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MORE ON THE GAUSS BICENTENNIAL COMMEMORATION

As announced in the last issue of EUREKA, the two-hundredth anniversary of the birth of Karl Friedrich Gauss will be commemorated in Ottawa on April 30, 1977. One of the highlights of the occasion will be a Gauss Competition, open to Ottawa area high school students. The competition consists of an essay (approximately 2000 words) in English or French, or a display (not too big!) on any aspect of Gauss' scientific work. Money and book prizes will be awarded. Entries should be submitted, no later than March 15, 1977, to:

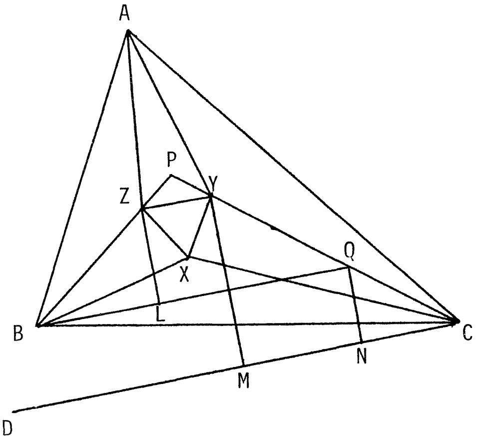
Dr. Kenneth S. Williams, Secretary,
Ottawa Gauss Bicentennial Committee,
c/o Department of Mathematics, Room 813, Arts Tower,
Carleton University,
Ottawa, Ont., K1S 5B6.

A DIRECT GEOMETRICAL PROOF OF MORLEY'S THEOREM

EUCLIDE PARACELSO BOMBASTO UMBUGIO, Guyazuela

MORLEY'S THEOREM. The intersections of the adjacent internal angle trisectors of a triangle are the vertices of an equilateral triangle.

Proof.[1] Extend BZ and CY to meet at P (see figure). On the segment PC, let $PQ = PB$, and let L be the projection of Z on BQ. Construct CD parallel to BQ and let M, N denote the projections of Y, Q on CD.



Since the angles PBQ, PQB, and DCP are equal, we have

$$\frac{YM}{YC} = \frac{ZL}{ZB} = \frac{QN}{QC} \quad \text{or} \quad \frac{YM - ZL}{YC - ZB} = \frac{QN}{QC}.$$

But $YM - ZL = QN$; hence $YC - ZB = QC$. But $YC - YQ = QC$. Therefore $YQ = ZB$; and since $PQ = PB$, it follows that $PZ = PY$. Then, since X is the incenter of triangle BCP, PX bisects the angle BPC, and triangles XYP and PZX are congruent. So $ZX = XY$.

Similarly, it can be shown that $ZY = ZX = XY$.

Q.E.D. et N.F.C.[2]

---

[1]This proof was communicated by the renowned problemist, Professor Euclide Paracelso Bombasto Umbugio, Guyazuela, to Dr. LEON BANKOFF, Los Angeles, California, who kindly translated it for us. The original proof was written in Esperanto, which Dr. Bankoff speaks like a native. Professor Umbugio is known primarily as a numerologist; this is one of his rare excursions in geometry.

[2]*N.F.C.* is the abbreviation of *Ne Fronti Crede*, the Latin equivalent of "Don't believe everything you see." Dr. Bankoff says that, to avoid embarrassment for the good professor, he took the liberty of adding *N.F.C.* to his *Q.E.D.* Those familiar with Professor Umbugio's published papers will recognize the need for this minor addendum.

EXTENSIONS OF TWO THEOREMS OF GROSSMAN

DANIEL SOKOLOWSKY, Antioch College

1. *Introduction.*

In a recent article [7], L. Sauvé recounts a history of various theorems which were unified by H. Grossman [4] into the following:

THEOREM 1. *The following four properties are equivalent:*

(a) *A circle is externally tangent to the four sides of a complete quadrilateral (see Figure* 1*).*

(b) $AB + BF = AD + DF$.

(c) $AC + CF = AE + EF$.

(d) $DC + CB = BE + ED$.

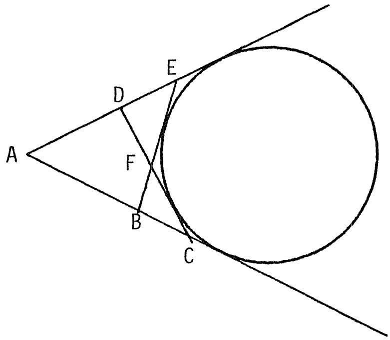


Figure 1

THEOREM 2. *The following four properties are equivalent:*

(a) *A circle is internally tangent to the four sides of a complete quadrilateral (see Figure* 2*).*

(b) $AB - BF = AD - DF$.

(c) $AC - CF = AE - EF$.

(d) $DC - CB = BE - ED$.

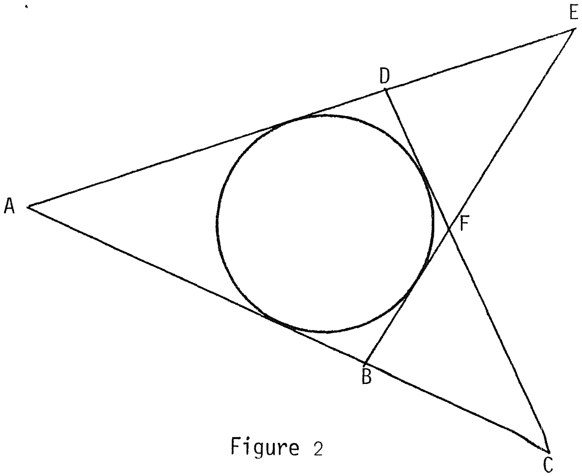


Figure 2

As Sauvé explains in [7], the equivalences which comprise these theorems were "disseminated in bits and pieces in various books and journals." Grossman appears to have been the first to organize them into the coherent forms of Theorems 1 and 2.

The main theorem of this article (Theorem 3 in Section 2) extends these theorems by adding to each of them a fifth property equivalent to the other four (property (a) in each is reworded in Theorem 3 into an equivalent more convenient form). The proofs of the two sets of equivalences are so similar that they are presented in Theorem 3 as two parts of a single theorem.

2. *The main theorem.*

We will make use of the following easily proven

LEMMA. A circle K intercepts chords AB and CD on two lines which meet at F (see Figure 3). If AF = DF, then AB = CD.

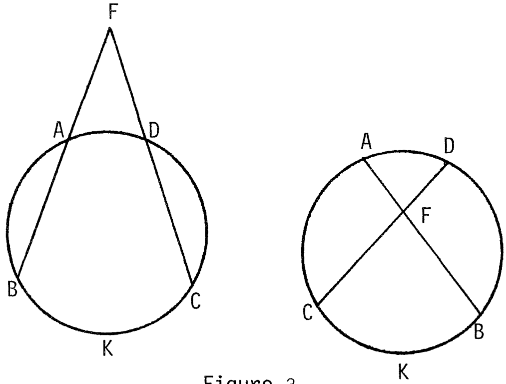


Figure 3

THEOREM 3. *Triangles* $T_1 = ABE$, $T_2 = ACD$ *have a common angle* A, *and sides* BE *and* CD *have a common point* F. *Let*

$$P_1 = AB + AE + BE, \quad P_2 = AC + AD + CD,$$
$$D_1 = AB + AE - BE, \quad D_2 = AC + AD - CD;$$

then

(i) *the following properties are equivalent:*

(a) T_1 *and* T_2 *have a common excircle (see Figure* 1*).*

(b) AB + BF = AD + DF.

(c) AC + CF = AE + EF.

(d) DC + CB = BE + ED.

(e) $P_1 = P_2$.

