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Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, Ontario, Canada K1S 1N3.

Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Nghi Chung.

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This issue is dedicated to the memory of
Professor Viktors Linis

HOMOGENEOUS COORDINATES AND THE LHUILIER THEOREM

DAN PEDOE

Augustus Ferdinand Möbius (1790-1868) introduced barycentric coordinates in 1827 [1]. It is agreed, even by those who disparage their use (see [2, p. 712]), that they were the first homogeneous coordinates systematically used in geometry. The Möbius idea, in plane Euclidean geometry for example, is to imagine masses p, q, and r attached to the respective vertices A, B, and C of a given (nondegenerate) triangle in the plane, and to consider the uniquely defined centroid of these masses. If this be the point P, then P lies in the plane and is defined not only by the triad (p,q,r), but equally by the triad (kp,kq,kr) for any $k \neq 0$. Noting that, if AP intersects BC in the point A', then we have BA' : A'C = $r : q$, since A' is the centroid of masses q at B and r at C, we see that, if P is given, then the ratios $p : q : r$ are uniquely determined, and the *barycentric coordinates* (p,q,r) of P constitute a system of homogeneous coordinates for the plane. We immediately allow all values of (p,q,r) except $(0,0,0)$, and the vertices of triangle ABC, the *triangle of reference*, can be assigned the coordinates $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. We note that the point $(0,-1,1)$ on the line BC is the *point at infinity* for that line, and the three points at infinity on the respective sides of triangle ABC have coordinates (x,y,z) that satisfy the equation $x + y + z = 0$. This is an homogeneous equation, and we shall see that the equation of any line in barycentric coordinates is an homogeneous equation. But first let us see the connection between barycentric coordinates and areal coordinates.

If [XYZ] denotes the signed area of triangle XYZ, then for the configuration considered in the last paragraph we have

$$[PAB] : [PCA] = BA' : A'C = r : q,$$

and we can therefore work with the area ratios defined by the point P,

$$[PBC] : [PCA] : [PAB] = p : q : r.$$

Since [PBC] + [PCA] + [PAB] = [ABC], we can introduce the *areal coordinates* (x,y,z) of P as

$$x = \frac{[PBC]}{[ABC]}, \qquad y = \frac{[PCA]}{[ABC]}, \qquad z = \frac{[PAB]}{[ABC]},$$

where we have the identical relation

$$x + y + z = 1.$$

We rarely use actual areal coordinates, but the above identical relation poses an apparent paradox. Points on the line at infinity $x + y + z = 0$ do not have numerical areal coordinates, but there is an infinity of number triads, such as $(1,1,-2)$, which satisfy the equation, and represent points at infinity. Once we establish that the equation of any line in the plane is homogeneous, the line at infinity ceases to be an outstanding line, and we shall have succeeded in democratising Euclidean geometry by immersing the Euclidean plane in a projective plane in which any two distinct lines always meet in a point, and the notion of parallel lines has vanished. Homogeneous coordinates are the simplest method for an analytical investigation of projective geometry. If, as seems to be the case nowadays, projective geometry is no longer taught in some countries obsessed with the "new" mathematics, it follows that few students have ever been exposed to homogeneous coordinates, and this is the raison d'être of this article.

Our next step is to show the connection between Euclidean vectors and barycentric coordinates. Let O be the origin of vectors and, as above, let P have barycentric coordinates (p,q,r) with respect to triangle of reference ABC. The result

$$q\vec{OB} + r\vec{OC} = (q+r)\vec{OA'},$$

where BA' : A'C = $r : q$, is easily proved by resolving the vector $\vec{OB}$ along the sides OA' and A'B of triangle OA'B and vector $\vec{OC}$ along the sides OA' and A'C of triangle OA'C. It follows that

$$p\vec{OA} + q\vec{OB} + r\vec{OC} = (p+q+r)\vec{OP}.$$

We pause here to give a useful application of this theorem to complex numbers (considered as vectors). We use the notation of Crux 702, recently discussed in this journal [1982: 323; 1983: 144], where this application plays an important role. Let α,β,γ be the respective affixes of the points A,B,C in the Argand diagram, and let l,m,n be three real numbers such that $l+m+n = 1$. If a point M in the plane satisfies

$$[MBC] = l[ABC], \qquad [MCA] = m[ABC], \qquad [MAB] = n[ABC],$$

then its affix z satisfies

$$l\alpha + m\beta + n\gamma = (l+m+n)z = z,$$

and conversely.

To obtain the equation of a line in homogeneous coordinates, we now introduce *trilinear coordinates*. These are the signed lengths of perpendiculars from a point P onto the sides of triangle ABC; and if these are (α,β,γ), and the side lengths of triangle ABC are a,b,c, we have the identical relation

$$a\alpha + b\beta + c\gamma = 2[ABC].$$

Using trilinear coordinates and Euclidean geometry, we can find the equation of a line.

Let the point R divide the segment PQ so that $r\mathrm{PR} = q\mathrm{RQ}$, and suppose we are given the trilinear coordinates $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ of P and Q. We wish to show that the trilinear coordinates of R are

$$(x,\ y,\ z) = \left(\frac{q\alpha_2+r\alpha_1}{q+r},\ \frac{q\beta_2+r\beta_1}{q+r},\ \frac{q\gamma_2+r\gamma_1}{q+r}\right).$$

This is done by dropping perpendiculars from R onto the sides of the triangle of reference, and using the properties of similar triangles. If we eliminate the ratios $q/(q+r)$ and $r/(q+r)$, we find that the equation of the line joining the two points P and Q is

$$\begin{vmatrix} x & y & z \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

The equation of a line is therefore a homogeneous linear equation, of the form $ux+vy+wz = 0$ in trilinear coordinates, and also in areal coordinates, the connection between trilinear coordinates (x, y, z) and areal coordinates (α, β, γ) being $xa = 2\Delta\alpha$, $yb = 2\Delta\beta$, $zc = 2\Delta\gamma$, where $\Delta = [\mathrm{ABC}]$. The same line is given by the equation $kux+kvy+kwz = 0$, $k \neq 0$, so we can attach the homogeneous coordinates $[u, v, w]$ to a line, and the way is now clear to the Principle of Duality (see [4]).

For a more detailed discussion of homogeneous coordinates, with applications, see [5]; and for an historical discussion, with some account of the resistance to homogeneous coordinates shown by Cauchy and Gauss, see [3].

We now come to the theorem that is generally known as the Lhuilier Theorem. According to F. G.-M. [8], it was proved by the French Swiss mathematician Simon Lhuilier (1750-1840) in the *Annales de Gergonne*, tome II, 1811-1812, pp. 293 ff. Since the same issue of this journal also contained six other solutions, by Encontre, Tédenat, Pilatte, Penjon, Rochat, and Legrand, it is not clear who originally proposed the problem. As given in Dörrie [9], the Lhuilier Theorem states that:

The sections of an arbitrary three-edged prism include all the possible forms of triangles.

Dörrie contains a fair amount of material but, like more modern books on mathematical morsels, hors d'oeuvres, and other chewables, he gives methods in geometry which can be improved. It should also be noted that the translator was not a mathetician, since area and volume are confused at least 8 times from page 283 onwards.

Dörrie shows that, if we are given a triangular prism with an orthogonal section ABC, then we can find a plane cutting the three parallel generators in a triangle A'B'C' of a given shape. The proof is Euclidean, and fairly long. We shall offer a proof which uses areal coordinates.

We use the notion that the shape of a triangle A'B'C' is determined by the ratios $a' : b' : c'$ of its sides, and these ratios in turn are determined by the ratios in which the sides are divided by the points of contact of the incircle with the sides. If the points of contact with B'C', C'A', A'B' are respectively L', M', N', and we have

$$B'L' : L'C' = q : r, \quad C'M' : M'A' = r : p, \quad A'N' : N'B' = p : q,$$

we know that

$$\begin{aligned} q : r &= a' - b' + c' : a' + b' - c', \\ r : p &= a' + b' - c' : -a' + b' + c', \\ p : q &= -a' + b' + c' : a' - b' + c', \end{aligned}$$

so that

$$a' : b' : c' = q+r : r+p : p+q.$$

We already have a given triangle in the problem, triangle ABC. We show that we can construct an *ellipse* to touch the sides of the triangle in points which divide the sides in any given ratios $q : r$, $r : p$, and $p : q$, where we assume that p,q,r are all positive. We can write down the equation of the ellipse in homogeneous coordinates as

$$p^2x^2 + q^2y^2 + r^2z^2 - 2qryz - 2rpzx - 2pqxy = 0.$$

This intersects the side BC, which has the equation $x = 0$, where $(qy - rz)^2 = 0$. Hence this conic touches BC, in the point L, say, and if we now assume that our coordinates are areal coordinates, we have BL : LC $= q : r$ since the point of contact L has coordinates $(0,r,q)$. Similarly, we have CM : MA $= r : p$ and AN : NB $= p : q$.

