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Crux Mathematicorum

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

G.W. Sands
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4

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Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

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THE OLYMPIAD CORNER
No. 93
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first set of problems selected this month come from D. Vathis, Chalcis, Greece. They are from the Entrance Examination for the Polytechnic School of Athens 25 years ago (i.e. 1962). (How many Canadian high school graduates today could tackle them, I wonder.)

1. Prove that

$$(x-1/2)(x+1/2) < x^2 < (x-23/48)(x+25/48)$$

for all real $x \geq 6$. Then prove that

$$\frac{2}{11} > \sum_{i=6}^{\infty} \frac{1}{i^2} > \frac{48}{265}$$

and calculate $\sum_{i=1}^{\infty} \frac{1}{i^2}$ accurate to three decimal places.

2. Determine those odd natural numbers n such that the common roots of

$$f(x) = (x+1)^n - x^n - 1$$

and

$$h(x) = (x+1)^{n-1} - x^{n-1}$$

contain the roots of $x^2 + x + 1$.

3. Prove that the polynomial

$$f_n(x) = x \sin a - x \sin(na) + \sin(n-1)a$$

is exactly divisible by

$$h(x) = x^2 - 2x \cos a + 1$$

where a is a real number and n is a natural number ≥ 2.

*

The next four problems are from the 1986 Annual High School Competition of the Greek Mathematical Society. I again thank D. Vathis of Chalcis, Greece for translating the problems and sending them in to me.

1. Let F: $\mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x+t) = \frac{F(x)\cos\theta - \sin\theta}{F(x)\sin\theta + \cos\theta},$$

where θ and t are constants, t a positive integer. Prove that:

(i) If $\theta = \pi/n$, $n = 1,2,3,\ldots$, then F is a periodic function with period nt.

(ii) If $\theta = k\pi$, where k is an irrational number, then F is not a periodic function with period nt, for any natural number n.

2. The following two propositions are well known:

(i) (Archimedean Property) Let x be a real number. Then there is an integer n greater than x.

(ii) If x is a real number, then there is a unique integer k_0 such that $k_0 \le x < k_0 + 1$.

Using (i) and (ii) (or otherwise), show that if a, b are real numbers and $a < b$, there exists a rational p such that $a < p < b$.

3. It is a well known proposition that if O, A, B, C are given points in space, and the real numbers k_1, k_2 and k_3, not all zero, can be found such that

$$k_1 + k_2 + k_3 = 0$$

$$k_1 \cdot \vec{OA} + k_2 \cdot \vec{OB} + k_3 \cdot \vec{OC} = 0$$

then the points A, B, and C are collinear.

(i) Write and prove the converse of this proposition.

(ii) Using this proposition, or otherwise, prove the following assertion: if a straight line meets the sides AB, AC of a triangle ABC in the points B_1, C_1 respectively, and meets the median AM in the point M_1, then

$$\frac{AB}{AB_1} + \frac{AC}{AC_1} = 2\frac{AM}{AM_1}.$$

4. Find all integer solutions $k,l,m \le 3$ for which the system

$$x + y = kz$$
$$y + z = lx$$
$$z + x = my$$

has nontrivial solutions.

[*Editor's note*: Problems 2 and 3 are standard in many textbooks, and solutions won't be published in this column unless they somehow offer something really new.]

*   *   *

The next group of questions are problems posed at the first round of the 23rd Spanish Mathematical Olympiad (1986–87) and were forwarded thanks to Professor Francisco Bellot, Valladolid, Spain.

23rd Spanish Mathematical Olympiad
First Round (Valladolid, 28 & 29 November 1986)
Day 1

1. The lengths of the sides of a triangle are 6, 8, 10. Show that there is a unique line which simultaneously bisects the area and the perimeter of the triangle. [*Editor's note*: This was also problem 1 of the 1985 Canadian Mathematics Olympiad. A solution appears on [1985: 272].]

2. Find all x, y, z (real numbers) such that

$$xyz = \frac{x^3 + y^3 + z^3}{3}.$$

3. Given a triangle ABC, let B' be a point on the side AB which is different from either A or B; let C' be the point of intersection of AC with the parallel to BC which passes through B'; let x be the circle centred at B which passes through C, x' the circle centred at B' and passing through C'; and D, the other point of intersection of x with the line AC. If t is the tangent to x at C, and t' the tangent to x' at D, show that

(i) t and t' intersect (at T, say), and
(ii) the triangle CTD is isosceles.

4. Consider the equations

$$x^2 + bx + c = 0$$

and

$$x^2 + b'x + c' = 0$$

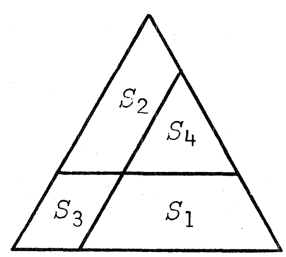
where b, c, b' and c' are integers such that

$$(b - b')^2 + (c - c')^2 > 0.$$

Show that if the equations have a common root, then the second roots are distinct integers.

Day 2

5. Given an equilateral triangle of side 1, determine the distances to the sides of two parallels drawn such that the areas S_1, S_2, S_3, S_4 are in arithmetical progression (see figure). (No decimals, please.)



6. Let $a_1, a_2, \ldots, a_n$ be n distinct real numbers. Calculate those points x on the real line which minimize $\sum_{i=1}^{n} |x - a_i|$, the sum of the distances from x to the a_i, in the cases $n = 3$ and $n = 4$.

7. In a triangle ABC with opposite sides a, b, c respectively, show that if

$$a + b = (a \tan A + b \tan B)\tan(C/2)$$

then the triangle is isosceles.

8. In a right triangle ABC, with centroid G, consider the triangles AGB, GBC, CGA. If the *sides* of ABC are all natural numbers, show that the *areas* of AGB, BGC, CGA are even natural numbers.

* * *

Please send in copies of your local and national competitions, (translated, if possible into English). By including these we can give a more representative impression of what is happening around the world. It has also been brought to my attention that some teachers and high school students would appreciate seeing some "beginner's" problems and favourite chestnuts that might not get used on a national contest because they are too "well known". If there is interest in having part of the column devoted to such problems from time to time, please let me know. To make such a section run I'll need the readership's help. Submit proposed problems *with solutions* and some indication of the *level of difficulty* (novice, intermediate, etc.), and I'll see if a beginner's page can become a semi–regular feature of the column.

* * *

We return now to solutions to problems posed in 1985 in Volume 10. These were all problems proposed for the 26th I.M.O. in Finland.

11. [1985: 305] *Proposed by Iceland.*

If

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0,$$

where $x_{i+n} = x_i$ and all $x_i^2 = 1$, prove that n is divisible by 4.

Solution by Ed Doolittle, The University of Toronto, Ontario, Canada.

Set $a_i = x_i x_{i+1} x_{i+2} x_{i+3}$. Clearly n must be *even* since each $a_i = 1$ must be paired with an $a_i = -1$. Now suppose $n = 2k$, k odd. Then k of the a_i must be 1 with the other k equal to -1 and we have

$$1 = \left[\prod_{i \text{ odd}} x_i\right]^2 = \left[\prod_{i \text{ odd}} x_i\right]\left[\prod_{i \text{ odd}} x_{i+4}\right]$$

$$= \prod_{i \text{ odd}} (x_i x_{i+4}) = \prod_{i \text{ odd}} (a_i a_{i+1}) = \prod a_i = -1,$$

a contradiction. Therefore k must be even, and n divisible by 4.

13. [1985: 305] *Proposed by Italy.*

Two persons X and Y play a game with a die. Whenever the toss of the die results in a 1 or a 2, X wins the toss; otherwise Y wins the toss. The game is won by the first player who wins two consecutive tosses. Determine the probabilities that each of the players wins the game within 5 tosses. Also, determine the probabilities of winning for each player if there is no restriction on the number of tosses.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

A player wins the game on the nth toss, $n \geq 2$, if and only if he wins the nth, $(n-1)$st and $(n-j)$th toss for all odd $j \geq 1$, and only these tosses. Therefore, if X_n and Y_n denote the probabilities that X and Y, respectively, win on the nth toss, we have

$$X_n = \frac{1}{3}\left[\frac{1}{3}\right]^{[n/2]}\left[\frac{2}{3}\right]^{[(n-1)/2]}$$

and

$$Y_n = \frac{2}{3}\left[\frac{2}{3}\right]^{[n/2]}\left[\frac{2}{3}\right]^{[(n-1)/2]},$$

for $n \geq 2$, where $[x]$ is the integer part of x.

The probability that X wins by the 5th toss is

$$\sum_{i=2}^{5} X_i = \frac{1}{9} + \frac{2}{27} + \frac{2}{81} + \frac{4}{243} = \frac{55}{243}.$$

The probability that Y wins by the 5th toss is

$$\sum_{i=2}^{5} Y_i = \frac{4}{9} + \frac{4}{27} + \frac{8}{81} + \frac{8}{243} = \frac{176}{243}.$$

The probability of X winning is

$$\sum_{i=2}^{\infty} X_i = \sum_{k=1}^{\infty} (X_{2k} + X_{2k+1})$$

$$= \sum_{k=1}^{\infty} \left\{ \left[\frac{1}{3}\right]^{k+1}\left[\frac{2}{3}\right]^{k-1} + \left[\frac{1}{3}\right]^{k+1}\left[\frac{2}{3}\right]^{k} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{5}{27}\left[\frac{2}{9}\right]^{k-1} = \frac{5}{21}.$$

Similarly the probability of Y winning is $\frac{16}{21}$.

