MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH.

Mathematical Spectrum Award for Volume 15

We remind our readers that each year two prizes are available to contributors who are still at school or are students in colleges or universities. A prize of £20 is for an article published in the magazine and another of £10 is for a letter or the solution of a problem.

There were no articles in Volume 15 by authors eligible for the £20 prize. The £10 prize has been awarded to M. G. Sykes for his contributions to the Problems and Solutions section. We look forward to further contributions from readers.

The Proportions of Quadratic Equations with Real and Non-Real Roots

JOHN HEY, *Bob Jones University, South Carolina*

John Hey is an undergraduate whose main subject is chemistry. He wrote this article after attending a course on mathematical statistics.

The question may be asked: what is the probability that a random quadratic equation has real roots? Several computer-generated random samples have indicated that the ratio τ of quadratic equations with real roots to those with non-real roots is very close to the golden ratio $(1+\sqrt{5})/2$, which is approximately 1·6180. However, no specific expression for τ appears to be known. In this article, we shall obtain such an expression.

Consider the quadratic equation

$$ax^2 + bx + c = 0$$

with a, b, c real numbers and $a \neq 0$. We first identify this quadratic equation with the point (a, c, b) in three-dimensional space relative to rectangular coordinate axes. The quadratic equation has real roots if and only if $b^2 - 4ac \geq 0$. The equation $b^2 - 4ac = 0$ describes a surface in the three-dimensional space; Figure 1 illustrates this surface in the first octant bounded by the values $a = m$, $b = m$, $c = m$, where m is some fixed positive real number. We shall comment on the choice of m later. We note that the section of this surface in the plane $a = m$ is a parabola, as is that in the plane $c = m$, whereas its section in the plane $b = m$ is a rectangular hyperbola. Moreover the surface is symmetrical about the ac-plane and also about the plane $a = -c$. It lies in four of the eight quadrants, and the parts of the surface in each of these four quadrants are similar.

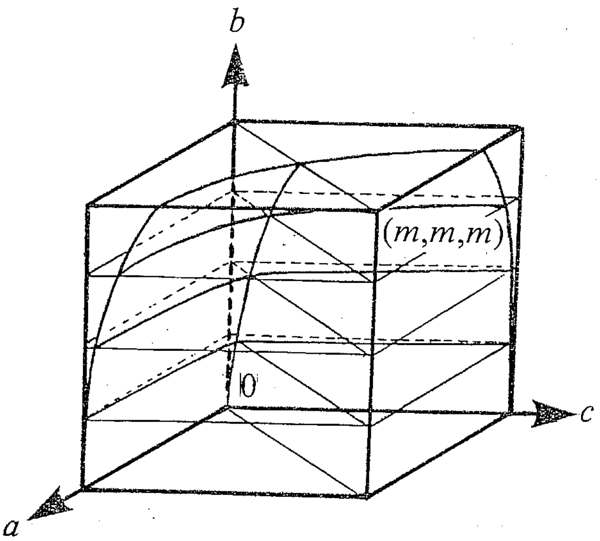


Figure 1. First octant of the region defined by $b^2 = 4ac$, $a, b, c = \pm m$.

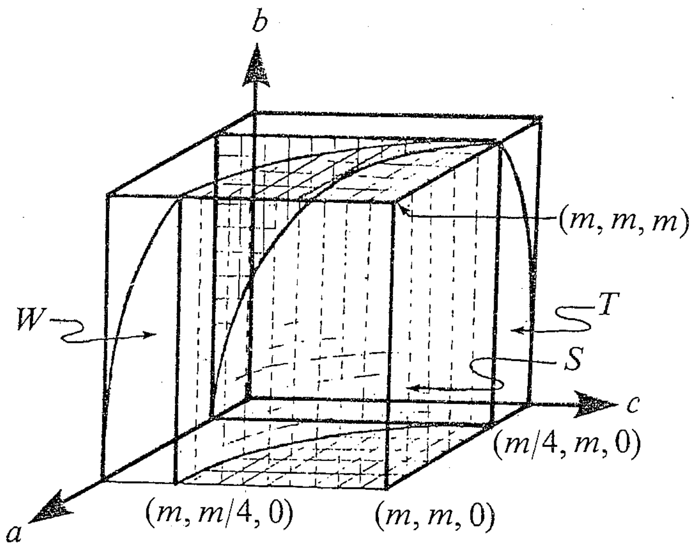


Figure 2. Subregions S, T, and W.

Consider for the moment those quadratic equations $ax^2 + bx + c = 0$ for which $-m \leq a, c, b \leq m$. These are given by the points in the cube

$$U_m = \{(a, c, b) | -m \leq a, c, b \leq m\}$$

with volume $8m^3$. The surface $b^2 = 4ac$ divides U_m into two regions R_m, C_m. The points in R_m are those for which $b^2 \geq 4ac$. These lie on the same side of the surface as the positive b-axis or on the surface itself, and represent the equations with real roots. The points in C_m are those for which $b^2 < 4ac$. These lie on the opposite side of the surface to the positive b-axis, and represent the equations with non-real roots. If we assume that a, b and c are uniformly and independently distributed in the ranges $-m \leq a, c, b \leq m$, so that the selection of a random quadratic equation is equivalent to selecting a random point in the three-dimensional space, where

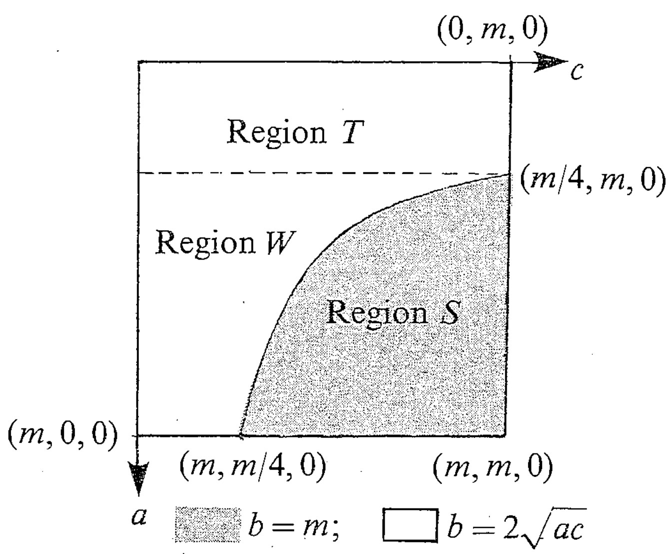


Figure 3. Projection of Figure 2 on the a–c plane.

$-m \leq a, c, b \leq m$, then the proportions of equations with real roots and with non-real roots will be obtained from the volumes of the regions R_m and C_m. To work out the volume of C_m, we divide the region into three parts S, T, W as shown in Figures 2 and 3. Now

$$\text{Volume } S = \int_0^m db \int_{m/4}^m da \int_{m^2/4a}^m dc = \int_0^m db \int_{m/4}^m \left(m - \frac{m^2}{4a}\right) da = \frac{3m^3}{4} - \frac{m^3}{2}\ln 2,$$

$$\text{Volume } T = \int_0^m dc \int_0^{m/4} da \int_0^{2\sqrt{(ac)}} db$$

$$= 2\int_0^m \sqrt{c}dc \int_0^{m/4} \sqrt{a}da = 2\left(\frac{2}{3}m^{3/2}\right)\left[\frac{2}{3}\left(\frac{m}{4}\right)^{3/2}\right] = \frac{m^3}{9},$$

$$\text{Volume } W = \int_{m/4}^m da \int_0^{m^2/4a} dc \int_0^{2\sqrt{(ac)}} db$$

$$= 2\int_{m/4}^m \sqrt{a}da \int_0^{m^2/4a} \sqrt{c}dc = \frac{4}{3}\int_{m/4}^m \sqrt{a}\left(\frac{m^2}{4a}\right)^{3/2} da$$

$$= \frac{m^3}{6}\int_{m/4}^m \frac{1}{a}\,da = \frac{m^3}{3}\ln 2.$$

Thus the volume of C_m is

$$\frac{(31 - 6\ln 2)m^3}{36}.$$

If we compare this to the volume of U_m, we see that the probability that this quadratic equation has non-real roots is given by

$$P(C_m) = \left(\frac{4(31 - 6\ln 2)m^3}{36}\right) \Big/ 8m^3 = \frac{31 - 6\ln 2}{72},$$

or approximately 0·3728. Thus the probability of real roots is given by

$$P(R_m) = 1 - P(C_m) = \frac{41 + 6\ln 2}{72},$$

or approximately 0·6272.

The proportion of these quadratic equations with real roots compared to those with non-real roots is given by

$$\tau_m = \frac{P(R_m)}{P(C_m)} = \frac{41 + 6\ln 2}{31 - 6\ln 2},$$

or approximately 1·6825. Note that this is independent of m, which justifies our original restriction to those quadratic equations $ax^2 + bx + c = 0$ for which $-m \leq a, c, b \leq m$, and indicates that our result is quite general, irrespective of the value of m.

A computer subroutine was written to test $P(C_m)$. This routine selected, from a random number table, integer values for a, b, c, and tested to determine whether the roots of the quadratic equation were real or non-real. The table gives the results for various values of m, using 1000 random trials. In general, the values were close to the calculated value.

m	$P(R_m)$
100	0·658
1000	0·635
10000	0·655
100000	0·629

The fact that these probabilities are independent of m is surprising. Simple analysis, much like that presented in this article, shows that the family of quadratic equations of the form $x^2 + px + q = 0$ has a very different probability structure. In that case, $P(C_m)$ and $P(R_m)$ depend on m and $P(C_m) \to 0$ as $m \to \infty$. Readers may like to carry out the analysis for themselves; it will involve only working out various areas in the plane, so only single integrals are involved. Thus, it is not possible to simplify the analysis by dividing $ax^2 + bx + c = 0$ by a to give an equation of the form $x^2 + px + q = 0$, since $p = b/a$ will not be uniformly distributed, as are a, b, c.

This analysis could also be extended to cover cubic equations. These have been studied by computational methods, but it would be interesting to find an analytic value for τ_m for the cubic case.

Acknowledgements

Special thanks are due to Dr Gary Guthrie for bringing this problem to the author's attention, and to Rick Jones for his valuable suggestions. Thanks are also due to Paul Hanna for his help with the graphics.

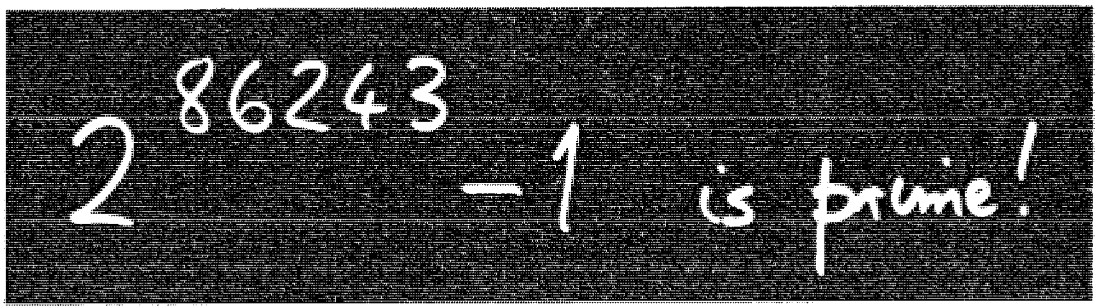
Drilling a Cube

In an article in Volume 16 Number 1, we printed J. Jabłkowski's solution to a problem posed by R. D. Kitchen in a letter published in 1973. The problem was to find 'in three or four lines without calculus' the volume of the residue when a 2-inch cube is drilled along its three axes by a drill of radius 1 inch. We have received further solutions from Basil Rennie, Bruce Andrews and E. H. Lockwood. Unfortunately, since none of these is a 'three- or four-line solution', shortage of space prevents our printing these nevertheless attractive solutions.

