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# ON A PROBLEM OF JANOUS CONCERNING A POWER MEAN INEQUALITY IN A TRIANGLE

by

Marcin E. Kuczma

Given an acute-angled triangle $\Delta$, let $h_1$, $h_2$, $h_3$ be the altitudes, $d_1$, $d_2$, $d_3$ be the distances from the circumcenter to the sides, and define for any real $t$

$$f(t,\Delta) = \frac{M_t(d_1,d_2,d_3)}{M_t\left(\frac{h_1}{3},\frac{h_2}{3},\frac{h_3}{3}\right)}$$

where $M_t$ denotes the $t$th order power mean

$$M_t(u,v,w) = \left(\frac{u^t + v^t + w^t}{3}\right)^{1/t} \quad \text{for} \quad t \neq 0 , \qquad M_0(u,v,w) = (uvw)^{1/3} .$$

Walther Janous [3] posed the following question:

*Problem* 1. *Find all exponents $t$ such that $f(t,\Delta) \leq 1$ holds for all acute triangles $\Delta$.*

This was motivated by the observation that

$$f(0,\Delta) \leq 1 \qquad \text{for all acute } \Delta , \tag{1}$$

whereas

$$f(1,\Delta) \geq 1 \qquad \text{for all acute } \Delta . \tag{2}$$

As Janous remarked, (1) is essentially equivalent to a result of Mitrinović and Pečarić [5]. As a reference to (2) we might quote Garfunkel's *Monthly* problem [2] (the reference given in [3], via a more general inequality [1], item 12.3, is a somewhat roundabout way). [*Editor's note*: this reference was in fact added to Janous's solution by the editor, who is often roundabout!]

Since

$$h_i = 2R \sin A_{i-1} \sin A_{i+1} \qquad \text{(cyclically)}$$

and

$$d_i = R \cos A_i$$

($A_1,A_2,A_3$ and $R$ denoting the angles and circumradius of $\Delta$), statements (1) and (2) can be equivalently rewritten as

$$8 \prod \sin^2 A_i \geq 27 \prod \cos A_i , \tag{1$'$}$$

$$3 \sum \cos A_i \geq 2 \sum \sin A_i \sin A_{i+1} \tag{2$'$}$$

((2$'$) is the form in which the statement appears in [2]).

These two facts, (1) and (2), indicate that there may be some border value of $t$, depending on $\Delta$, at which $f(\cdot,\Delta) - 1$ changes sign. And this suggests the next problem, in a sense dual to the former:

*Problem* 2. *Find all exponents $t$ such that $f(t,\Delta) \geq 1$ holds for all acute triangles $\Delta$.*

In this note we give an answer to Problem 1 and a conjecture concerning Problem 2.

PROPOSITION. (a) *For every non–equilateral acute–angled triangle $\Delta$ there exists a number $\tau = \tau(\Delta) \in (0,1)$ such that*

$$f(t,\Delta) \begin{cases} > 1 & \text{for } t > \tau , \\ = 1 & \text{for } t = \tau , \\ < 1 & \text{for } t < \tau . \end{cases}$$

(b) *The bounds of $\tau(\cdot)$ taken over all non–equilateral acute triangles $\Delta$ satisfy*

$$\inf_\Delta \tau(\Delta) = 0 , \qquad \sup_\Delta \tau(\Delta) \in [\xi,1] ,$$

*where $\xi = 0.793...$ is the real root of the equation*

$$2^t + 2^{t/2-1} = 3^t .$$

The assertion concerning inf $\tau$ evidently yields the answer to Problem 1: inequality $f(t,\Delta) \leq 1$ holds for *all* acute triangles $\Delta$ if and only if $t \leq 0$.

The assertion concerning sup $\tau$ is hardly satisfactory; after all, sup $\tau$ is an absolute constant, not depending on anything. If we knew its actual value, we would have an answer to Problem 2.

*Proof of Proposition.* (a) Let $\Delta$ be a fixed triangle of angles $A_1 \leq A_2 \leq A_3 < \pi/2$. Then

$$h_1 \geq h_2 \geq h_3 \qquad \text{and} \qquad d_1 \geq d_2 \geq d_3 . \tag{3}$$

We claim that

$$h_1 \leq 3d_1 , \qquad h_3 \geq 3d_3 . \tag{4}$$

Since $A_1 \leq \pi/3$, we get $d_1 = R \cos A_1 \geq R/2$, so that $h_1 \leq R + d_1 \leq 3d_1$. To obtain the second inequality in (4) we observe that

$$h_3 - 3d_3 = 2R \sin A_1 \sin A_2 - 3R \cos A_3 = R(\cos(A_2 - A_1) - 2 \cos A_3) ,$$

so it suffices to prove

$$\cos(A_2 - A_1) \geq 2 \cos A_3 .$$

The conditions imposed on the $A_i$'s yield

$$0 \leq A_2 - A_1 \leq 3A_3 - \pi < \pi/2$$

and $0 < \cos A_3 \leq 1/2$, whence

$$\cos(A_2 - A_1) \geq \cos(3A_3 - \pi) = \cos A_3(3 - 4\cos^2 A_3) \geq 2\cos A_3 ,$$

as needed. This settles (4).

Since the triangle is not equilateral, inequalities (1) and (2) are strict; this can be seen by examining the proofs given in [5] and [2] (as for (1), see also Remark 2 below). Hence, by the continuity of $M_t(u,v,w)$ with respect to $t \in \mathbb{R}$, there exists $\tau \in (0,1)$ (*a priori*, not necessarily unique) for which $f(\tau,\Delta) = 1$. This means that

$$\left(\frac{h_1}{3}\right)^\tau + \left(\frac{h_2}{3}\right)^\tau + \left(\frac{h_3}{3}\right)^\tau = d_1^\tau + d_2^\tau + d_3^\tau .$$

In view of (4), $(h_1/3)^\tau \leq d_1^\tau$ and

$$\left(\frac{h_1}{3}\right)^\tau + \left(\frac{h_2}{3}\right)^\tau = \sum \left(\frac{h_i}{3}\right)^\tau - \left(\frac{h_3}{3}\right)^\tau \leq \sum d_i^\tau - d_3^\tau = d_1^\tau + d_2^\tau .$$

These relations together with (3) show that vector $((h_1/3)^\tau, (h_2/3)^\tau, (h_3/3)^\tau)$ is majorized by $(d_1^\tau, d_2^\tau, d_3^\tau)$.

Choose $t \neq 0, \tau$ and consider $g(x) = x^{t/\tau}$ for $x > 0$. This function is strictly convex if $t < 0$ or $t > \tau$ and strictly concave if $0 < t < \tau$. By the majorization theorem, see e.g. [4, Ch. 3, C1], we obtain

$$\sum \left(\frac{h_i}{3}\right)^t = \sum g\left(\left(\frac{h_i}{3}\right)^\tau\right) \begin{cases} < \sum g(d_i^\tau) = \sum d_i^t & \text{when } t \notin [0,\tau] , \\ > \sum g(d_i^\tau) = \sum d_i^t & \text{when } t \in (0,\tau) . \end{cases} \tag{5}$$

The inequalities are strict because the two vectors do not coincide ($h_i = 3d_i$ for $i = 1,2,3$ holds in the equilateral triangle alone). We see that $\tau$ is unique. For $t < 0$, the inequality gets reversed after raising to power $1/t$. Consequently, relations (5) (accompanied by strict inequality (1)) yield exactly our assertion.

(b) Let $\Delta_\theta$ be the isosceles triangle of angles $2\theta$, $\pi/2 - \theta$, $\pi/2 - \theta$, with $\theta \in (0,\pi/4)$. Then

$$f(t,\Delta_\theta) = \frac{3}{2}\left(\frac{(\cos 2\theta)^t + 2(\sin\theta)^t}{(\cos\theta)^{2t} + 2(\sin 2\theta\cos\theta)^t}\right)^{1/t} \quad \text{for } t \neq 0 .$$

For any fixed $t > 0$ this expression tends to $3/2$ as $\theta \to 0$ and tends to the limit

$$3(2^t + 2^{t/2-1})^{-1/t} \tag{6}$$

as $\theta \to \pi/4$. The value of (6) is less than 1 if and only if $t < \xi$. Thus, for any $t_0$ between 0 and $\xi$ we can find a triangle $\Delta_{\theta_1}$ for which $f(t_0,\Delta_{\theta_1}) > 1$ and a triangle $\Delta_{\theta_2}$ for which $f(t_0,\Delta_{\theta_2}) < 1$. For these triangles, $\tau(\Delta_{\theta_1}) < t_0 < \tau(\Delta_{\theta_2})$. Since $t_0$

can be chosen arbitrarily in $(0,\xi)$, the bounds of $\tau$ satisfy $\inf \tau = 0$, $\sup \tau \geq \xi$; the proof is finished. $\square$

*Remark* 1. There is strong heuristic evidence for conjecturing that the upper bound of $\tau$ actually equals $\xi$; i.e., that $f(\xi,\Delta) \geq 1$ for all $\Delta$. The quantity $f(\xi,\Delta)$ considered as a function of angles $A_i$ on the set

$$\mathscr{A} = \left\{ (A_1,A_2,A_3) : 0 \leq A_i \leq \pi/2 \;,\; \sum A_i = \pi \right\}$$

has value 1 at $(\pi/3,\pi/3,\pi/3)$ and is $\geq 1$ at each "boundary" point ($A_i = \pi/2$ for some $i$). Attaining an absolute minimum $< 1$ at some other point *inside* $\mathscr{A}$ would violate what is sometimes called the "principle of insufficient reason".

