Mathematical Spectrum

2004/2005 Volume 37 Number 1

- The soccer world cup: Poisson versus Pascal
- Mathematics for the wastepaper bin
- Sorting sequences
- Slow runners also have a chance

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Diophantine equations, Mersenne primes and sixes and sevens

One of our readers, Bob Bertuello, wrote in excitedly at having discovered a way of solving an equation such as

$$19x + 140y = 1 \,.$$

We had better say straight away that what we mean by 'solving' is to find *integers* x, y that satisfy the equation. Otherwise it is rather easy: put in any number x and

$$y = \frac{1 - 90x}{140} \,.$$

But of course, even if x is an integer, y may not be.

In fact, the solution of such an equation probably goes back to Diophantus of Alexandria about AD 250; hence, the title of this article. Incidentally, David Burton in reference 1 states a problem about Diophantus that probably originated in the 4th century:

> His boyhood lasted $\frac{1}{6}$ of his life; his beard grew after $\frac{1}{12}$ more; after $\frac{1}{7}$ more he was married, and his son was born 5 years later; the son lived to half his father's age, and the father died 4 years after his son. How old was Diophantus when he died?

But to get back to our original equation, Diophantus used an algorithm in the seventh book of Euclid's *Elements* from about 350 BC. Let's first set out how to find a solution and then explain what's going on:

$$\begin{array}{rrcrl}
(1, 0) & 19 & \searrow & 140 & (0, 1) \\
(-14, 2) & \underline{14} & \nwarrow & \underline{133} & (7, 0) \\
(15, -2) & 5 & \searrow & 7 & (-7, 1) \\
(-44, 6) & \underline{4} & \nwarrow & \underline{5} & (15, -2) \\
(59, -8) & 1 & & 2 & (-22, 3)
\end{array}$$

$$19 \times 59 + 140(-8) = 1 \,.$$

Let me explain. Start with 19, top left. We can write $19 = 19 \times 1 + 140 \times 0$, which explains the $(1, 0)$ to the left of 19. Similarly $140 = 19 \times 0 + 140 \times 1$, top right. Now follow the first arrow. Divide 140 by 19: $19 \times 7 = 133$ and the remainder is 7. Now $133 = 19 \times 7 + 140 \times 0$ (hence the $(7, 0)$) and

$$\begin{aligned}
7 &= 140 - 133 \\
&= (19 \times 0 + 140 \times 1) - (19 \times 7 + 140 \times 0) \\
&= 19 \times (0 - 7) + 140(1 - 0) \\
&= 19(-7) + 140 \times 1 \,.
\end{aligned}$$

That explains the $(-7, 1)$ next to 7.

Next follow the second arrow. Divide 19 by 7; hence the 14 with its pair $(-14, 2)$ with the remainder 5 with its pair $(1 - (-14), 0 - 2) = (15, -2)$. Now follow the third arrow. Divide 7 by 5 to give remainder 2, with its pair $(-22, 3)$. Finally follow the fourth arrow and divide 5 by 2 to give remainder 1, with its pair $(59, -8)$. And, lo and behold,

$$1 = 19 \times 59 + 140(-8)\,.$$

It may seem a bit complicated at first, but you soon get the hang of it. This is the famous *Euclid's algorithm* to find the highest common factor of two numbers, in our case 19 and 140. The highest common factor is the last non-zero remainder, in our case 1. (We didn't carry out the last division to divide 2 by 1 to give remainder 0 — it's hardly worth it!)

Of course, if you just want the highest common factor, you can forget about the figures in brackets, but if you want a solution of the equation, they're the crucial things. I learned this way of setting out the algorithm, keeping track of the numbers in brackets as you go, from a former colleague, Sally Baker.

We haven't quite finished. Euclid's algorithm finds one solution, but are there others? And can we find them all? That is now easy. Let (x, y) be any solution (in integers, of course). Then

$$19x + 140y = 1\,.$$

Also

$$19 \times 59 + 140(-8) = 1\,,$$

our solution found using Euclid's algorithm. Hence,

$$19x + 140y = 19 \times 59 + 140(-8)\,,$$

so

$$19(x - 59) = 140(-8 - y)\,.$$

Now 19 divides $140(-8 - y)$, yet it is coprime to 140 (i.e. 19 and 140 have highest common factor 1); Euclid's algorithm has shown that. Hence 19 divides $-8 - y$, that is,

$$-8 - y = 19k \quad \text{for some integer } k\,.$$

Even the justification of this step uses Euclid's algorithm; you can't escape it! Now

$$19(x - 59) = 140 \times 19k\,,$$

so

$$x - 59 = 140k\,,$$

and

$$x = 59 + 140k\,.$$

Hence,

$$(x, y) = (59 + 140k, -8 - 19k)$$

for some integer k. Every solution (in integers) must be of this form. And these *are* all solutions, because

$$\begin{aligned}19(59+140k)+140(-8-19k) &= \big(19\times 59+140(-8)\big)+(19\times 140-140\times 19)k\\ &= 1+0\\ &= 1\,.\end{aligned}$$

Hence $(x, y) = (59+140k, -8-19k)$, where k is an integer, gives all the solutions of the equation in integers.

Magic, isn't it! If we jump to 17th century France, Descartes showed how to represent points in a plane relative to two perpendicular axes. Now the equation $19x+140y=1$ is the equation of a straight line, and we have found all the points with integer coordinates, so called *lattice points*, that lie on that line. We had better stop before we get into deeper water!

To change the subject completely, I have spotted on the web (reference 2) that a 41st Mersenne prime has been discovered, i.e. a prime number of the form 2^n-1, named after Father Marin Mersenne (1588–1648), a French monk. It is

$$2^{24\,036\,583}-1\,.$$

Finally, a reader, James Whiteman, has spotted the following problem on the web (reference 3): is there a multiple of 2^{666} with 666 digits consisting of only 6s and 7s? A beastly question indeed!

References

1 D. M. Burton, *Elementary Number Theory*, 3rd edn (McGraw-Hill, New York, 1997).

2 Mersenne primes: history, theorems and lists, http://www.utm.edu/research/primes/mersenne/.

3 Clifford Pickover group, http://groups.yahoo.com/group/CliffordPickover/.

A new format for *Mathematical Spectrum*

In case you do not recognize your favourite mathematics magazine, this really is the *Mathematical Spectrum* of old! We have produced this new format which we hope makes it easier to read; with less on the page, it looks less intimidating.

We are looking especially for small items which will fill part of a page, rather like this. It does not matter how simple; the simpler the better, up to a point. Something that has intrigued you, perhaps. Send it in, and we will publish it if we think that readers will be similarly interested. You might even win one of our annual prizes. Even if you don't, you have the honour of seeing your names in print in *Mathematical Spectrum*!

The Editor

Maths for the Wastepaper Bin

P. GLAISTER

An intriguing story from the 16th century about authorship is linked
to the mathematical properties of a wastepaper bin

1. Introduction

Figure 1 shows the cross-section of a plastic wastepaper bin in the shape of a cylinder which is open at the top and, conveniently, closed at the base. A standard problem in mechanics is to determine the location of the centre of gravity, G, of such a shape on the assumption that the base and curved side are made of the same material. Denoting by $\bar{x} = \mathrm{OG}$, r, ℓ and ρ, the height of the centre of gravity above the base, the radius of the base, the height of the cylinder and the density of the material respectively, then taking horizontal moments about the base gives

$$\pi r^2\rho \cdot 0 + 2\pi r\ell\rho \cdot \tfrac{1}{2}\ell = (\pi r^2\rho + 2\pi r\ell\rho)\bar{x}\,,$$

that is,

$$\bar{x} = \frac{\ell^2}{2\ell + r}\,. \tag{1}$$

For example, when $\ell = r = 1$, the height and base radius are the same and $\bar{x} = \frac{1}{3}$.

Note that the height of the centre of gravity determines the stability of the bin, an important consideration when my clumsy children are on the move!

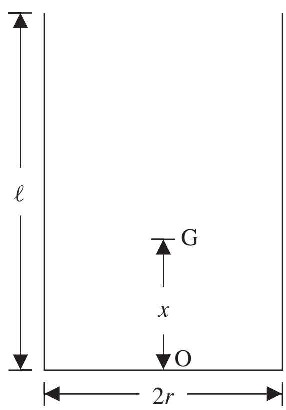


Figure 1

2. The problem

An interesting variation on this problem is to determine what happens to the location of G if the bin has its opening at the top widened while maintaining the length of the side. The shape is now that of the frustum of a hollow cone which is closed at one end. This is also a standard design for a wastepaper bin, and figure 2 shows the cross-section of a typical case. We keep the

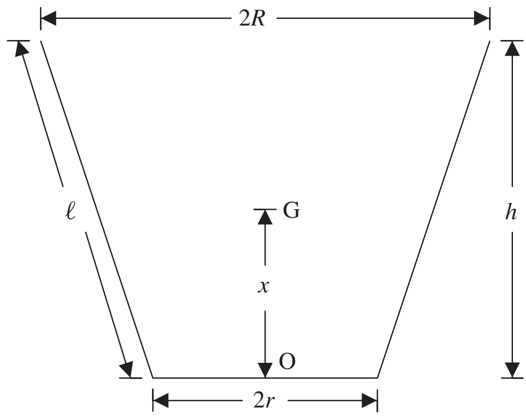


Figure 2

same notation as before, with ℓ now denoting the slant height, which we assume is the same as the vertical height in figure 1, and denote by h the vertical height. In addition, denoting the radius of the top by R, we have by Pythagoras

$$\ell = \sqrt{(R-r)^2 + h^2}\,. \tag{2}$$

To determine the height of the centre of gravity we again take horizontal moments about the base, giving

$$\pi r^2 \rho \cdot 0 + \pi(R+r)\ell\rho \cdot \tfrac{1}{3}h\frac{2R+r}{R+r} = \big(\pi r^2\rho + \pi(R+r)\ell\rho\big)\bar{x}\,, \tag{3}$$

where we have used the fact that the surface area of the frustum of the cone is $\pi(R+r)\ell$ and the height of the centre of gravity above the base is $\frac{1}{3}h(2R+r)/(R+r)$, both of which we leave as an exercise. Simplifying (3) yields

$$\bar{x} = \frac{\frac{1}{3}h\ell(2R+r)}{r^2 + \ell(R+r)} \tag{4}$$

and, using (2),

$$\bar{x} = \frac{\frac{1}{3}\ell(2R+r)\sqrt{\ell^2 - (R-r)^2}}{r^2 + \ell(R+r)} \tag{5}$$

as a function of R, where ℓ and r are considered fixed. In the case of a cylindrical bin $R = r$ and (5) returns the formula in (1).

We are now in a position to investigate our problem: starting with the bin shown in figure 1, if the opening at the top is widened slightly to give a bin of the form shown in figure 2 where the base r and the length of the side ℓ remain fixed, what happens to the centre of gravity? We focus on the specific example above, i.e. $\ell = r = 1$, and the formula in (5) becomes

$$\bar{x} = \frac{\frac{1}{3}(2R+1)\sqrt{1-(R-1)^2}}{R+2} = \frac{(2R+1)\sqrt{R(2-R)}}{3(R+2)}\,. \tag{6}$$

In the case of a cylindrical bin, $R = r = 1$ and $\bar{x} = \frac{1}{3}$ as found from (1).

3. Two-dimensional analogy

Before we consider the graph of $\bar{x}$ against R we note that, as the top is widened slightly, the side is lowered and we might therefore expect that the centre of gravity is lowered from its value of $\frac{1}{3}$ above the base. This is certainly the case for a two-dimensional shape of the form in figure 2. The formula corresponding to (5) in this case is

$$\bar{x} = \frac{\frac{1}{2}\ell\sqrt{\ell^2 - (R-r)^2}}{r+\ell}, \tag{7}$$

which we leave readers to prove. With $\ell = r = 1$, (7) gives $\bar{x} = \frac{1}{4}\sqrt{1-(R-1)^2} = \frac{1}{4}\sqrt{R(2-R)}$, which clearly has a maximum value of $\frac{1}{4}$ when $R = 1$, and is the special case of a two-dimensional shape of the form in figure 1.

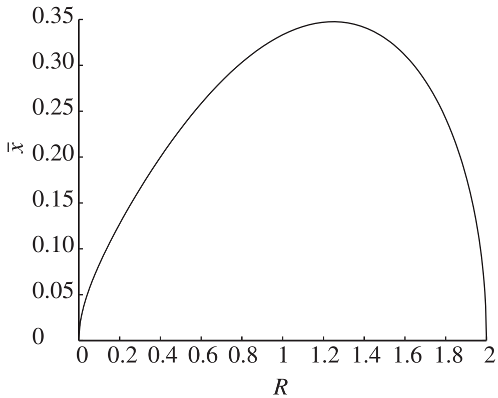


Figure 3

4. The solution

Returning to the formula in (6) for the frustum-shaped bin, figure 3 shows a graph of $\bar{x}$ against R and we see that, despite our original expectation, the centre of gravity is not at its maximum when the bin is cylindrical in shape. Therefore, starting with the original cylindrical bin with centre of gravity $\frac{1}{3}$ above the base, as the top is widened the centre of gravity initially rises to a maximum of approximately 0.35 above the base when $R \approx 1.25$, before falling again back past the original height when $R \approx 1.48$, and then falls further still until the side is horizontal. On the other hand, if the top is narrowed, then the centre of gravity falls until the side is horizontal.