(ii) *the following properties are equivalent:*

(a) T_1 *and* T_2 *have a common incircle (see Figure* 2*).*

(b) AB - BF = AD - DF.

(c) AC - CF = AE - EF.

(d) DC - CB = BE - ED.

(e) $D_1 = D_2$.

Proof. We begin by showing that, in (i), (b) implies (a) and (e). Thus suppose AB + BF = AD + DF.

Let X be a point on line BE on the side of B opposite from E with XB = AB (see Figure 4), and let Y be on line CD on the side of D opposite from C with $YD = AD$. Let K denote the circumcircle of ΔAXY and O its center. Let WX, YZ, AR, AS denote the chords in which K intercepts lines BE, CD, AB, AD, respectively.

By the lemma, we have

$$AR = WX, \quad AS = YZ, \quad WX = YZ,$$

the last being a consequence of

$$FX = BX + BF = AB + BF = AD + DF = DY + DF = FY.$$

Thus $AR = AS = WX = YZ$. These four chords, having equal length, are equidistant from O. Thus O is the center of a circle K' tangent to the lines along which the chords lie, and hence tangent to the lines along which lie the sides of T_1. Clearly, line OB bisects $\angle ABX$ which is an exterior angle of T_1 since X is exterior to its side BE. Thus K' is an excircle of T_1. A similar argument shows that K' is also an excircle of T_2. Thus (b) implies (a).

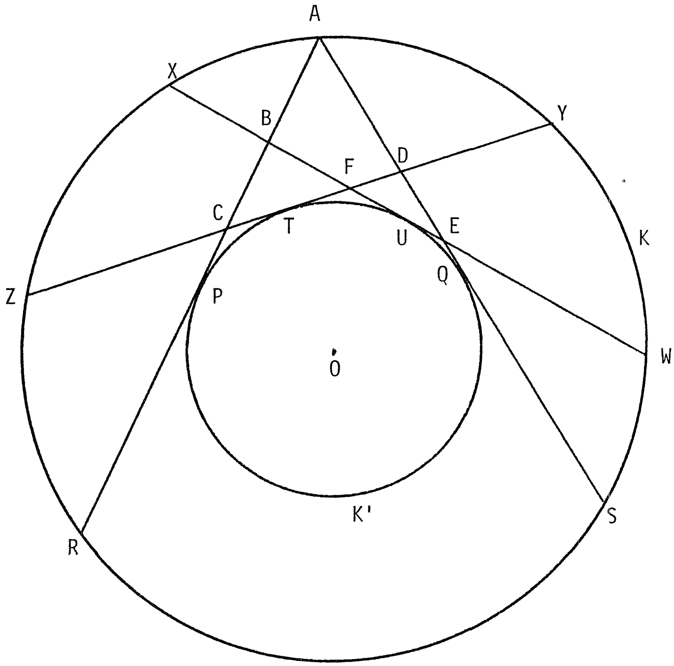


Figure 4

Let AC and CD touch K' at P and T, respectively. Then $CP = CT$ and

$$AP = \tfrac{1}{2} AR = \tfrac{1}{2} YZ = TZ,$$

whence $ZC = AC$, and similarly $EA = EW$. Now we have

$$\begin{aligned} P_1 &= AB + AE + BE = BX + EW + BE = WX \\ &= YZ = ZC + DY + CD = AC + AD + CD \\ &= P_2, \end{aligned}$$

and so (b) implies (e).

The implication (b) $\Longrightarrow$ (c)[1] now follows from

$$P_1 - (AB + BF) = P_2 - (AD + DF).$$

[1]This is the implication known as Urquhart's Theorem (see Section 3). See Sokolowsky [9] for a "no-circle" proof of this theorem. For a different approach, see also Eustice [3] and Williams [12].

Similar proofs would show that (c) implies (a), (e), and (b).

Next we show that (e) implies (b) (hence also (a) and (c)). Thus suppose $P_1 = P_2$.

Let Y lie on line CD on the side of D opposite from C with $YD = AD$ (see Figure 4), and let Z lie on line CD on the side of C opposite from D with $ZC = AC$. Then

$$YZ = AC + AD + CD = P_2.$$

Let K_1 denote the circumcircle of ΔAYZ and let AR and AS denote the chords in which K_1 intercepts lines AB and AD respectively.

Since $AD = DY$ and $AC = CZ$, it follows from the lemma that

$$AR = YZ = AS = P_2.$$

Let X lie on line BE on the side of B opposite from E with $XB = AB$, and let W lie on line BE on the side of E opposite from B with $AE = EW$. Then

$$WX = AB + AE + BE = P_1.$$

Let K_2 denote the circumcircle of ΔAWX and let AR' and AS' denote the chords in which K_2 intercepts lines AB and AD, respectively.

Since $AB = BX$ and $AE = EW$, it follows from the lemma that

$$AR' = WX = AS' = P_1.$$

Since $P_1 = P_2$, it follows that R' coincides with R, S' coincides with S, K_2 coincides with K_1, and the points A, X, Z, R, S, W, Y all lie on K_1. We now have

$$AR = AS = YZ = WX,$$

and so there is a circle K' touching these four equal chords. If YZ and WX touch K' at T and U, respectively, we have $TF = UF$ and

$$TY = \tfrac{1}{2} YZ = \tfrac{1}{2} WX = UX;$$

hence

$$FX = UX - UF = TY - TF = FY.$$

Since

$$FX = BX + BF = AB + BF, \quad FY = DY + DF = AD + DF,$$

we conclude that $AB + BF = AD + DF$, and so (b) is obtained.[2]

We have only left in (i) to show that (a) implies (b) - (e). To do this, let K' denote the common excircle given in (a), let O denote its center, and let K denote the circle with center O and radius OA. Using Figure 4 and arguments similar to those used earlier, the reader will have no difficulty in completing the proofs.

The proofs of the equivalences in (ii) follow similar lines. For example, to show that (b) implies (a) and (e), in Figure 5 let X lie on BE *on the same side* of B as E with $XB = AB$, let Y lie on CD *on the same side* of D as C, with $YD = AD$, and let

[2] Corresponding to the "no-circle" proof of (b) $\Longrightarrow$ (c) referred to in the footnote 1 on p. 165, there is a "no-circle" proof of (e) $\Longrightarrow$ (b). I invite readers to find such a proof in Problem 171 on p. 170 of this issue.

K denote the circumcircle of ΔAXY. The rest of the proof parallels exactly the proof of the corresponding implications of (i).

3. *Urquhart's Theorem and Pedoe's Two-Circle Theorem.*

The implication (b) $\Longrightarrow$ (c) of Theorem 3(i) was discovered by M.L. Urquhart (see [2]) who called it "the most elementary theorem of Euclidean Geometry," considering it so because its statement involves only the concepts of "straight line" and "distance." Investigating Urquhart's Theorem, D. Pedoe [5] found, via point reciprocation (see C.V. Durell [1]) that it was equivalent to the following:

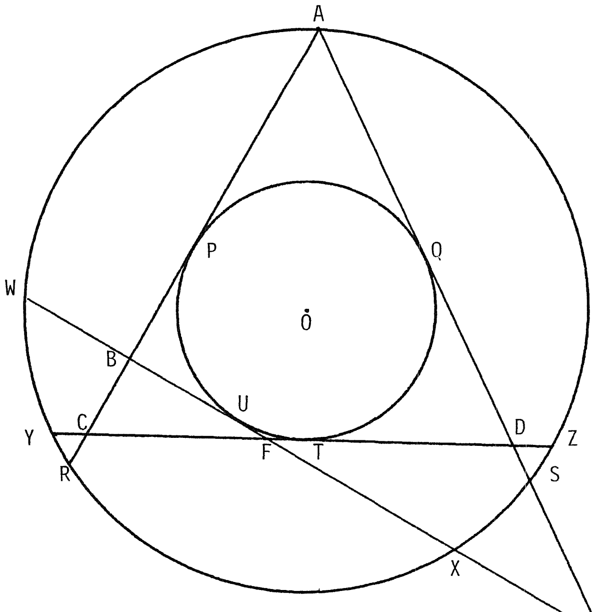


Figure 5

PEDOE'S TWO-CIRCLE THEOREM. ABCD *is a parallelogram and a circle* K *touches* AB *and* BC *(see Figure* 6*). Then* AC *is the radical axis of* K *and of a corresponding circle* K' *which touches* AD *and* DC.