*Remark* 2. Vector majorization techniques provide also a quick proof of (1), quite different from that given in [5] and [3]. We again resort to the basic monograph [4]. A function $F$ defined on a subset of $\mathbb{R}^3$ is called *Schur–convex* if $F(\mathbf{x}) \leq F(\mathbf{y})$ whenever vector $\mathbf{x}$ is majorized by $\mathbf{y}$; when strict inequality holds unless $\mathbf{y}$ is a permutation of $\mathbf{x}$, $F$ is called *strictly Schur–convex.* If $F$ is (strictly) Schur–convex and (strictly) decreasing in each argument and if $g$ is a (strictly) concave function of a real variable, then $G(x_1,x_2,x_3) = F(g(x_1),g(x_2),g(x_3))$ is (strictly) Schur–convex [4, Ch 3 B2]. The function $F(u,v,w) = -uvw$ is strictly Schur–convex on $\{(u,v,w) : u,v,w > 0\}$ [4, Ch. 3 F1]; the function $g(\theta) = -\sin^2\theta/\cos\theta = \cos\theta - (\cos\theta)^{-1}$ is strictly concave in $(0,\pi/2)$; the vector $(A_1,A_2,A_3)$ (angles of a triangle) majorizes $(\pi/3,\pi/3,\pi/3)$. Hence

$$-g(A_1)g(A_2)g(A_3) \geq -(g(\pi/3))^3 \;,$$

and this is nothing else than (1′) (i.e. (1)); equality holds only for $A_1 = A_2 = A_3 = \pi/3$.

*References*:


[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen 1969.

[2] J. Garfunkel, Problem E2029, *American Math. Monthly*, 74 (1967) 1133; solution by M.G. Greening 75 (1968) 1122.

[3] W. Janous, Comment on Problem 1199, *Crux Mathematicorum* 15 (1989) 14.

[4] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[5] D.S. Mitrinović, J.E. Pečarić, Problem 1199, *Crux Mathematicorum* 12 (1986) 283; solution 14 (1988) 87.


Institute of Mathematics
University of Warsaw

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# THE OLYMPIAD CORNER
## No. 115
R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

The first problems we pose are those from the Indian Mathematical Olympiad for 1989, for which we thank Dr. Shailesh Shirali, Director of Studies of the Rishi Valley School, Andrah Pradesh, India.

### 1989 INDIAN MATHEMATICAL OLYMPIAD

1. Prove that the polynomial
$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$
cannot be expressed as a product
$$f(x) = p(x)q(x)$$
where $p(x)$, $q(x)$ are both polynomials with integral coefficients and with degree $< 4$.

2. Let $a$, $b$, $c$, $d$ be any four real numbers, not all equal to zero. Prove that the roots of the polynomial
$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$
cannot all be real.

3. Let $A$ denote a subset of the set $\{1,11,21,31,\ldots,541,551\}$ having the property that no two elements of $A$ add up to 552. Prove that $A$ cannot have more than 28 elements.

4. Determine, with proof, all the positive integers $n$ for which (i) $n$ is not the square of any integer, and (ii) $[\sqrt{n}]^3$ divides $n^2$. (Notation: $[x]$ denotes the largest integer that is less than or equal to $x$.)

5. Let $a$, $b$, $c$ denote the sides of a triangle. Show that the quantity
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$
must lie between the limits 3/2 and 2. Can equality hold at either limit?

6. Triangle $ABC$ is scalene with angle $A$ having a measure greater than 90 degrees. Determine the set of points $D$ that lie on the extended line $BC$, for which

$$|AD| = \sqrt{|BD| \cdot |CD|}\ ,$$

where $|BD|$ refers to the (positive) distance between $B$ and $D$.

7. Let $ABC$ be an arbitrary acute angled triangle. For any point $P$ lying within this triangle, let $D$, $E$, $F$ denote the feet of the perpendiculars from $P$ onto the sides $AB$, $BC$, $CA$ respectively. Determine the set of all positions of the point $P$ for which the triangle $DEF$ is isosceles. For which $P$ will the triangle $DEF$ become equilateral?

* * *

The second set of problems we give this month are from the 38th Bulgarian Mathematical Olympiad, 3rd Round, written April 15, 1989. Thanks go to Jordan Tabov, Sofia, Bulgaria for forwarding these problems for use in the Corner.

38TH BULGARIAN MATHEMATICS OLYMPIAD

3rd Round

1. Let $p$ and $q$ be prime numbers for which the number

$$\sqrt{p^2 + 7pq + q^2} + \sqrt{p^2 + 14pq + q^2}$$

is an integer. Prove that $p = q$. (6 *points*; *O. Mushkarov*)

2. Prove that for every positive integer $n$ the equation $x \cos x = 1$ has a unique solution $x_n$ in the interval $[2n\pi,(2n + 1)\pi]$. Determine $\lim\limits_{n\to\infty}(x_{n+1} - x_n)$. (7 *points*; *S. Grozdev*)

3. A triangle $ABC$ is given. A line parallel to the base $AB$ intersects the sides $AC$ and $BC$ respectively at the inner points $M$ and $P$. Let $D$ be the point of intersection of the lines $AP$ and $BM$. Prove that the line joining the orthocentres of the triangles $ADM$ and $BDP$ is perpendicular to the line $CD$. (7 *points*; *B. Lazarov*)

4. A convex $n$–gon $(n > 3)$ possesses the following property: there are $n - 2$ of its diagonals each of which cuts the $n$–gon in two parts of equal area. Find all $n$–gons having this property. (6 *points*; *S. Grozdev*)

5. A truncated triangular pyramid is given with base $ABC$ and lateral edges $AA_1$, $BB_1$, and $CC_1$. Let $A_2$, $B_2$, and $C_2$ be the midpoints of the edges $B_1C_1$, $C_1A_1$, and $A_1B_1$, respectively. Prove that if $AA_2 \perp B_1C_1$ and $BB_2 \perp C_1A_1$, then

(a) $CC_2 \perp A_1B_1$;

(b) the line joining the orthocentre of the triangle $ABC$ and the circumcentre of the triangle $A_1B_1C_1$ is perpendicular to the base. (7 *points*; *J. Tabov*)

**6.** Let $0 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5$ be real numbers, and let

$$s = a_1 + a_2 + a_3 + a_4 + a_5$$

and

$$p = \prod_{1 \le i < j \le 5} (s - 2(a_i + a_j)).$$

Prove that if $a_1 + 3a_2 + a_4 \ge 3a_3 + a_5$, then $p \le (a_1a_2a_3a_4a_5)^2$. (7 *points*; *V. Goranko*)

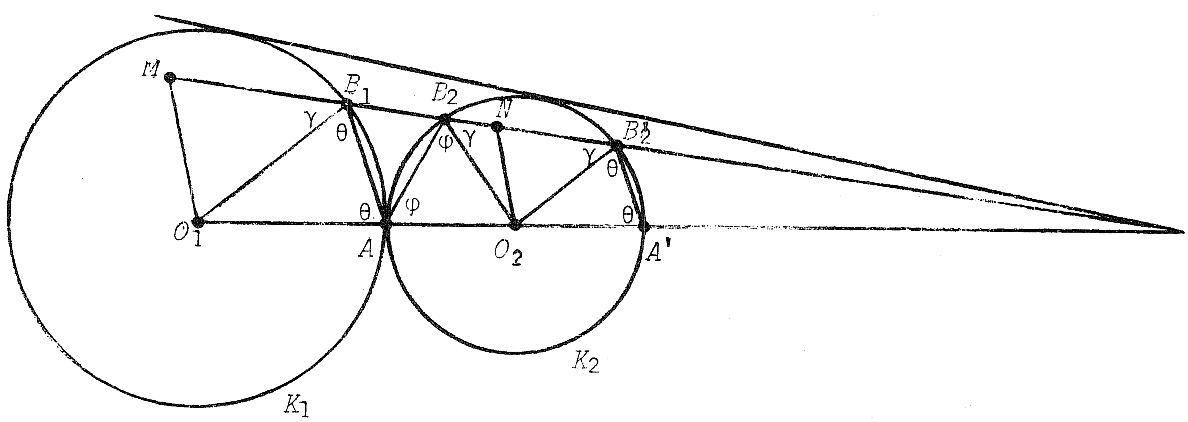
* * *

The first two solutions this month are to problems from the April 1981 number.

**B-1.** [1981: 114] *Bulgarian competitions.*

Two given circles $K_1$ and $K_2$, with centers $O_1$ and $O_2$ and different radii, are tangent externally at point $A$. Also given inside one of the circles is a point $M$ which does not lie on the line of centers $O_1O_2$. Show how to determine a line $l$ through $M$ such that, for some triangle $AB_1B_2$ with $B_1 \in l \cap K_1$ and $B_2 \in l \cap K_2$, the circumcircle of the triangle is tangent to line $O_1O_2$.

*Solution by Anonymous.* [See [1990: 34]]




Let $O_1O_2$ intersect $K_2$ again in $A'$. Draw $O_2N$ parallel to $O_1M$ and $A'N$ parallel to $AM$. Then $l$ is the line passing through $M$ and $N$. [In fact, $l$ intersects $O_1O_2$ at the external centre of similitude of the two circles.– Ed.] Note that $AB_1$ is parallel to $A'B_2'$. Let

$$\gamma = \angle MB_1O_1 = \angle B_2B_2'O_2 = \angle B_2'B_2O_2 ,$$
$$\theta = \angle O_1AB_1 = \angle O_1B_1A = \angle O_2B_2'A' = \angle O_2A'B_2' ,$$

and

$$\varphi = \angle O_2AB_2 = \angle O_2B_2A .$$

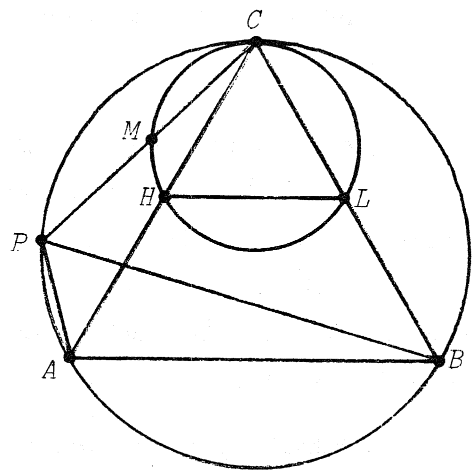
Since $\angle AO_2B_2 + \angle B_2O_2B_2' + \angle B_2'O_2A' = 180°$, it follows that $\gamma + \theta + \varphi = 180°$ so that $\angle AB_1B_2 = \varphi = \angle O_2AB_2$ and $\angle AB_2B_1 = \theta = \angle O_1AB_1$. Thus the circumcircle of $AB_1B_2$ is tangent to $O_1O_2$.

