

Mathematicorum

Crux

Published by the Canadian Mathematical Society.



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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 16 #4

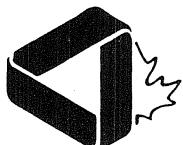
April / Avril

1990

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Canadian Mathematical Society



Société mathématique du Canada

Founding Editors: Léopold Sauvé, Frederick G.B. Maskell
Editor: G.W. Sands
Managing Editor: G.P. Wright

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1989 subscription rate for ten issues is \$17.50 for members of the Canadian Mathematical Society and \$35.00 for non-members. Back issues: \$3.50 each. Bound volumes with index: volumes 1 & 2 (combined) and each of volumes 3, 7, 8, 9 and 10: \$10.00. (Volumes 4, 5 & 6 are out-of-print). All prices quoted are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

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ACKNOWLEDGEMENT

The support of the Departments of Mathematics and Statistics of the University of Calgary and Carleton University, and of the Department of Mathematics of the University of Ottawa, is gratefully acknowledged.

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Published by the Canadian Mathematical Society
Printed at Carleton University

THE OLYMPIAD CORNER
No. 114
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The new problems we present are from the 8th annual *American Invitational Mathematics Examination* (A.I.M.E.) written March 20, 1990. The time allowed was three hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588-0322.

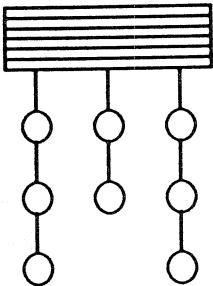
1990 AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1. The increasing sequence 2,3,5,6,7,10,11,... consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence.
2. Find the value of $(52 + 6\sqrt{43})^{3/2} - (52 - 6\sqrt{43})^{3/2}$.
3. Let P_1 be a regular r -gon and P_2 be a regular s -gon ($r \geq s \geq 3$) such that each interior angle of P_1 is $\frac{59}{58}$ as large as each interior angle of P_2 . What is the largest possible value of s ?
4. Find the positive solution to
$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0.$$
5. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $n/75$.
6. A biologist wants to calculate the number of fish in a lake. On May 1 she catches a random sample of 60 fish, tags them, and releases them. On September 1 she catches a random sample of 70 fish and finds that 3 of them are tagged. To calculate the number of fish in the lake on May 1, she assumes that 25% of these fish are no longer in the lake on September 1 (because of

death and emigrations), that 40% of the fish present on September 1 were not in the lake on May 1 (because of births and immigrations) and that the numbers of untagged fish and tagged fish in the September 1 sample are representative of the total population. What does the biologist calculate for the number of fish in the lake on May 1?

7. A triangle has vertices $P = (-8,5)$, $Q = (-15,-19)$ and $R = (1,-7)$.

The equation of the bisector of $\angle P$ can be written in the form $ax + 2y + c = 0$. Find $a + c$.

8. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured. A marksman is to break all eight targets according to the following rules: (1) The marksman first chooses a column from which a target is to be broken. (2) The marksman must then break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken?
- 

9. A fair coin is to be tossed ten times. Let i/j , in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

10. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C ?

11. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n - 3$ consecutive positive integers.

12. A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form

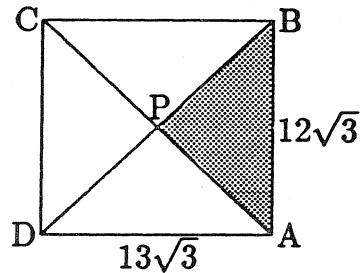
$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6},$$

where a , b , c , and d are positive integers. Find $a + b + c + d$.

13. Let $T = \{9^k : k \text{ is an integer, } 0 \leq k \leq 4000\}$. Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

14. The rectangle $ABCD$ at the right has dimensions $AB = 12\sqrt{3}$ and $BC = 13\sqrt{3}$.

Diagonals \overline{AC} and \overline{BD} intersect at P . If triangle ABP is cut out and removed, edges \overline{AP} and \overline{BP} are joined, and the figure is then creased along segments \overline{CP} and \overline{DP} , we obtain a triangular pyramid, all four of whose faces are isosceles triangles. Find the volume of this pyramid.



15. Find $ax^5 + by^5$ if the real numbers a, b, x and y satisfy the equations
 $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, $ax^4 + by^4 = 42$.
 * * *

The first solutions we give are responses to the appeals to clean up the archives.

1. [1981: 46] 1978 Romanian Mathematical Olympiad (Final Round – 11th Class)

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a real function defined by $f(x) = 0$ if x is irrational and $f(p/q) = 1/q^3$ if p and q are integers with $q > 0$ and p/q irreducible. Show that f has a derivative at each irrational point $x_0 = \sqrt{k}$, where k is a natural number that is not a perfect square.

Solution by R.K. Guy, Department of Mathematics and Statistics, The University of Calgary.

Since x_0 is irrational, $f(x_0) = 0$. The derivative, if it exists, is

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

The numerator is zero, except when $x_0 + h = p/q$ and $f(x_0 + h) = 1/q^3$. As x_0 is a quadratic irrational, \sqrt{k} , it can be approximated to order 2 but to no higher order [see Theorem 188 of Hardy & Wright, *Introduction to the Theory of Numbers*, 4th Ed.], i.e.

$$|h| = \left| \sqrt{k} - \frac{p}{q} \right| > \frac{K}{q^2}.$$

[The continued fraction for a quadratic surd is periodic, so its partial quotients, a_n , are bounded, $0 < a_n < M$, and it suffices to take $K = (M + 2)^{-3}$.] Thus

$$\left| \frac{f(x_0 + h) - f(x_0)}{h} \right| < \frac{1/q^3}{K/q^2} = \frac{1}{Kq}.$$

On the other hand, there's an infinity of good rational approximations to \sqrt{k} , satisfying

$$\left| \frac{p}{q} - \sqrt{k} \right| < \frac{1}{q^2}$$

(in fact $< 1/\sqrt{5}q^2$ [Hardy & Wright, Theorem 193]), so that $q \rightarrow \infty$ as $h \rightarrow 0$, and

$$\frac{1}{Kq} = \frac{(M+2)^3}{q} \rightarrow 0.$$

Thus the derivative exists and is equal to zero at all points $x_0 = \sqrt{k}$, where k is not a perfect square.

*

1. [1981: 75] *Second Selection Test, 1978 Romanian I.M.O. Team.*

Consider in the xy -plane the infinite network defined by the lines $x = h$ and $y = k$, where h and k range over the set \mathbb{Z} of all integers. With each node (h,k) we try to associate an integer $a_{h,k}$ which is the arithmetic mean of the integers associated with the four lattice points nearest to (h,k) , that is,

$$a_{h,k} = \frac{1}{4}(a_{h-1,k} + a_{h+1,k} + a_{h,k-1} + a_{h,k+1})$$

for any $h, k \in \mathbb{Z}$.

- (a) Prove that there is a network for which the *nodal numbers* $a_{h,k}$ are not all equal.
- (b) Given a network with at least two distinct nodal numbers $a_{h,k}$, show that, for any natural number n , there are in the network nodal numbers greater than n and nodal numbers less than $-n$.

Solution by R.K. Guy, Department of Mathematics and Statistics, The University of Calgary.

- (a) Setting $a_{h,k} = h + k$ gives such a network.
- (b) If there are two distinct nodal numbers, there must be two neighbouring such, say a, b with $a > b$. By the arithmetic mean property of the nodal numbers, there is a neighbouring nodal number c of a with $c > a$ and a neighbouring nodal number d of b with $d < b$. Continuing, we obtain a doubly infinite path $\dots < f < d < b < a < c < e < \dots$ of integers which contains numbers $> n$ and $< -n$ for all natural numbers n .

*

8. [1981: 77] *Third Selection Test, 1978 Romanian I.M.O. Team.*

Solve the following problem, first reformulating it in set-theoretic language:

A certain number of boys and girls are at a party, and it is known which

boys are acquainted with which girls. This acquaintance relationship is such that, for any subset M of the boys, the subset of the girls acquainted with at least one boy of M is at least as large as M . Prove that, simultaneously, every boy can dance with a girl of his acquaintance.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

This is a classical result due to P. Hall (1935). In the literature it is sometimes referred to as the Marriage Theorem which states that a family of sets A_1, A_2, \dots, A_n has a system of distinct representatives (*SDR*) if and only if the following condition (known as the marriage condition) is satisfied:

$$(*) \quad \text{For each } k = 1, 2, \dots, n \text{ and for each choice of } i_1, i_2, \dots, i_k \text{ with } 1 \leq i_1 < i_2 < \dots < i_k \leq n, |A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}| \geq k.$$

In the present case, if we let B and G denote the sets of boys and girls at the party, respectively, and for each $b \in B$ let G_b denote the set of girls acquainted with b , then we are to prove that the family $\{G_b \mid b \in B\}$ has an *SDR* under the assumption that for $M \subset G$,

$$\left| \bigcup_{b \in M} G_b \right| \geq |M| ,$$

which is clearly equivalent to $(*)$ above.

A proof of P. Hall's theorem can be found in many books on combinatorics, see for example R. Brualdi, *Introductory Combinatorics*, p. 155.

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Next we give two comments from Edward T.H. Wang regarding past solutions.

