Mathematical Spectrum

1999/2000 Volume 32 Number 1

- Throwing elliptical shields
- A generalized Argand diagram
- Sums of powers
- Catalan numbers

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column.

A Generalized Argand Diagram

GUIDO LASTERS and DAVID SHARPE

This shows how a novel use of complex numbers will solve a problem in geometry.

In two recent articles in *Mathematical Spectrum* (Volume 29 Number 3 pages 51–53 and Volume 30 Number 3 pages 58–60), certain geometrical problems were solved by the introduction of an abelian group on a circle. Specifically, given a circle C and a straight line l not meeting the circle, a point of C is chosen as the neutral element e and the binary operation is as illustrated in figures 1 and 2, figure 2 being the special case of figure 1 when $a = b$, in which case the chord ab becomes the tangent to the circle at a. The usual convention is adopted that parallel lines meet at infinity. The proof that this binary operation is associative relied on Pascal's Theorem on conics; the remaining axioms of an abelian group are easily verified.

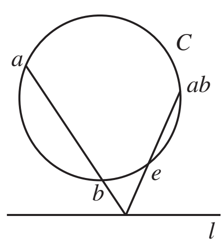


Figure 1.

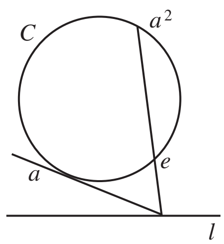


Figure 2.

A problem that does not seem to yield easily to this group is the following:

> *construct a triangle inscribed in C such that each side meets the tangent to C through the opposite vertex on the given line l* — see figure 3.

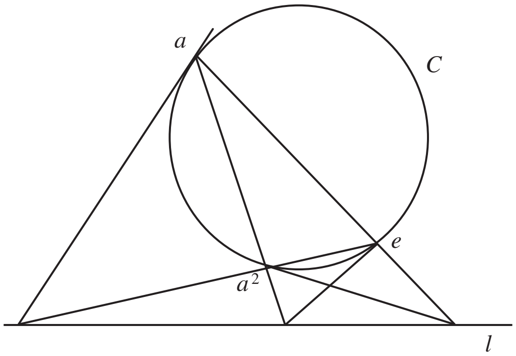


Figure 3.

If we choose one of the vertices as the neutral element of the group and denote another vertex by a, then the third vertex is a^2 and $a^3 = e$, so we have a cyclic group $\{e, a, a^2\}$ of order 3. The problem is: how can we locate the points a, a^2?

The problem is easily solved if l is 'the line at infinity'; we choose an equilateral triangle, in which case each tangent is parallel to the opposite side and so may be said to meet it on the line at infinity — see figure 4.

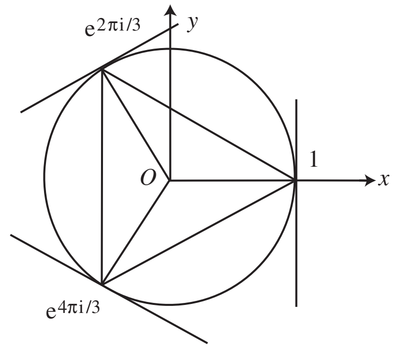


Figure 4.

This special case suggests a possible way of approaching the general problem. If we choose the radius of the circle to be one unit of distance and rectangular axes centred on the centre of the circle with the x-axis through one of the vertices of the triangle then, in the representation of complex numbers on an Argand diagram, the three vertices are 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$, or 1, $-\frac{1}{2} + i(\sqrt{3}/2)$, $-\frac{1}{2} - i(\sqrt{3}/2)$, as in figure 4.

We now start with the given circle C and line l in a plane Π, where l does not meet C. We choose another plane Π' which meets Π and to which l is parallel, and a point P lying in the plane parallel to Π' and containing l, but not lying on l — see figure 5. We now project the points of Π into Π' from the point P. The line l will project to the line at infinity in Π' and the circle C will project to an ellipse E. Also, straight lines in Π are projected into straight lines in Π'.

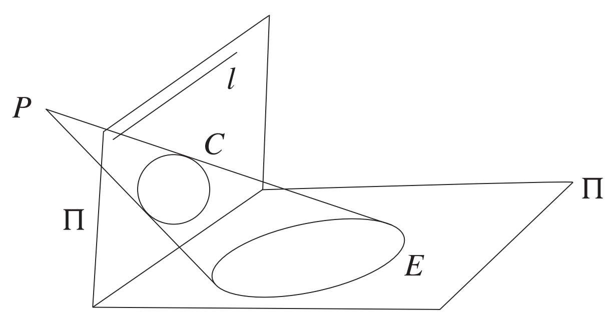


Figure 5.

The next step is to represent the complex numbers in Π'. We choose as our axes Oxy the axes of the ellipse E. However, because E is not in general a circle, we choose different scales along the real and imaginary axes so that the lengths of both semi-axes of the ellipse are one unit — see figure 6. We call E the 'unit ellipse'. The general complex number $x + \mathrm{i}y$ can be represented by the point (x, y) in Π'. This is a sort of 'squashed Argand diagram'!

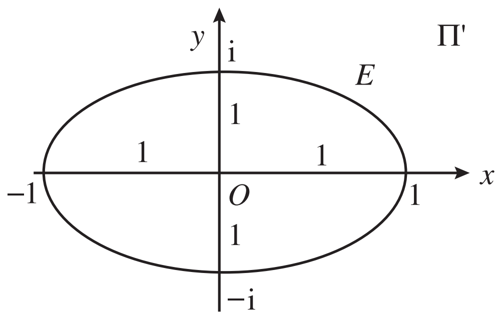


Figure 6.

If we now transfer back to the plane Π by reversing our projection, every complex number is represented by a point in Π. The unit ellipse E in Π' transfers back to the circle C, so the complex numbers $e^{\mathrm{i}\theta}$ lie on C; in particular $1, -1, \mathrm{i}, -\mathrm{i}$ lie on C (see figure 7). The tangents to the ellipse E in Π' at 1 and -1 are parallel, so in Π the tangents to C at 1 and -1 meet on the line l, as do those at i and $-$i. The straight lines joining $1, -1$ and $\mathrm{i}, -\mathrm{i}$ in Π' meet at the origin, so those in Π meet at the point representing the complex number O. This is no longer (in general) the centre of the circle C. This representation is what our title refers to as 'a generalized Argand diagram'. We should perhaps note that the points on the line l do not represent complex numbers, since this line corresponds to the 'line at infinity' in Π.

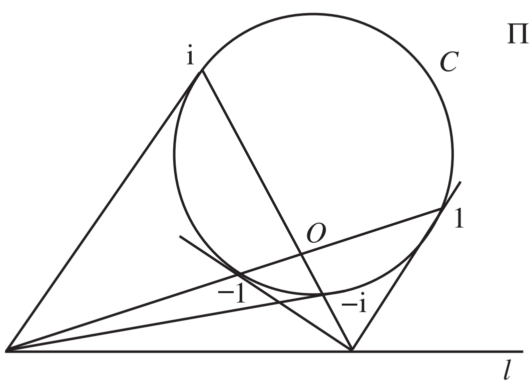


Figure 7.

To solve our geometrical problem, we need to locate the points on C representing $e^{2\pi\mathrm{i}/3}$ and $e^{4\pi\mathrm{i}/3}$, or $-\frac{1}{2}+\mathrm{i}(\sqrt{3}/2)$ and $-\frac{1}{2} - \mathrm{i}(\sqrt{3}/2)$. To do this, we first consider how addition and multiplication of complex numbers is represented in Π.

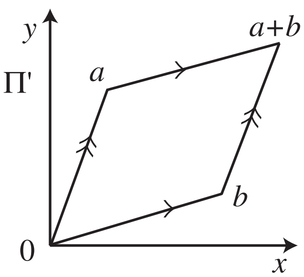


Figure 8.

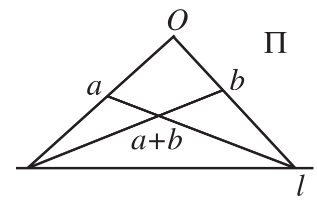


Figure 9.

In Π', addition obeys the usual parallelogram law, even though the scales are different in the real and imaginary directions — see figure 8. When we transfer this back to Π, the rule for addition is shown in figure 9. In both figures, a, b, O are taken to be non-collinear. There is no problem in Π' when a, b, O are collinear. To locate $a + b$ in Π when O, a, b are collinear, it is necessary to employ a third complex number c not on the line Oab and use the associative law

$$a + (b + c) = (a + b) + c,$$

as in figure 10. Notice that we have employed the associative law in Π; since it holds in Π', it also holds when we transfer back to Π.

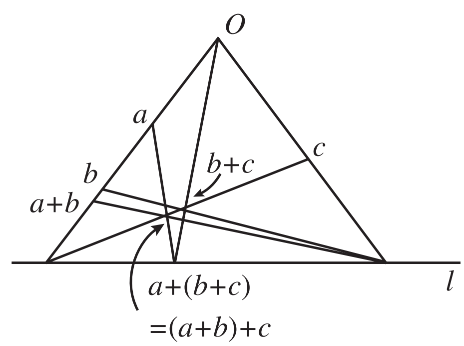


Figure 10.

Representation of multiplication in the Argand diagram is less easy to realize. We confine ourselves to complex numbers of unit modulus. In the usual Argand diagram, multiplication is represented as in figure 11, where the arrows denote parallel lines. The lines are parallel because their slopes are

$$\frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta} \qquad \text{and} \qquad \frac{\sin(\theta+\phi)}{\cos(\theta+\phi) - 1},$$

and both of these are equal to

$$-\cot\frac{\theta+\phi}{2},$$

as a little trigonometry will show. When we use different scales on the two axes, the unit circle becomes a unit ellipse, but exactly the same calculation holds, and again the two lines are parallel — see figure 12.

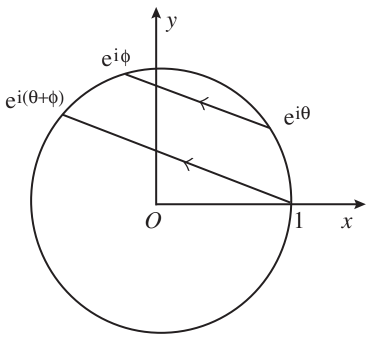


Figure 11.

If we now transfer this back to Π, the rule for multiplication in Π is precisely the one that we have previously used to define an abelian group on C, as in figure 13. This renders redundant our need to employ Pascal's Theorem to verify the associative law, since the multiplication of complex numbers is associative.

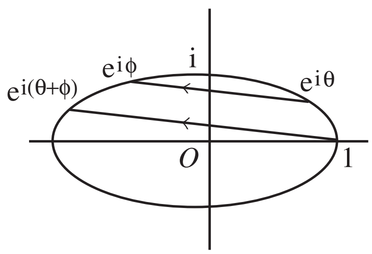


Figure 12.

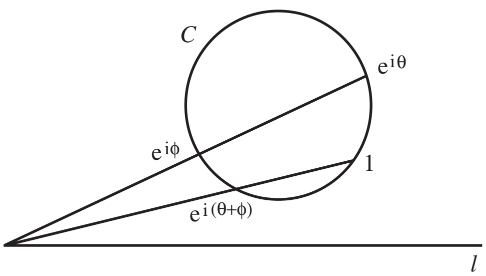


Figure 13.

To solve our problem, it still remains to locate the points $e^{2\pi i/3}$ and $e^{4\pi i/3}$ or $-\frac{1}{2} \pm i(\sqrt{3}/2)$, on C. If we can locate the point $-\frac{1}{2}$, then, following on from figure 7, figure 14 shows how to locate $-\frac{1}{2} \pm i(\sqrt{3}/2)$; if P denotes the point where the tangent to C at 1 meets l, then the line through P and $-\frac{1}{2}$ will meet C at $-\frac{1}{2} \pm i(\sqrt{3}/2)$. This is because, in Π', the line joining $-\frac{1}{2} \pm i(\sqrt{3}/2)$ is parallel to the imaginary axis. The problem now is to locate the point $-\frac{1}{2}$, knowing where -1 and O are. Or, more generally, given the points O and a, where is the point $\frac{1}{2}a$? This is the problem we now consider.

First consider the reverse construction: given O and a, find 2a. We choose c not on the line joining O and a and locate $a+c$ as in figure 15. Then add a to give $2a+c$. Now $2a$ can be located as in the figure.

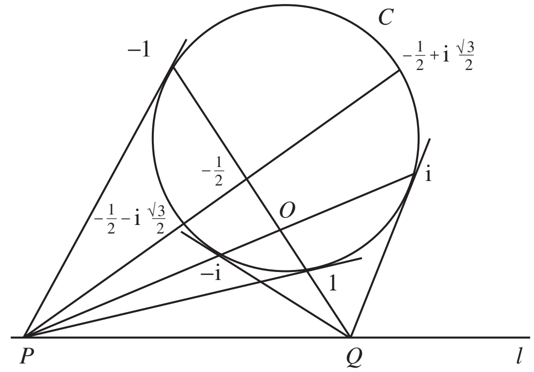


Figure 14.

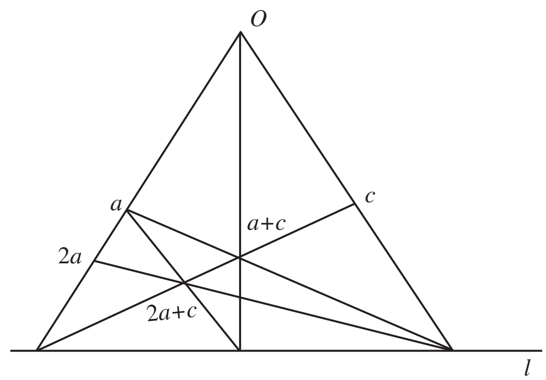


Figure 15.