**B-6.** [1981: 115] *Bulgarian competitions.*

Let $K$ be one of the arcs into which a given circle is divided by a chord $AB$ and let $C$ be the midpoint of $K$. Let $P$ be an arbitrary point on $K$ and let $M$ be a point on segment $PC$ such that $PM = |PA - PB|/2$. Find the set of all possible points $M$ for all points $P$ of $K$.

*Solution by Anonymous.* [See [1990: 34].]

If $P$ is at $C$, then $M$ is at $C$ as well. By symmetry, we need only consider the case $PB > PA$. Let $H$ on $AC$ and $L$ on $BC$ be such that $AH = BL = AB/2$. Let $PC$ intersect the circumcircle of $HCL$ at $M$. We claim that $2PM = PB - PA$. Note that $HL$ is parallel to $AB$. Since $\angle HLM = \angle HCM = \angle ABP$, $ML$ is parallel to $PB$. Next, $\angle HML = \angle HCL = \angle APB$ and $HM$ is parallel to $AP$. By similar triangles,




$$\frac{PC}{AC} = \frac{PM}{AH} = \frac{2PM}{AB}$$

or

$$2PM = \frac{PC\cdot AB}{AC} .$$

On the other hand, by Ptolemy's theorem

$$PB\cdot AC = PA\cdot BC + PC\cdot AB$$

so that

$$PB - PA = \frac{PC\cdot AB}{AC} = 2PM .$$

Thus, the locus of $M$ is the circular arc from $H$ through $C$ to $L$.

* * *

As promised, here are the numerical answers for the 1990 AIME. These problems and their official solutions are copyrighted by the Committee on the

American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Detailed solutions, and additional copies of the problems may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A., 68588–0322.

| | | | | | |
|---|---|---|---|---|---|
| 1. | 528 | 2. | 828 | 3. | 117 |
| 4. | 013 | 5. | 432 | 6. | 840 |
| 7. | 089 | 8. | 560 | 9. | 073 |
| 10. | 144 | 11. | 023 | 12. | 720 |
| 13. | 184 | 14. | 594 | 15. | 020 |

* * *

For solutions to the problems given in the October and November 1988 numbers of the Corner, readers are referred to the publication *An Olympiad Down Under: A Report on the 29th International Mathematical Olympiad in Australia* published by the Australian Mathematics Foundation Ltd., School of Information Sciences and Engineering, Canberra College of Advanced Education (ISBN 0–731651189). This publication offers solutions to the problems we listed as well as to several more that we did not. I would like to thank those readers who submitted solutions to these problems even though they are not being reproduced here. Solutions were received from Nicos Diamantis, Patras, Greece; George Evagelopoulos, law student, Athens, Greece; Guo–Gang Gao, Départment d'IRO, L'Université de Montréal; the late J.T. Groenman, Arnhem, The Netherlands; L.J. Hut, Groningen, The Netherlands; Bob Prielipp, University of Wisconsin–Oshkosh; Toshio Seimiya, Kawasaki, Japan; Robert E. Shafer, Berkeley, California; D.J. Smeenk, Zaltbommel, The Netherlands; David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

While cross–checking the problems with *An Olympiad Down Under*, I noticed one correction:

11. [1988: 226] *Proposed by Morocco.*

Find the largest natural number $n$ such that, if the set $\{1,2,\dots,n\}$ is arbitrarily divided into two non–intersecting subsets, then one of the subsets contains three distinct numbers such that the product of two of them equals the third.

*Correction:* Replace "largest" by "least".

A reader also sent in an interesting historical note about one of the problems.

2. [1988: 257] *Proposed by Czechoslovakia.*

In a given tetrahedron $ABCD$, let $K$ and $L$ be the centres of edges $AB$ and $CD$ respectively. Prove that every plane that contains the line $KL$ divides the tetrahedron into two parts of equal volume.

*Comment by Professor O. Bottema, Delft, The Netherlands.*

This problem was solved by the French mathematician Bobillier in 1827. See my article "A theorem of Bobillier on the tetrahedron", *Elemente Der Mathematik* 24 (1969) pp. 6–10.

*

I received comments about a solution published in the October 1988 number of the Corner. Both George Tsintsifas and the late G.R. Veldkamp pointed out methods of simplifying the solution to problem 5 [1987: 39] which was given in [1988: 228–230]. Both employed Apollonian circles.

5. [1987: 39; 1988: 228] *Final Round of the Bulgarian Olympiad 1985.*

Let $ABC$ $(AC \neq BC)$ be a triangle with $\gamma = \angle ACB$ a given acute angle, and let $M$ be the midpoint of $AB$. Point $P$ is chosen on the line segment $CM$ so that the bisectors of the angles $PAC$ and $PBC$ meet at a point $Q$ on $CM$. Find the angles $APB$ and $AQB$.

*Comment by George Tsintsifas, Thessaloniki, Greece.*

The following simplification of the given proof seems easier. Using the figure of page 229, we have

$$\frac{PQ}{QC} = \frac{AP}{AC} = \frac{BP}{BC} \tag{1}$$

(because of the internal bissectrice theorem). Also from the similar triangles $AMC$, $NMB$ and $AMN$, $CNB$ we have

$$\frac{AC}{BC} = \frac{BN}{AN} \cdot \tag{2}$$

From (1) and (2) we have

$$\frac{AC}{BC} = \frac{AP}{BP} = \frac{BN}{AN} \cdot$$

That is, the points $C$, $P$, $N$ lie on two Apollonian circles, symmetric with respect to the point $M$. Therefore $MP = MN$. From here the solution is as in the last paragraph of the published proof.

Richard K. Guy writes with a comment on a solution given in the November 1988 number of the Corner.

2. [1986: 97; 1988: 133, 260] *1985 Spanish Mathematical Olympiad, 1st Round.*

Let $n$ be a natural number. Prove that the expression

$$(n + 1)(n + 2)\cdots(2n - 1)(2n)$$

is divisible by $2^n$.

*Comment by R.K. Guy, The University of Calgary.*

The "induction free" proof is no shorter than that given in May. But worse, it doesn't obtain the nice "best possible" result of that proof:

$$2^n \| (n + 1)(n + 2)\cdots(2n)$$

(i.e. $2^n$ divides but not $2^{n+1}$).

The classical proof which follows is short, "induction free" and gives the "best possible" result.

The exact power of 2 dividing $(2n)!/n!$ is

$$\begin{aligned} &|\text{even \#'s in } \{1,\cdots,2n\}| + |\text{multiples of 4 in } \{1,\cdots,2n\}| \\ &\quad + |\text{multiples of 8 in } \{1,\cdots,2n\}| + \cdots \quad - \{|\text{even \#'s in } \{1,\cdots,n\}| \\ &\quad + |\text{multiples of 4 in } \{1,\cdots,n\}| + \cdots\} \\ &= n + \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \cdots - (\lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \cdots) = n\ . \end{aligned}$$

*   *   *

We turn now to solutions to problems from the *1st Nordic Mathematical Olympiad* [1988: 289].

1. Nine foreign journalists meet at a press conference. Each of them speaks at most three different languages, and any two of them can speak a common language. Show that at least five of them speak the same language.

*Solution adapted from one by John Morvay, Springfield, MO, U.S.A.*

Assume that no language is spoken by more than four people. This will require that

(1) each journalist speaks exactly three languages

(since if some journalist speaks at most two languages the Pigeon Hole Principle gives at least four of the other eight who share one of his languages), and

(2) there is no language with fewer than three speakers
(since if some language is spoken by at most two journalists, say $A$ and $B$, at least four of the seven other journalists must speak one of the two remaining languages spoken by $A$, say).

To determine the number of languages, by (1) we must have $3x + 4y = 9\cdot 3$, where $x$ and $y$ are the number of languages with 3 and 4 speakers respectively. There are now three cases: $x = 9$, $y = 0$; $x = 5$, $y = 3$; and $x = 1$, $y = 6$.

Since each pair of journalists can speak a common language, we must also have

$$\binom{3}{2}x + \binom{4}{2}y \geq \binom{9}{2} ,$$

or $x + 2y \geq 12$. This leaves only the case $x = 1$, $y = 6$.

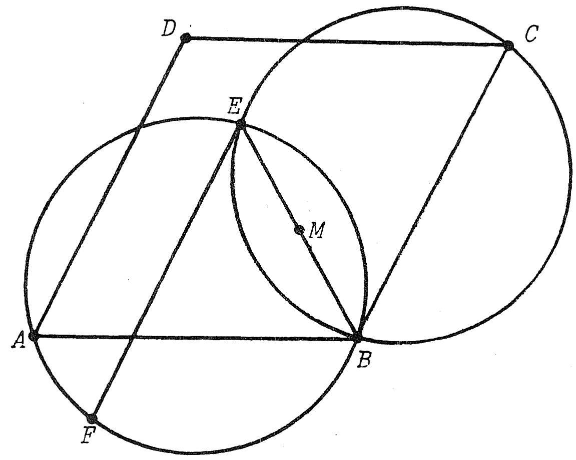
Without loss of generality let $l_0$ be the language spoken by exactly three of the group and $l_1,\cdots,l_6$ be those spoken by exactly four journalists. Also suppose that $l_0$ is spoken by each of $j_1$, $j_2$ and $j_3$ and the remainder of the journalists are $j_4,j_5,\cdots,j_9$. Now for each $i = 0,1,2$ six journalists are partitioned into two groups of 3 according to the language spoken with $j_i$, and the six languages $j_1,\cdots,j_6$ are partitioned in groups of two amongst $j_1$, $j_2$, and $j_3$, so that without loss $j_1$ speaks $\{l_0,l_1,l_2\}$, $j_2$ speaks $\{l_0,l_3,l_4\}$ and $j_3$ speaks $\{l_0,l_5,l_6\}$. We may also suppose that $j_4$, $j_5$ and $j_6$ speak $l_1$ while $j_7$, $j_8$ and $j_9$ speak $l_2$. Notice that it cannot be the case that $j_4$, $j_5$, $j_6$ also speak $l_3$, for then $j_7$, $j_8$ and $j_9$ must speak $l_4$ forcing all six of $j_4,\cdots,j_9$ to speak the same language. Without loss we have $l_3$ spoken by $j_4$, $j_7$ and $j_8$, and $l_4$ spoken by $j_5$, $j_6$ and $j_9$. But now $j_5$, $j_6$, $j_8$ and $j_9$ must speak a common language and this must be one of $l_5$ and $l_6$ so that one of these is spoken by 5 people (including $j_3$).

