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CONTENTS

THE HEINE-BOREL THEOREM

R.B. KILLGROVE

1. *Introduction.*

An earlier article in this journal [10] introduced the concept of a complete ordered plane, which is a generalization of the Euclidean plane. In that paper, a topology was introduced by using open triangular regions instead of open disks for defining neighborhoods. Suppose these open regions are used to define boundedness just as open disks are used in the Euclidean plane. One question that then arises is: Is the Heine-Borel Theorem still true? That is, is it true that *the compact sets are precisely those which are closed and bounded*? This question has been answered in the negative [7]. However, we will show in this paper that the Heine-Borel Theorem holds if more general polygonal regions are used instead of just triangular ones. But first we discuss some known results about the Heine-Borel Theorem in arbitrary topological spaces.

2. *The Heine-Borel Theorem in arbitrary topological spaces.*

Let X be a nonempty set of *points* and τ a family of subsets of X, including the empty set and X itself. Then (X,τ) is a *topological space* if and only if τ is closed under arbitrary unions and finite intersections. The members of τ are the *open sets* of the topology, and complements of open sets are its *closed sets*. The *closure* of a set S (here and in the sequel, "set" will always mean a subset of the universe X) can be defined either as (a) the smallest closed set which contains S, that is, the intersection of all closed sets containing S; or (b), *boundary points* having been introduced in the usual way, as the union of S and its boundary points. The *indiscrete topology* of X is the one where $\tau = \{\emptyset,X\}$, and the *discrete topology* is the one where τ consists of all subsets of X. Thus there can be many different topologies for the same universe X.

A set S is *compact* if and only if, for any family Γ of open sets whose union contains S, there is a finite subfamily of Γ whose union also contains S. This subfamily is called a *finite refinement* of Γ.

A useful way of obtaining topologies for a universe X is to use bases. A *base* is a family β of sets such that (i) for each point $p \in X$ there is a $B \in \beta$ such that $p \in B$; and (ii) for every pair $A,B \in \beta$ and each point $p \in A \cap B$, there is a $C \in \beta$ such that $C \subseteq A \cap B$ and $p \in C$. We can then take for the nonempty sets of the topology τ all the arbitrary unions of members of β.

On the other hand, if we already have a topology τ in place, it is useful to know how to create a base for *that* topology. We then say that β is a *base* *for* (X,τ) if and only if β is a family of *open* sets such that (i) for each point $p \in X$, there is a $B \in \beta$ such that $p \in B$; and (ii) if $A \in \tau$, $B \in \beta$, and $p \in A \cap B$, then there is a $C \in \beta$ such that $C \subseteq A \cap B$ and $p \in C$.

Hindman [4] defines his notion of boundedness thus: a set S is *basically bounded* if and only if, for every base of the topology, it is contained in a finite union of base elements. He then obtains the following nice theorem.

Hindman's Theorem. In *any* topological space, every closed and basically bounded set is compact.

To obtain our version of the Heine-Borel Theorem (italicized in the introduction), we also need the converse of Hindman's Theorem. It is true that compact sets are basically bounded, but unfortunately there are topological spaces in which some compact sets are not closed.

Wilansky [15] defines a topological space to be *KC* if and only if every compact set is closed. He then shows that Hausdorff implies *KC* but not conversely, as well as *KC* implies T_1 but not conversely. (A space is T_1 if and only if each singleton set is closed; and a space is *Hausdorff* if and only if for each pair of distinct points p and q there are disjoint open sets U and V such that $p \in U$ and $q \in V$.)

3. *The metric approach to the Heine-Borel Theorem.*

It is well known [6, pp. 118-124] that in a metric space boundedness is defined in terms of open balls, and these also form a base for a topology. It is also known that there are metric spaces in which the Heine-Borel Theorem fails. The fact that the Heine-Borel Theorem holds in a metric space if and only if the closures of the open balls are compact follows from

Bray's Lemma [7]. In a Hausdorff space with a base such that

(1) closures of base elements are compact,

(2) each finite union of base elements is contained in a base element,

then the Heine-Borel Theorem holds for base bounded.

(A set S is *base bounded* if and only if it is contained in some base element (of the given base).)

A topological space is *second countable* if and only if there is some countable base for the space. A space (X,τ) is *locally compact* if and only if, for each point $p \in X$, there is a $U \in \tau$ such that $p \in U$ and the closure of U is compact. A space is *regular* if and only if, for each point p and each closed set C such that $p \notin C$, there are disjoint open sets U and V such that $p \in U$ and $C \subseteq V$.

With these definitions in place, the following two relevant theorems will be meaningful.

Busemann's Theorem [1]. There is a metric for which the Heine-Borel Theorem holds if and only if the topological space is Hausdorff, locally compact, and second countable.

The second theorem is parallel to Busemann's.

Bray-Whitnall-Killgrove Theorem [9]. A space is regular and has a base for which the Heine-Borel Theorem holds for base bounded if and only if the space is Hausdorff and locally compact.

Note that "second countable" is a condition in the first but not in the second theorem. For an example where the second theorem applies but not the first, consider the space (X,τ), where X is an uncountable set and τ is the discrete topology. Then only the finite sets are compact. There are metrics, but none for which the Heine-Borel Theorem holds.

The first theorem is proved in [1]. For the second theorem and for Bray's Lemma, the proofs (so far unpublished) depend largely on the results of Wilansky [15] mentioned above and on the following well-known results: regular and T_1 imply Hausdorff, closed subsets of compact sets are compact, and Hausdorff and locally compact imply regular.

4. *Characterizing the Heine-Borel Theorem for base bounded.*

A topological space is *locally CK* if and only if for each point p there is an open set U such that $p \in U$ and every closed subset of U is compact. We are now in a position to state the

Characterization Theorem [9]. There is a base for which the Heine-Borel Theorem holds for base bounded if and only if the space is *KC* and locally *CK*.

Here we merely give an outline of the (unpublished) proof. The "only if" part is trivial. For the "if" part, locally *CK* allows us a base, namely, all open sets whose closed subsets are compact. The only tricky part is to show that this base is closed under finite unions. From induction, it suffices to show that, if A and B are any open sets whose closed subsets are compact, and if C is a closed subset of $A \cup B$, then C is compact. If $C-A$ is empty, then C is a closed subset of A, and hence compact. Now suppose $C-A$ is not empty, and let Γ be a family of open sets whose union contains C. Then $C-A$ is a closed subset of B, and hence there is a finite subfamily of Γ whose union W contains $C-A$. Now $C-W$ is a closed subset of A, so there is a finite subfamily of Γ whose union contains $C-W$. Now every point in C is contained in some member of one of these two subfamilies, so they collectively form a finite refinement of Γ. □

It is clear that locally compact implies locally *CK*, but the following counterexample [3] disproves the converse. Let $X = \{(x,y) | \ y \geq 0\}$. The base members are of two kinds: (1) open circular disks entirely contained in the open upper half-plane $y > 0$, and (2) sets of the form $D \cup \{p\}$, where p is a point on the x-axis and D is the intersection of any open disk centered at p with the open upper half-plane. (This is a standard example of a Hausdorff space which is not regular, and hence not locally compact.) In this space, all the action off the x-axis is as usual; hence we need only consider the base members of the form $D \cup \{p\}$. If a closed subset of such a base member does not contain the center p, then it is closed and bounded (in the usual way), where the action is as usual, and hence it is compact. Suppose a closed subset C of such a base member does contain the center p. For any family Γ of open sets whose union contains C, the point p is contained in some base member V which is itself contained in an open set of Γ. Now $C-V$ has a finite refinement of Γ, and thus C is compact.

We have so far discussed only a few of the approaches to boundedness in the literature. For more information, consult Hu [5] and Lambrinos [11]-[13].

5. *Polygonal base members for complete ordered planes.*

We now turn our attention to the original problem, that of obtaining a Heine-Borel Theorem for complete ordered planes. We assume (and will frequently use without explicit reference) the general background for complete ordered planes as given in [10]. For an even more general approach, the reader may consult the works of Salzmann (e.g., [14]). In [7] we showed that the Heine-Borel Theorem does hold for complete ordered affine planes when we use open triangular regions to bound sets. (For a study of affine planes, see [8].) In a private communication, L.A. Rubel suggested that the Heine-Borel Theorem might hold in any complete ordered plane if more general polygonal regions are used instead of just triangular ones. The rest of this paper is devoted to showing that he is correct!

