Mathematical Spectrum

1995/6 Volume 28 Number 1

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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year, consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column.

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Volume 28, Issues 1, 2 and 3 (postage included) costs
£8.50 or US$ 14.00 or $A 19.00

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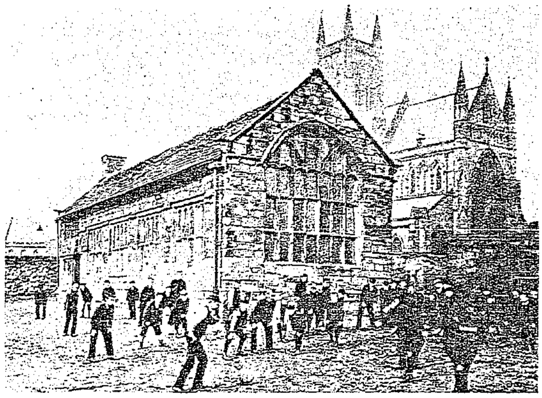
The Revd Thomas Penyngton Kirkman FRS (1806–1895)
Schoolgirl parades—but much more!

HAZEL PERFECT

After long years of neglect, Thomas Kirkman's wide-ranging contributions to mathematics are being recognized today. This article provides some account of his mathematical work, his other writings, and the background to them.

'I am a reckless and incorrigible sceptic'. Thus did Kirkman once describe himself; and scepticism is perhaps not an entirely unsuitable attribute in a man who was mathematician, theologian and philosopher. History has not been kind to Kirkman the mathematician. Only in fairly recent years has some reassessment been made of his considerable contributions to mathematical research (references 2, 3). His name, on the other hand, has long been popularly linked to one particular mathematical problem, the problem of the fifteen schoolgirls, a circumstance, of course, ensuring that the name of this neglected scholar was still remembered. The year 1995, being the centenary of his death, allows us to offer here a brief pen-portrait of a man of many parts.

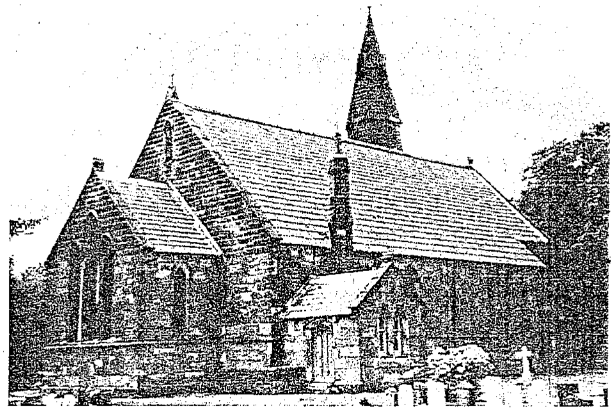
Thomas Penyngton Kirkman was born in Bolton, Lancashire in March 1806. He was one of several children, the only boy, in a non-academic family, his father being a dealer in cotton and cotton waste. It would seem that the business was of some significance, since it is listed in the early, rather selective, Bolton directories. So the family may have been reasonably prosperous. Kirkman's early education was at Bolton Grammar School. It exists today as Bolton School and dates from the early sixteenth century, being one of the oldest schools in Lancashire. According to an admission register for the years 1808–1831, young Thomas Kirkman entered the school in 1814 when he was eight years old, and he remained there until he was fourteen. In Kirkman's day tuition was free to the sons of parishioners. There would have been probably about 100 boys in the school, 20 senior boys being taught separately by the headmaster. Instruction in Latin and Greek was given, but little in mathematics, probably some arithmetic but no algebra or Euclid. Kirkman is recorded as being by far the best scholar in the school. Both the headmaster and the vicar urged Kirkman's father to allow him to remain at school and apply to Cambridge University. His father, however, was stubbornly unwilling for this, and Kirkman went to work in his father's office. There he continued his own private study of the classics, and also began to teach himself French and German.



The school building as it was in Kirkman's time—just before its demolition in 1880

Kirkman worked for nine years in the office. However, he was evidently of a scholarly turn of mind and at twenty-three years old, by undertaking private tutoring, he was able so to finance himself that he could enrol as a student at Trinity College, Dublin. It should be remembered that Oxford and Cambridge were the only universities in England at that time, and many English students went to Dublin to study. Many of these took 'extern' or 'steamboat' degrees, which meant that they attended no lectures or tutorials but only turned up twice a year for four years for their examinations. Though information about Kirkman's years at Trinity College is scanty, it does seem that he attended some lectures. In his second year he is recorded as being 'remarkably diligent', and in 1833, his final year, he was marked present at some of the Hebrew and divinity classes. The prescribed subjects for the BA degree then were mathematics, philosophy, classics and science, but there is no record of Kirkman's attendance, for instance, at any mathematics lectures. Like many other students, he had no room in college. We do not know just when Kirkman became particularly interested in mathematics. He was over forty when his first mathematical paper was published (triggered off, as we shall see, by a problem set in the *Lady's and Gentleman's Diary*). However, it is hard to believe that he could not have been influenced to some

The Revd T. P. Kirkman



Christ Church, Croft

extent by the drastic reforms in the mathematics syllabus, which had been introduced when Bartholomew Lloyd became professor in 1812. Until then the course had been very basic, but Lloyd was aware of the advances on the continent, in France in particular; French textbooks were introduced and new ones written so that instruction could be given in the continental methods. In a very short period of time the study of mathematics 'leapt a chasm of a hundred years'. Kirkman graduated from Trinity College in 1833 as a 'moderator', something like a wrangler in Cambridge. He stayed in Ireland for a further year as tutor to an Irish baronet.

In 1834 Kirkman returned to England and took holy orders the following year. Sixty years of life still remained to him and, of these, over fifty-three were to be spent in the little village of Croft near Warrington. Kirkman moved there in 1839 after holding curacies in Bury and Lymm. Croft remains a village today, and a walk down Lady Lane to the Church can still evoke the past, although the new now jostles uncomfortably with the old. The Church to which Kirkman came, first as curate-in-charge, and then as its first rector in 1845, dates from 1832. It was one of about 600 new churches built at that time following a decision by Parliament that the best way to improve the poor conditions in England after the Industrial Revolution was to bring the church to the people. It is an imposing building seating over 500, and dominated by its spire designed by one of the foremost architects of the day. The Old Rectory, which was Kirkman's home, stands next to the Church and was adapted from an earlier house already on that site. Kirkman's biographers suggest that his parochial work was small. However, his eldest son tells us that, in the course of this work, 'he formed, out of the roughest material, a parish choir of boys and girls who could sing at sight any four-part song set before them'; and Kirkman himself speaks of teaching elementary mathematics to a class of village children. That he showed concern for the educational needs of his older parishioners is also evident from the evening lectures delivered to them in the village school. In 1841 Kirkman married Eliza Ann Wright from Runcorn and, over the years, the Kirkmans had seven children. The income from the benefice was not large, but was augmented by property inherited by his wife.

Here, then, in a small village community, we find the Revd Mr Kirkman, far removed from the centres of mathematical research and from his eminent contemporaries Arthur Cayley, William Rowan Hamilton, Augustus de Morgan, P. G. Tait and others. In spite of this, Kirkman wrote extensively in mathematics. For many years subsequently his work was largely neglected, and only some fifteen years ago did a critical appraisal of his mathematics appear (reference 2). This goes into much more detail than is possible or appropriate here. We shall very simply try to indicate something of its scope and the background against which it was produced. Kirkman's contributions to mathematical research lie principally in four main areas: supremely in combinatorics, in hypercomplex numbers, in the theory of groups then in its infancy, and in the theory of knots stemming from his lifelong work in polyhedra.

In order to gain a little insight into the significance of Kirkman's work in the field of combinatorics, it is convenient to begin with one or two definitions from what is now called 'design theory' (in Kirkman's day it was called 'tactic'). This is a subject which has proved over the years to have important applications, particularly to certain types of statistical experimentation. At Rothamsted in the 1930s the theory of designs began to be widely exploited in the conduct of agricultural trials. Mathematically, it remains today a challenging and difficult field of research. By a *t-design* with parameters (v, k, λ), or a t-(v, k, λ) design, we shall understand a family of subsets of size k (often called blocks) of a set of v elements with the property that each subset of size t occurs in just λ of the blocks. In the case $t = 2$, $k = 3$, $\lambda = 1$, a t-design is called a *Steiner triple system*; so this is a collection of triples of an underlying set with the property that every pair of elements lies in just one triple. Figure 1 provides a pictorial representation of a 2-$(7, 3, 1)$ design. The blocks are the seven lines in the figure (one of them curved). Clearly every pair of elements of the underlying set of seven points lies on a unique line. This design is, in fact, a *finite projective plane*, since it has the 'dual' property that every two lines meet in just

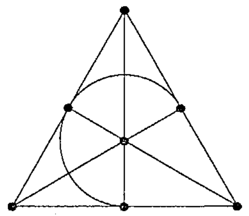


Figure 1

one point. Probably Kirkman's most enduring mathematical work lies in the area of design theory; and it all began with some problems posed in the *Lady's and Gentleman's Diary* in the mid 1840s. The first one, as set, was too general and too difficult, and it was replaced by a special case. In the terminology which we have introduced this was essentially to find the number of triples in a Steiner triple system on v elements. Of more significance is the problem of finding for which v such a system exists. (When it does, the number of triples is easily seen to be $\frac{1}{6}v(v-1)$.) In the first mathematical paper which Kirkman wrote, he solved this fundamental problem: such a system exists if and only if either of $v-1$, $v-3$ is divisible by 6. The difficulty lies in proving the sufficiency of the conditions, and Kirkman gave a constructive proof of this. The famous 'schoolgirls problem' was simply an offshoot of this work. Much has been written about this latter problem (see, for example, reference 1), and so here we can treat it briefly. Kirkman sent the problem in the following form to the *Lady's and Gentleman's Diary*, where it appeared in 1850:

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

One possible schedule (where the fifteen girls are labelled $0,\dots,6,0',\dots,6',P$) is shown in table 1 below.

Table 1

Sunday	01′3′	35′2′	26′0′	156	*P*44′
Monday	12′4′	46′3′	30′1′	260	*P*55′
Tuesday	23′5′	50′4′	41′2′	301	*P*66′
Wednesday	34′6′	61′5′	52′3′	412	*P*00′
Thursday	45′0′	02′6′	63′4′	523	*P*11′
Friday	56′1′	13′0′	04′5′	634	*P*22′
Saturday	60′2′	24′1′	15′6′	045	*P*33′

It is not too difficult to find solutions by trial and error; the solution in table 1 was found rather more systematically by a geometrical argument using a projective 3-space analogous to the plane in figure 1 (see reference 4). In particular, the schedule provides an example of a Steiner triple system on 15 elements. But it has a further property called 'resolvability'; namely that the 35 triples can be split into 7 sets of 5, in each set of which each of $0,\dots,6,0',\dots,6',P$ appears just once. By rights, Steiner triple systems should have been called Kirkman triple systems, since Kirkman anticipated Steiner by some five or six years. The name of Kirkman is, however, reserved for resolvable systems, in the sense that the $\frac{1}{6}v(v-1)$ triples can be split into $\frac{1}{2}(v-1)$ sets of $\frac{1}{3}v$ so that each of the v elements appears just once in each set. It was first proved in 1968 that resolvable systems exist whenever $v-3$ is divisible by 6 (and only then). Over the years Kirkman continued to make fundamental discoveries in tactic, and his methods foreshadow many of today's basic constructions. The most we can do here is to cite one or two further instances of his results. We met an example of a projective plane in figure 1. Finite projective planes in general are 2-$(v,k,1)$ designs with the additional property of duality (which implies that $v = k^2-k+1$); $k-1$ is called the *order* of the plane. Kirkman established the existence of projective planes of every prime order, and he demonstrated further that planes of orders 4 and 8 exist. He remarked that he could not construct a plane of order 6 and thought its existence 'improbable'; this negative result was established in 1907. It is known today that projective planes of every prime-power order exist, but whether there are other possible orders is still an unsolved problem. Some idea of the difficulty of these problems may be inferred from the fact that only in 1988 was the non-existence of a plane of order 10 established. Not all of Kirkman's results on designs were restricted to the case $t = 2$; for instance, a family of 3-designs with parameters $(2^n,4,1)$ was first established by him. We shall have occasion below to remark on Kirkman's use of combinatorial methods in his other fields of research: Kirkman was a natural combinatorialist.