Desargue's Theorem

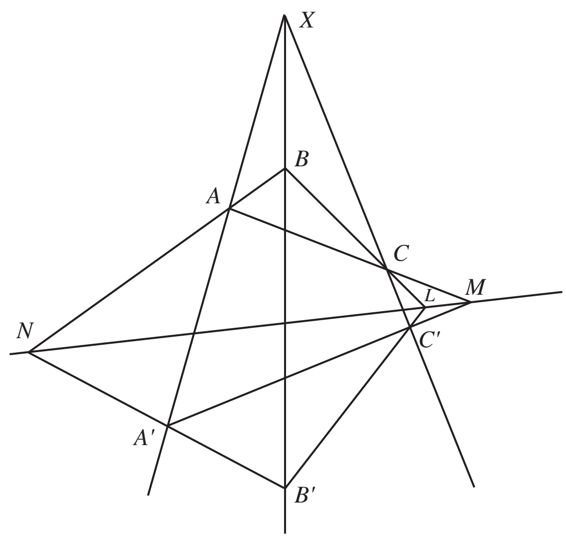


Figure 16.

AA', BB', CC' meet in a point X if and only if L, M, N are collinear; or, two triangles have their vertices in perspective if and only if they have their sides in perspective.

It proves more difficult to reverse this construction. To do so we make two uses of Desargue's Theorem on triangles in perspective — see figure 16. We start with a, again choose c not on Oa, and construct $a+c$. We then construct the straight lines O, $a+c$ and ac, meeting in b (say) and draw the line Pb produced to meet Oa in a' (see figure 17). We claim that $a' = \frac{1}{2}a$. For this to be the case, the construction in figure 15 starting with a' must yield a. Thus, if we construct the line a', $a+c$ to meet l in F (say), then F, $a'+c$ and O must be collinear.

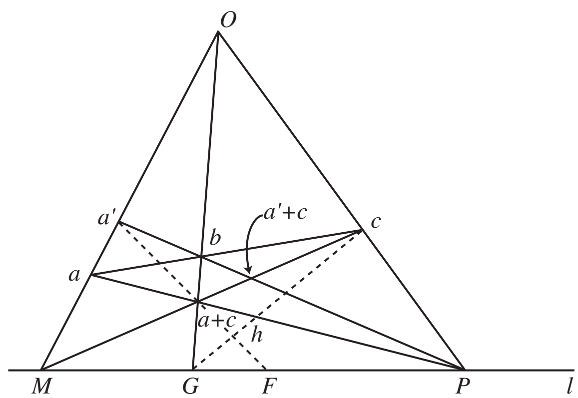


Figure 17.

Construct the straight line Gc to cross aP at h, as in figure 17. We now make two uses of Desargue's Theorem. Consider the triangles $Ma'F$ and cPh. The sides Ma' and cP meet at O, MF and ch meet at G and $a'F$, Ph meet at $a+c$. These three points, O, G and $a+c$, are collinear. Desargue's Theorem tells us that these triangles have their vertices in perspective, i.e. Mc, $a'P$ and Fh meet in a point. Thus $a'+c$, F and h are collinear.

For our second application of Desargue's Theorem, we consider triangles $a(a+c)a'$ and cGP. Their vertices are in perspective because ac, $(a+c)G$ and $a'P$ meet at b. Hence their sides are in perspective, so that h, O and F are collinear. But we also know that $a'+c$, F and h are collinear. Hence F, $a'+c$ and O are collinear, which is what was required.

To return to the problem with which we began: we are given a circle C and a straight line l not meeting C, and are asked to construct a triangle inscribed in the circle so that the tangent to C at each vertex meets the opposite side of the triangle on l. The construction is as follows — see figure 18.

1. Choose a point on C whose tangent to C is not parallel to l, and denote it by 1.
2. Draw the tangent to C at 1 to meet l in P.
3. Draw the other tangent to C through P, touching C at the point denoted by -1.
4. Join 1 and -1 to meet l in Q. (We need to make sure that $1, -1$ is not parallel to l, which would mean resiting 1.)

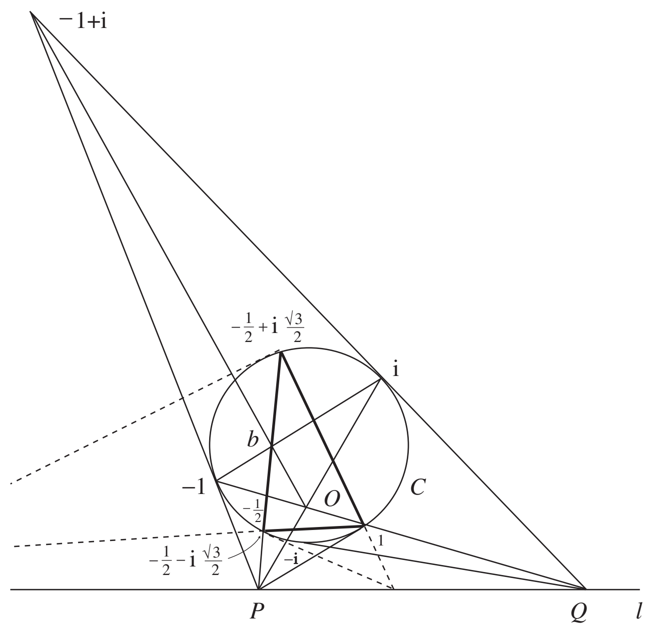


Figure 18.

5. Draw the tangents to C through Q to touch C at i and $-$i.
6. Draw the straight line through $-$i and i (which also passes through P — see figure 14) to meet the line joining -1 and 1 at O.
7. Use the construction illustrated in figure 17 above to locate the point $-\frac{1}{2}$. We take $a = -1$. If, for example, we take $c = \mathrm{i}$, we first find $-1+\mathrm{i}$, which is the point of intersection of the tangents at -1 and i. Now join -1, i and $-1+\mathrm{i}$, O to cross at b. Now Pb meets -1, O at $-\frac{1}{2}$ and C at $-\frac{1}{2} \pm \mathrm{i}(\sqrt{3}/2)$ (see figure 14). Then the triangle with vertices at $1, -\frac{1}{2}+\mathrm{i}(\sqrt{3}/2)$ and $-\frac{1}{2} - \mathrm{i}(\sqrt{3}/2)$ has the required properties.

No doubt this construction can be justified without reference to a generalized Argand diagram, but we claim that this way of representing complex numbers in the plane sheds interesting light on this and perhaps also similar problems.

The authors continue their EU collaboration; ***Guido Lasters*** *lives in Tienen, Belgium, and* ***David Sharpe*** *teaches at the University of Sheffield in the UK and is editor of* Mathematical Spectrum.

The 18th Century Chinese Discovery of the Catalan Numbers

P. J. LARCOMBE

A Chinese academic got there first in spotting the significance of the Catalan numbers.

Consider the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, \dots\} = \{1, 1, 2, 5, 14, 42, 132, 429, \dots\}$ generated most easily by the expression for the general $(r+1)$th term

$$c_r = \frac{1}{r+1}\binom{2r}{r}, \quad r = 0, 1, 2, \dots, \tag{1}$$

with which most of us are conversant. One might well suppose that the sequence originated with Eugène Catalan, for it is in an 1838 paper on mathematical aspects of (triangulated) polygon division that (1) is first given in this form and from which the sequence was to take its name. Readers of *Mathematical Spectrum* may have seen the equation

$$c_{n-1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} 2^{n-1}, \quad n > 1, \tag{2}$$

in an article by Vun and Belcher (reference 1) in which the Catalan numbers are shown to arise in three different ways, one of these being the polygon dissection problem. In fact Leonhard Euler had already identified the numbers $c_1, c_2, c_3, \dots$, in this geometrical setting in the middle of the previous century (see references 2 and 3). A readable account of the problem is available in H. Dörrie's book *100 Great Problems of Elementary Mathematics: Their History and Solution* (Dover Publications, New York, 1965) as Problem No. 7. Many people will, however, not have access to a 1988 paper by Luo (reference 4) who asserts that the full sequence was known prior to this by Antu Ming (c.1692–1763), a Chinese scholar with a wide variety of scientific and mathematical interests. Since Luo's article is published in Chinese this fact is not common knowledge in the Western World. I wish here, therefore, to set down some of the historical details he provides and to call attention to a novel formulation of the numbers therein which is due to Ming.

At the beginning of the 18th century a French Jesuit, Pierre Jartoux, brought with him to China the three expansions

$$\begin{aligned}
\pi &= 3\left\{1 + \frac{1^2}{4 \cdot 3!} + \frac{(1 \cdot 3)^2}{4^2 \cdot 5!} + \frac{(1 \cdot 3 \cdot 5)^2}{4^3 \cdot 7!} + \cdots\right\},\\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,\\
\operatorname{versin}(x) &= 1 - \cos(x)\\
&= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots,
\end{aligned}$$

all of them without proof. The second and third (valid for all values of x) are immediately recognisable as Maclaurin series and are credited to James Gregory (1638–1675), while the first is attributed to Isaac Newton (1642–1726). Ming saw these results, but, suspecting that Western mathematicians would be unwilling to share their derivations, he set about obtaining them for himself. In this he succeeded, using a mixture of arithmetic and (imported) Euclidean geometry as his basic tools, and he went on to derive six further expansions of other trigonometric functions. Applying recurrence methods systematically, his algorithms have the striking feature of program and calculation, and he unquestionably laid a foundation for the operation of infinite series in China which constitutes an important contribution to the country's historical development in mathematics. His methods were, of course, not rigorous by today's standards, and the question of what was actually meant by a proof there at that time is discussed by Jami (reference 5) in relation to Ming's work. Luo states that Ming discovered the Catalan numbers through his geometric models, and he highlights one or two associated representations of the function $\sin(m\alpha)$ as power series in $\sin(\alpha)$ in which they appear. Ming dealt with the values $m = 2, 3, 4, 5, 10, 10^2, 10^3, 10^4$, and found that the series for $\sin(3\alpha)$ and $\sin(5\alpha)$ terminated whilst his other expansions were infinite. We see, for example, how the full Catalan sequence occurs in each of the following two results:

$$\begin{aligned}
\sin(2\alpha) &= 2\left\{\sin(\alpha) - \sum_{n=1}^{\infty}\left[\frac{c_{n-1}}{2^{2n-1}}\right]\sin^{2n+1}(\alpha)\right\}\\
&= 2\sin(\alpha) - \sin^3(\alpha) - \tfrac{1}{4}\sin^5(\alpha)\\
&\qquad -\tfrac{1}{8}\sin^7(\alpha) - \cdots,
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\sin(4\alpha) &= 2\left\{2\sin(\alpha) - 5\sin^3(\alpha)\right.\\
&\qquad \left. + \sum_{n=1}^{\infty}\left[\frac{8c_{n-1} - c_n}{4^n}\right]\sin^{2n+3}(\alpha)\right\}\\
&= 4\sin(\alpha) - 10\sin^3(\alpha) + \tfrac{7}{2}\sin^5(\alpha)\\
&\qquad + \tfrac{3}{4}\sin^7(\alpha) + \cdots.
\end{aligned} \tag{4}$$

We do not concern ourselves with the convergence properties of these series. It is readily seen from (1) that (3) and (4) agree with an alternative formulation of $\sin(m\alpha)$ (m integer) which is said to be Euler's, namely,

$$
\begin{aligned}
\sin(m\alpha) &= m\sin(\alpha) - \left[\frac{m(m^2-1^2)}{3!}\right]\sin^3(\alpha) \\
&\quad + \left[\frac{m(m^2-1^2)(m^2-3^2)}{5!}\right]\sin^5(\alpha) \\
&\quad - \left[\frac{m(m^2-1^2)(m^2-3^2)(m^2-5^2)}{7!}\right] \\
&\quad \times \sin^7(\alpha) + \cdots,
\end{aligned}
$$

from which it is immediately obvious why the series form of $\sin(m\alpha)$ is only infinite for m even; for odd m the sum contains $\frac{1}{2}(m+1)$ terms in odd powers of $\sin(\alpha)$ up to and including the $\sin^m(\alpha)$ term.

Among other findings of Ming are well known recurrence formulae for Catalan numbers which are often assumed in the West to have first been established later in the 18th century or even more recently. Also of interest is a method devised by Ming for generating the sequence which seems to have escaped the notice of the vast majority of combinatorialists and historians of mathematics. In the present brief discussion it is described, for convenience, with reference to polynomial multiplication which is equivalent to the vector based method developed by Luo when interpreting Ming's work.

