Crux

Mathematicorum

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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER
No. 107

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

First up this month is the 30th I.M.O. which was held this year at Braunschweig, West Germany, July 13–24th. My particular thanks to Richard Nowakowski and Bruce Shawyer for having supplied me with the information.

This year a record 291 students from 50 countries took part in the contest. The maximum team size was again six students and all but four of the countries sent a full team.

The six problems of the competition were assigned equal weights of seven points each (the same as in the last eight I.M.O.'s) for a maximum possible individual score of 42 (and a maximum possible team score of 252). For comparison see the last eight I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207] and [1988: 193].

This year first place (gold) medals were awarded to students with scores from 38 to 42. There were 20 gold medals awarded, 10 for perfect papers and a further six for a score of 41. The youngest participant to write a perfect paper was a 14 year old girl from the U.S.S.R.! The second place (silver) medals were awarded to papers with scores from 30–37. There were 59 silver medals awarded (compared to 48 in 1988). Third place (bronze) medals were awarded to the 68 students with a score in the range 18–29. This compares to 66 in 1988. In addition, honourable mention was given to any student receiving full marks on at least one problem.

Congratulations to the following 20 gold medalists. Unfortunately I have only the last names and cannot give the initials.

		Score
Cizek	Czechoslovakia	42
Cuong	Vietnam	42
Ellenberg	U.S.A.	42
Göring	East Germany	42
Ivanov	U.S.S.R.	42
Luo	China	42
Moroiann	Romania	42
Siebert	East Germany	42
Todorovic	Yugoslavia	42

Zenker	East Germany	42
Härterich	West Germany	41
Huo	China	41
Jiang	China	41
Malinnikova	U.S.S.R.	41
Such	Czechoslovakia	41
Yu	China	41
Chau	Vietnam	40
Ivanov	U.S.S.R.	39
Todorov	Bulgaria	39
Banica	Romania	38

The international jury (comprised of the team leaders from the participating countries) set out to give a paper that was slightly more approachable for the students. The number of near perfect papers, and the increase in the number of gold and silver medals, would suggest they succeeded, but the large number of marks under ten (87) indicates that not all participants would agree it was such a success. The ranges for all medals were set higher than last year. Nevertheless 50.5% of the participants received medals, compared to 49% in 1988 and 50.6% in 1987. Several countries showed marked improvement, notably Greece, Hong Kong, Iran, Italy, Singapore and Turkey.

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. Team scores are obtained by adding up the individual scores of the members. These totals, as well as a breakdown of the medals awarded per country, is given in the following table. Congratulations to China on a convincing victory.

Rank	Country	Score (Max.252)	Prizes 1st	2nd	3rd	Total Prizes
1	China	237	4	2	–	6
2	Romania	223	2	4	–	6
3	U.S.S.R.	217	3	2	1	6
4	East Germany	216	3	2	1	6
5	U.S.A.	207	1	4	1	6
6	Czechoslovakia	202	2	1	3	6
7	Bulgaria	195	1	3	2	6
8	West Germany	187	1	3	2	6
9	Vietnam	183	2	1	3	6
10	Hungary	175	–	4	1	5
11	Yugoslavia	170	1	3	1	5
12	Poland	157	–	3	3	6
13	France	156	–	1	5	6
14	Iran	147	–	2	3	5
15	Singapore	143	–	–	4	4
16	Turkey	133	–	1	4	5
17	Hong Kong	127	–	2	1	3
18	Italy	124	–	1	2	3
19	Canada	123	–	1	3	4

20–21	Great Britain	122	–	2	1	3	
20–21	Greece	122	–	1	3	4	
22–23	Australia	119	–	2	2	4	
22–23	Colombia	119	–	1	2	3	
24	Austria	111	–	2	1	3	
25	India	107	–	4	–	4	
26	Israel	105	–	2	1	3	
27	Belgium	104	–	–	3	3	
28	Korea	97	–	1	–	1	
29	Netherlands	92	–	1	1	2	
30	Tunisia	81	–	1	–	1	
31	Mexico	79	–	–	1	1	
32	Sweden	73	–	–	2	2	
33–34	Cuba	69	–	–	1	1	
33–34	New Zealand	69	–	–	2	2	
35	Luxembourg	65	–	1	1	2	(Team of 3)
36–37	Brazil	64	–	–	3	3	
36–37	Norway	64	–	–	1	1	(Team of 4)
38	Morocco	63	–	–	1	1	
39	Spain	61	–	–	1	1	
40	Finland	58	–	–	–	0	
41	Thailand	54	–	–	1	1	
42	Peru	51	–	–	–	0	
43	Philippines	47	–	1	–	1	
44	Portugal	39	–	–	–	0	
45	Ireland	37	–	–	–	0	
46	Iceland	33	–	–	–	0	(Team of 4)
47	Kuwait	31	–	–	–	0	
48	Cyprus	24	–	–	–	0	
49	Indonesia	21	–	–	–	0	
50	Venezuela	6	–	–	–	0	(Team of 4)

This year the Canadian team slid to 19th place, just one point behind Italy (who moved up from 35th place to 18th this year). However the team total only changed by one point. Several team members will still be eligible next year and the experience may improve our performance. The team members, scores, and the leaders are as follows:

Philip Jong	35	Silver
Jeff Higham	26	Bronze
Hugh Thomas	22	Bronze
Ian Goldberg	21	Bronze
Andrew Chow	11	Honourable Mention
James Law	8	Honourable Mention

Leaders: Richard Nowakowski, Dalhousie University
Georg Gunther, Sir Wilfred Grenfell College
Bruce Shawyer, Memorial University of Newfoundland (Observer)

The U.S.A. team ranked 5th with a score of 207. Individual performances were as follows:

Jordan Ellenberg	Gold
Andrew Kresch	Silver
Samuel Kutin	Silver
Jeffrey Vanderkam	Silver
Samuel Vandervelde	Silver
David Carleton	Bronze

Leaders: Gerald Heuer, Concordia College, Moorhead, Minn.
Gregg Patruno, First Boston Corporation, N.Y.

The next few Olympiads are:

1990	Beijing, China
1991	Sweden
1992	U.S.S.R.
1993	Turkey

The Canadian Mathematical Society is seeking funding and help to host the 1995 Olympiad. Bruce Shawyer heads the organizational committee.

*

Next, we give the problems of this year's I.M.O. competition. Solutions to these problems, along with those of the 1989 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1989* which may be obtained for a small charge from:

Dr. W.E. Mientka
Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, U.S.A. 68588

THE 30th INTERNATIONAL MATHEMATICAL OLYMPIAD

Braunschweig, West Germany

First Day

July 18, 1989

Time: $4\frac{1}{2}$ hours

1. Prove that the set $\{1,2,\ldots,1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1,2,\ldots,117$) such that

(i) each A_i contains 17 elements;

(ii) the sum of all the elements in each A_i is the same.

[*Editor's note:* Compare this with *Crux* 1382 [1988: 268]!]

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;

(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

(i) no three points of S are collinear, and

(ii) for every point P of S there are at least k points of S equidistant from P.

Prove that

$$k < \frac{1}{2} + \sqrt{2n} .$$

Second Day

July 19, 1989

Time: $4\frac{1}{2}$ hours

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}} .$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1,x_2,\ldots,x_{2n})$ of the set $\{1,2,\ldots,2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1,2,\ldots,2n - 1\}$. Show that, for each n, there are more permutations with property P than without.

*　　　　*　　　　*

To finish this month's Corner we give solutions to the 21st *Canadian Mathematics Olympiad* (1989). The problems were given in the last issue of *Crux* [1989: 161]. The "official solutions" come from R. Nowakowski, Dalhousie University, who was chairman of the Canadian Mathematics Olympiad Committee of the Canadian Mathematical Society.

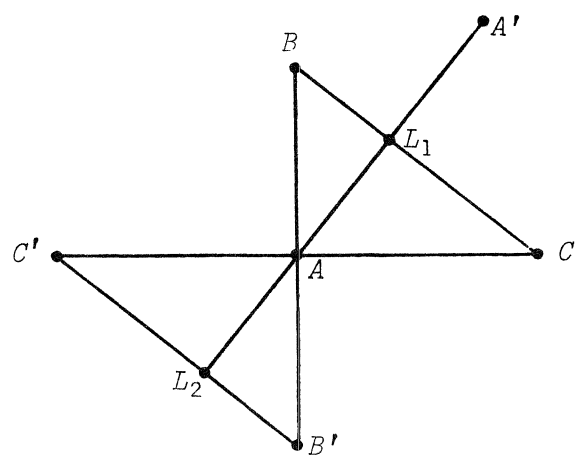
1. The integers $1,2,\ldots,n$ are placed in order so that each value is either strictly bigger than all the preceding values or is strictly smaller than all preceding values. In how many ways can this be done?

