Mathematical Spectrum

Volume 9 1976/77 *Number* 3

A Magazine of *Contemporary Mathematics*

Published by the *Applied Probability Trust*

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 9 of *Mathematical Spectrum* consists of three issues, of which this is the third. The first issue was published in September 1976, and the second in January 1977.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

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The Editor, *Mathematical Spectrum*,
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Counting in Cuneiform

MICHAEL ROAF
Gonville and Caius College, Cambridge

Counting is an integral part of all languages and it is not surprising that numbers appear in the earliest written documents; these inscriptions have been found in Mesopotamia and are over 5000 years old. Although the numbers can be understood, the other signs are still undeciphered. By the Old Babylonian period, as archaeologists call the period between 1900 and 1600 BC, very sophisticated developments had taken place in mathematics in Mesopotamia. Our knowledge of these developments comes from the clay tablets excavated during the last hundred years, and more discoveries are being made every year. The mathematical texts are only a fraction of all the cuneiform texts found so far, but they are an extremely interesting fraction. The tablets are frequently damaged and their terminology is still poorly understood, but sufficient of the mathematics can be interpreted to demonstrate the considerable abilities of Old Babylonian mathematicians.

Babylonians wrote their numbers with a rectangular-ended stylus impressed on a clay tablet: a vertical wedge stood for the unit and a sloping wedge for ten. These were repeated as often as necessary to write the other numbers as is shown in Figure 1. (The right-hand column of Figure 2 gives the numbers from 1 to 15.)

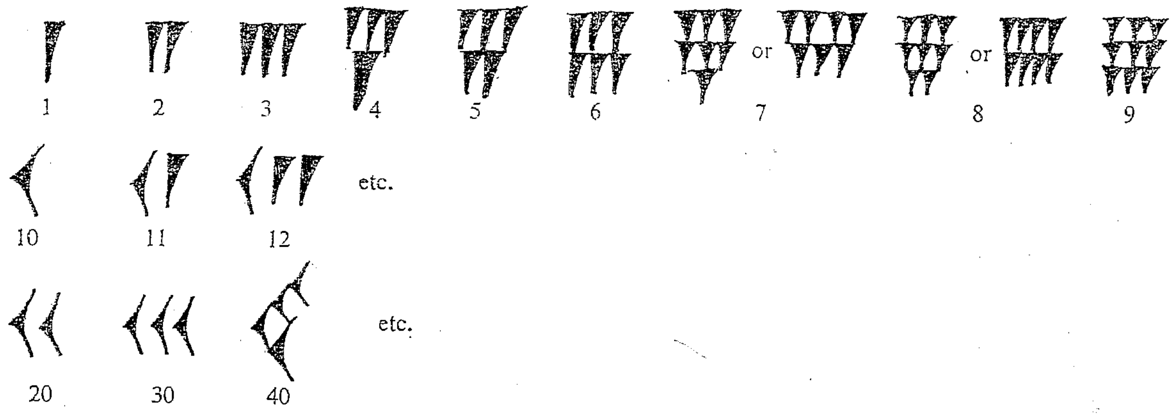

Figure 1

The most important feature of the Babylonian number system was that it used a place-value notation with a base of 60: 𒁹 could stand for 1, 60, 60 × 60, 60 × 60 × 60, etc. or even 1/60, 1/60 × 60, etc. The particular value was inferred from the context. We use a similar decimal place-value notation, so that in 111·11 the symbol 1 stands for 10 × 10, 10, 1, 1/10, 1/10 × 10 according to its position. This is greatly superior to the number systems of the Egyptians, Greeks and Romans where new symbols were added for the higher numbers (e.g. I, V, X, L, C, D, M, etc.). Using just two symbols repeated in various orders the Babylonians could write all the integers. The place-value notation was employed politically by a later Assyrian king. When the chief temple of Babylon, that of the god Marduk, was destroyed by the Assyrian king Sennacherib, Marduk had foretold 'seventy years as the period of the city's desolation'. Only eleven years later Sennacherib's successor Esarhaddon restored the temple, and to justify this he claimed that the god had 'turned the tablet upside down and ordered the city's restoration in the eleventh year'. For if one turns upside down (or reverses) the Babylonian 70, 𒁹𒌋, one gets 11, 𒌋𒁹.

In modern transcriptions of Babylonian numbers, commas are used to separate the figures and a semicolon for the sexagesimal point. So 1,59 means $60 + 59 = 119$ and 1;59 means $1\frac{59}{60}$. But since the sexagesimal point was not indicated on the tablets, these two numbers were written identically. Further, a sign for zero was not invented until the Seleucid period (3rd century BC) and so 1,59 could also have meant 1,0,59 ($= 60 \times 60 + 59$) or even 1,50,9 ($= 60 \times 60 + 50 \times 60 + 9$) though in this last case one would normally expect a gap between the 50 and the 9. Clearly such ambiguity was not practical for daily commercial transactions, and in business accounts the hundreds and thousands were normally expressly stated. Thus in a business document 119 would be written as 1 hundred and 19 not 1,59. It was only in the mathematical and astronomical texts that the sexagesimal place-value notation was consistently used.

The chief advantages of this system are that multiplication was extremely easy both for integers and for fractions, and that many calculations were simplified because the sexagesimal base has so many factors.

It is not known why the Babylonians chose 60 as their base rather than, for example, 10. It may have been connected with their systems of weights and measures, which were largely sexagesimal, or it may have been its divisibility. The base 60 is sufficiently practical to have survived to the present day in our divisions of time and of angle. The divisibility of 60 seems to have led the Babylonians to partition the integers into two classes. For convenience, we will call the integers in the two classes regular and irregular numbers respectively. The reciprocal of a regular number has a finite sexagesimal expansion, and regular numbers may be expressed as $2^{\alpha}3^{\beta}5^{\gamma}$. Irregular integers such as 7, 11, 13, 14, 17, etc., intrigued the Babylonians and special problems were constructed using these difficult numbers.

Most of the Old Babylonian mathematical texts that are preserved came from the scribal schools, and many were exercises copied out by the students. They are of two sorts, table texts and problem texts. The table texts were the tools used in every-day calculations; they list reciprocals for both regular and irregular numbers, multiplication tables, squares, square roots, cubes and even selected logarithms to the bases 2 and 16; though the Babylonians probably did not appreciate the additive properties of logarithms.

The problem texts cover many topics: problems in plane geometry and solid geometry, linear equations, quadratic equations and so on. All these problems are solved in an algebraic or arithmetical way. There is no use of the geometric approach characteristic of Euclid and other Greek mathematicians. The Babylonian problem texts are often phrased in terms of practical problems such as digging a canal, estimating the number of bricks in a given volume or calculating the area of a field; but it is clear that the problem texts are essentially theoretical. For example, a typical problem deals with the solution of a quadratic equation but is expressed in terms of fields and lengths, where field stands for x^2 and the length for x. Such a problem is translated below. The exact meanings of the particular words are uncertain, but the mathematical procedure is quite clear.

A field: I added four lengths and the field: 41,40.	$4x + x^2 = \frac{25}{36}\left[= \frac{41}{60} + \frac{40}{60^2}\right]$
You take the four lengths; the reciprocal of 4 is 15.	$x + \frac{x^2}{4}$
You multiply 15 × 41,40. You take 10,25.	$= \frac{25}{144}\left[= \frac{10}{60} + \frac{25}{60^2}\right]$
You add 1, the goer out: 1,10,25. It is the square of 1,5.	$1 + x + \frac{x^2}{4} = 1\frac{25}{144} = \left(\frac{13}{12}\right)^2$
You subtract 1, the goer out, which you added: 5. By two	$\frac{x}{2} = \frac{1}{12}$
You multiply. It is the square of 10.	$x = \frac{1}{6}$

Some of the achievements of Babylonian mathematicians almost 4000 years ago are astounding. They normally used the value of 3 for π but they also estimated it as 3;7,30 or 3·125 in decimals. They calculated $\sqrt{2}$ as 1;24,51,10 which is correct to three sexagesimal places. You may try squaring this in sexagesimal notation to verify its accuracy.

One text stands out as exceptional amongst all the Old Babylonian mathematical texts. It is in the Plimpton Library of Columbia University in New York City and its catalogue number is Plimpton 322. It was first translated by Professors Neugebauer and Sachs in their work on *Mathematical Cuneiform Texts* (reference 2). (Previously it had been identified as a 'commercial account'.) Plimpton 322 (shown in Figure 2) is neither a table text nor a problem text although it is clearly the answer to some problem and is in the form of a table. The tablet is broken along the left edge and the whole of the first column is missing. The size of the preserved part is 12·7 by 8·8 cm. Four columns are preserved; each of these has a heading and contains fifteen numbers. The reverse is not inscribed. The heading of the last column might be translated as 'its name' and is just a serial numbering of the rows from 1 to 15. The second and third columns are headed 'ibsi of the width' and 'ibsi of the diagonal' and list sets of integers forming the sides of Pythagorean triangles. (The term 'ibsi' might be translated 'number'.) The missing left-hand column probably contained the measurements of the other side of each triangle. The heading of the first extant column reads '*takiltum* of the diagonal which has been subtracted such that the width . . .'. The exact translation is uncertain but an examination of the numbers in the column shows that these equal the squares of the ratio of the diagonal and the third side. These numbers are arranged in order so that they decrease from 2 to $1\frac{1}{3}$ in almost equal stages. There are a few errors in the figures; most of these can be explained as slips in the scribe's calculations or in his copying. It is very surprising how few mistakes the Babylonian scribes made, far fewer than I have made in checking them. The translation of Plimpton 322 is shown in Figure 3; Table 1 is the reconstructed version in decimals.

