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CONTENTS

FIRST GET THE NAME RIGHT

R. ROBINSON ROWE, Sacramento, California

Right at the outset, in this issue of EUREKA commemorating the bicentennial of the birth of Carl Friedrich Gauss, we should all know, to avoid causing any unnecessary unrest to the great man's shade, how to pronounce his name correctly. The following anecdote may help to achieve that purpose.

When I was in high school, back in 1912, our physics teacher was upset because students, being unfamiliar with the German language, usually pronounced *Gauss* to rhyme with *sauce*, or even *gauze*. He gave us a bit of doggerel to help us remember the correct pronunciation. This was a long time ago, so my memory may have altered some details, but it ran approximately as follows:

The meticulous Gauss
Had a lauss
And a mauss
In his hauss,
But he kept out the grauss
And the sauss
And the causs,
'Cause his spauss
Would have kicked 'em *herauss*.

I would like to be able to say that my physics teacher was named Rouse, pronounced Rauss, but in fact his name was Borgers.

Editor's Note.

We are publishing the above bit of doggerel with tongue in cheek, intending no slur upon one of the greatest names in mathematics and, indeed, in world history. Gauss lived in an age when many had mice by the score and lice by the thousand. Nearby Hameln had its rats piped into the Weser five hundred years before, but mice were running up the clock in Gauss' time. So with only one mouse and one louse he could boast that his house was relatively vermin-free.

His was a happy home with each of his two successive spouses (spice?) and his six children — in a life which has been described as normal, quiet, and serene. He was not a touchy person as, say, Newton was, and he would not, we think, have taken offense at a well-intentioned and harmless whimsy. We can, at any rate, seek reassurance in the fact that today, having been interred more than one hundred years, he is sure to have a great sense of humus.

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GAUSS AND NUMBER THEORY

The arithmetical work of Gauss

KENNETH S. WILLIAMS, Carleton University

1. *Introduction.*

Carl Friedrich Gauss was born on April 30, 1777, in Brunswick (Braunschweig), Germany [1]. His mathematical genius was recognized at an early age and, at the age of fifteen, Gauss entered the Caroline College in Brunswick. While there, he began his research into that area of mathematics known as the theory of numbers, sometimes called the higher arithmetic, which is that branch of mathematics concerned with the properties of the integers $0, \pm 1, \pm 2, \ldots$. This research culminated in the *Disquisitiones Arithmeticae,* first published in 1801 when Gauss was only twenty-four. After its publication, Gauss' researches broadened to include astronomy, geodesy and electro-magnetism, both from the theoretical and practical points of view. But number theory always remained his first love. In the preface to *Disquisitiones Arithmeticae,* Gauss relates how he became interested in number theory:

"Engaged in other work I chanced on an extraordinary arithmetic truth. Since I considered it so beautiful in itself and since I suspected its connection with even more profound results, I concentrated on it all my efforts in order to understand the principles on which it depended and to obtain a rigorous proof. When I succeeded in this I was so attracted by these questions that I could not let them be ..."

The *Disquisitiones Arithmeticae* is divided into seven sections. It was written in Latin and it is an inexplicable fact that no English translation was made of it until 1966 [2]. The first three sections are devoted to the theory of congruences, the fourth to the theory of quadratic residues, the fifth and sixth to the theory of binary quadratic forms, and the final section to the arithmetical theory of the nth roots of unity (cyclotomy).

Many parts of this work had been considered before by such mathematicians as Fermat, Euler, Lagrange, Legendre and others; but Gauss treated the subject from his own point of view and added much that was completely new. Experimenting with the integers themselves and making use of his photographic memory and fantastic powers of mental calculation, Gauss discovered general theorems, which even he could only prove with great difficulty. In this article, I would like to describe a few of the beautiful arithmetical results discovered by Gauss.

2. *The law of quadratic reciprocity.*

One of Gauss' most important contributions to number theory was his rediscovery of the law of quadratic reciprocity. This had been discovered earlier by Euler and Legendre, but Gauss was the first to give a complete proof of it. In order to describe this law, we must first explain some notation.

If the difference $a-b$ of two integers a and b is divisible by a nonzero integer m, we say that a and b are *congruent modulo* m, and write this as

$$a \equiv b \pmod{m}.$$

Thus we have

$$13 \equiv 3 \pmod{5}, \quad 27 \equiv -1 \pmod{7}.$$

The elementary properties of congruences are given by Gauss in the early sections of *Disquisitiones Arithmeticae*. In particular he gave a necessary and sufficient condition for the linear congruence

$$ax \equiv b \pmod{m}$$

to be solvable, where a, b, m are given integers, namely, that the greatest common divisor of a and m divide b. With this criterion the congruence $2x \equiv 1 \pmod{4}$ is easily seen to be insolvable, whereas $3x \equiv 1 \pmod{7}$ is solvable (with solution $x \equiv 5 \pmod{7}$).

From the general linear congruence, it was only natural that Gauss should next turn his attention to quadratic congruences, the simplest of which is

$$x^2 \equiv a \pmod{p}, \quad p \text{ an odd prime.}$$

Experimenting with particular values of p and a , Gauss rediscovered a beautiful reciprocity between a pair of congruences of the above type. This relationship had been observed earlier by Euler and Legendre, but neither of them could prove it. Gauss noticed that if p and q are both primes then the congruences

$$x^2 \equiv p \pmod{q}, \quad y^2 \equiv q \pmod{p},$$

are both solvable or both insolvable, unless both p and q are congruent to 3 modulo 4, in which case one is solvable but the other is not. This is the famous law of quadratic reciprocity, which Gauss so loved that he gave six different proofs of it. Nowadays every university student studying elementary number theory learns this theorem.

Let us look at a few numerical examples. First, take $p=5$, $q=13$. Since $5 \equiv 13 \equiv 1 \pmod{4}$, both of

$$x^2 \equiv 13 \pmod{5}, \quad y^2 \equiv 5 \pmod{13}$$

must be solvable, or neither is solvable. Since none of $0^2, 1^2, 2^2, 3^2, 4^2$ is congruent to 13 modulo 5 the latter eventuality is the true one. Next take $p=13$, $q=17$. As $13 \equiv 17 \equiv 1 \pmod{4}$, both of

$$x^2 \equiv 13 \pmod{17}, \quad y^2 \equiv 17 \pmod{13}$$

must be solvable, or neither is solvable. Clearly both congruences are solvable, since the second has the solution $y \equiv 2 \pmod{13}$. Finally take $p = 3$, $q = 7$. As $7 \equiv 3 \pmod 4$, exactly one of

$$x^2 \equiv 3 \pmod 7, \qquad y^2 \equiv 7 \pmod 3$$

is solvable. Since the second has the solution $y \equiv 1 \pmod 3$, the first must be insolvable.

