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Special
MORLEY
Issue

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A TOPICAL INTEGRAL

LEON BANKOFF, Los Angeles, California

The readers of EUREKA may find it interesting at this time to evaluate with me the following integral:

$$I = \left(\frac{2fe}{a} \int_0^r x dx \right) y + x \int_0^{fs} dx.$$

I found it in an old notebook of mine and no longer recall its source.

We have

$$\begin{aligned} I &= \frac{2fe}{a} \left[\frac{x^2}{2} \right]_0^r y + x[x]_0^{fs} = \frac{2fe}{a} \left(\frac{r^2}{2} - 0 \right) y + x(fs - 0) \\ &= \frac{fer^2y}{a} + xfs. \end{aligned}$$

Now the well-known relation $f = ma$ yields

$$I = mer^2y + x(ma)s = merry + xmas.$$

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PRESENTING THE MORLEY ISSUE OF EUREKA

This issue of EUREKA is devoted to the celebrated Morley Theorem. The following brief quotation is from Coxeter and Greitzer [1], a book which has on its cover a three-colour representation of the Morley configuration:

One of the most surprising theorems in elementary geometry was discovered about 1904 by Frank Morley (the father of Christopher Morley, whose novel, *Thunder on the Left*, has a kink in its time sequence that appeals particularly to geometers). He mentioned it to friends in Cambridge, England, and published it twenty years later in Japan. [2]

Because of its historical interest, Morley's own proof of his theorem is given in the first of the articles that follow, exactly as it appeared in [2]. (The printing is too poor to reproduce it photographically.) It is an understatement to say that Morley's proof is hard to understand. In the article which immediately follows, Dan Pedoe attempts to shed some light on Morley's convoluted thinking. The remaining articles on Morley's Theorem culminate in what is almost certainly the most complete list of Morley references in existence.

REFERENCES

1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, Random House and The L.W. Singer Co., 1967, p. 47.
2. Frank Morley, On the intersections of the trisectors of the angles of a triangle, *Journal of the Mathematical Association of Japan for Secondary Education*, 6 (December 1924), 260 - 262.

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ON THE INTERSECTIONS OF THE TRISECTORS
OF THE ANGLES OF A TRIANGLE.

By

Professor FRANK MORLEY.

(From a letter directed to Prof. T. Hayashi.)

Dear Professor Hayashi:—

I have not published the theorem [The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle]¹. It arose from the consideration of cardioids. I noticed, in the Transactions of the *American Mathematical Society*, vol. 1, p. 115, that certain chains of theorems were true for any number of lines in a plane, when one replaces the intersection of the lines taken two at a time (1) by the centre of a circle touching the lines taken 3 at a time and (2) by the centre of a cardioid touching the lines taken 4 at a time, and so on.

So I was led to think on the cardioids touching 3 lines.

The cardioid is mapped on the unit circle by an equation.

$$x = 2t - t^2,$$

x a complex number, t a complex number such that $|t| = 1$. The tangent at t is

$$x - 3t + 3t^2 - \bar{x}t^3 = 0,$$

where \bar{x} is the conjugate of x . The 3 tangents from a point x are then such that

$$t_1 t_2 t_3 = x/\bar{x}.$$

Whence if θ_i are the angles which these tangents make with any fixed line, and ϕ the angle of x itself,

$$3\phi = \theta_1 + \theta_2 + \theta_3 \dots\dots\dots(1)$$

¹This enunciation of the theorem has been added here by Prof. T. Hayashi.

The image y of any points x in the tangent is given by

$$y - 3t + 3t^2 - \bar{x}t^3 = 0.$$

Thus the image of the centre $x = 0$ is

$$y = 3(t - t^2).$$

Hence, if

$$y = 2pe^{i\omega}, \quad \text{so that} \quad \bar{y} = 2pe^{-i\omega},$$

we have

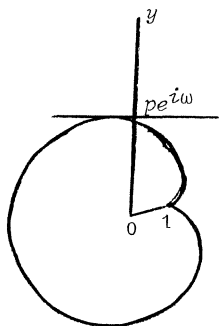
$$4p^2 = 9(1-t)(1-1/t),$$

$$e^{2i\omega} = -t^3,$$

$$t + 1/t = -2 \cos 2\omega/3,$$

$$\text{and} \quad p = 3 \sin \omega/3 \dots (2)$$

This is the line-equation of the cardioid. The equation $p = a \sin \mu\omega$ for any cycloidal curve is given in some of the older books (for instance, in Edwards, Differential Calculus), so that we might begin



with equation (2).

If then p_1, p_2, p_3 are perpendiculars from the centre on 3 tangents, and $\omega_1, \omega_2, \omega_3$ the angles of these perpendiculars, since

$$\sum_{i=1}^3 \sin \frac{\omega_i}{3} \sin \frac{\omega_2 - \omega_3}{3} = 0,$$

we have

$$\sum_{i=1}^3 p_i \sin \frac{\omega_2 - \omega_3}{3} = 0.$$

Replacing $\omega_2 - \omega_3$ by the angle A_1 of the triangle of tangents, but bearing in mind that in (3) the angles must have a sum congruent to 0, we get for the locus of centres 9 lines, such as

$$p_1 \sin \frac{\pi - A_1}{3} + p_2 \sin \frac{\pi - A_2}{3} + p_3 \sin \frac{-\pi - A_3}{3} = 0,$$

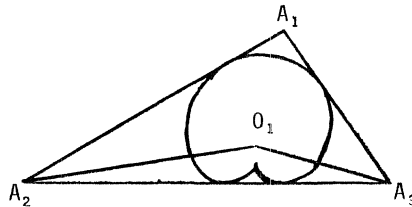
$$p_1 \sin \frac{2\pi - A_1}{3} + p_2 \sin \frac{\pi - A_2}{3} + p_3 \sin \frac{-2\pi - A_3}{3} = 0.$$

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But from (1) considering those cardioids whose centres are at a great distance (so that the triangle behaves like a point), we see that the 9 lines have only 3 directions, given by

$$3\phi = \theta_1 + \theta_2 + \theta_3.$$

They are thus 3 sets of 3 parallel lines, forming equilateral triangles. The centre changes from one line to another when one of the lines is a double tangent.