Solution.

Construct the sequence backwards. The last term must be 1 or n and each preceding term must be either the largest or the smallest of those numbers left. That is, in each position, except the first, there are two choices and in total there are 2^{n-1} such sequences.

2. Let ABC be a right–angled triangle of area 1. Let A', B', and C' be the points obtained by reflecting A, B, C, respectively, in their opposite sides. Find the area of $\Delta A'B'C'$.

Solution.



Let $\angle BAC = 90^\circ$. By the property of reflections, $CA = C'A$, $BA = B'A$, and CAC' and BAB' are collinear. Hence

$$\angle B'AC' = 90^\circ,$$

and $\Delta AB'C'$ is the image of ΔABC under a rotation by 180° about A. Thus BC is equal in length and parallel to $B'C'$. Let L_1 be the point of intersection of AA' and BC, and let L_2 be the point at which AA', if extended, meets $B'C'$. Now the line segments $A'L_1$, AL_1, and AL_2 are all equal in length. Also $A'L_2$ is perpendicular to $B'C'$, and AL_1 is perpendicular to BC. The altitude of $\Delta A'B'C'$ is thus three times that of ΔABC and the bases are the same length. Therefore the area of $\Delta A'B'C'$ is three times that of ΔABC, i.e. it has area 3.

3. Define $\{a_n\}_{n=1}^{\infty}$ as follows: $a_1 = 1989^{1989}$, and a_n for $n > 1$ is the sum of the digits of a_{n-1}. What is the value of a_5?

Solution.

The sum of the digits of a multiple of 9 is also a multiple of 9. Since 1989^{1989} is divisible by 9 then all of a_2, a_3, a_4 and a_5 are multiples of 9. Since a_1 is less than $(10^4)^{1989}$ and so has fewer than $4 \cdot 1989$ digits, it follows that a_2 is less than

$$9 \cdot 4 \cdot 1989 < 36 \cdot 2000 = 72{,}000$$

and so has at most 5 digits. Therefore, a_3 is strictly less than $5 \cdot 9 = 45$ and in turn a_4 is strictly less than $2 \cdot 9$. But a_4 is a multiple of 9 and so must be 9. Hence a_5 is also 9.

4. There are 5 monkeys and 5 ladders and at the top of each ladder there is a banana. A number of ropes connect the ladders; each rope connects two ladders. No two ropes are attached to the same rung of the same ladder. Each monkey starts at the bottom of a different ladder. The monkeys climb up the ladders but each time they encounter a rope they climb along it to the other end of the rope and then continue climbing upwards. Show that, no matter how many ropes there are, each monkey gets a banana.

Solution.

Consider the problem in the following way. Regard the bottom and top of each ladder and the points at which the ropes are tied to the ladders as vertices; call these latter vertices interior vertices. Regard the ropes and the parts of the ladders between vertices as edges. The ladder edges have a direction (upward) associated to them. The rules say that a monkey may travel in either direction along a rope edge but must follow the direction given for a ladder edge. The bottom and top vertices have no rope edges. The interior vertices have one rope edge, one ladder edge coming into the vertex, and one ladder edge leaving the vertex. The rules also stipulate that on entering an edge there is only one way out; moreover **there is only one way into the vertex that lets a monkey leave along a given edge.**

Suppose that two monkeys reach the same banana. Then they both traversed the last edge in the same direction. Therefore, by the rules, they must have crossed the preceding edge, and, in fact, all the same edges and in the same direction. This means that they must have started at the bottom of the same ladder, which they did not.

Suppose a monkey got trapped in a cycle. The vertices of the cycle must be interior vertices. Therefore there is an interior vertex, say A, at which the monkey entered the cycle. Let e be the edge crossed to get to A the first time and f the first edge crossed in the cycle. But now the only legitimate way to cross f again in the same direction is to use e to get to A, but e is not in the cycle and so is not used again. Therefore, no monkey can be trapped in a cycle and hence each monkey gets a banana.

5. Given are the numbers $1,2,2^2,\dots,2^{n-1}$. For a specific permutation $\sigma = x_1,x_2,\dots,x_n$ of these numbers we define

$$S_1(\sigma) = x_1, \quad S_2(\sigma) = x_1 + x_2, \quad S_3(\sigma) = x_1 + x_2 + x_3, \quad \dots$$

and $Q(\sigma) = S_1(\sigma)S_2(\sigma)\dots S_n(\sigma)$. Evaluate $\sum 1/Q(\sigma)$ where the sum is taken over all possible permutations.

Solution.

In general, consider given numbers $a_1,a_2,\dots,a_n$. For a specific permutation $\sigma = x_1,x_2,\dots,x_n$ of these numbers we define

$$S_1(\sigma) = x_1, \quad S_2(\sigma) = x_1 + x_2, \quad S_3(\sigma) = x_1 + x_2 + x_3, \dots,$$

and $Q(\sigma) = S_1(\sigma)S_2(\sigma) \dots S_n(\sigma)$. Then we claim that

$$\sum \frac{1}{Q(\sigma)} = \frac{1}{a_1a_2\dots a_n}.$$

We proceed by induction. For $n = 1$, the result is obvious. Assume that the result holds for $n = k$, for some positive integer $k \geq 1$. Consider the case $n = k + 1$. Note that, for any permutation σ, $S_{k+1}(\sigma)$ is just the constant sum $a_1 + \cdots + a_{k+1}$. So

$$\sum \frac{1}{Q(\sigma)} = \frac{1}{S_{k+1}(\sigma)} \sum \frac{1}{Q^*(\sigma)}$$

where $Q^*(\sigma) = S_1(\sigma)\cdots S_k(\sigma)$.

For each permutation σ,

$$S_k(\sigma) = (a_1 + a_2 + \cdots + a_{k+1}) - b$$

where b is one of $a_1,a_2,\cdots,a_{k+1}$. Consider those permutations where $b = a_1$. This means the $S_1(\sigma)$, $S_2(\sigma)$, $\cdots$, $S_k(\sigma)$ are formed from the k numbers $a_2,\dots,a_{k+1}$ and so, by the induction hypothesis, if we sum the $1/Q^*(\sigma)$'s just over these permutations then we obtain

$$\frac{1}{a_2a_3\cdots a_{k+1}} = \frac{a_1}{a_1a_2\cdots a_{k+1}}.$$

In general, if $b = a_i$ then $S_1(\sigma),S_2(\sigma),\cdots,S_k(\sigma)$ are formed from $a_1,a_2,\cdots,a_{k+1}$ except for a_i, and so, by induction, for these permutations the sum of the $1/Q^*(\sigma)$'s is

$$\frac{a_i}{a_1a_2\cdots a_{k+1}}.$$

So we have

$$\sum \frac{1}{Q(\sigma)} = \frac{1}{S_{k+1}(\sigma)} \sum \frac{1}{Q^*(\sigma)}$$

$$= \frac{1}{a_1 + \cdots + a_{k+1}} \left(\frac{a_1}{a_1 a_2 \cdots a_{k+1}} + \frac{a_2}{a_1 a_2 \cdots a_{k+1}} + \cdots + \frac{a_{k+1}}{a_1 a_2 \cdots a_{k+1}} \right)$$

$$= \frac{1}{a_1 a_2 \cdots a_{k+1}} .$$

Thus the general result is true by induction, and our answer for the case $a_i = 2^{i-1}$ of the problem is

$$\sum \frac{1}{Q(\sigma)} = \frac{1}{2^{0+1+\cdots+(n-1)}} = 2^{-n(n-1)/2}.$$

*

The results of the 21st Canadian Mathematics Olympiad are given below. Two hundred and fifty students wrote the contest. The students in the top four prize categories are

Lapell, Eli Michael	First Prize
Thomas, Hugh	Second Prize
Schach, Chris	Third Prize
Amano, Etsuko	Fourth Prize
Elliot, Jeff	Fourth Prize
Hsu, Leon C.	Fourth Prize
Kountourogiannis, Dimitri	Fourth Prize
Yang, Peter	Fourth Prize

The following table gives a breakdown of the marks awarded for each question. Clearly Question 5 was extremely difficult for almost all contestants.

	Mark											
Question	0	1	2	3	4	5	6	7	8	9	10	Avg
1	155	3	5	1	8	9	14	7	6	2	40	2.77
2	69	29	9	7	3	2	2	2	26	20	81	5.26
3	166	5	12	1	0	3	8	0	25	12	18	2.33
4	161	38	11	2	14	3	4	5	4	7	1	1.2
5	224	17	4	0	0	2	0	1	0	1	1	0.24

The average grade was 11.8 out of a possible 50.

I include this breakdown to encourage those readers who solved even one of these problems!

* * *

Keep sending me your nice solutions to current and past problems, as well as contest materials.