Sir William Rowan Hamilton, the Astronomer Royal of Ireland, and born the year before Kirkman, was a great figure at Trinity College, Dublin. His discovery of quaternions in 1843 had made him famous in the mathematical world. Quaternions, which provide a generalization of complex numbers, are expressions of the form $a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k}$, where the coefficients are real numbers and the 'imaginary units' i, j, k satisfy the relations

$$\begin{aligned} \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 &= -1, & \mathrm{jk} = -\mathrm{kj} &= \mathrm{i}, \\ \mathrm{ki} = -\mathrm{ik} &= \mathrm{j}, & \mathrm{ij} = -\mathrm{ji} &= \mathrm{k}. \end{aligned} \tag{1}$$

A significant point was the non-commutativity of their multiplication. Obviously Kirkman was aware of Hamilton's work, and he became interested in the subject. He was never one to shy away from generality, and his concern was in the extension of the notion to linear expressions with $2n-1$ rather than one or three units. Kirkman called them *pluquaternions*; today we would recognize them as hypercomplex numbers. A problem about products of sums of squares was occupying mathematicians at this time, and Kirkman and others (notably Cayley) recognized a connection with pluquaternions with 2^m-1 units. The following simple analysis with ordinary complex numbers

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= (a+\mathrm{i}b)(a-\mathrm{i}b)(c+\mathrm{i}d)(c-\mathrm{i}d) \\ &= [(a+\mathrm{i}b)(c+\mathrm{i}d)][(a-\mathrm{i}b)(c-\mathrm{i}d)] \\ &= (ac-bd)^2+(ad+bc)^2\end{aligned}$$

establishes that the product of 2 sums of 2 squares is itself a sum of 2 squares. A similar result that the product of 2 sums of 4 squares is itself a sum of 4 squares can be proved most easily using quaternion analysis. With rather more difficulty the result can be extended from 4 to 8 squares; and a natural conjecture was the extension to 2^m squares. The conjecture is, however, false: a product of 2 sums of 16 squares is not in general equal to a sum of 16 squares. Kirkman appreciated that this was because a multiplication of pluquaternions with $2^m - 1$ units by means of defining relations similar to (1) above is not possible when $m > 3$. In 1857, Kirkman was elected a Fellow of the Royal Society, partly for his work in connection with quaternions. Hypercomplex systems now play an important role in algebra, and much work has been done in this area in more recent years. As a postscript, we note that Kirkman met Hamilton at any rate on one occasion when Hamilton visited him at Croft Rectory after the British Association meeting in Manchester in 1861. The two men continued to be in touch by letter, and one cannot but feel for Kirkman when he writes that 'I wish I had the good fortune to be nearer to such a mathematician as you; for it would be of immense advantage to have the profit of conversation with such a man. It is a great loss to me to live cut off from all scientific intercourse'.

It would seem that Kirkman was motivated to begin his researches in the theory of groups when he read of the prize offered for a memoir in this field by the Academy of Sciences in Paris. The particular problem to be investigated was proposed in 1858, and memoirs were to be submitted in 1860. The mid nineteenth century was, of course, the very early days of group theory. The theory of permutation groups (or substitution groups) was being developed by Cauchy and others to tackle questions in the theory of equations, and at this time 'a group' usually meant 'a group of permutations'. It was only in 1854, with Cayley's introduction of the multiplication table of a finite group, that abstract group theory was born. So Kirkman was working when group theory was young. Also, not only was Kirkman working alone in an isolated village, but also in a country not yet fully abreast with the ideas fermenting on the continent of Europe, notably in France. Yet his ability to read in other languages than English, and also the enlightened outlook which he had met at Trinity College, Dublin, would have made him to some extent conversant with what was happening abroad. For applications to the theory of equations, their solvability by radicals for instance, some of the important notions in group theory are that of a normal subgroup, of a simple group as one possessing no proper normal subgroups, and of a solvable group. In his own terminology, and in papers of great length and complexity, Kirkman was investigating these ideas. Here, we can make just two observations about his own original contributions to the subject. First, it seems clear that Kirkman, in his attempts at a census of finite groups, did discover and construct genuinely new groups, which were only rediscovered some years later. Second, and perhaps more important, were his methods. It would be a great mistake to draw a definite line between Kirkman's purely combinatorial work and his contributions to the theory of groups; for his methods were largely combinatorial rather than, as we should say, group-theoretical. Kirkman makes an interesting point when, after comparing his own methods of construction of some particular groups with earlier group-theoretic methods, he predicts that 'This may perhaps ripen into a complete tactical [combinatorial] theory of groups'. We may mention, in passing, an observation about the cosets of a subgroup which would surely have pleased Kirkman. The result that the left and right cosets have a common system of representatives, first established by heavy group-theoretic means, was seen by van der Waerden in 1927 to depend only upon a purely combinatorial result about partitions of sets. Three memoirs were submitted for the French prize of 1860, two from French mathematicians later to become eminent in the field, and one from Kirkman. However, no prize was awarded. Kirkman remarks with some bitterness that 'Not the briefest summary was vouchsafed of what the competitors had added to science, although it was confessed that all had contributed results both new and important ... '.

Kirkman's work on polyhedra (he always called them polyedra) is to some extent a story of frustration and rejection. The Academy in Paris had set another prize problem in 1858, this time for submission in 1861, and the subject was polyhedra. Kirkman had been working in this area for a few years by then, and he intended to submit a memoir. His disappointment about the earlier prize made him change his mind, and instead he sent the material he had prepared for the competition to the Royal Society. Kirkman had rare talent as a combinatorialist and, as we have already seen, he did not fear to tackle problems of great generality. In this field, where little had previously been done beyond the study of the Platonic solids, he set himself to enumerate, classify, and construct various large classes of polyhedra. Cayley spoke of the problem as one of 'extreme difficulty' and further remarked 'I am not aware that it has been discussed elsewhere than in Mr Kirkman's valuable series of papers on the subject ... '. The manuscript submitted by Kirkman to the Royal Society was very long indeed; the editors evidently found it to be completely indigestible, and they agreed to publish only the first two of the twenty-one sections. We can best quote Kirkman's own words to describe his outrage at this rejection. He speaks of his work 'being flung aside as a troublesome cumbrance' and, in reference to a letter from de Morgan, says 'That letter was the only evidence ... that any competent judge, dead or alive, ever tried to read six pages of mine on this subject printed or in my manuscript'. The manuscript remains in the archives of the Royal Society. In connection with his work on the classification of polyhedra, Kirkman began to study

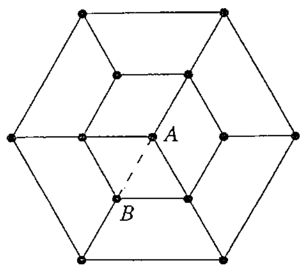


Figure 2

closed polygons on a polyhedron. They are called Hamiltonian circuits today. A Hamiltonian circuit is a closed path which passes exactly once through each vertex; and one of the unsolved problems of modern graph theory is that of finding necessary and sufficient conditions for their existence. Their name belies the fact that Kirkman anticipated Hamilton in the study of these paths, and again Kirkman's approach was the more general; Hamilton only considered the case of a dodecahedron. It is worth pausing to state the following negative proposition of Kirkman:

> If every face of a polyhedron has an even number of sides, and the number of its vertices is odd, then no Hamiltonian circuit exists.

Kirkman gives a very pretty illustration of this. 'If we cut in two the cell of a bee by a section of its six parallel edges we have a [polyhedron with thirteen vertices] whose faces are one hexagon and nine quadrilaterals. No closed 13-gon can be drawn'. Figure 2 shows a projection of this polyhedron. If, however, one more edge such as AB is added, then it is simple to construct a Hamiltonian circuit. A proof of Kirkman's theorem, rather simpler than his own, depends on the following two statements, which the reader may like to investigate:

(a) a polyhedron in which every face is even can have its vertices coloured red and blue (R and B) so that every edge joins R and B;

(b) a circuit in such a polyhedron must be of the form $RB.BR.RB \ldots BR$, and so has an even number of edges: so, in particular, a Hamiltonian circuit must have an even number of edges, and the polyhedron therefore has an even number of vertices.

Kirkman worked on polyhedra over a period of more than thirty years—he was still working in his eighties. Towards the end of this time he was in correspondence with P. G. Tait, who was researching into the theory of knots (in order to develop Lord Kelvin's idea of vortex atoms). Tait's biographer tells us that 'Kirkman's intimate knowledge of the properties of polyhedra suggested to him (Kirkman) a mode of attack on knots quite distinct from that developed by Tait'. The collaboration led to important advances in the theory. Kirkman's eldest son, in his obituary notice of his father, recalled that 'he used to sigh at the gulf of time which separated his own day from that of the possible use of his work in applied Science'. Here, then, in his work with Tait, was one application which may have given Kirkman some gratification.

In theology, and on the edge of theology and philosophy, Kirkman wrote at length. Some of his articles were reprints of lectures and sermons, and many of them were written in the period from the middle 1860s to the early 1870s when there was some lessening of his mathematical output. A book *Philosophy without Assumptions* was published in 1876 when Kirkman was seventy. This had a twofold purpose: first, the construction of a philosophy in which nothing is taken for granted; and, second, the destruction of the modern materialist philosophy. It was reviewed at some length by one of the leading churchmen of the day, Cardinal Manning, who commended its author for a 'vigorous creation of his own mind'. Though the reviewer had misgivings about the sufficiency of the philosophy in its positive aspect, he concluded that 'It is certainly the most formidable assault that has been delivered of late along the whole line of sceptical and materialist philosophy'; and he went on, 'I can only hope that the authorities so unrelentingly summoned to combat will not decline the passage of arms because of the jaunty defiance of Mr Kirkman's trumpet'. Few, however, took up the challenge.

In connection with Kirkman's theological writings, mention should be made of the 'Colenso affair'. John William Colenso was Bishop of Natal, and incidentally was also a mathematician. In the 1860s and 1870s he was writing his chief work of biblical criticism on the Pentateuch and the book of Joshua. This treatise challenged the fundamentalist position in regard to the literal truth of the scriptures, and aroused a storm of controversy in the Anglican Church. The work was condemned as heretical, and the Bishop of Cape Town tried to depose Dr Colenso; but the Privy Council revoked this. Kirkman, himself a Broad Churchman, spoke and wrote in support of Colenso. Through the years 1865 and 1866 Kirkman gave a series of lectures to his parishioners in Croft School. In the first of these, 'Truth against Tradition', he says 'Dr Colenso ... has revealed to the people his real honest convictions upon the books of Moses. ... [His work] sets candidly before us some of the long-known and settled results of the sciences ... '. And for emphasis, Kirkman goes on, 'What harm has been done to our love and reverence for the Scriptures by the discoveries of Copernicus and Galileo in astronomy? None whatever. It has merely happened that what our forefathers believed to be sacred history and sacred science, we now see to be sacred poetry'.

Kirkman was a great lover of words; he used them pointedly and he coined some memorable ones. We do not have to dig too deeply into his writings to find examples of the telling, sometimes caustic comment and the fitting analogy. It may be fair to say (as one biographer does) that he was over fond of 'stylistic gimmickry'; and he has been reprimanded for the 'perfection of his sarcasm', which no doubt did not endear him to everyone. His well-known paraphrase, in his

Philosophy without Assumptions, of the philosopher Herbert Spencer's definition of evolution is witness to this: 'Evolution is a change from a no-howish, untalkaboutable, all-alikeness, to a somehowish and in-general-talkaboutable not-all-alikeness, by continuous somethingelseifications, and sticktogetherations'. (Spencer's definition reads: 'Evolution is a change from an indefinite, incoherent, homogeneity to a definite, coherent, heterogeneity, through continuous differentiations and integrations'. First Edition of *First Principles*, 1862). In the same book we find his remark about the nature of mathematics: 'In mathematics we have 'must-follows' in plenty but no 'must-be's' ... '. How splendidly succinct and to the point! Kirkman had a lot of fun with words; throughout the years he contributed mathematical problems to the *Educational Times* and, although many contained serious mathematics, these were often expressed in quaint verses. The story of Old King Cole with his 'fiddlers five abreast' served to illustrate a resolvable 2-$(5^2, 5, 1)$ design. A little puzzle then current was rather prettily versified by Kirkman as

> Baby Tom of baby Hugh
> The nephew is and uncle too;
> In how many ways can this be true?'

In the character of 'Uncle Penyngton' he wrote a curious little book on mnemonical aids in mathematics in the belief that, if a formula is first presented to a student in a form in which he can readily chant it, then it is likely to remain fixed in his memory—'the ear is ever grateful for the humblest jingle'.

The Revd Mr Kirkman retired from his parish in 1892 at the age of eighty-six, and he and his wife moved to Bowdon, near Altrincham. He died in February 1895, his wife only twelve days later. There was very little national recognition of his achievements. In the *Proceedings of the Royal Society* there is a notice of his death, but no obituary was published. *The Times*, in a brief paragraph, recorded that 'he was exceptionally gifted as a mathematician'. Locally he was highly esteemed and much loved by his parishioners, who attended his funeral in the parish church of Croft in large numbers. The beautiful east window, which can be seen today, is in his memory.

A sceptic in some matters Kirkman may well have been. Certainly he did not take the opinions of others on trust, and he was a thinker of great originality and penetration. As a mathematician he was accustomed to the rigours of logical reasoning. He was also a parish priest with strong religious convictions, who could say with simplicity that 'my belief in God rests upon foundations in my soul which are deeper and nobler than those of logic'.

