MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AT SCHOOLS COLLEGES AND UNIVERSITIES

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Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 14 of *Mathematical Spectrum* will consist of three issues, of which this is the second. The first issue was published in September 1981 and the third will appear in May 1982.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH.

Mathematical Spectrum Awards for Volume 13

We remind our readers that each year two prizes are available to contributors who are still at school or are students in colleges or universities. A prize of £20 is for an article published in the magazine and another of £10 is for a letter or the solution of a problem.

There were no articles in Volume 13 by authors eligible for the £20 prize. The £10 prize has been awarded to Bob Bertuello for his letter *First and last digits of large numbers* (Volume 13, page 30). We look forward to further contributions from readers.

Four Paradoxes

BENGT KLEFSJÖ, *University of Luleå*

Dr Klefsjö teaches mathematics and mathematical statistics at the University of Luleå in the north of Sweden. He is very interested in educational problems and his main research interests are reliability and quality control.

In Volume 13, Number 2 of *Mathematical Spectrum* the editor asked readers to send in their favourite mathematical paradoxes. Here are some of mine (earlier published together with some more fallacies in the Swedish journal *Elementa* during 1971–1973).

1. A paradox in integration

Let F denote an indefinite integral of $f(x) = 1/x$, $x > 0$. By using integration by parts we get that

$$F(x) = \int 1 \cdot \frac{1}{x}\,dx = x \cdot \frac{1}{x} - \int x \cdot \left(-\frac{1}{x^2}\right) dx = 1 + F(x).$$

Accordingly, it follows that $0 = 1$.

2. A paradox in imaginary numbers

One often hears that the imaginary number i is the square root of -1, i.e. $i = \sqrt{-1}$. The calculation

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i.i = -1$$

shows that this must be taken with greatest caution.

3. A paradox in the binomial theorem

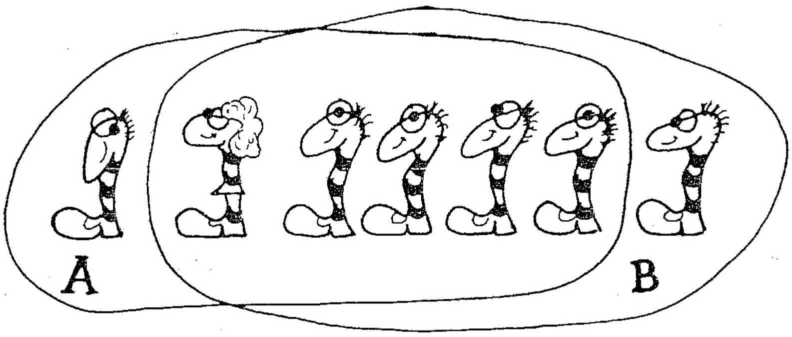
The binomial theorem says that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + b^n.$$

For $n=0$ the left-hand side is equal to $(a+b)^0 = 1$. In the right-hand side all terms containing $\binom{n}{k}$ will disappear and only $a^0 + b^0 = 2$ is left. This means that $1=2$.

4. A paradox in induction

We shall prove, by using induction, that in every set containing n persons, each member takes the same size shoe. For $n=1$ this is obvious. Now suppose that the statement is true for every set containing $n=p$ persons, and suppose that we have a set with $n=p+1$ persons. (We illustrate the situation for $p=6$.) All the people in A take the same size shoe, and the same is true for B.



Since

is a member of both the sets all the $p+1$ people take the same size shoe. The statement now follows by the principle of induction.

> A question asked in the 1981 Hungarian Mathematical Olympiad:
>
> Which numbers among the 2-digit numbers have the largest number of positive divisors?
>
> See if you can come up with the answer.

A Survey of Mathematical Puzzles I

KEITH AUSTIN, *University of Sheffield*

Keith Austin is a Lecturer in Pure Mathematics at the University of Sheffield. He has presented a monthly collection of mathematical puzzles on Sheffield's independent radio station and writes Brain-teasers for the *Sunday Times*. He is interested in the question of whether puzzles can be used to improve mental fitness in the same way that exercises are used to improve physical fitness.

Introduction

If you look in any newsagent's window nowadays, you will see a good number of magazines and paperback books devoted to puzzles for adults and children. The three articles in this series will describe some of the types of puzzles you might come across. Initially we shall consider puzzles which are not strictly mathematical, but we shall soon move on, and by the end of this first part we shall be considering chess problems, which are genuine mathematical puzzles. The second and third articles will consider other types of mathematical puzzles.

The various types of puzzle will be illustrated by examples, and the answers will be given at the end of the article.

Science fiction is currently very popular, particularly in the cinema. On the other hand there is very little mathematical fiction. For example, the only film of this type which comes to mind is 'Magic Town'. In this Capraesque story, a small town in America is found to be a perfect statistical sample for the whole country. When we contrast the almost total non-existence of mathematical fiction with the multiplicity of mathematical puzzles, we are led to the conclusion that the mathematical equivalent of science fiction is the mathematical puzzle.

1. General knowledge questions

These range from the easy:

What is the capital of France?

to the hard:

What is the capital of the Azerbaijan Soviet Socialist Republic?

I have included these questions because they represent the other end of the puzzle spectrum from the mathematical type. No thinking other than searching through the memory is involved.

Moving away from the straightforward questions of this type, we have the cryptic general knowledge question, as in BBC Radio's 'Round Britain Quiz'.

Example. When did Robin Hood have a sword fight with Sherlock Holmes?

Answer. In the film 'The Adventures of Robin Hood', Errol Flynn as Robin Hood fought Basil Rathbone, who is the definitive Sherlock Holmes of the cinema.

Incidentally, Alan Wheatley, Peter Cushing and Douglas Wilmer are other Sherlock Holmes's who have fought Robin Hood.

Although the memory still has to be searched for the relevant fact, the question also involves a puzzling element, which requires some deductive mental activity. This leads into our next type of puzzle.

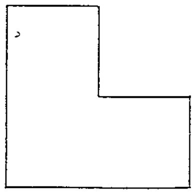
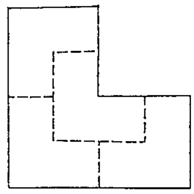
2. Riddles

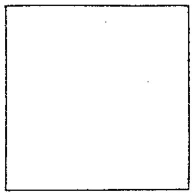
1. When is a door not a door?
2. Why did the chicken cross the road?
3. Which word is always pronounced incorrectly?
4. How many months contain 28 days?
5. I have 2 coins of total value 6 p. One is not 5 p. What are they?
6. A lily in a pond doubles its size each day. After 8 days it covers the pond. When did it cover half the pond?
7. One costs 7 p, 12 cost 14 p, 117 cost 21 p. What are they?
8. If 128 teams enter for a knockout tournament, how many matches will be played all together in the tournament?
9. In how many ways can 6 letters be put into 6 envelopes so that just one letter goes in the wrong envelope?
10. Why is a clock that never goes better than one that loses 1 minute per day?
11. You have to break a 6 × 8 bar of chocolate into 48 pieces. Break it along one of the lines. Take one of the pieces and break that along a line. And so on. What is the least number of breaks that you need to make?
12. Two cyclists, 20 miles apart, start to cycle towards each other, each at 10 m.p.h. A fly, on the nose of one, flies at 20 m.p.h. to the nose of the other, turns round and flies back to the nose of the first, and so on. How far has the fly flown when the cyclists meet?
13. What is the next term in each of the sequences:

O, T, T, F, F, S, S, __

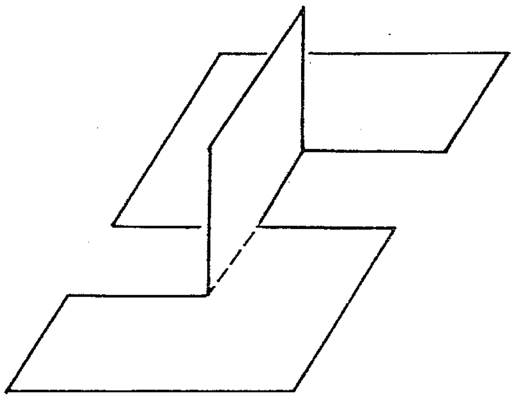
14. A hunter goes 1 mile South, 1 mile East, 1 mile North and finds himself back where he started. A bear appears. What colour is it?
15. I take a chess board and cut off two diagonally opposite corner squares. I have 31 dominoes, each the size of 2 squares of the chess board. Can I fit them on to the mutilated chess board?
16. I have a pint of water and a pint of milk. I take a spoonful of the milk and add it to the water and mix thoroughly. I take a spoonful of the mixture and add it to the milk. Is there more milk in the water or more water in the milk?
17. Rearrange the letters of NEW DOOR to make one word.
18. Three men dine in a restaurant and the bill comes to £15. They each give the waiter £5. The cashier tells him the bill was £10 and gives him £5. The waiter returns £1 to each man and keeps £2 for himself. Each man has paid £4 and

the waiter has £2 so totalling £14. But there was £15 originally. Where has the other £1 gone?