Consider in particular the cardioids which lie inside the triangle. Let O_1 be the centre of a cardioid with double tangent A_2A_3 . We have from (1)

$$\text{angle } A_3A_2O_1 = A_2/3,$$

$$\text{angle } O_1A_3A_2 = A_3/3,$$

and we have seen that the 3 lines O_1O_2 , O_2O_3 , O_3O_1 form an equilateral triangle.

That was the argument. Verification is naturally a much simpler matter. If you think above worth printing I shall be very pleased to have it appear in a Japanese journal.

Further should the matter of the memoir referred to be of interest I shall be glad to send a copy, with a correction, for the use of "direction lines" there is not clear.

With high regards,

sincerely yours.

(Sign)

NOTES ON MORLEY'S PROOF OF HIS THEOREM ON ANGLE TRISECTORS

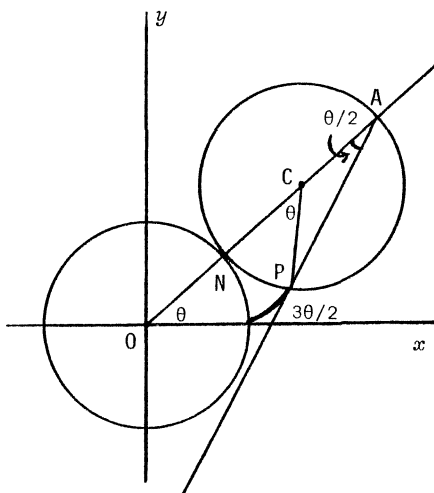
DAN PEDOE, University of Minnesota

Frank Morley was a remarkable geometer, but there cannot be many who have found it easy to follow his thoughts. His *Inversive Geometry* (Frank Morley and F.V. Morley, Chelsea, 1954) is a book with the most remarkable insights, but I, at any rate, find it almost incomprehensible. The following notes on Morley's proof of his celebrated theorem are very tentative first steps towards the elucidation of his work.

In this paper, written for a Japanese journal devoted to secondary education, Morley changes his notation a number of times. He uses θ for the angle made by a tangent to the cardioid with the x -axis (p. 260), then changes to ω on p. 261, and he then uses ω for the angle made with the x -axis by the perpendicular from the centre of the cardioid onto a tangent. I shall use the notation I am accustomed to, and shall then identify my results with Morley's.

We consider the first statement, p. 260: "The cardioid is mapped on the unit circle by an equation $x = 2t - t^2$, x a complex number, t a complex number such that $|t| = 1$."

We suppose that a cardioid is traced out by a point P fixed on the circumference of a unit circle (see figure) which rolls without slipping on a fixed unit circle. The centre O of the fixed circle, taken as origin of coordinates, is called the centre of the cardioid. If N is the point of contact, and ON meets the rolling circle again in A, then N is the instantaneous centre of motion for the rolling circle, so that the tangent at P to the cardioid is perpendicular to NP, and is therefore PA.



If θ is the angle $\text{NO}x$, the coordinates (x_P, y_P) of P are:

$$x_P = 2 \cos \theta - \cos 2\theta,$$

$$y_P = 2 \sin \theta - \sin 2\theta,$$

so that

$$\begin{aligned} x &= x_p + iy_p = 2(\cos \theta + i \sin \theta) - (\cos \theta + i \sin \theta)^2 \\ &= 2t - t^2, \end{aligned}$$

where $t = \cos \theta + i \sin \theta$, and $|t| = 1$.

The tangent at P to the cardioid is the line PA, and the affix of A is $3t$. Hence the equation of the tangent is:

$$\begin{vmatrix} X & \bar{X} & 1 \\ 3t & 3\bar{t} & 1 \\ 2t - t^2 & 2\bar{t} - \bar{t}^2 & 1 \end{vmatrix} = 0,$$

where X is any point on the tangent at $x = 2t - t^2$. (Morley uses the same x for both points.) After expansion, and use of the equation $t\bar{t} = 1$, which leads to

$$t(\bar{t} + 1) = 1 + t, \quad \bar{t}(t + 1) = 1 + \bar{t},$$

we obtain the equation

$$X - 3t + 3t^2 - \bar{X}t^3 = 0.$$

The three tangents to the cardioid from a point X correspond to points t_1 , t_2 , t_3 on the unit circle which are such that

$$t_1 t_2 t_3 = \frac{X}{\bar{X}}.$$

Hence, if $X = r(\cos \phi + i \sin \phi)$,

$$t_1 t_2 t_3 = \cos 2\phi + i \sin 2\phi,$$

and therefore

$$\theta_1 + \theta_2 + \theta_3 = 2\phi.$$

If ψ is the angle made by the tangent at P with Ox , the figure shows that this angle is $3\theta/2$, the tangent being the line PA, and hence the angles ψ_i made by the tangents which pass through the point X satisfy the relation:

$$\begin{aligned} \psi_1 + \psi_2 + \psi_3 &= \frac{3}{2}(\theta_1 + \theta_2 + \theta_3) \\ &= 3\phi. \end{aligned}$$

This is Morley's fundamental equation (1), p. 260.

