Crux Mathematicorum

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

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EDITORIAL

Have you ever wondered why there are 360 degrees in a circle? Or 7 days in one week? What about other counting conventions? What about the very foundation of our universally adopted counting system – the decimal system? Sure, all humans (with minor exceptions) have 10 fingers, but we also have 10 toes, so why not a base 20 system? In fact, cultures around the world have experimented with various systems: Mayans used base 20, Babylonians used base 60, people in New Guinea still have a base-27 counting system, where they use the names of 27 body parts as their numbers. Those who speak French realize that the language inherited a mixed use of base 10 and 20: 70=soixante-dix=$60 + 10$, 80=quatre-vingts=$4 \cdot 20$ all the way up to 99=quatre-vingt-dix-neuf=$(4 \cdot 20) + 19$. Bases 8 and 16, and of course base 2, are still used for computation (in fact, year 2016 looks quite nice in its base 2 representation: 11111100000).

So which system is the best? So-called Dozenal Societies will tell you it is the base-12 system and it can be hard to disagree with them. After all, 12 has many divisors (unlike 10, which creates messy fractions the moment division is at work), is not too small or too big (unlike 2 and 16 resulting in too few or too many digits) and already occurs naturally in many real-life situations: there are 12 months in a year, 24 hours in a day, 12 eggs in a dozen, etc. Plus, you can still use your fingers for counting: simply use the thumb as a pointer and count each of the three phalanges on the other four fingers totalling 12 on each hand. Better yet, you can use your other hand to denote the number of completed dozens and hence use your fingers to count up to 144. In fact, you can even afford to lose both thumbs and not suffer counting-wise.

Of course, changing the current system from the monopolizing base 10 seems nearly impossible. But one can still ask a question: if you could pick a system, what would you pick as a base?

Kseniya Garaschuk

THE CONTEST CORNER

No. 34

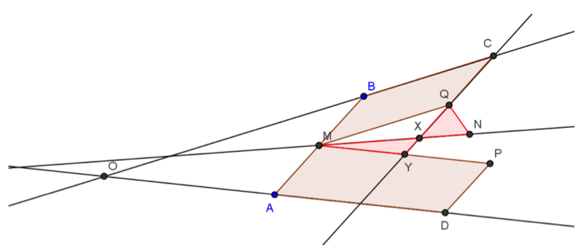
Robert Bilinski

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 juillet 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

CC166. Soit $WXYZ$ un carré. Trois droites parallèles d, d' et d'' passent respectivement par X, Y et Z. La distance entre d et d' vaut 5 tandis que la distance entre d et d'' vaut 7. Quelle est l'aire du carré ?

CC167. Les droites BC et AD sont sécantes en O, avec B et C du même côté de O, ainsi que A et D. En outre, $|BC| = |AD|$, $2|OC| = 3|OB|$ et $|OD| = 2|OA|$. Les points M et N sont les milieux respectifs de $[AB]$ et $[CD]$. Les quadrilatères $ADPM$ et $BCQM$ sont des parallélogrammes. La droite CQ coupe MN et MP respectivement en X et Y. Démontrer que les triangles MXY et QXN ont même aire.



CC168. Six étudiants de différents pays européens participent à un cours Erasmus ensemble. Chacun d'entre eux parle exactement 2 langues. Angela parle allemand et anglais ; Ulrich, l'allemand et l'espagnol ; Carine, le français et l'espagnol ; Dieter, l'allemand et le français ; Pierre, le français et l'anglais alors que Rocio parle espagnol et anglais. Si on choisit 2 personnes au hasard, quelle est la probabilité qu'ils parlent une langue commune ?

CC169. Quelle est la valeur de la base b si

$$\log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} = 110.$$

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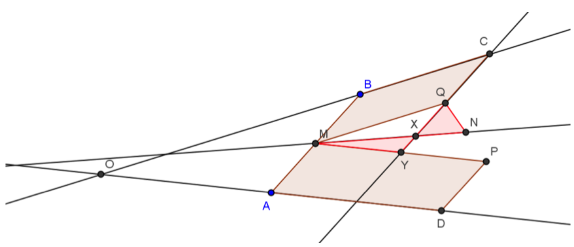
CC170. La somme de 35 entiers est S. On change 2 chiffres d'un des entiers and la nouvelle somme est T. La différence $S-T$ est toujours divisible par lequel des 5 nombres $2, 5, 7, 9$ ou 11 ?

CC166. Let $WXYZ$ be a square. Three parallel lines d, d' and d'' pass respectively through X, Y and Z. The distance between d and d' is 5 and the distance between d and d'' is 7. What is the area of the square ?

CC167. The lines BC and AD intersect at O, with both B and C on the same side of O, and the same goes for A and D. Among other properties, we have $|BC| = |AD|$, $2|OC| = 3|OB|$ and $|OD| = 2|OA|$. Points M and N are the respective middle points of segments $[AB]$ and $[CD]$. Quadrilaterals $ADPM$ and $BCQM$ are parallelograms. The line CQ cuts MN and MP respectively at X and Y. Show that triangles MXY and QXN have the same area.



CC168. Six students from different European countries participate in an Erasmus course together. Each student speaks exactly two languages. Angela speaks German and English ; Ulrich, German and Spanish ; Carine, French and Spanish ; Dieter, German and French ; Pierre, French and English and Rocio Spanish and English. If we choose 2 people at random, what is the probability that they speak a common language ?

CC169. What is the value of base b if

$$\log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} = 110.$$

CC170. The sum of 35 integers is S. We change 2 digits of one of the integers and the new sum is T. The difference $S-T$ is always divisible by which of the 5 numbers $2, 5, 7, 9$ or 11 ?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent initialement dans 2014 : 40(4), p. 142–143.

CC116. Does n^2 have more divisors $1 \pmod 4$ or $3 \pmod 4$?

Originally problem A3 from 2002 Mexican Math Olympiad.

We received one correct solution and one incorrect submission. We present the solution of S. Muralidharan.

We will show that there are more $1 \pmod 4$ divisors than $3 \pmod 4$ divisors for n^2.

Let

$$n = 2^{l_0} p_l^{1_1} p_2^{1_2} \cdots p_k^{1_k} q_l^{m_1} q_2^{m_2} q_r^{m_r}$$

be the prime decomposition of n where $p_1, p_2, \ldots, p_k$ are odd primes congruent to $1 \pmod 4$ and $q_1, q_2, \ldots, q_r$ are odd primes congruent to $3 \pmod 4$.

Since multiplying an odd divisor of n^2 by powers of 2 yields divisors congruent to $2 \pmod 4$ or $0 \pmod 4$, we can assume that $l_0 = 0$, that is, we can assume that n is odd.

The divisors of n^2 congruent to $1 \pmod 4$ that involve only the primes $p_1, p_2, \ldots, p_k$ are the terms in the expansion of

$$(1 + p_1 + \cdots + p_1^{2l_1})(1 + p_2 + \cdots + p_2^{2l_2}) \cdots (1 + p_k + \cdots + p_k^{2l_k})$$

and hence the number of such divisors equals

$$L_1 = (2l_1 + 1)(2l_2 + 1) \cdots (2l_k + 1).$$

Consider the terms of even powers of x in the expansion of

$$(1 + xq_1 + \cdots + x^{2m_1} q_1^{2m_1})(1 + xq_2 + \cdots + x^{2m_2} q_2^{2m_2})(1 + xq_r + \cdots + x^{2m_r} q_r^{2m_r}).$$

It is clear that the coefficient of each such term is a divisor of n^2 and is congruent to $1 \pmod 4$. Since multiplying two numbers congruent to $3 \pmod 4$ yields a number congruent to $1 \pmod 4$, it follows that all the divisors of n^2 that involve only the primes $q_1, q_2, \cdots, q_r$ will appear in the above expansion as a coefficient of an even power of x. Thus we need to count how many terms in the above expansion will contain even powers of x. That is clearly equal to the number of even terms in the expansion of

$$D(x) = (1 + x + \cdots + x^{2m_1})(1 + x + \cdots + x^{2m_2})(1 + x + \cdots + x^{2m_r}).$$

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The number of terms with even powers in the expansion of $D(x)$ is $\frac{D(1)+D(-1)}{2}$ and hence equal to

$$L_2 = \frac{(2m_1+1)(2m_2+1)\cdots(2m_r+1)+1}{2}.$$

Since multiplying a divisor of n^2 that involves only the primes p_i by a divisor of n^2 that involves only the primes q_j we obtain a divisor of n^2, it follows that the number of divisors of n^2 that are congruent to 1 (mod 4) is

$$L_1L_2 = L_1\frac{(2m_1+1)(2m_2+1)\cdots(2m_r+1)+1}{2} \tag{1}$$

The total number of divisors of n^2 (we have assumed that n is odd) is

$$L = L_1(2m_1+1)(2m_2+1)\cdots(2m_r+1).$$

Hence the number of divisors of n^2 congruent to 3 (mod 4) is

$$\begin{aligned} L - L_1L_2 &= L_1\Big((2m_1+1)\cdots(2m_r+1) - \frac{(2m_1+1)\cdots(2m_r+1)+1}{2}\Big) &(2)\\ &= L_1\frac{(2m_1+1)(2m_2+1)(2m_r+1)-1}{2} &(3)\end{aligned}$$

From (1)–(3), it is clear that n^2 has more divisors 1 (mod 4) than 3 (mod 4).

CC117. In a triangle ABC with $BC = 3$, choose a point D on BC such that $BD = 2$. Find the value of $AB^2 + 2AC^2 - 3AD^2$.

Originally #2 from the final round of 2010 South African school Olympiads.

We received ten correct submissions. We present the solution by S. Muralidharan.

We will prove a more general result : in a triangle ABC, if D is a point on BC such that $BD : DC = m : n$, then

$$nAB^2 + mAC^2 - (m+n)AD^2 = \frac{mn}{m+n}BC^2.$$

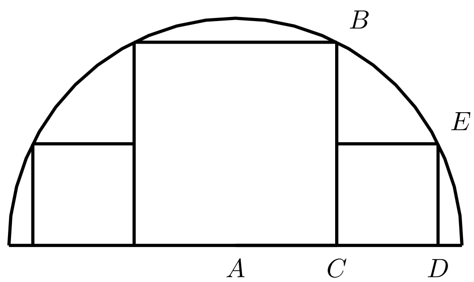
Let $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. Then $\overrightarrow{AD} = \dfrac{n\mathbf{b}+m\mathbf{c}}{m+n}$ and we have

$$\begin{aligned} &nAB^2 + mAC^2 - (m+n)AD^2\\ &\qquad = n\mathbf{b}^2 + m\mathbf{c}^2 - (m+n)\left(\frac{n\mathbf{b}+m\mathbf{c}}{m+n}\right)^2\\ &\qquad = \frac{1}{m+n}\big((m+n)(n\mathbf{b}^2+m\mathbf{c}^2) - (n^2\mathbf{b}^2 + m^2\mathbf{c}^2 + 2mn\mathbf{b}\cdot\mathbf{c})\big)\\ &\qquad = \frac{mn}{m+n}(\mathbf{b}^2 + \mathbf{c}^2 - 2\mathbf{b}\cdot\mathbf{c})\\ &\qquad = \frac{mn}{m+n}(\mathbf{c}-\mathbf{b})^{\mathbf{2}}\\ &\qquad = \frac{mn}{m+n}BC^2.\end{aligned}$$

For the given problem, $BC = 3$, $BD : DC = 2 : 1$ and hence

$$AB^2 + 2AC^2 - 3AD^2 = \frac{2}{3}BC^2 = 6.$$

CC118. If 2 small squares of side 2 and a bigger square are inscribed into a semi-circle, find the side of the larger square.



Originally #17 from the qualifying round of 2011 South African school Olympiads.

We received nine correct solutions. We present the solution of Scott Brown.

In the diagram shown above, let point A be the center of the diameter of the semi-circle. Let points B and E be the vertices of the larger square and smaller square respectively such that the line segment AB and AE are the radii of the semicircle. Let $CD = 2$ and $AC = x$. It follows that $BC = 2x$ and $AD = x + 2$.

