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- What became of the women?
- Three proofs of Pythagoras' theorem
- How a group can do geometry

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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And What Became of the Women?

CAROLINE SERIES

'*. . . the University would think the examination of young ladies a matter altogether beyond its sphere of duty.*' — Oxford Local Examination Delegacy, 1863.

Hail the triumph of the corset
Hail the fair Philippa Fawcett.
Victress in the fray.
Crown her queen of hydrostatics
And the other Mathematics
Wreathe her brow in bay.
(See reference 9, p. 30.)

The September 1996 issue of *Mathematical Spectrum* contained an article 'What became of the Senior Wranglers?' by D. O. Forfar. In that article, only one woman, Philippa Fawcett, is mentioned. Since during the period covered by the article, 1753–1909, women were not allowed to take degrees at Cambridge, and since the women's colleges at Girton and Newnham were only established in 1869 and 1871 respectively, this omission is scarcely surprising. When one considers the lamentable state of women's education up till the late years of the nineteenth century, and the enormous public prejudice which existed against women studying science or mathematics, it is really much more surprising that, towards the end of that period, there were several women who, morally at least, did attain Wranglerhood. At the time their achievements were hailed as turning points in the struggle for women's education. Today their stories can still inspire.

In 1890, Philippa Fawcett scored the highest mark of all candidates in Part 1 of the Mathematical Tripos. She was placed 'above the Senior Wrangler'. Her triumph, in mathematics, that last bastion of superiority of the male mind, was spectacular. There was lengthy discussion and comment in national papers in England and abroad. It would have been hard to think of a more effective or timely challenge to popular prejudice. Women's powers of reasoning could no longer be said to be inferior to men's.

Shortly before Philippa's birth in 1868, the pressure for some provision for education of women at Oxford and Cambridge was growing. Henry Sidgwick, Professor of Moral Philosophy at Cambridge, and one of the driving forces in this movement, started a series of meetings, often held in the Fawcetts' drawing room, to make plans. Her mother's diary records: 'Philippa was aged about two at this time, old enough to be brought in at the tea-drinking stage at the end of the proceedings and to toddle about in her white frock and blue sash amongst the guests.'(reference 9, p. 11.)

Anne Jemima Clough, born in 1820 and brought into contact with the campaign for women's education by her brother, the poet Arthur Clough, had been running a scheme of lectures for senior girls which by 1869 had spread to some 25 centres in the north of England. At a momentous meeting held in the Fawcett's house in December 1869, it was decided to set up a similar scheme of lectures in Cambridge, and in the spring of 1870 a series of lectures was attended by 70–80 women. Provision needed to be made for a 'hall or lodging' for women wishing to attend from outside Cambridge. Proceeding with the utmost discretion, the following summer Sidgwick rented and furnished a house at his own expense and invited Miss Clough to take charge with the first five students. This was the beginning of Newnham College.

Philippa Fawcett (reference 9).

Emily Davies, born in 1830, had been a tireless campaigner for women's education for many years. It was she who inspired Philippa's aunt Elizabeth Garrett Anderson to study medicine seriously and who stood by her side through ten years of struggle which paved the way for the admission of women into the medical profession. Described by a contemporary as 'a rather dim little person with mouse coloured hair and conventional manners', she was single-minded and ruthless, inflexible in her view that women could only challenge men's intellectual dominance if they matched them at their own tests. It was Emily Davies who persuaded the Cambridge Local Examination Syndicate to agree to a trial examination for girls. Despite having only six weeks to prepare, the performance of the 83 girls was found comparable to that of the boys in all subjects except arithmetic. (The quality of teaching improved so much as a result of this embarrassing discovery that within three years no inferiority could be detected!)

As a result of Emily Davies' efforts, Girton College for women opened in 1869 at Hitchin, midway between Cambridge and London, also with five students. One of the first scholars, Miss Woodhead, daughter of a Quaker grocer, studied mathematics. She was tutored by Mr Stuart, later Cambridge Professor of Mechanics, and Mr J. L. Moulton, Senior Wrangler and Smith's prizeman, later Lord Justice Moulton, who, it is recorded, poured 'amazing illuminations on elementary mathematics'.

Notwithstanding Miss Davies' efforts, the University Council refused to admit Girton students to University examinations, although they 'carefully abstained from expressing any disapproval of the Examiners' examining the students in their private capacity and in a clandestine way'. Thus in 1872 the first three candidates for the Tripos, including Miss Woodhead, were chaperoned into Cambridge by Miss Davies and took the examination in the sitting room of the University Arms. Despite the papers arriving an hour late (the runner was given the wrong address), the three passed with flying colours. When the news reached Hitchin, elated young women climbed onto the roof and rang the alarm bell so loudly that fire engines were got out.

In 1873 Girton College moved to its present location on the edge of Cambridge, where already more than half the professors admitted women to their lectures. Special lectures were still given under the Sidgwick arrangement, and other lecturers cycled out to Girton to coach. In a very short time, Newnham and Girton students began to challenge old prejudices and in particular to challenge men in the examinations. Selected girls were allowed to take the University examinations, but this was only by special permission, not by right, nor were their names to be included in the lists of results. Nevertheless, between 1874 and 1881, 21 students had entered for a Tripos examination and all had succeeded, four having been placed in the first class.

In 1876, the 18-year-old Charlotte Angas Scott was awarded a scholarship to Girton to study mathematics 'on the basis of home tutoring.' Charlotte had had no formal schooling but her father, Caleb Scott, a dynamic man and president of a nonconformist college near Manchester, had doubtless encouraged his daughter's studies. The entering class contained 11 girls.

Charlotte took the Mathematics Tripos in January 1880. Campaigning by Emily Davies had gained the girls permission to sit the same examinations as the men (in different rooms, of course). Their results would be read out after the men's, but in mathematics each candidate would be assigned a place relative to the ordered list of male candidates. The news leaked out that Charlotte had been placed eighth. Women were not allowed at the ceremony at which the results were read out, but when it came to the eighth on the list the undergraduates called out 'Scott of Girton, Scott of Girton' and there was such an uproar that the poor man's name could not be heard.

Charlotte Scott was the first woman Wrangler. Her achievement, and 'in a man's subject' at that, made a deep public impression. Recalling the event at her retirement celebrations 45 years later, Professor Harkness, who at the time had been a schoolboy in Cambridge, said that he believed her achievement marked the turning point in England from 'the theoretical feminism of Mill and others to the practical educational and political advances of the present time.' So strongly was public opinion aroused that a petition with over 10,000 signatures to grant women the right to sit examinations and be admitted to degrees was presented to the Cambridge authorities. Arthur Cayley, a leading Cambridge mathematician, was one of the main supporters and Charlotte's triumph cited as one of the main grounds. After a year of public pressure, the University voted in 1881 to grant women the right to be examined and to have their names on the official class lists, though in a separate table from the men. The other request in the petition fared worse: women were not eligible for Cambridge degrees until 1948.

Charlotte Scott with her entering class at Girton. She is standing third from the left (reference 4).

Despite Charlotte's achievement, it was still widely believed that mathematics was peculiarly incompatible with female thought processes. In fact 34 out of the 40 girls who entered the first preliminary trial for the Cambridge Local examination in arithmetic failed. As Henry Fawcett reputedly remarked: 'he did not imagine if the Universities were opened to women they would produce any Senior Wranglers.' He had not reckoned with his own daughter.

Philippa Fawcett came of a distinguished family. Her father Henry rose to be Postmaster General under Gladstone and was the man responsible for introducing the parcel post. Her mother Millicent, later Dame Millicent, was one of the leaders of the non-violent campaign for women's votes. Philippa herself was, in the words of one her Newnham contemporaries, 'modest and retiring almost to a fault'. She lived a very regular and quiet life and was coached by Mr E. W. Hobson of Christ's, a Senior Wrangler himself and judged to be the second best coach. She also played hockey. Philippa did outstandingly well in the exams she sat in the second year, with 75 more marks than the top Trinity man. Everyone anticipated a brilliant result in the Tripos.

The scene in the Senate when the results were to be announced is recorded in a letter written by Philippa's second cousin Marion: '...the gallery was crowded with girls and a few men...The floor was thronged by undergraduates... All the men's names were read first, the Senior Wrangler was much cheered... At last the man who had been reading shouted 'Women'. The undergraduates yelled 'Ladies' and for some moments there was a great uproar. A fearfully agitating moment for Philippa it must have been; the examiner could not attempt to read the names until there was a lull. Again and again he raised his cap, but he would not say 'ladies' instead of 'women' and quite right I think... At last he read Philippa's name, and announced she was 'above the Senior Wrangler'. There was great and prolonged cheering; many of the men turned towards Philippa, who was sitting in the gallery with Miss Clough, and raised their hats. When the examiner went on with the other names there were cries of 'Read Miss Fawcett's name again' but no attention was paid to this. I don't think any other women's names were heard, for the men were making such a tremendous noise...'(reference 9, p. 28.)

On her arrival back at College, Philippa was greeted by a crowd of fellow students and carried into Hall. Flowers, letters and telegrams poured in throughout the day. That evening there was an impromptu college feast and she was carried three times round a bonfire on the hockey pitch. The triumphal lay, whose first verse heads this article, was composed in her honour. The story made the lead in the *Telegraph* the next day: 'Once again has woman demonstrated her superiority in the face of an incredulous and somewhat unsympathetic world... And now the last trench has been carried by Amazonian assault, and the whole citadel of learning lies open and defenceless before the victorious students of Newnham and Girton. There is no longer any field of learning in which the lady student does not excel.'(reference 9, p. 30.)

The last (moral) Wrangler of this period, Grace Chisholm Young, was born in 1868. Her father, a distinguished civil servant but already almost sixty when she was born, retired when Grace was only seven, and took an active role in supervising her education at home. Grace won a scholarship to Girton in 1889. She became a Wrangler in Part 1 of the Tripos in 1892. Immediately afterwards she and a friend Isabel Maddison went to Oxford and sat for the final honours school in Mathematics, according to Dame Mary Cartwright (see postcript), 'just to show'. Grace obtained the highest marks of all students that year and became the first person of either sex to obtain a First in any subject at both Oxford and Cambridge. Grace and Isabel were the first women to take finals in Mathematics at Oxford, and it seems that no woman did so again until Dame Mary herself in 1923.

What did these three women do afterwards?

Grace Emily Chisholm (reference 7).

Charlotte Scott began lecturing at Girton and began work on a doctorate under Cayley. Since Cambridge did not grant advanced degrees to women, she took a B.Sc. from the University of London by external examination in 1882, and a Ph.D. by the same route in 1885. By a great stroke of good fortune, she was almost immediately offered the job of Head of the Mathematics Department at the newly founded American women's college Bryn Mawr, where she remained until her retirement in 1925. Scott came to be widely recognised as a mathematician. A first edition of *American Men of Science* shows her name starred. Her text on analytic geometry was reprinted after thirty years. She never married, but supported and encouraged generations of women mathematicians, many of whom went on to teach all over the United States. She played a leading role in American mathematical life and was widely respected as a scholar and teacher, a wise and gifted administrator, and a rock of integrity. On her retirement she returned to England and is buried in St Giles' churchyard, Cambridge.

