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# YET ANOTHER PROOF OF ROUTH'S THEOREM

James S. Kline and Daniel J. Velleman

Let $ABC$ be a triangle with unit area. Points $D$, $E$, and $F$ are chosen on sides $BC$, $CA$, and $AB$ respectively, and the lines $AD$, $BE$, and $CF$ are drawn, forming a new triangle $GHI$ inside $ABC$ as in Figure 1. Let

$$r = \frac{BD}{DC}, \quad s = \frac{CE}{EA}, \quad \text{and} \quad t = \frac{AF}{FB}.$$

Then Routh's Theorem says that *the area of triangle GHI is*

$$\frac{(rst-1)^2}{(rs+r+1)(st+s+1)(tr+t+1)}.$$

A number of proofs of this theorem can be found in the literature (see [1]–[7]). We present here a new proof of Routh's Theorem.

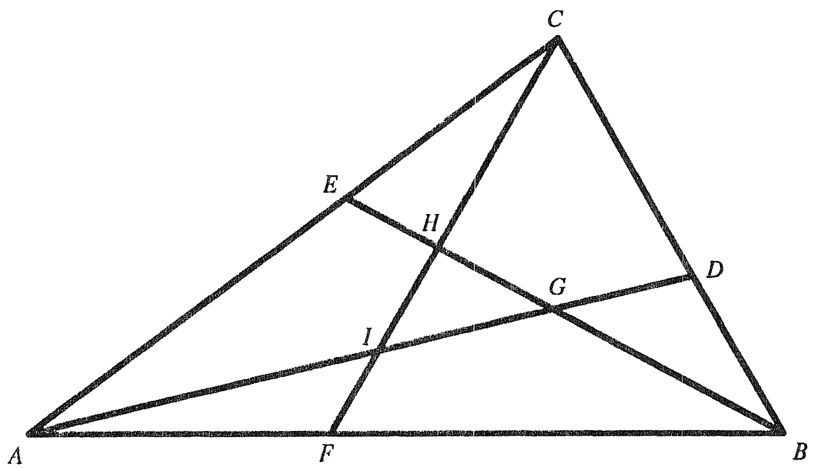


Figure 1

We begin with a well-known problem that is a special case of Routh's Theorem. Suppose that $r = s = t = 1/2$. Thus, $D$ is one third of the way from $B$ to $C, E$ is one third of the way from $C$ to $A$, and $F$ is one third of the way from $A$ to $B$. In this case Routh's Theorem tells us that the area of $GHI$ will be $1/7$. An elegant proof of this can be found in [8].

For our proof, we begin by dividing each side of triangle $ABC$ into seven equal pieces and then drawing lines through the division points parallel to the sides of $ABC$, cutting the triangle into a grid of 49 congruent smaller triangles (see Figure 2). It turns out that the points $G, H$, and $I$ lie on intersections of this grid. To see why, consider first the point $G$, which is the intersection of $AD$ and $BE$. Since $D$ is one third of the way from $B$ to $C, AD$ must cross $JK$ at a point that is one third of the way from $J$ to $K$. But clearly this is also the point where $LM$ crosses $JK$. Similarly, since $E$ is one third of the way from $C$ to $A, BE$ crosses $LM$ at a point that is one third of the way from $M$ to $L$, and this is also where $JK$ crosses $LM$. Thus $G$ must be the intersection of $JK$ and $LM$. Similar reasoning shows that $H$ and $I$ are intersection points in the grid.

It is now clear from Figure 2 that the interior of triangle $GHI$ consists of one small triangle (dark shading) plus halves of three parallelograms (lighter shading), each of which is made up of four small triangles. Since the area of each small triangle is $1/49$, the area of $GHI$ is

$$\frac{1}{49} + \frac{3}{2}\left(\frac{4}{49}\right) = \frac{1}{7}.$$

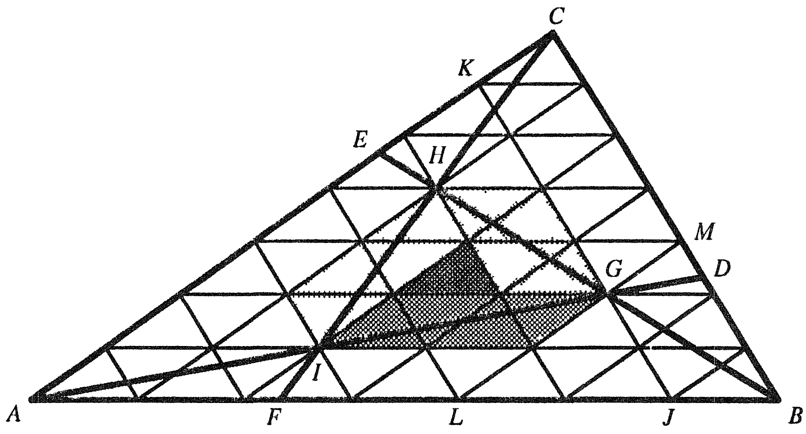


Figure 2

It is interesting to note that the dark triangle in Figure 2 is bounded by the line through $G$ and the point one third of the way from $I$ to $H$, the line through $H$ and the point one third of the way from $G$ to $I$, and the line through $I$ and the point one third of the way from $H$ to $G$. In other words, it is formed from $GHI$ by the mirror image of the process by which $GHI$ was formed from $ABC$. Of course its area, $1/49$, is $1/7$ the area of triangle $GHI$.

This proof can be used to motivate a general proof of Routh's Theorem. The key idea from this proof that we borrow for the more general proof is the idea of adding lines parallel to the sides of $ABC$ that pass through the points $G, H$, and $I$.

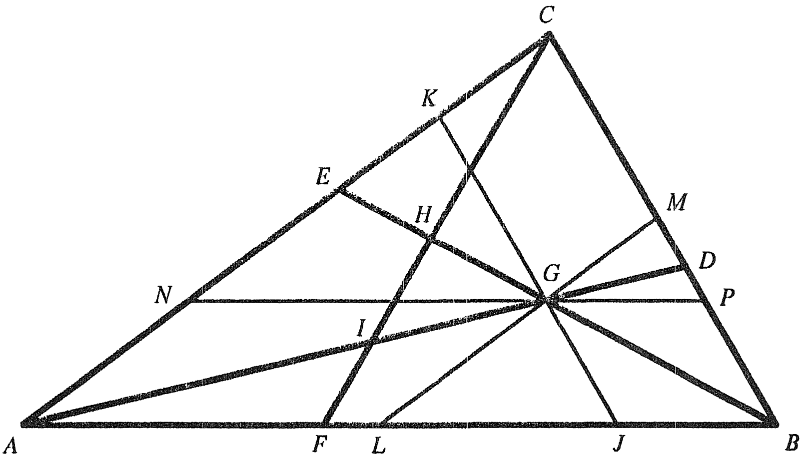


Figure 3

In Figure 3 such lines have been added through $G$. Clearly

$$\frac{JG}{GK} = \frac{BD}{DC} = r,$$

so $GK = JG/r$. Also, since triangles $LJG$ and $GPM$ are similar,

$$\frac{PM}{JG} = \frac{MG}{GL} = \frac{CE}{EA} = s,$$

so $PM = s \cdot JG$. Thus

$$\begin{aligned} BC &= BP + PM + MC = JG + PM + GK = JG + s \cdot JG + \frac{JG}{r} \\ &= \left(\frac{rs + r + 1}{r}\right) \cdot JG, \end{aligned}$$

so

$$\frac{JG}{BC} = \frac{r}{rs + r + 1}\,.$$

This is also the ratio of the altitudes of triangles $ABG$ and $ABC$, and since they have the same base, it is also the ratio of their areas. But $ABC$ has area 1, so we have

$$\text{area of triangle } ABG = \frac{r}{rs + r + 1}\,.$$

Similar reasoning, using lines parallel to the sides of $ABC$ passing through $H$ and $I$, leads to the formulas:

$$\text{area of triangle } BCH = \frac{s}{st + s + 1}\,, \qquad \text{area of triangle } CAI = \frac{t}{tr + t + 1}\,.$$

But then since triangle $ABC$ is the disjoint union of triangles $ABG, BCH, CAI$, and $GHI$, we have

$$\begin{aligned}\text{area of triangle } GHI &= 1 - \frac{r}{rs+r+1} - \frac{s}{st+s+1} - \frac{t}{tr+t+1} \\ &= \frac{(rst-1)^2}{(rs+r+1)(st+s+1)(tr+t+1)}.\end{aligned}$$

**References:**

[1] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley and Sons, New York, (1969), pp. 211, 219-220.

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[3] David C. Kay, *College Geometry*, Holt, Rinehart and Winston, New York, (1969), pp. 205-207.

[4] M. S. Klamkin and A. Liu, Three more proofs of Routh's Theorem, *Crux Mathematicorum*, 7(1981), pp. 199-203.

[5] Z. A. Melzak, *Companion to Concrete Mathematics*, John Wiley and Sons, New York, (1973), pp. 7-9.

[6] Ivan Niven, A new proof of Routh's Theorem, *Mathematics Magazine*, 49(1976), pp. 25-27.

[7] E. J. Routh, *A treatise on analytical statics, with numerous examples, Vol. 1*, 2nd ed., Cambridge University Press, London, (1896), p. 82.

[8] Hugo Steinhaus, *Mathematical Snapshots*, revised and enlarged edition, Oxford University Press, New York, (1960), p. 11.

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# THE SKOLIAD CORNER

## No. 2

R. E. WOODROW

For this month's contest problems we give the 15 problems of the 1994 Cariboo College Junior Mathematics Contest. The contest was forwarded to us by the organizer Dr. J. Totten. The first twenty years of the contest with solutions are available in a book at \$14.95 plus handling through the University College of the Cariboo Bookstore, Box 3010, Kamloops, B.C., V2C 5N3 (tel: (604) 828-5141 for details). The collection was reviewed previously [1993: 139].

## UNIVERSITY COLLEGE OF THE CARIBOO
## JUNIOR PRELIMINARY 1994
### Wednesday, March 16, 1994
**Time: 45 minutes**

**1.** The number of chords which can be drawn using four distinct points on the circumference of a circle is:

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6.

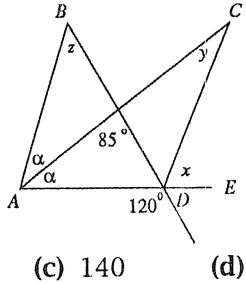
**2.** Sammy bought two 10-inch diameter pizzas. Sarah says her single pizza has at least as much material as both of Sammy's combined. If this is true, then the smallest diameter, in inches, that Sarah's pizza could have is:

(a) $5\sqrt{2}$ (b) $5\pi$ (c) $10\sqrt{2}$ (d) $20\sqrt{2}$ none of these.

**3.** One number is ten greater than another, and their average is at most 24. The largest value that the smaller of the two numbers could have is:

(a) $9\frac{1}{2}$ (b) $10\frac{1}{2}$ (c) 12 (d) 19 (e) 22.

**4.** If $\angle BDC = \angle CDE$, then the sum of the measure of the angles marked $x$, $y$, and $z$ is:




(a) 195 (b) 145 (c) 140 (d) 135 (e) none of these.

**5.** The number of sides in a polygon with 20 diagonals is:

(a) 5 (b) 6 (c) 7 (d) 8 (e) none of these.

