MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Message from the Editor

We are pleased to announce that from now on each issue of *Mathematical Spectrum* will contain four additional pages. This extra space will enable us to offer our readers a wider selection from the many interesting contributions we receive. We feel sure this will be a welcome development.

The price of an annual volume, which has remained unchanged since September 1985, will now be £4·80. This rise will cover both the cost of the additional pages and inflationary increases over the past three years. The dollar prices for subscribers in America and Australia (US$11·00 and $A16·00) will remain unchanged next year; the sterling equivalent of these dollar rates will be £6·00.

An order form for Volume 21 is enclosed with this issue, except in the case of those whose subscriptions are already covered for this volume, i.e. up to August 1989. We look forward to the continuing support of all our readers.

Editor

Iteration

DERMOT ROAF, *Exeter College, Oxford*

The author was an undergraduate at Christ Church, Oxford and a graduate student in Cambridge. He is Mathematics Fellow at his college with a particular interest in theoretical physics. His hobbies include bell-ringing (see *Mathematical Spectrum*, Volume 7, pages 60–66) and politics: he is currently Alliance Spokesman on Education on the 'balanced' Oxfordshire County Council.

1. Introduction

Many mathematical calculations involve repeating the same operation over and over again until sufficient accuracy has been achieved. This article will discuss the iterative procedures which can be written in the form $x_{n+1} = f(x_n)$ (where $f(x)$ is a function of x which is continuous in regions of interest) with some initial value of x_0. Under what conditions does the sequence of values of x_n tend to a limit (i.e. come as close as we please to some definite value even if it never actually takes that value just as the sequence $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots$ tends to the limit 2)? This sequence can be defined by $f(x) = 1+\frac{1}{2}x$ (with $x_0 = 1$). Work it out! And how quickly does the sequence converge (i.e. how many repetitions are needed to achieve any specified accuracy)?

Although the ideas in this article apply to functions of complex numbers and matrices and, indeed, to integral equations like

$$x_{n+1} = \int f(x_n, t)\, dt,$$

the article will discuss only simple real functions.

You may like to investigate the following sequences before reading further. Pencil and paper and an ordinary calculator will suffice as there is no need to do more than ten iterations. A microcomputer or a programmable calculator would be almost as easy to use. Look at

$$f(x) = 0.1x+2 \qquad \text{(with } x_0 = 1\text{)},$$

$$f(x) = 10x+2 \qquad \text{(with } x_0 = 1 \text{ and with } x_0 = -0.2222\text{)},$$

$$f(x) = 0.1+x^3 \qquad \text{(with } x_0 = 0 \text{ and with } x_0 = 1\text{)},$$

$$f(x) = x - \frac{x^3-x+0.1}{3x^2-1} \qquad \text{(with } x_0 = 0 \text{ and with } x_0 = 1\text{)}.$$

Finally look at $f(x) = Ax(1-x)$ (with $x_0 = 0.5$ and the following different values of A: 2.5, 3.2, 3.5, 3.83). This will require more iterations and a microcomputer would be appropriate. This function has been discussed in two recent articles in *Mathematical Spectrum*. G. Rowlands (reference 3) and Keith Devlin (reference 1) both described the 'chaos' of certain values of A and the 'bifurcations' for other values.

This article attempts to analyse the underlying mathematics and explain why some of these strange results occur.

2. Simple forms

Suppose we look at $f(x) = ax+b$ with $x_0 = 1$. The successive values are $a+b$, a^2+ab+b^2, $a^3+a^2b+ab+b$, and, after n iterations, $a^n+b(1+a+a^2+\ldots+a^{n-1})$, which gives the limit $b/(1-a)$ (if $|a| < 1$).

So, if we choose as a simple example $a = 0.1$ and $b = 2$, we have 1.0, 2.1, 2.21, 2.221, 2.2221, etc.—a sequence which tends to the value 2.22222 recurring, or $\frac{20}{9}$. Of course we can see that, if there is a limit, x_∞ say, then this is the solution of $x = f(x)$, or, in this case, $x = ax+b$, which gives $x = b/(1-a)$; or, with the particular values we chose for a and b, $2/(1-0.1) = \frac{20}{9}$.

What if we had chosen $a = 10$ and $b = 2$? Then the sequence would have been 1, 12, 122, 1222, etc., which clearly increases indefinitely; it does not tend to $2/(1-10) = -\frac{2}{9}$. But if we had started with $x_0 = -\frac{2}{9}$, then $x_1 = -\frac{20}{9}+2 = -\frac{2}{9}$, so we would stay at this value for ever. But with $x_0 \neq -\frac{2}{9}$ the magnitude of x_n increases without limit, however close to $-\frac{2}{9}$ x_0 may be (try -0.2222, for example).

If x_∞ is a solution of $x = f(x)$, to which we hope our sequence will lead, then a sufficient condition for convergence is that

$$|f(x)-f(x_\infty)| \leqslant K|x-x_\infty|$$

(where K is some fixed number less than 1) for x near to x_∞. For then

$$|x_n-x_\infty| = |f(x_{n-1})-f(x_\infty)| \leqslant K|x_{n-1}-x_\infty|,$$

so the distance between successive values of x_n and x_∞ is reduced by multiplication by at most K each time.

Because we do not know x_∞ (at least when we start) the convergence condition is difficult to use; instead we could ask that, for all values of x and y in the region considered, the function f should satisfy the condition

$$|f(x)-f(y)| \leqslant K|x-y|$$

(where, again, $K < 1$). This is known as a 'Lipschitz condition' and, if $f(x)$ is differentiable, it is equivalent to $|f'(x)| \leqslant K < 1$ for all x in the region. So in the case $f(x) = ax+b$, we have $f'(x) = a$ and this condition is satisfied if $|a| < 1$.

For further reading, you may like to try Chapter 2 of *Essentials of Numerical Analysis with Pocket Calculator Demonstrations* by P. Henrici (reference 2).

3. Roots of an equation

Suppose we want to find the roots of $x^3-x+0.1 = 0$. We can see that one of them is close to 0.1. If we write the equation in the form $x = 0.1+x^3$, we have it in iterative form (with $f(x) = 0.1+x^3$). If we restrict ourselves to the region $|x| \leqslant \frac{1}{2}$, then $|f'(x)| = 3x^2 \leqslant \frac{3}{4}$ and the condition is satisfied. Try it now with $x_0 = 0$. The results are 0, 0.1, 0.101, 0.101 0303, etc., tending to 0.101 0313.

If we had started with $x_0 = 1$, we would have 1, 1.1, 1.431, 3.0303, 27.928, with divergence. So how do we find the other roots (which, by inspection, are at approximately ± 1)? The cubic can be rearranged as

$$(x-1)(x+1)x = -0.1 \quad \text{or} \quad x = 1-\frac{0.1}{x(x+1)}.$$

Now near $x = 1$ this $f(x)$ has derivative less than 0.1 in magnitude. So with $x_0 = 1$, we have the sequence 1, 0.95, 0.946, etc., which is convergent to 0.945 649.

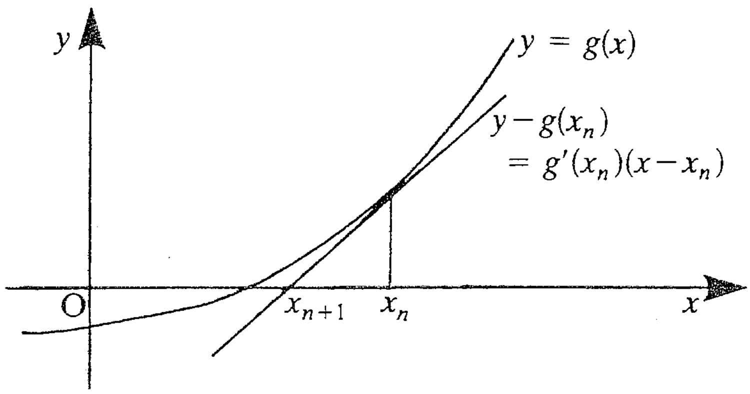
This form is unsuitable for the root near $x = -1$ where the derivative is infinite. We must rewrite the equation again. Using a similar method we have

$$f(x) = -1+\frac{0.1}{x(1-x)}.$$

With $x_0 = -1$, we now have the sequence -1, -1.05, -1.046, etc., converging to $-1.046\,681$.

The rearrangement of these equations is the challenge; to avoid it, we can use Newton's method for differentiable equations.

4. Newton's method



To solve $g(x) = 0$ we take the tangent at the first guess and use its intersection with $y = 0$ (see the figure). This gives

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \quad \text{or} \quad f(x) = x - \frac{g(x)}{g'(x)}\,.$$

This has derivative gg''/g'^2, so near the root (where $g = 0$), we can expect $|f'|$ to be small, provided g' is non-zero. Try this for our cubic:

$$x_{n+1} = x_n - \frac{x_n^3 - x_n + 0.1}{3x_n^2 - 1}$$

with initial values 0, 1 and -1.

You will see that the convergence with this method is much faster. In fact we have 'quadratic' convergence in which (for x near to x_∞)

$$|f(x) - f(x_\infty)| \leqslant L|x - x_\infty|^2$$

(where L is some fixed number). In this case, if it takes, say, two steps to improve the accuracy from 10^{-1} to 10^{-2}, it will take two more steps to improve the accuracy to 10^{-4} and another two to 10^{-8} etc. This method is often applied to finding square roots from $g(x) \equiv x^2 - A = 0$, so $g'(x) = 2x$, giving $f(x) = \frac{1}{2}(x + A/x)$. Try this with $A = 2$ and $x_0 = 1$.

