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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

Dear ***Crux*** readers,

Welcome to Volume 40!

This year ***Crux*** is celebrating its 40th anniversary and we are happy to be going strong.

As with most new volumes, this volume will see some changes in the Editorial Board. I would like to sincerely thank all the editors who have contributed to ***Crux*** in the past and I would like to welcome my new editors to Volume 40. I am very glad to see Robert Bilinski, who was the Skoliad editor for Volumes 31-34, step into the positions of Book Reviews Editor as well as Contest Corner Editor. I am also excited about supplementing my Guest Editors portion of the Board; Joseph Horan, Alejandro Erickson and Robin Koytcheff – I hope you find this job rewarding and inspiring.

As I mentioned in the last issue, all the back files of ***Crux*** are now available online, so you can access all the materials of the last 40 years. Happy reading!

Happy 40th Anniversary, and here is to 40 more!

Kseniya Garaschuk

THE CONTEST CORNER

No. 21

Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-contest@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_CCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 June 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

CC101. Find all pairs of whole numbers a and b such that their product ab is divisible by 175 and their sum $a+b$ is equal to 175.

CC102. In pentagon $ABCDE$, angles B and D are right. Prove that the perimeter of triangle ACE is at least $2BD$.

CC103. Let a and b be two rational numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{ab}$ is also rational. Prove that $\sqrt{a}$ and $\sqrt{b}$ must also be rationals.

CC104. Compare the area of an incircle of a square to the area of its circumcircle.

CC105. Knowing that $3.3025 < \log_{10} 2007 < 3.3026$, determine the left-most digit of the decimal expansion of 2007^{1000}.

. . . .

CC101. Déterminer toutes les paires d'entiers non négatifs a et b dont le produit ab est divisible par 175 et la somme $a+b$ est égale à 175.

CC102. Dans le pentagone $ABCDE$, les angles B et D sont droits. Démontrer que le périmètre du triangle ACE est supérieur ou égal à $2BD$.

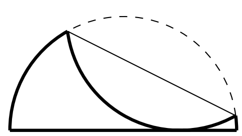
CC103. Soit deux nombres rationnels a et b tels que $\sqrt{a}+\sqrt{b}+\sqrt{ab}$ est aussi rationnel. Montrez que $\sqrt{a}$ and $\sqrt{b}$ sont aussi des rationnels.

CC104. Comparer l'aire du cercle inscrit dans un carré à l'aire du cercle circonscrit au carré.

CC105. En partant du fait que $3.3025 < \log_{10} 2007 < 3.3026$, déterminer le premier chiffre à gauche dans l'écriture décimale de 2007^{1000}.

CONTEST CORNER SOLUTIONS

CC51. A semicircular piece of paper with radius 2 is creased and folded along a chord so that the arc is tangent to the diameter as shown in the diagram. If the contact point of the arc divides the diameter in the ratio 3 : 1, determine the length of the crease.



Originally problem 5 of 1997 Invitational Mathematics Challenge, Grade 11.

Solved by L. Bobo; R. Girard; R. Hess; S. Muralidharan; A. Plaza; N. Stanciu; and T. Zvonaru. We present the solution by S. Muralidharan.

Let $O(0,0)$ be the centre of the semi-circle with radius 2. Let C be the centre of the circle formed by extending the folded portion. Let R be the point of tangency. Let DE be the length of the crease. Let S be the point where the line joining the centres intersects the crease.

The folded circular portion also has radius 2 and since it touches the diameter at the point $(1,0)$, its centre is at $C(1,2)$. Since these are equal circles, the line joining their centres and the common chord (the crease) bisect each other. Thus, $OS = \frac{OC}{2} = \frac{\sqrt{5}}{2}$ and hence

$$DE = 2DS = 2\sqrt{OD^2 - OS^2} = 2\sqrt{4 - \frac{5}{4}} = \sqrt{11}.$$

CC52. There are some marbles in a bowl. Alphonse, Beryl and Colleen each take turns removing one or two marbles from the bowl, with Alphonse going first, then Beryl, then Colleen, then Alphonse again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can Beryl and Colleen work together and force Alphonse to lose ?

Originally 2002 Canadian Open Mathematics Challenge, problem B3b.

Solved by R. Hess ; and S. Muralidharan. We present the solution by Richard Hess.

We claim that Beryl and Colleen can force Alphonse to lose for all N except $N = 2, 3, 4, 7$, or 8.

At $N = 2$, Alphonse leaves 1.

At $N = 3$ or 4, Alphonse leaves 2.

At $N = 7$ or 8, Alphonse leaves 6 after which Beryl and Colleen must leave 2, 3 or 4.

For $N = 5$ or 6, regardless of what Alphonse takes, Beryl and Colleen can work it so that when Alphonse's turn arrives there is only one marble left.

For $N = 9$ or 10, Alphonse must leave 7, 8 or 9 from which Beryl and Colleen can force 5 or 6.

For $N = 4k$ where $k > 2$, Alphonse must leave either $4k - 1$ or $4k - 2$ from which Beryl and Colleen can force $4(k - 1) + 1$ or $4(k - 2) + 2$.

For $N = 4k + 1$, Alphonse must leave either $4k$ or $4k - 1$ from which Beryl and Colleen can force $4(k - 1) + 2$ or $4(k - 1) + 1$.

For $N = 4k + 2$, Alphonse must leave either $4k + 1$ or $4k$ from which Beryl and Colleen can force $4(k - 1) + 2$ or $4(k - 1) + 1$.

For $N = 4k + 3$, Alphonse must leave either $4k + 2$ or $4k + 1$ from which Beryl and Colleen can force $4(k - 1) + 2$.

In all cases for $N \geq 11$, Alphonse will always be faced with a new value of the form $4t + 1$ or $4t + 2$ on his next turn eventually forcing him to $N = 5$ or 6 and a loss.

CC53. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

Originally problem 8 of 2009 Sun Life Financial Repêchage Competition.

Solved by R. Hess ; and T. Zvonaru. We present the solution by Titu Zvonaru.

Starting with the identity, $3^{n+1} = 3^n + 3^n + 3^n$, we want to find a positive integer, n, so that $7 \mid (n + 1)$, $6 \mid n$, $5 \mid n$, and $4 \mid n$. The latter three relations combine to $60 \mid n$, and we can quickly find that $n = 300$ works. Thus we can write $(3^{75})^4 +$

$(3^{60})^5 + (3^{50})^6 = (3^{43})^7$. Multiplying by m^{420} for any positive integer, m, gives us an infinity of quadruples

$$(a, b, c, d) = (m^{105}3^{75}, m^{84}3^{60}, m^{70}3^{50}, m^{60}3^{43})$$

of positive integers, each of which is a solution to the equation from the statement.

CC54. Let k, l, m, n be positive integers such that $k + l + m \geq n$. Prove the following relation for binomial coefficients

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

The summation in the left-hand side runs over all ordered partitions of n into three integers p, q, r such that $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq r \leq m$.

Originally 2004 Memorial University Undergraduate Mathematics Competition, Question 3.

Solved by J. L. Díaz-Barrero; S. Muralidharan; A. Plaza; and D. Văcaru. We present the solution by S. Muralidharan below.

We will prove the result by counting the number of ways of forming a team of n people from a group containing k Canadians, l Americans and m Australians.

The number of teams in which there are p Canadians, q Americans and r Australians where $0 \leq p \leq k$, $0 \leq q \leq l$ and $0 \leq r \leq m$ and $p + q + r = n$ is $\binom{k}{p}\binom{l}{q}\binom{m}{r}$. Thus the total number of teams is $\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r}$.

On the other hand a team of n people from the group containing $k + l + m$ people can be formed in $\binom{k+l+m}{n}$ ways. It follows that,

$$\sum_{p+q+r=n} \binom{k}{p}\binom{l}{q}\binom{m}{r} = \binom{k+l+m}{n}.$$

CC55. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

Originally 2011 APICS Math Competition, Question 3.

Solved by Š. Arslanagić; M. Coiculescu; J. L. Díaz-Barrero; R. Hess; D. E. Manes; S. Muralidharan; P. Perfetti; G. Tsapakidis; D. Văcaru; and T. Zvonaru. We present two solutions.

Solution 1 by José Luis Díaz-Barrero.

Let $S = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$. Then

$$\begin{aligned} S &= 2\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma}\right) + \left(\frac{\alpha-1}{1-\alpha} + \frac{\beta-1}{1-\beta} + \frac{\gamma-1}{1-\gamma}\right) \\ &= 2\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma}\right) - 3. \end{aligned}$$

Since α, β, γ are the roots of $P(x) = x^3 - x - 1$, then $1-\alpha, 1-\beta, 1-\gamma$ are the roots of $P(1-x) = -x^3 + 3x^2 - 2x - 1$ or the roots of $Q(x) = x^3 - 3x^2 + 2x + 1$. Hence, $\frac{1}{1-\alpha}, \frac{1}{1-\beta}$ and $\frac{1}{1-\gamma}$ are the roots of $Q(\frac{1}{x}) = 0$ or $x^3 + 2x^2 - 3x + 1$. By Viète formulae, we then have

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = -2,$$

so $S = 2(-2) - 3 = -7$.

Solution 2 composed of solutions by S. Muralidharan and Titu Zvonaru.

We set $y = \dfrac{1+x}{1-x}$. Then $x = \dfrac{y-1}{y+1}$ and then we have

$$\begin{aligned} \left(\frac{y-1}{y+1}\right)^3 - \frac{y-1}{y+1} - 1 &= 0, \\ (y-1)^3 - (y-1)(y+1)^2 - (y+1)^3 &= 0, \\ -6y^2 - 2 - (y^2-1)(y+1) &= 0, \\ y^3 + 7y^2 - y + 1 &= 0. \end{aligned}$$

The last equation has roots $y_1 = \frac{1+\alpha}{1-\alpha}, y_2 = \frac{1+\beta}{1-\beta}$ and $y_3 = \frac{1+\gamma}{1-\gamma}$. Hence, by Viète's formula, we have

$$y_1 + y_2 + y_3 = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma} = -7.$$

THE OLYMPIAD CORNER

No. 319

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-olympiad@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 May 2015***, although late solutions will also be considered until a solution is published.*

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The editor thanks Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, for translations of the problems.

OC161. The altitude BH dropped onto the hypotenuse AC of a right triangle ABC intersects the angle bisectors AD and CE at Q respectively P. Prove that the line passing through the midpoints of segments $[QD]$ and $[PE]$ is parallel to the line AC.

OC162. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $k, m, n \in \mathbb{N}$ we have

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

OC163. Let $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ and S be the set of all subsets of A. Find the number of functions $f : S \to B$ such that

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S\,.$$

OC164. Find all triples (m, p, q) where m is a positive integer and p, q are primes such that

$$2^m p^2 + 1 = q^5\,.$$

OC165. Let O be the circumcenter of acute ΔABC, and let H be its orthocenter. Let $AD \perp BC$, and let EF be the perpendicular bisector of AO, with D, E on the side BC. Prove that the circumcircle of ΔADE passes through the midpoint of OH.

. . . .

