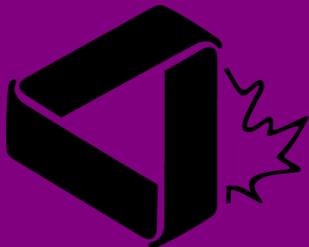


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This issue is dedicated

to the memory of

Frederick G.B. Maskell

THE OLYMPIAD CORNER: 61

M.S. KLAMKIN

I start my column this month with a list of 24 problems proposed at various 1984 Olympiads in the Soviet Union, for which I thank the sender, who forgot to identify himself. I solicit from readers elegant solutions to all of these problems.

1. *From the Leningrad 1984 Olympiad.*

Diametrically opposite points A and B are chosen on a circle which touches the sides of an angle with vertex O (neither A nor B being a point of tangency). The tangent to the circle at B intersects the sides of the angle in C and D and the line OA in E. Prove that the segments BC and DE are equal in length.

2. *From the Leningrad 1984 Olympiad (proposed by S.V. Fomin).*

From a sheet of squared paper measuring 29×29 , 99 squares, each of which consists of 4 unit squares, have been cut out. Prove that one more 2×2 square may be cut out.

3. *From the 1984 Leningrad Olympiad (proposed by Yu. I. Ionin and A.V. Smirnov).*

Prove that if the sum of the plane angles at the summit of a pyramid is more than 180° , then each lateral edge of the pyramid is shorter than half the perimeter of its base.

4. *From the 1984 Leningrad Olympiad (proposed by S.V. Fomin).*

The five integers a, b, c, d, e are chosen so that their sum and the sum of their squares are both divisible by the odd number p . Prove that

$$a^5 + b^5 + c^5 + d^5 + e^5 - 5abcde$$

is also divisible by p .

5. *From the 1984 Leningrad Olympiad (proposed by A.S. Merkuriev).*

In the sequence $1, 0, 1, 0, 1, 0, \dots$, each term beginning with the seventh equals the last digit of the sum of the previous six terms. Prove that the numbers $0, 1, 0, 1, 0, 1$ do not appear in that order in the sequence.

6. *From the 1984 Leningrad Olympiad (proposed by S.E. Rushkin).*

Two students take turns in writing their choice of one of the signs $+$, $-$, or \times between each pair of consecutive integers in the series

$$1 \ 2 \ 3 \ \dots \ 100.$$

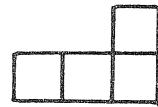
Prove that there is a strategy such that the student who wrote the first (and hence the last) sign can ensure that, after the ninety-nine signs have been filled in, the arithmetical result is (a) odd, (b) even.

7. From the 1984 Leningrad Olympiad (proposed by A.S. Merkuriev).

Prove that the integer a can be represented in the form x^2+2y^2 , where x and y are integers, if $3a$ can be represented in the same form.

8. From the 1984 Leningrad Olympiad (proposed by I.V. Itenberg).

Prove that at least 8 different colours are needed to paint the (infinite) squared plane so that the 4 squares of each L-shaped tetromino (see figure) have 4 different colours.



9. From the 1984 Tournament of Towns Olympiad (proposed by A.B. Pechkovski).

In a ballroom dance class 17 boys and 17 girls are lined up in parallel rows so that 17 couples are formed. It so happens that the difference in height between the boy and the girl in each couple is not more than 10 cm. Prove that if the boys and the girls were placed in each line in order of decreasing height, then the difference in height in each of the newly formed couples would still be at most 10 cm.

10. From the 1984 Tournament of Towns Olympiad (proposed by I.F. Sharygin).

Six altitudes are constructed from the three vertices of the base of a tetrahedron to the opposite sides of the three lateral faces. Prove that all three straight lines joining two base points of the altitudes in each lateral face are parallel to a certain plane.

11. From the 1984 Tournament of Towns Olympiad (proposed by A.V. Andjans).

The two pairs of consecutive natural numbers (8,9) and (288,289) have the following property: in each pair, each number contains each of its prime factors to a power not less than 2.

(a) Find one more pair of consecutive numbers with that property.

(b) Prove that there are infinitely many such pairs.

12. From the 1984 Tournament of Towns Olympiad (proposed by V.G. Il'ichev).

An infinite (in both directions) sequence of rooms is situated on one side of an infinite hallway. The rooms are numbered by consecutive integers and each contains a grand piano. A finite number of pianists live in these rooms. (There may be more than one of them in some of the rooms.) Every day some two pianists living in adjacent rooms (the k th and $(k+1)$ st) decide that they interfere with each other's practice, and they move to the $(k-1)$ st and $(k+2)$ nd rooms, respectively. Prove that these moves will cease after a finite number of days.

13. From the 1984 Tournament of Towns Olympiad (proposed by A.V. Zelevinsky).

Let $p(n)$ be the number of partitions of the natural number n into natural summands. The diversity of a partition is by definition the number of different

summands in it. Denote by $q(n)$ the sum of the diversities of all the $p(n)$ partitions of n .

(For example, $p(4) = 5$, the five distinct partitions of 4 being

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1;$$

and $q(4) = 1+2+1+2+1 = 7.$)

Prove that, for all natural numbers n ,

(a) $q(n) = 1 + p(1) + p(2) + p(3) + \dots + p(n-1),$

(b) $q(n) \leq \sqrt{2n} \cdot p(n).$

14. From the 1984 Tournament of Towns Olympiad (proposed by A.V. Andjans).

Prove that, for any natural number n , the graph of any increasing function $f: [0,1] \rightarrow [0,1]$ can be covered by n rectangles each of area $1/n^2$ whose sides are parallel to the coordinate axes.

15. From the 1984 Moscow Olympiad.

Let $X = (x_1, x_2, \dots, x_n)$ be a sequence of n nonnegative numbers whose sum is 1.

(a) Prove that $x_1x_2 + x_2x_3 + \dots + x_nx_1 \leq 1/4,$

(b) Prove that there exists a permutation $Y = (y_1, y_2, \dots, y_n)$ of X such that

$$y_1y_2 + y_2y_3 + \dots + y_ny_1 \leq 1/n.$$

16. From the 1984 Moscow Olympiad.

Prove that the area of the triangle which is the intersection of a cube of edge 1 with an arbitrary plane tangent to the sphere inscribed in the cube does not exceed $1/2$.

17. From the 1984 all-Union Olympiad (proposed by A.A. Fomin).

(a) The product of n integers is equal to n , and their sum is 0. Prove that n is divisible by 4.

(b) Prove that, for every natural number n divisible by 4, there exist n integers with product n and sum 0.

18. From the 1984 All-Union Olympiad (proposed by N. Agakharov).

Two boys paint the twelve edges of a white cube in two colours: red and green. The first chooses any three edges and paints them red; the second chooses some other three edges and paints them green; then, once more, the first paints three edges red, and, finally, the second paints the last three edges green. Can the first boy make all four edges of a face red?

19. From the 1984 All-Union Olympiad (proposed by A.B. Bolotov).

Find $xy + 2yz + 3zx$ if x, y, z are positive numbers for which

$$x^2 + xy^2 + \frac{y^2}{3} = 25, \quad \frac{y^2}{3} + z^2 = 9, \quad \text{and} \quad z^2 + zx + x^2 = 16.$$

20. From the 1984 All-Union Olympiad (proposed by I.K. Zhik and I.V. Voronovich).

The numbers +1 or -1 are written in each of the nine unit squares of a 3×3 sheet of squared paper. Then the number in each unit square is replaced by the product of the numbers of its neighbours (two unit squares are *neighbours* if they have a common side). Prove that after a finite number of such operations each unit square will contain a +1.

21. From the 1984 All-Union Olympiad (proposed by L.P. Kuptsov).

Three circles C_1, C_2, C_3 of radii r_1, r_2, r_3 in the plane are exterior to one another and $r_1 > r_2, r_1 > r_3$. The two tangents to C_3 are drawn from the intersection point of the exterior tangents to C_1 and C_2 , and the two tangents to C_2 from the intersection point of the exterior tangents to C_1 and C_3 . Prove that these two pairs of tangents (to C_3 and C_2) form a quadrilateral into which a circle can be inscribed. Find its radius.

22. From the 1984 All-Union Olympiad (proposed by A.V. Andjans).

A math teacher wrote the quadratic trinomial $x^2 + 10x + 20$ on the blackboard. Then each student in turn either increased by 1 or decreased by 1 either the constant term or the coefficient of x . Finally the trinomial $x^2 + 20x + 10$ appeared. Will a quadratic trinomial with integer roots necessarily appear on the blackboard in the process?

23. From the 1984 All-Union Olympiad (proposed by Yu. Mikheyev).

For what integers m and n do we have

$$(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n \quad \text{and} \quad (a + b\sqrt{d})^m = (b + a\sqrt{d})^n,$$

where a and b are relatively prime integers, while d is a natural number not divisible by the square of a prime?

24. From the 1984 All-Union Olympiad (proposed by O.R. Musin).

$n > 3$ natural numbers are written around a circle so that the ratio of the sum of the two neighbours of any number to the number itself is a natural number. Prove that the sum of all such ratios is

- (a) not less than $2n$,
- (b) not more than $3n$.

Finally, I present solutions to the problems set at the Annual Michigan Mathematics Prize Competitions for 1979, 1980, and 1981, which I published earlier in this column [1984: 41-43].

1. [1984: 41] From the 1979 Michigan Mathematics Prize Competition.