[*Editor's note.* Solutions to this problem were also received from Curtis Cooper, Department of Mathematics, Central Missouri State University, Warrensburg, MO; and from George Evagelopoulos, law student, Athens, Greece.]

2. Let $ABCD$ be a parallelogram in the plane. Draw two circles with common radius $R$, one through the points $A$ and $B$, and the other through the points $B$ and $C$. Let $E$ be the second point of intersection of the two circles. Assume that $E$ does not coincide with any vertex of the parallelogram. Show that the circle through the points $A$, $D$ and $E$ also has radius $R$.

*Solution by Botand Köszegi, student, Halifax West High School.*

In the figure, denote the midpoint of $EB$ by $M$, and let $F$ lie on circle $ABE$ such that $EF$ is parallel to $BC$. We know that the two circles are symmetrical to the point $M$ because their radius is $R$. So the image of $C$ to point $M$ is on the line $EF$, because $E$ is the image of $B$ and $EF \| BC$. And it is also on the circle $ABE$ (the two circles are symmetrical), so it is the point $F$. So $FE$ and $BC$ are equal and parallel. Therefore, translating the circle $ABE$ by $\overrightarrow{BC}$, the result passes through $D$, $E$ and $C$ (with $A$ going to $D$ and $B$ to $C$). And certainly the result has the same radius $R$, so we proved what we wanted.




**3.** Let $f\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function with $f(2) = a > 2$ and $f(mn) = f(m)f(n)$ for all $m,n \in \mathbb{N}$. Determine the least possible value for $a$.

*Solution by the editors.*

$f(n) = n^2$ satisfies all the properties and has $f(2) = 4$ so the smallest value for $a$ is 3 or 4. To rule out 3, suppose for a contradiction that $f$ satisfies the conditions and has $f(2) = 3$. Let $f(3) = b$. Now $2^3 = 8 < 9 = 3^2$ so $3^3 < b^2$ giving $5 < b$. Next $3^3 = 27 < 32 = 2^5$ so $b^3 < 3^5 = 243$. Since $7^3 = 343$ we must have $b = 6$. But now $3^8 = 6561 < 8192 = 2^{13}$. We would then require $3^8 \cdot 2^8 < 3^{13}$ giving $256 = 2^8 < 3^5 = 243$, which is a contradiction. Therefore the smallest value of $a$ is 4.

**4.** Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\,.$$

*Solution by Botand Köszegi, student, Halifax West High School.*

Since $a/b$, $b/c$, and $c/a$ are positive, we can apply the arithmetic mean, root–mean–square inequality to deduce

$$A = \frac{a/b + b/c + c/a}{3} \le \sqrt{\frac{a^2/b^2 + b^2/c^2 + c^2/a^2}{3}} = B\,. \qquad (1)$$

Now from the geometric mean, root–mean–square inequality,

$$1 = \sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} \le \sqrt{\frac{a^2/b^2 + b^2/c^2 + c^2/a^2}{3}} = B\,. \qquad (2)$$

Thus

$$A = A \cdot 1 \le B^2,$$

and the result is immediate.

*Notes*: (i) The two sides are equal if $a = b = c$.

(ii) By similar methods one can prove that if $x_1, x_2, \cdots, x_l$ are positive and $x_1 \cdot x_2 \cdots x_l = 1$ then $x_1^n + x_2^n + \cdots + x_l^n \le x_1^m + x_2^m + \cdots + x_l^m$ for $0 \le n \le m$.

[*Editor's note.* A similar solution was submitted by George Evagelopoulos, law student, Athens, Greece. A direct solution by induction was given by David Vaughan, Department of Mathematics, Wilfrid Laurier University, while M.A. Selby, Department of Mathematics, The University of Windsor, gave a solution employing Lagrange multipliers.]

* * *

That completes solutions to the 1st Nordic Olympiad, and this number of the Corner. Send in your problem sets and nice solutions!

* * *

# BOOK REVIEW

*Polynomials* by E.J. Barbeau, Springer–Verlag, New York, 1989. Hardcover, 441 + xxii pages. *Reviewed by Andy Liu, University of Alberta.*

A telegraphic review of this book would be

$$\text{"Mathematics}\big|_{\text{polynomials}}\text{" !}$$

It offers a comprehensive overview of the whole spectrum of undergraduate mathematics, unified by the study of a single topic. Polynomials constitute a major part of the high school mathematics curriculum, but many very natural questions about them are either glossed over or totally ignored. (What do we mean by solving an equation exactly? Can all equations be solved exactly? If not, which ones cannot, and what can one do about them?) The author takes off from there and skillfully guides the reader on a working tour in search of answers and further questions. The itinerary includes visits, which are more than cursory, to numerical analysis, calculus, real variables, abstract algebra, number theory, complex variables, topology and linear algebra. A minimal amount of text is interspersed among numerous exercises, problems and explorations. They are of varying levels of difficulty, so the reader will enjoy a certain amount of success in solving them.

There are also more challenging ones (many taken from *Crux Mathematicorum*!) from which greater satisfaction may be derived. Answers, solutions and comments are provided. This well–written book serves as a valuable link between high school and university mathematics. It is specially recommended for well motivated high school students, while university students may gain a refreshing perspective of what they are studying.

* * *

# PROBLEMS

*Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **December 1, 1990**, *although solutions received after that date will also be considered until the time when a solution is published.*

**1541**. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

$I_1$ is the excenter of $\Delta A_1A_2A_3$ corresponding to side $A_2A_3$. $P$ is a point in the plane of the triangle, and $A_1P$ intersects $A_2A_3$ in $P_1$. $I_2$, $P_2$, $I_3$, $P_3$ are analogously defined. Prove that the lines $I_1P_1$, $I_2P_2$, $I_3P_3$ are concurrent.

**1542***. *Proposed by Murray S. Klamkin, University of Alberta.*

For fixed $n$, determine the minimum value of

$$C_n = |\cos\theta| + |\cos 2\theta| + \cdots + |\cos n\theta|.$$

It is conjectured that min $C_n = [n/2]$ for $n > 2$.

**1543**. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Show that the circumradius of a triangle is at least four times the inradius of the pedal triangle of any interior point.

**1544**. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

One root of $x^3 + ax + b = 0$ is $\lambda$ times the difference of the other two roots ($|\lambda| \neq 1$). Find this root as a simple rational function of $a$, $b$ and $\lambda$.

**1545**. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A sphere is said to be inscribed into the skeleton of a convex polyhedron if it is tangent to all the edges of the polyhedron. Given a convex polyhedron $P$ and a point $O$ inside it, suppose a sphere can be inscribed into the skeleton of each pyramid spanned by $O$ and a face of $P$.

(a) Prove that if every vertex of $P$ is the endpoint of exactly three edges then there exists a sphere inscribed into the skeleton of $P$.

(b) Is this true without the assumption stated in (a)?

**1546**. *Proposed by Graham Denham, student, University of Alberta.*

Prove that for every positive integer $n$ and every positive real $x$,

$$\sum_{k=1}^{n} \frac{x^{k^2}}{k} \geq x^{n(n+1)/2}.$$

**1547**. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let $P$ be an interior point of a parallelogram $ABCD$, such that $\angle ABP = 2\angle ADP$ and $\angle DCP = 2\angle DAP$. Prove that $AB = PB = PC$.

**1548***. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $a_1$, $a_2$ be given positive constants and define a sequence $a_3, a_4, a_5, \ldots$ by

$$a_n = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}, \quad n > 2 .$$

Show that $\lim_{n\to\infty} a_n$ exists and find this limit.

**1549**. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In quadrilateral $ABCD$, $E$ and $F$ are the midpoints of $AC$ and $BD$ respectively. $S$ is the intersection point of $AC$ and $BD$. $H$, $K$, $L$, $M$ are the midpoints of $AB$, $BC$, $CD$, $DA$ respectively. Point $G$ is such that $FSEG$ is a parallelogram. Show that lines $GH$, $GK$, $GL$, $GM$ divide $ABCD$ into four regions of equal area.

**1550**. *Proposed by Miha'ly Bencze, Brasov, Romania.*

Let $A = [-1,1]$. Find all functions $f : A \to A$ such that

$$|xf(y) - yf(x)| \geq |x - y|$$

for all $x$, $y \in A$.

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# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1423***. [1989: 73] *Proposed by Murray S. Klamkin, University of Alberta.*

Given positive integers $k$, $m$, $n$, find a polynomial $p(x)$ with real coefficients such that

$$(x-1)^n \mid (p(x))^m - x^k .$$

What is the least possible degree of $p$ (in terms of $k$, $m$, $n$)?