If we now turn to p. 262 of Morley's paper, and consider the locus of the centre of cardioids which touch the sides of a given triangle ABC, we now prove that *the points at infinity on this locus coincide with the points at infinity on the sides of a determinate equilateral triangle.*

Let O be the centre of a cardioid touching the sides of triangle ABC which is at a great distance from the triangle, and let X be a point near the triangle. The tangents from X to the mammoth cardioid must be very nearly parallel to the sides of triangle ABC ("so that the triangle behaves like a point"). Hence, in equation (1), the angle ϕ which determines the direction of OX is one-third of the sum of the ψ_i , where the ψ_i correspond to the sides of triangle ABC .

Since any ψ_i is taken modulo 2π , and we are dividing by 3, we obtain 3 directions for ϕ which differ by $2\pi/3$.

If we know that the centre locus consists entirely of straight lines, these must form sets which are parallel to the sides of a determinate equilateral triangle.

To prove that the centre locus consists of lines, Morley obtains the pedal (p, ψ) equation of a cardioid. From the figure, if OQ is the perpendicular from O onto AP ,

$$p = OQ = OA \sin \frac{\theta}{2} = 3 \sin \frac{\psi}{3}. \quad (2)$$

Morley remarks that this can be obtained directly.

Now there is some very clever manipulation! Morley wishes to move from equation (2) to a trilinear equation for the locus of the centre. It is easily verified that for any P, Q and R ,

$$\sin P \sin (Q - R) + \sin Q \sin (R - P) + \sin R \sin (P - Q) = 0,$$

so that

$$\sin \frac{\psi_1}{3} \sin \frac{\psi_2 - \psi_3}{3} + \dots + \dots = 0.$$

Using (2), we can write this

$$p_1 \sin \frac{\psi_2 - \psi_3}{3} + p_2 \sin \frac{\psi_3 - \psi_1}{3} + p_3 \sin \frac{\psi_1 - \psi_2}{3} = 0,$$

where p_1, p_2, p_3 are the perpendiculars from the centre of the cardioid onto three tangents which make angles ψ_1, ψ_2 and ψ_3 with Ox .

This must be Morley's equation (3), the (3) having been omitted in his paper. These perpendiculars p_i are the *trilinear coordinates* (see Appendix) of O with respect to the triangle ABC , and the $(\psi_i - \psi_j)/3$ are related to the angles of the triangle. The fundamental relation connecting trilinear coordinates is

$$ap_1 + bp_2 + cp_3 = 2\Delta,$$

where a, b and c are the sides of the triangle, and Δ its area.

The equation (3) in trilinear coordinates is that of a line, and hence

Morley has proved that the locus of the centre of a cardioid which touches the sides of a given triangle is a set of lines.

In the last paragraph of p. 261 Morley interchanges the angles made by the tangents with the x -axis with the angles made by the perpendiculars onto the tangents, and remarks: "...bearing in mind that in (3) the angles must have a sum congruent to 0..." (which is not too clear), he obtains 9 lines for the locus of the centre.

Verifying, in trilinear coordinates, that these 9 lines are parallel in sets of 3 would be a formidable task, since lines are parallel in these coordinates if they intersect on the line at infinity,

$$ap_1 + bp_2 + cp_3 = 0.$$

But, as we have seen, Morley overcomes this difficulty by using equation (1) in a remarkable manner.

Finally, how does all this fit in with the trisectors of the angles of triangle ABC? Once again, on p. 262, a brilliant geometrical statement: "The centre changes from one line to another when one of the lines is a double tangent." The use of "line" is ambiguous, since it is used for the centre locus and for the lines forming the triangle of tangents, but the figure shows what he means. Using equation (1) again, Morley shows that the vertices of his equilateral triangle are the intersections of trisectors of the angles of triangle ABC.

Morley himself adds the final touch: "Verification is naturally a much simpler matter."

Appendix. All that need be said about "trilinear coordinates" is that it is not difficult to show that a linear equation in (p_1, p_2, p_3) represents a line. In fact, if the sides of the triangle are taken in the normal form as

$$x \cos \alpha_i + y \sin \alpha_i - q_i = 0 \quad (i = 1, 2, 3),$$

then we can take

$$p_i = x_p \cos \alpha_i + y_p \sin \alpha_i - q_i$$

for the trilinear coordinates (p_1, p_2, p_3) of $P = (x_p, y_p)$, and therefore a linear homogeneous equation in the p_i produces a linear equation in the (x_p, y_p) .

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Where else?

The Managing Editor of EUREKA, F.G.B. Maskell, resides at 1332 *Morley Blvd.*, Ottawa, Ontario.

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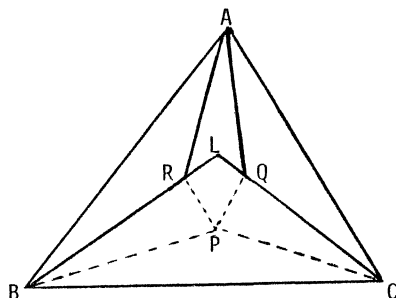
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ROBSON'S PROOF OF MORLEY'S THEOREM

Editor's remarks.

As a welcome relief for those readers who have, in the preceding pages, just struggled through the intricacies of Morley's own proof, I reproduce below a proof of Morley's Theorem by A. Robson which was published in *The Mathematical Gazette*, 11 (1922-1923), pp. 310-311. Robson's proof is thought by many (in particular by Dan Pedoe) to be one of the shortest and best of the known proofs, although perhaps not everyone would be willing to call it an elementary proof.

The figure at the right is that referred to at the beginning of Robson's proof, and the notes which follow the proof were supplied by Dan Pedoe.



660. [K¹. 1. c.] *Morley's Theorem* (v. Note 621).

In the figure, *Gazette*, vol. xi. p. 85, let BRL cut AQ in U ; AQ produced cuts BP in N and CP in V ; CP cuts AR in M ; QM cuts RN in O .

