Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

New World Record for π

French programmer Fabrice Bellard has claimed a new world record by calculating π to 2 700 billion decimal places. The previous record of 2 577 billion was held by Daisuke Takahashi of the University of Tsukuba in Japan in August 2009. Whereas Takahashi used a supercomputer with 94.2 teraflops of processing speed, Bellard employed a desktop costing less than £2000. The calculations took Mr Bellard's machine 131 days to complete compared to Takahashi's 29 days. Mr Bellard estimates that his method is 20 times more efficient that Takahashi's. It takes up more than 1 000 gigabytes of hard drive. Downloading it would take 10 days and reciting it would take 49 000 years. Mr Bellard is a digital television software engineer in Paris, and admits he is 'not especially interested in the digits of π'. The calculation was about the programming challenge: 'Arbitrary precision computation is one of my hobbies'.

The record for memorizing the expansion of π is currently held by a Chinese student, Lu Chao, who recited the first 67 890 digits in 2005. Why can't he learn the 67 891st?!

Reference

1 C. Smyth, Pi, a mathematical story that would take 49,000 years to tell, *The Times*, 7 January 2010.

Numerical coincidences

$$
\begin{aligned}
660 &= 6! - 60,\\
1395 &= 15 \times 93,\\
145 &= 1! + 4! + 5!,\\
407 &= 4^3 + 0^3 + 7^3,\\
144 &= (1+4)! + 4! = (1+\sqrt{4})! \times 4!,\\
387\,420\,489 &= 3^{87+420-489},\\
41\,096 \times 83 &= 3\,410\,968,\\
8 \times 86 &= 688,\\
111_{16} &= 333_9,\\
222_{67} &= \overline{14}\,\overline{14}\,\overline{14}_{25},\\
\overline{10}\,\overline{10}\,44_{(13)} &= \overline{11}\,\overline{11}^2_{(13)},\\
4\,4\,\overline{10}\,\overline{10}_{(13)} &= 77^2_{(13)},\\
7778^2 - 2223^2 &= 55\,555\,555,\\
888\,889^2 - 111\,112^2 &= 777\,777\,777\,777.
\end{aligned}
$$

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Numbers by the Dots

RUSSELL EULER, CATHY GEORGE and JAWAD SADEK

A formula, in closed form, is given for positive integers that are triangular and centred triangular numbers. Also, a difference equation that yields all of those numbers is given.

Figurate numbers probably originated with the Pythagorean society (see reference 1, page 58). These positive integers comprise the total number of dots in various regular polygons (such as triangles, squares, pentagons, hexagons, and so on). Many interesting properties of figurate numbers have been discovered (see reference 2). It is known that the n th triangular number, t_n, is given by

$$t_n = \frac{n(n+1)}{2},$$

and the n th square number is given by

$$s_n = n^2,$$

for $n \geq 1$. The first eight triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36. The first six square numbers are 1, 4, 9, 16, 25, 36. So, 1 and 36 are both triangular numbers and square numbers. In fact, it is known that there exist infinitely many numbers that are both triangular and square numbers. In reference 3, it is shown that the n th triangular square number is

$$\frac{(17+12\sqrt{2})^n + (17-12\sqrt{2})^n - 2}{32}.$$

The purpose of this article is to derive a similar result for the triangular numbers that are also centred triangular numbers. In addition, we provide a difference equation that yields all of those numbers.

Geometrically, *centred triangular numbers*, T_n, consist of a central dot (or vertex), $T_1 = 1$, with three dots around it and then additional dots in the gaps between adjacent dots (see figure 1).

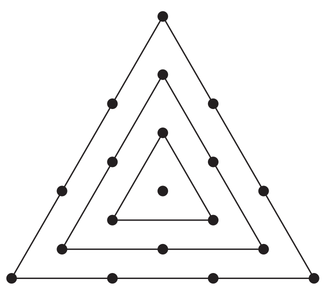

Figure 1

Table 1

Triangular numbers	Centred triangular numbers
$t_1 = 1$	$T_1 = 1$
$t_4 = 10$	$T_3 = 10$
$t_{16} = 136$	$T_{10} = 136$
$t_{61} = 1\,891$	$T_{36} = 1\,891$
$t_{229} = 26\,335$	$T_{133} = 26\,335$
$t_{856} = 366\,796$	$T_{495} = 366\,796$

So, the number of vertices on each edge of each triangle is one larger than the number of vertices on the edge of the preceding smaller triangle. Hence, T_n can be defined recursively by

$$\begin{aligned} T_1 &= 1, \\ T_n &= T_{n-1} + 3(n-1), \qquad \text{for } n \geq 2. \end{aligned} \tag{1}$$

The first four centred triangular numbers are 1, 4, 10, 19. By solving (1), we obtain

$$\begin{aligned} T_n &= 1 + 3(n-1) + 3(n-2) + \cdots + 3 \\ &= 1 + \frac{3n(n-1)}{2} \\ &= \frac{3n^2 - 3n + 2}{2}, \quad \text{for } n \geq 1. \end{aligned}$$

Table 1 displays the first six triangular numbers that are centred triangular numbers.

Theorem 1 *There exist infinitely many triangular numbers that are centred triangular numbers.*

The triangular numbers in theorem 1 are given by

$$\tfrac{3}{16}\big[\tfrac{4}{3} + (2+\sqrt{3})^{2k+1} + (2-\sqrt{3})^{2k+1}\big],$$

where $k \geq 0$.

To derive this formula, we use the following lemma. The lemma is useful in solving equations stemming from similar problems.

Lemma 1 *Let c, d, e, x, and y be nonzero numbers. If $ec + d = 1$, then the equation $x^2 - dy^2 = c$ can be rewritten as*

$$\frac{(dy-x)^2}{ec^2} - \frac{d(y-x)^2}{ec^2} = 1.$$

The proof of lemma 1 is a straightforward algebraic manipulation.

Proof of theorem 1 For the triangular number t_n to be a centred triangular number T_m, we require

$$n^2 + n = 3m^2 - 3m + 2.$$

Solving for the positive integer n yields

$$n = \frac{-1 + \sqrt{12m^2 - 12m + 9}}{2}.$$

This requires that $12m^2 - 12m + 9 = i^2$, where i is a positive integer that is also a multiple of 3. Thus,

$$12m^2 - 12m + 9 = 9b^2,$$

where $i = 3b$. By algebraic manipulation, this can be written as

$$(2m-1)^2 - 3b^2 = -2. \tag{2}$$

Let $a = 2m - 1$. By lemma 1, taking $d = 3$, $c = -2$, and $e = 1$, (2) can be rewritten as

$$\left(\frac{3b-a}{2}\right)^2 - 3\left(\frac{a-b}{2}\right)^2 = 1. \tag{3}$$

This is Pell's equation all of whose solutions are known (see reference 2). We note here that $(3b-a)/2$ and $(a-b)/2$ are both integers because (2) implies that $a = 2m - 1$ and b have the same parity. Let

$$A = \frac{3b-a}{2} \quad \text{and} \quad B = \frac{a-b}{2},$$

and write (3) as $A^2 - 3B^2 = 1$. Since $(2, 1)$ is the smallest nontrivial positive solution, the general formulae for all the solutions of this Pell equation are given by

$$A_k = \frac{(2+\sqrt{3})^k}{2} + \frac{(2-\sqrt{3})^k}{2},$$
$$B_k = \frac{(2+\sqrt{3})^k}{2\sqrt{3}} - \frac{(2-\sqrt{3})^k}{2\sqrt{3}},$$

for $k \geq 0$ (see reference 2, page 544).

It follows that all the solutions to (3) are given by

$$\begin{aligned} a_k &= A_k + 3B_k \\ &= \frac{(2+\sqrt{3})^k}{2} + \frac{(2-\sqrt{3})^k}{2} + \frac{\sqrt{3}(2+\sqrt{3})^k}{2} - \frac{\sqrt{3}(2-\sqrt{3})^k}{2} \\ &= \frac{(2+\sqrt{3})^k}{2}(1+\sqrt{3}) + \frac{(2-\sqrt{3})^k}{2}(1-\sqrt{3}) \end{aligned}$$

and

$$\begin{aligned} b_k &= A_k + B_k \\ &= \frac{(2+\sqrt{3})^k(3+\sqrt{3})}{6} + \frac{(2-\sqrt{3})^k(3-\sqrt{3})}{6}. \end{aligned}$$

Since $m = (a+1)/2$, all the values of m that satisfy (2) are given by

$$\begin{aligned} m_k &= \frac{a_k + 1}{2} \\ &= \frac{1}{2} + \frac{(2+\sqrt{3})^k}{4}(1+\sqrt{3}) + \frac{(2-\sqrt{3})^k}{4}(1-\sqrt{3}). \end{aligned}$$

This formula gives all of the indices of centred triangular numbers that are also triangular numbers. To determine a formula for the actual numbers, we use

$$\begin{aligned} T_{m_k} &= \frac{3m_k^2 - 3m_k + 2}{2} \\ &= \tfrac{3}{16}\left[\tfrac{4}{3} + (2+\sqrt{3})^{2k+1} + (2-\sqrt{3})^{2k+1}\right]. \end{aligned}$$

To get the indices of the triangular numbers that are also centred, we write

$$\begin{aligned} n_k &= \frac{1 + \sqrt{12m_k^2 - 12m_k + 9}}{2} \\ &= \frac{-1 + 3b_k}{2} \\ &= -\frac{1}{2} + \frac{3(A_k + B_k)}{2} \\ &= -\frac{1}{2} + \frac{(2+\sqrt{3})^k}{4}(3+\sqrt{3}) + \frac{(2-\sqrt{3})^k}{4}(3-\sqrt{3}). \end{aligned}$$

We can easily check, by using the formula for triangular numbers, that

$$\begin{aligned} t_{n_k} &= \frac{n_k(n_k+1)}{2} \\ &= \tfrac{3}{16}\left[\tfrac{4}{3} + (2+\sqrt{3})^{2k+1} + (2-\sqrt{3})^{2k+1}\right] \\ &= T_{m_k}. \end{aligned}$$

The result from theorem 1 permits us to prove an interesting observation about recurrence relationships satisfied by the indices of both triangular and centred triangular numbers.

Corollary 1 *Let $u_{k+1} = \Delta m_k = m_{k+1} - m_k$ and $v_{k+1} = \Delta n_k = n_{k+1} - n_k$, where m_k and n_k are the subscripts of centred triangular numbers and triangular numbers respectively, such that $T_{m_k} = t_{n_k}$. Then $(u_k)_{k\geq 1}$ satisfies the recurrence relation*

$$\begin{aligned} u_{k+2} - 4u_{k+1} + u_k &= 0, \\ u_1 = 2, \qquad u_2 &= 7, \end{aligned}$$

and $(v_k)_{k\geq 1}$ satisfies the recurrence relation

$$\begin{aligned} v_{k+2} - 4v_{k+1} + v_k &= 0, \\ v_1 = 3, \qquad v_2 &= 12. \end{aligned}$$

The proof of corollary 1 uses standard methods to solve recurrence equations and it will be omitted.

Acknowledgement

The authors thank the Editor for important suggestions.

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1 H. W. Eves, *An Introduction to the History of Mathematics* (Holt, Rinehart and Winston, New York, 1964).

2 K. H. Rosen, *Elementary Number Theory and Its Applications*, 5th edn. (Pearson, Boston, MA, 2005).

3 Problem E 954, *Amer. Math. Monthly* **58** (1951), p. 568.

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Jawad Sadek *is professor of mathematics at Northwest Missouri State University. His interests include Complex Analysis and helping his students discover the joy of mathematical research.*

$$\begin{aligned} 4^2 &= 16, \\ 34^2 &= 1156, \\ 334^2 &= 111556, \\ 3334^2 &= 11115556, \\ 33334^2 &= 1111155556, \end{aligned}$$

and so on.

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$$\begin{aligned} 7^2 &= 49, \\ 67^2 &= 4489, \\ 667^2 &= 444889, \\ 6667^2 &= 44448889, \\ 66667^2 &= 4444488889, \end{aligned}$$

and so on.

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What is the Probability that the Final Person on the Aircraft Sits in his own Seat?

PAUL BELCHER

Let us suppose that there are n $(n \geq 2)$ passengers who are to board an aircraft with n seats. The first person to board who we will call 'Freddy the Forgetful' has lost his boarding card and thus sits in a seat at random. The other $n-1$ sensible people have their boarding cards and will sit in their own seat if that is available. If their seat is already occupied then they will take an empty seat at random. The n passengers board one by one. We wonder what is the probability that the nth person, who we will call 'Neven', sits in his correct seat. We will label the passengers 1 to n and the seats $\overline{1}$ to $\overline{n}$.

Figure 1 shows the probability tree for the case when $n = 4$. So the probability that Neven sits in his correct seat is

$$\frac{1}{4} + \frac{1}{4 \times 3} + \frac{1}{4 \times 3 \times 2} + \frac{1}{4 \times 2} = \frac{1}{2}.$$

Similar trees for $n = 2$, 3, 5, and 6 also give the probability that Neven sits in his correct seat as $\frac{1}{2}$, which leads us to the conjecture that this is always the case.

