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Volume 6 of *Mathematical Spectrum* will consist of two issues, the second of which will be published in the Spring of 1974.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Hicks Building, The University, Sheffield S3 7RH.

A Summary of Results of the Mathematical Spectrum—1973 Questionnaire

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At the time of going to press, only some 4% of our readers had returned replies to the Questionnaire included in Volume 5 No. 2 issued last spring. One cannot draw reliable conclusions about the entire readership from so small a sample; but personally, I found the remarks of the small group of respondents fascinating. While hoping that they are broadly representative, I am unable to assert this with any confidence.

The occupations of the respondents were:

Schoolteachers	41%	Professional workers	14%
Lecturers	17%	Others	5%
Students at school and university	23%		

Here are their answers to our questions.

PART I—READER SATISFACTION

1. How well do you think that the Editorial Board has met its avowed aim of entertaining and instructing mathematics students?
   Well: 75% Moderately: 23% Badly: 1% No answer: 1%
2. Do you consider the average article in *Mathematical Spectrum* is
   Too hard: 19% About right: 77% Easy: 4%
3. Do you think the balance of topics discussed is
   Good: 62% Average: 37% Bad: 1%
4. Rank of the topics in order of respondents' interests:
   1. Pure Mathematics
   2. Mathematical Problems
   3. Statistics
   4. Applied Mathematics
   5. Computing Science
   6. Operational Research
   7. History
   8. Book Reviews
   9. Letters
   10. Careers and Miscellaneous
5. What would make *Mathematical Spectrum* a magazine of wider readership?
   Simpler articles: 27% More puzzles and entertainments: 42%
   Closer relationship to current syllabuses: 27% No answer: 4%

These percentages are only approximate, as some readers gave more than one reply. In addition, 25% of these respondents provided other suggestions; many emphasized the need for real applications of mathematics in science and industry, and requested more discussion of the teaching aspects of mathematics.

On the whole, the results of Part I bear out the general impressions of the Editors on the tastes and requirements of readers.

PART 2—COSTS AND CIRCULATION

6. Do you think your library will take out an order?
   Yes: 56% No: 20% No answer or not applicable: 24%
7. Would you recommend a personal copy of *Mathematical Spectrum* to your fellow students or colleagues as a good buy?
   Yes: 86% No: 10% No answer: 4%
8. Would you prefer a cheaper form of printing with a saving in price to the present high quality printing?
   Yes: 11% No: 28% No answer: 57%

Replies to Part 2 indicate that our pricing policy is proving acceptable to readers; most of them appear to prefer our present high quality printing.

PART 3—PERSONAL COMMENTS

These were varied and most instructive. They ranged from advice on the type of article (theoretical, practical, geometric, computer oriented, historical, research oriented) desired by readers, to suggestions that we provide model solutions to examination papers and Olympiad-type problems. Some suggested a greater frequency of issues, others felt that readers might submit titles of interest on which articles could be written. Many put forward the opinion that readers could contribute more to the magazine.

One overwhelming impression emerged from these suggestions: that our respondents take the dissemination of mathematics and the progress of mathematical education very seriously. I wish to thank them all, and assure them that the Editors will do all in their power to implement their valuable ideas and to meet their criticisms.

The Numerical Analysis of a Simple Game

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1. Introduction: description of the game

There are many games in which the players pay and receive penalties according to an agreed set of rules. Every player will have a choice of strategies, or particular lines of play; how can each determine the best strategy to follow? Unfortunately, most games are so complex that a full analysis of strategies is virtually impossible. But it may be instructive to examine a very simple game in order to discover which is the best strategy for each player. This analysis will involve a determination of the likely result of each combination of the strategies which players may adopt.

Suppose two players A and B each start with 3 units of capital. They alternately throw a tetrahedral die numbered from 1 to 4. Player A is allocated numbers 1 and 2, and player B numbers 3 and 4. If player A throws one of B's numbers, A pays B a penalty which depends on B's choice of strategy; the roles are reversed for B's throw. There are two strategies, but the player's choice of strategy must be declared at the beginning of the game. The strategies are as follows:

Strategy I. One number has a penalty of two units and the other has zero penalty;

Strategy II. Both numbers have a penalty of one unit.

A player's own numbers carry a zero penalty.

Let us first consider the penalties for Strategy I. If a player throws his own numbers (2 cases out of 4), he will pay zero penalty. But he may throw a '2 penalty' number (1 case out of 4) in which event he will pay a penalty of 2; alternatively, he may throw a '0 penalty' number (1 case out of 4) and pay zero penalty. Thus, the probability of a zero penalty is $\frac{3}{4}$, while the probability of a penalty of 2 is $\frac{1}{4}$.

For Strategy II, if a player throws his own numbers (2 cases out of 4), he will pay zero penalty. He may, however, throw a '1 penalty' number (2 cases out of 4) and pay a penalty of 1. Thus, in this case, the probability of a zero penalty is $\frac{1}{2}$, while the probability of a penalty of 1 is $\frac{1}{2}$.

2. Matrix representation and equations

A very convenient way of representing and manipulating this information about particular strategies is by a transition matrix. Suppose that B throws the die, then we may represent the possible change in A's capital, depending on his choice of Strategy I or Strategy II, by the respective matrices:

Strategy I: (2, 0)

$$T_1 = \begin{array}{c|ccccccc} \nearrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 2 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 4 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 5 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array},$$

Strategy II: (1, 1)

$$T_2 = \begin{array}{c|ccccccc} \nearrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}.$$

If A throws the die, the change in his capital, depending on B's choice of strategy, will be given by

$$
\text{Strategy I: } (2,0) \qquad\qquad\qquad \text{Strategy II: } (1,1)
$$

$$
T_3 = \begin{array}{c|ccccccc}
\nearrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\
3 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\
4 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\
5 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}, \qquad
T_4 = \begin{array}{c|ccccccc}
\nearrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}.
$$

The numbers down the side of each matrix denote A's capital *before* a certain throw; those along the top denote his capital *after* the throw. The elements of the matrix give the probability that A, having started with a certain capital, will finish with another. For example, consider the second row of T_1, where A has an initial capital of 1. There is a probability of $\frac{3}{4}$ that A will not receive a penalty and will finish with a capital of 1 (state 1), but there is a probability of $\frac{1}{4}$ that he will receive a penalty of 2 and will finish with a capital of 3 (state 3).

If a row vector M_n

$$M_n = [p_0(n), p_1(n), p_2(n), p_3(n), p_4(n), p_5(n), p_6(n)],$$

with $\sum_{i=0}^{6} p_i(n) = 1$, represents the state of A after n throws, where $p_i(n)$ is the probability that A is in state i, then

$$M_{n+1} = M_n T_\delta \quad (\delta = 1, 2, 3, 4).$$

Initially A has a capital of 3 units, so that

$$M_0 = (0, 0, 0, 1, 0, 0, 0).$$

Suppose B throws first; then

$$M_1 = M_0 T_\alpha \tag{1}$$

where T_α is the transition matrix of the strategy selected by A for B's penalties, which may be either T_1 or T_2. A then has the next throw and

$$M_2 = M_1 T_\beta = M_0 T_\alpha T_\beta \tag{2}$$

where T_β is the transition matrix of the strategy selected by B for A's penalties, which may be either T_3 or T_4. We see that

$$T_\alpha T_\beta = T_1 T_3,\ T_1 T_4,\ T_2 T_3 \text{ or } T_2 T_4.$$

These matrices are

$$
T_1 T_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{16} & \frac{5}{8} & 0 & \frac{3}{16} & 0 & 0 & 0 \\
\frac{3}{16} & 0 & \frac{5}{8} & 0 & \frac{3}{16} & 0 & 0 \\
0 & \frac{3}{16} & 0 & \frac{5}{8} & 0 & \frac{3}{16} & 0 \\
0 & 0 & \frac{3}{16} & 0 & \frac{9}{16} & 0 & \frac{1}{4} \\
0 & 0 & 0 & \frac{3}{16} & 0 & \frac{9}{16} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
T_1 T_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 \\
0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\
0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & 0 \\
0 & 0 & 0 & \frac{3}{8} & \frac{3}{8} & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$T_2 T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{8} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_2 T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It should be noted that the matrices are not symmetrical about the diagonal; they are all different and the sum of the elements in each row is 1.

Let $T_\alpha T_\beta = T$. If B and A throw again, then, since the players choose their strategies at the start of the game and are not allowed to change them, Equations (1) and (2) show that

$$M_4 = M_2 T = M_0 T^2.$$

In general

$$M_{2n} = M_0 T^n.$$

As this is a constant-sum game, i.e., at any time the sum of A's capital and B's capital is 6 units, the likely distribution of one player's capital also determines that of his opponent. The probability $p_0(n)$ that a player has 0 units is the probability that he has lost, while $p_6(n)$ is the probability that he has won.

3. Method of solution

The required solution to the game is the vector R describing the situation when a player has either won or lost, so p_1 to p_5 are zero and $p_0 + p_6 = 1$; further play cannot alter this vector.

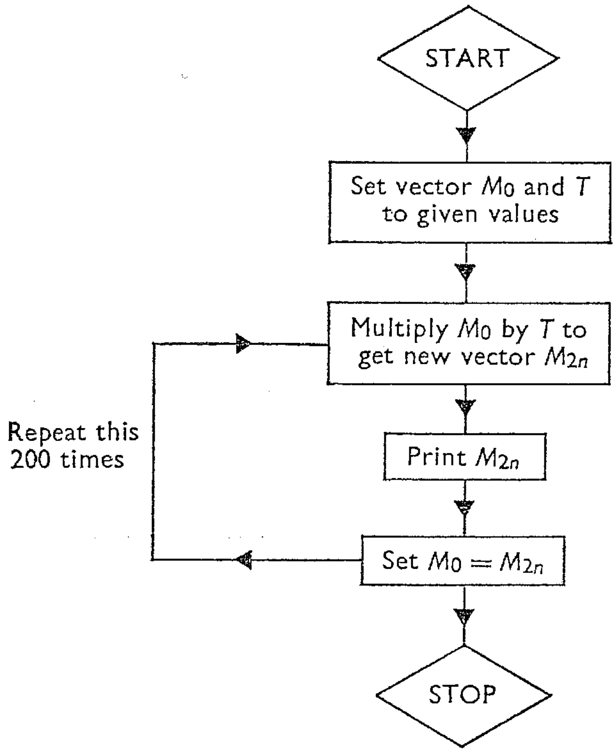
The standard method of finding R is by substitution. As further play does not alter R, it is unchanged by multiplication by T so that

$$R = RT. \tag{3}$$

The values of p_0 to p_6 can be determined from this equation. Unfortunately, in this particular case any vector of the form $((1-x), 0, 0, 0, 0, 0, x)$ will satisfy Equation (3).

After every multiplication of M_n by T, the probabilities $p_1(n)$ to $p_5(n)$ are reduced and $p_0(n) + p_6(n)$ tends nearer to one. A good solution may therefore be obtained by repeatedly multiplying M_0 by T until $p_1(n)$ to $p_5(n)$ are satisfactorily close to zero. This solution will then approximately satisfy Equation (3).

Matrix multiplication is a lengthy process by hand, but is a relatively simple task for a computer. The flowchart below shows the basis of the program which was used to evaluate our approximate solution. The row vector and the transition matrix were fed in on paper tape.



After 200 multiplications $p_1(400)$ to $p_5(400)$ were zero to at least 8 decimal places.

From these results we can construct a table giving the probability that A will win for each combination of strategies.