* * *

MINI-REVIEWS

by

ANDY LIU

[This is the third of Andy's sets of short reviews to appear in *Crux*. The quality of the books treated below is of course universally recognized. But the appearance of these reviews at this time should also be taken as a tribute to the man himself, Martin Gardner, whose 75th birthday occurs on October 21! – Ed.]

MARTIN GARDNER'S SCIENTIFIC AMERICAN SERIES

For over twenty years, Martin Gardner had a monthly column in *Scientific American* called "Mathematical Games". Despite the title, it covered a wide range of topics, from the very elementary to the frontiers of current research, but always with that delightful element of play. It was written in a lively style indicative of the strong literary background of Martin Gardner. When the subject under discussion was difficult, Martin Gardner took considerable effort to smooth out the path and guide the reader gently along. The column had and still has enormous influence in the mathematical community of North America and beyond.

Unfortunately for his readers, Martin Gardner retired in 1981. His column was replaced briefly by "Metamagical Themas" (an anagram of "Mathematical Games") written by D. Hofstadter. Now, A.K. Dewdney's "Computer Recreations" occupies that distinguished spot in *Scientific American.*

Fortunately, anthologies of Martin Gardner's columns have appeared regularly. To date, thirteen volumes have been published, with enough columns left over for at least five more books. If a school library can afford only one set of books on popular mathematics, this is the one!

***The Scientific American Book of Mathematical Puzzles and Diversions*, Simon & Schuster, 1959. (hardcover & paperback, 178 pp.)**

Topics covered are hexaflexagons, magic with a matrix, ticktacktoe, probability paradoxes, the icosian game and the Tower of Hanoi, curious topological models, the game of hex, Sam Loyd: America's greatest puzzlist, mathematical card tricks, memorizing numbers, polyominoes, fallacies, nim and tac tix, left or right, as well as two collections of short problems.

The 2nd Scientific American Book of Mathematical Puzzles and Diversions, Simon & Schuster, 1961. (hardcover & paperback, 251 pp.)

Topics covered are the five Platonic solids, tetraflexagons, Henry Ernest Dudeney: England's greatest puzzlist, digital roots, the soma cube, recreational topology, phi: the golden ratio, the monkey and the coconuts, mazes, recreational logic, magic squares, James Hugh Riley Shows Inc., eleusis: the induction game, origami, squaring the square, mechanical puzzles, probability and ambiguity, as well as two collections of short problems.

Martin Gardner's New Mathematical Diversions from Scientific American, Simon & Schuster, 1966. (hardcover & paperback, 253 pp.)

Topics covered are the binary system, group theory and braids, the games and puzzles of Lewis Carroll, paper cutting, board games, packing spheres, the transcendental number pi, Victor Eigen: mathemagician, the four–color map theorem, Mr. Apollinax visits New York, polyominoes and fault–free rectangles, Euler's spoilers: the discovery of an order 10 Graeco–Latin square, the ellipse, the 24 color squares and the 30 color cubes, H.S.M. Coxeter, bridg–it and other games, the calculus of finite differences, as well as three collections of short problems.

The Magic Numbers of Dr. Matrix, Prometheus Books, 1985. (hardcover & paperback, 326 pp.) (This is an enlarged version of *The Numerology of Dr. Matrix*, Simon & Schuster, 1967.)

This is different from any other book in this series. It centres around a mystic character Dr. Irving Joshua Matrix, a professional numerologist. Dr. Matrix columns appeared to be Martin Gardner's own favourite, as he returned to them periodically until the good Doctor's untimely (and temporary) demise in the last episode.

The Unexpected Hanging and Other Mathematical Diversions, Simon & Schuster, 1969. (hardcover & paperback, 255 pp.)

Topics covered are the paradox of the unexpected hanging, knots and Borromean rings, the transcendental number e, geometric dissections, Scarne on gambling, the church of the fourth dimension, a matchbox game–learning machine, spirals, rotations and reflections, peg solitaire, flatlands, Chicago Magic Convention, tests of divisibility, the eight queens and other chessboard diversions, a loop of string, curves of constant width, reptiles: replicating figures on the plane, as well as three collections of short problems.

Martin Gardner's 6th Book of Mathematical Diversions from Scientific American, University of Chicago Press, 1983. (paperback, 262 pp.) (This is a reprint of *Martin Gardner's 6th Book of Mathematical Games from Scientific American*, W.H. Freeman and Company, 1971.)

Topics covered are the helix, Klein bottles and other surfaces, combinatorial theory, bouncing balls in polygons or polyhedra, four unusual board games, sliding–block puzzles, parity checks, patterns and primes, graph theory, the ternary system, the cycloid: Helen of geometry, mathematical magic tricks, word play, the Pythagoras Theorem, limits of infinite series, polyiamonds, tetrahedra, the lattice of integers, infinite regress, O'Gara the mathematical mailman, extraterrestrial communication, as well as three collections of short problems.

Mathematical Carnival, Alfred A. Knopf, 1975/Vintage Books, 1977. (hardcover/paperback, 274 pp.)

Topics covered are sprouts and Brussels sprouts, penny puzzles, aleph–null and aleph–one, hypercubes, magic stars and polyhedra, calculating prodigies, tricks of lightning calculators, the art of M.C. Escher, card shuffles, Mrs. Perkins' quilt and other square–packing problems, the numerology of Dr. Fliess, random numbers, the rising hourglass and other physics puzzles, Pascal's Triangle, jam, hot and other games, cooks and quibble–cooks, Piet Hein's superellipse, how to trisect an angle, as well as one collection of short problems.

Mathematical Magic Show, Alfred A. Knopf, 1977/Vintage Books, 1978. (hardcover/paperback, 284 pp.)

Topics covered are nothing, game theory, guess–it and foxholes, factorial oddities, double acrostics, playing cards, finger arithmetic, Möbius bands, polyhexes and polyaboloes, perfect, amicable and sociable, polyominoes and rectification, knights of the square table, colored triangles and cubes, trees, dice, everything, as well as three collections of short problems.

Mathematical Circus, Alfred A. Knopf, 1979/Vintage Books, 1981. (hardcover/paperback, 272 pp.)

Topics covered are optical illusions, matches, spheres and hyperspheres, patterns of induction, elegant triangles, random walks and gambling, random walks on the plane and in space, Boolean Algebra, can machines think?, cyclic numbers, dominoes, Fibonacci and Lucas numbers, simplicity, solar system oddities, Mascheroni constructions, the abacus, palindromes: words and numbers, dollar bills, as well as two collections of short problems.

Wheels, Life and Other Mathematical Amusements, **W.H. Freeman and Company, 1983. (hardcover & paperback, 261 pp.)**

Topics covered are wheels, Diophantine analysis and Fermat's Last Theorem, alephs and supertasks, nontransitive dice and other probability paradoxes, geometrical fallacies, the combinatorics of paper folding, ticktacktoe games, plaiting polyhedra, the game of Halma, advertising premiums, Salmon on Austin's dog, nim and hackenbush, Golomb's graceful graphs, chess tasks, slither, $3X + 1$ and other curious questions, mathematical tricks with cards, the game of life, as well as three collections of short problems.

Knotted Doughnuts and Other Mathematical Entertainments, **W.H. Freeman and Company, 1986. (hardcover & paperback, 278 pp.)**

Topics covered are coincidence, the binary Gray code, polycubes, Bacon's cipher, doughnuts: linked and knotted, Napier's bones, Napier's abacus, sim, chomp and racetrack, elevators, crossing numbers, point sets on the sphere, Newcomb's paradox, look–see proofs, worm paths, Waring's problems, cram, bynum and quadraphage, the I Ching, the Laffer curve, as well as two collections of short problems.

Time Travel and Other Mathematical Bewilderments, **W.H. Freeman and Company, 1988. (hardcover & paperback, 295 pp.)**

Topics covered are time travel, hexes and stars, tangrams, nontransitive paradoxes, combinatorial card problems, melody–making machines, anamorphic art, six sensational discoveries, the Csaszar polyhedron, dodgem and other simple games, tiling with convex polygons, tiling with polyominoes, polyiamonds and polyhexes, curious maps, magic squares and cubes, block packing, induction and probability, Catalan numbers, fun with a pocket calculator, tree–planting problems, as well as two collections of short problems.

Penrose Tiles to Trapdoor Ciphers, **W.H. Freeman and Company, 1989. (hardcover & paperback, 311 pp.)**

Topics covered are Penrose tiling, Mandelbrot's fractals, Conway's surreal numbers, the Oulipo, Wythoff's nim, mathematical induction and colored hats, negative numbers, cutting shapes into n congruent parts, trapdoor ciphers, hyperbolas, the new Eleusis, Ramsey theory, from burrs to Berrocal, Sicherman dice, the Kruskal count and other curiosities, Raymond Smullyan's logic puzzles, the return of

Dr. Matrix (appropriately in the thirteenth anthology), as well as two collections of short problems.

