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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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THE OLYMPIAD CORNER

No. 131

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The problems for this month continue the problems proposed to the jury for the 32nd I.M.O., but not used at that event at Sigtuna, in Sweden. I hope you find them interesting and will send me your elegant solutions. Thanks go to Andy Liu, University of Alberta, and to Georg Gunther, Wilfred Grenfell College, Corner Brook, Newfoundland, for sending the problems to me.

13. *Proposed by Japan.*

For an acute triangle ABC, M is the midpoint of the segment BC, P is a point on the segment AM such that $PM = BM$, H is the foot of the perpendicular line from P to BC, Q is the point of intersection of the segment AB and the line passing through H that is perpendicular to PB, and finally R is the point of intersection of the segment AC and the line passing through H that is perpendicular to PC. Show that the circumcircle of ΔQHR is tangent to the side BC at the point H.

14. *Proposed by Spain.*

In the triangle ABC, with $\angle A = 60°$, a parallel IF to AC is drawn through the incenter I of the triangle, where F lies on the side AB. The point P on the side BC is such that $3BP = BC$. Show that $\angle BFP = \angle B/2$.

15. *Proposed by Poland.*

Let a, b, c be integers and p an odd prime number. Prove that if $f(x) = ax^2 + bx + c$ is a perfect square for $2p - 1$ consecutive integer values of x then p divides $b^2 - 4ac$.

16. *Proposed by Hong Kong.*

Find all positive integer solutions x, y, z of the equation $3^x + 4^y = 5^z$.

17. *Proposed by Bulgaria.*

Find the highest degree k of 1991 for which 1991^k divides the number

$$1990^{1991^{1992}} + 1992^{1991^{1990}}.$$

18. *Proposed by Ireland.*

Let α be the positive root of the equation $x^2 = 1991x + 1$. For natural numbers m and n define

$$m * n = mn + [\alpha m][\alpha n]$$

where $[x]$ is the greatest integer not exceeding x. Prove that for all natural numbers p, q and r,

$$(p * q) * r = p * (q * r).$$

19. *Proposed by the U.S.A.*

Real constants a, b, c are such that there is exactly one square all of whose vertices lie on the cubic curve $y = x^3 + ax^2 + bx + c$. Prove that the square has side $\sqrt[4]{72}$.

20. *Proposed by India.*

An odd integer $n \geq 3$ is said to be "nice" if and only if there is at least one permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that the n sums

$$\begin{gathered} a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n, \\ a_2 - a_3 + a_4 - \cdots - a_n + a_1, \\ a_3 - a_4 + a_5 - \cdots - a_1 + a_2, \\ \vdots \\ a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1} \end{gathered}$$

are all positive. Determine the set of all "nice" integers.

21. *Proposed by Czechoslovakia.*

Let $n \geq 2$ be a natural number and let the real numbers $p, a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n$ satisfy $1/2 \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$ $(i = 1, \ldots, n)$, and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Prove the inequality

$$\sum_{i=1}^n b_i \prod_{\substack{j=1 \\ j \neq i}}^n a_j \leq \frac{p}{(n-1)^{n-1}} .$$

22. *Proposed by Poland.*

Determine the maximum value of the sum

$$\sum_{i<j} x_i x_j (x_i + x_j)$$

over all n-tuples $(x_1, \ldots, x_n)$ satisfying $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$.

23. *Proposed by Finland.*

We call a set S on the real line R *super-invariant*, if for any stretching A of the set by the transformation taking x to $A(x) = x_0 + a(x - x_0)$ there exists a translation B, $B(x) = x + b$, such that the images of S under A and B agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$, and for any $t \in S$ there is an $u \in S$ such that $B(t) = A(u)$. Determine all super-invariant sets.

24. *Proposed by Bulgaria.*

Two students A and B are playing the following game. Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that the referee asks student A: "Can you tell the integer written by the other student?" If A answers "no", the referee puts the same question to the student B. If B answers "no", the referee puts the question back to A, etc. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer "yes".

* * *

We now turn to solutions to problems for the May 1991 number of *Crux.* I find no solutions on file for the 40th Mathematical Olympiad in Poland, so here is a good opportunity (and challenge) for our readers. The solutions we give are to the problems of the *1989 Swedish Mathematical Competition, Final Round* [1991: 130]. No solution has been received for the last problem of this contest.

1. Let n be a positive integer. Prove that the integers $n^2(n^2+2)^2$ and $n^4(n^2+2)^2$ can be written in base n^2+1 with the same digits but in opposite order.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Stewart Metchette, Culver City, California; Bob Prielipp, University of Wisconsin–Oshkosh; and by Chris Wildhagen, Rotterdam, The Netherlands.

Since

$$\begin{aligned} n^2(n^2+2)^2 &= n^2(n^4+4n^2+4) = n^6+4n^4+4n^2 \\ &= (n^6+3n^4+3n^2+1)+(n^4-1)+n^2 \\ &= 1(n^2+1)^3+0(n^2+1)^2+(n^2-1)(n^2+1)+n^2\cdot 1 \end{aligned}$$

and

$$\begin{aligned} n^4(n^2+2)^2 &= n^4(n^4+4n^2+4) = n^8+4n^6+4n^4 \\ &= (n^8+3n^6+3n^4+n^2)+(n^6+n^4-n^2-1)+1 \\ &= n^2(n^2+1)^3+(n^2-1)(n^2+1)^2+0(n^2+1)+1, \end{aligned}$$

the desired result holds.

2. Determine all continuous functions f such that $f(x)+f(x^2)=0$ for all real numbers x.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Chris Wildhagen, Rotterdam, The Netherlands.

Substituting $(-x)$ for x we have $f(-x)+f(x^2)=0$ so f is an even function. So it suffices to consider $x\geq 0$. Now $f(0)=0=f(1)$. Also, it easily follows by induction that $f(x)=(-1)^n f(x^{2^n})$. For $x\in(0,1)$ we then have that as $x\to\infty$, $x^{2^n}\to 0$ so that $\lim_{n\to\infty}(-1)^n f(x)=0$, by continuity, so that $f(x)=0$. For $y\in(1,\infty)$ we substitute $y=x^{2^n}$ to obtain $f(y)=(-1)^n f(y^{1/2^n})$. Now as $n\to\infty$ $y^{1/2^n}\to 1$ and it follows again that $f(y)=0$. Therefore $f(x)\equiv 0$.

3. Find all positive integers n such that $n^3-18n^2+115n-391$ is the cube of a positive integer.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Pavlos Maragoudakis, student, University of Athens, Greece; Stewart Metchette, Culver City, California; and by Chris Wildhagen, Rotterdam, The Netherlands.

Let $P(n) = n^3 - 18n^2 + 115n - 391$.

First, we prove that $P(n) < (n-5)^3$ for all $n \in N$.

$$\begin{aligned} P(n) < (n-5)^3 &\Leftrightarrow n^3 - 18n^2 + 115n - 391 < n^3 - 15n^2 + 75n - 125 \\ &\Leftrightarrow 3n^2 - 40n + 266 > 0 \end{aligned}$$

which is true for all n since the discriminant $(-40)^2 - 4 \cdot 3 \cdot 266 = 4(400 - 3 \cdot 266) < 0$.

Next, we find for which positive integers n it holds that $(n-7)^3 < P(n)$.

$$\begin{aligned} (n-7)^3 < P(n) &\Leftrightarrow n^3 - 21n + 147n - 343 < n^3 - 18n^2 + 115n - 391 \\ &\Leftrightarrow (3n+4)(n-12) = 3n^2 - 32n - 48 > 0 \\ &\Leftrightarrow n > 12. \end{aligned}$$

But $(n-6)^3$ is the only cube of a positive integer in $((n-7)^3, (n-5)^3)$, so if n is a solution with $n > 12$ then $P(n) = (n-6)^3$. But $P(n) = (n-6)^3 + 7(n-25)$. Thus for $n > 12$ the only solution is $n = 25$.

Now we need only check n with $0 < n \leq 12$. But for $0 < n \leq 10$, from $P(n) = (n-6)^3 + 7(n-25)$, it is easily seen that n is not a solution. This leaves

$$P(11) = 5^3 + 7(11-25) = 125 - 7 \cdot 14 = 3^3$$

and

$$P(12) = 6^3 + 7(12-25) = 216 - 7 \cdot 13 = 5^3.$$

Thus, all solutions are 11, 12, 25.

4. Let $ABCD$ be a regular tetrahedron. Where on the edge BD is the point P situated if the edge CD is tangent to the sphere with diameter AP?

Soluton by Seung-Jin Bang, Seoul, Republic of Korea.

We may assume

$$A = (0,-1,0), \quad B = (\sqrt{3},0,0), \quad C = (0,1,0), \quad D = (\sqrt{3}/3, 0, 2\sqrt{6}/3).$$

Since P lies on the edge BD, we have

$$P = (1-t)(\sqrt{3},0,0) + t(\sqrt{3}/3, 0, 2\sqrt{6}/3) = (\sqrt{3} - (2\sqrt{3}/3)t, 0, (2\sqrt{6}/3)t)$$

for some $t \in [0,1]$. The radius and center of the sphere with diameter AP are $(1/2)|A-P| = \sqrt{t^2 - t + 1}$ and $(\sqrt{3}/2 - (\sqrt{3}/3)t, -1/2, (\sqrt{6}/3)t)$, respectively. The line CD has direction $(a,b,c) = (\sqrt{3}/3, -1, 2\sqrt{6}/3)$ and passes through $C = (x_1, y_1, z_1) = (0,1,0)$. The square of the distance between a point $(x_0, y_0, z_0) = (\sqrt{3}/2 - (\sqrt{3}/3)t, -1/2, (\sqrt{6}/3)t)$ and the line CD is given by

$$\begin{aligned} &\left(\begin{vmatrix} y_0 - y_1 & z_0 - z_1 \\ b & c \end{vmatrix}^2 + \begin{vmatrix} z_0 - z_1 & x_0 - x_1 \\ c & a \end{vmatrix}^2 + \begin{vmatrix} x_0 - x_1 & y_0 - y_1 \\ a & b \end{vmatrix}^2 \right) \\ &= \frac{1}{4} \left(\begin{vmatrix} -\frac{3}{2} & \frac{\sqrt{6}}{3}t \\ -1 & \frac{2\sqrt{6}}{3} \end{vmatrix}^2 + \begin{vmatrix} \frac{\sqrt{6}}{3}t & \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3}t \\ \frac{2\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{vmatrix}^2 + \begin{vmatrix} \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3}t & -\frac{3}{2} \\ \frac{\sqrt{3}}{3} & -1 \end{vmatrix}^2 \right) \\ &= \frac{1}{4}(3t^2 - 8t + 8). \end{aligned}$$

(See S.M. Selby, Standard Mathematical Tables, The Chemical Rubber Company, 1973, p. 381, for example.) It follows that $t^2-t+1=(1/4)(3t^2-8t+8)$ and $t=2\sqrt{2}-2$. Now we obtain that the equation of the sphere with diameter AP is

$$\left(x-\frac{7\sqrt{3}}{6}+\frac{2\sqrt{6}}{3}\right)^2+\left(y+\frac{1}{2}\right)^2+\left(z-\frac{4\sqrt{3}}{3}+\frac{2\sqrt{6}}{3}\right)^2=(\sqrt{10}-\sqrt{5})^2$$

and the contact point between the sphere and the line is $(\sqrt{6}/6, 1-(\sqrt{2}/2), 2\sqrt{2}/3)$. Since $P=((7\sqrt{3}-4\sqrt{6})/3, 0, (8\sqrt{3}-4\sqrt{6})/3)$, we have $BP=4(\sqrt{2}-1)$ and $PD=2(\sqrt{2}-1)^2$. From this we get $PD/BP=(\sqrt{2}-1)/2$. The point should cut the edge in ratio $(\sqrt{2}-1)/2$.