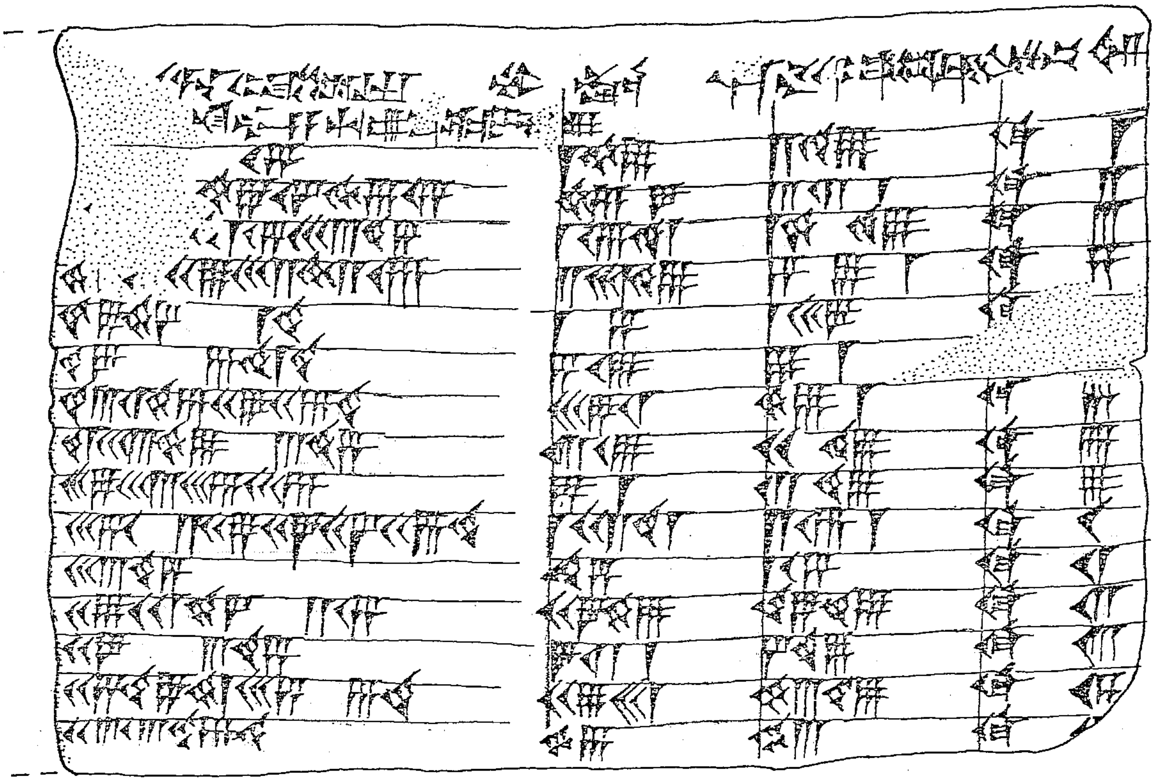

Figure 2. Autograph copy of the tablet Plimpton 322 (copied from Neugebauer and Sachs, reference 2, Plate 25). Reproduced by permission of the American Oriental Society. The left-hand part of the tablet is missing. Damaged areas are stippled.

The numbers in the first extant column (*takiltum* . . .) all have a finite sexagesimal expansion and all the numbers in the missing left-hand column are regular. What is more, these missing numbers are mostly simple, while the lengths of the other two sides are more complex. In fact this is always the case with Pythagorean numbers and a well-known theorem states that all relatively prime Pythagorean triples are uniquely represented in the form

$$2pq, p^2 - q^2, p^2 + q^2,$$

where p and q are relatively prime integers, not both odd, and $p > q$. Table 2 lists the values of the sides and diagonal of the triangle and the square of their ratio for all the regular values of q such that this square lies between 2 and 1·38. The table contains exactly those sets of numbers recorded in Plimpton 322, rearranged in a different order.

We are now in a position to consider what the scribe was trying to do. He has used the fact that all numbers of the form $2pq$, $p^2 - q^2$, $p^2 + q^2$, where p and q are relatively prime, are Pythagorean. He has taken those with p and q regular integers, with q lying between 0 and 60 and which have $((p^2 + q^2)/2pq)^2$ lying between 2 and 1;20. The two exceptions are line 11: 1, 45, 1,15 (this is the standard form for the 3, 4, 5 triangle in Babylonian texts) and 15 where the numbers can be calculated from $p = 9, q = 5$ giving 90, 56, 106 which can be reduced to 45, 28, 53: the scribe recorded this as 56, 53, probably having forgotten to divide the first term by two. Then the scribe rearranged these numbers so that the ratio of the sides decreases.

And all this was done more than three and a half millennia ago. What an achievement!

*	[*ta-ki-i*]*l-ti si-li-ip-tim* [*ša in-*]*na-as-sà-hu-ú-*[*m*]*a* sag *i- . . . -ú*	íb-si$_8$ sag	íb-si$_8$ *si-li-ip-tim*	mu-bi-ím
[2,0]	[1,59],15	1,59	2,49	ki-1
[57,36]	[1,56,56],58,14,50,6[a],15	56,7	3,12,1[g]	ki-2
[1,20,0]	[1,55,7],41,15,33,45	1,16,41	1,50,49	ki-3
[3,45,0]	[1],5[3,1]0,29,32,52,16	3,31,49	5,9,1	ki-4
[1,12]	[1],48,54,1,40	1,5	1,37	ki[-5]
[6,0]	[1],47,6,41,40	5,19	8,1	[ki-6]
[45,0]	[1],43,11,56,28,26,40	38,11	59,1	ki-7
[16,0]	[1],41,33,59[b],3,45	13,19	20,49	ki-8
[10,0]	[1],38,33,36,36	9,1[d]	12,49	ki-9
[1,48,0]	[1],35,10,2,28,27,24,26,40	1,22,41	2,16,1	ki-10
[1,0]	[1],33,45	45	1,15	ki-11
[40,0]	[1],29,21,54,2,15	27,59	48,49	ki-12
[4,0]	[1],27,0,3,45[c]	7,12,1[e]	4,49	ki-13
[45,0]	[1],25,48,51,35,6,40	29,31	53,49	ki-14
[45 or 1,30][f]	[1],23,13,46,40	56[f]	53[f]	ki-1[5]

* The left-hand column is not preserved and the zeros, included for clarity, would not have been written.

Notes to transliteration.

Words in italic script are in the Babylonian language; words in roman script are in the earlier Sumerian language but would have been read in Babylonian, just as today we may write i.e. (for *id est*) and read it as 'that is'. In the cuneiform script some signs represent more than one syllable (polyphony) and some syllables can be represented by more than one sign (homophony). In the transcription these are written with various accents or suffices; thus *si*, *sí*, *si̋*, *si*$_4$, *si*$_5$, etc.

Numbers and letters in square brackets are not preserved.

a. 50,6 written like 56.
b. 59 error for 45,14.
c. written like [1],27,3,45.
d. 9,1 error for 8,1.
e. 7,12,1 (the square of 2,41) error for 2,41.
f. 56 error for 28 (half 56) or 53 error for 1,46 (twice 53). See above.
g. 3,12,1 error for 1,20,25. I can see no explanation for this error.

Figure 3. Transliteration of the tablet Plimpton 322.

TABLE 1. Reconstructed version of Plimpton 322 in decimals (to 4 decimal places)

*	*takiltum* ...	ibsi of the width	ibsi of the diagonal	its name
120	1·9875	119	169	1
3456	1·9492	3367	4825	2
4800	1·9188	4601	6649	3
13500	1·8862	12709	18541	4
72	1·8150	65	97	5
360	1·7852	319	481	6
2700	1·7200	2291	3541	7
960	1·6927	799	1249	8
600	1·6427	481	769	9
6480	1·5861	4961	8161	10
60	1·5625	45	75	11
2400	1·4984	1679	2929	12
240	1·4500	161	289	13
2700	1·4302	1771	3229	14
45	1·3872	28	53	15

TABLE 2. A method for constructing the numbers in Plimpton 322

q	p	ibsi of the width $p^2 - q^2$	* $2pq$	ibsi of the diagonal $p^2 + q^2$	*takiltum* ... $\left(\frac{p^2+q^2}{2pq}\right)^2$	its name line in Plimpton 322
1	2	3	4	5	1·5625	11
4	9	65	72	97	1·8150	5
5	9	56	90	106	1·3872	15
5	12	119	120	169	1·9875	1
8	15	161	240	289	1·4500	13
9	20	319	360	481	1·7852	6
12	25	481	600	769	1·6427	9
15	32	799	960	1249	1·6927	8
25	48	1679	2400	2929	1·4984	12
25	54	2291	2700	3541	1·7200	7
27	50	1771	2700	3229	1·4302	14
27	64	3367	3456	4825	1·9492	2
32	75	4601	4800	6649	1·9188	3
40	81	4961	6480	8161	1·5861	10
54	125	12709	13500	18541	1·8862	4
					to 4 decimal places	

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For a general survey see:
1. O. Neugebauer, *The Exact Sciences in Antiquity* (Dover, New York, 1969, 2nd edn).
For primary sources see:
2. O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts*, American Oriental Society Series Volume 29 (American Oriental Society and American Schools of Oriental Research, New Haven, 1945).
3. F. Thureau-Dangin, *Textes Mathématiques Babyloniens* (Brill, Leiden, 1938).

The Effect of Bonus Points on the Cricket County Championship

ANDREW BOSI
University of Sheffield

1. Introduction

There have been numerous innovations in the reckoning of Champion Cricket County over the past 75 years, but none has caused such interest amongst statisticians as the bonus points system introduced in 1968. The fascination of this system was due to the possibility that any team could win, right up to the last match. This emphasizes a weakness which has since been eradicated by a further change in the rules: it was possible to obtain more points by amassing a mammoth total quickly than by winning a game.

One may well ask, what are the points for? Are they designed to find the best cricketing side, or simply to encourage 'brighter cricket'? The latter aim was obviously in mind when the system was devised, and the general public seems well satisfied with its effects. In this article, I am concerned with the best cricketing sides, the sort that win the Ashes (the fictitious Anglo-Australian cricket trophy); hence column A in Table 1, the actual championship positions, does not enter into my calculations. These are concerned solely with columns B and C.

The Football League awards two points for a win and one for a draw in its football games; this has the merits of both simplicity and fairness. Cricket and

TABLE 1. Ranking data for counties, 1900

Scoring system	A actual	B (W−L)/(matches played)	C bonus points system of 1968
Derbyshire	13	11	14
Essex	10	10	10
Glamorgan (formed 1921)	—	—	—
Gloucestershire	8	$5\frac{1}{2}$	5
Hampshire	15	15	15
Kent	3	3	4
Lancashire	2	2	2
Leicestershire	14	14	13
Middlesex	8	$5\frac{1}{2}$	6
Northamptonshire (formed 1905)	—	—	—
Nottinghamshire	5	4	7
Somerset	11	13	8
Surrey	8	8	3
Sussex	3	7	11
Warwickshire	6	9	9
Worcestershire	12	12	12
Yorkshire	1	1	1

(Data of rankings for all counties in years 1900–14, 1957–73 may be obtained from the author.)

simplicity are less compatible. However, in the 1900's a similar simple system was used for cricket: 1 for a win (W), 0 for a draw (D) and −1 for a loss (L). Unfortunately the system was slightly spoilt by dividing each team's points by the number of matches in which there had been a result, and basing the championship on the resultant average. This meant that W1, L0, D15 took precedence over W15, L1, D0. For column B of Table 1 I have calculated this basic figure, (W − L)/(matches played). This is fair when everyone plays the same number of games, as they have done since the early 1960s.

'Contrived' results, which would make nonsense of such a championship, have been excluded. A side bowling the opposition out twice to win has done so on merit. The losing side may make a 'business' declaration, for example to give the other side 20 minutes batting in fading light. Such results are not 'contrived'. When, however, there are three declarations in a four-innings match after only a few wickets have fallen, leaving the second side a target of about 120–150 runs in 70 to 80 minutes, the result tells us nothing of the two sides' ability in three-day cricket. Ties in the Table were resolved by considering the first-innings lead in matches.

Let us consider column C of Table 1; this gives the estimated positions under the bonus points system of 1968. In this system points are awarded as follows:

for a win: 10;

for a tie: 5 to each side;

for a draw in which scores are level and three innings completed: 5 points to the side batting fourth.

In addition, bonus points may be scored in the first 85 overs of each side's first innings in the following manner:

for every 25 runs over 150: 1 batting point;

for every 2 wickets: 1 bowling point.

