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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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CUBES OF NATURAL NUMBERS IN ARITHMETIC PROGRESSION

K.R.S. Sastry

INTRODUCTION The well known and the pretty identity

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2$$

may be looked upon as follows: $1, 2, 3, \cdots, k, \cdots$ is an arithmetic progression of natural numbers; define $S_k = 1^3 + 2^3 + 3^3 + \cdots + k^3$, then S_k is a square number for each $k = 1, 2, 3, \cdots$. A natural number of the form $\frac{1}{2}k(k+1)$ is called a *triangular* number. Each S_k is therefore the square of the triangular number $\frac{1}{2}k(k+1)$. A little experimentation immediately leads to an equally pretty identity

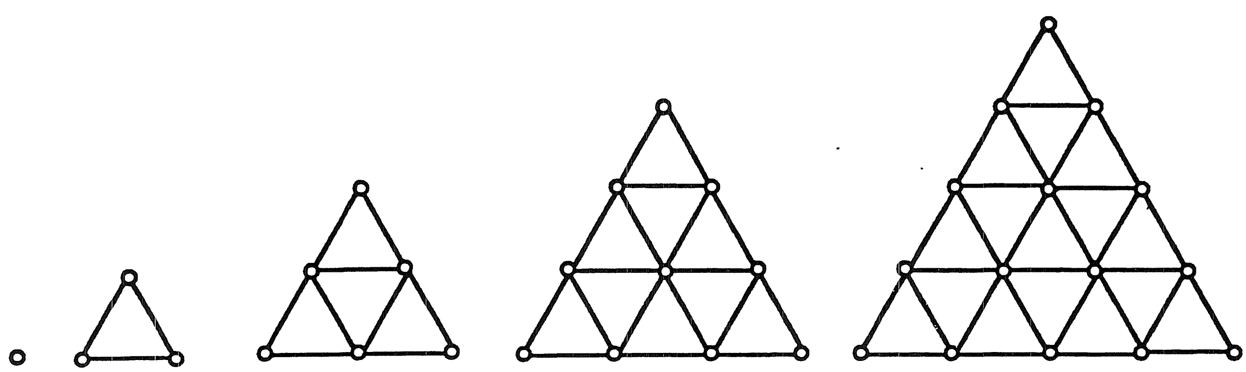
$$S_k = 1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1).$$

This time we observe that the arithmetic progression $1, 3, 5, \cdots, 2k-1, \cdots$ yields the triangular number $S_k = \frac{1}{2}(2k^2 - 1)(2k^2)$ for each $k = 1, 2, 3, \cdots$. Triangular numbers and square numbers are particular instances of the more general type of numbers called *polygonal* or *n-gonal* numbers. The name polygonal numbers of side n and rank r, or n-gonal numbers of rank r, given to the sequence of numbers defined by the equation

$$P(n,r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}; \quad n \geq 3, \ r = 1, 2, 3, \cdots \tag{1}$$

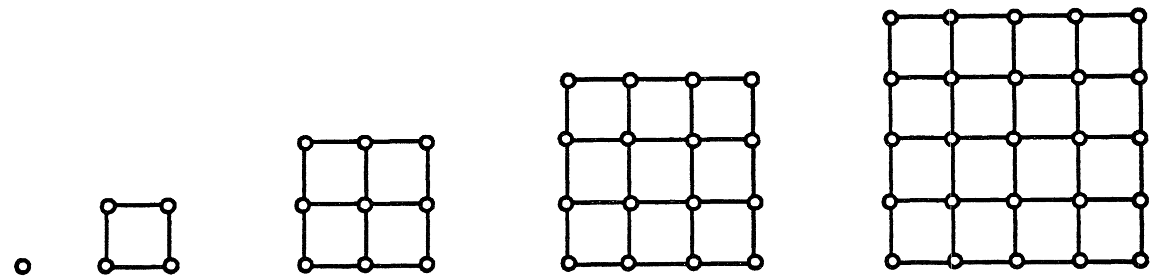
comes from the fact that these numbers of objects may be used to form n-gons, (see [1,2] for details). For example, we have the sequence of

triangular numbers for $n = 3$



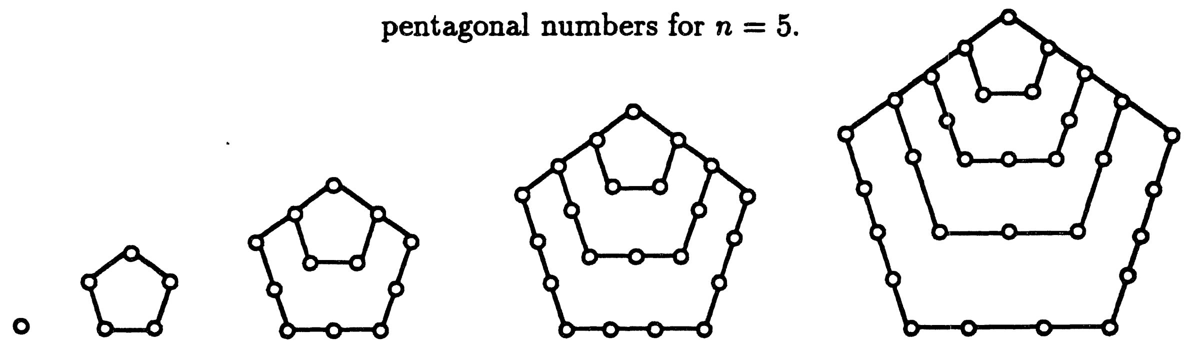
1 3 6 10 15

square numbers for $n = 4$



1 4 9 16 25

pentagonal numbers for $n = 5$.



1 5 12 22 35

THE PROBLEM Given the preceding observations regarding polygonal numbers, a natural question to ask would be:

Let $a_1, a_2, \cdots, a_k = a_1 + (k-1)d, \cdots$ be an arithmetic progression of natural numbers. Define $S_k = S_k(a_1, d) = a_1^3 + a_2^3 + \cdots + a_k^3$. What n-gonal numbers are obtainable as S_k for each $k = 1, 2, 3, \cdots$? (*)

The answer to the question just raised is provided by the following

THEOREM. Given an arithmetic progression $1 = a_1, a_2, a_3, \cdots$ with common difference $d = 1$ or of the form $2e$, the sequences $\{S_k\}$ defined by (*) are n-gonal numbers for $n = 3, 4$, or $4e + 2$ where $e = 1, 2, 3, \cdots$.

In the proof of the theorem, we make use of the following well known identities (these may be established by induction).

$$\sum_{i=1}^{k} i = \frac{1}{2}k(k+1); \quad \sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1); \quad \sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2. \qquad (2)$$

PROOF OF THE THEOREM By definition, 1 is an n-gonal number for all $n \geq 3$. Therefore, $S_1 = a_1^3$ is n-gonal if $a_1 = 1$. Hence $a_1 = 1$. For convenience we consider the sequences $\{a_{k+1}\}$ and $\{S_{k+1}\}$, $k = 0, 1, 2, \cdots$. Now

$$S_{k+1} = \sum_{i=0}^{k} a_{i+1}^3 = 1 + \sum_{i=1}^{k}(1+id)^3.$$

Expanding $(1+id)^3$ and using the identities (2) in the preceding equation, after some algebra, we see that

$$S_{k+1} = \frac{1}{4}(k+1)[k^2(k+1)d^3 + 2k(2k+1)d^2 + 6kd + 4].$$

With some effort, the latter factor further factorizes giving

$$\begin{aligned} S_{k+1} &= \frac{1}{4}(k+1)(kd+2)[k^2d^2 + kd(d+2) + 2] \\ &= \frac{1}{4}[k^2d + k(d+2) + 2][k^2d^2 + kd(d+2) + 2]. \end{aligned} \tag{3}$$

It follows that

(i) S_{k+1} is a square if $k^2d + k(d+2) + 2 = k^2d^2 + kd(d+2) + 2$. This yields $d = 1$ because d is a natural number and k is non-negative; and

(ii) if we write $k^2d + k(d+2) + 2 = x$ then $k^2d^2 + kd(d+2) + 2 = dx + 2 - 2d$ so that (3) can be put in the form

$$S_{k+1} = \frac{1}{4}x(dx + 2 - 2d) = \frac{1}{2}\left(\frac{dx^2}{2} - (2d-2)\frac{x}{2}\right).$$

Comparing this last expression for S_{k+1} with (1) for $P(n,r)$ we see that S_{k+1} can not represent n-gonal numbers for all $k \geq 0$ if d is odd. Put $d = 2e$ in (3) so that

$$S_{k+1} = [ek^2 + (e+1)k + 1][2e^2k^2 + 2e(e+1)k + 1]. \tag{4}$$

In (4) if we write $ek^2 + (e+1)k + 1 = r$ then $2e^2k^2 + 2e(e+1)k + 1 = 2er - (2e-1)$ and

$$S_{k+1} = r[2er - (2e-1)] = [(4e+2) - 2]\frac{r^2}{2} - [(4e+2) - 4]\frac{r}{2}.$$

This last expression shows that S_{k+1} is an n-gonal number of side $n = 4e + 2$ and rank r for all $k = 0, 1, 2, \cdots$, where $e = 1, 2, 3, \cdots$.

To complete the proof of the Theorem, we point out that the hexagonal number $P(6,r) = 2r^2 - r = \frac{1}{2}(2r-1)(2r)$ is the triangular number $P(3, 2r-1)$.

FURTHER PROBLEMS AND DIRECTIONS A glance at the listed references suggests a number of interesting problems involving n-gonal numbers. For example:

(1) Derive formulas to generate three consecutive triangular numbers so that their sum is (i) a square (such as 15, 21, 28 in which $15 + 21 + 28 = 64 = 8^2$)
(ii) a triangular number (such as 36, 45, 55 in which $36+45+55 = 136 = \frac{1}{2} \times 16 \times 17$).

(2) Derive formulas to generate three triangular numbers in arithmetic progression.

For a greater challenge,

(3) Determine natural numbers u and v so that the sum of the first u triangular numbers is the vth triangular number.

Generalizations of the above problems may be considered in which "triangular" is replaced by "n-gonal". [*Editor's note.* See also the author's proposal *Crux* 1756, this issue.]