Let polynomials $M_n(x)$ $(n \geq 1)$ be defined by $M_1(x) = x$, $M_2(x) = x^2$, and thereafter according to the equation

$$
M_{p+1}(x) = \left\{2\sum_{k=1}^{p-1} M_k(x) + M_p(x)\right\} M_p(x), \quad p \geq 2.
$$

Thus, for instance,

$$
\begin{aligned}
M_3(x) &= 2x^3 + x^4, \\
M_4(x) &= 4x^4 + 6x^5 + 6x^6 + 4x^7 + x^8, \\
M_5(x) &= 8x^5 + 20x^6 + \cdots + 8x^{15} + x^{16}.
\end{aligned}
$$

It is easily proved by induction that

$$
M_p(x) = \sum_{k=p}^{2^{p-1}} \alpha_k^{(p)} x^k, \quad p \geq 1,
$$

where $\alpha_p^{(p)}, \ldots, \alpha_{2^{p-1}}^{(p)} > 0$. In other words, the degree of $M_p(x)$ is 2^{p-1} and the lowest power of x it contains is x^p. Ming's surprising result is that for $p \geq 3$,

$$
\sum_{k=1}^{p} M_k(x) = \sum_{k=1}^{p} c_{k-1} x^k + R_p(x),
$$

where the remainder $R_p(x)$ is a polynomial whose lowest power of x is x^{p+1} and highest power is $x^{2^{p-1}}$. Verification of this can be made by performing the necessary mathematical operations on any of the mainstream computer algebra systems which lend themselves to such a task. A proof lies beyond the remit of this article. The above compares with the standard identity

$$
\tfrac{1}{2}\left[1 - \sqrt{1-4x}\,\right] = \sum_{k=1}^{\infty} c_{k-1} x^k,
$$

which follows easily from the binomial series

$$
(1-t)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-t)^k
$$

with $4x$ replacing t, since in view of (2), for $k \geq 1$,

$$
\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}} c_{k-1}.
\end{aligned}
$$

Around 1730, Ming started to write a book which included his analysis containing the Catalan numbers. It was completed by his student, Chen Jixin, in 1774, but not actually published until more than sixty years later, over seventy years after Ming's death. As Luo emphasises, therefore, Ming's achievements in connection with the Catalan sequence should not be determined by the date that his work appeared formally in print but by the period during which he worked on the results and formally wrote them up. Since the latter predates Euler's comments on the numbers as being the enumerative solutions to triangular decompositions of polygons (as a 1751 letter of his to Christian Goldbach shows; see reference 2), it is Ming who deserves recognition as the true discoverer of the sequence. During Ming's lifetime Chinese mathematicians had not yet begun to use symbolic mathematical notation, and his calculations, expressed in words, have been re-written in the syntax of contemporary mathematics by Luo who has performed a valuable service to the scientific community.

It is stressed once again that Ming's mathematical work was to a large extent underpinned by particular aspects of Western scientific knowledge introduced into China by Jesuit missionaries during the 17th and 18th centuries. Throughout this period the most significant branch of mathematics to which China was exposed was Euclidean geometry, since until then there was no comparable axiomatic deductive system; conventional Chinese geometry had instead been based on the right-angled triangle. In arithmetic and algebra, areas usually considered more familiar to Chinese custom, the Jesuits' innovations had more to do with mathematical methods than with concepts. By the beginning of the 17th century an important part of China's internal mathematical heritage had been lost or had become incomprehensible. Thus, in learning Western science, the Chinese not only rejuvenated their mathematics but also rediscovered their own history, identifying some of the ancient methods with those they took from the Jesuits. In this context Ming's book, *Ge Yuan Mi Lu Jie Fa*, reflects the manner in which mathematical information was absorbed and assimilated in China, being characteristic of the era in terms of both its contents and the history of its composition. Jami (reference 5), whilst not mentioning explicitly Ming's awareness of the Catalan numbers, provides us with an informative general commentary on his ground-breaking work.

Acknowledgement

The author is grateful to Dr H. L. Le (Department of Mathematics, University of Nottingham) for providing a translation of reference 4, and to the referee in helping to prepare the manuscript for publication.

References

1. I. Vun and P. Belcher, Catalan numbers, *Mathematical Spectrum* **30** (1997/8), pp. 3–5.
2. P. J. Larcombe and P. D. C. Wilson, On the trail of the Catalan sequence, *Mathematics Today* **34** (1998), pp. 114–117.
3. P. J. Larcombe, On pre-Catalan Catalan numbers: Kotelnikow (1766), *Mathematics Today* **35** (1999), p. 25.
4. J. J. Luo, Antu Ming, the first inventor of Catalan numbers in the world, *Neimenggu Daxue Xuebao* **19** (1988), pp. 239–245 (in Chinese).
5. C. Jami, Western influence and Chinese tradition in an eighteenth century Chinese mathematical work, *Historia Mathematica* **15** (1988), pp. 311–331.

***Peter Larcombe** is a senior lecturer in mathematics at the University of Derby. He became interested in the Catalan numbers through working with an undergraduate student on her final year project a few years ago, and has been researching their history since then.*

Sums of Powers

ROGER COOK

Fermat's Last Theorem tells us that when $k > 2$ a kth power cannot be expressed as a sum of two kth powers; so when can a kth power be expressed as a sum of kth powers?

The Pythagorean equation

$$x^2 + y^2 = z^2$$

is probably the most familiar Diophantine equation. If x and y have a common factor d then d also divides z, so the solution is a multiple of a *primitive* solution where the highest common factor of x and y is 1. In a primitive solution x and y cannot both be even; if they were both odd we would have $z^2 \equiv 2 \mod 4$, which is not possible. Thus in a primitive solution, one of x and y, x say, is even and the other is odd. The general primitive solution is then given by

$$x = 2ab, \quad y = a^2 - b^2, \quad z = a^2 + b^2,$$

where a and b are integers satisfying $a > b > 0$, not both odd and having highest common factor 1.

Since the Fermat equation

$$x^k + y^k = z^k$$

has no solution in positive integers when $k > 2$, a natural question to ask next is: 'When can a kth power be expressed as a sum of kth powers?' From work on Waring's problem we know that all positive integers can be expressed as a sum of s kth powers provided that s is large enough. If we just want to express all large integers in this way then it is known that $s = Ck \log k$ terms are sufficient, with a positive constant C. It is probably true that $s = 4k$ terms are sufficient but that is far from what can be proved at present. Here we will look at what is known about expressing a kth power as the sum of a relatively small number of kth powers. We begin with the question: 'What do we know about equations

$$x^k + y^k + z^k = t^k \text{?'} \tag{1}$$

Cubic equations

Two simple solutions of

$$x^3 + y^3 + z^3 = t^3 \tag{2}$$

in positive integers are

$$x = 3, \quad y = 4, \quad z = 5, \quad t = 6$$

and

$$x = 1, \quad y = 6, \quad z = 8, \quad t = 9.$$

According to Dickson (reference 4, p. 550), the first of these solutions had already been noted by P. Bungus in 1591. In 1753 L. Euler obtained a family of solutions by means of the substitution

$$x = p + q, \quad y = p - q, \quad z = r - s, \quad t = r + s,$$

which reduces the equation to

$$p(p^2 + 3q^2) = s(s^2 + 3r^2).$$

He was then able to go on to find the complete solution to the equation in rationals; see reference 8 (Theorem 235):

Rewriting equation (2) in the form

$$x^3 + y^3 = u^3 + v^3, \tag{3}$$

apart from the trivial solutions $x = -y, u = -v$ *and* $x = u, y = v$ *(or vice versa), all rational solutions of (3) are given by*

$$\begin{aligned} x &= c\big(1 - (a - 3b)(a^2 + 3b^2)\big), \\ y &= c\big((a + 3b)(a^2 + 3b^2) - 1\big), \\ u &= c\big((a + 3b) - (a^2 + 3b^2)^2\big), \\ v &= c\big((a^2 + 3b^2)^2 - (a - 3b)\big), \end{aligned}$$

where a, b, c are any rational numbers with $c \neq 0$.

The situation is more complicated when we look for integer solutions. Although the family of rational solutions described above contains all the integer solutions, no simple description of the integer solutions is known. Hardy and Wright (reference 8, Section 13.7) state that Ramanujan found a family of integer solutions given by

$$\begin{aligned} x &= 3a^2 + 5ab - 5b^2, \quad & y &= 4a^2 - 4ab + 6b^2, \\ z &= 5a^2 - 5ab - 3b^2, \quad & t &= 6a^2 - 4ab + 4b^2. \end{aligned}$$

For example, taking $a = 2$ and $b = 1$ gives the solution (17, 14, 7, 20).

If we look for all integer solutions, positive or negative, then equations (2) and (3) are equivalent. One solution of (3) is (1, 12, 9, 10), where both sides of the equation equal 1729. There is a famous story, recounted by C. P. Snow (reference 7, p. 37), that when Ramanujan was ill in a nursing home, G. H. Hardy came to visit him. Hardy remarked that the number on the taxi was 1729, which did not appear to be a very interesting number. Ramanujan corrected him, pointing out that it was the smallest positive integer which could be expressed as the sum of two positive cubes in two different ways.

Finally for cubes we include a result from Sierpinski (reference 11, p. 393):

For any integer $n > 2$ *there exists a positive cube which is the sum of n different cubes of positive integers.*

The examples

$$3^3 + 4^3 + 5^3 = 6^3$$

and

$$11^3 + 12^3 + 13^3 + 14^3 = 20^3$$

show that the result holds for $n = 3$ and 4. Now suppose that the result holds for a particular integer $n > 2$. Then there exist positive integers $a_1 < a_2 < \cdots < a_n < a_0$ such that

$$a_1^3 + a_2^3 + \cdots + a_n^3 = a_0^3.$$

Therefore

$$(6a_0)^3 = (3a_1)^3 + (4a_1)^3 + (5a_1)^3 + (6a_2)^3 + \cdots + (6a_n)^3$$

and

$$3a_1 < 4a_1 < 5a_1 < 6a_2 < \cdots < 6a_n ,$$

so the result also holds for $n + 2$. Since it holds for $n = 3$ and 4 it holds for all $n > 2$.

Quartic equations

The smallest nontrivial solution of the equation

$$x^4 + y^4 = u^4 + v^4 \tag{4}$$

in distinct positive integers is (133, 134, 158, 59). Euler treated the equation (4) using the substitution

$$x = p + q, \quad y = r - s, \quad u = p - q, \quad v = r + s,$$

which reduced it to

$$pq(p^2 + q^2) = rs(r^2 + s^2),$$

and he was then able to find solutions such as

$$12\,231^4 + 2903^4 = 10\,381^4 + 10\,203^4;$$

see Dickson (reference 4, p. 644). His formulae are a particular case of the parametric solution of degree 7:

$$\begin{aligned} x &= a^7 + a^5b^2 - 2a^3b^4 + 3a^2b^5 + ab^6, \\ u &= a^7 + a^5b^2 - 2a^3b^4 - 3a^2b^5 + ab^6, \\ y &= a^6b - 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7, \\ v &= a^6b + 3a^5b^2 - 2a^4b^3 + a^2b^5 + b^7 \end{aligned}$$

(see reference 8, Section 13.7). Taking $a = 1$ and $b = 2$ gives the solution (133, 134, 158, 59).

Euler also considered the equation

$$x_1^k + x_2^k + \cdots + x_s^k = y^k. \tag{5}$$

When $s = 2$ and $k > 2$, Fermat's Last Theorem asserts that there are no solutions in positive integers. Euler conjectured that there should be no solutions in positive integers when $s < k$. This conjecture was shown to be false in 1966 when Lander and Parkin (reference 10) discovered that

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

This is known to be the only solution with $y \leq 765$. More recently Noam Elkies (reference 5) found infinitely many solutions of degree 4. After he had been shown this, Roger Frye of Thinking Machines Corporation found the smallest solution

$$95\,800^4 + 217\,519^4 + 414\,560^4 = 422\,481^4,$$

which is the unique solution in the range

$$0 \leq x_1 \leq x_2 \leq x_3 < x_4 < 1\,000\,000.$$

It is not known whether there are any counterexamples to Euler's conjecture with higher powers.

When $k = 4$ some solutions to (5) with $s = 4$ are known, the smallest being

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4,$$

found by R. Norrie in 1911 (see Guy, reference 6, p. 79). When $s = 5$ an infinite family of solutions is provided by the identity

$$(4x^4 - y^4)^4 + 2(4x^3y)^4 + 2(2xy^3)^4 = (4x^4 + y^4)^4.$$

The examples

$$4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4$$

and

$$14^4 + 24^4 + 34^4 + 49^4 + 58^4 + 84^4 = 91^4$$

show that there are fourth powers which are the sums of 5 and 6 different fourth powers respectively. Our next result is a simple modification of one in Sierpinski (reference 11, p. 394):

For any integer $n > 2$ there exists a fourth power which is the sum of n different fourth powers.

Let S be the set of n for which the statement is true; then 3 and 4 belong to S. Observe that any set T of positive integers which contains 3 and 4 and which contains $m + n - 1$ whenever it contains m and n must contain all integers $n > 2$, since it will contain $m + 2$ whenever it contains m. Now suppose m and n belong to S. Then

$$a_0^4 = a_1^4 + \cdots + a_m^4,$$
$$b_0^4 = b_1^4 + \cdots + b_n^4,$$

for some

$$a_1 < a_2 < \cdots < a_m < a_0,$$
$$b_1 < b_2 < \cdots < b_n < b_0.$$

Hence

$$(a_0b_0)^4 = (a_1b_1)^4 + \cdots + (a_1b_n)^4 + (a_2b_0)^4 + (a_3b_0)^4 + \cdots + (a_mb_0)^4,$$

where

$$a_1b_1 < \cdots < a_1b_n < a_2b_0 < \cdots < a_mb_0,$$

so that $m + n - 1$ belongs to S, as required.

Finally for fourth powers, we observe that

$$8^4 + 9^4 + 17^4 = 3^4 + 13^4 + 16^4.$$

Higher powers

The example of Lander and Parkin (reference 10) shows that the equation (5) has a solution when $k = 5$ and $s = 4$. When $s = k = 5$ the equation (5) has an infinite family of solutions provided by

$$(75y^5 - x^5)^5 + (x^5 + 25y^5)^5 + (x^5 - 25y^5)^5 + (10x^3y^2)^5 + (50xy^4)^5 = (x^5 + 75y^5)^5,$$

and all the powers are positive when

$$0 < 25y^5 < x^5 < 75y^5$$

(see reference 8, p. 333). Taking $x = 2$ and $y = 1$ gives

$$43^5 + 57^5 + 7^5 + 80^5 + 100^5 = 107^5.$$

The example

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$$

shows that there exist fifth powers which are the sum of six different fifth powers. Now let S denote the positive integers n such that there exists a fifth power which is the sum of n different positive fifth powers. Just as for fourth powers, if m and n belong to S then so does $m + n - 1$. Since 4, 5 and 6 belong to S we then see that (cf. reference 11, p. 395):

For any integer $n > 3$, there exists a fifth power which is the sum of n distinct positive fifth powers.

