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This issue is dedicated to

Professor W.J. BLUNDON,
Memorial University of Newfoundland,

in appreciation for his long and valuable services to mathematics and to Canadian mathematical education.

CONTENTS

THE OLYMPIAD CORNER: 3

MURRAY S. KLAMKIN

1. *Some coming (and some past) events.*

The following is a list of some competitions open to high school students, with comments and addresses for further information. The deadlines for entries of some of them are past, but there's always next year.

(a) *Annual High School Mathematics Examination* (Canada and U.S.A.), March 1, 1979.

This examination is limited to pre-calculus mathematics with emphasis on intermediate algebra and plane geometry. It is a multiple choice type of examination and the time allowed is 80 minutes.

Sample problem (1978): *In a room containing N people, $N > 3$, at least one person has not shaken hands with everyone else in the room. What is the maximum number of people in the room that could have shaken hands with everyone else?*

(A) 0 (B) 1 (C) $N-1$ (D) N (E) none of these.

For information: Dr. Walter E. Mientka, Executive Director,
Annual High School Mathematics Examination,
917 Oldfather Hall,
University of Nebraska,
Lincoln, Nebraska 68588.

(b) *Competitions open to all high school students.*

(A) *Junior Mathematics Contest* (February 27, 1979): for students in Grades 9 - 11 in Canadian (and some U.S.A.) schools. There are 30 questions. (B) *The Euclid Contest* (April 24, 1979): aimed primarily at Grade 12 students, it involves both multiple choice and written solutions. (C) *The Descartes Competition* (April 24, 1979): designed for students in their last year of secondary school and used to determine scholarship recipients in the Faculty of Mathematics at the University of Waterloo. (Students may not write both (B) and (C).)

For information: Professor R.G. Dunkley,
Faculty of Mathematics,
University of Waterloo,
Waterloo, Ontario N2L 3G1.

(c) *Alberta High School Prize Examination*, March 8, 1979.

This examination involves both multiple choice (60 minutes) and written solutions (110 minutes).

Sample problem (1978): *36 points are placed inside a square whose sides have length 3. Show that there are 3 points which determine a triangle of area no greater than ½.*

For information: Professor G.J. Butler,
Provincial Exam Chairman,
Department of Mathematics,
University of Alberta,
Edmonton, Alberta T6G 2G1.

(d) *Eleventh Canadian Mathematical Olympiad*, May 2, 1979.

This Olympiad, now consisting of 5 problems to be done in 3 hours, was established in the autumn of 1968 by the Education Committee of the Canadian Mathematical Congress (now the Canadian Mathematical Society). Provincial competitions had already existed for many years in all the provinces and, with these firmly established, it was the right time to start on a national competition. Candidates are chosen by quota for each province from the students who performed well in the provincial competitions. Also, high school principals may nominate candidates who, for some good reason, did not participate in the provincial competition but nevertheless seem to be of Olympiad quality.

Sample problem (1977): *If* A, B, C, D *are four points in space such that*

$$\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2,$$

prove that A, B, C, D *lie in a plane.*

For information: Professor John Burry, Acting Chairman,
Canadian Mathematical Olympiad Committee,
Memorial University of Newfoundland,
St. John's, Newfoundland A1B 3X7.

The present chairman of the Canadian Mathematical Olympiad Committee (now on leave) is Professor W.J. Blundon, one of the great names in Canadian mathematical education. It was under his chairmanship that the Committee decided to pay each year for more than 30 subscriptions to *Crux Mathematicorum* to be sent to the *teachers* of all the Canadian Olympiad candidates who receive either a prize or an honourable mention. I have very pleasantly intersected with him in Problem Solving and Mathematical Competitions, as well as in the field of Geometric Inequalities. His paper "Inequalities associated with the triangle", *Canadian Mathematical Bulletin*, 8 (1965) 615-626, was a pioneering and important one on the structure of triangle inequalities.

I am very pleased that the editor has decided to honour Professor Blundon's long and valuable services to Canadian mathematical education by dedicating this issue of *Crux Mathematicorum* to him. It is therefore appropriate to insert here the following brief biography of Professor Blundon. It was written by Professor E.R. Williams, Memorial University of Newfoundland.

Jack Blundon was born on February 20, 1916 at St. Anthony, Newfoundland, not far from the site where archaeologists have discovered the remains of an ancient Norse settlement dating back over a thousand years. His father, after whom he was named, was a member of the Newfoundland Constabulary, and so, at an early age, his family moved first to Spaniards Bay, Newfoundland and then to Heart's Content where he completed his high school education, graduating in 1932 with the highest academic standing of any student on the island of Newfoundland.

He attended Memorial University College in St. John's from September 1932 to June 1933. In January 1934 he began his teaching career at a one-room all-grade school at St. Phillips, Newfoundland. His salary was $36 per quarter or $144 per year, from which he had to pay food and lodging. Although the effects of the Great Depression ravaged the land, he felt that he deserved an increase in salary. As a result, he approached the Chairman of the local School Board, who also happened to be the parish priest, for a salary increase. The clergyman, however, didn't think that an increase was justified on the grounds that school teachers only worked for five hours a day, five days a week. Professor Blundon's salary was increased, however, when he reminded the clergyman that by this line of reasoning, a clergyman only works two hours a day for one day a week.

Professor Blundon enrolled in a teacher training program at the Newfoundland Normal School from September 1934 to June 1935. During 1935-36 he taught school at Bishop Field College in St. John's and then spent seven years teaching high school at Curling and Corner Brook on the west coast of Newfoundland. In 1944, he became a School Supervisor with the Newfoundland Department of Education and survived many harrowing experiences travelling by dogteam during winter and by boat during summer to visit schools in the numerous small fishing communities that ring the northwest coast of Newfoundland.

Professor Blundon completed a London External B.A. degree with first class honours in Mathematics in 1943. He completed the M.A. degree at Columbia University in 1947-48. In September 1948, he joined the faculty of Memorial University of Newfoundland assuming the position of Head of Mathematics, a position which he retained until August 1976. He saw the department grow from one full-time and two-part time faculty members in 1948 to in excess of fifty full-time members in 1976.

Professor Blundon spent the years 1955-56 and 1966-67 at University College, London, where he worked with Davenport, Rogers, Few and others. His main research interest has been on multiple packing and covering problems. He also has a great interest in problem solving and has been a regular contributor to the problem section of the *American Mathematical Monthly* and several other noted journals. He has maintained a keen interest in the high school mathematics curriculum having served for many years on the Newfoundland School Mathematics Curriculum Committee.

Professor Blundon is a member of a number of professional organizations. He was a member of the Council of the Canadian Mathematical Society for a total of eight years and is presently Chairman of the Olympiad Committee which oversees the Canadian Mathematics Olympiad. He is also past Chairman of the Mathematics Sub-Committee of the Atlantic Provinces Committee on Science.

Professor Blundon is still a very active member of the mathematics community and his contributions to mathematics in Canada and more particularly in Newfoundland during the past forty-five years are worthy of much more recognition than can be expressed in these few short paragraphs.

(e) *Eighth U.S.A. Mathematical Olympiad*, May 1, 1979.

This Olympiad, consisting of 5 problems to be done in 3 hours, was established in the autumn of 1971 by the Olympiad Sub-Committee of the Mathematical Association of America. The purpose of the Olympiad was to attempt to discover secondary school students with superior mathematical talent, students who possessed mathematical creativity and inventiveness as well as competence in computational techniques. Candidates are chosen from the top 100 students in the Annual High School Mathematics Examination (U.S.A. and Canada), plus a small number of students with special recommendations.

