Dissecting Triangles into Isosceles Triangles

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Problem 2 of Part II of the 1993-1994 Alberta High School Mathematics Competition (see *CRUX* [20:65]) goes as follows:

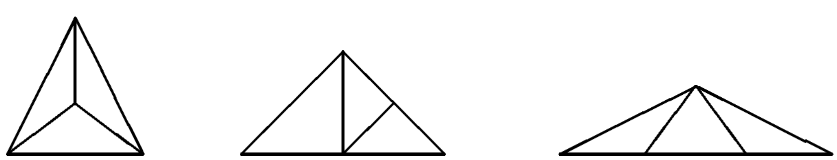
> *An isosceles triangle is called an* **amoeba** *if it can be divided into two isosceles triangles by a straight cut. How many different (that is, not similar) amoebas are there?*

All three authors wrote that contest. Afterwards, they felt that the problem would have been more meaningful had they been asked to cut non-isosceles triangles into isosceles ones.

We say that a triangle is **n-dissectible** if it can be dissected into n isosceles triangles where n is a positive integer. Since we are primarily interested in the minimum value of n, we also say that a triangle is **n-critical** if it is n-dissectible but not m-dissectible for any $m < n$. The isosceles triangles themselves are the only ones that are 1-dissectible, and of course 1-critical.

Note that, in the second definition, we should not replace "not m-dissectible for any $m < n$" by "not $(n-1)$-dissectible". It may appear that if a triangle is n-dissectible, then it must also be m-dissectible for all $m > n$. However, there are two exceptions. The solution to the contest problem, which motivated this study, shows that almost all 1-dissectible triangles are not 2-dissectible. We will point out later that some 2-dissectible ones are not 3-dissectible.

On the other hand, it is easy to see that all 1-dissectible triangles are 3-dissectible. Figure 1 illustrates the three cases where the vertical angle is acute, right, and obtuse respectively.



(a) (b) (c)

Figure 1.

We now find all 2-dissectible triangles. Clearly, such a triangle can only be cut into two triangles by drawing a line from a vertex to the opposite side, as illustrated in Figure 2.

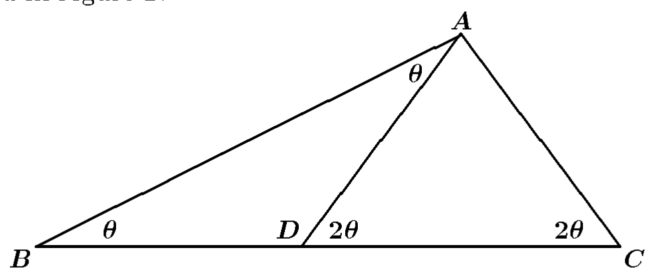


Figure 2.

Note that at least one of $\angle ADB$ and $\angle ADC$ is non-acute. We may assume that $\angle ADB \geq 90°$. In order for BAD to be an isosceles triangle, we must have $\angle BAD = \angle ABD$. Denote their common value by θ. By the Exterior Angle Theorem, $\angle ADC = 2\theta$. There are three ways in which CAD may become an isosceles triangle.

Case 1.
Here $\angle ACD = \angle ADC = 2\theta$, as illustrated in Figure 2. Then $\angle CAD = 180° - 4\theta > 0°$. This class consists of all triangles in which two of the angles are in the ratio 1:2, where the smaller angle θ satisfies $0° < \theta < 45°$. Of these, only the $(36°, 72°, 72°)$ triangle is 1-dissectible, but it turns out that every triangle here is 3-dissectible.

Case 2.
Here, $\angle CAD = \angle ADC = 2\theta$. Then $\angle CAB = 3\theta$ and $\angle ACD = 180° - 4\theta > 0°$. This class consists of all triangles in which two of the angles are in the ratio 1:3, where the smaller angle θ satisfies $0° < \theta < 45°$. Of these, only the $(36°, 36°, 108°)$ and the $(\frac{180}{7}°, \frac{540}{7}°, \frac{540}{7}°)$ triangles are 1-dissectible. It turns out that those triangles for which $30° < \theta < 45°$, with a few exceptions, are not 3-dissectible.