Acknowledgements

I have much appreciated the kindness and help which I have received from several people: members of staff in the libraries in Bolton, Warrington, Trinity College Dublin and the University of Sheffield; the Revd W. R. D. Alexander and Mrs Sheila Carr from Christ Church, Croft; Professor J. V. Luce from Trinity College, Dublin; and Mr A. J. Mitchell from Bolton School. I thank Mrs C. Brown for permission to use the photograph of Bolton Grammar School, which appeared in *The History of Bolton School* by the late Mr W. E. Brown.

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***Hazel Perfect**, University of Sheffield, is on the advisory board of Mathematical Spectrum and has contributed articles on a number of occasions. Her main mathematical interests are in combinatorics.*

Peter O'Halloran

The death has been announced of Peter O'Halloran. He was founder of the Australian Mathematics Competition, in which schools throughout Australia competed. Although this has since been copied the world over, the Australian competition is the largest of its kind with more than half a million entrants.

He believed in making mathematics fun for schoolchildren. Shortly before his death, he was given the World Cultural Council's award of 'Educator of the Year'.

When is the product of two 2-digit numbers equal to the product of their 'reverses', e.g.

$$63\times 24 = 36\times 42?$$

Andrew Robertshaw
(Student at the University of Sheffield)

Which integer is this?

$$t+t+t+t+t+t$$

K. R. S. Sastry
(Dodballapur, India)

Thinking About a Problem

TANG FU-SU

How do you solve problems in mathematics? Using a particular problem, the author considers how to tackle mathematical problems in general.

Introduction

Since 1986, a delegation of Chinese middle school children has formally participated in the International Mathematical Olympiad (IMO) each year and has done extremely well. The members of the delegation are selected at the Chinese Mathematics Olympiad Winter Camp. The test used in the selection competition in the winter camp is similar to that of the IMO. It is designed to assess mathematical thinking, methodology and ability.

In this article we will discuss the following problem which is selected from the First Chinese Mathematics Olympiad Winter Camp (1986).

Problem 1. Given the numbers 1, 1, 2, 2, 3, 3, ..., 1986, 1986, is it possible to permute these numbers so that there is one number between the two 1s, there are two numbers between the two 2s, three numbers between the two 3s, ... and 1986 numbers between the two 1986s?

This is a problem about existence or impossibility. The common method of solving this kind of problem is a constructive method. If you can construct a permutation which satisfies all requirements of the problem then existence is proved. If every permutation fails to satisfy all requirements of the problem then impossibility is proved. But it is very troublesome to solve the problem using this method because there are 3972 numbers and the number of all permutations of these is very large ($= 3972!/2^{1986}$).

Considering a similar problem with fewer numbers

To begin with, we try considering a similar problem (or several problems) with fewer numbers. For example, we consider a problem with four numbers $(1,1,2,2)$.

Problem 2. Given the numbers 1, 1, 2, 2, is it possible to permute them so that there is one number between the two 1s and two numbers between the two 2s?

Obviously, it is impossible. Only one permutation can satisfy the latter requirement (that there are two numbers between the two 2s):

$$2, \quad 1, \quad 1, \quad 2.$$

But this permutation does not satisfy the first requirement.

We can reach this conclusion another way. All permutations of the four numbers $1,1,2,2$ are as follows:

$$1,1,2,2; \qquad 1,2,1,2; \qquad 1,2,2,1;$$
$$2,1,1,2; \qquad 2,1,2,1; \qquad 2,2,1,1.$$

None of these six permutations satisfies all the requirements.

We consider another similar problem with six numbers $(1,1,2,2,3,3)$.

Problem 3. Given the numbers 1, 1, 2, 2, 3, 3, is it possible to permute them so that there is one number between the two 1s, two numbers between the two 2s, and three numbers between the two 3s?

It is possible. The following permutation satisfies all the requirements:

$$2, 3, 1, 2, 1, 3.$$

Generalisation

The process above can be regarded as a process of generalisation and specialisation.

We can generalise the original problem as follows.

Problem $*$. Given the numbers 1, 1, 2, 2, 3, 3, ..., n, n ($n \in \mathbb{N}$), is it possible to permute them so that there is one number between the two 1s, two numbers between the two 2s, three numbers between the two 3s, ... and there are n numbers between the two ns? Prove your statement.

Problems 1, 2 and 3 are special cases of problem $*$ when $n = 1986$, $n = 2$ and $n = 3$.

Specialisation

Considering the special cases of problem $*$ for $n = 1$, 2, 3, 4, 5, 6, ... gives us table 1. The real problem now is, for which values of n is this possible?

Analysing and solving the original problem

We now return to solve our original problem 1.

Obviously, it is very troublesome to solve this problem by making permutations. We try to analyse the construction of the permutation which satisfies all requirements of the problem.

Table 1

n	Numbers	Permutation
1	1, 1	impossible
2	1, 1, 2, 2	impossible
3	1, 1, 2, 2, 3, 3	2, 3, 1, 2, 1, 3
4	1, 1, 2, 2, 3, 3, 4, 4	2, 3, 4, 2, 1, 3, 1, 4
5	1, 1, 2, 2, 3, 3, 4, 4, 5, 5	did not find
6	1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6	did not find
7	1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7	4, 5, 6, 7, 1, 4, 1, 5, 3, 6, 2, 7, 3, 2
8	1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8	4, 5, 6, 7, 8, 4, 1, 5, 1, 6, 3, 7, 2, 8, 3, 2

Table 2

n	Permutation
3	2, 3, 1, 2, 1, 3
4	2, 3, 4, 2, 1, 3, 1, 4
7	4, 5, 6, 7, 1, 4, 1, 5, 3, 6, 2, 7, 3, 2
8	4, 5, 6, 7, 8, 4, 1, 5, 1, 6, 3, 7, 2, 8, 3, 2
11	6, 7, 8, 9, 10, 11, 5, 6, 1, 7, 1, 8, 5, 9, 4, 10, 3, 11, 2, 4, 3, 2
12	6, 7, 8, 9, 10, 11, 12, 6, 5, 7, 1, 8, 1, 9, 5, 10, 4, 11, 3, 12, 2, 4, 3, 2
15	8, 9, 10, 11, 12, 13, 14, 15, 7, 8, 1, 9, 1, 10, 5, 11, 7, 12, 6, 13, 5, 14, 4, 15, 3, 6, 2, 4, 3, 2
16	8, 9, 10, 11, 12, 13, 14, 15, 16, 8, 7, 9, 1, 10, 1, 11, 5, 12, 7, 13, 6, 14, 5, 15, 4, 16, 3, 6, 2, 4, 3, 2
19	10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 9, 10, 5, 11, 1, 12, 1, 13, 5, 14, 9, 15, 7, 16, 8, 17, 4, 18, 6, 19, 7, 4, 3, 8, 2, 6, 3, 2
20	10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 10, 9, 11, 5, 12, 1, 13, 1, 14, 5, 15, 9, 16, 7, 17, 8, 18, 4, 19, 6, 7, 4, 3, 8, 2, 6, 3, 2

Suppose there is a permutation (*) of the 3972 numbers which satisfies all the requirements of the problem, i.e. for every number k ($k \in \mathbb{N}$; $1 \leqslant k \leqslant 1986$), there are k numbers between the two k's. We number the places in the permutation as 1, 2, 3, ..., 3972. If the first k to appear is in the ith place, then the other k is in the $(i+k+1)$th place. The sum of the numbers of k's two places is $2i+k+1$. Hence we get the following lemma.

Lemma. If k is even, then the sum of the numbers of k's two positions is odd; if k is odd, then the sum of the numbers of k's two positions is even.

There are 993 even numbers and 993 odd numbers among 1, 2, 3, ..., 1986. Thus the sum of the numbers of all places is odd. On the other hand, the numbers of all places are 1, 2, 3, ..., 3972 and the sum of these numbers is

$$1+2+3+\cdots+3972 = \tfrac{1}{2}\times 3972(1+3972) = 1986\times 3973,$$

which is even. Thus there is a contradiction. This contradiction indicates that the assumption is false and gives the following conclusion: it is impossible to permute the given numbers so as to satisfy the requirements.

An extension of the problem

In a similar manner to the above proof of the original problem, we can prove the statement:

> When $n = 4k-3$ or $n = 4k-2$ ($k \in \mathbb{N}$), the answer to the general problem * is that it is impossible.

Proof. First we prove the statement when $n = 4k-3$. Suppose there is a permutation of the numbers $(1,1,2,2,3,3,\ldots,4k-3,4k-3)$ such that, for every m ($m \in \mathbb{N}$; $1 \leqslant m \leqslant 4k-3$), there are m numbers between the two m.

We number the places of the permutation as 1, 2, 3, ..., $2(4k-3)$. There are $2k-2$ even numbers and $2k-1$ odd numbers among the numbers $1,2,\ldots,4k-3$. It follows from the lemma that the sum of the numbers of all the places is even. On the other hand, the numbers of all places are $1,2,3,\ldots,2(4k-3) = 8k-6$, and the sum of these numbers is

$$1+2+3+\cdots+(8k-6) = \tfrac{1}{2}(8k-6)(1+8k-6) = (4k-3)(8k-5),$$

which is odd. Thus there is a contradiction. This contradiction shows that, when $n = 4k-3$, it is impossible to make the required permutation.

When $n = 4k-2$, the statement can be proved similarly.

What about $n = 4k-1$ or $n = 4k$ ($k \in \mathbb{N}$)?

We see from table 1 that, when $n = 3$, 4, 7 or 8 (i.e. $k = 1$ or 2), there are permutations which satisfy the requirements of the problem. Table 2 extends this to $k = 1$, 2, 3, 4 and 5.

We conjecture that:

> When $n = 4k-1$ or $4k$ ($k \in \mathbb{N}$), there is a permutation which satisfies all requirements of the general problem *.

Can you prove or refute this?

Acknowledgement

I am grateful to Dr David Green for his help in writing this article.

Further reading

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The Vacillating Mathematician and Linear Difference Equations

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A particular problem leads to the solution of a difference equation.

Introducing the vacillating mathematician

It is a commonplace idea that a good problem illustrates a general theory and, conversely, placing a problem in the context of a general theory illuminates the problem. We illustrate this idea afresh by considering the following problem proposed by Zeev Barel in a recent problem column (reference 2).

> A mathematician walks along the real line. He starts at a and intends to reach b. Halfway through he recalls that he has forgotten something at a and turns back. But halfway to a he decides to go to b anyway and turns around, only to change his mind again, halfway to b. He continues in this manner for ever.
>
> Find all the cluster points of the resulting sequence of stops.

We recall here that a cluster point (or limit point) of a sequence is defined to be the limit of a convergent subsequence of the sequence. For example, the sequence $a_n = (-1)^n$ has cluster points -1 and 1. We refer the reader to an analysis text such as Rudin (reference 6) for more examples and basic results on cluster points.

Without loss of generality we can assume that $a = 0$ and $b = 1$. If we denote the nth stop by a_n, a_n is the average of the $(n-1)$th stop with 1 if n is odd and with 0 if n is even. Therefore, the stops satisfy the difference equation

$$a_n = \tfrac{1}{2}(a_{n-1}) + \tfrac{1}{2}\delta(n) \quad (n > 0,\ a_0 = 0),$$

where $\delta(n) = 1$ if n is odd and 0 if n is even.

Thus, by analogy with the perhaps more familiar terminology of differential equations, what we have here is a 'first-order non-homogeneous' linear difference equation, 'first-order' because a_n is related only to the previous term in the sequence and 'non-homogeneous' because of the presence of the $\frac{1}{2}\delta(n)$ term. Among the examples used in textbook discussions of the theory of linear equations, perhaps the most trivial is that of a geometric progression ($a_n = ra_{n-1}$) and perhaps the favourite is that of the Fibonacci sequence ($a_n = a_{n-1} + a_{n-2}$). For some other examples, see Lax (reference 5) or Strang (reference 7). Our problem therefore gives us an example of a linear difference equation only slightly more complicated than the geometric progression and, as we will see below, one with a different flavour from that of the Fibonacci sequence. A more complicated difference equation with a similar non-homogeneous term ($a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 2 - \delta(n)$ with initial conditions $a_0 = 0$, $a_1 = 1$ and $a_2 = 5$) occurs in the problem of counting the number of triangles in an equilateral triangle of side n which is covered by a grid of unit equilateral triangles. The history of this problem and the difference equation are discussed extensively by Larsen (reference 4).

To get an idea of how the example works, we can look at some well-known methods of solving a linear difference equation as applied to this problem.