Solve for x and y , if

$$\frac{1}{x^2} + \frac{1}{xy} = \frac{1}{9} \quad \text{and} \quad \frac{1}{y^2} + \frac{1}{xy} = \frac{1}{16}.$$

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

Adding and subtracting the two equations give

$$\left(\frac{1}{x} + \frac{1}{y}\right)^2 = \frac{25}{144} \quad \text{and} \quad \left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{7}{144}.$$

Thus

$$\frac{1}{x} + \frac{1}{y} = \pm \frac{5}{12} \quad \text{and} \quad \frac{1}{x} - \frac{1}{y} = \pm \frac{7}{60},$$

from which we easily find that $(x, y) = (\frac{15}{4}, \frac{20}{3})$ or $(-\frac{15}{4}, -\frac{20}{3})$.

*

2. [1984: 41] From the 1979 Michigan Mathematics Prize Competition.

Find positive integers p and q , with q as small as possible, such that

$$\frac{7}{10} < \frac{p}{q} < \frac{11}{15}.$$

Solution by Paul Wagner, Chicago, Illinois.

From the hypotheses, $11q-1 \geq 15p$ and $10p \geq 7q+1$. Thus

$$11q - 1 \geq \frac{3}{2}(7q + 1), \quad \text{or} \quad q \geq 5.$$

By inspection, $q = 5$ and 6 are not suitable, but $q = 7$ is, and correspondingly $p = 5$.

*

3. [1984: 41] From the 1979 Michigan Mathematics Prize Competition.

Define $\alpha_1 = 2$ and $\alpha_{n+1} = \alpha_n^2 - \alpha_n + 1$ for all positive integers n . If $i \neq j$, prove that α_i and α_j have no common prime factor.

Solution by Florentin Smarandache, Lycée Sidi El Hassan Lyoussi, Sefrou, Maroc.

Let $Q(x) = x^2 - x + 1$. If $1 \leq i < j$, we have

$$\alpha_j = Q^{(j-i)}(\alpha_i),$$

where the parenthesized exponent indicates iteration, e.g., $Q^{(2)}(x) = Q(Q(x))$.

If a prime p divides α_i , then (all congruences being modulo p)

$$\alpha_{i+1} = Q(\alpha_i) \equiv 1,$$

$$\alpha_{i+2} = Q(\alpha_{i+1}) \equiv Q(1) = 1,$$

and, by induction,

$$\alpha_j = Q(\alpha_{j-1}) \equiv Q(1) = 1.$$

Therefore α_i and α_j have no common prime factor.

Comment by M.S.K.

More generally, α_1 could be any integer and $Q(x)$ any polynomial with integral coefficients such that $Q(0) = Q(1) = 1$.

*

4, [1984: 41] *From the 1979 Michigan Mathematics Prize Competition.*

A number of points are given in the interior of a triangle. Connect these points, as well as the vertices of the triangle, by segments that do not cross one another until the interior is subdivided into smaller disjoint regions that are all triangles. It is required that each of the given points be always a vertex of any triangle containing it. Prove that the number of these smaller triangular regions is always odd.

Solution by M.S.K.

Let n be the number of smaller triangles and i the number of interior points. We determine in two ways the sum of the angles of the n smaller triangles. First, this sum is $n\pi$. Next, the sum of the angles at each of the i interior points is 2π , and the sum of the remaining angles at the three boundary points is π . Thus

$$n\pi = 2\pi i + \pi, \quad \text{or} \quad n = 2i + 1. \quad \square$$

A similar proof (see [1]) shows that, if there are i points inside a convex m -gon, then the number of smaller triangles is $n = 2i + m - 2$. See [2] for a topological proof and connections with Euler's formula and Pick's theorem.

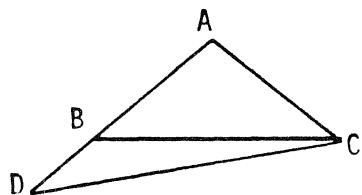
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2. R.M. Gaskell, M.S. Klamkin, and P. Watson, "Triangulations and Pick's Theorem", *Mathematics Magazine*, 49 (1976) 35-37.

*

5, [1984: 41] *From the 1979 Michigan Mathematics Prize Competition.*

$\triangle ABC$ is an isosceles triangle with $\angle B = \angle C = 40^\circ$. AB is extended to D so that $AD = BC$ (see figure). Prove that $\angle BCD = 10^\circ$.



I. Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Let $\angle BCD = \theta$. It is clear that $0 < \theta < 40^\circ$. Letting $AB = 1$, we have $BC = AD = 2 \sin 50^\circ$. Then, by the law of sines,

$$\frac{AD}{\sin(40^\circ + \theta)} = \frac{1}{\sin(40^\circ - \theta)}.$$

Thus $2 \sin 50^\circ \sin(40^\circ - \theta) = \sin(40^\circ + \theta)$. This equation is satisfied for $\theta = 10^\circ$, and this is the only solution since $\sin(40^\circ - \theta)$ decreases while $\sin(40^\circ + \theta)$ increases with θ in the interval $0 < \theta < 40^\circ$.

II. Comment by Dimitris Vathis, Chalcis, Greece.

This is Problem 877 proposed by H.J. Webb in the September 1973 issue of *Mathematics Magazine*. A synthetic solution by J.W. Wilson appeared on page 172 of the May 1974 issue.

*

1. [1984: 41] From the 1980 Michigan Mathematics Prize Competition.

On an escalator moving at constant speed a woman in a hurry walks up 9 steps as she travels from one floor to the next higher. A man in an even greater hurry runs 25 steps up the same escalator and reaches the top in half the time the woman took. How many steps does the escalator have between the two floors?

Solution by John Morvay, Dallas, Texas.

Let n be the number of steps on the escalator between floors. If the man travels the same length of time as the woman, he will cover two intervals between consecutive floors. Hence $n-9 = 2(n-25)$, so $n = 41$.

*

2. [1984: 41] From the 1980 Michigan Mathematics Prize Competition.

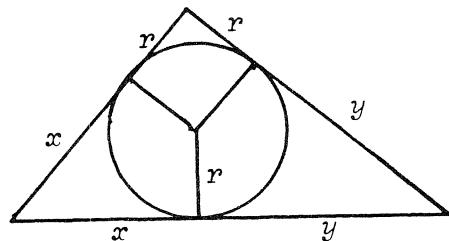
A circle inscribed in a right triangle divides the hypotenuse at its point of contact into segments of lengths x and y . Find, in terms of x and y ,

- (a) the area of the right triangle,
- (b) the diameter of the circle,

I. Solution by Paul Wagner, Chicago, Illinois.

Let A and r be the area and inradius, respectively. It is clear from the figure that

$$2A = (x+r)(y+r) = r\{(r+x) + (x+y) + (y+r)\},$$



from which follows

$$r^2 + (x+y)r - xy = 0.$$

Thus

$$(b) \quad 2r = -(x+y) + \sqrt{x^2+6xy+y^2},$$

$$(a) \quad A = \frac{1}{2}(x+r)(y+r) = \frac{1}{2}\{(r^2+(x+y)r-xy) + 2xy\} = xy.$$

II. *Comment by Dimitris Vathis, Chalcis, Greece.*

Part (a) of this problem is well known but few people are aware that it goes back to Archimedes. This result appears in Propositions 10,11,12 in his essay "On the regular heptagon in a circle". (See Carl Schöy, *Die Trigonometrischen Lehren des Persischen Astronomen Al-Biruni*, ed. J.P.H. Wieleitner, Hannover, 1927, Kapitel I and IV, s. 74-83.) E. Stamatis asserts that Propositions 10,11,12 as well as Theorems 1 to 13 of this essay are part of the lost book of Archimedes entitled "On the right triangles". (See his four-volume work in Greek, *The Works of Archimedes*, Athens, Greece, 1974.)

*

3. [1984: 42] From the 1980 Michigan Mathematics Prize Competition.

Solve the system

$$\frac{1}{x^2} - \frac{1}{xy} = 30, \quad \frac{1}{y^2} + \frac{1}{xy} = 28.$$

Solution by Gali Salvatore, Perkins, Québec.

With $u = 1/x$ and $v = 1/y$, the system becomes

$$\begin{cases} u^2 - uv = 30 \\ v^2 + uv = 28 \end{cases}$$

and eliminating the constants yields

$$14u^2 - 29uv - 15v^2 \equiv (2u - 5v)(7u + 3v) = 0.$$

From $v = 2u/5$ and $v = -7u/3$ we obtain, respectively,

$$(u, v) = (\pm 5\sqrt{2}, \pm 2\sqrt{2}) \quad \text{and} \quad (u, v) = (\pm 3, \mp 7),$$

with similarly placed signs corresponding. Finally, the solutions (x,y) are obtained by taking reciprocals.

*

4. [1984: 42] From the 1980 Michigan Mathematics Prize Competition.

Consider the infinite sequence of positive integers

$$49, 4489, 444889, \dots,$$

in which each number after the first is obtained by inserting the digits 4 and 8 (in that order) into the middle of the preceding number. Prove that all these numbers are perfect squares.

Comment by M.S.K.

This problem appears on page 114 of Ross Honsberger's *Mathematical Morsels* (M.A.A., 1978) with a solution by Ivan Niven and (independently) by Brian Lapcevic.

