{i=1}^{n} \log(i-t)$$

with respect to t, obtaining

$$\frac{d}{dt} M(t) = M(t) \sum_{i=1}^{n} \frac{1}{i-t} = M(t)\Psi(t). \tag{8}$$

Now the formula of Leibniz applied to (8) yields

$$M^{(k+1)}(t) = \sum_{j=0}^{k} \Psi^{(j)}(t) M^{(k-j)}(t) \binom{k}{j},$$

from which we get

$$M^{(k+1)}(0) = \sum_{j=0}^{k} \Psi^{(j)}(0) M^{(k-j)}(0) \binom{k}{j}. \tag{9}$$

From (7) and (6), we have

$$M^{(k-j)}(0) = (k-j)!\mu_{k-j} \quad \text{and} \quad \Psi^{(j)}(0) = j!\nu_{j+1},$$

and then (9) gives the recurrence relation

$$(k+1)!\mu_{k+1} = k! \sum_{j=0}^{k} \nu_{j+1}\mu_{k-j}$$

or

$$\mu_{k+1} = \frac{1}{k+1} \sum_{j=0}^{k} \nu_{j+1}\mu_{k-j}\ . \tag{10}$$

With $\mu_0 = 1$ and $\mu_1 = \nu_1$, as noted earlier, (10) may be used to evalute μ_k for $k = 2,3,4,\ldots$. In particular, we find that

$$\mu_2 = \tfrac{1}{2}(\nu_1^2 + \nu_2)$$

and

$$\mu_3 = \frac{1}{6}(\nu_1^3 + 3\nu_1\nu_2 + 2\nu_3).$$

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* * *

NOTES ON NOTATION: IV

LEROY F. MEYERS

Have you ever observed that the equals sign has several different interpretations? Compare the following two arguments.

(A) From trigonometry we know that

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}\,. \tag{1a}$$

If

$$x = x + \pi, \tag{2a}$$

then

$$\tan\frac{x+\pi}{2} = \frac{\sin(x+\pi)}{1+\cos(x+\pi)}, \tag{3a}$$

or

$$-\cot\frac{x}{2} = \frac{-\sin x}{1-\cos x}\,. \tag{4a}$$

Hence (from (1a) and (4a))

$$\tan\frac{x}{2} - \cot\frac{x}{2} = \frac{\sin x}{1+\cos x} - \frac{\sin x}{1-\cos x}\,. \tag{5a}$$

(B) Suppose we want to solve the equation

$$x^2 - 2x = 0. \tag{1b}$$

If

$$x = x + 1, \tag{2b}$$

then

$$(x+1)^2 - 2(x+1) = 0, \tag{3b}$$

or

$$x^2 - 1 = 0. \tag{4b}$$

Hence (from (1b) and (4b))

$$x^2 - 2x = x^2 - 1, \quad \text{or} \quad x = \tfrac{1}{2}. \tag{5b}$$

Why should we accept (A) but not (B)? Certainly, equation (1a) is an *identity*, true for all x (except those for which one side or the other is undefined), whereas equation (1b) is a *conditional equation* to be solved for x, so that we wish to *find* those values of x which make it true. Equations (4a) and (5a) are true (where defined), whereas if x is chosen to make (1b) true, then (4b) and (5b) are false.

However, a closer look at the arguments shows that equations (2a) and (2b) are given in the form of *assumptions*, because of the preceding word "If". Since these assumptions are obviously false for all x, *any* statement will follow from them. Hence equations (4) and (5) *should* be accepted as conclusions in argument (B) as well as in argument (A)! In fact, *as stated*, both arguments are correct!

But why are we uneasy? The fact is that we really are not interested in assuming a false statement such as (2a) or (2b). Instead, we want to *replace* each x in (1a) or (1b) by $x + \pi$ or $x + 1$, respectively. Since (1a) is true for (almost) all x, the result of replacement, (3a), is also true for (almost) all x, and so (4a) and (5a) will follow logically.

In contrast, the result of replacing x by $x+1$ in (1b) is a statement, (3b), which is not true for the same values of x as (1b), and so the argument producing (4b) and (5b) is invalid.

Thus, in the usual notation, there are two uses for the equals sign. The primary use is in *stating* that the left side *is* equal to the right side. The secondary use is in indicating that the left side is to be *replaced* by the right side. The distinction is recognized in several computer programming languages. Compare the two programs, in each of

FORTRAN		ALGOL 60	
X = X + 1	(6a)	X := X + 1;	(6b)
.....		.....	
IF (X.EQ.X+1) GOTO 9	(7a)	if X = X + 1 then goto *next*;	(7b)

In programs (6a) and (6b), the contents of the memory location named X are to be increased by (the contents of the memory named) 1. In other words, a *replacement* is made. On the other hand, in programs (7a) and (7b) a *test* is made to find out whether or not the contents of the location named X *are* equal to the contents of the location named X, plus (the contents of the location named) 1; various actions are taken, depending on the result of the test. In this case, the condition is a *statement* whose truth is to be tested. Note that the two programming languages use the simple = in different ways, and so are forced to use other symbols for the other use of =. Since I prefer to keep the primary use of = as making a *statement* of equality, I use := for the secondary use, that of indicating a replacement, since a colon followed by an equals sign is not otherwise used in mathematics. With this notation, (2a) and (2b) become

$$x := x + \pi \qquad (2'a) \qquad\qquad x := x + 1. \qquad (2'b)$$

Why (again) should we accept (A) but not (B)? Whereas replacement of an expression by something known to be equal to it preserves truth, replacement in general does not preserve truth of statements. However, if an expression Q which is to replace an expression P satisfies the same conditions as P, then replacement will preserve truth (but not necessarily falsity). Statement (1a) is true for all x (other than odd multiples of π), and should be marked as such in the argument. Hence statement (3a) is true for all x (other than those for which $x+\pi$ is an odd multiple of π). Thus statements (1a) and (3a) can be combined to yield (5a) for all x (except multiples of π). On the other hand, statement (1b) is treated as an *assumption* about x, with a fixed but unknown value of x. It is not known in advance whether the same assumption can be made about $x+1$. In fact, (1b) is true just when x is 0 or 2, whereas (3b) is true just when $x + 1$ is 0 or 2, i.e., just when x is -1 or 1. Since the values of x making (1b) true are not the same as the values of x making (3b) true, no conclusion can be drawn by combining the two statements.

Part of the reason for confusion between the two uses of = is that often the argument will be correct however the symbol is interpreted. Equality (as an assumption or as something already deduced) implies replacement. Thus, the assumption $x = y + 1$ (if y is not already in use) allows the replacement $x := y + 1$ in (1b), to yield $(y+1)^2 - 2(y+1) = 0$, or $y^2 - 1 = 0$. This can be combined with (1b) to yield the innocuous $x^2-2x = y^2-1$.