For $n \geq 3$, let $A_1, A_2, \ldots, A_n$ be n points, no three collinear, situated so that, for $i = 1,2,\ldots,n$, the points A_j not on the line A_iA_{i+1} (with $A_{n+1} = A_1$) all lie in the same one of the two open half-planes determined by that line. The intersection of the n open half-planes containing the points will be called an *open (convex) polygonal region* with vertices $A_1, A_2, \ldots, A_n$. If the corresponding closed half-planes are used instead, then we call their intersection a *closed (convex) polygonal region*. These regions are finite intersections of convex sets, so they are themselves convex, and in the sequel "region" will always mean "convex region". Also, being finite intersections of open [resp. closed] sets, they are also open [resp. closed]. Since they include the open triangular regions, it is easy to establish that *the open polygonal regions form a base for the topology generated by the open triangular regions*. But we have the Hausdorff property. Thus, in order to establish the Heine-Borel Theorem via Bray's Lemma, we need only show that closures of open polygonal regions are compact and that finite unions of open polygonal regions are contained in open polygonal regions.

Let P be any point of the open polygonal region with vertices $A_1, A_2, \ldots, A_n$, where $n > 3$. If P lies on line A_1A_3, it follows from the above definition that ωA_1PA_3 (in the notation of [10]). If P is not on A_1A_3, we claim that there is a point Q on A_2P such that ωA_1QA_3. Now P and A_2 are either on the same side of A_1A_3 or on opposite sides. In the former case, P is in open triangular region $A_1A_2A_3$, and a second formulation of open triangular regions (given in [10]) establishes our claim. In the latter case, by Hilbert's Separation Theorem, there is a Q on A_1A_3 such that ωA_2QP. From P,Q,A_3 on the same side of A_1A_2, we have not ωQA_1A_3; similarly from P,Q,A_1 on the same side of A_2A_3, we have not ωA_1A_3Q, and our claim is established. Now consider triangles $A_1A_iA_{i+1}$, $i = 3,4,\ldots,n-1$. If A_2P intersects A_1A_i for some i in a point Q_i, and if $\omega A_1Q_iA_i$, then by the Pasch axiom one of three things happens: (1) A_2, P, A_{i+1} are collinear; (2) there is an S on A_2P such that ωA_iSA_{i+1}; or (3) there is a Q_{i+1} on A_2P such that $\omega A_1Q_{i+1}A_{i+1}$. If Q_n exists, then $\omega A_1Q_nA_n$ and ωA_2PQ_n; if (1) terminates the process, then ωA_2PA_{i+1}; if (2) terminates the process, then ωA_iSA_{i+1} and ωA_2PS. In all cases we then have:

Entrapment Theorem. If the vertices of a polygonal region (open or closed) are in a convex set, then the entire region is in the convex set.

An equally important theorem is the following.

Containment Theorem. Given any finite set of points, not all collinear, there is a closed polygonal region containing the set and all its vertices are points of the set.

Outline of proof. Let n be the number of noncollinear points (so that $n \geq 3$). The theorem holds for $n = 3$. We will assume that it holds for all $n < k$ for some $k \geq 4$ and show that it also holds for $n = k$. The only case that needs to be investigated is the one in which no proper subset of the k points consists of the vertices of a closed region containing all the k points. It follows that each subset of k-1 points consists of the vertices of a closed region which does not contain the kth point. Pick such a region having k-1 of the points as vertices. Then at least one of the half-planes whose intersection forms the region fails to contain the kth point (call it A_k). Call A_1 and A_{k-1} the two vertices whose join determines that half-plane, and let $A_1, A_2, \ldots, A_{k-1}$ be the consecutive vertices of the region. Suppose A_k is on the side of A_1A_2 opposite from A_{k-1}. Now A_k and A_2 are on opposite sides of A_1A_{k-1}, so by the Crossing Theorem A_1 is an interior point of the open triangular region $A_2A_{k-1}A_k$, and by the Entrapment Theorem A_1 is interior to the region whose vertices are (in some order) A_2, A_3, ..., A_{k-1}, A_k. But the case under consideration requires A_1 to be outside this region. This contradiction shows that A_k is on the same side of A_1A_2 as A_{k-1}, and so all A_i for $i \geq 3$ are on the same side of A_1A_2. A similar argument shows that A_k, A_1, A_2, ..., A_{k-3} are all on the same side of $A_{k-2}A_{k-1}$. Suppose that there is an index j, $2 \leq j \leq k-3$, such that A_k is not on the same side of A_jA_{j+1} as the remaining $k-3$ points. Then there are points R and S on A_jA_{j+1} such that ωA_1RA_k and ωA_kSA_{k-1}. On A_1A_{k-1} is a point Q such that ωA_jQA_k. That $\omega A_1A_{k-1}Q$ is false follows from our result concerning line $A_{k-1}A_{k-2}$; that $\omega A_{k-1}A_1Q$ is false follows similarly from the result concerning A_1A_2, unless $j = 2$, in which case A_1 would be interior to triangle $A_2A_kA_{k-1}$. Therefore we must have ωA_1QA_{k-1}. Now consider triangle $A_1A_kA_{k-1}$. By Fano's Theorem, there is no T on RS ($= A_jA_{j+1}$) such that ωA_1TA_{k-1}. Now, applying the Pasch axiom to triangle A_1QA_k, we find that A_jA_{j+1} must intersect A_kQ in A_j with ωA_kA_jQ, and we have the desired contradiction.

Now suppose that, for some $i \notin \{1,k-1,k\}$, we have A_i and A_{k-1} on different sides of A_1A_k. Then there is an X on A_1A_k such that ωA_iXA_{k-1}. We already have conditions that ensure the existence of a Y such that ωA_iYA_k and ωA_1YA_{k-1}. From the Crossing Theorem, there is a Z such that ωA_1ZX and ωA_iZY. But clearly $Z = A_k$, and we have a contradiction. Similarly, A_kA_{k-1} determines a half-plane which contains the remaining A_i's. □

An *interior point* of an open or closed polygonal region with vertices $A_1, A_2, \ldots, A_n$ is a point of the region that is not on any of the closed intervals A_iA_{i+1} (the *sides* of the region if it is closed). That there is at least one interior point for every region follows from the fact that, for every point Q such that ωA_2QA_3, there is a point P such that ωA_1PQ. Here P is an interior point of the region.

Let Ψ be a closed polygonal region, and let Ω be the set of its interior points. Then Ω is an open polygonal region having the same vertices as Ψ in the same order. We wish to show that Ψ is $\bar{\Omega}$, the closure of Ω, that is, that every point on a side of Ψ is a boundary point of both regions. Choose a point P on side A_iA_{i+1} of Ψ (possibly even $P = A_i$) and a point $Q \in \Omega$. Let U be any open set such that $P \in U$; then there is an open triangular region T such that $P \in T$ and $T \subseteq U$. Line PQ intersects T in an open interval, say BE. Choose points C and D such that ωBCP and ωPDE. We assume that the labelling is such that ωQPC and either ωQDP or ωDQP. Then C is exterior to both Ψ and Ω, and either D is interior to both or else $Q \in$ BE. Therefore P is a boundary point of both regions.

Corollary. Every closed polygonal region is contained in an open polygonal region.

Outline of proof. Let P be an interior point of the given closed polygonal region with n vertices A_i. By Axiom O_3 there are points B_i such that ωPA_iB_i. By the Containment Theorem, there is a closed polygonal region Ψ all of whose vertices are among the $2n+1$ points $P, A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$. By the Entrapment

Theorem, P is an interior point of Ψ. Now, relative to Ψ, each B_i is either an interior point, or a point on one side, or else it is a vertex. In any case, the A_i are interior points of Ψ, so by the Entrapment Theorem the original closed region is contained in an open polygonal region (which consists of the interior points of Ψ). □

We are now in a position to establish condition (2) of Bray's Lemma. Suppose we have a finite union of open polygonal regions. By the Containment Theorem, the vertices of those regions lie in a closed polygonal region. By the Entrapment Theorem, the union of those regions lies in this closed region. By the above Corollary, this union lies in an open polygonal region.