		B's choice of strategy	
		I(2, 0): T_3	II(1, 1): T_4
A's choice of strategy	I(2, 0): T_1	0·533	0·517
	II(1, 1): T_2	0·567	0·545

4. Analysis of results

The analysis of the results is based on Von Neumann's Theory of Games, although in this case the correct answer may be obtained by inspection. Whichever strategy B chooses, A has a better chance of winning by choosing Strategy II (1, 1), and similarly B loses less heavily by choosing Strategy II (1, 1). The probability of A's winning with this combination of strategies is 0·545 and is called the value of the game. This is a stable solution or saddle-point because there is no advantage gained by letting the opponent declare his choice first.

Figure 1 shows, for the combination of both A and B playing Strategy II (T_2 and T_4), how the probability that A has won after a certain number of throws increases with the number of throws. It was plotted from the values of $p_6(n)$, $n = 1, \ldots, 400$, obtained from the computer, and tends to the value of the game.

This method of finding the optimum strategy can be extended to other games, especially other model games which have very simple rules.

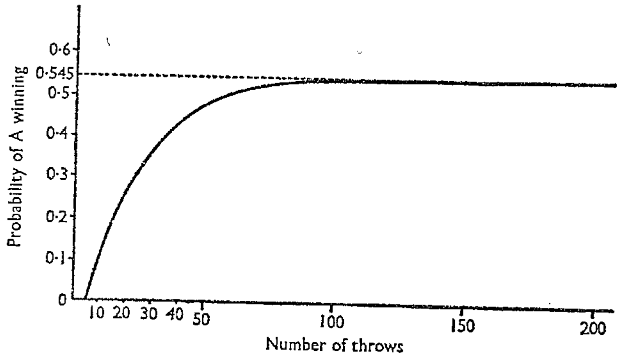


Figure 1

Acknowledgments

The author would like to thank Mr A. Corbett for writing the program and Kingston Polytechnic for the use of their computer. He would also like to thank Professor J. Gani for assistance in preparing this article for publication.

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Statistics in School and Society

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At a recent conference on 'The Teaching of Statistics in Schools' held on 17 March 1973 at the University of Sheffield, the following statements were made about statistics by different speakers.

> "Statistics is the collection and arrangement of numerical facts or data, whether relating to human affairs or to natural phenomena."
>
> "Statistics is often treated in a grossly formal way, with little regard to the fundamental topic of 'data analysis'. There is a tendency to treat the subject as 'exercise fodder' for pure mathematical ideas such as integration."

"In statistics one way to develop intuition is to ask students to estimate probabilities or to predict events and see how their guesses improve when they are asked to accept bets based on the associated odds."

These quotations (references 1, 2 and 3) indicate the variety of viewpoints put forward during the conference. Below is one person's impression of what went on—the issues raised, topics discussed and points made—prepared from memory and augmented by comparison with notes from some of the speakers.

1. Why a conference on the teaching of statistics in schools?

The initial impetus for this meeting came from a group of members of the local branch of The Mathematical Association. Their dissatisfaction with the current state of statistical teaching in schools was shown by the remarks of Dr Barnett and Mr Durran at the conference, to be a reflection of views held by the profession generally. For some time now, we heard, a number of teachers have been aware that serious problems beset the present-day teaching of statistics. Many teachers have no previous experience of the subject and attempt to learn it from books: but books at this level have been scarce, and good books still are. The statistics syllabuses of examining boards are loaded with a combination of new mathematics and a wealth of new concepts for which the students (and often the teachers) are totally unprepared. Consequently the usefulness of such excessively formal courses to school-leavers has been questioned; indeed, some university teachers have even suggested that when not actually a hindrance, previous experience of statistics is seldom of much use to the newly-enrolled university student.

If only a fraction of the previous remarks were true, how could a conference which raised these issues and sought possible ways of improving the situation fail to be worthwhile?

2. Do we need statistics in schools? It's already in society!

Although not specifically raised at the conference, this obvious question was implicitly discussed and some tentative answers offered. Professor Gani opened the conference by referring to a videotape (reference 4) in which he had analysed the articles of a statistical nature in a (random) issue of *The Times*. In that issue 13 such items were found; these consisted, roughly, of 6 on social statistics, 3 on operational research and general statistics and 4 on actuarial and business statistics. They included articles on the income distribution of New York dustmen, a comparison between the eating habits of the United States and the Soviet Union, the distribution of the amounts of money left in wills, and the patterns of readership of different newspapers and journals in an Oxford college common room. This simple and novel approach to a lesson made this point clear: in order to be able to make sense of this sort of material in a daily paper, some previous acquaintance with statistics was necessary.

At the conference it occurred to me that perhaps not everyone reads *The Times*; but Mr Merlane's address contained a large number of ideas, possibly at a less complex level, whose understanding also required a knowledge of elementary probability and statistics. While it is clearly not necessary for all of us to know how to plan a system of traffic lights, how to decide whether a school needs a traffic warden, or how to analyse a simple gambling game in order to determine a fair entrance fee, he indicated how such material could be learned by school students in an enjoyable manner. The insight gained into such phenomena is valuable not only to young students, but also to adults, in understanding the world about us.

In his address, Mr Durran discussed the possibility of school courses with some statistical content; he spoke of a 'Statistics with Mathematics' rather than a 'Mathematics with Statistics' course, and went on to wonder about the desirability of teaching 'Statistics with Biology' or similar composite courses. Indeed, almost all other sciences—biological, social, physical—as well as the mathematical ones, have a statistical component and he felt that it was time this was meaningfully catered for.

3. Some aspects of the teaching and learning of statistics

3a. Concept formation

We have already mentioned, and the point was repeatedly raised at the conference, that a major difficulty for both teachers and students is the problem of absorbing new concepts. In junior school and daily life, experience is gained which reinforces the concepts and intuitive understanding associated with such things as number, geometric ideas, dimension, weight, time and so on. Such experience is vital before these topics are discussed abstractly; but it seems to be considered that somehow such ideas as randomness, average, probability distribution, simulation, null hypothesis, estimate and many others can be adequately dealt with *for the first time* in purely mathematical terms.

This is, of course, not true; indeed it is the experience of many teachers that such topics require a far more refined intuitive understanding, which takes much longer to acquire. Concrete experience is essential, symbols on paper are almost irrelevant. During the conference, a number of ways of providing such experience were mentioned. These ranged from simple coin-tossing—with betting, to add an element of seriousness (or frivolity!)—to the more elaborate experiments and projects mentioned below.

Again a clear point emerged: such activities are not only extremely valuable (rather than frivolous as might appear on first sight), but they are probably essential before an adequate understanding of the subject can be gained.

3b. Let the statistics shine through

Another point which was made with some force was the following. If we are to study some statistical idea, then let us do so unashamedly; let us avoid clouding the issue by covering pages with numerical calculations, with integration theory or with algebraic manipulations. In discussing concepts, theory and examples should be kept to a minimum. For example, it is not necessary to introduce the notions of significance testing via normal distribution theory (including the t- and F-tests), as is often done. Very simple examples can be constructed and are preferable. Here, Mr Durran felt that non-parametric tests and rapid computational methods—the so-called quick and dirty methods of Tukey and others—were quite adequate, if not more desirable because of their greater flexibility.

Mr Durran drew a lesson from an examination question which clearly required students to perform a rather difficult but standard test. In the given situation, however, the test was obviously inapplicable and a student with the tiniest amount of insight (and certainly any 'practical researcher') would not have dreamt of carrying out any test at all. A *look* at the data was enough! Here was an illustration of Dr Barnett's point that the mathematical part of the statistics course had taken over, while the element of statistical analysis had disappeared.

3c. Practical work in statistics

Practical work, with both real-life and artificial situations, has an important role to play in learning statistics. In addition to forming the bed-rock of a student's statistical intuition, it offers excellent opportunities for statistics to be meaningfully illustrated. We conclude by listing some of the many examples given by Mr Merlane, and also directing the reader to reference 5.

Some simple practical work in statistics

Elementary

A statistical fair: roll-a-penny, fishing, guess the weight, simple fruit machines, snakes and ladders. (Odds may be offered, real money used and the proceeds given to charity!)

General

How many Ford cars are there with a K registration?
How should 100 tiles be apportioned in French *Scrabble*?
How are T.V. advertising charges determined?
How many packets of cereal X need to be bought before a full set of the enclosed coupons is obtained?

Probability models

Traffic plans, pattern recognition, queuing problems, petrol sales, journey times. (Simulation can be carried out by using dice or random numbers or even a computer.)

Statistical problems

Election forecasting, population models, stock control, parasite–host problems in biology and distributions in physical gases.

Many further ideas are mentioned in reference 6.

4. Where next?

The conference we have been talking about is one of many which have been held over the last few years all over Britain. Such conferences are held jointly or separately by organisations with an interest in the teaching of statistics in schools: for example, The Mathematical Association, The Royal Statistical Society, The Institute of Mathematics and its Applications and The Association of Teachers of Mathematics. We might also mention the wide variety of similar activities (in-service training courses, liaison group meetings) organised by institutes of education, university departments and similar bodies. All point to the widespread wish on the part of teachers to improve their knowledge and understanding of statistics.

Examining boards are also reviewing statistics syllabuses, considering the possibilities of new types of courses, and operating pilot experiments in these. For this reason it seems possible to end this article on a much more optimistic note than we began: the teaching of statistics is being widely discussed, issues are being raised and different views considered. We can only envy the students taking the subject in the next few years: they will be benefiting from these continuing efforts at improvement.

References

1. From the *Shorter Oxford Dictionary,* quoted by Professor J. M. Gani in the opening remarks to the Conference, which was held jointly by the Sheffield and District Branches of The Mathematical Association and The Royal Statistical Society. See 2, 3 below.

2. From Dr V. D. Barnett's address 'A broad view of developmental needs in school statistics—examinations, syllabuses, teaching problems'.

3. A composite quotation from the addresses 'Present-day statistics syllabuses' by Mr J. H. Durran and 'In a comprehensive school' by Mr G. Merlane.

4. Videotape on *Introduction to Statistics* available from Sheffield University Television Service.

5. a) Schools Council/Reading University *Sixth-form Mathematics Curriculum Project.* Newsletter No. 5; Sixth-form Discussion Paper 6/72; Reading Area News No. 6, No. 7, No. 9.

b) R. D. Harrison, An activity approach to the teaching of statistics and probability. I, II, III. *Mathematics Teaching* **34** (1966), 31–38; **35** (1966), 52–61; **36** (1966), 57–65.

c) Douglas Kaye, Investigations in and around the game of craps. *Mathematics Teaching* **41** (1967), 9–16.

d) R. Buxton, Probability and its measurement. *Mathematics Teaching* **49** (1969), 4–13; **50** (1970), 56–61.

e) Gabriel Goldstein, A classroom approach to simulation. *Mathematics Teaching* **49** (1969), 14–20.

f) Peter Wells, Decaying pennies. *Mathematics Teaching* **53** (1970), 2–3.

g) Freda Conway, A graphical approach to regression analysis. *Mathematics Teaching* **53** (1970), 7–11.

h) R. H. Clarke, Let's form a queue. *Mathematics Teaching* **56** (1971), 28–33.

6. G. H. Jowett and H. M. Davies, Practical experimentation as a teaching method in statistics. *J. R. Statist. Soc.* A **123** (1960), 10–36.

How Rigorous can a Proof be?

RAMESH KAPADIA
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In a recent article in *Mathematical Spectrum* (Volume 4, pages 63–68), Professor Schwarzenberger argued that rigour was not the only test for a mathematical proof; 'convincing-ness' was also important. Because I wondered whether absolute rigour was a meaningful concept, I decided to carry out an empirical test. Four proofs of Pythagoras' theorem were listed together with a questionnaire, asking the reader to rank the proofs according to the following criteria:

(i) Which is the most rigorous?
(ii) Which is the most convincing (to you)?
(iii) Which is the one you would use to convince a non-mathematical friend of the truth of the theorem?

