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Finite Lists and the Propositional Calculus

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Part 1

I was spending the weekend at Woodful Towers when wealthy old Sir Joshua Woodful was horribly murdered in the library. You probably read about it—it was in all the papers at the time. Immediately I called in my old friend Atlas Pierrot who was passing through the nearby village at the time on a walking holiday.

So it was that we were all gathered in the library with Inspector Hoey on a stormy night. The wind howled as we sat gazing at the large stain on the carpet. Atlas spoke, "I have made a complete list of the suspects". He placed a piece of paper on the table, on which was written, 'The Squire, The Vicar, The Cook, The Butler, The Maid, The Gardener'.

"Now for the alibis", said the inspector. "The vicar was taking evensong at the time of the murder."

Atlas crossed the name off his list.

"The cook was at the Women's Institute."

Atlas crossed off the name.

"The squire and the gardener were playing darts at the Pig and Whistle."

Two more names were crossed off.

"The maid was at the Roxy cinema at Woodfulton", I added.

A smile flickered across Atlas's face as he crossed off another name.

"Well, inspector, when we have eliminated the impossible whatever remains, however improbable, must be the truth."

"You sound like Mr Holmes", remarked the inspector.

"So the Butler did it", I gasped in astonishment. Just then the lights went out.

Afterwards I talked to Atlas about this method of making a list of all the possibilities and working through them.

"There is the restriction on its use in that there must only be a finite number of possibilities", he explained. "However, many situations do meet this condition, in particular the mathematical model of the logic of propositions known as the propositional calculus."

"Does this model exactly mirror the real situation?" I asked.

"No. To do this requires a complex model and so we compromise and select the most realistic simple model.

"We first have simple propositions or sentences which are statements of fact and are either true or false; e.g.,

'snow is white' is true,

'a cube has 7 faces' is false.

Next we select the logical words we want to have in our model. In this model we have

not, and, or, if–then.

We use these to build up complex sentences; e.g.,

'if snow is white then a cube has 7 faces', 'not snow is white'."

"Just a minute, shouldn't that be 'snow is not white'?"

"Yes, but the model would be much more complex if it had to consider the correct grammatical position of the 'not' so we always put it at the front.

"Similarly we depart from the common usage by using brackets in the model, e.g.,

'if snow is white then (a cube has 7 faces or London is in England)'.

This is to avoid the ambiguity which would result if the brackets were dropped."

"Dash it all, I would never use a sentence like that."

"Correct, but it's the same old problem. If the model only included sentences we commonly use, then it would have to be very complex, so we simply allow all possible sentences.

"This then produces a new problem. How do we decide whether a sentence is true or false if nobody ever uses it?"

"Deuced difficult, what!"

"In fact the answer is forced on us by our insistence on a simple model. Let me show you. What have the following sentences got in common?

'Snow is white and a cube has 7 faces.'

'London is in England and a rabbit is not an animal.'

'The world is round and 5 is bigger than 12.'"

"They all have the form

'a true sentence' and 'a false sentence'."

"Right. So in order to keep the model simple we insist that all sentences of this form behave in the same way, that is either they are all true or all false. We put a similar condition on other forms of sentences."

"Hold on! You still have to decide which they are all going to be, true or false."

"We will do that now. There are 14 different forms to consider, so I will leave you to work some out for yourself.

"The first form is

not 'a true sentence'.

A typical example of this is

'not snow is white'."

"That is false and I would expect any sentence of this form to be false."

"So we put in our model that all sentences of this form are false. I will leave the sentences of the form

not 'a false sentence'

to you and also the 4 'and' sentences.

"What about sentences of the form

'a true sentence' or 'a true sentence'?"

"Well, when I ask Babs if she wants to go to the Savoy or the Ritz I jolly well don't mean both. So I think these sentences are false."

"But what about that notice which says that you can get into the Test Match for half price if you are a student or a pensioner. You would make old Edgar Witherspoon pay full price just because he is attending evening classes during his retirement.

"The trouble is that 'or' has two meanings and so to keep our model simple we select one and it happens to be the one which allows both parts of the sentence to be true. We select this meaning, as mathematicians use this one."

"So

'snow is white or a cube has 6 faces'

is true."

"That is correct. The other 3 'or' cases are quite straightforward and I will leave them to you. Finally we come to 'if–then'.

"Let's take an example, say,

'if snow is white then a cube has 7 faces'."

"Well I would never start a sentence 'if snow is white then . . .' since I know snow is white already."

"So you have to consider sentences where you don't know whether they are true or false. For example, do you know where your friend Algy is at the moment?"

"No."

"So what do you think of the sentence

'if Algy is in London, then Algy is in England'?"

"It is true."

"Good. Now consider the four cases:

'Algy is in London' is true and 'Algy is in England' is true,
'Algy is in London' is true and 'Algy is in England' is false,
'Algy is in London' is false and 'Algy is in England' is true,
'Algy is in London' is false and 'Algy is in England' is false.

Which are possible?"

"The 1st, 3rd and 4th are possible, the 2nd is not. In fact I think this is what I meant by saying the 'if–then' sentence was true."

"So you were assuming

if true then true,
if false then true,
if false then false

are all true and

if true then false

is false."

Then I went away and worked out the cases Atlas had left. When I returned I asked Atlas,

"How do we use these results?"

"Let us consider the case of bootlegging I am working on at the moment. The facts we have are these:

Al or Frank or Bugs or Johnny was involved,
if Frank was involved then (Bugs was and Al was not),
if Johnny was involved then Bugs was not,
if Frank was not involved then (Johnny and Al were),
if Al was not involved then (Johnny was and Bugs was not).

All we have to do is to write down the list of possibilities and check each one against the facts. If we are lucky we will rule out all but one possibility and that will be our solution. If more than one possibility is left then we need more facts to rule out all but one of these possibilities which remain.

"We appear to have 16 possibilities:

Al was involved.	Frank was involved.	Bugs was involved.	Johnny was involved.
true	true	true	true
true	true	true	false
true	true	false	true

etc.

Let us take a typical possibility, say,

true false true false.

"The first fact is a number of sentences joined by 'or'. You have worked out that when we have two sentences joined by 'or' the whole is true provided at least one of the two is true and false if they are both false. Well this extends to any number of sentences. So, as 'Al was involved' is true the first fact is true.

"The next fact begins 'if Frank was involved . . .' and 'Frank was involved' is false and a sentence beginning 'if false then . . .' is always true so the second fact is true. The same argument shows the third fact is true.

"However the fourth fact begins 'if Frank was not involved . . .' and 'not Frank was involved' is true so we have 'if true then . . .' which means we have to look at the second half of the sentence. Now 'Johnny and Al were' is only true if 'Johnny was involved' and 'Al was involved' are both true which is not the case. So we have 'if true then false' which is false so the fourth fact does not hold in this possibility so we can rule out this possibility. I will leave you to consider the other 15 possibilities and work out who was involved in the bootlegging.

"By the way, what was the film you took the maid to see?"

"It was a story of police detection in San Francisco. You know the kind.

"It is 1.00 a.m.

"We are in an office at police headquarters. The lieutenant looks out of the window at the lights of the city spread out below.

" 'Somewhere out there is a man who has killed and will kill again unless we get to him first. But how can we find him?'

" 'There is one method that might work,' replies the chief, 'get out the telephone directories and pencils—lots of pencils.' "

A little while after his triumph at Woodful Towers, Atlas remarked that he now thought me ready for a rather more formal introduction to the Propositional Calculus. He thereupon presented me with a manuscript which I reproduce in full.

Part 2

The propositional calculus involves the symbols & (and), $\vee$ (or), $\rightarrow$ (if... then), $\neg$ (not), and letters p, q, r, s,... which stand for sentences (such as 'snow is white', '5 is bigger than 12' etc.). The letters, in fact, perform the same function as the familiar x, y, z,... do in algebra. Since, then, their job is to indicate places where a sentence may be put, they are called *sentence variables* or *sentence place-holders.* Our model is concerned with those rows of these symbols which become sentences when the sentence place-holders are replaced by sentences. Such a row is called a *well-formed formula* or *wff.*

Examples. $(p \& q)$, $(\neg p \vee \neg\neg(p \rightarrow \neg q))$ are wff's.
$p\&,q\vee\rightarrow rq\neg$ are not wff's.

So far we have the part of the model concerned with structure. We now come to the part concerned with meaning. This involves two further symbols T (true) and F (false) called *truth values* and four evaluation lists called *truth tables.*

p	$\neg p$
T	F
F	T

p	q	$(p \& q)$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$(p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$(p \rightarrow q)$
T	T	T
T	F	F
F	T	T
F	F	T

We have already seen (in Part 1) how these are arrived at.

The parts of the model are now assembled and we are ready to perform the basic calculations. Suppose we are given a wff and to each sentence place-holder in the wff is assigned a truth value. Then we can use the above truth tables to produce a single truth value in the way illustrated by the following example.

wff: $((p \& q) \vee (\neg(p \& \neg\neg q) \rightarrow r))$.

Assignment of truth values: T to p, F to q, F to r.

Replace the place-holders by the truth values:

$$((\mathrm{T} \& \mathrm{F}) \vee (\neg(\mathrm{T} \& \neg\neg \mathrm{F}) \rightarrow \mathrm{F})).$$

Look for any expressions which can be evaluated by the truth tables, $(\mathrm{T} \& \mathrm{F}) = \mathrm{F}$, $\neg \mathrm{F} = \mathrm{T}$. Replace each of these expressions by the single truth value, so simplifying the whole expression:

$$(\mathrm{F} \vee (\neg(\mathrm{T} \& \neg \mathrm{T}) \rightarrow \mathrm{F})).$$

Repeat the procedure. Since $\neg \mathrm{T} = \mathrm{F}$, we obtain

$$(\mathrm{F} \vee (\neg(\mathrm{T} \,\&\, \mathrm{F}) \rightarrow \mathrm{F}));$$

and further

$$(\mathrm{F} \vee (\neg \mathrm{F} \rightarrow \mathrm{F})),$$

$$(\mathrm{F} \vee (\mathrm{T} \rightarrow \mathrm{F})),$$

$$(\mathrm{F} \vee \mathrm{F}),$$

$$\mathrm{F}.$$

Now we can think of a wff as a machine or function with inputs, in this case three, p, q, r. We input the assigned truth values and get out a truth value, in this case F. Because there is only a finite number of possible inputs we can list them and against each input give the resulting output. Thus we obtain a complete description of the wff working as a truth machine. We extend the use of the term 'truth table' to include such a list.

Example. The truth table for the wff $((p \,\&\, q) \vee (\neg(p \,\&\, \neg\neg q) \rightarrow r))$.

Input			Output
p	q	r	$((p \,\&\, q) \vee (\neg(p \,\&\, \neg\neg q) \rightarrow r))$
T	T	T	
T	T	F	
T	F	T	
T	F	F	F
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The method used to obtain the given entry can be used to complete the table.

Now that we have the basic parts of the propositional calculus we can ask, how does it fit into the general field of mathematics and logic? The following points go some way to answering this question.

1. One can enlarge the model by adding parts to model the logic of the phrases 'for all...such that' and 'there exists...such that'. The new model is called the *restricted predicate calculus*.