19. To divide  into 4 equal parts we divide it so: 

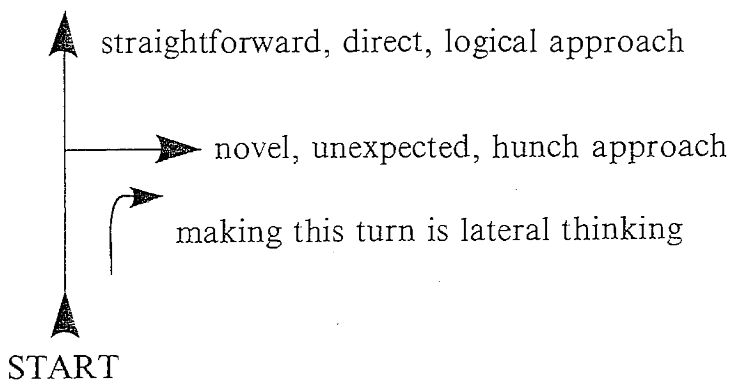
Divide  into 5 equal parts.

20. How can the following model be made from one piece of paper with cutting but no glueing?



I have used the word riddle here to cover a large range of puzzles. However, they all have one thing in common—they involve a twist. For example, in the NEW DOOR question we should read 'one word' as ONE WORD. In the cyclist question, leave the fly and first consider the cyclists and when they meet. These questions cannot generally be solved by hard work and logically plodding steadily in one direction, but need 'lateral thinking'. This means we have to turn off—change direction—look at the question in a different way.

The name lateral thinking comes from the following diagram:



The mental move away from the logical approach consists of a step in the dark—a move made without any clear reason, just a hunch that it might be worth trying something different. All that is needed is the vision to see the alternative route and the faith to be prepared to go in a direction which leads you know not where. It is always a gamble as it might be a wild goose chase; but, on the other hand, 'it might just work' as they say in the old movies.

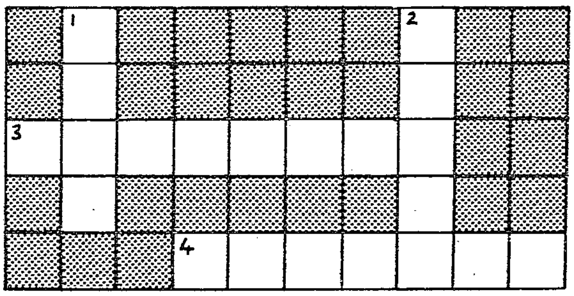
Take the chessboard problem. The direct approach is to try fitting the dominoes on to it. The lateral thinking approach is to remember that the squares of a chessboard are coloured. That seems a crazy irrelevant idea but in fact it answers the question.

3. Crossword puzzles

These fall into two classes, direct and cryptic. The former require knowledge of various types such as general facts, synonyms and quotations.

Cryptic puzzles appear to have the same format as the direct puzzles but the difference is that the clues are riddles. It is necessary first to untwist the riddle, but when that has been done it often sorts out into two separate clues to the answer. These two separate clue threads are twisted together to make the cryptic clue according to precise rules. This use of a set of rules occurs many times in mathematical puzzles. One of the cryptic rules is that the clue must contain a direct clue to the answer.

In the following puzzle all the answers are mathematical.



Across
3. Mix the whole of the large tin (8)
4. Sun just touches to brown fellow (7)

Down
1. Approach 10 differentiations (4)
2. So 55 exponentials find the answer (5)

A way to understand cryptic crosswords is to study the one in your daily paper together with its answer the following day.

If we consider the problem of filling an empty crossword puzzle with words without bothering with the clues, then we see that the only requirement is a knowledge of the list of English words. Similarly, this list is all that is needed to play Scrabble or to solve Ladder Words. The Ladder Word puzzle was invented by Lewis Carroll, and the following is a typical example:

Make a sequence of three-letter words so that the first word is CAT, the last is DOG and to move from one word to the next in the sequence we have only to change one letter.

4. IQ tests

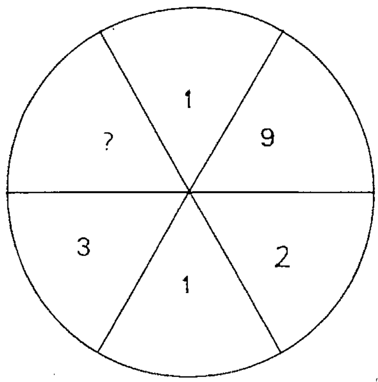
Some of the common types of questions are as follows.

(a) Find the next or missing term in a sequence.

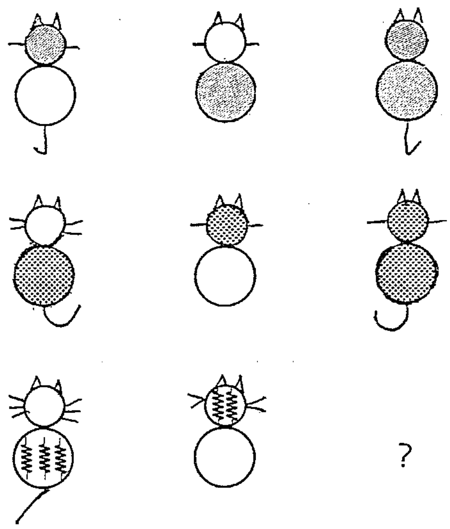
A, Y, C, W, E, U, ?

1, 5, 11, 19, ?, 41, . . .

(b) Find the missing element in a design.



(c) Problems based on finding the pattern in diagrams.



(d) Find the odd one out.

Cheshire, Bedlington, Manx, Persian, Siamese.

(e) Find the word which completes the two partial words.

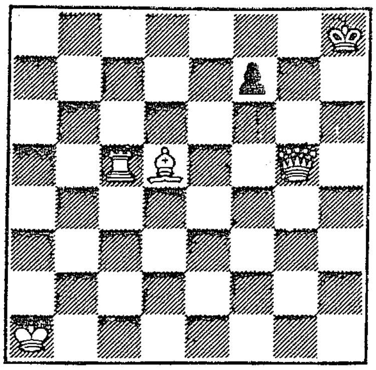
FRI(___)EAR

There is no systematic approach which guarantees the solution of these questions, rather it is a matter of letting the mind range freely over all aspects of the given information until it hits on the answer. The mental skill is in providing enough guidance to the mind so that it looks in the right place for the answer.

5. Chess problems

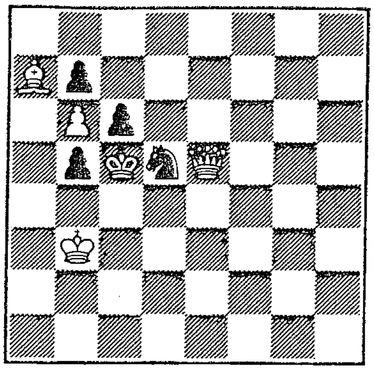
Chess problems provide us with our first collection of genuine mathematical puzzles. They require no knowledge beyond the rules of the game.

Example 1



White to play and mate in 2.

Example 2 A helpmate problem

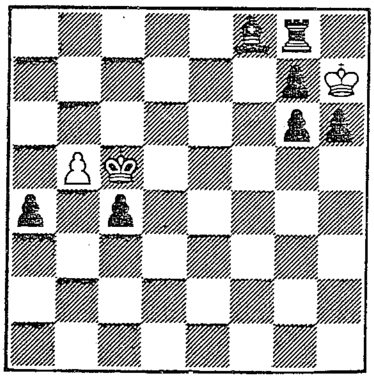


White to play and mate in 2, but Black plays to help White.

Chess problems have a long history and appear in many forms. Last year I entered for the Lloyd's Bank British Chess Problem Solving Championship and I was very impressed by the problems I was sent to solve. Although the form of these problems seems very limited, the composers are able to produce masterpieces of cunning and psychological deception. If anyone requires mental exercise then I would recommend chess problems. They can be found in newspapers and magazines, particularly of course in chess magazines. Other examples besides the above are mate in 3, mate in 4, and so on, and self-mate, where White tries to make Black mate him and Black tries not to. Also there are problems in fairy chess which involve novel pieces, novel boards and novel conditions. For example, there is the grasshopper who hops over any other piece in the same horizontal, vertical or diagonal line; or the board may be curved into the form of a cylinder or even a doughnut; or black may be subject to the maximummer condition, i.e. he always makes his geometrically longest move.

Another popular type of chess problem is that of retrograde analysis; from a given position we have to deduce something about the past history of the game.

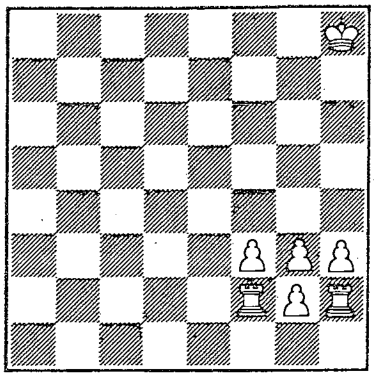
Example



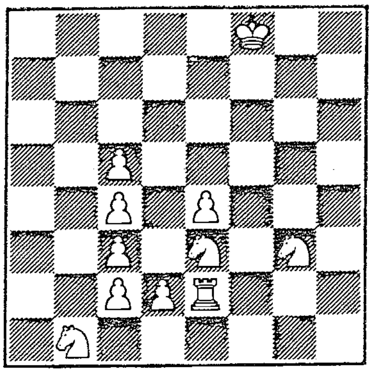
Who made the last move?

There are many mathematical puzzles which involve some aspect of chess but which probably should not be described as chess problems. For example, there is the riddle given above involving the chess board. Similarly we have the puzzle of placing eight queens on a board so that no two are *en prise*, or the knight's tour puzzle where we place a knight on a chess board and move it 63 times so that it visits every square of the board.