The law of quadratic reciprocity is not easy to prove. Gauss gave his first proof at the age of nineteen, a difficult proof depending on an induction argument. It is given in *Disquisisitiones Arithmeticae* (Art. 125 - 145). His second proof (Art. 262) used the theory of quadratic forms, his third and fifth proofs (which are similar to each other) depended on half-systems of residues modulo p, and the fourth and sixth proofs made use of roots of unity. Since Gauss' time, his proofs have been modified or improved, and new proofs have been discovered. In 1963, Murray Gerstenhaber estimated in [3] that there were about 152 proofs in existence.

Gauss' law of quadratic reciprocity is of such fundamental importance in mathmatics that it has become the foundation stone on which a lot of mathematics is built, including, in particular, more general reciprocity laws. Gauss himself wrote two memoirs on the theory of biquadratic reciprocity, and formulated a law of biquadratic reciprocity, for which he never published a proof. The first proofs of it were given by Jacobi and Eisenstein. These mathematicians also obtained the corresponding law of cubic reciprocity, and Eisenstein went on to give even more general laws. These results have been extended in this century to very deep laws by Artin, Hasse, Takagi, and others.

3. *The Gaussian integers.*

In order to formulate his law of biquadratic reciprocity, Gauss found it necessary to extend the idea of the ordinary integers to include numbers of the form $x + iy$, where x, y are ordinary integers and $i = \sqrt{-1}$. These numbers are now known as the *Gaussian integers* and they enjoy many of the properties of the ordinary integers, including the property of unique factorization into primes. Considered as a Gaussian integer, 5 is no longer a prime since $5 = (1 + 2i)(1 - 2i)$, and the factorization is complete since $1 \pm 2i$ are themselves prime Gaussian integers. However, 3 remains a prime. Thus the rules of ordinary arithmetic apply to the Gaussian integers. Gauss had thus invented a higher arithmetic that could be brought to bear upon problems which remained intractable with ordinary arithmetic, and which turned out to be a foundation stone for algebraic number theory.

4. *Algebraic number theory.*

Algebraic number theory is concerned with "integers" even more general than the Gaussian integers. For example, if m is a squarefree integer, one can define a domain of integers I_m as follows: if $m \equiv 2, 3 \pmod 4$ set

$$I_m = \{x + y\sqrt{m} \ : \ x, y \text{ ordinary integers}\}$$

(this includes the Gaussian integers when $m = -1$), and if $m \equiv 1 \pmod 4$ set

$$I_m = \{x + y\,\frac{1+\sqrt{m}}{2} \ : \ x, y \text{ ordinary integers}\}.$$

(There are technical reasons, which we will not go into, for replacing $\sqrt{m}$ by $\frac{1+\sqrt{m}}{2}$ when $m \equiv 1 \pmod 4$.)

The elements of I_m have all the properties of the ordinary integers except possibly that of unique factorization into primes. For example, the ordinary integer 10 has the unique prime factorization $10 = 2 \cdot 5$; but as an element of I_{10} it has the two prime factorizations $10 = 2 \cdot 5 = \sqrt{10} \cdot \sqrt{10}$, since 2, 5, and $\sqrt{10}$ are all primes in I_{10}. Thus for certain values of m (such as $m = -1$), I_m has the unique factorization property while for other m (such as $m = 10$) it does not. It is very important to know for which m the unique factorization property holds, since then the arithmetic is very much simpler. Gauss conjectured that there are infinitely many positive m for which I_m has the unique factorization property. All the evidence is in favour of this conjecture, but it remains unproved to this day. When m is negative, Stark [4], showed in 1967 that there are only finitely many m for which I_m has this property, namely, $m = -1, -2, -3, -7, -11, -19, -43, -67, -163$. Such are the mysteries of arithmetic!

Now let us look at a simple problem of arithmetic which can be solved using the integers of I_{-2}. Suppose we wish to find all ordinary integers x and y such that $y^3 = x^2 + 2$.

As the cube of an integer can never be congruent to 2 modulo 4, x must be odd, and an easy argument shows that this means that the integers $x + \sqrt{-2}$ and $x - \sqrt{-2}$ of I_{-2} have no commom factor. Hence, as $y^3 = (x + \sqrt{-2})(x - \sqrt{-2})$, the unique factorization property forces each of $x + \sqrt{-2}$ and $x - \sqrt{-2}$ to be the cube of an integer in I_{-2}, say,

$$x + \sqrt{-2} = (a + b\sqrt{-2})^3, \qquad (1)$$

$$x - \sqrt{-2} = (a - b\sqrt{-2})^3,$$

where a, b are ordinary integers. This implies that

$$y = a^2 + 2b^2.$$

Equating the imaginary parts of (1), we obtain

$$1 = 3a^2b - 2b^3 = b(3a^2 - 2b^2),$$

and so b divides 1, i.e., $b = \pm 1$. The possibility $b = -1$ is easily ruled out, and $b = 1$ leads to $a = \pm 1$, giving $x = \pm 5$, $y = 3$, as the only solutions.

5. *Gaussian sums.*

As a final example of Gauss' contribution to arithmetic, I will mention the problem of the sign of the so-called Gaussian sum. This sum is extremely important in many areas of mathematics.

Let p be an odd prime and let ζ_p be a pth root of unity, that is, a complex number $(\neq 1)$ such that $(\zeta_p)^p = 1$. Thus, when $p = 3$, there are 2 cube roots of unity namely $\frac{1}{2}(-1 \pm i\sqrt{3})$. It is easily seen that the following finite series

$$1 + \zeta_p + \zeta_p^{\,2} + \ldots + \zeta_p^{\,p-1}$$

is a geometric progression of p terms, whose sum is

$$\frac{\zeta_p^{\,p} - 1}{\zeta_p - 1} = \frac{1 - 1}{\zeta_p - 1} = 0, \qquad \text{as } \zeta_p \neq 1.$$

In his research Gauss was confronted with the analogous sum — known as the Gaussian sum:

$$1 + \zeta_p + \zeta_p^{\,2^2} + \ldots + \zeta_p^{\,(p-1)^2},$$

where the exponents are now the squares of the integers $0, 1, 2, \ldots, p-1$. This sum is, of course, not a geometric progression and its evaluation took Gauss many years. When $p = 3$, with $\zeta_3 = \frac{1}{2}(-1 + i\sqrt{3})$, the sum is

$$1 + \zeta_3 + \zeta_3^{\,4} = 1 + 2\zeta_3 = i\sqrt{3}\,.$$

Indeed in general it is not difficult to prove that if we denote this sum by G_p then

$$G_p = \begin{cases} \pm\sqrt{p}\ , & \text{if } p \equiv 1 \pmod 4, \\ \pm i\sqrt{p} & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

It was thus a question of determining which sign holds. After a great deal of effort Gauss' genius enabled him to prove that the plus sign holds in all cases. Although Gaussian sums have since been generalized, no such beautiful closed form expression has been discovered for the generalizations.

Gauss' name lives everywhere in mathematics and in number theory in particular.

