MATHEMATICAL SPECTRUM

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Logic — Friend or Foe?

KEITH AUSTIN, *University of Sheffield*

Keith Austin is a Lecturer in Pure Mathematics at the University of Sheffield. He believes that, just as few people fancy a raw egg, so few people find delight in raw mathematics in its original state. Instead, the mathematics should be 'cooked' or treated in some way so as to bring out the most appetizing side of its flavour, and then garnished with other ingredients so that it is attractive to the mind's palate.

'Here is the greengrocer's bill, Mr Holmes.'

'Thank you, Mrs Hudson. Have a look at this, Watson.'

I read the bill. It was for five pounds of potatoes, two pounds of peas, three pounds of brussels sprouts and four pounds of carrots, and came to four shillings and threepence.†

'Here are three others for the previous weeks. Mrs Hudson was assured that the prices per pound have not changed. What is the price of peas, Watson?'

'I will let w, x, y and z be the prices per pound of the four vegetables, in pence. Then the four bills can be expressed as equations so:

$$\begin{aligned} 5w+2x+3y+4z &= 51 \quad (1), \\ 3w+3x+5y+6z &= 65 \quad (2), \\ 21w+4x+3y+6z &= 113 \quad (3), \\ 7w+2x+y+2z &= 40 \quad (4). \end{aligned}$$

'Now I calculate $-10\times(1)+5\times(2)+2\times(3)-(4)$ and find $x = 1$.'

'True, Watson, but I calculate $2\times(1)-(2)-(3)+2\times(4)$ and find $x = 4$.'

Before I could reply, a young woman rushed into the room, threw herself at Holmes' feet and burst into tears. Mrs Hudson and I were able to calm her with the aid of a cup of tea and she then told us her story. Wenda, for that was her name, came from one of the aristocratic families of Borosnia, and had four sisters, Victoria, Xanda, Yonda and Zelda. The five girls had fallen in love with five brothers, Albert, Bertrand, Charles, Darcy and Edward, from another Borosnian family of high standing. Unfortunately there was a 200-year feud between the two families. However, it was eventually agreed that the marriages could take place on the understanding that none of the sisters was ever to meet any of the brothers apart from her husband.

†Watson is of course using the old British currency, where 12 pence = 1 shilling.

All had gone well until the Borosnian revolution, when the five couples had escaped to England. The young woman had unfortunately been lost overboard while crossing the Channel and had lost her memory, although it had now partially returned. She had certain information regarding her sisters and their husbands but needed the help of the great detective, Sherlock Holmes, to find the name of her husband.

She had seven clues. The first was that Bertrand or Charles had married Wenda or Zelda. I abbreviated all the clues as follows:

(1) $B \vee C - W \vee Z$; (2) $A \vee B - W \vee X$; (3) $C - X \vee Y \vee Z$;

(4) $B \vee C \vee D - V$; (5) $A \vee C \vee E - W \vee Y$; (6) $B \vee D - Y \vee Z$;

(7) $A \vee E - V \vee Z$.

'Perhaps I can help, Holmes,' I said. 'From (3) and (4) we have (8) $B \vee D - V$ and from (7) and (8) we have (9) $A \vee E - Z$. Now (6) and (9) give (10) $B \vee D - Y$, (5) and (10) give (11) $A \vee C \vee E - W$, (3) and (11) give (12) $A \vee E - W$, (8) and (10) give (13) $B - V \vee Y$, (9) and (12) give (14) $A - W \vee Z$, (2) and (13) give (15) $A - W \vee X$ and, finally, (14) and (15) give (16) $A - W$, so Wenda is married to Albert.'

'True enough, but (1) and (9) give (17) $B \vee C - W$, and (3) and (17) give (18) $B - W$, so Wenda is married to Bertrand.'

At this Wenda fainted.

'Quick, Watson, there is no time to lose. Send a message to Inspector Lestrade at Scotland Yard to come here immediately with a score of his best men. If I am not very much mistaken, we must act with all haste if we are to find Wenda's husband and catch Professor Moriarty.'

'Moriarty, but how?' I gasped.

'No one but that arch-fiend would misuse logic in such an evil way to mock this young woman in distress.'

'Were we wrong then in our use of logic to find Wenda's husband?'

'Our deductions were correct, but we overlooked one thing.'

'What was that?'

'We had assumed that all the given information was correct. In fact, as we reached the ridiculous situation of Wenda's being married to two brothers, we can conclude that the given information was not correct.'

'This sounds very like the matter of the greengrocer's bills.'

'Well done, Watson. Moriarty has overstretched himself with those bills. That greengrocer can bear a little investigating. Ah! Here is Lestrade with his men now. Come, Watson, the game's afoot.'

It was dawn as Holmes and I walked back through the streets of London, after seeing Wenda reunited with her husband and Moriarty's henchmen put behind bars.

'You know, Watson, logic is just like many of man's discoveries; it can be used for good or evil. In the hands of someone like Moriarty it can be used to deceive men into believing that which is false.'

Just then the sun shone through the thinning mists across the city and lit on the dome of St. Paul's.

'There must always be a Guiding Hand so that logic becomes our ally in the search for Truth and Justice.'

Miss Fatland 1985

The following problem was set in a problem-solving competition at the Royal Wolverhampton School and has been sent to us by J.N. MacNeill.

In Fatland, it is considered physically beautiful to be the shape of a snooker ball, that is, to be spherical.

"Spherical it beautiful," said Helen, eating her spaghetti sandwich.

"I'm more spherical than you are," said Julia, eating her potato risotto.

"Don't eat with your mouths full," said someone else. "I mean don't speak with your mouths full. In any case, you're either spherical or you're not; you can't be *more* spherical."

"Well ..." Julia thought. "I'm *nearer* to being spherical than Helen is."

"No you're not," countered Helen, "because my circumference is bigger than your circumference!" [In Fatland, the word 'waist' is almost obscene.]

"I agree that you're bigger than me all round," said Julia with a tinge of envy, "but we're not talking about *size*; we're talking about *shape*, and my shape is more like a sphere than yours is!"

"How can you say that? How can you tell," demanded Helen, "if one shape is more like a sphere than another is?"

Helen's question is a good one.

Invent a way of deciding which of two given solid shapes is nearer to being spherical. It does not matter if your method would be difficult to use in the case of complicated shapes like Julia and Helen.

Any resemblance to any living person is entirely to be expected.

Mid-Square Sequences

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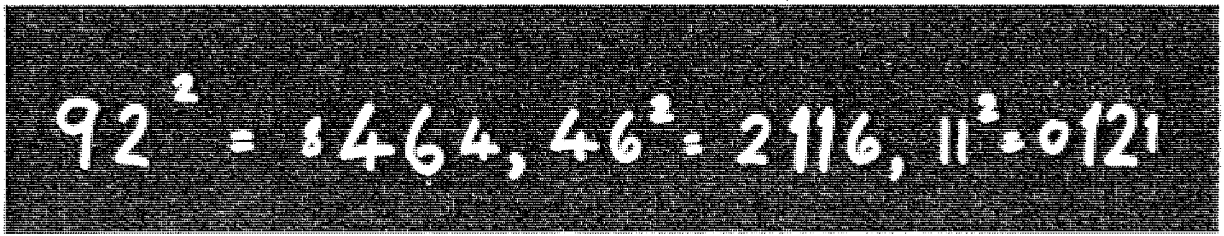
H.J. Godwin taught in the Universities of Wales (at Swansea) and London (at Royal Holloway College) from 1945 to 1982. He was Professor and Head of the Department of Statistics and Computer Science at Royal Holloway College from 1968 to 1982. His main interest is number theory and, in particular, the application of computers to it.



It is one thing to have a computer and another thing to find something useful to do with it. This article describes an area of mathematics about which little is known, and where exploration by computer could yield interesting results.

Since $92^2 = 8464$, $46^2 = 2116$, $11^2 = 0121, \ldots$, the numbers 92, 46, 11, 12, ... form what may be termed a *mid-square sequence*. At one time these were considered as a source of pseudo-random numbers (i.e. numbers that appear to be random, but are in fact determinate) but they are unsatisfactory in that capacity for a reason that will appear later. Nevertheless such sequences are interesting number-theoretic objects, and investigations of them could yield facts upon which eventually a proper theory might be built.

For a given starting value x we obtain the next term as follows: square x, divided by 10, and then take the remainder when the quotient (neglecting the remainder) is divided by 10^2. We can generalise this by replacing 10 by a given integer $b > 1$, which we call the *base*. To make the rule more concise we can use two established notations. For any number x, $[x]$ means the largest integer not greater than x, and, for integers l, m and n ($n \geqslant 1$), $l \equiv m \pmod{n}$ means that n divides $l - m$. We shall in fact, given m and n, always choose l so that $0 \leqslant l \leqslant n-1$, i.e. so that l is just the remainder when m is divided by n. Then the sequence is obtained, starting with x_1, by defining $x_2 \equiv [x_1^2/b] \pmod{b^2}$, $x_3 \equiv [x_2^2/b] \pmod{b^2}$, and so on. We shall call x_2 the *parent* of x_1 and x_1 an *offspring* of x_2, and similarly for other consecutive terms. This is justified by the fact that a term has only one parent, but a parent can have more than one offspring. For example, with $b = 10$, we have 84 as the parent of $29, 43$ and 62.

We shall call parents, parents' parents, etc. *ancestors*, and offspring, offspring's offspring, etc. *descendants* in accordance with everyday usage.

Since we take remainders on division by b^2, and there are only b^2 of these (namely $0, 1, \ldots, b^2-1$) we must, as we continue a sequence, eventually reach a term we have had before. For example, the sequence used in the introduction is $92, 46, 11, 12, 14, 19, 36, 29, 84, 5, 2, 0, 0, \ldots$ and the sequence beginning with 79 is $79, 24, 57, 24, \ldots$. Hence every sequence ends in a repeated term or set of terms that we call a *loop*; the number of terms in the loop may be only 1 as in the sequence beginning with 92, or more than 1 as in the other example. If $b = 22$ and we start with 177, we eventually reach a loop of 39 numbers of which the smallest is 28.