6. *At last! The Heine-Borel Theorem.*

In this final section, we will establish that every closed polygonal region is compact. We will then have in place all the conditions of Bray's Lemma and will be able to confirm the educated guess of L.A. Rubel, announced earlier, that *in any complete ordered plane the Heine-Borel Theorem holds when sets are bounded by open polygonal regions*. Our strategy will be to show that every closed polygonal region is a finite union of closed triangular regions, and then to show, as was done in Frand's Master's Thesis (referred to in [7]), that every closed triangular region is compact.

For $n \geq 4$, we establish that a closed polygonal region of n vertices is the union of a closed polygonal region of n-1 vertices and a closed triangular region. The desired result will then follow by induction. Let the closed polygonal region have vertices $A_1, A_2, \ldots, A_n$. We show that $A_1, A_2, \ldots, A_{n-1}$ are also the vertices of a closed polygonal region. This follows from the fact that if some A_i lies on the wrong side of A_1A_{n-1}, then it is an interior point of triangular region $A_1A_{n-1}A_n$ and thus also an interior point of the n-sided region, which is a contradiction. By the Entrapment Theorem, the n-sided region contains the (n-1)-sided one and the triangular one. Now any boundary point of the n-sided region is a boundary point of one of the two smaller regions. An interior point of the n-sided region lies on A_1A_{n-1} or else it lies in one of the two smaller regions. If the point is on A_1A_{n-1}, then it is a boundary point of both smaller regions. Thus every closed polygonal region is the union of closed triangular regions.

It is well known that a set is compact if and only if for any family Γ of *base members* there is a finite refinement of Γ. Here we will use only triangular regions for base members. An argument (not given here) can be pieced out from Forder [2] and Kelley [6] to show that closed intervals are compact.

Frand's Lemma. Let Γ be a family of open triangular regions whose union contains a closed interval AB, and let θ be the union of the members of a finite refinement of Γ. If a point C is not on line AB, then there is a point D such that ωBDC and the closed triangular region ABD is contained in θ.

Proof. Suppose θ contains only one base member. If C is an interior point of the base member, then by the Entrapment Theorem we are done. If C fails to be so conveniently situated, then either it is on a side or else it is an exterior point, in which case there is a point on a side between B and C. In any event the interior point D can be found and, as before, we are done.

Suppose the lemma is true for any finite refinement having k or fewer base members, and consider a finite refinement with $k+1$ base members. We need only consider the case where, any one of the base members having been removed, the union of the remaining base members no longer contains the closed interval AB. The base member containing B does not contain A; therefore A and B are on opposite sides of some side of the base member unless A is on that side. In any event, there is a point E on that side of the base member such that E = A or ωAEB. The closed interval AE is contained in a finite union θ' of k base members. Again we can guarantee that, if any one of the members is removed, then the union of the remaining members no longer contains the closed interval AE. The base member containing E does not contain B and, as before, there is a point F on one of its sides such that F = B or ωBFE. Let the point M satisfy ωFME. Then θ' contains the closed interval AM and, by the induction hypothesis, there is a point G such that ωMGC and the closed triangular region AMG is contained in θ'. Moreover, such a G can be chosen so that it is in the base member containing B as well. Let H be such that ωAGH and ωBHC. If H is in the base member containing B, then the closed polygonal region with vertices B,M,G,H is also in this base member, and so the closed triangular region ABH is in θ, so we can let D = H. If H is not in the base member containing B, then there is a point D which is in that base member and satisfies ωBDH. Now there is a point N such that ωAND and ωMNG. Then the closed triangular region AMN and the closed polygonal region MNDB are in θ, and their union is the closed triangular region ABD. □

Now suppose some closed triangular region ABC is not compact. Then it is contained in a family Γ of base members for which there is no finite refinement. Since the closed interval AB is compact, there is for it a finite refinement of Γ. By the lemma, there is a point D such that ωBDC and such that the above refinement is also a finite refinement for the closed triangular region ABD. Now we consider points P such that ωBPC and such that there are finite refinements for the closed triangular regions ABP. Let α consist of all such points P, of point B itself, and of all points Q such that ωQBC; and let β consist of all the other points of line BC. It can be shown that the hypotheses of Dedekind's Axiom hold. Hence there is a point Z such that ωXZY for any $X \in \alpha-\{Z\}$ and any $Y \in \beta-\{Z\}$. The closed interval AZ is compact. By the lemma, there are points X and Y such that ωBXZ and ωZYC, and such that there is a finite refinement for each of the closed triangular regions AXZ and AZY. Thus there are finite refinements for closed triangular regions ABX, ABZ, and ABY. But $Y \in \beta$, and this is the desired contradiction.

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* * *

NOTES ON NOTATION: VII

LEROY F. MEYERS

"Consider the distinct numbers

$$7(3 + 4) \quad \text{and} \quad -7(2 - 9).\text{"}$$

"Distinct? Don't they both have the value 49?"

"But don't you write: 'Consider the distinct points

$$A(3, 4) \quad \text{and} \quad B(2, -9).\text{',}$$

where 'A' and '(3, 4)' are just two names for the same point? Why not do the same thing with numbers?"

"But numbers aren't points! When expressions for two numbers are written down one after the other, the value of the entire expression is the product of the numbers, but ..."

"What about '23'? Surely you don't mean '6' by that!"

"Of course not! When the expressions for the numbers consist entirely of digits (except for a possible sign or decimal point), then I use parentheses or a dot to separate the numbers. As I was about to say, however, since numbers aren't points, no confusion can result by writing 'A(3, 4)'. After all, we don't multiply points."

"But don't you use 'AB' to denote the distance between the points A and B—or perhaps the segment or line determined by them? Here I interpret 'A(3, 4)', for example, as the segment with endpoints A and (3, 4)."

"I never thought of that. What do you use?"

"I'd write: 'Consider the distinct points A = (3, 4) and B = (2, -9).'"

"But that's ungrammatical. 'A = (3, 4)' is a sentence with the verb read 'equals', or 'is equal to'. It can't be put into apposition with the noun 'points'."

"In this context, I read '=' as 'equal to', without 'is', or as 'which is equal to'. In fact, I use the latter with continued equalities and similar mathematical statements, such as '$x < y = z$', which I read as 'x is less than y, which is equal to z'."

"So you use different readings of the same symbol?"

"Unfortunately, yes. Of course, '$x < y = z$' is really an abbreviation for '$x < y$ and $y = z$', and so could be read as 'x is less than y and y equals z'. Some of my students don't realize this, and write the solution of the inequality $x^2 > 1$ as '$-1 > x > 1$', when they mean '$-1 > x$ *or* $x > 1$'.

"What about 'Let $x = 3$.'?"

"Here again I use a different translation: 'Let x equal 3.' or 'Let x be equal to 3.', with a different form of the verb. I'm somewhat uneasy about this, but see no convenient way to avoid it if I don't want to write down the word 'equal' explicitly. While this is seldom confusing, I find myself often confused by the similar-looking 'Define $x = a + b$.'. On first reading, I expect this to be followed by a definition of the entire expression '$x = a + b$'. Usually, however, the author means: 'Define "x" to be "$a + b$".', or 'Define: $x = a + b$.', or "Make the definition: $x = a + b$.'; in the last two cases, the simple sign '=' may be replaced by the more explicit ':=' or '::='."

"I see. So now what's this about '7(3 + 4)' and '-7(2 - 9)'?"

"I was just pulling your goatee."

Mathematics Department, The Ohio State University, 231 West 18 Avenue, Columbus, Ohio 43210.

* * *

THE OLYMPIAD CORNER: 49

M.S. KLAMKIN

First a correction: the number of students who participated in the Twenty-Fourth International Mathematical Olympiad was 186, not 192 as I reported earlier [1983: 205].

I shall later give solutions to several problems proposed here earlier, but first I present two new problem sets. The first consists of the problems set at the 1983 Austrian Mathematical Olympiad. They were given to me by Thomas Mulgassner and translated from the German by Andy Liu, to both of whom my thanks. The second is a set of problems proposed in the April 1983 issue of *Kvant*, a Russian journal in mathematics and physics for secondary schools. For all of these problems I invite readers to send me elegant solutions for possible publication later in this column.