Case 3.
Here, $\angle ACD = \angle CAD$. Then their common value is $90° - \theta$ so that $\angle CAB = 90°$. This class consists of all right triangles. Of these, only the $(45°, 45°, 90°)$ triangle is 1-dissectible, and it turns out that every triangle here is 3-dissectible.

We should point out that while our three classes of 2-dissectible triangles are exhaustive, they are not mutually exclusive. For instance, the $(30°, 60°, 90°)$ triangle appears in all three classes, with the same dissection. The $(20°, 40°, 120°)$ triangle is the only one with two different dissections.

We now consider 3-dissectible triangles. Suppose one of the cuts does not pass through any vertices. Then it divides the triangle into a triangle and a quadrilateral. The latter must then be cut into two triangles, and this cut must pass through a vertex. Hence at least one cut passes through a vertex.

Suppose no cut goes from a vertex to the opposite side. The only possible configuration is the one illustrated in Figure 1(a). Since the three angles at this point sum to $360°$, at least two of them must be obtuse. It follows that the three arms have equal length and this point is the circumcentre of the original triangle. Since it is an interior point, the triangle is acute. Thus all acute triangles are 3-dissectible.

In all other cases, one of the cuts go from a vertex to the opposite side, dividing the triangle into an isosceles one and a 2-dissectible one. There are quite a number of cases, but the argument is essentially an elaboration of that used to determine all 2-dissectible triangles. We leave the details to the reader, and will just summarize our findings in the following statement.

Theorem.
A triangle is 3-dissectible if and only if it satisfies at least one of the following conditions:

1. It is an isosceles triangle.
2. It is an acute triangle.
3. It is a right triangle.
4. It has a $45°$ angle.
5. It has one of the following forms:
   (a) $(\theta,\ 90° - 2\theta,\ 90° + \theta)$, $0° < \theta < 45°$;
   (b) $(\theta,\ 90° - \frac{3\theta}{2},\ 90° + \frac{\theta}{2})$, $0° < \theta < 60°$;
   (c) $(\theta,\ 360° - 7\theta,\ 6\theta - 180°)$, $30° < \theta < 45°$;
   (d) $(2\theta,\ 90° - \frac{3\theta}{2},\ 90° - \frac{\theta}{2})$, $0° < \theta < 60°$;
   (e) $(3\theta,\ 90° - 2\theta,\ 90° - \theta)$, $0° < \theta < 45°$;
   (f) $(180° - 2\theta,\ 180° - \theta,\ 3\theta - 180°)$, $60° < \theta < 90°$;
   (g) $(180° - 4\theta,\ 180° - 3\theta,\ 7\theta - 180°)$, $30° < \theta < 45°$.
6. Two of its angles are in the ratio $p{:}q$, with the smaller angle strictly between $0°$ and $r°$, for the following values of p, q and r:

p	1	1	1	1	1	1	2	3	3
q	2	3	4	5	6	7	3	4	5
r	60	30	22.5	30	22.5	22.5	60	67.5	67.5

The fact that all right triangles are 2-dissectible is important because every triangle can be divided into two right triangles by cutting along the altitude to its longest side. Each can then be cut into two isosceles triangles by cutting along the median from the right angle to the hypotenuse, as illustrated in Figure 3. It follows that all triangles are 4-dissectible, and that there are no n-critical triangles for $n \geq 5$.

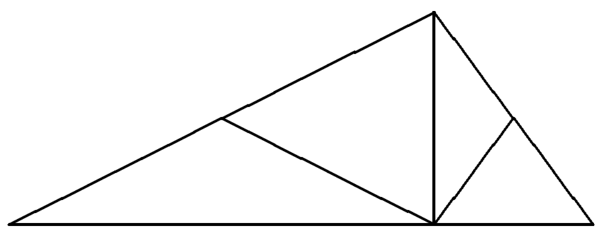


Figure 3.

We can prove by mathematical induction on n that all triangles are n-dissectible for all $n \geq 4$. We have established this for $n = 4$. Assume that all triangles are n-dissectible for some $n \geq 4$. Consider any triangle. Divide it by a line through a vertex into an isosceles triangle and another one. By the induction hypothesis, the second can be dissected into n triangles. Hence the original triangle is $(n + 1)$-dissectible.

