Mathematical Spectrum

A magazine for students and teachers of mathematics
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and for everyone interested in mathematics

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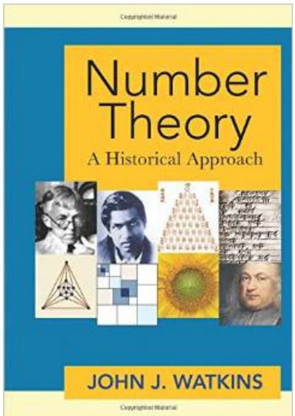
Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Number Theory: A Historical Approach



Long-suffering students are familiar with the standard 'theorem–proof' textbook, all a bit dry and formal, with little to set the pulse racing. This book is something different. To start with, the author has a student-friendly style. Secondly, the approach is historical. So, for example, you meet Pythagorean triples almost on the first page. The chapter headings emphasize this approach; for example, Chapter 2 Euclid, Chapter 4 Diophantus, Chapter 5 Fermat, Chapter 7 Euler and Lagrange, Chapter 8 Gauss, and Chapter 11 Sophie Germain. The downside is that the first proof discovered of a result is seldom the best. The proof of a result is often paired down over time, its essence extracted and superfluous parts avoided. Nevertheless, this approach gives an insight into how Number Theory has developed which the drier, more formal approach does not. This book may not be the one for the absolute beginner, but for anyone with some knowledge of Number Theory it pays rich dividends. It will need to be read with a clipboard, pen, and plenty of paper handy. The exercises are extensive and far from standard and for many will prove the highlight. Here is problem 3.27 to get the little grey cells working.

> The four-digit number 2310 contains each of the four digits 0, 1, 2, and 3 exactly once, and 2310 is the product of the first five primes: $2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$. Without multiplying it out, explain how you can quickly tell that the ten-digit number
>
> $$2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29$$
>
> cannot contain each of the digits $0, 1, 2, \ldots, 9$ exactly once.

Readers may struggle with the proofs of Paul Erdős and Sophie Germain, but there is no law that says you have to read and understand everything. Even the author omits justification of some of Srinivasa Ramanujan's amazing results, leaving the reader to gasp in amazement and admiration.

I noted very few errors. The author needs to revisit the proof of Theorem 12.7 (p. 346), where he seems to have overlooked that, when the greatest common divisor of two positive integers is expressed as a linear combination of the two integers with integer coefficients, one of the coefficients will be negative. I also noted the following small errors:

- p. 438, line 13 should refer to Problem 12.39, not 12.38,
- p. 441, in line 4 the x^7 term is missing,
- p. 452, Ramanujan was born in 1887, not 1877,
- p. 461, in Problem 15.20, 13 should be 11.

And I, a Brit, was amused to read of J. J. Sylvester as 'America's first great mathematician' (p. 446).

It would have been nice to have a longer list of primes than those less than 1000 at the end, and perhaps the smallest prime divisor for the non-primes except in obvious cases, but that is probably me being lazy, which is against the spirit of this unusual book.

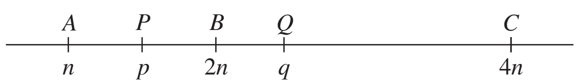
This book is strongly recommended to all willing to put in some effort who need the flame of enthusiasm to be ignited or reignited.

Reference

1 J. J. Watkins, *Number Theory: A Historical Approach* (Princeton University Press, 2014).

Bertrand's postulate

For every integer $n \geq 2$, there is a prime number p between n and $2n$, and so also a prime number q between $2n$ and $4n$.



Denoting the points representing

$$n,\ 2n,\ 4n,\ p,\ q$$

on the real line by

$$A,\ B,\ C,\ P,\ Q,$$

respectively, is it possible for Q to be in the same position fractionally between B and C as P is between A and B; precisely, is it possible that

$$\frac{PB}{AP} = \frac{QC}{BQ}\ ?$$

Tienen, Belgium **Guido Lasters**

The Euler Line of a Triangle

GUIDO LASTERS and DAVID SHARPE

We demonstrate the well-known *Euler line* of a triangle using vectors.

The orthocentre H, the centroid G, and the circumcentre K of a triangle lie on a straight line in the order H, G, K, as illustrated in figure 1. This is called the *Euler line* of the triangle. Furthermore, $HG : GK = 2 : 1$. All this is well known.

In this article we will present what we consider to be a simple demonstration of this result by the use of vectors. If the aim is to go straight to the Euler line, the trick is to choose the orthocentre H as the origin, but by taking any point as the origin we obtain a more general result; O does not even need to be in the plane of the triangle.

So we start with a triangle ABC and choose any point O as the origin. Let A, B, C be denoted by vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, relative to the origin. Let A', B', C' be the midpoints of the sides BC, CA, AB, so that A', B', C' are the points $(\boldsymbol{b}+\boldsymbol{c})/2$, $(\boldsymbol{c}+\boldsymbol{a})/2$, $(\boldsymbol{a}+\boldsymbol{b})/2$ respectively. Let N be the point $(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})/2$. Then

$$\overrightarrow{A'N} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{2} - \frac{\boldsymbol{b}+\boldsymbol{c}}{2} = \frac{\boldsymbol{a}}{2},$$

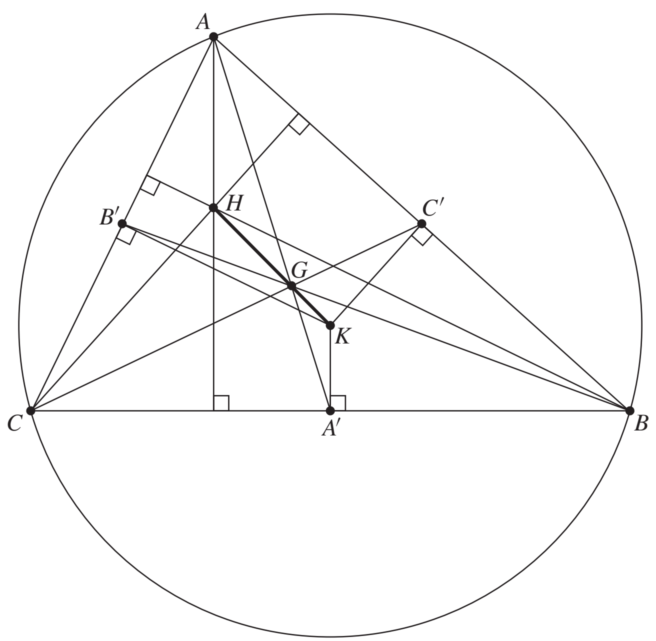


Figure 1 The Euler line.

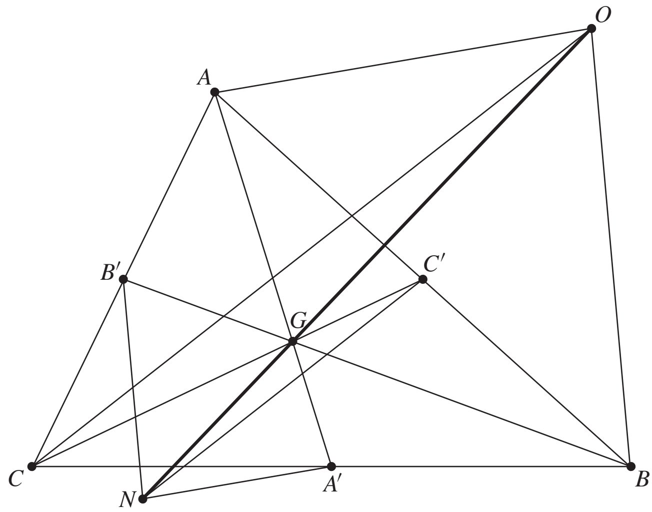


Figure 2

so that $A'N$ is parallel to OA and $A'N = \frac{1}{2}OA$. Similarly, $B'N$ is parallel to OB and $B'N = \frac{1}{2}OB$ and $C'N$ is parallel to OC and $C'N = \frac{1}{2}OC$. Thus, the straight lines through A' parallel to OA, through B' parallel to OB, and through C' parallel to OC meet in the single point N. Moreover, the centroid G of the triangle is at

$$\frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{3},$$

so that O, G, and N lie on a straight line in that order in the ratio $OG : GN = 2 : 1$. This is illustrated in figure 2.

This is very reminiscent of the Euler line. To obtain this we choose the orthocentre H of the triangle as the origin – see figure 1 again. Now $A'N$ is parallel to HA, so it is the perpendicular bisector of side BC; and similarly for $B'N$ and $C'N$. Thus, N is the circumcentre K of the triangle and we have the Euler line HGK with $HG : GK = 2 : 1$. Moreover,

$$A'K = \tfrac{1}{2}HA, \qquad B'K = \tfrac{1}{2}HB, \qquad C'K = \tfrac{1}{2}HC.$$

Alternatively, we could choose O to be the centre of the inscribed circle of the triangle. This is the point where the angle bisectors of the triangle meet. Now the lines through the midpoints of the sides which are parallel to the angle bisectors of the opposite angles meet in the point N and O, G, N lie on a straight line in that order with $OG : GN = 2 : 1$; an alternative Euler line! Moreover,

$$A'N = \tfrac{1}{2}OA, \qquad B'N = \tfrac{1}{2}OB, \qquad C'N = \tfrac{1}{2}OC.$$

This is illustrated in figure 3.

As an extension of this, we could have a tetrahedron $ABCD$ in three dimensions. Relative to some origin O, let A, B, C, D be denoted by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ respectively, and let

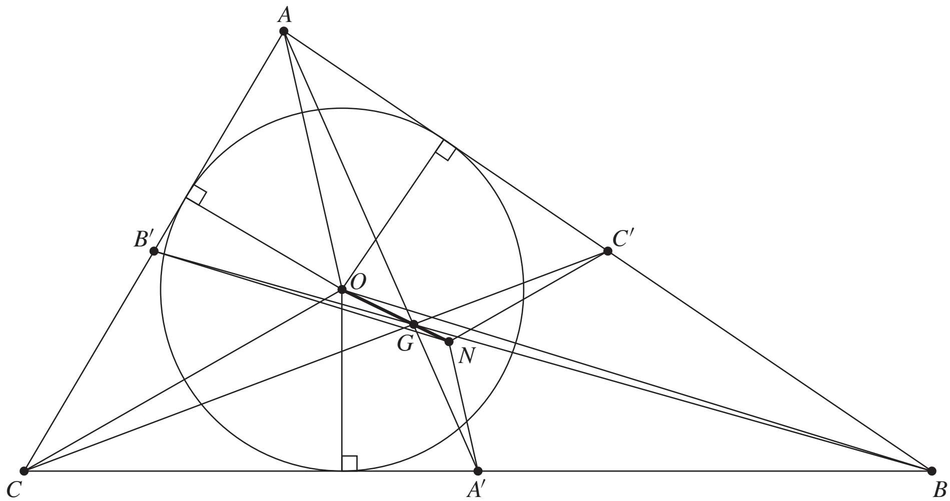


Figure 3

N be the point

$$\frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}{3}.$$

If A' denotes the centroid of the face BCD, then as above we see that $\overrightarrow{A'N} = \boldsymbol{a}/3$, so that the straight line through A' parallel to OA passes through N, and similarly for the other faces. Thus, the straight lines through the centroids of the faces parallel to the lines joining the chosen origin to the opposite vertices meet in the point N. Moreover, if G is the centroid of A, B, C, D, so that

$$\overrightarrow{OG} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}{4},$$

then O, G, N lie on a straight line in that order and $OG : GN = 3 : 1$. Also, $A'N = \frac{1}{3}OA$, $B'N = \frac{1}{3}OB$, $C'N = \frac{1}{3}OC$, and $D'N = \frac{1}{3}OD$. It is but a small step to a polyhedron with n vertices, in which case

$$OG : GN = (n-1) : 1.$$

We are grateful to Camilla Jordan who has produced interactive GeoGebra files showing the Euler line on the *Mathematical Spectrum* website, http://ms.appliedprobability.org/. One shows the Euler line as the user moves the triangle. The other shows the line OGN which hinges on the centroid as the user moves the point O.

The authors began their collaboration in 1997 with their article 'From Pascal to Groups' in Mathematical Spectrum. They have retired from teaching, ***Guido Lasters*** *from a school in Tienen, Belgium, and* ***David Sharpe*** *from the University of Sheffield, UK.*

The Lazy Man's Binomial Distribution

MARTIN GRIFFITHS and WILLIAM CASBOLT

In this article we consider the approximation of one particular probability model by another for which the associated sampling procedure requires a little less effort. Since this approximation arises through sheer laziness rather than for any sound mathematical reasons, we wanted to investigate the price we would have to pay in terms of the sizes of the relative errors that arise under various scenarios associated with these distributions. Both exact and asymptotic results are obtained.

