Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- Symmetry and the Nine-Point Circle
- A Trip from Trig to Triangle
- 600 Years of Prague's Horologe
- Prime Decades

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

The 21st Transilvanian Hungarian Mathematical Competition

This was held on 4th and 5th of February this year (2011), and the organizer, Mihaly Bencze, has kindly sent us the problems that were set, to test our little grey cells. So here they are. Please do not send your solutions to, or request solutions from, us. Happy problem-solving!

Problems for the 9th form

Problem 1 Prove that for $a, b, c, d \in \mathbb{R}$

$$a + b + c + d - a^2 - b^2 - c^2 - d^2 \leq 1.$$

Olosz Ferenc, Szatmárnémeti

Problem 2 Compare the following numbers

$$A = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{\text{2011 piece of 2−s}} \quad \text{and} \quad B = \underbrace{3^{3^{\cdot^{\cdot^{3}}}}}_{\text{2010 piece of 3−s}} .$$

Demeter Albert, Kolozsvár

Problem 3 Find the natural number solutions of the following equations:

(a) $20x^2 + 11y^2 = 2011$,

(b) $20x^2 - 11y^2 = 2011$.

Kacsó Ferenc, Marosvásárhely

Problem 4 In the parallelogram $ABCD$ the measure of the angle BAD is $45°$ and the measure of the angle ABD is $30°$. Show that the distance from point B to diagonal (AC) is equal to $AD/2$.

Olosz Ferenc, Szatmárnémeti

Problem 5 In the parallelogram $ABCD$ we have $AB > AD$. Let E and F be points on the sides (AB) and (CD) respectively, such that

$$\frac{EB}{AB} = \frac{DF}{DC} = \frac{1}{n}, \quad \text{with } n \in \{2, 3, 4, \ldots\},$$

and let G_1 and G_2 be the centroids of the triangles ADE and BCF respectively. If $G_1G_2 \cap EB = \{K\}$, show that $KA = EB$.

Olosz Ferenc, Szatmárnémeti

Problem 6 What is the next year when there are four Friday the 13ths?

* * *

Problems for the 10th form

Problem 1 Find the minimal value of

$$E(x) = (3 - 2\tan x)^2 + (3 + 2\cot x)^2,$$

for $x \in \mathbb{R} \setminus \{k\pi/2 \mid k \in \mathbb{Z}\}$.

Kovács Béla, Szatmárnémeti

Problem 2 (a) Show that for $x > 1$ we have

$$2^{(\log_2 x)^{1/2}} = x^{(\log_x 2)^{1/2}}.$$

(b) Find the real solutions of the following equation:

$$2^{(2x\log_2 x)^{1/2}} + x^{(2x\log_x 2)^{1/2}} = 2^x + x^2.$$

Longáver Lajos, Nagybánya

Problem 3 Find those complex numbers z for which

$$z^2 + \left(\frac{2z}{z-2}\right)^2 = 5.$$

Kovács Béla, Szatmárnémeti

Problem 4 Let M and N be the midpoints of the sides AB and CD, respectively, of a quadrilateral $ABCD$. If the lengths of the diagonals AC and BD are equal to $2\sqrt{3}$ and the measure of their angle is $60°$, compute the length of segment MN.

Dávid Géza, Székelyudvarhely

Problem 5 Let $ABCD$ and $MNPQ$ be two squares with sides of unity length (see figure 1). We place the squares one over the other so that the covering is complete. Square $ABCD$ will be fixed and $MNPQ$ will be rotated around its center. Find the minimum area of the common part of the squares.

András Szilárd, Kolozsvár

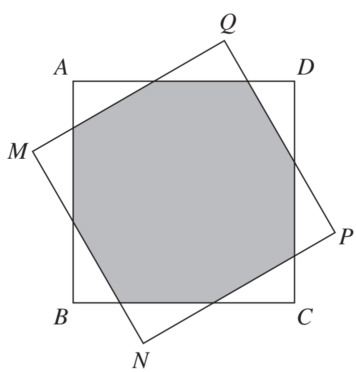

Figure 1

Problem 6 At most how many elements can that set contain which has the property that among each of its five elements there are three in geometric sequence.

András Szilárd, Kolozsvár

Problems for the 11th form

Problem 1 Find those natural numbers x, y, z for which

$$xy + yz + zx = 3(x + y + z) + 1.$$

Kovács Béla, Szatmárnémeti

Problem 2 If $A \in \mathcal{M}_2(\mathbb{C})$, and $\det A = \alpha$, prove that

$$\det(A^2 + A - \alpha I_2) + \det(A^2 + \alpha I_2) = \alpha(1 + 4\alpha).$$

Bencze Mihály, Brassó

Problem 3 Let $(x_n)_{n\geq 0}$ be a sequence defined recursively by $x_0 = 2$, $x_1 = 3$, and $x_{n+2} = 3x_{n+1} - x_n$ for $n \geq 0$. Does such a term of this sequence exist which is a complete square?

Kacsó Ferenc, Marosvásárhely

Problem 4 Triangle ABC is nonequilateral. Let A_1 be the symmetrical of A relative to B, B_1 be the symmetrical of B relative to C, and C_1 be the symmetrical of C relative to A. Show that if H and O, respectively H_1 and O_1, are the orthocenter and circumcenter of the triangles ABC and $A_1B_1C_1$, respectively, then OO_1HH_1 is a trapezoid.

Bencze Mihály, Brassó

Problem 5 Ten billiard balls are placed as can be seen in figure 2. At least how many billiard balls have to be taken away so that among the remaining balls there aren't three such balls whose centers form an equilateral triangle.

Demeter Albert, Kolozsvár

Problem 6 At most how many regions can be determined in the plane by 2011 circles and 1102 lines.

Demeter Albert, Kolozsvár

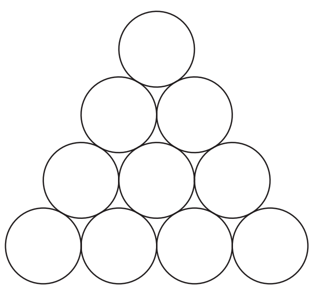

Figure 2

Problems for the 12th form

Problem 1 Prove that there is no a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf'(x) - f(x) = x, \quad \text{for all } x \in \mathbb{R}.$$

Kacsó Ferenc, Marosvásárhely

Problem 2 In a group $(G, \cdot)$ there are 2010 elements and there exist three elements a, b, and c different from the identity element e of G, and pairly different, such that $a^2 = b^2 = c^2 = e$. Show that group $(G, \cdot)$ is noncommutative.

Szilágyi Judit, Kolozsvár

Problem 3 In how many different ways can a promenade in the form of a $2 \times n$ rectangle be paved with squares 1×1 of two colors so that there are no squares which have the same color as two of its adjoining squares. Two squares are adjoining if they have a common side.

Nagy Eörs, Marosvásárhely and András Szilárd, Kolozsvár

Problem 4 Let $ABCD$ be a square, and let $M \in (AD)$ and $N \in (BC)$ such that $AM = BN$. Let $P \in (MN)$ such that

$$\frac{MP}{PN} = \left(\frac{AM}{MD}\right)^2.$$

Show that $AP \perp PB$ and find the locus of point P when M describes (AD).

Csapó Hajnalka, Csíkszereda

Problem 5 Let ABC be a triangle and let M, N, and P be the common points of the incircle with the sides BC, CA, and AB respectively. Show that if D is the midpoint of the side BC and $AD \cap NP = \{E\}$, then $ME \perp BC$.

Dávid Géza, Székelyudvarhely

Problem 6 Find the smallest positive integer m for which the following assertion is true: among each m consecutive positive integers there exists one number such that summing all proper divisors of this number the sum is not less than $\frac{4}{3}$ of that number.

Demeter Albert and András Szilárd, Kolozsvár

Missing out terms may not matter!

$$\frac{\log((m+1)/m)^m}{\log((m+1)/m)^{m+1}} = \frac{((m+1)/m)^m}{((m+1)/m)^{m+1}},$$

$$\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \sin\left(\frac{n+1}{2}\right)\alpha \sin\frac{n}{2}\alpha \Bigg/ \sin\frac{\alpha}{2},$$

$$\alpha + 2\alpha + \cdots + n\alpha = \left(\frac{n+1}{2}\right)\alpha\frac{n}{2}\alpha \Bigg/ \frac{\alpha}{2}.$$

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One Coincidence After Another!

ANDREW PERCY and ALISTAIR CARR

1. What a coincidence!

Euclid presents a proof of the Pythagorean theorem in his Proposition I.47 of *The Elements* (see, for example, reference 1) with the aid of a diagram which has been called 'Euclid's windmill' (see figure 1). Is it a coincidence in Euclid's diagram that the segments $\overline{AL}$, $\overline{FC}$ and $\overline{KB}$ appear to intersect at a point, despite this not playing a role in his proof? We think not, and will show that this coincidence is not by chance! According to Heath (see reference 1, pp. 367, 368) Heron, commenting on Euclid's *The Elements*, offered a proof some time between 178 AD and 284 AD, relying on three lemmas. More recently, Walser (see reference 2, p. 80) offers the coincidence as a 'Homage to Pythagoras' but leaves the proof to the imagination of the reader. But one may further ask if it is significant that Walser, writing in 2004, does not specify his triangle to be right-angled, although he clearly draws it to be so and pays homage to Pythagoras – a name universally associated with right-angled triangles? Is Walser aware that the proof may be generalized to any $\triangle ABC$? We will prove this generalization here.

The coincidence result for a general $\triangle ABC$ is mentioned in Vecten's letter to Gergonne (see reference 3) around 1816 but Vecten deems the proof too simple to need explanation. The elementary proof we give here uses an insightful progression of coincidences which we hope might please both Vecten and Walser.

2. A coincidence: the circumcentre

The perpendicular bisectors of each side of a triangle coincide at a point called the circumcentre. In figure 2 we extend the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ until they intersect at T. Then, since any two right-angled triangles with the same length of hypotenuse and a common side

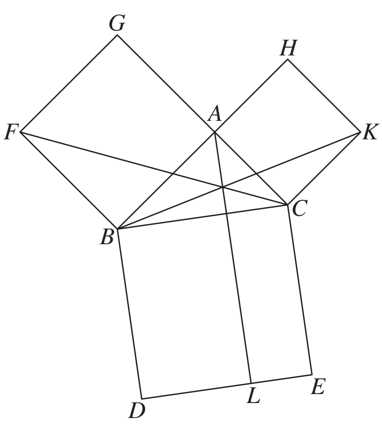

Figure 1 Euclid's windmill.

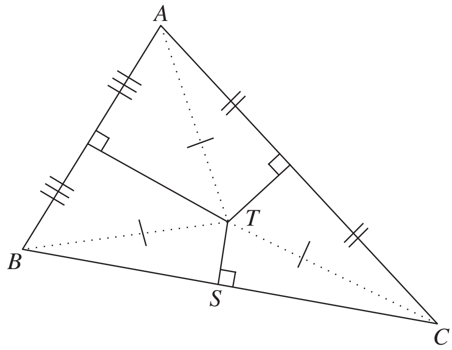

Figure 2 The circumcentre.

must be congruent, we see that $\overline{ST}$ is the perpendicular bisector of $\overline{BC}$. Of course, diagrams can sometimes cloud reality. If the angle at A is sufficiently large then the circumcentre will not lie within the triangle, but in all cases the method of proof is valid since the segments $\overline{BT}$ and $\overline{CT}$ will always form congruent right-angled triangles with the perpendicular bisector of $\overline{BC}$ and the two segments $\overline{BS}$ and $\overline{CS}$.

3. Another coincidence: the orthocentre

The orthocentre is the point of intersection of the altitudes of a triangle. To demonstrate this coincidence begin with $\triangle ABC$ and construct parallelograms $ACBZ$, $AYCB$ and $ACXB$ as in figure 3 (or figure 38 of reference 2). Construct the altitude at A and notice that it extends to become a perpendicular bisector of $\triangle XYZ$. Similarly the altitudes from B and C extend as perpendicular bisectors of $\triangle XYZ$ which are coincident by our previous result in Section 2. Again, although the orthocentre need not lie within the triangle, in all cases the method of proof is valid since the circumcentre of $\triangle XYZ$ so constructed must always be the orthocentre of $\triangle ABC$.

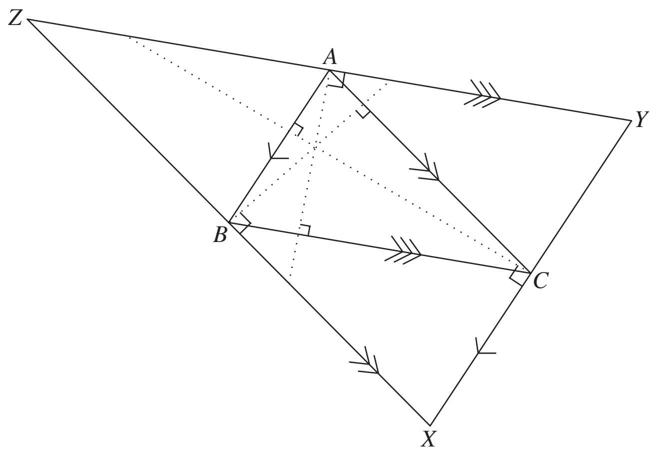

Figure 3 The orthocentre.

