Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 20 #3

March
mars
1994

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Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

SUBSCRIPTION INFORMATION

Crux Mathematicorum is published monthly (except July and August). The subscription rates for ten issues are detailed on the inside back cover. Information on Crux Mathematicorum back issues is also provided on the inside back cover.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum.

ACKNOWLEDGEMENTS

The support of the Department of Mathematics & Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

REMERCIEMENTS

Nous rendons hommage à l'appui du département de mathématique et statistique de l'Université de Calgary et du département de mathématiques de l'Université d'Ottawa.

Printed at / imprimé à: Synergy Print and Copy, Ottawa.

ISSN 0705-0348

Publication Mail Registration No. 5432

ON THE DISTRIBUTION OF ZERO POINTS OF THE REAL-VALUED FUNCTION $1 - \sum_{i=1}^{n} a_i/(x + a_i)$

Masakazu Nihei

1. INTRODUCTION Let $f(x)$ be a polynomial with real coefficients. If a is a real number such that $f(a) = 0$, then a is called a "real zero point" (in what follows we will call it simply a zero point). Alternatively, a is called a real root or real solution of the equation $f(x) = 0$.

In general, we can not expect to obtain, in concrete form, the solutions of an equation of fifth degree or more (see [4; p. 218] for example). In view of this fact, it would be a highly intriguing proposition to find intervals that will contain one and only one of the real roots of a given equation (i.e., separation of real roots).

To begin with, we will confine our consideration to the separation of real roots of a particular cubic equation expressed in a comparatively pretty form as follows:

$$(x+2)(x+3) + 2(x+1)(x+3) + 3(x+1)(x+2) = (x+1)(x+2)(x+3). \tag{1}$$

That is, we seek the real numbers x for which equation (1) holds. Let

$$f(x) = (x+1)(x+2)(x+3) - (x+2)(x+3) - 2(x+1)(x+3) - 3(x+1)(x+2);$$

then the function $f(x)$ is continuous on the open interval $\mathbb{R} = (-\infty, \infty)$. Since $f(-3) = -6 < 0$ and $f(-2) = 2 > 0$, $f(x)$ has at least one zero point on the open interval $(-3, -2)$ by virtue of the Intermediate Value Theorem (see [1] or [2]). Likewise, we can see that $f(x)$ has at least one zero point on the open interval $(-2, -1)$.

We can also see that $f(x)$ has one and only one positive zero point, once we have taken into account the fact that $f(3) = -18 < 0$, $f(4) = 8 > 0$ and that (1) is a cubic equation.

In this note, we will generalize the above-mentioned problem. It is of special interest to determine as precisely as possible the interval containing positive zero points.

2. RESULTS Since the equation (1) does not have $-1, -2, -3$ as its roots, division of both sides of (1) by $(x+1)(x+2)(x+3)$ will not bring about a change when studying how its roots are separated. This means that it suffices to study how the zero points of the real-valued function

$$f(x) = 1 - \frac{1}{x+1} - \frac{2}{x+2} - \frac{3}{x+3} \tag{2}$$

are distributed in order to study how the roots of the equation (1) are separated.

The following result shows the separation of the roots of the equation (1) generalized in the form of (2).

THEOREM: Let $0 < a_1 < a_2 < \cdots < a_n$, $n \geq 2$, $\sum_{i=1}^{n} a_i = s$. Then the real-valued function

$$f(x) = 1 - \sum_{i=1}^{n} \frac{a_i}{x + a_i}$$

has one zero point on each of the open intervals $(-a_i, -a_{i-1})$ $(i = 2, \ldots, n)$, and further, one positive zero point on $(s - a_n, s(n-1)/n)$.

Now, we will prepare three lemmas for our proof of this theorem, the first of which is well-known, i.e., arithmetic means dominate geometric means, and geometric means dominate harmonic means.

Lemma 1: For any n positive numbers $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n} .$$

Equality holds in each inequality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof: As the proof of the inequality on the left side is well known, see [3; p. 52–59] for example, we will prove only the inequality on the right side. By the arithmetic-mean–geometric-mean inequality, we obtain

$$\frac{1/a_1 + 1/a_2 + \cdots + 1/a_n}{n} \geq \sqrt[n]{\frac{1}{a_1 a_2 \cdots a_n}} = \frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}} .$$

Hence we have the desired result. □

Lemma 2: Keeping $x > 0$ fixed, we let $a_1, \ldots, a_n$ vary under the condition that $a_1 + \cdots + a_n = s$ $(a_1 \leq \cdots \leq a_n)$. In this situation, $f(x)$ becomes minimum precisely when $a_1 = \cdots = a_n = s/n$.

Proof: Let $g(x) = \sum_{i=1}^{n} x/(a_i + x)$. Then $f(x) = g(x) + 1 - n$. In fact,

$$g(x) + 1 - n = \sum_{i=1}^{n} \left(\frac{x}{a_i + x} - 1 \right) + 1 = \sum_{i=1}^{n} \frac{-a_i}{a_i + x} + 1 = f(x).$$

Therefore all we have to do is to show that $g(x)$ becomes minimum when $a_1 = \cdots = a_n$. Because the arithmetic mean dominates the harmonic mean (Lemma 1), we have

$$g(x) = \sum_{i=1}^{n} \frac{x}{a_i + x} \geq \frac{n^2}{(a_1 + \cdots + a_n + nx)/x} = \frac{xn^2}{s + nx}$$

which is the value of $g(x)$ when $a_1 = \cdots = a_n$, and equality holds if and only if $a_1 = \cdots = a_n$. Therefore $g(x)$ is larger than the right hand member, provided that $a_1 < \cdots < a_n$. This completes the proof of Lemma 2. □

Lemma 3: The sum of the roots of a monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is $-a_{n-1}$.

The proof is left to the reader. (A high school student might want to go over this point with his or her teacher.)

Proof of the Theorem: $f(x)$ is monotone increasing and continuous at points other than $x = -a_i$ $(i = 1, \cdots, n)$, and $f(x)$ varies from $-\infty$ to ∞ on $(-a_i, -a_{i-1})$, where $i = 2, \cdots, n$. Moreover, $f(0) = 1 - n < 0$, and since $f(x) \to 1$ as $x \to +\infty$, the function $f(x)$ has a zero point on each of the open intervals $(-a_i, -a_{i-1})$ and $(0, \infty)$.

We are now left with the problem of determining the positive zero point λ. Let x be the fixed value $x = s(n-1)/n$. By Lemma 2, the minimum value for $f(x)$ occurs with $a_1 = a_2 = \cdots = a_n$, which in this case yields $f(s(n-1)/n) = 0$ (each $a_i = s/n$). If not all of the a_i coincide, then $f(s(n-1)/n) > 0$, again by Lemma 2, and the positive zero point λ is strictly less than $s(n-1)/n$.

It remains to be proved that $\lambda > s - a_n$. To do this, we multiply $f(x)$ by $\prod_{i=1}^{n}(x + a_i)$ to obtain a polynomial $q(x)$ of degree n. Observe that $q(\alpha) = 0$ if and only if $f(\alpha) = 0$, and so q has roots $\alpha_1, \cdots, \alpha_{n-1}, \lambda$. Order the roots so that $\alpha_1 < \cdots < \alpha_{n-1} < 0 < \lambda$, where $-a_{n-i+1} < \alpha_i < -a_{n-i}$ for each $i = 1, 2, \cdots, n-1$. In the polynomial $q(x)$, the coefficient on the term x^{n-1} is

$$(a_1 + \cdots + a_n) - (a_1 + \cdots + a_n) = 0$$

and so the sum of the roots of $q(x)$ is zero (by Lemma 3). Thus,

$$\lambda = -\sum_{i=1}^{n-1} \alpha_i > \sum_{i=1}^{n-1} a_{n-i} = s - a_n.$$

Hence, our theorem is proved. □

Corollary: $f(x) = 1 - \sum_{i=1}^{n} i/(x+i)$ has a zero point on each of $(-i, -i+1)$ $(i = 2, \cdots, n)$, and just one such point on $(n(n-1)/2, (n^2-1)/2)$.

3. FURTHER PROBLEMS

Finally, let us propose two problems pertaining to the theorem stated above.

Problem 1: Can we find an interval containing the positive zero point of the function $f(x)$ (in the theorem stated above) more precisely (by elementary methods)?

Problem 2: Investigate how the zero-points of the function

$$f(x) = 1 - \sum_{i=1}^{n} \frac{b_i}{x + a_i} \quad (a_i > 0, b_i > 0)$$

are distributed.

Acknowledgement: The author would like to thank Mr. Isao Ashiba, the editor and the referee for their helpful suggestions.

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[1] E.W. Swokowski, *Functions and Graphs*, 3rd edition, Prindle, Weber & Schmidt, Boston, Massachusetts, 1980.

[2] G.H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, 1967.

[3] E. Beckenbach and R. Bellman, *An Introduction to Inequalities*, Random House and the L.W. Singer Company, 1961.

[4] R.C. Thompson, *Elementary Modern Algebra*, Scott, Foresman and Company, Glenview, Illinois, 1974.

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THE OLYMPIAD CORNER

No. 153

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The Olympiad Contest we give this issue is the 9th Balkan Mathematical Olympiad, under the auspices of the Greek Mathematical Society. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, who collected this contest, and many others, when he was team leader at the Moscow I.M.O.