*

1983 AUSTRIAN MATHEMATICAL OLYMPIAD

First day, June 15 — 4½ hours

1. For natural numbers x, let $Q(x)$ be the sum and $P(x)$ the product of the digits of x (in base ten). Show that, for each natural number n, there exist infinitely many natural numbers x such that

$$Q(Q(x)) + P(Q(x)) + Q(P(x)) + P(P(x)) = n.$$

2. Let x_1, x_2, x_3 be the roots of

$$x^3 - 6x^2 + ax + a = 0.$$

Determine all real numbers a such that

$$(x_1 - 1)^3 + (x_2 - 2)^3 + (x_3 - 3)^3 = 0.$$

Also, for each such a, determine the corresponding values of x_1, x_2, x_3.

3. Let P be any point in the plane of a triangle ABC, and let A'B'C' be the cevian triangle of the point P for the triangle ABC (with A' = AP ∩ BC, etc.). If the vertices of triangle A"B"C" are defined by

$$\vec{AA'} = \vec{A'A''}, \quad \vec{BB'} = \vec{B'B''}, \quad \vec{CC'} = \vec{C'C''},$$

show that

$$[A''B''C''] = 3[ABC] + 4[A'B'C'],$$

where the square brackets denote the signed area of a triangle.

Second day, June 16 — 4½ hours

4. The sequence $\{x_n\}$ is defined as follows: $x_1 = 2$, $x_2 = 3$, and

$$x_{2m+1} = x_{2m} + x_{2m-1}, \qquad m \geq 1;$$
$$x_{2m} = x_{2m-1} + 2x_{2m-2}, \qquad m \geq 2.$$

Determine x_n (as a function of n).

5. Let N be the set of natural numbers. For all $(a,b) \in N \times N$, find all the solutions $(x,y) \in N \times N$ of the equation

$$x^{a+b} + y = x^a y^b.$$

6. Let π_1 and π_2 be two planes in Euclidean space R^3. These planes effect a partition of the *reduced space* $S \equiv R^3 - (\pi_1 \cup \pi_2)$ into several components. Show that, for any cube in R^3, at least one of the components of S has a nonempty intersection with at least three faces of the cube.

*

PROBLEMS FROM *KVANT* (April 1983)

M796. *Proposed by L.D. Kurliandchik.*

Find $\angle APB$ if P is a point inside a square ABCD such that

$$PA : PB : PC = 1 : 2 : 3.$$

M797. *Proposed by D.B. Fuchs.*

It is well known that the last digit of the square of an integer is one of the following: 0, 1, 4, 5, 6, 9. Is it true that any finite sequence of digits may appear before the last one, that is, for any sequence of n digits $\{a_1, a_2, \ldots, a_n\}$ there exists an integer whose square ends with the digits $a_1a_2\ldots a_nb$, where b is one of the digits listed above?

M798. *Proposed by S.V. Fomin.*

$4k$ points on a circle are painted alternately red and blue; then the $2k$ red points are joined pairwise by k red line segments, and the $2k$ blue points by k blue segments. If no three of the segments are concurrent, prove that there are at least k intersection points of red line segments with blue ones.

M799. *Proposed by S.S. Vallander.*

(a) Find a solution of the equation

$$3^{x+1} + 100 = 7^{x-1},$$

and prove that it is unique.

(b) Find two solutions of the equation

$$3^x + 3^{x^2} = 2^x + 4^{x^2},$$

and prove that there are no other solutions.

M800. (a) The nodes of a square lattice are all the points of the plane both coordinates of which are integers. One of the nodes is the origin O. For each of the other nodes P, construct the line in which O and P are symmetric, that is, the perpendicular to OP at its midpoint. These lines partition the plane into little parts (triangles and convex polygons). To each of them assign an integer (its *rank*) as follows: the part containing O (which is a square) is of rank 1, the parts which have a common side with it are of rank 2, the remaining parts which have a common side with parts of rank 2 are of rank 3, etc. Prove that the total area of all parts of rank r is the same for all positive integers r.

(b) Is a similar statement true for an arbitrary lattice consisting of parallelograms (in particular, of rhombuses with 60° angles)? For a lattice consisting of regular hexagons?

*

4. [1983: 138] Determine the greatest common divisor of $n^2 + 2$ and $n^3 + 1$, where $n = 9^{753}$.

Solution by Noam D. Elkies, student, Columbia University.

Let (A,B) be the required g.c.d., where $A = n^3 + 1$ and $B = n^2 + 2$. From

$$(2n+1)A - (2n^2+n-4)B = 9,$$

it follows that $(A,B)|9$. However, $A \equiv 1 \pmod 9$, and so $(A,9) = 1$. Therefore $(A,B) = 1$.

K-1, [1983: 72] Each side of a given triangle is divided into three equal parts. The six points of division are the vertices of two triangles whose intersection is a hexagon. find the area of the hexagon in terms of the area S of the given triangle.

Solution by Noam D. Elkies, student, Columbia University.

Since the ratio of two areas is an affine invariant, it suffices to assume that the given triangle is equilateral, of side length, say, a. Then the hexagon of intersection is easily seen to be regular and to have a side length of $(\sqrt{3}/9)a$. Hence the area K of the hexagon is

$$K = 6\left(\frac{\sqrt{3}}{9}\right)^2 S = \frac{2S}{9}.$$

K-2, [1983: 72] The sequence $(a_1,a_2,\ldots,a_n)$ is a permutation of $(1,2,\ldots,n)$.

(a) Prove that $|a_1-a_2| + |a_2-a_3| + \ldots + |a_n-a_1| \geq 2n-2$.

(b) For how many distinct permutations of $(1,2,\ldots,n)$ does equality hold in (a)?

Solution by Noam D. Elkies, student, Columbia University.

We will write $a_{n+m} = a_m$ whenever $1 \leq m \leq n$. As a consequence, the permutation $(a_1, a_2, \ldots, a_n)$ can be represented equally well by

$$(a_\alpha, a_{\alpha+1}, \ldots, a_{\omega-1}, a_\omega, a_{\omega+1}, \ldots, a_{\alpha-1}), \qquad (1)$$

where $a_\alpha = 1$, $a_\omega = n$, and $1 \leq \alpha,\omega \leq n$.

(a) We have

$$\begin{aligned}
|a_1-a_2| + |a_2-a_3| + \ldots + |a_n-a_1| &= \sum_{j=\alpha}^{\omega-1}|a_j-a_{j+1}| + \sum_{j=\omega}^{\alpha-1}|a_j-a_{j+1}| \\
&\geq \left|\sum_{j=\alpha}^{\omega-1}(a_j-a_{j+1})\right| + \left|\sum_{j=\omega}^{\alpha-1}(a_j-a_{j+1})\right| \\
&= |a_\alpha - a_\omega| + |a_\omega - a_\alpha| \\
&= |1 - n| + |n - 1| \\
&= 2n - 2.
\end{aligned}$$

(b) It is clear from part (a) that equality occurs just when the a_j increase monotonically from $a_\alpha = 1$ to $a_\omega = n$, then decrease monotonically from a_ω back to a_α.

In (1), the number α can be chosen in n ways; and, for each such choice, each of the $n-2$ numbers k, $2 \le k \le n-1$, can be placed in exactly two ways: each can be placed somewhere in

$$A \equiv (a_{\alpha+1}, \ldots, a_{\omega-1})$$

or else somewhere in

$$B \equiv (a_{\omega+1}, \ldots, a_{\alpha-1}),$$

and the number ω is uniquely determined once one of the sequences A or B has been chosen. If we consider as equivalent all permutations (1) which differ only by the order of the elements in A or B, then there are $n \cdot 2^{n-2}$ different equivalence classes. And in each equivalence class there is exactly one permutation for which equality holds in (a): it is the permutation where the elements of A are in monotonically increasing order and those of B are in monotonically decreasing order.

Thus the required number is $n \cdot 2^{n-2}$.

Comment by M.S.K.

As a rider, find the maximum value of the sum in (a) and the number of permutations for which this maximum is attained.

K-3. [1983: 73] Let $k > 2$ be a given natural number. Does there exist an infinite set E of natural numbers such that, for every finite subset A of E,

$$\sum_{a_i \in A} a_i \neq b^k$$

for any natural number b?

Solution by Noam D. Elkies, student, Columbia University.