5. Two observations

Further examination of the centre of gravity in (4) in the form

$$\bar{x} = \frac{\text{total moment}}{\text{total mass}}$$

(as derived from (3)) shows that

$$\begin{aligned}\text{total moment} &= \tfrac{1}{3}\pi\rho h\ell(2R+r) = \tfrac{1}{3}\pi\rho\ell(2R+r)\sqrt{\ell^2-(R-r)^2} \\ &= \tfrac{1}{3}\pi\rho(2R+1)\sqrt{1-(R-1)^2} = \tfrac{1}{3}\pi\rho(2R+1)\sqrt{R(2-R)}\end{aligned}$$

and

$$\text{total mass} = \pi\rho\big(r^2 + \ell(R + r)\big) = \pi\rho(R + 2)$$

in the case $\ell = r = 1$. Thus, the mass is monotonically increasing, whereas the moment increases and then decreases as the top is widened, as shown in figure 4 in which the moment and mass (divided by $\pi\rho$) have been plotted against R, and these two combine to give the behaviour of the centre of gravity as shown in figure 3.

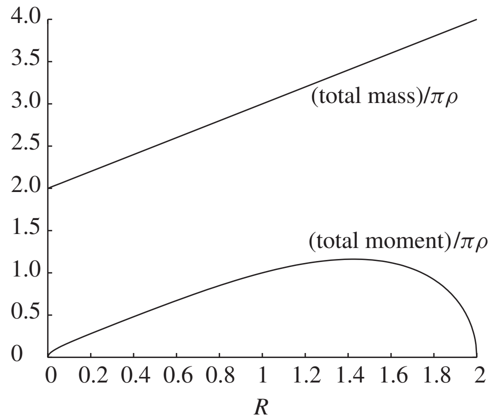


Figure 4

At this point readers may think that it is the presence of a base to the bin which makes the centre of gravity attain its maximum height when $R > r$ and not when the bin is cylindrical. To investigate this we need only examine the centre of gravity in (4) or (5) for a bin with no base. In this case the denominator in (5) does not contain the term r^2, and with $\ell = r = 1$ we have

$$\bar{x} = \frac{(2R+1)\sqrt{R(2-R)}}{3(R+1)}. \tag{8}$$

The corresponding graph of $\bar{x}$ against R is shown in figure 5 and we see that the centre of gravity attains a maximum height of approximately 0.51 above the base when $R \approx 1.14$, and thus it is not the presence of the base which accounts for this behaviour.

6. Some variations

A number of variations on this problem can be considered, including the case when the bin has a lid as well as a base, possibly both made of a different material to that of the side. We leave readers to prove and then investigate the general formula

$$\bar{x} = \frac{KR^2h + \frac{1}{3}h\ell(2R+r)}{kr^2 + KR^2 + \ell(R+r)} = \frac{(KR^2 + \frac{1}{3}\ell(2R+r))\sqrt{\ell^2 - (R-r)^2}}{kr^2 + KR^2 + \ell(R+r)}, \tag{9}$$

where K and k represent the ratio of the density of the lid and the base, respectively, to that of the side, including the determination of the location of the maximum.

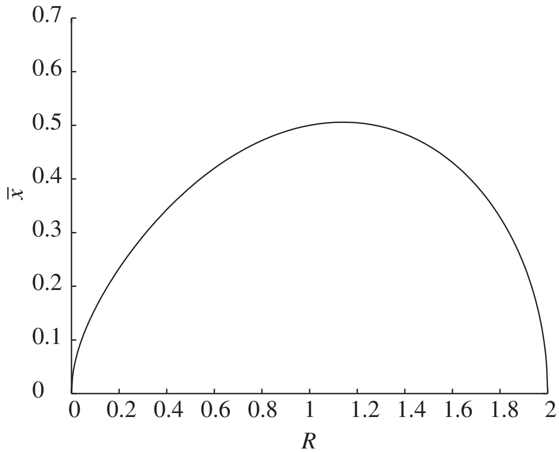


Figure 5

7. A cubic diversion

An interesting diversion is to determine the exact location of the maximum point shown in figures 3 and 5. We consider equation (9) in the special case where $K = 0$, so that the bin has no lid but does have a base with density k relative to that of the side, and again take $\ell = r = 1$ so that

$$\bar{x} = \frac{(2R+1)\sqrt{R(2-R)}}{3(R+1+k)}. \tag{10}$$

This includes the case with no base $(k = 0)$ as given by (8) and the case where the base is made of the same material as the side $(k = 1)$ as given by (6). Differentiating (10) with respect to R and simplifying gives

$$\bar{x}'(R) = \frac{1 + k + (4+5k)R - (2+4k)R^2 - 2R^3}{3\sqrt{R(2-R)}\,(R+1+k)^2}, \tag{11}$$

so that the maximum occurs when the radius R satisfies the cubic equation

$$f(R) \equiv 2R^3 + (2+4k)R^2 - (4+5k)R - (1+k) = 0 \tag{12}$$

corresponding to when the numerator in (11) vanishes. Readers may like to consider a simple numerical iteration to solve (12). Instead we use a result that is often overlooked, namely the formulae for the roots of a cubic equation; there is an interesting story concerning the authorship of these involving a number of protagonists (for example, see reference 1).

We begin by noting that $f(0) = -1 - k \le -1 < 0$ and $f(\frac{3}{2}) = (17+2k)/4 \ge \frac{17}{4} > 0$, so that (12) has at least one (positive) root in $(0, \frac{3}{2})$. It is a little harder to establish that this is the only positive root of (12) and that the other two are real and negative. The simplest way of doing this is to use the formulae for the roots based on cosines (for example, see references 2 or 3). Summarising these formulae, the cubic equation

$$R^3 + aR^2 + bR + c = 0$$

has three distinct, real roots if the 'discriminant' $D = P^2 + Q^3$ is less than 0, where $P = (9ab - 27c - 2a^3)/54$ and $Q = (3b - a^2)/9$, and the roots in this case are given by

$$R_n = -\tfrac{1}{3}a + 2\sqrt{-Q}\cos\left(\frac{\theta + 2n\pi}{3}\right), \qquad n = 0, 1, 2, \ldots,$$

where

$$\theta = \cos^{-1}\left(\frac{P}{\sqrt{-Q^3}}\right). \tag{13}$$

Readers unfamiliar with the solution of cubic equations can verify directly that these three values, R_1, R_2 and R_3, are the roots by checking that

$$R_1 + R_2 + R_3 = -a\,, \qquad R_1R_2 + R_1R_3 + R_2R_3 = b \quad \text{and} \quad R_1R_2R_3 = -c\,.$$

For the cubic equation (12) we have $a = 1 + 2k$, $b = -\frac{1}{2}(4 + 5k)$, $c = -\frac{1}{2}(1 + k)$, and after some algebra we find that

$$D = -\tfrac{1}{432}(197 + 892k + 1380k^2 + 850k^3 + 164k^4) < 0$$

since $k \geq 0$. The condition $D < 0$ ensures that $Q < 0$ and that $P/\sqrt{-Q^3}$ lies between -1 and $+1$, so the roots are real. Since the sum of the roots is $-a = -(1 + 2k)$, which is negative, and the product of the roots is $-c = \frac{1}{2}(1 + k)$, which is positive, one root of (12) has to be positive and the other two negative when $k \geq 0$.

In the special case $k = 1$, which we began with, we have $a = 3$, $b = -\frac{9}{2}$, $c = -1$, giving $P = -\frac{11}{4}$, $Q = -\frac{5}{2}$ and $D = -\frac{129}{16} < 0$, and the positive root is given by (13) with $n = 0$ as $R = -1 + \sqrt{10}\cos(\frac{1}{3}\cos^{-1}(-11/5\sqrt{10})) \approx 1.248$ as identified earlier from figure 3. The corresponding special case when $k = 0$ gives $R \approx 1.1397$ as identified earlier from figure 5 and which we leave readers to check.

8. Further work

We leave as a final exercise the case when the bin is partially full, say with a uniform material of prescribed volume and density. This could equally be thought of as a bucket of water, or even a glass of beer. Clearly the range of values for the radius of the top has to be restricted so that the vessel is large enough to contain its contents, which is particularly important for the latter kind of liquid refreshment!

References

1 G. Flegg, *Numbers — Their History and Meaning* (Penguin, London, 1984).
2 Wolfram Research, Cubic equation, available at http://mathworld.wolfram.com/CubicEquation.html
3 G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra* (Macmillan, London, 1953).

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis and perturbation methods, as well as mathematics and science education. His children consider the title of this article particularly apt because, in their opinion, the wastepaper bin is exactly what maths is for!

I'm Tired: Can We Stop Now?

SLOBODAN RADOMAN and PAUL BELCHER

To put interest and a random element into our running training, we run intervals as follows. First we generate a random number between 0 and 1. We then run that distance in miles and take a minute's rest. We then repeat the process, continuing to generate random numbers between 0 and 1 and running that distance, until the total length of all the intervals we have run is greater than 1 mile. How many intervals, on average, will we have to run before the total distance is greater than 1?

Let N be the random number of runs that are required to first make the total length of the runs greater than 1. Clearly, $\mathrm{P}(N = 1) = 0$. Let the N random numbers of runs $X_1, X_2, \ldots, X_N$ generated take the respective values $x_1, x_2, \ldots, x_N$.

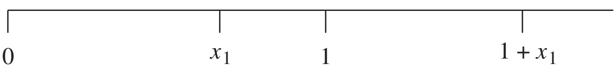


Figure 1

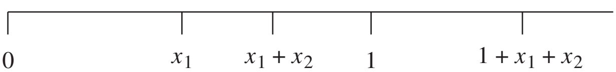


Figure 2

For N to equal 2, we can consider the lengths of runs shown in figure 1. We would require $x_1 + x_2$ to lie in the interval $]1, 1 + x_1]$. So the value x_2 lies in the interval $]1 - x_1, 1]$ and, since the run X_2 has the uniform distribution on $[0, 1]$, the probability of this is x_1. Thus,

$$\mathrm{P}(N = 2) = \int_0^1 x_1 \,\mathrm{d}x_1 = \tfrac{1}{2}.$$

For N to equal 3, the runs X_1 and X_2 of lengths x_1 and x_2 would be as in figure 2. We then obtain

$$\begin{aligned}
\mathrm{P}(N = 3) &= \int_0^1 \int_0^{1-x_1} (x_1 + x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \int_0^1 \left[x_1 x_2 + \frac{x_2^2}{2}\right]_0^{1-x_1} \mathrm{d}x_1 \\
&= \int_0^1 \left(\frac{1}{2} - \frac{x_1^2}{2}\right) \mathrm{d}x_1 \\
&= \frac{1}{2}\left[x_1 - \frac{x_1^3}{3}\right]_0^1 = \frac{1}{3}\,.
\end{aligned}$$

Continuing in this fashion gives

$$\mathrm{P}(N=4)=\tfrac{1}{8}\,, \qquad \mathrm{P}(N=5)=\tfrac{1}{30}\,, \qquad \mathrm{P}(N=6)=\tfrac{1}{144}\,.$$

Rewriting these to search for a pattern, we have

$$\begin{aligned}
\mathrm{P}(N=1)&=\frac{0}{1!}\,, &\qquad \mathrm{P}(N=2)&=\frac{1}{2!}\,,\\
\mathrm{P}(N=3)&=\frac{2}{3!}\,, &\qquad \mathrm{P}(N=4)&=\frac{3}{4!}\,,\\
\mathrm{P}(N=5)&=\frac{4}{5!}\,, &\qquad \mathrm{P}(N=6)&=\frac{5}{6!}\,,
\end{aligned}$$

and so we conjecture that

$$\mathrm{P}(N=n)=\frac{n-1}{n!}\,.$$

We now attempt to prove this conjecture.

If $N>n$, we require that

$$x_1+x_2+\cdots+x_n\le 1\,.$$

So

$$\mathrm{P}(N>n)=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\cdots\int_0^{1-x_1-x_2-\cdots-x_{n-1}} 1\,\mathrm{d}x_n\ldots\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,.$$

We now apply the substitution

$$y_i=1-x_1-x_2-\cdots-x_i\,, \qquad 1\le i\le n\,.$$

Now, in general,

$$\begin{aligned}
\int_0^{1-x_1-x_2-\cdots-x_{i-1}} f(x_i)\,\mathrm{d}x_i &= \int_{y_{i-1}}^0 f(x_i(y_i))(-1)\,\mathrm{d}y_i\\
&=\int_0^{y_{i-1}} f(x_i(y_i))\,\mathrm{d}y_i\,.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{P}(N>n)&=\int_0^1\int_0^{y_1}\int_0^{y_2}\cdots\int_0^{y_{n-1}} 1\,\mathrm{d}y_n\ldots\mathrm{d}y_3\,\mathrm{d}y_2\,\mathrm{d}y_1\\
&=\frac{1}{n!}
\end{aligned}$$

and, since $\mathrm{P}(N=n)=\mathrm{P}(N\le n)-\mathrm{P}(N\le n-1)$, we have

$$\begin{aligned}
\mathrm{P}(N=n)&=\left(1-\frac{1}{n!}\right)-\left(1-\frac{1}{(n-1)!}\right)\\
&=\frac{n-1}{n!}\,,
\end{aligned}$$

as required.

We are now in a position to find the expected value of N, $\mathrm{E}(N)$:

$$\begin{aligned}\mathrm{E}(N) &= \sum_{n=1}^{\infty} n\frac{n-1}{n!}\\ &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!}\\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\\ &= \mathrm{e}\,.\end{aligned}$$

We now find the variance of N, $\operatorname{var}(N)$:

$$\begin{aligned}\operatorname{var}(N) &= \sum_{n=1}^{\infty} n^2\frac{n-1}{n!} - \big(\mathrm{E}(N)\big)^2\\ &= \sum_{n=2}^{\infty} \frac{n}{(n-2)!} - \mathrm{e}^2\,.\end{aligned}$$

Now, $\mathrm{e}^x x^2 = \sum_{i=0}^{\infty} x^{i+2}/i!$, so differentiating gives

$$x^2\mathrm{e}^x + 2x\mathrm{e}^x = \sum_{i=0}^{\infty}(i+2)\frac{x^{i+2}}{i!}\,,$$

and putting $x = 1$ then gives

$$3\mathrm{e} = \sum_{i=0}^{\infty} \frac{i+2}{i!}.$$

So $\operatorname{var}(N) = 3\mathrm{e} - \mathrm{e}^2$.

