Indian Team Selection Test 2004

Practice Tests

Day 1

1 Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to BC, CA, AB, respectively. Prove that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

2 Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

3 For positive real numbers a, b, c find the minimum value of

$$\frac{a^2+b^2}{c^2+ab}+\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}.$$

4 Given a permutation $\sigma = (a_1, a_2, a_3, \dots a_n)$ of $(1, 2, 3, \dots n)$, an ordered pair (a_j, a_k) is called an inversion of σ if $a \le j < k \le n$ and $a_j < a_k$. Let $m(\sigma)$ denote the no. of inversions of the permutation σ. Find the average of $m(\sigma)$ as σ varies over all permutations.

Day 2

1 Prove that in any triangle ABC,

$$0 < \cot\left(\frac{A}{4}\right) - \tan\left(\frac{B}{4}\right) - \tan\left(\frac{C}{4}\right) - 1 < 2\cot\left(\frac{A}{2}\right).$$

2 Find all triples (x, y, n) of positive integers such that $(x+y)(1+xy) = 2^n$.

3 Suppose the polynomial $P(x) = x^3 + ax^2 + bx + c$ has only real zeroes and let $Q(x) = 5x^2 - 16x + 2004$. Assume that $P(Q(x)) = 0$ has no real roots. Prove that $P(2004) > 2004$.

4 Let f be a bijection of the set of all natural numbers on to itself. Prove that there exist positive integers a and d such that

$$f(a) < f(a+d) < f(a+2d).$$

Selection Tests

Day 1

1 A set A_1, A_2, A_3, A_4 of 4 points in the plane is said to be Athenian set if there is a point P of the plane satsifying

(i) P does not lie on any of the lines A_iA_j for $1 \le i < j \le 4$;

(ii) The line joining P to the mid-point of the line A_iA_j is perpendicular to the line joining P to the mid-point of A_kA_l, (i, j, k, l are distinct).

(a) Find all Athenian sets in the plane.

(b) For a given Athenian set, find the set of all points P in the plane satisfying (i) and (ii).

2 Determine all integers a such that $a^k + 1$ is divisible by 12321 for some k.

3 The game of *pebbles* is played on an infinite board of lattice points (i, j). Initially there is a *pebble* at $(0,0)$. A move consists of removing a *pebble* from point (i, j) and placing a *pebble* at each of the points $(i+1, j)$ and $(i, j+1)$ provided both are vacant. Show taht at any stage of the game there is a *pebble* at some lattice point (a,b) with $0 \le a+b \le 3$.

Day 2

1 Let ABC be a triangle, and P a point in the interior of the triangle. Let D, E, F be the feet of the perpendiculars from P to the sides BC, CA, AB. Assume that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Let I_a, I_b, I_c be the excenters of triangle ABC. Show that P is the circumcenter of triangle $I_aI_bI_c$.

2 Show that the only solutions of the equation $p^k + 1 = q^m$, in positive integers $k, q, m > 1$ and prime p are

(i) $(p, k, q, m) = (2, 3, 3, 2)$

(ii) $k = 1, q = 2$, and p is a prime of the form $2^m - 1$, $m \in \mathbb{N} \setminus \{1\}$

3 Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x+y) = f(x)f(y) - c\sin x \sin y \quad \text{for all reals } x, y,$$

where $c > 1$ is a given constant.

Day 3

1 Let ABC be a triangle and I its incentre. Let ρ_1 and ρ_2 be the inradii of triangles IAB and IAC respectively.

(a) Show that there exists a function $f : (0, \pi) \mapsto \mathbb{R}$ such that

$$\frac{\rho_1}{\rho_2} = \frac{f(\angle C)}{f(\angle B)}.$$

(b) Prove that

$$2(\sqrt{2} - 1) < \frac{\rho_1}{\rho_2} < \frac{1+\sqrt{2}}{2}.$$

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2 Define a function $g : \mathbb{N} \mapsto \mathbb{N}$ by the following rule:

(i) g is nondecrasing;

(ii) for each n, $g(n)$ is the number of times n appears in the range of g.

Prove that $g(1) = 1$ and $g(n+1) = 1 + g(n+1-g(g(n)))$ for all $n \in \mathbb{N}$.

3 Two runners start running along a circular track of unit length from the same starting point and in the same direction, with constant speeds v_1 and v_2 respectively, where v_1 and v_2 are two distinct relatively prime natural numbers. They continue running till they simultneously reach the starting point. Prove that

(a) At any given time t, at least one of the runners is at a distance not more than

$$\frac{[\frac{v_1+v_2}{2}]}{v_1+v_2}$$

units from the starting point.

(b) There is a time t such that both the runners are at least $\frac{[\frac{v_1+v_2}{2}]}{v_1+v_2}$ units away from the starting point. (All disstances are measured along the track).

Day 4

1 Let $x_1, x_2, x_3, \dots .x_n$ be n real numbers such that $0 < x_j < \frac{1}{2}$. Prove that

$$\frac{\prod\limits_{j=1}^{n} x_j}{\left(\sum\limits_{j=1}^{n} x_j\right)^n} \leq \frac{\prod\limits_{j=1}^{n}(1-x_j)}{\left(\sum\limits_{j=1}^{n}(1-x_j)\right)^n}.$$

2 Find all primes $p \geq 3$ with the following property: for any prime $q < p$, the number

$$p - \left\lfloor \frac{p}{q} \right\rfloor q$$

is squarefree (i.e. is not divisible by the square of a prime).

3 For each point with integer coordinates in a plane, consider a circular disk centered at this point and having the radius $\frac{1}{1000}$.

(a) Prove that there exists an equilateral triangle whose vertices lie in the interior of different disks;

(b) Show that every equilateral triangle whose vertices lie in the interior of different disks has a sidelength > 96.

Day 5

1 Let ABC be an acute-angled triangle and Γ be a circle with AB as diameter intersecting BC and CA at $F(\neq B)$ and $E(\neq A)$ respectively. Tangents from E and F to Γ intersect at P. Show that the ratio of the circumcentre of triangle ABC to that of EFP is a rational number.

2 Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ and $Q(x) = x^2 + px + q$ be two real polynomials. Suppose that there exist an interval (r,s) of length greater than 2 such that both $P(x)$ and $Q(x)$ are negative for $x \in (r,s)$ and both are positive for $x > s$ and $x < r$. Show that there is a real x_0 such that $P(x_0) < Q(x_0)$

3 An integer n is said to be good if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented in infinitely many ways as a sum of three distinct good integers whose product is the square of an odd integer.