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The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Augustin-Louis Cauchy

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The year 1989 marks the bicentenary of the birth of the great French mathematician Cauchy. He is remembered for the introduction of more rigorous methods of argument into the foundations of the differential and integral calculus and the theory of infinite series, and for his creation of the theory of analytic functions of a complex variable. He also made important contributions to the basic ideas of elasticity theory and to the early stages of the theory of groups. He was one of the most prolific mathematicians of all time; in the volume of his work he was exceeded only by Leonhard Euler, in the eighteenth century.

Augustin-Louis Cauchy was born in Paris on 21 August 1789, just a few weeks after the French Revolution's outbreak was signalled by the fall of the Bastille on 14 July. His father Louis-François Cauchy was secretary to a high police official; realising that he was in a dangerous position, he soon moved to a safer occupation. When the Terror reached its peak in April 1794, Louis-François left Paris, moving with his wife and family to a small country property he owned at Arcueil. They had a bad time that summer, living mainly on produce that they grew themselves, and Augustin-Louis had an attack of smallpox; he became a timid and studious child, and acquired a lasting horror of anything that savoured of revolution. When the Terror came to an end with the fall of Robespierre, the family returned to Paris; by 1800 Louis-François had become secretary to the Senate, a well-paid position that brought with it an official residence in the Luxembourg palace.

Augustin-Louis was taught at home by his father until he was 13 years old, and so had opportunities of meeting the great mathematicians Lagrange and Laplace, who were members of the Senate. His mathematical ability was already showing itself; Lagrange is said to have declared in 1801 that in due course 'he would surpass all of us poor geometers.'

After a successful school career, young Cauchy decided to compete in the entrance examination for the Ecole Polytechnique, which had been founded in 1795 to train future engineers. In the 1805 competition he came second out of 293 candidates, of whom 125 were successful. After his two years' course there in mathematics, mechanics, physics and chemistry, he went on to the Ecole des Ponts et Chaussées (school of bridges and roads) to be trained as a civil engineer. He graduated with flying colours in 1809 and was assigned to work on the new naval harbour and fortifications at Cherbourg. Besides putting strenuous efforts into his professional work, he devoted much of his spare time to mathematical research. In the end his health, which had always been delicate, broke down, and he returned to Paris on sick leave in the autumn of 1812.

Among the results he obtained during his stay in Cherbourg were several about polyhedra; he showed that the only regular polyhedra were the five convex ones known to the ancient Greeks, Kepler's star-dodecahedron and the three star-polyhedra discovered by Poinsot in 1809, and he showed that a convex polyhedron with rigid faces is itself rigid. He proved a conjecture stated by Fermat in the 17th century that every positive integer is the sum of not more than n polygonal numbers of order n, i.e. numbers of the form

$$k+\tfrac{1}{2}k(k-1)(n-2) \quad (k = 1,2,3,\ldots);$$

for instance, the pentagonal numbers are $1, 5, 12, 22, \ldots$.

He also wrote an important memoir on the theory of substitutions (or, as we should say today, of permutation groups); as a sample, he proved that if $f(x_1, x_2, \ldots, x_n)$ is a polynomial in n variables and takes more than two different values when the variables are permuted in any way, then it must take at least p different values, where p is the largest prime divisor of n. In fact, more is true; it was proved later that, if $n > 4$, then the polynomial must take at least n different values. The paper contained a number of results that were to become fundamental in the theory of groups; it also included the first comprehensive account of the theory of determinants.

All these results were presented to the Academy of Sciences in Paris, and made a considerable impression on the leading Paris mathematicians. The possibility of a career as a professional mathematician seemed to be opening before Cauchy.

We must now return briefly to Cauchy's student days at the Ecole Polytechnique; most of the aspirant engineers held liberal and anticlerical views, but a Catholic revival was beginning to emerge. In view of his solid Catholic education and his horror of revolutionary ideas, it is not surprising that he leaned to the Catholic side; in 1808 he became a member of a fervently Catholic society called the Congregation of the Holy Virgin. In April 1814 came the defeat and abdication of the Emperor Napoleon, and the restoration of the exiled Bourbon heir as King Louis XVIII. Under the new regime many influential courtiers and members of Parliament belonged to the Congregation, so that Cauchy had friends in high places, and his prospects of obtaining an academic position were much improved;

in December 1815 he was appointed to an adjoint professorship at the Ecole Polytechnique.

The new government reorganised the Academy of Sciences; in March 1816 it promulgated a decree removing two of its most distinguished members, Lazare Carnot and Gaspard Monge. Carnot had played an important part in the defence of revolutionary France against its invaders, and Monge had been an intimate friend of the Emperor. By the same decree, Cauchy was named as a member of the Academy; he accepted the nomination without hesitation. This action tarnished his reputation in scientific circles, and it was many years before he lived it down. Later in the same year he was made a full professor at the Ecole Polytechnique.

While waiting for the appointment, Cauchy worked spasmodically on some civil engineering projects in the neighbourhood of Paris, but with frequent spells of leave. He also wrote two important mathematical memoirs. The first of these, submitted in August 1814, was a long paper on definite integrals. Euler, Laplace and others had evaluated some definite integrals, such as

$$\int_0^\infty \frac{\cos bx}{a^2+x^2}\,\mathrm{d}x = \frac{\pi}{2a}\mathrm{e}^{-ab} \quad (a > 0,\ b > 0),$$

by methods involving imaginary changes of variables, and doubts had been raised, especially by Poisson, about the validity of such techniques. Possibly at Laplace's suggestion, Cauchy looked for and found a new approach that made it possible to use such devices legitimately under appropriate conditions on the functions involved. This paper was the first step in a long journey that eventually led to Cauchy's creation of the theory of analytic functions of a complex variable; it contained results equivalent to special cases of what later became known as Cauchy's theorem and the residue theorem. The second paper arose from the announcement by the Academy of Sciences of the theory of surface waves on a fluid of infinite depth as the subject for its chief mathematical prize for the year 1815; Cauchy submitted a memoir for the competition in October of that year, and was awarded the prize of 3000 francs.

The time from 1816 to 1830 was a very busy one in Cauchy's career. He was teaching at the Ecole Polytechnique, he gave lectures at the Collège de France in 1817 and again from 1824 to 1830, and he was assistant professor of mechanics at the Sorbonne from 1823 to 1830. In 1818 he married Aloïse de Bure; they had two daughters, both of whom ultimately married into the aristocracy. His father-in-law was a publisher, and much of Cauchy's mathematical work appeared under his auspices.

One of Cauchy's major achievements in this period was his reconstruction of the foundations of analysis. In the eighteenth century a function was thought of as being an analytic expression; it might be a polynomial, a rational function or an explicit algebraic function; it might involve logarithms, exponentials or trigonometric functions. It might also involve infinite series or products, and it was generally assumed that the formal rules of algebra applied to these irrespective of any considerations of convergence. Another assumption was that any identity holding for a range of values of the variables involved in it was universally valid, even when complex numbers (then usually called 'imaginary quantities') were substituted for the variables. This principle, usually described as 'the generality of analysis', was frequently appealed to.

In his lectures on analysis at the Ecole Polytechnique and in his books covering the same material, the *Analyse algébrique* (1821) and the *Calcul infinitésimal* (1823), Cauchy carried through his reconstruction. He rejected the principle of the generality of analysis, pointing out cases where its use could lead to erroneous results, and he rejected the use of divergent series for similar reasons. He developed the first comprehensive theory of limits, and used it as the basis for his account of derivatives and integrals, the central concepts of the calculus, and he devised a battery of tests for the convergence of infinite series. He was the first to prove existence theorems for the solutions of differential equations. His achievements eventually brought about a revolution in the exposition of the calculus and other branches of analysis.

It should be mentioned, however, that his lectures on analysis were not very popular with his students at the Ecole Polytechnique; most of his hearers found them too difficult, he tended to ignore the official syllabus, and his lectures often overran their allotted time, so that not enough time was left for work on illustrative examples and for other parts of the course. Only a few of the brighter members of the audience had any idea of what he was trying to do.

Among the numerous subjects to which Cauchy contributed in this period was the theory of elasticity, of which he was one of the principal pioneers, showing how stress and strain and the relations between them can be properly expressed in mathematical form. He tried to apply this work to the theory of light, which he thought of as being a vibration in an elastic medium called the ether; in this he was a precursor of James Clerk Maxwell.

He was so prolific that the existing periodicals were unable to find room for everything he produced. He overcame this difficulty by publishing a periodical of his own, the *Exercices de Mathématiques*; this appeared monthly for more than four years (1826–1830) and consisted entirely of papers written by himself.

In 1830 there was a dramatic break in Cauchy's career. In that year King Charles X, who had succeeded his brother Louis XVIII, was overthrown and replaced by Louis-Philippe, who belonged to a different branch of the royal family. Since many supporters of the new regime were anticlerical, wishing to diminish the influence of the Catholic church, Cauchy feared that the country would collapse into chaos, as had happened in the troubled times of the 1789 revolution. All public servants were required to take an oath of allegiance to the new king, but Cauchy declared that he had taken an oath to the Bourbons, and he would not go back on it. He abandoned his teaching posts before he was actually dismissed, and left the country. After a fruitless interlude in Switzerland, he went to Turin, where the King of Sardinia (a kingdom that also included Piedmont and Genoa) appointed him to a chair of theoretical physics at the university. The story is told that when the King asked him about the state of higher education in France, Cauchy said that he had expected such a question, so that he had prepared a memorandum on the subject; he took the paper from his pocket and proceeded forthwith to read it aloud to the King.

During his stay in Turin Cauchy made some further important contributions to complex function theory; in particular, he proved that an analytic function can be expanded in a power series whose radius of convergence is equal to the distance from the centre of the expansion to the nearest singularity of the function in the complex plane. This result explains why, for instance, the series

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\dots$$

converges only when $|x| < 1$; although the function is well behaved for all real x, it has a singularity when $x = \sqrt{-1}$.