5. Chaos

We now look at the iteration of $f(x) = Ax(1-x)$ for different values of A. Clearly if there is a limit point, it must be at $x = 0$ or $x = (A-1)/A$. The derivative is $A(1-2x)$, so has the values A and $2-A$, respectively, at these points. So for $|A| < 1$ we have a limit point at $x = 0$ and for $1 < A < 3$ we have a limit point at $(A-1)/A$. Calculate these sequences for a few values of

A between 0 and 3 with $x_0 = 0.5$, say. As A approaches 3, the convergence slows up. But now try $A = 3.2$. We do not expect convergence, but what we get is a sequence oscillating between two 'cluster' points (0.513 0445 and 0.799 4555). We have x_{n+2} close to x_n but different from x_{n+1}. Now

$$x_{n+2} = Ax_{n+1}(1-x_{n+1}) = A^2x_n(1-x_n)(1-Ax_n+Ax_n^2).$$

Putting $x_{n+2} = x_n = x$, we have the quartic

$$A^3x^4-2A^3x^3+(1+A)A^2x^2+(1-A^2)x = 0$$

with roots at $x = 0$ and at $x = (A-1)/A$ (we know that starting with *exactly* these values we will never move from them). So we can factorise the quartic to give

$$x(Ax+1-A)[A^2x^2-(1+A)Ax+1+A] = 0,$$

which has the new roots $[1+A\pm\sqrt{(A-3)(1+A)}]/2A$ (which link up at $A = 3$ with the root $(A-1)/A = \frac{2}{3}$).

Now if the two cluster points are α and β [satisfying the quadratic equation so that $\alpha+\beta = (1+A)/A$ and $\alpha\beta = (1+A)/A^2$], then the derivative

$$\frac{dx_{n+2}}{dx_n} = \frac{dx_{n+2}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} = A^2(1-2x_{n+1})(1-2x_n),$$

is, approximately,

$$A^2[1-2(\alpha+\beta)+4\alpha\beta] = A^2-2A(1+A)+4(1+A) = 4+2A-A^2,$$

which has magnitude less than 1 when $(A-1)^2$ lies between 4 and 6, i.e. for A lying in the range $3 < A < 1+\sqrt{6} = 3.449\,4897$ (and for some negative values of A as well).

For A slightly larger than 3.4495 the oscillations are between four cluster points. A diagram of the way the cluster points split up is to be found in reference 3. The diagram is developed further in reference 1. As A increases the cluster points increase to eight (at 3.544) and then sixteen (at 3.564). But with A greater than about 3.570 there is chaos and no apparent pattern. But beyond $A = 3.83$ (do I mean $1+\sqrt{8}$?) we find a three-fold repetition—and as we increase further we find six- and twelve-fold repetitions and then chaos again.

Of course for a cycle of N cluster points, the sequence will converge if the product of the derivatives $A(1-2x)$ at the N cluster points has magnitude less than some $K < 1$. This means that at least one of these points has to be near $x = \frac{1}{2}$. So if we start with $x_0 = \frac{1}{2}$ and try to find a value of A for which $x_N = \frac{1}{2}$, we will then be in a region of A with an N-cycle. For example $A \cong 3.74$ for $N = 5$. You may like to think how you could find such values of A.

Similar results will be found for $f(x) = A \sin x$. Devlin describes how to use iteration on a microcomputer to produce 'beauty from chaos'. Investigation of these patterns makes mathematics an experimental science. But, as it is a science, we have a duty to explain what we find. I hope you and your microcomputer enjoy it!

References

1. K. Devlin, 'Beauty from chaos', *Mathematical Spectrum* **18** (1985/86), 65–69.
2. P. Henrici, *Essentials of Numerical Analysis with Pocket Calculator Demonstrations*, (Wiley, London, 1982).
3. G. Rowlands, 'Chaos and strange attractors', *Mathematical Spectrum* **15** (1982/83), 19–26.

Probability and Gambling

Arthur Pounder of Manchester has sent the following problems which he set at the Maths Club of St. Peter's Grammar School, Prestwich, Manchester, when he was on the staff there. You might like to try them.

1. *The problem of Chevalier de Méré*

Two men A and B were gambling. They paid equal stakes of 6 francs each so that 12 francs were in the 'pot'. Both A and B were equally skilful at the game (as in tossing a fair coin). The first person to gain 3 points wins the pot. Unfortunately, the game had to finish without either player gaining the requisite 3 points. At the end of the game the score was A 2 and B 1. The Chevalier de Méré asked the question of Blaise Pascal: 'What is the *fairest* way of sharing the 12 francs?' The correspondence between Blaise Pascal and Pierre de Fermat then formed the basis of probability theory. Well, what is the fairest way?

2. *The three-cards problem*

Three cards are coloured as follows: A is white on both sides, B is black on both sides and C is black on one side and white on the other. The cards are placed in a box. One is chosen at random and placed on a table. If the card shows white uppermost, what are the odds that it also shows white underneath?

3. *Winning on the horses*

Four horses take part in a race. Their odds against winning are 5 to 1, 4 to 1, 3 to 1 and 2 to 1. If you have £57 to gamble, how can you bet to be certain of winning and how can you be sure of making the same profit no matter which horse wins?

Colouring Space

MIKLOS BONA, *University of Eötvös Lóránd, Budapest*

The author, aged 19, is a student. He participated in the International Mathematical Olympiads of 1985 and 1986 and was awarded a third prize. His favourite branches of mathematics are the theory of graphs and their uses and combinatorial geometry.

I would like to write about a generalization of the following simple colouring problems. Each point of a plane is either coloured red (R) or blue (B).

1. Prove that there exists an isosceles right-angled triangle whose vertices are the same colour.

2. Prove that there exists an equilateral triangle whose vertices are the same colour.

First consider problem 1. Suppose there is no such isosceles right-angled triangle and consider an isosceles right-angled triangle XYZ with the right angle at X. There are two possibilities: either Y and Z have the same colour (say red) or they have different colours (say Y red and Z blue, with X blue). It is easy to derive a contradiction in each case from figures 1 and 2; the order in which the points are considered is indicated by the subscripts. In figure 1 we have the monochromatic isosceles right-angled triangle YZR_4 and in figure 2 we have the monochromatic triangle ZB_2B_5. Note that we have used only eight points.

Now consider problem 2. Suppose that no such equilateral triangle exists and consider the equilateral triangle XYZ. We may suppose that X is coloured red and Y and Z are coloured blue. We now consider two possibilities shown in figures 3 and 4, according as the point T (where X, Z and T are collinear, with $XZ = ZT$) is blue or red. In figure 3, the equilateral triangle YTB_4 is monochromatic and in figure 4 the equilateral triangle ZB_2B_3 is monochromatic. This time we have used only nine points.

These problems become much more difficult when we use three colours in three dimensions. Our analogue of problem 1 is to prove that there is an isosceles right-angled triangle whose vertices are the same colour. We consider two cases. The first is where there is a line segment PQ whose endpoints are coloured the same colour, say green (G), and whose midpoint A is coloured another colour, say blue. Consider a square $CDEF$ whose plane is perpendicular to this segment, whose centre is A and whose diagonals are of length PQ. If one of its vertices is green, then that vertex together with P and Q will give a green right-angled isosceles triangle. Otherwise its vertices are blue or red. If three are of one colour, again a monochromatic triangle exists, so we can suppose that two are blue and two are red. Since A is blue,