5. Assume that $x_1,\ldots,x_5$ are positive real numbers such that $x_1<x_2$ and assume that x_3,x_4,x_5 are all greater than x_2. Prove that if $\alpha>0$, then

$$\frac{1}{(x_1+x_3)^\alpha}+\frac{1}{(x_2+x_4)^\alpha}+\frac{1}{(x_2+x_5)^\alpha}<\frac{1}{(x_1+x_2)^\alpha}+\frac{1}{(x_2+x_3)^\alpha}+\frac{1}{(x_4+x_5)^\alpha}\,.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Murray S. Klamkin, University of Alberta. We give Klamkin's solution and generalization.

Since $(x_1+x_2)^{-\alpha}-(x_1+x_3)^{-\alpha}$ is decreasing in x_1, it suffices to show that

$$(x_2+x_3)^{-\alpha}+(x_2+x_4)^{-\alpha}+(x_2+x_5)^{-\alpha}\le(2x_2)^{-\alpha}+(x_2+x_3)^{-\alpha}+(x_4+x_5)^{-\alpha}.$$

Since $(x_4+x_5)^{-\alpha}-(x_2+x_5)^{-\alpha}$ is increasing in x_5, it now suffices to show that

$$(x_2+x_3)^{-\alpha}+(x_2+x_4)^{-\alpha}+(2x_2)^{-\alpha}\le(2x_2)^{-\alpha}+(x_2+x_3)^{-\alpha}+(x_4+x_2)^{-\alpha},$$

and this is valid, completing the proof.

More generally we show that

$$F(x_1+x_3)+F(x_2+x_4)+F(x_2+x_5)<F(x_1+x_2)+F(x_2+x_3)+F(x_4+x_5)$$

where F is a strictly convex function over positive real numbers. One can show that, for the six permutations $x_5\ge x_4\ge x_3$, $x_5\ge x_3\ge x_4$, etc., the vector $(x_1+x_2, x_2+x_3, x_4+x_5)$, when its components are arranged in decreasing order, majorizes the vector $(x_1+x_3, x_2+x_4, x_2+x_5)$ when its components are arranged in decreasing order. Consequently, inequality (1) follows by the majorization inequality [1989: 36] or [1].

Reference:

[1] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

*

Problem 3 of the 1991 Asian Pacific Mathematics Competition generated solutions from four readers, and I will depart from normal practice and give two of them.

3. [1991: 131] *1991 Asian Pacific Mathematics Olympiad.*
Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be positive real numbers such that

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k.$$

Show that

$$\sum_{k=1}^{n} \frac{(a_k)^2}{a_k + b_k} \geq \frac{1}{2} \sum_{k=1}^{n} a_k.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Murray S. Klamkin, University of Alberta; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Klamkin's solution which starts from the weaker hypothesis $\sum a_k \geq \sum b_k$.

Applying the Cauchy inequality to $c_k = a_k/\sqrt{a_k + b_k}$ and $d_k = \sqrt{a_k + b_k}$ we obtain

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + b_k} \cdot \sum_{k=1}^{n} (a_k + b_k) \geq \left(\sum_{k=1}^{n} a_k \right)^2$$

with strict inequality unless for some λ, $a_k = \lambda(a_k + b_k)$ giving $b_k = ((1-\lambda)/\lambda)a_k$, for all k. It suffices for the inequality to show that

$$\left(\sum_{k=1}^{n} a_k \right)^2 \geq \frac{1}{2} \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} (a_k + b_k)$$

or that $\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} b_k$.

It follows that equality in the original inequality implies that $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$ and that $b_k = \mu a_k$ for $\mu = (1-\lambda)/\lambda$, above. But then $\mu = 1$ and there is equality if and only if $a_k = b_k$ for all k.

[*Editor's note.* The solutions of Bang and of Wang avoided use of the Cauchy inequality. Here is Wang's argument.]

First note that

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + b_k} = \sum_{k=1}^{n} \frac{b_k^2}{a_k + b_k} \quad \text{since} \quad \sum_{k=1}^{n} \frac{a_k^2 - b_k^2}{a_k + b_k} = \sum_{k=1}^{n} (a_k - b_k) = 0.$$

Thus

$$4\sum_{k=1}^{n} \frac{a_k^2}{a_k + b_k} - 2\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{2a_k^2 + 2b_k^2}{a_k + b_k} - (a_k + b_k) \right) = \sum_{k=1}^{n} \frac{(a_k - b_k)^2}{a_k + b_k} \geq 0.$$

The inequality follows immediately, and is an equality if and only if $a_k = b_k$ for each k.

* * *

We next turn to the June number of the Corner, and solutions from our readers to problems of the *15th All Union Mathematical Olympiad – Tenth Grade* [1991: 163–164].

1. Find natural numbers $a_1 < a_2 < \ldots < a_{2n+1}$ which form an arithmetic sequence such that the product of all terms is the square of a natural number.

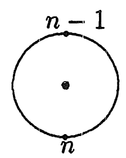
Solutions by George Evagelopoulos, Athens, Greece; and by Pavlos Maragoudakis, student, University of Athens, Greece.

The numbers $w, 2w, \ldots, (2n+1)w$ form an arithmetic sequence for any $w \in \mathbf{N}$. Now $p = w \cdot 2w \cdot \ldots \cdot (2n+1)w = (2n+1)!w^{2n+1}$. If we now choose $w = (2n)!$ then

$$p = ([(2n+1)!]^{n+1})^2 .$$

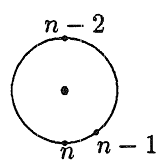
2. The numbers $1, 2, \ldots n$ are written in some order around the circumference of a circle. Adjacent numbers may be interchanged provided the absolute value of their difference is larger than one. Prove in a finite number of such interchanges it is possible to rearrange the numbers in their natural order.

Solutions by George Evagelopoulos, Athens, Greece; and by Pavlos Maragoudakis, student, University of Athens, Greece. We give Maragoudakis' solution.



From $n-1$ we can reach n in two ways. By moving $n-1$ (if $n-2$ does not lie on the arc from n to $n-1$ moving in the natural order (either clockwise or counterclockwise)) or by moving n otherwise, in a finite number of steps we can have n and $n-1$ adjacent and in the "natural" order.

In the new picture we now distinguish two cases:



A) $n-3$ does not lie on the arc between $n-2$ and n which contains $n-1$. In this case it is easy to move $n-2$ next to $n-1$ so that $n-2$, $n-1$ and n are touching.

B) $n-3$ lies on the arc between n and $n-2$ which contains $n-1$. In this case we move $n-1$ along this arc until it is the neighbour of $n-2$, and then move n adjacent to $n-1$.

Notice that in either case we preserve the natural order (clockwise or counterclockwise) established in the first step. By repeating this procedure a finite number of times we rearrange the numbers in natural order (clockwise, or counterclockwise).

3. Let ABC be a right triangle with right angle at C and select points D and E on sides AC and BC, respectively. Construct perpendiculars from C to each of DE, EA, AB, and BD. Prove that the feet of these perpendiculars are on a single circle.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; by George Evagelopoulos, Athens, Greece; and by D.J. Smeenk, Zaltbommel, The Netherlands.

Let $\alpha = \angle CED$, $\beta = \angle CBD$, and $\gamma = \angle DEA$. Also, let P, Q, R, S be the feet of the perpendiculars from C to AB, AE, ED and BD, respectively. The four quadrilaterals $CQPA$, $BPSC$, $CRSD$, and $CRQE$ are then cyclic with diameters AC, BC, CD, and CE, respectively. So, we have:

In $CQPA$: $\angle CQP = 180° - A$; $\angle AQP = \angle ACP = 90° - A = B$; $\angle QPA = 180° - \angle QCA$; $\angle QPC = \angle QAC = 90° - \angle QCA = 90° - (\alpha+\gamma)$, since $\angle QCA = \angle CEA$, both being the complement of $\angle EAC$.

In $BPSC$: $\angle BPS = 180° - \angle BCS$; $\angle CPS = 90° - \angle BCS = \beta$; $\angle PSC = 180° - B$; and $\angle PSB = 90° - B = A$.

In $CRSD$: $\angle RSD = 180° - \angle RCD = 180° - \alpha$; $\angle RSC = \angle RDC = 90° - \alpha$; $\angle CRS = 90° + \angle DRS = 90° + \beta$; $\angle DRS = \beta$.

In $CRQE$: $\angle CRQ = 180° - (\alpha+\gamma)$; $\angle ERQ = 90° - (\alpha+\gamma)$.
Then

$$\angle QRS = 180° - (\angle ERQ + \angle DRS) = 180° - (90° - (\alpha+\gamma) + \beta) = 90° + \alpha + \gamma - \beta$$

and

$$\angle QPS = \angle QPC + \angle CPS = 90° - (\alpha+\gamma) + \beta = 180° - \angle QRS.$$

Thus $PQRS$ is cyclic, as claimed.

4. Let $a \geq 0$, $b \geq 0$, $c \geq 0$ and $a+b+c \leq 3$. Prove

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3}{2} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\,.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by George Evagelopoulos, Athens, Greece; and by Pavlos Maragoudakis, student, University of Athens, Greece.