Before giving a proof of this theorem, we point out some of the relations between it and the preceding theorems in this article. Observe that the Two-Circle Theorem allows for the following possibilities:

I. AC intersects neither K nor K' (AC, K, K' have no point in common).

II. AC intersects K and K' in two distinct points.

III. AC is the common tangent of K and K' (AC, K, K' have one point in common).

It can be shown (although we will not digress to do so here) that the conditions of Theorem 3(i) (which are equivalent to: points B, D and C, E in Figures 1 and 4 lie on confocal *ellipses* with foci A, F) are equivalent to case I above. Similarly, the conditions of Theorem 3(ii) (which are equivalent to: points B, D and C, E of Figures 2 and 5 lie on confocal *hyperbolas* with foci A, F) are equivalent to case II. Finally, case III is equivalent to the simultaneous realization of conditions (i) and (ii) of Theorem 3 (which evidently occurs if and only if AB = AD and BF = DF).[3]

[3]The author is indebted to D. Pedoe for his help in developing these conditions.

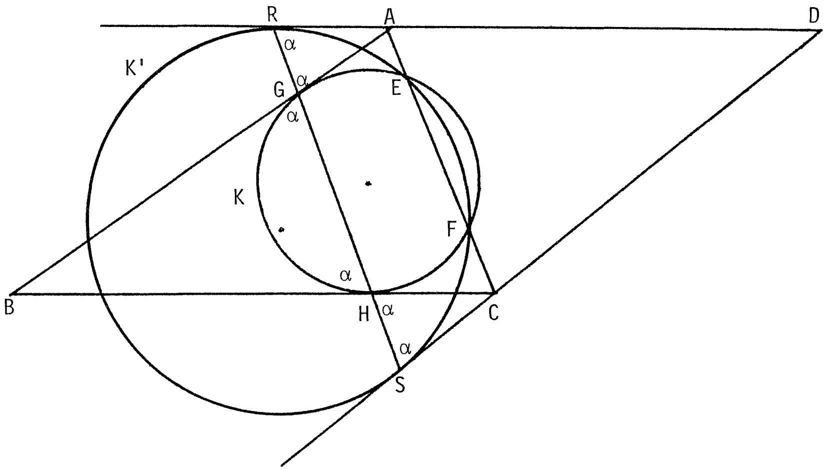


Figure 6

Proof of the Two-Circle Theorem.[4]

Let K touch AB at G and BC at H. Let line GH meet AD at R and CD at S. It is easy to check that each of the six angles indicated in Figure 6 is equal to α; hence $\triangle$ DRS is isosceles and DR = DS. Thus there is a circle K' touching DA at R and DC at S. The point A is on the radical axis, L, of K and K' since the lengths of its tangents AG and AR to K and K' are equal. C is also on L for the same reason. Hence line AC is the radical axis, L, of K and K'.

4. *Applications.*

Theorem 3 is useful in the solution of various construction problems. Using the notation P_1, P_2, D_1, D_2 as defined in the theorem, consider the following:

(a) F is a point on the side BE of $\triangle$ ABE. Construct a line L through F, meeting AB, AE at, say, C and D, such that $P_1 = P_2$.

Using Theorem 3(i), it suffices to find a point D on AE for which AD + DF = AB + BF. On ray AE lay off AP = AB + BF. The perpendicular bisector of PF meets AE at the required point D.

---

[4]Rennie's Lemma, which was used by G. Szekeres [11] in an earlier proof of Urquhart's "Elementary" Theorem and again by D. Pedoe [5] in a proof of the Two-Circle Theorem, is not used here. The lemma states: Given a circle K and two points A and C outside K, the line AC will be a tangent if the distance AC equals the difference between (or the sum of) the length of a tangent from A and the length of a tangent from C.

For another proof of the Two-Circle Theorem, see K.S. Williams [13].

(b) Through a given point F within an angle A, construct a line L meeting the sides of the angle at, say, B and E, such that P_1 is a minimum (see also Sauvé [8]).

The solution of (a) tells us that for each line BE through F there is a corresponding line CD through F for which $P_1 = P_2$. Playing a "Pólya-like" hunch [6], one might suspect that the solution line L is one which is its own correspondent. From Figure 4 we see that this occurs if and only if F lies on the minor arc of a circle K' tangent to the sides of angle A. Constructions for obtaining this circle can be found in Sokolowsky [10] and Sauvé [8]. The latter reference also contains a proof that the line L tangent to this circle at F is the line which minimizes P_1.

Theorem 3(ii) suggests obvious counterparts to problems (a) and (b). Thus consider:

(c) F is a point on side BE of ΔABE. Construct a line through F, meeting AB, AE at, say, C and D, such that $D_1 = D_2$.

(d) Through a point F within an angle A, construct a line L meeting the sides of the angle at, say, B and E, for which D_1 is a maximum.

Using Theorem 3(ii), the reader will have no difficulty in adapting the methods used for (a) and (b) to solve (c) and (d).

The Two-Circle Theorem suggests the following problem:

(e) ABCD is a parallelogram. A circle K is tangent to AB and CD, say at G and H, respectively. Construct a circle K' tangent to AD and CD such that AC is the radical axis of K and K'.

The proof given above of the Two-Circle Theorem shows that K' is the circle tangent to AD and CD at the points R and S (see Figure 6) where GH meets AD and CD.

5. *Acknowledgments*.

The author is indebted to D. Pedoe, University of Minnesota, who provided much inspiration and encouragement in the writing of this article, as well as to the editor for his painstaking efforts and valuable suggestions.

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13. K.S. Williams, Solution to Problem 139, *Eureka*, Vol. 2 (1976), pp. 158-159.

* * *

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 161.

For the problems given below, solutions, if available, will appear in EUREKA *Vol.* 3, *No.* 1, *to be published around Jan.* 15, 1977. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Jan.* 1, 1977.

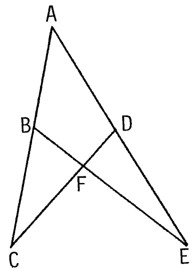
170. [1976: 136] *Correction.*

In (a) and (b) replace the word *into* by *onto*.

171. *Proposed by Dan Sokolowsky, Yellow Springs, Ohio.*

Let P_1 and P_2 denote, respectively, the perimeters of triangles ABE and ACD shown in the figure. Without using circles, prove that

$$P_1 = P_2 \Longrightarrow AB + BF = AD + DF.$$



172. *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School. Bayside, N.Y.*

Find all sets of five positive integers whose sum equals their product. Prove that you have obtained all solutions.

173. *Proposed by Dan Eustice, The Ohio State University.*

For each choice of n points on the unit circle ($n \geq 2$), there exists a point on the unit circle such that the product of the distances to the chosen points is greater than or equal to 2. Moreover, the product is 2 if and only if the n points are the vertices of a regular polygon.

174. *Proposed by Leroy F. Meyers, The Ohio State University.*

The function whose value at each rational number is 1 and at each irrational number is 0 is known to be discontinuous on the entire real line R. Describe a function which is defined on R and is continuous and differentiable at each point in a set E (specified below), but is discontinuous at each point not in E.