*

5, [1984: 42] *From the 1980 Michigan Mathematics Prize Competition.*

The sequence {100, 55, 45, 10, 35} has five terms, and each term starting with the third is the difference of the preceding two. A sequence terminates when the next term would be negative (since $10 - 35 = -25$, the above example terminates with 35). Zero terms are permitted. Find a positive integer B such that the sequence {100, B , ...} formed as indicated has the maximum number of terms. Generalize by showing how to find a positive integer B that will maximize the length of the sequence { A , B , ...} formed as above, where A is any given positive integer.

Solution by M.S.K.

Let the sequence {100, B , ..., y , x } be of maximal length and contain n terms. Then $0 \leq y < x$, the inverted sequence is

$$\{x, y, x+y, x+2y, x+3y, \dots, B, 100\},$$

and an easy induction shows that the k th term of the inverted sequence is

$$xf(k-2) + yf(k-1), \quad k = 1, 2, \dots, n,$$

where $\{f(j)\}$ is the Fibonacci sequence

$$f(-1) = 1, \quad f(0) = 0, \quad f(j) = f(j-1) + f(j-2), \quad j = 1, 2, 3, \dots.$$

Thus

$$100 = xf(n-2) + yf(n-1) \tag{1}$$

and

$$B = xf(n-3) + yf(n-2). \tag{2}$$

If we solve (1) in positive integers x and y with the argument n as large as possible, then the required value of B will follow from (2). (The method fails if the first term of the sequence is some $f(n)$. But then we take $B = f(n-1)$ and the maximal sequence has $n+2$ terms.)

The largest Fibonacci number not exceeding 100 is $f(11) = 89$, and the largest not exceeding $100-89$ is $f(6) = 8$, so the best representation of 100 is

$$100 = f(11) + f(6) + f(4).$$

To reduce this to the form (1), we note that

$$f(11) = f(10) + f(9) = 2f(9) + f(8) = 3f(8) + 2f(7) = 5f(7) + 3f(6),$$

50

$$\begin{aligned} 100 &= 5f(7) + 4f(6) + f(4) = 4f(7) + 5f(6) + f(5) + f(4) \\ &= 6f(6) + 4f(7). \end{aligned} \quad (3)$$

It follows from (1) and (3) that the sequence of maximal length contains 8 terms, and from (2) that it occurs for

$$B = 6f(5) + 4f(6) = 6 \cdot 5 + 4 \cdot 8 = 62.$$

More generally, for the sequence starting with A, B , the next few terms are

$$A - B, \quad -A + 2B, \quad 2A - 3B, \quad -3A + 5B.$$

If we label this sequence $\{u_{-1}, u_0, u_1, \dots\}$, then, inductively,

$$u_n = (-1)^{n+1}f(n)A + (-1)^nf(n+1)B, \quad n = -1, 0, 1, \dots.$$

Given A , the problem is to determine the B which maximizes the n for which $u_{n-1} \geq 0$, $u_n > 0$, and $u_{n+1} < 0$, that is, for which

$$\begin{aligned} (-1)^nf(n-1)A + (-1)^{n-1}f(n)B &\geq 0, \\ (-1)^{n+1}f(n)A + (-1)^nf(n+1)B &> 0, \\ (-1)^nf(n+1)A + (-1)^{n-1}f(n+2)B &< 0. \end{aligned}$$

For $n \geq 1$, these inequalities are equivalent to

$$(-1)^ng(n-1)A + (-1)^{n-1}B \geq 0, \quad (4)$$

$$(-1)^{n+1}g(n)A + (-1)^nB > 0, \quad (5)$$

$$(-1)^ng(n+1)A + (-1)^{n-1}B < 0, \quad (6)$$

where $g(n) = f(n)/f(n+1)$. It is known that the sequence $\{g(2m)\}$ is strictly increasing, the sequence $\{g(2m-1)\}$ is strictly decreasing, and both converge to the limit $\varphi = (\sqrt{5}-1)/2$.

For $n = 2m$, (4) and (6) yield

$$g(2m-1)A \geq B > g(2m+1)A > \varphi A, \quad (7)$$

and (5) is then automatically satisfied; and for $n = 2m-1$, (4) and (6) yield

$$g(2m-2)A \leq B < g(2m)A < \varphi A, \quad (8)$$

and again (5) is automatically satisfied. It is clear from (7) and (8) that the B which maximizes the n satisfying (4)-(6) is either

$$B = [\varphi A] + 1 \quad \text{or} \quad B = [\varphi A],$$

where the square brackets denote the greatest integer function.

In particular, for $A = 100$ we get

$$B = [\lceil A \rceil] + 1 = 62 \rightarrow \{100, 62, 38, 24, 14, 10, 4, 6\} \rightarrow n = 8,$$

$$B = [\lceil A \rceil] = 61 \rightarrow \{100, 61, 39, 22, 17, 5, 12\} \rightarrow n = 7,$$

so we must take $B = [\lceil A \rceil] + 1$.

But for $A = 99$ we get

$$B = [\lceil A \rceil] + 1 = 62 \rightarrow \{99, 62, 37, 25, 12, 13\} \rightarrow n = 6,$$

$$B = [\lceil A \rceil] = 61 \rightarrow \{99, 61, 38, 23, 15, 8, 7, 1, 6\} \rightarrow n = 9,$$

and here we must take $B = [\lceil A \rceil]$.

*

1. [1984: 42] From the 1981 Michigan Mathematics Prize Competition.

A canoeist is paddling upstream in a river when she passes a log floating downstream. She continues upstream for a while, paddling at a constant rate. She then turns around and goes downstream paddling twice as fast. She catches up to the same log two hours after she first passed it. How long did she paddle upstream?

Solution by H. Kaye, Brooklyn, N.Y.

Relative to an observer on the moving log, we can ignore the velocity of the stream. Then, if t is the time of paddling upstream after passing the log, we have $t = 2(2 - t)$, and so $t = 4/3$ hours.

*

2. [1984: 42] From the 1981 Michigan Mathematics Prize Competition.

Let $g(x) = 1 - \frac{1}{x}$ and define

$$g_1(x) = g(x), \quad g_{n+1}(x) = g(g_n(x)), \quad n = 1, 2, 3, \dots.$$

Evaluate $g_3(3)$ and $g_{1982}(1982)$.

Solution by H. Kaye, Brooklyn, N.Y.

The sequence $\{g_n(x)\}$ is periodic with period 3 and

$$g_1(x) = 1 - \frac{1}{x}, \quad g_2(x) = -\frac{1}{x-1}, \quad g_3(x) = x.$$

Thus

$$g_3(3) = 3, \quad \text{and} \quad g_{1982}(1982) = g_2(1982) = -\frac{1}{1981}.$$

*

3. [1984: 42] From the 1981 Michigan Mathematics Prize Competition.

Let Q denote a quadrilateral $ABCD$ whose diagonals AC and BD intersect. If each diagonal bisects the area of Q , prove that Q must be a parallelogram.

Solution by Paul Wagner, Chicago, Illinois.

Let the diagonals intersect at P. It is then immediate that $[ABP] = [CDP]$ and $[DAP] = [BCP]$, where the square brackets denote area. Hence

$$AP \cdot BP = CP \cdot DP \quad \text{and} \quad DP \cdot AP = BP \cdot CP.$$

From this follows $AP = CP$, $BP = DP$, and these imply that ABCD is a parallelogram.

*

4. [1984: 42] *From the 1981 Michigan Mathematics Prize Competition.*

Given that (a_1, a_2, \dots, a_7) and (b_1, b_2, \dots, b_7) are two permutations of the same seven integers, prove that the product

$$(a_1 - b_1)(a_2 - b_2) \dots (a_7 - b_7)$$

is always even.

Solution by Florentin Smarandache, Lycée Sidi El Hassan Lyoussi, Sefrou, Maroc.

More generally, let $\alpha = \{a_1, a_2, \dots, a_{2n+1}\}$ be a sequence of integers, and let $\beta = \{b_1, b_2, \dots, b_{2n+1}\}$ be a permutation of α . We show that the following product is always even:

$$P = (a_1 + \varepsilon_1 b_1)(a_2 + \varepsilon_2 b_2) \dots (a_{2n+1} + \varepsilon_{2n+1} b_{2n+1}),$$

where each ε_i is an odd integer.

Suppose, on the contrary, that P is odd. Then each factor of P is odd and each contains one odd and one even summand. Corresponding to each even [odd] a_i , there is therefore an odd [even] $a_j = b_i$. Hence α contains the same number of even and odd integers. This contradicts the fact that α contains $2n+1$ integers. Therefore P is even. \square

This generalization appears [but with the ε_i restricted to ± 1 (M.S.K.)] with solution in my book [1], where it is stated that it generalizes a special case that appeared in [2].

REFERENCES

1. Florentin Smarandache, *Problèmes avec et sans... Problèmes*, Somipress, Fès, Maroc, 1983, p. 22.
2. D. Gerll et G. Girard, *Les olympiades internationales de mathématique*, Hachette, Paris, 1976, p. 105.

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5. [1984: 43] *From the 1981 Michigan Mathematics Prize Competition.*

In analyzing the pecking order in a finite flock of chickens, we observe

that for any two chickens exactly one pecks the other. We decide to call chicken K a *king* provided that, for any other chicken X , K pecks X or K pecks a third chicken Y who in turn pecks X . Prove that every such flock of chickens has at least one king. Must the king be unique?

Comment by M.S.K.

Theorem 1 in [1] gives a very simple proof that every flock of chickens has at least one king and, in a corollary, that there may be more than one king. This reference also contains a flock of other interesting results about pecking orders, as well as an extensive list of references to the relevant literature.