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GAUSS' ROLE IN THE DEVELOPMENT OF NON-EUCLIDEAN GEOMETRY

CYRIL W.L. GARNER, Carleton University

To understand Gauss' part in the development of non-Euclidean geometry, we must review quickly the history of Euclid's famous Fifth Postulate. When Euclid (c. 330 - 275 B.C.) laid the axiomatic foundation for the study of geometry, he was particularly careful in his definition of parallel lines: lines which are in the same plane but do not meet when produced indefinitely in both directions. In order to prove the many properties of parallelism, he enunciated the following *Postulate (V)*:

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Even Euclid's first commentators held that this postulate was too complicated, too "unnatural", to be accepted without proof. During the next 2000 years, almost every mathematician attempted to prove it from the preceding four. Even Gauss tried it — but they all failed. The most common mistake was to substitute unintentionally an axiom equivalent to the famous Fifth Postulate. Martin ([5], pp. 276 - 8) lists 26 such equivalent postulates — there are many more. The most widely used alternative is known as Playfair's Axiom:

> Given a point P and a line ℓ not passing through P, there exists precisely one line incident with P which is parallel to ℓ.

It is impossible to exaggerate the enormous amount of time, energy and creative thought that went into "proving" this axiom. Now, of course, we know that the axiom is independent of the others, and must be assumed in an axiom system for Euclidean geometry; a non-Euclidean geometry, usually called hyperbolic geometry, can be developed by assuming the negation of the Fifth Postulate. Moreover, with the development in the late 19th century of models of hyperbolic geometry in Euclidean geometry, and of models of Euclidean geometry in hyperbolic, we see that the two geometries are relatively consistent; also, the introduction of Cartesian coordinates into Euclidean geometry makes that geometry as consistent as the real number system.

While this is all very clear now, in Gauss' time the question of parallels was a burning one in every mathematician's mind. Gerolamo Saccheri (1667 - 1733) had attempted to derive a contradiction by assuming the negation of the Fifth Postulate (in an equivalent form) and had derived many theorems which are now standard theorems of hyperbolic geometry, without realizing he was developing a consistent system, different from Euclid's. Unable to arrive at a contradiction by valid mathematical reasoning, he had finally concluded that since his theorems exhibited properties which were "repugnant to the nature of the straight line", the negation of the Fifth Postulate must be false.

Saccheri's work attracted little interest at the time of its publication, but it did influence mathematicians of Gauss' time and of the preceding generation. In addition to Gauss himself, Lambert (who died the year Gauss was born), Legendre (1752 - 1833), Fourier (1768 - 1830) and Wachter (1792 - 1817) are some of the prominent men who studied the "theory of parallels" and were influenced by Saccheri.

Then in 1829 Nikolai Ivanovich Lobatchewsky (1792 - 1856) published the first account of non-Euclidean geometry — he assumed the negation of Euclid's Fifth Postulate and developed hyperbolic geometry. Written in Russian and severely criticized by his Russian colleagues, it was unknown outside of Russia until 1840 when he published a treatise in German which Gauss saw and praised.

Almost simultaneously, and certainly independently, János Bolyai (1802 - 1860) published in 1832 his "Science of Absolute Space" as an appendix to a work by his father Wolfgang Bolyai (see [1]). In this work he developed what is now called "absolute geometry", free of any postulate concerning parallels. In his own words, he "created a new universe out of nothing". But it is at this point that Gauss enters, in one of his most disappointing roles as "prince of mathematics".

Bolyai's father, Wolfgang, was a close friend of Gauss — they had been students together at Göttingen 35 years previously, and had maintained an active correspondence since then. Wolfgang eagerly sent off a copy of the book to Gauss, asking for his opinion particularly of his son's Appendix. Gauss' answer was, in the extreme, disappointing, particularly to the young János who clearly expected Gauss to publicize his work. In his letter, Gauss wrote that while he was most impressed by the son's work, he had anticipated all his results by some 30 years, but had not published them. This was actually quite typical of Gauss, as he was a perfectionist in his published work, and in this remarkable letter to Bolyai, he compliments János on his clear writing, and thanks him for saving him the bother of writing up his own work for publication. Those interested in reading this letter should consult Bonola ([1], pp. 100 - 1) or Greenberg ([4], pp. 144 - 6).

But what is most amazing is that Gauss, despite his great reputation, was afraid to make public his discoveries in non-Euclidean geometry. As he relates in an 1829 letter to F.W. Bessel, he feared "the howl from the Bœotians (i.e. dull, obtuse individuals) if he were to publish his revolutionary discoveries". Worse than this he did not even support or publicly commend the efforts of his friend's son. The effect on János Bolyai was crushing — he was unwilling to admit that others could have arrived at non-Euclidean geometry earlier than himself, and even suspected his father of communicating his discoveries to Gauss before sending him the Appendix; he accused Gauss of wishing to claim the honour of discovery for himself. This disappointment plunged him into such a depression that he did no further mathematics. In Gauss' defence, it is clear from his papers and private letters that he had anticipated both Bolyai's and Lobatchewsky's work. He had a clear grasp of the importance of a "natural constant" of length in hyperbolic geometry (as expressed in an 1824 letter to Taurinus, with a request that the communication be kept confidential — see [4], p. 146), that the geometry of any infinitesimal region is Euclidean, and many other deep results. His particularly beautiful proof that the area of a triangle in hyperbolic geometry is proportional to the difference of the sum of its angles from 180° is standard in any textbook on non-Euclidean geometry (see e.g. Coxeter [3], pp. 297 - 9). Although Gauss published nothing at all on the subject, Coolidge ([2], p. 70) claims that his work was a direct result of Saccheri's investigations. But Martin ([5], p. 309) writes that "the only influence Gauss had on the inventions of Lobachewsky and Bolyai was that each author knew that Gauss had failed earlier at the problem of proving the parallel postulate". Had Gauss dared to publish his own work, or even to support Bolyai's work, his place in the history of non-Euclidean geometry would be a more admirable one.

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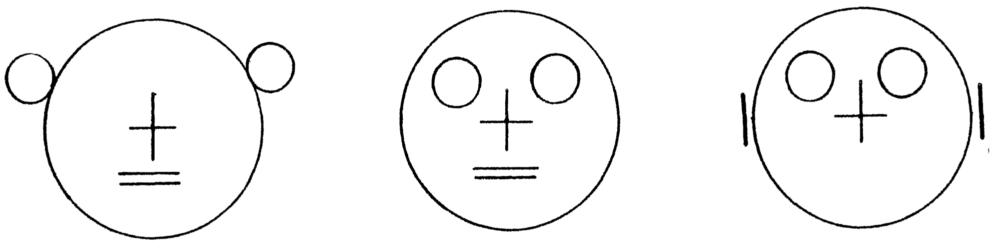
VARIATIONS ON A THEME BY BANKOFF IV

