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- Brahmagupta
- Geometry of p-adic fields
- Dynamic programming

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Brahmagupta

CHRIS PRITCHARD

After the Greeks, European mathematics slumbered for many centuries during which progress was made first in India and China and then in the Arab world. The greatest of the Indian mathematicians was Brahmagupta.

Introduction

Brahmagupta was born in 598, possibly in the Rajasthani town now known as Bhinmal, and died no earlier than the year 665. He lived and worked to the south of his native area, in Ujjain, an astronomical centre in what is now Madhya Pradesh, and he was fortunate in enjoying the patronage of a person of wealth. Whether this patron was Vayaghramukha, the reigning king of the Gurjaras whose capital was Bhinmal itself, is not known. The question of Brahmagupta's caste remains unresolved too, largely because of the form of his name. The *gupta* ending suggests membership of the Vaisya, the third highest of the social classes. At the time, the opportunity for even an intellectually gifted Indian to engage in contemplation fell almost exclusively to those belonging to the highest of the social classes, the Brahmins: so the patronage he enjoyed may have proved vital. In the eyes of Brahmagupta, mathematical activity has an aesthetic quality which demands that questions of little practical value be posed 'simply for pleasure', as he put it.

Brahmagupta's masterpiece, *Brahma-sphuta-siddhanta* (The Revised System of Brahma), composed in 628, consists of twenty-four chapters which are devoted in the main to astronomy and written in verse. It is clear throughout that he was fully familiar with the work of Ptolemy, the second century Alexandrian astronomer. Two chapters on mathematics were translated from Sanskrit into English by Colebrooke in 1817. The twelfth, *Ganita*, or 'arithmetical calculation', considers mathematical series and some aspects of geometry; the eighteenth, *Kuttaka*, loosely translated as 'algebra' but more literally as 'multiplier', offers solutions to first- and second-order indeterminate equations. In a second astronomical work, *Khanda Khadyaka*, Brahmagupta made use of methods equivalent to Stirling's interpolation formula to obtain the sines of intermediate angles from a table of sines.

Geometry

In common with other Indian geometers, Brahmagupta stated and extended the results of the Greeks and Chinese, though paying scant attention to definitions, axioms or formal proof. He gave Heron's formula for the area of a triangle given the lengths of all three sides. In extending the formula to quadrilaterals with sides a, b, c, d and semi-perimeter s,

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

he failed to state explicitly that the result is true for cyclic quadrilaterals only, though his facility with cyclic quadrilaterals is such that there can be little doubt that he must have known this well. Unfortunately, the twelfth century Indian mathematician, Bhaskara, added to the confusion by declaring that there was no quadrilateral, cyclic or otherwise, for which the formula was valid.

Brahmagupta produced two further cyclic quadrilateral formulae:

(a) if x and y are the diagonals of a cyclic quadrilateral of sides a, b, c and d, then

$$x^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}$$

and (1)

$$y^2 = \frac{(ab+cd)(ac+bd)}{ad+bc};$$

(b) if the elements of two Pythagorean triples (a,b,c) and (A,B,C), (so that $a^2+b^2=c^2$ and $A^2+B^2=C^2$) are combined to form products aC, cB, bC and cA representing the lengths of the sides of a quadrilateral (dubbed *Brahmagupta's trapezium* at a time when the word 'trapezium' was used with greater freedom than today) then that quadrilateral is cyclic.

As an example of the latter result, take the triples $(3,4,5)$ and $(7,24,25)$, which yield a cyclic quadrilateral with sides of length 75, 120, 100 and 35; or equivalently, 15, 24, 20 and 7. The triple $(3,4,5)$ taken with itself or any multiple of itself yields the sides 15, 20, 20 and 15 of a kite and, because the products form two symmetric pairs, this occurs for all other triples taken in self-combination.

The proof that the diagonals of Brahmagupta's trapezium are perpendicular, is worth giving. First it is necessary to use the cosine rule to prove a lemma.

Lemma. The diagonals of a quadrilateral are perpendicular if and only if the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair of opposite sides.

Proof. Construct a quadrilateral with sides of length a, b, c and d and diagonals of length $t+u$ and $v+w$, intersecting at angle ϕ, as in figure 1. Then, by the cosine rule,

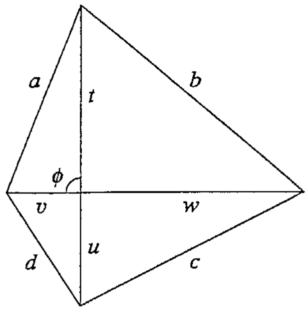

Figure 1

$$a^2 = v^2+t^2-2vt\cos\phi \quad \text{and} \quad c^2 = u^2+w^2-2uw\cos\phi,$$

whence

$$a^2+c^2 = v^2+t^2+u^2+w^2-2(vt+uw)\cos\phi.$$

Similarly

$$b^2+d^2 = v^2+t^2+u^2+w^2-2(tw+uv)\cos(180°-\phi)$$

or

$$b^2+d^2 = v^2+t^2+u^2+w^2+2(tw+uv)\cos\phi.$$

Thus $a^2+c^2 = b^2+d^2$ if and only if $\cos\phi = 0$, i.e. $\phi = 90°$.

Consider a quadrilateral with sides of length aC, cB, bC and cA, constructed from the Pythagorean triples (a,b,c) and (A,B,C). One pair of opposite sides has lengths aC and bC, and the other pair of opposite sides has lengths cB and cA. Now

$$\begin{aligned}(aC)^2+(bC)^2 &= (a^2+b^2)C^2\\ &= c^2C^2\\ &= c^2(A^2+B^2)\\ &= (cA)^2+(cB)^2.\end{aligned}$$

This confirms that the diagonals of Brahmagupta's trapezium meet at right angles.

We are in a position to calculate the sides, diagonals and area of the Brahmagupta trapezium generated by the Pythagorean triples $(3,4,5)$ and $(5,12,13)$. The sides are $aC = 39$, $cB = 60$, $bC = 52$ and $cA = 25$. Applying (1) above, the squares of the diagonals are given by

$$x^2 = \frac{(39\times25+60\times52)(39\times52+60\times25)}{39\times60+52\times25} = 3969,$$

$$y^2 = \frac{(39\times60+52\times25)(39\times52+60\times25)}{39\times25+60\times52} = 3136,$$

whence $x = 63$ and $y = 56$. Brahmagupta's extension to Heron's formula yields an area of 1764, a figure confirmed directly by noting that the area of the figure is half the product of the diagonals. At this point, Brahmagupta idiosyncratically undermined his own work by invoking an outmoded rule of thumb for calculating the 'gross' area of a quadrilateral, 'the product of the arithmetic means of the opposite sides', to give: Area $= \frac{1}{4}(39+52)(60+25) = 1933\frac{3}{4}$.

Number theory

Before Brahmagupta, mathematicians had been satisfied, by and large, to use indeterminate equations to find single solutions to practical problems. While considering the second-order form $ax^2+c = y^2$, where c may be positive or negative, Brahmagupta discovered a way of generating an infinite sequence of solutions usually from an initial solution spotted by inspection. When $c = 1$ the form reduces to the so-called Pell equation $ax^2+1 = y^2$. (The name of John Pell (1610–85), the English number theorist who in 1668 gave the factors of the first 100 000 integers, was carelessly attached to the equation by Euler!) The method he employed, a derivative of Euclid's algorithm, can be gleaned from this relatively straightforward example: solve

$$3x^2+1 = y^2.$$

By inspection $x_1 = 1$, $y_1 = 2$ satisfies the equation.

Now, combining this solution initially with itself and thereafter with subsequent solutions, we have the following scheme:

Cross products

$x_1 = 1$ $y_1 = 2 \longrightarrow 2$
$x_1 = 1$ $y_1 = 2 \longrightarrow 2$

Sum of cross products = 4.

Thus $x_2 = 4$ and, hence by substitution, $y_2 = 7$.

Cross products

$x_1 = 1$ $y_1 = 2 \longrightarrow 8$
$x_2 = 4$ $y_2 = 7 \longrightarrow 7$

Sum of cross products = 15.

Thus $x_3 = 15$ and, hence by substitution, $y_3 = 26$.

Continuing in like fashion, with x_1 and y_1 taken in combination with x_3 and y_3, then with x_4 and y_4, and so on, gives the solution set

$$\{(1,2),\ (4,7),\ (15,26),\ (56,97),\ (209,362),\ (780,1351),\ (2911,5042),\ (10\,864,18\,817),\ \ldots\}.$$

Readers may wish to try to prove for themselves that if $3x_1^2+1 = y_1^2$ and $3x_2^2+1 = y_2^2$ then $3(x_1y_2+x_2y_1)^2+1$ is a perfect square.

Examples of how Brahmagupta extended the method to the case in which taking $x_1 = 1$ leads to a non-integral value for y_1 are given in reference 2. Another Indian mathematician, Jayadeva, discovered in about 1000 AD that, unless the value of c is ±1, ±2 or ±4, it breaks down and has to be replaced by a *trial-and-improvement* technique.

Trigonometry

The seeds of trigonometry were sown by Hipparchus about 150 BC when he studied the relationships between the angles at the centre of a circle and the lengths of the chords that subtended them. The systematic construction of tables of chord lengths for angle sizes rising to 180° by half-degrees was given by Ptolemy after 100 AD. Early Indian mathematicians, notably Aryabhata, tended to relate the half-chord AM to half the angle which the full chord subtended, angle AOM (figure 2).

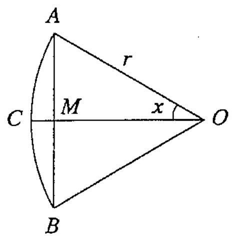

Figure 2

They calculated three quantities:

$$AM = r\sin x,$$

$$OM = r\cos x,$$

$$CM = r(1-\cos x) \quad \text{or} \quad r\,\mathrm{vers}\,x,$$

where 'vers' is the diminutive of 'versine'. (This last quantity, which the Arabs referred to as *sahen* or arrow for obvious reasons, passed into Latin as *sagitta* before fading into obscurity.) The greatest advance which Brahmagupta made to trigonometry is to be found in *Khanda Khadyaka*, composed in 665 when he was 67 years old. It was here that he demonstrated a quadratic interpolation formula, equivalent to Stirling's formula of degree two, for calculating values of $y = 150\sin x$ for angles between multiples of 15°. Given three values of x, say x_{n-1}, x_n and x_{n+1}, their corresponding function values y_{n-1}, y_n and y_{n+1}, the first- and second-order differences Δy and $\Delta^2 y$, with h the interval between the given angles,

$$\begin{aligned} y_{n+rh} &\approx y_n + \tfrac{1}{2}(\Delta y_{\frac{1}{2}(n-1)} + \Delta y_{\frac{1}{2}(n+1)})\frac{r}{1!} \\ &\quad + \tfrac{1}{2}\Delta^2 y_n\left(\frac{r(r-1)}{2!} + \frac{(r+1)r}{2!}\right) \\ &= y_n + \tfrac{1}{2}r(\Delta y_{\frac{1}{2}(n-1)} + \Delta y_{\frac{1}{2}(n+1)}) + \tfrac{1}{2}r^2\Delta^2 y_n. \end{aligned}$$

Table 1 is used here to demonstrate Brahmagupta's method of calculating $150\sin 72°$. Now 72 is $\frac{4}{5}$ of the 'distance' between 60 and 75, labelled r in the formula. So with $n = 60$ and $h = 15$, while y_n, $\Delta y_{\frac{1}{2}(n-1)}$, $\Delta y_{\frac{1}{2}(n+1)}$ and $\Delta^2 y_n$ are the values linked by the lozenge in table 1, Brahmagupta's interpolation formula yields:

$$\begin{aligned} 150\sin 72° &= 129.9 + \tfrac{1}{2}\times\tfrac{4}{5}(23.8+15) + \tfrac{1}{2}\times\tfrac{16}{25}(-8.8) \\ &= 142.604. \end{aligned}$$

The accurate value to three decimal places is 142.658. Equivalently, Brahmagupta's method gives a value of 0.9507 for $\sin 72°$ whereas its true value to four decimal places is 0.9511.

Table 1

x	y	Δy	$\Delta^2 y$
0	0		
		38.8	
15	38.8		−2.6
		36.2	
30	75		−5.1
		31.1	
45	106.1		−7.3
		23.8	
60	129.9		−8.8
		15.0	
75	144.9		−9.9
		5.1	
90	150		

Epilogue

The *Mahasiddhanta*, a work of *circa* 700 AD based largely on the *Brahma-sphuta-siddhanta*, became the basis of the *Zij al-Sindhind al-Kabir* of al-Fazari in the late eighth century. In due course this was translated into Latin by Adelard of Bath in 1126 and was then available to European mathematicians. Further refinement of Brahmagupta's method for solving indeterminate equations was made by Jayadeva and Bhaskaracharya. Some Europeans then became aware of the method through reading Fibonacci's reworking of the mathematics of Al-Fakhri while others made direct use of a translation of the *Arithmetica* of Diophantus. Solutions based on the deployment of continued fractions were developed by Fermat, Euler and Galois, but especially by Lagrange. Indian trigonometry, refined by the Arabs, first appeared in the West in 1464, in a book of Regiomontanus entitled *De triangulis omni modis*.

Exercises

1. Find, using Brahmagupta's method, the solutions of

   (a) $8x^2+1 = y^2$, (b) $15x^2+1 = y^2$.

2. Use the Brahmagupta interpolation formula to calculate:

   (a) $150\sin 53°$, (b) $150\sin 29°$, (c) $150\sin 67°$.

   In each case compare your answer with the calculator value.

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3. D. Pingree, (i) Brahmagupta, (ii) History of Mathematical Astronomy in India, in *Dictionary of Scientific Biography*, ed. C. C. Gillispie (Scribner, New York, 1981).

