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# CRUX MATHEMATICORUM

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*Editor: Léo Sauv , Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.*

*Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.*

*Typist-compositor: Nancy Makila.*

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For Vol. 4 (1978), the support of Algonquin College and of the Samuel Beatty Fund is gratefully acknowledged.

## A PROBLEM OF PIETER NIEUWLAND<sup>1</sup>

O. BOTTEMA

Pieter Nieuwland (1764-1794) is well-known in the history of Dutch literature, mainly for his poem *Orion*. He was a gifted and many-sided young man who at an early age was appointed as a lecturer for mathematics, astronomy and navigation at the "Illustrious School" of Amsterdam, and in 1793 as a professor at the University of Leyden. He could not, in his short life, contribute many important achievements to science, but his name is connected with the solution of a curious problem, on the boundary line between serious mathematics and mathematical recreations. It reads as follows: is it possible to make a hole in a given cube in such a way that a cube of equal size (or a somewhat larger one) may pass through it?

J.H. van Swinden [1, p. 512] tells us that the problem had been formulated by "Prince Rupert from the illustrious house of the palatine counts, an exceedingly clever man," a great-grandson of William the Silent. Wallis gave "a solution, making use of the geometric rules."

To Nieuwland goes the credit for determining the dimension of the largest cube which can pass through a given one. His solution is published in an appendix to van Swinden's text [1, pp. 608-610] and in this way it has become widely known [2]. It must be said that the comments of van Swinden are clearly written and that he has added some good figures, but the appendix itself is not easy reading.

Nieuwland's proof is reproduced below. To simplify matters we use the terminology of trigonometry and elementary analytic geometry. The proof is not completely satisfactory because the author restricts himself to holes of a certain (symmetric) type. His plan is to construct a prismatic tube with a square cross section, of which he supposes that two of its faces are parallel to a diagonal plane of the cube and that all four faces are equally distant from its center.

Let ABCD-EFGH be the given cube, with edge  $2a$ , and let ACGE be the above-mentioned diagonal plane (see Figure 1). We take a Cartesian frame as indicated, with the origin at the center; the diagonal plane is  $x = 0$ , the upper plane is  $z = a$  and the lower plane  $z = -a$ .

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<sup>1</sup>We published a few months ago [1977: 244] a historical article on Prince Rupert's Cubes by Sahib Ram Mandan. The proof of the result therein described, which was not included in the earlier article, is given here in an article reprinted, with the permission of the author and the editors, from *Euclides*, 45 (1969-70) 105-109. The article was translated from the Dutch by the author.

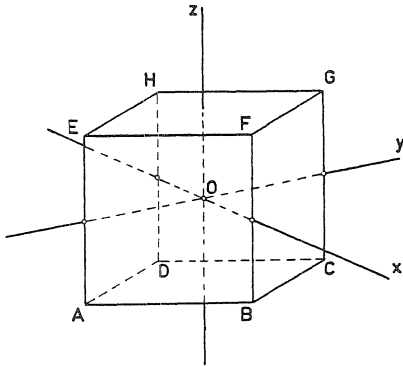


Figure 1

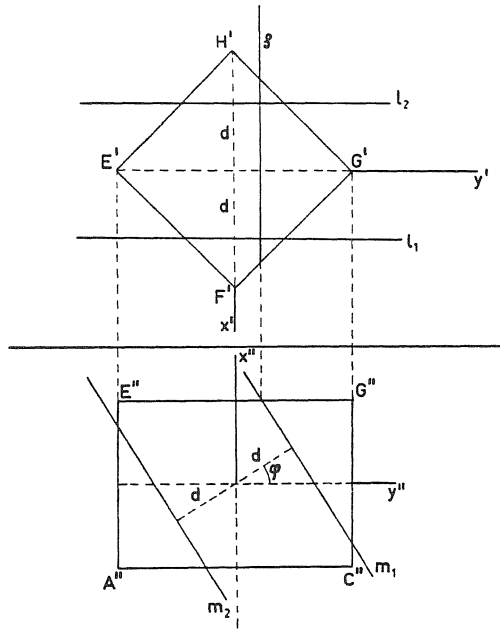


Figure 2

Let  $2d$  be the side of the square section of the tube. The equations of two of its faces,  $V_1$  and  $V_2$ , are  $x = d$  and  $x = -d$ . The other two,  $W_1$  and  $W_2$ , must be perpendicular to the diagonal plane and their common distance to  $O$  must be  $d$ . Their equations are therefore

$$y \cos \phi + z \sin \phi = d, \quad y \cos \phi + z \sin \phi = -d, \quad (1)$$

respectively. Here  $\phi$  is the angle between  $OY$  and the normal from  $O$  onto  $W_1$ ; it is the parameter at our disposal to make  $d$  as large as possible. Obviously, we may restrict ourselves to the interval  $0 \leq \phi \leq \pi/2$ .

The intersections of  $V_1$ ,  $V_2$  and the upper plane are  $l_1$ ,  $l_2$  (see Figure 2), which are parallel to  $EG$  and at the distance  $d$  from it. The intersections of  $W_1$ ,  $W_2$  and the diagonal plane are  $m_1$ ,  $m_2$  at the distance  $d$  from  $O$ ; and that of  $W_1$  and the upper plane is  $s$ . If  $m_1$  intersects the lower plane at a point between  $A$  and  $C$  (and therefore  $m_2$  the upper plane at a point between  $E$  and  $G$ ), we are not very successful. In this case the intersections of the tube and the upper and lower planes are rectangles, and the best value of  $d$  would be that for  $\phi = 0$ , which is  $d = \frac{1}{2}a\sqrt{2}$ . We shall therefore investigate the situation shown in Figure 2, where  $m_1$

intersects CG and  $m_2$  intersects AE. Then from the upper plane all is taken away which lies between  $l_1$  and  $l_2$  and to the left of  $s$ , with a similar situation in the lower plane. To be sure that the cube shall not be disconnected, the intersections  $S_1, S_2$  of  $s$  and  $l_1, l_2$  must lie inside the upper face of the cube. The equations of  $s$  are

$$z = a, \quad y \cos \phi + z \sin \phi = d.$$

For  $S_1$  we have

$$x = x_1 = d, \quad y = y_1 = \frac{d - a \sin \phi}{\cos \phi}.$$

The condition is therefore  $x_1 + y_1 \leq a\sqrt{2}$ . In the extreme case, with  $S_1$  on FG, we obtain

$$(a\sqrt{2} - d) \cos \phi + a \sin \phi = d, \quad (2)$$

which has a solution if and only if

$$(a\sqrt{2} - d)^2 + a^2 \geq d^2 \quad (3)$$

or

$$d \leq \frac{3}{4}a\sqrt{2}. \quad (4)$$

Hence we have Nieuwland's theorem:

*The largest cube which can pass (in the indicated way) through a hole in a given cube has an edge equal to  $3/4$  of a face diagonal of the latter.*

In the extreme case, equation (2) has the solution  $\phi = \phi_0$ , with  $\cos \phi_0 = \frac{1}{3}$ . If  $d = a$ , that is, if the second cube is as large as the given cube, then equation (2) reads

$$(\sqrt{2} - 1) \cos \phi + \sin \phi = 1, \quad (5)$$

with the elegant solution  $\phi = \pi/4$ . Then we have for  $m_1$  the equation  $y + z = a\sqrt{2}$ , which implies that  $W_1$  passes through the midpoint of CG (and  $W_2$  through the midpoint of AE), a remark not mentioned by Nieuwland.

Figure 3 tries to give an impression of the situation (with  $d = a$ ). From the upper and lower planes pentagons are cut off, and quadrilaterals from the vertical faces. The edges of the hole form skew heptagons.

A philosophical essay was recently published on the meaning of holes in sculptural art since Zadkine. A material model of Nieuwland's configuration would be an acceptable piece in any museum of modern art; in the catalogue it could be called *Terre neuve*, in honor of the man who seems to have been the first to solve the problem.