In 1891, Philippa Fawcett, together with the Senior Wrangler, G.T. Bennett, was placed in the top division of the first class of Part II. Philippa was awarded a scholarship at Newnham which gave her a further year of study. During this year she made her only contribution to research, a long paper on the motion of helical bodies in liquid. For the next 14 years she was a Newnham College lecturer. Then, following a trip to South Africa, she was invited to take up a post as a lecturer in a normal school in Johannesburg where she trained mathematics teachers. In 1905 she returned to England as principal assistant to the Director of Education in the newly formed London County Council. (She was, remarkably, offered this job without interview and at the same salary as a man.) She continued in this post till her retirement in 1934. She died in 1948, two months after her eightieth birthday and one month after Cambridge women were finally granted degrees.

After the Oxford finals, Grace Chisholm returned to Cambridge and completed Part II of the Tripos. There was no possibility of a woman getting a Fellowship at Cambridge. However at just that time, as part of an experiment, a small group of women were to be recruited to study at Göttingen under Felix Klein, one of the leading mathematicians of the day. So as to be on the safe side and so as to establish no unwelcome precedents, the women were to be foreigners, and, just to be sure, their subject would be mathematics. As Klein explained later : 'Mathematics had here rendered a pioneering service to the other disciplines. With it matters are, indeed, most straightforward. In mathematics, deception as to whether real understanding is present or not is least possible.'

Thus Grace Chisholm became one of three women admitted to Göttingen in 1893. They had to behave very discreetly: 'We are to go to Prof. Klein's private office before the regular time for changing classes so as not to meet the students in the halls and from there we are to go into the class.'

All went smoothly and Grace became the first woman in East Prussia to gain a Ph.D., which she did *magna cum laude* in 1895. She returned to England and married one of her former tutors, W. H. Young. Following a visit of Klein in 1897, they decided to 'throw up filthy lucre, go abroad, and devote ourselves to research.' Until their marriage, Young seems not to have done any research. Together, however, they began to publish many papers, and it seems probable that Grace's contributions, even to those written under her husband's name alone, were considerable. Their work was strongly influenced by the new ideas with which Grace had come into contact in Germany, and had in turn a strong influence in England, helping to establish the new standard of rigour which was the foundation of Cambridge's reputation as a world centre of pure mathematics.

Grace had very many interests outside mathematics and even found time to achieve her ambition of studying medicine. She brought up six children, two of whom themselves became mathematicians. Her husband died in 1942, two years before Grace herself.

Postscript

In the second half of the 20th century, education for women has become the norm. There is now a good sprinkling of women mathematicians in university posts although not so many in the senior ranks. Only two have achieved the distinction of being a Fellow of the Royal Society: Dame Mary Lucy Cartwright (1900–1998), Mistress of Girton 1949–68, whose work straddles the 20th century and was foundational for the modern theory of chaos and, much more recently, Dusa McDuff (1945–), elected an FRS in 1994 in recognition of her work on symplectic geometry.

It was Dusa's inspiration to organise a meeting of women mathematicians in London in September 1995 which was attended by 50 women from all over Britain. There was a lively day of talks and 'BWM Day' has become an annual event, taking place in 1998 for the first time outside London. BWM (no prizes for decipherment!) is loosely affiliated to a larger organisation European Women in Mathematics, founded in 1986, which organises various activities for professional women mathematicians including a biennial meeting somewhere in Europe. For more information, see the EWM web page: `http://www.math.helsinki.fi/EWM`

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We gratefully acknowledge permission to reproduce the following illustrations: Philippa Fawcett (p. 49), by permission of The Principal and Fellows, Newnham College, Cambridge; Charlotte Scott (p. 50) and Grace Emily Chisholm (p. 51), by permission of The Mistress and Fellows, Girton College, Cambridge.

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Three New Proofs of the Pythagorean Theorem

MICKEY SPRADLIN and MICHELLE WATKINS

Have you ever actually seen a proof of Pythagoras' Theorem? Here are three!

For the past three thousand years, numerous proofs of the Pythagorean Theorem have been found. In Dr Elisha Loomis' book *Pythagorean Proposition* (NCTM 1968), there are three hundred and seventy proofs. This does not mean that the proofs have been exhausted. Here we have three more.

Proof 1. (Michelle Watkins) As in figure 1, two congruent right triangles ABC, DEF are positioned such that A, C, D, E are collinear and B, E, F are collinear. Name the intersection point of FD and BC as G. Draw BH perpendicular to DF.

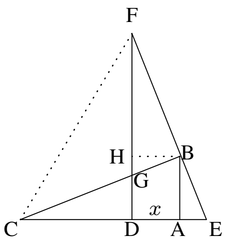

Figure 1.

Let AB $=$ DE $= a$, AC $=$ DF $= b$, and BC $=$ EF $= c$. Let DA $= x$. Then BH $= x$, FH $=$ DF $-$ DH $=$ DF $-$ AB $= b - a$, and CE $=$ CA $+$ AE $=$ CA $+$ (DE $-$ DA) $= b + a - x$.

Since the two triangles BHF and DEF are similar, the ratio equality BH/FH $= a/b$ is true. Hence

$$x/(b-a) = a/b, \quad x = a(b-a)/b.$$

Now look at triangle CEF. We use two ways to find its area.

Area of CEF $=$ DF $\times$ CE$/2 = b(a+b-x)/2$;
Area of CEF $=$ BC $\times$ EF$/2 = c^2/2$.
Therefore

$$\begin{aligned} c^2 &= b(a+b-x) \\ &= b(a+b-a(b-a)/b) \\ &= ab + b^2 - ab + a^2 \\ &= a^2 + b^2. \end{aligned}$$

The proof is complete.

Proof 2. (Michelle Watkins) Let triangles ABC, BDE be two congruent right triangles and put them in a position in figure 2 such that BD is perpendicular to AC, and hence DE is parallel to AC. Let BC $=$ ED $= a$, AB $=$ BD $= b$, AC $=$ BE $= c$. Extend BA and BC to meet the extension of ED at F, G respectively. Since AC, ED are parallel, angles ACB and BED are equal. Therefore angles BED and FGB are equal too. The two triangles BED and BDG are congruent. We have BE $=$ BG $= c$, DE $=$ DG $= a$.

Since triangles BFD and ABC are similar, the following ratio equalities are true:

BF$/$BD $=$ AC$/$BC, or BF$/b = c/a$, BF $= bc/a$;
FD$/$BD $=$ AB$/$BC, or FD$/b = b/a$, FD $= b^2/a$.

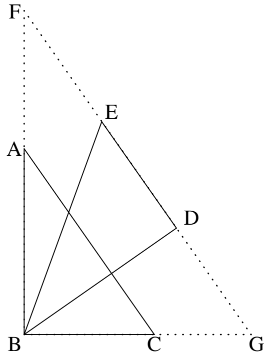

Figure 2.

Now we use two ways to find the area of triangle FBG.
Area of FBG $=$ BF $\times$ BG$/2 = (bc/a)c/2 = bc^2/(2a)$;
Area of FBG $=$ BD $\times$ FG$/2 = b$(FD $+$ DG)$/2$
$= b(b^2/a + a)/2$.
Therefore,

$$\begin{aligned} bc^2/a &= b(b^2/a + a), \\ c^2 &= a^2 + b^2. \end{aligned}$$

The proof is complete.

Proof 3. (Mickey Spradlin) Let ABC be a right triangle such that BC $= a$, AC $= b$, AB $= c$ and C is a right angle. Extend CB to a point D. Use AD as the diameter to draw a circle, which will pass through point C since angle C is a right angle. Extend AB to meet the circle at point E. Connect DE. Then angle AED is also a right angle. See figure 3.

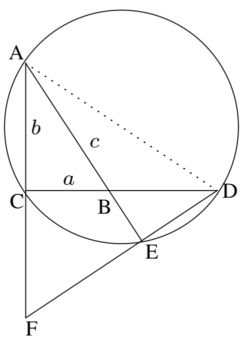

Figure 3.

Triangles ABC and BDE are similar. Hence DE/BE $=$ b/a, BD/BE $= c/a$, or

$$\mathrm{DE} = (b/a)\mathrm{BE}, \quad \mathrm{BD} = (c/a)\mathrm{BE}. \tag{1}$$

Now extend AC, DE to meet at point F. Then triangles AEF and ABC are similar, so AE/EF $= b/a$, or $(c + \mathrm{BE})/\mathrm{EF} = b/a$. Thus

$$ac + a\mathrm{BE} = b\mathrm{EF}. \tag{2}$$

Since triangles DCF and ABC are similar, DF/DC $= c/b$, or $(\mathrm{EF} + \mathrm{DE})/(a + \mathrm{BD}) = c/b$. Therefore

$$b\mathrm{EF} + b\mathrm{DE} = ac + c\mathrm{BD}. \tag{3}$$

Equations (2) and (3) yield

$$ac + a\mathrm{BE} = ac + c\mathrm{BD} - b\mathrm{DE}. \tag{4}$$

Replacing DE, BD in (4) by (1), we have

$$a\mathrm{BE} = c(c/a)\mathrm{BE} - b(b/a)\mathrm{BE}. \tag{5}$$

Now we have

$$a = \frac{c^2}{a} - \frac{b^2}{a},$$

giving $a^2 + b^2 = c^2$.

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Some Algebraic Rules in Differential Calculus

THOMAS BIER

The rule for differentiating a product extends to Leibniz's theorem for the nth derivative of a product. But how does the chain rule extend?

The product rule for differentiation

Assume $f(x), g(x), h(x)$ are functions with a suitable domain and range which can be differentiated often enough. Their ith derivatives are denoted by $f^{(i)}, g^{(i)}, h^{(i)}$. Assume $n \geqslant 1$ is any integer.

It is well known that there is a rule for taking the nth derivative of the product of such functions which, for two functions, takes the form of the binomial formula:

$$(f \cdot g)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} \cdot g^{(n-i)} \tag{1}$$

$$= \sum_{a_1+a_2=n} \frac{n!}{a_1!a_2!} f^{(a_1)} g^{(a_2)}.$$

For three functions the formula is

$$(f \cdot g \cdot h)^{(n)} = \sum \frac{n!}{a_1!a_2!a_3!} f^{(a_1)} g^{(a_2)} h^{(a_3)}\ , \tag{2}$$

where the last sum is taken over all non-negative integer solutions of the equation $a_1 + a_2 + a_3 = n$. For a given n there are only finitely many such solutions of this equation, and thus only finitely many terms in the sum. Of course, there is a similar formula for the nth derivative of a product of m such functions. We want to discuss here the nth derivative of the composition of two functions.