Solutions

1. (a) $\{(1,3), (6,17), (35,99), (204,577), (1189,3363),\ldots\}$;
   (b) $\{(1,4), (8,31), (63,244), (496,1921), (3905,15124),\ldots\}$.
2. (a) 119.7, 119.8; (b) 73.48, 72.72; (c) 138.0, 138.08. □

***Chris Pritchard** taught in London and the Western Isles before moving to Callander as Principal Teacher of Mathematics. Interested primarily in the history of the mathematical sciences, especially statistics, he has been researching the history of the normal curve on and off for the last twelve years.*

Prime Bernoulli Numbers

JOSEPH McLEAN

The author continues his search for prime Bernoulli numbers begun in Volume 25, Number 1.

The triangular Bernoulli numbers ${}^nB_r$ were defined originally by A. W. F. Edwards (*Mathematical Spectrum*, Volume 23, Number 4) as the partial sums of the horizontal rows of Pascal's triangle, and may be given by the formula

$$ {}^nB_r = \sum_{i=0}^{r} \binom{n}{i}. $$

The low frequency of primes when r is odd previously encouraged me to prove the following result (see *Mathematical Spectrum*, Volume 25, Number 1, pp. 8–10).

Proposition 1. For the Bernoulli number ${}^nB_r$ to be a prime when r is odd, then n must be even and such that $n+1$ divides $r!$.

This reduces greatly the number of n to be considered as providing possible primes for a given odd r and so speeds up the search time. In the remainder of this article, all Bernoulli numbers with fewer than 120 digits have been proved prime, while larger numbers await verification. With a search limit of $n < 2^{16}$ the results in table 1 are obtained.

Note that there are no Bernoulli primes at all for $r = 3$, 5, 7 or 11 and none within the search limit for $r = 15$, 27, 31 and 55.

Alternatively, for each even n we can search exhaustively through all odd $r < n$. Note that if $n+1$ is a prime there can be no solutions since proposition 1 requires that $n+1$ divides $r!$. In fact, there are no prime Bernoulli numbers with odd r at all for $n < 56$. For higher values of n, the results in table 2 are obtained, with no more primes for $n \leqslant 540$.

If we now consider the case of r even, where the severe restriction on n required for possible primes does not apply, it is of interest to locate the smallest n for each even r such that ${}^nB_r$ is a prime. The results for $r \leqslant 200$ are presented in table 3.

The largest of the Bernoulli numbers in table 3, for $r = 188$, has 440 digits.

Note that ${}^nB_{n-1} = 2^n - 1$, and so Bernoulli numbers of this form and which are prime are Mersenne primes. Such primes have been the subject of much study for many years and it is known that there are only 30 for $n < 200\,000$.

Additionally of interest is the occurrence of any adjacent tuples, i.e. for a given r, sequences ${}^nB_r$, ${}^{n+1}B_r, \ldots$, of consecutive Bernoulli numbers which are all prime. With all the following results, the search limit on n is $2^{16} = 65\,536$.

Table 1

Odd r	Even n for which ${}^nB_r$ is a prime
9	314
13	62, 64
17	62, 64, 90, 1364, 1682
19	56, 152, 1728, 9944
21	246
23	56, 64, 1784
25	64, 90, 220, 224, 1274, 2430, 57 056
29	1310, 3552
33	934, 1770, 19 574, 49 724
35	252, 424, 1704, 7292
37	494, 896, 17 574, 22 184
39	1704
41	560, 620, 32 174, 36 974
43	764, 2944, 3392, 6764, 15 344, 33 004
45	128, 174, 11 824
47	5680, 6272, 16 064, 17 328
49	1014, 38 398, 41 012
51	436
53	6272, 9656
57	17 916, 46 654, 49 342
59	54 144
61	8126, 59 454
63	128
65	734, 1880, 2386, 3812, 4304, 4990, 6992, 12 494, 12 504, 22 490, 43 354, 46 664, 59 736
67	84, 324, 328, 2536, 7136, 9540, 25 584
69	454, 464, 1014, 1112, 20 294, 37 110, 37 880, 41 846, 59 240
71	2008, 17 384, 43 616, 47 328
73	110, 122, 860, 1024, 5840, 8670, 9854, 13 166, 16 496, 43 482, 43 644
75	364, 1024, 50 812

Table 2

Even n	Odd r for which ${}^nB_r$ is prime	Even n	Odd r for which ${}^nB_r$ is prime
56	19, 23	320	259
62	13, 17	324	67
64	13, 17, 23, 25	328	67
84	67	344	87
90	17, 25	364	75
110	73	376	81, 293
122	73	412	385
128	45, 63	424	35
152	19	436	51
174	45	440	183
176	169	444	169
220	25	446	177, 189, 293, 385
224	25, 27	454	69
246	21, 145, 149, 193	464	69, 257
248	85, 129, 165	476	387
252	35	480	133
254	169, 185	492	201
286	273	494	37

Table 3

r	n	r	n	r	n	r	n
2	3	52	181	102	1264	152	250
4	5	54	56	104	377	154	3775
6	7	56	64	106	107	156	1280
8	10	58	191	108	366	158	10496
10	27	60	61	110	127	160	244
12	13	62	128	112	246	162	944
14	16	64	81	114	116	164	432
16	17	66	87	116	125	166	256
18	19	68	70	118	128	168	174
20	30	70	87	120	2297	170	176
22	31	72	91	122	10752	172	8109
24	26	74	80	124	2176	174	255
26	60	76	94	126	127	176	191
28	157	78	127	128	221	178	1271
30	31	80	119	130	464	180	1207
32	162	82	896	132	991	182	440
34	180	84	373	134	231	184	2814
36	53	86	1784	136	224	186	13307
38	184	88	89	138	2783	188	15360
40	188	90	764	140	143	190	2304
42	620	92	3965	142	3216	192	467
44	174	94	3711	144	246	194	488
46	448	96	874	146	243	196	718
48	50	98	127	148	2264	198	2512
50	759	100	245	150	3743	200	237

For $r = 6$ there are 81 adjacent pairs, or doublets. Those with $n < 10\,000$ are: $n = 55$, 591, 1207, 2327, 3799, 4423, 5159, 6407, 7127, 8663, 8983, 9103, 9183, and no triplets.

For $r = 8$ there are 184 doublets. Those with $n < 2000$ are: $n = 217$, 252, 431, 524, 681, 682, 810, 1007, 1050, 1226, 1261, 1449, 1487, 1533, 1583, 1710.

For $r = 8$ there are 13 triplets: $n = 681$, 5357, 9166, 9225, 12 201, 24 475, 24 699, 35 084, 45 449, 45 450, 49 163, 52 459, 60 365 and the quartet beginning with $n = 45\,449$.

For $r = 10$ there are 18 pairs. Those with $n < 10\,000$ are: $n = 27$, 47, 491, 5935, 6475, 7871, 8303, and no triplets.

For $r = 12$ there are 28 pairs. Those with $n < 10\,000$ are: $n = 13$, 14, 1791, 3166, 5085, 8991, 9151, 9758 and one triplet beginning with $n = 13$.

For $r = 14$ there are five pairs, for $n = 527$, 9567, 32 303, 36 447, 48 815, and no triplets.

For $r = 16$ there are 36 pairs. Those with $n < 10\,000$ are: $n = 57$, 82, 147, 148, 248, 2229, 2683, 2808, 2809, 3004, 5401, 7060, 8563, 9749 and two triplets beginning with $n = 147$ and $n = 25\,435$.

For $r = 18$ there are 20 pairs. Those with $n < 10\,000$ are: $n = 567$, 4639, 4991, 5143, and no triplets.

For $r = 20$ there are 15 pairs. Those with $n < 10\,000$ are: $n = 950$, 1375, 4605 5055, 9949, and no triplets.

For $r = 22$ there are four pairs, for $n = 119$, 215, 7455, 10 815, and no triplets.

For $r = 24$ there are 10 pairs, for $n = 57$, 58, 219, 2459, 3421, 6111, 16 447, 22 172, 39 065, 44 379 and the triplet beginning with $n = 57$.

The results in table 4 give some idea of the much larger occurrence of primes for even r, to the search limit of $n < 65\,536$.

Table 4

r	Number of primes	Number of doublets	Number of triplets	Number of quartets
2	6739	659	0	0
4	3940	381	27	1
6	1624	81	0	0
8	2387	184	13	1
10	751	18	0	0
12	695	28	1	0
14	306	5	0	0
16	884	36	2	0
18	519	20	0	0
20	514	15	0	0
22	228	4	0	0
24	457	10	1	0

□

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The Smarandache function and the number of primes up to x

For a positive integer k, $S(k)$ denotes the smallest natural number n such that $k \mid n!$. Then $S(k) \leqslant k$ and $S(k) = k$ if and only if $k = 1$, 4 or a prime number. Hence, for $x \geqslant 4$, the number of primes less than or equal to x is

$$\sum_{k=2}^{x} \left[\frac{S(k)}{k}\right] - 1,$$

where [ ] denotes the integral part.

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First Steps in the Geometry of p-adic Fields

I. SH. SLAVUTSKII

Everyone meets the task of finding a mistake in the 'proof' that all triangles are isosceles, for example. However, there exist domains where the above statement is true!

We consider $\mathbb{Q}$, the field of rational numbers, and two metrics on it. The first of them is the usual absolute value, and the second one is the p-adic valuation. It is known that the modulus of rational numbers, r and s, satisfies the conditions

(a) $|r| \geqslant 0$ and $|r| = 0 \Leftrightarrow r = 0$,
(b) $|rs| = |r| \times |s|$,
(c) $|r+s| \leqslant |r| + |s|$,

from which it follows easily that

(d) $|\pm 1| = 1$,
(e) $|r/s| = |r|/|s|$ $(s \neq 0)$,
(f) $|r| - |s| \leqslant |r-s|$, (and also $||r| - |s|| \leqslant |r-s|$).

This measure on the rationals is the most important one but is not the only one.

Now fix p, a prime integer. For $r \in \mathbb{Q}$ $(r \neq 0)$ there is the unique representation

$$r = \frac{a}{b}p^k,$$

where $b > 0$, $a, k \in \mathbb{Z}$ and $(ab, p) = 1$. For example, with $p = 2$, $\frac{24}{7} = \frac{3}{7} \times 2^3$ and $\frac{7}{24} = \frac{7}{3} \times 2^{-3}$. Choose a real number λ with $0 < \lambda < 1$ and introduce

$$|r|_p = \begin{cases} \lambda^k & (\text{if } r \neq 0), \\ 0 & (\text{if } r = 0), \end{cases} \tag{1}$$

the *p-adic norm* (or *p-adic valuation*) on $\mathbb{Q}$. We normalize it by writing $|p|_p = 1/p$, i.e. $\lambda = 1/p$, and throughout this article $| \ |_p$ will be the p-adic valuation.

For example, if $p = 5$ then $|\frac{25}{126}|_5 = \lambda^2 < 1$, $|\frac{126}{25}|_5 = \lambda^{-2} > 1$, $|\frac{23}{126}|_5 = 1$ and $|0|_5 = 0$. But if $p = 3$ then $|\frac{25}{126}|_3 = \lambda^{-2} > 1$, $|\frac{126}{25}|_3 = \lambda^2 < 1$, $|\frac{25}{127}|_3 = 1$ and $|0|_3 = 0$.

As with the usual absolute value, the p-adic value has the properties (a)–(c), and hence also satisfies (d)–(f). Consider (c), and let

$$r_1 = \frac{a_1}{b_1}p^{k_1}, \qquad r_2 = \frac{a_2}{b_2}p^{k_2}.$$

Then $k_1 \leqslant k_2$ implies $|r_1|_p \geqslant |r_2|_p$ and

$$|r_1 + r_2|_p = \left| p^{k_1}\left(\frac{a_1}{b_1} + p^{k_2-k_1}\frac{a_2}{b_2}\right)\right|_p$$

$$\leqslant |p^{k_1}|_p \leqslant |r_1|_p + |r_2|_p, \tag{*}$$

which proves property (c). The remark in brackets in property (f) becomes

$$||r_1|_p - |r_2|_p|_\infty \leqslant |r_1 - r_2|_p.$$

(Note that $| \ |_\infty$ will denote the ordinary absolute value of real or complex numbers.) In fact, we have

$$|r_1|_p = |r_1 - r_2 + r_2|_p \leqslant |r_1 - r_2|_p + |r_2|_p,$$

so that

$$|r_1|_p - |r_2|_p \leqslant |r_1 - r_2|_p.$$

Also

$$|r_2|_p - |r_1|_p \leqslant |r_2 - r_1|_p = |r_1 - r_2|_p$$

and the result follows.

Note that we have not only proved the third property of the p-adic valuation but also obtained a stronger result:

$$\left|\sum_i r_i\right|_p \leqslant \max_i\{|r_i|_p\}. \tag{2}$$

We can prove a more precise result. From now on we shall drop the subscript p and write $| \ |_p$ simply as $| \ |$.

Lemma. If $|r_1| \neq |r_2|$ then

$$|r_1 + r_2| = \max\{|r_1|, |r_2|\}. \tag{3}$$

The lemma follows immediately from (*). Indeed, for definiteness, put $|r_1| > |r_2|$, i.e. $k_1 < k_2$. Then

$$|r_1 + r_2| = \lambda^{k_1} = \max\{|r_1|, |r_2|\}.$$

On the field of rational numbers we shall consider the p-adic metric $| \ |$ defined above. Denote this domain by $\mathcal{P}$.

Call to mind how we have worked in $\mathbb{C}$, the complex field. Complex numbers are associated with the points of the complex plane. If $z_1, z_2 \in \mathbb{C}$, then $|z_1 - z_2|_\infty$ is the distance between the corresponding points of the complex plane. The equation $|z-a|_\infty = R$, the inequalities $|z-a|_\infty \leqslant R$ and $|z-a|_\infty < R$ correspond to the circle and the closed and open discs with centre a and radius R, respectively.