Variation No. 9 by Charles W. Trigg



See no evil Hear no evil Speak no evil

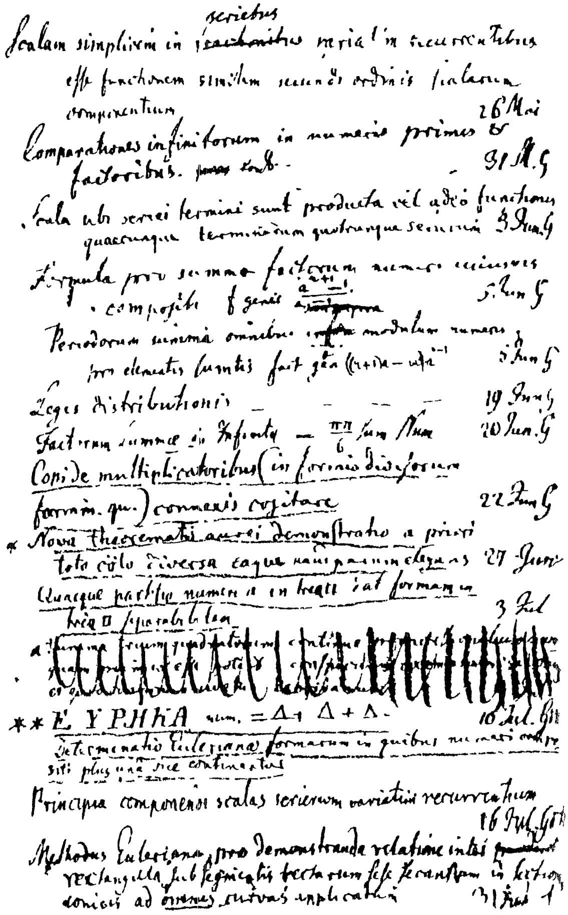
The "Three Monkeys" are very old and must surely have been known to Gauss. The middle "monkey" also happens to be the theme by Leon Bankoff, whose original caption was: $0 + 0 = 0$.

GAUSS, THE FOUNDER OF EUREKA?

TINCA TINCA

The widely held belief that Archimedes was the founding father of EUREKA has been challenged recently. A special committee under the auspices of UNESCO comprising experts from the fields of archeology, anthropology, history, sociology, psychology (para-, meta-, and freudian), and explicitly excluding mathematicians, has examined the available evidence and has produced some new findings. Some of its conclusions follow.

Its studies of folk customs going back to 3000 B.C. show that jumping out of bathtubs was not known among the natives. A Levi-Straussian hypothesis that this action may have been a subconscious re-enactment of an old myth (Athene jumping out of the head of Zeus) was dismissed. No archeological evidence for a bathtub which could be attributed to Archimedes was found in Sicily. The local ordinance records of Syracuse explicitly forbid streaking. The historical sources, notably Plutarch and Seneca, must be rejected as definitely Roman-imperialistically biased. The psychologists who contributed most to the investigations failed to agree. The closest tangible result was claimed by a parapsychologist who reportedly had contacted the ghost of Archimedes. When asked: "How about EUREKA?", the ghost answered: "That's Greek to me!"; and when told: "It is", it vanished and could not be apprehended again.


ΕΥΡΗΚΑ num. = Δ + Δ + Δ

Page 2 of the Diary: 1796, July 10, Göttingen

Dismayed by these negative results which undoubtedly would sadden the Editor and the faithful readers of our venerable publication, I remembered the slogan of 1977: Back to Gauss! And *eureka*! (I beg your pardon) there it was On the second page of Gauss' diary (see the reproduced evidence). The message is somewhat cryptic but it can be easily deciphered: the three Δs probably refer to the undergraduate mathematics journal *Delta*. Clearly, in Gauss' opinion, three issues of *Delta* can hardly match one number of EUREKA. One may be pardoned a small feeling of disappointment in the fact that only two stars are attached to our journal. However, a perusal of Gauss' diary shows that it is still the highest accolade bestowed by him.

Having fathered EUREKA at the tender age of nineteen years, two months, and ten days, Gauss finally receives a fitting tribute in this special issue.

... A last minute flash from Syracuse (Sicily, not N.Y.): a graffito has been found on a bathroom wall: φεῦ παπαῖ, ψυχρός εἰμι.[1]

Back to Archimedes?

Editor's Note.

The author of this article, whose real name is very familiar to readers of this journal, adopted for the nonce the pseudonym *Tinca Tinca*, being no doubt fearful of receiving obscene telephone calls from supporters of the Archimedean hypothesis for the founding of EUREKA.

There is a rationale behind the author's choice of *Tinca Tinca*, and some astute readers may be able to deduce his real name: if so, they are invited to send the name to the Editor, with reasons for their choice. In a month or two, after the hubbub has died down, the Editor will reveal the author's name for all the world to see, along with the names of readers who were able to deduce it on their own.

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GAUSS AND EASTER DATES

VIKTORS LINIS, University of Ottawa

In 1800 Gauss published a short article [1] giving a simple algorithm for the calculation of the Easter dates for the Gregorian as well as the Julian calendars. In contrast to the earlier methods with mystifying names like: Golden Number (year's number in the Metonic lunar cycle of 19 years, Metonic year = 235 lunar months), epact (excess of solar over lunar year on January 1), Easter limits (earliest and latest admissible dates), etc. Gauss' algorithm is a simple application of modular arithmetic (although the congruence symbol is not used in Gauss' article).

The algorithm for the Gregorian calendar in the present-day notation is as follows: denote by Y, M, D, the year, month, day of the Easter date; let $[\ldots]$ denote the greatest integer function and $\equiv$ the congruence symbol. Find the following ten values:

$k = [Y/100]$　　　　$a \equiv Y \pmod{19}$

$p = [(13 + 8k)/25]$　　　　$b \equiv Y \pmod{4}$

[1] Damn it, I'm cold.

$q = [k/4]$

$m \equiv 15 - p + k - q \pmod{30}$

$n \equiv 4 + k - q \pmod{7}$

$c \equiv Y \pmod{7}$

$d \equiv 19a + m \pmod{30}$

$e \equiv 2b + 4c + 6d + n \pmod{7}$

Then the Easter date is given by:

(i) if $d + e \leq 9$ then $M = 3$ and $D = 22 + d + e$

(ii) if $d = 29$, $e = 6$ then $M = 4$, $D = 19$

(iii) if $d = 28$, $e = 6$, $a > 10$ then $M = 4$, $D = 18$

(iv) in all other cases $M = 4$, $D = d + e - 9$.

In the Julian calendar the m and n values are fixed: $m = 15$, $n = 6$; the remaining calculations are the same.

In a later article [2] Gauss gave an algorithm for the computation of the Jewish Easter dates (and the New Year) as well.

Professor Carl-Eric Fröberg (University Lund, Sweden) has made statistical calculations concerning the relative frequencies of Easter dates over one *period* (see Problem 231 in this issue) [3]. In decreasing order these are:

April 19	3867×10^{-5}
April 18	3463
March 29, April 1,3,5,8,10,12,15,17	3383
March 30, 31, April 6,7,13,14,20	3325
March 28, April 2,4,9,11,16	3267
March 27	2900
April 21	2850
April 22	2417
March 26	2333
March 25	1933
April 23	1867
April 24	1450
March 24	1425
March 23	950
April 25 (latest date)	737
March 22 (earliest date)	483

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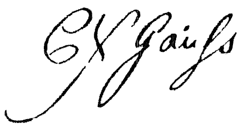
* * *

Gauss' signature.



PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should be mailed to the editor no later than September 1, 1977.