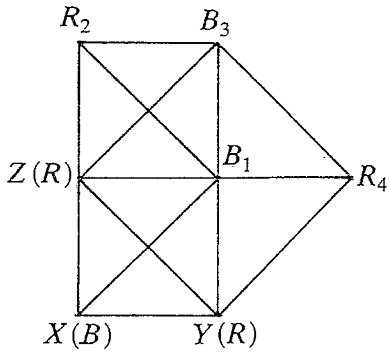


Figure 1

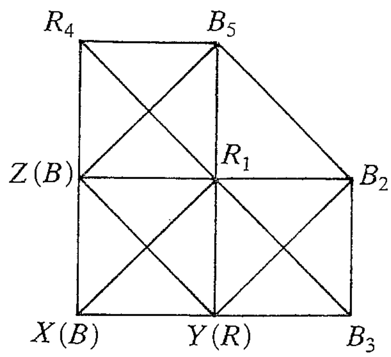


Figure 2

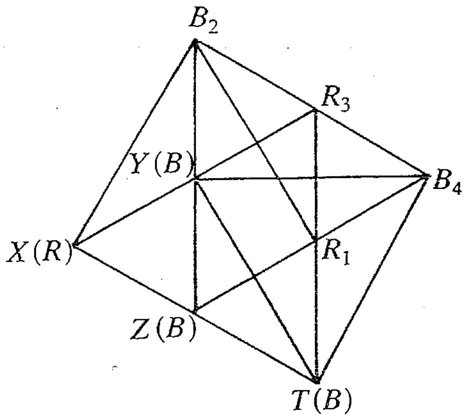


Figure 3

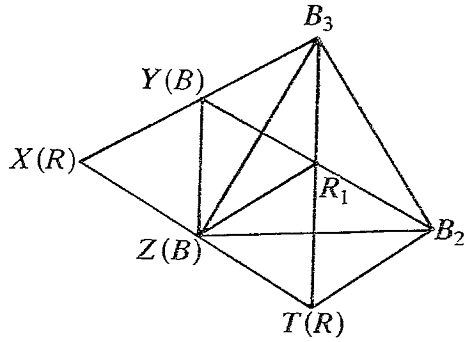


Figure 4

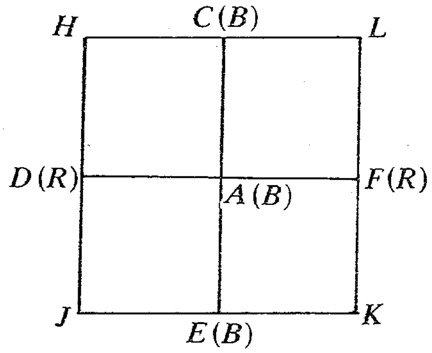


Figure 5

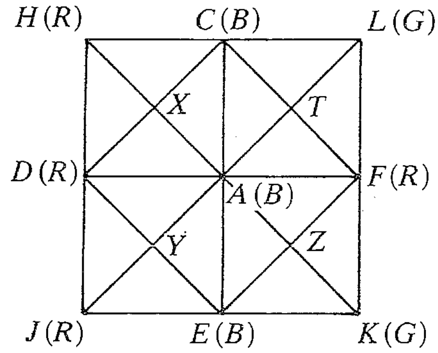


Figure 6

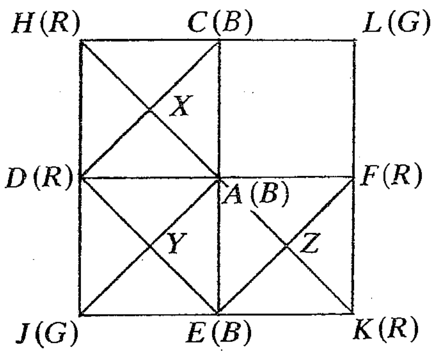


Figure 7

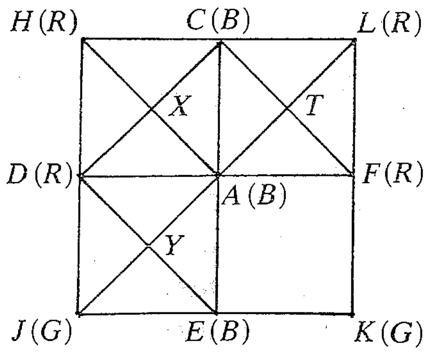


Figure 8

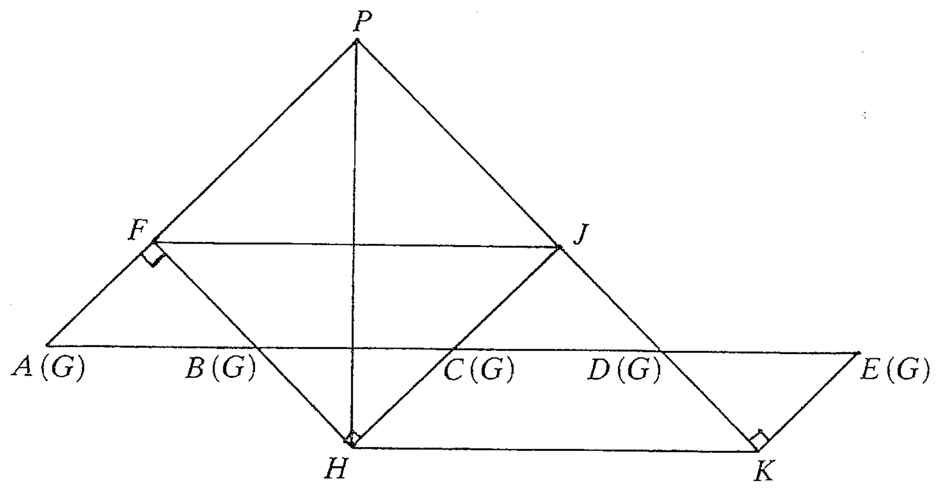


Figure 9

we may suppose that the colouring is as shown in figure 5, since otherwise a blue triangle with one vertex at A certainly exists. We now only need to consider the case in which two of the points H, J, K and L are red and two are green, since all the other possibilities readily give a monochromatic triangle. If H is green, then one of J and L is red, and we can reflect in either DF or CE to bring a red vertex into the top left-hand corner. Thus we may suppose that H is red. We consider separately the cases in which J, K and L are also red, as shown in figures 6, 7 and 8, respectively.

In figure 6, we may suppose that X and Y are green, since otherwise there is a monochromatic triangle. If Z is blue or green, there are monochromatic triangles (AEZ and XYZ) and, similarly, if T is blue or green. If Z and T are both red, then TZF is a red triangle. In figure 7, if X and Z are blue or red it is easy to find a monochromatic triangle, so we suppose that X and Z are green. According now as Y is blue, green or red, we have the monochromatic triangle AYE, XYZ or HYF. The argument in figure 8 is similar to that in figure 7.

Our second case is when no such line segment PQ exists. Thus every segment whose endpoints are the same colour has its midpoint also in that colour. Consider two distinct points A and E (our labelling is independent of the previous labelling) with the same colour, say green. In figure 9, $AB = BC = CD = DE$. Then C, B and D are also coloured green. If any of the points F, H, J and K is green, then we have a monochromatic triangle. Suppose that none of them is green and take F to be red. Then H must be blue, J red and K blue because of our condition. If P is green this gives the green triangle ADP, if P is red we have the red triangle FJP and if P is blue we have the blue triangle HKP.

I have not managed to prove the three-dimensional analogue of problem 2. If we use the same two cases as in problem 1, the second case is easy. The first case, however, produces many interesting problems. Perhaps readers could help.

Factorizing Large Numbers

IAN STEWART, *University of Warwick*

Ian Stewart is a Reader in Mathematics at the University of Warwick. He has taught in Germany, New Zealand and the United States. His research interests are in singularity theory and non-linear dynamics. His applied colleagues think he is a pure mathematician and his pure colleagues think he is an applied mathematician. He writes a lot, mostly about mathematics or computing but also science fiction: his more unorthodox efforts include three comic books about mathematics, published in French. He is European editor of the *Mathematical Intelligencer*.

It's not easy to factorize large numbers into primes. In 1903 F. N. Cole astonished the American Mathematical Society when he found the factorization $193\,707\,721 \times 761\,838\,257\,287$ of the 67th Mersenne number $2^{67}-1$. He said it took him 'three years of Sundays'. And Mersenne numbers, those of the form 2^n-1, have a special form that makes factorization easier than usual. Nowadays we can do a lot better than Mr Cole. Using a computer, the largest Mersenne number yet factorized (reference 1) is $2^{257}-1$, which equals

$$535\,006\,138\,814\,359 \times$$
$$1155\,685\,395\,246\,619\,182\,673\,003 \times$$
$$374\,550\,598\,501\,810\,936\,581\,776\,630\,096\,313\,181\,393.$$

Whew!

Of course, 'using a computer'. Those things are whizzes at calculating. Isn't it easy to factorize large numbers on a computer?

Surprisingly, no. For example, in 1984 the Sandia National Laboratories in Albuquerque, New Mexico issued a press release: *Mathematical Milestone—record-setting push on the factoring frontier*. The number concerned had just 69 digits. The work was a milestone because that number had no special features making it easy to factorize. It was the largest such number to be factorized by a general-purpose method.

Today, the best methods can guarantee to factorize numbers of up to about 80 digits, but no further. That doesn't mean it's impossible for larger numbers; just that you need plenty of luck. The difficulty of factorizing numbers is very irregular and depends on a lot of things that are more subtle than mere size.

Oddly enough, it is much easier to *test* a number to see whether or not it is prime (reference 2). Given a fast computer, such as a Cray X-MP, it is currently feasible to decide whether or not a 200-digit number is prime. But if the computer says that it is not prime, the method does not tell us what the prime factors are—and we'd need to be lucky to find them! *Factorization is harder than primality testing.*

Why is factorization so hard?

The first difficulty is that there is very little pattern to the way numbers factorize. For example,

$$120 = 2^3\times3\times5, \quad 121 = 11^2, \quad 122 = 2\times61, \quad 123 = 3\times41, \quad \ldots.$$

There's no connection between the factors of one number and those of numbers nearby.

It's also important not to underestimate 'large' numbers. A number such as 10 000 000 000 000 000 001 is easy enough to write down, but very hard to think about. For example, the obvious method of factorizing a number N is to try possible divisors 2, 3, 5, 7, ... in turn. It is enough to go up to $\sqrt{N}$ (because, if $N = mn$, then either m or n is at most $\sqrt{N}$). If N has, say, 100 digits, then $\sqrt{N}$ has 50 digits, and the number of trials is so large that the computation is likely to take longer than the lifetime of the universe! Indeed, billions of universes might be born and die before the computer found the answer.

There are short cuts, of course: you can rule out the divisor 2 if the final digit is odd, and 5 if it not 0 or 5, for example; and if the sum of digits is divisible by 3, so is the number. But the effect of using tricks like this, to remove a few possible divisors from the front of the list, is swamped by the sheer size of the rest. So something more clever than brute force is needed and that's where the mathematics gets interesting.

Here's an example where a little extra knowledge goes a long way. The number 68 718 821 377 looks hard to factorize, but if you know that it equals $327\,679^2 - 196\,608^2$ you can use the formula $m^2-n^2 = (m+n)(m-n)$ to find the factors 524 287 and 131 071. But you aren't usually that lucky.

There are subtler methods. For example, if a number N can be written as a sum of two squares in two different ways, say

$$N = a^2+b^2 = c^2+d^2,$$

then

$$N = \frac{(ac+bd)(ac-bd)}{(a+d)(a-d)}$$

and, after cancelling all factors in the denominator, this is *a product of two integers*. For example,

$$1000\,009 = 235^2+972^2 = 1000^2+3^2,$$

whence

$$1000\,009 = \frac{237\,916 \times 232\,084}{232 \times 238} = 293 \times 3413.$$

One popular practical method is J. M. Pollard's $(p-1)$-*method*. This is a little complicated to explain (see reference 1) but is based on another number-theoretic fact. If p is a prime divisor of N and $p-1$ divides a number Q, then p divides $a^Q - 1$ for any a not divisible by p and, hence, divides the highest common factor of N and $a^Q - 1$. Highest common factors can be found very quickly, *without* resolving the numbers into primes, by a method called the *Euclidean algorithm*. The idea is to try special test numbers Q that are products of lots of small factors. If N has a prime divisor p such that $p-1$ is a product of lots of small factors, then $p-1$ will divide a suitable Q, so such a p will rapidly be found.

For example, this method found the factor $p = 267\,009\,173\,510\,848\,737$ of $N = 3^{136}+1$. Here

$$p-1 = 2^7 \times 3^2 \times 7^2 \times 17^2 \times 19 \times 569 \times 631 \times 23\,993$$

and all its factors are very small in comparison to the original number N. A variant, the $(p+1)$-method, has been used to find the factor $225\,974\,065\,503\,889$ of $10^{102}+1$.

There are two basic kinds of factorization method. The first kind consists of 'general' methods that do not depend on the sizes of the factors being sought. The continued fraction method and the quadratic sieve (reference 1) are examples, but I don't want to go into details about these. The second kind finds 'small' factors, or factors with special features. Examples of this type are trial division, the $(p-1)$- and $(p+1)$-methods, another method due to Pollard known as the ρ method and a new method recently invented by Hendrik Lenstra of the University of Amsterdam.

Lenstra's method (reference 3) is interesting because it draws its inspiration from a branch of mathematics that has not, until now, been associated with prime factorization, namely, geometry. In the nineteenth century, geometers made extensive studies of 'elliptic curves', rather wiggly curves in the plane with equations like $y^2 = x^3 - 49x + 101$. They found deep connections with number theory and the theory of complex functions. Lenstra realised that some of the theory of elliptic curves could be used to produce a variation on Pollard's $(p-1)$-method. But now, instead of $p-1$ having lots of small factors, it is enough that $p-t$ should have small factors for a whole set of 'magic numbers' t.