Addresses of publishers:

Simon and Schuster, 1230 Avenue of the Americas, New York, NY 10020.
Prometheus Books, 700 E. Amherst Street, Buffalo, NY 14215.
University of Chicago Press, 5801 Ellis Avenue 4/F, Chicago, IL 60637.
W.H. Freeman and Company, 4419 West 1980 S., Salt Lake City, UT 84104.
Alfred A. Knopf and Vintage Books are divisions of Random House, 1265 Aerowood Drive, Mississauga, Ontario L4W 1B9.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1990,** *although solutions received after that date will also be considered until the time when a solution is published.*

1461. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c, r, R, s be the sides, inradius, circumradius, and semiperimeter of a triangle and let a', b', c', r', R', s' be similarly defined for a second triangle. Show that

$$(4ss' - \sum aa')^2 \geq 4(s^2 + r^2 + 4Rr)(s'^2 + r'^2 + 4R'r'),$$

where the sum is cyclic.

1462*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of a triangle, prove or disprove that

$$\sqrt{2}\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq \sqrt{\sin(A/2)} + \sqrt{\sin(B/2)} + \sqrt{\sin(C/2)}\ ,$$

with equality when $A = B = C$.

1463. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Prove that if n and r are integers with $n > r$ then

$$\sum_{k=1}^{n} \cos^{2r}(k\pi/n) = \frac{n}{4^r}\binom{2r}{r} .$$

1464. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$.

(a) Prove that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} \qquad (1)$$

if and only if the centroids G, G' of the two triangles coincide.

(b) Prove that if (1) holds, and either the circumcenters O, O' or the orthocenters H, H' of the triangles coincide, then ΔABC is equilateral.

(c)* If (1) holds and the incenters I, I' of the triangles coincide, characterize ΔABC.

1465. *Proposed by Murray S. Klamkin, University of Alberta.*

$A_1,A_2,\ldots,A_n$ and $B_1,B_2,\ldots,B_n$ are two sets of points on an n-dimensional unit sphere of center O such that each of the two sets of vectors $\mathbf{A}_i$ and $\mathbf{B}_i$ ($\mathbf{A}_i$ denoting the vector from O to A_i, etc.) are orthonormal. Show that for any $0 \le r \le n$ the two simplexes

$$OA_1\cdots A_rB_{r+1}\cdots B_n \qquad \text{and} \qquad OB_1\cdots B_rA_{r+1}\cdots A_n$$

have equal volumes.

1466. *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

On the sides of $\Delta A_1A_2A_3$, and outside $\Delta A_1A_2A_3$, we draw similar triangles $A_3A_2B_1$, $A_1A_3B_2$ and $A_2A_1B_3$, with geocentres G_1, G_2 and G_3 respectively. The geocentres of triangles $A_1B_3B_2$, $A_2B_1B_3$, $A_3B_2B_1$ and $A_1A_2A_3$ are Γ_1, Γ_2, Γ_3 and G, respectively. It is known that G is the geocentre of $\Delta B_1B_2B_3$ as well (see *Mathematics Magazine* 50 (1985) 84–89). Show that Γ_1G_1 has midpoint G, length $\frac{2}{3}|A_1B_1|$, and is parallel to A_1B_1.

1467. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Find the locus of a point P in the interior of square $A_1A_2A_3A_4$ such that

$$\angle PA_1A_2 + \angle PA_2A_3 + \angle PA_3A_4 + \angle PA_4A_1 = \pi.$$

1468. *Proposed by Jordi Dou, Barcelona, Spain.*

Prove that the midpoints of the sides and diagonals of a quadrilateral lie on a conic.

1469. *Proposed by Andy Liu, University of Alberta.*

The following is an excerpt from U.I. Lydna's *Medieval Justice.*

"Justice on Pagan Island was administered by a Tribunal of seven Chiefs, at most one from each tribe. In a trial, each Chief would deliver a preliminary verdict of "innocent" or "guilty". There was no problem when unanimity was achieved, but it was felt that the final verdict of any case in between should be left to the Big Chief in the Sky.

Accordingly, the following procedure was established. A pouch and a token were made for each Chief on the Tribunal, both bearing the tribal insignia of that Chief. The tokens were put into the pouches at random, not necessarily one token in each pouch. (The pouches were made in such a way that, without opening them, the defendant could not tell how many tokens, if any, were inside.) After each Chief had delivered a verdict, the pouches were brought before the defendant, who then chose a number of them equal to the number of "innocent" verdicts received. The defendant opened the chosen pouches and kept the tokens inside. Any pouch bearing the same insignia as one of the defendant's tokens was also opened and the token(s) inside added to the defendant's collection. This continued until no further pouches could be opened. Finally, the defendant would be deemed innocent if all seven tokens had been collected, and guilty otherwise."

Suppose a defendant received k "innocent" verdicts, $0 \le k \le 7$. What is his probability of acquittal?

1470*. *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \ge 6$. Computer calculations show that $a_6, a_7, \ldots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1345. [1988: 140] *Proposed by P. Erdos, Hungarian Academy of Sciences, and Esther Szekeres, University of New South Wales, Kensington, Australia.*

Given a convex n–gon $X_1X_2\cdots X_n$ of perimeter p, denote by $f(X_i)$ the sum of the distances of X_i to the other $n - 1$ vertices.

(a) Show that if $n \geq 6$, there is a vertex X_i such that $f(X_i) > p$.

(b) Is it true that for n large enough, the *average value* of $f(X_i)$, $1 \leq i \leq n$, is greater than p?

I. *Solution to (a) by János Pach, Hungarian Academy of Sciences.*

We prove somewhat more. Given a straight line s in the plane, let $d_s(X_i,X_j)$ denote the distance between the orthogonal projections of X_i and X_j into s. Of course $d_s(X_i,X_j) \leq d(X_i,X_j)$, the distance between X_i and X_j.

Claim. There exist an i and a straight line s such that

$$\sum_{j=1}^{n} d_s(X_i,X_j) \geq \frac{p}{2\sin(\pi/n)} .$$

Thus for $n \geq 6$ we have part (a).

We need the following well–known

Lemma. Let P be a convex n–gon whose vertices are contained in a disc of radius r. Then

$$\text{perimeter}(P) \leq 2nr\sin(\pi/n),$$

with equality if and only if P is a regular n–gon inscribed in a circle of radius r. □

Now let Q denote the barycenter of $X_1,\ldots,X_n$ [i.e.

$$Q = \frac{\mathbf{X}_1 + \cdots + \mathbf{X}_n}{n} ,$$

where $\mathbf{X}$ denotes the vector from some fixed origin to a point X], and let s_i be the line connecting X_i and Q $(1 \leq i \leq n)$. Then, by the properties of the barycenter,

$$\sum_{j=1}^{n} d_{s_i}(X_i,X_j) \geq nd(X_i,Q) \qquad (1)$$

for every i.

Assume the claim is false. Then in particular we obtain that, for every i,

$$\sum_{j=1}^{n} d_{s_i}(X_i,X_j) < \frac{p}{2\sin(\pi/n)}$$

and thus, by (1),

$$d(X_i,Q) < \frac{p}{2n\,\sin(\pi/n)}\,,$$

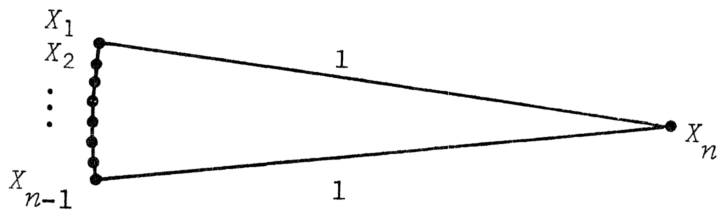
i.e. all vertices of the n–gon are within a disc of radius

$$r < \frac{p}{2n\,\sin(\pi/n)}\,.$$

This contradicts the lemma.

II. *Solution to (b) by the editor.*

The answer is no. Let $X_1,X_2,\cdots,X_{n-1}$ be very close together and all at distance 1 from X_n. Then $p \approx 2$ while



$$\sum_{i=1}^{n} f(X_i) \approx \underbrace{1 + 1 + \cdots + 1}_{n-1} + (n-1) = 2n - 2 \not\geq np.$$

Incidentally, it is easy to show that (a) fails for n = 3 or 4. Can any reader determine what happens for n = 5?

* * *

<u>1348</u>*. [1988: 141] *Proposed by Murray S. Klamkin, University of Alberta.*

Two congruent convex centrosymmetric planar figures are inclined to each other (in the same plane) at a given angle. Prove or disprove that their intersection has maximum area when the two centers coincide.

Solution by P. Penning, Delft, The Netherlands.