2. The basic parts of the propositional calculus described above and the corresponding parts of the restricted predicate calculus form the working logic which the mathematician picks up incidentally in his training and which he uses every day to construct his sequences of mathematical deductions.

3. A similar remark applies to the computer programmer, although his use is more explicit and more formal and precise than that of the mathematician.

4. A lot of work is done in simply studying the model. This proceeds in the same way as any branch of pure mathematics by asking questions which seem interesting and trying to answer them. One important example of this is the adding to the model of a structure to model the idea of deduction. For example,

from 'if $\triangle ABC$ is right-angled at A, then $AB^2+AC^2 = BC^2$'

and '$\triangle ABC$ is right-angled at A'

we deduce '$AB^2+AC^2 = BC^2$'.

This is added to the model as the rule

from $((p \rightarrow q)$ and $p)$ deduce q.

One can now take a selection of wff's and see what other wff's can be deduced from them. This is just the sort of thing that is done in axiomatic mathematics (e.g., group theory) where one begins with a set of axioms and aims to deduce interesting consequences.

To demonstrate the scope of the propositional calculus we discuss one of its important theorems and a rather surprising deduction from it.

Suppose we are trying to find a positive integer x to satisfy all of the following sequence of conditions,

$$x \neq 1, \quad x \neq 2, \quad x \neq 3, \quad x \neq 4, \ldots$$

If we take any finite set of the conditions, e.g.,

$$x \neq 2, \quad x \neq 5, \quad x \neq 13,$$

then we can find an x, say 1, to satisfy all the conditions in the set. However, when we put all the infinite number of conditions together we find there is no x which satisfies them all. Thus we have a situation where every finite part is satisfiable but the infinite whole is not.

Can we have a similar situation with the propositional calculus? Suppose we have an infinite sequence of wff's, e.g.,

$$(p_1 \& p_3), \quad \neg p_2, \quad (p_4 \rightarrow \neg p_1), \quad (p_1 \& p_1) \vee p_7, \quad \ldots$$

which involves the sentence place-holders $p_1, p_2, p_3, \ldots$ (of which there may be an infinite number). We have to try to assign truth values, T or F, to the p_i's so as to give every wff in the sequence the truth value T. The *compactness theorem of the propositional calculus* says that if, for every finite set of wff's from the sequence we can find suitable truth values, then we can find suitable truth values for the whole infinite sequence.

The proof of this theorem goes roughly as follows. If we do an exhaustive search for possible suitable truth values

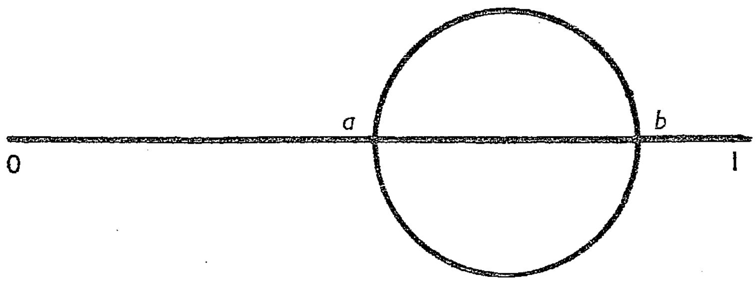
for p_1, for p_1 and p_2, for p_1, p_2 and p_3, ...

in turn, either we will succeed in finding suitable truth values for the whole infinite sequence or we will be stopped at some point in our search. This point will provide us with a finite set of wff's which cannot all be given the truth value T simultaneously. This situation contrasts with the above situation involving integers. The reason for the difference is that for the x there was an infinite number of possible choices of value, viz. 1, 2, 3, 4,... whereas for each p_i there is only a finite number, viz. T, F.

The compactness theorem may be used to obtain a result about sets (which is usually called the Heine–Borel theorem). This may be stated in the following form. A line, one metre long, is covered by an infinite number of coins. (These coins may be of varying size and some of them may be very small.) It is then possible to remove all but a finite number of the coins and still have the line covered. By the way, when we talk of a point of the line being covered, we mean that the point lies underneath the interior of the coin, not just its rim.

To apply the compactness theorem we list the conditions which a point, say x, of the line would have to satisfy if it was *not* to be covered by the original coins. To do so we express x as an infinite decimal $x_0 \cdot x_1 x_2 x_3 \ldots$; each of $x_1, x_2, x_3, \ldots$ takes one of the values $0, 1, \ldots, 9$; x_0 is 0 or 1 and if $x_0 = 1$, then $x_1, x_2, x_3, \ldots$ are all 0.

Suppose a coin is placed so



where a, b are the decimals $0 \cdot a_1 a_2 a_3 \ldots$, $0 \cdot b_1 b_2 b_3 \ldots$ respectively. If x is not covered by this coin, then we have the conditions

$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{x_1 \geqslant b_1\},$$
$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{(x_1 = b_1) \to (x_2 \geqslant b_2)\},$$
$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{[(x_1 = b_1) \,\&\, (x_2 = b_2)] \to (x_3 \geqslant b_3)\},$$
$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{[(x_1 = b_1) \,\&\, (x_2 = b_2) \,\&\, (x_3 = b_3)] \to (x_4 \geqslant b_4)\},$$
$$\vdots$$
$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{x_1 \geqslant b_1\},$$
$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{(x_1 = b_1) \to (x_2 \geqslant b_2)\},$$
$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{[(x_1 = b_1) \,\&\, (x_2 = b_2)] \to (x_3 \geqslant b_3)\},$$

and so on.

Repeat this for each coin and put together all the conditions on x we so obtain. (The conditions are simpler if $a < 0$ or $b > 1$.) Next replace each inequality as in the following example:

Replace $(x_4 \geqslant 6)$ by $[(x_4 = 6) \vee (x_4 = 7) \vee (x_4 = 8) \vee (x_4 = 9)]$.

Then add to the conditions the following which ensure that each x_i is given exactly one value and that the special case of $x_0 = 1$ is taken care of:

$$(x_0 = 0) \vee (x_0 = 1),$$
$$(x_0 = 0) \rightarrow \neg (x_0 = 1),$$
$$(x_0 = 1) \rightarrow \neg (x_0 = 0),$$
$$(x_0 = 1) \rightarrow (x_1 = 0),$$
$$(x_0 = 1) \rightarrow (x_2 = 0),$$
$$\vdots$$
$$(x_1 = 0) \vee (x_1 = 1) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9),$$
$$(x_1 = 0) \rightarrow \neg [(x_1 = 1) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9)],$$
$$(x_1 = 1) \rightarrow \neg [(x_1 = 0) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9)],$$
$$\vdots$$
$$(x_1 = 9) \rightarrow \neg [(x_1 = 0) \vee (x_1 = 1) \vee \ldots \vee (x_1 = 8)],$$
$$(x_2 = 0) \vee (x_2 = 1) \vee (x_2 = 2) \vee \ldots \vee (x_2 = 9),$$
$$(x_2 = 0) \rightarrow \neg [(x_2 = 1) \vee (x_2 = 2) \vee \ldots \vee (x_2 = 9)],$$

and so on.

We now have a complete set of conditions on x and they are made up of &, $\vee$, $\rightarrow$, $\neg$ and the equations

$$(x_0 = 0),\ (x_0 = 1),\ (x_1 = 0),\ (x_1 = 1),\ (x_1 = 2), \ldots,\ (x_1 = 9),\ (x_2 = 0),\ (x_2 = 1), \ldots$$

Replace these equations by the sentence place-holders

$$p_1, p_2, p_3, p_4, p_5, \ldots, p_{12}, p_{13}, p_{14}, \ldots$$

respectively. The set of conditions becomes a set of wff's. Then finding an x to satisfy the conditions is equivalent to assigning truth values, T or F, to the p_i so as to give all the wff's the truth value T.

It follows from the compactness theorem that if we cannot find an uncovered point x then there is a finite set of the conditions which cannot be satisfied. This finite set must have come from some finite set of coins and these coins must cover the line.

Further reading

A. Basson and D. O'Connor, *Introduction to Symbolic Logic* (University Tutorial Press, 1959).

Certain Graphs Arising in Genetics

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1. Introduction

Genetics, the study of the transmission of the characteristics of plants, animals and men from one generation to the next, gives rise to many interesting mathematical problems. Among these are new inequalities, new algebras and new patterns of chance events. In this article we shall examine only some problems of graph theory, an exciting modern branch of mathematics.

Three different problems will be discussed: the first two arise in a genetical context, while the final one, which is of relevance to genetics, arises in the study of marriage rules for primitive tribes. These examples have been chosen to illustrate various types of graph.

2. What is a graph?

The term 'graph' is used here in a sense somewhat different from that of a curve diagram with which the reader is already familiar. A graph in our sense resembles an electrical wiring diagram, or a street map. There is a set N of nodes (points or vertices), and a set of arcs (lines or edges) A joining together certain pairs of nodes. For the street map, each street corresponds to an arc, each street corner to a node. We may write down the details of a particular graph as $G = (\{1,2,3\}, (1,2), (1,3))$; this describes a graph with nodes labelled 1, 2 and 3 and arcs joining 1 to 2 and 1 to 3, as in Figure 1.

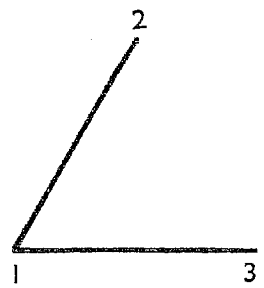


Figure 1

We shall consider two types of graph; an undirected graph as described in Figure 1, and a directed graph (or digraph) which we obtain if we treat the arc $(1,2)$ as different from the arc $(2,1)$. This we should need to do in a street map if all streets were one way streets. Other examples of undirected graphs are shown in Figures 2 and 3, and of digraphs in Figures 5 and 6.

It should be emphasised that, unlike in a street map, the lengths of the arcs are of no importance.

3. Phenograms

For some simple characteristics in plants and animals, such as shoot colour in garden peas, normal or dumpy wing in fruit flies, an individual's type is determined by only two *genes*. For our purposes, genes can be thought of as particles, whose exact nature is not relevant. These genes may be of n different types $A_1, A_2, \ldots, A_n$, say, so that each individual will have one of the

$$\binom{n+1}{2} = n(n+1)/2$$

possible pairs. This pair constitutes the *genotype* of the individual. However, two individuals with different genotypes may have identical appearances; we then say that their physical type, or *phenotype*, is the same. Thus if there are two types of genes A_1 and A_2, there will be three distinct genotypes A_1A_1, A_1A_2 and A_2A_2, but there may be one, two or three phenotypes. The correspondence between the set of genotypes and the set of phenotypes can be represented by a graph, in which each node represents a genotype, and two genotypes which produce the same phenotype have their nodes joined by an arc. Thus for the case $n = 2$, with genes of type A_1 and A_2 only, Figure 2 gives the complete set of five possibilities. The genotypes for which both genes are of the same type, such as A_1A_1 and A_2A_2, will have black discs at their nodes, while the others, such as A_1A_2, will have open discs.