The probability-generating function of the sequence $\mathrm{P}(N = n)$ is given by

$$f(x) = \sum_{n=2}^{\infty} \mathrm{P}(N = n)x^n = \sum_{n=2}^{\infty}(n-1)\frac{x^n}{n!}\,.$$

This article is an amended version of the Extended Essay that ***Slobodan Radoman****, an 18-year-old Atlantic College student from Montenegro, produced for the International Baccalaureate.* ***Paul Belcher*** *was his supervisor.*

Sorting Sequences

JONATHAN SMITH

Take any sequence of numbers, e.g. $17, 54, 23, 31, 17, 3$. There are various algorithms to sort these numbers into ascending order (with the lowest number on the left). With a sorting algorithm there need to be a certain number of comparisons made between numbers and a certain number of swaps made. Only adjacent numbers can be compared or swapped. The number of comparisons needed gives some idea of the efficiency of the algorithm, but the number of swaps is the same regardless of the algorithm.

For readers who are familiar with sorting algorithms, sorting the above sequence into ascending order using the bubble sort algorithm requires 15 comparisons and 10 swaps. Using the shuttle sort algorithm requires 14 comparisons and 10 swaps. Notice that the number of swaps is the same, as expected.

Given A distinct numbers there are $A!$ ways of arranging them in a sequence. If we treat the left-hand number as the most significant, followed by the next one in and so on, we can arrange the sequences in an order and then label them from 0 to $(A! - 1)$ in ascending order, giving us the sequence number. For example, if we take sequences with three numbers, there are $3! = 6$ possible sequences. We will number these 0 to 5 as in table 1.

Table 1 Sequences of three numbers.

Sequence	Sequence number
1 2 3	0
1 3 2	1
2 1 3	2
2 3 1	3
3 1 2	4
3 2 1	5

The aim of this article is to show how to work out the sequence number of a given sequence and, conversely, how to work out the sequence with a given sequence number.

1. Reducing the problem to standard form

The numbers in a sequence can be in any order, but it is important that they are the numbers $\{1, 2, 3, \ldots, A\}$. If they are not, then replace the lowest number with 1, the next lowest with 2 and so on.

If there are any duplicate digits, then these should be assigned numbers in ascending order, starting from the left-hand side. Thus, the sequence $17, 54, 23, 31, 17, 3$ becomes $2, 6, 4, 5, 3, 1$.

2. How many swaps are needed?

To do this we need to put the sequence number, N, into what I call 'base factorial'. In a standard base, b, each digit starting from the right represents b^0, b^1, b^2 and so on, i.e. if $N = \cdots + n_2 b^2 + n_1 b^1 + n_0$, then $N = (\ldots n_2 n_1 n_0)_b$. However, in base factorial, each digit represents 0!, 1!, 2!, 3! and so on. Thus, if $N = \cdots + n_2 2! + n_1 1! + n_0 0!$, then $N = (\ldots n_2 n_1 n_0)_{\mathrm{F}}$ (where F denotes 'factorial'), with $0 \leq n_i < i + 1$. To convert N to base factorial, start with N and divide successively by $1, 2, 3, 4, \ldots$ and take the remainders in reverse order. The remainder n_0 is always 0, because all integers are divisible by 1. The number of swaps needed is then the sum of $n_0, n_1, n_2, \ldots$.

Example 1 Take the sequence $2, 3, 1$, which has $N = 3$:

$$\begin{array}{r|l l}
1 & \underline{3} & \\
2 & \underline{3} & n_0 = 0 \\
3 & \underline{1} & n_1 = 1 \\
 & 0 & n_2 = 1
\end{array}$$

Thus $(3)_{10} = (110)_{\mathrm{F}}$.

The number of swaps needed to sort $2, 3, 1$ is the sum of the digits of $(110)_{\mathrm{F}}$, that is, 2. A glance at $2, 3, 1$ shows this to be true.

We don't need to know the sequence itself to find the number of swaps required, just the sequence number.

Example 2 How many swaps are needed to sort a sequence with sequence number $N = 947$?

$$\begin{array}{r|r l}
1 & \underline{947} & \\
2 & \underline{947} & n_0 = 0 \\
3 & \underline{473} & n_1 = 1 \\
4 & \underline{157} & n_2 = 2 \\
5 & \underline{39} & n_3 = 1 \\
6 & \underline{7} & n_4 = 4 \\
7 & \underline{1} & n_5 = 1 \\
 & 0 & n_6 = 1
\end{array}$$

Therefore, $947 = (1141210)_{\mathrm{F}}$ and the number of swaps needed is $1+1+4+1+2+1+0 = 10$.

However, from the sequence number we can tell more than just the number of swaps needed. We can derive the sequence itself.

Example 3 Which nine-number sequence is represented by $N = 947$?

This is another use for the base factorial expression: $(947)_{10} = (1141210)_{\mathrm{F}}$. We will need nine digits in our base factorial number: $(1141210)_{\mathrm{F}} = (001141210)_{\mathrm{F}}$.

We will call our nine digits $A, B, C, D, E, F, G, H, I$. These need to be written with the base factorial expression for 947 underneath:

$$\begin{array}{ccccccccc}
A & B & C & D & E & F & G & H & I \\
0 & 0 & 1 & 1 & 4 & 1 & 2 & 1 & 0.
\end{array}$$

Another property of the base factorial expression for 947 is that, for each digit, starting from the right-hand side, it gives the number of moves to the right that will put this sequence into ascending order. This is in effect a modified insertion sort algorithm. So we start by moving the H one place to the right:

$$\begin{array}{ccccccccc} A & B & C & D & E & F & G & I & H \\ 0 & 0 & 1 & 1 & 4 & 1 & 2 & & \end{array}$$

Now we move the G two places to the right:

$$\begin{array}{ccccccccc} A & B & C & D & E & F & I & H & G \\ 0 & 0 & 1 & 1 & 4 & 1 & & & \end{array}$$

Continue until we have used up all the digits. We now have

$$A\ B\ I\ C\ D\ F\ H\ G\ E$$

which we know is in ascending order. Therefore, $A = 1$, $B = 2$, $I = 3$, $C = 4$, $D = 5$, $F = 6$, $H = 7$, $G = 8$, $E = 9$. Since the original sequence was $A, B, C, D, E, F, G, H, I$, we now know that sequence number 947 is $1, 2, 4, 5, 9, 6, 8, 7, 3$.

Can we do the reverse? Can we obtain the sequence number from the sequence itself?

Example 4 Take the sequence $6, 1, 3, 7, 5, 2, 4$. What is the sequence number, N?

This is answered as follows. Starting with the right-most 'sequence' of one number, that is, 4, we ask: how many moves does the 4 need to make in order to put the sequence 4 into ascending order? The answer is zero. We now still have the sequence $6, 1, 3, 7, 5, 2, 4$.

Now consider the sequence 2, 4. How many moves does the 2 need to make in order to put the sequence 2, 4 into ascending order? Again, zero. We now still have $6, 1, 3, 7, 5, 2, 4$.

How many moves does the 5 need to make in order to put the sequence $5, 2, 4$, into ascending order? Two moves. We now have $6, 1, 3, 7, 2, 4, 5$.

Continue like this until the sequence is in ascending order. We can write it as in table 2, noting the number of moves needed in each case.

Table 2 Finding the sequence number.

Sequence	Moves
6 1 3 7 5 2 4	0
6 1 3 7 5 2 4	0
6 1 3 7 2 4 5	2
6 1 3 2 4 5 7	3
6 1 2 3 4 5 7	1
6 1 2 3 4 5 7	0
1 2 3 4 5 6 7	5

If we note the number of moves in reverse order, this is the base factorial representation of N, $(5013200)_\mathrm{F}$. Converting this to base 10, we find that

$$\begin{aligned} N &= 0 \times 0! + 0 \times 1! + 2 \times 2! + 3 \times 3! + 1 \times 4! + 0 \times 5! + 5 \times 6! \\ &= 3646\,. \end{aligned}$$

Therefore, the sequence number of $6, 1, 3, 7, 5, 2, 4$ is 3646.

We give one final example.

Example 5 Take the ten-digit sequence represented by $N = 39247$. How many swaps are needed to sort this into ascending order? What is the sequence?

$$
\begin{array}{r|r l}
1 & \underline{39247} & \\
2 & \underline{39247} & n_0 = 0 \\
3 & \underline{19623} & n_1 = 1 \\
4 & \underline{6541} & n_2 = 0 \\
5 & \underline{1635} & n_3 = 1 \\
6 & \underline{327} & n_4 = 0 \\
7 & \underline{54} & n_5 = 3 \\
8 & \underline{7} & n_6 = 5 \\
\multicolumn{1}{r}{} & 0 & n_7 = 7
\end{array}
$$

So, $39247 = (75301010)_{\mathrm{F}}$. Therefore, the number of swaps needed is $(7 + 5 + 3 + 0 + 1 + 0 + 1 + 0) = 17$.

There are ten digits, so $(75301010)_{\mathrm{F}}$ becomes $(0075301010)_{\mathrm{F}}$. Let the ten digits be A, B, C, D, E, F, G, H, I, J:

A	B	C	D	E	F	G	H	I	J
0	0	7	5	3	0	1	0	1	0

Now we start the swapping and we get successively

$$
\begin{aligned}
&A\ B\ C\ D\ E\ F\ G\ H\ I\ J\,,\\
&A\ B\ C\ D\ E\ F\ G\ H\ J\ I\,,\\
&A\ B\ C\ D\ E\ F\ G\ H\ J\ I\,,\\
&A\ B\ C\ D\ E\ F\ H\ G\ J\ I\,,\\
&A\ B\ C\ D\ E\ F\ H\ G\ J\ I\,,\\
&A\ B\ C\ D\ F\ H\ G\ E\ J\ I\,,\\
&A\ B\ C\ F\ H\ G\ E\ J\ D\ I\,,\\
&A\ B\ F\ H\ G\ E\ J\ D\ I\ C = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\,.
\end{aligned}
$$

Therefore, the sequence $A, B, C, D, E, F, G, H, I, J$ is $1, 2, 10, 8, 6, 3, 5, 4, 9, 7$.

Jonathan Smith *was a sixth-form student studying maths at Gresham's School in Norfolk. After taking a 'Gap Year', he will go to Cambridge to study engineering. He has always been interested in number theory, but recently has been enjoying discreet mathematics. He says that, being fairly modern, discreet mathematics still has much to reveal and is accessible to all.*

The Soccer World Cup: Poisson Versus Pascal

JOHN C. B. COOPER

Various statistical studies have sought to demonstrate that goal scoring in soccer may be modelled by the Poisson distribution while others have suggested that the negative binomial distribution provides a superior fit. Examples include references 1–8.

For the Poisson distribution[1], the probability that a team scores y goals over a fixed time interval of 90 minutes is given by

$$\mathrm{P}(y) = \frac{\mathrm{e}^{-\lambda}\lambda^{y}}{y!},$$

where $y = 0, 1, 2, 3\ldots, \lambda$ is the mean number of goals per match and e is the exponential. The mean and variance of this distribution are both equal to λ.

For the negative binomial distribution, sometimes referred to as the Pascal distribution[1] the probability of y goals is given by:

$$\binom{x+y-1}{y} p^{x} q^{y},$$

where $y = 0, 1, 2, 3\ldots$, with $x > 0$, $0 < p < 1$ and $q = 1 - p$.

The mean and variance of this distribution are given by

$$\mu = \frac{xq}{p} \quad \text{and} \quad \sigma^2 = \frac{xq}{p^2}.$$

Knowledge of the sample mean (m) and the sample variance (s^2) allows us to estimate the two parameters x and p as follows:

$$\hat{x} = \frac{m^2}{s^2 - m} \quad \text{and} \quad \hat{p} = \frac{m}{s^2}.$$

The Poisson distribution is valid where the probability of an event occurring in a given unit of time is the same for all such units of time, i.e. if the probability of a goal and hence the expected number of goals for a particular team were constant from one game to another. Intuitively, this is unrealistic since a team varies its players and tactics and plays against opposition of varying calibre on grounds of varying quality. In addition, there is some evidence that, even when two teams are of similar calibre, when one is trailing by, say, 3–0, it is more likely to lose heart than when losing 1–0. In such circumstances, the occurrence of goals may be better modelled by the negative binomial distribution.

[1]These distributions are named after the French mathematicians Blaise Pascal (1623–1662) and Siméon Poisson (1781–1840).

Table 1 World Cup summary statistics

Year	Goals	m	s^2	m/s^2	Poisson χ^2		Neg. bin. χ^2
1930	70	1.94	3.44	0.56	13.21*	>	4.53**
1934	70	2.06	1.88	1.11	2.11		n/a
1938	75	2.08	3.19	0.65	6.47	>	3.64**
1950	88	2.01	3.27	0.61	4.13	<	5.53
1954	138	2.65	5.53	0.48	10.13*	>	2.97**
1958	126	1.81	2.42	0.74	1.18	<	1.31
1962	89	1.39	1.89	0.74	2.61	>	0.67**
1966	86	1.34	1.41	0.95	0.31	=	0.31
1970	87	1.36	1.79	0.76	3.83	>	1.79**
1974	97	1.28	2.52	0.51	3.61	>	1.48**
1978	100	1.32	1.87	0.71	4.34	>	3.01**
1982	144	1.38	2.22	0.62	7.28	>	5.82**
1986	127	1.22	1.54	0.79	0.31	<	1.11
1990	115	1.11	1.23	0.91	2.93	<	4.42
1994	138	1.33	1.37	0.97	0.45	=	0.45
1998	170	1.33	1.41	0.94	0.44	<	0.81
2002	158	1.23	1.31	0.94	0.72	>	0.66**

To test this, statistical data for every soccer World Cup tournament from 1930 to 2002 were collected and both the Poisson and negative binomial distributions were fitted. The results and goodness of fit χ^2 statistics are shown in table 1. From these results, it may be observed that:

1. The Poisson distribution provides an inadequate fit in only two years (1930 and 1954), as indicated by *, while the negative binomial distribution provides an adequate fit in every year apart from 1934 when it could not be fitted since $m > s^2$.