Unfortunately, it is not possible to calculate with complete accuracy results for 1900 as, not surprisingly, no one recorded the scores after 85 overs. Indeed 'fall of wicket' was not then recorded by Wisden (reference 1). I have therefore made two assumptions, one of which is plausible, the other less so. The first is that there is a uniform rate of scoring, and the second, a uniform striking rate of wicket-taking. Of course, where the innings lasted less than 85 overs, no problem arises.

There is no easy way of estimating batting points; publication of runs per 100 balls is another recent innovation. Thus for each innings of more than 85 overs duration it was necessary to multiply runs per over by 85, subtract 150 and divide by 25. Ten matches was considered a sufficient number from which to estimate a team's bonus points. The easiest method was in fact to select six home matches per team by systematic sampling. Occasionally this yielded less than four away matches for some county: in this case further away matches were chosen at random. The positions were tabulated as in Table 1.

Before proceeding with the analysis, the limitations of this investigation should be stressed. The first is the difference between pre-war and post-war cricket. Leg-theory and defensive field placings were unknown, as indeed were over-rates of 12 an hour. Tactics might not be discussed over lunch, because the captain was an amateur and his most experienced colleague a senior professional. Pitches were not covered, nurtured and dressed, and the quicker over-rates and more balanced attacks meant that each type of bowler bowled on each type of pitch.

The second difference is that, whereas in 1900 the captain's first priority was to win and his second not to lose, or possibly the other way round, today a defeat is no different from a draw except for the effect on one's opponents. A. A. Thompson once wrote that whatever the system, the first team will remain first and the last team will always be last: but 'always' should not be included in the statistician's vocabulary.

2. Statistical analysis

Having decided on the compilation of the tables of data, and settled for rankings (position in table) instead of points obtained, the next step was to consider possible hypotheses. We consider these first in the language of cricket and then translate them into statistical hypotheses.

(i) Since the introduction of bonus points, there has been a change in emphasis in cricket games and so more difference between B and C.

(ii) Further to (i), bonus points have affected the designation of eventual cricket County Champions.

We proceed as follows to test the first hypothesis. The data of rankings for each year is subdivided into five groups: 1900–9 when the championship was based on (W − L)/(W + L); 1910–14 when points for first-innings lead were awarded; 1957–62 when the first steps towards 'brighter cricket' were made with the introduction of a very simple bonus points scheme; 1963–67 when the Championship reverted to points for wins and first-innings leads; and 1968–73 when the 85-overs bonus points came into being. There are minor differences within the groups, but none is great enough to alter a captain's strategy.

This hypothesis is basically about the correlation between the two columns B and C for each year. Accordingly, Spearman's rank correlation coefficient ($r = 1 - (6\,\Sigma\, d^2/n)$) was calculated for each year. It should be noted that n, the number of counties in the Championship, was 15 from 1900–4, 16 from then until 1914 and 17 for the post-war years, as Northamptonshire and Glamorgan entered the Championship in 1905 and 1921 respectively. The resulting coefficients were placed in five columns according to the year. A null hypothesis H_0 that there is no difference between columns can be tested against H_1, that the fifth-column coefficients are significantly lower. If μ_i denotes the mean of the ith column, $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$, while $H_1: \mu_1 = \mu_2 = \mu_3 = \mu_4 > \mu_5$.

2.1. *Carrying out a test.* The first hypothesis tested was (ii). We first noted the numbers of occurrences of a different champion being proclaimed under systems B and C.

We may regard such an occurrence as a 'success', and a year in which both systems give the same champion as a 'failure' of our hypothesis. We shall also assume that the successes are binomially distributed.

TABLE 2. Occurrences of different champions under B and C

Years	Number of occurrences	Percentage	
1900–9 (10 years)	2	20%	26·7%
1910–14 (5 years)	2	40%	
1957–62 (6 years)	3	50%	47·1%
1963–67 (5 years)	2	40%	
1968–73 (6 years)	3	50%	
		Overall percentage 37·5%	

We first test whether an increase in probability has taken place at the introduction of bonus points in 1968. The years 1900–67 will form Group A, and 1968–73 Group B. We use the pooled estimate technique as follows.

Successes in Group A: 9 in 26 $(\hat{p}_A = 9/26)$
Successes in Group B: 3 in 6 $(\hat{p}_B = 1/2)$
Successes in both groups: 12 in 32 $(\hat{p} = 12/32 = 3/8)$.

We consider the hypothesis $H_0: p_A = p_B$. Using the 5% level of significance in this one-sided test, we reject H_0 if the standardized variable Z_0 below is greater than 1·64; we see that

$$Z_0 = (\hat{p}_B - \hat{p}_A)\Big/\sqrt{\left\{\hat{p}\hat{q}\left(\frac{1}{n_A}+\frac{1}{n_B}\right)\right\}} = \frac{2}{13}\Big/\sqrt{\left\{\frac{3}{8}\cdot\frac{5}{8}\left(\frac{1}{26}+\frac{1}{6}\right)\right\}} < 1$$

so that the hypothesis that $p_A = p_B$ is tenable.

2.2. *Further tests of hypothesis.* A crude test of hypothesis (ii) would be to look at the proportion (or percentage) of times that column B produces a different champion from column C of the same year, calculating the percentage within each of the five groups outlined for hypothesis (i). This leaves problems in deciding the power of the test, and what corresponds to 5% significance. There are only six bonus points years and 25 other years. If p_i is the probability of a different winner as estimated in this way for the ith group, we test $H_0: p_1 = p_2 = p_3 = p_4 = p_5$ against the alternative $H_1: p_1 = p_2 = p_3 = p_4 < p_5$.

Possibly more appropriate would be a goodness-of-fit test. We would hope to show that a uniform distribution with a single jump at the year 1968 is a better fit than a completely uniform, linear or step distribution at 1914/1957.

This hypothesis differs from (i) in that we seek to place a stronger emphasis on differences in first place rather than differences throughout the table. The eighth team may move up or down, but the top team may only be replaced lower down; hence the relevance of A. A. Thompson's remarks quoted earlier. We cannot therefore reject our null hypothesis that there is no increase in probability of success. There is insufficient evidence that a change took place altering the probability for 1968–1973. The small sample size of Group B lessens the effectiveness of this analysis.

A test of fit for the hypothesis of a distribution with one step at 1914/1957 was similarly inconclusive. These results are in line with the general opinion of cricketers. From a mathematical point of view a team placed first can only move in one direction whereas those in the middle can move up or down. This is an over-simplification, but it is not completely irrelevant.

The next test tried was the measurement of the degree of correlation between B and C. For this, Spearman's rank correlation coefficient r_s was used, and the results are given in Table 3.

TABLE 3. Spearman's rank correlation coefficients, 1900–1973

Year	$\Sigma \lvert d \rvert$	Σd^2	r_s	Year	$\Sigma \lvert d \rvert$	Σd^2	r_s
1900	23	$86\frac{1}{2}$	0·846	1957	24	54	0·934
1901	19	82	0·854	1958	42	156	0·809
1902	36	122	0·782	1959	50	224	0·725
1903	22	$71\frac{1}{2}$	0·872	1960	20	84	0·894
1904	34	106	0·811	1961	38	124	0·848
1905	36	112	0·835	1962	38	140	0·828
1906	21	$52\frac{1}{2}$	0·923	1963	24	58	0·929
1907	36	136	0·800	1964	26	$69\frac{1}{2}$	0·915
1908	30	120	0·824	1965	48	212	0·740
1909	28	96	0·859	1966	30	94	0·885
1910	25	$78\frac{1}{2}$	0·885	1967	54	270	0·669
1911	18	28	0·959	1968	32	102	0·875
1912	22	54	0·921	1969	56	354	0·566
1913	28	84	0·876	1970	48	248	0·696
1914	22	56	0·918	1971	44	178	0·782
				1972	32	$142\frac{1}{2}$	0·825
				1973	34	122	0·850

The reader should note that $n = 15$ until 1904, $n = 16$ for 1905–1914 and $n = 17$ since 1957; all values of r_s are highly significant. Note also that some of the lowest figures are associated with the dry-weather years 1959, 1969, 1970; but 1973 is a glaring exception.

2.3. *Analysis of variance.* It was decided to carry out an analysis of variance on the above coefficients. This seemed a more intuitive approach than setting up a contingency table of rankings, and is also less laborious. The coefficients of correlation are presented in their predetermined groups in Table 4.

Details of the analysis of variance are presented in Table 5.

In what follows y_5 represents the mean correlation coefficient in column 5 (i.e. 0·766); y_A the mean for all other columns.

We now test the hypothesis (i) about change of emphasis after the introduction of bonus points. The null hypothesis is H_0: there is no difference between columns, while the alternative is H_1: column 5 of Table 4 has a smaller mean. If y_A denotes the mean for columns 1 to 4 of Table 4, we note that

$$(y_A - y_5)\Big/\sqrt{\left\{\left(\frac{1}{n_A} + \frac{1}{n_5}\right) 0{\cdot}0068\right\}} \sim t_{27},$$

so we reject H_0 if the test statistic is greater than 2·052. We find that

$$t = \left(\frac{22{\cdot}144}{26} - \frac{4{\cdot}594}{6}\right) \Big/ \sqrt{\left\{\left(\frac{1}{26} + \frac{1}{6}\right) 0{\cdot}0068\right\}}$$
$$= 2{\cdot}15\sqrt{1{\cdot}1} > 2{\cdot}15 > 2{\cdot}052 = t_{27}\,(5\%)$$

so we reject H_0 in favour of H_1.

Tests on the 'difference between k proportions', looking at the proportion of teams each year whose position in the table was altered by changing the scoring system, failed to indicate any significant difference between years or between groups of years.

We conclude that there has been a significant decline in the degree of correlation between teams' positions since the introduction of bonus points. Thus the bonus points system has succeeded in influencing the way in which the game is played. The balance has been shifted in favour of teams playing 'brighter cricket' and away from good cricketing sides.

TABLE 4. Rank correlation coefficients in five groups

1900–9	1910–14	1957–62	1963–67	1968–73
0·846	0·885	0·934	0·929	0·875
0·854	0·959	0·809	0·915	0·566
0·872	0·921	0·725	0·740	0·696
0·782	0·876	0·897	0·885	0·782
0·811	0·918	0·848	0·669	0·825
0·835		0·828		0·850
0·923				
0·800				
0·824				
0·859				

TABLE 5. Analysis of variance

						Total
Sum down columns of Table 4	8·406	4·559	5·041	4·138	4·594	26·738
No. coeff. in column	10	5	6	5	6	32
Mean corr. coeff.	0·841	0·912	0·840	0·828	0·766	
(Column sum)^2	70·661	20·874	25·412	17·123	21·105	
$\dfrac{(\text{Column sum})^2}{\text{No. in column}}$	7·066	4·157	4·219	3·425	3·517	22·341

Source	Sum of squares	Degrees of freedom	Mean square	F	
Treatments (year difference)	0·0424	4	0·0106	1·56	($F_{4,27}(5\%) = 2{\cdot}73$)
Residual	0·1837	27	0·0068		
Total (corrected)	0·2261	31			

3. Further work: improvements on existing methods

The limitations of this analysis have already been outlined. Many of the difficulties cannot be satisfactorily resolved, but further work on the problem is in progress. The first area of doubt in the estimation is 'When is a win contrived?' Occasionally it is 'business' to declare with two or three wickets down, but in these instances, the side doing this is usually in an impregnable position. There were in fact few games in which there was any doubt about whether the win was contrived or not.