Announcement

For the information of readers, we are saddened to note the deaths of two mathematicians who are well-known to readers of *CRUX*.

Leroy F. Meyers died last November. He was a long time contributor to *CRUX*. See the February 1996 issue of the *Notices of the American Mathematical Society*.

D.S. Mitrinović died last March. He is well-known for his work on Geometric Inequalities, in particular, being co-author of *Geometric Inequalities* and *Recent Advances in Geometric Inequalities*. See the July 1995 issue of the *Notices of the American Mathematical Society*.

THE SKOLIAD CORNER

No. 13

R.E. Woodrow

This issue we feature the Eleventh W.J. Blundon Contest, written February 23, 1994. This contest is sponsored by the Department of Mathematics and Statistics of Memorial University of Newfoundland and is one of the "Provincial" contests that receives support from the Canadian Mathematical Society

THE ELEVENTH W.J. BLUNDON CONTEST

February 23, 1994

1. (a) The lesser of two consecutive integers equals 5 more than three times the larger integer. Find the two integers.

(b) If $4 \leq x \leq 6$ and $2 \leq y \leq 3$, find the minimum values of $(x-y)(x+y)$.

2. A geometric sequence is a sequence of numbers in which each term, after the first, can be obtained from the previous term, by multiplying by the same fixed constant, called the **common ratio**. If the second term of a geometric sequence is 12 and the fifth term is $81/2$, find the first term and the common ratio.

3. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.

4. $ABCD$ is a square. Three parallel lines l_1, l_2 and l_3 pass through A, B and C respectively. The distance between l_1 and l_2 is 5 and the distance between l_2 and l_3 is 7. Find the area of $ABCD$.

5. The sum of the lengths of the three sides of a right triangle is 18. The sum of the squares of the lengths of the three sides is 128. Find the area of the triangle.

6. A palindrome is a word or number that reads the same backwards and forwards. For example, 1991 is a palindromic number. How many palindromic numbers are there between 1 and $99,999$ inclusive?

7. A graph of $x^2-2xy+y^2-x+y = 12$ and $y^2-y-6 = 0$ will produce four lines whose points of intersection are the vertices of a parallelogram. Find the area of the parallelogram.

8. Determine the possible values of c so that the two lines $x - y = 2$ and $cx + y = 3$ intersect in the first quadrant.

9. Consider the function $f(x) = \frac{cx}{2x+3}$, $x \neq -3/2$. Find all values of c, if any, for which $f(f(x)) = x$.

10. Two numbers are such that the sum of their cubes is 5 and the sum of their squares is 3. Find the sum of the two numbers.

Last issue we gave the problems of Part I of the Alberta High School Mathematics Competition, which was written Tuesday November 21, 1995. This month we give the solutions. How well did you do?

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION

Part I: Solutions

November 21, 1995 (Time: 90 minutes)

1. A circle and a parabola are drawn on a piece of paper. The number of regions they divide the paper into is at most

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7.

Solution. **(D)** The parabola divides the plane into two regions. The circle intersects the parabola in at most four points, so that it is divided by the parabola into at most four arcs. Each arc carves an existing region into two.

2. The number of different primes $p > 2$ such that p divides $71^2 - 37^2 - 51$ is

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4.

Solution. **(D)** We have $71^2 - 36^2 - 51 = (71 + 37)(71 - 37) - 51 = 3 \cdot 17(36 \cdot 2 - 1)$.

3. Suppose that your height this year is 10% more than it was last year, and last year your height was 20% more than it was the year before. By what percentage has your height increased during the last two years?

A. 30 **B.** 31 **C.** 32 **D.** 33 **E.** none of these.

Solution. **(C)** Suppose the height was 100 two years ago. Then it was 120 a year ago and 132 now.

4. Multiply the consecutive even positive integers together until the product $2 \cdot 4 \cdot 6 \cdot 8 \cdots$ becomes divisible by 1995. The largest even integer you use is

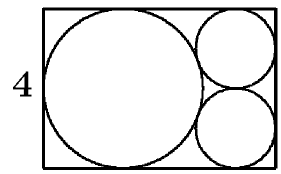
A. between 1 and 21 **B.** between 21 and 31

C. between 31 and 41 **D.** bigger than 41

E. non-existent, since the product never becomes divisible by 1995.

Solution. (C) All factors of 1995 are distinct and odd, with the largest one being 19. Hence the last even number used is 38.