The use of := may clear up some confusion in the use of "dummy letters" (or "bound variables", in the terminology of logic). For example, if the function f is defined so that $f(x) = e^x$ for all real x, and it is desired to integrate f from 0 to $x+1$, then the steps in the integration process are often written with a change of (dummy) letter of integration:

$$\int_0^{x+1} e^t dt = [e^t]_{t:=0}^{x+1} = e^{x+1} - e^0 = e^{x+1} - 1.$$

However, there is no need for such a change, since the use of := in symbols like $[F(x)]_{x:=a}^{b}$ merely indicates that a replacement is to be carried out, and that equality is not assumed. Hence we may write:

$$\int_0^{x+1} e^x dx = [e^x]_{x:=0}^{x+1} = e^{x+1} - e^0 = e^{x+1} - 1.$$

Clarity would be increased by using := in connection with summation symbols and the like, since the dummy letter is to be replaced by successive integers:

$$\sum_{k:=1}^{k+1} a_k = a_1 + \ldots + a_{k+1} .$$

A word of caution: some of my students have seen me write $x := a$ on the board but have misunderstood my explanation. They have copied it as $x_i = a$.

One final comment. In sloppy mathematical writing, the equals sign is often used merely as an abbreviation for the word(s) "is", "equals", or "is equal to". This can lead to confusion, as in: The square of 2 = 4. In almost all cases, I associate the symbols first, and then combine them with surrounding words. Rather than introduce parentheses, I prefer to use words to separate symbols: The square of 2 is 4.

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GEOMETRY A POSTERIOR(I)

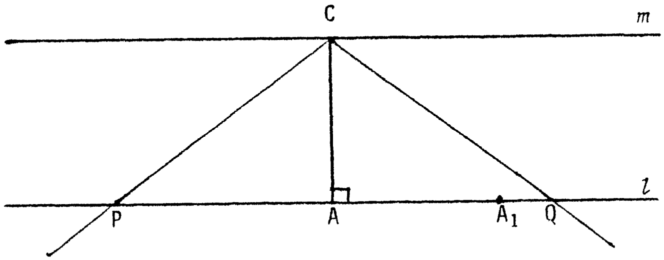
DAN PEDOE

Sometimes I come across what purport to be "geometrical" arguments which are so weird and wonderful that the inevitable question poses itself: with what part of his anatomy did the author construct such fundament-al arguments? (Delicacy dictates omission of the "or her" possibility.)

I can only reproduce part of the argument which appeared in a paper I was asked to referee, because I was denied access to the rest, so I do not know the name of the author. There is abundant internal evidence, however, that the author is Marxist. Hegel's writings are invoked, and I also know that certain "philosophical" journals are publishing Marxist discussions on the Zeno paradox, which our present author wishes to clarify. I was allowed to read another paper which "explained" the Zeno paradox by the use of overlapping neighborhoods on a line, the author, not the present one, having some acquaintance with topology.

I shall reproduce only one of the diagrams given to prove that *all points on a given line l in an Euclidean plane coincide with one another,* since readers of this journal will be able to follow the "reasoning" without too much difficulty. At the end of the "geometrical" arguments, the result reached by the author is "confirmed" by the use of Dedekind's Principle of Continuity!

"We begin with a line l on which two points A and A_1 are chosen (see figure). At A a perpendicular to l is drawn, and on it a point C is selected such that the segments AC and AA_1 are congruent. Let m be a line through C which is free to rotate about this point. Assume that m is at first parallel to l.



If, as shown in the figure, m is now rotated counterclockwise through a small angle of θ degrees, it will intersect l at point P; while if m is rotated clockwise θ degrees, it will intersect l at point Q.

As θ is decreased to zero, P and Q move boundlessly in opposite directions. Therefore, when m is parallel to l, m will be said to intersect l at plus infinity $(+\infty)$, and at minus infinity $(-\infty)$.

Now m intersects l at infinity (whether plus or minus) precisely when m doesn't intersect l at any point at all. Consequently infinity is at first simply an indefinite negation of the points on l and m: it is defined negatively as *not* being one of these points. However, in negating a determinate something, namely the points on l and m, the infinite must itself be a determinate something; this something also being a point: a *point at infinity*. In our case, since we distinguish between minus infinity and plus infinity, there will be two such points: S_∞ and T_∞.

We now have S_∞ and T_∞ negating the points which lie on l and m. However, this negation is itself negated as soon as S_∞ and T_∞ are adjoined to lines l and m and treated as if they were themselves *bona fide* points of Euclidean space. When this is done, S_∞ and T_∞ lose their status as points at infinity and are respectively transformed into the points S and T.

Let L be the line extending l and containing S and T. Also let M be the line extending m such that when m is parallel to l, M intersects L at S and T. L and M are straight lines both of which contain points S and T. Therefore, since two points determine a line, L and M must coincide. But in this case, A and C must also coincide, for it is at C that M intersects the line perpendicular to L at A. Further, since segment AA_1 is by construction congruent to segment AC, A_1 must also coincide with A. However, A_1 is an arbitrary point on l, therefore *all points on l must coincide with* A.

That l is but a point relative to L can also be proven to be a consequence of Dedekind's principle of continuity which states that if the points on a line are divided into two classes so that every point in the first class lies to the left of every point in the second class, then there is one and only one point which divides the line into these two classes."

I omit the "proof". The statement is clearly incomplete.

The author does consider the possibility of there being only one point at infinity, and not two. He says, in a footnote:

"If we assume that $S_\infty = T_\infty$, then instead of two points at infinity there will be only one. In that case L and M will be closed curves or circles with the point at infinity indistinguishable from any other point which lies on the circumference of L or M. Such a result is also reached by Hegel when he states:

'The image of the "progress to infinity" is the straight line, the infinite still remaining at its two limits and there only where the line is not; now the line is Determinate Being, which passes on to this its contradictory, that is, into the indeterminate. But as true infinity, turned back upon itself, it has for image the circle, the line which has reached itself, and wholly present and having neither beginning nor end.'"

This extract is, we are told, from Hegel's *Science of Logic*.

In conclusion, the scatological feeling induced by the paper I have quoted from may have been induced by the author's first paragraph. He writes (and this is Part II of his paper, and I was denied access to Part I):

"Gulliver and the Lilliputian emperor may have been overly hasty in their conclusions about each other's water. For it will be shown later in this article that their temperature measurements need not disprove the validity of the law of transformation of quantity into quality. In order to do so, however, it is first necessary to deal with certain matters involving the infinite."

My Heritage edition of *Gulliver's Travels* appears to contain no mention of temperature measurements of urine (can any reader locate the reference?), but it does reveal the likelihood that our unknown author is a Big-Endian, which explains the title and first paragraph of this note.