For $x \in R$, $x/(1+x^2) \leq 1/2$ since $0 \leq (x-1)^2/(2(1+x^2))$. So

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}\,.$$

We also have

$$\begin{aligned} 3^2 &\leq \left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right)(1+a+1+b+1+c) \\ &\leq \left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right)(3+3) = 6. \end{aligned}$$

Thus

$$\frac{3}{2} = \frac{9}{6} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$$

and the result follows.

5. Prove, for any real number c, that the equation

$$x(x^2-1)(x^2-10) = c$$

cannot have five integer solutions.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by George Evagelopoulos, Athens, Greece; and by Pavlos Maragoudakis, student, University of Athens, Greece. We give Bang's solution.

Let x_1, x_2, x_3, x_4, x_5 be all integer solutions of the equation $x^5-11x^3+10x-c=0$. From the relation between roots and coefficients we obtain

$$x_1+x_2+x_3+x_4+x_5=0,$$
$$\sum_{i<j} x_i x_j=-11,$$

and

$$x_1x_2x_3x_4x_5=c.$$

Now

$$\begin{aligned}(x_1+x_2+x_3+x_4+x_5)^2 &= x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+2\sum_{i<j} x_i x_j \\ &= x_1^2+x_2^2+x_3^2+x_4^2+x_5^2-22.\end{aligned}$$

This yields $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=22$. If $c=0$ then all integer solutions are $x=0,\pm1$. Suppose $c\neq 0$. Then $x_i\neq 0,\pm1$ and

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq 5\cdot 4=20.$$

Moreover, if $|x_i|\geq 3$ for some i, then $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2>22$. It follows that there cannot be five integer solutions for $x(x^2-1)(x^2-10)=c$.

6. A rook is placed on the lower left square of a chessboard. On each move, the rook is permitted to jump one square either vertically or horizontally. Prove the rook may be moved so that it is on one square once, one square twice, ..., one square 64 times, and

(a) so that the last move is to the lower left square,

(b) so that the last move is to a square adjacent (along an edge) to the lower left square.

Solutions by George Evagelopoulos, Athens, Greece; and by Pavlos Maragoudakis, student, University of Athens, Greece. We give the solution of Evagelopoulos.

(a) It is impossible! To come back to the same square, we need an even number of horizontal moves and an even number of vertical moves, namely the total number of moves has to be an even number! The number of visits to all squares of the chessboard is $1+2+3+\cdots+64=(64\cdot 65)/2=2080$. At the start the rook is already placed on the square a_1 (which means that it has already made a visit). Thus the remaining 2079 visits have to be made with an even number of moves, an impossibility!

(b) It is possible! Number the squares of the chessboard with numbers 1–64, as in the figure.

60	59	52	51	44	43	16	15
61	58	53	50	45	42	17	14
62	57	54	49	46	41	18	13
63	56	55	48	47	40	19	12
64	35	36	37	38	39	20	11
33	34	29	28	25	24	21	10
32	31	30	27	26	23	22	9
1	2	3	4	5	6	7	8

Starting at the square numbered k, the rook may be moved to the square numbered m $(m \geq k)$ by the sequence $k, k+1, \ldots, m-1, m$. This is symbolized $k \to m$. The opposite move is denoted $m \to k$.

The desired run, which starts out at the square numbered 1 and ends up at the square numbered 32 is as follows:

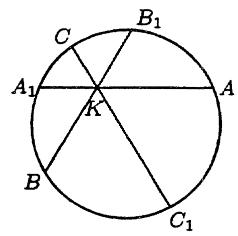
$$1 \to 64 \to 1 \to 63 \to 2 \to 62 \to 3 \to 61 \to 4 \to \cdots \to 30 \to 34 \to 34 \to 31 \to 33 \to 32.$$

It is obvious that the rook will visit

(i) the square numbered k, $2k$ times, for $1 \leq k \leq 32$, and

(ii) the squares numbered with m, $(129 - 2m)$ times for $33 \leq m \leq 64$.

7. Three chords, AA_1, BB_1, CC_1 to a circle meet at a point K where the angles B_1KA and AKC_1 are 60°, as shown. Prove that



$$KA+KB+KC = KA_1+KB_1+KC_1.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; by George Evagelopoulos, Athens, Greece; and by D.J. Smeenk, Zaltbommel, The Netherlands.

Let O be the center of the circle. All angles with vertex K are equal to 60°. Then, if we denote the angle $\angle AKO$ by θ, we have $\angle BKO = 120° - \theta$, and $\angle CKO = 120° + \theta$, and

$$KA = \frac{AA_1}{2} + KO \cdot \cos\theta; \quad KA_1 = \frac{AA_1}{2} - KO\cos\theta;$$
$$KB = \frac{BB_1}{2} + KO\cos(120° - \theta); \quad KB_1 = \frac{BB_1}{2} - KO\cos(120° - \theta);$$
$$KC = \frac{CC_1}{2} + KO\cos(120° + \theta); \quad KC_1 = \frac{CC_1}{2} - KO\cos(120° + \theta).$$

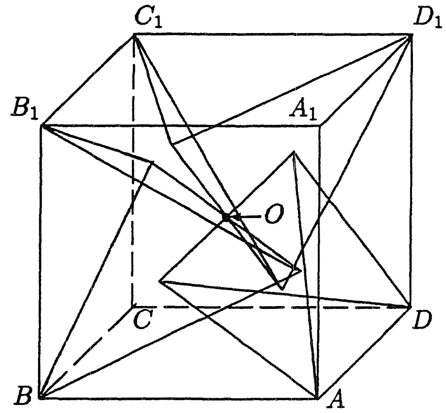
The result follows since

$$\cos\theta + \cos(120° - \theta) + \cos(120° + \theta) = 0.$$

Remark. It is curious that a related problem, *Crux* 1658 (see [1992: 181]), appears in the same issue.

8. Is it possible to put three regular tetrahedra, each having sides of length one, inside a unit cube so that the interiors of the tetrahedra do not intersect (the boundaries are allowed to touch)?

Solution by George Evagelopoulos, Athens, Greece.



Yes, it is possible as indicated in the figure. The edges AD, BB_1, and C_1D_1 of the cube are also edges of the tetrahedra and the opposite lying edges of the tetrahedra pass through the center O of the cube and are bisected by it.

* * *

That is all for this number. Send me your nice solutions as well as your contests.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

World Compendium of Mathematics Competitions, edited by Peter O'Halloran, University of Canberra. Published by the Australian Mathematics Foundation, 1992, ISBN 0-646-09564-1, softcover, 242 pages, US$10.00. *Reviewed by Murray Klamkin, University of Alberta.*

In 1968, the N.C.T.M. published a 41-page report by H. L. Grover [2] on various aspects of a number of high school mathematics contests being held in the United States. In [1], Hans Freudenthal treated quite a number of high school competitions throughout the world, as well as a few intercollegiate ones. He included some problems from the Israeli Olympiads and a fair number from the Dutch Olympiads. Both of these reports also contain a large number of pertinent references to the competition and problem literature. In [3], there are more references and information on competitions and sources of problems. For still more of the latter, see the highly recommended book edited by Stan Rabinowitz [4], which will soon be reviewed here.

In the present volume, Professor O'Halloran gives brief summaries of 17 international, 110 national and 104 regional competitions of many countries around the world. A number of these are on the collegiate level. Each of these summaries is given attractively, one to a page, in black, green and white colours. They are organized as follows:

(1) Contests are first categorized as International, National or Regional.

(2) Within these categories, contests are partitioned into Senior (16 years or older), Junior/Senior (all ages) and Junior (15 years or younger).

(3) Within these groupings, contests are sequenced by country in alphabetical order.

(4) Within each country grouping, the contests are in alphabetical order.

As a condensed sample of one of the page summaries, we have the following:

NATIONAL JUNIOR/SENIOR

MEXICO (in Spanish)

OLIMPIADAS DE MATEMATICUS

Information correct as of 1989.

The competition aims to identify Mexican Olympiad team members and promote mathematics in Mexico. There are three steps in this competition: state, national and the training/selection process of the Mexican team. In 1989, Mexico had its third Mexican Mathematical Olympiad. It has participated in the XXIXth I.M.O. and the Ibero-American Mathematical Olympiad.

Age of participants:	15–19 years.
School level of participants:	High school.
Number of entries:	3500.
Contact address:	Carlos Bosch-Giral, Depto. Matematicas, ITAM, Rio Hondo 1, TIZAPAN 01000, MEXICO D. F., MEXICO.

To help locate a particular contest, there is an index at the end giving the countries and continents in alphabetical order. Under each heading, the contests given therein are also in alphabetical order, with their page location numbers. For instance, if one looks up Canada, USA and Canada/USA, one finds a listing of 15, 21 and 2 contests respectively.

References:

[1] H. Freudenthal, ICMI Report on Mathematical Contests in Secondary Education (I), *Educ. Studies Math.* **2** (1969) 80–114.

[2] H. L. Grover, *School Mathematics Contests*, NCTM, Reston, VA, 1968.

[3] M. S. Klamkin, Olympiad Corners 3, 4 and 8, *Crux Mathematicorum* **5** (1979) 62–69, 102–107 and 220–227.

[4] S. Rabinowitz, *Index to Mathematical Problems 1980–1984*, MathPro Press, Westford, MA, 1992.

*   *   *   *   *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1771*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle and u, v, w be non-negative real numbers such that $u + v + w = 1$. Prove that

$$\sum ubc - s\sum vwa \geq 3Rr,$$

where s, R, r are the semiperimeter, circumradius and inradius of the triangle, and the sums are cyclic.

1772. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

The equation $x^3 + ax^2 + (a^2 - 6)x + (8 - a^2) = 0$ has only positive roots. Find all possible values of a.

1773. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle inscribed in a circle with radius R. Let L, M and N be the midpoints of the arcs AB, BC and CA, not containing C, A and B, respectively. Let E and F be the feet of the perpendiculars from M to AB and AC, respectively. Suppose that $\overline{AB} \neq \overline{AC}$ and $\overline{LE} = \overline{NF}$. Prove that $\overline{NF} = R$.

1774*. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the smallest $\lambda \geq 0$ such that

$$2(x^3 + y^3 + z^3) + 3xyz \geq (x^\lambda + y^\lambda + z^\lambda)(x^{3-\lambda} + y^{3-\lambda} + z^{3-\lambda})$$

for all non-negative x, y, z.