5. A rectangle contains three circles as in the diagram, all tangent to the rectangle and to each other. If the height of the rectangle is 4, then the width of the rectangle is



A. $3+2\sqrt{2}$ **B.** $4+\frac{4\sqrt{2}}{3}$ **C.** $5+\frac{2\sqrt{2}}{3}$
D. 6 **E.** $5+\sqrt{10}$.

Solution. (A) Let O be the centre of the large circle, P that of one of the small circles, and Q the point of tangency of the small circles. Then $\angle PQO = 180°$, $PQ = 1$ and $OP = 1+2$. By Pythagoras' Theorem, $OQ = 2\sqrt{2}$.

6. Mary Lou works a full day and gets her usual pay. Then she works some overtime hours, each at 150% of her usual hourly salary. Her total pay that day is equivalent to 12 hours at her usual hourly salary. The number of hours that she usually works each day is

A. 6 **B.** 7.5 **C.** 8
D. 9 **E.** not uniquely determined by the given information.

Solution. (E) Suppose Mary Lou usually works x hours per day but y on that day. All we know is $x+\frac{3}{2}(y-x)=12$ or $x+3y=24$.

7. A fair coin is tossed $10,000$ times. The probability p of obtaining at least three heads in a row satisfies

A. $0 \le p < \frac{1}{4}$ **B.** $\frac{1}{4} \le p < \frac{1}{2}$ **C.** $\frac{1}{2} \le p < \frac{3}{4}$ **D.** $\frac{3}{4} \le p < 1$ **E.** $p = 1$.

Solution. (D) Partition the tosses into consecutive groups of three, discarding the last one. If we never get 3 heads in a row, none of the 3333 groups can consist of 3 heads. The probability of this is $(\frac{7}{8})^{3333}$, which is clearly less than $\frac{1}{4}$. In fact,

$$\left(\frac{7}{8}\right)^{12} = \frac{117649^2}{8^{12}} < \frac{131072^2}{2^{36}} = \frac{1}{8}.$$

8. In the plane, the angles of a regular polygon with n sides add up to less than n^2 degrees. The smallest possible value of n satisfies:

A. $n < 40$ **B.** $40 \le n < 80$ **C.** $80 \le n < 120$
D. $120 \le n < 160$ **E.** $n \ge 160$.

Solution. (E) The sum of the angles is exactly $(n-2)180°$. From

$$180 < \frac{n^2}{n-2} = n + 2 + \frac{4}{n-2} < n + 3,$$

we have $n > 177$.

9. A cubic polynomial P is such that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$ and $P(4) = 5$. The value of $P(6)$ is

A. 7 **B.** 10 **C.** 13 **D.** 16 **E.** 19.

Solution. (C) By the Binomial Theorem, $P(5) = 4P(4) - 6P(3) + 4P(2) - P(1) = 9$. It follows that $P(6) = 5P(5) - 10P(4) + 10P(3) - 5P(2) + P(1) = 16$.

10. The positive numbers x and y satisfy $xy = 1$. The minimum value of $\frac{1}{x^4} + \frac{1}{4y^4}$ is

A. $\frac{1}{2}$ **B.** $\frac{5}{8}$ **C.** 1 **D.** $\frac{5}{4}$ **E.** no minimum.

Solution. (C) We have

$$\frac{1}{x^4} + \frac{1}{4y^4} = \left(\frac{1}{x^2} - \frac{1}{2y^2}\right)^2 + \frac{1}{x^2y^2} \geq 1,$$

with equality if and only if $x^2 = 2y^2$.

11. Of the points $(0,0)$, $(2,0)$, $(3,1)$, $(1,2)$, $(3,3)$, $(4,3)$ and $(2,4)$, at most how many can lie on a circle?

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7.