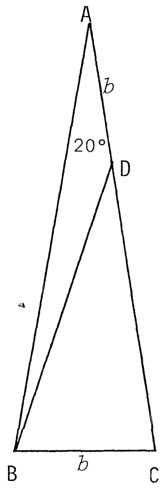
(a) $E = \{0\}$;

(b) E is a finite set;

(c) E is denumerable.

175. *Proposed by Andrejs Dunkels, University of Luleå, Sweden.*

Consider the isosceles triangle ABC in the figure, which has a vertical angle of 20°. On AC, one of the equal sides, a point D is marked off so that |AD| = |BC| = b. Find the measure of $\angle$ABD .



176. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Soit $f{:}R \to R$ une fonction différentiable paire. Montrer que sa dérivée f' n'est pas paire, à moins que f ne soit une fonction constante.

177. *Proposed by Kenneth S. Williams, Carleton University.*

P is a point on the diameter AB of a circle whose centre is C. On AP, BP as diameters, circles are drawn. Q is the centre of a circle which touches these three circles. What is the locus of Q as P varies?

178. *Proposed by Gali Salvatore, Ottawa, Ontario.*

Prove or disprove that the equation $ax^2 + bx + c = 0$ has no rational root if a, b, c are all odd integers.

179. *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

The equation $5x + 7y = c$ has exactly three solutions (x,y) in positive integers. Find the largest possible value of c.

180. *Proposed by Kenneth S. Williams, Carleton University.*

Through O, the midpoint of a chord AB of an ellipse, is drawn any chord POQ. The tangents to the ellipse at P and Q meet AB at S and T, respectively. Prove that $AS = BT$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

75. [1975: 71; 1976: 10] *A solution to this problem was recently received from* KENNETH M. WILKE, *Topeka, Kansas.*

126. [1976: 41, 123] *Late solution by* ANDRÉ BOURBEAU, *École Secondaire Garneau, Vanier, Ont.*

131. [1976: 67, 141] *Late solution by* CHARLES W. TRIGG, *San Diego, California.*

132. [1976: 67, 142] *Proposed by Léo Sauvé, Algonquin College.*

If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ for $\theta = \alpha, \beta, \gamma$, prove that the normals to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

at the points of eccentric angles α, β, γ are concurrent if and only if

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

II. *Comment by Dan Pedoe, University of Minnesota.*

The problem of the three normals to an ellipse rang a chime in my memory, and I turned up Salmon's *Conic Sections*, which has one most interesting result, and then Smith's *Conic Sections*, on which I was raised, where I had seen the result.

In Salmon, the 1900 edition (Longmans Green, London), p. 220, Exercise 10, there occurs this result:

The area of the triangle formed by the three normals is

$$\frac{(a^2-b^2)^2}{4ab}\cdot\tan\frac{\beta-\gamma}{2}\cdot\tan\frac{\gamma-\alpha}{2}\cdot\tan\frac{\alpha-\beta}{2}\cdot P,$$

where

$$P = \{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\}^2.$$

The result is ascribed to Mr. Burnside. This gives a necessary and sufficient condition for concurrency.

Smith obtains it essentially by multiplying two determinants together, although he uses equations. One of the points stressed in Somerville's *Analytical Conics* is that the eccentric angles of the feet of the four normals which pass through a point add to an odd multiple of π. Where a circle cuts an ellipse they add to an even multiple of π, so there are numerous aspects to the problem.

134. [1976: 68, 151] *Proposed by Kenneth S. Williams, Carleton University.*

ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

III. *Solutions by Leon Bankoff, Los Angeles, California.*

(a) *Geometric solution.*

Draw $QR \parallel BC$ and draw RC, PS, AS, as shown in Figure 1. Note that $AR = RC$ in the isosceles ΔARC.

Since CP bisects $\angle RCA$ and AS bisects $\angle RAC$, we have

$$\frac{AP}{PR} = \frac{AC}{RC} = \frac{AC}{AR} \quad \text{and} \quad \frac{AC}{AR} = \frac{CS}{RS}.$$

Thus $\dfrac{AP}{PR} = \dfrac{CS}{RS}$, so that $PS \parallel AC$ and $PR = RS = RQ$.
Hence $\angle RPQ = \angle RQP = 50°$ and $\angle PQA = 30°$.

(b) *Simplified trigonometric solution.*

From the same Figure 1,

$$\frac{PR}{RC} = \frac{\sin 10°}{\sin 30°} = 2\sin 10° = \frac{\sin 20°}{\cos 10°} = \frac{\sin 20°}{\sin 80°}$$
$$= \frac{RQ}{AR} = \frac{RQ}{RC}.$$

Hence $PR = RQ$, $\angle RPQ = \angle RQP = 50°$, and so $\angle PQA = 30°$.

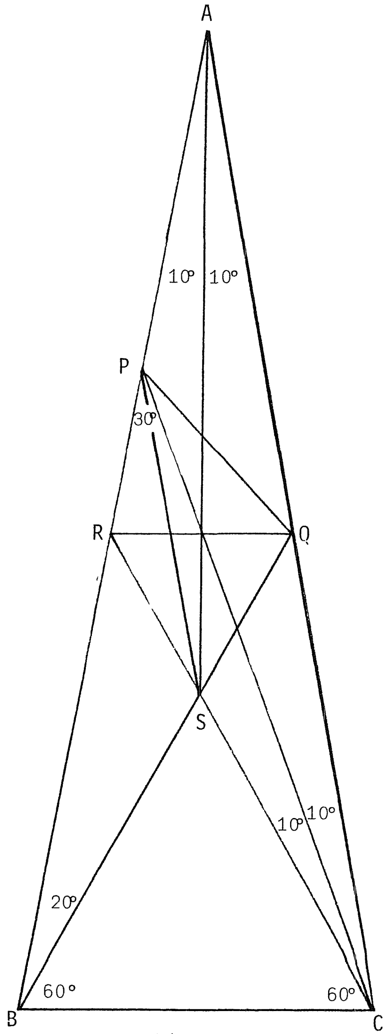


Figure 1

IV. *Solution by Dan Sokolowsky, Yellow Springs, Ohio.*

Honsberger's "old chestnut" discussed in the editor's comment [1976: 152-153] has $\angle QBC = 60°$, $\angle SCB = 50°$, as shown in Figure 2, and concludes that $\angle SQR = 30°$. This can be used to give a quick solution to our own problem, in which $\angle QBC = 60°$, $\angle PCB = 70°$, which shows that the two problems are intimately related.

Since $\angle BSC = \angle BRC = 50°$ and $\angle SPR = \angle SQR = 30°$, it follows that SBCR and PSRQ are cyclic quadrilaterals. Hence $\angle BRS = 50°$, $\angle PQS = \angle PRS = 80°$, and it follows immediately that $\angle PQA = 30°$.