(4) A referee observes that (3),

$$S_{k+1} = \frac{1}{4}[k^2d + k(d+2) + 2][k^2d^2 + kd(d+2) + 2]$$

can be n-gonal occasionally for odd d. For example $d = 3, k = 1$ yields $S_2 = 65 = P(8,5)$. Determine odd values of d and values of k for which S_{k+1} is an n-gonal number.

Analogous investigations may be undertaken to examine what n-gonal numbers are obtainable as sums of k consecutive natural numbers, k consecutive squares, cubes or higher powers of natural numbers.

The author thanks the editor and the referee for their suggestions.

References

[1] A.H. Beiler, *Recreations in the Theory of Numbers*, Dover, N.Y. (1964) 185–199.
[2] L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, N.Y. (1971) 1–39.

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Editor's note. The above article is the first one to be handled by *Crux*'s new "editor in charge of articles", Denis Hanson. It is hoped that readers will see to it that this feature becomes a regular part of *Crux*, by supplying Denis with articles which are **short, interesting**, and at an **elementary** level ("think high school" is probably a good guideline). Send all such articles to:

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Department of Mathematics and Statistics,
University of Regina,
Regina, Saskatchewan S4S 0A2,
Canada.

* * * * *

THE OLYMPIAD CORNER

No. 136

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we begin with the *Canadian Mathematics Olympiad* for 1992, which we reproduce with the permission of the Canadian Olympiad Committee of the Canadian Mathematical Society. My thanks to Ed Barbeau for sending me the contest. We will discuss the "Official" solutions in the next issue.

1992 CANADIAN MATHEMATICS OLYMPIAD

April, 1992 (Time: 3 hours)

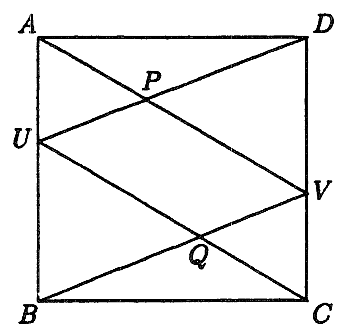
1. Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n+1$ is not an odd prime.

2. For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds.

3. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.



4. Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

5. A deck of $2n+1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n+1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k-1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

* * *

The next set of problems are from the twenty-first annual *United States of America Mathematical Olympiad*, written April 30. These problems are copyrighted by the Committee on the American Mathematics Competition of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588–0322. As always, we welcome your original "nice" solutions and generalizations.

21st UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

April 30, 1992 (Time: 3.5 hours)

1. Find, as a function of n, the sum of the digits of

$$9 \times 99 \times 9999 \times \cdots \times (10^{2^n} - 1),$$

where each factor has twice as many digits as the previous one.

2. Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

3. For a nonempty set S of integers, let $\sigma(S)$ be the sum of the elements of S. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \leq 1500$, there is a subset S of A for which $\sigma(S) = n$. What is the smallest possible value of a_{10}?

4. Chords $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $AA' = BB' = CC'$.

5. Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial

$$(\cdots((z - a_1)^2 - a_2)^2 \cdots - a_{1991})^2 - a_{1992}.$$

* * *

For a variant on the type of competition we give a contest aimed at undergraduates. The problem proposers were A.R. Blass, D.G. Dickson, R.K. Guy, M. Hochster, D.W. Masser, J.E. McLaughlin, and H.L. Montgomery. Note there are more proposers than problems! Thanks go to R.K. Guy for giving us this contest.

8th ANNUAL UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

April 6, 1991

1. Let $f(x)$ be a continuous function defined on the closed interval $[0,1]$, and suppose that $f(0) = f(1)$. Show that there is a real number $a \in [0,1/2]$ such that $f(a) = f(a+1/2)$.

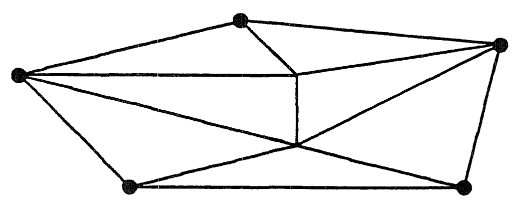
2. Suppose that $a_0, a_1, \ldots, a_n$ are integers with $a_n \neq 0$, and let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Suppose that x_0 is a rational number such that $P(x_0) = 0$. Show that if $1 \leq k \leq n$ then

$$a_k x_0 + a_{k+1} x_0^2 + \cdots + a_n x_0^{n-k+1}$$

is an integer.

3. Let $A = [a_{ij}]$ be an $n \times n$ matrix all of whose entries are ± 1. Suppose that the various columns of A are orthogonal to each other in the sense that if $1 \leq j < k \leq n$ then $\sum_{i=1}^{n} a_{ij} a_{ik} = 0$. Let S denote the difference between the number of $+1$'s in the matrix and the number of -1's. Show that $|S| \leq n^{3/2}$.

4. Suppose that a polygon with n vertices is decomposed into a union of finitely many non-overlapping triangles. Show that the number of interior edges is at least $n-3$. In the example depicted, $n = 5$ and there are 8 interior edges.



5. Let $\mathcal{P}$ denote a convex polygon in the plane whose centroid is C. For each edge e of $\mathcal{P}$ let l_e denote the line through e, and let C_e denote the point on l_e such that the line through C_e and C is perpendicular to l_e. Show that there is an edge e of $\mathcal{P}$ such that C_e lies on the edge e, not just on the line through e.

6. In a group of $2n$ people, each person has at least n friends. Show that it is possible to seat these people around a circular table in such a way that each pair of neighbors are friends. (We assume that if B is a friend of A then A is a friend of B.)

* * *

The only problem set for which we discuss solutions this month is the *1989 Singapore Mathematical Society Interschool Mathematical Competition* [1991: 66–67].

1. Let $n \geq 5$ be an integer. Show that n is a prime if and only if $n_i n_j \neq n_p n_q$ for every partition of n into 4 positive integers, $n = n_1 + n_2 + n_3 + n_4$, and for each permutation (i,j,p,q) of $(1,2,3,4)$.

Solutions by Margherita Barile, Genova, Italy; and by Michael Selby, University of Windsor.

If $n \geq 6$ is an even integer, then $n = 2m$, where $m \geq 3$, and $n = (m-1) + (m-1) + 1 + 1$ with $(m-1)\cdot 1 = (m-1)\cdot 1$. If $n \geq 5$ is an odd composite, we have $n = (2k+1)(2l+1) = 2k + 2l + 1 + 4kl$. Here $2k \cdot 2l = 4kl \cdot 1$.

To prove the converse, suppose $n \geq 5$ and n is such that $n = n_1 + n_2 + n_3 + n_4$, with (possibly after relabeling) $n_1 n_2 = n_3 n_4$. Since n_1 divides $n_3 n_4$ we can write $n_3 = k_1 a$, $n_4 = k_2 b$ with $k_1 k_2 = n_1$. We now have $n_1 n_2 = k_1 k_2 ab$. This gives $n_2 = ab$. Now $n = k_1 k_2 + ab + k_1 a + k_2 b = (k_1 + b)(k_2 + a)$, where $k_1 + b$, $k_2 + a > 1$. This shows that n is not prime, and proves the result.

2. Given arbitrary positive numbers a, b and c, prove that at least one of the following inequalities is false:

$$a(1-b) > \frac{1}{4}\,, \quad b(1-c) > \frac{1}{4}\,, \quad c(1-a) > \frac{1}{4}\,.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Margherita Barile, Genova, Italy; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Stephen D. Hnidei, Windsor, Ontario; Murray S. Klamkin, University of Alberta; Michael Selby, University of Windsor; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give the solution by both Selby and Smeenk.

Suppose each inequality is true. Then

$$a(1-b)b(1-c)c(1-a) > \frac{1}{64}\,.$$

The left-hand side is equal to $a(1-a)b(1-b)c(1-c)$. Since the maximum value of $f(x) = x(1-x)$ is $1/4$ the left-hand side is at most $1/64$, a contradiction.

Both Klamkin and Wang use the arithmetic-mean geometric-mean inequality, and point out a generalization.

Let $a_1, a_2, \ldots, a_n$ be positive, $n \geq 1$, and let π be a permutation of $\{1, 2, \ldots, n\}$. Then $a_i(1 - a_{\pi(i)}) \leq 1/4$ for at least one i.

Proof. If any $a_i \geq 1$, the conclusion is clear. Thus we may assume $0 < a_i < 1$ for all i. By the A.M.-G.M. inequality

$$\frac{1}{2} = \frac{1}{2n}\sum_{i=1}^{n}(a_i + (1 - a_{\pi(i)})) \geq \left(\prod_{i=1}^{n} a_i(1 - a_{\pi(i)})\right)^{1/2n} > \left(\prod_{i=1}^{n}\frac{1}{4}\right)^{1/2n} = \frac{1}{2}\,,$$

a contradiction.

Editor's note. Murray Klamkin points out that one can extend the problem to show that at least one of the inequalities

$$a_i a_{i+r}(1 - a_{i+s}) > 2^2/3^3 \quad \text{for } i = 1, 2, \ldots, n \ (a_{n+i} = a_i)$$

and at least one of the inequalities

$$a_i a_{i+r} a_{i+s}(1 - a_{i+t}) > 3^3/4^4 \quad \text{for } i = 1, 2, \ldots, n \ (a_{n+i} = a_i)$$

are false. One uses the fact that the maximum value of $a_i^m(1 - a_i)$ is $m^m/(m+1)^{m+1}$ from the A.M.-G.M. inequality.