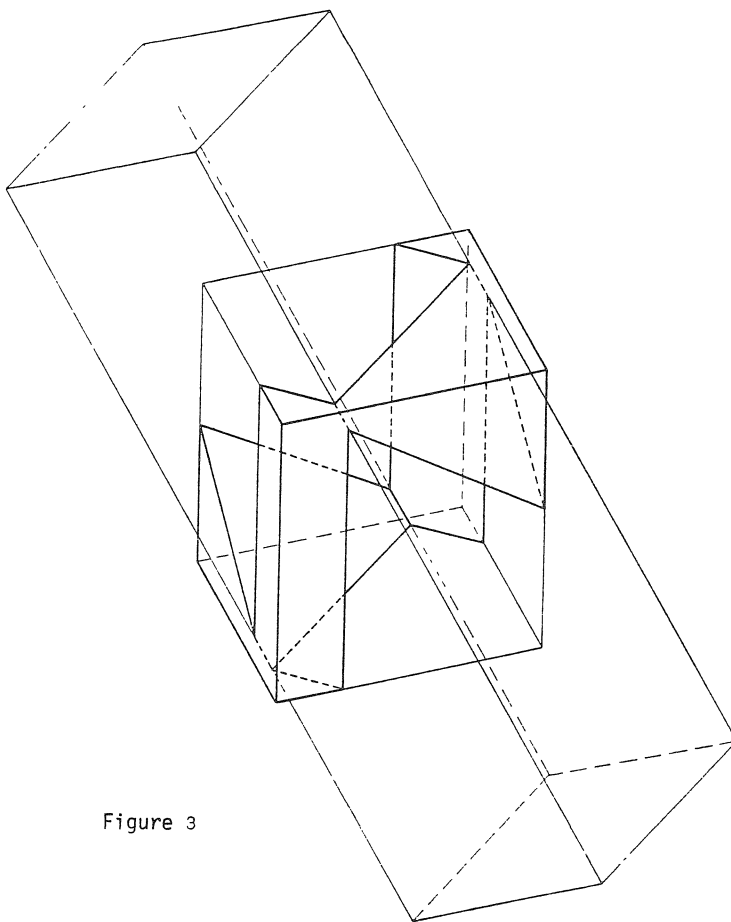


Figure 3

#### REFERENCES

1. J.H. van Swinden, *Grondbeginsels der meetkunde* (Principles of Geometry), Second Edition, Groningen, 1816.
2. M. Cantor was familiar with van Swinden's book (a German edition having been published in 1834). He gives some attention to the problem in his *Vorlesungen über Geschichte der Mathematik* (3. Band, 2. Auflage, Leipzig, 1901, p. 25), but Nieuwland's name is not mentioned. This is, however, done by M. Simon, *Ueber die Entwicklung der Elementar-Mathematik im XIX Jahrhundert* (Leipzig, 1906, p. 213) and in the *Encyklopädie der mathematischen Wissenschaften* (Band III, 1, Heft 6, Leipzig, 1920, Zacharias, p. 1134).

Technische Hogeschool Delft, Julianalaan 134, Delft, Nederland.

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## ALGORITHMS AND POCKET CALCULATORS:

### SQUARE ROOTS: I

CLAYTON W. DODGE

Have you ever wondered what happens inside a pocket calculator when you press a function button? Just how does a calculator find  $\sqrt{x}$  or  $\sin x$  or  $\ln x$ ? It is tempting to believe there is a little man inside who rapidly looks up the desired value in a table and then displays the result. Besides, then we can blame *him* when *we* get a wrong answer.

Alas! There is no such little man (or woman); only some very tiny electronics. And if such electronics were to be replaced by vacuum tubes, using the technology of 1948 in place of that of 1978, the works of a modest pocket calculator would fill a fair-sized house and would use enough electric power to keep an uninsulated house quite warm in the coldest winter weather. But thanks to today's microelectronics, more than 7000 transistors can be printed on one tiny silicon chip smaller than your little fingernail, to be powered by just milliwatts. In fact, nearly all of the  $\frac{1}{2}$  watt required to power a pocket calculator is used to light the display.

We shall examine how square roots are found in computers and calculators, but first let us review two square root algorithms, the long division method used in high schools and Newton's "divide-and-average" method.

Let us calculate  $\sqrt{18468.81}$  by the long division method. Recall that we first divide the digits into sets of twos starting at the decimal point, obtaining

$$\sqrt{1\ 84\ 68.81}.$$

Now we find the largest integer whose square does not exceed the number formed by the first group (or two groups) of digits. In this case  $13^2 < 184$  and  $14^2 > 184$ , so we choose 13. (We could have chosen  $1^2 = 1$ , using just the 1 in the first group, but using the 184 shortens the work.) Write the 13 above the 184 and subtract its square from the 184:

$$\begin{array}{r} 1\ 3 \\ \sqrt{1\ 84\ 68.81} \\ \underline{1\ 69} \\ 15 \end{array}$$

We have found that

$$130 < \sqrt{18468.81} < 140$$

since

$$130^2 = 16900 < 18468.81 < 19600 = 140^2.$$

Hence, letting  $n = 18468.81$  and  $a = 130$ , we seek  $b$  so that

$$(a+b)^2 = n.$$

The division process so far indicates that

$$n - a^2 = (a+b)^2 - a^2 = 2ab + b^2 = 1568.81,$$

so we must now try to find an appropriate value for  $b$ .

We shall find just one digit at a time, as is customary. Then  $b$  is a digit, so  $(a+b)^2$  will be an integer and we bring down just the next group of two digits. We wish to find as large a value of  $b$  as possible such that

$$b(2a+b) = 2ab + b^2 \leq 1568.$$

Since  $b$  is small relative to  $a$ , we first find  $b$  such that

$$b(2a+0) = 2ab \leq 1568.$$

That is, we use  $2a = 260$  as a trial divisor; we write  $2 \cdot 13 = 26$ , with a space beside the 6 for another digit, as the trial divisor:

$$\begin{array}{r} 1 \quad 3 \\ \sqrt{1 \quad 84 \quad 68.81} \\ 1 \quad 69 \\ 26 \quad 6 \quad | \quad 15 \quad 68 \end{array}$$

The quotient 6 we write over the next group of digits, beside the 26, and immediately below it:

$$\begin{array}{r} 1 \quad 3 \quad 6. \\ \sqrt{1 \quad 84 \quad 68.81} \\ 1 \quad 69 \\ 266 \quad 6 \quad | \quad 15 \quad 68 \\ \times 6 \quad | \end{array}$$

Now multiply the 266 by 6 and subtract the product from 1568. Unfortunately, as occurs occasionally because our trial divisor was too small, the product  $266 \cdot 6 = 1596$  is larger than 1568, so  $b$  was taken too large; we must reduce it. Thus we try  $b = 5$ :

$$\begin{array}{r} 1 \quad 3 \quad 5. \\ \sqrt{1 \quad 84 \quad 68.81} \\ 1 \quad 69 \\ 265 \quad 5 \quad | \quad 15 \quad 68 \\ \times 5 \quad | \quad 13 \quad 25 \\ \hline 2 \quad 43 \end{array}$$

Now we have found that

$$135^2 = 18225 < 18468.81 < 18496 = 136^2$$



and

$$18468.81 - 135^2 = 243.81,$$

because

$$(a+b)^2 = a^2 + b(2a+b) = 16900 + 1325 = 18225.$$

We apply the process again, this time taking  $a=135$  and  $b$  a whole number of tenths. We wrote the 265 and 5 in convenient positions for multiplying. Now we make use of these same positions to calculate the new trial divisor  $2a$ , since

$$2a = 2 \cdot 135.0 = 270.0 = 265 + 5.$$

Thus, just add the previous divisor, the old  $2a+b$ , to the old  $b$  value to get the new trial divisor. We have, after bringing down the next group of digits from the first line:

$$\begin{array}{r} 1 \quad 3 \quad 5. \\ \sqrt{1 \quad 84 \quad 68.81} \\ 1 \quad 69 \\ \hline 265 \overline{) 15 \quad 68} \\ + 5 \quad 13 \quad 25 \\ \hline 270 \overline{) 2 \quad 43 \quad 81} \end{array}$$

The quotient when 24381 is divided by 2700 is greater than 9, so we try  $b=9$ . Again write the 9 in its three locations, multiply 2709 by 9, and subtract the product from the 24381. We see that the root obtained is exact,<sup>1</sup> so the process terminates:

$$\begin{array}{r} 1 \quad 3 \quad 5. \quad 9 \\ \sqrt{1 \quad 84 \quad 68.81} \\ 1 \quad 69 \\ \hline 265 \overline{) 15 \quad 68} \\ 5 \quad 13 \quad 25 \\ \hline 2709 \overline{) 2 \quad 43 \quad 81} \\ 9 \quad 2 \quad 43 \quad 81 \\ \hline 0 \end{array}$$

We have found that

$$(135 + .9)^2 = 135^2 + .9 \cdot (270 + .9) = 18225 + 243.81 = 18468.81,$$

$$\text{so } \sqrt{18468.81} = 135.9.$$

I recall being told in high school that this algorithm was based on the equation

$$(a+b)^2 = a^2 + 2ab + b^2,$$

---

<sup>1</sup>It is common to say that it "comes out even." It, of course, does not: it comes out odd in this case. We prefer to say that it "is exact." Thus 2 divides 6 exactly, but not evenly; 2 divides 8 evenly.

but I believe that no further explanation was given; it was not until several years later that I satisfied myself of the truth of that assertion. So the above illustration is offered primarily for those who find themselves in that same position. It may be instructive to develop a corresponding algorithm for cube roots based, of course, upon the equation

$$(a+b)^3 = a^3 + b \cdot (3a^2 + 3ab + b^2).$$

Next time we shall examine Newton's method applied to square roots.