Consider two centrosymmetric figures F_1 and F_2, located in such a way that their centers of inversion coincide with a given point O. Arbitrary points on F_1 and F_2 are denoted by the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively. Because of the centro–symmetry we have

Situation I: $\mathbf{R}_1,\ -\mathbf{R}_1 \in F_1;\ \ \mathbf{R}_2,\ -\mathbf{R}_2 \in F_2.$

Now shift F_2 over a (small) vector $\mathbf{s}$. We have

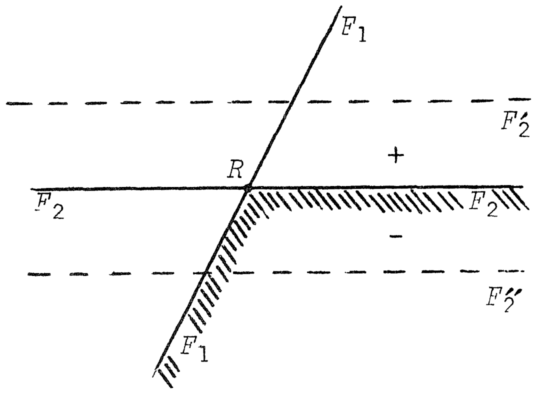
Situation II: $\mathbf{R}_2 + \mathbf{s},\ -\mathbf{R}_2 + \mathbf{s} \in F_2.$

Next, starting with situation I, shift F_2 over $-\mathbf{s}$. We have

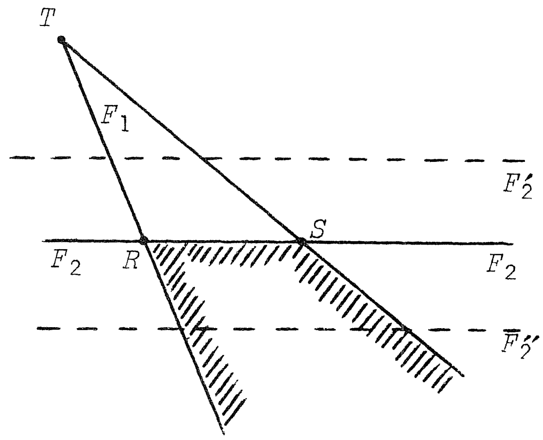
Situation III: $\mathbf{R}_2 - \mathbf{s},\ -\mathbf{R}_2 - \mathbf{s} \in F_2.$

Now we observe that situation III is obtained from situation II by an inversion with respect to O. The area that F_1 and F_2 have in common does not change under inversion, so shifting F_2 over $\mathbf{s}$ and over $-\mathbf{s}$ leads to the same area of intersection with F_1. Accordingly the intersection of F_1 and F_2 is extreme when $\mathbf{s} = \mathbf{0}$ (possibly equal to the intersection for $\pm\mathbf{s}$). Our conclusion is: ANY two centrosymmetric figures have an extreme in the area of their intersection when their centers coincide.

To determine whether the extreme is a maximum or minimum, we need to consider what happens at the points of intersection of F_1 and F_2. Consider a point of intersection R in situations I and II. (The shifted position is indicated with a prime.) The shift apparently leads to a gain in area (+) in this case. Next invert the region and consider the intersection point $-R$. The boundary of F_2 now shifts in the opposite direction to F_2'' and leads to a loss in area (–). Note that the total change is clearly a loss whenever the angle at the point of intersection and inside the common area is obtuse. In most cases with convex figures this angle will be obtuse, and any shift will lead to a loss (or no change) in the common area and consequently a maximum in the area of intersection for $s = 0$.



However, an acute angle is conceivable. We shall do a "worst–case" analysis in this case. Consider two neighbouring intersection points R, S of F_1 and F_2. TR and TS are tangent to F_1 at R and S. F_2 between R and S must lie on or in ΔRTS, because F_2 is convex. The smallest angle at R occurs if F_2 is straight between R and S. The angle is then acute, and the total change in area in the vicinity of R, when F_2 is shifted to F_2', is a gain. However the loss at S is larger so that there is a net loss in total. It must be remarked that the sum of the relevant angles (inside the common area) is equal to $\pi + \angle RTS$ (or larger if F_2 is not straight); apparently there are never two such neighbouring angles which are acute (for convex figures). There remains the possibility that one acute angle (and its inverted counterpart) remains after the pairs have been chosen. However this is not possible, because the total number of points of intersection is always a multiple of 4. To see this, choose a point P on F_1, outside F_2, with inverted point P'. In going from P to P' along F_1, one passes as



many times from outside to inside F_2 as from inside to outside F_2. So there are an even number of intersections between P and P' (which is just half of F_1). We conclude: for centrosymmetric convex figures, the area of intersection is a maximum for $s = 0$.

[*Editor's comments*: Penning, it seems, was the only reader who could get anywhere with this problem, and his result appears correct. Unfortunately the argument used is awkward and (to the editor's taste) a bit lacking in rigour. Is there a "nicer" proof?]

* * *

<u>1349</u>. [1988: 141] *Proposed by Josep Rifa i Coma, Institut "Jaume Callis", Barcelona, Spain.*

(a) Show that, if n is an even positive integer,

$$x^n(y - z) + y^n(z - x) + z^n(x - y) = 0 \tag{1}$$

has no solution in distinct nonzero real numbers.

(b) Show that (1) does have a solution in distinct nonzero real numbers if $n = 3$.

I. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

Rearranging (1) gives

$$x^n(y - z) - x(y^n - z^n) + yz(y^{n-1} - z^{n-1}) = 0. \tag{2}$$

Since each term is divisible by $y - z$, it is clear that there are solutions with $y = z$, and by symmetry there are also solutions with $x = y$ and with $x = z$. Since we are seeking solutions with distinct values for x, y and z, we will assume that $y \neq z$. Dividing (2) by $y - z$ gives

$$x^n - x(y^{n-1} + y^{n-2}z + \cdots + z^{n-1}) + yz(y^{n-2} + y^{n-3}z + \cdots + z^{n-2}) = 0. \tag{3}$$

(a) Let n be even. If (3) is viewed as a polynomial in x, Descartes' rule of signs [or differentiation] tells us that there are at most two real roots. But we already know of two roots, namely $x = y$ and $x = z$. Therefore there are no other real roots, and no solutions to (1) in distinct real numbers.

(b) Let n be odd. Now Descartes' rule tells us that there are at most three real roots. Since complex roots must come in pairs, there will always be an odd number of real roots; since we already know of two, there must be a third.

When $n = 3$, the third root is $-(y + z)$, as may be easily proved by the multiplication

$$(x - y)(x - z)(x + y + z) = x^3 - x(y^2 + zy + z^2) + yz(y + z).$$

Thus we have

$$x + y + z = 0, \quad x \neq y, \quad x \neq z, \quad y \neq z, \quad x, y, z \neq 0$$

as a solution to (1) when $n = 3$. This is not, however, as one might hope, a solution to (1) when $n = 5$.

II. *Solution by David Vaughan, Wilfrid Laurier University, Waterloo, Ontario.*

(a) If, say, $x = y$ then (1) becomes identically 0, so as long as at least two of x, y, z are equal they will be a solution.

Now assume (x,y,z) is a solution of (1) with x, y, z distinct. First suppose $x = 0$. Then (1) becomes

$$y^n z = z^n y.$$

Since neither y nor z is 0, this reduces to

$$y^{n-1} = z^{n-1},$$

and since n is even we know y must equal z, a contradiction. Thus suppose none of x, y, z are zero. We can divide (1) by x^{n+1} and set

$$u = \frac{y}{x} \neq 1, \qquad v = \frac{z}{x} \neq 1$$

to get

$$u - v + u^n(v - 1) + v^n(1 - u) = 0,$$

or

$$(u - 1)(v^n - 1) = (v - 1)(u^n - 1),$$

which (since u, $v \neq 1$) reduces to

$$v^{n-1} + \cdots + v + 1 = u^{n-1} + \cdots + u + 1. \tag{4}$$

Set

$$f(w) = w^{n-1} + \cdots + w + 1 = \begin{cases} \dfrac{w^n - 1}{w - 1}, & w \neq 1 \\ n, & w = 1. \end{cases}$$

Then

$$f'(w) = \begin{cases} \dfrac{(n - 1)w^n - nw^{n-1} + 1}{(w - 1)^2}, & w \neq 1 \\ \dfrac{n(n - 1)}{2}, & w = 1. \end{cases}$$

Now the sign of f' depends only on the sign of

$$g(w) = (n - 1)w^n - nw^{n-1} + 1.$$

Note that

$$\lim_{w\to+\infty} g(w) = \lim_{w\to-\infty} g(w) = +\infty,$$

$$g'(w) = (n-1)nw^{n-2}(w-1) = 0 \quad \text{only for} \quad w = 0, 1,$$

and

$$g(0) = 1, \quad g(1) = 0.$$

Thus $g(w) > 0$ for all $w \neq 1$, so $f'(w) > 0$ for all w. Thus f increases, so that $f(u) = f(v)$ if and only if $u = v$, and (4) is satisfied only if $u = v$, i.e. $y = z$. Hence in all cases when n is even, if (x,y,z) is a solution to (1) then at least two of x, y, z must be equal.