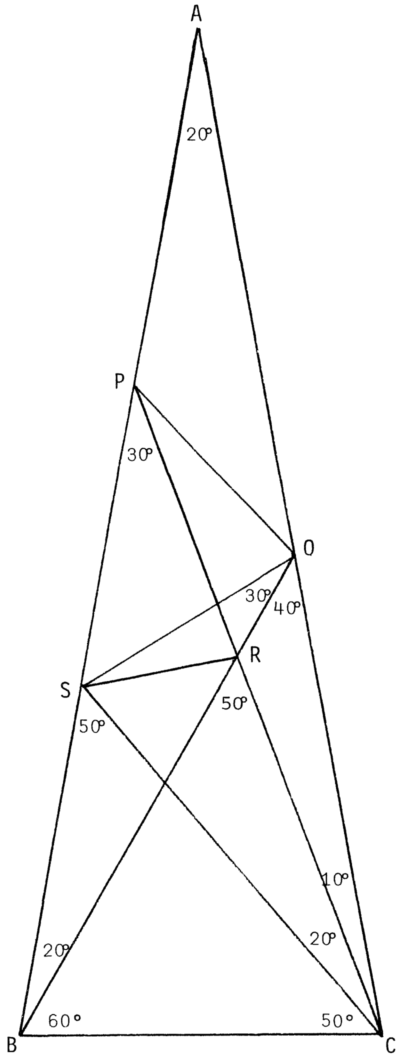
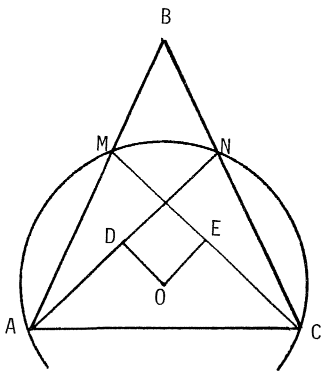


Figure 2

141. [1976: 93] *Proposed by Leon Bankoff, Los Angeles, California.*

What is wrong with the following "proof" of the Steiner-Lehmus Theorem? (See figure below and [1976: 92]for the source.)



If in a triangle two angle bisectors are equal, then the triangle is isosceles.

At the midpoints of the angle bisectors, I erect two perpendiculars which meet in O; with O as center and AO as radius, I describe a circle which will evidently pass through the points, A, M, N, C.

Now the angles MAN, MCN are equal since the measure of each is arc $\frac{MN}{2}$; hence BAC = ACB, and triangle ABC is isosceles.

I. *Solution by R. Robinson Rowe, Naubinway, Michigan.*

Leon's problem appealed to me because I fear that if he hadn't called attention to a gap in the published "proof," I would have accepted it as *écrit ergo est*. Once alerted, however, I could see that André had not proved that his perpendiculars DO and EO were of equal length, so that although AO = NO and MO = CO, he might have AO ≠ CO and he might strike circular arcs AN and MC which did not coincide.

II. *Solution by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

In the spirit of the "Popular Misconceptions" article in the October, 1975 issue of EUREKA [1975: 69-70], this "proof" actually establishes the stronger "theorem" that all triangles are isosceles. In fact, previous proofs of this statement are much more complex. Thus, this new proof is a quite valuable addition to the traditional geometry course. When combined with the "fact" that all angles are right angles, we can easily be convinced that the sum of the three angles of a triangle is indeed 270°, thereby disproving Euclid's Parallel Postulate. Tom Lehrer would have a field day!

"Clearly," though, AO = ON and MO = OC do *not* imply AO = OM. However, I understand some local angle trisectors have proved this implication is occasionally valid, hence always valid, and so all right angles are obtuse, clearly.

The adverb "evidently," as used in the proposal, would be a welcome addition to the "Brief Dictionary of Phrases Used in Mathematical Writing," by the celebrated French mathematical lexicologist H. Pétard,[1] of the Society for Useless Research, which was published in the *American Mathematical Monthly*, Vol. 73 (1966), pp. 196-197.

Also solved by WALTER BLUGER, *Department of National Health and Welfare;* HIPPOLYTE CHARLES, *Waterloo, Québec;* CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence;* F.G.B. MASKELL, *Algonquin College;* CHARLES W. TRIGG, *San Diego, California;* KENNETH S. WILLIAMS, *Carleton University; and the proposer.*

142. [1976: 93] *Proposed by André Bourbeau, École Secondaire Garneau.*

Find 40 consecutive positive integral values of x for which $f(x) = x^2 + x + 41$ will yield composite values only.

I. *Solution by Kenneth S. Williams, Carleton University.*

We prove a general result which contains the required question as a special case.

---

[1] *Editor's suggestion to Pétard.* "Evidently," in this context, means: I can't prove it but my son, the publisher, won't know the difference, and it should hold up for a few editions anyway. After that, who cares? (See [1976: 92].)

THEOREM. Let $f(x) = a_0x^r + a_1x^{r-1} + \ldots + a_{r-1}x + a_r$ *be a polynomial of degree* $r \geq 1$ *with nonnegative integral coefficients and such that* $f(x) \geq 2$ *for* $x = 0,1,2,\ldots,k-1$. *Then there exist* k *consecutive positive integral values of* x *for which* $f(x)$ *is composite.*

It would be very easy to modify the proof and relax the conditions on f.

Proof. Set $N = f(0) \cdot f(1) \cdot \ldots \cdot f(k-1)$, where it is clear that each factor $f(\ell) \geq 2$ for $0 \leq \ell \leq k-1$. Then $f(x)$ is composite for the k consecutive values

$$x = N,\ N+1,\ N+2,\ \ldots,\ N+k-1,$$

since, for $\ell = 0,1,\ldots,k-1$,

$$\begin{aligned} f(N+\ell) &= a_0(N+\ell)^r + a_1(N+\ell)^{r-1} + \ldots + a_r \\ &= \text{mult. of } N + (a_0\ell^r + a_1\ell^{r-1} + \ldots + a_r) \\ &= \text{mult. of } f(\ell) + f(\ell) \\ &= \text{mult. of } f(\ell) > f(\ell). \end{aligned}$$

II. *Solution by F.G.B. Maskell, Algonquin College.*

It is clear that, for $n = 0,1,2,\ldots,39$, we have

$$f(n) \leq f(39) = 1601;$$

hence, for the 40 consecutive values

$$x_n = (1601)! + n, \quad n = 0,1,2,\ldots,39, \tag{1}$$

$f(x_n)$ is composite since

$$\begin{aligned} f(x_n) &= x_n^2 + x_n + 41 \\ &= (1601)!\{(1601)! + 2n + 1\} + f(n) \\ &= \text{mult. of } f(n) > f(n). \end{aligned}$$

The numbers given by (1) are quite large, since $(1601)! \approx 8.439993 \times 10^{4436}$.

III. *Solution by R. Robinson Rowe, Naubinway, Michigan.*

It has been well publicized[1] that the classic $f(x) = x^2 + x + 41$ generates primes for $0 \leq x < 40$, and it is clear that $f(x)$ is composite when x is a multiple of 41. Since $x^2 + x = x(x+1)$, $f(x)$ is also composite if $x+1$ is a multiple of 41, or $x = 41n - 1$.

For the integer α, $0 \leq \alpha < 40$, let

$$A = f(\alpha) = \alpha^2 + \alpha + 41,$$

so that A is prime, and compute

$$f(A + \alpha) = A(A + 2\alpha + 2)$$

---

[1] *Editor's suggestion to H. Pétard.* The phrase "it has been well publicized" is also a worthy candidate for inclusion in your *Dictionary* (see p. 175 in this issue). It means, of course: I can't find the reference.

and

$$f(mA+\alpha) = A(m^2A + 2m\alpha + m + 1), \tag{1}$$

both of which are composite. Note that (1) shows that any multiple of A increased by α will generate a composite.

Now let

$$N = \prod_{x=0}^{38} (x^2+x+41), \tag{2}$$

so that N is the product of the primes 41, 43, 47, ..., 1523. By the first paragraph, 41 is a factor of $f(N-1)$ and of $f(N)$; furthermore, by (1),

with $\alpha = 1$, $43 | f(N+1)$,

with $\alpha = 2$, $47 | f(N+2)$,

. . . .

with $\alpha = 38$, $1523 | f(N+38)$.

Thus $f(x)$ is composite for the 40 consecutive values

$$x = N-1,\ N,\ N+1,\ \ldots,\ N+38.$$

I was relieved that the problem had not asked for the least set, since N as given by (2) is a very large number. In fact $N \approx 3.054811505 \times 10^{98}$, and the 40 composite values of $f(x)$ will all have 197 digits beginning with

93,318,733,... .