231. *Proposed by Viktors Linis, University of Ottawa.*

Find the period P of the Easter dates based on the Gaussian algorithm (see pp. 102 - 103 in this issue), that is, the smallest positive integer P satisfying the conditions:

$$D(Y+P)=D(Y) \quad \text{and} \quad M(Y+P)=M(Y)$$

for all Y, where D and M are the day and month functions of year number Y.

232. *Proposed by Viktors Linis, University of Ottawa.*

Given are five points A, B, C, D, E in the plane, together with the segments joining all pairs of distinct points. The areas of the five triangles BCD, EAB, ABC, CDE, DEA being known, find the area of the pentagon ABCDE.

The above problem with a solution by Gauss was reported by Schumacher [*Astronomische Nachrichten*, Nr. 42, November 1823]. The problem was given by Möbius in his book (p. 61) on the Observatory of Leipzig, and Gauss wrote his solution in the margins of the book.

233. *Proposed by Viktors Linis, University of Ottawa.*

The three points (1), (2), (3) lie in this order on an axis, and the distances $[1,2]=a$ and $[2,3]=b$ are given. Points (4) and (5) lie on one side of the axis, and the distance $[4,5]=2c>0$ and the angles $(415)=v_1$, $(425)=v_2$, $(435)=v_3$ are also known. Determine the position of the points (1), (2), (3) relative to (4) and (5).

Gauss gave a solution to this problem which was found in a book on navigation [*Handbuch der Schiffahrtskunde* von C. Rümker, 1850, p. 76].

234. *Proposed by Viktors Linis, University of Ottawa.*

Prove that

$$\cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{4\pi}{13}\ldots\cos\frac{2^{n-1}\pi}{13}=\pm\frac{1}{2^n}$$

Gauss' remark: inspect a polygon!

235. *Proposed by Viktors Linis, University of Ottawa.*

Prove Gauss' *Theorema Elegantissimum:* If

$$f(x) = 1 + \frac{1}{2}\cdot\frac{1}{2}xx + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}x^6 + \ldots,$$

show that

$$\sin\phi\, f(\sin\phi)\, f'(\cos\phi) + \cos\phi\, f(\cos\phi)\, f'(\sin\phi) = \frac{2}{\pi\sin\phi\cos\phi}\ .$$

(Gauss actually wrote xx, but feel free to write x^2 if you prefer.)

236. *Proposed by Viktors Linis, University of Ottawa.*

Solve the cryptarithmic subtraction:

$$\begin{array}{r} \text{GAUSS} \\ -\ \text{DIED} \\ \hline 1855 \end{array}$$

237. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

It is a well-known theorem (due to Gauss and F. Lucas) that if $f(z)$ is a polynomial with complex coefficients, then the zeros of the derivative $f'(z)$ all lie in the convex hull of the zeros of $f(z)$.

Prove or disprove the following converse: Suppose a closed set E in the complex plane has the property that if a polynomial has all its zeros in E then the derivative also has all its zeros in E ; then F is convex.

238. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution to this cryptarithm having both 1 and 7 represented among the letters:

$$\begin{array}{r} \text{CARL} \\ +\ 1777 \\ \hline \text{GAUSS} \end{array}$$

239. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve this addition cryptarithm. There is a unique solution in which each of the date digits 1, 5, 7, 8 is represented by a letter:

$$\begin{array}{r} \text{CARL} \\ 1777 \\ 1855 \\ \hline \text{GAUSS} \end{array}$$

240. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution for this base ten cryptarithm:

$$\begin{array}{r} \text{CARL} \\ \times\quad \text{F} \\ \hline \text{GAUSS} \end{array}$$

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

142. [1976: 93, 175] *Proposed by André Bourbeau, École Secondaire Garneau.*

Find 40 consecutive positive integral values of x for which $f(x) = x^2 + x + 41$ will yield composite values only.

V. *Comment by Harry L. Nelson, Livermore, California.*

It is my belief, based on a computer search lasting one minute only, that the longest string of primes generated by the formula

$$f(x) = x^2 + x + 41$$

(not counting the 40 already known for $x = 0, 1, \ldots, 39$) is the group of 13 from $x = 219$ to $x = 231$ (see Table 1). It seems unlikely, however, that the proof of this will ever be found.

x	$f(x)$	x	$f(x)$	x	$f(x)$
218	71•673	223	49993	228	52253
219	48221	224	50441	229	52711
220	48661	225	50891	230	53171
221	49103	226	51343	231	53633
222	49547	227	51797	232	47•1151

Table 1.

Deciding it might be more fruitful to search for long strings of consecutive composite values of $f(x)$, I made another computer search and found that the smallest string of 38 consecutive composite values was that from $x = 176955$ to $x = 176992$ (see Table 2).

x	$f(x)$	Factorization	x	$f(x)$	Factorization
176954	31312895111	Prime	176974	31319973691	1063•2647•11131
176955	31313249021	41•763737781	176975	31320327641	6271•4994471
176956	31313602933	$41\cdot61^2\cdot205253$	176976	31320681593	39619•790547
176957	31313956847	47•666254401	176977	31321035547	130121•240707
176958	31314310763	409•1091•70177	176978	31321389503	739•42383477
176959	31314664681	148747•210523	176979	31321743461	307•102025223
176960	31315018601	13877•2256613	176980	31322097421	24943•1255747
176961	31315372523	151•797•260209	176981	31322451383	2917•10737899
176962	31315726447	83•377297909	176982	31322805347	$151^2\cdot499\cdot2753$
176963	31316080373	53•590869441	176983	31323159313	661•47387533
176964	31316434301	383•81766147	176984	31323513281	55207•567383
176965	31316788231	61•647•793493	176985	31323867251	80231•390421
176966	31317142163	113•277142851	176986	31324221223	43•5393•135077
176967	31317496097	$173^2\cdot1046393$	176987	31324575197	347•6829•13219
176968	31317850033	85621•365773	176988	31324929173	26959•1161947
176969	31318203971	251•124773721	176989	31325283151	43•728494957
176970	31318557911	53•590916187	176990	31325637131	131•173•1382237
176971	31318911853	131•1301•183763	176991	31325991113	307•3433•29723
176972	31319265797	23671•1323107	176992	31326345097	7499•4177403
176973	31319619743	1061•2699•10937	176993	31326699083	Prime

Table 2.