To show that this conic is an ellipse, we investigate its intersections with the *line at infinity* $x + y + z = 0$. We obtain a quadratic whose discriminant turns out to be

$$-4pqr(p + q + r).$$

Since p,q,r are all positive, our conic does not intersect the line at infinity, and is therefore an ellipse.

We now consider the elliptical cylinder generated by lines passing through points of the ellipse and perpendicular to the plane ABC. Any plane which is not perpendicular to the plane ABC will cut the elliptical cylinder in an ellipse, and the triangular prism in a triangle A'B'C' whose sides touch the ellipse, and the points of contact are on the generators of the elliptical cylinder which pass through the points of contact of triangle ABC with its inscribed ellipse. It is a simple property of parallel projection that if collinear points B, L, and C are projected into the collinear points B', L', and C', then BL : LC = B'L' : L'C'. Hence all that we now have to show is that there is a plane section of the elliptical cylinder which is a circle, and this section, if we choose the ellipse inscribed in triangle ABC as we have indicated, will cut the triangular prism in a triangle of the given shape A'B'C'.

Since the plane ABC is orthogonal to the generators of the elliptical cylinder, a plane through the major axis of the ellipse we have been considering will cut the elliptical cylinder in an ellipse which has a principal axis along the major axis, and coinciding with it. But since we can increase the size of the other principal axis of our section without bound, as we increase the angle between the cutting plane and the plane ABC we can find a plane section which cuts the elliptical cylinder in a circle. Any plane parallel to this final plane will cut the elliptical cylinder in a circle and the triangular prism in a triangl e A'B'C' of the given shape. Since we can find an orthogonal section of a given triangular prism, the section ABC in the Dörrie statement need not be an orthogonal section.

Over forty years ago, the attractions of geometry providing some degree of comfort when England was almost overwhelmed by the dark uncertainties of World War II, I proved that any given triangle ABC can be projected orthogonally into a triangle A'B'C' of given shape [6]. I had not then come across the Lhuilier Theorem. I obtained a quadratic equation which gave the size of the sides of triangle A'B'C', and knowing, by geometric arguments similar to those given above, that the roots of this quadratic were real, I obtained a two-triangle inequality which related the sides a,b,c and area Δ of any given triangle ABC to the sides a',b',c' and area Δ' of any other given triangle A'B'C'. The inequality reads:

$$a^2(-a'^2 + b'^2 + c'^2) + b^2(a'^2 - b'^2 + c'^2) + c^2(a'^2 + b'^2 - c'^2) \geq 16\Delta\Delta',$$

with equality if and only if the two triangles are similar.

Bottema and Klamkin, both well known to readers of this journal, showed in [10] that Neuberg [11] had obtained the necessity of the condition, but not the sufficiency, and labelled the inequality the Neuberg-Pedoe inequality. It has attracted much attention recently, especially in mainland China, and a remarkable extension to two n-simplices in Euclidean n-space has been discovered by the two Chinese geometers who gave the remarkable solution to the rusty compass problem [12]. This has been accepted for publication in the *Bulletin of the Australian Mathematical Society*.

As a final note on Dieudonné's remark, "Who ever uses barycentric coordinates?", which I mention on page 712 of [2], I refer the reader to an application given in [7]. Neuberg represented a triangle ABC by the point (a^2, b^2, c^2) with respect to an equilateral triangle XYZ. This produces another proof of the above inequality. Sir Alexander Oppenheim rediscovered the method and obtained other inequalities by its use.

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1956 East River Terrace, Minneapolis, Minnesota 55414.

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PUBLISHING NOTE

Geometry and the Visual Arts, by Dan Pedoe, has just been published by Dover Publications, Inc., 180 Varick Street, New York, N.Y. 10014 (296 pp., \$5.95 US). This is an unabridged and corrected republication of the work first published in 1976 under the name *Geometry and the Liberal Arts* by Penguin books, Ltd., Harmondsworth, Middlesex, England, and by St. Martin's Press, New York, N.Y.

Other books by Dan Pedoe still available from Dover: *The Gentle Art of Mathematics* (143 pp., \$2.50 US); and *Circles, A Mathematical View* (96 pp., \$3.50 US).

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A PROBLEM OF INTEREST

EDWARD L. COHEN

It is well known in bankers' circles that if the annual interest rate i (in percent) is divided into 72 we obtain approximately the number of years required to double the principal. The problems we attempt to solve here are:

1. Why is 72 such a good choice for doubling the principal?

2. Can we find such constants H_q for the approximate trebling ($q = 3$), quadrupling ($q = 4$), etc., of the principal?

The approach uses logarithms (base e), limits, derivatives, approximations, and finance in one application, and could furnish a useful exercise in first-year calculus and economics courses.

Let P = principal, n = number of years, i = annual interest rate (in percent). If the interest is compounded annually, it is well known that the amount after n years is

$$P\left(1 + \frac{i}{100}\right)^n.$$

To double the principal, we must therefore have

$$2P = P\left(1 + \frac{i}{100}\right)^n \qquad \text{or} \qquad 2 = \left(1 + \frac{i}{100}\right)^n,$$

and solving for n gives

$$n = \frac{\log 2}{\log(1 + i/100)}. \tag{1}$$

We wish to find a constant H_2 which, when divided by i, gives, for "reasonable" values of i, a "reasonable" approximation of the number of years required to double the principal, that is, such that $H_2/i \approx n$, where n is given by (1). If we set $n = A_2/i$ in (1), then A_2 is a function of i given by

$$A_2(i) = \frac{i \log 2}{\log(1 + i/100)}.$$

Comparing

$$\frac{1}{A_2(i)} = \frac{\log(1 + i/100)}{i \log 2} = \frac{1}{\log 2} \cdot \frac{\log(100+i) - \log 100}{i}$$

with the derivative of the logarithm function,

$$\lim_{i\to 0} \frac{\log(x+i) - \log x}{i} = \frac{1}{x},$$

we find that

$$\lim_{i \to 0} A_2(i) = 100 \log 2 \approx 69.31.$$

We observe that the number 72 mentioned earlier is not far away from 69.31, and it will become clear as we go along why it is convenient to take $H_2 = 72$. But first we generalize.

Let

$$A_q(i) = \frac{i \log q}{\log(1 + i/100)}.$$

The tables in the text give, for each of $q = 2, 2\frac{1}{2}, 3, 4, 5$, the values of $A_q(i)$ and $A_q(i)/i$ for selected values of i, then the values of H_q/i for two or three prospective values of H_q, and, finally, the relative percentage error e_{H_q} that results when H_q is used for all i instead of $A_q(i)$. The limit of $A_q(i)$ as $i \to 0$ gives a value (of theoretical interest) for 0%. Otherwise $A_q(i)/i$ yields the exact (to the stated accuracy) number of years required to q-tuple the principal at i% compounded annually. The prospective H_q's featured are some that are most easily divisible by i as well as sensible. At the end of the article, we indicate our choice, made on the basis of the tables, for the most convenient values of the H_q's.

Table 1. $q = 2$

i%	$A_q(i)$	$A_q(i)/i$	$72/i$	$75/i$	e_{72}	e_{75}
0%	69.31					
5%	71.03	14.21	14.40	15.00	1.4%	5.6%
6%	71.37	11.90	12.00	12.50	.9%	5.1%
10%	72.73	7.27	7.20	7.50	1.0%	3.1%
12%	73.40	6.12	6.00	6.25	1.9%	2.2%
15%	74.39	4.96	4.80	5.00	3.2%	.8%
18%	75.38	4.19	4.00	4.17	4.5%	.5%
20%	76.04	3.80	3.60	3.75	5.3%	1.4%

Table 2. $q = 2\frac{1}{2}$

i%	$A_q(i)$	$A_q(i)/i$	$90/i$	$100/i$	e_{90}	e_{100}
0%	91.63					
5%	93.90	18.78	18.00	20.00	4.2%	6.5%
6%	94.35	15.73	15.00	16.67	4.6%	6.0%
10%	96.14	9.61	9.00	10.00	6.4%	4.0%

12%	97.02	8.09	7.50	8.33	7.2%	3.1%
15%	98.34	6.56	6.00	6.67	8.5%	1.7%
18%	99.65	5.54	5.00	5.56	9.7%	.4%
20%	100.51	5.03	4.50	5.00	10.5%	.5%