2. The χ^2 statistics indicate that the negative binomial distribution provides a superior fit for nine of the 17 years as indicated by **, the same degree of fit as the Poisson distribution in two years and an inferior fit for the remaining six years.

3. For the two years where $m/s^2 \geq 0.95$, the χ^2 statistics indicate the same degree of fit confirming that, if the mean and variance are the same, the two distributions fit equally well.

Accordingly, we may conclude that the negative binomial distribution is only slightly superior to the Poisson distribution in fitting these particular data.

References

1 M. J. Moroney, *Facts from Figures*, 3rd edn (Penguin, London, 1956).

2 C. Reep, R. Pollard and B. Benjamin, Skill and chance in ball games, *J. R. Statist. Soc. A* **134** (1971), pp. 623–629.

3 R. Pollard, B. Benjamin and C. Reep, Sport and the negative binomial distribution, in *Optimal Strategies in Sports* (North-Holland, Amsterdam, 1977).

4 D. J. Colwell and J. R. Gillett, Is goal scoring a Poisson process?, *Math. Gazette* **65** (1981), pp. 30–32.

5 R. Pollard, Goal scoring and the negative binomial distribution, *Math. Gazette* **69** (1985), pp. 45–57

6 M. Baxter and R. Stevenson, Discriminating between the Poisson and negative binomial distributions: an application to goal scoring in association football, *J. Appl. Statist.* **15** (1988), pp. 347–354.

7 S. Chu-Chun-Lin, Rendezvous of the Poisson and exponential distributions at the World Cup of soccer, *Teaching Statist.* **21** (1999), pp. 60–62.

8 G. Simpson, How random was the World Cup?, *Teaching Statist.* **21** (1999), p. 7.

Dr John Cooper *is senior lecturer in Financial Economics at Glasgow Caledonian University and holds visiting professorships in the USA, Peru and Hungary. His research interests include the application of mathematical and statistical methods in financial decision-making. Sadly, he has always been hopeless at soccer!*

Two conjectures about prime numbers

1. Among any five consecutive odd numbers (for example, 25, 27, 29, 31, 33), at least one of them is prime.
2. For every integer $n > 1$, in the sequence of the first n odd integers, there is a pair of primes which are equal distances from the centre of the sequence, for example

$$3 \quad 5 \quad 7 \quad | \quad 9 \quad 11 \quad 13$$

Razi Petrochemical Complex,
Bandar, Imam, Khozestan, Iran

Seyamack Jafari

[For the first conjecture, readers might like to consider the number $11! + 3$. Maybe you could then prove that, for all integers $n > 1$, there are n consecutive odd integers none of which is prime. But the second conjecture … ! — Ed.]

Simple Properties of a Circle

GUIDO LASTERS

In this article we shall demonstrate how to draw a tangent to a circle through a given point and how one can 'do arithmetic on a circle'. We shall leave readers to fill in some of the calculations, which involve simple coordinate geometry. Some background can be found in references 1 and 2.

1. Drawing a tangent

Consider a circle, taken as the unit circle, and a point T outside the circle. Choose coordinate axes so that T is the point $(t, 0)$, with $t > 1$; see figure 1.

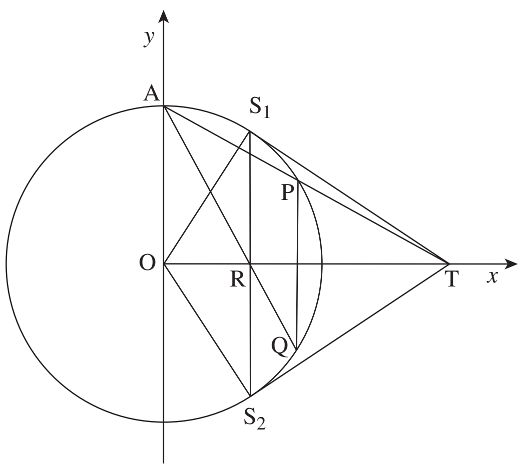


Figure 1

If A is the point $(0, 1)$, then the straight line AT meets the unit circle again at the point P with coordinates $(2t/(t^2+1), (t^2-1)/(t^2+1))$. Construct the straight line through P parallel to the y-axis to meet the circle again at the point Q with coordinates

$$\left(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\right).$$

Draw the straight line AQ, which crosses the x-axis at the point R with coordinates $(1/t, 0)$. Draw the line through R parallel to the y-axis to meet the circle at the points S_1 and S_2 with coordinates $(1/t, \pm\sqrt{t^2-1}/t)$. Then S_1T has slope $-1/\sqrt{t^2-1}$ and OS_1 has slope $\sqrt{t^2-1}$, so they are at right-angles and S_1T is tangent to the circle, as is S_2T. A similar construction holds for an ellipse when T is on one of its axes.

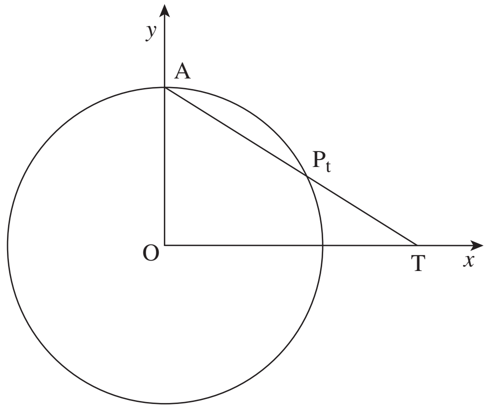


Figure 2

2. Arithmetic on a circle

We show how to define addition and multiplication of points on the circumference of a circle. We again take the unit circle, choose coordinate axes and let T be the point $(t, 0)$, which is no longer assumed to lie outside the circle. As before, A is the point $(0, 1)$ and the straight line AT is drawn to meet the circle again at a point which we now call P_t, with coordinates $(2t/(t^2+1), (t^2-1)/(t^2+1))$; see figure 2.

Let a and b be numbers. In figure 3, the straight line P_aP_b meets the line $y = 1$ at the point X with coordinates $(2/(a+b), 1)$. (If $a = b$, then P_aP_b is taken to be the tangent to the curve at P_a. If $a = -b$, then P_aP_b is parallel to the line $y = 1$.) Thus, P_0P_{a+b} also meets the line $y = 1$ at X. This gives us a way to locate P_{a+b} from P_a and P_b. First find X and then join X to $P_0 = (0, -1)$. The point P_{a+b} will be the point where this line again crosses the unit circle (see figure 3). So we have defined the sum of two points on the circle.

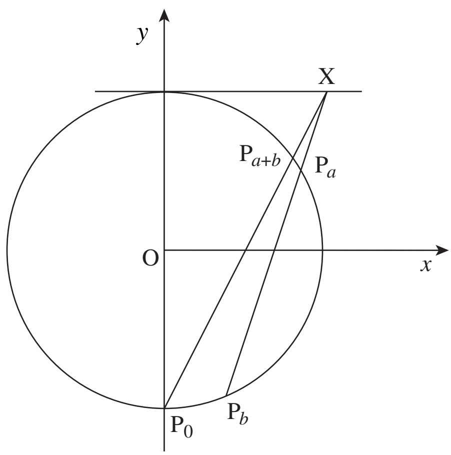


Figure 3

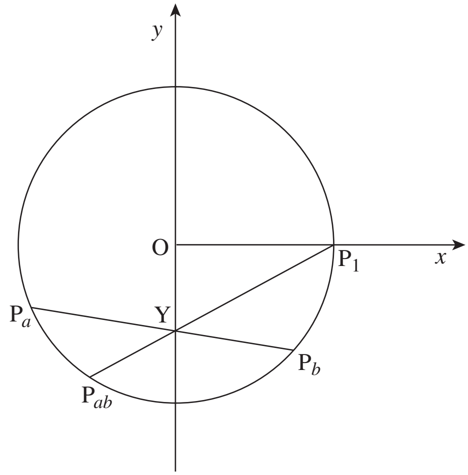


Figure 4

We can locate P_{ab} in a similar way. The straight line P_aP_b meets the y-axis at the point Y with coordinates

$$\left(0, \frac{ab+1}{ab-1}\right).$$

(If $ab = 1$, P_aP_b is parallel to the y-axis.) Thus P_1P_{ab} also meets the y-axis at Y. Hence, to locate P_{ab} from P_a and P_b, first find Y and then join Y to $P_1 = (1, 0)$. The point P_{ab} will be the point where this line again crosses the unit circle (see figure 4). In this way we can define the product of two points on the circle.

References

1 P. Fjelstad and P. Hammer, A picture of real arithmetic, *College Math. J.* **31** (2000), pp. 56–60.
2 G. Lasters and D. Sharpe, From Pascal to groups, *Math. Spectrum* **29** (1996/7), pp. 51–53.

The author teaches mathematics at a school in Leuven in Belgium. He tries to discover new links to interest his students. The circle is, except for the straight line, the most elementary geometrical figure, and the laws of addition and multiplication are the simplest concepts in elementary algebra. To link the two may interest them! He is grateful to Paul Fjelstad of Northfield, Minnesota, who gave him the basic idea for this article.

Slow Runners Also Have a Chance

M. A. KHAN

1. Introduction

In order to offset the inherent advantage of superior horses over their weaker competitors in a race, a handicapping system is usually adopted.

The system consists of penalizing front runners with a *handicapping weight*, this weight being proportionately lower for slower horses. The object is to equalize the chances of all the runners in the long run. The penalty weight depends on the horses' relative positions in a previous race and the margin of separation at the finish; for simplicity, in this paper we will consider only a fixed-scale handicapping system.

2. Allocation of rating points

Before imposing the handicapping, the horses are usually rated according to their running time over a particular distance, say 1600 metres. The rating point scale may be based upon an average running time, which is of the order of about 100 seconds over a trip of 1600 metres. Thus, if we choose to allot $100 - 10t$ rating points to a horse, if its running time is $100 + t$ seconds over this distance, then there will be an increase of 10 points for every decrease of 1 second in its running time and vice versa. Also, if J is the rating of some horse in a race and S is the sum of the rating points of all the participants in the race, then the probability of the horse winning the race is J/S.

Table 1 gives the expected performances of four horses, A, B, C and D, in the first race based upon their timings in the initial trial runs.

Table 1 The expected performances of four horses in the first race.

Horse	Timing (seconds)	Rated points	Probability of winning
A	98	120	$a(1) := \frac{120}{400} = 0.30$
B	99	110	$b(1) := \frac{110}{400} = 0.275$
C	101	90	$c(1) := \frac{90}{400} = 0.225$
D	102	80	$d(1) := \frac{80}{400} = 0.20$

Note that the sum of the winning probabilities of all the horses is 1, as must always be the case.

Since the running times do not vary by more than $\pm 2\%$ from the average of 100 seconds over this distance, all four horses are initially treated as equals, and so they are allotted the same handicapping weight in the first official race.

3. Handicapping mechanism

The handicapping weight W of a race horse consists of the jockey, the saddle and some extra adjustable weight in the form of lead balls. The total may initially be about 50 kg. It is obvious that an increase in W causes the running time of the horse to increase, while a reduction in W causes a decrease in running time. Suppose that we choose a unit weight w kg such that an alteration of w kg causes a change of 1 second in the running time.

The handicapping system is based upon subtracting a penalty of K rating points from the winner, along with a total increase of K rating points for the losers divided equally among them. The handicapping system is illustrated in table 2, where $q = K/S$.

Table 2 Handicapping system for a total of r runners showing the alterations to ratings and weights and their effect on the performances of the winner and the losers. The changes apply to the next race.

Result	Change in rating	Corresponding change in W	Effective change in the running time (seconds)	Corresponding change in the probability
Winner	$-K$	$+0.1Kw$	$+0.1K$	$-q$
Loser	$+\dfrac{K}{r-1}$	$-0.1\dfrac{Kw}{r-1}$	$-0.1\dfrac{K}{r-1}$	$+\dfrac{q}{r-1}$

From table 2 it will be observed that the handicapping system is such that the sum of various parameters for all the horses remains constant throughout.

In the example of table 1, with $K = 30$ and $w = 4$ the handicapping weight of the winner will be increased by $3w = 12$ kg, while that of each of the losers will be reduced by $\frac{12}{3} = 4$ kg. The probability that the winner will win the next race is reduced by $q = \frac{30}{400} = 0.075$ and that of each of the losers is increased by $q/3 = 0.075/3 = 0.025$. These changes will be applicable to the second race.

4. Developing a general recurrence relation

We can now proceed to determine the probability of winning of horse A in the $(n+1)$th race based upon its result in the nth race, when it competes with three other runners, B, C and D. The analysis gives rise to the following two possibilities:

(i) A may win the nth race with probability $P(n) := a(n)$ and then win the $(n+1)$th race with the reduced probability $U(n+1) := a(n) - q$.

(ii) A may lose the nth race with probability $\bar{P}(n) := 1 - a(n)$ and then win the $(n+1)$th race with the increased probability $V(n+1) := a(n) + q/3$.