Sample problem (1977): *If a, b, c, d, e are positive numbers bounded by p and q, that is,*

$$0 < p \le a,b,c,d,e \le q,$$

prove that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \le 25 + 6(\sqrt{p/q} - \sqrt{q/p})^2$$

and determine when there is equality.

For information: Professor S.L. Greitzer, Chairman,
U.S.A. Mathematical Olympiad Committee,
350-A Lafayette Road,
Metuchen, New Jersey 08840.

(f) *XXI International Mathematical Olympiad*, July 2 - 3, 1979, England.

This Olympiad was started by Romania in 1959. It is a two-day examination, with 3 problems to be done in 4½ hours each day. Each year, the host country sends out invitations to various countries which choose an 8-person team based on the results of their national Olympiads. Canada has been invited several times but has never participated because of a lack of funds for training and travel.

Sample problem (1977): *In a finite sequence of real numbers, the sum of any* *seven successive terms is negative and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.*

We will be glad to receive and publish information about other mathematical competitions.

2. *Now for some action.*

PRACTICE SET 5 (3 hours)

5-1. A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original position?

5-2. In a non-recent edition of Ripley's *Believe It Or Not*, it was stated that the number

$$N = 526315789473684210$$

is a *persistent* number, that is, if multiplied by any positive integer the resulting number always contains the ten digits 0, 1, 2,..., 9 in some order with possible repetitions.

(a) Prove or disprove the above statement.

(b) Are there any persistent numbers smaller than the above number?

5-3. In a regular (equilateral) triangle, the circumcenter O, the incenter I, and the centroid G all coincide. Conversely, if any two of O, I, G coincide, the triangle is equilateral. Also, for a regular tetrahedron, O, I, and G coincide. Prove or disprove the converse result that if O, I, and G all coincide for the tetrahedron, the tetrahedron must be regular.

SOLUTIONS TO PRACTICE SET 2

2-1. It is easy to see that there exists an infinite family of ellipses which can be inscribed in a given square. Prove, however, that only *one* ellipse can be inscribed in a given regular pentagon.

Solution.

The only inscribed ellipse in the regular pentagon is the inscribed circle. If there were another inscribed ellipse, it would have to intersect the inscribed circle in at least five points. But this is impossible since the maximum number of points of intersection between two distinct ellipses is four. To prove this, we can eliminate y^2 between the two second-degree equations for the ellipses and solve for y. Then, substituting back, we obtain a fourth-degree polynomial equation in x.

More generally, for n-dimensional Euclidean space, if p algebraic hypersurfaces have degrees $d_1, d_2, \ldots, d_p$ and have only a finite number of common points, then that number is at most $d_1 d_2 \ldots d_p$. This is Bezout's Theorem.

2-2. Determine all pairs of rational numbers (x,y) such that $x^3+y^3 = x^2+y^2$.

Solution.

Suppose (x,y) is a solution. If $x=0$, we get $y = 0$ or 1 and the solutions $(0,0)$, $(0,1)$. If $x \neq 0$, there is a unique rational number $m \neq -1$ such that $y = mx$, and substitution in the given equation yields

$$x = \frac{1+m^2}{1+m^3}, \qquad y = \frac{m(1+m^2)}{1+m^3}. \tag{1}$$

Conversely, it is easily verified that (1) yields a solution for any rational $m \neq -1$.

It is clear that this solution would remain valid if the word "rational" were replaced throughout by "real".

2-3. Three unequal disjoint circles are given on a large (planar) card. If the centers of the circles are collinear, show that it is always possible to fold the card along two straight lines such that the three circles lie on a common sphere.

Solution.

Suppose a sphere with center O exists which, after a suitable fold of the card, contains circles (A) and (B) (see Figure 1). Since OA and OB are perpendicular to

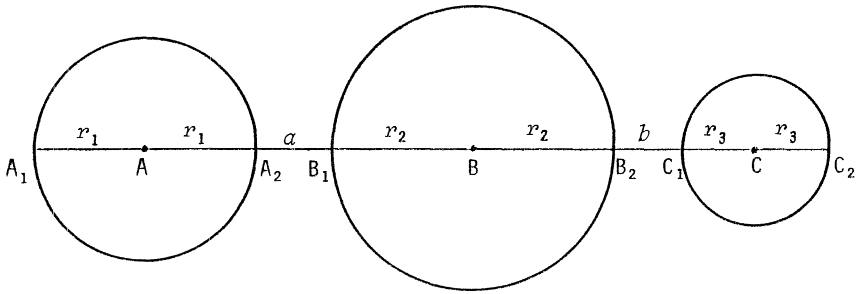


Figure 1

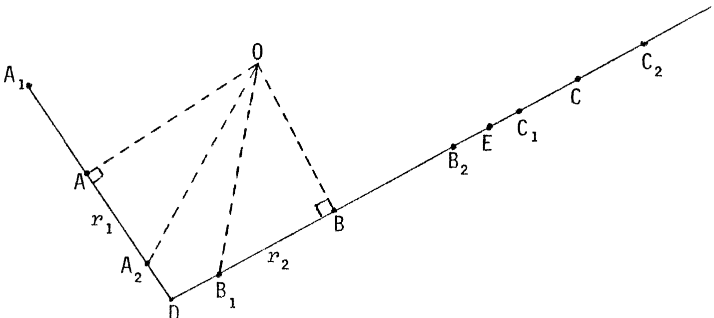


Figure 2

the planes of circles (A) and (B), respectively, the fold must be along a line perpendicular to the line of centers AB, and it must go through a point D between A_2 and B_1 (see Figure 2). If $A_2D = x < a$, then $DB_1 = a - x$ and

$$OA^2 + r_1^2 = OB^2 + r_2^2, \tag{1}$$

$$OA^2 + (r_1 + x)^2 = OB^2 + (r_2 + a - x)^2, \tag{2}$$

whence, by subtraction, $2x(a + r_1 + r_2) = a(a + 2r_2)$.

Conversely if, at a point D on A_2B_1 such that

$$A_2D = x = \frac{a(a + 2r_2)}{2(a + r_1 + r_2)}, \tag{3}$$

we make a fold in the prescribed manner resulting in a suitably restricted but arbitrary angle ADB < 180°, the perpendiculars at A and B will meet in a point O. Retracing our steps above, (3) is found to be equivalent to

$$2r_1x + x^2 = 2r_2(a - x) + (a - x)^2. \tag{4}$$

Now (2) holds for any x, i.e. regardless of the position of D on A_2B_1, but together with (4) it implies (1). Thus $OA_2 = OB_1 = r$, say, and the circles (A) and (B) lie on the sphere with center O and radius r.

It should now be obvious that if, at a point E on B_2C_1 such that

$$B_2E = \frac{b(b + 2r_3)}{2(b + r_2 + r_3)},$$

we make a fold in the prescribed manner in such a way that the perpendicular through C meets O, then $OC_1 = OB_2 = OB_1 = r$, and circle (C) lies on the same sphere as (A) and (B).