So the legs of right triangle ABC are x and $2x$. In the other right triangle ADE the legs are 2 and $x + 2$. Since the radii are the same then

$$x^2 + (2x)^2 = 2^2 + (x + 2)^2.$$

Solving for x, we obtain 2. Therefore the side length of the larger square is 4.

CC119. When the tens digit of a three digit number abc is deleted, a two digit number ac is formed. How many numbers abc are there such that $abc = 9ac + 4c$? (For example, $245 = 9 \times 25 + 4 \times 5$.)

Originally #20 from the qualifying round of 2014 South African school Olympiads.

We received six correct solutions, all with the same approach. We feature the slightly shortened solution by Matei Coiculescu.

Expanding $abc = 9ac + 4c$, we have

$$100a + 10b + c = 9(10a + c) + 4c,$$

where a, b, c are digits. Simplifying this equation, we get :

$$5(a + b) = 6c.$$

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However, since a, b, c are digits, c must be 5. Therefore, there are 6 three digit numbers which satisfy the property, namely

$$605, 515, 425, 335, 245, 155.$$

CC120. Suppose A is a 2-digit number and B is a 3-digit number such that A increased by B% equals B reduced by A%. Find all possible pairs (A, B).

Originally #2 from the final round of 2014 South African school Olympiads.

We received three correct solutions, and four incorrect submissions omitting some possible pairs (A, B). We present the solution of Titu Zvonaru.

We have

$$\begin{aligned} A + \frac{AB}{100} = B - \frac{AB}{100} &\Leftrightarrow AB = 50(B - A) \\ &\Leftrightarrow (50 - A)(50 + B) = 2500 \\ &\Leftrightarrow (50 - A) = \frac{2500}{(50 + B)}. \end{aligned}$$

Since B is a three digit number we deduce,

$$0 \leq 50 - A = \frac{2500}{(50 + B)} \leq \frac{2500}{50 + 100} < 17.$$

Trying for $50 - A$ the positive divisors of 2500 less than 17 (that is 1, 2, 4, 5, and 10) we obtain the pairs $(A, B) = (46, 575), (45, 450)$, and $(40, 200)$.

THE OLYMPIAD CORNER

No. 332

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

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La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC226. Dans un triangle ABC, soit D le point sur le segment BC pour lequel $AB + BD = AC + CD$. Sachant que le points B, C et les centres de gravité des triangles ABD et ACD sont situés sur un même cercle, démontrer que $AB = AC$.

OC227. Dans un sac, il y a 1007 boules noires et 1007 boules blanches, numérotées au hasard de 1 à 2014. À chaque étape, on prend une boule du sac et on la place sur la table. Toujours à la même étape, on peut ensuite choisir deux boules de couleurs différentes sur la table et les placer dans un autre sac, ce qui nous permet de gagner un nombre de points égal à la différence non négative des numéros sur ces boules. Combien de points peut-on s'assurer de gagner après 2014 étapes ?

OC228. Soit k un entier strictement positif et m un entier positif impair. Démontrer qu'il existe un entier non négatif n tel que le nombre $m^n + n^m$ admette au moins k diviseurs premiers distincts.

OC229. Déterminer toutes les fonctions f ($f : \mathbb{R}^+ \to \mathbb{R}^+$) pour lesquelles

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

pour tous $x, y \in \mathbb{R}^+$.

OC230. Déterminer tous les réels strictement positifs a, b et c qui vérifient le système d'équations

$$a\sqrt{b} = a + c, b\sqrt{c} = b + a, c\sqrt{a} = c + b.$$

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OC226. In a triangle ABC, let D be the point on the segment BC such that $AB+BD = AC+CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

OC227. In a bag there are 1007 black and 1007 white balls, which are randomly numbered 1 to 2014. In every step we draw one ball and put it on the table ; also if we want to, we may choose two different colored balls from the table and put them in a different bag. If we do that we earn points equal to the absolute value of their differences. How many points can we guarantee to earn after 2014 steps ?

OC228. Let k be a nonzero natural number and m an odd natural number . Prove that there exists a natural number n such that the number $m^n + n^m$ has at least k distinct prime factors.

OC229. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$,

$$f\left(\frac{y}{f(x+1)}\right) + f\left(\frac{x+1}{xf(y)}\right) = f(y)$$

OC230. Find, with justification, all positive real numbers a, b, c satisfying the system of equations :

$$a\sqrt{b} = a + c, b\sqrt{c} = b + a, c\sqrt{a} = c + b.$$

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section apparaissent initialement dans 2014 : 40(4), p. 147–148.

OC166. Let $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Find the maximum value of

$$\sum_{n=1}^{10}(na_n^2 - n^2 a_n)\,.$$

Originally problem 3 from day 2 of the 2012 Korea National Olympiad.

No solutions were submitted.

OC167. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-2)f(y) + f(y+2f(x)) = f(x+yf(x))$$

for all $x, y \in \mathbb{R}$.

Originally problem 2 from day 1 of the 2012 Spain National Olympiad.

We received two correct solutions. We present the solution by Joseph Ling.

It is easy to verify that (1) $f(x) = 0$ for all x and (2) $f(x) = x-1$ for all x are solutions to

$$(x-2)\,f(y) + f(y+2f(x)) = f(x+yf(x))\,. \tag{*}$$

We claim that there are no others.

Suppose that f is not identically zero. We first observe that $f(2) \neq 0$. For if $f(2) = 0$, then letting $x = 2$ in $(*)$, we arrive at $f(y) = 0$ for all y. Next, we observe that $x = 1$ is the only number that can possibly be a root of $f(x) = 0$. For if $f(x_0) = 0$, then $x_0 \neq 2$, and letting $x = x_0$ in $(*)$, we arrive at $(x_0 - 1)\,f(y) = 0$ for all y. Since f is not identically zero, this implies that $x_0 = 1$.

Suppose now x is a number such that $x \neq 2$ and $f(x) \neq 1$. Then letting

$$y = \frac{x - 2f(x)}{1 - f(x)},$$

we have

$$y + 2f(x) = x + yf(x)\,,$$

and so, $(*)$ is reduced to $(x-2)\,f(y) = 0$. Since $x \neq 2$, $f(y) = 0$, and so, $y = 1$. It follows that $f(x) = x - 1$.

We now prove that the above condition $f(x) \neq 1$ is redundant by showing that $x \neq 2 \Longrightarrow f(x) \neq 1$. Suppose x is a number such that $x \neq 2$ and $f(x) = 1$. We

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will derive a contradiction. Letting $y = 0$ in $(*)$ and noting that $f(x) = 1$, we get a unique candidate

$$x = x^* = 2 + \frac{1 - f(2)}{f(0)}.$$

Thus, with the possible exceptions of $x = 2$ and $x = x^*$, we always have $f(x) = x - 1$. In particular, letting y be any number such that none of y, $y + 2$, and $x^* + y$ coincide with 2 or x^*, we have $f(y) = y - 1$, $f(y + 2) = y + 1$, and $f(x^* + y) = x^* + y - 1$. Using these and the assumption that $f(x^*) = 1$, we let $x = x^*$ and simplify $(*)$ into $(x^* - 2)(y - 2) = 0$. Since $y \neq 2$, $x^* = 2$, a contradiction. This completes the proof that no number x can satisfy $x \neq 2$ and $f(x) = 1$ simultaneously.

It follows that $x = 2$ is the only possible exception to the rule $f(x) = x - 1$.

Recall that $f(2) \neq 0$. Choosing y to be such that none of y, $y+2f(2)$, and $2+yf(2)$ coincide with 2, we have $f(y + 2f(2)) = y + 2f(2) - 1$ and $f(2 + yf(2)) = 1+yf(2)$. Using these and letting $x = 2$ in $(*)$, we arrive at $(f(2) - 1)(y - 2) = 0$. Since $y \neq 2$, $f(2) = 1 = 2 - 1$. This means that $f(x) = x - 1$ with no exceptions.

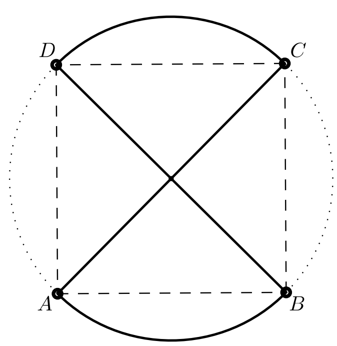
OC168. Let $ABCD$ is a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB + \angle CPD = 180^\circ.$$

Originally problem 5 from the 2012 Italy Math Olympiad.

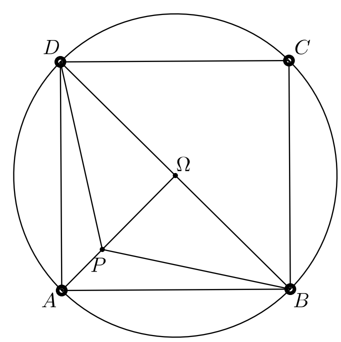
We received four correct solutions and one incorrect submission. We present the solution by Michel Bataille.

Let Γ be the circumcircle of the square $ABCD$. We show that the required locus is the union of the line segments AC, BD and the short arcs $\widehat{AB}$ and $\widehat{CD}$ of Γ (the points A, B, C, D being excluded).



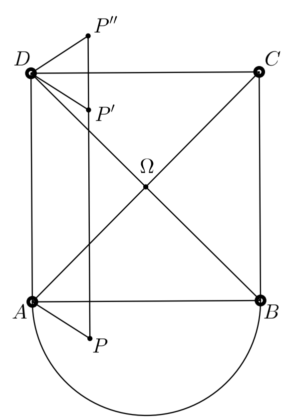
We first show that any point P of these sets satisfies the condition $\angle APB + \angle CPD = 180^\circ$. As a chord of Γ, a side of the square subtends an angle of 135° if the vertex is on the short arc and of 45° if the vertex is on the long arc. It follows that for a point P on either of the short arcs $\widehat{AB}, \widehat{CD}$, we have $\angle APB + \angle CPD = 45^\circ + 135^\circ = 180^\circ$.

Let P be on the line segments AC or BD. Without loss of generality, we may suppose that P is on $A\Omega$, where Ω is the centre of the square. Let $t = \angle PBD = \angle PDB$. Then, $\angle CPD = \angle \Omega PD = 90^\circ - t$ and $\angle APB = 180^\circ - \angle BP\Omega = 180^\circ - (90^\circ - t) = 90^\circ + t$. Thus, $\angle APB + \angle CPD = 180^\circ$.



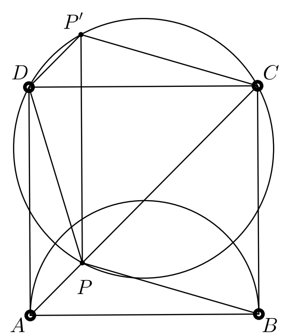
Conversely, suppose that P satisfies $\angle APB + \angle CPD = 180^\circ$. If $\angle APB = \angle CPD = 90^\circ$, then $P = \Omega$, a point of the diagonals. From now on, we suppose that $\angle APB$ and $\angle CPD$ are different from 90°. One of these angles then is obtuse, say $\angle APB > 90^\circ$. The point P is interior to the circle with diameter AB and clearly not on the side AB. We distinguish the two cases when in addition P is interior to the square or not.

Suppose that P is not interior to the square. Let P' be such that $APP'D$ is a parallelogram and P'' the reflection of P' in CD. Then, $\angle DP''C = \angle APB$, hence $\angle DP''C + \angle CPD = 180^\circ$. Therefore C, D, P, P'' are concyclic. On the other hand, the triangles APB and $DP''C$ are congruent and so $DP'' = AP$. Since AD and PP'' are parallel, the quadrilateral $APP''D$ is an isosceles trapezium.



As such, it has a circumcircle and so A is on the circle through C, D, P, P''. Thus, this circle is the circumcircle Γ of $ABCD$ and we conclude that P is on the short arc $\widehat{AB}$ of Γ.

Suppose that P is interior to the square. Again, let P' be such that $APP'D$ is a parallelogram. Then, $\angle DP'C = \angle APB$, hence $\angle DP'C + \angle CPD = 180°$. Therefore C, D, P, P' are concyclic. It follows that $\angle DPP' = \angle DCP' = \alpha$ where $\alpha = \angle ABP$. Since AD and PP' are parallel, we deduce $\angle ADP = \angle DPP' = \alpha$. Now, let $\theta = \angle APB$ and $\theta' = \angle APD$.