18. [1985: 306] *Proposed by Rumania.*

For $k \geq 2$, let $n_1, n_2, \ldots, n_k$ be positive integers such that

$$n_2 | (2^{n_1} - 1)\ ,\ n_3 | (2^{n_2} - 1)\ ,\ \ldots\ ,\ n_k | (2^{n_{k-1}} - 1)\ ,\ n_1 | (2^{n_k} - 1).$$

Prove that $n_1 = n_2 = \cdots = n_k = 1$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Suppose that some $n_i = 1$. Since $n_{i+1} | (2^{n_i} - 1)$ we find $n_{i+1} = 1$, then $n_{i+2} = 1$, and similarly $n_j = 1$ for all $1 \leq j \leq k$.

Hence we assume that $n_i \neq 1$ for all i, $1 \leq i \leq k$. Then n_k contains prime factors. Let P_k be the smallest prime factor of n_k. Since $n_k | (2^{n_{k-1}} - 1)$ we have $2^{n_{k-1}} \equiv 1 \bmod P_k$. Let S_k be the smallest positive integer x such that $2^x \equiv 1 \bmod P_k$. Then $S_k | n_{k-1}$ and $S_k | (P_k - 1)$ by Euler's theorem. In particular, $1 < S_k \leq P_k - 1 < P_k$ and so n_{k-1} has a prime factor less than P_k. Call it P_{k-1}.

By the same reasoning $2^{n_{k-2}} \equiv 1 \bmod P_{k-1}$ and so n_{k-2} has a prime factor $P_{k-2} < P_{k-1}$. Continuing in this fashion we obtain a sequence of primes

$$P_k > P_{k-1} > \cdots > P_2 > P_1$$

where $P_i | n_i$. But then $2^{n_k} \equiv 1 \bmod P_1$ and so n_k has a prime factor $P < P_1 < P_k$ contradicting the choice of P_k. Thus each $n_i = 1$.

19. [1985: 306] *Proposed by Rumania.*

Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}\,]$ for $n = 1,2,3,\ldots$ (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

Solution by Ed Doolittle, University of Toronto, Ontario, Canada.

Let $\{x\}$ denote the fractional part of the real number x. Note that $\{2^n\sqrt{2}\} > 1/2$ for infinitely many values of n because there are infinitely many 1's in the binary expansion of $\sqrt{2}$, and multiplying any number by 2^n shifts that number's binary expansion by n places, so that a 1 can be shifted into the first decimal place for infinitely many n.

Now

$$\begin{aligned}
\{2^n\sqrt{2}\} > \tfrac{1}{2} > 1 - \frac{1}{\sqrt{2}} &\Rightarrow 1 - \{2^n\sqrt{2}\} < \frac{1}{\sqrt{2}} \\
&\Rightarrow [2^{n+1} + (1 - \{2^n\sqrt{2}\})\sqrt{2}\,] = 2^{n+1} \\
&\Rightarrow [(2^n\sqrt{2} + 1 - \{2^n\sqrt{2}\})\sqrt{2}\,] = 2^{n+1} \\
&\Rightarrow [[2^n\sqrt{2} + 1]\sqrt{2}\,] = 2^{n+1}.
\end{aligned}$$

This gives infinitely many k of the form $[2^n\sqrt{2} + 1]$ for which a_k is an integral power of 2.

[*Editor's note: The problem was also solved by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.*]

<u>20</u>. [1985: 306] *Proposed by Sweden.*

Two equilateral triangles are inscribed in a circle with radius r. If A is the area of the set consisting of all points interior to both triangles, prove that $2A \geq r^2\sqrt{3}$.

Solution by Ed Doolittle, The University of Toronto, Toronto, Ontario.

The hexagon of minimum area with inscribed circle of radius R is the regular hexagon with area $2\sqrt{3}R^2$. [For a proof apply Jensen's inequality with $\Sigma \tan \theta_i$, $\Sigma\, \theta_i = \pi$.] The set consisting of the interior points of both triangles is the interior of a hexagon with inscribed circle of radius $R = r/2$. Thus $2A \geq r^2\sqrt{3}$.

<u>21</u>. [1985: 307] *Proposed by Sweden.*

If a, b, c are real numbers such that

$$\frac{1}{bc - a^2} + \frac{1}{ca - b^2} + \frac{1}{ab - c^2} = 0,$$

prove that

$$\frac{a}{(bc - a^2)^2} + \frac{b}{(ca - b^2)^2} + \frac{c}{(ab - c^2)^2} = 0.$$

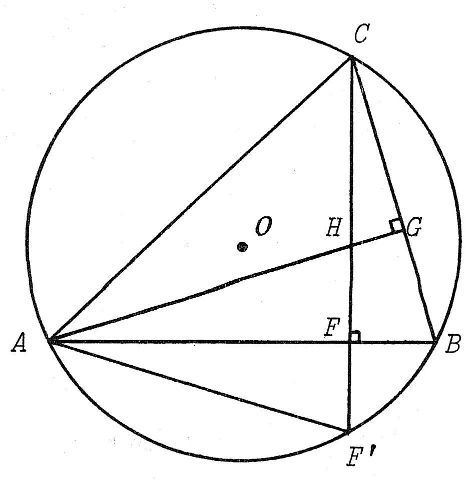
Comment by George Evagelopoulos, Athens, Greece.

This problem already appeared and was solved in the Olympiad Corner. See problem 3–3 [1979: 106–107].

<u>22</u>. [1985: 307] *Proposed by Spain.*

Show how to construct a triangle ABC given the side AB, the length of segment OH and the fact that OH is parallel to AB, where O and H are the circumcenter and orthocenter, respectively, of the triangle.

Solution by Ed Doolittle, The University of Toronto, Toronto, Ontario.



Analysis. In ΔABC, let F be the foot of the perpendicular from C, and F' the second intersection of the altitude with the circumcircle. Then $\overline{HF} = \overline{FF'}$ so that the reflection of H in AB is F', and $\overline{OA} = \overline{OB} = \overline{OF'}$. [To see this easily, let the foot of the perpendicular from A to BC be at G. In the right triangle AGB, $\angle HAF = \angle GAB$ is the complement of $\angle ABC$. On the other hand $\angle F'AF$ is the complement of $\angle AF'F$, which also equals $\angle ABC$ because of chord AC. Thus ΔHAF

and $\Delta F'AF$ are congruent.] Since OH is parallel to AB, OF' bisects FC' at the point P, where C' is the midpoint of AB. Then

$$\overline{OP} = \overline{OF'}/2 = \overline{OA}/2 = \overline{OB}/2.$$

Construction. Lay down AB. Locate the midpoint C' of AB, and the point F on AB at distance $\overline{OH}$ from C', on the side of B. Find the midpoint P of FC'. Locate the points M and N which divide PB in the ratio $\pm$ 1:2. Construct the circle on diameter MN. Then all points X on the circle are such that $\overline{XP} = \overline{XB}/2$. Since $\overline{OP} = \overline{OB}/2$, O lies on the intersection of the circle and the perpendicular through C'. The circumcircle can now be constructed, and C lies at the intersection of the circumcircle and the perpendicular to AB through F.

<u>23</u>. [1985: 307] *Proposed by Spain.*

Find all triples (x,y,z) of positive integers such that

$$1/x + 1/y + 1/z = 4/5.$$

Solution by Ed Doolittle, The University of Toronto, Toronto, Ontario; George Evagelopoulos, Law student, Athens, Greece; and Bob Prielipp, University of Wisconsin-Oshkosh, Wisconsin.

Without loss of generality we may assume that $x \le y \le z$ in any solution (x,y,z). Thus $3/x \ge 4/5$, so $x \le 15/4$. Now $x = 1$ is clearly impossible so we have $x = 2$ or $x = 3$.

If $x = 2$,

$$1/y + 1/z = 3/10.$$

Hence $2/y \ge 3/10$, so $y \le 20/3$. It follows that y = 4, 5, or 6, since $3/10 < 1/3$. Direct calculation shows that the only solutions of this kind are (2,4,20) and (2,5,10).

If $x = 3$,

$$1/y + 1/z = 7/15.$$

Hence $2/y \ge 7/15$, so $y \le 30/7$. It follows that $y = 3$ or $y = 4$. It is now easy to check by calculation that there are NO solutions of this type. One can now easily write down the twelve triples obtained by permuting the entries of (2,4,20) and (2,5,10).

<u>24</u>. [1985: 307] *Proposed by the Soviet Union.*

Decompose the number $5^{1985} - 1$ into a product of three integers each of which exceeds 5^{100}.

Solution by Curtis Cooper, Central Missouri State University, Missouri.

From the factorization

$$(5^{5h} - 1) = (5^h - 1)(5^{2h} + 3\cdot 5^h + 1 - 5^{(h+1)/2}(5^h + 1))\cdot$$
$$\cdot(5^{2h} + 3\cdot 5^h + 1 + 5^{(h+1)/2}(5^h + 1))$$

(see (7) on page *liv* of [1]) we have

$$5^{1985}-1=(5^{397}-1)(5^{794}-5^{596}+3\cdot 5^{397}-5^{199}+1)\cdot$$
$$\cdot(5^{794}+5^{596}+3\cdot 5^{397}+5^{199}+1).$$

Reference:

[1] John Brillhart, D.H. Lehmer, J.L. Selfridge, Bryant Tuckerman, and S.S. Wagstaff, Jr., *Factorization of* $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ *up to high powers*, Contemporary Mathematics, Vol. 22, American Mathematical Society, Providence, R.I., 1983.

25. [1985: 307] *Proposed by the Soviet Union.*

34 countries participated in a Jury session of the I.M.O., each country being represented by the leader and the deputy leader of the team. Before the meeting, some of the participants greeted each other by shaking hands, but no team leader shook hands with his or her deputy. After the meeting, the leader of the Illyrian team asked every other participant (including her own deputy) the number of people they had shaken hands with, and all the answers she got were different. How may people did the Illyrian team greet?