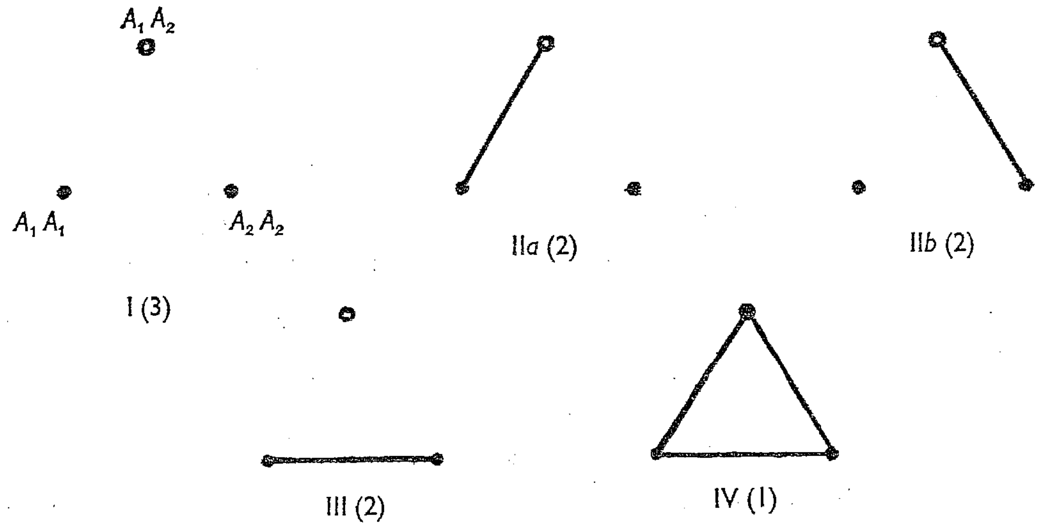


Figure 2

In Figure 2, the number in brackets below each graph is the number of distinct phenotypes. We see that in graph IV, no distinct phenotypes could be detected; I has three distinct types; II*a* and II*b* are essentially the same, since interchanging A_1 and A_2 in II*a* gives II*b*, and vice-versa. Since A_1A_1 and A_1A_2 are indistinguishable in II we say A_1 is dominant to A_2, the presence of A_1 masking the presence of A_2. There are thus only four distinct possibilities, or *phenograms*, for $n = 2$.

How many phenograms will there be for $n = 3, 4, 5\ldots$? The answer to this can be obtained using certain theorems of Pólya; for details, the reader is referred to

Cotterman (1953), and Hartl and Maruyama (1968) who also list all phenograms for $n = 3$. Here we discuss some simpler problems. The reader may be interested to know that the number of phenograms for $n = 3$ and $n = 4$ are 52 and 5525 respectively.

Let us denote by $f(i, m)$ the number of phenograms with i phenotypes and $m = \binom{n+1}{2}$ genotypes. It is clear that $f(m, m) = 1$ and $f(1, m) = 1$ for any m, but other values of $f(i, m)$ are more difficult to evaluate.

Problem 1

What values does $f(m-1, m)$ take? Figure 3 illustrates the set of phenograms for $m = 6$, $i = 5$, $f(5, 6) = 4$.

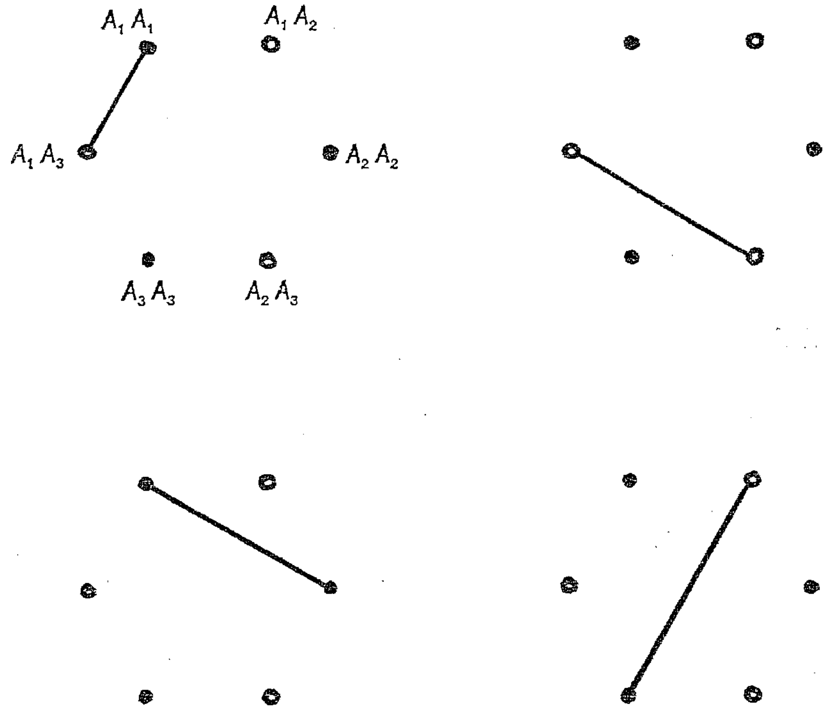


Figure 3

It is also instructive to study $f(2, m)$, though this is more difficult than the above problem.

Problem 2

Find $f(2, m)$ for $m = 6$ ($n = 3$), and $m = 10$ ($n = 4$).

4. Hierarchies

Having examined the idea of a phenogram, one is naturally led to inquire whether all phenograms are biologically reasonable. This question is obviously beyond the scope of the present article, but suppose we decide we do not wish to permit cases where A_1 is dominant to A_2 ($A_1A_1 \equiv A_1A_2$), A_2 is dominant to A_3

$(A_2A_2 \equiv A_2A_3)$, and A_3 is dominant to A_1, or any similar apparently self-contradictory set. We insist instead that if A_1 is dominant to A_2, and A_2 is dominant to A_3, then this implies that A_1 is dominant to A_3.

We might simply decide to divide the set of available types $A_1, A_2, \ldots, A_n$ into levels, such that each A_i was at one, and only one, level, and was dominant to all the A_j on lower levels. Our system would then be defined if we stated how many A_i's there were in each level. Thus with $n = 1$ there is only one system. For $n = 2$ we could have both genes in the same level, or each in a different level, and we could represent these two possibilities as (2) and (1, 1) respectively. For $n = 3$ we could have all three genes in the same level, written as (3), two in a higher level and one in a lower (2, 1), one in a higher and two in a lower level, (1, 2), or each in a separate level, (1, 1, 1). In general there would be 2^{n-1} different systems for n genes. The contents of our brackets define the partitions of n into positive integers and will occur again below. However, a cursory glance at a genetics textbook will show that there are many systems which cannot be fitted into the above scheme. One such is the system with $n = 4$ for which $A_1A_1 \equiv A_1A_4 \equiv A_1A_2$, $A_2A_2 \equiv A_2A_4$, $A_3A_3 \equiv A_3A_4$, A_1A_3, A_2A_3 and A_4A_4 are the six distinct phenotypes; this system occurs in humans.

Our next attempt, which I believe encompasses all known systems, permits each A_i to be in a sequence of levels, and to be dominant to any A_j which occurs *only* at lower levels. Thus the above system will be written as $\begin{array}{c} A_1A_3 \\ \hline A_2A_3 \\ \hline A_4 \end{array}$ indicating that A_1 is dominant to A_2 and A_4, A_3 is dominant to A_4 and A_2 is dominant to A_4.

There are only two distinct systems with $n = 2$, and only five with $n = 3$ (compared with 4 and 52 for phenograms); these are shown in Figure 4, with the number of phenotypes in brackets.

$$A_1\,A_2 \qquad \begin{array}{c} A_1 \\ \hline A_2 \end{array}$$

I (3) II (2)

$$A_1\,A_2\,A_3 \qquad \begin{array}{c} A_1\,A_2 \\ \hline A_2\,A_3 \end{array} \qquad \begin{array}{c} A_1\,A_2 \\ \hline A_3 \end{array} \qquad \begin{array}{c} A_1 \\ \hline A_2\,A_3 \end{array} \qquad \begin{array}{c} A_1 \\ \hline A_2 \\ \hline A_3 \end{array}$$

I (6) II (5) III (4) IV (4) V (3)

Figure 4

We might also represent these hierarchies by digraphs, with a node for each A_i, and an arrow from A_i to A_j if A_i is dominant to A_j. Figure 5 shows the digraphs for $n = 3$, numbered as for the corresponding cases in Figure 4.

We see that if P is the number of phenotypes, G the number of genotypes and A the number of arcs in digraph then $P = G - A$.

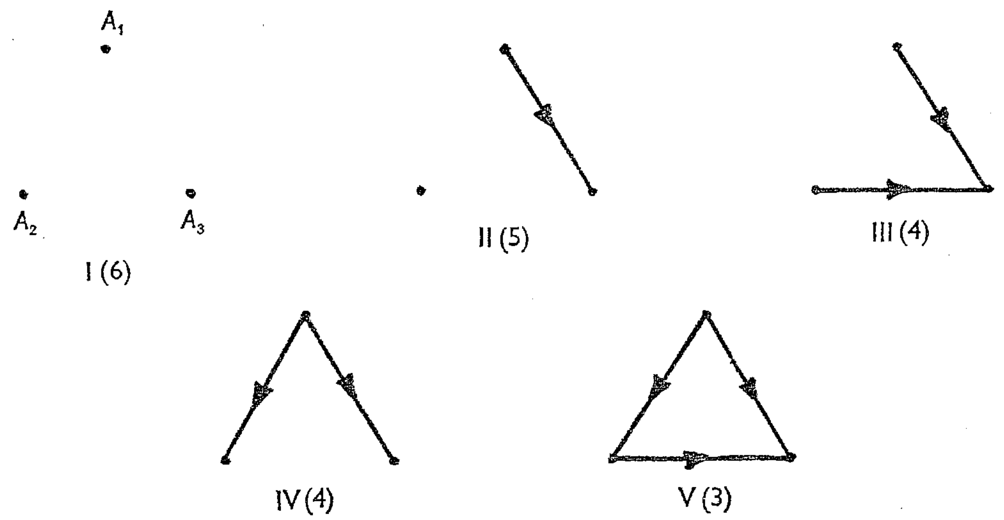


Figure 5

We can write our hierarchies as

$$\begin{array}{llll} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{array}$$

where a_{jk} is the number of A_i's in level j which are first represented in level k. This representation simplifies the construction of complete sets of hierarchies for any n. We need to derive one or two simple rules concerning the a_{jk}'s. Clearly we do not want the A_i's of any level to contain all those of any other level, since this would simply add an extra level to the system without adding any information. This can be accomplished by making sure that for every j the jth level has at least one A_i not in the level above, i.e., by making a_{jj} positive, and at least one A_i not in the level below, i.e., by making $a_{j+1,k} < a_{j,k}$ for at least one k. We also note that $(a_{11}, a_{22}, \ldots, a_{pp})$ is a partition of n.