The sheet was distributed to all members of staff and research students in the mathematics department at Nottingham, about sixty people in all. It was also given to first-year undergraduates, but their response was too small to assess. The proofs were (briefly) as follows:

(A) The standard proof given in Euclid (and written out in Schwarzenberger's article) for which the appropriate diagram is shown in Figure 1.

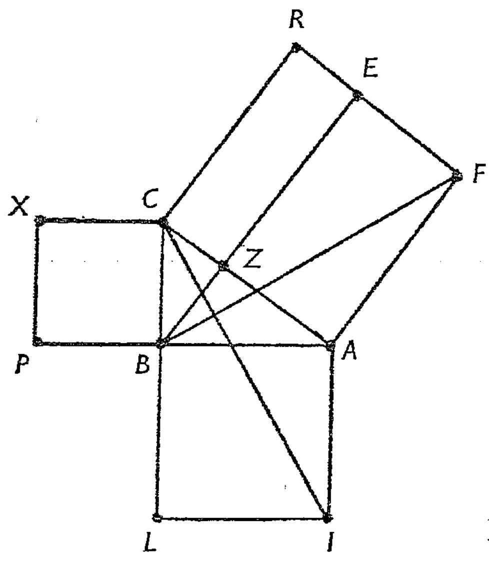


Figure 1

(B) The Chinese visual proof illustrated in Figure 2 (see Schwarzenberger's article for other versions).

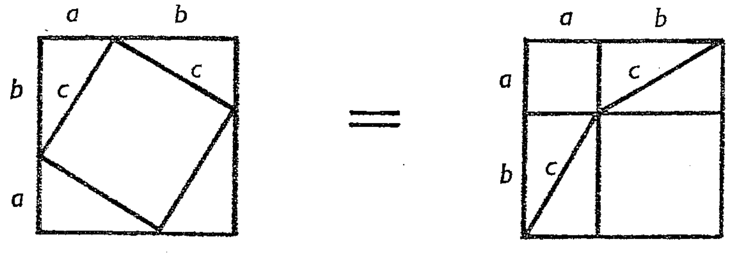


Figure 2

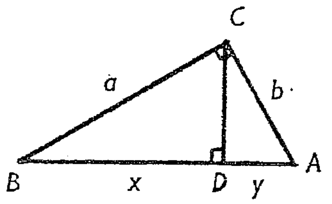
(C) The proof using inner products of vectors. For vectors l, m, defining the inner product (l/m) in the usual way, we have

$$d(l,m)^2 = (l-m/l-m)$$

and $(l/m) = 0$ if l, m are perpendicular. Let $A = (x, 0)$, $B = (x, y)$, $C = (0, y)$, $O = (0, 0)$, $OA = l$, $OC = AB = m$ so that $OB = l+m$ and $(l/m) = 0$. Then

$$\begin{aligned}(l+m/l+m) &= (l/l)+(l/m)+(m/l)+(m/m)\\ &= (l/l)+(m/m).\end{aligned}$$

(D) The proof by dropping a perpendicular from the right-angled vertex and using similar triangles, as indicated in Figure 3.



$x+y=c$

Figure 3

Not surprisingly, proof (B) was the almost unanimous first choice for the last question. Some people found the proofs equally convincing but (B) was the most popular choice with (A), (C), (D) equal second.

Table I

	Rankings																	
Proof	Staff						Post-graduates											
A	1	2	2	3	1	2	1	1	1	1	3	2	3	2	1	2	1	2
C	3	1	3	1	3	3	3	3	2	2	1	1	1	3	3	1	3	1
D	2	3	1	2	2	1	2	2	3	3	2	3	2	1	2	3	2	3
	$\chi^2 = 1$						$\chi^2 = 2{\cdot}7$											
	$\chi^2 = 2{\cdot}1$ overall																	

The answers to the first question turned out to be most revealing. Apart from placing (B) last (the 'anti-picture' lobby), the rankings were more or less random, as shown by Table 1. A Friedmann two-way analysis of variance was applied to the data. The null hypothesis was that rankings had been assigned by people on a random basis. For the staff $\chi^2 = 1{\cdot}0$, for the post-graduates $\chi^2 = 2{\cdot}7$, and overall $\chi^2 = 2{\cdot}1$. The value at the 5% level of significance is 7·82 so the null hypothesis must be accepted. Rigour is not the universally accepted criterion it is made out to be.

What is rigour? Applying the usual standard strictly, a proof is considered to be rigorous provided that no implicit unstated assumption is made in the demonstration. Exactly how does one search for unstated assumptions in a proof? By their nature they are not easy to find. This is reminiscent of the famous verification principle propounded by the logical positivist philosophers to banish metaphysics from philosophy: a statement is meaningful only if, in principle, it can be verified empirically. Not only does this exclude metaphysics but it also rejects science; thus the verification principle is meaningless.

The same sort of analysis applies to the absolutely rigorous proof, which perhaps deserves the famous aphorism 'only the completely meaningless is perfect'. Personally, I believe that rigour should be viewed as a more flexible concept to be applied locally rather than globally in mathematics.

How the Lion Tamer was Saved*

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A lion tamer enters his lion's cage. He notices at once that the beast is in an ugly mood. The door has slammed shut behind him and cannot be opened from inside, and there is nobody within earshot. He knows that he and the lion can move at the same maximum speed. He is, naturally, anxious to discover whether (a) there is a strategy for him enabling him to escape the lion for ever and ever, irrespective of what the lion might do, or whether (b) there is a strategy for the lion, assumed of infinite intelligence, following which he could get his man whatever the latter might do trying to evade capture. If, the man reflects ruefully, the cage had been the whole infinite plane then he could escape by moving at maximum speed on a straight line leading directly away from the lion's present position. Alas, there is the wall of the cage to contend with, and it very much looks as if he is doomed to serve as food for the magnificent King of the Animal World.

The object of this note is to cheer up the lion tamer by proving (Theorem 1) that he can escape the lion for ever and ever. Moreover (Theorem 2), had the encounter taken place in the middle of the Sahara Desert, then the man could plant a palm tree and move in such a way that (i) he remains always in the shade of his tree, no matter how small the shady area should be, (ii) his path converges, in the strict mathematical sense, to a limiting point M_{lim}, and all the time the lion must go hungry. Needless to add that the man will be careful not to disclose to the lion the exact position of M_{lim}. This last remark is nonsense because the man himself does not know the position of M_{lim}; this depends on the lion's moves.

* This article originally appeared in the *Ontario Secondary School Mathematics Bulletin* (October 1972), and is reprinted here with the permission of the Editor.

For simplicity's sake let us assume that both lion and man are mathematical points and that each can alter course and velocity at will and without restriction, as long as their velocities remain between 0 and $v_{\max}$ inclusive, where $v_{\max}$ is a fixed positive number.

Theorem 1. Given a positive number p, there is a strategy for the man following which he will escape the lion for ever and will never stray further than p units of length from his initial position.

Theorem 2. Given a positive number r, there is a strategy for the man such that if he follows that strategy then (i) *he will never be further than r units of length from his initial position,* (ii) *his path will converge, in the strict mathematical sense of the term, to a limiting point, without necessarily ever reaching it, and the lion will never catch him.*

Theorem 1 is due to A. S. Besicovitch; his original proof is published in reference 1. I shall give a version of his proof which, though using the same basic idea, might be easier to follow. Theorem 2 is new, and it will be proved in detail except for part (ii) which requires a rather more sophisticated mode of reasoning.

Proof of Theorem 1. I use the term 'line' to denote a 'both ways infinite straight line'. Let $|PQ|$ denote the distance between two points P, Q. Let L_0, M_0 be the initial positions of lion and man respectively. Of course, we assume that $L_0 \neq M_0$. We are given a number $p > 0$.

1. The man selects M_1, an arbitrary point strictly inside the cage, such that $|M_0M_1| < \frac{1}{2}p$ and $|M_0M_1| < \frac{1}{2}|M_0L_0|$. Moreover, the whole straight segment M_0M_1 is to lie in the cage. Since M_1 is strictly inside the cage, there is a positive number q such the $q < \frac{1}{2}p$, and the whole circular disk, centre M_1 and radius q, lies in the cage. Now the man moves from M_0 to M_1 in a straight line and at maximum speed $v_{\max}$. Before setting out, however, he reflects that on his way from M_0 to M_1 the lion cannot get him, in view of the fact that for every pair of simultaneously adopted positions M, L of man and lion, we have (Figure 1)

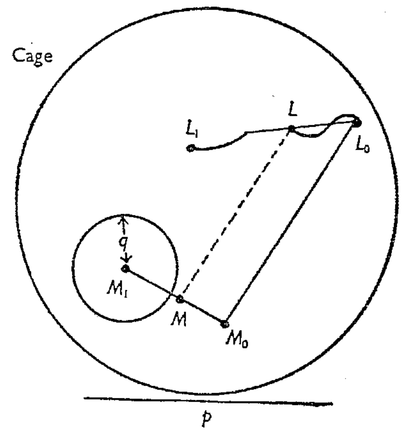

Figure 1

$$|ML| \geqslant |M_0L_0| - |M_0M| - |L_0L| \geqslant |M_0L_0| - |M_0M| - |M_0M|$$
$$\geqslant |M_0L_0| - 2|M_0M_1| > 0.$$

Whilst the man runs from M_0 to M_1, the lion moves from L_0 to L_1 along any path whatsoever. Before continuing from M_1, the man proves Lemma 1.

Lemma 1. If $a_n = 1/n$ *for* $n = 1, 2, 3, \ldots$ *then*

$$a_1 + a_2 + a_3 + \ldots = \infty$$

and

$$a_1^2 + a_2^2 + a_3^2 + \ldots \leqslant 2.$$

Proof of Lemma 1. For every $k = 1, 2, 3, \ldots$, we have

$$a_{k+1} + a_{k+2} + \ldots + a_{2k} \geqslant k a_{2k} = \tfrac{1}{2}.$$

Therefore the infinite series $a_1 + a_2 + a_3 + \ldots$ contains infinitely many mutually disjoint blocks of consecutive terms such that the terms of each block add up to at least $\frac{1}{2}$. Clearly, this makes the series divergent. In contrast, if $N = 2, 3, \ldots$ then

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_N^2 \leqslant 1 + \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{(N-1)N}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right)$$

$$= 2 - \frac{1}{N} < 2.$$

Having proved Lemma 1, the man carefully marks the position of M_1 on the floor of the cage using for the purpose a piece of chalk which the lion has kindly tossed over to him.

2. Suppose that the man has reached safely, without being eaten up, the positions $M_0, M_1, \ldots, M_n$, for some positive integer n. Let L_n be the lion's position at the time the man is at M_n. The man hurriedly proves Lemma 2.

Lemma 2. If the man moves at maximum speed in a fixed sense along the line l *through* M_n *perpendicular to* $M_n L_n$, *he cannot be caught.*

Proof of Lemma 2. Let, during the manoeuvre in question, lion and man be at L^1, M^1 respectively (Figure 2). Then

$$|M^1 L^1| \geqslant |M^1 L_n| - |L_n L^1| \geqslant |M^1 L_n| - |M_n M^1| > 0.$$

It is vital, for the lion tamer no less than for our mathematical integrity, that the last inequality sign is $>$ and not $\geqslant$.