The chain rule

The first derivative for the composition of two functions is the chain rule

$$(g(f(x)))' = g'(f(x)) \cdot f'(x). \tag{3}$$

If we differentiate again, we get

$$(g(f(x)))'' = g''(f(x)) \cdot (f'(x))^2 + g'(f(x)) \cdot f''(x). \tag{4}$$

The pattern for general n is not hard to find. We shall continue to call it the *chain rule* and it is also known under the name *formula of Faa di Bruno,* i.e.

$$(g(f(x)))^{(n)} = \sum_{i=1}^{n} g^{(i)}(f(x))$$

$$\times \sum \frac{n!}{a_1!a_2!\ldots a_n!} \left(\frac{f'}{1!}\right)^{a_1} \left(\frac{f''}{2!}\right)^{a_2} \cdots \left(\frac{f^{(n)}}{n!}\right)^{a_n}, \tag{5}$$

where the (inner) sum for a given i is taken over all non-negative integer solutions of now **two** equations

$$a_1 + \ a_2 + \cdots + \ a_n = i\,, \tag{6}$$

$$a_1 + 2a_2 + \cdots + na_n = n\,. \tag{7}$$

This formula is well known, it is listed in the relevant handbook (reference 1, in 24.1.2.II.C.) In a similar form, it was given by the Italian mathematician M. Faa di Bruno as early as 1855. Some proofs require a substantial amount of combinatorial counting; see the exercises in reference 5. The following proof is similar to reference 4.

In particular, for $n = 1$ there is only one solution $a_1 = 1$ of equations (6) and (7) and consequently we recover equation (3). Readers may like to check that for $n = 2$ we recover equation (4), and then work out the case $n = 3$ both by differentiating (4) and by using (5).

The composition of power-series and a proof of the chain rule

In order to prove the chain rule (5) we need a formula that generalizes the binomial formula in the same way that (2) generalized (1). Thus, for numbers $w_1, w_2, \ldots, w_n$, we can take their sum to the ith power and find the multinomial theorem, namely that

$$(w_1 + w_2 + \cdots + w_n)^i = \sum \frac{i!}{a_1!a_2!\ldots a_n!} w_1^{a_1} w_2^{a_2} \ldots w_n^{a_n}, \tag{8}$$

where the sum is taken over all non-negative integer solutions of the equation

$$a_1 + a_2 + \cdots + a_n = i. \tag{9}$$

As an illustration, let us work out how many terms we should have in the expansion of $(w_1 + w_2 + w_3)^3$. We find for $i = n = 3$ the following triples a_1, a_2, a_3 :

$$300, 030, 003; 210, 120, 201, 102, 021, 012; 111,$$

showing us that there should be exactly 10 terms in this expansion, with coefficients 1 for the first 3 terms, 3 for the next 6 terms, and a coefficient 6 for the last term:

$$w_1^3 + w_2^3 + w_3^3 + 3w_1^2 w_2 + 3w_1 w_2^2 + 3w_1^2 w_3 + 3w_1 w_3^2 + 3w_2^2 w_3 + 3w_2 w_3^2 + 6w_1 w_2 w_3.$$

We now prove (5) and it is clearly sufficient to do so at the point 0. So let $f(x), g(y)$ be functions which have an expansion as in the Taylor formula about 0 and it is no loss of generality to assume $f(0) = 0$. Then these functions have power-series expansions

$$f(x) = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots , \tag{10}$$

$$g(y) = d_0 + d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4 + \cdots , \tag{11}$$

with

$$c_j = \frac{f^{(j)}(0)}{j!} \text{ and } d_i = \frac{g^{(i)}(0)}{i!}. \tag{12}$$

The composition $g(f(x))$ also has a power series expansion

$$g(f(x)) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots , \tag{13}$$

with $b_0 = d_0$. To find b_n for $n \geqslant 1$ we first write

$$g(f(x)) = d_0 + d_1 f(x) + d_2 (f(x))^2 + \cdots + d_n (f(x))^n + \cdots \tag{14}$$

and, on using (10), we note that the power x^n arises only from

$$d_1(c_1 x + c_2 x^2 + \cdots + c_n x^n) + d_2(c_1 x + c_2 x^2 + \cdots + c_n x^n)^2 + \cdots + d_n (c_1 x + c_2 x^2 + \cdots + c_n x^n)^n. \tag{15}$$

But, by (8),

$$(c_1 x + c_2 x^2 + \cdots + c_n x^n)^i = \sum \frac{i!}{a_1!a_2!\ldots a_n!} (c_1 x)^{a_1} (c_2 x^2)^{a_2} \ldots (c_n x^n)^{a_n}, \tag{16}$$

where the sum is taken over all non-negative integer solutions of $a_1 + a_2 + \cdots + a_n = i$, and so the coefficient of x^n in (16) is

$$\sum \frac{i!}{a_1!a_2!\ldots a_n!} c_1^{a_1} c_2^{a_2} \ldots c_n^{a_n}, \tag{17}$$

where the sum is taken over all non-negative integer solutions of (6) and (7). Hence the coefficient of x^n in (15), and so in (14) and in (13), is

$$\sum_{i=1}^{n} d_i \sum \frac{i!}{a_1!a_2!\ldots a_n!} c_1^{a_1} c_2^{a_2} \ldots c_n^{a_n}, \tag{18}$$

where the summation without index in (17) and in (18) is taken over all non-negative integer solutions of the equations (6) and (7). Since $b_n = (g \circ f)^{(n)}(0)/n!$, it follows by (12) that

$$(g \circ f)^{(n)}(0) = \sum_{i=1}^{n} g^{(i)}(f(0)) \times \sum \frac{n!}{a_1!a_2!\ldots a_n!} \left(\frac{f'(0)}{1!}\right)^{a_1} \left(\frac{f''(0)}{2!}\right)^{a_2} \cdots \left(\frac{f^{(n)}(0)}{n!}\right)^{a_n}, \tag{19}$$

where the inner sum ranges over all the tuples $(a_1, a_2, \ldots, a_n)$ of non-negative integers satisfying (6) and (7).

Partitions and the Lah identity

In the chain rule the range of summation is described in terms of certain integer solutions of two linear equations. Another and perhaps more familiar way to state the range of summation involves the notion of a (number-) *partition* of an integer n. This is a decomposition of n in the form

$$n = n_1 + n_2 + \cdots + n_i \text{ with } 0 < n_1 \leqslant n_2 \leqslant \ldots \leqslant n_i. \tag{20}$$

The partition is then said to be of *length* i. The integers n_j are the *parts* of the partition. Given a partition of n of length i, we let for $1 \leqslant k \leqslant n$

$$a_k = |\{j \mid n_j = k\}|,$$

i.e. a_k is the number of parts which are equal to k. We shall call the n-tuple $(a_1, a_2, \ldots, a_n)$ the *type* of the partition. As a simple numerical example, the unique partition of 17 of type $(4,1,1,2,0,0,0,0,0,0,0,0,0,0,0,0,0)$ is of length 8 and has 4 parts equal to 1, 1 part equal to 2 and 3 each, and 2 parts equal to 4; thus it is given by

$$17 = 1+1+1+1\ +\ 2\ +\ 3\ +\ 4+4.$$

Now let (20) be a partition of n of length i. Then the type of the partition $(a_1, a_2, \ldots, a_n)$, is such that

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \text{total number of parts } n_j \text{ of the} \\ &\quad\ \text{partition} = i\,, \text{ and} \\ a_1 + 2a_2 + \cdots + na_n &= \text{sum of all parts } n_j \text{ of the partition} \\ &= n\,. \end{aligned}$$

Thus $(a_1, a_2, \ldots, a_n)$ satisfies (6) and (7).

Conversely, suppose that $(a_1, a_2, \ldots, a_n)$, where $a_k \geqslant 0$ for $k = 1, 2, \ldots, n$, is a solution of (6) and (7); and, for $k = 1, 2, \ldots, n$, take a_k parts $n_j = k$ if $a_k > 0$. Then the total number of parts n_j is $a_1 + a_2 + \cdots + a_n = i$, and the sum of all the n_j is $a_1 + 2a_2 + \cdots + na_n = n$, i.e.

$$n = n_1 + n_2 + \cdots + n_i$$

is a partition of n of length i. Hence there is a one to one correspondence between the partitions of n of length i and the non-negative integer solutions of (6) and (7), so that the number of these solutions is equal to the number of partitions of n of length i.

As was shown in reference 7, the integer coefficient

$$\frac{n!}{a_1!a_2!\ldots a_n!(1!)^{a_1}(2!)^{a_2}\ldots(n!)^{a_n}}, \qquad (21)$$

can be explained in terms of partitions. Indeed, for any set N with $n = |N|$ elements, a *set partition* of N is an expression

$$N = N_1 \cup N_2 \cup \ldots \cup N_i\,, \qquad (22)$$

with $N_a \neq \emptyset$ and $N_a \cap N_b = \emptyset$ for $a \neq b$. Again we shall call the N_a the *parts* of the set partition, but note that the parts themselves are now (non-empty) sets. The integer i is the *length* of the set partition. The associated tuple of integers $(a_1, a_2, \ldots, a_n)$ defined by $a_k = |\{j \mid |N_j| = k\}|$ (so that a_k is the number of sets N_j with k elements) is called the *type* of the set partition. Two set partitions of the same set N, (22) and

$$N = M_1 \cup M_2 \cup \ldots \cup M_i,$$

are regarded as equal if and only if

$$\{N_1, N_2, \ldots, N_i\}\ =\ \{M_1, M_2, \ldots, M_i\}.$$

It can be shown that the number of distinct set partitions of N of type $(a_1, a_2, \ldots, a_n)$ is given by (21). We illustrate the proof by considering the type $(a_1, a_2, a_3, 0, \ldots, 0)$ with $a_i > 0$ for $i = 1, 2, 3$. Then we have

$$N = (N_1^1 \cup \ldots \cup N_{a_1}^1) \cup (N_1^2 \cup \ldots \cup N_{a_2}^2) \cup (N_1^3 \cup \ldots \cup N_{a_3}^3),$$

where each N_k^j has $|N_k^j| = j$ elements, $j = 1, 2, 3$. The first bracket involves a choice of a_1 elements from N, for which there are

$$\frac{n(n-1)\ldots(n-a_1+1)}{a_1!}$$

possibilities. The second bracket involves a choice of a_2 2-subsets from the remaining $n - a_1$ elements, for which there are

$$\frac{(n-a_1)(n-a_1-1)}{2!}\cdot \\ \ldots \cdot \frac{(n-a_1-2a_2+2)(n-a_1-2a_2+1)}{2!} \cdot \frac{1}{a_2!}$$

possibilities. The third bracket involves a choice of a_3 3-subsets from the remaining $n - a_1 - 2a_2$ elements (in our case this is all there is left), for which there are

$$\frac{(n-a_1-2a_2)(n-a_1-2a_2-1)(n-a_1-2a_2-2)}{3!}\cdot \\ \ldots \cdot \frac{3\cdot 2\cdot 1}{3!} \cdot \frac{1}{a_3!}$$

possibilities. The product of these three terms counts the total number of possibilities and works out to be

$$\frac{n!}{a_1!a_2!a_3!(1!)^{a_1}(2!)^{a_2}(3!)^{a_3}}.$$

Let us now get back to the original formula (5) and apply it in the special case $f(x) = 1/x$. Then $f^{(k)}(x) = (-1)^k k! x^{-k-1}$ and so the product of the derivatives is

$$\begin{aligned} &(f')^{a_1}(f'')^{a_2}\cdots(f^{(n)})^{a_n} \\ &= (-x^{-2})^{a_1}(2!x^{-3})^{a_2}(-3!x^{-4})^{a_3}\cdots((-1)^n n! x^{-n-1})^{a_n} \\ &\qquad = (-1)^{a_1+2a_2+\cdots+na_n}(1!)^{a_1}(2!)^{a_2}\cdots \\ &\qquad\qquad (n!)^{a_n}x^{-(2a_1+3a_2+\cdots+(n+1)a_n)}. \end{aligned} \qquad (23)$$