When $k \geq 5$ no nontrivial solutions of the equation

$$x^k + y^k = u^k + v^k$$

are known. The equation

$$x_1^5 + x_2^5 + x_3^5 = y_1^5 + y_2^5 + y_3^5$$

was first solved by A. Moessner (see Lander, reference 9, p. 1070), who gave the solution

$$49^5 + 75^5 + 107^5 = 39^5 + 92^5 + 100^5.$$

In 1952 Swinnerton-Dyer (reference 12) found a family of solutions to the equation; both Moessner's and Swinnerton-Dyer's solutions also satisfy the auxiliary condition

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3.$$

Lander (reference 9) states that the smallest solution which does not satisfy this auxiliary condition is

$$26^5 + 85^5 + 118^5 = 53^5 + 90^5 + 116^5.$$

With more variables we have the example

$$19^5 + 34^5 + 37^5 + 46^5 = 27^5 + 29^5 + 32^5 + 48^5.$$

In 1976 Brudno (reference 3) gave a two parameter solution to the corresponding equation for sixth powers; one particular solution is

$$3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6.$$

Andrew Bremner (references 1, 2) has a geometric approach to these problems which generates all the parametric families of solutions which satisfy some auxiliary conditions, but these auxiliary conditions are not too restrictive.

References

1. A. Bremner, A geometric approach to equal sums of fifth powers, *J. Number Theory* **13** (1981), pp. 337–354.
2. A. Bremner, A geometric approach to equal sums of sixth powers, *Proc. London Math. Soc.* **43** (1981), pp. 544–581.
3. S. Brudno, On generating infinitely many solutions of the diophantine equation $A^6 + B^6 + C^6 = D^6 + E^6 + F^6$, *Math. Comp.* **30** (1976), pp. 646–648.
4. L. E. Dickson, *History of the Theory of Numbers*, Vol. 2 (Hafner, New York, 1934).
5. N. Elkies, On $A^4 + B^4 + C^4 = D^4$, *Math. Comp.* **51** (1988), pp. 825–835.
6. R. K. Guy, *Unsolved Problems in Number Theory*, Section D1 (Springer, Berlin, 1981).
7. G. H. Hardy, *A Mathematician's Apology*, with a foreword by C. P. Snow (CUP, Cambridge, 1967).
8. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Chapters 13 and 21 (OUP, Oxford, 1988).
9. L. J. Lander, Geometric aspects of diophantine equations involving equal sums of like powers, *Amer. Math. Monthly* **75** (1968), pp. 1061–1073.
10. L. J. Lander and T. R. Parkin, Counterexample to Euler's conjecture on sums of like powers, *Bull. Amer. Math. Soc.* **72** (1966), p. 1079.
11. W. Sierpinski, *Elementary Theory of Numbers*, Chapter XI (Warsaw, 1964).
12. H. P. F. Swinnerton-Dyer, A solution of $A^5 + B^5 + C^5 = D^5 + E^5 + F^5$, *Proc. Camb. Phil. Soc.* **48** (1952), pp. 516–518.

Roger Cook *is a member of the Department of Pure Mathematics at the University of Sheffield, as well as being an editor of* Mathematical Spectrum. *His mathematical interests include number theory and combinatorics. Outside mathematics he wastes far too much time maintaining a web site! Between two drafts of this article he attended several events at the Gilbert and Sullivan Festival in Buxton, which inspired him to think of matters mathematical.*

Throwing Elliptical Shields on Floorboards

P. GLAISTER

In my recent article 'Playing Cards with Buffon' (reference 1) I described a generalisation of the classical needle problem of Buffon. Instead of determining the probability that a needle will cross a crack when dropped onto floorboards of constant width, the article considered the case where the dropped object was a rectangular playing card, and the corresponding probability in that case was derived. This included the needle problem as a special case. A simple alternative to the original needle problem concerns the tossing of a coin, and a natural generalisation of this problem leads to some worthwhile mathematics. This includes the appearance of a special kind of integral which occurs in other areas of mathematics, and also leads to an unexpected and interesting observation. We begin with a brief description of the alternative to the needle problem.

Suppose an elliptical shield is dropped onto a floor comprising wooden planks (floorboards). The problem is to determine the probability, p, that the shield crosses a crack between the planks. This includes the circle (coin) as a special case. Figure 1 shows an ellipse with semi-major axis a and semi-minor axis b, where $a > b$, on a typical plank of width w. We assume initially that the largest dimension of the ellipse is less than the width of the board, i.e. $2a \le w$, so that the ellipse will fit between the cracks for any orientation. The probability that the ellipse will cross a crack can be determined from

$$p = 1 - q,$$

where q is the probability that the ellipse does *not* cross a crack. To determine q we consider the distance over which the centre of the ellipse can land so that it does not cross a crack. Figure 2 shows the required distance as a dotted line, and we see that this distance depends on the *orientation* of the ellipse.

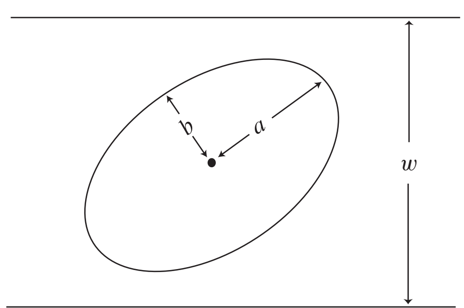


Figure 1. An ellipse on floorboards.

To assign a description of a particular orientation we measure the anti-clockwise angle between the horizontal and the semi-major axis. Denoting this angle by θ, we see that for an acute angle $0 \le \theta \le \frac{1}{2}\pi$, the length of the dotted line in figure 2 is

$$w - 2k,$$

where the distance k is to be determined. To determine k it is first necessary to transform the standard Cartesian equation of an ellipse by rotating the axes through an angle θ, as follows. Suppose that the equation of the ellipse shown in figure 2 and referred to X and Y axes is $X^2/a^2 + Y^2/b^2 = 1$, where the X axis is at an angle θ to the horizontal. The equation of this ellipse when referred to new axes, x and y, where the x axis is horizontal, is obtained by replacing X by $x\cos\theta + y\sin\theta$ and Y by $y\cos\theta - x\sin\theta$. Having done this, a tangent line $y = k$ can be sought by forcing the corresponding quadratic equation for the x coordinate of the point of tangency to have equal roots, which leads to the expression

$$k = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,.$$

We leave readers to check this formula. The total distance over which the centre of the ellipse can land is w.

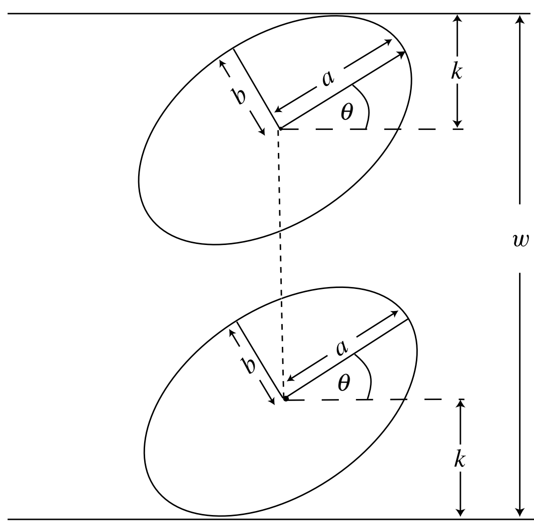


Figure 2. Possible positions of an ellipse on floorboards.

Now, since there is a different length for *each* orientation, we must calculate q from

$$q = \frac{\left(\begin{array}{c}\text{sum of outcomes for which the ellipse}\\ \text{does not cross a crack}\end{array}\right)}{\text{sum of possible outcomes}}$$
$$= \frac{\left(\begin{array}{c}\text{sum (over } \theta\text{) of distance over which the centre of}\\ \text{the ellipse can land so that the ellipse does}\\ \text{not cross a crack}\end{array}\right)}{\left(\begin{array}{c}\text{sum (over } \theta\text{) of possible distance over which the}\\ \text{centre of the ellipse can land}\end{array}\right)}.$$

To calculate the sum over θ we could either consider θ to range from 0 to 2π, and determine the corresponding distance to that above in the case $\frac{1}{2}\pi \le \theta \le \pi$, etc., or use symmetry to only sum over θ only in the range $0 \le \theta \le \frac{1}{2}\pi$ and then multiply the answer by 4. However, since sums over θ occur in both the numerator and denominator, it is unnecessary to multiply by 4. Thus

$$q = \frac{\text{sum over } 0 \le \theta \le \frac{1}{2}\pi \text{ of } w - 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}{\text{sum over } 0 \le \theta \le \frac{1}{2}\pi \text{ of } w},$$

and the 'sum' is determined by integrating (with respect to θ) between 0 and $\frac{1}{2}\pi$, so that

$$q = \frac{\int_0^{\frac{1}{2}\pi}\left[w - 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\right]\mathrm{d}\theta}{\int_0^{\frac{1}{2}\pi} w\,\mathrm{d}\theta} = \frac{\frac{1}{2}\pi w - 2\int_0^{\frac{1}{2}\pi}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\mathrm{d}\theta}{\frac{1}{2}\pi w}. \quad (1)$$

The probability that the ellipse does cross a crack is then

$$p = 1 - q = \frac{4}{\pi w}\int_0^{\frac{1}{2}\pi}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,\mathrm{d}\theta. \quad (2)$$

In the special case of a circle of radius $a = b$, then (2) becomes

$$p = \frac{4}{\pi w}\int_0^{\frac{1}{2}\pi} a\,\mathrm{d}\theta = \frac{2a}{w}. \quad (3)$$

The floorboards in a typical house are 140 mm wide, so that, using a 2 pence coin of radius 13 mm, we find that $p = 26/140 = 13/70 \approx 0.186$.

In the general case, however, (2) cannot be determined in terms of elementary functions and is an example of a *complete elliptic integral*. (For a description of such integrals see reference 2, p. 243, for example.) Note that *incomplete* elliptic integrals are of the form in (2) where the upper limit is α, say, where $0 < \alpha < \frac{1}{2}\pi$. Elliptic integrals occur in many areas of mathematics. For example, the period of oscillation of a pendulum can be expressed in terms of a complete elliptic integral (see reference 2, p. 302, for example). A numerical integration method is frequently used to calculate elliptic integrals.

Having determined the formula for the probability in (2), I recalled that one occurrence of an elliptic integral is in the determination of the length of the perimeter of an ellipse. Referring to the ellipse shown in figure 3 with semi-major and semi-minor axes a and b, respectively, where $a \ge b$, the length of the perimeter of the ellipse is given by

$$P = 4\int_0^{\frac{1}{2}\pi}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t, \quad (4)$$

where

$$x = a\cos t, \quad y = b\sin t \quad (5)$$

represent parametric equations of the ellipse with Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(see reference 2, pp. 276–279, for example). Substituting (5) into (4) yields

$$P = 4\int_0^{\frac{1}{2}\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\ \mathrm{d}t, \tag{6}$$

a complete elliptic integral. (It is intriguing that, while the length of the perimeter of the ellipse cannot be determined in terms of elementary functions, the area is easily shown to be the product πab. Note also that, in the case of a circle where $a = b$, the integral in (6) gives the perimeter $P = 2a\pi$, as usual.)

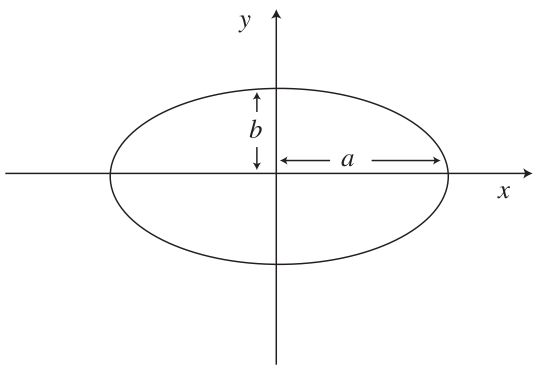


Figure 3. An ellipse with semi-major axis a and semi-minor axis b, $a \geq b$.

We now observe that the elliptic integral appearing in the formula for the probability p in (2) is precisely that occurring in the formula for the length of the perimeter P in (6), both for an ellipse with semi-major axis a and semi-minor axis b. Moreover, from (2) and (6) we have that

$$p = \frac{4}{\pi w} \cdot \frac{P}{4} = \frac{P}{\pi w}\,, \tag{7}$$

i.e.

$$p = \frac{1}{\pi}\left(\frac{\text{perimeter of ellipse}}{\text{width of boards}}\right), \tag{8}$$

an unexpected result. Furthermore, this is also true in the case of a circle of radius $a = b$, as can be seen from (3) which can be rewritten as

$$p = \frac{1}{\pi}\frac{2a\pi}{w} = \frac{1}{\pi}\left(\frac{\text{perimeter of circle}}{\text{width of boards}}\right). \tag{9}$$

One further special case of interest can also be determined from this analysis. By letting $b \to 0$, then the ellipse becomes a line of length $2a$, and equation (2) becomes

$$p = \frac{4}{\pi w}\int_0^{\frac{1}{2}\pi} a \sin\theta\ \mathrm{d}\theta = \frac{4a}{\pi w}\,. \tag{10}$$

This is the classical needle problem of Buffon in which a needle of length $\ell \leq w$ is dropped onto floorboards of width w. The standard result is $p = 2\ell/\pi w$ and with the needle represented by a line of length $\ell = 2a$, then this result coincides with that in equation (10) (see reference 3, for example).

In summary, a complete elliptic integral occurs naturally in the solution of an elliptical probability problem and, as a consequence, leads to the following interesting property:

> The probability that an ellipse crosses a crack when dropped onto floorboards of width w depends only on w and the perimeter of the ellipse, and not on the individual semi-major and semi-minor axes a and b. (Note that the ellipse must fit between the cracks for any orientation.)