Taking both these possibilities into account, and applying the laws of multiplication and addition of probabilities, the probability $a(n+1)$ that A wins the $(n+1)$th race can be written as

$$\begin{aligned} a(n+1) &= P(n)U(n+1) + \bar{P}(n)V(n+1) \\ &= a(n)[a(n) - q] + [1 - a(n)]\Big[a(n) + \frac{q}{3}\Big] \\ &= \Big(1 - \frac{4q}{3}\Big)a(n) + \frac{q}{3}. \end{aligned} \tag{1}$$

The solution to the recurrence relation (1) can be easily found by writing it in the form

$$a(n+1) - \frac{1}{4} = \left(1 - \frac{4q}{3}\right)\left(a(n) - \frac{1}{4}\right). \tag{2}$$

Now, on setting

$$a(n+1) - \frac{1}{4} = u(n+1),$$

$$a(n) - \frac{1}{4} = u(n)$$

and

$$1 - \frac{4q}{3} = Q,$$

which is assumed to be less than 1, (2) takes the form

$$u(n+1) = Qu(n). \tag{3}$$

Now, for $n = 1, 2, \ldots$ in (3), we have

$$u(2) = Qu(1),$$

$$u(3) = Qu(2) = Q^2u(1),$$

$$\vdots$$

Proceeding in this manner, we easily deduce that

$$u(n) = Q^{n-1}u(1),$$

implying that

$$a(n) - \frac{1}{4} = \left(1 - \frac{4q}{3}\right)^{n-1}\left(a(1) - \frac{1}{4}\right),$$

so that

$$a(n) = \left(1 - \frac{4q}{3}\right)^{n-1}\left(a(1) - \frac{1}{4}\right) + \frac{1}{4}. \tag{4}$$

A similar solution applies to the probabilities of winning $b(n)$, $c(n)$ and $d(n)$ of the remaining three horses B, C and D in terms of their initial probabilities $b(1)$, $c(1)$ and $d(1)$. Also, if Q is a positive fraction less than 1, so that $1 > 1 - 4q/3 > 0$, then it easily follows from (4) that $a(n) \to \frac{1}{4}$ as $n \to \infty$ because $Q^{n-1} \to 0$.

The result given in (4) can be easily generalised to r horses $A_1, A_2, \ldots, A_r$ whose probabilities of winning the first race based upon their initial trial runs are $p_1(1), p_2(1), \ldots, p_r(1)$. In such a case the probability $p_i(n)$ of some horse A_i winning the nth race can be easily shown to be

$$p_i(n) = \left[1 - \frac{rq}{r-1}\right]^{n-1}\left[p_i(1) - \frac{1}{r}\right] + \frac{1}{r}.$$

As before, $q = K/S$ is the probability reduction factor. Since the value of K usually lies between 10 and 30, based on a point rating of 100 for the average running time, it follows that q is a small fraction and consequently $1 - rq/(r-1) = Q$ is also a fraction less than 1

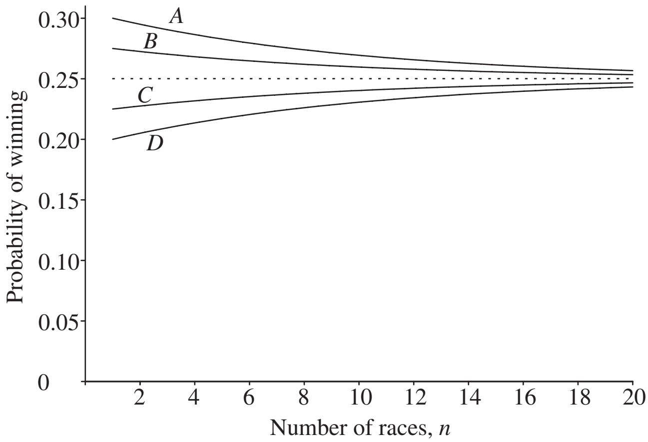


Figure 1 The probability of winning of the horses A, B, C and D of table 1 in the first 20 races given by the formula $p(n) = (0.9)^{n-1}(p(1) - \frac{1}{4}) + \frac{1}{4}$.

and so $Q^{n-1} \to 0$ as $n \to \infty$. Thus, the probability of winning of all the horses eventually approaches the average value $1/r$. Figure 1 shows the probability of winning of the four horses in table 1 for the first 20 races, where $q = 0.075$. It may be pointed out that a higher value of K brings the probabilities of winning of the participants to the average value $1/r$ faster as the number of races increases. In the same way, a lower value of K makes the convergence slower.

M. A. Khan *joined the Indian Railways in 1963 as an Electrical Engineer, having graduated from Muslim University Aligarh in 1958. After serving the railways for about 31 years, he retired as a Deputy Director from the Research Design and Standards Organisation in 1994. Whenever free, he likes to enjoy a game of duplicate Bridge.*

A curious sequence

I chanced across the curious sequence given by

$$u_{n+2} = \frac{1}{u_n u_{n+1}^{\phi}},$$

where ϕ is the golden ratio. Whatever initial values are chosen for u_1, u_2, it repeats after five terms. Can you show this?

Paston College, Norfolk **Jonny Griffiths**

Smith Numbers

SHYAM SUNDER GUPTA

1. Introduction

A composite integer N whose digit sum $S(N)$ is equal to the sum of the digits of its prime factors $S_p(N)$ is called a Smith number (see reference 1). For example, 85 is a Smith number because the digit sum of 85 is $S(85) = 8 + 5 = 13$, which is equal to the sum of the digits of its prime factors, $S_p(85) = S_p(17\times5) = 1 + 7 + 5 = 13$.

Albert Wilansky named Smith numbers after his brother-in-law Herald Smith's telephone number with this property which was $4\,937\,775 = 3\times5\times5\times65\,837$. Wilansky claimed that there are 360 Smith numbers less than 10 000, which is not correct as there are 376. It is now known that there are infinitely many Smith numbers (see reference 2 and also references 3 and 4). The 49 Smith numbers below 1000 are

4, 22, 27, 58, 85, 94, 121, 166, 202, 265, 274, 319, 346, 355, 378, 382, 391,
438, 454, 483, 517, 526, 535, 562, 576, 588, 627, 634, 636, 645, 648, 654, 663,
666, 690, 706, 728, 729, 762, 778, 825, 852, 861, 895, 913, 915, 922, 958, 985.

2. Computation of all Smith numbers below 10^9

A computer program in FORTRAN has been developed to investigate Smith numbers. The objective of the investigation was to find the proportion of Smith numbers as compared to that of prime numbers up to a given limit and the maximum number of prime factors of a Smith number. Let the number of Smith numbers up to x be denoted by $N_S(x)$ and the number of primes up to x by $\pi(x)$. The values of $N_S(x)$ and $\pi(x)$ up to $x = 10^m$ for $m = 1, 2, 3, \ldots, 9$ are tabulated in table 1, along with the ratio $r = \pi(x)/N_S(x)$.

Table 1

x	$N_S(x)$	$\pi(x)$	$\pi(x)/N_S(x)$
10^1	1	4	4.000 00
10^2	6	25	4.166 67
10^3	49	168	3.428 57
10^4	376	1 229	3.268 62
10^5	3 294	9 592	2.911 96
10^6	29 928	78 498	2.622 89
10^7	278 411	664 579	2.387 04
10^8	2 632 758	5 761 455	2.188 37
10^9	25 154 060	50 847 534	2.021 44

It can be seen from table 1 that the percentage of Smith numbers decreases with an increase in x. As an example, the percentage of Smith numbers up to 10^5 is 3.294% which decreases to 2.9928% for 10^6. The proportion of primes also decreases with an increase in x. It can be observed that the ratio r decreases with an increase in x for $x = 10^m$ and $m > 2$, which indicates that the decrease in Smith numbers is at a slower rate than the decrease in prime numbers.

In the range from 10^m to 10^{m+1}, the distribution of Smith numbers generally presents a rising trend. So generally $N_S(2\times10^m) - N_S(10^m)$ is lower than $N_S(10^{m+1}) - N_S(9\times10^m)$. As examples, for $m = 7$,

$$N_S(2 \times 10^7) - N_S(10^7) = 527\,739 - 278\,411 = 249\,328\,,$$
$$N_S(10^8) - N_S(9 \times 10^7) = 2\,632\,758 - 2\,353\,482 = 279\,276\,,$$

and, for $m = 8$,

$$N_S(2 \times 10^8) - N_S(10^8) = 5\,029\,891 - 2\,632\,758 = 2\,397\,133\,,$$
$$N_S(10^9) - N_S(9 \times 10^8) = 25\,154\,060 - 22\,497\,704 = 2656356\,.$$

This behaviour of the distribution of Smith numbers is different from that of prime numbers, which shows a falling trend. So it is likely that, for large x, $N_S(x)$ may exceed $\pi(x)$ or, in other words, the ratio r may be less than 1. We state this as the following conjecture.

Conjecture *There exists a number x such that $N_S(x) = \pi(x)$ and also $N_S(10^m) > \pi(10^m)$ for a value of m such that $10^m > x$.*

It may be an interesting study to find the smallest value of x (which may be quite large, more than 10^{18}) for which $N_S(x) = \pi(x)$.

3. Highly decomposable Smith numbers

Let the number of prime factors (counting multiplicity) of a number n be denoted by $\Omega(n)$. Then n is termed a *highly decomposable number* if, for every $n_1 < n$, $\Omega(n_1) < \Omega(n)$. It is obvious that the ith highly decomposable number is 2^i, where $i = 1, 2, 3, \ldots$. Similarly, a Smith number N is called a *highly decomposable Smith number* if $\Omega(N_1) < \Omega(N)$ for every Smith number $N_1 < N$. In other words, a Smith number which sets a record for the number of prime factors among the smaller Smith numbers is a highly decomposable Smith number. For $n = 2, 3, 4, \ldots,$ the smallest Smith number with n prime factors (not necessarily distinct) is tabulated in table 2. It was found that the maximum number of prime factors of a Smith number up to 10^9 is 24. Table 2 gives us the first entries in the sequence of highly decomposable Smith numbers. It may be noted that, except for the number 27, all other highly decomposable Smith numbers given are even. It may be interesting to find a counterexample or to prove that this is indeed the case.

4. Fibonacci Smith numbers

The Fibonacci numbers which are also Smith numbers can be termed Fibonacci Smith numbers. During computations of Smith numbers, it was found that the smallest Fibonacci Smith number is

$$F_{31} = 1\,346\,269 = 557 \times 2417\,.$$

Table 2

n	Smallest Smith number with n prime factors
2	4
3	27
4	636
5	378
6	729
7	648
8	576
9	2 688
10	17 496
11	44 928
12	75 776
13	168 960
14	765 952
15	319 488
16	958 464
17	5 537 792
18	5 963 776
19	2 883 584
20	5 767 168
21	7 077 888
22	279 969 792
23	544 997 376
24	778 567 680

Two more Fibonacci Smith numbers have been found. These are

$$\begin{aligned} F_{77} &= 5\,527\,939\,700\,884\,757 \\ &= 13 \times 89 \times 988\,681 \times 4\,832\,521\,, \\ F_{231} &= 844\,617\,150\,046\,923\,109\,759\,866\,426\,342\,507\,997\,914\,076\,076\,194 \\ &= 2 \times 13 \times 89 \times 421 \times 19\,801 \times 988\,681 \times 4\,832\,521 \\ &\quad \times 9\,164\,259\,601\,748\,159\,235\,188\,401\,. \end{aligned}$$

It may be an interesting study to find more Fibonacci Smith numbers.

References

1 A. Wilansky, Smith numbers, *Two-Year College Math. J.* **13** (1982), p. 21.
2 W. L. McDaniel, The existence of infinitely many k-Smith numbers, *Fibonacci Quart.* **25** (1987), pp. 76–80.
3 S. Yates, Smith numbers congruent to 4 (mod 9), *J. Recreational Math.* **19** (1987), pp. 139–141.
4 U. Dudley, Smith numbers *Math. Magazine* **67** (1994), pp. 62–65.

The author is Chief Engineer of Railways in Kolkata, India. His main hobby is the theory of numbers and many of his contributions have been published and are also available on the Internet (at http://www.shyamsundergupta.com/).

Mathematics in the Classroom

Game, set and match

As I write, the media is in a state of euphoria as a result of the progression of Tim Henman to the quarter finals of the French Open tennis tournament. It seems that it is over thirty years since a British male has gone so far in this tournament and the occurrence of what appears to be a rare event is certainly a cause for much celebration. It also augurs well for British prospects in the forthcoming tennis championship to be played at Wimbledon in late June when much of the nation is totally focused for two weeks on the outcome of racquet on ball, with hope springing eternal that another rare event may occur, i.e. a British player might win!

So it was with tennis in mind that I picked up a newly-acquired book, *Taking Chances: Winning with Probability* and found myself immediately drawn to a section entitled 'Anyone for tennis?'. Here ideas of probability are used to find the relationship between a player winning a point and the player winning a game. Remember that in tennis a game is won by the first player who wins four points and is ahead by at least two points. To add to the confusion, you need to be aware that it is also possible for a set, or indeed a match, to be won by the player who wins fewer points.

Points and games

Haigh argues convincingly for the relationship between p, the probability that player A wins a point, and G, the probability that player A wins the game. He does this by considering the probability, D, that A wins the game from deuce, considering all possibilities from deuce: A wins both of the next two points (probability p^2) and hence wins the game; A loses both of the next two points (probability q^2 where $q = 1 - p$) and hence loses the game; or A wins one of the next two points, loses the other (with probability $2pq$) and is hence back to deuce. This leads to the recurrence relation

$$D = p^2 + 2pqD,$$

which gives

$$D = \frac{p^2}{1-2pq} = \frac{p^2}{p^2+q^2}$$

as $(p+q)^2 = 1$.

Listing all the ways in which A can win the game, we see that

1. A wins four consecutive points with probability p^4;
2. A wins three of the first four points and then the fifth with probability $\mathrm{C}_3^4 p^4 q$;
3. A wins three of the first five points and then the sixth with probability $\mathrm{C}_3^5 p^4 q^2$;
4. A wins three of the first six points and then wins from deuce with probability $\mathrm{C}_3^6 p^3 q^3 D$.