**6.** A solution of 60 litres of sugar and water is 20% sugar (by volume). The number of litres of water which must be added to make a solution which is 5% sugar (by volume) is:

(a) 240 (b) 180 (c) 120 (d) 15 (e) none of these.

**7.** If each of two adjacent angles whose sum is 96° is bisected, the number of degrees in the angle formed by the two bisectors is:

(a) 36 (b) 48 (c) 96 (d) 132 (e) none of these.

**8.** The number of ways one may place 3 books, "a", "b", and "c" on a shelf so that, reading from left to right, the books will <u>not</u> be in alphabetical order is:

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of these.

**9.** A class of 31 students was organized for a tennis tournament. After each girl in the class was paired with a boy as partner, there were 7 boys left. The number of girls in the class was:

(a) 9 (b) 11 (c) 13 (d) 15 (e) none of these.

**10.** Each side of an equilateral triangle is 4 cm longer than the side of a square. If the sum of the perimeters of the two figures is 96 cm, each side of the triangle has length:

(a) 10 (b) 12 (c) 15 (d) 16 (e) $17\frac{1}{7}$.

**11.** At noon a plane leaves an airport and flies due west at 540 km/h. At 1 pm a second plane leaves the same airport and also flies due west at 720 km/h. The time of day when the second plane overtakes the first is:

(a) 2 pm (b) 3 pm (c) 3:30 pm (d) 4 pm (e) 4:30 pm.

**12.** If the radius of a circle is decreased by 6 cm, the area of the circle is one-ninth what it was. The original radius, in cm, was:

(a) 9 (b) 8 (c) 7 (d) $\frac{9}{2}$ (e) none of these.

**13.** A deck 3 metres wide surrounds a rectangular pool which is twice as long as it is wide. If the area of the deck is 360 $m^2$, the dimensions of the pool, in metres, are:

(a) $9 \times 18$ (b) $6 \times 12$ (c) $12 \times 24$ (d) $18 \times 36$ (e) none of these.

**14.** If one side of a triangle has length 5 cm and the second side is 8 cm, the length of the third side may lie between:

(a) $1\frac{3}{5}$ and 40 (b) 2 and 8 (c) $1\frac{1}{2}$ and $6\frac{1}{2}$ (d) 6 and 14 (e) 3 and 13.

**15.** One candle will burn completely in 4 hours, while a second candle of the same length requires 5 hours to burn completely. If the candles are lit at the same time, the number of hours that will pass before one of them is three times as long as the other is:

(a) $3\frac{7}{11}$ (b) 3 (c) 2 (d) $1\frac{1}{3}$ (e) none of these.

* * *

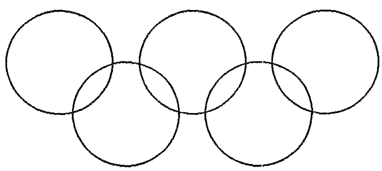
In the last number we gave the first round of the 1994–1995 Alberta High School Prize Examination. The contest was written November 15 by 978 students in grades 10, 11 and 12. The winner was a grade 11 student from Queen Elizabeth High School in Calgary, Derek Kisman, who led his team of three (including Chris Haines and Vienna Ng) to first place as a school. Here are the solutions:

| | | | |
|---|---|---|---|
| 1. *a* | 5. *c* | 9. *d* | 13. *c* |
| 2. *c* | 6. *e* | 10. *a* | 14. *c* |
| 3. *d* | 7. *c* | 11. *d* | 15. *c* |
| 4. *b* | 8. *b* | 12. *c* | 16. *d* |

* * *

In the past we gave some problem sets in the Olympiad Corner that were more appropriate for the Skoliad Corner. In this number we give reader's solutions to some of the problems of the 1993 Japanese Mathematical Olympiad for Elementary School Students [1993: 285].

**1.** There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



*Solution by Stewart Metchette, Culver City, California.*

For the five rings, we have:

$$a + b = b + c + d = d + e + f = f + g + h = h + i = N. \qquad (1)$$

Since we are dealing with the nine non-zero decimal digits we have $\sum_1^9 j = 9(10)/2 = 45$. The five regions sum to a common $N$ for $45/5 = 9$ but then one pair must be $9 + 0$ or one triplet $9 + 0 + 0$, which isn't allowed. So $N > 9$. Since $a + b = h + i$, there must be at least *two* pairs of decimal digits that sum to $N$. For $10 \le N \le 15$ we have

$$N = 9 + a = 8 + (1 + a) = \cdots, \quad \text{for} \quad 1 \le a \le 6$$

while $N = 16 = 9 + 17$ and $N = 17 = 9 + 8$ only. So $N \le 15$.

From 1:

$$a + b = b + c + d \quad \text{or} \quad a = c + d \tag{2}$$

and

$$h + i = f + g + h \quad \text{or} \quad i = f + g \tag{3}$$

The five central digits must equal $45 - 2N$:

$$(c + d) + e + (f + g) = a + e + i = 45 - 2N.$$

So we have

| $N$ | $2N$ | $45 - 2N$ | $a, e, i$ |
|---|---|---|---|
| 10 | 20 | 25 | $9, 8, -$ no digit available |
| 11 | 22 | 23 | $9, 8, 6; \ldots$ |
| 12 | 24 | 21 | $9, 8, 4; \ldots$ |
| 13 | 26 | 19 | $9, 8, 2; \ldots$ |
| 14 | 28 | 17 | $9, 7, 1, \ldots$ |
| 15 | 30 | 15 | $9, 5, 1; \ldots$ |

So $11 \le N \le 15$.

**4.** On each of three cards was written a whole number from 1 to 10. These cards were shuffled and dealt to three people who recorded the numbers on their respective cards. The cards were collected, and the process was repeated again. After a few times, the three people computed the totals of their numbers. They turned out to be 13, 15 and 23. What were the numbers on the cards?

*Solution by John Morvay, Springfield, Missouri.*

Now $13 + 15 + 23 = 51 = 3 \cdot 17$ so each person received three cards. Consider the triple $(x, y, z)$ $x \le y \le z$ with $x + y + z = 17$. Now the total 23 shows $z \ge 8$. If $z = 10$, then the third 10 can be used for at most one of 13, 15 whence $y \ge 5$ and $x = 3$. But then one player has at least $10 + 3 + 3 = 16$, a contradiction. If $z = 8$ then $y = 7$ (for $8 + 8 + 7 = 23$), but the triple $(2, 7, 8)$ cannot produce a sum of 13 or 15. This shows $z = 9$. If $x > 3$ someone must have a total of at least $9 + 4 + 4 = 17$. This shows $x \le 3$. Thus $9 + 9 + 5 = 23$ makes $y = 5$ and $x = 3$, and the cards were distributed $(3 + 5 + 5 = 13)$, $(3 + 3 + 9 = 15)$ and $(5 + 9 + 9 = 23)$.

**5.** Each of $A$, $B$, $C$, $D$ and $E$ was told in secrecy a different whole number from 1 to 5. The teacher asked $A$: "whose number is the largest?" $A$ said: "I don't know." The teacher then asked $B$: "Is $C$'s number larger than yours?" $B$ said: "I don't know." The teacher asked $C$: "Is $D$'s number larger than yours?" $C$ said: "I don't know." The teacher then asked $D$: "Is $B$'s number larger than yours?" $D$'s answer was not recorded. Finally, the teacher asked $B$ again: "Is $C$'s number larger than yours?" $B$ said: "No." From the above information, is it possible to deduce which number was told to whom, and was $D$'s answer "Yes", "No" or "I don't know"?

*Solution by John Morvay, Springfield, Missouri.*

After the answer of $A$ we conclude $A \neq 5$. After the answer of $B$ we conclude $B \neq 5$ and $B \neq 1$. After the answer of $C$ we conclude $C \neq 5$ and $C \neq 1$. $D$'s answer was not recorded. After the second answer of $B$ we conclude $B = 4$ (since $C$ must have one of 2, 3, or 4 and as $C < B$, so $B = 4$).

This is the most we can establish.

Possible configurations are:

$$A = 2, \quad B = 4, \quad C = 3, \quad D = 1, \quad E = 5,$$

$$A = 1, \quad B = 4, \quad C = 2, \quad D = 3, \quad E = 5,$$

$$A = 1, \quad B = 4, \quad C = 2, \quad D = 5, \quad E = 3.$$

* * *

Send me your Skoliad problem sets, nice solutions, and any suggestions for the future of this column.

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# THE OLYMPIAD CORNER

## No. 162

R. E. WOODROW

*All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

We begin this number with the six problems of the Final Round of the 6th Korean Mathematical Olympiad, which was written April 17–18, 1993. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Nfld., the Canadian Team leader at the 34th I.M.O. at Istanbul, who collected several sets of problems and passed them on to me.

## 6th KOREAN MATHEMATICAL OLYMPIAD

### Final Round — April 17–18, 1993

### Day 1

**1.** Let there by a $9 \times 9$ array of white squares. Find the largest positive integer $n$ satisfying the following property: There always remains either a $1 \times 4$ or $4 \times 1$ array of white squares no matter how you choose $n$ out of 81 white squares and color them black.

**2.** Let $ABC$ be a triangle with $\overline{BC} = a$, $\overline{CA} = b$, $\overline{AB} = c$. Find the point $P$ for which

$$a \cdot \overline{AP}^2 + b \cdot \overline{BP}^2 + c \cdot \overline{CP}^2$$

is minimal, and find the minimum.

**3.** Find the smallest positive integer $x$ for which $\dfrac{7x^{25} - 10}{83}$ is an integer.

**Day 2**

**4.** An integer is called a Pythagorean number if it is the area of a right triangle whose sides are of integral lengths, say, $x, y, z \in N$ such that $x^2 + y^2 = z^2$. Prove that for each positive integer $n$ $(n > 12)$, there exists a Pythagorean number between $n$ and $2n$.

**5.** Let $n$ be a given natural number. Find all the continuous functions $f(x)$ satisfying:

$$\binom{n}{0}f(x)+\binom{n}{1}f(x^2)+\binom{n}{2}f(x^{2^2})+\cdots+\binom{n}{n-1}f(x^{2^{n-1}})+\binom{n}{n}f(x^{2^n}) = 0.$$

**6.** Let $ABC$ be a triangle with $\overline{BC} = a$, $\overline{CA} = b$, $\overline{AB} = c$. Let $D$ be the midpoint of the side $BC$, and let $E$ be the point on $BC$ for which the line segment $AE$ is the bisector of the angle $A$. Let the circle passing through $A$, $D$, $E$ intersect with the sides $CA$, $AB$ at $F$, $G$, respectively. Finally, let $H$ be the point on $AB$ for which $\overline{BG} = \overline{GH}$, i.e., $\overline{BH} = 2\overline{BG}$. Prove that the triangles $EBH$ and $ABC$ are similar and find the ratio $\frac{\Delta EBH}{\Delta ABC}$ of their areas.

* * *

We turn to solutions from the readers to problems posed for the 33*rd International Mathematical Olympiad in Moscow* [1993: 222–223; 255–256]. We do not give those solutions which are similar to those published as official solutions by the organizers in Moscow.

**2.** *Proposed by China.*

Let $\mathbb{R}^+$ be the set of all non-negative real numbers. Two positive real numbers $a$ and $b$ are given. Suppose that a mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the functional equation

$$f(f(x)) + af(x) = b(a + b)x.$$

Prove that there exists a unique solution of this equation.