Since $(a_1, a_2, \ldots, a_n)$ satisfies (6) and (7), the power of (-1) in (23) is n, while the power of x is $-(n+i)$. Thus

$$\begin{aligned} &(f')^{a_1}(f'')^{a_2}\ldots(f^{(n)})^{a_n} \\ &\qquad = (-1)^n(1!)^{a_1}(2!)^{a_2}\cdots(n!)^{a_n}x^{-n-i}, \end{aligned} \qquad (24)$$

and consequently, for any given value of i, the inner sum of (5) reduces to the expression

$$(-1)^n\frac{n!}{a_1!\ldots a_n!}x^{-n-i}. \qquad (25)$$

This gives the following form of the chain rule:

$$g\left(\frac{1}{x}\right)^{(n)} = (-1)^n \sum_{i=1}^{n} g^{(i)}\left(\frac{1}{x}\right) x^{-n-i} \left(\sum \frac{n!}{a_1! \dots a_n!}\right). \tag{26}$$

If we choose the function $g(x) = x^a$ for any real a, we have

$$g\left(\frac{1}{x}\right)^{(n)} = \frac{d^n}{dx^n}\, x^{-a} = (-1)^n a(a+1) \dots (a+n-1) x^{-a-n} \tag{27}$$

and

$$g^{(i)}(x) = a(a-1) \dots (a-i+1) x^{a-i}, \tag{28}$$

so that

$$g^{(i)}\left(\frac{1}{x}\right) = a(a-1) \dots (a-i+1) x^{-a+i}. \tag{29}$$

Hence, substituting in (26) and cancelling the term $(-1)^n x^{-n-a}$ from both sides, we obtain

$$\begin{aligned} &a(a+1) \dots (a+n-1) \\ &\quad = \sum_{i=1}^{n} \frac{n!}{i!} \left(\sum \frac{i!}{a_1! \dots a_n!}\right) a(a-1) \dots (a-i+1), \end{aligned} \tag{30}$$

where the sum in the brackets is taken as usual over all non-negative integer solutions of the equations (6) and (7).

On the other hand it is known and well documented, see references 6 or 2, p.86f, that the coefficients given by expressing the rising factorial $x(x+1) \dots (x+n-1)$ in terms of the falling factorials $1, x, x(x-1), x(x-1)(x-2) \dots,$ $x(x-1) \dots (x-n+1)$ are just the *signless Lah numbers*

$$L'_{n,k} = \frac{n!}{k!} \binom{n-1}{k-1}:$$

$$x(x+1) \dots (x+n-1) = \sum_{k=1}^{n} L'_{n,k} x(x-1) \dots (x-k+1). \tag{31}$$

Comparing the equations (30) and (31) we find an identity

$$\sum \frac{n!}{a_1! \dots a_n!} = \frac{n!}{i!} \binom{n-1}{i-1}, \tag{32}$$

where the sum is taken over all solutions of the equations (6) and (7), or alternatively over all partitions of the integer n into i parts, $a_1, \dots, a_n$ being the type of the partition.

Finally, if we substitute the identity (32) back into (26), we obtain the expression

$$g\left(\frac{1}{x}\right)^{(n)} = (-1)^n \sum_{i=1}^{n} \frac{n!}{i!} \binom{n-1}{i-1} g^{(i)}\left(\frac{1}{x}\right) x^{-n-i}. \tag{33}$$

Readers are invited to prove (33) directly. More challenging projects would be to take either $f(x)$ or $g(x)$ to be an exponential, a logarithm, a trigonometric function, etc.

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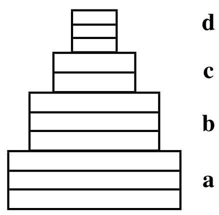
Solution to Braintwister 4

(Piles of money)

Answer. 42

Solution.

Let the pile consist of a 2p coins, b 10p coins on top of those, c 20p coins on top of those, and d 1p coins on top. Trying the moves with some piles you will soon see that each of the 2p coins needs only 1 move, each of the 10p coins needs 2 moves, each 20p coin needs 4 and each 1p needs 8. So the total number of moves is $a+2b+4c+8d$ and the value of the coins is $2a+10b+20c+d$. Since these are equal, we deduce that $7d = a+8b+16c$. In addition there are 10 coins and so $a+b+c+d = 10$. Eliminating d gives $8a+15b+23c = 70$ and you can quickly deduce that $c = 1, b = 1, a = 4$ and $d = 4$, giving a total value of 42 pence (and 42 moves).



VICTOR BRYANT

How a Group can do Geometry

G. LASTERS, R. PEETERMANS and D. SHARPE

Groups are of no use, are they? Read on.

1. A group constructed

In reference 1, a group-theoretical construction was introduced on the points of the circumference of a circle, or more generally an ellipse.

Specifically, let $\mathcal{C}$ be a circle and let l be a straight line not meeting $\mathcal{C}$. The elements of the group G are the points of the circumference of the circle. A point e of $\mathcal{C}$ is designated. For points a, b of $\mathcal{C}$, we define their product ab as shown in figure 1. In case $a = b$, the chord joining a and b becomes the tangent to $\mathcal{C}$ at a, as in figure 2. Parallel lines are taken to meet at infinity. In reference 1, it was shown that this construction defines an Abelian group with neutral element e. The only tricky part of the verification is the associative law $a(bc) = (ab)c$, and this is proved using a theorem of Pascal for conics.

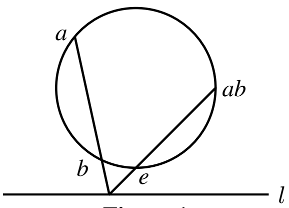


Figure 1.

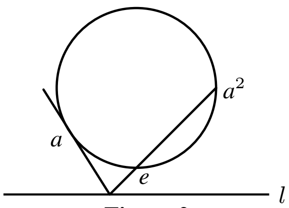


Figure 2.

We show in this article how this group-theoretical construction can be used to solve the following problem.

Let $\mathcal{C}$ be a circle and let l be a straight line not meeting $\mathcal{C}$. Let ABC be a triangle inscribed in $\mathcal{C}$. Construct another inscribed triangle $A'B'C'$ such that BC and $B'C'$ meet on l, CA and $C'A'$ meet on l, AB and $A'B'$ meet on l.

We first note that, for $a, b, c, d \in G, ab = cd$ precisely when ab and cd meet on l as in figure 3. Also, there is an element $f \in G$ other than e such that $f^2 = e$, as in figure 4. For $a \in G$, write $a' = fa$. Then $a'^2 = f^2a^2 = ea^2 = a^2$, so the tangents to $\mathcal{C}$ at a and a' meet on l as in figure 5; this determines a', given a.

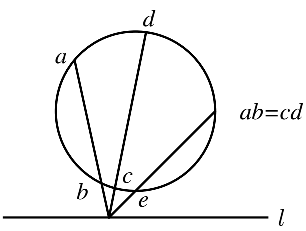


Figure 3.

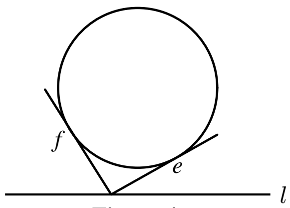


Figure 4.

We can now carry out our construction. Denote the points A, B, C by a, b, c and put $a' = fa, b' = fb, c' = fc$. Then $b'c' = f^2bc = bc$, and similarly $c'a' = ca$ and $a'b' = ab$. Hence BC and $B'C'(= b'c')$ meet on l, as do CA and $C'A'$; also AB and $A'B'$. The construction is shown in figure 6. It is worth remarking that, by Desargue's Theorem, AA', BB' and CC' are concurrent.

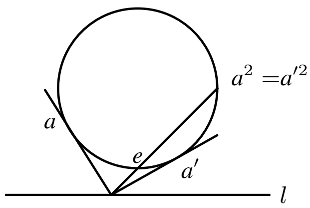
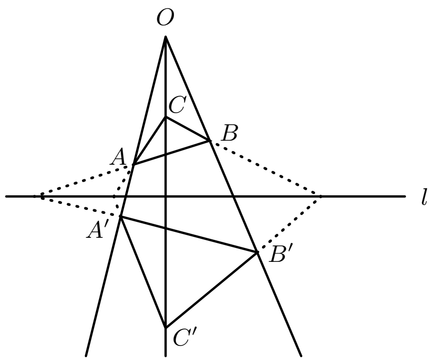


Figure 5.

Desargue's Theorem



AA', BB', CC' meet ar a point O if and only if BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ meet on a line l. (See reference 2, p. 152.)

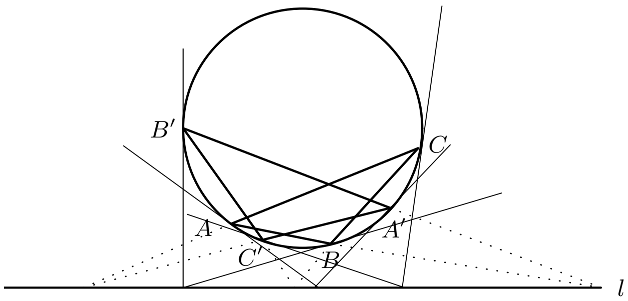


Figure 6.

2. A dual construction

The group constructed in section 1 can be dualized in the following way. Again, let $\mathcal{C}$ be a circle, but now let L be a point not lying on $\mathcal{C}$. The elements of the dual Abelian group G' are the tangents to $\mathcal{C}$. A tangent e of $\mathcal{C}$ is designated. The product of tangents a and b is shown in figure 7, with the special case when $a = b$ shown in figure 8. As before, parallel lines are taken to meet at infinity. Then e is the neutral element of G', clearly $ab = ba$ for all $a, b \in G'$ and figure 9 shows how a^{-1} can be found, given a.

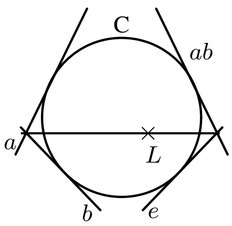


Figure 7.

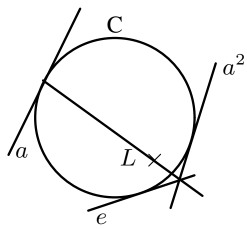


Figure 8.

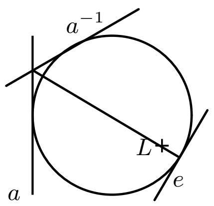


Figure 9.