4. [1988: 35; 1989: 267] *10th Austrian-Polish Mathematics Competition.*

Does the set $\{1, 2, \dots, 3000\}$ contain a subset A of 2000 elements such that $x \in A$ implies $2x \notin A$?

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The readers may be interested in my article "On double-free sets of integers", *Ars Combinatoria* 28 (December 1989) pp. 97–100, in which I generalize this problem.

A set S of integers is called *double-free* (D.F.) if $x \in S$ implies $2x \notin S$. Let $N_n = \{1, 2, \dots, n\}$ and $f(n) = \max\{|A| : A \subset N_n \text{ is D.F.}\}$. We prove that $f(n) = \lceil n/2 \rceil + f(n/4)$. Here $\lceil x \rceil$ is the least integer greater than or equal to x .

5. [1988: 102; 1990: 40] *Tenth Atlantic Provinces Mathematics Competition.*

Show that

$$\sqrt{3} \leq \exp \left[\int_{\pi/6}^{\pi/2} \frac{\sin x}{x} dx \right] \leq 3.$$

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

A solution which may be "closer" to the one foreseen by the setters of the contest is the following. Let I denote the integral. Then from $\sin x \leq 1$ we get

$$I \leq \int_{\pi/6}^{\pi/2} \frac{1}{x} dx = \ln \frac{\pi}{2} - \ln \frac{\pi}{6} = \ln 3$$

and thus $\exp(I) \leq 3$. On the other hand, from Jordan's inequality $(\sin x)/x \geq 2/\pi$ we get

$$I \geq \int_{\pi/6}^{\pi/2} \frac{2}{\pi} dx = \frac{2}{3}$$

and thus $\exp(I) \geq e^{2/3} > \sqrt{3}$ as $e^4 > 27$.

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The next solutions are to problems which were given in the June 1988 number of the Corner.

- M1043.** [1988: 165] *1987 KVANT.*

Can the set of all integers be partitioned into three subsets so that for any integer n the numbers n , $n - 50$, $n + 1987$ will belong to three different subsets? (S.V. Konyagin)

Solution by George Evangelopoulos, Law student, Athens, Greece.

No! Let us suppose that the partition of the set of all integers into three subsets, which satisfy the hypothesis, is possible. We will write $m \sim k$ if the integers m and k belong to the same subset. We shall prove that

$$n \sim n + 1937 \text{ and } n \sim n - 150$$

for any integer n . From this it will follow that

$$0 \sim 1937 \sim 2 \cdot 1937 \sim \dots \sim 50 \cdot 1937 = 646 \cdot 150 - 50 \sim 645 \cdot 150 - 50 \sim \dots \sim -50 .$$

That is to say, $0 \sim -50$, which contradicts the hypothesis.

Let us call a triple (a, b, c) of integers *representative* if it contains one number from each subset. Then the triples $(n - 50, n, n + 1987)$, $(n - 100, n - 50, n + 1937)$ and $(n + 1937, n + 1987, n + 2 \cdot 1987)$ are representative for each n . In particular, from these three triples we have that $n - 50 \not\sim n + 1987$, $n + 1937 \not\sim n - 50$ and

$n + 1987 \neq n + 1937$, and from the first one we have that $n \neq n - 50$ and $n \neq n + 1987$. It follows that $n \sim n + 1937$.

Thus in the second triple we can replace the number $n + 1937$ with the number n , which means that the triple $(n - 100, n - 50, n)$ is representative. If we substitute $n - 50$ for n , we obtain the triple $(n - 150, n - 100, n - 50)$, which also is representative. Comparing the last two triples we see that $n \sim n - 150$. This completes the proof.

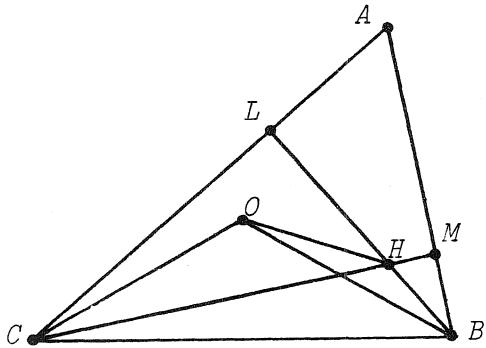
[*Editor's note:* This problem was also solved by Curtis Cooper, Central Missouri State University.]

M1046. [1988: 165] 1987 KVANT.

In the acute triangle ABC the angle A is 60° . Prove that one of the bisectors of the angle between the two altitudes drawn from the vertices B and C passes through the circumcircle of the triangle. (V. Progrebnyak, 10th form student, Vinnitsa)

Correction and solution by George Evangelopoulos, Law student, Athens, Greece.

The problem should say that one of the altitudes passes through the *centre* of the circumcircle. Let us suppose that the altitudes BL and CM meet at the point H and that O is the circumcentre of the triangle ABC . Then the central angle BOC is twice the inscribed angle BAC , namely $\angle BOC = 120^\circ$. Also $\angle BHC = \angle LHM = 180^\circ - 60^\circ = 120^\circ$. Consequently the points B , H , O and C are concyclic. The point H is, without loss, on the arc BO of this circle. Then $\angle CHO = \angle CBO = (180^\circ - 120^\circ)/2 = 30^\circ$ (the triangle BOC is isosceles!). Further $\angle CHL = 180^\circ - 120^\circ = 60^\circ$. So OH is the bisector of the angle CHL , which is formed by the altitudes BL and CM .



[*Editor's note:* This problem was also solved by a trigonometric argument by the late J.T. Groenman of Arnhem, The Netherlands.]

M1047. [1988: 165] 1987 KVANT.

In a one round chess tournament no less than $3/4$ of the games concluded at some moment were drawn. Prove that at this moment at least two players had the same number of points. (Miklos Bona, gymnasium student, Hungary)

Solution by George Evangelopoulos, Law student, Athens, Greece.

In this problem it is useful to use the following system of rating: 1 point for a victory, 0 for a draw, and -1 for defeat. We denote by n the number of players taking part in the tournament and we put $k = n/2$ for n even and $k = (n-1)/2$ for n odd.

Suppose that at the given moment all the players have different results. Then among them there will be k players with positive results or k players with negative results. It suffices to consider the first case. For all these k players to obtain a different number of points, since each can have no more points than his number of wins, the total number of victories for these players is not less than $1 + 2 + \dots + k = k(k+1)/2$. The total number of games played in the entire tournament is $n(n-1)/2$, so at the moment the fraction of wins out of the total number possible is at least

$$\frac{k(k+1)}{n(n-1)} > \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

This means that the number of draws is less than $3/4$, contradicting the hypothesis.

M1051. [1988: 165] 1987 KVANT.

The lower left hand corner of an 8×8 chessboard is occupied by nine pawns forming a 3×3 square. A pawn can jump over any other pawn landing symmetrically, if the corresponding square is empty. Using such moves, is it possible to reassemble the 3×3 square

- (a) in the upper left hand corner?
- (b) in the upper right hand corner?

(Ya.E. Briskin)

Solution by George Evangelopoulos, Law student, Athens, Greece.

In both cases (a) and (b) the answer is negative. To see this colour the rows of the checkerboard in alternating colours, black and white, beginning with black for the bottom row. It is obvious that when a pawn jumps, it always lands on a square of the same colour. But the initial position has 6 black and 3 white squares and either final position has 6 white and 3 black squares.

M1056. [1988: 165] 1987 KVANT.

In each little square of a 1987×1987 square table there is a number no greater than 1 in absolute value. In any 2×2 square of the table the sum of the numbers is 0. Prove that the sum of all the numbers in the table is no greater than 1987. (A.S. Merkuriev)

Solution by George Evangelopoulos, Law student, Athens, Greece.

We divide the square into 994 disjoint regions: the first is the upper left corner square, the second is the 3×3 square in the upper left corner without the corner square, the third is the 5×5 square minus the 3×3 square, and so forth.

We claim that the sum of the numbers in any region except the first one is at most 2. This is because beginning at the left and at the top we can cover any region except the first two by disjoint 2×2 squares leaving a 3×3 square minus the top left corner, which is identical to the second region. This shape can be covered by two 2×2 squares which overlap on the centre square A , plus the lower right corner B . The sum for this shape is then the sum for the two squares plus the number in square B minus the number in A , giving $0 + 0 + 1 + 1 = 2$ as an upper bound.

Consequently the sum of all the squares is at most $1 + 2 \cdot 993 = 1987$.

M1057. [1988: 165] 1987 KVANT.

Two players in turn write natural numbers on a board. The rules forbid writing numbers greater than p and divisors of previously written numbers. The player who has no move loses.

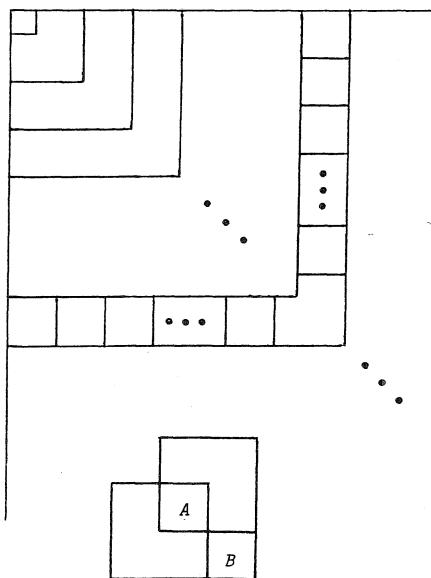
(a) Determine which of the players has a winning strategy for $p = 10$ and describe this strategy.