This greatly increases the chance of the method succeeding. The idea is to pick one particular t, corresponding to one particular elliptic curve, and hunt for a factor. If that choice of t doesn't find a factor quickly, then select a new value of t, corresponding to a new elliptic curve, and try again. And so on. To *guarantee* a factorization you may need a long calculation, but often you strike it rich quickly.

Until Lenstra got his idea, nobody had suspected any connections of this kind between geometry, number theory and efficient methods for prime factorization. It is a dramatic example of how a new idea can transform a piece of pure mathematics, studied for its own interest, into something with useful applications.

Even Lenstra's method needs to be improved if we want to factorize large numbers really quickly. Using all known methods in combination, the chances of factorizing a randomly chosen 250-digit number are about 20%. Does there exist a really efficient method for finding the prime factors of large numbers? The question is wide open.

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Interesting numbers

We shall prove the following (interesting?) proposition. By 'natural numbers' we mean, of course, the numbers $1, 2, 3, \ldots$.

Proposition. All natural numbers are interesting.

Proof. Suppose that there exist uninteresting natural numbers. These can be arranged in order, say $b_1 < b_2 < b_3 < \ldots$. Now b_1 is the smallest uninteresting natural number. But this property makes b_1 interesting. Contradiction!

Notes

(*a*) This proof is a simple example of proof by contradiction, or *reductio ad absurdum*. One of my students responded (quite correctly!) with: 'That's ridiculous'.

(*b*) Since the proof depends on the property of the natural numbers that every non-empty subset has a least member, it does not extend to the real numbers. Perhaps some real numbers are not interesting. If so, which?

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The Luck of the Draw

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Introduction

Randomisation has long been used in society as a means of ensuring that duties and rewards are fairly allocated. One common use of randomisation is to list individuals or groups in an arbitrary order, especially in cases where this ordering may have important consequences. The selection of a team to bat first in a cricket match, the barrier draw for a horse race, and the Draft Lottery used in the USA in the early 1970s to select conscripts for service in the Vietnam war are examples of this.

Another example is the ordering of candidates on a ballot paper. Under most voting systems, the candidate or group of candidates listed first on a ballot paper may have a considerable advantage over others. Randomisation provides a method of ordering which is fair (and seen to be fair): each candidate has an equal chance of obtaining the coveted first position on the ballot paper.

The Commonwealth Electoral Act in Australia has required, since 1949, that the order of candidates on the Senate ballot paper should be determined by a 'random draw'. However, in the 1975 election this draw appeared to show a systematic favouring of one particular party. We examine how such non-randomness should be measured, what might have caused it, and whether a different ordering of candidates would have led to a different outcome.

The 1975 Senate ballot draw

Voting in Australian elections is compulsory. To record a valid vote, a voter must place a number against *every* candidate on the ballot paper, indicating his order of preference. On the Senate ballot paper, the candidates are arranged in groups, identified (prior to 1984) only by the letters $A, B, C, \ldots$. Each group corresponds to a different political party; the order of the candidates within each group is chosen by that party.

It is not unusual for the Senate ballot paper to comprise a total of 50 or more candidates in up to 10 groups. To ensure that their vote is valid, most voters follow the 'how-to-vote' card of their preferred party. But a few voters fill in their ballot paper sequentially, column by column from left to right. It is this 'donkey vote' which makes the *order* of the groups on the ballot paper important.

In November 1975, a continuing political crisis in Australia was resolved by the Governor-General dismissing the Australian Labour Party (ALP) Government and appointing the leader of the Liberal Party (LP)/National Country Party (NCP) coalition as caretaker Prime Minister. A simultaneous election for both Houses of Parliament was held on 13 December 1975. There were 10 Senators to be elected in each of the six states and two in each of the territories (Australian Capital Territory and Northern Territory). Of interest to us is the outcome of the random draw for the eight Senate ballot papers, summarised in table 1.

Table 1

Summary of outcome of draw for positions on 1975 Senate ballot paper

State	Position of LP/NCP	Position of ALP	Number of groups	Probability E_1	Probability E_2	Probability E_3
New South Wales	2	8	10	$\frac{1}{2}$	$\frac{17}{90}$	$\frac{1}{10}$
Victoria	1	6	8	$\frac{1}{2}$	$\frac{13}{56}$	$\frac{1}{8}$
Queensland	2	6	7	$\frac{1}{2}$	$\frac{11}{42}$	$\frac{1}{7}$
S. Australia	1	3	9	$\frac{1}{2}$	$\frac{5}{24}$	$\frac{1}{9}$
W. Australia†	7, 1	10	11	$\frac{1}{3}$	$\frac{9}{55}$	$\frac{1}{11}$
Tasmania†	1, 2	5	6	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{6}$
Australian Capital Territory	1	2	4	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{4}$
Northern Territory	1	3	3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$

Now imagine you are an ardent ALP supporter. You notice that in every state and territory, both members of the LP/NCP coalition precede the ALP. Not only that: at least one member of the coalition is in first or second position in every case. All eight allegedly random draws appear to favour the LP/NCP coalition. You ask yourself, how could this have happened? Was the draw for ballot positions truly random?

Probability calculations can help us answer this question. Let us *assume*, for the moment, that the draw was truly random: with N groups, all $N!$ permutations of the groups are equally likely. We then compute the

†In Western Australia and Tasmania, the LP and NCP had separate groups of candidates, but directed their preferences to each other. In all other states, the LP/NCP candidates were in a single group.

probability, under this assumption, of obtaining an ordering of groups as extreme as that actually obtained, or even more so. If this 'significance probability' P^* is reasonably large, we have no reason to doubt our assumption. But if this probability is very small, we may suspect our assumption is not correct.

For future reference, let us define, for each ballot paper, events E_1, E_2 and E_3 as follows.

E_1: both LP/NCP groups precede the ALP group,

E_2: at least one of the LP/NCP groups is in first or second position, and both precede the ALP group,

E_3: one of the LP/NCP groups is in first position, and both precede the ALP group.

It is straightforward to compute the probabilities of these events, assuming a truly random draw (see Problem 20.12 on Page 95). These probabilities are listed in table 1. All eight draws took place simultaneously but in different cities across Australia, presumably independently of one another. We now combine the probabilities in table 1, under this assumption.

Events E_1 and E_2 occur in all eight cases. The probability that event E_1 occurs in all eight cases is given by $(\frac{1}{2})^6(\frac{1}{3})^2 = \frac{1}{576}$. Similarly, the probability that event E_2 occurs in all eight cases is 1 in 46 000, approximately.

Looking more closely at table 1, we see an even more extreme outcome: E_3 has occurred in six out of the eight draws. The probability of E_2 occurring in New South Wales and Queensland while E_3 occurs in all other states or territories is 1 in 1151 000, approximately.

The significance probability P^* is widely used to measure how consistent is a given set of data with a specified hypothesis (in our case, of eight independent random draws). The smaller the value of P^*, the stronger is our evidence against the hypothesis. In most scientific research, a hypothesis is regarded as untenable if the computed P^* is less than 0.01%.

Our final P^*-value of 1 in 1151 000 provides *apparently* overwhelming evidence against our hypothesis. But there are two fallacies in our argument. First, we would have been equally sceptical of the hypothesis if event E_3 had occurred in *any* six of the eight draws. We would have been even more sceptical if E_3 had occurred *more* than six times. So a more appropriate P^*-value is the probability P_0^* that event E_3 occurs in *at least* six of the eight draws while E_2 occurs in all the rest. This turns out to be 1 in 39 500 (see Problem 20.13), still extremely small.

A more serious fallacy is that we have decided what we mean by 'extreme' only *after* looking at the data. Ask yourself: if you had not seen table 1 but had been told only that there was some doubt about the randomisation method, would you have thought of events E_1, E_2 or E_3 as being relevant?

In almost any random permutation you will find some sort of systematic pattern, if you look hard enough. Because it is one of thousands of possible patterns, such a pattern will generally have a very low *a priori* probability. If you generate a series of random permutations, you are likely to find a *different* apparent pattern in each. By computing the significance probability as if this observed pattern were specified in advance, you will most likely conclude, in every case, that the permutation was *not* generated at random.

To avoid this fallacy, we must decide *before* looking at the data what we mean by 'extreme'. To do this, we ask two questions. What sort of non-randomness matters in practice? If the ballot draws were not truly random, what sort of non-randomness would we expect to find?

We look at these questions in turn.

What really matters

In Senate elections voters' preferences are allocated according to the proportional representation method, treating each state or territory as a multi-member constituency. The first step is to compute the 'quota': if k candidates are to be elected, this is the total number of valid votes divided by $k+1$. All candidates with first-preference votes in excess of the quota are automatically elected, and their surplus votes are transferred to the remaining candidates, in proportion to voters' preferences. This process is repeated until no remaining candidate has a quota. Then the candidate with the *least* first-preference votes is excluded, and his votes are transferred to the remaining candidates according to voters' preferences. This process of exclusion and transfer of votes continues until the required number of candidates is elected.

Usually each of the two main political parties gains about 45% of all valid votes, and hence at least four successful candidates out of the ten vacancies. The final composition of the Senate depends on the distribution in each state of the last two quotas of transferred votes between the last three remaining candidates. Generally, these candidates belong to the three most popular parties in terms of first-preference votes. Let us denote these parties by the letters X, Y and Z.

What matters is the *relative* positions of these parties on the ballot paper. If one of these draws the first position, it naturally obtains the bonus of the donkey vote. If instead the party in first position is a minor one whose candidates are subsequently excluded, the donkey vote is transferred eventually to whichever of parties X, Y and Z precedes the other two. In either case, the donkey vote is transferred within that group to their last remaining candidate.

How big is the donkey vote? We get a rough idea by comparing the proportion of first-preference votes obtained by a minor party in different states and hence at different positions on the ballot paper. For example, in

1983 the Socialist Workers Party gained only 0.03% of first preference votes when not in first position, but 0.73% when in first, suggesting a donkey vote of 0.7%. Similar comparisons using other minor parties and other Senate elections gave consistent estimates of between 0.7% and 0.9%.