Mathematics Department, University of Maine, Orono, Maine 04473.

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### POTPOURRI

1. The Tenth Canadian Mathematical Olympiad will be held on Wednesday, May 3, 1978 at schools throughout Canada. The Canadian Mathematical Congress has decided once again to award one-year subscriptions to CRUX MATHEMATICORUM to the *teachers* of all Olympiad candidates who receive either a prize or an honourable mention. After the 1977 Olympiad, the Congress awarded gift subscriptions to the teachers of 34 candidates from 33 schools.

2. The Fifth Annual Conference of the Ontario Association for Mathematics Education will be held on May 5-7, 1978 at McMaster University. Registration fee is \$10 for O.A.M.E. members. Send registration requests to:

OAME '78,  
Dept. of Mathematics,  
McMaster University,  
Hamilton, Ont. L8S 4K1.

3. The Department of Mathematics and Statistics of Miami University, Oxford, Ohio, announces a Sixth Annual Conference to be held on September 29-30, 1978. Topic: Applications of statistics and mathematics. Featured speakers: George Carrier (Harvard University), Victor Klee (University of Washington), Frederick Mosteller (Harvard University). For more information, write to:

Charles Holmes or Elwood Bohn,  
Department of Mathematics and Statistics,  
Miami University,  
Oxford, Ohio 45056.

4. Extract from a letter recently sent to his subscribers by the Editor of *The Antioch Review*: "*The Antioch Review* is small enough to be innovative in content and design, to be diverse. Our advertising is minimal and discrete." (After all, who wants a magazine with continuous, wall-to-wall advertising?)

5. L.F. Meyers makes the following contribution to mathematical epidemiology. He found it in *Calculus*, by Lipman Bers with Frank Karal, Second Edition, Holt, Rinehart and Winston, 1976, p. 110: "If  $f''(x_0) = 0$  and  $f'$  changes sign at  $x_0$ , we say that  $(x_0, f(x_0))$  is a *point of inflection* of the graph." Meyers adds that one of his students once wrote, in answer to a question: "Inflection points, none."

6. L.F. Meyers also writes that reference [4] in [1978 30] can be completed by replacing (1973?) by (1973) 31-32.

(Continued on page 120)

# PROBLEMS - - PROBLÈMES

*Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (\*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.*

331. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Here is a problem to welcome CRUX MATHEMATICORUM.

Each distinct letter in this alphametic stands for a particular but different digit. It is certainly appropriate, but this unique TITLE is truly odd! So what do you make of it?

WELL  
WELL  
A  
NEW  
TITLE

332. *Proposed by Leroy F. Meyers, The Ohio State University.*

In the quadratic equation

$$A(\sqrt{3} - \sqrt{2})x^2 + \frac{B}{\sqrt{2} + \sqrt{3}}x + C = 0,$$

we are given:

$$A = \sqrt[4]{49 + 20\sqrt{6}};$$

B = the sum of the geometric series

$$8\sqrt{3} + (8\sqrt{6})(3^{-\frac{1}{2}}) + 16(3^{-\frac{1}{2}}) + \dots;$$

and the difference of the roots is

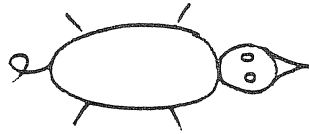
$$(\sqrt[6]{6}) \log 10 - 2 \log \sqrt{5} + \log \sqrt{\log 18 + \log 72},$$

where the base of the logarithms is 6. Find the value of C.

(This was the first of three problems in a final examination, 3 hours long, for Dutch high school students in 1916.)

333. *Proposed by R. Robinson Rowe, Sacramento, California.*

The World War I COOTIE, lousy vector of trench fever, popularized a simple but hilarious game by that name in the early 1920's. Five or more players, each with pad and pencil, cast a single die in turn. Rolling a 6, a player sketched a "body" on the pad (see figure) and on later turns added a head with a 5, four legs with a 4, the tail with a 3. Having the head, he could add two eyes with a 2 and a proboscis or nose with a 1. Having all six, he yelled "COO000TIEEEEE!" and raked in the pot.



What was the probability of capturing a COOTIE in just six turns?

334. *Proposed by Philip D. Straffin, Beloit College, Beloit, Wisconsin.*

Let  $A, B, C$  be three fixed noncollinear points in the plane, and let  $X_0$  be the centroid of  $\triangle ABC$ . Call a point  $P$  in the plane *accessible* from  $X_0$  if there is a sequence of points  $X_0, X_1, \dots, X_n = P$  such that  $X_{i+1}$  is closer than  $X_i$  to at least two of the points  $A, B, C$  ( $i = 0, 1, \dots, n-1$ ). Characterize the set of points in the plane which are accessible from  $X_0$ .

(One application is to the effect of agenda control on committee decisions. In this interpretation  $A, B, C$  are the preferred points of the three committee members in a two-dimensional policy space, and the  $X_i$  are proposals to be voted on by majority rule.)

335. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Trouver une condition nécessaire et suffisante pour que l'équation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

ait l'une de ses racines égale au carré de l'autre.

336. *Proposed by Viktors Linis, University of Ottawa.*

Prove that if in a convex polyhedron there are four edges at each vertex, then every planar section which does not pass through any vertex is a polygon with an even number of sides.

337. *Proposed by V.G. Hobbes, Westmount, Québec.*

If  $p$  and  $q$  are primes greater than 3, prove that  $p^2 - q^2$  is a multiple of 24.

338. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Can one locate the center of a circle with a VISA card?

339\*. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Is  $\binom{37}{2} = 666$  the only binomial coefficient  $\binom{n}{r}$  whose decimal representation consists of a single digit repeated  $k(\geq 3)$  times?

340. *Proposed by Léo Sauvé, Algonquin College (editor).*

(This is designed to help those readers who find it hard to shake the habit of sending in only answers to proposed problems.)

Find a problem whose answer is  $\frac{22}{7} - \pi$ .

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## S O L U T I O N S

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

248. [1977: 131, 154; 1978: 26] (Corrected) *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Circle (Q) is tangent to circles (O), (M), (N), as shown in Figure 1 [1978: 26], and FG is the diameter of (Q) parallel to diameter AB of (O). W is the radical center of circles (M), (N), (Q). Prove that WQ is equal to the circumradius of  $\triangle PFG$ .

III. *Comment by the proposer.*

As proved by Dodge in solution I, the stated property holds for any three circles (M), (N), (Q), externally tangent in pairs, whether or not there exists, as in Figure 1, a circle (O) to which all three are internally tangent (see text and Figure 2 [1978: 27]).

But the stated property is even more general, since it holds equally well for all other cases of mutual tangency of the three circles (M), (N), (Q). Specifically, the property also holds when

(a) (M) and (N) are both internally tangent to (Q) and externally tangent to each other;

(b) (M), say, and (Q) are both internally tangent to (N) and externally tangent to each other.

These facts can be established in several ways, but Dodge's method, with minor modifications, is perfectly adequate for the job.

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256. [1977: 155; 1978: 53] *Proposed by Harry L. Nelson, Livermore, California.*  
Prove that an equilateral triangle can be dissected into five isosceles

triangles,  $n$  of which are equilateral, if and only if  $0 \leq n \leq 2$ . (This problem was suggested by Problem 200.)

