Indian Team Selection Test 2005

Day 1

1 Let ABC be an acute triangle and F its Fermat point, that is, the interior point of ABC such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each one of triangles ABF, BCF and CAF, the Euler line (the line connecting its circumcenter and its centroid) is drawn. Prove that these three lines pass through one common point.

2 Prove that one can find $n_0 \in \mathbb{N}$ such that $\forall m \geq n_0$, there are three positive integers a, b , c such that

(i) $m^3 < a < b < c < (m+1)^3$;

(ii) abc is the cube of an integer.

3 If a, b ,c are three positive real numbers such that $ab+bc+ca=1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

Day 2

1 An n-gon inscribed in a circle ($n \geq 4$) is partitioned into $n-2$ triangles using non-intersecting diagnols. Prove that the sum of the inradii of the triangles is constant.

2 Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation

$$\tau(an) = n$$

doesn't have a positive integer solution n.

3 There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

(i) Each pair of students are in exactly one club.

(ii) For each student and each society, the student is in exactly one club of the society.

(iii) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Find all possible values of k.

Day 3

1 Let $0 < a < b$ be two rational numbers. Let M be a set of positive real numbers with the properties:

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(i) $a \in M$ and $b \in M$;

(ii) If $x \in M$ and $y \in M$, then $\sqrt{xy} \in M$.

Let M^*denote the set of all irrational numbers in M. Prove that for every c,d such that $a < c < d < b$, M^* contains an element m with the property $c < m < d$

2 Find all $f : \mathbb{N} \to \mathbb{N}$ satisfying:

$$(f^2(m) + f(n)) \mid (m^2 + n)^2.$$

3 A merida path of order $2n$ is a lattice path in the first quadrant of xy plane joining $(0,0)$ to $(2n,0)$ using three kinds of steps $U = (1,1)$, $D = (1,-1)$ and $L = (2,0)$ (U joins (x,y) to $(x+1,y+1)$ etc.). An ascent in a merida path is a maximal string of consecutive steps of the form U. If $S(n,k)$ denotes the number of merdia paths of order $2n$ with exactly k ascents, compute $S(n,1)$ and $S(n,n-1)$.

Day 4

1 Let $ABCD$ be a convex quadrilateral. The lines parallel to AD and CD through the orthocenter H of $\triangle ABC$ intersect AB and BC respectively at P and Q. Prove that the perpendicular through H to the line PQ passes through the orthocenter of $\triangle ACD$.

2 Given real numbers a, α, β, σ, and ρ such that $\sigma, \rho > 0$ and $\sigma\rho = \frac{1}{16}$, prove that there are integers x and y such that

$$-\sigma \le x + \alpha(ax + y + \beta) \le \rho.$$

3 Consider a matrix of the size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, and the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of entries not exceeding C in absolute value.

Day 5

1 For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X. Prove that the radical axis of the incircles of the triangles ABX and ACX passes through a point independent of X.

2 Determine all positive integers $n > 2$, such that

$$\frac{1}{2}\varphi(n) \equiv 1 \pmod 6.$$

3 For real numbers a, b, c, d not all equal to 0, define a real function $f(x) = a + b\cos 2x + c\sin 5x + d\cos 8x$. If $f(t) = 4a$ for some real t, prove that there exists a real number s such that $f(s) < 0$.

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