Then BP , BL are isogonal, and so are CP , CL ;

$\therefore AP$, AL are also isogonal;

$\therefore A(BRLU) = A(CVPM)$;

$\therefore N(BRLU) = Q(CVPM) = Q(PMCV)$,

and these pencils have a common ray; \therefore their corresponding rays have collinear intersections, *i.e.* P , O , L are collinear.

As R is the in-centre of ANB , $ARN = 90^\circ + \frac{1}{3}B$.

As Q is the in-centre of AMC , $RMQ = 90^\circ - \frac{1}{3}A - \frac{1}{3}C$;

\therefore the difference, *viz.* $ROM = 60^\circ$.

Similarly the other angles at O are 60° ; since they have a common base and equal angles at each of its extremities, the triangles ORL , OQL are congruent, and so are the triangles PRL , PQL .

The College, Marlborough.

A. ROBSON.

Notes on Robson's Proof of the Morley Theorem.

The proof uses the idea of *isogonal rays*. If two rays through the vertex of an angle make equal angles with its sides, they are said to be isogonal. They

are then mirror images in the bisector of the angle. The theorem used by Robson on isogonals is:

If three lines from the vertices of a triangle are concurrent, their isogonals are also concurrent.

He then uses the idea of projective pencils, pencils with equal cross ratios, and the theorem that if two projective pencils with distinct vertices have a self-corresponding ray, the three intersections of corresponding rays are collinear. He also uses the theorem that in a cross ratio the interchange of a pair of elements together with the interchange of the other pair does not affect the value of the cross ratio. Robson's proof is as short as anyone could desire, and it avoids elaborate initial constructions.

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A LIST OF REFERENCES TO THE MORLEY THEOREM

1. INTRODUCTION

In anticipation of this Morley issue of EUREKA, the editor had asked Professor Charles W. Trigg to prepare an extensive list of references to the Morley Theorem, a task for which his encyclopedic knowledge of the literature of geometry made him particularly well fitted. Professor Trigg agreed and eventually compiled a list of 107 items. He acknowledges with thanks that several particularly hard-to-find items were supplied to him by Dr. Leon Bankoff.

Coincidentally, Professor C.O. Oakley was at the same time preparing a list of 116 Morley references which was submitted and accepted by the *American Mathematical Monthly* for publication in March or April 1978.

Professor H.S.M. Coxeter found out about the two solitary Morley archeologists and effected a *rapprochement* between them. The intersection of their two lists contained 75 common items; hence the union of the two consisted of 148 items. It was thought highly desirable by all concerned to have the complete list of 148 items published together. We are indebted to the editor of the *American Mathematical Monthly* for the singular privilege of publishing here Professor Oakley's list, in advance of its publication in the *Monthly*. The list appears below in Section 2. This is followed, in Section 3, by a supplementary list containing only the 32 items from Professor Trigg's list that do not already appear in Professor Oakley's list.

Together the two lists form what is almost certainly the most complete assemblage of Morley references in existence.

2. A LIST OF REFERENCES TO THE MORLEY THEOREM

C.O. OAKLEY, Haverford College

The following letter-coding of the reference numbers should be clear and, we trust, useful. They give some indication of the mathematical nature of the references.

- B. Book
- CC. Mathematics associated with Clifford chains
- CS. Complete solution (for all 18 Morley triangles)
- CV. Proof using complex variables
- G. Proof by geometry
- IP. Indirect proof
- PG. Proof by projective geometry
- PP. Proposed problem (Morley, or related)
- PPS. Proposed problem solved
- R. Related material
- T. Proof by trigonometry

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AN ELEMENTARY GEOMETRIC PROOF OF THE MORLEY THEOREM

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The celebrated Morley Theorem can be stated as follows:

THEOREM 1. The intersections of adjacent trisectors of the interior angles of a triangle are the vertices of an equilateral triangle.

As we shall see, this statement is also true for the intersections of the trisectors of the exterior angles of a triangle as well as for those of its reflex angles.

The truth of Theorem 1 can be demonstrated on the basis of a simple lemma which is stated below; but we indicate first how we are led to the lemma since this will show at once how Theorem 1 follows as a corollary.

In $\triangle ABC$ let V_s denote the trisector of angle V adjacent to side s , where $V = A, B, C$ and $s = a, b, c$ (see Figure 1). Let B_a, C_a meet at X and B_c, C_b meet at R . Then X is the incenter of $\triangle RBC$ and so is at a common distance r from BR and CR . Reflect X about BR to P (on AB) and about CR to Q (on AC). Then $XP = XQ = 2r$, and BR, CR are the perpendicular bisectors of XP, XQ respectively.

Let O denote the center of the circumcircle K of $\triangle APQ$ and let w denote the arc of K subtended by angle A . Letting $A = 3\alpha, B = 3\beta, C = 3\gamma$, so that $\alpha + \beta + \gamma = 60^\circ$, we have $w = \angle POQ = 6\alpha$.