Now let r, s and t be distinct numbers in $\mathbb{Q}$, let $R = \lambda^l$, with $\lambda = |p|$ (for example, $|p| = 1/p$) and let $l \in \mathbb{Z}$. Consider a triangle with the vertices r, s and t and the sides $|r-s|$, $|s-t|$ and $|t-r|$. Consider also $C(a,R)$, the circle $|x-a| = R$, and closed (or open) discs in $\mathcal{P}$. For example, the open disc $D(a,R)$ is the set of numbers x which satisfy the condition $|x-a| < R$ with the radius R belonging to the collection of values of norms $\{\lambda^l : l \in \mathbb{Z}\}$. Here a is the centre of the disc $D(a,R)$. The geometry of $\mathcal{P}$ has some surprising properties.

Theorem 1. Every triangle (in $\mathcal{P}$) is isosceles.

For example, let $|r-s| < |s-t|$. Then $|t-r| = |(t-s)+(s-r)| = \max\{|t-s|, |s-r|\} = |t-s|$ and

$$|t-r| = |s-t|.$$

So, all triangles are isosceles and the base of any triangle does not exceed its other sides.

Theorem 2. Any point of a closed disc (including points of its circumference) may be taken as the centre of the disc.

Let $\bar{D}(a,R) = \{x : |x-a| \leqslant R\}$ and $b \in \bar{D}(a,R)$, so that

$$|a-b| \leqslant R.$$

Consider the disc $\bar{D}(b,R) = \{x : |x-b| \leqslant R\}$. We show that $\bar{D}(a,R) = \bar{D}(b,R)$. If $x \in \bar{D}(a,R)$, then $|x-a| \leqslant R$, so that $|x-b| = |(x-a)+(a-b)| \leqslant \max\{|x-a|, |a-b|\} \leqslant R$ and $\bar{D}(a,R) \subseteq \bar{D}(b,R)$. By symmetry, $\bar{D}(b,R) \subseteq \bar{D}(a,R)$. Therefore the discs $\bar{D}(a,R)$ and $\bar{D}(b,R)$ coincide.

Remark. This result also holds with 'closed' replaced by 'open'.

Corollary. If two closed discs have a common point, they are 'concentric'. In particular, closed discs of equal radius with a common point coincide.

Theorem 3. The distance between a fixed point exterior to a disc and any point of the disc is a constant.

Let $\bar{D} = \{x : |x-a| \leqslant R\}$ be a disc with centre a and let b be a fixed point exterior to this disc $\bar{D}$, so that $|b-a| > R$. Then, for any point $x \in \bar{D}$,

$$\begin{aligned}|x-b| &= |x-a+a-b| \\ &= \max\{|x-a|, |a-b|\} = |b-a|\end{aligned}$$

because $|x-a| \leqslant R < |b-a|$.

Finally, we give one more strange theorem for the domain $\mathcal{P}$. The symbol $\bigcup_i A_i$ denotes the union of a family A_i of points of the domain $\mathcal{P}$.

Theorem 4. Every circle $C(a,R) = \{x : |x-a| = R\}$ is a union of the disjoint open discs with the same radius R.

First take the fixed point $b \in C(a,R)$, so that

$$|b-a| = R, \tag{4}$$

and note that all points of the open disc

$$D(b,R) = \{x : |x-b| < R\} \tag{5}$$

are on the circle $C(a,R)$. In fact, let $x \in D(b,R)$. By conditions (4) and (5),

$$\begin{aligned}|x-a| &= |x-b+b-a| \\ &= \max\{|x-b|, |b-a|\} = |b-a| = R,\end{aligned}$$

so $x \in C(a,R)$. Hence

$$\bigcup_{b \in C(a,R)} D(b,R) \subseteq C(a,R),$$

and the converse inclusion is clear. Recall that, by theorem 2 for open discs, two discs of the type (5) with the same radius R coincide if they have a common point.

We conclude by proposing a few simple problems.

1. $|p|_p = 1/p$, where p is a prime. We proved that
$$||a|_p - |b|_p|_\infty \leqslant |a-b|_p.$$
Is it true that
$$||a|_p - |b|_p|_p < |a-b|_p?$$
2. Define a square with vertices a, b, c and d by $|a-b| = |b-c| = |c-d| = |d-a|$ and $|a-c| = |b-d|$. Prove that the length of a diagonal does not exceed the length of a side of a square.
3. If three sides of a quadrilateral are equal then the fourth side does not exceed them.
4. If a triangle has its vertices on the circle $C(a,R)$, then no side of the triangle exceeds R.
5. If r and s are interior points of the disc $D(a,R)$, then $|r-s| < R$.
6. If D_1 and D_2 are two disjoint discs with the centres a_1 and a_2, respectively, then $\max|x_1-x_2| = |a_1-a_2|$ for $x_1 \in D_1$ and $x_2 \in D_2$.
7. Give the definition of a rhombus (diamond) and study relations betweeen its sides and diagonals.

For a more thorough treatment of p-adic rational fields see the references.

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An Extension of a Fermat Problem

K. R. S. SASTRY

The author, whose name is familiar to *Spectrum* readers, now lives in Dodballapur, Bangalore District, India. Here he shows that altering the context of a problem alters the nature of the solution.

The pupils of Pythagoras arranged objects like sea-shells in the form of triangles and discovered the sequence of triangular numbers $1, 3, 6, 10, \ldots, \frac{1}{2}r(r+1)$, ... (figure 1).

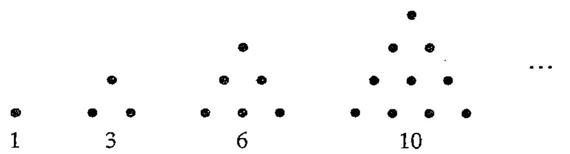

Figure 1

Likewise, they discovered the sequence of square numbers $1, 4, 9, 16, \ldots, r^2, \ldots$ (figure 2).

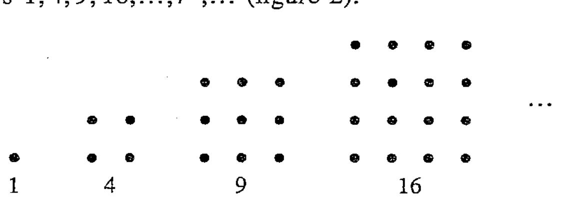

Figure 2

More generally, a *polygonal number* of side $n \geqslant 3$, or n-gonal number, is given by the formula

$$P(n,r) = \tfrac{1}{2}(n-2)r^2 - \tfrac{1}{2}(n-4)r \quad (r = 1, 2, 3, \ldots).$$

Observe that $P(3,r)$ is the sequence of triangular numbers and $P(4,r)$ that of square numbers. The sequence of pentagonal numbers is given (figure 3) by

$$P(5,r) = \tfrac{1}{2}r(3r-1) \quad (r = 1, 2, 3, \ldots).$$

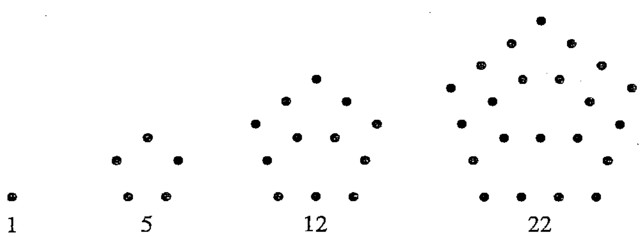

Figure 3

Euler solved the Fermat problem that no triangular number greater than 1 can be a cube or a biquadrate (i.e. a fourth power) (reference 1, page 10). However, if we replace 'triangular' number by 'n-gonal' number and ask the question:

> Can an n-gonal number greater than 1 be a cube or a biquadrate?

the answer is somewhat mixed.

Firstly, for $n = 4$ a square number can be a cube, 64 for example. Also, a square number can be a biquadrate, for example 81. In fact, we have an infinity of square numbers that are cubes given by k^6 and that are biquadrates given by k^4 ($k = 2, 3, 4, \ldots$).

Also, when $r = 2$ then $P(n,2) = n$. Hence when $n = k^3$ then $P(k^3, 2) = k^3$. Likewise $P(k^4, 2) = k^4$. Thus the second polygonal number can be a cube or a biquadrate. In light of these observations we rephrase Fermat's problem as follows:

> Let $n > 4$. Can an n-gonal number be a cube or a biquadrate for some $r > 2$?

We give a partial answer to this question.

Theorem 1. If n is an even integer greater than 8, then there is an n-gonal cube $P(n,r)$ with $r > 2$.

Proof. We have to solve the equation

$$\tfrac{1}{2}(n-2)r^2 - \tfrac{1}{2}(n-4)r = \lambda^3.$$

This suggests a neat solution for $\lambda = r$. If $\lambda = r$ then

$$\tfrac{1}{2}(n-2)r^2 - \tfrac{1}{2}(n-4)r = r^3$$

factorizes to

$$r(r-1)[2r-(n-4)] = 0.$$

Since $r \neq 1, 0$, we have $r = \frac{1}{2}(n-4)$. Furthermore,

$$P(n, \tfrac{1}{2}[n-4]) = (\tfrac{1}{2}[n-4])^3$$

and

$$\tfrac{1}{2}(n-4) > 2.$$

Thus the theorem follows.

In the next theorem we show that n-gonal numbers can be biquadrates.

Theorem 2. If $n = 2k^2 + 2k + 4$ $(k > 2)$, then the n-gonal number $P(n,r)$ is a biquadrate for $r = k$.

This is easily verified. We are now left with several interesting open problems.

When $n = 6$ then the hexagonal number $2r^2 - r = \frac{1}{2}(2r-1)(2r)$ is also a triangular number. As Euler showed, this cannot be a cube or a biquadrate.

(a) $n = 8$. Can an octagonal number $P(8,r)$ be a cube for some $r > 2$?

(b) Are there even integers n $(n > 6)$ with $n \neq 2k^2 + 2k + 4$, such that the n-gonal number $P(n,r)$ is a biquadrate for some $r > 2$?

(c) Suppose n is an odd integer greater than 3. Is there an n-gonal number $P(n,r)$ which is a cube or a biquadrate for some n and $r > 2$?

Polygonal numbers are a rich source of problems from the very elementary to the very advanced. Consult the references for a taste of their rich variety.

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Self-reference Integers

K. R. S. SASTRY

In this short article, the author offers a greater challenge than the year-puzzle we have been running for years.

A very popular problem found in school mathematics magazines, school mathematics competitions and so on is the so-called year-puzzle familiar to *Spectrum* readers:

> Use the digits of the year, say 1995, in that order to express the integers 1, 2, 3, ..., using the operations +, −, ×, ÷, √, ! and concatenation.

The article 'Primorial, Factorial and Multifactorial Primes', see reference 1, introduces the concept of a *multifactorial* to the reader as a generalization of the factorial, !. For example,

$$7!! = 7!_2 = 7{\times}5{\times}3{\times}1; \quad 7!_4 = 7\cdot 3.$$

In general,

$$n!_k = 1 \quad \text{for } n \leqslant 1;$$

otherwise

$$n!_k = n(n-k)!_k \quad (k = 1, 2, \ldots, n).$$

Now the question is: why use the digits of a year to express natural numbers? As one mathematician put it, it is God who gave the integers, not the imperfect year. So we use the digits of an integer n itself to express n. Hence we have the following definition.

Definition. A natural number n is said to have *self-reference* if n can be expressed in terms of its single digits in their order of occurrence using the operations +, −, ×, ÷, ( ), √, exponentiation and multifactorial (including the factorial !).

Here are some examples of self-reference integers.

1. Every single-digit number 0, 1, 2, ..., 9 has self-reference trivially.
2. $127 = -1+2^7$.
3. $362\,880 = (-3-6+2+8+8+0)!$.
4. $66 = 6!!+6!_3$.
5. $80 = 8!_3+0$.
6. $18 = (1+8)!_7$, $27 = (2+7)!_6$, $36 = (3+6)!_5$, $45 = (4+5)!_4$.

The above examples suggest a number of very challenging problem proposals. The basic problem is analogous to the year-puzzle:

P_1. Find as many integers greater than 10 as you can that have self-reference.

As you do P_1, look for self-reference integers with special appeal. For instance:

P_2. Is example 4 unique? Or is there another integer made of a single repeating digit expressible in terms of the multifactorials of its digits?

Look at example 5. Actually, the self-reference integer 80 induces self-reference to the entire string of ten integers beginning with 80, namely 80, 81, 82, ..., 89. This suggests:

P_3. Does there exist a string of more than ten consecutive integers having self-reference? Is it possible to determine the maximum length of such a string of self-reference integers?

Example 6 lists four self-reference integers in arithmetic progression with common difference greater than 1. Hence,

P_4. Determine five or more self-reference integers in an arithmetic progression of common difference greater than 1.

P_5. Are there three self-reference integers in geometric progression?

Finally:

P_6. Answer questions P_1, P_2, ..., P_5 in a base other than ten.

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Dynamic Programming and Two Puzzles of Dudeney

DAVID K SMITH

A puzzle posed by the king of mathematical puzzles, Henry Dudeney, is used to introduce a branch of operational research known as dynamic programming.

Introduction

The name of Henry Ernest Dudeney is well known to enthusiasts for the lighter side of mathematics. At the turn of the century Dudeney collected, invented and published puzzles and problems of many different kinds, and several of these retain the capacity to fascinate and amuse people today. Some of Dudeney's ideas have stimulated research as well. (An article on Dudeney has appeared earlier in *Mathematical Spectrum* (reference 4).) Among his other inventions, he presented a simple 'puzzle game' which he called the '37 Puzzle Game'; this deceptively easy puzzle has not received the attention that it deserves, as it is of interest to the mathematician, the computer scientist and the operational researcher.