Any number of circles can thus be fitted onto the same sphere, provided their centers are collinear. The circles need not be unequal, as the proposal says, but they must be disjoint. The sphere itself is determined by the first fold at D where angle ADB is arbitrary but, as mentioned earlier, must be suitably restricted. This simply means that angle ADB must be large enough so that the resulting sphere is large enough to accommodate even the largest circle.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

LAPLACE

Mathématicien, Astronome, Physicien,... et Ministre français

(suite)

NICOLE et JEAN DHOMBRES

2. *Le point fixe: l'explication du Système du Monde*

Quiconque s'est frotté aux sciences a entendu prononcer le nom de Laplace:

—d'évidence l'astronome, ne serait-ce que pour l'hypothèse cosmogonique de la nébuleuse primitive de Laplace, le Newton français dont la mort symboliquement survint un siècle, presque jour pour jour, après celle de son illustre devancier;

—d'évidence l'ingénieur, ne serait-ce que pour la transformée de Laplace, qui justifiait avant qu'il soit conçu le calcul symbolique d'Heaviside;

—d'évidence le physicien, ne serait-ce que pour l'étude de la capillarité où les formules de Laplace n'ont point eu besoin de grandes améliorations et furent bien utiles lors de la détermination de la charge de l'électron par Millikan;

—d'évidence le mécanicien et le mathématicien, ne serait-ce que pour l'équation caractéristique du potentiel de la gravitation, la nullité d'un laplacien;

—d'évidence le probabiliste, le statisticien ou l'utilisateur de méthodes statistiques, pour qui la loi normale de Laplace-Gauss est d'usage constant.

A cette énumération, on pourrait imaginer Laplace comme un touche-à-tout de génie, un sur-doué aux curiosités multiformes, aux goûts changeants, mais transformant toute matière en or pur. Ce n'est pas là le caractère de Laplace, qui apparaît comme l'homme d'une unique réflexion: la mécanique céleste, "car la constance imperturbable des vues a toujours été le trait principal de son génie", dira le mathématicien Fourier dans son éloge de Laplace.

Continuité et persévérance dans le travail, dans la réflexion centrée sur un but unique. Les événements politiques, les honneurs du jour ou les drames familiaux, rien ne détruit la cadence de ses publications. Les intérêts qu'il porte à la chimie, aux probabilités, ou à la physique sont tous ordonnés en fait autour de l'astronomie car claironne Laplace lui-même[20]:

"L'Astronomie, par la dignité de son objet et par la perfection de ses théories, est le plus beau monument de l'esprit humain, le titre le plus noble de son intelligence".

Cette dignité est l'objet d'une profession de foi qui veut répondre à certain texte célèbre de Pascal[20]:

"Séduit par les illusions des sens et de l'amour-propre, l'homme s'est regardé longtemps, comme le centre du mouvement des astres, et son vain orgueil a été puni par les frayeurs qu'ils lui ont inspirées. Enfin, plusieurs siècles de travaux ont fait tomber le voile qui cachait à ses yeux, le système du monde. Alors il s'est vu sur une planète presque imperceptible dans le système solaire dont la vaste étendue n'est elle-même, qu'un point insensible dans l'immensité de l'espace. Les résultats sublimes auxquels cette découverte l'a conduit, sont bien propres à le consoler du rang qu'elle assigne à la terre; en lui montrant sa propre grandeur, dans l'extrême petitesse de la base qui lui a servi pour mesurer les cieux".

Fidélité qui engage aussi une méthodologie scientifique puisqu' "il y a extrêmement loin de la première vue du ciel, à la vue générale par laquelle on embrasse aujourd'hui les états passés et futurs du système du monde. Pour y parvenir, il a fallu observer les astres pendant un grand nombre de siècles; reconnaître dans leurs apparences, les mouvements réels de la terre; s'élever aux lois des mouvements planétaires, et de ces lois, au principe de la pesanteur universelle; redescendre enfin de ce principe, à l'explication complète de tous les phénomènes célestes, jusque dans leurs moindres détails. Voilà ce que l'esprit humain a fait dans l'Astronomie. L'exposition de ces découvertes et de la manière la plus simple dont elles ont pu naître et se succéder, aura le double avantage d'offir un grand ensemble de vérités importantes, et la vraie méthode qu'il faut suivre dans la recherche des lois de la nature".[20]

L'outil, et ce n'est qu'un outil, est l'analyse mathématique dont les applications au système du monde manifestent "la puissance de ce merveilleux instrument sans lequel il eût été impossible de pénétrer un mécanisme aussi compliqué dans ses effets qu'il est simple dans sa cause".

Ce mécanisme épistémologique détermine une idéologie scientifique fascinante par l'ampleur oratoire que lui donne Laplace... mais qui repose de fait sur l'unicité de la solution d'une équation différentielle linéaire du second ordre, la vitesse et la position initiales étant fournies:

"Une intelligence qui, pour un instant donné, connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux".

Pour la mécanique céleste, la grandeur de Laplace fut bien d'avoir mis en évidence ce déterminisme réglé. La situation se présentait en effet de la façon suivante. Si la gravitation universelle expliquait à merveille l'allure générale des phénomènes célestes, la finesse et le nombre des observations avaient indiqué des perturbations: des accélérations ou des ralentissements tant des mouvements de la lune que des planètes, ou par exemple des satellites de Jupiter, etc... . Y-avait-il périodicité de ces phénomènes, c'est-à-dire stabilité du système ou au contraire le système courrait-il à sa désorganisation? Laplace comprend qu'il faut affiner les approximations newtoniennes des calculs, les développer en séries entières jusqu'à un terme plus avancé en jouant de l'énormité de la masse du soleil par rapport à celle des autres planètes, en un mot combiner l'analyse mathématique à une approximation valide des calculs. Laplace développa assez les calculs et les approximations pour prouver toutes les périodicités compatibles avec les observations. Nous n'en dirons pas plus ici. Mais on devine les conséquences philosophiques vite exploitées, se déchargeant bien vite du fardeau de la critique des approximations faites.

Effectivement, Newton lui-même avait prévu, comme dira Leibnitz, une sorte d' "angelus rector" chargé de "décrasser de temps en temps" le système solaire pour le remonter "comme un horloger son ouvrage". Laplace réduit l'ange au chômage! D'où l'aphorisme de Laplace en réponse à Napoléon qui lui objectait narquoisement n'avoir point vu Dieu dans son système:

"Sire, je n'avais pas besoin de cette hypothèse..."

C'est que Laplace, dans le discours scientifique, déteste d'autant plus le mélange des genres et des styles qu'il le pratique dans ses relations avec le pouvoir comme nous allons le voir. L'intrusion d'un ange, d'un argument métaphysique, le hérisse aussitôt. C'est un laïc. Citons un passage très caractéristique de sa première leçon à l'Ecole Normale de l'An III:

"Leibnitz crut voir dans l'arithmétique binaire l'image de la création. Il imagina que l'unité pouvait représenter Dieu, et zéro, le néant; et que l'Etre suprême avait tiré du néant tous les êtres de cet univers, de même que l'unité avec le zéro exprime tous les nombres dans ce système de numération.

Cette idée plut tellement à Leibnitz, qu'il en fit part au jésuite Grimaldi, président du tribunal des mathématiques à la Chine, dans l'espérance que cet emblème de la création convertirait au christianisme l'empereur d'alors, qui aimait particulièrement les mathématiques. Ce trait nous rappelle le commentaire de Newton sur l'apocalypse.

Quand vous voyez les écarts d'aussi grands hommes, écarts qui sont dûs aux impressions reçues dans l'enfance, vous sentez combien un système d'éducation libre de préjugés est utile aux progrès de la raison humaine, et qu'il est beau d'être appelés, comme vous l'êtes, à la présenter à vos concitoyens dans toute sa pureté et dégagée des nuages qui l'ont trop souvent obscurcie".