3. (a) Show that

$$\tan\left(\frac{\pi}{12}\right) = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}\,.$$

(b) Given any thirteen distinct real numbers, show that there exist at least two, say x and y, which satisfy the inequality

$$0 < \frac{x-y}{1+xy} < \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}\,.$$

Comments and solutions by Seung-Jin Bang, Seoul, Republic of Korea; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Evagelopoulos, Athens, Greece; Stephen D. Hnidei, Windsor, Ontario; Beatriz Margolis, Paris, France; Bob Prielipp, University of Wisconsin–Oshkosh; D.J. Smeenk, Zaltbommel, The Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

(a) Let $t = \tan(\pi/12)$. Then using the double-angle formula for $\tan\alpha$ we get

$$\frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \frac{2t}{1-t^2}\,,$$

which simplifies to $t^2 + 2\sqrt{3}\,t - 1 = 0$. Since $t > 0$, we obtain

$$t = \frac{1}{2}(-2\sqrt{3} + \sqrt{16}) = 2 - \sqrt{3} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}\,.$$

(b) Let x_i denote the thirteen numbers, $1 \le i \le 13$ and let θ_i $-\pi/2 < \theta_i < \pi/2$ be such that $x_i = \tan\theta_i$. Then

$$\frac{x_i - x_j}{1 + x_i x_j} = \frac{\tan(\theta_i) - \tan(\theta_j)}{1 + \tan\theta_i \tan\theta_j} = \tan(\theta_i - \theta_j).$$

Since $\tan\theta$ is strictly increasing on $(-\pi/2, \pi/2)$, the proposed inequality is equivalent to the statement that given any thirteen angles in $(-\pi/2, \pi/2)$ there exist at least two, say θ_i and θ_j, such that $0 < \theta_i - \theta_j < \pi/12$. This follows by the Pigeon-hole principle partitioning $(-\pi/2, \pi/2)$ into 12 subintervals of length $\pi/12$.

Editor's Note. Bang, Covas, Selby (in his solution to number 2) and Wang all pointed out the similarity to problem 5 of the 16th Canadian Olympiad [1984: 181]. Wang points out that it was also problem E3121 in the *Monthly* (Vol. 92, 1985, p. 736) solved in (Vol. 94, 1987, 880–881). There they generalize to $n \ge 4$ distinct reals and obtain at least two with

$$0 < \frac{x-y}{1+xy} < \tan\left(\frac{\pi}{n-1}\right)\,.$$

4. There are n participants in a conference. Suppose (i) every 2 participants who know each other have no common acquaintances, and (ii) every 2 participants who do not know each other have exactly 2 common acquaintances. Show that every participant is acquainted with the same number of people in the conference.

Solution by Margherita Barile, Genova, Italy.

Let α be one of the participants in the conference, and let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set of his acquaintances. If $n = 2$ then $r = 1$ and both participants each know one person. If $n > 2$, then $n \geq 4$ and it is easy to check that $r \geq 2$.

Since the elements of A have α as common acquaintance, they pairwise do not know each other; hence, in particular α_1 and α_2 have exactly one further common acquaintance β_1. Call B_1 the set of the acquaintances of β_1. Since α and β_1 have α_1 and α_2 as common acquaintances, and by the hypothesis they cannot have more, we have $A \cap B_1 = \{\alpha_1, \alpha_2\}$. Now suppose $\beta_2 \neq \beta_1$, with $\beta_2 \notin A$ and call B_2 the set of acquaintances of β_2. Then $A \cap B_2 = \{\alpha_i, \alpha_j\}$ for some distinct $i, j \in \{1, \ldots, r\}$. If $i = 1$ and $j = 2$ (or vice versa), then α_1 and α_2 would have three common acquaintances, namely α, β_1, β_2, and this contradicts the hypothesis. Therefore $A \cap B_1 \neq A \cap B_2$. This actually shows there is a bijective correspondence between the ordered pairs of elements of A and the $n - r - 1$ participants α does not know. Hence $r(r-1)/2 = \binom{r}{2} = n - r - 1$ and so

$$r^2 + r + 2 - 2n = 0. \tag{1}$$

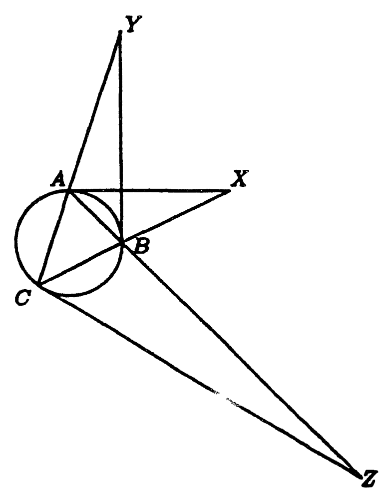
Solving this quadratic we get the solutions

$$r_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{8n-7}, \quad r_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{8n-7}\,.$$

Since r_2 is negative, we discard it. So $r_1 = (-1 + \sqrt{8n-7})/2$ and this is independent of the choice of α.

[*Editor's note.* The condition (1) yields $n = \frac{r(r+1)}{2} + 1$ as a necessary condition, but it is not sufficient. The reader can check that for $r = 3$, $n = 7$ there can be no solution.]

5. In the following diagram, ABC is a triangle, and X, Y, and Z are respectively the points on the sides CB, CA and BA extended such that XA, YB, and ZC are tangents to the circumcircle of ΔABC. Show that X, Y and Z are collinear.



Solutions and comments by Seung-Jin Bang, Seoul, Republic of Korea, and D.J. Smeenk, Zaltbommel, The Netherlands. We give the Euclidean proof submitted by Smeenk.

Let α, β and γ be the angles of ΔABC at A, B and C, respectively. Then $\angle BAX = \gamma$ and $\angle BXA = \beta - \gamma$. Applying the law of sines to ΔBAX,

$$\frac{c}{\sin(\beta-\gamma)} = \frac{XB}{\sin\gamma} \Rightarrow XB = \frac{c\sin\gamma}{\sin(\beta-\gamma)}\,. \tag{1}$$

Similarly from ΔCAX,

$$XC = \frac{b\sin\beta}{\sin(\beta-\gamma)}\,. \tag{2}$$

Now from (1) and (2) and the law of sines for ΔABC,

$$\frac{XB}{XC} = \frac{c^2}{b^2}\,. \tag{3}$$

In the same way

$$\frac{YC}{YA} = \frac{a^2}{c^2} \quad \text{and} \quad \frac{ZA}{ZB} = \frac{b^2}{a^2}\,. \tag{4}$$

From (3) and (4),

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB} = \frac{c^2}{b^2}\cdot\frac{a^2}{c^2}\cdot\frac{b^2}{a^2} = 1.$$

It follows that X, Y and Z are collinear by Ceva's theorem.

Editor's note. Bang provided an analytic proof, as well as pointing out a solution published in *TRIGON* 27, 2(1989), p. 13. He also pointed out that the problem is a (very) special case of Pascal's Theorem on the inscribed hexagon $A_1A_2A_3A_4A_5A_6$ of a conic section. Let $P = A_1A_2 \cap A_4A_5$, $Q = A_2A_3 \cap A_5A_6$, $R = A_3A_4 \cap A_6A_1$. Then P, Q, R are collinear. (See H.S.M. Coxeter and S.L. Greitzer, *Geometry revisited*, MAA, 1967, pp. 74–76) The given problem takes $A_1 = A_2 = A$, $A_3 = A_4 = B$, $A_5 = A_6 = C$, and the conic section is a circle.

* * *

That pretty much finishes matters for the March 1991 number. We end with a comment on a solution given there.

3. [1989: 100; 1991: 69] *1987 Hungarian National Olympiad.*

Determine the minimum of the function $f(x) = \sqrt{a^2+x^2} + \sqrt{(b-x)^2+c^2}$ where a, b, c are positive numbers.

Comment by Murray S. Klamkin, University of Alberta.

An immediate solution with generalizations follows by Minkowski's inequality. Here

$$f(x) \geq [(a+c)^2 + (x+b-x)^2]^{1/2} = [(a+c)^2 + b^2]^{1/2}$$

and with equality if and only if $a/c = x/(b-x)$. One generalization is that in the above all the powers of 2 are replaced by powers of p and all the powers of $1/2$ are replaced by powers of $1/p$, where $p > 1$. Another generalization is

$$(a^p + x^p + y^p)^{1/p} + (b^p + (c-x)^p + (d-y)^p)^{1/p} \geq ((a+b)^p + c^p + d^p)^{1/p}.$$

* * * * *

BOOK REVIEWS

Edited by ANDY LIU, University of Alberta.

Mathematics and Informatics Quarterly. Editorial Board: George Berzsenyi, Petar Kenderov, Mark Saul, Jordan Tabov and Willie Yong. Sponsored by the Bulgarian Academy of Sciences and the Union of Bulgarian Mathematicians. Published by Science, Culture and Technology Publishing, Singapore. ISSN 0218-2513. Subscription address: Prof. George Berzsenyi, Department of Mathematics, Box 121, Rose-Hulman Institute of Technology, Terre Haute, IN 47803-3999, USA. Subscription Rates: US\$12 (student), US\$18 (individual), and US\$30 (institution), surface mail. *Reviewed by Andy Liu.*

This international journal began publication in 1991, with three pilot issues in the first volume. The intended audience are high school and undergraduate students. Each issue contains several very readable articles on mathematics and theoretical computing science.

Regular features include warm-up problems (with solutions in the same issue), forgotten theorems, and a problem section under the title "International Mathematical Talent Search". The reviewer is very impressed with the quality of this publication, and recommends it highly to all readers.

Here is a sample problem from the IMTS. "On an 8 by 8 board, we place d dominoes, each covering two adjacent squares, so that no more dominoes can be placed on the remaining squares. What is the smallest value of d for which the above statement is true?"

*

Tournament of the Towns, 1984–1989, Questions and Solutions. Edited by Peter J. Taylor (Australian Mathematics Competition, University of Canberra, P.O. Box 1, Belconnen, A.C.T. 2616, Australia). Softcover, 177 pages, \$A 21.00. *Reviewed by Murray S. Klamkin, University of Alberta.*

In Olympiad Corner 78 [1986: 197–202], I had discussed the Putnam Intercollegiate Competition as well as some other ones. However, I had not mentioned the Tournament of the Towns Competition which to me now is one of the premier mathematics competitions in the world for secondary school students. Many of the problems would even be challenging to Putnam contestants.