Table 3. $q = 3$

i%	$A_q(i)$	$A_q(i)/i$	$110/i$	$115/i$	$120/i$	e_{110}	e_{115}	e_{120}
0%	109.86							
5%	112.59	22.52	22.00	23.00	24.00	2.3%	2.1%	6.6%
6%	113.13	18.85	18.33	19.17	20.00	2.8%	1.7%	6.1%
10%	115.27	11.53	11.00	11.50	12.00	4.6%	.2%	4.1%
12%	116.33	9.69	9.17	9.58	10.00	5.4%	1.1%	3.2%
15%	117.91	7.86	7.33	7.67	8.00	6.7%	2.5%	1.8%
18%	119.48	6.64	6.11	6.39	6.67	7.9%	3.7%	.4%
20%	120.51	6.03	5.50	5.75	6.00	8.7%	4.6%	.4%

Table 4. $q = 4$

i%	$A_q(i)$	$A_q(i)/i$	$140/i$	$144/i$	$150/i$	e_{140}	e_{144}	e_{150}
0%	138.63							
5%	142.07	28.41	28.00	28.80	30.00	1.5%	1.4%	5.6%
6%	142.75	23.79	23.33	24.00	25.00	1.9%	.9%	5.1%
10%	145.45	14.55	14.00	14.40	15.00	3.7%	1.0%	3.1%
12%	146.79	12.23	11.67	12.00	12.50	4.6%	1.9%	2.2%
15%	148.78	9.92	9.33	9.60	10.00	5.9%	3.2%	.8%
18%	150.76	8.38	7.78	8.00	8.33	7.1%	4.5%	.5%
20%	152.07	7.60	7.00	7.20	7.50	7.9%	5.3%	1.4%

Table 5. $q = 5$

i%	$A_q(i)$	$A_q(i)/i$	$160/i$	$175/i$	$180/i$	e_{160}	e_{175}	e_{180}
0%	160.94							
5%	164.93	32.99	32.00	35.00	36.00	3.0%	6.1%	9.1%
6%	165.73	27.62	26.67	29.17	30.00	3.5%	5.6%	8.6%
10%	168.86	16.89	16.00	17.50	18.00	5.2%	3.6%	6.6%
12%	170.42	14.20	13.33	14.58	15.00	6.1%	2.7%	5.6%
15%	172.73	11.52	10.67	11.67	12.00	7.4%	1.3%	4.2%
18%	175.03	9.72	8.89	9.72	10.00	8.6%	0.0%	2.8%
20%	176.55	8.83	8.00	8.75	9.00	9.4%	.9%	2.0%

Summary. We have tried to choose amongst the prospective H_q's those that seem easily divisible by most of the interest rates. Our preferences are:

$$H_2 = 72, \quad H_{2\frac{1}{2}} = 100, \quad H_3 = 120, \quad H_4 = 150, \quad H_5 = 180.$$

These constants give reasonable results.

Problem 1. If $rA_s(i) = A_t(i)$, what is the relationship amongst r, s, and t?

Problem 2. Show that $H_7 = 200$ and $H_{10} = 250$ produce satisfactory values for the rates of interest used in the tables.

Department of Mathematics, University of Ottawa, 585 King Edward Avenue, Ottawa, Ontario, Canada K1N 9B4.

* * *

CORRIGENDUM

In my article "Some Boolean Inequalities from a Triangle" [1983: 128-131], it was stated on page 129 that the inequality of Theorem 2.01 is reversed if $0 < m,n < 1$. This is incorrect: the inequality remains true in this case also.

J.L. BRENNER

* * *

THE PUZZLE CORNER

Puzzle No. 38: Rebus (*6 2)

In mathematics, we may list
MY WHOLE a number theorist.

Puzzle No. 39: Rebus (7)

S

It's usual, as you can see:
All tangents pair with KEY.

Puzzle No. 40: Terminal deletion (8)

In music numbers: "slower, faster"—FIRST directs.
The *SECOND diagram shows numbers all complex.

Puzzle No. 41: Enigma (4)

Four. None there.
Round. It's square.

ALAN WAYNE, Holiday, Florida

* * *

THE OLYMPIAD CORNER: 46

M.S. KLAMKIN

I give three new problem sets this month.

The first consists of the problems set at the First Annual American Invitational Mathematics Examination (AIME), which took place on March 22, 1983. (It was dedicated to the memory of Professor John H. Staib, the first AIME Chairman.) Of the 407 133 students who wrote the American High School Mathematics Examination (AHSME), those who achieved a score of at least 95 were invited to take the AIME, and 1822 did so. The AIME is an "answer only" type of examination, like the 1983 Netherlands Invitational Mathematics Examination given last month in this column [1983: 138]. The answers are all integers in the interval [0, 999], so the results can be machine-graded quickly and various kinds of statistical information are easily obtained. The answers are given at the end of this column.

Our next problem set consists of those given at the Twelfth U.S.A. Mathematical Olympiad (USAMO), which took place on May 3, 1983. The participants were the 53 students who scored at least 10 (out of 15) on the AIME. The top scorers on the USAMO will constitute the U.S.A. team for the 24th International Mathematical Olympiad (IMO) due to take place in July, 1983 in Paris, France.

Correspondence about the examination questions and solutions of the AHSME, the AIME, and the USAMO should be addressed to the respective examination committee chairmen:

Dr. Stephen B. Maurer
Alfred P. Sloan Foundation
630-5th Avenue
New York, N.Y. 10111,

Professor George Berzsenyi
Department of Mathematics
Lamar University
Beaumont, Texas 77710,

and myself. Solutions to various AHSME, AIME, USAMO, and IMO examinations, as well as subscription information about the new high school journal *Arbelos*, edited by Professor S.L. Greitzer, can be obtained from

Dr. W.E. Mientka
Executive Director, AHSME & AIME
University of Nebraska
917 Oldfather Hall
Lincoln, Nebraska 68588.

The third problem set, which I am able to present through the courtesy of Jim Williams, University of Sydney, consists of the problems given at the 1983 Australian Mathematical Olympiad, whose format is the same as that of the IMO.

Finally, I present results and solutions for the Fifteenth Canadian Mathematics Olympiad, the problems of which appeared in this column last month [1983: 139].

FIRST ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

March 22, 1983 — Time: $2\frac{1}{2}$ hours

1. Let x, y and z all exceed 1 and let w be a positive number such that

$$\log_x w = 24, \quad \log_y w = 40 \quad \text{and} \quad \log_{xyz} w = 12.$$

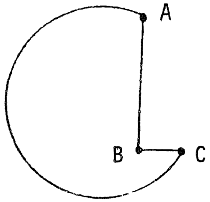
Find $\log_z w$.

2. Let $f(x) = |x-p| + |x-15| + |x-p-15|$, where $0 < p < 15$. Determine the minimum value taken by $f(x)$ for x in the interval $p \le x \le 15$.

3. What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

4. A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.



5. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of their cubes is 10. What is the largest real value that $x + y$ can have?

6. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

7. Twenty-five of King Arthur's knights are seated at their customary round table. Three of them are chosen—all choices of three being equally likely—and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

8. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

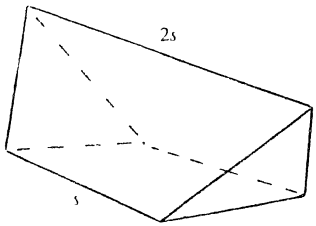
9. Find the minimum value of

$$f(x) = \frac{9x^2\sin^2 x + 4}{x\sin x}$$

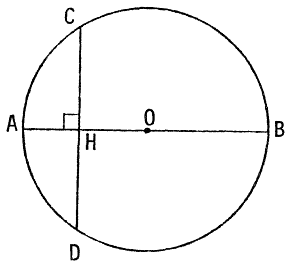
for $0 < x < \pi$.

10. The numbers 1447, 1005 and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

11. The solid shown has a square base of side length s. The upper edge is parallel to the base and has length $2s$. All other edges have length s. Given that $s = 6\sqrt{2}$, what is the volume of the solid?

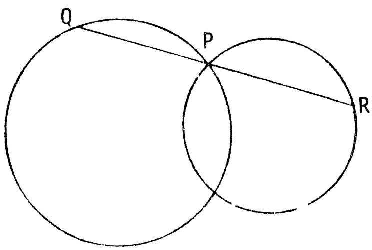


12. Diameter AB of a circle has length a 2-digit integer (base ten). Reversing the digits gives the length of the perpendicular chord CD. The distance from their intersection point H to the center O is a positive rational number. Determine the length of AB.

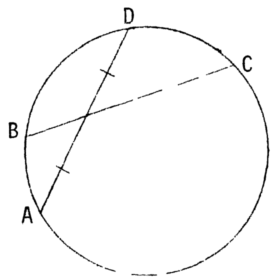


13. For $\{1,2,3,\ldots,n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1,2,4,6,9\}$ is $9-6+4-2+1=6$ and for $\{5\}$ it is simply 5.) Find the sum of all such alternating sums for $n = 7$.