Dudeney's '37 Puzzle Game' was described by the author as follows (reference 3):

> Here is a beautiful new puzzle game, absurdly easy to play but quite fascinating. To most people it will seem to be practically a game of chance—equal for both players—but there are pretty subtleties in it, and I will show how to win with certainty.
>
> Place the five dominoes, 1, 2, 3, 4, 5, on the table. (*Dudeney illustrates this with the dominoes* (1,0), (2,0) *etc.*) There are two players, who play alternately. The first player places a coin on any domino, say the 5, which scores 5; then the second player removes the coin to another domino, say to the 3, and adds that domino, scoring 8; then the first player removes the coin again, say to the 1, scoring 9; and so on. The player who scores 37, or forces his opponent to score more than 37, wins. Remember, the coin must be removed to a different domino at each play.

Needless to say, the game doesn't need dominoes, and can be played with pencil and paper. It is perhaps easier to think of it in the way that Berlekamp et al describe it in *Winning Ways* (reference 2), where it is presented as a more general game in the family of Nim-related games. The game is identified by the value of the integer Y.

> From a single heap [of points or tokens, such as the matchsticks used in Nim] either player may subtract any number from 1 to Y, except that the immediately previous deduction may not be repeated, and you win if you can always move but at some stage your opponent cannot.

You will win in any game if you can leave the heap empty, or if you take one point away and leave a heap of one for your opponent (who cannot play because he is forbidden from repeating the 'immediately preceding deduction'). Dudeney's 'beautiful puzzle' is this game played with $Y = 5$ and a heap which starts with 37 points. (The rules for deciding that a game has finished are equivalent; in Dudeney's version, the winner either raises the total to 37 or forces the opponent to exceed a total of 37; in Berlekamp's version, being able to reach 37 is expressed as being able to make a move, and forcing the opponent to exceed 37 is expressed as placing the opponent in a position where no move is permissible.) In this article, we shall take the second version, referring to a heap of points, although it will continue to be described as 'Dudeney's game'.

When Y is an even number, the game is easy to solve. Suppose $Y = 4$, and the players are 'Black' and 'White'. If the heap has reached 5 points after Black moved by taking away 1 point, then White cannot win; taking 2 points allows Black to win by removing 3 points; White taking 3 points allows Black to win; White taking 4 points also allows Black a win. If Black had taken 2, 3 or 4 points to reach the heap size of 5 points, nothing that White can do will prevent a lost game. A heap size of 5 is secure for Black, no matter what the last move had been. In the same way, heaps of size 10, 15, 20 etc. will be secure. Other positions are secure, but only for a restricted set of (immediately) preceding moves; for instance a heap size of 3 preceded by the removal of three points. (White must take away 1 or 2 points, and then Black can remove whatever is left, and win.)

Secure heap sizes reflect the game of 'Nim' and provide a useful key to the understanding of many take-away games, Dudeney's included. In general, where Y is even, a player wins by reducing the heap to one whose size is a multiple of $(Y+1)$. This will always be a secure position. An optimal strategy, from then on, is simple: the opponent takes away X points and the first player subtracts $(Y+1-X)$. This number must be different from X, no matter what value X is. Eventually the heap will be empty after a move by the first player. If the game starts with a heap which is a multiple of $(Y+1)$, then the second player can win with optimal play. Playing the game optimally will mean that, during a game, every secure position occurs.

When $Y = 3$, secure positions are at the multiples of $(Y+1)$ as well. Playing the game is slightly more complicated. Some secure positions may be skipped. If Black leaves a heap of 4 points, the next pair of moves would be one of: 1 then 3; 2 then 1; 3 then 1. The second of these leaves White unable to move, despite there being points on the table. With a heap of 8 points left by Black, the subsequent play might 'skip over' a heap of 4 points with the removal of 2 points, then 1 followed by 2 and then 3, or 3 and then 2.

$Y = 3$ is exceptional. For other odd values of Y safe positions are not always at the multiples of $(Y+1)$. If $Y = 5$, as in Dudeney's puzzle, a heap with 6 points is only secure if Black's move was the removal of 3 points. If the preceding move was anything else, then White removes 3 points, forcing Black to take either 1 point or 2 points, and allowing White to win. So what is a secure position for Black to reach, which will be safe whatever play had led to it? The answer is a heap of 7 points. No matter what move removing points has led to this heap, there is no move by White which can prevent Black from winning. (Try it and see!) But although a heap of 7 points is a secure position, and the pattern of secure positions for Y even is that they are multiples of a fixed value, we cannot assume that the next secure position for Black is a heap of 14. (It isn't!) Now is the chance for the reader to try to find an optimal strategy for the game before a discussion of systematic ways of looking at it.

Game strategy

The secure heaps for Black in Dudeney's game have 7, 13, 20, 26 and 33 points. However, in a game, it may not always be possible to achieve them all, and in the solution which Dudeney published, he gave instances of play by the second player in which some of these heap sizes were avoided. There is clearly a regular pattern for the secure positions, alternately increasing in size by 6 or 7. If the game is extended further, these intervals between secure positions are repeated. The fact that the two intervals are $Y+1$ and $Y+2$ is no coincidence.

Similar patterns of secure positions appear for other values of Y. Berlekamp (reference 2) presents the table (similar to table 1) for selected (odd) values of Y. This table shows the gaps between successive secure positions, referred to as 'pearls'. These are spaced at intervals $E = Y+1$ and $D = Y+2$. In the 37-game with $Y = 5$, the gaps are given by the entry (5 modulo 8), shown as (*DE*).

Table 1 shows that different values of Y (when Y is odd) can produce a variety of patterns of gaps between 'pearls'. Some, like Dudeney's game, have gaps which are alternately D and E, and others have the same secure positions as when Y is even. There are longer patterns ($Y = 23$ gives the pattern *DDEDDDDEE*, corresponding to the pearls 25, 50, 74, 99, 124, 149, 174, 198 and 222 before the gaps repeat and the next pearls are 222+25, 222+50, 222+74, 222+99 etc.). For $Y = 55$ and $Y = 95$, the first gaps are different from the later pattern. ($Y = 55$ has pearls 57, 114, 114+56, 114+56+57, 114+56+57+56 = 283, 283+56, 283+56+57, 283+56+57+56 etc.)

Table 1. Berlekamp's secure positions as 'pearl-strings'. (The items in parentheses are repeated.)

Y	Pearlstring
3 modulo 8	(*E*)...
5 modulo 8	(*DE*)...
7	(*DEE*)...
9	(*DDE*)...
15 modulo 32	(*E*)...
17 modulo 32	(*DDE*)...
23	(*DDEDDDDEE*)...
25 modulo 32	(*DE*)...
31 modulo 128	(*DEE*)...
33	(*DDDEDDE*)...
39 modulo 128	(*DEE*)...
41	(*DDDEDE*)...
55	DD(*EDE*)...
63 modulo 128	(*E*)...
65 modulo 128	(*DDDE*)...
7 modulo 64 and ⩾ 71	(*DE*)...
73 modulo 128	(*DDEDE*)...
87 modulo 128	(*DDE*)
95	DDEE(*DDE*)...
97	(*DDEDDDE*)...
103	(*DE*)...
105 modulo 128	(*DE*)...

Dudeney's problem is of interest because it is small but the solution—as the author wrote—has some subtleties which are not immediately obvious.

Dynamic programming

In the nineteen-fifties and sixties, the subject of dynamic programming was developed as an approach to solving sequential problems, where decisions are actually or effectively taken one after another. Many mathematical problems are like this. Commercial decisions are often taken in succession to one another, separated by periods of time where the business may change a little. Retailers assess their stock at regular intervals and so make a series of decisions about whether to place orders or not. Batsmen in cricket make decisions about whether they should attempt a single on each of the six balls in an over so as to try to change who is facing the bowling. Bidders in an auction make a succession of decisions about whether or not to increase their bids until they decide to drop out or successfully purchase the item. Players in many games face similar problems, as they make decisions one after another. This was noted by Richard Bellman, the pioneer of dynamic programming (DP), who described how the approach of sequential decisions could be used for finding an optimal strategy for a range of games (reference 1). The best decisions depend on many factors. The retailer must count the stock; the batsman observes the ferocity of the bowling and the number of balls still to come.

Dynamic programming works by breaking up a large problem of mathematical decision-making ('what is the best way to control the stock of Easter eggs for the next three months?') into a set of smaller ones ('what should be done today with only ten eggs on the shelf and four weeks to go until Easter?'). Each of these small problems is one of finding the best value of a function, which depends on two terms named by Richard Bellman. First, there is the state, which summarizes the information available at the time of making the decision. The state may have several components, or just one. Then there is the stage (the number of decisions to be made). Breaking up the problem means that the function values are linked by a recurrence relation (just as the Fibonacci series is defined by a recurrence relation) allowing one to find the best value (for a particular state and stage) by looking at the consequences of each decision that might be taken. ('If I don't order any more eggs, next week I may have sold out, or there may be some left over. If I order more eggs, it will cost me money, but there is a smaller chance of selling out and disappointing the customers.') The stages of a problem appear in a fixed order since each decision will reduce the number of future decisions by one. The states of most problems are either integers or one of a small set of possible values. Making a decision will help determine the state at the next stage, and the principle of dynamic programming is to choose the best possible decision using the recurrence relation. Fuller descriptions of dynamic programming are given, for example, in references 5 and 6.

To work, DP calculates a numerical value for the best decision for a specified state and stage. For the retailer, it will be a mixture of the expected profit from having goods to sell and the cost of buying more and looking after them. For the cricketer, it will be some measure of the chance of the batting team gaining a win. For the auction-bidder, the value may be personal ('Can I afford it and do I want it at the price?') or commercial ('What profit will I make when I sell it to a customer?').

In one of his early papers, Richard Bellman looked at a board game and the best way to play it. He assumed that it would be possible to know exactly the value of every position that might occur in the course of a game. He defined a function $f(p)$ as the value of a game for the player (White) about to move with the game in position p. $f(p) = 1$ if p is a winning position, $f(p) = -1$ if it is a losing position and $f(p) = 0$ if a draw is the best result. Then a move (W) by White has the effect of transforming p to $T_W(p)$. White will have several possible moves $W1, W2, \ldots$, so there will be several different states which might be reached at the next stage, $T_{W1}(p), T_{W2}(p), \ldots$. Each of these will be followed by a move ($B1$ or $B2$ or ...) by Black which transforms the position to $T_{B1}(T_{W1}(p))$ or $T_{B1}(T_{W2}(p))$ or $T_{B2}(T_{W1}(p))$ and so on. T_W and T_B are operators on the set of positions. (In this case, the number of decisions still to be made is not important.) Playing in the best way possible, Black should strive to minimise the value to White of this resulting position, and White should select a move so that Black's best move is frustrated, and try to maximise the value. Hence, the expression:

$$f(p) = \max_{T_W}\{\min_{T_B}\{f(T_B(T_W(p)))\}\}$$

(where T_W and T_B are restricted to a set of permitted operations, T_{W1}, T_{W2}, etc and T_{B1}, T_{B2}, etc, corresponding to the moves which may take place). White moves first and then Black. Black looks at the moves which White may make and chooses the one which makes the function after the two players' moves as small as possible; but White knows this, and so will have chosen a move to make the function at this stage as large as possible.

Bellman used chess as an example, and commented:

> As it stands, however, the equation is useless computationally because of the dimensionality barrier, and it is useless analytically because of our lack of knowledge of the intrinsic structure of chess.

The 'dimensionality barrier', often referred to as the 'curse of dimensionality', follows from the amount of information needed to describe a situation in dynamic programming completely. (This often requires two or more dimensions.) In chess, a full description of the state will involve a list of the locations of all pieces, or of the pieces which occupy each square on the board. (Either 32 or 64 dimensions!) Even at the end of a game, with, say, four pieces on the board, there will be over a million possible positions of the pieces, and the number of states is significantly more than the number of positions, since it is necessary to identify every piece (in addition to the two kings), because of the different moves that are allowed by the rules of the game.

Dudeney's game, solved by dynamic programming

Dudeney's game is clearly one where players make decisions in sequence. The decision that each player needs to make is 'what number to choose next?'. As the players look at the position in a game, they will only consider the number of points in the heap (s) and the move (m) which led to this heap. Earlier moves in the game and the starting position will be irrelevant. So it is possible to try to find a solution to this problem (and to any of the family of 'take-away' games) using Bellman's approach.

When White is about to make a move, the information which describes the position (the state) is two-dimensional. First there is the number (s) in the heap; second, there is the move (m) leading to this, which will be forbidden for White. So we can write $p = (s, m)$, and White takes w from the heap, followed by Black taking b; then Bellman's recurrence relation becomes:

$$f(s,m) = \max_{w \neq m} \{\min_{b \neq w} \{f(s-w-b, b)\}\}.$$

Inside the parentheses, Black is making a move b which is different from White's, aiming to make the function $f(s-w-b, b)$ as small as possible. White, of course, is trying to find a move which makes this as large as possible, no matter what move Black will use. White is forbidden to use the move m.

If White is to play first and do his best, we need to know the values of $f(37, m)$ for any preceding move m. If they are all -1, then White cannot win. If there is at least one which is $+1$, White can win by optimal play.

To solve this we need some values for the function corresponding to the end of the game. The game will have finished when $s = 0$, and that will mean that Black is the winner. Black will have lost when there is no move allowed after White's next move. So it will be impossible to find a move b in the expression above. The value of the function then can be described, either by listing all the losing positions (s, m) or by imagining that Black plays and ends up with s negative, i.e. with $s < 0$ and $s+m \geqslant 0$. So we can write $f(0, m) = -1$ for all values m, and $f(s, m) = 1$ for all $s < 0$ with $s+m \geqslant 0$.

Now the recurrence relation can be used to calculate $f(1, m)$ for all possible m from these, but it is easier in this case to look at the way that the game will end. If m is 1, then the position is a losing one giving $f(1,1) = -1$. For other values of m, White can play a '1' and win (so $f(1,2) = f(1,3) = f(1,4) = f(1,5) = +1$). Similarly, $f(2, m)$ can be found, and all values of m give winning positions for White, either by playing a '2' if possible or by playing a '1' from the position $(2, 2)$ and presenting the opponent with the losing position $(1, 1)$.