We may examine the effect of the donkey vote by computing the final distribution of votes for each of the 3! orderings of parties X, Y and Z. With a donkey vote of only 0.9% the outcome of the election can still depend on the relative order of parties X, Y and Z, even when the support for the two main parties differs by as much as 5%, as was the case in some states in 1975. Further details are given in reference 1.

What sort of non-randomness would we expect?

In each state or territory, the draw for positions on the Senate ballot paper used a variation of the familiar idea of 'drawing names out of a hat'. For each group of candidates, a slip of paper identifying the group was sealed inside an envelope. All envelopes were put into a box, which was 'shaken'. The envelopes were withdrawn one by one; the first envelope defined the first group on the ballot paper, the second envelope the second group and so on.

This procedure will produce a random draw *only* if the envelopes are well mixed before being withdrawn. This is unlikely. As a comparison, the U.S. 1970 Draft Lottery was performed by inserting 365 capsules (one for each day of the year) into a box, starting first with all capsules for January, then all for February, and so on. The box was then shaken by being carried up and down three flights of stairs. Although the capsules were cylindrical with hemispherical ends, they were still not properly mixed: those capsules put in last tended to come out first (see reference 2). Envelopes do not mix as well as capsules, and so would also tend to come out in the reverse order of insertion.

Unfortunately, we don't know how the envelopes were inserted. One possibility is that they were inserted in alphabetical order by party name. In such a case, with poor mixing the LP/NCP envelope, put in almost last, would tend to come out first. Look again at table 1! Alternatively, the envelopes may have been inserted in alphabetical order by surname of the first candidate in the group. To check either possibility, we may list the parties in alphabetical order either by party name or first candidate name, and compare each list with the order on the ballot paper, by computing rank-correlation coefficients. Negative coefficients indicate a tendency for the envelopes to come out in reverse alphabetical order. Only one of the 16 coefficients listed in table 2 is significantly different from zero, but it is positive. Our conjecture is not supported. But since we do not know how the envelopes were inserted, we cannot really work out why the 1975 draw was so extreme.

Table 2

Rank correlation coefficients: position on ballot paper *versus* alphabetical order

State	Groups	Ranked by name of party	Ranked by name of first candidate
New South Wales	10	−0.406	0.527
Victoria	8	−0.595	−0.071
Queensland	7	−0.429	0.107
S. Australia	9	0.429	0.702
W. Australia	11	0.109	0.473
Tasmania	6	0.114	0.208
Australian Capital Territory	4	0.400	0.400
Northern Territory	3	−0.500	0.500

Conclusion

The best way to assess the extent of any non-randomness in the Senate ballot draw is to compute the significance probability based on event E_1, that both LP/NCP groups precede the ALP group. This event is well-defined prior to any draws and is what matters in practice. In 1975, event E_1 occurred on all eight ballot papers, with a corresponding significance probability P^* of 1 in 576. A P^*-value as low as this generally implies that the hypothesis in question is untenable. So our calculations provide strong evidence that the 1975 draw was not truly random. Although we cannot verify it, this non-randomness was most likely caused by poor mixing of the envelopes in the box.

Since the same method of randomisation was used in all 15 Senate elections between 1949 and 1983, we might expect to find the same sort of non-randomness throughout, although perhaps not to the same extent as in 1975. Mrs A. G. Harcourt and I, in a submission to the 1983 Joint Senate Committee on Electoral Reform, looked at this possibility, essentially by noting the occurrence or otherwise of event E_1 in every state in every election. We found no evidence of non-randomness, apart from the 1975 election. What really happened in 1975 remains a mystery.

Mrs Harcourt and I recommended that the draw for the ballot positions should employ a double randomisation technique, using numbered balls in a spherical container. In this method, the groups of candidates are first listed in an arbitrary order. The first draw of numbered balls assigns a number to each group, while the second draw determines the order of the groups, now identified solely by number. For example, if the first ball drawn in the second draw is numbered 7, the first group on the ballot paper is that assigned '7' at the first draw. Our recommendation was accepted, and is incorporated in Section 106B of the Australian Electoral Act 1983. It was used for the first time in the 1984 election with success.

References and further reading

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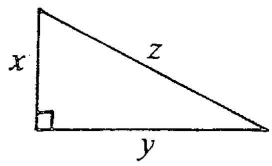
Pythagorean Triangles



A Pythagorean triangle is a right-angled triangle whose sides are of integer length. There are formulae which give the sides of all such triangles as follows:

$$x = 2kst, \qquad y = k(s^2 - t^2), \qquad z = k(s^2 + t^2),$$

where k, s and t are integers with $k > 0$, $s > t > 0$, s and t coprime and one of s and t even and the other odd. (Thus, for example, $k = 1$, $s = 2$, $t = 1$ gives $x = 4$, $y = 3$, $z = 5$ and $k = 1$, $s = 3$, $t = 2$ gives $x = 12$, $y = 5$, $z = 13$.) Here is the question: how many Pythagorean triangles are there with perimeter 180?

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Some Fascinating Formulae of Ramanujan

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A recent article in *Mathematical Spectrum* (Volume 20 Number 1, pages 1–8) considered the life and work of the great Indian mathematician Ramanujan. I have selected a number of formulae from his notebooks (see the reference) which may interest readers. In each case, I have indicated where these occur in his notebooks. The proofs are not difficult and readers are invited to prove them for themselves; it is the results themselves that I find so intriguing.

1. $(a+1)(b+1)(c+1)+(a-1)(b-1)(c-1) = 2(a+b+c+abc).$

   (Vol. 1, p. 240.)

2. $\{6m^2+(3m^3-m)\}^3+\{6m^2-(3m^3-m)\}^3 = \{6m^2(3m^2+1)\}^2.$

   (Vol. 1, p. 202.)

3. $\{x+\frac{1}{2}x^2+\frac{21}{64}x^3+\frac{31}{128}x^4+\dots\}^{11} = x^{11}+\frac{11}{2}x^{12}+\frac{1111}{64}x^{13}+\frac{111\,111}{2688}x^{14}+\dots.$

   (Vol. 1, p. 244.)

4. $2^{1/3} = \frac{5}{4}(1+\frac{24}{1000})^{1/3} = \frac{63}{50}(1+\frac{188}{1000\,000})^{-1/3}.$ (Vol. 2, p. 384.)

It is not suggested that the pattern continues in formula 3. From formulae 4, we can obtain the approximations

$$\tfrac{5}{4}(1+\tfrac{1}{3}\times\tfrac{24}{1000}) = 1.26, \qquad \tfrac{63}{50}(1-\tfrac{1}{3}\times\tfrac{188}{1000\,000}) = 1.259\,921\,04$$

to $2^{1/3}$. These compare with the approximation

$$1.259\,921\,049\,894\,873\,164\,767\,208$$

which Ramanujan himself gave.

It would be interesting to find formulae similar to formulae 4 for $5^{1/3}$, $7^{1/3}$ and so on.

Reference

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Safe Driving Speeds on Newly Surfaced Roads

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The present article was conceived when one of the authors almost had his car windscreen broken.

1. Introduction

Cheshire County Council, in common with many British local authorities recommend that drivers limit their speed to 20 m.p.h. (about 32 km/h) when travelling over roads which have been newly dressed with a bituminous binder and stone chippings. One unavoidable snag with this method of maintenance is that some surplus stones remain on the road until they are swept up some hours after laying. The advisory limit is intended to avoid the situation where stones are caused to fly into the path of other vehicles and so cause damage to paintwork and even breakage of windscreens. In this article we present a simple mathematical model which describes some aspects of this situation, and we argue that a 'safe' speed is not significantly lower than that recommended.

2. Assumptions

1. The wheel does not skid. Due to a variety of factors such as the nature of the tyre, the tackiness of the binder, the weight and size of the vehicle, stones are thrown up at random speeds not greater than the speed of the vehicle. We therefore assume the simplest case in which stones are projected with the speed of the vehicle.
2. In practice it is observed that the direction of projection is random, but in this simple model we assume that a stone remains in the vertical plane containing the direction of motion of the vehicle.
3. Vehicles travel at constant speed, their separation from the vehicle in front being not less than that recommended in the British Highway Code.
4. Air resistance is negligible.
5. Drivers are not unduly worried by stones hitting the front bumper, grill, etc.

3. The theory of the bounding parabola

Suppose a projectile is fired from a point O with fixed speed V in any direction. Which regions of the plane of projection are accessible? Elementary work shows that, if the particle is fired at an angle θ to the horizontal, the equation of the trajectory of the projectile is

$$y = x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta), \tag{1}$$

where g is the gravitational acceleration. A point (X, Y) is then accessible if we can find an angle or angles satisfying

$$\tan^2\theta - \frac{2V^2}{gX}\tan\theta + \frac{2V^2 Y}{gX^2} + 1 = 0, \tag{2}$$

which is a rearrangement of (1). If this quadratic for $\tan\theta$ has complex roots then the point (X, Y) is inaccessible. If there are two real roots then the point (X, Y) is accessible with two angles of projection. The condition for equal roots gives the dividing curve between the two regions, i.e.

$$x^2 = \frac{2V^2}{g}\left(\frac{V^2}{2g} - y\right). \tag{3}$$

This curve is known as the *bounding parabola* (figure 1).

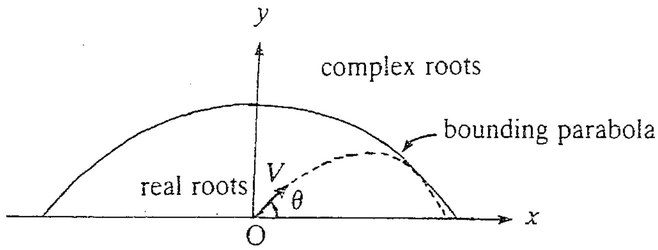

Figure 1

4. Solution

A stone is projected from the origin O at time $t = 0$ and moves along a typical trajectory as shown in figure 2. The front of the bonnet of the following car has coordinates (S, h). We will argue as follows. There is a danger of the bonnet, windscreen or top of the car being hit by the stone if FC has crossed the bounding parabola, i.e. if FC is within the bounding parabola at time t_{E} = time of flight of the stone from O to E.