Solution. (C) Since $(0,0)$, $(1,2)$ and $(2,4)$ are collinear, a circle passes through at most two of them. Since $(2,0)$, $(3,1)$, $(3,3)$ and $(4,3)$ are not concyclic, a circle passes through at most three of them. The circle with centre $(1,2)$ and passing through $(0,0)$ also passes through $(2,0)$, $(3,1)$, $(3,3)$ and $(2,4)$.

12. The number of different positive integer triples (x,y,z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124 \quad \text{is:}$$

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** none of these.

Solution. (B) Subtraction yields $24 = x + y^2 - x^2 - y = (y-x)(y+x-1)$. Note that one factor is odd and the other even, and that the first is smaller than the second. Hence either $y - x = 1$ and $y + x - 1 = 24$, or $y - x = 3$ and $y + x - 1 = 8$. They lead to $(x,y,z) = (12,13,57)$ and $(3,6,-85)$ and $(3,6,-85)$ respectively. However, we must have $z > 0$.

13. Which of the following conditions does *not* guarantee that the convex quadrilateral $ABCD$ is a parallelogram?

A. $AB = CD$ and $AD = BC$ **B.** $\angle A = \angle C$ and $\angle B = \angle D$

C. $AB = CD$ and $\angle A = \angle C$ **D.** $AB = CD$ and AB is parallel to CD

E. none of these.

Solution. **(C)** Let $\angle ABD = \angle BAD = \angle DCB = 40°$ and $\angle CBD = 80°$. Then $ABCD$ is not a parallelogram. Let E on BC be such that $\angle CDE = 40°$. Then triangles BAD and CDE are congruent, so that $AB = CD$. It is easy to see that the other three conditions do guarantee parallelograms.

14. How many of the expressions

$$x^3 + y^4, \quad x^4 + y^3, \quad x^3 + y^3, \quad \text{and} \quad x^4 - y^4,$$

are positive for all possible numbers x and y for which $x > y$?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4.

Solution. **(A)** We have $x^3 + y^4 < 0$ for $x = -\frac{1}{4}$ and $y = -\frac{1}{3}$. Each of the other three expressions is negative if $x = 0$ and $y < 0$.

15. In triangle ABC, the altitude from A to BC meets BC at D, and the altitude from B to CA meets AD at H. If $AD = 4$, $BD = 3$ and $CD = 2$, then the length of HD is

A. $\frac{\sqrt{5}}{2}$ **B.** $\frac{3}{2}$ **C.** $\sqrt{5}$ **D.** $\frac{5}{2}$ **E.** $\frac{3\sqrt{5}}{2}$.

Solution. **(B)** Note that $\angle CAD = 180° - \angle BCA = \angle CBE$. Hence triangles CAD and HBD are similar, so that

$$\frac{HD}{BD} = \frac{CD}{AD}.$$

16. Which of the following is the best approximation to

$$\frac{(2^3-1)(3^3-1)(4^3-1)}{(2^3+1)(3^3+1)(4^3+1)} \cdots \frac{(100^3-1)}{(100^3+1)}?$$

A. $\frac{3}{5}$ **B.** $\frac{33}{50}$ **C.** $\frac{333}{500}$ **D.** $\frac{3{,}333}{5{,}000}$ **E.** $\frac{33{,}333}{50{,}000}$.

Solution. **(C)** The given expression factors into

$$\frac{(2-1)(2^2+2+1)(3-1)(3^2+3+1)\cdots(100-1)(100^2+100+1)}{(2+1)(2^2-2+1)(3+1)(3^2-3+1)\cdots(100+1)(100^2-100+1)}.$$

Since $((n+2)-1) = n+1$ and $(n+1)^2-(n+1)+1 = n^2+n+1$, cancellations yield

$$\frac{(2-1)(3-1)(100^2+100+1)}{(2^2-2+1)(99+1)(100+1)} = \frac{10101}{15150}.$$

That completes the Skoliad Corner for this issue. Send me contest materials, as well as your comments, suggestions, and desires for future directions for the Skoliad Corner.