We call the number of terms before a sequence begins to repeat the *length* of the sequence, and denote the length, starting from x, by $l(x)$. In the above examples with $b = 10$ we have $l(92) = 12$ and $l(79) = 3$. If we are trying to generate pseudo-random numbers we do not want $l(x)$ to be small, but, as will be seen below, $l(x)$ rarely exceeds $2b$, and it is this fact that led to the abandonment of mid-square sequences as a source of pseudo-random numbers.

In everyday speech we refer to a collection of ancestors and offspring as a family tree, and we define the set of numbers all leading to the same loop as a *tree*. (This term is also used, in the same sense, in computer science, for any group of objects that can be put into this ancestor and descendant relationship.) We now show that, for any $b > 1$, there must be at least two trees. First, 0 leads to itself and so is a loop of length 1 at the base of one tree, and, second b leads to itself and is a loop of length 1 at the base of a different tree. For $b = 2$ and $b = 4$ these are the only trees, but for all other values of b that have been examined there are further trees. Let us denote by n_0 the number of elements in the tree with loop 0 and by n_b the number of elements in the tree with loop b. If $x < b$ then $x^2/b < b$ and so the parent of x is $[x^2/b]$; since also $x^2/b < x$ we have $[x^2/b] < x$. Hence if we start with $x < b$ we eventually reach 0, and so the tree with loop 0 certainly contains $b-1, b-2, \ldots, 1, 0$, and we have $n_0 \geqslant b$. On the other hand, if $x = b^2-b$, then $[x^2/b] = b^3-2b^2+b \equiv b \pmod{b^2}$ and so the tree with loop b contains not only b but also b^2-b, which is different from b if $b \geqslant 3$. Hence, for $b \geqslant 3$, we have $n_b \geqslant 2$. For quite a lot of values of b (see below for details) $n_b = 2$, but the only value of b for which n_0 is known to be b is 3.

If x is a multiple of b, say $x = kb$, then the parent of x is the remainder when k^2b is divided by b^2 and so is also a multiple of b. Hence the loop of the tree to which x belongs contains only multiples of b and we call such a tree a *multiple tree*, the others being *non-multiple trees*. For a few values of b (see below) there are no non-multiple trees.

The average length of a sequence is $[l(0)+l(1)+\ldots+l(b^2-1)]/b^2$; we shall denote this by $a(b)$. This is the important number from the point of view of constructing pseudo-random sequences. Unless it is large one soon gets into a loop and the sequence is definitely non-random.

The number of trees for a given base b will be denoted by $T(b)$, the maximum length of a loop by $c(b)$, and the largest value of $l(x)$ by $m(b)$.

At this stage the reader may like to verify that $a(10) = 5.77$, $T(10) = 5$, $c(10) = 2$ and $m(10) = 15$ (this maximum being attained for $x = 42$ and $x = 69$). (The base 10 is not particularly interesting, but it is an easy one from the point of view of computation—one needs only a table of squares and then the parent of each term can be read directly.) It will be noticed that, with $b = 10$, the number 50 forms a tree on its own. The Swedish mathematician B. Jansson, in a book published in 1966, defined such a tree as a *samoan of order 1*, a *samoan of order m* being a tree with a loop of m numbers and no other members. He raised the question of whether samoans of order greater than 1 exist; as we shall see later, they do.

The loop-lengths of non-multiple trees do not seem to follow any simple pattern, but those for multiple trees do, and the reader may like to investigate this.

We now a give a summary of the results for $3 \leqslant b \leqslant 200$. We have $n_b = 2$ for $b = 7, 9, 11, 18, 22, 23, 27, 46, 47, 67, 86, 94, 103, 121, 162, 163$. The ratios n_0/b^2 and n_b/b^2 vary widely; n_0/b^2 has least value 0.10 (for $b = 57$) and largest value 0.84 (for $b = 174$), whereas n_b/b^2 has greatest value 0.70 (for $b = 68$) but is less than 0.1 for 154 values of b. Further, $a(b)/b$ varies from 0.23 (for $b = 120$) to 1.12 (for $b = 22$). Also, $m(b)/b$ varies from 0.65 (for $b = 40$) to 2.89 (for $b = 189$). Only 13 values of b have $m(b) \geqslant 2b$. Next, $T(b)/\sqrt{b}$ has least value 0.39 (for $b = 167$) and greatest value 2.56 (for $b = 30$). We find that $c(b)$ varies much more widely than does $T(b)$ and $c(b)/\sqrt{b}$ lies between 0.10 (for $b = 96$) and 13.78 (for $b = 165$). There are no non-multiple trees for $b = 4, 6, 12, 14, 51, 84$ and 138 (and for $b = 2$ also). There are 99 samoans of order 1, 8 of order 2, 6 of order 3, and one of order 6, namely the loop 170, 390, 2055, 2307, 734, 1804, with $b = 74$.

There are many questions that can be asked and to which a study of bases greater than 200 may provide answers. For example:

(1) Does $a(b)/b$ ever exceed the value it attains for the comparatively small value $b = 22$?

(2) Are there any more bases having no non-multiple trees?

(3) Are there samoans of orders 4 or 5 or greater than 6?

We now turn to the practical problem of investigating mid-square sequences using a computer. A lot depends on what computer is available, and in particular on its word-length (i.e. how big is the largest integer that can be held in one location) and on its storage capacity.

We can keep down the size of integers to some extent by calculating the parent of x as follows. Let $x = pb+q$ with $0 \leqslant q \leqslant b-1$ and let $r \equiv q^2 \pmod{b}$ and $s \equiv p^2 \pmod{b}$. Then the parent of x is the remainder when $sb+2pq+(q^2-r)/b$ is divided by b^2.

We can now consider in turn each value of x such that $0 \leqslant x \leqslant b^2-1$, forming the sequence beginning with x until we have gone through a loop. This will give us $l(x)$ and eventually we shall find how many trees there are, and what their loop-lengths are. However this is a very wasteful procedure because the same value of x can be encountered many times, once when we start from it and on other occasions when we reach it having started from another number. If sufficient storage is available, it pays to allot a set of $2b^2$ locations (the technical term for this is an *array*), two to each of the numbers $0, \ldots, b^2-1$, and set their values initially to zero. Having pursued the sequence beginning with x, we know $l(y)$ for each term y that we have met, and we set the yth pair of elements of the array to $l(y)$ and to a number representing the loop, e.g. its smallest member. When we come to choose another x we need consider only those for which the value of l in the array is still zero, because the others have already been dealt with. When the last x has been used we have all the values of l in store, and it is easy to evaluate $a(b)$ and $m(b)$. There is one difficulty: usually if y is the parent of x then $l(y) = l(x)-1$, but if x and y are in a loop then $l(y) = l(x)$. Hence it may be desirable to do a preliminary study to find out how many trees there are [i.e. run through the x's without recording $l(x)$] and to deal with the numbers in the loops before doing the main calculation.

Finally, when the numbers get too large for the space and time available, there are always mid-cube sequences to be investigated. These are easier to handle because the parent of x is $[x^3/b] \pmod{b}$ and so only numbers less than b, instead of less than b^2, have to be considered. For example, with $b = 10$, $7, 4, 6, 1, 0$ is a sequence. There is just one tree, and the average length is 2.9. It seems that no study has been made of mid-cube sequences, and so the field is wide open for exploration.

Did you know that every integer is the sum of five cubes? For example,

$$7 = (-57)^3+(-55)^3+7^3+56^3+56^3.$$

Can you prove this?

The Best Ways of Living

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The author is Professor of Zoology at the University of Leeds. Most of his research has been in the field of animal mechanics, the application of engineering mechanics to the study of animals. Recently he has been studying the mechanics of walking and running of people, dogs, dinosaurs and the larger African mammals, largely by film analysis and mathematical modelling.

The structure and behaviour of animals are moulded by two very powerful optimizing processes. First, evolution by natural selection tends to make animals inherit the best possible set of characteristics from their parents. Secondly, the process of learning enables animals to adjust their behaviour in the light of experience to give the best possible chances of success in finding food, escaping enemies etc.. Zoologists, realizing this, have recently been making much use of optimization theory. They have been checking whether animals actually optimize the particular characteristics which their theories say should be optimized. Here are two examples.

The larvae of ladybirds feed on greenfly and other aphids, earning the gratitude of gardeners (reference 2). They eat the soft flesh, leaving the outer case which encloses the aphid's body like a suit of armour. The more they eat of a particular aphid, the harder it is to get more and it would be very difficult to get the last trace of flesh; it is better to give up at some stage and look for another aphid. So much is fairly obvious, but it is harder to decide exactly when is the best time for giving up. Do ladybirds choose their giving-up times, so as to get the highest possible rate of food intake?

Figure 1 (a) shows the results of an experiment in which starved ladybird larvae were each given an aphid to eat. The aphid was weighed at intervals of ten minutes, to discover how much had been eaten. In experiments (i),

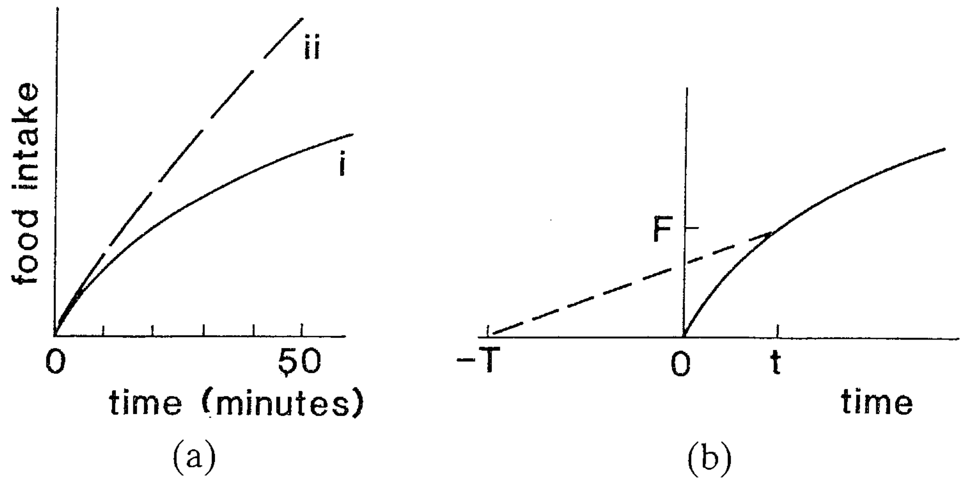


Figure 1 (a). Graphs of food intake against time for ladybird larvae given (i) a single aphid and (ii) an aphid that is replaced every ten minutes. (Based on reference 2)

Figure 1 (b). The geometric construction discussed in the text. The curve is curve (i) from (a)

the same aphid was returned for successive ten-minute periods, and the rate of food intake declined as the remaining flesh became harder to get. In experiments (ii) a new aphid was supplied every ten minutes and the initial rate of food intake was almost maintained, showing that the ladybird was not getting satiated.