*Solution by Robert B. Israel, University of British Columbia.*

If $p$ is a polynomial in $x$, $p^m - x^k$ is divisible by $(x-1)^n$ if and only if

$$p^m = x^k + O((x-1)^n) \text{ as } x \longrightarrow 1.$$

[*Editor's note*: this means that $p^m = x^k + r(x)$, where

$$|r(x)| \le M|x-1|^n \tag{1}$$

for some constant $M$ and for $x$ sufficiently close to 1.] This is true if and only if

$$p = x^{k/m} + O((x-1)^n).$$

[*Editor's note*: Len Bos contributes the following elaboration for the editor, who regrets not paying more attention as a student during analysis class. We have

$$p = x^{k/m} + \left[\left(1 + \frac{r(x)}{x^k}\right)^{1/m} - 1\right]x^{k/m} . \tag{2}$$

Applying the Mean Value Theorem to the function $f(t) = (1+t)^{1/m}$, $t > -1$, yields that

$$|(1+t)^{1/m} - 1| = |t|\cdot\frac{1}{m}(1+c)^{1/m - 1}$$

for some $c$, $|c| < |t|$, so with $t = r(x)/x^k$ we get

$$\left|\left(1 + \frac{r(x)}{x^k}\right)^{1/m} - 1\right| = \frac{|r(x)|}{x^k}.\frac{1}{m}(1+c)^{1/m - 1}$$

for some $c$ between 0 and $r(x)/x^k$. For $x$ sufficiently close to 1 this means

$$\left|\left(1 + \frac{r(x)}{x^k}\right)^{1/m} - 1\right|x^{k/m} \le K|x-1|^n$$

for some constant $K$, by (1). Thus from (2) we get the result.]

This means that $p$ consists of the terms of the Taylor series of $x^{k/m}$ about $x = 1$ up to order $(x-1)^{n-1}$, plus any combination of higher powers of $x - 1$. That Taylor series is

$$1 + \sum_{j=1}^{n-1} \frac{k/m(k/m - 1)\cdots(k/m - j + 1)}{j!}(x - 1)^j \ . \qquad (3)$$

If $k$ is not divisible by $m$, all coefficients of the Taylor series are nonzero, so the least degree of $p$ is $n - 1$. If $k$ is divisible by $m$, the coefficients for $j > k/m$ are zero, so the least degree of $p$ is $\min(n - 1, k/m)$.

*Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, who gave the correct answer for the second part without proof; and by the proposer, who answered the second part only in the case that m does not divide into k.*

*It appears that the series* (3) *can also be written as*

$$\frac{k(k - m)(k - 2m)\cdots(k - (n - 1)m)}{(n - 1)!\,m^{n-1}} \sum_{i=0}^{n-1} \frac{(-1)^{n-1-i}\binom{n-1}{i}}{k - im}x^i \ ;$$

*can any reader supply a proof?*

*   *   *

<u>**1425**</u>. [1989: 73] *Proposed by Jordi Dou, Barcelona, Spain.*

Let $D$ be the midpoint of side $BC$ of the equilateral triangle $ABC$ and $\omega$ a circle through $D$ tangent to $AB$, cutting $AC$ in points $B_1$ and $B_2$. Prove that the two circles, distinct from $\omega$, which pass through $D$ and are tangent to $AB$, and which respectively pass through $B_1$ and $B_2$, have a point in common on $AC$.

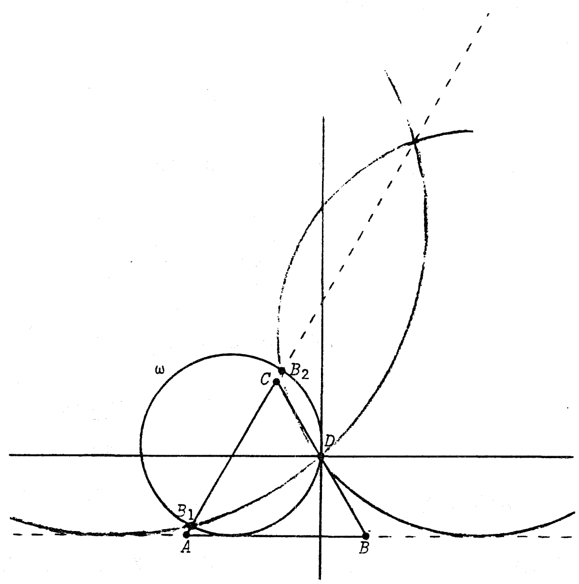
I. *Solution by P. Penning, Delft, The Netherlands.*

Introduce a rectangular coordinate system such that point $D$ is the origin and $AB$ is the line $y = -1$. All circles passing through $D$ and tangent to $AB$ are then given by

$$x^2 + y^2 - 4mx + (1 - 4m^2)y = 0 \ . \qquad (1)$$

Line $AC$ we denote by $l : x = py + q$, where

$$q = -2p = -2/\sqrt{3} \ . \qquad (2)$$




Now let $\omega$ be given by the value $m = m_0$ in (1), and let its intersections with $l$ be $B_1(x_1,y_1)$ and $B_2(x_2,y_2)$. Let the other two circles be given by $m = m_1$ (intersections with $l$ are $B_1$ and $P(x_3,y_3)$) and by $m = m_2$ (intersections with $l$ are $B_2$ and $Q(x_3',y_3')$). The result should be that $x_3 = x_3'$ and $y_3 = y_3'$, for $p$ and $q$ given by (2).

First eliminate $x$ from circle (1) and line $l$; we get

$$(p^2 + 1)y^2 + (1 + 2pq - 4mp - 4m^2)y + q(q - 4m) = 0 . \tag{3}$$

There are six equations of this type, each concerning one of the above values of the parameter $m$ and a point of intersection of the corresponding circle with line $l$. With $m = m_1$ and point $B_1$ we get

$$(p^2 + 1)y_1^2 + (1 + 2pq - 4m_1p - 4m_1^2)y_1 + q(q - 4m_1) = 0 \tag{4}$$

and with $m = m_1$ and point $P$ we get

$$(p^2 + 1)y_3^2 + (1 + 2pq - 4m_1p - 4m_1^2)y_3 + q(q - 4m_1) = 0 .$$

Subtracting and dividing by $y_1 - y_3$ yields

$$(p^2 + 1)(y_1 + y_3) + 1 + 2pq - 4m_1p - 4m_1^2 = 0 .$$

Similarly for $m = m_2$,

$$(p^2 + 1)(y_2 + y_3') + 1 + 2pq - 4m_2p - 4m_2^2 = 0 .$$

Subtracting these two,

$$(p^2 + 1)(y_3 - y_3') = (p^2 + 1)(y_2 - y_1) + 4(m_1 - m_2)(p + m_1 + m_2) . \tag{5}$$

Now with $m = m_0$ and point $B_1$, (3) becomes

$$(p^2 + 1)y_1^2 + (1 + 2pq - 4m_0p - 4m_0^2)y_1 + q(q - 4m_0) = 0 . \tag{6}$$

Subtracting (4) and dividing by $4(m_1 - m_0)$ yields

$$py_1 + (m_1 + m_0)y_1 + q = 0$$

or

$$m_1 + m_0 = -p - \frac{q}{y_1} . \tag{7}$$

Similarly for point $B_2$,

$$m_2 + m_0 = -p - \frac{q}{y_2} . \tag{8}$$

Using (7) and (8) to eliminate $m_1$ and $m_2$ in favor of $m_0$, (5) becomes

$$(p^2 + 1)(y_3 - y_3') = (p^2 + 1)(y_2 - y_1) + 4\left(\frac{q}{y_2} - \frac{q}{y_1}\right)\left(p - 2m_0 - 2p - \frac{q}{y_1} - \frac{q}{y_2}\right)$$

or

$$\frac{y_3 - y_3'}{y_2 - y_1} = 1 + \frac{4q[(p + 2m_0)y_1y_2 + q(y_1 + y_2)]}{y_1^2y_2^2(p^2 + 1)} . \tag{9}$$

From (6), $y_1$ and similarly $y_2$ are the roots of the quadratic

$$(p^2 + 1)y^2 + (1 + 2pq - 4m_0p - 4m_0^2)y + q(q - 4m_0) = 0 .$$

Thus

$$y_1 + y_2 = \frac{-1 - 2pq + 4m_0p + 4m_0^2}{p^2 + 1} , \qquad y_1y_2 = \frac{q(q - 4m_0)}{p^2 + 1} ,$$

and (9) now becomes

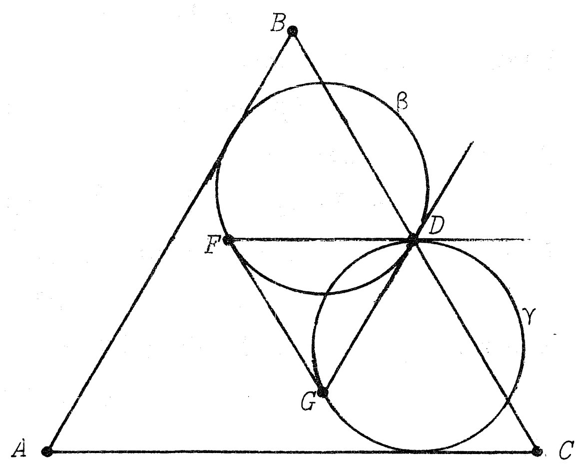
$$\frac{y_3 - y_3'}{y_2 - y_1} = 1 + \frac{4[(p + 2m_0)(q - 4m_0) - 1 - 2pq + 4m_0p + 4m_0^2]}{(q - 4m_0)^2}$$

$$= \frac{(q - 4m_0)^2 + 4(2m_0 q - pq - 4m_0^2 - 1)}{(q - 4m_0)^2}$$

$$= \frac{q^2 - 4pq - 4}{(q - 4m_0)^2}\,.$$

Hence $y_3 = y_3'$ if $q^2 - 4pq - 4 = 0$. Line $l$ given by (2) satisfies this condition, but apparently there are an infinite number of lines $l$ for which the given property holds.

II. *Solution by the proposer.*




An inversion $I$ of centre $D$ transforms the lines $AB$, $AC$ into circles $\beta$, $\gamma$ of equal radius and meeting at $D$ at an angle of $60^\circ$ [i.e. the tangents to $\beta$ and $\gamma$ at $D$ are parallel to $AB$ and $AC$ respectively].

There is a triangle such that $\beta$ is its circumcircle and $\gamma$ is one of its excircles. To see this it is sufficient to note that the tangent to $\gamma$ at $D$ cuts $\beta$ again at $F$, and that the tangent to $\beta$ at $F$ is also tangent to $\gamma$ (at a point $G$ such that $\Delta DFG$ is equilateral). Then we have that the degenerate triangle $DFF$, with $FF$ tangent to $\gamma$, has the above property.