OC161. L'altitude BH vers l'hypotenuse AC du triangle ABC intersecte les bissectrices de AD et CE à Q et P respectivement. Démontrer que la ligne passant par les mipoints des segments $[QD]$ et $[PE]$ est parallèle à la ligne AC.

OC162. Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{R}$ telles que pour tout $k, m, n \in \mathbb{N}$, l'inégalité qui suit est valide

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

OC163. Soit $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ et S l'ensemble de tous les sous ensembles de A. Déterminer le nombre de fonctions $f : S \to B$ telles que

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ pour tout } A_1, A_2 \in S\,.$$

OC164. Déterminer tous les triplets (m, p, q) où m est un entier positif et p, q sont des nombres premiers tels que

$$2^m p^2 + 1 = q^5\,.$$

OC165. Soit O le centre du cercle circonscrit du triangle aïgu ΔABC et soit H l'orthocentre. Soit $AD \perp BC$ et soit EF la bissectrice perpendiculaire de AO, avec D et E sur le côté BC. Démontrer que le cercle circonscrit de ΔADE passe par la mipoint de OH.

OLYMPIAD SOLUTIONS

OC101. Let n, k be positive integers so that $1 < k < n-1$. Prove that the binomial coefficient $\binom{n}{k}$ is divisible by at least two distinct primes.

Originally question 5 from the 2011 Estonia Team Selection Test, Day 2.

No solution was received to this problem. We give the official solution from Estonian Math Competitions 2010/2011*, The Gifted and Talented Development Centre, Tartu, 2011.*

We can assume without loss of generality that $2k \leq n$, otherwise we interchange k with $n-k$.

Assume by contradiction that

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

for some prime p and some integer n. Write every number in the numerator in the form $n - i = p^{\alpha_i} s_i$ with $p \nmid s_i$, where $0 \leq i \leq k-1$.

First let us observe that we have $s_i \neq s_j$. Indeed, assume by contradiction that $s_i = s_j$ for some $i < j$. Then, as $p^{\alpha_i} s_i = n - i > n - j = p^{\alpha_j} s_j$ we get $\alpha_i \geq 1 + \alpha_j$. Therefore

$$n \geq p^{\alpha_i} s_i \geq p p^{\alpha_j} s_i = p(n-j) > p(n-k) \geq 2(n-k)\,,$$

which contradicts $2k \leq n$.

This shows that the k terms $s_0, s_1, \ldots, s_{k-1}$ at the top are pairwise distinct.

Moreover, as the numerator contains at least two consecutive integers, at least one of these is not divisible by p. Therefore, there exists some j so that $s_j = n - j > n - k \geq k$.

As the elements $s_0, s_1, \ldots, s_{k-1}$ are pairwise distinct, and at least one of them is strictly greater than k, we have

$$s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k\,.$$

Moreover, as

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{1 \cdot 2 \cdot \ldots \cdot k} = p^l \,,$$

we get

$$\prod p^{\alpha_i} s_i = p^l 1 \cdot 2 \cdot \ldots \cdot k \Rightarrow s_1 \cdot .. \cdot s_k | p^l 1 \cdot 2 \cdot ... \cdot k$$

As each s_i is not divisible by p, $s_1 \cdot \ldots \cdot s_k$ is relatively prime with p^l. Therefore

$$s_1 \cdot \ldots \cdot s_k | 1 \cdot 2 \cdot \ldots \cdot k$$

But this contradicts $s_0 s_1 \cdot \ldots \cdot s_{k-1} > 1 \cdot 2 \cdot \ldots \cdot k$.

As we got a contradiction, our assumption is wrong, therefore $\binom{n}{k}$ cannot be a power of a prime.

OC102. Let $\mathbb{N}$ denote the set of all nonnegative integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that

1. $0 \leq f(x) \leq x^2$ for all $x \in \mathbb{N}$
2. $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}$ with $x > y$.

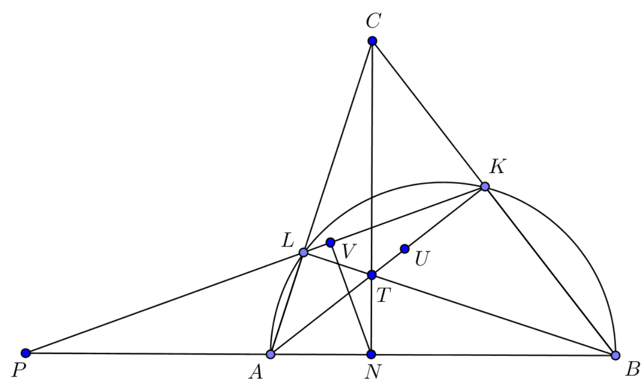
Originally question 5 from the 2011 South Africa National Olympiad.

No solution was received to this problem.

OC103. Let K and L be points on a semicircle with diameter AB. Denote the intersection of AK and BL as T and let N be the point such that N is on segment AB and line TN is perpendicular to AB. If U is the intersection of the perpendicular bisectors of AB and KL and V is a point on KL such that angles UAV and UBV are equal, then prove that NV is perpendicular to KL.

Originally question 3 from 2011 Croatia Team Selection Test, Day 2.

Solved by Š. Arslanagić; and D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru. We give the solution of the latter group.



Denote the intersection of AL and BK by C. Then AK and BL are altitudes in the triangle ABC, and therefore T is the orthocenter in this triangle. This implies that C, T, N are collinear.

The condition $\angle UAV = \angle UBV$ means that the quadrilateral $ABUV$ is cyclic.

Let M be the midpoint of AB. We will prove that $MUVN$ is cyclic, which shows that $NV \perp KL$.

If $KL \parallel AB$, then ABC is isosceles, in which case M coincides with N and U coincides with V, so $NV \perp KL$.

Otherwise, let P be the intersection of AB and KL. By symmetry, we can assume that A is between P and B.

As usual we denote by A, B, C respectively a, b, c the angles respectively the sides in the triangle ABC.

By applying the Menelaus theorem in triangle ABC with transversal $P-L-K$ we obtain

$$\frac{PA}{PB}\frac{KB}{KC}\frac{LC}{LA}=1 \Leftrightarrow \frac{PA}{c+PA}c\cos(B)b\cos(C)\frac{a\cos(C)}{c\cos(A)}=1 \Leftrightarrow \frac{PA}{c+PA}=\frac{b\cos(A)}{a\cos(B)}.$$

Therefore

$$PA=\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\,.$$

As $ABUV$ is cyclic, we have

$$PU\cdot PV=PA\cdot PB\,.$$

To complete the proof, we need to show that

$$PU\cdot PV=PM\cdot PN\,.$$

We have

$$\begin{aligned}
PU\cdot PV &= PM\cdot PN \Leftrightarrow\\
PA\cdot PB &= PM\cdot PN \Leftrightarrow\\
PA\cdot(c+PA) &= (PA+b\cos(A))(PA+\frac{c}{2}) \Leftrightarrow\\
PA^2+cPA &= PA^2+bPA\cos(A)+PA\frac{c}{2}+\frac{c}{2}b\cos(A) \Leftrightarrow\\
PA\frac{c}{2} &= bPA\cos(A)+\frac{c}{2}b\cos(A) \Leftrightarrow\\
\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\frac{c}{2} &= b\frac{bc\cos(A)}{a\cos(B)-b\ cos(A)}\cos(A)+\frac{bc\cos(A)}{2} \Leftrightarrow\\
\frac{c}{2(a\cos(B)-b\ cos(A))} &= \frac{2b\cos(A)}{2(a\cos(B)-b\ cos(A))}+\frac{a\cos(B)-b\ cos(A)}{2(a\cos(B)-b\ cos(A))} \Leftrightarrow\\
c &= 2b\cos(A)+a\cos(B)-b\ cos(A) \Leftrightarrow\\
c &= b\cos(A)+a\cos(B)
\end{aligned}$$

which is true.

OC104. Given a triangle ABC, let D be the midpoint of the side AC and let M be the point on the segment BD so that $BM : MD = 1 : 2$. The rays AM and CM intersect the sides BC and AB at E respectively F. We know that $AM \perp CM$. Prove that the quadrangle $AFED$ is cyclic if and only if the median from A in ΔABC meets the line EF at a point situated on the circumcircle of ΔABC.

Originally question 3 from the 2011 Romania Team Selection Test, Day 4.

Solved by D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; and O. Geupel. We give the solution of Oliver Geupel.

Let $a = BC$, $b = CA$, and $c = AB$.

By Apollonius' theorem, the squared length of the triangle median BD is

$$BD^2 = \frac{2(a^2+c^2)-b^2}{4}\,.$$

By the assertions $BM : MD = 1 : 2$ and $AM \perp CM$, we have $\frac{2}{3}BD = DM = \frac{b}{2}$. It follows that

$$13b^2 = 8(a^2+c^2). \tag{1}$$

Using barycentric coordinates, we have

$$\begin{aligned} M &= \frac{1}{6}A + \frac{2}{3}B + \frac{1}{6}C \\ E &= \frac{4}{5}B + \frac{1}{5}C \\ F &= \frac{1}{5}A + \frac{4}{5}B. \end{aligned}$$

Therefore $BE = \frac{a}{5}$, $BF = \frac{c}{5}$, $EF = \frac{b}{5}$, and $EF \parallel AD$.

We show that the quadrilateral $AFED$ is cyclic if and only if

$$b^2 = 2c^2 \quad \text{and} \quad a^2 = \frac{9}{4}c^2. \tag{2}$$

By the law of cosines in $\triangle CDE$ we have

$$\begin{aligned} DE^2 &= CE^2 + CD^2 - 2CE\cdot CD\cos C \\ &= \frac{16}{25}a^2 + \frac{1}{4}b^2 - 2\cdot\frac{4}{5}a\cdot\frac{1}{2}b\cdot\frac{a^2+b^2-c^2}{2ab} \qquad (3) \\ &= \frac{6}{25}a^2 - \frac{3}{20}b^2 + \frac{2}{5}c^2. \qquad (4) \end{aligned}$$

The trapezoid $AFED$ with $EF \neq AD$ is cyclic if and only if it is isosceles, i.e. $DE = \frac{4}{5}c$. By (3), this is equivalent to

$$8a^2 = 5b^2 + 8c^2\,.$$

In view of (1), it simplifies to (2). Consequently, quadrilateral $AFED$ is cyclic if and only if (2).

It suffices to show that the median from A meets the line EF at a point situated on the circumcircle of $\triangle ABC$ if and only if (2) holds.

Let N be the midpoint of BC and let P be the intersection of lines EF and AN.

By Apollonius' theorem the squared length of the median AN is

$$AN^2 = \frac{2(b^2+c^2)-a^2}{4}\,.$$

By Menelaus' theorem applied to the line EF in $\triangle ABN$ we have

$$\frac{PN}{PN+AN} = \frac{BF}{AF}\cdot\frac{NE}{BE} = \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8},$$

whence $PN = \dfrac{3}{5}AN$.

