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The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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The Shakespeare of Mathematics

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'Mathematics is, thanks to him, like a vast and beautiful palace of which he has renewed the foundations, adorned the heights, and in which one cannot take a step without finding some monument of his genius to admire.' Thus spoke Jean Baptiste Delambre, secretary of the French Academy of Sciences, following the death of the world's leading mathematician, Joseph Louis Lagrange, in 1813. Many of the monuments in the palace described by Delambre still bear their creator's name: Lagrangian, Lagrange's equations Lagrange multipliers, Lagrange interpolation, Lagrange remainder, Lagrange's theorem in group theory; some do not: the calculus of variations, the variation of parameters; a few are misleadingly attributed: Wilson's theorem. To commemorate the two hundred and fiftieth anniversary of Lagrange's birth, we tell the story of this gentle giant of mathematics, who was born an Italian, became German by adoption, and died a Frenchman!

Joseph Louis Lagrange, one of the most versatile mathematical geniuses that the world has ever known, was born in Turin on 25 January 1736, the eldest of eleven children. His father was a senior government official of French ancestry, his mother the only daughter of a wealthy Italian doctor. The well-to-do family suffered heavy financial reverses while Lagrange was still a boy, and this threw him upon his own resources from an early age. Later in life, he saw this misfortune as a blessing in disguise, reflecting philosophically: 'Had I been rich, I might not have fallen prey to mathematics'.

Little is known for certain about Lagrange's conversion to mathematics, although it is believed that a memoir of Edmund Halley (of comet fame), on the use of algebra in optics, first revealed to him his true vocation. What is certain is that, by the age of 19, this youthful genius had raised himself to the forefront of living mathematicians, making one of the greatest discoveries in the calculus since its invention by Newton and Leibniz: the calculus of variations. He sent a summary of his results on the subject to Euler, the then reigning prince of mathematicians, who had done some pioneering work of his own on this new branch of the calculus. Euler received the letter with enthusiasm, generously withholding publication of his own findings so that all the credit might fall on the young Italian. Lagrange's achievements were quickly recognized both at home and abroad: in September 1755 he was made professor of mathematics at the Royal Artillery School in Turin, and in September 1756, at the incredibly young age of 20, he was elected a foreign member of the renowned Berlin Academy. One of the most illustrious careers in the annals of mathematics had begun.

In 1757 a youthful band of Turin scientists, among them Lagrange, formed a learned society that developed into the Royal Academy of Sciences of Turin. A prime aim of this society was the publication of a journal, the *Miscellanea Taurinesia*. Its first volume, appearing in 1759, contained no fewer than three articles by Lagrange on topics as diverse as: the propagation of sound, including a discussion of the problem of vibrating strings; the theory of maxima and minima, out of which grew the calculus of variations; the application of differential calculus to probability theory. From then until the age of 26, he filled the pages of the *Miscellanea* with a series of highly original contributions, rapidly establishing himself as the foremost mathematician of his generation.

Lagrange's reputation had been purchased at great personal cost; his single-minded devotion to study took its toll on him, both mentally and physically, and at the age of 26 he had a breakdown. Although he heeded the doctor's advice to regulate his diet and work practices, he never fully recovered, and for the rest of his life he suffered from intermittent bouts of depression.

When he did again put pen to paper it was to write on the problem of libration of the moon: Why does the moon always present the same face to the earth? For his essay, Lagrange was awarded the highly coveted Grand Prize of the French Academy of Sciences for 1764, a success that he repeated in the years 1766, 1772, 1774 and 1780 for his application of mathematics to astronomy—a truly remarkable achievement.

Lagrange felt mathematically isolated at Turin and longed to make the acquaintance of the French mathematicians with whom he had been corresponding. Thus when his friend the Marquis Caraccioli, newly appointed ambassador to England, invited Lagrange to accompany him to London, with a stopover in Paris, Lagrange jumped at the opportunity. He arrived at the French capital in November 1763 to the warmest of welcomes from the scientific society. Too warm perhaps, for after attending a banquet held in his honour, he fell seriously ill and was unable to continue the journey. During convalescence in Paris, Lagrange formed the one deep friendship of his life, that with d'Alembert, the celebrated French intellectual. When fully restored to health, he returned to Turin by way of Geneva, where he dined with Voltaire, eventually arriving home in May 1764.

Lagrange's days at Turin were now numbered. Euler resigned as director of mathematics at the Berlin Academy in 1766, suggesting Lagrange as his replacement, a recommendation endorsed by d'Alembert. Thereupon, Frederick the Great wrote modestly to Lagrange that 'the greatest king in Europe wishes to have the greatest mathematician in Europe at his side'. Lagrange accepted the invitation and departed from his native city on

Joseph Louis Lagrange 1736–1813

21 August 1766, never to return. He travelled to Berlin by way of Paris and London, reaching his destination at the end of October.

Within a year of arriving in Berlin, Lagrange married his cousin, Vittoria Conti. In a letter of unusual candour to d'Alembert he sketches the background to his marriage thus: 'Being in a foreign country, without friends and with a delicate health, I invited one of my relatives to come and take care of me. I did not inform you of it, because the matter seemed of so little importance, that it was not worthwhile to speak of it to you.' Alas, events did not turn out as Lagrange had anticipated. It was *he* who was called on to minister to her, when she succumbed to a chronic illness from which she eventually died in 1783.

During twenty glorious years at Berlin Lagrange, then at the height of his powers, produced one great memoir after another in rapid succession on a wide spectrum of topics. Two subjects in which he made outstanding contributions during the period were number theory and the theory of equations. In the former, he found all integer solutions to Pell's equation $Nx^2+1 = y^2$, where N is a non-square integer; gave the first proof of Wilson's theorem, namely that $(p-1)!+1$ is divisible by p for any prime p, and showed that every positive integer is a sum of four squares. In the latter, he analysed the methods whereby his predecessors had solved quadratic, cubic and quartic equations. He discovered that, in each case, the solution had been obtained using an 'auxiliary equation' of lower degree, but when he applied similar methods to the quintic equation, he found that its 'auxiliary equation' had degree six. This led him to believe that there was no purely algebraic way of solving a quintic equation. In these studies Lagrange considered the number of values assumed by a rational function when its variables are permuted—such investigations led to the theory of permutation groups and galois theory. The theorem that the order of a subgroup of a finite group divides the order of the group was all but established, and is today known as Lagrange's theorem.

Lagrange's researches on mechanics culminated in his masterpiece, the *Mécanique analytique*, which represents mechanics as a branch of analysis. The work was written in Berlin, but published in Paris in 1788. It contains the general equations of motion of a dynamical system now known as Lagrange's equations. Lagrange, an extreme example of the analyst, prided himself that from the beginning to the end of the book there was not a single diagram! Not only is Lagrange to be remembered as one of the most original and prolific mathematicians of all time, but also for the beauty of his writings, and nowhere is this more true than in the *Mécanique*; its sheer eloquence and grandeur moved the great physicist, Sir William Rowan Hamilton, to refer to it as 'a scientific poem by the Shakespeare of mathematics.'

After the death of Frederick the Great in August 1786, Lagrange felt ill at ease in Berlin and resigned his position there to accept Louis XVI's invitation to join the French Academy of Sciences. On arrival in Paris, he was greeted by Marie Antoinette, accommodated in a fine apartment at the Louvre and awarded a handsome salary. Despite such a friendly reception, he lapsed into a deep depression and lost his interest in mathematics; even his first printed copy of the *Mécanique analytique*, the toil of a quarter-century, lay on his desk unopened for two years. The pathetic figure cut by the sad, lonely Lagrange aroused compassion in the sixteen-year-old daughter of one of his colleagues, the astronomer Lemonnier; she fell in love with him, and in 1792 she married him. Despite the nearly forty-year disparity in their ages, their union was a happy one.

The outbreak of the French Revolution shook Lagrange out of his lethargy and served to rekindle his interest in mathematics. Throughout the Revolution, he was treated with the greatest respect by the authorities. Indeed, when in September 1793 a decree was issued ordering all foreigners out of the country, he was specifically exempted from it by name. That same year he was retained as president of a commission overseeing the reform of weights and measures, even when distinguished French academicians Laplace, Lavoisier and Coulomb had been 'purged' from it. It was no small part due to Lagrange's influence that the metric system was perfected and adopted by the commission in 1799.

Lagrange was appointed professor of mathematics first at the short-lived École Normale in 1795, and two years later at the newly established École Polytechnique, the earliest success of which was to revitalise his interest in analysis. His lectures on this subject formed the basis of his book, *Théorie des fonctions analytiques*, of 1797. This work attempted to place the differential calculus on a sound footing by banning all mention of differentials, infinitesimals and limits, none of which at that time had been defined satisfactorily. He 'achieved' this by defining the derivatives $f'(x), f''(x), \ldots$ of a function f to be the coefficients of $h, h^2/2!, \ldots$ in the Taylor series expansion of $f(x+h)$; it is indeed to Lagrange that we are indebted for the notation $f'(x), f''(x), \ldots$. Although we now know that Lagrange's approach was doomed to failure, the book is to be remembered as the first 'theory of functions of a real variable' and for the first appearance of the Lagrange form of remainder in a Taylor series.