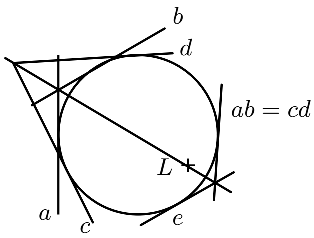


Figure 10.

We note that, for $a, b, c, d \in G'$, $ab = cd$ precisely where the point of intersection of a and b, the point of intersection of c and d, and L are collinear (see figure 10).

The associative law $a(bc) = (ab)c$ follows from Brianchon's Theorem for conics, which is the dual of Pascal's Theorem (see reference 2, page 61). This says that, given six tangents to a conic (in our case a circle), the lines joining opposite vertices of the hexagon formed by these tangents are concurrent. Applying this to figure 11, we see that the line joining the intersection of a and bc and the intersection of ab and c passes through L. From figure 10, this tells us that $a(bc) = (ab)c$.

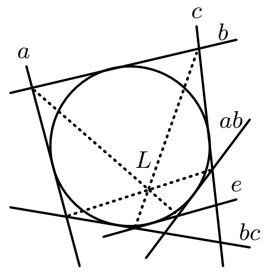


Figure 11.

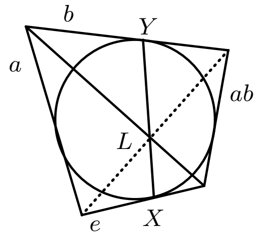


Figure 12.

We shall use this group G' to establish two geometrical constructions, the dual of that given in reference 1 and the dual of that given in section 1 of the present article.

1. Given a circle $\mathcal{C}$ and a point L not lying on $\mathcal{C}$, construct a quadrilateral circumscribing the circle whose opposite vertices intersect at L.

We refer to figure 12. Choose any tangent to the circle and label it e. Denote its point of contact with the circle by X. Construct the straight line XL to meet the circle again at Y and construct the tangent b at Y. Then, in the group $G', b^2 = e$. Now let a be *any* tangent and construct the tangent ab. Then the tangents a, b, ab, e form a quadrilateral of the required form. The reason for this is that $(ab)b = ab^2 = ae$.

Note that the dual of this argument is a simpler construction than the one given in reference 1, using an arbitrary a.

2. Given a circle, a point L not lying on the circle and a triangle ABC circumscribing the circle, construct another triangle $A'B'C'$ circumscribing the circle so that AA', BB' and CC' all meet at L.

We denote the sides of a triangle ABC by a, b, c ($BC = a, CA = b, AB = c$) and choose an arbitrary tangent e. Figure 13 shows how to find $f \in G'$ other than e, such that $f^2 = e$. Put $a' = fa, b' = fb, c' = fc$. Then a', b', c' are different from a, b, c and $a'^2 = f^2a^2 = a^2$, so the line joining the points where a and a' touch the circle, passes through L (see figure 14) and a' can be constructed from a without reference to e. And similarly for b, b' and c, c'. The tangents a', b', c' form the triangle $A'B'C'$ (see figure 15).

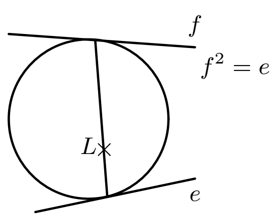


Figure 13.

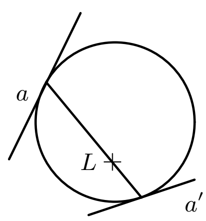


Figure 14.

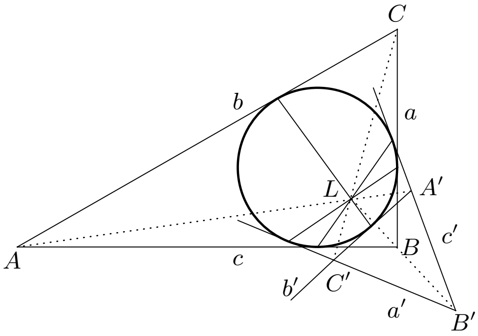


Figure 15.

Now $b'c' = f^2bc = bc$, so the line joining A and A' passes through L. Similarly, BB' and CC' pass through L.

Both problem 2 and its dual in section 1 extend to n-gons.

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Biography *The first two authors are mathematics teachers in Belgium, and are interested in the use of very simple algebra in geometry. They have introduced the use of groups in geometry even to primary school children, and claim that the youngsters are capable of thinking abstractly and receive the ideas enthusiastically. The third author has difficulty in introducing groups at all to students at the University of Sheffield in the UK. He is editor of* Mathematical Spectrum.

Braintwister

5. The years in question

My daughter started teaching last year. As an arithmetic exercise she asked her class to express the year as the sum of two or more consecutive positive integers, and most of them came up with 1997 = 998 + 999. If she asks the same question this year she might get, for example, 1998 = 665 + 666 + 667 or 1998 = 498 + 499 + 500 + 501.

Luckily there will be a possible answer during each year of her teaching career but in one of those years the class will find it *very* hard because the sum will have to involve over thirty terms. And, in a year shortly after her retirement, it will actually be impossible to express the year as a sum in this way.

To which two years am I referring?

(The solution will be published next time.)

VICTOR BRYANT

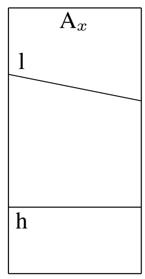
Find a thousand positive integers whose sum is equal to their product.

PETER DERLIEN
(University of Sheffield)

A geometrical construction

In Volume 27 Number 3 Page 67, G. Lasters proposed the following geometrical construction problem.

> A straight line h drawn on a piece of paper is called the 'horizon'. Straight lines are said to be 'parallel' if they intersect on the horizon (produced). Given a straight line l and a point A on the paper not lying on l, how would you construct the straight line through A parallel (in the above sense) to l? The construction should be confined to the piece of paper.



Here is a possible construction. Let X be the intersection of h and l (off the paper). Construct a straight line h_1 parallel to h and half the distance of h from A and a line l_1 parallel to l half the distance of l from A. Let X_1 be the intersection of h_1 and l_1. Then X_1 lies on AX and $AX_1 = \frac{1}{2}AX$. Repeat the construction with h, l replaced by h_1, l_1 to obtain h_2, l_2, then h_3, l_3 etc. The intersection X_n of h_n, l_n lies on AX and $AX_n = \frac{1}{2^n}AX$. Eventually X_n lies on the paper and AX_n produced is the desired line.

Two Interesting Ratios in a Dushy's Cuboid

DUSHY TISSAINAYAGAM

Enlarging a cuboid leads to some interesting mathematics.

Consider a square with sides of unit length (figure 1). Attach a rectangle of unit area to the *right* side of the square. The length of one side of this unit-area rectangle should be equal to the length of the adjoining side of the square (figure 2). Now, attach another rectangle of unit area on *top* of the 2×1 rectangle, again making the lengths of the abutting sides equal (figure 3). Keep attaching rectangles of unit area always alternating between the top and the right side of the previous rectangle (figure 4 and figure 5).

An interesting question arises. Assume that the lower left corners of successive rectangles are all fixed at $(0, 0)$. After the nth attachment, if (x_n, y_n) are the coordinates of the *moving* corner of the rectangle that is diagonally opposite to the fixed corner, what value does the ratio x_n/y_n approach? This question is answered in reference 1. Surprisingly, it is $\pi/2$.

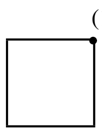


Figure 1.

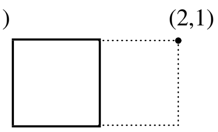


Figure 2.

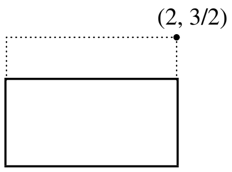


Figure 3.

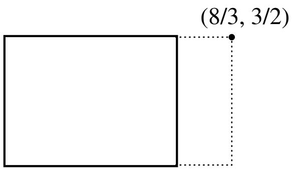


Figure 4.

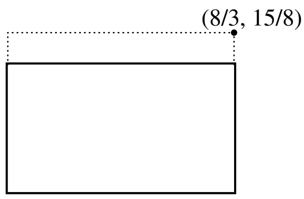


Figure 5.

Instead of a square to begin with, we may consider a cube with sides of unit length and keep attaching cuboids of unit *volume*. The repeated attachments are done on the mutually perpendicular faces of the previous cuboid, starting with the face $x = \text{constant} > 0$, then $y = \text{constant} > 0$, then $z = \text{constant} > 0$ and continued in this order (figures 6, 7 and 8). The cube thus being transformed will be modestly called *Dushy's cuboid*.

We will call the 1st, 2nd and 3rd attachments along the x, y and z axes respectively the *first batch* of attachments. Likewise, the nth batch of attachments will consist of the $3n - 2$, $3n - 1$ and $3n$th attachments. We will also take the moving vertex of the cuboid, diagonally opposite to the vertex that is fixed at the origin, to obtain coordinates (x_n, y_n, z_n) after the nth batch of attachments. Initially,

$$x_0 = y_0 = z_0 = 1.$$

After the first batch of attachments,

$$x_1 = x_0 + \frac{1}{y_0 z_0},$$
$$y_1 = y_0 + \frac{1}{z_0 x_1},$$
$$z_1 = z_0 + \frac{1}{x_1 y_1}.$$

After the nth batch of attachments,

$$x_n = x_{n-1} + \frac{1}{y_{n-1} z_{n-1}},$$
$$y_n = y_{n-1} + \frac{1}{z_{n-1} x_n},$$
$$z_n = z_{n-1} + \frac{1}{x_n y_n}.$$

The volume of the cuboid after the nth batch of attachments is given by

$$x_n y_n z_n = x_{n-1} y_{n-1} z_{n-1} + 3 = \ldots = x_0 y_0 z_0 + 3n = 3n + 1.$$

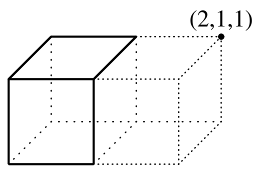


Figure 6.

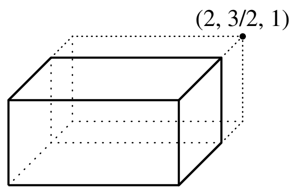


Figure 7.

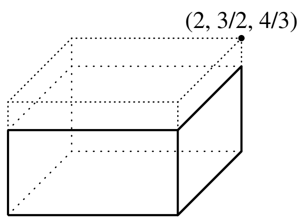


Figure 8.