I note also that (2) omits $x = 39$, $A = 1601$. Not only would this have added a 41st consecutive composite; it would also connect the set to $N+40$ and $N+41$, which both yield composite values, making a set of 43.

A very interesting sequel to a classic.

IV. *Comment by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

Among the polynomial generators of primes, $f(x) = x^2+x+41$ is perhaps the best known. In Beiler [1] can be found a list of the 101 values of $f(x)$ from $x = 0$ to $x = 100$. The accompanying discussion is fascinating. The fact that $f(x)$ is prime from $x = 0$ to $x = 39$ is totally overshadowed by the fact that $x^2 - 79x + 1601$ is prime for all x from 0 to 79. Although I have not checked it completely, I believe that $g(x) = x^2 - x + 41$ is just as "prime" as $f(x)$.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono; and by the proposer. Two incorrect solutions were received.*

Editor's comments.

1. Dodge and the proposer both showed that $f(x)$ was composite for

$$x = u,\ u+1,\ u+2,\ \ldots,\ u+39,$$

where

$$u = f(0) \cdot f(1) \cdot \ldots \cdot f(39) \approx 4.89075 \times 10^{101}.$$

Dodge also showed that $f(x)$ is never divisible by any prime less than 41.

2. If Conrad has not yet completed his investigation of the function $g(x) = x^2 - x + 41$, he (and other readers) can find much interesting information about it in Stevens [3].

3. When I first published this problem, in the May 1976 issue of EUREKA, it was new to me and no information about its source or history was volunteered by the proposer. Since then, by patient digging (editor: a poor drudge. Samuel Johnson's *Dictionary*, 1755), I've been able to uncover the following information about this problem:

(a) It occurs as a problem, with solution, in Honsberger [2]. The same reference also contains a proof that $f(x) = x^2 + x + 41$ is never a square, except for $f(40) = f(-41) = 41^2$.

(b) It appeared as a problem in one of the 1975 issues of the *Ontario Secondary School Mathematics Bulletin*, without any proposer's name. The problem editor of the *Bulletin* is the same Professor Honsberger.

(c) A solution of the problem referred to in (b) appeared in the May 1976 issue of the *Bulletin*. The solver was, *surprise*! *surprise*! none other than our own proposer.

4. It would be interesting if some computer nut were to make a search and discover the smallest set of 40 consecutive integers x for which $f(x)$ is composite.

REFERENCES

1. Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover, 1964, pp. 219-220.

2. Ross Honsberger, *Mathematical Gems II*, The Mathematical Association of America, 1976, pp. 37, 163-164.

3. D.C. Stevens, Solution to Problem E1477, *American Mathematical Monthly*, Vol. 69 (1962), p. 234.

143. [1976: 93] *Proposé par Léo Sauvé, Collège Algonquin.*

On donne

$$f(n) = x^n + y^n + z^n,$$

où (x,y,z) est un triplet de nombres complexes tels que $f(n) = n$ pour $n = 1,2,3$. Montrer que le triplet (x,y,z) n'est pas réel et calculer $f(4)$, $f(5)$, et $f(6)$.

Adapted from the solutions submitted independently by Clayton W. Dodge, University of Maine at Orono; and Gali Salvatore, Ottawa, Ont.

We have given that

$$f(1) = \Sigma x = 1, \tag{1}$$
$$f(2) = \Sigma x^2 = 2, \tag{2}$$
$$f(3) = \Sigma x^3 = 3. \tag{3}$$

From $(\Sigma x)^2 = \Sigma x^2 + 2\Sigma yz$, we get

$$\Sigma yz = -\frac{1}{2}, \tag{4}$$

and from $\Sigma x^3 - 3xyz = (\Sigma x)(\Sigma x^2 - \Sigma yz)$, we get

$$xyz = \frac{1}{6}. \tag{5}$$

Finally, from $(\Sigma xy)^2 = \Sigma y^2z^2 + 2xyz(\Sigma x)$, we get

$$\Sigma y^2z^2 = -\frac{1}{12},$$

which shows that the triplet (x,y,z) is not real.

Suppose now that we know $f(n)$, $f(n-1)$, and $f(n-2)$ in addition to equations (1)-(5) above. Then

$$\begin{aligned} f(n) &= f(1)\cdot f(n) = (x+y+z)(x^n+y^n+z^n) \\ &= f(n+1) + yz(y^{n-1}+z^{n-1}) + zx(z^{n-1}+x^{n-1}) + xy(x^{n-1}+y^{n-1}) \\ &= f(n+1) + yz\{f(n-1) - x^{n-1}\} + zx\{f(n-1) - y^{n-1}\} + xy\{f(n-1) - z^{n-1}\} \\ &= f(n+1) + (\Sigma yz)f(n-1) - xyz\, f(n-2) \\ &= f(n+1) - \frac{1}{2}f(n-1) - \frac{1}{6}f(n-2) \end{aligned}$$

by (4) and (5). Solving for $f(n+1)$ gives

$$f(n+1) = f(n) + \frac{1}{2}f(n-1) + \frac{1}{6}f(n-2). \tag{6}$$

Using (6) together with (1)-(3), we find

$$f(4) = \frac{25}{6}, \quad f(5) = 6, \quad f(6) = \frac{103}{12}, \quad f(7) = \frac{221}{18},$$

and so forth.

It is remarkable that even though the triplet (x,y,z) is not real, all the integral power sums

$$f(n) = x^n + y^n + z^n$$

are real and rational, and that includes negative powers since, if (6) is rewritten as

$$f(n-2) = -3f(n-1) - 6f(n) + 6f(n+1),$$

then (1)-(3) gives, for $n = 2,1,0,-1,\ldots$

$$f(0) = 3, \quad f(-1) = -3, \quad f(-2) = -3. \quad f(-3) = 45, \quad f(-4) = -135,$$

and so forth.

Also solved by WALTER BLUGER, *Department of National Health and Welfare (partial solution);* LEROY F. MEYERS, *The Ohio State University;* KENNETH S. WILLIAMS, *Carleton University; and the proposer. A comment was submitted by* STEVEN R. CONRAD, *Benjamin N. Cardozo High School, Bayside, N.Y.*

Editor's comments.

1. To arrive at the result, Meyers used Newton's identities (see, for example, [1, 4]) for which he also gave a proof by induction.

2. It is amusing to note that $f(5) = 6$ can be obtained instantly by substituting $u = -1$ in the identity

$$\frac{x^5+y^5+z^5+u^5}{5} = \frac{x^3+y^3+z^3+u^3}{3} \cdot \frac{x^2+y^2+z^2+u^2}{2},$$

which holds whenever $x+y+z+u = 0$. This identity can be found in [2].

3. In his comment, Conrad referred to a problem he had proposed in the *American Mathematical Monthly* for which a solution by O.G. Ruehr was published in [3]. In our notation, the problem asks for $f(n+1)$ given

$$f(k) = x_1^k + x_2^k + \ldots + x_n^k \quad \text{and} \quad f(k) = k \text{ for } k = 1,2,\ldots,n.$$

Ruehr obtained the explicit formula

$$f(n+1) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \binom{n+1}{k+1}.$$

When $n = 3$, this formula gives $f(4) = \frac{25}{6}$, in accordance with our own result. Ruehr's proof is very complicated. I hope one of our readers can find a simpler one.