1775. *Proposed by P. Penning, Delft, The Netherlands.*

Find the radius of the smallest sphere (in three-dimensional space) which is tangent to the three lines $y = 1, z = -1$; $z = 1, x = -1$; $x = 1, y = -1$; and whose centre does *not* lie on the line $x = y = z$.

1776. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Given $0 < x_0 < 1$, the sequence $x_0, x_1, \ldots$ is defined by

$$x_{n+1} = \frac{3}{4} - \frac{3}{2}\left|x_n - \frac{1}{2}\right|$$

for $n \geq 0$. It is easy to see that $0 < x_n < 1$ for all n. Find the smallest closed interval J in [0,1] so that $x_n \in J$ for all sufficiently large n.

1777. *Proposed by Guy N. Bloodsucker, Stirling Pound, Banffshire, Scotland.*

Find all positive integer solutions of $x^2 + y^2 = n!$.

1778. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Find all real numbers $x_1, x_2, \ldots, x_{1778}$ satisfying the system of equations

$$3 + 2x_{i+1} = 3|x_i - 1| - |x_i|, \quad i = 1, 2, \ldots, 1778 \quad (x_{1779} = x_1).$$

1779. *Proposed by Christopher Bradley, Clifton College, Bristol, U.K.*

Two circles C_1 and C_2 are given with the centre A of circle C_1 lying on C_2. BC is the common chord. The chord AD of C_2 meets BC at E. From D lines DF and DG are drawn tangent to C_1 at F and G. Prove that E, F, G are collinear.

1780. *Proposed by Jordan Stoyanov, Queen's University, Kingston, Ontario.*

Prove that, for any natural number n and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$(1 - \sin^2\alpha_1 \sin^2\alpha_2 \ldots \sin^2\alpha_n)^n + (1 - \cos^2\alpha_1 \cos^2\alpha_2 \ldots \cos^2\alpha_n)^n \geq 1.$$

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1357*. [1988: 175; 1989: 243] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Isosceles right triangles $AA'B$, $BB'C$, $CC'A$ are constructed outwardly on the sides of a triangle ABC, with the right angles at A', B', C', and triangle $A'B'C'$ is drawn. Prove or disprove that

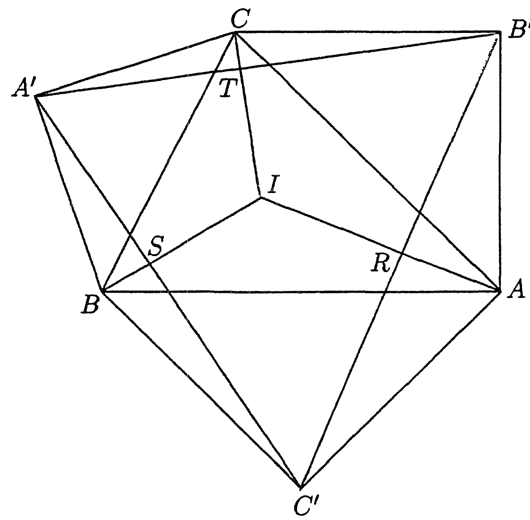
$$\sin A' + \sin B' + \sin C' \geq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2},$$

where A', B', C' are the angles of $\Delta A'B'C'$.

II. *Partial solution by Johannes Waldmann, student, University of York, Great Britain.*

Change notation as in the diagram, and let $\alpha = \angle CAB$, $\alpha' = \angle C'A'B'$, and so forth. Then the problem is to prove or disprove that

$$\sum \sin \alpha' \geq \sum \cos \frac{\alpha}{2} \,. \tag{1}$$



Here we show that (1) is true at least in case ABC is non-obtuse.

We have

$$\cos \frac{\alpha}{2} = \cos\left(\frac{\pi}{2} - \frac{\beta+\gamma}{2}\right) = \sin\left(\frac{\beta+\gamma}{2}\right) ,$$

so (1) is in fact

$$\sum \sin \alpha' \geq \sum \sin\left(\frac{\beta+\gamma}{2}\right) .$$

Since $\sum \alpha' = \pi$ and also $\sum(\beta+\gamma)/2 = \pi$, and since $\sum \sin\varphi$ is Schur-concave (see Chapter 3 of Marshall and Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, 1979, especially Proposition C.1), we only need to show that

$$(\alpha', \beta', \gamma') \prec \left(\frac{\beta+\gamma}{2}, \frac{\gamma+\alpha}{2}, \frac{\alpha+\beta}{2}\right) . \tag{2}$$

From now on, let $\alpha \leq \beta \leq \gamma \leq \pi/2$. Let a denote $\overline{BC}$, and a' denote $\overline{B'C'}$, and so forth. Thus $a \leq b \leq c$. We write F for the area of ABC. With $\cos\angle B'AC' = \cos(\pi/2 + \alpha) = -\sin\alpha$ and $2F = bc\sin\alpha$ we obtain $a'^2 = b^2/2 + c^2/2 + 2F$ from the law of cosines in $B'AC'$. Rephrasing this as

$$a'^2 = \frac{a^2+b^2+c^2}{2} + 2F - \frac{a^2}{2}$$

shows that $c' \leq b' \leq a'$ and so $\gamma' \leq \beta' \leq \alpha'$. To prove (2) it is therefore sufficient to show

$$\alpha' \leq \frac{\beta+\gamma}{2} \quad \text{and} \quad \gamma' \geq \frac{\alpha+\beta}{2} , \tag{3}$$

since obviously

$$\frac{\alpha+\beta}{2} \leq \frac{\alpha+\gamma}{2} \leq \frac{\beta+\gamma}{2} .$$

Let I be the incentre of ABC. Denote the intersection of IA and $B'C'$ by R, and let S and T be defined analogously. Now consider the angles of the quadrilateral $A'TIS$. Note that

$$\angle TIS = \angle CIB = \pi - \frac{\beta+\gamma}{2}$$

and $\angle SA'T = \alpha'$. Further we have

$$\angle ISA' = \angle BSC' \geq \pi/2,$$

since $BC' \geq BA'$ because of $c \geq a$ and BI is the angle-bisector of $\angle A'BC'$. Likewise, $\angle ITA' \geq \pi/2$. When applying these two inequalities in computing the sum of the angles of $A'TIS$, we get

$$\alpha' + \frac{\pi}{2} + \frac{\pi}{2} + \left(\pi - \frac{\beta+\gamma}{2}\right) \leq 2\pi,$$

and the left hand side of (3) is established. The right hand side can be proved by similar considerations in the quadrilateral $C'RIS$.

So (1) has been proved for non-obtuse triangles ABC. In case ABC is obtuse ($\gamma > \pi/2$), one is still able to show the right hand side of (3), but for the left hand side of this statement the argument given above fails. This is due to the fact that the point S is situated outside the triangle. However it seems very sound to me that (3) and therefore (1) still hold.

* * * * *

1576. [1990: 240; 1991: 285] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

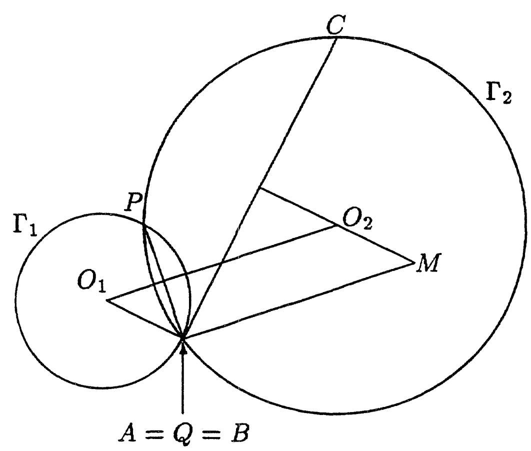
Circles Γ_1 and Γ_2 have a common chord PQ. A is a variable point of Γ_1. AP and AQ intersect Γ_2 for the second time in B and C respectively. Show that the circumcentre of ΔABC lies on a fixed circle. (This problem is not new. A reference will be given when the solution is published.)

III. *Comment by J. Chris Fisher, Clemson University, Clemson, South Carolina.*

Here is an answer to the editor's challenge [1991: 286] following the two solutions to this problem, namely to use solution I to deduce that the locus of the circumcenter M of ΔABC has the same radius as Γ_1. It's easy! Just use a special position of the roving point A; for example, let $A = Q$:

Rule to locate B: join A ($= Q$) to P; B is where AP meets Γ_2 again — namely Q.

Rule to locate C: join A to Q; here $A = Q$ so the resulting line QC is the tangent to Γ_1 at Q. What is important is that $O_1Q \perp AC$ ($= QC$).



As $A \to Q$ the circumcircle of ABC approaches the circle tangent to PQ at Q and containing C. Its center M is where the perpendicular bisector of QC, which passes through O_2, meets the perpendicular to PQ at Q. Since $O_1O_2 \perp PQ$ (PQ is the common chord), it follows that O_2O_1QM is a parallelogram, so that $O_2M = O_1Q =$ the radius of Γ_1, as required.

Two proofs of this result were also sent in by K.R.S. Sastry, Addis Ababa, Ethiopia.

* * * * *

1672. [1991: 237] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that for positive real numbers a, b, c, x, y, z,

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq 3\left(\frac{xy+yz+zx}{x+y+z}\right),$$

and determine when equality holds.

I. *Solution by Murray S. Klamkin, University of Alberta.*

We change notation by letting $(a,b,c) \Rightarrow (u,v,w)$. Then by letting

$$y+z=a^2, \quad z+x=b^2, \quad x+y=c^2,$$

we get $x+y+z=(a^2+b^2+c^2)/2$ and

$$\begin{aligned} xy+yz+zx &= \frac{x(y+z)+y(z+x)+z(x+y)}{2} = \frac{1}{2}\sum\left(\frac{b^2+c^2-a^2}{2}\right)a^2 \\ &= \frac{1}{4}(a^2+b^2+c^2)^2-\frac{1}{2}(a^4+b^4+c^4)=4F^2, \end{aligned}$$

where F is the area of an acute triangle of sides a, b, c (for the last equality see [1986: 252]). Thus the inequality can be rewritten as

$$\frac{ua^2}{v+w}+\frac{vb^2}{w+u}+\frac{wc^2}{u+v} \geq \frac{24F^2}{a^2+b^2+c^2} \tag{1}$$

where a, b, c are sides of an acute triangle of area F.