(b) Determine which of the players has a winning strategy for $p = 1000$.
(D.V. Fomin)

Solutions by Duane M. Broline, Eastern Illinois University, Charleston, and by George Evangelopoulos, Law student, Athens, Greece.

(a) The first person wins. His first play should be 6. After that the only plays are 4, 5, 7, 8, 9, 10. We group these in pairs (4,5), (7,9) and (8,10). It is clear that these give a reply to the subsequent plays of the second player.

(b) The first player has a winning strategy for any positive integer p . Suppose for a contradiction that no first play of the first player wins. Then in response to a first play of 1 the second player must be able to play a number n after which no play of the first player guarantees the win. But now if the first



player instead first plays n the available plays for both players in the subsequent play are exactly those plays for the game above and player I is now in the same situation as II was then. Thus he must win, a contradiction.

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Send in your nice solutions and Olympiads.

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M I N I - R E V I E W S

by

ANDY LIU

RAYMOND SMULLYAN'S LOGIC SERIES

While this series deals only with logic, it more than makes up by its tremendous depth. While the reader may find parts of the books in this series difficult, they will not find them difficult to read. Each book takes the form of a series of logic puzzles, presented in very attractive settings. The reader can reasonably expect to have success with the earlier ones. As confidence increases, the reader will discover the skill of the author in paving a smooth path towards some very important results in logic and mathematics, particularly those associated with the name Kurt Gödel.

What is the Name of this Book?, Prentice-Hall, 1978. (hardcover & paperback, 241 pp.)

A distinguishing feature of Raymond Smullyan's logic puzzles is that not all of the information provided is to be taken at its face value. This is exemplified by his favourite characters, the knights who always tell the truth, and the knaves who always lie. Before one can utilize a statement made by one of them, it is important to know if it is made by a knight or a knave. Very often, the clues are contained in the statement itself.

The knights and knaves made their debut in this wonderful book (what is its name again?), along with other denizens, from Alice (of Wonderland fame) to Count Dracula. In solving numerous intriguing logic puzzles, the reader gets an enjoyable lesson in propositional logic, an introduction to some logical paradoxes and curiosities, and a proof of a form of Gödel's famous Incompleteness Theorem.

The Chess Mysteries of Sherlock Holmes, Alfred A. Knopf, 1979. (hardcover & paperback, 171 pp.)

This and the next volume are quite different from the other books in the series in that logic is exercised over the chessboard. The reader needs to know the rules of chess, but being a good player is not essential. In fact, this is more often a handicap, as the moves that are made in the games in these books, though perfectly legal, are hardly what one would describe as good moves. The object is not to win but to deduce the past history of a game, based on the current position and possibly some additional information. In this volume, Watson serves as the narrator, with Sherlock Holmes as the chessboard detective.

The Chess Mysteries of the Arabian Knights, Alfred A. Knopf, 1981. (hardcover & paperback, 170 pp.)

In this volume, the characters are the White King Haroun Al Rashid and his entourage, as well as the opposing camp headed by the Black King Kazir. The setting is that of the "Tales of the Arabian Nights". The style of the narrative and the charming illustrations by Greer Fitting add to the authenticity.

The Lady or the Tiger?, Alfred A. Knopf, 1982. (226 pp.)

Although the knights and knaves make only a brief appearance in the first part of this book, the other characters are unmistakeably knight-like or knave-like, though with fascinating variations. The reader is also introduced to meta-puzzles, or puzzles about puzzles. The second half of the book is a novelty called a "mathematical novel". The step-by-step unveiling of the "Secret of the Monte Carlo Lock" is an absorbing study in combinatorial logic.

Alice in Puzzleland, William Morrow, 1982/Penguin Books, 1984. (hardcover/paperback, 182 pp.)

This book celebrates the one-hundred-and-fiftieth anniversary of the birth of Lewis Carroll, of whom Raymond Smullyan has been described as a modern version. In this volume, Alice and other Carrollian characters are reunited to entertain the reader with logic and meta-logic puzzles. There are also some elementary mathematical problems. The illustrations by Greer Fitting are simply gorgeous.

To Mock a Mockingbird, Alfred A. Knopf, 1985. (246 pp.)

The first third of this book consists of the basic knight-knave type of puzzles and more meta-puzzles. The remaining part takes the reader on another tour of the realm of combinatorial logic. This second "mathematical novel" shares several common characters with the "Mystery of the Monte Carlo Lock" while adding many

more, most of which are birds that can talk.

Forever Undecided, Alfred A. Knopf, 1987/Oxford Univ. Press, 1988.
(hardcover/paperback, 257 pp.)

This is the most challenging yet of Raymond Smullyan's books on logic puzzles. After a review of propositional logic, the reader is gradually introduced to the subject of modal logic, where the principal notions are that of a proposition being possibly true as opposed to being necessarily true.

Addresses of publishers:

Prentice-Hall, 1870 Birchmont Road, Scarborough, Ontario M1P 2J7

Alfred A. Knopf, a division of Random House, 1265 Aerowood Drive, Mississauga, Ontario L4W 1B9

William Morrow, 105 Madison Avenue, New York, NY 10016

Penguin Books, 2801 John St., Markham, Ontario L3R 1B4

Oxford University Press, 70 Wynford Drive, Don Mills, Ontario M3C 1J9

P R O B L E M S

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1990, although solutions received after that date will also be considered until the time when a solution is published.

1531. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\frac{v+w}{u} \cdot \frac{bc}{s-a} + \frac{w+u}{v} \cdot \frac{ca}{s-b} + \frac{u+v}{w} \cdot \frac{ab}{s-c} \geq 4(a+b+c),$$

where a, b, c, s are the sides and semiperimeter of a triangle, and u, v, w are positive real numbers. (Compare with *Crux* 1212 [1988: 115].)

1532. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine all (possibly degenerate) triangles ABC such that

$$(1 + \cos B)(1 + \cos C)(1 - \cos A) = 2 \cos A \cos B \cos C.$$

1533. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

For any integers $n \geq k \geq 0$, $n \geq 1$, denote by $p(n,k)$ the probability that a randomly chosen permutation of $\{1, 2, \dots, n\}$ has exactly k fixed points, and let

$$P(n) = p(n,0)p(n,1)\cdots p(n,n).$$

Prove that

$$P(n) \leq \exp(-2^n n!).$$

1534. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Triangle $H_1H_2H_3$ is formed by joining the feet of the altitudes of an acute triangle $A_1A_2A_3$. Prove that

$$\frac{s}{r} \leq \frac{s'}{r'},$$

where s , s' and r , r' are the semiperimeters and inradii of $A_1A_2A_3$ and $H_1H_2H_3$ respectively.

1535. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let P be a variable point inside an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Through P draw two chords with slopes b/a and $-b/a$ respectively. The point P divides these two chords into four pieces of lengths d_1 , d_2 , d_3 , d_4 . Prove that $d_1^2 + d_2^2 + d_3^2 + d_4^2$ is independent of the location of P and in fact has the value $2(a^2 + b^2)$.

1536* *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

This being problem 1536 in volume 16 of *Crux*, note that the numbers 16 and 1536 have the following nice property:

$$\begin{array}{r} 96 \\ 16 \overline{) 1536} \\ 144 \\ \hline 96 \\ 96 \\ \hline 0 \end{array} \quad \left(\text{in Europe: } 1536 : 16 = \begin{array}{r} 96 \\ 0 \end{array} \right).$$

That is, 1536 is exactly divisible by 16, and upon dividing 16 into 1536 via "long division" the last nonzero remainder in the display is equal to the quotient $1536/16 = 96$. Assuming *Crux* continues to publish 100 problems each year, what will be the next *Crux* volume and problem numbers to have the same property?

1537. *Proposed by Isao Ashiba, Tokyo, Japan.*

ABC is a right triangle with right angle at A . Construct the squares $ABDE$ and $ACFG$ exterior to ΔABC , and let P and Q be the points of intersection of CD and AB , and of BF and AC , respectively. Show that $AP = AQ$.

1538. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Find all functions $y = f(x)$ with the property that the line through any two points $(p, f(p)), (q, f(q))$ on the curve intersects the y -axis at the point $(0, -pq)$.

1539*. *Proposed by D.M. Milosević, Pranjani, Yugoslavia.*

If α, β, γ are the angles, s the semiperimeter, R the circumradius and r the inradius of a triangle, prove or disprove that

$$\sum \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \leq \left(\frac{2R - r}{s} \right)^2,$$

where the sum is cyclic.

1540. *Proposed by Len Bos and Bill Sands, University of Calgary.*

For k a positive odd integer, define a sequence $\langle a_n \rangle$ by: $a_0 = 1$ and, for $n > 0$,

$$a_n = \begin{cases} a_{n-1} + k & \text{if } a_{n-1} \text{ is odd ,} \\ a_{n-1}/2 & \text{if } a_{n-1} \text{ is even ,} \end{cases}$$

and let $f(k)$ be the smallest $n > 0$ such that $a_n = 1$. Find all k such that $f(k) = k$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1348*. [1988: 141; 1989: 210] *Proposed by Murray S. Klamkin, University of Alberta.*

Two congruent convex centrosymmetric planar figures are inclined to each other (in the same plane) at a given angle. Prove or disprove that their intersection has maximum area when the two centers coincide.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

The statement is true for any two bounded convex centrosymmetric

sets, congruent or not.