Observe that the point P lies on the circumcircle of triangle ABC if and only if $BN \cdot CN = AN \cdot PN$. This is successively equivalent to

$$\frac{a^2}{4} = \frac{3}{20}(2(b^2+c^2)-a^2)$$

and to $4a^2 = 3(b^2+c^2)$. In view of (1), this simplifies to (2).

This completes the proof.

OC105. Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer $a, 1 < a < \frac{n}{k}+1$, such that $n \mid a^2 - a$.

Originally question 2 from 2011 China Team Selection, Quiz 3, Day 1.

Solved by Oliver Geupel whose solution is presented below.

Let $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ be the prime factorization of n.

Then, the condition $n \mid a(a-1)$ holds if and only if there are numbers $b_1, b_2, \ldots, b_k \in \{0,1\}$ such that

$$\begin{aligned} &a \equiv b_1 \pmod{p_1^{e_1}} \\ &a \equiv b_2 \pmod{p_2^{e_2}} \\ &\ldots \\ &a \equiv b_k \pmod{p_k^{e_k}}\,. \end{aligned} \tag{5}$$

For $i, j \in \{1, 2, \ldots, k\}$, define

$$b_{ij} = \begin{cases} 1 & \text{if } i < j, \\ 0 & \text{if } i \geq j. \end{cases}$$

For $j = 1, \ldots, k$, let $\mathcal{S}_j$ denote the following system of simultaneous congruences :

$$\begin{aligned}
&a \equiv b_{1j} \pmod{p_1^{e_1}}\\
&a \equiv b_{2j} \pmod{p_2^{e_2}}\\
&\cdots\\
&a \equiv b_{kj} \pmod{p_k^{e_k}}\,.
\end{aligned}$$

By the Chinese remainder theorem, there are k numbers $a_1, \ldots, a_k$, such that for $j = 1, \ldots, k$ it holds $2 \le a_j \le n$, and $a = a_j$ solves the simultaneous congruences $\mathcal{S}_j$.

The k closed intervals

$$\begin{aligned}
I_1 &= \left[2,\ \frac{n}{k}\right],\\
I_2 &= \left[\frac{n}{k} + 1,\ 2 \cdot \frac{n}{k}\right],\\
I_3 &= \left[2 \cdot \frac{n}{k} + 1,\ 3 \cdot \frac{n}{k}\right],\\
&\ldots,\\
I_k &= \left[(k-1) \cdot \frac{n}{k} + 1,\ n\right]
\end{aligned}$$

constitute a disjoint partition of the interval $[2, n]$.

If at least one of the numbers a_j, $1 \le j \le k$ belongs to I_1, then this number has the required property.

Otherwise, there are two distinct numbers a', a'' among $a_1, \ldots, a_k$ that belong to the same interval I_j by the Pigeonhole principle.

But then, one of the numbers $|a' - a''|$ and $|a' - a''| + 1$ satisfies a system of simultaneous congruences of the form (5) and belongs to I_1.

This completes the proof.

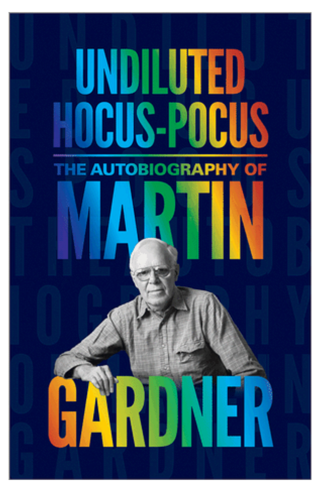
BOOK REVIEWS

Robert Bilinski

Undiluted Hocus-Pocus : The Autobiography of Martin Gardner with foreword by Persi Daiconis and an afterword by James Randi
ISBN 9780691159911 (ebook : ISBN 9781400847983)
Princeton University Press, 2013, hardcover $24.95 (US), 288 pages

Reviewed by ***Chelluri C.A. Sastri****, Dalhousie University*

Mathematicians know about Martin Gardner mainly through his books on games and puzzles – what is called recreational mathematics – and his popular *Mathematical Games* column in *Scientific American.* But this was only one facet of his life and work, for he was passionate about many things besides mathematics : magic, philosophy, literature (fantasy and poetry in particular), religion, and debunking pseudoscience. What is really striking is that, in terms of formal education, he just had an undergraduate degree in philosophy from the University of Chicago, where he didn't take even a single course in mathematics. By his own admission, he didn't know calculus well. How then could he produce such wonderful work and simultaneously earn the respect and friendship of several distinguished mathematicians and computer scientists – John Horton Conway, Donald Knuth, and Roger Penrose, to name a few – *and* lure youngsters to mathematics and keep them interested ? (Along the way, he took under his wing the gifted thirteen-year old Persi Diaconis – he was a magician at the time – and nurtured him to the point of helping him get into graduate school at Harvard.) *How* did he do it ? The book under review provides some clues but doesn't completely answer the question. (The book *Magical Mathematics* by Diaconis and Ron Graham does provide some insight.) The title, by the way, refers to his love of magic.



Given his background and his enormous intellectual appetites, the only way Gardner could accomplish what he did was to teach himself what he wanted to write about. Indeed, he was the ultimate autodidact. And, he was indefatigable ; he worked full-time for nearly a month on the material for each column, and kept it up for at least 25 years. He says, "I had to struggle to understand what I wrote, and this helped me write in ways that others could understand." It is much like someone learning a topic by teaching a course on it, except that Gardner did it for a long time. Anyway, this explains, to some extent at least, his popularity with young readers. His appeal to professional mathematicians, however, is not so easy to explain. But as noted earlier, his writings did appeal to several of them. Some of them even contributed to his column, and he says that those contributions were far superior to anything he could write – he may have been modest in saying that – and were a major reason for the growing popularity of the column.

Gardner must have enjoyed the work he put in, given the challenges it posed and the thrills it afforded : he says at the beginning of the chapter on *Scientific American* that his association with it was the second luckiest event in his life, with the first being meeting Charlotte, his future wife. His first contribution to the magazine was an article on a mathematical toy, a large cloth structure called Hexahexaflexagon. The first "hexa" referred to the six inventors, mostly graduate students at Princeton, while the second "hexa" referred to the six faces that could be seen by flexing the structure in specific ways. It was such a hit that the publisher asked him if there was more material like that he could write about. He said yes, and that was how his column was born. As time went on, it became so popular that many of the one million readers of *Scientific American* said in surveys that the column was the main reason they read the magazine.

The article on hexahexaflexagons was followed by ones on Hex and the Soma cube, both games invented by a Danish magician and poet named Piet Hein, except that Hex was independently invented by John Nash also. Gardner talks a little bit about these and some other things he wrote, but refrains from making any boastful claims about his column. However, the entry on him in Wikipedia says that his column provided the first introduction of many subjects to wider audiences, notable among them being Polyominoes, Rep-tiles, the Superellipse, Pentominoes, Fractals, Conway's Game of Life, Tangrams, Penrose tiles, Public-key cryptography, Hofstadter's Gödel, Escher, and Bach, and finally the Monster group. Wikipedia also quotes him as saying, "I just play all the time and am fortunate enough to get paid for it."

Martin Gardner was born in 1914 in Tulsa, Oklahoma, into a well-to-do family. His father was a successful petroleum geologist, and his mother, who had once been a Montessori teacher, loved colors and painted. Gardner shared her love of colors and painting ; he later became a caricaturist, with some of his caricatures appearing in *Phoenix*, the University of Chicago's humor magazine. His interest in science began with a curious experiment : teaching his neighbor's dog to overturn a bucket with one of his paws. He loved to read, and grew fond of fantasies – the Oz books by L. Frank Baum, and *Alice in Wonderland* in particular. He loved them so much that he later published annotated versions of them. In fact, he published annotated versions of many books, including *Casey at the Bat*. He didn't like high school – he felt it was like four years in prison – but that's where his interest in recreational math through games and puzzles began. He learned some magic tricks from his father and was on his way to creating some tricks of his own. His father let him have a small laboratory at home. Around this time, he fell in love with the writings of G.K. Chesterton, whose story The *Colored Lands* made a deep impression on Gardner because of the role colors play in it. He read quite a bit of poetry and wrote poems himself, some of which were published. His best friend in high school was John Shaw, a Catholic who went to a different school, but who shared Gardner's love of books, in particular the Sherlock Holmes stories. They used to play practical jokes on each other. Shaw later became a bookseller.

After graduating from high school, Gardner attended the University of Chicago, where he studied philosophy. The intellectual atmosphere at the university, the personalities involved, and life in Chicago in general affected him profoundly, so much so that he devotes more than four chapters to its description. Two central characters in this narrative are Robert Hutchins, the president of the university, and Mortimer Adler, a friend of Hutchins and professor of philosophy. Among the many anecdotes Gardner tells about them and their activities is an amusing one about the "madman theory," according to which it is good for a university to have a faculty member who is mad because opposition to his crazy opinions stimulates students into thinking seriously about fundamental questions. Gardner adds that Adler was the University of Chicago's madman.

Gardner became a co-editor of *Comment*, the campus magazine, and contributed several articles to it. He honed his writing skills there, and that led him eventually to decide to make his living by writing. And he became a prolific writer. Apart from books on math and magic and the annotated editions of fantasies mentioned earlier, he wrote short stories, including children's stories and science fiction, a novel about religion, books on popular science, books on philosophy, including one co-authored with Rudolf Carnap and variously titled as the *Philosophical Foundations of Physics* and *An Introduction to the Philosophy of Science*, books of essays, and the list goes on. In all, he wrote or edited more than a hundred books including anthologies of poetry and books attacking pseudoscience. And many of them are still in print. He had an impish sense of humor and sometimes wrote under one of three pseudonyms : Armand T. Ringer (an anagram of his name), George Groth, and Uriah Fuller. In addition to all this, Cambridge University Press collected his columns for *Scientific American* and published them in fifteen volumes. In his own estimation, his best books were *The Whys of a Philosophical Scrivener*, and *The Night is Large*, whose title comes from a play by Lord Dunsany, another of his favorite writers.

Gardner was a skeptic through and through, never accepting received wisdom without questioning it. It is this spirit that led him to launch ferocious attacks on pseudoscience. However, being a shy person, he didn't engage in public debates like William Buckley and Gore Vidal but did it in print instead, where he never pulled a punch. At least one dispute ended up in court, with Gardner winning it. In most cases, the opponents simply slunk away. His targets included, among others, Uri Geller, the Israeli spoon bender, and L. Ron Hubbard, the father of Dianetics and founder of Scientology. He was friends with Carl Sagan, the astrophysicist, whose motto was that extraordinary claims required extraordinary evidence. Together with James Randi, a famous magician and skeptic, and some other friends, Gardner started an organization called the Committee for Scientific Investigation of Claims of the Paranormal or CSICOP. The organization has since shortened its acronym to CSI and publishes a periodical called the *Skeptical Inquirer*. Gardner contributed several articles to it. He held strong opinions in other areas too and never hesitated to express them ; in fact, he could be quite combative. For instance, concerning

Freud, he says that whenever Freud said anything that made sense, it was not original, and when he was original, he spouted baloney. One wonders what present day psychologists and psychiatrists would say about that. Similarly, Gardner says that he would rather read some of the poems he anthologized, which may not be well known, than the "vapid verse" of William Carlos Williams or the poetry of Ezra Pound. He says similar things about modern art.