Solution by John Morvay, Dallas, Texas; and by Daniel Ropp, Washington University, St. Louis, MO.

Since each participant does not greet himself or his teammate, each participant may greet 0,1,2,...,66 people. As each of the 67 participants asked by the Illyrian team leader gives a different response we may label those participants $P_0,\dots,P_{66}$, the subscripts corresponding to the number of handshakes attested to by each P_i. P_{66} must greet everyone except his team leader and himself. He does not greet P_0, so he must greet $P_1,P_2,\dots,P_{65}$ and the Illyrian leader. This means that P_0 and P_{66} are teammates. Notice too that for $1 \le i \le 65$, P_i greets $i-1$ people amongst $P_1,\dots,P_{65}$ and the Illyrian team leader. In this way we may proceed by induction to show that for $j \le 32$, P_j and P_{66-j} are from the same team and for $j < i < n-j$, P_j shakes $i-j-1$ hands amongst $P_{j+1},\dots,P_{n-j-1}$ and the Illyrian team leader. Now it follows that the Illyrian deputy is P_{33} and both the leader and the deputy of the Illyrian team greeted each of $P_{34},P_{35},\dots,P_{66}$ for a total of 66 people.

26. [1985: 307] *Proposed by Turkey.*

Determine the smallest positive integer n such that

(i) n has exactly 144 distinct positive divisors, and

(ii) there are ten consecutive integers among the positive divisors of n.

Solution by Ed Doolittle, The University of Toronto, Toronto, Ontario; and also by Daniel Ropp, Washington University, St. Louis, MO.

Among any ten consecutive integers at least one is divisible by 5, at least one is divisible by 7, at least one by 8 and at least one by 9. Thus for n to satisfy (ii) it must be the case that n is divisible by $r = 2^3\cdot 3^2\cdot 5\cdot 7$. Conversely, if r divides n then n is divisible by

k for $1 \le k \le 10$, and so satisfies (ii). Therefore we seek the least multiple of r with exactly 144 positive integral divisors. Recall that $\Pi\, P_i^{a_i}$ has $\Pi(1 + a_i)$ divisors where the P_i are distinct primes. If $n = qr$ where q is an integer and q has 2 or more prime factors other than 2, 3, 5 or 7 then n has at least $(1 + 3)(1 + 2)(1 + 1)(1 + 1)(1 + 1) = 192$ factors. Thus, to satisfy (i) n must have the form

$$n = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}p^{a_5}$$

where $a_5 \ge 0$ and p is a prime different from 2, 3, 5 or 7. We wish to minimize n, so we must have $p = 11$. From the remarks on (ii) above, $a_1 \ge 3$, $a_2 \ge 2$, $a_3 \ge 1$, $a_4 \ge 1$. Also we must have $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5$, for if $a_i < a_{i+1}$ say, the integer n' obtained by interchanging a_i and a_{i+1} is smaller than n, and would also satisfy (i) and (ii). The problem is then to minimize

$$n = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}$$

over all ordered 5–tuples (a_1,a_2,a_3,a_4,a_5) where $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5$, $a_1 \ge 3$, $a_2 \ge 2$, $a_3 \ge 1$, $a_4 \ge 1$ such that

$$\prod_{i=1}^{5}(1 + a_i) = 144.$$

By considering the possibilities for a_5, then a_4, then a_3, etc., one finds the only such 5–tuples are:

$$(3,2,2,1,1),\quad (5,2,1,1,1),\quad (3,3,2,2,0),\quad (5,3,2,1,0),$$
$$(7,2,2,1,0),\quad (5,5,1,1,0),\quad (8,3,1,1,0),\quad (11,2,1,1,0).$$

By inspection, the corresponding value of n is least for the 5–tuple (5,2,1,1,1), and the solution to the problem is then

$$n = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 110880.$$

<u>27</u>. [1985: 307] *Proposed by the U.S.A.*

Determine the range of $w(w + x)(w + y)(w + z)$, where x, y, z, w are real numbers such that

$$x + y + z + w = x^7 + y^7 + z^7 + w^7 = 0.$$

Solution by Ed Doolittle, The University of Toronto, Toronto, Ontario.

Let w, x, y, z be the roots of the quartic polynomial

$$t^4 + p_1t^3 + p_2t^2 + p_3t + p_4 = 0.$$

Let $S_n = x^n + y^n + z^n + w^n$. Then

$$S_1 = -p_1 = 0$$
$$S_2 = p_1^2 - 2p_2 = -2p_2$$
$$S_3 = -p_1^3 + 3p_1p_2 - 3p_3 = -3p_3$$

$$S_4 = -p_1S_3 - p_2S_2 - p_3S_1 - 4p_4 = 2p_2^2 - 4p_4$$
$$S_5 = -p_1S_4 - p_2S_3 - p_3S_2 - p_4S_1 = 5p_2p_3$$
$$S_7 = -p_1S_6 - p_2S_5 - p_3S_4 - p_4S_3 = 7p_3(p_4 - p_2^2).$$

Now $S_7 = 0$ implies $p_3 = 0$ or $p_4 = p_2^2$. But $p_4 = p_2^2$ entails $S_4 = -2p_2^2 \leq 0$ giving $S_4 = 0$ and $x = y = z = w = 0$, and the value zero is given to the expression. Thus we assume $p_3 = 0$. Now

$$\begin{aligned}(w + x)(w + y)(w + z) &= w^3 + w^2(x + y + z) + p_3 \\ &= w^2(x + y + z + w) + p_3 = 0 + 0 = 0.\end{aligned}$$

Hence

$$w(w + x)(w + y)(w + z) = w \cdot 0 = 0.$$

In any event the range of the expression is $\{0\}$.

One sees that the w factor in the given product could be dropped. In this form the question occurs as example 7, p.179 of W.L. Ferrar, *Higher Algebra*, Oxford, 1943.

* * *

With this solution we finish the backlog of solutions to problems given in Vol.11 (1985). The list of problems from that volume which appear not to be solved in any numbers so far is produced below. Naturally we welcome your elegant solutions to any of these problems.

[1985: 37–39]	12, 13, 16, 17, 21–23, 25
[1985: 71–73]	34, 37, 39–42, 44–46, 48
[1985: 102–105]	53, 55, 58–60, 62, 64–66, 68–70, 73, 74
[1985: 168]	2 (1984 Dutch)
[1985: 169]	3 (1985 Australian)
[1985: 170]	3 (34th Bulgarian)
[1985: 237–238]	4–6, 9–11 (1981 Leningrad)
[1985: 238–239]	2, 3, 5, 6 (1984 Bulgarian)
[1985: 239]	2 (1983 Greek)
[1985: 270]	2, 3, 5
[1985: 272]	10
[1985: 304–308]	1–5, 7, 9, 12, 14–17, 28–30.

Please send your solutions and problem sets directly to me.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **October 1, 1988,** *although solutions received after that date will also be considered until the time when a solution is published.*

1321. *Proposed by Jordi Dou, Barcelona, Spain.* (Dedicated in memoriam to Léo Sauvé.)

The circumcircle of a triangle is orthogonal to an excircle. Find the ratio of their radii.

1322. *Proposed by M.S. Klamkin, University of Alberta, and D.J. Newman, Temple University.*

Show that closed form expressions exist for the two integrals

$$\int t^{p-1}(1-t)^{q-1}dt \quad \text{and} \quad \int t^{p-1}(1+t)^{q-1}dt$$

for the special case $p + q = 1$, p rational.

1323. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be the right triangle with $\angle A = 1$ radian and $\angle C = \pi/2$. Let I be the incenter, O the midpoint of AB, and N the midpoint of OC. Is ΔNIO an acute, obtuse, or right triangle?

1324. *Proposed by Len Bos, University of Calgary.*

Let $S_i(x_1,x_2,\ldots,x_n)$ be the ith elementary symmetric function, i.e. $S_0 = 1$, $S_1 = x_1 + x_2 + \cdots + x_n$, $S_2 = \sum_{i<j} x_i x_j$, etc. Let J be the $n \times n$ Jacobian matrix of $S_1,\ldots,S_n$ with respect to $x_1,\ldots,x_n$; i.e.

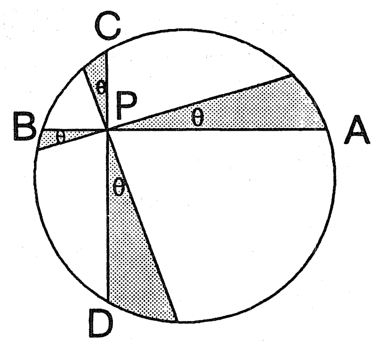
$$J_{ij} = \frac{\partial S_i}{\partial x_j}.$$

Show that

$$\det J = \prod_{i<j}(x_j - x_i),$$

the Vandermonde determinant.

1325. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*



Let P be any point inside a unit circle. Perpendicular chords AB and CD pass through P. Two other chords passing through P form four angles of θ radians each as shown in the figure. Prove that the area of the shaded region is 2θ.

1326. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Solve

$$\cos^4 2\theta + 2\sin^2 2\theta = 17(\sin\theta + \cos\theta)^8$$

for $0 \le \theta \le 360^\circ$.

1327. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let x_1, x_2, x_3 be the distances of the vertices of a triangle from a point P in the same plane. Let r be the inradius of the triangle, and p be the power of the point P with respect to the circumcircle of the triangle. Prove that

$$x_1x_2x_3 \ge 2rp.$$

1328. *Proposed by Sharon Reedyk and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Use a combinatorial argument to establish the (known) identity

$$\begin{bmatrix} n \\ 0 \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ m \end{bmatrix}\begin{bmatrix} n-m \\ 0 \end{bmatrix} = 2^m\begin{bmatrix} n \\ m \end{bmatrix}.$$

1329. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let ABC be a triangle , and let congruent circles C_1, C_2, C_3 be tangent to half lines AB and AC, BA and BC, CA and CB, respectively.