A second area of doubt, where error is almost certainly incurred, is the assumption of a uniform striking rate (wickets per over) for the purposes of estimating bowling points. If the innings is declared closed with a few wickets in hand this assumption is often plausible; otherwise points are awarded as follows.

TABLE 6. Award of points

$\mathscr{L}$, Length of innings (overs)	Number of bowling points
$\mathscr{L} \leqslant 85$	5
$85 < \mathscr{L} < 106$	4
$106 \leqslant \mathscr{L} < 140$	3
$140 \leqslant \mathscr{L} < 210$	2

In practice, sides bowling more than 95 overs are in most instances gaining points, though it is possible that they could lose. Hence the effect of bowling points, and with it the value of good bowling, is being diminished. In fact the errors are not great, especially in the context of relative placings in the reconstructed Championship table. There are two possible ways in which this part of the analysis could be improved.

The first is to estimate the score of 85 overs assuming uniform scoring rate, and then estimate how many wickets had fallen at that point, assuming that the batsmen all scored at the same rate and extras were evenly distributed. The second is to investigate the distribution of fall of wickets. Once a satisfactory distribution had been found, a table similar to Table 6 could be compiled and bonus points estimated as quickly as before. I believe, though I have no proof of my conjecture, that if we take a team's completed total and divide it by two, on average four wickets will be down at that point. (Certainly the mode is four.) In assuming uniform striking rate we anticipate that at this point the fifth wicket has just fallen. Another improvement would be to increase the number of matches used in estimating bonus points.

To end on a fitting note, we may quote from an article in the centenary issue of *Wisden* in 1963, reviewing a century of County Championship cricket. 'Over the years the competition has witnessed the fundamental irreconcilability of rewarding a victor appropriately and of apportioning merit in a drawn match.'

4. Acknowledgements

I should like to record my gratitude to Dr D. N. Shanbhag of the University of Sheffield for his assistance with the statistical side of the project, and to Mr Joe Lister, Secretary of the Yorkshire County Cricket Club, for his kind cooperation in compiling the data.

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Perfect and Pandiagonal Magic Hyper-cubes

PATRICK BROOKE
Winchester College

1. Introduction

A $q \times q$ matrix whose elements are the integers $1, 2, \ldots, q^2$ is called a *magic square* of order q if the sums of the elements in all rows and columns are the same (and hence equal to $\frac{1}{2}q(q^2+1)$). I call a magic square *perfect* if the elements in each of the two principal diagonals have the same sum as the elements in each row and column; and a magic square is *pandiagonal* if the sum of the elements in *every* diagonal, broken or unbroken, is equal to the row and column sum.† A broken diagonal is illustrated below:

$$
\begin{array}{ccccc}
\cdot & \cdot & \cdot & * & \cdot \\
\cdot & \cdot & \cdot & \cdot & * \\
* & \cdot & \cdot & \cdot & \cdot \\
\cdot & * & \cdot & \cdot & \cdot \\
\cdot & \cdot & * & \cdot & \cdot
\end{array}
$$

It is clear how a magic hyper-cube in $n(\geqslant 2)$ dimensions is defined. A perfect magic hyper-cube is one in which the sum property holds also for all unbroken

† The nomenclature is not standard. My terminology is intended to distinguish between the three types of magic squares that have been considered. However it differs from that used by Rouse Ball (reference 1) who treats the terms 'perfect' and 'pandiagonal' as synonyms with the meaning that I have assigned to the latter term.

diagonals; and in a pandiagonal magic hyper-cube it holds for the broken diagonals as well as the unbroken ones. In a $q \times q \times q$ cube the unbroken diagonals are the 4 space diagonals and the $6q$ diagonals parallel to a face of the cube which start at one edge and finish at the opposite one.

Examples of perfect magic cubes are given in references 1, 2a and 3. These are the only perfect magic cubes that I have been able to find, though in reference 2b Gardner refers to a 7th-order perfect magic cube. Andrews (reference 4) investigates magic cubes in which the sum property does not hold for space diagonals, and Collings (reference 5) describes a method for producing magic cubes in which the sum property holds only for diagonals that pass through the centre. So far no general method appears to have been obtained for producing perfect magic cubes. In Section 2 I describe a method for constructing n-dimensional pandiagonal magic hyper-cubes of arbitrarily large prime order, some variants of the construction are given in Section 3, and in Section 4 I show how to construct n-dimensional perfect magic hyper-cubes whose orders are multiples of nth powers. In each case n can be any integer greater than 1. It was Misra's article (reference 6) in a previous number of *Mathematical Spectrum* that drew my attention to the problem. Although Misra dealt only with magic squares (in which the sum property does not necessarily hold for the diagonals), my method owes a great deal to his.

In this article I shall make considerable use of arithmetic modulo p. This is 'clock arithmetic'. Two integers m, n are equivalent modulo p if $m - n$ is a multiple of p; we write

$$m \equiv n \pmod p.$$

It is easy to verify that, if

$$m \equiv n \pmod p \quad \text{and} \quad r \equiv s \pmod p,$$

then

$$m \pm r \equiv n \pm s \pmod p.$$

The term 'arithmetic progression modulo p' is interpreted in the natural way. For instance 1, 3, 0, 2, 4 is an arithmetic progression modulo 5 with common difference 2, since $3 - 1$, $0 - 3$, $2 - 0$ and $4 - 2$ are all equivalent to 2 modulo 5.

The arithmetic of matrices is naturally extended for hyper-cubes. Thus, if λ is a scalar, the scalar multiple $\lambda\mathbf{A}$ of the hyper-cube $\mathbf{A}$ is the hyper-cube obtained by multiplying each element of $\mathbf{A}$ by λ. To add two hyper-cubes $\mathbf{A}, \mathbf{B}$ we add the elements in corresponding positions and put the sum in that position; the resulting sum hyper-cube is denoted by $\mathbf{A} + \mathbf{B}$.

2. Construction of n-dimensional pandiagonal magic hyper-cubes

The basis of the construction is an elementary hyper-cube $\mathbf{A}_0$ which is obtained as follows.

Let p and $d_1, d_2, \ldots, d_n$ be positive integers. The elements of $\mathbf{A}_0$ are taken from the set $\{0, 1, \ldots, p - 1\}$. First 0 is placed at one corner of the hyper-cube. The n edges which meet at this corner are the axes of the hyper-cube, with their positive directions leading away from the chosen corner. The elements of the kth axis are then taken so as to form an arithmetic progression modulo p with common difference d_k. Internal elements of the hyper-cube are equivalent modulo p to the sum of the corresponding values on each axis. Thus, if the coordinates of an element are $(x_1, x_2, \ldots, x_n)$, then the value v, say, of the element is equivalent modulo p to

$$d_1x_1 + d_2x_2 + \cdots + d_nx_n. \tag{1}$$

The next element in the direction of the kth axis has its kth coordinate equal to $x_k + 1$, and so its value is equivalent to $v + d_k$. Hence the elements of every row in the direction of the kth axis form an arithmetic progression modulo p with common difference d_k.

If $n = 2, p = 5, d_1 = 1, d_2 = 2$, $\mathbf{A}_0$ is the matrix

$$\begin{array}{r|ccccc}
 & 3 & 4 & 0 & 1 & 2 \\
 & 1 & 2 & 3 & 4 & 0 \\
d_2 = 2 & 4 & 0 & 1 & 2 & 3 \\
 & 2 & 3 & 4 & 0 & 1 \\
 & 0 & 1 & 2 & 3 & 4 \\
\hline
 & \multicolumn{5}{c}{d_1 = 1}
\end{array}$$

Lemma 1. If p is a prime and $0 < d_k < p$ $(k = 1, 2, \ldots, n)$, then each row of $\mathbf{A}_0$ contains each of the integers $0, 1, \ldots, p - 1$ just once.

Proof. Since each row contains p elements, the lemma can be false only if two elements in different places are equal. Let the first element of a given row be u and the difference (mod p) between terms be d. The rth and sth elements are then equivalent to $u + (r - 1)d$ and $u + (s - 1)d$, respectively. If these elements are equal, then

$$u + (r - 1)d \equiv u + (s - 1)d \pmod{p}$$

and therefore
$$(r - s)d = mp,$$

for some integer m. Since p is prime, this means that p divides at least one of $r - s$ and d. But $0 < d < p$ and so p does not divide d; on the other hand, since $|r - s| < p$, p can divide $r - s$ only if $r - s = 0$. Thus $r = s$ and the lemma is proved.

Lemma 2. Suppose that the prime p and the positive integers $d_1, d_2, \ldots, d_n$ are such that

$$p > d_1 + d_2 + \cdots + d_n,$$

and the sums of the elements of the $2^n - 1$ non-empty subsets of $\{d_1, d_2, \ldots, d_n\}$ all take different values (between 1 and $p - 1$). Then every diagonal of $\mathbf{A}_0$, broken or unbroken, contains each of the integers $0, 1, \ldots, p - 1$ just once.

Proof. A step along a given diagonal is made by changing the coordinates by the addition of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, where

$$\varepsilon_k = \begin{cases} 1 & \text{if the projection of the diagonal on the } k\text{th axis is in the positive direction,} \\ -1 & \text{if the projection of the diagonal on the } k\text{th axis is in the negative direction,} \\ 0 & \text{if the diagonal is perpendicular to the } k\text{th axis.} \end{cases}$$

Hence such a step changes the value of an element by the amount

$$d = \sum_{k=1}^{n} \varepsilon_k d_k,$$

which is independent of the element. Thus the elements of the diagonal form an arithmetic progression modulo p with common difference d. If $0 < |d| < p$, then the diagonal will contain each of the integers $0, 1, \ldots, p-1$ just once; this can be proved in exactly the same way as in Lemma 1.

The condition that

$$0 < \left| \sum_{k=1}^{n} \varepsilon_k d_k \right| < p$$

for all possible choices of the ε_k is clearly equivalent to the hypothesis of the lemma.

Note. The second condition of the lemma is satisfied if $d_k \geqslant 2d_{k-1}$ $(k = 2, \ldots, n)$. Thus the entire hypothesis is satisfied when $d_k = 2^{k-1}$ $(k = 1, 2, \ldots, n)$ and $p > 2^n$.

It is of interest to note that, as the number $2^n - 1$ of non-empty subsets of $\{d_1, d_2, \ldots, d_n\}$ must not exceed $p - 1$, $p \geqslant 2^n$; and since p is a prime, $p > 2^n$.

Given the hyper-cube $\mathbf{A}_0$, we form the hyper-cubes $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ by rotating $\mathbf{A}_0$ about the principal diagonal through the zero corner. If X_k is the kth axis of $\mathbf{A}_0$, the axes of $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ are

$$\begin{array}{l} X_1, X_2, \ldots, X_{n-1}, X_n, \\ X_n, X_1, \ldots, X_{n-2}, X_{n-1} \\ \dots\dots\dots\dots\dots\dots \\ X_2, X_3, \ldots, X_n, \quad X_1, \end{array}$$

respectively.