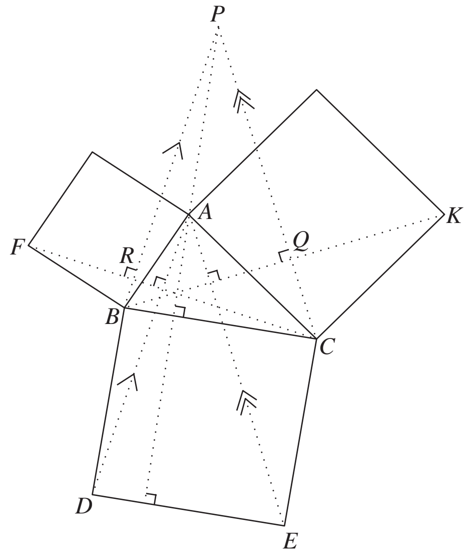

Figure 4 The coincident point in Euclid's windmill.

4. Just by coincidence...

Finally we consider the coincidence shown in Euclid's windmill, but we don't assume that $\triangle ABC$ is a right-angled triangle (see figure 4).

Since $\overline{CA}$ and $\overline{CK}$ are the same length, as are $\overline{CB}$ and $\overline{CE}$, rotating $\triangle CKB$ anticlockwise about C through a right angle results in $\triangle CAE$, with $\overline{AE}$ perpendicular to $\overline{KB}$ (a result known to Vecten (see reference 3)). A similar rotation is made for $\triangle BFC$ to give $\triangle BAD$, with $\overline{AD}$ perpendicular to $\overline{FC}$. We drop the altitude from A without assuming that it also passes through the intersection of $\overline{FC}$ and $\overline{KB}$.

Now translate $\triangle ADE$ so that $\overline{DE}$ overlies $\overline{BC}$ creating $\triangle PBC$. Since $\overline{PC}$ is parallel to $\overline{AE}$ and $\overline{PB}$ is parallel to $\overline{AD}$, the angles at Q and R are right angles.

We see that the coincident point of the altitudes of $\triangle PBC$ is also the coincidence point sought. The orthocentre of $\triangle PBC$ as constructed will always give the coincidence of the altitude extended from A and the lines $\overline{FC}$ and $\overline{KB}$ even if this point is outside $\triangle ABC$.

References

1 Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (Dover, New York, 1956).

2 Hans Walser (translated from the original German by Peter Hilton and Jean Pedersen), *99 Points of Intersection* (MAA, Washington, DC, 2006).

3 M. Vecten, Extrait d'une lettre au rédacteur des annales, *Mathématiques Pures et Appliquées (Gergonne's Annales)* **7** (1816–1817), pp. 321–324.

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Using a Sledgehammer to Crack Some Nuts

MARTIN GRIFFITHS

1. Introduction

As mathematicians, we are generally used to proving results 'from the bottom up' so to speak. In other words, in order to prove what might be regarded as a relatively difficult theorem, we start with slightly more elementary results (often called *lemmas*) and gradually build up layer upon layer of complexity as we head towards our goal. I thought it might be fun to commit a form of mathematical heresy, and to try to do things the other way round; hence the title of this article. When doing something such as this there is of course always the potential for circular arguments to arise. These occur when theorem X is used to prove theorem Y that was itself assumed in order to prove theorem X! We note here that it is not necessary to assume either of the 'nuts' considered in this article in order to prove the 'sledgehammer'.

The 'sledgehammer' we shall be using is the prime number theorem (PNT), one of the crowning achievements of 19th century mathematics. It provides us with a certain amount of information on how the primes are distributed amongst the integers, and was first proved by Charles de la Vallée Poussin and Jacques Hadamard (apparently independently) in 1896. With $\pi(n)$ denoting the number of primes no greater than n, the PNT is stated as follows:

$$\pi(n) \sim \frac{n}{\ln n}.$$

Such a result is known as an *asymptotic relation*. An alternative way of expressing this is

$$\lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1. \tag{1}$$

A full analytic proof and a brief sketch of an elementary proof of the PNT can be found in reference 1.

It occurred to me that it would indeed be interesting to see which of the more straightforward results concerning primes are actually implied by the PNT. Therefore, using this grand theorem, and assuming no other results concerning the primes, we shall attempt to give proofs of two of these 'nuts'; namely Bertrand's postulate and the divergence of the series of reciprocals of the primes. It is worth noting, however, that although these results might be regarded as straightforward in comparison to the PNT, it is by no means a trivial matter to prove either of them using elementary methods.

2. A useful result

We shall first find it useful to obtain a rough estimate of the size of the kth prime, p_k say. This can, in the spirit of this paper, indeed be achieved via the PNT. Using (1) we know that for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$1 - \varepsilon < \frac{\pi(n)\ln n}{n} < 1 + \varepsilon,$$

for all $n \geq N$. Thus, so long as $0 < \varepsilon < 1$,

$$\ln[n(1-\varepsilon)] < \ln \pi(n) + \ln \ln n < \ln[n(1+\varepsilon)], \tag{2}$$

for all $n \geq N$. Furthermore, assuming that $0 < \varepsilon < 1$, it is straightforward to show that

$$\lim_{n\to\infty} \frac{\ln[n(1-\varepsilon)]}{\ln n} = \lim_{n\to\infty} \frac{\ln[n(1+\varepsilon)]}{\ln n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\ln \ln n}{\ln n} = 0.$$

Using these results with (2) gives

$$\lim_{n\to\infty} \frac{\ln \pi(n)}{\ln n} = 1,$$

which, in conjunction with (1), implies that

$$\lim_{n\to\infty} \frac{\ln \pi(n)}{\ln n} \frac{\pi(n) \ln n}{n} = 1.$$

Finally, with $n = p_k$ we have

$$\lim_{k\to\infty} \frac{k \ln k}{p_k} = 1, \tag{3}$$

telling us that, for large k, the kth prime is roughly equal to $k \ln k$.

Note that this result does not imply that the kth prime will always be close to $k \ln k$ in absolute terms. Rather, it tells us that the relative error in using $k \ln k$ for p_k tends to zero as k increases without limit.

3. Summing the reciprocals of the primes

We all know that the geometric series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots$$

and the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots$$

converge (to 2) and diverge respectively. However, the behaviour of the series of reciprocals of the primes,

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \cdots,$$

is probably less well known. It does in fact diverge, and we utilise the PNT to prove this here.

From (3) we know that there exists some $N \in \mathbb{N}$ such that $p_n < 2n \ln n$ for all $n \geq N$. Then for any $M > N$ we have

$$\begin{aligned} \sum_{n=1}^{M} \frac{1}{p_n} &= \sum_{n=1}^{N-1} \frac{1}{p_n} + \sum_{n=N}^{M} \frac{1}{p_n} \\ &> \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{2} \sum_{n=N}^{M} \frac{1}{n \ln n}. \end{aligned}$$

Now

$$\sum_{n=N}^{M} \frac{1}{n\ln n} > \int_{N}^{M+1} \frac{1}{x\ln x}\,\mathrm{d}x$$
$$= \ln\ln(M+1) - \ln\ln N,$$

from which we see that

$$\sum_{n=1}^{M} \frac{1}{p_n} > \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{2}(\ln\ln(M+1) - \ln\ln N),$$

and hence that

$$\sum_{n=1}^{M} \frac{1}{p_n} \to \infty \quad \text{as } M \to \infty.$$

An elementary proof of this result appears as Theorem 1.13 in reference 1.

We can actually go a little further than this and obtain an asymptotic relation for $\sum_{n=1}^{M} 1/p_n$. Result (3) tells us that for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$(1-\varepsilon)n\ln n < p_n < (1+\varepsilon)n\ln n,$$

for all $n \geq N$. Thus for any $M > N$ it is true that

$$\begin{aligned}\sum_{n=1}^{M} \frac{1}{p_n} &> \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{1+\varepsilon}\sum_{n=N}^{M} \frac{1}{n\ln n} \\ &> \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{1+\varepsilon}\int_{N}^{M+1} \frac{1}{x\ln x}\,\mathrm{d}x \\ &= \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{1+\varepsilon}(\ln\ln(M+1) - \ln\ln N), \end{aligned} \tag{4}$$

and similarly

$$\sum_{n=1}^{M} \frac{1}{p_n} < \sum_{n=1}^{N-1} \frac{1}{p_n} + \frac{1}{1-\varepsilon}(\ln\ln M - \ln\ln(N-1)), \tag{5}$$

provided that $0 < \varepsilon < 1$. From (4) and (5) it follows that

$$\sum_{k=1}^{n} \frac{1}{p_k} \sim \ln\ln n.$$

4. Bertrand's postulate

We first note that Bertrand's postulate (BP) is a theorem (conjectured by Joseph Bertrand in 1845 but first proved by Pafnuty Chebyshev in 1850) rather than a postulate, but its name just seems to have stuck over the years. It says that if you were to pick any $n \in \mathbb{N}$ then there will definitely be at least one prime p satisfying $n < p \leq 2n$.

Another point to be made here is that BP is quite different in character to the previous result. Putting things rather simplistically, the result concerning the divergence of the series of reciprocals of the primes tells us what will happen in the long term, while BP provides information about a certain property of the positive integers at all stages. The PNT is another 'long-term' result, and for this reason we cannot expect it to lead to a complete proof of BP.

In our quest for a partial result, let us suppose that there exists some $m \in \mathbb{N}$ such that there is no prime p satisfying $m < p \le 2m$. Then $\pi(m) = \pi(2m)$ and we have

$$\frac{\pi(m)\ln m}{m} \bigg/ \frac{\pi(2m)\ln 2m}{2m} = \frac{2\ln m}{\ln 2m}. \tag{6}$$

If we suppose further that there exist arbitrarily large values of m for which there is no prime p satisfying $m < p \le 2m$ then, noting that

$$\lim_{m\to\infty} \frac{2\ln m}{\ln 2m} = 2,$$

it must be the case that for any $n \in \mathbb{N}$ there exists some $m > n$ such that the left-hand side of (6) is as close as we like to 2. This would imply that it is not true that

$$\lim_{n\to\infty} \frac{\pi(n)\ln n}{n} = 1,$$

thereby contradicting the PNT. From this we are able to establish the fact that there exists some $N \in \mathbb{N}$ such that if $n > N$ then there is a prime p satisfying $n < p \le 2n$. Thus the PNT implies only an extremely weak version of BP. It tells us that BP must be true from some point on, but does not give any indication whatsoever as to where this point is!

For interested readers, a detailed and elementary proof of BP, exploiting various mathematical properties of the central binomial coefficients, can be found in reference 2.

References

1 T. M. Apostol, *Introduction to Analytic Number Theory* (Springer, New York, 1976).
2 M. Griffiths, *The Backbone of Pascal's Triangle* (United Kingdom Mathematics Trust, Leeds, 2008).

Martin Griffiths *joined the teaching profession following a career in the British Army. He is now both Head of Mathematics at a grammar school in Colchester and a part-time Lecturer in Mathematics at the University of Essex. His doctorate was in the field of epidemic modelling, although he has a far wider range of mathematical interests. Indeed, in addition to his contributions to 'Mathematical Spectrum', over 40 of his articles have been published (or accepted for publication) in journals as diverse as 'The Mathematical Gazette', 'Journal of Mathematical Biology', and 'The Fibonacci Quarterly'. Many of these articles arose as a consequence of ideas that originated in the classroom or lecture theatre. He is also the author of a book about the central binomial coefficients, published by the United Kingdom Mathematics Trust and aimed at able 16–20-year-old students and their teachers. He is currently Reviews Editor of 'The Mathematical Gazette'.*

Functions Satisfying Two Trigonometric Identities

DENIS BELL

In this note we study two familiar identities in trigonometry, the addition formula for the sine function and the Pythagorean identity. We characterize the set of functions satisfying these identities. This leads to a surprising conclusion.

Two fundamental identities in trigonometry are the Pythagorean identity,

$$\sin^2 x + \cos^2 x = 1,$$

and the addition formula for the sine function,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Of course, using calculus these formulae are expressible in terms of the single function $s(x) = \sin x$ as

$$s^2(x) + s'^2(x) = 1 \tag{1}$$

and

$$s(x+y) = s(x)s'(y) + s'(x)s(y). \tag{2}$$

It is natural to ask *which other functions satisfy these equations*? In this article we explore this question. Clearly, the constant functions 1 and -1 satisfy (1) and the zero function satisfies (2), so we exclude these trivial solutions from the discussion.

Henceforth, all functions are assumed to be real-valued with domain $\mathbb{R}$. The function f is said to be C^k $(k = 1, 2)$ if f is k-times differentiable with continuous kth order derivative.