Before continuing our story, we take a brief look at Lagrange the man. The traits observed by all who knew him were his even temper, modesty and dedication to his work. He instinctively avoided every quarrel. In a letter to his close friend d'Alembert, he once advised 'in every case peace is better than war'—a philosophy that kept Lagrange's head on his shoulders throughout the Revolution. Although born an Italian, all his works are written in French and he became a French citizen when he settled in Paris. Certainly France regarded him as one of her greatest sons. Napoleon referred to him as the 'lofty pyramid of the mathematical sciences' and made him a senator, Count of the Empire and a Grand Officer of the Legion of Honour, but such wordly honours sat uncomfortably on Lagrange. He never allowed his portrait to be painted, maintaining that it was a man's *work* which deserved preservation. The portrait accompanying this article was obtained by stealth at a meeting of the Institut!

Among the students who attended Lagrange's lectures at the École Normale was the young Joseph Fourier, who was himself destined to become a world-famous mathematician. He has left us the following recollections of his teacher:

> He has a strong Italian accent. He dresses quietly, in black or brown. Everyone knows that he is an extraordinary person, but one needs to have seen him to recognize him as a great one. Some of what he says excites ridicule. The other day he said 'There are a lot of important things to be said on this subject, but I shall not say them.'

Not all of Lagrange's observations were so empty. When he learned that his friend, the celebrated chemist Lavoisier, had gone to the guillotine, even he was provoked into exclaiming: 'It took them only a moment to lay low that head, a hundred years will not suffice to produce its like again.' That he was possessed of a wry sense of humour can be gleaned from the following quotation: 'Newton was the greatest genius that ever lived, and the most fortunate, for we cannot find more than once a system of the world to establish.'

Lagrange's final years were spent in preparing a revised edition of his *Mécanique analytique*. The treatise was expanded so much that it had to be published in two volumes; the first of these appeared in 1811, but the manuscript for the second was incomplete at the time if his death. Although now well into his seventies, he was in full possession of his mental faculties and his capacity for work remained undiminished. In February 1813, he suffered a severe fainting attack during which he fell, cutting his head on the edge of a table, and he was found by his wife lying unconscious on the study floor. He quickly recovered from the fall, but the faints continued. On 8th April some colleagues called to bring him a new imperial decoration, the grand cross of the Order of Reunion. He told them how two days earlier he was convinced that he was dying and went on to say:

> Oh, death is not to be feared, and it comes without pain, it is a last function which is neither distressing nor disagreeable. ... I wanted to die, but my wife did not wish it. At that moment I would have preferred a wife less good, less eager to revive my strength, who would let me die peacefully. I have had a career, I have obtained some celebrity in mathematics. I have never hated anyone, I have done no harm, and there must be an end, but my wife did not wish it.

His wish was soon granted. The farewell meeting with his friends exhausted him and his condition rapidly deteriorated. He died on the morning of 10 April 1813, but the monuments to his genius are to be seen everywhere in that palace of mathematical ideas which he helped to furnish so lavishly.

A knight on a chessboard

In Volume 18 Number 2 we asked whether it is possible for a knight to visit each square of a chessboard exactly once such that, having reached the last square, one more move will take it back to its starting square. Eddie Kent, of Stoke Newington, London, has sent us the following solution attributed to a pensioned Moravian officer named Wenzclides, long before computers. Note that all row sums and all column sums are 260.

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

How Obvious Is It?

HAZEL PERFECT, *University of Sheffield*

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1. Looking at things in the right way

How obvious is it that, in a set of objects not all of the same colour and not all of the same shape, there will be at least two which differ in both colour and shape? I remember attending a popular lecture in mathematics some years ago at which the speaker began by posing this question to the audience. (A solution is given at the end of this article.)

The question 'How obvious is it?' is a subjective one: what *you* regard as obvious *I* may not, and vice versa. Also, to a considerable extent, it depends on *how you look at it* whether something is obvious or not. Let us consider a few examples which may illustrate these points.

(1) To the 10-year-old Carl Friedrich Gauss it was soon obvious that the sum of the numbers $1, 2, \ldots, 100$ is equal to $\frac{1}{2} \times 100 \times 101$ $(= 5050)$. His teacher at the Volksschule in Brunswick had put this calculation to his class of small boys in the hope, I suppose, of giving himself a quiet half-hour. But the young Gauss looked at the problem 'in the right way' and saved himself the labour of tedious calculations. He noticed that

$$\begin{aligned} &1 + \ \ 2 + \ldots + 99 + 100 \\ = \ &100 + 99 + \ldots + \ \ 2 + \ \ \ 1 \end{aligned}$$

and that the sum of the two terms in each of the 100 columns is 101; from which the result follows at once.

(2) A square board of size $8\,\text{cm} \times 8\,\text{cm}$ has two opposite corner squares of size $1\,\text{cm} \times 1\,\text{cm}$ cut off. Can it be exactly covered by non-overlapping dominoes each of size $1\,\text{cm} \times 2\,\text{cm}$? If you transform the original board into a chessboard you will see almost immediately that the answer to the question is 'No'. For each domino must cover two squares of opposite colours whereas two squares of the same colour have been cut off the board leaving 32 of one colour and only 30 of the other.

(3) There are many results about the concurrence of lines associated with a triangle. The most obvious is probably that the perpendicular bisectors of its sides are concurrent (at the centre of its circumscribing circle). How

obvious is it that the altitudes of a triangle are concurrent? In figure 1 a triangle PQR has been constructed through the vertices of the triangle ABC and with corresponding sides parallel as shown. The altitudes of ABC are just the perpendicular bisectors of the sides of PQR; and the conclusion is immediate.

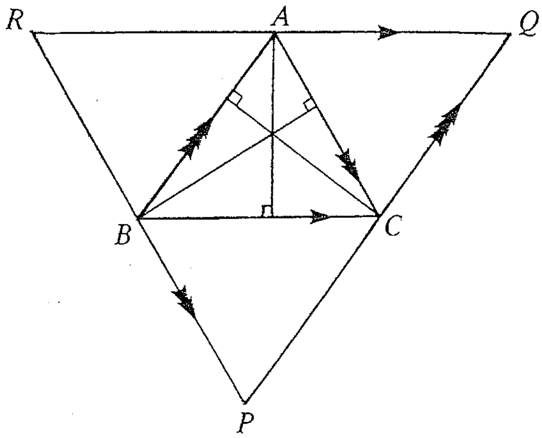

Figure 1

(4) Mr A climbs to the top of a mountain by a certain footpath and comes down to the foot of the mountain by the same route on the following day. On each occasion he starts at 8 o'clock in the morning. Why is it that, however often he halts or hurries ahead, there will be one point on the path that he reaches at exactly the same time of day? (A solution is given at the end of this article).

Mathematics is often described as a deductive science, and the classical example of deductive reasoning is provided by Euclid's 'Elements'. Without too much inaccuracy, perhaps problem-solving in mathematics may be said to consist in deducing the less obvious from the more obvious. In the next two sections we shall look at two assertions which most of us would regard as obvious—the pigeon-hole principle and the intermediate-value theorem—and we shall find that some much less obvious results can be deduced quite readily from them.

2. Deductions from the pigeon-hole principle

The pigeon-hole principle states that *if* $n+1$ *letters are posted in* n *pigeon-holes then at least one pigeon-hole must contain more than one letter*. Just as obvious is the following slight generalization: *if* $mn+1$ *letters are posted in* n *pigeon-holes then at least one pigeon-hole must contain more than* m *letters*.

We examine just a few varied deductions.

(1) There are n people present at a party and some of them shake hands with each other. We assert that it must always happen that at least two people shake the same number of hands. (We assume that no one shakes hands with himself.)

Suppose s ($\leqslant n$) of the people shake hands each with at least one other person. Call these people $a_1, \ldots, a_s$, and suppose a_i shakes hands with exactly j_i people (who must be among $a_1, \ldots, a_s$), $1 \leqslant i \leqslant s$. Each j_i evidently satisfies $1 \leqslant j_i \leqslant s-1$ and, as there are s numbers j_i, at least two are equal.

(2) Every sequence $(a_1, a_2, \ldots, a_{n^2+1})$ of n^2+1 real numbers has a monotonic sequence of length $n+1$, or longer [i.e. a subsequence $(b_1, \ldots, b_{n+1})$ such that $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_{n+1}$ or $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{n+1}$].

For each i, let a_i^* denote the length of the longest increasing subsequence which ends in a_i. If some $a_i^* \geqslant n+1$ there is nothing further to prove, so let us assume that $1 \leqslant a_i^* \leqslant n$ for all i. By the pigeon-hole principle (the second form with $m = n$), it follows that there must be at least $n+1$ integers i for which the corresponding a_i^* are equal, say $a_{i_1}^* = \ldots = a_{i_{n+1}}^*$, where $i_1 < \ldots < i_{n+1}$. From this it follows that $a_{i_1} > \ldots > a_{i_{n+1}}$. Otherwise if, say, $a_{i_s} \leqslant a_{i_{s+1}}$ then $a_{i_{s+1}}$ could be adjoined to the subsequence ending in a_{i_s} to form a longer increasing sequence; which would contradict $a_{i_s}^* = a_{i_{s+1}}^*$.

Somewhat harder is the following interesting application to the approximation of irrational numbers by rationals. The result, which is due to Dirichlet, asserts something considerably stronger than just the existence of rational numbers arbitrarily close to a given irrational number.

(3) If h is irrational, there are infinitely many rational numbers p/q in lowest terms such that

$$\left| h - \frac{p}{q} \right| < \frac{1}{q^2}.$$

We show first that, for a given positive integer k, there is an approximation p/q in lowest terms with $0 < q \leqslant k$ such that

$$\left| h - \frac{p}{q} \right| < \frac{1}{qk}$$

(and it is at this crucial stage that we use the pigeon-hole principle). We shall use the notation $[x]$ to denote the integral part of x, i.e. the least integer not exceeding x. Consider then the set of real numbers

$$0,\ h-[h],\ 2h-[2h],\ \ldots,\ kh-[kh] \tag{1}$$

together with the set of k intervals

$$0 \leqslant x < \frac{1}{k},\ \frac{1}{k} \leqslant x < \frac{2}{k},\ \ldots,\ 1-\frac{1}{k} \leqslant x < 1.$$

Each of the $k+1$ numbers (1) lies in one of these intervals, and so there must be an interval which contains at least two of them. So, for some r and s with $k \geqslant r > s \geqslant 0$,

$$|(rh-[rh])-(sh-[sh])| < \frac{1}{k}.$$

Put $q = r-s$ and $p = [rh]-[sh]$; then $0 < q \leqslant k$ and

$$\left|h-\frac{p}{q}\right| < \frac{1}{qk}.$$

If p and q have a proper common factor we may obviously cancel it out without destroying this inequality.

To prove the original assertion, let us assume that only a *finite* number of approximations p/q of h satisfy

$$\left|h-\frac{p}{q}\right| < \frac{1}{q^2}$$

(at least one exists since $1/q^2 \geqslant 1/qk$). Each one gives rise to a strictly positive error since h is not rational. Let e be the smallest error which arises and choose $k > 1/e$. By what we have just proved, we can find an approximation p/q $(0 < q \leqslant k)$ such that

$$\left|h-\frac{p}{q}\right| < \frac{1}{qk} \leqslant \frac{1}{k} < e.$$

Thus

$$\left|h-\frac{p}{q}\right| < \frac{1}{q^2} \quad \text{and} \quad \left|h-\frac{p}{q}\right| < e;$$

which contradicts the minimality of e. Therefore our assumption of finiteness must be incorrect.

3. Deductions from the intermediate-value theorem

The statement contained in the intermediate-value theorem is intuitively obvious†: namely that, *if a continuous function (i.e. a function whose graph is a continuous curve) takes both positive and negative values, then somewhere in between it must take the value zero.*

We choose to deduce here two geometrical propositions from this theorem.

(1) Given a closed curve $\mathscr{C}$ it is possible to circumscribe a square about it (tangentially as in figure 2).

† And, when proper definitions have been formulated, can be rigorously proved.

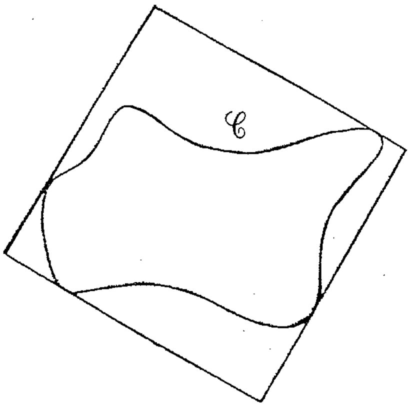

Figure 2

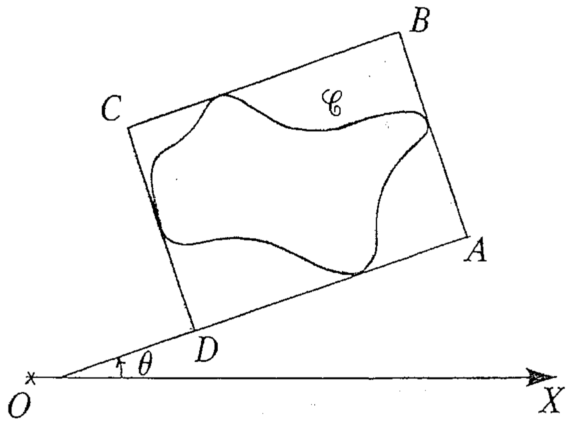

Figure 3

Take a given direction in the plane, making an angle θ (say) with $\overrightarrow{OX}$ and circumscribe a rectangle $ABCD$ about $\mathscr{C}$ so that one pair of opposite sides is parallel to the given direction. Certainly this is possible and the rectangle is uniquely determined (see figure 3). Put $AB = a(\theta)$ and $BC = b(\theta)$. Corresponding to each angle θ is a uniquely determined number $a(\theta)-b(\theta)$ and, as θ varies continuously through 90°, $a(\theta)-b(\theta)$ will vary continuously from its original value $c(\theta)$ (say) to the value $c(\theta+90) = -c(\theta)$; for the variable rectangle 'slides round $\mathscr{C}$' until AB is along BC and $a(\theta)$ and $b(\theta)$ are interchanged. Somewhere between the initial and final positions of the rectangle is a position at which $c(\theta) = 0$, by the intermediate-value theorem. In this position $ABCD$ is a square.

(2) Given two closed curves lying in the same plane, it is possible to find a straight line which simultaneously bisects the two areas $\mathscr{A}$ and $\mathscr{B}$ inside the curves.

First of all (and you should fill in the details of the argument), by the intermediate-value theorem, there exists a straight line bisecting the area $\mathscr{A}$ and parallel to any given direction, and it is unique. Consider then the line $l(\theta)$ (say) which makes an angle θ with $\overrightarrow{OX}$ and which bisects $\mathscr{A}$ (see figure 4).

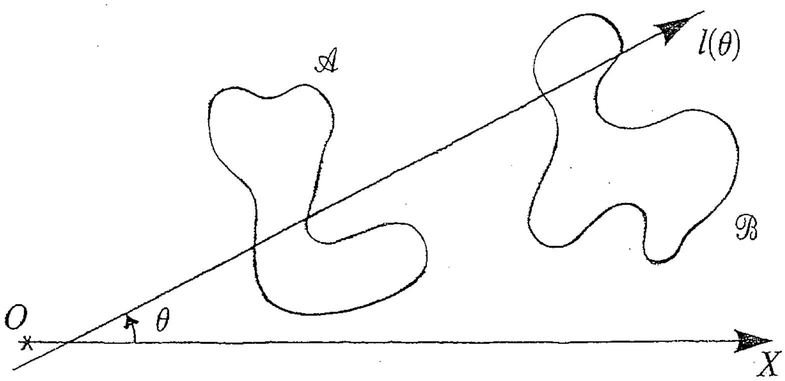

Figure 4

Let the parts of $\mathscr{B}$ to the left of $l(\theta)$ and to the right of $l(\theta)$ be $\mathscr{B}_1(\theta)$ and $\mathscr{B}_2(\theta)$†. As θ varies through 180°, $\mathscr{B}_1(\theta)-\mathscr{B}_2(\theta)$ varies from the value $\alpha(\theta)$ (say) to the value $-\alpha(\theta)$. So somewhere in between, again by the intermediate-value theorem since the variation is continuous, it takes the value zero. In this position the line $l(\theta)$ simultaneously bisects $\mathscr{A}$ and $\mathscr{B}$.

The well-known 'ham-sandwich' theorem, which states that in three dimensions there is a plane which simultaneously bisects the volumes inside three closed surfaces (two slices of bread and the ham in between!) is a generalization of (2); but its proof requires a stronger result than the intermediate-value theorem.

4. Is it really obvious?

Some assertions which are 'obvious' turn out to be wrong! Surely it is obvious that there are twice as many positive integers as there are even positive integers, and many more positive rational numbers than there are positive integers. But both of these assertions are incorrect. Two sets are said to be equal if their members can be paired off one-to-one. On the basis of this definition, these three sets—of positive integers, even positive integers, positive rational numbers—are all equal. The pairing off of even positive integers with positive integers is simple:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \ldots & n & \ldots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2 & 4 & 6 & \ldots & 2n & \ldots \end{array}$$

To pair off the positive rational numbers with the positive integers, we need only write them out in a single list, and figure 5 shows how this may be done. For convenience we write (p,q) for p/q and delete from the list repetitions $(2p,2q)$, $(3p,3q)$, ... of (p,q). There are many more, much more surprising and unexpected, results connected with infinite sets, which conflict with our intuitive notions of what is obvious.

$$\begin{array}{ccccccc} (1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) \;\ldots \\ & & \downarrow & & \uparrow & & \downarrow \\ (2,1) & \leftarrow & (2,2) & & (2,3) & & (2,4) \;\ldots \\ \downarrow & & & & \uparrow & & \downarrow \\ (3,1) & \rightarrow & (3,2) & \rightarrow & (3,3) & & (3,4) \;\ldots \\ & & & & & & \downarrow \end{array}$$

Figure 5

† We view this along the positive direction of $l(\theta)$.

Again, surely it is obvious that there are two sides to a piece of paper! But what about the Möbius strip? You can make one by cutting a strip of paper, twisting it *once* and glueing the ends together. If you take a pencil and trace round the strip lengthwise, you will readily convince yourself that it has only one side.

In another field, we recall that for many centuries mathematicians believed that the axioms of Euclidean geometry contained intuitively obvious statements about the physical world. Therefore Euclidean geometry as a whole, which is built upon these axioms, must (they concluded) provide a true, if idealized, description of physical space. Only in the eighteenth and nineteenth centuries, when unsuccessful attempts were made to deduce one of these axioms (the parallel postulate†) from the remaining ones, did they begin to question these 'obvious' conclusions. Eventually the parallel postulate was seen to be independent of the other axioms, and self-consistent geometries different from Euclidean geometry (the non-Euclidean geometries), in which the parallel postulate is replaced by a different axiom, were subsequently created and studied. Following the development of the theory of relativity, these non-Euclidean geometries were used to describe aspects of physical space.

In conclusion, a further thought: we are perhaps inclined to say that something is obvious when what we really mean is that it is familiar. For instance, how obvious really are the multiplication tables of arithmetic? Even more basically, how obvious is the notion of a set of objects?

Solutions to the problems in Section 1

(Paragraph 1) Take an object A of colour c_1 and shape s_1. There exists an object B of colour $c_2 \neq c_1$. If B has shape other than s_1 there is nothing further to prove. Suppose, therefore, that B has shape s_1. There exists an object C of shape $s_2 \neq s_1$. If C has colour c_1, then B and C satisfy the conclusion. If C has colour c_2, then A and C satisfy the conclusion. If C has colour $c_3 \neq c_1, c_2$, then A and C and B and C both satisfy the conclusion.

(Problem (4)) Let Mr A find a friend Mr B who will do a 'carbon copy' of Mr A's downhill journey at the same time that Mr A is ascending the mountain. Mr A and Mr B must meet somewhere! Equally well, you might regard the problem as another application of the intermediate-value theorem.

† In one form this states that, if P is a point not lying on a straight line l, there is exactly one straight line through P parallel to l.

On Doodles and 4-Regular Graphs

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1. Introduction

The next time you are doodling on a piece of paper, let your pen come back to its starting point and then consider what kind of mathematical figure you have drawn.

If you have always let your pen trace a smooth continuous curve, and whenever you have met a previous point on the curve you have crossed directly over the old curve, it is probable that you have drawn what graph-theorists call a 4-regular graph: figure 1 is an example. In this case n, the number of crossing points $= 9$, m, the number of edges (an edge is a curve joining two adjacent crossing points) $= 18$ and r, the number of regions (one is shown shaded, 34589) $= 11$, including the region of the plane exterior to the graph. Such diagrams are called *4-regular graphs* because every crossing point (these are usually called *vertices*, or *nodes* has four edges adjoined to it.

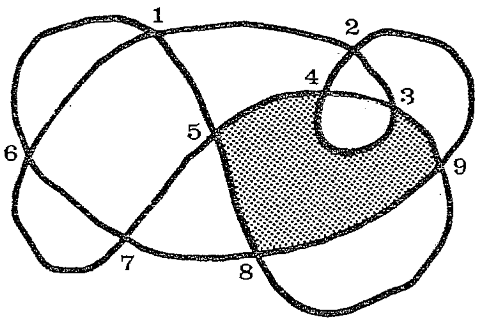


Figure 1

You might like to think what you would have to do whilst doodling if the result were not to be a 4-regular graph; and conversely, whether there are 4-regular graphs which are not doodles! You might also like to think of how many *small* regions (one with few edges) you have managed to draw in your doodle.

Let us say a region is a *k-gon* if it is bounded by k edges. Then the smallest region possible is 1-gon or loop; then a 2-gon; then a 3-gon; and so on. In the doodle shown in figure 1 there are four 2-gons and three 3-gons. There are also two 4-gons, one 5-gon, and the outer-region 6-gon.

The purpose of this article is to discuss the possibility of bounds on the frequencies of small regions in 4-regular graphs. Simple question to ask are:

(i) Can one draw a doodle without a 2-gon?

(ii) Can one draw a doodle without a 3-gon?

(iii) Can one draw a doodle without either a 2-gon or a 3-gon?

Take a pencil and paper and see if you can answer yes or no to these questions. Don't cheat by looking at the answers at the end of the article! Answering (i) is quite hard; answering (ii) is easy. Answering (iii), and more complex questions on small regions, requires some mathematical analysis which we shall now give.

2. Leonhard Euler (1707–1783)

Euler was the father of graph theory, and hence of *topology* (a branch of mathematics which used to be called *analysis situs*, or 'analysis of position'). He was one of the most prolific mathematicians that has ever lived. Arago, a scientist who was contemporary with him, said, 'Euler calculated without effort, as men breathe or as eagles sustain themselves in the wind'. (An account of Euler's life is to be found in Volume 15 Number 3 of *Mathematical Spectrum*.)

In 1736 he gave a solution of the famous seven-bridge problem of the city of Königsberg. Inhabitants of the city wished to know if it was possible to walk around two islands and the main river banks of their city (see figure 2), in one stroll (a 'dawdle', not a 'doodle'!), crossing all seven bridges exactly once. Using reasoning which is now a fundamental part of graph theory, Euler proved that such a stroll was impossible.

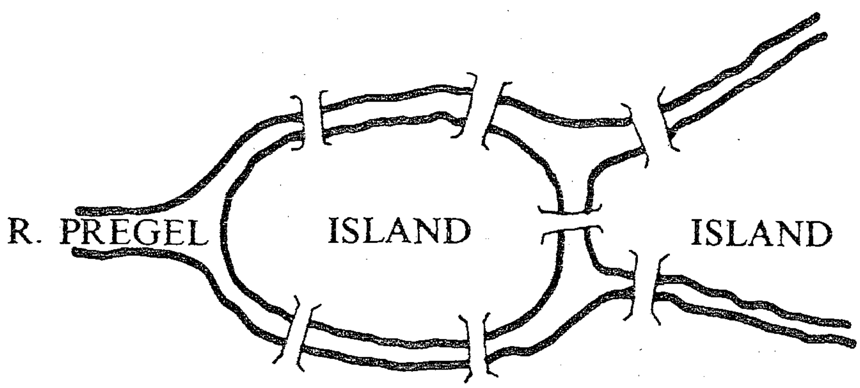


Figure 2

Later he proved a result which is a basis for very many theorems in topology. He showed that, under certain quite general conditions, the numbers of vertices (n), edges (m) and faces (f) of a solid figure bounded by plane faces, called a *polyhedron*, are related by the equation $n-m+f = 2$. Translating this to graphs drawn in a plane (replacing f by r, the number of regions) we have the relationship

$$n-m+r = 2. \tag{1}$$

This is known as *Euler's equation*, and a proof may be found in any elementary text on topology. The student will be interested to know, on looking up the rather short proof, that Euler (who, remember, calculated as men breathe) took some two years to arrive at a satisfactory proof. No present-day student should mock Euler for this, but rather learn that inventing new mathematics is a vastly different thing from reading about mathematics that men have mulled over for centuries or millenia.

As well as result (1), we shall need the following basic result about graphs:

$$m = \text{total number of edges} = \tfrac{1}{2}(\text{sum of the degrees of all vertices}),$$

where the *degree of a vertex* is the number of edges adjacent to the vertex. Thus if the graph has n vertices and their degrees are $\{d_1, d_2, \ldots, d_n\}$, we can write

$$m = \tfrac{1}{2}\sum_{i=1}^{n} d_i. \tag{2}$$

Let us see that (1) and (2) work out with the doodle shown in figure 1:

$$n-m+r = 9-18+11 = 2 \quad [\text{as in (1)}];$$

and, since the doodle is 4-regular, all the nine vertices have degree 4 and so

$$m = 18 = \tfrac{1}{2}(\text{sum of the degrees}) = \tfrac{1}{2}\times 9\times 4 \quad [\text{as in (2)}].$$

Note from (2) that, if G is *any* 4-regular graph, its number of edges is

$$m = \tfrac{1}{2}(n\times 4) = 2n. \tag{3}$$

3. An inequality theorem on small regions

We now prove an inequality, in order to answer questions of the type we asked at the beginning of the article. It actually deals with a more general case, namely that of d-regular graphs, where d is any positive integer.

Theorem. Let G be a d-regular graph drawn in a plane. Let it have n, m and r vertices, edges and regions, respectively, and let its regions have frequency distribution $\{f_1, f_2, \ldots, f_k, \ldots\}$, where f_k is the number of regions bounded by exactly k edges (i.e. f_k is the number of k-gons in G). Then for any positive integer j,

$$\sum_{k=1}^{j} (j+1-k)f_k \geqslant n[\tfrac{1}{2}d(j+1)-(j+d+1)]+2(j+1). \tag{4}$$

Proof. Since the degree of each vertex is d, (2) gives

$$m = \tfrac{1}{2}dn;$$

then Euler's equation gives

$$n-m+r = n-\tfrac{1}{2}dn+r = 2,$$

from which

$$r = 2+n(\tfrac{1}{2}d-1). \tag{5}$$

Now each edge occurs in exactly two regions; therefore

$$2m = dn = 1f_1+2f_2+3f_3+\ldots+tf_t, \tag{6}$$

assuming that a region having the most edges is a t-gon.

For a positive integer j, the total number of k-gons with $k > j$ is

$$r-\sum_{k=1}^{j} f_k,$$

and each of these has $j+1$ or more edges. Therefore, from (6),

$$dn \geqslant 1f_1+2f_2+3f_3+\ldots+jf_j+(j+1)\left(r-\sum_{k=1}^{j} f_k\right). \tag{7}$$

Inserting into (7) the expression found for r in (5), and rearranging terms, we obtain the result of the theorem.

4. Small regions of 4-regular graphs

Armed with the above theorem, we can now answer the questions we posed about doodles and 4-regular graphs. When $d = 4$, the inequality (4) reduces to

$$\sum_{k=1}^{j} (j+1-k)f_k \geqslant (j-3)n+2(j+1). \tag{8}$$

Let us suppose that our doodles have no loops in them, so that $f_1 = 0$, and discuss the cases $j = 2$, $j = 3$ and $j > 3$.

(a) $j = 2$ *and* $f_1 = 0$. With these values, inequality (8) is

$$f_2 \geqslant 6-n. \tag{9}$$

Therefore all loopless 4-regular planar graphs which have 5 or fewer vertices have at least one 2-gon. For $n \geqslant 6$ our theorem tells us nothing about f_2.

(b) $j = 3$ *and* $f_1 = 0$. With these values the theorem gives

$$2f_2+f_3 \geqslant 8. \tag{10}$$

An immediate corollary of this (but a weaker result) is

$$f_2 + f_3 \geqslant 4. \tag{11}$$

It is interesting to note that both (10) and (11) are independent of n, the number of crossings in the doodle. Thus from (11) we deduce that *any* doodle with no loops (strictly, *any* loopless planar 4-regular graph) must have at least four regions having two or three edges. This answers—in the negative—question (iii) that we asked in the introduction.

The other two questions were, can a doodle be drawn (i) without a 2-gon? or (ii) without a 3-gon. Both these questions can be answered in the affirmative, as the examples in figures 3 and 4 show. (On page 233 of *Graphical Enumeration*, by F. Harary and E. M. Palmer (Academic Press, New York, 1973) it is conjectured that $f_2 \geqslant 2$ always. Our figure 3 shows that this conjecture is false: there is in fact an infinity of 4-regular graphs for which $f_2 = 0$.)

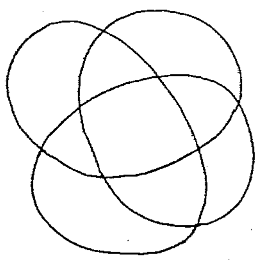


Figure 3

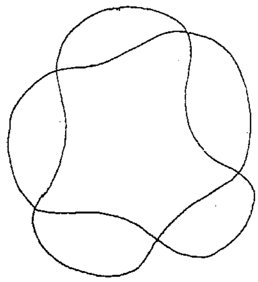


Figure 4

The final case for discussion is the following.

(c) $j > 3$ *and* $f_1 = 0$. The theorem result (8) now has the following corollary [cf. (10) and (11)]:

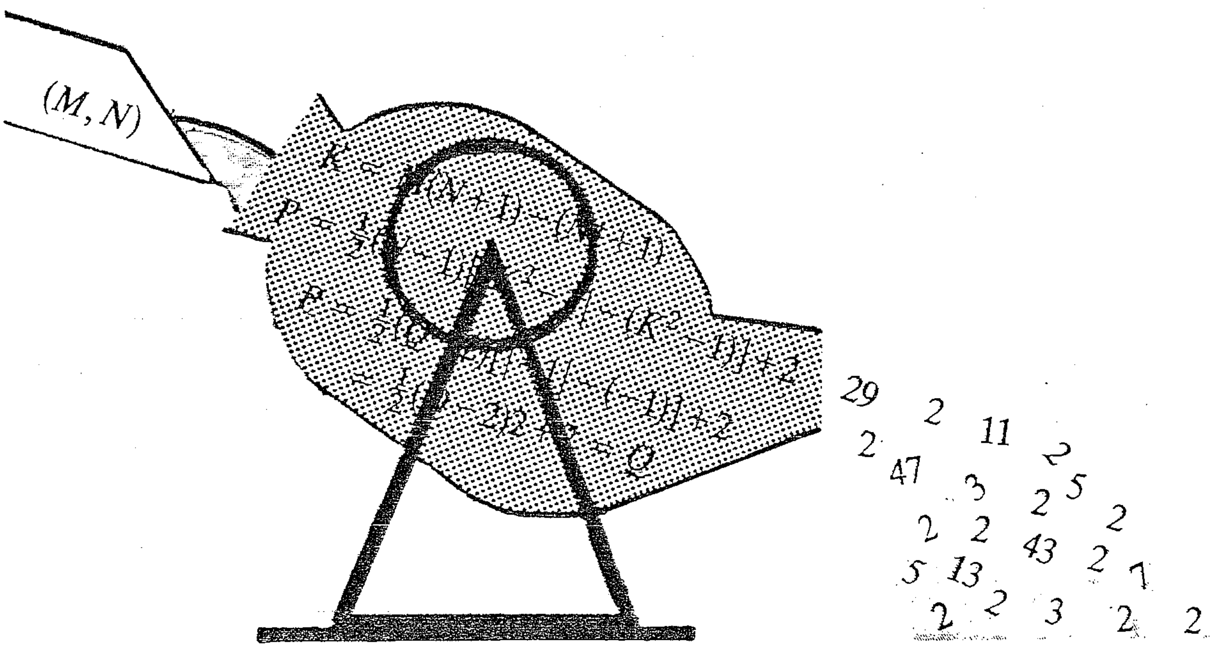
$$f_2 + f_3 + \ldots + f_j \geqslant \frac{j-3}{j-1}n + 2\frac{j+1}{j-1}. \tag{12}$$

Suppose we put $j = 5$ in this, and consider what kinds of doodles can be drawn. We find that

$$f_2 + f_3 + f_4 + f_5 \geqslant \tfrac{1}{2}n + 3 = \tfrac{1}{2}r + 2. \tag{13}$$

This tells us that, no matter how long we go on doodling, making n as large as we please (say $n = 10^6$, or even $n = 10^{20}$—which would take us several lifetimes and not a few pencils!) and trying to avoid making small regions, more than half the regions we produce will be smaller than 6-gons.

Seconds of Prime Beef



Arthur Pounder of Manchester writes about the formula Keith Devlin gives in his article in Volume 18 Number 2 p.33 which generates all the prime numbers and no others. The formula was given in two parts, First put

$$K = M(N+1)-(N!+1),$$

and then calculate

$$P = \tfrac{1}{2}(N-1)[|K^2-1|-(K^2-1)]+2.$$

As different positive whole number values of M, N are inserted, P gives all the primes and no other numbers. Try it out! Most of the time you will get $P = 2$ but, if you are patient, other primes will appear. For example, $M = 1$ and $N = 2$ give $P = 3$; $M = 5$ and $N = 4$ give $P = 5$; $M = 103$ and $N = 6$ give $P = 7$; and you will have to wait until $M = 329\,891$ and $N = 10$ before you obtain $P = 11$.

Our correspondent asks: how does it work? It goes back to a result attributed to Sir John Wilson (1770), who was a judge, but the same result was probably known to Leibniz a century before that. The result is that, for a positive integer N, $N+1$ is prime if and only if $N! \equiv -1 \pmod{N+1}$. For example, put $N = 6$. Then $N+1$ is prime and $N! = 720 \equiv -1 \pmod 7$. On the other hand, put $N = 5$. Then $5! = 120 \equiv 0 \pmod 6$.

So, suppose we want to obtain a given prime number Q out of our formula. First put $N = Q-1$. The $N+1$ is prime, so, by Wilson's theorem, $N! \equiv -1 \pmod{N+1}$. This means that $N!+1$ is divisible by $N+1$, so there is a whole number M such that $M(N+1)-(N!+1)$ is zero, i.e. $K = 0$. For example, when $Q = 7$ we take $N = 6$ and $M = 103$. But now

$$P = \tfrac{1}{2}(Q-2)[|-1|-(-1)]+2 = \tfrac{1}{2}(Q-2)2+2 = Q.$$

Thus the formula gives every prime number.

But suppose we choose N such that $N+1$ is not prime. Then $N! \not\equiv -1 \pmod{N+1}$ and, whatever integer M we choose, $K \neq 0$. But then $K^2-1 \geqslant 0$ and

$$P = \tfrac{1}{2}(N-1)[(K^2-1)-(K^2-1)]+2 = 2,$$

so we shall obtain the prime value 2. This explains why the formula usually gives the value $P = 2$. Nevertheless, it is true that the formula will give all prime values and no others, albeit very inefficiently, as M and N vary.

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In Volume 18 Number 2 we challenged readers to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$, !, brackets and concatenations (e.g. 98). We had lots of replies, but the following six numbers have resisted the combined onslaught of all our readers: 68, 69, 73, 87, 99 and 100. The number 25 proved difficult, but yielded to Anne Watts of Pinner, $1+\{[(\sqrt{9})\times 8]\div 6\}!$, and Mike Wenble of Langley, $1+[(\sqrt{9})!-8+6]!$ Guy Willard, of The Haberdashers' Aske's School, Elstree, has sent the following amusing identities:

1. $f_1+f_9-f_8-f_6 = 1-9+8+6$, $f_1+f_9-f_8+f_6 = -1+9+8+6$, where f_i is the ith term of the Fibonacci sequence $1,1,2,3,5,8,13,21,34,\ldots$;
2. $\displaystyle\int_1^8\int_9^6 \mathrm{d}x\,\mathrm{d}y = -19-8+6;$
3. $(1+9)^2 = 8^2+6^2;$
4. $1^3 = 9^3-8^3-6^3;$
5. $\binom{9}{1}\cdot\binom{8}{6} = (19+8-6)\cdot(1+9+8-6)$, where $\binom{n}{r}$ denotes the binomial coefficient.

Stochastic Compartmental Models

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M . J. Faddy has been a lecturer in the Department of Statistics at the University of Birmingham since 1970, with occasional absences lecturing overseas. He is interested in mathematical models of random phenomena, particularly in the biological sciences, and is not ashamed to admit to using computers to aid his understanding of such models.

1. Introduction

During their lives, individuals seem to pass in an orderly fashion through the various stages or compartments of, say, education, employment and retirement. However, taken collectively, such individuals spend varying amounts of time in these compartments; statisticians model this aggregate behaviour of a population in terms of random variables describing the varying amounts of time spent by individuals in each compartment. It has been found that these stochastic models give reasonable predictions for large populations.

There are many examples of compartmentalised behaviour: Bartholomew (reference 1) describes the type of social mobility referred to above (chapters 2, 3), while Chiang (reference 2) discusses the progress of a disease passing through stages of increasing severity, up to death, with possible remission to a less severe stage, or complete recovery (chapters 4, 5).

This article is concerned with a class of stochastic (compartmental) models of the type described above; it explores their mathematical structure to reveal why certain forms of solution arise, and discusses under what circumstances the mathematical arguments break down. Consider a system of compartments containing individuals moving randomly between the compartments, or out of the system altogether, over continuous time. Much of the essential theory can be derived from a system with only one compartment; this establishes the structure of the solution from which generalisations (e.g. new individuals entering the compartments from outside the system) follow.

2. The case of the single compartment

Individuals in the single compartment may only move out of the system, as there are no other compartments; let $\mu\,\delta t$ be the probability of such a movement for a single individual in a 'short' time-interval $(t, t+\delta t)$. Denote by $p(t)$ the probability that a single individual present in the compartment initially at time 0 is still there at time $t > 0$, then

$$p(t+\delta t) = p(t)\times(1-\mu\,\delta t)$$

or, letting $\delta t \to 0$,

$$\frac{d}{dt}p(t) = -\mu p(t). \tag{2.1}$$

The solution of (2.1) with $p(0) = 1$ is clearly

$$p(t) = e^{-\mu t}. \tag{2.2}$$

Probability (2.2) refers to a single individual; if N individuals are present in the compartment at time 0 and they behave *independently* of one another, then the number n of individuals present at time t will be *binomially* distribed:

$$\binom{N}{n}[p(t)]^n[1-p(t)]^{N-n}, \quad n = 0,1,\ldots,N. \tag{2.3}$$

This process, starting with N individuals, is often called a *simple death process*.

A generalisation, allowing new individuals to enter the compartment from outside the system may now be considered: suppose the probability of one such individual entering the compartment in a 'short' time interval $(t, t+\delta t)$ is $\lambda\,\delta t$, where the entries in non-overlapping time-intervals are independent. Then the probability $p_n(t)$ of n new individuals entering in the time interval $(0,t)$ is $e^{-\lambda t}(\lambda t)^n/n!$. This can readily be seen by splitting the time interval $(0,t)$ into K sub-intervals each of length t/K, when the probability $p_n(t)$ is binomial of the form

$$\binom{K}{n}\left(\frac{\lambda t}{K}\right)^n\left(1-\frac{\lambda t}{K}\right)^{K-n}.$$

When $K \to \infty$ it is well known that

$$p_n(t) \to \frac{(\lambda t)^n}{n!}e^{-\lambda t}. \tag{2.4}$$

Probabilities (2.4) for $n = 0,1,2,\ldots$ make up the familiar Poisson distribution with mean λt and, for obvious reasons, the entry process is called a Poisson process of rate λ.

A further interesting property of this Poisson process concerns *conditional* entry times: if only one individual is known to have entered the compartment in the time interval $(0,t)$, then the actual time of entry T will have conditional distribution

$$P\{T < s \mid 1 \text{ entry in } (0,t)\}$$
$$= \frac{P\{1 \text{ entry in } (0,s)\} \times P\{0 \text{ entries in } (s,t)\}}{P\{1 \text{ entry in } (0,t)\}}$$

$$= \frac{(\lambda s)e^{-\lambda s}e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}}, \quad \text{from (2.4)}$$

$$= \frac{s}{t} \quad (0 < s < t).$$

In other words, this conditional distribution of T is *uniform* on the interval $(0, t)$, with probability density function $1/t$ $(0 < s < t)$.

Now after entering the compartment, probability (2.2) will describe the individual's subsequent activities, so conditional on one entry in the time interval $(0, t)$ the individual will still be in the compartment at time t with probability

$$q(t) = \int_0^t \frac{1}{t} p(t-s)\, ds = \frac{1-e^{-\mu t}}{\mu t}. \tag{2.5}$$

It can further be shown [Cox and Lewis (reference 3), chapter 2] that, conditional on k individuals entering the compartment during the time interval $(0, t)$, their times of entry will be *independently* uniformly distributed on $(0, t)$, so that their number n in the compartment at time t will be binomially distributed

$$\binom{k}{n}[q(t)]^n[1-q(t)]^{k-n}, \quad n = 0, 1, \ldots, k. \tag{2.6}$$

But the k entries follow a Poisson distribution with mean λt, so the *unconditional* probability distribution of the number of individuals in the compartment at time t, that entered during the time interval $(0, t)$, will be

$$\sum_{k=n}^{\infty} \binom{k}{n}[q(t)]^n[1-q(t)]^{k-n}\frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{e^{-\lambda t q(t)}[\lambda t q(t)]^n}{n!}, \quad n = 0, 1, \ldots, \infty.$$

This is a Poisson distribution with mean

$$\lambda t q(t) = \frac{\lambda}{\mu}(1-e^{-\mu t}). \tag{2.7}$$

Thus, for a single compartment with N individuals present at time 0, and new individuals entering in a Poisson process, the number of individuals present at time t may be written as the sum of two independent random variables: one binomially distributed (2.3) and the other Poisson distributed with mean (2.7).

3. Some generalisations

From (2.2) the probability that an individual in the compartment at time 0 is still there at time t is $e^{-\mu t}$, so if T represents the random amount of time spent by the individual in the compartment, it follows that

$$e^{-\mu t} = P(T > t) = \int_t^{\infty} f(x)\,dx,$$

where $f(x)$ is the probability density function of T. It is clear that $f(t) = \mu e^{-\mu t}$ $(0 < t < \infty)$ and so T is *exponentially* distributed: this is a consequence of the assumption that $\mu\,\delta t$ is the probability of an individual leaving the compartment in *any* 'short' time interval of length δt. An arbitrary probability density function $f(t)$ $(0 < t < \infty)$ to describe the individuals' compartmental residence times may be accommodated by noting that, in general

$$p(t) = \int_t^{\infty} f(x)\,dx \tag{3.1}$$

is the probability that an individual in the compartment at time 0 is still there at time $t > 0$. Hence binomial probabilities (2.3) with $p(t)$ given by (3.1) for the number of individuals in the compartment at time t follow from the assumption of *independent* behaviour of the N individuals initially present. Poisson entry of new individuals may also be accommodated by simply adjusting (2.5) and (2.7) to incorporate $p(t)$ defined by (3.1).

However, entry of new individuals by any process other than the Poisson cannot be accommodated simply, as this would induce some *dependence* among the individuals' conditional entry times and so the argument leading to (2.6) would break down. Incidentally, a consequence of the Poisson entry process with rate λ is that the random times between successive individuals' entries are independent, and exponentially distributed with density function $\lambda e^{-\lambda t}$, so that any departure from this will result in an entry process other than the Poisson.

Introducing more compartments results in the number of individuals initially in department i being *multinomially* distributed across the various compartments j $(= 1, 2, \ldots, m)$ with probabilities

$$p_{ij}(t) = P\{\text{individual in } j \text{ at time } t \mid \text{individual in } i \text{ at time } 0\}, \tag{3.2}$$

instead of the simple binomial distribution (2.3). Poisson entry of individuals at rate λ_i into compartment i $(= 1, 2, \ldots, m)$ gives further numbers of individuals in the compartments at time t following independent Poisson distributions with means

$$\lambda_i \int_0^t p_{ij}(t-s)\,ds,$$

as in (2.5) and (2.7).

If one were to summarise the structure of these models, one would conclude that the key assumption is that of *independent* behaviour of individuals within the compartments. This ties in with *Poisson* entry processes, leading to standard distributional results.

4. A simple example

Most practical examples [Bartholomew (reference 1), Chiang (reference 2)] involve many compartments resulting in quite complicated expressions for the probabilities (3.2), dependent upon many parameters necessary to specify the inter-compartmental movement. A practical example involving a single compartment does arise from a simple experiment, where each individual in a class (compartment) is set some task and the number of individuals who have yet to complete this task is observed at certain fixed times. Table 1 displays some data from a single compartment.

Table 1

Time (minutes)	Observed number of individuals	Mean number of individuals (standard deviation) from binomial distribution (2.3)
0	20	20 (0)
5	12	13.4(2.1)
10	8	9.0(2.2)
15	7	6.0(2.0)
20	4	4.0(1.8)

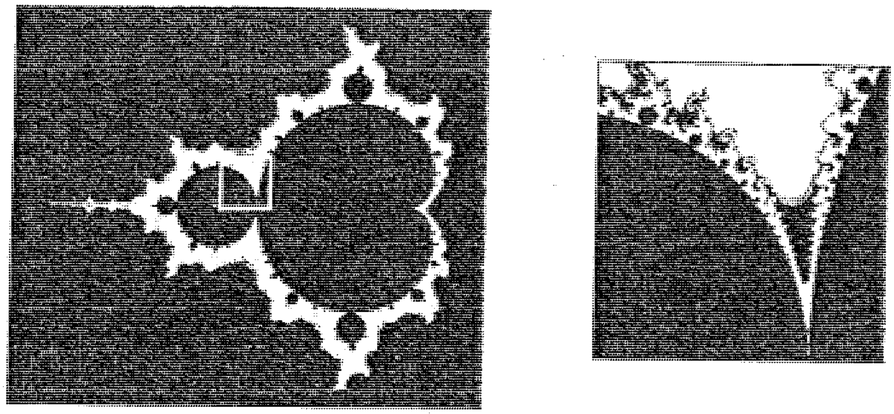
The value of μ used in calculating the probability $p(t)$ from (2.2) was 0.08, obtained from a rough straight line drawn through a plot of the points log(observed number) against time. Close agreement between the observed numbers and the theoretical binomial distribution is shown in table 1. Any substantial differences between these observed numbers and the binomial distribution would be evidence against one or more assumptions of the model: e.g. exponentially distributed compartmental residence times or independent behaviour of the individuals within the compartment.

References

1. D. J. Bartholomew, *Stochastic Models for Social Processes*, 2nd edn (Wiley, London, 1973).
2. C. L. Chiang, *Introduction to Stochastic Processes in Biostastistics* (Wiley, New York, 1968).
3. D. R. Cox and P. A. W. Lewis, *The Statistical Analysis of Series of Events* (Methuen, London, 1966).

Computer Column

MIKE PIFF



The Mandelbrot Set with a close up of 'Sea-Horse Valley'

This program will draw the Mandelbrot set† on a BBC micro, and save it as a file called "PICTURE". To reload it, change to MODE 1 and *LOAD "PICTURE". To obtain better definition, change line 60 to MAXITS% = 100, say. Blow-ups of selected parts can also be obtained by changing line 80, which sets the limits in the x and y directions. Particularly interesting are enlargements of the regions round the points and deep cusps of the Mandelbrot set.

```
 10 MODE1
 20 VDU 23,1,0;0;0;0;
 30 VDU 19,3,10,0,0,0
 40 VDU 19,0,6,0,0,0
 50 CLG
 60 MAXITS%=30
 70 XCRSNESS%=4:YCRSNESS%=4
 80 XLEFT=-2.0:XRIGHT=0.7:YLOWER=-1.5:YUPPER=1.5
 90 MINCOL%=0:MAXCOL%=319:MINROW%=0:MAXROW%=255
100 OFSET=FNOFFSET(XLEFT,XRIGHT,1+MAXCOL%)
110 XLEFT=XLEFT-OFSET:XRIGHT=XRIGHT-OFSET
120 OFSET=FNOFFSET(YLOWER,YUPPER,1+MAXROW%)
130 YLOWER=YLOWER-OFSET:YUPPER=YUPPER-OFSET
140 DIM XCOORD(MAXCOL%),YCOORD(MAXROW%)
150 COLGAP=(XRIGHT-XLEFT)/(MAXCOL%+1):TEMP=XLEFT-COLGAP
160 FOR I%=MINCOL% TO MAXCOL%
170 TEMP=TEMP+COLGAP
180 XCOORD(I%)=TEMP
190 NEXT I%
200 ROWGAP=(YUPPER-YLOWER)/(MAXROW%+1):TEMP=YLOWER-ROWGAP
210 FOR I%=MINROW% TO MAXROW%
220 TEMP=TEMP+ROWGAP
230 YCOORD(I%)=TEMP
240 NEXT I%
250 FOR I%=MINCOL% TO MAXCOL%
260 FOR J%=MINROW% TO MAXROW%
270 GCOL 0,FNOUT(XCOORD(I%),YCOORD(J%),MAXITS%):PLOT69,XCR
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† See *Mathematical Spectrum* Volume 18 Number 3, pages 65–69.

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SNESS%*I%,YCRSNESS%*J%
  280 NEXT J%
  290 NEXT I%
  300 *SAVE "PICTURE" 3000 8000
  310 VDU 23,1,1;0;0;0;
  320 END
  330 DEF FNOUT(X,Y,MAXITS%)
  340 LOCAL I%,TEMP,X1,Y1,HLT
  350 I%=0:X1=0:Y1=0
  360 REPEAT
  370 I%=I%+1:TEMP=(X1-Y1)*(X1+Y1)+X:Y1=2*X1*Y1+Y:X1=TEMP:HL
T=(ABS(X1)+ABS(Y1)>4)
  380 UNTIL (I%=MAXITS%) OR HLT
  390 IF HLT THEN I%=(I% MOD 3) ELSE I%=3
  400 =I%
  410 DEF FNOFFSET(MIN,MAX,GRID%)
  420 LOCAL TEMP
  430 TEMP=GRID%*MIN/(MIN-MAX)
  440 IF (MIN>0)OR(MAX<0) THEN =0 ELSE =TEMP-INT(TEMP)
```

Perfect squares which use the first nine natural numbers

In Volume 18 Number 2 L. B. Dutta gave 13 such perfect squares. Now Guy Willard has sent what he tells us are all 30 such numbers.

$11\,826^2 = 139\,854\,276$	$23\,439^2 = 549\,386\,721$
$12\,363^2 = 152\,843\,769$	$24\,237^2 = 587\,432\,169$
$12\,543^2 = 157\,326\,849$	$24\,276^2 = 589\,324\,176$
$14\,676^2 = 215\,384\,976$	$24\,441^2 = 597\,362\,481$
$15\,681^2 = 245\,893\,761$	$24\,807^2 = 615\,387\,249$
$15\,963^2 = 254\,817\,369$	$25\,059^2 = 627\,953\,481$
$18\,072^2 = 326\,597\,184$	$25\,572^2 = 653\,927\,184$
$19\,023^2 = 361\,874\,529$	$25\,941^2 = 672\,935\,481$
$19\,377^2 = 375\,468\,129$	$26\,409^2 = 697\,435\,281$
$19\,569^2 = 382\,945\,761$	$26\,733^2 = 714\,653\,289$
$19\,629^2 = 385\,297\,641$	$27\,129^2 = 735\,982\,641$
$20\,316^2 = 412\,739\,856$	$27\,273^2 = 743\,816\,529$
$22\,887^2 = 523\,814\,769$	$29\,034^2 = 842\,973\,156$
$23\,019^2 = 529\,874\,361$	$29\,106^2 = 847\,159\,236$
$23\,178^2 = 537\,219\,684$	$30\,384^2 = 923\,187\,456$

Letters to the Editor

Dear Editor,

Sources in recreational mathematics

I have embarked on a project to find the sources of classical problems in recreational mathematics. Some of these problems are quite new, e.g. the twelve coins problem of 1945. Nonetheless, it can be difficult to find their sources. For example, I have no source for the problem of the forty unfaithful wives or the problem of getting a correct answer from someone who is either a liar or a truthteller. Other problems are much more ancient than I had expected, e.g. the monkey and the coconuts problem is a thousand years older than I initially knew. Many of the older problems act as historical markers, showing the transmission of mathematics in time and space. For example, the Chinese remainder theorem and the 100 fowls problem both start in China, then move to India and apparently pass through the Arabs to Alcuin, Fibonacci and medieval Europe. Many of the problems have large gaps in their history, e.g. the geometric progression $1+7+49+\ldots$ occurs in the Papyrus Rhind and in Fibonacci. I have found especial problems in trying to find the connections between the Hindus and medieval Europe.

The initial object of this project was to produce a book of sources, translated into English with annotation, for the series in Recreational Mathematics that I am editing for Oxford University Press. However, it now appears that the first stage must be the preparation of an annotated bibliography of the material. I am putting the information into a computer file, arranged by subject (about 220 subjects so far) and then chronologically within each subject. This is presently 110 pages long. I also have prepared a file of Queries and Problems relating to this material, presently 17 pages long, and I have a draft of a paper which outlines the project and some of the material.

I would be delighted to hear from anyone interested in this project, particularly anyone able to provide information.

I am also compiling a list of mathematical monuments and have a computer file of a draft article on this. I would be happy to hear from anyone who knows of a mathematical monument.

Yours sincerely,

DAVID SINGMASTER
Polytechnic of the South Bank,
London SE1 0AA.

Dear Editor,

Curiosities about prime numbers

I include a number of curiosities which might amuse readers.