In my generalization of *Crux* 1051 [1986: 252], I had shown that

$$\frac{ua^{4n}}{v+w}+\frac{vb^{4n}}{w+u}+\frac{wc^{4n}}{u+v} \geq 8F_n^2, \quad 0 \leq n \leq 1, \tag{2}$$

where F_n is the area of a triangle with sides a^n, b^n, c^n. We now show that (2) for $n=1/2$ is a stronger inequality than (1), i.e.,

$$8F_{1/2}^2 \geq \frac{24F^2}{a^2+b^2+c^2}.$$

This inequality follows immediately from the two known inequalities $a^2+b^2+c^2 \geq 4F\sqrt{3}$ ([1], item 4.4) and

$$4F_{1/2}^2 \geq F\sqrt{3} \tag{3}$$

([1], item 10.3). There is equality if and only if $a=b=c$ and $u=v=w$.

More generally, using the Oppenheim generalization

$$\sqrt[n]{4F_n/\sqrt{3}} \geq 4F/\sqrt{3}$$

([2], p. 104; see also [1988: 116]) of the Finsler–Hadwiger inequality (3), we have from (2) that

$$\frac{ua^{4n}}{v+w}+\frac{vb^{4n}}{w+u}+\frac{wc^{4n}}{u+v}\geq\frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^{2n}\geq\frac{3}{2}\left(\frac{16F^2}{a^2+b^2+c^2}\right)^{2n} \tag{4}$$

for $0\leq n\leq 1$. By specializing the values of u,v,w and a,b,c, we can obtain a number of known inequalities and some apparently new ones, e.g., $a=b=c$ gives

$$\frac{u}{v+w}+\frac{v}{w+u}+\frac{w}{u+v}\geq\frac{3}{2}$$

(also noted on [1986: 253]), and $u=v=w$ gives

$$a^{4n}+b^{4n}+c^{4n}\geq 3\left(\frac{16F^2}{3}\right)^n. \tag{5}$$

References:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

[2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic, Dordrecht, 1989.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

The following two inequalities may deserve some interest for their own sake. Let n be an integer greater than 1.

PROPOSITION 1. *For any nonnegative numbers $a_1,\ldots,a_n$, $u_1,\ldots,u_n$ with $\sum_{i=1}^n a_i=s$ and all $a_i<s$ (this means that at least two a_i's have to be positive),*

$$\sum_{i=1}^n\frac{a_iu_i}{s-a_i}\geq\frac{2}{n-1}\sum_{1\leq i<j\leq n}\sqrt{u_iu_j}\;-\;\frac{n-2}{n-1}\sum_{i=1}^n u_i; \tag{6}$$

equality holds if and only if

$$\frac{\sqrt{u_1}}{s-a_1}=\cdots=\frac{\sqrt{u_n}}{s-a_n}.$$

PROPOSITION 2. *For any nonnegative numbers $u_1,\ldots,u_n$ with $\sum_{i=1}^n u_i>0$,*

$$\frac{2}{n-1}\sum_{1\leq i<j\leq n}\sqrt{u_iu_j}\geq 2n\Big(\sum_{i=1}^n u_i\Big)^{-1}\sum_{1\leq i<j\leq n}u_iu_j\;-\;(n-2)\sum_{i=1}^n u_i; \tag{7}$$

equality holds if and only if either all u_i's are equal, or $u_{i_0}=0$ for some i_0, other u_i's being equal.

Postponing their proofs for now, note that Propositions 1 and 2 combined result in

COROLLARY 1. *For every a_i,u_i satisfying the conditions of Propositions* 1 *and* 2*,*

$$\sum_{i=1}^n\frac{a_iu_i}{s-a_i}\geq 2n\Big(\sum_{i=1}^n u_i\Big)^{-1}\sum_{1\leq i<j\leq n}u_iu_j\;-\;\frac{n(n-2)}{n-1}\sum_{i=1}^n u_i, \tag{8}$$

with equality only for $a_1=\cdots=a_n$, $u_1=\cdots=u_n$.

COROLLARY 2. *For any nonnegative numbers $a_1, \dots, a_n$, $x_1, \dots, x_n$ with at least two a_i's positive and $\sum_{i=1}^n x_i > 0$,*

$$\frac{a_1(x_2+\cdots+x_n)}{a_2+\cdots+a_n} + \frac{a_2(x_1+x_3+\cdots+x_n)}{a_1+a_3+\cdots+a_n} + \cdots + \frac{a_n(x_1+\cdots+x_{n-1})}{a_1+\cdots+a_{n-1}} \geq \frac{2n}{n-1}\Big(\sum_{i=1}^n x_i\Big)^{-1} \sum_{1\le i<j\le n} x_i x_j, \quad (9)$$

equality holding only for $a_1 = \cdots = a_n$, $x_1 = \cdots = x_n$.

Proof of Corollary 2 *(from Corollary* 1*).* Denote $\sum_{i=1}^n x_i$ by z and set $u_i = z - x_i$ in (8). Then we have

$$\mathrm{LHS}(8) = \mathrm{LHS}(9)$$

and, in view of $\sum_{i=1}^n u_i = (n-1)z$,

$$\begin{aligned}\sum_{1\le i<j\le n} u_i u_j &= \frac{1}{2}\Big(\Big(\sum_{i=1}^n u_i\Big)^2 - \Big(\sum_{i=1}^n u_i^2\Big)\Big) = \frac{1}{2}\Big((n-1)^2 z^2 - \sum_{i=1}^n (z^2 - 2zx_i + x_i^2)\Big) \\ &= \frac{1}{2}\Big((n-1)^2 z^2 - \Big(nz^2 - 2z^2 + (z^2 - 2\sum_{1\le i<j\le n} x_i x_j)\Big)\Big) \\ &= \frac{(n-1)(n-2)}{2} z^2 + \sum_{1\le i<j\le n} x_i x_j,\end{aligned}$$

so that

$$\mathrm{RHS}(8) = \frac{2n}{(n-1)z}\Big(\frac{(n-1)(n-2)}{2} z^2 + \sum_{1\le i<j\le n} x_i x_j\Big) - \frac{n(n-2)}{(n-1)} \cdot (n-1)z = \mathrm{RHS}(9).$$

Remark. For $n = 3$, (9) is just the proposed inequality.

Inequality (8) is more general than (9) because a tuple of nonnegative u_i's need not be representable by nonnegative x_i's via $u_i = (\sum_{j=1}^n x_j) - x_i$; e.g., for $n = 3$, the three numbers u, v, w can be written as $u = y + z$, $v = z + x$, $w = x + y$ ($x, y, z \geq 0$) if and only if they are the lengths of sides of a triangle.

Proof of Proposition 1.

$$\begin{aligned}\Big(\sum_{i=1}^n \sqrt{u_i}\Big)^2 &= \Big(\sum_{i=1}^n \sqrt{s-a_i}\cdot\sqrt{\frac{u_i}{s-a_i}}\Big)^2 \le \Big(\sum_{i=1}^n (s-a_i)\Big)\Big(\sum_{i=1}^n \frac{u_i}{s-a_i}\Big) \\ &= (n-1)\sum_{i=1}^n \frac{su_i}{s-a_i} = (n-1)\sum_{i=1}^n \Big(u_i + \frac{a_i u_i}{s-a_i}\Big);\end{aligned}$$

hence

$$(n-1)\sum_{i=1}^n \frac{a_i u_i}{s-a_i} \geq \Big(\sum_{i=1}^n \sqrt{u_i}\Big)^2 - (n-1)\sum_{i=1}^n u_i = 2\sum_{1\le i<j\le n} \sqrt{u_i u_j} - (n-2)\sum_{i=1}^n u_i,$$

and this is (6). In estimating we used the Cauchy–Schwarz inequality, which turns into equality if and only if the two vectors are proportional: $(\sqrt{u_i/(s-a_i)}) \sim (\sqrt{s-a_i})$, i.e., when $(\sqrt{u_i}) \sim (s-a_i)$.

To prove Proposition 2 we need a lemma.

LEMMA. *Let $A, B, C, D > 0$ be any constants and let $u, v \geq 0$ vary so that the sum $u+v$ remains constant. The expression*

$$Auv + B(u+v) - C\sqrt{uv} - D(\sqrt{u}+\sqrt{v}) \tag{10}$$

is maximized either when $u = v$ or when one of u, v is 0.

Proof of Lemma. Since we can arbitrarily manipulate with A, B, C, D, we may assume that the fixed value of $u+v$ is 2. Write $u = 1+t$, $v = 1-t$, $t \in [0,1]$. The expression (10) becomes

$$A(1-t^2) - C\sqrt{1-t^2} - D(\sqrt{1+t}+\sqrt{1-t}) + (\text{a constant}) =: \varphi(t)\,.$$

Then

$$\varphi'''(t) = 3Ct(1-t^2)^{-5/2} + \tfrac{3}{8}D((1-t)^{-5/2} - (1+t)^{-5/2}) > 0\,,$$

so φ' is convex. As $\varphi'(0) = 0$, we see that φ' either has a constant sign in $(0,1)$ or changes sign just once, from minus to plus. This means that φ is either monotonic in $[0,1]$ or piecewise monotonic, first decreasing and then increasing. Anyway, the maximum of φ occurs at an endpoint, whence the claim.

Proof of Proposition 2. By homogeneity, we may assume $\sum_{i=1}^n u_i = n$ and restate claim (7) as

$$(n-1)\sum_{1\leq i<j\leq n} u_i u_j \; - \sum_{1\leq i<j\leq n} \sqrt{u_i u_j} \leq \frac{n(n-1)(n-2)}{2}\,. \tag{11}$$

Denote the expression on the left by $F(u_1, \ldots, u_n)$. We are facing the problem of maximizing F on the $(n-1)$-simplex

$$\Delta = \{(u_1, \ldots, u_n) \,:\, u_1 \geq 0, \ldots, u_n \geq 0,\; u_1 + \cdots + u_n = n\}.$$

Suppose F attains its maximum on Δ at some point $(c_1, \ldots, c_n)$ inside Δ; i.e., such that $c_1 > 0$, ... , $c_n > 0$. Keeping $c_3, \ldots, c_n$ fixed and letting u_1, u_2 vary so that $u_1 + u_2 = c_1 + c_2$ we obtain an expression of type (10), which is known to reach its maximal value at $u_1 = c_1 > 0$, $u_2 = c_2 > 0$. By the lemma, $c_1 = c_2$. In the same way we show that any two c_i's must be equal. So $(1, \ldots, 1)$ is the only point *inside* Δ where F can attain its maximum.