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I LOVE A (LONGER) PARADE

Donald Cross's "Squares on Parade" in this journal [1981: 162] shows only a tiny sampling of two, three, or four squares (or three biquadrates) adding up to 19059138. According to Landau [1], the number of ways in which a positive integer can be expressed as the sum of four squares is equal to 24 times the sum of the positive odd divisors of the integer if the integer is even, but to only 8 times the sum if the integer is odd. Hence the number of ways to express

$$19059138 = 2\cdot 3^4\cdot 7^6$$

as a sum of four squares is

$$24\cdot\frac{3^5-1}{3-1}\cdot\frac{7^7-1}{7-1} = 398594328.$$

However, this count includes sums made by taking squares of both positive and negative numbers, and by rearranging summands. Hence each "standard" sum of four squares can occur up to 16 times (for sign changes) and 24 times (for rearrangement). Hence there are at least

$$\frac{398594328}{16\cdot 24} = 1038006.0625$$

"standard" sums. In fact, there are somewhat more of them than this, because zeros in the sum are not affected by sign changes, and equal numbers are not affected by rearrangements. Here are a few of the more than one million ranks of up to four squares in the parade which Cross did not salute as they passed by:

$$\begin{aligned} 21^4 + 42^4 + 63^4 &= \ldots \\ &= 1^2 + 121^2 + 4364^2 \\ &= 3^2 + 16^2 + 152^2 + 4363^2 \\ &= 11^2 + 12^2 + 152^2 + 4363^2 \\ &= 12^2 + 12^2 + 75^2 + 4365^2 \\ &= \ldots \quad . \end{aligned}$$

Since the only odd prime divisors of 19059138, namely 3 and 7, are congruent to 3 modulo 4, and they occur to even powers, there is exactly one "standard" decomposition into two squares, as in Cross's last line. Note that Cross's second line includes a second decomposition into three biquadrates: $3^4 + 54^4 + 57^4$.

REFERENCE

1. Edmund Landau, *Vorlesungen über Zahlentheorie*, S. Hirzel, Leipzig, 1927 (reprinted as *Elementare Zahlentheorie* by Chelsea, New York, 1946), Vol. 1, p. 113, Satz 172.

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THE OLYMPIAD CORNER: 28

M.S. KLAMKIN

In the Olympiad Corner: 26 [1981: 171-177], the second sentence on page 174 should be amended to read: The problems and solutions were prepared jointly by the Mathematics Departments of the University of Alberta and the University of Calgary.

As I explained last month in this column, the more than 100 problems proposed by the participating countries for the 1981 International Mathematical Olympiad were first filtered down to 19 by the host country. The 6 problems finally used in the Olympiad were then selected from these 19 problems by the team leaders of the countries involved. These 6 problems having been listed in last month's column, it may be of interest to readers to know what the remaining 13 "filtered" problems were. They are given below. I urge readers to send me solutions to these 13 problems, and I may later publish here some of the more elegant solutions received.

1981 I.M.O. PROPOSED BUT UNUSED PROBLEMS

1. A sphere is tangent to the edges AB, BC, CD, DA of a tetrahedron ABCD at the points E, F, G, H, respectively. The points E, F, G, H are the vertices of a square. Prove that if the sphere is tangent to the edge AC, then it is also tangent to the edge BD. (Bulgaria)

2. Find the minimum value of

$$\max\{a+b+c,\ b+c+d,\ c+d+e,\ d+e+f,\ e+f+g\}$$

subject to the constraints

(a) $a,b,c,d,e,f,g \geq 0$;

(b) $a+b+c+d+e+f+g = 1$. (Canada)

3. Let $\{f_n\}$ be the Fibonacci sequence $\{1,1,2,3,5,\ldots\}$.

(a) Find all pairs (a,b) of real numbers such that, for each n, $af_n + bf_{n+1}$ is a number of the sequence.

(b) Find all pairs (u,v) of positive real numbers such that, for each n, $uf_n^2 + vf_{n+1}^2$ is a number of the sequence. (Canada)

4. A cube is assembled with 27 white cubes. The larger cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside? Give an approximation of the size of your answer. (Colombia)

5. Let P and Q be polynomials over the complex field, each of degree at least 1. Let

$$P_k = \{z \in C \mid P(z) = k\}, \qquad Q_k = \{z \in C \mid Q(z) = k\},$$

and assume that $P_0 = Q_0$ and $P_1 = Q_1$. Prove that $P = Q$. (Cuba)

6. A sequence $\{a_n\}$ is defined by means of the recursion formula

$$a_1 = 1; \quad a_{n+1} = \frac{1}{16}(1 + 4a_n + \sqrt{1+24a_n}), \quad n = 1,2,3,\ldots .$$

Find an explicit formula for a_n. (Federal Republic of Germany)

7. Determine the smallest natural number n having the property that, for every integer p, $p \geq n$, it is possible to subdivide (partition) a given square into p squares (not necessarily equal). (France)

8. On a semicircle with unit radius four consecutive chords AB, BC, CD, DE with lengths a, b, c, d, respectively, are given. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4. \quad \text{(Netherlands)}$$

9. Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1}, \quad k = 0,1,\ldots,n.$$

Determine $P(n+1)$. (Romania)

10. Prove that a convex pentagon ABCDE with equal sides and for which the interior angles satisfy $A \geq B \geq C \geq D \geq E$ is regular pentagon. (Romania)

11. A sequence $\{u_n\}$ of real numbers is given by u_1 and, for $n \geq 1$, by the recurrence relation

$$4u_{n+1} = \sqrt[3]{64u_n + 15}.$$

Describe, with proof, the behavior of u_n as $n \to \infty$. (United Kingdom)

12. Several equal spherical planets are in outer space. On the surface of each planet there is a region which is invisible from any of the remaining planets. Prove that the sum of the areas of all these regions is equal to the area of the surface of one planet. (U.S.S.R.)

13. A finite set of unit circles is given in a plane and the area of their union is S. Prove that there exists a subset of mutually disjoint circles such that the area of their union is greater than $2S/9$. (Yugoslavia)

Editor's note: All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 3: Alphametic

$$\begin{array}{r} \text{SNARK} \\ \text{A} \\ \hline \text{BOOJUM} \end{array}$$

BOOJUM: the words sounded frighteningly cruel,
An impossible product to me;
I should have ignored a single small rule—
For the B was a zero, you see.

HANS HAVERMANN, Weston, Ontario

Puzzle No. 4: Rebus (6 2 5)

$\circ$ O $\propto$ $\circ$ O $\propto$

A lot were hid
By Captain Kidd.

ALAN WAYNE, Holiday, Florida

Answer to Puzzle No. 2 [1981: 219]: Cosecant; can, coset.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

669. *Proposed by Charles W. Trigg, San Diego, California.*

The digits from 1 to 9 are arranged in a 3×3 array. This square array can be considered to consist of a 2×2 square and a 5-element L-shaped gnomon in four ways. If the sums of the elements in the four corner 2×2 squares are the same, the square is said to be *gnomon-magic*, and the common sum is the magic constant. [Charles W. Trigg, "Another Type of Third-Order Magic Square", *School Science and Mathematics*, 70 (May 1970) 467.] Such a square is

$$\begin{matrix} 9 & 4 & 8 \\ 2 & 1 & 3 \\ 6 & 7 & 5 \end{matrix}$$

with magic constant 16. Is there a nine-digit gnomon-magic square with four odd corner digits?

670. *Proposed by O. Bottema, Delft, The Netherlands.*

The points A_i, $i = 1,2,\ldots,6$, no three of which are collinear, are the vertices of a hexagon. X_0 is an arbitrary point other than A_2 on line A_1A_2. The line through X_0 parallel to A_2A_3 intersects A_3A_4 in X_1; the line through X_1 parallel to A_3A_6 intersects A_6A_1 in X_2; the line through X_2 parallel to A_5A_6 intersects A_4A_5 in X_3; and the line through X_3 parallel to A_2A_5 intersects A_1A_2 in X_4.

(a) Prove the following closure theorem: if $X_0X_1X_2X_3X_4$ is closed (that is, if X_4 coincides with X_0) for some point X_0, then it is closed for any point X_0.