(*a*) The first two sets of primes below contain all the Hindu/Arabic numerals just once as well as having the lowest element summation I can find. The last two sets are as above but without zero.

$$\{2,3,5,67,89,410\}, \qquad \{2,5,7,61,83,409\},$$

$$\{2,3,5,41,67,89\}, \qquad \{2,3,5,47,61,89\}.$$

The summation of each of the first two sets is a pleasant 567, with 207 for the last two sets. It is amusing to note that the difference between the sets with and without zero is 360. The element product of the last set is an interesting 7654 890.

(*b*) I looked for primes which did not start with an even number or 5 which when reversed would be higher and composite, and then for some which would be lower and composite. Here are some examples of both types:

(i) $103,\ 301 = 7\times43$, (ii) $331,\ 133 = 7\times19$.

Starting with 331 and 133, we have the magnificence of an 'Aztec style' pyramid:

Prime	Composite	
331	133	$= 7\times19$
3331	1333	$= 31\times43$
33331	13333	$= 67\times199$
333331	133333	$= 151\times883$
3333331	1333333	$= 23\times29\times1999$
33333331	13333333	$= 13\times1025641$

But $333333331 = 17\times19607843$ and $133333333 = 7\times59\times113\times2857$.

For (i) we have an interesting group

$$103,\ 301 = 7\times43, \qquad 109,\ 901 = 17\times53,$$

$$173,\ 371 = 7\times53, \qquad 137,\ 731 = 17\times43,$$

and we notice that $301\times901 = 3.4\times43\times35\times53$.

(*c*) Some little observations involving the first few primes:

$$2\times3 = 1+2+3,$$

$$2\times3\times5\times7 = 1+2+\ldots+20,$$

$$2+3+5 = 1+2+3+4,$$

$$2+3+5+7+11 = 1+2+\ldots+7.$$

We notice that the first and the last are perfect numbers, i.e. each one is equal to the sum of its positive divisors (excluding the number itself).

Yours sincerely,

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Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

19.1. (Submitted by J. N. MacNeill, Royal Wolverhampton School)
A prime pair is a pair of prime numbers which differ by 2, e.g. 11 and 13. If p, q is a prime pair, what happens when p^2+q^2 is divided by 72 and why?

19.2. (Submitted by Ruth Lawrence, St. Hugh's College, Oxford)
A bag contains n balls, labelled 1 to n, and r balls are picked at random from the bag, one by one, without replacement. What is the probability that the numbers of the balls picked form an arithmetic progression (a) in the order in which they are picked and (b) after a possible rearrangement (e.g. 5, 1, 13, 9 when $r = 4$).

19.3. (Submitted by Peter Ivády, Budapest)
Let $0 < x \leqslant \pi$. Show that

$$\frac{2+\cos x}{3} \leqslant \frac{2(1-\cos x)}{x^2}.$$

Solutions to Problems in Volume 18, Number 2

18.4. Expressed in Roman digits, the number IV ($=4$) increases by $\frac{1}{2}$ when the right-hand digit is moved to the front, i.e. giving VI ($=6$). Find a number which does this when expressed in the so-called Arabic numerals (i.e. $1, 2, 3, \ldots$).

Solution by John Catlow (Chatham House School, Ramsgate)

Let the number be n, and denote by k the number of digits of n. Also, let the right-hand digit be b and let the residue after removing this digit be A. Then

$$n = 10A+b.$$

When we remove b to the front, we obtain

$$\tfrac{3}{2}n = 10^{k-1}b+A,$$

so that

$$3(10A+b) = 2(10^{k-1}b+A),$$
$$28A = (2\times 10^{k-1}-3)b.$$

The left-hand side has a factor 4 and $2\times 10^{k-1}-3$ is odd, so 4 must divide b. Since $1 \leqslant b \leqslant 9$, this means that either $b = 4$ or $b = 8$. Also the left-hand side is divisible by 7, so that $7|(2\times 10^{k-1}-3)$. We therefore look for a number of the form $2\times 10^{k-1}-3$, i.e. 17, 197, 1997, etc., with a factor of 7. The lowest such number is 199 997. This gives $A = 28\,571$ when $b = 4$ and $A = 57\,142$ when $b = 8$. Thus two numbers with the desired property are 285 714 and 571 428.

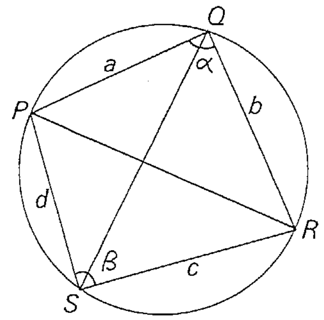
Also solved by Ruth Lawrence (St. Hugh's College, Oxford), Bob Bertuello (The Open University), who also points out that $428\,571\times\frac{1}{3} = 142\,857$ and $714\,285\times 0.8 = 571\,428$, Ashley Brooks (University of Warwick), Richard Wheatley (St. John's

College, Cambridge), Magnus Olsson (University of Lund, Sweden), Guy Willard (The Haberdashers' Aske's School, Elstree), Jean Corriveau (Brandon University, Canada), Imre Merényi (Babes-Bolyai University, Romania) and László Cseh (Babes-Bolyai University, Romania).

18.5. Determine the lengths of the diagonals of a cyclic quadrilateral in terms of the lengths of its sides.

Solution by John Catlow (Chatham House School, Ramsgate)

By the cosine formula



$PR = x$

$QS = y$

$$x^2 = a^2+b^2-2ab\cos\alpha = c^2+d^2-2cd\cos\beta.$$

But $PQRS$ is cyclic, so that $\alpha+\beta = 180°$ and $\cos\alpha = -\cos\beta$. Hence

$$cdx^2 = cd(a^2+b^2)-2abcd\cos\alpha,$$
$$abx^2 = ab(c^2+d^2)+2abcd\cos\alpha,$$

whence

$$\begin{aligned}(ab+cd)x^2 &= ab(c^2+d^2)+cd(a^2+b^2)\\ &= (ad+bc)(ac+bd),\end{aligned}$$

which gives

$$x = \sqrt{\frac{(ad+bc)(ac+bd)}{ab+cd}}.$$

Similarly,

$$y = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}.$$

Also solved by Philip Wadey (University of Exeter), Ruth Lawrence (St. Hugh's College, Oxford), Ashley Brooks (University of Warwick), Guy Willard (The Haberdashers' Aske's School, Elstree), Robert Haynes (Hautlieu School, Jersey) and Imre Merényi (Babes-Bolyai University, Romania).

18.6. Determine the probability that two whole numbers chosen at random have highest common factor 1. (You may need to use the fact that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^3}+\ldots = \frac{\pi^2}{6}.)$$

Solution by Ruth Lawrence (St. Hugh's College, Oxford)

Let p be the required probability. Given any two whole numbers, m and n, they have a highest common factor, d, say. Thus

$$
\begin{aligned}
1 &= \sum_{d=1}^{\infty} (\text{probability that two whole numbers chosen at random have h.c.f. } d) \\
&= \sum_{d=1}^{\infty} P[m,n \text{ chosen at random are (i) both divisible by } d, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(ii) } m/d, n/d \text{ have h.c.f. } 1] \\
&= \sum_{d=1}^{\infty} \frac{1}{d^2}p \quad \begin{array}{l}\text{(because the probability that a number} \\ \text{chosen at random is divisible by } d \text{ is } 1/d).\end{array} \\
&= \tfrac{1}{6}\pi^2 p.
\end{aligned}
$$

Hence $p = 6/\pi^2$ (about 0.607 93).

Also solved by Ashley Brooks (University of Warwick), Guy Willard (The Haberdashers' Aske's School, Elstree) and László Cseh (Babes-Bolyai University, Romania).

Book Review

The Psychology of Learning Mathematics By RICHARD R. SKEMP. Penguin Books, Second Edition 1986. Pp. 295. £4.95 paperback.

On page 132 we read: 'This [referring to a critical examination of some of our ideas about mathematics] can take us from a stage of simplicity (in which we do things without thinking about why they work) to one of doubt (in which we become aware of much that we do not understand) before we emerge again into a new state of simplicity resulting from comprehension.' I find this to be an admirably succinct description of the stages which the student of mathematics passes through, and I remember, in my own experience, especially the uncomfortable middle stage of doubt. If you recognise this pattern in your own mathematical progress and are just now experiencing some fundamental doubts and uncertainties, then I think that this book will be of help to you. The author is both a progressive mathematician and a professional psychologist, and he sets out to examine the mental activities involved in learning mathematics; an appreciation of which is of course especially important for teachers of mathematics— and maybe Part A is more for them than for the general student. In Part B the ideas of Part A are applied to basic topics of mathematics, and it is here that you will perhaps find most of interest. The author pertinently remarks in Chapter 1 '... an answer has more meaning to someone who has first asked a question'—as any good teacher surely knows! If you have been asking basic questions about, for instance, numbers and the validity of the (familiar but not obvious) rules of arithmetic, you will find some of them answered here, and there is much that will set you thinking further.

This is a second edition of Professor Skemp's book, making its appearance fifteen years after the first edition and after many reprints. I recommend it as good reading between, say, sixth form and university.

University of Sheffield — HAZEL PERFECT

CONTENTS

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