Thus equal perimeters imply equal probabilities. This is true for all ratios a/b, including a circle where $a/b = 1$, and a needle where $a/b = \infty$. (Note that a needle of length $2a$ has perimeter $2.2a = 4a$ by considering it as an infinitely thin ellipse as $b \to 0$.)

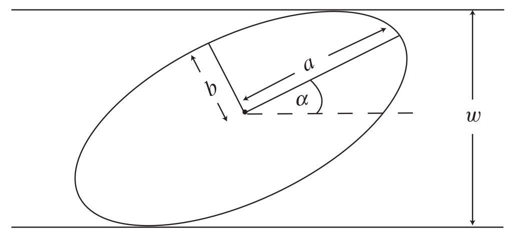


Figure 4. A large ellipse on floorboards.

Finally, we note that, if the ellipse cannot fit between the cracks for all orientations, then it is still possible to determine the probability, as follows. In the case where the ellipse can only be rotated through an angle α before crossing a crack, as shown in figure 4, then the probability q is given by equation (1) where the upper limit in the integral on the numerator changes to α, where $\alpha \in (0, \frac{1}{2}\pi)$. Since the distance from the top to the bottom of the ellipse is $2\sqrt{(a^2 \sin^2\alpha + b^2\cos^2\alpha)}$ using the result proved earlier, then α is given by $2\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} = w$, and thus

$$\sin\alpha = \sqrt{\frac{w^2/4 - b^2}{a^2 - b^2}} \quad \text{and} \quad \cos\alpha = \sqrt{\frac{a^2 - w^2/4}{a^2 - b^2}}\,.$$

(Note that, in this case, $b < \frac{1}{2}w < a$, whereas the case already discussed is $a \leq \frac{1}{2}w$. If $b \geq \frac{1}{2}w$ then the ellipse will always cross a crack and the probability $p = 1$.) Therefore, in this case

$$\begin{aligned} p &= 1 - \frac{\int_0^\alpha w - 2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\ \mathrm{d}\theta}{\int_0^{\frac{1}{2}\pi} w\ \mathrm{d}\theta} \\ &= 1 - \frac{2\alpha}{\pi} + \frac{4}{\pi w}\int_0^\alpha \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\ \mathrm{d}\theta, \end{aligned} \tag{11}$$

where

$$\alpha = \sin^{-1}\frac{\sqrt{w^2/4 - b^2}}{\sqrt{a^2 - b^2}}\,,$$

and this time involves an incomplete elliptic integral. This result is no longer of the form given in (8), but does match up with the corresponding expression in (2) evaluated for $a = \frac{1}{2}w$ when we let $a \to \frac{1}{2}w$ in (11) and where $\alpha = \sin^{-1}(1) = \frac{1}{2}\pi$. Furthermore, if we let $b \to 0$ in (11) we obtain the expression

$$\begin{aligned} p &= 1 - \frac{2}{\pi}\sin^{-1}(\tfrac{1}{2}w/a) + \frac{4}{\pi w}\int_0^{\sin^{-1}(\frac{1}{2}w/a)} a\sin\theta \,\mathrm{d}\theta \\ &= 1 - \frac{2}{\pi}\sin^{-1}(\tfrac{1}{2}w/a) + \frac{4}{\pi w}\left(a - \sqrt{a^2 - w^2/4}\right) \\ &= 1 - \frac{2}{\pi}\sin^{-1}(w/\ell) + \frac{2}{\pi}\left(\ell/w - \sqrt{(\ell/w)^2 - 1}\right), \quad (12) \end{aligned}$$

where $\ell = 2a$, representing the classical probability that a needle of length $\ell = 2a$ will cross a crack when dropped onto floorboards of width w in the case where $\ell > w$ (see reference 3, for example).

References

1. P. Glaister, Playing cards with Buffon, *Mathematical Spectrum* **30** (1997/98), pp. 8–9.
2. R. Courant, *Differential and Integral Calculus*, Vol. 1 (Blackie & Son, London, 1937).
3. J. Gani, The problem of Buffon's needle, *Mathematical Spectrum* **13** (1980/81), pp. 14–18.

Paul Glaister *lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, perturbation methods, as well as mathematics and science education. Although his two children have shown some interest in dropping objects onto floorboards from a probability viewpoint, they are definitely less keen on the practical aspect of picking them up again!*

Colin plays a game in which three standard dice are rolled and a score is obtained by adding up all the distinct three-figure numbers which can be made from the numbers coming up on the dice. For instance, if the dice come up 1, 4 and 6, then the score is

$$146 + 164 + 416 + 461 + 614 + 641 = 2442,$$

but if the dice come up 5, 5 and 5, then the score is just 555.

- In the long term what would be the average score?
- In the long term, what would be the ratio of above-average to below-average scores?

The 'expected' total score from 12 attempts is defined as 12 times the long-term average for one attempt.

- Is it possible for the actual total score from 12 attempts to be the 'expected' total score from 12 attempts? How do you know?

JOHN MACNEILL
(University of Warwick)

Which of the following is more true than the others?

A $11 + 2 = 12 + 1$

B $(1011)_2 + (10)_2 = (1100)_2 + (1)_2$

C Eleven plus two = Twelve plus one

IAN RICHARDS
Penwith Sixth Form College,
Penzance.

Braintwister

9. Secret society

Today I am going to tell you about a secret society: there is an integer known only to the members. They all do their arithmetic correctly, but before writing or stating a number, they have to add the society's secret number to it.

There are 19 members in the society. I asked the first member how many members there were. He whispered the answer to another member, who whispered it to another, and so on until each member had passed the answer on once. The final member announced that there were 67 members.

What is the society's secret number?

(The solution will be published next time.)

VICTOR BRYANT

Mathematics in the Classroom

Probability is an essential branch of mathematics taught in a lot of disciplines at university level. An important category of its exercises involves the calculation of specific constants that enable a given function to be a *probability density* function.

Even the definition of a probability density function can cause consternation among the audience, because it involves a part of calculus that is not easy to grasp, namely *improper integrals*. Such abstract ideas are not always well received, especially when presented to an audience consisting of people orientated to applications, as are the students at the Hellenic Naval Academy and the Technological Education Institute of Athens.

I well remember the difficulties encountered when trying to teach improper integrals (see reference 1) last year in a calculus course that usually precedes probability theory. What remains in my mind is the reluctance shown by the majority of students when it came to the presentation of such important functions as Gamma and Beta. So now I grasp the opportunity to combine probability exercises with that specific part of calculus.

A Gamma–Beta Algebra

Modelling mathematical concepts in an appropriate way can fascinate students and help to motivate them to master the accompanying methodology. The definition of a Beta function is simply stated here, i.e.

$$B(\gamma, \delta) = \int_0^1 \chi^{\gamma-1}(1-\chi)^{\delta-1}\,\mathrm{d}\chi. \tag{1}$$

Its association with the Gamma function is described by the relationship:

$$B(\gamma, \delta) = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}\,. \tag{2}$$

The only information regarding the Gamma function given to the students is that

$$\Gamma(n) = (n-1)! \tag{3}$$

for positive integers n. I deliberately do not refer back to last year's calculus to remind them that the Gamma function can be expressed as

$$\Gamma(\alpha) = \int_0^\infty \chi^{\alpha-1}\mathrm{e}^{-\chi}\,\mathrm{d}\chi \qquad \text{for } \alpha > 0.$$

Instead I make a connection with our new material, the probability densities, and give the following exercise:

Find an appropriate value of c such that the following function

$$f(\chi) = \begin{cases} c\chi^2(1-\chi), & 0 < \chi < 1 \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density.

This can be solved by elementary means or by calculating $B(3, 2)$ using (2) and (3).

What I emphasize at this point is the recognition of a Beta integral in the above exercise, or more specifically the identification of the parameters that are essential to the definition of a Beta integral. Once this link has been established, it is relatively straightforward to employ relations (2) and (3), this last part enabling students to feel involved in the learning procedure. I am aware of their satisfaction in solving the exercise and I even have some of them asking:

> *Could you restate the definition of the Gamma function on its whole domain (not only for positive integers)?*

The Gamma–Beta algebra did achieve its goal, i.e. not only did it give them a way to handle certain probability exercises but it also helped them overcome their fear and their initial aversion to important improper integrals.

Kyriakos I. Petakos
Hellenic Naval Academy and
Technological Educational Institute of Athens

Reference

1. P. Gillett, *Calculus and Analytic Geometry* (D.C. Heath and Company, 1984).

Computer Column

Programming in Prolog

What is your favourite programming language? Basic? C? Fortran? Perhaps you are proficient in several languages and use the appropriate one for the task in hand (after all, a language that is suitable for scientific applications is not necessarily a good choice for commercial applications). Whichever language you prefer, it is probably a *procedural* language.

In a procedural language you, the programmer, must specify in detail the precise steps the computer processor must go through in order to solve the problem under consideration. Is there an alternative approach? Surely it would be more natural to spend programming effort describing *what* you want the computer to do rather than *how* to do it?

The ideal would be to use *logic programming*, and have the computer translate the logical thinking of the programmer into an efficient program. At present, the closest we have to this ideal is Prolog (the name derives from 'Programming in Logic'). When writing a program in Prolog you still have to take account of *some* procedural details; you are not programming in 'pure logic'. Nevertheless, the emphasis is on 'what to do' rather than 'how to do it'. This is a *declarative* approach to programming.

There are several reasons why you might want to learn Prolog. A Prolog program is often much simpler and shorter than an equivalent program written in a procedural language; Prolog is ideally suited to building knowledge-based and expert systems; it lends itself well to artificial intelligence applications; and it is an excellent tool for developing logical thinking. A word of warning, though: Prolog will seem very strange to you if you are already familiar with a procedural language like Basic. There are no `repeat ... until` loops; no `if ... then` conditions; no `goto` jumps.

Writing a Prolog program consists of:

- declaring some *facts* about objects and the relationships between those objects,
- declaring some *rules* about objects and the relationships between those objects,
- asking *questions* about objects and their relationships.

For instance, if you want to tell Prolog the fact that sneezing is a symptom of a cold you would use the following standard form:

```
symptom(cold, sneezing).
```

This fact asserts a relationship (i.e. `symptom`) between two objects (i.e. `cold` and `sneezing`). One can imagine a long list of similar facts that might be used in a medical expert system:

```
symptom(cold, sneezing).
symptom(cold, blocked_nose).
symptom(cold, cough).
symptom(cold, headache).
symptom(flu, shivers).
symptom(flu, temperature).
symptom(flu, muscle_aches).
symptom(flu, headache).
symptom(green_monkey_disease,
                          headache).
```

A fact is always, unconditionally, true; a *rule* specifies things that are true only if some condition is satisfied. For example, we could add the following rules to our medical expert system:

```
healthy(person) :- likes(person,
                              jogging).
unhealthy(person) :- likes(person,
                              smoking).
```

The symbol `:-` can usually be read as 'if'. So we could read the first rule above as: 'a person is healthy if that person likes jogging'. (Note that a Prolog rule does not necessarily correspond to reality.)

Once you have a knowledge base of facts and rules, you can ask questions about them. In Prolog, a question looks just like a fact except that it is preceded by the symbol `?-`. For example:

```
?- symptom(X, sneezing),
         symptom(X, blocked_nose),
         symptom(X, cough).
```

asks: is there a disease `X` which causes the symptoms of sneezing and a blocked nose and a cough? Prolog searches through the database of facts and rules, and responds with:

```
X = cold.
```

If you wanted to know all the symptoms of flu, you would simply ask:

```
?- symptom(flu, X).
```

These very simple examples should give you an idea of how different Prolog is from other languages. If you wish to give Prolog a try, you can find further information (including the location of various public-domain implementations of the language) at: `http://www.comlab.ox.ac.uk/archive/logic-prog.html`.

Stephen Webb

Letters to the Editor

Dear Editor,

On circular primes

I have recently looked at properties of a natural number

$$n = n_1 n_2 \dots n_k$$

of k digits in base 10 which are preserved by the operation R of clockwise rotation:

$$\mathrm{R}(n) = n_k n_1 n_2 \dots n_{k-1}\,.$$

Starting with n we get k natural numbers

$$n, \mathrm{R}^2(n), \mathrm{R}^3(n), \dots, \mathrm{R}^{k-1}(n)$$

with $\mathrm{R}^{k-1}(n) = n_2 n_3 \dots n_k n_1$.

We consider those numbers which remain prime for any cyclic rotation of their digits, which will be called *circular primes*.

If n contains the digits 0, 2, 4, 6, 8 or 5, we can rotate them to the last place so that n is not a circular prime (assuming $n > 9$ — we shall assume that n contains at least two digits). Thus any circular prime > 9 can consist only of the digits 1, 3, 7, 9. One example of such a number is 9311 since

$$1193,\ 3119,\ 9311 \text{ and } 1931$$

are all primes. We call 1193 a *primeval* circular prime since it is the smallest number in this set of primes.

The following sequence of primeval circular primes

$$\begin{gathered} 2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 37,\ 79,\ 113,\ 197, \\ 199,\ 337,\ 1193,\ 3779,\ 11\,939,\ 19\,937, \\ 193\,939,\ 199\,933,\ \mathrm{R}_{19},\ \mathrm{R}_{23},\ \mathrm{R}_{317},\ \mathrm{R}_{1031}, \end{gathered}$$

where R_n indicates the *repunit* consisting of n '1s', are all the primeval circular numbers known up to now. Here are some heuristics which suggest that the list is finite if we exclude the repunit primes.