As an example consider $n = 4$. We first write down the $(a_{11}, a_{22}, \ldots, a_{pp})$ terms, namely 4; 3, 1; 2, 2; 2, 1, 1; 1, 3; 1, 2, 1; 1, 1, 2; and 1, 1, 1, 1. Now we look at a particular one, 2, 1, 1 say. We first note that a_{21} is 0 or 1 (leaving out an allele from level one), then we go on to get a_{31} and a_{32}. If $a_{21} = 0$ we will have to put $a_{31} = 0$ and $a_{32} = 1$ since $a_{22} = 1$; if $a_{21} = 1$ we may have $a_{31} = 0$, $a_{32} = 0$; $a_{31} = 0$, $a_{32} = 1$ or $a_{31} = 1$, $a_{32} = 0$. There are thus four possibilities

$$\begin{array}{l} 2 \\ \hline 0\ 1 \\ \hline 0\ 0\ 1 \end{array} \quad \begin{array}{l} 2 \\ \hline 1\ 1 \\ \hline 1\ 0\ 1 \end{array} \quad \begin{array}{l} 2 \\ \hline 1\ 1 \\ \hline 0\ 1\ 1 \end{array} \quad \begin{array}{l} 2 \\ \hline 1\ 1 \\ \hline 0\ 0\ 1 \end{array}$$

The reader is left to supply the corresponding hierarchies in terms of the A_i's, the phenograms and the digraphs.

We should also like to count the number of possible hierarchies. If we are told that there are 2 levels, and i_{11} is given, then there are i_{11} possibilities.

Problem 3

How many possibilities are there if we have 3 levels with i_{11} and i_{22} given? How many if we have 4 levels with i_{11}, i_{22} and i_{33} given?

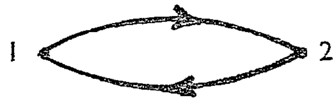
Problem 4

How many phenotypes are there for a hierarchy with p levels all i_{jk} given? The answer can be expressed in terms of n, the i_{jj}'s and the $r_j = \sum_{k=1}^{j} i_{jk}$ the number of all alleles in level j, for $j = 1, 2, \ldots, p$.

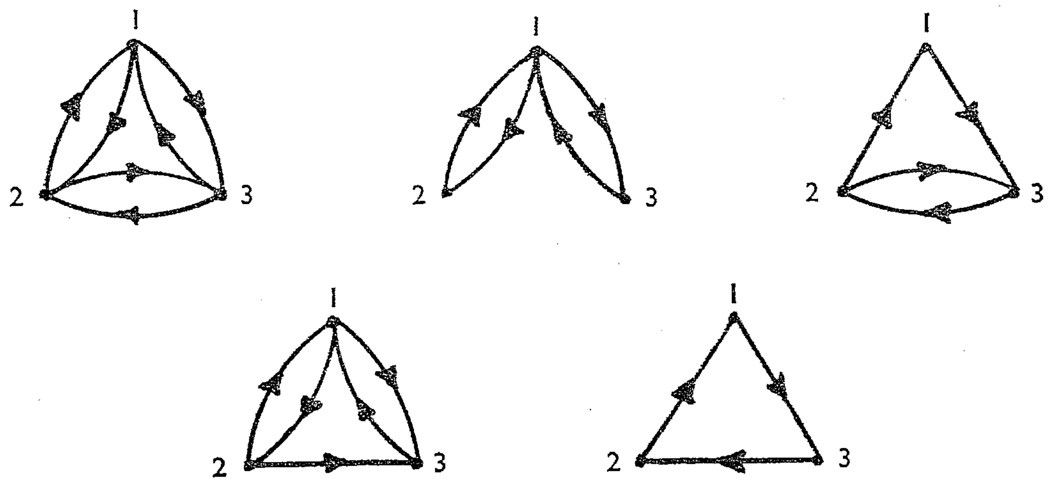
5. Marriage systems

We now consider a further problem on which graphs may throw some light. Most societies have a variety of rules specifying whom one may, or may not, marry. Those in advanced societies are usually based on biological relatedness; one may not marry one's sibling or parent. Primitive societies, on the other hand, tend to have rules based on classificatory relatedness. The population is divided into groups of people or clans, and the rules state whom the members of one clan may marry in terms of the other clans (see, for example, Fox (1967)).

As an example suppose there are four clans labelled 1, 2, 3 and 4. The rules might state that the following marriages are possible (1×2), (2×1), (3×4) and (4×3), where in each case $(i\times j)$ indicates that a male of clan i marries a female of clan j.



Marriage rules for 2 clans



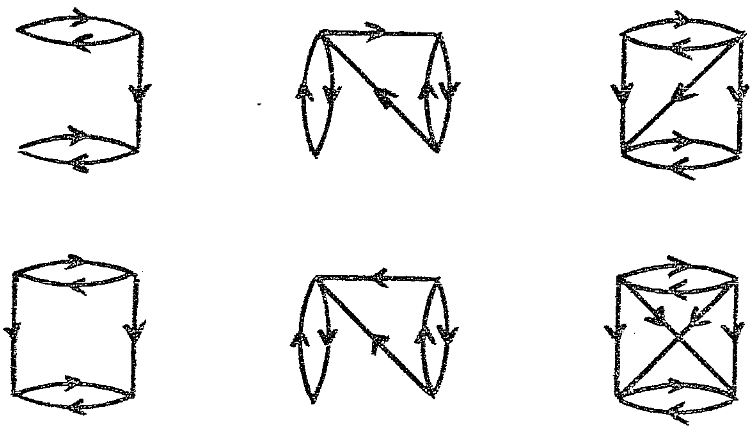
Marriage rules for 3 clans

Figure 6

Systems of this kind can be represented by digraphs, with an arrow from node i to node j if $(i \times j)$ is permitted. Figure 6 shows all the possible 'reasonable' systems with 2 and 3 clans.

There is only one reasonable system for 2 clans, and 5 for 3 clans, out of 3 and 16 possible digraphs with those number of nodes. What is the special feature of the digraphs here? What has been done for reasons of balance is to make sure that every clan has a supply of marriage partners for both its men and its women, i.e., that every node has at least one input and at least one output.

For 4 clans we find that out of 218 possible digraphs 80 have this property. However, 6 of these have another undesirable feature; these are illustrated in Figure 7.



Undesirable marriage rules for 4 clans.

Figure 7

In each case illustrated in Figure 7 the system would rapidly break down because there would be a net flow of females into one section of the graph. Any such group would be heading for social friction and a rapid reappraisal of their rules.

It is for this reason that a more restrictive criterion of an acceptable system must be formulated: it is that any subset of nodes must have at least one input and one output. If this is done, we achieve a 'reasonable' marriage rule.

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Some Remarks on Blow's Game

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1. Introduction

The article by David Blow in a recent issue of *Mathematical Spectrum* (Vol. 6, No. 1, pp. 2–7) raises many interesting issues. A game involving choice of strategies and chance outcomes is given an enterprising analysis, and the best strategies for the two players are decided. In Blow's article one part of the calculation is done by matrix iteration on a computer, and it is extremely instructive to do it this way. The method described also has the advantage of being a general one which applies to a wide range of problems, but for the problem in question a simpler alternative is available, and we describe below how this type of calculation can be done by pencil and paper. The analysis in Section 5 below should be read referring to Sections 2 and 3 of Blow's article.

2. A simple example

To introduce the technique we will start by considering a simple example. Suppose that two players toss a die. If you throw a 6 then you win, otherwise you pass the die to the other player and he has a turn. Calculate the two players' respective chances of winning.

The game may be described by means of a flow diagram. There are four states, 1: A to go, 2: B to go, 3: A wins, 4: B wins, and these are represented by the nodes of Figure 1. At each throw in the game there are various probabilities of making a

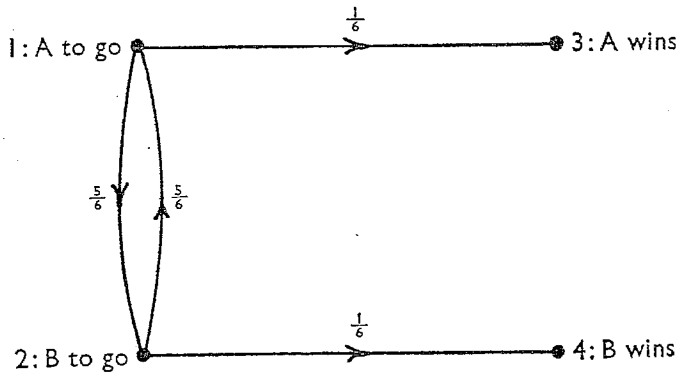

Figure 1. Flow diagram for a simple die game.

transition from one state to another. If A is to throw, there is a probability $\frac{1}{6}$ that he wins with the throw, and a probability of $\frac{5}{6}$ that the next throw is B's. Similar probabilities apply to B.

The diagram can be described by a transition matrix M of probabilities indicating the passage from one of the states, 1, 2, 3, 4 at one throw to another of these, as the arrow suggests. The probabilities 1 indicate that once you have got into either of states 3 or 4, you will stay in them.

$$M = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ \frac{5}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

The matrix M is constructed according to exactly the same conventions as in Blow's article. If A starts, this corresponds to adopting an input row vector $(1,0,0,0)$, and the probabilities of various outcomes after 1, 2, 3,... moves are obtained by post-multiplying this vector by successive powers of M. The respective chances of A and B ultimately winning are given by the third and fourth positions of the vector $(1,0,0,0)\,M^n$, where we take the limit as n tends to infinity.

There is however a more convenient method, in which we draw up a smaller matrix, omitting the rows and columns corresponding to the two states 'A wins' and 'B wins'. These states at which the game stops can be called *absorbing states*; but we will come increasingly to think of our diagrams as representing water flowing through a network of pipes and call them *sinks*. This leaves us with the *reduced matrix* Q consisting of the first 2 rows and columns of M:

$$Q = \begin{bmatrix} 0 & \frac{5}{6} \\ \frac{5}{6} & 0 \end{bmatrix}.$$

Let us refer once more to Figure 1. We have been interpreting the numbers on it as probabilities—the probability of a transition from one state to another as a result of a single throw in the game. What we want to know is, if a large number of games start in the state 'A to go' what fractions end up in states 'A wins' and 'B wins' respectively? Putting the question another way, if we pump water steadily into the top left-hand node, what amounts flow out at the two right-hand nodes? It is convenient to redraw the diagram as in Figure 2. We are now indicating a unit flow into the input node, and we are calling the total outflow from the first node x, and the total outflow from the second y. We must introduce algebraic unknowns at some stage, and this is a convenient way of doing it. In particular examples, you may later be able to work out short cuts for yourself, but for the moment we will write down what is a standard method.

To study these systems we need to apply two basic laws:

(i) at each node the total outflow is equal to the total inflow;

(ii) at each node the total outflow is partitioned between the various pipes in the proportions stated in the data of the question.

Rule (ii) has been used in constructing Figure 2. We now use Rule (i). Equating outflow to inflow for the first node,

$$x = \tfrac{5}{6}y + 1,$$

and for the second node

$$y = \tfrac{5}{6}x.$$

From this we easily obtain

$$x = \tfrac{25}{36}x + 1,$$

so that

$$x = \tfrac{36}{11} \quad \text{and} \quad y = \tfrac{30}{11}.$$

The outflow to the first sink is $\frac{1}{6}x = \frac{6}{11}$, and to the second sink $\frac{1}{6}y = \frac{5}{11}$. So A stands to win $\frac{6}{11}$ of the games, and B stands to win $\frac{5}{11}$.