Having proved Lemma 2, the man moves at maximum speed along l through a distance $\frac{1}{2} a_n q$, the sense along l being such that, unless the previously marked point M_1 happens to lie on the line $M_n L_n$, the man initially is getting closer to M_1 (Figure 3). (He has a momentary fear that, while on this tack, he might collide with the bars of the cage and so knock himself out. Fortunately, Lemma 3 will set his mind at rest on that score.)

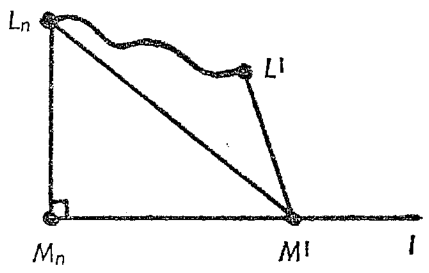

Figure 2

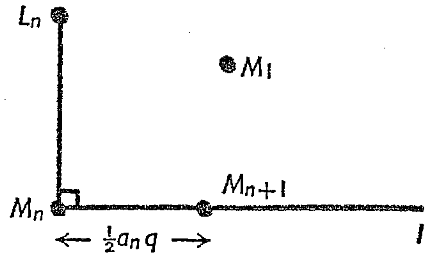

Figure 3

By following this rule the man will be safe from the lion, in view of Lemma 2. Also, his path $M_0 M_1 M_2 M_3 \ldots$ is of total length $|M_0 M_1| + \frac{1}{2}q(a_1 + a_2 + \ldots)$ which, by Lemma 1, is infinite. Since v_{max} is finite, the man can keep going for ever and ever. It only remains to prove that M_n really does lie inside the cage. This will follow if we can show that $|M_n M_0| < p$ for $n = 1, 2, \ldots$.

Lemma 3. For $n = 1, 2, \ldots, |M_{n+1} M_1|^2 - |M_n M_1|^2 \leqslant (\frac{1}{2}a_n q)^2$.

Proof of Lemma 3. Denote by d_n the distance of M_1 from the line $M_n L_n$.

Case 1. $d_n \leqslant \frac{1}{2}a_n q$ (Figure 4).

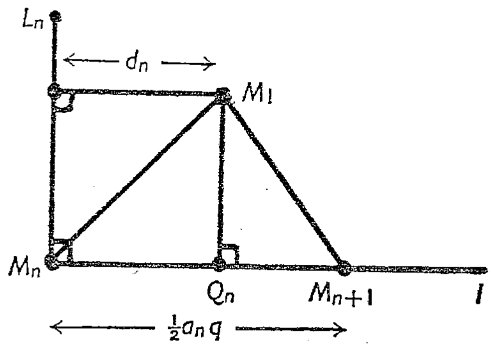

Figure 4

Then

$$|M_{n+1} M_1|^2 = |Q_n M_1|^2 + |Q_n M_{n+1}|^2 \leqslant M_n M_1|^2 + (\tfrac{1}{2}a_n q)^2.$$

Case 2. $d_n > \frac{1}{2}a_n q$ (Figure 5).

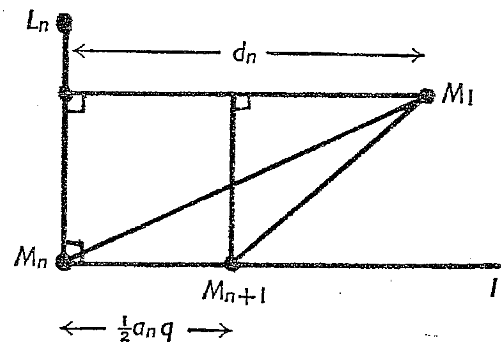

Figure 5

Then

$$|M_{n+1} M_1|^2 \leqslant |M_n M_1|^2 < |M_n M_1|^2 + (\tfrac{1}{2}a_n q)^2.$$

This proves Lemma 3.

On adding the inequalities obtained from Lemma 3 by substituting, in turn, for n the values 1, 2,..., $m-1$, we find, by Lemma 1,

$$\begin{aligned} |M_m M_1|^2 &\leqslant (\tfrac{1}{2}q)^2(a_1^2+\ldots+a_{m-1}^2) \\ &< (\tfrac{1}{2}q)^2(a_1^2+a_2^2+\ldots) \leqslant \tfrac{1}{2}q^2 < q^2; \\ |M_m M_1| &< q. \end{aligned}$$

Hence, by our choice of q, the whole path $M_0 M_1 M_2 \ldots$ lies in the cage, and we have indeed saved the lion tamer's life. Finally, we observe that

$$|M_0 M_m| \leqslant |M_0 M_1| + |M_1 M_m| < \tfrac{1}{2}p + q < p.$$

Proof of Theorem 2. Making use of a convenient *Fata Morgana*, the man imagines himself and the lion in some cage, and he applies Theorem 1, with varying initial positions M_0 and varying values of p. Let him start at the point M^0 and apply Theorem 1 with $M_0 = M^0$ and $p = \frac{1}{2}r$. He follows the correct strategy until he has in all moved through a length s, where s is arbitrary but fixed and positive. Suppose that he has by then got to the position M^1. He now changes his strategy, applying what he gets from Theorem 1 by putting $M_0 = M^1$ and $p = \frac{1}{4}r$. Again, he follows this new strategy until he has run through a total length s, reaching the point M^2 in the process. He now switches to $M_0 = M^2$ and $p = \frac{1}{8}r$, following this strategy until he has covered the length s, and so on. The lion will not be able to get him, and since the total length to be covered by him is $s+s+s+\ldots$, he can go on for ever. Let him stop at some stage, say at a point P on the run from M^k to M^{k+1}, and let him consider his position, taking zero time for the operation. He will notice that

$$\begin{aligned} |M^0 P| &\leqslant |M^0 M^1| + |M^1 M^2| + \ldots + |M^{k-1} M^k| + |M^k P| \\ &\leqslant \tfrac{1}{2}r + \tfrac{1}{4}r + \ldots + 2^{-k} r + 2^{-k-1} r < r, \end{aligned}$$

so that he will be able to enjoy the shade of his palm tree for ever and ever, provided r is small enough.

There remains the proof that his path will converge. At every stage beyond M^k, say between M^a and M^{a+1}, his position Q will be such that

$$\begin{aligned} |M^k Q| &\leqslant |M^k M^{k+1}| + |M^{k+1} M^{k+2}| + \ldots + |M^{a-1} M^a| + |M^a Q| \\ &\leqslant r((\tfrac{1}{2})^{k+1} + (\tfrac{1}{2})^{k+2} + \ldots + (\tfrac{1}{2})^a + (\tfrac{1}{2})^{a+1}) < r(\tfrac{1}{2})^k, \end{aligned}$$

which can be made arbitrarily small if only k is big enough. It now follows from a theorem in analysis, *Cauchy's General Principle of Convergence*, that the path converges to a limit point. This principle is somewhat intricate to state and goes rather beyond the scope of this note.

Moral: there is no need to despair, as long as you can shrink your size to zero and can move with unrestricted acceleration.

Reference

1. J. E. Littlewood, *A Mathematician's Miscellany* (Methuen, London, 1953), 135–136.

Approximate Methods in Mathematics

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This article will be concerned with a few examples which illustrate a general approach to problems rather than any specific mathematical technique. This approach has been particularly successful in the applied[1] rather than the pure side of mathematics. Basically it consists of using, in any procedure for a detailed solution, as much of the information about the problem as can be obtained from physical insight or experience about similar problems, or in fact from any source.

All the techniques discussed below have rigorous mathematical justifications. The point I wish to make is that very often an important technique is first developed by heuristic reasoning with a rigorous proof suggested later. Most of the exciting developments in the area of which Example 3 is an illustration came about this way.

1. As a first trivial example suppose we want the value of $\sqrt{10}$. Since $3^2 = 9$ and $4^2 = 16$, $\sqrt{10}$ must lie between 3 and 4 and probably closer to 3 than 4. Write $\sqrt{10} = 3+x$ where we expect x to be small (it clearly cannot be as big as 1). Then, squaring both sides, we have

$$10 = (3+x)^2 = 9+6x+x^2.$$

But, with x small, we can as a first approximation neglect x^2 compared with $6x$ and so $10 \doteqdot 9+6x$ which gives $x \doteqdot \frac{1}{6}$ which is indeed large compared with $\frac{1}{6}^2$. Thus a first approximation is $\sqrt{10} = 3{\cdot}167$. A more accurate approximation is obtained by using the same procedure on $3\frac{1}{6}$. Writing

$$10 = (3\tfrac{1}{6}+y)^2 \doteqdot (\tfrac{19}{6})^2+\tfrac{19}{3}y,$$

gives $y \doteqdot -0{\cdot}004$ and hence $\sqrt{10} = 3{\cdot}163$. The procedure clearly gives the answer quickly to whatever degree of accuracy required.

As a method, the above is applicable to other root calculations without much increase in numerical complexity. You should work out, for example, $13^{\frac{1}{3}}$ and $25^{\frac{2}{5}}$.

2. As a second example involving the roots of algebraic equations let us find the roots of $0{\cdot}01x^3-x+1 = 0$ or more generally $\varepsilon x^3-x+1 = 0$ where ε is a small (compared with 1) positive parameter. The formal solutions of cubic equations can be found, of course, but the procedure is rather tedious. The following procedure is much simpler.

A first thought is to say that εx^3 is small because $0 < \varepsilon \ll 1$ and so a first approximation is $-x+1 \doteqdot 0$, that is, $x \doteqdot 1$. To get a better approximation we write $x = 1+\delta$ where δ is small compared with 1. Substitution into the equation

[1] I do *not* mean mechanics here. In spite of the emphasis on it at school, it represents only a minute portion of what is considered applied mathematics. Calculus and differential equations are now considered to be mainly applied mathematics.

gives

$$\varepsilon(1+3\delta+3\delta^2+\delta^3)-(1+\delta)+1 = 0.$$

Since ε and δ are small we can neglect $\varepsilon\delta$, $\varepsilon\delta^2$, and $\varepsilon\delta^3$ compared with ε and δ and so the last equation gives $\varepsilon-\delta = 0$ which gives $x \doteqdot 1+\varepsilon$, as a more accurate solution: the systematic procedure is now clear and the next approximation will have an ε^2-term.

However, the solution $x \doteqdot 1+\varepsilon$ is only one of the three solutions of the cubic. What are the other two? The other solutions must involve the term εx^3 since when we neglect it we obtain only one solution. If εx^3 is not small like ε then x^3 must be sufficiently large for εx^3 *not* to be negligible. With x^3 large, so is x. But if x is large we can neglect 1 compared with x in the equation and so the other two solutions must come from the approximate form $\varepsilon x^3 - x \doteqdot 0$. This gives $x = 0$ or $\varepsilon x^2 \doteqdot 1$. Clearly $x \neq 0$, since it violates our assumption that x is large, and so we are left with $x \doteqdot \pm 1/\sqrt{\varepsilon}$ as the other two solutions. Better approximations are obtained in the same way as above by writing $x = \pm(1/\sqrt{\varepsilon})+\delta$ where now the magnitude of δ is small compared with $1/\sqrt{\varepsilon}$ (which is large).

Thus to a first approximation the three solutions are $x = 1,\ 1/\sqrt{\varepsilon},\ -1/\sqrt{\varepsilon}$. The smaller the value of ε, the more accurate are these solutions. In case you might be tempted to feed the algebraic equation directly into a computer, if I choose $\varepsilon = 10^{-40}$ say, the computer would find it very hard indeed to obtain solutions without help which involves some *a priori* knowledge of these.

You might like to examine the following examples: (i) $\varepsilon x^2 - x + 1 = 0$, for which you have an exact answer; (ii) $\varepsilon x^4 - x + 1 = 0$; and (iii) $x^3 - \varepsilon x + 1 = 0$.