*Solutions by George Evagelopoulos, Athens, Greece; and Waldemar Pompe, student, University of Warsaw, Poland. We give Pompe's solution.*

Let $x_0 \geq 0$ and let $x_n = f^{(n)}(x_0) = f(f^{n-1}(x_0))$, $f^{(0)}(x_0) = x_0$. Since $f : \mathbb{R}^+ \to \mathbb{R}^+$, $x_n \geq 0$ for all $n \in \mathbb{N}$. Putting $x_n$ into the given equation we obtain

$$x_{n+2} + ax_{n+1} = b(a+b)x_n$$

for starting values $x_0$ and $x_1 = f(x_0)$. Solving this recurrence we get

$$x_n = Ab^n + B(-1)^n(a+b)^n$$

for some real constants $A$ and $B$. If $B \neq 0$, then taking $n$ large enough we would get $|Ab^n| < |B(-1)^n(a+b)^n|$ (because $|b| < |a+b|$). Therefore, for some large $n$, $x_n$ would be negative. Hence $B = 0$, which gives $x_n = Ab^n$ and, in particular, $x_0 = A$, $f(x_0) = x_1 = Ab = bx_0$, for any $x_0 \geq 0$. This gives a unique solution $f(x) = bx$.

4. *Proposed by Colombia.*

Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let $O_1, O_2, O_3, O_4$ be the centroids of the triangles constructed on $AB, BC, CD, DA$ respectively. Show that $O_1O_3$ is perpendicular to $O_2O_4$.

*Solutions by George Evagelopoulos, Athens, Greece; by Geoffrey Kandall, Hamden, Connecticut; and by Waldemar Pompe, student, University of Warsaw, Poland. We first give Pompe's solution.*

**A nice lemma:**

Let $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ (with $A_1B_1 = B_1C_1$, $C_1D_1 = D_1A_1$ and $A_2B_2 = B_2C_2$, $C_2D_2 = D_2A_2$) be two directly similar deltoids with the same orientation in the plane. Then $A_1A_2 = C_1C_2$ if and only if $B_1B_2 \| D_1D_2$.

**Proof of the nice lemma:**

We consider two cases:

*Case* (i): (Figure 1.)

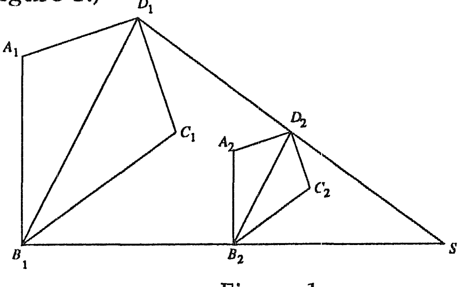



Figure 1

Assume that the lines $B_1D_1$ and $B_2D_2$ are parallel. Then either $A_1B_1C_1D_1$ is translated from $A_2B_2C_2D_2$ or both deltoids are homothetic (with the scale $k \neq 1$ and a centre $S$) to each other. In the first situation $A_1A_2 = C_1C_2$ and $B_1B_2 \| D_1D_2$ so the lemma is true. In the second situation $A_1A_2/C_1C_2 = k \neq 1$ and $B_1B_2 \cap D_1D_2 = S$, so the lemma is again true.

*Case* (ii): (Figure 2.)

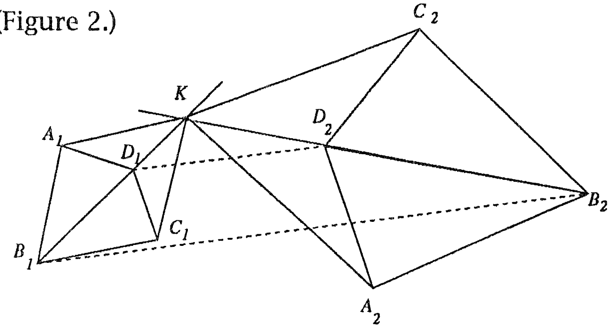



Figure 2

Assume that the lines $B_1D_1$ and $B_2D_2$ are not parallel and let $K = B_1D_1 \cap B_2D_2$. Assume that $A_1A_2 = C_1C_2$. Since $A_1K = C_1K$ and $A_2K = C_2K$ triangles $A_1A_2K$ and $C_1C_2K$ are congruent. Thus $\angle A_1KA_2 = \angle C_1KC_2$. Consider the rotation about the point $K$ with the directed angle $\angle A_2KA_1$ and then a homothety with the centre $K$ and the scale $A_2K/A_1K = k$. Using these two transformations $A_2$ goes to $A_1$ and $C_2$ goes to $C_1$. Since the deltoids are directly similar, $B_2$ goes to $B_1$ and $D_2$ goes to $D_1$. Therefore $D_2K/D_1K = B_2K/B_1K$ $(= k)$, which shows that $B_1B_1 \| D_1D_2$.

In the same way we can prove the converse, but since we won't need it, we can leave the proof to the reader as an easy exercise.

See Figure 3. Let $K$, $L$, $M$, $N$ be the midpoints of the segments $AB$, $BC$, $CD$, $DA$ respectively. Since $AC = BD$ we can use the nice lemma on the pair of deltoids $AO_1BK$ and $CO_3DM$. Then we get

$$MK \| O_1O_3. \tag{1}$$

Using the nice lemma on the pair of deltoids $BO_2CL$ and $DO_4AN$ we obtain

$$LN \| O_2O_4. \tag{2}$$

Thus since $AC = BD$, $KLMN$ is a rhombus, which gives $KM \perp LN$. From (1) and (2) it follows that also $O_1O_3 \perp O_2O_4$, which was to be shown.

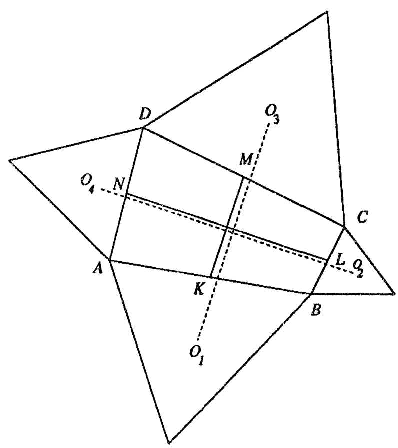



Figure 3

*Remarks:*

(1) The assumption about convexity of $ABCD$ was not necessary.

(2) The equilateral triangles must be constructed either all outwardly, or all inwardly, or the two opposite outwardly and the remaining two opposite inwardly.

(3) Using the lemma, many generalizations may be given.

*Generalization and solution by G. Kandall.*

This problem can be generalized as follows: Let $ABCD$ be a convex quadrilateral such that $AC = BD$. Isosceles triangles $O_1AB$, $O_2BC$, $O_3CD$, $O_4DA$ are constructed (outwardly) so that $\angle O_1AB = \angle O_1BA = \angle O_3CD = \angle O_3DC = \theta$ and $\angle O_2BC = \angle O_2CB = \angle O_4AD = \angle O_4DA = \varphi$. ($\varphi, \theta$ need not be equal.) We will show that $O_1O_3 \perp O_2O_4$.

Let $M_1$, $M_2$, $M_3$, $M_4$ be the midpoints of $AB$, $BC$, $CD$, $DA$ respectively. Let $\overrightarrow{AM_1} = \overrightarrow{M_1B} = \mathbf{p}$, $\overrightarrow{BM_2} = \overrightarrow{M_2C} = \mathbf{q}$, $\overrightarrow{CM_3} = \overrightarrow{M_3D} = \mathbf{r}$, $\overrightarrow{DM_4} = \overrightarrow{M_4A} = \mathbf{s}$, $\overrightarrow{M_1O_1} = \mathbf{p}'$, $\overrightarrow{M_2O_2} = \mathbf{q}'$, $\overrightarrow{M_3O_3} = \mathbf{r}'$, $\overrightarrow{M_4O_4} = \mathbf{s}'$. Obviously $\mathbf{p} \cdot \mathbf{p}' = \mathbf{q} \cdot \mathbf{q}' = \mathbf{r} \cdot \mathbf{r}' = \mathbf{s} \cdot \mathbf{s}' = 0$.

Quadrilateral $M_1M_2M_3M_4$ is a rhombus (here we use $AC = BD$), so $M_1M_3 \perp M_2M_4$. But $\overrightarrow{M_1M_3} = \mathbf{q} - \mathbf{s}$ (this follows easily from $\overrightarrow{M_1M_3} = \mathbf{p} + 2\mathbf{q} + \mathbf{r} = -\mathbf{p} - 2\mathbf{s} - \mathbf{r}$) and analogously $\overrightarrow{M_2M_4} = \mathbf{r} - \mathbf{p}$, hence $(\mathbf{q} - \mathbf{s}) \cdot (\mathbf{r} - \mathbf{p}) = 0$.

Let $T$ be the linear transformation that rotates a vector counterclockwise by 90°. This linear transformation preserves lengths, so obviously $\mathbf{v} \circ \mathbf{w} = T(\mathbf{v}) \circ T(\mathbf{w})$ for any vectors $\mathbf{v}$, $\mathbf{w}$. Let $\alpha = \tan\theta$, $\beta = \tan\varphi$. It is easy to see that $T(\mathbf{p}) = \frac{1}{\alpha}\mathbf{p}'$, $T(\mathbf{p}') = -\alpha\mathbf{p}$, $T(\mathbf{q}) = \frac{1}{\beta}\mathbf{q}'$, $T(\mathbf{q}') = -\beta\mathbf{q}$, $T(\mathbf{r}) = \frac{1}{\alpha}\mathbf{r}'$, $T(\mathbf{r}') = -\alpha\mathbf{r}$, $T(\mathbf{s}) = \frac{1}{\beta}\mathbf{s}'$, $T(\mathbf{s}') = -\beta\mathbf{s}$. We have $\overrightarrow{O_1O_3} = (-\mathbf{p}' + \mathbf{r}') + (\mathbf{q} - \mathbf{s})$, $\overrightarrow{O_2O_4} = (-\mathbf{q}' + \mathbf{s}') + (\mathbf{r} - \mathbf{p})$, so $\overrightarrow{O_1O_3} \circ \overrightarrow{O_2O_4} = (-\mathbf{p}' + \mathbf{r}') \circ (-\mathbf{q}' + \mathbf{s}') - (\mathbf{p} \circ \mathbf{r} + \mathbf{r}' \circ \mathbf{p}) + (\mathbf{q} \circ \mathbf{s}' + \mathbf{s} \circ \mathbf{q}')$.

Now

(i) $(-\mathbf{p}' + \mathbf{r}) \circ (-\mathbf{q}' + \mathbf{s}') = T(-\mathbf{p}' + \mathbf{r}) \circ T(-\mathbf{q}' + \mathbf{s}') = (\alpha\mathbf{p} - \alpha\mathbf{r}) \circ (\beta\mathbf{q} - \beta\mathbf{s}) = \alpha\beta(\mathbf{p} - \mathbf{r}) \circ (\mathbf{q} - \mathbf{s}) = 0$.

(ii) $\mathbf{p}' \circ \mathbf{r} + \mathbf{r}' \circ \mathbf{p} = T(\mathbf{p}') \circ T(\mathbf{r}) + T(\mathbf{r}') \circ T(\mathbf{p}) = -\alpha\mathbf{p} \circ \frac{1}{\alpha}\mathbf{r}' - \alpha\mathbf{r} \circ \frac{1}{\alpha}\mathbf{p}' = -(\mathbf{p}' \circ \mathbf{r} + \mathbf{r}' \circ \mathbf{p})$ so

$$\mathbf{p}' \circ \mathbf{r} + \mathbf{r}' \circ \mathbf{p} = 0.$$

(iii) $\mathbf{q} \circ \mathbf{s}' + \mathbf{s} \circ \mathbf{q}' = 0$ as in (ii).