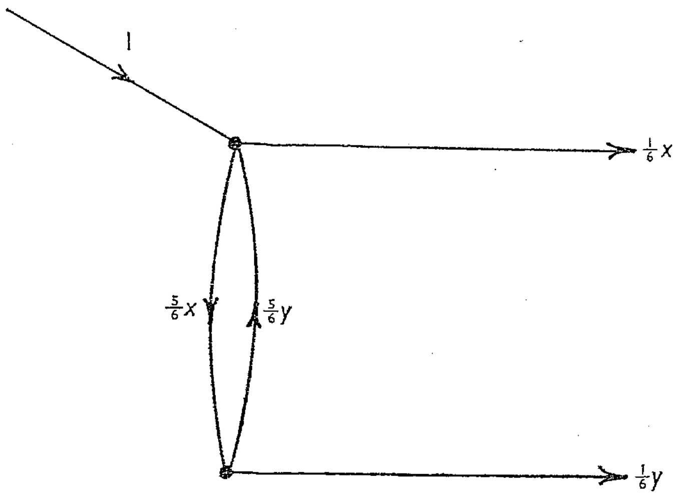

Figure 2. Input–output diagram.

3. An interesting modified game

Suppose we now change the rules: A is to win on the first throw if he throws a 6 (only), thereafter either player wins if he throws a 5 or a 6, otherwise he passes the die to his opponent. The appropriate diagram is now Figure 3. The total outflow from each of the 3 nodes is x, y, z respectively. The related reduced matrix is

$$Q = \begin{bmatrix} 0 & \frac{5}{6} & 0 \\ 0 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \end{bmatrix}.$$

This time, equating the outflows and inflows at the nodes we get

$$x = 1, \quad y = \tfrac{5}{6}x + \tfrac{2}{3}z, \quad z = \tfrac{2}{3}y.$$

These equations have solutions

$$x = 1, \quad y = \tfrac{3}{2}, \quad z = 1.$$

Now the outflow to the 'A wins' sink is $\frac{1}{6}x + \frac{1}{3}z = \frac{1}{2}$, and the outflow to the 'B wins' sink is $\frac{1}{3}y = \frac{1}{2}$.

Note that the game is fair: the two players stand an equal chance of winning.

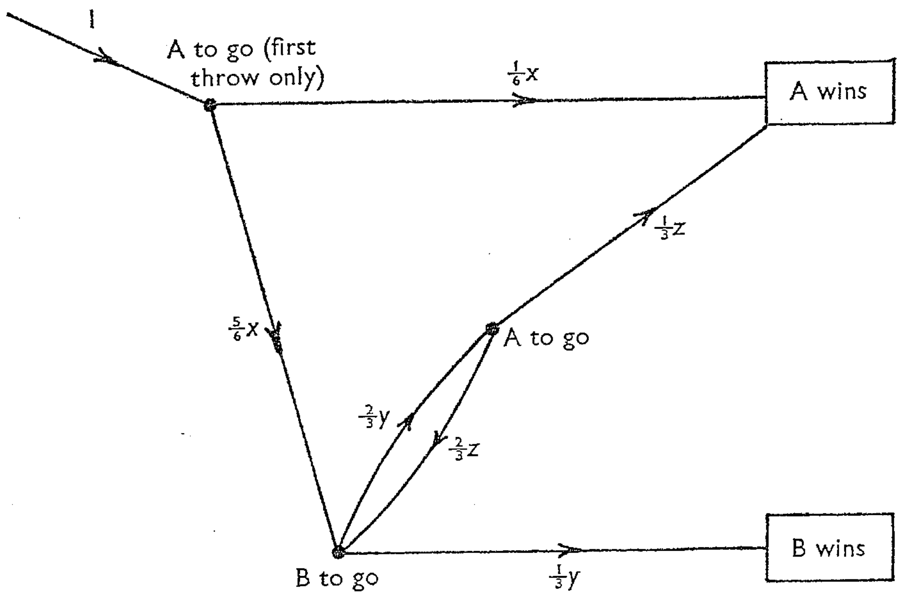

Figure 3. Flow diagram for modified die game.

4. Some theory

We have drawn up the matrices Q, but we have not explained how they help solve our problems. Looking back at the simultaneous equations which had to be solved in each problem we see that they can be put into a very convenient form using matrix notation.

For the second example, the equations are $\boldsymbol{x} = \boldsymbol{x}Q+\boldsymbol{u}$, where $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{u} = (1, 0, 0)$. In the first example, the equations are of the same form and in every example which follows we will be able to reduce the equations to this form. The equation is easy to solve in general; since

$$\boldsymbol{x}(I-Q) = \boldsymbol{u},$$

then provided $I-Q$ is not singular,

$$\boldsymbol{x} = \boldsymbol{u}(I-Q)^{-1}.$$

From the components of $\boldsymbol{x}$ the required information is easily calculated. It is possible to get a general formula for this, but we do not need it.

The matrix $(I-Q)^{-1}$ is called the *Leontief inverse* of Q. Leontief is an American who recently won a Nobel prize for his contributions to economic theory. This matrix plays a key role in some of his economic work. It is clear that we might argue that since

$$\boldsymbol{x} = \boldsymbol{x}Q+\boldsymbol{u},$$

then

$$\boldsymbol{x} = (\boldsymbol{x}Q+\boldsymbol{u})\,Q+\boldsymbol{u} = \boldsymbol{u}(I+Q)+\boldsymbol{x}Q^2,$$

$$\boldsymbol{x} = \boldsymbol{u}(I+Q+Q^2)+\boldsymbol{x}Q^3.$$

We may continue in this way, and if $xQ^n \to 0$ as $n \to \infty$, which is certainly the case in the earlier examples, and if certain formulae of ordinary algebra apply to matrices, as we may show they do, then

$$x = u(I+Q+Q^2+\ldots) = u(I-Q)^{-1},$$

which is the same result as before. We do not need to argue this way, but the reader may wish to consider this argument as a problem, as it shows a connection with the iteration used by Blow.

As a final piece of theory we might consider the total amount of water contained in the pipes of the network. In the first example this is $x+y = 6$, and in the second it is $x+y+z = 3\frac{1}{2}$.

What is the significance of these numbers? They give the average number of moves in a game! We will not prove this, but once more leave it as an investigation for the reader. Note that we do not include the one unit of water in the inflow pipe, but we do add the water in the pipes going out to the sinks.

5. Blow's example

The solution of Blow's example involves doing four calculations of this type on the matrices which he denotes by T_1T_3, T_1T_4, T_2T_3 and T_2T_4. His matrices are unreduced. We will not carry out all four calculations in detail, but we will first consider T_2T_4. Reducing the matrix given in Blow's article we obtain

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

The corresponding flow diagram is given in Figure 4, where the total outflow from the respective nodes is x, y, z, u, v. This time each node has a closed loop associated with it. The flow in and out of this loop occurs on both sides of the equations when Rule (i) is applied, and so the term can either be included on both sides or can be omitted from both. Omitting the loop terms, the equations from the 5 nodes are

$$\tfrac{1}{2}x = \tfrac{1}{4}y,$$

$$\tfrac{1}{2}y = \tfrac{1}{4}x+\tfrac{1}{4}z,$$

$$\tfrac{1}{2}z = \tfrac{1}{4}y+\tfrac{1}{4}u+1,$$

$$\tfrac{1}{2}u = \tfrac{1}{4}z+\tfrac{1}{4}v,$$

$$\tfrac{3}{4}v = \tfrac{1}{4}u.$$

It is relatively easy to evaluate x in terms of y and each in terms of z, and to do the same with v and u, and substitute these values in the middle equation. This leads eventually to the results

$$x = \tfrac{20}{11}, \quad y = \tfrac{40}{11}, \quad z = \tfrac{60}{11}, \quad u = \tfrac{36}{11}, \quad v = \tfrac{12}{11}.$$

The probability that A wins is $\frac{1}{2}v = \frac{6}{11}$ and that B wins is $\frac{1}{4}x = \frac{5}{11}$. Since $\frac{6}{11} = 0{\cdot}545$, these results check with Blow's calculation.

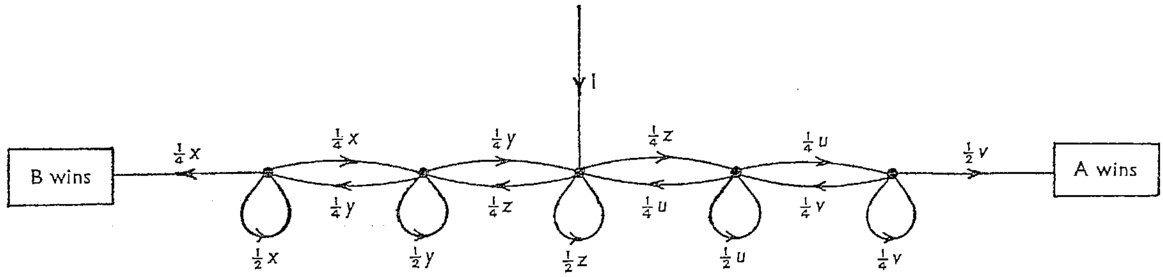

Figure 4. Flow diagram for Blow's example.

In this example it will be seen that the work can be shortened by expressing some of the unknowns in terms of the others on the way, calculating these on the diagram as you go. Anyone who is interested can find out the tricks by experimenting.

A similar calculation can be carried out in Blow's other three cases. With $T_1 T_4$ and $T_2 T_3$ the equations are a little more complicated, but they can be solved without undue trouble. The case of $T_1 T_3$ is very interesting, because if a diagram is drawn it will be found that the network splits up into two quite separate portions. No water flows through one portion at all, so that half of the numbers involved do not enter into the calculation. This shows the practical value of the diagrams very convincingly. The average length of game in the case calculated above is $x+y+z+u+v = \frac{168}{11}$ moves.

6. Further problems for solution

Other problems can be solved using these techniques. The first gives a warning against gambling on equal terms with people who are richer than you are.

Problem 1. Two men have capital p and q units respectively, and they gamble by tossing a coin for one unit at a time. The game goes on until one or the other is bankrupt. Show that their chances of winning are in the ratio $p : q$. (Experiment with particular numbers rather than worrying about the general case.)

Problem 2. To promote sales a firm gives away a souvenir with each purchase. There are 10 in a set. Show that, with the usual assumptions about each souvenir being equally likely, the average number of purchases needed in order to acquire a set is

$$10(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{10}).$$

(This is a matter of drawing the appropriate network and calculating its total capacity.)

Problem 3. Here is another game to analyse by Blow's methods. Two players agree to play a game in which a coin is tossed repeatedly. The first player chooses a sequence of three results, for example he could choose Head–Tail–Head. The second player also chooses a sequence, for example he could choose Head–Tail–Tail.

The coin is then tossed until one of the players' sequences is produced. Thus with the examples given, the sequence TTHHTH results in a win for the first player. If the other player had said TTH, THH or HHT then on this occasion he would have been the winner.

The problem is to decide on the best strategy. The interesting thing is that if the second player is informed of the first player's bet *he can always place a bet which stands a better chance of winning*. At first sight this may appear contrary to common sense; you are invited to show by analysis that this is indeed the case. If both players choose in ignorance of the other's bet, then it is a different problem —and this you may investigate for yourself.

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English Church Bell Ringing

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1. Introduction

English bell-ringers believe that the best sound a bell can make is when the bell and its clapper are both swinging freely. In a campanile, on the other hand, the bells are fixed and sounded by hammers.