(b) We can always find solutions of (1) if two of x, y, z are equal or if one is zero and the other two are distinct and nonzero. If (x,y,z) is to be a solution of (1) with x, y, z distinct and nonzero, then equivalently (4) has a solution with $u \neq 1 \neq v \neq u$. But if n is odd,

$$\lim_{w\to+\infty} f(w) = \lim_{w\to-\infty} f(w) = +\infty.$$

Thus there must be a number C, sufficiently large so that $f(w) = C$ has at least two solutions, one positive and one negative, with the positive solution greater than 1. Thus (4) has solutions with $u \neq 1 \neq v \neq u$ so that for any odd n (1) has solutions (x,y,z) all distinct and nonzero.

III. *Solution by Murray S. Klamkin, University of Alberta.*

(a) More generally we can replace the equation by

$$F(x,y,z) \equiv G(x)(y-z) + G(y)(z-x) + G(z)(x-y) = 0$$

where $G(t)$ is a strictly convex function.

Since the equation is cyclic, we can assume without loss of generality that either (i) $x \geq y \geq z$ or (ii) $z \geq y \geq x$. For (i) we now show that $F(x,y,z) \geq 0$, with equality if and only if at least two of x, y, z are equal. First, we have

$$F(x,x,z) = F(x,z,z) = 0.$$

Next, it follows easily that, as a function of y, F is concave. Hence $F(x,y,z) > 0$ for $x > y > z$. Case (ii) is similar except F is convex so that $F < 0$ for $z > y > x$.

Incidentally, it is a known result (p.16 of D. Mitrinovic, *Analytic Inequalities*, Springer–Verlag, Heidelberg, 1970) that a real function G defined on an interval I is convex if and only if

$$G(x)(y-z) + G(y)(z-x) + G(z)(x-y) \geq 0$$

for any three points $x > y > z$ of I.

(b) Since

$$P \equiv x^3(y-z) + y^3(z-x) + z^3(x-y)$$

vanishes for any one of the conditions $x = y$, $y = z$, or $z = x$,

$$(x-y)(y-z)(z-x)$$

must be a factor of P. Since P is homogeneous of degree 4, the remaining factor must be homogeneous of degree 1, i.e. $x + y + z$. Thus all the distinct non–zero solutions of $P = 0$ are given by $x + y + z = 0$, for example $(x,y,z) = (1,2,-3)$.

Comment. It is a known result that if

$$P_n(x,y,z) = x^n(y - z) + y^n(z - x) + z^n(x - y) ,$$

where $n = 2,3,\ldots$, then

$$P_n(x,y,z) = -(x - y)(y - z)(z - x)H_n(x,y,z)$$

where $H_n(x,y,z)$ is the complete homogeneous polynomial of degree $n - 2$ in x, y, z, i.e. it consists of all possible terms of degree $n - 2$ and each term has coefficient 1. For example, $H_2 = 1$, $H_3 = x + y + z$ (as above), and

$$H_4 = x^2 + y^2 + z^2 + yz + zx + xy.$$

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; ROBERT E. SHAFER, Berkeley, California; and the proposer. J.T. GROENMAN, Arnhem, The Netherlands, and RICHARD I. HESS, Rancho Palos Verdes, California, solved (b) *and part of* (a).

Janous also extended (b) *to all odd positive integers n.*

* * *

1350. [1988: 141] *Proposed by Peter Watson–Hurthig, Columbia College, Burnaby, British Columbia.*

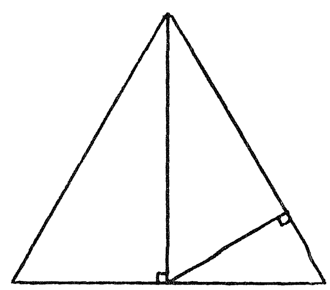
(a) Dissect an equilateral triangle into three polygons that are similar to each other but all of different sizes.

(b) Do the same for a square.

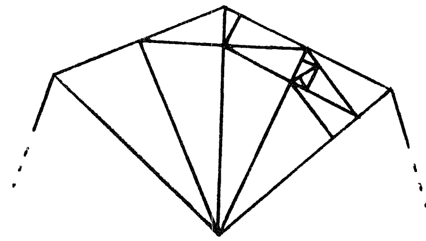
(c)* Can you do the same for any other regular polygon? (Allow yourself more than three pieces if necessary.)

I. *Solution by L.F. Meyers, The Ohio State University.*

(a) Every nonisosceles right triangle can be dissected into two noncongruent triangles which are similar to the original triangle. In particular, an equilateral triangle can be dissected into two 30–60–90 triangles, and one of these can be dissected into two more such.

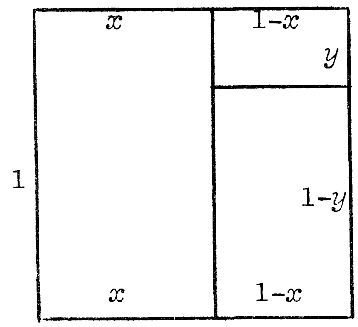


(c) A regular n–gon which is not a square can be dissected into $2n$ congruent



nonisosceles right triangles by drawing lines from the center of the n–gon to its vertices and the midpoints of its sides. These right triangles can then be dissected into as many noncongruent, but similar, right triangles as needed. [How many triangles? – Ed.]

(b) A square of side 1 can be dissected into three similar but noncongruent rectangles of dimensions



$$x \times 1, \quad (1-x) \times (1-y), \quad (1-x) \times y,$$

as in the figure. The requirement of similarity yields the equations

$$\frac{x}{1} = \frac{1-x}{1-y} = \frac{y}{1-x},$$

which imply

$$y = x(1-x) \quad \text{and} \quad x(1-y) = 1-x,$$

so that

$$x^3 - x^2 + 2x - 1 = 0,$$

whose only real root is given approximately by

$$x = 0.569840291.$$

Hence

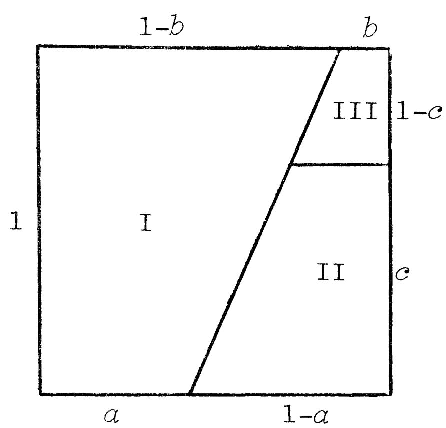
$$y = 0.2451223338.$$

The pair (x,y) really does work.

II. *Solution to* (b) *by the proposer.*

[The proposer solved part (a) as above.]

Comparing three pairs of corresponding sides shows that trapezoid I will be similar to trapezoid II if and only if



$$\frac{1}{c} = \frac{1-b}{1-a} = \frac{a}{x}$$

and that trapezoid II will be similar to trapezoid III if and only if

$$\frac{c}{1-c} = \frac{1-a}{x} = \frac{x}{b}.$$

One nice solution comes from choosing $c = 2/3$ thereby making $a = 3/7$, $b = 1/7$, $x = 2/7$.