II. *Solution by the proposer.*

Johnson has shown [1978: 53] that the dissection is possible for  $n = 0, 1, 2$  and impossible for  $n = 5$ . We consider the cases  $n = 3, 4$ .

Suppose an equilateral triangle is dissected into five triangles, three of which are equilateral and have a vertex in common with the original triangle. (This *must* occur, in particular, when  $n = 4$ , if such a dissection is possible.) The remaining two triangles together must form a convex polygonal region of  $p$  sides, with  $p = 3, 4, 5$ , or  $6$ , as shown in Figure 1.

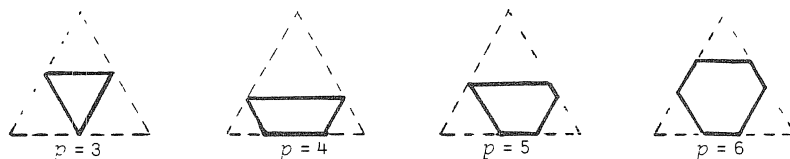


Figure 1

The cases  $p = 5, 6$  can be immediately excluded, since the polygonal region cannot be dissected into two triangles of any kind. If the two remaining triangles are isosceles (equilateral or not), then the case  $p = 3$  is clearly impossible; and the case  $p = 4$  (see Figure 2, where it is assumed that the original triangle has a side of length 1) would require a common solution to the two incompatible equations

$$x = 1 - x, \quad x = 1 - 2x.$$

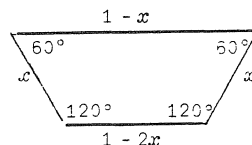


Figure 2

We conclude that the case  $n = 4$  is impossible, and that if the case  $n = 3$  is possible then exactly one angle of the original triangle must be divided in the dissection.

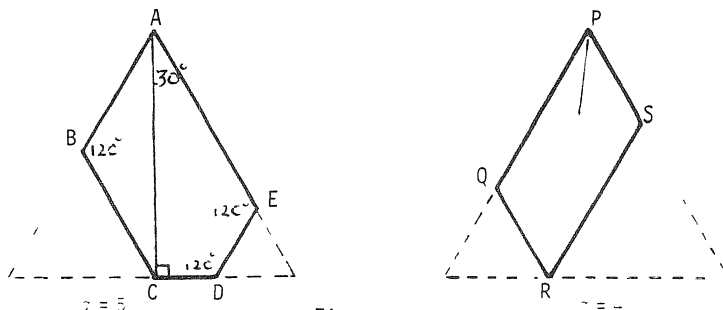


Figure 3

For the case  $n=3$ , cutting off two corner equilateral triangles leaves a convex polygonal region of  $q$  sides, with  $q=4$  or  $5$ , as shown in Figure 3. If  $q=5$ , the dividing line issued from A must be a diagonal of the pentagon terminating in, say, C. Then  $\triangle ABC$  must be isosceles and quadrilateral ACDE has the angles shown in the figure. This quadrilateral must now be divided by a diagonal into two triangles, one of which is equilateral, and this is clearly impossible. If  $q=4$ , the remaining quadrilateral PQRS is a parallelogram, and it is easy to see that the desired dissection is impossible whether or not the dividing line emanating from P terminates in RS or not. The case  $n=3$  is therefore impossible, and the proof is complete.

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265. [1977: 190; 1978: 74] *Late solution: Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

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267. [1977: 190; 1978: 76] *Proposed by John Veness, Cremorne, N.S.W., Australia.*

Some products, like  $56 = 7 \cdot 8$  and  $17820 = 36 \cdot 495$ , exhibit consecutive digits without repetition. Find more (if possible, all) such products  $c = a \cdot b$  which exhibit without repetition four, five, ..., ten consecutive digits.

II. *Solution by CDC-7600 (your friendly neighborhood computer), Livermore, California.*

I had this problem for breakfast this morning. After a few burps, caused by the new name of this journal, and a series of muted intestinal rumblings, I am able to announce that the list given below contains all the answers omitted from the previous compilation [1978: 76-77]. There are...oh, about 30 or so new answers (I lose track when counting above 25).

3876 = 19 • 204	6372 = 59 • 108	8024 = 59 • 136
4508 = 23 • 196	7056 = 18 • 392	8340 = 12 • 695
4807 = 19 • 253	7098 = 13 • 546	8463 = 7 • 1209
4890 = 15 • 326	7290 = 15 • 486	8607 = 19 • 453
5278 = 13 • 406	7362 = 18 • 409	8970 = 26 • 345
5408 = 32 • 169	7580 = 15 • 492	9072 = 54 • 168
6042 = 38 • 159	7452 = 69 • 108	9480 = 15 • 632
6052 = 34 • 178	7638 = 19 • 402	9702 = 63 • 154
6290 = 34 • 185	7920 = 48 • 165	9720 = 15 • 648

$$9802 = 13 \cdot 754$$

$$26910 = 78 \cdot 345$$

$$36508 = 4 \cdot 9127$$

$$32890 = 46 \cdot 715$$

$$58401 = 63 \cdot 927$$

I would like to thank Mr. Harry L. Nelson, my master (he thinks), for lending me his typewriter and stationery to submit these not very nourishing results.

*Editor's comment.*

I have in past issues sung the refrain "Solutions, not merely answers" in several different sharps and flats (and the malady lingers on). But there are a few problems (the present one is a good example) where it is quite appropriate to send in only answers. These are usually search-type problems with a large number of answers in which the method of solution is repetitive and not particularly illuminating. The interest here is in the answers themselves and not in how they were obtained. These problems are usually most efficiently handled by a computer search.

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275. [1977: 227] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Given are the points  $P(a,b)$  and  $Q(c,d)$ , where  $a, b, c, d$  are all rational. Find a formula for the number of lattice points (integral coordinates) on segment  $PQ$ .

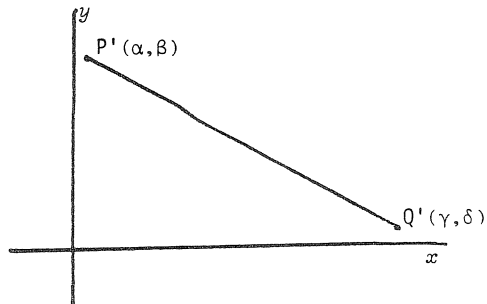
*Solution by Gali Salvatore, Ottawa, Ontario.*

Let  $n$  be the required number of lattice points. If the segment  $PQ$  is horizontal or vertical, then  $n$  can be quickly determined by inspection, so we assume the segment is oblique.

If  $h$  and  $k$  are integers, each of the transformations

$$x = x' + h, \quad y = y' + k, \quad x = -x', \quad y = -y' \quad (1)$$

clearly preserves all lattice points. Given the rational points  $P$  and  $Q$  (we do not assume their coordinates to be as in the proposal, since we reserve  $a, b, c$  for another use later), we can apply, if necessary, one or more of the transformations (1) so that the resulting rational points  $P'(\alpha, \beta)$  and  $Q'(\gamma, \delta)$  are as shown in the figure, where the first quadrant segment  $P'Q'$  has a negative slope and





$$0 \leq \alpha < 1 \quad \text{and} \quad 0 \leq \delta < 1.$$

Since  $P'$  and  $Q'$  are rational points, the line  $P'Q'$  has a unique equation of the form

$$ax + by = c, \quad (2)$$

where  $a, b, c$  are positive integers whose g.c.d. is 1. The required number  $n$  is the number of solutions in *strictly positive* integers of the Diophantine equation (2). We can assume  $a$  and  $b$  to be relatively prime, since otherwise  $n = 0$ .