In this case, it is known that *given any tangent $t$ to $\gamma$ cutting $\beta$ in two points, the other tangents to $\gamma$ through these points concur in a point on $\beta$.*

It is clear that the given problem is the above property transformed by $I$.

[*Editor's note.* The proposer gives the more general property: *If two conics $\phi$, $\varphi$ admit a triangle inscribed in $\phi$ and circumscribed about $\varphi$, then any triangle inscribed in $\phi$ with two sides tangent to $\varphi$ will also have its third side tangent to $\varphi$.* He derives this as a corollary of the known result that *the six sides of two triangles inscribed in a conic are tangent to another conic.* For this see, e.g., Theorem 134, p. 204 of C.V. Durell, *Projective Geometry*, MacMillan & Co., London, 1926.]

*There were no other solutions submitted for this problem. The editor would like to see another solution, though.*

* * *

**1426**. [1989: 74] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Prove that if $n$ is a positive integer then

$$512 \mid 3^{2n} - 32n^2 + 24n - 1 .$$

I. *Solution by Nicos D. Diamantis, student, University of Patras, Greece.*

For $n = 1$, we have the result, since $3^{2\cdot 1} - 32\cdot 1^2 + 24\cdot 1 - 1 = 0$. We suppose that the result is true for a natural number $k$, i.e.

$$512 \mid 3^{2k} - 32k^2 + 24k - 1 .$$

We have then

$$3^{2k} = 32k^2 - 24k + 1 + 512l$$

for some integer $l$, so that

$$\begin{aligned} 3^{2(k+1)} &= 9\cdot 32k^2 - 9\cdot 24k + 9 + 512\cdot 9l \\ &= 32(k+1)^2 - 24(k+1) + 1 + 2^8k^2 - 2^8k + 512\cdot 9l , \end{aligned}$$

and then

$$3^{2(k+1)} - 32(k+1)^2 + 24(k+1) - 1 = 256k(k-1) + 512\cdot 9l . \qquad (1)$$

Since $k(k-1)$ is even, the right side of (1) is divisible by 512, and therefore

$$512 \mid 3^{2(k+1)} - 32(k+1)^2 + 24(k+1) - 1 .$$

So by induction the result is true for every $n \in \mathbb{N}$.

II. *Solution by Daniel B. Shapiro, Ohio State University.*

The problem can be restated as:

$$3^{2n} \equiv 32n^2 - 24n + 1 \bmod 512 .$$

There is a direct proof by induction which does not indicate how such congruences are generated. Here is a proof which motivates more general results:

By the Binomial Theorem,

$$3^{2n} = 9^n = (1+8)^n = 1 + 8\binom{n}{1} + 8^2\binom{n}{2} + 8^3\binom{n}{3} + \cdots + 8^n.$$

Reducing modulo $512 = 8^3$, we get

$$3^{2n} \equiv 1 + 8n + 64\cdot\frac{n(n-1)}{2} = 1 - 24n + 32n^2 \bmod 512 . \qquad \square$$

We discuss the general problem of representing an exponential function as a polynomial mod $k$. Here $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Z}^+$ is the set of positive integers.

*Definition.* Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ be a function and $k$ a positive integer. We say that $f$ *equals a polynomial mod* $k$ if there is a polynomial $g \in \mathbb{Z}[x]$ such that $f(n) \equiv g(n)$ mod $k$ for every $n \in \mathbb{Z}^+$.

THEOREM. *Let $p$ be a prime and $a$, $m$ positive integers. Then $a^x$ equals a polynomial mod $p^m$ if and only if either $a \equiv 0$ mod $p^m$ or $a \equiv 1$ mod $p$.*

*Proof of "only if".* Suppose there is a polynomial $g(x)$ with $a^n \equiv g(n)$ mod $p^m$ for every positive integer $n$. Certainly if $r \equiv s$ mod $p^m$ then $g(r) \equiv g(s)$ mod $p^m$. Then the sequence $g(1), g(2), g(3), \ldots$ is periodic with period $p^m$. If $p \mid a$ then $g(n) \equiv a^n \equiv 0 \bmod p^m$ for all $n \geq m$. The periodicity implies that this congruence holds for all $n$ and hence $a \equiv 0 \bmod p^m$. If $p \nmid a$ then the minimal period of the sequence $a^n \bmod p$ is the order of $a$, which divides $\varphi(p) = p - 1$. On the other hand the minimal period of the same sequence $g(n) \bmod p$ must divide $p$, so this minimal period must be 1. Then $a^1 \equiv a^2 \equiv a^3 \equiv \cdots \bmod p$, and cancelling $a$, we find $a \equiv 1 \bmod p$.

*Proof of "if".* If $a \equiv 0 \bmod p^m$ the claim is trivial. Suppose that $a \equiv 1 \bmod p$, say $a = 1 + rp$ for some positive integer $r$. By the Binomial Theorem we have

$$a^n = (1 + rp)^n = \sum_{k=0}^{n} (rp)^k \binom{n}{k} .$$

The polynomial

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

has rational number coefficients, but takes integer values whenever an integer is substituted for $x$. Generally, $\binom{x}{k}$ might not equal a polynomial mod $p^m$ since the denominators can involve $p$. For instance, $x(x-1)/2$ does not equal a polynomial mod 4. However,

*Claim.* $p^k\binom{x}{k}$ *equals a polynomial mod* $p^m$.

The power of $p$ dividing $k!$ is well–known to be at most $k$. In fact, if $k! = p^s u$ where $p \nmid u$, it is easy to check that

$$s = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \cdots$$

where $[x]$ represents the greatest integer $\leq x$. Then

$$s \leq \frac{k}{p} + \frac{k}{p^2} + \frac{k}{p^3} + \cdots = \frac{k}{p}\left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{p-1} \leq k ,$$

by summing the geometric series. Since $u$ is relatively prime to $p$ there exists a positive integer $v$ with $uv \equiv 1 \bmod p^m$. Then

$$\frac{p^k}{k!} = \frac{p^{k-s}}{u} \equiv p^{k-s}v \bmod p^m$$

and the claim follows.

Therefore

$$a^n = \sum_{k=0}^{n} r^k p^k \binom{n}{k}$$

does equal a polynomial mod $p^m$. □

Many nice examples of such congruences can be generated. For example,

$$10^n \equiv 405n^2 + 333n + 1 \bmod 729$$

for every positive integer $n$.

The ideas here provide a motivation for understanding some $p$-adic analysis. For instance if $f : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ (where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers), we try to express $f(x)$ as a convergent series of the type

$$\sum_{k=0}^{n} a_n \binom{x}{k} .$$

It turns out that if $f$ is *continuous* (relative to the $p$-adic topology) then $f$ can be expressed as such a "Mahler expansion" where the coefficients $a_n \in \mathbb{Z}_p$ satisfy $\lim_{n\to\infty} a_n = 0$ in $\mathbb{Z}_p$. Conversely if $\{a_n\}$ is a sequence of $p$-adic integers with limit zero then the series above does converge uniformly on $\mathbb{Z}_p$, and the limit is a continuous function. It is not surprising that if $a \equiv 1 \bmod p$ then the function $a^n$ extends to a continuous function $a^x$ on $\mathbb{Z}_p$, and the Mahler expansion is what we expect:

$$a^x = \sum_{k=0}^{n} (a-1)^k \binom{x}{k} .$$

On the other hand, if $a \not\equiv 1 \bmod p$ then $a^n$ cannot be extended to a continuous function on $\mathbb{Z}_p$.

These results on $p$-adic analysis appear in Mahler's short book [2] on pages 51–60. A different approach to the "$p$-adic interpolation" of $f(x) = a^x$ is presented in Koblitz's book [1] on pages 26–28.

I am grateful to Professor W. Sinnott for simplifying the proof of the theorem by telling me about Mahler expansions.

*References*:

[1] Neal Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Graduate Texts in Math., Springer-Verlag, 1977.

[2] Kurt Mahler, *Introduction to p-adic Numbers and Their Functions*, Cambridge Univ. Press, 1973.

*Also solved by RAUL F.W. AGOSTINO, Rio de Janeiro, Brazil; WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; EDWARD L. COHEN, University of Ottawa; GRAHAM DENHAM, student, University of Alberta; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; MATHEW ENGLANDER, Toronto, Ontario; HERTA*

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*Does any reader know of a* Crux *problem with more solvers than this one?*

* * *

1427. [1989: 74] *Proposed by David Singmaster, Polytechnic of the South Bank.*

In *Numerous Numerals* (NCTM, 1975), James Henle defines the "fracimal" $.a.b.c.$ ... to be the number

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \cdots ,$$

where $a,b,c,\ldots$ is a finite sequence of integers, each greater than 1. Which rational numbers can be represented as fracimals?

*Solution by S.J. Maltby, student, Calgary.*

Represent the fracimal by $.a_1.a_2.\cdots.a_n.$

Clearly, 0 is the minimum since $1/a_1 > 0$ for all $a_1 \in \mathbb{Z}^+$ (0 may or may not be a fracimal, depending on whether the sequence $a_1,a_2,\cdots,a_n$ may have $n = 0$). Also, $a_i \geq 2$ for all $i$, so

$$\sum_{i=1}^{n} \prod_{j=1}^{i} \frac{1}{a_j} \leq \sum_{i=1}^{n} 2^{-i} = 1 - 2^{-n} < 1 .$$

So a rational which may be represented by a fracimal must be in [0,1).