1. Introduction

The second author, a student at the University of Essex, approached the first author, a lecturer at the same institution, concerning something he had noticed whilst doing some exercises associated with the binomial distribution. In particular, one question asked for the probability that when five cards are drawn, at random and with replacement, from a pack of playing cards, exactly three of them are hearts. If we let X denote the random variable representing the number of hearts that appear, then $X \sim \mathrm{B}(5, \frac{1}{4})$. The answer is given by

$$\mathrm{P}(X = 3) = \tbinom{5}{3}\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^2 = \tfrac{45}{512}.$$

The student wondered what would happen if, instead of replacing the cards, we took the lazy option, and did not bother to put them back into the pack. Would it make much difference to the answer?

On letting Y be the number of hearts arising from this without-replacement scenario when five cards are removed at random from the pack, his subsequent calculation was as follows:

$$\begin{aligned}\mathrm{P}(Y = 3) &= \tfrac{13}{52}\cdot\tfrac{12}{51}\cdot\tfrac{11}{50}\cdot\tfrac{39}{49}\cdot\tfrac{38}{48} + \tfrac{13}{52}\cdot\tfrac{12}{51}\cdot\tfrac{39}{50}\cdot\tfrac{11}{49}\cdot\tfrac{38}{48} + \tfrac{13}{52}\cdot\tfrac{12}{51}\cdot\tfrac{39}{50}\cdot\tfrac{38}{49}\cdot\tfrac{11}{48} \\ &\quad + \tfrac{13}{52}\cdot\tfrac{39}{51}\cdot\tfrac{12}{50}\cdot\tfrac{11}{49}\cdot\tfrac{38}{48} + \tfrac{13}{52}\cdot\tfrac{39}{51}\cdot\tfrac{12}{50}\cdot\tfrac{38}{49}\cdot\tfrac{11}{48} + \tfrac{13}{52}\cdot\tfrac{39}{51}\cdot\tfrac{38}{50}\cdot\tfrac{12}{49}\cdot\tfrac{11}{48} \\ &\quad + \tfrac{39}{52}\cdot\tfrac{13}{51}\cdot\tfrac{12}{50}\cdot\tfrac{11}{49}\cdot\tfrac{38}{48} + \tfrac{39}{52}\cdot\tfrac{13}{51}\cdot\tfrac{12}{50}\cdot\tfrac{38}{49}\cdot\tfrac{11}{48} + \tfrac{39}{52}\cdot\tfrac{13}{51}\cdot\tfrac{39}{50}\cdot\tfrac{12}{49}\cdot\tfrac{11}{48} \\ &\quad + \tfrac{39}{52}\cdot\tfrac{38}{51}\cdot\tfrac{13}{50}\cdot\tfrac{12}{49}\cdot\tfrac{11}{48} \\ &= \binom{5}{3}\frac{13\cdot 12\cdot 11\cdot 39\cdot 38}{52\cdot 51\cdot 50\cdot 49\cdot 48}. \end{aligned} \tag{1}$$

He was intrigued by the fact that $\binom{5}{3}$ appeared in his expression for $\mathrm{P}(X = 3)$ and in his expression for $\mathrm{P}(Y = 3)$, and took delight in being shown that (1) can in fact be written as

$$\frac{\binom{5}{3}\binom{47}{10}}{\binom{52}{13}}. \tag{2}$$

We then calculated the relative error in using $\mathrm{P}(Y=3)$ for $\mathrm{P}(X=3)$. This is given by

$$\left|\frac{\mathrm{P}(X=3)-\mathrm{P}(Y=3)}{\mathrm{P}(X=3)}\right| \approx 0.072.$$

At a little over 7%, this might be regarded as a reasonable approximation. Indeed, the student did initially term this the 'nearly binomial distribution'. We soon found, however, that in some cases it gave really rather poor approximations to the binomial distribution. This led us eventually to change its name to the 'lazy man's binomial distribution', before discovering that it did in fact already have an official name: the *hypergeometric distribution* (see references 1 and 2).

An alternative derivation of the expression for $\mathrm{P}(Y=3)$ goes as follows. Suppose that the 52 cards are laid out in a line, and arranged such that the left-most five cards contain exactly three hearts. The right-most 47 cards must therefore contain exactly 10 hearts. The diagram below gives one such possibility, where H denotes a heart and N represents a card that is not a heart (noting that not all of the 47 right-most letters have been written down explicitly):

(NHHNH) (NHNNN . . . HHNNHNNNN).

The number of ways of arranging the three Hs and two Ns on the left is given by $\binom{5}{3}$ while that for the 10 Hs and 37 Ns on the right is $\binom{47}{10}$. Thus, there are $\binom{5}{3}\binom{47}{10}$ ways of arranging the cards such that exactly three of the five cards on the left are hearts. The total number of arrangements of the 52 cards, comprising 13 Hs and 39 Ns, is $\binom{52}{13}$, from which the result follows.

It is worth pointing out here that the numerical expression (2) may be rearranged as follows:

$$\frac{\binom{5}{3}\binom{47}{10}}{\binom{52}{13}} = \frac{5!}{3!\,2!}\cdot\frac{47!}{10!\,37!}\cdot\frac{13!\,39!}{52!} = \frac{5!\,47!}{52!}\cdot\frac{39!}{2!\,37!}\cdot\frac{13!}{3!\,10!} = \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}}.$$

This may be explained, from a probabilistic point of view, by noting that the numerator of the expression on the right gives the number of ways of choosing three hearts (from 13) and two cards that are not hearts (from 39), while the denominator represents the number of ways of selecting five cards from the entire pack of 52. This and the argument in the previous paragraph each lead to the probability that exactly three hearts appear when five cards are taken at random from a pack without replacement.

The lazy man's binomial distribution can be generalised. Suppose that a pack comprises n playing cards in total, of which m belong to some particular suit S. We pick k cards without replacement from the pack. Let Y be the number of these cards that belong to S. Using an argument similar to that given in the alternative derivation of $\mathrm{P}(Y=3)$, we have

$$\mathrm{P}(Y=y) = \binom{k}{y}\binom{n-k}{m-y} \Big/ \binom{n}{m}.$$

Our aim here is, using a very much simplified version of a pack of cards, to ascertain how good the approximations are in a number of different situations. This is an investigation into what price, in terms of accuracy, we have to pay for our laziness.

2. Some exact and some asymptotic results

In order to keep the mathematical manipulations reasonably straightforward, we consider here an alternative pack of $2n$ cards comprising n hearts and n cards that are not hearts. Let $X_{2n,j}$

and $Y_{2n,j}$ denote the number of hearts that appear when j cards are drawn at random from this pack with and without replacement, respectively. For the sake of simplicity once more, we are only concerned with the comparisons of probabilities arising from these two distributions by way of the event 'exactly half the cards drawn are hearts'. It may be assumed, therefore, that j is even, $j = 2k$, say. We use $r(2n, 2k)$ to denote the relative error in using $Y_{2n,2k}$ for $X_{2n,2k}$.

As an introductory example, suppose that eight cards are drawn from the pack. We then have, for example

$$\mathrm{P}(X_{2n,8} = 4) = \tbinom{8}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4 = \tbinom{8}{4}\left(\tfrac{1}{2}\right)^8 \tag{3}$$

and

$$\mathrm{P}(Y_{2n,8} = 4) = \binom{8}{4}\binom{2n-8}{n-4} \Bigg/ \binom{2n}{n}.$$

Stirling's asymptotic formula for $n!$ (see references 3 and 4) is given by

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n,$$

and from this we may obtain

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}}.$$

It follows that

$$\mathrm{P}(Y_{2n,8} = 4) \sim \binom{8}{4}\frac{2^{2(n-4)}}{\sqrt{(n-4)\pi}} \cdot \frac{\sqrt{n\pi}}{2^{2n}} = \frac{1}{2^8}\binom{8}{4}\sqrt{\frac{n}{n-4}}. \tag{4}$$

In fact, since

$$\lim_{n\to\infty}\sqrt{\frac{n}{n-4}} = 1,$$

we have from (3) and (4) that

$$\lim_{n\to\infty} r(2n, 8) = \lim_{n\to\infty}\left|\frac{\mathrm{P}(X_{2n,8} = 4) - \mathrm{P}(Y_{2n,8} = 4)}{\mathrm{P}(X_{2n,8} = 4)}\right| = 0.$$

For this particular situation, therefore, we see that the relative error in using the lazy man's distribution for the binomial distribution tends to zero as n increases without limit. It is in fact clear that this result is true for any fixed even positive integer j.

Now, rather than keeping the number of hearts drawn constant, let the number drawn be directly proportional to the pack size $2n$. In particular, suppose this time that n of the $2n$ cards are chosen at random. What is the probability that exactly half of them are hearts? We have, remembering that n is assumed to be even,

$$\mathrm{P}\left(X_{2n,n} = \frac{n}{2}\right) = \binom{n}{n/2}\left(\frac{1}{2}\right)^n$$

and

$$\mathrm{P}\left(Y_{2n,n} = \frac{n}{2}\right) = \binom{n}{n/2}\binom{2n-n}{n-n/2} \Bigg/ \binom{2n}{n} \sim \binom{n}{n/2} \cdot \frac{2^n}{\sqrt{n\pi/2}} \cdot \frac{\sqrt{n\pi}}{2^{2n}} = \binom{n}{n/2}\frac{\sqrt{2}}{2^n}.$$

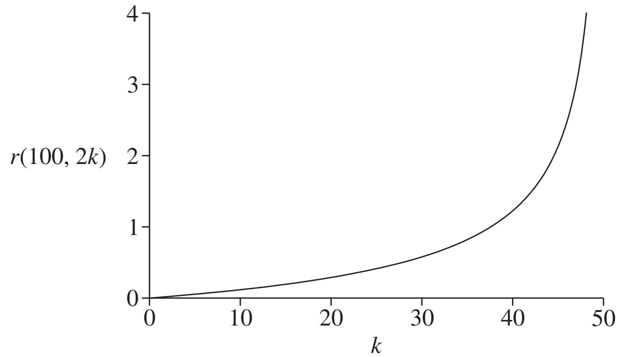


Figure 1 A graph showing the sizes of the relative errors that arise when the lazy man's distribution is used as an approximation to the binomial distribution in order to calculate the probability that k hearts are obtained when $2k$ cards are drawn at random from a pack of 100 cards.

Therefore,

$$\lim_{n\to\infty} r(2n, n) = \lim_{n\to\infty} \left| \frac{\mathrm{P}(X_{2n,n} = n/2) - \mathrm{P}(Y_{2n,n} = n/2)}{\mathrm{P}(X_{2n,n} = n/2)} \right| = \sqrt{2} - 1.$$

In this case, therefore, the relative error tends to $\sqrt{2} - 1$ as n increases without limit.

Above we considered the behaviour of the approximations as n increased without limit. We now look at the situation in which we vary k while holding n constant. Suppose that $2k$ cards are taken from the pack. Let us consider the probabilities that exactly k of these cards are hearts. We have

$$\mathrm{P}(X_{2n,2k} = k) = \binom{2k}{k}\left(\frac{1}{2}\right)^{2k}$$

and

$$\mathrm{P}(Y_{2n,2k} = k) = \binom{2k}{k}\binom{2(n-k)}{n-k} \bigg/ \binom{2n}{n},$$

so

$$\begin{aligned} r(2n, 2k) &= \left| \frac{\mathrm{P}(X_{2n,2k} = k) - \mathrm{P}(Y_{2n,2k} = k)}{\mathrm{P}(X_{2n,2k} = k)} \right| \\ &= \left| \left(\binom{2n}{n} - 2^{2k}\binom{2(n-k)}{n-k} \right) \bigg/ \binom{2n}{n} \right|. \end{aligned} \tag{5}$$

A particular instance of this may be seen in figure 1. We have set $2n = 100$ and then plotted the relative error arising from the probability that exactly k hearts result when $2k$ cards have been drawn. The approximations are reasonably good for small values of k, but the relative error increases dramatically as k approaches 50. It is indeed possible to pay a heavy price for one's laziness.

The distributions considered here are discrete, although, for the sake of clarity, we have joined up the dots in order to form the curve given in figure 1. Incidentally, the graph was plotted using MATHEMATICA®, which allows continuous versions of binomial coefficients to be plotted via gamma functions.

The interested reader might like to select his or her own value of k and then calculate $r(100, 2k)$ from the expression given in (5). This may then be checked visually by way of the graph shown in figure 1.

3. Closing comments

Of course, from figure 1 alone it is not possible to tell which of $\mathrm{P}(X_{100,2k} = k)$ or $\mathrm{P}(Y_{100,2k} = k)$ is the larger. For the more general situation given by (5), we see that it is a matter of deciding which is the largest out of $\binom{2n}{n}$ or $2^{2k}\binom{2(n-k)}{n-k}$ when $0 \le k < n$.