Method of summation

We can repeatedly use the difference equation and sum the resulting finite series. This method of course depends on the special form of the problem and works well for a geometric progression, but not for the Fibonacci sequence. In the case of the vacillating mathematician,

$$a_n = \frac{\delta(n)}{2} + \frac{\delta(n-1)}{2^2} + \frac{\delta(n-2)}{2^3} + \ldots + \frac{\delta(1)}{2^n}$$

$$= \begin{cases} \dfrac{1}{2} + \dfrac{1}{2^3} + \dfrac{1}{2^5} + \ldots + \dfrac{1}{2^n} = \dfrac{1}{2}\,\dfrac{1-(\frac{1}{2})^{n+1}}{1-(\frac{1}{2})^2} & (n \text{ odd}), \\[2ex] \dfrac{1}{2^2} + \dfrac{1}{2^4} + \dfrac{1}{2^6} + \ldots + \dfrac{1}{2^n} = \dfrac{1}{2^2}\,\dfrac{1-(\frac{1}{2})^n}{1-(\frac{1}{2})^2} & (n \text{ even}). \end{cases}$$

Therefore, a_n approaches $\frac{1}{3}$ as $n \to \infty$ through even values and approaches $\frac{2}{3}$ as $n \to \infty$ through odd values.

This is essentially the method used in the solution published by Gantner (reference 3), without however identifying the underlying difference equation.

Generating function

One of the standard methods for solving linear difference equations is to set up a generating function, and it works quite well on this example. Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Then

$$xf(x) = \sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=2}^{\infty} a_{n-1} x^n$$

and

$$f(x) = \sum_{n=1}^{\infty} \{\tfrac{1}{2}a_{n-1} + \tfrac{1}{2}\delta(n)\}x^n = \tfrac{1}{2}xf(x) + \frac{1}{2}\,\frac{x}{1-x^2}.$$

Therefore

$$(1-\tfrac{1}{2}x)f(x) = \frac{1}{2}\,\frac{x}{1-x^2}.$$

Solving for $f(x)$ we get

$$f(x) = \frac{1}{2}\,\frac{x}{(1-\frac{1}{2}x)(1-x^2)}.$$

A partial-fraction decomposition of this and expansion of the resulting terms in geometric series give

$$f(x) = \sum_{n=1}^{\infty} \{(-\tfrac{1}{3})(\tfrac{1}{2})^n + \tfrac{1}{2} - \tfrac{1}{6}(-1)^n\}x^n.$$

Therefore

$$a_n = -\tfrac{1}{3}(\tfrac{1}{2})^n + \tfrac{1}{2} - \tfrac{1}{6}(-1)^n$$

and the cluster points are $\frac{1}{3}$ and $\frac{2}{3}$.

Eigenvalues and eigenvectors

For readers familiar with eigenvalues and eigenvectors there is an elegant solution obtained from a homogeneous equation derived from the original non-homogeneous one. This method also brings out the central role eigenvalues play in such problems. See Lax (reference 5) and Strang (reference 7).

We start with

$$a_n = \tfrac{1}{2}a_{n-1} + \tfrac{1}{2}\delta(n), \qquad a_{n+2} = \tfrac{1}{2}a_{n+1} + \tfrac{1}{2}\delta(n+2).$$

Subtracting the first equation from the second, we get the homogeneous linear difference equation

$$a_{n+2} = \tfrac{1}{2}a_{n+1} + a_n - \tfrac{1}{2}a_{n-1} \quad (n > 2)$$

with initial conditions $a_0 = 0$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{4}$. We can write this in matrix form as

$$X_0 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \qquad X_n = AX_{n-1} \quad (n \geqslant 1),$$

with

$$X_n = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}.$$

The eigenvalues of A are $\frac{1}{2}$, 1 and -1, and

$$E^{(1)} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, \qquad E^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad E^{(3)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

are corresponding eigenvectors. Writing X_0 as a linear combination of eigenvectors, we have

$$\begin{aligned} X_n = A^n X_0 &= A^n(-\tfrac{1}{3}E^{(1)} + \tfrac{1}{2}E^{(2)} - \tfrac{1}{6}E^{(3)}) \\ &= -\tfrac{1}{3}(\tfrac{1}{2})^n E^{(1)} + \tfrac{1}{2}E^{(2)} - \tfrac{1}{6}(-1)^n E^{(3)}. \end{aligned}$$

Therefore, as in the last section,

$$a_n = -\tfrac{1}{3}(\tfrac{1}{2})^n + \tfrac{1}{2} - \tfrac{1}{6}(-1)^n$$

and the cluster points are $\frac{1}{3}$ and $\frac{2}{3}$.

Lax (reference 5) also has an example of a linear difference equation for which some modifications are necessary because of repeated eigenvectors. I have checked that this method, with modifications for repeated eigenvalues and non-homogeneity, works for the problem of counting triangles mentioned above, but with horrendous computations. This adds one more to the multitude of methods discussed by Larsen (reference 4) for counting triangles.

Exponential generating function

Yet another way of setting up a generating function is to use an exponential generating function as in Larsen (reference 4). Let

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

Then

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \{\tfrac{1}{2}a_n + \tfrac{1}{2}\delta(n+1)\} \frac{x^n}{n!} \\ &= \tfrac{1}{2}f(x) + \tfrac{1}{4}(e^x + e^{-x}). \end{aligned}$$

Therefore we get a differential equation

$$f'(x) - \tfrac{1}{2}f(x) = \tfrac{1}{4}(e^x + e^{-x})$$

whose solution can be obtained by any of the standard methods for solving differential equations as

$$\begin{aligned} f(x) &= ce^{\frac{1}{2}x} + \tfrac{1}{2}e^x - \tfrac{1}{6}e^{-x} \\ &= \sum_{n=0}^{\infty} \{c(\tfrac{1}{2})^n + \tfrac{1}{2} - \tfrac{1}{6}(-1)^n\} \frac{x^n}{n!}. \end{aligned}$$

As $a_0 = 0$, $c = -\frac{1}{3}$ and we get once more

$$a_n = -\tfrac{1}{3}(\tfrac{1}{2})^n + \tfrac{1}{2} - \tfrac{1}{6}(-1)^n$$

and the cluster points are $\frac{1}{3}$ and $\frac{2}{3}$.

Can we do more?

In all the above methods we have first found the explicit formula for a_n and then taken the limit as $n \to \infty$. All that we want to find out is the behaviour of the sequence of stops in the limit, so it should not be necessary to compute a_n explicitly. A clue as to how we can get the cluster points without computing a_n lies in the fact that it does not matter what the initial point is. Indeed, if we repeat the computations in all of the above methods with an arbitrary a_0 instead of 0, the only difference in the formula for a_n would be a term $a_0(\frac{1}{2})^n$. This term approaches 0 as $n \to \infty$. We can show this directly and get what is perhaps the most elegant solution *and* a generalization of the problem.

Method of generalization

Suppose our vacillating mathematician starts at any arbitrary point and alternately goes half way to 1 and 0. If $\{c_n\}$ and $\{d_n\}$ are two such sequences of stops, with any two starting points c_0 and d_0,

$$c_n - d_n = \tfrac{1}{2}(c_{n-1} - d_{n-1}) = \ldots = (\tfrac{1}{2})^n(c_0 - d_0) \to 0$$

as $n \to \infty$. Therefore these two sequences have the same cluster points. If we start from $\frac{1}{3}$, the stops alternate between $\frac{1}{3}$ and $\frac{2}{3}$. Therefore, the cluster points are $\frac{1}{3}$ and $\frac{2}{3}$.

Some concluding remarks

1. As can easily be seen from this solution, the problem generalizes to higher dimensions also. A generalization to two dimensions could be the following. Suppose we have a mathematically trained (and hence indecisive? or just plain greedy?) donkey at any point in the plane, and the donkey alternates between going half way to each of two haystacks. Then, it is doomed to a sequence of stops with cluster points at the two points which trisect the line segment joining the two haystacks. (Our donkey is perhaps related to the animal historically known as Buridan's ass—see box.)

Buridan's ass

Jean Buridan was a 14th-century French scholastic philosopher who studied under William of Ockham and was rector of the University of Paris in 1328 and 1340. He wrote on topics in logic, metaphysics, physics and ethics, but he is best-known for his contribution to the theory of choice. The memorable image of the donkey standing equidistant between two bales of hay who dies of starvation because he has no reason to prefer one bale to the other has become associated with Buridan's name, although it did not originate with him.

2. An interesting exercise in this context would be to show without using the explicit formula for the nth term of a Fibonacci sequence that the limit of the ratio a_{n+1}/a_n of the Fibonacci sequence is $\frac{1}{2}(1+\sqrt{5})$ irrespective of the choice of initial values a_0 and a_1. For a method that makes use of the explicit formula see Azarian (reference 1).

Acknowledgement

I would like to thank Alex Kleiner for improvements in the presentation of this article.

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Intercepting Missiles

J. GILDER

A simple strategy to shoot down a missile that is traversing the parabola of safety.

It is well known that a projectile moving under the action of a uniform gravitational field traverses a parabola, and that if we launch a shell from a fixed point O with fixed initial velocity u in all possible directions in a fixed vertical plane we produce a family of parabolas. It is perhaps less well known that this family has an envelope which is also a parabola, the parabola of safety, and that this parabola can be traversed by a projectile, a missile say. This article shows that there is a simple rule that enables a gunner at O to fire a shell so as to intercept such a missile.

First, some revision. Let x and y be coordinates measured from O respectively horizontally northwards and vertically upwards. Then the equations of motion under constant acceleration tell us that after time t the coordinates of our shell will be

$$x = ut\sin\theta, \qquad y = ut\cos\theta - \tfrac{1}{2}gt^2,$$

where g is the acceleration due to gravity and our shell has been launched at an angle of declination θ clockwise from the vertical. Now $y = 0$ when $x = 0$ and $x = u^2\sin 2\theta/g$. The maximum value of the last expression is u^2/g, achieved when $\theta = \frac{1}{4}\pi$. We can call u^2/g the *range* of our gun at O. If our gun is fired vertically upwards, the energy equation gives the maximum height reached above O as half the range.

Eliminating t from our equations gives

$$y = x\cot\theta - \frac{gx^2\operatorname{cosec}^2\theta}{2u^2}.$$

If we adopt a unit of distance so that the range $u^2/g = \frac{1}{2}$ unit, we obtain

$$y = x\cot\theta - x^2(1+\cot^2\theta).$$

Regarding this as a quadratic in $\cot\theta$, the discriminant tells us that $y \leqslant \frac{1}{4}-x^2$, i.e. no trajectories are above the parabola $y = \frac{1}{4}-x^2$. That this is the envelope can be checked by finding the intersection between this parabola and our general trajectory. Doing so gives

$$\tfrac{1}{4} = x\cot\theta - (x\cot\theta)^2,$$

giving the *repeated* root $x = \frac{1}{2}\tan\theta$ and correspondingly

$$y = \tfrac{1}{4}(1-\tan^2\theta).$$

Thus each trajectory *touches* the parabola $y = \frac{1}{4}-x^2$, which confirms that this is indeed the envelope. At the point of intersection,

$$\frac{x}{y} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta.$$

Thus a shell launched at an angle of declination θ intersects the envelope at a point whose angle of declination is 2θ, the time of flight being $\sec\theta/2u$. We thus have the situation depicted in figure 1.

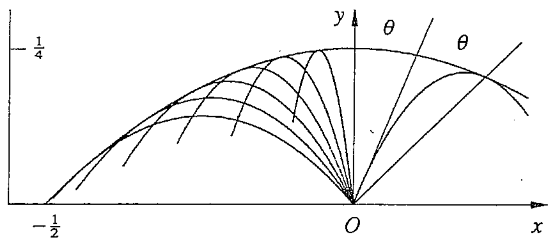


Figure 1

Now the envelope has gradient 1 at $(-\frac{1}{2},0)$ and, if a missile traverses this parabola, then to achieve a range of 1 unit it must pass through $(-\frac{1}{2},0)$ with a velocity of $u\sqrt{2}$, i.e. with a vertical component of velocity equal to u. It will therefore pass through $(0,\frac{1}{4})$ as well as $(\frac{1}{2},0)$, and so such a missile will indeed traverse the envelope.

If the missile is intercepted at $x = \frac{1}{2}\tan\theta$ then, as the horizontal component of the missile's velocity is u, the missile's x-coordinate at the moment of firing the shell from O is given by

$$x = \tfrac{1}{2}\tan\theta - u\frac{\sec\theta}{2u} = \frac{\sin\theta - 1}{2\cos\theta}$$

and correspondingly

$$y = \tfrac{1}{4}-x^2 = \frac{1}{4}\left(1-\frac{\sin^2\theta - 2\sin\theta+1}{\cos^2\theta}\right)$$
$$= \frac{2\sin\theta - 2\sin^2\theta}{4\cos^2\theta}$$
$$= \frac{\sin\theta(1-\sin\theta)}{2\cos^2\theta}.$$

Therefore $y/(-x) = \tan\theta$!

Thus, at the instant of firing the shell, the missile is at an elevation of θ above the negative x-axis. So our gunner has merely to mount his sights at right angles to the gun barrel. The situation is as shown in figure 2, where the dots represent positions at successive moments.