3. Rather similar ideas to those used in Example 2 can be applied to differential equations. I should add again that there is a rigorous mathematical justification for the procedure which is described practically below.

Consider the simple first-order equation

$$\varepsilon\frac{dy}{dx}+y = 1 \quad \text{with } y(0) = 0, \quad 0 < \varepsilon \ll 1, \tag{1}$$

for $x \geqslant 0$. We might try neglecting $\varepsilon(dy/dx)$ because ε is small. If we do this $y(x) \equiv 1$, which does *not* satisfy the boundary condition. We might argue that it will be valid for most of the domain of x except near $x = 0$ where it is clearly not valid. Near $x = 0$, we conclude that $\varepsilon(dy/dx)$ is not small and in fact must be of comparable size to y or 1. If we write $y(x) = 1+Y(x)$ then

$$\varepsilon\frac{dy}{dx}+Y = 0 \quad \Rightarrow \quad Y(x) = Ae^{-x/\varepsilon},$$

where A is an undetermined constant. Since $y(0) = 0$, we must have $Y(0) = -1$ and so $A = -1$.

The solution is then

$$y(x) = 1-e^{-x/\varepsilon}. \tag{2}$$

Except near $x = 0$, $e^{-x/\varepsilon}$ is exponentially small and so $y(x) = 1$, namely the solution we obtained by neglecting $\varepsilon(dy/dx)$, which is indeed a good approximation *except* near $x = 0$. From (2), $[dy/dx]_{x=0} = -1/\varepsilon$ and so $y(x)$ is very steep near $x = 0$. This steep region is called a singular region or boundary layer. Figure 1 illustrates the solution for $0 < \varepsilon \ll 1$.

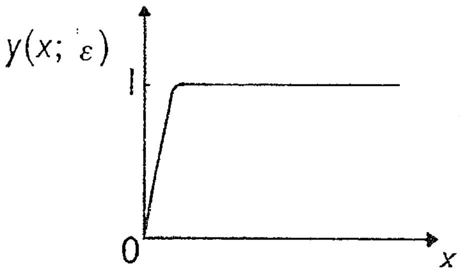

Figure 1

The idea used in this example has a wide applicability: the subject is called singular perturbation theory. In its most obvious form (when dealing with differential equations) the class of equations is recognized by the appearance of a small parameter multiplying the *highest* derivative in the equation.

As another example, which we can also solve exactly, although with more algebra, consider

$$\varepsilon\frac{d^2y}{dx^2} - y + 1 = 0, \quad y(-1) = 0 = y(1), \quad 0 < \varepsilon \ll 1 \tag{3}$$

for $-1 \leqslant x \leqslant 1$.

We expect $y = 1$ to be the solution away from $x = \pm 1$ where it does not satisfy the boundary conditions. Write

$$y = 1 + Y_{-1} + Y_1, \tag{4}$$

where we expect $Y_{-1}(x)$ and $Y_1(x)$ to be important only near $x = -1$ and $x = 1$ respectively. Near $x = -1$ then the approximate form of (3) with (4) is

$$\varepsilon\frac{d^2Y_{-1}}{dx^2} - Y_{-1} = 0. \tag{5}$$

In view of the boundary at $x = -1$, it is convenient to write the solution of (5) as

$$Y_{-1}(x) = A\exp\{-(x+1)/\varepsilon^{\frac{1}{2}}\} + B\exp\{(x+1)/\varepsilon^{\frac{1}{2}}\}.$$

Our assumption was that $Y_{-1}(x)$ was negligible except near $x = -1$. This implies that $B = 0$ (otherwise $Y_{-1}(x)$ would grow exponentially). We further require from (3) that $y(-1) = 1 + Y_{-1}(-1) = 0$ and so $A = -1$ giving, near $x = -1$,

$$y(x) = 1 - \exp\{-(x+1)/\varepsilon^{\frac{1}{2}}\}.$$

Similarly near $x = 1$, we get $Y_1(x) = -\exp\{(x-1)/\varepsilon^{\frac{1}{2}}\}$. Thus the 'asymptotic' solution for $0 < \varepsilon \ll 1$ is

$$y(x) = 1 - \exp\{-(x+1)/\varepsilon^{\frac{1}{2}}\} - \exp\{(x-1)/\varepsilon^{\frac{1}{2}}\}. \tag{6}$$

Note that the exponentials are *not* negligible only near $x = -1$ and $x = 1$, and $y(x) \doteqdot 1$ elsewhere. Again there are singular regions (or boundary layers) near the boundaries and the solution looks like Figure 2.

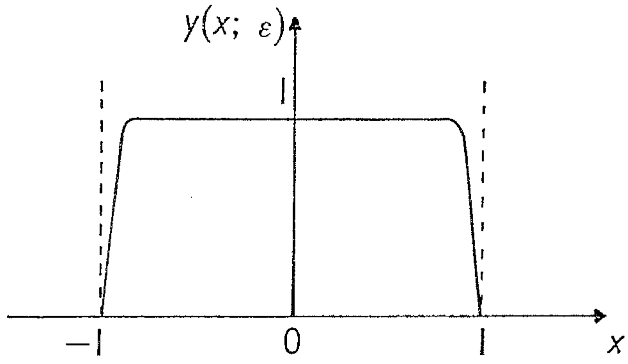

Figure 2

Here the singular regions are less steep for the same ε since from (6),

$$\left[\frac{dy}{dx}\right]_{x=\mp 1} = \pm\frac{1}{\sqrt{\varepsilon}}.$$

A shock wave from a supersonic aircraft is an example of a steep region in the solution of the equations (the Navier–Stokes equations) governing the motion of the air. The small parameter essentially represents the small viscosity of air.

4[2]. As another example I would like to describe a procedure for evaluating certain integrals by divergent series! I shall describe in detail the approximate evaluation of the integral

$$I(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \tag{7}$$

for large positive x; it does not have a closed-form solution. As $x \to \infty$ clearly $I(x) \to 0$ since the range of integration tends to zero as does the integrand.

If we integrate (7) by parts successively we get

$$\begin{aligned} I(x) &= \left[\frac{-e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \\ &= \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \\ &= \frac{e^{-x}}{x} + \left[\frac{e^{-t}}{t^2}\right]_x^\infty + 2\int_x^\infty \frac{e^{-t}}{t^3}\,dt, \end{aligned}$$

[2] This section is harder than the previous ones, and is included for the benefit of advanced students. The following section 5 is in no way dependent on it.

and so on, to obtain

$$\begin{aligned} I(x) = e^{-x}\left[\frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}-\frac{3!}{x^4}+\ldots+\frac{(-1)^{n+1}(n-1)!}{x^n}\right] \\ +(-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt. \end{aligned} \tag{8}$$

Let us write, for convenience,

$$\begin{aligned} S_n(x) &= e^{-x}\left[\frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}-\ldots+\frac{(-1)^{n+1}(n-1)!}{x^n}\right], \\ r_n(x) &= (-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt, \end{aligned} \tag{9}$$

and so (8) becomes

$$I(x) = S_n(x)+r_n(x). \tag{10}$$

The series $S_n(x)$ for a *fixed* x is divergent—the nth term tends to infinity as $n\to\infty$. Since the integral $I(x)$ in (7) is finite this means that $r_n(x)$ must also be infinite as $n\to\infty$ to cancel with the infinite part of $S_n(x)$ so that $r_n(x)+S_n(x)$ in (10) is finite.

Let us now suppose that n is fixed and x becomes large. From (9) the magnitude of $r_n(x)$, denoted by $|r_n(x)|$ is such that

$$|r_n(x)| = \left|(-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt\right| = n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt. \tag{11}$$

Now in the integral in (11), $x\leqslant t$ and so if we replace $1/t^{n+1}$ by its greatest value $1/x^{n+1}$ we get

$$|r_n(x)| < n!\int_x^\infty \frac{e^{-t}}{x^{n+1}}\,dt$$

and so we get an upper bound on $|r_n(x)|$ since

$$|r_n(x)| < \frac{n!}{x^{n+1}}\int_x^\infty e^{-t}dt = \frac{n!\,e^{-x}}{x^{n+1}}.$$

But this is the magnitude of the next $(n+1)$th term in the series for $S_n(x)$. Thus the ratio of the remainder $r_n(x)$ to the last term in $S_n(x)$ is such that

$$\left|r_n(x)\Big/(-1)^n\frac{e^{-x}(n-1)!}{x^n}\right| < \frac{n!\,e^{-x}}{x^{n+1}}\cdot\frac{x^n}{(n-1)!\,e^{-x}} = \frac{n}{x},$$

which is small for *n fixed* and x sufficiently large compared with n. Thus for a fixed *finite n* the accuracy of approximating $I(x)$ by $S_n(x)$ increases with increasing x. The error is of the order of the last term neglected in $S_n(x)$.

Thus although $S_n(x)$ is a divergent series as $n\to\infty$ it nevertheless gives a very accurate approximation to $I(x)$ for a *fixed n*. We call this type of approximation an *asymptotic* one and denote it by the symbol $\sim$ and write

$$I(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \sim e^{-x}\left\{\frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}-\ldots\right\}. \tag{12}$$

For example, if we take $x = 10$ and choose $n = 4$ we have

$$\int_{10}^\infty \frac{e^{-t}}{t}\,dt \sim e^{-10}\left\{\frac{1}{10}-\frac{1}{10^2}+\frac{2}{10^3}-\frac{6}{10^4}\right\}.$$

Comparing this approximation with the exact value of this integral the error is less than 3×10^{-3} %.

5. As a final illustration (see the reference given at the end of the article) which demonstrates that care must be taken and, where possible, rigour must prevail, let us try to solve (pretending that it is not trivial)

$$x+10y = 21, \quad 5x+y = 7. \tag{13}$$

If we argue that the coefficient of x in the first equation is small compared with the coefficient 10 of y we might neglect, as a first approximation, x compared with $10y$ and so $10y \doteqdot 21$ and $y \doteqdot 2{\cdot}1$. The second equation then gives $5x \doteqdot 7-2{\cdot}1$ and so $x \doteqdot 0{\cdot}98$. These are reasonably close to the exact answer to (13), namely $x = 1$, $y = 2$. In each situation x is small compared with $10y$. Let us now consider

$$0{\cdot}01x+y = 0{\cdot}1, \quad x+101y = 11, \tag{14}$$

which on using the same argument gives $y \doteqdot 0{\cdot}1$ and $x \doteqdot 11-0{\cdot}1(101) = 0{\cdot}9$. Again $0{\cdot}01x \doteqdot 0{\cdot}009$ is small compared with y which is $0{\cdot}1$. However, the exact solution of (14) is $x = -90$ and $y = 1$. Our approximate solution is nowhere near this, even though our procedure seemed reasonable and even seemed to be verified *a posteriori*. The point is that Equations (14) are almost a set of equations which cannot be solved, that is, a 'singular' problem.

So as to appreciate what is happening in this last example you should examine in detail the following equations

$$\varepsilon x+y = 0{\cdot}1 \quad \text{and} \quad x+101y = 11,$$

as ε varies from zero to say $0{\cdot}1$. You should then be in a position to make some general remarks on the procedure for solving systems of algebraic equations and explain why the wrong solution was obtained for (14) but not (13).

A final comment about the viewpoint of this article is that in examining any problem, whether it is evaluating an integral, setting up model equations for the spread of malaria, or trying to predict (as is possible) where a smoke particle will be deposited in the lung, the essential feature of the method used is the stripping away of inessentials, and the use of a flexible rather than a dogmatic approach.