To this end, let

$$f(n, k) = 2^{2k}\binom{2(n-k)}{n-k}.$$

First, note that $f(n, 0) = \binom{2n}{n}$. Also,

$$\begin{aligned} f(n, k+1) - f(n, k) &= 2^{2(k+1)}\binom{2(n-k-1)}{n-k-1} - 2^{2k}\binom{2(n-k)}{n-k} \\ &= 2^{2k}\binom{2(n-k-1)}{n-k-1}\left(4 - \frac{2(n-k)(2n-2k-1)}{(n-k)^2}\right) \\ &= 2^{2k}\binom{2(n-k-1)}{n-k-1}\frac{2}{n-k} \\ &> 0. \end{aligned}$$

This shows, via an inductive-type argument, that

$$\mathrm{P}(Y_{2n,2k} = k) \ge \mathrm{P}(X_{2n,2k} = k),$$

for $0 \le k < n$.

Note that we have not considered the use of absolute errors here. This is because they would not have given us much in the way of useful information. After all, we have been mainly looking at situations in which n increases without limit, or is at least very large. In such cases each of the individual probabilities in the two distributions tends to zero, or is very small. The absolute errors, therefore, will always be very small. On the other hand, small absolute errors do not necessarily imply small relative errors. To take an extreme example, it is indeed possible for an absolute error to tend to zero while the corresponding relative error increases without limit.

References

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Ahmed's Integral: The Maiden Solution

ZAFAR AHMED

In 2001–2002, I proposed a new definite integral in *The American Mathematical Monthly* (see references 1 and 2), which later came to be known as *Ahmed's integral*. In the meantime, this integral has been mentioned in mathematical encyclopaedias and dictionaries and further it has also been cited and discussed in several books and journals. In particular, a Google search with the keyword 'Ahmed's integral' throws up more than 60 listings. Here I present the maiden solution for this integral.

My proposal of evaluating the following integral

$$\int_0^1 \frac{\tan^{-1}\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\,\mathrm{d}x \tag{1}$$

was published in 2001 (see reference 1), when it was thrown open to be solved within the next six months. Subsequently, 20 authors and two problem solving groups proposed correct solutions. The solutions to (1) by two authors, Kunt Dale and George L. Lamb, Jr., were published in 2002 (see reference 2). *The American Mathematical Monthly* usually prefers to publish the solutions of other solvers rather than that of the proposer.

This integral is now known as *Ahmed's integral*. A Google search for 'Ahmed's integral' brings more than 60 listings to view. It has been included in encyclopaedias and dictionaries. Various solutions, extensions, properties, and connections of this integral have been discussed in a variety of ways. This integral is very well discussed in two very interesting books on integrals (see reference 3, pp. 17–20, and reference 4, pp. 190–194) and mentioned in another (see reference 5, p. 277). This analytically solvable integral also serves as a test model for various new methods of high precision numerical (quadratures) integrations (see reference 6).

Since in recent times this integral has evoked considerable attention, I propose to present the maiden solution that was sent along with the proposal in reference 1.

Let us call the integral (1) I and use $\tan^{-1} z = \pi/2 - \tan^{-1}(1/z)$ to split I as $I = I_1 - I_2$. Using the substitution $x = \tan\theta$, we can write

$$I_1 = \frac{\pi}{2}\int_0^{\pi/4} \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{2-\sin^2\theta}},$$

which can be evaluated as $I_1 = \pi^2/12$ by using the substitution $\sin\theta = \sqrt{2}\sin\phi$. Next we use the representation

$$\frac{1}{a}\tan^{-1}\frac{1}{a} = \int_0^1 \frac{\mathrm{d}x}{x^2+a^2} \qquad (a \neq 0)$$

to express

$$I_2 = \int_0^1\int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2)(2+x^2+y^2)}.$$

Furthermore, I_2 can be rewritten as

$$\begin{aligned} I_2 &= \int_0^1 \int_0^1 \frac{1}{1+y^2}\left(\frac{1}{1+x^2} - \frac{1}{2+x^2+y^2}\right) \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{(1+y^2)(1+x^2)} - \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{(1+y^2)(2+x^2+y^2)}. \end{aligned} \tag{2}$$

Utilizing the symmetry of the integrands and the domains for x and y, the second integral in (2) is equal to I_2. This leads to

$$2I_2 = \int_0^1 \int_0^1 \frac{\mathrm{d}x\, \mathrm{d}y}{(1+y^2)(1+x^2)} = \left(\int_0^1 \frac{\mathrm{d}x}{1+x^2}\right)^2 = \frac{\pi^2}{16}.$$

Eventually, we get

$$I = \frac{\pi^2}{12} - \frac{\pi^2}{32} = \frac{5\pi^2}{96}.$$

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Derek Collins

We are sorry to report the death of Derek Collins. Derek was for some years our Applied Mathematics editor and latterly was on the advisory board of *Mathematical Spectrum*. He taught at The University of Sheffield, for part of the time as Head of Applied Mathematics. He had a special interest in a project to develop ways of introducing Applied Mathematics projects in schools. Derek had a keen interest in the theatre; he claimed to have seen on the stage every one of Shakespeare's plays!

A Fibonacci Curiosity

THOMAS KOSHY and ZHENGUANG GAO

Using recursion, we establish that the sums $\sum_{k\geq 0,\, k \text{ even}} \binom{m-k}{k}$ and $\sum_{k\geq 0,\, k \text{ odd}} \binom{m-k}{k}$, which occur in the well-known Lucas formula, are equal if and only if $m = 3n - 1$.

Introduction

Fibonacci numbers continue to marvel the mathematical community with their ubiquity and charming properties (see references 1–3). They are still a fertile ground for exploration and experimentation.

Fibonacci numbers F_n are often defined recursively as follows:

$$\mathrm{F}_1 = 1 = \mathrm{F}_2,$$
$$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}, \qquad n \geq 3.$$

Lucas' formula

In 1876, the French mathematician, François Édouard Anatole Lucas (1842–1891), after whom the well-known Lucas numbers are named, discovered an explicit formula for F_n:

$$\mathrm{F}_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \tag{1}$$

where $\lfloor x \rfloor$ denotes the *floor* of the real number x, and $n \geq 0$. For example,

$$\begin{aligned} \mathrm{F}_{11} &= \sum_{k=0}^{5} \binom{10-k}{k} \\ &= \binom{10}{0} + \binom{9}{1} + \binom{8}{2} + \binom{7}{3} + \binom{6}{4} + \binom{5}{5} \\ &= 89. \end{aligned}$$

It follows by the Lucas formula that F_{n+1} can be found by computing the sum of the elements along the northeast diagonal n of Pascal's triangle, where $n \geq 0$.

Interestingly, (1) can be rewritten as

$$\mathrm{F}_{n+1} = \sum_{k\geq 0,\, k \text{ even}} \binom{n-k}{k} + \sum_{k\geq 0,\, k \text{ odd}} \binom{n-k}{k}. \tag{2}$$

This aroused our curiosity: is it possible for the two sums on the right-hand side of (2) to be equal? If yes, can we identify the corresponding values of n?

To begin with, notice that

$$\sum_{k\geq 0,\, k \text{ even}} \binom{7-k}{k} = 11 \neq 10 = \sum_{k\geq 0,\, k \text{ odd}} \binom{7-k}{k},$$
$$\sum_{k\geq 0,\, k \text{ even}} \binom{8-k}{k} = 17 = \sum_{k\geq 0,\, k \text{ odd}} \binom{8-k}{k}.$$

To answer the above questions, we let

$$\mathrm{E}_n = \sum_{k\geq 0,\, k \text{ even}} \binom{n-k}{k} \quad \text{and} \quad \mathrm{O}_n = \sum_{k\geq 0,\, k \text{ odd}} \binom{n-k}{k}.$$

Then, by Pascal's identity, we have

$$\begin{aligned}\mathrm{E}_{n-1} + \mathrm{O}_{n-2} &= \sum_{k\geq 0} \binom{n-2k-1}{2k} + \sum_{k\geq 1} \binom{n-2k-1}{2k-1} \\ &= \sum_{k\geq 0} \binom{n-2k}{2k} \\ &= \mathrm{E}_n .\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{O}_{n-1} + \mathrm{E}_{n-2} &= \sum_{k\geq 0} \binom{n-2k-2}{2k+1} + \sum_{k\geq 0} \binom{n-2k-2}{2k} \\ &= \sum_{k\geq 0} \binom{n-2k-1}{2k+1} \\ &= \mathrm{O}_n .\end{aligned}$$

Now let $\mathrm{D}_n = \mathrm{E}_n - \mathrm{O}_n$. Then we have

$$\begin{aligned}\mathrm{D}_{n-1} - \mathrm{D}_{n-2} &= (\mathrm{E}_{n-1} - \mathrm{O}_{n-1}) - (\mathrm{E}_{n-2} - \mathrm{O}_{n-2}) \\ &= (\mathrm{E}_{n-1} + \mathrm{O}_{n-2}) - (\mathrm{O}_{n-1} + \mathrm{E}_{n-2}) \\ &= \mathrm{E}_n - \mathrm{O}_n \\ &= \mathrm{D}_n .\end{aligned}$$

Thus, D_n satisfies the recurrence $\mathrm{D}_n = \mathrm{D}_{n-1} - \mathrm{D}_{n-2}$, where $\mathrm{D}_0 = 1-0 = 1$, $\mathrm{D}_1 = 1-0 = 1$, and $n \geq 2$.

This recursive definition enables us to compute the sequence $\{\mathrm{D}_n\}_{n\geq 0}$:

$$\underbrace{1\ 1\ 0\ \ -1\ \ -1\ 0}\ \underbrace{1\ 1\ 0\ \ -1\ \ -1\ 0}\cdots .$$

Clearly, the sequence $\{\mathrm{D}_n\}_{n\geq 0}$ is periodic with period 6; so the pattern continues for ever.

It now follows that

$$\mathrm{D}_n = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \pmod 6, \\ -1 & \text{if } n \equiv 3 \text{ or } 4 \pmod 6, \\ 0 & \text{if } n \equiv 2 \text{ or } 5 \pmod 6. \end{cases}$$

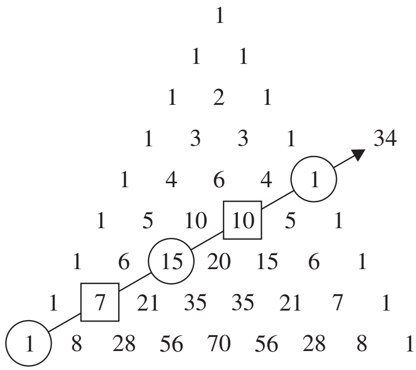

Figure 1 Pascal's triangle.

But $n \equiv 2$ or $5 \pmod 6$ if and only if $n \equiv 2 \pmod 3$. Consequently, $\mathrm{D}_n = 0$ if and only if $n \equiv 2 \pmod 3$. In other words, $\mathrm{E}_n = \mathrm{O}_n$ if and only if $n \equiv 2 \pmod 3$. Thus,

$$\sum_{k\geq 0}\binom{3n-2k-1}{2k} = \sum_{k\geq 0}\binom{3n-2k-2}{2k+1}.$$

By the Lucas formula, $\mathrm{E}_{3n-1} + \mathrm{O}_{3n-1} = \mathrm{F}_{3n}$. Consequently, $\mathrm{E}_{3n-1} = \frac{1}{2}\mathrm{F}_{3n} = \mathrm{O}_{3n-1}$. Thus,

$$\sum_{k\geq 0}\binom{3n-2k-1}{2k} = \sum_{k\geq 0}\binom{3n-2k-2}{2k+1} = \frac{1}{2}\mathrm{F}_{3n}.$$

For example, let $n = 3$. Then we have

$$\sum_{k\geq 0}\binom{8-2k}{2k} = \binom{8}{0} + \binom{6}{2} + \binom{4}{4} = 17 = \frac{1}{2}\mathrm{F}_9.$$

This represents the sum of the circled numbers in Pascal's triangle; see figure 1.

Similarly,

$$\sum_{k\geq 0}\binom{7-2k}{2k+1} = 17 = \frac{1}{2}\mathrm{F}_9;$$

see the boxed numbers in figure 1.

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Palindromic Pell Walks

THOMAS KOSHY

Pell walks are a nice way of interpreting Pell numbers and Pell–Lucas numbers combinatorially. In this article, we will investigate such walks whose alphabetic representations are palindromic.

Introduction

Pell numbers and Pell–Lucas numbers are an interesting source for creativity, exploration, and fecundity. Like Fibonacci numbers and Lucas numbers, they too have interesting applications to combinatorics (see reference 1) especially to the study of lattice paths (see references 2 and 3), as we will see shortly.

Pell numbers P_n and Pell–Lucas numbers Q_n satisfy the second-order recurrence $x_n = 2x_{n-1} + x_{n-2}$ and the same initial condition $x_1 = 1$, where $n \geq 3$, when $x_2 = 2$, $x_n = \mathrm{P}_n$, and when $x_2 = 3$, $x_n = \mathrm{Q}_n$ (see references 2, 4, 5, and 6). They can also be defined explicitly by *Binet-like formulas* as follows:

$$\mathrm{P}_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad \mathrm{Q}_n = \frac{\gamma^n + \delta^n}{2},$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the solutions of the equation $x^2 = 2x + 1$. Table 1 shows the first ten Pell numbers and Pell–Lucas numbers.