When Cauchy had been in Turin for a couple of years, he was invited to become one of the tutors of the young Duke of Bordeaux, the grandson and heir of the exiled King Charles X of France. He accepted the appointment as a loyal duty to the Bourbon dynasty, whose legitimacy he had always upheld. He remained at his post for five years, first at Prague and then at Görz (now called Gorizia), and was rewarded with the title of Baron.

In 1838 he returned to Paris; since he firmly refused to take an oath of allegiance to the reigning king, he could not be appointed to any public office. For the next ten years he continued his mathematical research and regularly took part in the meetings of the Academy of Sciences (no oath was required for this). He did some more work on the theory of light, obtained new results in the theory of substitutions, and laid sounder foundations for complex function theory. He made an unsuccessful attempt to prove Fermat's last theorem, which states that the equation

$$x^n + y^n = z^n$$

has no solutions in non-zero integers x, y and z, when $n \geqslant 3$; the problem remains unsettled to this day.

In 1836 the Academy of Sciences began to publish its weekly *Comptes Rendus* (Proceedings), and Cauchy published much of his work in it; in fact, he sent in so much that he would almost have filled it by himself if the Academy had not decided to limit the length of papers to a maximum of four pages. By then, indeed, he had got into the bad habit of publishing his ideas as soon as he thought of them, without taking the trouble to sort them out into a coherent story.

Cauchy's life went on in this fashion until the 1848 revolution overthrew Louis Philippe and established the Second Republic. This time the revolution was to his advantage; oaths of allegiance were temporarily out of fashion, and he was appointed as professor of mathematical astronomy at the Sorbonne. When, however, Napoleon III became Emperor in 1851, oaths of allegiance were required again; Cauchy withdrew from teaching for some months. All ended well for him, though; two distinguished scientists were excused from taking the oath—one was Cauchy (on the extreme right of politics) and the other was the physicist Arago (on the extreme left). As a result, Cauchy was able to continue his teaching and research undisturbed for the rest of his life.

It is said that Cauchy contributed a large part of his salary to Catholic charities (in which he had always been active) and to local charities in the little town of Sceaux, where he had a country house. He died there on 23 May 1857, at the age of 67.

As we have seen, Cauchy was in many ways a stiff and unbending character, who adhered firmly to his own (not very popular) principles. Bertrand said of him that everybody respected him and nobody really liked him; this may be an exaggeration, for he had some close friends among those who shared his religious and political views.

His mathematical work remains influential to this day, and his results are continually quoted in teaching and research. His collected works occupy 27 quarto volumes; these began to appear in 1882, and the final volume was published in 1974.

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Recurring Decimals

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I recently came across J. M. H. Peters' article 'Recurring decimals' (*Mathematical Spectrum* **17**, pages 40–41), which treats recurring decimal representations for rational numbers of the form $1/A$ (A any positive integer). Two points immediately attracted my attention.

First, Peters gives a method (using 'nine's complements') for evaluating $\frac{1}{17}$ with his 10-digit calculator, which displays

$$\tfrac{1}{17} = 0.058\,823\,529.$$

I, however, am not so lucky. My eight-digit calculator presents

$$\tfrac{1}{17} = 0.058\,8235;$$

although, if I compute $1/0.17$, I do get $5.882\,3529$ (without, of course, much reliance that the final 9 is not rounded up). However, my son is certainly underprivileged. His calculator only has a six-figure display, so the best he could deduce (via $1/0.17$) is

$$\tfrac{1}{17} = 0.058\,823(5)$$

which does not take the recurring pattern sufficiently far for Peters' method to be applied.

I remember this sort of problem back in the days when I was equipped with only a slide rule, and a bit of ingenuity could get the answer from just a 'two good-figure' display. Let me demonstrate this approach, with a 'three good-figure calculator' (the old 'four-figure' tables?), which avoids the need for any subtlety.

First, calculate $r/17$, to the limit of certain accuracy, for $r = 1$ to 16 (excluding 10):

$\frac{1}{17} = 0.0588$	$\frac{4}{17} = 0.235$	$\frac{7}{17} = 0.411$	$\frac{11}{17} = 0.647$	$\frac{14}{17} = 0.823$
$\frac{2}{17} = 0.117$	$\frac{5}{17} = 0.294$	$\frac{8}{17} = 0.470$	$\frac{12}{17} = 0.705$	$\frac{15}{17} = 0.882$
$\frac{3}{17} = 0.176$	$\frac{6}{17} = 0.352$	$\frac{9}{17} = 0.529$	$\frac{13}{17} = 0.764$	$\frac{16}{17} = 0.941.$

Then, invoking the property* that all $r/17$ involve the same cyclic

*This property holds for any $1/A$ which has a recurrence pattern of (maximum length) $A-1$ terms; so we are, in fact, assuming that $\frac{1}{17}$ has a 16-digit repeating cycle. Had we chosen instead to study $\frac{1}{13}$ say, we would have needed to modify our method, as $\frac{1}{13}$ ($= 0.\dot{0}7692\dot{3}$) only has a six-figure recurrence pattern.

arrangement of digits in their recurring decimals, we get the recurring cycle for 1/17 by following the only possible trail amongst these truncated $r/17$ values (which each provide three consecutive digits of the recurrence; except for the first, which gives four). This then yields:

r	
1	**0588**
15	882
14	823
4	235
6	352
9	529
5	294
16	941
7	411
2	117
3	176
13	764
11	647
(8	470)
(12	705)

where the $r = 8$ and $r = 12$ contributions at the very end are redundant, as we already have the maximum length of the recurrence cycle ($17-1 = 16$ digits) with the 7 obtained from $r = 11$.

As another example, let us compute $\frac{1}{29}$ with a normal eight-digit display calculator. For these longer strings, we should be able to get by with fewer $r/29$, so we try using just the first eight:

$$\tfrac{1}{29} = 0.034\,482\,75(9) \qquad \tfrac{4}{29} = 0.137\,9310(3) \qquad \tfrac{7}{29} = 0.241\,3793(1)$$

$$\tfrac{2}{29} = 0.068\,965\,51(7) \qquad \tfrac{5}{29} = 0.172\,4137(9) \qquad \tfrac{8}{29} = 0.275\,8620(7)$$

$$\tfrac{3}{29} = 0.103\,4482(8) \qquad \tfrac{6}{29} = 0.206\,8965(5).$$

Then we get:

r		r	
1	**03448275**	5	**1724137**
8	2758620	4	1379310
6	2068965	3	1034482
2	0689655		

using all r from 1 to 8, except 7. Clearly, the second partial cycle leads (with overlap) into the first, giving a total length of 28 digits (the maximum possible period, $29-1$). So, combining (by stitching the second on to the end of the first), we get:

$$\tfrac{1}{29} = 0.\dot{0}34\,482\,758\,620\,689\,655\,172\,413\,793\dot{1},$$

and, of course, it is straightforward to check this result by multiplying sections of it by 29 and adding. For example:

$$29 \times 344\,8275 = 999\,999\,75 \quad \text{and} \quad 29 \times 862\,068 = 249\,999\,72,$$

giving

$$29 \times 344\,827\,586\,2068 = \frac{999\,999\,752\,499\,9972}{999\,999\,999\,999\,72},$$

and so on.

Peters also tentatively conjectured that, for any recurring-decimal cycle $n_1 \ldots n_r n_{r+1} \ldots n_{2r}$ (of even length), $n_s + n_{r+s} = 9$ for all $s \in \{1, \ldots, r\}$. This interesting result is indeed true, as we now show, *if and only if* A divides 10^r+1, where $1/A$ is the division which gives rise to $0.\dot{n}_1 \ldots n_r n_{r+1} \ldots \dot{n}_{2r}$. (Of course, testing that 10^r+1 is indeed divisible by A, and then using nine's complements, cuts down considerably on the space needed to establish the $\frac{1}{17}$ and $\frac{1}{29}$ recurrences in our earlier examples.)

We shall in fact prove a more general result than this. Let R be a positive rational number, not necessarily an integer. We may always write

$$R = m.m_1 \ldots m_t \dot{n}_1 \ldots \dot{n}_{2r}, \tag{1}$$

where m is a non-negative integer, $m_1, \ldots, m_t, n_1, \ldots, n_{2r}$ are integers between 0 and 9, and r or t may be zero. For, if the recurrence is of odd length, $\dot{n}_1 \ldots \dot{n}_r$ say, then this is equivalent to $\dot{n}_1 \ldots n_r n_1 \ldots \dot{n}_r$. Also, not every n_i is 9.

Theorem. For R written in the form (1),

$$n_s + n_{r+s} = 9 \quad \text{for all } s \in \{1, \ldots, r\}$$

if and only if $10^t(10^r+1)R$ is an integer.