Consequently $\overrightarrow{O_1O_3} \circ \overrightarrow{O_2O_4} = 0$.

*Remark.* The conclusion remains valid if one or both pairs of opposite isosceles triangles are constructed inwardly. Only minor modifications of the above proof are required.

**6.** *Proposed by Iran.*

Let $f(x)$ be a polynomial with rational coefficients and $\alpha$ be a real number such that $\alpha^3 - \alpha = (f(\alpha))^3 - f(\alpha) = 33^{1992}$. Prove that for each $n \geq 1$

$$(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = 33^{1992},$$

where $f^{(n)}(x) = f(f(\ldots f(x)))$, and $n$ is a positive integer.

*Solutions by George Evagelopoulos, Athens, Greece; and by Waldemar Pompe, student, University of Warsaw, Poland. We give Pompe's solution.*

Let $g(x) = x^3 - x$. It is easy to see that if $c > 2\sqrt{3}/9$, then the equation $g(x) = c$ has a real solution which is unique. We have

$$g(\alpha) = g(f(\alpha)) = 33^{1992}.$$

But, since $33^{1992} > 2\sqrt{3}/9$, $f(\alpha) = \alpha$. Thus $f^{(n)}(\alpha) = \alpha$ and we get

$$(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = \alpha^3 - \alpha = 33^{1992},$$

which completes the proof.

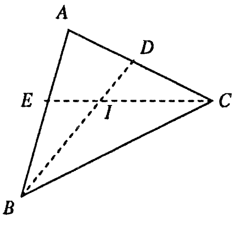
**9.** *Proposed by Japan.*

In a triangle $ABC$, let $D$ and $E$ be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides $AC$, $AB$, respectively. Determine the angles $\angle A$, $\angle B$, $\angle C$, if

$$\angle BDE = 24^\circ, \quad \angle CED = 18^\circ.$$

*Solutions by George Evagelopoulos, Athens, Greece; and by D. J. Smeenk, Zaltbommel, The Netherlands.*

Let $I$ be the incenter of triangle $ABC$. Let $D'$ be the projection onto $AC$ of $I$, and let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$, as usual. Now $\angle ADB = \frac{1}{2}\beta + \gamma$, so




$$ID = \frac{r}{\sin(\frac{1}{2}\beta + \gamma)} \quad \text{and} \quad IE = \frac{r}{\sin(\beta + \frac{1}{2}\gamma)} \qquad (1)$$

Applying the law of sines to triangle $IDE$ we have

$$\frac{\sin(\beta + \frac{1}{2}\gamma)}{\sin(\frac{1}{2}\beta + \gamma)} = \frac{\sin 18^\circ}{\sin 24^\circ}. \qquad (2)$$

As $\angle BDE = 24^\circ$ and $\angle CED = 18^\circ$ we have $\angle DIE = 138^\circ = 90^\circ + \frac{1}{2}\alpha \Rightarrow \alpha = 96^\circ$. Thus

$$\beta + \gamma = 84^\circ \quad \text{so} \quad \gamma = 84^\circ - \beta \quad \text{and} \quad \frac{1}{2}\gamma = 42^\circ - \frac{1}{2}\beta. \qquad (3)$$

Now using (3) in (2) we get

$$\frac{\sin(42^\circ + \frac{1}{2}\beta)}{\sin(84^\circ - \frac{1}{2}\beta)} = \frac{\sin 18^\circ}{\sin 24^\circ}.$$

Expanding and doing some calculations gives

$$\tan\frac{1}{2}\beta = \frac{\sin 18^\circ \sin 84^\circ - \sin 24^\circ \sin 42^\circ}{\sin 24^\circ \cos 42^\circ + \sin 18^\circ \cos 84^\circ},$$

and this equals $\tan 6^\circ$

[To see this we have

$$\frac{\sin 18^\circ \sin 84^\circ - \sin 24^\circ \sin 42^\circ}{\sin 24^\circ \cos 42^\circ + \sin 18^\circ \cos 84^\circ} = \frac{\sin 6^\circ}{\cos 6^\circ}$$

just in case

$$\sin 18^\circ \sin^2 84^\circ - \sin 24^\circ \sin 42^\circ \cos 6^\circ$$
$$= \sin 6^\circ \sin 24^\circ \cos 42^\circ + \sin 18^\circ \sin^2 6^\circ,$$

$$\sin 18^\circ [\cos^2 6^\circ - \sin^2 6^\circ]$$
$$= \sin 24^\circ [\sin 6^\circ \cos 42^\circ + \cos 6^\circ \sin 42^\circ],$$

$$\sin 18^\circ \cos 12^\circ = \sin 24^\circ \sin 48^\circ,$$
$$\sin 18^\circ = 2 \sin 12^\circ \sin 48^\circ,$$
$$\sin 18^\circ = \cos 36^\circ - \cos 60^\circ,$$
$$\sin 18^\circ = -2 \sin^2 18^\circ + \frac{1}{2}.$$

giving the equivalent condition

$$2 \sin^2 18^\circ + \sin 18^\circ - \frac{1}{2} = 0.$$

This is the same as $\sin 18^\circ = -\frac{1}{4}(-1 + \sqrt{5})$, which is true.]

So $\frac{1}{2}\beta = 6^\circ$, $\beta = 12^\circ$ and $\alpha = 96^\circ$, $\beta = 12^\circ$, $\gamma = 72^\circ$.

**12.** *Proposed by South Korea.*

Prove that $n = \dfrac{5^{125} - 1}{5^{25} - 1}$ is a composite number.

*Solutions by Seung-Jin Bang, Seoul, Korea; and by George Evagelopoulos, Athens, Greece. We give Bang's solution.*

Let $x = 5^{25}$, then

$$\begin{aligned}
5^{125} - 1 &= x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1) \\
&= (x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 - 5x^3 - 10x^2 - 5x)(x - 1) \\
&= ((x^2 + 3x + 1)^2 - 5x(x + 1)^2)(x - 1) \\
&= ((x^2 + 3x + 1)^2 - (5^{13}(x + 1))^2)(x - 1) \\
&= (x^2 + 3x + 1 + 5^{13}(x + 1))(x^2 + 3x + 1 - 5^{13}(x + 1))(x - 1)
\end{aligned}$$

which implies $\frac{5^{125}-1}{5^{25}-1}$ is a composite number, completing the proof.

* * *

That completes the file of "new" solutions to the unused I.M.O. problems. The next solutions on file are to problems of the 1993 Japanese Mathematical Olympiad for Elementary School Students. As they seem more appropriate for the Skoliad Corner, I will end this number here and refer the reader to the "new" column this month for the Answers. Send me your Olympiads and nice solutions.

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# PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before* **September 1, 1995,** *although solutions received after that date will also be considered until the time when a solution is published.*

**2011.** *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABC$ is a triangle with incenter $I$. $BI$ and $CI$ meet $AC$ and $AB$ at $D$ and $E$ respectively. $P$ is the foot of the perpendicular from $I$ to $DE$, and $IP$ meets $BC$ at $Q$. Suppose that $IQ = 2IP$. Find angle $A$.

**2012.** *Proposed by K.R.S. Sastry, Dodballapur, India.*

Prove that the number of primitive Pythagorean triangles (integer-sided right triangles with relatively prime sides) with fixed inradius is always a power of 2.

**2013.** *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given is a convex $n$-gon $A_1A_2\ldots A_n$ ($n \ge 3$) and a point $P$ in its plane. Assume that the feet of the perpendiculars from $P$ to the lines $A_1A_2, A_2A_3, \ldots,$ $A_nA_1$ all lie on a circle with centre $O$.

(a) Prove that if $P$ belongs to the interior of the $n$-gon, so does $O$.

(b) Is the converse to (a) true?

(c) Is (a) still valid for nonconvex $n$-gons?

**2014.** *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Show that the polynomial

$$2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$$

has $x + y + z$ as a factor.

(b)* Is the remaining factor irreducible (over the complex numbers)?

**2015.** *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*

Prove that

$$(\sin A + \sin B + \sin C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{\pi},$$

where $A$, $B$, $C$ are the angles (in radians) of a triangle.

**2016.** *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Recall that $0.\overline{19}$ stands for the repeating decimal $0.19191919\ldots$, for example, and that the period of a repeating decimal is the number of digits in the repeating part. What is the period of

$$\text{(a) } 0.\overline{19} + 0.\overline{199}\,, \qquad \text{(b) } 0.\overline{19} \times 0.\overline{199}\,?$$

**2017.** *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

We are given a fixed circle $\kappa$ and two fixed points $A$ and $B$ not lying on $\kappa$. A variable circle through $A$ and $B$ intersects $\kappa$ in $C$ and $D$. Show that the ratio

$$\frac{AC \cdot AD}{BC \cdot BD}$$

is constant. [This is not a new problem. A reference will be given when the solution is published.]

**2018.** *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

How many permutations $(x_1, \ldots, x_n)$ of $\{1, \ldots, n\}$ are there such that the cyclic sum $\sum_{i=1}^{n} |x_i - x_{i+1}|$ (with $x_{n+1} = x_1$) is (a) a minimum, (b) a maximum?

**2019.** *Proposed by P. Penning, Delft, The Netherlands.*

In a plane are given a circle $C$ with a diameter $\ell$, and a point $P$ within $C$ but not on $\ell$. Construct the equilateral triangles that have one vertex at $P$, one on $C$, and one on $\ell$.

**2020.** *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

Let $a, b, c, d$ be **distinct** real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \qquad \text{and} \qquad ac = bd.$$

Find the maximum value of

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b}.$$

* * * * *

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1827**. [1993: 78; 1994: 57] *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D.M. Milošević, Pranjani, Yugoslavia.*

Let $a, b, c$ be the sides, $A, B, C$ the angles (measured in radians), and $s$ the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{bc}{A(s-a)} \geq \frac{12s}{\pi},$$

where the sums here and below are cyclic.

(ii)* It follows easily from the proof of *Crux* 1611 (see [1992: 62] and the correction on [1993: 79]) that also

$$\sum \frac{b+c}{A} \geq \frac{12s}{\pi}.$$

Do the two summations above compare in general?