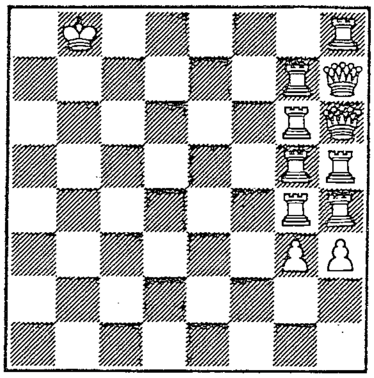
However, there are other puzzles which, although not of the standard form of a chess problem, could still be described as such. For example, there is the puzzle of finding the shortest chess game which ends in stalemate. Another example is that of showing that chess is not transitive, that is, we can have players A, B, C such that A beats B, B beats C and yet C beats A. To see this, suppose A, B, C have the following arrays of pieces, given as if they were sitting at the bottom of the board and had white pieces,



A (sitting here)



B (sitting here)

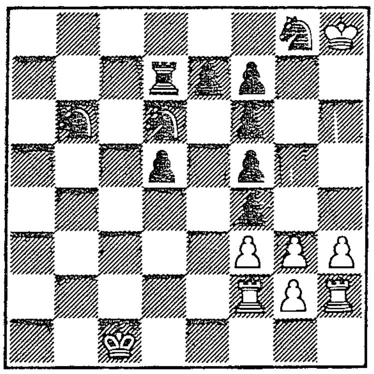


C (sitting here)

Let us consider what happens when they play each other. We shall allow either player to go first and each player uses his best strategy.

A (White) v B (Black)

B (Black)

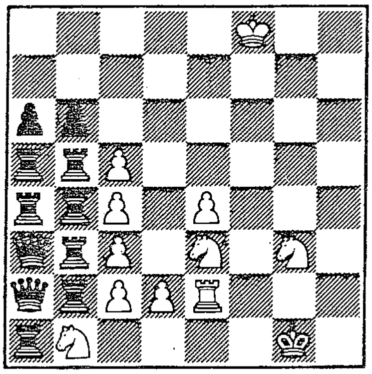


A (White)

A wins by moving R–R1.

B(White) v C (Black)

C (Black)

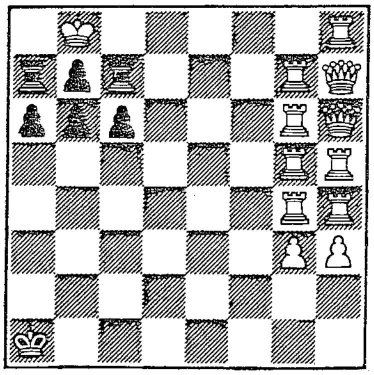


B (White)

B wins by moving R–N2.

C (White) v A (Black)

A (Black)



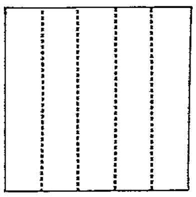
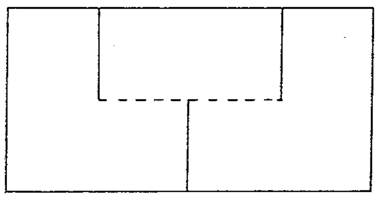
C (White)

C wins by taking the two black rooks immediately and then mating the black king.

1. General knowledge questions

Paris, Baku

2. Riddles

1. When it is ajar.
2. To get to the other side.
3. Incorrectly.
4. 12.
5. 1 p and 5 p—the other is 5 p.
6. After 7 days.
7. House numbers.
8. 127—one team goes out in each match.
9. No ways.
10. It is correct twice a day, the other much less often.
11. 47 breaks—each break increases the number of pieces by 1, cf. riddle 8.
12. 20 miles—the cyclists take 1 hour to meet.
13. E—for eight. 88—8 and its mirror image.
14. White—the hunter started at the North Pole.
15. No—each domino covers a square of each colour and there are only 30 squares of one colour.
16. They are the same—the milk in the water has excluded the same amount of water, which has gone into the milk.
17. ONE WORD
18. There is no other £1—we should not add the waiter's £2 to the £12 paid by the men but subtract it to give £10, which is the amount of the bill.
19. 
20. Cut the paper so 

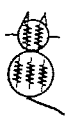
    Fold the paper along the dotted line. The *twist* is that one L-shaped piece is turned right over.

3. Crossword puzzles

The Cryptic Crossword. Across. 3. INTEGRAL ('mix' means anagram), 4. TAN-GENT. *Down.* 1. TEN-D, 2. SO-LV-E (LV = 55).

The ladder word. CAT, COT, DOT, DOG (there are other answers.)

4. IQ tests

(a) G, 29 (b) 4 (c)  (d) Bedlington (e) END.

5. Chess problems

Example 1.

White		Black		White	
R–B6	if	P–B3	then	Q–N8	(mate)
	if	P–B4 or K–R2	then	R–KR6	(mate)

Example 2.

White	Black	White
Q–K4	N × P	Q–K5

Note the different forms of the answers. In Example 2, as the players are cooperating, we only have a single sequence of moves. However, in Example 1, where the players are opposing each other, in order to show that White can be sure to win, we have to consider all the possibilities for Black and give the appropriate play for White in each case.

This latter situation occurs in many puzzles based on games. If the answer involves more moves by Black, then every possibility must be considered at each move and the answer can be very complicated.

Example of retrograde analysis. Black made the last move. We show this by showing White could not have made the last move. If White has just moved his pawn then before his move Black was in check, which is not possible. If White has just moved his king then it was in check from the rook. Yet that rook has not just reached that square, so we have an impossible situation. Thus Black made the last move.

Fifth International Congress on Mathematics Education

The International Commission on Mathematical Instruction (ICMI) has accepted an invitation from the Australian Academy of Science to hold the Fifth International Congress on Mathematical Education at the University of Adelaide in August 1984.

A principal objective of the congress will be to facilitate both professional and personal contact amongst its participants. In particular, the organisers seek to encourage existing working groups in mathematics education to meet at the congress and to encourage overseas participants to visit Australian colleagues in their home educational institutions. It is expected that the program will span all levels of education and discuss problems of general interest while recognising different cultural perspectives. The official language of the congress is English.

The ICME 5 Organising Committee in Australia expects to issue a first announcement by May 1982, and a second one by May 1983. The committee requests comment from prospective participants which might assist it in planning congress activities; responses received before July 1982 will be especially helpful. Please write to ICME 5, Wattle Park Teachers' Centre, 424 Kensington Road, Wattle Park, SA 5066, Australia.

Where to Park the Car

M. G. J. VAN DER BURG, *University of York*

The author graduated from King's College London where he also obtained his Ph.D. in the area of general relativity. Since he moved to York in 1964 his interests have moved towards numerical analysis. He has given talks on various mathematical topics in schools and is involved in lecturing on computing and numerical methods to teachers.

On busy Saturday mornings many of us get into our cars and drive down the High Street, which has kerbside parking all the way along, in order to shop at the local supermarket. We always hope that the parking space right outside the entrance to the supermarket will be vacant, and decline to park in any free space we pass before reaching the shop. In practice this means that we frequently have to park well beyond the shop and have a long walk back. We all have a natural resistance to parking a long way before reaching the shop, and so the question arises: from what point before reaching our destination should we park in the first free space that we pass in order to minimise our expected walking distance?

As a first attempt at this problem we assume that the High Street is infinitely long with parking spaces as shown in Figure 1, and that we pass by any free space more than n parking spaces before the shop. We also assume that the probability that any parking space is free when we reach it is p, and hence that the probability that it is occupied is $q = 1 - p$, the same for all the parking spaces.

The probability that we actually park at the first possible stop, n spaces before the shop, is therefore p. The probability that we park $n - 1$ spaces before the shop is equal to the probability that the first space we consider is occupied and that the next space is free, that is pq. The probability that we park $k < n$ spaces before the shop is the probability that the first $n - k$ spaces we consider are occupied and the next one is free, that is pq^{n-k}. Similarly, the probability that we park k spaces after passing the shop is pq^{n+k}. What we have, in effect, is a geometric distribution for the parking process.

The expected walking distance to the shop, in units of the length of a parking space, is then

$$E(n) = n \,.\, p + (n-1) \,.\, pq + (n-2) \,.\, pq^2 + \cdots + 2 \,.\, pq^{n-2} + 1 \,.\, pq^{n-1} + 0 \,.\, pq^n$$
$$+ 1 \,.\, pq^{n+1} + 2 \,.\, pq^{n+2} + 3 \,.\, pq^{n+3} + \cdots.$$

The series can easily be summed by splitting it into two parts and expressing it in the form

$$E(n) = pq^{n+1}\left\{-\frac{d}{dq}(q^{-1} + q^{-2} + \cdots + q^{-n}) + \frac{d}{dq}(q + q^2 + q^3 + \cdots)\right\}.$$

Summing the two geometric series, one finite and the other infinite, we obtain

$$E(n) = n - p^{-1}q(1 - 2q^n).$$

We then seek the value of n which minimises $E(n)$. Firstly

$$E(0) = p^{-1}q, \qquad E(1) = p^{-1}(1 - 2q + 2q^2),$$

so that $E(1) > E(0)$ if $q < \frac{1}{2}$, $p > \frac{1}{2}$. Hence the driver who never considers parking before he reaches the shop has the best strategy if $p > \frac{1}{2}$, which means that on average more than half the parking spaces are vacant. In general, $E(n) < E(n-1)$ if

$$n - p^{-1}q(1 - 2q^n) < n - 1 - p^{-1}q(1 - 2q^{n-1}),$$

which gives $q^n > \frac{1}{2}$. Hence the walking distance is a minimum when $n = N$, where

$$q^N > \tfrac{1}{2} > q^{N+1},$$

or, alternatively, N is the integer part of $-\ln(2)/\ln(q)$. Some sample values of N as a function of p are

p	0·25	0·1	0·05	0·01	0·001	0·0001
N	2	6	13	68	138	6931

It is interesting to note that $N < E(N) < N + 1$, so that with the best strategy we expect to walk between N and $N + 1$ parking-space lengths to the shop. This is because, on occasion, we shall end up parking a very long way beyond the shop.