9th BALKAN MATHEMATICAL OLYMPIAD

Athens, Hellas, May 4–9, 1992

1. (Bulgaria). Let m and n be positive integers and

$$A(m,n) = m^{3^{4n}+6} - m^{3^{4n}} - m^5 + m^3.$$

Find every n such that $A(m,n)$ is divisible by 1992 for every m.

2. (Cyprus). Prove that for each positive integer n

$$(2n^2 + 3n + 1)^n \geq 6^n (n!)^2.$$

3. (Greece). Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB respectively (different from A, B, C). If $AFDE$ is inscribable in a circle show that:

$$\frac{4(DEF)}{(ABC)} \leq \left(\frac{EF}{AD}\right)^2.$$

4. (Romania). For every integer $n > 3$ find the minimum positive integer $f(n)$ such that every subset of the set $A = \{1, 2, 3, \ldots, n\}$ which contains $f(n)$ elements contains elements $x, y, z \in A$ which are pairwise relatively prime.

* * *

As a pre-Olympiad contest this month we give the problems of Part II of the Alberta High School Mathematics Competition for 1994. My thanks go to the chairman of the contest board, Alvin Baragar of The University of Alberta, for furnishing a copy.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION, PART II

February 8, 1994

1. Find all polynomials $P(x)$ that satisfy the equation

$$P(x^2) + 2x^2 + 10x = 2xP(x+1) + 3.$$

2. An isosceles triangle is called an *amoeba* if it can be divided into two isosceles triangles by a straight cut. How many *different* (i.e., not similar) amoebas are there?

3. (a) Show that there is a positive integer n so that the interval

$$\left(\left(n + \frac{1}{1994}\right)^2, \left(n + \frac{1}{1993}\right)^2\right)$$

contains an integer N.

(b) Find the smallest integer N which is contained in such an interval for some n.

4. $ABCDE$ is a convex pentagon in the plane. Through each vertex draw a straight line which cuts the pentagon into two parts of the same area. Prove that for some vertex, the line through it must intersect the "opposite side" of the pentagon. (Here the opposite side to vertex A is the side CD, the opposite side to B is DE, and so on.)

5. Let a, b, c be real numbers. Their pairwise sums $a+b$, $b+c$ and $c+a$ are written on three round cards and their pairwise products ab, bc and ca are written on three square cards. We call (a, b, c) a *tadpole* if we can form three pairs of cards, each consisting of one round card and one square card with the same number on both. An example of a tadpole is $(0, 0, 0)$.

(a) Find all possible tadpoles of the form (a, a, a).

(b) Prove that there is a tadpole that is not of the form (a, a, a). (You do not have to find the actual values of a, b, and c.)

* * *

We next start the solutions readers have submitted to problems from the 1993 numbers of the Corner. The first solutions are to some of the problems of the *1991–92 A.H.S.M.C. Part II* [1993: 4].

1. The Committee to Halt Excessive Amount of Photocopying (CHEAP) is itself accused of over-expenditure in photocopying, even though it never makes more than one copy of anything. The new committee set up to investigate this accusation makes, for each of its 13 members, a photocopy of everything CHEAP has photocopied, so that it can study whether the expenditure has been justified. Each committee is charged 7 cents per page for the first 2000 pages and 5 cents per page thereafter. It turns out that the photocopying expenditure of the new committee is 10 times that of CHEAP. How many photocopies did CHEAP make? Find all possible solutions.

Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.

Let A_1, A_2 be, respectively, the number of photocopies made by CHEAP and the new committee, and P_1, P_2 be the respective costs (in cents). We have

$$A_2 = 13A_1 \tag{1}$$

$$P_2 = 10P_1 \tag{2}$$

There are three cases:

i) $A_1 > 2000$.

Here $P_1 = 2000 \cdot 7 + (A_1 - 2000) \cdot 5 = 5A_1 + 4000$, $P_2 = 2000 \cdot 7 + (A_2 - 2000) \cdot 5 = 65A_1 + 4000$, by (1). Now, (2) gives

$$65A_1 + 4000 = 50A_1 + 40000 \Rightarrow A_1 = 2400.$$

ii) $A_1 \leq 2000$ and $A_2 > 2000$.

Here $P_1 = 7A_1$, $P_2 = 2000 + (A_2 - 2000) \cdot 5 = 65A_1 + 4000$ so

$$65A_1 + 4000 = 70A_1 \Rightarrow A_1 = 800.$$

iii) $A_1 \leq 2000$, and $A_2 \leq 2000$.

Here $P_2 = 7A_2 = 7 \cdot 13A_1 = 13 \cdot 7A_1 = 13P_1 > 10P_1$ giving no solutions.

Therefore CHEAP made either 800 or 2400 copies.

2. The base of a tub is a square with sides of length 1 metre. It contains water 3 centimetres deep. A heavy rectangular block is placed in the tub three times. Each time, the face that rests on the bottom of the tub has a different area. When this is done, the water in the tub ends up being 4 centimetres, 5 centimetres and 6 centimetres deep. Find the dimensions of the block.

Solutions by John Morvay, Springfield, Missouri; and by Panos E. Tsaoussoglou, Athens, Greece.

Let $x < y < z$ be the length of the sides of the block. From the data $4xy + 100 \times 100 \times 3 = 100 \times 100 \times 4$ so $xy = 25 \times 10^2$. Also $5xz + 100 \times 100 \times 3 = 100 \times 100 \times 5$ so $xz = 40 \times 10^2$. Finally $6yz + 100 \times 100 \times 3 = 100 \times 100 \times 6$ so $yz = 50 \times 10^2$. Thus $(xyz)^2 = 25 \times 40 \times 50 \times 10^6 = 5 \times 10^{10}$. Thus $xyz = 10^5\sqrt{5}$. This gives $z = 40\sqrt{5}$, $y = 25\sqrt{5}$, and $x = 20\sqrt{5}$.

4. Suppose x, y and z are real numbers which satisfy the equation $ax + by + cz = 0$, where a, b and c are given positive numbers.

(a) Prove that $x^2 + y^2 + z^2 \geq 2xy + 2yz + 2xz$.

(b) Determine when equality holds in (a).

Solution by Panos E. Tsaoussoglou, Athens, Greece.

Now $cz = -(ax + by)$.

(a) It is sufficient to prove

$$x^2 + y^2 = \frac{(ax + by)^2}{c^2} \geq \frac{2xy - 2(x + y)(ax + by)}{c}, \quad c > 0.$$

This gives

$$(a + c)^2x^2 + 2[(a + c)(b + c) - 2c^2]xy + (c + b)^2y^2 \geq 0.$$

Because $(a + c)^2 > 0$ it is sufficient to show

$$\Delta = [(a + c)(b + c) - 2c^2]^2 - (a + c)^2(c + b)^2 \leq 0.$$

But

$$\begin{aligned}\Delta &= [(a + c)(b + c) - 2c^2 - (a + c)(c + b)][(a + c)(b + c) - 2c^2 + (a + c)(c + b)] \\ &= -2c^2(ab + ac + bc) \cdot 2 \leq 0.\end{aligned}$$

(b) In case of equality

$$x^2 + y^2 + z^2 - 2x(y + z) - 2yz = 0 \tag{1}$$

then

$$x^2 - 2x(y + z) + (y - z)^2 = 0.$$

Now $\Delta_1 = (y + z)^2 - (y - z)^2 = 4yz \geq 0$.

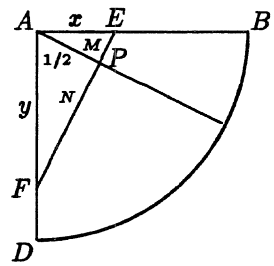
So y and z are both nonpositive or both nonnegative. By symmetry all three of x, y, z are nonpositive or nonnegative and we may assume the latter. Factoring (1) gives $(\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{y} + \sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y} - \sqrt{z}) = 0$. This holds if $\sqrt{x} + \sqrt{y} = \sqrt{z}$, or $\sqrt{x} + \sqrt{z} = \sqrt{y}$ or $\sqrt{y} + \sqrt{z} = \sqrt{x}$. Of course if we also use the condition $ax + by + cz = 0$, $a, b, c \in \mathbb{R}^+$ we see that the only solution is $x = y = z = 0$ since $x, y, z \geq 0$ or $x, y, z \leq 0$.

5. *ABCD* is a square piece of paper with sides of length 1 metre. A quarter-circle is drawn from B to D with centre A. The piece of paper is folded along EF, with E on

AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF could have.

Solution by Seung-Jin Bang, Albany, California; and by Panos E. Tsaoussoglou, Athens, Greece. We give Tsaoussoglou's solution.

Let the perpendicular from A to EF meet EF at P. Since folding along EF puts A on the circle, AP is half the radius. Let $EP = M$, $FP = N$, $AE = x$ and $AF = y$.



Then $M^2 = x^2 - \frac{1}{4}$, $N^2 = y^2 - \frac{1}{4}$ and the area of AEF is given by

$$\frac{1}{4}(\sqrt{x^2 - 1/4} + \sqrt{y^2 - 1/4}).$$

Also $x^2 + y^2 = (M+N)^2 = x^2 - 1/4 + y^2 - 1/4 + 2\sqrt{x^2 - 1/4}\sqrt{y^2 - 1/4}$ giving $\sqrt{x^2 - 1/4}\sqrt{y^2 - 1/4} = 1/4$. Applying the Arithmetic-Geometric mean inequality we get that the minimum area is $1/2\sqrt{1/4} = 1/4$. This occurs when $\sqrt{x^2 - 1/4} = \sqrt{y^2 - 1/4}$, i.e. $x = y$ and $\sqrt{x^2 - 1/4} = 1/4$ so $x = y = \sqrt{2}/2$.