Citation

As was announced in the February 1996 issue of *CRUX*, Professor Ron Dunkley was appointed to the **Order of Canada**. This honour was bestowed on Ron by the Governor-General of Canada, the His Excellency The Right Honourable Roméo LeBlanc, in mid-February, and we are pleased to publish a copy of the official citation:

Professor Ronald Dunkley, OC

A professor at the University of Waterloo and founding member of the Canadian Mathematics Competition, he has dedicated his career to encouraging excellence in students. He has trained Canadian teams for the International Mathematics Olympiad, authored six secondary school texts and chaired two foundations that administer significant scholarship programs. An inspiring teacher, he has stimulated interest and achievement among students at all levels, and provided leadership and development programs for teachers across the country.

THE OLYMPIAD CORNER

No. 173

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

The first Olympiad problems that we give in this issue are the problems of the Selection Tests for the Romanian Team to the 34th International Mathematical Olympiad. My thanks go to Georg Gunther, Sir Wilfred Grenfell College for collecting them and sending them to us while he was Canadian team leader at the IMO at Istanbul, Turkey.

SELECTION TESTS FOR THE ROMANIAN TEAM, 34th IMO.

Part I — Selection Test for Balkan Olympic Team

1. Prove that the sequence $\mathrm{Im}\,(z_n)$, $n \geq 1$, of the imaginary parts of the complex numbers $z_n = (1+i)(2+i)\cdots(n+i)$ contains infinitely many positive and infinitely many negative numbers.

2. Let ABC be a triangle inscribed in the circle $\mathcal{O}(O, R)$ and circumscribed to the circle $\mathcal{J}(I, r)$. Denote $d = \frac{Rr}{R+r}$. Show that there exists a triangle DEF such that for any interior point M in ABC there exists a point X on the sides of DEF such that $MX \leq d$.

3. Show that the set $\{1, 2, \ldots, 2^n\}$ can be partitioned in two classes such that none of them contains an arithmetic progression with $2n$ terms.

4. Prove that the equation $x^n + y^n = (x+y)^m$ has a unique integer solution with $m > 1$, $n > 1$, $x > y > 0$.

Part II — First Contest for IMO Team

1st June, 1993

1. Find the greatest real number a such that

$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}} > a$$

is true for all positive real numbers x, y, z.

2. Show that if x, y, z are positive integers such that $x^2 + y^2 + z^2 = 1993$, then $x + y + z$ is not a perfect square.

3. Each of the diagonals AD, BE and CF of a convex hexagon $ABCDEF$ determine a partition of the hexagon into quadrilaterals having the same area and the same perimeter. Does the hexagon necessarily have a centre of symmetry?

4. Show that for any function $f : \mathcal{P}(\{1, 2, \ldots, n\}) \to \{1, 2, \ldots, n\}$ there exist two subsets, A and B, of the set $\{1, 2, \ldots, n\}$, such that $A \neq B$ and $f(A) = f(B) = \max\{i \mid i \in A \cap B\}$.

Part III — Second Contest for IMO Team

2nd June, 1993

1. Let $f : (0, \infty) \to \mathbb{R}$ be a strict convex and strictly increasing function. Show that the sequence $\{f(n)\}_{n \geq 1}$, does not contain an infinite arithmetic progression.

2. Given integer numbers m and n, with $m > n > 1$ and $(m, n) = 1$, find the **gcd** of the polynomials $f(X) = X^{m+n} - X^{m+1} - X + 1$ and $g(X) = X^{m+n} + X^{n+1} - X + 1$.

3. Prove that for all integer numbers n, with $n \geq 6$, there exists an n-point set M in the plane such that every point P in M has at least three other points in M at unit distance to P.

4. For all ordered 4-tuples (n_1, n_2, n_3, n_4) of positive integer numbers with $n_i \geq 1$ and $n_1 + n_2 + n_3 + n_4 = n$, find the 4-tuples for which the number $\dfrac{n!}{n_1!n_2!n_3!n_4!}2^l$, where

$$l = \binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} + \binom{n_4}{2} + n_1n_2 + n_2n_3 + n_3n_4,$$

has a maximum value.

Part IV — Third Contest for IMO Team

3rd June, 1993

1. The sequence of positive integers $\{x_n\}_{n \geq 1}$ is defined as follows: $x_1 = 1$, the next two terms are the even numbers 2 and 4, the next three terms are the three odd numbers 5, 7, 9, the next four terms are the even numbers 10, 12, 14, 16 and so on. Find a formula for x_n.