Of course this analysis can only be of use if we know the value of p, presumably from our experience of previous visits to the High Street. Also, in a real situation, parking spaces are easier to find on the fringes of a shopping centre than in the middle of the High Street. Nevertheless this model gives an interesting application of simple ideas in probability theory and indicates how one can proceed to tackle a more realistic situation. In more complicated cases the mathematical models become easier to solve if we change from a discrete to a continuous distribution of parking spaces, so that instead of having to sum series we evaluate integrals.

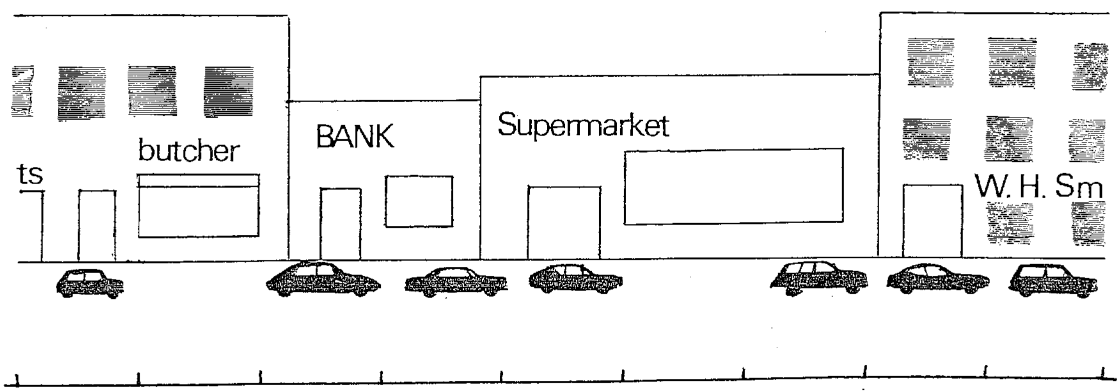


Figure 1. High Street with kerbside parking spaces and the supermarket.

Lie Transformation Groups

W. D. HALFORD, *Massey University*

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1. Introduction

This is the second of two articles which attempt to give a brief elementary introduction to the study of Lie groups; the first was published in Volume 14 Number 1 of *Mathematical Spectrum*. Lie group theory has helped greatly in understanding and advancing many areas of pure and applied mathematics and science. It may not be too much to claim that the development of Lie groups and Lie algebras has revolutionized the treatment of differential geometry and parts of theoretical physics.

Here we shall approach the concept of a Lie group via the transformation groups which we looked at in the first article and then we shall take a quick look at Lie groups in the context of differential equations. But first we reflect on the origin of such groups and their creator.

2. Biographical details

The Norwegian mathematician Marius Sophus Lie was born in Nordfjordeid in 1842, the sixth and youngest child of a Lutheran pastor. At school the young Sophus apparently showed no special interest in mathematics, but took both mathematics and philology at the University of Christiania (now Oslo) in preparation for secondary school teaching.

At a meeting of the Berlin Mathematical Society in 1869 he met the German mathematician Felix Klein (Klein is best known for his *Erlangen Programm*, the main idea of which was that each of the various geometries known at the time (such as metric geometry, projective geometry, line geometry, affine geometry, inversive geometry) could be characterized by an appropriate group of transformations. Lie and Klein went together to Paris in 1870 and struck up a very close friendship which, although deeply tested on one or two occasions, lasted throughout their lives. These two men were greatly inspired by the work of some young French mathematicians, notably Camille Jordan, Gaston Darboux and Evariste Galois. It was the work of Galois, however, on the connection between finite groups and algebraic equations which influenced Lie most in his early researches. (It was Galois who invented the name 'group' for the algebraic structure bearing this name.)

When the Franco-Prussian war broke out in 1870 Klein returned to Germany, and Lie, who was very keen on hiking and mountaineering, decided to hike to Italy.

While resting one day and making some mathematical notes in his notebook, he was arrested by a French soldier who thought Lie was a German spy and that his notes were French military plans written in secret code. During imprisonment for a month at Fontainebleau Lie discovered 'contact transformations', which gave great insight via group theory into the Hamiltonian theory of dynamics (an enormously rich theory invented by the Irish mathematician William Rowan Hamilton at about the time of Lie's birth). Lie used his contact transformations to show in his first paper how a 1–1 correspondence could be established between intersecting lines (in line geometry) and tangential spheres (in sphere geometry).

In 1871 Lie became an assistant tutor at the University of Christiania and in the following year obtained his doctor's degree and was appointed as an 'extraordinary' professor of mathematics. In 1873 he began his work on continuous transformation groups, a work devoted primarily to the task of using group-theoretical techniques to advance the study of differential equations. In fact, Lie stated that 'differential equations constitute the most important branch of modern mathematics'. After nine years of work in collaboration with Ernst Engel of Germany, Lie published the massive three-volume work *Theorie der Transformationsgruppen* (1893). It was preceded by *Differentialgleichungen* (Differential Equations) in 1891 and followed by *Vorlesungen über Kontinuierliche Gruppen* (Lectures on Continuous Groups) in 1896, and *Geometrie der Berührungstransformationen* (Geometry of Contact Transformations) in 1896.

During this intensely active period of writing Lie was appointed (in 1886) to the chair of mathematics at the University of Leipzig, where he succeeded his old friend Klein. However, he became homesick and so melancholic that in 1898 he returned to Christiania to a special post created for him. He lived only six months after this, dying in 1899 at the age of 56. Although honoured by his *alma mater*, Lie was disappointed by the lack of any general recognition of the value of his work and became somewhat embittered by this in his later years.

It is said that Lie 'conformed to the conventional idea of the genius. He was shy and withdrawn, but nevertheless outspoken on occasion and given to temper tantrums . . . was careless in dress and sometimes appeared in class without collar or necktie, or even with his hat on . . . Lie's students respected him, but gossiped about his eccentricities. Nevertheless the doctoral candidates at Leipzig flocked to Lie, who assigned easy topics for research and provided a tremendous amount of assistance. But pupils found him personally crude and were wont to contrast him with the suave Klein. They pronounced the latter a "cultured gentleman" but said "the best you can say for Lie is that he is a good fellow". The consensus today is that he was a very good fellow indeed.'†

Lie's books and papers have greatly influenced the course of modern mathematics, especially through the subsequent work of the French mathematician Élie Cartan and the German mathematical physicist Hermann Weyl. The theory of

† E. E. Kramer, *The Nature and Growth of Modern Mathematics* (Hawthorn Books, New York, 1970).

Lie groups and Lie algebras which has been developed has profoundly influenced the theory of differential equations, algebra, geometry, analysis, topology and theoretical physics. The early accounts of Lie theory were 'local' in the sense that the arguments used depended basically on existence and uniqueness theorems for differential equations which hold in some neighbourhood of a point. The most modern treatments of Lie group theory usually start from the notion of an abstract topological group and expound the theory in the advanced terminology of fibre bundles, thereby creating a 'global' theory, with local Lie groups as a special case. The introduction which we give below is local, and is based on the discussion of transformation groups in the first article of this series.

3. Lie transformation groups

Each of the six continuous transformation groups I–VI described in Section 2 of the first article has a property which characterizes these groups as *Lie groups*. That property is: the parameters of the product (resultant) transformation are 'analytic' functions of the parameters of the component transformations. Let $\gamma_1, \ldots, \gamma_r$ be the parameters of the product transformation. Let $\alpha_1, \ldots, \alpha_r$ $\beta_1, \ldots, \beta_r$ be the parameters of the two component transformations. Then because of the closure property of the continuous groups, each γ will be a continuous function of the α's and β's. Let us write this symbolically as

$$\gamma_i = \varphi_i(\alpha;\beta) \qquad i = 1, \ldots, r.$$

To require that the φ's be analytic means that each function φ_i can be differentiated any number of times with respect to the α's (the β's being kept fixed in the process) or the β's (the α's being fixed), and that the derivatives themselves are continuous functions.

Thus, for the linear group in two dimensions (Section 2.4 of the earlier article) the parameters of the component transformations are a_1, b_1, c_1, d_1 (these are the α's) and a_2, b_2, c_2, d_2 (the β's). Equations (9) give the product parameters a', b', c', d' (the γ's) as continuous functions of the component parameters; for example

$$a' = a_1a_2 + b_2c_1 \equiv \varphi_1(a_1, b_1, c_1, d_1;\, a_2, b_2, c_2, d_2)$$

is the first of equations (9) which expresses $a'(=\gamma_1)$ as a function φ_1 of the α's and β's. We can differentiate φ_1 (partially) as many times as we like with respect to a_1 (treating all other variables a_2, b_2, c_1 as fixed in the expression for a'):

$$\partial\varphi_1/\partial a_1 = a_2, \qquad \partial^2\varphi_1/\partial a_1^2 = 0, \qquad \partial^3\varphi_1/\partial a_1^3 = 0, \ldots,$$

where the ∂ notation indicates partial differentiation. Each derivative is a continuous function. Therefore φ_1 is an analytic function. And so on.