Theorem 1. Suppose that p and $d_1, d_2, \ldots, d_n$ satisfy the conditions of Lemma 2 and that also

$$\begin{vmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \dots & \dots & \dots & \dots \\ d_2 & d_3 & \ldots & d_1 \end{vmatrix}$$

is not a multiple of p. If $\mathbf{M}$ is defined by

$$\mathbf{M} = \sum_{k=0}^{n-1} p^k \mathbf{A}_k.$$

and $\mathbf{M}^+$ is the hyper-cube obtained by adding 1 to each element of $\mathbf{M}$, then $\mathbf{M}^+$ is a pandiagonal magic hyper-cube.

The construction of $\mathbf{M}^+$ is illustrated below for the case $n = 2, p = 5, d_1 = 1, d_2 = 2$.

$\mathbf{A}_0$	$\mathbf{A}_1$	$\mathbf{M}$	$\mathbf{M}^+$
3 4 0 1 2	4 1 3 0 2	23 9 15 1 12	24 10 16 2 13
1 2 3 4 0	3 0 2 4 1	16 2 13 24 5	17 3 14 25 6
4 0 1 2 3	2 4 1 3 0	14 20 6 17 3	15 21 7 18 4
2 3 4 0 1	1 3 0 2 4	7 18 4 10 21	8 19 5 11 22
0 1 2 3 4	0 2 4 1 3	0 11 22 8 19	1 12 23 9 20

Note that ordinary addition is used to obtain $\mathbf{M}$ and $\mathbf{M}^+$, not addition modulo p.

Proof of Theorem 1. Every row and diagonal (broken or unbroken) of $\mathbf{A}_0$ contains each of the numbers $0, 1, \ldots, p-1$ just once, and so the sum of the elements in each row or diagonal is

$$\sum_{i=0}^{p-1} i = \tfrac{1}{2}p(p-1).$$

But each row or diagonal of $\mathbf{A}_k$ ($k = 1, \ldots, p-1$) is also a row or diagonal of $\mathbf{A}_0$. Hence every row sum and every diagonal sum of each of the hyper-cubes $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ is $\tfrac{1}{2}p(p-1)$.

If the elements of $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ in a particular position are $a_0, a_1, \ldots, a_{n-1}$, respectively, then the corresponding element in $\mathbf{M}^+$ is

$$1 + \sum_{k=0}^{n-1} p^k a_k. \tag{2}$$

It follows that the sum of the elements in each row or diagonal of $\mathbf{M}^+$ is

$$p.1 + \sum_{k=0}^{n-1} p^k \tfrac{1}{2}p(p-1) = p + \tfrac{1}{2}p(p-1)\frac{p^n - 1}{p-1} = \tfrac{1}{2}p(p^n + 1).$$

Hence $\mathbf{M}^+$ is pandiagonal if its elements are the integers $1, 2, \ldots, p^n$.

Using the expression (2) for the elements of $\mathbf{M}^+$ and noting that $0 \leqslant a_k \leqslant p-1$ for $k = 0, 1, \ldots, n-1$, we first see that each element of $\mathbf{M}^+$ lies between 1 and

$$1 + (p-1)\sum_{k=0}^{n-1} p^k = 1 + (p-1)\frac{p^n - 1}{p-1} = p^n.$$

It remains to show that the elements of $\mathbf{M}^+$ are all different.

Let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be the coordinates of two elements, e and f, say, of $\mathbf{M}^+$. Denote by a_k and b_k $(k = 1, 2, \ldots, n)$ the elements of $\mathbf{A}_k$ in the corresponding positions, so that

$$e = 1 + \sum_{k=0}^{n-1} p^k a_k, \qquad f = 1 + \sum_{k=0}^{n-1} p^k b_k.$$

Thus the 'digits' of e and f when expressed to base p are $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$, respectively. Now suppose that $e = f$. Since expressions to base p are unique, $a_k = b_k$ for $k = 0, 1, \ldots, n-1$. If the a_k, b_k are expressed in coordinates as in (1), we obtain equations for the a_k, b_k which can be written in matrix form:

$$\begin{bmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} \equiv \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \cdot \\ b_{n-1} \end{bmatrix}$$

$$\equiv \begin{bmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{bmatrix},$$

i.e.

$$\begin{bmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{bmatrix} \equiv 0 \pmod{p}.$$

But we have stipulated that

$$\begin{vmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{vmatrix} \not\equiv 0 \pmod{p}$$

and so $x_k - y_k \equiv 0 \pmod{p}$ for $k = 1, 2, \ldots, n$. Here we are using the analogue for arithmetic modulo p of a well-known result on the solution of linear equations. This analogue is proved in precisely the same way as the familiar theorem because p is a prime and therefore the modulo p arithmetic includes unique division as well as addition, subtraction and multiplication. We now note that $|x_k - y_k| \leqslant n < p$ and that therefore the equivalence $x_k - y_k \equiv 0 \pmod{p}$ implies the identity $x_k = y_k$ $(k = 1, 2, \ldots, n)$. Thus e and f are in the same position. The proof of the theorem is now complete.

Corollary. For every integer $n \geqslant 2$, there exists at least one pandiagonal magic hyper-cube of prime order $p > 2^n$.

Proof. Let $d_k = 2^k - 1$ $(k = 1, 2, \ldots, n)$. We have already remarked that, when $p > 2^n$, the conditions of Lemma 2 are satisfied. Also

$$\begin{vmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & \ldots & 2^{n-1} \\ 2^{n-1} & 1 & \ldots & 2^{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ 2 & 2^2 & \ldots & 1 \end{vmatrix}$$

and, subtracting half the $(k + 1)$th column from the kth column $(k = 1, 2, \ldots, n - 1)$ we see that the determinant has the same value as

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & 2^{n-1} \\ \frac{1}{2}(2^n - 1) & 0 & \ldots & 0 & 2^{n-2} \\ 0 & \frac{1}{2}(2^n - 1) & \ldots & 0 & 2^{n-3} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & \ldots & \frac{1}{2}(2^n - 1) & 1 \end{vmatrix}.$$

The value of this determinant is found by expansion in terms of the first row; it is

$$(-1)^{n-1}2^{n-1}\{\tfrac{1}{2}(2^n - 1)\}^{n-1} = (-1)^{n-1}(2^n - 1)^{n-1},$$

which is not divisible by p since $p > 2^n$. Hence p and the d_k satisfy the conditions of Theorem 1 and the resulting hyper-cube $\mathbf{M}^+$ is pandiagonal.

3. Variants of the basic construction

(a) *Hyper-cubes of composite order.* An n-dimensional perfect magic hyper-cube $\mathbf{N}$ of order pq can be constructed from two n-dimensional perfect magic hyper-cubes $\mathbf{P}$, of order p, and $\mathbf{Q}$, of order q. This is done by replacing each element e of $\mathbf{P}$ by a hyper-cube $\mathbf{Q}(e)$; the latter is obtained from $\mathbf{Q}$ by adding $(e - 1)q^n$ to each element. Thus $\mathbf{Q}(e)$ contains each of the integers

$$(e - 1)q^n + 1, (e - 1)q^n + 2, \ldots, (e - 1)q^n + q^n = eq^n$$

just once. Since e takes every value between 1 and p^n exactly once, $\mathbf{N}$ contains each of the numbers $1, 2, \ldots, (pq)^n$ just once. Hence $\mathbf{N}$ is a perfect magic hyper-cube if the sum of the elements in each row and in each unbroken diagonal is $\frac{1}{2}pq\{(pq)^n + 1\}$.

Consider a particular row or unbroken diagonal of **N**. This can be made up of rows or unbroken diagonals of hyper-cubes **Q**(e) with the elements e lying in appropriate rows or unbroken diagonals of **P**. Let the elements of the requisite row or diagonal of **P** be $e_1, e_2, \ldots, e_p$ and let the elements in the row or diagonal of **Q** be $f_1, f_2, \ldots, f_q$. Then the sum of the elements in the row or diagonal of **N** is

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=1}^{q}\{q^n(e_i-1)+f_j\} &= \sum_{i=1}^{p}\{q^{n+1}(e_i-1)+\tfrac{1}{2}q(q^n+1)\}\\ &= \sum_{i=1}^{p}(q^{n+1}e_i-\tfrac{1}{2}q^{n+1}+\tfrac{1}{2}q)\\ &= q^{n+1}\tfrac{1}{2}p(p^n+1)-\tfrac{1}{2}pq^{n+1}+\tfrac{1}{2}pq\\ &= \tfrac{1}{2}pq\{(pq)^n+1\},\end{aligned}$$

as required. Thus **N** is a perfect magic hyper-cube.

The method can be extended to produce a perfect magic hyper-cube of order $p_1p_2\ldots p_r$ from perfect magic hyper-cubes of orders $p_1, p_2, \ldots, p_r$, respectively.

We have confined ourselves to perfect magic hyper-cubes because the unbroken diagonals of a composite hyper-cube are so easily obtained from the unbroken diagonals of the original hyper-cubes. However it may be shown that broken diagonals can be treated in the same way, and so pandiagonal magic hyper-cubes of composite order result from the use of pandiagonal magic hyper-cubes in the construction.

(b) *Associative pandiagonal magic hyper-cubes.* An associative hyper-cube is one in which the sum of any two elements symmetrically opposite the centre is constant. When the order of the hyper-cube is odd, this sum is equal to twice the central element.

Let $\mathbf{A}_0$ be an elementary hyper-cube of order p of the kind used in Section 2. Since p is odd, $\mathbf{A}_0$ contains a central element. We now add, modulo p, a constant to each element of $\mathbf{A}_0$ to produce a hyper-cube $\mathbf{B}_0$ with central element $\tfrac{1}{2}(p-1)$. This process preserves the property that every row and diagonal contains each of the integers $0, 1, \ldots, p-1$ just once, for the modulo p addition of a constant to the elements of a permutation of $0, 1, \ldots, p-1$ merely produces another such permutation. We now show that $\mathbf{B}_0$ is associative.

Denote the element at the origin of $\mathbf{B}_0$ by t. Since the coordinates of the central element are $(\tfrac{1}{2}(p-1), \ldots, \tfrac{1}{2}(p-1))$, the coordinates of any two points symmetrically placed about the centre are

$$(\tfrac{1}{2}(p-1)-x_1, \tfrac{1}{2}(p-1)-x_2, \ldots, \tfrac{1}{2}(p-1)-x_n)$$

and

$$(\tfrac{1}{2}(p-1)+x_1, \tfrac{1}{2}(p-1)+x_2, \ldots, \tfrac{1}{2}(p-1)+x_n).$$

The sum s of the elements at these points is then equivalent, modulo p, to

$$\{t + d_1(\tfrac{1}{2}(p-1) - x_1) + \cdots + d_n(\tfrac{1}{2}(p-1) - x_n)\}$$
$$+ \{t + d_1(\tfrac{1}{2}(p-1) + x_1) + \cdots + d_n(\tfrac{1}{2}(p-1) + x_n)\}$$
$$= 2\{t + \tfrac{1}{2}(p-1)(d_1 + \cdots + d_n)\}.$$

Now

$$t + \tfrac{1}{2}(p-1)(d_1 + \cdots + d_n) \qquad (= t + d_1\tfrac{1}{2}(p-1) + \cdots + d_n\tfrac{1}{2}(p-1))$$

is equivalent, modulo p, to the central element which has the value $\frac{1}{2}(p-1)$. Thus

$$s \equiv p - 1 \pmod{p},$$

i.e.

$$s = (p-1) + mp,$$

where m is an integer. But s is the sum of two integers, each of which lies between 0 and $p-1$, and so $s = p - 1$. This proves that $\mathbf{B}_0$ is associative.