(a) Determine the locus of the circumcentre P of ΔDEF, where D, E, F are the centres of C_1, C_2, C_3.

(b) If C_1, C_2, C_3 all pass through the same point, show that their radius ρ satisfies

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{R}$$

where r and R are the inradius and circumradius of ΔABC.

1330. *Proposed by M.A. Selby, University of Windsor, Windsor, Ontario.*

Find all positive integers j such that $[\sqrt{j}]$ divides into j, where $[x]$ is the greatest integer less than or equal to x.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

__1109__. [1986: 13; 1987: 322] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with orthocentre H. A rectangular hyperbola with centre H intersects line BC in A_1 and A_2, line CA in B_1 and B_2, and line AB in C_1 and C_2. Prove that the points P, Q, R, the midpoints of A_1A_2, B_1B_2, C_1C_2, respectively, are collinear.

II. *Solution by Jordan B. Tabov, Sofia, Bulgaria.*

It is known, and quoted on [1987: 332], that the midpoint of a chord of a hyperbola is also the midpoint of the segment intercepted on the chord by the asymptotes of the hyperbola. We give a short proof for the rectangular hyperbola.

Lemma. Suppose that in a Cartesian system of coordinates the line $ax + by + c = 0$ intersects the hyperbola $xy = 1$ in two points D and E. Then the midpoint of the segment DE has coordinates $(-c/2a,-c/2b)$.

Proof. The coordinates (x_1,y_1) and (x_2,y_2) of D and E satisfy the system

$$ax + by + c = 0$$
$$xy = 1$$

and consequently the equation

$$(ax + by + c)x - b(xy - 1) = 0,$$

which simplifies to

$$ax^2 + cx + b = 0.$$

Hence x_1 and x_2 are the two roots of this equation, and so

$$\frac{x_1 + x_2}{2} = -\frac{c}{2a}.$$

Similarly,

$$\frac{y_1 + y_2}{2} = -\frac{c}{2b}. \quad \square$$

To solve our problem, choose a Cartesian system of coordinates with origin H and axes coinciding with the asymptotes of the given rectangular hyperbola so that its equation is $xy = 1$. Let A and B have coordinates (x_1,y_1) and (x_2,y_2), respectively. Then the equation of the line AB is

$$(y_2 - y_1)(x - x_1) = (x_2 - x_1)(y - y_1),$$

and according to the lemma the midpoint of C_1C_2 has coordinates

$$\left[\frac{m}{2(y_2 - y_1)}, \frac{-m}{2(x_2 - x_1)}\right],$$

where

$$m = x_1y_2 - y_1x_2.$$

Since $AC \perp BH$, the equation of AC is

$$(x - x_1, y - y_1)\cdot(x_2, y_2) = 0,$$

or

$$x_2x + y_2y - n = 0$$

where

$$n = x_1x_2 + y_1y_2,$$

and according to the lemma the midpoint of the segment B_1B_2 has coordinates $(n/2x_2, n/2y_2)$. Similarly the midpoint of A_1A_2 has coordinates $(n/2x_1, n/2y_1)$.

Three points with coordinates (p_1,q_1), (p_2,q_2), (p_3,q_3) are collinear if and only if

$$(p_1 - p_3)(q_2 - q_3) = (q_1 - q_3)(p_2 - p_3).$$

Therefore it remains to prove that

$$\left[\frac{m}{2(y_2 - y_1)} - \frac{n}{2x_1}\right]\left[\frac{n}{2y_2} - \frac{n}{2y_1}\right] = \left[\frac{-m}{2(x_2 - x_1)} - \frac{n}{2y_1}\right]\left[\frac{n}{2x_2} - \frac{n}{2x_1}\right].$$

This equality is equivalent successively to

$$\left[\frac{m}{y_2 - y_1} - \frac{n}{x_1}\right]\left[\frac{y_2 - y_1}{y_1y_2}\right] = \left[\frac{-m}{x_2 - x_1} - \frac{n}{y_1}\right]\left[\frac{x_2 - x_1}{x_1x_2}\right],$$

$$\frac{m}{y_1y_2} - \frac{n(y_2 - y_1)}{x_1y_1y_2} = \frac{-m}{x_1x_2} - \frac{n(x_2 - x_1)}{x_1x_2y_1},$$

$$m(x_1x_2 + y_1y_2) = n[x_2(y_2 - y_1) - y_2(x_2 - x_1)],$$

and finally

$$mn = nm,$$

and consequently is true.

* * *

<u>1137</u>*. [1986: 79, 177 (revised); 1987: 228] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s}$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

II. *Solution (completed) by Wolfgang Gmeiner, Millstatt, Austria and the proposer.*

The "missing link" in our previous argument [1987: 228] was to prove that

$$(x + y + z)\left[\frac{1}{\sqrt{x^2 + xy + y^2}} + \frac{1}{\sqrt{x^2 + xz + z^2}} + \frac{1}{\sqrt{y^2 + yz + z^2}}\right] > 4 + \frac{2}{\sqrt{3}} \qquad (1)$$

for all x, y, $z > 0$ (see (4) on [1987: 229]). We now establish this inequality.

Set

$$w_1 = \sqrt{x^2 + xy + y^2}\ , \quad w_2 = \sqrt{y^2 + yz + z^2}\ , \quad w_3 = \sqrt{z^2 + zx + x^2}\ .$$

Since (1) is homogeneous, we may assume $x + y + z = 1$. The inequality to be proved then is

$$\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} > 4 + \frac{2}{\sqrt{3}}. \tag{2}$$

Thus we will discuss the function

$$F(x,y,z,\lambda) = \frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \lambda(x + y + z - 1)$$

by the method of Lagrange multipliers.

Necessary conditions for critical points with $0 < x,y,z < 1$ are

$$\frac{\partial F}{\partial x} = -\frac{1}{2}\left[\frac{2x + y}{w_1^3} + \frac{2x + z}{w_3^3}\right] + \lambda = 0, \tag{3}$$

$$\frac{\partial F}{\partial y} = -\frac{1}{2}\left[\frac{2y + z}{w_2^3} + \frac{2y + x}{w_1^3}\right] + \lambda = 0, \tag{4}$$

$$\frac{\partial F}{\partial z} = -\frac{1}{2}\left[\frac{2z + x}{w_3^3} + \frac{2z + y}{w_2^3}\right] + \lambda = 0. \tag{5}$$

Adding the equations (3), (4) and (5) we get

$$\lambda = \frac{1}{2}\left[\frac{x + y}{w_1^3} + \frac{y + z}{w_2^3} + \frac{z + x}{w_3^3}\right] .$$

Inserting this expression in (3) and using $y + z = 1 - x$, we obtain

$$x\left[\frac{1}{w_1^3} + \frac{1}{w_2^3} + \frac{1}{w_3^3}\right] = \frac{1}{w_2^3}. \tag{6}$$

Similarly, from (4) and (5) there follow

$$y\left[\frac{1}{w_1^3} + \frac{1}{w_2^3} + \frac{1}{w_3^3}\right] = \frac{1}{w_3^3}, \tag{7}$$

$$z\left[\frac{1}{w_1^3} + \frac{1}{w_2^3} + \frac{1}{w_3^3}\right] = \frac{1}{w_1^3}. \tag{8}$$

From (6), (7), (8) we get

$$xw_2^3 = yw_3^3 = zw_1^3\ ,$$

or squaring,

$$x^2(y^2 + yz + z^2)^3 = y^2(z^2 + zx + x^2)^3 = z^2(x^2 + xy + y^2)^3. \tag{9}$$

As inequality (2) is symmetric we may and do assume $0 < x \leq y \leq z < 1$. We put $y = ax$ and $z = bx$ where $1 \leq a \leq b$. Then the second equality in (9) becomes

$$a^2(b^2 + b + 1)^3 = b^2(a^2 + a + 1)^3. \tag{10}$$

By differentiation it is easily checked that the function

$$f(t) = \frac{(t^2+t+1)^3}{t^2}$$

strictly increases for $t \geq 1$. Thus (10) yields $a = b$, i.e. $y = z$, as necessary for critical points in the interior of the considered region. Inserting this in (2), we have to prove

$$\frac{2}{w_1} + \frac{1}{y\sqrt{3}} > 4 + \frac{2}{\sqrt{3}}$$

where $x = 1 - 2y$, i.e.

$$\frac{2}{\sqrt{3y^2 - 3y + 1}} + \frac{1}{y\sqrt{3}} > 4 + \frac{2}{\sqrt{3}}, \tag{11}$$

where $0 < y < 1/2$. The transformation

$$y = \frac{1}{2} - w$$

changes (11) to

$$\frac{2\sqrt{3}}{\sqrt{12w^2+1}} + \frac{1}{1-2w} > 2\sqrt{3} + 1, \tag{12}$$

that is, $\ell(w) > r(w)$ for $0 < w < 1/2$ where

$$\ell(w) = \frac{1}{1-2w}, \qquad r(w) = 2\sqrt{3} + 1 - \frac{2\sqrt{3}}{\sqrt{12w^2+1}}.$$

Now

$$\ell'(w) = \frac{2}{(1-2w)^2}, \qquad r'(w) = \frac{24\sqrt{3}w}{(12w^2+1)^{3/2}},$$

and we claim the following: there exists precisely one w_0, $0 < w_0 < 1/6$, such that

(i) $\ell'(w) > r'(w)$ for $0 < w < w_0$,

(ii) $r'(w) > \ell'(w)$ for $w_0 < w < 1/6$,

(iii) $\ell'(w) > r'(w)$ for $1/6 < w < 1/2$.