More generally, we prove that there exists an infinite set E of natural numbers such that, for every finite subset A of E,

$$\sum_{a_i \in A} a_i \neq u_j, \qquad j = 1,2,3,\ldots,$$

where $\{u_1,u_2,u_3,\ldots\}$ is any given strictly increasing infinite sequence of integers that has $u_1 > 0$ and has arbitrarily large gaps. A result slightly stronger than the proposed one will then follow if we take $u_j = j^k$ for some fixed integer $k \ge 2$.

We generate the required infinite set $E = \{e_1,e_2,e_3,\ldots\}$ by an inductive procedure. For e_1, we take any positive integer not among the u_j (the gap property ensures its existence). Having chosen

$$E_n \equiv \{e_1, e_2, \ldots, e_n\}$$

for some $n \geq 1$, we choose a u_j such that $u_j \geq e_n$ and

$$u_{j+1} - u_j \geq \sum_{i=1}^{n} e_i + 2$$

(this is possible because of the gap property), and then set $e_{n+1} = u_j + 1$. We must now show that the resulting infinite set E has the desired property.

The empty set is a finite subset of E, and the sum of its elements, which we take to be zero, is not among the u_j. Since every nonempty finite subset A of E is a subset of E_n for some $n \geq 1$, it suffices to show that for every subset E_n, $n \geq 1$, the sum of the elements is not among the u_j. This is certainly true for the subset $E_1 = \{e_1\}$, by the choice of e_1. If $n > 1$, we have

$$e_n = u_r + 1 \qquad \text{and} \qquad u_{r+1} - u_r \geq \sum_{i=1}^{n-1} e_i + 2$$

for some r; hence

$$u_r < e_n < \sum_{i=1}^{n} e_i = e_n + \sum_{i=1}^{n-1} e_i \leq (u_r+1) + (u_{r+1}-u_r-2) = u_{r+1} - 1 < u_{r+1}.$$

Thus the sum of the elements of E_n lies strictly between u_r and u_{r+1}, and so is not among the u_j. This completes the proof.

K-4. [1983: 73] Some turtles are creeping in the plane in different (constant) directions, but at the same speed. Prove that the turtles will eventually be located at the vertices of a convex polygon, no matter what their initial positions were.

Solution by Noam D. Elkies, student, Columbia University.

It is a safe assumption that the number of turtles, say n, is finite. The desired result holds trivially if $n = 1$, 2, or 3, so we assume that $n > 3$. All vectors used in the solution will be position vectors with origin at some fixed point O in the plane.

The turtles having been arbitrarily numbered from 1 to n, let the position vector of the kth turtle at time $t \geq 0$ be $\vec{r}_k(t)$, and let its constant velocity be $\vec{v}_k$. We then have

$$\vec{T}_k(t) = \vec{T}_k(0) + t\vec{v}_k, \qquad k = 1,2,\ldots,n.$$

Let P(t), P'(t), and P" denote the simple polygons whose vertices are the endpoints of $\vec{T}_k(t)$, $\vec{T}_k(t)/t$, and $\vec{v}_k$, respectively, $k = 1,2,\ldots,n$, at any time $t > 0$. (Note that consecutive vertices of P(t), and the corresponding vertices of P'(t) and P", need not correspond to consecutive values of k.)

Since the $\vec{v}_k$ are all distinct and $|\vec{v}_k| = v$ is the same for all k, it follows that the vertices of P" are n distinct points on a circle with center O and radius v. So P" is a nondegenerate convex n-gon, that is, it has n interior angles, all less than 180°. Since P(t) and P'(t) are homothetic polygons, P(t) is a nondegenerate convex n-gon if and only if P'(t) is also a nondegenerate convex n-gon. Now P(t) is a nondegenerate n-gon for any t (hence also P'(t)), since if two turtles were due to pass over the same point at the same time, one would have to slow down to let the other pass, contradicting the fact that each turtle has a constant velocity. Now let $V_k'(t)$ and V_k'' denote the measures of the kth interior angle of P'(t) and P", respectively. Since $V_k'(t)$ varies continuously with t and

$$\lim_{t\to\infty} \vec{T}_k(t)/t = \vec{v}_k, \qquad k = 1,2,\ldots,n,$$

it follows that

$$\lim_{t\to\infty} V_k'(t) = V_k'' < 180^\circ, \qquad k = 1,2,\ldots,n;$$

hence P'(t) is eventually convex, and so is P(t).

We conclude that P(t) is eventually a nondegenerate convex n-gon. □

The same result follows if the endpoints of the n distinct vectors $\vec{v}_k$ are the vertices of a strictly convex n-gon, cyclic or not. Requiring that $|\vec{v}_k|$ be the same for all k is just one way to achieve this.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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RARA AVIS

If you wish to upset the law that all crows are black, you must not seek to show that no crows are: it is enough if you prove one single crow to be white.

WILLIAM JAMES (1842-1910), quoted in *TIME*, November 14, 1983.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

881. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find the unique solution to the following "areametic", where A, B, C, D, E, N, and R represent distinct decimal digits:

$$\int_{\mathrm{B}}^{\mathrm{D}} \mathrm{C}x^{\mathrm{N}}dx = \mathrm{AREA}.$$

882. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

The interior surface of a wine glass is a right circular cone. The glass, containing some wine, is first held upright, then tilted slightly but not enough to spill any wine. Let D and E denote the area of the upper surface of the wine and the area of the curved surface in contact with the wine, respectively, when the glass is upright; and let D_1 and E_1 denote the corresponding areas when the glass is tilted. Prove that

$$\text{(a)}\quad E_1 \geq E, \qquad \text{(b)}\quad D_1 + E_1 \geq D + E, \qquad \text{(c)}\quad \frac{D_1}{E_1} \geq \frac{D}{E}.$$

883. *Proposed by J. Tabov and S. Troyanski, Sofia, Bulgaria.*

Let ABC be a triangle with area S, sides a,b,c, medians m_a,m_b,m_c, and interior angle bisectors t_a,t_b,t_c. If

$$t_a \cap m_b = \mathrm{F}, \qquad t_b \cap m_c = \mathrm{G}, \qquad t_c \cap m_a = \mathrm{H},$$

prove that

$$\frac{\sigma}{S} < \frac{1}{6},$$

where σ denotes the area of triangle FGH.

884. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

The eccentric warden revisited (see Crux 722 [1983: 89]).

A prison warden has n prisoners in n cells, one prisoner per cell, with all cells initially closed. He also has a secret function

$$f: \{1,2,\ldots,n\} \to \{1,2,3,\ldots\}$$

with the following property: For $k = 1,2,\ldots,n$, on day k each cell $k,2k,3k,\ldots$ is reversed $f(k)$ times (from open to closed or vice versa). Thus, on day 1 all cells are reversed $f(1)$ times; on day 2, cells $2,4,6,\ldots$ are reversed $f(2)$ times; etc. Each reversal (from open to closed or vice versa) is counted once towards the $f(k)$ times. The end result is that, after the n days have elapsed, cell k has been reversed a total of exactly k times, $k = 1,2,\ldots,n$.

Find all functions f with this property.

885. *Proposed by Charles W. Trigg, San Diego, California.*

Planes are drawn perpendicular to the four space diagonals of a cube at their trisection points. What is the nature of the solid bounded by these planes? What is the volume of the solid in terms of the edge, e, of the cube.

886*. *Proposed by A.W. Goodman, University of South Florida.*

Prove that

(a) $$\sum_{k=1}^{n-1} (-1)^{k+1}(n-k)^2 = \frac{n(n-1)}{2};$$

(b) $$\sum_{k=1}^{n-1} (-1)^{n-k-1}k^2(n-k)^2 = \frac{n}{4}\{1+(-1)^n\}.$$

887. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

$A_1A_2A_3A_4$ is an isosceles trapezoid, with $A_4A_3 \parallel A_1A_2$, whose circumcircle has center O. The midpoints of the segments A_4A_3 and A_1A_2 are U and V, respectively; and l_4, the Wallace-Simson line of A_4 with respect to triangle $A_1A_2A_3$, intersects UV in S.

Prove that (a) $l_4 \parallel OA_3$, and (b) US = OV.

888. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

(a) Find all solutions in natural numbers of the system

$$x + y = zw, \qquad xy = z + w.$$

(b) Show that the system has infinitely many solutions in integers.

889. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

$A_1A_2\ldots A_n$ is a regular n-gon ($n \ge 3$) inscribed in a circle of radius r; M is the midpoint of the arc A_1A_n; and, for $i = 1,2,\ldots,n$, P_i is the orthogonal projection of A_i upon a fixed diameter D of the circle. Prove the following:

(a) $\displaystyle\sum_{i=1}^{n} \overrightarrow{A_iP_i} = \vec{0}$;

(b) $\displaystyle\sum_{i=2}^{n} A_1A_i = 2r\cot\frac{\pi}{2n}$ and $\displaystyle\prod_{i=2}^{n} A_1A_i = nr^{n-1}$;

(c) if $n = 2m$, then $\displaystyle\prod_{i=1}^{m} MA_i = \sqrt{2}r^m$ and $\displaystyle\prod_{i=2}^{m} A_1A_i = \sqrt{m}r^{m-1}$;

(d) if $n = 2m+1$, then $\displaystyle\prod_{i=1}^{m} MA_i = r^m$.

890. *Proposed by Leroy F. Meyers, The Ohio State University.*

Construct triangle ABC, with straightedge and compass, given the lengths b and c of two sides, the midpoint M_a of the third side, and the foot H_a of the altitude to that third side.

* * *

THE PUZZLE CORNER

Puzzle No. 47: Rebus (3-6)

L^2

In Canto I is written:
"I dug the REBUS pitkin;"
It will be found
In Ezra Pound.

Puzzle No. 48: Deletion (10, 9)

In every FIRST the count of feet is four;
A LAST has three quadrillion feet, and more.

ALLAN WAYNE, Holiday, Florida

Answer to Puzzle No. 42 [1983: 256]: A case of mistaken identity.

Answer to Puzzle No. 43 [1983: 256]: The characteristic.

Answer to Puzzle No. 44 [1983: 256]: The Constant Symbol.

Answer to Puzzle No. 45 [1983: 256]: Factory into hands (Factor Y into H and S).

Answer to Puzzle No. 46 [1983: 256]: Coversines (C over S in E's).

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

83. [1975: 84; 1976: 28] *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que le produit de deux, trois ou quatre entiers positifs consécutifs n'est jamais un carré parfait.

II. *Comment by Leroy F. Meyers, The Ohio State University.*

Let $P_n = m(m+1)\ldots(m+n-1)$. This problem shows that P_2, P_3, and P_4 are never squares. In a comment following the solution, the editor listed eight additional properties of P_n (which he labeled (a) to (h)), all of which he found in Dickson [1]. It may interest readers to know of other properties of P_n discovered since 1920 (when [1] was first published). I wish to thank Jack Tull, who supplied reference [7], which led to references [2]-[6].

(i) P_n is never a square [3,4].

(j) Given $l > 1$, there is a c such that, for $n > c$, P_n is never an lth power [5 and unpublished Rigge].

(k) There is a c such that, for all $l > 1$ and $n > c$, P_n is never an lth power [6 and unpublished Erdös-Siegel].

Results obtained between those mentioned in Dickson [1] and Erdös [4] are described in [2], which gives many references, whereas [7] is a summary of results. Erdös also considers when a binomial coefficient can be a power.

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2. Richard Obláth, "Über Produkte aufeinander folgender Zahlen", *Tôhoku Mathematical Journal*, 38 (1933) 73-92.

3. Olov Rigge, "Über ein diophantisches Problem", *Skandinaviske Matematikerkongres*, 9 (1938) 155-160.

4. P. Erdös, "Note on products of consecutive integers", *Journal of the London Mathematical Society*, 14 (1939) 194-198.

5. _______, _______II, *ibid.*, 245-249.

6. _______, _______III, *Indagationes Mathematicae*, 17 (1955) 85-90.

7. _______, *On the product of consecutive integers*, Report of the Institute in the Theory of Numbers, Boulder, Colorado, June 21-July 17, 1959.

* * *

502. [1980: 15; 1981: 22] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Given $n > 3$ points in the plane, no three collinear, we are interested in "triangulating" their convex hull, that is, in covering it with nonoverlapping triangles, each having three of the given points as vertices.

(a) For a fixed set of points, there are several ways of triangulating. Do they all give the same number of triangles?

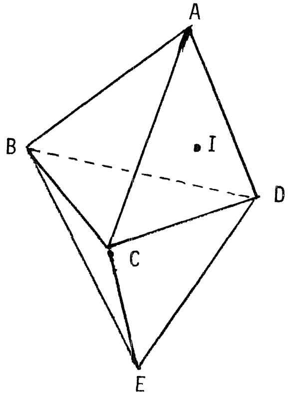
(b) For fixed n, different sets of n points may be triangulated with different numbers of triangles. What bounds can be given for the number of triangles?

III. *Comment by M.S. Klamkin and Andy Liu, University of Alberta.*

In comment II [1981: 23], the following statement is given without proof: *Suppose n points are given in general position in space (no four in a plane) whose convex hull has q vertices, where* $4 \le q \le n$. *Then the tetrahedration of the interior of the convex hull contains*

$$\tau(n,q) = 3n - q - 7$$

tetrahedra.



This statement is incorrect if $n > 4$. For a counterexample, consider the hexahedron ABCDE shown in the figure, in which $n = q = 5$. There are two distinct tetrahedrations: one into the two tetrahedra ABCD and EBCD; and one into the three tetrahedra AEBC, AECD, and AEDB. So here $\tau(5,5) = 2$ or 3. For a counterexample in which $n > q$, suppose the point I in the figure lies in the interior of ABCD $\cap$ AECD. Then we have two distinct tetrahedrations: one into EBCD and four tetrahedra with vertex I; and one into AEBC, AEDB, and four tetrahedra with vertex I. So $\tau(6,5) = 5$ or 6.

* * *

526. [1980: 78; 1981: 87, 279] *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

The following are examples of *chains* of lengths 4 and 5, respectively:

25, 225, 1225, 81225

25, 625, 5625, 75625, 275625.

In each chain, each link is a perfect square, and each link (after the first) is obtained by prefixing a single digit to its predecessor.

Are there chains of length n for $n = 6,7,8,\dots$?

III. *Comment by Leroy F. Meyers, The Ohio State University.*

I believe that some emendations are needed in solution II [1981: 280-282]. Luckily, they do not affect the result.

On page 280, there is no justification for the assumption that $a_1 \le a_2$. If we want to have $\beta + \alpha \ge \beta - \alpha$, then either $2^i5^j > 2^{k-i}5^{k-j}$ (with probably no restriction on a_1 and a_2), or $i = j = k/2$ (with $a_1 \ge a_2$, not the other way around!), or $2^i5^j < 2^{k-i}5^{k-j}$ (in a very restricted case, with $a_1 \ge 2a_2$). Hence display (3) should read simply "$a_1a_2 = a$" (that is, the triple inequality and "and" should be deleted).

On page 281, the line beginning with "1.234" should instead begin with "0.137"; the two lines beginning with "1.111" should begin with "0.370"; the line beginning with "0.555" should begin with "0.185"; the line beginning with "-0.3658" should begin with "-1.0485"; the line beginning with "0.0654" should begin with "-0.6178"; the first line should have "(9,9)" in place of "(3,9)"; and the inequality in the line beginning with "ever" should read

$$\text{"}0.370\cdot 10^{k/2} > 3^{k-1} > 2^{i}\text{"}.$$

Inequality (8) turns out to be all right.

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749*. [1982: 137; 1983: 190] *Proposed by Ram Rekha Tiwari, Radhaur, Bihar, India.*

Solve the system

$$\frac{yz(x+y+z)(y+z-x)}{(y+z)^2} = a^2$$

$$\frac{zx(x+y+z)(z+x-y)}{(z+x)^2} = b^2$$

$$\frac{xy(x+y+z)(x+y-z)}{(x+y)^2} = c^2.$$

Comment by Leroy F. Meyers, The Ohio State University.

This problem is a restatement of Crux 454(b), by the same proposer. In effect, it asks for the lengths of the sides x, y, and z of a triangle whose angle bisectors have lengths a, b, and c. See the editorial comments appended to the solution to Crux 454 [1980: 125-127].