Combining these four possibilities, it is left to the reader to show that

$$G = p^4 + 4p^4q + 10p^4q^2 + 20p^3q^3D,$$

which can be simplified to

$$G = \frac{p^4 - 16p^4q^4}{p^4 - q^4}, \qquad p \neq q.$$

Trial values of p will show how G increases as p increases. For example, if $p = 0.6$, then $G = 0.74$ and, if $p = 0.8$, then $G = 0.98$. Haigh points out that data from the 1997 Wimbledon Championship showed that on service Henman won 75% of points and 22 service games out of 23, which supports the formula well. You may like to run a check on it using this summer's data when they become available.

Clearly the formula cannot take the value $p = 0.5$, and symmetry suggests that in this case $G = 0.5$ also. To see how G behaves close to $p = 0.5$, p is written as $0.5 + \delta p$, where δp is small. Again it is left to the reader to show that

$$G = 0.5 + \frac{5\delta p}{2} - \cdots ,$$

where terms of order $(\delta p)^3$ and higher have been omitted as negligible. As Haigh points out, this means that any small improvement on a p value that is close to 0.5 will yield an improvement of 2.5 times that amount in the probability of winning the game.

Games and sets

To consider this situation, we need to have two probabilities of winning a point, namely s if the player is serving and r if the player is receiving. The tie-break rule makes this rather complicated, but if we assume no tie-break rule and that the winner of a set is the first player to win at least six games and be at least two ahead of the opponent, then Haigh tells us that the probabilities of winning a set are dependent on r and s as set out in table 1.

Table 1

	r				
s	0.5	0.4	0.3	0.2	0.1
0.5	0.500	0.354	0.221	0.113	0.038
0.6	0.646	0.500	0.347	0.200	0.079
0.7	0.779	0.653	0.500	0.327	0.150
0.8	0.887	0.800	0.673	0.500	0.275
0.9	0.962	0.921	0.850	0.725	0.500

Not surprisingly, the average of r and s is a key value: if this average exceeds 0.5, then the probability of winning a set will also be in excess of 0.5.

Extensions

All this and much more is described in Haigh's book, where the probabilities involved in other sporting activities such as darts, snooker and squash-type games are explored. Similarly, probabilities relating to strategies in TV games such as 'Who Wants to be a Millionaire?', 'The Weakest Link' and 'Blockbusters' are discussed in a very accessible and interesting way.

This is only a small taster of the book, but enough perhaps to convince you that it is a very worthwhile read.

Carol Nixon

Taking Chances: Winning with Probability. By John Haigh. Oxford University Press, 2003. Paperback, 388 pages, £9.99 (ISBN 0-19-852663-6).

Computer Column

Quantum computers

Last time, we examined one vision of the future of computing — DNA computers — and found them to be something of a stop-gap solution. This time, we are going to delve instead into the mysterious world of quantum computers. As computers become ever smaller, quantum effects become ever more important; for traditional computers, this is a problem to be fought against. Quantum computers, by contrast, aim to turn this to our advantage.

In the quantum world, nothing is certain; rather than allowing us to predict exactly what will happen, the laws of quantum mechanics can only assign probabilities. It is tempting simply to write this off as being an artefact of our own ignorance, and hope that a better theory could give us exact answers, but it turns out that matters are altogether stranger than this: when we're not looking, *all* possibilities happen together, and only resolve into a definite outcome when we decide to take a look.

The most famous illustration of this is a thought experiment due to Erwin Schrödinger, in which a cat is sealed in a box, together with a phial of poison gas attached to a probabilistic release mechanism. Common sense tells us that, if we wait a while before opening the box, there's a chance that the mechanism will have activated (killing the cat), and a chance that it won't have. Normally, we work with probabilities because we don't have complete knowledge of the apparatus involved — in this case, of the workings of the release mechanism. If we had a good model of the mechanism, we would expect to be able to say for certain whether the cat would be alive or dead when we opened the box.

Common sense, however, isn't always right. Quantum mechanics tells us that, far from being down simply to our lack of knowledge, probabilities are a basic part of the description of quantum systems. It no longer even makes sense to say that the system *is* in one state or another, despite our not knowing what that state is: all possibilities exist simultaneously until we choose to look. Schrödinger's mechanism actually contained a radioactive atom, and was set to release the gas if the atom decayed — an event which is intrinsically unpredictable. According to the laws of quantum mechanics, until we open the box and take a look, the atom will have *both* decayed and not decayed, resulting in the cat being *both* dead and alive — a state known as a *superposition*.

It is difficult to know which aspect of this experiment is the most curious: the idea of the cat being both dead and alive, or that the simple act of looking at it should change matters. Yet, versions of this experiment have been carried out (not involving cats!) and all agree that these strange superpositional states do indeed exist. Some interpretations of the theory even go further, and argue that the superposition does not go away when we look in the box: we simply join in as well. The result of the experiment is then a superposition of the state where we find a dead cat and the state where we find a live cat.

If this all seems a bit hard to swallow, don't worry: even Richard Feynman quipped that, 'I think I can safely say that nobody understands quantum mechanics.' However, it's all true (so far as we know), and without it we couldn't have quantum computers.

There are two basic tricks that quantum computers can pull that conventional computers cannot. The first is to take advantage of the fact that it can be in a superposition of a large number of different states at once, like Schrödinger's cat, and that manipulating the computer as a whole is equivalent to manipulating each of the individual states *in parallel*. For example, if you wanted to factorise a number, you could get a quantum computer to do it by simultaneously trying all possible divisors. Unfortunately, though, this isn't quite as magical as it seems: what we would get out at the end would be a superposition of all possible answers, and just looking at it would cause the computer to fall into one of the states at random. If we set it the task of factorising 24, for example, it would be as likely to fall into the state 'tried dividing by 7, failed' as to fall into the state 'tried dividing by 6, succeeded'. The end result would be no more useful than picking a random number and trying to divide 24 by it, hoping that we've guessed right. This sort of problem dogs most naïve algorithms: it is easy to make a quantum computer do an apparently unlimited amount of work, but very hard to get it to spit the answer out at the end.

Quantum computers would still not be very useful, therefore, if it wasn't for their second party piece: under the right conditions, different states in a superposition can *interfere* with each other, like waves. When two water waves arrive at the same position, they can either reinforce each other or cancel out (if the peak of one wave meets the trough of the other), and the same is true of quantum states. (For this reason, quantum states are often described as 'probability waves'.) With care, this interference can be used to filter out the right answer from all the possibilities, but it's not easy! In 1982, Feynman showed that quantum computers could in principle be more powerful than conventional ones — because they would be more efficient at performing simulations of quantum systems — but it took until 1996 for anyone to come up with an algorithm which could exploit the fact.

The first useful quantum algorithm, designed by Lov Grover, answers the following question: given a list of N unsorted items, is a particular item in the list? Classically, there's nothing for it but to start at the beginning of the list and work your way through until you either find the item in question or reach the end of the list, taking a maximum of N steps. Grover's quantum algorithm, by contrast, could do the job in $\sqrt{N}$ steps! To see how this works, it is easier to look at a simpler algorithm, designed by David Deutsch.

Deutsch's algorithm considers just about the simplest possible function: one which takes a binary input and gives a binary output. There are only four possible functions: two 'constant' functions ($f(0) = 0,\ f(1) = 0$ and $f(0) = 1,\ f(1) = 1$) and two 'balanced' functions ($f(0) = 1, f(1) = 0$ and $f(0) = 0, f(1) = 1$). Classically, finding out whether such a function was constant or balanced would require two function evaluations, ($f(0)$ and $f(1)$), but Deutsch was able to show that a quantum computer could do them both simultaneously. He even designed an experimentally feasible setup to demonstrate the idea, shown in figure 1.

In Deutsch's experiment, a laser beam enters at input 1, and is split in two by a half-silvered mirror. The phases of the waves in each half of the beam are then shifted, in one case by an amount proportional to $f(0)$ and in the other by an amount proportional to $f(1)$. After that, the two halves are recombined by a second half-silvered mirror.

The trick here is that the two beams interfere at the second half-silvered mirror in such a way that the beam only reaches one of the outputs (depending on whether f is balanced or constant). This seems like a bit of a cheat (since there are two phase shifters, each recording the result of one function evaluation). However, quantum mechanics tells us that the experiment could equally well be performed using just a single photon of light (or even a single electron), with the particle genuinely 'going both ways' through the experiment — its state would temporarily be a superposition of 'in the left branch' and 'in the right branch'.

The point is that, with care, interference can allow us to combine the results of simultaneous calculations in such a way that we still get a useful answer out. Here, the interference at the second half-silvered mirror destroyed the information that the two beams had gathered

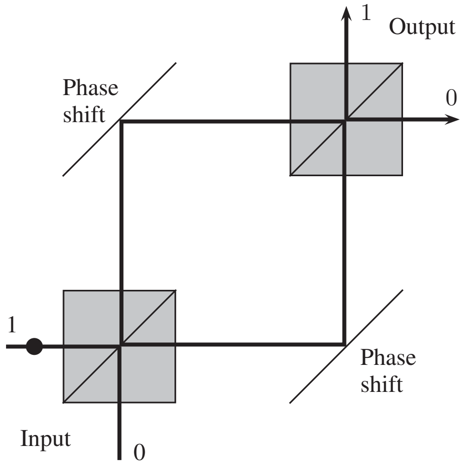


Figure 1 Deutsch's experiment.

about the values of $f(0)$ and $f(1)$ individually, but replaced it with information about a more global property of the function (whether it was constant or balanced). This is where quantum computers can shine: they can be very good at determining global properties of a problem. In Grover's algorithm, interference is used in a very similar way to pull out the global information about whether or not the given item is in the list.

This has been a bit of a whistle-stop tour, but I hope I have been able to give you some kind of a flavour for how strange the quantum world really is. Although quantum computers cannot perform miracles (as it might seem at first sight), they can potentially do much that conventional computers could never manage. Actually building a working model is proving to be a real challenge — superpositions are very delicate, and collapse incredibly easily — but the first, tiny computers have already been built by companies such as IBM. More powerful algorithms than Grover's have also been found, such as one by Peter Shor, which can efficiently factorise large numbers (potentially allowing the RSA codes used in internet transactions to be broken). Quantum computers are almost certainly coming, but it may be a while before you find one on your desktop!

Peter Mattsson

Perfect squares

Start with a number and add its digits. Then do the same with the resulting number and so on. If the number you start with is a perfect square, you will end up with 1, 4, 7 or 9. Try it with 254 016, 254 765 and 255 025. Two of these are perfect squares and one isn't; which is not the perfect square? Can you prove this property of a perfect square?

Shyam Lal College (E)
Delhi University

Vinod Tyagi

Letters to the Editor

Dear Editor

The Fibonacci sequence

Referring to the letter of A. G. Summers in Volume 35, Number 3, I think that I can provide a proof for his results (i) and (ii), which can be rewritten in the following form.

The Fibonacci sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$. Let p be prime.

(a) If p is of the form $5k \pm 1$ (last digit 1 or 9), then

$$p \mid u_p - 1 \quad \text{and} \quad p \mid u_{p-1}\,.$$

(b) If p is of the form $5k \pm 2$ (last digit 3 or 7), then

$$p \mid u_p + 1 \quad \text{and} \quad p \mid u_{p+1}\,.$$

For the proof, I will use the theory of the congruences. I recall just one relation: if a and b are any two integers and p is a prime, then

$$(a \pm b)^p \equiv a^p \pm b^p \pmod{p}\,. \tag{1}$$

In addition, I will use the theory of quadratic residues. I recall some definitions and some relations whose proofs can be found in any book on number theory.

For all a such that $gcd(a, m) = 1$, we call a a *quadratic residue modulo* m if there exists an integer x such that $x^2 \equiv a \pmod{m}$; if no such x exists, then a is called a *quadratic nonresidue*.

Introduce the Legendre symbol $(\frac{a}{p})$. If p is an odd prime, then $(\frac{a}{p}) = 1$ if a is a quadratic residue modulo p, $(\frac{a}{p}) = -1$ if a is a quadratic nonresidue modulo p and $(\frac{a}{p}) = 0$ if p divides a.

Euler's criterion states that

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}\,. \tag{2}$$

The Gauss reciprocity theorem states that, if p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{[(p-1)/2][(q-1)/2]}\,. \tag{3}$$

Finally, I recall the Binet formulae for the Fibonacci and Lucas numbers. If $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then the nth Fibonacci number is

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{4}$$

and the nth Lucas number is

$$l_n = \alpha^n + \beta^n\,. \tag{5}$$

It is well known that

$$l_n = u_{n-1} + u_{n+1}\,. \tag{6}$$

And now the proof. Let p be an odd prime other than 5. Then, from (4) and (1),

$$\begin{aligned} u_p &= \frac{\alpha^p - \beta^p}{\alpha - \beta} \\ &\equiv \frac{(\alpha-\beta)^p}{\alpha-\beta} = (\alpha-\beta)^{p-1} = 5^{(p-1)/2} \pmod{p}\,, \end{aligned}$$

whence, from (2),

$$u_p \equiv \left(\frac{5}{p}\right) \pmod{p}\,. \tag{7}$$

To evaluate $(\frac{5}{p})$, from (3) we have

$$\left(\frac{5}{p}\right)\left(\frac{p}{5}\right) = (-1)^{p-1} = 1\,,$$

so $(\frac{5}{p})$ and $(\frac{p}{5})$ are either both 1 or both –1.