We now prove that all rationals in this interval may be represented. Let such a rational be $p/q$ in lowest terms. We proceed by induction on $p$. For $p = 1$, let

$n = 1$ and $a_1 = q$. Now assume it is true for all $p \le k$. For $p = k + 1$, let $r \ge 2$ be the integer such that

$$\frac{1}{r-1} > \frac{p}{q} > \frac{1}{r}.$$

Then $pr > q > p(r-1)$, so $p > pr - q > 0$. Now consider

$$\frac{p}{q} - \frac{1}{r} = \frac{pr - q}{qr}.$$

Since $pr - q < p < q$, by hypothesis $(pr - q)/q$ has a fracimal representation $.b_1.b_2.\cdots.b_m$. If we now let $a_1 = r$, $a_i = b_{i-1}$ for $2 \le i \le m + 1$, we have the fracimal

$$\begin{aligned}.a_1.a_2.\cdots.a_{m+1} &= .r.b_1.b_2.\cdots.b_m\\ &= \frac{1}{r} + \frac{1}{rb_1} + \frac{1}{rb_1b_2} + \cdots + \frac{1}{rb_1b_2\cdots b_m}\\ &= \frac{1}{r} + \frac{1}{r}\left(\frac{pr-q}{q}\right) = \frac{p}{q}.\end{aligned}$$

So $p/q$ has a fracimal representation for $p = k + 1$. Therefore by induction, any $p/q$ between 0 and 1 has a fracimal representation.

*Note.* Suppose we allow infinite fracimals (i.e. the sequence $a_1, a_2, \cdots$ may or may not be infinite). Then any real number in $[0,1)$ may be represented. Let $x$ be such a real. Let $c_i = 0$ or $1$ $(i = 1,2,\cdots)$ such that

$$x = \sum_{i=1}^{\infty} \frac{c_i}{2^i}$$

(i.e. $.c_1c_2c_3\cdots$ is the binary representation of $x$). Let $d_1, d_2, d_3, \cdots$ be the (possibly finite) sequence of $i$'s such that $c_i = 1$. Then a fracimal for $x$ is

$$.2^{d_1}.2^{d_2-d_1}.2^{d_3-d_2}.\ \cdots = \frac{1}{2^{d_1}} + \frac{1}{2^{d_2}} + \cdots = x.$$

*Also solved by BRIAN CALVERT, Brock University, St. Catharines, Ontario; NICOS D. DIAMANTIS, student, University of Patras, Greece; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN and ANDY LIU, University of Alberta; ROBERT SEALY, Mount Allison University, Sackville, New Brunswick; C. WILDHAGEN, Breda, The Netherlands; and the proposer. There was one incorrect solution submitted.*

*Klamkin and Liu mention a paper of R. Cohen (Egyptian fraction expansions,* Math. Magazine 46 (1973) 76–80) *containing stronger results. In particular Cohen shows that every number between 0 and 1 has a unique fracimal representation in which the sequence $a,b,c,\cdots$ is nondecreasing.*

*   *   *

1428*. [1989: 74] *Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.*

$A_1A_2A_3$ is a triangle with sides $a_1$, $a_2$, $a_3$, and $P$ is an interior point with distances $R_i$ and $r_i$ $(i = 1,2,3)$ to the vertices and sides, respectively, of the triangle. Prove that

$$\left(\sum a_1R_1\right)\left(\sum r_1\right) \geq 6 \sum a_1r_2r_3$$

where the sums are cyclic.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We shall derive this inequality from

$$R_1 + R_2 + R_3 \geq 6r \tag{1}$$

([1], item 12.14) by certain transformations ([2], p. 295). Here $r$ is the inradius of $\Delta A_1A_2A_3$. From (1) we get via reciprocation

$$\sum a_1r_1R_1 \sum \frac{1}{r_1} \geq 12F = 6 \sum a_1r_1 \tag{2}$$

where $F$ is the area of $\Delta A_1A_2A_3$. From (2) we get via isogonal conjugation

$$\sum \frac{a_1R_1}{k} \sum r_1 \geq 6 \sum \frac{a_1}{r_1}$$

where $k = r_1r_2r_3$, i.e.

$$\sum a_1R_1 \sum r_1 \geq 6 \sum a_1r_2r_3 \ . \tag{3}$$

Done!

From (3) we get by another reciprocation

$$\sum a_1R_1 \sum R_2R_3 \geq 6 \sum a_1r_1R_1^2 \ .$$

*References*:

[1] Bottema et al, *Geometric Inequalities*, Groningen, 1968.

[2] Mitrinović et al, *Recent Advances in Geometric Inequalities*, Dordrecht, 1989.

*Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MURRAY S. KLAMKIN, University of Alberta; and GEORGE TSINTSIFAS, Thessaloniki, Greece.*

*All solvers reduced the problem to inequality* (1) *above. Klamkin's solution was in fact the same as the above. Noting that both* Crux 1422 *and this problem are merely dual inequalities of known inequalities, he reminds proposers to first check their proposed inequalities for equivalence to one of the numerous inequalities listed in the references* [1] *and* [2] *given above. As an example he gives several inequalities dual to* *the inequality of* Crux 1273 (*one was already given by Janous on* [1988: 277]).

*　　　　*　　　　*

**1429***. [1989: 74] *Proposed by D.S. Mitrinovic, University of Belgrade, and J.E. Pečaric, University of Zagreb.*

(a) Show that

$$\sup \sum \frac{x_1^2}{x_1^2 + x_2 x_3} = n - 1 ,$$

where $x_1, x_2, \cdots, x_n$ are $n$ positive real numbers ($n \geq 3$), and the sum is cyclic.

(b) More generally, what is

$$\sup \sum \frac{x_1^{r+s}}{x_1^{r+s} + x_2^r x_3^s} ,$$

for natural numbers $r$ and $s$?

I. *Solution by Murray S. Klamkin, University of Alberta.*

We claim that the answer to part (b) is $\leq n - 1$, for any real numbers $r$ and $s$.

Note that the summand in (b) can be rewritten as $1/(1 + u_1)$, where

$$u_1 = \frac{x_2^r x_3^s}{x_1^{r+s}} , \quad \text{etc.},$$

so that

$$\prod_{i=1}^{n} u_i = 1 .$$

Thus the above claim is a consequence of the following more general result:

*Let* $n > 2$ *and let* $u_1, u_2, \cdots, u_n$ *be positive real numbers such that* $u_1 u_2 \cdots u_n = 1$. *Set*

$$S_n = \frac{1}{1 + u_1} + \frac{1}{1 + u_2} + \cdots + \frac{1}{1 + u_n} .$$

*Then* $1 < S_n < n - 1$, *where both bounds are best possible.*

The proof is by induction on $n$. For $n = 3$ and the upper bound, we must prove

$$\frac{1}{1 + u_1} + \frac{1}{1 + u_2} + \frac{1}{1 + u_3} < 2 ,$$

or

$$(1 + u_2)(1 + u_3) + (1 + u_3)(1 + u_1) + (1 + u_1)(1 + u_2) < 2(1 + u_1)(1 + u_2)(1 + u_3) ,$$

which follows whenever $u_1 u_2 u_3 \geq 1$ by expanding out. For $n = 3$ and the lower bound, we must prove

$$\frac{1}{1 + u_1} + \frac{1}{1 + u_2} + \frac{1}{1 + u_3} > 1 ,$$

or

$$(1 + u_2)(1 + u_3) + (1 + u_3)(1 + u_1) + (1 + u_1)(1 + u_2) > (1 + u_1)(1 + u_2)(1 + u_3) ,$$

which also follows by expanding out, this time whenever $u_1u_2u_3 \leq 1$.

Now for the upper bound of $S_n$, we assume without loss of generality that $u_1 \geq u_2 \geq \cdots \geq u_{k+1}$. Our inductive hypothesis is $S_k < k - 1$ for $k > 2$ and (here) $u_1u_2 \cdots u_k \geq 1$. Then if $u_1u_2 \cdots u_{k+1} \geq 1$, also $u_1u_2 \cdots u_k \geq 1$, and

$$S_{k+1} = S_k + \frac{1}{1 + u_{k+1}} < k - 1 + 1 = k .$$

For the lower bound of $S_n$, we assume without loss of generality that $u_n \geq u_{n-1} \geq \cdots \geq u_1$. Our inductive hypothesis is $S_k > 1$ for $k > 2$ and (here) $u_1u_2 \cdots u_k \leq 1$. Then if $u_1u_2 \cdots u_{k+1} \leq 1$, also $u_1u_2 \cdots u_k \leq 1$, and

$$S_{k+1} = S_k + \frac{1}{1 + u_{k+1}} > 1 .$$

Hence $1 < S_n < n - 1$ for all $n > 2$. That these bounds are sharp follows by choosing

$$u_1 = u_2 = \cdots = u_{n-1} = \epsilon , \qquad u_n = 1/\epsilon^{n-1} \tag{1}$$

for the upper bound and choosing

$$u_1 = u_2 = \cdots = u_{n-1} = 1/\epsilon , \qquad u_n = \epsilon^{n-1}$$

for the lower bound, where $\epsilon$ is arbitrarily small.

As a further extension which can be proven in the same way, we have

$$0 < \sum_{i=1}^{n} \frac{1}{1 + a_i + b_i + \cdots + z_i} < n - 1 , \tag{2}$$

where

$$\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} b_i = \cdots = \prod_{i=1}^{n} z_i = 1$$

and all these parameters are positive. Here as long as there are at least two sets of parameters, say $a_i$ and $b_i$, then (2) is valid for all $n > 1$. In particular, to get arbitrarily close to the lower bound 0 for

$$\frac{1}{1 + a_1 + b_1} + \frac{1}{1 + a_2 + b_2} + \frac{1}{1 + a_3 + b_3} ,$$

we can let $a_2 = b_1 = a_3 = b_3 = 1/\epsilon$ and $a_1 = b_2 = \epsilon^2$.

It is to be noted that the above result also solves the Olympiad Corner problem 1986.2 [1988: 3]:

$$1 < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \cdots + \frac{a_n}{a_n + a_1} < n - 1 \tag{3}$$

for $n > 2$ and any positive numbers $a_1, a_2, \cdots, a_n$.