But a swinging bell has a certain minimum period, so it is impossible to ring tunes. English ringers are 'change-ringers'.

This article will describe (in an elementary way) the mechanics and the permutations of English bell-ringing.

2. Mechanics of a swinging bell

A bell in an English tower swings freely; the rigid clapper inside the bell swings freely too and hits the bell at the top of its swing. The bell is swung by pulling a rope attached to a wheel fixed to the bell. The mass of the clapper is much less than that of the bell and therefore the behaviour of the system is primarily determined by the simple swinging of the bell.

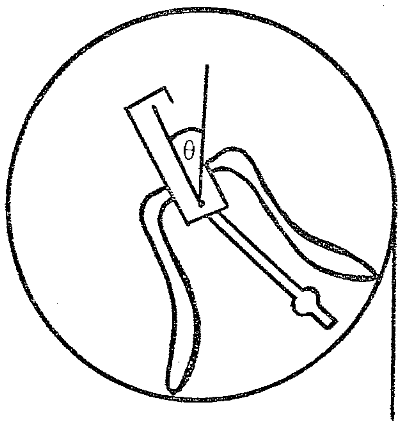

Diagram 1

The bell is hung on a horizontal axis, so that in stable equilibrium it hangs mouth downwards. When the bell swings at angular speed $d\theta/dt$ any particular part of it which is distance r from the axis has speed $r\,d\theta/dt$, so if its mass is δm, its kinetic energy is $\frac{1}{2}r^2(d\theta/dt)^2\delta m$. Summing these elements over the whole bell we have a total kinetic energy

$$T = \sum\tfrac{1}{2}r^2(d\theta/dt)^2\delta m = \tfrac{1}{2}(d\theta/dt)^2\sum r^2\delta m.$$

Denoting the 'moment of inertia' $\sum r^2\delta m$ by J we have $T = \frac{1}{2}J(d\theta/dt)^2$. If the centre of mass is a distance h from the axis and the bell has mass M and has swung through an angle θ from the position of equilibrium, then the potential energy is

$$V = Mgh(1-\cos\theta).$$

Thus the total energy is

$$E = T + V = \tfrac{1}{2}J(d\theta/dt)^2 + Mgh(1-\cos\theta). \tag{1}$$

Since the total energy is constant, i.e., $dE/dt = 0$,

$$J\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} + Mgh\sin\theta\frac{d\theta}{dt} = 0.$$

This gives the differential equation

$$J\frac{d^2\theta}{dt^2} + Mgh\sin\theta = 0. \tag{2}$$

Now, if θ is small, we can replace $\sin\theta$ by θ and (2) becomes

$$J\left(\frac{d^2\theta}{dt^2}\right) + Mgh\,\theta = 0. \tag{3}$$

We therefore have simple harmonic motion with solution

$$\theta = \theta_0\cos(\omega t + \alpha),$$

where $J\omega^2 = Mgh$ and θ_0, α are arbitrary constants. This gives the period for small oscillations of the bell from the stable equilibrium position (mouth downwards), namely,

$$(2\pi/\omega) = 2\pi(J/Mgh)^{\frac{1}{2}}.$$

But if θ is large we have the more complicated equation (2). To solve this we return to the energy equation (1)

$$\tfrac{1}{2}J(d\theta/dt)^2 + Mgh(1-\cos\theta) = \text{constant} = E_0, \quad \text{say},$$

so that,

$$\left(\frac{d\theta}{dt}\right) = \left[\frac{2(E_0 - Mgh + Mgh\cos\theta)}{J}\right]^{\frac{1}{2}}.$$

Thus

$$dt = \frac{J^{\frac{1}{2}}d\theta}{(2E_0 - 4Mgh\sin^2\frac{1}{2}\theta)^{\frac{1}{2}}}$$

and

$$t = \int_{u=0}^{t} du = \int_{\phi=\theta_0}^{\theta}\frac{J^{\frac{1}{2}}d\phi}{(2E_0 - 4Mgh\sin^2\frac{1}{2}\phi)^{\frac{1}{2}}}, \tag{4}$$

which gives an expression for t as an integral with upper limit θ. This can be tabulated numerically, but one particular case is worth noting.

If $E_0 = 2Mgh$, i.e., if the total energy is exactly enough to raise the bell to the unstable equilibrium position (mouth up, $\theta = \pi$), the time taken to reach this from the stable equilibrium position (mouth down, $\theta = 0$) is

$$t = \int_0^{\pi}\frac{J^{\frac{1}{2}}d\phi}{(2E_0)^{\frac{1}{2}}\cos\frac{1}{2}\phi} = \left(\frac{J}{2E_0}\right)^{\frac{1}{2}}\int_0^{\pi}\sec\tfrac{1}{2}\phi\,d\phi = \left[\left(\frac{2J}{E_0}\right)^{\frac{1}{2}}\log\left|\tan\left(\frac{\phi}{4}+\frac{\pi}{4}\right)\right|\right]_0^{\pi},$$

which is infinite. Therefore the bell would take an infinite time to reach the unstable equilibrium position.

On the other hand, if $E_0 = 2Mgh\sin^2\frac{1}{2}\phi_0$, so that the bell has sufficient energy to rise to $\theta = \phi_0$, then it takes a time

$$t = \int_0^{\phi_0}\left(\frac{J}{4Mgh}\right)^{\frac{1}{2}}\frac{d\phi}{(\sin^2\frac{1}{2}\phi_0-\sin^2\frac{1}{2}\phi)^{\frac{1}{2}}} = \left(\frac{J}{2Mgh}\right)^{\frac{1}{2}}\int_0^{\phi_0}\frac{d\phi}{(\cos\phi-\cos\phi_0)^{\frac{1}{2}}}$$

to do so. The integrand is finite except near $\phi = \phi_0$ where, writing $\phi = \phi_0-\psi$, we have the integrand

$$\left(\frac{J}{4Mgh}\right)^{\frac{1}{2}}[\sin\tfrac{1}{2}(\phi+\phi_0)\sin\tfrac{1}{2}(\phi_0-\phi)]^{-\frac{1}{2}} \simeq \left(\frac{J}{4Mgh}\right)^{\frac{1}{2}}\operatorname{cosec}^{\frac{1}{2}}\phi_0(\tfrac{1}{2}\psi)^{-\frac{1}{2}}.$$

Thus the integral behaves like $\int\psi^{-\frac{1}{2}}d\psi = [2\psi^{\frac{1}{2}}]$ which remains finite near $\psi = 0$. (The term $\operatorname{cosec}^{\frac{1}{2}}\phi_0$ will be finite for $\phi_0 < \pi$.) Hence if the bell has insufficient energy to rise to the unstable equilibrium position, it will fall back again in a finite time.

We therefore see that altering the energy of the bell near the value which would lift it to the unstable equilibrium position will change the period of its swing. But the period will not fall below $2\pi(J/Mgh)^{\frac{1}{2}}$ (the period for small oscillations).

3. Change-ringing: how to ring all possible orders

Now a bell swinging through large angles sounds twice in each complete cycle. Its energy can be changed by pulling on the rope, so the time of its next sound can be adjusted. In practice it is fairly easy to change the period of a one ton bell by about 20% (and a lighter bell by rather more) while it rings a comfortable 25 to 30 sounds per minute.

With, say, an octave of eight bells this means an interval between successive bells of about $\frac{1}{4}$ second. Now most tunes require a given bell to sound more often than once every two seconds and this is impossible for bells being swung. It is for this reason that campaniles are needed for tune-ringing.

English ringers cannot ring tunes but would like to change the order in which the bells are rung, as 'rounds' (the descending octave) are rather dull. Even 'call-changes' (an arbitrary order repeated indefinitely) are dull. So English ringers became 'change-ringers' some 300 years ago. Here the intention is that no order is repeated. For example, on eight bells, Diagram 2 (at the end of the article) gives sixteen different orders starting and ending with 'rounds'. The bell with the highest note is called the treble or 'one', the next 'two' and so on down to the lowest note 'tenor' or 'eight'. In fact there are clearly $8\times7\times6\times5\times4\times3\times2\times1 = 40{,}320$ different orders on eight bells (which would take about 20 hours to ring). The rules for determining which order or row comes when are called 'methods'. Diagram 2 is the method called 'original' or 'plain hunting'. Each bell follows the same pattern of two sounds or 'blows' at the front, one blow successively in each position until it gets to the eighth (last) place where it has two blows and then returns to the front, one place at a time.

Because of the difficulty of making large changes in the period of a bell's swing, we restrict changes to moving a bell one place forward, one place backwards or leaving the bell in the same place. We also try to avoid repetition (even of musically attractive orders like 1 3 5 7 2 4 6 8).

How can we obtain all the possible rows (the 'extent') on, say, eight bells? It is easiest to start with fewer bells. Producing the extent on two bells is easy— 1 2, 2 1, 1 2. On three bells, starting with 1 2 3, we must go on either to 1 3 2 or 2 1 3, as other orders would require bells to move more than one place. If we start 1 2 3, 2 1 3 we must then have either 1 2 3 (again!) or 2 3 1. We do not want the immediate repetition 1 2 3 so we have 1 2 3, 2 1 3, 2 3 1. We continue with 3 2 1 (the other possible order 2 1 3 would repeat) and thus we find the complete extent: 1 2 3, 2 1 3, 2 3 1, 3 2 1, 3 1 2, 1 3 2, 1 2 3. There are six different rows and six changes (two essentially different changes).

If we had started 1 2 3, 1 3 2 we would have completed the extent in the opposite order.

On four bells we can start in several ways, but if we try to move as many bells as possible each time we might change the outer pairs and the inner pair alternately thus: 1 2 3 4, 2 1 4 3, 2 4 1 3, 4 2 3 1, 4 3 2 1, 3 4 1 2, 3 1 4 2, 1 3 2 4. Now continuing we would return to 1 2 3 4 without completing the extent. To avoid this we could go to 1 3 4 2, but then continue changing outer and inner pairs 3 1 2 4, 3 2 1 4, 2 3 4 1, 2 4 3 1, 4 2 1 3, 4 1 2 3, 1 4 3 2, and then, to avoid repetition, 1 4 2 3, 4 1 3 2, 4 3 1 2, 3 4 2 1, 3 2 4 1, 2 3 1 4, 2 1 3 4, 1 2 4 3 and, finally, 1 2 3 4. All these twenty-four rows are different. We may notice that the rows with 1 in front are rung in the order 1 2 3 4, 1 3 2 4, 1 3 4 2, 1 4 3 2, 1 4 2 3, 1 2 4 3, 1 2 3 4. The bells 2, 3 and 4 are following the pattern that 1, 2 and 3 followed in the extent on three bells. Moreover it is easy to see that the row 1 *l m n* is either followed by the rows *l* 1 *n m*, *l n* 1 *m*, *n l m* 1 or preceded by these rows.

Therefore in this method, called 'Plain Bob Minimus', we can check that no row is rung twice by making sure that the rows with 1 in front are all different.

On five bells we can follow a similar technique, as in Diagram 3. Whenever 1 returns to the front, we make the 2, 3, 4 and 5 follow the rules for the extent on four bells.