Then I discovered that the snotty editor had condescendingly written in [1976: 178]: "It would be interesting if some computer nut were to make a search and discover the smallest set of 40 consecutive integers x for which $f(x)$is composite." Without in any way admitting that my middle name is Filbert, I hopped once more on my trusty computer Rosinante and soon came upon the smallest string of 40 consecutive composite values of $f(x)$, which begins at $x = 1081296$ (see Table 3). Furthermore, on the way from Table 2 to Table 3, I scrutinized the landscape very carefully, but could espy no other string of 38 or 39 consecutive composite values.

x	$f(x)$	Factorization
1081295	1169199958361	Prime
1081296	1169202120953	3607•324148079
1081297	1169204283547	397•39343•74857
1081298	1169206446143	56923•20540141
1081299	1169208608741	829•1410384329
1081300	1169210771341	151•7743117691
1081301	1169212933943	20261•57707563
1081302	1169215096547	53•18133•1216603
1081303	1169217259153	281•4160915513
1081304	1169219421761	367•3185883983
1081305	1169221584371	113•10347093667
1081306	1169223746983	19963•58569541
1081307	1169225909597	1867•626259191
1081308	1169228072213	3187•366874199
1081309	1169230234831	53•22060947827
1081310	1169232397451	227•5150803513
1081311	1169234560073	6373•183466901
1081312	1169236722697	1493•7187•108967
1081313	1169238885323	223•1031•5085571
1081314	1169241047951	240997•4851683
1081315	1169243210581	91807•12735883
1081316	1169245373213	172507•6777959
1081317	1169247535847	83•14087319709
1081318	1169249698483	227•5150879729
1081319	1169251861121	43•263•103391269
1081320	1169254023761	197•5935299613
1081321	1169256186403	151•7743418453
1081322	1169258349047	43•251•8387•12917
1081323	1169260511693	973823 • 1200691
1081324	1169262674341	71•46817•351763
1081325	1169264836991	10369•112765439
1081326	1169266999643	47•24878021269
1081327	1169269162297	911•1283500727
1081328	1169271324953	4219•8353•33179
1081329	1169273487611	263•45953•96749
1081330	1169275650271	83•14087658437
1081331	1169277812933	47•7237•3437647
1081332	1169279975597	25793•45333229
1081333	1169282138263	41•2003•14239181
1081334	1169284300931	41•28519129291
1081335	1169286463601	71•16468823431
1081336	1169288626273	Prime

Table 3.

Editor's comment.

Our heartfelt thanks to Rosinante and to Harry L. (Filbert) Nelson, Executive Editor of the *Journal of Recreational Mathematics*. In the current (April 1977) *Scientific American*, Martin Gardner writes glowingly of Harry Nelson and *J.R.M.* The encomium is well deserved.

*　　　　　*　　　　　*

154. [1976: 110,159,197,225; 1977: 20] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

1	2	3	4	5	6	7	8	9	10	11	12
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	3	5	9	7	15	11	27	25	21	13	45

We observe that $\ell = 17 = p_7$.

IV. *Comment by Harry L. Nelson, Livermore, California.*

In his comment III [1977: 20 - 22], the proposer notes that the existence of two pairs of consecutive primes p_{r-1}, p_r, p_n, p_{n+1} such that

$$\sqrt{p_n} < p_{r-1} < p_r < \sqrt{p_{n+1}}$$

would invalidate his equivalent Conjectures 1 and 2. Such pairs of primes do not exist. In fact, the following is true: when p_n and p_{n+1} are consecutive primes, we have always

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1, \tag{1}$$

the maximum value being 0.67087 which occurs for $\sqrt{11} - \sqrt{7}$.

Editor's comment.

It would be helpful to have either a proof or a reference for relation (1). Nelson also asks the following interesting question, which is only distantly, if at all, connected with our problem: are there consecutive primes p_n and p_{n+1} such that

$$p_{n+1} - p_n = 2i, \qquad i = 1, 2, 3, \ldots . \tag{2}$$

He says that (2) is known to hold for $1 \le i \le 50$, and he is prepared to send on request a list of such primes. Interested readers should write to: Harry L. Nelson, 4259 Emory Way, Livermore, California 94550.

There is in existence a formula which gives explicitly the difference $p_{n+1} - p_n$ in terms of $p_1, \ldots, p_{n-1}$. It occurs in a paper entitled "Formulae for the nth prime" presented around 1966 by J.M. Gandhi and published in the *Proceedings of the Washington State University Conference on Number Theory,* March 1971, pp. 96 - 111. This formula may perhaps be of help in answering the interesting question raised by Nelson in (2). The reference was sent to me by Kenneth M. Wilke, 600 First National Bank Tower, Topeka, Kansas 66603, who would undoubtedly honour readers' requests for additional information on this matter.

* * *

196. [1976: 220] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Montrer que, si $|a_i| < 2$ pour $1 \le i \le n$, alors l'équation

$$1 + a_1 z + \ldots + a_n z^n = 0$$

n'a pas de racine à l'intérieur du disque $|z| = \frac{1}{3}$.

La réciproque est-elle vraie?

Solution du proposeur.

On a, pour $|z| < \frac{1}{3}$,

$$|a_1 z + \ldots + a_n z^n| < 2\left(\frac{1}{3} + \ldots + (\tfrac{1}{3})^n\right) = 1 - \frac{1}{3^n} < 1,$$

et l'équation donnée n'a pas de racine. La réciproque est fausse, car est elle contredite par l'équation $1 + \frac{5}{2}z = 0$.

* * *

197. [1976: 220] *Proposed by Charles W. Trigg, San Diego, California.*

In the octonary system, find a square number that has the form $aaabaaa$.

I. *Solution by the proposer.*

In the octonary system, odd square numbers terminate in 1, and even square numbers terminate in 0 or 4. Hence $a = 1$ or 4, whereas b may have any one of eight values.

By the standard procedure for extracting square roots,

$$\sqrt{1110111} = 1042+, \qquad \sqrt{1117111} = 1046+,$$

so for possible odd solutions we need only test

$$(1043)^2 = 1110311, \qquad (1045)^2 = 1114531,$$

neither of which is of the proper form.

We repeat the process to find possible even solutions. Since

$$\sqrt{4440444} = 2105+, \qquad \sqrt{4447444} = 2107+,$$

the only possible solution is

$$(2106)^2 = 4441444,$$

which is of the proper form and is thus the unique solution.

II. *Comment by Michael P. Closs, University of Ottawa.*

I suspect that this problem is a hoax, and thus entirely suitable for an April issue of this journal.

The only octonary system I have come across is that of the Yuki Indians of California (which the proposer should know, being himself from California). A check of the Yuki number sequence (see [1]) reveals the following squares:

1	*pa-wi*	
4	*o-mahat*	"two-forks"
9	*hutcam-pawi-pan*	"beyond-one-hang"
16	*hui-co(t)*	"middle-none"
64	*omahat-tc-am-op*	"two-fork-pile-at"

It is clear that none of these number words have the form *aaabaaa*. Moreover it is known from inspection of the Yuki number sequence that the higher numerals are of a compound form and that the elements entering into these compounds are the following: *sul, luk, coi, al-a-wa, kitc, poi, pat, pan pa, hutcam, mikas*. It is thus evident that no square in the Yuki sequence could possibly have the form *aaabaaa*.

Unless the proposer can show that the Yuki were not unique in possessing an octonary system, I am afraid that his problem has no solution.