The Biggest Prime Number in the World

KEITH DEVLIN, *University of Lancaster*

The author took his B.Sc. at King's College, London, in 1968 and his Ph.D. at the University of Bristol in 1972. He has held positions at the Universities of Aberdeen, Manchester, Heidelberg, Bonn and Toronto, and is currently Reader in Mathematics at the University of Lancaster. His main mathematical interest is set theory and he has written over thirty research papers and five textbooks, including *Sets, Functions and Logic* (Chapman and Hall) for beginning university students. More recently he has been interested in the development of fast numerical algorithms for computer usage. In 1982 he gave the first of the London Mathematical Society Popular Lectures. This article first appeared in the *Guardian* on 12 May 1983.



A prime number is any whole number which can only be divided (without recourse to remainders or fractions) by the numbers 1 and itself. For example, 2, 3, 5, 7, 11 are all primes. Although the prime numbers have been studied by mathematicians (both professional and recreational) since ancient times it is only over the last few years that interest has been shown from other quarters.

Recent developments in cryptology (the science of making and breaking secret codes) have tended to involve more and more aspects of the branch of mathematics known as number theory, and in particular the properties of prime numbers. Not surprisingly therefore, large communications and data organisations such as IBM and the Bell Telephone Company now provide extensive funding for research into prime numbers, and it is widely believed (though not, of course, acknowledged) that less academic agencies such as the CIA are also highly involved in such matters. So it is unlikely that only a handful of ivory-towered mathematicians will show interest in the recent announcement that a new prime has been discovered, a prime immeasurably larger than any known beforehand.

So large is this number that it would be pointless trying to represent it in the way numbers are usually expressed, using a string of the digits 0 to 9. Fortunately, mathematicians have a special notation for describing numbers of this magnitude. Using this notation the number in question looks quite tame; it is $2^{86243}-1$. That this number is prime was discovered by David Slowinski of the U.S.A. As you might imagine, he had more than a £5 pocket calculator to help him with his calculation. In fact he made use of the world's most powerful computer, the giant Cray-1 machine at the Cray Research Laboratories. Even with this incredible computing power, it took the machine 1 hour 3 minutes and 22 seconds simply to check that the above number is indeed a prime. Months of computing were required to find this number in the first place.

It is not hard to explain what the notation used above means. To obtain Slowinski's number, you take the number 2 and multiply it by itself 86243 times, and then, as a final fillip, you subtract 1. The result is a number with precisely 25962 digits when written out in the normal way.

How can we begin to comprehend the size of such a monster? To get some idea, let's look at the apparently insignificant number 2^{64}. This can be visualised as follows. Imagine an ordinary chessboard. If we number the squares on this chessboard starting in the top left-hand corner and proceeding row by row down to the bottom right-hand corner, using the numbers 1, 2, 3 and so on, the last square we number will get the number 64.

Now imagine that we start putting ten-pence pieces on to the squares of the chessboard. On square number 1 we put two coins, on square 2 we put four, on square 3 we put eight, and so on, on each successive square putting exactly twice as many coins as on the previous one.

On the last square we will form a pile of exactly 2^{64} ten-pence pieces. How high do you think this pile will be. Six feet? Fifty feet? More? Wait for it. The pile will be about 37 million million kilometres high! So the pile would stretch way beyond the moon (a mere 400000 kilometres away) and the sun (150 million kilometres from Earth), and in fact would reach the nearest star, Proxima Centauri. And that is only for 2^{64}. To reach Slowinski's new prime, you have to double up the pile of coins a further 86179 times. You would have left the entire universe long before you got there.

Why should anyone be interested in such huge numbers? There are various answers to such a question. To the mathematician, the way the prime numbers are distributed amongst all the numbers is an extremely interesting question in its own right. No one can say just where the next prime number will be. With small numbers, there appear to be lots of primes about. For instance, of the numbers less than 25 the numbers 2, 3, 5, 7, 11, 13, 17, 19, 23 are all primes. But as soon as you start looking at much larger numbers, the primes become much less frequent, though they do not appear to follow any particular pattern.

Besides this perhaps esoteric interest, like almost all pure research there are various useful offshoots from the work. For instance, simply to get the computer to handle a number with 86243 binary digits, like Slowinski's, an entire discipline of computer science known as multi-precision arithmetic has had to be developed, and you can bet your last prime that the CIA (amongst others) are interested in that.

In a forthcoming issue of *Mathematical Spectrum*, Keith Devlin will describe in more detail how large primes are useful in devising 'unbreakable' codes.

The Case of the Crossed Line

JOHN PYM, *University of Sheffield*

John Pym has been trying to find a job as a secret agent. Unfortunately, if he advertises under his own name and phone number, he merely receives abusive calls telling him he's not very secret, while if he gives a false name and number his phone never rings and he has no chance to act. Until he solves this dilemma he is working as a Professor of Pure Mathematics at the University of Sheffield.

As I strolled through the foyer, a soggy rendering of 'Happy birthday to you' issued from the doorman's cubicle. 'Morning, Frank,' I called with the tone of bored bonhomie which greets the standing jokes of friends. As I travelled to the twenty-second floor, I considered the unquenchable ambition in the human spirit. For the price of an air ticket anyone can travel at hundreds of miles an hour for hours on end, yet some people spend the best years of their lives striving to travel at twenty-one miles an hour for about ten seconds, on foot. Frank's great achievement was 'When the saints go marching in' on a set of tuned spittoons (the sound was awful, but the spectacle amazing) and his eyes were fixed on far horizons: the late Beethoven quartets.

I let myself into the office marked *Charles Smith* and found a girl waiting for me: early twenties, dressed in grey, blouse up to the neck, ankle-length skirt, sensible shoes, even a hat. I saw at once that this was a woman I could respect. I offered her my hand. 'Yoggert,' I said, 'Bert Yoggert.'

'Yoghourt?' (You could hear the 'h'.) 'That's not what it says on the door.'

I waved my left hand vaguely in the air. 'Somewhere on the other side of *that* door,' I said, 'are four thousand million people, and not one of 'em's called EMERGENCY EXIT. Now what can I do for you?'

I'd lost her sympathy. 'You're supposed to be the detective, clever clogs! You tell me.'

Civilly, I offered her a chair and moved to my own behind my desk. 'Well, I'd say you were the owner of a fair-sized diamond which you keep displayed in a room you call the library on account of someone having nicked a book from it once. Of course, you wouldn't want to leave the gem unprotected, and my guess is you installed a super-sophisticated computer-controlled anti-burglar system. Then I'd say that last night you were stunned to find an intruder in this library, but that you have been unable to interest the police in the affair as the fellow ran off without taking anything. Except they—the police—did advise you to get one of those conventional jobs which reacts every time the neighbourhood cats pass through by making enough din to thoroughly wake you, while it plays sweet lullabies to the night shift at the local cop shop. So now you've come to see what private enterprise can do. That'll be £13·50 plus VAT. My accountant will send you the bill.'

'Rubbish!' she cried, standing up. 'I'm not paying you for what I already knew.'

'Madam,' I said coolly, 'I deal in information, most of it given verbally. Once spoken—or rather, heard—it cannot be taken back. Anyone could claim that they knew it already. I must insist that my clients pay for what I tell them. If you don't want to know the answer to a question, you shouldn't ask.'

But another matter was already troubling her. 'How did you know all that?'

'If you didn't like my fee for being a detective,' I grinned, 'you'll loathe my fee for teaching you how to be a detective.'

The prominent feature of the library was a huge television screen: the diamond's pale gleam was lost in the dazzling spectacle of its alleged protector. The picture was a brown rectangle ('That's the house') surrounded by a yellow circle. This represented the line of defence, of course unmarked on the broad lawns outside. 'That's funny,' she said. 'It's coming from the alpine garden.' She turned to the window to see who it was. A red spot (an intruder) appeared at the top right corner of the screen, and as it moved it traced out a red line, indicating its path; a second red spot appeared on the yellow circle; it indicated where the computer calculated the line and circle would meet. Then she screamed. 'Oh! It's a pussy. Quick! Switch it off! Quickly! Switch ——.' I figured I needed to see this machine in action, so I put on a soothing but very slow voice. 'Certainly,' I said. 'But would you be so kind as to tell me where the switch is?' Alas, I was too late. At the moment when the line met the circle, a load of rubbish descended from somewhere in the sky and buried the cat. The rubbish shuddered a little and the cat crawled out. Then shot off fast as it spotted the two dogs, one coming from either side of the house. It just made the nearest tree and sat scared peering at the dogs barking at the bottom.

My client sat sobbing while I worked out that on the roof there must be some machine for the precision hurling of rubbish. 'Quick!' I said. 'Where's the switch?' She looked up at me with large sad eyes and said 'Poor pussy' in a tone that meant 'Your turn now.' A new red line was crossing the screen. It stopped as the postman paused to watch the dogs. I stood in the library window gesticulating to catch his attention. When he saw me, he grinned and changed course to walk across the lawn. The red spot on the circle changed position as the computer recalculated the critical point. He was too far away to hear; I tried semaphore: Get back beware sophisticated anti-intruder device. I had no flags; my frantic signalling evoked only a friendly wave which disappeared in a load of rubbish and came out as a clenched fist. Then he saw the two new dogs.

Now, to the field of battle.
The man, alone. Two dogs, each swifter than the fastest ball
man ever bowled, hurtle across the turf
curving towards the twin stumps of his legs. His sole defence
(no sturdy bat) a bag of Royal Mail. What's this? He spins?
Has he mistook the game? He swings his bag round like a hammer-thrower.
Now to this idiot whirl the first dog comes —— and goes!
The Royal Mail has got him in the guts
and swept him up and on: he's hooked for six!
Only the cat thinks this dishevelled splay-legged one-time enemy
caught in her tree is worth a hostile spit.

A sound of tearing cloth. The second dog
has half a trouser leg, and so pink muscle now
tempts the white fangs. The cur goes in to bite ——
too late! The Royal Mail's dumped on its rump. It turns
to find its enemy: but where? and suddenly
its left eye's covered by a sticker which
tells all the world —— and now the right one too tells all ——
that it was opened by the Post Office
looking for a good read. The cur, in its blind panic, yelps.
The Royal Mail whirls round the postman's head
twice more, then dips. A new howl arcs into the air,
plays out its pained parabola until
it crashes through a window and becomes
a snivel lying in a heap of glass
on a library floor.

The postman stands alone. He takes two letters from his bag,
quietly tears them up and casts the pieces to the wind,
a hero's sacrifice. Then briefly indicates
that there will be no further post today,
nor yet tomorrow, nor the next month, next year——
and turns and strides away across the lawn.

She was now crying for the dogs too, and I wasn't feeling good either. I found a bottle and a couple of glasses. 'Here,' I said, 'take a swig of this. It'll pull you together.'