Equivalent results for tournaments of at least three players, where player X "pecks" player Y by beating him or her (or by calling Y "chicken" if he or she is afraid to play), can be found in [2].

REFERENCES

1. Stephen B. Maurer, "The King Chicken Theorems", *Mathematics Magazine*, 53 (1980) 67-80.
2. S. Gudder, *A Mathematical Journey*, McGraw-Hill, New York, 1976, pp. 273-275.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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OUR BELOVED FRED IS GONE

Frederick G.B. Maskell, M.A. (Oxon), Lt. Col. Royal Canadian Engineers (ret.), Professor at Algonquin College (ret.), Managing Editor of *Crux Mathematicorum* (ret.), died of cancer, in his 71st year, at home with his family, on January 28, 1985. He was the beloved husband of Monica and dear father of Mary, John, and Elizabeth. His funeral service was held at St. Mark's Anglican Church, in Ottawa, on January 31, 1985.

There are no words to express adequately our sense of loss.

A letter from his son John appears on the last page of this issue.

PROBLEMS -- PROBLÈMES

Problmem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should preferably be mailed to the editor before June 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

1001. Proposed by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

In the following exact cryptarithmetic division, each X can be any of the decimal digits except the single digit represented by S. Restore the digits.

$$\begin{array}{r} \text{XSXX} \\ \text{XSX)XXXSXXXS} \\ \text{XSX} \\ \hline \text{XXXX} \\ \text{XXXS} \\ \hline \text{XXXX} \\ \text{XSX} \\ \hline \text{XXXS} \\ \text{XXXS} \end{array}$$

1002. Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.

Let m and n be given natural numbers, where $m \leq n$. Evaluate the sum

$$\sum_{j=1}^m \frac{\binom{m}{j}}{\binom{n}{j}} \cdot j.$$

- 1003*. Proposed by M.S. Klamkin, University of Alberta.

Without using tables or a calculator, show that

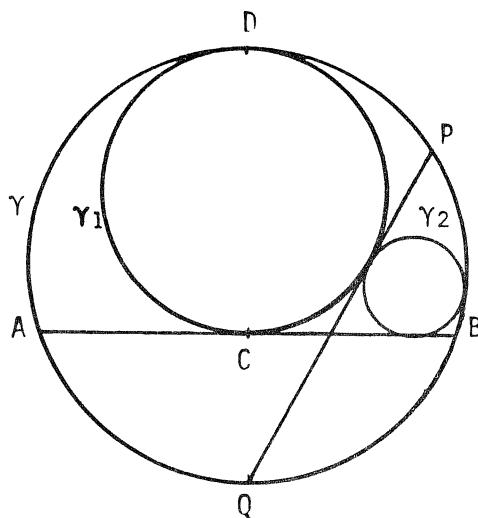
$$\ln 2 > \left(\frac{2}{5}\right)^{2/5}.$$

1004. Proposed by O. Bottema, Delft, The Netherlands.

There exists a right triangle with perimeter p and area F if and only if the positive numbers p and F satisfy what condition?

1005. Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.

A chord AB divides a circle γ into two segments. A circle γ_1 , of radius r_1 , is inscribed in one of the segments, tangent to AB at its midpoint C and to the arc at D, as shown in the figure. A circle γ_2 , of radius r_2 , is then inscribed in the mixtilinear triangle CBD. The common interior tangent to γ_1 and γ_2 meets circle γ in P and Q. Find the length of PQ in terms of r_1 and r_2 .



1006. Proposed by Hans Havermann, Weston, Ontario.

Given a base-ten positive integer of two or more digits, it is possible to spawn two smaller base-ten integers by inserting a space somewhere within the number. We call the left offspring thus created the *farmer* (*F*) and the value of the right one (ignoring leading zeros, if any) the *ladder* (*L*). A number is called *modest* if it has an *F* and an *L* such that the number divided by *L* leaves remainder *F*. (For example, 39 is modest.)

Consider, for $n > 1$, a block of n consecutive positive integers all of which are modest. If the smallest and largest of these are a and b , respectively, and if $a-1$ and $b+1$ are not modest, then we say that the block forms a *multiple berth* of size n . A multiple berth of size 2 is called a set of *twins*, and the smallest twins are {411, 412}. A multiple berth of size 3 is called a set of *triplets*, and the smallest triplets are {4000026, 4000027, 4000028}.

(a) Find the smallest quadruplets.

(b)* Find the smallest quintuplets. (There are none less than 25 million.)

1007. Proposed by Leroy F. Meyers, The Ohio State University.

It is known that every positive rational number can be written as the sum of finitely many reciprocals of distinct positive integers (the Egyptian fraction decomposition). Show that every positive real number can be written as the sum of infinitely many reciprocals of distinct positive integers.

1008. Proposed by Jordan F. Tahov, Sofia, Bulgaria.

A circle γ of diameter AB and two real numbers x and y are given. A variable point N ranges over γ . Find the locus of a point M on the line AN such that $AM = |xAN + yBN|$.

1009. Proposed by Allan Wm. Johnson Jr., Washington, D.C.

Prove that every pandiagonal fourth-order magic square can be written in the form

$A+B+C$	$D+B-C$	$D-P+C$	$A-B-C$
$D-B-E$	$A-B+E$	$A+B-E$	$D+B+E$
$A+B-C$	$D+B+C$	$D-B-C$	$A-B+C$
$P-P+F$	$A-P-E$	$A+P+F$	$D+P-F$

1010.* Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Are there integers $k \neq 1$ such that the sequence $\{3n^2k + 3nk^2 + k^3\}$, n an integer, contains infinitely many squares? If the answer is yes, determine all such k .

(The case $k = 1$ is dealt with in Crux 873 [1984: 335].)

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TO HONOUR FRED'S MEMORY

At the request of Fred Maskell and his family, mourners at his wake and obsequies were invited to send, instead of flowers, memorial donations to one of the following, all of which were dear to Fred's heart:

Crux Mathematicorum
Algonquin College
200 Lees Avenue
Ottawa, Ontario, Canada K1S 0C5

Moral Re-Armament in Canada
Suite 302, 141 Somerset St. W.
Ottawa, Ontario, Canada K2P 2H1

St. Mark's Church Memorial Fund
1606 Fisher Avenue
Ottawa, Ontario, Canada K2C 1X6

Readers who would like to honour Fred's memory may wish to do likewise,

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SOLUTIONS

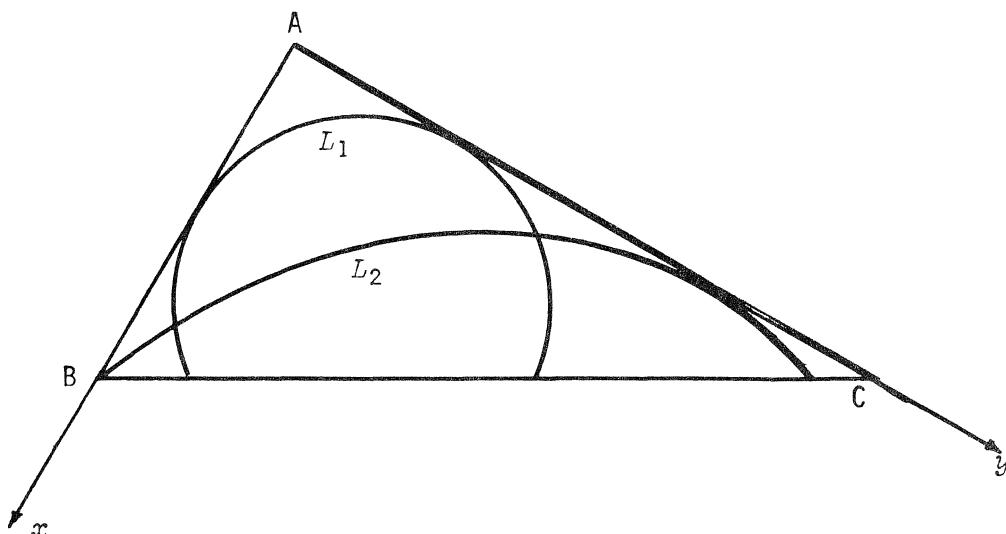
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

808, [1983: 22; 1984: 123] Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

Find the length of the largest circular arc contained within the right triangle with sides $a > b \geq c$.

II. Solution by Jordi Dou, Barcelona, Spain.

We assume that ABC is a right triangle with $A > B \geq C$. Of the circular arcs L contained within the triangle, it seems clear that one of maximal length will be of one of the two types illustrated in the figure.



Type 1. Arcs L_1 tangent to AB and AC, with endpoints on the hypotenuse BC (the incircle is of this type).

Type 2. Arcs L_2 tangent to AC with one endpoint at B, the other endpoint being on the hypotenuse BC (the arc tangent to AC at C, about which the proposer made a plausible conjecture [1984: 123], is of this type).

Arcs of type 1 have their centres on the segment IB' of the bisector of angle A, where I is the incentre and $BB' = BA$. Arcs of type 2 have their centres on the arc $B'C'$ of the parabola with focus B and directrix AC, where C' is the intersection of the perpendicular bisector of BC with the perpendicular to AC at C.