One can describe most of the global properties of a Lie group of transformations by knowing its local properties, i.e. its properties only in the vicinity of the identity element. Thus it is usually sufficient to look at the *infinitesimal transformations* of the group in that piece of the group manifold which is continuously connected to the identity (see Section 3 of the first article). Loosely speaking, a finite transformation (say, a rotation about the origin through an angle t) can be regarded as a sequence of infinitesimal transformations (say, a sequence of infinitesimal rotations through angles Δt, the product (resultant) of which is the rotation through the angle t). It is the continuous character of the group which allows us to take this view, and the nature of the local Lie group is represented essentially by the infinitesimal transformation. Using the calculus we can obtain a (partial) differential operator which 'generates' the infinitesimal transformation. The group is discussed then in a small neighbourhood of the identity transformation, but any time we wish to obtain the finite transformation equations (which we wrote down in the examples in Section 2 of the first article) we can do so by employing a certain technique known as *exponentiation*. It goes beyond the scope and intent of this paper to develop the operator calculus involved in describing the infinitesimal transformations, but it is not beyond the level of second-year university mathematics to make some inroads into this subject.

4. Lie groups and differential equations

In order to proceed more than just a few steps into this area of work, the original breeding ground of Lie group theory, we need to have on hand the full apparatus of partial differential calculus. However, we can get a feeling for what is happening and a glimpse at Sophus Lie's main contribution to mathematics with only a rudimentary knowledge of calculus and differential equations.

The study of differential equations addresses itself to the problem: given a relation between certain derivatives of a function, find the function. The derivatives may be ordinary (the relation then being called an ordinary differential equation), or they may be partial (when the relation is a partial differential equation). For example, find the function $y(x)$ given that

$$\frac{dy}{dx}=\frac{y^2}{x^2}. \tag{1}$$

This first-order ordinary differential equation has the solution

$$y=\frac{ax}{a+x}, \tag{2}$$

where c is a constant, as we may verify by substitution into the given equation (1). This solution can be expressed parametrically by

$$x=a(1-v)/v, \qquad y=a(1-v), \qquad v\neq 0, \tag{2a}$$

since elimination of the parameter v yields (2). A second example is the first-order ordinary differential equation

$$\frac{dy}{dx}=\frac{x^2-xy+y^2}{xy} \tag{3}$$

where the general solution $y(x)$ is given by

$$(y-x)\,e^{y/x}=c, \tag{4}$$

where c is a constant. In terms of the parameter v, which can take all real values except $v = 1$, this solution may be expressed as

$$x = c\,e^{-v}/(v-1), \qquad y = cv\,e^{-v}/(v-1). \tag{4a}$$

As a third example consider the ordinary differential equation

$$(x^2 + y^2)\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left(y - x\frac{dy}{dx}\right)^2. \tag{5}$$

This can be solved to yield the solution

$$x^2 + y^2 = c^2 \tag{6}$$

where c is a constant. This solution can be expressed parametrically by

$$x = c\cos u, \qquad y = c\sin u, \qquad 0 \le u \le 2\pi. \tag{6a}$$

The usual approach to a first course of study of differential equations is to begin with a list of special types of first-order equations and work through the particular methods which yield the general solutions in exact (closed) form. For example, to obtain the solution of the differential equation (3) above, we may make the substitution $y = rx$, where r is a function of x. Then (3) becomes

$$\frac{dr}{dx} = \frac{1-r}{rx}. \tag{7}$$

This is a differential equation in which *the variables r and x are separable*, and has the solution $x(r-1)\,e^r = c$, where c is a constant. Now, using $r = y/x$, we obtain the solution (4). This method works for all first-order differential equations of the 'homogeneous' form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \tag{8}$$

where f is an arbitrary function. Equation (1) is another example of this type, but the substitution $y = rx$ is unnecessary in this case because the variables x and y are immediately separable. The collection of methods for solving all types of first-order differential equations appears to be a bag of tricks, the methods seeming to have little in common. There is, of course, nothing wrong with having a compendium of techniques for obtaining solutions to differential equations with recognizable forms, because obtaining solutions is the name of the game for practitioners. However, the mathematician desires more than this—he prefers a theoretical structure which brings together these seemingly disparate *ad hoc* methods. Similar remarks apply to differential equations of all types and orders.

Lie discovered in about 1870 that the various types of solvable first-order differential equations are characterized by the groups of transformations which leave their solutions invariant (i.e. their solutions retain the same form under the group action). For example, the solution (2) of the differential equation (1) is invariant under the group of scale transformations or dilatations

$$x_1 = \lambda x, \qquad y_1 = \lambda y, \qquad \lambda \text{ constant.}$$

This is easily seen by replacing x and y in (2a) by λx and λy respectively, for then $y_1/x_1 = y/x = v$. Similarly the solution (4) of equation (3) is invariant under the group of dilatations. Indeed, solutions to all first-order 'homogeneous' differential equations of the form (8) are invariant under this group. Again, the solution (6) of the differential equation (5) is invariant under the rotation group in two dimensions since the transformations

$$x_1 = x\cos t - y\sin t, \qquad y_1 = x\sin t + y\cos t \tag{9}$$

yield directly $x_1^2 + y_1^2 = x^2 + y^2 = c^2$. The differential equation (5) is of the form

$$y - x\frac{dy}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} . f(x^2 + y^2),$$

where f is an arbitrary function. It is a fact that solutions to all equations of this type are invariant under the group of rotations.

A significant aspect of Lie's work is this: he developed a method for telling whether the solution of a given differential equation is invariant under a given transformation group or not *without a knowledge of the solution.* We shall not describe the method here because it involves the use of the differential operator which generates the infinitesimal transformations of the given group.

One of Lie's most important achievements in the study of first-order differential equations by group methods is the following result. Given a differential equation, suppose we find that the *differential equation itself* is invariant under a particular transformation group; then the solution of the equation can be found immediately in one of the following two ways (we shall not prove these results):

(i) By introducing so-called *canonical coordinates* r and s in terms of which the 1-parameter transformation equations in the coordinates x and y become the translation equations

$$r_1 = r, \qquad s_1 = s + \tau,$$

where τ is the parameter. Then the differential equation is reduced to the form

$$\frac{ds}{dr} = F(r)$$

i.e. the variables are separated and we can now solve by quadratures. ('Solution by quadratures' means finding $\int F(r)dr$ either exactly or by numerical methods.)

Example. The differential equation (3) is invariant under the group of dilatations

$$x_1 = \lambda x, \qquad y_1 = \lambda y, \qquad \lambda \text{ constant} \tag{10}$$

since

$$\frac{dy_1}{dx_1}=\frac{\lambda dy}{\lambda dx}=\frac{x^2-xy+y^2}{xy}=\frac{(x_1^2-x_1y_1+y_1^2)/\lambda^2}{x_1y_1/\lambda^2}=\frac{x_1^2-x_1y_1+y_1^2}{x_1y_1}.$$

It can be shown that the canonical coordinates for this group are

$$r=y/x, \qquad s=\log x. \tag{11}$$

(You will recognize the first of these from the work which we did to get equation (7).) So we get, using (10),

$$r_1=y_1/x_1=\lambda y/\lambda x=y/x=r$$

and

$$s_1=\log x_1=\log(\lambda x)=\log x+\log\lambda=s+\tau,$$

where $\tau=\log\lambda$. Using the first of the relations (11), the differential equation becomes (7). From the second of the relations (11) we obtain $ds=dx/x$, and so finally the differential equation (3) may be written, in canonical coordinates,

$$\frac{ds}{dr}=\frac{r}{1-r}.$$

This can be solved immediately:

$$s=\int\frac{rdr}{1-r}=-r-\log|1-r|.$$

In terms of the x, y-coordinates, this is just (4).

(ii) The second method is to write down an *integrating factor* for the given differential equation in terms of the tangent vectors to the orbits. An *orbit* (of a transformation group) is the path, or locus, of a point (x_0, y_0) under the action of the group. For example, the orbits of the rotation group in two dimensions are concentric circles in the xy-plane, since the equations of the group transform the point (x_0, y_0) to the point (x, y) where $x^2+y^2=x_0^2+y_0^2=$ constant. We shall not describe the integrating factor method here; the interested reader may consult the literature (e.g. references (b), (d) or (h) listed in the next section).

Besides the above techniques, Lie showed how to find the whole family of first-order differential equations which are invariant under a given group. Again, we shall not elaborate here. Putting everything together, we can now construct a table in which the entries take the form:

Transformation group G	General form of differential equation invariant under G	Canonical coordinates

The task of solving a particular first-order differential equation is now a simple one. We use the table to find the appropriate canonical coordinates which enable us to express the equation in 'variables separable' form, and solve by quadratures.

In the case of an ordinary differential equation of order higher than the first (i.e. one involving second and higher derivatives of the function which we wish to find), if it is invariant under a one-parameter group the methods of Lie can be used to reduce the order of the equation. Partial differential equations involve functions of two or more independent variables and their derivatives; for such equations the Lie theory may be used to reduce the number of independent variables, and in some cases its use may achieve a reduction from a partial to an ordinary differential equation.