If a denotes the side of the square, we have $\frac{a}{\sin\theta'} = \frac{AP}{\sin\alpha} = \frac{a}{\sin\theta}$, hence $\theta' = \theta$ or $\theta' = 180° - \theta$. In the former case, we deduce $\angle PAD = \angle PAB$, hence P is on AC. In the latter case, we have $\angle BPC = \theta$ (since $\angle BPC + \angle APD = 180°$ as well) and in a similar way, P is on BD. The proof is complete.

OC169. Find all positive integers $n \geq 2$ such that for all integers $0 \leq i, j \leq n$ the numbers $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have same parity.

Originally problem 1 from the 2012 Iran National Math Olympiad TST.

We received no solutions.

OC170. Let ABC be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments BC, respectively AC at D, respectively E. Prove that

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA)\,.$$

Originally problem 1 from day 2 of the 2012 Romanian TST.

We received one correct solution. We give the solution by Titu Zvonaru.

Let $a = BC$, $b = CA$ and $c = AB$. By the Angle Bisector Theorem, we have

$$DC = \frac{ab}{b+c} \quad \text{and} \quad EC = \frac{ab}{a+c}.$$

By the Cosine Law on triangle DEC, we have

$$DE^2 = \frac{a^2b2}{(b+c)^2} + \frac{a^2b^2}{(a+c)^2} - 2\frac{a^2b^2}{(a+c)(b+c)}\cos(C).$$

By the Cosine law on triangle ABC,

$$\begin{aligned} DE^2 &= \frac{a^2b2}{(b+c)^2} + \frac{a^2b^2}{(a+c)^2} - 2\frac{a^2b^2}{(a+c)(b+c)} \cdot \frac{a^2+b^2-c^2}{2ab} \\ &= \frac{ab(a^2bc + ab^2c + 3abc^2 - a^3c - b^3c - a^2c^2 - b^2c^2 + ac^3 + bc^3 + c^4)}{(a+c)^2(b+c)^2}. \end{aligned}$$

Next, using the inequalities :

$$\begin{aligned} (c+\sqrt{ab})^2 &\le (c+a)(c+b), \\ 2abc^2 &\le a^2c^2 + b^2c^2, \\ a^2bc + ab^2c &\le a^3c + b^3c, \end{aligned}$$

we have that

$$\begin{aligned} DE^2 &\le \frac{ab(a^3c + b^3c + abc^2 + a^2c^2 + b^2c^2 - a^3c - b^3c - a^2c^2 - b^2c^2 + ac^3 + bc^3 + c^4)}{(a+c)^2(b+c)^2} \\ &= \frac{ab(abc^2 + bc^3 + ac^3 + c^4}{(a+c)^2(b+c)^2} \\ &= \frac{abc^2(a+c)(b+c)}{(a+c)^2(b+c)^2} \\ &\le \frac{abc^2}{(c+\sqrt{ab})^2}. \end{aligned}$$

Taking square roots, it suffices to prove that

$$\frac{c\sqrt{ab}}{c+\sqrt{ab}} \le (3-2\sqrt{2})(c+2\sqrt{ab}).$$

This inequality is true if and only if

$$\begin{aligned} c\sqrt{ab} &\le (3-2\sqrt{2})(c+2\sqrt{ab})(c+\sqrt{ab}) \\ (3+2\sqrt{2})c\sqrt{ab} &\le (3+2\sqrt{2})(3-2\sqrt{2})(c+2\sqrt{ab})(c+\sqrt{ab}) \\ 3c\sqrt{ab} + 2\sqrt{2}c\sqrt{ab} &\le c^2 + 3c\sqrt{ab} + 2ab \\ 0 &\le (c-\sqrt{2ab})^2 \end{aligned}$$

and the last inequality clearly holds thus the string of inequalities holds. Equality holds if and only if $a = b$ and $c^2 = 2a^2$, that is, triangle ABC is an isosceles right angled triangle with angle BCA as the right angle.

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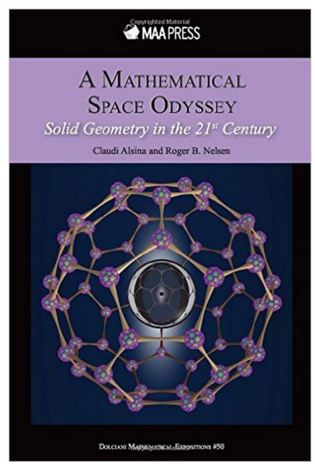
BOOK REVIEWS

Robert Bilinski

A mathematical space odyssey : Solid geometry in the 21st century
by Claudi Alsina and Roger B. Nelsen
978-0-88385-358-0, hardcover, 273 pages
Published by MAA Press, 2015

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

This book marks the authors fifth collaboration for the MAA Press. I am guessing they know each other through their common research interests since one is a teacher in Portland, Oregon while the other is based in Catalonia, Spain. One can find the 62-page resumé of Claudi Alsina on the web : he has a very long career in math and math education and has published several books in Spanish and Catalan, while Roger Nelson is known for his two previously published "Proofs without words" books.



Surely, we live in a math world where algebra is the queen and geometry fell by the wayside but I have always wondered why there does not exist a book as popular or widely known as Euclid's Elements for the 3D world. Even if this book does not quite make it there, it is moving in the right direction. The authors make quite an interesting and far-reaching (and probably incomplete on purpose) survey of the results in 3D geometry with a modern outlook that transcends Euclidean geometry. They cover stereometry, combinatorics of 3D patterns, weird 3D shapes, cuts, projections, 3D fractals, optimized 3D shapes and 3D origami. The listed theorems range from classical Greek results to results from the 21st century with each subject covered in a blitzkrieg mode rather than in-depth.

On top of all of the different geometric points of view, it was fun to see the algebraic equivalents and "fresh" geometric equivalents of known inequalities. The representation chapter as well as the chapters on combinatorics and optimization are probably most useful for math Olympians as pure 3D geometry problems become more rare, so seeing these objects in a more general setting is important.

As a book published in the "Dolciani Mathematical Expositions" collection of the MAA, it attains its goal of serving as a permanent reference and refresher for teachers : it promotes math and is accessible to a pre-calculus readership. Both authors are teachers of higher mathematics, but, as the authors themselves put it, "courses in solid geometry have largely disappeared from American high schools and colleges". The book has a few tidbits for problem solvers as one of the authors, Claudi Alsina, has been involved in various mathematical contests, but the aim of the book is much larger : revive a lost art. One cannot do that without making people practitioners of 3D geometry, so each chapter ends with some pretty nifty challenges. The book does contain some proofs (of the easier parts), but many of the explanations are dense, so one has to be comfortable with algebra and good at interpreting complex geometrical figures to completely grasp the book.

A nice feature of the book is a transversal aspect of the exposition. Alsina is a math teacher in an architectural faculty, which probably got him interested in 3D geometry and also made him familiar with the application of said geometry to architecture, so the book often uses physical objects such as buildings as examples. There are also many historical references making for an interesting read.

Overall, this was quite an interesting book, a fun and very quick read. I was in a time crunch when I read it so I did not do much more than read the challenge problems at the end of each chapter, but I will get back to them in my leisure time as they did pique my interest. The book has a bit of everything for everyone and should definitely be taken as an introductory book on the subject of 3D geometry. As an older reference, I recommend Marcel Bergers 2-volume *Geometry* (which is an English translation from the French of his 5-volume *Géométrie*). Sadly, I cannot recommend a newer must-have on the subject, but one can probably find a lot of magnificent books in the French school of geometry. Good reading !

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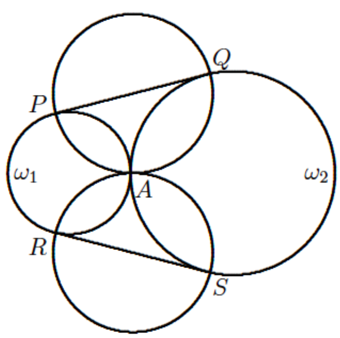
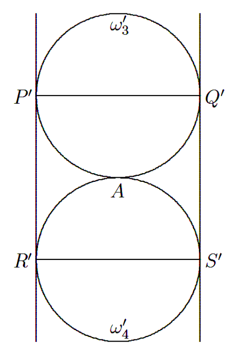
Application of Inversive Methods to Euclidean Geometry : solutions

Andy Liu

In this part, we present the solutions to ten problems in *Application of Inversive Methods to Euclidean Geometry* by Andy Liu appearing in the previous issue (Volume 41 (3), p. 114–118). Solution to problem 1 appears in the original article.

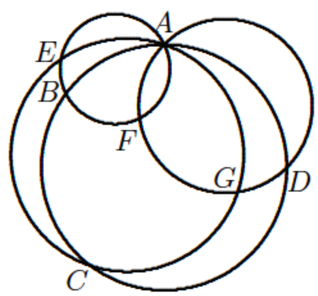
Problem 2 (below left)
Two circles ω_1 and ω_2 are tangent externally to each other at A. A common exterior tangent touches ω_1 at P and ω_2 at Q. The other common exterior tangent touches ω_1 at R and ω_2 at S. Prove that the circumcircles of triangles PAQ and RAS are tangent to each other.

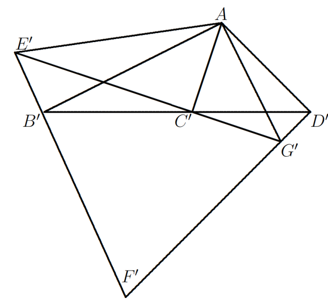




Solution to Problem 2
Invert with respect to A (above right). Then ω_1 and ω_2 become parallel lines $P'R'$ and $Q'S'$. PQ and RS become circles ω_3' and ω_4', tangent to both $P'R'$ and $Q'S'$. The circumcircles of triangles PAQ and RAS become diameters $P'Q'$ and $R'S'$ of ω_3' and ω_4'. These diameters are orthogonal to PR and are therefore parallel to each other. Hence the two circumcircles are also tangent to each other.

Problem 3 (below left)
AB, AC and AD are three chords on a circle. Circles with AB and AC as diameters intersect at E, circles with AB and AD as diameters intersect at F, and circles with diameters AC and AD intersect at G. Prove that E, F and G are collinear.





Solution to Problem 3

Invert with respect to A (above right). Then the original circle becomes the line $B'C'D'$. The other three circles become the lines $E'B'F'$, $E'C'G'$ and $F'G'D'$, and they are orthogonal to AB', AC, and AD, respectively.

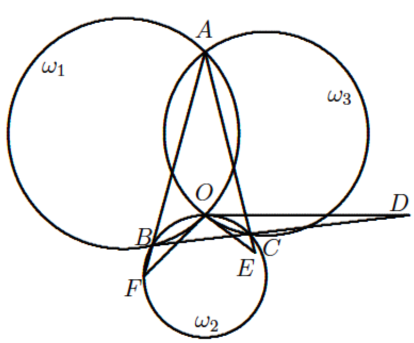
Hence $AE'B'C'$, $AB'F'D'$ and $AC'G'D'$ are cyclic quadrilaterals, so that $\angle B'E'C' = \angle B'AC'$, $\angle B'AF' = \angle B'D'F'$ and $\angle C'AG' = \angle C'D'G'$. It follows that

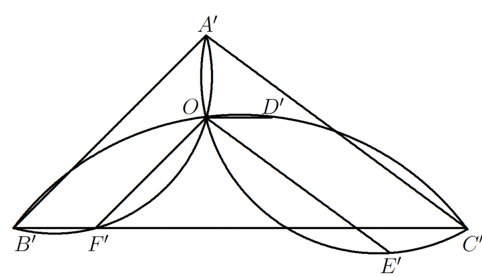
$$\begin{aligned}\angle F'AG' &= \angle B'AG' - \angle B'AF' \\ &= \angle B'AG' - \angle C'D'G' \\ &= \angle B'AG' - \angle B'D'F' \\ &= \angle B'AC' \\ &= \angle F'E'G'.\end{aligned}$$

Hence A, E', F' and G' are concyclic, so that E, F and G are collinear.