Consider the general probability tree for n passengers. We will attempt to count the number of different paths in this tree. One path is when Freddy sits in his own seat and thus everyone else does as well. For all the other paths Freddy does not sit in his own seat and thus at least one of the remaining $n-1$ passengers does not sit in their own seat. For these remaining $n-1$ passengers, let the number of them that do not sit in their own seat be i. For each i

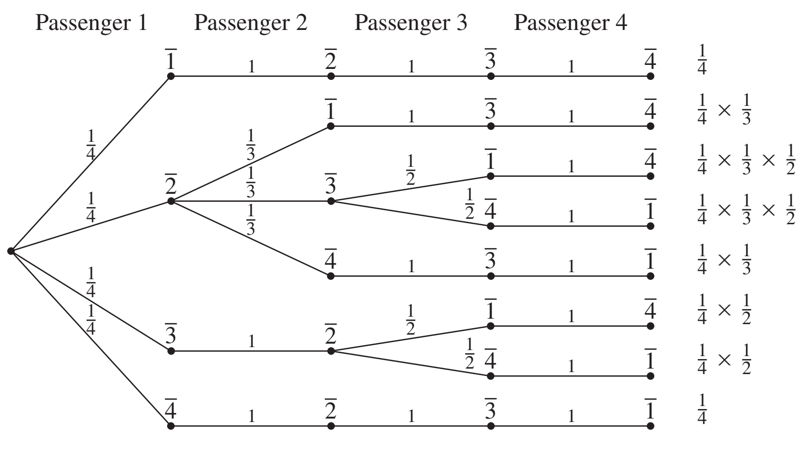

Figure 1

($1 \le i \le n-1$), the number of different paths will be C_i^{n-1} as the i people are chosen from the remaining $n-1$ passengers. So the total number of different paths will be

$$1 + C_1^{n-1} + C_2^{n-1} + \cdots + C_{n-1}^{n-1} = \sum_{i=0}^{n-1} C_i^{n-1} = 2^{n-1},$$

using the summation property of a row of Pascal's triangle that can be proved using the binomial theorem.

We claim that these paths can be put into pairs, where each path in the pair has the same probability, but one has Neven sitting in his correct seat, whereas the other does not. If we look at the probability tree in figure 1, we see that the claim is true in the case when $n = 4$ with four such pairs. Working in general, let us first consider the following two paths.

(i) Freddy sits in his own seat and thus everyone else, including Neven, does; the probability of this will be $1/n$.

(ii) Freddy sits in Neven's seat, so the other $n-2$ people will sit in their own seats and then Neven will sit in Freddy's seat; the probability of this will also be $1/n$.

So this is one pair satisfying our claim.

For the other paths, Freddy will not sit in his own seat and at least one person other than Neven will also not sit in his own seat. Now for the $n-1$ people excluding Neven but including Freddy, let us suppose that k ($2 \le k \le n-1$) of them sit in the wrong seats. Let the last person of these $n-1$ people to sit in the wrong seat be passenger l, who we will call 'Lasty'. After Lasty sits down the rest of the passengers will sit in their correct seats, with the possible exception of Neven. The sequence of people sitting in the wrong seats always ends when someone sits in Freddy's seat. The tree will split at Lasty with:

(i) probability $1/(n-l+1)$ that he (Lasty) sits in Freddy's seat and so Neven and all the remaining passengers sit in their own seats, and

(ii) probability $1/(n-l+1)$ that he (Lasty) sits in Neven's seat and so all the other remaining passengers sit in their own seat with the exception of Neven who sits in Freddy's seat.

With these two paths the probability was of course the same up until the split at Lasty and so both these paths have the same probability. So these pairs also satisfy our claim.

Thus we can now see that all the paths in the tree can be divided into 2^{n-2} pairs, where the two paths in a pair both have the same probability but one path leads to Neven sitting in his own seat whereas the other path does not. So as the probability of Neven sitting in his correct seat is the same as the probability of him not sitting in his own seat, we have proved our conjecture that the probability that the final person sits in his correct seat is $\frac{1}{2}$.

Having obtained this probability of $\frac{1}{2}$, we can also prove the result by strong induction. Let $P(n)$ be the probability that the last (i.e. nth) person sits in his correct seat in the case when the aircraft has n seats. We will prove that $P(n) = \frac{1}{2}$ for $n \ge 2$.

Step 1 With $n = 2$, if Freddy sits in his own seat then so will Neven and if Freddy sits in Neven's seat then Neven will sit in Freddy's. So $P(2) = \frac{1}{2}$.

Step 2 We assume the result for all integers greater than or equal to 2 and smaller than or equal to k and attempt to prove the result for $k+1$. With $k+1$ passengers there are $k+1$

possible seats for Freddy to sit in and they will each have a probability of $1/(k+1)$. There are three different cases to consider.

(i) Freddy sits in his own seat and thus Neven will also sit in his own seat.

(ii) Freddy sits in Neven's seat and so Neven cannot.

(iii) Freddy sits in the seat of passenger l, where $2 \le l \le k$, giving $k-1$ possibilities for l. Then all the passengers from 2 to $l-1$ will sit in their own seat and passenger l will sit at random. We now ignore the first $l-1$ passengers and consider passenger l as the new 'Freddy' and Freddy's seat as the one now corresponding to passenger l. So we have passengers l to $k+1$ and seats $\overline{1}$ plus $\overline{l+1}$ to $\overline{k+1}$, giving $k+1-(l-1) = k-l+2$ passengers and seats. We now have a problem identical to the original one with the first person, who was originally labelled as l, sitting at random and we are asking for the probability that the last person sits in his correct seat. As $2 \le l \le k$ then $2 \le k-l+2 \le k$ and so $P(k-l+2) = \frac{1}{2}$, by the induction hypothesis.

Considering all three cases, multiplying the probability of that case occurring by the probability of the last person sitting in the correct seat and summing, we have

$$P(k+1) = \frac{1}{k+1} \times 1 + \frac{1}{k+1} \times 0 + \frac{k-1}{k+1} \times \frac{1}{2} = \frac{1}{2},$$

as required, completing the induction proof.

Let us now denote the sum of the fractions that are the probabilities of the tree branches that lead to Neven sitting in his correct seat by $F(n)$. Then we have the combinatorial formula

$$F(n) = \frac{1}{n} + \sum_{2 \le i \le n-1} \frac{1}{ni} + \sum_{2 \le i < j \le n-1} \frac{1}{nij} + \cdots + \frac{1}{n(n-1)\cdots 2} = \frac{1}{2}, \qquad n \ge 2. \qquad (1)$$

All these fractions have n on the bottom and then, multiplied together, all possible combinations (including none) of numbers from the set $\{2, 3, \ldots, n-1\}$. The first term of $1/n$ is from the branch where Freddy sits in his own seat and thus everyone else does. The second term is where Freddy sits in someone else's seat (but not Neven's) and this person then sits in Freddy's seat. The factor of $1/n$ is from the choices that Freddy has. Suppose that he sits in the seat of passenger r $(2 \le r \le n-1)$, then all the passengers from 2 to $r-1$ sit in their own seats. The probability that passenger r then sits in Freddy's seat is $1/(n-r+1)$. To keep the notation simple we then let $i = n-r+1$ $(2 \le r \le n-1)$. The third term is when there are two people (neither of them Neven) and Freddy in the wrong seats. The factor of $1/n$ is from the choices that Freddy has. Suppose that he sits in the seat of passenger r $(2 \le r \le n-1)$ and that passenger r sits in the seat of passenger s $(r < s \le n-1)$, who then sits in Freddy's seat. The probability that passenger r sits in the seat of passenger s is $1/(n-r+1)$. The probability that passenger s sits in Freddy's seat is $1/(n-s+1)$. To keep the notation simple we then let $i = n-s+1$ and $j = n-r+1$ $(2 \le i < j \le n-1)$. The rest of the terms follow this pattern with the last term being the one with $n-1$ passengers (including Freddy) sitting in the wrong seats and Neven sitting in his correct seat. There are 2^{n-2} terms in the sum for $F(n)$, each being a fraction with a numerator of 1.

Multiplying $F(n)$ by $n!$ and reversing the order we also have

$$\frac{n!}{2} = 1 + \sum_{2 \le i \le n-1} i + \sum_{2 \le i < j \le n-1} ij + \cdots + (n-1)(n-2)\cdots 2. \tag{2}$$

We now attempt to prove (1) directly by induction, for $n \in \mathbb{N},\ n \geq 2$, without considering the aircraft story.

Step 1 If $n = 2$ the left-hand side of (1) is just the fraction $\frac{1}{2}$, as required.

Step 2 We assume the result for $n = k$ and attempt to prove for $n = k+1$. The fractions in (1) for $F(k)$ all have k in the denominator and then, multiplied together, all possible combinations (including none) of numbers from the set $\{2, 3, \ldots, k-1\}$. We can remove this factor of k in the denominator of each fraction by multiplying (1) by k. So considering the fractions in the expression for $kF(k)$, obtained by multiplying (1) for $F(k)$ by k, none of them will have k in the denominator. If we then add the equations for $F(k)$ and $kF(k)$ together, on the right-hand side we will have those fractions both with k and without k in the denominator. If we then divide this new equation by $k+1$, on the right-hand side we will have fractions all having $k+1$ in the denominator, with the other factors being all possible combinations (including none) from the set $\{2, 3, \ldots, k\}$. So

$$F(k+1) = \frac{1}{k+1}(F(k) + kF(k)) = F(k) = \frac{1}{2},$$

completing the induction proof.

To illustrate this induction step we will show the step from 4 to 5:

$$\begin{aligned}
F(4) &= \frac{1}{4} + \frac{1}{4\cdot 2} + \frac{1}{4\cdot 3} + \frac{1}{4\cdot 3\cdot 2}, \\
4F(4) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3\cdot 2}, \\
F(4) + 4F(4) &= \frac{1}{4} + \frac{1}{4\cdot 2} + \frac{1}{4\cdot 3} + \frac{1}{4\cdot 3\cdot 2} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3\cdot 2}, \\
\frac{1}{5}(F(4) + 4F(4)) &= \frac{1}{5\cdot 4} + \frac{1}{5\cdot 4\cdot 2} + \frac{1}{5\cdot 4\cdot 3} + \frac{1}{5\cdot 4\cdot 3\cdot 2} + \frac{1}{5} + \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3} \\
&\quad + \frac{1}{5\cdot 3\cdot 2} \\
&= \frac{1}{5} + \frac{1}{5\cdot 2} + \frac{1}{5\cdot 3} + \frac{1}{5\cdot 4} + \frac{1}{5\cdot 2\cdot 3} + \frac{1}{5\cdot 2\cdot 4} + \frac{1}{5\cdot 3\cdot 4} \\
&\quad + \frac{1}{5\cdot 4\cdot 3\cdot 2} \\
&= F(5).
\end{aligned}$$

An alternative induction proof, as kindly suggested by the Editor, would be to prove (2) by induction. In this version the step from k to $k+1$ is achieved by multiplying the expression for $k!/2$ by $k+1$ on both sides and then rearranging the terms on the right-hand side to give the required result for $(k+1)!/2$.

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Stirling Numbers

M. A. KHAN

The numbers in the triangular array

$$\begin{array}{cccccc} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ . & . & . & . & . & . \end{array}$$

are called the *Stirling numbers* after the Scottish mathematician James Stirling (1692–1770), who first considered them. The array can be built up row-by-row. If $S(n, k)$ denotes the number in the nth row and kth column, where $1 \leq k \leq n$, then

$$S(n, 1) = S(n, n) = 1 \tag{1}$$

and

$$S(n, k) = kS(n-1, k) + S(n-1, k-1), \tag{2}$$

for $1 < k < n$. They provide the coefficients of successive terms when r^n is expressed as the sum of 'falling factorials'. Thus

$$\begin{aligned} r &= r, \\ r^2 &= r + r(r-1), \\ r^3 &= r + 3r(r-1) + r(r-1)(r-2), \end{aligned}$$

and, in general,

$$r^n = \sum_{k=1}^{n} S(n, k) r(r-1) \cdots (r-k+1). \tag{3}$$

We can prove (3) by induction on n. When $n = 1$, it says

$$r = S(1, 1)r,$$

which is clear by (1). Now let $n > 1$ and assume inductively that

$$r^{n-1} = \sum_{k=1}^{n-1} S(n-1, k) r(r-1) \cdots (r-k+1).$$

Then

$$
\begin{aligned}
r^n &= \sum_{k=1}^{n-1} S(n-1,k)r(r-1)\cdots(r-k+1)r \\
&= \sum_{k=1}^{n-1} S(n-1,k)[r(r-1)\cdots(r-k+1)k + r(r-1)\cdots(r-k+1)(r-k)] \\
&= \sum_{k=1}^{n-1} kS(n-1,k)r(r-1)\cdots(r-k+1) \\
&\quad + \sum_{k=2}^{n} S(n-1,k-1)r(r-1)\cdots(r-k+1) \\
&= S(n-1,1)r + \sum_{k=2}^{n-1}(kS(n-1,k)+S(n-1,k-1))r(r-1)\cdots(r-k+1) \\
&\quad + S(n-1,n-1)r(r-1)\cdots(r-n+1) \\
&= S(n-1,1)r + \sum_{k=2}^{n-1} S(n,k)r(r-1)\cdots(r-k+1) \\
&\quad + S(n-1,n-1)r(r-1)\cdots(r-n+1) \\
&= \sum_{k=1}^{n} S(n,k)r(r-1)\cdots(r-k+1)
\end{aligned}
$$

because $S(n-1,1) = 1 = S(n,1)$ and $S(n-1,n-1) = 1 = S(n,n)$ from (1). This proves the inductive step.