5. Further reading

We have merely opened the door on a theory which is, pure mathematically, extremely rich and influential and which has, from the point of view of applications, yet to reach its full vast potential. We have not, for example, concerned ourselves with the representation of groups by linear transformations, an application of great importance in physics. We have not said anything about Lie algebras and their connection with Lie groups; a wealth of material lies in store here. Below we list a few books for future reading. Some will be accessible from the sort of background which we have established in these two articles together with some pure mathematics courses at university. Others are mentioned because of their high reputations as textbooks at advanced undergraduate or at graduate level. It should be noted that there is a vast amount of literature on Lie groups and Lie algebras and their applications. The short list which follows is a selection from many good books available.

(a) G. G. Hall, *Applied Group Theory* (Longmans, London, 1967).
An introduction to those aspects of group theory are of great importance in physical applications. A brief but very readable treatment once some knowledge of vector spaces has been acquired. Includes a chapter on continuous groups.

(b) A. Cohen, *An Introduction to the Lie Theory of One-Parameter Groups* (Stechert, New York, 1931).
Very much in the spirit of Lie's original work. A good straightforward introduction to the Lie group treatment of differential equations.

(c) J. G. F. Belinfante and B. Kolman, *A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods* (Society for Industrial and Applied Mathematics, Philadelphia, 1972).
An excellent survey which introduces the modern concepts and methods of Lie theory used in current applications. Mathematical prerequisites are kept to a minimum, but the reader would require a reasonable knowledge of linear algebra. Includes useful material on computational methods using a digital computer. Contains a very good bibliography.

(d) G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations* (Springer-Verlag, New York, 1974).

This book carries the flavour of Lie's own exposition. Besides developing the general theory of similarity methods for differential equations it devotes a considerable amount of space to the solutions of the heat equation, the axially-symmetric wave equation, and other differential equations arising in areas such as stochastic processes.

(e) R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).
Aimed at graduate-level physics students. An easy-to-read book which illustrates the theory generously with worked examples and diagrams. Although it does not attempt a complete coverage of the theory and its applications, it nevertheless contains a wealth of information. Contains an excellent bibliography.

(f) M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, Ma, 1962).
A book which has done sterling service to the physics community. Designed for the graduate student who knows quite a lot about quantum mechanics but who is unfamiliar with group theory. Contains some useful chapters on continuous groups and group representations.

(g) A. A. Sagle and R. E. Walde, *Introduction to Lie Groups and Lie Algebras* (Academic Press, New York, 1973).
A modern introduction written for the beginning graduate student in mathematics who has a good knowledge of modern algebra, topology and analysis. An excellent book.

(h) M. Ackerman and R. Hermann, *Sophus Lie's* 1880 *Transformation Group Paper* (Math Sci Press, Brookline, Ma, 1975).
A translation of Lie's original, with a historical and mathematical commentary. There are several chapters which provide an introduction to modern notation and ideas as a setting against which Lie's paper can be studied. Further chapters provide details of recent applications and extensions of the theory. Addressed to mathematical graduates. However, the translation of Lie's paper, which takes up the greater part of the book, is more widely readable.

The five books listed next are acknowledged to be 'classics' of their kind, intended for a graduate readership.

(i) H. Weyl, *Theory of Groups and Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1931).

(j) H. Weyl, *The Classical Groups* (Princeton University Press, Princeton, NJ, 1946).

(k) C. Chevalley, *The Theory of Lie Groups* (Princeton University Press, Princeton, NJ, 1946).

(l) N. Jacobson, *Lie Algebras* (Wiley Interscience, New York, 1962).

(m) S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press, New York, 1962).

Letters to the Editor

Dear Editor,

Roots of polynomials

I am about to start my second year in the sixth form. Whilst investigating the relationship between the roots and coefficients of a polynomial, I came across an interesting phenomenon.

Consider a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (n \geq 1, a_n \neq 0)$$

with real coefficients. If we differentiate $f(x)$ $n-1$ times, we get

$$\frac{d^{n-1}f(x)}{dx^{n-1}} = a_n n!x + a_{n-1}(n-1)!,$$

and this polynomial has the one root $x = -(a_{n-1}/na_n)$. Now $f(x)$ has n complex roots. If we denote these by $\alpha_1, \ldots, \alpha_n$, we have

$$f(x) = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

If we look at the coefficient of x^{n-1} in $f(x)$, we have

$$a_{n-1} = -a_n(\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

Thus *the root of the* $(n-1)$*th derivative of* $f(x)$ *is*

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n},$$

which is just the arithmetic mean of the roots of $f(x)$.

For example, consider a quadratic polynomial

$$y = x^2 + ax + b$$

with real coefficients but no real roots. It will have a conjugate pair of complex roots $\alpha \pm i\beta$ (α, β real), so the root of the derivative will be $x = \alpha$. Now

$$\begin{aligned} y &= (x-\alpha-i\beta)(x-\alpha+i\beta) \\ &= (x-\alpha)^2 + \beta^2, \end{aligned}$$

so, when $x = \alpha, y = \beta^2$ and the vertex of the parabola $y = x^2 + ax + b$ is at the point (α, β^2).

I should like to know if any other sixth-form students could find a use for this result.

Yours sincerely,

NIGEL MCCANN

(Immingham School, South Humberside)

Dear Editor,

When squares are triangles

In Volume 13, Number 3, Doug Averis and David Sharpe showed how to find all numbers which are both square and triangular. Their result can be carried a stage further to give a difference equation which will give these numbers. Explicitly, in the notation of the article,

$$l_{n+1} + 2\sqrt{2}k_{n+1} = (3 + 2\sqrt{2})(l_n + 2\sqrt{2}k_n),$$

which gives

$$l_{n+1} = 3l_n + 8k_n,$$
$$k_{n+1} = l_n + 3k_n.$$

The l's can be eliminated to give

$$k_{n+2} = 6k_{n+1} - k_n.$$

(The l's also satisfy the same difference equation.) With $k_1 = 0$, $k_2 = 1$, this formula will give the square triangular numbers, and is to be found in *Recreations in the Theory of Numbers* by A. H. Beiler (published by Dover), together with much else on Pell's equation.

Yours sincerely,
P. J. O'Grady
(Warwick School, Warwick CV34 6PP)

Dear Editor,

Partitions of sets of integers

While looking through some past issues of *Mathematical Spectrum*, I noticed M. D. Sandford's letter in Volume 9, Number 1. He asks, given a positive integer n, what is the largest integer m such that the set $\{1, 2, \ldots, m\}$ can be partitioned into n subsets in such a way that no subset contains 3 integers in arithmetic progression? In fact this is a well-known combinatorial mathematics problem. A theorem of Van der Waerden states that given any positive integer n and positive integers $t_1, t_2, \ldots, t_n$ then there exists an integer m with the property that in any partition of the set $\{1, 2, \ldots, m\}$ into subsets S_i, $i = 1, 2, \ldots, n$, at least one subset, say S_j, contains an arithmetic progression of $t_j + 1$ terms. The smallest such number m, which will be denoted by $W(n; t_1, t_2, \ldots, t_n)$ is called the Van der Waerden number. These numbers have been the subject of much research and are known only in a small number of cases. Using this notation, Sandford's letter gives the values of $W(2; 2, 2)$ and $W(3; 2, 2, 2)$ and speculates on the value of $W(n; 2, 2, \ldots, 2)$.

A recent paper on this topic, by M. D. Beeler and P. E. O'Neil, 'Some new Van der Waerden numbers' appears in *Discrete Mathematics*, Volume 28, Number 2 (November 1979), pp. 135–146. Interested readers are referred to that paper as it is also an excellent survey of known results. All the following information comes from the paper. For $n = 2$ the only known values of $W(2; t_1, t_2)$ are:

$t_1 \backslash t_2$	2	3	4	5	6	7	8	9
2	9	18	22	32	46	58	77	97
3		35	55	73				
4			178					

In addition, $W(3;2,2,2)=27$, $W(3;2,2,3)=51$ and $W(4;2,2,2,2)=76$. This last fact, which is due to Beeler and O'Neil, answers the specific question posed by Sandford and disproves his conjecture that $W(n;\ 2,\ 2,\ldots,\ 2)=3^n$.

It may be of interest to exhibit one of the partitions of the set $\{1,2,\ldots,75\}$ into four subsets in which no 3 integers are in arithmetic progression, which was found by Beeler and O'Neil.

$\{1,2,8,10,11,13,17,27,34,35,38,39,45,47,48,50,54,64,71,72\}$

$\{3,15,16,19,21,25,30,32,33,40,52,53,56,58,62,67,69,70,75\}$

$\{4,5,12,22,26,28,29,31,37,41,42,49,59,63,65,66,68,74\}$

$\{6,7,9,14,18,20,23,24,36,43,44,46,51,55,57,60,61,73\}$

The solution has some interesting properties. It is immaterial which subsets contain the integers 1, 38 and 75 subject only to the proviso that they obviously may not all be contained within the same subset. Hence 59 further solutions may be constructed from the example above. In addition note that the pair of integers i and $i+37$, $i=2,3,\ldots,36$ are always contained in the same subset.

Finally, a comment about how the results were obtained. They are computational and not analytical, thus answering another query in the original letter. For $W(4;2,2,2,2)$ the computer time is given as a staggering 1200 hours! One can only speculate on the values of $W(4;2,2,2,3)$ or $W(5;2,2,2,2,2)$ and the length of time required to compute them.