14. In the adjoining figure, two circles of radii 8 and 6 are drawn with their centers 12 units apart. At P, one of the points of intersection, a line is drawn in such a way that the chords QP and PR have equal length. Find the square of the length of QP.



15. The adjoining figure shows two intersecting chords in a circle, with B on minor arc AD. Suppose that the radius of the circle is 5, that $BC = 6$, and that AD is bisected by BC. Suppose further that AD is the only chord starting at A which is bisected by BC. It follows that the sine of the central angle of minor arc AB is a rational number. If this number is expressed as a fraction m/n in lowest terms, what is the product mn?



TWELFTH U.S.A. MATHEMATICAL OLYMPIAD

May 3, 1983 — Time: $3\frac{1}{2}$ hours

1. On a given circle, six points A, B, C, D, E and F are chosen at random, independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, i.e., having no common points.

2. Prove that the roots of $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ cannot *all* be real if $2a^2 < 5b$.

3. Each set of a finite family of subsets of a line is a union of two closed intervals. Moreover, any three of the sets of the family have a point in common. Prove that there is a point which is common to at least half of the sets of the given family.

4. Six segments S_1, S_2, S_3, S_4, S_5 and S_6 are given in a plane. These are congruent to the edges AB, AC, AD, BC, BD and CD, respectively, of a tetrahedron ABCD. Show how to construct a segment congruent to the altitude of the tetrahedron from vertex A with straightedge and compass.

5. Consider an open interval of length $1/n$ on the real number line where n is a positive integer. Prove that the number of irreducible fractions p/q, with $1 \le q \le n$, contained in the given interval is at most $(n+1)/2$.

*

AUSTRALIAN MATHEMATICAL OLYMPIAD

Paper I: March 15, 1983 — Time: 4 hours

1. Each positive rational number occupies an infinite number of positions in the following pattern

$$\frac{1}{1};\ \frac{2}{1},\ \frac{1}{2};\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3};\ \frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4};\ \frac{5}{1},\ \frac{4}{2},\ \frac{3}{3},\ \frac{2}{4},\ \frac{1}{5};\ldots\ .$$

For instance the number $\frac{2}{3}$ occupies positions 9, 42, ...

a) Find the first five positions occupied by the number $\frac{1}{2}$.

b) Find an expression for the nth occurrence of the number $\frac{1}{2}$.

c) Find an expression for the first occurrence of the number $\frac{p}{q}$, where p and q are relatively prime and $p < q$.

2. ABC is a triangle and P is a point inside it. Angle PAC = angle PBC. The perpendiculars from P to BC, CA meet these sides at L, M respectively, and D is the midpoint of AB. Prove that DL = DM.

3. A box contains p white balls and q black balls. Beside the box there is a pile of black balls. Two balls are taken out from the box. If they are of the same colour, a black ball from the pile is put into the box. If they are of different colours, the white ball is put back into the box. This procedure is repeated until the last pair of balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

Paper II: March 16, 1983 — Time: 4 hours

4. Find all pairs of natural numbers (n,k) for which $(n+1)^k - 1 = n!$

5. (a) Find the rearrangement $\{a_1,a_2,\ldots,a_n\}$ of $\{1,2,\ldots,n\}$ which maximises

$$a_1a_2 + a_2a_3 + \ldots + a_na_1 = Q.$$

(b) Find the rearrangement that minimises Q.

6. The right triangles ABC and AB_1C_1 are similar and have opposite orientation. The right angles are at C and C_1, and $\angle CAB = \angle C_1AB_1$. M is the point of intersection of the lines BC_1 and CB_1. Prove that if the lines AM and CC_1 exist, then they are perpendicular.

*

The Fifteenth Canadian Mathematics Olympiad took place on Wednesday, May 4, 1983, with 269 students participating from 177 schools across Canada. Among the prizewinners were:

First prize ($1000): William James RUCKLIDGE, Toronto French School, Toronto, Ontario.

Second prize ($750): Martin PIOTTE, Collège des Eudistes, Montréal, Québec.

Third prize ($400): Michael CLASE, Memorial University, St. John's, Newfoundland.

Mike Sean MOLLOY, Osgoode Township High School, Metcalfe, Ontario.

Fourth prize ($200): David CONIBEAR, Merivale High School, Nepean, Ontario.

Neale GINSBURG, Loyalist Collegiate & Vocational Institute, Kingston, Ontario.

William Thomas WHITE, Thousand Islands Secondary School, Brockville, Ontario.

The problems and the official solutions are given below. They were prepared by the Olympiad Committee of the Canadian Mathematical Society, consisting of:

G. BUTLER, Chairman, University of Alberta;
T. LEWIS, University of Alberta;
A. LIU, University of Alberta;

M.S. KLAMKIN, University of Alberta;
J. SCHAER, University of Calgary;
J.C. FISHER, University of Regina;
J. WILKER, University of Toronto;
G. LORD, Office of Population Research, Princeton, New Jersey;
P. ARMINJON, Université de Montréal;
E. WILLIAMS, Memorial University of Newfoundland.

FIFTEENTH CANADIAN MATHEMATICS OLYMPIAD

May 4, 1983 — Time: 3 hours

1. Find all positive integers w,x,y,z which satisfy $w! = x! + y! + z!$.

Solution.

Without loss of generality, assume $x \le y \le z$. Then $w \ge z+1$ so that

$$(z+1)z! = (z+1)! \le w! = x! + y! + z! \le 3z!.$$

Hence $z \le 2$. A quick check shows that $x = y = z = 2$ and $w = 3$ form the only solution.

2. For each real number r, let T_r be the transformation of the plane that takes the point (x,y) into the point $(2^r x,\ r2^r x + 2^r y)$. Let F be the family of all such transformations, i.e., $F = \{T_r\colon r \text{ is a real number}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.

Edited solution.

A curve $y = f(x)$, assumed to be defined for every real x, remains unchanged by every T_r if and only if, for all real r and x,

$$f(2^r x) = r2^r x + 2^r y = r2^r x + 2^r f(x). \tag{1}$$

For $x = 0$, (1) yields $f(0) = 0 + 2^r f(0)$ for all r, and so $f(0) = 0$. We will find the curve for which $f(1) = \alpha$ and $f(-1) = -\beta$, where α and β are arbitrary constants. It follows from (1) that, for all r,

$$f(2^r) = r2^r + 2^r\alpha \tag{2}$$

and

$$f(-2^r) = -r2^r - 2^r\beta. \tag{3}$$

For any $x \ne 0$, let $r = \log_2|x|$. If $x > 0$, then $x = 2^r$ and (2) becomes $f(x) = x\log_2 x + \alpha x$; and if $x < 0$, then $x = -2^r$ and (3) becomes $f(x) = x\log_2(-x) + \beta x$.

In summary, all the solution functions are given, for arbitrary α and β, by

$$y = f(x) \equiv \begin{cases} x\log_2 x + \alpha x, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ x\log_2(-x) + \beta x, & \text{if } x < 0. \end{cases}$$

3. The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?

Solution.

The answer is no. Let S be an equilateral triangle and T an isosceles right triangle, both of area 4. The lines joining pairwise the midpoints of the sides of each triangle divide it into four triangles of area 1. Folding along these lines, S turns into a regular tetrahedron with positive volume while T turns into a degenerate tetrahedron (in fact a square) with zero volume. Yet both tetrahedra have faces all of area 1. (To avoid quibbling, one may reduce the right angle in T slightly so that the resulting tetrahedron is not degenerate. Its altitude and consequently its volume can be made arbitrarily close to zero.)

4. Prove that for every prime number p there are infinitely many positive integers n such that p divides $2^n - n$.

Solution.

For $p = 2$, p divides $2^n - n$ for every even n. Hence we may assume that $p \neq 2$. Let m be any positive integer and let $n = (mp-1)(p-1)$. Then we have

$$n = mp^2 - mp - p + 1 \equiv 1 \pmod{p}.$$

On the other hand,

$$2^n = (2^{p-1})^{mp-1} \equiv 1^{mp-1} = 1 \pmod{p}$$

by Fermat's Theorem. Hence $2^n - n \equiv 0 \pmod{p}$ for infinitely many values of n.

5. The geometric mean (G.M.) of k positive numbers $a_1, a_2, \ldots, a_k$ is defined to be the (positive) kth root of their product. For example, the G.M. of 3, 4, 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all nonempty subsets of S.

Solution.