(b) Show that closure takes place if and only if the six points A_i lie on a conic.

671. *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

The following alphametic is dedicated to the editors of the Problem Department in the *Two-Year College Mathematics Journal*: Erwin Just, Sam Greenspan, and Stan Friedlander. Given that one of the names is prime, solve

$$\begin{array}{r} \text{SAM} \\ \text{STAN} \\ \underline{\text{ERWIN}} \\ \text{PRIME} \end{array}.$$

672. *Proposed by Jordi Dou, Barcelona, Spain.*

Given four points P,A,B,C in a plane, determine points A',B',C' on PA,PB,PC, respectively, such that

$$\frac{AA'}{PA'} = t\alpha, \quad \frac{BB'}{PB'} = t\beta, \quad \frac{CC'}{PC'} = t\gamma,$$

where α,β,γ are given constants, and such that the hexagon ABCA'B'C' is inscribed in a conic.

(This generalizes Crux 485 [1980: 256], which corresponds to the special case $\alpha = \beta = \gamma = 1$, $t = k/(k-1)$.)

673*. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

Determine for which positive integers n the following property holds: if m is any integer satisfying

$$\frac{n(n+1)(n+2)}{6} \le m \le \frac{n(n+1)(2n+1)}{6},$$

then there exist permutations $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ of $(1,2,\ldots,n)$ such that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = m.$$

(See Crux 563 [1981: 208].)

674. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a given triangle and let A'B'C' be its medial triangle (A' being the midpoint of BC, etc.). In the medial triangle, the internal angle bisectors meet the opposite sides in R, S, T (R being on B'C', etc.).

If the points P and Q divide the perimeter of the original triangle ABC into two equal parts, prove that the midpoint of segment PQ lies on the perimeter of triangle RST.

675. *Proposed by Harry D. Ruderman, Hunter College, New York, N.Y.*

ABCD is a skew quadrilateral and P,Q,R,S are points on sides AB,BC,CD,DA, respectively. Prove that PR intersects QS if and only if

$$AP\cdot BQ\cdot CR\cdot DS = PB\cdot QC\cdot RD\cdot SA.$$

676. *Proposed by William Moser, McGill University, Montréal, Québec.*

Gertie, the secretary, was so angry with her boss that she maliciously put every one of the six letters she had typed into an envelope addressed to one of the others. In fact Drab received the letter of the man to whom Gertie had mailed Crumb's letter; and the man whose letter was sent to Fatso received the letter of the man to whom Epsilon's letter was mailed. She did not mail to Axworthy the letter of the man to whom she mailed Bilk's letter. Whose letter did Crumb receive?

(I found this problem among the papers of my late brother Leo Moser (1921-1970). There was nothing to indicate prior publication.)

677. *Proposed by E.J. Barbeau, University of Toronto.*

Prove that there do not exist four distinct square integers in arithmetic progression.

(This problem is apparently due to Fermat (1640), but the proofs mentioned in Dickson's *History of the Theory of Numbers* (Vol. II, p. 440) either have obscure references, are unclear, or appear unsatisfactory.)

678. *Proposed jointly by Joe Dellinger and Ferrell Wheeler, students, Texas A & M University, College Station, Texas.*

For a given fixed integer $n \geq 2$, find the greatest common divisor of the integers in the set $\{a^n - a \mid a \in Z\}$, where Z is the set of all integers.

679*. *Proposed by Bob Prielipp, University of Winsconsin-Oshkosh.*

The equation $x^2+y^3 = 2z^4$ has the solutions (1,1,1) and (239,1,13). Does it have infinitely many solutions (x,y,z) consisting of relatively prime positive integers?

680. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Find interesting sets of three distinct real numbers such that their product is equal to their sum.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

471. [1979: 228; 1980: 195] Late solution: E.A. VAN DER MOLEN, Hengelo Ov, The Netherlands.

* * *

540. [1980: 114; 1981: 127] *Proposed by Leon Bankoff, Los Angeles, California.*

Professor Euclide Paracelso Bombasto Umbugio has once again retired to his *tour d'ivoire* where he is now delving into the supersophisticated intricacies of the works of Grassmann, as elucidated by Forder's *Calculus of Extension*. His goal is to prove Neuberg's Theorem:

If D, E, F *are the centers of squares described externally on the sides of a triangle* ABC, *then the midpoints of these sides are the centers of squares described internally on the sides of triangle* DEF.

Help the dedicated professor emerge from his self-imposed confinement and enjoy the thrill of hyperventilation by showing how to solve his problem using only high-school, synthetic, Euclidean, "plain" geometry.

II. *Comment by J. Chris Fisher, University of Regina.*

This result is a special case of the following

THEOREM. Given an arbitrary n-gon $A = A_1A_2\ldots A_n$, *let B be the n-gon whose vertices* B_i *are the centres of squares erected externally on the sides* A_iA_{i+1} *of A (where* $A_{n+k} = A_k$*). Then the midpoints of the diagonals* A_iA_{i+2} *of A are the centres of squares erected internally on the sides of B.*

Of course, the triangle case implies the theorem for n-gons, so the proof given earlier [1981: 127] suffices to establish the above theorem.

* * *

556. [1980: 184; 1981: 189] *Proposed by Paul Erdős, Mathematical Institute, Hungarian Academy of Sciences.*

Every baby knows that

$$\frac{(n+1)(n+2)\ldots(2n)}{n(n-1)\ldots2\cdot1}$$

is an integer. Prove that for every k there is an integer n for which

$$\frac{(n+1)(n+2)\ldots(2n-k)}{n(n-1)\ldots(n-k+1)} \qquad (1)$$

is an integer. Furthermore, show that if (1) is an integer, then $k = o(n)$, that is, $k/n \to 0$.

Joint solution by Benji Fisher, student, Bronx H.S. of Science, Bronx, N.Y.; and Jeremy Primer, student, Columbia H.S., Maplewood, N.J.

Let N and D denote the numerator and denominator, respectively, of (1). For prime p and positive integer x, let $a = \nu_p(x)$ denote the largest integer such that $p^a|x$. The first part of the problem will follow from the fact, which we will establish, that for every k there is an integer n such that $\nu_p(D) \le \nu_p(N)$ holds for every prime p.

The value of k being fixed, we first consider primes $p \le k$. Since there are

k indicated factors in D, we have $D \le n^k$ and hence

$$\nu_p(D) \le k \log_p n = k \frac{\log n}{\log p} \le \frac{k}{\log 2} \log n. \qquad (2)$$

On the other hand, since there are $n-k$ indicated factors in N, we have

$$\nu_p(N) \ge \left[\frac{n-k}{p}\right] \ge \left[\frac{n-k}{k}\right] > \frac{n}{k}-2. \qquad (3)$$

Clearly, as n increases, the last expression in (3) eventually exceeds that in (2), that is, there is an integer $M = M(k) > 2k$ such that $\nu_p(D) < \nu_p(N)$ holds for all $n \ge M$ and all $p \le k$.