Reference

L. E. Segel, Simplification and scaling. *Society for Industrial and Applied Mathematics (SIAM) Reviews* **14** (1972), 547–571.

Letters to the Editor

Dear Editor,

How best to arrange supports for book shelves

Those teaching statics rarely have the time or the inclination to include even a brief account of the deflection of beams. The subject is usually considered to be too dreary or too difficult—in fact fit only for engineers! To show that it need not be dreary I outline below a problem that is both interesting and of practical applicability, and one that can be tackled by anyone who has understood the underlying assumptions relating to the equilibrium of a slightly elastic beam.

The problem is how to arrange a given number of supports on the same level so that the maximum downward deflection of a uniformly loaded beam resting on them is as small as possible. (Where does one locate the two supports for a shelf to hold back-numbers of *Mathematical Spectrum*, in order to avoid an unsightly sag at the centre or at the ends?) The supports divide the beam into sections. Our problem may be taken to be equivalent to arranging the supports so that the maximum downward deflection in each section is equal.

In elementary statics we learn how to determine the shearing force and bending moment at any point of a beam in an equilibrium position. Further, in most standard texts it is shown that an approximation to the bending moment of a slightly deflected beam is EI/R, where E is the modulus of elasticity of its material, I is the moment of inertia of its cross-section, and R is its radius of curvature, provided that the breadth and depth of the beam are small compared with its length. By equating these two expressions for the bending moment we may determine an approximation to the deflection.

To solve our problem let us take the x axis along the undeflected direction of the beam and the y axis vertical. For any point P of the beam we may now equate the two expressions for the bending moment. For small deflections EI/R may be approximated by $\pm EI(d^2y/dx^2)$, and this is equated to the sum of the moments about P for the section of the beam in which P lies. On integrating the resulting equation twice we obtain the deflection y. Let the length of the beam be $2a$, and denote by k the weight per unit length divided by EI. I summarize below the solution of the problem with various numbers of supports.

The case of one support is, of course, trivial: it must be at the centre, and the maximum deflection is $\frac{1}{8}ka^4$ which occurs at the ends.

In the case of two supports, it is clear that they should be on opposite sides of the centre, equidistant from it. Let the distance of each support to its nearest end be b. Then the maximum deflection in the middle section will occur at the centre, and, by the above method, is found to be $k(a-b)^2(5a^2-10ab-b^2)/24$. Similarly, the deflection at the ends is $kb(-8a^3+24a^2b-12ab^2-b^3)/24$. Equating these quantities we find that $4b^3-12a^2b+5a^3=0$, and hence, approximately, $b=0{\cdot}446298a$. The maximum deflection is approximately $0{\cdot}0043156ka^4$.

When we have three or more supports, the analysis is simplified if we make use of Clapeyron's theorem of three moments: this is a linear relation between the moments at three adjacent supports. In these cases it is not always easy to determine just where the maximum deflection occurs. We could, of course, attempt to solve the equation $dy/dx=0$, but a simpler way to find the maximum deflection is to tabulate y for various P. With three supports, one support is at the centre and the other two are on opposite sides of it at a distance c (say) from their nearest end. By a simple iterative technique, based on trial and error, we may determine the value of c for which the maximum deflection in each of the four sections is equal. We find that, approximately, $c = 0{\cdot}284746a$, and that the maximum deflection is approximately $0{\cdot}0007152ka^4$.

For four supports we may use bi-variate iteration to find that the supports should be at distances which are approximately $0{\cdot}209070a$ and $0{\cdot}734228a$ from each end. In this case the maximum deflection is approximately $0{\cdot}0002079ka^4$.

Unfortunately, with more than four supports the amount of computation required is rather heavy, and at this stage the discerning bibliophile might well decide that two shorter shelves would be a better proposition!

Yours sincerely,

F. D. BURGOYNE

(University of London King's College)

* * *

Dear Editor,

A simple solution of the biquadratic equation

Your readers may be interested in a simple solution of the biquadratic equation. This method of obtaining the solution is different from both Ferrari's method and Descartes' method.

The standard form of the biquadratic equation is

$$ax^4+4bx^3+6cx^2+4dx+e = 0. \tag{1}$$

Let α, β, γ and δ be the roots of (1) and let

$$\alpha+\beta = P, \quad \gamma+\delta = Q,$$
$$\alpha\beta = R, \quad \gamma\delta = S.$$

Then

$$P+Q = -4b/a, \tag{2}$$
$$t+PQ = 6c/a, \tag{3}$$
$$QR+PS = -4d/a, \tag{4}$$

and

$$RS = e/a, \tag{5}$$

where

$$t = \alpha\beta+\gamma\delta = R+S. \tag{6}$$

Now, solving Equations (2) and (4) for P and Q,

$$P = \frac{4(d-bR)}{a(R-S)}, \quad Q = -\frac{4(d-Sb)}{a(R-S)}.$$

But

$$t = (6c/a)-PQ,$$

i.e.,

$$t = \frac{6c}{a}+\frac{16}{a^2}\frac{(d-Sb)(d-Rb)}{(R-S)^2}.$$

Using (5) and (6) to eliminate R and S we have

$$a^3t^3-6a^2ct^2+4a(4bd-ae)\,t-8(2ad^2+2eb^2-3ace) = 0. \tag{7}$$

Let $t = t_1$ be a root of (7). Then

$$\alpha\beta+\gamma\delta = t_1, \tag{8}$$
$$\alpha\beta-\gamma\delta = [(\alpha\beta+\gamma\delta)^2-4\alpha\beta\gamma\delta]^{\frac{1}{2}}$$
$$= [(at_1^2-4e)/a]^{\frac{1}{2}}. \tag{9}$$

Solving (8) and (9) we can find $\alpha\beta$ and $\gamma\delta$. Now using

$$\alpha+\beta+\gamma+\delta = -4b/a \tag{10}$$

and

$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = -4d/a, \tag{11}$$

we can find $\alpha+\beta$ and $\gamma+\delta$. Hence the solution of (1) is obtained as the roots of the two quadratic equations

$$x^2-(\alpha+\beta)\,x+\alpha\beta = 0$$

and

$$x^2-(\gamma+\delta)\,x+\gamma\delta = 0.$$

Example. Solve $x^4+3x^3+x^2-2 = 0$.

Here $a = 1$, $b = \frac{3}{4}$, $c = \frac{1}{6}$, $d = 0$ and $e = -2$. From Equation (7) we have

$$t = -1 \quad \text{or} \quad t^2-2t+10 = 0.$$

Letting $t_1 = -1$ we use Equations (8)–(11) to obtain

$$\alpha\beta = 1, \quad \gamma\delta = -2$$

and

$$\alpha+\beta = -1, \quad \gamma+\delta = -2.$$

Thus the required roots are the roots of the equations

$$x^2+x+1 = 0 \quad \text{and} \quad x^2+2x-2 = 0.$$

These roots are ω, ω^2 (the imaginary cube roots of unity) and $-1\pm\sqrt{3}$.

Yours sincerely,

A. B. Patel
(V. S. Patel College of Arts and Science,
Billimora, India)

Reference

S. Barnard and J. M. Child, *Higher Algebra* (Macmillan, London, 1960).

Problems Submitted by Readers

Dear Editor,

Bisecting the interior area of a triangle

I have a problem in elementary geometry to be solved by straight edge and compass construction only. The problem is this: given any triangle and any point, construct a straight line through the point which bisects the area of the triangle.

I had a solution but lost it, and cannot now reconstruct it. Perhaps one of your readers can help.

Yours sincerely,

W. H. Carter (Major)
(69 Viceroy Court, Lord Street,
Southport, Lancs. PR8 1PW)

Dear Editor,

I have been told that a solution without calculus, in 'three or four lines', exists for the following.

A metal cube with 2 inch edges is drilled perpendicularly through the midpoint of one of its faces by a cylindrical drill of radius 1 inch. The drilling is then repeated along the other two perpendicular axes of the cube. In this way, a circular cylinder of radius 1 inch is pierced along each of the three axes of the cube. What is the volume remaining?

I should be most interested to know whether any reader of *Mathematical Spectrum* knows of the 'easy solution' I have referred to.

Yours sincerely,

R. D. Kitchen
(Lady Manners School,
Bakewell, Derbyshire DE4 1JA)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

6.1. (Submitted by B. G. Eke, University of Sheffield.) The odd integer n can be expressed as a sum of squares in two different ways, say $n = x^2+y^2 = u^2+v^2$, where $x<u$ and x, u are even. Prove that there are positive integers a, r, b, s such that
(i) $u-x = 2ar, y-v = 2as$,
(ii) r and s are relatively prime,
(iii) $(x+ar)r = (y-as)s = brs$.
Deduce that n cannot be prime.

6.2. In a gathering of people, some shake hands with others. Show that there are two people who shake hands the same number of times.

6.3. Show that, for every positive integer n, there is a finite set of points in the plane with the property that every point of the set is distant one unit from exactly n points of the set.

6.4. (Submitted by R. J. Webster, University of Sheffield.) Show that every positive integer has a multiple which has decimal form $99\ldots900\ldots0$.

Solutions to Problems in Volume 5, Number 2

5.5. Show that there exist irrational numbers a, b such that a^b is rational.

Solution by M. Ram Murty (Carleton University, Ottawa) and V. Kumar Murty (Woodruffe High School, Ottawa)

Let p be a prime number. Then $p^{1/p}$ is irrational. Put $x = p^{1/p}$. If x^x is rational, then we put $(a, b) = (x, x)$. If x^x is irrational, then we put $(a, b) = (x^x, p/x)$, since $(x^x)^{p/x} = p$. This gives an infinite number of solutions. (In particular, either $(\sqrt{2}, \sqrt{2})$ or $((\sqrt{2})^{\sqrt{2}}, \sqrt{2})$ will do for the pair (a, b).)

M. Ram Murty and V. Kumar Murty go on to ask two further questions: I. Do there exist irrational numbers a, b such that a^b is irrational? II. Do there exist numbers a, b such that a is rational, b is irrational and a^b is irrational? To answer I, if x^x is irrational, we put $(a, b) = (x, x)$. If x^x is rational, we put $(a, b) = (x, x+1)$, since $x^{x+1} = (x^x)\,x$, which will be irrational. To answer II, if p^x is irrational, we put $(a, b) = (p, x)$. If p^x is rational, we put $(a, b) = (p, x+(1/p))$, since $(p)^{x+(1/p)} = xp^x$, which will be irrational.

5.6. Let p, q be real numbers with

$$\frac{1}{p} - \frac{1}{q} = 1 \quad \text{and} \quad 0 \leqslant p \leqslant \tfrac{1}{2}.$$

Show that

$$p + \tfrac{1}{2}p^2 + \tfrac{1}{3}p^3 + \ldots = q - \tfrac{1}{2}q^2 + \tfrac{1}{3}q^3 - \ldots.$$

Solution by D. J. Blow (Kingston Grammar School, Kingston-upon-Thames)

From

$$\frac{1}{p} - \frac{1}{q} = 1,$$

we obtain

$$\frac{1}{1-p} = 1+q,$$

and $0 \leqslant p \leqslant \frac{1}{2}$, $0 \leqslant q \leqslant 1$. Now

$$-\log(1-p) = \log(1+q),$$

and we can expand these as power series to give

$$p + \tfrac{1}{2}p^2 + \tfrac{1}{3}p^3 + \ldots = q - \tfrac{1}{2}q^2 + \tfrac{1}{3}q^3 - \ldots.$$

Also solved by J. D. Barrow (University of Durham), Katherine Barton (Brighton and Hove High School), P. Cheung (The Grammar School, Ebbw Vale), Susan Street (Barton Peveril Grammar School), John Hill (John Leggott Sixth Form College, Scunthorpe), M. Ram Murty and V. Kumar Murty.