The Tournament of the Towns Competition originated in the then Soviet Union in 1979 with three towns and eventually expanded to many towns throughout the world. In particular, due to the dedicated efforts of Andy Liu, Edmonton is a participating town. The Tournament is conducted annually in two rounds with two levels of papers within each round; Junior papers for students in years 8, 9, 10 and Senior papers for students in years 11 and 12. Within each round there are two papers, a training version and a main version, spaced about a week apart, and each consisting of 4 to 7 problems to be done in 4 or 5 hours. Students are awarded points based on their best responses to three questions. For a more detailed history of the Tournament, how it is scored and a description of other activities, including proposed new activities, see [1], [2], [3], as well as the preface of the book.

The book consists of twenty rounds of questions and their solutions. The solutions are due to a number of contestants as well as a number of colleagues of the editor, all of whom are acknowledged. Some of these (e.g., S. Bilchev, A. Liu, D. Singmaster, J. Tabov) should be familiar to readers of *Crux.*

The following is a sampling of some of the problems.

Junior Questions

1. On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g., if a grey and a brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

2. A median, a bisector and an altitude of a certain triangle intersect at an inner point O. The segment of the bisector from the vertex to O is equal to the segment of the altitude from the vertex to O. Prove that the triangle is equilateral.

3. Find all real solutions of the system of equations

$$(x+y)^3 = z, \quad (y+z)^3 = x, \quad (z+x)^3 = y.$$

4. Let $a_1, a_2, \ldots, a_n$ be an arrangement of the integers $1, 2, \ldots, n$. Let

$$S = \frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n}.$$

Find a natural number n such that among the values of S for all arrangements $a_1, a_2, \ldots, a_n$, all the integers from n to $n + 100$ appear.

Senior Questions.

1. Prove that the area of a unit cube's projection on any plane equals the length of its projection on the perpendicular to this plane.

2. We are given 101 rectangles with sides of integer lengths not exceeding 100. Prove that among these 101 rectangles there are 3 rectangles, say A, B and C, such that A will fit inside B and B inside C.

3. Is there a power of 2 such that it is possible to rearrange the digits giving another power of 2?

4. There are 1988 towns and 4000 roads in a certain country (each road connects two towns). Prove that there is a closed path passing through no more than 20 towns.

The solutions given to all the problems (a number of them have multiple solutions) are in general quite good and should be very instructive, especially to secondary school students. Nevertheless there are a number of solutions which could have been elaborated on (no doubt, one can say this about almost all problem collections). I give two examples.

Prove that for all values of a, $3(1 + a^2 + a^4) \geq (1 + a + a^2)^2$.
The solution which was by Calvin Li, a student in Edmonton, is nice but rather special. One should have at least included an editorial note that the result is a special case of Cauchy's inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2.$$

Prove that from any set of seven natural numbers (not necessarily consecutive) one can choose three, the sum of which is divisible by three.
Again there are much more general results (see *Crux* 1596 [1992: 24]).

With respect to improvements and generalizations of the solutions of these problems, the editor, Peter Taylor, informed me that he would be glad to receive any and that they will likely be incorporated in subsequent editions of the book.

My hat is off to all the many people who have cooperated (see book and [1] for names) in the running and in the setting of the excellent problems of the Tournament of the Towns Competition as well as in the preparation of the book. This book should be in the hands of all readers of *Crux* and all students who compete in mathematical competitions.

As a final note, anyone who is interested in organizing the Tournament of the Towns Competition in his town and wants information as to how to proceed can contact one of the following:

Turnir Gorodov, *Kvant*, Gorogo 32/1, Moscow, Russia 103006;

Prof. Andy Liu, Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada, Phone (403) 492–3527;

Dr. P.J. Taylor, Head, Faculty of Information Sciences and Engineering, University of Canberra, P.O. Box 1, Belconnen, A.C.T. 2616, Australia.

References:

[1] N.N. Konstantinov, J.B. Tabov and P.J. Taylor, Birth of the Tournament of the Towns, *Mathematics Competitions* (Journal of the World Federation of National Mathematics Competitions) 4 (1991) 28–41.

[2] P.J. Taylor, Tournament of the Towns corner, ibid., pp. 91–96.

[3] N.N. Konstantinov, The Tournament of Towns, *Quantum* 1 (Jan. 1990) 50–51, 1 (Nov.–Dec. 1990) 51–52, 61–63.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **January 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1751. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with incenter I and circumcenter O. Suppose that $AB < AC$ and $IO = \frac{1}{2}(AC - AB)$. Prove that

$$\text{area}(\Delta IAO) = \frac{1}{2}[\text{area}(\Delta BAO) - \text{area}(\Delta CAO)].$$

1752*. *Proposed by Murray S. Klamkin, University of Alberta.*

If A and B are positive integers and p is a prime such that $p \mid A$, $p^2 \nmid A$ and $p^2 \mid B$, then the arithmetic progression

$$A, A+B, A+2B, A+3B, \ldots$$

contains no terms which are perfect powers (squares, cubes, etc.). Are there any infinite non-constant arithmetic progressions of positive integers, with no term a perfect power, which are *not* of this form?

1753. *Proposed by Jordi Dou, Barcelona, Spain.*

Given points P_i, $i = 1,2,3,4$, and line ℓ in the plane, find a pair of points X, Y so that points $X_i = XP_i \cap \ell$ and $Y_i = YP_i \cap \ell$ are symmetric on ℓ (i.e., the midpoints of X_iY_i coincide). Also prove that for all such X, Y the lines XY pass through a fixed point.

1754*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be positive integers such that $2 \le k < n$, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n x_i = 1$. Prove or disprove that

$$\sum x_1x_2\ldots x_k \le \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1, x_2, \ldots, x_n$. [The case $k = 2$ is known — see inequality (1) in the solution of *Crux* 1662, this issue.]

1755. *Proposed by Dan Pedoe, Minneapolis, Minnesota, and J. Chris Fisher and Robert E. Jamison, Clemson University, Clemson, South Carolina.*

(a) If $T_1 = \Delta A_1B_1C_1$ is a triangle inscribed in $T = \Delta ABC$, and each side of T_1 is one-half the corresponding side of T, prove that T_1 is the medial triangle of T (the triangle whose vertices are the midpoints of the sides of T).

(b) Define $\Delta A_1B_1C_1$ to be *properly inscribed* in ΔABC if the two triangles have the same orientation and A_1, B_1, C_1 are on the *lines* BC, CA, AB respectively. Prove that if $\Delta A_1B_1C_1$ is one such triangle, then there is exactly one other properly inscribed triangle congruent to it, with one exception. The exception occurs when the given inscribed triangle is the *pedal triangle* of some point M (i.e., when its vertices are the feet of the perpendiculars from M onto the respective sides of ΔABC); in this case there is no other inscribed triangle congruent to $\Delta A_1B_1C_1$.

1756. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

For positive integers $n \geq 3$ and $r \geq 1$, the *n-gonal number of rank r* is defined as

$$P(n,r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2} \,.$$

Call a triple (a, b, c) of natural numbers, with $a \leq b < c$, an *n-gonal Pythagorean triple* if $P(n,a) + P(n,b) = P(n,c)$. When $n = 4$, we get the usual Pythagorean triple.

(i) Find an n-gonal Pythagorean triple for each n.

(ii) Consider all triangles ABC whose sides are n-gonal Pythagorean triples for some $n \geq 3$. Find the maximum and the minimum possible values of angle C.

1757. *Proposed by Avinoam Freedman, Teaneck, N.J.*

Let $A_1A_2A_3$ be an acute triangle with sides a_1, a_2, a_3 and area F, and let $\Delta B_1B_2B_3$ (with sides b_1, b_2, b_3) be inscribed in $\Delta A_1A_2A_3$ with $B_1 \in A_2A_3$, etc. Show that for any $x_1, x_2, x_3 > 0$,

$$(x_1a_1^2 + x_2a_2^2 + x_3a_3^2)(x_1b_1^2 + x_2b_2^2 + x_3b_3^2) \geq 4F^2(x_2x_3 + x_3x_1 + x_1x_2).$$

1758. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

In Hilbert High School there are an infinite number of lockers, numbered by the natural numbers: $1, 2, 3, \ldots$. Each locker is occupied by exactly one student. The administration decides to rearrange the students so that the lockers are all still occupied by the same set of students, one to a locker, but in some other order. (Some students might not change lockers.) It turns out that the locker numbers of infinitely many students end up higher than before. Show that there are also infinitely many students whose locker numbers are lower than before.

1759. *Proposed by Isao Ashiba, Tokyo, Japan.*

A is a fixed point on a circle, and P and Q are variable points on the circle so that $AP + PQ$ equals the diameter of the circle. Find P and Q so that the area of ΔAPQ is as large as possible.

1760. *Proposed by Ray Killgrove, San Diego State University.*

Find all positive integers B so that $(111)_B = (aabbcc)_6$, where a, b, c represent distinct base 6 digits, $a \neq 0$.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1528*. [1990: 75; 1991: 152] *Proposed by Ji Chen, Ningbo University, China.*

If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, prove or disprove that

$$\frac{a^2}{(a^2+1)^2} + \frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} \le \frac{16}{25}.$$

II. *Solution by Hui-Hua Wan, Ningbo University, China, and the proposer.*

Let $a \le b \le c \le d$. If $a \ge 1/8$, then

$$(48a-4)(a^2+1)^2 - 125a^2 = (2a-1)^2(12a^3+11a^2+32a-4) \ge 0,$$

so

$$\frac{a^2}{(a^2+1)^2} \le \frac{48a-4}{125}$$

and thus

$$\frac{a^2}{(a^2+1)^2} + \frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} \le \frac{48(a+b+c+d)-16}{125} = \frac{16}{25}.$$

Assume therefore that $a < 1/8$. Note that for $x > 0$,

$$(540x+108)(x^2+1)^2 - 2197x^2 = (3x-2)^2(60x^3+92x^2+216x+27) \ge 0,$$

so

$$\frac{x^2}{(x^2+1)^2} \le \frac{540x+108}{2197}.$$

Thus

$$\frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} \le \frac{540(b+c+d)+324}{2197} = \frac{108}{169} - \frac{540a}{2197}.$$

But

$$\frac{a^2}{(a^2+1)^2} < a^2 < \frac{a}{8} < \frac{540a}{2197},$$

hence

$$\frac{a^2}{(a^2+1)^2} + \frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} < \frac{108}{169} < \frac{16}{25}.$$