As the heap becomes larger, it becomes harder to consider all the possibilities in such an *ad hoc* way, and the recurrence relationship becomes useful. When the heap has size 3 and the last play was m, White plays $w \neq m$ and Black plays $b \neq w$. (Common sense means that $w \leqslant 3$.) The two states that the game passes through are $(3-w, w)$ and $(3-w-b, b)$, for each of which the function value is already known. So we can draw a table of the possible moves, as in table 2. This table shows all positions that the two players can reach from the position $(3, 4)$. White chooses, if possible, a row from the table where there is no entry of -1. If White were to make any move corresponding to a row with -1 in it, then this would allow Black to respond with the move for that column and White would lose. So White should make the move $w = 3$ and is sure of a win. Hence $f(3, 4) = +1$. Repeating this process for every possible state of the game is straightforward, though potentially tedious. Computer help is useful! Once $f(3, m)$ has been calculated, $f(4, m)$ can be found, then $f(5, m)$ and so on. The big problem ('What is the best way to play Dudeney's game?') has been broken into a set of smaller problems ('What is the best way to play and win from a given position?'). Eventually, a table can be created of all the state values, as shown in table 3. The 'pearls' at $s = 7$, 13, 20, 26 and 33 (where White will lose whatever move is made) can be seen there. Other losing positions are apparent. $(1, 1)$ (already mentioned), $(3, 3)$, $(4, 4)$ and $(5, 5)$ all lead to a loss for White. $(6, 3)$ is also a losing position, as White cannot win with one move and cannot force Black into a losing position. These isolated losing positions close to the end of the game differ from those described below when both players have several moves to make.

The table of function values shows repeated patterns of isolated losing positions in the game. Those

Table 2. The possible moves for the two players from the position (3,4) with the resulting positions and their function values. (White moves first, and the function is +1 if White can win or has won and is −1 for a win for Black.)

		b = 1	b = 2	b = 3
w	1		$f(0,2) = -1$	$f(-1,3) = 1$
	2	$f(0,1) = -1$		$f(-2,3) = 1$
	3	$f(-1,1) = 1$	$f(-2,2) = 1$	

Table 3. Values of $f(s,m)$ for the 37-game (lines mark the secure positions; between double lines, the patterns of values are almost identical (see text)).

s	m = 1	2	3	4	5
1	−1	1	1	1	1
2	1	1	1	1	1
3	1	1	−1	1	1
4	1	1	1	−1	1
5	1	1	1	1	−1
6	1	1	−1	1	1
7	−1	−1	−1	−1	−1
8	1	1	1	1	1
9	1	1	1	1	1
10	1	1	1	1	1
11	1	1	1	−1	1
12	1	1	1	1	−1
13	−1	−1	−1	−1	−1
14	−1	1	1	1	1
15	1	1	1	1	1
16	1	1	−1	1	1
17	1	1	1	1	1
18	1	1	1	1	−1
19	1	1	−1	1	1
20	−1	−1	−1	−1	−1
21	−1	1	1	1	1
22	1	1	1	1	1
23	1	1	1	1	1
24	1	1	1	−1	1
25	1	1	1	1	−1
26	−1	−1	−1	−1	−1
27	−1	1	1	1	1
28	1	1	1	1	1
29	1	1	−1	1	1
30	1	1	1	1	1
31	1	1	1	1	−1
32	1	1	−1	1	1
33	−1	−1	−1	−1	−1
34	−1	1	1	1	1
35	1	1	1	1	1
36	1	1	1	1	1
37	1	1	1	−1	1

for White between $s = 7$ and $s = 20$ are repeated (with a shift of the origin) between $s = 20$ and $s = 33$, with one exception. White loses at (11,4) and (24,4); at (12,5) and (25,5); at (14,1) and (27,1); at (16, 3) and (29,3); at (18,5) and (31,5); at (19,3) and (32,3). The repetition continues for $s > 33$. The exception is that (8,1) is a winning position, but (21,1) is a losing position, as is (34,1). In (8,1) White wins by playing a 4, a move which leads to (4,4) and defeat for Black, because this is so close to the end of the game.

We asked for $f(37,m)$; the line for $s = 37$ has values +1 except for $m = 4$, so we deduce that White can win from the start of the game with optimal play. His move should be to take away 4, leading to a heap of 33. After that, White should always move to a position where the function is −1, possibly using the isolated positions.

Other ways to solve the puzzle

Dynamic programming is only one way of solving Dudeney's puzzle. It could be solved by completely checking all possible games from all possible positions, which would be extremely time-consuming. It could be solved using the ideas of computer-assisted games, where the progress of the game is represented by a 'game tree', with all positions given a numerical value, associated (either explicitly or implicitly) with the chance of winning from that position. Or it could be solved by experimenting with smaller problems and proving that there is a pattern of secure positions.

Conclusion

Dudeney's 'beautiful new puzzle' is easy to understand, and yet has many subtleties. It deserves to be better known. We have shown how to find a solution using dynamic programming, following an approach which can be applied more widely to other games.

The title of this article refers to two puzzles; the second is historical: 'How did Dudeney solve the problem, nearly a century ago, without dynamic programming and game trees?' We don't know; probably he used a systematic method which might be recognisable as a precursor of these inventions of the mid-twentieth century.

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Paradox in probability (1)

Two gamblers, *A* and *B*, sit at a table in a Mississippi stern-wheel paddle steamer. *A* tosses two silver dollars and they lie on the table hidden from *B*. Two other passengers, *C* and *D*, pass by. *C* says to *D*: 'When I see the head on a silver dollar it reminds me of what a wonderfully good likeness of George Washington the Mint has got for their coins.' *A* looks up and says: 'Excuse me, sir, but you have rather upset our game; my friend *B* here was going to bet on whether the two coins were both showing the same face or showing one head and one tail. We have always thought that the probability of the coins showing the same was exactly a half, but now you have given him the extra information that there is one head showing. And that must surely change the odds.'

C answers: 'My dear chap, you have no need to worry at all. *B* knows only that one of the coins shows George Washington's head and that the other one is equally likely to show head or tail. And so you can safely just carry on as usual.' But *D* broke in: 'When the coins were tossed, there were four possible outcomes, HH, HT, TH and TT, each with probability one quarter. The extra information that there was at least one head simply eliminates the possibility TT, leaving the other three outcomes all having probability one third. With the extra information, *B*'s estimate for the probability of the two coins showing the same is therefore reduced from one half to one third.'

Basil Rennie
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Determining Sums Recursively

P. GLAISTER

In this article we show how to determine sums of certain finite series in a recursive manner.

Consider the series

$$S_n = \sum_{i=1}^{n} a_i \quad (n \geqslant 1) \tag{1}$$

and assume that a simple formula can be found for S_n for all positive integers n. The aim is to determine

$$T_n = \sum_{i=1}^{n} ia_i \quad (n \geqslant 1). \tag{2}$$

One approach is to write out T_n explicitly as follows:

$$\begin{aligned}
T_n &= 1\times a_1+2\times a_2+3\times a_3+\cdots+n\times a_n \\
&= a_1+a_2+a_3+\cdots+a_n \\
&\qquad +a_2+a_3+\cdots+a_n \\
&\qquad\qquad +a_3+\cdots+a_n \\
&\qquad\qquad\qquad \ddots \quad \vdots \\
&\qquad\qquad\qquad\qquad +a_n \\
&= S_n+(S_n-a_1)+[S_n-(a_1+a_2)]+\cdots \\
&\qquad\qquad +[S_n-(a_1+a_2+\cdots+a_{n-1})] \\
&= S_n+(S_n-S_1)+(S_n-S_2)+\cdots+(S_n-S_{n-1}) \\
&= nS_n-(S_1+S_2+\cdots+S_{n-1})
\end{aligned}$$

and hence

$$T_n = nS_n - \sum_{i=1}^{n-1} S_i \quad (n \geqslant 2). \tag{3}$$

(Note that if $n = 1$ then $T_1 = a_1 = S_1$.) Thus, assuming that (1) can be determined explicitly, equation (3) gives a formula for T_n.

As a first example, suppose $a_i = 1$ for all i, so that

$$S_n = \sum_{i=1}^{n} 1 = n \quad (n \geqslant 1)$$

and

$$\begin{aligned}
T_n = \sum_{i=1}^{n} i &= nS_n - \sum_{i=1}^{n-1} S_i \\
&= n\times n - \sum_{i=1}^{n-1} i \\
&= n^2-\left(\sum_{i=1}^{n} i-n\right) \\
&= n^2-(T_n-n) \\
&= n^2+n-T_n \quad (n \geqslant 2),
\end{aligned}$$

and rearranging

$$T_n = \tfrac{1}{2}(n^2+n) = \tfrac{1}{2}n(n+1) \quad (n \geqslant 1), \tag{4}$$

where, since $T_1 = 1 = \frac{1}{2}\times 1\times(1+1)$, we have included $n = 1$.

If we now let $a_i = i$ and define

$$U_n = \sum_{i=1}^{n} i^2,$$

then employing (3) with T and S replaced by U and T, we have

$$\begin{aligned}
U_n = \sum_{i=1}^{n} i\times i &= nT_n - \sum_{i=1}^{n-1} T_i \\
&= n\times\tfrac{1}{2}n(n+1) - \sum_{i=1}^{n-1} \tfrac{1}{2}i(i+1) \\
&= \tfrac{1}{2}n^2(n+1) - \tfrac{1}{2}\sum_{i=1}^{n-1} i^2 - \tfrac{1}{2}\sum_{i=1}^{n-1} i \\
&= \tfrac{1}{2}n^2(n+1) - \tfrac{1}{2}\left(\sum_{i=1}^{n} i^2-n^2\right)-\tfrac{1}{2}T_{n-1} \\
&= \tfrac{1}{2}n^2(n+1)-\tfrac{1}{2}(U_n-n^2)-\tfrac{1}{4}(n-1)n \\
&= \tfrac{1}{2}n^3+\tfrac{3}{4}n^2+\tfrac{1}{4}n-\tfrac{1}{2}U_n
\end{aligned}$$

and rearranging we have

$$\begin{aligned}
U_n &= \tfrac{2}{3}(\tfrac{1}{2}n^3+\tfrac{3}{4}n^2+\tfrac{1}{4}n) \\
&= \tfrac{1}{6}(2n^3+3n^2+n) \\
&= \tfrac{1}{6}n(n+1)(2n+1) \quad (n \geqslant 1),
\end{aligned}$$

where $U_1 = \frac{1}{6}\times(1+1)\times(2+1)$. Similarly one can determine $\sum_{i=1}^{n} i^3$, which we leave as an exercise.

As a further example consider $a_i = F_i$, the ith Fibonacci number, defined by

$$F_1 = F_2 = 1; \qquad F_i = F_{i-1}+F_{i-2} \quad (i \geqslant 3),$$

the first few of which are 1, 1, 2, 3, 5, 8, 13, 21, etc. In this case, we have no immediate formula for the 'starting' series

$$S_n = \sum_{i=1}^{n} F_i \quad (n \geqslant 1).$$

However, this is not a difficult matter since

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} F_i \\
&= \sum_{i=1}^{n} (F_{i+2}-F_{i+1}) \\
&= (F_{n+2}+F_{n+1}+\cdots+F_4+F_3) \\
&\qquad -(F_{n+1}+F_n+\cdots+F_3+F_2)
\end{aligned}$$

$$= F_{n+2} - F_2$$
$$= F_{n+2} - 1. \qquad (5)$$

To obtain an explicit formula for S_n it is necessary to know F_{n+2}, for which we could use the well-known formula

$$F_n = \frac{a^n - b^n}{\sqrt{5}} \quad (n \geqslant 1),$$

where $a = \frac{1}{2}(1+\sqrt{5})$ and $b = \frac{1}{2}(1-\sqrt{5})$.

Having obtained S_n we can now employ (3) to determine

$$T_n = \sum_{i=1}^{n} iF_i = nS_n - \sum_{i=1}^{n-1} S_i = n(F_{n+2}-1) - \sum_{i=1}^{n-1} (F_{i+2}-1)$$
$$= nF_{n+2} - n - \sum_{i=1}^{n-1} F_{i+2} + \sum_{i=1}^{n-1} 1$$
$$= nF_{n+2} - n - \sum_{j=3}^{n+1} F_j + n - 1$$
$$= nF_{n+2} - 1 - (S_{n+1} - F_1 - F_2)$$
$$= nF_{n+2} - 1 - (F_{n+3} - 1 - 1 - 1)$$
$$= nF_{n+2} - F_{n+3} + 2 \quad (n \geqslant 2), \qquad (6)$$

which is also true for $n = 1$ by direct evaluation of both sides. Similarly, $\sum_{i=1}^{n} i^2 F_i$ can be determined. The proof of formulae (5) and (6) using induction is a good exercise.

□

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The Loneliness of the Factorions

CLIFF PICKOVER

Number is the bond of the eternal continuance of things. **Plato**

Factorions are numbers that are the sum of the factorial values for each of their digits. For example, 145 is a factorion because it can be expressed as

$$145 = 1! + 4! + 5!.$$

Two tiny factorions are

$$1 = 1! \quad \text{and} \quad 2 = 2!.$$

The largest known factorion is 40 585; it can be written as

$$40\,585 = 4! + 0! + 5! + 8! + 5!.$$

(40 585 was discovered in 1964 by R. Dougherty using a computer search.)

Can you end the loneliness of the factorions? Do any others exist?

In fact these four factorions are the *only* ones known.

Various proofs have been given to show that 40 585 is the largest possible factorion. To indicate why this is true we need only realize that if

$$A = a_r a_{r-1} \ldots a_1 a_0$$
$$= a_r \times 10^r + \cdots + a_1 \times 10 + a_0$$

is a factorion, then

$$10^r \leqslant a_r \times 10^r \leqslant A = a_r! + \cdots + a_1! + a_0!$$
$$\leqslant 9!\,(r+1) = 362\,880(r+1).$$

Thus $r \leqslant 6$. In fact, if $r = 6$ then we easily find that $a_6 = 1$ and $A \leqslant 1999\,999$. (A little more work shows that $A \leqslant 999\,999$.) It is then straightforward to check on a computer that 40 585 is the last factorion.