III. *Solution to part* (i) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We employ the A.M.-G.M. inequality and obtain

$$\sum \frac{bc}{A(s-a)} \geq 3\sqrt[3]{\frac{(abc)^2}{ABC(s-a)(s-b)(s-c)}}. \tag{1}$$

Now, $ABC \leq [(A+B+C)/3]^3 = (\pi/3)^3$, whence by (1)

$$\sum \frac{bc}{A(s-a)} \geq \frac{9}{\pi}\sqrt[3]{\frac{(abc)^3}{(s-a)(s-b)(s-c)}} = \frac{9}{\pi}\sqrt[3]{\frac{16F^2R^2}{F^2/s}} = \frac{9}{\pi}\sqrt[3]{16R^2 s}, \tag{2}$$

where $F$ is the area and $R$ the circumradius of $\Delta ABC$, and we've used

$$abc = 4RF \quad \text{and} \quad F^2 = s(s-a)(s-b)(s-c).$$

Finally, using the known inequality $s \leq 3\sqrt{3}R/2$ (item 5.3 of Bottema et al, *Geometric Inequalities*) we obtain from (2)

$$\sum \frac{bc}{A(s-a)} \geq \frac{9}{\pi}\sqrt[3]{16\left(\frac{2s}{3\sqrt{3}}\right)^2 s} = \frac{12s}{\pi}.$$

*Editor's note.* Janous also proved the converse inequality

$$\sum \frac{bc}{A(s-a)} \leq \frac{1}{3}\left(\sum \frac{1}{A}\right)\left(s + \frac{(4R+r)^2}{s}\right),$$

by applying Chebyshev's inequality to

$$\frac{1}{A} \geq \frac{1}{B} \geq \frac{1}{C} \quad \text{and} \quad \frac{bc}{s-a} \leq \frac{ca}{s-b} \leq \frac{ab}{s-c},$$

and giving a fairly lengthy demonstration of the equality

$$\sum \frac{bc}{s-a} = s + \frac{(4R+r)^2}{s}. \tag{3}$$

Does anyone have a short proof of (3)?

*     *     *     *     *

**1843.** [1993: 140; 1994: 113] *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D.M. Milošević, Pranjani, Yugoslavia.*

Let $a, b, c$ be the sides, $A, B, C$ the angles (measured in radians), and $s$ the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{a}{2A(s-a)} \geq \frac{9}{\pi}.$$

(ii)* It is obvious that also

$$\sum \frac{1}{A} \geq \frac{9}{\pi}.$$

Do these two summations compare in general?

II. *Solution to part* (i) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The proof is almost the same as my proof of *Crux* 1827 (i), namely:

$$\begin{aligned}\sum \frac{a}{2A(s-a)} &= \frac{3}{2}\sqrt[3]{\frac{abc}{ABC(s-a)(s-b)(s-c)}} \\ &= \frac{3}{2\sqrt[3]{ABC}}\sqrt[3]{\frac{4RF}{Fr}} = \frac{3}{2\sqrt[3]{ABC}}\sqrt[3]{\frac{4R}{r}} \\ &\geq \frac{3}{2\sqrt[3]{ABC}}\sqrt[3]{4\cdot 2} = \frac{3}{\sqrt[3]{ABC}} \geq \frac{3}{(A+B+C)/3} = \frac{9}{\pi}.\end{aligned}$$

For further interpolation, we first recall that

$$\sum_{i=1}^{n} \frac{S-\lambda_i}{\lambda_i} x_i^2 \geq 2\sum_{i<j} x_i x_j \tag{1}$$

where $\lambda_1, \dots, \lambda_n > 0$, $S = \lambda_1 + \cdots + \lambda_n$, and $x_1, \dots, x_n$ are real numbers. Indeed, (1) is equivalent to

$$S\sum_{i=1}^{n} \frac{x_i^2}{\lambda_i} \geq 2\sum_{i<j} x_i x_j + \sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} \sqrt{\lambda_i}\cdot\frac{x_i}{\sqrt{\lambda_i}}\right)^2,$$

which is true by the Cauchy-Schwarz inequality. Now using (1) for $n = 3$ and

$$\lambda_1 = s - a, \quad \lambda_2 = s - b, \quad \lambda_3 = s - c,$$

$$x_1 = \frac{1}{\sqrt{A}}, \quad x_2 = \frac{1}{\sqrt{B}}, \quad x_3 = \frac{1}{\sqrt{C}},$$

we get (since $S = \lambda_1 + \lambda_2 + \lambda_3 = s$)

$$\sum \frac{a}{(s-a)A} \geq 2 \sum \frac{1}{\sqrt{BC}} \left( \overset{\text{A.M.-G.M.}}{\geq} \frac{6}{\sqrt[3]{ABC}} \geq \frac{18}{\pi} \right).$$

More generally, let us put

$$x_1 = \frac{1}{A^t}, \quad x_2 = \frac{1}{B^t}, \quad x_3 = \frac{1}{C^t},$$

where $t > 0$. Then we infer

$$\sum \frac{a}{(s-a)A^t} \geq 2 \sum \frac{1}{(BC)^{t/2}} \left( \geq \frac{6}{(ABC)^{t/3}} \geq \frac{2 \cdot 3^{t+1}}{\pi^t} \right).$$

Finally, we give the converse inequality

$$\sum \frac{a}{2A(s-a)} \leq \frac{1}{3} \left( \frac{2R}{r} - 1 \right) \cdot \sum \frac{1}{A}. \tag{2}$$

Because of symmetry we may and do assume $a \leq b \leq c$. Then

$$\frac{1}{A} \geq \frac{1}{B} \geq \frac{1}{C} \quad \text{and} \quad \frac{a}{s-a} \leq \frac{b}{s-b} \leq \frac{c}{s-c}.$$

Hence we get via Chebyshev's inequality that

$$\sum \frac{a}{2A(s-a)} \leq \frac{1}{6} \sum \frac{1}{A} \cdot \sum \frac{a}{s-a}.$$

But it is known (e.g. page 19 of [2] or item 22 p. 54 of [1]) that

$$\sum \frac{a}{s-a} = \frac{4R - 2r}{r}.$$

Thus (2) follows.

*References:*

[1] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities.*

[2] V.P. Soltan and S.J. Meidman, *Identities and Inequalities for Triangles*, (in Russian), Kishinev, 1982.

[*Editor's note.* Inequality (2) should be compared with the inequality arising from part (ii), namely

$$\sum \frac{1}{A} \leq \sum \frac{a}{2A(s-a)},$$

which was proved by Ardila and Kuczma, Ardila's very nice proof having appeared on [1994: 113]. All three summations are equal when $ABC$ is equilateral.]

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**1919.** [1994: 49] *Proposed by H.N. Gupta, University of Regina.*

In Lotto 6/49, six balls are randomly drawn (without replacement) from a bin holding balls numbered from 1 to 49. Find the expected value of the $k$th lowest number drawn, for each $k = 1, 2, \ldots, 6$.

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We consider more generally "Lotto $n/N$" (where $n$ balls are drawn from balls numbered from 1 to $N$, $n \le N$), and show that the expected value of the $k$th lowest number drawn equals

$$E(k) = \frac{k(N+1)}{n+1}, \qquad k = 1, 2, \ldots, n.$$

Indeed, $x$ is the $k$th lowest number for $\binom{x-1}{k-1}\binom{N-x}{n-k}$ draws (since for each of the $k-1$ numbers drawn from the $x-1$ numbers less than $x$ there are $n-k$ numbers drawn from the $N-x$ numbers greater than $x$). Thus, the probability that $x$ is the $k$th lowest number equals

$$\frac{\binom{x-1}{k-1}\binom{N-x}{n-k}}{\binom{N}{n}}, \qquad k \le x \le N-n+k,$$

so that

$$\sum_{x=k}^{N-n+k} \binom{x-1}{k-1}\binom{N-x}{n-k} = \binom{N}{n}.$$

Note that this argument provides a neat proof of the familiar identity

$$\sum_{x=b}^{a-c} \binom{x}{b}\binom{a-x}{c} = \binom{a+1}{b+c+1}.$$

It follows that

$$E(k) = \sum_{x=k}^{N-n+k} \frac{x\binom{x-1}{k-1}\binom{N-x}{n-k}}{\binom{N}{n}} = k\sum_{x} \frac{\binom{x}{k}\binom{N-x}{n-k}}{\binom{N}{n}} = \frac{k\binom{N+1}{n+1}}{\binom{N}{n}} = \frac{k(N+1)}{n+1},$$

as claimed.

In Lotto 6/49 the expected values are therefore

$$E(k) = \frac{50k}{7} = \frac{50}{7}, \quad \frac{100}{7}, \quad \ldots, \quad \frac{300}{7}.$$

*Editor's comments by Chris Fisher.* The proposer supplied a solution that was quite similar to our featured solution. He then showed that such an approach (together with a measure of perseverance) leads to a formula for the variance,

$$V(k) = \frac{k(n-k+1)}{n}V(1) = \frac{k(n-k+1)}{(n+1)^2(n+2)}(N+1)(N-n)\,.$$

He also presented a second, more elegant solution, but (as Fermat once said) there is not enough space here to reproduce it; he has promised to develop his ideas into a small note for *Crux.*

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; KEITH EKBLAW, Walla Walla, Washington; RICHARD I. HESS, Rancho Palos Verdes, California (two solutions); KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer (two solutions). One incorrect solution was received.*

*     *     *     *     *

**1920.** [1994: 49] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $a$, $b$, $c$ be the sides of a triangle.

(a) Prove that, for any $0 < \lambda \le 2$,

$$\frac{1}{(1+\lambda)^2} < \frac{(a+b)(b+c)(c+a)}{(\lambda a+b+c)(a+\lambda b+c)(a+b+\lambda c)} \le \left(\frac{2}{2+\lambda}\right)^3,$$

and that both bounds are best possible.

(b)* What are the bounds for $\lambda > 2$?

*Solution by Ji Chen, Ningbo University, China.*

Let

$$f(a,b,c) = \frac{(a+b)(b+c)(c+a)}{(\lambda a+b+c)(a+\lambda b+c)(a+b+\lambda c)};$$

then

$$f(1,1,1) = \left(\frac{2}{2+\lambda}\right)^3, \quad f(0,1,1) = \frac{1}{(1+\lambda)^2}, \quad f(2,1,1) = \frac{9}{(1+\lambda)(3+\lambda)^2}\,.$$

We claim that: when $\lambda > 1+\sqrt{13}$,

$$f(1,1,1) \le f(a,b,c) < f(0,1,1); \tag{1}$$

when $1+\sqrt{5} < \lambda \le 1+\sqrt{13}$,

$$f(2,1,1) < f(a,b,c) < f(0,1,1); \tag{2}$$

when $3 < \lambda \le 1 + \sqrt{5}$,

$$f(2,1,1) < f(a,b,c) \le f(1,1,1); \tag{3}$$

when $0 < \lambda \le 3$,

$$f(0,1,1) < f(a,b,c) \le f(1,1,1) \tag{4}$$

[this implies the inequality of part (a)]; and when $-1 \le \lambda < 0$,

$$f(1,1,1) \le f(a,b,c) < f(0,1,1). \tag{5}$$

Inequalities (1) to (5) are equivalent to certain cubic triangle inequalities. For example, the left inequality of (1) is equivalent to

$$\left(\frac{2}{2+\lambda}\right)^3 (\lambda a + b + c)(a + \lambda b + c)(a + b + \lambda c) \le (a+b)(b+c)(c+a)$$

[as long as $\lambda \ge -1$, all factors involved are nonnegative]. These cubic triangle inequalities all follow by a theorem of K. B. Stolarsky (Cubic triangle inequalities, *Amer. Math. Monthly* 78 (1971) 879–881).

*Editor's comments.* Here are some more details of this proof. Stolarsky's theorem referred to is: *if $P(x_1, x_2, x_3)$ is a real symmetric form of degree* 3, *and $P(1,1,1), P(1,1,0)$ and $P(2,1,1)$ are all nonnegative, then $P(a,b,c) \ge 0$* (where $a, b, c$ are the edges of a triangle).

So if we set

$$P_1(a,b,c) = \prod(b+c) - \left(\frac{2}{2+\lambda}\right)^3 \prod(\lambda a + b + c),$$

where the products are cyclic over $a, b, c$, we get

$$\begin{aligned} P_1(1,1,1) &= 8 - \left(\frac{2}{2+\lambda}\right)^3 (2+\lambda)^3 = 0\,, \\ P_1(1,1,0) &= 2 - \left(\frac{2}{2+\lambda}\right)^3 \cdot 2(1+\lambda)^2 = \frac{2\lambda(\lambda^2 - 2\lambda - 4)}{(2+\lambda)^3}\,, \\ P_1(2,1,1) &= 18 - \left(\frac{2}{2+\lambda}\right)^3 (2+2\lambda)(3+\lambda)^2 = \frac{2\lambda(\lambda^2 - 2\lambda - 12)}{(2+\lambda)^3}\,. \end{aligned}$$

Thus $P_1(1,1,0) \ge 0$ if $1 - \sqrt{5} \le \lambda \le 0$ or $\lambda \ge 1 + \sqrt{5}$, while $P_1(2,1,1) \ge 0$ if $1 - \sqrt{13} \le \lambda \le 0$ or $\lambda \ge 1 + \sqrt{13}$, so both $P_1(1,1,0)$ and $P_1(2,1,1)$ are nonnegative if $1 - \sqrt{5} \le \lambda \le 0$ or $\lambda \ge 1 + \sqrt{13}$. By Stolarsky's theorem this gives the left hand inequalities in (1) and (5). Also, both $P_1(1,1,0)$ and $P_1(2,1,1)$ are nonpositive if $0 \le \lambda \le 1 + \sqrt{5}$, so Stolarsky's theorem applied to $-P_1(a,b,c)$ yields the right hand inequalities of (3) and (4).

Similarly, letting

$$P_2(a,b,c) = \prod(b+c) - \frac{1}{(1+\lambda)^2} \prod(\lambda a + b + c),$$

we have

$$
\begin{aligned}
P_2(1,1,1) &= 8 - \frac{(2+\lambda)^3}{(1+\lambda)^2} = \frac{-\lambda(\lambda^2-2\lambda-4)}{(1+\lambda)^2}, \\
P_2(1,1,0) &= 2 - \frac{2(1+\lambda)^2}{(1+\lambda)^2} = 0, \\
P_2(2,1,1) &= 18 - \frac{(2+2\lambda)(3+\lambda)^2}{(1+\lambda)^2} = \frac{2\lambda(3-\lambda)}{(1+\lambda)},
\end{aligned}
$$

so all three of these are nonnegative when $0 \le \lambda \le 3$, and we get the left hand inequality in (4). By considering $-P_2$ we get the right hand inequalities of (1), (2) and (5). Finally, letting

$$P_3(a,b,c) = \prod(b+c) - \frac{9}{(1+\lambda)(3+\lambda)^2}\prod(\lambda a + b + c),$$

we have

$$
\begin{aligned}
P_3(1,1,1) &= 8 - \frac{9(2+\lambda)^3}{(1+\lambda)(3+\lambda)^2} = \frac{-\lambda(\lambda^2-2\lambda-12)}{(1+\lambda)(3+\lambda)^2}, \\
P_3(1,1,0) &= 2 - \frac{18(1+\lambda)^2}{(1+\lambda)(3+\lambda)^2} = \frac{2\lambda(\lambda-3)}{(3+\lambda)^2}, \\
P_3(2,1,1) &= 18 - \frac{9(2+2\lambda)(3+\lambda)^2}{(1+\lambda)(3+\lambda)^2} = 0,
\end{aligned}
$$

and all three are nonnegative when $3 \le \lambda \le 1+\sqrt{13}$, giving the left hand inequalities of (2) and (3).

For the convenience of the readers, here is Stolarsky's proof of the above theorem. Let

$$P(a,b,c) = A(a^3+b^3+c^3) + B(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2) + Cabc$$

where $A$, $B$, $C$ are constants, and define

$$c_1 = 2A+2B = P(1,1,0), \qquad c_2 = 3A+5B+C, \qquad c_3 = \frac{3}{2}c_1 + c_2,$$

$$c_6 = 3A+6B+C = P(1,1,1), \qquad c_4 = -\frac{3}{2}c_1 - c_2 + 4c_6, \qquad c_5 = 2c_6.$$

By hypothesis, $c_1 \ge 0$, $c_1 + c_2 = \frac{1}{2}P(2,1,1) \ge 0$, and $c_6 \ge 0$. Hence $c_3 \ge 0$ and $2c_3 + c_4 = c_3 + 4c_6 \ge 0$. Assume without loss of generality that $a \ge b$ and $a \ge c$. Then $a$, $b$, $c$ will vary over the side lengths of all triangles when

$$a = x+y+z, \qquad b = x+y, \qquad c = y+z,$$

and $x$, $y$, $z$ vary over all nonnegative real numbers. Also $a = b = c$ if and only if $x = z = 0$. The conclusion of the theorem now follows from the identity

$$
\begin{aligned}
P(a,b,c) = {}& [c_1(x+z)(x-z)^2 + (c_1+c_2)(x^2z+xz^2)] \\
& + [c_3(x-z)^2 + (2c_3+c_4)xz]y + [c_5(x+z)]y^2 + c_6y^3,
\end{aligned}
$$

since the right side is nonnegative. (The editor verified this identity using Maple. Evidently Stolarsky used a different method to find it!)

*Both parts also solved by GERD BARON, Technische Universität Wien, Austria; G. P. HENDERSON, Campbellcroft, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; and WALDEMAR POMPE, student, University of Warsaw, Poland. Part (a) only solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MURRAY S. KLAMKIN, University of Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

*Actually, Tsaoussoglou finds the correct bounds for* $0 < \lambda \le 3$. *Also, Klamkin conjectures the correct bounds for* $\lambda > 2$ *without proof.*

*Henderson in fact goes on to find lower and upper bounds corresponding to values of* $\lambda$ *less than* $-1$. *Letting* $g(a,b,c)$ *be the reciprocal of the given expression, that is,*

$$g(a,b,c) = \frac{(\lambda a + b + c)(a + \lambda b + c)(a + b + \lambda c)}{(a+b)(b+c)(c+a)},$$

*he obtains*

$$\frac{(1+\lambda)(3+\lambda)^2}{9} \le g(a,b,c) \le \left(\frac{2+\lambda}{2}\right)^3 \qquad \text{for } 1-\sqrt{5} \le \lambda \le -1,$$

$$\frac{(1+\lambda)(3+\lambda)^2}{9} \le g(a,b,c) \le (1+\lambda)^2 \qquad \text{for } 1-\sqrt{13} \le \lambda \le 1-\sqrt{5},$$

$$\left(\frac{2+\lambda}{2}\right)^3 \le g(a,b,c) \le (1+\lambda)^2 \qquad \text{for } \lambda \le 1-\sqrt{13}.$$

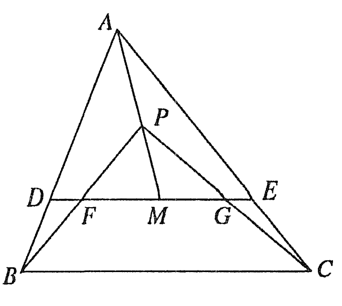
*This could now be proved as above, as the possibility that* $\lambda a + b + c$, *etc. may be negative no longer causes problems when passing to the symmetric form of degree* 3. *Both Henderson and Konečný solved the problem for the reciprocal* $g(a,b,c)$ *rather than the proposer's original expression.*

* * * * *

**1921.** [1994: 74] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$D$ and $E$ are points on sides $AB$ and $AC$ of a triangle $ABC$ such that $DE \| BC$, and $P$ is an interior point of $\Delta ADE$. $PB$ and $PC$ meet $DE$ at $F$ and $G$ respectively. Let $O_1$ and $O_2$ be the circumcenters of $\Delta PDG$ and $\Delta PFE$ respectively. Prove that $AP \perp O_1O_2$.

*Solution by Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain.*




In order to prove that $AP \perp O_1O_2$, it suffices to show that $AP$ is the radical axis of the circles $(O_1)$ and $(O_2)$, because in this case $AP$ would be perpendicular to the line of centers $O_1O_2$. Suppose $AP$ intersects $DE$ at the point $M$. The power of $M$ with respect to $(O_1)$ is $MD \cdot MG$, and with respect to $(O_2)$ is $MF \cdot ME$. Therefore we must show $MD \cdot MG = MF \cdot ME$, or

$$\frac{MD}{MF} = \frac{ME}{MG}\,. \tag{1}$$

By the Menelaus theorem applied to triangle $AME$ with the line $CGP$,

$$\frac{CE}{EA} \cdot \frac{AP}{PM} \cdot \frac{MG}{GE} = 1; \tag{2}$$

and the same theorem applied to triangle $AMD$ with the line $BFP$ gives us

$$\frac{BD}{DA} \cdot \frac{AP}{PM} \cdot \frac{MF}{FD} = 1. \tag{3}$$

Moreover, $AE/EC = AD/BD$, and therefore from (2) and (3) we obtain $MG/GE = MF/FD$. By a property of proportions, this means $MG/ME = MF/MD$, and (1) follows.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; ASHISH KR. SINGH, Kanpur, India; and the proposer.*

* * * * *

**1922.** [1994: 74] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
The function $f$ is defined on nonnegative integers by: $f(0) = 0$ and

$$f(2n+1) = 2f(n) \quad \text{for } n \ge 0, \qquad f(2n) = 2f(n) + 1 \quad \text{for } n \ge 1.$$

(a) Let $g(n) = f(f(n))$. Show that $g(n - g(n)) = 0$ for all $n \ge 0$.

(b) For any $n \ge 1$, let $r(n)$ be the least integer $r$ such that $f^r(n) = 0$ (where $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n))$, etc.). Compute

$$\liminf_{n\to\infty} \frac{n}{2^{r(n)}}\,.$$

*Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

(a) Most of the solution can be worked out suitably in binary notation. To prepare for this, we first prove the following lemma.

LEMMA. Let

$$L(n) = 2^{\lfloor \log_2(2n) \rfloor}, \quad n \ge 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\le x$. Then $f(n) = L(n) - n - 1$.

*Proof.* Since

$$1 + \lfloor \log_2(2n) \rfloor = \lfloor 1 + \log_2(2n) \rfloor = \lfloor \log_2(4n) \rfloor,$$

we have $2L(n) = L(2n)$. Furthermore, we claim that $L(2n+1) = L(2n)$. Suppose $\lfloor \log_2(2n) \rfloor < \lfloor \log_2(2n+1) \rfloor$. Then letting $\lfloor \log_2(2n) \rfloor = k$, we have

$\log_2(2n) < k+1 \leq \log_2(2n+1)$ which implies $2n < 2^{k+1} \leq 2n+1$ or $n < 2^k \leq n + 1/2$, clearly an impossibility.

Since $f(1) = 0$ and $L(1) = 2$, $f(n) = L(n) - n - 1$ holds for $n = 1$. Assume the formula holds up to $2n-1$ for some $n \geq 1$. Then

$$f(2n+1) = 2f(n) = 2L(n)-2n-2 = L(2n)-2n-2 = L(2n+1)-(2n+1)-1$$

and

$$f(2n) = 2f(n) + 1 = L(2n) - 2n - 1.$$

This completes the proof of the lemma. □

Since $g(0) = f(f(0)) = f(0) = 0$, $g(0-g(0)) = 0$. Assume henceforth that $n \geq 1$. The binary representation of $n$ can be written in the form

$$n = \underbrace{1 \cdots 1}_{s}\ \underbrace{0 \cdots 0}_{t}\ \underbrace{1* \cdots *}_{u},$$

where each $*$ is either 0 or 1, and $s \geq 1$, $t, u \geq 0$. Since $L(n)-1$ is just a string of 1's in binary notation, and since $n + f(n) = L(n) - 1$ by the lemma, we see that $f$ acts on $n$ by interchanging all 1's and 0's in the binary expansion of $n$. In other words,

$$f(n) = \underbrace{1 \cdots 1}_{t}\ \underbrace{0*' \cdots *'}_{u}$$

where each $*'$ is $1 - *$. Thus,

$$g(n) = f(f(n)) = \underbrace{1* \cdots *}_{u};$$

i.e., $g(n)$ is the original final block of $u$ digits in $n$. As a result,

$$n - g(n) = \underbrace{1 \cdots 1}_{s}\ \underbrace{0 \cdots 0}_{t+u}.$$

Since the last expression contains no final block of digits after the initial blocks of 1's and 0's, we have $g(n - g(n)) = 0$ for all $n \geq 1$.

(b) The answer is $2/3$. To see this, it is necessary to examine more closely the blocks of binary digits of $n$. Since

$$n = \begin{cases} \underbrace{1 \cdots 1}_{s_1}\ \underbrace{0 \cdots 0}_{s_2} \cdots \underbrace{1 \cdots 1}_{s_m} & \text{if } m \text{ is odd,} \\ \underbrace{1 \cdots 1}_{s_1}\ \underbrace{0 \cdots 0}_{s_2} \cdots \underbrace{0 \cdots 0}_{s_m} & \text{if } m \text{ is even,} \end{cases}$$

and since $f$ always wipes out the leading block of 1's, it is clear that $r(n) = m$. Since $n$ has at least $r(n)$ digits and $2^{r(n)}$ has $1 + r(n)$ digits, $n/2^{r(n)}$ will exceed 1 unless $n$ has exactly $r(n)$ digits; i.e., unless $n$ is in the sequence (in base 2) $1, 10, 101, 1010, 10101, \ldots$ . So the limit inferior of $n/2^{r(n)}$ is the limit of the sequence (in base 2)

$$\frac{1}{10}, \quad \frac{10}{10^2}, \quad \frac{101}{10^3}, \quad \ldots,$$

which is (in base 10)

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{2}{3}.$$

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; LEROY F. MEYERS, The Ohio State University, Columbus; PAUL PENNING, Delft, The Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's; C. WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Part (a) only solved by FRANCISCO PIMENTEL, Fortaleza, Brazil; and ASHISH KR. SINGH, Kanpur, India.*

*The binary representation approach was also used by Hess, Penning, Pye, and the proposer.*

* * * * *

**1923.** [1994: 74] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

In triangle $ABC$, cevians $AD, BE, CF$ are equal and concur at point $P$. Prove that

$$PA + PB + PC = 2(PD + PE + PF).$$

*Solution by Himadri Choudhury, student, Hunter High School, New York.*

Let $AD = BE = CF = x$. Note that

$$\frac{PD}{AD} = \frac{[BPC]}{[ABC]}, \quad \frac{PE}{BE} = \frac{[APC]}{[ABC]}, \quad \frac{PF}{CF} = \frac{[APB]}{[ABC]},$$

where $[XYZ]$ denotes the area of triangle $XYZ$. Summing, we get

$$\frac{PD + PE + PF}{x} = \frac{[ABC]}{[ABC]} = 1,$$

so $PD + PE + PF = x$. Now note that

$$AP = x - PD = PE + PF,$$

$$BP = x - PE = PD + PF,$$

$$CP = x - PF = PD + PE.$$

Summing these we get

$$AP + BP + CP = 2(PD + PE + PF)$$

as desired.

*Also solved (often the same way) by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.*

*Heuver and Klamkin mention that similar results hold in higher dimensions. For example, if equal length cevians $AE, BF, CG, DH$ of a tetrahedron $ABCD$ concur at $P$, then*

$$PA + PB + PC + PD = 3(PE + PF + PG + PH),$$

*which can be proved as above.*

*Penning notes the result, "shown a long time ago", that every (nonequilateral) triangle contains two real points P satisfying the condition of the problem and which do not lie on a side. He gives some references, including:*

*O. Bottema, "On some remarkable points of a triangle",* Nieuw Archief voor Wiskunde **19** (1971) 46–57.

*G. R. Veldkamp, "Gelijke, concurrente hoektransversalen in een driehoek",* Nieuw Tijschrift voor Wiskunde 59 (1971) 67–76.

*The Veldkamp article also includes further references.*

* * * * *

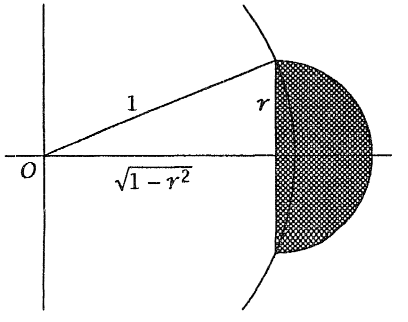
**1924.** [1994: 74] *Proposed by Jisho Kotani, Akita, Japan.*

A large sphere of radius 1 and a smaller sphere of radius $r < 1$ overlap so that their intersection is a circle of radius $r$, i.e., a great circle of the small sphere. Find $r$ so that the volume inside the small sphere and outside the large sphere is as large as possible.

*Solution by Robert P. Sealy, Mount Allison University, Sackville, New Brunswick.*

The answer is $r = 2/\sqrt{5}$.

We may consider the larger sphere as the surface generated by revolving the curve $x^2 + y^2 = 1$ around the $x$-axis. Then $V$, the volume inside the smaller sphere but outside the larger sphere, is given by




$$V = \frac{2}{3}\pi r^3 - \pi\int_{\sqrt{1-r^2}}^{1}(1 - x^2)\,dx.$$

Then [by the Fundamental Theorem of Calculus]

$$\begin{aligned}\frac{dV}{dr} &= 2\pi r^2 + \pi\frac{d}{dr}\left(\int_1^{\sqrt{1-r^2}}(1-x^2)\,dx\right)\\ &= 2\pi r^2 + \pi(1-(1-r^2))\frac{d}{dr}\sqrt{1-r^2}\\ &= 2\pi r^2 + \pi r^2\frac{-r}{\sqrt{1-r^2}} = 2\pi r^2 - \frac{\pi r^3}{\sqrt{1-r^2}}\,,\end{aligned}$$

which equals 0 if $2\sqrt{1-r^2} = r$ or $r = 2/\sqrt{5}$. Apply the first derivative test to see that we have obtained a maximum.

*Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; FRANCISCO PIMENTEL, Fortaleza, Brazil; and the proposer. One incorrect solution was sent in.*

*Sealy was the only solver to use the Fundamental Theorem of Calculus. Almost all other solvers either evaluated the integral (which of course is easy), or already knew the formula for the volume of a spherical cap, and then differentiated.*

* * * * *

**1925.** [1994: 74] *Proposed by Ignotus, Godella, Spain.*

Let $n$ be a $k$-digit positive integer and let $v(n)$ be the set of $k$ "right-truncations" of $n$: for example, $v(1994) = \{1994, 199, 19, 1\}$. Show that there are infinitely many $n$ such that $v(n)$ is a complete set of residues mod $k$. (This problem was inspired by *Crux* problems 1884 and 1886 [1994: 231, 235].)

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The infinitely many $n$'s come up via $k = 10, 100, 1000, \ldots$ by the following procedure (exemplified for $k = 10$ and 100):

$$\begin{aligned}k = 10: &\quad n = 9876543210\\ k = 100: &\quad n = 99889787796867669585756559484746454493837363\,5\\ &\qquad\;\; 34333928272625242322918171615141312119080706\,05\\ &\qquad\;\; 0403020100.\end{aligned}$$

*Editor's note.* What Walther has done for $k = 100$ is create a 100-digit number whose 99 2-digit substrings $(99, 98, 88, 89, 97, 78$, etc) are all different, with the only 2-digit string missing being 09, which is handled by the initial 9 of the number. Thus the right-truncations of $n$ modulo 100 will account for all numbers from 0 to 99, each occurring exactly once. The proposer used the same idea, though his (her?) 100-digit number $n$ was different. Neither (understandably!) gave an example for $k = 1000$, and the editor isn't quite convinced that examples exist for every $k = 10^t$ because of the larger overlap involved among the $t$-digit substrings of $n$ as $t$ gets larger. Can some reader supply a neat proof of this?

*Also solved by the proposer. One other reader apparently misunderstood the problem, which the editor fears wasn't stated as clearly as it might have been.*

*The proposer (an enigmatic and well-travelled figure) composed the problem and his solution "on a shaky train". Both he and Janous wonder about solutions where k is not a power of 10. Erudite colleague Richard Guy notes that the "number"*

$$0\,1\,\underbrace{0\,0\,\ldots\,0}_{p-2}$$

*is a "solution" whenever 10 is a primitive root of p (for example, p = 7). He mentions Artin's conjecture that if g is a positive integer which is not a square, then there are infinitely many primes p so that g is a primitive root of p. (This conjecture turns up in* F9, *page* 248 *of Guy's* Unsolved Problems in Number Theory, *Second Edition, Springer,* 1994.*)*

* * * * *

**1926.** [1994: 74] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

On sides $BC, CA, AB$ of $\Delta ABC$ are chosen points $A_1, B_1, C_1$ respectively, such that $\Delta A_1B_1C_1$ is equilateral. Let $o_1, o_2, o_3$ and $O_1, O_2, O_3$ be respectively the incircles and incentres of triangles $AC_1B_1, BA_1C_1, CB_1A_1$. If $O_1C_1 = O_2C_1$, show that

(a) $B_1O_3 = B_1O_1$ and $A_1O_2 = A_1O_3$;

(b) three external common tangents to the pairs of circles $o_1, o_2$; $o_2, o_3$; $o_3, o_1$, different from the sides of $\Delta ABC$, have a common point.

*Combination of solutions by Federico Ardila, student, MIT, Cambridge, Massachusetts; and the proposer.*

(a) Since $C_1O_1 = C_1O_2$ and

$$\begin{aligned}\angle O_1C_1O_2 &= \angle O_1C_1B_1 + 60^\circ + \angle O_2C_1A_1 \\ &= \frac{1}{2}(\angle AC_1B_1 + 120^\circ + \angle BC_1A_1) = \frac{1}{2}(240^\circ) = 120^\circ, \qquad (1)\end{aligned}$$

$O_2$ is the clockwise rotation of $O_1$ with center $C_1$ and angle of rotation $120^\circ$ (assuming points $A, B, C$ are oriented counterclockwise around triangle $ABC$). Therefore since triangle $A_1B_1C_1$ is equilateral, the clockwise rotation of point $O_2$ with center $A_1$ through an angle of $120^\circ$ and the counterclockwise rotation of point $O_1$ with center $B_1$ through an angle of $120^\circ$ must coincide. An easy way to see this is to construct exterior equilateral triangles upon the sides of triangle $A_1B_1C_1$, and consider the relative positions of points $O_1, O_2, O_3$ with respect to these three triangles. [*Editor's note.* See Figure 1. Rotate equilateral triangle $A'B_1C_1$ 120° about $C_1$, and it coincides with $\Delta A_1B'C_1$, with $O_1$ coinciding with $O_2$. Similarly, when $\Delta A_1B'C_1$ is rotated 120° about $A_1$ it coincides with $\Delta A_1B_1C'$, and if $\Delta A_1B_1C'$ is rotated 120° about $B_1$ it coincides with $\Delta A'B_1C_1$; thus the point that $O_2$ goes to under the former rotation must go to $O_1$ under the latter. Reversing the latter rotation gives the result.]

Call this point of "coincidence" $Q$. Clearly $Q$ satisfies

$$\angle O_1B_1Q = \angle O_2A_1Q = 120°$$

and the above condition determines $Q$ uniquely. But now notice that $O_3$ satisfies the above condition, because analogously to (1)

$$\angle O_1B_1O_3 = \angle O_2A_1O_3 = 120°.$$

Therefore $Q$ and $O_3$ must be the same point. But $Q$ satisfies

$$O_1B_1 = B_1Q \qquad \text{and} \qquad O_2A_1 = A_1Q$$

whence

$$O_1B_1 = B_1O_3 \qquad \text{and} \qquad O_2A_1 = A_1O_3$$

as we wished to prove.

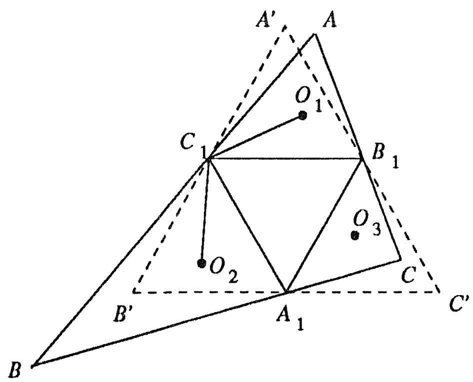



Figure 1

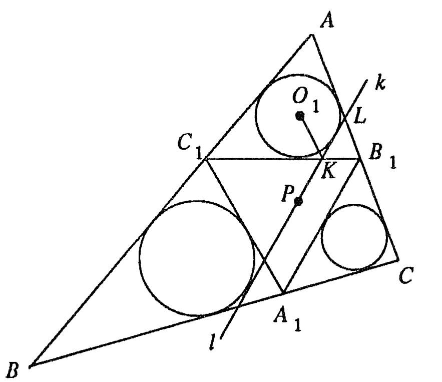



Figure 2

(b) Notice that since $\angle O_2C_1A_1 + \angle O_1C_1B_1 = 60°$ and $O_2C_1 = O_1C_1$, the reflection of point $O_2$ around $A_1C_1$ and the reflection of point $O_1$ around $B_1C_1$ are the same point, say $P$. Analogously, $P$ is also the reflection of $O_3$ around $A_1B_1$. We will show that $P$ is a common point of the three given tangents. Let $k$ be a tangent to $o_1$ passing through $P$ and intersecting the segments $B_1C_1$ and $B_1A$ in points $K$ and $L$ respectively (see Figure 2). Note that

$$\angle LKO_1 = \angle O_1KC_1 = \angle C_1KP \quad \text{and} \quad \angle LKO_1 + \angle O_1KC_1 + \angle C_1KP = 180°\,,$$

which means that $\angle PKC_1 = 60° = \angle A_1B_1C_1$. Therefore the lines $k$ and $A_1B_1$ are parallel. Let $l$ be a tangent to $o_2$ passing through $P$ and intersecting segments $A_1C_1$ and $A_1B$. The same proof shows that $l$ and $A_1B_1$ are parallel. Therefore $k = l$. Hence, we have shown that the common external tangent to $o_1$ and $o_2$ that is not a side of $\Delta ABC$ passes through $P$. An analogous proof shows that the other external tangents defined in (b) pass through $P$.

*Also solved by TOSHIO SEIMIYA, Kawasaki, Japan; and (part* (a) *only) D. J. SMEENK, Zaltbommel, The Netherlands.*

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**1927.** [1994: 74] *Proposed by Rolf Kline, Edmonton, Alberta.*

Suppose that, for three consecutive years, a certain provincial government reduces what it spends annually on education. The percentage decreases year by year are $a$, $b$ and $c$ percent, where $a, b, c$ are positive integers in arithmetic progression. Suppose also that the amounts (in dollars) the government spends on education during these same three years are three positive integers in harmonic progression. Find $a$, $b$ and $c$.

*Solution by Tim Cross, Wolverley High School, Kidderminster, U. K.*

The government spending in each year is given by:

| Year | Spending |
|---|---|
| 0 | $N$ |
| 1 | $N_1 = N\left(1 - \frac{a}{100}\right)$ |
| 2 | $N_2 = N\left(1 - \frac{a}{100}\right)\left(1 - \frac{b}{100}\right)$ |
| 3 | $N_3 = N\left(1 - \frac{a}{100}\right)\left(1 - \frac{b}{100}\right)\left(1 - \frac{c}{100}\right)$ |

Now $a$, $b$, $c$ in arithmetic progression implies

$$c = 2b - a,$$

and $N_1, N_2, N_3$ in harmonic progression implies

$$\frac{1}{N_2} = \frac{1}{2}\left(\frac{1}{N_1} + \frac{1}{N_3}\right).$$

Cancelling the factor $N(1 - \frac{a}{100})$ in the denominators of the second equation yields

$$\frac{2}{1 - \frac{b}{100}} = 1 + \frac{1}{(1 - \frac{b}{100})(1 - \frac{c}{100})} = 1 + \frac{1}{(1 - \frac{b}{100})(1 - \frac{2b-a}{100})},$$

which implies

$$2\left(1 - \frac{2b - a}{100}\right) = \left(1 - \frac{b}{100}\right)\left(1 - \frac{2b - a}{100}\right) + 1,$$

$$200(100+a-2b) = (100-b)(100+a-2b) + 10000,$$

and thus

$$(100+a-2b)(100+b) = 10000.$$

Since $1 \le a, b \le 99$ we have $101 \le 100+b \le 199$, and the only (positive integer) factorization of 10000 with one factor between 101 and 199 is $80 \times 125$. Hence $100+b = 125$ and $100+a-2b = 80$, so

$$\mathbf{b = 25}, \qquad \mathbf{a = 30}, \qquad \mathbf{c = 20}.$$

It may be of interest to note that a unique solution also exists in the case when $N_1$, $N_2$ and $N_3$ (the three annual amounts spent) are in *arithmetic* progression. Since in this case $N_2 - N_1 = N_3 - N_2$ we have, again cancelling a factor of $N(1 - \frac{a}{100})$ throughout,

$$\frac{b}{100} = 1 - \frac{b}{100} - 1 = \left(1 - \frac{b}{100}\right)\left(1 - \frac{c}{100} - 1\right) = \left(1 - \frac{b}{100}\right)\left(\frac{2b-a}{100}\right),$$

which implies

$$100b = (100-b)(2b-a)$$

and hence

$$2b^2 - (100+a)b + 100a = 0.$$

For $b$ to be rational we must have

$$(100+a)^2 - 4 \cdot 2 \cdot 100a = a^2 - 600a + 10000 = k^2$$

for some integer $k$, i.e.,

$$(300-a)^2 - 80000 = k^2. \tag{1}$$

As $1 \le a \le 99$ and $300 - a \ge \sqrt{80000}$, $a$ must be at most 17. Trial gives the only suitable value of $a$ to be 15, whence $2b^2 - 115b + 1500 = 0$ and so $b = 20$ (being an integer) and $c = 25$. [*Editor's note.* Here is an alternative way to get $a = 15$. Since $a \ge 1$, from (1) we get $k^2 \le 299^2 - 80000 = 9401$, so $k \le 96$. Rewriting (1) as

$$(300-a-k)(300-a+k) = 80000,$$

we see that we need to factor 80000 into two factors which differ by at most 192. The only such factorization is $80000 = 250 \times 320$, which yields $k = 35$ and $a = 15$.]

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, State University of New York, Oneonta; LEROY F. MEYERS, The Ohio State University, Columbus;*

*FRANCISCO PIMENTEL, Fortaleza, Brazil; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

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**1929.** [1994: 75] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Define a sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 6$ and

$$a_{n+1} = \left\lfloor \frac{5}{4}a_n + \frac{3}{4}\sqrt{a_n^2 - 12} \right\rfloor$$

for all $n \geq 1$, where $\lfloor \quad \rfloor$ denotes the greatest integer function. Prove that $a_n \equiv 1 \bmod 10$ for all $n \geq 2$.

*Solution by Bill Correll Jr., student, Denison University, Granville, Ohio.*

$a_2$ is easily seen to be 11. Suppose $a_n = 10k + 1$ for some positive integer $k$. Then we have

$$10k + 1 > \sqrt{100k^2 + 20k - 11} = \sqrt{a_n^2 - 12} > 10k,$$

which implies

$$\frac{30k+3}{4} > \frac{3}{4}\sqrt{a_n^2 - 12} > \frac{30k}{4},$$

$$\frac{50k+5}{4} + \frac{30k+3}{4} > \frac{5}{4}a_n + \frac{3}{4}\sqrt{a_n^2 - 12} > \frac{50k+5}{4} + \frac{30k}{4} = \frac{80k+5}{4},$$

and finally

$$20k + 2 > \frac{5}{4}a_n + \frac{3}{4}\sqrt{a_n^2 - 12} > 20k + 1.$$

Thus

$$a_{n+1} = 20k + 1.$$

This completes the proof by induction.

*Also solved by H. L. ABBOTT, University of Alberta; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; FRANCISCO PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; CORY PYE, student, Memorial University of Newfoundland, St. John's; ASHISH KR. SINGH, Kanpur, India; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo,* *Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH WILKE, Topeka, Kansas; and the proposer.*

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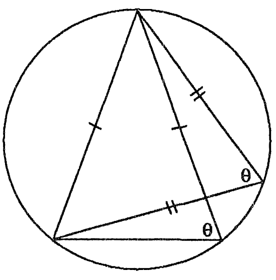
**1930.** [1994: 75] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

$T_1$ is an isosceles triangle with circumcircle $K$. Let $T_2$ be another isosceles triangle inscribed in $K$ whose base is one of the equal sides of $T_1$ and which overlaps the interior of $T_1$. Similarly create isosceles triangles $T_3$ from $T_2$, $T_4$ from $T_3$, and so on. Do the triangles $T_n$ approach an equilateral triangle as $n \to \infty$?

*Solution by Himadri Choudhury, student, Hunter High School, New York.*

Note that the base angle of $T_n$ is equal to the angle opposite the base of $T_{n+1}$ (as the figure indicates). Therefore if $\theta$ is the base angle for $T_n$ then the base angle for the next triangle ($T_{n+1}$) is




$$\frac{180^\circ - \theta}{2} = 90^\circ - \frac{\theta}{2}\,.$$

Suppose now that $\theta$ is the base angle for $T_1$. Then the base angle for $T_n$ is

$$90 - \frac{90}{2} + \frac{90}{4} - \frac{90}{8} + \cdots + (-1)^{n-2}\frac{90}{2^{n-2}} + (-1)^{n-1}\frac{\theta}{2^{n-1}}\,.$$

Note that the limit as $n \to \infty$ of the above is

$$\frac{90}{1 + 1/2} = 60^\circ$$

by the formula for the sum of an infinite geometric series. Since each $T_n$ is isosceles, the angles of $T_n$ do approach $60^\circ$ as $n \to \infty$.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; FRANCISCO PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; D. J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

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