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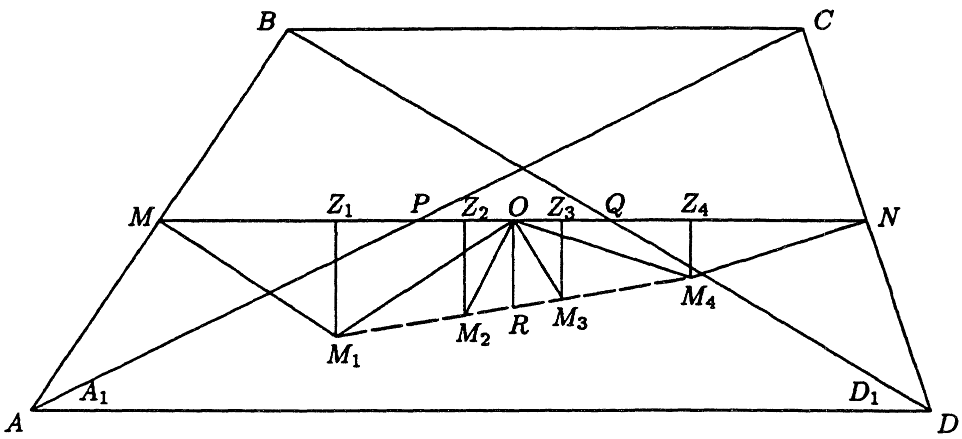
1655. [1991: 172] *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be a trapezoid with $AD \parallel BC$. M, N, P, Q, O are the midpoints of AB, CD, AC, BD, MN, respectively. Circles m, n, p, q all pass through O, and are tangent to AB at M, to CD at N, to AC at P, and to BD at Q, respectively. Prove that the centres of m, n, p, q are collinear.

I. *Solution by P. Penning, Delft, The Netherlands.*



Let length $AD = 4l$ and length $BC = 4m$, where without loss of generality $l \geq m$. The centres are denoted by M_1, M_2, M_3, M_4 and the projections thereof on MN by Z_1, Z_2, Z_3, Z_4, respectively. Then $MN = 2(l+m)$ and

$$PQ = MN - MP - NQ = 2(l+m) - 2m - 2m = 2(l-m),$$

so

$$OZ_1 = MZ_1 = NZ_4 = OZ_4 = \frac{m+l}{2},$$
$$PZ_2 = Z_2O = OZ_3 = Z_3Q = \frac{l-m}{2}.$$

Let $\angle BAD = A$, $\angle CDA = D$, $\angle CAD = A_1$, $\angle BDA = D_1$, and let the distance between AD and BC be h. By projecting B and C on AD it is easily seen that

$$h\cot A + h\cot D_1 = 4l = h\cot A_1 + h\cot D \tag{1}$$

and

$$h\cot A_1 - h\cot A = 4m = h\cot D_1 - h\cot D,$$

so

$$\frac{4(l+m)}{h} = \cot A_1 + \cot D_1, \quad \frac{4(l-m)}{h} = \cot A + \cot D. \tag{2}$$

Now

$$\angle Z_1M_1O = \angle Z_1M_1M = \angle BMZ_1 = A$$

and similarly

$$\angle Z_2M_2O = A_1, \quad \angle Z_3M_3O = D_1, \quad \angle Z_4M_4O = D,$$

so

$$Z_1M_1 = \cot A \cdot \frac{m+l}{2}, \quad Z_4M_4 = \cot D \cdot \frac{m+l}{2}. \tag{3}$$

The angle x between M_1M_4 and MN therefore satisfies

$$\tan x = \frac{Z_4M_4 - Z_1M_1}{Z_1Z_4} = \frac{\cot D - \cot A}{2}.$$

By (2) and (3), midpoint R of M_1M_4 satisfies

$$\begin{aligned} OR &= \frac{Z_1M_1 + Z_4M_4}{2} = (\cot A + \cot D)\frac{m+l}{4} \\ &= \frac{h(\cot A + \cot D)(\cot A_1 + \cot D_1)}{16}. \end{aligned}$$

Treating the centres M_2 and M_3 in a similar way leads to the angle x' between M_2M_3 and MN satisfying

$$\tan x' = \frac{Z_3M_3 - Z_2M_2}{Z_2Z_3} = \frac{\cot D_1 - \cot A_1}{2}.$$

Thus (by (1), say) $x = x'$, i.e., M_1M_4 and M_2M_3 make the same angle with MN. Moreover, the midpoint R' of M_2M_3 satisfies

$$\begin{aligned} OR' &= \frac{Z_2M_2 + Z_3M_3}{2} = (\cot A_1 + \cot D_1)\frac{l-m}{4} \\ &= \frac{h(\cot A_1 + \cot D_1)(\cot A + \cot D)}{16} = OR, \end{aligned}$$

so M_2M_3 and M_1M_4 have the same midpoint. Therefore all four centres are collinear.

II. *Solution by the proposer.*

Let π be the parabola tangent to AB and CD and tangent to MN at O. The polar of $E = AB \cap CD$, which passes through the points B_1, D_1 of contact of the parabola with AB and CD, is parallel to MN (because O is the midpoint of MN). It follows by similar triangles that $NC/MA = ED_1/B_1E$, which implies [via properties of cross-ratios: compare, e.g., with Dan Pedoe's article "A parabola is not an hyperbola" [1979: 122–124]] that AC is also tangent to π. It is known that $(*)$ *the circumcircles of the triangles whose sides are tangent to π pass through the focus F of π.* Consider the variable triangles whose sides are AB, x, y, where x and y are tangent to π at points X, Y on π which tend to O. All the circumcircles of these triangles pass through F and these circles tend to the circle m. Therefore m passes through F. Analogously n, p and q pass through F. Therefore their centres lie on the perpendicular bisector of OF.

Remarks. (i) It is clear that the circumcircles of all $\binom{5}{3} = 10$ triangles that can be formed from three of the five tangents AB, CD, AC, BD, MN pass through F.

(ii) The Simson theorem (the locus of all points whose projections on the sides of a triangle are collinear is the circumcircle of the triangle) gives us a proof of property (*). The projections of the focus on the tangents are on the tangent at the vertex.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and JOSÉ YUSTY PITA, Madrid, Spain.

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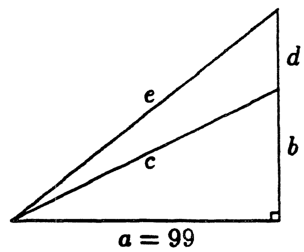
1657. [1991: 172] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Pythagoras the eternal traveller reached Brahmapura on one of his travels. "Dear Pythagoras", told the townspeople of Brahmapura, "there on the top of that tall vertical mountain 99 *Brahmis* away resides Brahmagupta, who exhibits unmatched skills in both travel through the air and mathematics". "Aha", exclaimed Pythagoras, "I must have a chat with him before leaving." Pythagoras made his way directly to the foot of the mountain. No sooner had he reached it than he found himself in a comforting magic spell that flew him effortlessly to the summit where Brahmagupta received him. The two exchanged greetings and ideas. "You will have noticed that the foot and summit of this mountain forms with the town a triangle that is Pythagorean [an integer-sided right triangle—*Ed.*]", Brahmagupta remarked. "I can rise high vertically from the summit and then proceed diagonally to reach the town thus making yet another Pythagorean triangle and in the process equalling the distance covered by you from the town to the summit." Tell me, dear *Crux* problem solver, how high above the mountain did Brahmagupta rise?

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let $a = 99$ be the distance from the town to the mountain, b the height of the mountain, c the hypotenuse of the first Pythagorean triangle, d the extent of Brahmagupta's rise, and e the hypotenuse of the second Pythagorean triangle. Then it is clear from the problem statement that $e = a+b-d$. But from the Pythagorean theorem,



$$a^2 + (b+d)^2 = e^2 = (a+b-d)^2 = a^2 + 2a(b-d) + (b-d)^2,$$

whence

$$d = \frac{ab}{2b+a} = \frac{99b}{2b+99}\,. \tag{1}$$

Now let us look at the Pythagorean triangles that have 99 as one of the legs. Note that 99, being odd, must be of the form $K(m^2-n^2)$, where K, m and n are positive integers, $(m,n) = 1$, and $m \equiv n+1 \bmod 2$. There are just seven such triangles:

k	$m-n$	$m+n$	m	n	$b=2Kmn$	$c=K(m^2+n^2)$	d
1	9	11	10	1	20	101	14.244
33	1	3	2	1	132	165	36
3	3	11	7	4	168	195	38.234
11	1	9	5	4	440	451	44.494
9	1	11	6	5	540	549	45.344
3	1	33	17	16	1632	1635	48.043
1	1	99	50	49	4900	4901	49.005

Here d has been calculated using (1). Note that d is an integer only when $b = 132$. Also note that $b + d = 132 + 36 = 168$, which is just equal to the longer leg of the triangle on the next line. Thus Brahmagupta rose 36 *Brahmis*.

Also solved by CHARLES ASHBACHER, Hiawatha, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J.A. MCCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Bunuel, Madrid, Spain; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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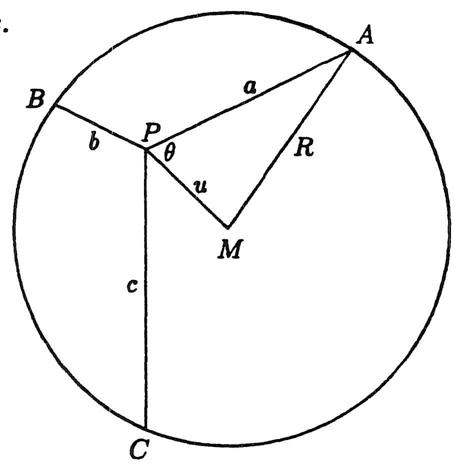
1658. [1991: 172] *Proposed by Avinoam Freedman, Teaneck, New Jersey.*

Let P be a point inside circle O and let three rays from P making angles of 120° at P meet O at A, B, C. Show that the power of P with respect to O is the product of the arithmetic and harmonic means of PA, PB and PC.