It is well known that the sequence of prime powers has density 0, so there are arbitrary large gaps in it as far down the line as we choose to go. We can, and do, choose $n \ge M$ such that none of

$$n,\ n-1,\ \ldots,\ n-k+1 \qquad (4)$$

is a prime power. Now let p be any prime greater than k. Then at most one of the numbers (4) is a multiple of p. If none of them is, then $\nu_p(D) = 0 \le \nu_p(N)$. Suppose one of them is, say $n-l = \alpha p^\beta$, where $(\alpha,p) = 1$ and $\beta \ge 1$. Then $\nu_p(D) = \beta$. Since $n-l$ is not a prime power, we have $\alpha \ne 1$ and so $\alpha \ge 2$. Therefore

$$(\alpha+1)p^\beta = \frac{\alpha+1}{\alpha}(n-l) \le \frac{3n}{2} < 2n-k.$$

But $p > k$ also implies that $(\alpha+1)p^\beta = n-l+p^\beta > n$. It follows that $(\alpha+1)p^\beta$ is one of the indicated factors in N, and so $p^\beta | N$. We conclude that, for the chosen value of n, $\nu_p(D) \le \nu_p(N)$ holds for every prime p.

To prove the second part of the problem, we will henceforth assume that $n = n(k)$ is the smallest integer for which (1) is an integer. Let $\varepsilon > 0$ be given. There is an $M = M(\varepsilon)$ such that there is a prime between x and $(1+\varepsilon)x$ whenever $x \ge M$. (This is a well-known consequence of the prime number theorem.) We will show that $k/n < 2\varepsilon$ whenever $k \ge 2M$. Fix some such k. Since $n > k$ (except for the trivial case $k = 1$), we have

$$n-\frac{k}{2} > \frac{k}{2} \ge M$$

and there is a prime p such that

$$n-\frac{k}{2} < p < (1+\varepsilon)(n-\frac{k}{2}).$$

Since $2p > 2n-k$, we have $p|D$ but $p \nmid N$ unless $n < p < (1+\varepsilon)(n-(k/2))$, in which case

$$n < (1+\varepsilon)(n-\frac{k}{2}) \quad \text{and} \quad \frac{k}{n} < \frac{2\varepsilon}{1+\varepsilon} < 2\varepsilon.$$

The required conclusion, $k/n \to 0$, follows.

Editor's Comment.

This remarkable solution, the only one we have received, is the work of two members of the winning U.S.A. team in the 1981 International Mathematical Olympiad, in which our two solvers each obtained a perfect score of 42. The solution itself appears to be a by-product of the publicity generated by the I.M.O. Fisher, who sent the solution to the editor, wrote: "Jeremy and I started working on this when a reporter from National Public Radio wanted to have on tape the solution of a problem as it was worked out by two math whizzes. Jeremy selected the problem randomly from the bound 1980 Volume of *Crux* I had gotten for winning the U.S.A. Mathematical Olympiad. The reporter was disappointed, because he was able to capture on his tape only our (nearly complete) solution of the first part of the problem. We finished it later."

Perhaps our two bright young solvers should consider a career in show biz. Having them crack unsolved problems on prime time television could do a lot for mathematics.

* * *

560. [1980: 185] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Take a complete quadrilateral. On each of the three diagonals as diameter, draw a circle. Prove that these three circles are coaxal.

Comment by Leon Bankoff, Los Angeles, California.

This is the classic Gauss-Bodenmiller Theorem, the proof of which serves as a springboard for three important corollaries:

(a) The midpoints of the three diagonals of the complete quadrilateral are collinear on what is called the *Gauss-Newton line* or the *diagonal center line.*

(b) The orthocenters of the four triangles of the complete quadrilateral are collinear on the radical axis of the circles whose diameters are the three diagonals.

(c) The Gauss-Newton line is perpendicular to the radical axis of the three circles.

These corollaries stem from the application of general coaxal-circle theory; they arise as incidental steps in the proof of the main Gauss-Bodenmiller Theorem.

Several different types of proof appear in the literature, each elegant in its own fashion, but each dependent on a particular level of sophistication for comprehension. As an example, we take a look at Lachlan's treatment of the problem [14]. Despite its elegant brevity, his proof would not convince a student unfamiliar with the manipulation of pencils in involution. Lachlan paraphrases the problem as follows: *Show that the circles described on the diagonals* AA', BB', CC' *of a tetragram as diameters have two common points.* His solution goes like this:

In the complete quadrilateral whose diagonals are AA', BB', and CC', let the circle described on diagonal BB' cut the circle described on CC' in P and P'. Then

BPB' and CPC' are right angles; therefore, since P{AA',BB',CC'} is a pencil in involution, APA' is a right angle.

This constitutes Lachlan's proof, which tacitly assumes that the reader will immediately recognize that the intersecting circles have two points in common, thus rendering them coaxal. This proof, though valid for intersecting circles, ignores the case for nonintersecting circles. The same omission mars the proofs given in Catalan [3], Daus [6], Johnson [13], Maxwell [15], M'Clelland [16], and Shively [19].

One easily accessible proof which combines elegance and generality with an economy of space and concepts is that given by Eves [10]. He proves the Gauss-Bodenmiller Theorem *and* the three corollaries given above in about a dozen lines, and his proof, which is valid for nonintersecting as well as for intersecting circles, requires nothing more than the definitions of the radical axis and coaxal circles.

Solutions were submitted by J.T. GROENMAN, Arnhem, The Netherlands; SAHIB RAM MANDAN, Bombay, India; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. References were sent by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; DAN PEDOE, University of Minnesota; and DAN SOKOLOWSKY, California State University at Los Angeles.

Editor's comment.

We have 20 references for this problem, but all it takes is one reference to make a problem "well known". Our proposer himself discovered that his problem was well known, but he was too late to have it withdrawn: shortly after his problem was published here, he found it in Askwith [1].

Pedoe noted another interesting property of the Gauss-Bodenmiller configuration, which Durell [7] ascribes to Plücker: *The director circles of all conics inscribed in the quadrilateral belong to the same coaxal system.*

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561. [1980: 211; 1981: 206] Late solution: E.A. VAN DER MOLEN, Hengelo Ov, The Netherlands.

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573. [1980: 251] *Proposed by Charles W. Trigg, San Diego, California.*

In Crux 430 [1980: 52], attention was called to the fact that the decimal digits of 8^3 sum to 8. Is there another power of 2, say $P > 8$, and a positive integer k such that the decimal digits of P^k sum to P?

I. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

Let $P = 2^m$. A necessary condition for the digit sum of P^k to be equal to P is that

$$2^{mk} \equiv 2^m \pmod 9 \quad \text{or} \quad 2^{m(k-1)} \equiv 1 \pmod 9.$$

Since 2 belongs to the exponent 6 modulo 9, we must have $m(k-1) \equiv 0 \pmod 6$. Thus the only values of m and k we need consider are:

$m = 4$ ($P = 16$), $k = 4,7,10,13,16,19,22,25,\ldots,$

$m = 5$ ($P = 32$), $k = 7,13,19,25,31,37,43,49,\ldots,$

$m = 6$ ($P = 64$), $k = 2,3,4,5,6,7,8,9,\ldots,$

$m = 7$ ($P = 128$), $k = 7,13,19,25,31,37,43,49,\ldots,$

$m = 8$ ($P = 256$), $k = 4,7,10,13,16,19,22,25,\ldots,$

etc. Checking in a table of powers of 2 (in the *CRC Standard Mathematical Tables*, for example), we quickly find that $(m,k) = (6,6)$ yields a solution:

$$64^6 = 68\,719\,476\,736$$

has a digit sum of 64.