Despite his avowed skepticism, Gardner believed in God. He says that he and his wife were philosophical theists, meaning that they believed in a personal god, but not of any particular religion, and hoped for an afterlife. He says that it's a matter of the heart rather than the head and that the belief made him happier. He also believed in something called mysterianism, which holds that just as calculus is beyond the understanding of chimpanzees, whose DNA is very close to ours, there exist truths far beyond our capacity to comprehend. As an example, he mentions consciousness and/or freewill, which have so far defied the attempts at explanation by neuroscientists as well as philosophers. His contention is that a scientific explanation is simply not possible. He says that there are less radical mysterians such as his friend Roger Penrose, who believe that some day scientists will unlock the secrets. Be that as it may, it is remarkable that he felt so confident in expressing his views. Indeed, he remained intellectually alive and alert to the very end of his life, in 2010. After reading this book, one cannot but come away with the impression that his long life was chock-full of ideas and creativity and that he was a popularizer of science and recreational mathematics without parallel.

Wikipedia says that because of his shyness Gardner declined many honors when he found out that a public appearance would be required. However, a puzzle collector named Tom Rodgers persuaded him in 1993 to attend an evening devoted to Gardner's puzzle-solving efforts, called a *Gathering for Gardner*. Even though Gardner is gone, the gatherings, known as G4Gn, where n stands for the number of the gathering, continue to be held biennially.

FOCUS ON...

No. 10

Michel Bataille

Some Sequences of Integrals

Introduction

In problem **2520** [2000 : 115], Paul Bracken considered the asymptotic behaviour of some integrals of the form $\int_0^1 (1+ax+bx^2)^n\,dx$ as $n\to\infty$ and a partial solution was given a year later [2001 : 218]. Recently, I proposed two problems on the same topic, derived from results I had obtained when solving this problem (see **3604** [2011 : 46,49 ; 2012 : 32] and **3642** [2011 : 235,237 ; 2012 : 202]). Maybe it is time to complete the work initiated by problem **2520** with a general study of the sequences $\{I_n\}$ where $I_n = \int_0^1 (ax^2+bx+c)^n\,dx$ and $ax^2+bx+c>0$ for all $x\in[0,1]$. In what follows, the purpose is to determine a "simple" sequence $\{\omega_n\}$ such that $I_n \sim \omega_n$ as $n\to\infty$, meaning that $\lim_{n\to\infty} I_n/\omega_n = 1$. For convenience, we will drop "as $n\to\infty$" after the symbol $\sim$.

A lemma

First, we give a quick proof of the following result : If $r\in(0,1]$ and $s\geq r$, then

$$\int_0^r (s^2-x^2)^n\,dx \;\sim\; \frac{s^{2n+1}}{2}\cdot\sqrt{\frac{\pi}{n}}.$$

The substitution $x=su$ reduces the question to showing that

$$\int_0^\rho (1-u^2)^n\,du \;\sim\; \frac{1}{2}\cdot\sqrt{\frac{\pi}{n}},$$

where $\rho\in(0,1]$. Since $\lim_{n\to\infty}\sqrt{n}\int_\rho^1 (1-u^2)^n\,du = 0$ if $\rho<1$ (the integral being less than $(1-\rho^2)^n$), all finally amounts to proving that

$$\int_0^1 (1-u^2)^n\,du \;\sim\; \frac{1}{2}\cdot\sqrt{\frac{\pi}{n}}. \qquad (1)$$

With the help of the substitution $u=1-2t$, we calculate

$$\int_0^1 (1-u^2)^n\,du = 2^{2n+1}\int_0^{1/2} t^n(1-t)^n\,dt = 2^{2n}\int_0^1 t^n(1-t)^n\,dt = 2^{2n}\frac{(n!)^2}{(2n+1)!}$$

and Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ easily leads to (1).

From now on, we set $\phi(x) = ax^2 + bx + c$ where a, b, c are real numbers ($a \neq 0$) and we assume that $\phi(x) > 0$ for all $x \in [0,1]$ (so that $c > 0$ and $a + b + c > 0$). As above, $I_n = \displaystyle\int_0^1 (\phi(x))^n\, dx$.

The case when ϕ is decreasing on $[0,1]$

Here are the results in that case :

$$\text{if } b \neq 0, \quad I_n \sim \frac{c^{n+1}}{n|b|} \qquad \text{and if } b = 0, \quad I_n \sim \frac{c^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \tag{2}$$

If $b = 0$, then $a < 0$ and $I_n = |a|^n \displaystyle\int_0^1 \left(\frac{c}{|a|} - x^2\right)^n dx$, so the lemma directly gives the announced result.

Now, suppose that $b \neq 0$. Since $\phi(x) = c\left(1 + \frac{b}{c}x + \frac{a}{c}x^2\right)$, it suffices to prove that

$$\lim_{n\to\infty} n \cdot \int_0^1 (1 + \beta x + \alpha x^2)^n = \frac{1}{|\beta|}$$

whenever $\beta \neq 0$ and $x \mapsto \psi(x) = 1 + \beta x + \alpha x^2$ is positive and decreasing on $[0,1]$ (this implies $\beta < 0$). Let $\varepsilon \in (0, |\beta|)$. Since $\lim_{x\to 0^+} \frac{1-\psi(x)}{x} = |\beta|$, we can choose $\delta \in (0,1)$, small enough to ensure that for $x \in [0, \delta]$,

$$0 < 1 - (|\beta| + \varepsilon)x \leq \psi(x) \leq 1 - (|\beta| - \varepsilon)x. \tag{3}$$

Because $0 \leq n \cdot \displaystyle\int_\delta^1 (\psi(x))^n\, dx \leq n(\psi(\delta))^n$ and $0 < \psi(\delta) < 1$, we have

$$\lim_{n\to\infty} n \cdot \int_\delta^1 (\psi(x))^n\, dx = 0.$$

Let $J_n = \displaystyle\int_0^1 (\psi(x))^n\, dx$ and $K_n = \displaystyle\int_0^\delta (\psi(x))^n\, dx$. From (3), we obtain

$$\frac{n}{n+1} \cdot \frac{1 - \rho_1^{n+1}}{|\beta| + \varepsilon} \leq nK_n \leq \frac{n}{n+1} \cdot \frac{1 - \rho_2^{n+1}}{|\beta| - \varepsilon}$$

where $\rho_1 = 1 - (|\beta| + \varepsilon)\delta, \quad \rho_2 = 1 - (|\beta| - \varepsilon)\delta$ are in $(0,1)$.

Observing that $nJ_n = nK_n + n\displaystyle\int_\delta^1 (\psi(x))^n\, dx$, we readily deduce that

$$\limsup_{n\to\infty}\ nJ_n \leq \frac{1}{|\beta| - \varepsilon} \quad \text{ and } \quad \liminf_{n\to\infty}\ nJ_n \geq \frac{1}{|\beta| + \varepsilon}$$

and since this holds for all $\varepsilon \in (0, |\beta|)$, $\frac{1}{|\beta|} \leq \liminf\limits_{n\to\infty}\ nJ_n \leq \limsup\limits_{n\to\infty}\ nJ_n \leq \frac{1}{|\beta|}$.

The result follows.

The other cases

(a) If ϕ is increasing on $[0,1]$, the change of variables $x = 1-y$ shows that

$$I_n = \int_0^1 (a+b+c-(2a+b)y+ay^2)^n \, dy$$

and applying (2) gives

$$\text{if } 2a+b \neq 0, \quad I_n \sim \frac{(a+b+c)^{n+1}}{n(2a+b)}, \tag{4}$$

$$\text{if } 2a+b = 0, \quad I_n \sim \frac{(c-a)^{n+\frac{1}{2}}}{2} \cdot \sqrt{\frac{\pi}{n|a|}}. \tag{5}$$

(b) If ϕ attains its minimum on $(0,1)$, then $a>0$ and $0 < \frac{-b}{2a} < 1$ so that $b<0$. Also note that $\Delta = b^2 - 4ac < 0$. Let $\mu = \frac{\sqrt{|\Delta|}}{2a}$. Then,

$$I_n = a^n \int_0^1 \left(\left(x + \frac{b}{2a} \right)^2 + \mu^2 \right)^n dx = a^n \int_{\frac{b}{2a}}^{1+\frac{b}{2a}} (y^2+\mu^2)^n \, dy$$

$$= a^n \left(\int_0^{1+\frac{b}{2a}} (y^2+\mu^2)^n \, dy + \int_0^{\frac{-b}{2a}} (y^2+\mu^2)^n \, dy \right).$$

Now, from $(4,5)$ we obtain

$$\int_0^k (y^2+\mu^2)^n \, dy = k \int_0^1 \left(k^2x^2+\mu^2\right)^n \, dx \sim \frac{(\mu^2+k^2)^{n+1}}{2kn}$$

for positive k. It is then straightforward to deduce

$$\text{if } |b| < a: \quad I_n \sim \frac{(a+b+c)^{n+1}}{n(2a+b)},$$

$$\text{if } |b| > a: \quad I_n \sim \frac{c^{n+1}}{n|b|},$$

$$\text{if } |b| = a: \quad I_n \sim \frac{2c^{n+1}}{n|b|}.$$

(c) There remains the case when ϕ attains its maximum on $(0,1)$. Then

$$a<0, \quad 0 < \frac{-b}{2a} < 1 \quad \text{and} \quad \Delta = b^2-4ac > 0.$$

Completing the square as in the previous case, we arrive at

$$I_n = |a|^n \left(\int_0^{1+\frac{b}{2a}} (\nu^2-y^2)^n \, dy + \int_0^{\frac{-b}{2a}} (\nu^2-y^2)^n \, dy \right),$$

where $\nu = \frac{\sqrt{\Delta}}{2|a|}$. Using the lemma, we obtain

$$I_n \sim \sqrt{\frac{\pi}{n|a|}} \left(c - \frac{b^2}{4a} \right)^{n+\frac{1}{2}}.$$

Exercise

The obvious one is to find an alternative solution to **3604** and **3642** with the help of the results established above.

26th Tournament of Towns : A Square from Similar Rectangles

Olga Zaitseva

Seventy of the brightest young people from different parts of the world were brought together to spend nine days immersing themselves in current streams of mathematical research : the 26th Tournament of Towns Summer Conference took place August 2–11, 2014, in Russia. The conference was held at the Centre for the Gifted Children, 20 km away from Kaliningrad (former Königsberg).

The location of the Summer Conferences changes every year but the structure stays the same. For each conference, a group of world-renowned mathematicians and educators (authors of projects, called the Jury) creates several topics for investigation. No topic has ever been repeated. After the participants arrive to the conference, the authors of projects give orientation lectures and hand out materials on each topic. Each participant then chooses one project to work on and authors of projects become the participants' mentors.