Indeed, $\ell'(w) > r'(w)$ is equivalent to

$$p(w) = (12w^2+1)^3 > 432w^2(1-2w)^4 = q(w).$$

The function $q(w)$ attains its maximum on the interval $(0,1/2)$ at $w = 1/6$, and $p(1/6) = q(1/6)$, and $p(w)$ is increasing. Hence (iii) follows. In order to show the existence of w_0 we consider the function

$$d(w) = \log p(w) - \log q(w), \qquad 0 < w < 1/6.$$

Then

$$d'(w) = \frac{2(24w^2 + 6w - 1)}{w(12w^2+1)(1-2w)}.$$

The only relevant solution of $d'(w) = 0$ is

$$w_1 = \frac{-3+\sqrt{33}}{24}.$$

It is easily checked that $d(w)$ attains its only minimum on the interval $(0,1/6)$ at $w = w_1$ and, furthermore, $d(w_1) < 0$, i.e. $p(w_1) < q(w_1)$. From $p(0) > q(0)$ and $p(1/6) = q(1/6)$ we thus conclude (i) and (ii).

The proof of (12), i.e. (11), is now finished as follows: (i) and $\ell(0) = r(0)$ yield $\ell(w) > r(w)$ for $0 < w < w_0$; (ii) and the easily checked $\ell(1/6) > r(1/6)$ imply $\ell(w) > r(w)$ for $w_0 < w < 1/6$ (for if there would exist a value $\hat{w}$, $w_0 < \hat{w} < 1/6$, such that $\ell(\hat{w}) = r(\hat{w})$ then (ii) would give $\ell(1/6) < r(1/6)$); and (iii) finally gives $\ell(w) > r(w)$ for $1/6 < w < 1/2$. Therefore $\ell(w) > r(w)$ for $0 < w < 1/2$ as claimed in (12).

For (2) to be proved we still have to consider the boundary of the region. Let e.g. $z = 0$. Then $y = 1 - x$ and (2) becomes

$$h(x) := \frac{1}{\sqrt{x^2 - x + 1}} + \frac{1}{1 - x} + \frac{1}{x} \geq 4 + \frac{2}{\sqrt{3}}.$$

By symmetry, we may and do restrict ourselves to the case $0 < x \leq 1/2$. Since $h(1/2) = 4 + 2/\sqrt{3}$ we have only to show that $h(x)$ decreases on $(0,1/2)$, i.e. that

$$h'(x) = \frac{1 - 2x}{2(x^2 - x + 1)^{3/2}} + \frac{1}{(1 - x)^2} - \frac{1}{x^2} < 0$$

on $(0,1/2)$. Equivalently, we have to show

$$x^2(1 - x)^2 < 2(x^2 - x + 1)^{3/2} \tag{13}$$

on $(0,1/2)$. If we let $\alpha = x(1 - x)$ and note that $0 < \alpha < 1/4$ then (13) follows from the easily verified inequality

$$\alpha^4 < 4(1 - \alpha)^3, \qquad 0 < \alpha < 1/4.$$

Remark. This proof (in a slightly modified form) is part of a paper "A pair of general triangle inequalities" by the two solvers and coming soon in the Swiss journal *Elemente der Mathematik.*

III. *Comments by the editor.*

LEN BOS, University of Calgary, contributes the following interpretation of inequality (2) above. Put

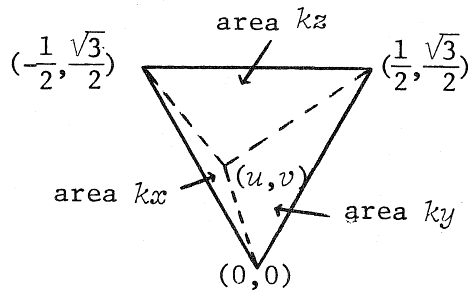
$$x = \frac{v}{\sqrt{3}} + u, \qquad y = \frac{v}{\sqrt{3}} - u, \qquad z = 1 - \frac{2v}{\sqrt{3}}. \tag{14}$$

Then (2) becomes

$$\frac{1}{\sqrt{u^2 + v^2}} + \frac{1}{\sqrt{(u + 1/2)^2 + (v - \sqrt{3}/2)^2}} + \frac{1}{\sqrt{(u - 1/2)^2 + (v - \sqrt{3}/2)^2}} > 4 + \frac{2}{\sqrt{3}}. \tag{15}$$

The left side of (15) is just the sum of the reciprocals of the distances of interior point (u,v) from the vertices $(0,0)$, $(\pm\, 1/2, \sqrt{3}/2)$ of an equilateral triangle. Equality occurs in (15) if (u,v) is the midpoint of any of the sides of this triangle. Thus (15) can be phrased as: *if three equal point charges are placed at the vertices of an equilateral triangle, then the midpoints of the sides have minimum potential among all points inside or on the triangle.* There is a drawing on page 54 of Sir James Jeans, *The Mathematical Theory of Electricity and Magnetism*, Cambridge University Press (5th ed.), 1960, showing equipotentials for such a situation, which suggests that one need only consider the centre and the midpoints of the sides to find the point of minimum potential. Does any reader have a more definite reference for this result?

Incidentally, REX WESTBROOK, University of Calgary, noticed that Bos's transformation (14) amounts to choosing (u,v) to be the point inside the triangle with normalized barycentric coordinates (x,y,z).



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1148. [1986: 108] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find the triangle of smallest area that has integral sides and integral altitudes.

Solution.

The smallest such triangle is the right triangle with sides 15, 20, 25, altitudes 12, 15, 20, and area 150.

Editor's comment.

This answer was given by several readers (see below). The proposer verified it by noting that any smaller triangle has altitudes at least 1 and thus sides at most 300, and then used a computer to show there are no smaller triangles with the required properties. Wilke sent in a fairly long proof containing one step the editor wasn't able to follow. The others more or less just gave the answer. Can someone find a nice number–theoretic argument, the less computer help the better?

Found by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Klamkin notes that the following related comment appears on page 200 of L.E. Dickson, History of the Theory of Numbers Vol II: "*Of several triangles with integral sides, area, and one altitude, the least appears to have the sides 4, 13, 14, area 24, and altitude (to side 4) 12.*" *He also mentions another apparently open problem: find the triangle of smallest area that has integral sides, area, circumradius, and inradius.*

* * *

1165*. [1986: 178; 1987: 306] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

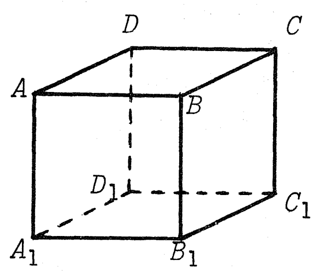
For fixed $n \geq 5$, consider an n–gon P imbedded in a unit cube.

(i) Determine the maximum perimeter of P if n is odd.

(ii) Determine the maximum perimeter of P if it is convex (which implies it is planar).

(iii) Determine the maximum volume of the convex hull of P if also $n < 8$.

Solution by Jordi Dou, Barcelona, Spain.



(i) Let d_3 denote a space diagonal (of length $\sqrt{3}$), d_2 denote a face diagonal (of length $\sqrt{2}$), and d_1 denote an edge (of length 1) of the cube. A polygon P_n of n sides (n odd) imbedded in the cube can have a maximum of $n-2$ sides d_3; this maximum is attained when

$$P_n = AC_1AC_1 \ldots C_1AD_1BD_1BD_1 \ldots D_1BA$$

for example, the remaining two sides being d_2 and d_1, for a perimeter of

$$(n-2)\sqrt{3} + \sqrt{2} + 1. \tag{1}$$

P_n could also have $n-3$ sides d_3 and the three remaining sides d_2; for example

$$P_n = AC_1AC_1 \ldots C_1AB_1CA,$$

for which the perimeter is

$$(n-3)\sqrt{3} + 3\sqrt{2}. \tag{2}$$

Since $2\sqrt{2} > \sqrt{3} + 1$, (2) is larger than (1), so the maximum perimeter is $(n-3)\sqrt{3} + 3\sqrt{2}$.

(ii) If P_n is convex, its perimeter will not exceed the perimeter of the intersection of the plane of P_n with the cube. The cross–section of maximum perimeter is formed by one of the six planes passing through a pair of opposite edges of the cube, the resulting rectangle having perimeter $2(1 + \sqrt{2})$. Thus for $n \geq 5$ there exists P_n whose perimeter p_n satisfies

$$p_n > 2(1 + \sqrt{2}) - \epsilon$$

where ϵ is an arbitrary positive number. The value $2(1 + \sqrt{2})$ is only attained for $n = 4$.

(iii) If P_n contains a vertex V of the cube, and all three vertices adjacent to X, then removing X from P_n will cause a decrease of $1/6$ in the volume of the convex hull of P_n. A polygon missing two adjacent vertices of the cube means a loss of volume in the convex hull of $1/2$. The required maximum volumes are given in the following table.

n	volume	P_n	missing vertices
4	2/6	AB_1CD_1A	B, D, A_1, C_1
5	3/6	$ACB_1D_1C_1A$	B, D, A_1
6	4/6	$ABCB_1C_1D_1A$	D, A_1
7	5/6	$ABCB_1C_1D_1A_1A$	D

* * *

1179. [1986: 206; 1988: 22] *Proposed by Jack Garfunkel, Flushing, New York.*

Squares are erected outwardly on each side of a quadrilateral $ABCD$.

(a) Prove that the centers of these squares are the vertices of a quadrilateral $A'B'C'D'$ whose diagonals are equal and perpendicular to each other.