The hyper-cubes $\mathbf{B}_1, \ldots, \mathbf{B}_{n-1}$ are formed by rotation of $\mathbf{B}_0$ in the same way as the $\mathbf{A}_i$ were formed from $\mathbf{A}_0$. It follows that all the $\mathbf{B}_i$ are associative, and so are the hyper-cubes $\mathbf{N}, \mathbf{N}^+$ which correspond to the $\mathbf{M}, \mathbf{M}^+$ of Section 2. Hence $\mathbf{N}^+$ is the desired associative pandiagonal magic hyper-cube. Its central element is

$$1 + \sum_{k=0}^{n-1} \tfrac{1}{2}(p-1)p^k = 1 + \tfrac{1}{2}(p-1)\frac{p^n - 1}{p-1} = \tfrac{1}{2}(p^n + 1);$$

and the sum for opposite pairs of elements is twice this, i.e. $p^n + 1$.

It is clear that two associative perfect magic hyper-cubes can be combined according to the method of Section 3(a) to produce an associative perfect magic hyper-cube of composite order.

(c) *Perfect magic hyper-cubes of small order*. Here, as in Section 2, the symbols $\varepsilon_1, \ldots, \varepsilon_n$ can take the values 0, 1, -1. Suppose that the prime p and the positive integers $d_1, \ldots, d_n$ satisfy the following conditions.

I. $p > d_1 + \cdots + d_n$;

II. $\Sigma_{k=1}^n \varepsilon_k d_k = 0$ for $\varepsilon_k = \varepsilon'_k$ $(k = 1, \ldots, n)$, where all the ε'_k are non-zero; and $\Sigma_{k=1}^n \varepsilon_k d_k \neq 0$ for any other combination of $\varepsilon_1, \ldots, \varepsilon_n$ in which not every ε_k is 0.

III. the determinant

$$\begin{vmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ d_2 & d_3 & \ldots & d_1 \end{vmatrix}$$

is not a multiple of p.

Let $\mathbf{A}_0$ be the elementary hyper-cube corresponding to $p, d_1, \ldots, d_n$. There is a unique unbroken (space) diagonal corresponding to $(\varepsilon'_1, \ldots, \varepsilon'_n)$ and in this diagonal all the elements are the same, since $\Sigma_{k=1}^n \varepsilon'_k d_k = 0$. By adding, modulo p, a constant to all the elements of $\mathbf{A}_0$ we can obtain a hyper-cube $\mathbf{B}_0$ such that all the elements in this diagonal are $\frac{1}{2}(p-1)$, so that their sum is $\frac{1}{2}p(p-1)$. We see as in (b) that the elements in every row and unbroken diagonal of $\mathbf{B}_0$ also have sum

$\frac{1}{2}p(p-1)$; however the broken diagonals in the direction $(\varepsilon'_1, \ldots, \varepsilon'_n)$ need not have this sum. It follows that the hyper-cube $\mathbf{N}^+$ constructed from $\mathbf{B}_0$ in the usual way is perfectly magic, but is not necessarily pandiagonal. Since the central element of $\mathbf{A}_0$ is $\frac{1}{2}(p-1)$, $\mathbf{N}^+$ is also associative. An argument like that used after the proof of Lemma 2 shows that $p \geqslant 2^n - 1$.

A $7 \times 7 \times 7$ perfect magic cube has been produced in this way and is exhibited in Figure 1.

(d) *Derivation of new hyper-cubes.* In the expression

$$\mathbf{M} = \sum_{k=0}^{n-1} p^k \mathbf{A}_k$$

the $\mathbf{A}_k$ can be permuted to give different magic hyper-cubes $\mathbf{M}^+$. The proof of Theorem 1 still applies since permutation of the rows of the determinant

$$\begin{vmatrix} d_1 & d_2 & \ldots & d_n \\ d_n & d_1 & \ldots & d_{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ d_2 & d_3 & \ldots & d_1 \end{vmatrix}$$

leaves the value unchanged or multiplies it by -1.

If $(a_0, a_1, \ldots, a_{p-1})$ is a permutation of $(0, 1, \ldots, p-1)$, replace the elements of $\mathbf{A}_0$ by $a_0, a_1, \ldots, a_{p-1}$, respectively. The resulting hyper-cube $\mathbf{B}_0$ leads also to a pandiagonal magic hyper-cube $\mathbf{N}^+$. This process may be used on the hyper-cubes in (b) and (c) as well, provided that the permutation leaves $\frac{1}{2}(p-1)$ unchanged.

4. Perfect magic hyper-cubes of even order

The magic hyper-cubes constructed in Theorem 1 are of odd order since there the order p is a prime greater than 2^n. Hence the construction in Section 3(a) for composite order magic hyper-cubes so far only guarantees the existence of odd order magic hyper-cubes. However the method developed in the next theorem yields perfect magic hyper-cubes whose orders may be even as well as odd.

Theorem 2. Let a, b be integers the sole restriction on which is that the combination of a being even and b being odd is not allowed. Then there exists an n-dimensional perfect magic hyper-cube of order ab^n.

Proof. The basis of the construction is once again a suitable elementary hyper-cube $\mathbf{A}_0$. I shall fully describe $\mathbf{A}_0$ and its properties, but I shall not go into all the details of the proof that the corresponding hyper-cube $\mathbf{M}^+$ is perfectly magic.

First the set $\{0, 1, \ldots, ab^n - 1\}$ is split up into ab^{n-1} subsets $S_0, S_1, \ldots, S_r$, where $r = ab^{n-1} - 1$. The partition is such that each S_i contains b elements and the sum of these is the same for all the subsets. Since

$$1 + 2 + \cdots + (ab^n - 1) = \tfrac{1}{2}ab^n(ab^n - 1),$$

the common sum is $\frac{1}{2}b(ab^n - 1)$ (an integer, in view of the condition on a and b). When b is even, the required partition is easily carried out, for it is only necessary to ensure that the integers j and $ab^n - 1 - j$ are assigned to the same set S_i. When b is odd, the general method of partition is much more complicated and it is best to omit its description; however in practice, when ab^n is fairly small, trial and

	343	10	76	135	152	211	277
	64	130	196	206	272	331	5
	184	201	260	326	49	59	125
[1]	255	321	37	54	113	179	245
	32	98	108	174	233	250	309
	103	162	228	294	304	27	86
	223	282	299	15	81	147	157

	100	166	225	291	308	24	90
	220	286	296	19	78	144	161
	340	14	73	139	149	215	274
[2]	68	127	193	210	269	335	2
	188	198	264	323	46	63	122
	259	318	41	51	117	176	242
	29	95	112	171	237	247	313

	256	322	38	55	114	180	239
	33	92	109	175	234	251	310
	104	163	229	288	305	28	87
[3]	224	283	300	16	82	141	158
	337	11	77	136	153	212	278
	65	131	190	207	273	332	6
	185	202	261	327	43	60	126

	69	128	194	204	270	336	3
	189	199	265	324	47	57	123
	253	319	42	52	118	177	243
[4]	30	96	106	172	238	248	314
	101	167	226	292	302	25	91
	221	287	297	20	79	145	155
	341	8	74	140	150	216	275

	218	284	301	17	83	142	159
	338	12	71	137	154	213	279
	66	132	191	208	267	333	7
[5]	186	203	262	328	44	61	120
	257	316	39	56	115	181	240
	34	93	110	169	235	252	311
	105	164	230	289	306	22	88

	31	97	107	173	232	249	315
	102	168	227	293	303	26	85
	222	281	298	21	80	146	156
[6]	342	9	75	134	151	217	276
	70	129	195	205	271	330	4
	183	200	266	325	48	58	124
	254	320	36	53	119	178	244

	187	197	263	329	45	62	121
	258	317	40	50	116	182	241
	35	94	111	170	236	246	312
[7]	99	165	231	290	307	23	89
	219	285	295	18	84	143	160
	339	13	72	138	148	214	280
	67	133	192	209	268	334	1

$n = 3, p = 7;$
$d_1 = 1, d_2 = 2; d_3 = 3;$
element at origin of A_0: 6

Figure 1. A $7 \times 7 \times 7$ perfect magic cube.

error are quite sufficient. It may be noted that several partitions of $\{0, 1, \ldots, ab^n - 1\}$ are likely to be possible and that these will lead to different hyper-cubes M^+.

Denote the members of S_i $(i = 0, 1, \ldots, r(=ab^{n-1} - 1))$ by $c_{i,0}, c_{i,1}, \ldots, c_{i,b-1}$. Along its first axis A_0 is now given the values

$$c_{0,0}, c_{0,1}, \ldots, c_{0,b-1}; c_{1,0}, c_{1,1}, \ldots, c_{1,b-1}; \ldots; c_{r,0}\, c_{r,1}, \ldots, c_{r,b-1}.$$

To describe the other axes it is best to introduce a special notational device. If an element u is repeated m times, we write $m \times (u)$; likewise $m \times (u_1, \ldots, u_j)$ means that the contents of the bracket are repeated m times. The second, third, ..., nth axes of $\mathbf{A}_0$ can now be written

$$ab^{n-2} \times (b \times (c_{0,0}), b \times (c_{0,1}), \ldots, b \times (c_{0,b-1})),$$

$$ab^{n-3} \times (b^2 \times (c_{0,0}), b^2 \times (c_{0,1}), \ldots, b^2 \times (c_{0,b-1})),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a \times (b^{n-1} \times (c_{0,0}), b^{n-1} \times (c_{0,1}), \ldots, b^{n-1} \times (c_{0,b-1})).$$

The value of an element of $\mathbf{A}_0$ with coordinates $(x_1, x_2, \ldots, x_n)$, say, is a c_{ij} and is calculated as follows. Let the elements at $(x_1, 0, \ldots, 0)$, $(0, x_2, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, x_n)$ be $c_{i_1,j_1}, c_{0,j_2}, \ldots, c_{0,j_n}$, respectively. Then

$$i = i_1$$

and

$$j \equiv j_1 + j_2 + \cdots + j_n \pmod b.$$

Thus, in a 3-dimensional cube of order 8 $(a = 1, b = 2)$ an internal element which corresponds to the elements $c_{2,1}, c_{0,0}, c_{0,1}$ on the three axes has the value $c_{2,0}$. An example of an $\mathbf{A}_0$ is given in Figure 2.

$$\begin{bmatrix} 0 & 7 & 6 & 1 & 4 & 3 & 5 & 2 \\ 0 & 7 & 6 & 1 & 4 & 3 & 5 & 2 \\ 7 & 0 & 1 & 6 & 3 & 4 & 2 & 5 \\ 7 & 0 & 1 & 6 & 3 & 4 & 2 & 5 \\ 0 & 7 & 6 & 1 & 4 & 3 & 5 & 2 \\ 0 & 7 & 6 & 1 & 4 & 3 & 5 & 2 \\ 7 & 0 & 1 & 6 & 3 & 4 & 2 & 5 \\ 7 & 0 & 1 & 6 & 3 & 4 & 2 & 5 \end{bmatrix}$$

Figure 2. An elementary hyper-cube for $n = 2$, $a = b = 2$, $ab^n = 8$.
$c_{0,0} = 7$, $c_{0,1} = 0$; $c_{1,0} = 1$, $c_{1,1} = 6$; $c_{2,0} = 3$, $c_{2,1} = 4$; $c_{3,0} = 2$, $c_{3,2} = 5$.