5.7. A toy boat floats in a bath with a brick as cargo. The brick is taken out of the boat and placed at the bottom of the bath. Does the water level in the bath rise or fall, and by how much?

Solution by Richard Amas (The Grammar School, Ebbw Vale)

Let the mass of the brick be M, the mass of the boat be m and the density of the brick be d. In the case when the brick is in the boat, the weight of water displaced will be $(M+m)\,g$, so the volume of water displaced will be $M+m$. In the case when the brick is at the bottom of the bath, the volume of water displaced by the boat will be m, but the volume of water displaced by the brick will be M/d. Hence the difference in the volume of water displaced in the two cases is

$$M-\frac{M}{d}.$$

If we assume that the bath has uniform cross-sectional area A, this means that the water level will fall by

$$\frac{M}{A}\left(1-\frac{1}{d}\right) \text{ units.}$$

Also solved by J. D. Barrow (University of Durham).

5.8. Let f be a polynomial with complex coefficients and leading coefficient 1. Show that there exists a complex number z with modulus 1 such that

$$|f(z)| \geqslant 1.$$

Solution

Put

$$f = x^n + a_1 x^{n-1} + \ldots + a_n.$$

We may suppose that $n \geqslant 1$. Then, for a complex number z,

$$|f(z)| = |z^n|\,|1+w|,$$

where

$$w = a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}.$$

Thus, when $|z| = 1$,

$$|f(z)| = |1+w|.$$

It is sufficient to show that there exists z such that $|z| = 1$ and $-(\frac{1}{2}\pi) \leqslant \arg w \leqslant (\frac{1}{2}\pi)$. Put

$$u = \exp\left(\frac{2\pi i}{n+1}\right),$$

so that $|u| = 1$. Also put

$$\begin{aligned} w_1 &= a_1 u^{-1} + a_2 u^{-2} + \ldots + a_n u^{-n}, \\ w_2 &= a_1 u^{-2} + a_2 u^{-4} + \ldots + a_n u^{-2n}, \\ &\cdots \\ w_{n+1} &= a_1 u^{-(n+1)} + a_2 u^{-2(n+1)} + \ldots + a_n u^{-n(n+1)}. \end{aligned}$$

Then

$$\begin{aligned} w_1 + w_2 + \ldots + w_{n+1} &= a_1 u^{-1} \frac{1-u^{-(n+1)}}{1-u^{-1}} + a_2 u^{-2} \frac{1-u^{-2(n+1)}}{1-u^{-2}} \\ &\quad + \ldots + a_n u^{-n} \frac{1-u^{-n(n+1)}}{1-u^{-n}} \\ &= 0. \end{aligned}$$

Hence, for some r, $-(\frac{1}{2}\pi) \leqslant \arg w_r \leqslant (\frac{1}{2}\pi)$ and the complex number u^r has the required properties.

Book Reviews

Principles of Objective Testing in Mathematics. W. G. FRASER and J. M. GILLAM. Heinemann Educational Books Ltd, London, 1972. Pp. 150. £1·80.

As far as I am aware this book, one of a series for the different subjects of the School Curriculum, breaks new ground. Because of this it merits a rather more extended review than usual. In fact it ranges over a wider area than is at first apparent from the title because it attempts to set up a framework of objectives within which the more specific aim of the book can be accomplished.

The word 'objective' is a tricky one here since it is being used in more than one sense. Firstly, what is an objective test? Most of us would, perhaps, after glancing at the book, decide that it is more or less synonymous with a multiple choice question. To take an example at random from the book:

If x and y are two prime numbers and each is greater than 7, which of the following is true?

A $x \times y$ is a prime number
B $x - y$ is a prime number
C $x - y$ is a whole number
D $x + y$ is an odd number
E $x \times y$ is an odd number

Such a question is objective in that there is no subjectivity involved in deciding whether an answer is right or wrong. Strictly, however, a question such as 'What is the product of 7 and 8?' is objective too, and there are occasional questions of this type in the book —they are suitable for testing on small groups.

The other meaning of objective is the more difficult one. Our objectives here are our answers to the questions 'Why do we teach . . .?' Most teachers in my experience deliberately avoid answering questions of this type, but they are important, and they need to be answered. Moreover, the teacher who is making some attempt to answer them is one who is striving to improve his teaching.

The plan of the book, then, is the logical result of asking this type of question, so that in defining what we mean by testing and objective, we can then proceed to the detail of test construction. Any intending (or practising) teacher could do worse than study the first four chapters to find out some of the theory behind mathematics teaching; at least he will learn some useful jargon. The first chapter is 'The Purpose of Examinations' to which the answers given are the stock ones:

(a) Measurement of Achievement
(b) Selection
(c) Prognosis
(d) Diagnosis
(e) Motivation
(f) Instrument of Teaching (i.e. forces teacher to think of objectives)

The second chapter continues the first, but with special reference to mathematics. This will be fairly well known to most and needs little comment here.

Chapter 3 deals with Educational Objectives and is partly a summary of Bloom's 'Taxonomy of Educational Objectives', particularly Handbook 1 on The Cognitive Domain (Taxonomy = Classification; Cognitive Domain = The Area of Thought). To summarise further would be pointless. In Chapter 4, however, the authors attempt to carry on where Bloom left off, by providing specific objectives in the teaching of mathematics. This summary seems to me a most useful one because it relates each detailed objective to specific examples of questions designed to test that objective, thus illustrating both the objective in question and the strength of objective testing in pin-pointing that objective.

The taxonomy of mathematical objectives in the cognitive domain is worth repeating:

A (i) Knowledge and Information
- (a) of terminology
- (b) of specific facts
- (c) of conventions, classifications and categories dealing with (b)
- (d) of principles and generalizations

(ii) Techniques and Skills

B Comprehension
- (a) Translation
- (b) Interpretation
- (c) Extrapolation

C Application

D Higher Abilities
- (a) to analyse
- (b) to structure (synthesise)
- (c) to make value judgements of information
- (d) to solve by generalization, evaluation, proof, induction, inference

Each category is illustrated by a number of multiple choice questions and some discussion.

There is not space to give the taxonomies for the affective and psychomotor domains, and as these are less relevant to the main theme of the book they are only treated briefly.

The second section of the book deals with types of questions and with the construction of questions. One's response to these chapters is that objective testing is too difficult to be carried out by a single teacher, working with a single class. While this is true, it is no reason to reject what is said about testing in the book, and it can still be used to improve our own examination techniques.

Each item consists of a stem, which poses a problem, and responses (usually 4 or 5) which list the possible answers, the correct one being the key and the remainder being distractors. The distractors are as important as the key, and one way to obtain suitable distractors is to give the question to a class and note the wrong responses. Such items would be gathered over an extended period by a team, since up to 100 items might be needed in a test.

When items have been gathered the test must be compiled. This is perhaps the most difficult part of all, because the decisions to be made are subjective. The syllabus must be split into areas with weights. An initial split at traditional 'O' level might be to divide 100 items thus:

Arithmetic	30
Algebra	30
Geometry	30
Trigonometry	10

Items have also to be split between the abilities A–D above, maybe

A	Knowledge and Information	25
B	Comprehension	30
C	Application	30
D	Higher Abilities	15

A matrix is then compiled to combine the two, and questions are allotted as described in the two-way matrix. Clearly, many more items will be needed than are actually used. Unused items can go into a 'bank' for future use.

The remainder of the book deals with the analysis of the test and its items which must be done before it can be used in an examination situation. Most readers will require some background reading before they are happy about these chapters and they do not need comment here. The closing chapter draws the threads together and lists some of the pros and cons of objective testing under the general title of Assessment. My own summary of the points here is:

In favour
- (i) wide syllabus coverage
- (ii) rapidity of marking
- (iii) objectivity of marking
- (iv) balanced syllabus coverage/reduced subjectivity of question choice
- (v) can be used to test higher abilities (C and D)
- (vi) pre-testing increases reliability
- (vii) useful in diagnosis of teaching deficiencies
- (viii) little exam time spent in writing

Limitations
- (i) difficulties of construction
- (ii) random guessing of solutions
- (iii) can encourage coaching
- (iv) difficult to test
  - (a) creative ability
  - (b) fluency of expression
  - (c) ability to select and organise ideas
  - (d) ability to synthesise

To cover these faults the authors suggest that assessment in mathematics should be based on

(a) an external traditional examination
(b) an external objective examination
(c) teacher's assessment

A book of this sort cannot please everyone all the time. In the early chapters there is a tendency to make unsubstantiated statements of the type '. . . it is unlikely that boys are different in ability from girls' and 'Objective testing . . . has a less harmful influence on the educational system than traditional types . . .'.

Most of the many items given as examples are excellent, but two annoyed me; firstly

'When $10+4\times 3-10\div 2$ is simplified the answer is . . .'.

Surely BODMAS is forgotten in schools today, and would anyone really write down an expression like that? Secondly the graph on page 63 is wrong; the maximum value of the cubic function is 4/27, not approximately 1. Either the units on the axes should be omitted and the question rephrased, or the graph should be drawn correctly.

Nevertheless this will be a useful book. Obviously examiners and teacher training establishments will find it useful, but teachers too can learn a lot from it. How should it be used? I would suggest that one of the most fruitful uses for it would be in a teachers' centre. If teachers could meet once a week for a term, or a year, and construct some objective tests for themselves, they would learn much about their subject and in time a set of objective tests could be built up which would ease the strain of examination time considerably. If they could get as far as publishing their results, even in duplicated form, they would be benefiting mathematics teaching and teachers everywhere. This is, therefore, a book to initiate work rather than to read and put aside. The authors are to be congratulated on their book, and have filled a gap in the literature, even if one's overall impression is of the difficulties rather than the advantages. I hope teachers will use this work widely and fruitfully in the next few years, and will experiment with this type of examination in their schools; all too often school examinations are thrown together in a hurry without any real plan; and here is an excellent corrective.

University of Nottingham K. E. SELKIRK

The School Mathematics Project—The First Ten Years. By BRYAN THWAITES. Cambridge University Press, 1972. Pp. 266. £3·00, hardback; £1·50 paperback.

'What is Modern Mathematics?' asked a puzzled parent during a recent parent–teacher confrontation on television. Of course he did not receive an answer, because the term Modern Mathematics has so many different meanings. To the university mathematician it probably means Mathematics which has been published during the last hundred years; to a secondary school teacher, reluctantly re-learning his subject, it means 'Sets, Matrices and Vectors and all that sort of stuff', while the primary school teacher may use the same term to mean teaching Mathematics by 'discovery' methods. To the child it can mean anything from having fun and games with sand and water to yet another dreary lesson on number bases. It is not surprising that parents are a little confused by it all.

The most common use of the term may well be as a description of the syllabus reform which has taken place in secondary school Mathematics in the last ten years; in this the most influential force has been the School Mathematics Project. This book provides a clear record of the project, its philosophy and its publications. The phenomenal spread of the project is well illustrated by the increase in the number of candidates for Ordinary Level SMP Mathematics, printed on page 213; this is a growth function *par excellence*! Even more startling perhaps is the statement on page 224 by Tim Wheatley of Cambridge University Press that 'around half the schools in England are using SMP books'.

The annual reports which are collected together in this book make fascinating reading, as they trace not only the development of the project, but the changing pattern of English Secondary School education during the decade. Thus in 1961 the reports are chiefly interested in 'the evolution of a syllabus for the whole grammar school range'. Later the emphasis changes to the sixth forms, but by 1967 interest centres on the A to H CSE course (arguably the most successful venture to date).