Let $S = \{a_1, a_2, \ldots, a_n\}$. Then the G.M. of S is $(a_1 a_2 \ldots a_n)^{1/n}$. Now consider a particular a_i. It occurs in $\binom{n-1}{k-1}$ subsets of S each with cardinality k, with exponent $1/k$ in each of the corresponding G.M.'s. Hence the exponent of a_i in the G.M. of the G.M.'s is given by

$$\frac{1}{2^n-1}\sum_{k=1}^{n}\binom{n-1}{k-1}\cdot\frac{1}{k} = \frac{1}{2^n-1}\sum_{k=1}^{n}\binom{n}{k}\cdot\frac{1}{n} = \frac{1}{n}\left\{\frac{1}{2^n-1}\left(\sum_{k=0}^{n}\binom{n}{k} - 1\right)\right\} = \frac{1}{n},$$

the same as that in the G.M. of S.

*

ANSWERS TO THE FIRST ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
60	15	20	26	4	35	57	61	12	432	288	65	448	130	175

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

CORRIGENDUM

It has been brought to the attention of the author of "The Geometry of Dürer's Conchoid" [1983: 32-37] that the second line of the article should begin with "the 16th century artist Albrecht Dürer...".

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

851. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the doubly true decimal alphametic

תשעים = עשר - עשר × עשר

which means TEN × TEN - TEN = NINETY in Hebrew. (*Goyim* readers will find it convenient to rewrite the alphametic as ABC × ABC - ABC = DECBF.)

This same Hebrew alphametic appeared in the *Journal of Recreational Mathematics* 15 (1982-1983) 136, proposed by Meir Feder, who asked for a solution in base 7.

852. *Proposed by Jordi Dou, Barcelona, Spain.*

Given are three distinct points A,B,C on a circle. A point P in the plane has the property that if the lines PA,PB,PC meet the circle again in A',B',C', respectively, then A'B' = A'C'.

Find the locus of P.

853. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let f be a real-valued function, defined for all $x \geq 0$, such that

$$\lim_{x\to\infty} f(x) \qquad \text{and} \qquad \int_0^r \frac{f(x)}{x}\,dx$$

both exist, the second for all $r > 0$. Evaluate

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx,$$

where $a > b > 0$, and deduce the value of

$$\int_0^\infty \frac{e^{ax} - e^{bx}}{x(e^{ax}+1)(e^{bx}+1)}\,dx.$$

(This problem was suggested by Problem A-3 on the 1982 William Lowell Putnam Mathematical Competition.)

854. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

For $x,y,z > 0$, let

$$A = \frac{yz}{(y+z)^2} + \frac{zx}{(z+x)^2} + \frac{xy}{(x+y)^2}$$

and

$$B = \frac{yz}{(y+x)(z+x)} + \frac{zx}{(z+y)(x+y)} + \frac{xy}{(x+z)(y+z)}.$$

It is easy to show that $A \leq \frac{3}{4} \leq B$, with equality if and only if $x = y = z$.

(a) Show that the inequality $A \leq 3/4$ is "weaker" than $3B \geq 9/4$ in the sense that

$$A + 3B \geq \frac{3}{4} + \frac{9}{4} = 3.$$

When does equality occur?

(b) Show that the inequality $4A \leq 3$ is "stronger" than $8B \geq 6$ in the sense that

$$4A + 8B \leq 3 + 6 = 9.$$

When does equality occur?

855. *Proposed by Christian Friesen, student, University of New Brunswick.*

Let N be the set of natural numbers. For each $n \in N$, prove the existence of a polynomial $f_n(x_1,x_2,\ldots,x_n)$ such that the mapping $f_n\colon N^n \to N$ is a bijection.

856. *Proposed by Jack Garfunkel, Flushing, N.Y.*

For a triangle ABC with circumradius R and inradius r, let $M = (R-2r)/2R$. An inequality $P \geq Q$ involving elements of triangle ABC will be called *strong* or *weak*, respectively, according as

$$P - Q \leq M \quad \text{or} \quad P - Q \geq M.$$

(a) Prove that the following inequality is strong:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \geq \frac{3}{4}.$$

(b) Prove that the following inequality is weak:

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \geq \sin B \sin C + \sin C \sin A + \sin A \sin B.$$

857. *Proposed by Leroy F. Meyers, The Ohio State University.*

(a) Given three positive integers, show how to determine algebraically (rather than by a search) the row (if any) of Pascal's triangle in which these integers occur as consecutive entries.

(b) Given two positive integers, can one similarly determine the row (if any) in which they occur as consecutive entries?

(c)* The positive integer k occurs in the row of Pascal's triangle beginning with $1, k, \ldots$. For which integers is this the only row in which it occurs?

858. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c. For $n = 0,1,2,\ldots$, let P_n be a point in the plane whose distances d_a, d_b, d_c from sides a,b,c satisfy

$$d_a : d_b : d_c = \frac{1}{a^n} : \frac{1}{b^n} : \frac{1}{c^n}.$$

(a) A point P_n being given, show how to construct P_{n+2}.

(b) Using (a), or otherwise, show how to construct the point P_n for an arbitrary given value of n.

859. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let ABC be an acute-angled triangle of type II, that is (see [1982: 64]), such that $A \leq B \leq \pi/3 \leq C$, with circumradius R and inradius r. It is known [1982: 66] that for such a triangle $x \geq \frac{1}{4}$, where $x = r/R$. Prove the stronger inequality

$$x \geq \frac{\sqrt{3} - 1}{2}.$$

860*. *Proposed by Anders Lönnberg, Mockfjärd, Sweden.*

The sequence $\{s_n\}_{n=1}^{\infty}$ is defined by

$$s_n = \sum_{m=1}^{n} (-1)^{m-1} m^{n-(m-1)},$$

so that

$$s_1 = 1^1 = 1, \quad s_2 = 1^2-2^1 = -1, \quad s_3 = 1^3-2^2+3^1 = 0, \quad s_4 = 1^4-2^3+3^2-4^1 = -2,$$

etc. Does $s_n = 0$ ever occur again for some $n > 3$?

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

712. [1982: 47; 1983: 56] Late solutions: LEON BANKOFF, Los Angeles, California; et ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec.

* * *

727. [1982: 78; 1983: 115] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let t_b and t_c be the symmedians issued from vertices B and C of triangle ABC and terminating in the opposite sides b and c, respectively. Prove that $t_b = t_c$ if and only if $b = c$.

II. *Comment by the proposer.*

It is well known that a triangle is isosceles if and only if it has two equal internal angle bisectors. (The "only if" part is trivial, and the "if" part is the well-known and nontrivial Steiner-Lehmus Theorem.) The situation is entirely different when "internal angle bisectors" is replaced by "external angle bisectors". The "only if" part remains true and trivial, but the "if" part is no longer true: there are nonisosceles triangles with two equal external angle bisectors. Such triangles are known as *pseudo-isosceles triangles* (one example is the *Emmerich triangle*, with angles 132°, 36°, and 12°). See [1976: 19-24] in this journal for details and references.

There is a very intimate connection between symmedians and internal angle bisectors on the one hand, and exsymmedians and external angle bisectors on the other. A *symmedian* through a vertex of a triangle is the reflection of the median in the internal angle bisector at that vertex, and an *exsymmedian* is the reflection of the exmedian in the external angle bisector. (An *exmedian* is a line through a vertex parallel to the opposite side of the triangle.)

Conside now the following two theorems:

Theorem 1. The present problem.

Theorem 2. The present problem with symmedians t_b, t_c replaced by exsymmedians T_b, T_c.

In the light of the first two paragraphs of this comment, one would expect Theorem 1 to be true and Theorem 2 to be false. Theorem 1 is indeed true, as attested by my own solution and by references [1]-[3] given earlier [1983: 115]. And Theorem 2 is indeed false, in spite of the Wulczyn reference [4] given earlier [1983: 116], wherein the theorem is "proved" to be true. More precisely, $b = c$ implies $T_b = T_c$, but $T_b = T_c$ does *not* imply $b = c$. This was proved recently by O. Bottema and the author in *Nieuw Tijdschrift voor Wiskunde*, 70 (1983) 143-151. We showed that

$$T_b = T_c \iff (b^2 - c^2)(2a^2 - b^2 - c^2) = 0.$$

Nonisosceles triangles in which $2a^2 = b^2+c^2$ have $T_b = T_c$. We will call such triangles *quasi-isosceles* (by analogy with pseudo-isosceles). We showed how to construct infinitely many such triangles with integral sides, and derived several of their properties.

Editor's comment.

Readers are invited to try and find the error in the Wulczyn solution. (It may be a significant fact that no solution other than Wulczyn's was submitted.) If they find it, they should inform the Problem Editor of *Pi Mu Epsilon Journal*, where this solution appeared. If they *don't* find it, then it may be Bottema and our proposer who are "in Dutch". Readers should then scrutinize *their* solution (or ask them for a translation: the original appeared in Dutch).