Now the expression for x_n can be written recursively, as follows,

$$x_n = x_{n-1} + \frac{x_{n-1}}{3n-2} = \left(\frac{3n-1}{3n-2}\right) x_{n-1}$$
$$= \frac{(3n-1)(3n-4)\ldots 8.5.2}{(3n-2)(3n-5)\ldots 7.4.1}.$$

Similarly,

$$z_n = z_{n-1} + \frac{z_n}{3n+1} = \Big(\frac{3n+1}{3n}\Big) z_{n-1} = \frac{(3n+1)(3n-2)\dots 10.7.4}{(3n)(3n-3)\dots 9.6.3}\,.$$

Also,

$$y_n = \frac{3n+1}{x_n z_n} = \Big(\frac{3n}{3n-1}\Big) y_{n-1} = \frac{(3n)(3n-3)\dots 9.6.3}{(3n-1)(3n-4)\dots 8.5.2}.$$

Now form the fractions, i.e.

$$\begin{aligned}
\alpha_n &= \frac{y_n}{x_n} = \frac{(3n/(3n-1))y_{n-1}}{((3n-1)/(3n-2))x_{n-1}} \\
&= \big[1 - \frac{1}{(3n-1)^2}\big]\alpha_{n-1} \\
&= \big[1 - \frac{1}{(3n-1)^2}\big]\big[1 - \frac{1}{(3n-4)^2}\big]\dots\big[1-\tfrac{1}{5^2}\big]\big[1-\tfrac{1}{2^2}\big], \\
\beta_n &= \frac{z_n}{y_n} = \frac{((3n+1)/3n)z_{n-1}}{(3n/(3n-1))y_{n-1}} = \big[1 - \frac{1}{(3n)^2}\big]\beta_{n-1} \\
&= \big[1 - \frac{1}{(3n)^2}\big]\big[1 - \frac{1}{(3n-3)^2}\big]\dots\big[1-\tfrac{1}{6^2}\big]\big[1-\tfrac{1}{3^2}\big].
\end{aligned}$$

We notice that $0 < \alpha_n < \alpha_{n-1} < \dots < \alpha_1 < \alpha_0 = 1$ for all integral $n \geqslant 0$. As $\{\alpha_n\}$ is a monotonically decreasing sequence that is bounded below, it converges. The same applies to $\{\beta_n\}$. To proceed further we need some results from complex function theory. The first is the following formula,

$$\sin\ \pi z = \pi z \prod_{n=1}^{\infty}\Big(1 - \frac{z^2}{n^2}\Big); \tag{2}$$

and the second is Weierstrass's expression for the gamma function $\Gamma(z)$, namely

$$\frac{1}{z\Gamma(z)} = \frac{1}{\Gamma(1+z)} = \mathrm{e}^{\gamma z}\prod_{n=1}^{\infty}(1+\frac{z}{n})\mathrm{e}^{-z/n}, \tag{3}$$

where γ is Euler's constant, that is

$$\gamma = \lim_{n\to\infty}\Big(1 + \tfrac12 + \tfrac13 + \dots + \frac1n - \log\ n\Big) = 0.5772\dots.$$

Taking $z = 1/3$ in (1), we have

$$\lim_{n\to\infty}\beta_n = \prod_{n=1}^{\infty}\big[1 - \frac{1}{(3n)^2}\big] = \frac{\sin\pi/3}{\pi/3} = \frac{3\sqrt3}{2\pi} = 0.8269\dots.$$

To deal with α_n we use (2) with $z = -1/3$, giving $\Gamma(2/3)$, and $z = -2/3$, giving $\Gamma(1/3)$, so as to obtain

$$\lim_{n\to\infty}\alpha_n = \prod_{n=1}^{\infty}\big[1 - \frac{1}{(3n-1)^2}\big] = \prod_{n=1}^{\infty}\frac{(1-(2/3n))}{(1-(1/3n))^2} = \frac{\Gamma^2(2/3)}{\Gamma(1/3)} = 0.6845\dots.$$

In our 3-dimensional cuboids the ratio z_n/x_n of the length of the smallest side to that of the largest is $\alpha_n\beta_n$, which decreases and tends to $\lim_{n\to\infty}\alpha_n \cdot \lim_{n\to\infty}\beta_n = 0.5660\dots$.

What happens in arbitrary m-dimensional space? By means of steps analogous to those in three dimensions we find that the ratio $\rho_n(m)$ of the smallest side to the largest side decreases, tending towards

$$R(m) = \frac{m\Gamma(2/m)}{\Gamma^2(1/m)}$$

as $n \to \infty$. We note that, since $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, $R(m) = 2/\pi$, i.e. the value obtained in [1].

From (2), with $z = 2/m$ and $z = 1/m$, we deduce that

$$R(m) = \tfrac12\prod_{n=1}^{\infty}\big[1 + \frac{1}{nm(nm+2)}\big].$$

Hence $R(m)$ decreases as m increases and

$$\lim_{m\to\infty}R(m) = \tfrac12.$$

Thus $R(m) > 1/2$ for all m, and since, as has been noted, $\rho_n(m)$ decreases as n increases, $\rho_n(m) > 1/2$ for all m, n. In other words, in a Dushy's cuboid of any dimension, the longest side is always less than twice the length of the shortest.

References

1. L. Short and J.P. Melville, An unexpected appearance of π, *Mathematical Spectrum* (1992/3) **25** pp.65–70.

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Mathematics in the Classroom

Understanding the Central Limit Theorem

There are few theorems that occur on A-level Maths syllabuses these days, but one that does remain and causes students much agonising over its interpretation is the central limit theorem. To understand it requires a grasp of the concept of a sampling distribution, something which does not come naturally to most students. That random variables exist and can be measured is not a problem. That a set of such measurements will have a mean and a variance is also readily understood, but the idea that these sample statistics are themselves random variables with their own distributions (and also parameters) is a leap too far for the average student. To try to ease their comprehension, we have devised the following simple exercise to demonstrate how these ideas work.

The exercise is designed around readily available data, namely car numbers. Specifically, our random variable is *the sum of the digits in a car number*. So before we start we need some understanding of the population of this random variable. Clearly the value of our random variable can range from 1 to 27, and assuming all digits are equally likely to occur in a number plate (of no more than 3 digits) the reader is left to verify the following probability distribution for the sum of digits in any 1, 2 or 3 digit number, as follows.

Sum of numbers on number plate	No. of ways of making the sum
1	3
2	6
3	10
4	15
5	21
6	28
7	36
8	45
9	55
10	63
11	69
12	73
13	75
14	75
15	73
16	69
17	63
18	55
19	45
20	36
21	28
22	21
23	15
24	10
25	6
26	3
27	1
Total	999

Calculation of the mean (μ) and variance (σ^2) of this distribution shows that μ=13.5 and σ^2=24.6. So we know precisely the population mean and variance (a very unreal situation!)

Each student is asked to collect 50 car numbers and to sort them into 10 samples each of size 5. For each car number, the sum of the digits is found, then the mean and variance of this sum is calculated for each of the 10 samples. A section of a typical student's results might look like the following table.

Sample number	Car number	Sum of digits	Sample mean	Sample variance
1	921	12		
	634	13		
	688	22		
	457	16		
	363	12	15.0	14.40
2	30	3		
	620	8		
	477	18		
	215	8		
	359	17	10.8	33.36
3	211	4		
	185	14		
	333	9		
	792	18		
	68	14	11.8	23.36
4	227	11		
	246	12		
	476	17		
	385	16		
	279	18	14.8	7.76
5	468	18		
	352	10		
	235	10		
	257	14		
	545	14	13.2	8.96

The class results are then pooled, which usually results in about 200 sample means and 200 sample variances, each obtained from samples of size 5. Frequency distributions are drawn up for the 200 sample means and the 200 sample variances.

Sampling distribution of the means

We know from expectation algebra that the mean and variance of the sample mean, based on samples of size n, are μ and σ^2/n respectively. Hence our sampling distribution of the sample means should provide estimates of μ (13.5) and $\sigma^2/5$ (24.6/5). In addition to this, the central limit theorem asserts that, under certain conditions, the sample mean will be normally distributed. Those conditions include the requirement that the sample size be large, but in the case of an underlying distribution which is symmetrical, this condition is not essential. For this situation, the sampling distribution of the sample mean comes out normal every time, even though we have used samples of only size 5.

Sampling distribution of the sample variances
A-level syllabuses require very little about this sampling distribution, but this is a useful opportunity to demonstrate that sample variance is *not* an unbiased estimator for population variance, a result which contradicts all our intuition. The sampling distribution of the variances never looks symmetrical, and is clearly not centred around the population variance, as demonstrated by calculating its mean. This sample mean is invariably found to be underestimating the population variance. Application of the correction factor $n/(n-1)$ (in this case 5/4) will adjust the sample mean to give a much better estimate of the population variance (known to be 24.6). Hence a demonstration of the surprising equation: $E\,(\text{sample variance}) = (n-1)\sigma^2/n$ is achieved, where E represents expected value.

So, at the end of the exercise we have not only obtained two sampling distributions which always support the known theory, but we have a much better grasp of what the notion of a sampling distribution is all about. This has always worked well for me.

Carol Nixon

Letters to the Editor

Dear Editor,

Adjoining consecutive numbers

In Volume **30**, p. 5, K.R.S. Sastry gave the example of consecutive numbers 183 and 184 which when adjoined give a perfect square:

$$183184 = 428^2.$$

He asked for another consecutive pair with this property. I have obtained the following answers using a Casio fx-7700 GE programmable calculator, searching for numbers smaller than 10000^2:

$$328329 = 573^2, \qquad 528529 = 727^2,$$
$$715716 = 846^2, \quad 60996100 = 7810^2.$$

Alternatively, with the larger number first:

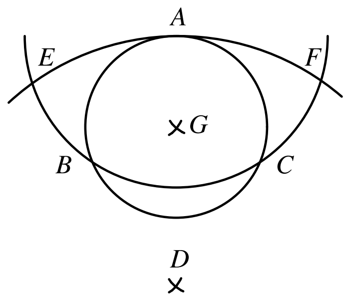
$$8281 = 91^2, \quad 82428241 = 9079^2, \quad 98029801 = 9901^2\,.$$

Yours sincerely,
BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG)

Dear Editor,

Problem 29.8

There is a shorter solution (fewer arcs) than the one given in Volume **30** p. 23, which has a more direct proof. Unfortunately I do not have a reference for this. With centre A on the given circle, draw arc BC (radius r). With radius r, centres B and C, draw arcs intersecting at D. With radius $AD (= R)$, centre D, draw an arc giving intersections E and F on arc BC. With radius AE (i.e. r) and centres E, F draw arcs intersecting at G. Then G is the centre of the given circle.



To prove this, first prove that isosceles triangles EAG and DAE are similar (equiangular). Therefore,

$$\frac{AG}{r} = \frac{r}{R}\,.$$

Hence triangles AGB and ABD are similar (sides in same ratio and included angles equal). Therefore,

$$AG = BG\,.$$

Similarly $AG = CG$, and G is the centre.

Yours sincerely,
A.K. JOBBINGS
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Dear Editor,

A variant of Rolle's theorem

The familiar theorem of Rolle is that, if the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists c in (a, b) such that $f'(c) = 0$. A natural question is what happens if $f(a) \neq f(b)$. One answer is, of course, the Mean Value Theorem to the effect that there exists c such that $f'(c) =$

$(f(b) - f(a))/(b - a)$. However, another answer may also be given.

Theorem. Suppose that (i) the function is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and (ii) f is not a strictly monotonic function on $[a, b]$. Then there exists c in (a, b) such that $f'(c) = 0$.