Otherwise F must be maximized at some point of the $(n-2)$-dimensional boundary of Δ. Repeating the argument with respect to each $(n-2)$-dimensional face of Δ, then to faces of lower dimensions, we arrive at the conclusion that F takes its maximum at the centroid of some face, i.e., at a point

$$(\underbrace{n/k, \ldots, n/k}_{k}, \underbrace{0, \ldots, 0}_{n-k}),$$

where $1 \leq k \leq n$. Now, the value of F at such a point is

$$(n-1)\sum_{i<j\leq k}\left(\frac{n}{k}\right)^2 - \sum_{i<j\leq k}\frac{n}{k} = \binom{k}{2}\cdot\frac{n}{k}\cdot\left((n-1)\frac{n}{k}-1\right) = \frac{n(k-1)(n^2-n-k)}{2k}.$$

We have to show that this value does not exceed the right-hand side of (11); this is equivalent to

$$k^2-(2n-1)k+(n^2-n)\geq 0. \tag{12}$$

The quadratic trinomial on the left has $k=n$ and $k=n-1$ for its roots. So (12) does hold, with strict inequality for all k smaller than $n-1$.

This settles (11), hence also (7), together with the statements concerning the occurrences of equality in (7).

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; G.P. HENDERSON, Campbellcroft, Ontario; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Penn State Harrisburg; and the proposer. There was one incorrect solution sent in.

The proposer also found the generalization (4) *and the special case* (5) *from solution* I. *He then noted that by the general means inequality (and using item* 4.10 *of Bottema* [1]*), for* $n>1$

$$\sqrt[n]{\frac{a^{4n}+b^{4n}+c^{4n}}{3}} \geq \frac{a^4+b^4+c^4}{3} \geq \frac{16F^2}{3},$$

from which it follows that (5) *holds for any* $n>0$. *He asks whether* (4) *also holds for any* $n>0$.

* * * * *

1678. [1991: 238] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Show that

$$\sqrt{s}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \leq \sqrt{2}(r_a+r_b+r_c),$$

where a,b,c are the sides of a triangle, s the semiperimeter, and r_a,r_b,r_c the exradii.

I. *Solution by Stephen D. Hnidei, student, University of Windsor.*

Using the fact that

$$2s\sqrt{3} = (a+b+c)\sqrt{3} \leq 2(r_a+r_b+r_c)$$

(inequality 5.29, page 55 of Bottema et al, *Geometric Inequalities*), it is enough to show

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \leq \sqrt{6s}. \tag{1}$$

By squaring both sides, this is equivalent to

$$\sqrt{ab}+\sqrt{ac}+\sqrt{bc} \leq a+b+c. \tag{2}$$

Using that

$$a+b\geq 2\sqrt{ab}, \quad a+c\geq 2\sqrt{ac}, \quad b+c\geq 2\sqrt{bc}$$

for positive values of a, b, c, we establish (2).

II. *Solution by Murray S. Klamkin, University of Alberta.*
More generally, we show that, for $n \geq 1/2$ or $n \leq 0$,

$$2^n(r_a^{2n} + r_b^{2n} + r_c^{2n}) \geq s^n(a^n + b^n + c^n). \tag{3}$$

The case $n = 1/2$ is the required inequality.

Since $r_a = rs/(s-a)$, etc., and $F^2 = (rs)^2 = s(s-a)(s-b)(s-c)$, where r is the inradius and F the area of the triangle, (3) becomes

$$\frac{2^n(rs)^{2n} \cdot \sum(s-b)^{2n}(s-c)^{2n}}{[(s-a)(s-b)(s-c)]^{2n}} \geq s^n(a^n + b^n + c^n),$$

where the sum is cyclic over a, b, c, or

$$\left(\frac{2s}{F^2}\right)^n \sum(s-b)^{2n}(s-c)^{2n} \geq a^n + b^n + c^n. \tag{4}$$

Case (i): $1/2 \leq n \leq 1$.

By the power mean inequality, for $n \geq 1/2$,

$$\left(\frac{2s}{F^2}\right)^n \sum(s-b)^{2n}(s-c)^{2n} \geq 3\left(\frac{2s}{F^2}\right)^n \left(\frac{1}{3}\sum(s-b)(s-c)\right)^{2n} = 3(2s)^n\left(\frac{4F'^2}{3F}\right)^{2n}, \tag{5}$$

where F' is the area of a triangle of sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, and F' satisfies

$$4F'^2 = \frac{1}{4}(2\sum bc - \sum a^2) = \sum(s-b)(s-c)$$

(e.g., see p. 102 of [2]). Now from (5) and the Finsler–Hadwiger inequality $4F'^2 \geq F\sqrt{3}$ (item 10.3, page 91 of [1]), to prove (4) it suffices to show that

$$\frac{3(a+b+c)^n}{3^n} = \frac{3(2s)^n}{3^n} \geq a^n + b^n + c^n,$$

which follows by the power mean inequality for $n \leq 1$.

Case (ii): $n > 1$.

For any n, (4) is equivalent to

$$2^n \sum(s-b)^{2n}(s-c)^{2n} \geq [(s-a)(s-b)(s-c)]^n(a^n + b^n + c^n). \tag{6}$$

Letting $a = y+z$, $b = z+x$, $c = x+y$ where x, y, z are nonnegative numbers, we get $s = x+y+z$ and $s-a = x$, etc., so (6) becomes

$$2^n[(yz)^{2n} + (zx)^{2n} + (xy)^{2n}] \geq [yz(xy+xz)]^n + [zx(yz+yx)]^n + [xy(zx+zy)]^n.$$

We now simplify by letting $yz = u$, $zx = v$, $xy = w$ to give

$$2^n(u^{2n} + v^{2n} + w^{2n}) \geq u^n(v+w)^n + v^n(w+u)^n + w^n(u+v)^n. \tag{7}$$

Since

$$\left(\frac{v+w}{2}\right)^n \le \frac{v^n+w^n}{2}, \quad \text{etc.},$$

for $n \ge 1$, to prove (7) it suffices to show that

$$u^{2n}+v^{2n}+w^{2n} \ge v^n w^n + w^n u^n + u^n v^n,$$

or

$$(u^n-v^n)^2+(v^n-w^n)^2+(w^n-u^n)^2 \ge 0,$$

which is true.

Case (iii): $n<0$ (the case $n=0$ is trivial).

In (7), if we replace (u,v,w) by $(1/u,1/v,1/w)$ and change n to $-n$, we have to show that

$$u^{2n}+v^{2n}+w^{2n} \ge (2uvw)^n[(v+w)^{-n}+(w+u)^{-n}+(u+v)^{-n}]$$

for $u,v,w,n>0$. Since $2^n(vw)^{n/2}(v+w)^{-n} \le 1$, etc., it suffices to show that

$$u^{2n}+v^{2n}+w^{2n} \ge u^n(vw)^{n/2}+v^n(wu)^{n/2}+w^n(uv)^{n/2},$$

which follows by using Hölder's inequality.

Comments. (a) There is equality in (3) only for the equilateral triangle.

(b) for $n \ge 1$, (7) can be extended to any number of variables. Here we just sketch the extension to four variables, i.e.,

$$3^n(t^{2n}+u^{2n}+v^{2n}+w^{2n}) \ge \sum t^n(u+v+w)^n$$

where the sum is cyclic over t, u, v, w. Since by the power mean inequality,

$$\left(\frac{u+v+w}{3}\right)^n \le \frac{u^n+v^n+w^n}{3}, \quad \text{etc.}$$

for $n \ge 1$, it suffices to show that

$$3(t^{2n}+u^{2n}+v^{2n}+w^{2n}) \ge 2(t^n u^n+t^n v^n+t^n w^n+u^n v^n+u^n w^n+v^n w^n),$$

which is equivalent to the obvious inequality

$$\sum (t^n-u^n)^2 \ge 0,$$

the sum being symmetric over t, u, v, w. For the case $n<0$, we change (t,u,v,w) to $(1/t,1/u,1/v,1/w)$ and n to $-n$ and proceed as before.

(c) By choosing $u=1-2\varepsilon$, $v=1-2\varepsilon$, $w=1$ where ε is an arbitrarily small positive number, it can be shown that (7) and so (3) *fails* whenever $0<n<1/5$. We leave as an open problem:

Does (3) *hold for* $1/5 \le n < 1/2$*?*

References:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

[2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

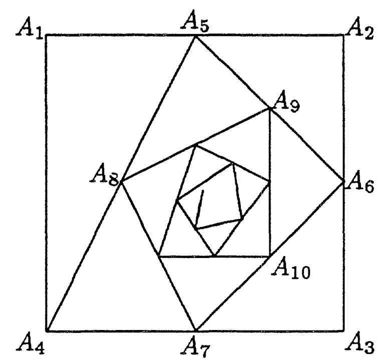
Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; VEDULA N. MURTY,Penn State Harrisburg; BOB PRIELIPP, University of Wisconsin–Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

It would be nice to have a unified proof of (7) *for the case* $n \geq 1/2$, *and also to settle it for the remaining interval* $1/5 \leq n < 1/2$.

* * * * *

1679. [1991: 238] *Proposed by Len Bos and Bill Sands, University of Calgary.*

$A_1A_2A_3A_4$ is a unit square in the plane, with $A_1(0,1)$, $A_2(1,1)$, $A_3(1,0)$, $A_4(0,0)$. A_5 is the midpoint of A_1A_2, A_6 the midpoint of A_2A_3, A_7 the midpoint of A_3A_4, A_8 the midpoint of A_4A_5, and so on. This forms a spiral polygonal path $A_1A_2A_3A_4A_5A_6A_7A_8\ldots$ converging to a unique point inside the square. Find the coordinates of this point.

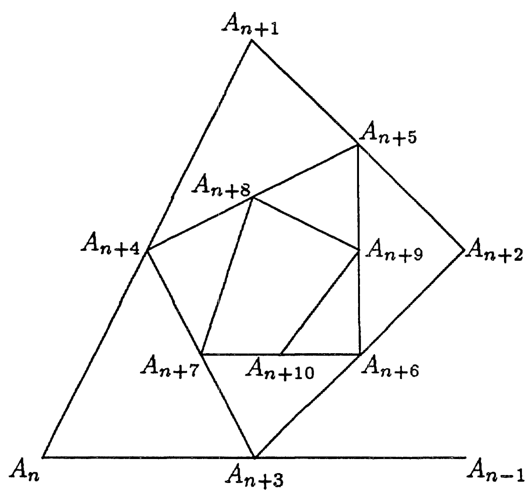


Solution by the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen.