We can use (3) to find expressions for $\sum_{r=1}^{n} r^m$, the sum of the mth powers of the first n natural numbers, for a given m. We first rewrite (3) as

$$r^n = \sum_{k=1}^{n} S(n,k)\binom{r}{k}k!,$$

where

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!}$$

is the familiar binomial coefficient. We can say that $\binom{r}{k} = 0$ when $k > r$. Now,

$$
\begin{aligned}
\sum_{r=1}^{n} r^m &= \sum_{r=k}^{n}\sum_{k=1}^{m} S(m,k)\binom{r}{k}k! \\
&= \sum_{k=1}^{m} k!\,S(m,k)\sum_{r=k}^{n}\binom{r}{k}. \qquad (4)
\end{aligned}
$$

But

$$\begin{aligned}\binom{n+1}{k+1} - \sum_{r=k}^{n} \binom{r}{k} &= \binom{n+1}{k+1} - \binom{n}{k} - \binom{n-1}{k} - \cdots - \binom{k}{k} \\ &= \binom{n}{k+1} - \binom{n-1}{k} - \binom{n-2}{k} - \cdots - \binom{k}{k} \\ &= \binom{n-1}{k+1} - \binom{n-2}{k} - \binom{n-3}{k} - \cdots - \binom{k}{k} \\ &= \cdots \\ &= \binom{k+1}{k+1} - \binom{k}{k} \\ &= 0,\end{aligned}$$

so that (4) gives the formula

$$\sum_{r=1}^{n} r^m = \sum_{k=1}^{m} k!\, S(m,k) \binom{n+1}{k+1}.$$

Thus,

$$\begin{aligned}\sum_{r=1}^{n} r &= S(1,1)\binom{n+1}{2} \\ &= \tfrac{1}{2} n(n+1), \\ \sum_{r=1}^{n} r^2 &= S(2,1)\binom{n+1}{2} + 2!\, S(2,2)\binom{n+1}{3} \\ &= \frac{1}{2}(n+1)n + 2\frac{(n+1)n(n-1)}{3 \times 2 \times 1} \\ &= \tfrac{1}{6} n(n+1)(3 + 2(n-1)) \\ &= \tfrac{1}{6} n(n+1)(2n+1), \\ \sum_{r=1}^{n} r^3 &= S(3,1)\binom{n+1}{2} + 2!\, S(3,2)\binom{n+1}{3} + 3!\, S(3,3)\binom{n+1}{4} \\ &= \tfrac{1}{2}(n+1)n + 6 \times \tfrac{1}{6}(n+1)n(n-1) + 6\tfrac{1}{24}(n+1)n(n-1)(n-2) \\ &= \tfrac{1}{4}(n+1)n(2 + 4(n-1) + (n-1)(n-2)) \\ &= \tfrac{1}{4}(n+1)n(n^2+n) \\ &= (\tfrac{1}{2} n(n+1))^2, \\ \sum_{r=1}^{n} r^4 &= S(4,1)\binom{n+1}{2} + 2!\, S(4,2)\binom{n+1}{3} + 3!\, S(4,3)\binom{n+1}{4} + 4!\, S(4,4)\binom{n+1}{5},\end{aligned}$$

which boils down to

$$\sum_{r=1}^{n} r^4 = \tfrac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1).$$

As a further example of the use of (3),

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{n^4}{n!} &= \sum_{n=1}^{\infty}\frac{n+7n(n-1)+6n(n-1)(n-2)+n(n-1)(n-2)(n-3)}{n!} \\
&= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=2}^{\infty}\frac{7}{(n-2)!}+6\sum_{n=3}^{\infty}\frac{1}{(n-3)!}+\sum_{n=4}^{\infty}\frac{1}{(n-4)!} \\
&= \mathrm{e}+7\mathrm{e}+6\mathrm{e}+\mathrm{e} \\
&= 15\mathrm{e}.
\end{aligned}
$$

There is also a combinatorial way of defining Stirling numbers. Consider, for example, the set $\{1, 2, 3, 4\}$. We can partition this into two nonempty subsets in seven ways:

$$
\begin{gathered}
\{1\},\ \{2,3,4\}, \\
\{2\},\ \{1,3,4\}, \\
\{3\},\ \{1,2,4\}, \\
\{4\},\ \{1,2,3\}, \\
\{1,2\},\ \{3,4\}, \\
\{1,3\},\ \{2,4\}, \\
\{1,4\},\ \{2,3\},
\end{gathered}
$$

and $S(4, 2) = 7$. In general, $S(n, k)$, for $1 \le k \le n$, is the number of ways of partitioning an n-element set into k nonempty subsets. Anticipating this result, we will denote this number by $S(n, k)$. Clearly $S(n, 1) = S(n, n) = 1$ as in (1). Let $1 < k < n$. The number of partitions of $\{1, \dots, n\}$ into k nonempty subsets, one of which is $\{n\}$, is the same as the number of partitions of $\{1, \dots, n-1\}$ into $k-1$ nonempty subsets, i.e. it is $S(n-1, k-1)$. If $\{n\}$ is not a subset in the partition, then we partition $\{1, \dots, n-1\}$ into k nonempty subsets, in $S(n-1, k)$ ways, and then insert n into any one of these subsets, giving $kS(n-1, k)$ partitions which do not include $\{n\}$. Hence,

$$S(n,k) = kS(n-1,k) + S(n-1,k-1),$$

which is (2).

References

1 M. R. Spiegel, *Probability and Statistics* (McGraw Hill, New York).

2 H. S. Hall and S. R. Knight, *Higher Algebra* (H. M. Publishers, Agra, 2002).

M. A. Khan *retired in 1994 from his position as deputy director in research, design, and standards with Indian Railways. Mathematics is his hobby.*

What is $\log_a(x^{1/\log_a x})$, where $a > 1$, $x > 0$, and $x \neq 1$?

Moldova State University **Stefan Alexei**

Caught up in a Box

PRITHWIJIT DE

This article is a consequence of my encounter with an article entitled ‘Diophantine boxes’ written by the famous Irish mathematician and mathematical humorist, Des MacHale (see reference 1). Therein, the author discussed problems related to the determination of the dimensions of rectangular boxes with integer length, breadth, and height having volumes equal to their surface areas. Thus, if x, y, and z denote the length, breadth, and height of the box, then the problem reduces to solving the following equation in positive integers:

$$2(xy + yz + zx) = xyz. \tag{1}$$

In order to solve (1) write it as

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}. \tag{2}$$

Since (2) is symmetric in x, y, and z we may assume without loss of generality that $x \geq y \geq z$. This gives

$$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{3}{z} \quad \Longrightarrow \quad z \leq 6.$$

Also it is evident from (2) that $1/z < \frac{1}{2}$, implying $z \geq 3$. Combining the two inequalities we can write $3 \leq z \leq 6$. As the range of z is now completely specified, (2) may be solved in x and y after fixing a value of z in its range and repeating the exercise for every such value in it. Let us solve (2) by this method. First, let z be equal to 3. Now, (2) reduces to

$$(x - 6)(y - 6) = 36.$$

Since $x - 6$ and $y - 6$ are positive integers and $x \geq y$, we find that

$$(x, y) \in \{(42, 7), (24, 8), (18, 9), (15, 10), (12, 12)\}.$$

So there are five rectangular boxes with their surface area the same as the volume; their dimensions and surface areas are shown in table 1.

We now proceed to consider the other cases. If $z = 4$ then (2) leads to

$$(x - 4)(y - 4) = 16.$$

Table 1

x	y	z	Area
42	7	3	882
24	8	3	576
18	9	3	486
15	10	3	450
12	12	3	432

Table 2

x	y	z	Area	$x+y+z$	Diagonal
42	7	3	882	52	
24	8	3	576	35	
18	9	3	486	30	
15	10	3	450	28	
12	12	3	432	27	
20	5	4	400	29	21
12	6	4	288	22	14
8	8	4	256	20	12
10	5	5	250	20	
6	6	6	216	18	

The solution set is $\{(20, 5), (12, 6), (8, 8)\}$, and we end up with three boxes of dimensions $20 \times 5 \times 4$, $12 \times 6 \times 4$, and $8 \times 8 \times 4$ respectively. The case when $z = 5$ is a bit tricky for we arrive at the following equation when we set out to solve (2):

$$3xy = 10(x + y).$$

From this we obtain $x = 10y/(3y - 10)$. Now, note that as $z = 5$ and $x \geq y \geq z$ we must have $x \geq y \geq 5$. We also have an upper bound for y, namely $y \leq \frac{20}{3}$, since

$$\frac{2}{y} \geq \frac{1}{x} + \frac{1}{y} = \frac{3}{10}.$$

Thus, we see that y can take only two possible values, 5 and 6. Now, y cannot be equal to 6 because if it is then 8 (which is equal to $3y-10$) has to divide 60 (which is equal to $10y$), which is not possible. Hence, y must be 5 and we can verify easily that $x = 10y/(3y - 10) = 10$. The only value in the range of z that is yet to be considered is 6. If $z = 6$ then we are left with the following equation:

$$(x - 3)(y - 3) = 9.$$

The solutions are $(x, y) = (12, 4)$ and $(x, y) = (6, 6)$. The constraint $y \geq z$ rules out the first solution. Hence, in this case the only solution is a cube of side 6 units.

This appears to be an opportune time to summarize the results obtained so far in a tabular form – see table 2. The last column in table 2 shows the length of the diagonal of the box. We see that there are three boxes, $20 \times 5 \times 4$, $12 \times 6 \times 4$, and $8 \times 8 \times 4$, which not only have integral edges but also have integral diagonals. Curiously, the lengths of the diagonals differ from the sums of the lengths of the edges for the three boxes by the same amount, 8. Thus, $(x, y, z) = (20, 5, 4), (12, 6, 4), (8, 8, 4)$ satisfy the system of simultaneous equations

$$2(xy + yz + zx) = xyz,$$
$$(x + y + z) - \sqrt{x^2 + y^2 + z^2} = 8,$$

where x, y, and z are positive integers. Also, the length of the box is always a multiple of its height.

Let us tweak the problem a little. Suppose that, instead of a closed box, we are given an open-top box whose surface area is equal to its volume and we have to find the dimensions assuming that they are positive integers. With the same notation as before, the problem boils down to finding integral solutions to

$$xy + 2yz + 2zx = xyz,$$

i.e.

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{2z} = \frac{1}{2}. \tag{3}$$

Observe that (3) is not symmetric in x, y, and z; hence, the technique employed to solve (1) cannot be emulated in order to solve (3). However, we have the following bounds at once:

$$x \geq 3, \qquad y \geq 3, \qquad z \geq 2.$$

These follow by observing that

$$\frac{1}{x} < \frac{1}{2}, \qquad \frac{1}{y} < \frac{1}{2}, \qquad \frac{1}{2z} < \frac{1}{2}.$$

Also from (3),

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}\left(1 - \frac{1}{z}\right) < \frac{1}{2}.$$

Now, we may assume that $x \geq y$ and use this fact to derive

$$\frac{2}{x} \leq \frac{1}{x} + \frac{1}{y} < \frac{1}{2},$$

leading to $x \geq 5$. We also observe that

$$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{2z} \leq \frac{2}{y} + \frac{1}{2z}.$$

This leads to

$$\frac{1}{2z} \geq \frac{1}{2} - \frac{2}{y}.$$

Now,

$$z \geq 2 \quad \Longrightarrow \quad \frac{1}{2z} \leq \frac{1}{4} \quad \Longrightarrow \quad \frac{1}{2} - \frac{2}{y} \leq \frac{1}{4} \quad \Longrightarrow \quad \frac{2}{y} \geq \frac{1}{4} \quad \Longrightarrow \quad y \leq 8.$$

Recall that $y \geq 3$. Hence, we have $3 \leq y \leq 8$, and we can find the complete solution set of (3) by substituting in values of y in succession and solving (3) in x and z. First, let us fix the value of y to be 3. We need to solve

$$\frac{1}{x} + \frac{1}{2z} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

From this we get

$$\frac{1}{2z} = \frac{x-6}{6x} \quad \Longrightarrow \quad z = \frac{3x}{x-6}.$$

Table 3

x	y	z	Area	$x+y+z$	Diagonal
7	3	21	441	31	
24	3	4	288	31	
8	3	12	288	23	
9	3	9	243	21	
15	3	5	225	23	
12	3	6	216	21	
5	4	10	200	19	
12	4	3	144	19	13
6	4	6	144	16	
8	4	4	128	16	
20	5	2	200	27	
5	5	5	125	15	
12	6	2	144	20	
6	6	3	108	15	9
8	8	2	128	18	

Therefore,

$$z = 3 + \frac{18}{x-6}.$$

For z to be an integer, $x-6$ has to be a factor of 18 and so the possible values of x are 7, 8, 9, 12, 15, and 24. The corresponding values of z are 21, 12, 9, 6, 5, and 4 respectively. If $y=4$ then it follows that $z = 2+8/(x-4)$ and the possible values of x are 5, 6, 8, and 12. When $y=5$ we see that $z = 5x/(3x-10)$. Also $z \le 5$ because

$$\frac{1}{2z} \ge \frac{1}{2} - \frac{2}{5} = \frac{1}{10}.$$

It turns out that z can take on two values, 2 and 5. The corresponding values of x are 20 and 5. When $y=6$, $2z = 3+9/(x-3)$ and, as $2z$ is an integer, $x \in \{4, 6, 12\}$. But $x \ge y$. Hence, $x=6$ and $x=12$. This gives $z=3$ and $z=2$. No solution exists when $y=7$. In this case $z = 1+(2x+14)/(5x-14)$ and we obtain from $5x-14 \le 2x+14$ that $x \le 9$. Hence, the possible values of x are 7, 8, and 9. But none of them make z an integer. Hence, there is no solution. If $y=8$ then $z = 1+(x+8)/(3x-8)$. This yields $3x-8 \le x+8$, i.e. $x \le 8$. But $x \ge 8$, so that $x=8$ and $z=2$. We present the entire list of solutions in table 3.