These rules can clearly be extended to any number of bells. We have therefore shown that it is possible to generate all rows without moving any bell more than one place at a time.

The way in which one row is followed by another may be described by specifying the positions of the bells which do *not* change their positions. Thus the change from 1 3 5 2 7 4 8 6 to 3 1 5 7 2 8 4 6 would be denoted by 38. A change in which all bells move is labelled as x. So Diagram 2 can be described as x – 1 8 – x – 1 8 – x – 1 8 – x – 1 8 – x – 1 8 – x – 1 8 – x – 1 8 – x – 1 8; and Diagram 3 as 5 – 1 – 5 – 1 – 5 – 1 – 5 – 1 – 5 – 1 2 5 – 5 – 1...1 2 5...1 2 3...1 2 3...1 2 5...1 2 5...1 2 5... 1 2 3...1 2 5...1 2 5...1 2 5...1 2 3. But as we add more bells the rules given above will lead to more varied changes between rows. For example on eight bells we will have the basic pattern of Diagram 2, but the final 1 8 will be replaced usually by 1 2 and at other times by 1 2 3 8 or 1 2 3 4 or 1 2 3 4 5 8 or 1 2 3 4 5 6.

4. Pattern of change-ringing: 'methods' and their names

Change-ringers are not too worried about a complicated basic pattern, but they would like to reduce the number of possible different alterations to the pattern. For example we can obtain the full extent on eight bells with Diagram 2 provided that the last change (1 8) is replaced as a general rule by 1 2 and occasionally by 1 4 (known as a 'bob') and by 1 2 3 4 (a 'single'). We can then tell our band of ringers to ring $x - 1\ 8 - x - 1\ 8 - x - 1\ 8 - x - 1\ 8 - x - 1\ 8 - x - 1\ 8 - x - 1\ 8 - x - 1\ 2$ and repeat indefinitely except that when the conductor calls 'bob' the 1 2 is to be replaced by 1 4 and when he (or she) calls 'single' the 1 2 is to be replaced by 1 2 3 4. This method is called 'Plain Bob Major'. Other methods can be described in this way by the places made at each change and by the effects of bobs and singles. Each method will have its own name.

Changes on five bells are called 'Doubles' (because two pairs can change at once—even though frequently only one pair changes); on seven 'Triples', on nine 'Caters' and on eleven 'Cinques'. Normally these changes are rung on an even number of bells with the tenor last every time.

Changes on four bells are called 'Minimus'; on six 'Minor'; on eight 'Major'; on ten 'Royal' and on twelve 'Maximus'. The 'method' of changing is often the same on several different numbers. For example Grandsire Doubles (shown in Diagram 4) is 3 – 1 – 5 – 1 – 5 – 1 – 5 – 1 – 5 – 1 (repeated twice more) while Grandsire Caters is 3 – 1 – 9 – 1 – 9 – 1 – 9 – 1 – 9 – 1 – 9 – 1 – 9 – 1 – 9 – 1 – 9 – 1 (repeated six times more). (Write these out and you will see that they are very similar.)

Some methods are more complicated than others and have names like Kent Treble Bob or Cambridge Surprise. These add to the exotic flavour of change-ringing (even though it is a peculiarly English art).

5. Avoiding repetitions: 'proof' of methods

Each method will have its own problems of 'proof', i.e., making certain that the same row is not rung twice.

One of the important ideas in proof is the even or odd nature of a row. A row is said to be even if it can be generated from 'rounds' (1 2 3...) by an even number of simple transpositions (or interchange of two bells). Thus, on eight bells, 1 3 5 2 7 4 8 6 is even, because we can go from 1 2 3 4 5 6 7 8 to 1 3 2 4 5 6 7 8 to 1 3 5 4 2 6 7 8 to 1 3 5 2 4 6 7 8 to 1 3 5 2 7 6 4 8 to 1 3 5 2 7 4 6 8 to 1 3 5 2 7 4 8 6 in six simple transpositions. An odd row is one which is not even. We will not prove that this definition is satisfactory, but if you examine rows you will find that exactly half of all possible rows are even and so the other half are odd.

If we assume that being even or odd is an intrinsic property of a row we can, Diagram 2

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
2	4	1	6	3	8	5	7
4	2	6	1	8	3	7	5
4	6	2	8	1	7	3	5
6	4	8	2	7	1	5	3
6	8	4	7	2	5	1	3
8	6	7	4	5	2	3	1
8	7	6	5	4	3	2	1
7	8	5	6	3	4	1	2
7	5	8	3	6	1	4	2
5	7	3	8	1	6	2	4
5	3	7	1	8	2	6	4
3	5	1	7	2	8	4	6
3	1	5	2	7	4	8	6
1	3	2	5	4	7	6	8
1	2	3	4	5	6	7	8

Diagram 3

1	2	3	4	5
2	1	4	3	5
2	4	1	5	3
4	2	5	1	3
4	5	2	3	1
5	4	3	2	1
5	3	4	1	2
3	5	1	4	2
3	1	5	2	4
1	3	2	5	4
1	3	5	2	4
3	1	2	5	4
3	2	1	4	5
.	.	.	.	.
1	5	3	4	2
1	5	4	3	2
.	.	.	.	.
.	.	.	.	.
1	4	5	2	3
1	4	2	5	3
.	.	.	.	.
.	.	.	.	.
1	2	4	3	5
1	2	4	5	3
.	.	.	.	.
.	.	.	.	.
1	4	2	3	5
1	4	3	2	5
.	.	.	.	.
.	.	.	.	.
1	3	4	5	2
1	3	5	4	2
.	.	.	.	.
.	.	.	.	.
1	5	3	2	4
1	5	2	3	4
.	.	.	.	.
.	.	.	.	.
1	2	5	4	3
1	2	5	3	4
.	.	.	.	.
.	.	.	.	.
1	5	2	4	3
1	5	4	2	3
.	.	.	.	.
.	.	.	.	.
1	4	5	3	2
1	4	3	5	2
.	.	.	.	.
.	.	.	.	.
1	3	4	2	5
1	3	2	4	5
.	.	.	.	.
.	.	.	.	.
1	2	3	5	4
1	2	3	4	5

Diagram 4

1	2	3	4	5
2	1	3	5	4
2	3	1	4	5
3	2	4	1	5
3	4	2	5	1
4	3	5	2	1
4	5	3	1	2
5	4	1	3	2
5	1	4	2	3
1	5	2	4	3
1	2	5	3	4

repeated
to give

1 2 4 5 3

and
again
to give

1 2 3 4 5

At a bob

5	1	4	2	3
1	5	4	3	2
1	4	5	2	3

At a single

5	1	4	2	3
1	5	4	3	2
1	5	4	2	3

for example, see that it will not be possible to ring all 120 changes on five bells in the method called 'Grandsire Doubles' by the use of 'bobs' alone. This method is given in Diagram 4 and can be described in place notation as 3 – 1 – 5 – 1 – 5 – 1 – 5 – 1 – 5 – 1 repeated. At a bob the last 5 is replaced by 3; at a single the last 5 is replaced by 3 and the last 1 is replaced by 1 2 3. It is easy to see that unless a single is rung, every change has two pairs interchanging and so every row is even. Hence only the 60 even rows can be rung and singles are needed if we want to ring the extent of 120 changes.

Similar more complex Group Theory considerations enable us to make deductions about other methods. Some methods have unsolved problems. No one knows whether, for example, 5040 changes of 'Stedman Triples' (on seven bells) can be rung without the use of 'singles'. Fermat's last theorem and the four-colour map problem seem simple but are difficult; bell-ringers have similar problems—and await more mathematicians to show interest—and genius—in solving them.

Letter to the Editor

Dear Editor,

In the article 'How the Lion Tamer was Saved' in Volume 6, No. 1, pp. 14–18, Richard Rado describes how a lion tamer, trapped inside a cage with an angry lion who can run at the same top speed as he can, can nevertheless stay away from the lion indefinitely by exercising sufficient daring and cleverness and infinitely fast footwork. It is interesting to note that if the lion tamer initially panics and runs to the edge of the cage and subsequently never leaves the edge (but possibly runs around it), then even fast footwork cannot save him, for the lion then has a simple strategy to catch him, namely: run to the centre of the cage and then advance as quickly as possible towards the lion tamer, all the while staying on the straight line joining the centre of the cage to the lion tamer.

However, while this strategy is very simple it takes a bit of calculation to see that it actually succeeds, for if the lion tamer runs around the edge of the cage at top speed, and the lion moves out in a spiral matching the lion tamer's angular velocity, then the lion's radial component of velocity decreases to zero as he nears the edge and it is not immediately clear whether or not he will actually reach the edge (and hence the lion tamer) in a finite length of time. Nevertheless, a bit of calculus shows that he does. If the cage has radius R, the top speed of the runners is V, and s is the distance from the centre of the cage to the lion at any time, then we are interested in the time required for s to increase from 0 to R. Now to match the lion's angular velocity, the lion's component of velocity perpendicular to the radius is $(s/R)V$ or less if the lion tamer is not running at full speed V; consequently the lion's radial component of velocity (that is, ds/dt) is at least $\{V^2-(s/R)^2V^2\}^{\frac{1}{2}}$. Thus, regarding time t as a function of s, we have

$$dt/ds \leqslant 1/V\{1-(s/R)^2\}^{\frac{1}{2}}$$

and so the time T required for the lion to get from the centre to the edge of the cage is

$$T \leqslant \int_0^R 1/V\{1-(s/R)^2\}^{\frac{1}{2}}ds.$$

This is an improper integral, but fortunately for the lion a convergent one. It converges to $\frac{1}{2}\pi(R/V)$, so that is the maximum time the lion needs.

Finally, note that R/V is the time required to run at velocity V straight from the centre to the edge of the cage, and $\frac{1}{2}\pi \approx 1{\cdot}5$, so the lion tamer's panic running around the edge of the cage can increase his life span about 50%.

Yours sincerely,

N. M. Rice

(Department of Mathematics, Queen's University, Kingston, Ontario)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

7.4. (Submitted by T. J. Fletcher.) Two players A and B begin with capital of p and q units respectively, and they gamble by tossing a coin; at each toss one unit of capital is transferred from the loser to the winner of that toss. (If, say, A calls 'heads', then heads he wins, tails he loses.) The game continues until one or the other is bankrupt. Compare A's and B's chances of winning. (See the article in this issue by T. J. Fletcher.)

7.5. Two men stand on the edges of two cliffs, the heights of the cliffs above sea level being the same. The cliffs are separated by a deep chasm. The men point loaded pistols directly at each other (the pistols may not be of the same make) and each fires at the same moment. Show that the bullets collide.

7.6. (Submitted by B. G. Eke, University of Sheffield.) Show that, among any ten consecutive positive integers, at least one is relatively prime to all the others.

Solutions to Problems in Volume 6, Number 2

6.5. Show that, given a finite number of points in the plane which do not all lie on the same straight line, there exists a straight line passing through exactly two of the points.

Solution by M. Ram Murty and V. Kumar Murty (Carleton University, Ottawa)

Draw all straight lines which join pairs of the given set of points; denote this (finite) collection of straight lines by S. Consider one of the points A. We can draw a straight line l through A which meets all the members of S and does not pass through any other point of the set. Consider all the points of intersection of l with the members of S. Not all the members of S can pass through A because not all the points lie on the same straight line. Hence there exists a point of intersection M different from A such that there are no intersection points between A and M. Select a member l' of S which passes through M. If l' passes through only two points of the set we are done, so assume that it passes through at least three points, say B, C, D. We consider two cases.