Lemma. Let l, l' be two parallel lines. Suppose A and B are bounded convex sets comprised between l and l' , each of them touching both l and l' . Let M be the set of midpoints of all segments with one end in A and the other in B . Then

$$\text{area}(M) \geq \frac{1}{2}(\text{area}(A) + \text{area}(B)). \quad (1)$$

Proof. M contains, in particular, midpoints of all segments parallel to l and l' having one end in A and the other in B . Let M_0 be the set of all these midpoints. The intersection of M_0 with any line parallel to l and l' is a segment whose length equals the arithmetic mean of the lengths of the respective sections of A and B ; they are nonempty because A and B spread from l to l' . By the Cavalieri principle (i.e. by integration), the area of M_0 is half the sum of the areas of A and B (their possible overlapping is no obstacle). So (1) follows. \square

Proof of the statement. Let F and G be the two figures under consideration; suppose they have a common center of symmetry. Let $F_{\mathbf{v}}$ denote the figure F shifted by a nonzero vector \mathbf{v} . The claim is that

$$\text{area}(F \cap G) \geq \text{area}(F_{\mathbf{v}} \cap G).$$

Denote by $F_{-\mathbf{v}}$ the shift of F by vector $-\mathbf{v}$. Consider the sets

$$A = F_{\mathbf{v}} \cap G, \quad B = F_{-\mathbf{v}} \cap G.$$

These are bounded congruent convex sets, mutually symmetric with respect to the center of symmetry of F and G . Thus they have two parallel common supporting lines and hence satisfy the conditions of the lemma. Choose any point X from the midpoint set M defined in the lemma. Thus X is the midpoint of PQ , where $P \in A$ and $Q \in B$. Since $P, Q \in G$, $X \in G$ also. Since $P \in F_{\mathbf{v}}$ and $Q \in F_{-\mathbf{v}}$, there exists a parallelogram PP_0QQ_0 with $P_0, Q_0 \in F$ and $\overrightarrow{P_0P} = \overrightarrow{QQ_0} = \mathbf{v}$, and so $X = \text{midpoint}(P_0Q_0) \in F$. This shows that $M \subseteq F \cap G$. Since A and B are congruent, by the lemma we obtain

$$\text{area}(F \cap G) \geq \text{area}(M) \geq \text{area}(A) = \text{area}(F_{\mathbf{v}} \cap G). \quad \square$$

Remark. Instead of the simple lemma above, we might employ the inequality of Brunn–Minkowski (e.g. [1], [2]):

$$\sqrt{\text{area}(M)} \geq \frac{1}{2}(\sqrt{\text{area}(A)} + \sqrt{\text{area}(B)}), \quad (2)$$

holding for any bounded convex planar sets A, B (M denoting their midpoint set). The claim is hence derived in exactly the same way as above.

Note that (2) follows immediately from (1); thus the lemma provides a quick proof of the Brunn–Minkowski inequality under the assumption that the planar sets A, B have two parallel common supporting lines. Without this assumption, the

Brunn-Minkowski inequality is not as elementary as the lemma. Resorting to it is advantageous in one respect: namely, (2) holds in space of arbitrary dimension n , with square roots replaced by n th roots. Consequently the statement of the problem is also true for convex centrosymmetric sets in n -space.

References:

- [1] V.V. Buldygin and A.B. Harazishvili, *Brunn-Minkowski's Inequality and its Applications* (in Russian), Kiev, 1985.
- [2] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987. (Translation of *Theorie der Convexen Körper*.)

[Editor's note: This more general result was obtained earlier by P. Penning [1989: 210]. Kuczma's proof seems more convincing to the editor, though.]

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1414. [1989: 47] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum value of the sum

$$\sqrt{\tan \frac{B}{2} \tan \frac{C}{2} + \lambda} + \sqrt{\tan \frac{C}{2} \tan \frac{A}{2} + \lambda} + \sqrt{\tan \frac{A}{2} \tan \frac{B}{2} + \lambda}$$

where A, B, C are the angles of a triangle and λ is a nonnegative constant. (The case $\lambda = 5$ is item 2.37 of O. Bottema et al, *Geometric Inequalities*.)

I. *Solution by Svetoslav J. Bilchev, Technical University, Russe, Bulgaria.*

We shall start from the well known inequality

$$\left(\frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{3} \right)^2 \leq \frac{x + y + z}{3}, \quad (1)$$

where $x, y, z > 0$, with equality when $x = y = z$. (1) is equivalent to

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{3(x + y + z)}. \quad (2)$$

In (2) let

$$x = \tan \frac{B}{2} \tan \frac{C}{2} + \lambda, \quad y = \tan \frac{C}{2} \tan \frac{A}{2} + \lambda, \quad z = \tan \frac{A}{2} \tan \frac{B}{2} + \lambda.$$

Then we will have

$$\sum \sqrt{\tan \frac{B}{2} \tan \frac{C}{2} + \lambda} \leq \sqrt{3\left(3\lambda + \sum \tan \frac{B}{2} \tan \frac{C}{2}\right)}.$$

But it is also well known that

$$\sum \tan \frac{B}{2} \tan \frac{C}{2} = 1,$$

where A, B, C are the angles of an arbitrary triangle. Hence it follows that

$$\sum \sqrt{\tan \frac{B}{2} \tan \frac{C}{2} + \lambda} \leq \sqrt{3(3\lambda + 1)}.$$

Thus the required maximum value is $\sqrt{3(3\lambda + 1)}$, attained when $A = B = C = \pi/3$.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In [1], p. 93, item 3.11, we have proved even more (see also [2]): Let n and m be nonnegative integers with m odd, let $A, B, C \in \mathbb{R}$ with $A + B + C = m\pi$, and let $u, v, w \in \mathbb{R}$ be such that

$$\tan \frac{(2n+1)B}{2} \tan \frac{(2n+1)C}{2} + u \geq 0, \text{ etc.}$$

Then

$$\sum \left(\tan \frac{(2n+1)B}{2} \tan \frac{(2n+1)C}{2} + u \right)^\alpha \begin{cases} \leq 3^{1-\alpha}(1 + \sum u)^\alpha & \text{if } 0 < \alpha < 1, \\ \geq 3^{1-\alpha}(1 + \sum u)^\alpha & \text{if } \alpha < 0 \text{ or } \alpha > 1. \end{cases}$$

This follows via Jensen's inequality from the fact that $f(x) = x^\alpha$ is concave for $0 < \alpha < 1$ and convex for $\alpha < 0$ or $\alpha > 1$, and from

$$\sum \tan \frac{(2n+1)B}{2} \tan \frac{(2n+1)C}{2} = 1.$$

Putting $\alpha = 1/2$, $n = 0$, $m = 1$, $u = v = w = \lambda$, for the given expression we get the maximum $\sqrt{3 + 9\lambda}$, attained at $A = B = C = \pi/3$.

References:

- [1] D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.
- [2] D.S. Mitrinović, J.E. Pečarić, and W. Janous, Some trigonometric inequalities, RAD JAZU (Zagreb) 428 (1987) 103–127.

III. *Solution by the proposer.*

More generally, we determine the maximum of the sum

$$S = \sum F\left[\lambda + G\left(\tan \frac{B}{2} \tan \frac{C}{2}\right)\right]$$

where F is increasing, F and G are concave, and the summations here and subsequently are symmetric over A, B, C . By Jensen's inequality,

$$\begin{aligned} S &\leq 3F\left[\lambda + \frac{1}{3} \sum G\left(\tan \frac{B}{2} \tan \frac{C}{2}\right)\right] \\ &\leq 3F\left[\lambda + G\left(\frac{1}{3} \sum \tan \frac{B}{2} \tan \frac{C}{2}\right)\right] \\ &= 3F(\lambda + G(1/3)). \end{aligned} \tag{3}$$

For the special case $G(x) = x$, F need not be increasing. If either G is strictly concave or else $G(x) = x$ and F is strictly concave, then there is equality if and only if $A = B = C$. This applies to the special case of the given problem where $F(x) = \sqrt{x}$ and $G(x) = x$.

If instead F and G are convex and F is decreasing, then inequality (3) is reversed.

For the special case $G(x) = x$, it can be shown by using the majorization inequality that $2F(\lambda) + F(\lambda + 1)$ is the *minimum* value of S for F concave (and the maximum value of S for F convex). This extreme occurs for degenerate triangles obtained by taking the limit as ϵ goes to 0 for the angles

$$A = \pi - \epsilon - \epsilon^2, \quad B = \epsilon, \quad C = \epsilon^2.$$

Then we have

$$\tan \frac{A}{2} \tan \frac{B}{2} = 1, \quad \tan \frac{B}{2} \tan \frac{C}{2} = \tan \frac{C}{2} \tan \frac{A}{2} = 0.$$

Since for any nondegenerate triangle ABC ,

$$\sum \tan \frac{B}{2} \tan \frac{C}{2} = 1,$$

we have

$$\left(\tan \frac{A}{2} \tan \frac{B}{2}, \tan \frac{B}{2} \tan \frac{C}{2}, \tan \frac{C}{2} \tan \frac{A}{2} \right) \prec (1,0,0)$$

and so

$$S = \sum F\left(\lambda + \tan \frac{B}{2} \tan \frac{C}{2}\right) \geq 2F(\lambda) + F(\lambda + 1)$$

for F concave. So for the original sum, lower and upper bounds are given by

$$2\sqrt{\lambda} + \sqrt{\lambda + 1} \leq \sum \sqrt{\lambda + \tan \frac{B}{2} \tan \frac{C}{2}} \leq \sqrt{3(3\lambda + 1)}.$$

In particular, for $\lambda = 5$ the lower and upper bounds are

$$6.921625697 \quad \text{and} \quad 6.928203202.$$

Also solved by JACK GARFUNKEL, Flushing, N.Y.; and VEDULA N. MURTY, Pennsylvania State University at Harrisburg. One other reader sent in the correct maximum value without proof.