There are fifteen open-top boxes with the surface area the same as the volume. Interestingly, there are two boxes with integral diagonals and the dimensions of these boxes satisfy the following system of simultaneous equations:

$$xy + 2(yz+zx) = xyz,$$
$$(x+y+z) - \sqrt{x^2+y^2+z^2} = 6,$$

where x, y, and z are positive integers.

For a rectangular box with surface area A and volume V, we define R as the ratio of area to volume, i.e. $R = A/V$. Then the problems discussed here are solutions to the equation

$R = 1$. It is natural to ask about the range of values of R. For ease of distinction between the ratios of the surface-areas to volumes of a closed-top box and an open-top box, we will refer to the ratios as R^{c} and R^{o}, where the superscripts 'c' and 'o' represent closed-top and open-top respectively. Observe that

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = R^{\mathrm{c}},$$
$$\frac{2}{x} + \frac{2}{y} + \frac{1}{z} = R^{\mathrm{o}},$$

and it follows that $R^{\mathrm{c}} \le 6$ and $R^{\mathrm{o}} \le 5$ because $x \ge 1$, $y \ge 1$, and $z \ge 1$. But the question whether R^{c} and R^{o} assume all integer values between their lower and upper limits still remains and can only be answered by exhibiting solutions of the equations above in integers. Let us do that, then, by setting $R^{\mathrm{c}} = 2$. So we need to solve

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 2.$$

As this equation is symmetric in x, y, and z, we may assume without loss of any generality that $x \ge y \ge z \ge 1$. Emulating the method used to solve (2) we obtain $2 \le z \le 3$, implying that z can take on only two values, 2 and 3. Using these values of z to solve the equation in x and y yields the following solutions: $(x, y, z) = (6, 3, 2), (4, 4, 2), (3, 3, 3)$. Next, put $R^{\mathrm{c}} = 3$. We need to solve

$$\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 3,$$

where $x \ge y \ge z \ge 1$. We see at once that $6/z \ge 3$, implying that $z \le 2$. Now, by putting $z = 1$ and $z = 2$ in succession and solving the equation, we obtain $(x, y, z) = (6, 3, 1), (4, 4, 1), (2, 2, 2)$. When $R^{\mathrm{c}} = 4$ and $R^{\mathrm{c}} = 6$, we need to solve

$$\frac{1}{2x} + \frac{1}{2y} + \frac{1}{2z} = 1, \tag{4}$$
$$\frac{1}{3x} + \frac{1}{3y} + \frac{1}{3z} = 1. \tag{5}$$

We recognize these two equations as special cases of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1, \tag{6}$$

which we have already solved. So in order to solve (4) we only have to consider those solutions of (6) for which x, y, and z are all even, and to solve (5) we need those solutions for which x, y, and z are all multiples of 3. Hence, $(x, y, z) = (2, 2, 1)$ is the only solution to (4) and $(x, y, z) = (1, 1, 1)$ is the only solution to (5). The case $R^{\mathrm{c}} = 5$ yields $z = 1$ and $2/x + 2/y = 3$. Since $2/y \ge 2/x$ we must have $4/y \ge 3$ giving us $y \le \frac{4}{3}$. This forces y to be 1 and x to be 2. Table 4 presents all the solutions for $R^{\mathrm{c}} = 2, 3, 4, 5, 6$.

Now we consider the other equation involving R^{o}. Set $R^{\mathrm{o}} = 2$ and observe that, by assuming $x \ge y$ we get

$$\frac{2}{y} \ge \frac{1}{x} + \frac{1}{y} = 1 - \frac{1}{2z}.$$

Table 4

R^{c}	x	y	z	$x+y+z$	Diagonal
2	6	3	2	11	7
2	4	4	2	10	6
2	3	3	3	9	
3	6	3	1	10	
3	4	4	1	9	
3	2	2	2	6	
4	2	2	1	5	3
5	2	1	1	4	
6	1	1	1	3	

From this it follows that

$$\frac{1}{4} \geq \frac{1}{4z} \geq \frac{1}{2} - \frac{1}{y},$$

yielding $y \leq 4$. Also, $2/y < 2$, implying $y \geq 2$. Thus, $2 \leq y \leq 4$. Solving the equation for every possible value of y we obtain $(x, y, z) = (3, 2, 3), (4, 2, 2), (6, 3, 1), (4, 4, 1)$. If $R^{o} = 3$, then we obtain, after a bit of algebra,

$$1 \geq \frac{1}{z} \geq 3 - \frac{4}{y},$$

and from this we obtain $y \leq 2$. Therefore, the possible values of y are 1 and 2. The solutions in this case are $(x, y, z) = (4, 1, 2), (3, 1, 3), (2, 2, 1)$. Arguing in a similar vein for $R^{o} = 4$ and $R^{o} = 5$, we obtain the complete set of values of the dimensions of an open-top box. The reader may verify that this is indeed the case. They are presented in table 5.

As a by-product of these deductions we also obtain solutions to the following system of equations in positive integers. The first pair of equations is

$$xy + yz + zx = xyz,$$
$$x + y + z - \sqrt{x^2 + y^2 + z^2} = 4,$$

Table 5

R^{o}	x	y	z	$x+y+z$	Diagonal
2	3	2	3	8	
2	6	3	1	10	
2	4	2	2	8	
2	4	4	1	9	
3	3	1	3	7	
3	4	1	2	7	
3	2	2	1	5	3
4	2	1	1	4	
5	1	1	1	3	

whose solutions are $(x, y, z) = (6, 3, 2), (4, 4, 2)$. The second pair of equations is

$$xy + yz + zx = 2xyz,$$
$$x + y + z - \sqrt{x^2 + y^2 + z^2} = 2,$$

whose solution is $(x, y, z) = (2, 2, 1)$, and the final pair is

$$2(yz + zx) + xy = 3xyz,$$
$$x + y + z - \sqrt{x^2 + y^2 + z^2} = 2,$$

whose solution is $(x, y, z) = (2, 2, 1)$.

Reference

1 D. MacHale, Diophantine boxes, *Math. Gazette* **84** (2000), pp. 211–215.

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Number patterns

$$987\,654\,321 \times 9 = 8\,888\,888\,889,$$
$$987\,654\,321 \times 18 = 17\,777\,777\,778,$$
$$987\,654\,321 \times 27 = 26\,666\,666\,667,$$
$$987\,654\,321 \times 36 = 35\,555\,555\,556,$$
$$987\,654\,321 \times 45 = 44\,444\,444\,445,$$
$$987\,654\,321 \times 54 = 53\,333\,333\,334,$$
$$987\,654\,321 \times 63 = 62\,222\,222\,223,$$
$$987\,654\,321 \times 72 = 71\,111\,111\,112,$$
$$987\,654\,321 \times 81 = 80\,000\,000\,001.$$

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Trigonometric Functions and Fibonacci and Lucas Arrays

THOMAS KOSHY

Introduction

Fibonacci numbers F_n and Lucas numbers L_n continue to be an intriguing source of fun and excitement, and offer boundless opportunities for experimentation, conjecturing, and exploration. They are often defined recursively as

$$\begin{aligned} \mathrm{F}_1 &= 1, \\ \mathrm{F}_2 &= 1, \\ \mathrm{F}_n &= \mathrm{F}_{n-1} + \mathrm{F}_{n-2} \end{aligned}$$

and

$$\begin{aligned} \mathrm{L}_1 &= 1, \\ \mathrm{L}_2 &= 3, \\ \mathrm{L}_n &= \mathrm{L}_{n-1} + \mathrm{L}_{n-2}, \end{aligned}$$

where $n \geq 3$. Using the recurrence relations, Fibonacci and Lucas numbers can be extended to zero subscript, i.e. $\mathrm{F}_0 = 0$ and $\mathrm{L}_0 = 2$.

Binet's formulas

Using Binet's formulas, both Fibonacci and Lucas numbers can be defined explicitly as

$$\begin{aligned} \mathrm{F}_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ \mathrm{L}_n &= \alpha^n + \beta^n, \end{aligned}$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ are the solutions of the quadratic equation $x^2 = x + 1$ (see reference 1).

Fibonacci and Lucas numbers occur in numerous unexpected places (see reference 1). This article presents two such occurrences.

Vieta's investigation of $2\cos(nx/2)$

Around 1591, the French mathematician Franciscus Vieta (1540–1603) discovered that both $2\cos(nx/2)$ and $2\sin(nx/2)$ can be expanded in terms of $2\cos(x/2)$ (see reference 2). Both result from the following two trigonometric identities:

$$\begin{aligned} \cos A + \cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}, \\ \sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}. \end{aligned}$$

For example,

$$\cos\frac{(n+1)x}{2}+\cos\frac{(n-1)x}{2}=2\cos\frac{nx}{2}\cos\frac{x}{2},$$

so

$$\cos\frac{(n+1)x}{2}=2\cos\frac{nx}{2}\cos\frac{x}{2}-\cos\frac{(n-1)x}{2}.$$

This in turn helps us define $u_n = 2\cos(nx/2)$ recursively as follows:

$$\begin{aligned} u_0 &= 2,\\ u_1 &= 2\cos\frac{x}{2},\\ u_n &= u_1 u_{n-1} - u_{n-2}, \end{aligned}$$

where $n \geq 2$, as Vieta did.

Thus we have

$$\begin{aligned} u_0 &= 2,\\ u_1 &= u_1,\\ u_2 &= u_1^2 - 2,\\ u_3 &= u_1^3 - 3u_1,\\ u_4 &= u_1^4 - 4u_1^2 + 2,\\ u_5 &= u_1^5 - 5u_1^3 + 5u_1,\\ u_6 &= u_1^6 - 6u_1^4 + 9u_1^2 - 2,\\ u_7 &= u_1^7 - 7u_1^5 + 14u_1^3 - 7u_1,\\ &\vdots \end{aligned}$$

as desired.

A hidden treasure

The preceding array of equations contains a hidden treasure. To see it, we arrange the absolute values of the coefficients on the right-hand side in a triangular array A, as in figure 1. Each row of this array begins with a 1, except row 0; each element $A(n, j)$ can be obtained by adding the element $A(n-1, j)$ to the element $A(n-2, j-1)$ diagonally across from it in the previous row; see figure 1.

Accordingly, array A can be defined recursively as follows:

$$\begin{aligned} &A(0,0) = 2,\\ &A(n,0) = 1, \qquad n \geq 1, \\ &A(n,j) = A(n-1,j) + A(n-2,j-1), \qquad n, j \geq 2. \end{aligned} \tag{1}$$

Thinking of u_n as a polynomial in u_1 of degree n, and noticing that the power of u_1 decreases by 2 from left to right, figure 1 can be employed to find the expansion of u_n for any n.

j	0	1	2	3	4	row sums
$n=0$	2					2
$n=1$	1					1
$n=2$	1	2				3
$n=3$	1	3				4
$n=4$	1	4	2			7
$n=5$	1	5	5			11
$n=6$	1	6	9	2		18
$n=7$	1	7	14	7		29
						$\uparrow$ L_n

Figure 1

Two interesting observations are as follows: row $2n$ ends in a 2, and rows $2n$ and $2n+1$ contain the same number of elements, namely $n+1$.