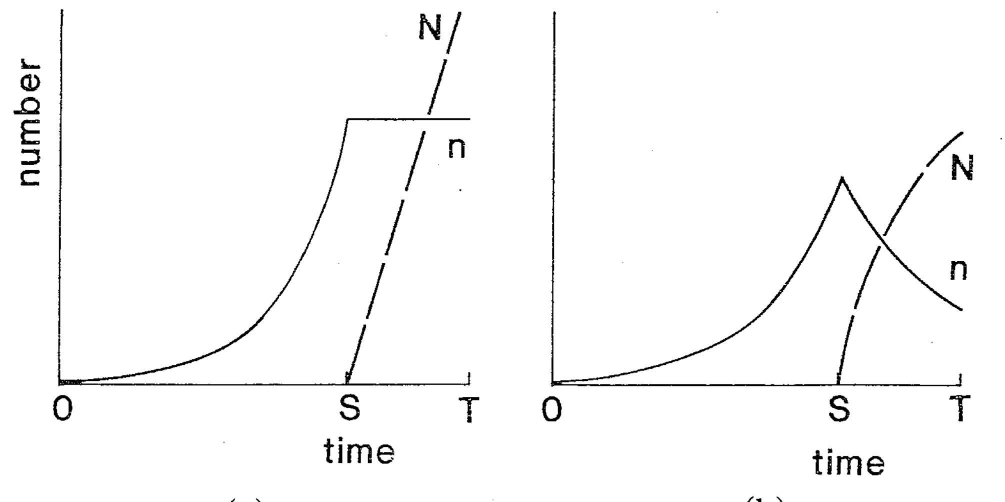
Let a ladybird take (on average) time T to find a new aphid. Let it feed on this aphid for time t, obtaining a quantity F of food. Its mean rate of food intake is $F/(t+T)$. This is also the gradient of the broken line in figure 1 (b). The problem of maximizing the rate of food intake for a given search time T is equivalent to giving the broken line the greatest possible gradient. Plainly, this can be done by choosing the feeding time t so that the broken line is tangential to the feeding curve. It was shown in this way that for a search time of 15 minutes, the optimum feeding time was 20 minutes. For a search time of 30 minutes, the optimum feeding time was 35 minutes.

Does the ladybird behave as the theory suggests? It is easier to test it in laboratory experiments than in the less controlled conditions of the garden. Ladybird larvae were put in trays of standard size with different numbers of aphids, to give different search times. (It took longer to find an aphid when there were only a few in the tray.) They were watched for several hours, and search times and feeding times were noted as they ate a series of aphids. When aphids were plentiful and search times were short, mean feeding times were almost exactly the theoretical optimum. When aphids were scarce and search times were long, feeding times were just a little longer than the corresponding optimum.

Now for a second problem, again about insects (reference 3). Wasps live in colonies, each occupying its own nest and consisting of a queen and her offspring. The queen can produce two kinds of offspring, workers and reproductives. Workers act as food collectors and nursemaids for their younger brothers and sisters, but are incapable of breeding, and die when winter comes. Reproductives may survive the winter and will be capable of breeding by the spring, but do no work. (Female reproductives become queens.) A queen could in principle produce workers or reproductives or both at each stage of the season. What should her policy be?

We shall adopt the hypothesis that the queen behaves in such a way as to produce as many reproductives as possible by the end of the season. Such behaviour seems likely to be favoured by natural selection. What behaviour is required? A queen who produces only reproductives will not produce many, because she will have no workers to help look after them. It seems likely that the best policy will involve producing both workers and reproductives. It can be shown (for the proof, see reference 1) that the best policy is to produce workers only until a critical day and then to switch suddenly to producing reproductives only. When is the critical day?

Let the season last from time 0 to time T. Let a colony consist, at any time t, of n workers and N reproductives. Let it produce workers only until time S, and thereafter reproductives only [figure 2(a)].



(a) (b)

Figure 2. Schematic graphs showing how the numbers of workers (n) and reproductives (N) in a wasp colony would change during a season (a) if there were no deaths and (b) if deaths occurred

We shall assume that:

(i) the rate of production of offspring is proportional to the number of workers available to rear them;

(ii) no deaths occur until after time T.

From time 0 until time S, workers only are being produced and, by assumption (i),

$$\frac{\mathrm{d}n}{\mathrm{d}t} = rn,$$

where r is a constant. Hence, since $n = 1$ when $t = 0$ (the queen is counted here as a worker, because she has to rear the first few offspring herself),

$$n = \exp(rt)\dagger$$

and the number of workers eventually produced is

$$n_S = \exp(rS).$$

From time S to time T reproductives are produced. By assumption (i),

$$\frac{\mathrm{d}N}{\mathrm{d}t} = Rn_S = R\exp(rS),$$

where R is a constant. Production at this rate continues for a period $T-S$, so the final number of reproductives is

$$N_T = (T-S)R\exp(rS).$$

We want to know the optimum changeover time, that is, the value of S that maximizes N_T. Now

$$\frac{\mathrm{d}N_T}{\mathrm{d}S} = R\{-\exp(rS)+(T-S)r\exp(rS)\}$$

which is zero for the optimum, so that, for the optimum,

$$T-S = \frac{1}{r}.$$

Life is not as simple as that. Workers die, so that n increases less fast than suggested before time S, and declines after time S. The rate of production of reproductives, $\mathrm{d}N/\mathrm{d}t$, declines as workers die. Figure 2 (b) is more realistic than figure 2 (a) and the equations should be made correspondingly more complicated.

The constant r and the mortality rate were measured for a species of hornet (large wasps) living in Israel. It was calculated that the optimum value of $T-S$ was 27 days, so that colonies should switch from producing workers to producing reproductives 27 days before the end of the season. It was observed, however, that they made the switch on average 40 days before the end of the season.

†'exp' stand for 'exponential', so that $\exp(rt) = e^{rt}$.

There are two possible reasons for the discrepancy. Either these hornets behave suboptimally or we have formulated the optimization problem incorrectly. A possible fault in our model is the implicit assumption that the date when egg production must end can be predicted a month in advance. That date actually depends on the weather and is correspondingly unpredictable. A model that gave T a probability distribution instead of a fixed value might be more realistic.

Many applications of optimization theory in zoology have concerned feeding behaviour or reproductive strategy, like our two examples. Others have asked questions including: What is the best strength for a bone? (Too weak a bone will break and too strong a bone is cumbersome.) At what speed should you change from walking to running? Is it better to fight a rival or to avoid the risk of getting hurt? and many others. The mathematical techniques that have been used have included calculus, linear programming, the theory of games and the Pontryagin method. Examples of all of these are included in reference 1.

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In Volume 17 Number 2, we suggested that you should try multiplying 142 857 successively by 2, 3, 4, 5 and 6 to see what happens. Richard Dobbs of Magdalen College, Oxford, has sent in the number

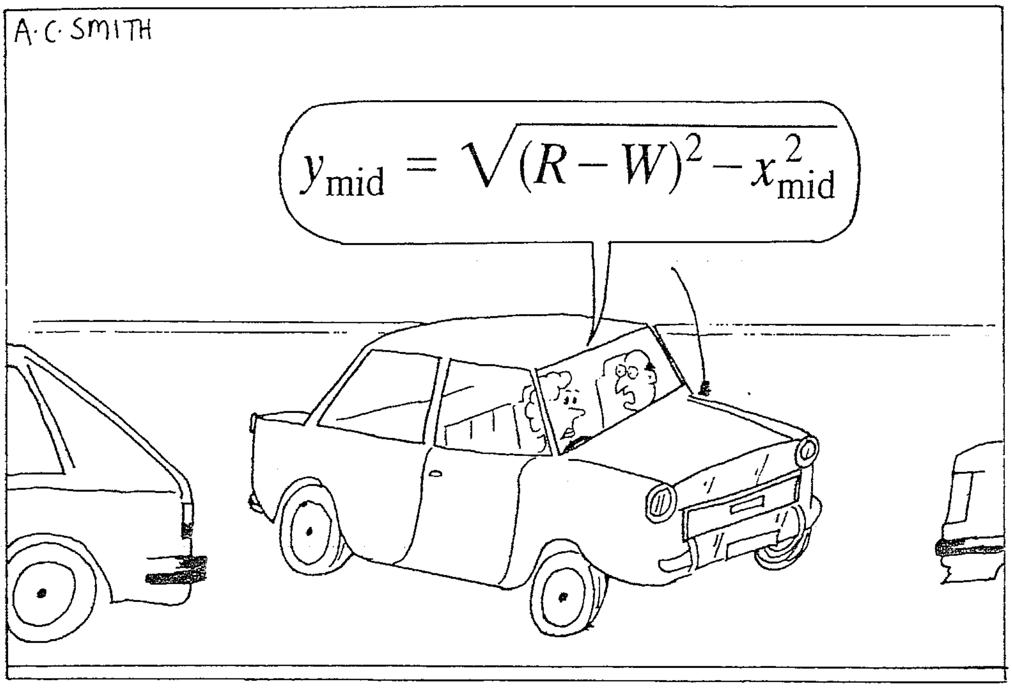
$$0588\,235\,294\,117\,647$$

and suggests multiplying it by 2, 3 up to 16. A similar thing happens. (It is important to retain the 0 at the beginning or it will not work!) Are there any other numbers with similar properties?

Mathematics of Simple Car-Parking

ANNA E. HART, *Liverpool Polytechnic*

The author read mathematics at Cambridge, and then spent some time in industry. She is now a lecturer in the Department of Mathematics at Liverpool Polytechnic. Her interests include mathematical modelling.



Parking a car constitutes a bad dream for some drivers. It is really a matter of common sense and simple geometry, but some (especially this author) tend to panic at the wheel, and turn relatively simple manoeuvres into a nightmare.

We park our car on a concrete area behind the house, and access is via a narrow entry. Usually I drive up the entry and reverse on to the concrete area, and reverse the procedure to come out. However, this involves reversing out on to a fairly busy road. The other week I tried the operation the other way round, reversing up the entry and driving forward into the parking space. The result was that I had a nasty encounter with a wooden post. Subsequently, to get out was a problem too! Calculating the mathematics of 'how' is easier, and safer, than experimentation, and these are some of the results.

Figure 1 shows the parking space and obstacles, together with an indication of the usual mode of entry.

Sid Dunn (see the reference) describes how the steering mechanism of a car works. When the two front wheels turn they move so that their horizontal axes always intersect on the line through the rear axle. If this point of intersection is X, then X will move along the axle line as the steering wheel

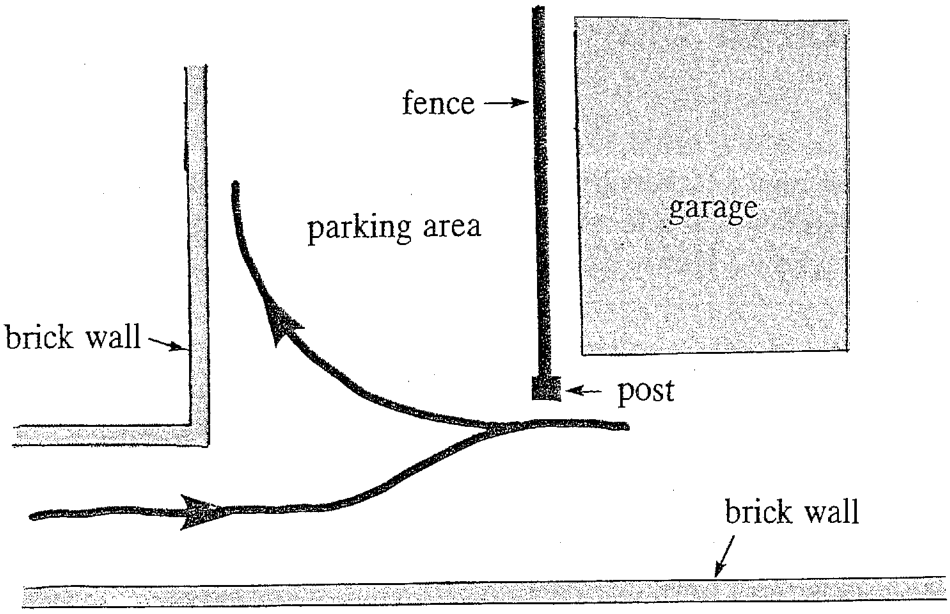


Figure 1

turns. The minimum turning radius for a car (available from the car handbook) is the minimum distance from X to the further rear wheel road contact point (see figure 2). It determines the maximum steering capacity of the car. Once the steering wheel is on full lock the car will move in such a way that the further rear wheel travels in a circle centre X and radius R. X is constant until the steering wheel is moved again. Let the minimum turning radius be R, the length of the car be L and the width of the car be W.

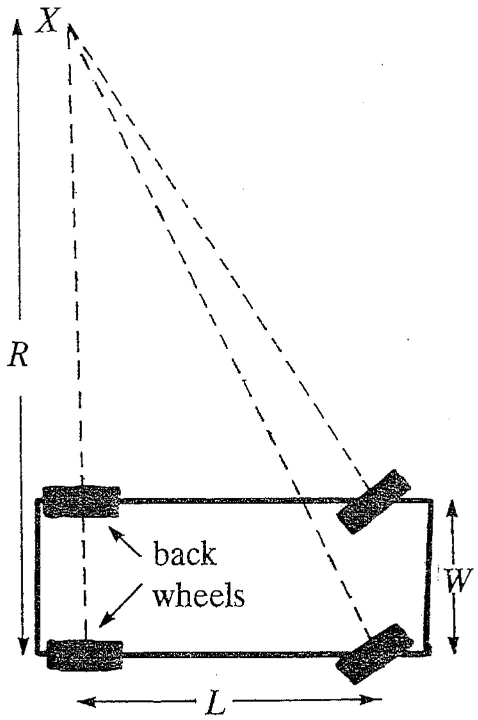


Figure 2

Our problem is to turn the car through 90°. We shall consider reversing into the parking space. If the steering wheel is on full lock then the further rear wheel travels on the circumference of a circle radius R, the nearer rear wheel travels on the circumference of a circle $R-W$ and the further front wheel travels on the circumference of a circle radius $\sqrt{R^2+L^2}$ (see figure 3).

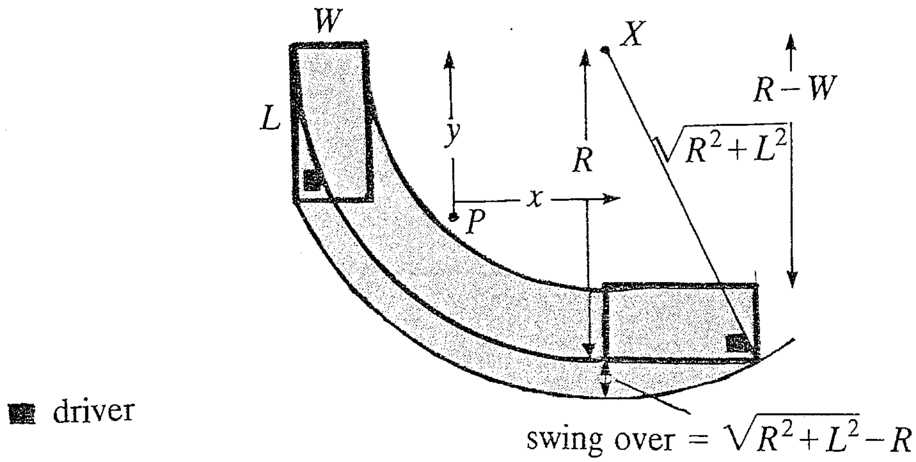


Figure 3

The car therefore covers the shaded area between two circles, and for a comfortable ride this must be obstacle-free. As the car turns, the front wheel swings over by an amount $\sqrt{R^2+L^2}-R$, as shown, so the car must initially be *at least* this distance away from the brick wall. Now the wooden post P is fixed in space, but X is determined by the initial position of the car. Once I turn the wheel on full lock then X remains fixed. I therefore need to decide on the position of X relative to P in order to manoeuvre the car successfully. This position of X then defines the starting position of the car. The post P must lie inside the inner circle. If its coordinates relative to X are (x, y), then $x^2+y^2 < (R-W)^2$. If w_1 and w_2 are widths of the entry and parking space respectively, then w_1 must be at least $W+\sqrt{R^2+L^2}-R = w_{\text{min}}$. If $w_1 = w_{\text{min}}$ then $w_2 > R$.

My usual parking solution is to position the back of the car in line with the fence, and turn the wheel on full lock (see figure 4, $X = X_0$). However, the final position of the car is rather near the brick wall. Using our results it is easy to calculate how I could maximise this distance from the wall (to make it easier to get out of the car) or position the car in the centre of the concrete area.

In order to finish away from the wall I need to choose X_1 in order to maximise x, i.e. to minimise y (see figure 5). Now

$$y_{\text{min}} = \sqrt{R^2+L^2}-w_1 \quad \text{and so} \quad x_{\text{max}} = \sqrt{(R-W)^2-y_{\text{min}}^2}.$$