It is noticeable that the line joining missile and shell appears to have a constant gradient of $-\tan\theta$ for

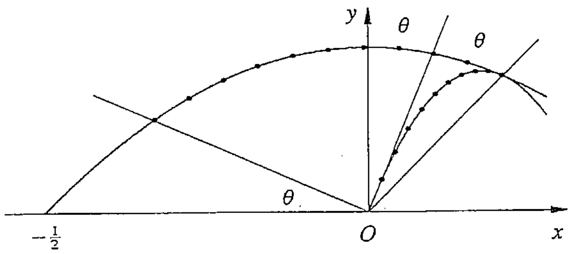


Figure 2

all time. Indeed this can be checked algebraically and bears out the nautical adage that two vessels with a constant relative bearing are on a collision course.

By considering the energy equations of the two projectiles, one can show that, at the moment of collision, the velocity of the shell is $\sin\theta$ times the velocity of the missile.

Two extreme cases can be easily checked. If the gun is fired vertically upwards when the missile is at $(-\frac{1}{2},0)$ then, as both the shell and the missile have the same vertical velocity, they will achieve their maximum height simultaneously, i.e. they will collide at $(0,\frac{1}{4})$. If the gun is fired when the missile is at $(0,\frac{1}{4})$ then it must be fired horizontally to match the missile's horizontal speed, but in this case you should not hold your breath while waiting for the collision! □

__John Gilder__ is a lecturer in pure mathematics at UMIST in Manchester, whose special interest is in combinatorics. One of his relaxations is listening to jazz of the late 1950s—this article was written mainly to the sound of Chet Baker and the Gerry Mulligan quartet.

A Theorem on the Face Angles of a Polyhedron

JOHN MACNEILL

Theorem. The sum of all the face angles of a polyhedron with E edges and F faces is $E-F$ revolutions.

Proof. Suppose that there are n_k k-sided faces, $k=3$, 4, 5, ... Then $F=\sum n_k$ and, since each edge is shared by two faces, $E=\frac{1}{2}\sum kn_k$. Now the sum of the interior angles of a k-sided face is $\frac{1}{2}(k-2)$ revolutions.

So the sum in revolutions of all the face angles is equal to

$$\begin{aligned}\tfrac{1}{2}\textstyle\sum n_k(k-2) &= \tfrac{1}{2}\textstyle\sum kn_k - \sum n_k\\ &= E-F,\end{aligned}$$

as required.

The theorem has an interesting corollary, which I actually discovered first. Before stating this corollary, two definitions are made.

Definition. The *buckle angle* of a vertex of a polyhedron is one revolution minus the sum of all the face angles at that vertex. So the buckle angle of a vertex where the polyhedron is concave or convex is positive. However, the buckle angle of a vertex where the polyhedron is neither concave nor convex (as may occur for instance on a stellated polyhedron) may be positive, zero or negative.

Definition. The *Euler characteristic* of a polyhedron with F faces, E edges and V vertices is $F-E+V$. It is known that every polyhedron which is topologically equivalent to a sphere has an Euler characteristic of 2 and every polyhedron which is topologically equivalent to a torus (anchor ring) has an Euler characteristic of 0.

Corollary. The sum in revolutions of the buckle angles of a polyhedron is the Euler characteristic of the polyhedron.

Proof. Working in revolutions,

$$\begin{aligned}&\text{sum of buckle angles}\\ &\quad= \text{sum for all vertices of}\\ &\qquad(1-\text{sum of face angles at vertex})\\ &\quad= V-\text{sum of all face angles}\\ &\quad= V-(E-F)\quad\text{(by the theorem)}\\ &\quad= F-E+V\end{aligned}$$

as required. □

__John MacNeill__ is an analyst programmer at the University of Warwick.

An Illuminating Program

R. S. THODY

What a school mathematics project can lead to.

My daughter was wandering around the house muttering what seemed to be dire oaths under her breath as she made drinks and disappeared again into her bedroom. Being a concerned parent and worrying about her mental health, I asked gingerly if there was anything wrong.

'Oh, no. I'm just sorting out my maths homework. It's an investigation.' 'An investigation? What's it all about?' (Quality time for parent's brownie points!)

'Well, there is this council and they are putting up lights along the promenade. The lights are linked up to a computer. There are a thousand lights and a thousand programs on the computer. Program 1 turns all the lights on. Program 2 turns off all the even-numbered lights. Program 3 alters the condition of any light whose number divides by 3. Program 4 alters the condition of any light whose number divides by 4 ... and so on ....'

'The problem is how many lights are on after all the programs have been run, which are they and is there any pattern to the numbers.'

'Yes, very good', I said, and walked away.

The trouble is, of course it stays in your head and you keep thinking about it.

'Well, if program 1 turns them on and then program 2 turns off the evens then after two programs all the odds are on! Brilliant, I've cracked this. Now all those divisible by 3 are going to be turned on ... no off ... no the opposite of whatever they are now after program 2. Soooooooo ... that means that 6 will be off now and on then and 9 will be on then and off now, so 1 is on, 2 is off, 3 is, ah!, 3 is on but now off again and so is 4 ....'

'No. I need to write this down. I'll draw little circles for the lights and then alter them after each program. I'll try four programs and see what happens, and then ten programs with ten lights ... and so on. I wonder what happens if I run fewer programs than lights or more programs than lights.'

You can see I was really getting into this.

By now I was launched into the investigation and eventually came up with the pattern of progressive squares.

All this set me thinking that as they had got a computer with a thousand programs on it why didn't somebody write one more program and then they could solve the problem on the computer! This would of course defeat the object of the thinking out bit of the exercise as the answer would be given. If, however, the program was designed to deal with as many lights and as many programs as required, then some purpose might be served as the pattern of numbers would be seen on the screen. It could also let students check very quickly differing numbers of combinations of programs and lights. For example students could see what would happen if they used only ten lights but ran twenty programs as against running ten programs for the same number of lights. They might realise that any program numbered after the last light's number will have no effect whatsoever on the number of lights switched on. They will also see that running fewer programs than there are lights will not 'activate' the progressive squares of numbers. This may then give the clue to the answer that any light which has an even number of divisors will be turned off and any which has an odd number of divisors will be turned on.

Further examination of the pattern could indicate that, provided the same number of programs as lights are run, the number of lights on remains the same until the next perfect square is reached and a formula dealing with integers is necessary.

I therefore decided that, although I am not a programmer, I would write a GWBASIC program to deal with the problem. It may be that the program is not particularly sophisticated, but it does work and in QBASIC also!

For anyone who is interested in using the program as a teaching aid, be my guest, it is printed below. You do not have to type in my details, but some acknowledgement would be appreciated! □

```
10 CLS
20 CLEAR
30 PRINT TAB(25) "THIS PROGRAM HAS BEEN DESIGNED by"
40 FOR N = 1 TO 4:PRINT
50 NEXT N
60 PRINT TAB(26) "R. S. THODY, M.Sc., F.C.A., M.I.M."
70 FOR N = 1 TO 4 :PRINT:NEXT N
80 PRINT TAB(21) "PRINCIPAL LECTURER IN ACCOUNTING AND FINANCE"
90 FOR N = 1 TO 4:PRINT:NEXT N
100 PRINT TAB(25) "DE MONTFORT UNIVERSITY, LEICESTER"
110 FOR N = 1 TO 20000:NEXT N
120 CLS
```

```
130 PRINT "The program is designed to calculate the number of lights either on or off to"
131 PRINT "answer the following mathematics investigation. The program has been"
132 PRINT "adapted to deal with any number of lights and any number of programs."
140 PRINT
150 PRINT "At a seaside town the council has decided to put up a number of lights along"
151 PRINT "the sea front. The whole system is to be linked to a computer which can have as"
152 PRINT "many numbered programs as you like. Program number 1 switches all the lights on."
160 PRINT "Program number 2 switches all even numbered lights off."
161 PRINT "Program number 3 alters the condition of any light which has a number"
162 PRINT "divisible by 3. Program number 4 alters the condition of any light which has"
163 PRINT "a number divisible by 4 ... and so on ... ."
180 PRINT "The idea is to find out how many lights are on or off, and which they are,"
181 PRINT "after running any number of programs."
190 PRINT "Obviously the more lights and programs you have the longer will it take to"
191 PRINT "come up with an answer."
200 FOR N = 1 TO 2000:NEXT N
220 CLEAR
230 FOR N = 1 TO 3:PRINT:NEXT N
240 PRINT "Enter how many lights there are in total."
241 PRINT "If you wish to quit the program enter 9999."
250 INPUT L
260 IF L = 9999 THEN CLS:GOTO 630
270 PRINT "Enter the number of programs you wish to run."
280 INPUT K
290 CLS
300 FOR N = 1 TO 2:PRINT:NEXT N
305 PRINT "You have just asked for ";L;" LIGHTS and ";K;" PROGRAMS."
310 PRINT "Is this information correct?"
320 PRINT "Enter Y or N":A$ = INPUT$(1)
330 IF A$ = "n" THEN 220 ELSE IF A$ = "N" THEN GOTO 220
340 IF A$ = "y" THEN GOTO 360 ELSE IF A$ = "Y" THEN GOTO 360
350 CLS:PRINT "Please enter Y or N. You have just pressed ";A$:GOTO 300
360 DIM LIGHT(L)
365 PRINT K " programs running through Light No: "
370 FOR N = 1 TO L
375 PRINT N;
380 LIGHT(N) = 0
390 FOR T = 1 TO K
400 Y = N/T
410 Z = INT(Y)
420 IF Y = Z THEN IF LIGHT(N) = 0 THEN LIGHT(N) = 1 ELSE IF LIGHT(N) = 1 THEN LIGHT(N) = 0
430 NEXT T
440 NEXT N
450 LIGHT = 0
460 FOR N = 1 TO L
470 LIGHT = LIGHT + (LIGHT(N))
480 NEXT N
490 FOR N = 1 TO 3:PRINT:NEXT N
500 PRINT "LIGHTS ON = " LIGHT "THEREFORE LIGHTS OFF = "(L) - LIGHT
510 PRINT
520 PRINT "The following lights are on: "
530 FOR N = 1 TO L
540 IF LIGHT(N) = 1 THEN PRINT N;
550 NEXT N
560 PRINT
570 PRINT "Do you want to try another combination?  Please enter Y or N."
580 A$ = INPUT$(1)
590 IF A$ = "N" THEN GOSUB 630 ELSE IF A$ = "n" THEN GOSUB 630
600 IF A$ = "Y" THEN 220 ELSE IF A$ = "y" THEN GOTO 220
610 CLS:PRINT "Please enter Y or N.  You have just pressed ";A$:GOTO 570
630 CLS:PRINT "O.K.  Perhaps another day.  Bye."
640 END
```

__Roy Thody__ is principal lecturer in accounting and finance at De Montfort University, Leicester. He is a chartered accountant who has served in articles, the Royal Navy, industry and education. His interest in mathematics stems from having taught financial management for more years than he cares to remember. He keeps sane by playing the melodeon and dancing the morris.

Graphic Calculators and the Mandelbrot Set

JAMES HORTH

Having encountered the Mandelbrot set and other similar work in the 'Complex Numbers' unit of the 16–19 Further Mathematics course, I decided to have a go at programming my Casio FX 7000 to plot images of the set. I found that the calculator, although reluctantly slow, was fully able to complete the task.

The Mandelbrot set is the set of complex numbers which when iterated by a certain mapping again and again leaves the numbers still bounded. Since these numbers are complex, they can be expressed in the form $A+\mathrm{i}B$, where i is the square root of -1. These values of A and B can then be used to plot each number on Cartesian axes and the resulting plot is known as an Argand diagram. The familiar image of Mandelbrot 'bugs' repeating on all scales are the result of 'zooming in' on minute portions of the Argand diagram.

Programming the Mandelbrot set

The Mandelbrot set is

$$M = \{C \in \mathbb{C} : Z_0 = 0, Z_{n+1} = Z_n^2 + C \text{ is bounded}\}.$$

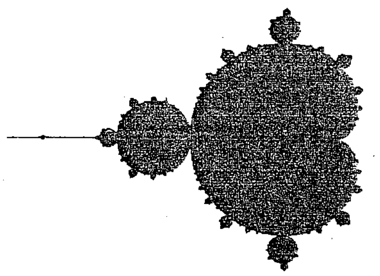
For example, if $C = 0$ we have $Z_1 = Z_2 = 0$. One can see that subsequent iterations will continue to output 0. Since this is a bounded sequence, we deduce that the origin is in the Mandelbrot set. If we take $C = 1$,

$$Z_1 = 0^2+1 = 1, \quad Z_2 = 1^2+1 = 2, \quad Z_3 = 2^2+1 = 5.$$

The sequence continues: 26, 677, 458 330, 2.1×10^{11}, …, rapidly tending towards infinity. Thus 1 is not in the set. In order to program this formula for complex numbers, exactly the same process of squaring takes place. However, my calculator was unfortunately unable to cope with complex numbers by itself and so I had to tell it how to deal with the real and imaginary coefficients from scratch.

If we start with the complex number $A+\mathrm{i}B$, then squaring this has this result:

$$(A+\mathrm{i}B)^2 = A^2+2\mathrm{i}AB+(\mathrm{i}B)^2 = A^2+2\mathrm{i}AB-B^2.$$



The Mandelbrot set. ©1985 MAP ART. Reproduced with permission.