Au demeurant, n'étant entré dans aucun détail de mécanique céleste, ayant lâchement ignoré les probabilités et le déterminisme aussi bien que la calorimétrie, le mathématicien que je suis ne peut résister au plaisir d'indiquer en quelques mots une technique de Laplace, appelée à porter son nom, *la transformée de Laplace*, appliquée à la résolution d'équations différentielles ou aux dérivées partielles. Je le ferai dans un cas particulièrement élégant. Il s'agit de résoudre les équations différentielles linéaires du second ordre dont les coefficients sont des polynômes du premier degré. Prenons en exemple l'équation

$$t\frac{d^2x}{dt^2} + (n+1)\frac{dx}{dt} + x = 0, \tag{1}$$

laquelle provient d'ailleurs de l'équation de Bessel

$$s^2\frac{d^2J}{ds^2} + s\frac{dJ}{ds} + (s^2 - n^2)J = 0 \tag{2}$$

en posant $J = s^n x$ et $4t = s^2$. L'idée pour résoudre (1) est de représenter x comme résultant de l'intégration d'une fonction ϕ le long d'un chemin C orienté du plan complexe selon

$$x(t) = \int_{C^+} e^{zt}\phi(z)dz.$$

Il s'agit de déterminer astucieusement et le chemin C et la fonction ϕ. L'équation différentielle (1) devient

$$\int_{C^+}(tz^2 + (n+1)z + 1)e^{zt}\phi(z)dz = 0.$$

Une intégration par parties permet de remplacer la multiplication par t selon une dérivation portant sur ϕ:

$$\int_{C^+} tz^2 e^{zt}\phi(z)dz = \int_{C^+}\frac{d(z^2\phi(z)e^{zt})}{dz}\,dz - \int_{C^+} e^{zt}\frac{d(z^2\phi(z))}{dz}\,dz.$$

D'où l'équation

$$0 = \Big[z^2\phi(z)e^{zt}\Big]_{C^+} + \int_{C^+}\left((1+(n+1)z)\phi(z) - \frac{d(z^2\phi(z))}{dz}\right)e^{zt}dz.$$

Si la fonction $z^2e^{zt}\phi(z)$ est univoque dans le plan complexe, le terme tout intégré s'annule en choisissant un chemin fermé. Il suffit donc d'annuler l'intégrant:

$$(1+(n+1)z)\phi(z) = \frac{d(z^2\phi(z))}{dz},$$

dont l'intégration est facile selon

$$\operatorname{Log} z^2\phi(z) = -\frac{1}{z} + \operatorname{Log} z^{n+1},$$

soit

$$\phi(z) = \lambda z^{n-1}e^{-1/z}$$

à une constante multiplicative près que l'on choisit évidemment aujourd'hui de la forme $\lambda = 1/2i\pi$, car

$$x(t) = \frac{1}{2i\pi}\int_{C^+} e^{zt-1/z}z^{n-1}dz. \tag{3}$$

Mais la théorie de Cauchy des fonctions holomorphes n'était pas née. Poursuivons cependant un instant, éclairant une démarche du passé par les connaissances d'aujourd'hui.

Comme la fonction $e^{zt-1/z}$ possède un seul point singulier, d'ailleurs essentiel, en 0, et pour obtenir autre chose que la solution triviale $x \equiv 0$, nous prendrons un chemin fermé C dont l'indice par rapport à 0 est 1. Auquel cas $x(t)$ apparaît comme le coefficient d'ordre $-n$ dans le développement en séries de Laurent de

$$e^{zt-1/z} = e^{zt}e^{-1/z},$$

évidemment de la forme

$$\frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} + \dots .$$

Revenons à Laplace qui intègre (3) avec avantage suivant un cercle centré à l'origine et de rayon $1/\sqrt{t}$, car alors $dz/z = id\theta$ et

$$x(t) = \frac{t^{-n/2}}{2\pi}\int_{-\pi}^{+\pi} e^{2i\sqrt{t}\sin\theta} e^{in\theta} d\theta.$$

En revenant à une solution de l'équation de Bessel (2), et pour s'adapter à la normalisation classique en prenant $\lambda = 1/2^n \cdot 1/2i\pi$, on obtient

$$J_n(s) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{is\sin\theta} e^{in\theta} d\theta,$$

ce qui fournit une représentation intégrale de la n-ième fonction de Bessel J_n et la fonction génératrice de ces mêmes fonctions grâce à la théorie que Fourier développera au tout début du XIXème siècle:

$$e^{is\sin\theta} = \sum_{k=-\infty}^{+\infty} J_k(s) e^{ik\theta}.$$

Comme autre illustration d'un beau travail de Laplace, peut-être moins connu, je mentionnerai brièvement, mais avec plaisir, la capillarité. Il s'agit d'expliquer les effets de surface, à la séparation de deux milieux. Cette tension superficielle provoquant par exemple la forme de ménisque concave de la surface extérieure du mercure dans un tube à essais ou encore la montée de l'eau dans un tube étroit, en contradiction avec le principe des vases communicants. Avant Laplace, Clairaut ou Jurin avaient certes précisé les phénomènes de surface, mais ils n'avaient pu découvrir "l'explication de la loi de cette ascencion". Laplace élabore la théorie à grands renforts de longs et pénibles calculs. Ô certes, il ne veut pas nous épargner les équations et ne cherche pas des raisonnements géométriques intrinsèques. Foin de l'élégance mathématique!

Il parvient à montrer, par une classique analyse infinitésimale des forces en présence, que la différence de pression de chaque côté d'une surface de liquide, une portion de bulle de savon par exemple, est en chaque point proportionnelle à ce qu'on appelle la courbure moyenne de la surface considérée. Rappelons que cette courbure moyenne est la trace de l'endomorphisme associé à la deuxième forme quadratique fondamentale sur une surface: cette trace est la somme des courbures principales: $1/R_1 + 1/R_2$.

Ainsi, si des deux côtés la pression reste la même, par exemple pour une fine nappe non fermée, la courbure moyenne doit être nulle. Ce sont les surfaces dites minimales dont on peut avoir de saisissantes images en circonscrivant une pellicule d'eau savonneuse à un léger fil déformé à loisir.

(suite et fin au prochain numéro)

NOTES

(22) Le 11 Nivôse, on pouvait traverser la Seine à pied tant la couche de glace était profonde.

(23) "En appelant les premiers géomètres, les premiers physiciens, les premiers naturalistes du monde au professorat, la Convention jeta sur les fonctions enseignantes un éclat inaccoutumé et dont nous ressentons encore les effets" constatera F. Arago beaucoup plus tard dans son éloge de Fourier.

(24) Fourier, dans son *Eloge historique de M. le marquis de Laplace* (15 juin 1829), ajoutera, narquois, que dégagé certes de l'analyse, le livre de Laplace n'est à proprement parler qu' "une table des matières d'un traité mathématique".

(25) Extrait d'un discours de Barère du 13 Prairial an III, au Comité de Salut Public, annonçant la création d'une Ecole Normale.

(26) Bien sûr, les leçons données à l'Ecole Normale par Laplace et Lagrange contenaient tous les matériaux nécessaires pour faire un cours de mathématiques remarquable à l'orée du XIXème siècle. Un Ministre de l'Intérieur citera explicitement Laplace à ce sujet dans une circulaire destinée à toutes les écoles centrales de France.

"Préférez dans l'enseignement les méthodes générales, attachez vous à les présenter de la manière la plus simple, et vous verrez en même temps qu'elles sont presque toujours les plus faciles".