Consider numbers n with d digits, i.e.

$$10^{d-1} \le n \le 10^d - 1. \qquad (*)$$

The Prime Number Theorem states (roughly) that the probability of a randomly chosen number of the size of n being prime is about

$$p(n) \approx \frac{1}{\ln n} \approx \frac{1}{d \cdot \ln 10}\,.$$

The number of d-digit integers n in the range $(*)$ which consist only of the digits 1, 3, 7, or 9 is 4^d, so we might expect roughly

$$\frac{4^d}{d \cdot \ln 10}$$

of these to be prime. In fact, this estimate has to be increased since such numbers are not chosen at random; they are specifically chosen not to be divisible by 2 or 5 so the expected number has to be multiplied by $2/1 \times 5/4 = 5/2$ to give the expected number

$$\frac{5}{2}\frac{4^d}{d \ln 10} = 1.086\dots\frac{4^d}{d}\,.$$

Any d-digits circular prime which is not a repunit must generate d distinct numbers by cycling. The probability that these are all primes is roughly

$$\left(\frac{5}{2}\right)^d \left(\frac{1}{d \ln 10}\right)^d,$$

and so we expect roughly

$$4d\left(\frac{5}{2}\right)^d \left(\frac{1}{d \ln 10}\right)^d = \left(\frac{10}{d \ln 10}\right)^d$$

d-digit circular primes. Since

$$\sum_{d=1}^{\infty} \left(\frac{10}{d \ln 10}\right)^d,$$

converges, we should only expect a finite number of such circular primes. The table compares the actual number of circular primes with the estimate.

d	Estimated	Actual
1	4.34294	4
2	4.71529	4
3	3.03381	4
4	1.38962	2
5	0.49439	2
6	0.14381	2
7	0.03538	0
8	0.00754	0
9	0.00141	0
10	0.00023	0
11	0.000036	0
12	0.000005	0

The same observations also hold for the integers written in bases other than 10.

Yours sincerely,

F. Russo
(Micron Technology Italy,
Via Pacinotti 3/4,
67051 Avezzano (Aq),
Italy)

Dear Editor,

'Calculus Unequalled' by P. Glaister

This interesting article in Volume 31, pages 38–39, considered, *inter alia*, solving the equation

$$c^x = x^c \quad (c > 0) \tag{$*$}$$

by constructing the sequence

$$x_{n+1} = \mathrm{e}^{px_n} \quad \text{where} \quad p = \frac{\ln c}{c} \Rightarrow \mathrm{e}^p = c^{1/c}.$$

If e^p is replaced by α and x_n by a_n then it is seen that this is the same as

$$a_{n+1} = \alpha^{a_n}.$$

A discussion of this sequence is to be found in my article 'More about an infinite exponential', Volume 27, pages 54–56. Its behaviour is more curious than one might at first imagine. It was shown in that article that the sequence $\{a_n\}$ has one of four types of behaviour. Specifically:

$0 < \alpha < \mathrm{e}^{-\mathrm{e}} \Rightarrow$ it converges to a 2-cycle, i.e. $\{a_{2n+1}\}$ and $\{a_{2n}\}$ both converge, but the two limits are different;

$\mathrm{e}^{-\mathrm{e}} < \alpha < 1 \Rightarrow$ it converges to the unique solution of $X = \alpha^X$;

$1 < \alpha < \mathrm{e}^{1/\mathrm{e}} \Rightarrow$ it converges to the smaller of the two roots of $X = \alpha^X$;

$\alpha > \mathrm{e}^{1/\mathrm{e}} \Rightarrow$ it increases without limit.

Figure 1 shows how p varies with c and also explains the definition of the number $c^\star$. Note that, since the graph has a maximum (of amount $1/\mathrm{e}$) at e, if $1 < c < \mathrm{e}$ then $c^\star > \mathrm{e}$ and if $c > \mathrm{e}$ then $1 < c^\star < e$. In terms of c it follows (after some manipulation) that:

$0 < c < \frac{1}{\mathrm{e}} \Rightarrow \{x_n\}$ converges to a 2-cycle,

$\frac{1}{\mathrm{e}} < c < 1 \Rightarrow \{x_n\}$ converges to c, the unique solution of equation $(*)$,

$1 < c < \mathrm{e} \Rightarrow \{x_n\}$ converges to c if $x_0 < c^\star$;

$\mathrm{e} < c \Rightarrow \{x_n\}$ converges to $c^\star$ if $x_0 < c$.

In every case it is being assumed that $x_0 > 0$.

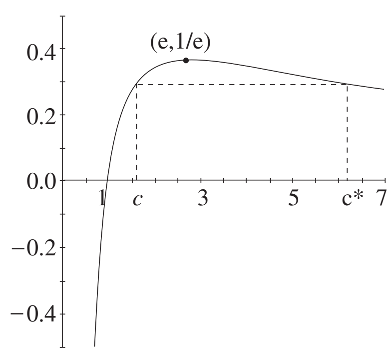


Figure 1.

The fact that the sequence has a 2-cycle is 'well known'. In other words, it was discovered a long time ago. An editorial comment to Bertuello's note (Volume 27, page 22) referred to Bromwich, *Infinite Series* (Macmillan, London, 1949). In this book (see page 23, example 11) the solution of equation $(*)$ is discussed and there are references to Eisenstein (1844) and Seidel (1870). This latter article is credited with the discovery of the 2-cycle. That sequences can undergo period-doubling is now truly well known, but the case considered here is a little odd in that usually period-doubling is the precursor to yet more complicated behaviour, leading eventually to chaos. This observation suggests considering the complex form of the sequence, i.e.

$$z_{n+1} = w^{z_n} \quad (n \geq 0),$$

where z_0 and w are complex numbers. Given the interest that has been shown in sequences defined by simple recurrence relations, it is unlikely that the behaviour of this last sequence has not been investigated. However, this is very far from my own field of interest and I do not know of any such work. Perhaps a reader (or editor!) might know more and be prepared to enlighten us.

Yours sincerely,

GLENN T. VICKERS
(School of Mathematics and Statistics,
University of Sheffield,
Sheffield S3 7RH)

Dear Editor,

Checking multiplication

Bob Bertuello's letter in Volume 31, page 68, prompts me to offer the following methods for checking answers to multiplication sums.

Split each number into pairs thus:

$$46\ 37 \times 9\ 14 = 4\ 23\ 82\ 18.$$

Sum pairs on each side.

$$46 + 37 = \mathbf{83};$$
$$9 + 14 = \mathbf{23};$$
$$4 + 23 + 82 + 18 = 1\,\mathbf{27};$$
$$\text{repeat to get } 1 + 27 = \mathbf{28}.$$

Now $83 \times 23 = 19\,09$ and $19 + 9 = \mathbf{28}$.

This method is a generalization of the rule of 9s; in fact it is essentially modulo 99 arithmetic. A similar method works modulo 101, like the well-known rule of 11s. For the product above $(37 - 46) \times (14 - 9) = -9 \times 5 = -45$, whilst $18 - 82 + 23 - 4 = -45$.

The method can be extended to blocks of any number of digits. It can also be generalized to work to other moduli.

(1) For example apply the rule of 8s to the same sum. First note that $8 = 10 - (2) \times 1$. Evaluate $4 \times 2^3 + 6 \times 2^2 + 3 \times 2 + 7$. The best way is to use 'nesting' and reduce modulo 8 as you go, viz.

$$4 \times 2 + 6 = 14 = 6.$$

Then $6 \times 2 + 3 = 15 = 7$ and $7 \times 2 + 7 = 21 = \mathbf{5}$. Similarly 914 reduces to **2**. Then $5 \times 2 = \mathbf{2}$. On the right;

$$\begin{aligned} 4 \times 2 + 2 &= 2;\\ 2 \times 2 + 3 &= 7;\\ 7 \times 2 + 8 &= 6;\\ 6 \times 2 + 2 &= 6;\\ 6 \times 2 + 1 &= 5;\\ 5 \times 2 + 8 &= \mathbf{2}. \end{aligned}$$

(2) Now a more significant example working modulo 997. First note that $997 = 1\,000 - (3) \times 1$. Apply to $123\,456 \times 789\,012 = 97\,408\,265\,472$. Evaluate $123 \times 3 + 456 = \mathbf{825}$; and $789 \times 3 + 12 = 2379 = 2 \times 3 + 379 = \mathbf{385}$. Now

$$\begin{aligned} 825 \times 385 &= 33 \times 25 \times 385 = 33 \times 9625\\ &= 33 \times (9 \times 3 + 625) = 33 \times 625 = 21516\\ &= 21 \times 3 + 516 = \mathbf{579}. \end{aligned}$$

And $97 \times 3^3 + 408 \times 3^2 + 265 \times 3 + 472$. Using nesting:

$$\begin{aligned} 97 \times 3 + 408 &= 699;\\ 699 \times 3 + 265 &= 2362 = 2 \times 3 + 362 = 368;\\ 368 \times 3 + 472 &= 1576 = 1 \times 3 + 576 = \mathbf{579}. \end{aligned}$$

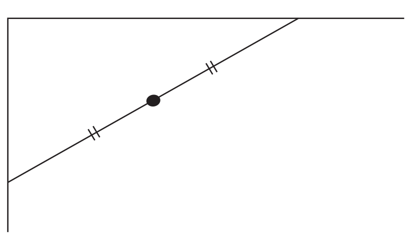
Yours sincerely,

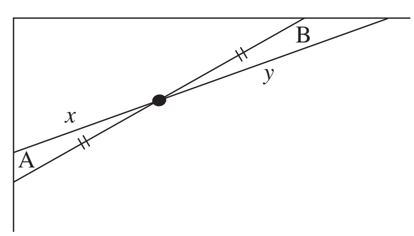
A. SUMMERS
(57 Conduit Road,
Stamford,
Lincs PE9 1QL)

Solution to Braintwister 8
(Cutting corners)

Answer: The fence is 26 metres long and the triangle has area 120 square metres.

Solution:
You can use calculus, but geometry is neater. It turns out that the new fence must have the post at its midpoint (as shown on the left).

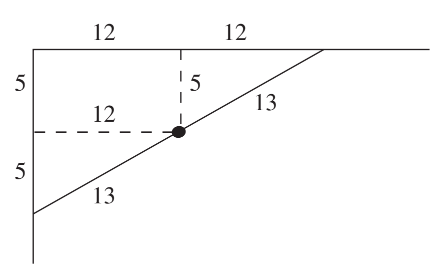




For, given any other line (as on the right, with $x < y$, say) triangle A is smaller than triangle B, and so the area cut off by the new line is larger than before.

Hence the situation is as follows:

length of fence: 26 m
area of triangle: $120\,\mathrm{m}^2$



VICTOR BRYANT

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

32.1 Let $n \geq 1$ be an integer and let $\theta_1, \ldots, \theta_n$ be positive real numbers such that $\theta_1 + \cdots + \theta_n < \frac{1}{2}\pi$.

Prove that

$$\frac{(1-\sin\theta_1)(1-\sin\theta_2)\cdots(1-\sin\theta_n)}{(1-\cos\theta_1)(1-\cos\theta_2)\cdots(1-\cos\theta_n)} \geq \frac{1-\sin(\theta_1+\theta_2+\cdots+\theta_n)}{1-\cos(\theta_1+\theta_2+\cdots+\theta_n)}.$$

(Submitted by Hassan Shah Ali, Tehran)

32.2 Find the sum of the infinite series

$$\frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \cdots + (-1)^{n+1}\frac{n}{(n+1)!} + \cdots.$$

(Submitted by J. A. Scott, Chippenham)

32.3 Let $n \geq 1$ be an integer, let $a_1, \ldots, a_n$ be positive real numbers and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers smaller than 1. Prove that

$$\frac{\lambda_1\lambda_2\ldots\lambda_n}{\lambda_2\lambda_3\ldots\lambda_n a_1 + \lambda_1\lambda_3\ldots\lambda_n a_2 + \cdots + \lambda_1\lambda_2\ldots\lambda_{n-1}a_n} + \frac{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)}{[(1-\lambda_2)(1-\lambda_3)\cdots(1-\lambda_n)a_1 + (1-\lambda_1)(1-\lambda_3)\cdots(1-\lambda_n)a_2 + \cdots + (1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_{n-1})a_n]} \leq \frac{1}{a_1+a_2+\cdots+a_n}.$$

(Submitted by Zhang Yun, Jinchang City, China)

32.4 Let $n > 1$ be an integer and let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Prove that

$$(4n-4)\,\alpha_1\alpha_n \leq (\alpha_1+\cdots+\alpha_n)^2,$$

and determine when equality occurs.

(Submitted by Hassan Shah Ali, Tehran)

Solutions to Problems in Volume 31 Number 2

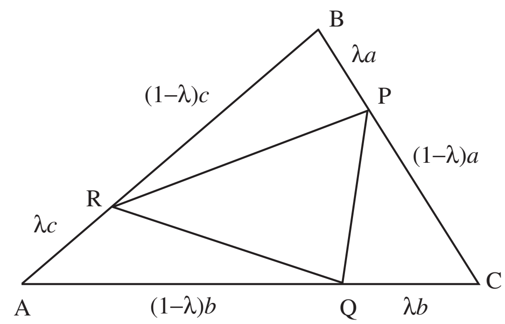
31.5 ABC is a fixed triangle.

(i) P, Q, R are points on the sides BC, CA, AB respectively such that BP/BC = CQ/CA = AR/AB. When is the area of triangle PQR minimal?

(ii) P′, Q′, R′ are points on the sides BC, CA, AB respectively such that P′ is the midpoint of BC and R′Q′ is parallel to BC. When is the area of triangle P′Q′R′ maximal?