After time t_{E} the position of FC is

$$S - Vt_{\mathrm{E}}$$

and so the vehicle is in danger of being hit if

$$S - Vt_{\mathrm{E}} < X_{\mathrm{E}}. \tag{4}$$

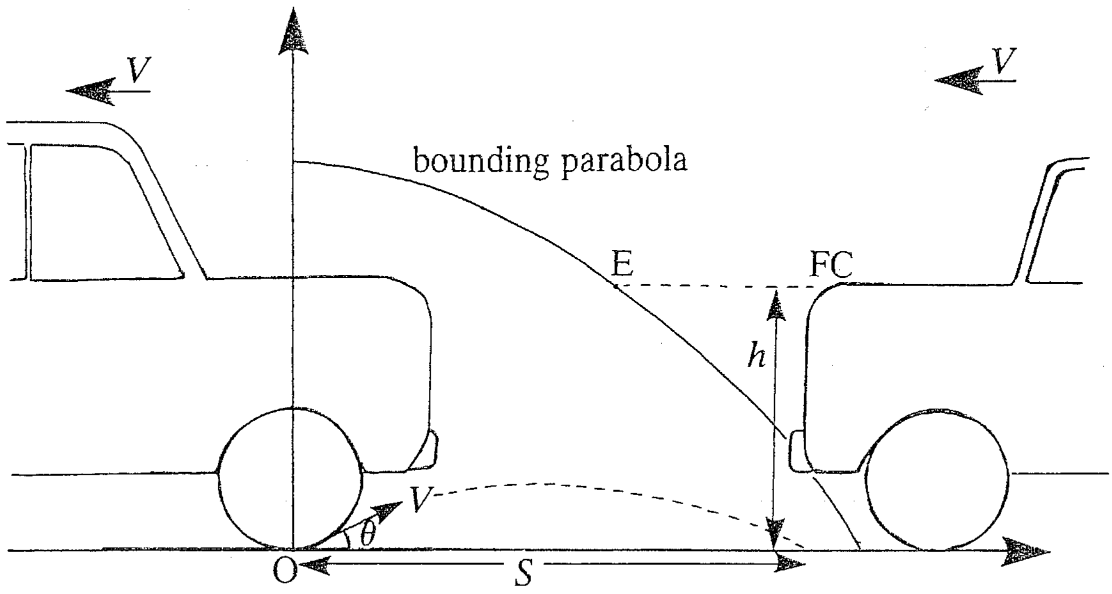

Figure 2

An expression for S is now required and experience shows this to be related to the speed of travel V. The Highway Code contains shortest stopping distances at a range of speeds, and we have assumed that drivers are at least adhering to these recommendations. An empirical model has been developed (see reference 1) which fits the Highway Code data and provides an equation relating the separation distance to speed of travel. This equation can be written

$$S = 0.682V + 0.076V^2, \tag{5}$$

where S is in metres and V is in metres per second.

Since E lies on the bounding parabola its x-coordinate is obtained from (3) as

$$X_E = \sqrt{a(a-2h)}, \quad \text{where } a = \frac{V^2}{g}. \tag{6}$$

Furthermore, the equal roots of (2) are

$$\tan\theta = \frac{a}{x}, \tag{7}$$

whence

$$\cos\theta = \frac{x}{\sqrt{a^2+x^2}}$$

and, since $x = Vt\cos\theta$, we obtain

$$t_E = \frac{\sqrt{a^2+X_E^2}}{V}. \tag{8}$$

Combining (4), (6) and (8), we have the result that the following vehicle is in danger of being hit by the stone if

$$S-\sqrt{2a(a-h)} < \sqrt{a(a-2h)}. \tag{9}$$

Therefore, from (5),

$$\sqrt{g}(0.682+0.076V) \leqslant \sqrt{\frac{V^2}{g}-2h+\sqrt{2\left(\frac{V^2}{g}-h\right)}}. \tag{10}$$

5. Interpretation of the result

If the equality given by (10) is solved numerically for $h = 0.75\,\mathrm{m}$ (typical of many popular makes of car) we find that, if a driver exceeds 5.531 m/s (12.372 m.p.h.) he is in danger of being hit.

It might be argued that, even though the car remains outside the bounding parabola at time t_E, at some later time when it is within the bounding parabola it could still be hit by a stone which has taken time $t_h > t_\mathrm{E}$ to reach height h. It can easily be shown that the time taken for a stone to rise and then fall to height h is

$$t_h = \frac{V\sin\theta+\sqrt{V^2\sin^2\theta-2gh}}{g}, \tag{11}$$

which is an increasing function of θ. Taking $V = 5.531\,\mathrm{m/s}$ and $h = 0.75\,\mathrm{m}$, equations (6) and (7) imply that θ_E, the angle of projection which takes a stone to the point E on the bounding parabola, is about 54°. Consequently a greater time to reach a height h must arise from any angle of projection greater than this.

However, observation shows that the wheel arches and rubber flaps of the vehicle constrain angles to be less than 54°. We therefore conclude that if vehicles travel at 5.531 m/s with a separation of at least that recommended by the Highway Code then there is no danger of being hit. The value obtained of 5.531 m/s is rather lower than the advisory limit recommended by local authorities, but in practice drivers are reluctant to slow down to 20 m.p.h. and would almost certainly ignore requests to slow down even further.

Suppose now that we no longer assume a Highway Code separation. Consider a stone projected from O as before as in figure 2. The time it takes to reach height h as shown is given by (11). Its x-coordinate is then

$$x_h = V\cos\theta\, t_h. \tag{12}$$

If the following car has separation S_1 given by

$$S_1 = x_h + Vt_h \tag{13}$$

then a stone projected at angle θ will reach the bonnet at time t_h. Our procedure is as follows. Choose V and calculate the smallest angle $\theta_{\min}$ that can cause a stone to reach height h at this speed, and then gradually increase

θ from this smallest value up to $\frac{1}{2}\pi$, at each stage calculating S_1 from (13). Simultaneously we calculate

$$S = 0.682V + 0.076V^2, \tag{14}$$

the Highway Code separation. We then gradually increase V and repeat the whole procedure. It soon becomes apparent that at very low speeds $S_1 < S$ for all angles of projection. This means that if a vehicle travels at the recommended separation it must be outside the range of the furthest stone to reach height h. Eventually, as V is increased, there comes a point when $S_1 > S$ for some angles of projection, meaning that a vehicle travelling at the recommended separation can be hit on the front of the bonnet by a stone at some angle. This searching problem is easily handled on a microcomputer using equations (13) and (14). For $h = 0.75$ m it reveals that, if V is less than about 5.17 m/s a car travelling at the recommended separation is safe. Above this speed there is the possibility of some stone hitting the car.

Again, a value of 5.17 m/s is unrealistic in practice, but we can argue that, instead of allowing θ to vary from θ_{min} to $\frac{1}{2}\pi$, we constrain θ to lie between θ_{min} and 35°, say. The searching routine so modified yields a speed of approximately 8 m/s or 18 m.p.h.. Measurement of the overhang of the boot on many makes of car leads us to the conclusion that 35° is a maximum value, stones projected at a higher angle hitting the underside of the vehicle. Many saloon-type cars have a much smaller maximum. (We are of course neglecting tractors, etc.)

This model, into which we have incorporated some measure of reality, leads us to the conclusion that, for most types of vehicle travelling at the Highway Code recommended separation, a safe driving speed is not significantly lower than that recommended by the local authorities.

6. Conclusions

We have described only one aspect of the problem, i.e. when stones are thrown directly behind the vehicle. As we have already stated, in practice stones are not confined to the vertical plane containing the direction of motion of the vehicle, but are scattered in a variety of directions, for example into the path of an overtaking vehicle, or a vehicle travelling in the opposite direction; nor is the speed of the stone necessarily the same as the speed of the vehicle. These additional aspects provide scope for further work. However, we believe that future work will reduce the maximum speed recommended for safety. Study of these additional aspects could provide useful material for students' projects.

Reference

1. D. N. Burghes, I. Huntley and J. MacDonald. *Applying Mathematics* (Ellis Horwood, Chichester, 1982).

Little Julie's Confusing Calculations

Those of you who have seen 'Guys and Dolls' will remember that Big Jule, the high-rolling dice player from Chicago, has a pair of dice whose spots have worn off—but he remembers where they were! My neighbour, Little Julie, has a simple four-function calculator that she has used so much that the signs have worn off the four operation keys. Unfortunately she can't remember which key is which.

'Look at this equation,' said Little Julie last Saturday morning. '$a*a*a*a*a = a$.'

'What are those symbols?' I demanded.

'Oh, those are the operations. Since I can't remember which is which on my calculator, I use the same blob for all four operations. Of course, I used each of the keys just once in my calculation.'

'But then you must get different answers depending on which order you push the buttons.'

'Maybe, but I tried a value of a and it worked for all 24 orders of the operations!'

After some calculation, I found that she was right. What was Little Julie's value of a ? How many of the 24 cases have just this one solution?

A week later, Little Julie showed me her microcomputer. Needless to say, she had worn off the keys on her keyboard. Further, her video had contracted the same disease and printed $*$ for each of the operations. Now computer languages operate differently from calculators. For example, $1+2\times3\div4-5$ is evaluated as $[\{(1+2)\times3\}\div4]-5 = 9/4-5 = -11/4$ on a calculator, but it is evaluated as $1+(2\times3/4)-5 = -5/2$ on a computer. That is, computers use the conventional algebraic precedence rules (BODMAS), but (most) calculators do immediate operations.

However, once again, Little Julie had tried all 24 orders of the four operations in her equation and found that her value of a worked in all cases! What was her value of a this time and how many of the 24 cases have just this one solution?

DAVID SINGMASTER
Polytechnic of the South Bank

Is n^2-n+41 prime for all natural numbers n?

Computer Column

MIKE PIFF

Van der Pol's equation

In the last column (Volume 20 Number 2), we looked at the effect a predator and prey have on one another's populations and saw that, in certain conditions, the populations would oscillate wildly. As the prey flourishes, so the predator increases and causes the prey to decay, and so it likewise decays, and so on. In one model, these oscillations were damped, so that the two populations reached a steady state.