I. *Solution by P. Penning, Delft, The Netherlands.*



Let the lengths of PA, PB, PC be a, b, c respectively, R the radius of O and M its centre. Let u be the distance PM. Denote angle MPA by θ. Then $\angle MPB = \theta + 120°$ and $\angle MPC = \theta + 240°$. Applying the cosine rule in triangles MPA, MPB, MPC gives

$$R^2 - u^2 = a^2 - 2ua\cos\theta,$$
$$R^2 - u^2 = b^2 - 2ub\cos(\theta + 120°),$$
$$R^2 - u^2 = c^2 - 2uc\cos(\theta + 240°).$$

Divide these equations by a, b, c respectively and add:

$$(R^2 - u^2)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = a + b + c - 2u[\cos\theta + \cos(\theta + 120°) + \cos(\theta + 240°)]$$
$$= a + b + c.$$

Since the power of P is equal to $R^2 - u^2$, the proof follows.

I note that the result apparently holds also for n rays from P that make equal angles of $360°/n$. The sum of the cosines

$$\cos\theta + \cos\left(\theta + \frac{360°}{n}\right) + \cos\left(\theta + \frac{2\cdot 360°}{n}\right) + \cdots + \cos\left(\theta + \frac{(n-1)360°}{n}\right)$$

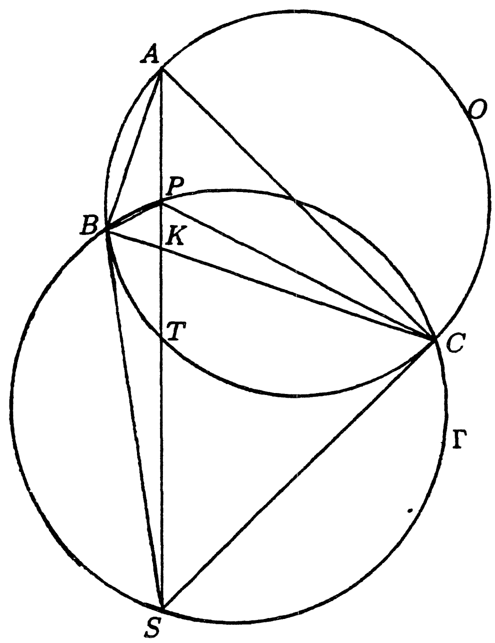
is still zero [e.g., being the x-component of the vector sum $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = \mathbf{0}$ where $A_1A_2\ldots A_n$ is a regular n-gon with centre O, $|OA_i| = 1$, and $\overrightarrow{OA_1}$ makes angle θ with the x-axis.— *Ed.*].

II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

We shall prove the following generalization.

ABC is a triangle inscribed in circle O, and P is a point inside ABC. We denote $\angle BPC = \alpha$, $\angle CPA = \beta$, $\angle APB = \gamma$. Let x be the arithmetic mean of $PA\sin\alpha$, $PB\sin\beta$ and $PC\sin\gamma$, and let y be the harmonic mean of $PA/\sin\alpha$, $PB/\sin\beta$ and $PC/\sin\gamma$. Then xy is equal to the power of P with respect to O.

We denote the circumcircle of ΔPBC by Γ. Let K, T and S be the intersections (other than A and P) of AP with BC, O and Γ, respectively. Then we have



$$\left.\begin{array}{l} \angle BSC = \pi - \alpha, \\ \angle SBC = \angle SPC = \pi - \beta, \\ \angle SCB = \angle SPB = \pi - \gamma. \end{array}\right\} \qquad (1)$$

By Ptolemy's theorem we get

$$PB \cdot SC + PC \cdot SB = PS \cdot BC.$$

By (1) and the law of sines for ΔSBC, this becomes

$$PB\sin\beta + PC\sin\gamma = PS\sin\alpha.$$

Therefore

$$\begin{aligned} x &= \frac{PA\sin\alpha + PB\sin\beta + PC\sin\gamma}{3} = \frac{PA\sin\alpha + PS\sin\alpha}{3} \\ &= \frac{AS\sin\alpha}{3} = \frac{(AK + KS)\sin\alpha}{3}. \end{aligned} \qquad (2)$$

Because area(ΔPBK) + area(ΔPKC) = area(ΔPBC), we get

$$PB \cdot PK\sin\gamma + PK \cdot PC\sin\beta = PB \cdot PC\sin\alpha. \qquad (3)$$

Therefore we have from (3)

$$\begin{aligned}
&PB\cdot PC\sin\alpha + PA\cdot PC\sin\beta + PA\cdot PB\sin\gamma \\
&\qquad = PB\cdot PC\sin\alpha + PA\left(\frac{PB\cdot PC\sin\alpha}{PK}\right) \\
&\qquad = \frac{PB\cdot PC\sin\alpha(PA+PK)}{PK} = \frac{PB\cdot PC\cdot AK\sin\alpha}{PK}\,. \qquad (4)
\end{aligned}$$

Then we get from (2) and (4)

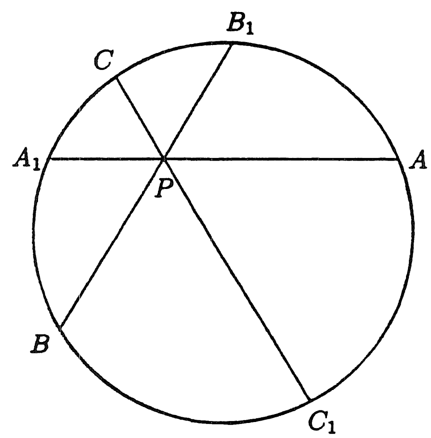
$$\begin{aligned}
xy &= \frac{(AK+KS)\sin\alpha}{3}\cdot\frac{3}{\sin\alpha/PA+\sin\beta/PB+\sin\gamma/PC} \\
&= \frac{(AK+KS)\sin\alpha\cdot PA\cdot PB\cdot PC}{PB\cdot PC\sin\alpha + PA\cdot PC\sin\beta + PA\cdot PB\sin\gamma} \\
&= \frac{PA\cdot(AK+KS)\cdot PK}{AK}\,. \qquad (5)
\end{aligned}$$

Because $AK\cdot KT = BK\cdot KC = PK\cdot KS$, we have

$$(AK+KS)PK = AK\cdot PK + AK\cdot KT = AK\cdot PT.$$

Hence we obtain from (5) that $xy = PA\cdot PT$. This implies that xy is equal to the power of P with respect to O.

Comment. This problem may also be easily proved by problem 7 of the 15th All Union Mathematical Olympiad — 10th Grade [1991: 163]. Let A_1, B_1, C_1 be the second intersections of AP, BP, CP with O. Then by problem 7 [1991: 163], we have



$$PA+PB+PC = PA_1+PB_1+PC_1.$$

Denoting the power of P with respect to O by k^2, we have

$$PA\cdot PA_1 = PB\cdot PB_1 = PC\cdot PC_1 = k^2.$$

Denoting the arithmetic and harmonic means of PA, PB, PC by x and y respectively, we have

$$x = \frac{1}{3}(PA+PB+PC), \quad \frac{1}{y} = \frac{1}{3}\left(\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}\right).$$

Therefore

$$\begin{aligned}
\frac{k^2}{y} &= \frac{1}{3}\left(\frac{PA\cdot PA_1}{PA}+\frac{PB\cdot PB_1}{PB}+\frac{PC\cdot PC_1}{PC}\right) \\
&= \frac{1}{3}(PA_1+PB_1+PC_1) = \frac{1}{3}(PA+PB+PC) = x.
\end{aligned}$$

Hence we obtain $xy = k^2$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (a second solution); D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There was one incorrect solution sent in.

Penning's generalization to n rays was also predicted by the proposer.

Several solvers pointed out that the power of a point interior to a circle is usually defined to be negative, so in this problem we should really take the absolute value.

Festraets-Hamoir and Smeenk also mentioned the related Olympiad Corner problem on [1991: 163]. *Klamkin gave a generalization similar to (but not as elegant as) Seimiya's, and also said that the problem was known, but could not supply a reference.*

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1660. [1991: 172] *Proposed by Isao Ashiba, Tokyo, Japan.*

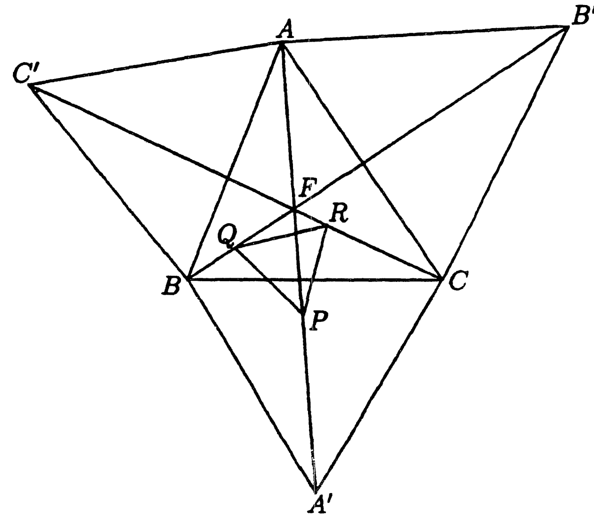
Construct equilateral triangles $A'BC, B'CA, C'AB$ exterior to triangle ABC, and take points P, Q, R on AA', BB', CC', respectively, such that

$$\frac{AP}{AA'} + \frac{BQ}{BB'} + \frac{CR}{CC'} = 1.$$

Prove that ΔPQR is equilateral.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.