Theorem 1 *Suppose that s is a C^1 function satisfying (1). Then s consists locally of functions of the form $\sin(x+\phi)$, for some constant ϕ, 1, -1 pieced together in a continuous fashion. If s is C^2, then s has the form*

$$s(x) = \sin(x+\phi). \tag{3}$$

Proof Note that since s is assumed to be real-valued, (1) implies that $|s(x)| \le 1$ for all x. Suppose that $|s(x_0)| < 1$ for some x_0. Then $|s(x)| < 1$ for x in an open interval I containing x_0. The following argument discusses the behaviour of s on the interval I and characterizes s on this interval.

Because s' is continuous, (1) implies that s' is one of the functions $\sqrt{1-s^2}$ or $-\sqrt{1-s^2}$. We then have

$$s'' = \frac{-ss'}{\sqrt{1-s^2}} \qquad \left(\text{or } \frac{ss'}{\sqrt{1-s^2}}\right),$$

and it follows that s is C^2 on I. Differentiating in (1), we obtain

$$s'(s+s'') = 0. \tag{4}$$

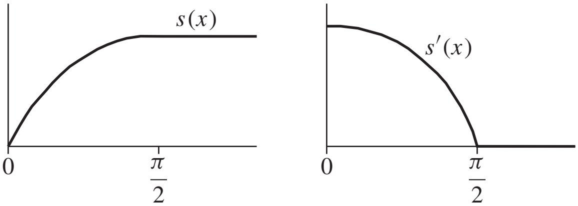

Figure 1

Since $|s| < 1$, (1) implies that $s' \neq 0$ and (4) then implies that $s'' = -s$. It follows from the uniqueness theorem for linear second-order differential equations that s has the form

$$s(x) = a \sin x + b \cos x, \qquad x \in I,$$

where $a^2 + b^2 = 1$ by (1).

Using the addition formula for $\sin$, we can write $s(x)$, $x \in I$, in the form (3), where $\cos \phi = a$ and $\sin \phi = b$. It is easy to see that piecing together these sine functions and the constant functions 1 and -1 in a continuous fashion produces a C^1 function satisfying (1) and that this is the most general such solution.

Finally, we observe that if s is composed of more than one of these components then it will fail to be C^2 at the join point(s). An example is the function

$$s(x) = \begin{cases} \sin x, & x \leq \dfrac{\pi}{2}, \\ 1, & x > \dfrac{\pi}{2}, \end{cases}$$

where s' is continuous but nondifferentiable at $x = \pi/2$ (see figure 1).

Now consider the addition formula (2). Here we have the following result.

Theorem 2 *Suppose that s is a C^1 function satisfying (2). If $s(0) \neq 0$ then s has the form $s(x) = c\mathrm{e}^{x/2c}$, for some nonzero constant c. If $s(0) = 0$ then s has one of the following forms: $s(x) = x$, $s(x) = c \sin(x/c)$, or $s(x) = c \sinh(x/c)$.*

Proof Setting $y = 0$ in (2) gives

$$s(x) = s(x)s'(0) + s'(x)s(0). \tag{5}$$

Suppose first that $s(0) \neq 0$. Note that in this case $s'(0) \neq 1$ otherwise (5) implies that s is constant; hence, $s \equiv 0$ by (2). We may therefore write (5) in the form

$$s'(x) = \alpha s(x)$$

where $\alpha \neq 0$. This has solution $s(x) = c\mathrm{e}^{\alpha x}$. Substituting into (2) and solving for α gives $\alpha = 1/2c$ as stated.

We next consider the case $s(0) = 0$. Equation (5) now implies $s'(0) = 1$, otherwise we would again have $s \equiv 0$. Choose y_0 such that $s(y_0) \neq 0$. Setting $y = y_0$ in (2), we conclude that $x \mapsto s(x)$ is C^2. Differentiating (2) in x then in y gives the equations

$$s'(x + y) = s'(x)s'(y) + s''(x)s(y),$$
$$s'(x + y) = s'(x)s'(y) + s''(y)s(x).$$

It follows that

$$s''(x)s(y) = s''(y)s(x).$$

Thus

$$s''(x) = \beta s(x), \tag{6}$$

where $\beta = s''(y_0)/s(y_0)$, with y_0 as above. Solving (6) with the initial conditions $s(0) = 0$ and $s'(0) = 1$ and using (2) we obtain

1. $s(x) = x$, if $\beta = 0$,
2. $s(x) = c\sin(x/c)$, where $c = \sqrt{-1/\beta}$, if β is negative,
3. $s(x) = c\sinh(x/c)$, where $c = \sqrt{1/\beta}$, if β is positive.

This completes the proof.

Remark We note that in the case $s(0) = 0$, the hyperbolic solutions in Theorem 2 can be obtained from $c\sin(x/c)$ by replacing c by $-\mathrm{i}c$ and using the identity $\sin(\mathrm{i}x) = \mathrm{i}\sinh x$. Furthermore, the remaining solution $s(x) = x$ arises as $\lim_{c\to\infty} c\sin(x/c)$. Thus the three solution types satisfying $s(0) = 0$ can be subsumed into a single family.

Finally, combining Theorems 1 and 2, we have our main result.

Theorem 3 *The only C^1 function s satisfying both identities (1) and (2) is* $\sin x$.

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Cancelling

$$\frac{19}{95} = \frac{1}{5},$$

$$\frac{3544}{7531} = \frac{344}{731},$$

$$\frac{2666}{6665} = \frac{266}{665} = \frac{26}{65} = \frac{2}{5},$$

$$\frac{143185}{17018560} = \frac{1435}{170560}.$$

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Symmetry and the Nine-Point Circle

JINGCHENG TONG and SIDNEY KUNG

Using two kinds of symmetries, we produce the nine-point circle and show that it has three additional lesser-known points.

In this article, using two kinds of symmetries, we produce the *nine-point circle* (the *Feuerbach circle*) and show that it has three additional lesser-known points. Our method is relatively simple in comparison with those given elsewhere (see references 1–5), and it allows us to disclose the symmetric property of the well-known circle.

We give two definitions on symmetry and a lemma.

Definition 1 Points P and P' are *central symmetric* about point I if I is on PP' and $PI = IP'$ (see figure 1).

Definition 2 Points Q and Q' are *axial symmetric* about line l if l is the perpendicular bisector of QQ' (see figure 2).

Lemma 1 *Let circle C, centre O, and circle C', centre O', be two congruent circles with RS as the common chord and $OO' \cap RS = U$ (see figure 3). If T is on circle C, then*

(i) *the central symmetric point T' of T about U is on circle C',*

(ii) *the axial symmetric point T'' of T about RS is on circle C'.*

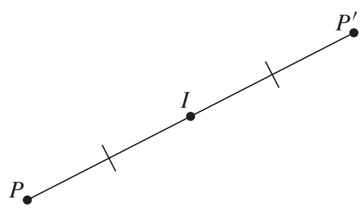

Figure 1

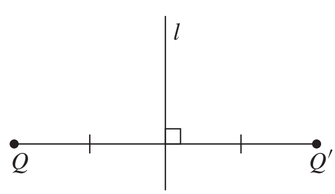

Figure 2

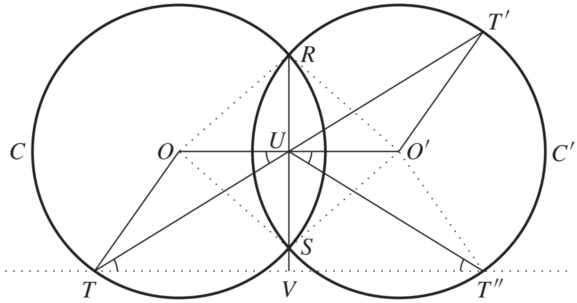

Figure 3

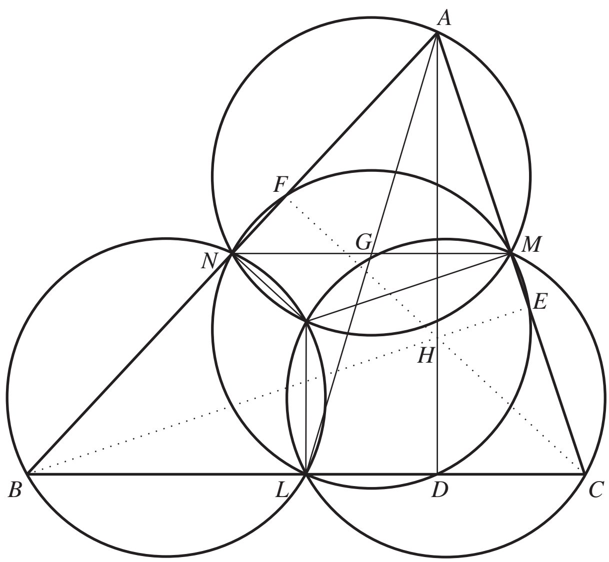

Figure 4

Proof Refer to figure 3. The figure $ORO'S$ is a rhombus, so $OU = UO'$ and OO' is perpendicular to RS. We see that $\triangle OTU \cong \triangle O'T'U$ (side-angle-side), so that $O'T' = OT$. (The symbol '$\cong$' denotes 'is congruent to'.) Hence (i) holds. Let $TT'' \cap RS = V$. Then $\triangle TUV \cong \triangle T''UV$ (side-angle-side). Thus, $TU = T''U$ and $\angle TUV = \angle T''UV$. Hence $\angle OUT = \angle O'UT''$ and $\triangle OTU \cong \triangle O'T''U$. Therefore, $O'T'' = OT$ and we have (ii).

Consider any triangle ABC (see figure 4). Denote by L, M, and N the mid-points of the sides BC, AC, and AB respectively; D, E, and F are the feet of the altitudes on these sides, and H is the orthocentre.

Triangles AMN and LMN are congruent, so their circumcircles are congruent with MN as the common chord. Also D is the axial symmetric point of A about MN. Hence, by lemma 1, D is on the circumcircle of $\triangle LMN$. In a similar manner, E and F are also on the circumcircle of $\triangle LMN$.

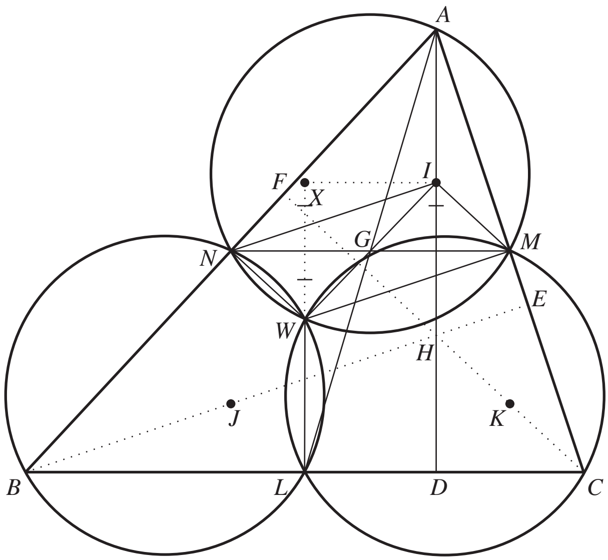

Figure 5

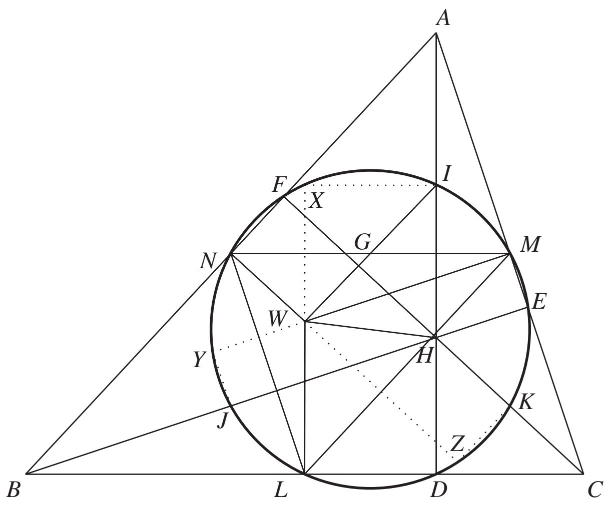

Figure 6

Refer to figure 5. Let W be the circumcentre of $\triangle ABC$. Since WM and WN are perpendicular to CA and AB respectively, W is on the circumcircle of $\triangle ANM$. Similarly, W is on the circumcircles of triangles BLN and CML.

Let I be the mid-point of AH. Since $NI \| BH \| WM$ and $IM \| FC \| NW$, the figure $NIMW$ is a parallelogram. Hence $IG = GW$. Thus, I is the central symmetric point of W about G, so it must be on the circumcircle of $\triangle LMN$. In a similar fashion, we see that the mid-points J and K of BH and CH respectively are on the circumcircle of $\triangle LMN$.

We have now found the nine points of the nine-point circle, namely L, M, N, D, E, F, I, J, and K. Our next step is to find three lesser-known points on the nine-point circle.

At I, we draw the perpendicular to AD intersecting LW produced at X (see figure 6). Since $\angle LDI = 90°$, IL (not shown) is a diameter of the nine-point circle. Also, $\angle LXI = 90°$ so X lies on it. Similarly, Y (or Z), which is the intersection of the perpendicular bisector of BH (or CH) and MW produced (or NW produced), is on it as well.