Pell and Pell–Lucas numbers satisfy numerous delightful properties (see reference 2). For example, $\mathrm{P}_n + \mathrm{Q}_n = \mathrm{P}_{n+1}$ and $2\mathrm{P}_n + \mathrm{Q}_n = \mathrm{Q}_{n+1}$. They can be confirmed using induction or the Binet-like formulas.

Next we present some basic vocabulary about lattice paths for clarity.

Lattice paths

A *lattice point* on the cartesian plane is a point (x, y) with integral coordinates x and y. A *lattice path* is a sequence of connected horizontal, vertical, or diagonal steps $\overline{X_k X_{k+1}}$, emanating from the origin, where both X_k and X_{k+1} are lattice points. The number of unit steps in the path is its *length*.

Table 1 The first ten Pell numbers and Pell–Lucas numbers.

n	P_n	Q_n	n	P_n	Q_n
1	1	1	6	70	99
2	2	3	7	169	239
3	5	7	8	408	577
4	12	17	9	985	1 393
5	29	41	10	2 378	3 363

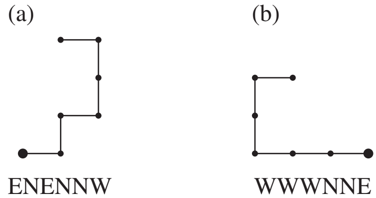

Figure 1

For example, figure 1 shows two lattice paths of length 6, where the thick dot indicates the origin. The path in figure 1(a) consists of one step in the easterly direction (E), followed by one in the northerly direction (N), one in the easterly direction, two in the northerly direction, and one in the westerly direction (W); it is denoted by the *word* ENENNW. Likewise, the path in figure 1(b) is denoted by WWWNNE.

Pell walks

Pell walks are a special class of lattice paths (see references 2, 3, 6, and 7). Pell numbers and Pell–Lucas numbers pop up a number of times in the study of Pell walks. This article investigates a special class of Pell walks. This endeavour yields additional delightful occurrences of the Pell family.

Starting at the origin on the cartesian plane, a Pell walk of *length* n consists of n E, N, or W-steps such that no W-step follows immediately an E-step or vice versa; that is, *no* Pell word can contain EW or WE as a subword. Figure 2 shows the possible paths of length n, where $0 \leq n \leq 3$. It is known that there are exactly Q_{n+1} Pell walks of length n (see reference 6).

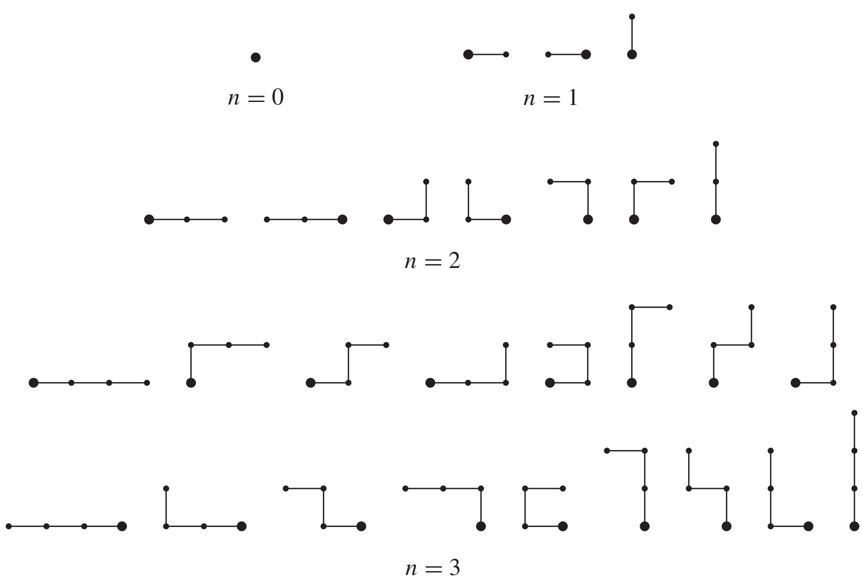

Figure 2

Next we introduce the concept of a palindromic Pell walk. To this end, a *palindrome* is a word w whose reflection about the vertical line through the middle is itself; that is, $w = w^{\mathrm{R}}$, where w^{R} denotes the reverse of w. For example, *mom* and *madam* are palindromic, whereas 2342 and *word* are not.

Palindromic Pell walks

A *palindromic Pell walk* (PPW) is a Pell walk with the property that the corresponding Pell word is palindromic. Table 2 shows the PPWs of length n, where λ denotes the null word and $0 \le n \le 4$; see figure 3 also.

It appears from table 2 that the number of PPWs is a Pell–Lucas number. The following theorem confirms this observation.

Theorem 1 *The number of PPWs of length n is $\mathrm{Q}_{\lfloor (n+3)/2 \rfloor}$, where $\lfloor x \rfloor$ denotes the floor of the real number x, that is, the largest integer less than or equal to x.*

Proof Let $w = w_1 w_2 \cdots w_n$ be a PPW of length n.

Table 2 Palindromic Pell walks.

n	Palindromic Pell walks	Number of such walks
0	λ	1
1	E, N, W	3
2	EE, NN, WW	3
3	EEE, ENE, NEN, NNN, NWN, WNW, WWW	7
4	EEEE, ENNE, NEEN, NNNN, NWWN,WNNW, WWWW	7

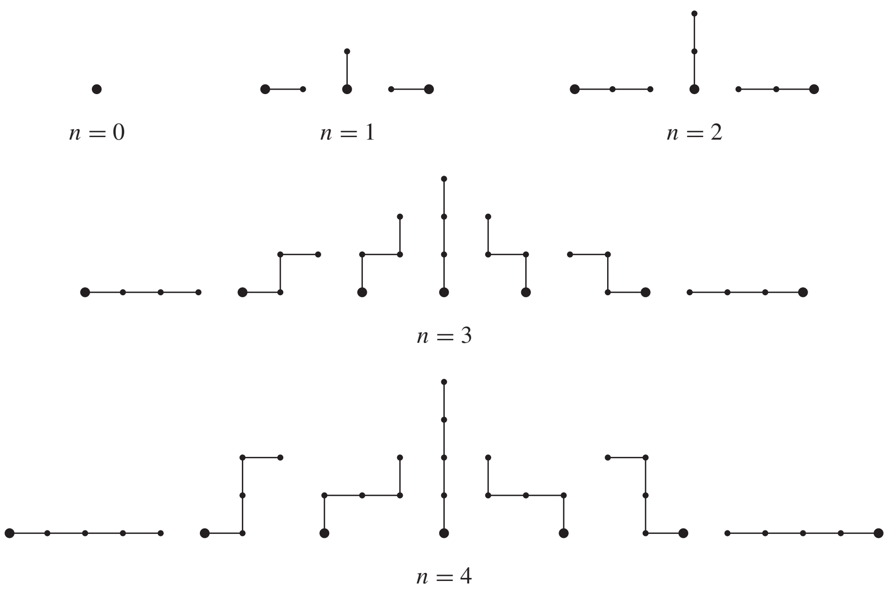

Figure 3

Case 1 Let $n = 2k$, an even integer, where $k \geq 0$. Since $w = w_1 \cdots w_k w_{k+1} \cdots w_{2k} = (w_1 \cdots w_k)(w_1 \cdots w_k)^{\mathrm{R}}$, the number of such PPWs equals the number of Pell walks $w_1 \cdots w_k$ of length k. But there are Q_{k+1} Pell walks of length k (see reference 6). Consequently, there are $\mathrm{Q}_{k+1} = \mathrm{Q}_{(n+2)/2} = \mathrm{Q}_{\lfloor (n+3)/2 \rfloor}$ PPWs of length n.

Case 2 Suppose that $n = 2k + 1$. Then

$$w = w_1 \cdots w_k w_{k+1} w_{k+2} \cdots w_{2k+1} = (w_1 \cdots w_k) w_{k+1} (w_1 \cdots w_k)^{\mathrm{R}}.$$

Therefore, the number of PPWs of length $2k + 1$ equals the number of Pell walks $w_1 \cdots w_{k+1}$ of length $k + 1$, namely $\mathrm{Q}_{k+2} = \mathrm{Q}_{(n+3)/2} = \mathrm{Q}_{\lfloor (n+3)/2 \rfloor}$.

Thus, in both cases, the formula works.

As an example, there are $\mathrm{Q}_{\lfloor (5+3)/2 \rfloor} = \mathrm{Q}_4 = 17$ PPWs of length 5:

EEEEE, EENEE, ENENE, ENNNE, ENWNE,
NEEEN, NENEN, NNENN, NNNNN, NNWNN, NWNWN, NWWWN,
WNENW, WNNNW, WNWNW, WWNWW, WWWWW.

Next we will find the number of PPWs that begin with N.

PPWs beginning with N

The number of PPWs of length n beginning with N is the same as the number of PPWs of length $n - 2$. By theorem 1, there are $\mathrm{Q}_{\lfloor (n+1)/2 \rfloor}$ such PPWs. Thus, we have the following result.

Corollary 1 *There are $\mathrm{Q}_{\lfloor (n+1)/2 \rfloor}$ PPWs of length n beginning with N.*

As an example, let $n = 5$. There are $\mathrm{Q}_3 = 7$ PPWs of length 5 beginning with N:

NEEEN, NENEN, NNENN, NNNNN, NNWNN, NWNWN, NWWWN.

Theorem 1 has another interesting byproduct, as the next corollary shows.

Corollary 2 *There are $\mathrm{Q}_{\lfloor (n-1)/2 \rfloor}$ PPWs of length n beginning with NN.*

Proof The number of PPWs of length n beginning with NN equals the number of PPWs of length $n - 4$. By theorem 1, there are $\mathrm{Q}_{\lfloor (n-1)/2 \rfloor}$ such PPWs. (This follows by corollary 1 also.)

As an example, there are $\mathrm{Q}_{\lfloor (6-1)/2 \rfloor} = \mathrm{Q}_2 = 3$ PPWs of length 6 that begin with NN:

NNEENN, NNNNNN, NNWWNN.

Next we will find the number of PPWs beginning with E.

PPWs beginning with E

Let e_n denote the number of PPWs of length n beginning with E. Then, by symmetry, the number of PPWs of length n beginning with W also equals e_n (see figure 3). Consequently, by theorem 1 and corollary 1, we have

$$2e_n + Q_{\lfloor (n+1)/2 \rfloor} = Q_{\lfloor (n+3)/2 \rfloor},$$

$$2e_n = Q_{\lfloor (n+3)/2 \rfloor} - Q_{\lfloor (n+1)/2 \rfloor} = \begin{cases} Q_{(n+2)/2} - Q_{n/2} & \text{if } n \text{ is even,} \\ Q_{(n+3)/2} - Q_{(n+1)/2} & \text{otherwise.} \end{cases}$$

Suppose that n is even. Then $Q_{(n+2)/2}$ and $Q_{n/2}$ are consecutive Pell–Lucas numbers. Since

$$Q_{m+1} - Q_m = 2P_m,$$

it follows that $e_n = P_{n/2}$. On the other hand, let n be odd. Then $Q_{(n+3)/2}$ and $Q_{(n+1)/2}$ are consecutive Pell–Lucas numbers; so $e_n = P_{(n+1)/2}$. Thus,

$$e_n = \begin{cases} P_{n/2} & \text{if } n \text{ is even,} \\ P_{(n+1)/2} & \text{otherwise} \end{cases}$$
$$= P_{\lfloor (n+1)/2 \rfloor}.$$

We now have the following result.

Theorem 2 *There are* $P_{\lfloor (n+1)/2 \rfloor}$ *PPWs of length* n *beginning with E.*

As an example, there are $e_5 = P_3 = 5$ PPWs of length 5 beginning with E:

EEEEEE, EENEE, ENENE, ENNNE, ENWNE;

and there are $e_6 = P_3 = 5$ such walks of length 6:

EEEEEE, EENNEE, ENEENE, ENNNNE, ENWWNE.

Since the number of PPWs of length n beginning with EE equals the number of PPWs of length $n-2$ beginning with E, the next result follows from theorem 2.

Corollary 3 *There are* $P_{\lfloor (n-1)/2 \rfloor}$ *PPWs of length* n *beginning with EE.*

As an example, there are $P_{\lfloor (7-1)/2 \rfloor} = P_3 = 5$ PPWs of length 7 beginning with EE:

EEEEEEE, EEENEEE, EENENEE, EENNNEE, EENWNEE.

It also follows by theorem 2 that the number of PPWs of length n beginning with W is also $P_{\lfloor (n+1)/2 \rfloor}$. Consequently, by corollary 1 and theorem 2, we have

$$\begin{pmatrix} \text{total number} \\ \text{of PPWs of length } n \end{pmatrix} = 2 \begin{pmatrix} \text{number of PPWs} \\ \text{beginning with E} \end{pmatrix} + \begin{pmatrix} \text{number of PPWs} \\ \text{beginning with N} \end{pmatrix},$$
$$Q_{\lfloor (n+3)/2 \rfloor} = 2P_{\lfloor (n+1)/2 \rfloor} + Q_{\lfloor (n+1)/2 \rfloor}.$$

The next two results give the number of PPWs of length n beginning with EN and NE.