For the maximum area take $x = 1$ (or $y = 1$). Then $\sqrt{3}/2 \cdot \sqrt{y^2 - 1/4} = 1/4$ and $y = \sqrt{3}/3$. In both cases $x = 1, y = \sqrt{3}/3$, $x = \sqrt{3}/3, y = 1$ the area is $\sqrt{3}/6$.

* * *

The next solutions are to problems of the *1991 British Mathematical Olympiad* [1993: 4–5].

1. Prove that the number

$$3^n + 2 \times 17^n$$

where n is a non-negative integer, is never a perfect square.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; Seung-Jin Bang, Albany, California; J. Brenner, Palo Alto, California; Pavlos Maragoudakis, Pireas, Greece; Stewart Metchette, Culver City, California; Waldemar Pompe, student, University of Warsaw, Poland; Bob Prielipp, University of Wisconsin–Oshkosh; Dale Shoults, student, The University of Calgary; D.J. Smeenk, Zaltbommel, The Netherlands; Panos E. Tsaoussoglou, Athens, Greece; and Chris Wildhagen, Rotterdam, The Netherlands. We give Pompe's solution.

If $n = 0$, we have $3^0 + 2 \times 17^0 = 3$, and it is not a perfect square. If $n \geq 1$, we have:

$$\begin{array}{rcr} 17 \equiv 1 \bmod 8 & & 3^2 \equiv 1 \bmod 8 \\ 17^2 \equiv 1 \bmod 8 & \text{and} & 3^{2k} \equiv 1 \bmod 8 \\ 2 \times 17^n \equiv 2 \bmod 8 & & 3^{2k+1} \equiv 3 \bmod 8 \end{array}.$$

Hence

$$3^n + 2 \times 17^n \equiv \begin{cases} 3 \bmod 8, & \text{if } n \text{ is even} \\ 5 \bmod 8, & \text{if } n \text{ is odd.} \end{cases}$$

But, if k is a positive integer

$$k^2 \equiv \begin{cases} 0 \bmod 8, & \text{if } k \text{ is divisible by } 4 \\ 4 \bmod 8, & \text{if } k \text{ is even but not divisible by } 4 \\ 1 \bmod 8 & \text{if } k \text{ is odd.} \end{cases}$$

Therefore $3^n + 2 \times 17^n$ cannot be a perfect square.

2. Find all positive integers k such that the polynomial $x^{2k+1} + x + 1$ is divisible by the polynomial $x^k + x + 1$. For each such k specify the integers n such that $x^n + x + 1$ is divisible by $x^k + x + 1$.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Pavlos Maragoudakis, Pireas, Greece; by Dale Shoults, student, The University of Calgary; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Bang's solution to a slightly more general problem.

We determine the ordered pairs (n, k) such that $x^n + x + 1 = (x^k + x + 1)p(x)$, where $p(x)$ is a nonzero polynomial in x.

It is clear that $n \geq k$ and $n = k$ is a trivial possibility. So now we assume $n > k$, and let ω be a root of the equation $x^k + x + 1 = 0$. Then $\omega \neq 0$ and $\omega^n + \omega + 1 = 0$ giving $\omega^n - \omega^k = \omega^k(\omega^{n-k} - 1) = 0$. This implies that $\omega^{n-k} = 1$.

Since $|\omega|^{n-k} = |\omega^{n-k}| = 1$ we have $|\omega| = 1$ and from $1 = |\omega|^k = |\omega^k| = |\omega + 1|$ we obtain that the real part of ω, $\mathrm{Re}\omega = -1/2$. Now we have $k \geq 2$ and that $\omega = (-1+\sqrt{3}i)/2$ or $\omega = (-1 - \sqrt{3}i)/2$ where $i = \sqrt{-1}$. If ω is a multiple root of $x^k + x + 1 = 0$ then $k\omega^{k-1} + 1 = 0$ and $\omega^{k-1} = -1/k$ contradicting $|\omega^{k-1}| = 1$. It follows that ω is a simple root. Since there are only two possibilities for ω, $k = 2$. Now let $n \equiv l \bmod 3$ where $0 \leq l < 3$. From $\omega^n + \omega + 1 = \omega^l + \omega + 1 = 0$ we see that $l = 2$, and $n \equiv 2 \bmod 3$.

It follows that $(n, k) = (k, k)$ or $(2m + 2, 2)$, m a positive integer.

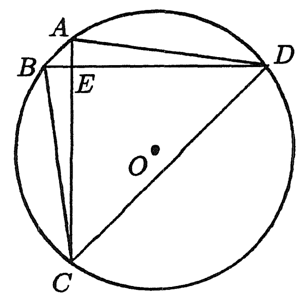
Answer (i) $k = 2$, (ii) $(n, k) = (k, k)$ or $(3m + 2, 2)$.

3. $ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E. Prove that if AC is perpendicular to BD then

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. \qquad (*)$$

Is it true that if $(*)$ holds then AC is perpendicular to BD? Give a reason for your answer.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Pavlos Maragoudakis, Pireas, Greece; by Dale Shoults, student, The University of Calgary; by D.J. Smeenk, Zaltbommel, The Netherlands; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Ardila's solution.



Suppose $AC \perp BD$. By Pythagoras

$$EA^2 + EB^2 + EC^2 + ED^2 = \frac{1}{2}(AB^2 + BC^2 + CD^2 + DA^2).$$

Let O be the circumcentre of $ABCD$. By the law of cosines

$$AD^2 = AO^2 + OD^2 - 2AO \cdot OD \cos \angle AOD = 2r^2 - 2r^2 \cos \angle AOD, \text{ etc.}$$

So we have

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2 - r^2(\cos \angle AOD + \cos \angle DOC + \cos \angle COB + \cos \angle BOA).$$

But $\angle AOD + \angle BOC = \overset{\frown}{AD} + \overset{\frown}{BC} = 2(\angle ACD + \angle BDC) = 2(90^\circ) = 180^\circ$. So $\cos \angle AOD + \cos \angle BOC = \cos \angle DOC + \cos \angle BOA = 0$ and $(*)$ follows.

AC may not be perpendicular to BD even when $(*)$ holds. For example, let $ABCD$ be a rectangle that is not a square. Then E is the centre of the circle and $EA = EB = EC = ED = r$, so $(*)$ holds, but $AC \not\perp BD$.

4. Find, with proof, the minimum value of $(x+y)(y+z)$ where x, y, z are positive real numbers satisfying the condition $xyz(x+y+z) = 1$.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Pavlos Maragoudakis, Pireas, Greece; by Waldemar Pompe, student, University of Warsaw, Poland; by Henry Ricardo, Tappan, New York; by Dale Shoults, student, The University of Calgary; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Pompe's two solutions.

Solution I. Consider a triangle ABC with sides $a = y+z$, $b = x+z$, $c = x+y$ respectively. (Such a triangle of course exists because $a + b \geq c$, etc.) Then $s = x+y+z$, $s - a = x$, $s - b = y$, and $s - c = z$. According to *Heron's formula*

$$\text{Area } = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz(x+y+z)} = 1.$$

But on the other hand

$$\text{Area } = \frac{1}{2}ac\sin B = \frac{1}{2}(x+y)(y+z)\sin B.$$

Thus

$$(x+y)(y+z) = \frac{2}{\sin B} \geq 2.$$

Equality holds, for example, when $x = z = 1$ and $y = \sqrt{2} - 1$. Therefore 2 is the desired minimum value.

Solution II. We will use the Arithmetic Mean–Geometric Mean inequality. We have

$$\frac{(x+y)(y+z)}{2} = \frac{y(x+y+z) + xz}{2} \geq \sqrt{xyz(x+y+z)} = 1.$$

Equality holds, for example, when $x = z = 1$ and $y = \sqrt{2} - 1$, giving 2 as the minimum value.

5. Find the number of permutations (arrangements) $p_1, p_2, p_3, p_4, p_5, p_6$ of 1, 2, 3, 4, 5, 6 with the property: for no integer n, $1 \leq n \leq 5$, do $p_1, p_2, \ldots, p_n$ form a permutation of $1, 2, \ldots, n$.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Dale Shoults, student, The University of Calgary; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's solution.

Let N be any integer greater than 1. For $1 \leq n \leq N$, let a_n be the number of permutations $P_1, P_2, \ldots, P_n$ of $1, 2, \ldots, n$ with the property:
For no integer k with $1 \leq k \leq n-1$ do $P_1 P_2 \ldots P_k$ form a permutation of $1, 2, \ldots, k$.

Then it is clear that the following recurrence relation holds:

$$N! = \sum_{n=1}^{N} a_n (N-n)!$$

Since $a_1 = 1$ it follows that $a_2 = 1$, $a_3 = 3$, $a_4 = 13$, $a_5 = 71$ and $a_6 = 461$.

6. Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by $2xy$ then x is a perfect square.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Stewart Metchette, Culver City, California; by Waldemar Pompe, student, University of Warsaw, Poland; by Dale Shoults, student, The University of Calgary; by Panos E. Tsaoussoglou, Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We first give Ardila's solution.