Solution by Harriet Robjent (Gresham's School, Holt)

(i)



$$\begin{aligned}\text{Area}\,\triangle\,\text{PQR} &= \text{Area}\,\triangle\,\text{ABC} - \text{Area}\,\triangle\,\text{AQR} \\ &\quad - \text{Area}\,\triangle\,\text{BRP} - \text{Area}\,\triangle\,\text{CPQ} \\ &= \tfrac{1}{2}ab\sin C - \tfrac{1}{2}\lambda c(1-\lambda)b\sin A \\ &\quad - \tfrac{1}{2}\lambda a(1-\lambda)c\sin B - \tfrac{1}{2}\lambda b(1-\lambda)a\sin C \\ &= \tfrac{1}{2}ab\sin C - \tfrac{1}{2}\lambda(1-\lambda) \\ &\quad \times (bc\sin A + ca\sin B + ab\sin C) \\ &= \triangle(\text{ say}),\end{aligned}$$

so

$$\begin{aligned}\frac{\mathrm{d}\triangle}{\mathrm{d}\lambda} &= -\tfrac{1}{2}(1-\lambda-\lambda)(bc\sin A + ca\sin B + ab\sin C) \\ &= 0 \qquad \text{when } \lambda = \tfrac{1}{2}\end{aligned}$$

and

$$\frac{\mathrm{d}^2\triangle}{\mathrm{d}\lambda^2} = bc\sin A + ca\sin B + ab\sin C > 0,$$

so $\lambda = \frac{1}{2}$ gives a minimum value for $\triangle$, i.e. when P, Q, R are the midpoints of the sides of the triangle.

(ii)
$$\begin{aligned}\text{Area}\,\triangle\,\text{P}'\text{Q}'\text{R}' &= \text{Area}\,\triangle\,\text{ABC} - \text{Area}\,\triangle\,\text{AQ}'\text{R}' \\ &\quad - \text{Area}\,\triangle\,\text{BR}'\text{P}' - \text{Area}\,\triangle\,\text{CP}'\text{Q}' \\ &= \tfrac{1}{2}ab\sin C - \tfrac{1}{2}\lambda^2 bc\sin A \\ &\quad - \tfrac{1}{2}\tfrac{1}{2}a(1-\lambda)c\sin B \\ &\quad - \tfrac{1}{2}\tfrac{1}{2}a(1-\lambda)b\sin C \\ &= \triangle'(\text{ say}),\end{aligned}$$

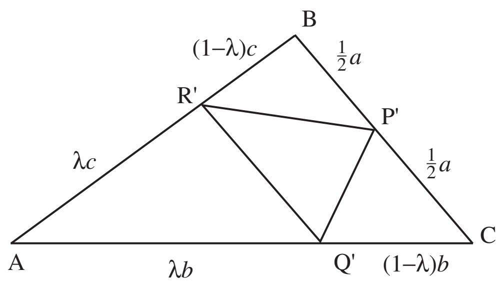
so

$$\frac{\mathrm{d}\triangle'}{\mathrm{d}\lambda} = -\lambda bc\sin A + \tfrac{1}{4}ac\sin B + \tfrac{1}{4}ab\sin C$$
$$= (-\lambda + \tfrac{1}{2})bc\sin A$$
$$\left(\text{since } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$
$$= 0 \qquad \text{when } \lambda = \tfrac{1}{2},$$

and

$$\frac{\mathrm{d}^2\triangle'}{\mathrm{d}\lambda^2} = -bc\sin A < 0,$$

so $\lambda = \frac{1}{2}$ gives a maximum value for $\triangle'$, i.e. when Q$'$, R$'$ are the midpoints of the sides of the triangle.



(Ed:

$$\triangle = (\text{Area} \triangle \text{ABC})\big(1 - 3\lambda(1-\lambda)\big)$$
$$= (\text{Area} \triangle \text{ABC})\big(3(\lambda - \tfrac{1}{2})^2 + \tfrac{1}{4}\big),$$

which is minimal when $\lambda = \frac{1}{2}$. Also

$$\triangle' = (\text{Area} \triangle \text{ABC})\big(1 - \lambda^2 - (1-\lambda)\big)$$
$$= (\text{Area} \triangle \text{ABC})\big(\tfrac{1}{4} - (\tfrac{1}{2} - \lambda)\big)^2,$$

which is maximal when $\lambda = \frac{1}{2}$. This avoids the use of calculus.)

Also solved by Jeremy Young (Nottingham High School) using areal coordinates, Mark Brimicombe (Christ Church, Oxford), Chun Chung Tang (Impington Village College, Cambridgeshire).

31.6 Evaluate $\sum_{k=1}^{n}[k2^{1/k}]$, where $[\alpha]$ denotes the integer part of α.

Solution (independently) by Chun Chung Tang and Simon Pickett (Gresham's School, Holt)

$[k2^{1/k}] = k$ if $k2^{1/2} < k+1$, i.e. if $2^{1/k} < 1 + \frac{1}{k}$, i.e. if $2 < (1+\frac{1}{k})^k$. Now, by the binomial theorem,

$$\left(1+\frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \cdots > 2 \qquad \text{if} \quad k \geq 2.$$

Hence

$$\sum_{k=1}^{n}[k2^{1/k}] = 2 + 2 + 3 + 4 + \cdots + n$$
$$= \tfrac{1}{2}n(n+1) + 1$$
$$= \tfrac{1}{2}n^2 + \tfrac{1}{2}n + 1.$$

Also solved by Jeremy Young.

31.7 Show that the sequence $\{u_n\}$, where $u_n = [n\sqrt{3}]$, contains infinitely many perfect squares.

Solution by Jeremy Young

We use the notation $[x]$, $\{x\}$ for the integer part and the fractional part of x respectively, so that $\{x\} = x - [x]$. Put $n = \left[\frac{m^2}{\sqrt{3}}\right] + 1$, where $m \in \mathbb{N}$. Then

$$u_n = [n\sqrt{3}] = \left[\left(\left[\frac{m^2}{\sqrt{3}}\right] + 1\right)\sqrt{3}\right]$$
$$= \left[\left(\frac{m^2}{\sqrt{3}} - \left\{\frac{m^2}{\sqrt{3}}\right\} + 1\right)\sqrt{3}\right]$$
$$= \left[m^2 + \left(1 - \left\{\frac{m^2}{\sqrt{3}}\right\}\right)\sqrt{3}\right]$$
$$= m^2 + \left[\left(1 - \left\{\frac{m^2}{\sqrt{3}}\right\}\right)\sqrt{3}\right].$$

Thus $u_n = m^2$ if

$$\left[\left(1 - \left\{\frac{m^2}{\sqrt{3}}\right\}\right)\sqrt{3}\right] = 0,$$

i.e. if

$$1 - \left\{\frac{m^2}{\sqrt{3}}\right\} < \frac{1}{\sqrt{3}},$$

i.e. if

$$\left\{\frac{m^2}{\sqrt{3}}\right\} > 1 - \frac{1}{\sqrt{3}}.$$

It is sufficient to show that this is true for infinitely many values of m.

Let $v_m = \left\{\frac{m^2}{\sqrt{3}}\right\}$ and observe that

$$v_{m+1} = \left\{v_m + \left\{\frac{2m+1}{\sqrt{3}}\right\}\right\}.$$

If

$$1 - \frac{1}{\sqrt{3}} < \left\{\frac{2m+1}{\sqrt{3}}\right\} < \frac{1}{\sqrt{3}},$$

and $v_m \leq 1 - \frac{1}{\sqrt{3}}$, then

$$1 - \frac{1}{\sqrt{3}} < v_m + \left\{\frac{2m+1}{\sqrt{3}}\right\} < 1$$

so $v_{m+1} > 1 - \frac{1}{\sqrt{3}}$. This shows that, if

$$1 - \frac{1}{\sqrt{3}} < \left\{\frac{2m+1}{\sqrt{3}}\right\} < \frac{1}{\sqrt{3}},$$

then either $v_m > 1 - \frac{1}{\sqrt{3}}$ or $v_{m+1} > 1 - \frac{1}{\sqrt{3}}$, so one of the u_ns corresponding to m and $m+1$ is a perfect square.

It now suffices to show that there are infinitely many m such that

$$1 - \frac{1}{\sqrt{3}} < \left\{\frac{2m+1}{\sqrt{3}}\right\} < \frac{1}{\sqrt{3}}.$$

An increase of m to $m+1$ causes $\frac{2m+1}{\sqrt{3}}$ to increase by $\frac{2}{\sqrt{3}}$ so $\left\{\frac{2m+1}{\sqrt{3}}\right\}$ increases by $\frac{2}{\sqrt{3}} - 1$, which is the width of the interval into which we want it to fall. Further, $\frac{2m+1}{\sqrt{3}} \neq k - \frac{1}{\sqrt{3}}$ for any integer k, so $\left\{\frac{2m+1}{\sqrt{3}}\right\}$ cannot equal $1 - \frac{1}{\sqrt{3}}$. Thus, as m increases, every time $\left\{\frac{2m+1}{\sqrt{3}}\right\}$ comes round to the interval $\left(1 - \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, it cannot jump right over it, but must take exactly one value within it. This shows that there are infinitely many values of m such that

$$1 - \frac{1}{\sqrt{3}} < \left\{\frac{2m+1}{\sqrt{3}}\right\} < \frac{1}{\sqrt{3}}$$

and so proves the result.

31.8 Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \left(\mathrm{e} - \left(1 + \frac{1}{n}\right)^n\right)^\alpha$$

for different values of the positive real number α.

Solution

L'Hopital's rule is as follows: Suppose that (i) f, g are differentiable in (a, b), (ii) $f(x), g(x) \to 0$ as $x \to a$, (iii) $g'(x) \neq 0$ in (a, b), (iv) $f'(x)/g'(x) \to c$ as $x \to a$. Then $f(x)/g(x) \to c$ as $x \to a$.

Let

$$f(x) = \mathrm{e} - (1+x)^{1/x}, \qquad g(x) = x$$

in $(0, \infty)$. Then g clearly satisfies conditions (i) and (iii) of L'Hopital's rule. Also

$$f(x) = \mathrm{e} - \mathrm{e}^{\frac{\log(1+x)}{x}} \to 0 \qquad \text{as } x \to 0$$

since

$$\frac{\log(1+x)}{x} = 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \cdots \to 1$$

as $x \to 0$, so that (ii) is satisfied. Finally,

$$\begin{aligned}
f'(x) &= -\exp\left(\frac{\log(1+x)}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\frac{\log(1+x)}{x} \\
&= -\exp\left(\frac{\log(1+x)}{x}\right)\frac{x\frac{1}{1+x} - \log(1+x)}{x^2} \\
&= -\exp\left(\frac{\log(1+x)}{x}\right)\left(\frac{1}{1+x} - \frac{\log(1+x)}{x}\right)\frac{1}{x} \\
&= -\exp\left(\frac{\log(1+x)}{x}\right)\Big[\left(1 - x + x^2 - \cdots\right) \\
&\qquad - \left(1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \cdots\right)\Big]\frac{1}{x} \\
&= -\exp\left(\frac{\log(1+x)}{x}\right)\left(-\tfrac{1}{2} + \tfrac{2}{3}x - \cdots\right) \\
&\to \tfrac{1}{2}\mathrm{e}
\end{aligned}$$

as $x \to 0$. Thus

$$\frac{f'(x)}{g'(x)} \to \tfrac{1}{2}\mathrm{e}$$

as $x \to 0$ and so (iv) is satisfied with $c = \frac{1}{2}\mathrm{e}$. Hence

$$\frac{f(x)}{g(x)} \to \tfrac{1}{2}\mathrm{e}$$

as $x \to 0$. Putting $x = \frac{1}{n}$ we therefore see that

$$\left[\mathrm{e} - \left(1 + \frac{1}{n}\right)^n\right] \Big/ \frac{1}{n} \to \tfrac{1}{2}\mathrm{e}$$

as $n \to \infty$. So

$$\left[\mathrm{e} - \left(1 + \frac{1}{n}\right)^n\right]^\alpha \Big/ \frac{1}{n^\alpha} \to \left(\tfrac{1}{2}\mathrm{e}\right)^\alpha$$

as $n \to \infty$ and, by the limit form of the comparison test, the series $\sum_{n=1}^{\infty}\left[\mathrm{e} - \left(1+\frac{1}{n}\right)^n\right]^\alpha$ and $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ both converge or both diverge.

Hence the series converges when $\alpha > 1$ and diverges when $\alpha \leq 1$.

Vulgar repetition

EVE/DID = .TALKTALKTALKTALKTALK... (recurring decimal problem by B. M. Jones).

Each letter stands for a different digit. The solution is $242/303 = .7986798679867986\ldots$.
Here is another one of the same kind for you to solve.

IS/IT = .BEANSBEANSBEANSBEANSBEANS....

JOHN MACNEILL
(University of Warwick)

Reviews

Elementary Mathematical Models: Order Aplenty and a Glimpse of Chaos. By DAN KALMAN. MAA, Washington, 1997. Pp. xvi+345. Paperback \$32.50 (ISBN 0-88385-707-3).

Dan Kalman focuses on the power of maths to model growth in various sytems, both discrete and continuous. This EMM book has been clearly written and well laid out, with helpful diagrams where they are required. The reader is led through arithmetic growth, finds his way to exponential growth, and ends with a simple consideration of logistic models and how chaos can arise within them. A running theme is the study of difference equations and how these lead to a general equation for modelling growth.

Often, more general mathematical principles follow on in context from the discussions on growth. For example, the reader is introduced to the concepts of rational functions, asymptotes and factorisation by first considering polynomial growth models. At the end of each chapter there are exercises on the topics covered, testing both mechanical ability and understanding.

The EMM course has a well-defined target audience of students in America, but unfortunately the same cannot be true in Britain. EMM goes beyond GCSE requirements, but I think it will also be inadequate for any A-level course that requires some mathematics, where a good A-level textbook is a better reference. This volume, then, is perhaps most suited to the non-mathematician who wishes to discover something about the most basic applications of mathematics.