REFERENCES

1. G. Chrystal, *Algebra*, Chelsea, 1952, Vol. I, pp. 436-438.

2. H.S. Hall, S.R. Knight, *Higher Algebra*, Macmillan, 1891, p. 444.

3. O.G. Ruehr, Solution to Problem E2487, *American Mathematical Monthly*, Vol. 82 (1975), pp. 764-765.

4. Louis Weisner, *Introduction to the Theory of Equations*, Macmillan, 1938, pp. 114-115.

144. [1976: 94] *Proposed by Viktors Linis, University of Ottawa.*

In a triangle ABC, the medians AM and BN intersect at G. If the radii of the inscribed circles in triangles ANG and BMG are equal, show that ABC is an isosceles triangle.

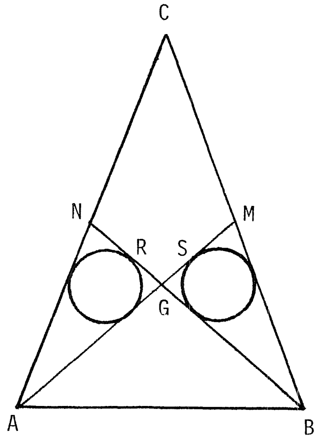
(I found this problem in DELTA No. 1, 1976, a Polish journal for the popularization of physics and mathematics.)

Editor's comment.

The two solvers mentioned below submitted solutions in English in which the mathematical content was practically identical, so that each solution has an equal right to be presented. So as not to appear to favour the wording of one over that of the other, I have written below a solution in French which embodies their common ideas.

Solution soumise indépendamment par Leon Bankoff, Los Angeles, Californie; et Leroy F. Meyers, The Ohio State University.

Les triangles ANG et BMG ont même aire, qui est $\frac{1}{6}$ de celle du triangle ABC (voir la figure ci-jointe.) Puisque les rayons de leurs cercles inscrits sont égaux, il en est de même de leurs demi-périmètres $AN + GR$ et $BM + GS$. Or, trivialement, les tangentes communes intérieures à deux cercles égaux sont égales et se bissectent; donc $GR = GS$, d'où $AN = BM$ et $AC = BC$. *Voilà!*



Also solved by WALTER BLUGER, *Department of National Health and Welfare;* STEVEN R. CONRAD, *Benjamin N. Cardozo High School, Bayside, N.Y.;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence;* GEORGE W. MASKELL, *Huddersfield, England; and* KENNETH S. WILLIAMS, *Carleton University.*

145. [1976: 94] *Proposed by Walter Bluger, Department of National Health and Welfare.*

A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 collinear points are equal.

Construct a magic pentagram with the ten smallest possible positive primes.

I. *Comment by Charles W. Trigg, San Diego, California.*

In [2] Harry Langman discusses magic "cross-pentagons" (pentagrams). He concludes, "Carrying out a somewhat more elaborate, but similar, analysis for prime numbers, we find a solution with minimal sum of all the primes. Here the line sum is 72." He leaves the solution to the reader.

II. *Partial solution by the proposer.*

I begin with the configuration given in Figure 1, which has a line sum of 24, but in which the numbers are not all prime. Then I perform successively the operations

(a) Subtract 1 from outer set and subtract 2 from inner set.

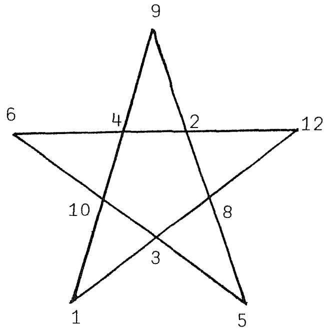


Figure 1

(b) Multiply all numbers by 6.

(c) Add 13 to outer set and add 5 to inner set.

The result is the configuration of Figure 2, which has a line sum of 144, and in which all numbers are primes.

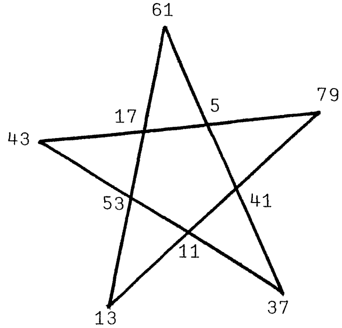


Figure 2

Editor's comment.

Although the proposer's solution with line sum 144 is totally overshadowed by Langman's minimal solution with line sum 72, the method he uses is interesting and may be helpful to readers in trying to discover exactly what Langman's minimal solution is. I'll gladly publish it if anyone finds it.

Note that the configuration in Figure 1 may well be the one with minimal line sum for arbitrary positive integers, since Dongre in [1] and Richards in [3] both prove that no magic pentagram exists which includes only the integers 1,2,...,10. However a magic hexagram does exist which includes only the integers 1,2,...,12; it can be found in [1].

REFERENCES

1. N.M. Dongre, Solution to Problem E2265, *American Mathematical Monthly*, Vol. 78 (1971), p. 1025.

2. Harry Langman, *Play Mathematics*, Hafner, New York, 1962, pp. 80-83.

3. Ian Richards, Impossibility, *Mathematics Magazine*, Vol. 48 (1975), pp. 249-262.

146. [1976: 94] *Proposé par Jacques Marion, Université d'Ottawa.*

Montrer qu'il n'existe pas de fonction rationnelle $R(z)$ telle que $R(n) = n!$ pour tout nombre naturel n.

Solution de Leroy F. Meyers, The Ohio State University.

Posons

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 z^r + a_1 z^{r-1} + \ldots + a_r}{b_0 z^s + b_1 z^{s-1} + \ldots + b_s},$$

où P et Q sont des polynômes sans diviseur commun et où les coefficients initiaux a_0 et b_0 ne sont pas nuls. Puisque $\lim\limits_{n\to\infty} \frac{n^{r-s}}{n!} = 0$, on arriverait à la contradiction

$$1 = \lim_{n\to\infty} \frac{P(n)/Q(n)}{R(n)} = \lim_{n\to\infty} \frac{a_0 n^{r-s}/b_0}{n!} = 0.$$

Aussi résolu par KENNETH S. WILLIAMS, *Université Carleton; et par le proposeur.*

147. [1976: 94] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

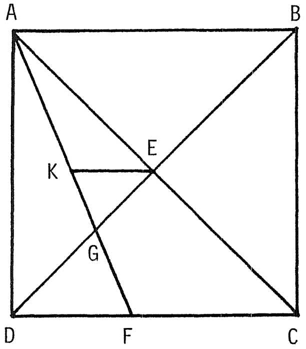
In square ABCD, $\overline{AC}$ and $\overline{BD}$ meet at E. Point F is in $\overline{CD}$ and $\angle CAF = \angle FAD$. If $\overline{AF}$ meets $\overline{ED}$ at G and if $EG = 24$, find CF.

(I wrote this question originally for the Bergen County, N.J. Math League.)

Solution by Leon Bankoff, Los Angeles, California.

Draw $EK \parallel DC$ as shown in the figure. In right triangle AEG, $\angle AGE = 67\frac{1}{2}°$; and $\angle EKG = \angle KAE + \angle AEK = 67\frac{1}{2}°$. Hence triangle KEG is isosceles and

$$FC = 2KE = 2EG = 48.$$



Also solved by LEON BANKOFF, *Los Angeles, California (three more solutions);* ANDRÉ BOURBEAU, *École Secondaire Garneau (two solutions);* WALTER BLUGER, *Department of National Health and Welfare;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont. (two solutions);* CLAYTON W. DODGE, *University of Maine at Orono (two solutions);* G.D. KAYE, *Department of National Defence;* CHARLES W. TRIGG, *San Diego, California (two solutions);* KENNETH S. WILLIAMS, *Carleton University; and the proposer (two solutions).*

Editor's comment.

In one of his solutions, de Peiza showed more generally that if ABCD is a rectangle with $AB:BC = m$ and if $EG = k$, then

$$CF = 2k\left(1 + \frac{m^2 - 1}{\sqrt{m^2 + 1}}\right).$$

In a note accompanying his proposal, the proposer said that this problem was also used at the first Annual Long Island Mathematics Championship held on April 28, 1976, and that it was apparently the most difficult question on the contest questionnaire, since only four correct solutions were received.