This time, we shall look at a model, not of a damped oscillation, but of a set of equations which always dampen down to the same oscillation, no matter what initial conditions we start them at. Suppose we have the differential equation

$$\ddot{x} = -x + f'(x)\dot{x},$$

where $f(x)$ is a known function of x. In our example, we shall take $f(x) = x - kx^3$. Putting $\dot{y} = -x$, we can write our equation as a pair of equations

$$\dot{x} = y + f(x), \qquad \dot{y} = -x$$

and then try to solve these numerically. In fact, I have shifted the origin so that the program in Volume 20 Number 2 can easily be modified to solve these. Delete lines 10, 15, 40 and 60, and change lines 110, 260 and 270 to

```
110 DT=5E-2
260 DEF FNF(X,Y)=X+Y-1000-(X-600)^3/40000
270 DEF FNG(X,Y)=600-X
```

Because the final oscillation is so stable, we can use quite a large time step `DT` in approximating the equation.

Try solving other differential equations with this program. You may have to re-centre the origin to draw the solutions. For instance, any differential equation of the form $\ddot{x} = f(x, \dot{x})$ can be written as

$$\dot{x} = y, \qquad \dot{y} = f(x, y),$$

and so

$$FNF(X, Y) = Y, \qquad FNG(X, Y) = f(x, y).$$

Be careful, though, to use a very small `DT`, until you are sure that making it larger does not cause the shape of your solutions to deteriorate.

Letter to the Editor

Dear Editor,

On sums of two powers

In *Mathematical Spectrum* Volume 19 Number 2 Joseph McLean reported on his experiments with sums of two powers.

There is a well-known lemma in the theory of numbers about the number of different representations of an integer N as a sum of two squares:

The natural number N with the prime-factor representation

$$N = 2^e p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$$

has no primitive representation $N = x^2 + y^2$ if $e > 1$ or if $p_i \equiv 3 \pmod 4$ for some i. N has 2^{s-1} different primitive representations if $e = 0$ or $e = 1$ and $p_i \equiv 1 \pmod 4$ for all i. See the reference, chapter 3. (A representation $N = x^2 + y^2$ is called primitive if the highest common divisor of x and y is 1.)

From this lemma one can immediately obtain the number of all primitive representations and also the number of all representations. The table shows some examples.

Number	s	Primitive representations	All representations
13×17	2	2	2
$5\times13\times17$	3	4	4
$5\times5\times13\times17$	3	4	6
$5\times5\times5\times13\times13$	2	2	6
$5\times13\times17\times29$	4	8	8
$5\times5\times5\times13\times17\times29$	4	8	16
$5\times13\times17\times29\times37$	5	16	16
5^n	1	1	$[\frac{1}{2}(n+1)]$

I shall now give two applications of these 'sums of powers'. Here are two problems:

1. Place different natural numbers on the vertices of a rectangle, chosen so that the sums of the integers on the extremities of each side of the rectangle will be squares (cubes). Find the solution with the smallest sum of all four integers.

Solution. (*a*) For squares: Denote the numbers on the vertices by a, b, c and d and put

$$N = a+b+c+d = (a+b)+(c+d) = (a+d)+(b+c).$$

Now choose N to be the smallest number with two representations as a sum of two squares:

$$N = 85 = 4+81 = 36+49.$$

Put

$$a+b = 4, \qquad a+d = 36, \qquad b+c = 49, \qquad c+d = 81.$$

These equations are satisfied by $a = 1$, $b = 3$, $c = 46$ and $d = 35$.

(*b*) For cubes: Take

$$N = 4104 = 2^3+16^3 = 9^3+15^3$$
$$= 8+4096 = 729+3375.$$

We can take $a = 1$, $b = 7$, $c = 728$ and $d = 3368$ and there are more possible solutions.

2. Place different integers (natural numbers) on the vertices of a tetrahedron so that the integers on each edge sum to a square (cube).

Solution. Let the integers be a, b, c and d and take a number N that is the sum of two squares in three ways. If we choose the smallest possible integer, $N = 325$, we obtain

$$N = 324+1 = 289+36 = 225+100.$$

Put

$$a+b = 1, \qquad a+c = 36, \qquad a+d = 100,$$
$$b+c = 225, \qquad b+d = 289, \qquad c+d = 324.$$

These equations are satisfied by

$$a = -94, \qquad b = 95, \qquad c = 130, \qquad d = 194.$$

If we want only natural numbers, we may choose for example

$$N = 5785 = 69^2+32^2 = 67^2+36^2 = 59^2+48^2$$
$$= 4761+1024 = 4489+1296 = 3481+2304$$

and solve the system of equations

$$a+b = 4761, \qquad a+c = 4489, \qquad a+d = 3481,$$
$$b+c = 2304, \qquad b+d = 1296, \qquad c+d = 1024.$$

These are satisfied by

$$a = 3473, \qquad b = 1288, \qquad c = 1016, \qquad d = 8.$$

Reference

H. Gupta. *Selected Topics in the Theory of Numbers* (Abacus Press, Cambridge, Mass., 1980).

Yours sincerely,
HANS ENGELHAUPT
(Franz-Ludwig-Gymnasium,
Bamberg,
W. Germany.)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

20.9. (Submitted by Stanley Rabinowitz, Massachusetts, USA)
Prove that $15^n - 2^{3n+1} + 1$ is divisible by 98 for all positive integers n.

20.10. (Submitted by Gregory Economides, Lower Sixth, Royal Grammar School, Newcastle upon Tyne)
Let x_1, x_2, x_3 and x_4 be positive real numbers. Show that

$$\frac{x_1+x_3}{x_1+x_2} + \frac{x_2+x_4}{x_2+x_3} + \frac{x_3+x_1}{x_3+x_4} + \frac{x_4+x_2}{x_4+x_1} \geqslant 4,$$

and determine when equality occurs.

20.11. (Submitted by Seung-Jin Bang, Seoul, Korea)
Let

$$F(x, y, z) = \frac{1-z}{xz+y}(1-\mathrm{e}^{-(xz+y)}) + \frac{1-z}{xz+y-x}(\mathrm{e}^{-(xz+y)} - \mathrm{e}^{-x}) + \frac{1}{x}(1-\mathrm{e}^{-x}),$$

where $x, y > 0$, $0 < z < 1$ and $xz+y \neq x$. Show that $F(x, y, z)$ is positive.

20.12. (Submitted by R. M. Clark, Monash University)
A set of $n > 3$ objects is to be arranged in random order. One object is labelled A, one L and one N; the rest are unlabelled. Assuming that all $n!$ orderings are equally likely, find the probabilities of the events

E_2: at least one of either L or N is in first or second position, and both L and N precede A;

E_3: either L or N is in first position, and both L and N precede A.

Repeat the calculations when there are just two labelled objects, one labelled A and one labelled L/N.
(These probabilities are relevant to Table 1 on page 79.)

20.13 (Submitted by R. M. Clark, Monash University)
Show that the ordinary binomial distribution cannot be used to compute P_0^* as defined on page 80.

Solutions to Problems in Volume 20 Number 1

20.1. Three primes of the form p, $p+2$, $p+4$ are said to form a 'prime triple'. How many prime triples are there?

Solution by Amites Sarkar (Winchester College)

Since p, $p+2$ and $p+4$ are all incongruent modulo 3, one of them is divisible by 3. Since 3 is the only prime which is divisible by 3, there exists but one such triple, $\{3, 5, 7\}$.

Also solved by Adrian Hill (Trinity College, Cambridge), Martin Anthony (University of Glasgow), Bramwell Hayes (Penwith Sixth Form College, Penzance), Nicholas O'Shea (Gresham's School, Holt), Peter Bevin (King Edward's School, Birmingham), Keith Gordon (Edgware), John Holcomb (St. Bonaventure University, USA) and Eddie Cheng (Memorial University of Newfoundland).

20.2 Evaluate the integral

$$\int_0^x \frac{t^{n-1}}{\sqrt{t^{2n}+a}}\,\mathrm{d}t,$$

where a is a positive real number and n is a positive integer.

Solution by Martin Anthony

Denote the given integral by I and apply the substitution $t^n = \sqrt{a}\tan\theta$. Then

$$\begin{aligned} I &= \frac{1}{n}\int_0^{\tan^{-1}(x^n/\sqrt{a})} \frac{\sqrt{a}\sec^2\theta}{\sqrt{a}\sec\theta}\,\mathrm{d}\theta \\ &= \frac{1}{n}[\ln|\sec\theta+\tan\theta|]_0^{\tan^{-1}(x^n/\sqrt{a})} \\ &= \frac{1}{n}\ln(x^n+\sqrt{a+x^{2n}})-\frac{1}{2n}\ln a. \end{aligned}$$

[An alternative substitution is $t^n = \sqrt{a}\sinh\theta$, to give $I = (1/n)\sinh^{-1}(x^n/\sqrt{a})$.]

Also solved by Adrian Hill, Amites Sarkar, Kate Croudace (Wymondham College), Nicholas O'Shea, Keith Gordon, John Holcomb and Eddie Cheng.

20.3 A curve lies in the first quadrant; the origin lies on the curve, as does the point P. R_1 is the region enclosed by the arc OP, the vertical through P and the x-axis; R_2 is the region enclosed by the arc OP, the horizontal through P and the y-axis, and n is a positive constant. Find an equation for the curve if, for all positions of P,

(a) the area of R_2 is n times the area of R_1;

(b) the volume swept out when R_2 is rotated through one revolution about the y-axis is n times the volume swept out when R_1 is rotated through one revolution about the x-axis.

Solution by Adrian Hill

(a)
$$n\int_0^x f(u)\,\mathrm{d}u = \int_0^{f(x)} f^{-1}(v)\,\mathrm{d}v.$$

We differentiate this to give

$$nf(x) = xf'(x) \Rightarrow \frac{n}{x} = \frac{f'(x)}{f(x)}.$$

Integrating this with respect to x gives

$$n\log x+\log c = \log f(x) \Rightarrow f(x) = cx^n,$$

where c is a constant.

(b)
$$n\pi\int_0^x f(u)^2\,\mathrm{d}u = \pi\int_0^{f(x)} f^{-1}(v)^2\,\mathrm{d}v.$$

We differentiate this to give

$$nf(x)^2 = x^2f'(x) \Rightarrow \frac{n}{x^2} = \frac{f'(x)}{f(x)^2}.$$

Integrating this with respect to x gives

$$-\frac{n}{x} - c = -\frac{1}{f(x)} \Rightarrow f(x) = \frac{x}{n+cx},$$

where c is a constant.

20.4. Five raffle tickets are drawn at random without replacement from a set of 149 numbered from 1 to 149 consecutively. Find the mean and the variance of their sum.