Theorem 3 *There are* $Q_{\lfloor (n-1)/2 \rfloor}$ *PPWs of length* n *beginning with EN.*

Proof The number of PPWs of length n beginning with EN equals the number of PPWs of length $n-4$. So the given result follows by theorem 1.

As an example, there are $Q_2 = 3$ PPWs of length 5 beginning with EN:

ENENE, ENWNE, ENWNE;

and $Q_3 = 7$ such walks of length 8:

ENEEEENE, ENENNENE, ENNEENNE,
ENNNNNNE, ENNWWNNE, ENWNNWNE, ENWWWWNE.

Theorem 4 *There are* $P_{\lfloor (n-1)/2 \rfloor}$ *PPWs of length* n *beginning with NE.*

Proof The desired result is the number of PPWs of length $n-4$ that begin with N or E (*not* W); so, by corollary 1 and theorem 2, it equals

$$Q_{\lfloor (n-3)/2 \rfloor} + P_{\lfloor (n-3)/2 \rfloor} = P_{\lfloor (n-1)/2 \rfloor}.$$

As an example, there are $P_2 = 2$ PPWs of length 5 beginning with NE: NEEEN and NENEN; and $P_2 = 2$ such paths of length 6 beginning with NE: NEEEEN and NENNEN.

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Exploring the New Zealand Puzzle

KEITH BRANDT and SERGE NEVSKY

The *New Zealand puzzle* consists of 16 square pieces and a 4×4 grid. The puzzle pieces are to be placed on the grid one at a time according to some rules. The puzzle is computationally difficult, in that there are millions of ways to place the pieces on the board. We describe a computer program that solves the puzzle; we then look for variations of the puzzle that have fewer solutions.

Dedicated to our mothers, Jan Robertson and Natalia Valenti

1. Introduction

It is one of the world's hardest puzzles. Or so say the instructions that come with it. The New Zealand puzzle was developed by a German company named Philos-Spiele, http://philos-spiele.de. The puzzle consists of a square wooden board with a 4×4 grid on it. Painted on each square of the grid is one of five themes – a kiwi bird, a fern, a sheep, a silhouette of a map of New Zealand, and the letters NZ. Along with the board are 16 square wooden pieces, each with one of the five themes on it, that are to be placed on the board. The puzzle board and sample pieces are shown in figure 1. There are four NZ pieces and three pieces each for the other themes, and the layout of themes on the board is shown in figure 2.

The challenge is to place the 16 pieces one at a time so that no piece is ever placed on or next to (horizontally, vertically, or diagonally) a square that has the same theme. Once a piece is placed on a square, the theme of that square becomes that of its piece, and no other pieces may be placed on that square. Thus, a solution consists of a list of 16 allowable moves (which will necessarily fill the board). Part of the difficulty in solving the puzzle lies in the fact that the layout of themes on the board changes each time a piece is placed.

Figure 1 The New Zealand puzzle.

sheep	kiwi	sheep	map
NZ	map	NZ	fern
kiwi	fern	kiwi	map
map	sheep	NZ	fern

Figure 2 The layout of themes.

We solve the puzzle with a computer program that uses a recursive backtracking approach (also known as depth-first search – see references 1 or 2 for background). To our surprise, there are over 14 000 solutions. We describe our solution to the original puzzle and then look for variations that have fewer solutions. We enumerate and solve all possible variations of the puzzle (with 16 pieces, five themes, and a 4×4 grid).

2. Counting the possibilities

To solve the puzzle, the program must consider all possible ways to place the pieces on the board – at least until it reaches a configuration where no other pieces may be placed. We use elementary counting tools (again, see references 1 or 2) to find the number of ways to place the pieces on the board if there were no restrictions. Recall that, for $0 \leq k \leq n$, the binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

is the number of subsets of k objects from a collection of n objects. First choose where the various themes will go; then choose the order in which to place the pieces on the board. From the 16 locations on the board, choose four to host the NZs. From the 12 locations that remain, choose three to host the ferns. Continue in this fashion to choose locations for the kiwis, sheep, and maps as well. At this point we have decided where each of the 16 pieces will go, but we have not yet placed them on the board. To follow the rules of the puzzle, we must place the pieces one at a time. Thus, there are $16!$ ways to set these pieces down. Via the multiplication principle, we have

$$\tbinom{16}{4}\tbinom{12}{3}\tbinom{9}{3}\tbinom{6}{3}\tbinom{3}{3}16! = 1.407 \times 10^{22}$$

different ways a player could lay the pieces on the board.

Of course, there are restrictions as to how the pieces may be placed, so the value obtained above is indeed a gross overestimate. We can check by hand that there are six ways to place the first piece and 32 ways to place the first two pieces. If we assume there are as few as four choices for each of the first ten pieces placed, there would be over one million ways to place the pieces on the board.

3. Solving the puzzle

We use a recursive algorithm that backtracks whenever the board reaches a configuration where no additional pieces may be placed. We choose an alphabetical ordering for the five themes

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

Figure 3 Locations on the board.

(0 for fern, 1 for kiwi, 2 for map, 3 for NZ, and 4 for sheep) and use counters to keep track of how many pieces of each theme are currently available. These counters are initially set according to the distribution of themes on the 16 pieces. The ordering for the 16 locations on the board is shown in figure 3.

The heart of the program is the function PUT, which systematically runs through themes and locations. When it finds a location for an available theme, it places a piece in that location and updates the variables that keep track of the board and the number of pieces available for that theme. When it reaches a state when none of the available themes can be placed, the most recent piece placed is removed and the search continues. PUT is outlined below. Here i and j represent the current theme and location under consideration respectively, and n gives the number of pieces that have been placed on the board.

```
function PUT(n)
if n=16, board is complete. Output solution.
else
   i=0
   while i<5
      if pieces of theme i remain
         j=0
         while j<16
            if location j is available and no neighbouring
            locations have theme i
               place piece with theme i in location j
               update counters and other variables
               PUT(n+1)
               remove piece with theme i from location j
               reset counters and other variables
            j=j+1
      i=i+1
```

The program finds 14 925 solutions, all of which give the same final arrangement of themes on the board. One solution, the first in our ordering of the themes, is shown in table 1.

To find the first solution, the function PUT is called 4 701 967 times. For the exhaustive search that finds all solutions, PUT is called 7 395 682 times. Note that the value 7 395 682 is an upper bound to the number of different ways we may place pieces on the board until either solving the puzzle or reaching a dead end.

Table 1 One solution to the puzzle.

Step	Theme	Location
1	NZ	12
2	map	13
3	sheep	8
4	sheep	10
5	kiwi	7
6	kiwi	9
7	fern	4
8	fern	6
9	kiwi	15
10	fern	14
11	NZ	3
12	NZ	5
13	map	0
14	map	2
15	NZ	11
16	sheep	1

4. Variations

As mentioned earlier, we were surprised to find such a large number of solutions. In this section, we describe our efforts to find variations of the puzzle that have fewer solutions.

4.1. Changing the distribution of pieces

Recall that the puzzle comes with four NZ pieces and three pieces each of the other themes. Imagine a puzzle that comes with four pieces of each theme, and the instructions ask you to find any 16 of them that can be placed on the board according to the rules. It is very easy to modify our program to solve this generalized version of the puzzle. Simply set the counters for each theme to the initial value of 4, giving the program more pieces to work with. This approach, however, opens up a can of worms in the sense that now many different piece distributions are possible. To indicate a particular distribution of pieces, we use the notation

$$[a_0, a_1, a_2, a_3, a_4],$$

where a_k gives the number of pieces with theme k. The original puzzle has piece distribution $[3, 3, 3, 4, 3]$. Our generalized puzzle solver finds over 229 000 solutions spread out over the original distribution as well as distributions $[4, 4, 2, 3, 3]$, $[3, 3, 3, 3, 4]$, and $[4, 3, 3, 2, 4]$. The distribution $[4, 4, 2, 3, 3]$ has over 191 000 solutions (not what we are looking for), but the other two are interesting. Distribution $[3, 3, 3, 3, 4]$ has 14 925 solutions (same as the original puzzle), and distribution $[4, 3, 3, 2, 4]$ has 7 688 solutions. Thus, by simply packaging the pieces in different ways, the makers of the puzzle could have three variations that all use the original layout of themes on the board. What is interesting to us is that one of these variations has roughly half as many solutions as the original puzzle.

4.2. New boards

Another way to create a variation of the puzzle is to consider different boards, and it is quite easy to write a program to generate them. First note that in the original board, adjacent locations never share the same theme. Thus, in a sense, the layout of the board already satisfies the rules that players must follow when they place pieces on the board. In our search for new boards, we consider only boards that have this property.

We have discussed the distribution of themes on the puzzle pieces. We must also consider the distribution of themes on a board, for which we use the notation

$$(a_0, a_1, a_2, a_3, a_4),$$

where a_k gives the number of locations on the board that have theme k. The original board has distribution $(3, 3, 4, 3, 3)$. For our purposes, it is enough to know the *distribution type* of a board, which we define to be its distribution with the entries sorted. Thus, the distribution type gives general information about the distribution of themes on a board without paying attention to which theme is which. For distribution types, we use the notation

$$\langle b_0, b_1, b_2, b_3, b_4 \rangle,$$

where $b_k \leq b_{k+1}$. As we consider new boards, we will look closely at their distribution types.

4.2.1. Generating boards To generate boards, we use a recursive function PUTTHEME, which, not surprisingly, resembles the function PUT that solves the puzzle. The function systematically considers theme i in location n, while checking that none of the neighbouring locations have already been assigned theme i.

```
function PUTTHEME(n)
if n=16, board is complete. Output board.
else
   i=0
   while i < 5
      if no neighbouring locations have theme i
         put theme i in location n
         PUTTHEME(n+1)
         remove theme i from location n
      i = i+1
```

The function PUTTHEME generates 935 040 boards, whose various distribution types are shown in table 2. We decided not to consider boards of type $\langle 0, 4, 4, 4, 4 \rangle$ and $\langle 1, 3, 4, 4, 4 \rangle$, as we did not feel that they fit with the original spirit of the puzzle.

The rest of this section is dedicated to minimizing the number of boards to consider. Our approach is to identify and remove redundant boards that are equivalent to boards that have already been considered. Two boards are equivalent if one can be transformed into the other by some basic operations. We will see that if two boards are equivalent, the puzzles that they host will, for our purposes, be identical. Next, we describe these operations and refine our notion of equivalence of boards.

Table 2 All possible boards.

Distribution type	Number of boards
$\langle 0, 4, 4, 4, 4\rangle$	840
$\langle 1, 3, 4, 4, 4\rangle$	54 240
$\langle 2, 2, 4, 4, 4\rangle$	61 440
$\langle 2, 3, 3, 4, 4\rangle$	537 840
$\langle 3, 3, 3, 3, 4\rangle$	280 680

4.2.2. Reduced boards Given any board, we may produce an equivalent board by swapping two of its themes. For example, consider the two boards given in figure 4. Board B is obtained from Board A by swapping sheep and map. However, these boards differ only in the names assigned to the locations; their basic structure is identical. If we stick with the alphabetical ordering of themes for Board A, but switch sheep and map in the ordering of themes for Board B (0 for fern, 1 for kiwi, 2 for sheep, 3 for NZ, and 4 for map), both boards look like the board in figure 5.

If we run our generalized puzzle solver program (using four pieces of each theme) on either board with any ordering of the themes, we will find the same number of puzzles (with various piece distributions) and the same number of solutions. (Note that the corresponding puzzle

Board A (original board)

sheep	kiwi	sheep	map
NZ	map	NZ	fern
kiwi	fern	kiwi	map
map	sheep	NZ	fern

Board B

map	kiwi	map	sheep
NZ	sheep	NZ	fern
kiwi	fern	kiwi	sheep
sheep	map	NZ	fern

Figure 4 Swapping themes.

4	1	4	2
3	2	3	0
1	0	1	2
2	4	3	0

Figure 5 Structure of the original board.

0	1	0	2		fern	kiwi	fern	map
3	2	3	4		NZ	map	NZ	sheep
1	4	1	2	=	kiwi	sheep	kiwi	map
2	0	3	4		map	fern	NZ	sheep

Figure 6 Original board reduced.

for Board B has the same piece distribution as the original puzzle, and one of its solutions is identical to the one presented in table 1, with the map pieces and sheep pieces swapped.) Thus, these two boards are equivalent. We now use swapping to develop the idea of a reduced board.

Note that the themes in locations 0 and 1 must be different. Thus, we may fix them to be themes 0 (fern) and 1 (kiwi) respectively. Furthermore, we may assume that the theme in location 2 is either 0 (fern) or 2 (map). Note that the theme in location 2 cannot be 1 (kiwi). If it is 0 (fern), we make no changes since that theme is already in location 0. However, if it is 3 (NZ) or 4 (sheep), we may swap it with theme 2 (map). We generalize our reasoning and introduce the concept of a reduced board. A board is *reduced* if it satisfies the following:

- theme 0 is in location 0,
- for $n \geq 1$, if themes 0 up to k are used in locations 0 up to $n-1$, then the theme in location n is less than or equal to $k+1$.