Only four? We got seventeen. The readers of EUREKA are prepared to take on the Long Island high school students any time.

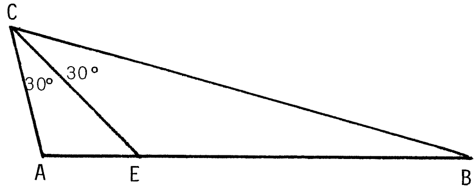
148. [1976: 94] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

In $\triangle ABC$, $\angle C = 60°$ and $\angle A$ is greater than $\angle B$. The bisector of $\angle C$ meets $\overline{AB}$ in E. If CE is a mean proportional between AE and EB, find $\angle B$.

(I wrote this question originally for the New York City Senior Interscholastic Mathematics League.)

I. *Solution by Walter Bluger, Department of National Health and Welfare.*

From the hypothesis and the law of sines, we have (see figure)



$$1 = \frac{CE}{AE} \cdot \frac{CE}{EB} = \frac{\sin(B+60°)}{\sin 30°} \cdot \frac{\sin B}{\sin 30°}$$

$$= 4\sin(B+60°)\sin B\,;$$

hence

$$\frac{1}{2} = 2\sin(B+60°)\sin B = \cos 60° - \cos(2B+60°) = \frac{1}{2} - \cos(2B+60°).$$

Thus $\cos(2B+60°) = 0$ and $2B+60° = 90°$ or $270°$. The second possibility must be rejected since $B < A$, and the first yields $B = 15°$.

II. *Comment by Leon Bankoff, Los Angeles, California.*

The triangle ABC is determined by the vertices 1, 2, and 9 (or their cyclic permutations) of a regular dodecagon. The angles B, C, A are in the ratio 1:4:7.

Also solved by LEON BANKOFF, *Los Angeles, California (solution as well);* ANDRÉ BOURBEAU, *École Secondaire Garneau;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence;* LEROY F. MEYERS, *The Ohio State University;* CHARLES W. TRIGG, *San Diego, California (ten (count 'em) solutions); and the proposer.*

149. [1976: 94] *Proposed by Kenneth S. Williams, Carleton University.*

Find the last two digits of 3^{1000}.

I. *Solution by G.D. Kaye, Department of National Defence.*

Since

$$3^{1000} = (1+80)^{250} = 1 + 250 \cdot 80 + \text{terms containing factor } 80^n \text{ with } n \geq 2,$$

it is clear that the last two digits of 3^{1000} are 01.

II. *Solution by F.G.B. Maskell, Algonquin College.*

Since

$$3^{1000} = (10-1)^{500}$$

$$= 10000N - \frac{500 \cdot 499 \cdot 498}{3!} \cdot 10^3 + \frac{500 \cdot 499}{2!} \cdot 10^2 - 500 \cdot 10 + 1$$

$$= 10000(N - 50 \cdot 499 \cdot 83) + 5000(2495 - 1) + 1\,,$$

it follows that the last four digits of 3^{1000} are 0001.

Also solved by LEON BANKOFF, *Los Angeles, California;* WALTER BLUGER, *Department of National Health and Welfare;* STEVEN R. CONRAD, *Benjamin N. Cardozo H.S., Bayside, N.Y.;* RADFORD DE PEIZA, *Woburn C.I., Scarborough, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* G.W. MASKELL, *Huddersfield, England;* R. ROBINSON ROWE, *Naubinway, Michigan;* CHARLES W. TRIGG, *San Diego, California (three solutions); and the proposer (two solutions). Late solution by* ANDRÉ BOURBEAU, *École Secondaire Garneau.*

Editor's comment.

Many of the solutions used congruences and were based on the fact that $3^{20} \equiv 1 \pmod{100}$. But the solutions given above are just as short, they are more elementary and, in fact, require less calculation.

Conrad noted that the problem occurs, with a solution, in *The USSR Olympiad Problem Book*, p. 128. The reference was given to him by a colleague, Irwin Kaufman of South Shore H.S., Brooklyn, N.Y.

150. [1976: 94, 95] *Proposed by Kenneth S. Williams, Carleton University.*

If $[x]$ denotes the greatest integer $\le x$, it is trivially true that

$$\left[\left(\tfrac{3}{2}\right)^k\right] > \frac{3^k - 2^k}{2^k} \quad \text{for } k \ge 1,$$

and it seems to be a hard conjecture (see [1]) that

$$\left[\left(\tfrac{3}{2}\right)^k\right] \ge \frac{3^k - 2^k + 2}{2^k - 1} \quad \text{for } k \ge 4.$$

Can one find a function $f(k)$ such that

$$\left[\left(\tfrac{3}{2}\right)^k\right] \ge f(k)$$

with $f(k)$ between $\dfrac{3^k - 2^k}{2^k}$ and $\dfrac{3^k - 2^k + 2}{2^k - 1}$?

I. *Solution by Walter Bluger, Department of National Health and Welfare.*

Yes, Virginia, there is such a function. If we set

$$R = \left(\tfrac{3}{2}\right)^k = \left(1 + \tfrac{1}{2}\right)^k = 1 + \frac{k}{2} + \ldots + \frac{1}{2^k},$$

then the last term represents the last digit in the binary representation of R, and hence

$$[R] + 1 > R + \frac{1}{2^{k+1}} . \tag{1}$$

We also have

$$\frac{3^k - 2^k}{2^k} < R - 1 + \frac{1}{2^{k+1}} < \frac{3^k - 2^k + 2}{2^k} < \frac{3^k - 2^k + 2}{2^k - 1} . \tag{2}$$

Thus if we set $f(k) = \left(\tfrac{3}{2}\right)^k - 1 + \dfrac{1}{2^{k+1}}$, then, from (1) and (2),

$$\left[\left(\tfrac{3}{2}\right)^k\right] > f(k) \quad \text{and} \quad \frac{3^k - 2^k}{2^k} < f(k) < \frac{3^k - 2^k + 2}{2^k - 1} ,$$

as required.

II. *Comment by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

The function $g(k) = \left[\left(\frac{3}{2}\right)^k\right]$ is a nice prime-generating instrument, since

$$g(2) = 2,\ g(3) = 3,\ g(4) = 5,\ g(5) = 7,\ \ g(6) = 11,\ g(7) = 17,$$

but, *oops*!, $g(8) = 25$.

REFERENCE

1. G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford University Press 1960, p. 337, condition (f).

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LETTER TO THE EDITOR

Dear editor:

I wonder if you know of these *Eurekas*:

1. *Eureka*, Comics Magazine, published in Milan, Italy, 1967 to date. Circulation 100,000. Wit and humor.

2. *Eureka*, published in Paris, France, 1971 to date. Literature.

3. *Eureka*, the California Foreign Language Bulletin. Complete file in the library of the University of the Pacific, Stockton, California.

4. *Eureka*, the journal of the Cambridge Archimedean Society, mentioned by G.H. Hardy in the preface of *A Mathematician's Apology*, 1940.

CHARLES W. TRIGG,
San Diego, California.

Editor's comment.

I an grateful to Mr. Trigg for informing us of these namesakes of our journal. I have been told by several people who should know that *Eureka* No. 4 above has been defunct for many years.

At present, our own EUREKA is sadly lacking in wit and humour. So I propose that we take over or merge with *Eureka* No. 1 above. We could then bring you, along with your regular dose of mathematical vitamins, a comic supplement every month.

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LIES, DAMNED LIES, AND ...

The words of liars
blush, but a statistician's
figures are shameless

W.H. AUDEN

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