Quasi-isosceles triangles have been extensively studied in the past under other names. They have been variously called *automedian triangles* (because their medians m_a, m_b, m_c can be rearranged by translation to form a triangle inversely similar to the original triangle) and *RMS triangles* (because $a = \sqrt{(b^2+c^2)/2}$, the *root-mean-square* of b and c). Much of this discussion has taken place right here in this journal. See Crux 210 [1977: 10, 160, 196; 1978: 13, 193], Crux 309 [1978: 12, 200], and Crux 313 [1978: 35, 207], where other references are given.

What? You don't have our back volumes? Too bad. *We* still have some, and will gladly exchange them for some of your money-money, if *you* still have some.

* * *

735. [1982: 107; 1983: 123] Late partial solution by ELWYN ADAMS, Gainesville, Florida.

* * *

743. [1982: 135] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with centroid G inscribed in a circle with center O. A point M lies on the disk ω with diameter OG. The lines AM, BM, CM meet the circle again in A', B', C', respectively, and G' is the centroid of triangle A'B'C'. Prove that

(a) M does not lie in the interior of the disk ω' with diameter OG';

(b) [ABC] $\le$ [A'B'C'], where the brackets denote area.

Solution by the proposer.

First we set up some machinery to deal with the problem. Let γ be the circumcircle of triangle ABC. We assume for now that M is any point not on γ. We recall a well-known formula of Leibniz: If the points $P_1, P_2, \ldots, P_k$ have masses $m_1, m_2, \ldots, m_k$, respectively, and if G is the centroid of the system, then for every point P we have

$$\sum_{i=1}^{k} m_i |\vec{PP_i}|^2 = m|\vec{PG}|^2 + \frac{1}{m} \sum_{i<j} m_i m_j |\vec{P_iP_j}|^2,$$

where $m = \Sigma m_i$. For the system consisting of the vertices of triangle ABC, each with unit mass, we obtain, with P = M,

$$MA^2 + MB^2 + MC^2 = 3MG^2 + \tfrac{1}{3}(a^2+b^2+c^2), \tag{1}$$

where a,b,c are the sides of the triangle; and with P = O we get

$$3R^2 = 3OG^2 + \tfrac{1}{3}(a^2+b^2+c^2), \tag{2}$$

where R is the radius of γ. If $p(P)$ denotes the power of a point P with respect to γ, we get from (2)

$$p(G) = OG^2 - R^2 = -\frac{1}{9}(a^2+b^2+c^2). \tag{3}$$

Since M does not lie on γ, we have, using directed segments,

$$p(M) = MA \cdot MA' = MB \cdot MB' = MC \cdot MC' \ne 0,$$

from which follows

$$\frac{MA}{MA'} + \frac{MB}{MB'} + \frac{MC}{MC'} = \frac{MA^2 + MB^2 + MC^2}{p(M)}. \tag{4}$$

Now $p(M) = OM^2 - R^2$; hence, from (1), (3), and (4),

$$S \equiv \frac{MA}{MA'} + \frac{MB}{MB'} + \frac{MC}{MC'} = -3 + 3 \cdot \frac{OM^2 + MG^2 - OG^2}{p(M)};$$

and therefore, since the three summands of S all have the same sign, we have, for any point M not on γ,

$$|S| = \left|\frac{MA}{MA'}\right| + \left|\frac{MB}{MB'}\right| + \left|\frac{MC}{MC'}\right| = \left|-3 + 3\cdot\frac{OM^2+MG^2-OG^2}{p(M)}\right|. \qquad (5)$$

We conclude from (5) that $|S| \le 3$ if M lies on the (closed) disk ω, with equality if and only if M lies on the boundary of ω, and $|S| > 3$ if M lies outside ω.

Proceeding likewise with triangle A'B'C', centroid G', and disk ω', we obtain

$$|S'| \equiv \left|\frac{MA'}{MA}\right| + \left|\frac{MB'}{MB}\right| + \left|\frac{MC'}{MC}\right| = \left|-3 + 3\cdot\frac{OM^2+MG'^2-OG'^2}{p(M)}\right|,$$

and $|S'| \le 3$ if M lies on the (closed) disk ω', with equality if and only if M lies on the boundary of ω', and $|S'| > 3$ if M lies outside ω'.

Everything is now in place to solve our problem.

(a) If M lies on the disk ω, then $|S| \le 3$, and $|S'| \ge 3$ now follows from the well-known inequality

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9,$$

which holds for all $x,y,z > 0$. Therefore M lies outside or on the boundary of ω'. In any case, M is not an interior point of ω'.

(b) Again, suppose that M lies on the disk ω, and let a',b',c' be the sides of triangle A'B'C'. From the similarity of triangles BMC and C'MB', we get $|MB| : |MC'| = a : a'$. From this and two similar results, we get

$$\left|\frac{MA}{MA'}\right|\cdot\left|\frac{MB}{MB'}\right|\cdot\left|\frac{MC}{MC'}\right| = \frac{abc}{a'b'c'} = \frac{[ABC]}{[A'B'C']}.$$

From the A.M.-G.M. inequality, we now have

$$\sqrt[3]{\frac{[ABC]}{[A'B'C']}} = \sqrt[3]{\left|\frac{MA}{MA'}\right|\cdot\left|\frac{MB}{MB'}\right|\cdot\left|\frac{MC}{MC'}\right|} \le \frac{1}{3}\left(\left|\frac{MA}{MA'}\right|+\left|\frac{MB}{MB'}\right|+\left|\frac{MC}{MC'}\right|\right) = \frac{|S|}{3} \le 1,$$

and so $[ABC] \le [A'B'C']$, with equality if and only if M = O. □

More generally, let G be the centroid of the points $P_1,P_2,\ldots,P_k$ on a hypersphere γ with center O in Euclidean space R^n. For any point M not on γ, let the lines MP_i meet γ again in P_i', and let G' be the centroid of the k points P_i'. Finally, let ω and ω' be the balls with diameters OG and OG', respectively. If

$$S = \frac{MP_1}{MP_1'} + \frac{MP_2}{MP_2'} + \cdots + \frac{MP_k}{MP_k'},$$

where the segments are directed, then we can show as above that $|S| \le k$ if M lies in ω, with equality if and only if M lies on the boundary of ω, and $|S| > k$ if M lies outside ω.

Proceeding from this, it is easy to show as above that if M is in ω, then it is not an interior point of ω'. Also, for $n = 2$, if $P_1P_2\ldots P_k$ is a regular polygon inscribed in circle γ, then $|S| \geq k$ for any point M not on γ.

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744. [1982: 136] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

(a) Prove that, for all nonnegative integers n,

$$5 \mid 2^{2n+1} + 3^{2n+1},$$
$$7 \mid 2^{n+2} + 3^{2n+1},$$
$$11 \mid 2^{8n+3} + 3^{n+1},$$
$$13 \mid 2^{4n+2} + 3^{n+2},$$
$$17 \mid 2^{6n+3} + 3^{4n+2},$$
$$19 \mid 2^{3n+4} + 3^{3n+1},$$
$$29 \mid 2^{5n+1} + 3^{n+3},$$
$$31 \mid 2^{4n+1} + 3^{6n+9}.$$

(b) Of the first eleven primes, only 23 has not figured in part (a). Prove that there do not exist polynomials f and g such that

$$23 \mid 2^{f(n)} + 3^{g(n)}$$

for all positive integers n.

I. *Solution by Kenneth W. Spackman, University of Kentucky at Lexington.*

For part (a), we simply note that, for all nonnegative integers n,

$$2^{2n+1} + 3^{2n+1} \equiv 4^n(2+3) \equiv 0 \pmod{5},$$
$$2^{n+2} + 3^{2n+1} \equiv 2^n(2^2+3) \equiv 0 \pmod{7},$$
$$2^{8n+3} + 3^{n+1} \equiv 3^n(2^3+3) \equiv 0 \pmod{11},$$
$$2^{4n+2} + 3^{n+2} \equiv 3^n(2^2+3^2) \equiv 0 \pmod{13},$$
$$2^{6n+3} + 3^{4n+2} \equiv 13^n(2^3+3^2) \equiv 0 \pmod{17},$$
$$2^{3n+4} + 3^{3n+1} \equiv 8^n(2^4+3) \equiv 0 \pmod{19},$$
$$2^{5n+1} + 3^{n+3} \equiv 3^n(2+3^3) \equiv 0 \pmod{29},$$
$$2^{4n+1} + 3^{6n+9} \equiv 16^n(2+3^9) \equiv 0 \pmod{31},$$

the last being a consequence of $2+3^9 = 5 \cdot 31 \cdot 127$.