(27) Motion du 8 mai 1790 par laquelle "le roi est supplié" d'écrire à S.M. Britannique afin que des membres de la Royal Society s'associent à ceux de l'Académie des Sciences pour calculer la longueur du pendule simple qui bat la seconde au niveau de la mer à la latitude moyenne de 45 degrés. C'était l'idée "naturelle" de Huygens au XVIIème siècle, reprise par Turgot au XVIIIème. Cette motion ne parle pas du système décimal. La réponse anglaise fut un courtois silence.

(28) On est bien éloigné de la phrase magnifiquement impérialiste, de l'Auguste de la Chine, l'Empereur Jaune, au troisième siècle avant notre ère. "Un seul système de poids, de longueurs, d'essieux et de caractères d'écriture".

(29) Il reste à écrire, et les documents ne sont pas nombreux, une histoire des relations du courtisan Laplace et du souverain Napoléon. Cette histoire est plus entrelacée qu'on ne l'imagine souvent. On le fera dans un travail ultérieur.

Université de Nantes et Université d'Ottawa,
Conseiller scientifique, Ambassade de France au Canada, 42 Sussex, Ottawa, Ont. K1M 2C9.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

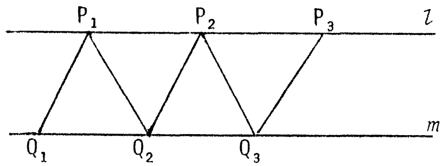
421. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Solve the following four decimal alphametics, which prove that *no* fences make good neighbors:

$$\begin{array}{rcrcrcr}
 & & & & \text{THE} & & \text{LES}\\
 & & \text{UNITED} & & \text{UNITED} & & \text{ETATS-}\\
\text{UNITED} & & \text{STATES} & & \text{STATES} & & \text{UNIS}\\
\underline{\text{STATES}}\,, & & \underline{\text{AND}}\,, & & \underline{\text{AND}}\,, & & \underline{\text{ET}}\,.\\
\text{CANADA} & & \text{CANADA} & & \text{CANADA} & & \text{CANADA}
\end{array}$$

422. *Proposed by Dan Pedoe, University of Minnesota.*

The lines l and m are the parallel edges of a strip of paper and P_1, Q_1 are points on l and m, respectively (see figure). Fold P_1Q_1 along l and crease, obtaining P_1Q_2 as the crease. Fold P_1Q_2 along m and crease, obtaining P_2Q_2. Fold P_2Q_2 along l and crease, obtaining P_2Q_3. If the process is continued indefinitely, show that the triangle $P_nP_{n+1}Q_{n+1}$ tends towards an equilateral triangle.



(I don't know the origin of this problem. A student asked me to prove it many years ago.)

423. *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y.*

In a triangle ABC whose circumcircle has unit diameter, let m_a and t_a denote the lengths of the median and the internal angle bisector to side a, respectively. Prove that

$$t_a \le \cos^2\frac{A}{2}\cos\frac{B-C}{2} \le m_a.$$

424*. *Proposed by J. Walter Lynch, Georgia Southern College, Statesboro, Georgia.*

Is it possible to make a convex object out of homogeneous material that will be at rest in exactly one position? (For example, a cube would be at rest in 6 positions, a hemisphere in two, a sphere in infinitely many.)

425. *Proposed by Gali Salvatore, Perkins, Québec.*

Let $x_1, x_2, \ldots, x_n$ be the zeros of the polynomial

$$P(x) = x^n + ax^{n-1} + a^{n-1}x + 1, \quad n \ge 3$$

and consider the sum

$$\sum_{k=1}^{n} \frac{x_k + 2}{x_k - 1}.$$

Find all values of a and n for which this sum is defined and equal to $n - 3$.

426. *Proposed by Charles W. Trigg, San Diego, California.*

There are two positive integers less than 10^{10} for each of which

i) its digits are all alike;

ii) its square has a digit sum of 37.

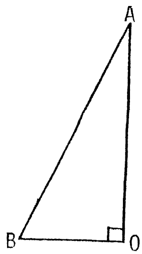
Find them and show that there are no others.

427. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A corridor of width a intersects a corridor of width b to form an "L". A rectangular plate is to be taken along one corridor, around the corner and along the other corridor with the plate being kept in a horizontal plane. Among all the plates for which this is possible, find those of maximum area.

428. *Proposed by J.A. Spencer, Magrath, Alberta.*

Let AOB be a right-angled triangle with legs $OA = 2OB$ (see figure). Use it to find an economical Euclidean construction of a regular pentagon whose side is not equal to any side of ΔAOB. "Economical" means here using the smallest possible number of Euclidean operations: setting a compass, striking an arc, drawing a line.



429. *Proposed by M.S. Klamkin and A. Liu, both from the University of Alberta.*

On a $2n \times 2n$ board we place $n \times 1$ polyominoes (each covering exactly n unit squares of the board) until no more $n \times 1$ polyominoes can be accommodated.

What is the maximum number of squares that can be left vacant?

This problem generalizes Crux 282 [1978: 114].

430. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) For $n = 1$, 8^{-n} equals a decimal fraction whose digits sum to 8. Prove that 8^{-n} for $n = 2,3,4,\ldots$ never again equals a decimal fraction whose digits sum to 8.

(b) The cube of 8 has decimal digits that sum to 8. For $n = 4,5,6,\ldots$, is there another 8^n whose decimal digits sum to 8?

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

355. [1978: 160] *Proposed by James Gary Propp, Great Neck, N.Y.*

Given a finite sequence $A = (a_n)$ of positive integers, we define the family of sequences

$$A_0 = A; \qquad A_i = (b_r), \quad i = 1,2,3,\ldots ,$$

where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}.

For example, if $A = A_0 = (2,4,2,2,4,5)$, then $A_1 = (3,2,1)$, $A_2 = (1,1,1)$, $A_3 = (3)$, and $A_4 = (1) = A_5 = A_6 = \ldots$.

The *degree* of a sequence A is the smallest i such that $A_i = (1)$.

(a) Prove that every sequence considered has a degree.

(b) Find an algorithm that will yield, for all integers $d \geq 2$, a shortest sequence of degree d.

(c) Let $\Lambda(d)$ be the length of the shortest sequence of degree d. Find a formula, recurrence relation, or asymptotic approximation for $\Lambda(d)$.

(d) Given sequences A and B, define C as the concatenation of A and B. Find sharp upper and lower bounds on the degree of C in terms of the degrees of A and B.

I. *Solution of part (a) by Leroy F. Meyers, The Ohio State University.*

For any considered finite sequence $A = A_0$, let λA_i and σA_i denote respectively the length and the sum of the terms of A_i. Then

$$\sigma A_{i+1} = \lambda A_i \leq \sigma A_i, \qquad i = 0,1,2,\ldots$$

with equality just when all terms in A_i are 1. Since a strictly decreasing sequence of positive integers must be finite, whereas the sequence (σA_i) is infinite, there must be a smallest nonnegative integer k such that $\sigma A_{k+1} = \sigma A_k$, and then

$$A_k = (1,1,\ldots,1), \quad A_{k+1} = (\lambda A_k), \quad \text{and} \quad A_{k+2} = (1).$$

Thus the degree of A is k+2, unless $\lambda A_k = 1$, in which case the degree of A is k.