It is known from Problem 179 [1977: 54-55] that if (2) has exactly  $n$  solutions in positive integers, then

$$(n-1)ab + a + b \leq c \leq (n+1)ab.$$

Thus, upper and lower bounds for  $n$  are given by

$$\frac{c}{ab} - 1 \leq n \leq \frac{c - a - b}{ab} + 1,$$

which can be written more symmetrically as

$$\frac{c - ab}{ab} \leq n \leq \frac{(a-1)(b-1) + (c-1)}{ab}. \quad (3)$$

If (3) does not determine  $n$  uniquely, we can narrow it down still further. With the restrictions assumed for  $a, b, c$ , equation (2) always has infinitely many solutions in integers, and we can use standard Diophantine techniques to find the solution  $P_0(x_0, y_0)$  of least positive abscissa  $x_0$ . If  $y_0 \leq 0$ , then  $n = 0$ . If  $y_0 > 0$ , then  $n \geq 1$  and the remaining  $n-1$  solutions are given by

$$P_t = (x_0 + bt, y_0 - at), \quad t = 1, \dots, n-1$$

so we must have

$$y_0 - a(n-1) > 0 \quad \text{and} \quad y_0 - an \leq 0.$$

Thus

$$\frac{y_0}{a} \leq n < 1 + \frac{y_0}{a},$$

which determines  $n$  uniquely as the smallest integer greater than or equal to  $y_0/a$ . In terms of the *ceiling* of  $x$ ,

$$\lceil x \rceil = \text{the smallest integer } \geq x,$$

(a notation due to K.E. Iverson), our result is

$$n = \left\lceil \frac{y_0}{a} \right\rceil. \quad (4)$$

For example, if the transformed points are  $P' \left( \frac{3}{5}, \frac{274}{25} \right)$  and  $Q' \left( \frac{219}{9}, \frac{1}{4} \right)$ , the equation of line  $P'Q'$  is

$$2x + 5y = 56.$$

Thus  $a = 2$ ,  $b = 5$ ,  $c = 56$ , and (3) gives  $4.6 \leq n \leq 5.9$ , so  $n = 5$ . But in general one would have to resort to (4) to find  $n$ .

A comment was received from HERMAN NYON, Paramaribo, Surinam.

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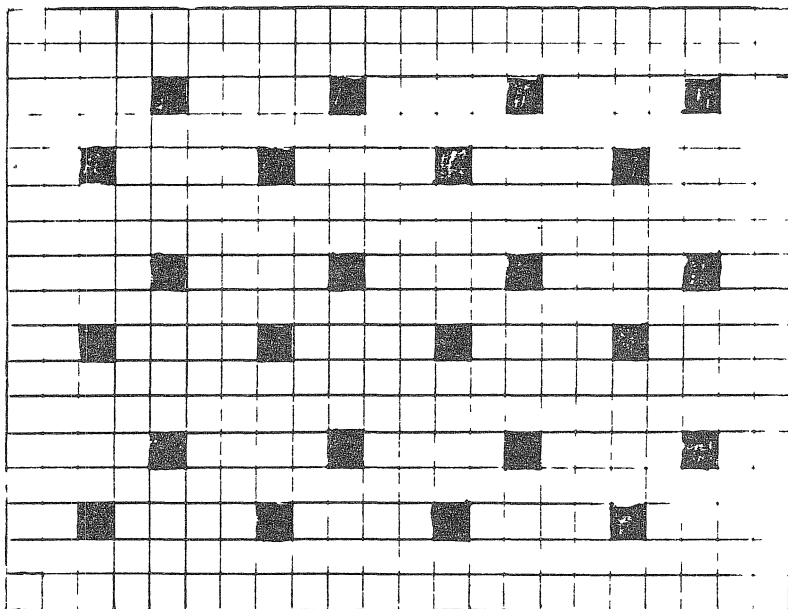
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276. [1977: 227] *Proposed by Sidney Penner, Bronx Community College, Bronx, N.Y.*

How many unit squares must be deleted from a  $17 \times 22$  checkerboard so that it is impossible to place a  $3 \times 5$  polyomino on the remaining portion of the board? (A  $3 \times 5$  polyomino covers exactly 15 squares of the board.)

*Solution by Paul J. Campbell for the Beloit College Solvers, Beloit, Wisconsin.*

Since 24 nonoverlapping  $3 \times 5$  polyominoes can be placed on a  $17 \times 22$  checkerboard (see Problem 135 [1976: 153]), the required number of squares to be deleted is at least 24. And 24 deletions suffice, as can be seen from the figure below, where  $4 \times 4$  blocks are bounded by appropriately placed deletions.



Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; M.S. KLAMKIN and A. LIU, University of Alberta (jointly); and the proposer. One incorrect solution was received.

*Editor's comment.*

Let  $f(m,n)$  be the smallest number of squares that must be deleted from an  $m \times n$  checkerboard so that it is impossible to place a  $3 \times 5$  polyomino on the remaining portion of the board. Our problem shows that  $f(17,22) = 24$ , and Klamkin and Liu showed as well that  $f(15,20) = 21$ . It would be interesting to have a formula giving  $f(m,n)$  explicitly in terms of  $m$  and  $n$ . To assist readers in discovering and checking such a formula, I give below a table giving all values of  $f(m,n)$  for  $1 \leq m,n \leq 15$ . The table was prepared by Klamkin and Liu. When such a formula has been found, we can then address ourselves to the task of finding a more general one for an  $r \times s$  polyomino, and then on into three dimensions!

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3
4	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3
5	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5
6	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6
7	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6
8	0	0	1	1	2	3	3	4	4	5	6	6	7	7	8
9	0	0	1	1	3	3	3	4	4	6	6	6	7	7	9
10	0	0	2	2	3	4	4	5	6	6	7	8	8	9	10
11	0	0	2	2	3	4	4	6	6	7	8	8	10	10	11
12	0	0	2	2	4	4	4	6	6	8	8	8	10	10	12
13	0	0	2	2	4	5	5	7	7	8	10	10	12	12	13
14	0	0	2	2	4	5	5	7	7	9	10	10	12	12	14
15	0	0	3	3	5	6	6	8	9	10	11	12	13	14	15

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277. [1977: 227] Proposed by R. Robinson Rowe, Naubinway, Michigan.

Literally, EUREKA is multipowered; find its roots

$\sqrt{\text{EUREKA}} = \text{UEA}$  (UEA are alternates of eUrEkA),

$\sqrt[3]{\text{EUREKA}} = \text{RT}$  (RT is the cube RT),

$\sqrt[4]{\text{EUREKA}} = ?$

*Solution by Clayton W. Dodge, University of Maine at Orono.*

If EUREKA represents the same number in each part of the problem, it must be a perfect 12th power. But 3 is the only natural number whose 12th power has six digits, and  $3^{12} = 531441$  is not of the proper form. We conclude that the three parts of the problem are unrelated to one another.

$\sqrt{\text{EUREKA}} = \text{UEA}$ . Here we must have  $317 \leq \text{UEA} \leq 987$ ; and  $A^2 \equiv A \pmod{10}$  implies  $A = 1, 5, \text{ or } 6$ , but not 0 since  $K \neq A$ . Thus  $321 \leq \text{UEA} \leq 986$ . Observing that the second digit of UEA equals the first digit of  $(\text{UEA})^2$ , consulting a table of squares shows that we need only consider

UEA = 415, 416, 521, 526, 631, 641, 645, 751, 756, 871, 875, and 876.

Of these, only 756 satisfies all the requirements, and so the unique answer is

$$\sqrt{571536} = 756.$$

$\sqrt[3]{\text{EUREKA}} = \text{RT}$ . Here we require  $47 \leq \text{RT} \leq 98$ ; and since  $T^3 \equiv A \pmod{10}$ , we must have, respectively,

$$T = 2, 3, 7, \text{ or } 8 \quad \text{and} \quad A = 8, 7, 3, \text{ or } 2.$$

Checking a table of cubes of two-digit numbers ending in 2, 3, 7, or 8 in the indicated range shows that only  $\text{RT} = 57$  is satisfactory, and the unique answer is

$$\sqrt[3]{185193} = 57.$$

$\sqrt[4]{\text{EUREKA}} = ?$  Here it seems easiest just to try each ? such that  $18 \leq ? \leq 31$ , and we quickly find that only  $? = 22$  is satisfactory, giving the unique solution

$$\sqrt[4]{234256} = 22.$$

Finally, it is worth noting that  $\sqrt[n]{\text{EUREKA}}$  is never an integer for any  $n > 4$ .

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; ROBERT S. JOHNSON, Montréal, Québec; VIKTORS LINIS, University of Ottawa; HERMAN NYON, Paramaribo, Surinam; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

*Editor's comment.*

A few solvers misunderstood the parenthetical remark in the second part of the proposal. It did not mean that RT was a cube (in which case there is no solution), but rather that RT was the cube ROOT.