1415. [1989: 47] Proposed by G.P. Henderson, Campbellcroft, Ontario.

Given the system of differential equations

$$\begin{aligned} \dot{x}_1 &= -(c_{12} + c_{13})x_1 + c_{12}x_2 + c_{13}x_3 \\ \dot{x}_2 &= c_{21}x_1 - (c_{21} + c_{23})x_2 + c_{23}x_3 \\ \dot{x}_3 &= c_{31}x_1 + c_{32}x_2 - (c_{31} + c_{32})x_3, \end{aligned}$$

where the c 's are positive constants, show that $\lim_{t \rightarrow \infty} x_i(t)$ is a weighted average,

independent of i , of the initial values $x_1(0), x_2(0), x_3(0)$.

Solution by Kee-Wai Lau, Hong Kong.

The characteristic equation of the system is

$$\begin{vmatrix} m+c_{12}+c_{13} & -c_{12} & -c_{13} \\ -c_{21} & m+c_{21}+c_{23} & -c_{23} \\ -c_{31} & -c_{32} & m+c_{31}+c_{32} \end{vmatrix} = 0 ,$$

or

$$m[m^2 + (c_{12} + c_{13} + c_{21} + c_{23} + c_{31} + c_{32})m + (c_{12}c_{23} + c_{13}c_{21} + c_{13}c_{23} + c_{12}c_{31} + c_{12}c_{32} + c_{13}c_{32} + c_{21}c_{31} + c_{21}c_{32} + c_{23}c_{31})] = 0 ,$$

whose roots are 0, m_1 , m_2 , say. Since the c 's are positive constants, either m_1 and m_2 are both negative or they are complex conjugates with negative real parts.

If m_1 and m_2 are distinct, we have that the solution to the system is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = P \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + Q \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} e^{m_1 t} + R \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} e^{m_2 t} \quad (1)$$

where P , Q , R are constants, and the three column vectors on the right are linearly independent eigenvectors corresponding to the eigenvalues 0, m_1 , m_2 , respectively. (In case m_1 , m_2 are complex then $r_i = \bar{q}_i$.) It follows that

$$\begin{aligned} P + q_1 Q + r_1 R &= x_1(0) \\ P + q_2 Q + r_2 R &= x_2(0) \\ P + q_3 Q + r_3 R &= x_3(0) . \end{aligned}$$

So [by (1) and Cramer's Rule]

$$\lim_{t \rightarrow \infty} x_i(t) = P = \frac{(q_2 r_3 - q_3 r_2)x_1(0) + (q_3 r_1 - q_1 r_3)x_2(0) + (q_1 r_2 - q_2 r_1)x_3(0)}{(q_2 r_3 - q_3 r_2) + (q_3 r_1 - q_1 r_3) + (q_1 r_2 - q_2 r_1)} ,$$

as required.

If $m_1 = m_2 < 0$, we have the solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} e_1 + f_1 t \\ e_2 + f_2 t \\ e_3 + f_3 t \end{pmatrix} e^{m_1 t} . \quad (2)$$

Hence

$$\begin{aligned} K + e_1 &= x_1(0) \\ K + e_2 &= x_2(0) \\ K + e_3 &= x_3(0) . \end{aligned} \quad (3)$$

If $e_1 = e_2 = e_3$ then $x_1(0) = x_2(0) = x_3(0)$ so that $x_i(t)$ is in fact constant. [Editor's note. From the original system, $\dot{x}_i(0) = 0$. From (2),

$$x_i(t) = x_i(0) + f_i t e^{m_1 t}$$

so that

$$\dot{x}_i(t) = f_i e^{m_1 t} (1 + m_1 t) .$$

Thus $0 = \dot{x}_i(0) = f_i$, so $\dot{x}_i(t) = 0$. Sorry for the interruption.] If $e_1 \neq e_2$ say, then from (2) and (3) follows

$$\lim_{t \rightarrow \infty} x_i(t) = K = \frac{e_2 x_1(0) - e_1 x_2(0)}{e_2 - e_1} .$$

This completes the solution of the problem.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; and the proposer.

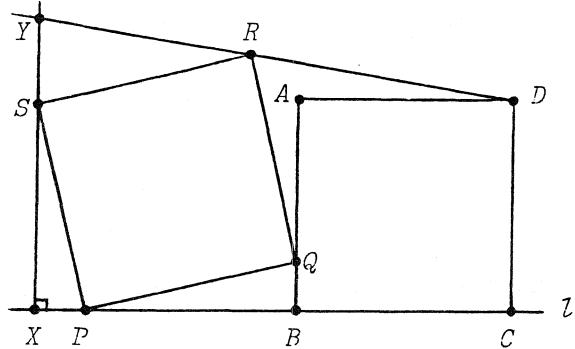
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1416. [1989: 48] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

In the figure, the unit square $ABCD$ and the line l are fixed, and the unit square $PQRS$ rotates with P and Q lying on l and AB respectively. X is the foot of the perpendicular from S to l . Find the position of point Q so that the length XY is a maximum.



Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let $\angle QPB = \theta$ and let B be the origin.

Then we see that

$$X = (-\sin \theta - \cos \theta, 0)$$

and

$$R = (-\sin \theta, \sin \theta + \cos \theta).$$

The line RD can be expressed by

$$y - 1 = \frac{1 - (\sin \theta + \cos \theta)}{1 + \sin \theta}(x - 1).$$

It follows that the length XY is

$$\begin{aligned} f(\theta) &= 1 + \frac{1 - (\sin \theta + \cos \theta)}{1 + \sin \theta} \cdot (-\sin \theta - \cos \theta - 1) \\ &= 1 + \frac{\sin 2\theta}{1 + \sin \theta}. \end{aligned}$$

Note that

$$f'(\theta) = \frac{2(1 - \sin \theta - \sin^2 \theta)}{1 + \sin \theta}.$$

This means that $f(\theta)$ has a maximum when $1 - \sin \theta - \sin^2 \theta = 0$, i.e.

$$\sin \theta = \frac{\sqrt{5} - 1}{2}.$$

Hence the length XY is maximized when $BQ = (\sqrt{5} - 1)/2$.

Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium,

Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; MICHAEL RUBINSTEIN, student, Princeton University; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

The problem was taken from the 1847 Japanese mathematics book Juntendo Sanpu.

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1417. [1989: 48] Proposed by G.R. Veldkamp, De Bilt, The Netherlands.

O, A, B, C, D are five points in space, no four on the same plane, with $\angle AOB = \angle COD = 90^\circ$. Let p be the line through O intersecting AC and BD , and let q be the line through O intersecting AD and BC . Prove that $p \perp q$.

I. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

p is the intersection of planes OAC and OBD , which have respective normal vectors $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{d}$ (we put $\mathbf{a} = \overrightarrow{OA}$, etc.). Consequently p has the direction vector

$$\mathbf{d}_p = (\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d}) .$$

Similarly q has the direction vector

$$\mathbf{d}_q = (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) .$$

We now have to show that $\mathbf{d}_p \cdot \mathbf{d}_q = 0$. Using the familiar identity

$$(\mathbf{t} \times \mathbf{u}) \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{t} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w}) - (\mathbf{t} \cdot \mathbf{w})(\mathbf{u} \cdot \mathbf{v}) ,$$

and noting

$$\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{c} \cdot \mathbf{d} ,$$

we get

$$\begin{aligned} \mathbf{d}_p \cdot \mathbf{d}_q &= [(\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d})] \cdot [(\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})] \\ &= [(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{d})][(\mathbf{b} \times \mathbf{d}) \cdot (\mathbf{b} \times \mathbf{c})] - [(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c})][(\mathbf{b} \times \mathbf{d}) \cdot (\mathbf{a} \times \mathbf{d})] \\ &= [-(\mathbf{a} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{a})][-(\mathbf{b} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b})] - [-(\mathbf{a} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{b})][-(\mathbf{b} \cdot \mathbf{d})(\mathbf{d} \cdot \mathbf{a})] \\ &= 0 . \end{aligned}$$

II. Solution by P. Penning, Delft, The Netherlands.

Introduce a rectangular coordinate system, with O as origin, and axes such that

$$A = (a, 0, 0) , \quad B = (0, b, 0) , \quad C = (x, y, z) , \quad D = (u, v, w) .$$

Since OC and OD are perpendicular, one must require

$$xu + yv + zw = 0 . \tag{1}$$

With l and m variable, any point on AC or BD will have respective forms

$(a(1-l) + lx, ly, lz)$ and $(mu, b(1-m) + mv, mw)$.