II. *Solution of part (b) by Eugene Levine, Adelphi University, Garden City, N.Y.*

Any sequence considered can be put in *canonical form* as follows: first, we rename the terms so that 1 is the most frequent term, 2 is the second most frequent, and so on, where ties may be decided indifferently; second, we arrange the terms in decreasing order; and third, we tack an unending string of zeros at the end, with the stipulation that the length of such a sequence is the number of nonzero terms it contains. Thus

$$(2,4,2,2,4,5) \longrightarrow (1,2,1,1,2,3) \longrightarrow (3,2,2,1,1,1) \longrightarrow (3,2,2,1,1,1,\bar{0}).$$

Notice that the process preserves length and degree as long as the degree is not 1. From now on, all sequences will be assumed canonical.

Here is the desired algorithm. Suppose M' is a shortest canonical sequence of degree d-1 and let M be the unique canonical sequence such that $M_1 = M'$. Then M is the shortest canonical sequence of degree d. This procedure, applied repeatedly to

$$(1,1,\bar{0}) \text{ yields } (2,1,\bar{0}),\ (2,1,1,\bar{0}),\ (3,2,1,1,\bar{0}),\ (4,3,2,2,1,1,1,\bar{0}),\ldots .$$

We show that these are in fact shortest sequences of their respective degrees.

LEMMA. For $d \geq 2$, let $M = (m_n)$ be the canonical sequence of degree d derived by the above algorithm, and let $A = (a_n)$ be any canonical sequence of degree d. Then, for all positive integers n, we have $m_n \leq a_n$.

An immediate consequence of the inequalities $m_n \leq a_n$ is that $\lambda M \leq \lambda A$ [in the notation of solution I], and this validates our algorithm.

Proof. It is clear that $(1,1,\bar{0})$ is the shortest canonical sequence of degree 2, so the lemma is true for $d = 2$. We assume the lemma holds for $d = q - 1$ and show by contradiction that it must hold for $d = q$. Let j be the least integer such that $m_j > a_j$. If $j = 1$, then $m_1 > a_1$; but the first term of a canonical sequence is the number of distinct nonzero terms it contains, which equals the length of the next sequence in its family. Thus $M_1 = M'$ contains more nonzero terms than $A_1 = A'$, and there exists an n such that $m'_n > 0$ and $a'_n = 0$. But since M' and A' are of degree

q-1, our inductive hypothesis has been contradicted. If $j > 1$, then

$$a_j < m_j \le m_{j-1} \le a_{j-1} \le a_j + 1;$$

therefore $a_j = a_{j-1} - 1$ and $m_j = m_{j-1} = a_{j-1} = v$, say. Since $m_1 \ge m_2 \ge \ldots \ge m_j = v$, there are at least j terms of M that are greater than or equal to v, and so

$$m'_v + m'_{v+1} + m'_{v+2} + \ldots \ge j.$$

Also, since $a_j = v - 1$, there are at most j-1 terms of A that are greater than or equal to v, and so

$$a'_v + a'_{v+1} + a'_{v+2} + \ldots \le j - 1.$$

Combining these two inequalities, we get

$$m'_v + m'_{v+1} + m'_{v+2} + \ldots > a'_v + a'_{v+1} + a'_{v+2} + \ldots .$$

Yet by the inductive hypothesis $m'_n \le a'_n$ for all $n \ge v$. This contradiction completes the proof of the lemma.

III. *Comment on part (c) by the proposer.*

Here are the first few terms of Λ:

2, 2, 3, 4, 7, 14, 42, 213, 2837, 175450, 139759600,... .

Meyers also submitted a partial solution of part (b) and comments on parts (c) and (d).

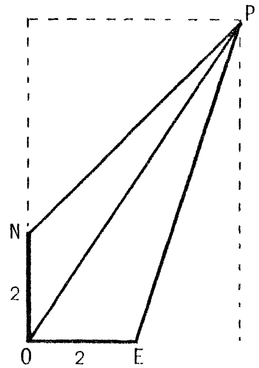
Editor's comment.

Parts (c) and (d) of this problem remain open. For a related problem by the same proposer, see *Mathematics Magazine*, 51 (May 1978) 194, Problem 1047.

* * *

356. [1978: 160] *Proposed by the late R. Robinson Rowe, Sacramento, California.*

Jogging daily to a landmark windmill P on the northeasterly horizon (see figure), Joe wondered how far it was. Directly (path OP), his time was 25 minutes; jogging first 2 miles due North (path ONP) took 30 minutes, and jogging first 2 miles due East (path OEP) took 35 minutes. How far was Joe's jog (path OP)?



Solution by Allan Wm. Johnson Jr., Washington, D.C.

We assume a (locally) flat earth and introduce a coordinate system with origin at O and positive x and y directions (scaled in miles) along OE and ON, respectively. Let the coordinates of P be (x,y). If Joe's constant jogging rate is r mi/min, then

$$OP = 25r, \qquad NP = 30r - 2, \qquad EP = 35r - 2.$$

The windmill is then located at the common intersection of the three circles

$$\begin{aligned} x^2 + y^2 &= (25r)^2, \\ x^2 + (y-2)^2 &= (30r-2)^2, \\ (x-2)^2 + y^2 &= (35r-2)^2. \end{aligned}$$

These equations are easily solved to yield

$$x = 5r(7 - 30r), \qquad y = \frac{5r(24 - 55r)}{4}, \tag{1}$$

where r has one of two values:

$$r_1 = \frac{936 - 56\sqrt{66}}{3485} \approx 0.13804 \text{ mi/min} \approx 8.2821 \text{ mi/hr}$$

or

$$r_2 = \frac{936 + 56\sqrt{66}}{3485} \approx 0.39912 \text{ mi/min} \approx 23.947 \text{ mi/hr}.$$

The corresponding values of OP are

$$25r_1 \approx 3.4509 \text{ mi} \qquad \text{and} \qquad 25r_2 \approx 9.9781 \text{ mi}. \tag{2}$$

From (1), the corresponding values of x and y are

$$x_1 \approx 1.97317, \qquad y_1 \approx 2.83112 \tag{3}$$

and

$$x_2 \approx -9.92563, \qquad y_2 \approx 1.02185. \tag{4}$$

It appears from (4) that P is in a *northwesterly* direction, which is not what the proposal says. On the other hand, while (3) shows that P is in a northeasterly direction, it also shows that E must be *further east* than P, which is not what the figure in the proposal shows. If the proposer is wrong about the figure, he might be wrong as well about the direction of the windmill, so I am inclined to let both of the answers (2) stand.

Also solved by P.R. BEESACK, Carleton University, Ottawa; CECILE M. COHEN, John F. Kennedy H.S., New York City; STEVE CURRAN for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; J.A.H. HUNTER, Toronto, Ontario; ROBERT S. JOHNSON, Montréal, Québec; N. KRISHNASWAMY, student, Indian

Institute of Technology, Kharagpur, India; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; G. RAMANAIAH, Madras Institute of Technology, India; BASIL C. RENNIE, James Cook University of North Queensland, Australia; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, New Mexico (three solutions); and the proposer. One incorrect solution was received.

Editor's comment.

Most solvers dismissed the second answer in (2) with the remark that a jogging speed of nearly 24 mi/hr is impossible for a human being. But the proposal does not specify that the jogger is human. Joe, for example, could be a horse. What's that you say? Horses don't jog? I was hoping you'd say that, dear reader. My dictionary gives, among the various meanings of *jog* [1]:

5. to cause (a horse) to go at a steady trot.
9. to run or ride at a steady trot.
12. a steady trot, as of a horse.

Who ever heard of a horse named Joe, you say? Well, if we can have a streetcar named Desire... . Stop, I know what you're going to say next: the proposal says that "Joe wondered how far it was", and can a horse do that? A horse might not only be able to *wonder* how far it has travelled, it might even be able to *calculate* the distance. For example, there was in the late 1800s a horse named *Muhamed* who could "extract cube roots in his head" [2], and nothing more complicated than square roots occurs in our problem.