If the starting term of our sequence was $C+\mathrm{i}D$ then each subsequent value of A and B is given by

$$A_{n+1} = A_n^2 - B_n^2 + C, \qquad B_{n+1} = 2A_nB_n + D.$$

It is then a simple matter to make the calculator iterate the sequence of figures and test whether it exceeds certain boundaries, or remains within them after a fixed number of iterations. It should be obvious that the more iterations performed, the closer the figure formed resembles the ideal figure (which would require an infinite number of calculations for each point to be plotted). The listing for the program I wrote runs as follows.

The Mandelbrot set

```
"Y MAXIMUM"?->K
"Y MINIMUM"?->L
"X MAXIMUM"?->N
"X MINIMUM"?->M
"NO. OF ITERATIONS"?->Q
Range M+50, N+50, 0, L+50, K+50, 0
M->C
K->D
Goto 0

Lbl 1
C+(N-M)/95->C
Lbl 0
0->P
C>N=>D-(K-L)/65->D
C>N=>M->C
D<0=>D>-K=>-K-D
D<L=>Goto 4
C->Z:D->J
Goto 2

Lbl 2
P+1->P
Z²-J²+C->A
2ZJ+D->B
A->Z:B->J
Abs B>2=>Goto 1
Abs A>3=>Goto 1
P>Q=>Goto 3
Goto 2

Lbl 3
Plot C+50, D+50
Plot C+50, 50-D
Goto 0

Lbl 4
"END OF PLOT"
```

After typing in the program, run it, and try the range parameters as follows.

```
Y MAXIMUM?0.9
Y MINIMUM?-0.9
X MAXIMUM?0.55
X MINIMUM?-2.1
```

For ten iterations it can take about half an hour to plot the image; for 100, you may have to leave the calculator running overnight! If you wish to look closer at small regions of the set, the 'zoom' function cannot be used, as would be usual for plots of normal functions; instead you must use the 'trace' function to find the maximum and minimum points of the smaller area to be looked at. Having noted down these figures, 50 must then be subtracted from each as, in order to keep the x and y axes away from the image, I first had to translate the image by 50 in both the x and y directions. When we rerun the program with these new coordinates, it is likely that more iterations will be needed to produce an image of comparable definition to the image preceding, which was on a larger scale.

Julia sets

The program I have given can also be modified to produce images of a related set of fractals, known as Julia sets, after the Frenchman Gaston Julia who first examined them. We again use the $Z_{n+1} \to Z_n^2 + C$ iteration, but this time, instead of C being the value of the point which is being examined, it is a constant value for the whole set. The Julia set for the value C is defined by

$$J_C = \{Z \in \mathbb{C} : Z_0 = Z, Z_{n+1} = Z_n^2 + C \text{ is bounded}\}.$$

There are thus an infinite number of different Julia sets, many only very subtly different, one for each value of C. In order to decide whether a point is in a particular Julia set, one first picks the complex number C for that set and then assigns for the value of Z_0 the value of the point that is being examined. So, for example, if I were to examine whether, for the Julia set defined by $C = 0.5$, the point 0 were in the set, then:

$$Z_0 = 0, \qquad Z_1 = 0^2 + 0.5 = 0.5,$$
$$Z_2 = 0.5^2 + 0.5 = 0.75, \quad Z_3 = 0.75^2 + 0.5 = 1.0625.$$

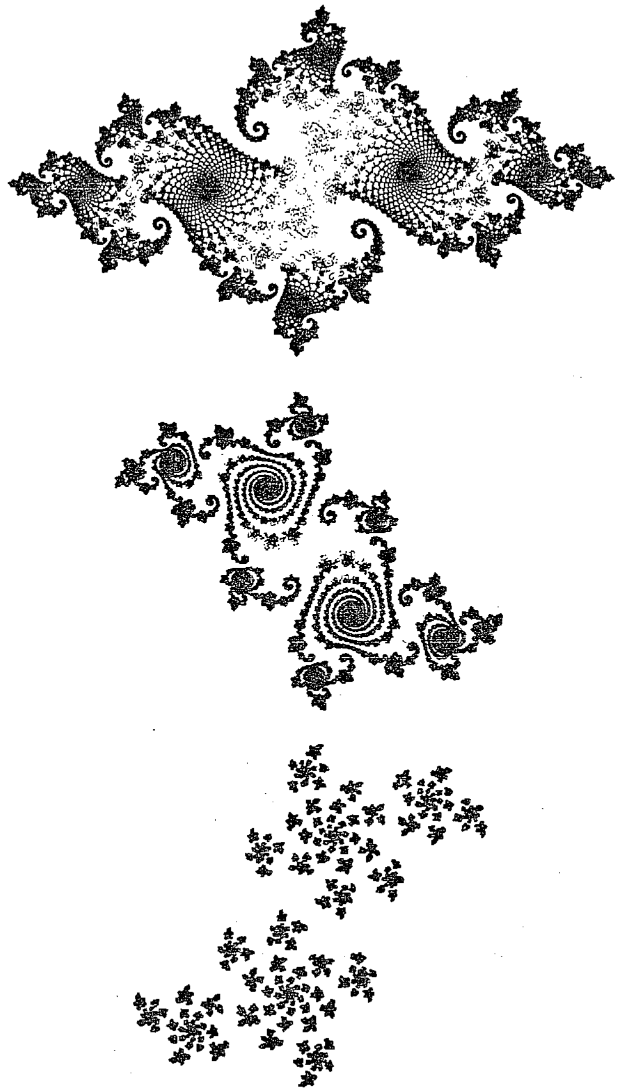
The sequence continues 1.629, 3.15, 10.4, 110, 1200, $1.44 \times 10^8, \ldots$. The point 0 is therefore not inside the set.

In order to convert the program to drawing Julia sets the following lines must be altered or added to the Mandelbrot program. After `"NO. OF ITERATIONS"?->Q` add

```
"COORDINATES OF SET:"
"REAL VALUE"?->X
"IMAGINARY VALUE"?->Y
```

Replace `Z²-J²+C->A` with `Z²-J²+X->A`, `2ZJ+D->B` with `2ZJ+Y->B` and `Plot C+50, 50-D` with `Plot 50-C, 50-D`.

Finally, the maximum and minimum x/y values should be set to 1.2 and -1.2, respectively. With this program in the memory, the Julia sets themselves can be explored. One of my particular favourite images is the Julia set with $C = -0.11 + 0.6557i$.



Examples of Julia sets. ©1985 MAP ART. Reproduced with permission.

Further exploration

There are many interesting connections between the Mandelbrot and Julia sets, not least the endless fractal properties on all scales. One such property is that the Mandelbrot set is a 'map' for the Julia sets. Any Julia set with coordinates inside the Mandelbrot set is 'connected', whilst Julia sets with coordinates corresponding to points outside the Mandelbrot set are 'totally disconnected'. This property becomes apparent in Julia sets with coordinates near the Mandelbrot boundary only at large magnifications. However, as the coordinates move farther and farther outside the boundary, the sets become visibly more fragmented, until at a distance of about 5 from the origin, the set is a very sparse set of points.

By varying the initial mapping from $Z \to Z^2 + C$ to other mappings and working out the new mapping's effect on the real and imaginary parts of Z, you can program your own iteration. However, if the mapping is to be altered, there are a few things which must be changed from the Mandelbrot program. The second plot line must be removed (`Plot C+50, 50-D` in the Mandelbrot program), as must the line reading `D<0=>D>-K=>-K-D`, as these lines speed up the plotting of the Mandelbrot and Julia sets relying on their *symmetrical* properties; other mappings may not have such symmetries and so leaving them in could plot a misleading diagram.

Further values for nice-looking images of Julia sets are suggested in the 16–19 Complex Numbers text, but in my opinion it is only through our own exploration that we can really appreciate the complexity and subtle variety of the Julia and Mandelbrot sets. □

When this article was written ***James Horth*** *was a sixth-form student at Ampleforth College, York.*

Computer Column

The World Wide Web

For over a year this site has had a World Wide Web page. However, we have been keeping quiet about it, because the Applied Probability Trust, those wonderful people who make *Mathematical Spectrum* possible, were not on WWW! However, now they are, so here is the address:

`http://www.shef.ac.uk/~apt.`

We have a linked page for each publication, including *Mathematical Spectrum.*

From here, you can branch off to the School of Mathematics and Statistics (SoMaS) page at Sheffield, `http://www.shef.ac.uk/~ms`, which currently holds details about undergraduate and postgraduate opportunities at Sheffield, links to other interesting sites, road maps, etc., plus pages on staff members (rudimentary at present). Or you could branch off to the University home page and explore other departments here, or even other mathematics departments round the world.

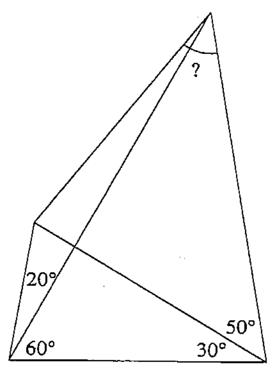
I am sure that readers of this magazine will be well aware of the explosive interest in the WWW over the past year. You may not be aware, though, of the steps involved in making information available this way. As an example, the bulk of the information about SoMaS was available in Postscript files, comprising our undergraduate brochure, which I stripped of Postscript markup last Easter, substituting HTML markup and links, and later re-coding the tables when HTML table coding became stable. Postgraduate information was in LaTeX form—all mathematicians use LaTeX/TeX don't they?—and the markup there was easily converted to HTML.

Then there were the various images, in this case mostly done using Paintbrush and TeX, and converted to transparent `.gif` format.

As you will see as you watch any page, things change rapidly, so keep watching SoMaS and the APT for any new developments. **Mike Piff**

$$\int_{\int_{\int_{e^{1994}}^{e^{1995}}\frac{1}{x}\,dx}^{\int_{1994}^{1995}dx}\frac{1}{x}\,dx}^{\int_{\int_{e^{1994}}^{e^{1995}}\frac{1}{x}\,dx}^{\int_{1}^{e^{e^{1995}}}\frac{1}{x}\,dx}\frac{1}{x}\,dx} dx = ?$$

Farshid Arjomandi
(Physics Student,
University of California,
Santa Barbara)



Can any reader find the angle shown without using trigonometry?

Junji Inaba
(Student, William Hulme's
Grammar School, Manchester)

Mathematics in the Classroom

The aim of this regular feature is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

Educational software

The extent to which IT is used in the teaching of A-level Mathematics is likely to be very variable. Most of us teachers feel an obligation to regularly update our own IT skills but the rate at which new packages are coming on to the market coupled with pressures of time, both conspire against this. We also feel the need to ensure that students have an appropriate experience of the technology available, but what constitutes appropriate is not always clear. If the package is to be used as a teaching tool in which the drudgery is removed from the mathematics, then it is important that the student remains aware of what the equations mean and how to interpret the results, otherwise disaster could occur in the examination room where computers do not yet have a place.

However, achieving these objectives is not easy because I do not consider myself to be a very computer-literate teacher. So, if a package is to work for me, it must be simple to grasp without needing the investment of hours of time to achieve its mastery. Therefore when some literature arrived describing MathPlus as a package that had been 'developed to fill a niche in the education market for secondary-school students aged 14 years and up until second-year degree students', which furthermore claimed to 'create, solve and manipulate equations on screen easily and accurately', it was an interesting prospect. With the promise that 'you do not need to learn a programming language to use it' alongside the description of an 'intuitive package which includes more than 250 math functions', it became impossible to refuse the invitation to try it out.

MathPlus was introduced by Waterloo Maple Software, known for their development work with a product called Maple. (Maple V version 3 was well reviewed in a recent issue of *Personal Computer World* (June 1995) where it was described as a cheerful and excellent package which every school and college with a serious maths department should have available on computer in the library, notwithstanding reservations about the documentation.) Although I have not yet tried out all 250 maths functions available in MathPlus, in the areas that I have explored, I have found it to be a very user-friendly, versatile, intriguing and fascinating package with pleasing colourful graphics.

It will generate graphs speedily and then allow you to rotate three-dimensional graphs simply by clicking and dragging the mouse in the desired direction so that they can be viewed from all angles, an impossibility with just pen and paper. It will enable all sorts of calculations to be carried out including those which require abstract rather than numerical answers. It can integrate, including multiple integrals and by parts integration (but requires the user to choose the $u(x)$ and $dv(x)/dx$ functions, hence demanding an understanding of the process), and differentiate both fully and partially. It will create a Taylor series approximation of a smooth function, solve first-order differential equations and linear second- and higher-order ones and can even cope with Bessel's differential equation. The inversion of matrices and evaluation of determinants is covered and, for more advanced work, it will calculate Fourier transforms and generate Chebyshev polynomials.

It is certainly very easy to use and the producers are justly proud of their 'click and solve' technology which enables you to move expressions and terms around the screen. Solutions are carried out with each step labelled automatically as performed.