Using induction, it can be shown that

$$A(n, j) = \frac{n}{n-j}\binom{n-j}{j}, \tag{2}$$

where $n > 0$ and $0 \le j \le \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to the real number x. For example,

$$A(6, 2) = \frac{6}{4}\binom{4}{2} = 9$$

and

$$A(7, 3) = \frac{7}{4}\binom{4}{3} = 7.$$

Interestingly, in 1966 Draim and Bicknell arrived at the same array A when they investigated the sums of like powers of the solutions

$$r = \frac{p + \sqrt{p^2 + 4q}}{2}$$

and

$$s = \frac{p - \sqrt{p^2 + 4q}}{2}$$

of an arbitrary quadratic equation $x^2 = px + q$ (see references 1 and 3). For example,

$$\begin{aligned}
r^0 + s^0 &= 2,\\
r^1 + s^1 &= p,\\
r^2 + s^2 &= p^2 + 2q,\\
r^3 + s^3 &= p^3 + 3pq,\\
r^4 + s^4 &= p^4 + 4p^2q + 2q^2.
\end{aligned}$$

More generally, using induction Draim and Bicknell showed that

$$r^n + s^n = \sum_{j=0}^{\lfloor n/2 \rfloor} A(n, j) p^{n-2j} q^j. \tag{3}$$

Row sums

Interestingly, every row sum in figure 1 is a Lucas number:

$$\sum_{j=0}^{\lfloor n/2 \rfloor} A(n, j) = \mathrm{L}_n. \tag{4}$$

For example,

$$\begin{aligned}\sum_{j=0}^{\lfloor 5/2 \rfloor} A(5, j) &= \sum_{j=0}^{2} A(5, j) \\ &= 1 + 5 + 5 \\ &= 11 \\ &= \mathrm{L}_5.\end{aligned}$$

Equation (4) follows intuitively from the recursive definition of u_n by noticing that the sum of the absolute values of the coefficients in u_n is equal to the sum of the absolute values of the coefficients in u_{n-1} and that in u_{n-2}. It can be established formally using (1), induction, and the recursive definition of Lucas numbers. It also follows from (3) by Binet's formula for L_n, since when $p = q = 1$, $r = \alpha$ and $s = \beta$.

An explicit formula for $\mathbf{L}_n$

Thus, by (2) and (4), we have yet another explicit formula for L_n:

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} = \mathrm{L}_n.$$

For example,

$$\begin{aligned}\mathrm{L}_7 &= \sum_{j=0}^{3} \frac{7}{7-j} \binom{7-j}{j} \\ &= \frac{7}{7}\binom{7}{0} + \frac{7}{6}\binom{6}{1} + \frac{7}{5}\binom{5}{2} + \frac{7}{4}\binom{4}{3} \\ &= 1 + 7 + 14 + 7 \\ &= 29;\end{aligned}$$

see figure 1.

Powers of Lucas numbers

In 1970, Umansky developed formulas for the first eight powers of the Lucas number L_n (see references 1 and 3). For example,

$$\begin{aligned}
\mathrm{L}_n^1 &= \mathrm{L}_n, \\
\mathrm{L}_n^2 &= \mathrm{L}_{2n} + 2(-1)^n, \\
\mathrm{L}_n^3 &= \mathrm{L}_{3n} + 3(-1)^n \mathrm{L}_n, \\
\mathrm{L}_n^4 &= \mathrm{L}_{4n} + 4(-1)^n \mathrm{L}_n^2 - 2, \\
\mathrm{L}_n^5 &= \mathrm{L}_{5n} + 5(-1)^n \mathrm{L}_n^3 - 5\mathrm{L}_n.
\end{aligned}$$

Hoggatt's formula for L_n^m

A few months after Umansky's discovery, H. E. Hoggatt, Jr. (1921–1980), a co-founder of *The Fibonacci Association* and *The Fibonacci Quarterly*, made a fascinating observation: the absolute values of the various coefficients on the right-hand side are the same as the entries in array A. In fact, using the strong version of induction, he established the following formula for L_n^m, where $m \geq 1$:

$$\begin{aligned}
\mathrm{L}_n^m &= \mathrm{L}_{mn} + \sum_{j=1}^{\lfloor m/2 \rfloor} A(m, j)(-1)^{nj+j-1} \mathrm{L}_n^{m-2j} \\
&= \mathrm{L}_{mn} + \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{m}{m-j} \binom{m-j}{j} (-1)^{nj+j-1} \mathrm{L}_n^{m-2j}
\end{aligned}$$

(see reference 4).

Vieta's investigation of $2\sin(nx/2)$

Returning to Vieta's discovery, expressing

$$v_n = 2 \sin \frac{nx}{2}$$

in terms of

$$u_1 = 2 \cos \frac{x}{2}$$

also yields interesting dividends. To this end, first notice that v_n can also be defined recursively as follows:

$$\begin{aligned}
v_0 &= 0, \\
v_1 &= 2 \sin \frac{x}{2}, \\
v_n &= u_1 v_{n-1} - v_{n-2},
\end{aligned}$$

where $n \geq 2$.

j	0	1	2	3	4	row sums
$n=0$	0					0
$n=1$	1					1
$n=2$	1					1
$n=3$	1	1				2
$n=4$	1	2				3
$n=5$	1	3	1			5
$n=6$	1	4	3			8
$n=7$	1	5	6	1		13
$n=8$	1	6	10	4		21
						$\uparrow$ F_n

Figure 2

Thus, we have

$$
\begin{aligned}
v_0 &= 0, \\
v_1 &= v_1, \\
v_2 &= v_1 u_1, \\
v_3 &= v_1(u_1^2 - 1), \\
v_4 &= v_1(u_1^3 - 2u_1), \\
v_5 &= v_1(u_1^4 - 3u_1^2 + 1), \\
v_6 &= v_1(u_1^5 - 4u_1^3 + 3u_1), \\
v_7 &= v_1(u_1^6 - 5u_1^4 + 6u_1^2 - 1), \\
&\ \vdots
\end{aligned}
$$

again as Vieta discovered.

Since v_1 is a factor of both v_0 and v_1, it follows by induction that v_1 is a factor of v_n for every n.

As before, the absolute values of the coefficients in v_n/v_1 on the right-hand side can be used to form the triangular array B in figure 2.

The recursive definition of this array B remains the same as the one for array A, except that $B(0,0) = 0$. The degree of v_n/v_1 is $n-1$; row $2n-1$ and row $2n$ contain the same number of elements, namely n.

For example, row 7 has $\lceil 7/2 \rceil = 4$ elements and row 8 has $\lceil 8/2 \rceil = 4$ elements.

Row sums

This time, every row sum is a Fibonacci number (see figure 2):

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} B(n, j) = \mathrm{F}_n. \tag{5}$$

For example,

$$\begin{aligned}\sum_{j=0}^{\lfloor 6/2 \rfloor} B(7, j) &= \sum_{j=0}^{3} B(7, j) \\ &= 1 + 5 + 6 + 1 \\ &= 13 \\ &= \mathrm{F}_7.\end{aligned}$$

Equation (5) follows intuitively from the recursive definition of v_n and can be established using induction.

Finally, it is worth noting that array A (excluding row 0) occurs in the study of the topological indices of cycloparaffins C_nH_{2n} and array B (except the two top rows) in the study of the topological indices of paraffins C_nH_{2n+2} (see reference 1).

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Where the cycle terminates

Start with a 3-digit number $N_1 = abc$, let S_1 be the sum of its digits and let $N_2 = N_1 - S_1 + 26$. Repeat this with N_2, and so on. Try an example. What are the possible numbers at which this sequence stabilises?

Lucknow, India **M. A. Khan**

The Story of *le pli cacheté*, Number 11.668, or did Wolfgang Doeblin's Discoveries Include a Version of Itô's Formula?

DAVID FORFAR

> 'I can say, to give an idea of Doeblin's stature, that one can count on the fingers of one hand the mathematicians who, since Abel and Galois, have died so young and left behind a body of work so important'. Paul Lévy.

There is a famous formula in the theory of stochastic processes of the 'diffusion' type (such as Brownian motion or the diffusion of heat) which shows that a function of the underlying diffusion is itself a diffusion process, and the formula specifies the parameters governing the diffusion of the function in relation to the parameters governing the underlying diffusion. (In 1828, Robert Brown, FRS (1773–1858), a Scottish botanist, protégé of Sir Joseph Banks, documented the 'random motion' of pollen grains in water. No correct explanation was proffered for 77 years until Einstein, in 1905, in a famous paper on 'Brownian motion', showed that not only did the 'random motion' of the pollen particles provide good evidence for the existence of molecules in 'random motion' but also that the size of the molecules themselves could be calculated from the observed statistics of the motion of the pollen grains.)

The insightful aspect of the formula is that the function diffuses in the way the formula describes. Itô's formula involves a second derivative of the function (and not just a first derivative, as the normal rule for differentiation of a function would suggest).

The formula was discovered by the Japanese mathematician Kiyosi Itô. Kiyosi Itô was born on 7 September 1915 and died on 10 November 2008 in Kyoto, Japan.

Itô published the formula between 1942 and 1944 in Japanese, in Japanese mathematical journals, which were, not surprisingly, not read and probably not obtainable by western mathematicians (see references 1, 2, and 3) until the appearance of references 4 and 5. It is regarded as something of a touch of genius to have discovered the formula, which is now named 'Itô's fomula'.

But, unbeknown to Itô, or to the mathematical community at large, Wolfgang Doeblin (born on 17 March 1915, less than six months earlier than Itô) had discovered a version of the formula in 1938 or 1939, but had not had the chance to publish it because of the exigencies of the Second World War.

Wolfgang Doeblin was born in Berlin in 1915, the son of Alfred Döblin, the famous German author of, among other things, *Berlin Alexanderplatz*, and a doctor by profession. As Wolfgang's father was Jewish and his books banned by the Nazis, the family fled to France from Berlin in 1933 after the burning of the Reichstag building and the ascent to power of Hitler. In June 1940, Alfred Döblin fled with his wife to the United States.

Wolfgang enrolled at the École Polytechnique and, after his undergraduate studies there, he carried out mathematical research in Paris in probability theory. Wolfgang was fortunate with this choice because the theory of random (stochastic) processes was a new, exciting and rapidly expanding field and there was, in Paris, a powerful school of probability under the French probabilists Maurice Fréchet (1878–1973), Paul Lévy (1886–1971), and Emile Borel (1871–1956): the school continues to flourish to this day. Doeblin rapidly established himself as a probabilist of exceptional depth and power having submitted, by the time he was 24 (in 1939), thirteen papers to various mathematical journals and thirteen *Notes* to the *Comptes Rendus de l'Académie des Sciences* as well as his doctoral thesis.

By 1936, Wolfgang Doeblin had become a naturalized Frenchman, having adopted the name Vincent Doeblin. In 1939, he was drafted into the French Army as a *soldat téléphoniste*. He passed himself off as someone from Alsace, no doubt because of his German accent. An extraordinary aspect is that Vincent's older brother Peter had been drafted into the German Army and was fighting against the French!

Vincent Doeblin's heroic efforts in 1940 resulted in his being awarded the *Croix de Guerre avec Palmes* and the Military Medal. However, Vincent Doeblin was aware of the fate that would befall him if he was to fall into the hands of the Nazis, who would have discovered his religion and identity.

The French *Académie des Sciences* allows manuscripts to be deposited with them, under their *pli cacheté* system. The aim of this system is that discoveries that cannot be published (for whatever reason) can be given a definite date of discovery. The rules are as follows.

1. The name and address of the author must be stated on the *pli*.

2. The date of acceptance by the *Académie* is then inscribed on the *pli* and it is given a number (e.g. 11.668), which is sent to the author with a receipt.

3. The author can retract the *pli* on presentation of the receipt.

4. With evidence of the receipt, the author can require, after one year, the opening of the *pli* and, if they wish, the *Académie* can publish the contents.

5. After the author's death, the inheritors of his estate can demand, on production of appropriate legal documents of good title, the opening of the *pli*.

6. The *Académie* has the right to open the *pli* after 100 years.

In 1940, while in the army, Vincent Doeblin bought a simple jotter and scribbled down his mathematical discoveries, sending these to the French *Académie des Sciences* under their *pli cacheté* system. We surmise that the story of Evariste Galois, writing furiously, in 1832, the night before his fatal duel, to set down his profound mathematical discoveries (saying 'I have not time, I have not time…') would not have been lost on Vincent Doeblin.

In June 1940, Doeblin's military unit became surrounded by the German Army and he sensed imminent surrender. Doeblin tried to escape but there was no way out. On 21 June 1940, at the age of 25, Vincent Doeblin took his own life with a bullet to the head.

In 2000, some 60 years after it was deposited with the *Académie* in 1940, the existence of Doeblin's *pli* was discovered by the French statistician Bernard Brû, and opened thanks to Vincent's younger brother, Claude Doblin's, exercising his rights according to the rules of the *pli cacheté* system.

There, in paragraph 15 of the *pli*, stands a version of Itô's formula, with the proof of the formula, as the French probabilist Marc Yor averred in reference 6, '*tout à fait rigoureuse*' i.e. completely rigorous.

As a result of the *pli cacheté* and his published results on Markov chains (particularly the Doeblin coupling method), Wolfgang Doeblin, despite having died at age 25, is now recognised as one of the foremost probabilists of the twentieth century. An excellent DVD (see reference 7) about the short life of Wolfgang Doeblin includes technical details of his solution of Kolmogorov's equation, his use of trajectories, and his version of Itô's formula.

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Exercise for the readers of *Mathematical Spectrum*

The N random variables, X_i (for all i from 1 to N) are independent and each takes the three values $\sqrt{2/N}$, 0, $-\sqrt{2/N}$, with probabilities of $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$.