A more fruitful avenue of research may be the search for factorions 'of the second kind' which are formed by the *product* of the factorial values for each of their digits. Additionally, there are hypothetical factorions 'of the third kind' formed by grouping digits. For example, a factorion of the third kind might have the form

$$abcdef = (ab)! + c! + d! + (ef)!,$$

where each letter represents a digit.

To date, I am unaware of the existence of factorions of the second or third kind, and I would be interested in hearing from readers who can find any. (I should point out Herve Bronninan from Princeton University has recently found some magnificent factorions in other bases, most notably 519 326 767, which in base 13 is written as 8.3.7.9.0.12.5.11 and is equal to $8! + 3! + 7! + 9! + 0! + 12! + 5! + 11!$. You can interpret this base 13 number as:

$$8\times13^7 + 3\times13^6 + 7\times13^5 + 9\times13^4$$
$$+ 0\times13^3 + 12\times13^2 + 5\times13^1 + 11\times13^0.)$$

Digressions: Narcissistic numbers

Dik T. Winter from Amsterdam is an expert on a somewhat related class of numbers which are the sums of powers of their digits. In other words, these are N-digit numbers which are equal to the sum of the Nth powers of their digits. For example,

$$153 = 1^3+5^3+3^3.$$

Variously called narcissistic numbers, 'numbers in love with themselves', Armstrong numbers or perfect digital variants, these kinds of numbers have fascinated number theorists for decades. For example, the English mathematician G. H. Hardy (1877–1947) noted that 'There are just four numbers, after unity, which are the sums of the cubes of their digits .... These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician.' I gave 153 as an example of such a number. Can you find the other three?

The largest narcissistic number discovered to date is the incredible 39-digit number:

$$115\,132\,219\,018\,763\,992\,565\,095\,597\,973\,971\,522\,401.$$

(Each digit is raised to the 39th power.) Can you beat the world record? What would G. H. Hardy have thought of this multidigit monstrosity?

The frequency of occurrence of narcissistic numbers varies according to the base of the number system in which one conducts a search. For example, in our standard (base 10) system there are 88 known numbers of this type, while in base 4 there are only 11. Table 1 shows some numbers from Dik T. Winter. The number of digits for the largest number known is in parentheses.

As one searches for larger and larger narcissistic numbers, will they eventually run out, as in the case of the lonely factorials?

Finally, Kevin S. Brown writes that he knows of only three occurrences of $n!+1 = m^2$, namely

$$25 = 4!+1 = 5^2,$$

$$121 = 5!+1 = 11^2,$$

$$5041 = 7!+1 = 71^2.$$

We do not know if there are any others. Perhaps these 'Brown numbers' will be as lonely as the factorions. Prolific mathematician Paul Erdős long ago conjectured that there are only three such numbers. Erdős offers a cash prize for a proof of this—see *Mathematical Spectrum* Volume 27 Number 2 pages 43–44.

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Table 1

Base	Total	Digits	Largest
2	1	(1)	1
3	5	(3)	122
4	11	(4)	3 303
5	17	(14)	14 421 440 424 444
6	30	(18)	105 144 341 423 554 535
7	59	(23)	12 616 604 301 406 016 036 306
8	62	(29)	11 254 613 377 540 170 731 271 074 472
9	58	(30)	104 836 124 432 728 001 478 001 038 311
10	88	(39)	115 132 219 018 763 992 565 095 597 973 971 522 401
11	134	(45)	123 44A A12 A72 180 342 291 2A8 AA4 963 568 083 A26 845 6A4
12	87	(51)	150 793 46A 6B3 B14 BB5 6B3 958 98B 966 29A 8B0 151 534 4B4 B07 14B

($A = 10$, $B = 11$.)

Clifford Pickover is a research staff member at the IBM Thomas J. Watson Research Center in Yorktown Heights, New York. He is the author of numerous popular books on mathematics, art and science, including 'Chaos in Wonderland: Visual Adventures in a Fractal World' and 'Mazes for the Mind: Computers and the Unexpected'. He in an associate editor for 'Computers and Graphics' and 'Computers in Physics', and an editorial-board member for 'Speculations in Science and Technology', 'Leonardo' and 'YLEM'.

How would you find the centre of a circle?

?

Are there any real irrational numbers such that x^y is rational?

Mathematics in the Classroom

The aim of this regular feature is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

Coursework

The new modular 'A' level course that we teach offers many opportunities for coursework, with modules containing up to 40% of the total marks allocated to coursework tasks. Teaching methods at GCSE level certainly seem to develop in students the skills necessary for successful coursework and we usually find that students have had good prior experiences of this mode of learning, so set about coursework with some enthusiasm as a welcome relief from the academic theory of the course. The mechanics and statistics modules lend themselves very readily to the gathering of data and consequent testing of mechanical principles or statistical hypotheses. But one of the pure mathematics modules also carries a coursework task which focuses on the solution of equations by numerical methods, and as a core module, must be taken by everyone on the course. Even if this were not the case, this task constitutes an interesting investigation from which a student can learn much about this topic.

The task

Students are asked to investigate the solution of equations using three methods: systematic search for change of sign; fixed-point iteration after re-arranging the equation $f(x) = 0$ into the form $x = g(x)$; and finally by use of the Newton–Raphson method. For each of these three methods the student must find an equation for which the method works and one for which the method fails. Graphic calculators have proved invaluable in this exercise for first of all demonstrating that a chosen equation has at least one root, and secondly when used to write simple programs to produce iterative sequences that do (or do not) converge to the required root.

Change-of-sign methods. It is easy enough to find examples of equations whose solutions can be identified by a change-of-sign method, and almost equally easy to find one that has no solution even though the function changes sign, leading to the hope of the presence of a nearby root. (Anything of the form $f(x) = 1/(x+a)$ will achieve this.) For a more interesting example of this kind, investigate the function

$$f(x) = \frac{x}{2\ln x - 1} - 1.$$

This has no roots (as a graphic calculator will confirm) but changes sign between $x = 1$ and $x = 2$. Application of interval bisection or linear interpolation will show a convergence to $x = 1.649$, which is the asymptote rather than the root.

Fixed-point iteration after re-arranging $f(x) = 0$ into the form $x = g(x)$. An equation $f(x) = 0$ will frequently re-arrange to $x = g(x)$ in more than one way. One arrangement may lend itself to locating a solution whilst another may well not (as $g'(x)$ should lie between -1 and $+1$ for success). As an illustration, consider the function

$$f(x) = x - \ln x - 2.$$

$f(x) = 0$ will then re-arrange to $x = \ln x + 2$, or $x = e^{x-2}$, so we have a choice for $g(x)$. Change-of-sign methods (and a glance at the graphic calculator) suggest that the root lies between $x = 3$ and $x = 4$, probably closer to $x = 3$. For $g(x) = \ln x + 2$, $g'(x) = \frac{1}{3}$ at $x = 3$. For $g(x) = e^{x-2}$, $g'(x) = e$ at $x = 3$. So, if $-1 \leqslant g'(x) \leqslant 1$, we need to choose $g(x) = \ln x + 2$. Now check that, after eight iterations of $x_n = g(x_{n-1})$ starting at $x = 3$, we arrive at the root being at $x = 3.146$ (to 3 decimal places).

If in any doubt about the wisdom of choosing $g(x) = e^{x-2}$, we see that if $x_1 = 3$, one iteration gives $x_2 = e$ which is outside our known interval of $[3, 4]$. A graph of $y = x$ and $y = e^{x-2}$ demonstrates why this method fails.

Newton–Raphson method. This usually turns out to be everyone's favourite method because of the rapid convergence in successful cases. If we consider a function of the form $f(x) = x - 3\ln x$, then it can be readily shown that this function has a root in $[1, 2]$ and another in $[4, 5]$. Using the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and starting with $x_1 = 2$, we soon see that $x_2 = 1.8411$, $x_3 = 1.8427$, $x_4 = 1.8571$ and $x_5 = 1.8572$, giving rapid convergence to the root at 1.857 (to 3 decimal places). Locating the root in $[4, 5]$ is left as an exercise for the reader.

But Newton–Raphson does not always work either. For example, consider the function $f(x) = 7x^3 - 5x^2 - 1$. A glance at the graph confirms that there is only one root and that is in $[0, 1]$. Further, interval bisection reduces this to the interval $[0.5, 1]$. One iteration of the Newton–Raphson formula starting with $x_1 = 0.5$ yields $x_2 = 6$, so we know that this is not going towards the root. Further investigation of the graph explains why this has occurred. We have started the iteration at a point very close to a turning point where the tangent is relatively flat, so the tangent-sliding process which should move nearer the root is actually moving well away from it.

Finally

Inventing functions whose roots can be located by the various methods can be lots of fun, if occasionally frustrating. Finding functions whose roots resist these standard methods of location helps students to realise under what conditions the methods are useful and under what conditions they can be misleading—all valuable experiences. It is equally valid to start with functions whose roots are readily found by solving equations, and then seeing which numerical methods will lead to the same solutions. It would be interesting to hear from anyone else who has experienced this coursework task, especially when it comes to ensuring that students come up with their own individual examples—not so easy when your cohort size is in excess of 200!

Carol Nixon

Computer Column

Sunrise and sunset

We all know that the sun rises and sets because the earth rotates on its axis once every 24 hours, and that it rises and sets at different times because of the tilt of the earth's axis. But these simple facts hide a wealth of minor details which make the calculation more complex. For instance, at the time of writing, the sun rises here at about 6.25 am and sets at about 6.10. If the simple model were correct, even allowing for the slight longitude shift of Sheffield from the Greenwich meridian, the two times would be more symmetrically placed about 12 noon.

The first complication is that the earth in fact does *not* rotate in 24 hours but in about 23 hours 56 minutes. (This is the reason why the star field shifts across the sky throughout the year.) The other 4 minutes are taken up by the earth in turning through a small angle θ to face the sun again, after moving an angle θ along its orbit.

The second complication is that θ is not just $2\pi/365$ radians. Alright, it is more like $2\pi/365.25$ radians anyway, because of the correction of one day every leap year. The problem is that if the earth traversed a circular orbit then θ would be constant, but as it traverses an ever-so-slightly elliptical orbit, so θ varies as the year progresses. In fact, the sun isn't at its highest point in the sky at noon every day; sometimes it is earlier and sometimes later.

Mathematically, we can calculate when sunrise and sunset will be on the assumption that the earth is a sphere traversing a perfectly circular orbit. We can then add or subtract a correction called the *equation of time* which modifies these figures, both in the same direction, and we have a pretty good estimate of when the local clock times of sunrise and sunset will be.

Assuming you are at Greenwich, or on one of the boundaries between time zones, it is a simple calculation:

$$\cos a \cos l = \sin s, \qquad \sin t \cos s = \sin a.$$

Here a is an intermediate variable called the *azimuth* of the sun, l is latitude in radians, s is the angle in radians of the sun above the equator and t is the time (in radians!) of sunset, measured from noon. These formulae translate into the calculation:

```
sintime:=
      sqrt(sqr(cos(latitude))-sqr(sin(sun)))
           /(cos(sun)*cos(latitude));
time:=arctan(sintime/sqrt(1-sqr(sintime)));
time:=time/pi*12;
if sun>0 then time:=12-time;
fudgefactor:=1/15; (*4 minutes*)
time:=time+fudgefactor;
```

to translate `time` into the hour of sunset. (OK, I cheated a bit!)

But wait! We know our latitude, but what is this `sun` variable? How do we calculate that? Well, we can approximate the motion of the sun above and below the equator by a sine wave, of period 1 year and amplitude 23.5°, measured from March 21st. *That* is an easy calculation!

Finally, we must add a correction to these figures; the equation of time is here plotted in minutes against radians, where the year is measured in radians starting from March 21st. After this correction, if t is the sunset time calculated before and e is the equation of time in hours, then the sun sets at about $t+e+\frac{1}{15}L$ and rises at about $12-t+e+\frac{1}{15}L$, where L is your longitude in degrees. (360° longitude equals 24 hours.)

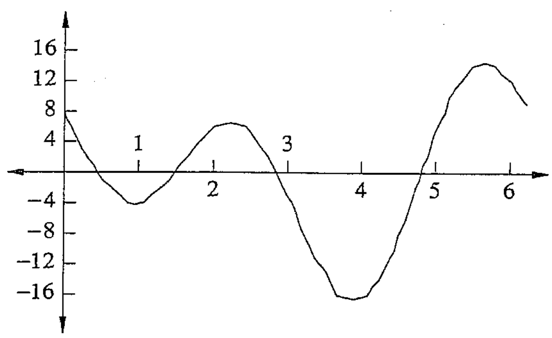

Figure 1. The equation of time.

You now have the information you need to write a program. This program should be accurate to within a few minutes in the Northern hemisphere, provided you don't stray over the arctic circle or enter daylight saving time or British Summer Time as we call it .... If you want the finished program, it is available via `http://www.shef.ac.uk/~ms/staff/piff`.

Mike Piff

Letters to the Editor

Dear Editor,

Primes of the form $x^y + y^x$

In Volume 28, Number 1, pages 20–21, Kinichiro Kashihara asked how many primes have the form $2^n + n^2$. I have found a more general problem in the reference. Consider the Smarandache expression $x^y + y^x$, where $(x, y) = 1$ and $x, y \geqslant 2$. Jose Castillo conjectured that there are only a finite number of primes of this form. For example, $3^2 + 2^3 = 9 + 8 = 17$ is prime, but $4^5 + 5^4 = 1024 + 625 = 1649 = 17 \times 97$ is not prime.