Although the informal contacts between participants and mentors are ongoing, there is a designated date when participants need to submit their solutions. The Jury checks the progress, removes solved parts, and supplies new information for the projects. If needed, the members of the Jury deliver lectures on methods that could be used in the research. After the deadline, authors of projects analyze the results, evaluate the progress of participants and overall teams' progress and designate directions in which the work can be continued.

This year, the Jury devised seven topics for investigation – a record number. The list includes : *A Square from Similar Rectangles, Algorithms and Labyrinths, Division of Segment, Combinatorial Geometry and Graph Colourings : from Algebra to Probability, De Bruijn Sequences and Universal Cycles, On the Poncelet Theorem*, and *Point-Line Incidences*.

Let us consider one of the topics, *A Square from Similar Rectangles*, in more detail. This topic has received much interest lately due to revealed connections with physics, harmonic analysis and the theory of probability. In addition, many introductory problems are accessible to an average high school student and we would like to use this opportunity to introduce several such problems to the reader.

1. Form a square from $m \times n$-rectangles, where m and n are integers.
2. A designer gets an order to manufacture two different frames for the square window. Figure 1 shows the suggested designs. Can all the panes in either design in Figure 1 be similar rectangles ?
3. Is it possible to dissect a square into three similar rectangles, no two of which are the same ?

Readers may connect the name of the city with the famous problem about the seven bridges of Königsberg (`http://en.wikipedia.org/wiki/Seven_Bridges_of_K\"onigsberg`).

FIGURE 1: The two suggested window frame designs.

4. Is it possible to dissect a square into five squares?

5. Each drawer's door of a cabinet is in the shape of a square (see Figure 2). Is the cabinet necessarily in the shape of a square?

FIGURE 2: The drawers' design.

The next set of problems (problems 6–12) require more inquisitive brain work and are hard even for an A+ student. The main part of the project starts with formulation of the following result :

Theorem 1 (Freiling–Laczkovich–Rinne–Szekeres (1994)) *For $r > 0$ the following three conditions are equivalent :*

1. *A square can be tiled by rectangles with side ratio r ;*

2. *For some positive rational numbers c_i the following equality holds*

$$c_1 r + \cfrac{1}{c_2 r + \cfrac{1}{c_3 r + \cdots + \cfrac{1}{c_n r}}} = 1;$$

3. *The number r is a root of a nonzero polynomial with integer coefficients, such that all complex roots of the polynomial have positive real parts.*

Problems 13–30 constitute the core part of the project. Participants are asked to prove implications of Theorem 1 and a number of auxiliary results. To work on these tasks, one must apply some knowledge of polynomials, complex variables and mathematical theory of electrical circuits. Nine teams (sixteen students) were working on the project and the maximal advance was achieved by a team from Taiwan. A single unsolved problem from this section was problem 23 that asked to prove implication $1 \implies 3$ in Theorem 1. The following question was included even though it is an open problem.

Problem 1 *When can a cube be dissected into parallelepipeds similar to a given one?*

One of the most significant consequences of the conference is that the work on unanswered problems continues after the conference is over. In addition to continuing to investigate those problems that were not answered during the conference, the mentors also suggest that the participants look for elementary proofs of their recently proved theorems.

If you would like to try out some of these problems as well as problems associated with other projects, the materials of the 26th Summer Conference Tournament of Towns can be found at `http://www.turgor.ru/lktg/2014/`.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

Submissions of proposals. *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions should be received by the editor by **1 May 2015**, *although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

3901. *Proposed by D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Let $A, B \in M_n(\mathbb{R})$ with $\det A = \det B \neq 0$. If $a, b \in \mathbb{R} \setminus \{0\}$, prove that

$$\det(aA + bB^{-1}) = \det(aB + bA^{-1}).$$

3902. *Proposed by Michel Bataille.*

Let ABC be a triangle with $AB = AC$ and $\angle BAC \neq 90^\circ$ and let O be its circumcentre. Let M be the midpoint of AC and let P on the circumcircle of $\triangle AOB$ be such that $MP = MA$ and $P \neq A$. The lines l and m pass through A and are perpendicular and parallel to PM, respectively. Suppose that the lines l and PC intersect at U and that the line PB intersect AC at V and m at W. Prove that U, V and W are not collinear and that l is tangent to the circumcircle of $\triangle UVW$.

3903. *Proposed by George Apostolopoulos.*

Consider a triangle ABC with an inscribed circle with centre I and radius r. Let C_A, C_B and C_C be circles internal to ABC, tangent to its sides and tangent to the inscribed circle with the corresponding radii r_A, r_B and r_C. Show that $r_A + r_B + r_C \geq r$.

3904. *Proposed by Abdilkadir Altintaş.*

Let ABC be an equilateral triangle and let D, E and F be the points on the sides AB, BC and AC, respectively, such that $AD = 2$, $AF = 1$ and $FC = 3$. If the triangle DEF has minimum possible perimeter, what is the length of AE ?

3905. *Proposed by Jonathan Love.*

A sequence $\{a_n : n \geq 2\}$ is called *prime-picking* if, for each n, a_n is a prime divisor of n. A sequence $\{a_n : n \geq 2\}$ is called *spread-out* if, for each positive integer k, there is an index N such that, for $n \geq N$, the k consecutive entries $a_n, a_{n+1}, \cdots, a_{n+k-1}$ are all distinct. For example, the sequence

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

is spread-out. Does there exist a prime-picking spread-out sequence ?

3906★. *Proposed by Titu Zvonaru and Neculai Stanciu.*

If $x_1, x_2, \ldots, x_n$ are positive real numbers, then prove or disprove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

for all positive integers n.

3907. *Proposed by Enes Kocabey.*

Let $ABCDEF$ be a convex hexagon such that $AB+DE = BC+EF = FA+CD$ and $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Let the midpoints of the sides AF, CD, BC and EF be M, N, K and L, respectively, and let $MN \cap KL = \{P\}$. Show that $\angle BCD = 2\angle KPN$.

3908. *Proposed by George Apostolopoulos.*

Prove that $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ for each integer $n \geq 3$.

3909. *Modified proposal of Victor Oxman, Moshe Stupel and Avi Sigler.*

Given an acute-angled triangle together with its circumcircle and orthocentre, construct, with straightedge alone, its circumcentre.

Editor's Comment. The Poncelet-Steiner Theorem (1833) states that whatever can be constructed by straightedge and compass together can be constructed by straightedge alone, given a circle and its centre ; but Steiner showed that given only the circle and a straightedge, the centre cannot be found. (This shows that the orthocentre must be given in the present problem ; it cannot be constructed with the straightedge and circumcircle !) Details can be found in texts such as A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), or on the internet by googling the Poncelet-Steiner Theorem.

3910. *Proposed by Paul Yiu.*

Two triangles ABC and $A'B'C'$ are homothetic. Show that if B' and C' are on the perpendicular bisectors of CA and AB respectively, then A' is on the perpendicular bisector of BC, and the homothetic center is a point on the Euler line of ABC.

. . . .

3901. *Proposé par D. M. Bătinetu-Giurgiu and Neculai Stanciu.*

Soient $A, B \in M_n(\mathbb{R})$ telles que $\det A = \det B \neq 0$. Si $a, b \in \mathbb{R}^*$, démontrer que

$$\det\left(aA + bB^{-1}\right) = \det\left(aB + bA^{-1}\right).$$

3902. *Proposé par Michel Bataille.*

Soit ABC un triangle tel que $AB = AC$ et $\angle BAC \neq 90^\circ$; soit O le centre de son cercle circonscrit. Soit M le mipoint de AC et soit P sur le cercle circonscrit de $\triangle AOB$ tel que $MP = MA$ and $P \neq A$. Les lignes l et m passent par A et sont perpendiculaire et parallèle à PM respectivement. Les lignes l et PC intersectent à U, puis que la ligne PC intersecte AC à V et m à W. Démontrer que U, V et W sont non colinéaires et que l est tangente au cercle circonscrit de $\triangle UVW$.

3903. *Proposé par George Apostolopoulos.*

Considérer un triangle ABC avec cercle inscrit de centre I et rayon r. Soient CA, CB et CC les cercles internes à ABC, tangents à ses côtés et au cercle inscrit, avec rayons correspondants r_A, r_B et r_C. Démontrer que $r_A + r_B + r_C \geq r$.

3904. *Proposé par Abdilkadir Altintaş.*

Soit ABC un triangle équilatéral et soient D, E et F les points sur les côtés AB, BC et AC respectivement, tels que $AD = 2, AF = 1$ et $FC = 3$. Si le triangle DEF a le périmètre le plus petit possible, quelle est la longueur de AE ?

3905. *Proposé par Jonathan Love.*

Une suite $\{a_n : n \geq 2\}$ est dite *premier-cueillante* si, pour tout n, a_n est un diviseur premier de n. Une suite $\{a_n : n \geq 2\}$ est dite *écartée* si, pour tout entier positif k, il existe un entier N tel que, pour $n \geq N$, les k valeurs consécutives $a_n, a_{n+1}, \cdots, a_{n+k-1}$ sont distinctes. Par exemple, la suite

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \cdots\}$$

est écartée. Existe-t-il une suite à la fois premier-cieuillante et écartée ?

3906$\star$. *Proposé par Titu Zvonaru et Neculai Stanciu.*

Si $x_1, x_2, \ldots, x_n$ sont des nombres réels positifs, prouver vrai ou prouver faux que

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_n^2}{x_1} \geq \sqrt{n(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

quelque soit n entier positif.

3907. *Proposé par Enes Kocabey.*

Soit $ABCDEF$ un hexagone convexe tel que $AB + DE = BC + EF = FA + CD$, puis que $AB \parallel DE, BC \parallel EF, CD \parallel AF$. Les mipoints des côtés AF, CD, BC et EF sont dénotés M, N, K et L, respectivement ; aussi, $MN \cap KL = \{P\}$. Démontrer que $\angle BCD = 2\angle KPN$.

3908. *Proposé par George Apostolopoulos.*

Démontrer que $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ pour tout entier $n \geq 3$.

3909. *Proposition modifiée de Victor Oxman, Moshe Stupel et Avi Sigler.*

Étant donné un triangle aïgu, son cercle circonscrit et son orthocentre, construire le centre du cercle circonscrit à l'aide de la règle.

Commentaire de l'éditeur. Selon le théorème Poncelet-Steiner (1883), étant donné un seul cercle et son centre, toute construction possible à l'aide de règle et compas est possible à l'aide de règle ; mais Steiner a aussi démontré qu'étant donné seulement un cercle et la règle, le centre ne peut pas être construit. (Ceci montre que l'orthocentre doit être donné dans le problème ci-haut, car il ne peut pas être construit à partir de règle et le cercle circonscrit.) Les détails se trouvent notamment dans le livre A. S. Smogorzhevskii, *The Ruler in Geometrical Constructions*, (Blaisdell 1961), ou aussi en googlant "Poncelet-Steiner Theorem".

3910. *Proposé par Paul Yiu.*

Deux triangles ABC et $A'B'C'$ sont homothétiques. Démontrer que si B' et C' se trouvent sur les bissectrices perpendiculaires de CA et AB respectivement, il ensuit que A' se trouve sur la bissectrice perpendiculaire de BC et que le centre homothétique est un point sur la ligne d'Euler de ABC.