'Ugh!' she said. 'I don't drink.'

'It's milk,' I said with a touch of moral superiority.

In the university car park my Ferrari stood out like a sore thumb in the pile of bicycles. I guessed academics were discovering the pleasures of a new austerity. I wandered over to the mathematics department. 'Hi,' I said to the fellow I called on, 'You won't remember me, but I need to consult you again. Basically, my problem is, if I'm outside a circle, can I get in without crossing the boundary? Without jumping over, that is.'

'It sounds as if you need the intermediate value theorem,' he said. 'What it says is ——'

I held up my hand to stop him. 'Look,' I said. 'Last time I came I couldn't afford to pay; now I can. A hundred pounds an hour O.K.?' For a moment his eyes wouldn't focus. As I got out my stop-watch, I said 'Perhaps you could begin by saying *Yes* or *No*.'

'Well, no ——'

I stopped him again. 'I didn't really need the *Well*,' I said, 'but I can afford to be generous. Here's your 3 p. Can I have a receipt? I'll be putting it down to expenses.'

So I went to the University Library and turned up the intermediate value theorem. It wasn't too tough. When you'd got rid of the jargon, it said something like this. You draw a straight line (call it L) right across a sheet of paper, and take a point P above it and a point Q below it. Put your pencil on P and draw another line, any line—straight, crooked, wiggly, zig-zag—to Q obeying just this rule: your pencil must not be lifted off the paper ('continuous' is the word they use). Then this line must intersect L at least once, maybe more often if you kind of back-track. There was a proof too, but I reckoned I didn't need that.

So here was the theory behind my client's anti-burglar device. You didn't need much nous to change L into a circle, put P outside and Q inside. Then anyone moving from P to Q must cross the circle (the device worked up to a height of fifty feet, so a normal crook wouldn't jump over) and get clobbered. So what went wrong? The case didn't look very good. I found a campus coffee bar and sat brooding over my glass of milk (glass!—I'm not one of those fresh food fanatics who keeps a cow tethered in the corner of his dining room, but plastic revolts me) watching the soured cream of tomorrow's intellect debating whether to hold a sit-in until their just demands were met or only until the Disco for Democracy on Tuesday. The stuff coming from behind me was nearer what I liked to hear: about the nature of reason and proof, the meaning of truth and falsehood. '—— instance, the intermediate value theorem may well not be true.'

I was round in a flash, with my deerstalker politely raised. 'Madam,' I said, "I apologise for inadvertently overhearing, but the intermediate value theorem is a passionate interest of mine.'

I evoked an impressive display of incisors rather than a smile. 'It is always a great joy to discover a new, keen pupil.'

Things didn't look promising. 'I would be particularly interested to learn why you thought it might be false.'

'I didn't say it was false!' Now I had anger to contend with. 'I said it was not true. If you're going to make a profession out of eavesdropping, you'll need to hook a brain up to those ears. All right. What do you think of the completeness axiom for real numbers?'

I'd never heard of it, but you don't pass an exam if you don't give answers. 'I reckon it'll be the next popular craze when the cube goes out of fashion,' I said.

Then she exploded. 'I've suffered some ham-fisted attempts to pick me up in my time,' she said, 'but ——.' So for the first time, I looked at her. If I hadn't reformed, I'd have spent the rest of this article describing her, and all the adjectives would have been superlatives. By the time I was listening again, she was saying '—— creep!' and storming off. Things were desperate. I turned to the girl she was with.

'How about you?'

'She's my tutor. I've got to go.'

A final attempt: 'I'll buy you dinner.'

She shouted over her shoulder as she ran off. 'Seven-thirty, here.'

Obviously I needed to go into the fine details of the crime. 'So you were awakened by the thuds of two or three loads of rubbish, which you thought was a small gang being scared off. Then silence for a while. Then you heard a noise down here —— what sort of noise?'

'A chuckle,' my client said.

I frowned. 'Do you mean a sort of gloat, you know, *Heh*, *heh*, *heh*, as the crook gazed triumphantly on the diamond?'

She shook her head. 'A chuckle.'

'Why,' I asked,' should an ace diamond thief be chuckling on the job?'

'When I got here,' she explained, 'he was reading the *Beano*.'

I sat down suddenly. 'You don't look like you take the *Beano*.'

'Of course I don't.' She was riled. 'But my father did. From the first issue till the day he died. That day, it was——' she began to cry softly '——delivered late. His last moments could have been that much happier. I cancelled it at once. But the old ones are all kept over here. The intruder had got out a pile and was reading through them. Of course, he was drunk ——.'

'Drunk?' I said.

'He staggered away. I thought he was going to run into the wall, but he lurched sideways and went through the french window. And he didn't put down the *Beano* he was reading ——'

'Your *Beanos* in order?' I interrupted.

'Of course,' she said.

'And there's one missing,' I mused. 'How long between the thuds and the chuckle?'

'Difficult to say. Half an hour?'

I let her witter on a bit while I did the calculation. Say four or five minutes to get in, then two minutes per copy—about thirteen copies. 'Would the missing copy be for 8 May 1954?' I interrupted.

She went to look. 'No,' she said, with some pleasure I thought. Then 'It was for 1 May.' Then she twigged it wasn't a bad trick to be just one week out. 'How ——?'

'You heard of Leo Baxendale?' She shook her head. 'He's just the greatest in his line there's ever been. Invented 'The Bash Street Kids' in February 1954. That's what your intruder couldn't resist; there cannot be many criminals of such taste and discretion.'

Seven fifty-five. A girl was hurrying across the campus when an arm came out of nowhere, spun her round and left us both walking towards my car. 'Oh,' she said, 'am I late?' Then she looked at me. 'Where are we going?' I told her Windsor Brown's. She stopped dead. She was wild. 'Look! The jerks who invite me out usually mean a Chinese take-away walking along the pavement. I'm not dressed for Windsor Brown's. I don't even own anything I could wear to Windsor Brown's. You leave me alone!' She had on jeans that had first hit the jumble sales before she was born and a T-shirt that originally read '*Hands off Vietnam*' but, after a long series of triumphant deletions ending with '*Zimbabwe*' now had '*Afghanistan*' squeezed in at the bottom. I rubbed my hand in a puddle and printed three muddy hands on my dress shirt; I cut a hole in the knee of my dinner suit and a slit in the jacket; I took off my bow tie and wrapped it loosely round her neck. 'Let's go,' I said. 'The only item of dress Windsor ever cares about is a cheque book.'

Windsor Brown's was one of those places where you can hardly see the candles for the gloom. 'It hasn't got prices'—— she was surprised by the menu.

'Windsor's an Artist,' I said. 'He doesn't believe Art can be equated to dross like money. In return for the boost he gives our souls, his customers struggle to keep him in the style to which he feels he is entitled.'

'Ooh!' she said. 'You mean I can have what I like and it won't make any difference?'

Windsor could cook. For five minutes, she was engrossed with an amazed delight in something that had lived as a happy and fantastic creature drifting in the seas' warm currents, till Windsor's genius had reduced it to an amorphous pulp. Then she remembered I was there. 'What's that?' she asked. *Croutons citronnés sous parapluie* wasn't actually on the menu. 'Looks like dry toast,' she said. 'Hey, isn't *parapluie* umbrella?'

'Sure,' I said. 'Windsor wants to squeeze lemon over it because it's a starter. I can't stand lemon. Solution: he keeps the juice off with an umbrella. He's a perfectionist. But if you're going to drink the wine that fast, you'd better tell me about the intermediate value theorem while you still can.'

'You really want to know?'

'Try me.'

'You asked for it. Well, I'm not sure I've really got it, but it's something like this. Dr Randall—that's my tutor, she's brilliant—is a constructivist. Constructivists think a proof is only any good if it tells you exactly how to do what the theorem claims. For example, the theorem that you can find a (complex number) solution for any quadratic equation is fine, because you can actually write the roots down using the ordinary formula. But the usual proofs that every polynomial equation has a complex solution are no good because they don't tell you how you could go about finding one. And if there's no method given for calculating a solution, constructivists say it might as well not exist. Of course, they wouldn't say the theorem was false; they'd say that it hadn't been properly (meaning constructively) proved, and so may not be true. In fact, in the case of the polynomial equation, I think there are other proofs which are OK, but the intermediate value theorem hasn't been constructively proved, and no one thinks it can be. Bored yet?'

She poured a glass of wine down her throat. It's a lubricant: the words slip out faster but get battered in the process. 'Go on,' I said.

'So you might say constructivists were rather like computers.' She giggled. 'You can't calculate it, they won't believe it. Course, they're theoretical computers. Wouldn't want a theorem that wasn't true today true tomorrow cause they'd invented bigger machine.'

'Right. But what would a computer do if it was faced with the intermediate value theorem?'

Another glass of wine. I crossed my fingers. I was edgy—I normally keep them loose in case I have to go for a gun. 'Thick,' she said. 'Computer's thick. Computer wouldn't know interwhatsit wasn't a constructive theorem. Would jus try to find point lines crossed. Newton's method, I expect.' She cleared a space in the centre of

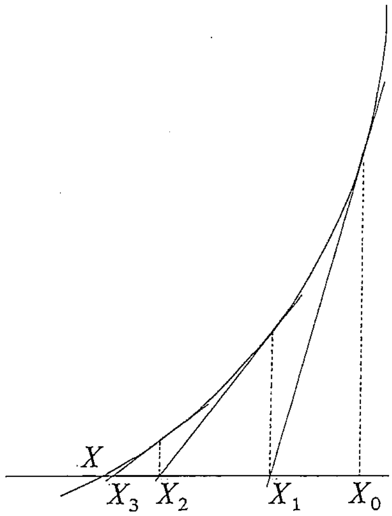


Figure 1. Newton's method.
To find the point X where the curve crosses the line: start with a point X_0; draw the tangent to the curve at the point vertically above X_0; this cuts the line at X_1; draw the tangent at the point above X_1; this cuts the line at X_2; and so on. The points $X_0, X_1, X_2, X_3, \ldots$ get closer to X, and X can be found to any desired accuracy.

the table with a sweep of her forearm, dipped her finger in her gravy—oh, it had some saucy name—and drew a wobbly version of Figure 1 on the cloth. 'Only works for decent graphs. Others, silly computer fails hopeless.'

I raised an eyebrow and a waiter appeared at my elbow. Windsor only hires guys who can see in the dark. 'Another bottle for the lady.' I said.

'Lady?' she was surprised. 'Wassin your glass? An wasson your plate? Looks burnt.'

I toasted her in life-enhancing milk. '*Pain flambé!*' I said. 'He makes it in the back yard. He covers the umbrella in a mixture of petrol and gin, and tosses the bread through the flame.' She made a face. 'If you won't suffer for Art,' I said, 'what's life for?'

'You don eat much,' she said.

I explained, 'I gave up gluttony when I gave up wine and women.'

She dropped her knife and fork. 'So wass I doin ere?'

'I wanted to know about constructivism.'

'Heck,' she said despondently. 'An I was gonna play ard to get.'