There are infinitely many composite numbers of this form. For, put $x = 2pn + 1$ and $y = 2pm - 1$, where p is any prime and m and n are natural numbers. Because $x^y \equiv 1 \pmod{p}$ and $y^x \equiv -1 \pmod{p}$, therefore $x^y + y^x \equiv 0 \pmod{p}$, so $x^y + y^x$ is divisible by p and is clearly greater than p.

Yours sincerely,
PETER CASTINI
(Student at Arizona State University)

Reference

1. F. Smarandache, Properties of the Numbers, *University of Craiova Conferences* (1975).

Dear Editor,

Wondrous Numbers/The 1996 puzzle

Re Junji Inaba's problem on page 33 in Volume 28 Number 2, the numbers generated by the equations Junji postulates are well known; they are called 'Wondrous Numbers'. A good starting point is Mike Mudge's Numbers Count column in the December 1995 issue of Personal Computer World. Mike also discusses a generalised version of the equations.

As a new subscriber, your '1996 puzzle' is new to me, although I have played with many similar things in the past. However, it strikes me that in this form the problem could be completely solved by someone expert in a suitable AI language—PROLOG seems the obvious example. May I challenge your readership to write a suitable program?

Yours sincerely,
ALAN D. COX
(Pen-y-Maes, Ostrey Hill,
St Clears, Dyfed SA33 4AJ)

Dear Editor,

The domino problem

In Volume 28, Number 2, page 44 of *Mathematical Spectrum*, your correspondent, George Jellis, investigates the number $T_{m,n}$ of ways of covering an $m \times n$ rectangle by 1×2 dominoes. He obtains recurrences for this quantity when $m \leqslant 4$. This problem has been completely solved by P. W. Kasteleyn (The statistics of dimers on a lattice I: The number of dimer arrangements on a quadradic lattice, *Physica* **27** (1961), 1209–1225). His formula is

$$T_{m,n} = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos^2\frac{j\pi}{m+1} + 4\cos^2\frac{k\pi}{n+1}\right)^{1/4},$$

which yields, for instance, $T_{8,8} = 12\,988\,816$. Kasteleyn's methods are algebraic, involving the calculation of an $\frac{1}{2}mn \times \frac{1}{2}mn$ determinant related to the incidence matrix of the graph with vertices corresponding to the mn squares in the rectangle, and edges joining vertices corresponding to adjacent squares.

Yours sincerely,
ROBIN CHAPMAN
(Department of Mathematics,
University of Exeter,
North Park Road, Exeter EX4 4QE)

Paradox in probability (2)

At the *Goldfinger* casino there is a new game. An assistant, A, on the stage shows the patrons three cards, a King, a Queen and a Jack, he shuffles them and then puts one in a box and the other two face down on a table. Another assistant, B, looks at these two cards and then holds one up for all the patrons to see; it is (in a typical game) the Queen. The patrons are then invited to place bets on whether the card in the box is the King.

The method of betting is that the house offers, at \$8 each, 'King' tickets, which will pay \$18 if the card in the box is the King. Also they offer 'non-King' tickets at \$11 each; these will pay \$18 if the card turns out not to be the King.

Some of the patrons think that the probability of the King is $\frac{1}{3}$ and so they cheerfully buy 'non-King' tickets, thinking that they have a $\frac{2}{3}$ probability of a return of \$18 for the \$11 bet, an expectation of \$1 gain. Others think that the King and Jack are equally likely and they cheerfully buy 'King' tickets, thinking that they have a $\frac{1}{2}$ probability of a return of \$18 for the \$8 bet, an expectation of \$1 gain. The proprietor knows that by selling one ticket of each kind he is certain of a gain of \$1.

Another patron, Z, is more cautious. He approaches B and bribes him for the full story of what goes on. B tells him that A's shuffling is perfectly genuine and that his (B's) instructions are to look at both cards on the table and to show one of them, but if one of them is the King he must show the other, because to show that the King was not the card in the box would make the betting pointless. How does Z bet?

Basil Rennie
(Adelaide)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

28.9 A polynomial function of degree n is such that $p(x) \geqslant 0$ for all x. Prove that

$$p(x)+p'(x)+p''(x)+\cdots+p^{(n)}(x) \geqslant 0$$

for all x.
(Submitted by Junji Inaba, student at William Hulme's Grammar School, Manchester.)

28.10 Let P be a point on one of the sides of triangle ABC, not equal to A, B or C. Reflect P across each of the other two sides to P′ and P″. What is the minimum value of the radius of the circumscribed circle of triangle PP′P″ as P travels round the perimeter of triangle ABC?
(Submitted by Noah Rosenberg, student at Rice University, Houston.)

28.11 Let P and Q be points on the sides AB and AC of triangle ABC, respectively, and let D be the point of intersection of BQ and CP. Prove that AD bisects PQ if and only if PQ is parallel to BC.
(Submitted by Sam and Jim Yu, students at University of South Dakota.)

28.12 Prove that no term of the Fibonacci sequence ends in 1996.
(Submitted by Toby Gee, student at The John of Gaunt School, Trowbridge.)

Solutions to Problems in Volume 28 Number 1

28.1 *Pre-Fermat's last theorem!* The positive integers x, y and z satisfy the equation $x^n+y^n = z^n$, where n is a positive integer greater than 1. Prove that, if the differences between z and x and z and y are greater than 1, then neither x nor y can be prime.

Solution by Can Minh, University of California, Berkeley

Suppose that $z-y > 1$ and x is prime. Then

$$x^n = z^n-y^n = (z-y)(z^{n-1}+z^{n-2}y+\cdots+y^{n-1}).$$

Then $z-y = x^k$ and

$$z^{n-1}+z^{n-2}y+\cdots+y^{n-1} = x^{n-k}$$

for some natural number $k < n$. Then

$$x^{n-k} > z^{n-1} = (x^k+y)^{n-1} > x^{k(n-1)},$$

so $n-k > nk-k$, i.e. $n > nk$. This is impossible. Hence, if $z-y > 1$ then x cannot be prime. By symmetry, if $z-x > 1$ then y cannot be prime.

Also solved by Junji Inaba, Toby Gee, Mansur Boase (St Paul's School, London) and David Bibby (Ysgol Rhiwabon, Ruabon, Clwyd, Wales).

28.2 Prove that

$$x\ln a \geqslant \ln x+\ln(\ln a)+1,$$

with $a > 1$ and $x > 0$, and determine when equality occurs.

Solution by Toby Gee

Let $f(x) = x\ln a-\ln x-\ln(\ln a)-1$. Then $f'(x) = \ln a-(1/x)$, so that $f'(x) > 0$ whenever $x > 1/\ln a$ and $f'(x) < 0$ whenever $x < 1/\ln a$. Thus f has a unique minimum at $x = 1/\ln a$, namely 0.

Also solved by Timothy Hawkins (Malvern College), Denis Pertuio (Austin Peay State University, Tennessee), Sam and Jim Yu, Can Minh and David Bibby.

28.3 An *exotic number* is a natural number which can be expressed using each of its own digits just once and the operations +, −, ×, ÷, √, !, brackets and powers. Thus, for example, $120 = [(2+1)!-0!]!$ is exotic. Show that, if $n_1n_2\cdots n_r$ is exotic, then so is $n_1n_2\cdots n_r9765\,625$, so that there are infinitely many exotic numbers.

Solution by Toby Gee/Junji Inaba

$$\begin{aligned} n_1n_2\cdots n_r9765\,625 &= n_1n_2\cdots n_r\times 10^7+5^{10} \\ &= n_1n_2\cdots n_r\times\left(9+\frac{6}{6}\right)^7+5^{2\times 5}. \end{aligned}$$

Thus, if $n_1n_2\cdots n_r$ is exotic, so is $n_1n_2\cdots n_r9765\,625$.

Also solved by David Bibby and Noah Rosenberg (Rice University, Houston, Texas.)

Filip Sajdak, who proposed the problem, points out that 10^n is exotic whenever $n \geqslant 16$. For, then,

$$10^n = [(1+0!)(0!+0!+0!+0!+0!)]^s$$

where $s = (0!+0!)(0!+0!)(0!+0!)(0!+0!)+0+0+t$ and t is the sum of $n-16$ terms each of the form 0!.

28.4 Find all natural numbers n such that 2^n+n^2 is a perfect square.

Solution by Toby Gee

Suppose $2^n+n^2 = m^2$ for some natural number m. Then $2^n = (m+n)(m-n)$, so $m+n = 2^{n-a}$ and $m-n = 2^a$ for some number a with $n-a > a$, i.e. $2a < n$. Then

$$n = \tfrac{1}{2}(2^{n-a}-2^a) = 2^{a-1}(2^{n-2a}-1).$$

If n is odd then $a = 1$ and $n = 2^{n-2}-1$. This is impossible. If n is even then, for all $n \geqslant 14$,

$$(2^{n/2}+1)^2 > 2^n+n^2 > (2^{n/2})^2$$

and 2^n+n^2 is strictly between consecutive perfect squares. A check of the even numbers less than 14 shows that $n = 6$ is the only solution.

Also solved by Junji Inaba and David Bibby.

Can Minh asks whether the following is true: for a prime number p, the equation $p^n+n^p = m^p$ has no solutions for natural numbers m and n. □

Reviews

Mathematics—The Science of Patterns. By KEITH DEVLIN. W. H. Freeman and Co, Oxford, 1995. Pp. 216. Hardback £19.95 (ISBN 0-7167-5047-3).

The first thing that one notices about this book is how attractive it is. It is a hardback, fully illustrated in colour, with reasonably large, clear print. The next thing that one notices is the wide variety of topics covered; we move from $(a+b)^2 = a^2+2ab+b^2$, to irrational numbers, to the prime-number theorem in the space of a few pages. One begins to wonder for whom these topics are intended; whilst this is never made entirely clear, one is given the impression that the intended reader is the 'interested layman', with very little mathematical knowledge. Unfortunately, unlike Devlin's previous book, *Mathematics: The New Golden Age*, there is nothing here that assumes more knowledge, a position which is often irritating.

The choice of topics is very wide indeed, from differential equations to Fourier analysis to the Mandelbrot set and the Riemann hypothesis. Inevitably, however, with so many areas to cover, depth loses out to breadth; one feels somewhat cheated to be given the result that $\zeta(2) = \frac{1}{6}\pi^2$, with no hint of how this is to be established; or to be told that if you doubt Desargues' theorem, you should 'draw a few further figures until you are convinced'. In all honesty, apart from a few gems (such as Conway's convex solid that only tiles space non-periodically), there is very little to interest the mathematician; and I have to say that I think that few beginners would be glad to have advanced theorems thrust upon them, with no indications of the stories (and proofs!) behind them. For instance, I have always found the story of Euler's discovery of $\zeta(2)$ as fascinating as his initial argument for its correctness.

Perhaps the most telling, and frustrating, part of the book is the bibliography. The recommended books include those by Ivars Peterson and Ian Stewart's *The Problems of Mathematics*, but there are no 'real mathematics' books; anyone wanting to read up on, say, the Riemann hypothesis, is given no indications of where to turn.

In conclusion, whilst this book looks very nice, it is not cheap; for a lot less one could purchase the far superior *Problems of Mathematics* or, and in my opinion best of all, *Mathematics: The New Golden Age*, a book that covers fewer topics in more detail, and that is not afraid to include 'real mathematics'. Only buy this book if you really want to; it might look nice on a school library shelf, but an interested student would learn far more from one of the books referred to above.

Student at The John of Gaunt School, Trowbridge — TOBY GEE

The Lighter Side of Mathematics: Proceedings of the Eugène Strens Memorial Conference on Recreational Mathematics and its History. Edited by RICHARD K. GUY AND ROBERT E. WOODROW. The Mathematical Association of America, Washington DC, 1994. Pp. viii+367. Paperback $38.50 (ISBN 0-88385-516-X).

Eugène Strens (1899–1980) was a Dutch enthusiast of recreational mathematics, chess and 'ex libris', the book-plates pasted inside the front covers of books, bearing the owner's name or crest. In 1986 Strens' collection of material on mathematical recreations was presented to the University of Calgary; the conference held to celebrate this event is recorded in this book.

The articles cover a variety of aspects of recreational mathematics, from Frieze Patterns, Ten-Pin Bowling Scores and Rubik's Cube to the Ancient English Art of Change Ringing and the Confessions of a Puzzlesmith. There is also a description of the Displays of Combinatorial Games, Puzzles and Toys which formed part of the conference. The articles are placed under three headings, Tiling & Colouring, Games & Puzzles and People & Pursuits.

Here is a taste of the puzzles in the book.

> I have taken a large rectangle and divided it up into a number of smaller rectangles, not necessarily all the same size. I will not let you see what I have done but I will tell you that each of the smaller rectangles has at least one side of integer length. You have to write out a proof which shows that the large rectangle has at least one side of integer length. (14 proofs are given.)
>
> I have taken a piece of squared graph paper and selected five grid points. I will not let you see what I have done. You have to write out a proof which shows that there are two of the five points such that the straight line from one point to the other goes through a grid point. (Hint: use the pigeon-hole principle.)

This is a most interesting and entertaining book. However I would see it as a purchase for the library rather than for an individual and as an addition to an established collection of recreational mathematics books rather than as a first book for a new collection.

University of Sheffield — KEITH AUSTIN

Lion Hunting and Other Mathematical Pursuits: A Collection of Mathematics, Verse and Stories by Ralph P. Boas, Jr. Edited by GERALD L. ALEXANDERSON AND DALE H. MUGLER. Mathematical Association of America, Washington DC, 1995. (Dolciani Mathematical Expositions, Volume 15). Pp. xii+308. Paperback $35.00 (ISBN 0-88385-323X).

R. P. Boas (1912–1992) was an eminent and prolific American mathematician whose research was mainly in the fields of real and complex analysis. He was also a devoted teacher, an able administrator and an amused observer of the world around him; and it was these latter characteristics that led to the bulk of this book's contents.