(1) What are the means, standard deviations, and moment generating functions of

$$\text{(a) } X_i, \qquad \text{(b) } |X_i|, \qquad \text{(c) } X_i^2, \qquad \text{(d) } X_i^3, \qquad \text{(e) } |X_i|^3?$$

(2) What are the means, standard deviations, and moment generating functions of

$$S_N = \sum_{i=1}^{i=N} X_i, \qquad T_N = \sum_{i=1}^{i=N} |X_i|, \qquad U_N = \sum_{i=1}^{i=N} |X_i|^2,$$

$$V_N = \sum_{i=1}^{i=N} X_i^3, \qquad W_N = \sum_{i=1}^{i=N} |X_i|^3?$$

(3) As $N \to \infty$ show, by considering moment generating functions, that the distribution of S_N becomes in the limit a normal distribution. What are the distributions (in the limit) of T_N, U_N, V_N, and W_N? Show that T_N 'blows up', U_N gets nearer and nearer to the number 1 and V_N and W_N get nearer to 0, as N becomes increasingly large, i.e. U_N (although a random variable) gradually ceases to become random, as N becomes increasingly large and becomes (in the limit) the constant 1 and V_N and W_N become the constant 0.

(4) If X_i takes the values $\{(2/N)^{1/3}, 0, -(2/N)^{1/3}\}$ with probabilities $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and each time step t/N, a new independent random variable is generated creating a process $S_N = \sum_{i=1}^{i=N} X_i$, show that S_N has a mean of 0 but infinite standard deviation (as $N \to \infty$) and therefore the process 'blows up' as $N \to \infty$.

(5) List all the types of distribution you know that have the property that when the independent random variables X_i $(i = 1, \ldots, N)$ all have the same type of distribution, then the random variable $S_N = \sum_{i=1}^{i=N} X_i$ (i.e. the convolution of all the X_i) also has the same type of distribution (i.e. distributions that are closed under convolutions). Consider both discrete and continuous distributions. Hint: use the following fact:

$$e^x = 1 + x + \frac{x^2}{2} e^{\theta x} \quad \text{for all } x, \text{ where } 0 \le \theta < 1.$$

Solutions

- The variables (a), (b), (c), (d), and (e) take values

$$\left(\sqrt{\frac{2}{N}}, 0, -\sqrt{\frac{2}{N}}\right), \quad \left(\sqrt{\frac{2}{N}}, 0\right), \quad \left(\frac{2}{N}, 0\right),$$
$$\left(\left(\sqrt{\frac{2}{N}}\right)^3, 0, -\left(\sqrt{\frac{2}{N}}\right)^3\right), \quad \left(\left(\sqrt{\frac{2}{N}}\right)^3, 0\right)$$

with probabilities

$$\left(\tfrac{1}{4}, \tfrac{1}{2} \cdot \tfrac{1}{4}\right), \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad \left(\tfrac{1}{4}, \tfrac{1}{2} \cdot \tfrac{1}{4}\right), \quad \left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

- The means of variables (a), (b), (c), (d), and (e) are

$$0, \quad \frac{1}{\sqrt{2N}}, \quad \frac{1}{N}, \quad 0, \quad \frac{\sqrt{2}}{N\sqrt{N}};$$

their standard deviations are

$$\frac{1}{\sqrt{N}}, \quad \frac{1}{\sqrt{2N}}, \quad \frac{1}{N}, \quad \frac{2}{N\sqrt{N}}, \quad \frac{\sqrt{2}}{N\sqrt{N}}.$$

- For the variables S_N, T_N, U_N, V_N, and W_N the means are N multiplied by the values for the means in the previous solution and the standard deviations are $\sqrt{N}$ multiplied by the values for the standard deviations in the previous solution (as the expectation is a linear function and the variance is a linear function if the variables are independent). The means are

$$0, \qquad \sqrt{\frac{N}{2}}, \qquad 1, \qquad 0, \qquad \frac{\sqrt{2}}{\sqrt{N}};$$

the standard deviations are

$$1, \qquad \frac{1}{\sqrt{2}}, \qquad \frac{1}{\sqrt{N}}, \qquad \frac{2}{N}, \qquad \frac{\sqrt{2}}{N}.$$

- The moment generating functions of (a), (b), (c), (d), and (e) are

$$\frac{1}{4}\exp\left(t\sqrt{\frac{2}{N}}\right)+\frac{1}{2}+\frac{1}{4}\exp\left(-t\sqrt{\frac{2}{N}}\right),$$

$$\frac{1}{2}+\frac{1}{2}\exp\left(t\sqrt{\frac{2}{N}}\right), \qquad \frac{1}{2}+\frac{1}{2}\exp\left(t\frac{2}{N}\right),$$

$$\frac{1}{4}\exp\left(t\left(\sqrt{\frac{2}{N}}\right)^3\right)+\frac{1}{2}+\frac{1}{4}\exp\left(-t\left(\sqrt{\frac{2}{N}}\right)^3\right), \qquad \frac{1}{2}+\frac{1}{2}\exp\left(t\left(\sqrt{\frac{2}{N}}\right)^3\right).$$

- The moment generating functions of S_N, T_N, U_N, V_N, and W_N are the above raised to the Nth power.

- By using the hint, we see that, in the limit as $N \to \infty$, the moment generating functions become

$$\exp\left(\tfrac{1}{2}t^2\right), \qquad \text{'blows up'}, \qquad \exp(t), \qquad 1, \qquad 1.$$

  S_N becomes a normal distribution, as the moment generating function for a normal distribution is $\exp(\frac{1}{2}t^2)$. T_N 'blows up' and U_N gets nearer and nearer to 1 (as $\exp(t) = \exp(1*t)$) and V_N and W_N get nearer to 0 (as $\exp(0) = 1$) as N becomes increasingly large i.e. U_N (although a random variable) gradually ceases to become random and becomes (in the limit) the constant 1 and V_N and W_N become the constant 0.

- In mathematical terms, we say U_N converges to 1 (and V_N and W_N converge to zero) in 'mean square' (or in the 'L^2 norm') as the mean of the distribution of U_N is 1 and the standard deviation of the distribution of U_N tends to zero.

- Discrete distributions: binomial with respect to the number of trials, Poisson with respect to the parameter. Continuous distributions: normal, gamma with respect to the degrees of freedom.

The moral of the story is that if you add up an infinite number of infinitely small random variables (like X_i^2), the result can be certainty, not randomness! Put crudely, this is what Itô's formula boils down to.

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The Big Mac® and the Gamma Distribution

JOHN C. B. COOPER

Introduction

The probability density function of the gamma distribution is given by

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x},$$

where $x > 0$, $\alpha > 0$, $\beta > 0$, and $\Gamma(\alpha)$ is Euler's gamma function, i.e.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \mathrm{e}^{-x}\,\mathrm{d}x.$$

The mean and variance of the distribution are $\mu = \alpha/\beta$ and $\sigma^2 = \alpha/\beta^2$ (see the Appendix), from which it can easily be demonstrated that

$$\alpha = \frac{\mu^2}{\sigma^2} \quad \text{and} \quad \beta = \frac{\mu}{\sigma^2}. \tag{1}$$

The mode is given by $(\alpha - 1)/\beta$.

The gamma distribution is positively skewed and the parameter α governs its shape. When $\alpha = 1$, the gamma distribution reduces to the exponential distribution, namely $f(x) = \beta \mathrm{e}^{-\beta x}$, and the density function assumes its maximum value at $x = 0$.

For all other values of $\alpha > 0$, $f(x)$ is zero at $x = 0$, rises to a maximum, and thereafter declines. As α increases, the skewness decreases and the distribution becomes more peaked, approaching the normal distribution in the limit. Graphs of various gamma distributions and the exponential distribution may be viewed in references 1 and 2.

As an example of a gamma distribution, let us assume that $\alpha = 2$ and $\beta = 0.5$ so that the probability density function is $f(x) = 0.25x\mathrm{e}^{-0.5x}$ with $\mu = 4$ and $\sigma^2 = 8$. We may now calculate the probability that, say, $X \le 5$ from the probability distribution function as follows:

$$\mathrm{P}(X \le 5) = \int_0^5 f(x)\,\mathrm{d}x = \int_0^5 0.25x\mathrm{e}^{-0.5x}\,\mathrm{d}x = [-0.5\mathrm{e}^{-0.5x}(x+2)]_0^5 = 0.71.$$

Empirical application

McDonald's, the fast food chain, operates restaurants in over 100 countries. Probably its most famous offering is the Big Mac, an identical product wherever it is produced and consumed. (The author can confirm that the Big Mac in Scotland is indistinguishable from the Big Mac in Hungary!)

Table 1 Working time required to purchase a Big Mac in 70 countries. Source: Union Bank of Switzerland and author's calculations.

Minutes	Observed frequency	Observed percentage	Theoretical probability	Expected frequency
1–20	28	0.400	0.360	25
20–30	10	0.143	0.161	11
30–40	9	0.129	0.127	9
40–50	7	0.100	0.096	7
50–60	4	0.057	0.071	5
60–70	3	0.043	0.053	4
70–80	4	0.057	0.037	3
80–90	1	0.014	0.028	2
90–100	1	0.014	0.019	1
> 100	3	0.043	0.048	3
Total	70	1.000	1.000	70

As part of its ongoing research into the purchasing power of earnings worldwide, the Union Bank of Switzerland periodically calculates the working time required to buy certain homogeneous commodities in 70 cities. One commodity so examined is the Big Mac. The bank computes this time by dividing the local price of the hamburger by a weighted average of the net hourly pay of 13 different occupations. The relevant figures for 2003 are shown in table 1 and figure 1 (see reference 3). A brief perusal suggests that these times are positively skewed. The times actually range from 185 minutes in Nairobi to 10 minutes in Chicago, Los Angeles, Miami, and Tokyo. Indeed the mode is 10 minutes while the mean time is 36.9429 minutes with a standard deviation of 31.3160.

To establish whether the gamma distribution would provide a reasonable fit to this data, estimates for the two parameters α and β were calculated by substituting $\mu = 36.9429$ and $\sigma = 31.3160$ into (1) to obtain $\alpha = 1.3916$ and $\beta = 0.0377$. Interestingly, $(\alpha - 1)/\beta = 10.3873$ is a good estimate of the mode of 10.

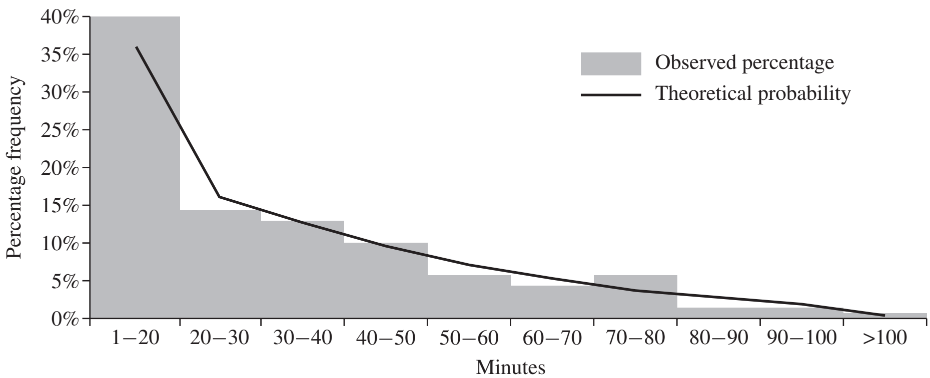

Figure 1 Working time required to purchase a Big Mac in 70 countries.

Thereafter, theoretical probabilities were obtained using the freely available Simple Interactive Statistical Analysis (SISA) software package; these are shown in table 1 together with the theoretical frequency of occurrence of the various times. Clearly this particular gamma distribution provides a remarkably good fit to the observed data.

Appendix

The kth moment of the gamma distribution is given by

$$\mathrm{E}(X^K) = \int_0^\infty x f(x)\,\mathrm{d}x = \int_0^\infty x^k \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x}\,\mathrm{d}x.$$

This may be rewritten as

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} \int_0^\infty \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} x^{\alpha+k-1} \mathrm{e}^{-\beta x}\,\mathrm{d}x = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k}.$$

Therefore

$$\mathrm{E}(X) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\beta} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta} = \frac{\alpha}{\beta}$$

and

$$\mathrm{E}(X^2) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\beta^2} = \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta^2} = \frac{\alpha(\alpha+1)}{\beta^2}.$$

Thus

$$\mu = \mathrm{E}(X) = \frac{\alpha}{\beta} \quad \text{and} \quad \sigma^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{\alpha}{\beta^2}.$$

References

1 J. E. Freund, *Mathematical Statistics with Applications* (Pearson, Upper Saddle River, NJ, 2004).

2 I. B. Hossack, J. H. Pollard and B. Zehnwirth, *Introductory Statistics with Application in General Insurance* (Cambridge University Press, 1999).

3 Union Bank of Switzerland, *Prices and Earnings* (Zurich, 2003).

John Cooper *is a senior lecturer in Financial Economics at Glasgow Caledonian University and holds visiting professorships in the USA, Peru, and Hungary. His research interests include the application of mathematical and statistical methods in financial decision making.*

A faux Pas(cal)?

What is the next number in the sequence

$$1,\ 11,\ 121,\ 1331,\ 14641?$$

Christchurch,
New Zealand

John Mahony

Mathematics in the Classroom

Is π in π?

Once Y9 students (age 13–14) are confident with the idea of π and in using it to calculate the area and circumference of a circle it is a good idea to introduce them to the irrational properties of this special number. A good starter is to engage the students in π's decimal expansion by doing a 'my birthday' search. The excellent website, reference 1, provides the perfect vehicle for this. I begin with my own date of birth, 18 November 1952, to give students the opportunity to 'beat' its position in the decimal expansion.