One important misprint caught my eye: my colleague Ian Warburton is said to teach at a non-existent 'Nottingham Grammar School'.

The value of this book is somewhat limited by the fact that it contains very little new material. As a record of this important project, however, its value will increase as the years go by; this alone should justify its place in the library of every staff-room and college of education.

Nottingham High School PETER HORRIL

Advanced Pure Mathematics. A Revision Course. By R. G. MEADOWS and R. DELBOURGO. Penguin Books Ltd, Harmondsworth, Middlesex, 1971. Pp. 128. £2·00.

This is a revision book covering for the most part traditional aspects of Pure Mathematics which are examined at G.C.E. A-level by the various boards, and as such its reader must be able to distinguish between the requirements of different boards and projects.

Each chapter is made up of the following sections:

(i) statements of various theoretical results and formulae in a particular topic;
(ii) worked examples using the foregoing theory;
(iii) a number of set questions, followed by
(iv) hints and solutions to (iii).

The questions used have been selected from recent A-level papers of different examination boards in England and Wales. Its nine chapters are: Algebra; Trigonometry, Complex Numbers and Hyperbolic Functions; Further Topics in Algebra; Co-ordinate Geometry; Further Geometry; Vector Analysis; Differentiation and Integration; Further Calculus Topics; Differential Equations.

The accuracy of the content of this book leaves much to be desired and the text has suffered from inadequate proof reading. There are, apart from a liberal sprinkling of errors (upwards of fifty!) particularly in the first four chapters, many instances of careless statements. Examples are:

(i) page 10—proof by induction 'for integral n' rather than 'positive integral n';
(ii) page 12—no restrictions stated on the Binomial expansion;
(iii) page 35—language used, 'the first expression has the same roots as';
(iv) page 37—a real classic, '$\cos \theta/2 < \pm\frac{1}{2}$';
(v) page 72—the shear matrices are described as translations whilst the stretch matrices are described as shears;
(vi) page 119, no. 3(c)—the integration should yield

$$\int \frac{dv}{f(v)-v} = \log_e(x+c);$$

(vii) page 89—lists $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec}^2 x$.

Inevitably some answers quoted are wrong and question H5, page 34, provides a worked example solved by a distinctly poor method and indicating a lack of appreciation of the bookwork at the beginning of the question.

There are examples of inconsistencies, including the use of 'identity' and 'equals' signs. Also the natural logarithm is shown sometimes with the base e and sometimes without it.

There are omissions in the theory presented including:

(i) the question of sign on the area of a triangle; the usual definition of radical axis; major, minor, conjugate and transverse axes;
(ii) change of variable with corresponding change of limits in a definite integral;
(iii) for a polynomial equation with real coefficients any complex roots occur in conjugate pairs.

Even with all corrections made we doubt whether an 'average student' working on his own would gain any further insight into the subject from using this text, or be more adequately prepared for an A-level examination in mathematics.

We consider that the size of type used is far too small for easy reading of the text and that the general layout is congested. Probably an attempt has been made to cram too much material into a book of 128 pages in order to produce what is hopefully a 'competitive' volume. In relation to what the book can offer to a student, we feel that at £2·00 it is over-priced. Certainly we could find no use for this book in our sixth forms.

Springfield Secondary School
Jarrow, Co. Durham

G. ADIE
S. MARR

Statistics and Probability. By S. E. HODGE and M. L. SEED. Blackie and Chambers, London, 1972. Pp. 264. £2·00.

This introductory text is intended primarily for sixth formers and covers most of the A-level syllabuses. However, the authors feel that it will also provide a useful grounding for university and college students who require some basic statistics.

Topics are introduced in a fairly conventional order. The opening three chapters contain the basic ideas of probability including a mention of Markov chains, and the following three chapters treat descriptive statistics. The notion of probability distribution is then introduced, and the standard distributions and their interrelationships discussed. Chapter 11 covers sampling theory and the Central Limit Theorem, followed by hypothesis testing for large and small samples and goodness-of-fit tests. Non-parametric tests are also mentioned here and the book concludes with a treatment of correlation and regression.

On the whole, I enjoyed this book. It is written in a very readable style (although the English could perhaps be improved in places) and contains many helpful diagrams and interesting examples. New ideas are introduced through such examples before the relevant theory is discussed. Revision exercises are strategically placed throughout the book; a large number of the problems have been taken from past A-level papers. In addition the book contains plenty of ideas for gathering experimental data, and I would fully endorse the view of the authors that it is more interesting to handle data that have been personally collected. Of special interest were one or two worked examples where statistical analysis arrives at a conclusion in opposition to that suggested by simple inspection of the data.

However, there were one or two points that I found unsatisfactory. Non-standard notation is used in parts which could result in confusion, notably: $\mathscr{P}$ and $\mathscr{E}$ for the Poisson and exponential density functions respectively; x denoting both a random variable and the values it can assume; and 'estimate' used to mean estimator and estimate (although in the latter two cases, the authors point out their deviation from the more usual notation). Flow charts for calculating mean and variance are given without any explanation, thus seeming meaningless and inappropriate. Of a more serious nature, diagrams of the chi-squared and t distributions are incorrectly drawn as neither meets the x-axis as depicted in this book. Finally, most calculations and derivations are written out in full when they could perhaps have been left to the reader to work through.

In general, this is a book which will appeal to the average sixth former, and which he could use largely without supervision. It may therefore be deserving of a place on the library shelf.

12 Barnfield Avenue, Sheffield S10 5TA C. M. NIXON

Problem Solvers Series. Edited by L. MARDER. George Allen & Unwin Ltd, London. £1·90, hardback; £0·80, paperback.

4. Analytical Mechanics. By D. F. LAWDEN. 1972. Pp. 78.

7. Vector Fields. By L. MARDER. 1972. Pp. 81.

These two further books in the series continue the idea of the earlier numbers in providing a source of worked examples to supplement an undergraduate course of lectures or a standard textbook. Both do this well and should be found useful by first or second year honours mathematics students and also by those reading engineering or physics.

Analytical Mechanics deals with systems of particles and rigid body motions. There are examples on moments and products of inertia including principal axes, plane motions for particles and bodies, Euler's equations and tops, Lagrange's equations, small oscillations and normal coordinates. Hamilton's equations are mentioned briefly.

The reader will find it useful that a lot of theory is explained in the text; several topics often included in a lecture course are done as examples here.

Vector Fields is a sequel to *Vector Algebra* (No. 3 in this series). It contains solved problems on scalar and vector fields; line, surface and volume integrals; vector formulae and identities; the theorems of Gauss, Stokes and Green; curvilinear coordinates and Cartesian tensors.

The book would have been more complete had an example or two on double integration and the use of Jacobians been included before plunging into surface integration on page 17. The final chapter on Cartesian tensors *in vacuo* as presented appears out of place. One feels the subject should either be done using Cartesian tensors throughout (which is elegant but uncommon), or alternatively omitted altogether.

Both the paperback editions provide very good value and should have a wide appeal to students. It is a pity the editor of the series did not devise different systems for the numbering of problems, sections and equations; e.g., problem (2.3) bears no relation to section 2.3 and does not use equation (2.3)!

University of Durham D. H. WILSON

Problem Solvers Series. Edited by L. MARDER. George Allen & Unwin Ltd, London. £1·90, hardback; £0·80, paperback.

8. Matrices and Vector Spaces. By F. BRICKELL. 1972. Pp. 88.

This book is the eighth in a series designed to provide an inexpensive source of fully solved problems in a wide range of mathematical topics. The volume is intended for students of science and engineering at universities and polytechnics, and for first year students in mathematics.

The chapters deal with algebra of matrices including determinants, elementary matrices and rank, linear transformations, and real quadratic forms. A final section deals with vector spaces.

The book consists essentially of worked examples, but includes definitions and some background theory. It is expected to be used in conjunction with lecture notes or existing texts. Certainly, it would be difficult to read without any background material, particularly as knowledge of a number of related topics is required in some of the examples.

A wide range of problems is solved in which standard results are complemented by numerical applications. Some answers are given more fully than others, but in all cases essential steps are given. There is a fair amount of cross-reference between problems, and at a glance it is not always easy to distinguish between 'text' and solutions. Each chapter concludes with a small number of exercises.

My overall impression is that this book will prove of value to the student; it will also serve partly as further reading for the able pupil studying some of these topics in school.

Nottingham High School I. C. WARBURTON

Problems in Applied Mathematics. By G. R. H. BOYS. Edward Arnold (Publishers) Ltd, London, 1972. Pp. 165. £1·00.

People always have strong views about this kind of collection of problems. If you are one of those who like them, then you will probably be attracted to this book which consists of problems on Applied Calculus, Vectors and the usual Mechanics in Part I, and Probability and Statistics in Part II.

The problems themselves are taken largely from modern A-level examinations, in particular those set on the MEI and the SMP syllabuses, and have the property of being imaginative, stimulating and demanding. It is possible that they might make a refreshing change from the rather dry exercises found in most of the standard school texts but it is unlikely that they would be appropriate for first year undergraduates studying mathematics.

University of Durham HUGH NEILL

Games Playing with Computers. By A. G. BELL. George Allen & Unwin Ltd, London, 1972. Pp. 204. £5·25.

The greater part of this book deals with card and board games, the inevitable section on Nim appearing early. Other standard topics dealt with are Noughts and Crosses, the

Knight's Tour and Chess, to which some 25 pages of the book are devoted. The chapter on Heuristics is extremely well written and useful examples are given. One failing noticed was that a number of the diagrams and tables were insufficiently described in the text. Numerous practical examples are given, such as Chess, Draughts and Kalah. Also there are some less common applications of computers to games which include Snakes and Ladders and 'Barricellian symbio-organisms', a branch of cellular automata.

Science Sixth Form — STEVE JOWETT
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Notes on Contributors

J. Gani is Professor of Statistics and Head of the Department of Probability and Statistics in the University of Sheffield. He has taught in universities in this country as well as in Australia and the U.S.A. He is very interested in mathematical education and has managed *Mathematical Spectrum* since 1968. His other main interest is the application of probability to problems in biology.

David Blow attended Kingston Grammar School, where he gained an Open Scholarship to Corpus Christi College, Cambridge. After these examinations the Senior Mathematics Master encouraged him to write this article whose subject was prompted by the author's interest in bridge and similar games, including Monopoly. His other interests include classical music and reading.

T. P. Speed is Lecturer in Probability and Statistics at the University of Sheffield. He was born and educated in Australia and came to Sheffield after a period as Teaching Fellow and Lecturer at Monash University, Melbourne. His main research interests are algebra and probability theory; he is also very active in mathematical education.

Ramesh Kapadia is working on a research project on mathematical education at Nottingham University. He is one of the Editors of *Manifold*, a mathematics magazine published at the University of Warwick, where he did his first degree in pure mathematics.

Richard Rado retired two years ago from the Chair of Mathematics in the University of Reading, but one cannot imagine him retiring from the active pursuit of mathematics and mathematical research. He is a familiar figure at conferences and seminars the world over, and he continues to pour out a stream of papers on an exceptionally wide range of topics. Combinatorial mathematics in all its forms lies at the heart of his interests, although he is equally at home in algebra, analysis, geometry, and the theory of numbers. In 1972, Richard Rado was awarded the Senior Berwick Prize by the London Mathematical Society.

J. D. Murray is Fellow and Tutor in Mathematics at Corpus Christi College, Oxford, and Reader in Mathematics in the University. He spent many years in America at the Universities of Harvard and Michigan and latterly at New York University, where he was Professor of Mathematics. His research interests are primarily in differential equations, fluid mechanics, and mathematical modelling in the bio-medical sciences.

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