This problem has an interesting history. It appeared recently, with different numbers, in [3], where it was said to have been suggested by J. Levitt, Ministry of Education, Toronto; and a solution by Charles W. Trigg, who solved it once more for us, was published in [4]. But it goes much further back. The information in the next paragraph was supplied by Johnson, our featured solver, and by Meyers.

The problem goes back at least to 1939, when it appeared, in a letter from D.R.C., Sacramento, California, to the editors of *Popular Science* [5]. Two months later [6], a letter from W.L.B., Chicago, Illinois, was published giving just the answer to the problem but no indication of a solution. Thirty-eight years later, in 1977, the same magazine published a letter about the problem, extracts of which are given below [7]:

> For more than 37 years I have tried, on and off, to solve a problem that appeared in *Popular Science*.
> All right, I give up. What's the answer?...
> It certainly looked easy, and I started to work on it. By 3 a.m. I

had filled a lot of sheets of paper, both sides, with notations...

...Over the years, I probably have shown the problem to more than 500 people...

Now I'd like to challenge *Popular Science*. If your readers of today cannot solve this little problem (and prove the answer) how about your finding the answer, if there is one, editors?

R.F. DAVIS, Sun City, Arizona.

The editors caved in. A few months later [8], they published an article by Darrell Huff entitled "The Incredible Tale of the 37-Year Puzzle", which included solutions to the problem by various methods.

REFERENCES

1. *The Random House Dictionary of the English Language*, The Unabridged Edition, Random House, New York, 1967, p. 769.

2. David Wallechinsky and Irving Wallace, *The People's Almanac*, Doubleday & Co., 1975, p. 698.

3. *Ontario Secondary School Mathematics Bulletin*, 14 (May 1978) 15.

4. ________, 14 (September 1978) 24.

5. *Popular Science*, September 1939, p. 14.

6. ________, November 1939.

7. ________, February 1977, pp. 6, 10.

8. ________, October 1977, pp. 26 ff.

* * *

357, [1978: 160] *Proposed by Leroy F. Meyers, The Ohio State University.*

In a certain multiple-choice test, one of the questions was illegible, but the choice of answers was clearly printed. Determine the true answer(s).

(a) All of the below.

(b) None of the below.

(c) All of the above.

(d) One of the above.

(e) None of the above.

(f) None of the above.

Solution by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.

The truth of	is then contradicted by	therefore
(a) or (f)	(e)	(a) and (f) are false
(c)	(b)	(c) is false
(b)	(d)	(b) is false
(d)	the falseness of (a), (b), (c)	(d) is false

So the only answer that can be true is (e), provided the truth of (e) leads to a consistent set of truth values and the falsity of (e) does not. This is in fact the case, so (e) is the only true answer.

Also solved by LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas; STEVE CURRAN for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; G.P. HENDERSON, Campbellcroft, Ontario; ROBERT S. JOHNSON, Montréal, Québec; the following students of JACK LeSAGE and ARVON KYER, Eastview Seconday School, Barrie, Ontario: KEVIN HANCOCK, MARK BRIGHAM, and JEFF CLARK; F.G.B. MASKELL, Collège Algonquin, Ottawa; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY PRIMER, student, Columbia H.S., Maplewood, N.J.; G. RAMANAIAH, Madras Institute of Technology, India; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One incorrect solution was received.

Editor's comment.

The proposer reports that he first saw this problem as a "problem of the week" put up by a graduate student in the mathematics lounge at Ohio State University about ten years ago.

* * *

358. [1978: 161] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum of x^2y, subject to the constraints

$$x+y+\sqrt{2x^2+2xy+3y^2} = k \text{ (constant)}, \qquad x,y \ge 0.$$

Solution by the proposer.

By the A.M.-G.M. inequality, we have

$$x+y = \frac{x}{2}+\frac{x}{2}+y \ge 3\left(\frac{x^2y}{4}\right)^{1/3},$$

with equality if and only if $x/2 = y$, or $x = 2y$; and similarly

$$\begin{aligned} 2x^2+2xy+3y^2 &= \frac{2x^2}{8}+\ldots+\frac{2x^2}{8}+\frac{2xy}{4}+\ldots+\frac{2xy}{4}+y^2+y^2+y^2 \\ &\ge 15\left\{\left(\frac{2x^2}{8}\right)^8\left(\frac{2xy}{4}\right)^4(y^2)^3\right\}^{1/15} \\ &= 15\left(\frac{x^2y}{4}\right)^{2/3}, \end{aligned}$$

with equality if and only if $2x^2/8 = 2xy/4 = y^2$, or $x = 2y$. Thus

$$k = x+y+\sqrt{2x^2+2xy+3y^2} \ge \sqrt[3]{x^2y}\left(\frac{3}{\sqrt[3]{4}}+\frac{\sqrt{15}}{\sqrt[3]{4}}\right)$$

and the required maximum, which is attained whenever $x = 2y$, is

$$\max x^2y = \frac{4k^3}{(3+\sqrt{15})^3}.$$

I had proposed this problem at a practice session for the 1977 U.S.A. Mathematical Olympiad Team.

Also solved by P.R. BEESACK, Carleton University, Ottawa; G.P. HENDERSON, Campbellcroft, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec (partial solution); N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India (partial solution); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and KENNETH M. WILKE, Topeka, Kansas.

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359. [1978: 161] *Proposed by Charles W. Trigg, San Diego, California.*

Construct a third-order additive magic square that contains three prime elements and has a magic constant of 37.

Solution by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.

It is well-known and easy to show that the central element of a third-order magic square is one-third the magic constant; so here it must be $37/3 = k$, say. Customarily, the elements of a magic square are distinct positive integers; but here the definition is obviously meant to be relaxed to allow positive fractions. We will relax it no further and assume that the answers we seek all contain nine distinct positive rational numbers, at least three of which (the three primes) are integers.

If no corner cell were integral, then two cells in the same horizontal or vertical line as the center would have to be integral; but this would not yield an integral sum for that line. Therefore some corner cell must contain an integer, say A. If we agree that a magic square is not changed by a rotation or a reflection, then we may assume that A is in the top left corner. Two cases will now be considered.

1. If a cell adjacent to A contains an integer, say B (we may assume it to be the right of A), then the remaining cells must be filled in as in Figure 1. Here, apart from A and B, only $3k-A-B$ can be integral; so the elements of the first row must be distinct odd primes. The positivity of the remaining elements now requires

$$2 < A,B < 2k < 25, \quad 12 < k < A+B < 3k-1 = 36, \quad 24 < 2k < 2A+B < 4k < 50.$$

Only ten triples $(A,\ B,\ 3k-A-B)$ of distinct primes (not counting symmetries) satisfy these conditions: they are listed in Table 1. These values, when substituted in Figure 1, yield ten satisfactory answers to our problem.