The Yuki were led to their octonary system by counting the spaces between the fingers rather than the digits themselves. If we count the spaces between the digits of the form *aaabaaa* we find that there are 3 spaces on each side of the *b*. Consequently, the Yuki might interpret the form *aaabaaa* as a cryptogram for 3×3 or 9 which is a square. However, at best, this is only a weak solution to the problem and I doubt if the Yuki would have approved of it.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* G.D.KAYE, *Department of National Defence, Ottawa;* ANDRÉ LADOUCEUR, *École secondaire De La Salle, Ottawa;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* H.L. RIDGE, *University of Toronto;* R. ROBINSON ROWE, *Sacramento, California;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer (second solution). Late solution by* F.G.B. MASKELL, *Algonquin College, Ottawa.*

REFERENCE

1. Roland B. Dixon and A.L. Kroeber, "Numeral Systems of the Languages of California", *American Anthropologist*, Vol. 9 (1907), pp. 663 - 690.

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198. [1976: 220] *Proposed by Gali Salvatore, Ottawa, Ont.*

Without using an acre of paper, find the coefficient of x^8 in the expansion of the polynomial

$$P = (1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^6.$$

Solution by L.F. Meyers, The Ohio State University.

$$P = \left((1+x)^{-2} + 8x^7 - 9x^8 + \dots\right)^6$$

$$= (1+x)^{-12} + 6(1+x)^{-10}(8x^7 - 9x^8 + \dots) + \text{terms of degree} \ge 14.$$

The expansion of P contains only three terms in x^8; their coefficients are 75582, -54, and -480. (To keep down the acreage, the first of these was calculated on the neighbor's lot). The required coefficient is thus $75582 - 54 - 480 = 75048$.

Also solved by W.J. BLUNDON, *Memorial University of Newfoundland;* ROLAND H. EDDY, *Memorial University of Newfoundland;* R.S. JOHNSON, *Montréal, Québec (answer only);* G.D. KAYE, *Department of National Defence, Ottawa;* MURRAY S. KLAMKIN, *University of Alberta;* F.G.B. MASKELL, *Algonquin College, Ottawa;* R. ROBINSON ROWE, *Sacramento, California;* CHARLES W. TRIGG, *San Diego, California; and the proposer. Two incorrect solutions were received.*

Editor's comment.

Klamkin, using essentially the same method as in our featured solution, found that the coefficient of x^{2n+2} in the expansion of

$$\left(1 - 2x + 3x^2 - \dots + (2n+1)x^{2n}\right)^m$$

was

$$\binom{-2m}{2n+2} + m(2n+2)\binom{-2m}{1} + m(2n+1),$$

which reduces to 75048 when $n = 3$ and $m = 6$.

Our solver, L.F. Meyers, calculated that his solution took up $4 \times 8\frac{1}{2}$ in.^{2} or 5.42×10^{-6} acre, and expert editing reduced this to the 1.35×10^{-6} acre it occupies on this page; while Rowe's solution, by his own admission, took up an entire (Lilliputian) acre.

Edith Orr, EUREKA's resident poetess and arbiter of linquistic propriety, took exception to the proposer's use of the anachronistic *acre*, since the metric system is now legal in Canada. Indeed, she fairly hectared the proposer about it.

Miss Orr, who is Chairperson[1] of the local GO METRIC Committee, feels strongly about such matters. She should be patient and realize that long-ingrained habits are hard to break, and that in Canada, if the truth be told, the metric system is just inching along.

* * *

199. [1976: 220] *Proposed by H.G. Dworschak, Algonquin College, Ottawa, Ont.*

If a quadrilateral is circumscribed about a circle, prove that its diagonals and the two chords joining the points of contact of opposite sides are all concurrent.

I. *Essence of the solutions submitted by Murray S. Klamkin, University of Alberta; Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India; and Charles W. Trigg, San Diego, California.*

This theorem is true not only for a circle but for any conic.

Brianchon's Theorem states that the lines joining the opposite vertices of a hexagon circumscribed about a conic are concurrent. The quadrilateral in Figure 1 can be considered as a degenerate circumscribed hexagon in two ways: either AFBCHD, whereupon FH passes through P, the intersection of AC and BD; or ABGCDE, whereupon AC, BD, and GE are concurrent at P. Hence the diagonals and the joins of the opposite points of tangency of the quadrilateral are concurrent.

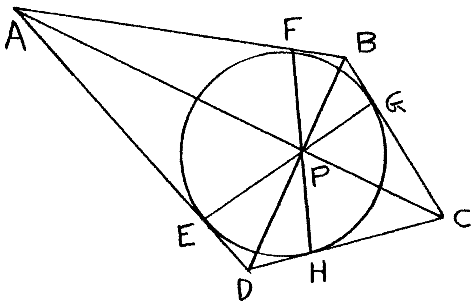

Figure 1.

II. *Composite solution made up from those submitted by W.A. McWorter, Jr., The Ohio State University; and Basil C. Rennie, James Cook University of North Queensland, Australia.*

Any quadrilateral is a perspective projection of a parallelogram. In the preimage of this projection, which preserves intersections, the inscribed circle is an inscribed ellipse. Thus we have an ellipse inscribed in a parallelogram. The chords joining the points of contact of opposite sides go through the centre of the

[1]It is only with the greatest reluctance that I use this barbarous neologism, in deference to Miss Orr's sensitive feelings. Before it becomes more firmly entrenched in the language, I would like to suggest a less obnoxious substitute: *meet head* has no sexist overtones and would be appropriate in most cases.

ellipse, and so do the diagonals of the parallelogram. Why? If it isn't obvious read on. The ellipse and its circumscribing parallelogram are both unchanged by rotation of half a turn about the centre, each of the two diagonals and the two chords is invariant under this transformation, and the only invariant lines are those through the centre.

Incidentally, this method shows that "circle" could be replaced by "ellipse" in the problem.

III. *Solution by Joseph D.E. Konhauser, Macalester College, St. Paul, Minnesota.*

This theorem is usually proved by an appeal to Brianchon's Theorem, but I give here an elementary proof which uses only the rudiments of high school geometry.

Let the vertices of the quadrilateral be A, B, C, D and let the points of tangency of the inscribed circle with the sides of the quadrilateral be P, Q, R, S, as shown in Figure 2. Let X be the point of intersection of diagonal AC with QS. The ratio of the areas $|AXS|$ and $|CXQ|$ of triangles AXS and CXQ is

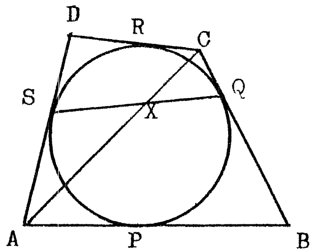

Figure 2.

$$\frac{|AXS|}{|CXQ|}=\frac{\frac{1}{2}XS\cdot XA\sin AXS}{\frac{1}{2}XQ\cdot XC\sin CXQ}=\frac{XS\cdot XA}{XQ\cdot XC}. \tag{1}$$

Lines SD and QC are tangent to the inscribed circle at the endpoints S and Q of chord SQ, so angles ASX and CQX are supplementary and have equal sines; hence

$$\frac{|AXS|}{|CXQ|}=\frac{\frac{1}{2}SA\cdot SX\sin ASX}{\frac{1}{2}QC\cdot QX\sin CQX}=\frac{SA\cdot SX}{QC\cdot QX}. \tag{2}$$

From (1) and (2),

$$\frac{XA}{XC}=\frac{SA}{QC}.$$

Let Y be the point of intersection of diagonal AC with PR. An argument like the one above, applied to triangles AXP and CXR, yields

$$\frac{YA}{YC}=\frac{PA}{RC}.$$

But $AP = AS$ and $CR = CQ$, so

$$\frac{XA}{XC}=\frac{YA}{YC}$$

and X = Y. To complete the solution, apply the preceding argument to diagonal BD.