Thus, we have shown that there are twelve special points lying on the circumcircle of $\triangle LMN$, and that the twelve points are vertices of three rectangles. The centres of the rectangles will be the centre of the circle. These interesting features are not seen in the traditional proofs of the nine-point circle theorem.

We believe that more interesting results could be found by exploring the 3-dimensional case (for a sphere) with properly defined planar symmetry.

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Number patterns

$$4 \times 4 = 16,$$
$$34 \times 34 = 1156,$$
$$334 \times 334 = 111556,$$
$$3334 \times 3334 = 11115556,$$
$$33334 \times 33334 = 1111155556,$$

and so on.

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A Trip from Trig to Triangle

MICHEL BATAILLE

A selection of identities and inequalities for the triangle, derived from the properties of a trigonometric function.

Introduction

The purpose of this article is to offer the reader a short excursion into the territories of trigonometry and geometric inequalities. The first stage, the trigonometric part, will unify the proofs of some frequently used identities for the angles of a triangle (see reference 1 for a recent occurrence in *Mathematical Spectrum*). Surprisingly, the auxiliary function that provides these results also unifies the proofs of four beautiful, easy-to-memorize inequalities. This will naturally lead us to the last stage of this article: a selection of known or lesser known inequalities for the general triangle, proved by means of the results previously established.

Some trigonometric identities

The following function ϕ of three variables will serve our unifying aim:

$$\phi(x, y, z) = \sin x \cos(y - z) + \sin y \cos(z - x) + \sin z \cos(x - y). \tag{1}$$

Using well-known trigonometric formulas, we first deduce interesting alternative expressions for this function. For example, $\sin a \cos b = \frac{1}{2}(\sin(a + b) + \sin(a - b))$ with $a = x$ and $b = y - z$ yields

$$\sin x \cos(y - z) = \tfrac{1}{2}(\sin(x + y - z) + \sin(z + x - y)).$$

Transforming the other two terms in the same way and adding up give the following identity:

$$\phi(x, y, z) = \sin(x + y - z) + \sin(y + z - x) + \sin(z + x - y).$$

On the other hand, if in (1) we expand $\cos(y - z)$ as $\cos y \cos z + \sin y \sin z$ (and similarly $\cos(z - x)$, $\cos(x - y)$), we obtain

$$\begin{aligned}\phi(x, y, z) &= 3 \sin x \sin y \sin z + \sin x \cos y \cos z + \sin y \cos z \cos x + \sin z \cos x \cos y \\ &= 3 \sin x \sin y \sin z + \cos z \sin(x + y) + \sin z \cos(x + y) + \sin x \sin y \sin z,\end{aligned}$$

and so

$$\phi(x, y, z) = \sin(x + y + z) + 4 \sin x \sin y \sin z.$$

As a corollary, we single out two useful particular cases:

$$\text{if } x + y + z = \pi, \qquad \phi(x, y, z) = \sin(2x) + \sin(2y) + \sin(2z) = 4 \sin x \sin y \sin z, \tag{2}$$

$$\text{if } x + y + z = \frac{\pi}{2}, \qquad \phi(x, y, z) = \cos(2x) + \cos(2y) + \cos(2z) = 1 + 4 \sin x \sin y \sin z. \tag{3}$$

Some identities for the angles of a triangle

Let ABC be any triangle and, as usual, $A = \angle BAC$, $B = \angle CBA$, and $C = \angle ACB$ be its angles. Since $A + B + C = \pi$, (2) provides the following identities:

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$
$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \tag{4}$$

(the latter by taking $x = \pi/2 - A/2$, $y = \pi/2 - B/2$, $z = \pi/2 - C/2$), while (3) gives

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, \tag{5}$$
$$\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$$

(taking successively $x = A/2$, $y = B/2$, $z = C/2$ and $x = \pi/2 - A$, $y = \pi/2 - B$, $z = \pi/2 - C$). The reader is invited to find less common identities from other choices of the triple (x, y, z).

Various relationships between the elements of a triangle can be derived from these identities. For later use, here is an example connecting the inradius r and the circumradius R of triangle ABC. We shall make use of the basic results

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2F} \quad \text{(law of sines)} \qquad \text{and} \qquad 2F = (a + b + c)r,$$

where a, b, and c are the sides opposite to angles A, B, and C respectively and F is the area of ΔABC. First, we readily deduce that

$$2Rr(\sin A + \sin B + \sin C) = \frac{1}{2R}((2R)^3 \sin A \sin B \sin C),$$

and then, substituting (4) on the left-hand side and the formula $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ on the right-hand side, we obtain

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \tag{6}$$

Note that from (5), this can also be written as

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}.$$

Four beautiful inequalities

As we shall now see, interesting inequalities can be found, using the function ϕ again. This is the case of the following batch of four:

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \geq \cos A + \cos B + \cos C, \tag{7}$$
$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} \geq \sin A + \sin B + \sin C, \tag{8}$$
$$\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \geq \sin A \sin B \sin C, \tag{9}$$
$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \geq \cos A \cos B \cos C. \tag{10}$$

The inequalities (7) and (8) date back to 1973, when the famous problemist Leon Bankoff proposed (7) as a problem in *Mathematics Magazine* (see reference 2). A unified proof is given here, using the following property immediately deduced from the very definition of ϕ.

If $\sin x$, $\sin y$, and $\sin z$ are nonnegative, then $\phi(x, y, z) \le \sin x + \sin y + \sin z$. Then (7) and (8) quickly follow through

$$\phi\left(\frac{A}{2}, \frac{B}{2}, \frac{C}{2}\right) \le \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$$

and

$$\phi\left(\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2}\right) \le \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) + \sin\left(\frac{\pi}{2} - \frac{B}{2}\right) + \sin\left(\frac{\pi}{2} - \frac{C}{2}\right),$$

using (3) and (2).

Also, $\phi(A, B, C) \le \sin A + \sin B + \sin C$ combined with (2) and (4) gives (9). As for (10), it obviously holds if the triangle is not acute-angled. Otherwise, $\sin(\pi/2 - A)$, $\sin(\pi/2 - B)$, and $\sin(\pi/2 - A)$ are nonnegative; hence,

$$\phi\left(\frac{\pi}{2} - A, \frac{\pi}{2} - B, \frac{\pi}{2} - C\right) \le \cos A + \cos B + \cos C,$$

and we finally get (10) making use of (3) and (5).

More inequalities in the triangle

Keeping the notations introduced above, we start with a famous inequality, namely Euler's inequality:

$$R \ge 2r. \tag{11}$$

With the help of previous results, we can give a quick proof. From (9),

$$\sin A \sin B \sin C = 8\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right) \le \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2};$$

hence,

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le 1$$

and Euler's inequality (11) follows from (6).

Note that with (5), we obtain $\cos A + \cos B + \cos C \le \frac{3}{2}$. Replacing A, B, and C by $\pi/2 - A/2$, $\pi/2 - B/2$, and $\pi/2 - C/2$, which are also the angles of a triangle, gives the stronger inequality

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}. \tag{12}$$

(For another proof of (12), see reference 3.)

A natural companion to the latter is

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \le \frac{3\sqrt{3}}{2},$$

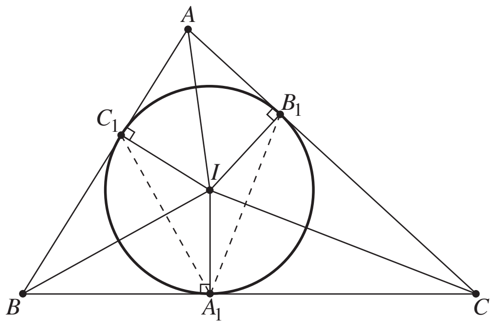

Figure 1

that we shall prove in a rather unexpected way. Let $q(ABC)$ denote the ratio of the semi-perimeter of ΔABC to its circumradius, that is $q(ABC) = \sin A + \sin B + \sin C$ (from the law of sines).

Suppose that the incircle is tangent to the sides BC, CA, and AB at A_1, B_1, and C_1 respectively, and call $\Delta A_1B_1C_1$ the in-triangle of ΔABC (see figure 1). Then, observing that, for example, $A_1 = \angle B_1A_1C_1 = \frac{1}{2}\angle B_1IC_1$ (where I is the incentre), we see that the angles of $\Delta A_1B_1C_1$ are $A_1 = \pi/2 - A/2$, $B_1 = \pi/2 - B/2$, and $C_1 = \pi/2 - C/2$.

From (8), it follows that $q(A_1B_1C_1) \geq q(ABC)$. Now, let $A_0 = A$, $B_0 = B$, and $C_0 = C$, and for a positive integer n, let $\Delta A_nB_nC_n$ be the in-triangle of $\Delta A_{n-1}B_{n-1}C_{n-1}$. The sequence $(q(A_nB_nC_n))$ is increasing and an easy induction shows that the angles of $\Delta A_nB_nC_n$ are

$$A_n = \frac{\pi}{3} + \frac{(-1)^n}{2^n}\left(A - \frac{\pi}{3}\right), \qquad B_n = \frac{\pi}{3} + \frac{(-1)^n}{2^n}\left(B - \frac{\pi}{3}\right), \qquad C_n = \frac{\pi}{3} + \frac{(-1)^n}{2^n}\left(C - \frac{\pi}{3}\right).$$

Clearly,

$$\lim_{n\to\infty} q(A_nB_nC_n) = \lim_{n\to\infty} (\sin A_n + \sin B_n + \sin C_n) = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2},$$

and so

$$q(A_1B_1C_1) = \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}.$$

Note in passing that we have also obtained the well-known inequality $a + b + c \leq 3\sqrt{3}R$.

To end our journey, we propose a nice result, slightly stronger than Euler's inequality. Consider the excenters I_1, I_2, and I_3 of triangle ABC and let r' denote the inradius of triangle $I_1I_2I_3$ (see figure 2). We prove that $R \geq r' \geq 2r$.

Let I_1 be the centre of the A-excircle. Since $\angle I_1BC = \pi/2 - B/2$ and $\angle I_1CB = \pi/2 - C/2$, we have

$$\angle I_2I_1I_3 = \angle BI_1C = \frac{1}{2}(B + C) = \frac{\pi}{2} - \frac{A}{2}.$$

Thus, the angles of the triangle $I_1I_2I_3$ are

$$\frac{\pi}{2} - \frac{A}{2}, \qquad \frac{\pi}{2} - \frac{B}{2}, \qquad \frac{\pi}{2} - \frac{C}{2}.$$

Let temporarily R' be the circumradius of $\Delta I_1I_2I_3$.

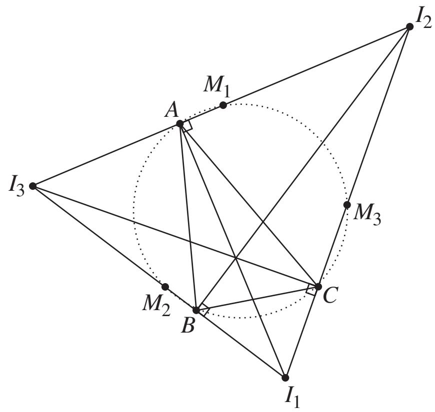

Figure 2

Using (6), (3), (7), (5), and (6) again in succession, we have

$$\begin{aligned}\frac{r'}{R'} &= 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)\\ &= \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1\\ &\geq \cos A + \cos B + \cos C - 1\\ &= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\ &= \frac{r}{R},\end{aligned}$$

so that $r' \geq rR'/R$.

To complete the proof, we notice that A, B, and C are the feet of the altitudes of $\Delta I_1I_2I_3$ (since the internal and external bisectors of an angle are perpendicular). It follows that the circumcircle of ΔABC is the nine-point circle of $\Delta I_1I_2I_3$. This circle, which also passes through the midpoints M_1, M_2, and M_3 of I_2I_3, I_3I_1, and I_1I_2 (see figure 2 and the preceding article). Since triangle $M_1M_2M_3$ is similar to triangle $I_1I_2I_3$ with scale factor $\frac{1}{2}$, $R = \frac{1}{2}R'$ and the inequality $r' \geq rR'/R$ becomes $r' \geq 2r$. Moreover, from (11) applied to $\Delta I_1I_2I_3$, we have $R' \geq 2r'$, hence $2R \geq 2r'$, and finally $R \geq r' \geq 2r$, as desired.

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Routh's Theorem Revisited

AYOUB B. AYOUB

The author provides a new proof of Routh's theorem.

Introduction

Towards the end of the 17th century, the Italian engineer Giovanni Ceva proved the theorem that now carries his name. It is stated as follows.

Ceva's theorem *If D, E, and F are points on the sides BC, CA, and AB of triangle ABC then the line segments AD, BE, and CF meet in one point if and only if*

$$\frac{AF}{FB}\,\frac{BD}{DC}\,\frac{CE}{EA} = 1$$

(see figure 1).

The line segments AD, BE, and CF joining the vertices A, B, and C with points D, E, and F on opposite sides of the triangle ABC are called *cevians* in honour of Ceva.

Routh's theorem

In 1896, the English mathematician E. J. Routh introduced a generalization of Ceva's theorem (see reference 1, p. 82). Routh's theorem can be stated as follows.