A	B	$3k - A - B$
$4k - 2A - B$	k	$2A + B - 2k$
$A + B - k$	$2k - B$	$2k - A$

Figure 1

A	B	$3k - A - B$
11	3	23
13	5	19
11	7	19
13	7	17
7	11	19
7	13	17
7	17	13
5	19	13
7	19	11
3	23	11

Table 1

2. If no cell adjacent to A is integral, it is easily seen that the only cells that *could* be integral are those a (chess) knight's move from A. Calling one of them C, the remaining cells must be filled in as in Figure 2. Here again,

A	$2k - C$	$k - A + C$
$2k - 2A + C$	k	$2A - C$
$k + A - C$	C	$2k - A$

Figure 2

C	A	$2A - C$
3	5	7
3	7	11
3	11	19
3	13	23
5	11	17
7	13	19
11	17	23

Table 2

only three cells can be integral: those in the triple $(C, A, 2A - C)$. We note that these numbers are in arithmetic progression. They must therefore be distinct odd primes, and positivity requires

$$2 < C,A,2A-C < 2k < 25.$$

We can assume $C<A$, and the solutions for which $C>A$ can then be obtained by a reflection in the principal diagonal. There are seven triples of primes in arithmetic progression in the specified range: these are listed in Table 2. These values, when substituted in Figure 2, yield seven additional answers to our problem.

To conclude, excluding symmetries our problem has exactly seventeen distinct solutions.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ALLAN Wm. JOHNSON Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH M. WILKE, Topeka, Kansas (two solutions); and the proposer. One incorrect solution was received.

Editor's comment.

No solver found all 17 distinct solutions. Our featured solver came closest with 16, and the editor was only too happy to supply the last one, which he cribbed from another solution.

* * *

360. [1978: 161] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Montrer directement (c'est-à-dire sans faire appel à un théorème plus général) que, pour $n=1,2,3,\ldots$, la mantisse de $\sqrt{n}$,

$$\{\sqrt{n}\} = \sqrt{n} - [\sqrt{n}],$$

est dense dans l'intervalle $(0,1)$.

Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.

If

$$0 < x < x+\varepsilon < 1, \tag{1}$$

we select any integer $m > 1/2\varepsilon$; then the interval from $(m+x)^2$ to $(m+x+\varepsilon)^2$ is of length $> 2\varepsilon m > 1$ and therefore contains an integer n satisfying

$$x < \sqrt{n} - m < x+\varepsilon.$$

It now follows from (1) that $m = [\sqrt{n}]$, and *voilà*!

Also solved by CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; F. DAVID HAMMER, Santa Cruz, California; LEROY F. MEYERS, The Ohio State University; JACQUES PROPP, Harvard College, Cambridge, Massachusetts; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; et par le proposeur.

Editor's comment.

The method used in our featured solution, with a slight modification too obvious to mention, would show that, for $n = 1,2,3,\ldots$, the mantissas of $\sqrt[q]{n}$ are dense in $(0,1)$ for any fixed integer $q > 1$. Satyanarayana proved this fact directly by another method.

* * *

361. [1978: 191] *Proposed by the late R. Robinson Rowe, Sacramento, California.*

Find MATH in the two-stage alphametic

$$\begin{array}{r} \mathrm{MH}\cdot\mathrm{M}\cdot\mathrm{AT/H} = \mathrm{MATH} \\ \underline{\mathrm{MATH}} \\ \mathrm{A}xxx \\ xxxxx \\ xxxxx \\ \underline{xxxxx} \\ xx\mathrm{MATH}xx \end{array}$$

in which the x's need not be distinct from M, A, T, or H.

Solution by Charles W. Trigg, San Diego, California.

MATH^2 has 8 digits, so $\mathrm{MATH} > 3162$. The first partial product establishes that (M,A,H) = (4,9,2) or (3,7,2). However, $42 \cdot 4 \cdot 9\mathrm{T}/2 > 49\mathrm{T}2$ for all T, and $32 \cdot 3 \cdot 7\mathrm{T}/2 = 37\mathrm{T}2$ only if T = 9; so MATH = 3792 and the unique reconstruction is

$$\begin{array}{r} 32\cdot 3\cdot 79/2 = 3792 \\ \underline{3792} \\ 7584 \\ 34128 \\ 26544 \\ \underline{11376} \\ 14379264 \end{array}$$

Not used in the solution, the symmetrical placing (even the occurrence) of MATH in the product is a serendipity.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ALLAN Wm. JOHNSON Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; HARRY L. NELSON, Livermore, California; HERMAN NYON, Paramaribo, Surinam; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Three solvers pointed out that

$$\mathrm{MATH} \times \mathrm{MATH} = xx\mathrm{MATH}xx \qquad (1)$$

has the unique solution MATH = 3792, so the additional conditions are redundant. But all three of them had to slug it out by computer. The proposer was aware of this. He wrote: "Although the problem could have been posed [as in (1)] with a unique solution, it would not have the logical type of solution applicable to the two-stage problem, as posed above."

* * *

362. [1978: 191] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

In Crux 247 [1977: 131; 1978: 23, 37] the following inequality is proved:

$$\frac{1}{2n^2}\,\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_n}\le\frac{a_1+\dots+a_n}{n}-\sqrt[n]{a_1\dots a_n}\le\frac{1}{2n^2}\,\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_1}.$$

Prove that the constant $1/2n^2$ is best possible.

Solution by Paul R. Beesack, Carleton University, Ottawa.

In Crux 247 [1978: 23], Basil C. Rennie proved the following more general "weighted" result: *if* $0<m\le t_r\le M$ *and* $w_r\ge 0$ *for* $r=1,2,\dots,n$ *with* $\Sigma w_r=1$, *while*

$$A=\Sigma w_r t_r,\qquad G=\Pi t_r^{w_r},\qquad S=\Sigma w_r t_r^2,$$

then

$$\frac{S-A^2}{2M}\le A-G\le\frac{S-A^2}{2m}. \tag{1}$$

As noted by Rennie, this includes Williams' inequality as a special case by taking all $w_r=1/n$; hence it suffices to prove here that the constants $1/2M$ and $1/2m$ in (1) are best possible.

Take any $a\in[m,M]$ and set

$$t_1=\dots=t_{n-1}=a,\qquad t_n=(1+t)a, \tag{2}$$

where $t\ge 0$ if $a=m$ and $t\le 0$ if $a=M$. Then we easily compute

$$A=a(1+w_nt),\qquad G=a(1+t)^{w_n},\qquad S=a^2\{1+w_n(t^2+2t)\},$$

and

$$S-A^2=w_n(1-w_n)a^2t^2.$$

Without loss of generality, we may assume that $0<w_n<1$; then, for $t\ne 0$,

$$\frac{A-G}{S-A^2}=\frac{1+w_nt-(1+t)^{w_n}}{aw_n(1-w_n)t^2}\equiv f(t).$$

By L'Hospital's Rule, applied twice, we find

$$\lim_{t\to 0} f(t) = \frac{1}{2a}. \tag{3}$$

On taking $a = M$ and letting $t \to 0^-$ in (3), we see that the constant $1/2M$ in (1) cannot be improved; and similarly on taking $a = m$ and letting $t \to 0^+$, it follows that the constant $1/2m$ in (1) is best possible.

Remark. Using the t_r given by (2), it is possible to show (with somewhat more work) that equality is attained in either place in (1) only if $t = 0$. This raises the question whether such equality can be attained in general only when all the t_r are equal.

Also solved by the proposer.

* * *

LULLABY FOR A MATHEMATICIAN'S CHILD[1]

Sleep, my little extension in time,
Sleep.
Covered with integrals
Among intervals
Sleep.

Father is hunting infinity
He will bring it home
For you and me. Someday.
Sleep.

Sleep, my little extension in time,
Sleep.
Eigenvalues clad in mantoids
Isomorphisms onto
They shall not harm you.
Sleep.

Tomorrow the tangential plane
Tomorrow the little subsets.
1,2,3,...one,two, three, infinity
And beware of Bourbaki.
Sleep.

ISOTTA CESARI

* * *

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