Thus, in a reduced board, no theme occurs until all of its predecessors occur. For example, the original puzzle board (figure 5) is not reduced. It is easily reduced, however, by swapping themes 0 (fern) and 4 (sheep). The resulting reduced board is shown in figure 6.

By looking through the locations on a board in order and swapping themes as necessary, we can reduce any board. The function `REDUCE` inputs a board and returns a reduced version of it. We represent the board as a list of themes, and we use the function `SWAP`, which inputs a board and returns a new board with two of its themes swapped.

```
function REDUCE(board):
   if board[0]>0:
      board = SWAP(board,0,board[0])
   for k = 1 to 15:
      let m = maximum of themes in locations 1 up to k
      if board[k] > m+1:
         board = swap(board,m+1,board[k])
   return board
```

In a nutshell, the reduced version of a board is a particular permutation of its themes. Furthermore, since reducing a board amounts to a sequence of swaps, we know that any board is equivalent to a reduced board. (In the language of algebra, any permutation can be written as the product of two-cycles. See reference 3, p. 108.) Thus, we can greatly shorten the list of boards to study by considering only reduced boards.

4.2.3. Flips and rotations We use one last tool to shorten our list a bit further. For example, if we simply rotate a board 90 degrees clockwise, we will get an equivalent board. Indeed, any rotation or flip from the dihedral group of the square (see reference 3, p. 31) will result in an equivalent board. The dihedral group consists of all symmetries of a square, and it has eight elements. Consider a square in the Cartesian plane centered at the origin (shown in figure 7). The elements of the dihedral group, which are listed in table 3, correspond to all of the various flips and rotations of the square.

Now we make our notion of equivalent boards more precise. Two boards are equivalent if one can be obtained from the other by a sequence of swaps and operations from the dihedral group. Again, what matters for us is that equivalent boards will host the same puzzles.

We can now describe our approach to generate our shortest possible list of boards to consider. We modify the function PUTTHEME to produce a culled list of reduced boards. Each time PUTTHEME generates a new board, it is reduced and compared to all other boards on the list. Additionally, all flips and rotations from the dihedral group are applied to the board, and again the results are reduced. If none of these variations is already on the list, the board is added. In

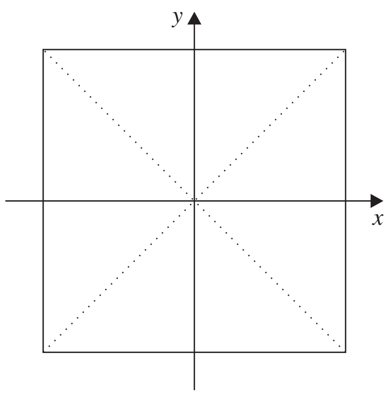

Figure 7 Symmetries of the square.

Table 3 Elements of the dihedral group.

Element	Description
I	identity
ρ	rotation 90 degrees clockwise
ρ^2	rotation by 180 degrees
ρ^3	rotation by 270 degrees clockwise
h	flip across line $y = 0$
v	flip across line $x = 0$
d_1	flip across line $y = x$
d_2	flip across line $y = -x$

Table 4 Culled list of boards.

Distribution type	Number of boards
$\langle 2, 2, 4, 4, 4\rangle$	70
$\langle 2, 3, 3, 4, 4\rangle$	574
$\langle 3, 3, 3, 3, 4\rangle$	306

the end, we found 950 boards. A summary of the boards and their distribution types is given in table 4.

5. Putting it all together

Our next task is study all puzzles that can live on the 950 boards identified in table 4. Once again, we need to look at piece distributions. As we did for board distributions, we define the piece *distribution type*, which is the piece distribution with the entries sorted. We use the notation $\{a_0, a_1, a_2, a_3, a_4\}$, where $a_k \leq a_{k+1}$. To review our use of notation, recall the features of the original puzzle. It has piece distribution $[3, 3, 3, 4, 3]$ and board distribution $(3, 3, 4, 3, 3)$. Thus, it comes with four NZ pieces and three pieces each for the other themes, and the original board contains four squares with map and three squares each for the other themes. Its piece distribution type is $\{3, 3, 3, 3, 4\}$ and its board distribution type is $\langle 3, 3, 3, 3, 4\rangle$.

There are five piece distributions of type $\{3, 3, 3, 3, 4\}$, since there are five choices for the theme with four pieces. For type $\{2, 3, 3, 4, 4\}$, there are five choices for the theme with two pieces, and $\binom{4}{2} = 6$ choices for the theme with three pieces (and the remaining themes will then have four pieces); thus, there are 30 piece distributions of this type. Finally, for type $\{2, 2, 4, 4, 4\}$, there are $\binom{5}{2} = 10$ choices for the theme with two pieces (and the remaining themes will have four pieces). Thus, there are 45 possible puzzles for each board we consider; we summarize them in table 5.

Recall that for the original board, there were only four piece distributions that resulted in puzzles with solutions. Thus, for each board, we should expect to see a good number of piece distributions that lead to puzzles with no solutions. Fortunately, we do not need to solve all puzzles individually. After running our generalized puzzle solver on all 950 boards, we found scores of puzzles that have fewer solutions than the original. We found a handful of puzzles with fewer than 1000 solutions and roughly 100 boards that host no puzzles with solutions.

The smallest number of solutions is 169, but the corresponding puzzle is not interesting. Due to patterns in the board, we can easily solve the puzzle by hand. Our favourite puzzle

Table 5 Possible puzzles per board.

Piece distribution type	Puzzles per board
$\{2, 2, 4, 4, 4\}$	10
$\{2, 3, 3, 4, 4\}$	30
$\{3, 3, 3, 3, 4\}$	5

fern	kiwi	map	fern
map	NZ	sheep	kiwi
kiwi	fern	map	NZ
sheep	NZ	kiwi	sheep

Figure 8 Puzzle with 747 solutions.

has 747 solutions. Its pieces consist of four kiwis, four NZs, three ferns, three maps, and two sheep, and it lives on the board shown in figure 8. We invite the reader to solve it.

6. Computational aspects

We wrote our programs in PYTHON®. Using the IDLE platform (see http://wiki.python.org/moin/IDLE) on a 2.4 GHz laptop, our original puzzle solver runs in about 5 minutes. Using the original board, our generalized puzzle solver runs in about 10 minutes (over 12 million calls of the function `PUT`). When running through all 950 boards, we ran into computational difficulties. Assuming 10 minutes per board, we would expect the following:

$$(950 \text{ boards}) \times (10 \text{ minutes}) = 9\,500 \text{ minutes} \approx 6.6 \text{ days}.$$

However, some of the boards require significantly more than 10 minutes to process, and the resulting computation time was far greater than 6.6 days. We then turned to the PYPY interpreter (see http://pypy.org), which was designed partly to increase speed. Using PYPY in place of the default PYTHON implementation (known as CPYTHON), we were able to complete our exhaustive search of all 950 boards in roughly one day. In our case, PYPY did not require that any modifications be made to the program, only that the interpreter be switched from CPYTHON to PYPY.

Consider the following anecdote as evidence of PYPY's increase in performance: the original puzzle solver runs in 4 minutes and 46 seconds using the default implementation of the PYTHON language. The same program runs in just 24 seconds using PYPY, which is almost a 12-fold increase in performance. The PYPY website attributes the speed increase to its Just-in-Time compiler, advertising increase in speed by a multiple of 6.25 over CPYTHON based on their own benchmarks. The performance increase in our program exceeds even their benchmark.

7. Conclusion

The New Zealand puzzle is accessible, yet challenging, and its solution and variations use a rich mix of tools from discrete mathematics, algebra, and computer science. We feel there is much left to study regarding the puzzle, and we share some further questions for those who might be interested.

- Find a 'nightmare' puzzle that has no solutions, yet 15 of 16 available pieces may be placed.

- Consider more variations of the puzzle. Perhaps use a larger board and more themes, or change board to be a cylinder, torus, etc.
- Change the rules so that pieces may be stacked on one another.
- For the original puzzle, all solutions result in the same final arrangement of themes. Is there a puzzle where this is not the case?
- Design a double puzzle that comes with 32 pieces. A solution consists of a final arrangement with two pieces on each location.
- Is it possible to shorten further the list of boards to consider or to streamline the solution process? As we reviewed the results from our exhaustive search, we observed many puzzles that have the same number of solutions.

Acknowledgement We thank Tammo Reisewitz for sharing the New Zealand puzzle with us and for challenging us to solve it. We would also like to thank Mark J. Johnson and John Koelzer, who provided significant technical and computational assistance.

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Serge Nevsky *is a web developer in the Kansas City area who received his bachelor's degree from Rockhurst University in 2012 with a double major in mathematics and French. His interests include art, music, and computer science. This article is based on Serge's senior research project.*

Fun with dates

The Indian Mathematical Olympiad is held on the first Sunday of February each year, so in 2014 it was held on 02/02/2014. If we successively add the digits, we reach the number 2, the same as the number we reach (trivially) for the day. This works for every day in February 2014. For which other months and years does this work?

Nagoon College, India **Upam Sarmah**

A Van Aubel Theorem Revisited

PAUL GLAISTER

The lines joining the midpoints of the squares on opposite sides of a plane quadrilateral are equal in length and perpendicular. This result, due to van Aubel, is proved by making novel use of the vector product.

One of van Aubel's theorems (see reference 1) can be stated as follows.

Theorem 1 *Given the plane quadrilateral $OABC$ shown in figure 1, denote by P, Q, R, and S the centres of the squares on the sides OA, AB, BC, and CO, respectively. Then the lines joining the midpoints of the squares on opposite sides of the quadrilateral, namely PR and SQ, are equal in length and perpendicular.*

In reference 2 a number of proofs using different techniques are given, one of which is purely geometrical in nature. This particular proof is illustrated very clearly using an animation shown in reference 3 for which an interesting preliminary result is needed (see reference 4), namely that the lines joining the midpoint of OB to P and Q are equal in length and perpendicular (and similarly for the other pairs of adjacent squares).

In this article we give a simple proof of the theorem which makes novel use of the vector product.

First we introduce k as the unit vector perpendicular to the plane containing the quadrilateral with direction up out of the plane, and for simplicity define $a = OA$, $b = OB$, and $c = OC$.

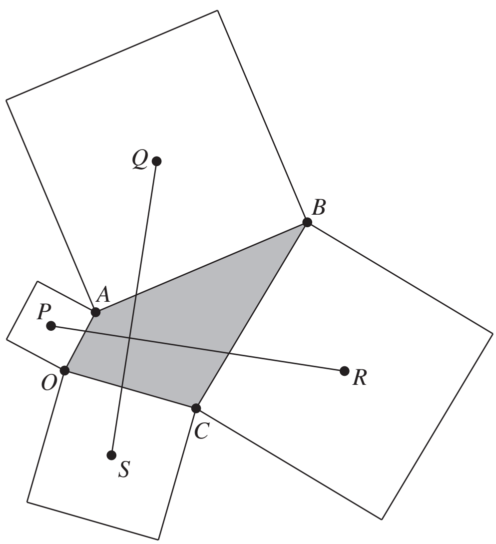

Figure 1

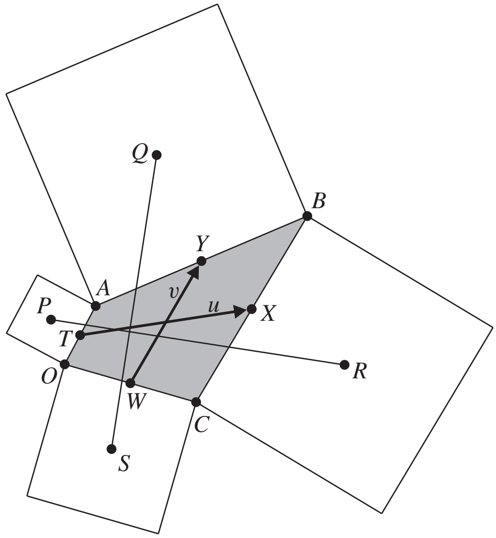

Figure 2

We therefore have

$$OP = \tfrac{1}{2}a - \tfrac{1}{2}a \times k,$$

since k is perpendicular to a and P is the centre of the square on the side OA. Similarly,

$$OS = \tfrac{1}{2}c + \tfrac{1}{2}c \times k.$$

We also have

$$OQ = \tfrac{1}{2}(a+b) - \tfrac{1}{2}AB \times k = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(b-a) \times k,$$

and similarly

$$OR = \tfrac{1}{2}(b+c) - \tfrac{1}{2}(c-b) \times k.$$

Combining these results we have

$$PR = \tfrac{1}{2}(b+c-a) + \tfrac{1}{2}(a+b-c) \times k$$

and

$$SQ = \tfrac{1}{2}(a+b-c) - \tfrac{1}{2}(b+c-a) \times k.$$

Denoting by

$$u = \tfrac{1}{2}(b+c-a) \quad \text{and} \quad v = \tfrac{1}{2}(a+b-c),$$

then PR and SQ can be written as

$$PR = u + v \times k \quad \text{and} \quad SQ = v - u \times k,$$

where u and v are shown in figure 2 as the vectors from the midpoint T of OA to the midpoint X of BC, and from the midpoint W of OC to the midpoint Y of AB, respectively.