To find p , one has to adjust l and m so that these two points lie on a straight line through O , i.e.,

$$\frac{a(1-l) + lx}{mu} = \frac{ly}{b(1-m) + mv} = \frac{lz}{mw},$$

leading to

$$[a(1-l) + lx]mw = mulz$$

and so

$$l = \frac{aw}{aw + zu - xw}, \quad 1 - l = \frac{zu - xw}{aw + zu - xw}.$$

The point where p intersects AC will now be

$$\left(\frac{a(zu - xw)}{aw + zu - xw} + \frac{awx}{aw + zu - xw}, \frac{awy}{aw + zu - xw}, \frac{awz}{aw + zu - xw} \right)$$

so that a vector in the direction of p is given by

$$\mathbf{p} = (zu, wy, wz).$$

In a similar way, a vector in the direction of q is given by

$$\mathbf{q} = (wx, zv, zw).$$

The scalar product $\mathbf{p} \cdot \mathbf{q}$ is

$$zw(xu + yv + zw) = 0$$

because of (1). So p is perpendicular to q .

III. Solution by the proposer.

We introduce a plane W parallel to p and q , and intersecting OA , OB , OC , OD at A' , B' , C' , D' respectively. Since p contains O and intersects AC , p and $A'C'$ lie in the same plane, so $p \parallel A'C'$. Similarly $p \parallel B'D'$, so $A'C' \parallel B'D'$. Similarly $A'D' \parallel q \parallel B'C'$. Therefore $A'C'B'D'$ is a parallelogram with $A'B'$ and $C'D'$ as its diagonals. Let S be their intersection. Since $\angle AOB = 90^\circ$, we have $\overline{OS} = \overline{SA'} = \overline{SB'}$. Similarly $\overline{OS} = \overline{SC'} = \overline{SD'}$. Hence $\overline{A'B'} = \overline{C'D'}$. $A'C'B'D'$ is therefore a rectangle. It follows that $p \perp q$.

Also solved by JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; and TOSHIO SEIMIYA, Kawasaki, Japan. The proposer also sent in another solution, similar to solution I.

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1418. [1989: 48] Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.

Given that

$$\frac{\cos^3\theta}{\cos(\alpha - 3\theta)} = \frac{\sin^3\theta}{\sin(\alpha - 3\theta)} = m,$$

show that $\cos \alpha = 2m - m^{-1}$.

Solution by S.A. Obaid, San Jose State University.

The given equation may be rewritten as

$$\cos^3\theta = m \cos(\alpha - 3\theta) = m \cos 3\theta \cos \alpha + m \sin 3\theta \sin \alpha, \quad (1)$$

$$\sin^3\theta = m \sin(\alpha - 3\theta) = -m \sin 3\theta \cos \alpha + m \cos 3\theta \sin \alpha. \quad (2)$$

Equations (1) and (2) give

$$\cos^6\theta + \sin^6\theta = m^2[\cos^2(\alpha - 3\theta) + \sin^2(\alpha - 3\theta)] = m^2. \quad (3)$$

But cubing $\cos^2\theta + \sin^2\theta = 1$ and using (3) yields

$$3 \sin^2\theta \cos^2\theta = 1 - m^2. \quad (4)$$

Since equations (1) and (2) are linear in $\cos \alpha$ and $\sin \alpha$, Cramer's rule can be used to obtain

$$\cos \alpha = (\cos^3\theta \cos 3\theta - \sin^3\theta \sin 3\theta)m^{-1}. \quad (5)$$

Using the identities

$$\cos 3\theta = \cos \theta(1 - 4 \sin^2\theta), \quad \sin 3\theta = 3 \sin \theta - 4 \sin^3\theta,$$

(5) reduces to

$$\begin{aligned} \cos \alpha &= [\cos^4\theta - 3 \sin^4\theta + 4 \sin^2\theta(\sin^4\theta - \cos^4\theta)]m^{-1} \\ &= [\cos^4\theta - 3 \sin^4\theta + 4 \sin^2\theta(\sin^2\theta - \cos^2\theta)]m^{-1} \\ &= [(\cos^2\theta + \sin^2\theta)^2 - 6 \sin^2\theta \cos^2\theta]m^{-1} \\ &= (1 - 6 \sin^2\theta \cos^2\theta)m^{-1}. \end{aligned}$$

By virtue of (4), this equation becomes

$$\cos \alpha = [1 - 2(1 - m^2)]m^{-1} = 2m - m^{-1}.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; V.N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; EICHI WATANABE, Rio de Janeiro, Brazil; and the proposer.

Bellot Rosado points out that the problem appears as an example on p. 84 of Hobson's A Treatise on Plane Trigonometry, Cambridge Univ. Press, 1921.

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1419. [1989: 48] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Points $A, F_1, F_2, B, D_1, D_2, C, E_1, E_2$ lie (in that order) on a circle, so that chords $AD_1, AD_2; BE_1, BE_2; CF_1, CF_2$ trisect angles A, B, C , respectively, of $\triangle ABC$. Let $D_1E_2 \cap D_2F_1 = P, E_1F_2 \cap E_2D_1 = Q, F_1D_2 \cap F_2E_1 = R$.

- (a) Prove that ΔPQR is equilateral.
- (b) Show that ΔPQR has side length $\frac{m_e - m}{3}$,

where m, m_e are the side lengths of the equilateral Morley triangles formed by the interior and exterior trisectors, respectively, of the angles of ΔABC .

Solution by Dan Sokolowsky, Williamsburg, Virginia.

- (a) Let $A = 3\alpha, B = 3\beta, C = 3\gamma$, so that $\alpha + \beta + \gamma = 60^\circ$. (1)

So

$$\begin{aligned}\angle QPR &= \frac{1}{2}(\widehat{D_1D_2} + \widehat{E_2A} + \widehat{AF_1}) \\ &= \frac{1}{2}(2\alpha + 2\beta + 2\gamma) = 60^\circ.\end{aligned}$$

We can show likewise that $\angle PQR = \angle QRP = 60^\circ$, so ΔPQR is equilateral.

- (b) Let the given circle have center O and (for convenience) radius 1. We first compute a side, say PQ , of ΔPQR . Since

$$\angle D_1OF_1 = \widehat{D_1F_1} = \widehat{D_1B} + \widehat{BF_2} + \widehat{F_2F_1} = 2\alpha + 4\gamma,$$

we have

$$D_1F_1 = 2 \sin\left(\frac{1}{2}\angle D_1OF_1\right) = 2 \sin(\alpha + 2\gamma). \quad (2)$$

In $\triangle D_1PF_1$, $\angle D_1PF_1 = 120^\circ$ and $\angle D_1F_1P = \widehat{D_1D_2}/2 = \alpha$, so using (1), (2) and the sine law,

$$\begin{aligned}D_1P &= \frac{D_1F_1 \sin \alpha}{\sin 120^\circ} = \frac{4 \sin(\alpha + 2\gamma) \sin \alpha}{\sqrt{3}} \\ &= \frac{2}{\sqrt{3}}[\cos 2\gamma - \cos(2\alpha + 2\gamma)] = \frac{2}{\sqrt{3}}[\cos 2\gamma - \cos(120^\circ - 2\beta)].\end{aligned}$$

A similar calculation using $\triangle E_2QF_2$ gives

$$E_2Q = \frac{2}{\sqrt{3}}[\cos 2\gamma - \cos(120^\circ - 2\alpha)].$$

Also, $\angle D_1OE_2 = 4\alpha + 4\beta$, so

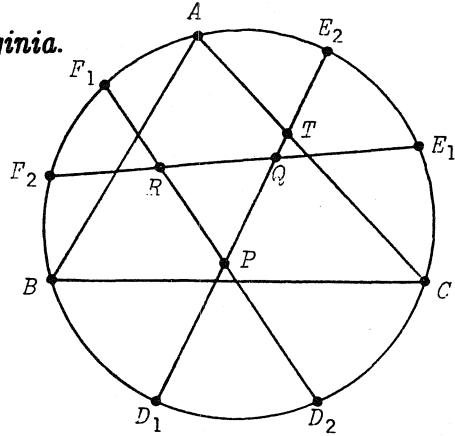
$$D_1E_2 = 2 \sin(2\alpha + 2\beta) = 2 \sin(120^\circ - 2\gamma).$$

Then using the readily verified identity

$$2 \sin(120^\circ - 2\gamma) - \frac{4}{\sqrt{3}} \cos 2\gamma = \frac{2}{\sqrt{3}} \cos(120^\circ - 2\gamma)$$

we obtain

$$\begin{aligned}PQ &= D_1E_2 - D_1P - E_2Q \\ &= \frac{2}{\sqrt{3}}[\cos(120^\circ - 2\alpha) + \cos(120^\circ - 2\beta) + \cos(120^\circ - 2\gamma)].\end{aligned}$$



It is well known that

$$m = 8 \sin \alpha \sin \beta \sin \gamma .$$

The calculation is given in [1], p. 163. A similar calculation gives

$$m_e = 8 \sin(60^\circ - \alpha) \sin(60^\circ - \beta) \sin(60^\circ - \gamma) . \quad (3)$$

Using the identities

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y) ,$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y) ,$$

and (1), m and m_e can be rewritten respectively as

$$m = 2[\sin(60^\circ - 2\alpha) + \sin(60^\circ - 2\beta) + \sin(60^\circ - 2\gamma)] - \sqrt{3} ,$$

$$m_e = 2[\sin 2\alpha + \sin 2\beta + \sin 2\gamma] - \sqrt{3} .$$

Then, using the identity

$$\sin x - \sin(60^\circ - x) = \sqrt{3} \cos(120^\circ - x) ,$$

we obtain

$$m_e - m = 2\sqrt{3}[\cos(120^\circ - 2\alpha) + \cos(120^\circ - 2\beta) + \cos(120^\circ - 2\gamma)] = 3PQ ,$$

so

$$PQ = \frac{m_e - m}{3} .$$

It is known [2] that the corresponding sides of the inner and exterior Morley triangles of ΔABC are parallel, and that one such pair of corresponding sides, call them P_1Q_1 and P_2Q_2 , meet AC at an angle $2\alpha + \beta$. We note that if PQ meets AC at T (see the above figure),

$$\angle CTD_1 = \frac{1}{2}(\widehat{D_1C} + \widehat{AE_2}) = 2\alpha + \beta ,$$

so that the sides of ΔPQR are parallel to the corresponding sides of the inner and exterior Morley triangles.