The theorem is quite easily proved by use of a lemma, similar to the Intermediate Value Theorem for continuous functions, which deserves to be better known.

Lemma. If f is differentiable on (a, b) and $[p, q] \subset (a, b)$, then, in (p, q), f' takes every value C between $f'(p)$ and $f'(q)$.

Proof. We may suppose that $f'(p) < C < f'(q)$. Now put

$$g(x) = f(x) - Cx.$$

Then g is continuous on $[p, q]$ and therefore attains its infimum in $[p, q]$ at some point c in $[p, q]$. But

$$g'(p) = f'(p) - C < 0 \text{ and } g'(q) = f'(q) - C > 0,$$

and so c is not p or q, i.e $c \in (p, q)$. Also $g'(c) = 0$, i.e. $f'(c) = C$.

Proof of the theorem. Assume that the hypothesis of the theorem holds. If also $f'(x) \neq 0$ for all $x \in (a, b)$, then one of the following statements is true.

(i) $f'(x) > 0$ for all $x \in (a, b)$;

(ii) $f'(x) < 0$ for all $x \in (a, b)$;

(iii) there exist $r, s \in (a, b)$ such that $f'(r) > 0$ and $f'(s) < 0$.

If (i) holds, then, by the Mean Value Theorem, f strictly increases on $[a, b]$ and this contradicts (ii). Hence (i) is false. Similarly (ii) is false. If (iii) holds, it follows from the lemma that there exists c in (a, b), such that $f'(c) = 0$ and we again have a contradiction. Therefore $f'(x)$ cannot be non-zero in all of (a, b).

Yours sincerely,

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Dear Editor,

Sums of powers of integers

During the last few years *Mathematical Spectrum* has published several articles on sums $\sum_{k=1}^{p} k^p$, where p is a positive integer. The latest, by T.H. Fay and K.R.S. Sastry in Volume **30**, pp. 10–13, which deals with slightly more general sums, also has a useful bibliography. However, none of these articles mentions the following interesting results:

$$\frac{n^{p+1}}{p+1} \leqslant \sum_{k=1}^{n} k^p \leqslant \frac{(n+1)^{p+1}}{p+1}, \tag{1}$$

and

$$\Big(\sum_{k=1}^{n} k^p\Big)/n^{p+1} \to 1/(p+1) \text{ as } n \to \infty. \tag{2}$$

To prove (1) we use the function $[x]$, the integral part of x, i.e. the largest integer less than or equal to x. Since $[x] = k$ for $k \leqslant x < x + 1$,

$$\int_1^{n+1} [x]^p \mathrm{d}x = \sum_{k=1}^{n} \int_k^{k+1} [x]^p \mathrm{d}x = \sum_{k=1}^{n} k^p.$$

But $x - 1 \leqslant [x] \leqslant x$ for all x and so

$$\int_1^{n+1} (x-1)^p \mathrm{d}x \leqslant \sum_{k=1}^{n} k^p \leqslant \int_1^{n+1} x^p \mathrm{d}x \leqslant \int_0^{n+1} x^p \mathrm{d}x.$$

Integration now yields (1), and (2) quickly follows.

Yours sincerely,

ZERONG HE
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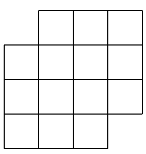
An $n \times n$ chessboard has two of its diagonally opposite corner squares removed. Is it possible to cover it with non-overlapping dominoes each of which covers two squares?



The function $f(x)$ is defined for x in the range $-3.5 \leqslant x \leqslant 3.18$. For what values of x is the function $f([|1 - x|])$ defined, where [ ] denotes the integral part?

ANAND KUMAR
(Patria, India)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

30.9 There are n ballot papers, n_1 of which are coloured 1, n_2 coloured 2, ..., n_r coloured r. What is the probability that two ballot papers picked out at random are coloured i and j? (The order of selection is irrelevant.)

(Submitted by Ice Risteski, Skopje, Macedonia)

30.10 (i) Find all pairs of real numbers whose sum is equal to their product.
(ii) Find all triples of real numbers whose sum is equal to their product.
(iii) For $n > 3$, find all n-tuples of positive real numbers whose sum is equal to their product.

(Submitted by Peter Derlien, University of Sheffield; (ii) was submitted independently by Seyamack Jafari, Ahwaz, Iran)

30.11 Prove that, if p is a prime number, then $L_p \equiv 1 \pmod p$, where L_n denotes the nth Lucas number defined by $L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n$.

(Submitted by Mansur Boase, St Paul's School, London)

30.12 Show that no prime number can be written as the sum of two squares in two different ways.

(Submitted by Farshid Arjomandi, Santa Barbara, CA, USA)

Solutions to Problems in Volume 30 Number 1

30.1 What is the probability that the six numbers in the UK national lottery on a given payout day do not include two consecutive numbers? (The winning numbers are an unordered random choice of six distinct numbers from 1 to 49.)

Solution by Andrew Lobb, St Olave's Grammar School, Orpington.

Choose integers x_1 to x_6 such that $x_{n+1} > x_n, x_1 \geqslant 1$ and $x_6 \leqslant 44$. Now $x_1, x_2+1, x_3+2, x_4+3, x_5+4, x_6+5$ lie between 1 and 49 and no two are consecutive. The number of such choices is $\binom{44}{6}$ and the total number of choices is $\binom{49}{6}$. Hence the chance of the six numbers not including two consecutive numbers is

$$\binom{44}{6} \Big/ \binom{49}{6} = \frac{22919}{45402}\,.$$

Also solved by Jeremy Young, Nottingham High School.

30.2 If

$$\begin{aligned} x_1 + x_2 + \cdots + x_{10} &= 1, \\ x_1^2 + x_2^2 + \cdots + x_{10}^2 &= 2, \\ &\vdots \\ x_1^{10} + x_2^{10} + \cdots + x_{10}^{10} &= 10, \end{aligned}$$

what is $x_1^{11} + x_2^{11} + \cdots + x_{10}^{11}$?

Solution by Tim Raine, Gresham's School, Holt.

Generally, denote the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0\,,$$

by $\alpha, \beta, \gamma, \cdots$, and write

$$S_r = \alpha^r + \beta^r + \gamma^r + \cdots\,.$$

Newton's Identities are:

$$\begin{aligned} S_1 + p_1 &= 0 \\ S_2 + p_1 S_1 + 2p_2 &= 0 \\ S_3 + p_1 S_2 + p_2 S_1 + 3p_3 &= 0 \\ &\vdots \\ S_n + p_1 S_{n-1} + p_2 S_{n-2} + \cdots + np_n &= 0 \end{aligned}$$

and, when $k > 0$,

$$S_{n+k} + p_1 S_{n+k-1} + p_2 S_{n+k-2} + \cdots + p_n S_k = 0.$$

Let $x_1, x_2, \cdots, x_k$ be the roots of the equation

$$x^{10} + p_1 x^9 + p_2 x^8 + \cdots + p_{10} = 0\,.$$

Then,

$$\begin{aligned} &1 + p_1 = 0 \\ &2 + p_1 + 2p_2 = 0 \\ &3 + 2p_1 + p_2 + 3p_3 = 0 \\ &4 + 3p_1 + 2p_2 + p_3 + 4p_4 = 0 \\ &5 + 4p_1 + 3p_2 + 2p_3 + p_4 + 5p_5 = 0 \\ &6 + 5p_1 + 4p_2 + 3p_3 + 2p_4 + p_5 + 6p_6 = 0 \\ &7 + 6p_1 + 5p_2 + 4p_3 + 3p_4 + 2p_5 + p_6 + 7p_7 = 0 \\ &8 + 7p_1 + 6p_2 + 5p_3 + 4p_4 + 3p_5 + 2p_6 + p_7 + 8p_8 = 0 \\ &9 + 8p_1 + 7p_2 + 6p_3 + 5p_4 + 4p_5 + 3p_6 + 2p_7 \\ &\qquad + p_8 + 9p_9 = 0 \\ &10 + 9p_1 + 8p_2 + 7p_3 + 6p_4 + 5p_5 + 4p_6 + 3p_7 \\ &\qquad + 2p_8 + p_9 + 10p_{10} = 0, \end{aligned}$$

which give

$$p_1 = -1, p_2 = -\tfrac{1}{2!}, p_3 = -\tfrac{1}{3!}, p_4 = \tfrac{1}{4!},$$
$$p_5 = \tfrac{29}{5!}, p_6 = \tfrac{151}{6!}, p_7 = \tfrac{1091}{7!},$$
$$p_8 = \tfrac{7841}{8!} p_9 = \tfrac{56519}{9!}, p_{10} = \tfrac{396271}{10!}.$$

Now

$$S_{11} + 10p_1 + 9p_2 + 8p_3 + 7p_4 + 6p_5 + 5p_6 + 4p_7 + 3p_8 + 2p_9 + p_{10} = 0,$$

which gives

$$S_{11} = \frac{42359239}{10!} = 11.673 \text{ to 3 decimal places.}$$

30.3 Find a natural number N such that $\sqrt{N}$ is of the form $M.1997\ldots$.

Solution by Andrew Lobb.

We want N such that

$$M + \frac{1997}{10^4} \leqslant \sqrt{N} < M + \frac{1998}{10^4}.$$

Set $M = 10^4$. Then we want

$$10^8 = 2 \times 1997 + \frac{1997^2}{10^8} \leqslant N < 10^8 + 2 \times 1998 + \frac{1998^2}{10^8}.$$

Thus two possible N are 100003995, 100003996.

30.4 Let L_n denote the nth Lucas number, defined by $L_{n+2} = L_{n+1} + L_n$ with $L_0XS = 2, L_1 = 1$, and let F_n denote the nth Fibonacci number, defined by $F_{n+2} = F_{n+1} + F_n$ with $F_1 = F_2 = 1$. Prove that, for all n, L_n is not divisible by any Fibonacci number F_a with $a \geqslant 5$.

Solution by Kieran Gillick, Gresham's School, Holt.

First, for all $n \geqslant 2$,

$$L_n = F_a L_{n+1-a} + F_{a-1} L_{n-a}\,, \tag{1}$$

for $2 \leqslant a \leqslant n$. This can be proved by induction on a.

Suppose that F_a is a factor of L_n for some n and some $a \geqslant 5$, Suppose that $n \geqslant a$. It follows from (1) that F_a divides $F_{a-1}L_{n-a}$. But

$$F_{a-2}L_{n-a} + F_{a-1}L_{n-a} = F_a L_{n-a}\,,$$

so now F_a divides $F_{a-2}L_{n-a}$.XS Continuing in this way, we see that F_a divides $F_{a-3}L_{n-a}, F_{a-4}L_{n-a}, \cdots, F_1 L_{n-a}$, i.e. F_a divides L_{n-a}. Hence F_a divides $L_{n-2a}, L_{n-3a}, \cdots$ until F_a divides L_n for some n with $0 \leqslant n < a$. Since $a \geqslant 5, n \neq 0$ or 1. Now, by induction on N, we can show that

$$L_N = F_{N+1} + F_{N-1}$$

for all $N \geqslant 2$. Thus $L_N < 2F_{N+1}$ for $N \geqslant 2$. Hence $L_n \leqslant L_{a-1} < 2F_a$ and F_a divides L_n, so $L_n = F_a$. But, from

$$L_n = F_{n+1} + F_{n-1},$$

we have

$$F_{n+1} < L_n < F_{n+1} + F_n = F_{n+2},$$

so L_n lies strictly between two consecutive Fibonacci numbers and so cannot be a Fibonacci number.