We are now in a position to prove van Aubel's theorem by showing that $PR = u + v \times k$ and $SQ = v - u \times k$ are equal in length and perpendicular.

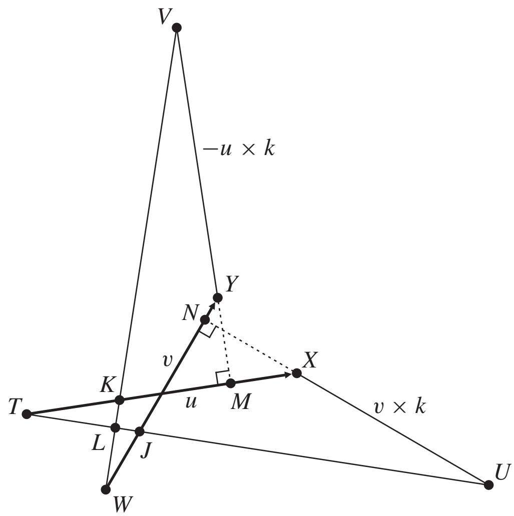

Figure 3

In figure 3 we have removed all points from figure 2 except T, X, W, and Y, noting that $u = TX$ and $v = WY$, and enlarged the diagram to make it easier to read. First we have that XU represents $v \times k$, which is perpendicular to WY, and of magnitude $|v|$ since k is a unit vector perpendicular to v. Similarly, YV represents $-u \times k$, which is perpendicular to TX and of magnitude $|u|$ since k is also perpendicular to u.

The vectors representing $u + v \times k$ and $v - u \times k$ are TU and WV, respectively. To complete the proof we use congruent triangles as follows. Referring to figure 3 we have $YV = TX$ and $XU = WY$. Also, TX is perpendicular to VM and UN is perpendicular to WY, so that

$$\angle WYV = 180^\circ - \angle WYM = 180^\circ - \angle TXN = \angle TXU;$$

hence, $\triangle TXU$ is congruent to $\triangle VYW$. Therefore, $TU = WV$.

In addition,

$$\angle YWV = \angle TUX,$$

so that

$$\begin{aligned}\angle TLK &= \angle WLU \\ &= 180^\circ - (\angle YWV + \angle WJL) \\ &= 180^\circ - (\angle TUX + \angle UJN) \\ &= 180^\circ - (\angle TUX + 90^\circ - \angle TUX) \\ &= 90^\circ;\end{aligned}$$

hence, TU is perpendicular to WV.

Van Aubel's theorem is therefore proved since $TU = u + v \times k = PR$ and $WV = v - u \times k = SQ$ are equal in length and perpendicular.

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Letters to the Editor

Dear Editor,

The AM–GM inequality and Bernoulli's inequality are equivalent

Recently (see reference 1) Lech Maligranda proved that the inequality of arithmetic and geometric means (i.e. the AM–GM inequality) and Bernoulli's inequality are equivalent. We now present Maligranda's proof of this nice result.

For a real number $x \geq -1$ and a positive integer n, Bernoulli's inequality states that

$$(1+x)^n \geq 1 + nx.$$

Substituting $1 + x = y$, this can be written equivalently as

$$y^n \geq n(y-1) + 1, \quad \text{for all } y \geq 0. \tag{1}$$

For n positive numbers $x_1, x_2, \ldots, x_n$, the AM–GM inequality states that

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{1/n} = G_n. \tag{2}$$

Applying (1) for

$$y = \frac{A_n}{A_{n-1}} > 0$$

we get

$$\left(\frac{A_n}{A_{n-1}}\right)^n \geq \frac{nA_n - (n-1)A_{n-1}}{A_{n-1}} = \frac{x_n}{A_{n-1}} \qquad \text{or} \qquad A_n^n \geq x_n A_{n-1}^{n-1}. \tag{3}$$

Using (3) repeatedly we obtain

$$A_n^n \geq x_n A_{n-1}^{n-1} \geq x_n x_{n-1} A_{n-2}^{n-2} \geq \cdots \geq x_n x_{n-1} \cdots x_2 x_1 = G_n^n,$$

and subsequently we obtain (2).

To show the converse (i.e. (1)) we work as follows. For $n = 1$ we have equality in (1). For $n \geq 2$ we distinguish two subcases:

(i) $0 \leq y \leq 1 - 1/n$,

(ii) $y > 1 - 1/n$.

For case (i) we can see that

$$y^n \geq 0 \geq 1 + n(y-1)$$

and subsequently we obtain (1). For case (ii) we can see that

$$n(y-1) + 1 > 0.$$

Applying the AM–GM inequality for the numbers

$$n(y-1)+1, \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}},$$

we have

$$\begin{aligned} y^n &= \left(\frac{(n(y-1)+1) + 1 + 1 + \cdots + 1}{n}\right)^n \\ &\geq (n(y-1)+1) \cdot 1 \cdot 1 \cdots 1, \end{aligned}$$

and subsequently we obtain (1).

Reference

1 L. Maligranda, The AM–GM inequality is equivalent to the Bernoulli inequality, *Math. Intelligencer* **34** (2012), pp. 1–2.

Yours sincerely,

Sprios P. Andriopoulos
(Third High School of Amaliada
Eleia
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Dear Editor,

Sums of squares

Write

$$S_n = 1^2 + 2^2 + \cdots + n^2, \qquad A_n = 1 + 2 + \cdots + n.$$

Then we have

$$\begin{aligned} S_{2n} &= 1^2 + 2^2 + \cdots + (2n)^2 \\ &= (1^2 + 3^2 + \cdots + (2n-1)^2) + (2^2 + 4^2 + \cdots + (2n)^2) \\ &= \sum_{k=1}^{n} (2k-1)^2 + 4S_n \\ &= 4S_n - 4A_n + n + 4S_n \\ &= 8S_n - 4 \times \tfrac{1}{2} n(n+1) + n \\ &= 8S_n - n(2n+1). \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} S_{2n} &= (1^2 + 2^2 + \cdots + n^2) + ((n+1)^2 + (n+2)^2 + \cdots + (2n)^2) \\ &= S_n + \sum_{k=1}^{n} (n+k)^2 \\ &= S_n + n^2 \times n + 2nA_n + S_n \\ &= 2S_n + n^3 + 2n \times \tfrac{1}{2} n(n+1) \\ &= 2S_n + n^2(2n+1). \end{aligned} \tag{2}$$

If we equate (1) and (2) we obtain

$$6S_n = n^2(2n+1) + n(2n+1),$$

which gives the formula

$$S_n = \tfrac{1}{6} n(n+1)(2n+1).$$

Yours sincerely,

Abbas Rouhol Amini
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Sirjan
Iran)

Dear Editor,

A filigree sphere

Regarding the article entitled ‘A filigree sphere’ (Volume 46, Number 3, pp. 99–106), method 2 and method 3 in section 2, as well as subsection 3.1, can be unified by considering the positions of the loops. If we call the position of the loops λ, with $\lambda = \pm 1$ at the ends, then method 2 corresponds to $\lambda = \frac{1}{3}$, method 3 corresponds to $\lambda = 1$, and subsection 3.1 corresponds to negating λ.

If we take $\lambda = 0$, we get an icosidodecahedron, cuboctahedron, and tetratetrahedron (more commonly known as an octahedron).

We can go further. Taking $\lambda = 3$, we get a quasitruncated icosahedron, quasitruncated dodecahedron, quasitruncated octahedron, quasitruncated cube, and a quasitruncated tetrahedron (see reference 1), of which only the quasitruncated cube is considered to be a 'proper' polyhedron.

Furthermore, starting with one of the four Kepler–Poinsot polyhedra would generate further polyhedra.

Reference

1 P. Taylor, *The Simpler? Polyhedra* (Nattygrafix, Ipswich, 1999).

Yours sincerely,

Douglas Boffey
(14 Honister Walk
Woodlands
Netherley
Liverpool L27 5RY
UK)

Dear Editor,

A reciprocal sum of Fibonacci products

In a letter by M. A. Khan (Volume 46, Number 2, p. 92) it was shown that

$$\sum_{i=1}^{n} \frac{1}{F_{2i}F_{2i+2}} = \frac{F_{2n}}{F_{2n+2}},$$

by the use of Binet's formula for F_n. This can be avoided with the well-known identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^2$$

in the form

$$F_{2i-1}F_{2i+1} - F_{2i}^2 = (-1)^{2i} = 1,$$

from which

$$F_{2i-2}F_{2i+2} = F_{2i}^2 - 1$$

can be derived. This can be rewritten as

$$\frac{1}{F_{2i}F_{2i+2}} = \frac{F_{2i}}{F_{2i+2}} - \frac{F_{2i-2}}{F_{2i}},$$

and the summation obtained by adding from $i = 1$ to n.

A further result can be obtained from

$$\sum_{i=1}^{n} \frac{1}{F_{2i}F_{2i+2}} + \sum_{i=1}^{n} \frac{1}{F_{2i-1}F_{2i+1}} = \sum_{i=1}^{2n} \frac{1}{F_i F_{i+2}} = 1 - \frac{1}{F_{2n+1}F_{2n+2}},$$

as

$$\frac{1}{F_i F_{i+2}} = \frac{1}{F_i F_{i+1}} - \frac{1}{F_{i+1}F_{i+2}}.$$

Hence,

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{F_{2i-1}F_{2i+1}} &= 1 - \frac{1}{F_{2n+1}F_{2n+2}} - \frac{F_{2n}}{F_{2n+2}} \\ &= \frac{F_{2n+1}F_{2n+2} + (F_{2n}F_{2n+2} - F_{2n+1}^2) - F_{2n}F_{2n+1}}{F_{2n+1}F_{2n+2}} \\ &= \frac{F_{2n}}{F_{2n+1}}. \end{aligned}$$

Yours sincerely,

Robert J. Clarke
(44 Webb Court
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UK)

Dear Editor,

$$k^2 = t^2 - 5u^2$$

In my article 'Finding arbitrarily many integer solutions to the equation $k^2 - 5 = t^2 - 5u^2$' (Volume 46, Number 3, pp. 119–121), I asked whether, if we are free to choose the integer k, the equation

$$k^2 - 5 = t^2 - 5u^2$$

can have arbitrarily many integer solutions. I offered a tentative 'yes' as my answer. After a discussion prompted by the article, Professor Shaun Stevens discovered the following elegant argument.

Pell's equation $x^2 - 5y^2 = 1$ has the fundamental solution $x = 9,\ y = 4$, meaning that every solution in positive integers (x_n, y_n) is given by

$$x_n + y_n\sqrt{5} = (9 + 4\sqrt{5})^n$$

(with n any positive integer).

This means that

$$k^2 - 5 = (k^2 - 5)(x_n^2 - 5y_n^2) = (k + \sqrt{5})(x_n + y_n\sqrt{5})(k - \sqrt{5})(x_n - y_n\sqrt{5}).$$

If we define positive integers t_n, u_n by

$$t_n + u_n\sqrt{5} = (k + \sqrt{5})(x_n + y_n\sqrt{5}),$$

we get infinitely many solutions (t_n, u_n) to $k^2 - 5 = t^2 - 5u^2$ (they are all distinct because the (x_n, y_n) are all distinct for all natural numbers n).

Yours sincerely,
Jonny Griffiths
(Paston Sixth Form College
Norfolk
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Dear Editor,

The divergence of $\{\cos(n)\}$

I enjoyed the variety of proofs that Kurt Fink and Jawad Sadek gave for the divergence of $\{\cos(n)\}$ in their article (see Volume 47, Number 2, pp. 68–70). To their collection, I offer the following short proof which admits immediate generalisations.

Starting with the trigonometric identity

$$2\cos\theta\cos\phi - \cos(\theta+\phi) = \cos(\theta-\phi)$$

in the form

$$2\cos\theta\cos\phi - 2\cos^2\tfrac{1}{2}(\theta+\phi) + 1 = \cos(\theta-\phi),$$

consider the following choices for θ and ϕ.

(i) $\theta = n+2$, $\phi = n$. If $\{\cos(n)\}$ converges to a limit L then, letting $n \to \infty$, the left-hand side tends to 1 whereas the right-hand side takes the fixed value $\cos 2$; a contradiction since $\cos 2 \neq 1$.