My client and I were in the library at nine-thirty the next morning when Charles Falconer staggered into view across the lawn. 'That's him!' she cried. 'Quick! Get him.' Then 'Ugh, he's drunk already.' Charles did look in a bad way. He lurched this way and that, a few steps forward, backwards, sideways. Occasionally he fell down. But, however erratically, he progressed towards the library window. 'Relax,' I told her, 'I asked him to come. Just watch the machine.' The wild red line which recorded Charles's unpredictable route was anything but decent, and the spot which recorded where the computer thought he would cross the circle was jumping all over the place, and often disappeared altogether. When the red line neared the circle, Charles

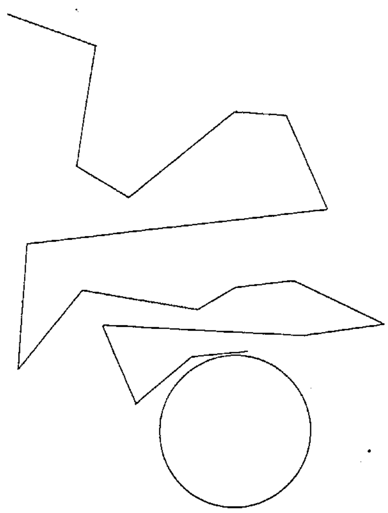


Figure 2. An indecent path.
A person travelling along this path is at no time moving in a direction which is towards any point of the circle although the path gets very close to the circle. A relevant feature of this path is that it does not have tangents at its turning-points.

seemed to veer off at a tangent, then lurched sideways and was inside; the machine thought fast, but not fast enough: Charles stood and smiled as the rubbish missed him. The dogs rushed over and rooted in the rubbish. Not only was Charles clean, but he didn't have the aniseed flavour with which my client thoughtfully laced her missile. Twice more Charles pulled this trick. Then he bowed politely to us, and walked away, straight as a die.

'Your computer's a constructivist,' I said.

That evening we were sitting in Charles Falconer's flat on the third floor of the tower block we'd built on the site of Frank Whitfield's old house. Charles and I were roaring with laughter over a pile of old *Beanos*. This was annoying Frank who was picking out soulful melodies on his spittoons. 'I don't see why Charles didn't just walk off with the lot when he had 'em in his hand,' he said grumpily.

'I'm not a common thief,' said Charles. 'My forte is the confidence trick.'

'Anyway,' I explained, 'it would have left her very unhappy, she might have insisted the police investigate, and it could have been difficult. This way, she thinks I'm great, she knows her house is impregnable to rational men but not to drunks, and we are enjoying a gift from a grateful client of mine!' I waved a *Beano* at him. 'It's like you say, Frank. Since Charles and I became partners, every day's a birthday.'

Notes and references

The characters in this story inhabit a strange world: its lines and circles are the usual mathematical abstractions (i.e. without breadth) but its computers cannot do instantaneous calculations. However, constructivists really do exist, perhaps not many compared with the total number of mathematicians, but the proportion is growing. Their basic philosophy could be taken to be the 'moral' of the story, that unless a proof tells you how to do the relevant calculation, it lacks something. A proper exposition of their outlook is given in Chapter 1 of the book *Foundations of Constructive Analysis* by Errett Bishop (McGraw-Hill, 1967). The present author is not a constructivist, and he hopes he has not misrepresented their views. There is a difference, for constructivists, between a proposition being *not true* (which means that there can be no constructive proof) and being *false* (which means that there is a constructive proof which begins with the proposition and ends with an undeniable contradiction, like $0 = 1$).

The *intermediate value theorem* is one of the key results of real analysis (the 'theory' of the calculus). Its proof depends on the *axiom of completeness* (also called the *least upper bound principle* and other aliases) which is one of the basic assumptions mathematicians usually make about numbers (roughly speaking, it says that numbers and infinite decimal expansions are the same thing—here, for example, $0{\cdot}5 = 0{\cdot}5000\ldots$) and which constructivists reject. Any introductory text on analysis will contain discussions (from the ordinary mathematician's viewpoint) of these two statements; they are also in Ian Stewart's *Concepts of Modern Mathematics* (Penguin, 1975).

The *Beano* has been published weekly by D. C. Thompson since July 1938. Leo Baxendale drew for it for about ten years, until 1962; 'The Bash Street Kids' (and his other creations) were taken over by less inspired hands after he left. His autobiography, *A Very Funny Business*, (Duckworth, 1978) provides a look behind the scenes of comic production. The conflict between the postman and the dogs was written when the author was under the influence of *War Music*, an exciting reworking of part of the *Iliad*, by Christopher Logue (Jonathan Cape, 1981).

Previous episodes in the career of the main protagonist have been recorded in *Mathematics Teaching* (**55**, 1971, p. 21) and *Mathematical Spectrum* (**13**, 1980/81, pp. 33–40). He had always assumed he was working for the forces of justice and right, but the subtly anti-social nature of his activities was brought home to him by a letter in *Spectrum* (**13**, 1980/81, p. 91). His reform dates from its publication.

1984

Here is a problem for the year made famous by George Orwell. The problem is to represent the integers 1 to 100 in terms of the digits of 1984 in their correct order using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, !. For example,

$$21 = 19 + (8 \div 4), 49 = -1 + ((\sqrt{9})! \times 8) + \sqrt{4}.$$

We have been successful for all but two integers. We shall be interested to see your solutions.

Some Simple Mathematical Models in Ecology

J. D. MURRAY, *University of Oxford*

The author is Fellow and Tutor in Mathematics at Corpus Christi College, Oxford, and Reader in Mathematics in the University. He spent many years in America at the Universities of Harvard, Michigan, and New York, where he was Professor of Mathematics. His research interests are primarily in mathematical modelling in the biomedical sciences.

1. Introduction

Ecology is the study of the relationship between man and his environment, or rather the interrelation of living things and the environment which sustains them. It covers an incredibly wide spectrum of activities, for example the interaction of animal species in prey–predator situations or competing for the same food source. Epidemic diseases are also examples of interacting species: malaria is between man and mosquito. Population studies are, of course, major topics in ecology. The point is that this subject covers probably a wider range of phenomena and fields of study than any other. It is not surprising therefore that mathematicians, scientists, engineers, socialists, holy men and so on have become deeply involved with ecology and with a corresponding spectrum of relevance to the real world.

Here I would like to discuss, *from a simple and introductory point of view only*, one aspect of ecology in which mathematicians have become involved, namely population studies. In these, mathematical modelling has been of real practical use.

I shall consider here only time-dependent problems. Spatial effects are very important in such studies and are even more fascinating. There is much current research going on in this area.

The most important point about any model (in its final form) of a practical situation is that it must have some bearing on the real problem. For example, some attempts in the past to give an explanation for epidemics are:

1. They are caused by evil spirits and/or displeased gods. Many people continue to accept this explanation. This theory has been held by the largest number of people for the largest period of time!

2. Hippocrates, the Greek medical scientist (5th century BC) said that the spread of disease could be attributed to (i) temperament (ii) personal habits (iii) environment. In relation to (iii) he made some practical observations and said (in an essay on 'Air, Waters and Localities'), 'As to places looking to the West and which feel no winds from the East, but are exposed to those from the North and South, their position beyond all others is most favourable to disease'.

3. Astrology predetermines epidemics. In 1865 Alexander Howe in England published a book on 'A Theoretical Inquiry into the Physical Causes of Epidemic Diseases'. In it he gives his 'Laws of Pestilence' with 31 propositions. The following are typical:

Proposition 2. The length of the interval between successive periodic visitations (of an epidemic) corresponds with the period of a single revolution of the lunar node, and a double revolution of the lunar apse time.

Proposition 20. Cholera, the epidemic which is typical of this century, invariably travels from East to West, or in the same direction as the regression of the lunar node along the line of the elliptic.

4. In the 1960s a pure mathematician studying models for epidemics started with the statement that the microbes shall be considered members of a set defined on an appropriate Banach space.

From a relevance point of view 4 comes rather low down whereas 2 is not bad. In fact 2(i) has only been taken seriously in about the last 20 or so years.

What I wish to do here is to describe some simple *deterministic* (that is, including no random element), as opposed to *probabilistic* models (which include random elements) for population interaction. Models with random elements are, of course, highly relevant to real-life situations, particularly when questions of initial growth or extinction are concerned. Deterministic models are generally valid when the populations in question are not small. Models can be made as complicated as you wish by taking detailed account of medical progress, wars, pest control, vagaries of democracies or dictatorships, and so on. My purpose here is to discuss, for illustrative purposes, how simple models are set up. Although simple, they are, for certain practical situations, not entirely inadequate. Such simple models can give, and often have, given remarkable insight into the situations being studied, even if they are not quantitatively accurate. They can also serve the function of posing certain questions which should be answered by any field ecologist. They can also sometimes resolve differing practical views of specific situations.

2. Simple models for population growth

Let $n(t)$ be the number of individuals in a given population at time t. If B is the birth (plus immigration) number per unit time and D the death (plus emigration) number per unit time, we may reasonably say that the rate of change of n with time is given by

$$\frac{dn}{dt} = B - D. \tag{1}$$

The real difficulty now is to specify the dependence of B and D on the population n. The simplest is to consider

$$B = bn, \qquad D = dn, \tag{2}$$

where b and d are the birth and death rates per unit time. With (2) in (1) we have the first-order differential equation

$$\frac{dn}{dt} = (b - d)n, \qquad n = n_0 \quad \text{at } t = t_0, \tag{3}$$

where n_0 is the population at time t_0. This is a separable differential equation which can be integrated. From (3)

$$\int_{n_0}^{n} \frac{dn}{n} = (b-d) \int_{t_0}^{t} dt,$$

which gives

$$\ln [n(t)/n_0] = (b-d)(t-t_0),$$

and so

$$n(t) = n_0 \exp((b-d)(t-t_0)).$$

Clearly, if $b > d$, then $n \to \infty$ as $t \to \infty$, whereas if the birth rate is less than the death rate, i.e. $b < d$, then $n \to 0$ as $t \to \infty$, and the population becomes extinct.

Although this model is much too simplistic for most situations, it does fit certain phenomena involving populations, at least for part of their history.

Practically all models which mirror real situations must be *non-linear*. For single species we would then have in place of (3)

$$\frac{dn}{dt} = f(n), \tag{4}$$

where $f(n)$ is non-linear. When the population is fairly small the growth may well behave like (3). Since the population clearly cannot become infinite, we may reasonably say that, when the population is too large, it has an inhibitory effect. For example, if

$$\frac{dn}{dt} = f(n) = an - bn^2, \tag{5}$$

the an $(a > 0)$ represents the linear growth whereas $-bn^2$ $(b > 0)$ represents the negative or restraining effect when n becomes too large (a kind of over-crowding effect). The form (5) is called the *Logistic Law of Population*.