Routh's theorem *If AD, BE, and CF are not necessarily concurrent cevians of triangle ABC, and if $AF/FB = x$, $BD/DC = y$, and $CE/EA = z$, then the ratio of the area of the triangle OMN, formed by the cevians, to the area of triangle ABC is*

$$\frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)}$$

(see figure 2).

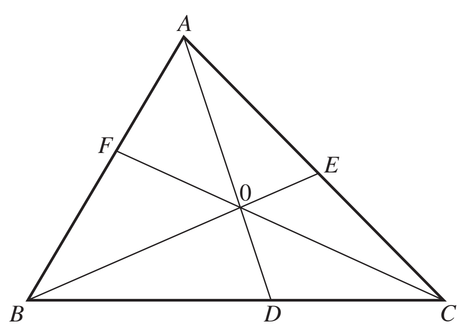

Figure 1

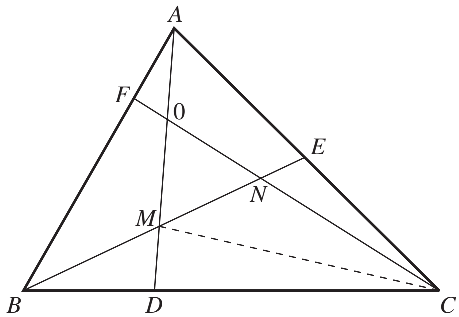

Figure 2

Thus the cevians AD, BE, and CF will meet in one point if and only if $xyz = 1$ and that is exactly the assertion of Ceva's theorem. So Ceva's theorem is a special case of Routh's theorem when the cevian triangle OMN degenerates to a point.

Proof of Routh's theorem

There are several proofs of this theorem (see references 2 (pp. 217–220), 3 (pp. 205–207), 4, 5, 6 (pp. 7–9), and 7), and here we introduce a new proof based on the following theorem.

Menelaus' theorem *If a transversal FED meets the sides AB, AC, and BC of triangle ABC at F, E, and D respectively, then*

$$\frac{AF}{FB}\frac{BD}{CD}\frac{CE}{EA} = 1$$

(see figure 3).

Figure 2 implies that

$$\frac{\triangle MNO}{\triangle ABC} = \frac{\triangle MNO}{\triangle MCO}\frac{\triangle MCO}{\triangle ADC}\frac{\triangle ADC}{\triangle ABC}, \tag{1}$$

where $\triangle ABC$ denotes the area of triangle ABC, and so on.

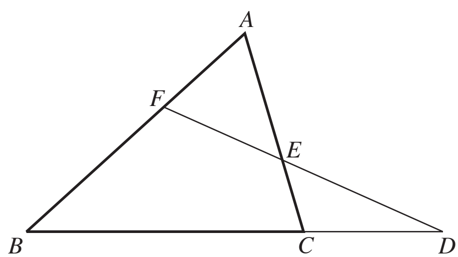

Figure 3

Each of the three ratios on the right-hand side of (1) involves two triangles sharing a height, so the ratio is equal to the ratio between the lengths of the bases. Consequently,

$$\frac{\triangle MNO}{\triangle ABC} = \frac{ON}{OC}\,\frac{MO}{DA}\,\frac{DC}{BC}. \tag{2}$$

Since $BD/DC = y$, then $BC/DC = (BD + DC)/DC = y + 1$, or

$$\frac{DC}{BC} = \frac{1}{y+1}. \tag{3}$$

To express MO/DA in terms of x, y, and z, we apply Menelaus' theorem to triangle ABD together with the transversal FOC to give

$$\frac{AF}{FB}\,\frac{BC}{DC}\,\frac{DO}{OA} = 1.$$

Hence

$$\frac{x}{1}\,\frac{y+1}{1}\,\frac{DO}{OA} = 1,$$

from which we have

$$\frac{DO}{OA} = \frac{1}{x(y+1)}.$$

Thus,

$$\frac{DA}{OA} = \frac{DO+OA}{OA} = \frac{1}{x(y+1)} + 1 = \frac{xy+x+1}{x(y+1)}. \tag{4}$$

Similarly, if we consider triangle ACD together with the transversal EMB, we get

$$\frac{AE}{EC}\,\frac{CB}{DB}\,\frac{DM}{MA} = 1,$$

from which we have

$$\frac{1}{z}\,\frac{y+1}{y}\,\frac{DM}{MA} = 1.$$

Hence,

$$\frac{DM}{MA} = \frac{yz}{y+1}.$$

Then,

$$\frac{DA}{MA} = \frac{DM+MA}{MA} = \frac{yz}{y+1} + 1 = \frac{yz+y+1}{y+1}. \tag{5}$$

From (4) and (5), we get

$$\frac{MO}{DA} = \frac{MA}{DA} - \frac{OA}{DA} = \frac{y+1}{zy+y+1} - \frac{x(y+1)}{xy+x+1} = \frac{(y+1)(1-xyz)}{(xy+x+1)(zy+y+1)}. \tag{6}$$

From (5) and (6) we have

$$\frac{MO}{MA} = \frac{MO}{DA}\,\frac{DA}{MA} = \frac{1-xyz}{xy+x+1}. \tag{7}$$

Rotating A, B, and C anticlockwise, (7) gives

$$\frac{ON}{OC} = \frac{1 - xyz}{zx + z + 1}. \tag{8}$$

Substituting (8), (6), and (3) in (2) yields

$$\frac{\triangle MNO}{\triangle ABC} = \frac{1 - xyz}{zx + z + 1} \, \frac{(y+1)(1 - xyz)}{(xy + x + 1)(yz + y + 1)} \, \frac{1}{y+1}.$$

Thus,

$$\frac{\triangle MNO}{\triangle ABC} = \frac{(xyz - 1)^2}{(xy + x + 1)(yz + y + 1)(zx + z + 1)}.$$

As an example of Routh's theorem, let the division ratios be $2 : 1$, $3 : 2$, and $4 : 3$. Then, according to Routh's theorem,

$$\frac{\triangle MNO}{\triangle ABC} = \frac{(4-1)^2}{(3+2+1)(2+1.5+1)(4+1)} = \frac{1}{15}.$$

While we gave Routh's theorem in terms of the division ratios of the sides, Routh himself gave it in terms of the lengths of the segments in which the sides are divided, i.e.

$$\frac{\triangle MNO}{\triangle ABC} = \frac{(AF \cdot BD \cdot CE - AE \cdot CD \cdot BF)^2}{(AB \cdot AC - AF \cdot AE)(BA \cdot BC - BF \cdot BD)(CB \cdot CA - CD \cdot CE)}$$

(see reference 1, p. 82). The interested reader may want to prove that the two forms are equivalent.

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600 Years of Prague's Horologe and the Mathematics Behind it

M. KŘÍŽEK, A. ŠOLCOVÁ and L. SOMER

The mathematical model of the astronomical clock of Prague was invented around 1410 by Jan Šindel. In honour of this great achievement we introduce and investigate a new term, the *Šindel sequence*. We show how it is connected with triangular numbers and the bellworks of the astronomical clock.

1. Historical remarks

In the centre of the Old Town in Prague, there is an astronomical clock – an interesting rarity visited by many tourists. We found that there is a surprising connection between this clock and triangular numbers. In this article we take note of special properties of these numbers that make the regulation of the bellworks more precise.

The mathematical model of the astronomical clock of Prague was invented by Jan Šindel (Johannes Andreae, circa 1375–1456). He was a professor at Prague University, which was founded in 1348. The clock was realized by Mikuláš (Nicolas) from Kadaň around 1410. It is placed inside an almost 60 meter high tower of the Old Town City Hall. The clock has two large dial-plates on the south wall of the tower.

The upper dial-plate is an astrolabe controlled by a clock mechanism (see figure 1). It represents a stereographic projection of the celestial sphere from its North Pole onto the tangent plane passing through the South Pole (see figure 2). The centre of the dial-plate thus corresponds to the South Pole of the celestial sphere. The smallest interior circle around the South Pole illustrates the Tropic of Capricorn, whereas the exterior circle illustrates the Tropic of Cancer.

Figure 1 The upper dial-plate of the astronomical clock in Prague.

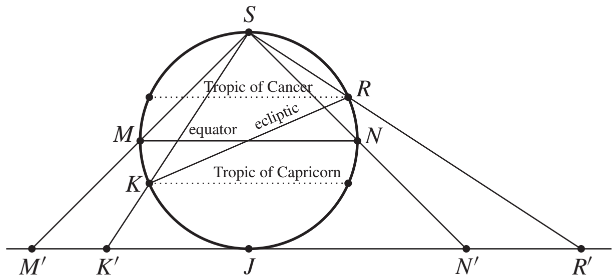

Figure 2 The stereographic projection of the celestial sphere from its North Pole. Notice that the diameter $|K'R'|$ of the projected ecliptic is larger than the diameter $|M'N'|$ of the projected equator, even though they have the same diameter $|KR| = |MN|$ on the celestial sphere.

The concentric circle between them corresponds to the equator of the celestial sphere (see figure 1).

An important property (known already to Ptolemy) of the stereographic projection is given in the following theorem.

Theorem 1 *Any circle on the sphere which does not pass through the North Pole (circles passing through the North Pole are mapped onto straight lines), is mapped onto a circle.*

Therefore, the ecliptic on the celestial sphere is projected onto a circle, which is represented by the gilded ring with zodiac signs along the ecliptic. However, its centre is not in the South Pole, but the ring rotates eccentrically around this pole (see figure 1). The astronomical clock also shows the approximate position of the Sun on the ecliptic, the motion of the Moon and its phases, the rising, culmination, and setting of the Sun, the Moon, and zodiac signs.

The gilded solar hand indicates the Central-European time in the ring of Roman numbers. The clock-hand with a small gilded asterisk shows the sidereal time (see figure 1). (The sidereal time is the time measured by the rotation of the Earth with respect to the stars. One day in sidereal time is equal to 23 hours 56 minutes and 4 seconds.) Twenty-four golden Arabic numerals are used for the ancient Czech time. Twelve black Arabic numerals denote the planetary hours of the Babylonian time measured from sunrise. The black circular area at the bottom of the dial-plane corresponds to the astronomical night, when the Sun is lower than $18°$ below the horizon.

2. What mathematics is hidden behind the astronomical clock of Prague?

The ingenuity and skill of clockmakers of the 15th century can be demonstrated by the following example. The bellworks of the astronomical clock contains a large gear with 24 slots (the first two are connected) at increasing distances along its circumference (see figure 3). This arrangement allows for a periodic repetition of 1–24 strokes of the bell each day. There is also a small auxiliary gear whose circumference is divided by six slots into segments of arc lengths 1, 2, 3, 4, 3, 2 (see figure 4). These numbers form a period which repeats after each revolution and their sum is $s = 15$.

At the beginning of every hour a catch rises, both gears start to revolve and the bell chimes. The gears stop when the catch simultaneously falls back into the slots on both gears. The bell strikes $1 + 2 + \cdots + 24 = 300$ times every day. Since this number is divisible by $s = 15$, the small gear is always at the same position at the beginning of each day.

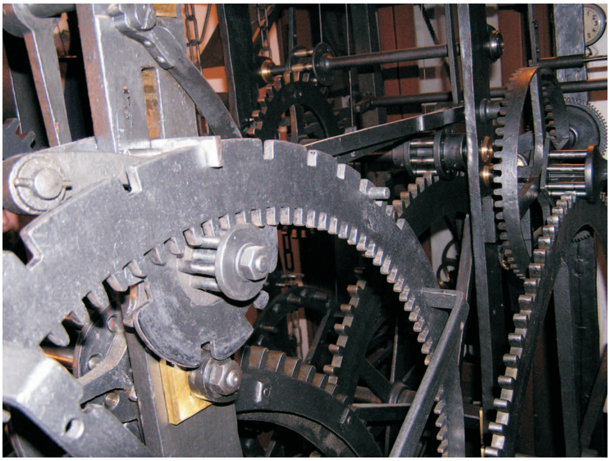

Figure 3 The bellworks of the Prague astronomical clock.

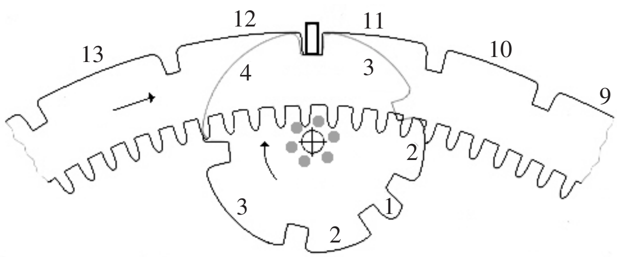

Figure 4 The number of bell strokes is denoted by the numbers $\ldots, 9, 10, 11, 12, 13, \ldots$ along the large gear. The small gear placed behind it is divided by slots into segments of arc lengths $1, 2, 3, 4, 3, 2$. The catch is indicated by a small rectangle on the top.

The large gear has 120 interior teeth which drop into a pin gear with six little horizontal bars that surround the centre of the small gear (see figures 3 and 4). The large gear revolves once per day and therefore the small gear revolves 20 times per day with approximately four times greater circumferential speed. Thus, the small gear makes the regulation of strokes sufficiently precise despite the wearing out of the slots on the large gear.