(b)* If squares are likewise erected on the sides of $A'B'C'D'$, with centers A'', B'', C'', D'', and this procedure is continued, will quadrilateral $A^{(n)}B^{(n)}C^{(n)}D^{(n)}$ tend to a square as n tends to infinity?

II. *Editor's comment.*

In response to the editor's request for further references to part (a), Clayton W. Dodge kindly points out R.L. Finney's article "Dynamic proofs of Euclidean theorems", in *Mathematics Magazine* (1970) 177–185. Theorem 1 of that article is just part (a). Finney gives what appear to be the original references:

M.H. van Aubel, Note concernant les centres des carrés construits sur les côtés d'un polygone quelconque, *Nouv. Corresp. Math.* 4 (1878) 40–44.

C.–A. Laisant, Sur quelques propriétés des polygones, *Assoc. Franc. Avanc. Sci. Le Havre* (1877) 142–154.

Finney's article inspired Exercise 20.22 on page 93 of Dodge's *Euclidean Geometry and Transformations*, Addison–Wesley, 1972.

The editor has since located yet another occurrence of part (a), this time in that most eminent of references, *Crux Mathematicorum* (*Eureka* as it was then known). See Problem 37 [1975: 62].

Incidentally, both spellings "von Aubel" and "van Aubel" have been used in print: "von" in Kelly's paper referred to on [1988: 24], "van" in Finney's paper and in Léo's remark on [1975: 62]. Surely some reader can set us all straight on which spelling is correct.

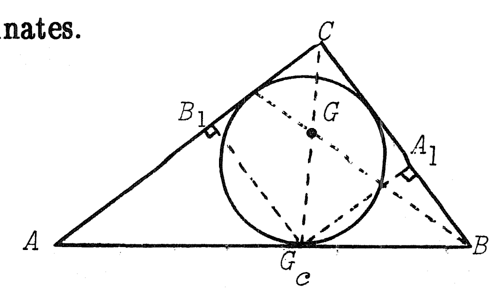
*   *   *

1198. [1986: 283] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with incenter I, Gergonne point G, and Nagel point N, and let J be the isotomic conjugate of I. Prove that G, N, and J are collinear.

Solution by the proposer.

We use normal homogeneous triangular coordinates.



Let G_c be the intersection of CG with AB, and B_1, A_1 the feet of the perpendiculars from G_c to AC, BC respectively. Since

$$G_cB = s - b,$$

where s is the semiperimeter, we have

$$G_cA_1 = (s - b)\sin B$$

and similarly

$$G_cB_1 = (s - a)\sin A.$$

Thus

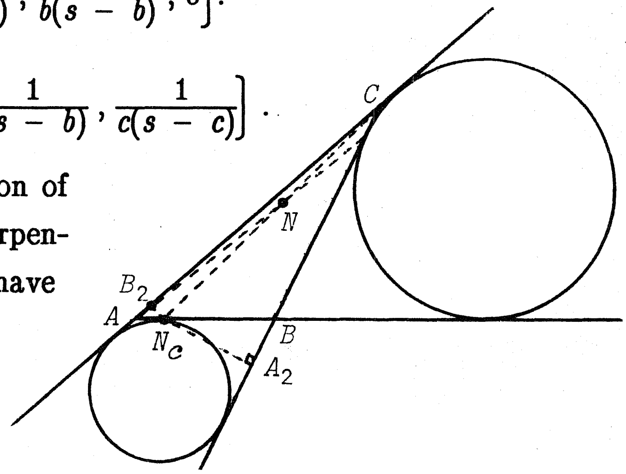
$$G_c = ((s - b)b\ ,\ (s - a)a\ ,\ 0),$$

or

$$G_c = \left[\frac{1}{a(s-a)}, \frac{1}{b(s-b)}, 0\right].$$

By symmetry,

$$G = \left[\frac{1}{a(s-a)}, \frac{1}{b(s-b)}, \frac{1}{c(s-c)}\right].$$



Similarly, letting N_c be the intersection of CN with AB, and B_2, A_2 the feet of the perpendiculars from N_c to AC, BC respectively, we have

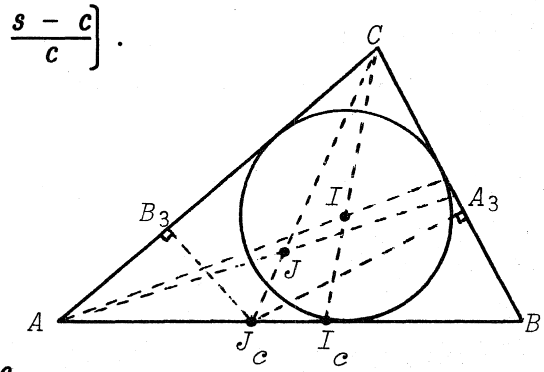
$$N_cB = s - a,$$
$$N_cA_2 = (s - a)\sin B,$$
$$N_cB_2 = (s - b)\sin A,$$

so

$$N_c = \left[\frac{s-a}{a}, \frac{s-b}{b}, 0\right]$$

and thus

$$N = \left[\frac{s-a}{a}, \frac{s-b}{b}, \frac{s-c}{c}\right].$$



Finally, let J_c, I_c be the intersections of AB with CJ, CI respectively, and B_3, A_3 the feet of the perpendiculars from J_c to AC, BC respectively. Then

$$BI_c{:}AI_c = a{:}b$$

so

$$AJ_c = BI_c = \frac{ac}{a+b},$$
$$BJ_c = AI_c = \frac{bc}{a+b}.$$

Thus

$$J_cA_3 = \frac{bc}{a+b}\sin B,$$
$$J_cB_3 = \frac{ac}{a+b}\sin A,$$

so

$$J_cA_3{:}J_cB_3 = b^2{:}a^2 = \frac{1}{a^2} : \frac{1}{b^2}.$$

Hence

$$J = \left[\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}\right].$$

Now

$$\begin{vmatrix} \frac{1}{a(s-a)} & \frac{1}{b(s-b)} & \frac{1}{c(s-c)} \\ \frac{s-a}{a} & \frac{s-b}{b} & \frac{s-c}{c} \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} \frac{1}{s-a} & \frac{1}{s-b} & \frac{1}{s-c} \\ s-a & s-b & s-c \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2(s-a)(s-b)(s-c)}\begin{vmatrix} (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \\ s-a & s-b & s-c \\ bc & ca & ab \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2(s-a)(s-b)(s-c)}\begin{vmatrix} s^2-bs-cs & s^2-cs-as & s^2-as-bs \\ s-a & s-b & s-c \\ bc & ca & ab \end{vmatrix}$$

$$= \frac{s}{a^2b^2c^2(s-a)(s-b)(s-c)}\begin{vmatrix} s-b-c & s-c-a & s-a-b \\ s-a & s-b & s-c \\ bc & ca & ab \end{vmatrix}$$

$$= \frac{s}{a^2b^2c^2(s-a)(s-b)(s-c)}\begin{vmatrix} 2s-a-b-c & 2s-a-b-c & 2s-a-b-c \\ s-a & s-b & s-c \\ bc & ca & ab \end{vmatrix}$$

$$= \frac{s}{a^2b^2c^2(s-a)(s-b)(s-c)}\begin{vmatrix} 0 & 0 & 0 \\ s-a & s-b & s-c \\ bc & ca & ab \end{vmatrix} = 0,$$

so G, N, J are collinear.

Also solved by D.J. Smeenk, Zaltbommel, The Netherlands.

* * *

<u>1199</u>*. [1986: 283] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

Prove that for acute triangles,

$$s^2 \le \frac{27R^2}{27R^2 - 8r^2}(2R+r)^2,$$

where s, r, R are the semiperimeter, inradius, and circumradius, respectively.

Solution by Jack Garfunkel, Flushing, N.Y.

This problem and *Crux* 1218 (also due to Mitrinovic and Pecaric) are the same; we prove both. *Crux* 1218 says: prove that

$$F_1 \le \frac{4F^3}{27R^4} \tag{1}$$

where F_1 is the area of the orthic triangle of an acute triangle of area F and circumradius R. The following can be found on p.191 of R.A. Johnson, *Advanced Euclidean Geometry*:

$$\frac{F_1}{F} = \frac{r_1}{R} \quad \text{and} \quad r_1 = 2R\prod \cos A$$

where r_1 is the inradius of the orthic triangle and the product is cyclic over the angles A, B, C of the acute triangle. Using the above, (1) becomes

$$\prod \cos A \le \frac{2F^2}{27R^4}$$

which, using

$$F = 2R^2\prod \sin A,$$

can be written

$$\prod \cos A \le \frac{8}{27}\prod \sin^2 A, \tag{2}$$

or

$$\prod \cot A \csc A \le \frac{8}{27}\,. \tag{3}$$

Since we are dealing with an acute triangle, the angles can be renamed:

$$A \to 90^\circ - \frac{A}{2}, \quad B \to 90^\circ - \frac{B}{2}, \quad C \to 90^\circ - \frac{C}{2}.$$

Then (3) becomes

$$\prod \tan\frac{A}{2} \sec\frac{A}{2} \le \frac{8}{27}\,.$$

Now

$$\prod \tan\left(\frac{A}{2}\right) = \frac{r}{s}, \quad \prod \sec\frac{A}{2} = \frac{4R}{s},$$

so we have to show that $2s^2 \ge 27Rr$, which is a known inequality (item 5.12 of Bottema, *Geometric Inequalities*). This proves *Crux* 1218.

To show *Crux* 1218 and 1199 are the same, use the identities

$$\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}, \quad \prod \sin A = \frac{rs}{2R^2}$$

in (2). With a little algebra we get *Crux* 1199.