II. *Solution by the proposer.*

There are exactly five powers of 2 between 2^9 and 2^{377} whose digit sums are themselves powers of 2. They are:

Powers	Digit Sums
2^{36}	$64 = 2^6$
2^{85}	$128 = 2^7$
2^{176}	$256 = 2^8$
2^{194}	$256 = 2^8$
2^{200}	$256 = 2^8$

The second and fourth can be eliminated because $7 \nmid 85$ and $8 \nmid 194$. The remaining three yield solutions to our problem. It can be verified that

$$64^6 = 68\,719\,476\,736$$
$$256^{22} = 95\,780\,971\,304\,118\,053\,647\,396\,689\,196\,894\,323\,976\,171\,195\,136\,475\,136$$
$$256^{25} = 1\,606\,938\,044\,258\,990\,275\,541\,962\,092\,341\,162\,602\,522\,202\,993\,782\,792\,835\,301\,376$$

have digit sums of 64, 256, and 256, respectively. □

The last one, 2^{200}, is the smallest power of 2 that contains at least four of each of the ten digits.

III. *Comment by Allan Wm. Johnson Jr., Washington, D.C.*

I found the example $64^6 = 68\,719\,476\,736$ in Madachy [1] and in Mohanty and Kumar [2], where are published lists of numbers having a power whose decimal digits sum to the number. These tables also furnish examples of P^k for P a power of a number other than 2:

$$9^2,\quad 27^3,\quad 25^4,\quad 36^4,\quad 36^5,\quad 27^7,\quad 81^9.$$

Also solved by HERMAN NYON, Paramaribo, Surinam; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

The solutions 256^{22} and 256^{25} were also found by Johnson with the help of a computer. Trigg's decimal representation of these numbers was carefully checked against Johnson's print-out.

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574. [1980: 251] *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Given five points A, B, C, D, E, construct a straight line l such that the three pairs of straight lines {AD,AE}, {BD,BE}, {CD,CE} intercept equal segments on l.

Solution by the Samsoe Problem Group, Silkeborg, Denmark.

We interpret the word "segments" in the proposal as meaning "directed segments". To eliminate some trivial and indeterminate cases, we will assume that of the five given points (assumed coplanar) no three are collinear. The line l = DE is an obvious trivial solution for any given set of points A,B,C,D,E.

To find the nontrivial solutions, if any, we let $l \neq$ DE be a line in the plane, and suppose that AD,AE,BD,BE,CD,CE intersect l in $A_1, A_2, B_1, B_2, C_1, C_2$, respectively. We will find necessary and sufficient conditions for the result

$$\overline{A_1A_2} = \overline{B_1B_2} = \overline{C_1C_2}. \tag{1}$$

We can assume that l is not parallel to DE, for otherwise (1) would hold if and only if A,B,C were collinear (on a line parallel to l).

We introduce an affine coordinate system in the plane with DE as first axis and l as second axis. Let D = $(a,0)$, E = $(b,0)$, and let the common value of the numbers (1) be the real number $c \neq 0$. If (x,y) represents the affine coordinates of one of the points A,B,C, say A, and if $A_1 = (0,r)$ and $A_2 = (0,s)$, then we have

$$\frac{x}{a} + \frac{y}{r} = 1 \quad \text{and} \quad \frac{x}{b} + \frac{y}{s} = 1,$$

or

$$r = \frac{ay}{a-x} \quad \text{and} \quad s = \frac{by}{b-x}.$$

Hence

$$\overline{A_1A_2} = c \iff c = \frac{by}{b-x} - \frac{ay}{a-x}$$
$$\iff y = \frac{c}{a-b}\cdot\frac{(a-x)(b-x)}{x}$$
$$\iff A \in H,$$

where H is a certain hyperbola containing D and E and having l as one of its asymptotes; and the same statement can be made about B and C. It follows that (1) holds if and only if the five given points A,B,C,D,E are on a hyperbola; and if they are, then the only nontrivial solutions l to our problem are the two asymptotes.

Now the five given distinct points A,B,C,D,E determine a unique conic. If this conic is not a hyperbola, then our problem has no (nontrivial) solution l. If the conic is a hyperbola, its two asymptotes are the only (nontrivial) solutions l. In the latter case, the asymptotes can be constructed by ruler and compass when only the five points A,B,C,D,E are known. See, for example, "Pascal Redivivus: II", by Dan Pedoe, in this journal [1979: 281-287; 1980: 96].

Also solved by CLAYTON W. DODGE, University of Maine at Orono; and the proposer.

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575. [1980: 251] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

If $n \geq 2$ is an integer and square brackets denote the greatest integer function, evaluate

$$S = \sum_{r=1}^{[n/2]} \sum_{k=1}^{n+1-2r} \left[\frac{n-r}{k+r-1}\right].$$

Solution by Jordi Dou, Barcelona, Spain.

Our solution will make use of the following formula, which is valid for any integer n and any natural number h:

$$\sum_{i=1}^{h} \left[\frac{n-i}{h}\right] = n-h. \tag{1}$$

The proof of (1) is simple. Of the h numbers $(n-i)/h$, only one is an integer, say $(n-q)/h$. Then we have

$$\sum_{i=1}^{h} \left[\frac{n-i}{h}\right] = q\cdot\frac{n-q}{h} + (h-q)\left(\frac{n-q}{h}-1\right) = n-h.$$

If we set $i = k+r-1$ and note that $k \leq n+1-2r$ is equivalent to $r \leq n-i$, then the $[n/2]\cdot[(n+1)/2]$ terms of S can be written in the form

$$S = \sum_{i=1}^{n-1} \sum_{r=1}^{n-i} \left[\frac{n-r}{i}\right] \equiv \sum_{i=1}^{n-1} S_i.$$

Now if $i \le n/2$, then S_i contains only i possibly nonvanishing terms, and

$$S_i = \left[\frac{n-1}{i}\right] + \left[\frac{n-2}{i}\right] + \ldots + \left[\frac{n-i}{i}\right] = n - i$$

by (1); while if $i > n/2$, then S_i contains $n - i$ terms each equal to 1. Thus $S_i = n - i$ for every i, and we have

$$S = \sum_{i=1}^{n-1} S_i = \sum_{i=1}^{n-1} (n-i) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \binom{n}{2}. \quad \square$$

We also have, more symmetrically,

$$S_0 = \sum_{r=1}^{n-1} \sum_{k=1}^{n-1} \left[\frac{n-r}{k+r-1}\right] = \binom{n}{2}$$

for, of the $(n-1)^2$ terms in S_0, those that are not already in S all vanish because

$$k > n+1-2r \iff k+r-1 > n-r.$$

Also solved by DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer. One incorrect solution was received.

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576. [1980: 251] *Proposed by Mats Röyter, student, Chalmers University of Technology, Sweden.*

Consider two n-digit numbers in base b, the digits of one being a permutation of the digits of the other. Prove that the difference of the numbers is divisible by $b - 1$.