Suppose (i) BR, CR meet w at Y, Z respectively;

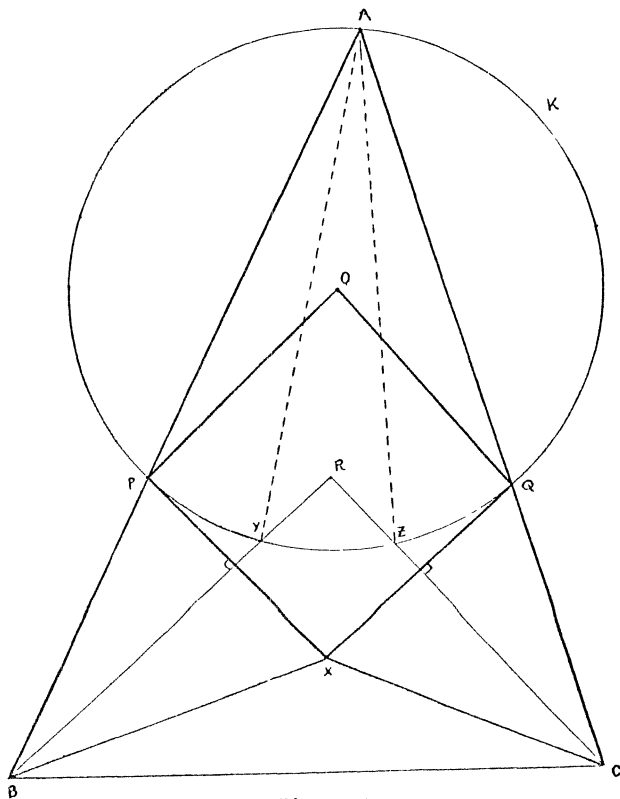


Figure 1

- (ii) Y and Z trisect w ;
- (iii) $\triangle XYZ$ is equilateral.

Then Theorem 1 would follow immediately.

Proving these suppositions is what the lemma is about. Their truth results from the fact that quadrilateral $OPXQ$ has two special properties:

- (i) it is symmetric about OX (as is obvious);
- (ii) its angles at P , Q , X are equal.

To see the latter, note that

$$\angle BXP = 90^\circ - \beta, \quad \angle CXQ = 90^\circ - \gamma, \quad \angle BXC = 180^\circ - (\beta + \gamma).$$

The sum of these angles is $360^\circ - 2(\beta + \gamma)$; hence $\angle PXQ = 2(\beta + \gamma) = 120^\circ - 2\alpha$. Since the angles at P and Q are equal, and that at O is 6α , it follows that the angles at P , Q , X are all equal to $120^\circ - 2\alpha$. Hence we state our lemma as follows:

LEMMA. In quadrilateral $OPXQ$ (see Figure 2), suppose $OP = OQ$, $XP = XQ$, and the angles at O and X are 6α and $120^\circ - 2\alpha$ respectively. Let K denote the circle with center O , radius OP , and let w denote the arc of K subtended by the angle at O .

Then the perpendicular bisectors of XP , XQ meet w at points Y , Z , respectively, such that Y , Z trisect w and $\triangle XYZ$ is equilateral.

Proof. Let XP , XQ meet circle K again at S , T respectively. From the hypotheses, we obviously have

$$\triangle OPX \cong \triangle OQX \quad \text{and} \quad \angle OPX = \angle OQX = 120^\circ - 2\alpha.$$

Then $\angle OTQ = \angle OQT = 60^\circ + 2\alpha$; hence $PX \parallel OT$. Similarly, $QX \parallel OS$, so $OSXT$ is a rhombus and $\angle TOS = \angle PXQ = 120^\circ - 2\alpha$.

Since $PX \parallel OT$, we have $\angle TOP = 60^\circ + 2\alpha = \angle OTQ$; hence $OPXT$ is an isosceles trapezoid. The perpendicular bisector of PX is then also the perpendicular bisector of OT , hence it meets circle K at two points Y , Y' (where we let Y denote the one on the same side of OT as XP).

Now X is interior to $\angle POQ$, so OX meets w at a point V . Clearly OX , and hence OV , bisect $\angle POQ$ as well as $\angle TOS$, so $\angle TOV = 60^\circ - \alpha$, while $\angle POV = \angle QOV = 3\alpha$. Since Y is on the perpendicular bisector of OT , we have $OY = TY$, so $\triangle TOY$ is equilateral and $\angle TOY = 60^\circ$. Since V lies on the same side of OT as X , we have $\angle YOY = \alpha$, and hence Y lies on w . For the same reasons $\angle POY = 2\alpha$ (thus Y trisects w), and Y lies on the same side of OX as P since $\angle POY = 3\alpha < 180^\circ$.

We can show similarly that the perpendicular bisector of QX meets w at a point Z which trisects w and is on the same side of OX as Q . Thus $PY = YZ = ZQ$.