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278. [1977: 227] Proposed by W.A. McWorter, Jr., The Ohio State University.

If each of the medians of a triangle is extended beyond the sides of

the triangle to  $\frac{4}{3}$  its length, show that the three new points formed and the vertices of the triangle all lie on an ellipse.

I. *Solution and comment by Dan Pedoe, University of Minnesota, Minneapolis.*

To remove all ambiguity in the wording of this problem, it should be understood that if the median AM, say, of triangle ABC is extended to A', then  $AA' = \frac{4}{3}AM$ .

An affine transformation will map triangle ABC onto an equilateral triangle, midpoints being mapped onto midpoints; and if G' is the centroid of the equilateral triangle, then a circle with centre G' evidently goes through the six designated points. Hence an ellipse, in the original figure, goes through the vertices of triangle ABC and the tips of the extended medians.

This problem is discussed in my film *Central Similarities*, which I made for the Minnesota College Geometry Project. There is an intimate connection between the ellipse discussed above and the ellipse which touches the sides of triangle ABC at their midpoints.<sup>1</sup> One arises from the other by a central similarity (homothety) with centre at the centroid of the triangle and a suitable scale factor.

II. *Comment by O. Bottema, Delft, The Netherlands.*

The factor  $\frac{4}{3}$  is the only one for which the theorem holds, and furthermore the ellipse concerned is Steiner's circumscribed ellipse, in which the tangents at the vertices are parallel to the opposite sides.

Also solved by LEON BANKOFF, Los Angeles, California; CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; ROBERT S. JOHNSON, Montréal, Québec; M.S. KLAMKIN, University of Alberta, Edmonton; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; and the proposer. Late solution by Y. de BRUYN, student, University of Toronto.

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279. [1977: 227] *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa, Ont.*

On donne sur une droite trois points distincts A, O, B tels que O est entre A et B, et  $AO \neq OB$ . Montrer que les trois coniques ayant deux foyers et un sommet aux trois points donnés sont concourantes en deux points.

*Solution de L.F. Meyers, The Ohio State University.*

L'ellipse ayant foyers A, O et un sommet B, et l'ellipse ayant foyers O, B et un sommet A se rencontrent en deux points distincts. P désignant l'un quelconque des points d'intersection, on a

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<sup>1</sup>See Problem 318 [1978: 36]. (Editor)

$$\overline{PA} + \overline{PO} = \overline{OA} + 2\overline{OB}$$

pour la première ellipse et

$$\overline{PB} + \overline{PO} = \overline{OB} + 2\overline{OA}$$

pour la seconde, d'où

$$\overline{PA} - \overline{PB} = \overline{OB} - \overline{OA},$$

et donc P est un point de l'hyperbole ayant foyers A, B et un sommet O.

Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; and the proposer.

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280. [1977: 227] *Proposed by L.F. Meyers, The Ohio State University.*

A jukebox has  $N$  buttons.

(a) If the set of  $N$  buttons is subdivided into disjoint subsets, and a customer is required to press exactly one button from each subset in order to make a selection, what is the distribution of buttons which gives the maximum possible number of different selections?

(b) What choice of  $n$  will allow the greatest number of selections if a customer, in making a selection, may press any  $n$  distinct buttons out of the  $N$ ? How many selections are possible then?

(Many jukeboxes have 30 buttons, subdivided into 20 and 10. The answer to part (a) would then be 200 selections.)

I. *Solution by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

We assume in both parts that a selection does not depend upon the order in which the buttons are pressed.

(a) Suppose the  $N$  buttons are partitioned into  $n$  sets of  $a_i$  buttons ( $i = 1, 2, \dots, n$ ), so that  $N = \sum a_i$ ; then the number of possible selections is  $S = \prod a_i$ . Unless  $N = 1$ , maximal  $S$  requires each  $a_i \geq 2$  for if, say,  $a_1 = 1$ , then combining the corresponding set with that of  $a_2$  would increase  $S$  since  $a_2 + 1 > 1 \cdot a_2$ . If any  $a_i = 4$ , the corresponding set can be divided into two sets of 2 buttons without affecting  $S$ , and we will assume this has been done in every case. If any  $a_i > 4$ , dividing the corresponding set into two sets of 2 and  $a_i - 2$  buttons would increase  $S$  since  $2(a_i - 2) = 2a_i - 4 > a_i$ . Hence for maximal  $S$  we must have each  $a_i = 2$  or 3; and furthermore there cannot be more than two sets of 2 buttons each since  $3^2 > 2^3$ .

Let  $N = 3q + r$ ,  $0 \leq r < 3$ . It follows from the above discussion that for maximal  $S$  it is necessary and sufficient to have:

- i) if  $r=0$ ,  $q$  sets of 3 buttons, giving  $S=3^q$ ;
- ii) if  $r=1$ ,  $q-1$  sets of 3 buttons and two sets of 2 buttons (or one set of 4 buttons), giving  $S=4 \cdot 3^{q-1}$ ;
- iii) if  $r=2$ ,  $q$  sets of 3 buttons and one set of 2 buttons, giving  $S=2 \cdot 3^q$ .

(b) Here the number of selections is given by the binomial coefficient  $\binom{N}{n}$  which, as is well-known, is greatest when  $n=\lfloor N/2 \rfloor$ , where the brackets denote the greatest integer function.

II. *Comment by M.S. Klamkin, University of Alberta, Edmonton.*

A variant of this problem would be to partition the  $N$  buttons into  $n$  (disjoint) subsets, and a selection would consist of pressing (without regard to order) exactly one button from each of *any* number (including zero) of the subsets. (Pressing no button at all would correspond to buying a few minutes of silence—a scarce commodity!—from the jukebox.)

[In the notation of solution I(a)], we have  $N=\sum a_i$  and

$$S = 1 + \sum_{i=1}^n a_i + \sum_{i < j} a_i a_j + \dots + a_1 a_2 \dots a_n$$

or

$$S = (1+a_1)(1+a_2)\dots(1+a_n).$$

The maximum occurs when  $a_i=1$  for all  $i$  and is equal to  $2^N$ . For a proof, note that

$$a_i > 1 \implies 1+a_i < (1+1)(1+(a_i-1)).$$

Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; M.S. KLAMKIN, University of Alberta, Edmonton (solution as well); VIKTORS LINIS, University of Ottawa; R. ROBINSON ROWE, Sacramento, California; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

*Editor's comment.*

Klamkin noted that both parts of this problem are known. For part (a) he refers to [1], and the equivalent of part (b) can be found in nearly every elementary algebra textbook. The proposer mentioned that a problem similar to part (a) occurred as Problem 4 of the 1976 International Olympiad. See [2] for the problem and a solution. Related material can be found in [3].

#### REFERENCES

1. L. Carlitz, solution to a problem of L. Moser, *Pi Mu Epsilon Journal*, 3 (1961) 232-233.
2. *Mathematics Magazine*, 49 (1976) 223, 263.
3. Leroy F. Meyers, "An integer construction problem", *American Mathematical Monthly*, 66 (1959) 556-561.

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281. [1977: 250] *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

Reconstituer les chiffres de l'addition décimale suivante:

$$\text{HUIT} + \text{HUIT} + \text{HUIT} = \text{DOUZE} + \text{DOUZE}.$$

*Solution by Charles W. Trigg, San Diego, California.*

The problem can be analyzed conveniently through the two additions

$$\begin{array}{r} \text{HUIT} \\ \text{HUIT} \\ \text{HUIT} \\ \hline abcdf \end{array} \quad \text{and} \quad \begin{array}{r} \text{DOUZE} \\ \text{DOUZE} \\ \hline abcdf \end{array}.$$

From the first addition,  $a = 1$  or  $2$ . From the second,  $a = 2$ ,  $D = 1$ ,  $0 < 5$ , so  $H > 5$ . Furthermore,  $f$  is even, so  $T$  is even, and the common sum and DOUZE are multiples of 3.