The following theorem and the adjoining table will give us more insight into part (a) as well as provide a proof for part (b).

p	$\operatorname{ord}_p 2$	$\operatorname{ord}_p 3$
5	4	4
7	3	6
11	10	5
13	12	3
17	8	16
19	18	18
23	11	11
29	28	28
31	5	30
37	36	18
41	20	8
43	14	42
47	23	23

THEOREM. Let $p > 3$ be prime. If the order modulo p of either 2 or 3 is even, then there exist linear polynomials f and g with integer coefficients such that

$$p \mid 2^{f(n)} + 3^{g(n)} \qquad (1)$$

for every nonnegative integer n.

On the other hand, if $\operatorname{ord}_p 2$ and $\operatorname{ord}_p 3$ are both odd, then there are no polynomials f and g (of any degree) with integer coefficients for which (1) holds for any nonnegative integer n.

Proof. Suppose $\operatorname{ord}_p 2 = 2a$, where a is a positive integer. Then $2^{2a} \equiv 1$ but $2^a \equiv -1$ (congruences throughout are modulo p). Thus

$$2^{2a} - 3^{p-1} \equiv 1 - 1 = 0,$$

and so either

$$2^a + 3^{(p-1)/2} \equiv 0 \qquad \text{or} \qquad 2^a - 3^{(p-1)/2} \equiv 0.$$

In the first case,

$$2^{(p-1)n+a} + 3^{(p-1)n+(p-1)/2} \equiv 0$$

for all nonnegative integers n. In the second case, we must have $3^{(p-1)/2} \equiv -1$, and so

$$2^{(p-1)n+2a} + 3^{(p-1)n+(p-1)/2} \equiv 0$$

for all nonnegative integers n. A similar argument goes through if $\operatorname{ord}_p 3$ is even.

We now assume that $\operatorname{ord}_p 2$ and $\operatorname{ord}_p 3$ are both odd; then their least common multiple, say m, is also odd. Suppose there exist polynomials f and g with integer coefficients and a nonnegative integer n such that $p \mid 2^a+3^b$, where $a = f(n)$ and $b = g(n)$. (If a or b is negative, then 2^{-1} or 3^{-1} must of course be calculated modulo p.) From $2^a + 3^b \equiv 0$, we obtain successively

$$2^a \equiv -3^b, \qquad 2^{am} \equiv (-1)^m 3^{bm}, \qquad 1 \equiv -1,$$

and the last is ruled out because $p > 3$. □

If it is assumed that the proposer meant "polynomials *with integer coefficients*", then part (b) of our problem now follows from the fact that, as shown in the table, $\operatorname{ord}_{23} 2$ and $\operatorname{ord}_{23} 3$ are both odd.

II. *Comment by R.B. Killgrove, University of South Carolina at Aiken.*

We give a counterexample to show that part (b) is incorrect as stated, that is, without the specification that the polynomials have integer coefficients. We claim that

$$23 \mid 2^{f(n)} + 3^{g(n)}$$

for all nonnegative integers n when

$$f(n) = 11n + 3 \quad \text{and} \quad g(n) = 11n + \frac{\ln 15}{\ln 3}.$$

Let $h(n) = 2^{f(n)} + 3^{g(n)}$. We have $h(0) = 2^3 + 15 = 23$, and so $23 \mid h(0)$. Suppose $23 \mid h(k)$ for some integer $k \geq 0$; then, since

$$2^{11} = 23 \cdot 89 + 1 \quad \text{and} \quad 3^{11} = 23 \cdot 7702 + 1,$$

we have

$$\begin{aligned} h(k+1) &= 2^{11(k+1)+3} + 3^{11(k+1)+\ln 15/\ln 3} \\ &= 2^{11} \cdot 2^{11k+3} + 3^{11} \cdot 3^{11k+\ln 15/\ln 3} \\ &= 23 \cdot 89 \cdot 2^{11k+3} + 23 \cdot 7702 \cdot 3^{11k+\ln 15/\ln 3} + h(k) \\ &= 23 \cdot 89 \cdot 2^{11k+3} + 23 \cdot 7702 \cdot 15 \cdot 3^{11k} + h(k); \end{aligned}$$

so $23 \mid h(k+1)$ and the induction is complete.

Also solved by CURTIS COOPER, Central Missouri State University; the COPS of Ottawa; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT S. JOHNSON, Montréal, Québec (partial solution); FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; R.B. KILLGROVE, University of South Carolina at Aiken; F.G.B. MASKELL, Algonquin College, Ottawa; L.F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; DAVID SINGMASTER, Polytechnic of the South Bank, London; MALCOLM A. SMITH, Georgia Southern College; DAVID R. STONE, Georgia Southern College; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Singmaster gave a generalization, roughly along the lines of the theorem in our solution I, to arbitrary base primes instead of just 2 and 3. He also outlined far-reaching generalizations to arbitrary modulus and even to arbitrary commutative rings with unit. A copy of his solution is available from the editor.

On the debit side, one solver, in proving a "stronger result" for part (b), started by saying that "It is clear that a and b must both be nonnegative integers for 2^a+3^b to be an integer". Clear as mud, as our comment II shows. And the COPS of Ottawa (whoever they are) incautiously crawled out on a limb by claiming, on the basis of unspecified evidence, that 41 is the next prime that has the same "defect" as 23. The editor sneakily extended the table in solution I to show that the correct answer is 47, not 41. Crack! We hope the COPS all land on their (flat) feet. There's no police like Holmes.

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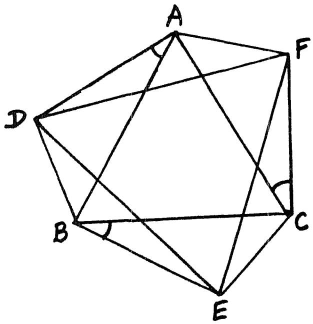
745. [1982: 136] *Proposed by Roger Izard, Dallas, Texas.*

In the adjoined figure, triangles ABC and DEF are both equilateral, and angles BAD, CBE, and ACF are all equal. Prove that triangles ABC and DEF have the same center.



Solution by the COPS of Ottawa.

Let BE ∩ CF = P, CF ∩ AD = Q, and AD ∩ BE = R. Triangle PQR is equilateral, for

$$\angle P = \angle QCB - \angle CBP = \angle ACB = 60^\circ,$$

with similar results at Q and R. It is very easy to show that two equilateral triangles, one of which is inscribed in the other, have the same centre. Hence triangles ABC, PQR, and DEF all have the same centre.

Also solved by JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; DAN PEDOE, University of Minnesota; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands (two solutions); DAN SOKOLOWSKY, California State University at Los Angeles (two solutions); GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer (two solutions).

Editor's comment.

A superficially different but equivalent problem by the same proposer appeared as Problem 1161 in *Mathematics Magazine*, 56 (1983) 46.

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746. [1982: 136] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given are two concentric circles and a triangle ABC inscribed in the outer circle. A tangent to the outer circle at A is rotated about A in the counterclockwise sense until it first touches the inner circle, say at P. The procedure is repeated at B and C, resulting in points Q and R, respectively, on the inner circle. Prove that triangle PQR is directly similar to triangle ABC.

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Let O be the center, and r and R $(r < R)$ the radii, of the two circles. Since

$$\angle AOP = \angle BOQ = \angle COR = \operatorname{Arccos}\frac{r}{R} \equiv \theta,$$

it follows that triangle ABC is transformed into triangle PQR by a rotation through θ about O followed by a shrinking of ratio $R : r$. Both of these transformations preserve direct similarity.

Also solved by the COPS of Ottawa; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; L.F. MEYERS, The Ohio State University; DAN PEDOE, University of Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; MALCOLM A. SMITH, Georgia Southern College; DAN SOKOLOWSKY, California State University at Los Angeles (two solutions); GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

Sokolowsky noted that the same conclusion follows even if AP,BQ,CR are not tangent to the inner circle, provided the directed angles AOP,BOQ,COR are all equal.

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747. [1982: 137] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle (with sides a,b,c in the usual order) inscribed in a circle with center O. The segments BC,CA,AB are divided internally in the same ratio by the points A_1,B_1,C_1, respectively, so that

$$\overline{BA_1} : \overline{A_1C} = \overline{CB_1} : \overline{B_1A} = \overline{AC_1} : \overline{C_1B} = \lambda : \mu,$$

where $\lambda + \mu = 1$. A line through A_1 perpendicular to OA meets the circle in two points, one of which, P_a, lies on the arc CAB; and points P_b, P_c are determined analogously by lines through B_1,C_1 perpendicular to OB, OC. Prove that

$$\overline{AP}_a^2 + \overline{BP}_b^2 + \overline{CP}_c^2 = a^2 + b^2 + c^2,$$

independently of λ and μ.

Investigate the situation if the word "internally" is replaced by "externally".

Solution by Jordi Dou, Barcelona, Spain.