References:

- [1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, M.A.A., 1967.
- [2] D. Sokolowsky, An elementary geometric proof of the Morley theorem, *Crux Mathematicorum* 3 (1977) 291–294.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

*As a reference for equation (3) for m_e , Bellot Rosado and Lopez Chamorro mention G. Loria, *Triangles équilatéraux dérivés d'un triangle quelconque*, Math. Gazette 23 (1939) 364–372. A more recent reference is K.R.S. Sastry, *Constellation Morley*, Math. Magazine 47 (1974) 15–22.*

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1420. [1989: 48] *Proposed by Shailesh Shirali, Rishi Valley School, India.*

If a, b, c are positive integers such that

$$0 < a^2 + b^2 - abc \leq c,$$

show that $a^2 + b^2 - abc$ is a perfect square. (This is a generalization of problem 6 of the 1988 I.M.O. [1988: 197].)

Solution by Robert B. Israel, University of British Columbia.

The assertion is trivial if $c = 1$, while for $c = 2$ we have

$$a^2 + b^2 - abc = (a - b)^2,$$

so we can assume $c > 2$. By symmetry we may assume $a \geq b$.

For fixed d with $0 < d \leq c$, consider the hyperbola C with equation

$$x^2 + y^2 - cxy = d.$$

Let C_1 be the branch of C with $x > y$. We assume that C_1 contains a point (a, b) with positive integer coordinates, and must show that d is a perfect square.

Since C_1 is increasing, the x (or y) coordinates provide an ordering for points of C_1 : namely, put $(x, y) < (x', y')$ if and only if $x < x'$ (if and only if $y < y'$). The linear transformation

$$T : (x, y) \mapsto (y, cy - x)$$

maps C onto itself, with inverse

$$T^{-1} : (x, y) \mapsto (cx - y, x).$$

Note that $T(\sqrt{d}, 0) = (0, -\sqrt{d})$, where both points lie on C_1 , and there are no fixed points on C . By continuity, T maps C_1 to itself, with $T(x, y) < (x, y)$ for $(x, y) \in C_1$. The points $T^n(x, y)$ have no limit on C_1 (as this would be a fixed point). Thus if $(x, y) \in C_1$ with $x > 0$ and $y > 0$, there is some least positive integer n such that

$$T^n(x, y) \leq (\sqrt{d}, 0),$$

and since $T^{n-1}(x, y) > (\sqrt{d}, 0)$ we must have

$$T^n(x, y) > (0, -\sqrt{d}).$$

Since T also preserves the property of having integer coordinates, and $(a, b) \in C_1$, we obtain a point (u, v) on C_1 with integer coordinates such that $0 < u \leq \sqrt{d}$ and $-\sqrt{d} < v \leq 0$. If d is not a square, this means that $u \geq 1$ and $v \leq -1$. But then

$$d = u^2 + v^2 - cuv \geq 2 + c,$$

contradicting the hypothesis.

Incidentally, this shows (after checking the case $c = 1$ separately) that we can replace " $\leq c$ " by " $\leq c + 1$ " in the hypothesis.

Also solved (in a similar way) by the proposer.

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1421. [1989: 73] *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with sides a, b, c . The escribed circle to the side a has centre I_a and touches a, b, c (produced) at D, E, F respectively. M is the midpoint of BC .

- (a) Show that the lines I_aD, EF and AM have a common point S_a .
- (b) In the same way we have points S_b and S_c . Prove that

$$\frac{\text{area}(\Delta S_a S_b S_c)}{\text{area}(\Delta ABC)} > \frac{3}{2}.$$

I. *Solution to (a) by Shailesh Shirali, Rishi Valley School, India.*

Let I_aD meet EF at Y , and let AY meet BC at X . We need to show that $BX = XC$. Now

$$\angle EI_aD = \angle ABC (\equiv B)$$

and

$$\angle FI_aD = \angle ACB (\equiv C);$$

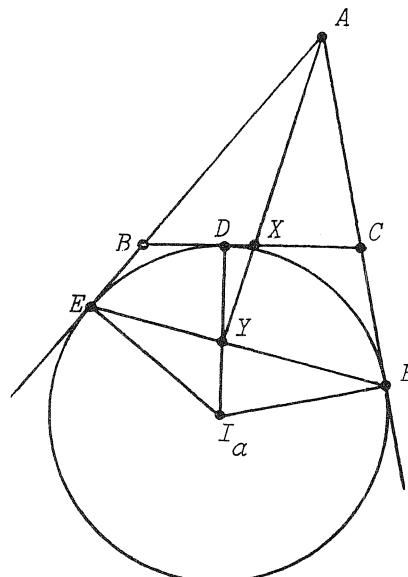
so from the isosceles triangles I_aEF and AEF ,

$$\frac{\sin B}{\sin C} = \frac{EY}{FY} = \frac{\sin BAX}{\sin CAX}.$$

Next,

$$\frac{BX}{CX} = \frac{BX/AX}{CX/AX} = \frac{\sin BAX/\sin B}{\sin CAX/\sin C} = 1,$$

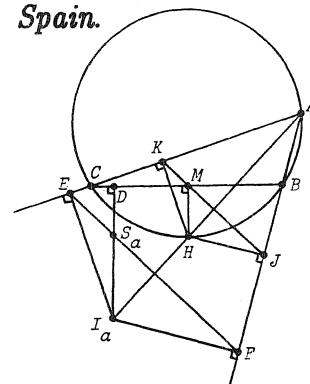
so $BX = CX$, as required.



II. *Solution to (a) by Jose Yusty Pita, Madrid, Spain.*

[Yusty Pita also solved part (b). – Ed.]

Drawing the Simson line JMK of the midpoint of the arc BC of the circumcircle of ΔABC , and letting $EF \cap I_aD = S_a$, then the figure I_aFS_aE is homothetic to $HJKM$, with center A . Thus S_aM passes through A .



III. *Solution to (b) by George Tsintsifas, Thessaloniki, Greece.*

[Tsintsifas also solved part (a). – Ed.]

Let AH be the altitude from A of ΔABC . Assuming $b \geq c$ we easily find

$$\begin{aligned} HM &= \frac{a}{2} - BH \\ &= \frac{a^2 - 2ac \cos B}{2a} = \frac{b^2 - c^2}{2a} \end{aligned}$$

and

$$MD = \frac{a}{2} - (s - b) = \frac{b - c}{2}.$$

Thus

$$\frac{MS_a}{AM} = \frac{MD}{HM} = \frac{a}{b + c}.$$

Letting G be the centroid of $\triangle ABC$,

$$\frac{GS_a}{GM} = 1 + \frac{MS_a}{GM} = 1 + \frac{3MS_a}{AM} = 1 + \frac{3a}{b + c} = \frac{3a + b + c}{b + c}.$$

With N the midpoint of AC , we similarly have

$$\frac{GS_b}{GN} = \frac{a + 3b + c}{a + c}.$$

Thus (writing $[XYZ]$ for the area of $\triangle XYZ$)

$$\frac{[S_a GS_b]}{[MGN]} = \frac{(3a + b + c)(a + 3b + c)}{(b + c)(a + c)},$$

and therefore

$$[S_a GS_b] = \frac{(3a + b + c)(a + 3b + c)}{(b + c)(a + c)} \cdot \frac{[ABC]}{12},$$

so that

$$[S_a S_b S_c] = \frac{[ABC]}{12} \sum \frac{(3a + b + c)(3b + c + a)}{(b + c)(c + a)}. \quad (1)$$

But we can write

$$\frac{(3a + b + c)(3b + c + a)}{(b + c)(c + a)} = \frac{(a + b + c)^2}{(b + c)(c + a)} + \frac{2(a + b)(a + b + c)}{(b + c)(c + a)} + \frac{4ab}{(b + c)(c + a)}. \quad (2)$$