We introduce a coordinate system with origin at A, x-axis along AB and y-axis along AC, as shown in the figure, and take for our unit of length

$$BC = a = 1, \quad \text{so that} \quad AC = b = \cos C \text{ and } AB = c = \sin C.$$

The equation of segment IB' is

$$y = x, \quad r \leq x < \sin C,$$

where $r = \sin C \cos C / (1 + \sin C + \cos C)$ is the inradius. The equation of parabolic arc $B'C'$ is

$$y = \sqrt{2x \sin C - \sin^2 C}, \quad \sin C \leq x \leq \frac{1}{2 \sin C}.$$

Thus for each x in the interval $r \leq x \leq 1/(2 \sin C)$ there is a corresponding circular arc $L(x)$ to be considered. For any such arc $L(x)$, let $M(x, y)$ be its centre, N the intersection with BC of the parallel to AB through M (so that \overline{NM} is positive or negative according as M lies below or above BC), M' the perpendicular projection of M upon BC , $R(x)$ its radius, $\alpha(x)$ the angle it subtends at its centre, and $\lambda(x)$ its length. It is clear that $R(x) = x$; $y = AT$, where T is the point of contact of L on AC ;

$$\overline{NM} = \overline{TM} - \overline{TN} = x - (AC-y) \tan C; \quad \overline{M'M} = \overline{NM} \cos C;$$

$$\alpha(x) = 2 \arccos(\overline{M'M}/x); \text{ and } \lambda(x) = x\alpha.$$

These results yield, for an arc $L_1(x)$ of type 1, that is, for $r \leq x \leq \sin C$,

$$\lambda_1(x) = 2x \arccos \left\{ \sin C + \cos C - \frac{\sin C \cos C}{x} \right\}; \quad (1)$$

and for an arc $L_2(x)$ of type 2, that is, for $\sin C \leq x \leq 1/(2 \sin C)$, the corresponding result is

$$\lambda_2(x) = 2x \arccos \left\{ \cos C - \frac{\sin C}{x} (\cos C - \sqrt{2x \sin C - \sin^2 C}) \right\}. \quad (2)$$

Let $\phi_1(x)$ denote the expression in braces in (1). Then $\phi_1(r) = -1$, $\lambda_1(r) = 2\pi r$, and $\lambda_1'(x) \rightarrow -\infty$ as $x \downarrow r$; so $\lambda_1(x)$ decreases rapidly from its initial value $\lambda_1(r)$, corresponding to the incircle. We then have also

$$\lambda_1(\sin C) = 2B \sin C.$$

Let $\phi_2(x)$ denote the expression in braces in (2). Then $\phi_2(\sin C) = \sin C$ and

$$\lambda_2(\sin C) = 2B \sin C = \lambda_1(\sin C),$$

as expected, for the arcs of types 1 and 2 coalesce for $x = \sin C$, corresponding to an arc with centre B' . We also have, for $x_1 = 1/(2 \sin C)$,

$$\phi_2(x_1) = \cos C, \quad \lambda_2(x_1) = \frac{C}{\sin C},$$

$$\phi_2'(x_1) = \frac{2 \sin^3 C}{\cos C}, \quad \lambda_2'(x_1) = 2(C - \tan C) < 0.$$

The inequality in the very last result is particularly instructive because it

disproves the proposer's conjecture that the arc with centre C' , which is tangent to AC at C , can have maximal length. For $\lambda_2(x)$ decreases to $\lambda_2(x_1)$ as $x \uparrow x_1$, so there is an $x_0 < x_1$ for which $\lambda_2(x_0)$ is maximal, and $\lambda_2(x_0) > \lambda_2(x_1)$.

On the other hand, it is evident that, for small values of C , $\lambda_1(r)$ is much too small to be maximal, and then the x_0 for which $\lambda_2(x_0)$ is maximal is a value of x for which $\lambda_2'(x)$ vanishes. So, for $0 < C \leq \pi/4$, $\lambda_{\max} \equiv \lambda_{\max}(C)$ is either $\lambda_1(r)$ or $\lambda_2(x_0)$ for some $x_0 > \sin C$. Of particular interest for our problem is the value $C = \psi$ for which

$$\lambda_{\max}(\psi) = \lambda_1(r) = \lambda_2(x_0), \quad r < x_0.$$

For then $\lambda_{\max}(C) = \lambda_1(r)$ if $C > \psi$ (the incircle is the maximal arc) and $\lambda_{\max}(C) = \lambda_2(x_0)$ if $C < \psi$ (an arc of type 2 is maximal). If $C = \psi$, then there are two arcs of equal maximal length: the incircle and an arc of type 2. With a pocket calculator we found

$$\psi \approx 0.43983479 \approx 25^\circ 12' 02.44''$$

and the corresponding values

$$r \approx 0.16530600, \quad x_0 \approx 0.99604054, \quad \lambda_{\max} = \lambda_1(r) = \lambda_2(x_0) \approx 1.03864825.$$

For $x_1 = 1/(2 \sin \psi) \approx 1.17428784$, we find that $\lambda_2(x_1) \approx 1.03298529 < \lambda_2(x_0)$.

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857. [1983: 179; 1984: 304] *Proposed by Leroy F. Meyers, The Ohio State University.*

(a) Given three positive integers, show how to determine algebraically (rather than by a search) the row (if any) of Pascal's triangle in which these integers occur as consecutive entries.

(b) Given two positive integers, can one similarly determine the row (if any) in which they occur as consecutive entries?

(c)* The positive integer k occurs in the row of Pascal's triangle beginning with 1, k , For which integers is this the only row in which it occurs?

II. *Comment on part (c)* by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

Lest my earlier answer to part (c)* [1984: 306] be considered too flippant, I present the argument that led to my remark, "Almost all".

Let x be a large positive number. We wish to determine the number of integers not greater than x that occur in that part of Pascal's triangle which excludes the first and last two entries in each row, that is, the number of entries $\binom{n}{m} \leq x$ such that $2 \leq m \leq n-2$. In fact, since most entries occur twice in the same row, we can confine our attention to the portion for which $2 \leq m \leq [n/2]$. For any column m ,

the number of such integers is

$$p(m) = n(m) - 2m + 1,$$

where $n(m)$ is the largest value of n for which $\binom{n}{m} \leq x$. To find an approximate value for $n(m)$, we solve for n the equation

$$x = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Since, as we will see later (in the table), most of the contributions come from the first few columns of Pascal's triangle, we may safely use the approximation $x = n^m/m!$ and Stirling's formula to obtain

$$n(m) \approx \sqrt[m]{m!x} \approx \frac{m}{e} \cdot \sqrt[m]{x}$$

and

$$\begin{aligned} p(m) &\approx \sqrt[m]{m!x} - 2m + 1 \\ &\approx \frac{m}{e} \cdot \sqrt[m]{x} - 2m + 1. \end{aligned} \tag{1}$$

For example, suppose $x = 10^6$. Then, from (1), i.e., without using Stirling's formula since m is always small, we obtain the results in the adjoining table. Thus no more than about 0.172% of the integers less than 10^6 occur in the part of interest of Pascal's triangle, and "almost all" do not.

m	$n(m)$	$p(m)$
2	1414	1411
3	181	176
4	70	63
5	41	32
6	30	19
7	24	11
8	21	6
9	19	2
		1720

To answer the question more precisely, the only integers i that appear in more than one row are those that can be represented as

$$i = \frac{n!}{m!(n-m)!} \quad \text{with} \quad 2 \leq m \leq [n/2].$$

But that is begging the question, isn't it?

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877. [1983: 242] Proposed by Charles W. Trigg, San Diego, California.

Prove or disprove the following statement: Every prime which is the reverse of a square integer is congruent to 1 modulo 6.

Examples: $61 = 10 \cdot 6 + 1$ is the reverse of 4^2 , and $12391 = 2065 \cdot 6 + 1$ is the reverse of 139^2 .

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

We will prove somewhat more than is asked: Every odd integer not divisible by 3 which is a permutation of the digits of a square is congruent to 1 modulo 6.

Every square is congruent to 0 or 1 modulo 3, and permuting its digits has no

effect on the remainder modulo 3. Therefore if the integer is not divisible by 3, it is congruent to 1 modulo 3. Thus it is congruent to 1 or 4 modulo 6, and if it is odd it must be congruent to 1 modulo 6.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; STEWART METCHETTE, Culver City, California; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; BASIL C. RENNIE, James Cook University of North Queensland, Australia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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878. [1983: 242] *Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.*

Let A be a real $n \times n$ matrix with nonzero entries. If A is singular (i.e., $\det A = 0$), does there always exist a real $n \times n$ matrix B such that $\det(AB+BA) \neq 0$?

Solution by Paul J. Campbell, Beloit College, Wisconsin.

The answer is yes if $n = 2$ and no if $n > 2$. For $n = 2$, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where $\det A = ad-bc = 0$ and (at least) $b \neq 0$. Then we easily find that

$$\det(AB+BA) = b^2 \neq 0.$$

For $n > 2$, let A be the $n \times n$ matrix all of whose entries are 1's. For any $n \times n$ matrix B , AB and BA have rank at most 1. Hence $AB+BA$ has rank at most 2, so $\det(AB+BA) = 0$.

More generally, we have the following

THEOREM. Let A be a real $n \times n$ matrix of rank r . Then there exists a real $n \times n$ matrix B such that $\det(AB+BA) \neq 0$ if and only if $r \geq n/2$.

Proof. Suppose $r < n/2$ (so that $\det A = 0$) and let B be any $n \times n$ matrix. Then each of AB and BA has rank at most r and their null spaces must have nonzero intersection, since the sum of their nullities exceeds n . This nonzero intersection is sent to the zero vector by $AB+BA$, and so $\det(AB+BA) = 0$.