Also solved by Andrew Lobb and Jeremy Young.

Figure of Speech Density Index

The phrase 'confident Niagara' occurs in an Alan Bennett play. In this two-word phrase we may discern three figures of speech:

(1) transferred epithet;
(2) metaphor;
(3) hyperbole.

Dividing the number of figures of speech (3) by the number of words (2) gives a value (1.5) which we shall call the Figure of Speech Density Index of the phrase.

You are challenged to devise a phrase which, in appropriate context, has a Figure of Speech Density Index larger than 1.5.

J.N. MacNeill

While it is not possible to duplicate the cube, i.e. to find a rational number whose cube is 2, it *is* possible to get very near to it:

$$\left(\frac{6064}{4813}\right)^3 = 1.999\,999\,986\,993\,559\ldots.$$

The sum of two irrational square roots of integers cannot be an integer, but again, a close approximation is possible:

$$\sqrt{23} + \sqrt{13272} = 119.999\,998\,114\,65\ldots.$$

L. P. Knight

Reviews

The Mathematical Olympiad Handbook: An Introduction to Problem Solving. By A. GARDINER. Oxford University Press, 1997. Pp. x+229. Paperback \$14.95 (ISBN 0-19-850105-6).

This book is a collection of problems from the first 32 British Mathematical Olympiad (BMO) papers, 1965–1996. Tony Gardiner is the author of over 250 publications on mathematics. He has been involved in the BMO competitions for many years and has been the leader of the UK team four times at the International Mathematical Olympiad (IMO). He is vice-president of the World Federation of National Mathematics Competitions, and received the Paul Erdös National Award in 1995 for his tireless work in promoting mathematics. There is surely not one young British mathematician today who is not indebted to him.

The book begins with some useful tips on problem solving and how to use the book. Part I, the 'Background', covers numbers, algebra, proof, elementary number theory, geometry and trigonometric formulae, with plenty of exercises, and ends with a comprehensive ten-page annotated bibliography of recommended books arranged by topic; Part II contains the BMO problems, and Part III 'Hints and outline solutions', followed by an appendix on the UK teams and results in the 1967–1996 IMOs. The style is 'reader-friendly' and everything is very clearly explained.

Most of the problems in the book are challenging, but accessible and well worth the effort required to solve them. The author begins by quoting David Hilbert: 'a mathematical problem should be difficult in order to entice us, yet not completely inaccessible lest it mock our efforts. It should be a guidepost on the tortuous path to hidden truths, ultimately rewarding us by the satisfaction of success in its solution.'

Two examples of problems from the book that require 'basic techniques' and effort, rather than 'one slick, unmotivated step' to solve, are given below:

> Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$
> (1993, 4).

This problem has several solutions, but only requires a knowledge of certain facts about angles, circles and tangents. The second example requires the use of a recurrence relation:

> The seven dwarfs walk to work each day in single file, with heights alternating *up-down-up-down-*... or *down-up-down-up-*.... If they all have different heights, for how long can they continue like this with a new order every day? What if Snow White always came along too?
> (1995, 5).

Hints and outline solutions are given to show the reader how to solve the problems, rather than actually solving them. This is because Gardiner believes in 99% perspiration. This he illustrated at the 1997 IMO award ceremony lecture at the Royal Society with the following tale: two rats fell into a milk churn; one gave up and drowned, while the other refused to give up and swam till the milk turned to butter and it was able to climb out.

This book is strongly recommended. It is a *sine qua non* for all those interested in solving problems and a must for every library.

Student, St Paul's School — MANSUR BOASE

Power Play. By EDWARD J BARBEAU. The Mathematical Association of America, 1997. Pp. 198. Paperback \$29.00. (ISBN 0-88385-523-2).

A new book by Barbeau, the author of the brilliant though expensive *Polynomials*, is always a pleasure and this is no exception. The book is somewhat unusual, dealing as it does with powers of integers. Whilst this may seem like an extremely narrow subject, the author has managed to find an amazing quantity of material in a large number of journals over the last 50 years. The book thus acts as a reference book to a remarkable number of results in the sidestreets of elementary number theory.

Power Play is much more than a glorified index, however. It contains a large number of exercises at a level from basic algebra to hard number theory; it is difficult to think of a student who would not find a variety of questions at a suitable level. There is also a considerable amount of theory on some well known topics, including Fibonacci numbers and the Pell equation. In fact, the treatment of Pell and Pell-like equations is both the most thorough and the most useful that I have seen in any book; the sections on Pell equations of third or higher degree is particularly unusual and interesting.

It is difficult to give a full idea of the wide range of subjects that are covered, but to give an indication, we have chapters on Pythagorean triples, Pell's equation, Equal Sums of Equal Powers and several others, including an amazing rag-bag of 'Interesting Sets'.

The only criticism I have is that for a book with such a large quantity of results, the index is extremely poor; whilst it is difficult to see a way of making it perfect, it could certainly be a lot better than the one provided.

In conclusion, this book is warmly recommended to everyone; teachers will find a wealth of 'extension material' and students will find a great deal of new and interesting results.

Student,
The John of Gaunt School, Trowbridge. — TOBY GEE

A Tour of the Calculus. By David Berlinski. Mandarin Paperbacks, 1997. Pp. 331. Paperback $7.99. (ISBN 0-7493-1629-2).

This delightful book is written mainly with non-mathematicians in mind, with introductions to number systems, functions, velocity and area presented in layman's language. Yet it is also entertaining for more advanced students to read. There are numerous anecdotes about the author himself and brief descriptions of famous mathematicians, from Newton and Leibniz, the founders of calculus, to Lagrange and Riemann who, unknown to many, also contributed a large amount to elementary calculus. Theorems, such as the Intermediate Value Theorem, are simply acknowledged in the text, and then formally proved in the appendix at the end of the chapter. The reader can then get as much information as possible without being too tied down with details, while those who wish to be more rigorous also find what they want.

Student, Winchester College. Pak-San Man

For All Practical Purposes. Introduction to Contemporary Mathematics. 4th edn. By COMAP. W.H. Freeman and Company, New York, 1996. Pp. xxii + 884. Hardback $29.95. (ISBN 0-7167-2841-9).

This book, apparently already established as a college text in America, has been promoted as an answer to the often encountered question 'Just what is mathematics used for?' The people of COMAP have attempted to achieve this by describing various problem areas where mathematical modelling has proved invaluable. Predictably we get Street Networks, Statistical Methods, Probability, Voting Systems, Game Theory and more. Each subject is dealt with in the greatest depth that will not scare off the non-mathematicians. The more mathematically-minded might find it frustrating to have to continue reading in the anticipation of getting somewhere definite or having a proof demonstrated of some of the assertions, only to find themselves at the end of the chapter.

Remember then that *For All Practical Purposes* is not intended for the mathematician (who may not care at all about practical purposes) but merely for the interested layman; an intention which it fulfils very adequately with the aid of colour, pictures and impressively clear diagrams. It focuses on the applications of mathematics and does not get bogged down with the rigour that maths entails, keeping the style flowing and providing continuous 'real world' examples of how techniques are used. Perhaps the features of this book that are most engaging are the Spotlights, which are generally page-long accounts (with photographs) of particular mathematicians and their work.

In conclusion, this book is not a worthwhile purchase for the serious mathematician; if he wishes to answer the question at the beginning of this review I would far more strongly recommend Hardy's *A Mathematician's Apology*. A purchase by a school library, however, would give interested students a lively text and the more able ones a book which will nudge them into further investigation of the topics covered. For this purpose COMAP have included lists of suggested readings at the end of each chapter.

Student,
St Olave's Grammar School, Orpington Andrew Lobb

Other books received

For All Practical Purposes. CD-ROM and Print Component. 4th edn. By COMAP. W.H. Freeman and Company, New York, 1998. Hardback $12.95 (ISBN 0-7167-3076-6).

This interactive study guide consists of a CD for use with a CD-ROM and an accompanying manual. It will probably introduce readers to many new topics. There are five parts: Part I Management Science, covering such topics as networks, scheduling problems and linear programming; Part II Statistics; Part III Coding Information; Part IV Social Choice and Decision Making; Part V On Size and Shape, covering such topics as growth rate, non-Euclidean geometry, symmetry and patterns, tiling.

Introduction to Non-Linear Systems. By J. Berry. Arnold, London, 1996. Pp. xi+212. Paperback $8.99 (ISBN 0-340-67700-7).

This volume is a useful addition to the growing collection of introductory texts for undergraduates under the title 'Modular Mathematics'. To quote from the cover: 'Since the popularization of chaos theory great interest has been generated in non-linear dynamical systems. This book presents an introduction to the basic mathematical concepts and techniques needed to describe and analyse these systems, and is aimed at students who have taken a first course in calculus. After reviewing differential equations, matrix algebra and iteration methods, first and second order continuous systems are discussed. Chapter 4 investigates discrete systems and the final chapter is a collection of investigations that can be explored as more open-ended tasks.'

MEI: Foundations of Advanced Mathematics. By Dave Faulkner, Roger Porkess, David Snell and Diana Cowey. Hodder & Stoughton Educational, 1997. Pp. 176. Paperback $9.99 (ISBN 0-340-65855-X).

This attractively produced volume is part of the MEI Structured Mathematics programme, and is intended as an access course to Advanced Level Mathematics or as a GNVQ mathematics unit. There are chapters on Calculations, Algebra, Graphs, Statistics and Probability and Trigonometry.

Calculus Connections: A Multimedia Adventure. Vol 2. By Douglas Quinney and Robert Harding. John Wiley & Sons Inc., 1997. CD-ROM. $27.50 (ISBN 0-471-13796-0).

Constitutions of Matter - Mathematically Modelling the Most Everyday of Physical Phenomena. By Martin H. Krieger. The University of Chicago Press, 1997. Pp. 365. Paperback $51.95 (ISBN 0-226-45304-9).

Error Correcting Codes. By John Baylis. Chapman and Hall, London, 1998. Pp. 240. Paperback. $24.99 (ISBN 0-412-78690-7).

Projective Geometry. By A. Beutelspacher and U. Rosenbaum. CUP, Cambridge, 1998. Pp. 272. Softback/Hardback. $15.95/$45.00 (ISBN 0-521-48364-6/0-521-48277-1).

P-automorphisms of Finite P-groups. By E.I. Khukhro. CUP, Cambridge, 1998. Paperback $24.95 (ISBN 0-521-59717-X).

Fundamental Ideas of Analysis. By M. Reid. John Wiley & Sons Ltd., 1998. Pp. 432. Hardback $24.95 (ISBN 0-471-15996-4)

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