First, for the sum

$$K_1 = \sum \frac{(a + b + c)^2}{(b + c)(c + a)} = \frac{2(a + b + c)^3}{(b + c)(c + a)(a + b)},$$

using the A.M.-G.M. inequality on $a + b$, $b + c$, $c + a$ we obtain

$$\frac{2(a + b + c)}{3} \geq \sqrt[3]{(a + b)(b + c)(c + a)} \quad (3)$$

so that

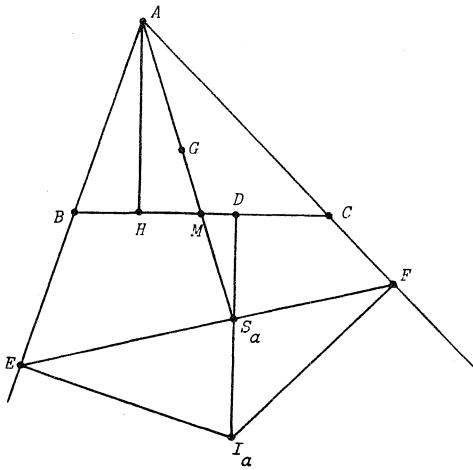
$$K_1 \geq \frac{27}{4}. \quad (4)$$

Also for

$$K_2 = 2 \sum \frac{(a + b + c)(a + b)}{(b + c)(c + a)} = 2(a + b + c) \sum \frac{a + b}{(b + c)(c + a)}$$

we have, by the A.M.-G.M. inequality and (3),

$$K_2 \geq \frac{6(a + b + c)}{\sqrt[3]{(a + b)(b + c)(c + a)}} \geq 9. \quad (5)$$



Finally, we have from the A.M.-G.M. inequality that

$$\sum ab(a + b) \geq 6abc$$

and thus

$$4 \sum ab(a + b) \geq 3 \left(\sum ab(a + b) + 2abc \right) \\ = 3(a + b)(b + c)(c + a);$$

therefore

$$K_3 = 4 \sum \frac{ab}{(a + c)(b + c)} = \frac{4}{(a + b)(b + c)(c + a)} \sum ab(a + b) \geq 3. \quad (6)$$

So from (1), (2), (4), (5), (6) we have

$$[S_a S_b S_c] \geq \frac{[ABC]}{12} \left(\frac{27}{4} + 9 + 3 \right) = \frac{25}{16}[ABC] > \frac{3}{2}[ABC].$$

IV. *Solution to (b) by Dan Sokolowsky, Williamsburg, Virginia.*

[Sokolowsky first solved part (a), and then derived equation (1) of Tsintsifas' proof (III). We pick up his solution at this point.]

Let

$$f_1 = abc, \quad f_2 = \sum ab(a + b), \quad f_3 = \sum a^3,$$

where the sums are cyclic over a, b, c . Then by expanding we find that

$$\rho = \frac{[S_a S_b S_c]}{[ABC]} = \frac{\sum (3a + b + c)(3b + c + a)(a + b)}{12(a + b)(b + c)(c + a)} \\ = \frac{f_3 + 3f_2 + 4f_1}{2(f_2 + 2f_1)}. \quad (7)$$

Thus, for a positive constant w , $\rho \geq w$ holds if and only if

$$f_3 - (2w - 3)f_2 - (4w - 4)f_1 \geq 0$$

which can be written

$$[kf_3 - (2w - 3)f_2] + [(1 - k)f_3 - (4w - 4)f_1] \geq 0 \quad (8)$$

for any k .

Now we have

$$a^3 + b^3 - ab(a + b) = (a + b)(a - b)^2 \geq 0$$

with equality if and only if $a = b$, and likewise

$$b^3 + c^3 - bc(b + c) \geq 0 \quad \text{and} \quad c^3 + a^3 - ca(c + a) \geq 0,$$

with equality if and only if $b = c$ and $c = a$ respectively. Adding, we obtain

$$2f_3 - f_2 \geq 0 \quad (9)$$

with equality if and only if $a = b = c$. Also, by the law of the arithmetic and geometric mean,

$$f_3 - 3f_1 \geq 0 \quad (10)$$

with equality if and only if $a = b = c$. By (9) and (10), (8) holds for k and w

satisfying the system

$$k = 2(2w - 3), \\ 3(1 - k) = 4w - 4$$

(and provided $0 < k < 1$). The solution of this system is $k = 1/4$, $w = 25/16$. Thus $\rho \geq 25/16$. Also when $w = 25/16$, $k = 1/4$, and $a = b = c$, equality holds in (8), so $\min \rho = 25/16$, occurring if and only if $a = b = c$.

We now find the best upper bound for ρ . By (7), $\rho \leq B$ for some constant B if and only if

$$[(2B - 3)f_2 - f_3] + (4B - 4)f_1 \geq 0. \quad (11)$$

But since

$$f_2 - f_3 = a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \geq 0,$$

(11) holds for $B = 2$. Hence $\rho \leq 2$. Considering the set of triangles ABC with $b = a$ fixed and $c < 2a$, one readily calculates from (7) that on this set $\lim_{c \rightarrow 0} \rho = 2$.

Hence $\text{lub } \rho = 2$, but $\rho = 2$ cannot be attained by nondegenerate triangles.

To summarize,

$$\frac{25}{16} \leq \frac{[S_a S_b S_c]}{[ABC]} < 2$$

for nondegenerate triangles ABC , both bounds being best possible.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposers.

Yusty Pita also found the sharp lower bound of 25/16.

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1422. [1989: 73] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2A_3$ be a triangle and M an interior point; $\lambda_1, \lambda_2, \lambda_3$ the barycentric coordinates of M ; and r_1, r_2, r_3 its distances from the sides A_2A_3, A_3A_1, A_1A_2 respectively. Set $A_iM = R_i, i = 1, 2, 3$. Prove that

$$\sum_{i=1}^3 \lambda_i R_i \geq 2 \left(\lambda_1 \cdot \frac{r_2 r_3}{r_1} + \lambda_2 \cdot \frac{r_3 r_1}{r_2} + \lambda_3 \cdot \frac{r_1 r_2}{r_3} \right).$$

Solution by Murray S. Klamkin, University of Alberta.

Since

$$\lambda_i = \frac{a_i r_i}{2F}$$

where F is the area of $\Delta A_1A_2A_3$, the inequality is equivalent to

$$\sum a_1 r_1 R_1 \geq 2 \sum a_1 r_2 r_3, \quad (1)$$

where the sums here and subsequently are cyclic over the indices. By the

transformation of isogonal conjugation, (1) is dual to

$$\sum a_1 R_1 \geq 2 \sum a_1 r_1 = 4F . \quad (2)$$

Since (2) is the known Steensholt inequality (see [1984: 326] and item 12.19 of [4]), (1) follows.

We can apply other transformations to (1) (see [1], [2], [3]) to obtain other inequalities. For instance, point reciprocation applied to (1) (or inversion to (2)) gives

$$R_1 R_2 R_3 \sum a_1 \geq 2 \sum a_1 r_1 R_1^2 , \quad (3)$$

which thus is also valid.

References:

- [1] M.S. Klamkin, An identity for simplexes and related inequalities, *Simon Stevin* 48 (1974–75) 57–64.
- [2] D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989, pp. 293–296.
- [3] M.S. Klamkin, Triangle inequalities via transforms, *Notices Amer. Math. Soc.*, January 1972, A–103, 104.
- [4] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOSE YUSTY PITA, Madrid, Spain; and the proposer.

Janous also obtained the equivalent inequality (3) above.

Janous notes that, by the identity

$$\sum a_1 r_2 r_3 = \frac{1}{2R} \sum a_1 r_1 R_1^2$$

([3], page 296 of [2], using the above references), from (1) the given inequality is equivalent to

$$\sum \lambda_1 R_1 \geq \frac{1}{R} \sum \lambda_1 R_1^2 .$$

He also applies the known inequality ([2], p. 319, item (35)]

$$\sum x^2 r_1 R_1 \geq \sum y z r_2 r_3 ,$$

where x, y, z are non-negative real numbers, in the case $x = \sqrt{a_1}, y = \sqrt{a_2}, z = \sqrt{a_3}$ to obtain the companion inequality

$$\sum \lambda_1 R_1 \geq 2 \sum * \sqrt{\lambda_2 \lambda_3} \cdot \sqrt{r_2 r_3} .$$

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1424. [1989: 73] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that

$$\sum a \tan A \geq 10R - 2r$$

for any acute triangle ABC , where a, b, c are its sides, R its circumradius, and r its inradius, and the sum is cyclic.

Solution by Kee-Wai Lau, Hong Kong.

Since

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

and

$$\sum \cos A = 1 + \frac{r}{R},$$

so the given inequality is equivalent to

$$\begin{aligned}\sum \sin A \tan A &\geq 5 - \frac{r}{R}, \\ \sum (\sin A \tan A + \cos A) &\geq 6,\end{aligned}$$

and finally

$$\sum \sec A \geq 6.$$

Now $\sec x$ is convex for $0 \leq x < \pi/2$, so

$$\sum \sec A \geq 3 \sec\left(\frac{A+B+C}{3}\right) = 6. \quad (1)$$

This proves the required inequality.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JACK GARFUNKEL, Flushing, N.Y.; MURRAY S. KLAMKIN, University of Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; BOB PRIELIPP, University of Wisconsin-Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; JOSE YUSTY PITA, Madrid, Spain; and the proposer.

Nearly all solvers used the above method. Usually for inequality (1) above they referred to item 2.45 of Bottema et al, Geometric Inequalities (which, it should be noted, is valid only for acute triangles.)

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The absolute deadline for receipt of applications is August 31, 1990.

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Department of Mathematics
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