Problem 4 (below left)

Three circles ω_1, ω_2 and ω_3 pass through O. B is the other point of intersection of ω_1 and ω_2, C is the other point of intersection of ω_2 and ω_3, and A is the other point of intersection of ω_3 and ω_1. The tangent to ω_2 at O intersects BC at D, the tangent at O to ω_3 intersects CA at E, and the tangent at O to ω_1 intersects AB at F. Prove that D, E and F are collinear.





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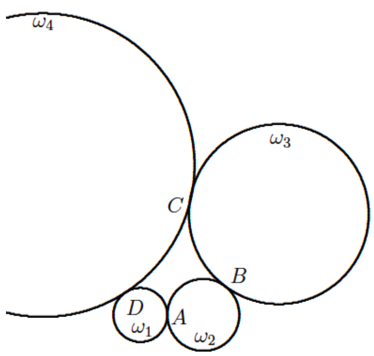
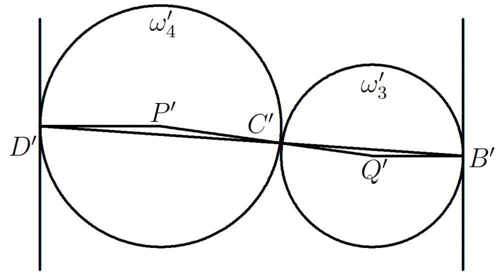
Solution to Problem 4
Invert with respect to O (above right). Then the three circles turn into triangle $A'B'C'$ while the tangent lines OD, OE and OF turn into themselves. Hence OD', OE' and OF' are parallel to $B'C'$, $C'A'$ and $A'B'$ respecitvely. Moreover, D', E' and F' lie on the circumcircles of triangles $OB'C'$, $OC'A'$ and $OA'B'$ respectively. Let Q be the circumcentre of triangle $A'B'C'$. Then Q lies on the perpendicular bisectors of OD', OE' and OF'. Hence O, D', E' and F' are concyclic. It follows that D, E and F are collinear.

Problem 5 (below left)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 touch at A, ω_2 and ω_3 touch at B, ω_3 and ω_4 touch at C and ω_4 and ω_1 touch at D. Prove that A, B, C and D are concyclic.





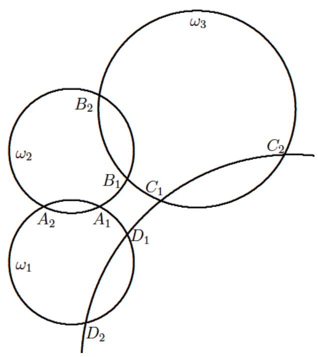
Solution to Problem 5
Invert with respect to A (above right). Then ω_1 and ω_2 become a pair of parallel lines, tangent to ω_4' and ω_3' at D' and B' respectively. These two circles are tangent to each other at C'. Let P' and Q' be the centres of ω_4' and ω_3' respectively. Then C' lies on $P'Q'$.

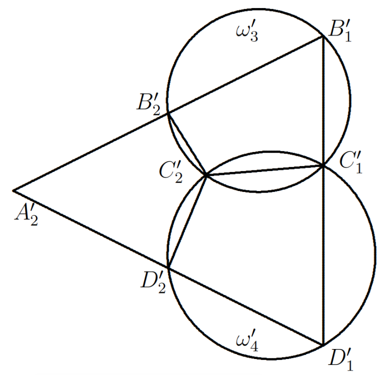
Since $C'D'$ and $C'B'$ are parallel, $\angle C'P'D' = \angle C'Q'B'$. Since $C'P' = D'P'$ and $C'Q' = B'Q'$,

$$\angle P'C'D' = \frac{1}{2}(180^\circ - \angle C'P'D') = \frac{1}{2}(180^\circ - \angle C'Q'B') = \angle Q'C'B'.$$

Hence C' also lies on $B'D'$, which means that A, B, C and D are concyclic.

Problem 6 (below left)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 intersect at A_1 and A_2, ω_2 and ω_3 intersect at B_1 and B_2, ω_3 and ω_4 intersect at C_1 and C_2, and ω_4 and ω_1 intersect at D_1 and D_2. Prove that if A_1, B_1, C_1 and D_1 are collinear or concyclic, then so are A_2, B_2, C_2 and D_2.



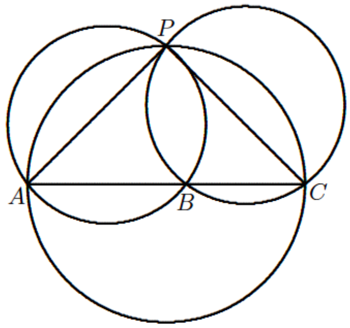


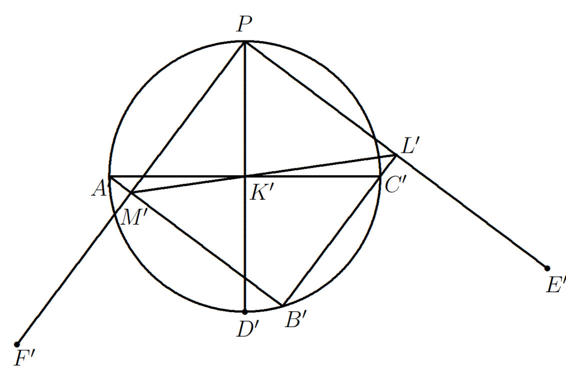
Solution to Problem 6

Invert with respect to A_1 (above right). Then $A_1B_1C_1D_1$, ω_1 and ω_2 become the sides of triangle $A'_2B'_1D'_1$. Since $B'_2B'_1C'_1C'_2$ is cyclic, $\angle A'_2B'_2C'_2 = \angle C'_2C'_1B'_1$. Similarly, $\angle A'_2D'_2C'_2 = \angle C'_2C'_1D'_1$. Since $A_1,\ B_1,\ C_1$ and D_1 are either collinear or concyclic, C'_1 lies on $B'_1D'_1$. Hence $\angle C'_2C'_1B'_1 + \angle C'_2C'_1D'_1 = 180°$. It follows that $\angle A'_2B'_2C'_2 + \angle A'_2D'_2C'_2 = 180°$, so that $A'_2,\ B'_2,\ C'_2$ and D'_2 are concyclic. Hence $A_2,\ B_2,\ C_2$ and D_2 are either collinear or concyclic.

Problem 7 (below left)

$A,\ B$ and C are three points on a line and P is a point not on this line. Prove that the circumcentres of triangles PAB, PBC and PCA are concyclic with P.





Solution to Problem 7

Let F, D and E be the respective circumcentres. Invert with respect to P (above right). Then the circles become the sides of triangle $A'B'C'$. The images D', E' and F' of the circumcentres are the reflections of P across the respective sides.

Hence the midpoints K', L' and M' of PD', PE' and PF' are the feet of perpendiculars from P to the sides of triangle $A'B'C'$. Since $A,\ B$ and C are collinear, P lies on the circumcircle of triangle $A'B'C'$. It follows that $M'K'L'$ is the Simson line of triangle $A'B'C'$, so that F', D' and E' are also collinear. Hence the circumcentres are concyclic with P.

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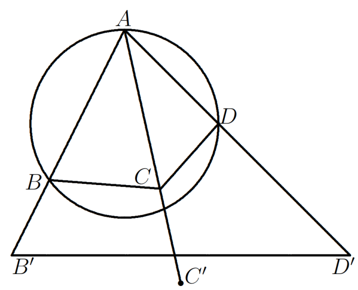
Problem 8
Prove Ptolemy's Inequality which states that $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ for any convex quadrilateral $ABCD$, with equality if and only if the quadrilateral is cyclic. (Hint : Because this is quantitative, expect to use the "polar-coordinate" definition of inversion.)

Solution to Problem 8
Invert with respect to A :



Now $B'D' = \dfrac{BD \cdot r^2}{AB \cdot AD}$, where r is the radius of inversion. Similarly,

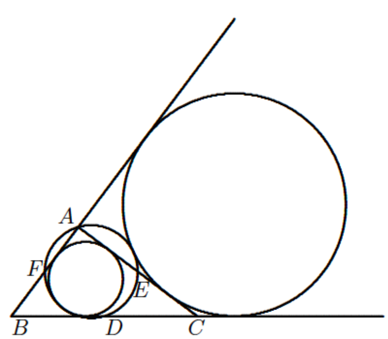
$$B'C' = \frac{BC \cdot r^2}{AB \cdot AC} \text{ and } C'D' = \frac{CD \cdot r^2}{AC \cdot AD}.$$

By the Triangle Inequality, $B'D' \leq B'C' + C'D'$. Substituting in this the above expressions, we have

$$\frac{BD}{AB \cdot AD} \leq \frac{BC}{AB \cdot AC} + \frac{CD}{AC \cdot AD},$$

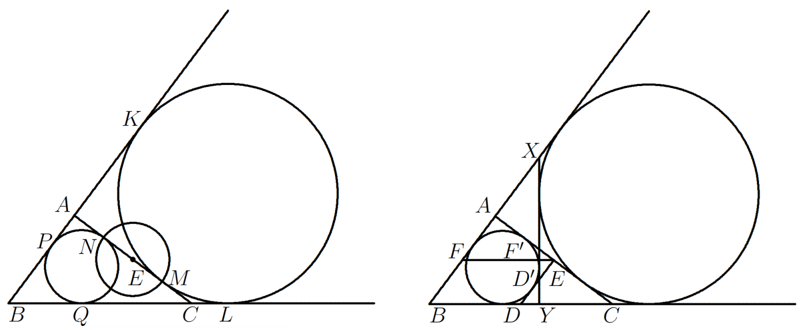
or $AC \cdot BD \leq BC \cdot AD + CD \cdot AB$. Equality holds if and only if C' is collinear with B' and D'. Since the circumcircle of triangle BAD turns into the line $B'D'$, this holds if and only if C is concyclic with A, B and D.

Problem 9 (below)
Prove that the circle which passes through the midpoints of the sides of a triangle is tangent to the triangle's incircle and excircles.



Solution to Problem 9

We shall prove that the midpoint circle is tangent to the incircle and the excircle facing B. By symmetry, it will be tangent to the other two excircles. Let BC, CA and AB be tangent to the excircle facing B at K, L and M, and the incircle at N, P and Q, respectively.



Now

$$AB + AC - BC = (AQ + BQ) + (AN + CN) - (BP + CP) = 2AN$$

and

$$AB + AC - BC = (BK - AK) + (AM + CM) - (BL - CL) = 2CM.$$

Hence $AN = CM$. Since E is the midpoint of AC, it is also the midpoint of NM. Invert with respective to E and choose $EM = EN = \dfrac{BC - AB}{2}$ as the radius of inversion. Then both circles are orthogonal to the circle of inversion, and coincide with their respective images.

Let XY be the other common interior tangent of these two circles, with X on AB and Y on BC. Let XY intersect DE at D' and EF at F'. If we can prove that D' and F' are the images of D and F respectively, then the midpoint circle inverts into the line XY, and the desired result follows. By symmetry, we have $BX = BC$ and $BY = BA$. Note that XFF' and XBY are similar triangles.

It follows that

$$\begin{aligned}
FF' &= \frac{BY \cdot XF}{BX} = \frac{BA(BX - BF)}{BC} = \frac{BA(2BC - BA)}{2BC},\\
EF' &= EF - FF' = \frac{BC}{2} - \frac{2BC \cdot BA - BA^2}{2BC} = \frac{(BC - BA)^2}{2BC},\\
EF \cdot EF' &= \frac{BC}{2} \cdot \frac{(BC - BA)^2}{2BC} = (\frac{BC - BA}{2})^2.
\end{aligned}$$

Hence F' is indeed the inversive image of F. From the similar triangles XBY and $D'DY$, we can deduce in an analogous manner that D' is in fact the inversive image of D.