When the small gear revolves it generates by means of its slots a periodic sequence whose particular sums correspond to the number of strokes of the bell at each hour as follows:

$$\begin{array}{l} 1\ 2\ 3\ 4\ \ \underbrace{3\ 2}_{5}\ \ \underbrace{1\ 2\ 3}_{6}\ \ \underbrace{4\ 3}_{7} \\ \underbrace{2\ 1\ 2\ 3}_{8}\ \ \underbrace{4\ 3\ 2}_{9}\ \ \underbrace{1\ 2\ 3\ 4}_{10}\ \ \underbrace{3\ 2\ 1\ 2\ 3}_{11}\ \ \underbrace{4\ 3\ 2\ 1\ 2}_{12} \\ \underbrace{3\ 4\ 3\ 2\ 1}_{13}\ \ \underbrace{2\ 3\ 4\ 3\ 2}_{14}\ \ \underbrace{1\ 2\ 3\ 4\ 3\ 2}_{15}\ \ldots \end{array} \tag{1}$$

In theorem 2 of reference 1 we showed that we could continue indefinitely in this way. However, not all periodic sequences have such a nice summation property. For instance, we immediately find that the period 1, 2, 3, 4, 5, 4, 3, 2 could not be used for such a purpose, since $6 < 4 + 3$. Also the period 1, 2, 3, 2 could not be used, since $2 + 1 < 4 < 2 + 1 + 2$.

The astronomical clock of Prague is probably the oldest clock still functioning that contains such an apparatus. Due to the beautiful summation property (1), the sequence 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, ... was called the *clock sequence* in reference 2.

3. Connection of the bellworks with the triangular numbers

Now we show how the *triangular numbers*

$$T_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}, \qquad k = 0, 1, 2, \ldots, \tag{2}$$

are related to the bellworks. We shall look for all periodic sequences that have a similar property to the sequence 1, 2, 3, 4, 3, 2 in (1), i.e. one that could be used in the construction of the small gear. Put $\mathbb{N} = \{1, 2, \ldots\}$.

A sequence $\{a_i\}_{i=1}^{\infty}$ is said to be *periodic* if there exists $p \in \mathbb{N}$ such that

$$a_{i+p} = a_i, \quad \text{for all } i = 1, 2, \ldots. \tag{3}$$

The smallest p satisfying (3) is called the *period length* and the associated sequence $a_1, \ldots, a_p$ is called the *period*.

The periodic sequence $\{a_i\}$ of positive integers is said be a *Šindel sequence* if, for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$T_k = \sum_{i=1}^{n} a_i. \tag{4}$$

The triangular number T_k on the left-hand side is equal to the sum $1 + \cdots + k$ of hours on the large gear, whereas the sum on the right-hand side expresses the corresponding rotation of the small gear (see figure 5). For the kth hour, we have

$$k = T_k - T_{k-1} = \sum_{i=m+1}^{n} a_i, \tag{5}$$

where $T_{k-1} = \sum_{i=1}^{m} a_i$. Since $a_i > 0$, the number n depending on k in (4) is unique. We also see that $a_1 = 1$ when $\{a_i\}$ is a Šindel sequence.

Our next theorem shows that condition (4) can be replaced by a much weaker condition containing only a finite number of ks. This enables us to perform only a finite number of arithmetic operations to check whether a given period $a_1, \ldots, a_p$ yields a Šindel sequence. From now on let

$$s = \sum_{i=1}^{p} a_i \tag{6}$$

denote the sum of the period.

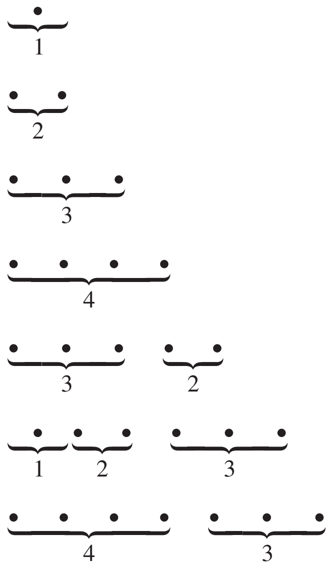

Figure 5 The bullets in the kth row indicate the number of strokes at the kth hour (see (5)). The numbers denote lengths of segments on the small gear.

Theorem 2 *A periodic sequence $\{a_i\}$ for s odd is a Šindel sequence if (4) holds for $k = 1, 2, \dots, (s-1)/2$.*

Proof The case $s = 1$ is trivial. So let $s \geq 3$ be odd and suppose that, for every $k \in \{1, 2, \dots, (s-1)/2\}$, there exists $n \in \mathbb{N}$ such that

$$T_k = \sum_{i=1}^{n} a_i. \tag{7}$$

We have to verify equality (7) for all $k \geq (s+1)/2$.

For $k = s-1$ we have

$$\begin{aligned} T_{s-1} &= 1 + 2 + \cdots + (s-1) \\ &= \tfrac{1}{2}(s-1)s \\ &= \tfrac{1}{2}(s-1)\sum_{i=1}^{p} a_i \\ &= (a_1 + \cdots + a_p) + \cdots + (a_1 + \cdots + a_p) \quad \text{with } \tfrac{1}{2}(s-1) \text{ brackets.} \end{aligned}$$

Suppose now that $k = s-1-k'$, where $1 \leq k' \leq (s-3)/2$ and $s > 3$. By assumption (7), there exists $n' \in \mathbb{N}$ such that

$$\frac{k'(k'+1)}{2} = \sum_{i=1}^{n'} a_i. \tag{8}$$

From (2) we observe that

$$T_k = T_{s-1-k'} = \frac{(s-1-k')(s-k')}{2} = \frac{s(s-1-2k')}{2} + \frac{k'(k'+1)}{2}. \tag{9}$$

Since $1 \le k' \le (s-3)/2$ and s is odd, it follows that $m = s-1-2k'$ is an even positive integer. Thus, by (9), (6), (8), and (3),

$$T_k = \frac{s-1-2k'}{2} \sum_{i=1}^{p} a_i + \sum_{i=1}^{n'} a_i = \sum_{i=1}^{pm/2+n'} a_i .$$

Next, let $k = qs + k'$ with $q \in \mathbb{N}$ and $0 \le k' < s$. Then by (2) and (6) we find that

$$T_k = \frac{(qs+k')(qs+k'+1)}{2} = sj + \frac{k'(k'+1)}{2} = \sum_{i=1}^{pj} a_i + T_{k'},$$

where $j = q(qs+1)/2 + qk'$ is an integer and $T_{k'} = 0$ for $k' = 0$. By our earlier observation in this proof $T_{k'} = \sum_{i=1}^{n'} a_i$ for some $n' \in \mathbb{N}$ when $0 < k' < s$.

Note that the number $(s-1)/2$ in theorem 2 cannot be reduced. To see this it is enough to consider the periodic sequence $\{a_i\}$ with the period 1, 2, 2, 1, 4, 1, 4 and $s = 15$. Here, $T_1, \ldots, T_6$ are all of the form $\sum_{i=1}^{n} a_i$, but T_7 is not.

The power of theorem 2 can be demonstrated on sequence (1) with $s = 15$. It is enough to check (4) only for $k \le (s-1)/2 = 7$ (see the first row of (1)) and the validity of (4) for $k > 7$ follows from theorem 2.

Acknowledgements

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Exploring Prime Decades Less Than Ten Billion

JAY L. SCHIFFMAN

Consider a sequence of ten consecutive integers of the form $\{a_0, a_1, \ldots, a_8, a_9\}$ with a_1, a_3, a_7, and a_9 all primes. Such a sequence is called a *prime decade*. This article will furnish an overview of prime decades less than ten billion.

1. Introduction

All primes with the exception of 2 and 5 necessarily terminate in the digits 1, 3, 7, or 9. Consider a sequence of ten consecutive integers of the form $\{a_0, a_1, \ldots, a_8, a_9\}$. A *prime decade* occurs if each of the integers terminating in the digits 1, 3, 7, and 9 is actually a prime. To cite an example, in the interval 10–19, each of the integers 11, 13, 17, and 19 is prime, yielding our initial prime decade. On the other hand, in the interval 40–49, while 41, 43, and 47 are primes, 49 is composite. Our goal is to introduce the reader to an easily-posed open problem in elementary number theory which has neat ramifications to a related problem; namely the famous twin-prime conjecture. (Twin primes are odd primes that differ by two, such as 11 and 13. Whether the number of twin prime pairs is infinite remains an open problem. In each prime decade, we have two pairs of twin primes.) We commence our exploration with table 1 generating the thirty-seven prime decades less than 10^5 with the distance to the next prime decade in parentheses.

Note that all prime decades must start when $p = 10k + 1$. In all other cases, we run into a multiple of 5. It is likewise the case that the initial integer p in any prime decade is such that $p \equiv 2 \pmod 3$ so that $p \equiv 11 \pmod{30}$. Hence a pair of prime decades must be a multiple of 30 apart.

As we peruse table 1, additional questions arise. While several prime decades are at a distance of 90 from one another, there is no pair of prime decades that is 30 apart, sometimes called a *twin prime decade*. Indeed such twin prime decades exist. We need to delve more deeply, and a MATHEMATICA® search renders the initial twin prime decade:

$$\{1006301, 1006303, 1006307, 1006309\}, \qquad \{1006331, 1006333, 1006337, 1006339\}.$$

What is even more striking is that the eight primes in the twin prime decade are consecutive. Unlike the prime decades {11, 13, 17, 19} and {101, 103, 107, 109} (there are primes between 19 and 101 such as 73), no primes exist between 1006309 and 1006331. If we seek to find the prime decades at distance 90 from one another which consist of consecutive primes, then there are a number of prime decades meeting this criteria. The initial such pair of prime decades is

$$\{1154454311, 1154454313, 1154454317, 1154454319\},$$
$$\{1154454401, 1154454403, 1154454407, 1154454409\}.$$

Table 1 The initial thirty-seven prime decades.

1	{11, 13, 17, 19}	(90)
2	{101, 103, 107, 109}	(90)
3	{191, 193, 197, 199}	(630)
4	{821, 823, 827, 829}	(660)
5	{1481, 1483, 1487, 1489}	(390)
6	{1871, 1873, 1877, 1879}	(210)
7	{2081, 2083, 2087, 2089}	(1170)
8	{3251, 3253, 3257, 3259}	(210)
9	{3461, 3463, 3467, 3469}	(2190)
10	{5651, 5653, 5657, 5659}	(3780)
11	{9431, 9433, 9437, 9439}	(3570)
12	{13001, 13003, 13007, 13009}	(2640)
13	{15641, 15643, 15647, 15649}	(90)
14	{15731, 15733, 15737, 15739}	(330)
15	{16061, 16063, 16067, 16069}	(1980)
16	{18041, 18043, 18047, 18049}	(870)
17	{18911, 18913, 18917, 18919}	(510)
18	{19421, 19423, 19427, 19429}	(1590)
19	{21011, 21013, 21017, 21019}	(1260)
20	{22271, 22273, 22277, 22279}	(3030)
21	{25301, 25303, 25307, 25309}	(6420)
22	{31721, 31723, 31727, 31729}	(3120)
23	{34841, 34843, 34847, 34849}	(8940)
24	{43781, 43783, 43787, 43789}	(7560)
25	{51341, 51343, 51347, 51349}	(3990)
26	{55331, 55333, 55337, 55339}	(7650)
27	{62981, 62983, 62987, 62989}	(4230)
28	{67211, 67213, 67217, 67219}	(2280)
29	{69491, 69493, 69497, 69499}	(2730)
30	{72221, 72223, 72227, 72229}	(5040)
31	{77261, 77263, 77267, 77269}	(2430)
32	{79691, 79693, 79697, 79699}	(1350)
33	{81041, 81043, 81047, 81049}	(1680)
34	{82721, 82723, 82727, 82729}	(6090)
35	{88811, 88813, 88817, 88819}	(9030)
36	{97841, 97843, 97847, 97849}	(1290)
37	{99131, 99133, 91137, 99139}	(1980)

2. Forbidden fruits

In relation to this section, a follow-up question concerns whether all distances that are multiples of 30 from a prime decade would at some point generate a new prime decade. For example, can two prime decades be 60 or 150 apart? The answer is in the negative, and the proof below relies on arithmetic modulo seven and is achieved via case analysis. Since p is prime, $p \not\equiv 0 \pmod 7$. In similar fashion, $p \not\equiv 1 \pmod 7$, otherwise $p + 6 \equiv 0 \pmod 7$. In an analogous manner, $p \not\equiv 5 \pmod 7$, otherwise $p + 2 \equiv 0 \pmod 7$. Finally $p \not\equiv 6 \pmod 7$, otherwise $p + 8 \equiv 0 \pmod 7$. Thus if $p \equiv 0, 1, 5, 6 \pmod 7$, this would contradict our assumption that $\{p, p+2, p+6, p+8\}$ comprises a prime decade. We next rule out the final cases, namely $p \equiv 2, 3, 4 \pmod 7$. If $p \equiv 2 \pmod 7$, then both $p + 68 \equiv 0 \pmod 7$ and $p + 152 \equiv 0 \pmod 7$. If $p \equiv 3 \pmod 7$, then $p + 60 \equiv 0 \pmod 7$ and $p + 158 \equiv 0 \pmod 7$. If $p \equiv 4 \pmod 7$, then both $p + 66 \equiv 0 \pmod 7$ and $p + 150 \equiv 0 \pmod 7$. Hence our assumption that $\{p, p+2, p+6, p+8\}$ constitutes a prime decade precludes

$$\{p+60, p+62, p+66, p+68\} \quad \text{and} \quad \{p+150, p+152, p+156, p+158\}$$

from being prime decades.