The solution of (5) with $n(t_0) = n_0$ is given by integrating (5) as follows:

$$\int_{n_0}^{n} \frac{dn}{n(a-bn)} = \int_{t_0}^{t} dt$$

and so

$$\int_{n_0}^{n} \frac{1}{a} \left\{ \frac{1}{n} + \frac{b}{a-bn} \right\} dn = (t-t_0),$$

that is

$$[\ln n - \ln (a-bn)]_{n_0}^{n} = a(t-t_0),$$

and so

$$\ln \left[\left(\frac{n}{a-bn} \right) \left(\frac{a-bn_0}{n_0} \right) \right] = a(t-t_0),$$

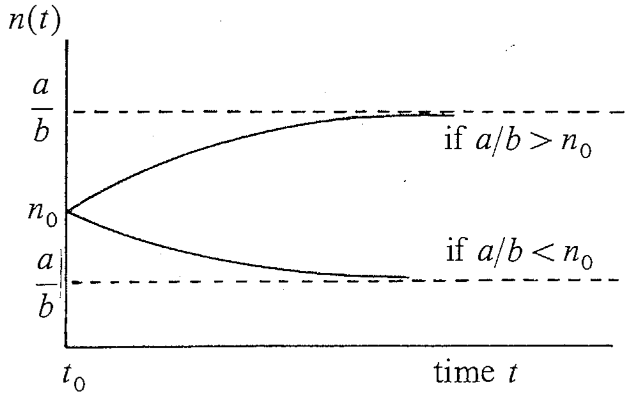


Figure 1

which gives

$$\frac{n(t)}{a - bn(t)} = \left(\frac{n_0}{a - bn_0}\right) e^{a(t - t_0)},$$

and finally

$$n(t) = \frac{n_0 a/b}{n_0 + [a/b - n_0]\, e^{-a(t - t_0)}} \tag{6}$$

which is illustrated in Figure 1. There is thus a saturation population as $t \to \infty$; the actual saturation values depend on the ratio of a/b. If a population is thought to obey such a logistic law, the parameters a and b can be determined from the data using (6). This solution is a reasonable fit for various population situations, for example (i) fruit fly populations, (ii) certain bacteria, (iii) rat populations (in the absence of predators) obey it for part of the time, and (iv) USA population from 1870 to 1910.

3. Interacting models: Prey–predator model

If we have several interacting species, for example, large fish which feed on small fish, man and malarial mosquito and so on, we require an equation for each species and hence we have a system of differential equations like

$$\frac{dn_i}{dt} = f_i(n_1, n_2, \ldots, n_k), \qquad i = 1, 2, \ldots, k, \tag{7}$$

where f_i represents birth, death, interaction with other species and so on.

In this section I shall describe a simple two-species prey–predator situation which was first proposed by Volterra [1] in 1926 when he was asked to try to explain the oscillations in fish catches in the Adriatic Sea. It is a non-linear model which, although now considered unrealistic (justifiably), is of fundamental importance in ecology since it is the stepping stone for many useful models. It also gave indications of certain general phenomena associated with population oscillations.

The practical situation was that the catches of two fish species in the Adriatic were observed to oscillate in a regular way. The oscillations of the two species had the same period but were out of phase. The fish consist of a large species, N say, which eat the smaller species, n say. It seemed as if the number of large ones, the predators, grew until there were not enough small ones, the prey, to give enough food to sustain this large population, and as a consequence the large fish started to decline in numbers. When it dropped to a sufficiently low level, the small fish started to multiply again, and when their population was high the larger fish, now with a plentiful food supply, started to produce more large fish again. The cycle was then repeated. Volterra's (1926) model system was

$$\frac{dN}{dt} = -aN + bNn, \tag{8}$$

$$\frac{dn}{dt} = cn - dNn. \tag{9}$$

In Equation (8) for the large fish the term bNn $(b > 0)$ represents the gain due to interaction with the small fish, whereas if there were no prey $(n = 0)$ the predators would simply decay exponentially to extinction (this is the $-aN$ $(a > 0)$ term).

In (9), the equation for the prey, the interaction term is negative $(d > 0)$, that is a loss term, whereas in the absence of predators the prey would simply grow exponentially $(c > 0)$.

To investigate the equations it is convenient to introduce dimensionless variables by

$$\left.\begin{aligned} U(\tau) &= \frac{d}{c}N, \qquad & u(\tau) &= \frac{b}{a}n, \\ \tau &= ct, \qquad & \alpha &= \frac{a}{c} \end{aligned}\right\} \tag{10}$$

and (8) and (9) become

$$U' = \alpha U(u - 1), \qquad u' = u(1 - U), \tag{11}$$

where the prime denotes differentiation with respect to τ and $\alpha > 0$ is a parameter. Let us suppose that at some time $\tau = \tau_0$ the populations are known, that is

$$U(\tau) = U_0, \qquad u(\tau) = u_0 \qquad \text{at } \tau = \tau_0. \tag{12}$$

We are interested only in positive solutions, of course.

Equations (11) cannot be solved simply. We can, however, obtain a relationship between U and u. The simplest way to get this is to divide the first equation of (11) by the second to obtain, using (12),

$$\frac{dU}{du} = \frac{\alpha U(u-1)}{u(1-U)}$$

$$\Rightarrow \int_{U_0}^{U} \frac{1-U}{U}\, dU = \alpha \int_{u_0}^{u} \frac{(u-1)}{u}\, du,$$

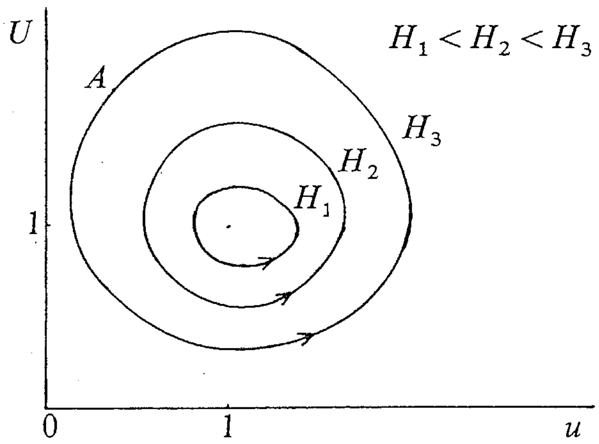


Figure 2

which on integration gives

$$du + U - \log u^{\alpha} U = \alpha u_0 + U_0 - \log u_0^{\alpha} U_0 = H, \tag{13}$$

where H is a constant determined by the initial conditions (12) and α. If we now plot U against u for various H, we get solutions as in Figure 2. For each H the U–u relation (13) gives a *closed* trajectory. The point is that, as time increases, U and u vary but they remain on the closed trajectory they started on (and determined by U_0, u_0 and α). For example, suppose the initial values U_0, u_0 give the point A on the H_3 trajectory. Since $U_0 > 1$ and $u_0 < 1$, the first equation of (11) shows that U decreases initially and so does u, from the second equation of (11). When U has decreased to 1, $u' = 0$ and, for longer time τ, u increases again. When $u = 1$ then $U' = 0$ and U thereafter increases again. In this way you can see that U and u simply move round and round the trajectory. This means the solutions are periodic in time. Also, the maximum of U does *not* occur at the maximum of u; that is, the population oscillations are *not* in phase. A typical time plot of U and u is shown in Figure 3 when $U_0 < 1$ and $u_0 > 1$.

Even this simple model (11) indicates the existence of oscillating populations in interacting situations. The ideas from this theory have in fact been used in the control of a specific citrus fruit pest in America. The natural predator of this pest insect is the ladybird beetle which was introduced by the farmers. The pest was kept down; the two interacting populations oscillated in time.

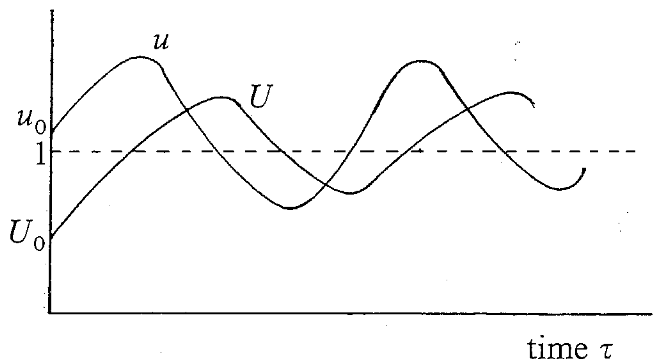


Figure 3. Periodic solutions of equations (11).

The reason the Volterra model is not realistic is that the system is not orbitally stable. Without going into the mathematical theory of this, it can be seen from Figure 2 that any small external perturbation would shift U and u onto another orbit with a different period; equations (11) can easily be solved on a simple programmable calculator. Any realistic model system of equations must be such as to possess a *limit cycle solution*. This is in essence a stable orbital solution which is unique for a whole range of different conditions.

Studies of more complex interacting species give some interesting and practical results. If two populations, for example, compete for the same food supply (the third population), then it can be shown that one species will become extinct. If it is the food, then so will the other two species. Much useful information has been obtained by such models on the optimal catches, or harvesting of forests and so on, which should be allowed. The problems in this line can be fascinating, practically and mathematically.

Oscillating systems are now a subject of very wide research by mathematicians, biologists, physicists, etc. It is believed that there is a strong connection between oscillating biological phenomena and morphogenesis, that is development of the shape of an organ or a whole organism.

The above examples of mathematical modelling give only the briefest introduction to the use of mathematics in ecology. In the general area of mathematical biology, the choice of subject is *very* wide indeed. It is an exciting field to be in at this stage, and is one of the fastest-growing areas of any branch of mathematics. Because of the wide variety of subjects it encompasses, it can accommodate the widest spectrum of mathematical taste—even some pure mathematicians can be of use and find it fascinating if they have an open and receptive mind!

Reference

1. V. Volterra, Variations and fluctuations in the number of individuals in interacting species, *R. C. Accad. Lincei* (6) 2 (1926), 31–113. (This paper, written in Italian, is translated in R. N. Chapman's *Animal Ecology* (New York, 1931).)

Colson News

E. Hillman, A. Paul and C. A. B. Smith are planning to circulate a small quarterly newsletter exploring the properties of negative digits, their advantages, and related topics with articles, puzzles, correspondence, beginning in January 1984. They estimate that £1 per year (25p per issue) should at present cover costs of printing, postage etc. Enquiries and subscriptions to Cedric A. B. Smith, The Galton Laboratory, University College London, 4 Stephenson Way, London NW1 2HE, England.

Professor Leon Mirsky

All connected with this magazine have been saddened by the death of Professor Leon Mirsky on 1 December 1983. He was involved with *Mathematical Spectrum* from its very conception, and his wise counsel as one of its editors has been invaluable ever since. His articles in *Spectrum*, 'From rule of thumb to abstract structure' in Volume 1, 'A case study in inequalities' in Volume 9 and 'A medley of squares' in Volume 10 are classics of their kind, and may be reread with profit any number of times. They display his wide mathematical knowledge, his absolute precision and his masterly use of the English language, familiar to his colleagues and students and to all who knew him.

Leon Mirsky's name has been well known to generations of university students in mathematics and the sciences through his book *An Introduction to Linear Algebra*, first published by Oxford University Press in 1955, still a mine of useful information. Latterly, he was one of the pioneers of combinatorial mathematics, and wrote one of the first books on one of its new branches: *Transversal Theory*, published by Academic Press in 1971. He also wrote numerous research papers on a wide variety of topics.

It is perhaps fitting that, although he officially retired in September 1983, Leon Mirsky should have continued right up to his death to teach a course on his beloved theory of numbers to final-year undergraduates in the University of Sheffield, where he spent most of his working life. His knowledge of mathematics was encyclopaedic, and he will be sadly missed by his colleagues in the mathematical fraternity the world over, and by his friends who will remember his stimulating and informed conversation on a wide range of subjects.