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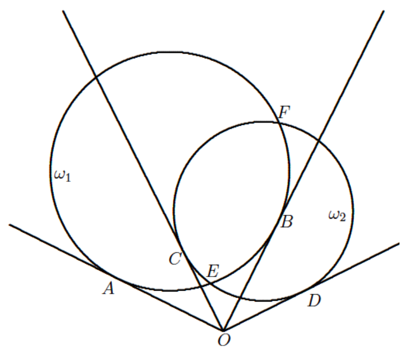
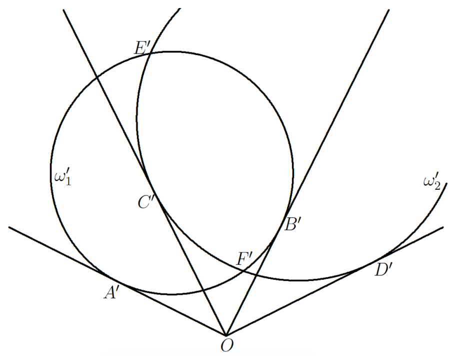
Problem 10 (below left)
From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.





Solution to Problem 10.

Let ω_1 be the circle tangent to OA at A and to OB at B. Let ω_2 be the circle tangent to OC at C and to OD at D. If ω_1 and ω_2 are of the same size, then both E and F will lie on the bisector of $\angle COB$, and the desired result follows immediately. Thus we may assume that ω_1 is larger than ω_2.

Invert with respect to O so that A and B coincide with their respective images A' and B' (above right). The rays OA, OC, OB and OD become the rays OA', OC', OB' and OD' respectively. The circle ω_1 coincides with its image ω_1' while the image of the circle ω_2 is another circle ω_2'. Note that the image E' of E is collinear with E and O, and the image F' of F is collinear with F and O.

We now go from the first diagram to the second in a different way. First we perform a reflection about the bisector of $\angle COB$, and then a dilation from O so that D is mapped into A'. Note that the rays OA, OC, OB and OD become the rays OD', OB', OC' and OA' respectively, while the circle ω_2 becomes the circle ω_1'. By inversion, $OD \cdot OD' = OA^2$. Since D is mapped into A', A is mapped into D' so that the circle ω_1 becomes ω_2'. It follows that F is mapped into E' while E is mapped into F'. Thus the rays OE and OF become each other, and the desired result follows.

On choosing the modulus

Y. I. Ionin and A.I. Plotkin

Practically every mathematical olympiad includes problems involving integers and their properties. These problems are often quite challenging, but there are several standard techniques that can be used to solve them. We will discuss one such technique in this article.

The idea is as follows : replace the numbers given in the problem by their remainders when divided by some carefully chosen integer, say n ; we then say we are taking these numbers *modulo* n and the number n itself is called the *modulus*. We start with a small example.

Example 1 *Are there integers x and y that satisfy the equation*

$$x^3 + x + 10y = 20004?$$

To answer this question, note that if we know the last digit of x, we can figure out the last digit of x^3 and (for any y) the last digit of $x^3 + x + 10y$. Let us construct a table of the last digits of x, y and $x^3 + x + 10y$:

x	0	1	2	3	4	5	6	7	8	9
x^3	0	1	8	7	4	5	6	3	2	9
$x^3 + x + 10y$	0	2	0	0	8	0	2	0	0	8

We can see that the number $x^3 + x + 10y$ ends in 0, 2 or 8 for any x and y and hence cannot equal 20004. $\square$

Such reasoning is probably familiar to you, but we will delve deeper into its meaning. The last digit of any number is its remainder when divided by 10. The solution of the above problem relies on the fact that to find the last digit of the number $x + x^3$ we only need to know the last digit of x. This is of course true not only for the remainders modulo 10, but for the remainders modulo any other integer. So one could shorten the previous solution by considering the given numbers modulo 5 instead of 10.

Example 2 *Find all positive integers n such that the integers $n + 1$, $n + 71$ and $n + 99$ are all prime.*

Consider the remainders of the given integers modulo 3. Note that 1, 71 and 99 all give us different remainders modulo 3 and therefore the numbers $n + 1$, $n + 71$ and $n + 99$ also all have different remainders modulo 3. However, there are only 3 possible remainders modulo 3, namely 0, 1 and 2. Therefore, one of the numbers is divisible by 3 and, since it has to be prime, it is equal to 3. As such, we must have that $n = 2$. $\square$

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We say that integers a and b are *equal modulo* m ($m \neq 0$) if a and b have the same remainder when divided by m; in other words, $a - b$ is divisible by m. We then write $a \equiv b \pmod{m}$. Now, suppose $a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$. Since

$$\begin{aligned}(a_1 + a_2) - (b_1 + b_2) &= (a_1 - b_1) + (a_2 - b_2),\\ (a_1 - a_2) - (b_1 - b_2) &= (a_1 - b_1) - (a_2 - b_2),\\ a_1a_2 - b_1b_2 &= a_1(a_2 - b_2) + b_2(a_1 - b_1),\end{aligned}$$

we get :

$$\begin{aligned}a_1 + a_2 &\equiv b_1 + b_2 \pmod{m},\\ a_1 - a_2 &\equiv b_1 - b_2 \pmod{m},\\ a_1a_2 &\equiv b_1b_2 \pmod{m}.\end{aligned}$$

Now, given an expression involving sum, difference and product of integers, we can replace them by their modulo m equivalents. In particular, we can find all the values of a given expression modulo m as we did in Example 1.

To further illustrate the use of this technique, consider the following sequence of lemmas :

a) a perfect square is equal to 0 or 1 modulo 3,

b) a perfect cube is equal to 0 or 1 modulo 4,

c) a perfect cube is equal to 0, 1 or -1 modulo 9,

d) a perfect cube is equal to 0, 1 or -1 modulo 7,

e) we have $n \equiv n^n \pmod{8}$ for all odd n.

Exercise 1 *Prove Lemmas a)–e).*

Exercise 2 *Prove that any natural number, written in base 10, is equal to the sum of its digits modulo 3 and modulo 9.*

Is 97531 a perfect square or a perfect cube? No. It is not a perfect square because it is equal to 3 modulo 4 and it is not a perfect cube because it is equal to 7 modulo 9 (since the sum of its digits equals to 25). As you can see, the success is often determined by the choice of the modulus.

Example 3 *Show that the sum of squares of 5 consecutive integers is not a perfect square.*

Let us try to approach this problem modulo 3. If the first of the numbers is divisible by 3, then their squares taken modulo 3 form the sequence 0, 1, 1, 0, 1 and hence their sum is equal to 0 modulo 3. We did not get a contradiction : a perfect square can be equal to 0 modulo 3. So maybe we did not pick the best modulus; let us try reasoning modulo 4. It is easy to see that in this case we get the sequence 0, 1, 0, 1, 0 (with the sum of 2) or the sequence 1, 0, 1, 0, 1 (with the sum of 3).

Either way, lemma b) above states that a perfect square cannot be equal to 2 or 3 modulo 4. □

Exercise 3 *Prove that none of the following numbers can be perfect squares :*

a) the sum of squares of 3 odd numbers,

b) a number whose base 10 representation consists of 20 ones and some zeros,

c) $2^2 + 4^4 + 6^6 + \cdots + 50^{50}$.

Exercise 4 *Find all natural numbers n such that* $1! + 2! + \cdots n!$ *is a perfect square.*

Exercise 5 *Prove that the number* $222\ldots2$ *consisting of 1982 twos cannot be written in the form* $xy(x+y)$ *for integers* x *and* y.

Exercise 6 *Show that the sum of squares of three consecutive integers cannot be a perfect cube.*

Let us now go through two more difficult examples.

Example 4 *Prove that the the number* $1^1 + 2^2 + 3^3 + \cdots + 1983^{1983}$ *cannot be written in the form* m^k *for natural numbers* m, k *with* $k \geq 2$.

Let us start with reasoning modulo 3. Since every integer is equal to -1, 0 or 1 modulo 3, then $a^n \equiv a^2 \pmod 3$ for even n and $a^n \equiv a \pmod 3$ for odd n. Therefore, the numbers 1^1, 2^2, 3^3, … taken modulo 3 form a periodic sequence of length 6 : $1, 1, 0, 1, -1, 0$. Using this periodicity, it is not hard to show that the sum is equal to 2 modulo 3 and therefore cannot be a perfect square.

How can we show that it cannot be a perfect cube, a fourth or fifth power and so on ? Modulo 4 comes to the rescue. The numbers 1^1, 2^2, 3^3, … taken modulo 4 form a periodic sequence 1, 0, -1, 0 from which we can deduce that the sum $1^1 + 2^2 + 3^3 + \cdots + 1983^{1983}$ is divisible by 4. Hence, if this sum can be written in the form m^k where $k \geq 3$, it would be divisible by 8. So let us consider this sum modulo 8. If n is even ($n \neq 2$), then n^k is divisible by 8. If n is odd, then by lemma d) $n^n \equiv n \pmod 8$ and hence

$$1^1 + 2^2 + 3^3 + \cdots + 1983^{1983} = 2^2 + 1 + 3 + \cdots + 1983 \pmod 8\,.$$

But this last sum is not divisible by 8 (it is equal to 984068). □

Example 5 *Consider two integer sequences* (x_n) *and* (y_n)*, where*

$$x_1 = x_2 = 10,\ y_1 = y_2 = -10,\ x_{n+2} = (x_{n+1}+1)x_n + 1,\ y_{n+2} = (y_{n+1}+1)y_n + 1$$

for all $n \in \mathbb{N}$. *Are there integers* k, l *such that* $x_k = y_l$ *?*

Let us try to pick a modulus for which both sequences are constant, that is equal to 10 and -10. If x_n, x_{n+1} and x_{n+2} are all equal to 10 modulo m, then 10 and $(10+1)10+1 = 111$ are also equal modulo m. Therefore $m = 101$. It is easy enough to check that the sequence (y_n) is constant at -10 when taken modulo 101. Therefore, $x_k = y_l$ is not possible for any values of k, l. □

We conclude with some exercises.

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Exercise 7 *Find all prime numbers* p *for which* $2p^4 - p^2 + 16$ *is a perfect square.*

Exercise 8 *Find all natural numbers* n *for which* $3^n + 55$ *is a perfect square.*

Exercise 9 *Suppose that* $n \in \mathbb{N}$ *is such that the number* $n^2 + 1$ *has 10 digits. Prove that the base 10 representation of* $n^2 + 1$ *has two equal digits.*

Exercise 10 *Suppose that* $n^{n+1} + (n+1)^n$ *is divisible by 3 for some* $n \in \mathbb{N}$. *Prove that* $n - 1$ *is divisible by 6.*

Exercise 11 *Prove that a perfect cube is equal to* $0, 1$ *or* -1 *modulo 7.*

Exercise 12 *Show that the equation* $15x^3 + 13y^6 = 101$ *has no integer solutions.*

Exercise 13 *Prove that for* $n \geq 2$ *the number* $1! + 2! + \cdots + n!$ *cannot be written in the form* m^k *for natural numbers* m, k *with* $m \geq 3$.

. . . .

This article originally appeared in Russian in Kvant, 1984 (6), p. 28–30. It has been translated and adapted with permission.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1 juillet 2016** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Un astérisque ($\star$) *signale un problème proposé sans solution.*

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

4031. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Démontrer que

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

F_n étant le $n^{\text{ième}}$ nombre de Fibonacci ($F_0=0, F_1=1$ et $F_{n+2}=F_n+F_{n+1}$ pour tout $n\geq 0$).