Proof. Since not every n_i is equal to 9,

$$n_1 \ldots n_r + n_{r+1} \ldots n_{2r} < 2 \times \underbrace{\{9 \ldots 9\}}_{r \text{ nines}} = 2(10^r - 1).$$

Now, $10^t(10^r+1)R$ is an integer

$$\Leftrightarrow \ 10^t(10^r+1)(R-m) \text{ is an integer}$$

$$\Leftrightarrow \ (10^r+1)(0.\dot{n}_1 \ldots \dot{n}_{2r}) \text{ is an integer.}$$

Also,

$$10^{2r}(0.\dot{n}_1\ldots\dot{n}_{2r}) = n_1\ldots n_{2r}.\dot{n}_1\ldots\dot{n}_{2r},$$

so that

$$(10^{2r}-1)(0.\dot{n}_1\ldots\dot{n}_{2r}) = n_1\ldots n_{2r}.$$

Hence

$$\Leftrightarrow\ n_1\ldots n_{2r} = (10^r-1)\times\text{an integer}$$

$$\Leftrightarrow\ 10^r n_1\ldots n_r + n_{r+1}\ldots n_{2r} = (10^r-1)\times\text{an integer}$$

$$\Leftrightarrow\ (10^r-1)n_1\ldots n_r + (n_1\ldots n_r + n_{r+1}\ldots n_{2r}) = (10^r-1)\times\text{an integer}$$

$$\Leftrightarrow\ n_1\ldots n_r + n_{r+1}\ldots n_{2r} = (10^r-1)\times\text{an integer} \tag{2}$$

$$\Leftrightarrow\ n_1\ldots n_r + n_{r+1}\ldots n_{2r} = 10^r-1 = \underbrace{9\ldots9}_{r\text{ nines}}$$

$$\Leftrightarrow\ n_s + n_{r+s} = 9 \quad \text{for } 1 \leqslant s \leqslant r.$$

If we assume Peters' conjecture for a particular reciprocal, calculation can be even shorter. As an example, return to $\frac{1}{29}$, and see how just the first four $r/29$ computed on our eight-digit (seven good-figure) display calculator are sufficient to establish the recurrence pattern. See figure 1.

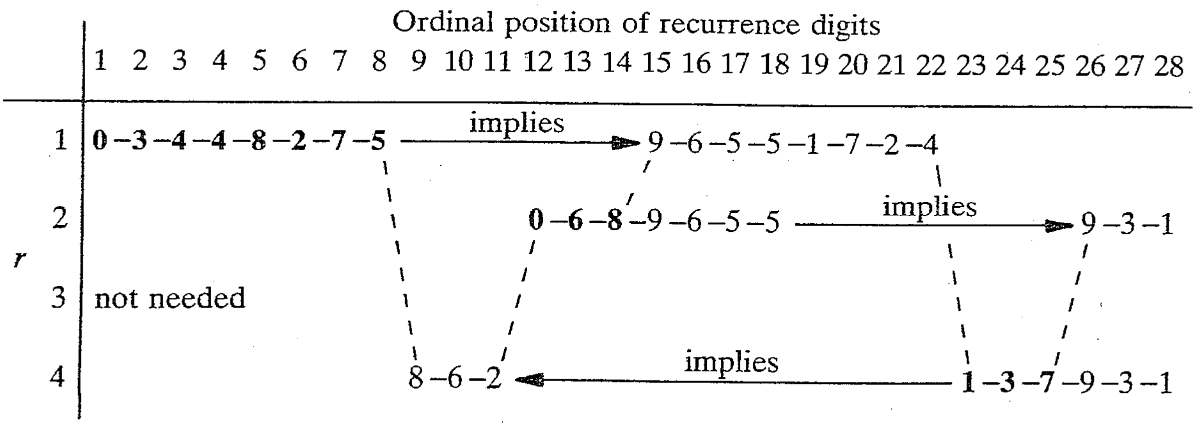

Figure 1. Derivation of the $\frac{1}{29}$ recurrence pattern from the first 'seven' figures of $r/29$ for $r = 1$, 2 and 4.

Postscript

I am grateful to Mr Peters for drawing my attention to correspondence in *Mathematical Spectrum* **17**, p. 89, where an alternative modification of his conjecture is referenced.

Get Lost!

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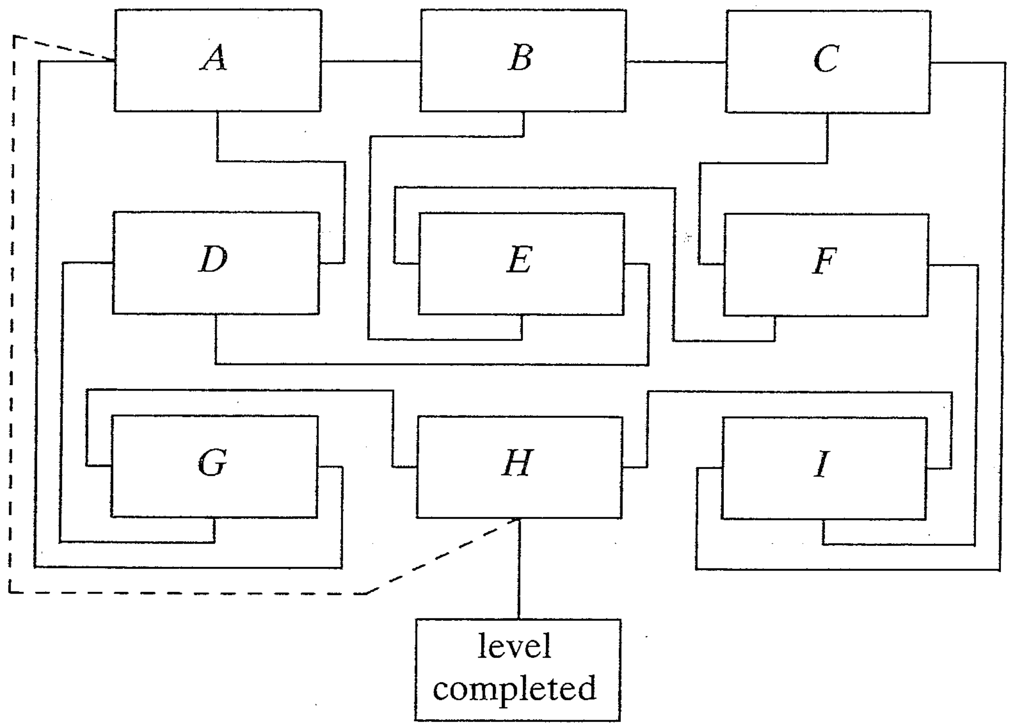
The computer game 'GET LOST!' is played by guiding an explorer through a maze of interconnected rooms. The four levels of the game all use the maze as shown in the map; there are nine rooms, labelled A to I for reference, each with three doors (left, bottom and right). The dotted line applies only to levels three and four.



During play the screen shows only one room at a time, the room where the explorer is. The screen never shows the journey between rooms—if the explorer leaves room G by the bottom door then room G vanishes from the screen and almost immediately room D appears, accompanied by the sound effect which is heard whenever the explorer changes room.

To complete a level of 'GET LOST!', simply guide the explorer out of room H by the bottom door. The snag is that *all rooms look the same*; at the start of each level the computer program puts the explorer in one of the nine rooms, but you cannot tell which room! Needless to say, if you play 'GET LOST!' many times you will find that the starting-room can be any one of the nine rooms and cannot be predicted.

On level one and level three when the explorer enters a room you see which door he comes in by, but on level two and level four when the explorer enters a room he appears in the middle of the room and you cannot tell which door he used. On level three and level four the bottom door in room H leads to the left door in room A until the explorer has visited all nine rooms; then it leads to the 'level completed' room. On level one and level two there is no need to visit every room.

For each of the four levels, invent a method of completing that level.

Powerful cycles

Malcolm Smithers has previously written to us about 'powerful numbers' (see Volume 16 Number 3 page 77) such as

$$153 = 1^3+5^3+3^3, \qquad 4150 = 4^5+1^5+5^5+0^5.$$

He has now sent us the '2-powerful cycle'

$$\{4, 16, 37, 58, 89, 145, 42, 20\}.$$

We have

$$\begin{array}{cccc} 4^2 = 16, & 1^2+6^2 = 37, & 3^2+7^2 = 58, & 5^2+8^2 = 89, \\ 8^2+9^2 = 145, & 1^2+4^2+5^2 = 42, & 4^2+2^2 = 20, & 2^2+0^2 = 4, \end{array}$$

which goes back to the beginning. Numbers may end up in a 2-powerful cycle. For example,

$$\begin{array}{ccc} 1^2+9^2+8^2+7^2 = 195, & 1^2+9^2+5^2 = 107, & 1^2+0^2+7^2 = 50, \\ 5^2+0^2 = 25, & 2^2+5^2 = 29, & 2^2+9^2 = 85, \end{array}$$

which now moves into the 2-powerful cycle above. An example of a 3-powerful cycle is

$$1^3+3^3+3^3 = 55, \qquad 5^3+5^3 = 250, \qquad 2^3+5^3+0^3 = 133,$$

giving $\{133, 55, 250\}$.

Sample Size—It Crops Up Like a Bad Penny

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The author runs the Brighton Statistics Consultancy Unit. His main research interest is with state-space time-series models. He enjoys drawing and painting, and helped with the present cover design of *Mathematical Spectrum*.

1. Introduction

This article examines the number of tosses required to determine whether a coin is biased. It argues that if the bias is small this number is likely to be so large that for most practical purposes it will not be easy to distinguish slightly biased coins from true coins.

Those engaged in statistical consultancy usually find that one type of question is regularly asked. It crops up in many different guises. 'Have I got enough data?' 'How many people do I need to interview?' 'Have I repeated the experiment enough times?' These questions are all concerned with sample size. Those asking the questions are often surprised to learn that there is not a straightforward answer concerning sample size for testing hypotheses. Sample size depends on the precision of the answers sought. It also depends on variation in the data and unknowns relating to the hypothesis being tested. These can sometimes be guessed from similar work, but often a pilot study is also required. The coin-tossing problem will serve as an example to highlight some general principles that can be applied.

Obtaining a good approximation for the sample size can be very important. If it is chosen too small then insufficient evidence will be available to draw any meaningful conclusions. If it is drawn larger than need be then time and effort will have been expended in gaining unnecessary precision. In the commercial environment, which aims for low costs and efficiency, these misjudgements must be minimised.