SOLUTIONS

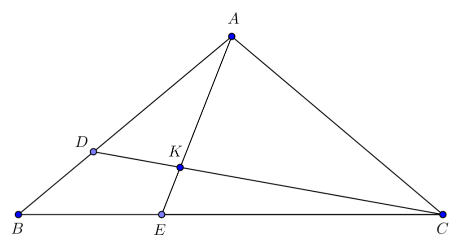
No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3801. *Proposed by George Apostolopoulos.*

Triangle ABC is isosceles with $AB = AC$ and $\angle A = 100^\circ$. Let D be the point on AB such that $\angle BCD = 10^\circ$ and let E be the point on BC such that $EC = AC$. Determine the point K on CD such that triangles KAD and KCE have equal areas.

Solved by AN-anduud Problem Solving Group ; Š. Arslanagić ; R. Barbara ; R. Barroso Campos ; M. Bataille ; C. Curtis ; O. Geupel ; N. Hodžić ; D. Jonsson ; V. Konečný ; O. Kouba ; K.W. Lau ; S. Malikić ; M. Modak ; C. Sánchez-Rubio ; Skidmore College Problem Group ; D. Smith ; N. Stanciu and T. Zvonaru ; E. Swylan ; D. Văcaru ; and the proposer. We present two solutions.

Solution 1, by Oliver Geupel.



As the point K moves along the segment CD from C to D, the value of $[KCE] - [KAD]$ is strictly increasing. Hence, there is a unique location of point K such that $[KAD] = [KCE]$. Denoting the intersection of the lines AE and CD by K, we prove that K has the required property.

Using the Law of Sines in triangle ACD, we get $\dfrac{AD}{\sin 30^\circ} = \dfrac{AC}{\sin 50^\circ}$. Hence

$$AD = \frac{AC}{2\sin 50^\circ}$$

and

$$BD = AB - AD = AB \cdot \frac{2\sin 50^\circ - 1}{2\sin 50^\circ}.$$

Also, using the Law of Sines in triangle ABC, we have

$$BE = BC - CE = AB \cdot 2\sin 50^\circ - AB.$$

Thus,

$$\frac{BE}{BD} = 2\sin 50^\circ = \frac{BC}{AB}.$$

Therefore, the lines AC and DE are parallel. Hence the triangles ACK and EDK are homothetic. We conclude

$$\frac{AK}{EK} = \frac{CK}{DK}$$

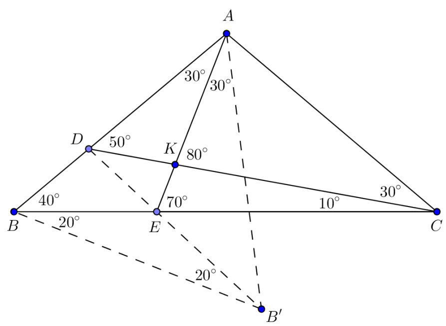
and

$$\frac{[KAD]}{[KCE]} = \frac{AK \cdot DK}{CK \cdot EK} = 1.$$

This completes the proof.

Solution 2, by Cristóbal Sánchez-Rubio.

Let B' be the symmetric point of B with respect to the AE-axis. Then, using the given information, we can establish the angle measures as shown in the figure :



We show that the desired point K is the meeting point of the lines AE and CD.

It is easy to see that the triangle ABB' is equilateral, the angles $\angle EBB'$ and $\angle EB'B$ are both of $20°$ and $\angle BEB' = 140°$. Therefore $\angle DEB = 40^o$ and the line DE is parallel to AC.

So the triangles DEK and AKC are similar and we have :

$$\frac{KE}{KA} = \frac{KD}{KC} \iff KE \cdot KC = KA \cdot KD$$

Since $\angle AKD = \angle EKC$, the areas of KAD and KCE are the same, which completes the proof.

Editor's Comment. Bataille noticed that this is problem 983, proposed by the same author in *The College Mathematics Journal*, Vol. 43, No 4, September 2012.

3802. *Proposed by Marcel Chiriţă.*

Solve the following system

$$\begin{gathered}\sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1} = 15 \\ 3^{2x+\sqrt{3y+1}} + 3^{3y+\sqrt{4z+1}} + 3^{4z+\sqrt{2x+1}} = 3^{30}\end{gathered}$$

for $x, y, z \in \mathbb{R}$.

Solved by AN-Anduud Problem Solving Group ; G. Apostolopoulos ; Š. Arslanagić ; M. Bataille ; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; R. Boukharfane ; M. Coiculescu ; C. Curtis ; J. L. Díaz-Barrero ; C. R. Diminnie ; R. Hess ; N. Hodžić ; S. Malikić ; P. Perfetti ; C. R. Pranesachar ; D. Smith ; D. Văcaru ; and the proposer. We present the solution by Chip Curtis.

For simplicity, let $u = \sqrt{2x+1}$, $v = \sqrt{3y+1}$, $w = \sqrt{4z+1}$ and then $p = u^2-1+v$, $q = v^2-1+w$ and $r = w^2-1+u$. Then by AM-GM we have :

$$\begin{aligned} 3^{30} &= 3^p + 3^q + 3^r \\ &\geq 3 \cdot \sqrt[3]{3^p 3^q 3^r} \\ &= 3 \cdot 3^{\frac{p+q+r}{3}} \end{aligned}$$

with equality if and only if $p = q = r$. On the other hand, by Cauchy-Schwartz

$$\begin{aligned} p+q+r &= (u+v+w) - 3 + 3(u^2+v^2+w^2) \\ &= 12 + (u^2+v^2+w^2) \\ &\geq 12 + \frac{(u+v+w)^2}{3} \\ &= 87 \end{aligned}$$

with equality if and only if $u = v = w$.

Hence for any triple (u, v, w) with $u + v + w = 15$, we have

$$3^{u^2-1+v} + 3^{v^2-1+w} + 3^{w^2-1+u} \geq 3^{30}$$

with equality if and only if $u = v = w$. Since their sum is 15, this implies that $u = v = w = 5$ and hence

$$x = 12, y = 8, z = 6.$$

3803. *Proposed by José Luis Díaz-Barrero.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+ab} + \sqrt{c^2+bc} \leq \sqrt{2}(a+b+c)\,.$$

Solved by A. Alt ; AN-anduud Problem Solving Group ; G, Apostolopoulos ; Š. Arslanagić ; D. Bailey, E. Campbell, and C. Diminnie ; M. Bataille ; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru ; R. Boukharfane ; C. Curtis ; J. G. Heuver ; N. Hodžić ; T. Karamfilova ; O. Kouba ; K. Lau ; S. Malikić (2 solutions) ; D. E. Manes ; P. McCartney ; C. Mortici ; S. Muralidharan ; P. Perfetti ; C.M. Quang ; Skidmore College Problem Group ; D. Smith ; G. Tsapakadis ; D. Văcaru (2 solutions) ; H. Wang and J. Wojdylo ; P. Y. Woo ; and the proposer. We present three solutions.

Solution 1, by AN-anduud Problem Solving Group.

By the AM-GM inequality, we have :

$$\begin{aligned}\sum \sqrt{a^2+ca} &= \frac{1}{\sqrt{2}} \sum \sqrt{2a(a+c)} \\ &\le \frac{1}{\sqrt{2}} \sum \frac{2a+(a+c)}{2} \\ &= \frac{1}{2\sqrt{2}} \sum (3a+c) = \sqrt{2}(a+b+c).\end{aligned}$$

Clearly, equality holds if and only if $a=b=c$.

Solution 2, by Kee-Wai Lau.

We have

$$\sqrt{a^2+ca} = \frac{1}{2\sqrt{2}}(3a+c-(\sqrt{2a}-\sqrt{a+c})^2) \le \frac{3a+c}{2\sqrt{2}}.$$

Similarly,

$$\sqrt{b^2+ab} \le \frac{3b+a}{2\sqrt{2}} \quad \text{and} \quad \sqrt{c^2+bc} \le \frac{3c+b}{2\sqrt{2}}.$$

The result follows by adding up the three inequalities.

Solution 3, by D. M. Bătineţu-Giurgiu, Neculai Stanciu and Titu Zvonaru.

By the Cauchy-Schwarz inequality, we have :

$$(\sqrt{a}\cdot\sqrt{a+c}+\sqrt{b}\cdot\sqrt{b+a}+\sqrt{c}\cdot\sqrt{c+b})^2 \le (a+b+c)(a+c+b+a+b+c)$$

or

$$\sqrt{a^2+ca}+\sqrt{b^2+ab}+\sqrt{c^2+bc} \le \sqrt{2}(a+b+c).$$

Editor's Comment. D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru proved the following generalization : let $m,n \in \mathbb{N}$ with $m \le n$, $b_j \in [0,\infty)$, $j=1,2,\ldots,m$, $x_k \in (0,\infty)$, $k=1,2,\ldots,n$. Let $B_m = \sum_{j=1}^m b_j$, $X_n = \sum_{k=1}^n x_k$ and $X_{k,m} = \sum_{i=0}^{m-1} b_i x_k x_{k+i}$ where the indices are taken modulo n and $k=1,2,\ldots,n$. Then

$$\sum_{k=1}^n \sqrt{X_{k,m}} \le \sqrt{B_m}\cdot X_n.$$

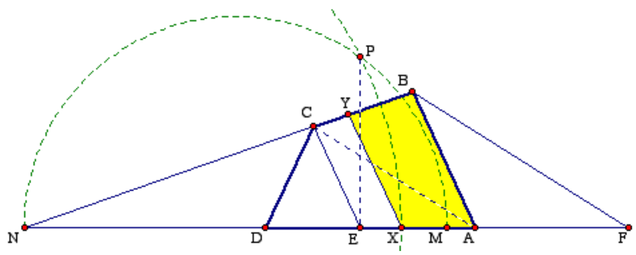
The inequality in the problem is the special case when $m=n=3$, $b_1=2$, $b_2=b_3=1$ and $(x_1,x_2,x_3)=(a,b,c)$.

3804. *Proposed by Václav Konečný.*

Let $ABCD$ be a convex quadrilateral. Construct, using only compass and straightedge, the line parallel to one side of the quadrilateral which bisects its area.

Solved by O. Geupel, C. Sánchez-Rubio, E. Swylan, P. Woo, and the proposer. We present the partial solution by Peter Woo, modified and completed by the editor.

We shall use square brackets to denote area. Label the given quadrilateral $ABCD$ so that D is further from AB than C is. Let E be the point of the side AD for which $CE||AB$, and F be the point on the line AD for which $BF||CA$. Note that the points lie in the order D, E, A, F and, because $[ABC] = [AFC]$, the area of triangle FCD equals the area of the given quadrilateral $ABCD$. Our goal is to construct the bisecting line XY parallel to AB with X on AD and Y falling on either BC or CD.



Construct M to be the midpoint of FD. Then $[MCD] = \frac{1}{2}[FCD] = \frac{1}{2}[ABCD]$. If M coincides with E, then MC is the bisecting line parallel to AB (so that $X = M$ and $Y = C$). Should E lie between D and M, then XY will necessarily lie between EC and AB (with Y on the segment BC); otherwise, EC will lie between XY and AB (and Y will lie on CD).