4032. *Proposé par Dan Stefan Marinescu et Leonard Giugiuc.*

Soit un triangle ABC ayant pour longueurs de côtés a, b et c. Soit r le rayon du cercle inscrit dans le triangle et r_a, r_b et r_c les rayons des cercles exinscrits du triangle. Démontrer que

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\geq 2\sqrt{3r(r_a+r_b+r_c)}.$$

4033. *Proposé par Salem Malikic.*

Soit $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ des réels strictement positifs et $x_1,\ldots,x_n$ des réels tels que $x_1+\cdots+x_n=1$ et $\alpha_ix_i+\beta_i\geq 0$ pour tout i ($i=1,2,\ldots,n$). Déterminer la valeur maximale de l'expression

$$\sqrt{\alpha_1x_1+\beta_1}+\sqrt{\alpha_2x_2+\beta_2}+\cdots+\sqrt{\alpha_nx_n+\beta_n}.$$

4034. *Proposé par Michel Bataille.*

Évaluer

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\sum_{k=0}^{2n}\frac{(-1)^k}{2n+1-k}\binom{2k}{k}\binom{4n-2k}{2n-k}.$$

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4035. *Proposé par Daniel Sitaru et Leonard Giugiuc.*

Déterminer toutes les solutions X de l'équation matricielle suivante, X étant une matrice 2×2 de nombres réels :

$$X^3 - 5X^2 + 6X = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}.$$

4036. *Proposé par Arkady Alt.*

Soit a, b et c des réels positifs ou nuls. Démontrer que

$$k(ab + bc + ca)(a + b + c) - (a^2c + b^2a + c^2b) \leq \frac{(3k - 1)(a + b + c)^3}{9}$$

pour tout réel $k \geq \frac{11}{24}$.

4037. *Proposé par Michel Bataille.*

Soit P un point quelconque sur le cercle γ inscrit dans le triangle ABC. Les perpendiculaires à BC, CA et AB, issues du point P, coupent γ de nouveau aux points respectifs U, V et W. Démontrer que l'aire du triangle UVW est indépendante de la position du point P sur γ.

4038. *Proposé par George Apostopoulos.*

Soit x, y et z des réels strictement positifs tels que $x + y + z = xyz$. Déterminer la valeur minimale de l'expression

$$\sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1}.$$

4039. *Proposé par Abdilkadir Altinaş.*

Soit un triangle ABC dans lequel $\angle CAB = 48°$ et $\angle CBA = 12°$ et soit D un point sur AB tel que $CD = 1$ et $AB = \sqrt{3}$. Déterminer $\angle DCB$.

4040. *Proposé par Ali Behrouz.*

Déterminer toutes les fonctions f ($f : \mathbb{N} \mapsto \mathbb{N}$) telles que

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2, \quad \forall a, b \in \mathbb{N}.$$

. . . .

4031. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Prove that

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

where F_n represents the nth Fibonacci number ($F_0=0, F_1=1$ and $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 0$).

4032. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Prove that in any triangle ABC with sides a, b and c, inradius r and exradii r_a, r_b, r_c, we have:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\geq 2\sqrt{3r(r_a+r_b+r_c)}.$$

4033. *Proposed by Salem Malikic.*

Let $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ be positive real numbers and $x_1,\ldots,x_n$ be real numbers such that $x_1+\cdots+x_n=1$ and $\alpha_ix_i+\beta_i\geq 0$ for all $i=1,\ldots,n$. Find the maximum value of

$$\sqrt{\alpha_1x_1+\beta_1}+\sqrt{\alpha_2x_2+\beta_2}+\cdots+\sqrt{\alpha_nx_n+\beta_n}.$$

4034. *Proposed by Michel Bataille.*

Evaluate

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\sum_{k=0}^{2n}\frac{(-1)^k}{2n+1-k}\binom{2k}{k}\binom{4n-2k}{2n-k}.$$

4035. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a and b be two real numbers such that $ab=225$. Find all real solutions (in real 2×2 matrices) to the matrix equation

$$X^3-5X^2+6X=\begin{pmatrix}15 & a\\ b & 15\end{pmatrix}.$$

4036. *Proposed by Arkady Alt.*

Let a, b and c be non-negative real numbers. Prove that for any real $k\geq\frac{11}{24}$ we have:

$$k(ab+bc+ca)(a+b+c)-(a^2c+b^2a+c^2b)\leq\frac{(3k-1)(a+b+c)^3}{9}.$$

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4037. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that the area of UVW is independent of the chosen point P on γ.

4038. *Proposed by George Apostopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find the minimum value of the expression

$$\sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1}.$$

4039. *Proposed by Abdilkadir Altinaş.*

In a triangle ABC, let $\angle CAB = 48^\circ$ and $\angle CBA = 12^\circ$. Suppose D is a point on AB such that $CD = 1$ and $AB = \sqrt{3}$. Find $\angle DCB$.

4040. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2 \quad \forall a, b \in \mathbb{N}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014: 40(4), p. 163–166, unless otherwise specified.

3931. *Proposed by Bill Sands.*

A class is given two math tests. Each student in the class must write either Test 1 or Test 2 , but could write both tests. It turned out that one-quarter of the students who wrote Test 1 got an A, that one-third of the students who wrote Test 2 got an A, and that the same number of students got A on the two tests. Also, one-half of all the students in the class got an A on at least one of the two tests. Prove that

a) every student wrote Test 1, and

b) no student got an A on both tests.

We received nine correct solutions and three incorrect submissions. We present the solution by Michael M. Parmenter.

Let

$$x = \#\text{ of students who wrote Test 1 only},$$
$$y = \#\text{ of students who wrote both Test 1 and Test 2},$$
$$z = \#\text{ of students who wrote Test 2 only}.$$

We know

$$\frac{1}{4}(x+y) = \frac{1}{3}(y+z),$$

giving

$$y = 3x - 4z. \tag{1}$$

We also know that

$$\frac{1}{2}(x+y+z) \le \frac{1}{4}(x+y) + \frac{1}{3}(y+z),$$

giving

$$3x + 2z \le y. \tag{2}$$

(a) Substituting (1) into (2), we obtain $3x + 2z \le 3x - 4z$. Since $z \ge 0$, we conclude that $z = 0$, as desired.

(b) From (a), we know that $z = 0$ and $y = 3x$. Hence the number of students obtaining an A in Test 1 is

$$\frac{1}{4}(x + 3x) = x,$$

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and the number of students obtaining an A in Test 2 is

$$\frac{1}{3}(3x+0)=x.$$

Finally, the number of students obtaining at least one A is

$$\frac{1}{2}(x+3x+0)=2x.$$

Therefore, the number of students obtaining an A on both tests is

$$x+x-2x=0,$$

as desired.

3932. *Proposed by Arkady Alt.*

Let x and y be natural numbers satisfying equation $x^2-14xy+y^2-4x=0$. Find $\gcd(x,y)$ in terms of x and y.

We received eight correct submissions and one incomplete solution. We present the solution by Oliver Geupel and a remark by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

We show that x is a perfect square and $\gcd(x,y)=2\sqrt{x}$. Rewriting the given equation as

$$\left(y-7x-2\sqrt{x(12x+1)}\right)\left(y-7x+2\sqrt{x(12x+1)}\right)=0,$$

we obtain

$$y=7x\pm 2\sqrt{x(12x+1)}.$$

Therefore, the product of the co-prime numbers x and $12x+1$ is a perfect square, which means that x and $12x+1$ are perfect squares, $x=z^2$ and $12x+1=u^2$ with relatively prime positive integers z and u. If the number x were odd then

$$u^2=12x+1\equiv 5 \pmod{8},$$

which is impossible. Thus, x is even and therefore z is even. Consequently,

$$\gcd(x,y)=\gcd\left(x,2\sqrt{x(12x+1)}\right)=\gcd(z^2,2zu)=2z.$$

Hence the result.

Remark, by Dionne Bailey, Elsie Campbell, and Charles Diminnie.

By rewriting the original equation in the form

$$(24x+1)^2-12\,(y-7x)^2=1$$

and considering solutions of the resulting Pell's Equation

$$u^2 - 12v^2 = 1$$

(with $u = 24x + 1$ and $v = |y - 7x|$), we can generate solutions of our equation. Some of the smaller solutions are listed in the following table. Note that $\gcd(x, y) = 2\sqrt{x}$ in each case.

x	y	d
4	56	4
784	56	56
784	10,920	56
152,100	10,920	780
152,100	2,118,480	780
29,506,624	2,118,480	10,864

3933. *Proposed by Dragoljub Milošević.*

Let $ABCDEFG$ be a regular heptagon. Prove that

$$\frac{AD^3}{AB^3} - \frac{AB + 2AC}{AD - AC} = 1.$$

Thirteen correct solutions were received. We present four solutions after some preliminaries and editor comments.

Preliminaries. Let $ABCDEFG$ be a regular heptagon having sides of length a, short diagonals (*e.g.* AC) of length b and long diagonals (*e.g.* AD) of length c. Let $\theta = \pi/7$, so that $a = 2R\sin\theta$, $b = 2R\sin 2\theta$ and $c = 2R\sin 3\theta$, where R is the circumradius.

Five solvers based their solution of the use of some of the relationships

$$a^2 + ac = b^2;$$

$$b^2 + ab = c^2;$$

$$a^2 + bc = c^2; \qquad (1)$$

$$ac + ab = bc. \qquad (2)$$

These can be verified by applying Ptolemy's theorem to the respective cyclic quadrilaterals $ABCD$, $ACEG$, $ADEG$, $ADFG$. However, one solver used the trigonometic representations for a, b and c to obtain (1) and (2).

Four solvers used trigonometry. The result is equivalent to

$$\sin^3 3\theta(\sin 3\theta - \sin 2\theta) = \sin^3\theta(\sin\theta + \sin 2\theta + \sin 3\theta). \qquad (3)$$

Three of these followed the strategy of Solution 4.

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One solver placed the vertices of the heptagon on the complex unit circle and reduced the equality to $7\zeta(\zeta^5+1)(\zeta^7+1)=0$ where ζ is a primitive seventh root of unity.

Solution 1, by Dag Jonsson and Ricard Peiró i Estruch (independently).

We have to show that

$$\frac{c^3}{a^3}-\frac{a+2b}{c-b}=1 \quad \text{or} \quad \frac{c^3}{a^3}=\frac{a+b+c}{c-b}.$$

From (1) and (2), we obtain that

$$\frac{c^3}{a^3}=\frac{c}{a^2}\left(a+\frac{bc}{a}\right)=\frac{1}{c-b}(a+b+c).$$

Solution 2, by Michel Bataille.

Observe that $\sin 3\theta=\sin 4\theta$ and that

$$\begin{aligned}
\sin^4 3\theta-\sin^4\theta &= (\sin 3\theta+\sin\theta)(\sin 3\theta-\sin\theta)(\sin^2\theta+\sin^2\theta)\\
&= 2\sin 2\theta\cos\theta\cdot 2\sin\theta\cos 2\theta(\sin^2 3\theta+\sin^2\theta)\\
&= \sin 2\theta\sin 4\theta(\sin^2 3\theta+\sin^2\theta)\\
&= \sin^3 3\theta\sin 2\theta+\sin^2\theta\sin 2\theta\sin 3\theta\\
&= \sin^3 3\theta\sin 2\theta+\sin^3\theta(2\sin 3\theta\cos\theta)\\
&= \sin^3 3\theta\sin 2\theta+\sin^3\theta(\sin 2\theta+\sin 4\theta)\\
&= \sin^3 3\theta\sin 2\theta+\sin^3\theta(\sin 2\theta+\sin 3\theta).
\end{aligned}$$

Rearranging the terms yields (3).

Solution 3, by AN-anduud Problem Solving Group.

Let AD and CE intersect at Q. Observe that $CQ=CE-EQ=b-a$. The triangles ACD and CQD, both with angles θ, 4θ, 2θ, are similar and so

$$\frac{c}{a}=\frac{a}{c-b}=\frac{b}{b-a}=\frac{a+b+c}{c}.$$

Therefore

$$\frac{c^3}{a^3}=\left(\frac{a+b+c}{c}\right)\left(\frac{a}{c-b}\right)\left(\frac{c}{a}\right)=\frac{a+b+c}{c-b}.$$

Solution 4, by Kee-Wai Lau.

Without loss of generality, let $a=1$ and $t=\cos(\theta)$. Then $b=2t$ and

$$c=1+2\cos(2\theta)=4t^2-1.$$

The equality to be derived is equivalent to

$$(4t^2-1)^3 - \frac{1+4t}{4t^2-2t-1} = 1.$$

First, note that

$$\begin{aligned} 0 &= \cos 3\theta + \cos 4\theta = (4t^3-3t) + 2(2t^2-1)^2 - 1 \\ &= 8t^4 + 4t^3 - 8t^2 - 3t + 1 = (t+1)(8t^3-4t^2-4t+1). \end{aligned}$$

Since $t \neq 1$, $8t^3 - 4t^2 - 4t + 1 = 0$.

$$\begin{aligned} &(4t^2-1)^3(4t^2-2t-1) - (1+4t) - (4t^2-2t-1) \\ &\qquad = 256t^8 - 128t^7 - 256t^6 + 96t^5 + 96t^4 - 24t^3 - 20t^2 + 1 \\ &\qquad = (2t+1)(8t^3-4t^2-4t+1)(16t^4-8t^3-4t^2+2t+1) = 0, \end{aligned}$$

from which the result follows.

3934. *Proposed by George Apostolopolous.*

Let a, b and c be the side lengths of a triangle. Prove that

$$\frac{a}{\sqrt[3]{4b^3+4c^3}} + \frac{b}{\sqrt[3]{4a^3+4c^3}} + \frac{c}{\sqrt[3]{4a^3+4b^3}} < 2.$$

We received 19 correct solutions, most very similar to the one featured here. We present the solution by Cao Minh Quang.

Since

$$\frac{b^3+c^3}{2} \geq \left(\frac{b+c}{2}\right)^2,$$

we have $\sqrt[3]{4(b^3+c^3)} \geq b+c$. From $a < b+c$, it follows that $a+b+c < 2(b+c)$. Thus,

$$\frac{a}{\sqrt[3]{4b^3+4c^3}} \leq \frac{a}{b+c} < \frac{2a}{a+b+c}.$$

Therefore,

$$\sum_{\text{cyclic}} \frac{a}{\sqrt[3]{4b^3+4c^3}} < \sum_{\text{cyclic}} \frac{2a}{a+b+c} = 2.$$

3935. *Proposed by Michel Bataille.*

For positive integers n, let $P_n(x) = x^n + \sum\limits_{k=1}^{n}(-1)^k(n-k+1)x^{n-k}$.

a) Prove that if $n \geq 3$, the polynomial P_n has a unique zero x_n in $(1,\infty)$ and find real numbers α, β such that $\lim\limits_{n\to\infty}(x_n - \alpha n) = \beta$.

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b) Prove that for all integers $n \geq 2$:

$$1-\frac{1}{4n^2} < x_{2n+1}-x_{2n} < 1+\frac{1}{2n+1} \quad \text{and} \quad 1-\frac{1}{2n-1} < x_{2n}-x_{2n-1} < 1+\frac{1}{4n^2}.$$

We received one correct solution and one incomplete submission. The solution presented below follows the solution by Nermin Hodžić, augmented by parts from the proposer's solution and by the editor.

First we prove by induction that

$$P_n(x) = \frac{x^{n+2} + (2-n)x^{n+1} - nx^n + (-1)^n}{(x+1)^2}.$$

For $n = 1$ the claim is obvious. Suppose the formula holds for n. Then

$$\begin{aligned}
&x^{n+1} + \sum_{k=1}^{n+1} (-1)^k (n+1-k+1) x^{n+1-k} \\
&= x^{n+1} + \sum_{k=1}^{n+1} (-1)^k x^{n+1-k} + \sum_{k=1}^{n+1} (-1)^k (n-k+1) x^{n+1-k} \\
&= (-1)^{n+1} + \sum_{k=1}^{n} (-1)^k x^{n+1-k} + x^{n+1} + x \sum_{k=1}^{n} (-1)^k (n-k+1) x^{n-k} \\
&= (-1)^{n+1} + x^{n+1} \sum_{k=1}^{n} \left(-\frac{1}{x}\right)^k + x\left(x^n + \sum_{k=1}^{n} (-1)^k (n-k+1) x^{n-k}\right) \\
&= (-1)^{n+1} + \frac{x^{n+1} + (-1)^n}{1+\frac{1}{x}} - x^{n+1} + \frac{x(x^{n+2} + (2-n)x^{n+1} - nx^n + (-1)^n)}{(x+1)^2} \\
&= \frac{x^{n+3} + (1-n)x^{n+2} - (n+1)x^{n+1} + (-1)^{n+1}}{(x+1)^2}.
\end{aligned}$$

Therefore the positive roots of $P_n(x)$ coincide with the positive roots of

$$Q_n(x) = x^{n+2} + (2-n)x^{n+1} - nx^n + (-1)^n = x^n(x^2 + (2-n)x - n) + (-1)^n.$$

We note that

$$Q_n'(x) = (n+2)x^{n+1} + (2-n)(n+1)x^n - n^2x^{n-1} = (n+2)x^{n-1}(x+1)\left(x - \frac{n^2}{n+2}\right).$$

Therefore $Q_n(x)$ is strictly decreasing on $(1, \frac{n^2}{n+2})$ and strictly increasing on $(\frac{n^2}{n+2}, \infty)$. Since

$$Q_n(1) = 3 + (-1)^n - 2n < 0$$

for all $n \geq 3$, there are no roots of $Q_n(x)$ in $(1, \frac{n^2}{n+2})$. Also, since $Q_n(x) \to \infty$ as $x \to \infty$, there is exactly one root in $(\frac{n^2}{n+2}, \infty)$. Now note that for this root we have

$$x_n^2 + (2-n)x_n - n = \frac{(-1)^{n+1}}{x_n^n}.$$

For n odd, the right side is bounded between 0 and 1, thus
$$\frac{n-2+\sqrt{n^2+4}}{2} < x_n < \frac{n-2+\sqrt{n^2+8}}{2},$$
whereas for n even, the right side is bounded between -1 and 0, so
$$n-1 < x_n < \frac{n-2+\sqrt{n^2+4}}{2}.$$
Therefore for all n, we have
$$1 = \lim_{n\to\infty}\frac{n-1}{n} \le \lim_{n\to\infty}\frac{x_n}{n} \le \lim_{n\to\infty}\frac{n-2+\sqrt{n^2+8}}{2} = 1,$$
which shows $\lim\limits_{n\to\infty}\frac{x_n}{n} = 1$ and $\alpha = 1$. Further
$$-1 = \lim_{n\to\infty}(n-1-n) \le \lim_{n\to\infty}(x_n - n) \le \lim_{n\to\infty}\left(\frac{n-2+\sqrt{n^2+8}}{2} - n\right) = -1,$$
which implies $\lim\limits_{n\to\infty}(x_n - n) = -1$ and thus $\beta = -1$.

For part b), we first consider the function $f(x) = \sqrt{x^2+4}$. Then $f'(x) = \frac{x}{\sqrt{(x^2+4)}}$ and $f''(x) = \frac{4}{(x^2+4)^{3/2}}$. Therefore $f(x)$ is concave up with minimum at 0 and thus has the property that for all $x \ge 0$
$$f'(x) > f(x) - f(x-1) > f'(x-1).$$
Now we can calculate
$$\begin{aligned} x_{2n+1} - x_{2n} &> \frac{1}{2}\left(2n-1+\sqrt{(2n+1)^2+4} - \left(2n-2+\sqrt{(2n)^2+4}\right)\right) \\ &= \frac{1}{2}(1+f(2n+1)-f(2n)) \\ &> \frac{1}{2}(1+f'(2n)) = \frac{1}{2}\left(1+\frac{2n}{\sqrt{4n^2+4}}\right). \end{aligned}$$
We need to show
$$\begin{aligned} & & 1-\frac{1}{4n^2} &< \frac{1}{2}\left(1+\frac{2n}{\sqrt{4n^2+4}}\right) \\ &\Longleftrightarrow & (4n^2-2)\sqrt{4n^2+4} &< 8n^3 \\ &\overset{n\ge 1}{\Longleftrightarrow} & (16n^4-16n^2+4)(4n^2+4) &< 64n^6 \\ &\Longleftrightarrow & 48n^2-16 &> 0, \end{aligned}$$
which clearly holds for $n \ge 2$. Similarly we obtain
$$\begin{aligned} x_{2n} - x_{2n-1} &< \frac{1}{2}\left(2n-2+\sqrt{(2n)^2+4} - \left(2n-3+\sqrt{(2n-1)^2+4}\right)\right) \\ &= \frac{1}{2}(1+f(2n)-f(2n-1)) \\ &< \frac{1}{2}(1+f'(2n)) = \frac{1}{2}\left(1+\frac{2n}{\sqrt{4n^2+4}}\right). \end{aligned}$$

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And we need to show

$$\begin{aligned} & \frac{1}{2}\left(1+\frac{2n}{\sqrt{4n^2+4}}\right) < 1+\frac{1}{4n^2} \\ \iff \quad & 8n^3 < (4n^2+2)\sqrt{4n^2+4} \\ \iff \quad & 64n^6 < (16n^4+16n^2+4)(4n^2+4) \\ \iff \quad & 0 < 128n^4+80n^2+16, \end{aligned}$$

which clearly holds. For the remaining two inequalities, we first prove that $x_n < n-1+\frac{1}{n}$ by showing that $Q_n\left(n-1+\frac{1}{n}\right) > 0$ for all $n \geq 3$. We calculate

$$Q_n\left(n-1+\frac{1}{n}\right) = \frac{1}{n^2}\left(n-1+\frac{1}{n}\right)^n + (-1)^n.$$

It is easily checked that this is positive if $n = 3$. For $n \geq 4$, we note that

$$Q_n\left(n-1+\frac{1}{n}\right) > \frac{1}{n^2}(n-1)^3 - 1,$$

so it suffices to check that

$$\frac{1}{n^2}(n-1)^3 > 1,$$

which is equivalent to

$$(n^3-4n^2)+(3n-1) > 0.$$

which clearly holds. Together with part a), we have $n-1 < x_n < n-1+\frac{1}{n}$ and can now finally show

$$x_{2n+1}-x_{2n} < \left(2n+\frac{1}{2n+1}\right)-(2n-1) = 1+\frac{1}{2n+1}$$

and

$$x_{2n}-x_{2n-1} > (2n-1)-\left(2n-2+\frac{1}{2n-1}\right) = 1-\frac{1}{2n-1}.$$

3936. *Proposed by Paul Bracken.*

Let $p \geq 1$ and suppose $\{x_k\}_{k=1}^n \in (0,1)$. Prove that

$$\prod_{k=1}^n (1-x_k^p) \leq e^{-(p+1)^{1/p}\sum_{k=1}^n x_k^{p+1}}.$$

There were five correct submissions for this problem. We present two solutions.

Solution 1, by Paolo Perfetti.

The inequality is

$$\prod_{k=1}^n (1-x_k^p)e^{(p+1)^{\frac{1}{p}}x_k^{p+1}} \leq 1$$

which is implied by

$$f(x) = (1 - x^p)e^{(p+1)^{1/p}x^{p+1}} - 1 \le 0, \qquad 0 < x < 1.$$

We have:

$$\begin{aligned} f'(x) &= e^{(p+1)^{\frac{1}{p}}} \left(-px^{p-1} + (1-x^p)(p+1)(p+1)^{\frac{1}{p}}x^p\right) = \\ &= x^p e^{(p+1)^{\frac{1}{p}}} \left(-\frac{p}{x} + (1-x^p)(p+1)^{\frac{p+1}{p}}\right) =: x^p e^{(p+1)^{\frac{1}{p}}} g(x), \end{aligned}$$

where $g(x)$ is defined by the expression in the brackets. Then:

$$g'(x) = \frac{p}{x^2} - px^{p-1}(p+1)^{\frac{p+1}{p}} \ge 0 \iff x \le (p+1)^{-\frac{1}{p}} =: x_0,$$

and thus the maximum of g occurs for $x = x_0$ and

$$g(x_0) = -p(p+1)^{\frac{1}{p}} + \left(1 - \frac{1}{p+1}\right)(p+1)^{\frac{p+1}{p}} = -p(p+1)^{\frac{1}{p}} + p(p+1)^{\frac{1}{p}} = 0$$

This means $g(x) \le 0$, and so $f'(x) \le 0$. Because of $\lim\limits_{x\to 0^+} f(x) = 0$, it follows $f(x) < 0$ for any $0 < x < 1$. This concludes the proof.