2. Testing for bias in a coin

Suppose a coin is tossed n times. Let the number of times that it lands heads up be denoted by the random variable X. If p is the probability of obtaining a head in one toss then X has the binomial distribution with

$$E(X) = np, \qquad \mathrm{Var}(X) = np(1-p).$$

For large n this distribution can be approximated by the normal distribution with the above mean and variance. If $\hat{p}$ is the proportion of heads obtained then, for large n,

$$\hat{p} = \frac{X}{n} \mathrel{\dot\sim} N\left(p, \frac{p(1-p)}{n}\right).$$

Suppose $p = \frac{1}{2}+\delta$ where δ is the unknown bias of the coin. A reasonable estimate of the bias is $\hat{\delta} = \hat{p}-\frac{1}{2}$, which for large n has the approximate distribution

$$N\left(p-\tfrac{1}{2}, \frac{p(1-p)}{n}\right).$$

It follows that

$$\hat{\delta} \mathrel{\dot\sim} N\left(\delta, \frac{\frac{1}{4}-\delta^2}{n}\right).$$

For any ordinary coin the bias, if it exists, is likely to be small. Thus the variance, $(\frac{1}{4}-\delta^2)/n$, can be approximated by $(4n)^{-1}$. Hence for large n and small bias δ,

$$(\hat{\delta}-\delta)2\sqrt{n} \mathrel{\dot\sim} N(0,1).$$

Suppose a hypothesis test is used to judge whether the coin is biased. Then the null hypothesis is $H_0 : \delta = 0$, that is, the coin is unbiased. Let the alternative hypothesis be $H_1 : \delta = \delta_0$, that is, the coin is biased with a bias δ_0 which we assume to be known. Suppose H_0 is true. Then $\hat{\delta}2\sqrt{n} \mathrel{\dot\sim} N(0,1)$. The test examines a realisation of

$$\hat{\delta}2\sqrt{n} = \left(\frac{x}{n}-\frac{1}{2}\right)2\sqrt{n},$$

where x is the observed number of heads. If this lies in one of the tails of the standardised normal distribution then the hypothesis $H_0 : \delta = 0$ will be rejected. Suppose the test is carried out at the $100\alpha\%$ level of significance. The coin is then deemed to be biased if the test statistic $\hat{\delta}2\sqrt{n}$ lies outside the limits $\pm Z_{\frac{1}{2}\alpha}$. The limits $\pm Z_{\frac{1}{2}\alpha}$ are determined from the standardised normal distribution by the equation

$$P(-Z_{\frac{1}{2}\alpha} < Z < Z_{\frac{1}{2}\alpha}) = 1-\alpha, \quad \text{where } Z \sim N(0,1).$$

Thus

$$P(|\hat{\delta}2\sqrt{n}| > Z_{\frac{1}{2}\alpha} \mid \delta = 0) \doteq \alpha. \tag{1}$$

This gives the probability of type-I error. It is the probability of rejecting H_0 when H_0 is true. Type-II error occurs if H_0 is not rejected when H_1 is true. If we choose the probability of type-II error to be β, then

$$P(-Z_{\frac{1}{2}\alpha} < \hat{\delta}2\sqrt{n} < Z_{\frac{1}{2}\alpha} \mid \delta = \delta_0) \doteq \beta. \tag{2}$$

By specifying values for α and β, the required sample size can be approximated from equations (1) and (2).

However, the calculation involves knowing δ_0, the actual bias of the coin. The estimation of this bias may be treated as a statistical problem, but can also be considered as one of applied mathematics. This involves setting up a model to represent the dynamics of the spinning coin and its subsequent motion on hitting the ground. Bias is introduced in a coin by changing the physical characteristics across its width, and my preliminary calculations show that there is little scope for this because the width is small compared with the diameter of the coin. A realistic assessment of bias on a British two-pence coin (which is 25 mm in diameter and 2 mm thick) was found to be 0.01 or less. Thus it is difficult to manufacture real coins of normal size with anything other than a very small bias.

3. Finding the sample size

The sample size can be found by considering the fact that, when δ_0 is small, $(\hat{\delta}-\delta_0)2\sqrt{n} = Z$ is approximately standard normal and using the two equations (1) and (2). Choose values for α and β. Equation (1) fixes $Z_{\frac{1}{2}\alpha}$. With this value $Z_{\frac{1}{2}\alpha}$ determined, n can then be approximated from equation (2). From equation (2) we can see that

$$P(-Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n} < Z < Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n}) \doteq \beta, \tag{3}$$

where $Z \sim N(0,1)$. Hence n can be found approximately by solving the equation

$$\int_{-Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n}}^{Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n}} \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}x^2)\, dx = \beta.$$

However, this difficult task can be bypassed by using an approximation.

Assume for the moment that $\delta_0 > 0$ so that, for some k,

$$-Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n} < -k^2. \tag{4}$$

For large values of k^2, greater than 3, say, the area under the normal curve to the left of $-Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n}$ is negligible. Then equation (3) can be approximated by $P(Z < Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n}) = \beta$. If Z_β is defined by the equation $P(Z < -Z_\beta) = \beta$, then, since

$$Z_{\frac{1}{2}\alpha}-\delta_0 2\sqrt{n} = -Z_\beta, \tag{5}$$

it follows that

$$n = \frac{(Z_{\frac{1}{2}\alpha}+Z_\beta)^2}{4\delta_0^2}. \tag{6}$$

From equations (4) and (5) it follows that $2Z_{\frac{1}{2}\alpha}+Z_\beta > k^2$. The approximation is only valid if α and β are chosen to satisfy this inequality. For typical values of α and β (i.e. $\alpha < 0.05$ and $\beta < 0.5$) the inequality, with $k = 3$, easily holds. Equation (6), which determines n, could also have been obtained by making the assumption that $\delta_0 < 0$. In this case $-Z_{\frac{1}{2}\alpha}+\delta 2\sqrt{n}$ is sufficiently large so that the area under the normal curve to the right of $Z_{\frac{1}{2}\alpha}+\delta_0 2\sqrt{n}$ is negligible. Values of $\frac{1}{4}(Z_{\frac{1}{2}\alpha}+Z_\beta)^2$, are given for various α and β in table 1.

Table 1. Values of $\frac{1}{4}(Z_{\frac{1}{2}\alpha}+Z_\beta)^2$

		Probability of type-I error, α		
		0.01	0.05	0.1
	0.05	4.5	3.2	2.7
Probability of type-II error, β	0.1	3.8	2.6	2.1
	0.2	3.0	2.0	1.5

The relationship between δ_0 and δ_0^{-2} is given in table 2.

Table 2. Values of δ_0^{-2}

δ_0	± 0.1	± 0.05	± 0.02	± 0.01	± 0.005
δ_0^{-2}	100	400	2500	10 000	40 000

By multiplying values of $\frac{1}{4}(Z_{\frac{1}{2}\alpha}+Z_\beta)^2$ from table 1 with values of δ_0^{-2} from table 2, estimates of the sample size n can be obtained.

When $\delta_0 = \pm 0.1$, $\alpha = 0.1$ and $\beta = 0.2$, the sample size is 150, but this increases rapidly as δ_0 becomes smaller. With $\delta_0 = \pm 0.02$, $\alpha = 0.1$ and $\beta = 0.2$, the required sample size is 3750. To perform this number of tosses would be a daunting task. However, with $\alpha = 0.05$ and $\beta = 0.05$ and with the perhaps more realistic $\delta_0 = \pm 0.01$, the sample size required rises to 32 000. If, with this α and β, δ_0 is taken to be ± 0.005, then we require $n = 128\,000$.

4. Concluding remarks

A very large number of tosses of a coin is required to distinguish a slightly biased from an unbiased coin, so much so that, from a purely practical point of view, two such coins are indistinguishable.

The question concerning sample size to detect some population characteristic is a recurring theme in statistics. The coin-tossing example to detect bias has illustrated ideas that may be generally applied in other contexts where sample size needs to be determined.

The Convergence of the Sequence $\left(\left\{1+\frac{1}{n}\right\}^n\right)$.

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In this article we shall describe two ways of proving that the sequence

$$\left(\left\{1+\frac{1}{n}\right\}^n\right) \quad (n = 1,2,3,\ldots)$$

has a limit which may be simpler than the proofs usually given. The first uses *Bernoulli's inequality*, namely,

$$(1+x)^n \geqslant 1+nx$$

for all positive integers n and all real numbers $x > -1$. This may be established easily using induction on n.

We write

$$A_n = \left(1+\frac{1}{n}\right)^n.$$

Then

$$\begin{aligned}
\frac{A_{n+1}}{A_n} = \frac{\left(1+\dfrac{1}{n+1}\right)^{n+1}}{\left(1+\dfrac{1}{n}\right)^n} &= \frac{n+2}{n+1}\left(\frac{n(n+2)}{(n+1)^2}\right)^n \\
&= \frac{n+2}{n+1}\left(1-\frac{1}{(n+1)^2}\right)^n \\
&\geqslant \frac{n+2}{n+1}\left(1-\frac{n}{(n+1)^2}\right) \\
&= \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}+\frac{1}{(n+1)^2}\right) \\
&= 1+\frac{1}{(n+1)^3} > 1.
\end{aligned}$$

Hence $A_1 < A_2 < A_3 < \ldots$. Next,

$$\frac{1}{\sqrt{A_{2n}}} = \left(\frac{2n}{2n+1}\right)^n = \left(1-\frac{1}{2n+1}\right)^n \geqslant 1-\frac{n}{2n+1} > \frac{1}{2}.$$

Thus $A_{2n} < 4$ for all n. Hence the sequence (A_n) is increasing and is bounded above by 4, so it has a limit.

The second proof uses the inequality between the geometric and arithmetic means of positive real numbers, namely that

the geometric mean < the arithmetic mean

when the numbers are not all the same. Let a and b be distinct positive real numbers, and consider the $n+1$ real numbers a and n copies of b. Then

$$(ab^n)^{1/(n+1)} < \frac{a+nb}{n+1}.$$

If we put $a = 1$ and $b = 1+(1/n)$, this gives

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1},$$

so that the sequence (A_n) is increasing. Now put $b = 1+(1/2n)$ and $a = 1/b^n$. This gives

$$1 < \frac{1+n\left(1+\dfrac{1}{2n}\right)^{n+1}}{(n+1)\left(1+\dfrac{1}{2n}\right)^n},$$

which simplifies to

$$\left(1+\frac{1}{2n}\right)^n < 2.$$

Thus $A_{2n} < 4$ for all n, so again we see that the sequence (A_n) is increasing and bounded above by 4. Hence the sequence has a limit.

Of course, neither method tells us what the limit is. But readers who do not already know might be able to guess if they try a few values of n on their calculators: $n = 10, 100, 1000, 10\,000$ should be more than enough.

Musings on an Interesting Sequence

CHRIS NASH, *King Edward's School, Birmingham*

Chris wrote this article whilst in the first year of the sixth form. He was a member of the British team for the 1988 International Mathematical Olympiad in Australia.

The following problem appeared in the 1986 International Mathematical Olympiad. The integers 2, 5 and 13 have the property that the product of every two of them is a perfect square plus 1. Prove that it is not possible to extend the sequence so that the property still holds. It is possible to solve this problem by means of an argument using even and odd numbers.

A superficially similar problem is considered in reference 2 problem 49, page 90. Consider a sequence of numbers in ascending order, $x_1, x_2, x_3, x_4, \ldots$, potentially infinite, such that all the x_i are positive integers and the product of every two of them is a perfect square minus 1. We shall study the case when $x_1 = 1$ and then find x_2, x_3, x_4 such that the condition is satisfied and the x's are the smallest possible. We shall obtain the values $1, 3, 8, 120$. We then ask: is this the end of the sequence, or does an x_5 exist so that the sequence still satisfies the condition? It would be futile to try to find x_5 by hand, as it must contain 'at least 100 digits' (to quote reference 2), so we consider turning to a computer. However, the task is still enormous; every possible x_5 from 121 to 10^{100} at least would have to be tested! Such a huge number will have to be decreased in some way by analysis of the problem.

Now x_5 must satisfy

$$x_5 = a^2 - 1 \tag{1}$$

$$3x_5 = b^2 - 1 \tag{2}$$

$$8x_5 = c^2 - 1 \tag{3}$$

$$120x_5 = d^2 - 1 \tag{4}$$

for some positive integers a, b, c and d. Equation (1) tells us that we only need to test numbers that are one less than a perfect square, but this scarcely helps. To test all numbers with fewer than 100 digits will still take about $\sqrt{10^{100}} = 10^{50}$ trials, which is still far too many. Instead, we consider equations (1) and (2) and eliminate x_5 to give

$$b^2 - 3a^2 = -2.$$

We must attempt to find solutions of it in positive integers. (An equation with integer coefficients which is to be solved in integers is called a *Diophantine equation*.)

The allied equation $x^2-3y^2 = 1$, with right-hand side 1, is an example of *Pell's equation* and there is a standard way of solving this in positive integers (see reference 1 pages 338, 339). The smallest solution is given by $x_1 = 2$, $y_1 = 1$. Then all solutions (x_n, y_n) are given by

$$x_n + y_n\sqrt{3} = (2+\sqrt{3})^n$$

for $n = 1, 2, 3, \ldots$. Since, then,

$$x_n - y_n\sqrt{3} = (2-\sqrt{3})^n,$$

we have that

$$x_n = \tfrac{1}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n], \qquad y_n = \tfrac{1}{6}\sqrt{3}[(2+\sqrt{3})^n - (2-\sqrt{3})^n].$$

It is an easy exercise to show that, if b and a are positive integers which satisfy the equation $x^2-3y^2 = -2$, then b and a are either both even or both odd and $\frac{1}{2}(b+3a)$ and $\frac{1}{2}(b+a)$ are positive integers which satisfy the equation $x^2-3y^2 = 1$. Hence, if b and a are positive integers satisfying $x^2-3y^2 = -2$, then

$$\tfrac{1}{2}(b+3a) = \tfrac{1}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n], \qquad \tfrac{1}{2}(b+a) = \tfrac{1}{6}\sqrt{3}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$$

for some positive integer n. We can solve these to give

$$b = \tfrac{1}{2}(\sqrt{3}-1)(2+\sqrt{3})^n - \tfrac{1}{2}(\sqrt{3}+1)(2-\sqrt{3})^n, \tag{5}$$

$$a = \tfrac{1}{6}(3-\sqrt{3})(2+\sqrt{3})^n + \tfrac{1}{6}(3+\sqrt{3})(2-\sqrt{3})^n. \tag{6}$$

It is easy to check that, for every positive integer n, these values do satisfy the equation $x^2-3y^2 = -2$, so this gives all positive integer solutions of this equation. We shall relabel the b and a in (5) and (6) as b_n and a_n.

Equation (6) is awkward to use to determine the a_n. However, it follows easily from

$$(2\pm\sqrt{3})^2 + 1 = 4(2\pm\sqrt{3})$$

that

$$a_{n+2} = 4a_{n+1} - a_n. \tag{7}$$

Since $a_1 = 1$ and $a_2 = 3$ from (6), we can now use (7) to give that

$$a_3 = 11, \qquad a_4 = 41, \qquad a_5 = 153, \qquad a_6 = 571,$$

and so on. Returning to equation (1), we see that these give the values 0, 8, 120, 1680, 23 408 and 326 040 for x_5. Thus 8 and 120 appear as solutions, which we have already seen, and these are the smallest possible values.

So what of x_5? We must now try the values $1680, 23\,408, \ldots$. They already satisfy equations (1) and (2), but equations (3) and (4) must also be satisfied. We now resort to the computer. Each a_n is approximately $2+\sqrt{3}$ times bigger than the previous one, so that the corresponding values of x_5 are each about $(2+\sqrt{3})^2 \approx 14$ times bigger than the previous one. Hence, to test all x_5 values less than 10^{100}, only about $\log_{14} 10^{100} \approx 90$ numbers need to be tested. This is quite an improvement on testing all 10^{100} numbers! Admittedly, the numbers do get large, but are now in fact easy to test.

The routine goes as follows:

1. Generate the next a-value.
2. Calculate $x = a^2-1$.
3. Test that both $8x+1$ and $120x+1$ are perfect squares. If not, return to 1.
4. You have found an x_5.

Once the computer software to handle large numbers was written, it was possible then to test every x_5 up to about 10^{1000} within a fairly short time. The main difficulty was in checking that numbers were perfect squares. One way would be to check that every prime factor of the number appears an even number of times. However, the factorization of large numbers becomes difficult, if not impossible (see references 3 and 4). The test eventually used was a method used in the extraction of the square root of a number by hand.

The program was written to handle numbers as large as $2^{4096} \approx 10^{1233}$. This range permitted the next 1070 possible values of x_5 to be tested overnight. (The program took two days to write.) For a standard home machine to handle and test so many large numbers, this was extremely fast, though perhaps a mainframe computer could have done the job in seconds! The computer generated each a-value in sequence, then calculated x and tested $8x+1$ and $120x+1$ for 'squareness'. *No solutions were found*. Hence, if x_5 does exist, it is at least 1234 digits long!

So where could we go from here? We could start with a different value of x_1 and look, for example, at the sequences $2, 4, 12, \ldots$ or $8, 120, 190, \ldots$. And, of course, even if we find x_5, there are always $x_6, x_7, \ldots$!

References

1. D. M. Burton, *Elementary Number Theory*. Revised Printing (Allyn and Bacon, 1980).
2. J. J. Clessa, *Micropuzzles* (Pan/Personal Computer News Computer Library, 1983).

3. Keith Devlin, Prime numbers and secret codes, *Mathematical Spectrum* **16** (1983/84), 65–67.
4. Ian Stewart, Factorizing large numbers, *Mathematical Spectrum* **20** (1987/88), 74–77.

Editorial Note. A. Baker and H. Davenport proved in the 1960s that, in fact, no such x_5 exists. The research paper containing their proof is in *The Collected Works of Harold Davenport* Volume IV (Academic Press, 1977) edited by B. J. Birch, H. Halberstam and C. A. Rogers, pages 1748–1756.

It makes you think

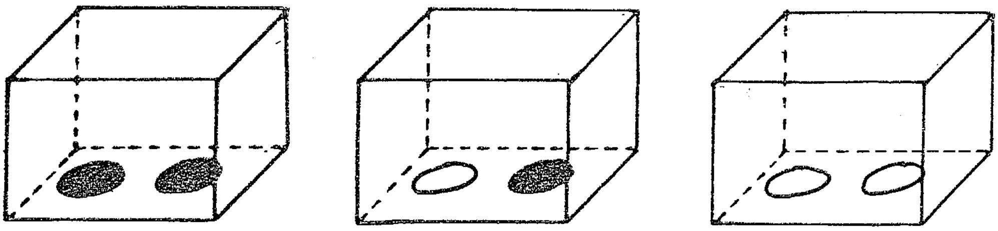
Here are two more puzzles that one of our readers, Arthur Pounder, set at his Maths Club at St. Peter's Grammar School, Prestwich, Manchester. Readers may well have come across them before. If not, why not try to solve them?

1. *The insurance agent.* A mother wishes to take out life insurance for her three daughters. When the insurance agent arrives, he wishes to discover their ages in years. The mother tells him that the product of their ages is 36, but he says that is not sufficient. 'The sums of their ages equals the number of the house opposite', she tells him. Again he tells her that is not sufficient. The mother then tells him: 'My eldest daughter plays the violin', upon which the agent departs satisfied. What are their ages?

2. *The logicians' Christmas party.* Five logicians hold a Christmas party. They start off by pulling their crackers and discover that they have three red party hats and two blue ones. They decide to play a game using the hats. Three of them, A, B and C, sit on chairs such that C can see B and A, B can see A and A can see neither B nor C. Each one is given a hat. They are then asked the following questions by the other two. 'C, do you know the colour of your hat?' C replies 'No'. Then B is asked: 'Do you know the colour of your hat?' B replies 'No'. Then A is asked: 'Do you know the colour of your hat?' A replies 'Yes'. What colour is A's hat?

Bertrand's Box Paradox

OLIVER D. ANDERSON, *University of Western Ontario*



The recent historical article by Angela Newing in *Mathematical Spectrum* Volume 21 Number 2 (1989) on H. E. Dudeney sparked memories of another pioneer, this time in probability teaching, J. L. F. Bertrand of the French Academy of Sciences. This year is the centenary of the publication of his *Calcul des Probabilités* and it is therefore perhaps appropriate to remind ourselves of one of the best-known chestnuts which originated there, the famous Bertrand's box paradox. (The stature of Bertrand can perhaps be assessed by remarking that the printing of the copy of his book, held here at the University of Western Ontario, is dated 1972. Not many authors run that long.)

Consider three boxes of identical external appearance. Each box contains two coins. In one box, both coins are gold; in another, they are both silver; in the remaining box, one is gold and one is silver. If a box is chosen at random, what is the probability that it contains unlike coins?

For a paradox, we need some conflict, and this is supplied by the following pair of contradictory solutions.

Solution 1. There are three possible cases, all equally likely, of which only one gives the required result of unlike coins. Hence the probability is $\frac{1}{3}$.

Solution 2. Remove a coin at random from one of the boxes, without letting anyone see it. If we now consider that box, it contains just one remaining coin which is either made of the same metal as the removed coin or it is not, giving the probability of unlike coins this time equal to $\frac{1}{2}$.

We do not intend to spell out how the paradox is resolved. We assume that readers of *Mathematical Spectrum* can do that for themselves. However, having done so, they might like to reflect on the fact that what is now simple for the thoughtful sixth former was a real challenge for graduate mathematicians just one hundred years ago.

Great Lengths and Hidden Powers

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The author took his B.Sc. at King's College, London, in 1968 and his Ph.D. at the University of Bristol in 1972. He has held positions at several British and overseas universities and is currently Reader in Mathematics at the University of Lancaster. He writes a regular mathematical column in the *Guardian*; this article is one which originally appeared there, and is reproduced here with permission.

Imagine you are lecturing on a difficult maths problem. You have tried to solve the problem yourself, so you know how hard it is. So you promise to pay $1,000 to anyone who can solve it.

Your money seems to be safe. But sitting in your audience is a tenacious individual who is prepared to go to great lengths to find a solution, including enlisting the help of a Cray supercomputer. Within two weeks, your listener returns...

Not only that but, as the man gently points out, it was not $1,000 that you offered but $10,000—a slip of the tongue that was recorded for posterity on videotape.

This recently happened to the British mathematician John Horton Conway.

Dr Conway, who moved from Cambridge to Princeton University in New Jersey in 1986, is well known to computer users as the inventor of the computer game 'Life'.

The problem he posed to the 500 or so professional mathematicians and scientists attending his lecture at a meeting held at AT&T's Bell Laboratories in Murray Hill, New Jersey, on 15 July 1988, was a typical Conway teaser, closely related to an old problem of Fibonacci.

In 1202, the Italian mathematician Fibonacci of Pisa introduced what is nowadays known as the Fibonacci sequence. This sequence starts off with two 1s, and thereafter grows according to the rule that each new number is the sum of its two predecessors. Thus, the first few numbers in this sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

Conway's problem concerned an amended version of the Fibonacci sequence, obtained as follows. Start off with two 1s, but now the rule for calculating each new number is: take the last number you have, call it n, and then add together the nth number in the sequence and the nth number from the end.

This gives you a sequence that starts off like this: 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7. To find the next number, count forward 7 places to the number 4, count back 7 places to the number 3, and add 4 to 3 to get 7. And so on.

The first thing you will discover if you work out a few more terms (it's easy to program on a micro) is that each new number in the sequence is never greater than the previous one by more than 1.

A bit harder to spot, but just as easy to check (by hand or computer) is that if you divide each number in the sequence by its position in the sequence the answer you get is always close to one half. This turns out to be no mere accident. Dr Conway managed to prove conclusively that the further along the sequence you go, this ratio gets steadily closer to one half.

The question Dr Conway asked his audience at Bell Labs was this: from which point in the sequence is this ratio always within 0.05 of one half?

Before you rush to your micro to find out, I should point out that that figure of 0.05 is a fairly tight bound, and at the time of his lecture Dr Conway was convinced there was no way of getting at the answer. Hence the \$10,000 wager.

But Conway reckoned without Dr Colin Mallows, another expatriate Brit, now working at Bell Labs. Bringing to bear all the modern techniques of statistical data analysis, pattern recognition, computer generated graphs, and a supercomputer, Dr Mallows had the solution within two weeks. The critical position in the sequence is number 3,173,375,556. So Conway had to take a deep breath and write out a cheque for \$10,000. But Mallows declared himself happy to accept the intended \$1,000 and keep the larger cheque as a souvenir.

Conway's \$10,000 Challenge

DAVID YATES

The author obtained his Ph.D. in psychology at the University of Lancaster. He is founder of the Science and Philosophy Research Group International, which has mainly studied problems in philosophical psychology. His own interests are in explaining consciousness and in solving puzzles.

In the preceding article, which I read in the *Guardian*, Keith Devlin wrote about John Conway's \$10,000 challenge concerning the sequence (s_n) of positive integers which can be defined by

$$s_1 = 1, \qquad s_2 = 1, \qquad s_n = s_{s_{n-1}} + s_{n-s_{n-1}} \quad \text{for } n > 2.$$

Conway's challenge was to find the smallest positive integer N for which

$$0.45 < \frac{s_n}{n} < 0.55 \tag{1}$$

for all $n \geqslant N$. Devlin gave the solution as $N = 3{,}173{,}375{,}556$, but surely the correct value is a mere 1490. Proving it also seems easier than Devlin implied.

I calculated the first 3251 terms on my BBC Micro, for reasons which will become clear, and obtained that

$$s_{1489} = s_{1490} = 819, \qquad s_{3251} = 1762.$$

Thus

$$\frac{s_{1489}}{1489} > 0.55 \quad \text{but} \quad 0.45 < \frac{s_{1490}}{1490} < 0.55.$$

I verified that (1) holds for all values of n from 1490 to 3251. Now consider an integer $k \geqslant 3251$ and assume inductively that (1) holds for all values of n from 1490 to k. It is not difficult to prove that

$$s_{n+1} = s_n \text{ or } s_n + 1$$

for all values of n; readers are invited to do just this in the Problems section (see page 28). Hence

$$s_{3251} \leqslant s_k \leqslant s_{3251} + (k - 3251)$$

i.e.,

$$1762 \leqslant s_k \leqslant k - 1489.$$

Thus

$$1762 \leqslant s_k < k \quad \text{and} \quad 1490 \leqslant k + 1 - s_k < k.$$

(Note that 3251 was chosen to give 1490 here.) Hence we can assume that (1) holds with $n = s_k$ and $n = k + 1 - s_k$, i.e.

$$0.45 < \frac{s_{s_k}}{s_k} < 0.55 \tag{2}$$

and

$$0.45 < \frac{s_{k+1-s_k}}{k+1-s_k} < 0.55. \tag{3}$$

We can deduce from (2) and (3) that

$$0.45 < \frac{s_{s_k} + s_{k+1-s_k}}{s_k + (k+1-s_k)} < 0.55,$$

i.e.

$$0.45 < \frac{s_{k+1}}{k+1} < 0.55,$$

which completes the inductive step. Hence (1) holds for all $n \geqslant 1490$ but not for $n = 1489$, so that $N = 1490$.

Computer Column

MIKE PIFF

Random number generators

If you are suspicious of your built-in BBC random-number generator, and so you should be, the program below will give a significant improvement in its performance.

The idea is to create a short table of random numbers, `rnd%` and then to randomly index into this table, select a value, and then replace this used value.

There is nothing peculiar about the number 113, it can be almost anything. The program has been set up to simulate `RND`, which returns a random integer, but it would be easy to simulate `RND(n)` or `RND(1)`, say, in a similar fashion. Indeed, for `RND(1)`, all we need to do is change the returned value to

```
(2^31+temp%)/2^32
```

say.

```
   1 crand%=113
   2 DIM rnd%(crand%)
  10 PROCinitrnd
  20 FOR i%=1 TO 20
  30 PRINT FNrnd
  40 NEXT
  50 END
4000 DEF PROCinitrnd
4010 LOCAL dummy%,i%
4020 dummy%=RND(-TIME)
4030 FOR i%=1 TO crand%
4040 dummy%=RND
4050 NEXT
4060 FOR i%=1 TO crand%
4070 rnd%(i%)=RND
4080 NEXT
4090 ENDPROC
5000 DEF FNrnd
5010 LOCAL temp%,i%
5020 i%=RND(crand%)
5030 temp%=rnd%(i%)
5040 rnd%(i%)=RND
5050 =temp%
```

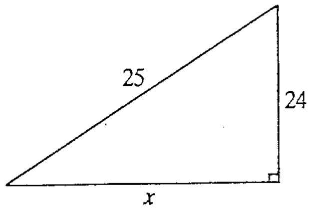
Letter to the Editor

Dear Editor,

Right-angled triangles

Whilst marking a student's GCSE mock examination paper, I came across the following oddity, which was new to me.

Question. Find the value of x?



Answer. $x = \sqrt{25+24} = \sqrt{49} = 7.$

I was amused by this, and tried to find other similar cases. To my surprise, this is generally true. Thus, if $24, 25$ are replaced by $4, 5$ or by $12, 13$ then

$$x = \sqrt{5+4} = \sqrt{9} = 3, \qquad x = \sqrt{13+12} = \sqrt{25} = 5,$$

respectively. A little algebra, which is left to the reader, will show why and when this works.

Yours sincerely,
S. Maloney
Cardinal Manning Boys' School.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

22.1 (Submitted by David Yates)
Show that

$$s_1 = 1, \qquad s_2 = 1, \qquad s_n = s_{s_{n-1}} + s_{n-s_{n-1}}, \quad \text{for } n > 2,$$

defines a sequence and that, for $n > 1$, $s_n = s_{n-1}$ or $s_{n-1}+1$.

22.2 (Submitted by A. D. Sands, University of Dundee)
A triangle with angles A, B and C and opposite sides of lengths a, b and c is *rational* if all the ratios a/b, b/c and c/a are rational. If $\cos A$, $\cos B$ and $\cos C$ are all rational, show that the triangle is rational.

22.3 (Submitted by Gregory Economides, Sixth Form of the Royal Grammar School, Newcastle upon Tyne)
The power P transferred from a cell of e.m.f. E and internal resistance r to a purely resistive load of resistance R is given by

$$P = \frac{E^2 R}{(R+r)^2}.$$

Find, *without calculus*, the value of R for which P attains its maximum value.

22.4 (Submitted by L. A. Fearnehough, Portsmouth Sixth Form College)
Evaluate the integral

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx.$$

Solutions to Problems in Volume 21 Number 2

21.5 Prove the identity

$$n! = A^n - \binom{n}{1}(A-1)^n + \binom{n}{2}(A-2)^n - \ldots + (-1)^n(A-n)^n,$$

where A is any number and n is a positive integer.

Solution by Gregory Economides (Royal Grammar School, Newcastle upon Tyne)

Let

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(A-k)^n. \qquad \text{(i)}$$

The coefficient of A^n on the right-hand side of (i) is given by

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0$$

and the coefficient of A^{n-r}, C_r $(r = 1, 2, \ldots, n)$ say, is given by

$$C_r = (-1)^r \binom{n}{r} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^r.$$

Let

$$f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

and define a sequence $f_n(x)$ by

$$f_1(x) = \frac{d}{dx}f(x) = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1},$$

$$f_2(x) = \frac{d}{dx}\{xf_1(x)\} = \sum_{k=1}^{n} k^2\binom{n}{k}x^{k-1},$$

$$\vdots$$

$$f_n(x) = \frac{d}{dx}\{xf_{n-1}(x)\} = \sum_{k=1}^{n} k^n\binom{n}{k}x^{k-1}.$$

Thus

$$C_r = (-1)^{r+1}\binom{n}{r}f_r(-1).$$

Also,

$$\begin{aligned} f_1(x) &= n(1+x)^{n-1}, \qquad f_2(x) = n(n-1)x(1+x)^{n-2}+n(1+x)^{n-1}, \\ f_3(x) &= n(n-1)(n-2)x^2(1+x)^{n-3}+n\{(3n-2)x+1\}(1+x)^{n-2}. \end{aligned} \tag{ii}$$

It can be shown by induction that, for $2 \leqslant r \leqslant n$,

$$f_r(x) = n(n-1)\dots(n-r+1)x^{r-1}(1+x)^{n-r}+(1+x)^{n-r-1}g_{r-2}(x), \tag{iii}$$

where $g_{r-2}(x)$ is a polynomial in x of degree at most $r-2$. Hence from (ii) and (iii) we have that

$$f_r(-1) = 0 \quad (r = 1,2,\dots,n-1), \qquad f_n(-1) = (-1)^{n-1}n!.$$

Thus

$$C_r = 0 \quad (r = 1,2,\dots,n-1) \quad C_n = n!.$$

Hence $f(n) = n!$.

21.6 Let $0 \leqslant x \leqslant \frac{1}{2}\pi$. Prove that

$$(\pi^2+24)x-4x^3 \leqslant 12\pi\sin x.$$

Solution by Gregory Economides

Let

$$f(x) = 12\pi\sin x+4x^3-(\pi^2+24)x.$$

Then

$$f'(x) = 12\pi\cos x+12x^2-(\pi^2+24),$$

$$f''(x) = -12\pi\sin x+24x = -12\pi\left(\sin x-\frac{2x}{\pi}\right).$$

Now $\sin x > 2x/\pi$ for $0 < x < \frac{1}{2}\pi$. (This may easily be seen from the graph of $\sin x$ between 0 and $\frac{1}{2}\pi$.) Hence $f''(x) < 0$ for $0 < x < \frac{1}{2}\pi$, and $f(x)$ is concave downwards for $0 < x < \frac{1}{2}\pi$. Also, $f(0) = 0 = f(\frac{1}{2}\pi)$. Hence $f(x) \geqslant 0$ for $0 < x < \frac{1}{2}\pi$, with equality if and only if $x = 0$ or $\frac{1}{2}\pi$.

21.7 The letters AACEINNOSTW can be rearranged and the resulting sequences of letters are ordered in a dictionary ordering (so that the second sequence is AACEINNOSWT). How far along the sequence do you have to go to reach a famous mathematician?

Solution by Colin Lindsay (The Open University)

There are

$\frac{1}{4}(4\times 10!)$	'words' with A, C or E before I, plus
$\frac{1}{4}(7\times 9!)$	'words' with IA, IC, IE, IN or IO before IS, plus
$\frac{1}{2}(1\times 5!)$	'words' with ISAACE before ISAACN, plus
$3\times 3!$	'words' with ISAACNEO etc. before ISAACNEW, plus
$2\times 2!$	'words' with ISAACNEWO etc. before ISAACNEWT, plus
$1\times 1!$	'words' with ISAACNEWTN before ISAACNEWTON.

These add up to 4263923, so that ISAACNEWTON is the 4263924th ordered sequence.

Also solved by Amites Sarkar (Winchester College), Gregory Economides and Dylan Gow (Oakham School).

A reader, who may prefer to remain anonymous, answered the question: 'As far as you are ABEL!'

21.8 Show that the function

$$f(x) = \frac{\int_0^x \sin^q t \, dt}{\int_0^x \sin^p t \, dt},$$

where $p > q$ and $0 < x \leqslant \frac{1}{2}\pi$, is strictly decreasing.

Solution by Gregory Economides

$$f'(x) = \frac{1}{\left(\int_0^x \sin^p t \, dt\right)^2}\left(\sin^q x \int_0^x \sin^p t \, dt - \sin^p x \int_0^x \sin^q t \, dt\right)$$

$$= \frac{\sin^q x}{\left(\int_0^x \sin^p t \, dt\right)^2} \int_0^x \sin^q t\,(\sin^{p-q} t - \sin^{p-q} x) \, dt.$$

Let $0 < x \leqslant \frac{1}{2}\pi$. For $0 < t < x$, the integrand in the integral in the numerator is negative. Since the other terms are positive, it follows that $f'(x) < 0$ for $0 < x \leqslant \frac{1}{2}\pi$, so that $f(x)$ is strictly decreasing for $0 < x \leqslant \frac{1}{2}\pi$.

Also solved by Dylan Gow and Amites Sarkar.

Readers might like to consider whether $\sin t$ can be replaced by other functions in the integrals.

The 1989 problem

This was to express the numbers 1 to 150 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\surd$, ! and concatenation (i.e. forming 19 from 1 and 9, for example). Our original correspondent, Mike Wenble, failed with five numbers: 132, 133, 136, 142, 148. David Mottram of Sandbach School, found

$$136 = (-1+9)\times(8+9).$$

He also proposed the following 'illegal' expressions:

$$132 = -1\times\sum[(\surd 9)!] + \sum(8+9),$$

$$133 = 1-\sum[(\surd 9)!] + \sum(8+9),$$

$$142 = \sum(19) - [8\times(\surd 9)!],$$

$$148 = 1-(\surd 9)! + \sum(8+9),$$

where $\sum n = 1+2+\ldots+n$.

Reviews

Invitation to Mathematics. By JOHN BOWERS. Basil Blackwell, Oxford, 1988. Pp. 202. Hardback £14·95 (ISBN 0-631-14641-5), paperback £4·95 (ISBN 0-631-14642-3).

This is an invitation to mathematics, not an introduction. Mathematics, for the author, is 'the Subject with a Difference' and this is a mathematics book with a difference. It is in no sense a textbook and is meant to stimulate rather than to expound, to entertain as well as to inform. Ideal for sixth-form students contemplating a mathematics or mathematics-related course in higher education, it will not make new converts to the subject but it will allow those with an interest to gain some ideas of its vast scope and extraordinary fascination.

The author stresses that the reader need only know the solutions to simultaneous and quadratic equations and Pythagoras' theorem, but this does not mean that it is always mathematically easy; some parts are certainly difficult and the sixth-form reader will learn much new mathematics. All the main areas are covered—axiomatic foundations and the need for proof, mechanics, the theory of numbers, geometry, algebra, calculus and analysis—but statistics is largely excluded because there is a separate *Invitation to Statistics* in the series.

A valuable introductory chapter describes how whole new areas of mathematics have developed at different periods in history and a biographical chapter at the end shows the wide diversity of people who have contributed to mathematics; emphasizing how many of these were not mathematicians in the ordinary sense at all, the author rightly points out that the community of mathematicians refers not only to the greatest practitioners, but also to the many teachers, researchers and users of mathematics in its various applications.

Though this is certainly a serious treatment of a very important subject, the style is engaging and unpretentious and there are also entertaining anecdotes and interesting problems throughout the text. The author makes clear that there is no immediate danger of computer science replacing mathematics as a subject and that no computer presently contemplated is capable of solving scientific or industrial problems, so mathematicians will be in demand for many years to come. The *Invitation to Mathematics* is a book which is certain to interest readers of *Mathematical Spectrum*, and to stimulate those who see the study of mathematics as playing some part in their future career.