Next we consider the sums of the elements in an arbitrary row of $\mathbf{A}_0$. Suppose that the row is parallel to the kth axis and that its first element is $c_{l,m}$. Then its first b^{k-1} elements are all $c_{l,m}$, its next b^{k-1} elements are $c_{l,m+1}$ (or $c_{l,0}$ if $m = b - 1$) and so on. Thus among the first b^k elements of this row each of the numbers $c_{l,0}, c_{l,1}, \ldots, c_{l,b-1}$ appears exactly b^{k-1} times. The sum of the first b^k elements is therefore

$$\tfrac{1}{2}b(ab^n - 1)b^{k-1} = \tfrac{1}{2}b^k(ab^n - 1);$$

and the sum of all the ab^n elements in the row is

$$\tfrac{1}{2}b^k(ab^n - 1)(ab^n/b^k) = \tfrac{1}{2}ab^n(ab^n - 1).$$

Unbroken diagonals may be treated similarly. For suppose that a given unbroken diagonal is not perpendicular to the kth axis, but is perpendicular to the jth axis for $j < k$. Then in all the other axes to which the diagonal is not perpendicular the values of the elements do not change within successive groups of b^k elements; and an argument of the kind used for rows will now show that the elements of the diagonal have the desired sum.

As in Section 2, the hyper-cubes $\mathbf{A}_1, \ldots, \mathbf{A}_{n-1}$ are obtained from $\mathbf{A}_0$ by rotation,

$$\mathbf{M} = \sum_{k=0}^{n-1} (ab^n)^k \mathbf{A}_k$$

and $\mathbf{M}^+$ is derived from $\mathbf{M}$ by adding 1 to each element. Every element of $\mathbf{M}^+$ lies between 1 and $(ab^n)^n$, and it must finally be shown that elements in different positions have different values. I will only outline the stages of the proof.

(i) Let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ be the coordinates of two elements, e and f, say, of $\mathbf{M}^+$. Suppose that $e = f$. Then, as in the proof of Theorem 1, the elements g_k, h_k in the $\mathbf{A}_k$ $(k = 0, 1, \ldots, n-1)$ which occupy the same positions are also equal. Hence, if in $\mathbf{A}_0$, $g_0 = c_{l,m}$ and $h_0 = c_{t,u}$, then $c_{l,m} = c_{t,u}$ and therefore $l = t$, $m = u$.

(ii) Since l is determined by the first coordinate x_1 of g_0, x_1 must lie within a certain range; and since $t = 1$, y_1 lies in the same range.

(iii) By also considering g_1 and h_1 in $\mathbf{A}_1, \ldots, g_{n-1}$ and h_{n-1} in $\mathbf{A}_{n-1}$, we see that, for every k, the ranges of x_k and y_k are the same.

(iv) In $\mathbf{A}_0$, two elements with the same 1st coordinate and whose 2nd, $\ldots$, nth coordinates lie in the ranges for $x_2(y_2), \ldots, x_n(y_n)$ have the same value.

(v) In view of (iv), the values of g_0, h_0 are determined by x_1, y_1, respectively. Since $g_0 = h_0$, $x_1 = y_1$.

(vi) The elements g_1, h_1 in $\mathbf{A}_1$ have the same first coordinate x_1. Reasoning as in (iv) and (v) shows that $x_2 = y_2$.

(vii) $x_k = y_k$ for all k, i.e. e and f have the same coordinates.

I am very grateful to Mr J. H. Durran for his help in the preparation of this article, to my mother and brother for checking the final draft, and to the editor of this magazine for the encouragement he has given me.

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Letter to the Editor

Dear Editor,

Partitions of sets of integers

M. D. Sandford's problem (*Mathematical Spectrum*, Volume 9, Number 1, p. 26) was to determine the function $F(n)$ defined as the largest number k such that the set of integers $\{1, 2, \ldots, k\}$ can be partitioned into n subsets, none of which contains a 3-term arithmetic progression. The values so far known are $F(1) = 2$, $F(2) = 8$ and $F(3) = 26$ (though for the last we have to take the word of a computer). Sandford suggested that the general formula might be $F(n) = 3^n - 1$, but I looked at the figures and wondered if it might be $2 + 6(n-1)^2$. These two formulae both agree for the three known values, but for larger values the former is much bigger; for instance for $F(4)$ the first gives 80 and the second 56, and the computer results at that stage could not decide between these two. However, I heard later from Sandford that the computer had made further progress, and partitioned 59 into four sets, thereby ruling out the quadratic formula for $F(n)$. Also he told me that he felt that a quadratic did not increase fast enough for large n and that therefore the nth-power formula seemed more plausible. At this stage I was faced with Sandford's intuition and the computer both telling me that a quadratic did not give big enough values. It had to be true, and so I set about proving it.

First to get a way of visualising the facts, a partition of $\{1, \ldots, m\}$ into n sets can be regarded as a brick or block of size $m \times n$ with crosses in some of the squares. For instance the partition of $\{1, \ldots, 8\}$ into $\{1, 2, 5, 6\}$ and $\{3, 4, 7, 8\}$ becomes the 8×2 block of Figure 1. Since $\{1, 2, 5, 6\}$, $\{3, 4, 7, 8\}$ is a partition of $\{1, \ldots, 8\}$, every column has exactly one cross. Neither $\{1, 2, 5, 6\}$ nor $\{3, 4, 7, 8\}$ contains three terms in arithmetic progression; and this is shown pictorially by the fact that in both the rows of Figure 1 no cross appears midway between two other crosses.

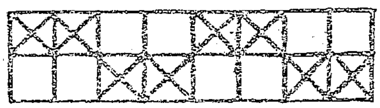

Figure 1

Theorem. Given an $m \times n$ block such that (i) every column contains exactly one cross and (ii) in each row there is no cross midway between two other crosses, we can construct a $52m \times 6n$ block with the same properties.

Proof. The construction is based on the 26×3 block produced by the RMCS computer and given in Sandford's letter. It is shown in Figure 2.

Take 52 copies of the $m \times n$ brick and put them in some of the places of a rectangular floor with 6 rows and 52 columns. The arrangement of crosses in this larger setting must

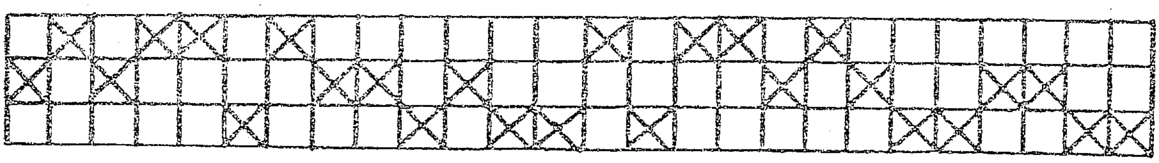

Figure 2

have properties (i) and (ii). For the disposition of bricks in the first row of the floor, the 26 odd-numbered spaces are alone eligible, and we place a brick in one of these spaces if there is a cross in the corresponding position of the 26×3 block. Now take any two crosses in the first n rows of crosses (i.e. the first row of bricks). If they are in the same brick there is no trouble. So suppose that they are in two different bricks, A and B. Then (because there is an odd number of brick-spaces between A and B) there is one brick-space exactly midway between, and this space is unoccupied (because the 26×3 block has been checked). Therefore there can be no cross midway between the chosen ones.

The other $5n$ rows (that is 5 rows of bricks) are filled up in the obvious way. The even-numbered positions in the second row of bricks are filled like the top row of the 26×3 block, then the odd-numbered positions in the third row like the second row of the 26×3 block, and so on. The pattern is as shown in Figure 3, each black square representing an $m \times n$ brick.

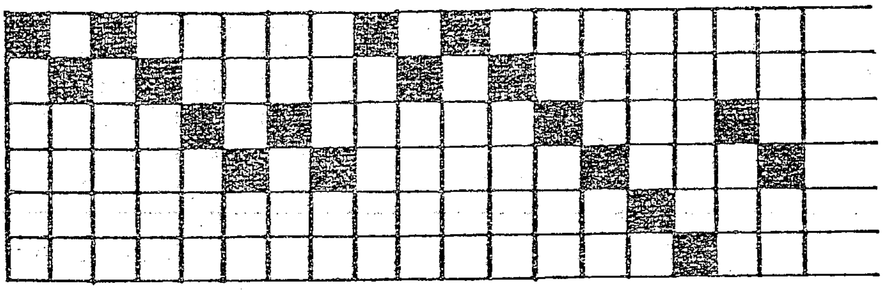

Figure 3

In this way we have produced a block of size $52m \times 6n$ with the required properties (i) and (ii). The theorem is therefore proved.

Now the result of the theorem can be expressed as $F(6n) \geqslant 52F(n)$. What does this tell us about the behaviour of $F(n)$ for large n?

Clearly $F(36n) \geqslant 52F(6n) \geqslant 52^2F(n)$ and in general $F(6^tn) \geqslant 52^tF(n)$ for any positive integer t. Putting $n = 2$ we have

$$F(2 \times 6^t) \geqslant F(2) \times 52^t = 8 \times 52^t > (2 \times 6^t)^{2 \cdot 2}$$

because $8 > 2^{2 \cdot 2}$ and $52 > 6^{2 \cdot 2}$.

This has shown that $F(n) > n^{2 \cdot 2}$ for $n = 2 \times 6^t$ $(t = 1, 2, \ldots)$, and so for arbitrarily large values of n.

Yours sincerely,
B. C. Rennie
(James Cook University of North Queensland)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

9.7. (Submitted by J. G. Brennan, University College of Swansea.)
The seven small circles in the figure all have unit radius. Find a quadratic equation, one of whose roots is the radius of the large circles. What is the geometrical significance of the other root of the quadratic equation?