Also solved by S.J. BILCHEV, Technical University and EMILIA VELIKOVA, Mathematikalgymnasium, Russe, Bulgaria; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and MURRAY S. KLAMKIN, University of Alberta.

Bilchev and Velikova, Janous, and Klamkin also noted the equivalence of problems 1199 *and* 1218.

*   *   *

<u>1200</u>. [1986: 283] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

In a certain game, the first player secretly chooses an n–dimensional vector $\mathbf{a} = (a_1,a_2,\ldots,a_n)$ all of whose components are integers. The second player is to determine $\mathbf{a}$ by choosing any n–dimensional vectors $\mathbf{x}_i$, all of whose components are also integers. For each $\mathbf{x}_i$ chosen, and before the next $\mathbf{x}_i$ is chosen, the first player tells the second player the value of the dot product $\mathbf{x}_i \cdot \mathbf{a}$. What is the least number of vectors $\mathbf{x}_i$ the second player has to choose in order to be able to determine $\mathbf{a}$? [Warning: this is somewhat "tricky"!]

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Two $\mathbf{x}_i$ are sufficient. Pick $\mathbf{x}_1 = \mathbf{a}$ and you learn what $\mathbf{a}\cdot\mathbf{a}$ is. Then pick

$$\mathbf{x}_2 = (1,M,M^2,M^3,\ldots,M^n)$$

where $M > 2\sqrt{\mathbf{a}\cdot\mathbf{a}} = 2Q$. Then

$$\mathbf{a}\cdot\mathbf{x}_2 = a_1 + a_2M + \cdots + a_nM^n.$$

The a_i can be determined by constructing

$$\begin{aligned} N &= \mathbf{a}\cdot\mathbf{x}_2 + Q(1 + M + \cdots + M^n) \\ &= (a_1 + Q) + (a_2 + Q)M + \cdots + (a_n + Q)M^n. \end{aligned}$$

Writing N in base M produces the ith digit $a_i + Q$ from which we subtract Q to get a_i.

Comments:

(1) If the components of $\mathbf{x}_i$ are not restricted to integers then we can do it with only one guess, such as

$$\mathbf{x}_1 = (1,e,e^2,\ldots,e^n).$$

Being given $\mathbf{a}\cdot\mathbf{x}_1$ to arbitrary accuracy allows us to test all possible $\mathbf{a}$ until a perfect match is found.

(2) If the components of $\mathbf{a}$ are known to be positive integers then the trick of making $\mathbf{x}_1 = \mathbf{a}$ is not needed. We can choose $\mathbf{x}_1 = (1,1,\ldots,1)$ and $Q = \mathbf{x}_1\cdot\mathbf{a}$ above.

(3) If the actual components of $\mathbf{x}_i$ must be known by the second player then I suspect it will take him n guesses, for example

$$\mathbf{x}_1 = (1,0,\ldots,0), \qquad \mathbf{x}_2 = (0,1,0,\ldots,0), \qquad \text{etc.}$$

Can anyone do better?

Also solved by the proposer, who believes that the "nontricky" version of this problem, when $\mathbf{a}$ *has positive integral components* (*as in Hess's comment* 2), *occurred as a U.S.S.R. Olympiad problem.*

The conjecture in (3) *above was also surmised by the proposer.*

* * *

1201*. [1987: 14] *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

Prove that

$$(x+y+z)\left[\frac{xc^2}{a^2}+\frac{ya^2}{b^2}+\frac{zb^2}{c^2}\right] \geq \left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right](a^2yz+b^2zx+c^2xy),$$

where a, b, c are the sides of a triangle and x, y, z are real numbers.

Solution by Murray S. Klamkin, University of Alberta.

The given inequality is just a disguised version of the known inequality [1]

$$x^2+y^2+z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C, \tag{1}$$

where A, B, C are the angles of the triangle. To motivate the transformation of the given inequality to (1), we first get rid of the denominators on the left hand side by letting

$$x = a^2x', \qquad y = b^2y', \qquad z = c^2z'.$$

This gives, on dropping the primes and collecting like terms, that

$$\begin{aligned} a^2c^2x^2 + b^2a^2y^2 + c^2b^2z^2 &\geq a^2(b^2+c^2-a^2)xy + b^2(c^2+a^2-b^2)yz + c^2(a^2+b^2-c^2)zx \\ &= 2abc(axy\cos A + byz\cos B + czx\cos C), \end{aligned}$$

since $2bc\cos A = b^2+c^2-a^2$, etc. Then by letting

$$x = by', \qquad y = cz', \qquad z = ax'$$

and dropping the primes, we get (1).

The more general inequality

$$x^2+y^2+z^2 \geq (-1)^{n+1}(2yz\cos nA + 2zx\cos nB + 2xy\cos nC) \tag{2}$$

is derived in [1] by expressing it as

$$[x+(-1)^n(y\cos nC + z\cos nB)]^2 + [y\sin nC - z\sin nB]^2 \geq 0.$$

There is equality if and only if

$$\frac{x}{\sin nA} = \frac{y}{\sin nB} = \frac{z}{\sin nC}.$$

Similarly, the proposers' inequality (a) in *Crux* 1181 [1988: 25] can be thought of as a disguised form of (2) for the case $n = 2$.

Reference:

[1] M.S. Klamkin, Asymmetric triangle inequalities, *Publ. Electrotechn. Fak. Ser. Univ. Beograd*, No.357–380 (1971) 33–44.

Also solved by S.J. BILCHEV, Technical University and EMILIA VELIKOVA, Mathematikalgymnasium, Russe, Bulgaria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

* * *

1202. [1987: 14] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let M_0, M_1 be lattice points and let M be a point such that M_0M_1M is an equilateral triangle. Let (a,b) be the coordinates of M reduced modulo 1. Prove that the set of all such pairs (a,b) is dense in the unit square $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$, where M_0, M_1 vary over all lattice points.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

It will be enough to consider the case

$$M_0 = (0,0), \qquad M_1 = (2s,2t), \qquad M = (s - t\sqrt{3}, t + s\sqrt{3})$$

where s, t are integers. Then we get for M the reduced coordinates

$$(a_t,b_s) = (\{-t\sqrt{3}\},\{s\sqrt{3}\}),$$

where $\{z\} = z - [z]$ is the "fractional part" of z. It is well–known that the sequence $(n\sqrt{3}, n \in \mathbb{N})$ is uniformly distributed modulo 1 (see e.g. Exercise 2.5 page 23 of [1]). Hence each of the sets $\{a_t : t \in \mathbb{N}\}$ and $\{b_s : s \in \mathbb{N}\}$ is dense in $[0,1]$. Thus the set of all pairs (a_t,b_s), $s,t \in \mathbb{N}$, is dense in the unit square (keeping in mind that s and t are allowed to vary independently).

Reference:

[1] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley, New York, 1974.

Also solved by the proposer.

* * *

1203. [1987: 14] *Proposed by Milen N. Naydenov, Varna, Bulgaria.*

A quadrilateral inscribed in a circle of radius R and circumscribed around a circle of radius r has consecutive sides a, b, c, d, semiperimeter s and area F. Prove that

(a) $2\sqrt{F} \le s \le r + \sqrt{r^2 + 4R^2}$;

(b) $6F \le ab + ac + ad + bc + bd + cd \le 4r^2 + 4R^2 + 4r\sqrt{r^2 + 4R^2}$;

(c) $2sr^2 \le abc + abd + acd + bcd \le 2r[r + \sqrt{r^2 + 4R^2}]^2$;

(d) $4Fr^2 \le abcd \le \frac{16}{9}r^2(r^2 + 4R^2)$.

Solution by Murray S. Klamkin, University of Alberta.

By Poncelet's Porism ([1], page 95) it follows that there are infinitely many such quadrilaterals and any point of the outer circle may be taken as a vertex. In view of the hypotheses, we have the known relations [2]

$$a + c = b + d = s$$
$$F = \sqrt{abcd} = rs,$$
$$ac + bd = ef,$$

where e, f are the diagonals.

(a) The left hand inequality is just the AM–GM inequality

$$\sqrt[4]{abcd} \le \frac{a + b + c + d}{4}.$$

The right hand inequality is known and appears in the solution of problem 488, pp.465–466 of *Nieuw Archief voor Wiskunde* XXVI (1978), by its proposers W.J. Blundon and R.H. Eddy. Briefly their proof is as follows. Let two consecutive angles of the quadrilateral be α and β. Then it follows that

$$\csc \alpha + \csc \beta = \frac{s}{2r} \tag{1}$$

and

$$4r^2(1 + \csc \alpha \csc \beta) = ac + bd = ef = 4R^2 \sin \alpha \sin \beta. \tag{2}$$

Hence

$$2 \csc \alpha \csc \beta = \frac{\sqrt{r^2 + 4R^2} - r}{r}. \tag{3}$$

Since

$$(\csc \alpha - 1)(\csc \beta - 1) \ge 0$$

with equality if and only if at least one of α, β is a right angle, (1) and (3) give

$$s \le r + \sqrt{r^2 + 4R^2}, \tag{4}$$

with equality if and only if at least one pair of opposite angles of the quadrilateral are right angles. It also follows from (2) and (3) that the product of the diagonals is

$$ef = 2r(r + \sqrt{r^2 + 4R^2}) \tag{5}$$

and so is the same for all possible quadrilaterals.