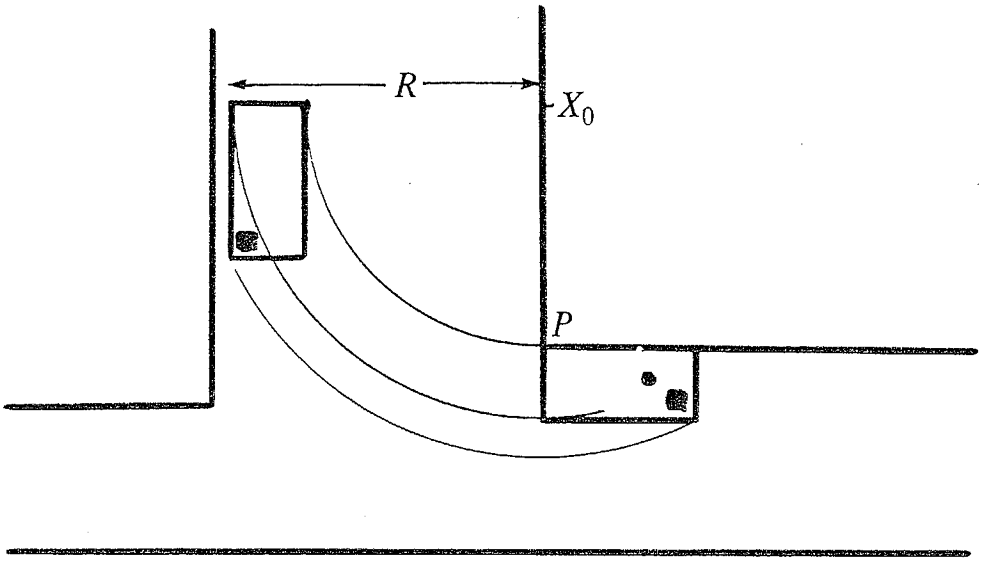


Figure 4

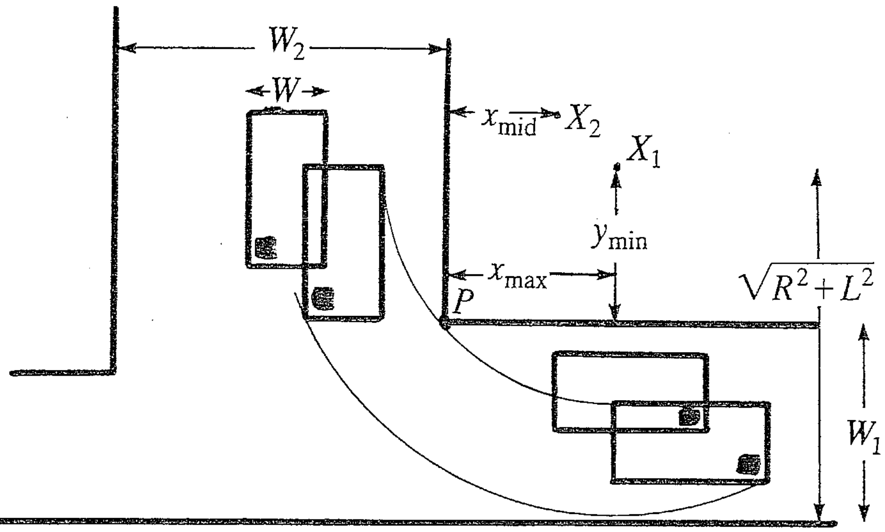


Figure 5

I should need to drive past the post until the back wheels were x_{max} away from it, and also position the car as near the wall as possible, i.e. distance $\sqrt{R^2+L^2}-R$ away from it.

In order to place the car in the centre of the parking space, I need

$$x_{mid} = R-\tfrac{1}{2}(W+w_2), \quad \text{so} \quad y_{mid} = \sqrt{(R-W)^2-x_{mid}^2},$$

giving X_2, from which it is possible to calculate the starting position of the car, if feasible.

When driving forwards the analysis is similar, but in reverse (see figure 6). It should be clear why it easier to reverse in, and also what mistake I made when I hit the post.

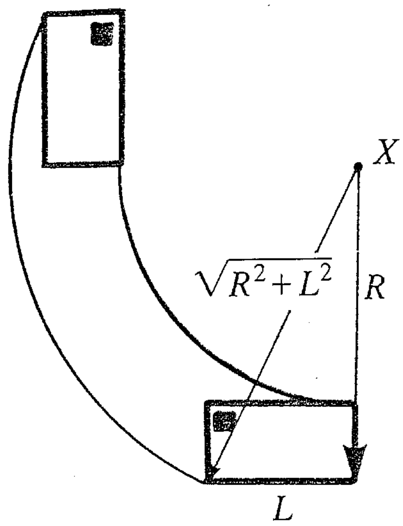


Figure 6

For those interested, Sid Dunn's chapter contains mathematics of more complicated manoeuvres, and further exercises for the enthusiast.

The above workings need slight adjustment to cater for the body of the car. Incidently, since this analysis I have had little problem in parking the car!

Reference

S. C. Dunn, Parking a car, in *Case Studies in Mathematical Modelling*, ed. D. T. G. James and J. J. McDonald, pp. 110–123 (Stanley Thornes, Cheltenham, 1981).

The Biggest Prime Number in the World

According to a recent BBC-2 Horizon programme, this title now goes not to $2^{86243}-1$ (see *Mathematical Spectrum* Volume 16 Number 2, page 37) but to

$$2^{132049}-1.$$

This, like the previous one, was discovered by David Slowinski in the U.S.A. The calculation took a week on a Cray Supercomputer.

MALCOLM SMITHERS
Student, Open University

Computer Column

MIKE PIFF

Here is a solution to the path problem I set a few issues back. On our Prime 9950, this program ran very slowly, even in compiled basic, taking almost a minute of CPU time.

```
100     REM PROGRAM TO PRINT OUT A PATH OF LENGTH PATHLENGTH OF THREE COLOURS
110     REM A,B AND C OF SLABS SUCH THAT NO SEQUENCE OF K SLABS IS IMMEDIATELY
120     REM FOLLOWED BY THE SAME COLOUR SEQUENCE.
130     REM
140     ORDA=64
150     ISFALSE=0
160     ISTRUE=1
170     FIRSTCOLOUR=1
180     LASTCOLOUR=3
190     PATHLENGTH=1000
200     DIM PATH(1000)
210     PATH(1)=FIRSTCOLOUR
220     PATH(2)=FIRSTCOLOUR+1
230     PATH(3)=FIRSTCOLOUR
240     I=3
250     REM
260     REM    CHECK FOR REPEATS IN PATH SO FAR
270     REM
280     OK=ISTRUE
290     FOR K=1 TO INT(I/2)
300        REM    SEE IF FINAL K SLABS HAVE JUST BEEN REPEATED
310        REPEATS=ISTRUE
320        M=I-K
330        REM    SEE IF [I-2K+1...I-K] THE SAME AS [I-K+1...I]
340        FOR L=0 TO K-1
350           IF PATH(I-L)=PATH(M-L) THEN 380
360           REPEATS=ISFALSE
370           GOTO 400
380           REM    NO REPEAT SO FAR
390        NEXT L
400        REM    CHECK IF PATTERN REPEATED
410        IF REPEATS=ISFALSE THEN 440
420        OK=ISFALSE
430        GOTO 470
440     REM    END OF K LOOP
450     NEXT K
460     REM
470     REM    END OF CHECK
480     REM
490     IF OK=ISTRUE THEN 570
500     REM         WALK BACK AND FIND FIRST STONE WHICH CAN BE CHANGED
510     IF PATH(I)<>LASTCOLOUR THEN 540
520     I=I-1
530     GOTO 500
540     REM    TRY NEXT STONE IN CURRENT POSITION
550     PATH(I)=PATH(I)+1
560     GOTO 260
570     REM    SEE IF FINISHED
580     IF I=PATHLENGTH THEN 630
590     REM    WALK ONE SLAB FORWARDS
600     I=I+1
610     PATH(I)=FIRSTCOLOUR
620     GOTO 260
630     REM                PATH FOUND. PRINT IT OUT
640     FOR I=1 TO PATHLENGTH
650        PRINT CHAR(PATH(I)+ORDA);
660     NEXT I
670     PRINT
680     END
```

Disappointed with the performance of this program, I translated it into Pascal and it then ran to completion in 10 CPU seconds.

```
program path(input,output);

const
   firstcolour=1;    lastcolour=3;
   minpathlength=1;    maxpathlength=1000;    zpathlength=0;

type
   colours=firstcolour..lastcolour;
   pathlengths=minpathlength..maxpathlength;
   zpathlengths=zpathlength..maxpathlength;
   paths=array[pathlengths]of colours;

var
   path:paths;
   position:pathlengths;
   blocklength,maxblocklength,marker,blockstart:zpathlengths;
   different,finished:boolean;
   orda:0..256;

begin{path}
   path[1]:=firstcolour;
   path[2]:=succ(firstcolour);
   path[3]:=firstcolour;
   position:=3;
   finished:=false;
```

```
repeat
   blocklength:=zpathlength;
   maxblocklength:=position div 2;
   repeat
      blocklength:=succ(blocklength);
      different:=false;
      blockstart:=position-blocklength;
      marker:=zpathlength;
      repeat
         if path[position-marker]<>path[blockstart-marker] then
            different:=true
         else
            marker:=succ(marker);
      until different or (marker=blocklength);
   until not different or (blocklength=maxblocklength);
   if different then
      if position=maxpathlength then
         finished:=true
      else
         begin
            position:=succ(position);
            path[position]:=firstcolour;
         end{if}
   else
      begin
         while path[position]=lastcolour do
            position:=pred(position);
         path[position]:=succ(path[position]);
      end{if};
until finished;
orda:=ord('a')-1;
for position:=minpathlength to maxpathlength do
   write(chr(orda+path[position]));
writeln;
end{path}.
```

In Fortran, it ran in about the same time as the Pascal version. Any moral in this?

```
      INTEGER PATH
      DIMENSION PATH(1000)
      INTEGER FIRSTC,LASTCO,MINPAT,MAXPAT,ZPATHL,POSITI,BLOCKL
      INTEGER MAXBLO,MARKER,BLOCKS
      LOGICAL DIFFER,FINISH
      FIRSTC=1
      LASTCO=3
      MINPAT=1
      MAXPAT=1000
      ZPATHL=0
      PATH(1)=FIRSTC
      PATH(2)=FIRSTC+1
      PATH(3)=FIRSTC
      POSITI=3
      FINISH=.FALSE.
1     CONTINUE
         BLOCKL=ZPATHL
         MAXBLO=INT(POSITI/2)
2        CONTINUE
            BLOCKL=BLOCKL+1
            DIFFER=.FALSE.
            BLOCKS=POSITI-BLOCKL
            MARKER=ZPATHL
3           CONTINUE
               IF(PATH(POSITI-MARKER).NE.PATH(BLOCKS-MARKER))GOTO 4
                  MARKER=MARKER+1
                  GOTO 5
4              CONTINUE
                  DIFFER=.TRUE.
5              CONTINUE
            IF(.NOT.DIFFER.AND.(MARKER.LT.BLOCKL))GOTO 3
         IF(DIFFER.AND.(BLOCKL.LT.MAXBLO))GOTO 2
         IF(DIFFER)GOTO 6
19          CONTINUE
            IF(PATH(POSITI).LT.LASTCO)GOTO 20
               POSITI=POSITI-1
               GOTO 19
20          CONTINUE
            PATH(POSITI)=PATH(POSITI)+1
            GOTO 30
6        CONTINUE
            IF(POSITI.EQ.MAXPAT)GOTO 7
               POSITI=POSITI+1
               PATH(POSITI)=FIRSTC
               GOTO 30
7           CONTINUE
               FINISH=.TRUE.
30       CONTINUE
      IF(.NOT.FINISH)GOTO 1
100   FORMAT(1000I1)
      DO 200 POSITI=1,1000
         WRITE(1,100) PATH(POSITI)
200   CONTINUE
      STOP
      END
```

A bit less intelligible than Basic or Pascal, wouldn't you agree? Happy programming!