D. W. SHARPE

Computer Column

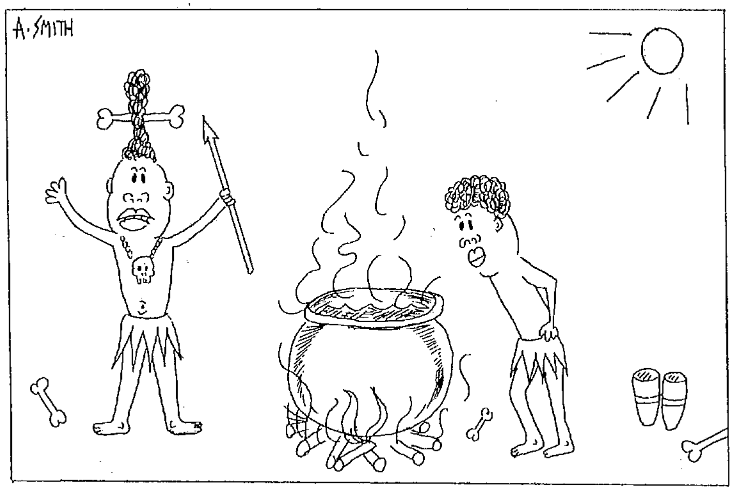
MICHAEL PIFF, *University of Sheffield*

Cannibals!

To start this column, I thought you might like to try the following game, which is in the 'pursuit' family. It is in fairly standard BASIC, and will need adapting only slightly to your micro. A routine to position the cursor at a given row and column must be written; alternatively, the screen can be re-drawn periodically using the included routine. If your micro does not clear its screen by printing a special character, that bit will have to be changed too. Finally, the dimensions of the island may have to be adjusted if they are too large, but the rest of the program should not then need to be altered, apart from perhaps changing the left and right boundaries of the island.

I have avoided any high-resolution graphics in this program. You may not have them, but if you do, the program is written so that it can easily be modified to use them.

I hope that future articles will be a little less serious than this one, if I don't get eaten in the meantime....




```
100  REM            CANNIBALS!
110  DEF FNS(X,Y)=X+SGN(Y-X)
120  DIM ISLAND(18,39),C(1000,2),COLINC(9),ROWINC(9)
130  DIM LEFTHAND(18),RIGHTHAND(18)
140  NRFROWS=18
150  NRFCOLS=39
160  MAXDIFF=INT(NRFROWS*NRFCOLS/5)
170  SPACE=1
180  SPACE$=' '
190  CANNIBAL=2
200  CANNIBAL$='C'
210  EXCANNIBAL$='c'
220  MAN=3
230  MAN$='x'
240  PIT=4
250  PIT$='o'
260  SEA=5
270  SEA$='~'
280  REM INSERT VALUE OF CLEARSCREEN CHARACTER
290  CLEARSCREEN$=CHAR(26)
300  MAT READ COLINC,ROWINC,LEFTHAND,RIGHTHAND
310  DATA -1,0,1,-1,0,1,-1,0,1,1,1,1,0,0,0,-1
320  DATA -1,-1,12,10,5,5,4,3,4,5,6,7,6,5,6
330  DATA 7,8,9,12,14,27,28,32,33,33,34,34,35
340  DATA 36,37,37,37,36,36,35,35,32,30
350  REM INSTRUCTIONS
360  PRINT "You are stranded on Cannibal Island!"
370  PRINT "If you don't lure all the cannibals"
380  PRINT "into pits, you will be eaten!"
390  PRINT "You may indicate the direction of your"
400  PRINT "moves as follows:"
410  PRINT 'NW N NE      Return = Stay where you are'
420  PRINT 'W  x  E      Q      = Quit'
430  PRINT 'SW S SE      R      = Redraw screen'
440  PRINT 'Any invalid move will be interpreted as no move'
450  REM ENTRY TO PROGRAM IF DIFFICULTY TO BE CHANGED
460  PRINT 'What level of difficulty?(1..';MAXDIFF;') ';
470  INPUT ' ',DIFFICULTY
480  IF DIFFICULTY<1 THEN DIFFICULTY=1
490  IF DIFFICULTY>MAXDIFF THEN DIFFICULTY=MAXDIFF
500  BOUND1=5*DIFFICULTY
510  BOUND3=NRFROWS*(NRFCOLS+1)
520  U=(BOUND3-BOUND1)/3
530  BOUND2=BOUND1+U*(2+DIFFICULTY/MAXDIFF)
540  REM ENTRY TO PROGRAM IF DIFFICULTY NOT TO BE CHANGED
550  NRFCANNIBALS=0
560  REM INITIALISE ISLAND
570  FOR ROW = 1 TO NRFROWS
580    FOR COL = 1 TO NRFCOLS
590      ISLAND(ROW,COL)=SEA
600    NEXT COL
610  NEXT ROW
620  REM PLACE MAN
630  MANROW = 2+INT((NRFROWS-2)*RND(0))
640  MANCOL =LEFTHAND(MANROW)+
      INT((RIGHTHAND(MANROW)-LEFTHAND(MANROW)+1)*RND(0))
650  ISLAND(MANROW,MANCOL) = MAN
660  REM PLACE CANNIBALS AND PITS
670  FOR ROW = 2 TO NRFROWS-1
680    FOR COL = LEFTHAND(ROW) TO RIGHTHAND(ROW)
690      IF (ROW=MANROW)AND(COL=MANCOL) THEN 870
700      ISLAND(ROW,COL)=PIT
710      R = BOUND3*RND(0)
720      IF R > BOUND2 THEN 870
730      IF R>BOUND1 THEN 820
740      REM A POTENTIAL CANNIBAL
750       IF NRFCANNIBALS >= BOUND1 THEN 870
760       IF(ABS(ROW-MANROW)<=2)AND(ABS(COL-MANCOL)<=2)THEN 850
770       NRFCANNIBALS = NRFCANNIBALS+1
780       C(NRFCANNIBALS,1) = ROW
790       C(NRFCANNIBALS,2) = COL
800       ISLAND(ROW,COL) = CANNIBAL
810       GOTO 870
820      REM A SPACE
830       ISLAND(ROW,COL) = SPACE
840       GOTO 870
850      REM NEAR TO MAN
860       IF RND(0)>0.6 THEN 820
870    REM END OF COL LOOP
880    NEXT COL
890  NEXT ROW
900  IF NRFCANNIBALS=0 THEN 540
910  REM CLEAR SCREEN AND PRINT ISLAND
```

```
920  PRINT CLEARSCREEN$
930  FOR ROW = 1 TO NRFROWS
940    FOR COL = 1 TO NRFCOLS
950      ON ISLAND(ROW,COL) GOTO 1100,990,960,1020,1050
960      REM MAN
970        MESSAGE$=MAN$
980        GOTO 1080
990      REM CANNIBAL
1000        MESSAGE$=CANNIBAL$
1010        GOTO 1080
1020      REM PIT
1030        MESSAGE$=PIT$
1040        GOTO 1080
1050      REM SEA
1060        MESSAGE$=SEA$
1070        GOTO 1080
1080      REM END OF ON STATEMENT
1090      GOSUB 2690
1100    REM END OF COL LOOP
1110    NEXT COL
1120 NEXT ROW
1130 REM PRINT HELP MATERIAL
1140 ROW=19
1150 COL=26
1160 MESSAGE$='Level '+STR$(DIFFICULTY)
1170 GOSUB 2690
1180 ROW=21
1190 MESSAGE$='NW N NE'
1200 GOSUB 2690
1210 ROW=22
1220 MESSAGE$='W  x  E'
1230 GOSUB 2690
1240 ROW=23
1250 MESSAGE$='SW S SE'
1260 GOSUB 2690
1270 REM START NEXT ROUND OF MOVES
1280 ROW=20
1290 COL=18
1300 MESSAGE$=STR$(NRFCANNIBALS)+' cannibal'
1310 IF NRFCANNIBALS=1 THEN MESSAGE$=MESSAGE$+'!'
        ELSE MESSAGE$=MESSAGE$+'s!'
1320 GOSUB 2690
1330 INPUT LINE 'Your move ',DIRECTION$
1340 IF (DIRECTION$='Q')OR(DIRECTION$='q') THEN 2640
1350 IF (DIRECTION$='R')OR(DIRECTION$='r') THEN 910
1360 DIRECTION=5
1370 IF (DIRECTION$='N')OR(DIRECTION$='n')THEN DIRECTION=8
1380 IF (DIRECTION$='S')OR(DIRECTION$='s')THEN DIRECTION=2
1390 IF (DIRECTION$='E')OR(DIRECTION$='e')THEN DIRECTION=6
1400 IF (DIRECTION$='W')OR(DIRECTION$='w')THEN DIRECTION=4
1410 IF (DIRECTION$='NE')OR(DIRECTION$='ne')THEN DIRECTION=9
1420 IF (DIRECTION$='NW')OR(DIRECTION$='nw')THEN DIRECTION=7
1430 IF (DIRECTION$='SE')OR(DIRECTION$='se')THEN DIRECTION=3
1440 IF (DIRECTION$='SW')OR(DIRECTION$='sw')THEN DIRECTION=1
1450 GOSUB 2740
1460 ROW = MANROW
1470 COL = MANCOL
1480 MESSAGE$ = SPACE$
1490 MANROW = MANROW+ROWINC(DIRECTION)
1500 MANCOL = MANCOL+COLINC(DIRECTION)
1510 ON ISLAND(MANROW,MANCOL) GOTO 1520,1610,1770,1690,1690
1520 REM NORMAL MOVE
1530  ISLAND(ROW,COL)=SPACE
1540  ISLAND(MANROW,MANCOL) = MAN
1550  GOSUB 2690
1560  ROW = MANROW
1570  COL = MANCOL
1580  MESSAGE$ = MAN$
1590  GOSUB 2690
1600  GOTO 1770
1610 REM HITS A CANNIBAL
1620  ISLAND(ROW,COL)=SPACE
1630  GOSUB 2690
1640  ROW=21
1650  COL=1
1660  MESSAGE$="Straight into the cannibal's mouth! "
1670  GOSUB 2690
1680  GOTO 2640
1690 REM FALLS INTO PIT OR SEA
1700  ISLAND(ROW,COL)=SPACE
1710  GOSUB 2690
1720  ROW=21
1730  COL=1
1740  MESSAGE$="Splash! You're dead!"
1750  GOSUB 2690
1760  GOTO 2640
1770 REM END OF ON STATEMENT
1780 REM SCAN THROUGH CANNIBALS AND MOVE THEM ONE AT A TIME
1790 CANNIBALINDEX = NRFCANNIBALS
1800 MOVESMADE=0
1810 REM CHECK THAT A CANNIBAL HAS BEEN MOVED!
1820 CANNIBALROW = C(CANNIBALINDEX,1)
1830 CANNIBALCOL = C(CANNIBALINDEX,2)
1840 ROW = CANNIBALROW
1850 COL = CANNIBALCOL
1860 MESSAGE$ = EXCANNIBAL$
1870 REM CALCULATE NEW COORDS FOR CANNIBAL
1880 NEWCROW=CANNIBALROW
1890 NEWCCOL=CANNIBALCOL
1900 REM CHANGE COORDS OF CANNIBAL
1910 ROWDIST=ABS(MANROW-CANNIBALROW)
1920 COLDIST=ABS(MANCOL-CANNIBALCOL)
1930 DIST=ROWDIST+COLDIST
1940 IF DIST<2 THEN DIST=2 - DIFFICULTY/MAXDIFF
1950 IF RND(0)*DIST<ROWDIST THEN
        NEWCROW=FNS(CANNIBALROW,MANROW)
1960 IF RND(0)*DIST<COLDIST THEN
        NEWCCOL=FNS(CANNIBALCOL,MANCOL)
1970 REM HANDLE FOUR POSSIBILITIES
1980 ON ISLAND(NEWCROW,NEWCCOL)GOTO 2340,2530,2150,1990,1990
1990 REM CANNIBAL FALLS INTO PIT
2000  MOVESMADE=MOVESMADE+1
2010  ISLAND(CANNIBALROW,CANNIBALCOL)=SPACE
2020  GOSUB 2690
2030  MESSAGE$=SPACE$
2040  GOSUB 2690
```

```
2050  IF CANNIBALINDEX=NRFCANNIBALS THEN 2090
2060  REM MOVE LAST CANNIBAL IN LIST TO CURRENT POSITION
2070   C(CANNIBALINDEX,1)=C(NRFCANNIBALS,1)
2080   C(CANNIBALINDEX,2)=C(NRFCANNIBALS,2)
2090  NRFCANNIBALS = NRFCANNIBALS-1
2100  ROW=21
2110  COL=1
2120  MESSAGE$='Splash goes a cannibal!'
2130  GOSUB 2690
2140  GOTO 2530
2150 REM CANNIBAL HITS MAN
2160  MOVESMADE=MOVESMADE+1
2170  ISLAND(CANNIBALROW,CANNIBALCOL)=SPACE
2180  ISLAND(NEWCROW,NEWCCOL)=CANNIBAL
2190  GOSUB 2740
2200  GOSUB 2690
2210  ROW=MANROW
2220  COL=MANCOL
2230  MESSAGE$=CANNIBAL$
2240  GOSUB 2690
2250  ROW=CANNIBALROW
2260  COL=CANNIBALCOL
2270  MESSAGE$=SPACE$
2280  GOSUB 2690
2290  ROW=21
2300  COL=1
2310  MESSAGE$="You've been eaten by a cannibal!"
2320  GOSUB 2690
2330  GOTO 2640
2340 REM CANNIBAL MOVES TO VACANT SPACE
2350  MOVESMADE=MOVESMADE+1
2360  ISLAND(CANNIBALROW,CANNIBALCOL) = SPACE
2370  ISLAND(NEWCROW,NEWCCOL) = CANNIBAL
2380  GOSUB 2690
2390  ROW = NEWCROW
2400  COL = NEWCCOL
2410  MESSAGE$ = CANNIBAL$
2420  GOSUB 2690
2430  ROW=CANNIBALROW
2440  COL=CANNIBALCOL
2450  MESSAGE$=SPACE$
2460  GOSUB 2690
2470  C(CANNIBALINDEX,1) = NEWCROW
2480  C(CANNIBALINDEX,2) = NEWCCOL
2490  ROW=22
2500  COL=1
2510  MESSAGE$='The cannibals are getting close!'
2520  IF DIST<4 THEN GOSUB 2690
2530 REM END OF ON STATEMENT
2540 REM LOOK AT NEXT CANNIBAL
2550 CANNIBALINDEX = CANNIBALINDEX-1
2560 REM SEE IF ALL CANNIBALS COVERED
2570 IF CANNIBALINDEX > 0 THEN 1820
2580 IF MOVESMADE=0 THEN 1780
2590 GOSUB 2740
2600 IF NRFCANNIBALS >0.5 THEN 1270
2610 ROW=21
2620 COL=1
2630 MESSAGE$='Well done, the cannibals are all dead!'
2640 REM ASK IF HE WANTS ANOTHER GAME
2650 INPUT LINE
       "Type 'Y' or 'y' if you want another game ",ANSWER$
2660 IF ANSWER$='Y' THEN 540
2670 IF ANSWER$='y' THEN 450
2680 GOTO 2900
2690 REM PRINT MESSAGE$ AT ROW,COL
2700 REM YOU WILL HAVE TO SUPPLY A ROUTINE TO DO THIS. IF NOT
2710 REM POSSIBLE, THEN THE SUBROUTINE BELOW WILL HAVE TO BE
2720 REM USED TO RE-PRINT THE ISLAND.
2730 RETURN
2740 REM BLANK OFF MESSAGES
2750 ROW=21
2760 COL=1
2770 MESSAGE$=SPA(50)
2780 GOSUB 2690
2790 PRINT MESSAGE$
2800 RETURN
2810 REM RE-PRINT ISLAND. NOT USED IN PROGRAM; A RESERVE.
2820 SYMBOLS$=SPACE$+CANNIBAL$+MAN$+PIT$+SEA$
2830 FOR ROW = 1 TO NRFROWS
2840   FOR COL = 1 TO NRFCOLS
2850     PRINT MID(SYMBOLS$,ISLAND(ROW,COL),1);
2860   NEXT COL
2870   PRINT
2880 NEXT ROW
2890 RETURN
2900 REM END OF GAME
2910 PRINT CLEARSCREEN$
2920 END
```