> The string **18111952** occurs at position 25 115 087 (counting from the first digit after the decimal point).

Following the starter, we investigate the position of some known sequences, which were first introduced in Y7 (age 11–12).

> The string **11235813** occurs at position 48 300 974.
>
> The string **1361015** occurs at position 5 505 126.
>
> The string **1491625** occurs at position 20 579.

The restrictions on the programming dictates that not all inputs work.

> The string **149162536** does not occur in the first 200 000 000 digits of π.

(Sorry! Don't give up, π contains lots of other cool strings.)

However, this stimulates discussion into whether the string occurs after 200 000 000 decimal places, and the temptation to try 'cool' strings.

> The string **00000000** occurs at position 172 330 850.
>
> The string **11111111** occurs at position 159 090 113.
>
> The string **10101010** occurs at position 73 631 390.

The lesson concludes with the following plenary, designed to stretch even the most able, and guaranteed to stimulate philosophical discussion. 'How much of π is in π?' 'Is π in π?'

> The string **31415926** occurs at position 50 366 472.
>
> The string **27182818** occurs at position 73 154 827.

If π is in π, then it behaves like a Mandlebrot set, a number contained within itself continuously. A mind-blowing thought!

There are many websites which can be used to display π. Reference 2 gives some further discussion points and can lead into data handling, a line graph to start.

> The first 1 000 000 decimal places contain: 99 959 zeros, 99 758 ones, 100 026 twos, 100 229 threes, 100 230 fours, 100 359 fives, 99 548 sixes, 99 800 sevens, 99 985 eights, and 100 106 nines.

References

1 http://www.angio.net/pi/piquery

2 http://newton.ex.ac.uk/research/qsystems/collabs/pi/

The Folkestone Academy **David Benjamin**

Letters to the Editor

Dear Editor,

Patterns in powers

The pattern formed by the squares of

$$9,\ 99,\ 999,\ 9999, \ldots$$

may be well known to readers: the respective squares are

$$81,\ 98\,01,\ 998\,001,\ 9998\,0001, \ldots.$$

The spaces are included to divide the integers into *blocks*. If we say that each square consists of two blocks of digits, the next square is obtained by adding a 9 and a 0 alternately in front of each block.

There are many other similar patterns for squares, for example of 8, 98, 998, ... But also the same method applies to higher powers. I have tabulated the powers up to the eighth for 9, 99, ... (see table 1). We notice that the method fails between 9^6 and 99^6, 9^7 and 99^7, and between 9^8 and 99^8.

Table 1

1	9	99	999	9999
2	81	98 01	998 001	9998 0001
3	729	97 02 99	997 002 999	9997 0002 9999
4	6561	96 05 96 01	996 005 996 001	9996 0005 9996 0001
5	59049	95 09 90 04 99	995 009 990 004 999	9995 0009 9990 0004 9999
6	531441	94 14 80 14 94 01	994 014 980 014 994 001	9994 0014 9980 0014 9994 0001
7	4782969	93 20 65 34 79 06 99	993 020 965 034 979 006 999	9993 0020 9965 0034 9979 0006 9999
8	43046721	92 27 44 69 44 27 92 01	992 027 944 069 944 027 992 001	9992 0027 9944 0069 9944 0027 9992 0001

Note too that for *even* powers, apart from 9^6 and 9^8, if the last block of digits is removed then the remaining blocks of digits are *palindromic*, i.e. the blocks read the same from left to right as from right to left.

I leave the reader with the following two questions.

1. When and why do the patterns fail?

2. Why do we get the palindrome with even powers?

To answer the first question it may help to consider cubes of 8, 98, and 998; or even the squares and cubes of 87, 987, and 9987.

Yours sincerely,

Alastair Summers
(57 Conduit Road
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Dear Editor,

Sums of rational cubes

It has been a long-standing problem in mathematical circles to find the sum of two rational cubes equal to a given integer (generally positive). It is a difficult nut to crack and it has been proved that not all numbers can be expanded in this way. Among the problems that cannot be solved are those for the integers 1, 3, 4, 5, 18, and 36.

Minimum solutions are generally desirable. The following are examples of solutions for certain numbers:

$$\begin{aligned}
119\,041 &= \left(\tfrac{679}{15}\right)^3 + \left(\tfrac{446}{15}\right)^3,\\
6\,017\,193 &= \left(\tfrac{1455}{8}\right)^3 + \left(\tfrac{81}{8}\right)^3,\\
16\,776\,487 &= \left(\tfrac{2427}{5}\right)^3 + \left(-\tfrac{2302}{5}\right)^3,\\
16\,776\,487 &= \left(\tfrac{1471}{6}\right)^3 + \left(\tfrac{761}{6}\right)^3,\\
24\,375\,176 &= \left(\tfrac{1382}{3}\right)^3 + \left(-\tfrac{1256}{3}\right)^3.
\end{aligned}$$

Although it is not possible to solve for the number 1, I have found the following approximate solution, which is correct to 10 decimal places:

$$\left(\tfrac{76504}{96389}\right)^3 + \left(\tfrac{76504}{96389}\right)^3 = 1 \quad \text{(to 10 decimal places)},$$

the actual value obtained being $1.000\,000\,000\,036\,022\,05\ldots$.

As an example of the difficulty of this type of problem, the French mathematician Adrien-Marie Legendre 'proved' that it was impossible to find two fractions the sum of whose cubes is equal to 6. The 'proof' was blown out of the water by the most famous English puzzlist, Henry Ernest Dudeney, who discovered the following (simple?) solution:

$$\left(\tfrac{17}{21}\right)^3 + \left(\tfrac{37}{21}\right)^3 = 6.$$

(See 'The Doctor of Physic' in *The Canterbury Puzzles and Other Curious Problems* by H. E. Dudeney.)

Yours sincerely,

Bob Bertuello
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Dear Editor,

Comparison of the rational mean and Newton–Raphson convergence

I am writing to extend the good work in Bob Bertuello's letter on *The rational mean* in Volume 39, Number 2, Page 80. I had never come across this method before and was most impressed!

I was keen to investigate why it worked so well, but in the course of doing so discovered, contrary to what Bob's Example 2 suggests, that it doesn't usually work well if P is not constant.

To remind readers, we are seeking the nth root of P. The rational mean method (RM for short) states that if x is an estimate for the nth root then

$$f(x) = \frac{(n+1)Px + (n-1)x^{n+1}}{(n-1)P + (n+1)x^n}$$

is a much better estimate. This is indeed the case if P is constant. To prove this, form a Taylor series centred on the root, α, we are trying to find. Noting that $\alpha^n = P$, we have

$$f(\alpha) = \frac{(n+1)P\alpha + (n-1)\alpha^{n+1}}{(n-1)P + (n+1)\alpha^n} = \frac{2n\alpha^{n+1}}{2n\alpha^n} = \alpha,$$
$$f'(x) = \frac{(n+1)(n-1)(x^n - P)^2}{((n-1)P + (n+1)x^n)^2}.$$

For $x = \alpha$, $x^n - P = 0$; hence, $f'(\alpha) = 0$. Since the factor $x^n - P$ is squared it follows that $f''(\alpha)$ also vanishes. Next use Leibniz's theorem for repeated differentiation of a product to find $f'''(\alpha)$ as follows. Since $\alpha^n - P = 0$ we only need the second derivative of $(x^n - P)^2$ multiplied by

$$\frac{(n+1)(n-1)}{((n-1)P + (n+1)x^n)^2},$$

which, with $x = \alpha$, is

$$2n^2\alpha^{2n-2}\frac{(n+1)(n-1)}{((n-1)P + (n+1)\alpha^n)^2} = 2n^2\alpha^{2n-2}\frac{(n+1)(n-1)}{(2nP)^2}$$
$$= \frac{(n+1)(n-1)}{2\alpha^2}.$$

Hence, if $x - \alpha$ is sufficiently small

$$f(x) \approx f(\alpha) + f'(\alpha)(x-\alpha) + \frac{1}{2}f''(\alpha)(x-\alpha)^2 + \frac{1}{6}f'''(\alpha)(x-\alpha)^3$$
$$= f(\alpha) + \frac{(n+1)(n-1)}{12\alpha^2}(x-\alpha)^3.$$

In other words the rate of convergence is of the third order, one order better than the Newton–Raphson rate (NR for short).

However, if P is a function of x there are another two terms in the expression for $f'(x)$, which therefore does not vanish when $x = \alpha$. These terms combine to give

$$\frac{4nP'x^{n+1}}{((n-1)P + (n+1)x^n)^2};$$

whence, $f'(\alpha) = P'\alpha/nP$, and so

$$f(x) \approx \alpha + \frac{P'\alpha}{nP}(x-\alpha)$$

for sufficiently small $x - \alpha$, in other words we only have first-order convergence. Note that P' is evaluated here at $x = \alpha$.

In Bob's Example 2, the equation to be solved was $x^5 + 3x^2 - 1000 = 0$. In this case

$$\frac{P'\alpha}{nP} = \frac{-6\alpha^2}{5(1000 - 3\alpha^2)} \approx -0.02,$$

since $\alpha \approx 4$. This means that once the iterates get close enough to the root, each RM iterate is approximately 50 times closer to the root than the previous one. On the other hand the NR method takes some time to 'settle down'. As a result the root correct to nine decimal places is reached sooner by the RM method. However, by the time we get to the 21st iterates, I estimate that the NR iterate is of the order of 10 billion times closer to the true root than the RM one! I should point out that this will only show up if the technology is good enough to work to 40 decimal places!

For interest, and comparison with Bob's Example 2, which favours RM, I have compiled table 1, which compares the two methods for this example and some other equations. In the last example failure is inevitable for RM because $|P'\alpha/nP|$ is greater than 1 (in fact it is about 15).

Table 1

Equation	RM: number of iterates to reach an accuracy of 9 decimal places	NR: number of iterates to reach an accuracy of 9 decimal places	Number of iterates before NR gives more accurate estimate than RM	Initial estimate	Root found given to 9 decimal places
$x^5 + 3x^2 - 1000 = 0$	9	19	21	1	3.943 217 720
$x^5 + 3x^4 - 1000 = 0$	60	17	16	1	3.519 253 369
$x^4 - x^2 - 1 = 0$	21	6	3	1	1.272 019 650
$x^2 - x - 1 = 0$	19	5	2	1	1.618 033 989
$x^3 + 3x + 1 = 0$	52	23	22	1	−3.103 803 403
$x^3 - 3x^2 + 7x + 4 = 0$	fails	6	–	1	−0.464 595 701

In addition to the examples given, I used an EXCEL® spreadsheet to do the necessary calculations on randomly generated monic polynomials of degrees between 2 and 5 which had integer coefficients numerically less than 6. I then took a sample of 20, eliminating cases with integer roots or no roots at all: in four cases one or both of the two methods failed, e.g. because of division by 0. In the remaining 16 cases NR achieved an accuracy of nine decimal places before RM; in just one case RM achieved an accuracy of four decimal places before NR. I should, however, point out that the two methods sometimes converged to different roots. In every case I took an initial estimate of 1.

To sum up: for extraction of roots the rational mean method is superior to Newton–Raphson, the convergence being of third order as opposed to second order. For solving polynomial equations in general, the Newton–Raphson method is usually superior to the rational mean method because the latter now only yields first-order convergence.

Yours sincerely,

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Dear Editor,

The sum of the first n squares

In Volume 38, Number 3, Page 107, I gave a geometrical method of obtaining the formula for the sum of the first n cubes. In Volume 41, Number 3, Page 137, I gave a method of obtaining the formula for the sum of the first n squares. Here I give a geometrical method of finding this sum.

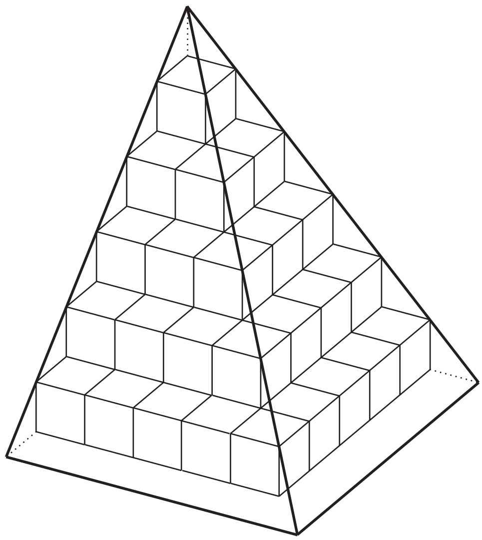

Figure 1

Consider a 'right step pyramid' consisting of one, four, nine, down to n^2 unit cubes as shown in figure 1. Its total volume is

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

The volume of the right pyramid is $\frac{1}{3}(n+1)^3$, so the volume of the step pyramid is

$$\begin{aligned}&\tfrac{1}{3}(n+1)^3 - 2 \times \tfrac{1}{2}[n + (n-1) + \cdots + 1] - (n+1)\tfrac{1}{3}\\ &\quad = \tfrac{1}{3}(n+1)^3 - \tfrac{1}{2}n(n+1) - \tfrac{1}{3}(n+1)\\ &\quad = \tfrac{1}{6}n(n+1)(2n+1).\end{aligned}$$

Yours sincerely,

Abbas Rooholamini Gugheri
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Iran)

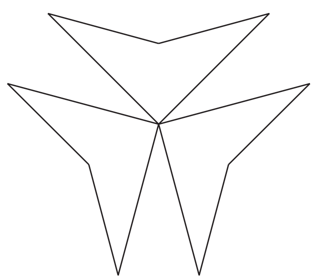
Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

42.9 Two 24-hour clocks are set going simultaneously at midnight, both showing the correct time. One gains 2 minutes per hour, the other loses 1 minute per hour. How long is it before they again show the same time simultaneously? How long is it before they again show midnight simultaneously?