Choosing your Error

EVANS W. CURRY, *Texas Tech University*
LAWRENCE HOTCHKISS, *The Ohio State University*
DERALD WALLING, *Texas Tech University*

Professor Curry earned his Ph.D. in sociology from Louisiana State University. His current interests are focused on the application of psychophysical models to measurement and theory. Lawrence Hotchkiss earned his Ph.D. in sociology from the University of Wisconsin. He is a Senior Research Specialist at the National Centre for Research and Training in Vocational Education at The Ohio State University. His current interests are focused on the relationships between education and the youth labour market. Professor Walling earned his Ph.D. in mathematics at Iowa State University. His current research interest lies in the area of mathematical modelling in psychophysics and in computer science.

1. Introduction

Many problems arise in the analysis of data where it is desirable to fit a function to a set of points in the plane. Given the N observations (x_i, y_i), $i = 1,2,\ldots,N$, where y_i may be dependent on x_i, it is necessary to find a function $f(x)$ such that $y = f(x)$ fits these data points as well as possible according to some selected criterion. Since there are errors of measurement in the data, as well as some random effects, the given points will not lie exactly on the curve $y = f(x)$. It is customary to assume that the x_i are error-free but the y_i are subject to error. Thus we may write $y_i = f(x_i)+\epsilon_i$, $i = 1,2,\ldots,N$, where ϵ_i is the error in y_i for the ith observation. We want to obtain values for the parameters in $f(x)$ so as to have the 'best' fit possible. To obtain this 'best' fit we shall minimize the error in terms of some criterion. The central issue here is that, even when the criterion has been established, care must be taken in its application.

2. Minima of what?

To illustrate the necessity of choosing the correct error term carefully once a criterion has been selected, we have taken $y = f(x)$ to be the power function

$$y = ax^b, \tag{1}$$

where the parameters a and b are constants to be estimated. Even though we have selected the power function, much of what is being said also applies to other well-known functions with parameters represented non-linearly which are used in curve fitting. One other commonly used function is $y = Ae^{-kx}$. We chose to use the power function for this illustration because it appears frequently in many disciplines where curve-fitting is common.

One criterion that is often used in fitting curves to data is the *method of least squares*. This method, attributed to the distinguished French mathematician Legendre (1752–1833), requires that the sum of the squared errors

$$\sum_{i=1}^{N} \epsilon_i^2$$

be a minimum. In the case of the points (x_i, y_i), $i = 1, 2, \ldots, N$, to be fitted by (1),

$$\sum_{i=1}^{N} \epsilon_i^2 = \sum_{i=1}^{N} (y_i - a x_i^b)^2 \tag{2}$$

is the sum to be minimized.

When applying the least-squares criterion to the power function (1), however, a commonly used procedure is to transform the variables so as to linearize the parameters. A standard transformation is to take the logarithm of both sides of (1) to obtain

$$\ln y = \ln a + b \ln x. \tag{3}$$

In equation (3), if we let $Y = \ln y$, $X = \ln x$ and $s = \ln a$, the expression (3) becomes

$$Y = s + bX, \tag{4}$$

where the parameters s and b are represented linearly.

To use the least-squares criterion to obtain parameter estimates for s and b in (4), we must minimize

$$I(s, b) = \sum_{i=1}^{N} \epsilon_i'^2 = \sum_{i=1}^{N} (Y_i - s - bX_i)^2, \tag{5}$$

where we have written

$$\epsilon_i' = (Y_i - s - bX_i) = (\ln y_i - \ln a - b \ln x_i) \tag{6}$$

for the new error in Y_i. To minimize (5), one method is to find the first partial derivatives of I with respect to s and b and to set them equal to 0 to identify the values of (s, b) leading to a minimum. Following this method, we obtain the equations

$$sN + b \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} Y_i \tag{7}$$

and

$$s \sum_{i=1}^{N} X_i + b \sum_{i=1}^{N} X_i^2 = \sum_{i=1}^{N} X_i Y_i. \tag{8}$$

Equations (7) and (8) can easily be solved simultaneously because they are linear in the unknowns s and b. It is for this very reason that one resorts to using logarithmic transformations, since linear equations then result when the first partial derivatives are set equal to 0.

Solving (7) and (8) simultaneously, we have the values of s and b that provide the best linear fit for the data

$$(X_i, Y_i) = (\ln x_i, \ln y_i), \quad i = 1, 2, \ldots, N. \tag{9}$$

The value b will then be known and the value of a can easily be determined from the expression $s = \ln a$. Solving for a and b, we have the estimates necessary to specify the function (1).

It is very important that users of this procedure realize that they did not minimize

$$\sum_{i=1}^{N} (y_i - ax_i^b)^2 \tag{10}$$

for the errors $\epsilon_i = y_i - ax_i^b$, $i = 1, 2, \ldots, N$, but instead minimized

$$\sum_{i=1}^{N} (\ln y_i - \ln a - b \ln x_i)^2 \tag{11}$$

for the errors $\epsilon_i' = (\ln y_i - \ln a - b \ln x_i)$, $i = 1, 2, \ldots, N$. This observation may not be vital when we want simply to determine the fit of some curve to observed data. *However*, if the goal is to determine the fit of the particular curve $y = ax^b$ to the data, and determine values of a and b such that expression (10) is a minimum, we have failed to achieve it. When parameter estimates of a transformed model $\ln y_i = \ln a + b \ln x_i$ are found by least squares, the values of a and b so determined will not generally minimize the sum of the squares of the errors for the original model $y_i = ax_i^b$ using the observed data set.

3. An example

To illustrate the different results obtained in using logarithms (or any other transformation) to linearize models in which the parameters are represented non-linearly, consider the data in table 1, and suppose we want to use the method of least squares to fit (1). If we use the logarithmic transformation illustrated above, we find that $\ln a$ is estimated as 1.171 655 and b as 0.439 24 so that

$$\ln y = 1.171\,655 + 0.439\,24 \ln x. \tag{12}$$

This would lead to the power function

$$y = 3.227\,33 x^{0.439\,24}. \tag{13}$$

Table 1. Observed data

x	y
170	2
170	203
2031	169
2031	247
32812	175
32812	259

We realize that, even though we have obtained a fit, we have not minimized (10) but have minimized (11). The expression found using the logarithmic transformation may be a very good fit, but if the intent is to minimize (10), the values of a and b that minimize (11) cannot be assumed to be good approximations to the values of a and b that minimize (10). The importance of this observation lies in the fact that, when curve fitting is used in many other disciplines, the users are assuming that, for each x_i, y_i is a normally distributed variate with mean ax_i^b and variance σ^2 independent of x_i. Their intent is to minimize this variance. Using *the method of maximum likelihood*, it can be shown that the associated variance is given by expression (10) divided by N. Hence, to minimize the variance, we *must* minimize (10). If we do this, we find that a is estimated as 73.432 43 and b as 0.109 93 so that

$$y = 73.432\,43x^{0.109\,93}, \tag{14}$$

a very different function from (13).

We have selected data to illustrate that these results can be quite different. The point is that one does not know *a priori* that the data may not be distributed in such a way as to produce similar results for the two fits. For the given data, the results are shown in table 2.

An indication of the relative fit of these data can be obtained by comparing the variance about the predicted values resulting from the two methods to the variance about $\bar{y}$, the mean of the y values. The variance about $\bar{y}$ is 7164.132, whereas the variance about the predicted values using (13) is 13 626.66 and using (14) is 5248.308. From these results, it is evident that one model only provides improved predictions over the mean. Model (13) is clearly inferior to (14), based on a minimum-variance criterion.

To obtain a minimum for (10), partial derivatives of (10) with respect to a and b can be found and can then be set equal to 0. Since the parameters in $f(x)$ are represented non-linearly, the resulting system of two equations in

Table 2. Values of y for the power and logarithmic functions

x	y using (13)	y using (14)
170	30.79963	129.1459
170	30.79963	129.1459
2031	91.56636	169.6306
2031	91.56636	169.6306
32812	310.78880	230.3218
32812	310.78880	230.3218

a and b will not be a linear system. Such systems can be very difficult to solve simultaneously, and logarithmic transformations are used to bypass this problem.

There was a time when logarithmic transformations were perhaps needed to allow users of these techniques to arrive at approximate values for the parameters with a minimum of computational difficulty. However, with the rapid development of electronic computation, this procedure is no longer so essential for the general curve-fitting problem.

There are numerical methods of obtaining the values that will minimize (10). One such method is to find initial estimates by using a grid search of possible parameter values. If we use these initial estimates, any one of several refinement procedures such as the Gauss–Newton, gradient or Marquardt procedures could be used to refine the initial estimates. The procedure NLIN (see the reference, pages 317–329) in the *Statistical Analysis System* employs these techniques.

4. Conclusions

While engaged in research on various applications of mathematics to the social sciences, we observed that authors of papers in research journals from various disciplines used logarithmic transformations widely when they should not have done so. Using the procedure of least squares, many obtained the minimum of some expression, often an expression transformed from the actual value to be minimized. They then simply assumed that they had found the maximum-likelihood estimators for the parameters of interest. This practice sometimes leads to the comparison of logarithmic transformed models with models whose parameters are represented linearly, an error which these authors failed to recognize.