Letters to the Editor

Dear Editor,

Beyond the calculator

The use of continued fractions to find the units and tenth digits of $(\sqrt{2}+\sqrt{3})^{1980}$ seems to be rather like using a sledge-hammer to crack a nut. We have $(\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$. Now, using the binomial theorem,

$$(5+2\sqrt{6})^{990}+(5-2\sqrt{6})^{990} = 2\cdot 5^{990}+2\binom{990}{2}5^{988}.24+\dots+2\cdot 24^{495},$$

and every term on the right-hand side except the last is a multiple of 10. For the last term, an odd power of 24 ends in 4, so $2\cdot 24^{495}$ has units digit 8. Thus 8 is the units digit of $(5+2\sqrt{6})^{990}$. Now $5-2\sqrt{6} \simeq 0\cdot 1$, so $(5-2\sqrt{6})^{990}$ is very small. If we now subtract this from both sides, we obtain

$$(\sqrt{2}+\sqrt{3})^{1980} = (5+2\sqrt{6})^{990} = \dots 7\cdot 9(99\dots 9)\dots.$$

Yours sincerely,

G. N. Randall
(50 Broomground, Winsley, Bradford-on-Avon, Wiltshire BA15 2JX)

Dear Editor,

Beyond the calculator

In the article by Hille and McPherson in Volume 16 Number 1, the authors determined the units, tens and tenths digits of $(5+2\sqrt{6})^{990}$ using continued fractions. But the use of partial fractions does not seem to be necessary as the following argument shows.

Let $a = 5+2\sqrt{6}$ and $b = 5-2\sqrt{6}$, so that $ab = 1$ and $a+b = 10$. We now consider the difference equation

$$u_{n+2} = 10u_{n+1} - u_n \qquad (n \geq 0)$$

with $u_0 = 2, u_1 = 10$. The solution is $u_n = a^n + b^n$. As $0 < b < 1$, it will suffice to determine the behaviour of a^n. We have

$$u_{n+2} \equiv -u_n \pmod{10},$$

so that $u_{2n+1} \equiv 0 \pmod{10}$ for all integers $n \geq 0$. Hence $u_{2n} \equiv 2(-1)^n \pmod{100}$ for all integers $n \geq 0$. In particular, $u_{990} \equiv -2 \equiv 98 \pmod{100}$. As $0 < b^{990} < 1$, we see that the integer part of a^{990} is congruent to 97 (mod 100), so that the units and tens digits of a^{990} are 7 and 9 respectively. Also, $b^{990} \simeq 2\cdot 32 \times 10^{-986}$, so the 'first few' terms after the decimal point in a^{990} are 9's.

Yours sincerely,

J. R. Hilditch
(164 Morton Way, London N14 7AL)

[Mr D. J. Orton of Portsmouth Grammar School has written making a similar point.—Ed.]

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

16.4. (Adapted from *Közepiskolai Matematikai Lapok*) How many squares are crossed by the diagonal of a 1984×1066 chessboard?

16.5. (From the magazine *Function*) Show that the product of n consecutive integers is divisible by $n!$.

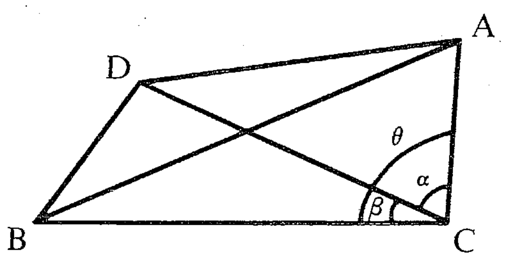
16.6. (A seasonal problem) A snowplough clears snow at a constant rate during a snowstorm in which snow is falling at a constant rate. Between 10 a.m. and 11 a.m. it travels 1 mile, and between 11 a.m. and noon it travels $\frac{1}{2}$ mile. At what time did the snowstorm begin?

Solutions to Problems in Volume 15, Number 3

The following solutions are by Ruth Lawrence of Huddersfield (now a student at St. Hugh's College, Oxford). We also received attractive solutions from Ian Wright (Winchester College), but these were too late to be included.

15.7. The points A, B, C are non-collinear and are such that $AB^2 \geq AC^2 + BC^2$. Prove that $CD^2 \leq AD^2 + BD^2$ for every point D in the plane in which A, B, C lie. Does this relation hold if D is not in this plane?

Solution 1



Write $\angle ACD = \alpha$, $\angle BCD = \beta$, $\angle ACB = \theta$. Then

$$AD^2 = AC^2 + CD^2 - 2AC\,.\,CD\cos\alpha$$

$$BD^2 = BC^2 + CD^2 - 2BC\,.\,CD\cos\beta,$$

whence

$$AD^2 + BD^2 - CD^2 = CD^2 - 2CD(AC\cos\alpha + BC\cos\beta) + (AC^2 + BC^2).$$

The right-hand side is a quadratic expression in CD with 'discriminant' (i.e. '$b^2 - 4ac$')

$$\begin{aligned}\Delta &= 4\{(AC\cos\alpha + BC\cos\beta)^2 - (AC^2 + BC^2)\}\\ &= 4\{2AC\,.\,BC\cos(\alpha+\beta) - (AC\sin\alpha - BC\sin\beta)^2\}\\ &= 4\{2AC\,.\,BC\cos\theta - (AC\sin\alpha - BC\sin\beta)^2\}\end{aligned}$$

(because D lies in the plane of A, B, C). But

$$AB^2 = AC^2 + BC^2 - 2AC \, . \, BC \cos\theta,$$

so that

$$\Delta = 4\{AC^2 + BC^2 - AB^2 - (AB \sin\alpha - BC \sin\beta)^2\}.$$

Since $AB^2 \geq AC^2 + BC^2$, this means that $\Delta \leq 0$. It follows that $AD^2 + BD^2 - CD^2$ cannot change its sign for different positions of D. If we take $D = C$, we see that it must be ≥ 0, as required.

If D is not in the plane of A, B, C, we may construct C' in the plane of A, B, D such that $AC' = AC$, $BC' = BC$ and $C'D \geq CD$. Then, because $AB^2 \geq AC^2 + BC^2$, we have $AB^2 \geq AC'^2 + BC'^2$, so, by the first part,

$$CD^2 \leq C'D^2 \leq AD^2 + BD^2,$$

and the relation holds in general. (Ed: The last part of Ruth Lawrence's solution requires a little thought. It may be better to look at the projection D' of D on to the plane ABC and use the first part together with Pythagoras' theorem.)