De la Salle Sixth Form College, Pendleton, Salford PETER ROWLANDS

Forever Undecided: A Puzzle Guide to Gödel. By RAYMOND SMULLYAN. Oxford University Press, 1988. Pp. xii+257. £5·95 (ISBN 0-19-282196-2).

Forever Undecided presents itself as an introduction to logical reasoning via a series of puzzles, starting from very basic ideas, then elaborating towards the celebrated theorems of Gödel. The most elementary concepts of logic are soon built up into complex ideas of undecidability, consistency and incompleteness which are behind Gödel's theorems.

The book begins with a wide selection of 'liar/truth-teller' problems to introduce the topics; the problems are claimed to be original but soon seem derivative because all are similar in vein. Nevertheless they do serve well to explain the ideas as they are introduced. Soon however the problems become thinner on the ground and as a result the text becomes dry; many ideas are developed too quickly towards the middle of the book. Indeed some of the text is very difficult to understand even after several readings. However, the reward for pressing on through the difficulties is to be found towards the book's end. The new techniques and technology are applied to yield very powerful and surprising results indeed; many of Gödel's theorems are included.

In conclusion, *Forever Undecided* should serve as a good introduction to logical reasoning, accessible to students of all levels but with things to offer for more advanced readers too. As a puzzle collection, the book may provide some intriguing diversions; as a 'textbook' it should stimulate the interested student to further reading in the subject.

King Edward's School, Birmingham CHRIS NASH

Mathematics and the Unexpected. By IVAR EKELAND. University of Chicago Press, 1988. Pp. 176. £15·95 (ISBN 0-226-19989-4).

This is an utterly absorbing book written by a distinguished mathematician with an outstanding gift for popularising his subject matter. His theme is the mathematics of time and he has written a coherent, concise and very readable book. Beginning with the celestial mechanics of Kepler and Newton he takes his reader through the heyday of classical determinism to Poincaré, perturbation theory, the study of instability and chaos, and on to a brief but lucid and balanced account of catastrophe theory. While the subject matter is far from trivial he has succeeded in making the mathematical demands on the reader very modest. Certainly any intelligent sixth former studying mathematics would be able to follow more than enough to enjoy and appreciate the book.

Like all good communicators, the author makes considerable use of images (such as Arnold's cat, Smale's horseshoe and Thom's cusp) so as to fix the key ideas in the reader's mind. He includes some interesting historical details and helpful diagrams. He recaps and restates his arguments sufficiently to ensure pellucidity but without becoming tedious. And without artificially imposing his theme on the material he succeeds in giving the book a pleasing coherence.

The relationships between mathematics, science and philosophy can appear daunting if not arid. In this book, however, some of these relationships are brought to life in surprising ways. The concluding chapter consists of philosophical reflections arising out of the book. Whether or not the reader agrees with it all (and this one does not) it is original and most stimulating. It reads like a commentary on the words of the writer of Ecclesiastes: 'God has also set eternity in the hearts of men; yet they cannot fathom what God has done from beginning to end.' (Eccl.3.11).

I thoroughly enjoyed and warmly recommend this book, which was originally published in French as *Le calcul, l'imprévu*.

Fettes College, Edinburgh CHRISTOPHER ASH

Learning and Doing Mathematics. By JOHN HEALEY MASON. Macmillan, Basingstoke, 1988. P. vii + 82. £5·95 (ISBN 0-333-44942-8).

This book aims to help students to solve problems in mathematics by illustrating methods which can be used whenever the student finds himself 'stuck'. There are many problems and exercises which can be solved using the techniques, and full solutions are supplied.

The book introduces ideas systematically without the need for any further mathematics to be learned. The concepts of specialization and generalization are introduced first and shown in operation, applied to problems in the text. Dr Mason then builds on these techniques to explain the importance of sound mathematical argument. The result is well-explained mathematical thinking. Full reasoning is shown in the solutions to all the problems; this is quite unusual and indeed a pleasant change from the usual mathematical texts, where correct results seem to come out of thin air. Instead here we have a clear picture of the reasoning used in each problem.

Based on a unit of the Open University Foundation Course in Mathematics, the book addresses itself to sixth-form and higher education students. However, maths students at this level may well be disappointed; being a Foundation Course, there are no attempts at A-level-style problems in the book. Indeed, the amount of mathematics required should be well within the reach of, say, GCSE pupils, where the many problems could provide a source of class investigations.

In conclusion, *Learning and Doing Mathematics* tries to help students by presenting ideas to be tried whenever they are confronted with a problem. The concepts are presented very clearly and illustrated well, but may be found more useful by pupils younger than intended.

Sixth Form, King Edward's School, Birmingham CHRIS NASH

Fifty Challenging Problems in Probability with Solutions. By FREDERICK MOSTELLER. Dover Publications Inc., New York, 1965. Reprint 1987. Pp. viii + 88. Paperback £3·15.

This book contains an interesting set of problems, which are drawn from a range of mathematical levels. Many are well-known questions. They are all chosen for their challenge and interest, but are refreshingly varied.

The problems posed and discussed range from the elementary 'Trials until First Success' to 'Buffon's needle'. Topics such as the Poisson distribution and the principle of symmetry when points are dropped on a line are dealt with in the text of the solution. The final problem, Molina's Urns, displays Fermat's famous conjecture in number theory as a probability problem.

The first section of the book contains the problems only; they are then restated and discussed individually in the solution section. Each question is clearly and fully explained and answered. The work promotes learning through problem solving by introducing principles as they are required.

The book is thoughtfully laid out and built to last. It is a work which could be used at intervals for entertainment or worked through methodically in order to gain an understanding of the principles which are explained. It is an interesting read and a source of intellectual enjoyment for those who do not necessarily already have a knowledge of probability.

Sixth Form, Gresham's School, Holt
(now at University College, Oxford) N. J. SHEA

Modelling with Projectiles. By DEREK HART and TONY CROFT, Ellis Horwood Ltd, Chichester, 1988. Pp. 152. £22·50 (ISBN 0-7458-0323-7).

The authors' aim is to provide a readable, vivid and stimulating account of projectile motion, and in this they have admirably succeeded. With a minimum of mathematical preliminaries the reader is taken on a guided tour of the subject, beginning with early Greek catapulting devices, concluding with the high-altitude ballistics of the present century, and visiting many fascinating highlights en route. The familiar ports of call such as military and sporting applications are there, as well as some new ones too, for example the harvesting of grain and safe-driving speeds on newly surfaced roads. My own particular favourite in the latter category concerns the Australian composer Percy Grainger, perhaps best known for his *In an English Country Garden*, who was reputed to have been able to throw a cricket ball over his house, and catch it on the other side by running through the house! The book is delightfully presented, with an attractive cover, an abundance of illustrations, relevant historical detail, helpful appendices and an annotated bibliography. Carefully selected exercises develop the student's feel for the subject, sometimes providing source material for projects and class discussion. This book, combining as it does the theory of projectiles with interesting practical applications, is unique of its kind and can be highly recommended to students and teachers alike, whether they be in sixth forms, colleges, polytechnics or universities. The authors have scored a direct hit!

University of Sheffield R. J. WEBSTER

Elementary Decision Theory. By H. CHERNOFF and L. E. MOSES. Dover Publications Inc., New York. Pp. xv+364. £7·15.

This book is an unabridged, corrected, republication of the edition which was first published by Wiley in 1959. It offers a first course in statistics from a decision-making point of view, introducing new topics by examples and relegating more mathematical concepts to appendices.

The fact that this book is being republished suggests that it has been very successful. However, we thought that the format appears old fashioned and that it would probably be unattractive to the readers for whom it was intended. The theory covered may not have changed in the last 30 years, but presentation techniques have improved considerably.

North Staffordshire Polytechnic D. J. COLWELL AND J. R. GILLETT

Concise Statistics. By M. G. GODFREY, E. M. ROEBUCK and A. J. SHERLOCK. Edward Arnold, London, 1988. Pp. 402. Paperback £8·95 (ISBN 0-7131-3591-3).

These three authors have produced a welcome addition to the small list of good A-level statistics books available. This is an attractively produced book with many worked solutions followed by exercises of original questions. A unique feature of this book is that it contains the outlines of worked solutions to all problems which makes it ideal for self-study. The book adequately covers most A-level statistics syllabuses. Tests of significance, the most demanding section of an A-level syllabus, are well covered and include t distributions and non-parametric tests.

I would thoroughly recommend this book to any reader looking for a course text suitable for class use.

Portsmouth Sixth Form College ALAN FEARNEHOUGH

Mathematical Modelling Courses. Editors J. S. BERRY, D. N. BURGHES, I. D. HUNTLEY, D. J. G. JAMES and A. O. MOSCARDINI. Ellis Horwood Ltd, Chichester, 1987. Pp. 281. £38·50 (ISBN 0-85312-931-2).

In this book you will find a welcome addition to a series of books on mathematical modelling published by Ellis Horwood. It is not a handbook of models but it contains a wide range of methods and experiences of teaching modelling. It includes work at all levels from secondary school to Master's degree for the specialist and non-specialist. The wide interest in this approach is seen in contributions from six different countries. It is written by enthusiasts of the modelling approach to teaching mathematics. They convey their enthusiasm to the reader but they take a realistic approach and do not underestimate the difficulties. There is no 'right' way to teach mathematical modelling; this book gives a whole series of approaches. The reader must select the method that suits the purpose of the class. There are many excellent hints in the book.

University of Sheffield D. M. BURLEY

In recent months we have received many enquiries about **Mathematical Spectrum** from students in Iran. One of these has subsequently written to ask for our help in finding a penfriend in Britain. Anyone interested is invited to contact:

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The University,
Sheffield S3 7RH, England

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