If you want to try this package out, it is offered on a 30-day money-back guarantee so that teachers can evaluate it for themselves. It has certainly provided my department with a lot of interest and fun, and although we have not yet made it accessible to students (many of whom are more computer-literate than we are and would no doubt unwrap the many facets of MathPlus much faster than we have) it seems well worth taking a look at, although you need to be clear on what you would be hoping to gain by using such packages.

If you are currently using a package that you value as a teaching tool, do write and tell us about it.

Technical details

MathPlus is currently available for Windows (3.1 and NT), Macintosh and PowerMac and will soon be available on CD-ROM and for Unix systems, and is priced at £199 for a single user. The UK distributors are Robinson Marshall Europe Plc, Nadella Building, Leofric Business Park, Progress Close, Coventry CV3 2TF.

Carol Nixon □

Letters to the Editor

Dear Editor

The sequence $\{a_n\}$ with $a_n = 2^n + n^2$ ($n \in \mathbb{N}$)
(see Problem 28.4)

I am interested in how many prime numbers there are in this sequence. Clearly a_n is not prime for even n. Also, for odd n not divisible by 3, $n^2 \equiv 1 \pmod 3$ so $a_n \equiv (-1)^n + 1 \equiv 0 \pmod 3$ and the only prime is $a_1 = 3$. Thus we need only consider those n of the form $n = 6k+3$ for $k = 0, 1, 2, \ldots$.

Among these values, $a_3 = 17$, $a_9 = 593$, $a_{15} = 32\,993$, $a_{21} = 2097\,593$ and a_{33} are all prime. However, there are infinitely many composite numbers in the sequence $\{a_{6k+3}\}$. To see this, we first note that all numbers in the sequence $\{a_{136h+3}\}$ ($h = 0, 1, 2, \ldots$) are divisible by 17. For $136 = 8\times 17$ and $2^8 \equiv 1 \pmod{17}$, so $2^{136h+3} + (136h+3)^2 \equiv 2^3 + 3^2 \equiv 0 \pmod{17}$. Thus, if k is a multiple of 68, then a_{6k+3} is a multiple of 17 ($= a_3$). There are also infinitely many multiples of a_9, a_{15}, a_{21} and a_{33} in this sequence.

If any readers know of a prime distribution in the sequence $\{a_n\}$, I would be very pleased to hear of it.

Yours sincerely,
KENICHIRO KASHIHARA
(Kamitsuruma 4-13-15,
Sagamihara, Kanagawa 228, Japan)

Dear Editor,

The Smarandache function—1

I read the letter by I. M. Radu that appeared in Volume 27 Number 2 stating that there is always a prime between $S(n)$ and $S(n+1)$ for all numbers $0 < n < 4801$, where $S(n)$ is the Smarandache function. $S(n)$ is defined as the smallest number n such that $S(n)!$ is divisible by n.

Since I have a computer program that computes the values of $S(n)$, I decided to investigate the problem further. The search was conducted up through $n < 1033\,197$ and four instances were found where there is no prime p such that $S(n) \leqslant p \leqslant S(n+1)$. They are as follows:

$$n = 224 = 2^5\times 7, \qquad S(n) = 8,$$
$$n = 225 = 3^2\times 5^2, \qquad S(n) = 10;$$

$$n = 2057 = 11^2\times 17, \qquad S(n) = 22,$$
$$n = 2058 = 2\times 3\times 7^3, \qquad S(n) = 21;$$

$$n = 265\,225 = 5^2\times 103^2, \qquad S(n) = 206,$$
$$n = 265\,226 = 2\times 13\times 101^2, \qquad S(n) = 202;$$

$$n = 843\,637 = 37\times 151^2, \qquad S(n) = 302,$$
$$n = 843\,638 = 2\times 19\times 149^2, \qquad S(n) = 298.$$

As can be seen, the first two values contradict the assertion made by I. M. Radu in his letter. Note that the last two cases involve pairs of twin primes. This may provide a clue in the search for additional solutions.

Yours sincerely,
CHARLES ASHBACHER
(Decisionmark, 200 2nd Ave Suite 300,
Cedar Rapids, IA 52401, USA)

Dear Editor

The Smarandache function—2

We show that, when $n \geqslant 27$ is a composite number, there are at least five prime numbers between $S(n)$ and n.

T. Yau proved that, when $n \geqslant 10$ is a composite number, $S(n) \leqslant \frac{1}{2}n$. To see this, we consider two cases. If $n = pq$, where $p, q > 1$ and $(p, q) = 1$, then $S(n) = \max\{S(p), S(q)\}$. Now $S(p) \leqslant p = n/q \leqslant \frac{1}{2}n$ and similarly $S(q) \leqslant \frac{1}{2}n$, so $S(n) \leqslant \frac{1}{2}n$. If $n = p^r$, where p is a prime and $r \geqslant 2$, then $S(n) \leqslant pr \leqslant \frac{1}{2}p^r = \frac{1}{2}n$. The last inequality requires $n \geqslant 10$.

The Bertrand–Tchebychev postulate/theorem tells us that there exists at least one prime between $\frac{1}{2}n$ and n. Hence, when $n \geqslant 10$ is composite, there is a prime between $S(n)$ and n. (We can check this directly for n down to 6. But note that $S(4) = 4$.) This was pointed out by Stuparu. But we can improve on this if we apply Breusch's theorem, which says that, when $n \geqslant 48$, there is a prime strictly between n and $\frac{9}{8}n$. Since $(\frac{9}{8})^5 < 2$ but $(\frac{9}{8})^6 > 2$, this means that there are at least five prime numbers between n and $2n$. Hence, when $n \geqslant 96$, there exist at least five prime numbers between $\frac{1}{2}n$ and n and so, when $n \geqslant 96$ is composite, there are at least five primes between $S(n)$ and n. We can now check the numbers down to 27. But $S(26) = 13$, and there are only four primes between 13 and 26.

Yours sincerely,
L. SEAGULL
(Glendale Community College,
Arizona, USA)

Dear Editor

The Smarandache function and the Fibonacci relationship

In Volume 26 Number 3 page 85, T. Yau asked for which triplets $n, n+1, n+2$ the Smarandache function satisfies the Fibonacci relationship

$$S(n) + S(n+1) = S(n+2).$$

Two solutions

$$S(9) + S(10) = S(11), \qquad \text{i.e. } 6+5 = 11;$$
$$S(119) + S(120) = S(121), \qquad \text{i.e. } 17+5 = 22$$

were found, but no general solution was given.

To investigate this problem further a computer program was written that tested all values for n up to 1000 000. This gave the solutions, with the prime factorisations of n, $n+1$ and $n+2$, shown in the box at the top of page 21.

I am unable to discern a pattern in these numbers that would lead to a proof that there is an infinite family of solutions. Perhaps another reader will be able to do so.

Yours sincerely,
CHARLES ASHBACHER
(Decisionmark, 200 2nd Ave Suite 300,
Cedar Rapids, IA 52401, USA)

$S(9)+S(10) = S(11)$ $6 + 5 = 11$	$9 = 3^2$, $10 = 2\times5$, $11 = 11$;
$S(119)+S(120) = S(121)$ $17 + 5 = 22$	$119 = 7\times17$, $120 = 2^3\times3\times5$, $121 = 11^2$;
$S(4900)+S(4901) = S(4902)$ $14 + 29 = 43$	$4900 = 2^2\times5^2\times7^2$, $4901 = 13^2\times29$, $4902 = 2\times3\times19\times43$;
$S(26\,243)+S(26\,244) = S(26\,245)$ $163 + 18 = 181$	$26\,243 = 7\times23\times163$, $26\,244 = 2^2\times3^8$, $26\,245 = 5\times29\times181$;
$S(32\,110)+S(32\,111) = S(32\,112)$ $26 + 197 = 223$	$32\,110 = 2\times5\times13^2\times19$, $32\,111 = 163\times197$, $32\,112 = 2^4\times3^2\times223$;
$S(64\,008)+S(64\,009) = S(64\,010)$ $127 + 46 = 173$	$64\,008 = 2^3\times3^2\times7\times127$, $64\,009 = 11^2\times23^2$, $64\,010 = 2\times5\times37\times173$;
$S(368\,138)+S(368\,139) = S(368\,140)$ $151 + 82 = 233$	$368\,138 = 2\times23\times53\times151$, $368\,139 = 3\times41^2\times73$, $368\,140 = 2^2\times5\times79\times233$;
$S(415\,662)+S(415\,663) = S(415\,664)$ $146 + 167 = 313$	$415\,662 = 2\times3\times13\times73^2$, $415\,663 = 19\times131\times167$, $415\,664 = 2^4\times83\times313$.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

28.1 *Pre-Fermat's last theorem!* The positive integers x, y and z satisfy the equation $x^n+y^n = z^n$, where n is a positive integer greater than 1. Prove that, if the differences between z and x and z and y are greater than 1, then neither x nor y can be prime.
(Submitted by Lee Talbot, student at De Montfort University.)

28.2 Prove that

$$x \ln a \geqslant \ln x + \ln(\ln a) + 1,$$

where $a > 1$ and $x > 0$, and determine when equality occurs.
(Submitted by Junji Inaba, student at William Hulme's Grammar School, Manchester.)

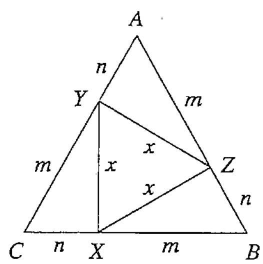
28.3 An *exotic number* is a natural number which can be expressed using each of its own digits just once and the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$, !, brackets and powers. Thus, for example, $120 = [(2+1)!-0!]!$ is exotic. Show that, if $n_1n_2\ldots n_r$ is exotic, then so is $n_1n_2\ldots n_r9765\,625$, so that there are infinitely many exotic numbers.
(Submitted by Filip Sajdak, student at the University of Auckland, New Zealand.)

28.4 Find all natural numbers n such that 2^n+n^2 is a perfect square.
(Submitted by Kenichiro Kashihara, Kanagawa, Japan.)

Solutions to Problems in Volume 27 Number 2

27.5 Let X, Y and Z be points on sides BC, CA and AB, respectively, of an equilateral triangle ABC with $BX = CY = AZ = m$ units and $XC = YA = ZB = n$ units, where m and n are integers that are prime. Prove that, if $\triangle XYZ$ has integral sides, then X, Y and Z are the midpoints of the sides of $\triangle ABC$.

Solution by Toby Gee, Frome Community College



Triangles AYZ, BZX, CXY are congruent, so $XY = YZ = ZX = x$ (say). The cosine rule in $\triangle CXY$ gives

$$\begin{aligned} x^2 &= m^2+n^2-2mn\cos 60^\circ \\ &= m^2+n^2-mn \\ &= (m-n)^2+mn, \end{aligned}$$

so

$$mn = x^2-(m-n)^2 = (x+m-n)(x-m+n).$$

We may assume without loss of generality that $m \geqslant n$. (If $n > m$, turn the diagram over.) Since x is an integer and m and n are prime, there are two cases.

Case 1: $x+m-n = m$ and $x-m+n = n$.
Then $x-n = 0 = x-m$, so $m = x = n$ and X, Y and Z are the midpoints of the sides of $\triangle ABC$.

Case 2: $x+m-n = mn$ and $x-m+n = 1$.
Then $x = mn+n-m$. Now $m+n > x$, and so $m+n > mn+n-m$, whence $2m > mn$ and $2 > n$. But n is prime, so this case is impossible.

Also solved by Junji Inaba, Vytautas Paskunas (Christ's Hospital School, Horsham), Richard Edlin (Westminster School), Can Minh (University of California, Berkeley).

27.6 Find all real-valued functions satisfying the functional equation

$$f(x)+f\left(\frac{1}{1-x}\right) = \frac{x}{x-1} \tag{1}$$

for all $x \in \mathbb{R}$ ($x \neq 0, 1$).

Solution by Vytautas Paskunas

Let $x \neq 0, 1$. Then $(x-1)/x \neq 0, 1$, so, if we replace x by $(x-1)/x$ in (1) we obtain

$$f\left(\frac{x-1}{x}\right)+f(x) = 1-x. \tag{2}$$

Subtract (1) from (2) to give

$$f\left(\frac{x-1}{x}\right)-f\left(\frac{1}{1-x}\right) = -x-\frac{1}{x-1}. \tag{3}$$

We now replace x by $(x-1)/x$ in (3) to give

$$f\left(\frac{1}{1-x}\right)-f(x) = -\frac{x-1}{x}+x. \tag{4}$$

We now subtract (4) from (1) to give

$$2f(x) = \frac{x}{x-1}+\frac{x-1}{x}-x. \tag{5}$$

Conversely, if f is given by (5), then it is easily verified that (1) holds.

Also solved by Junji Inaba, Toby Gee, Can Minh, Richard Edlin, Neil Pollock (University of Aberdeen).

27.7 What is the maximum possible degree of a polynomial in t given by the $n \times n$ determinant

$$\left|\begin{array}{ccc|ccc} t & & & & & \\ & \ddots & & & & \\ & & t & & & \\ \hline & & & 0 & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \end{array}\right| \begin{array}{c} \\ \\ \\ \uparrow \\ m \\ \downarrow \end{array}$$
$$\qquad\qquad\qquad \leftarrow \; m \; \rightarrow$$

where the unspecified entries are arbitrary constants?