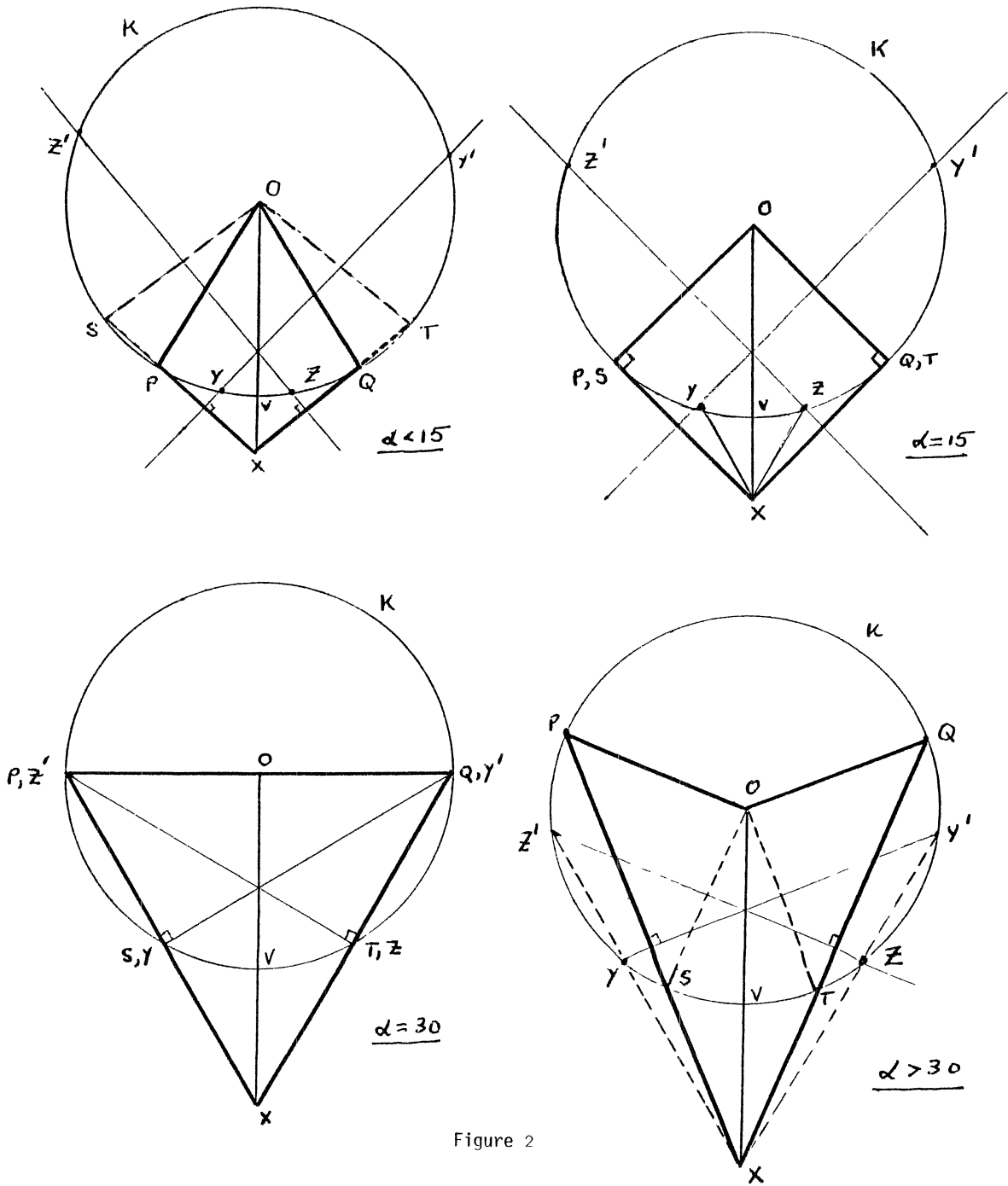


Figure 2

Finally, $PY = XY$ and $ZQ = ZX$, so that $XY = YZ = ZX$ and $\triangle XYZ$ is equilateral. This completes the proof of the lemma, and Theorem 1 follows.

Now, referring back to Figure 1, since $\angle YXZ = 60^\circ$ we have $\angle PXY = 30^\circ - \alpha$, and hence $\angle BXY = (90^\circ - \beta) + (30^\circ - \alpha) = 120^\circ - (\alpha + \beta)$. Since $\angle PBX = 2\beta$, XY meets AB at an angle of $2\alpha + \gamma$, BC at an angle of $\alpha + 2\gamma$, and AC at an angle of $|\alpha - \gamma|$. Similarly XZ meets AC at an angle of $2\alpha + \beta$, etc.

The same method as was used to prove Theorem 1 will also verify it when "interior angles" is replaced by "exterior angles" or "reflex angles." The reader will have no difficulty arriving at a suitably modified lemma for these two cases, and the proofs are virtually the same. Also, an angle count similar to that in the preceding paragraph will show that in all cases the corresponding sides of the three Morley triangles meet the sides of $\triangle ABC$ at the same angles, thereby proving the following

THEOREM 2. Corresponding sides of the three Morley triangles (interior, exterior, reflex) are parallel.

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THE BEAUTY AND TRUTH OF THE MORLEY THEOREM

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If a committee of mathematicians were assembled to judge a beauty contest involving geometrical theorems, it is almost certain that one of the chief contenders would be Morley's Triangle Theorem. Granted that beauty is in the eye of the beholder, it would be hard to find anyone who would deny that this elegant theorem deserves a high place of honor in the Geometrical Hall of Fame and Esthetic Excellence. Morley's Theorem arrived on the mathematical scene only three-quarters of a century ago and one cannot help but wonder how this newcomer happened to escape the notice of geometrical doodlers during the millennia following Euclid. A plausible explanation for this oversight may be that it was simply a matter of obeisance to a sort of taboo associated with Euclidean angle trisections—an erroneous identification of nonconstructibility with nonexistence. Whatever the reason for its late arrival, the Morley configuration has never been found in the mathematical literature before the turn of this century and, unless evidence to the contrary turns up from some unsuspected source, credit for its discovery must continue to rest with Frank Morley.

Those who confront the Morley configuration for the first time invariably react with an instantaneous display of awe and astonishment. After the initial

emotional excitement subsides, reason steps in and demands some verification for what our eyes and brain seem to perceive. Going along with Keats, we believe that "Beauty is truth, truth beauty," and we feel the need for cerebral evidence to confirm intuitive suspicions. Is the inner triangle really equilateral? Does this happen with all parent triangles? Is a convincing proof available? Does one need a familiarity with higher mathematics to achieve the intellectual satisfaction complementing our esthetic pleasure? While the mathematical sophisticate may revel in proofs by complex numbers, involution, or Brianchon's Theorem, we wonder what sort of proof we can offer the high school student, the bright and eager neophyte on the threshold of mathematical exploration and enlightenment.

These questions can now be answered more readily than ever before. The current issue of EUREKA presents the most comprehensive collection of references on the Morley Theorem ever to be assembled and the interested researcher should have no trouble gleaning from the list enough accessible sources to furnish a large variety of proofs. A few comments on the diversity of the demonstrations and the relative elegance of the various types of proofs may be helpful.