I. *Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

It is well known that *every natural number is congruent, modulo b-1, to the sum of its digits in base b.* So if the digits of N in base b are a permutation of those of N' in base b, then their digit sums are the same and their difference $N - N'$ is congruent to 0 modulo b-1, that is, $N - N'$ is divisible by b-1.

For completeness, we give a proof of the italicized statement:

$$\sum_{i=0}^{n-1} a_i b^i = \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} a_i(b^i - 1) \equiv \sum_{i=0}^{n-1} a_i \pmod{b-1},$$

since $b-1 \mid b^i - 1$ for $i = 0,1,2,\ldots$.

II. *Comment by Charles W. Trigg, San Diego, California.*

A related known result occurs in Problem 3145, *American Mathematical Monthly*, 33 (November 1926) 481: If a number of n digits in base b is divisible by any factor of $b^n - 1$, that divisibility is not altered by a cyclic permutation of the digits of the original number.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; SAMSOE PROBLEM GROUP, Silkeborg, Denmark; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Nearly all solutions received were more or less equivalent to our solution I. So the selection of a solution to feature was made on the basis of presentation rather than content.

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577. [1980: 252] *Proposed by R.B. Killgrove, California State University at Los Angeles.*

Explain why, in calculus courses, we are never asked to find the exact value of the arc length for the general sine curve $y = a\sin bx$ from $x = 0$ to $x = c$, where $c \le 2\pi/b$.

Solution by Mats Röyter, student, Chalmers University of Technology, Gothenburg, Sweden.

Since $y = a\sin bx$ and $y' = ab\cos bx$, the required arc length is

$$\int_0^c \sqrt{1 + y'^2}\,dx = \int_0^c \sqrt{1 + a^2b^2\cos^2 bx}\,dx$$

$$= \int_0^c \sqrt{1 + a^2b^2(1-\sin^2 bx)}\,dx$$

$$= \sqrt{1+a^2b^2}\int_0^c \sqrt{1 - \frac{a^2b^2}{1+a^2b^2}\sin^2 bx}\,dx$$

$$= \frac{\sqrt{1+a^2b^2}}{b}\int_0^{bc}\sqrt{1-\frac{a^2b^2}{1+a^2b^2}\sin^2\phi}\;d\phi.$$

Since $0 < bc \le 2\pi$ and $a^2b^2/(1+a^2b^2) < 1$, the arc length is given by an elliptic integral of the second kind, which, as every baby knows, cannot be evaluated exactly.

More precisely, the above shows merely that the question of finding the exact value of the arc length will never be answered, not that it will never be asked.

Also solved by the proposer.

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578. [1980: 252] *Proposed by S.C. Chan, Singapore.*

An unbiased cubical die is thrown repeatedly until a 5 and a 6 have been obtained. The random variable X denotes the number of throws required. Calculate $E(X)$, the expectation of X and $V(X)$, the variance of X.

Solution by the St. Olaf College Problem Solving Group, Northfield, Minnesota.

Let X_1 be the number of throws until either a 5 or a 6 appears, and let X_2 be the number of additional throws until the other one appears. Then $X = X_1+X_2$ is the total number of throws required, where X_1 and X_2 are independent geometric random variables with parameters $p_1 = 1/3$ and $p_2 = 1/6$, respectively. Since a geometric random variable with parameter p has expectation $1/p$ and variance $(1-p)/p^2$, we have

$$E(X) = E(X_1) + E(X_2) = 3 + 6 = 9$$

and

$$V(X) = V(X_1) + V(X_2) = 6 + 30 = 36.$$

Also solved by JORDI DOU, Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and LEROY F. MEYERS, The Ohio State University.

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579. [1980: 252] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

If n is a nonnegative integer, express in closed form the sum

$$H_n = \sum_{r+s+t=n} a^r b^s c^t,$$

which is the sum of all homogeneous products of degree n in a,b,c.

I. *Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

We will denote by $H_n(r)$ the sum of all the formally distinct homogeneous products of degree n in the r variables $a,b,c,\ldots$. We are asked to find $H_n(3)$, and the "formally distinct" requirement means that we must assume that a,b,c are all distinct.

Since

$$H_n(2) = a^n + a^{n-1}b + \ldots + ab^{n-1} + b^n = \frac{a^{n+1}-b^{n+1}}{a-b},$$

we have

$$\begin{aligned} H_n(3) &= \sum_{i=0}^{n} c^i H_{n-i}(2) \\ &= \frac{1}{a-b}\left\{a\cdot\frac{a^{n+1}-c^{n+1}}{a-c} - b\cdot\frac{b^{n+1}-c^{n+1}}{b-c}\right\} \\ &= \frac{a^{n+2}}{(a-b)(a-c)} + \frac{b^{n+2}}{(b-c)(b-a)} + \frac{c^{n+2}}{(c-a)(c-b)}. \end{aligned}$$

II. *Comment extracted from the solution of Leroy F. Meyers, The Ohio State University.*

Since $H_n(3)$ depends continuously on $a,b,c,$ the limiting case when $c = a$, say, can be found from

$$H_n(3) = -\frac{(b-c)a^{n+2} + (c-a)b^{n+2} + (a-b)c^{n+2}}{(b-c)(c-a)(a-b)}$$

by L'Hôpital's rule. The result is easily found to be

$$\lim_{c\to a} H_n(3) = \frac{(n+1)a^{n+2} - (n+2)a^{n+1}b + b^{n+2}}{(a-b)^2},$$

which is not the same as $H_n(2) = (a^{n+1}-b^{n+1})/(a-b)$. Similarly, we find

$$\lim_{\substack{c\to a\\ b\to a}} H_n(3) = \tfrac{1}{2}(n+1)(n+2)a^n,$$

which is different from both $\lim_{b\to a} H_n(2) = (n+1)a^n$ and $H_n(1) = a^n$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

Two solvers gave as their answer

$$H_n(3) = \frac{f^{(n)}(0)}{n!}, \text{ where } f(x) = \frac{1}{(1-ax)(1-bx)(1-cx)}.$$

Begging your pardon, sirs, this is begging the question: it is merely replacing one problem by an equivalent problem and solving neither. What still needs to be done is to find the coefficient of x^n in the Maclaurin expansion of $f(x)$. This is best accomplished by first splitting $f(x)$ into three partial fractions. This was the method used by Klamkin and Satyanarayana, who showed, more generally, that if the r variables are $a,b,c,\ldots,k,l,$ then

$$H_n(r) = \frac{a^{n+r-1}}{(a-b)(a-c)\ldots(a-l)} + \frac{b^{n+r-1}}{(b-a)(b-c)\ldots(b-l)} + \ldots + \frac{l^{n+r-1}}{(l-a)(l-b)\ldots(l-k)}.$$

This can also be proved by induction, the inductive step from $H_n(r)$ to $H_n(r+1)$ being performed as in the step from $H_n(2)$ to $H_n(3)$ in solution I.

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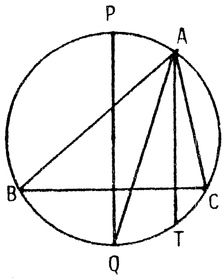
580. [1980: 252] *Proposed by Leon Bankoff, Los Angeles, California.*

In the figure, the diameter PQ ⊥ BC and chord AT ⊥ BC. Show that

$$\frac{AQ}{PQ} = \frac{AB + AC}{PB + PC} = \frac{TB + TC}{QB + QC}.$$



Solution by Roland H. Eddy, Memorial University of Newfoundland.