By (2), $(\frac{p}{5}) = p^2$ and there are two cases:

(a) If p is of the form $p = 5k \pm 1$, then p^2 is of the form $5h + 1$ and $p^2 \equiv 1 \pmod{5}$, so $(\frac{5}{p}) \equiv 1$ and, from (7),

$$u_p \equiv 1 \pmod{p} \quad \text{or} \quad p \mid u_p - 1\,. \tag{8}$$

From (5) and (1), we have

$$l_p = \alpha^p + \beta^p \equiv (\alpha+\beta)^p = 1 \pmod{p} \tag{9}$$

and, from (6),

$$l_p = u_{p-1} + u_{p+1} = 2u_{p-1} + u_p \equiv 1 \pmod{p}\,. \tag{10}$$

Subtracting (8) from (10) and dividing by 2, we have

$$u_{p-1} \equiv 0 \pmod{p} \quad \text{or} \quad p \mid u_{p-1}\,. \tag{11}$$

(b) If p is of the form $p = 5k \pm 2$, then p^2 is of the form $5h + 4$ and $p^2 \equiv -1 \pmod{5}$, so $(\frac{5}{p}) \equiv -1$ and, from (7),

$$u_p \equiv -1 \pmod{p} \quad \text{or} \quad p \mid u_p + 1\,. \tag{12}$$

From (9) and (6),

$$l_p = u_{p-1} + u_{p+1} = 2u_{p+1} - u_p \equiv 1 \pmod{p}\,. \tag{13}$$

Adding (12) and (13) and dividing by 2, we have

$$u_{p+1} \equiv 0 \pmod{p} \quad \text{or} \quad p \mid u_{p+1}\,.$$

This completes the proof.

Yours sincerely

Gian Paolo Almirante
(Via Forze Armate 260/7
20152 Milano
Italy)

Dear Editor

Three squares in arithmetic progression

Recent interest in the ladders problem in *Mathematical Spectrum* (see references 1 and 2) prompted the following observation. For positive integers $m_1 > n_1$, $m_2 > n_2$, put

$$\begin{aligned} u &= (m_1^2 + n_1^2)(m_2^2 - n_2^2), & v &= (m_1^2 - n_1^2)(m_2^2 + n_2^2), \\ w &= 2m_1n_1(m_2^2 + n_2^2), & x &= 2m_2n_2(m_1^2 + n_1^2), \\ y &= 2m_1n_1(m_2^2 - n_2^2), & z &= 2m_2n_2(m_1^2 - n_1^2). \end{aligned}$$

Then

$$\begin{aligned} u^2 + x^2 &= (m_1^2 + n_1^2)^2(m_2^2 + n_2^2)^2 = v^2 + w^2, \\ u^2 - y^2 &= (m_1^2 - n_1^2)^2(m_2^2 - n_2^2)^2 = v^2 - z^2, \end{aligned}$$

and

$$u^2 - v^2 = w^2 - x^2 = y^2 - z^2.$$

For example, $m_1 = 3$, $n_1 = 2$, $m_2 = 2$, $n_2 = 1$ give

$$39^2 - 25^2 = 60^2 - 52^2 = 36^2 - 20^2.$$

These suggested formulae give three perfect squares which are in arithmetic progression. For positive integers m and n with $m > n$, put

$$\begin{aligned} x &= 2mn - m^2 + n^2, \\ y &= m^2 + n^2, \\ z &= 2mn + m^2 - n^2. \end{aligned}$$

Then

$$y^2 - x^2 = 4mn(m^2 - n^2) = z^2 - y^2,$$

so that x^2, y^2, z^2 are in arithmetic progression. For example, $m = 2$ and $n = 1$ give 1^2, 5^2, 7^2 and $m = 3$ and $n = 2$ give 7^2, 13^2, 17^2 in arithmetic progression. I do not know whether all triples of perfect squares which are in arithmetic progression can be obtained in this way.

References

1 K. R. S. Sastry, Heron ladders, *Math. Spectrum* **34** (2001/2002), pp. 61–64.

2 Problem 33.2, The ladder problem, *Math. Spectrum* **33** (2000/2001), pp. 67–68.

Yours sincerely

Muneer Jebreel

(SS-Math-Hebron UNRWA
Field Education Officer
PO Box 19149
Jerusalem
Israel)

Dear Editor

Reversing the roles of exp *and* ln

It is usual to define

$$e^x = \lim_{n\to\infty}\Big(1+\frac{x}{n}\Big)^n \tag{1}$$

and to define $\ln y$ for $y > 0$ by

$$\ln y = x \quad \text{if } y = e^x .$$

Can these roles be reversed, so that we define $\ln y$ for $y > 0$ as a limit and then define e^x by $y = e^x$ if $x = \ln y$?

From (1),

$$y = \lim_{n\to\infty}\Big(1+\frac{\ln y}{n}\Big)^n \quad \text{for } y > 0 . \tag{2}$$

Thus, naïvely, for large n,

$$y \approx \Big(1+\frac{\ln y}{n}\Big)^n \quad \text{for } y > 0$$

so

$$\ln y \approx n(y^{1/n}-1) \quad \text{for } y > 0 .$$

We shall therefore prove that, for $y > 0$,

$$\ln y = \lim_{n\to\infty} n(y^{1/n}-1) .$$

Let $0 < \varepsilon' < y/2$. From (2) and since $\lim_{n\to\infty} y^{1/n} = 1$, there exists an n_0 such that, for $n > n_0$,

$$y^{1/n} < 2 \quad \text{and} \quad y-\varepsilon' < \Big(1+\frac{\ln y}{n}\Big)^n < y+\varepsilon' .$$

Thus, for $n > n_0$,

$$n[(y-\varepsilon')^{1/n}-1] < \ln y < n[(y+\varepsilon')^{1/n}-1] . \tag{3}$$

Now,

$$\begin{aligned}
&n[(y+\varepsilon')^{1/n}-1] - n[(y-\varepsilon')^{1/n}-1]\\
&\quad = ny^{1/n}\Big[\Big(1+\frac{\varepsilon'}{y}\Big)^{1/n} - \Big(1-\frac{\varepsilon'}{y}\Big)^{1/n}\Big]\\
&\quad = 2ny^{1/n}\Big[\frac{1}{n}\frac{\varepsilon'}{y} + \frac{(1/n)(1/n-1)(1/n-2)}{3!}\Big(\frac{\varepsilon'}{y}\Big)^3 + \cdots\Big]\\
&\quad \le 2y^{1/n}\Big(\frac{\varepsilon'}{y}\Big)\Big[1 + \frac{1\times 2}{3!}\Big(\frac{\varepsilon'}{y}\Big)^2 + \frac{1\times 2\times 3\times 4}{5!}\Big(\frac{\varepsilon'}{y}\Big)^4 + \cdots\Big]\\
&\quad \le 2y^{1/n}\Big(\frac{\varepsilon'}{y}\Big)\Big[1 + \Big(\frac{\varepsilon'}{y}\Big)^2 + \Big(\frac{\varepsilon'}{y}\Big)^4 + \cdots\Big]\\
&\quad = 2y^{1/n}\Big(\frac{\varepsilon'}{y}\Big)\frac{1}{1-(\varepsilon'/y)^2}\\
&\quad < 2\times 2\frac{\varepsilon'}{y}\frac{1}{1-\frac{1}{4}} = \frac{16\varepsilon'}{3y} .
\end{aligned}$$

Now let $\varepsilon > 0$ and put $\varepsilon' = \min\{3y\varepsilon/16, y/4\}$ to ensure that $16\varepsilon'/3y < \varepsilon$ and $\varepsilon' < y/2$. Then

$$n[(y+\varepsilon')^{1/n} - 1] - n[(y-\varepsilon')^{1/n} - 1] < \varepsilon .$$

Also, (3) holds and

$$n[(y-\varepsilon')^{1/n} - 1] < n[y^{1/n} - 1] < n[(y+\varepsilon')^{1/n} - 1] ,$$

so

$$|(ny - n(y^{1/n} - 1)| < \varepsilon \quad \text{for } n > n_0 .$$

Hence,

$$\ln y = \lim_{n\to\infty} n(y^{1/n} - 1) .$$

Yours sincerely
Chan Wei Min
(404 Pandan Gardens
12–24
Singapore 600404)

Dear Editor

Proving a mathematical identity by using physics

Mathematics is an indispensable tool in describing physical principles, but in some cases the reverse is also true: occasionally, by using the principles of physics, we can establish a mathematical identity.

For instance, consider the time taken by a projectile to hit an inclined plane (see figure 1). Using well-known principles of dynamics, we have

$$r\sin\beta = (u\sin\alpha)t - \tfrac{1}{2}gt^2 ,$$
$$(u\cos\alpha)t = r\cos\beta .$$

If we eliminate r, we obtain

$$t = \frac{2u(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{g\cos\beta} .$$

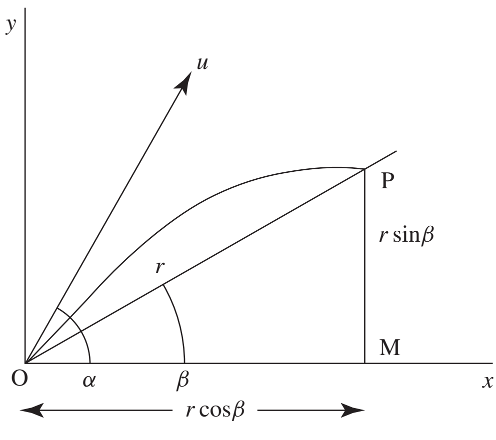


Figure 1

Now, if we look at it in another way, we note that the distance described in the same period of time perpendicular to the plan OP is zero. Since the initial component of velocity perpendicular to the plane is $u\sin(\alpha-\beta)$, and the acceleration in this direction is $-g\cos\beta$, by using the basic mechanics equation

$$0 = [u\sin(\alpha-\beta)]t - (\tfrac{1}{2}g\cos\beta)t^2 ,$$

we obtain

$$t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} .$$

On equating the values of t obtained by the two different methods, we obtain the well-known identity

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\,.$$

Yours sincerely

M. A. Khan
(C/o A. A. Khan
Manager Regional Office
Indian Overseas Bank
Ashok Marg
Lucknow
India)

Dear Editor

Integration simplified

My students and I enjoyed reading P. Glaister's article 'Integration simplified' in *Mathematical Spectrum*, Volume 35, Number 3. We have considered the problem proposed in that article from a slightly different perspective, and offer the following observations.

Suppose that $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial of degree $n \geq 0$ that satisfies

$$\int_{-1}^{1} p(x)\,\mathrm{d}x = p(1) - p(-1)\,.$$

By the fundamental theorem of calculus,

$$\int_{-1}^{1} p'(x)\,\mathrm{d}x = p(1) - p(-1)\,,$$

and so

$$\int_{-1}^{1} \{p(x) - p'(x)\}\,\mathrm{d}x = 0\,.$$

Writing out the derivative, substituting, and integrating, we obtain the following formulae:

$$(a_0 - a_1) + \frac{1}{3}(a_2 - 3a_3) + \frac{1}{5}(a_4 - 5a_5) + \cdots + \frac{1}{n-1}\big(a_{n-2} - (n-1)a_{n-1}\big) + \frac{1}{n+1}a_n = 0$$

when n is even, and

$$(a_0 - a_1) + \frac{1}{3}(a_2 - 3a_3) + \frac{1}{5}(a_4 - 5a_5) + \cdots + \frac{1}{n}(a_{n-1} - na_n) = 0$$

when n is odd.

From these equations, it is not difficult to see that the desired polynomials must form an n-dimensional subspace of the $(n+1)$-dimensional space of polynomials of degree n. Moreover, the equations make it a trivial task to list a set of basis polynomials for this subspace.

Yours sincerely

John J. Boncek
(Department of Mathematics and Physics
Troy State University Montgomery
PO Drawer 4419
Montgomery, AL 36103-4419
USA)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

37.1 Solve the sixes and sevens problem in the Editor's column (page 3 of this issue).

37.2 Let $x_i > 0$ for $i = 1, \ldots, n$. Prove that

$$\prod_{i=1}^{n} \frac{1}{n-1} \sum_{j=1}^{n-1} x_{i+j-1} \geq \prod_{i=1}^{n} x_i \,,$$

where $x_{n+i} = x_i$ for $i = 1, 2, \ldots, n-1$.

(Submitted by H. A. Shah Ali, Tehran, Iran)

37.3 Let k, n be integers such that $2 \leq k \leq n$. Determine the number of sequences of k integers $a_1, \ldots, a_k$ with $1 \leq a_1 < a_2 < \cdots < a_k = n$ and such that $a_2 - a_1, a_3 - a_2, \ldots, a_{k-1} - a_{k-2}$ are odd and $a_k - a_{k-1}$ is even.

(Submitted by Farshid Arjomandi, San Diego State University, California)

37.4 Let n be a natural number and α a non-zero real number. Evaluate

$$\lim_{x \to 0} \frac{1 - (\cos x \cos 2x \ldots \cos nx)}{\sin^2 \alpha x} \,.$$

(Submitted by Anand Kumar, Ramanujan School of Mathematics, Patna, India)

Solutions to Problems in Volume 36 Number 2

36.5 P is a point on an ellipse. Prove that every chord of the ellipse parallel to the tangent at P has its midpoint on the line OP, where O is the centre of the ellipse. Do the same for a hyperbola.

Solution by Puyuan Yu, Ampleforth College, York

Set the equation of the ellipse to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,.$$

If we differentiate, we obtain

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \,,$$

so the slope of the tangent at $\mathrm{P}(x_0, y_0)$ is $-b^2x_0/a^2y_0$. Consider a chord parallel to the tangent at P, with equation

$$y = -\frac{b^2x_0}{a^2y_0}x + d\,.$$

This meets OP, with equation $y = (y_0/x_0)x$, when

$$-\frac{b^2x_0}{a^2y_0}x + d = \frac{y_0}{x_0}x\,,$$

that is, when

$$x = \frac{a^2x_0y_0d}{a^2y_0^2 + b^2x_0^2}\,.$$

The chord meets the ellipse when

$$b^2x^2 + a^2\Big(-\frac{b^2x_0}{a^2y_0}x + d\Big)^2 = a^2b^2\,,$$

that is, when

$$\Big(b^2 + \frac{b^4x_0^2}{a^2y_0^2}\Big)x^2 - \frac{2b^2x_0d}{y_0}x + a^2d^2 - a^2b^2 = 0\,.$$

If x_1, x_2 denote the roots of this equation, then

$$\begin{aligned} x_1 + x_2 &= \frac{2b^2x_0d}{y_0}\Big/\Big(b^2 + \frac{b^4x_0^2}{a^2y_0^2}\Big) \\ &= \frac{2a^2x_0y_0d}{a^2y_0^2 + b^2x_0^2}\,, \end{aligned}$$

so the midpoint of the chord has x-coordinate

$$\frac{x_1 + x_2}{2} = \frac{a^2x_0y_0d}{a^2y_0^2 + b^2x_0^2}$$

and so lies on OP.