Let $x^2 + y^2 - x = k(2xy)$, so $y^2 = x(2ky + 1 - x)$. Now let $x = an^2$, where a is square free. It follows that for some integer m

$$2ky + 1 - x = am^2 \qquad \text{and} \qquad y^2 = a^2 n^2 m^2.$$

Since a is square free $a|y$. Write $y = ab$. Then $1 - x = am^2 - 2ky = am^2 - 2kab = a(m^2 - 2kb)$ and as $x = an^2$ so $1 = a(n^2 + m^2 - 2kb)$. Thus $a|1$ so $x = n^2$, as we wished to prove.

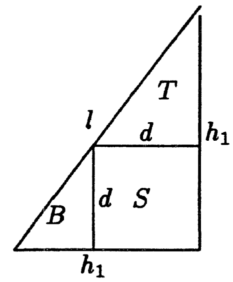
[*Editor's note. Next we give Wildhagen's generalization and solution.*]

We show that $2xy$ can be replaced by xy. Suppose that $xy|N$ where $N = x^2+y^2-x$, and $x, y \in \mathbb{N}$. Let p be any prime factor of x, and let $p^k \| x$, (i.e. $p^k | x$ and $p^{k+1} \not| x$). We want to show that k is even. Let $p^l \| y$. Clearly $l \geq 1$. Then $p^{k+l} \| xy$, hence $p^{k+l} | N$.

The obvious relations $P^k \| x^2 - x$ and $p^{k+1} | N$ imply $p^k \| y^2$ so $k = 2l$, an even number.

7. A ladder of length l rests against a vertical wall. Suppose that there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find *explicitly*, in terms of l and d, the height h from the ground that the ladder reaches up the wall.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Dale Shoults, student, The University of Calgary, Alberta; and by Chris Wildhagen, Rotterdam, The Netherlands. We use Shoults' solution.



For each possible pair of values of l and d, both h_1 and h_2 are possibilities for h. They are distinct unless the angle the ladder makes with the wall is 45°.

Since the area of the triangle B plus the area of triangle T plus the area of the square S equal the area of the large triangle

$$\frac{d(h_2-d)}{2}+\frac{d(h_1-d)}{2}+d^2=\frac{h_1h_2}{2}.$$

This simplifies to give $h_1h_2 = d(h_1+h_2)$ or $h_1+h_2 = h_1h_2/d$. (1)

Squaring 1 and rearranging gives

$$h_1^2+h_2^2=\frac{(h_1h_2)^2}{d^2}-2(h_1h_2). \qquad (2)$$

Applying Pythagoras to the large right triangle, $l^2 = h_1^2+h_2^2$. Combining with (2) gives

$$l^2=\frac{(h_1h_2)^2}{d^2}-2(h_1h_2). \qquad (3)$$

Using the quadratic formula on (3) to solve for h_1h_2 gives $h_1h_2 = d^2 + d\sqrt{d^2+l^2}$, (4) the positive sign is needed because $\sqrt{d^2+l^2} > d$ and the negative sign would make $h_1h_2 < 0$. Set $m = d+\sqrt{d^2+l^2}$. We have $h_1h_2 = dm$. (5) Combining (1) and (5) we have $h_1 + h_m$. Multiplying by h_2, substituting dm for h_1h_2 and rearranging gives $h_2^2 - mh_2 + dm = 0$.

Solving this for h_2, replacing m in terms of d and l, and simplifying gives the two possible values of h.

$$h=\frac{1}{2}\left[d+\sqrt{d^2+l^2}\pm\sqrt{l^2-2d^2-2d\sqrt{d^2+l^2}}\right].$$

* * *

Next some misprints that crept into his solutions and were spotted by Seung-Jin Bang, Albany, California.

In [1993: 6], 23rd line (solution of #5) "$f(n+2)-2^{n+3}f(n+1)=\ldots$" should read "$f(n+2)-2^{2n+3}f(n+1)=\ldots$"

In [1993: 9] 16th line (solution of #10) "the number of elements in X is m" should read "the number of elements in X is $4m$".

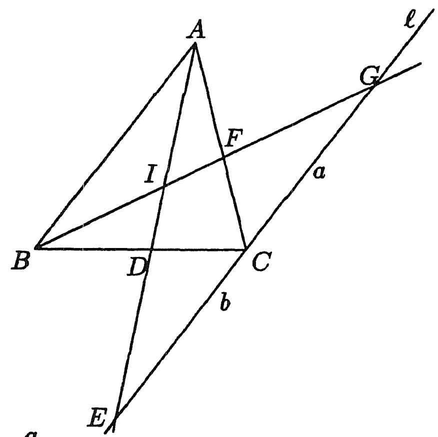
* * *

We conclude this number of the Corner with an elegant alternative solution to a problem discussed in the January 1993 number of the Corner.

8. [1991: 197; 1993: 8–9] *Proposed by Ireland.*

Let ABC be a triangle and ℓ the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line ℓ at E. Let the internal bisector of the angle at B meet the side AC at F and the line ℓ at G. If $GF = DE$ prove that $AC = BC$.

Solution by K.R.S. Sastry, Addis Ababa, Ethiopia.



If $AC \neq BC$ then suppose $AC > BC$. Then $\beta > \alpha$, $\beta/2 > \alpha/2$ and referring to the figure, we have $AI > BI$, $IE > IG$, whence $ID > IF$. Thus $AI + ID > BI + IF$, or $AD > BF$. (1)
Now ΔABD and ΔECD are similar, so

$$\frac{BD}{DC} = \frac{c}{b} = \frac{AD}{DE}.$$

Also $\Delta AFB \sim \Delta CFG$ giving $AF/FC = c/a = BF/FG$. Thus

$$\frac{AD}{DE} \times \frac{FG}{BF} = \frac{c}{b} \times \frac{a}{c} \quad \text{or} \quad \frac{AD}{BF} = \frac{a}{b} < 1.$$

This yields $AD < BF$, contradicting (1). The supposition $AC < BC$ also leads to a contradiction. We conclude $AC = BC$.

Remark. This is a variant of the famous Steiner–Lehmus (also infamous! — for the inumerable wrong proofs) theorem: If two internal angle bisectors of a triangle are equal, then the triangle is isosceles.

* * *

That concludes this number. Send me your Olympiad and pre-Olympiad contests as well as your nice solutions.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1921. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

D and E are points on sides AB and AC of a triangle ABC such that $DE \| BC$, and P is an interior point of ΔADE. PB and PC meet DE at F and G respectively. Let O_1 and O_2 be the circumcenters of ΔPDG and ΔPFE respectively. Prove that $AP \perp O_1O_2$.

1922. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

The function f is defined on nonnegative integers by: $f(0) = 0$ and

$$f(2n+1) = 2f(n) \quad \text{for } n \geq 0, \qquad f(2n) = 2f(n) + 1 \quad \text{for } n \geq 1.$$

(a) Let $g(n) = f(f(n))$. Show that $g(n - g(n)) = 0$ for all $n \geq 0$.

(b) For any $n \geq 1$, let $r(n)$ be the least integer r such that $f^r(n) = 0$ (where $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n))$, etc.). Compute

$$\liminf_{n \to \infty} \frac{n}{2^{r(n)}} \,.$$

1923. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In triangle ABC, cevians AD, BE, CF are equal and concur at point P. Prove that

$$PA + PB + PC = 2(PD + PE + PF).$$

1924. *Proposed by Jisho Kotani, Akita, Japan.*

A large sphere of radius 1 and a smaller sphere of radius $r < 1$ overlap so that their intersection is a circle of radius r, i.e., a great circle of the small sphere. Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

1925. *Proposed by Ignotus, Godella, Spain.*

Let n be a k-digit positive integer and let $v(n)$ be the set of k "right-truncations" of n: for example, $v(1994) = \{1994, 199, 19, 1\}$. Show that there are infinitely many n such that $v(n)$ is a complete set of residues mod k. (This problem was inspired by *Crux* problems 1884 and 1886 [1993: 264, 265].)

1926. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

On sides BC, CA, AB of ΔABC are chosen points A_1, B_1, C_1 respectively, such that $\Delta A_1B_1C_1$ is equilateral. Let o_1, o_2, o_3 and O_1, O_2, O_3 be respectively the incircles and incentres of triangles AC_1B_1, BA_1C_1, CB_1A_1. If $O_1C_1 = O_2C_1$, show that

(a) $B_1O_3 = B_1O_1$ and $A_1O_2 = A_1O_3$;

(b) three external common tangents to the pairs of circles o_1, o_2; o_2, o_3; o_3, o_1, different from the sides of ΔABC, have a common point.

1927. *Proposed by Rolf Kline, Edmonton, Alberta.*

Suppose that, for three consecutive years, a certain provincial government reduces what it spends annually on education. The percentage decreases year by year are a, b and c percent, where a, b, c are positive integers in arithmetic progression. Suppose also that the amounts (in dollars) the government spends on education during these same three years are three positive integers in harmonic progression. Find a, b and c.

1928. *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the tetrahedron $A_1A_2A_3A_4$, not necessarily regular, let a_i be the triangular face opposite vertex A_i ($i = 1,2,3,4$). Let Q be any point in the interior of a_1 and P a point on the segment A_1Q. For $i = 2,3,4$ let B_i be the point where the plane through P parallel to a_i meets the edge A_1A_i. Prove that

$$\frac{A_1B_2}{A_1A_2} + \frac{A_1B_3}{A_1A_3} + \frac{A_1B_4}{A_1A_4} = \frac{A_1P}{A_1Q} .$$

1929. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 6$ and

$$a_{n+1} = \left\lfloor \frac{5}{4}a_n + \frac{3}{4}\sqrt{a_n^2 - 12} \right\rfloor$$

for all $n \geq 1$, where $\lfloor\ \rfloor$ denotes the greatest integer function. Prove that $a_n \equiv 1 \bmod 10$ for all $n \geq 2$.

1930. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

T_1 is an isosceles triangle with circumcircle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3, and so on. Do the triangles T_n approach an equilateral triangle as $n \to \infty$?

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1826. [1993: 78] *Proposed by P. Penning, Delft, The Netherlands.*

(a) In a box we put two marbles, one black and one white. We choose one marble at random. If it is white, we put it back in the box, add an extra white marble to the box, shake the box, and draw again, continuing to replace the marble along with an extra white marble every time a white marble is chosen, until the black marble is chosen and the game ends. What is the average number of marbles chosen?

(b) What is the average number of marbles chosen if we add an extra white marble only after every *second* white marble that is chosen?

Solution by Marcin E. Kuczma, Warszawa, Poland.

Let the random variable X denote the number of marbles drawn, and let $E(X)$ denote the expected value of X. Then in case (a),

$$Pr(X=n)=\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{n-1}{n}\frac{1}{n+1}=\frac{1}{n(n+1)},$$

while in case (b),

$$Pr(X=2k-1)=\frac{1}{2}\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{3}{4}\frac{3}{4}\cdots\frac{k-1}{k}\frac{k-1}{k}\frac{1}{k+1}=\frac{1}{k^2(k+1)}$$

and

$$Pr(X=2k)=\frac{1}{2}\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{3}{4}\frac{3}{4}\cdots\frac{k-1}{k}\frac{k-1}{k}\frac{k}{k+1}\frac{1}{k+1}=\frac{1}{k(k+1)^2}.$$

Hence in case (a),

$$E(X)=\sum_{n=1}^{\infty}nPr(X=n)=\sum_{n=1}^{\infty}\frac{1}{n+1}=\infty,$$

while in case (b),

$$\begin{aligned}
E(X)&=\sum_{k=1}^{\infty}(2k-1)Pr(X=2k-1)+\sum_{k=1}^{\infty}2kPr(X=2k)\\
&=\sum_{k=1}^{\infty}\frac{2k-1}{k^2(k+1)}+\sum_{k=1}^{\infty}\frac{2}{(k+1)^2}=\sum_{k=1}^{\infty}\left(\frac{3}{k(k+1)}-\frac{1}{k^2}\right)+\sum_{k=1}^{\infty}\frac{2}{(k+1)^2}\\
&=\sum_{k=1}^{\infty}\left(\frac{3}{k}-\frac{3}{k+1}\right)-\sum_{k=1}^{\infty}\frac{1}{k^2}+2\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\\
&=3-\frac{\pi^2}{6}+2\left(\frac{\pi^2}{6}-1\right)=\mathbf{1}+\frac{\boldsymbol{\pi}^2}{\mathbf{6}},
\end{aligned}$$

using the known fact that $\sum_{k=1}^{\infty}(1/k^2)=\pi^2/6$. [*Editor's note*: for example, see the recent article "$6/\pi^2$" by Gareth Jones, in the December 1993 *Mathematics Magazine*, pages 290–298, especially §5.]

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington (with only an approximate value given for part (b)*); HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; BEATRIZ MARGOLIS, Paris, France; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposer.*

Both Hess and Janous proved the more general result that if an extra white marble is added after every kth white marble is drawn, then $E(X)=1+\zeta(k)$ *where* $\zeta(z)=\sum_{n=1}^{\infty}(1/n^z)$ *is the Riemann Zeta function.*

* * * * *

1828. [1993: 78] *Proposed by T. W. O. Richards, Cheddar Gorge, Great Britain.*

In the last century, the English mathematician Arthur Cayley introduced a permutation problem, loosely based on the card game Treize, which he called **Mousetrap**. Suppose that the numbers $1, 2, \ldots, n$ are written on n cards, one on each card. After shuffling (permuting) the cards, start counting the deck from the top card down. If the number on the card does not equal the count, then put that card at the bottom of the deck and continue counting. If the two are equal then put the card aside and start counting again from 1.

Let's say the game is **won** if all the cards have been put aside. In this case, form a new deck with the cards in the order in which they were set aside and play a new game with this deck. For example, if we start with $n = 5$ cards in the order 25143, we win:

251**4**3 → 3251 → 3**2**51 → 513 → 51**3** → 51 → **5**1 → **1**

and the new deck is 42351, which wins again:

4**2**351 → 351**4** → **351** → **13** → **3**

but now our deck, 24513, puts aside no cards at all. Is there an arrangement (using more cards, if necessary) which will give you three or more consecutive wins?

Solution by Richard I. Hess, Rancho Palos Verdes, California.

With one or two cards there is the repetitive arrangement:

$$1 \to 1 \to 1 \ldots ; \quad 12 \to 12 \to 12 \ldots .$$

With six cards, there is just one arrangement:

$$165342 \to 132564 \to 125346 \to 136524.$$

With eight cards, again there is just one arrangement:

$$52173846 \to 21463578 \to 72135648 \to 21435867.$$

With nine cards, there are 8 arrangements:

157362948; 157392486; 231765489; 469523718;

523149768; 543987261; 684523719; 756482319.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; P. PENNING, Delft, The Netherlands; A.N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; and the proposers. There was one incorrect solution sent in, probably due to misunderstanding the problem.

No theoretical approach was found either by the solvers or by the proposers. All the solutions were done by trial and error or by computer. 't Woord poses the question: is there some n such that there are no permutations (on at least 6 elements) that give n successive wins? Penning found the following frequencies for the number of arrangements that gave $0, 1, 2$ or 3 wins for $n = 3, 4, 5, 6, 7, 8$:

n	0 *wins*	1 *win*	2 *wins*	3 *wins*	*total*
3	4	2			6
4	18	4	2		24
5	105	14	1		120
6	636	72	11	1	720
7	4710	316	14		5040
8	38508	1730	81	1	40320

* * * * *

1831*. [1993: 112] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be any real numbers and let λ be an *odd* positive integer. Prove or disprove that

$$x(x+y)^\lambda + y(y+z)^\lambda + z(z+x)^\lambda \geq 0.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

The inequality is trivial for $\lambda = 1$, easy for $\lambda = 3$, difficult for $\lambda = 5$, false for $\lambda \geq 7$.

Counterexample for $\lambda \geq 7$:

$$x = -2, \quad y = -1, \quad z = 5.$$

The expression takes value $7 \cdot 3^\lambda - 4^\lambda$, which is negative for $\lambda \geq 7$.

Now the proof for $\lambda = 3$ and $\lambda = 5$.

The nonsingular linear substitution $u = y + z$, $v = z + x$, $w = x + y$ transforms the claimed inequality into

$$(v + w - u)w^\lambda + (w + u - v)u^\lambda + (u + v - w)v^\lambda \geq 0. \tag{1}$$

Since the expression in (1) is cyclic and invariant under $(u, v, w) \mapsto (-u, -v, -w)$, there is no loss of generality in assuming $u \geq 0$, $v \geq 0$.

Case $w \geq 0$. Let us regroup the terms in (1) as follows:

$$(u^{\lambda+1} + uv^\lambda - u^\lambda v) + (v^{\lambda+1} + vw^\lambda - v^\lambda w) + (w^{\lambda+1} + wu^\lambda - w^\lambda u).$$

Either we have $u^\lambda v \leq u^{\lambda+1}$ or $u^\lambda v \leq uv^\lambda$ (according as $u \geq v$ or $u \leq v$). Hence the first expression in parentheses is nonnegative. So are the other two, by cyclicity.

[Incidentally, we have shown that the proposed inequality is valid for *every* exponent $\lambda \geq 1$ when all three sums $u = y + z$, $v = z + x$, $w = x + y$ are nonnegative (or nonpositive, with λ an odd integer). Value 0 is attained when all three expressions in parentheses are zero; and this is easily seen to be the case only for $u = v = w = 0$, i.e., $x = y = z = 0$.]

Case $w < 0$. We will show that *strict* inequality holds in (1) (for $\lambda = 3$ and $\lambda = 5$). By homogeneity, it is enough to consider $w = -1$. Denote the resulting polynomial by $F(u, v)$ resp. $G(u, v)$ for $\lambda = 5$ resp. $\lambda = 3$:

$$F(u,v) = v^6 + (u+1)v^5 - (u^5+1)v + (u^6 - u^5 + u + 1),$$
$$G(u,v) = u^4 - (v+1)u^3 + (v^3+1)u + (v^4 + v^3 - v + 1)$$

(the specific grouping of terms in either case has its aim).

We first consider $G(u,v)$. Since $\partial^2 G/\partial u^2 = 6u(2u-v-1)$, we see that, for any fixed $v \geq 0$, the first order derivative $\partial G/\partial u$ is a decreasing function of $u \in [0,(v+1)/2]$ and an increasing function of $u \in [(v+1)/2,\infty)$. Thus $\partial G/\partial u$ (regarded as a function of u, for v fixed) assumes at $u=(v+1)/2$ its (strict) minimum value

$$\frac{\partial G}{\partial u}\Big(\frac{v+1}{2}, v\Big) = \frac{3}{4}(v-1)^2(v+1) \geq 0.$$

Consequently $\partial G/\partial u \geq 0$ for $u \in [0,\infty)$ and hence $G(u,v)$ is increasing in u. So

$$G(u,v) \geq G(0,v) = v^4+v^3-v+1 \begin{cases} > v^3 - v \geq 0 & \text{for } v \geq 1, \\ \geq -v+1 > 0 & \text{for } 0 \leq v < 1. \end{cases}$$

Thus $G(u,v) > 0$ for $u,v \geq 0$ is proved.