42.10 Banana cross – not quite a Maltese cross! Do a transverse cut on a banana and the regular pattern shown is obtained (idealised). If the long sides are 2 cm in length and the internal angles are 90° and 30°, what is the total internal area?



(Submitted by Bob Bertuello, Midsomer Norton, Bath, UK)

42.11 In an acute-angled triangle ABC, prove that

$$\frac{\tan^3 A}{\tan^2 B} + \frac{\tan^3 B}{\tan^2 C} + \frac{\tan^3 C}{\tan^2 A} \geq \tan A \tan B \tan C.$$

(Submitted by Anand Kumar, Ramanujan School of Mathematics, Patna, India)

42.12 Find all natural numbers n and k such that

$$n^2 + (n+1)^2 + \cdots + (n+k)^2 = (n+k+1)^2 + (n+k+2)^2 + \cdots + (n+2k)^2.$$

(Submitted by Abbas Rooholamini Gugheri, Sirjan, Iran)

Solutions to Problems in Volume 42 Number 1

42.1 A square sits in a right-angled triangle with circles of radii R, r and t touching as shown in figure 1. Find the relative dimensions of the right-angled triangle and the relationship between R and t.

Solution by Bob Bertuello, who proposed the problem

Consider figure 2. From ΔFPQ, $\tan\alpha = r/3r = \frac{1}{3}$, so

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{3}{4}.$$

Hence, ΔABC is a $3:4:5$ right-angled triangle. Thus, $AE : FD = 3r : 4r$, so the similar triangles FAE and BFD have dimensions in the ratio $3:4$ and $r = \frac{3}{4}R$. The right-angled triangle XYZ has sides of lengths r, $2r-t$, $r+t$, so that

$$(r+t)^2 = (2r-t)^2 + r^2,$$

which gives $t = \frac{2}{3}r$. Hence, $t = \frac{1}{2}R$.

Also solved by Abbas Rooholamini Gugheri, Sirjan, Iran.

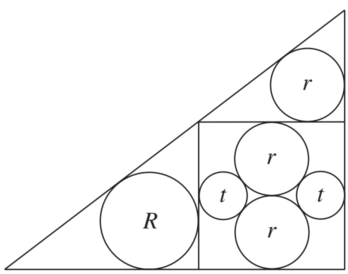


Figure 1

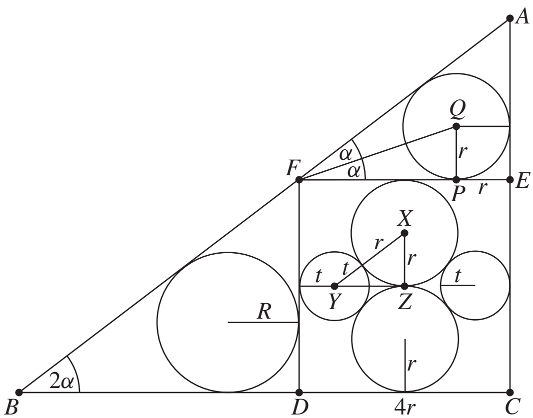


Figure 2

42.2 The normals to the curve $y = \sin nx$ at the points P and Q with x-coordinates $\pi/2n$ and $\pi/2n + h$, respectively, meet at M. Find the limiting position of M as $Q \to P$.

Solution by Guido Lasters, who proposed the problem

The normal at P has equation $x = \pi/2n$. The slope of the tangent at Q is $n\cos n(\pi/2n+h)$, so the slope of the normal at Q is

$$-\frac{1}{n\cos(\pi/2 + nh)} = \frac{1}{n\sin nh}$$

and the equation of the normal at Q is

$$y - \sin n\left(\frac{\pi}{2n} + h\right) = \frac{1}{n\sin nh}\left(x - \left(\frac{\pi}{2n} + h\right)\right).$$

The two normals meet at M with x-coordinate $\pi/2n$ and y-coordinate

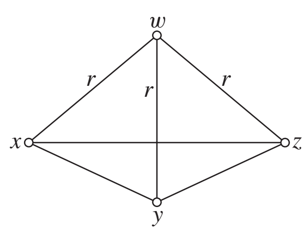
$$\begin{aligned}\sin n\left(\frac{\pi}{2n} + h\right) - \frac{h}{n\sin nh} &= \sin\left(\frac{\pi}{2} + nh\right) - \frac{nh}{n^2\sin nh}\\ &\to 1 - \frac{1}{n^2} \quad \text{as } h \to 0,\end{aligned}$$

so the limiting position of M is the point $(\pi/2n, 1 - 1/n^2)$.

42.3 Show that, given any partition into two subsets of the set of positive rational numbers smaller than 1 with denominators not exceeding 6, one of the subsets contains a, b, c such that $ab = c$.

Solution by Joshua Lam, who proposed the problem

Denote the two sets of any partition by A and B. Consider the 'graph' with six vertices, labelled 1 to 6, with every two of these vertices joined by an edge. Colour the edge joining m to n, where $m < n$, red or blue according to whether the fraction m/n lies in the set A or in B, respectively. Consider any vertex w (say). There are five edges leaving w, so there must be three red ones or three blue ones, say three red, labelled wx, wy, wz. If



one of xy, yz, xz is red, this means there is a red triangle. If xy, yz, xz are blue, then xyz is a blue triangle. Thus there is a monochromatic triangle lmn with $l < m < n$. Hence, the fractions l/m, m/n, l/n all lie in A or all in B, and

$$\frac{l}{m} \times \frac{m}{n} = \frac{l}{n}.$$

42.4 (i) Determine the number of permutations ϕ of $\{1, \ldots, n\}$ such that $\phi^2 = \mathrm{Id}$.

(ii) Determine the number of permutations ϕ of $\{1, \ldots, n\}$ such that $\phi^3 = \mathrm{Id}$.

(iii) Determine all permutations ϕ of $\{1, 2, 3, 4, 5, 6\}$ such that $\phi(i) + \phi^{-2}(i) = 7$ for all i.

Solution by M. N. Deshpande, Kavita Laghate, and M. R. Modak, who proposed the problem

(i) There is the identity. The number of 2-cycles $(a\,b)$ is $\binom{n}{2}$. The number of double 2-cycles (e.g. $(1\,2)(3\,4)$, $(1\,3)(2\,4)$, $(1\,4)(2\,3)$ from 1, 2, 3, 4) is $3\binom{n}{4}$. The number of triple 2-cycles is $5 \times 3\binom{n}{6}$. And so on. Thus, the total is

$$\sum_{k=0}^{[n/2]} (2k-1)(2k-3)(2k-5)\cdots 1\binom{n}{2k} = \sum_{k=0}^{[n/2]} \frac{(2k)!}{2k(2k-2)\cdots 2}\,\frac{n!}{(2k)!\,(n-2k)!}$$
$$= \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!\,(n-2k)!}.$$

(ii) There is the identity. The number of 3-cycles is $2\binom{n}{3}$ (e.g. $(1\,2\,3)$ and $(1\,3\,2)$ using 1, 2, 3). The number of double 3-cycles is

$$\binom{n}{6}\binom{6}{3}\frac{2 \times 2}{2}.$$

The number of triple 3-cycles is

$$\binom{n}{9}\binom{9}{3}\binom{6}{3}\frac{2 \times 2 \times 2}{3 \times 2}.$$

And so on. The sum of these simplifies to

$$\sum_{k=0}^{[n/3]} \frac{n!}{3^k k!\,(n-3k)!}.$$

(iii) It is not possible for ϕ to have a 1-cycle, i.e. for $\phi(i) = i$ for some i, for then $i + i = 7$. Suppose that ϕ contains a 2-cycle (i, j). Then $\phi(i) + \phi^{-2}(i) = 7$, so $j + i = 7$ and the possibilities are $(1\,6)$, $(2\,5)$, $(3\,4)$. If ϕ contains a 3-cycle $(i\ j\ k)$, then $\phi(i)+\phi^{-2}(i) = 7$, so $j + j = 7$, which is impossible. If ϕ contains a 4-cycle $(i\ j\ k\ l)$, then $\phi(i)+\phi^{-2}(i) = 7$, i.e. $j+k = 7$, and $\phi(j)+\phi^{-2}(j) = 7$, i.e. $k+l = 7$, so $j = l$, which is not so. Since ϕ does not have a 1-cycle, it cannot have a 5-cycle. Suppose that ϕ is a 6-cycle, say $\phi(1\,i\ j\,k\,l\,m)$. Then $\phi(1)+\phi^{-2}(1) = 7$, so $i+l = 7$, $\phi(i)+\phi^{-2}(i) = 7$, so $j+m = 7$, $\phi(m)+\phi^{-2}(m) = 7$, so $1+k = 7$ and $k = 6$, which gives $\phi = (1, i, j, 6, 7-i, 7-j)$ with $7-i \neq j$ and $7-j \neq i$, i.e. $i + j \neq 7$. Hence the possibilities are

$$(1\,6)(2\,5)(3\,4),\quad (1\,2\,3\,6\,5\,4),\quad (1\,3\,2\,6\,4\,5),\quad (1\,2\,4\,6\,5\,3),$$
$$(1\,4\,2\,6\,3\,5),\quad (1\,3\,5\,6\,4\,2),\quad (1\,5\,3\,6\,2\,4),\quad (1\,4\,5\,6\,3\,2),\quad (1\,5\,4\,6\,2\,3).$$

Reviews

Problems from Murray Klamkin. Edited by Andy Liu and Bruce Shawyer. MAA, Washington, DC, 2009. Hardback, 280 pages, $61.50 (ISBN 0-88385-828-8).

A volume of problems posed by the great Murray Klamkin during the time he spent in Canada. There are complete solutions, for which many readers will be grateful.

The Loom of God: Mathematical Tapestries at the Edge of Time. By Clifford A. Pickover. Plenum Press, New York, 1997. Hardback, 292 pages, $29.95 (ISBN 0-30645-411-4).

This book explores the connections between mathematics and religion over thousands of years, whilst providing the reader with an insight into some interesting mathematical results. All of the ideas covered are introduced through a simple story which runs throughout the entire book. Essentially, there are three characters who travel through time, examining the links between mathematics and religion. Although the story contributes very little itself towards the mathematical concepts, it does act as a somewhat entertaining complement to the heart of the book: the historical element.

There are numerous descriptions of how mathematics was used in many different societies throughout history, an example of which being the use of quipus by the Incas for storing information. Most importantly, though, religious beliefs, both past and present, are focussed upon highlighting the possible links with mathematics. Some of these links, however, are more questionable than others, such as the prominence of the number 40 in many religions, something which could easily be dismissed as coincidence. Despite this, I found the multitude of connections between religion and mathematics intriguing.

Different theories predicting the end of the human civilisation arise throughout the book. Several possibilities are analysed, ranging from the crisis of overpopulation and resulting depletion of natural resources to the near sudden destruction of the Earth by an asteroid's impact. The threat which each catastrophe poses to human life and the likelihood of each event occurring is taken into account, allowing each theory to be reviewed. I found these chapters to be particularly interesting, as they provide a particularly appealing assessment of the inevitable, whether it be due to divine or scientific causes.

Student, Nottingham High School **George Bignall**

Other books received

A Trigonometric Primer: From Elementary to Advanced Trigonometry. By Konstantine Zelator. Brainstorm Fantasia, Inc., Pittsburgh, 2005. Hardback, 885 pages, $48.00 (ISBN 0-9761810-1-0).

Statistical Practice in Business and Industry. Edited by Shirley Coleman, Tony Greenfield, Dave Stewardson and Douglas C. Montgomery. John Wiley, Chichester, 2008. Hardback, 433 pages, £55.00 (ISBN 0-470-01497-4).

Introduction to Meta-Analysis. By Michael Borenstein, Larry V. Hedges, Julian P. T. Higgins and Hannah R. Rothstein. John Wiley, Chichester, 2009. Hardback, 421 pages, £34.95 (ISBN 978-0-470-05724-7).

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Articles

Letters to the Editor

Reviews

Number patterns

$$
\begin{aligned}
1 \times 8 + 1 &= 9,\\
12 \times 8 + 2 &= 98,\\
123 \times 8 + 3 &= 987,\\
1234 \times 8 + 4 &= 9876,\\
12345 \times 8 + 5 &= 98765,\\
123456 \times 8 + 6 &= 987654,\\
1234567 \times 8 + 7 &= 9876543,\\
12345678 \times 8 + 8 &= 98765432,\\
123456789 \times 8 + 9 &= 987654321.
\end{aligned}
$$

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