A similar comment was received from R.C. LYNESS, Southwold, Suffolk, England.

Editor's comment.

It follows from the forgetful editor's earlier comment to Crux 454 that we can kiss this problem goodbye.

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763. [1982: 209] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are n points in general position in space (i.e., no four in a plane). By a *tetrahedration* of their convex hull is meant a partitioning of the convex hull into nonoverlapping tetrahedra each having four of the given points as vertices. Show that the number of edges in every tetrahedration is independent of the number of vertices of the convex hull.

(This problem was inspired by Crux 502 [1981: 22].)

Comments were received from JORDI DOU, Barcelona, Spain; and M.S. KLAMKIN and ANDY LIU, University of Alberta.

Editor's comment.

The stated conclusion is incorrect. The proposer's proof was based on an incorrect lemma (see comment III to Crux 502 on page 279 of this issue).

Let $e(n,q)$ denote the number of edges in a tetrahedration of the interior of the convex hull of n points, q of which lie in the convex hull. Dou gave counter-examples showing that some values of $e(n,q)$ are: $e(5,5) = 9$ or 10; $e(8,8) = 18$, 19, or 22; and $e(8,4) = 22$ or 24.

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767. [1982: 210] *Proposed by H. Kestelman, University College, London, England.*

Let $z_1, z_2, \ldots, z_k$ be complex numbers such that

$$z_1^s + z_2^s + \ldots + z_k^s = 0$$

for all $s = 1,2,\ldots,k$. Must all the z_r be 0?

I. *Solution by the proposer.*

They must all be 0. For suppose they are not, and assume $z_1, z_2, \ldots, z_q$ all nonzero and the rest 0. Let v_s denote the row vector $(z_1^s, z_2^s, \ldots, z_q^s)$, $1 \le s \le k$, and let $v_0 = (1, 1, \ldots, 1)$. Since $v_0, v_1, \ldots, v_k$ are vectors in C^q, and $k+1 > q$, these vectors are linearly dependent: let v_j be the first to be spanned by its predecessors. If

$$v_j = c_0v_0 + c_1v_1 + \ldots + c_{j-1}v_{j-1},$$

it follows, since $v_sv_0^T$ is 0 if $1 \le s \le k$ and is q if $s = 0$, that $c_0 = 0$. This implies that

$$z_r^{j-1} = \sum_{s=1}^{j-1} c_s z_r^{s-1}, \qquad r = 1,2,\ldots,q;$$

hence v_{j-1} is spanned by its predecessors, and this contradicts the definition of j. □

A proof can be constructed using the elementary symmetric functions of the roots of a polynomial, but the proof above seems more elementary in that it uses only linear independence and makes no appeal to factorizability.

II. *Solution by M.S. Klamkin, University of Alberta.*

The answer is yes, and this is a known result. See [1] for a proof based on the Vandermonde determinant. More generally, the author and D.J. Newman have shown [2] that, if the sum vanishes for

$$s = n,\ n+1,\ n+2,\ \ldots,\ n+k-1,$$

then all the $z_r = 0$. Another extension from [2] is the following: if

$$\sum_{i=1}^{k} z_i^s = \sum_{i=1}^{k} a_i^s, \qquad s = 1,2,\ldots,k, \quad a_i \text{ given},$$

then, aside from permutations, $(z_1,z_2,\ldots,z_k) = (a_1,a_2,\ldots,a_k)$. For the case $k = 3$, the author has conjectured [3] that, if

$$z_1^{s_i} + z_2^{s_i} + z_3^{s_i} = 0, \qquad i = 1,2,3,$$

where the s_i are relatively prime positive integers one of which is divisible by 2 and one by 3, then all the $z_r = 0$.

Also solved by STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

REFERENCES

1. Harley Flanders, "An application of the Vandermonde determinant", *American Mathematical Monthly*, 60 (1953) 708.

2. M.S. Klamkin and D.J. Newman, "Uniqueness theorems for power equations", *Elemente der Mathematik*, 25 (1970) 130-134.

3. Problem 6312* (proposed by M.S. Klamkin), *American Mathematical Monthly*, 87 (1980) 675; partial solution by Constantine Nakassis, *ibid.*, 89 (1982) 505.

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768. [1982: 210] *Proposed by Jack Garfunkel, Flushing, N.Y.; and George Tsintsifas, Thessaloniki, Greece.*

If A,B,C are the angles of a triangle, show that

$$\frac{4}{9}\Sigma \sin B \sin C \le \Pi \cos\frac{B-C}{2} \le \frac{2}{3}\Sigma \cos A,$$

where the sums and product are cyclic over A,B,C.

Solution by Vedula N. Murty, Pennsylvania State University, Capitol Campus.

Usina the known identities

$$\Pi \cos\frac{B-C}{2} = \frac{1}{4}(1 + \Sigma\cos(B-C)) = \frac{1}{4}(1 - \Sigma\cos A + 2\Sigma\sin B\sin C),$$

the proposed inequalities are easily shown to be equivalent to

$$27(1 + 2\Sigma\sin B\sin C) \le 99\Sigma\cos A \le 11(9 + 2\Sigma\sin B\sin C). \qquad (1)$$

With the usual meanings for R,r,s, and $x = r/R$, $y = s/R$, the known results

$$\Sigma\sin B\sin C = \frac{r^2 + s^2 + 4Rr}{4R^2} = \frac{1}{4}(x^2 + y^2 + 4x)$$

and

$$\Sigma\cos A = \frac{R + r}{R} = 1 + x$$

transform (1) into the equivalent

$$-x^2 + 14x \le y^2 \le -x^2 + \frac{10}{3}x + \frac{16}{3}. \qquad (2)$$

Expressed in terms of x and y, Blundon's "best quadratic inequalities" (see Crux 653 [1982: 190]),

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \le s^2 \le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr},$$

are found to be equivalent to

$$-x^2 + 10x + 2 - 2(1-2x)\sqrt{1-2x} \le y^2 \le -x^2 + 10x + 2 + 2(1-2x)\sqrt{1-2x}. \qquad (3)$$

If we add

$$-2(1-2x)(1 - \sqrt{1-2x}) \quad (\le 0)$$

to the left member of (3) and

$$2(1-2x)(\frac{5}{3} - \sqrt{1-2x}) \quad (\ge 0)$$

to its right member, we obtain (2). Hence (3) implies (2), and the proof is complete.

Equality occurs just when $1-2x = 0$, or $R = 2r$, that is, just when the triangle is equilateral.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposers.

* * *

769. [1982: 210] *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

For each positive integer n, let $S_n = x^n+y^n+z^n$, where x,y,z are real numbers. Given that $S_1 = 0$, express S_n as a polynomial in S_2 and S_3.

Solution by the proposer.

It will suffice to express S_n explicitly as a polynomial in $X = S_2/2$ and $Y = S_3/3$. To accomplish this, we will need the following known recurrence relation (see Crux 639 [1982: 145]):

$$S_{n+3} = \tfrac{1}{2}S_2S_{n+1} + \tfrac{1}{3}S_3S_n = XS_{n+1} + YS_n, \qquad n = 0,1,2,\ldots \quad . \tag{1}$$

Consider the generating function

$$S(t) = 3 + S_2t^2 + S_3t^3 + \ldots + S_nt^n + \ldots \quad .$$

Using (1), it is easy to show that

$$(1 - Xt^2 - Yt^3)S(t) = 3 - Xt^2,$$

and so

$$S(t) = \frac{3 - Xt^2}{1 - (X+Yt)t^2} = (3 - Xt^2)\sum_{k=0}^{\infty}(X+Yt)^k t^{2k}. \tag{2}$$

The value of S_n is found immediately by equating the coefficients of t^n in (2). According to the parity of n, we have

$$S_{2m} = \sum_{k=0}^{[m/3]}\left\{3\binom{m-k}{2k} - \binom{m-k-1}{2k}\right\}X^{m-3k}Y^{2k}$$

and

$$S_{2m+1} = \sum_{k=0}^{[(m-1)/3]}\left\{3\binom{m-k}{2k+1} - \binom{m-k-1}{2k+1}\right\}X^{m-3k-1}Y^{2k+1},$$

where the square brackets denote the greatest integer function. Using

$$3\binom{a}{b} - \binom{a-1}{b} = \frac{2a+b}{a}\binom{a}{b} = \frac{2a+b}{b}\binom{a-1}{b-1},$$

the value of S_n can be written more simply as

$$S_{2m} = \sum_{k=0}^{[m/3]}\frac{2m}{m-k}\binom{m-k}{2k}X^{m-3k}Y^{2k}$$

and

$$S_{2m+1} = \sum_{k=0}^{[(m-1)/3]}\frac{2m+1}{2k+1}\binom{m-k-1}{2k}X^{m-3k-1}Y^{2k+1}.$$

Also solved by CURTIS COOPER, Central Missouri State University; G.C. GIRI, Midnapore College, West Bengal, India; M.S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

There appears to be no reason to restrict x,y,z to real numbers. The formula for S_n shows however that, even for complex x,y,z, S_n is real for all n provided only that S_2 and S_3 be real.