Reference

1. Jane T. Helwig and Kathryn A. Council, editors, *SAS User's Guide*, 1979 edition, (SAS Institute, Inc., 1979).

Letters to the Editor

Dear Editor,

Recurring decimals

The author of 'Recurring Decimals' (Volume 17, Number 2) asks for a proof of his rule which states that if a recurring decimal has an even number of digits in its cycle, then each digit in the second half of the cycle is the nines-complement of the corresponding digit in the first half. For example $\frac{1}{7} = 0.\dot{1}4285\dot{7}$, where $1+8 = 4+5 = 2+7 = 9$. Such a proof does not exist, for the very good reason that the rule itself is not true. For example, $\frac{1}{21} = \dot{0}.04761\dot{9}$, $\frac{1}{33} = 0.\dot{0}\dot{3}$ and $\frac{1}{63} = 0.\dot{0}1587\dot{3}$. The last two examples also serve to show the falsity of the author's assertion that the digital root of the cycle is always 9.

Yours sincerely,

DAVID MARKS
(Student, Open University.
119 The Avenue, Ealing,
London W13 8JT)

[Author's reply:

Oh dear! I forgot the small print, quoted in reference 1 as: 'This law, however, holds good only if the denominator be (1) prime or (2) the product of primes, none of which by itself yields half the whole period ..., or (3) the product of primes in (2) with a power of 2 or 5 or both.'

The interested reader will find a further discussion of recurring decimals of reciprocal primes in references 2 and 3.

References

1. T. F. G. Dexter and A. H. Garlick, *Longman's Senior Arithmetic for Schools and Colleges*, page 118. (Longman, Green and Co., London, 1904).
2. L. F. Taylor, *Numbers*, page 32. (Faber and Faber, London, 1970).
3. G. T. Q. Hoare, Some reflections upon $1/P$ when P is prime. *Mathematics in School*, (1982) pp.4–6.

J. M. H. PETERS
(Liverpool Polytechnic)]

Dear Editor,

Integers round a circle

I write with reference to the problem set on page 35, Volume 17 Number 2 of the January 1985 issue of *Mathematical Spectrum*, i.e., arranging the integers 1 to 12 round a circle so that, for every three consecutive digits a, b and c, b^2-ac is divisible by $N = 13$.

The simplest starting sequence is to put $a = 1$, $b = 2$, $c = 4$ or, more generally, $a \equiv 1 \pmod{13}$, $b \equiv 2 \pmod{13}$, $c \equiv 4 \pmod{13}$. Then $a = 1+\alpha N$, $b = 2+\beta N$ and

$c = 4+\gamma N$, where α, β and γ are integers, and $b^2 - ac$ can be expressed in the form λN, where λ is also an integer. Hence $b^2 - ac$ is divisible by $N = 13$.

The same property also holds if we continue the sequence $1, 2, 4$ by doubling each term up to and including 2048 ($= 2^{11}$) and replacing these terms by modulo 13 division. We obtain the sequence

$$1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7.$$

If we arrange these numbers round a circle, e.g. on a clock-face, opposite numbers add up to 13.

The rule also works for certain other odd numbers N as well. The numbers 1 to 10 can be arranged so that $b^2 - ac$ is divisible by 11, i.e.,

$$1, 2, 4, 8, 5, 10, 9, 7, 3, 6.$$

It also holds for $N = 19$.

However, the rule fails for all even values of N and also for odd values of N if there exists some positive integer $n < N-1$ such that

$$2^n \equiv 1 \pmod N.$$

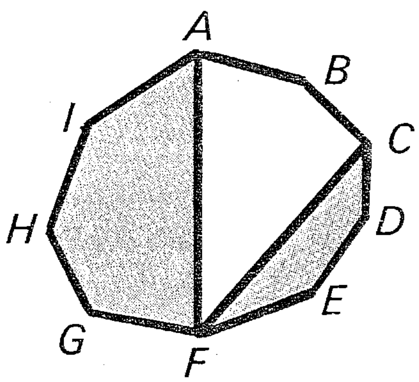
Thus for $N = 7, 9, 15, 17, 21$ or 23 the full cycle length is not obtained ($n = 3, 6, 4, 8, 6, 11$, respectively).

Yours sincerely,
H. RINDL
(City of Birmingham Polytechnic)

[Readers will find a list of all primes smaller than 1000 which have this property in Table 1, page 357 of D. M. Burton's *Elementary Number Theory* (revised printing) Allyn and Bacon 1980; they are those whose least primitive root is 2. For an arbitrary prime number N, instead of using powers of 2 use powers of its primitive root. For example, for $N = 7$ use powers of 3. A similar point has been made by Adrian Hill (The Royal Grammer School, High Wycombe) and Ruth Lawrence (St. Hugh's College, Oxford). Ed.]

Dear Editor.

Polygons of maximum area



In your last issue the problem is posed of proving that a quadrilateral with fixed sides achieves maximum area when disposed so as to be cyclic. I have only a clumsy algebraic proof of this, but thought it might be worth noting that the result is also true for polygons with an arbitrary number of sides. The proof is rather trivial, but appealing. Consider a general configuration and suppose that some set of points $FABC$ is not cyclic (i.e., we take a set of points of which three are consecutive, the fourth need not be). Then we can hold the shaded portions rigid and adjust $FABC$ until it is cyclic, thereby increasing the area. Therefore, if the area is already maximal, all such sets of points must be cyclic and, since there is only one circle through ABC, every other vertex must lie on it.

Yours sincerely,
P. L. ROE
(Cranfield Institute of Technology)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

17.7. (Submitted by P. V. Balfre, Salisbury College of Technology)
Let n be a positive integer and let m be the number of odd positive divisors of n. Show that there are exactly m different ways of expressing n as a sum of consecutive integers.

17.8. (Submitted by Kee-wai Lau, Hong Kong)
In a series of independent games between two players, each player has probability $\frac{1}{2}$ of winning each game. The series terminates as soon as either player has won three consecutive games. Find the probability that the series terminates just after the nth game.

17.9. (Submitted by Horst Alzer, Wuppertal, Federal Republic of Germany)

(i) Let f be a function defined for $0 < x < 1$ such that $f(x) > 0$ for all such x and such that

$$\frac{f(x)}{f(y)} + \frac{f(1-x)}{f(1-y)} \leqslant 2 \qquad (*)$$

for all x and y $(0 < x, y < 1)$. Show that f must be a constant function.

(ii) Show that there exist infinitely many non-constant functions f defined for $0 < x < 1$ such that $f(x) \neq 0$ for all such x and which satisfy condition (*) for all x and y $(0 < x, y < 1)$.

(iii) Determine all functions f defined for $0 < x < 1$ such that $f(x) > 0$ for all such x and such that

$$\frac{f(x)}{f(1-x)} + \frac{f(y)}{f(1-y)} \leqslant 2$$

for all x and y $(0 < x, y < 1)$.

Solutions to Problems in Volume 17, Number 1

17.1. Consider for example, the numbers 12 493 526 and 92 493 576. Now

$$(6+5+9+2)-(2+3+4+1) = 12, \qquad (6+5+9+2)-(7+3+4+9) = -1,$$

which have remainders 1 and 10, respectively, on division by 11, and these are also the remainders when the original numbers are divided by 11. Prove a general result on divisibility by 11 suggested by these examples.

Solution by Ruth Lawrence (St. Hugh's College, Oxford)

Consider a number m expressed to base 10 as $a_n a_{n-1} \dots a_0$, so that

$$m = a_0 + 10a_1 + 10^2 a_2 + \dots + 10^n a_n.$$

Since $10 \equiv -1 \pmod{11}$,

$$m \equiv a_0 - a_1 + a_2 - \dots + (-1)^n a_n \pmod{11},$$

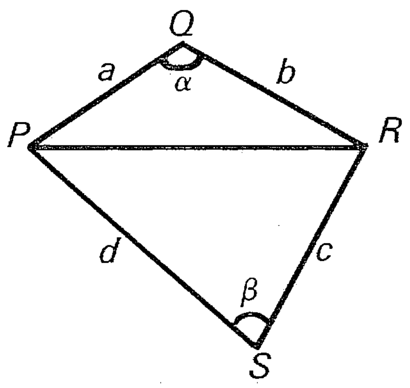
so m has the same remainder on division by 11 as does

$$(a_0 + a_2 + a_4 + \dots) - (a_1 + a_3 + a_5 + \dots).$$

Also solved by D. J. Brimicombe (University of Exeter), Nishad Gumaste (University of Warwick) and Philip Wadey (University of Exeter).

17.2. Given a quadrilateral with sides of fixed lengths, show that the maximum area is when the quadrilateral is cyclic.

Solution by Stephen Gourley (Totton College)



The area A of the quadrilateral is given by

$$A = \tfrac{1}{2}ab\sin\alpha + \tfrac{1}{2}cd\sin\beta.$$

By the cosine rule,

$$PR^2 = a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\beta.$$

So, if we differentiate with respect to α, we have

$$2ab\sin\alpha = 2cd\sin\beta\,\frac{d\beta}{d\alpha},$$

which gives

$$\frac{d\beta}{d\alpha} = \frac{ab\sin\alpha}{cd\sin\beta}.$$

Now

$$\begin{aligned}\frac{dA}{d\alpha} &= \tfrac{1}{2}ab\cos\alpha + \tfrac{1}{2}cd\cos\beta\frac{d\beta}{d\alpha}\\ &= \tfrac{1}{2}ab\cos\alpha + \tfrac{1}{2}ab\frac{\cos\beta\sin\alpha}{\sin\beta}\\ &= \tfrac{1}{2}ab\frac{\sin(\alpha+\beta)}{\sin\beta}\end{aligned}$$

and, for a maximum value of A, this must be zero. Thus $\alpha+\beta = 180°$ (since $\alpha+\beta = 0°$ or $360°$ will give zero area). Moreover, as $\alpha+\beta$ increases through the value $180°$, $dA/d\alpha$ changes from + sign to – sign ($\beta \neq 0°$ or $180°$). Hence there is a maximum when $\alpha+\beta = 180°$, i.e. when the quadrilateral is cyclic.

Also solved by D. J. Brimicombe (University of Exeter) and Ruth Lawrence (St. Hugh's College, Oxford).