For the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

the same argument applies; the x-coordinate of the midpoint of the chord is now

$$\frac{a^2x_0y_0d}{a^2y_0^2 - b^2x_0^2}\,.$$

36.6 Find a finite sequence of integers whose sum is 2004 such that the sum of every four consecutive numbers in the sequence is negative.

Solution by Puyuan Yu

Consider a sequence of the form

$$p\,p\,p\,n\,p\,p\,p\,n\,\ldots\,p\,p\,p\,n\,p\,p\,p\,,$$

where p is a positive and n a negative integer and there are m blocks of the form $p\ p\ p\ n$, with three ps at the end. We want

$$3p+n<0$$

and

$$m(3p+n)+3p=2004\,.$$

Try

$$3p+n=-1\,,$$

whence

$$3p-m=2004\,.$$

Let $m=3$. Then $p=669$ and $n=-2008$. This gives the sequence

$$669,\ 669,\ 669,\ -2008\ \text{(three times)},\ 669,\ 669,\ 669\,.$$

36.7 Let $x_1,\ldots,x_n$ $(n\geq 2)$ be real numbers such that $x_1+\cdots+x_n=1$. Determine the maximum value of $\sum_{1\leq i<j\leq n}x_ix_j$ and also when this maximum value is attained.

Solution by H. A. Shah Ali, who proposed the problem

We have

$$\sum_{i=1}^{n}(x_i-x_{i+1})^2\geq 0\,,$$

where $x_{n+1}=x_1$, so that

$$\sum_{i=1}^{n}x_ix_{i+1}\leq\sum_{i=1}^{n}x_i^2\,,$$

with equality if and only if $x_1=\cdots=x_n$.

Let σ be a permutation of $1,\ldots,n$. Then

$$\begin{aligned}x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)}&\leq\sum_{i=1}^{n}x_{\sigma(i)}^2\\&=\sum_{i=1}^{n}x_i^2\,,\end{aligned}$$

and summing these inequalities over all permutations gives

$$\sum_{\sigma}(x_{\sigma(1)}x_{\sigma(2)}+x_{\sigma(2)}x_{\sigma(3)}+\cdots+x_{\sigma(n)}x_{\sigma(1)})\leq n!\sum_{i=1}^{n}x_i^2\,.$$

Consider a pair i, j of integers with $1 \le i < j \le n$. There are $2 \times (n-2)!$ permutations which map $\{1, 2\}$ to $\{i, j\}$, similarly mapping $\{2, 3\}, \ldots, \{n, 1\}$ to $\{i, j\}$. Hence, the inequality can be written

$$2n(n-2)! \sum_{1 \le i < j \le n} x_i x_j \le n! \sum_{i=1}^{n} x_i^2 ,$$

so that

$$\sum_{1 \le i < j \le n} x_i x_j \le \frac{n-1}{2}\left(\left(\sum_{i=1}^{n} x_i\right)^2 - 2 \sum_{1 \le i < j \le n} x_i x_j\right),$$

whence

$$n \sum_{1 \le i < j \le n} x_i x_j \le \frac{n-1}{2} ,$$

or

$$\sum_{1 \le i < j \le n} x_i x_j \le \frac{n-1}{2n} .$$

The maximum value of the expression is $(n-1)/2n$, which is attained precisely when $x_1 = \cdots = x_n = 1/n$.

36.8 The sequence (u_n) has nth term

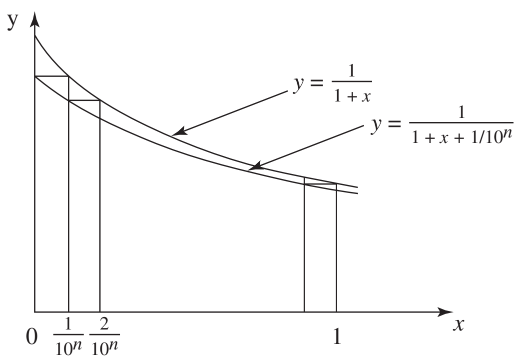
$$u_n = \mathrm{e}^{1/(10^n+1)} \mathrm{e}^{1/(10^n+2)} \ldots \mathrm{e}^{1/(2 \times 10^n)} .$$

What is its limit?

Solution by Bor-Yann Chen, University of California, Irvine

We have

$$\ln u_n = \frac{1}{10^n+1} + \frac{1}{10^n+2} + \cdots + \frac{1}{10^n+10^n} .$$



If we compare areas in the diagram, we have

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{1+x+1/10^n} &< \frac{1}{10^n}\left(\frac{1}{1+1/10^n} + \frac{1}{1+2/10^n} + \cdots + \frac{1}{1+10^n/10^n}\right) \\ &< \int_0^1 \frac{\mathrm{d}x}{1+x} ,\end{aligned}$$

so that

$$\left[\ln\left(1+x+\frac{1}{10^n}\right)\right]_0^1 < \ln u_n < [\ln(1+x)]_0^1\,,$$

that is,

$$\ln\left(2+\frac{1}{10^n}\right)-\ln\left(1+\frac{1}{10^n}\right) < \ln u_n < \ln 2\,.$$

As $n \to \infty$, $\ln(2+1/10^n)-\ln(1+1/10^n) \to \ln 2$, so that $\ln u_n \to \ln 2$, whence $u_n \to 2$.

For a solution using Euler's constant (see the Editor's column in Volume 36, Number 2),

$$\begin{aligned}\ln u_n &= \frac{1}{10^n+1}+\frac{1}{10^n+2}+\cdots+\frac{1}{2\times 10^n}\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2\times 10^n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10^n}\right),\end{aligned}$$

which we can write as

$$\left(\ln(2\times 10^n)+\gamma_{2\times 10^n}\right)-\left(\ln(10^n)+\gamma_{10^n}\right)=\ln 2+\gamma_{2\times 10^n}-\gamma_{10^n}\,.$$

Now $\gamma_{2\times 10^n}$ and γ_{10^n} both tend to Euler's constant γ as $n \to \infty$, so that $\ln u_n \to \ln 2$ as $n \to \infty$, whence $u_n \to 2$ as $n \to \infty$.

Stuart Hall (University of Warwick) also solved the problem, using measure theory.

Reviews

How to Enjoy Calculus. By Eli S. Pine. Universities Press, Hyderabad, 2002. Paperback, 137 pages, £14.95 (ISBN 8-17371-406-1).

Does the title sound a little cheesy? Well, at least Professor Pine does not put an exclamation mark at the end of it. I have to say his breezy style did win me round by the end of this zappy book. The opening lines give a flavour: 'Calculus, first of all, is wrongly named. A far truer and more meaningful name is SLOPE-FINDING, and that is what I shall call it.' (He adds AREA-FINDING to this later.)

I think he has a fair point, and throughout there is an open, frank intent to engage the reader and to debunk the subject of jargon. Sometimes he will use a sentence such as: 'The denominator, being very small, acts like Superman — it really wallops the numerator with a tremendous kick.' You will either find this irritating, or (like me) rather refreshing.

What of the mathematics? Well, to my eye, it is consistently explained well. If you want to start from scratch with this subject, you could do far worse than to adopt Professor Pine as a mentor. Sometimes I found him to be over-detailed (are five separate graphs necessary to find the gradient at $(1, 1)$ on $y = x$?), yet elsewhere not detailed enough (the quotient rule for differentiating is plucked out of thin air with no explanation). I felt too that the layout of the book was rather antiquated; it looks like a Durrell textbook some of the time. But the exposition doesn't really suffer as a result.

I was delighted by the examples chosen, each compelling in its simplicity. An isosceles triangle has two sides of length 10 cm, what is the largest area possible? What is the largest cone that can be inscribed in a sphere? The examples are never completely straightforward, but at the same time they are something that you might well ask out of curiosity.

Professor Pine hints at a denouement to the book: 'In having the ability to find these expressions for area and slope, something new and unexpected will happen. It will be just as startling and unexpected as a caterpillar turning into a butterfly.' I think by this he means the differential equation, which he bills as 'the Wedding of Area-Finding and Slope-Finding'. Again, excellent examples are chosen, on cooling and on light absorption. At this point, Professor Pine becomes almost prophetic: 'One day, not far from now, we will find the secret of eternal life, no small thanks to these very methods of analysis.' By this time, the man has become so likeable that you actually believe him.

Paston College, Norfolk **Jonny Griffiths**

Changing Core Mathematics. Edited by Chris Arney and Donald Small. MAA, Washington, DC, 2002. Paperback, 220 pages, $28.95 (ISBN 0-88385-172-5).

This collection of articles and papers aims to examine how the construction of the curriculum of the first two years of undergraduate mathematics in the USA should change over the next five to ten years. There is a lengthy introduction from the editors, followed by twenty-one papers, written by a range of engineers, physicists, and mathematicians from across America. As you might expect with the range of different disciplines represented, how to integrate mathematics into varied subject areas is a major concern. Throughout the book, calculus gets special attention in this regard, with careful consideration of what is required here and how best to teach it. There are also other areas of especial focus, including the technology now available for teaching and learning maths, its implications and its drawbacks. Instructional techniques are also studied, with some attempt to come up with the best tools for the various jobs involved.

In their opening pages, the editors offer an interesting historical overview of undergraduate maths teaching in America. They also go a long way to answering the aims of the project by publishing a possible framework for the years in question, which goes into considerable detail. Later writers vary in accessibility: some offer practical examples on diffraction and on using capacitor and resistor data, but I have to say that this book is often quite a hard read. It will only be the most ardent curriculum reformers who will claim to actually enjoy all of these pages, I fear. The text is enlivened, however, by some pithy quotations, my favourite being the following:

> 'Discovery consists of seeing what everybody has seen, and thinking what nobody has thought' (Albert Szent-Gyorgyi).

Paston College, Norfolk **Jonny Griffiths**

Conjecture and Proof. By Miklós Laczkovich. MAA, Washington, DC, 2001. Paperback, 140 pages, $24.95 (ISBN 0-88385-722-7).

This volume is based on a course of lectures given in Budapest for visiting North American students. It is to be hoped that the students did not cross the pond hoping for a holiday! The results are described as 'easily accessible', and the blurb wisely points out that 'easily accessible does not mean elementary'. The content is well above school level. As the title suggests, the book presents proofs of some of the seminal results in mathematics.

Part I is entitled 'Proofs of Impossibilities, Proofs of Non-existence', and proves such things as the irrationality of e and π and the impossibility of trisecting an arbitrary angle using a straight edge and compass. Part II is entitled 'Constructions, Proofs of Existence', starting disarmingly with the pigeonhole principle and ending with Russell's paradox and Borel sets.

The proofs are well explained, and I can imagine a keen student being enormously stimulated by attending the course in Budapest. The same effect could come by studying this volume, which is subtitled 'Classroom Resource Materials'. But teacher and students alike had better be prepared for hard thinking!

University of Sheffield **David Sharpe**

Fractals, Graphics, & Mathematics Education. Edited by Benoit Mandelbrot and Michael Frame. MAA, Washington, DC, 2002. Paperback, 206 pages, $39.95 (ISBN 0-88385-169-5).

The heart of this book is a collection of twelve case studies describing experiences of using fractals in a variety of classroom situations, ranging in age from early high school to undergraduate, and in format from public lectures, master classes, summer schools, lecture courses to seminars and individual projects. These contributions are 'topped and tailed' with six essays by the editors, together with an extremely helpful bibliography of books, articles, videos and websites.

Since Mandelbrot first peered at his eponymous, and now iconic, set in 1979, an enormous amount of work on fractals has taken place in areas ranging from the most abstract measure theory to the most concrete description of scales of 'roughness' in various applications. In his essays, Mandelbrot passionately makes the case for using fractals in the classroom: they are fun for a wide range of age groups and can 'hook' students into mathematics; they are visually appealing, unexpectedly complex and often counter-intuitive; they promote discovery learning; and they make us all learners in an area where it is often easier to raise interesting questions than answer them. Moreover, such 'experimental mathematics' conveys an accurate flavour of raw rather than polished mathematics, as one contributor puts it on page 48: '… our courses have shown to students that mathematics did not stop with the development of calculus in the 17th century but, like Professor Mandelbrot, is "alive and well", even today.' Most of the subject matter discussed has a familiar ring (for example, paper-folding and dragon curves, Sierpiński's gasket and the chaos game, Koch's snowflake, logistical growth with the bifurcation diagram and Feigenbaum's constant, the lakes of Wada, Newton's method, Julia and Mandelbrot sets), but these are personalised in the context of individual teaching experiences, including some practical advice on sources of material, classroom set-up, course outlines and even, in a few places, specific lesson plans.

Perhaps inevitably in a book which started life as a set of conference proceedings by fractal enthusiasts, the tone at times verges on the polemical, and the contents are rather bitty in character. But overall, it is a useful compendium of ideas and inspiration for anyone contemplating giving an introductory course (or even a one-off talk) on fractals, especially at the sixth form or early undergraduate level.

Tonbridge School, Kent **Nick Lord**

Other books received

Discrete Mathematics for Computing. By Peter Grossman. Palgrave Macmillan, Basingstoke, 2002. Paperback, 301 pages, £19.99 (ISBN 0-333-98111-1).

Mathematical Spectrum

2004/2005 Volume 37 Number 1

ISSN 0025-5653

Published by the Applied Probability Trust

Printed by MFK Chiltern Press Ltd, Bicester, Oxfordshire, UK