(b) The left hand inequality, as in (a), is an immediate consequence of the AM–GM inequality

$$\frac{1}{6}\sum ab \ge \sqrt{abcd},$$

where the sum is over all pairs from a, b, c, d. The right hand inequality follows from (4) and (5):

$$\begin{aligned}\sum ab &= (a + c)(b + d) + ac + bd \\ &= s^2 + ef \\ &\le (r + \sqrt{r^2 + 4R^2})^2 + 2r(r + \sqrt{r^2 + 4R^2}) \\ &= 4r^2 + 4R^2 + 4r\sqrt{r^2 + 4R^2}.\end{aligned}$$

(c) The left hand inequality can be strengthened to

$$8sr^2 \le \sum abc.$$

Again by the AM–GM inequality,

$$\sum abc \ge 4(abcd)^{3/4} = 4(rs)^{3/2}.$$

Since the least perimeter quadrilateral circumscribing a given circle is the square, we have $s \ge 4r$. Thus

$$\sum abc \geq 4(rs)^{3/2} \geq 8sr^2.$$

For the right hand inequality,

$$\sum abc = ac(b + d) + bd(a + c) = efs \leq 2r(r + \sqrt{r^2 + 4R^2})^2$$

from (4) and (5) again.

(d) The left hand inequality is equivalent to $s \geq 4r$ above. For the right hand inequality, we have

$$abcd = r^2s^2 \leq r^2(r + \sqrt{r^2 + 4R^2})^2,$$

so it is enough to prove

$$r + \sqrt{r^2 + 4R^2} \leq \frac{4}{3}\sqrt{r^2 + 4R^2}$$

or

$$9r^2 \leq r^2 + 4R^2$$

or

$$2r^2 \leq R^2,$$

which is known (#15.15 of [3]).

References:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960.

[2] C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell, London, 1953.

[3] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1968.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a) and half of (b)); and the proposer.

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<u>1204</u>. [1987: 15] *Proposed by Thomas E. Moore, Bridgewater State College, Bridgewater, Massachusetts.*

(a) Show that if n is an even perfect number, then $n - \phi(n)$ is a square (of an integer), where $\phi(n)$ is Euler's totient function.

(b) Find infinitely many n such that $n - \phi(n)$ is a square.

I. *Solution by Bob Prielipp, University of Wisconsin–Oshkosh.*

(a) If n is an even perfect number,

$$n = 2^{p-1}(2^p - 1)$$

where $2^p - 1$ is a Mersenne prime (and hence p is a prime number). Clearly $(2^{p-1}, 2^p - 1) = 1$ so

$$\phi(n) = \phi(2^{p-1})\phi(2^p - 1) = 2^{p-2}(2^p - 2) = 2^{p-1}(2^{p-1} - 1).$$

Thus

$$\begin{aligned} n - \phi(n) &= 2^{p-1}(2^p - 1) - 2^{p-1}(2^{p-1} - 1) \\ &= 2^{p-1}(2^p - 2^{p-1}) = (2^{p-1})^2. \end{aligned}$$

(b) Let p be an arbitrary prime number, and let $n = p^{2k-1}$ where k is an arbitrary positive integer. Then

$$n - \phi(n) = p^{2k-1} - p^{2k-2}(p-1) = p^{2k-2} = (p^{k-1})^2.$$

II. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

[Kierstead first proved part (a) as above.]

(b) Let n be of the form

$$n = p^{2a+1}q^{2b+1}$$

where p and q are primes. Then

$$\phi(n) = p^{2a}q^{2b}(p-1)(q-1)$$

and

$$n - \phi(n) = p^{2a}q^{2b}(p+q-1).$$

Now if $p + q - 1 = k^2$, we obtain

$$n - \phi(n) = (p^a q^b k)^2.$$

It remains only to find values of (p,q) such that $p + q - 1$ is a square. There are many such pairs, such as (2,3), (3,7), (3,23), etc.

III. *Comments on part (b) by Richard K. Guy, University of Calgary.*

(i) Take any solution n, and increase the exponent of any prime factor by two; the result is another solution.

(ii) If n is a solution of the form $3p$ where p is a prime, then $n - \phi(n) = p + 2$ is a square, so p is a prime of the form $k^2 - 2$. There are certainly infinitely many such primes, but no one can prove it!

Also solved by FRANK P. BATTLES and LAURA L. KELLEHER, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; DUANE M. BROLINE, Eastern Illinois University, Charleston, Illinois; HUGH M. EDGAR, San Jose State University, San Jose, California; HERTA T. FREITAG, Roanoke, Virginia; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; SIDNEY KRAVITZ, Dover, New Jersey; KEE–WAI LAU, Hong Kong; ROGER LEE, student, White Plains H.S., White Plains, N.Y.; M.M. PARMENTER, Memorial University of Newfoundland, St. John's; KENNETH M. WILKE, Topeka, Kansas; JURGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer.

All solvers, of course, solved part (a) the same way. Part (b) was solved by the above methods or special cases of them. To the great disappointment of the editor, nobody solved part (b) the obvious way, by showing there are infinitely many even perfect numbers.

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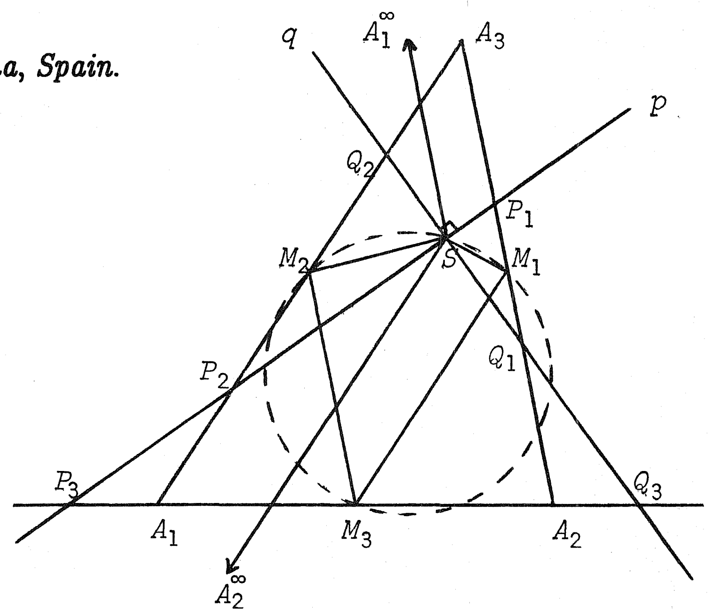
1205. [1987: 15] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let triangle $A_1A_2A_3$ have sides a_1, a_2, a_3 with respective midpoints M_1, M_2, M_3. Let lines p, q, with intersections with a_i or its extension denoted P_i, Q_i respectively, have the properties that M_i is the midpoint of P_iQ_i for each i and that $p \perp q$. Find the locus of the intersection point S of p and q.

Solution by Jordi Dou, Barcelona, Spain.



Denote by A_i^∞ the point at infinity of the line a_i. Since $\angle P_1SQ_1 = 90^\circ$, p and q are the bisectors of $\angle M_1SA_1^\infty$ and of $\angle M_2SA_2^\infty$. Thus the angles M_1SM_2 and $A_1^\infty SA_2^\infty$ are equal. Since

$$\angle M_1SM_2 = \angle A_1^\infty SA_2^\infty = 180^\circ - \angle A_3$$
$$= 180^\circ - \angle M_1M_3M_2,$$

the points M_1, M_2, M_3, S are concyclic. Therefore the locus of S is the 9–point circle of the triangle.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; and the proposer. Klamkin only gave the equation of the locus in trilinear coordinates.

* * *

1206. [1987: 15] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let X be a point inside triangle ABC, let Y be the isogonal conjugate of X and let I be the incenter of ΔABC. Prove that X, Y, and I colline if and only if X lies on one of the angle bisectors of ΔABC.

Solution by J.T. Groenman, Arnhem, The Netherlands.

We use normal triangular homogeneous coordinates. Let

$$X = (x_1,x_2,x_3), \qquad Y = \left[\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3}\right], \qquad I = (1,1,1).$$

They are collinear if and only if

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \frac{x_2}{x_3} - \frac{x_3}{x_2} - \frac{x_1}{x_3} + \frac{x_3}{x_1} + \frac{x_1}{x_2} - \frac{x_2}{x_1} = 0$$

$$\Leftrightarrow \quad x_1x_2^3 - x_1x_3^3 - x_1^3x_2 + x_2x_3^3 + x_1^3x_3 - x_2^3x_3 = 0$$

$$\Leftrightarrow \quad (x_1 - x_2)(x_3 - x_1)(x_2 - x_3) = 0$$

$$\Leftrightarrow \quad x_1 = x_2 \quad \text{or} \quad x_3 = x_1 \quad \text{or} \quad x_2 = x_3$$

$\Leftrightarrow$ X lies on one of the angle bisectors.

Also solved by JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; DAN PEDOE, Minneapolis, Minnesota; RENE SCHIPPERUS, student, University of Calgary; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * *

WORDS OF MILD ALARM FROM THE EDITOR

Despite its evident popularity and a recent modest increase in subscription rates, *Crux Mathematicorum* is currently losing money. This situation cannot continue indefinitely because our publisher, the Canadian Mathematical Society, requires that all of its publications break even. I urge you to help us find more subscribers. High school teachers can join the Canadian Mathematical Society for \$20, receive the *Notes of the CMS* as a privilege of membership, and then subscribe to *Crux* for just another \$15. A subscription without CMS membership costs \$30, surely a remarkably low price for a publication of such wide interest. In the back of this and future issues we will include a form, which can be conveniently removed and duplicated if necessary, and which your friends and colleagues can use to obtain a subscription to *Crux* with or without CMS membership. Please do what you can to ensure the future of *Crux Mathematicorum*.

And while I have your attention, remember that *Crux* is always, and especially lately, in need of nice accessible problems, especially in areas such as number theory, combinatorics, or high school mathematics. *Crux* has a small, but dependable, group of reader-proposers whose membership can be determined by anyone who regularly scans the problem section, and whom the editor excepts with gratitude from the next sentence. All other readers of *Crux* should resolve to contribute at least one problem sometime this year.

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