From the hundreds digits' columns, we have

$$2U + (0 \text{ or } 1) = 3U + (0, 1, \text{ or } 2) - 10k.$$

The only solutions of this equality are

$$2(9) + 0 = 3(9) + 1 - 10, \quad (1)$$

$$2(9) + 1 = 3(9) + 2 - 10, \quad (2)$$

$$2(8) + 0 = 3(8) + 2 - 10, \quad (3)$$

and

$$2(0) + (0 \text{ or } 1) = 3(0) + (0 \text{ or } 1). \quad (4)$$

From each of (1) and (2), we get  $U = 9$ ,  $O = 4$ ,  $H = 9 = U$ .

From (3), we get  $U = 8$ ,  $O = 4$ ,  $H = 9$ ,  $Z < 5$ , and  $T = 0, 2, \text{ or } 6$ . But if  $T = 6$ , then  $E = 4 = 0$  or  $E = 9 = H$ ; if  $T = 2$ , then  $E = 8 = U$ , or  $E = 3$  and  $Z = 2 = T$  (in order that  $3 \mid \text{DOUZE}$ ); and if  $T = 0$ , then  $E = 5$ ,  $Z = 3$ ,  $d = 7$ , and  $I = 9 = H$ .

From (4), we get  $U = 0$ ,  $O = 2$ ,  $H = 8$ , and  $T = 4$  or  $6$ . If  $T = 4$ , then  $E = 6$ ; and  $Z = 3$ ,  $I = 2 = 0$  or  $Z = 9$ ,  $I = 6 = E$ . If  $T = 6$ , then  $E = 4$  or  $9$ . But if  $E = 9$ , then  $Z = 3$  and  $I = 2 = 0$ ; so we must have  $E = 4$ ,  $Z = 5$ , and  $I = 3$ , and these are satisfactory in all respects.

Thus the unique solution is

$$8036 + 8036 + 8036 = 12054 + 12054 = 24108,$$

and the French will be surprised to learn that

$$\text{HUIT} + \text{HUIT} + \text{UIT} = \text{DOUZE} + \text{DOUZE} = \text{OEDUH}.$$



Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; the following students of JACK LeSAGE, Eastview Secondary School, Barrie, Ontario: BEV WALLBANK and NANCY WHETHAM (independently); F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; SIDNEY PENNER, Bronx Community College, New York; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

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282. [1977: 250] *Proposed by Erwin Just and Sidney Penner, Bronx Community College.*

On a  $6 \times 6$  board we place  $3 \times 1$  trominoes (each tromino covering exactly three unit squares of the board) until no more trominoes can be accommodated. What is the maximum number of squares that can be left vacant?

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Instead of considering the maximum number of squares, we consider, equivalently, the minimum number of trominoes required to *block* the board (so that no more trominoes can be placed).

At least two squares must be covered in each row and in each column. If exactly two squares are covered in a particular outside column (call it column 1), then they must be less than three squares apart (otherwise column 1 would need another square covered), so four other trominoes must be placed parallel to the two original trominoes in the four other rows so as to cover all third column squares (see Figure 1). Hence none of these six trominoes extends to column 6, and a seventh tromino is required to block the board. A similar result holds for rows.

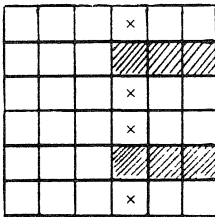


Figure 1

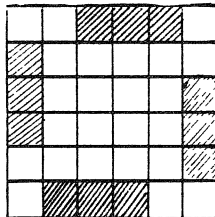


Figure 2

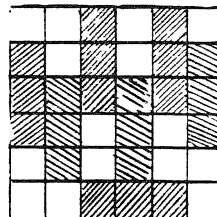


Figure 3

So, if six trominoes are to suffice, then at least three squares must be covered in each outside row and column. If, in a particular outside row or column, no three of the covered squares are covered by a single tromino, then the argument of the preceding paragraph shows that a seventh tromino is required. Hence each outside row or column must have at least three squares covered by a single tromino,

as shown in Figure 2. But this leaves an inner  $4 \times 4$  square which cannot be blocked by two trominoes. So at least seven trominoes are required, and Figure 3 shows that seven trominoes suffice, leaving the maximum number of 15 squares vacant.

II. *Comment by M.S. Klamkin and A. Liu, University of Alberta, Edmonton.*

The general problem here appears to be difficult. This would be to determine the minimum number of polyominoes of a given type which can block an  $m \times n$  board. This is dual to determining the maximum number of polyominoes which can be packed on the board. In particular, we have also shown that thirteen  $3 \times 1$  trominoes are required to block an  $8 \times 8$  board. Unfortunately, our proof is too brutal to be given here.

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; ROBERT S. JOHNSON, Montréal, Québec; M.S. KLAMKIN and A. LIU, University of Alberta, Edmonton (joint solution as well); HERMAN NYON, Paramaribo, Surinam; and the proposers.

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283.\* [1977: 250] *Proposed by A.W. Goodman, University of South Florida.*

The function

$$y = - \frac{2x \ln x}{1 - x^2}$$

is increasing for  $0 < x < 1$  and in fact  $y$  runs from 0 to 1 in this interval. Therefore an inverse function  $x = g(y)$  exists. Can this inverse function be expressed in closed form and if so what is it? If it cannot be expressed in closed form, is there some nice series expression for  $g(y)$ ? The series need not be a power series.

*Editor's comment.*

No solutions were received for this problem. It will remain open for the time being, and readers are urged to consider it again.

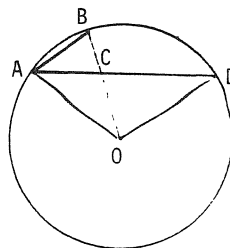
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284. [1977: 250] *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Given a sector AOD of a circle (see figure), can a straightedge and compass construct the line OB so that  $AB = AC$ ?



I. *Solution by Lai Lane Luey, Willowdale, Ontario.*

Suppose such a construction is possible. Then triangles ABC and OAB are both isosceles and have equal base angles at B; hence they are similar and have equal

vertex angle  $\theta$  (see adjoining figure). Now

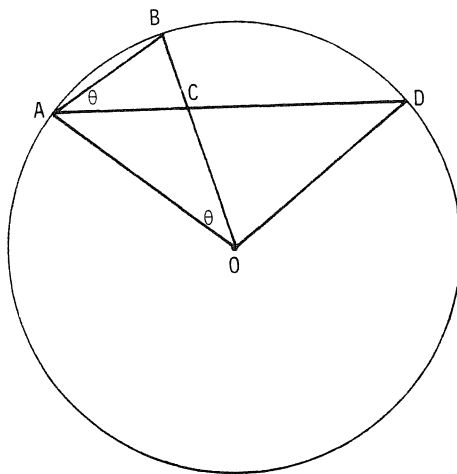
$$\angle BOD = 2\angle BAD = 2\angle AOB,$$

so OB trisects  $\angle AOD$ .

Since it is impossible in general to trisect a given angle with straightedge and compass, the answer to the problem is NO.

II. *Solution by Samuel L. Greitzer, Rutgers University.*

[As shown in solution I], all one has to do is to trisect  $\angle AOD$ , perhaps by using one of the hundreds of methods I have been receiving through the years, or by using a straightedge with a teeny weeny mark on it.



Also solved by PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; DAN PEDOE, University of Minnesota; SIDNEY PENNER, Bronx Community College, N.Y.; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; FRANK STOYLES, Algonquin College, Ottawa; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer. One incorrect solution was received.

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285. [1977: 251] *Proposed by Robert S. Johnson, Montréal, Québec.*

Using only the four digits 1, 7, 8, 9 (each exactly once) and four standard mathematical symbols (each at least once), construct an expression whose value is 109.