Consider first the case where E lies between M and D. Should the lines BC and AD meet in a point N, then we construct X so that NX is the geometric mean of NM and NE; that is, $NX^2 = NM \cdot NE$. For the usual Euclidean construction when E lies between N and M as in the figure, let P be either point where the perpendicular to AD through E meets the circle whose diameter is MN; then X is the point between M and E where the circle with centre N and radius NP intersects AD. (When N is on the other side so that M lies between E and N, use the same construction reversing the roles of E and M.) Let the line through X parallel to AB meet CB at Y. Because

$$\frac{[XYN]}{[ECN]} = \frac{NX^2}{NE^2} = \frac{NM}{NE} = \frac{[MCN]}{[ECN]},$$

we have $[XYN] = [MCN]$, whence

$$[XYCD] = [MCD] = \frac{1}{2}[ABCD],$$

as desired. This argument breaks down when the sides BC and AD are parallel (so that N is at infinity). A straightforward continuity argument places X at the midpoint of the segment ME; alternatively, one can argue directly (using the parallelograms $AFBC$ and $EABC$) to show that when X is the midpoint of ME, the base DM of triangle CDM is twice the length of the base XA of parallelogram $XABY$ while these two polygons have the same altitude.

It remains to construct the point X in the case where M lies between D and E. Here we want $[XYD] = [MCD]$, so that now DX is the geometric mean of DE

and DM ; that is, $DX^2 = DE \cdot DM$. The proof of the claim proceeds as before with D in the role of N :

$$\frac{[XYD]}{[ECD]} = \frac{DX^2}{DE^2} = \frac{DM}{DE} = \frac{[MCD]}{[ECD]},$$

whence $[XYD] = [MCD]$, as desired.

3805. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with incentre I. Let A' be on ray IA beyond A such that $A'A = BC$. Let B' and C' be similarly defined, such that $B'B = CA$ and $C'C = AB$. Prove that

$$\frac{[A'B'C']}{[ABC]} \geq (1+\sqrt{3})^2,$$

where $[\cdot]$ denotes the area.

Solved by M. Bataille ; N. Hodžic ; O. Kouba ; S. Malikić ; M. Modak ; C. R. Pranesacher ; and the proposer. We present the solution of Madhav Modak, modified by the editor.

We note that

$$[A'B'C'] = [A'IB'] + [B'IC'] + [C'IA'].$$

Since $\angle AIB = 180^\circ - \frac{1}{2}(A+B) = 90^\circ + \frac{1}{2}C$, we have with usual notation,

$$\begin{aligned} [A'IB'] =& \frac{1}{2}A'I \cdot B'I \sin(90^\circ + \frac{1}{2}C) \\ =& \frac{1}{2}(a+AI)(b+BI)\cos(C/2) \\ =& \frac{1}{2}(ab + b \cdot AI + a \cdot BI + AI \cdot BI)\cos(C/2). \end{aligned} \tag{1}$$

The Law of Sines for $\triangle AIB$ gives $AI/\sin(B/2) = c/\cos(C/2)$, so that

$$\frac{1}{2}b \cdot AI \cos(C/2) = \frac{1}{2}bc \sin A \cdot \frac{\sin(B/2)}{\sin A} = [ABC] \cdot \frac{\sin(B/2)}{\sin A}.$$

Similarly,

$$\frac{1}{2}a \cdot BI \cos(C/2) = [ABC] \cdot \frac{\sin(A/2)}{\sin B}.$$

Hence (1) can be written as :

$$[A'IB'] = [ABC] \cdot \frac{\cos(C/2)}{\sin C} + [ABC] \cdot \frac{\sin(B/2)}{\sin A} + [ABC] \cdot \frac{\sin(A/2)}{\sin B} + [AIB].$$

We have similar expressions for $[B'IC']$ and $[C'IA']$. Adding gives

$$\begin{aligned} [A'B'C'] =& [A'IB'] + [B'IC'] + [C'IA'] \\ =& [ABC] \cdot (E_1 + E_2 + 1), \end{aligned}$$

where

$$E_1 = \frac{\cos(A/2)}{\sin A} + \frac{\cos(B/2)}{\sin B} + \frac{\cos(C/2)}{\sin C},$$
$$E_2 = \frac{\sin(B/2)+\sin(C/2)}{\sin A} + \frac{\sin(C/2)+\sin(A/2)}{\sin B} + \frac{\sin(A/2)+\sin(B/2)}{\sin C}.$$

Hence

$$\frac{[A'B'C']}{[ABC]} = E_1 + E_2 + 1. \tag{2}$$

We now prove that $E_1 \geq 3$ and $E_2 \geq 2\sqrt{3}$, which will prove the claim.

- First, for E_1 we have

$$E_1 = \frac{1}{2\sin(A/2)} + \frac{1}{2\sin(B/2)} + \frac{1}{2\sin(C/2)} = \frac{1}{2}\sum_{\text{cyclic}} \csc(A/2).$$

  The convexity of $f(x) = \csc(x/2)$ on $(0,\pi)$ implies that

$$E_1 = \frac{1}{2}[f(A)+f(B)+f(C)] \geq \frac{3}{2}\cdot f[(A+B+C)/3] = \frac{3}{2}\cdot\csc(\pi/6) = 3.$$

- Next, by the *AM-GM* inequality,

$$\begin{aligned} E_2 &\geq 6\left[\frac{\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)}{\sin^2 A\sin^2 B\sin^2 C}\right]^{1/6} \\ &= 6\left[\frac{1}{64\cos^2(A/2)\cos^2(B/2)\cos^2(C/2)}\right]^{1/6} \\ &= 3\left[\frac{1}{\cos(A/2)\cos(B/2)\cos(C/2)}\right]^{1/3}. \end{aligned}$$

  Another application of the *AM-GM* inequality gives

$$\cos(A/2)\cos(B/2)\cos(C/2) \leq \frac{1}{27}\left[\sum_{\text{cyclic}}\cos(A/2)\right]^3,$$

  and the concavity of the function $g(x) = \cos(x/2)$ on $(0,\pi)$ gives

$$\sum_{\text{cyclic}}\cos(\frac{A}{2}) = g(A)+g(B)+g(C) \leq 3\cdot g[(A+B+C)/3)] = 3\cdot\cos(\pi/6) = \frac{3\sqrt{3}}{2}.$$

  Hence,

$$\cos(A/2)\cos(B/2)\cos(C/2) \leq \frac{3\sqrt{3}}{8},$$

  implying that $E_2 \geq 2\sqrt{3}$.

3806. *Proposed by Michel Bataille.*

Let triangle ABC with angles $\alpha, \beta, \gamma \neq 90^\circ$ be inscribed in a circle with centre O and radius R, and let U, V, W be the centres of the hyperbolas with parameter R, focus O and associated directrices BC, CA, AB, respectively. Prove that

$$[UVW] \times [ABC] = R^4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$

where $[\cdot]$ denotes the area.

Solved by C.R. Pranesachar ; D. Văcaru, and the proposer. We present the solution by C.R. Pranesachar.

Because the letters a, b, c are reserved for the sides of the given triangle ABC, we shall let

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$$

be the equation of the hyperbola whose centre is U, eccentricity is e_1, and directrix (corresponding to its focus O) is BC. If OU intersects the directrix BC at D, then $O = (a_1e_1, 0)$, $D = \left(\frac{a_1}{e_1}, 0\right)$, and

$$OD = OU - DU = \frac{a_1\left(e_1^2 - 1\right)}{e_1} = R\cos\alpha. \tag{1}$$

The parameter of the hyperbola (which equals half the length of its latus rectum) equals $R = \frac{b_1^2}{a_1}$, or (because $b_1^2 = a_1^2\left(e_1^2 - 1\right)$),

$$a_1\left(e_1^2 - 1\right) = R. \tag{2}$$

From (1) and (2) we have $e_1 = \sec\alpha$ and $a_1 = R\cot^2\alpha$, so that

$$OU = a_1e_1 = R\frac{\cos\alpha}{\sin^2\alpha}.$$

Similarly,

$$OV = R\frac{\cos\beta}{\sin^2\beta} \quad \text{and} \quad OW = R\frac{\cos\gamma}{\sin^2\gamma}.$$

But $\angle UOV = \alpha + \beta$ and $\sin(\alpha+\beta) = \sin\gamma$, so that $[OUV] = \frac{1}{2}OU \cdot OV \sin\gamma$, etc. Consequently,

$$[UVW] = [OUV] + [OVW] + [OWU] = \sum_{cyclic} \frac{R^2\cos\alpha\cos\beta}{2\sin^2\alpha\sin^2\beta}\sin\gamma.$$

Letting $F = [ABC]$ and using the formulae $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}, \sin\alpha = \frac{2F}{bc}$, etc., one has

$$[UVW] = \frac{R^2}{64F^3}\sum_{cyclic} c^2(b^2+c^2-a^2)(c^2+a^2-b^2).$$

Since $F = [ABC] = \frac{abc}{4R}$, we have after simplification,

$$[UVW]\times[ABC] = \frac{R^4}{4a^2b^2c^2}(a^6+b^6+c^6-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)+6a^2b^2c^2).$$

But

$$\begin{aligned}\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \\ &= \sum_{cyclic}\frac{(b^2+c^2-a^a)^2}{4b^2c^2} = \frac{\sum a^2(b^2+c^2-a^a)^2}{4a^2b^2c^2} \\ &= \frac{a^6+b^6+c^6-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)+6a^2b^2c^2}{4a^2b^2c^2}.\end{aligned}$$

Thus,

$$[UVW]\times[ABC] = R^4(\cos^2\alpha+\cos^2\beta+\cos^2\gamma),$$

as desired.

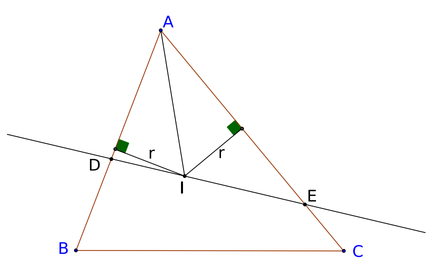
3807. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with incentre I through which an arbitrary line passes meeting sides AB and AC at the points D and E respectively. Show that

$$\frac{1}{r} \geq \frac{1}{AD} + \frac{1}{AE}$$

where r denotes the inradius of ABC.

Solved by A. Alt ; M. Amengual Covas ; AN-anduud Problem Solving Group ; Š. Arslanagić ; R. Barroso Campos ; M. Bataille ; R. Barbara ; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru ; C. Curtis ; P. De ; O. Geupel ; D. Jonsson ; O. Kouba ; S. Malikić ; M. Modak ; C.R. Pranesachar ; C. M. Quang ; C. Sánchez-Rubio ; E. Swylan ; G. Tsapakidis ; D. Văcaru ; P.Y. Woo ; and the proposer. We present the solution by Oliver Geupel.