We refer readers to Professor P. L. Roe's letter in this issue, which extends this from a quadrilateral to polygons with an arbitrary number of sides.

17.3. A destroyer sights the periscope of an enemy submarine and immediately heads towards it. The submarine retracts its periscope and the captain of the destroyer decides that the submarine will proceed in a straight line (of unknown bearing) with speed 10 knots. If the destroyer's speed is 30 knots, how should the captain steer in order to ensure that, at some subsequent time, the destroyer will pass over the submarine?

Solution by Ruth Lawrence (St Hugh's College, Oxford)

Let A and B be the initial positions of the submarine and destroyer, respectively, let t be the time measured from the sighting of the periscope, denote by $(r(t), \theta(t))$ the polar coordinates of the destroyer from the point A and the line AB, and let $d = \frac{1}{4}AB$.

The captain of the destroyer should first travel at full speed to the point on AB distant d from A and $3d$ from B. At this point $t = \frac{1}{10}d$ and the destroyer would pass over the submarine if the path of the submarine had bearing along AB. At all times, the distance of the submarine from A is $10t$ so that, if at time $t > \frac{1}{10}d$ the destroyer steers so that $r(t) = 10t$, then, if its bearing from A at any time subsequent to $\frac{1}{10}d$ coincides with that of the submarine, it will at that time pass over it. The radial and transverse components of the velocity of the destroyer are $\dot{r}$ and $r\dot{\theta}$, and its speed is 30 knots, so

$$\begin{aligned}900 &= \dot{r}^2 + r^2\dot{\theta}^2\\ &= 100 + 100t^2\dot{\theta}^2.\end{aligned}$$

So

$$\dot{\theta} = \frac{\sqrt{8}}{t}$$

This gives

$$\theta = \sqrt{8}\log\left(\frac{t}{t_0}\right).$$

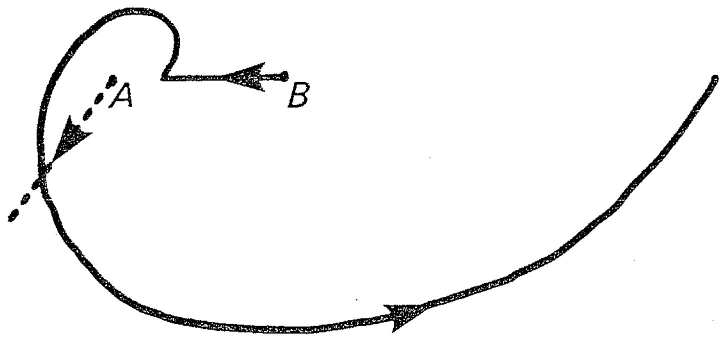
When $t = \frac{1}{10}d$, $\theta = 0$, so $t_0 = \frac{1}{10}d$. Since $\theta(t)$ is an increasing function, the destroyer will pass over the submarine.

Thus the path of the destroyer is given by

$$r(t) = \begin{cases} 4d-30t & (\text{for } 0 \leqslant t \leqslant \frac{1}{10}d), \\ 10t & (\text{for } t > \frac{1}{10}d), \end{cases}$$

$$\theta(t) = \begin{cases} 0 & (\text{for } 0 \leqslant t \leqslant \frac{1}{10}d), \\ \sqrt{8}\log\left(\frac{10t}{d}\right) & (\text{for } t > \frac{1}{10}d). \end{cases}$$

This is a straight line followed by a spiral.



Also solved by D. J. Brimicombe (University of Exeter).

Dividing by 7 or 13

On page 26 of Volume 17 Number 1, we included a problem sent in by Anthony Higgins. Start with a positive integer N and divide it by 50 to give quotient q_1 and remainder r_1. Then divide q_1 by 50 to give quotient q_2 and remainder r_2. Continue in this way until a zero quotient occurs. It turns out that N is divisible by 7 if and only if $r_1+r_2+r_3+\ldots$ is divisible by 7. Why? Philip Wadey (University of Exeter), Richard Dobbs (Magdalen College, Oxford) and Ruth Lawrence (St. Hugh's College, Oxford) all sent us explanations based on the fact that $50 = 7^2+1$, a fact also spotted by Iain Carey (Bramhall High School). Thus

$$N = 50q_1+r_1, \qquad q_1 = 50q_2+r_2, \qquad q_2 = 50q_3+r_3, \qquad \text{etc},$$

so that

$$\begin{aligned} N &= r_1+50r_2+50^2r_3+\ldots \\ &= r_1+(7^2+1)r_2+(7^2+1)^2r_3+\ldots \\ &= (r_1+r_2+r_3+\ldots)+7A \end{aligned}$$

for some integer A. Thus N and $r_1+r_2+r_3+\ldots$ have the same remainder on division by 7.

Richard Dobbs supplied a similar rule for division by 13. This comes from the fact that $25 = 2\times13-1$. This time we use successive division by 25 (or, rather, multiplication by 100 and division by 4). A similar argument to the above will give that the remainder when N is divided by 13 is the same as the remainder when $r_1 - r_2 + r_3 - r_4 + \dots$ is divided by 13. Can you see why the signs alternate this time? Alternatively, Ruth Lawrence proposes the use of successive division by 40. This is because $40 = 3 \times 13 + 1$. This time, the remainder when N is divided by 13 is the same as the remainder when $r_1 + r_2 + r_3 + r_4 + \dots$ is divided by 13.

Richard Dobbs points out that Problem 17.1 is a similar rule for division by 11, using the fact that $10 = 11 - 1$ and successive division by 10. He suggests similar rules for division by 17 and 19 using the facts

$$50 = 3 \times 17 - 1 \quad \text{and} \quad 400 = 21 \times 19 + 1.$$

Can you work out what these rules are?

Book Reviews

The Thread: A Mathematical Yarn. By PHILIP J. DAVIS. Harvester Press, Brighton, 1983. Pp. 140. £6·95.

Imagine Tristram Shandy—or, if you are not yet acquainted with Mr Sterne's account of that young gentleman's prehistory, history and opinions, then put it on your reading list for next year, or sometime: but not this year, because this year you must read, or dip into, or even save up for, this delightful, digressive, moving, eclectic account of—what?—well, whatever he chose to put in, by—well, imagine Tristram Shandy, with his mind most wonderfully sharpened by a life spent studying mathematics, and with sympathies broadened to comprise Tschebyscheff, Thaïs, St.Paphnutis, a Lama from Sikkim called Ted, and the diverse people who visit the Wall. And if you don't know where the Wall is or what it means, then you ought to, and this book may begin to tell you.

It contains no mathematics. I enjoyed it. I want to read more books by Professor Davis.

University of Sheffield C. J. KNIGHT

Mathematics in Sport. By M. STEWART TOWNEND. John Wiley and Sons, Chichester, 1984. Pp. 202. £12·95.

Most students and teachers are familiar with some form of sporting activity. This book concentrates on the construction and analysis of mathematical models of a wide range of sporting topics. The first three chapters discuss material derived from track and field athletics. Basic mechanical principles are used to study running, throwing and jumping. Chapter 4 uses statistical techniques to analyse exercises that have been designed to measure that vague criterion called fitness. Chapter 5 considers a miscellaneous group of sports ranging from darts to downhill skiing and gymnastics, while the penultimate chapter investigates the hydrodynamics and aerodynamics of sailing. The final chapter concentrates on those physical principles which govern ball games. The book is completed by a collection of computer programs, written in BASIC, which complement the mathematical analyses.

Division of the book into chapters based on different sports has led to a certain lack of logical development of the physical and mathematical principles. For example, it is assumed that the concept of moment of inertia is fully understood in Chapter 1 when models of running are discussed, yet the reader has to wait until Chapter 5 before the moment of inertia of a body about a given axis is defined. In a similar manner, in Chapter 1 it is assumed that a runner experiences a resistive force which is proportional to his or her velocity, while later in Chapter 3 it is stated that the resistive force, determined experimentally, is proportional to the square of the velocity. The linear regression between two variables is introduced in Chapter 1, but the reader has to wait until Chapter 4 before the evaluation of a regression line is explained. A paired *t*-test should have been used in Chapter 4 in the analysis of the Harvard step test data, and the concepts of statistical significance testing explained in more detail.

Apart from these and other small blemishes, I believe the material presented in this text offers interesting and stimulating alternatives to that found in more traditional applied mathematics courses. However, a certain degree of mathematical maturity is necessary before the modelling aspects can be fully appreciated. With careful guidance from a good and sympathetic teacher, this book could form an excellent base for critical discussions on how to begin to solve real problems by using mathematics.

University of Sheffield E. A. TROWBRIDGE

The Planiverse. By A. D. DEWDNEY. Picador, London, 1984. Pp. 272. £2·95.

Alexander Dewdney's 'Planiverse' is perhaps best categorised as *science faction*—who am I to call it fiction? 'Planiverse' relates an account of the accidental contact made by a group consisting of Dr Dewdney and his students with the inhabitants of the planet Arde which exists in a real two-dimensional universe, the Planiverse. This contact is made via the medium of a computer which had been programmed to simulate a hypothetical two-dimensional world.

The action of the book occurs on two levels. Firstly, there is the account of the minor day-to-day drama of the close-knit circle of human observers. They are forced, on the one hand, to maintain secrecy about their discovery for fear of establishment ridicule, with the consequent possibility of being dispossessed of their access to the mediating computer, whilst on the other hand they are trying to maximise their time of contact with the Planiverse. Secondly, and at the heart of the book, there is the intricate account of daily life on Arde, told primarily through the eyes of Yendred, an Ardean inhabitant. This account relates, in complex detail, how the various physical, chemical, biological and technological systems either must, or conceivably could, operate on two-dimensional Arde. My only query (and, sixth-formers, here is a chance to use the library!) is 'what would be the nature of an Ardean Huygen's principle?'

A convincing tale needs a convincing teller of tales, and Alexander Dewdney is certainly that. For my money, the Planiverse *does* exist!

University of Sheffield D. F. ROSCOE

CONTENTS

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