Also solved by F.G.B. MASKELL, *Algonquin College, Ottawa; and* JOSEPH D.E. KONHAUSER, *(second solution).*

Editor's comment.

Brianchon's Theorem can be found in most texts on advanced geometry. Some of the more accessible references are given below.

Solutions I and II are projective in nature. Coxeter and Greitzer write in [3]:

> Brianchon's proof employs the "duality" of points and lines, which belongs to projective geometry. However, in the case when the conic is a circle, the search for a Euclidean proof became a challenging problem. This challenge was successfully answered by A.S. Smogorzhevskii [6].

Coxeter and Greitzer then give Smogorzhevskii's proof of Brianchon's Theorem. The proof is based on the existence of three circles which, when taken in pairs, have radical axes that are the three diagonals of the circumscribed hexagon. And it is known that such radical axes must be concurrent (or parallel). A specialization of this proof to a degenerate hexagon (quadrilateral) would give us a non-projective proof for our problem. But our own solution III would seem to be a more elementary and more direct way of achieving this end.

Klamkin pointed out that Brianchon's Theorem with a still more degenerate hexagon (triangle) leads to the following theorem, which can be found in Dörrie [5]: *In every triangle circumscribed about a conic the lines connecting the vertices to the opposite points of contact meet in a point.*

REFERENCES

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, Wiley, 1969, pp. 254 - 255.

2. H.S.M. Coxeter, *Projective Geometry*, Blaisdell, 1964, pp. 83 - 84.

3. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library, Random House, 1967, pp. 77 - 79.

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EYPHKA! $1977 = \Delta + \Delta + \Delta$

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One of the best-known results proved by Gauss was that every positive integer is the sum of three triangular numbers (counting zero as a triangular number). Gauss noted the fact in his diary as follows (see page 101 in this issue):

$$** \text{EYPHKA} \quad \text{num.} = \Delta + \Delta + \Delta. \tag{1}$$

Triangular numbers are of the form $\frac{1}{2}n(n+1)$. If M is a given positive integer, then (1) means that there exist integers x, y, and z such that

$$M = \tfrac{1}{2}x(x+1) + \tfrac{1}{2}y(y+1) + \tfrac{1}{2}z(z+1)$$

or, equivalently, that

$$8M + 3 = (2x+1)^2 + (2y+1)^2 + (2z+1)^2.$$

Thus M is the sum of three triangular numbers if and only if $8M+3$ is the sum of three odd squares.

Using the theory of ternary quadratic forms (see [1]), Gauss showed that if a positive integer $n \equiv 0, 4,$ or $7 \pmod 8$ it can never be expressed as a sum of three odd squares, that it may or may not be so expressible if $n \equiv 1, 2, 5,$ or $6 \pmod 8$ and that it is always so expressible in at least one way if $n \equiv 3 \pmod 8$. Since $n = 8M + 3 \equiv 3 \pmod 8$, the result follows. Gauss also showed that the number of ways $n = 8M + 3$ can be expressed as the sum of three odd squares is in general $2^{m+2}h$, where m is the number of prime factors of n and h is the number of classes of binary quadratic forms of determinant $-n$.

For $M = 1977$, we have $n = 8M + 3 = 15819 = 3 \cdot 5273$, so $m = 2$ and it can be shown that $h = 1$. Thus $M = 1977$ should be expressible as the sum of three triangular numbers in $2^4 \cdot 1 = 16$ ways.

$M =$	Δ +	Δ +	Δ	$8M+3 =$	a^2 +	b^2 +	c^2
1977 =	1953 +	21 +	3	15819 =	125^2 +	13^2 +	5^2
	1770	171	36		119	37	17
	1653	171	153		115	37	35
	1596	378	3		113	55	5
	1596	276	105		113	47	29
	1596	210	171		113	41	37
	1326	630	21		103	71	13
	1326	351	300		103	53	49
	1275	666	36		101	73	17
	1275	351	351		101	53	53
	1176	780	21		97	79	13
	1176	630	171		97	71	37
	1035	666	276		91	73	47
	903	903	171		85	85	37

The table above gives all the ways $M = 1977$ can be expressed as the sum of three triangular numbers and the three odd squares whose sum is $8M+3$. Note that the table contains only 14 entries, since two of them must be counted twice because of squares (53 and 85) being alike.

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1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, 1971, Vol. II, p. 17.

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LETTER TO THE EDITOR

Dear Editor:

May I comment briefly on Clayton Dodge's article on the history of complex numbers [1977: 32-39]? His concluding references to high school teaching give me an excuse, since I'm principally interested in mathematics education (including the application of problem-solving, heuristics, etc., in the classroom).

The article seems to me to be simultaneously almost wholly right and yet fundamentally wrong. The bare facts (who did what when) I can't quarrel with; they seem to be well-established and agreed upon. I can't even dispute the main emphases of the narrative, though I find unnecessary and distasteful the parenthetical sneers of hindsight. It's the covert messages contained in the envelope I object to. I pick two, one is an epistemological point, the other a question of what is important in mathematics.

First, it is said that the "true meaning" of complex numbers is given by Gauss and Hamilton's definitions and operational rules. This is a horrid over-simplification. We only have to ask "why these particular rules?" and/or "when do we apply these rules?" to see that the meaning is not contained within the formal system. A word does not get put into a dictionary until it exists in speech or print, and it doesn't get into speech or print unless a meaning has been identified that calls for a new word. The achievement of the dictionary definition is not to settle the "true meaning" of the word but to display its various connections with other words already in the dictionary. The analogy with what happens in mathematics is not perfect, but is, I believe, close enough to be instructive.

Second, this piece is typical of many "popular" accounts of mathematics history in loading *all* its values on the side of increasing precision and clarification, of mathematicians being clear and not muddy, of being right and not wrong. Well, of course. But what about an equally sympathetic treatment of the adventurous, the exploratory, the "foolish" activities of mathematicians, since these are part of the history too - and not regrettable episodes but, frequently, essential energizings of the mathematical body? I'm afraid popular histories have done as much as poor textbooks to give learners a distorted view of what mathematics is really like. (I'd partially except Boyer, Carruccio, Eves, Kline and Struik, only one of which appears on Dr. Dodge's list.) When mathematicians write history it seems they are almost always concerned to suppress and deny part of the mathematical experience and present only what can be most admired in its approximation to perfection. The rest is included only as a frame for these jewels.

I suppose I am escalating a little extravagantly from a very short article(!) but although at times I want to be struck by the greatness of certain mathematical achievements, at other times I like to feel that mathematics is a human activity I might even share in.

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