It is intriguing that 60 and 150 set the framework for what I affectionately classify as the *forbidden fruits*, that is multiples of 30 that are prohibited from being the distances between prime decades. The functions

$$f(n) = 60 + 210n \quad \text{and} \quad g(n) = 150 + 210n \qquad (n \text{ a nonnegative integer})$$

generate the output values that are forbidden. The proof is immediate since $210 \equiv 0 \pmod 7$.

3. Distances between prime decades

Our next goal is to furnish the reader with the first integer in the initial prime decade that is at a given distance from the next prime decade. We ran through all multiples of 30 that were permissible to 30000 in my search using MATHEMATICA 7.0. Table 2 generates all distances to 3600. We will utilize the term *proper distance* to denote that there is no prime decade lying between the two prime decades having distance d from one another. For example, the prime decades

$$\{11, 13, 17, 19\} \quad \text{and} \quad \{101, 103, 107, 109\}$$

are at distance 90 from one another and this distance is proper; for there is no other prime decade which lies between $\{11, 13, 17, 19\}$ and $\{101, 103, 107, 109\}$. The distance of 30 between the prime decades

$$\{1006301, 1006303, 1006307, 1006309\} \quad \text{and} \quad \{1006331, 1006333, 1006337, 1006339\}$$

is likewise proper. In contrast, the distance of 420 between the prime decades

$$\{15641, 15643, 15647, 15649\} \quad \text{and} \quad \{16061, 16063, 16067, 16069\}$$

is not proper, for the prime decade $\{15731, 15733, 15737, 15739\}$ lies between the two given prime decades. The distances between prime decades are in columns 1, 3, and 5 of table 2 with the first integer in the initial prime decade having that distance in the very next column to the right (columns 2, 4, and 6).

Table 2 Distances between prime decades (columns 1, 3, and 5) and the first integer in the initial prime decade having that distance (columns 2, 4, and 6). Proper distances are indicated by '+'.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6
30+	1006301	90+	11	120+	1022381
180	11	210+	1871	240+	632081
300+	1121831	330+	15731	390+	1481
420	15641	450+	1068251	510+	18911
540+	284741	600	1481	630+	191
660+	821	720	101	750+	875261
810	11	840	3512231	870+	18041
930+	958541	960+	680291	1020+	299471
1050	821	1080+	663581	1140+	165701
1170+	2081	1230	152154881	1260	821
1290	191	1350+	79691	1380	101
1440+	1022501	1470	11	1500+	300491
1560+	294311	1590	1871	1650+	72506501
1680	191	1710+	463451	1770	101
1800+	419051	1860	11	1890	191
1920+	6054281	1980	101	2010+	6332861
2070	11	2100	18911	2130+	266681
2190+	3461	2220+	855731	2280+	67211
2310	15731	2340+	7159511	2400	3251
2430	821	2490+	661091	2520	299471
2550+	257861	2610+	467471	2640	821
2700+	389561	2730	13001	2760+	116531
2820+	2846861	2850	16061	2910+	119291
2940	394811	2970	18041	3030+	22271
3060	191	3120+	31721	3150	101
3180	15731	3240	11	3270	191
3330+	654161	3360	101	3390+	51633731
3450	11	3480+	1912451	3540	4262171
3570	2081	3600+	295871	–	–

4. Prime decades less than one billion 30 and 90 apart

It is of interest to note that there are 65 prime decades at the minimal distance of 30 from their successors below ten billion, while there are 127 similar prime decades of distance 90 from their successors in the same range. In table 3, we list the first member in the first prime decade for the outcomes less than one billion.

5. Next steps

In this article, the reader is introduced to a fascinating open problem in elementary number theory, as it remains unknown as to whether there are infinitely many prime decades. The resolution of this problem has great ramifications. If there are infinitely many prime decades, then since each prime decade contains two pairs of twin primes, an infinitude of twin prime pairs is formed (the prime decade conjecture is stronger), resolving an open problem in elementary

Table 3 Prime decades less than one billion at distances 30 and 90 apart.

Column 1 (30 apart)	Column 2 (30 apart)	Column 3 (90 apart)	Column 4 (90 apart)
1006301	179028761	11	431343461
2594951	211950251	101	518137091
3919211	255352211	15641	543062621
9600551	267587861	3512981	588273221
10531061	557458631	6655541	637272191
108816311	685124351	20769311	639387311
131445701	724491371	26919791	647851571
152370731	821357651	41487071	705497951
157131641	871411361	71541641	726391571
–	–	160471601	843404201
–	–	189425981	895161341
–	–	236531921	958438751
–	–	338030591	960813851
–	–	409952351	964812461
–	–	423685721	985123961

number theory which has existed for centuries. It would likewise be of interest to determine whether there are prime decades for every distance that is a multiple of thirty not falling under the forbidden fruits. While I have found this to be the case to 30000, it means I have infinitely many more miles to go before I sleep! MATHEMATICA and other appropriate computer algebra software should enable the mathematical community to obtain further inroads into unchartered territory, paving the way for new and exciting explorations as well as dynamic results in the theory of numbers.

References

1 Mathematica 7.0, Wolfram Research, Inc., Champaign, IL (2008).
2 http://mathworld.wolfram.com

Jay L. Schiffman *has taught mathematics at Rowan University for the past seventeen years. His research interests are in number theory, discrete mathematics, and the interface of mathematics with technology in the classroom as manifested via the use of graphing calculators, computer algebra systems, and mathematical websites. In addition, he is interested in professional development at the K-16 levels and enjoys traveling across the USA to conduct workshops and present papers at mathematical conferences.*

Spotted in a Christmas cracker

Peter picked one pepper more than Paul.
Pat picked one pepper more than Pam.
Peter and Paul picked 10 more peppers than Pat and Pam.
Peter, Paul, Pat and Pam picked 60 peppers.
How many peppers did Peter pick?

Another Proof of Carlson's Infinite Product Expansion for ln(*x*)

M. A. NYBLOM

By employing the hyperbolic tangent function and the logarithmic form of its inverse, an alternative proof is presented for an infinite product expansion of the natural logarithm function.

1. Introduction

The arithmetic–geometric mean inequality is one of the most well-known inequalities of classical analysis. This inequality, in the two variable case, states that for any positive numbers x and y their arithmetic mean $A(x, y) = (x + y)/2$ is always greater than or equal to their geometric mean $G(x, y) = \sqrt{xy}$, that is $\sqrt{xy} \leq (x + y)/2$, with equality holding if and only if $x = y$. In contrast, another yet less well-known definition for the mean of two positive numbers x and y is the logarithmic mean defined by $L(x, y) = (x - y)/(\ln(x) - \ln(y))$, for $x \neq y$, and $L(x, x) = x$. The logarithmic mean was introduced by Carlson (see reference 1) who showed that $L(x, y)$ separated the arithmetic–geometric means in that, for $x \neq y$,

$$\sqrt{xy} < (xy)^{1/4}\frac{\sqrt{x}+\sqrt{y}}{2} < L(x, y) < \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{x+y}{2}. \tag{1}$$

Later in reference 1 Carlson deduced, via the inequalities in (1), that the natural logarithm function, $\ln(x)$, could be expanded as an infinite product, valid for all positive x, as follows:

$$\ln(x) = (x - 1)\prod_{n=1}^{\infty}\frac{2}{1 + \sqrt[2^n]{x}}. \tag{2}$$

Since the publication of reference 1, two alternative proofs of (2) have appeared in references 2 and 3, the first of which by Levin was based on an existing infinite product expansion for a class of analytic functions, while the second by Osler most notably applied the difference of squares identity in the form $x - y = (x^{1/2} - y^{1/2})(x^{1/2} + y^{1/2})$ as the basis for his proof. Following in the spirit of Osler's approach, we intend in this article to present another elementary proof of (2) using nothing more than the double-angle identity for the hyperbolic tangent function, $\tanh(x)$, and the logarithmic form of the inverse hyperbolic tangent function,

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$$

valid for $|x| < 1$.

2. The elementary proof

We begin by deriving an infinite product formula involving the hyperbolic tangent function. To this end consider an arbitrary w in $(0, 1)$, then as the mapping $\tanh(\cdot) : (0, \infty) \to (0, 1)$ is a bijection, there must exist a unique ξ in $(0, \infty)$ such that $w = \tanh(\xi)$. Recalling the double angle identity $\tanh(2\theta) = 2\tanh(\theta)/(1+\tanh^2(\theta))$ observe, after rearranging and setting $\theta = \xi/2^r$, the following telescoping product:

$$\prod_{r=1}^{N}\left(1+\tanh^2\left(\frac{\xi}{2^r}\right)\right) = \prod_{r=1}^{N}\frac{2\tanh(\xi/2^r)}{\tanh(\xi/2^{r-1})} = \frac{2^N\tanh(\xi/2^N)}{\tanh(\xi)} = \frac{2^N}{\xi}\tanh\left(\frac{\xi}{2^N}\right)\frac{\xi}{\tanh(\xi)},$$

where N is an integer greater than unity. Now since $\xi/2^N \to 0$ as $N\to\infty$, we find, using l'Hôpital's rule, that $\lim_{N\to\infty}(2^N/\xi)\tanh(\xi/2^N)) = 1$. Thus we deduce that

$$\lim_{N\to\infty}\prod_{r=1}^{N}\left(1+\tanh^2\left(\frac{\xi}{2^r}\right)\right) = \frac{\xi}{\tanh(\xi)}. \tag{3}$$

Next, to relate the infinite product in (3) to the natural logarithm function, first set

$$x = \frac{1+w}{1-w}$$

and recall by definition that

$$\frac{\xi}{2^r} = \frac{1}{2^r}\tanh^{-1}(w) = \frac{1}{2^{r+1}}\ln\left(\frac{1+w}{1-w}\right) = \frac{1}{2^{r+1}}\ln(x),$$

and so substituting $\theta = \xi/2^r$ in $\tanh(\theta) = (\mathrm{e}^{2\theta}-1)/(\mathrm{e}^{2\theta}+1)$ yields

$$1+\tanh^2\left(\frac{\xi}{2^r}\right) = 1+\frac{\sqrt[2^{r-1}]{x}-2\sqrt[2^r]{x}+1}{\sqrt[2^{r-1}]{x}+2\sqrt[2^r]{x}+1} = 2\frac{(\sqrt[2^{r-1}]{x}+1)}{(\sqrt[2^r]{x}+1)^2},$$

while similarly substituting

$$\xi = \tanh^{-1}(w) = \frac{1}{2}\ln\left(\frac{1+w}{1-w}\right) = \frac{1}{2}\ln(x)$$

into $\xi/\tanh(\xi)$ yields

$$\frac{x+1}{x-1}\frac{\ln(x)}{2}.$$

Thus setting $x = (1+w)/(1-w)$ we see (3) transforms to

$$\lim_{N\to\infty}\prod_{r=1}^{N}\frac{2(\sqrt[2^{r-1}]{x}+1)}{(\sqrt[2^r]{x}+1)^2} = \lim_{N\to\infty}\frac{x+1}{\sqrt[2^N]{x}+1}\prod_{r=1}^{N}\frac{2}{\sqrt[2^r]{x}+1} = \frac{x+1}{x-1}\frac{\ln(x)}{2},$$

which upon taking limits and rearranging demonstrates the validity of (2) for x in $(1,\infty)$. Finally, since (2) holds for x in $(1,\infty)$ and trivially for $x=1$, observe by setting $x = 1/y$, with $0<y<1$, in (2) that

$$\begin{aligned}\ln\left(\frac{1}{y}\right) &= \lim_{N\to\infty}\frac{1-y}{y}\prod_{r=1}^{N}\frac{2\sqrt[2^r]{y}}{(1+\sqrt[2^r]{y})} \\ &= \lim_{N\to\infty}\frac{1-y}{y}y^{\sum_{r=1}^{N}1/2^r}\prod_{r=1}^{N}\frac{2}{(1+\sqrt[2^r]{y})} \\ &= (1-y)\prod_{r=1}^{\infty}\frac{2}{(1+\sqrt[2^r]{y})},\end{aligned}$$

which after multiplying by -1 shows that (2) also holds when $0<x<1$.

References

1 B. C. Carlson, The logarithmic mean, *Amer. Math. Monthly* **79** (1972), pp. 615–618.
2 A. Levin, A new class of infinite products generalizing Viete's product formula for π, *Ramanujan J.* **10** (2005), pp. 305–324.
3 T. J. Osler, Interesting finite and infinite products from simple algebraic identities, *Math. Gazette* **90** (2006), pp. 90–93.