As one would expect, the long list of references contains many duplications of previously published proofs, either through rediscovery or by worthy attempts at refinement and clarification. Some of the proofs suffer from excessive brevity and insufficient clarity; others are unwieldy and cumbersome and offend our sense of elegance. Some proofs are direct, others indirect. Some are primarily trigonometrical, others purely geometrical (elementary or advanced). Preference for one type of proof or another becomes a matter of individual taste.

A preliminary classification of proofs results in the establishment of two basic categories, direct and indirect. Here we use the term *indirect* not in its customary meaning of "reductio ad absurdum" but in the sense of a reversal of the usual sequence of steps from hypothesis to conclusion. Examples of indirect proofs are those by Naranengar, Chepmell, Boon, Grossman, Davis, Dobbs and Child, to name a few. These proofs start with the foreknowledge that the internal triangle is equilateral and, with subsequent constructions based on the known values of the surrounding angles, lead to the demonstration that certain rays emanating from the vertices of the inner triangle actually converge to form the outer triangle. This procedure is essentially a proof not of the theorem as stated by Morley but of its converse. An analogous situation occurs with the Steiner-Lehmus Theorem regarding two equal base angle bisectors [see EUREKA 2 (1976) 19-24]. The easily established converse does not provide a legitimate proof of the main theorem.

Among the direct proofs, the preponderance of those published are trigonometrical. We can account for this top-heaviness by noting that trigonometry is the ideal tool for handling submultiple angles. Theoretically we should be able to convert any trigonometrical argument to one involving synthetic geometry, but in practice this would surely result in a muddy proof. We find considerable variation in the trigonometrical approaches. Some are directed toward the computation of the sides of the equilateral triangle (e.g., Neuberg, Letac, Thébault), while others by-pass this method and go directly to the calculation of the angles surrounding the three pertinent intersections, leaving 60° for each vertex of the Morley triangle (e.g., Satyanarayana, Bankoff and others). Letac's solution (*Sphinx*, 1939) achieves a deceptive simplicity by merely omitting steps essential for a clear understanding—again a case of brevity at the expense of clarity.

Excellent direct geometrical solutions have been constructed using isogonal conjugates, cross ratios, Desargues' theorem, Menelaus' theorem, and complex numbers. More advanced mathematicians find these proofs particularly attractive since they combine concision with precision. Examples of these techniques may be found among the proofs devised by Neuberg, Thébault, Robson, Ghiocas and Lubin.

The ideal proof yet to be discovered should be one that adheres to synthetic geometry, that follows a direct path from hypothesis to conclusion, that is relatively easy to follow and is neither too long nor too involved. The existing proofs that approximate these ideals most closely are those by W.E. Philip and by B. Niewenglowski.

Numerous authors have reached out for extensions of Morley's original theorem and have wandered into ramifications involving external angle bisectors, angle quintisectors and nonequilateral Morley triangles. Vandeghen, for example, correlates the basic Morley Triangle with those constructed within the orthic triangle, foot-median triangle, excenter triangle and triangles formed by the excircle and incircle contacts, as well as the circumcircle tangential triangle at the vertices of the parent triangle.

The reader is encouraged to explore the various references with the hope of devising improvements, shortcuts and possibly new methods of attack. Morley's Theorem is still very young and we can surely expect novel methods of proof to loom up in the future as more and more geometrical aficionados try their hand at further refinement and clarification of this beautiful theorem.

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Because the Morley material has expanded to fill nearly all the space available, the problem section is rather rudimentary in this issue. We will make up for it in the next few issues.

PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

291. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Using soap, on a mirror, please trace

The apparent outline of your face;

Now explain (if you're wise)

Why it turns out "half size",

Using geometry as your base.

292. *Proposed by Charles W. Trigg, San Diego, California.*

Fold a square piece of paper to form four creases that determine angles with tangents of 1, 2, and 3.

293. *Proposed by David R. Stone, University of Kentucky, Lexington.*

For which b is the exponential function $y = b^x$ tangent to the given line $y = mx$? Conversely, given $y = b^x$, for which m is $y = mx$ tangent to $y = b^x$?

294.* *Proposed by Harry D. Ruderman, Hunter College, New York.*

Prove that there are infinitely many integers that cannot be expressed in the form $3ab + a + b$, where a and b are nonzero integers.

295. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If $0 < b \leq a$, prove that

$$a + b - 2\sqrt{ab} \geq \frac{1}{2} \frac{(a - b)^2}{a + b}.$$

296. *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Soit p un nombre premier. Montrer que $p^4 - 20p^2 + 4$ n'est pas un nombre premier.

297. *Proposed by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

A young lady went to the store to purchase four items. In computing her bill, the nervous clerk multiplied the four amounts together and announced that the bill was \$6.75. Since the young lady had added the four amounts mentally and obtained the same total, she paid her bill and left. Assuming that the prices for each item are distinct, what are the individual prices?

298. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

The equation $x^2 - 9x + 18 = 4$ has the property that, if the left side is factored, so that $(x - 3)(x - 6) = 4$, then one of the roots, $x = 7$, is found by illegally setting one of the factors equal to the constant on the right, $x - 3 = 4$. Unfortunately, the second root cannot be similarly found; it is not $x - 6 = 4$. Find all such quadratic equations in which both roots can be obtained by equating each factor in turn to the nonzero constant on the right.

I first heard of this problem in a lecture by Howard Eves some years ago.

299. *Proposed by M.S. Klamkin, University of Alberta.*

If

$$F_1 = (-r^2 + s^2 - 2t^2)(x^2 - y^2 - 2xy) - 2rs(x^2 - y^2 + 2xy) + 4rt(x^2 + y^2),$$

$$F_2 = -2rs(x^2 - y^2 - 2xy) + (r^2 - s^2 - 2t^2)(x^2 - y^2 + 2xy) + 4st(x^2 + y^2),$$

$$F_3 = -2rt(x^2 - y^2 - 2xy) - 2st(x^2 - y^2 + 2xy) + (r^2 + s^2 + 2t^2)(x^2 + y^2),$$

show that F_1 , F_2 and F_3 are functionally dependent and find their functional relationship. Also, reduce the five-parameter representation of F_1 , F_2 and F_3 to one of two parameters.