Yours sincerely,
TERRY S. GRIGGS
(Preston Polytechnic)

Dear Editor,

Stamps and coins: two partition problems

Since the publication of the above article (Volume 13, Number 2) a number of people have asked us for the number of ways to change £1 when the $\frac{1}{2}$p coin is no longer legal tender. Your readers may wish to know that this number is 2498, which is very much smaller than 64703, the corresponding number when $\frac{1}{2}$p is allowed. Nevertheless the new number is much bigger than the 292 ways of changing \$1 into coins of 1, 5, 10, 25 and 50 cents. We do not wish to enter into controversy about which is the better system of partitioning 100 units into coins! Incidentally our number 64703 has been confirmed by Dr David Singmaster who programmed a computer to evaluate the coefficient of t^{200} in the series expansion for

$$(1-t)^{-1}(1-t^2)^{-1}(1-t^4)^{-1}(1-t^{10})^{-1}(1-t^{20})^{-1}(1-t^{100})^{-1}.$$

He also told us that the number increases to 570583 if we include the obsolescent $2\frac{1}{2}$p coin.

Finally we give a sketch of how we arrive at 2498. The number concerned is the number of solutions to the equation $x_1 + 2x_2 + 5x_3 + 10x_4 + 50x_5 = 100$ in non-negative integers. We denote by $f(n)$ the number of solutions to the reduced equation $x_1 + 2x_2 + 5x_3 + 10x_4 = n$ so that our required answer is now $f(100) + f(50) + f(0)$. From our article we see that, if n is a multiple of 10, then

$$f(n) = \frac{A_4(n+1)(n+2)(n+3)}{6} + \frac{A_3(n+1)(n+2)}{2} + C(n+1) + D, \tag{1}$$

where A_4, A_3, C, D are constants (if n is not a multiple of 10 the values for C, D are different). We have, as in our article,

$$A_4 = \lim_{t\to 1} \frac{(1-t)^4}{(1-t)(1-t^2)(1-t^5)(1-t^{10})} = \frac{1}{100},$$

$$A_3 = -\lim_{t\to 1} \frac{d}{dt} \frac{(1-t)^4}{(1-t)(1-t^2)(1-t^5)(1-t^{10})} = \frac{7}{100}.$$

Now trivially $f(0) = 1$ and it is easy to verify by direct enumeration that $f(10) = 11$ so that we can now calculate C, D from (1) giving $C = \frac{13}{50}$, $D = \frac{33}{50}$. From (1) we now have $f(100) = 2156$, $f(50) = 341$, which gives our number 2498.

Yours sincerely

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Dear Editor,

A partition problem for coins

Some special properties of the British coinage offer an alternative approach to the partition problem described by C. M. and P. Shiu in Volume 13, Number 2.

Consider firstly the number of ways of making up n halfpence using $\frac{1}{2}$p, 1p, 2p coins. Up to $[\frac{1}{4}n]$ 2p coins may be used† and, if r 2p coins have been chosen, then there are $[\frac{1}{2}n] - 2r + 1$ possible choices for the number of 1p coins. The total is therefore the sum of an arithmetic progression:

$$\sum_{r=0}^{[\frac{1}{4}n]} ([\tfrac{1}{2}n] - 2r + 1) = ([\tfrac{1}{4}n] + 1)([\tfrac{1}{2}n] - [\tfrac{1}{4}n] + 1) = ([\tfrac{1}{4}n] + 1)([\tfrac{1}{4}n + \tfrac{1}{2}] + 1). \tag{A}$$

This takes the values

$$\tfrac{1}{16}(n+4)^2, \qquad \tfrac{1}{16}(n+3)^2, \qquad \tfrac{1}{16}(n+2)(n+6), \qquad \tfrac{1}{16}(n+1)(n+5)$$

respectively as $n \equiv 0, 1, 2, 3$ (Mod 4).

† $[\frac{1}{4}n]$ denotes the integer part of $\frac{1}{4}n$.

It is easier to see that the number of ways of making up $5n$ pence using 5p, 10p coins is

$$[\tfrac{1}{2}n] + 1. \tag{B}$$

Very fortunately, formulae (A) and (B) also give the number of ways of making up £$5n$ using £5, £10, £20 notes and the number of ways of making up $50n$ pence using 50p coins and £1 notes. So the number of ways of making up N halfpence using any units from the $\frac{1}{2}$p coin to the £20 note is

$$F(N) = \sum_{1000p+100q+10r+s=N} f(p/2)f(p/2+1)f(q)f(r)f(s/2)f(s/2+1)$$

where $f(n) = [\frac{1}{2}n] + 1$.

Using this formula, it is easy to write a computer program to evaluate $F(N)$ and reassuring to find that $F(200) = 64\,704$. However, using my best program, it takes a microcomputer about three hours to decide that there are $1{\cdot}26 \times 10^{14}$ ways of making up £50. It seems sensible, therefore, to use the formulae for $\sum r^2, \sum r^3$, etc. to obtain $F(N)$ as a polynomial for fairly round values of N. The number of ways of making up $10M$ pence using $\frac{1}{2}$p, 1p, 2p, 5p, 10p coins is

$$\sum_{10r+s=20M} f(r)f(s/2)f(s/2+1)$$

$$= \sum_{r=0}^{2M} ([\tfrac{1}{2}r]+1)([(20M-10r)/4]+1)([(20M-10r)/4+\tfrac{1}{2}]+1)$$

which gives

$$\tfrac{1}{6}(M+1)(25M^3+70M^2+49M+6) = G(M), \text{ say.}$$

Similarly, the number of ways of making up £P using $\frac{1}{2}$p, 1p, 2p, 5p, 10p, 50p, £1 units is

$$H(P) = (P+1)(50\,000P^5 + 203\,500P^4 + 253\,075P^3 + 83\,450P^2 - 7\,707P + 18)/18.$$

Note that $H(-1) = 0$, which proves the 'Irish Banker's Theorem':

You can't change a £1 overdraft!

Changes in our coinage would give an interesting twist to the problem. If a 20p coin were to be introduced, then the number of ways of making up N halfpence using any coins up to 20p would be

$$\sum_{10r+s=N} g(r)g(s), \qquad \text{where } g(n) = f(n/2)f(n/2+1),$$

the expression in formula (A). The number of ways of making up $20M$ pence using these coins would be

$$J(M) = \sum_{r=0}^{M} G(2r) \qquad \text{(where } G \text{ is as before)}$$

$$= (M+1)(240M^4 + 930M^3 + 1086M^2 + 363M + 18)/18.$$

The total number of ways of making up £1 using $\frac{1}{2}$p to £1 notes would now be

$$J(5) + (G(1) + G(3) + G(5)) + 2 = 98\,411 + (50 + 972 + 5176) + 2$$

$$= 104\,561.$$

Furthermore, if the government were to be consistent and introduce a £2 note as well as a 20p coin†, then the number of ways of making up N halfpence using any of the twelve units from $\frac{1}{2}$p to £20 would be

$$\sum_{1000p+100q+10r+s=N} g(p)g(q)g(r)g(s).$$

Apart from the tidier appearance of this expression, we should then have 513 269 191 ways of making up a 'fiver', instead of the paltry 105 706 537 available at present.

† Write to your M.P.

Yours sincerely,
P. J. O'Grady
(Warwick School, Warwick CV34 6PP)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

14.4. *The Obstinate Dog Problem* (Submitted by Lloyd Taylor, a student at the University of Nottingham, who writes that the problem was investigated by the Dutch mathematical physicist Huygens (1629–1695).) A man has his dog on a lead and walks in a straight line with the lead always taut. Initially the lead is perpendicular to the line along which the man walks. Describe the curve traced out by the dog.

14.5. (From the 1981 Hungarian Mathematical Olympiad) Solve the inequality

$$x(x+1)(x+2)(x+3) \geq \tfrac{9}{16}.$$

14.6. A polynomial in x has integer coefficients and leading coefficient 1, and it takes the value 1 for four distinct integer values of x. Show that there is no integer value of x for which the polynomial takes the value 24.

Solutions to Problems in Volume 13, Number 3

13.7. Construct a Turing machine which includes | and $\times$ among its tape symbols and which performs multiplication, i.e. if it is started with the string

$$\underbrace{|\,|\,|\cdots|\,|}_{\leftarrow n \rightarrow}\times\underbrace{|\,|\,|\cdots|\,|}_{\leftarrow m \rightarrow}$$

then, after operating for some time, it will eventually settle permanently into a situation where the string on the tape is

$$\underbrace{|\,|\,|\,|\cdots|\,|\,|}_{\leftarrow n\times m \rightarrow}$$

Solution

You will need to read Keith Austin's article on the life and work of Alan Turing in Volume 13, Number 3 to understand what Turing machines are and hence the solution to this problem. Since we received no solutions, we supply Keith Austin's solution here. A suitable Turing machine has tape symbols |, $\times$, a, c, d (B denotes blank) and states $S, T, U, V, W, R, Q, P, N, M$, where S is the starting state. The operating table is shown in Table 1.

TABLE 1

S |	$T\ a \rightarrow$	$Q\ c$	Q | $\leftarrow$
T |	T | $\rightarrow$	$Q\ \times$	$Q \times \leftarrow$
$T\ \times$	$U \times \rightarrow$	Q |	Q | $\leftarrow$
U |	$V\ c \rightarrow$	$Q\ a$	$S\ a \rightarrow$
V |	V | $\rightarrow$	$S\ \times$	$P \times \leftarrow$
$V\ B$	$W\ d \rightarrow$	$P\ a$	$P\ a \leftarrow$
$W\ B$	R | $\leftarrow$	$P\ B$	$N\ B \rightarrow$
$R\ d$	$R\ d \leftarrow$	$N\ a$	$N\ B \rightarrow$
R |	R | $\leftarrow$	$N\ \times$	$N\ B \rightarrow$
$R\ c$	$U\ c \rightarrow$	N |	$N\ B \rightarrow$
$V\ d$	$W\ d \rightarrow$	$N\ d$	$M\ B \rightarrow$
W |	W | $\rightarrow$	M |	M | 0
$U\ d$	$Q\ d \leftarrow$		

Situations requiring the other rows do not occur in the operations considered in the problem.