It is well-known that, with the conditions of the problem, lines AA', BB' and CC' are concurrent in the Fermat point F of ABC and form angles of 60° at F. (F minimizes the sum of the distances of a point in the plane of ABC from its three vertices, in the case that none of the angles of ABC is $> 120°$.) Also,



$$FA + FB + FC = AA' = BB' = CC';$$

thus from the equation of the problem we have

$$AP + BQ + CR = FA + FB + FC. \quad (1)$$

In the triangle FPQ we have $\angle PFQ = \angle PFR = 60°$ and $\angle QFR = 120°$ (the other possibilities are analogous), and so

$$FQ = FB - BQ, \quad FP = AP - FA, \quad FR = FC - CR.$$

The cosine law in the triangles PFQ and QFR yields

$$PQ^2 = FQ^2 + FP^2 - FP \cdot FQ \quad \text{and} \quad QR^2 = FQ^2 + FR^2 + FQ \cdot FR$$

respectively. For $PQ^2 = QR^2$ it must be that

$$FP(FP - FQ) = FP^2 - FP \cdot FQ = FR^2 + FQ \cdot FR = FR(FR + FQ),$$

and this is true because by (1) the left hand side equals

$$(AP - FA)(AP + BQ - FA - FB) = (FB + FC - BQ - CR)(FC - CR),$$

which equals the right hand side. Therefore $PQ = QR$. In an analogous manner $PR = QR$ and the triangle PQR is equilateral. [*Editor's note.* By (1), it can be assumed that P, Q, R lie as in the diagram.]

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. A second solution was sent in by Bellot Rosado and López Chamorro.

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1661. [1991: 207] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ΔABC is inscribed in a circle Γ_1. Let D be a point on BC produced such that AD is tangent to Γ_1 at A. Let Γ_2 be a circle which passes through A and D, and is tangent to BD at D. Let E be the point of intersection of Γ_1 and Γ_2 other than A. Prove that $\overline{EB} : \overline{EC} = \overline{AB}^3 : \overline{AC}^3$.

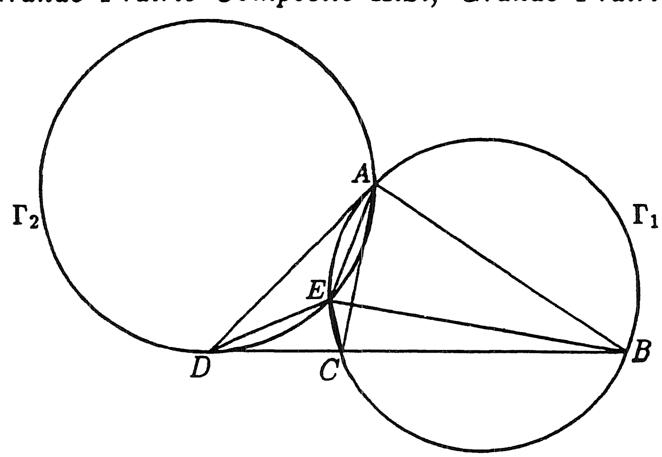
Solution by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.

The triangles DAB and DCA are similar and therefore

$$\frac{\text{area } \Delta DAB}{\text{area } \Delta DCA} = \frac{(AB)^2}{(CA)^2}.$$

Since the altitude for both triangles from A is the same, it follows that

$$\frac{DB}{DC} = \frac{(AB)^2}{(CA)^2}. \qquad (1)$$



Quadrilateral $CBAE$ is inscribed in circle Γ_1. On account of tangency of AD at A to Γ_1, $\angle ABE = \angle EAD$. Similarly, since DB is tangent at D to Γ_2, $\angle EAD = \angle EDC$, so $\angle ABE = \angle EDC$. Furthermore $\angle EAB = \angle ECD$ because $\angle EAB$ and $\angle ECB$ are supplementary. From this we conclude that $\Delta DCE \sim \Delta BAE$ and hence

$$\frac{DC}{EC} = \frac{BA}{EA}\,. \tag{2}$$

By Ptolemy's theorem, (1), and (2) we have

$$\begin{aligned} CA\cdot EB &= CB\cdot AE + CE\cdot BA \\ &= (DB - DC)\frac{BA\cdot EC}{DC} + CE\cdot BA \\ &= \frac{DB\cdot BA\cdot EC}{DC} = \frac{(AB)^3\cdot EC}{(CA)^2}\,, \end{aligned}$$

which results in $EB : EC = (AB)^3 : (AC)^3$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; P. PENNING, Delft, The Netherlands; SHAILESH A. SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

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1662. [1991: 207] *Proposed by Murray S. Klamkin, University of Alberta.*
Prove that

$$\frac{x_1^{2r+1}}{s-x_1} + \frac{x_2^{2r+1}}{s-x_2} + \cdots + \frac{x_n^{2r+1}}{s-x_n} \ge \frac{4^r}{(n-1)n^{2r-1}}(x_1x_2 + x_2x_3 + \cdots + x_nx_1)^r,$$

where $n > 3$, $r \ge 1/2$, $x_i \ge 0$ for all i, and $s = x_1 + x_2 + \cdots + x_n$. Also, find some values of n and r such that the inequality is sharp.

Combination of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Marcin E. Kuczma, Warszawa, Poland; and Pavlos Maragoudakis, student, University of Athens, Greece.

[*Editor's note.* In tackling this problem, the readers indulged in something of an orgy of interpolation! The beleaguered editor managed to string some of these results together, as follows. Needless to say, Janous, Kuczma and Maragoudakis each completely solved the original problem.]

$$\overbrace{\sum_{i=1}^{n}\frac{x_i^{2r+1}}{s-x_i}}^{\mathbf{A}} \ge \overbrace{\frac{n}{(n-1)s}\sum_{i=1}^{n}x_i^{2r+1}}^{\mathbf{B}} \ge \overbrace{\frac{1}{n-1}\sum_{i=1}^{n}x_i^{2r}}^{\mathbf{C}} \ge \overbrace{\frac{n^{1-r}}{n-1}\left(\sum_{i=1}^{n}x_i^2\right)^r}^{\mathbf{D}}$$

$$\ge \overbrace{\frac{n^{1-2r}}{n-1}\left(\sum_{i=1}^{n}x_i\right)^{2r}}^{\mathbf{E}} \ge \overbrace{\frac{4^r}{(n-1)n^{2r-1}}\left(\sum_{i=1}^{n}x_ix_{i+1}\right)^r}^{\mathbf{F}}$$

A $\ge$ **B** (Maragoudakis). Assuming for this case $x_1 \ge x_2 \ge \cdots \ge x_n$, we have also

$$\frac{1}{s-x_1} \ge \frac{1}{s-x_2} \ge \cdots \ge \frac{1}{s-x_n},$$

so by Chebyshev's inequality

$$n\sum_{i=1}^{n}\frac{x_i^{2r+1}}{s-x_i} \ge \sum_{i=1}^{n}x_i^{2r+1}\cdot\sum_{i=1}^{n}\frac{1}{s-x_i}.$$

By the arithmetic mean-harmonic mean inequality,

$$(n-1)s\sum_{i=1}^{n}\frac{1}{s-x_i} = \sum_{i=1}^{n}(s-x_i)\cdot\sum_{i=1}^{n}\frac{1}{s-x_i} \ge n^2,$$

and **A** $\ge$ **B** follows.

B $\ge$ **C**: equivalent to

$$n\sum_{i=1}^{n}x_i^{2r+1} \ge \sum_{i=1}^{n}x_i\cdot\sum_{i=1}^{n}x_i^{2r},$$

which is again Chebyshev's inequality.

C $\ge$ **D** $\ge$ **E**: equivalent to

$$\frac{1}{n}\sum_{i=1}^{n}x_i^{2r} \overset{(r\ge 1)}{\ge} \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)^r \ge \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^{2r},$$

and both inequalities are true by the power mean inequality. To handle any $r \ge 1/2$ we should include

B $\ge$ **D**: follows from

$$\frac{1}{n}\sum_{i=1}^{n}x_i^{2r+1} \ge \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)^{(2r+1)/2}$$

and

$$\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)^{1/2} \ge \frac{1}{n}\sum_{i=1}^{n}x_i,$$

again the power mean. (Thanks to the proposer for advice on this inequality.)
[*Editor's note.* **C** and **D** are the Janous and Kuczma interpolations, respectively.]

E $\geq$ **F** (Maragoudakis): equivalent to

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geq 4\sum_{i=1}^{n} x_i x_{i+1}, \quad n \geq 4, \tag{1}$$

which has appeared in *Crux* [1985: 284]! We also include (because it's interesting)

D $\geq$ **F** (Kuczma): follows from

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i x_{i+1}$$

which when multiplied by 2 is just

$$x_1^2 + \sum_{i=1}^{n-1}(x_i - x_{i+1})^2 + x_n^2 \geq 0,$$

and from

$$n^{1-r} \geq 4^r n^{1-2r} \quad \text{for } n \geq 4.$$

It follows from the above inequalities (and **D** $\geq$ **F** in particular) that equality holds in the problem if and only if $n = 4$ and $x_1 = x_2 = x_3 = x_4$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; and the proposer.

The known inequality (1) *was also used by the proposer, who gave the* Crux *reference (Maragoudakis gave a proof of* (1) *in his solution). The proposer had an interpolation as well, namely*

$$\frac{1}{(n-1)s}\left(\sum_{i=1}^{n} x_i^{r+1/2}\right)^2,$$

which fits in between **B** *and* **E**.

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1663*. [1991: 207] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles of a triangle, r its inradius and s its semiperimeter. Prove that

$$\sum\sqrt{\cot(A/2)} \leq \frac{3}{2}\sqrt{r/s}\sum\csc(A/2),$$

where the sums are cyclic over A, B, C.

Solution by Ji Chen, Ningbo University, China.

Using

$$\cot\frac{A}{2} = \frac{s-a}{r}, \quad \csc\frac{A}{2} = \sqrt{\frac{bc}{(s-b)(s-c)}}, \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

and putting $s-a=x^2$, $s-b=y^2$, $s-c=z^2$, the given inequality is equivalent to:

$$\left(\sum x^2\right)\left(\sum x\right) \le \frac{3}{2}\sum x\sqrt{(x^2+y^2)(x^2+z^2)}\,,$$

where the sums are cyclic over x, y, z. Squaring, this is equivalent to

$$\left(\sum x^2\right)^2\left(\sum x\right)^2 \le \frac{9}{4}\left(\sum x^2(x^2+y^2)(x^2+z^2) + 2\sum yz(y^2+z^2)\sqrt{(x^2+y^2)(x^2+z^2)}\right).$$

This in turn is implied by

$$\left(\sum x^2\right)^2\left(\sum x\right)^2 \le \frac{9}{4}\left(\sum x^2(x^2+y^2)(x^2+z^2) + 2\sum yz(y^2+z^2)(x^2+yz)\right),$$

which is true because

$$\begin{aligned}
9\sum x^2(x^2+y^2)(x^2+z^2) &+ 18\sum yz(y^2+z^2)(x^2+yz) - 4\left(\sum x^2\right)^2\left(\sum x\right)^2 \\
&= 5\sum x^6 - 8\sum(x^5y+x^5z) + 15\sum(x^4y^2+x^4z^2) - 8\sum x^4yz \\
&\qquad -16\sum y^3z^3 + 2\sum(x^3y^2z+x^3z^2y) + 3x^2y^2z^2 \\
&= \frac{1}{6}\sum(y-z)^4(x^2+15y^2+15z^2+4yz+8zx+8xy) \ge 0.
\end{aligned}$$

[Can anybody find a "nice" proof that the above expression is positive?— *Ed.*]

Also solved by MARCIN E. KUCZMA, Warszawa, Poland.