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Letters to the Editor

Dear Editor,

Message in a bottle

Prithwijit De, in his article 'Message in a bottle' (Volume 43, Number 2, pp. 50–52), discussed how to minimise the surface area of a cylindrical bottle, for which he gave the formula

$$A = 2\pi r^2 + \frac{2V}{r}$$

($V > 0$ is fixed, $r > 0$ is variable). He then follows most textbooks which regard minimisation as a nice opportunity to practise differentiation. But this is often unnecessary and the well-known result about arithmetic and geometric means will do the job.

For three quantities $a, b, c \geq 0$ this says

$$\tfrac{1}{3}(a + b + c) \geq \sqrt[3]{abc}, \tag{1}$$

with equality holding only when $a = b = c$. Putting $a = 6\pi r^2$ and $b = c = 3V/r$ gives

$$A = \frac{1}{3}\left(6\pi r^2 + \frac{3V}{r} + \frac{3V}{r}\right) \geq \sqrt[3]{6\pi r^2 \, . \, \frac{3V}{r} \, . \, \frac{3V}{r}}$$

or $A \geq \sqrt[3]{54\pi V^2}$, with equality holding only when $6\pi r^2 = 3V/r$, i.e. $r = \sqrt[3]{V/2\pi}$. This gives $\sqrt[3]{54\pi V^2}$ as the minimum value of A as $r > 0$ varies.

He then considers a more complicated shape involving an acute angle α and a quantity

$$f(\alpha) = \frac{3 - 2\cos\alpha}{3\sin\alpha}.$$

We need the minimum value of $f(\alpha)$. For this we *will* differentiate, but only once; textbooks urge you to do it twice. Here,

$$f(\alpha) = \frac{1}{\sin\alpha} - \frac{2}{3}\cot\alpha$$

so

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} = -\frac{\cos\alpha}{\sin^2\alpha} + \frac{2}{3\sin^2\alpha} = \frac{\frac{2}{3} - \cos\alpha}{\sin^2\alpha}.$$

If $\beta = \cos^{-1}\frac{2}{3}$, clearly $\sin\beta = \sqrt{5}/3$ and, for $0 \le \alpha < \beta$,

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} < 0;$$

but, for $\beta < \alpha \le \pi/2$,

$$\frac{\mathrm{d}f}{\mathrm{d}\alpha} > 0.$$

Thus the graph of $f(\alpha)$ is u-shaped with minimum value

$$f = \frac{3 - 2\cos\beta}{3\sin\beta} = \frac{\frac{5}{3}}{\sqrt{5}} = \frac{\sqrt{5}}{3}, \quad \text{when } \alpha = \beta. \tag{2}$$

His formula for the area of the new-shaped bottle is

$$A = \pi r^2(1 + f(\alpha)) + \frac{2V}{r}.$$

Then, from (2),

$$A \ge \pi r^2\left(1 + \frac{\sqrt{5}}{3}\right) + \frac{2V}{r}$$

with equality holding only when $\alpha = \beta$.

Now, with $a = \pi r^2(3 + \sqrt{5})$, and $b = c = 3V/r$, (1) gives

$$A \ge \sqrt[3]{\pi r^2(3+\sqrt{5})\,.\,\frac{3V}{r}\,.\,\frac{3V}{r}}$$

or

$$A \ge \sqrt[3]{9(3+\sqrt{5})\pi V^2}$$

with equality holding only when

$$\pi r^2(3+\sqrt{5}) + \frac{3V}{r}$$

or

$$r = \sqrt[3]{\frac{3V}{(3+\sqrt{5})\pi}},$$

and $\alpha = \beta$. So we have our minimum.

Yours sincerely,

Norman Routledge
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Dear Editor,

Pan-digit squares

The following nine-digit square uses all the digits $1, \ldots, 9$:

$$11826^2 = 139854276.$$

There is a history of investigation of such squares; a list of 30 was provided by Beiler in reference 1. A computer search confirms that these are all the possibilities.

Beiler followed the nine-digit list with eight squares of 10 digits, each using the full set of digits $0, \ldots, 9$ and without a leading zero. A computer investigation of this format produced 87 solutions and it is not clear how he made his selection. Two solutions are of particular interest. Firstly $58413^2 = 3412078569$ shows $0, \ldots, 4$ entirely in the first half of the square and $5, \ldots, 9$ in the second half, the only case of such a division. Also there is $45624^2 = 2081549376$. If this square is multiplied by four then the result, 8326197504, is another pan-digit square, and this is the only case of such a pair.

The computer investigation was then extended to 18-digit and 20-digit squares which use exactly two sets of digits each. The number of solutions is large. There are 25460 18-digit squares using $1, \ldots, 9$ twice each. As an example,

$$352669377^2 = 124375689473568129.$$

In this example each half of the square uses a full set of the nine digits. With this further restriction the computer counted 247 possibilities.

Also of interest are cases where the generator is also pan-digital. Listed in table 1 are the 28 solutions that meet this requirement.

For 20-digit squares using the set of $0, \ldots, 9$ twice each, the computer found 466901 solutions. Among these there are 532 where the generator is also pan-digital but these include the 28 examples of 18-digit squares of table 1 but with a single zero added to the end of each

Table 1 Double-9 pan-digital squares with pan-digital generators.

Generator	Square	Generator	Square
345918672	119659727638243584	749258163	561387794822134569
351987624	123895287449165376	754932681	569923352841847761
359841267	129485737436165289	759142683	576297613152438489
394675182	155768499286733124	759823641	577331965422496881
429715863	184655722913834769	762491835	581393798441667225
439516278	193174558626973284	783942561	614565938947238721
487256193	237418597616853249	784196235	614963734988175225
527394816	278145291943673856	845691372	715193896675242384
527498163	278254311968374569	891357624	794518413862925376
528714396	279538912537644816	914863275	836974811943725625
572493816	327749169358241856	915786423	838664772551134929
592681437	351271285764384969	923165487	852234516387947169
729564183	532263897116457489	928163754	861487954239372516
746318529	556991346728723841	976825431	954187922648335761

Table 2 Halved double-10 pan-digital squares with pan-digital generators.

Generator	Square
4253907186	18095726347102438596
5296031874	28047953610423951876

generator and two zeroes added to the squares. Further, there are 2525 where each half of the square shows the full $0, \ldots, 9$ set. Finally, combining these two restrictions there are just two examples, shown in table 2, where the generator is pan-digital and the square has a full set of digits in each half.

Less obviously interesting are 19-digit squares since the digit sets cannot be the same. However, using one zero and two sets of $1, \ldots, 9$, there are many solutions and there are 91 examples where a set each side of a central zero is achieved, for example

$$2096573841^2 = 4395621870765493281.$$

Note that in this example the generator is also pan-digital. With this constraint alone there are 184 solutions. Of these, there are just 10 with the zero in the centre and only the one above where the sets of digits are complete on each side.

Another pattern is to show separately the even and odd digits with a zero between them. Under this requirement there appears to be only one solution:

$$1399918338^2 = 1959771353068682244.$$

Reference

1 A. H. Beiler, *Recreations in the Theory of Numbers* (Dover, New York, 1964).

Yours sincerely,

Tom Marlow
(24 Saxon Way
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Base 13

A little-known tribe which is not superstitious has developed a base 13 number system using the following symbols:

$$0\ 1\ 2\ 3\ \mathrm{A}\ 4\ 5\ \mathrm{B}\ 6\ 7\ 8\ \mathrm{C}\ 9.$$

The fact that they use our symbols and positional notation is a mystery; it is suspected that a previous visit by missionaries may have something to do with it. What is $(\mathrm{AC})^2$ in their system?

Midsomer Norton, Bath, UK **Bob Bertuello**

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

44.1 Find all positive integers n such that

$$\log_{2009} n = \log_{2010} n + \log_{2011} n.$$

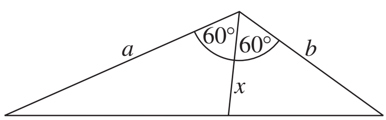
(Submitted by Abbas Rouholamini Gugheri, Sirjan, Iran)

44.2 What is

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7}?$$

(Submitted by Luo Qi, Guilin Normal College, Guangxi, P. R. China)

44.3 Find a relation connecting x, a, and b.



(Submitted by R. J. Webster, University of Sheffield, adapted from a problem of D. F. Ferguson (see *Math. Gazette* **155** (1921), p. 377). Ferguson stunned the mathematical world in 1946 when he found an error in the 528th place of William Shanks's then world record 707 decimal digit expansion of π, a record that had stood for over 70 years.)

44.4 Let A, B, C be the angles of a triangle with $A \le B \le C$. Prove that

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} \le \sqrt{3}$$

if and only if $B \le \pi/3$. (See the article 'A trip from trig to triangle' on pp. 19–23 of this issue.

(Submitted by Michel Bataille, Rouen, France)

Solutions to Problems in Volume 43 Number 2

43.5 Prove that all prime numbers are solitary – see the article ‘Lopsided numbers’ on pp. 53–54 of Volume 43 Number 2.

Solution by Henry Ricardo, Tappan, New York

Recall that $\sigma(n)$ denotes the sum of the positive divisors of the positive integers n. For $a > 0$ and $b > 1$, $\sigma(a)b < \sigma(ab)$ since 1 contributes to the right-hand side but not to the left-hand side. Now let p be prime and let n be a positive integer such that $\sigma(p)/p = \sigma(n)/n$. This is equivalent to $(p+1)n = p\sigma(n)$, so that p divides n, say $n = kp$ for some positive integer k. If $k > 1$, then $\sigma(n)/n = \sigma(kp)/kp > k\sigma(p)/kp = \sigma(p)/p$ by the inequality established earlier. Hence $k = 1$ and $n = p$, so all prime numbers are solitary.

43.6 For positive integers a and n, sum the finite series

$$aa! + (a+1)(a+1)! + (a+2)(a+2)! + \cdots + (a+n-1)(a+n-1)!$$

Solution by Bor-Yann Chen, University of California, Irvine

Since $aa! = (a+1)a! - a! = (a+1)! - a!$, the expression is equal to

$$((a+1)! - a!) + ((a+2)! - (a+1)!) + \cdots + ((a+n)! - (a+n-1)!) = (a+n)! - a!.$$

Also solved by Gian Almirante, Milan, Abbas Rouholamini, Sirjan, Iran, and Henry Ricardo.

43.7 The positive real numbers a, b, c are such that $a^2 + b^2 = c^2$, $c = b^2/a$, and $b - a = 1$. Determine a, b, c.

Solution by Bor-Yann Chen

We can write

$$a = c\cos\alpha, \qquad b = c\sin\alpha$$

for some real number α. Then

$$c = \frac{b^2}{a} = \frac{c\sin^2\alpha}{\cos\alpha},$$

so that

$$\sin^2\alpha = \cos\alpha,$$

or

$$1 - \cos^2\alpha = \cos\alpha,$$

so

$$\cos^2\alpha + \cos\alpha - 1 = 0,$$

so

$$\cos\alpha = \frac{-1 + \sqrt{1+4}}{2} = \frac{-1+\sqrt{5}}{2} = \frac{1}{\phi},$$

where $\phi = (1+\sqrt{5})/2$, the golden ratio. Then

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{\frac{\sqrt{5}-1}{2}} = \frac{1}{\sqrt{\phi}}.$$

Since $b - a = 1$,

$$c(\sin\alpha - \cos\alpha) = 1,$$

so that

$$c\left(\frac{1}{\sqrt{\phi}} - \frac{1}{\phi}\right) = 1$$

and

$$c = \frac{\phi}{\sqrt{\phi} - 1}.$$

Then

$$a = c\cos\alpha = \frac{1}{\sqrt{\phi} - 1} \quad \text{and} \quad b = c\sin\alpha = \frac{\sqrt{\phi}}{\sqrt{\phi} - 1}.$$

Also solved by Abbas Rouholamini Gugheri.

43.8 Determine all nondegerate triangles ABC in which

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}.$$

Solution by Prithwijit De, who proposed the problem

First note that, because

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$A : B : C = a : b : c$ (where a, b, c are the side-lengths and A, B, C are the angles of the triangle). Hence the sides of the triangle are proportional to A, B, C, so that $A < B + C = \pi - A$ so that $0 < A < \pi/2$, and similarly for B and C.

Consider the function f defined for $0 < x < \pi/2$ by

$$f(x) = \frac{\sin x}{x}.$$

Then

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0,$$

for $0 < x < \pi/2$, so that f is strictly decreasing. Since A, B, C all lie strictly between 0 and $\pi/2$,

$$\frac{\sin A}{A} = \frac{\sin B}{B} = \frac{\sin C}{C}$$

implies that $A = B = C$. Hence ABC must be an equilateral triangle. Clearly all equilateral triangles satisfy the condition.

The year is a sum of consecutive primes

$$2011 = 157 + 163 + 167 + 173 + 179 + 181 + 191 + 193 + 197 + 199 + 211.$$

Reference

1 https://twitter.com/mathematicsprof

The University of Sheffield **Harry Horton**

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