Suppose now that $r \geq n/2$ and let N be the null space of A . The rank and nullity of A sum to n , so we have $\dim N \leq n/2$. Thus the whole space \mathbb{R}^n is at least "twice as large as" N . Hence we may choose for B a nonsingular matrix that sends to N a subspace Q of \mathbb{R}^n such that Q and N have only the zero vector in common. Because B is nonsingular, the null spaces of AB and BA are Q and N , respectively. Since their null spaces have no nonzero vector in common, neither do the image spaces of AB and BA . Now suppose

$$(AB+BA)x = 0 \quad \text{and let} \quad y = ABx = -BAx = BA(-x).$$

Thus y is in the image spaces of both AR and RA , so $y = 0$. Then x must be in both null spaces, so $x = 0$. Consequently, $AR+RA$ is nonsingular and its determinant is nonzero.

Also solved by MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts (partial solution); BASIL C. RENNIE, James Cook University of North Queensland, Australia; and PETER ROSS, University of Santa Clara, California. The proposer submitted a comment.

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879. [1983: 242] *Proposed by Leroy F. Meyers, The Ohio State University.*

The U.S. Social Security numbers consists of 9 digits (with initial zeros permitted). How many such numbers are there which do not contain any digit three or more times consecutively?

I. *Solution by the proposer.*

A recursive solution is given for the generalized problem of finding how many n -digit numbers written in base b contain no digit k or more times consecutively, where b and k are integers, each greater than 1.

Let α_n be the number of n -digit numbers written in base b which contain no digit written k or more times consecutively, where k is fixed. Then such a number may be obtained from an $(n-1)$ -digit number with no digit occurring more than k times consecutively by adjoining at the end any of the $b-1$ digits which are not the last digit of the $(n-1)$ -digit number. Further numbers can be obtained (if $k > 2$) by adjoining two copies of any of the $b-1$ digits not the same as the last digit of an $(n-2)$ -digit admissible number. Additional numbers can be obtained in this way up to the adjunction of $k-1$ digits, all the same, to an $(n-k+1)$ -digit number. Thus we have the recurrence

$$\alpha_n = (b-1)\alpha_{n-1} + (b-1)\alpha_{n-2} + \dots + (b-1)\alpha_{n-k+1} \quad (1)$$

for $n \geq k$. Obviously, $\alpha_n = b^n$ if $1 \leq n < k$. The difference equation (1) can be solved explicitly (although not in general by radicals if $k > 5$), but direct working out of the recurrence is simpler.

In the stated problem, $k = 3$, $b = 10$, and $n = 9$. So

$$\alpha_1 = 10, \quad \alpha_2 = 100, \quad \alpha_j = 9(\alpha_{j-1} + \alpha_{j-2}), \quad j = 3, 4, \dots, 9,$$

from which the required answer is $\alpha_9 = 936\,845\,190$.

II. *Solution by Hayo Ahlborg, Benidorm, Alicante, Spain.*

In general, we will say that a number of 9 decimal digits is of a certain pattern if it consists of z_1 groups of one digit, z_2 groups of two consecutive equal digits, z_3 groups of three consecutive equal digits, ..., z_9 groups of nine conse-

z_9	z_8	z_7	z_6	z_5	z_4	z_3	z_2	z_1	N	
								9	$1 \cdot 10 \cdot 9^8$	
							1	7	$8 \cdot 10 \cdot 9^7$	
							2	5	$21 \cdot 10 \cdot 9^6$	
							3	3	$20 \cdot 10 \cdot 9^5$	
							4	1	$5 \cdot 10 \cdot 9^4$	936 845 190
							1	6	$7 \cdot 10 \cdot 9^6$	
							1	4	$30 \cdot 10 \cdot 9^5$	
							1	2	$30 \cdot 10 \cdot 9^4$	
							1	3	$4 \cdot 10 \cdot 9^3$	
							2	3	$10 \cdot 10 \cdot 9^4$	
							2	1	$12 \cdot 10 \cdot 9^3$	
							3		$1 \cdot 10 \cdot 9^2$	994 502 610
							1	5	$6 \cdot 10 \cdot 9^5$	
							1	3	$20 \cdot 10 \cdot 9^4$	
							1	1	$12 \cdot 10 \cdot 9^3$	
							1	2	$12 \cdot 10 \cdot 9^3$	
							1	1	$6 \cdot 10 \cdot 9^2$	
							2	1	$3 \cdot 10 \cdot 9^2$	999 540 000
								4	$5 \cdot 10 \cdot 9^4$	
								2	$12 \cdot 10 \cdot 9^3$	
								2	$3 \cdot 10 \cdot 9^2$	
								1	$6 \cdot 10 \cdot 9^2$	
									$2 \cdot 10 \cdot 9$	999 963 000
								3	$4 \cdot 10 \cdot 9^3$	
								1	$6 \cdot 10 \cdot 9^2$	
									$2 \cdot 10 \cdot 9$	999 997 200
								2	$3 \cdot 10 \cdot 9^2$	
								1	$2 \cdot 10 \cdot 9$	999 999 810
									$2 \cdot 10 \cdot 9$	999 999 990
								1	$1 \cdot 10 \cdot 1$	1000 000 000
1	1									

cutive equal digits, where adjacent (nonempty) groups contain different digits.

Thus each $z_i \geq 0$ and

$$z_1 + 2z_2 + 3z_3 + \dots + 9z_9 = 9.$$

The number of distinct patterns with z_1 singles, z_2 pairs, z_3 triples, etc., is

$$F = \frac{(z_1+z_2+\dots+z_9)!}{z_1!z_2!\dots z_9!}$$

and the total number of different numbers showing these patterns is

$$N = F \cdot 10 \cdot 9^{z_1+z_2+\dots+z_9-1}.$$

The table on the preceding page is now self-explanatory.

III. Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Consider the following problem: Ten equally talented athletes compete in the nine events of a Superstars competition. What is the probability that the same athlete wins three or more consecutive events?

This is a Markov process, and after n events there are three possible states:

(a) The winner of the n th event is not the same as the winner of the $(n-1)$ st event.

(b) The same athlete has won (exactly) the last two events.

(c) Some athlete has already won three consecutive events.

Let A_n , B_n , and C_n be the probabilities of states (a), (b), and (c), respectively, at the completion of n events. Then it is clear that

$$A_{n+1} = 0.9(A_n + B_n), \quad B_{n+1} = 0.1A_n, \quad C_{n+1} = C_n + 0.1B_n,$$

with $A_1 = 1$ and $B_1 = C_1 = 0$. From the recursion relations it is easy to show that

$$C_9 = 0.1 \sum_{i=1}^8 B_i = 0.01 \sum_{i=1}^7 A_i,$$

so it is necessary to calculate only A_2, \dots, A_7 and B_2, \dots, B_6 to obtain the result $C_9 = 0.06315481$. The probability that no athlete has won three consecutive events is therefore $1 - C_9 = 0.93684519$.

Now perform this competition a billion times, with a different outcome each time, and what do we have (other than ten tired athletes)? Why, the answer to the proposer's problem: 936 845 190.

Also solved CURTIS COOPER, Central Missouri State University at Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California; EDWIN M. KLEIN, University of Wisconsin-Whitewater; and KENNETH M. WILKE, Topeka, Kansas. Two incorrect solutions were received.

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880.* [1983: 242] Proposed by Clark Kimberling, University of Evansville, Indiana.

For a given triangle ABC, what curve is formed by all the points P in three-dimensional space satisfying

$$\angle BPC = \angle CPA = \angle APB?$$

Editor's comment.

No solution was received for this problem, which therefore remains open. The proposer noted that the curve intersects the plane of the triangle in the "first" isogonic centre (i.e., the solution to Fermat's minimization problem).

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881. [1983: 275] Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

Find the unique solution to the following "areametic", where A, B, C, D, E, N, and R represent distinct decimal digits:

$$\int_B^D Cx^N dx = \text{AREA}.$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

We have

$$\frac{C(D^{N+1} - B^{N+1})}{N + 1} = \text{AREA}.$$

It is seen immediately that

$$A \neq 0, \quad C \neq 0, \quad D > B, \quad C \neq 5 \text{ unless } N = 4, \quad (1)$$

and the four-digit AREA requires $N > 1$.

For $N = 2$, we try successively $D = 9, 8, 7, \dots$ and test the possible values of B and C, keeping in mind the restrictions (1). No solution is found.

We proceed likewise for $N = 3$, and the only solution we find is

$$D = 6, \quad B = 0, \quad C = 8, \quad \text{AREA} = 2592.$$

Similar attempts with $N = 4, 5, 6, \dots$ yield no other solutions, so the unique solution is

$$\int_0^6 8x^3 dx = 2592,$$

Also solved by FRANK BATTLES and LAURA KELLEHER, Massachusetts Maritime Academy, Buzzards Bay (jointly); RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College; and the proposer.