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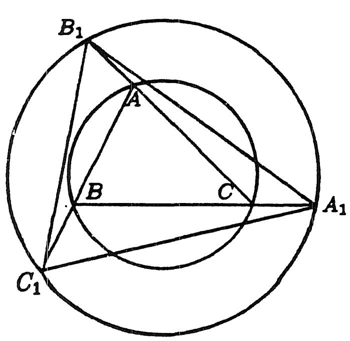
1664. [1991: 207] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.* (Dedicated to Jack Garfunkel.)

Consider two concentric circles with radii R_1 and R ($R_1 > R$) and a triangle ABC inscribed in the inner circle. Points A_1, B_1, C_1 on the outer circle are determined by extending BC, CA, AB, respectively. Prove that

$$\frac{F_1}{R_1^2} \ge \frac{F}{R^2},$$

where F_1 and F are the areas of triangles $A_1B_1C_1$ and ABC respectively, with equality when ABC is equilateral.

Solution by Murray S. Klamkin, University of Alberta.



Let $x = CA_1$, $y = AB_1$, $z = BC_1$. Then by the power of a point theorem,

$$x(a+x) = y(b+y) = z(c+z) = R_1^2 - R^2, \qquad (1)$$

where a, b, c are the sides of ΔABC. Since

$$[A_1B_1C_1] = [ABC] + [CA_1B_1] + [AB_1C_1] + [BC_1A_1]$$

(where $[ABC]$ = the area of ABC, etc.), and

$$\frac{\text{base } \Delta CA_1B_1}{\text{base } \Delta ABC} = \frac{x}{a}\,, \quad \frac{\text{altitude } \Delta CA_1B_1}{\text{altitude } \Delta ABC} = \frac{B_1C}{AC} = \frac{b+y}{b}\,, \text{ etc.,}$$

we have

$$F_1 = F\left(1 + \frac{x(b+y)}{ab} + \frac{y(c+z)}{bc} + \frac{z(a+x)}{ca}\right)\,,$$

so that the proposed inequality becomes

$$\frac{x(b+y)}{ab} + \frac{y(c+z)}{bc} + \frac{z(a+x)}{ca} \geq \frac{R_1^2 - R^2}{R^2}\,.$$

Using (1), it becomes

$$\frac{x}{yab} + \frac{y}{zbc} + \frac{z}{xca} \geq \frac{1}{R^2}\,.$$

By using the arithmetic-geometric mean inequality, it suffices to show that

$$3(abc)^{-2/3} \geq 1/R^2. \tag{2}$$

Since $abc = 4RF$, the latter is equivalent to the known inequality

$$3\sqrt{3}\,R^2 \geq 4F$$

[e.g., item 7.9 of Bottema et al, *Geometric Inequalities*], which corresponds to the maximum area triangle inscribed in a given circle being equilateral. Hence there is equality in the proposed inequality if and only if ABC is equilateral.

Comment. Using the solution of *Crux* 1113 [1987: 184], it follows that $s_1/R_1 \geq s/R$, where s and s_1 are the semiperimeters of triangles ABC and $A_1B_1C_1$, respectively. It would be of interest to determine whether or not the following inequalities also hold:

$$\frac{r_1}{R_1} \geq \frac{r}{R} \quad \text{and} \quad \frac{r_1}{s_1} = \frac{F_1}{s_1^2} \geq \frac{r}{s} = \frac{F}{s^2}\,,$$

where r and r_1 are the inradii. Note that if $r_1/s_1 \geq r/s$ were valid then coupled with $s_1/R_1 \geq s/R$ would give $r_1/R_1 \geq r/R$ and coupled with $(s_1/R_1)^2 \geq (s/R)^2$ would give $F_1/R_1^2 \geq F/R^2$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Seimiya's solution was similar to Klamkin's, but referred to item 5.27 *of Bottema for inequality* (2).

As well as Crux 1113 *mentioned above, another Garfunkel problem,* Crux 1603 [1992: 52] *also contains a similar configuration.*

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1665. [1991: 207] *Proposed by P. Penning, Delft, The Netherlands.*

1665^n ends in 5 for $n \geq 1$, and in 25 for $n \geq 2$. Find the longest string of digits which ends 1665^n for all sufficiently large n.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We have to deal with

$$1665^{n+1} \equiv 1665^n \bmod 10^N,$$

where N is greatest possible and n is large enough; that is,

$$5^n \cdot 3^{2n} \cdot 37^n \cdot 2^7 \cdot 13 = 1665^n \cdot 1664 \equiv 0 \bmod 10^N,$$

i.e.

$$5^n \cdot 2^7 \equiv 0 \bmod 10^N.$$

Thus $N = 7$ and $n \geq 7$, and (by calculating $1665^7 \bmod 10^7$) we get as "ending block"

$$2890625.$$

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; STAN HARTZLER, Messiah College, Grantham, Pennsylvania; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; J.A. MCCALLUM, Medicine Hat, Alberta; CORY PYE, student, Memorial University of Newfoundland, St. John's; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect and one incomplete answer sent in.

Festraets-Hamoir, Israel, and the proposer sent in solutions similar to the above.

* * * * *

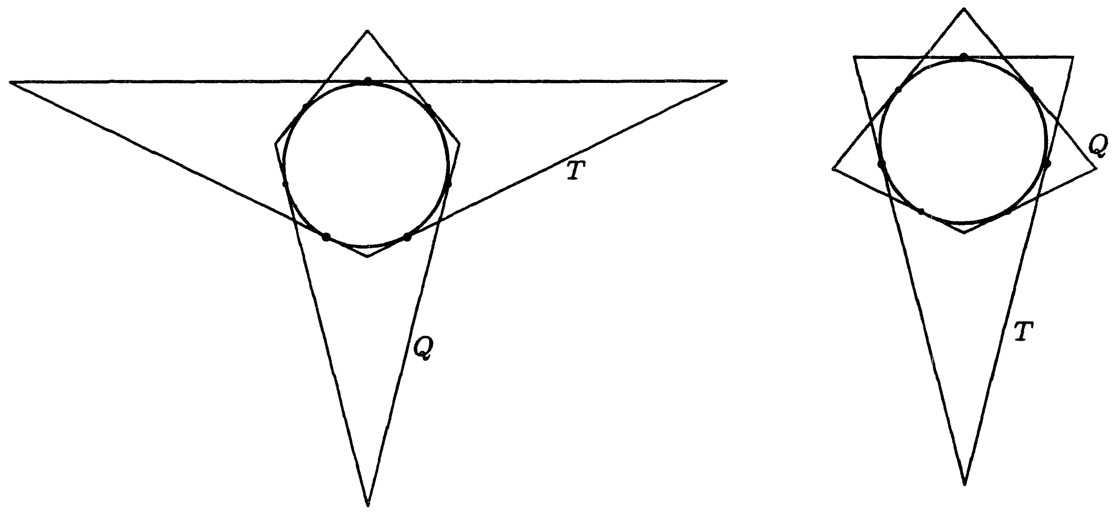
1666. [1991: 207] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

(a) How many ways are there to select and draw a triangle T and a quadrilateral Q around a common incircle of unit radius so that the area of $T \cap Q$ is as small as possible? (Rotations and reflections of the figure are not considered different.)

(b)* The same question, with the triangle and quadrilateral replaced by an m-gon and an n-gon, where $m, n \geq 3$.

Partial solution by Jordi Dou, Barcelona, Spain.

(a) It is clear that the three sides of T and the four of Q must be tangent to the unit circle ω at the seven vertices $P_1, P_2, \ldots, P_7$ of a regular heptagon inscribed in ω. Call the "distance" between two contact points (of T or Q with ω) the number of arcs of size $2\pi/7$ between them. Let P_i, P_j, P_k be the contact points of T with ω, and d_1, d_2, d_3 the distances P_iP_j, P_jP_k, P_kP_i, respectively. The number N of "different" triangles T is equal to the number of distinct sums $d_1 + d_2 + d_3 = 7$ with the condition $d_i \leq 3$. There only exist two solutions: $1 + 3 + 3 = 7$ and $2 + 2 + 3 = 7$, as shown below.



(b) *Comment.* For the case $n = 3$, the considerations expressed in (a) permit us to affirm that the number $N_{m,3}$ of solutions is equal to the number of distinct sums $d_1 + d_2 + d_3 = m + 3$ with the condition $d_i < (m+3)/2$. For example:

$$\begin{array}{ll}
N_{4,3} = 2 & (1+3+3,\ 2+2+3) \\
N_{5,3} = 1 & (2+3+3) \\
N_{6,3} = 3 & (1+4+4,\ 2+3+4,\ 3+3+3) \\
N_{7,3} = 2 & (2+4+4,\ 3+3+4) \\
N_{10,3} = 5 & (1+6+6,\ 2+5+6,\ 3+4+6,\ 3+5+5,\ 4+4+5)
\end{array}$$

For $n = 4$, we get for example $N_{7,4} = 13$.

Part (a) *also solved by the proposer.*

Part (b) *remains unsolved. The proposer notes that it is equivalent to: how many ways are there to arrange n 0's and m 1's equally spaced around a circle so that neither the 0's nor the 1's are contained in a semicircle? (Rotations and reflections are not counted as different.) Can anyone settle this problem, at least for $n = 3$, say?*

* * * * *

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