I. *Solution by Don Baker, Presidio Junior H.S., San Francisco, California.*

Using the symbols for division, subtraction, repeating decimal, and decimal point:

$$91 \div .\overline{7} - 8 = 109.$$

II. *Solution by Viktors Linis, University of Ottawa.*

Using the symbols for negative, addition, greatest integer function, and square root:

$$-19 + [\sqrt{8}]^7 = 109.$$

III. *Five solutions by F.G.B. Maskell, Algonquin College, Ottawa.*

Using the symbols for addition, subtraction, cosine function, and  $\pi$ :

$$91 + 7 + 8 - \cos \pi - \cos \pi - \cos \pi = 109,$$

$$97 + 8 + 1 - \cos \pi - \cos \pi - \cos \pi = 109,$$

$$98 + 1 + 7 - \cos \pi - \cos \pi - \cos \pi = 109,$$

$$87 + 19 - \cos \pi - \cos \pi - \cos \pi = 109,$$

$$89 + 17 - \cos \pi - \cos \pi - \cos \pi = 109.$$

IV. *Two more solutions by F.G.B. Maskell.*

Using the symbols for addition, subtraction, natural logarithm, and  $e$ :

$$97 + 18 - \ln e - \ln e - \ln e - \ln e - \ln e - \ln e = 109,$$

$$98 + 17 - \ln e - \ln e - \ln e - \ln e - \ln e - \ln e = 109.$$

V. *Two more solutions by F.G.B. Maskell.*

Using the symbols for the  $n$ th prime, addition, subtraction, and the vinculum:

$$7p_8 - \overline{p_9 + 1} = 109,$$

$$p_9 + \overline{87 - 1} = 109.$$

VI. *One more solution by F.G.B. Maskell.*

Using the symbols for summation, addition, equality, and a dummy variable:

$$\sum_{n=8}^{7+9} n + 1 = 109.$$

VII. *A last solution by F.G.B. Maskell.*

Using the symbols for summation, subtraction, equality, and a dummy variable:

$$\sum_{n=9}^{17} n - 8 = 109.$$

VIII. *Solution by Leroy F. Meyers, The Ohio State University.*

Using the symbols for subtraction, multiplication, greatest integer function, and  $e$ :

$$\left[ 7^e \right] - 1 \times 89 = 109.$$

IX. *Solution by Bob Prielipp, The University of Wisconsin-Dei kosn.*

Using the symbols for square root, Euler's totient, addition, and parentheses:

$$9\sqrt{\phi(8)}\phi(7) + 1 = 109.$$

X. *Solution by Harry D. Ruderman, Hunter College, New York.*

Using the symbols for logarithm, square root, addition, and greatest integer function:

$$\log_{[\sqrt{7}]} \log \sqrt{\sqrt{\dots \sqrt{1+8}}}^9 = 109,$$

where the dots indicate the presence in the expression of 109 radical signs.

XI. *Solution by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

Using the symbols for square root, greatest integer function, decimal point, and  $\pi$ :

$$\left[ \sqrt{19.87^\pi} \right] = 109.$$

XII. *Second solution by Kenneth M. Wilke.*

Using the symbols for square root, addition, greatest integer function, and  $e$ :

$$\left[ \sqrt{e^{\sqrt{87}}} + \sqrt{9+1} \right] = 109.$$

XIII. *Solution by the proposer.*

Using the symbols for square root, decimal point, negative, and addition:

$$\sqrt{\sqrt{\dots \sqrt{.1^{-87}}} + 9} = 109,$$

where the dots indicate the presence in the expression of 20 radical signs.

*Editor's comment.*

Except for that of the proposer, the solutions are given above in alphabetical order of the solvers' names, not in some fancied order of merit. Several additional solutions were rejected for not conforming strictly to the rules of the problem, or for using symbols that the editor arbitrarily classified as not "standard."

The inventiveness of our readers is indeed dazzling!

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286. [1977: 251] *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

This problem generalizes *Mathematics Magazine* Problem 939 (proposed May, 1975; solution May, 1976, p. 151) and our Problem 204 (proposed January, 1977; solution May, 1977, p. 140).

Find, for positive integers  $N \leq L \leq H$ :

- (a) the number of rectangular parallelepipeds (r.p.),
  - (b) the number of cubes,
  - (c) the number of different sizes of r.p.'s
- imbedded in a  $W \times L \times H$  r.p. made up of  $WLH$  unit cubes.

*Solution and comment by the proposer.*

(a) A sub-r.p. is determined by three pairs of parallel planes, each pair parallel to one of the three mutually orthogonal (pairs of) faces of the  $W \times L \times H$  r.p. There are  $W+1$  places for a plane to pass parallel to an  $L \times H$  face, and hence  $\binom{W+1}{2}$  choices for the pair in that direction. Similar reasoning for the other two directions leads to the conclusion that the number of imbedded r.p.'s is

$$\binom{W+1}{2} \binom{L+1}{2} \binom{H+1}{2} = \frac{1}{8} WLH(W+1)(L+1)(H+1).$$

(b) There are  $WLH$  cubes of side 1,  $(W-1)(L-1)(H-1)$  cubes of side 2 and, in general, the number of cubes of side  $i$  is

$$(W+1-i)(L+1-i)(H+1-i), \quad i = 1, 2, \dots, W.$$

Hence the total number of cubes is

$$\begin{aligned} \sum_{i=1}^W (W+1-i)(L+1-i)(H+1-i) &= \sum_{j=1}^W j(L-W+j)(H-W+j) \\ &= \sum_{j=1}^W j^3 + (L+H-2W) \sum_{j=1}^W j^2 + (L-W)(H-W) \sum_{j=1}^W j \\ &= \frac{1}{12} W(W+1)[6LH - (W-1)(2L+2H-W)]. \end{aligned}$$

(c) We count the number of sizes of sub-r.p.'s of all possible dimensions  $R \times S \times T$ , with  $R \leq S \leq T$ . There are four cases to consider. The count will be based on the well-known fact that the number of (unordered) selections, with repetitions allowed, of  $n$  elements from an  $m$ -element set is  $\binom{m+n-1}{n}$ .

*Case 1.*  $R, S, T \leq W$ . The number of sizes is  $\binom{W+2}{3}$ .

*Case 2.*  $R, S \leq W$ ;  $W < T \leq H$ . The number of sizes is  $(H-W) \binom{W+1}{2}$ .

*Case 3.*  $R \leq W$ ;  $W < S, T \leq L$ . The number of sizes is  $W \binom{L-W+1}{2}$ .

*Case 4.*  $R \leq W$ ;  $W < S \leq L$ ;  $L < T \leq H$ . The number of sizes is  $W(L-W)(H-L)$ .

Summing up the number of sizes in the four cases gives, after reduction,

$$\frac{1}{6} W[ (3H - L - W + 2)(2L - W + 1) - L(L - W) ]. \quad (1)$$

*Comment.*

In the light of our solution to part (a), we can generalize Dodge's solution to Problem 204 [1977· 140] and obtain the interesting identity

$$\sum_{m=1}^W \sum_{k=1}^L \sum_{j=1}^H (W+1-m)(L+1-k)(H+1-j) = \binom{W+1}{2} \binom{L+1}{2} \binom{H+1}{2}$$

which, of course, extends to an arbitrary number of summations.

If we set  $W=1$  in (1), we find that the number of sizes of rectangles in an  $L \times H$  grid ( $L \leq H$ ) is  $\frac{1}{2}L(2H - L + 1)$ , a result that could be added to those obtained in Problem 204; and setting  $W=L=1$  in (1) yields, to no one's surprise, the fact that the number of sizes of segments in a segment of length  $H$  is  $H$ .

Also solved by N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; and R. ROBINSON ROWE, Sacramento, California.

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(Continued from page 99)

7. The bound Volume 3 (1977) of CRUX MATHEMATICORUM will shortly be mailed to subscribers who ordered it along with their 1978 subscription. Other readers can obtain a copy by sending \$10 (in U.S. or Canadian funds) to the Managing Editor, whose address appears on the front page of this issue. Volumes 1-2 (1975-1976), bound together in a single volume, are also available at \$10 for the combined volume, but only a few copies are left: a new printing will soon be necessary.

These bound volumes constitute the permanent record of CRUX MATHEMATICORUM. They are reduced in size (to about  $6\frac{1}{2}'' \times 8\frac{1}{2}''$ ), include dozens of corrections of minor errors and misprints, and are provided with a full index.

8. V. Linis would like to share with us the following flattering comment on our profession. He found it in *The Selfish Gene*, by Richard Dawkins (Oxford University Press, 1976), p. 202:

"Of particular interest are 'subtle cheats' who appear to be reciprocating, but who consistently pay back slightly less than they receive. It is even possible that man's swollen brain, and his predisposition to reason mathematically, evolved as a mechanism of ever more devious cheating, and ever more penetrating detection of cheating in others."

9. E.J. Barbeau noted the following interesting factorizations:

$$\begin{array}{ll} 2342 = 2 \cdot 1171 & 7928 = 8 \cdot 991 \\ 2343 = 3 \cdot 11 \cdot 71 & 7929 = 9 \cdot 881 \end{array}$$

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