(ii) $\theta = (n+2)\alpha$, $\phi = n\alpha$. If $\{\cos(n\alpha)\}$ converges to a limit L then, letting $n \to \infty$, the left-hand side tends to 1 whereas the right-hand side takes the fixed value $\cos 2\alpha$. Convergence thus requires that $\cos 2\alpha = 1$ and hence $\alpha = k\pi$ for some integer k. But then $\cos(n\alpha) = \cos(nk\pi) = (-1)^{nk}$, so k must be even to ensure convergence. Thus, $\{\cos(n\alpha)\}$ converges if and only if $\alpha/2\pi$ is an integer.

(iii) $\theta = (n+2)\alpha + \beta$, $\phi = n\alpha + \beta$. If $\{\cos(n\alpha+\beta)\}$ converges to a limit L then, letting $n \to \infty$, the left-hand side tends to 1 whereas the right-hand side takes the fixed value $\cos 2\alpha$. As in (ii), convergence requires that $\alpha = k\pi$ for some integer k so that $\cos(n\alpha+\beta) = \cos(nk\pi+\beta) = (-1)^{nk}\cos\beta$. This converges if and only if either k is even or $\cos\beta = 0$.

Yours sincerely,
Nick Lord
(Tonbridge School
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UK)

Dear Editor,

Summing Fibonacci numbers

Readers may be familiar with a 'telescoping technique' for summing Fibonacci numbers. For example,

$$\begin{aligned}
F_1 + F_2 + \cdots + F_n &= (F_2 - F_0) + (F_3 - F_1) + (F_4 - F_2) \\
&\quad + \cdots + (F_{n-1} - F_{n-3}) + (F_n - F_{n-2}) + (F_{n+1} - F_{n-1}) \\
&= F_n + F_{n+1} - F_0 - F_1 \\
&= F_{n+2} - 1, \\
F_1 + F_3 + \cdots + F_{2n-1} &= (F_2 - F_0) + (F_4 - F_2) + (F_6 - F_4) \\
&\quad + \cdots + (F_{2n-2} - F_{2n-4}) + (F_{2n} - F_{2n-2}) \\
&= F_{2n} - F_0 \\
&= F_{2n}.
\end{aligned}$$

Another example follows from the identity

$$F_{4n+3} = F_{2n+1}F_{2n+4} - F_{2n-1}F_{2n+2},$$

which gives

$$\begin{aligned}
F_7 + F_{11} + \cdots + F_{4n+3} &= (F_3F_6 - F_1F_4) + (F_5F_8 - F_3F_6) \\
&\quad + \cdots + (F_{2n+1}F_{2n+4} - F_{2n-1}F_{2n+2}) \\
&= F_{2n+1}F_{2n+4} - F_1F_4 \\
&= F_{2n+1}F_{2n+4} - 3.
\end{aligned}$$

Yours sincerely,

M. A. Khan
(c/o A. A. Khan
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Munshi Puliya Branch
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India)

Nearly π

$$\sqrt[2^5+2]{2^3 \times (2^3+2)^{(2^4)}} = 3.141\,591\,444\,141\,992\,652\,182\,488\,412\,553\,1\ldots.$$

Malaga, Spain **Fernando Mancebo Rodriguez**

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

48.1 Express x and y in terms of a, b, and c (see figure 1).

(Submitted by Vivek v. v., United Arab Emirates)

48.2 A sequence is formed by adding to a given number A the terms of an arithmetic progression (e.g. 11, 13, 17, 23, 31, ...). Find the sum of the first n terms of such a sequence.

(Submitted by by Versha Lodhi, India)

48.3 In figure 2, $ABCD$ is a rectangle with $AB = 2BC$, angles PAB and PBA are $15°$, and angles DEP and CFP are right angles. What is the area of the quadrilateral $DEFC$?

(Submitted by by Subramanyam Durbha, Community College of Philadelphia, USA)

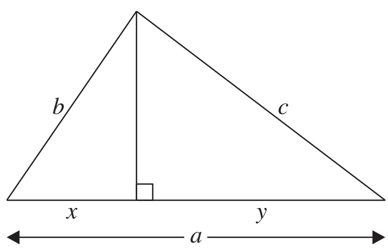

Figure 1

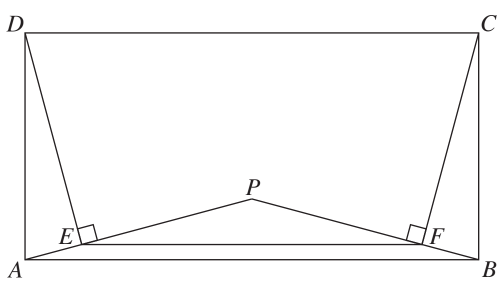

Figure 2

48.4 Prove that

$$\frac{2\tan 54^\circ}{\sqrt{(\tan 54^\circ)^2+1}}$$

is equal to the golden ratio.

(Submitted by by James Metz, Hawaii, USA)

Solutions to Problems in Volume 47 Number 2

47.5 For positive integers x, y in base 10, $x \sim y$ denotes the concatenation of x and y. For example, $27 \sim 38 = 2738$. Let $T_n = \frac{1}{2}n(n+1)$, $n \geq 1$, be the nth triangular number. Prove that there are infinitely many positive integers n such that $T_n \sim T_{n+1}$ is divisible by 5 and infinitely many n such that $T_n \sim T_{n+1}$ is divisible by 9.

Solution by Annanay Kapila, Year 12, Nottingham High School, UK

If $n = 5k-2$, where k is any positive integer, then

$$T_{n+1} = 5 \times \frac{k(5k-1)}{2}.$$

If k is even so is $k(5k-1)$; if k is odd, then $5k-1$ is even so $k(5k-1)$ is again even. Hence, T_{n+1} is a multiple of 5, so $T_n \sim T_{n+1}$ is also a multiple of 5. (A similar argument proves that $T_n \sim T_{n+1}$ is a multiple of 5 whenever $n = 5k-1$.)

If $n = 9k-1$, then

$$T_n = 9 \times \frac{(9k-1)k}{2}, \qquad T_{n+1} = 9 \times \frac{k(9k+1)}{2}.$$

A similar argument to the above shows that T_n and T_{n+1} are both divisible by 9, so that $T_n \sim T_{n+1}$ is divisible by 9 whenever $n = 9k-1$.

Also solved by Lucia Ma Li, IES Isabel de España, and Ángel Plaza, Universidad de Las Palmas, Gran Canaria, Spain.

47.6 Prove that every triangular number other than 1 can be written in the form a^2+b^2+a and every triangular number can be written in the form a^2+b^2-a for some positive integers a and b.

Solution by Annanay Kapila

For any positive integer k,

$$T_{2k} = \tfrac{1}{2}2k(2k+1) = k^2+k^2+k$$

and

$$T_{2k+1} = \tfrac{1}{2}(2k+1)(2k+2) = (2k+1)(k+1) = 2k^2+3k+1 = k^2+(k+1)^2+k,$$
$$T_{2k} = (k+1)^2+k^2-(k-1),$$
$$T_{2k-1} = \tfrac{1}{2}(2k-1)2k = k^2+k^2-k.$$

Also solved by Henry Ricardo, New York Math Circle, USA.

47.7 Find all positive integers n such that

$$3^n < \binom{2n}{n} < 3^{n+1},$$

where $\binom{2n}{n}$ denotes the binomial coefficient.

Solution by Annanay Kapila

By calculation, we see that the inequality is false for $n = 1, 2, 3, 4$ and true for $n = 5, 6, 7, 8, 9$. Denote $\binom{2n}{n}$ by c_n. Then we have

$$\frac{c_{n+1}}{c_n} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1}.$$

Consider the function f defined by

$$f(x) = \frac{2(2x+1)}{x+1}.$$

Now,

$$f'(x) = 2 \times \frac{2(x+1) - (2x+1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0,$$

for all $x > 0$, so the function f is strictly increasing for $x > 0$ (there are no vertical asymptotes here). Now,

$$\frac{c_2}{c_1} = 3$$

so that

$$\frac{c_{n+1}}{c_n} > 3$$

for all $n \geq 2$. Hence, if we find $k \geq 2$ such that $c_k > 3^{k+1}$, then

$$c_{k+1} = \left(\frac{c_{k+1}}{c_k}\right) c_k > 3^{k+2},$$

so that $c_n > 3^{n+1}$ for all $n \geq k$. Since $c_{10} = 184\,756$ and $3^{11} = 177\,147$, this means that $c_n > 3^{n+1}$ for all $n \geq 10$, so that result is false for $n \geq 10$.

[Tom Moore, who proposed the problem, avoided the use of calculus by noting that

$$\frac{c_{n+1}}{c_n} = 3 + \frac{n-1}{n+1} > 3,$$

for all $n \geq 2$.]

47.8 For positive integers a and b, prove that $(a - b)^2 = a + b$ if and only if a and b are consecutive triangular numbers.

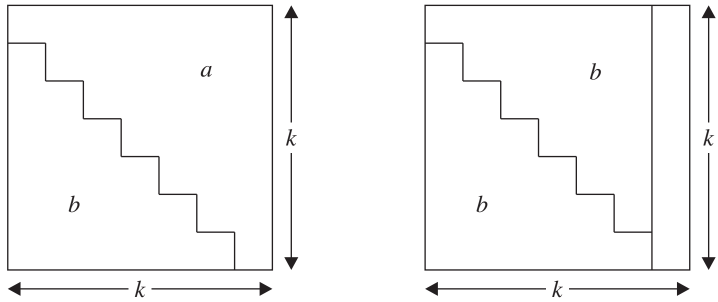

Figure 3

Solution by Annanay Kapila

Since a and b are interchangeable, we may suppose that $a > b$. First

$$T_{n+1} + T_n + \tfrac{1}{2}(n+1)(n+2) + \tfrac{1}{2}n(n+1) = (n+1)^2$$

and

$$(T_{n+1} - T_n)^2 = \left[\tfrac{1}{2}(n+1)(n+2) + \tfrac{1}{2}n(n+1)\right]^2 = (n+1)^2.$$

Conversely, suppose that

$$(a-b)^2 = a+b.$$

Write $a - b = n$ so that $a + b = n^2$. Then

$$a = \tfrac{1}{2}(n^2+n) = T_n, \qquad b = \tfrac{1}{2}(n^2-n) = T_{n-1}.$$

Also solved by Henry Ricardo.

Jonny Griffiths (Paston College, Norfolk, UK) offers a 'proof without words' (see figure 3), with $a > b$.

Numbers in different bases

To convert a number from any base to base 10, use *multiplication*. For example,

$$4071_8 = (1 \times 8^0) + (7 \times 8^1) + (0 \times 8^2) + (4 \times 8^3) = 2105.$$

To convert a number from base 10 to any other base, use *division*. For example, to convert 350 to base 8,

$$\begin{array}{r|rl} 8 & 35^{3}0 & \\ 8 & 43 & \text{rem } 6 \\ & 5 & \text{rem } 3 \end{array} \qquad 350 = 536_8.$$

Lyndon Barton

Reviews

Creating Symmetry: The Artful Mathematics of Wallpaper Patterns. By F. A. Farris. Princeton University Press, 2015. Hardback, 248 pages, £24.95 (ISBN 9780691161730).

There have been many accounts of the classification of wallpaper patterns into exactly 17 groups, according to the type of symmetry displayed. This beautifully illustrated book, however, offers something new. As indicated in the title, it deals with ways of *creating* symmetry, and to do this, uses an original approach involving some powerful mathematics, such as mappings of the complex plane to itself, group theory, and wave functions based on Fourier analysis. Useful advice will be found in the preface on ways to read the book, if you are an advanced undergraduate or a 'brave mathematical adventurer'.

Two factors increase the accessibility of the material. Firstly, the author is a very helpful guide, with a lucid writing style, who starts off slowly with an enticing study of a mystery closed curve with five-fold rotational symmetry, leading the reader to discover how it comes to have this property. Only later does he take the reader through rosettes as functions on the complex plane and then move on to frieze and wallpaper functions, leaving until later topics such as colour symmetry, polyhedral symmetry, and hyperbolic wallpaper.

The second factor that increases accessibility is the presence of so many superb colour pictures whose construction is explained in the text. These give visual delight and also help to prevent the accumulation of too much abstract explanation too quickly. For advanced undergraduates who have perhaps already encountered some of the relevant mathematical concepts, there could be great benefit in *seeing* these concepts applied in a creative way, and being given glimpses of territory as yet new to them.

The University of Sheffield **Peter Derlien**

Single Digits: In Praise of Small Numbers. By M. Chamberland. Princeton University Press, 2015. Hardback, 240 pages, £18.95 (ISBN 9780691161143).

Marc Chamberland is the creator of the YouTube channel *Tipping Point Math* which is designed to bring mathematics to a wider public. In this book he has presented an intriguing collection of 118 topics from many different mathematical areas, organized into nine chapters, one for each of the digits $1, 2, \ldots, 9$. Sensibly, within each chapter earlier sections tend to be easier than later, and the same trend is present across the nine chapters. Such is the diversity of material here that I cannot do justice to it in the limited space of this review. Suffice it to say that *Mathematical Spectrum* readers of varying mathematical background can expect to find here much that will get their attention and start them thinking.

The University of Sheffield **Peter Derlien**

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