Solution 2, by the proposer. Define the function $f(x) = x(1 - x^p)$. Then $f(x)$ has a unique maximum on $(0, 1)$ which occurs at $x_0 = (p + 1)^{-1/p} \in (0, 1/2]$. The largest the function can be is $f(x_0) = p(p + 1)^{-1-1/p}$. The result $f(x) \le f(x_0)$ implies the following inequality

$$\frac{1}{p}(p+1)^{1+1/p}x(1-x^p) \le 1.$$

Now rewrite this in the form

$$\frac{1}{1-x^p} \ge \frac{p+1}{p}(p+1)^{1/p}x.$$

Multiply this on both sides by x^{p-1} and integrate over $(0, s)$ where $0 < s < 1$ to obtain

$$-\frac{1}{p}\log(1-s^p) \ge \frac{(p+1)^{1/p}}{p}s^{p+1}.$$

This is equivalent to

$$\log(1-s^p) \le -(p+1)^{1/p}s^{p+1}.$$

Now replace s successively by x_k and add the resulting inequalities from $k = 1$ to n to give

$$\sum_{k=1}^{n} \log(1-x_k^p) \le -(p+1)^{1/p}\sum_{k=1}^{n} x_k^{p+1}.$$

Exponentiating both sides of this, the result follows.

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Editor's Comments. All solvers except for the proposer used basically the same idea and techniques: isolate the inequality that yields the desired inequality, do some differential calculus and optimization, and arrive at the conclusion. Two solutions used power series as an intermediate step; some used logarithms. The proposer's solution directly uses integration to obtain the sufficient inequality, which is perhaps an unexpected result.

Bataille noted that the inequality still holds for $p \in (0,1)$. The proposer's solution mentions that a particular value is bounded in a smaller region than $(0,1)$ using $p \geq 1$, but otherwise does not use the assumption.

3937. *Proposed by Marcel Chiriţă.*

If the vertices of a triangle are represented by the complex numbers a, b, c, and these numbers satisfy

$$\frac{a-b}{c-b} + \frac{c-a}{b-a} = 2\frac{b-c}{a-c},$$

then prove that the triangle is equilateral.

We received 13 correct submissions. We present the solution by Francisco Perdomo and Ángel Plaza.

The proposer tacitly assumes that the given triangle is non-degenerate. When we let $x = a - b$ and $y = b - c$, the given equation becomes

$$\frac{x}{-y} + \frac{-(x+y)}{-x} = 2\frac{y}{x+y}.$$

By clearing the denominators the equation reduces to $y^3 = x^3$. It follows that

- $y = x$, so that $b - c = a - b$, in which case $b = \frac{a+c}{2}$, so that b represents the midpoint of the segment joining the other two points and the given triangle would be degenerate; or
- $y = e^{2\pi i/3}x$, in which case $b - c = e^{2\pi i/3}(a - b)$, so that

  $$c - b = e^{-\pi i}\left(e^{2\pi i/3}(a-b)\right) = e^{-\pi i/3}(a-b),$$

  and a, b, c represent the vertices of an equilateral triangle labeled counterclockwise (because a rotation through -60° takes the vector $a - b$ to $c - b$; or
- $y = e^{4\pi i/3}x$, in which case $c - b = e^{\pi i/3}(a - b)$, and a, b, c represent the vertices of an equilateral triangle labeled clockwise.

3938. *Proposed by Francisco Javier García Capitán, modified by the editor.*

Let ABC be a triangle, O a circle and P a point on O. Find two points on O for which the sum $PA^2 + PB^2 + PC^2$ reaches its minimum and its maximum.

We received eleven correct solutions. We feature two of them.

Solution 1, by Michel Bataille.

Let G be the centroid of ΔABC. The Leibniz relation yields

$$PA^2 + PB^2 + PC^2 = 3PG^2 + GA^2 + GB^2 + GC^2$$

for any point P. Thus $PA^2 + PB^2 + PC^2$ and PG reach their extrema on O at the same points P. Now, let Ω be the center of the circle O and let ρ be its radius.

- If $\Omega = G$, then $PG = \rho$ for any point P on O and the sum $PA^2+PB^2+PC^2$ is constant on O. It reaches its minimum and maximum at every point of O.
- If $\Omega \neq G$, the line ΩG (a diameter of O) intersects O at two points P_1 and P_2, say with P_1 closer to G than P_2. Then $P_1G = \min PG : P \in O$ and $P_2G = \max PG : P \in O$ and the sum $PA^2 + PB^2 + PC^2$ reaches its minimum on O at P_1 and its maximum on O at P_2.

Solution 2, by Arkady Alt, San Jose, CA.

We will solve a more general problem. Let O be the circle $x^2 + y^2 = R^2$. Given $A_i = (x_i, y_i)$, $i = 1, 2, ..., n$ be points in the plane, find the points on O for which the sum $\sum_{i=1}^n PA_i^2$ reaches its minimum and maximum. Define

$$\begin{aligned} h(x,y) = \sum_{i=1}^n PA_i^2 &= \sum_{i=1}^n \left((x-x_i)^2 + (y-y_i)^2\right) \\ &= n(x^2+y^2) + X_2 + Y_2 - 2(xX - 1 + yY_1) \\ &= nR^2 + X_2 + Y_2 - 2(xX_1 + yY_1), \end{aligned}$$

where $X_2 = \sum_{i=1}^n x_i^2$, $X_1 = \sum_{i=1}^n x_i$, $Y_2 = \sum_{i=1}^n y_i^2$, and $Y_1 = \sum_{i=1}^n y_i$. By Cauchy's inequality,

$$|xX_1 + yY_1| \le \sqrt{x^2+y^2} \cdot \sqrt{X_1^2 + Y_1^2} = R \cdot \sqrt{X_1^2 + Y_1^2},$$

with equality if and only if $\frac{x}{X_1} = \frac{y}{Y_1}$ and $x^2 + y^2 = R^2$, that is, if and only if

$$(x,y) = \frac{R}{\sqrt{X_1^2 + Y_1^2}} \cdot (X_1, Y_1) \quad \text{or} \quad (x,y) = -\frac{R}{X_1^2 + Y_1^2} \cdot (X_1, Y_1).$$

Thus we have

$$\min h(x,y) = h\left(\frac{RX_1}{\sqrt{X_1^2+Y_1^2}}, \frac{RY_1}{\sqrt{X_1^2+Y_1^2}}\right) = nR^2 + X_2 + Y_2 - 2R \cdot \sqrt{X_1^2 + Y_1^2}$$

and

$$\max h(x,y) = h\left(-\frac{RX_1}{\sqrt{X_1^2+Y_1^2}}, -\frac{RY_1}{\sqrt{X_1^2+Y_1^2}}\right) = nR^2 + X_2 + Y_2 + 2R \cdot \sqrt{X_1^2 + Y_1^2}.$$

Let $G(x_0, y_0)$ be the centroid of the set $\{A_i\}_{i=1}^n$, and let d be the distance from the origin to G. Then $x_0 = \frac{X_1}{n}$, $y_0 = \frac{Y_1}{n}$, and $\sqrt{X_1^2 + Y_1^2} = nd$. Therefore, the

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minimum and maximum values of $h(x,y)$ occur at $P_* = (x_*, y_*)$ and $P^* = (x^*, y^*)$, respectively, where

$$x_* = \frac{Rx_0}{d},\ y_* = \frac{Ry_0}{d},\ x^* = -\frac{Rx_0}{d} \quad \text{and} \quad y^* = -\frac{Ry_0}{d}.$$

These are the points at which the line through G and the center of the circle intersect the circle.

3939. *Proposed by George Apostolopolous.*

Let a, b and c be positive real numbers such that $a^2 + b^2 + c^2 = 27$. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 - 3a + 9}} \leq 3.$$

There were 20 solutions. We present four different solutions. All sums in the solutions are cyclic with three terms.

Solution 1, by Arkady Alt and Henry Ricardo (independently).

Since $a^2 - 3a + 9 = (a-3)^2 + 3a \geq 3a$, then

$$\sum \frac{a}{\sqrt{a^2 - 3a + 9}} \leq \sum \sqrt{\frac{a}{3}} = \frac{1}{\sqrt{3}} \sum \sqrt{a}.$$

Since the power means $((1/3)\sum a^n)^{1/n}$ increase with n,

$$\left(\frac{1}{3}\sum \sqrt{a}\right)^2 \leq \left(\frac{1}{3}\sum a^2\right)^{1/2} = 3,$$

so that $\sum \sqrt{a} \leq 3\sqrt{3}$ and the result follows.

Solution 2, by John C. Heuver.

By the Cauchy-Schwarz Inequality and the inequality of the arithmetic and quadratic means,

$$\begin{aligned}
\sum \frac{a}{\sqrt{a^2 - 3a + 9}} &\leq \left(\sum a \cdot \sum \frac{a}{a^2 - 3a + 9}\right)^{1/2} \\
&\leq \left(\sum a \cdot \sum \frac{a}{3a}\right)^{1/2} = \left(\sum a\right)^{1/2} \\
&\leq \left[3\left(\frac{1}{3}\sum a^2\right)^{1/2}\right]^{1/2} = 3.
\end{aligned}$$

Solution 3, by Kee-Wai Lau and Paolo Perfetti (independently).

Recall Hölder's Inequality for finite sums,

$$\sum x_i y_i \leq \left(\sum x_i^p\right)^{1/p} \left(\sum y_i^q\right)^{1/q}$$

for positive p, q, x_i, y_i and $1/p + 1/q = 1$. Apply this to $(p, q) = (4, 4/3)$ and $y_i = 1$ to obtain

$$\begin{aligned}\sum \frac{a}{\sqrt{a^2-2a+9}} &\leq \frac{1}{\sqrt{3}} \sum \sqrt{a} \leq \frac{1}{\sqrt{3}} \left(\sum a^2\right)^{1/4} \left(\sum 1\right)^{3/4} \\ &= 3^{-1/2} \cdot 3^{3/4} \cdot 3^{3/4} = 3.\end{aligned}$$

Solution 4, by Titu Zvonaru.

First, we establish that

$$\frac{x}{\sqrt{x^2-3x+9}} \leq \frac{x^2}{36} + \frac{3}{4}$$

for $x \geq 0$. This is equivalent to

$$(x^4 + 54x^2 + 729)(x^2 - 3x + 9) \geq 1296x^2.$$

However, the difference between the two sides is equal to

$$\begin{aligned}x^6 - 3x^5 + 63x^4 - 162x^3 - 81x^2 - 2187x + 6561 \\ = (x-3)^2(x^4 + 3x^3 + 72x^2 + 243x + 729) \geq 0.\end{aligned}$$

Applying this inequality to a, b, c and adding yields that

$$\sum \frac{a}{\sqrt{a^2-3a+9}} \leq \frac{\sum a^2}{36} + 3 \cdot \frac{3}{4} = 3.$$

3940. *Proposed by Michał Kremzer.*

Find positive a and b so that $\dfrac{a+b}{a(\tan a + \tan b)} = 2015.$

We received five correct solutions and one incorrect submission to this problem. We present two of the solutions.

Solution 1, by Digby Smith.

Let n be any integer, and

$$a = b = \arctan(1/2015) + n\pi.$$

Then

$$\frac{a+b}{a(\tan(a) + \tan(b))} = \frac{2a}{2a\tan(a)} = \frac{1}{\tan(a)},$$

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and since $\tan(a) = \frac{1}{2015}$, the result follows.

Solution 2, by Joseph DiMuro.

Try to find a solution where $a = \frac{\pi}{4} + n\pi$ and $b = \frac{\pi}{4} + m\pi$ for n and m integers. Substituting into the equation and simplifying, we get that m and n have to satisfy

$$\frac{2m + 2n + 1}{4m + 1} = 2015.$$

For example, $m = 1$ and $n = 5036$ satisfy this equation, leading to the solution

$$a = \frac{5}{4}\pi \quad \text{and} \quad b = \frac{20145}{4}\pi.$$

Or, $m = 0$ implies $n = 1007$, leading to the solution of

$$a = \frac{\pi}{4} \quad \text{and} \quad b = \frac{\pi}{4} + 1007\pi.$$

Editor's Comments. In addition to providing a specific solution, Oliver Geupel used the Intermediate Value Theorem to show that, for any value of $a \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, there is a value of b which satisfies the equation. Therefore, the solutions presented above are just a few of the uncountably many options.

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