We note that PB = PC, QB = QC, AQ = PT, and use Ptolemy's Theorem to obtain

$$BC\cdot AQ = AB\cdot QC + AC\cdot QB = QB\cdot(AB+AC), \tag{1}$$
$$BC\cdot AQ = BC\cdot PT = PC\cdot TB + PB\cdot TC = PB\cdot(TB+TC), \tag{2}$$
$$BC\cdot PQ = PB\cdot QC + PC\cdot QB = QB\cdot(PB+PC) = PB\cdot(QB+QC). \tag{3}$$

The first of the desired equalities then follows from (1) and (3), and the second from (2) and (3).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; SAMSOE PROBLEM GROUP, Silkeborg, Denmark; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Five of the other solutions were roughly equivalent to our featured solution, so presentation was the deciding factor here. The remaining ten solutions were more or less heavily trigonometric.

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581. [1980: 283] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

The Soldier's Farewell

My love, since we must P A R T
There's A L O E in my soul;
Oh, hear the drumbeats R O L L
That T E L L how throbs my heart!

In the "word square" above, each word represents a four-digit decimal integer which is a perfect square. The letters are one-to-one images of the digits. Restore the digits.

Solution by Charles W. Trigg, San Diego, California.

The only four-digit square ending in 44 and with its first three digits distinct is 3844. Therefore L = 0, and RO and TE are two-digit squares.

The penultimate digit of a square is even unless the terminal digit is 6. A square ending in 5 ends in 25. No square ends in 2, 3, 7, or 8. Hence, from the six two-digit squares, only two $\begin{matrix}\mathrm{RO}\\ \mathrm{TE}\end{matrix}$ square arrays can be formed, namely: $\begin{matrix}25\\16\end{matrix}$ and $\begin{matrix}49\\16\end{matrix}$. Now the only square of the form $x096$ is 4096, which has A = R. The only square of the form $y056$ is 7056, and the only square of the form $z721$ is 3721. Thus the unique reconstruction is

3721
7056
2500
1600.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; J.A. McCALLUM, Medicine Hat, Alberta; FRED A. MILLER, Elkins, West Virginia; HERMAN NYON, Paramaribo, Surinam; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; DONVAL R. SIMPSON, Fairbanks, Alaska; ROBERT A. STUMP, Hopewell, Virginia; RAM REKHA TIWARI, Radhur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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582. [1980: 283] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In how many ways can five distinct digits A, B, C, D, E be formed into four decimal integers AB, CDE, EDC, BA for which the mirror-image multiplication

$$AB \cdot CDE = EDC \cdot BA$$

is true? (For example, the mirror-image multiplication $AB \cdot CD = DC \cdot BA$ is true for $13 \cdot 62 = 26 \cdot 31$.)

I. *Union of all the solution sets received.*

Due to the symmetry of equality and the commutativity of multiplication, we consider that any solution $AB \cdot CDE = EDC \cdot BA$ is identical to $BA \cdot EDC = CDE \cdot AB$. The problem then has at least 13 distinct solutions:

12•693 = 396•21	39•682 = 286•93
13•682 = 286•31	46•352 = 253•64
24•693 = 396•42	48•231 = 132•84
26•341 = 143•62	48•693 = 396•84
28•451 = 154•82	68•473 = 374•86
36•462 = 264•63	69•352 = 253•96
39•341 = 143•93	

II. *Comment by the proposer.*

Dickson's *History of the Theory of Numbers* (Vol. 1, p. 462) credits Witting for the mirror-image multiplications

$$102\cdot402 = 204\cdot201$$
$$213\cdot936 = 639\cdot312$$
$$213\cdot624 = 426\cdot312,$$

which motivated me to compute the examples

$$992\cdot483\cdot156 = 651\cdot384\cdot299$$
$$852\cdot473\cdot187 = 781\cdot374\cdot258$$
$$984\cdot652\cdot168 = 861\cdot256\cdot489.$$

None of these multiplications has factors composed of distinct digits, so I computed examples with distinct digits:

$$43\cdot6528 = 8256\cdot34$$
$$693\cdot2784 = 4872\cdot396$$
$$3516\cdot8274 = 4728\cdot6153$$
$$49368\cdot2751 = 1572\cdot86394$$
$$381472\cdot6509 = 9056\cdot274183.$$

In the last equation, every digit appears exactly once on each side!

Partial solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono (11 solutions); J.T. GROENMAN, Arnhem, The Netherlands (5 solutions); FRED A. MILLER, Elkins, West Virginia (8 solutions); HERMAN NYON, Paramaribo, Surinam (12 solutions); CHARLES W. TRIGG, San Diego, California (11 solutions); KENNETH M. WILKE, Topeka, Kansas (11 solutions); and the proposer (11 solutions).

Editor's comment.

Each solver had his own process allegedly designed to squeeze out all solutions, and some were quite elaborate Rube Goldberg contraptions. Yet in each case some solutions remained unsqueezed. And there is no guarantee that our 13 solutions constitute the complete solution set. There may still be more solutions waiting forlornly to be squeezed, in a place accessible only by computer, where the hand of man has never set foot.

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583. [1980: 283] *Proposed by Charles W. Trigg, San Diego, California.*

A man, being asked the ages of his two sons, replied: "Each of their ages is one more than three times the sum of its digits". How old is each son?

Solution by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.

It is clear that each son is at least 1 year old. Let the age of one of the sons be

$$A = \overline{a_n a_{n-1} \ldots a_1 a_0}, \quad 0 \le a_i \le 9, \; a_n \ne 0.$$

Then we have

$$10^n \le A = 1 + 3\sum_{i=0}^{n} a_i \le 1 + 27(n+1),$$

which can hold only if $n = 1$. Now the condition

$$10a_1 + a_0 = 1 + 3(a_1 + a_0)$$

is equivalent to $a_1 = (2a_0+1)/7$, for which the only solution is $(a_1, a_0) = (1,3)$. Since the only possible age for a son is 13, each son is 13 years old.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; FRED A. MILLER, Elkins, West Virginia; HERMAN NYON, Paramaribo, Surinam; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAVID R. STONE and WALTER LYNCH, Georgia Southern College, Statesboro, Georgia (jointly); ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Two incorrect solutions were received.

Editor's comment.

Most solvers *assumed* that the age of each son was a two-digit number, and then found the problem trivial (some even said unworthy of *Crux*, which is a left-handed compliment). But nothing in the proposal warrants such an assumption. The "man" in the proposal could be, for all we are told, a personage of the present or of the past, of fact or fiction or legend. He could be, for example, Methuselah, who lived to the ripe old age of 969 and may have, in his youth, fathered sons who eventually reached an age well into the hundreds. It must be *proved* that the age of each son is a two-digit number.

Also, most solvers inexplicably concluded that the two sons must be 13-year-old twins. Why, for heaven's sake? There are twelve months in a year, but the gestation period of a human being is only nine months; so the sons could well be both 13 years old and still not be twins. Even if they were born on the same day they might still not be twins: they might have different mothers, or they could be two of a set of triplets (the third being a daughter).

Moral: In *Crux*, even easy problems should be treated with respect.

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