For completeness, we start by showing that the sequence $A_1, A_2, A_3, \ldots$ does converge. We first define, for integers $n \geq 1$, $\mathcal{Q}_n$ to be the quadrangle $A_nA_{n+1}A_{n+2}A_{n+3}$ (interior included). Obviously $\mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \mathcal{Q}_3 \supset \cdots$. We denote the diameter of $\mathcal{Q}_n$ by d_n, and show that

$$d_{n+7} \leq d_n/2 \qquad (1)$$

for $n = 1, 2, \ldots$. This implies that $\bigcap_{n=1}^{\infty} \mathcal{Q}_n$ is a point singleton, and hence the convergence of $A_1, A_2, A_3, \ldots$ follows. To prove (1), consider $\mathcal{Q}_{n+7}$. Its diameter d_{n+7} is the greatest of



$$\overline{A_{n+7}A_{n+8}},\ \overline{A_{n+8}A_{n+9}},\ \overline{A_{n+9}A_{n+10}},\ \overline{A_{n+10}A_{n+7}},\ \overline{A_{n+7}A_{n+9}},\ \overline{A_{n+8}A_{n+10}}.$$

Now

$$\overline{A_{n+7}A_{n+8}} = \frac{1}{2}\,\overline{A_{n+3}A_{n+5}} \le \frac{1}{2}d_n, \quad \overline{A_{n+8}A_{n+9}} = \frac{1}{2}\,\overline{A_{n+4}A_{n+6}} \le \frac{1}{2}d_n,$$

$$\overline{A_{n+9}A_{n+10}} = \frac{1}{2}\,\overline{A_{n+5}A_{n+7}} \le \frac{1}{2}d_n, \quad \overline{A_{n+10}A_{n+7}} = \frac{1}{2}\,\overline{A_{n+6}A_{n+7}} \le \frac{1}{2}d_n,$$

$$\overline{A_{n+7}A_{n+9}} \le \frac{1}{2}(\overline{A_{n+4}A_{n+5}} + \overline{A_{n+3}A_{n+6}}) = \frac{1}{2}\left(\frac{1}{2}\,\overline{A_nA_{n+2}} + \frac{1}{2}\,\overline{A_{n+3}A_{n+2}}\right) \le \frac{1}{2}d_n,$$

$$\overline{A_{n+8}A_{n+10}} \le \frac{1}{2}(\overline{A_{n+4}A_{n+7}} + \overline{A_{n+5}A_{n+6}}) = \frac{1}{2}\left(\frac{1}{2}\,\overline{A_{n+4}A_{n+3}} + \frac{1}{2}\,\overline{A_{n+1}A_{n+3}}\right) \le \frac{1}{2}d_n,$$

and (1) follows.

We proceed to the main question. For $n = 1, 2, 3, \ldots$ let $A_n = (x_n, y_n)$. Then from $x_{n+4} = (x_n + x_{n+1})/2$ we deduce

$$\frac{1}{2}x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} = \frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}$$

for all $n = 1, 2, 3, \ldots$. Since

$$\frac{1}{2}x_1 + x_2 + x_3 + x_4 = 0 + 1 + 1 + 0 = 2,$$

it follows that

$$\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2 \tag{2}$$

for $n = 1, 2, 3, \ldots$. Similarly,

$$\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2} + 1 + 0 + 0 = \frac{3}{2}$$

entails

$$\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2} \tag{3}$$

for $n = 1, 2, 3, \ldots$. If $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, (2) and (3) imply $7x/2 = 2$ and $7y/2 = 3/2$, i.e.,

$$(x, y) = (4/7, 3/7).$$

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; J. CHRIS FISHER and ROBERT E. JAMISON, Clemson University, Clemson, South Carolina; R.K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; RICHARD MCINTOSH, University of Regina, and PETER MONTGOMERY, student, UCLA; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The

Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposers. Three incomplete solutions were sent in.

The solutions of Penning and Smeenk were very similar to the above (without the proof of convergence), and a few others were nearly as nice. The remaining solutions (including the proposers') were rather more involved, using eigenvalues and such.

Bang, Fisher and Jamison, Janous, and Wildhagen generalized the idea of midpoint, using the relation

$$A_{n+4} = \delta A_n + (1-\delta)A_{n+1},$$

where $0 < \delta < 1$ *is fixed, to generate the points of the spiral. The limit point is then*

$$\left(\frac{2}{3+\delta}, \frac{1+\delta}{3+\delta}\right),$$

which can also be proved as above. $\delta = 1/2$ *gives the original problem.*

Actually, Janous's solution is even more general, and is in the form of an article, "Punktiterationen auf Quadraten", to appear in Wissenschaftliche Nachrichten.

McIntosh and Montgomery, and independently Fisher and Jamison, came up with an attractive variation. Take the unit circle centred at the origin, and begin to inscribe a regular n-gon in it, starting at the point $(1,0)$ *and proceeding counterclockwise; however, instead of drawing the nth side, connect vertex n to the midpoint of the first side, then connect this midpoint to the midpoint of the second side, etc., obtaining a polygonal spiral. Then the limit point of this spiral is*

$$\left(-\frac{1}{2n-1}, 0\right).$$

This can be shown as above (except the convergence might be a bit messy?). The case $n = 4$ *is equivalent to the original problem.*

Fisher and Jamison combined these two ideas, starting a regular n-gon inscribed in the unit circle and spiralling inward, but using the $\delta, 1-\delta$ *division points instead of midpoints. They obtain the limit point*

$$\left(\frac{\delta-1}{\delta+n-1}, 0\right).$$

Can anyone see an easy reason (without calculation) why the limit point should always end up on the x-axis?

* * * * *

1680. [1991: 238] *Proposed by Zun Shan and Ji Chen, Ningbo University, China.*

If m_a, m_b, m_c are the medians and r_a, r_b, r_c the exradii of a triangle, prove that

$$\frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} + \frac{r_a r_b}{m_a m_b} \ge 3.$$

I. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

In what follows, all sums are (a,b,c)-cyclic, and we denote the semiperimeter $\frac{1}{2}\sum a$ by s. Since $r_a = \sqrt{s(s-b)(s-c)/(s-a)}$, etc., we get

$$r_b r_c = s(s-a), \quad r_c r_a = s(s-b), \quad r_a r_b = s(s-c).$$

Further, we have

$$\begin{aligned}
m_b m_c &= \frac{1}{2}\sqrt{2c^2+2a^2-b^2}\cdot\frac{1}{2}\sqrt{2a^2+2b^2-c^2} \\
&= \frac{1}{8}\sqrt{(4a^2+b^2+c^2)-(3b^2-3c^2)}\cdot\sqrt{(4a^2+b^2+c^2)+(3b^2-3c^2)} \\
&\le \frac{4a^2+b^2+c^2}{8} = \frac{(2a^2+2b^2+2c^2)+(2a^2-b^2-c^2)}{8} \\
&= \frac{(2a^2+2b^2+2c^2)^2-(2a^2-b^2-c^2)^2}{8[(2a^2+2b^2+2c^2)-(2a^2-b^2-c^2)]} \\
&\le \frac{(2a^2+2b^2+2c^2)^2}{8(3b^2+3c^2)} = \frac{(\sum a^2)^2}{6(b^2+c^2)}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\sum a^2\right)^2\left(\frac{1}{3}\left(\sum\frac{r_b r_c}{m_b m_c}\right)-1\right) &\ge \left(\sum a^2\right)^2\left(\frac{1}{3}\left(\sum\frac{s(s-a)\cdot 6(b^2+c^2)}{(\sum a^2)^2}\right)-1\right) \\
&= 2s\sum\left((s-a)(\sum a^2 \;-\; a^2)\right)-\left(\sum a^2\right)^2 \\
&= 2s\left(\sum a^2\right)\left(\sum(s-a)\right)-2s\left(\sum(s-a)a^2\right)-\left(\sum a^2\right)^2 \\
&= 2s\left(\sum a^2\right)s-2s\left(s\left(\sum a^2\right)-\left(\sum a^3\right)\right)-\left(\sum a^2\right)^2 \\
&= 2s\left(\sum a^3\right)-\left(\sum a^2\right)^2 = \left(\sum a\right)\left(\sum a^3\right)-\left(\sum a^2\right)^2 \\
&\ge 0
\end{aligned}$$

by Cauchy's inequality, and so

$$\sum\frac{r_b r_c}{m_b m_c} \ge 3,$$

with equality only if $a=b=c$.

Remark. A beautiful inequality! Especially in the light of $r_a r_b r_c \le m_a m_b m_c$ ([1], item 8.21). It should however be noticed that $M_k(m_a^t, m_b^t, m_c^t) \le M_k(r_a^t, r_b^t, r_c^t)$, for every $t,k\ge 1$ ([2], Ch. 8, D.2.a; of course M_k denotes the kth power mean). Maybe the inequality of the present problem could be hence derived in some way?

References:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1968.

[2] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, 1979.

II. *Solution by the proposers.*

Because

$$r_b r_c = s(s-a) = \frac{a+b+c}{2}\cdot\frac{b+c-a}{2} = \frac{(b+c)^2-a^2}{4}, \quad \text{etc.},$$

the given inequality becomes

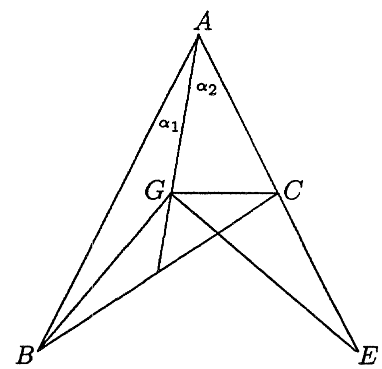
$$\sum \frac{(b+c)^2-a^2}{m_b m_c} \ge 12.$$

Applying Klamkin's median duality (see p. 109 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*), we need to prove

$$\sum \frac{(m_b+m_c)^2-m_a^2}{bc} \ge \frac{27}{4}\,. \tag{1}$$

First we observe that

$$|b-c| \ge \frac{2}{3}|m_b-m_c|, \quad \text{etc.} \tag{2}$$



Assuming without loss of generality that $b \le c$, we get $\alpha_2 \ge \alpha_1$; extending AC to E so that $AE = c$, it follows that $EG \ge BG$. So

$$c - b = CE \ge EG - CG \ge BG - CG = \frac{2}{3}(m_b - m_c),$$

as claimed.