It follows from $S_1 = 0$ that $Y = S_3/3 = xyz$. And if $xyz \neq 0$, we easily find that

$$S_{-1} = -\frac{X}{Y}, \qquad S_{-2} = \frac{X^2}{Y^2}, \qquad S_{-3} = \frac{3}{Y} - \frac{X^3}{Y^3}.$$

It would be interesting to have an explicit formula for S_n in terms of X and Y valid for all negative integers n. Failing that, it would be useful to have a recurrence relation from which the values of S_n can be generated successively for $n = -1, -2, -3, \ldots$.

A related problem of interest is Crux 143 [1976: 178].

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770. [1982: 210] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Let P be an interior point of triangle ABC. Prove that

$$PA\cdot BC + PB\cdot CA > PC\cdot AB.$$

Solution adapted from that of nearly all solvers.

If P is *any* point in the plane of triangle ABC, then by the ptolemaic inequality (see, e.g., [1]-[4], whose authors all submitted solutions) we have, in magnitude only,

$$PA\cdot BC + PB\cdot CA \geq PC\cdot AB.$$

Since the four points P,A,B,C are not collinear, the inequality is always strict except when P lies on the circumcircle of triangle ABC on the arc AB which does not contain vertex C, where equality holds.

Solutions were submitted by LEON BANKOFF, Los Angeles, California; O. BOTTEMA, Delft, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; DAN PEDOE, University of Minnesota; STANLEY RABINOWITZ, Nashua, New Hampshire; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer.

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968, p. 128.

2. Clayton W. Dodge, *Euclidean Geometry and Transformations*, Addison-Wesley, Reading, 1972, pp. 190, 194.

3. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, Boston, 1972, p. 132.

4. D. Pedoe, *Circles*, Dover, New York, 1979, p. 10.

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771. [1982: 245] *Proposed by Charles W. Trigg, San Diego, California.*

The letters in the adjoining cryptarithm are in one-to-one correspondence with the ten decimal digits. What is the highest the B's can FLY with the least HUM?

$$\begin{array}{r} WAX \\ HUM \\ \underline{FLY} \\ BBB \end{array}$$

Solution by Anneliese Zimmermann, Bonn, West Germany.

We have $W+H \geq 3$, so $F \leq 6$ in any solution. A solution containing simultaneously max-FLY and min-HUM, if such exists, must therefore be looked for first among those in which $H = 1$ and $F = 6$. In any such solution, we must have $W = 2$ and $B = 9$. Now $A+U+L \leq 9$, so $L \notin \{8,7\}$, and we look for the max-min solution among those for which $L = 5$ and $U = 0$. Since $X+M+Y > 10$ for any choice of the remaining digits, we must have $A = 3$. Finally, $\{X,M,Y\} = \{4,7,8\}$, and the max-min requirement is satisfied if and only if $M = 4$, $Y = 8$, and $X = 7$. The unique solution is

$$\begin{array}{r} 237 \\ 104 \\ \underline{658} \\ 999 \end{array}.$$

Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; DONALD C. FULLER, Gainesville Junior College, Gainesville, Florida; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One incorrect solution was received.

Editor's comment.

Feder referred to a similar problem in [1].

REFERENCE

1. George J. Summers, *Test Your Logic*, Dover, New York, 1972, Puzzle No. 36: "Three J's".

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772. [1982: 245] *Proposed by the editor.*

Find necessary and sufficient conditions on the real numbers a,b,c,d for the equation

$$z^2 + (a+bi)z + (c+di) = 0$$

to have exactly one real root.

(This is simply an exercise in proof-writing, not a great mathematical challenge. Solvers should strive for a proof that is correct, complete, concise, and linguistically as well as mathematically elegant.)

Solution by Gali Salvatore, Perkins, Québec.

We assume, as is customary, that roots of polynomial equations are counted according to their multiplicities and that, consequently, the expression "exactly one real root" means "one real and one imaginary root".

Suppose the given equation has exactly one real root. Then b and d cannot both be zero, for otherwise the equation would have two real or two imaginary roots, depending on the sign of the discriminant $a^2 - 4c$. If the real root is r, then, setting $z = r$ and separating the real and imaginary parts, we obtain

$$r^2 + ar + c = 0 \quad \text{and} \quad br + d = 0. \qquad (1)\text{-}(2)$$

Now we must have

$$b \neq 0, \qquad (3)$$

for otherwise also $d = 0$ from (2). Hence $r = -d/b$, and substituting this in (1) yields an equation equivalent to

$$b^2c + d^2 - abd = 0. \qquad (4)$$

We now show that the necessary conditions (3) and (4) are also sufficient. If $b \neq 0$, the given equation is equivalent to

$$(bz+d)(bz+ab-d+b^2i) + (b^2c+d^2-abd) = 0.$$

So if (4) also holds, then the equation has exactly one real root, $z = -d/b$.

Also solved by KENT D. BOKLAN, student, Massachusetts Institute of Technology; MARCO A. ETTRICK, New York Technical College, Brooklyn, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; OLIVIER LAFITTE, élève de Mathématiques Supérieures au Lycée Montaigne à Bordeaux, France; ROBERT C. LYNESS, Southwold, Suffolk, England; F.G.B. MASKELL, Algonquin College, Ottawa; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and the proposer.

Editor's comment.

Several of the solutions submitted did not, in the editor's opinion, satisfy all five of the proposer's criteria for excellence. One which came close was correct, complete, linguistically as well as mathematically elegant—and $4\frac{1}{2}$ pages long!

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774. [1982: 245] *Proposed by Bob Prielipp, University of Wisconsin-Oshkosh.*

Let $(G,\cdot)$ and $(G',\circ)$ both be finite groups of the same order. If, for each positive integer k, $(G,\cdot)$ and $(G',\circ)$ contain the same number of elements of order k, are the groups $(G,\cdot)$ and $(G',\circ)$ necessarily isomorphic?

Solution by R. B. Killgrove, University of South Carolina at Aiken.

The answer is no. The counterexamples which follow can be found in Hall [1]. Each group is of order p^3, where p is an odd prime, and in each case a,b,c are distinct group elements other than the group identity e.

The first group $(G,\cdot)$ has the defining relations

$$a^p = b^p = c^p = e,\ b\cdot c = c\cdot b,\ c\cdot a = a\cdot c,\ a\cdot b = b\cdot a;$$

and those of the second group $(G',\circ)$ are

$$a^p = b^p = c^p = e,\ b\circ c = c\circ b,\ c\circ a = a\circ c,\ a\circ b = b\circ a\circ c.$$

In each case, the identity is of order 1, and each of the other p^3-1 elements is of order p. But the groups are not isomorphic, for $(G,\cdot)$ is abelian while $(G',\circ)$ is not.

Also solved by ALAN EDELMAN, student, Yale University. In addition one incorrect solution was received.

REFERENCE

1. Marshall Hall, Jr., *The Theory of Groups*, Macmillan, New York, 1959, p. 52.

* * *

A CONSECUTIVE-PRIME MAGIC SQUARE

The sixteen consecutive primes 12553, ..., 12689 rearrange into a fourth-order magic square with magic sum 50478. This result was computed on a Model I Radio Shack TRS-80 microcomputer.

12553	12641	12671	12613
12583	12647	12611	12637
12689	12601	12619	12569
12653	12589	12577	12659

ALLAN WM. JOHNSON JR.
Washington, D.C.

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