Solution

The determinant is a sum of terms which are $\pm$ a product of terms one from each row and one from each column. To obtain a non-zero product, the terms chosen from the last m rows must come from the first $n-m$ colums, so m of the t's cannot be chosen. Thus the maximum number of t's that can be chosen to give a non-zero product is $n-2m$, so the degree cannot exceed $n-2m$. On the other hand,

$$\left|\begin{array}{ccc|ccc|ccc} t & & & & & & & & \\ & \ddots & & & & & & & \\ & & t & & & & & & \\ \hline & & & t & & & 1 & & \\ & & & & \ddots & & & \ddots & \\ & & & & & t & & & 1 \\ \hline & & & 1 & & & 0 & \cdots & 0 \\ & & & & \ddots & & \vdots & \ddots & \vdots \\ & & & & & 1 & 0 & \cdots & 0 \end{array}\right| \begin{array}{c} \\ \\ \\ \uparrow \\ m \\ \downarrow \\ \uparrow \\ m \\ \downarrow \end{array}$$
$$\qquad\qquad \leftarrow \; m \; \rightarrow \; \leftarrow \; m \; \rightarrow$$

where all unspecified entries are zero, is $(-1)^m t^{n-2m}$, and so has degree $n-2m$. Hence the maximum possible degree is $n-2m$.

27.8 Prove that, if $e^x/x = e^y/y$, where $y > x > 0$, then $xy < 1$.

Solution 1 by Vytautas Paskunas

Consider $f(t) = e^t/t$. Then

$$f'(t) = \frac{e^t(t-1)}{t^2},$$

so that $f'(t) = 0$ when $t = 1$, $f'(t) < 0$ for $t < 1$ and $f'(t) > 0$ for $t > 1$. Hence f has a minimum at $t = 1$. Thus, for x and y to be such that $e^x/x = e^y/y$, $0 < x < 1 < y$. Suppose that we have found x and y such that $e^x/x = e^y/y$ and $xy \geqslant 1$. Then $y \geqslant 1/x > 1$, so $f(y) \geqslant f(1/x)$, i.e. $e^y/y = e^x/x \geqslant xe^{1/x}$, i.e. $e^{x-(1/x)} \geqslant x^2$ or $x-(1/x) \geqslant 2\ln x$. Put

$$g(t) = t-\frac{1}{t}-2\ln t.$$

Then

$$g'(t) = 1+\frac{1}{t^2}-\frac{2}{t} = \left(\frac{1}{t}-1\right)^2 \geqslant 0$$

and $g'(t) > 0$ when $t \neq 1$, so that, as $0 < x < 1$, $g(x) < g(1) = 0$. Hence $x-(1/x) < 2\ln x$. Thus $xy < 1$.

Solution 2 by Toby Gee, not using calculus

Put $y = x+k$, so that $k > 0$. Then

$$\frac{e^x}{x} = \frac{e^{x+k}}{x+k},$$

so that

$$\frac{x+k}{x} = e^k,$$

whence $x = k/(e^k-1)$. Thus

$$xy = \frac{k}{e^k-1}\left(\frac{k}{e^k-1}+k\right) = \frac{k^2e^k}{(e^k-1)^2}.$$

Now

$$(e^k-1)^2-k^2e^k$$
$$= e^{2k}-2e^k-k^2e^k+1$$

$$= 1+\frac{2k}{1!}+\frac{(2k)^2}{2!}+\frac{(2k)^3}{3!}+\cdots$$
$$-2\left(1+\frac{k}{1!}+\frac{k^2}{2!}+\frac{k^3}{3!}+\ldots\right)$$
$$-k^2\left(1+\frac{k}{1!}+\frac{k^2}{2!}+\frac{k^3}{3!}+\cdots\right)+1$$
$$= a_4k^2+a_5k^5+\cdots,$$

where

$$a_n = \frac{1}{n!}[2^n-2-n(n-1)].$$

When $n = 4$, $2^4 > (4\times3)+2$. Suppose inductively that $2^n > n(n-1)+2$ for some $n \geqslant 4$. Then

$$2^{n+1} > 2n(n-1)+4$$

and

$$[2n(n-1)+4]-[(n+1)n+2] = n^2-3n+2$$
$$= (n-1)(n-2) > 0,$$

so that $2^{n+1} > (n+1)n+2$. Hence, by induction, $a_n > 0$ for all $n \geqslant 4$ and $xy < 1$.

Also solved by Junji Inaba, Can Minh, Richard Edlin, Neil Pollock. □

Reviews

An Equation that Changed the World: Newton, Einstein and the Theory of Relativity. By HARALD FRITZSCH. University of Chicago Press, 1994. Pp. 304. Hardback £23.95. (ISBN 0-226-26557-9).

There is a definite art of conversational 'gamesmanship' whereby one talker proceeds to wipe the floor with the other on some subject about which neither knows very much. Telltale excerpts from such conversations are immediately recognizable: "... as Sartre himself once said ...", "Mozart (*Amadeus that is*) was always considered greatest in his latter-day phase ..." and of course "... yes, but only simplistically, that all changed after Einstein's Relativity ...". Such phrases are well able to leave the unwary in a humbled silence. There have been many attempts to enlighten the population beyond this stage of perception in which Einstein is a crazed genius, his ideas horribly clever and utterly paradoxical. It would be fair to say that most have resulted in mass consternation to say the least. with hands on chins not as an expression of deep thought, but propping up a drooping jaw.

I therefore always view it as laudable when an author tries a novel approach, in order to cut away the murk of misunderstanding, baffle and bluff. And novel it most certainly is, to the point of necromancy! For our benefit are resurrected Albert Einstein and Isaac Newton! Newton of course is in the dark about relativity, and at first raises many objections, deftly dealt with by Einstein. Without realising it the reader, ignorant of the theories, learns along with Newton; a very flattering experience. The emergence of the 'gamma factor' (enabling calculations of alterations of mass, time and space) through discussion is very satisfying. I certainly felt for a moment that I had achieved it. Later on, it is Einstein who must sit back and learn from a fictional particle physicist about more recent developments: notably the development of nuclear weapons. One of my favourite features was Fritzsch's treatment of the men as real people rather than as vehicles for their theories: Einstein's horror at nuclear destruction and his quantum quibbles, Newton's rational argument and enthusiasm. I particularly enjoyed the hint of rivalry between the men when comparing the sizes of their namesake roads at the CERN laboratory, where some of the debate took place.

It was refreshing to discover a book which combined a good read with important physical principles. It had a good balance of discussing the theories with a certain amount of mathematical rigour, while also taking the necessary time to dwell on the consequences. The author certainly has a definite flair for communication. Lively and thought-provoking: I much recommend the book.

Sixth form of Ampleforth College, York JAMES HORTH

The Trisectors. By UNDERWOOD DUDLEY. The Mathematical Association of America, Washington, DC, 1994. Pp. xviii+184. Paperback $27.50 (ISBN 0-88385-514-3).

Well, what would you think if the editor of *Mathematical Spectrum* sent you a review copy of a book described on the back cover as 'a companion to *Mathematical Cranks*'?

The book is about some of the surprisingly many people who try to trisect an angle using only compasses and an unmarked straight edge (paper and pencil are also allowed), the famous problem dating back to the ancient Greeks. A theoretically exact solution was proved impossible by Wantzel in 1837. Underwood Dudley hopes his book will help reduce the time wasted by trisectors, their own time and the time of mathematicians and others whom trisectors try to interest in their work. The author takes as his model *A Budget of Paradoxes* (London, 1872) by Augustus de Morgan, who may take credit for reducing the time wasted by circle-squarers: the present work is a revision of Dudley's *The Budget of Trisections* (Springer-Verlag, 1987).

The first chapter gives numerous exact trisections where the conditions imposed are relaxed, for instance Archimedes' trisection which involves two marks on the straight edge. It ends with the words: 'Now you can trisect any angle anytime, anyplace, for anyone who asks. But no-one ever will.'

Underwood Dudley has assiduously collected trisections, which behaviour surely makes him unique. A trisector characteristically:

- is male (the two female trisectors in the book are not died-in-the-wool who-cares-if-it's-impossible-my-trisection's-exact-and-the-mathematical-establishment's-wrong trisectors);

- fails to understand what *impossible* means in mathematics (reasons for discounting Wantzel's proof of impossibility range from 'mine's not a trisection in the usual sense—the angle is increased in size by a factor of four-thirds and then this is quartered' and 'Wantzel used trigonometry but I use a straight edge and compasses' to 'blasphemy against the Great Mathematician' and an argument from Gödel's theorem invalidating all proofs of impossibility);
- knows little mathematics (with surprising exceptions such as a member of a university mathematics faculty who was eventually allowed to conduct a seminar on his trisection ...);
- think trisection is important (the author mentions the protractor);
- draws needlessly complicated diagrams (the construction can usually be simplified);
- loves writing letters about his trisection (to mathematicians, to politicians, to crowned heads of Europe; to Einstein, ...);
- may accuse mathematicians of
    - unthinking acceptance of established teaching on the impossibility of trisection;
    - fear of being undermined by the (double seismic) repercussions of the discovery of trisection;
    - conspiracy/mediocrity/inferiority;
- may greatly fear the theft of his discoveries.

The author's dealings with three selected trisectors are described at length. Throughout the book, the anonymity of the trisectors is preserved, as is the anonymity of mathematicians whom the author scolds for giving undue lack of discouragement to trisectors.

In the final chapter, about a hundred trisections are given, with an entertaining leavening of biographical details, quotations from trisectors, measures of error and apposite comments by the author. I wonder if any psychologist who studies cognitive dissonance has taken an interest in trisectors.

So here is a mathematically flavoured insight into humanity, a book far more readable than I was expecting and about which the author writes: 'The worst victim of mathematical anxiety can read this book with profit and dry palms. It is quite suitable to give as a present.'

MIS, University of Warwick — JOHN MACNEILL

Statistics. By ALAN GRAHAM. Teach Yourself Books, Hodder and Stoughton, London, 1994. Pp. x+283. Paperback £5.99. (ISBN 0-340-56181-5).

The author has attempted 'to present the basic ideas of statistics clear and simply'; he has largely succeeded in doing so. The 14 chapters and the appendix cover the following topics:

> introducing statistics; some basic mathematics; graphing data; choosing a suitable graph; summarising data; lies and statistics; choosing a sample; collecting information; reading tables of data; regression: describing relationships between things; correlation: measuring the strength of a relationship; chance and probability; probability models; deciding on differences; appendix: choosing the right statistical technique.

The style of the book is relaxed, with exercises at regular intervals in the text. A summary is provided at the end of each chapter, together with comments on the exercises.

The book can be recommended to all non-experts who wish to acquire an understanding of the key concepts and principles of statistics. After going through it, they should have no difficulty reading articles containing discussions of statistical data and graphs.

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Behavioral Decision Theory: A New Approach. By E. C. POULTON. Cambridge University Press, 1994. Pp. xix+314. Hardback £35.00. (ISBN 0-521-44368-7).

This book is concerned with some of the fallacies of behavioral decision theory. The author points out that an investigator studying a well-known fallacy may inadvertently introduce one of the simple biases which occur in the quantification of judgments. He then proceeds to analyse these in some detail. The book consists of the following 16 chapters:

> outline of heuristics and biases; practical techniques; apparent overconfidence; hindsight bias; small sample fallacy; conjunction fallacy; regression fallacy; base rate neglect; availability and simulation fallacies; anchoring and adjustment biases; expected utility fallacy; bias by frames; simple biases accompanying complex biases; problem questions; training; overview.

The book concludes with 8 pages of references and a very useful index. The author's aim is to help readers avoid the various fallacies described; this book should thus prove valuable to students and researchers in statistics, psychology and the social sciences.

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Episodes in Nineteenth and Twentieth Century Euclidean Geometry. By ROSS HONSBERGER. Mathematical Association of America, Washington DC, 1995. Pp. 163. Paperback $28.50. (ISBN 0-88385-639-5).

If you are unfamiliar with cleavers and splitters, the orthocentre, the Fuhrmann circle, the symmedian point, Miquel's theorem, Tucker circles, Brocard points, the orthopole, Menelaus' theorem, or Cevians—which probably includes just about all of us—then this volume by a well-known author will enlighten you. This book is one of a series from the Mathematical Association of America designed to introduce subjects outside the usual school syllabus to mathematics students and lay people.

PRIMUS. Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA. Annual subscription $40.00 in the USA, $44.00 elsewhere.

The subtitle of this quarterly magazine, which explains its title, is 'Problems, Resources and Issues in Mathematics Undergraduate Studies'. It has been going for four years. Although written with the North American market in mind, it is relevant to all involved in teaching undergraduate mathematics. □

Mathematical Spectrum

1995/6 Volume 28 Number 1

ISSN 0025-5653

Published by the Applied Probability Trust
Typeset by The Pi-Squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK