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CONTENTS

A FURTHER NOTE ON AREAS OF TRIANGLES INSCRIBED IN A GIVEN TRIANGLE

HOWARD EVES

In a recent issue of this journal [1979: 191-192], Dan Pedoe commented on various solutions of the old problem: *Two triangles whose vertices lie on the sides of a given triangle at equal distances from the midpoints of these sides are equal in area*. It is to be understood that the equal distances on a side of the triangle need not have the same lengths as the equal distances on any other side of the triangle. Two such inscribed triangles may be called a pair of *isotomic triangles* of the given triangle.

It does not seem to be generally known that this problem possesses a very simple solution that scarcely requires the use of pencil and paper.

Any triangle can be projected orthogonally into an equilateral triangle; do so for the given triangle. Since, under an orthogonal projection, equal distances on any given line project into equal distances on the image line, it follows that the pair of isotomic triangles of the given triangle project into a pair of isotomic triangles of the equilateral triangle. Therefore, since equality of areas is invariant under an orthogonal projection, it suffices to show that the areas of any pair of isotomic triangles of an equilateral triangle are equal. Now this is very easily accomplished by showing, with the aid of the familiar formula for the area of a triangle in terms of two sides and the included angle, that the sum of the areas of the three triangles cut off from the equilateral triangle by one of the isotomic triangles is equal to the sum of the areas of the three triangles cut off from the equilateral triangle by the other isotomic triangle.

Another triumph (though a minor one) for the transformation method!

Professor Emeritus, University of Maine.
Box 251, RFD 2, Lubec, ME 04652.

* * *

CROOKS WHAT?

For the benefit of recent subscribers, we reprint the following from an earlier issue [1978: 90]:

Crux mathematicorum is an idiomatic Latin phrase meaning: a puzzle or problem for mathematicians. The phrase appears in the Foreign Words and Phrases Supplement of ***The New Century Dictionary***, D. Appleton - Century Co., 1946, Vol. 2, p. 2438. It also appears in *Websters New International Dictionary*, Second Unabridged Edition, G. & C. Merriam Co., 1959, Vol. 1, p. 637.

THE EDITOR

PASCAL REDIVIVUS: II

DAN PEDOE

In *Pascal Redivivus: I* [3], I showed how to find the vertex and the focus of a parabola, given 5 points on the parabola, using only Euclidean constructions. I now show how to find the salient points of an ellipse, given 5 points on the curve, using only Euclidean constructions, and do the same for an hyperbola.

The efficacy of the Pascal Theorem has been demonstrated in my first note, and we can now assume that it is possible, using only a straightedge, to find the further intersection, with the unique conic which passes through 5 given points, of any line through one of the given points. Thus, suppose A, B, C, D, E are the 5 given points: if AB is not parallel to CD, another point F on the conic can be found such that AB is parallel to CF; and if BC is not parallel to DE, then we can find a point G on the conic such that BC is parallel to DG.

It is known (1) that the join of the mid-points of parallel chords of a conic lie on a diameter, a line through the centre of the conic; and (2) that the directions given by this diameter and the parallel chords produce a pair of *conjugate diameters*, each of which bisects chords of the conic parallel to the other (Figure 1).

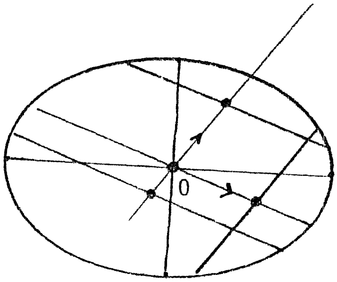

Figure 1

The first theorem enables us to find the centre of the ellipse through 5 given points by Euclidean construction. We assume that this has been done and call the centre O. To utilize the second theorem, and to show how to find the principal axes of the ellipse, we must turn aside for the moment and talk about *involutions* (see [2], Chapter IX).

The projective theory of conics, developed during the last century, shows that there is an intimate and remarkable connection between the theory of projective correspondences on a line and on a conic. On a line, if we use the parameter t, the map (correspondence) called a *projective* correspondence,

$$t' = At,$$

where A is a nonsingular matrix, gives us a one-to-one algebraic correspondence between points on the line, or between points on two distinct lines. If we have 4 points t_i $(i=1,\ldots,4)$ on the line, then the 4 image points $t_i' = At_i$ have the same cross ratio as the t_i, so that

$$(t_1t_2,\ t_3t_4) = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)} = (t_1't_2',\ t_3't_4').$$

This is a distinguishing feature of a projective correspondence. We can also discuss projective correspondences between pencils of lines, a line in one pencil being given by a parameter u, and the corresponding line in the same or another pencil being given by the parameter u', where there is a relation

$$u = u'B,$$

B being a nonsingular matrix. Again the cross ratios of 4 lines in one pencil and that of the 4 corresponding lines in the other pencil are equal. If we intersect 4 lines of a pencil with a line which does not pass through the vertex of the pencil, then the 4 points of intersection have the same cross ratio as the 4 corresponding lines.

Now let u_i and u_i' refer to corresponding lines in a projective relation between pencils of lines with distinct vertices U and U', and let P_i be the point of intersection of these lines (Figure 2). Then the point P_i moves on a conic which passes through the points U and U'. The points U and U' are not distinguished points on the conic, and if V and V' are any two distinct points on the conic, then the lines VP_i and $V'P_i$ are in a projective correspondence. Using the accepted symbol $\barwedge$ for a projective correspondence, we can write

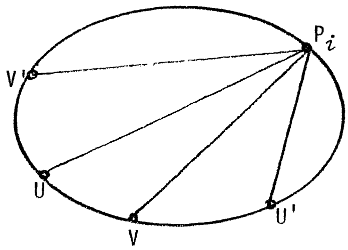

Figure 2

$$V(P_i) \barwedge V'(P_i),$$

where V and V' are any two points on the conic.

We can now talk of a projective correspondence $(P) \barwedge (P')$ *on the conic*. This means that if V and V' are any two points on the conic, then

$$V(P) \barwedge V'(P').$$

A proof of the Pascal Theorem follows very readily from these concepts, as do proofs of many other theorems. Another beautiful aspect of the conic is that it is a self-dual structure, and can be built up from its tangents as well as from its points, with a corresponding projective theory. All this can be found in [2].

In nonhomogeneous coordinates, a projective relation can be written

$$ptt' + qt + rt' + s = 0 \qquad (ps - qr \neq 0).$$

If $t \to t'$ and $t' \to t$ for any one pair of values of (t,t'), then $q = r$, and the same relation holds for any pair of corresponding values of (t,t'). The projective relation is then called an *involution*.

When this occurs on a conic, the joins of corresponding pairs t,t' pass through a uniquely determined point (Figure 3), which we call *the centre of the involution*.

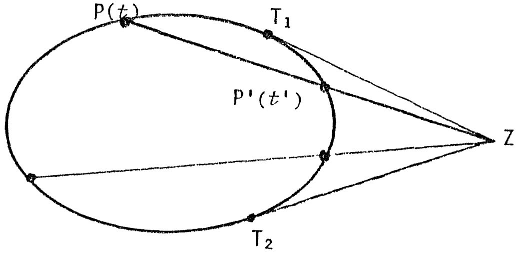

Figure 3

The fixed points of a projective correspondence are obtained by putting $t = t' = T$ in the equation above, and are given by

$$pT^2 + (q+r)T + s = 0.$$

In the case of an involution $q = r$, and these fixed points are distinct and form a harmonic range with any pair (t,t') of the involution. If we look at Figure 3, the fixed points of the involution cut on the conic by lines through the centre Z are the points of contact of tangents through Z to the conic. The whole theory of pole and polar can be derived from the relation $(T_1T_2, PP') = -1$.

After this brief diversion, we return to our problem. We have the centre O of the ellipse through the 5 given points. We also have two pairs of conjugate diameters. The projective theory shows that conjugate diameters are pairs in involution, the fixed diameters (which exist for the hyperbola but not for the ellipse) being the asymptotes of the conic.

If we draw a circle Σ to pass through O, the other intersections of pairs of conjugate diameters through O with the circle through O will give us pairs of points *on the circle* which are in involution. Their joins therefore pass through a uniquely determined point C_1, which we determine from two pairs of conjugate diameters of the ellipse (Figure 4). Any line through C_1 meeting the circle in X and X' gives us a pair of conjugate diameters OX, OX' of the ellipse.

The principal axes of the ellipse are the unique pair of conjugate diameters which are at right angles. Pairs of lines at right angles through O are in involution, and cut out pairs of points on the circle Σ whose join passes through the centre O_1 of the circle. Hence, joining the centre O_1 of the circle Σ to the point C_1 gives us points K and H on the circle such that OK and OH are both conjugate diameters of the ellipse, and at right angles, and these lines are the principal axes (Figure 4).

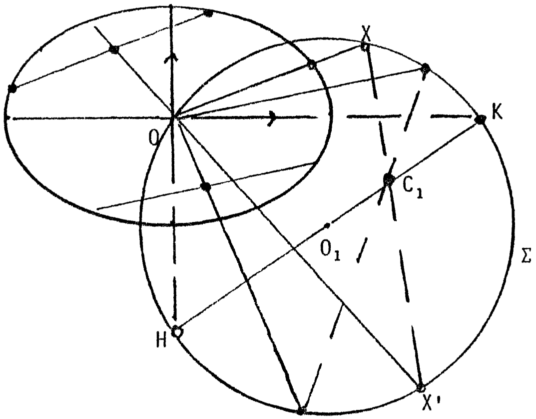

Figure 4

All this can be done more quickly than the explanation may suggest. We now have to distinguish between the major and the minor axes of the ellipse, and we can do this by finding the *equiconjugate diameters*. These are a pair of diameters which are conjugate, and also equal in length, making equal angles with the principal axes of the ellipse.

Equal diameters of the ellipse make equal angles with the principal axes, and are in involution. If we trace this involution on the circle Σ, we note that the principal axes are the fixed members of this involution, and therefore the centre of the involution on the circle Σ cut out by equal diameters of the ellipse is the intersection of the tangents to the circle at the points H and K (Figure 5). This point is at infinity in a direction perpendicular to the diameter HK of the circle. Hence, to obtain the equiconjugate diameters of the ellipse, we draw

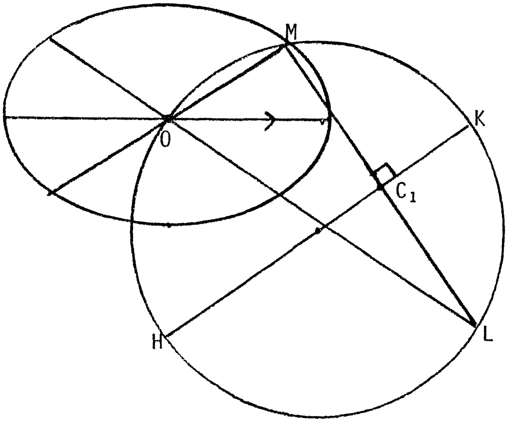

Figure 5

through C_1, the centre for conjugate diameters, a line perpendicular to the diameter KH of the circle Σ. If this cuts the circle at L and M, then OL and OM are the equiconjugate diameters of the ellipse (Figure 5).

We can now distinguish between the major and minor axes of the ellipse. If the lengths of these axes are $2a$ and $2b$, the equiconjugate diameters make an angle whose tangent is b/a with the major axis. Choose as that principal axis the one of the two which makes an angle less than 45° with the equiconjugate diameters. This is the major axis of the ellipse.

To obtain the vertices of the ellipse and the foci, we need the auxiliary circle. This is the circle on the major axis of the ellipse as diameter (Figure 6). If A is any point on the ellipse, and AN is perpendicular to the major axis, meeting the auxiliary circle in A', then AN/A'N = b/a. (The ellipse is, in fact, an orthogonal projection of the circle, and many properties of the ellipse are deduced immediately from this fact.) From A, a point on the ellipse, drop the perpendicular AN onto the major axis (Figure 7). Draw AW parallel to the axis to meet an equiconjugate diameter in W. Let WZ be perpendicular to the major axis, and mark off NA' = OZ on NA. Then

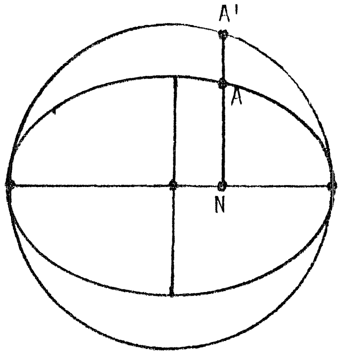

Figure 6

$$AN/A'N = AN/OZ = WZ/OZ = b/a,$$

so that A' is on the auxiliary circle, and the circle with centre O and radius OA' is the auxiliary circle (Figure 7). This cuts the major axis in the vertices of the ellipse on the major axis.

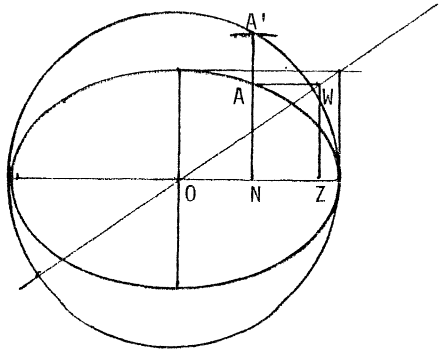

Figure 7

The equiconjugate diameters cut the tangents at the extremities of the major axis in points whose joins parallel to the major axis touch the ellipse at the extremities of the minor axis (Figure 8). Since the feet of perpendiculars from a focus onto a tangent to the ellipse lie on the auxiliary circle, we find the foci of the ellipse by

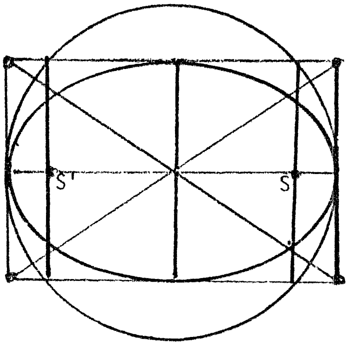

Figure 8

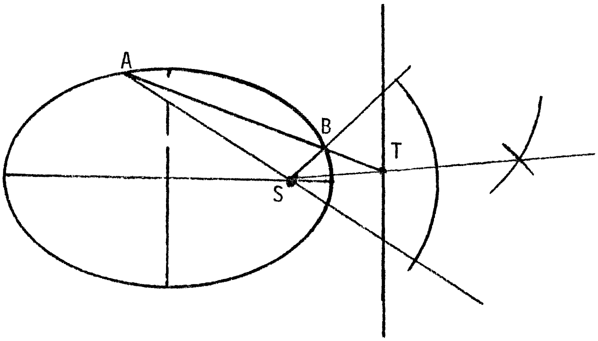

Figure 9

finding where the tangents at the extremities of the minor axis cut the auxiliary circle, and dropping perpendiculars onto the major axis.

Finally, the directrices are simply found by angle bisection, using the theorem [1, p. 101] that if S is a focus, and A and B are points on the ellipse, then if AB meets the directrix corresponding to S in T, then ST bisects the angle ASB externally (Figure 9). Bisect the angle ASB externally, and find T. Then the perpendicular from T onto the major axis is the directrix corresponding to the focus S.

The problem for a hyperbola, given 5 points on the curve, proceeds at first as for the ellipse. We determine the centre of the curve and the centre for conjugate diameters, but now the point C_1 in Figure 4 lies *outside* the circle Σ; and since the asymptotes are the fixed diameters of the involution generated by conjugate diameters, we obtain the asymptotes of the hyperbola by joining the centre O of the hyperbola to the points of contact of the tangents from C_1 to Σ. The principal axes of the hyperbola are OH and OK, where the line O_1C_1 cuts Σ in H and K, and the major axis of the hyperbola is that one of the two principal axes which makes an angle less than 45° with the asymptotes.

If we draw the tangent at a point A on the hyperbola to intersect the major axis at T, and N is the foot of the perpendicular from A onto the major axis, the theory of harmonic ranges tells us that $OT \cdot ON = a^2$, where a is the length of the semi-major axis of the hyperbola. We can easily find the length a by Euclidean construction, and the circle with centre O and radius a is the auxiliary circle, which intersects the major axis in the vertices of the hyperbola. A similar construction can be applied to the ellipse, instead of the one given.

As we have a tangent to the hyperbola at A, find where this intersects the auxiliary circle, and perpendiculars to the tangent at these points intersect the major axis at the foci of the hyperbola. The construction for the directrix corresponding to a focus proceeds as for the ellipse, but the points A and B must be on the same branch of the hyperbola. If they are on opposite branches, ST must bisect the angle ASB *internally*. Since we have the asymptotes, there are no foundational difficulties in deciding whether points are on the same branch or not.

This note and the preceding one involve a number of Euclidean constructions. A problem which has intrigued me for some time is the following:

Given 5 *points, which determine a unique conic through the points, what is the simplest geometric construction which will determine the nature of the conic: ellipse, parabola, or hyperbola?*

Very few writers deal with this problem. Möbius, whom I quote in [4], makes the following pre-Cantorian statement:

Given 5 *points chosen arbitrarily in a plane, the chance that the unique conic which can be drawn through these points is an hyperbola rather than an ellipse is* $\sqrt{\infty}$: 1.

Perhaps this can be verified from the constructions given above?

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4. ________, "Notes on the History of Geometrical Ideas, I. Homogeneous Coordinates," *Mathematics Magazine*, 48 (September 1975) 215-217.

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* * *

MATHEMATICAL SWIFTIES

"So you have been wasting all this time looking at the light reflected in your tea cup," Tom shouted caustically.

"The error is still too large," Tom sighed perceptibly.

"The Banach-Tarski theorem will not solve the packaging problem," Tom declared paradoxically.

"That shell will never buckle," Tom uttered inflexibly.

M.S.K.

* * *

POSTSCRIPT

Comment on "Maximum and Minimum of the Sum of Reciprocals of the Sides of a Triangle," by V.N. Murty and M. Perisastry [1979: 214-216]:

For elementary triangle inequalities, one should not ordinarily use calculus techniques. However, if they are used, both necessary and sufficient conditions should be established. The mistake made here was to consider only the necessary conditions. Checking the endpoint extrema would show that there are no nontrivial bounds.

That the lower bound

$$\sqrt{5}\sqrt[4]{5+2\sqrt{5}} \le 2S\sqrt{\Delta}$$

is obviously incorrect follows by considering the degenerate triangle $a=b=1$, $c=2$, which has zero area. That the upper bound is also incorrect follows by considering the limit in the case of an isosceles triangle whose base approaches zero. Here we can take

$$a=b=1, \qquad c=2\sin\theta, \qquad \Delta=\sin\theta\cos\theta;$$

and then

$$\Delta S^2 = \sin\theta\cos\theta\left(\frac{1}{1}+\frac{1}{1}+\frac{1}{2\sin\theta}\right)^2,$$

which approaches ∞ as $\theta \to 0$.

M.S. KLAMKIN,
University of Alberta.

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THE OLYMPIAD CORNER: 10

MURRAY S. KLAMKIN

The problems in this month's new Practice Set were taken from some past Hungarian Mathematical Olympiads and were translated into English by Frank J. Papp (Associate Editor, *Mathematical Reviews*).

PRACTICE SET 8

8-1. At midnight a truck starts from city A and goes to city B; at 2:40 a.m. a car starts along the same route from city B to city A. They pass at 4:00 a.m. The car arrives at its destination 40 minutes later than the truck. Having completed their business, they start for home and pass each other on the road at 2:00 p.m. Finally, they both arrive home at the same time. At what time did they arrive home?

8-2. Find all fourth-degree polynomials (with complex coefficients) with the property that the polynomial and its square each consist of exactly five terms.

8-3. Let n be a given natural number. Find nonnegative integers k and l so that their sum differs from n by a natural number and so that the following expression is as large as possible:

$$\frac{k}{k+l} + \frac{n-k}{n-(k+l)} .$$

*

SOLUTIONS TO PRACTICE SET 7

7-1. (a) Determine $F(x)$ if, for all real x and y,

$$F(x)F(y) - F(xy) = x + y.$$

(b) Generalize.

Solution.

(a) If F is any solution of the functional equation then, for $y = 0$ and any real x, we must have

$$F(0)\big(F(x) - 1\big) = x. \tag{1}$$

Now $F(0) \neq 0$ since (1) holds when $x \neq 0$; and $F(0) = 1$ since (1) holds also when $x = 0$. Hence, from (1) again,

$$F(x) = x + 1. \tag{2}$$

Conversely, (2) satisfies the given functional equation since

$$(x+1)(y+1) - (xy+1) = x + y.$$

(b) One extension would be to determine $F(x)$ if, for all real x, y, z,

$$F(x)F(y)F(z) - F(xyz) = yz + zx + xy + x + y + z.$$

The solution is similar. Any solution F of this functional equation must satisfy, for $y = z = 0$ and any real x,

$$F(0)\big(F(x)F(0) - 1\big) = x. \tag{3}$$

Here again, $F(0) \neq 0$ since (3) holds when $x \neq 0$; and $F(0) = \pm 1$ since (3) holds when

$x = 0$. Thus, from (3),

$$F(x) = x + 1 \qquad \text{or} \qquad F(x) = x - 1.$$

But only the first of these, $F(x) = x + 1$, satisfies the functional equation.

Another functional equation whose unique solution is given by $F(x) = x + 1$ is the following:

$$F(x)F(y)F(z) - F(y)F(z) - F(z)F(x) - F(x)F(y) - F(xyz) = -x - y - z - 3,$$

for all real x,y,z. (First set $x = y = z = 0$ and find $F(0) = 1$, -1, or 3; then eliminate the last two possibilities.)

7-2. $A_1A_2A_3A_4$ denotes a kite (i.e., $A_1A_2 = A_1A_4$ and $A_3A_2 = A_3A_4$) inscribed in a circle. Show that the incenters I_1, I_2, I_3, I_4 of the respective triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$ are the vertices of a square. (Jan van de Craats, The Netherlands.)

Solution.

Since $A_1A_2 = A_1A_4$, it follows that A_1A_3 is a diameter of the circle and an axis of symmetry of the kite. The incenters I_1 and I_3 lie on the axis of symmetry, and I_2 and I_4 are on either side of it. Hence $I_1I_2 = I_1I_4$ by symmetry, and it suffices to show that $I_1I_2I_3I_4$ is a rectangle. But this, by a happy coincidence, follows from a problem proposed in this journal last month (Crux 483(b) [1979: 265]), for which readers of this column are invited to send solutions to the editor.

7-3. Show that the polynomial equation with real coefficients

$$P(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_{n-3}x^3 + x^2 + x + 1 = 0$$

cannot have all real roots.

Solution.

All roots are nonzero. Their reciprocals, which we denote r_i, $i = 1,\ldots,n$, are roots of the equation

$$x^n + x^{n-1} + x^{n-2} + a_{n-3}x^{n-3} + \ldots + a_0 = 0.$$

Thus

$$\sum_i r_i = -1 \qquad \text{and} \qquad \sum_{i<j} r_ir_j = 1,$$

so

$$\sum_i r_i^2 = \left(\sum_i r_i\right)^2 - 2\sum_{i<j} r_ir_j = -1,$$

which implies that the roots cannot all be real.

Readers are invited to prove that the same result holds, more generally, if $P(x)$ is a polynomial with real coefficients consisting of at least three nonzero terms, and such that the coefficients (zero or not) of three consecutive powers of x are in geometric progression.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

491. *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

(Dédié au souvenir de Victor Thébault, jadis inspecteur d'assurances à Le Mans, France.)

Résoudre la cryptarithmie décimale suivante:

$$\text{UN} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} = \text{ONZE}.$$

492. *Proposed by Dan Pedoe, University of Minnesota.*

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}$AB are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b)* Is the construction possible when $r < \frac{1}{2}$AB?

493. *Proposed by R.C. Lyness, Southwold, Suffolk, England.*

(a) A, B, C are the angles of a triangle. Prove that there are positive x,y,z, each less than $\frac{1}{2}$, simultaneously satisfying

$$y^2 \cot\frac{B}{2} + 2yz + z^2 \cot\frac{C}{2} = \sin A,$$

$$z^2 \cot\frac{C}{2} + 2zx + x^2 \cot\frac{A}{2} = \sin B,$$

$$x^2 \cot\frac{A}{2} + 2xy + y^2 \cot\frac{B}{2} = \sin C.$$

(b)* In fact, $\frac{1}{2}$ may be replaced by a smaller $k > 0.4$. What is the least value of k?

494. *Proposed by Rufus Isaacs, Baltimore, Maryland.*

Let r_j, $j = 1,\ldots,k$, be the roots of a polynomial with integral coefficients and leading coefficient 1.

(a) For p a prime, show that

$$p \mid \sum_j (r_j^p - r_j).$$

(Note that the sum is an integer, since it is a symmetric polynomial of the roots, and hence a polynomial of the coefficients.)

This generalizes Fermat's Little Theorem.

(b)* Prove or disprove the corresponding extension of Gauss's generalization of Fermat's Theorem: for any positive integer n,

$$n \mid \sum_j \left(\sum_{d \mid n} r_j^d \mu(n/d) \right),$$

where μ is the Möbius function.

495. *Proposed by J.L. Brenner, Palo Alto, California; and Carl Hurd, Pennsylvania State University, Altoona Campus.*

Let S be the set of lattice points (points having integral coordinates) contained in a bounded convex set in the plane. Denote by N the minimum of two measurements of S: the greatest number of points of S on any line of slope 1, -1. Two lattice points are *adjoining* if they are exactly one unit apart. Let the n points of S be numbered by the integers from 1 to n in such a way that the largest difference of the assigned integers of adjoining points is minimal. This minimal largest difference we call the *discrepancy* of S.

(a) Show that the discrepancy of S is no greater than $N+1$.

(b) Give such a set S whose discrepancy is $N+1$.

(c)* Show that the discrepancy of S is no less than N.

496. *Proposed by E.J. Barbeau, University of Toronto.*

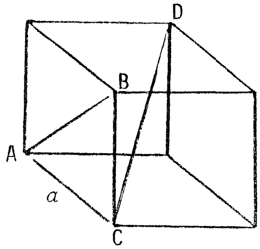
Solve the Diophantine equation

$$(x+1)^k - x^k - (x-1)^k = (y+1)^k - y^k - (y-1)^k$$

for $k = 2,3,4$ and $x \neq y$.

497*. *Proposed by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

Given is a cube of edge length a with diagonal CD, face diagonal AB, and edge CB, as shown in the figure. Points P and Q start at the same time from A and C, respectively, move at constant rates along AB and CD, respectively, and reach B and D, respectively, at the same time. Find the area of the surface swept out by segment PQ.



498. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let $a_i(t)$, $i = 1,2,3$, be given functions whose Wronskian, $w(t)$, never vanishes. Let

$$u(t) = \sqrt{\Sigma a_i^2} \quad \text{and} \quad v(t) = (\Sigma a_i^2)(\Sigma a_i'^2) - (\Sigma a_i a_i')^2.$$

Prove that the general solution of the system

$$x_1'/a_1 = x_2'/a_2 = x_3'/a_3$$

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

can be expressed in terms of

$$\int \frac{uw}{v}dt,$$

no other quadratures being required.

499. *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

A certain polyhedron has all its edges of unit length. An ant travels along the edges and, at each vertex it reaches, chooses at random a new edge along which to travel (each edge at a vertex being equally likely to be chosen). The expected (mean) length of a return trip from one vertex back to it is 6 for some vertices and 7.5 for the other vertices.

Calculate the volume of the polyhedron.

500. *Proposed by H.S.M. Coxeter, University of Toronto.*

Let 1, 2, 3, 4 be four mutually tangent spheres with six distinct points of contact 12, 13, ..., 34. Let 0 and 5 be the two spheres that touch all the first four. Prove that the five "consecutive" points of contact 01, 12, 23, 34, 45 all lie on a sphere (or possibly a plane).

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

400. [1978: 284; 1979: 242] *Proposed by Andrejs Dunkels, University of Luleå, Sweden.*

In the false bottom of a chest which had belonged to the notorious pirate Capt. Kidd was found a piece of parchment with instructions for finding a treasure buried on a certain island. The essence of the directions was as follows.

"Start from the gallows and walk to the white rock, counting your paces. At the rock turn left through a right angle and walk the same number of paces. Mark the spot with your knife. Return to the gallows. Count your paces to the black rock, turn right through a right angle and walk the same distance. The treasure is midway between you and the knife."

However, when the searchers got to the island they found the rocks but no trace of the gallows remained. After some thinking they managed to find the treasure anyway. How?

(This problem must be very old. I heard about it in my first term of studies at Uppsala.)

Editor's comment.

Leroy F. Meyers, The Ohio State University, was the first to point out that solution II [1979: 243] is not clearly stated. This was the result of an editing error. The solution should have read as follows:

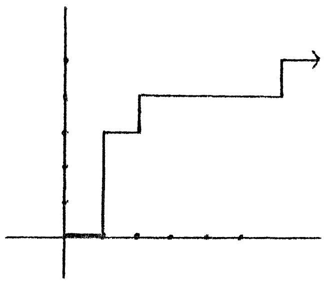
More generally, suppose the turns are, first, left through an angle θ then, later, right through an angle $\pi - \theta$, and [in the notation of solution I], WK = mGW and BY = mGB. Then

$$z_1 = 1 + me^{i\theta}(1 - z), \qquad z_2 = -1 + me^{-i(\pi - \theta)}(-1 - z) = -1 + me^{i\theta}(1 + z),$$

and the affix of T is $\frac{1}{2}(z_1 + z_2) = me^{i\theta}$, so the treasure can be found without the aid of the gallows.

408. [1979: 16, 277] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A *zigzag* is an infinite connected path in a Cartesian plane formed by starting at the origin and moving successively one unit right or up (see figure). Prove or disprove that for every zigzag and for every positive integer k, there exist (at least) k collinear lattice points on the zigzag.



(This problem was given to me by a classmate at City College of New York in 1971-72. Its origin is unknown to me.)

Comment by M.S. Klamkin, University of Alberta.

I had given this problem as a 5-"fribble" (a super milk shake) problem at the June 1979 Olympiad Practice Session at the U.S. Military Academy in West Point, N.Y. Subsequently, Michael Larsen (Lexington, Mass.), a high school student and a top scorer (40 and 39) on two International Mathematical Olympiads (40 is tops) sent me a valid solution. No one else including myself had succeeded with it. Here is a somewhat modified version of his proof:

Let (x_n,y_n) be the nth point in the zigzag. There are two cases:

i) There exists rational r such that $\{y_n - rx_n\}_{n=1}^{\infty}$ changes sign infinitely often. Since $|y_{n+1} - y_n| \le 1$ and $|x_{n+1} - x_n| \le 1$, it follows that infinitely often $|y_n - rx_n| \le 1 + r$. If $r = p/q$, then $|qy_n - px_n| \le p + q$ for infinitely many n. Hence, there are infinitely many n such that $qy_n - px_n = c$ for some c. These points (x_n,y_n) are collinear.

ii) For every rational r, either $y_n - rx_n$ is eventually nonnegative or eventually nonpositive. It is easy to see that $y_n - x_n/k$ is eventually nonnegative and $y_n - kx_n$ is eventually nonpositive, for otherwise we would have k collinear points parallel to the x- (resp. y-) axis. Thus there exists $\lambda > 0$ such that $y_n - rx_n$ is eventually nonnegative if $r < \lambda$ and $y_n - rx_n$ is eventually nonpositive if $r > \lambda$. Let

$$\frac{p}{q} < \lambda < \frac{p}{q} + \frac{1}{4kq}\ .$$

Then there exists N such that, for $n \ge N$,

$$y_n - (p/q)x_n \ \ge\ 0 \ \ge\ y_n - (p/q)x_n - x_n/4kq,$$

that is, $x_n/4k \ge qy_n - px_n \ge 0$ for $n > N$. If now $N \le n < 2N$, then $x_n \le 2N$, so the N integers

$$\{qy_n - px_n\}_{n=N}^{2N-1} \quad \text{are among the integers} \quad \left\{0,1,\ldots,\left[\frac{N}{2k}\right]\right\}.$$

By the pigeonhole principle, there exists an integer c such that $qy_n - px_n = c$ for at least k values of n. These k points are collinear. □

A. Meir (University of Alberta) has proposed the following interesting related problem:

For $k=1,2,\ldots$, let $F(k)$ denote the largest integer such that there exists a zigzag of length $F(k)$ (the number of lattice points in the zigzag) without k collinear points.

(a) Is $F(k)$ finite for all k?

(b) Determine $F(k)$ or at least find good bounds for $F(k)$. For example, $F(3)=4$.

Finally, one could also consider 3-dimensional zigzag problems analogous to the initial one and to the one of Meir.

Editor's comment.

Nobody else solved this problem. Michael Larsen, who is now 17, was only 15 when he obtained a perfect score on an International Olympiad. He too could say:

> Je suis jeune, il est vrai; mais aux âmes bien nées
> La valeur n'attend point le nombre des années.

*　　　　*　　　　*

410. [1979: 17] *Proposed by James Gary Propp, student, Harvard College, Cambridge, Massachusetts.*

Are there only finitely many powers of 2 that have no zeros in their decimal expansions?

Comment by Charles W. Trigg, San Diego, California.

Rudolph Ondrejka, in discussing "Non-zero Factors of 10^n," *Recreational Mathematics Magazine*, No. 6, December 1961, page 59, said "the largest value of 2^n with non-zero digits is 2^{86} (and this has been checked out to 2^{5000}), ..."

In a letter to me on May 7, 1976, Mr. Ondrejka reported that he had extended the range "to the 33219th power of 2; every power had at least one or more zeros."

The computer sheets containing the powers of 2 examined by Ondrejka would have made a pile 9 feet high. They have been bound into 75 volumes and are now in the possession of Long Island University. In discussing Ondrejka's investigation in *Mathematics of Computation*, 23 (April 1969), page 456, John W. Wrench remarked: "the total number of digits in all 75 volumes is 166 115 268. ... The selection of 33219 as the total number of entries in this immense tabulation was based on the author's plan to include all powers of 2 whose individual lengths do not exceed 10000 decimal digits."

Excerpt from *Mathematics on Vacation*, by Joseph S. Madachy, Scribner's, 1966, pages 127-128:

"Fred Gruenberger, a mathematician at the RAND Corporation in Santa Monica, California, programmed on IBM 1620 electronic computer to produce consecutive powers of 2 and type the exponent for those that lacked zeros. ... It went to 2^{10535} before the computer had to be cut off. 2^{10535} is a number with 3172 digits. These digits pass the frequency test of randomness, so there are about 317 each of 0's and 5's in the number. If a number containing n digits passes the frequency test of randomness it means that each of the 10 digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, appears approximately 10 per cent of the time in the given number. Roughly half of the 0's and 5's will produce a new 0 on the next power, and the empirical results indicate that the probability of a power of 2 higher than 86 having no 0's is quite low. The matter awaits an analytic proof, but the above analysis tends to discourage further empirical searching."

"For those who are interested in further empirical results, T. Charles Jones, a student at Davidson College in Davidson, North Carolina, reports that up to 2^{57134} (which has 17200 digits) the results are equally negative."

The 36 known values of 2^n that contain no zeros correspond to $n = 0$ to 9, 13 to 16, 18, 19, 24, 25, 27, 28, 31 to 37, 39, 49, 51, 67, 72, 76, 77, 81, and 86. The three largest values are:

$$2^{77} = 151\,115\,727\,451\,828\,646\,838\,272$$

$$2^{81} = 2\,417\,851\,639\,229\,258\,349\,412\,352$$

$$2^{86} = 77\,371\,252\,455\,336\,267\,181\,195\,264$$

Note that six of the triads in 2^{77}, which is also devoid of 9, are palindromic.

It is highly unlikely that any larger power of 2 is devoid of zeros.

Indeed, using a print-out of 2^n through $n = 376$, I found that the highest powers of 2 that do not contain the digits

0,	1,	2,	3,	4,	5,	6,	7,	8,	9

are

86,	91,	168,	153,	107,	71,	93,	71,	78,	108

respectively. In a letter of July 26, 1976, Ondrejka reported that these values held through $n = 1000$. He also observed that "2^{68} is the first power of 2 with all 10 digits represented. 2^{170} is the first power of 2 that has at least two of each digit."

To this we add that the smallest powers of 2 containing multiple sets of every digit are:

2^{177} with at least three of each digit,

2^{200} with at least four of each digit,

2^{209} with at least five of each digit,

2^{246} with at least six of each digit,

2^{291} with at least seven of each digit,

2^{318} with at least eight of each digit.

These numbers contain 54, 61, 63, 75, 88, and 96 digits, respectively.

$2^{51} = 2\,251\,799\,813\,685\,248$ is the smallest power to contain all nonzero digits.

$2^{59} = 576\,460\,752\,303\,423\,488$, which contains no 1 or 9, is the smallest power to contain at least two of every digit present.

The proposer also submitted a comment.

Editor's comment.

It is no accident that this problem was proposed by a student. The concept of the power of a prime is a simple one learned early in life, and it has always held endless fascination for the mathematically inclined among the young. Hans Albrecht Bethe, winner of the 1967 Nobel Prize in Physics, is quoted in *The New Yorker* of December 3, 1979 as saying: "When I was seven, I learned about powers, and filled a whole book with the powers of two and three."

Few will doubt that, barring misprints, the abundant information given in the above comment is all true. But much of it is unverifiable, except by those who can afford to spend a sabbatical year or two holed up at Long Island University.

Here is more true, though also unverifiable, information about powers of 2:

1. Powers of 2 have the following incestuous relationship: for each fixed $n \geq 0$, there are infinitely many powers of 2 whose first $[n \log 2] + 1$ digits are precisely those of 2^n.

2. There are infinitely many powers of 2 whose first million digits consist of the first million digits of π.

3. Though it is likely that every power of 2 above 2^{86} contains zeros, there are powers of 2 in which the first zero occurs as far down the line as we please. There are, for example, infinitely many powers of 2 whose first 9 million digits consist of the sequence 123456789 repeated a million times.

To see why these astounding facts are true, and to see why this list of mind-boggling properties could be stretched out at will, see Crux 50 [1975: 78-81][1].

[1]The bound volume for the years 1975-1976, now in its second printing, is still available. For more information, see the front page of this issue.

Finally, here is some *verifiable* information about a high power of 2. The number

$$2^{44497} = 854\ 509 \ldots 011\ 228\ 672$$

contains exactly 1332 zeros among its 13395 digits, and the locations of the first and last zeros are shown above. Thus 9.944% of the digits are zeros, about what one would expect from the frequency test of randomness. We have said that this information can be verified. To do so, write to Creative Publications, Inc., P. O. Box 10328, Palo Alto, California 94303, and order Catalogue Item No. 62041. This is a colourful 24×36 inch poster containing, in large type, all 13395 digits of the largest known prime, the 27th Mersenne prime, $2^{44497} - 1$, roped on April 8, 1979 by Harry Nelson and David Slowinski riding tandem on a CRAY-1 computer. When you receive the poster, add 1 to the last digit and start counting the zeros. And after? Stick the poster on the wall above your desk. It makes a wonderful conversation piece.

* * *

411. [1979: 46] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

"But you can't make arithmetic out of passion. Passion has no square root." (Steven Shagan, *City of Angels*, G.P. Putnam's Sons, New York, 1975, p. 16.)

On the contrary, show that in the decimal system

$$\sqrt{\text{PASSION}} = \text{KISS}$$

has a unique solution.

I. *Voyeuristic solution and clinical report by Clayton W. Dodge, University of Maine at Orono* [with *riposte* by Edith Orr].

In view of the number of digits in PASSION [that's it, keep your hands right there where I can see them], we must have $1000 < \text{KISS} < 3162$. Since $S \neq N$, we must also have $S \neq 0, 1, 5, 6$. Now all we have to do is to square each KISS with $K = 1, 2, 3$ and $S = 2, 3, 4, 7, 8, 9$. [Yes, we could program a calculator to do it, but what's your hurry?] As each KISS is squared [you'll have to pucker your lips, dear], look for the double S in PASSION. A near solution is $2877^2 = 8277129$. [Let's try again, shall we?] Finally [ahhh, all PASSION spent at last], we are left with the unique solution

$$\sqrt{4133089} = 2033.$$

Clinical report. Observe that 2033 is not prime but is the product of the two primes 19 and 107, representing the union of two individuals, each with its own uniqueness. Alas, their PASSION is neither perfect nor abundant, but sadly deficient. Send them to Masters and Johnson.

II. *Solution by Allan Wm. Johnson Jr., Washington, D.C.* [with a cameo appearance by the irrepressible Edith].

As we will see below, there is a unique solution in the decimal system, and in this solution $\mathrm{I}=0$ and $\mathrm{A}=1$. But there are infinitely many other bases B in which there is at least one solution, even with $\mathrm{I}=0$ and $\mathrm{A}=1$. This is shown by the identity

$$\{xB^3+(2x-1)B+(2x-1)\}^2 = x^2B^6+B^5+(2x-1)B^4+(2x-1)B^3+(B-2)B+(B-1),$$

where $B=(2x-1)^2+1$ and $x>1$. For $x=2$, we obtain the decimal solution

$$2033^2 = 4133089.$$

To show that it is unique, we can proceed as follows ... [but it's a lot more fun to do it as in solution I].

Also solved by HAYO AHLBURG, Benidorm, Spain; CECILE M. COHEN, John F. Kennedy H.S., New York, N.Y.; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, Ohio State University; HERMAN NYON, Paramaribo, Surinam; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Another first for *Crux Mathematicorum*: an X-rated solution! If you don't know Edith Orr, get acquainted with her by looking up [1977: 39, 129, 224, 225; 1978: 8-10]. Bound volumes for these years are still available.

* * *

412. [1979: 47] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

The sides BC, CA, AB of ΔABC are produced respectively to D, E, F so that CD = AE = BF. Show that ΔABC is equilateral if ΔDEF is equilateral.

(This problem was brought to my notice several years ago. I was told it was from a magazine. It is, of course, a converse of the trivial problem that ΔDEF is equilateral if ΔABC is equilateral.)

I. *Solution by Howard Eves, University of Maine.*

It suffices to prove that if ΔABC is not equilateral, then ΔDEF is not equilateral. Suppose, then, that ΔABC is not equilateral and so labeled that either $a\le b<c$ or $a<b\le c$. Denote the angles of ΔABC by α, β, γ and their supplements

by α', β', γ' (see Figure 1). Then $\alpha' \geq \beta'$ since $\alpha \leq \beta$. Since $a < c$, we have $BD < AF$ and there is a point G on AF such that $AG = BD$. In Δ's AEG and BFD, we have

$$AE = BF, \quad AG = BD, \quad \alpha' \geq \beta';$$

hence $EF > EG \geq FD$, and ΔDEF is not equilateral.

(One is reminded of the famous Steiner-Lehmus problem, which also is an evasive converse of a trivial problem.)

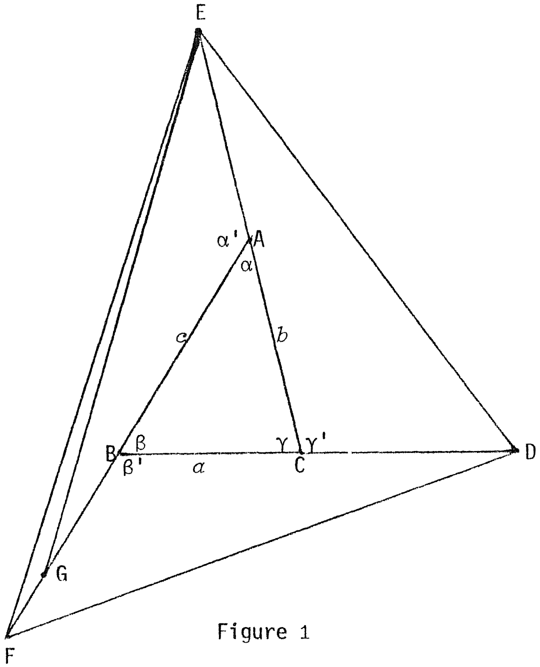


Figure 1

II. *Solution by A. Liu, University of Regina.*

Let the angles α, β, γ be as indicated in Figure 2 and assume, without loss of generality, that $a \geq b \geq c$. Then $BD \geq CE \geq AF$ and $\alpha \geq \beta \geq \gamma$. In particular, we have

$$\beta + \gamma \leq 2\alpha, \qquad (1)$$

with equality if and only if $\alpha = \beta = \gamma$. Now

$$\gamma + 60^\circ - \alpha = \angle BAC \geq \angle ABC = \alpha + 60^\circ - \beta;$$

hence $\beta + \gamma \geq 2\alpha$, and $\alpha = \beta = \gamma$ follows from (1). Thus $BD = CE = AF$ and $a = b = c$.

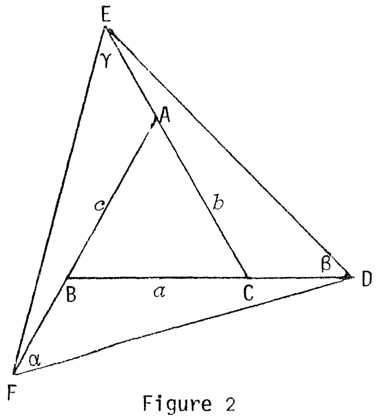


Figure 2

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain (two solutions); G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; DAN PEDOE, University of Minnesota; FREDERICK NEIL ROTHSTEIN, New Jersey Department of Transportation, Trenton, N.J.; and DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio.

Editor's comment

Sokolowsky proved the slightly stronger result: If ΔABC is not equilateral, then ΔDEF is not (even) isosceles.

Several solvers showed that the same result holds if the points D, E, F lie on the *segments* BC, CA, AB (that is, when DEF is inscribed in ABC). But, as Klamkin pointed out, this case has already been very extensively discussed in [1], with extensions to polygons and even to the hyperbolic plane. Many of the proofs given there are easily modified to fit our own problem.

REFERENCE

1. Problem 754, "Nested Equilateral Triangles," *Mathematics Magazine*, 43 (1970) 280-282; 44 (1971) 45-53, 173-178, 296.

* * *

413. [1979: 47] *Proposed by G.C. Giri, Research Scholar, Indian Institute of Technology, Kharagpur, India.*

If $a,b,c > 0$, prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

I. *Solution by Jeremy D. Primer, student, Columbia H.S., Maplewood, N.J.*

We apply three times the inequality

$$x+y+z \ge \sqrt{yz}+\sqrt{zx}+\sqrt{xy},$$

which follows from $(\sqrt{y}-\sqrt{z})^2+(\sqrt{z}-\sqrt{x})^2+(\sqrt{x}-\sqrt{y})^2 \ge 0$ and holds for all $x,y,z \ge 0$.

We obtain

$$\begin{aligned}\frac{a^8+b^8+c^8}{a^3b^3c^3} &= \frac{a^5}{b^3c^3}+\frac{b^5}{c^3a^3}+\frac{c^5}{a^3b^3}\\ &\ge \frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\\ &\ge \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\\ &\ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c},\end{aligned}$$

with equality if and only if $a=b=c$.

II. *Solution by M.S. Klamkin, University of Alberta.*

More generally, suppose we have given the positive numbers a_i, $i=1,\ldots,n$. For $r \ge 1$, the power mean inequality gives

$$(\Sigma a_i^r)/n \geq (\Sigma a_i/n)^r.$$

If σ_k denotes the kth elementary symmetric function of the a_i, then the normalized symmetric function P_k is defined by

$$P_k = \sigma_k/\binom{n}{k}, \qquad k = 1, 2, \ldots, n,$$

and the Maclaurin inequalities then assure us that

$$P_1 \geq P_2^{1/2} \geq P_3^{1/3} \geq \ldots \geq P_n^{1/n}.$$

Thus if $r_2 + r_3 + \ldots + r_n = r$, we have

$$(\Sigma a_i^r)/n \geq (\Sigma a_i/n)^r = P_1^r = \prod_{k=2}^{n} P_1^{r_k} \geq \prod_{k=2}^{n} (P_k^{1/k})^{r_k}.$$

For example, using a,b,c for the a_i when $n = 3$ and $r = 8$, we get

$$\frac{a^8 + b^8 + c^8}{3} \geq \left(\frac{a+b+c}{3}\right)^8 = P_1^8 = P_1^6 \cdot P_1^2 \geq P_3^2 \cdot P_2 = (abc)^2 \cdot \frac{bc + ca + ab}{3},$$

and the proposed inequality follows at once.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; ROLAND H. EDDY, Memorial University of Newfoundland; ROBERT S. JOHNSON, Montréal, Québec; J.A. McCALLUM, Medicine Hat, Alberta; V.N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, The University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DEBASHIS SEN, student, Indian Institute of Technology, Kharagpur, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The inequality in this problem is of the sort that teemed in the older treatises on algebra such as those of *Chrystal* and *Hall and Knight*. And, yup, it's there. Wilke found it in Chrystal [1] and the editor found it in Hall and Knight [2]. Eddy also found it in Mitrinović [3].

One solver also gave a relatively complicated proof of the similar inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{a^7 + b^7 + c^7 + d^7}{a^2b^2c^2d^2}.$$

But see how easily it follows from the method of solution II:

$$\frac{a^7 + b^7 + c^7 + d^7}{4} \geq \left(\frac{a+b+c+d}{4}\right)^7 = P_1^7 = P_1^4 \cdot P_1^3 \geq P_4 \cdot P_3 = abcd \cdot \frac{(bcd + cda + dab + abc)}{4},$$

and the desired inequality follows.

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1. G. Chrystal, *Algebra*, Chelsea, New York, 1964, Part II, p. 51, Ex. 22.

2. H.S. Hall and S.R. Knight, *Higher Algebra*, Macmillan, New York, 1964 printing, p. 511, Ex. 199.

3. D.S. Mitrinović, *Elementary Inequalities*, P. Noordhoff Ltd., Groningen, 1964, p. 121.

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414. [1979: 47] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

A few years ago a distinguished mathematician wrote a book saying that the theorems of Ceva and Menelaus were dual to each other. Another distinguished mathematician reviewing the book wrote that they were not dual. Explain why they were both right or, if you are feeling in a sour mood, why they were both wrong.

Editor's comment.

To set the background for this problem, in his book [1] Clayton W. Dodge writes (p. 17): "... two statements are called *duals* of one another [*sic*] if each is transformed into the other by the interchange of the words 'point' and 'line', and, of course, also such associated terms as 'collinear' and 'concurrent'." Later (p. 24), he writes: "Where Menelaus' theorem discusses the *collinearity* of three *points* on the *lines* (sides) of a triangle, Ceva's theorem considers the *concurrence* of three *lines* on (through) the *points* (vertices) of a triangle. So each is dual to the other."

Reviewing the book in [2], E.J.F. Primrose wrote: "This chapter [Chapter 1] contains the only point on which I would quarrel with the author, namely, his statement that the theorems of Ceva and Menelaus are dual. For one thing, Euclidean geometry itself is not self-dual (strictly, the theorems of Ceva and Menelaus belong to affine geometry, but this also is not self-dual); also, why should

$$(BL/LC)(CM/MA)(AN/NB) = -1 \qquad (1)$$

for Menelaus' theorem be the dual of

$$(BL/LC)(CM/MA)(AN/NB) = 1 \qquad (2)$$

for Ceva's theorem?"

I. *Comment by Clayton W. Dodge, University of Maine at Orono.*

This undistinguished mathematician notes that, curiously enough, the exact same thing that happened to the (unidentified) book in the proposal happened to him regarding his book [1]. The two theorems are duals in the sense that the terms "point" and "line" are interchanged in their statements, but not duals in their equations, so the distinguished reviewer of my book, E.J.F. Primrose, and I are both more or less correct. Of course, my statement of duality should have been qualified. And, naturally, the reviewer's judgment was impeccable when he concluded in [2] with: "I can recommend this as a good textbook for anyone teaching geometry from the standpoint of transformations."

II. *Comment by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

As affine theorems, the theorems of Menelaus and Ceva may or may not be duals, depending upon how one interprets the word "dual". But their counterparts in the projective plane are strictly duals of each other.

Let X, Y, Z be three collinear points on the sides BC, CA, AB, respectively, of a triangle ABC. Then three points L, M, N on these same respective sides are collinear if and only if the product of the three cross ratios

$$(B,C;\ X,L)(C,A;\ Y,M)(A,B;\ Z,N) = 1. \tag{3}$$

This is the projective form of the Menelaus theorem.

For the projective form of Ceva's theorem, let x, y, z be three concurrent lines through the vertices bc, ca, ab, respectively, of a trilateral abc. Then three lines l, m, n through these same respective vertices are concurrent if and only if the product of the three cross ratios

$$(b,c;\ x,l)(c,a;\ y,m)(a,b;\ z,n) = 1. \tag{4}$$

The duality is obvious. In particular, when X, Y, and Z are on the line at infinity, (3) reduces to the affine theorem (1). And if we let

$$bc = \mathrm{A},\ ca = \mathrm{B},\ ab = \mathrm{C};\quad ax = \mathrm{X},\ by = \mathrm{Y},\ cz = \mathrm{Z};\quad al = \mathrm{L},\ bm = \mathrm{M},\ cn = \mathrm{N},$$

then (4) is equivalent to

$$(C,B;\ X,L)(A,C;\ Y,M)(B,A;\ Z,N) = 1,$$

which reduces to the affine theorem (2) when X, Y, and Z are the midpoints of BC, CA, and AB.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; and by the proposer.

REFERENCES

1. Clayton W. Dodge, *Euclidean Geometry and Transformations*, Addison-Wesley, 1972.

2. *Mathematical Gazette*, 57 (January 1973) 139.

Editor's comment.

The equivalence of the projective theorems of Menelaus and Ceva is an automatic consequence of their duality. But it is not so easy to prove directly the equivalence of their affine forms. See Dan Pedoe's article in this journal [1977: 2-4].

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415. [1979: 47] *Proposed by A. Liu, University of Alberta.*

Is there a Euclidean construction of a triangle given two sides and the radius of the incircle?

Solution by Jeremy D. Primer, student, Columbia H.S., Maplewood, N.J.

We will show that given arbitrary sides $b,c > 0$ and inradius $r > 0$, the side a of a solution triangle, if one exists, cannot, in general, be found by Euclidean construction. This shows that the answer to the problem is NO, for otherwise such a construction as that required by the proposal would automatically yield side a for any b,c,r for which a solution triangle exists. We will need the following theorem which can be found in Dickson [1]:

It is impossible to construct with ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root, when the unit of length is given.

Suppose a solution triangle exists for given $b,c,r > 0$. Equating two area formulas, we get

$$rs = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$, and this is easily shown to be equivalent to

$$4r^2(a+b+c) = (b+c-a)(c+a-b)(a+b-c). \tag{1}$$

Conversely, given $b,c,r > 0$, if (1) has a positive root a_0, then the triangle with sides a_0,b,c will have inradius r. For our counterexample, we choose $b = 3$, $c = 2$, $r = \frac{1}{2}$. For these values, (1) is equivalent to

$$a^3 - 5a^2 + 10 = 0, \tag{2}$$

an equation with two positive roots a_i ($1 < a_1 < 2$ and $4 < a_2 < 5$), one negative root, and no rational root (one need only try $a = \pm1, \pm2, \pm5, \pm10$). Thus there exist two triangles with the given data but, by the theorem quoted above, neither can be constructed by ruler and compasses.

Also solved by HAYO AHLBURG, Benidorm, Spain; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; HOWARD EVES, University of Maine; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

Ahlburg and Dou showed, in effect, that if $-a_0$ is the negative root of (2), then a_0 is the third side (also not constructible) of the triangle with $b=3$, $c=2$, and *exradius* $r_a=\frac{1}{2}$.

REFERENCE

1. Leonard Eugene Dickson, *A New First Course in the Theory of Equations*, Wiley, New York, 1939, p. 33.

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416. [1979: 47] *Proposed by W.A. McWorter Jr., The Ohio State University.*

Let A_0BC be a triangle and a a positive number less than 1. Construct P_1 on A_0B so that $A_0P_1/A_0B = a$. Construct A_1 on P_1C so that $P_1A_1/P_1C = a$. Inductively construct P_{n+1} on A_nB so that $A_nP_{n+1}/A_nB = a$ and construct A_{n+1} on $P_{n+1}C$ so that $P_{n+1}A_{n+1}/P_{n+1}C = a$. Show that all the P_i are on a line and all the A_i are on a line, the two lines being parallel.

Solution by G.C. Giri, Research Scholar, Indian Institute of Technology, Kharagpur, India (revised by the editor).