From (2) we have

$$\begin{aligned}
4(m_b+m_c)^2 - 4m_a^2 &= 8m_bm_c + 4m_b^2 + 4m_c^2 - 4m_a^2 \\
&\ge 4m_b^2 + 4m_c^2 - 9(b-c)^2 + 4m_b^2 + 4m_c^2 - 4m_a^2 \\
&= 8m_b^2 + 8m_c^2 - 4m_a^2 - 9(b-c)^2 \\
&= 2(2c^2+2a^2-b^2) + 2(2a^2+2b^2-c^2) - (2b^2+2c^2-a^2) - 9(b-c)^2 \\
&= 9a^2 - 9b^2 - 9c^2 + 18bc,
\end{aligned}$$

and then, by the law of cosines,

$$\begin{aligned}
\sum \frac{(m_b+m_c)^2-m_a^2}{bc} &\ge \frac{9}{4}\sum \frac{a^2-b^2-c^2+2bc}{bc} = \frac{9}{4}\sum\frac{-2bc\cos A + 2bc}{bc} \\
&= \frac{27}{2} - \frac{9}{2}\sum \cos A \ge \frac{27}{2} - \frac{9}{2}\cdot\frac{3}{2} = \frac{27}{4},
\end{aligned}$$

which is (1).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and KEE-WAI LAU, Hong Kong.

Janous mentions the similar (recently obtained) inequality

$$\sum \frac{r_a}{m_a} \geq 3$$

reported by Mitronivić, Pečarić, Volenec, and Chen Ji in addenda to "Recent Advances in Geometric Inequalities"; see the Journal of Ningbo University **4** (1991) 79–145, *especially p.* 89.

Lau's solution uses the inequality

$$\frac{bc}{m_b^2 + m_c^2} + \frac{ca}{m_c^2 + m_a^2} + \frac{ab}{m_a^2 + m_b^2} \geq 2, \tag{3}$$

which he proposes to the readers, and for which he has a complicated proof. If the editor receives a nice proof of (3), *he will print it, along with Lau's solution of the original inequality.*

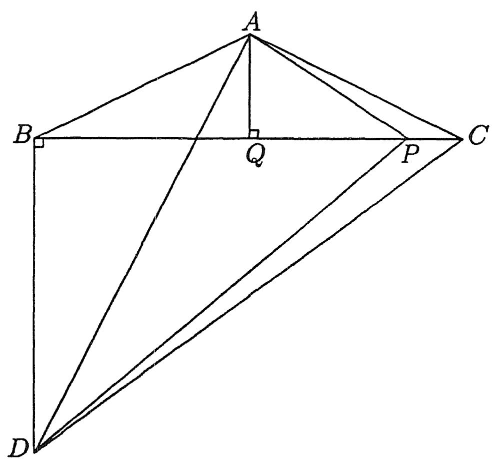
* * * * *

1681. [1991: 270] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $\overline{AB} = \overline{AC} < \overline{BC}$. Let P be a point on side BC such that $\overline{AP}^2 = \overline{BC} \cdot \overline{PC}$, and let CD be a diameter of the circumcircle of ΔABC. Prove that $\overline{DA} = \overline{DP}$.

Solution by Andy Liu, University of Alberta.

Let Q be the midpoint of BC. Then



$$\begin{aligned}(AP)^2 &= (AQ)^2 + (QC - PC)^2 \\ &= (AQ)^2 + (QC)^2 + (PC)^2 - 2QC \cdot PC \\ &= (AC)^2 + (PC)^2 - (AP)^2\end{aligned}$$

by the condition on P, so

$$2(AP)^2 - (PC)^2 = (AC)^2. \tag{1}$$

Hence, by the condition on P and by (1),

$$\begin{aligned}(DP)^2 &= (BD)^2 + (BC - PC)^2 = (BD)^2 + (BC)^2 + (PC)^2 - 2BC \cdot PC \\ &= (CD)^2 + (PC)^2 - 2(AP)^2 = (CD)^2 - (AC)^2 = (DA)^2\end{aligned}$$

or $DP = DA$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; BENO ARBEL, Tel-Aviv University, Israel; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI

DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; DAVID HANKIN, Brooklyn, N.Y.; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; ALBERT KURZ, student, Council Rock H.S., Newtown, Pennsylvania; KEE-WAI LAU, Hong Kong; JOSEPH M. LING, University of Calgary; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; CHRIS WILDHAGEN, Rotterdam, The Netherlands; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

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1682*. [1991: 270] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For a finite set S of natural numbers let

$$Alt(S) = x_1 - x_2 + x_3 - \cdots,$$

where $x_1 > x_2 > x_3 > \cdots$ are the elements of S in decreasing order. Determine

$$f(n) = \sum Alt(S),$$

where the sum is extended over all non-empty subsets S of $\{1, 2, \ldots, n\}$.

Solution by Albert Kurz, student, Council Rock H.S., Newtown, Pennsylvania.

Let P_n be the collection of all non-empty subsets of $\{1, 2, \ldots, n\}$, and note that P_n can be divided into three disjoint classes:

C_1, containing only $\{n\}$;

C_2, the class of all non-empty subsets of $\{1, 2, \ldots, n-1\}$;

C_3, the class of subsets of $\{1, 2, \ldots, n\}$ containing n and at least one of the elements of $\{1, 2, \ldots, n-1\}$.

Then

$$f(n) = \sum_{S \in C_1} Alt(S) + \sum_{S \in C_2} Alt(S) + \sum_{S \in C_3} Alt(S).$$

Now notice the one-to-one correspondence between members of C_2, of the form $\{x_1, x_2, \ldots\}$, and the members of C_3, of the form $\{n, x_1, x_2, \ldots\}$ (with $n > x_1 > x_2 > \ldots$ for both). Thus

$$\begin{aligned} f(n) &= \sum_{S \in C_1} Alt(S) + \sum_{S \in C_2} [Alt(S) + Alt(S \cup \{n\})] \\ &= n + \sum_{S \in C_2} [(x_1 - x_2 + x_3 - \cdots) + (n - x_1 + x_2 - x_3 + \cdots)] \\ &= n + \sum_{S \in C_2} n, \end{aligned}$$

and since there are $2^{n-1} - 1$ members of C_2,

$$f(n) = n + (2^{n-1} - 1)n = n2^{n-1}.$$

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Seoul, Republic of Korea; MANUEL BENITO, I.B. Sagasta, Logroño, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; WEIXUAN LI and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; JOSEPH M. LING, University of Calgary; CORY PYE, student, Memorial University of Newfoundland, St. John's; SHOBHIT SONAKIYA, Kanpur, India; DAVID R. STONE, Georgia Southern University, Statesboro; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Several solvers pointed out that one can let the given sum include $S = \emptyset$ (the empty set) by defining $Alt(\emptyset) = 0$. This would slightly simplify the above proof.

* * * * *

1683. [1991: 270] *Proposed by P. Penning, Delft, The Netherlands.*

Given is a fixed triangle ABC and fixed positive angles μ, ν such that $\mu + \nu < \pi$. For a variable line l through C, let P and Q be the feet of the perpendiculars from A and B, respectively, to l, and let Z be such that $\angle ZPQ = \mu$ and $\angle ZQP = \nu$ (and, say, the sense of QPZ is clockwise). Determine the locus of Z.

Composite solution by Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain; and Václav Konečný, Ferris State University, Big Rapids, Michigan.

Because $AP \perp l$, P is on the circle $\mathcal{C}_P$ whose diameter is AC; because $BQ \perp l$, Q is on the circle $\mathcal{C}_Q$ whose diameter is BC. The lines PZ that form with l the angle μ meet $\mathcal{C}_P$ in a point P' for which the arc CP' equals 2μ. P' is a fixed point since both C and μ are fixed. Analogously, lines QZ forming with l the angle ν intersect $\mathcal{C}_Q$ in Q' such that arc CQ' equals 2ν, and Q' is a fixed point. By definition the lines PP' and QQ' are two sides of the triangle QPZ, forming (at Z) an angle of $\pi - \mu - \nu$, so that $P'Q'$ subtends (from Z) an angle $\mu + \nu$ when Z is on one side of it, and the supplementary angle when Z is on the other side. Consequently, the locus of Z is the circle passing through P', Q', and the second point of intersection of $\mathcal{C}_P$ and $\mathcal{C}_Q$.

Also solved by JORDI DOU, Barcelona, Spain; J. CHRIS FISHER, University of Regina; L.J. HUT, Groningen, The Netherlands; MARCIN E. KUCZMA, Warszawa, Poland; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1686. [1991: 270] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 4/3$ and

$$a_{n+1} = \frac{3(5 - 7a_n)}{2(10a_n + 17)}$$

for $n \geq 0$. Find a formula for a_n in terms of n.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

In view of the fact that (by calculation) the members of the sequence seem to converge on $1/4$, alternately from above and below, the substitution $b_n = a_n - 1/4$ gives promise of simplifying the equation of the problem statement. [*Editor's note.* One might also notice that $x = 1/4$ is a fixed point, i.e., a solution of

$$\frac{3(5-7x)}{2(10x+17)} = x,$$

so that convergence of the sequence to $1/4$ is not surprising.] Accordingly, we obtain

$$b_{n+1} = \frac{3[5-7(b_n+1/4)]}{2[10(b_n+1/4)+17]} - \frac{1}{4} = -\frac{26b_n}{20b_n+39}\,, \quad b_0 = \frac{13}{12}\,. \tag{1}$$

Now the substitution $b_n = 1/c_n$ will get the sum into the numerator instead of the denominator, giving

$$c_{n+1} = -\frac{10}{13} - \frac{3}{2}c_n, \quad c_0 = \frac{12}{13}\,. \tag{2}$$

Finally, the substitution $d_n = c_n + 4/13$ gives

$$d_{n+1} = -\frac{3}{2}d_n, \quad d_0 = \frac{16}{13}\,. \tag{3}$$

Thus it is clear that

$$d_n = \left(-\frac{3}{2}\right)^n d_0 = \frac{16}{13}\left(-\frac{3}{2}\right)^n. \tag{4}$$

Back-substituting (4), (3), (2) and (1) into the equation of the problem statement gives

$$c_n = \frac{16}{13}\left(-\frac{3}{2}\right)^n - \frac{4}{13}\,, \quad b_n = \frac{13}{16(-3/2)^n - 4}\,,$$

and finally

$$a_n = \frac{(-3/2)^n + 3}{4(-3/2)^n - 1} = \frac{3^n + 3(-2)^n}{4 \cdot 3^n - (-2)^n}\,.$$

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Seoul, Republic of Korea; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, student, 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; LEROY F. MEYERS, The Ohio State University, Columbus; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Popular means of solution were by induction (having obtained the formula somehow), and by matrix methods. Many solvers considered the analogous recurrence with general coefficients.

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