Solution by Nicholas O'Shea

Consider choosing r tickets from n. If the ith ticket drawn has number a_i and the mean and variance are μ_r and V_r, then

$$\mu_1 = E(a_1) \quad \text{and} \quad V_1 = rE(a_1^2) - \mu_1^2.$$

It is easy to calculate $\mu_1 = (1/n)(1+2+3+\ldots+n) = \frac{1}{2}(n+1)$ and

$$V_1 = \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) - \mu_1^2 = \tfrac{1}{12}(n^2-1).$$

Now

$$\mu_r = E(a_1+a_2+\ldots+a_r) = rE(a_1) = r\mu_1 = \tfrac{1}{2}r(n+1)$$

and

$$\begin{aligned} V_r &= E(a_1+a_2+\ldots+a_r)^2 - \mu_r^2 \\ &= rE(a_1^2) + r(r-1)E(a_1a_2) - \tfrac{1}{4}r^2(n+1)^2, \end{aligned}$$

which is a quadratic in r. Now $V_r = 0$ for $r = 0$ and $r = n$ [sum definitely zero in one case and definitely $\frac{1}{2}n(n+1)$ in the other]. So $V_r = r(n-r)f(n)$ for some function $f(n)$. But

$$V_1 = \tfrac{1}{12}(n^2-1) = (n-1)f(n),$$

so

$$V_r = \tfrac{1}{12}r(n-r)(n+1).$$

With $n = 149$ and $r = 5$, we then have $\mu = 375$ and $V = 9000$.

We also received a correct solution of the mean from Bramwell Hayes. This question was set (as an optional question) in the 1986 Oxford Mathematics Finals and the only correct solution for the variance used very advanced methods. There were 28 further correct means and the other 21 Oxford Finalists who attempted the question did not even get the mean right. So the question is like Tom Lehrer's New Math—it's so simple that only a child can do it!

Reviews

The Problems of Mathematics. By IAN STEWART. Oxford University Press, 1987. Pp. ix + 257 with nine figures. £5·95 paperback, £17·50 hardback.

Ian Stewart has written a book which is both instructive and amusing, which is in itself a considerable feat. Naturally the reader who has some prior knowledge of mathematics will derive greater benefit from this book, but almost anyone who has, say, some idea of calculus and modern algebra will find that reading it will broaden his horizons and deepen his understanding, although it is probably too advanced for the average sixth-former.

The topics covered range in time from remote antiquity to the present day, as recently as 1985, and while it must be admitted that the older items have already been treated adequately by other writers, nevertheless a clear view of the unity of mathematics is afforded by presenting problems in their correct historical perspective.

After an introductory survey the author deals in turn with prime numbers, Fermat's last theorem, transcendental numbers, non-Euclidean geometry, set theory, non-standard analysis, group theory, topology of manifolds, the four-colour theorem, complex analysis, probability theory, cosmology, catastrophe theory, randomness versus determinism, fractals, quantum mechanics, theory of algorithms, computability, and the relationships between pure and applied mathematics—altogether a fascinating selection.

Valuable features of the book are a bibliography listing well over two hundred references for further reading, and a detailed index.

No book on mathematics is intended to be read at a single sitting but once started this one is hard to put down.

University College of Swansea J. G. BRENNAN

Knotted Doughnuts and other Mathematical Entertainments. By MARTIN GARDNER. W. H. Freeman and Company, New York, 1986. Pp. xiii + 278. £10·95 paperback.

For twenty-five years, from 1956 to 1981, Martin Gardner wrote a monthly column, called 'Mathematical Games' for the magazine *Scientific American*. Over the years he has published collections of these articles on mathematical recreations in book form. This is the eleventh such collection and contains twenty of the columns from the years 1972 to 1974 and one from 1981.

Where do mathematical recreations fit into the general spectrum of mathematics? If we compare mastering mathematics to fighting with a cobra then the day-to-day learning in the classroom may be likened to the situation of the sacrificial goat, tied so that it cannot escape, whereas the tackling of a mathematical puzzle or game resembles the action of the mongoose which is ready to take the initiative and fight back.

One topic from this book which you can seize by the neck and wrestle with until one of you yields is the following.

Suppose we have a clock with just a minute hand. Whenever the hand passes 4 a bell sounds and whenever the hand passes 8 a whistle sounds. The clock is laid on the table face up. The hand is freed from the mechanism and spun like a roulette wheel.

When it has come to rest, the mechanism is reconnected, without moving the hand, and the clock starts to go. What is the probability that the first sound we will hear is the bell? The answer is $\frac{2}{3}$, for that is the probability that the hand will stop its spin in the arc 8–12–4. Suppose now that we have two of these clocks and we spin each hand and then start them both going. What is the probability that the first sound we will hear is the bell? For eleven years it was believed that since the probability for each clock was $\frac{2}{3}$ that was also the probability for the pair. Then Donald Knuth, a computer scientist at Stanford University, found the true answer. The full story is given in the chapter entitled 'Elevators'.

I can thoroughly recommend that you buy this book, and any of the previous ten collections that you do not already own.

University of Sheffield — KEITH AUSTIN

Statistics. Problems and Solutions. By E. E. BASSETT, J. M. BREMNER, I. T. JOLLIFFE, B. JONES, B. J. T. MORGAN, P. M. NORTH. Edward Arnold, London, 1986. Pp. 227.

This is a book of problems, together with their solutions, which has been designed to give the reader a guided experience in solving problems in statistics. Each problem solution is followed by an extensive set of notes which discuss the techniques which have been used to solve the problem and offer alternative methods of solution.

The problems are classified into five areas: probability and random variables, probability distributions, data summarisation and goodness of fit, inference, analysis of structured data (which is, regression, correlation, contingency tables and time series).

Sixth-form students and first-year undergraduates at polytechnics and universities should find this a worthwhile book to use, alongside a first-level textbook on statistics.

North Staffordshire Polytechnic — D. J. COLWELL
J. R. GILLETT

From Number Theory to Secret Codes—A Computer Illustrated text. By T. H. JACKSON. Adam Hilger, 1987. Pp. 86. £15 including BBC or IBM software.

This is a fascinating text. 'Book' is the wrong word, because it is more than that. The author's aim is not to write a textbook on number theory. Thus it is not set out in the usual 'Theorem–Proof' way which can be so deadening. It reads rather like a novel, discussing prime numbers, factorization, continued fractions, modular arithmetic, and (the climax of the book) cryptography, or how to encipher a message, i.e. to write it in a secret form, for transmission, and then to decipher it.

The subject will come alive when you insert the disc for your BBC B (or IBM PC) and follow the instructions. Just type in a number and its prime factorization will appear on the screen. It will tell you the 89th prime. It will work out the highest common factors of two numbers before your very eyes using Euclid's algorithm, it will express $\sqrt{13}$ as a continued fraction, you will be able to do modular arithmetic, and, best of all, you will be able to encipher a message like 'Mathematics can be fun', and hopefully decipher it as well.

This book is strongly recommended to all sixth formers and students, especially those who feel they might be losing a bit of enthusiasm for mathematics. You'll need a BBC B computer with disc drive (or an IBM PC), and you'll be well away.

Number theory will never be the same again!

University of Sheffield DAVID SHARPE

A Simple Introduction to Numerical Analysis. By R. D. HARDING and D. A. QUINNEY. Adam Hilger, 1986. Pp. 125 + BBC Disk. £15·00.

A Mathematical Toolkit. By R. R. HARDING. Adam Hilger, 1986. Pp. 197 + BBC Disk. £15·00.

These two computer texts cover similar ground, but at a different pace.

The introduction covers fixed point iteration, recurrences, linear equations, numerical integration, and ordinary differential equations, each topic having a chapter of explanation in the book, and ready written illustrative programs to run on a BBC micro. The presentation is lucid and accurate, and the programs are user-friendly and well written.

In the Toolkit, there are additional topics of maxima and minima, curve-fitting. Fourier transforms, matrix eigenvalues and singular value decomposition, and least squares. The book and disk are designed for the applications programmer, and, mostly, procedures are provided, rather than complete programs, though some illustrative programs are included. The beginner would find the text tougher going, but, as an accompaniment to an undergraduate analysis course, this has to be extremely good value for its modest price.

In short, I can recommend either of these books without reservation. Buy either, and you have a high-quality product, well prepared, and covering the basics of mathematical computing up to intermediate level.

University of Sheffield MIKE PIFF

Spatial Structure and the Microcomputer. By A. N. BARRETT and A. L. MACKAY. Macmillan, 1987. Pp. ix + 204. £12·00. Paperback.

Most people will find something of interest in this book. Among the topics covered are some traditional ones such as matrices, coordinate systems and spherical trigonometry. However, there are some more unusual ones, such as Voronoi polyhedra (soap bubbles?), Penrose's aperiodic tiling of the plane, Julesz stereographic patterns, generalised inverses, and fast Fourier transforms. Listings are provided in various dialects of BASIC and APL, with sample screen dumps, so these should provide many happy hours of debugging for the reader!

I was surprised to find, on checking through the listing of a routine to invert a matrix, that no pivot strategy was being used, and so this routine could fail for even modestly sized matrices, because of excessive and avoidable rounding errors. So, don't expect numerical perfection from these programs, just sit back and admire the pictures!

University of Sheffield MIKE PIFF

L O N D O N   M A T H E M A T I C A L   S O C I E T Y

AN EVENING OF POPULAR LECTURES

Once again the London Mathematical Society is holding an evening of popular lectures. These lectures, in which prominent mathematicians speak on topics of current interest in a way that is accessible to a wide audience of teachers, sixth formers and people with a general interest in mathematics, have been a great success.

The 1988 lectures will be given by Professor D.G. Kendall, on **How should a mathematician think about shape?** and Professor M.V. Berry, on **Chaology.** The lectures will be at Imperial College, London, on 1 July at 7.30 p.m., and at the University of Liverpool on 11 July at 7.00 p.m.

Admission is free, by ticket obtainable in advance. For the London venue write to Miss Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V ONL, and for the Liverpool venue write to Dr I.R. Porteous, Department of Pure Mathematics, University of Liverpool, P.O. Box 147, Liverpool L69 3BX. A stamped addressed envelope would be much appreciated.

CONTENTS

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