Denoting area by $[\cdot]$, we have

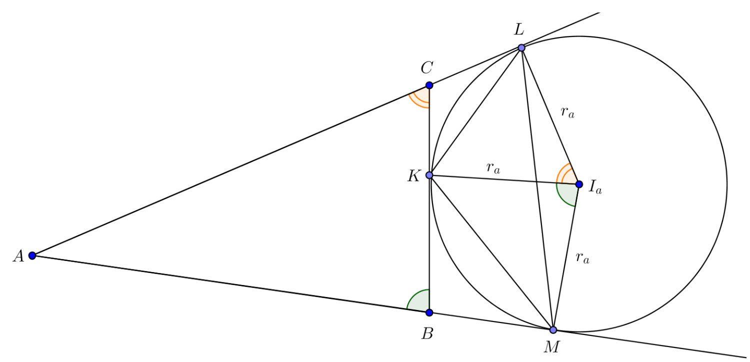
$$AD\cdot AE \geq AD\cdot AE\sin\angle A = 2[ADE] = 2[ADI]+2[AEI] = AD\cdot r + AE\cdot r.$$

Hence
$$r \leq \frac{AD \cdot AE}{AD + AE}$$
and the result follows immediately. The equality holds if and only if $\angle A$ is a right angle.

3808. *Proposed by Mehmet Şahin.*

Let ABC be a triangle with area Δ ; circumradius R ; exradii r_a, r_b, r_c ; and excenters I_a, I_b, I_c. The excircle with centre I_a touches the sides of ABC at K, L, and M. Let Δ_1 represent the area of triangle KLM and let Δ_2 and Δ_3 be similarly defined. Prove that
$$\frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{r_a + r_b + r_c}{2R}.$$

Solved by A. Alt ; M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; P. De ; O. Geupel ; J. Heuver ; O. Kouba ; S. Malikić ; C.R. Pranesachar ; C. Sánchez-Rubio ; G. Tsapakidis ; D. Văcaru ; P. Y. Woo ; T. Zvonaru ; and the proposer. We present a composite of similar solutions by Arkady Alt, Miguel Amengual Covas, and Oliver Geupel.



We use the common notation
$$a = BC, \quad b = CA, \quad c = AB, \quad 2s = a + b + c.$$

Since quadrilaterals MI_aKB, I_aLCK, and MI_aLA are cyclic, we have
$$\angle MI_aK = \angle B, \quad \angle KI_aL = \angle C, \quad \text{and} \quad \angle MI_aL = \angle 180^\circ - \angle A.$$

It follows that
$$\begin{aligned}
\Delta_1 &= [I_aKM] + [I_aLK] - [I_aLM] \\
&= \frac{r_a^2}{2}(\sin B + \sin C - \sin(180^\circ - A)) \\
&= \frac{r_a^2}{2}(\sin B + \sin C - \sin A) \\
&= \frac{r_a^2}{2} \cdot \frac{b + c - a}{2R} = \frac{r_a}{2R} \cdot r_a(s - a) = \frac{r_a}{2R}\Delta.
\end{aligned}$$

Analogously,

$$\Delta_2 = \frac{r_b}{2R}\Delta, \quad \Delta_3 = \frac{r_c}{2R}\Delta,$$

hence the result.

3809. *Proposed by Michel Bataille.*

For positive real numbers x, y, let

$$G(x,y) = \sqrt{xy}, \quad A(x,y) = \frac{x+y}{2}, \quad Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Prove that

$$G(x^x, y^y) \geq (Q(x,y))^{A(x,y)}\,.$$

Solved by AN-anduud Problem Solving Group ; R. Boukharfane ; C. Curtis ; P. Deiermann and H. Wang ; O. Kouba ; K. W. Lau ; P. Perfetti ; D. Smith ; and the proposer. One incorrect solution was received. We present the solution by Paolo Perfetti.

The given inequality is equivalent to

$$x^{\frac{x}{2}} x^{\frac{y}{2}} \geq \left(\sqrt{\frac{x^2+y^2}{2}}\right)^{\frac{x+y}{2}} \iff x^{\frac{2x}{x+y}} y^{\frac{2y}{x+y}} \geq \frac{x^2+y^2}{2},$$

which upon being divided by x^2 becomes

$$\frac{y^{\frac{2y}{x+y}}}{x^{\frac{2y}{x+y}}} \geq \frac{1}{2}\left(1+\left(\frac{y}{x}\right)^2\right). \tag{1}$$

Without loss of generality, we assume that $x \leq y$. Let $t = \frac{y}{x}$. Then $t \geq 1$, $\frac{2y}{x+y} = \frac{2t}{1+t}$ and (1) becomes

$$t^{\frac{2t}{1+t}} \geq \frac{1+t^2}{2} \iff \frac{2t}{1+t}\ln t \geq \ln\left(\frac{1+t^2}{2}\right). \tag{2}$$

To prove (2), let $f(t) = \frac{2t}{1+t}\ln t - \ln\frac{1+t^2}{2}$, $t \geq 1$. Then by routine calculations, we find :

$$f'(t) = 2\left(\frac{1-t^2+(1+t^2)\ln t}{(1+t)^2(1+t^2)}\right).$$

We claim that

$$1-t^2+(1+t^2)\ln t \geq 0 \quad \text{for all} \quad t \geq 1. \tag{3}$$

Let $h(t) = \ln t - \dfrac{t^2-1}{1+t^2} = \ln t - 1 + \dfrac{2}{1+t^2}$. Then

$$h'(t) = \frac{1}{t} - \frac{4t}{(1+t^2)^2} = \frac{(1-t^2)^2}{t(1+t^2)^2} \geq 0,$$

so $h(t)$ is an increasing function.

Since $h(1) = 0$, we have $h(t) \geq 0$, from which (3) follows. Hence, $f'(t) \geq 0$, which implies that $f(t)$ is an increasing function. Since $f(1) = 0$, we conclude that $f(t) \geq 0$ for all $t \geq 1$, which establishes (2) and completes the proof.

3810. *Proposed by Ovidiu Furdui.*

Let $k > 0$ be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} dxdy,$$

where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a.

Solved by Š. Arslanagić ; R. Boukharfane ; C. Curtis ; O. Geupel ; R. I. Hess ; O. Kouba ; J. Ling ; D. Stone and J. Hawkins ; and the proposer. One incorrect solution was received, although the error was of a purely algebraic variety. We present two solutions.

Solution 1, by Oliver Geupel.

For $0 \leq x \leq 1$, let $f(x) = \int\limits_0^1 \left\{ \frac{x}{y} \right\} dy$. Let $y = \frac{x}{t}$, then $dy = -\frac{x}{t^2}\, dt$ and we obtain :

$$\begin{aligned} f(x) = x \int_x^\infty \frac{\{t\}}{t^2}\, dt &= x \left(\int_x^1 \frac{dt}{t} + \lim_{n\to\infty} \sum_{\ell=1}^{n} \int_\ell^{\ell+1} \frac{t-\ell}{t^2}\, dt \right) \\ &= x \left(-\log x + \lim_{n\to\infty} \sum_{\ell=1}^{n} \left(\log(\ell+1) - \log \ell + \frac{\ell}{\ell+1} - 1 \right) \right) \\ &= -x \log x + x \cdot \lim_{n\to\infty} \left(\log(n+1) - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \right) \right) \\ &= -x \log x + x(1-\gamma). \end{aligned}$$

Hence, $f(x^k) = x^k(1-\gamma) - kx^k \log x$.

Let I be the integral to be evaluated. Then we have :

$$\begin{aligned} I = \int_0^1 f(x^k)\, dx &= \frac{1-\gamma}{k+1} - k \int_0^1 x^k \log x\, dx \\ &= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \int_u^1 x^k \log x\, dx \\ &= \frac{1-\gamma}{k+1} - k \cdot \lim_{u\to 0+} \left[\frac{1}{k+1} x^{k+1} \log x - \frac{x^{k+1}}{(k+1)^2} \right]_u^1 \\ &= \frac{1-\gamma}{k+1} + \frac{k}{(k+1)^2}, \end{aligned}$$

because $\lim\limits_{u\to 0+} u^{k+1}\log u = 0$ by l'Hôpital's rule.

Solution 2, by Chip Curtis slightly modified by the editor.

We consider this as an integral over a two-dimensional region, and perform a change of variables. Let

$$u = \frac{x^k}{y},\ \ v = y;$$

let $\alpha = \dfrac{1}{k}$, so that we obtain $x = (uv)^\alpha$. The Jacobian is given by :

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \alpha u^{\alpha-1}v^\alpha & \alpha u^\alpha v^{\alpha-1} \\ 0 & 1 \end{vmatrix} = \alpha u^{\alpha-1}v^\alpha.$$

The image $\mathcal{S}$ of the region $\mathcal{R} = [0,1]\times[0,1]$ is

$$\begin{aligned}\mathcal{S} &= \left\{(u,v) : 0 \le v \le 1 \text{ and } 0 \le u \le \frac{1}{v}\right\} \\ &= \{(u,v) : 0 \le u \le 1 \text{ and } 0 \le v \le 1\} \cup \left\{(u,v) : 1 \le u \le \infty \text{ and } 0 \le v \le \frac{1}{u}\right\}.\end{aligned}$$

The integral then becomes :

$$\begin{aligned} I(k) &= \alpha \iint_{\mathcal{S}} \{u\}u^{\alpha-1}v^\alpha \ dvdu \\ &= \alpha \int_0^1 \int_0^1 \{u\}u^{\alpha-1}v^\alpha \ dvdu + \alpha \int_1^\infty \int_0^{\frac{1}{u}} \{u\}u^{\alpha-1}v^\alpha \ dvdu \\ &= \alpha \int_0^1 \int_0^1 u^\alpha v^\alpha \ dvdu + \alpha \int_1^\infty \{u\}u^{\alpha-1}\left(\int_0^{\frac{1}{u}} v^\alpha \ dv\right)\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \alpha\int_1^\infty \{u\}u^{\alpha-1}\left(\frac{1}{(\alpha+1)u^{\alpha+1}}\right)\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \frac{\alpha}{\alpha+1}\int_1^\infty \{u\}u^{-2}\ du \\ &= \frac{\alpha}{(\alpha+1)^2} + \frac{\alpha}{\alpha+1}(1-\gamma) \\ &= \frac{k}{(k+1)^2} + \frac{1-\gamma}{k+1}, \end{aligned}$$

where the integral at the end of the computation is part of the one performed in the previous solution, presented above.

Editor's comments. Two main methods of proof were utilized. One method involved using a substitution ; some did a 1D swap just to compute the inside integral, and others used a full 2D change of variables, complete with Jacobian factor. The other method featured a computation of the branches of $\{\frac{x^k}{y}\}$ in the unit square, then summing the integrals over each region. The reader may wish to prove that the limit defining the Euler-Mascheroni number γ is convergent. Two comments regarding this problem were received, both stating that the problem and the proposer's solution appear (on pages 104 and 129, respectively) in the proposer's 2013 book, *Limits, Series, and Fractional Part Integrals*, published by Springer.

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(Bold font indicates featured solution.)

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