Solution 2. Ruth Lawrence's second solution uses vectors. As well as being very neat, it solves the general case when D may not lie in the plane of A, B, C straightaway. Let A, B, C have position vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ respectively relative to the origin D. We are given that

$$AB^2 \geq AC^2 + BC^2,$$

so

$$|\boldsymbol{a} - \boldsymbol{b}|^2 \geq |\boldsymbol{a} - \boldsymbol{c}|^2 + |\boldsymbol{b} - \boldsymbol{c}|^2,$$

which simplifies to

$$0 \geq 2|\boldsymbol{c}|^2 - 2\boldsymbol{a} \, . \, \boldsymbol{c} - 2\boldsymbol{b} \, . \, \boldsymbol{c} + 2\boldsymbol{a} \, . \, \boldsymbol{b},$$

or

$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - |\boldsymbol{c}|^2 \geq |\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}|^2 \geq 0.$$

Hence $|\boldsymbol{c}|^2 \leq |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$, or $CD^2 \leq AD^2 + BD^2$, as required.

15.8. Show that, for every positive integer n and every positive odd integer k, $1 + 2 + \cdots + n$ divides $1^k + 2^k + \cdots + n^k$.

Solution

Now $n + 1 - r \equiv -r \pmod{n+1}$, so that

$$\sum_{r=1}^{n} (n + 1 - r)^k \equiv \sum_{r=1}^{n} (-r)^k = -\sum_{r=1}^{n} r^k \qquad \text{because } k \text{ is odd.}$$

But $\sum_{r=1}^{n} (n + 1 - r)^k = \sum_{r=1}^{n} r^k$, so that $2\sum_{r=1}^{n} r^k \equiv 0 \pmod{n+1}$. Also

$$2\sum_{r=1}^{n} r^k = 2\sum_{r=1}^{n-1} r^k + 2n^k \equiv 0 \pmod{n},$$

using the previous part with $n - 1$ in place of n. Since n and $n + 1$ are coprime, this means that

$$2\sum_{r=1}^{n} r^k \equiv 0 \pmod{n(n+1)}.$$

But $n(n+1)$ is even, so that $\sum_{r=1}^{n} r^k \equiv 0 \pmod{\tfrac{1}{2}n(n+1)}$, and $1^k + 2^k + \cdots + n^k$ is divisible by $1 + 2 + \cdots + n$, as required.

15.9. Let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \le x_i \le 1$ for $i = 1, 2, \ldots, n$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \le \begin{cases} \dfrac{1}{4} & \text{when } n \text{ is even} \\[2ex] \dfrac{1}{4} - \dfrac{1}{4n^2} & \text{when } n \text{ is odd.} \end{cases}$$

Discuss when equality occurs.

Solution

Now $0 \le x_i \le 1$ so $0 \le x_i^2 \le x_i$ for $1 \le i \le n$, whence

$$\sum_{i=1}^{n} x_i^2 \le \sum_{i=1}^{n} x_i. \tag{1}$$

Put

$$E(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2, \qquad \alpha = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Then

$$E(x_1, \ldots, x_n) \le \alpha - \alpha^2 = \tfrac{1}{4}(1 - (2\alpha - 1)^2) \le \tfrac{1}{4}. \tag{2}$$

We now consider how to maximize E by varying just one of the x_i's, say x_n. We have

$$\begin{aligned} E(x_1, \ldots, x_n) &= \frac{1}{n}\left(x_n^2 + \sum_{i=1}^{n-1} x_i^2\right) - \left(\frac{x_n}{n} + \frac{1}{n}\sum_{i=1}^{n-1} x_i\right)^2 \\ &= \left(\frac{1}{n} - \frac{1}{n^2}\right)x_n^2 - \left(\frac{2}{n^2}\sum_{i=1}^{n-1} x_i\right)x_n + \text{terms independent of } x_n \\ &= \frac{1}{n^2(n-1)}\left\{(n-1)x_n - \sum_{i=1}^{n-1} x_i\right\}^2 + \text{terms independent of } x_n. \end{aligned}$$

Thus, for fixed values of $x_1, \ldots, x_{n-1}$, E attains a maximum value when

$$\left|x_n - \frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right|$$

is a maximum. Since we have the restriction $0 \le x_n \le 1$, this occurs when x_n is either 0 or 1.

Now consider $E(x_1, \ldots, x_n)$. If x_1 is not 0 or 1, we can increase the value of E by changing x_1 either to 0 or 1, keeping $x_2, \ldots, x_n$ fixed. Now move to x_2. A similar thing happens. We can deal with each variable in turn. It follows that $E(x_1, \ldots, x_n) \le E(x_1', \ldots, x_n')$ for some values $x_1', \ldots, x_n'$ which are 0 or 1. Now, when n is odd,

$$\left|\sum_{i=1}^{n} x_i' - \frac{n}{2}\right| = \left|\sum_{i=1}^{n} (x_i' - \tfrac{1}{2})\right| \ge \frac{1}{2},$$

so

$$\left(n\alpha' - \frac{n}{2}\right)^2 \ge \frac{1}{4} \qquad \left(\text{with } \alpha' = \frac{1}{n}\sum_{i=1}^{n} x_i'\right)$$

or $(2\alpha' - 1)^2 \ge 1/n^2$.

Thus, when n is odd, we obtain from (2) that

$$E(x_1,\ldots,x_n) \leq E(x'_1,\ldots,x'_n) \leq \tfrac{1}{4}(1-(2\alpha'-1)^2) \leq \tfrac{1}{4}\left(1-\frac{1}{n^2}\right).$$

When n is even, (2) gives straightaway that $E(x_1,\ldots,x_n) \leq \frac{1}{4}$.

To obtain equality, we need equality at (1), and this occurs if and only if each x_i is 0 or 1. When n is even, we also need $\alpha = \frac{1}{2}$ for equality, from (2), and equality occurs if and only if half of the x_i's are 0 and half are 1. When n is odd, equality occurs if and only if each x_i is 0 or 1 and $\sum_{i=1}^{n}(x_i - \frac{1}{2}) = \pm\frac{1}{2}$, i.e. if and only if $(n-1)/2$ of the x_i's are 1, $(n-1)/2$ are 0, and the remaining x_i is 0 or 1.

Book Reviews

The Higher Arithmetic (fifth edition). By H. Davenport. Cambridge University Press, 1982. Pp. 189. £12·00 hardback, £5·00 paperback.

The late Professor Davenport was an exceptionally attractive writer, and the new edition of his book (which appeared originally in 1952 and is now published under the imprint of Cambridge University Press) is to be warmly welcomed. For all its brevity and unpretentiousness it can be described as a classical account in a double sense. The material dealt with has been the subject of intensive study since the seventeenth century and many of the most illustrious mathematicians (Fermat, Legendre, Gauss, Jacobi, Dirichlet to name but a few) have contributed to it. And, secondly, the account given here by the author is so well-balanced, so pleasingly presented, and by now so firmly established that it can justly claim a position as a minor classic in its own right.

'Higher arithmetic' is, of course, the traditional name for the theory of numbers, that is, the study of properties of whole numbers. The statement of results in this domain is often childishly simple, but the construction of proofs calls not merely for great powers of inventiveness but also for understanding at a really deep level. Here the author confines himself to topics whose discussion requires almost nothing in the way of technical preparation, the chapter headings being (i) Factorization and the primes, (ii) Congruences, (iii) Quadratic residues, (iv) Continued fractions, (v) Sums of squares, (vi) Quadratic forms, (vii) Some Diophantine equations. This is a fairly standard fare for a book on elementary number theory, but the mode of writing is unusual. It reads less like a formal textbook and much more like an easy-flowing, informal essay; yet, amazingly, the standards of careful, logical argument are fully observed. Instruction is liberally laced with pleasure, and a mastery of the material deployed here will put one in a strong position to grapple with more advanced works.

It may well be that parts of the book are a little too exacting for many readers of *Mathematical Spectrum*—more mathematical experience is, perhaps, needed to appreciate fully the beauty and accuracy of the argument. For all that, they are urged to have a try: even an incomplete attempt is likely to arouse curiosity and to implant a wish for further study of the subject a little later in their mathematical careers.

University of Sheffield — L. Mirsky

Calculator Calculus. By GEORGE MCCARTY. E. and F. N. Spon Ltd., London, 1982. Pp. xiv + 254. £5·95 paperback.

Some idea of the contents of this book may be gained by listing the various chapter headings. Following a fourteen-page preface these are: 1. Squares, Square Roots and the Quadratic Formula; 2. More Functions and Graphs; 3. Limits and Continuity; 4. Differentiation, Derivatives, and Differentials; 5. Maxima, Minima, and the Mean Value Theorem; 6. Trigonometric Functions; 7. Definite Integrals; 8. Logarithms and Exponentials; 9. Volumes; 10. Curves and Polar Coordinates; 11. Sequences and Series; 12. Power Series; 13. Taylor Series; 14. Differential Equations. There follows an appendix entitled 'Some Calculation Techniques and Machine Tricks', and a section headed 'Reference Data and Formulas' which summarizes a number of elementary algebraic and geometric formulas and also includes a list of derivatives and integrals of the elementary functions of analysis.

The preface makes two important points. In the first place I quote from page x: 'There has been no attempt to be complete in the exposition of theory in this book, but the most important theorems are cited explicitly and illustrated numerically'. In other words no student can expect to grasp the subject by studying this book alone; he will also need to refer to a conventional text for a more rigorous treatment of the subject.

Secondly, the author warns a would-be reader that 'The recommended machine for our works is one which has buttons to calculate trig and log functions, displays at least eight digits and has an adaptor-recharger ... it is our experience that a student who attempts to do the work with a four- or seven-function machine will become distracted by the copious arithmetic and eventually will despair'.

The book is clearly set out, is endowed with ample diagrams and is refreshingly free from typographical errors. (A careful reading disclosed fewer than half-a-dozen.) Each chapter is followed by a collection of exercises and then by a collection of more difficult problems, of which the harder ones are starred and to which answers are provided.

The main general criticism I have to make is that the author uses terms before he has defined them. Thus (i) 'continuous graph' appears on p. 15 but a continuous function is not defined until p. 27; (ii) a closed interval $[a, b]$ appears on p. 57 but is not defined at all; (iii) Euler's constant γ appears on p. 29 but is not defined until p. 162, and so on. More specifically, the equation on p. 110, Exercise 4, line 5, is incorrect, and I suspect that a similar error has been made on p. 185, line 10.

To summarize, this book uses the calculator as a microscope to study very small numbers, with consequent illumination of limits, whether they arise as limits of iterative processes in the solution of equations, or as limits of sums in the integral calculus. One thing is certain: no student who has used his calculator to study the sequence $(1{\cdot}1)^{10}$, $(1{\cdot}01)^{100}$, $(1{\cdot}001)^{1000}, \ldots$ will ever fall into the error, so familiar to teachers of analysis, of writing

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1.$$

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J. G. BRENNAN

Linear Algebra: an Introduction (second edition). By A. O. MORRIS. Van Nostrand Reinhold Company Ltd., Wokingham, 1982. Pp. viii + 203. £4·75.

A second edition of this well-written book is welcome, and doubly so as it now contains answers to all the exercises. It is intended for first-year students in universities and polytechnics, but may well find readers also in the sixth form as more of the topics covered feature in A-level syllabuses.

CONTENTS

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