Student,
St Olave's Grammar School, Orpington. ANDREW LOBB

Magic Tricks, Card Shuffling and Dynamic Computer Memories. By S. BRENT MORRIS. MAA, Washington, 1998. Pp. 150. Paperback £16.95 (ISBN 0-88385-527-5).

S. Brent Morris is believed to have the only doctorate in the world on card shuffling; he has based this book around one type of shuffle — the 'perfect shuffle'.

The first couple of chapters are spent introducing the reader to this technique, and to some of its properties. For instance, did you know that shuffling a deck of cards eight times can restore it to its original order? The mathematics of the shuffle is developed from the first chapter, and, for every property mentioned, a mathematical explanation is offered.

Each chapter opens with a description of a card trick, the mathematics of which is discussed over the succeeding pages. The chapter finishes with a complete description of how to perform the trick, and how it works. This structure makes the book accessible to a wide audience; the mathematician can concentrate on the sometimes quite involved permutation mathematics, while the trainee magician can skip some of the more complicated proofs and explanations, and still learn the trick.

The entire book is presented in a very clear manner; the mathematics has been kept as intelligible as possible, and each new formula or theorem has a full explanation alongside it. The text is easy to read, being written in a very conversational style, and it is possible to skip the more complex sections without losing any of the meaning.

One of the most interesting sections is Morris' application of the perfect shuffle principle to the workings of computer memories; it is fascinating to see the same mathematics applied to electronics as is applied to card tricks!

This is a well-written book, which is easy to dip into and offers a new perspective on permutation mathematics. It is essentially a bit of fun, and, provided that is all you are looking for, I would heartily recommend it.

Student, Gresham's School, Holt. KIERAN GILLICK

Achilles in the Quantum Universe: The Definitive History of Infinity. By RICHARD MORRIS. Souvenir Press Ltd, London, 1998. Pp. xii+224. Hardback £18.99 (ISBN 0-285-63439-9).

Ever since Zeno proposed his famous 'proof' that motion is impossible since it involves an infinite series of acts, each halving the distance to the destination, many astronomers, mathematicians, physicists and philosophers have found to their cost that the concept of infinity needs to be treated with extreme caution. Adopting a chronological approach, this book is about how our understanding of infinite time and space has gradually evolved. It uses this as a background to outline the current theory of atomic structure, quantum mechanics, the speed of light, relativity, black holes, cosmology, models of our universe, and the possibility of other universes.

Although purely mathematical ideas such as limits, differential calculus and transfinite numbers are mentioned, the emphasis is on infinity in the physical sciences. The obligatory $e = mc^2$ aside, readers will find scarcely a single equation: as the eye-catching title suggests, this is part of a welcome trend of books introducing science in simple terms to the general public. The style is informal and readable, although occasional baseball analogies betray its American origins.

This book succeeds in providing a clear overview of some of the basic principles of current physics, and is therefore to be recommended to beginners with more interest than knowledge. However, the nature of the subject may prevent it from being understood in much depth by a non-specialist audience, and many of the ideas considered remain too challenging to be fully explained in a work such as this. Some recommendations for further reading (rather than simply a list of the author's numerous other works) might therefore have been a useful addition. Yet the final test of a book such as this is whether it leaves the reader wanting to discover more, and in this respect I certainly feel it succeeds.

Student, Nottingham High School. JEREMY YOUNG

The Jungles of Randomness: A Mathematical Safari. By IVARS PETERSON. Penguin Press, 1998. Pp. 239. Paperback £11.99 (ISBN 0140271724).

In *The Jungles of Randomness*, Ivars Peterson takes us on a fascinating journey into the order and disorder in the world around us and argues that only a fine line separates the two. He reveals patterns that are hidden in apparent randomness as well as the surprising chaos embedded in apparent order. Well, that's what it says on the backcover!

I have found that this book serves as an introduction into the exciting world of recent research, which applies modern mathematical techniques to a variety of seemingly complex, scientific questions. Although only lightly touching upon the mathematics behind the problems, the topics covered are, on the whole, extremely interesting and thought provoking. *The Jungles of Randomness* is certainly an accessible book for the non-mathematician, at whom it is clearly aimed, and it belongs in the growing niche of popular mathematics books.

A lively preface focusing on Arthur Stanley Eddington's (1882–1944) comment that 'if an army of monkeys were strumming on typewriters, they *might* write all the books in the British Museum', is followed by ten chapters, each on a different subject, but linked by the common theme of randomness. Each chapter begins with a quotation from a non-mathematician; a pleasant touch which keeps the book informal and encourages one to read on.

I enjoyed some chapters more than others, especially the one explaining the error-correcting codes present on barcodes, CDs and credit cards. These make it possible to drill a hole 3 mm wide in a CD and suffer no loss in sound quality! Another chapter described 'random walks' in one, two and three dimensions. An intriguing fact I discovered was that in three dimensions, even if a walker takes *infinitely* many steps (north, south, east, west, up and down) at random, the probability of ever returning to the origin is only 0.34. For one or two dimensions, the walker returns an infinite number of times! The final chapter, which particularly interested me, revealed amazing coincidences; the most startling being the 'fifteen-year-old boy [who] caught a ten-pound cod in a Norwegian fjord and presented the fish to his grandmother for preparation. When the grandmother opened up its stomach, she found inside a diamond ring — a family heirloom that she had lost while fishing in the fjord ten years earlier.' Apparently these chance occurrences are much more common than one would imagine. *The Jungles of Randomness* ends with the frightening thought that randomness even lies at the very foundations of pure mathematics.

The book contains something for everyone: the biologist would be fascinated in the chapter on virus shells with unique coat proteins, the ecologist in how Asian fireflies synchronise their flashes, the physicist in the different shapes of drums which produce indistinguishable sound waves, and the gambler with the probabilities concerning dice, coins, cards and slot machines. *The Jungles of Randomness* is filled with illustrations enhancing the text which could otherwise become dry at points.

I heartily recommend this book, not for the mathematical content, which is limited at times (although the main text is supplemented by a small appendix), but as a general interest book catering for a broad spectrum of people.

Student, Nottingham High School. ANDREW HOLLAND

Life's Other Secret. By IAN STEWART. Allen Lane (Penguin), UK, 1998. Pp. 285. Hardback £20 (ISBN 0-7139-9161-5).

The vast majority of readers of the *Mathematical Spectrum* will have come across Ian Stewart at some stage, either through some of his many books like *Does God Play Dice?* or his appearances on TV and radio.

His latest book explores the relationship between mathematics and biology. In Stewart's own words his aim is to convince us 'that wonderful as genes are, they are not the whole answer to the question of life. More radically, I am going to try to convince you that a full understanding of life depends on mathematics.' To do this we are effectively given a prospectus for 'biomathematics', pointing out where mathematical theory is being used and where there is scope for further development.

The examples given range from group theory and how animals walk and run to topology in DNA coiling via the formation of tiger stripes, as well as including more obvious topics such as neural networks and computer simulations of evolution.

Stewart has two main points: one is that, certainly amongst the general population and to an extent among biologists, too much stress is put on genetics being the complete answer. We need to remember that DNA is effectively a very long recipe and requires the role of physics and chemistry (and their mathematical laws) as the chefs. The other major premise of this book is that at present scientists usually expect too much when applying mathematics to biology and discard theories when they are not perfect matches for everything that is being modelled, whereas it would be more beneficial to take the useful information we can get and accept the limitations.

This, as is usual for a book by Ian Stewart, makes fascinating reading. He does occasionally overstress his point, which can be wearisome, although possibly necessary. All in all, however, *Life's Other Secret* is a fascinating look at a fairly new and rapidly developing area of mathematics and is well worth reading.

Student, Trinity Hall, Cambridge. IAN GLOVER

Mathematics: The New Golden Age, 2nd edition. By KEITH DEVLIN. Penguin, UK, 1998. Pp. 292. Paperback £9.99 (ISBN 0-14-025865-5).

This is a revised edition of Devlin's book which was originally published ten years ago. It has been updated to take account of some major recent advances, most notably the proof of Fermat's Last Theorem.

The book covers a wide variety of topics with eleven chapters on prime numbers and cryptography, group theory, set theory, chaos, algorithms, the four colour problem, the Riemann hypothesis, diophantine equations, number systems, topology and Fermat's Last Theorem. All of these are well written and pitched at a level accessible to an interested GCSE or A-level student without being condescending to more advanced readers.

Each chapter starts with a brief tour of the basic principles and history of the subject and then moves on to give a summary of recent research and where it might lead in the future. It is a good book for anyone considering a university course in mathematics, as most of the topics mentioned would be met again and extended in any undergraduate course. One feature that will be appreciated by readers of this magazine is Devlin's inclusion of a bibliography at the end of each chapter. This gives books suitable for those who wish to see the mathematics developed in more detail, and are at the right level for readers of *Mathematical Spectrum*.

For those who have not read it before, especially those considering studying mathematics at university, this is an interesting and entertaining book that is well worth reading. For those who have read it previously, there is insufficient new material to justify buying it, but they may wish to look briefly at the account of Wiles' proof of Fermat's Last Theorem and at the end of the topology chapter.

Student, Trinity Hall, Cambridge. IAN GLOVER

Twenty Years Before the Blackboard: The Lessons and Humor of a Mathematics Teacher. By MICHAEL STUEBEN with DIANE SANDFORD. MAA, Washington, 1998. Pp. 174. Paperback $29.50 (ISBN 0-88385-525-9).

'This book ... contains truths about teaching high school mathematics that took me nearly two decades to discover.' Among the truths are advice for teachers to 'learn your pupils' by studying them, compliment your students, do not whine, almost never be critical, if you have to make students do things try to make them want to do them and seek to build up a good relationship with your class.

What is the goal and purpose of these mathematics lessons? The author's answer is that students take into their lives, not so much the algebra and geometry, as self-discipline, responsible attitudes, persistence, love of learning, respect for others, honest self-analysis and the self-esteem that comes from meeting rigorous challenges. In the chapter on problem solving, we find that people do not learn important lessons from words: they learn from experience.

'Nothing is so unequal as the equal treatment of unequals.' The teacher must be seen as being fair, but in order to achieve this, the teacher must actually be more than fair in some cases. Teachers must be flexible; absolute rules are a device of convenience to escape the difficulty of decision when an exceptional case occurs.

'Never smile until after Christmas' is one of the truths in the chapter on humour in the classroom. We also find the following joke which illustrates indirect proof. A man comes to a psychiatrist and claims to be dead. The psychiatrist asks him, 'Do dead men bleed?' 'Of course not', replies the patient. The psychiatrist pricks the patient's finger with a needle and squeezes out a drop of blood. 'Okay, you've convinced me that I was wrong,' says the patient. 'Dead men do bleed.'

The chapter on mathematics education begins, 'The following criticisms are mostly cynical, uncomplimentary, and, I believe, truthful. In a different time, I might have been fired and blacklisted for distributing this material.'

The author quotes from a wide selection of people — from Isaac Asimov and Albert Schweitzer to H. L. Mencken and Albert Einstein. For example, we have Paul Halmos's advice to those lecturing on theorems, 'Proofs can be looked up. Contexts, histories, motivations and applications are harder to find — that's what teachers are really for.'

I have indicated the wisdom contained in this book by picking some of the truths from the first part, which is entitled 'Teaching'. The second, slightly shorter, part is 'The Scrapbook' and consists of examples of mathematical and academic humour, mnemonics and the author's favourite proofs. I quote one joke from this part.

> Two mathematicians were arguing in a restaurant about what mathematics the average person knew. When one went to the cloakroom the other called the waitress over and asked if she would reply to his next question by saying, 'x squared'. She agreed. When the absent mathematician returned, the other suggested they do a test. He called the waitress over and asked her if she knew what the integral of $2x$ was. She replied, 'x squared'. The mathematician turned to his friend, 'You see, people do know some mathematics'. The waitress continued, 'Plus a constant'.

Michael Stueben is to be congratulated and thanked for sharing the fruits of his twenty years as a teacher with us and I am glad of this opportunity to recommend his excellent book to others.

University of Sheffield. KEITH AUSTIN

Other books received

Essential Quantitative Methods for Business, Management and Finance. By LES OAKSHOTT. Macmillan Press Ltd., Basingstoke, UK, 1998. Pp. xx+297. Softback $14.99 (ISBN 0-333-72797-5).

Groups, Representations and Physics. By H. F. JONES. IOP Publishing, Bristol, 1998. Pp. viii + 326. Softback £23.00 (ISBN 0-7503-0504-5).

Incomplete Tilings. By PATRICK TAYLOR. Nattygrafix, Ipswich, 1998. Pp. 75. Softback £5.00 (ISBN 0-9516701-3-1).

Foundation mathematics for GCSE. By BRIAN SPEED, KEVIN EVANS AND KEITH GORDON. Collins Educational, London, 1998. Pp. 508. Softback £12.50 (ISBN 0-00322472-4).

Mathematics for GNVQ+GCSE. By LESLYE BUCHANAN AND BRIAN GAULTER. OUP, Oxford, 1998. Pp. 568. Paperback £13.00 (ISBN 0-19914-727-2).

Test Booster: Mathematics National Tests Key Stage 3. By KEVIN EVANS AND KEITH GORDON. Collins Educational, London, 1999. Pp. vi+122. Paperback £5.99 (ISBN 0-00-323520-3).

Mathematical Spectrum

1999/2000 Volume 32 Number 1

1 A generalized Argand diagram
GUIDO LASTERS and DAVID SHARPE

5 The 18th century Chinese discovery of the Catalan numbers
P. J. LARCOMBE

7 Sums of powers
ROGER COOK

10 Throwing elliptical shields on floorboards
P. GLAISTER

14 Mathematics in the classroom

15 Computer column

16 Letters to the editor

19 Problems and solutions

22 Reviews

ISSN 0025-5653

Published by the Applied Probability Trust
Printed by Pear Tree Press Ltd, Stevenage, Herts, UK