Let A_1', B_a, C_a be the orthogonal projections of A_1, B, C, respectively, upon AO. We have

$$\overline{AA_1'} = \overline{AB}_a + \overline{B_aA_1'} = \overline{AB}_a + \lambda\overline{B_aC_a} = \overline{AB}_a + \lambda(\overline{AC}_a - \overline{AB}_a) = \lambda\overline{AC}_a + \mu\overline{AB}_a.$$

Since $\overline{AC}_a = b\sin B = b^2/2R$ and $\overline{AB}_a = c\sin C = c^2/2R$, where R is the circumradius, we therefore have

$$\overline{AP}_a^2 = 2R\overline{AA_1'} = \lambda b^2 + \mu c^2.$$

Finally, with this and two similar results, we obtain the desired

$$\overline{AP}_a^2 + \overline{BP}_b^2 + \overline{CP}_c^2 = a^2 + b^2 + c^2. \tag{1}$$

In general, the line through A_1 perpendicular to OA meets the circle in two points, and either one can be used for P_a, not necessarily the one on the arc CAB; and similarly for P_b and P_c.

Since directed segments were used in the solution, the result (1) remains valid when A_1, B_1, C_1 are outside the circle, provided the points P_a, P_b, P_c exist. If $\overline{AA_1'} > 2R$, for example, then P_a does not exist, but in the relation

$$\overline{AP}_a^2 = \overline{AA_1'}^2 + \overline{A_1'P}_a^2 = \overline{AA_1'}^2 + (R^2 - \overline{OA_1'}^2), \tag{2}$$

which is valid when P_a exists, the right member always has a positive value because $\overline{AA_1'} > \overline{OA_1'}$. So if we consider that $\overline{AP}_a$ is *defined* by (2) when P_a does not exist, then from

$$\overline{OA_1'}^2 = (\overline{AA_1'} - R)^2 = \overline{AA_1'}^2 - 2R\overline{AA_1'} + R^2,$$

we conclude that $\overline{AP}_a^2 = 2R\overline{AA_1'}$ and the result (1) remains valid.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer.

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748. [1982: 137] *Proposed by H. Kestelman, University College, London, England.*

Let $a_1, a_2, \ldots, a_n$ be distinct complex numbers. For the *Vandermonde matrix*

$$M = \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix},$$

show that the elements of the jth column of M^{-1} are the coefficients in the polynomial f_j, of degree $n-1$, given by

$$f_j(t) = \prod_{r \neq j} \frac{t-a_r}{a_j-a_r}.$$

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Since the a_i are all distinct, M^{-1} exists and the function f_j exists for any j. For some chosen j, let $[c_0, c_1, \ldots, c_{n-1}]$ be the jth column vector of M^{-1}. The scalar product of the ith row vector of M and the jth column vector of M^{-1} is 1 if $i = j$ and 0 if $i \neq j$. Therefore if we set

$$\phi_j(t) = c_0 + c_1 t + \ldots + c_{n-1}t^{n-1},$$

then we have $\phi_j(a_j) = 1$ and $\phi_j(a_r) = 0$ if $r \neq j$. Since $f_j(t)$ also has this property, $\phi_j - f_j$ has degree less than n and has n zeros. Thus $\phi_j - f_j = 0$ and $\phi_j(t) \equiv f_j(t)$, as required.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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749*. [1982: 137] *Proposed by Ram Rekha Tiwari, Radhaur, Bihar, India.*

Solve the system

$$\frac{yz(x+y+z)(y+z-x)}{(y+z)^2} = a^2$$

$$\frac{zx(x+y+z)(z+x-y)}{(z+x)^2} = b^2$$

$$\frac{xy(x+y+z)(x+y-z)}{(x+y)^2} = c^2.$$

Editor's comment.

Not one solution or comment was received for this elementary problem. C'mon, readers.

* * *

750. [1982: 138] *Proposed by Anders Lönnberg, Mockfjärd, Sweden.*

For which real a does the graph of

$$y = x^{x^{x^a}}, \quad x > 0$$

have a point of inflection with horizontal tangent (that is, where $y' = 0$)?

Solution by the COPS of Ottawa.

Let $z = x^{x^a}$, so that $\ln z = x^a \ln x$ and $\ln y = z \ln x$. Differentiating with respect to x, we get

$$\frac{z'}{z} = \frac{x^a}{x}(1 + a\ln x)$$

and

$$\frac{y'}{y} = \frac{z}{x} + z'\ln x = \frac{z}{x}f(x), \tag{1}$$

where

$$f(x) = 1 + x^a(1 + a\ln x)\ln x; \tag{2}$$

and differentiating (1) gives

$$\frac{yy'' - y'^2}{y^2} = \frac{xz'-z}{x^2}f(x) + \frac{z}{x}f'(x). \tag{3}$$

Since $y > 0$ and $z > 0$ for all $x > 0$, we get from (1)

$$y' = 0 \iff f(x) = 0$$

and hence from (3)

$$y'' = y' = 0 \iff f(x) = f'(x) = 0.$$

Now

$$f'(x) = x^{a-1}g(x),$$

where

$$g(x) = (a \ln x)^2 + 3(a \ln x) + 1.$$

Therefore

$$f'(x) = 0 \iff g(x) = 0 \iff a \ln x = k \iff x = e^{k/a},$$

where $k = (-3\pm\sqrt{5})/2$. Now from (2)

$$f'(x) = f(x) = 0 \iff x - e^{k/a} = 1 + e^{k}(1+k)\frac{k}{a} = 0. \tag{4}$$

Furthermore, from $f''(x) = x^{a-1}g'(x) + (a-1)x^{a-2}g(x)$ we obtain

$$f''(e^{k/a}) = ae^{k-2k/a}(2k+3) \neq 0,$$

and this implies that $y''' \neq 0$ for $x = e^{k/a}$.

The curve has a point of inflection with horizontal tangent just when $y' = y'' = 0$ but $y''' \neq 0$, that is, just when $x = e^{k/a}$, where $k = (-3\pm\sqrt{5})/2$ and, from (4), $a = -k(1+k)e^{k}$. Explicitly, the required values of a are

$$-(2+\sqrt{5})\exp\left(-\frac{3+\sqrt{5}}{2}\right) \quad \text{and} \quad -(2-\sqrt{5})\exp\left(-\frac{3-\sqrt{5}}{2}\right).$$

Also solved by DAVID R. STONE, Georgia Southern College; and the proposer.

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THE PUZZLE CORNER

Answer to Puzzle No. 37 [1983: 135]: There are 276 (a dozen short of being 2·144) solutions to the initial premise. In the first part we equate this with TEN (the number of my toes) and get MODEST = 598702. For the second part, 277 = ODD, with MODEST either 327981 or 627891. Finally, if 276 = ONE, then MODEST must be 120648.

* * *

VIKTORS LINIS

1916 - 1983

On 7 July 1983, with members of the Senate and of the Faculty of Science and Engineering of the University of Ottawa in attendance, we said farewell to one of the founders of *Crux Mathematicorum*, Viktors Linis, who died suddenly on 2 July 1983 while visiting relatives in Milwaukee, Wisconsin.

Viktors Linis was born in Rostov, Russia, of Latvian parents. He graduated from the University of Latvia in Riga and subsequently came to Canada in 1948. He received his Doctorate of Philosophy from McGill University in 1954, and three years later became head of the Department of Mathematics at the University of Ottawa, which post he held until 1972.

He published papers on complex analysis and on the teaching and philosophy of mathematics. He was president of the Canadian Society for the History and Philosophy of Mathematics from 1975 to 1977, and in 1980 was honoured with an award for excellence in teaching by the Ontario Confederation of University Faculty Associations.

Viktors Linis was generous with his time, his knowledge and experience, and his counsel; never forward, but always approachable. He cared for teachers on the secondary side as well as on the postsecondary side of the secondary-postsecondary interface, and was known and respected throughout the community of high school mathematics teachers in the Ottawa area and beyond.

He actively supported mathematical associations at the national, provincial, and local levels, and was a founding member of the Carleton-Ottawa Mathematics Association in 1973. He was one of the group of teachers from both sides of the interface who, over a two-year period, undertook for the Ministry of Education a study of ways of improving communication across the interface on curriculum matters.

Throughout his life, he retained his love of his homeland, and was prominent in the Canadian Latvian and Baltic communities.

His last generous act before his unexpected death was to associate himself fully with a small group of other subscribers to *Crux Mathematicorum* in a bid to establish a local problem solving club—one which would give enjoyment to participants and at the same time provide a means to teachers of mathematics of keeping mathematically fit.

He will be greatly missed. To his widow and to his sister we extend our deepest sympathy and express our gratitude for the service which Viktors has given to the mathematical community in general and to *Crux Mathematicorum* in particular.

F.G.B. MASKELL