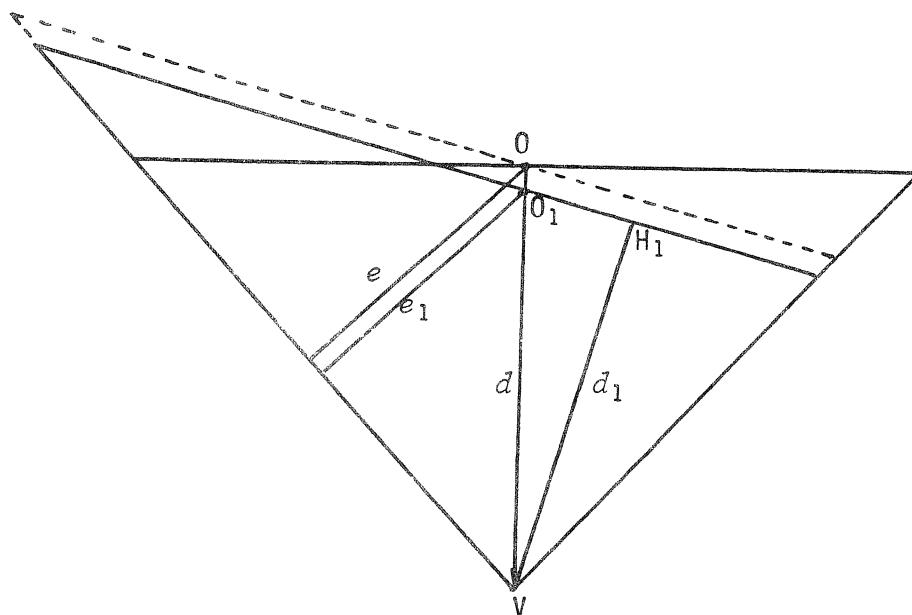
882. [1983: 275] Proposed by George Tsintsifas, Thessaloniki, Greece.

The interior surface of a wine glass is a right circular cone. The glass, containing some wine, is first held upright, then tilted slightly but not enough to spill any wine. Let D and E denote the area of the upper surface of the wine and the area of the curved surface in contact with the wine, respectively, when the glass is upright; and let D_1 and E_1 denote the corresponding areas when the glass is tilted. Prove that

$$(a) \quad E_1 \geq E, \quad (b) \quad D_1 + E_1 \geq D + E, \quad (c) \quad \frac{D_1}{E_1} \geq \frac{D}{E}.$$

Solution by Jordi Dou, Barcelona, Spain.

Let K and K_1 denote the wine cones in the upright and tilted positions, respectively. We assume for convenience that the direction of gravity is tilted, rather than the wine glass, so that K and K_1 are part of the same conical surface of revolution γ with vertex V . (The figure shows a section made by their common plane of symmetry.)



The cone K , whose volume we denote by $[K]$, has a circular base δ with centre O and area D , and lateral surface ϵ with area E . The cone K_1 , with volume $[K_1]$, has an elliptical base δ_1 with area D_1 and lateral surface ϵ_1 with area E_1 . It is clear that $[K] = [K_1]$.

Let the line VO intersect δ_1 in O_1 ; let $d = VO$ and $d_1 = VH_1$ be the altitudes of K and K_1 ; and let e and e_1 be the distances from O and O_1 to the generatrix of γ . Then

$$\frac{1}{3}dD = \frac{1}{3}eE = [K] = [K_1] = \frac{1}{3}d_1D_1 = \frac{1}{3}e_1E_1.$$

Before proceeding further, we need to show that $V_0 < V_1$. Let K' be the cone whose base δ' passes through 0 and is parallel to δ_1 , and let its lateral surface ϵ' have area E' . If we now let Δ be the area of the part of ϵ that does not belong to ϵ' , and Δ' the area of the part of ϵ' that does not belong to ϵ , then $E' = E + \Delta' - \Delta$. Since clearly $\Delta' > \Delta$, we therefore have $E' > E$, so

$$[K'] = \frac{1}{3}eE' > \frac{1}{3}eE = [K] = [K_1],$$

and from this $V_0 < V_1$ follows.

This last result implies that $e_1 < e$ and $d_1 < V_0 < V_1 = d$. Now $e_1E_1 = eE$ gives

$$E_1 > E \quad (\text{a})$$

and $d_1D_1 = dD$ gives $D_1 > D$, so

$$D_1 + E_1 > D + E. \quad (\text{b})$$

Now $e_1/V_0 = e/d$, so $e_1/d_1 > e/d$; and since $e_1/d_1 = D_1/E_1$ and $e/d = D/E$, we have finally

$$\frac{D_1}{E_1} > \frac{D}{E}. \quad \square \quad (\text{c})$$

More generally, let $D(\theta)$ and $E(\theta)$ denote the base and lateral surface areas of the wine cone after the glass has been tilted from the vertical through an angle $\theta \geq 0$ (restricted only so that there is no wine spillage). With a few obvious modifications, the above proof would show that

$$D(\theta), \quad E(\theta), \quad D(\theta) + E(\theta), \quad \text{and} \quad \frac{D(\theta)}{E(\theta)}$$

are all functions that increase strictly with θ .

Also solved by M.S. KLAMKIN, University of Alberta; and the proposer.

Editor's comment.

For another tilted wine glass problem, by R.C. Lyness, see Problem 1157 in *Mathematics Magazine*, 57 (1984) 48-49.

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883. [1983: 275] Proposed by J. Tabov and S. Troyanski, Sofia, Bulgaria.

Let ABC be a triangle with area S , sides a, b, c , medians m_a, m_b, m_c , and interior angle bisectors t_a, t_b, t_c . If

$$t_a \cap m_b = P, \quad t_b \cap m_c = Q, \quad t_c \cap m_a = R,$$

prove that $\sigma/S < 1/6$, where σ denotes the area of triangle PQR.

Solution by M.S. Klamkin, University of Alberta (revised by the editor).

Using barycentric coordinates and vectors with an origin outside the plane of triangle ABC (assumed for now to be nondegenerate), the incenter I and centroid G of the triangle are given by

$$\vec{I} = \frac{a\vec{A} + b\vec{B} + c\vec{C}}{a + b + c} \quad \text{and} \quad \vec{G} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}.$$

Then

$$\vec{P} = \vec{A} + u(\vec{I} - \vec{A}) = \vec{B} + v(\vec{G} - \vec{B}),$$

and equating the coefficients of \vec{A} and \vec{B} on both sides gives $v = 3c/(b+2c)$. Thus

$$\vec{P} = \frac{c\vec{A} + b\vec{B} + c\vec{C}}{b + 2c},$$

and similarly

$$\vec{Q} = \frac{a\vec{A} + a\vec{B} + c\vec{C}}{c + 2a} \quad \text{and} \quad \vec{R} = \frac{a\vec{A} + b\vec{B} + b\vec{C}}{a + 2b}.$$

Now we find

$$2\sigma = |\vec{Q} \times \vec{R} + \vec{R} \times \vec{P} + \vec{P} \times \vec{Q}| = f(a, b, c) |\vec{B} \times \vec{C} + \vec{C} \times \vec{A} + \vec{A} \times \vec{B}| = f(a, b, c) \cdot 2S,$$

where

$$f(a, b, c) \equiv \frac{bc^2 + ca^2 + ab^2 - 3abc}{(b+2c)(c+2a)(a+2b)}.$$

Thus $\sigma/S = f(a, b, c)$ for nondegenerate triangles ABC. We will show that $f(a, b, c) \leq 1/6$ holds for all triangles, even for degenerate ones (for which we assume that σ/S is defined by $f(a, b, c)$).

The inequality $f(a, b, c) \leq 1/6$ is easily shown to be equivalent to

$$2bc(a+b-c) + 2ca(b+c-a) + 2ab(c+a-b) + 21abc \geq 0,$$

and this is clearly true since $a+b-c \geq 0$, etc. Equality holds just when $abc = 0$, that is, just when ABC is a degenerate triangle with one side of length zero. \square

Alternatively, if $abc \neq 0$ (which does not exclude degeneracy of another kind), the inequality $f(a, b, c) < 1/6$ is equivalent to

$$\frac{b-c}{a} \cdot \frac{c-a}{b} \cdot \frac{a-b}{c} < \frac{27}{2}$$

and to

$$\frac{27}{2} + \frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} > 0,$$

each of which is clearly true since $b-c \leq a$, etc.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposers.

Editor's comment.

For related problems concerning sets of concurrent cevians, see Problems 588 [1981: 306], 685 [1982: 292], and 790 [1984: 60].

One unusual fact to emerge from the above solution is that, for the quantities

$$\frac{b-c}{a}, \quad \frac{c-a}{b}, \quad \frac{a-b}{c},$$

where a, b, c are arbitrary nonzero real (or even complex) numbers, their product is always equal to the negative of their sum. Compare with Problem 680 [1982: 284], where we were asked to find triples of real numbers whose product equals their sum.

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884. [1983: 276] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

The eccentric warden revisited (see Crux 722 [1983: 89]).

A prison warden has n prisoners in n cells, one prisoner per cell, with all cells initially closed. He also has a secret function

$$f: \{1, 2, \dots, n\} \rightarrow \{1, 2, 3, \dots\}$$

with the following property: For $k = 1, 2, \dots, n$, on day k each cell $k, 2k, 3k, \dots$ is reversed $f(k)$ times (from open to closed or vice versa). Thus, on day 1 all cells are reversed $f(1)$ times; on day 2, cells $2, 4, 6, \dots$ are reversed $f(2)$ times; etc. Each reversal (from open to closed or vice versa) is counted once towards the $f(k)$ times. The end result is that, after the n days have elapsed, cell k has been reversed a total of exactly k times, $k = 1, 2, \dots, n$.

Find all functions f with this property.

Solution by Edwin M. Klein, University of Wisconsin-Whitewater.