III. *Solution to* (b) *by Richard K. Guy, University of Calgary.*

An answer is that the rectangles

$$a \times 1, \quad a^2 \times a, \quad (a^2 + 1) \times (a^2 + 1 - a)$$

fit together to make a square, and are the same shape if

$$a^3 - 2a^2 + a - 1 = 0.$$

An amusing corollary is that (since $7/4$ is such a good approximation to the root) rectangles

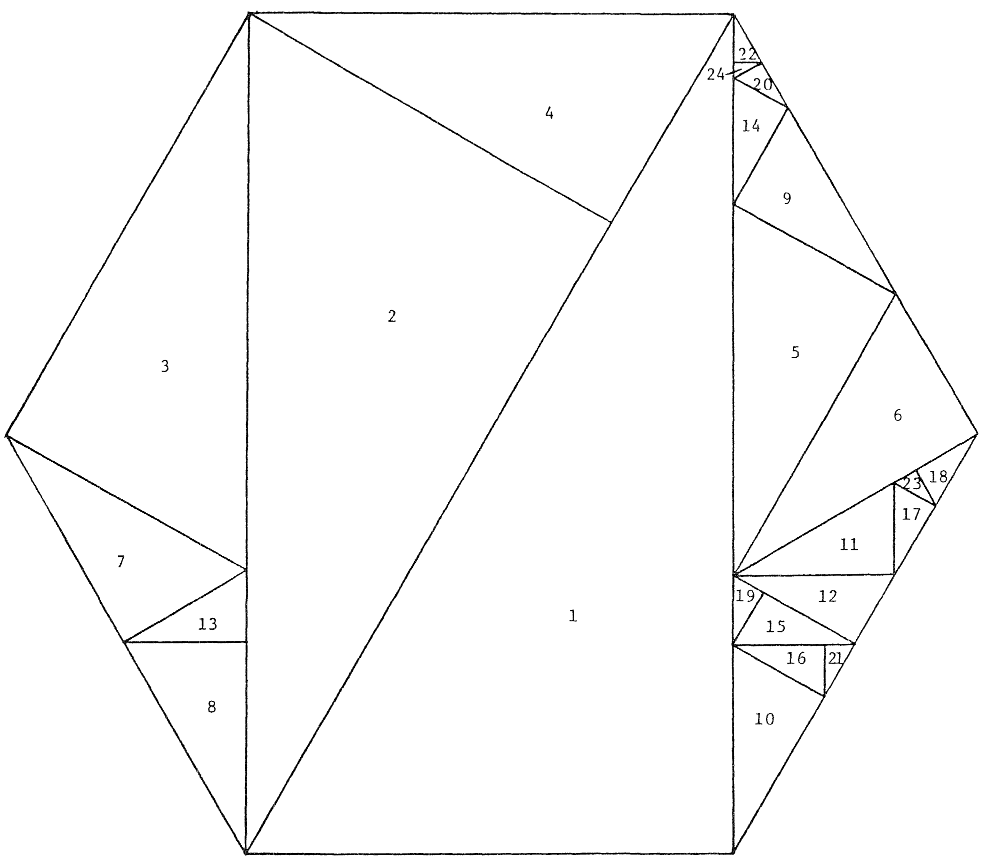
$$196 \times 112\ , \qquad 343 \times 196\ , \qquad 455 \times 260$$

are of the same shape and fit together to make a 455×456 rectangle.

IV. *Solution to* (c) *by Jordi Dou, Barcelona, Spain.*

[Dou also solved parts (a) and (b) as in solution I.]

A regular hexagon dissected into 24 similar triangles, all of different sizes:



triangle	area (hexagon=1)
1	1/3
2	1/4
3	1/9
4	1/12
5	4/81
6	1/27
7	1/36
8	1/48
9	4/243
10	1/64
11	1/81
12	1/108
13	1/144
14	4/729
15	1/192
16	1/256
17	1/324
18	1/432
19	1/576
20	1/729
21	1/768
22	1/972
23	1/1296
24	1/2916

Also solved (parts (a) *and* (b) *only) by RICHARD I. HESS, Rancho Palos Verdes, California; and P. PENNING, Delft, The Netherlands.* *MURRAY S. KLAMKIN, University of Alberta, solved all three parts,* (c) *for the hexagon.*

Readers are invited to try to reduce the number of similar but noncongruent polygons a regular n–gon can be dissected into, for $n > 4$. The numbers arising from Meyers' solution, whatever they are, are nowhere near 3.

* * *

1351. [1988: 174] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find eleven consecutive positive integers, the sum of whose squares is the square of an integer.

I. *Solution by Frank P. Battles and Laura L. Kelleher, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.*

Let u represent the middle (i.e. 6th) integer in the list of 11. We can readily reduce the problem to that of obtaining integer solutions to the equation

$$11(u^2 + 10) = X^2.$$

Clearly 11 divides X so we let $X = 11v$. Substituting, we can rewrite the above as the misnamed "Pell's" equation

$$u^2 - 11v^2 = -10$$

for which standard solution techniques exist (see for example Nagell's *Number Theory*, section 58). All solutions fall into two infinite families which may be described in the following way.

Let u_n be the rational part and v_n the coefficient of $\sqrt{11}$ in the expansions of

$$(1 + \sqrt{11})(10 + 3\sqrt{11})^n \qquad \text{or} \qquad (-1 + \sqrt{11})(10 + 3\sqrt{11})^n.$$

For $n = 2$, for example, we get from the first family $u_2 = 859$, $v_2 = 259$, which leads to the particular solution

$$854^2 + 855^2 + \cdots + 864^2 = (11 \cdot 259)^2.$$

As another approach we can find u_n and v_n, either recursively or in closed form, from the difference equations

$$u_{n+2} = 20u_{n+1} - u_n \ ,$$
$$v_{n+2} = 20v_{n+1} - v_n \ ,$$

where the conditions

$$u_0 = 1,\ v_0 = 1\ ;\quad u_1 = 43,\ v_1 = 13$$

determine the first family while

$$u_0 = -1,\ v_0 = 1\ ;\quad u_1 = 23,\ v_1 = 7$$

determine the second.

II. *Editor's comments.*

The following information was kindly contributed by Richard I. Hess and Stewart Metchette.

For which positive integers n do there exist n consecutive positive squares whose sum is a square? This problem, still unsolved, is discussed on pp. 319–322 of Dickson's *History of the Theory of Numbers* Vol. II, where results of Lucas, Martin, and Barisien are given. A more recent reference is a paper of Alfred [1] which gives some necessary conditions on n, and finds, with five unsettled cases, all values $n \leq 500$ with the above property. A follow–up paper of Philipp [2] handles these five cases. The complete list of such n then begins:

$$1, 2, 11, 23, 24, 26, 33, 47, 49, 50, 59, 73, 74, 88, 96, 97, \ldots .$$

The only value of $n > 1$ for which the *first* n positive squares $1^2, 2^2, \ldots, n^2$ sum to a square is $n = 24$. This fact turned up in Martin Gardner's column in the September 1966 *Scientific American* as the basis of a puzzle (see also pp. 139–149 of Gardner's *Mathematical Carnival*).

References:

[1] Brother U. Alfred, Consecutive integers whose sum of squares is a perfect square, *Math. Magazine* 37 (1964) 19–32.

[2] S. Philipp, Note on consecutive integers whose sum of squares is a perfect square, *Math. Magazine* 37 (1964) 218–220.

Also solved by AAGE BONDESEN, Royal Danish School of Educational Studies, Copenhagen; NICOS D. DIAMANTIS, student, University of Patras, Greece; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; JOHN FLATMAN, Timmins, Ontario; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

<u>1352</u>. [1988: 174] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine lower and upper bounds for

$$S_r = \cos^r A + \cos^r B + \cos^r C$$

where A, B, C are the angles of a non–obtuse triangle, and r is a positive real number, $r \neq 1, 2$. (The case $r = 1$ and 2 are known; see items 2.16 and 2.21 of Bottema et al, *Geometric Inequalities.*)

I. *Solution by the proposer.*

Case (i): $0 < r \le 1$. It is known ([1], item 2.16) that $1 \le S_1 \le 3/2$, with equality if and only if the angles are $(90^\circ,90^\circ,0^\circ)$ for the lower bound or $(60^\circ,60^\circ,60^\circ)$ for the upper bound. From the power mean inequality,

$$\left(\frac{S_r}{3}\right)^{1/r} \le \frac{S_1}{3} \le \frac{1}{2},$$

and since $S_r \ge S_1$ we have

$$1 \le S_r \le \frac{3}{2^r}$$

with equality as above.

Case (ii): $r \ge 2$. It is also known ([1], item 2.21) that $3/4 \le S_2 \le 1$, with equality if and only if the angles are $(90^\circ,90^\circ,0^\circ)$ for the upper bound or $(60^\circ,60^\circ,60^\circ)$ for the lower bound. Again from the power mean inequality,

$$\frac{1}{2} \le \left(\frac{S_2}{3}\right)^{1/2} \le \left(\frac{S_r}{3}\right)^{1/r},$$

and since $S_r \le S_2$ we have

$$\frac{3}{2^r} \le S_r \le 1$$

with equality as above.

Case (iii): $1 < r < 2$. Here $S_2 \le S_r \le S_1$ and

$$\frac{S_1}{3} \le \left(\frac{S_r}{3}\right)^{1/r} \le \left(\frac{S_2}{3}\right)^{1/2}.$$

Hence

$$\max\left\{\frac{3}{4}, \frac{1}{3^{r-1}}\right\} \le S_r \le \min\left\{\frac{3}{2}, 3^{1-r/2}\right\}.$$

It would be of interest to find the best bounds for this case.

Incidentally, the best lower bound for the case $r \le 0$ is given by

$$S_r \ge \frac{3}{2^r},$$

with equality only for angles $(60^\circ,60^\circ,60^\circ)$. This follows from

$$\cos A \cos B \cos C \le \frac{1}{8}$$

and the A.M.–G.M. inequality:

$$\frac{S_r}{3} \ge (\cos A \cos B \cos C)^{-r/3} \ge \frac{1}{2^r}.$$

Special cases of the latter inequality for $r = -1$ and -2 are also given in [1], items 2.45 and 2.46. However, it was not noted in [1] that the case for $r = -1$ is only valid for non–obtuse triangles.