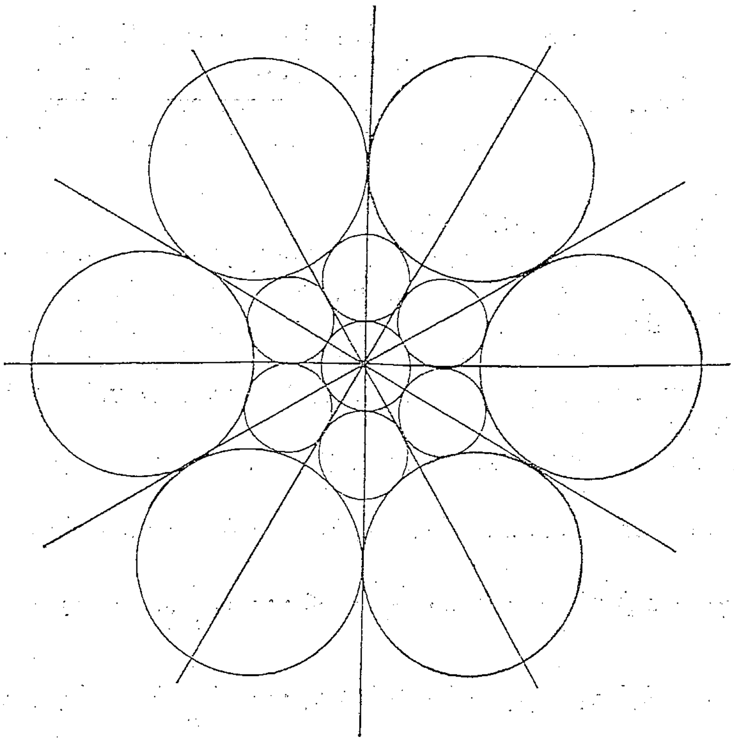

Figure 1

9.8. (Submitted by B. G. Eke, University of Sheffield.) Let S_1 denote the sequence of positive integers 1, 2, 3, 4, 5, 6, . . ., and define the sequence S_{n+1} in terms of S_n by adding 1 to those integers in S_n which are divisible by n. (Thus, for example, S_2 is 2, 3, 4, 5, 6, 7, . . . S_3 is 3, 3, 5, 5, 7, 7, . . .) Determine those integers n with the property that the first $n-1$ integers in S_n are n.

9.9. Let

$$u_n = \left(1 + \frac{1}{n}\right)^n, \qquad v_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

Show that the sequence (u_n) is strictly increasing, whereas (v_n) is strictly decreasing.

Solutions to Problems in Volume 9, Number 1

9.1. Prove that e^x cannot be expressed in the form $f(x)/g(x)$, where $f(x), g(x)$ are polynomials in x with real coefficients.

Solution by John Kleenan (Winchester College)

Assume that $e^x = f(x)/g(x)$, where $f(x), g(x)$ are polynomials in x with respective degrees m, n. Now

$$\frac{f(x)}{g(x)} = e^x = \frac{d}{dx}(e^x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2},$$

so

$$f(x)g(x) = g(x)f'(x) - f(x)g'(x).$$

But the degree of the left-hand side of this equation is $m + n$, whereas the degree of the right-hand side is less than $m + n$. This provides the required contradiction.

Also solved by Phil Nicklin (University of Birmingham).

9.2. A changing room has n lockers numbered 1 to n and all are locked. An attendant performs the sequence of operations $T_1, T_2, \ldots, T_n$, where T_k is the operation whereby the condition of being locked or unlocked is altered in the case of those lockers (and only those) whose numbers are divisible by k. Which lockers are unlocked at the end?

Solution by J. Pemberton (U.M.I.S.T.)

The locker n will be unlocked at the end if and only if n has an odd number of divisors. Now, whenever m is a divisor of n, so is (n/m). Thus, when n is not a perfect square, the divisors of n can be paired off and n has an even number of divisors. On the other hand, when n is a perfect square, we can pair off all divisors of n with the exception of $\sqrt{n}$ and n has an odd number of divisors. Thus the locker n will be unlocked at the end if and only if n is a perfect square.

Also solved by John Kleenan, Alan Burns (W. R. Tuson College, Preston).

9.3. The real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ $(n \geqslant 1)$ are such that

$$a_1 \leqslant \frac{1}{2}(a_1 + a_2) \leqslant \frac{1}{3}(a_1 + a_2 + a_3) \leqslant \cdots \leqslant \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

$$b_1 \leqslant \frac{1}{2}(b_1 + b_2) \leqslant \frac{1}{3}(b_1 + b_2 + b_3) \leqslant \cdots \leqslant \frac{1}{n}(b_1 + b_2 + \cdots + b_n).$$

Show that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leqslant n \sum_{k=1}^{n} a_k b_k.$$

Solution by Alan Burns

We establish the required inequality by induction on n, and note that it is true when $n = 1$. We now assume that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leqslant n \sum_{k=1}^{n} a_k b_k$$

when $a_1, \ldots, a_n, b_1, \ldots, b_n$ satisfy the given inequalities, and consider $a_1, \ldots, a_{n+1}$, $b_1, \ldots, b_{n+1}$ satisfying also the extra inequalities

$$\frac{1}{n}(a_1 + a_2 + \cdots + a_n) \leqslant \frac{1}{n+1}(a_1 + a_2 + \cdots + a_{n+1})$$

and the corresponding one for the b's. These can be rewritten as

$$a_{n+1} \geqslant \frac{a_1 + \cdots + a_n}{n}, \qquad b_{n+1} \geqslant \frac{b_1 + \cdots + b_n}{n},$$

from which we obtain

$$\left(a_{n+1} - \frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(b_{n+1} - \frac{1}{n}\sum_{k=1}^{n} b_k\right) \geqslant 0,$$

$$a_{n+1}b_{n+1} - \frac{1}{n}b_{n+1}\sum_{k=1}^{n} a_k - \frac{1}{n}a_{n+1}\sum_{k=1}^{n} b_k + \frac{1}{n^2}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \geqslant 0,$$

$$a_{n+1}\sum_{k=1}^{n} b_k + b_{n+1}\sum_{k=1}^{n} a_k \leqslant na_{n+1}b_{n+1} + \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right).$$

Also,

$$\begin{aligned}\left(\sum_{k=1}^{n+1} a_k\right)\left(\sum_{k=1}^{n+1} b_k\right) &= \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + a_{n+1}\sum_{k=1}^{n} b_k + b_{n+1}\sum_{k=1}^{n} a_k + a_{n+1}b_{n+1} \\ &\leqslant \frac{n+1}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + (n+1)a_{n+1}b_{n+1} \\ &\leqslant (n+1)\left(\sum_{k=1}^{n+1} a_k b_k\right)\end{aligned}$$

by the inductive hypothesis. This completes the inductive step.

Introduction to Probability: Theory and Applications. By R. L. Scheaffer and W. Mendenhall. Duxbury Press, North Scituate, Massachusetts, 1975. Pp. x+289. Price not stated.

The material covered is for a one-year course in mathematical probability theory for undergraduates. It is similar to, and no better or worse than, many other books written for the same readership. After an initial discursive chapter on modelling we move quickly into axiomatic probability theory, probability distributions, standard distributions, expectation, generating functions, multivariate distributions, functions of random variables and some limit theorems. An indication of the level of treatment is that the geometric, hypergeometric, gamma, beta and bivariate normal distributions are treated in the body of the text as standard. Several limit theorems are quoted without proof. The large number of examples (with answers) makes it useful as a textbook.

There are many references to what might be called 'pure' applications of probability. Under the usual simple assumptions we have road accidents and queues giving Poisson and exponential distributions, and birth processes giving negative binomial distributions. Such references to real life situations occur in the introductions and in the examples. They do help to lighten the text, but are not followed up in terms of true practicalities. To be truly practical we would have to compare, for example, these theoretical queues with real queues, questioning the assumptions, seeing the effects of changing the assumptions, etc. The authors' view of applications is also reflected in the fact that the chapter on statistical applications (touching on estimation, hypothesis testing, minimax and Bayes decisions) is only 24 pages long.

University of Sheffield — P. Holmes

Elementary Statistical Concepts. By Ronald E. Walpole. Collier-MacMillan, London, 1976. Pp. viii+231. £7·25 (Student edition £4·25).

The preface refers to this book as 'a simplified alternative to the author's *Introduction to Statistics*'. This means that it is an introduction to mathematical statistics with the mathematics played down. Illustrative examples have been substituted for mathematical proofs. The text is well laid out and readable.

The early chapters, being more discursive and slow moving, are more suitable for the general reader. Unfortunately this style is not maintained. By the time we get to probability there is only a brief general introduction. The rules for manipulation of probabilities are exemplified from equally likely cases and applied to relative frequency and subjective probabilities, with no indication as to why this should be admissible. There is an unfortunate ambiguity on page 57 when a defective fuse being 'replaced' means that the same defective fuse is put back. Similarly the statement on page 123 that 'we are 95% confident that our computed interval does in fact contain the true population parameter' is, at best, misleading.

The book seems to fall between two stools. It is not mathematical enough for the sixth form specialist, as it omits standard proofs and uses no calculus. On the other hand, it is not broad enough in scope for the more general student; it does not tie in the concepts with real applications, it does not help much in giving an intuitive understanding and feel for the concepts involved, and it uses no real data in any of its examples.

University of Sheffield — P. Holmes

Statistics in the Real World. By RICHARD J. LARSON and DONNA FOX STROUP. Collier-Macmillan, London, 1976. Pp. xxi+245. £4·50.

This is one of a small but growing number of books inspired by the 'Statistics by Example' approach to the teaching of the subject. The title is slightly misleading, for the book is not a comprehensive survey of applications of statistics. Its subtitle 'A book of examples' is a more accurate description. The range of applications considered is the most attractive feature of the book. There are problems based on fingerprints, bird calls, mineral dating, silver content of Byzantine coins, glacial flow, methods of stopping smoking, strikes in the USA 1891 to 1905, Lister's figures on using carbolic acid as an operating-room disinfectant, and others, to make up a total of fifty.

The chapter headings are: characterizing variability, the one-sample problem, the two-sample problem, the paired data problem, the correlation problem and non-parametric methods. From this it will be seen that there is a strong emphasis on significance testing and confidence intervals (which are correctly explained). Critical regions are introduced intuitively, but the test statistics are brought in with little or no explanation as to their appropriateness.

In their attempt to be readable the authors are liable to be misunderstood, and are occasionally wrong. References to a bell-shaped curve sometimes mean the normal distribution but sometimes mean only a symmetrical unimodal distribution. This book cannot be considered as a first book for the teaching of statistics, nor as the basic text book for any course. Its place is in the library as a reference book for sixth-form teachers. Here it could be a source of ideas for applications to be developed into projects by sixth-form pupils.

University of Sheffield P. HOLMES

Notes on Contributors

Michael Roaf took a first degree in mathematics at Oxford and then did graduate work in Middle-Eastern archaeology in London and Oxford. He is now a Bye-Fellow at Gonville and Caius College, Cambridge. His main topic of research is the great staircase at Persepolis in Iran, but he also digs up Bahrain during the winter.

Andrew Bosi attended St Chad's College, Wolverhampton, and had completed two years of the Probability and Statistics Honours Course at the University of Sheffield when he wrote the article in this issue. He is currently Treasurer of the Students' Union in Sheffield. He specialises in railway timetables as well as cricket statistics, and his other interests are philately and numismatism.

Patrick Brooke wrote the article on magic hyper-cubes just before taking his A-level examinations. Subsequently he won a third prize as a competitor at the 18th International Mathematical Olympiad. Having been awarded a scholarship in December 1976, he will, in October 1977, go to Gonville and Caius College, Cambridge, to read mathematics. One of his principal recreations is bell ringing.

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ISSN 0025–5653

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Printed in England by Galliard (Printers) Ltd, Great Yarmouth