To deal with $F(u,v)$, we will need a lemma.

Lemma. Let $\varphi(t)$ be a differentiable convex function in an interval J and let $\alpha, \beta \in J$ be such that $\alpha < \beta$,

$$\varphi'(\alpha) < 0 < \varphi'(\beta), \qquad \varphi(\alpha) + (\beta-\alpha)\varphi'(\alpha) > 0. \tag{2}$$

Then $\varphi(t) > 0$ in J.

Proof of the lemma. From the first condition of (2) we infer that the global minimum of φ over J occurs at some point between α and β. The second condition of (2) says that the tangent to the graph of φ at α is above the t-axis at β, hence also at every point of $[\alpha,\beta]$ (the slope $\varphi'(\alpha)$ being negative). Every point of the graph of φ lies above or on that tangent; the lemma results. $\square$

Now fix $u \geq 0$ arbitrarily and note that $F(u,v)$ is a convex function of variable $v \in J = [0,\infty)$. Thus, to prove $F(u,v) > 0$, it will be enough to find α and β (depending on u) so as to satisfy the conditions (2) of the lemma (for $\varphi(v) = F(u,v)$). Our choice will be the following:

$$\text{if } 0 \leq u \leq 9/8, \text{ take } \alpha = 1/2, \ \beta = 1; \qquad \text{if } u > 9/8, \text{ take } \alpha = u/2, \ \beta = 3u/5.$$

So we must verify

$$\frac{\partial F}{\partial v}\Big(u,\frac{1}{2}\Big) < 0 < \frac{\partial F}{\partial v}\big(u,1\big) \quad \text{and} \quad F\Big(u,\frac{1}{2}\Big) + \frac{1}{2}\cdot\frac{\partial F}{\partial v}\Big(u,\frac{1}{2}\Big) > 0 \quad \text{for } 0 \leq u \leq \frac{9}{8},$$

$$\frac{\partial F}{\partial v}\Big(u,\frac{u}{2}\Big) < 0 < \frac{\partial F}{\partial v}\Big(u,\frac{3u}{5}\Big) \quad \text{and} \quad F\Big(u,\frac{u}{2}\Big) + \frac{u}{10}\cdot\frac{\partial F}{\partial v}\Big(u,\frac{u}{2}\Big) > 0 \quad \text{for } u > \frac{9}{8}.$$

Standard calculation brings these six inequalities to the polynomial form:

$$-16u^5+5u-8 < 0 < -u^5+5u+10, \quad 64u^6-128u^5+76u+19 > 0 \quad \text{for } 0 \leq u \leq \frac{9}{8}, \tag{3}$$

$$-8u^5+5u^4-16 < 0 < 358u^5+2025u^4-3125, \quad 159u^6-300u^5+128u+320 > 0 \quad \text{for } u > \frac{9}{8}. \tag{4}$$

In (3), the second and the third inequalities are true at $u = 0$ and at $u = 9/8$; and since the functions involved are concave in $[0, 9/8]$, these two inequalities are satisfied in that interval.

The first inequality in (3) and the first two inequalities in (4) hold trivially.

To prove the third inequality of (4), we again use the lemma. Denoting the polynomial $159u^6 - 300u^5 + 128u + 320$ by $\varphi(u)$ we compute $\varphi''(u) = 30u^3(159u - 200)$; so $\varphi(u)$ is convex in the interval $[200/159, \infty)$. Choosing $\alpha = 3/2$ and $\beta = 8/5$ we can easily verify conditions (2). Therefore $\varphi(u) > 0$ for $u \geq 200/159$; in particular, $\varphi(200/159) > 0$. And since $\varphi(0) > 0$ and φ is concave in $[0, 200/159]$, we conclude $\varphi(u) > 0$ for all $u \geq 0$. This seems to complete the proof of $F(u, v) > 0$ for $u, v \geq 0$.

Thus, for $\lambda = 3$ and $\lambda = 5$ the proposed inequality is true for every triple of real numbers x, y, z, with equality only for $x = y = z = 0$.

Counterexamples for $\lambda \geq 7$ were also found by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD I. HESS, Rancho Palos Verdes, California; and A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands. There was one incorrect solution sent in.

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1832. [1993: 112] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

An old unsolved problem is: "is it possible that a box can have its sides, face diagonals, and space diagonal all of integer lengths, i.e., are there positive integers a, b, c such that

$$a^2 + b^2, \quad b^2 + c^2, \quad c^2 + a^2, \quad \text{and} \quad a^2 + b^2 + c^2$$

are all perfect squares?" What if we replace the squares by triangular numbers? For n a positive integer, let $t_n = n(n+1)/2$ be the nth triangular number.

(a) Find positive integers a, b, c such that

$$t_a + t_b, \quad t_b + t_c, \quad t_c + t_a$$

are all triangular numbers.

(b)* Is there such a solution so that $t_a + t_b + t_c$ is also a triangular number?

Solution to both parts by Richard I. Hess, Rancho Palos Verdes, California.

A small computer search found the cases in the table below, where $t_a + t_b$, $t_a + t_c$, $t_b + t_c$ and $t_a + t_b + t_c$ are all triangular.

a	b	c	$t_a + t_b$	$t_a + t_c$	$t_b + t_c$	$t_a + t_b + t_c$
11	14	14	$171 = t_{18}$	$171 = t_{18}$	$210 = t_{20}$	$276 = t_{23}$
230	741	870	$301476 = t_{776}$	$405450 = t_{900}$	$653796 = t_{1143}$	$680361 = t_{1166}$
609	779	923	$489555 = t_{989}$	$612171 = t_{1106}$	$730236 = t_{1208}$	$915981 = t_{1353}$
714	798	989	$574056 = t_{1071}$	$744810 = t_{1220}$	$808356 = t_{1271}$	$1063611 = t_{1458}$
1224	1716	3219	$2222886 = t_{2108}$	$5932290 = t_{3444}$	$6655776 = t_{3648}$	$7405476 = t_{3848}$

Both parts also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; and R.P. SEALY, Mount Allison University, Sackville, New Brunswick. Part (a) *solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution was received.*

Can anyone find an infinite family of solutions?

*   *   *   *   *

1833. [1993: 112] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

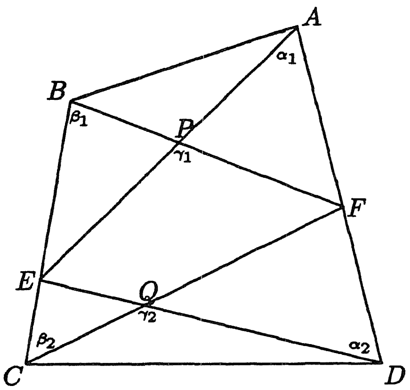
E and F are points on sides BC and AD, respectively, of a quadrilateral $ABCD$. Let $P = AE \cap BF$ and $Q = CF \cap DE$. Prove that E and F divide BC and AD (or BC and DA) in the same ratio if and only if

$$\frac{[FPQ]}{[EDA]} = \frac{[EQP]}{[FBC]},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.

Let $\alpha_1 = \angle EAD$, $\alpha_2 = \angle EDA$, $\beta_1 = \angle FBC$, $\beta_2 = \angle FCB$, $\gamma_1 = \angle EPF$, $\gamma_2 = \angle EQF$ as in the figure. Applying the law of sines we obtain



$$\frac{FP}{\sin\alpha_1} = \frac{AF}{\sin\gamma_1}, \qquad \frac{FQ}{\sin\alpha_2} = \frac{FD}{\sin\gamma_2},$$
$$\frac{EP}{\sin\beta_1} = \frac{EB}{\sin\gamma_1}, \qquad \frac{EQ}{\sin\beta_2} = \frac{EC}{\sin\gamma_2}. \tag{1}$$

Letting $F = \angle PFQ$ and $E = \angle PEQ$,

$$\frac{[FPQ]}{[EDA]} = \frac{[EQP]}{[FBC]} \iff \frac{FP\cdot FQ\sin F}{ED\cdot EA\sin E} = \frac{EP\cdot EQ\sin E}{FB\cdot FC\sin F}. \tag{2}$$

Using (1) together with

$$\frac{ED}{\sin\alpha_1} = \frac{AD}{\sin E} = \frac{EA}{\sin\alpha_2}, \quad \frac{FB}{\sin\beta_2} = \frac{BC}{\sin F} = \frac{FC}{\sin\beta_1}$$

we obtain that (2) is equivalent to

$$\frac{\dfrac{AF\sin\alpha_1}{\sin\gamma_1}\cdot\dfrac{FD\sin\alpha_2}{\sin\gamma_2}\cdot\sin F}{\dfrac{AD\sin\alpha_1}{\sin E}\cdot\dfrac{AD\sin\alpha_2}{\sin E}\cdot\sin E} = \frac{\dfrac{EB\sin\beta_1}{\sin\gamma_1}\cdot\dfrac{EC\sin\beta_2}{\sin\gamma_2}\cdot\sin E}{\dfrac{BC\sin\beta_2}{\sin F}\cdot\dfrac{BC\sin\beta_1}{\sin F}\cdot\sin F}.$$

Letting $AF/AD = \lambda$ and $BE/BC = \mu$ and cancelling the sines we see that this equality is equivalent to $\lambda(1-\lambda) = \mu(1-\mu)$, or

$$(\mu - \lambda)(\mu + \lambda - 1) = \mu^2 - \lambda^2 - \mu + \lambda = 0,$$

and this occurs if and only if $\mu = \lambda$ or $\mu = 1 - \lambda$, i.e.,

$$\frac{AF}{AD} = \frac{BE}{BC} \qquad \text{or} \qquad \frac{AF}{AD} = \frac{CE}{BC},$$

which we may interpret as E and F dividing BC and AD respectively in the same ratio, as we wished to prove.

Also solved by MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

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1834. [1993: 113] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Given positive numbers A, G and H, show that they are respectively the arithmetic, geometric and harmonic means of some three positive numbers x, y, z if and only if

$$\frac{A^3}{G^3} + \frac{G^3}{H^3} + 1 \le \frac{3}{4}\left(1 + \frac{A}{H}\right)^2.$$

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

If

$$x + y + z = 3A, \qquad xyz = G^3 \quad , \text{ and } \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{H}, \tag{1}$$

then

$$xy + yz + zx = \frac{3G^3}{H},$$

so (1) holds if and only if x, y, z are the solutions of

$$v^3 - 3Av^2 + \frac{3G^3}{H}v - G^3 = 0. \tag{2}$$

There are only positive solutions of (2), because $A, G, H > 0$. Rewrite (2) as

$$(v-A)^3 + \left(\frac{3G^3}{H} - 3A^2\right)(v - A) + A\left(\frac{3G^3}{H} - 3A^2\right) + A^3 - G^3 = 0;$$

then with the substitution $v = u + A$ one gets

$$u^3 - \left(3A^2 - \frac{3G^3}{H}\right)u + \left(-2A^3 + \frac{3AG^3}{H} - G^3\right) = 0.$$

All solutions of the equation $u^3 - pu + q = 0$ are real numbers if and only if the condition $q^2/4 \le p^3/27$ holds, so it follows that (1) is equivalent to

$$\left(-A^3 + \frac{3AG^3}{2H} - \frac{G^3}{2}\right)^2 - \left(A^2 - \frac{G^3}{H}\right)^3 \le 0$$

which (on multiplying out) is equivalent to

$$\frac{A^3}{G^3}+\frac{G^3}{H^3}+1\le\frac{3}{4}\left(1+\frac{A}{H}\right)^2. \tag{3}$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; FRANCISCO LUIS ROCHA PIMENTEL, Fortaleza, Brazil; and the proposer.

Klamkin points out problem 4248 *of* School Science and Mathematics, *solution on pages* 335–336 *of the November* 1991 *issue. This problem is to show that*

$$\left(1-\frac{H}{A}\right)^2\le 4\left(1-\frac{G^3}{A^2H}\right)\left(1-\frac{AH^2}{G^3}\right),$$

which is equivalent to (3).

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1835. [1993: 113] *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn-1}{n-1}$$

for $n=1,2,3,\ldots$.

Solution by Richard McIntosh, University of Regina.

For any function $f(x)$, define the difference operator Δ by $\Delta f(x)=f(x+1)-f(x)$. Then it can be found in every book on finite differences that

$$\Delta^n f(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+k),\quad n=0,1,2,\cdots$$

[here as usual $\Delta^n f(x)=\Delta(\Delta^{n-1}f(x))$ is defined recursively. —*Ed.*]. If $f(x)$ is a polynomial of degree $n-1$, then

$$0=\Delta^n f(0)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k).$$

In particular, for

$$f(k)=\binom{nk-1}{n-1}=\frac{(nk-1)(nk-2)\cdots(nk-n+1)}{(n-1)!}$$

we get

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{nk-1}{n-1}=0.$$

Therefore

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\binom{nk-1}{n-1} = -(-1)^n\binom{-1}{n-1} = -(-1)^n\frac{(-1)(-2)\cdots(-n+1)}{(n-1)!} = \mathbf{1}.$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One other reader sent in the correct solution without proof.

Both Ardila and Bellot point out that the result in the proposed problem is a special case of more general identities due to H. W. Gould. These identities can be found in the American Mathematical Monthly **85** (1978), 450–465 *and* **63** (1956), 84–91, *respectively. 't Woord's and Wildhagen's solutions are essentially the same as Solution* I *above. The proposer's solution is a combinatorial argument using inclusion–exclusion.*

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1836. [1993: 113] *Proposed by Jisho Kotani, Akita, Japan.*

Let $ABCD$ be a quadrilateral inscribed in a circle Γ, and let $AC \cap BD = P$. Assume that the center of Γ does not lie on AC or BD. Draw circles with diameters AB, BC, CD, DA, and let the areas of the moon-shaped regions inside these circles and outside Γ be F_1, F_2, F_3, F_4. M_1, M_2, M_3, M_4 are the midpoints of the sides of $ABCD$, and H_1, H_2, H_3, H_4 are the feet of the perpendiculars from P to the sides of $ABCD$. Prove that, if $F_1 + F_2 + F_3 + F_4 = \text{area}(ABCD)$, then M_1, M_2, M_3, M_4, H_1, H_2, H_3, H_4 are concyclic.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

Lemma. *Let K and L be two points in the plane. The locus of points P such that*

$$(KP)^2 + (LP)^2 = \textit{ constant}$$

is a circle with centre M, the midpoint of KL.

The lemma follows from the theorem of Pythagoras. [See, for example, Locus 10 on page 14 of Nathan Altshiller Court, *College Geometry*, or Theorem 97 on page 68 of R.A. Johnson, *Advanced Euclidean Geometry. —Ed.*]

Let a, b, c, d be the sides of the given quadrilateral (with $a = DA$, etc.), and let R and O be the radius and centre of Γ. We have

$$F_1 + F_2 + F_3 + F_4 + \pi R^2 = \frac{\pi}{2}\left(\frac{a}{2}\right)^2 + \frac{\pi}{2}\left(\frac{b}{2}\right)^2 + \frac{\pi}{2}\left(\frac{c}{2}\right)^2 + \frac{\pi}{2}\left(\frac{d}{2}\right)^2 + \text{ area}(ABCD)$$

and from the given condition we get

$$a^2 + b^2 + c^2 + d^2 = 8R^2.$$

We shall prove that the diagonals of $ABCD$ are perpendicular. Let the segment CC' be the diameter of Γ; because of the assumption that the diagonals do not pass through O, $C' \neq A$. Since CDC' and $C'BC$ are both right angles, we get

$$(CD)^2 + (C'D)^2 + (BC)^2 + (C'B)^2 = 8R^2 = a^2 + b^2 + c^2 + d^2,$$

or $(C'D)^2 + (C'B)^2 = a^2 + b^2$. According to the lemma, points A and C' lie on a circle centred at the midpoint M of BD (where $M \neq 0$ by assumption). This implies (by symmetry in OM) that $a = DA = BC'$. Therefore $\angle DCA = \angle C'CB$. Since $\angle CDB = \angle CC'B$ we get that the triangles DPC and $C'BC$ are similar, which implies (since $\angle CBC' = 90°$) that AC and BD are perpendicular.

Now our problem is to prove that if diagonals AC and BD are perpendicular, then M_1, M_2, M_3, M_4, H_1, H_2, H_3, H_4 are concyclic. Obviously $M_1M_2M_3M_4$ is a rectangle and has a circumcircle with diameters M_1M_3 and M_2M_4, so

$$\angle H_1PB = \angle PAB = \angle BDC = \angle M_3PD,$$

showing that H_1, P, M_3 lie on a line. [This is the theorem of Brahmagupta, e.g., see Theorem 276 on page 137 of Court's *College Geometry*, or Theorem 3.23 on page 59 of Coxeter & Greitzer, *Geometry Revisited.* See also the similar result pointed out by Jordi Dou on [1991: 53]!—*Ed.*] Therefore $\angle M_3H_1M_1 = 90°$ and hence H_1 lies on the circumcircle of $M_1M_2M_3M_4$. The same argument shows that H_2, H_3, H_4 lie on that circle too, which was to be shown.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The proposer adds that the argument can be reversed: If M_1, M_2, M_3, M_4 are concyclic then $F_1 + F_2 + F_3 + F_4 = \text{area}(ABCD)$.

Dou points out that the second part of the proof (the final paragraph above, starting with $AC \perp BD$) provides the solution to Crux 1866 [1993: 203]. *Bellot notes the relationship of our problem to number* 3 *from the* 1991 *British Math. Olympiad* [1993: 5]. *He gives two references that investigate further properties of "orthodiagonal quadrilaterals":*

[1] *Jordan Tabov, Simple properties of the orthodiagonal quadrilaterals,* Mathematics and Informatics, **1**:1 (*February* 1991), 1–5.

[2] *Agnis Andžāns, On the inscribed orthodiagonal quadrilaterals*, Mathematics and Informatics, **3**:1 (1993), 6–8.

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1837. [1993: 113] *Proposed by Andy Liu, University of Alberta.*

A function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be *strictly log-convex* if

$$f(x_1)f(x_2) \geq \left(f\left(\frac{x_1+x_2}{2}\right)\right)^2$$

for all x_1, $x_2 \in \mathbb{R}$, with equality if and only if $x_1 = x_2$. f is said to be *strictly log-concave* if the inequality is reversed.

(a) Prove that if f and g are strictly log-convex functions, then so is $f+g$.

(b)* Does the same conclusion hold for strictly log-concave functions?

Combination of solutions of Marcin E. Kuczma, Warszawa, Poland; and A.N. 't Woord, Eindhoven University of Technology, Eindhoven, The Netherlands.