More generally, suppose that, for arbitrary real a,b and for $n=0,1,2,\ldots$, we have

$$A_nP_{n+1} = a\cdot A_nB, \qquad P_{n+1}A_{n+1} = b\cdot P_{n+1}C. \tag{1}$$

(The proposed problem has $0<a=b<1$.) We interpret all points as position vectors emanating from some origin. With $\lambda = 1-a$ and $\mu = 1-b$, we get from (1)

$$P_{n+1} = aB + \lambda A_n, \qquad A_{n+1} = bC + \mu P_{n+1}, \qquad n=0,1,2,\ldots \tag{2}$$

Now (2) implies

$$P_{n+1} - P_n = \lambda(A_n - A_{n-1}), \qquad A_{n+1} - A_n = \mu(P_{n+1} - P_n), \qquad n=1,2,3,\ldots \tag{3}$$

Hence

$$P_{n+1} - P_n = \lambda\mu(P_n - P_{n-1}), \qquad n=2,3,4,\ldots \tag{4}$$

and

$$A_{n+1} - A_n = \lambda\mu(A_n - A_{n-1}), \qquad n=1,2,3,\ldots \tag{5}$$

It follows from (5) that the A_i are all on a line, from (4) that the P_i are all on a line, and from (3) that these two lines are parallel. (More precisely, to cover the cases where a or b is 0 or 1, there exist two parallel or coinciding lines, one of which contains all the A_i and the other all the P_i.)

It now follows by induction from (4) and (5) that

$$P_n - P_{n-1} = (\lambda\mu)^{n-2}(P_2 - P_1), \qquad n = 2,3,4,\ldots$$

and

$$A_n - A_{n-1} = (\lambda\mu)^{n-1}(A_1 - A_0), \qquad n = 1,2,3,\ldots$$

Hence

$$P_n - P_1 = \sum_{k=2}^{n}(P_k - P_{k-1}) = (P_2 - P_1)\sum_{k=2}^{n}(\lambda\mu)^{k-2}, \qquad n = 2,3,4,\ldots$$

and

$$A_n - A_0 = \sum_{k=1}^{n}(A_k - A_{k-1}) = (A_1 - A_0)\sum_{k=1}^{n}(\lambda\mu)^{k-1}, \qquad n = 1,2,3,\ldots$$

from which the positions of P_n and A_n can be determined directly. In particular, if

$$|\lambda\mu| = |(1-a)(1-b)| < 1,$$

the sequences $\{P_i\}$ and $\{A_i\}$ converge respectively to the points P and A defined by

$$P = P_1 + (P_2 - P_1)\cdot\frac{1}{1-\lambda\mu} = \frac{P_2 - \lambda\mu P_1}{1-\lambda\mu}$$

and

$$A = A_0 + (A_1 - A_0)\cdot\frac{1}{1-\lambda\mu} = \frac{A_1 - \lambda\mu A_0}{1-\lambda\mu}.$$

It now follows from (2) that

$$P = \frac{aB + (b-ab)C}{a+b-ab} \qquad \text{and} \qquad A = \frac{(a-ab)B + bC}{a+b-ab}.$$

Hence P and A lie on BC and divide it in the ratios

$$BP : PC = b(1-a) : a \qquad \text{and} \qquad BA : AC = b : a(1-b).$$

In particular, in the proposed problem where $0 < a=b < 1$, P and A are *isotomic points* (equidistant from the midpoint) of BC. Indeed, this remains true even if $0 < a=b < 2$, because then $|\lambda\mu| = (1-a)^2 < 1$.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

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417. [1979: 47] *Proposed by John A. Tierney, U.S. Naval Academy, Annapolis, Maryland.*

It is easy to guess from the graph of the folium of Descartes,

$$x^3 + y^3 - 3axy = 0, \quad a > 0$$

that the point of maximum curvature is $(3a/2, 3a/2)$. Prove it.

Partial solution by the proposer, completed by the editor.

We use the parametric form of the Cartesian equation in the proposal:

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}, \quad t \neq -1.$$

We first find

$$\frac{dx}{dt} = \frac{3a(1-2t^3)}{(1+t^3)^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{3at(2-t^3)}{(1+t^3)^2},$$

from which we get

$$\frac{dy}{dx} = \frac{t(2-t^3)}{1-2t^3} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2(1+t^3)^4}{3a(1-2t^3)^3}.$$

Now what we are looking for is the maximum value of $|\kappa|$, where

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} = \frac{2\varepsilon}{3a}\cdot R, \tag{1}$$

where $\varepsilon(=\pm 1)$ is the sign of $1-2t^3$ and

$$R = \frac{(1+t^3)^4}{\{(1-2t^3)^2 + t^2(2-t^3)^2\}^{3/2}}. \tag{2}$$

For future use, we observe at this point that κ is indeterminate when $1-2t^3=0$; but, with the help of (2), we find that $|\kappa| = 2/a$ when $t = 1/\sqrt[3]{2}$. Note that κ has a finite discontinuity at this point, because of the change of sign. Also evident from (1) and (2) are the following:

$$\lim_{t\to -1}\kappa(t) = 0, \qquad \lim_{t\to -\infty}\kappa(t) = 2/3a, \qquad \lim_{t\to +\infty}\kappa(t) = -2/3a.$$

This takes care of all the endpoint extrema.

We will now find the relative extrema of κ. We will first find those of R, and those of κ will then follow from (1). Since $R > 0$ for all $t \neq -1$, we can use logarithmic differentiation to obtain

$$\operatorname{sgn}\frac{dR}{dt} = \operatorname{sgn}\frac{t(t-1)(2t^6-3t^5-3t^4-14t^3-3t^2-3t+2)}{t^3+1}.$$

This yields the critical values $t=0$ and $t=1$, which we will investigate later. To find the remaining critical values of R, we are faced with the unappetizing task of finding the real roots of the equation

$$2t^6 - 3t^5 - 3t^4 - 14t^3 - 3t^2 - 3t + 2 = 0 \tag{3}$$

which (it is easily verified) has no rational root. Fortunately, this is a reciprocal equation equivalent to

$$2\left(t^3 + \frac{1}{t^3}\right) - 3\left(t^2 + \frac{1}{t^2}\right) - 3\left(t + \frac{1}{t}\right) - 14 = 0$$

which the substitution

$$u = t + \frac{1}{t} \tag{4}$$

reduces to $2u(u^2-3) - 3(u^2-2) - 3u - 14 = 0$ or

$$2u^3 - 3u^2 - 9u - 8 = 0,$$

which also has no rational root. It is easy to verify by elementary calculus that this equation has exactly one real root, which can be approximated (by the Newton-Raphson method or otherwise) to $u_0 \approx 3.258033949$. Hence (3) has exactly two real roots, which are found from (4) to be $t = (u_0 \pm \sqrt{u_0^2 - 4})/2$, that is,

$$t_1 \approx 2.9149782 \quad \text{and} \quad t_2 \approx 0.34305574.$$

We will now use the fact that $d\kappa/dt = (2\varepsilon/3a)dR/dt$ to identify the extrema of κ at the critical values $t = 0, 1, t_1, t_2$.

At	$d\kappa/dt$ changes sign from	yielding relative
$t = 0$	+ to -	maximum $\kappa = 2/3a$
$t = 1$	- to +	minimum $\kappa = -8\sqrt{2}/3a$
$t = t_1$	+ to -	maximum $\kappa \approx -0.529/a$
$t = t_2$	- to +	minimum $\kappa \approx 0.529/a$

If we compare the values of all the relative and endpoint extrema, we conclude that $|\kappa|$ attains its absolute maximum value $8\sqrt{2}/3a$ when $t = 1$. This occurs, to no one's surprise, at the point $(3a/2,\ 3a/2)$.

Also solved partially by W.J. BLUNDON, Memorial University of Newfoundland.

Editor's comment.

Readers will find it instructive to draw the unexpectedly tortuous graph of κ versus t, using the information given above. The proposer wrote that this seemed to be a problem one would expect to see in many calculus texts, and that he was surprised at not finding it in any of the many books he looked into.

Now we know why.

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A MERRY CHRISTMAS AND A HAPPY NEW PROBLEM-SOLVING YEAR.

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TITLES OF ARTICLES AND NOTES

LETTER TO THE EDITOR

PROBLEMS AND SOLUTIONS

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