It is given that

$$\sum_{d|k} f(d) = k, \quad k = 1, 2, \dots, n.$$

It is well known (see almost any textbook on number theory) that the only function with this property is Fuler's ϕ -function.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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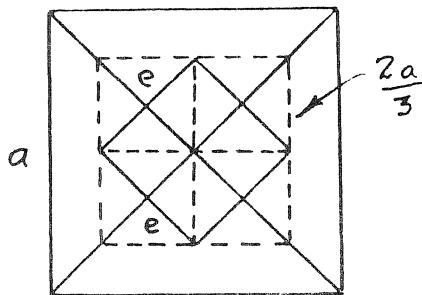
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885. [1983: 276] *Proposed by Charles W. Trigg, San Diego, California.*

Planes are drawn perpendicular to the four space diagonals of a cube at their trisection points. What is the nature of the solid bounded by these planes? What is the volume of the solid in terms of the edge, e , of the cube.

Solution by the proposer.

The solid formed by the trisecting planes is similar to the solid formed by planes through the vertices of the cube and perpendicular to the space diagonals. Their ratio of similitude is 1:3. It is well known that the outer solid is a regular octahedron with the vertices of the cube at the centroids of its faces. As is evident from the orthogonal projection of the octahedron, of edge a , in the figure, $e = a\sqrt{2}/3$. Now the volume of the larger octahedron is $a^3\sqrt{2}/3$. Hence the volume of the regular octahedron formed by the trisecting planes is



$$\left(\frac{1}{3}\right)^3 \left(\frac{3e}{\sqrt{2}}\right)^3 \frac{\sqrt{2}}{3} = \frac{e^3}{6}.$$

Also solved by the COPS of Ottawa (two solutions); JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

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886. [1983: 276] *Proposed by A.W. Goodman, University of South Florida.*

Prove that

$$(a) \sum_{k=1}^{n-1} (-1)^{k+1} (n-k)^2 = \frac{n(n-1)}{2};$$

$$(b) \sum_{k=1}^{n-1} (-1)^{n-k-1} k^2 (n-k)^2 = \frac{n}{4} \{1 + (-1)^n\}.$$

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

If we multiply both sides of part (a) by the constant $(-1)^n$, we obtain the equivalent

$$\sum_{k=1}^{n-1} (-1)^{n-k-1} (n-k)^2 = (-1)^n \frac{n(n-1)}{2}. \quad (1)$$

Let

$$f(n, m) = \sum_{k=1}^{n-1} (-1)^{n-k-1} k^m (n-k)^2. \quad (2)$$

We will prove by induction that, for $n \geq 2$,

$$f(n, 0) = (-1)^n \frac{n(n-1)}{2}, \quad (3)$$

$$f(n,1) = \frac{1 + (-1)^n(2n^2 - 1)}{8}, \quad (4)$$

$$f(n,2) = \frac{n}{4}\{1 + (-1)^n\}. \quad (5)$$

Observe that (3) and (5) are identical to (1) and part (b), respectively. Formula (4) is a bonus.

It is easily seen from (2) that $f(2,m) = 1$ for all m , and therefore (3)-(5) are all true for $n = 2$. Next we note that, for $n > 2$,

$$\begin{aligned} f(n,m) &= (-1)^n(n-1)^2 + \sum_{k=2}^{n-1} (-1)^{n-k-1} k^m (n-k)^2 \\ &= (-1)^n(n-1)^2 + \sum_{k=1}^{n-2} (-1)^{n-k-2} (k+1)^m (n-k-1)^2 \\ &= (-1)^n(n-1)^2 + \sum_{i=0}^m \binom{m}{i} f(n-1,i). \end{aligned}$$

Therefore

$$f(n,0) = (-1)^n(n-1)^2 + f(n-1,0) = (-1)^n(n-1)^2 + (-1)^{n-1} \frac{(n-1)(n-2)}{2}$$

$$= (-1)^n \frac{n(n-1)}{2},$$

$$\begin{aligned} f(n,1) &= (-1)^n(n-1)^2 + f(n-1,0) + f(n-1,1) \\ &= (-1)^n(n-1)^2 + (-1)^{n-1} \frac{(n-1)(n-2)}{2} + \frac{1 + (-1)^{n-1}\{2(n-1)^2 - 1\}}{8} \end{aligned}$$

$$= \frac{1 + (-1)^n(2n^2 - 1)}{8},$$

$$\begin{aligned} f(n,2) &= (-1)^n(n-1)^2 + f(n-1,0) + 2f(n-1,1) + f(n-1,2) \\ &= (-1)^n(n-1)^2 + (-1)^{n-1} \frac{(n-1)(n-2)}{2} + \frac{1 + (-1)^{n-1}\{2(n-1)^2 - 1\}}{4} \\ &\quad + \frac{n-1}{4}\{1 + (-1)^{n-1}\} \\ &= \frac{n}{4}\{1 + (-1)^n\}. \end{aligned}$$

Also solved by the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; FUNG WAI WA, Hong Kong; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HFSS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts (part (a) only); EDWIN M. KLEIN, University of Wisconsin-Whitewater; JACK LESAGE, Eastview Secondary School, Barrie, Ontario (part (a) only); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands

(part (a) only); JORDAN B. TABOV, Sofia, Bulgaria (part (b) only); KENNETH M. WILKE, Topeka, Kansas; and KENNETH S. WILLIAMS, Carleton, University.

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887. [1983: 276] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

$A_1A_2A_3A_4$ is an isosceles trapezoid, with $A_4A_3 \parallel A_1A_2$, whose circumcircle has center O. The midpoints of the segments A_4A_3 and A_1A_2 are U and V, respectively; and ℓ_4 , the Wallace-Simson line of A_4 with respect to triangle $A_1A_2A_3$, intersects UV in S.

Prove that (a) $\ell_4 \parallel OA_3$, and (b) $US = OV$.

I. *Solution to part (a) by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let the line through A_4 perpendicular to A_1A_2 meet the circumcircle again in P. It is well known [1] and easy to prove that line PA_3 (= line OA_3) is parallel to ℓ_4 .

II. *Solution to part (b) by nearly all solvers.*

Let A_4P meet A_1A_2 in Q. It follows from part (a) that triangles OUA_3 and SVQ are congruent. Hence $VS = UO$ and, consequently, $US = OV$.

Also solved by JORDI DOU, Barcelona, Spain; WALther JANous, Ursulinengymnasium, Innsbruck, Austria (also part (b)); STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; JORDAN B. TABOV, Sofia, Bulgaria; GEORGE TSINTSIFAS, Thessaloniki, Greece; DIMITRIS VATHIS, Chalcis, Greece; and the proposer.

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1. H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library No. 19, Mathematical Association of America, 1967, p. 43.

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888. [1983: 276] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

(a) Find all solutions in natural numbers of the system

$$x + y = zw, \quad xy = z + w.$$

(b) Show that the system has infinitely many solutions in integers.

Solution by Kenneth M. Wilke, Topeka, Kansas.

(a) It is clear from the symmetry of the system that it suffices to find all solutions (x,y,z,w) for which $x \leq y$ and $x \leq z \leq w$. All other solutions can then be found by permuting x and y , or z and w , or (x,y) and (z,w) .

The equations can be combined to give

$$(x-1)(y-1) + (z-1)(w-1) = 2.$$

For $x = 1$ we obtain only the solution $(1, 5, 2, 3)$; for $x = 2$, only the solution $(2, 2, 2, 2)$; and no solutions for $x > 2$.

(b) There are several infinite sets of integer solutions. One of them is

$$(x, y, z, w) = (0, -k^2, k, -k), \quad k \text{ any integer.}$$

Also solved by PAUL R. BEESACK, Carleton University, Ottawa; the COPS of Ottawa; J.T. GROENMAN, Arnhem, The Netherlands; F.D. HAMMER, Palo Alto, California; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; EDWIN M. KLEIN, University of Wisconsin-Whitewater; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; MALCOLM SMITH, Georgia Southern College at Statesboro; JORDAN B. TABOV, Sofia, Bulgaria; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer. A comment was received from BOB PRIELIPP, University of Wisconsin-Oshkosh.

Editor's comment.

Prieliipp found the problem, with a solution (inelegant for part (a), and with an error in part (b)), on pages 17 and 99 of W. Sierpiński's *250 Problems in Elementary Number Theory* (American Elsevier, New York, 1970).

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LETTER TO THE EDITOR

Dear Léo,

This is both a letter to you personally and, if you wish to consider it so, a letter to the Editor addressed to my father's many friends among *Crux* readers.

Dad received the news of his cancer with grace and fortitude. It was not a surprise but rather a confirmation of what he was already aware of within himself. His reaction was simple. "I don't know what God has in store for me. But, whatever it is, it will be good."

As his days unfolded, he shared with us what was going on within him. "I've never really been sick. I've never really known how to relate to sick people. I think God has something more for me to learn before I die." Later he said to Mum, "You've been through this before. Sickness I've never understood. Now I understand. My cancer is a gift. It is an opportunity to learn and grow."

And so Dad proceeded to prepare to die. As his body adjusted to the macrobiotic cancer diet, he was relieved of the pain in his legs and the ache in his abdomen. But as he continued to weaken and waste away, he recognized that his time was running out. He hoped. He feared. He prayed. Above all, he appreciated. He gave the best of his heart and mind to those around him even in his affliction.

Even on that last Sunday, January 27, his eyes and his mind were clear. The peacefulness within him was marvellous to share. On Monday morning, January 28, at 9:15, Dad turned off all the lights and went home.

He graduated.

JOHN MASKELL

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