(a) If f and g are strictly log-convex, then for any real numbers x_1, x_2,

$$\begin{aligned}
\left[(f+g)\left(\frac{x_1+x_2}{2}\right)\right]^2 &= \left[f\left(\frac{x_1+x_2}{2}\right)+g\left(\frac{x_1+x_2}{2}\right)\right]^2 \\
&= \left[f\left(\frac{x_1+x_2}{2}\right)\right]^2 + 2f\left(\frac{x_1+x_2}{2}\right)g\left(\frac{x_1+x_2}{2}\right) + \left[g\left(\frac{x_1+x_2}{2}\right)\right]^2 \\
&\leq f(x_1)f(x_2) + 2\sqrt{f(x_1)f(x_2)g(x_1)g(x_2)} + g(x_1)g(x_2) \\
&\leq f(x_1)f(x_2) + f(x_1)g(x_2) + f(x_2)g(x_1) + g(x_1)g(x_2) \\
&= [f(x_1)+g(x_1)][f(x_2)+g(x_2)] \\
&= (f+g)(x_1)\cdot(f+g)(x_2),
\end{aligned}$$

where the second inequality follows by the A.M.–G.M. inequality. Equality holds if and only if $x_1 = x_2$.

(b) The answer is NO. For example,

$$f(x) = e^{-x^2+2x} \quad \text{and} \quad g(x) = e^{-x^2-2x} \quad (= f(-x))$$

are strictly log-concave, because $\ln f(x)$ and $\ln g(x)$ are strictly concave. Yet

$$(f+g)(1)\cdot(f+g)(-1) = (f(1)+f(-1))^2 = (e+e^{-3})^2 > 4 = [(f+g)(0)]^2,$$

showing that $f+g$ is not log-concave.

Part (a) *also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; WALDEMAR POMPE, student, University of Warsaw, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

Only Kuczma and 't Woord solved part (b), *and their counterexamples were virtually the same.*

Klamkin notes that the result of part (a) *is known; see for example Theorem F, pages* 18–19 *of Roberts and Varberg,* Convex Functions, *Academic Press,* 1973.

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1838. [1993: 113] *Proposed by Stoyan Kapralov and Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Find all sequences $a_1 \le a_2 \le \cdots \le a_n$ of positive integers such that

$$a_1 + a_2 + \cdots + a_n = 26, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 62, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = 164.$$

Solution by Leonardo P. Pastor, Córdoba, Argentina.

Let $S = (a_i)$, $1 \le i \le n$, be a sequence of positive integers that satisfies the three equations. Note that for each a_i in the sequence, we have $1 \le a_i \le 5$. Otherwise the left hand side of the third equation would be greater than 164.

Let x, y, z, w and t be the non-negative integers representing the number of 1's, 2's, 3's, 4's and 5's in the sequence, respectively. Therefore, we should have

$$x + 2y + 3z + 4w + 5t = 26,$$

$$x + 4y + 9z + 16w + 25t = 62,$$

$$x + 8y + 27z + 64w + 125t = 164.$$

From this it is clear that the number of 5's, i.e. t, is 0 or 1 and that if $t = 1$ then $w = 0$. Also, we have that $w \le 2$. Solving this system in terms of w and t we find

$$x = 5 - 4w - 15t,$$

$$y = 3 + 6w + 20t,$$

$$z = 5 - 4w - 10t.$$

Thus the only solutions are $(t, w, x, y, z) = (0,0,5,3,5)$, $(t, w, x, y, z) = (0,1,1,9,1)$, $(t, w, x, y, z) = (0,2,-3,15,-3)$ and $(t, w, x, y, z) = (1,0,-10,23,-5)$. Clearly the last two do not solve the problem. Therefore the only sequences which satisfy the conditions of the problem are

$$(1,1,1,1,1,2,2,2,3,3,3,3,3) \quad \text{and} \quad (1,2,2,2,2,2,2,2,2,2,3,4).$$

Also solved by H.L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; MARGHERITA BARILE, student, Universität Essen, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, Wolverley High School, Kidderminster, U.K.; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; F. J. FLANIGAN, San Jose State University, San Jose, California; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; GOTTFRIED PERZ, *Pestalozzigymnasium, Graz, Austria; FRANCISCO LUIZ ROCHA PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposers. Two other readers found only one of the two solutions.*

*   *   *   *   *

1839. [1993: 113] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Notice that

$$122 = 11^2 + 1 = 12^2 - 22,$$

i.e., the (base 10) integer $N = 122$ can be partitioned into two parts (1 and 22), so that the first part is the difference between N and the greatest square less than N, and the second part is the difference between N and the least square greater than N. Find another positive integer with this property.

I. *Solution by Sam Baethge, Science Academy, Austin, Texas.*

Partition the number into two parts "a" and "b" and express the number as $a(10^k) + b$. Then

$$n^2 + a = a(10^k) + b \quad \text{and} \quad a(10^k) + b = (n+1)^2 - b.$$

If we eliminate b from this system we have

$$a(10^k) + 2(n^2 + a - a(10^k)) = (n+1)^2$$

or

$$(n-1)^2 = a(10^k - 2) + 2,$$

which has solutions for $a = 1$ and k any positive even integer. The given example is for $k = 2$ and $n = 11$. Others are:

$$\begin{array}{lll} k=4,\ n=101 & \text{and} & 10202 = 101^2 + 1 = 102^2 - 0202, \\ k=6,\ n=1001 & \text{and} & 1002002 = 1001^2 + 1 = 1002^2 - 002002, \\ k=8,\ n=10001 & \text{and} & 100020002 = 10001^2 + 1 = 10002^2 - 00020002, \end{array}$$

etc.

II. *Solution by P. Penning, Delft, The Netherlands.*

Write $N = 100a + b$ with $0 < b < 100$. Then

$$100a + b = x^2 + a = (x+1)^2 - b,$$

and so

$$2x + 1 = a + b \quad \text{and} \quad x^2 = 99a + b. \tag{1}$$

Introduce

$$2m = x - 1 \quad \text{and} \quad a = 2c + 1$$

[for note from (1) that

$$98a = x^2 - 2x - 1 = (x-1)^2 - 2, \tag{2}$$

from which it is easy to see that x and a must both be odd.—*Ed.*]. From (2) we get $49(2c+1) = 2m^2 - 1$ or

$$49c = m^2 - 25 = (m+5)(m-5),$$

which implies [since 49 and 10 are relatively prime]

$$m = 49v \pm 5 \tag{3}$$

where v is an integer. There are only two solutions within the constraint $0 < b < 100$, namely:

$$v = 0, \quad m = 5, \quad c = 0, \quad a = 1, \quad x = 11, \quad b = 22$$

which leads to the given example 122; and

$$v = 2, \quad m = 93, \quad c = 176, \quad a = 353, \quad x = 187, \quad b = 22$$

which leads to the example

$$35322 = 187^2 + 353 = 188^2 - 22. \tag{4}$$

[*Editor's note.* For instance, eliminating a in (1) yields

$$x^2 - 198x - 99 = -98b,$$

so $0 < b < 100$ means that

$$-9800 < x^2 - 198x - 99 < 0,$$

or

$$100 < (x-99)^2 < 9900,$$

or

$$10 < |x - 99| < 100,$$

so $0 < x < 89$ or $109 < x < 199$. Now (3) and $x = 2m + 1$ give

$$x = 98v + 1 \pm 10,$$

and the only integer solutions are $v = 0$, $x = 11$, and $v = 2$, $x = 187$, as claimed.]

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHARLES ASHBACHER, Cedar Rapids, Iowa; CHRISTOPHER BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain;

TIM CROSS, Wolverley High School, Kidderminster, U.K.; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J.A. MCCALLUM, Medicine Hat, Alberta; WALDEMAR POMPE, student, University of Warsaw, Poland; A.N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; and the proposer.

Engelhaupt, Guy, Hess and the proposer found both solutions I and II. The other solvers were nearly evenly divided between those with Solution I (usually finding the infinite family) and those with Solution II.

Guy and Hess give other solutions too, e.g.

$$180125042 = 13421^2 + 1801 = 13422^2 - 25042$$

and

$$395930202 = 19897^2 + 39593 = 19898^2 - 0202.$$

This last solution with (4) *hints at another infinite family(?). Guy's solution was in fact a complete analysis of the problem, finding "all" solutions in some sense.*

* * * * *

1840. [1993: 114] *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Let ΔABC be an acute triangle with area F and circumcenter O. The distances from O to BC, CA, AB are denoted d_a, d_b, d_c respectively. $\Delta A_1B_1C_1$ (with sides a_1, b_1, c_1) is inscribed in ΔABC, with $A_1 \in BC$ etc. Prove that

$$d_a a_1 + d_b b_1 + d_c c_1 \geq F.$$

Comment by Murray S. Klamkin, University of Alberta.

Since $d_a = R\cos A$, etc., where R is the circumradius, the inequality can be rewritten as

$$a_1\cos A + b_1\cos B + c_1\cos C \geq \frac{F}{R} = \frac{a\cos A + b\cos B + c\cos C}{2},$$

and in this form it is the same inequality as proposed by G. Tsintsifas in Problem E2968 in the *American Mathematical Monthly*, solution by O.P. Lossers on pages 361–362 of the May 1985 issue.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

The proposer's proof is similar to Lossers' in the Monthly, *and the solutions of Ardila and Kuczma appear longer.*

* * * * *

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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