*Remark.* Inequality (3) was derived by Zulauf [1]. He also showed [2] that if $x_1, x_2, \cdots, x_7$ are any nonnegative numbers, and $B_k = x_k + x_{k+1} \neq 0$ ($x_k = x_{k-7}$ for $k > 7$), then

$$(I_i): \quad \sum_{k=1}^{7} \frac{x_k}{B_{k+i}} \geq 3\,, \qquad i = 1,2,4,5,$$

$$(I_3): \quad \sum_{k=1}^{7} \frac{x_k}{B_{k+3}} \geq 2$$

(subscripts mod 7). Here $(I_2)$, $(I_3)$, and $(I_4)$ are best possible results whereas it is not known whether 3 is the greatest lower bound for $(I_1)$ and $(I_5)$. For further results on cyclic inequalities see [3] and the referred to papers therein.

*References*:

[1] A. Zulauf, Note on the expression $\sum x_k/(x_k + x_{k+1})$, *Math. Gazette* 43 (1959) 42.

[2] A. Zulauf, Note on some inequalities, *Math. Gazette* 43 (1959) 42–44.

[3] D.S. Mitrinović, *Analytic Inequalities*, Springer–Verlag, Heidelberg, 1970, pp. 132–138.

[*Editor's comment.* The alert reader will have noted that Klamkin's solution doesn't completely answer the question, in that it doesn't show there are positive real $x_i$'s such that the sum in (b) approaches $n - 1$ arbitrarily closely. One needs, for example, to show that the $u_i$'s given in (1) can be attained. When informed of this omission, Klamkin sent back an additional argument, which follows shortly. The only other reader to respond to this problem, *Walther Janous*, also derived the more general inequality $S_n < n - 1$ given by Klamkin and reduced the problem of showing this is best possible in the special case (b) to verifying that a certain $(n - 1) \times (n - 1)$ matrix is always nonsingular. But he didn't finish off the proof, and Sam Maltby, a student at the University of Calgary, has since done this. Klamkin's extra argument is very similar but a little simpler, so here it is.]

*Addendum by Murray S. Klamkin.*

That the upper bound $n - 1$ and the lower bound 1 for part (b) are sharp will follow as in (1) by choosing, for the upper bound,

$$\frac{x_2^r x_3^s}{x_1^{r+s}} = \frac{x_3^r x_4^s}{x_2^{r+s}} = \cdots = \frac{x_n^r x_1^s}{x_{n-1}^{r+s}} = \epsilon\,, \qquad \frac{x_1^r x_2^s}{x_n^{r+s}} = \frac{1}{\epsilon^{n-1}} \tag{4}$$

where $\epsilon$ is arbitrarily small. (For the lower bound we use the same set of equations with $\epsilon$ replaced by $1/\epsilon$.) Note that the last equation in (4) is redundant since it follows from the product of the other $n-1$ equations. It now remains to show that (4) always has a solution. Let $x_i = \epsilon^{t_i}$, so that (4) becomes the following linear set of $n-1$ equations in the unknowns $t_2,t_3,\cdots,t_n$, where $t_1$ can be arbitrary:

$$\begin{aligned}
rt_2 + st_3 \qquad\qquad &= 1 + (r+s)t_1 \\
-(r+s)t_2 + rt_3 + st_4 \qquad\qquad &= 1 \\
-(r+s)t_3 + rt_4 + st_5 \qquad &= 1 \\
\vdots \qquad\qquad & \\
-(r+s)t_{n-1} + rt_n &= 1 - st_1 \,.
\end{aligned}$$

The coefficient matrix is a tridiagonal one with all the super diagonal terms being $s$, all the main diagonal terms being $r$ and all the inferior diagonal terms being $-(r+s)$. We now show that the determinant of this matrix is never zero, so that there is a unique solution for $t_2,t_3,\cdots,t_n$ in terms of $t_1$. Letting $D_{n-1}$ denote the determinant of this matrix, it follows by expansion along the first column that

$$D_{n-1} = rD_{n-2} + s(r+s)D_{n-3} \,. \qquad (5)$$

Since the roots of the characteristic equation $x^2 - rx - s(r+s) = 0$ for (5) are $r+s$ and $-s$, and since $D_1 = r$ and $D_2 = r^2 + rs + s^2$, it follows that

$$(r+2s)D_n = (r+s)^{n+1} - (-s)^{n+1} \,.$$

Since $r$ and $s$ are positive, $D_n$ is never zero.

It would be of interest to find an explicit solution of the inverse of the above tridiagonal matrix since we would then have an explicit solution for the $x_i$'s. For example, here is an explicit solution for the case $n = 4$, $r = s = 1$:

$$x_2 = x_1\epsilon^{3/5}\,, \qquad x_3 = x_1\epsilon^{2/5}\,, \qquad x_4 = x_1\epsilon^{9/5}\,.$$

* * *

**1430**. [1989: 74] *Proposed by Mihaly Bencze, Brasov, Romania.*

*AD*, *BE*, *CF* are (not necessarily concurrent) Cevians in triangle *ABC*, intersecting the circumcircle of $\Delta ABC$ in the points *P*, *Q*, *R*. Prove that

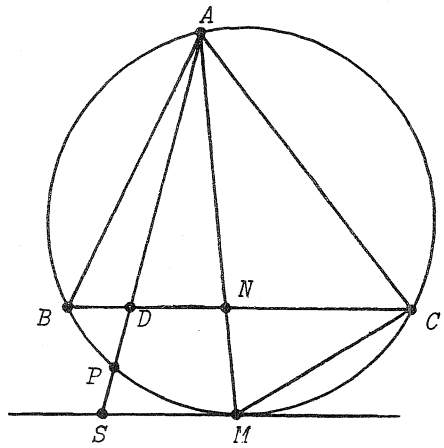
$$\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq 9 \,.$$

When does equality hold?

*Solution by Toshio Seimiya, Kawasaki, Japan.*

Let *AM* be the chord of the circumcircle of $\Delta ABC$ which bisects the interior angle *BAC* of $\Delta ABC$, cutting *BC* in *N*. Let the tangent to the circumcircle cut *AP* produced at *S*. Then $MS \| BC$ and $DS \geq DP$. Therefore

$$\frac{AD}{DP} \geq \frac{AD}{DS} = \frac{AN}{NM} . \qquad (1)$$




As $\angle BAN = \angle NAC$, we get

$$\frac{AB}{BN} = \frac{AC}{NC} = \frac{AB + AC}{BN + NC} = \frac{b + c}{a} .$$

From $\angle MAC = \angle BAM = \angle NCM$ we get $\Delta ACM \sim \Delta CNM$, therefore

$$\frac{AC}{NC} = \frac{AM}{CM} = \frac{CM}{NM} .$$

Thus

$$\frac{AM}{NM} = \frac{AM}{CM}\cdot\frac{CM}{NM} = \left(\frac{AC}{NC}\right)^2 = \frac{(b + c)^2}{a^2} .$$

Hence

$$\frac{AN}{NM} = \frac{AM}{NM} - 1 = \frac{(b + c)^2}{a^2} - 1 .$$

From (1) we get

$$\frac{AD}{DP} \geq \frac{(b + c)^2}{a^2} - 1 .$$

Similarly we get

$$\frac{BE}{EQ} \geq \frac{(c + a)^2}{b^2} - 1 , \qquad \frac{CF}{FR} \geq \frac{(a + b)^2}{c^2} - 1 .$$

Thus

$$\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq \frac{(b + c)^2}{a^2} + \frac{(c + a)^2}{b^2} + \frac{(a + b)^2}{c^2} - 3$$

$$= \left(\frac{b^2}{a^2} + \frac{a^2}{b^2}\right) + \left(\frac{c^2}{b^2} + \frac{b^2}{c^2}\right) + \left(\frac{a^2}{c^2} + \frac{c^2}{a^2}\right) + 2\left(\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}\right) - 3 . \qquad (2)$$

By the A.M.–G.M. inequality, we get

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} \geq 2\cdot\frac{b}{a}\cdot\frac{a}{b} = 2 , \qquad \frac{c^2}{b^2} + \frac{b^2}{c^2} \geq 2 , \qquad \frac{a^2}{c^2} + \frac{c^2}{a^2} \geq 2 ,$$

and

$$\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} \geq 3\cdot\sqrt[3]{\frac{bc}{a^2}\cdot\frac{ca}{b^2}\cdot\frac{ab}{c^2}} = 3 ,$$

equality holding when $a = b = c$. Hence from (2) we obtain

$$\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq 2 + 2 + 2 + 2\cdot 3 - 3 = 9 .$$

Equality holds when $\Delta ABC$ is equilateral and $AD$, $BE$, $CF$ are bisectors of angles $A$, $B$, $C$, respectively.

*Also solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; GEORGE TSINTSIFAS, Thessaloniki, Greece; JOSE YUSTY PITA, Madrid, Spain; and the proposer.*

*The result (1), that the minimum of $AD/DP$ occurs when $ADP$ is the angle bisector, was observed by several solvers. Yusty Pita's argument for this was especially neat: letting $X$, $Y$ lie on $BC$ such that $AX \perp BC$ and $PY \perp BC$, we have $AD/DP = AX/YP$, so we wish to maximize $YP$, which occurs when $P$ is at $M$ in the above figure.*

*Klamkin conjectures that the corresponding inequality for an $n$-dimensional simplex is*

$$\sum \frac{A_iD_i}{D_iP_i} \geq \frac{(n+1)^2}{n-1} \ .$$

* * *

# W.J. BLUNDON

Jack Blundon, Professor Emeritus of Memorial University of Newfoundland, and a frequent contributor to *Crux* in the period 1977–1985, died in March 1990 at the age of 74.

R.H. Eddy of Memorial University kindly sends in the following tribute to Professor Blundon: "I would like to make a few comments about his first love, problem solving. Even though Professor Blundon had a very well–defined research area, he liked to think of himself, above all, as a problem solver who particularly enjoyed exercises in number theory and geometry. He was not content with merely solving a particular problem, rather, he sought the most elegant solution. He painstakingly rewrote a solution several times in order to accomplish this goal. My own interest in problem solving was initiated and nurtured by him."

Professor Eddy also points out the obituary of Professor Blundon which appeared in the May 1990 issue of the *Canadian Mathematical Society Notes*. Here is mentioned his great interest in high school mathematics contests (he was for example Chairman of the Canadian Mathematics Olympiad for three years), as well as many other aspects of his life.

* * *

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