Appropriately the book begins with an autobiographical sketch by Boas. His parents both taught English literature and his own feeling for the language (by no means universal among mathematicians) may well have had its origin in their influence. When entering Harvard he intended to become a chemist, but a record bill from broken glassware helped to steer him towards mathematics. He graduated in 1933 and during the next few years he managed to fit in a Harvard PhD as well as a good deal of European travel. His first job was an instructorship at Duke University in North Carolina, where he met his wife who was similarly employed. Having decided that they would both have academic careers, they shared all household tasks. A former graduate student recalls that, when she and her husband visited them in the 1940s, they were astonished by this then unheard of practice. Parenthood made no difference to the Boas principle: mother and father took equal turns at looking after the children, though often ingenuity was needed to fit in also their professional duties. However, their greatest difficulty was finding permanent jobs in locations close enough to one another to allow daily travel from a common home. Eventually they settled near Chicago, with him at Northwestern University (in Evanston) and her at de Paul (Chicago).

The seemingly odd association in the book's title of lion hunting with mathematics refers to a rather elaborate mathematical joke. In the 1930s mathematicians liked to amuse themselves by devising ways of catching lions with the aid of mathematics. Here are two simple examples. (i) Place a spherical cage in a desert containing a lion, enter it and lock it. Now perform an inversion with respect to the cage. The lion is then inside the cage and you are outside it. (ii) Since the lion has non-zero mass, it has moments of inertia. Grab it during one of them. Boas and a British friend, Frank Smithies, took the joke further by publishing under a pseudonym a paper listing all lion hunting methods then known to them. These and some newer ones are given in the book.

There are several sections headed 'Recollections and Verse'. Many of the former are mildly mathematical. One that is technical in another sense concerns a friend who was an expert on wine. Boas once remarked to him that all *he* knew on the subject was that he was born in a good year for port. The friend hesitated barely perceptibly, then said 'Oh, I thought you were older than that'. From his 15-year stint as chairman of his department comes the story of two student deputations, the first complaining bitterly about one of their instructors, the second, from the same class, extolling his excellence.

The shortest piece is a quotation from another mathematician: Professors know lots of things—if you give them time to look them up. I am less enamoured of Boas' verse, though his sentiments are always admirable.

During his entire career, Boas constantly strove to invent new ways of teaching material that his students had found difficult. Many of his new approaches appear in this book; for instance, there are treatments of inverse functions, indeterminate forms and the fundamental theorem of algebra. My own favourite is a new proof of the universal chord theorem which says that, given a continuous curve C and a chord of C, there are arbitrarily short parallel chords.

With such a wealth and variety of material in the book, a review can provide only a glimpse of the contents. Professional mathematicians will derive most pleasure from the book, but anyone with an interest in mathematics should enjoy substantial parts of it. Since it is ideal for dipping into, it would make wonderful bedside reading were it not so hard to put down.

University of Sheffield H. BURKILL

Circles: A Mathematical View. By DAN PEDOE. The Mathematical Association of America, Washington, DC, 1995. Pp. 144. Paperback $18.95 (ISBN 0-88385-518-6).

This is a new edition of a mathematical classic (first published in 1957!). This edition now includes an introductory chapter that makes the book accessible to readers with little or no geometrical background. However, I feel that some prior knowledge of these topics is helpful.

The book covers most of the famous circle theorems (in particular, there is a very easy-to-follow proof of Feuerbach's theorem on the 9-point circle), constructions with compass alone, and inversion. The section on inversion is particularly impressive, as, unlike several other introductions to the subject, it is not afraid to use complex numbers. For example, bilinear mappings are presented in a form which makes a geometric analysis of their properties easy to understand. The latter parts of this chapter are perhaps a little difficult for A-level students; the section on non-Euclidean geometry is taken as far as the concept of distance, which is basically a proof that the distance function defines a metric space.

The last chapter, on the isoperimetric theorem, seems curiously out of place. It has little connection with the rest of the book, and it requires much more careful study. If one has the time and patience, however, it will undoubtedly reward the efforts required.

At the end of the book, there is a nice set of problems, with solutions, on all topics apart from the isoperimetric theorem.

The last chapter aside, this book would make an excellent second book on geometry, after, say, Coxeter and Greitzer's 'Geometry Revisited'.

Student at The John of Gaunt School TOBY GEE
Trowbridge

In Search of Infinity. By N. YA VILENKIN. Birkhäuser, Basel, 1995. Pp. 145. Hardback DM46 (ISBN 3-7643-3819-9).

Your reviewer first became fascinated by infinite sets on reading Vilenkin's original book 'Stories about Sets'. The present volume is an extension of that original book. Most readers who have the original will not need to invest in the expanded volume as the essentials are already there. Thus readers may prefer to start at Chapter 2, where infinite sets are considered in the inimitable way of the original with an infinite hotel which is full and yet can accommodate an infinity of infinite collections of extra guests. Of course, it depends what you mean by infinity, as you will see. There are infinities and infinities, and some are bigger than others. Which leads to all sorts of complications. What about the barber in the village who shaves all those and only those who do not shave themselves? Does he shave himself or not? And that is only the beginning.

There are other amazing things. Curves that fill squares, or which have prickles at every point. The writer takes his readers to the very limits of imagination and beyond, and challenges all our preconceptions. If you want to be excited and challenged, then read this book; or the previous 'Stories about Sets' if you can get hold of it.

University of Sheffield DAVID SHARPE

Groups. By C. JORDAN AND D. JORDAN. Edward Arnold, London, 1994. Pp. xi+207. Paperback £8.99 (ISBN 0-340-61045X).

This is a very well-written and well-presented introduction to groups going up to factor groups and ending with a classification of all groups of order up to 12. The first six chapters are easy to read; the degree of difficulty increases markedly in Chapter 7 with the introduction of group actions. The emphasis on group actions in the rest of the book may be rather sophisticated for some readers new to the subject, but throughout there are many examples and suggestions for further investigation, either individually or with others. The book closes with a list of thirteen projects which should keep readers and their fellow-students happy for hours.

The husband and wife team of authors clearly have a gift for exposition of sometimes quite sophisticated material. As they mention in the preface, mealtime conversation must have been somewhat different from that in most households! Perhaps the authors could be persuaded to write a sequel. There is a great need for a student-friendly text on Galois theory written at this high expository level, for example.

University of Sheffield — DAVID SHARPE

The Mathematical Experience. By PHILIP J. DAVIS, REUBEN HERSH AND ELENA ANNE MARCHISOTTO. Birkhäuser, Basel, 2nd revised edition, 1995. Pp. 500. Hardback DM78 (ISBN 0-8176-3739-7).

This is a new edition of a book that has become something of a mathematical classic over the fifteen years since its original publication. The new material consists of sets of exercises designed for a 'course in general mathematics appreciation'; apparently, nothing else has changed.

The book itself is a marvelous collection of mini-essays on a wide variety of mathematical issues, ranging from philosophy to proof, encompassing classic theorems and styles of teaching along the way. The aim of the book is to provide a view of what mathematics is and how it is done, and it comes closer to achieving this than any other book that I know of. It is therefore unsurprising that it is to be found in many school mathematics departments; if it isn't in your school, then it shouldn't be too hard to obtain it from a public library.

The question of whether this new edition is worth buying is another matter. The exercises are not very mathematical in content; they are for the most part short essays, and they frequently refer to other materials that are not readily available. Unless this is a set text for a course that you are studying, there is little point in investing in this edition as compared to the paperback edition which is, however, warmly recommended.

Student at The John of Gaunt School Trowbridge — TOBY GEE

Other books received

Arithmetricks. By EDWARD H. JULIUS. Wiley, Chichester, 1995. Pp. 142. Paperback £7.50 (ISBN 0-471-10639-9).
Readers of *Mathmatical Spectrum* will probably find the level of this book to be too elementary. But, in these days when we work out 2×3 on a calculator, perhaps we all need to get into the habit of doing mental arithmetic and of finding quick ways of doing it. So you might find a browse through this book entertaining and enlightening. It might make us all think!

Statistics 2. By NATIONAL EXTENSION COLLEGE. Collins Educational, London, 1995. Pp. 135. Paperback £9.50 (ISBN 0-00-322398-1).

Pure Mathematics 3 and 4. By NATIONAL EXTENSION COLLEGE. Collins Educational, London, 1995. Pp. 370. Paperback £15.95 (ISBN 0-00-322416-3).
These attractively produced volumes are in the series from the National Extension College covering the University of London modular mathematics syllabus for A and AS level.

Mechanics 2 (Advanced Modular Mathematics for A and AS level). By NATIONAL EXTENSION COLLEGE. Collins Educational, London, 1995. Pp. vi+162. Paperback £9.50 (ISBN 0-00-322400-7).
This student-centred volume from the National Extension College provides full coverage of the University of London syllabus module M2: further kinematics, further dynamics, uniform circular motion, work–energy principle, collisions, statics of rigid bodies. It has been extensively tested as part of the NEC's flexible-learning courses.

Mechanics 3. By JOHN BERRY, PAT BRYDEN, TED GRAHAM AND ROGER PORKESS. Hodder and Stoughton, London, 1995. Pp. 172. Paperback £7.99 (ISBN 0-340-57862-9).
This is the third Mechanics book written to support the MEI Structured Mathematics series and completes the basic Mechanics course started with Mechanics 1 and 2. Circular motion, elasticity and simple harmonic motion are introduced, while earlier work on centre of mass is extended by using calculus methods. The final chapter covers dimensional analysis. This book can also be used in conjunction with any mechanics course.

Exploring Mechanics. By THE MEW GROUP. Hodder and Stoughton, London, 1995. Pp. 96. Softback £41.99 (ISBN 0-340-64817-1).
This book presents problem-solving and investigative ideas for Mechanics. The photocopiable students' section encourages students to think about problems in physical and practical terms as a complement to theoretical approaches. The activities in this book can be used to support the subject core for Mathematics.

The teacher's section provides guidance on the problems in the students' section, plus photocopiable sheets of 'Hints and Nudges' for students. These accompany both short and long investigations.

Exploring Mechanics offers a fresh insight into the subject for all A-level and undergraduate students of Mathematics and Engineering.

Practical Mathematics using MATLAB. By GUNNAR BACKSTROM. Studentlittur, Lund, 1995. Pp. 165. Softback £18.95 (ISBN 0-86238-397-8).
This book is a guided tour of first-year university mathematics, intended to parallel conventional courses. Most of the applications are numeric and also illustrate elementary principles of programming. Students may work through the examples and the problems on a personal computer, either individually or in groups under supervision. The software used is available for the IBM PC and compatibles, for the Macintosh and for various workstations.

Pure Mathematics 3. By CATHERINE BERRY, VAL HANRAHAN AND ROGER PORKESS. Hodder and Stoughton, London, 1995. Pp. 188. Softback. £7.99 (ISBN 0-340-57864-5).
MEI Structured Mathematics is a series of student books that has been written to support the MEI scheme of the same name. Together all the books cover all the components in the scheme, adopting a practical, active approach designed to facilitate both independent and supported learning.

In *Pure Mathematics 3* further techniques in algebra, trigonometry, differentiation and integration are introduced. Other chapters cover parametric equations, vector geometry and differential equations. □

LONDON MATHEMATICAL SOCIETY

1996 POPULAR LECTURES

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London - Tuesday 18 June
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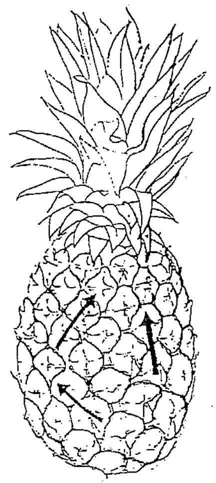

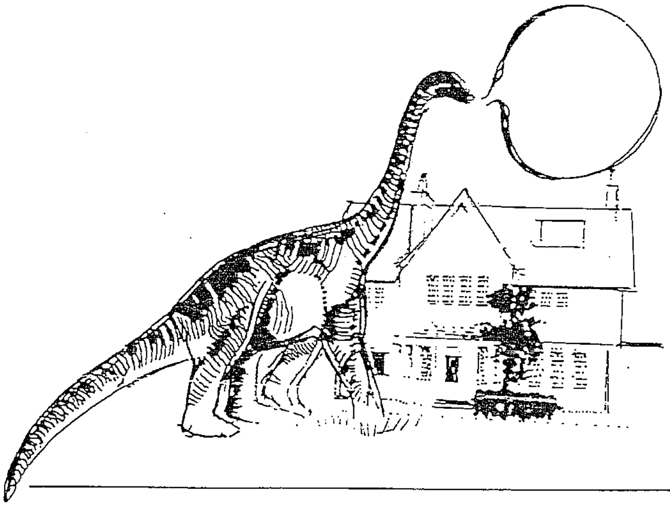

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"Simple ideas in mechanics, combined with homely experiments, show how mathematical thinking can tell us about the equilibrium of bubbles and balloons, and the motion of dinosaurs."

BIRMINGHAM UNIVERSITY Commences at 2.00 pm, 3.00-4.00 pm refreshments, ends at 5.00 pm. Haworth Lecture Theatre, Chemistry Building, University of Birmingham. Admission is free. Enquiries to Dr A.D. Gardiner, Department of Mathematics, University of Birmingham B15 2TT.

IMPERIAL COLLEGE LONDON Commences at 7.30 pm, 8.30 pm refreshments, ends at 10.00 pm. Lecture Theatre Room 220, The Mechanical Engineering Building, Imperial College, South Kensington, London SW7. Admission free, with ticket in advance. Apply by Friday 14 June to Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL. A stamped addressed envelope would be appreciated.

GLASGOW UNIVERSITY Commences at 2.00 pm, 3.00 pm refreshments, ends at 4.30 pm. Western Infirmary Lecture Theatre, University Place, University of Glasgow. Admission is free. Enquiries to Dr P. Heywood, Department of Mathematics & Statistics, Edinburgh University, James Clerk Maxwell Building, The King's Buildings, Edinburgh EH9 3JZ; e-mail: philip@maths.ed.ac.uk.

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