Case 1. This is when B, C, D all lie on the same side of l. Suppose that the order of the points on l' is B, C, D, M. Then the line through A and C will pass through only two points of the set, for, wherever we try to place a third, we shall be able to draw a line joining two points of the set which meets the chord AM, contrary to the definition of M.

Case 2. This is when two of B, C, D lie on one side of l and one on the other. Suppose that the order of the points on l' is B, M, C, D. Then the line joining A and D will pass through only two points of the set.

6.6. Let n be a positive integer which is not divisible by 2 or 5. Show that $1/n$ has decimal expansion of the form $0.\dot{a}_1 a_2 \dots \dot{a}_m$.

Solution. Suppose not. Then the decimal expansion of $1/n$ is of the form

$$\frac{1}{n} = 0.b_1 b_2 \dots b_r \dot{a}_1 a_2 \dots \dot{a}_m,$$

where $r, m \geqslant 1$ and $b_r \neq a_m$. Thus

$$n(0.b_1 b_2 \dots b_r \dot{a}_1 a_2 \dots \dot{a}_m) = 1,$$

$$n(b_1 b_2 \dots b_r) + n(0.\dot{a}_1 a_2 \dots \dot{a}_m) = 10^r, \tag{1}$$

$$n(b_1 b_2 \dots b_r a_1 a_2 \dots a_m) + n(0.\dot{a}_1 a_2 \dots \dot{a}_m) = 10^{r+m}. \tag{2}$$

We subtract (1) from (2) to give

$$n(b_1 b_2 \dots b_r a_1 a_2 \dots a_m) - n(b_1 b_2 \dots b_r) = 10^{r+m} - 10^r.$$

Hence

$$10 \mid n(a_m - b_r)$$

so that

$$10 \mid (a_m - b_r),$$

because 10 and n are relatively prime. But $-9 \leqslant a_m - b_r \leqslant 9$, so that $a_m = b_r$, which provides the required contradiction.

Also solved by Susan Street (Clare College, Cambridge), David Seal (Winchester College), M. Ram Murty and V. Kumar Murty.

6.7. Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

is not an integer when $n > 1$.

Solution by David Seal

Suppose that $n > 1$, let k be the least common multiple of $1, 2, \ldots, n$, and let p be the positive integer such that $2^p \leqslant n < 2^{p+1}$. Then k is divisible by 2^p but, since none of $1, 2, \ldots, n$ is divisible by 2^{p+1}, k is not divisible by 2^{p+1}. Moreover, none of $1, 2, \ldots, n$ except 2^p itself is divisible by 2^p. Hence all but one of the integers

$$\frac{k}{1}, \frac{k}{2}, \ldots, \frac{k}{n}$$

is even, so their sum is odd. But

$$\frac{k}{1}+\frac{k}{2}+\ldots+\frac{k}{n} = k\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

and k is even. It follows that

$$1+\frac{1}{2}+\ldots+\frac{1}{n}$$

cannot be an integer.

Also solved by M. Ram Murty and V. Kumar Murty.

6.8. From the series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots$$

delete every term whose denominator has a '3' in its decimal representation. Show that the modified series is convergent.

Solution by M. Ram Murty and V. Kumar Murty

There are $9^n - 1$ integers between 1 and $10^n - 1$ (inclusive) whose decimal expansions do not possess a '3'. For $n \geqslant 1$, write

$$t_n = \frac{1}{10^{n-1}}+\frac{1}{10^{n-1}+1}+\ldots+\frac{1}{10^n-1},$$

where the terms whose denominators possess a '3' in their decimal expansions have been omitted. There are $9^n - 9^{n-1}$ terms in this sum, so that

$$t_n < \frac{9^n - 9^{n-1}}{10^{n-1}} = 8\left(\frac{9}{10}\right)^{n-1}.$$

Since the infinite series

$$\sum_{n=1}^{\infty} 8\left(\frac{9}{10}\right)^{n-1}$$

is convergent, with sum 80, it follows that the given series is convergent, and that its sum does not exceed 80.

Also solved by David Seal.

Note on problem 6.3. We recall the problem:

> Show that, for every positive integer n, there is a finite set of points in the plane with the property that every point of the set is distant one unit from exactly n points of the set.

In Volume 6, Number 2 we published a solution of this problem by David Seal (Winchester College). His construction provided, for each integer $n \geqslant 2$, a set possessing 3.2^{n-2} points with the given property. Professor Paul Erdös asked whether 3.2^{n-2} can be replaced by a smaller number. David Seal has pointed out that this is indeed possible. We reproduce his argument below.

Suppose there is a finite set S of points in the plane with the given property for $n = k$. We shall obtain from S a set of points with the given property for $n = k+2$. As in the original argument, all but a finite number of directions d have the following property: when S is translated through one unit in the direction d to give another set of points T, then each point of T will be distant one unit from precisely one point of S (and so also each point of S will be distant one unit from exactly one point of T). It is therefore possible to choose three such directions d_1, d_2, d_3, where d_2 is obtained from d_1 and d_3 from d_2 by anticlockwise rotations of 60°.

We begin with the set S and translate it through one unit in the directions d_1 and d_2 to obtain the sets T and U. Then each point of S is distant one unit from exactly k points of S, one point of T and one point of U. Also, each point of T is distant one unit from precisely k points of T and one point of S. But the set U can be obtained from T by a translation of one unit in the direction d_3, so each point of T is distant one unit from precisely one point of U. Finally, each point of U is distant one unit from precisely k points of U, one point of S and one point of T. If we now unite S, T and U, we obtain a set with the given property for $n = k+2$.

When $n = 0$, we may take as our set a single point. If we now apply the above construction successively, we obtain a set of $3^{n/2}$ points with the given property for each even number n. When $n = 1$, we take as our set two points distant one apart and apply the construction successively. This provides a set of $2 \times 3^{(n-1)/2}$ points with the given property for each odd number n.

David Seal adds that he does not know whether these numbers can be improved upon, but they are certainly smaller that $3 . 2^{n-2}$ for each integer $n > 3$.

Book Reviews

Géométrie pour l'élève-professeur. By JEAN FRENKEL. Hermann, Paris, 1973. Pp. 353. About £4·50.

The purpose of this book is to revive interest in geometry among secondary-school teachers in mathematics. The author believes that linear algebra and geometry are intimately related and endeavours to provide a unified text relying on algebraic argument and geometric illustration. In this he has been successful, though at a level of difficulty which some may find hard to accept.

Part I of the book is concerned with affine geometry; it considers affine spaces, their applications, barycentres, and the fundamental theorem of affine geometry. Part II covers Euclidean geometry; it discusses among other topics bilinear and quadratic forms, the orthogonal group of a quadratic form, finite-dimensional Euclidean vector spaces, orthogonal transformations and isometries. Part III deals with finite-dimensional projective geometry; projective spaces, analytic projective geometry, the theory of collinearity and perspective as well as projections are studied in detail. There are two appendices; a bibliogrophy of some thiry-seven works and a three page index complete the book.

A useful feature of the work is the set of exercises interspersed in each of the eighteen chapters; there are also some complementary problems at the end of the book. The author states in his preface that he would wish to 'convince the reader that without geometry, algebra is blind; without algebra, geometry is paralytic'. Those who are looking for mathematical stimulus may find this text occasionally difficult but always rewarding.

C.S.I.R.O., Canberra — J. GANI

Computers and Computation (Readings from *Scientific American* with introduction by ROBERT R. FENICHEL and JOSEPH WEIZENBAUM). W. H. Freeman & Co, Reading. Pp. 283. £4·70, hardback; £2·10, paperback.

This book is a collection of reprints of about 25 articles which originally appeared in the magazine *Scientific American* over the period 1950 to 1971. One need hardly say, therefore, that the quality of content of the text is high; the numerous illustrations provide a very valuable supplement to the text. These include many photographs, both coloured and black and white, as well as diagrams both simple and complex, but always clear. Messrs Fenichel and Weizenbaum have written helpful introductory notes and have made the book as up to date as possible by revising the various articles, authors' biographies and bibliographies.

The articles are grouped (rather forcedly, I feel) into five sections. The first and longest deals with 'Fundamentals'—computer hardware, its nature and design; basic computing principles and techniques; various input and output systems and their use. The second section is broadly concerned with 'Artificial Intelligence'—machines playing chess; composing music; recognising patterns. The short third section has a more direct mathematical leaning, including articles on the solution by computer of problems in logic and probability. The fourth contains articles on solution by computer of the more 'real-life', practical problems in fields such as biochemistry, fluid mechanics and urban planning. The final section consists solely of four, more general essays, about the diverse uses of computers in such fields as education and business organisations.

Perhaps I could make a minor point. As the text is American rather than English, the English reader may experience the odd brief confusion or frustration upon meeting an unusual American spelling, idiom or jargon.

This book was not designed as a strict reference-type book and as such is unsuitable, with the possible exception of the section on 'Fundamentals'. But as a more general text, it is most admirable, and interesting. As a sixth former I have taken an interest in computers, and may pursue a career in this field. I am confident that when I have read this book very thoroughly I shall have a deeper knowledge of computers and computation; this will give me a more solid base from which to decide whether or not to pursue my inclination.

Shrewsbury School GERALD WEATHERSTON WILSON

Invitation to Mathematics. By WILLIAM H. GLENN and DONOVAN A. JOHNSON. Dover Publications, Inc, New York, 1973. Pp. 373. £1·75.

This edition is a republication of the 1962 original. Written in an easily readable style which makes it suitable for the layman interested in mathematics or the young student, it also offers many ideas for the teacher seeking a fresh approach.

The authors endeavour to show that mathematics is more than a combination of arithmetic, algebra, geometry, and its various other branches; they describe it as 'a way of thinking, a way of reasoning'. They show that a basic knowledge of the subject is essential to numerous areas of modern life, and indicate many fields of research in which a more advanced knowledge is indispensable.

The text is divided into six self-contained sections. Part I, Invitation to Mathematics, introduces the reader to numerous basic mathematical concepts and outlines some problems which have puzzled mathematicians throughout the ages. There are several one-page accounts of the work of famous mathematicians; this reviewer found their insertion at intervals in the main text rather distracting. Part II, The World of Measurement, explains in a fascinating way the origins of many familiar concepts of measurement, and outlines the use of measurement in the modern world. Part III, Adventures in Graphing, presents graphing as a story of mathematics in pictures, and shows how algebra and geometry can help each other. Part IV, Short Cuts in Computing, presents labour-saving tricks for speedy calculations, while Part V, Computing Devices, describes the development from the most primitive stages of the complex machines now used to free us from tedious computations. Part VI, The World of Statistics, discusses sampling and shows how to interpret collected data intelligently.

Each section includes exercises for which solutions are provided, and lists suggestions for further reading. There are also ideas for group projects.

This book should provide interest and enjoyment for the layman and non-specialist mathematician and would be a valuable addition to a school library.

University of Sheffield M. HITCHCOCK

Notes on Contributors

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