300. *Proposed by Léo Sauvé, Algonquin College (editor).*

The sine and cosine are known as transcendental functions, so one would expect that $\sin x$ and $\cos x$ would be transcendental numbers for most values of x . Does there exist a *dense* subset E of the reals such that $\sin x$ and $\cos x$ are both algebraic for every x in E ?

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

199. [1976: 220; 1977: 112] *Proposed by H.G. Dworschak, Algonquin College.*

If a quadrilateral is circumscribed about a circle, prove that its

diagonals and the two chords joining the points of contact of opposite sides are all concurrent.

Editor's comment.

I give below eight additional references to this problem. References [1]-[7] were sent in by Charles W. Trigg, San Diego, California, and [8] came from Murray S. Klamkin, University of Alberta.

REFERENCES

1. M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, 1968, p. 209.
2. John Casey, *A Sequel to Euclid*, Longmans-Green, 1884, p. 144.
3. E.A. Maxwell, *Geometry for Advanced Pupils*, Oxford, 1949, pp. 140-141.
4. William J. McClelland, *The Geometry of the Circle*, Macmillan, 1891, p. 126.
5. Charles Pierson, Solution of Problem 85, *The Pentagon*, 18 (Fall 1958) 42-43.
6. J.M. Wilson, Solution of Problem 2310 on p. 34 of Vol. VIII, *Mathematical Questions from the Educational Times*, edited by W.J. Miller, C.F. Hodgson & Son, London, 1868.
7. Solution of Geometry Problem 196, *American Mathematical Monthly*, 10 (May 1903) 138.
8. *Mathematics Magazine*, Nov-Dec 1962, pp. 289-294. (This contains nine proofs of the theorem.)

234. [1977: 104, 154, 257] *Late solution: DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.*

240. [1977: 105, 264] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution for this base ten cryptarithm:

$$\begin{array}{r} \text{CARL} \\ \times \text{ F} \\ \hline \text{GAUSS} \end{array}$$

Editor's comment.

R. Robinson Rowe and the proposer have both confirmed that the three solutions given in [1977: 264] are the only ones. The details of their calculations will be left to enrich the archives of this journal. This problem thus has the unique distinction of having exactly three unique solutions.

241. [1977: 130, 265] *Late solution: DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.*

Une omelette de douze oeufs!

LA MORT DE CONDORCET

Un soir de germinal an II (1794), un homme, visiblement épuisé, entre dans un cabaret de Clamart-le-Vignoble, dans la banlieue parisienne. Il commande une omelette. "De combien d'oeufs?", lui demande-t-on. Il hésite, puis répond: "De douze oeufs!". Devant ce chiffre pantagruélique, le propriétaire s'effare et va avertir le Comité de Surveillance installé dans l'église. Aussitôt, on arrête l'homme qui dit s'appeler Pierre Simon, mais en qui l'on reconnaît bientôt le mathématicien et philosophe Condorcet: celui-ci, bien avant la Révolution, a combattu—comme Voltaire—l'intolérance, la torture, l'injustice sociale. Député à la Législative et à la Convention, il a voulu transposer dans les faits sa lutte de plusieurs années. Hélas! il s'est bientôt rangé parmi les ennemis de Robespierre. Lors de la proscription des Girondins, il se solidarise avec eux et doit s'enfuir pour ne pas être arrêté. L'histoire de l'omelette le trahira: jeté en prison, on le trouvera mort au petit matin—suicide, a-t-on dit.

ALAIN DECAUX¹

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*

THE TWELVE DAYS OF CHRISTMAS

"On the first day of Christmas, my true love sent to me a partridge in a pear tree".

"On the second day of Christmas, my true love sent to me two turtle doves and a partridge in a pear tree".

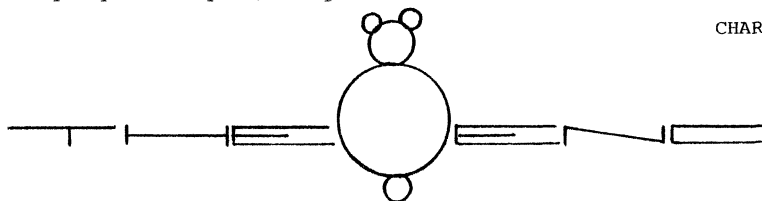
Thus the traditional cumulative song continues until "On the twelfth day of Christmas, my true love sent to me twelve drummers drumming, eleven pipers piping, ten lords a-leaping, nine ladies dancing, eight maids a-milking, seven swans a-swimming, six geese a-laying, five golden rings, four calling birds, three French hens, two turtle doves, and a partridge in a pear tree."

Apparently my true love considered that a partridge a day would keep the poachers away.

In the course of this (the "t" in the last word is a libation to the w-libbers) orgy of giving, my true love contributed 12 partridges (inseparable from their pear trees), 12 drummers, 22 each of turtle doves and pipers, 30 each of French hens and lords, 36 each of calling birds and ladies, 40 each of golden rings and maids, and 42 each of geese and swans.

My true love was strictly for the birds, giving 184 of them compared to 76 women, 40 rings, 30 lords, and 34 musicians. In 12 days there were 364 gifts bestowed. One for every day of the year, except—Christmas?

CHARLES W. TRIGG²



TRIGG

¹Extrait d'une critique littéraire parue dans *Histoire Pour Tous* (février 1963).

²Reprinted by permission from the *Wesley Palms Breezes* (Nov-Dec 1975), p. 12.

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