13.8. A confectioner's shop sells three sorts of fruit pie, apple, blackberry and cherry, and their quality is measured on a scale 1–6 (1 = low, 6 = high). The probabilities of the quality of each pie are given by

Pie	Quality	Probability
Apple	3	1
Blackberry	1	0·51
	5	0·49
Cherry	2	0·56
	4	0·22
	6	0·22

If you do not like cherries, which pie should you choose to maximize your probability of obtaining the best quality? If you like apple, blackberry and cherry pie equally, which should you now choose?

Solution

This problem is a good example of the surprising things that can happen in probabilistic situations. Suppose you do not like cherries. Then apple will be the best pie with probability 0·51 and blackberry with probability 0·49, so you should choose apple.

But suppose now that you like all three varieties equally. We tabulate the different possibilities and their probabilities below.

	Quality					
Apple	3	3	3	3	3	3
Blackberry	1	5	1	5	1	5
Cherry	2	2	4	4	6	6
Probability	$0{\cdot}51 \times 0{\cdot}56 = 0{\cdot}2856$	0·2744	0·1122	0·1078	0·1122	0·1078
Best pie	A	B	C	B	C	C

The probability that apple is best is 0·2856.
The probability that blackberry is best is $0{\cdot}2744 + 0{\cdot}1078 = 0{\cdot}3822$.
The probability that cherry is best is $0{\cdot}1122 + 0{\cdot}1122 + 0{\cdot}1078 = 0{\cdot}3322$.
Hence you should choose blackberry. Thus, if cherry is considered, choose blackberry rather than apple (or cherry). But remove cherry and the choice switches from blackberry to apple. Curious!

13.9. At a party consisting of at least four people, among any four guests there is one who has previously met the other three. Prove that, among any four guests, there is one who has previously met every other person in the party.

Solution

This is an exercise in logical thinking, and in being able to express your arguments logically. The problem was actually taken from the book *Introduction to Graph Theory* by B. Andrásfai (Adam Hilger, Bristol, 1977), where many similar problems and their solutions will be found.

We may assume that there are two people A, B at the party who have not previously met, since otherwise everyone previously knew everyone else and the problem is easily solved! We may also assume that there is another pair (one of whom may be A or B) who are also meeting for the first time, since otherwise everyone except A, B has previously met all his fellow-guests and again the problem is easily solved. If the second pair contains neither A nor B, then these two people C, D together with A, B contradict the hypothesis, since none of A, B, C, D has previously met the other three. Thus we can assume that the second pair are A, C (say). Let D be a fourth person at the party. We claim that D has previously met every other person at the party. If we apply the hypothesis to A, B, C, D, then D must have met A, B, C previously. Consider a fifth person E. If D had not previously met E, then A, B, C, E would contradict the hypothesis. This entertaining problem is now solved.

Book Reviews

Mathematical Models in the Social, Management and Life Sciences. By D. N. BURGHES and A. D. WOOD. Ellis Horwood Limited, Chichester, 1980. Pp. 287. Cloth £13·50. Paperback £5·90.

This interesting book aims at giving an appreciation of how mathematical models are formulated, solved and applied, plus a concise description of the basic mathematical techniques needed. As the title indicates, the models are taken from the non-physical sciences, and the book is primarily intended to provide those studying management, economics, planning, sociology and biology, with a practical course in applying mathematics. However, much of the material is within the grasp of an A-level mathematics student, and teachers at this level will find plenty of ideas here illustrating the importance of mathematical concepts in realistic situations.

An introductory chapter discusses the role and limitations of mathematical modelling, with a simple case study, and a glance at future trends. Each of the ten chapters which follow is built around a mathematical topic; sequences and series, limits and continuity, turning points and change, growth and decay, cycles and oscillations, difference equations, vectors and matrices, optimisation techniques, theory of games, catastrophe theory. There is usually an introductory problem, then a concise presentation of the relevant techniques, followed by the solution of the problem, further case studies using similar methods, and then problems both on the mathematical techniques and it applications. Finally there are appendices on number systems, integration, and partial differentiation, an extensive bibliography, answers to the problems, and an index.

The explanations of techniques seem to me to be the least satisfactory parts. The authors realise that they do not have space for full explanations; the problem then is that what they give is usually not enough for someone unfamiliar with the topic, but is not necessary for someone who knows it already. Perhaps all that is needed is a statement of the knowledge required plus suitable references. The presentation is sometimes rather sophisticated: for example, is the sort of reader for whom the book is intended likely to need (or appreciate) an 'ε, δ' treatment of 'obvious' limit properties? There are also rather too many errors in the text. Some are obvious slips unlikely to cause much trouble, but others are more serious; in particular on page 68 the first test for a maximum ($f'(x_0)$ decreases steadily with x) and the second ($f''(x_0) < 0$) is wrong.

The main interest of the book is in its wide-ranging collection of models. Some of these arise very naturally from the definition of the problem, as in the section dealing with financial matters such as annuities and mortgages where geometric sequences and series are inescapable. It is none the less valuable to have such examples so clearly set out. In other cases the choice of model is much more subjective, and here more discussion of reasons behind the choice would have been welcome. For example, we are told (page 41) that fitting a curve to the graph of company profits y against advertising expenditure x (both in £100000s) has produced the formula $y = (22x + 11)/(x + 2)$. Now presumably the data give a rising curve through $(0, 5\frac{1}{2})$ and approaching the $y = 22$ level (though this itself is open to question—might not consumer resistance cause a down-turn in profits if expenditure continues to increase?). I would hope that a student who has taken to heart the chapter on the aims and philosophy of modelling would want to know why a rational function like this has been used, how the constants have been found, and what alternatives are available. It is a measure of the authors' success that such questions come to mind, for this is a rich and stimulating book which, despite some shortcomings, can be enthusiastically recommended.

City of London School T. J. HEARD

Introductory Mechanics. By C. D. COLLINSON. Edward Arnold (Publishers) Ltd, London, 1980. Pp. 246. £7·95.

Mechanics: Essential Theory and Exercises. By P. J. HOLT. Hodder and Stoughton, Sevenoaks, 1980. Pp. 208. £2·95.

Introductory Mechanics is a book for the first-year undergraduate. The reader is taken systematically through Newtonian mechanics, working the whole time with vectors. After a thorough treatment of Newton's laws of motion, there are chapters on simple harmonic motion and damping, gravitation, central forces with application to orbits and Rutherford scattering, impacts and variable mass problems, statics of a rigid body, and the motion of a rigid body in two but not three dimensions. In addition there are chapters on relativity, and towards the end some more advanced work on fluid dynamics and on Newtonian cosmology. The book aims to present a course 'to interest and challenge those students who have studied the subject before and yet to remain within the grasp of those studying the subject for the first time.' I feel that only the first of these two aims succeeds. For the undergraduate who has already spent two years on the subject at school this is a most useful book. The text is surprisingly readable, and by and large avoids the common fault of books at this level, namely of going too fast to be readily understood by the average undergraduate, but even so it would have helped if the author had drawn more attention to the important results. However, after reading this book one is left with nothing but sympathy for the newcomer to the subject, expected to digest all this in one year. In the early stages of studying mechanics the student needs to go slowly, taking time to sort out the concepts involved. Perhaps the fault is with the universities, expecting everybody to reach this stage by the end of the first year, but *Introductory Mechanics* is not for the newcomer. For the schoolteacher, seeking a book to satisfy the demands of that annoying pupil who is always asking the awkward questions, or for the majority of undergraduates, however, this book will be of considerable help.

Mechanics: Essential Theory and Exercises takes the subject from the very elementary stages up to single-subject A level. In just 200 pages it highlights learnable formulae and illustrates the ideas with a large number of worked examples and exercises. Here is a book written by a schoolmaster who knows both his subject and his readers—the mechanics is covered well and on many occasions he draws attention to the mistakes which are commonly made by pupils. Most of the expected topics are dealt with at a brisk though manageable pace, the most obvious exception being the section on work done, energy, its conservation, elasticity and associated equations and six worked examples all in nine pages! But it is a surprise to find no mention of moments of inertia and the rotation of rigid bodies—this is in several single-subject A-level syllabuses. My main concern is that this very reasonably priced book may be unable to find a large enough market. It claims to be a 'useful revision course for examination candidates'. Certainly it is not substantial enough to become a standard school textbook for a two-year course, and few school budgets these days would have room for a class set of a revision book. But it ought to find its way into both public and school libraries, where it will be of use both to the teacher in search of ideas and worked examples, and to the pupil who wants an alternative to an incomprehensible teacher!

City of London School J. F. ALLMAN

Elementary Differential Equations (sixth edition). By EARL D. RAINVILLE and PHILLIP E. BEDIENT. Collier Macmillan, London, 1981. Pp. xiv + 529. Hard covers £10·50. Soft covers £7·95.

This book is an ideal reference book for a school or college library. The first half of the book tackles all the work usually found in British sixth-form syllabuses, while the second half shows where this is leading in college and university courses. The book is attractively produced and unlike some, contains answers.

University of Durham H. NEILL

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