Finally, it is to be noted that, by using the pedal angle transformation

$$A = \frac{\pi - A'}{2}, \qquad B = \frac{\pi - B'}{2}, \qquad C = \frac{\pi - C'}{2}$$

and dropping primes, S_r becomes

$$S_r' = \sin^r\left(\frac{A}{2}\right) + \sin^r\left(\frac{B}{2}\right) + \sin^r\left(\frac{C}{2}\right) .$$

As a result, all the inequalities above are valid with S_r replaced by S_r' but this time not only for non–obtuse triangles but for all triangles (possibly degenerate).

Reference:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1968.

II. *Partial solution by Richard I. Hess, Rancho Palos Verdes, California.*

To look for extrema of S_r consider

$$\begin{aligned}\frac{1}{r}dS_r &= -\cos^{r-1}A \sin A\, dA - \cos^{r-1}B \sin B\, dB + \cos^{r-1}C \sin C(dA + dB)\\ &= (\cos^{r-1}C \sin C - \cos^{r-1}A \sin A)dA\\ &\qquad\qquad + (\cos^{r-1}C \sin C - \cos^{r-1}B \sin B)dB,\end{aligned}$$

where (from $A + B + C = 180^\circ$) $dA + dB + dC = 0$ has been imposed. An extremum of S_r not at an edge point (A, B or C equals 0° or 90°) must have $dS_r = 0$, i.e.

$$\cos^{r-1}A \sin A = \cos^{r-1}C \sin C = \cos^{r-1}B \sin B.$$

By examining the function

$$f(x) = \cos^{r-1}x \sin x$$

and its derivative

$$f'(x) = \cos^{r-2}x(1 - r \sin^2 x)$$

one can see that $f(x)$ can have the same value for at most two values of x. It therefore follows that at an extremum of S_r, not an edge point, two of the angles A, B, C will be equal.

An extremum on an edge point occurs either with one angle (say C) equal to 90°, in which case

$$S_r = \cos^r A + \sin^r A,$$

or with one angle equal to 0° and the other two equal to 90°, in which case $S_r = 1$. In the first case

$$\frac{1}{r}S_r' = -\cos^{r-1}A \sin A + \sin^{r-1}A \cos A$$

equals 0 if and only if

$$\tan A = \tan^{r-1}A,$$

i.e.

$$\begin{array}{lll} A = 0^\circ,\ 45^\circ, \text{ or } 90^\circ & \text{if} & r > 1, \\ A = 45^\circ & \text{if} & 0 < r < 1. \end{array}$$

In any case two of A, B, C are again equal.

Thus to find the extrema of S_r one can put $A = B$ and examine the extrema of

$$\begin{aligned} S_r &= 2\cos^r A + \cos^r(180^\circ - 2A) \\ &= 2\cos^r A + (1 - 2\cos^2 A)^r \end{aligned} \tag{1}$$

where $45^\circ \le A \le 90^\circ$. Then

$$\frac{dS_r}{dA} = -2r\cos^{r-1}A \sin A + 4r(1 - 2\cos^2 A)^{r-1}\cos A \sin A.$$

This is 0 when

$$2\cos A = \left(\frac{\cos A}{1 - 2\cos^2 A}\right)^{r-1}, \tag{2}$$

in which case

$$\begin{aligned} S_r &= 2\cdot 2\cos^2 A(1 - 2\cos^2 A)^{r-1} + (1 - 2\cos^2 A)^r \\ &= (1 - 2\cos^2 A)^{r-1}(1 + 2\cos^2 A). \end{aligned}$$

The value $A = 60^\circ$ always satisfies (2), and in this case

$$S_r = 3/2^r. \tag{3}$$

Endpoints $A = 45^\circ$ and $A = 90^\circ$ give respectively

$$S_r = 2^{1-r/2} \tag{4}$$

and

$$S_r = 1. \tag{5}$$

Thus extrema of S_r come from (3), (4) and (5), plus any further solutions of (2). Considering only (3)–(5) we have the bounds:

$$\left.\begin{array}{lll} \text{for} & 0 < r \le \frac{\ln 9/4}{\ln 2} \approx 1.169925\,, & 1 \le S_r \le 3/2^r\,; \\ \text{for} & \frac{\ln 9/4}{\ln 2} \le r \le \frac{\ln 3}{\ln 2} \approx 1.5849625\,, & 1 \le S_r \le 2^{1-r/2}; \\ \text{for} & \frac{\ln 3}{\ln 2} \le r \le 2\,, & \frac{3}{2^r} \le S_r \le 2^{1-r/2}; \\ \text{for} & r \ge 2\,, & \frac{3}{2^r} \le S_r \le 1\,. \end{array}\right\} \tag{6}$$

For $r < 1$,

$$\left(\frac{\cos A}{1 - 2\cos^2 A}\right)^{r-1} = \left(\frac{1 - 2\cos^2 A}{\cos A}\right)^{1-r}$$

is zero at $A = 45^\circ$ and rises monotonically to ∞ at $A = 90^\circ$. Since $2 \cos A$ falls monotonically from $\sqrt{2}$ at $A = 45^\circ$ to 0 at $A = 90^\circ$, there is only one solution (namely $A = 60^\circ$) to (2) in this case. Thus the bounds

$$1 \leq S_r \leq 3/2^r$$

given above for this case are true (and best possible).

For $1 < r < 2$ there are solutions to (2) other than $A = 60^\circ$. These were numerically checked and found to give values of S_r falling between the above extrema.

Editor's note: Hess's arguments are deficient at this point. Here is an attempt to fill in some of the gap.

Let

$$g(x) = \frac{x}{1 - 2x^2}, \qquad 0 < x < \frac{1}{\sqrt{2}}.$$

Then we may rewrite (2) as

$$(g(x))^{r-1} = 2x.$$

One solution is $x = 1/2$ (corresponding to $A = 60^\circ$). We first verify that for $r > 2$ there are no other solutions. Note that

$$g'(x) = \frac{1 + 2x^2}{(1 - 2x^2)^2} > 0,$$

$$g''(x) = \frac{4x(2x^2 + 3)}{(1 - 2x^2)^3} > 0 \qquad \text{for} \quad 0 < x < \frac{1}{\sqrt{2}},$$

and thus g is increasing and concave on $(0,1/\sqrt{2})$. Hence, since $r > 2$,

$$h(x) = (g(x))^{r-1}$$

will also be increasing and concave on $(0,1/\sqrt{2})$, since

$$h'(x) = (r - 1)(g(x))^{r-2}g'(x) > 0$$

and

$$h''(x) = (r - 1)(r - 2)(g(x))^{r-3}(g'(x))^2 + (r - 1)(g(x))^{r-2}g''(x) > 0.$$

From

$$h(0) = 0 = 2\cdot 0,$$

it is clear that $h(x) = 2x$ will have only one positive solution, namely $x = 1/2$. Thus the above bound (6)

$$\frac{3}{2^r} \leq S_r \leq 1$$

is true and best possible if $r > 2$. This of course agrees with the proposer (see I).

It remains to verify Hess's last paragraph in the case $1 < r < 2$. Let x_0 be the solution, other than 1/2, of

$$h(x) = 2x , \qquad 0 < x < 1/\sqrt{2},$$

(i.e. (2)) in this case. (x_0 exists and is unique unless $r = 4/3$, when there is a double root at $x = 1/2$.) We must show that

$$f_r(x_0) = 2x_0^r + (1 - 2x_0^2)^r$$

(i.e. (1)) lies within the various bounds, depending on r, given above in (6). What seems to be true from numerical results, and what would be enough, is that

$$1 < f_r(x_0) < 2^{1-r/2} , \quad 1 < r < 2 \qquad (7)$$

and

$$f_r(x_0) < \frac{3}{2^r} , \quad 1 < r < \frac{4}{3} . \qquad (8)$$

Can any reader prove (7) and (8)?

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium, who obtained the above bounds for $0 < r \le 1$ *and* $r \ge 2$ *and bounds weaker than the proposer's for* $1 < r < 2$.

* * *

CHARLES W. TRIGG

Readers of *Crux Mathematicorum* will be saddened to learn of the death of Charles W. Trigg of San Diego, California on June 28 at the age of 91. Professor Trigg was well–known to all lovers of mathematics problems and recreational mathematics, his name having appeared over the years in practically every publication devoted to the subject. In particular he has been featured many times in *Crux*, from its second volume right up until a couple of years ago. The last instance was in [1987: 238] with the printing of his efficient solution to *Crux* 1149.

* * *

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