2. The triangle ABC is given and let D, E, F be three points such that $D \in (BC)$, $E \in (CA)$, $F \in (AB)$, $BD = CE = AF$ and $\widehat{BAD} = \widehat{CBE} = \widehat{ACF}$. Show that ABC is equilateral.

3. Let p be a prime number, $p \geq 5$, and $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$. Prove that for any partition with three subsets of $\mathbb{Z}_p^*$ there exists a solution of the equation

$$x + y \equiv z \bmod p,$$

each term belonging to a distinct member of the partition.

As a national Olympiad, we have the Final Round of the Czechoslovak Mathematical Olympiad, 1993.

CZECHOSLOVAK MATHEMATICAL OLYMPIAD 1993
Final Round

1. Find all natural numbers n for which $7^n - 1$ is a multiple of $6^n - 1$.

2. A 19×19 table contains integers so that any two of them lying on neighbouring fields differ at most by 2. Find the greatest possible number of mutually different integers in such a table. (Two fields of the table are considered neighbouring if they have a common side.)

3. A triangle AKL is given in a plane such that $|\angle ALK| > 90° + |\angle LAK|$. Construct an equilateral trapezoid $ABCD$, $AB \perp CD$, such that K lies on the side BC, L on the diagonal AC and the outer section S of AK and BL coincides with the centre of the circle circumscribed around the trapezoid $ABCD$.

4. A sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers is defined recursively by $a_1 = 2$ and $a_{n+1} =$ the sum of 10th powers of the digits of a_n, for all $n \geq 1$. Decide whether some numbers can appear twice in the sequence $\{a_n\}_{n=1}^{\infty}$.

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(-1) = f(1)$ and

$$f(x) + f(y) = f(x + 2xy) + f(y - 2xy)$$

for all integers x, y.

6. Show that there exists a tetrahedron which can be partitioned into eight congruent tetrahedra, each of which is similar to the original one.

Ah, the filing demons are at it again. When I attacked a rather suspicious looking pile of what I thought were as yet unfiled solutions to 1995 problems from the *Corner*, I found a small treasure-trove of solutions to various problem sets from 1994, and some comments of Murray Klamkin about earlier material that he submitted at the same time. The remainder of this column will be devoted to catching up on this backlog in an attempt to bring things up to the November 1994 issue. First two comments about solutions from 1992 numbers of the corner.

2. [1990: 257; 1992: 40] 1990 *Asian Pacific Mathematical Olympiad.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of the products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n, \quad \text{for} \quad k = 1, 2, \ldots, n-1.$$

Comment by Murray S. Klamkin, University of Alberta. It should be noted that the given inequality, as well as other stronger ones, follows from the known and useful Maclaurin inequalities, and that

$$\left[\frac{S_k}{\binom{n}{k}}\right]^{1/k}$$

is a non-increasing sequence in k, and with equality **iff** all the a_i's are equal.

2. [1992: 197] 1992 *Canadian Mathematical Olympiad.*

For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z),$$

and determine when equality holds.

Comment by Murray S. Klamkin, University of Alberta. It should be noted that the given inequality is the special case $\lambda = 1$ of Schur's inequality

$$x^\lambda(x-y)(x-z) - y^\lambda(x-z)(x-y) + z^\lambda(z-x)(z-y) \geq 0.$$

For a proof, since the inequality is symmetric, we may assume that $x \geq y \geq z$, or that $x \geq z \geq y$. Assuming the former case, we have

$$\begin{aligned} &x^\lambda(x-y)(x-z) - y^\lambda(y-z)(x-y) + z^\lambda(z-x)(z-y) \\ &\quad \geq x^\lambda(x-y)(y-z) - y^\lambda(x-z)(x-y) + z^\lambda(z-x)(z-y) \geq 0. \end{aligned}$$

We have assumed $\lambda \geq 0$. For $\lambda < 0$, we have

$$(yz)^{-\lambda}(x-y)(x-z) - (zx)^{-\lambda}(y-z)(x-y) + (xy)^{-\lambda}(x-z)(y-z) \geq$$

$$(yz)^{-\lambda}(x-y)(x-z) - (zx)^{-\lambda}(y-z)(x-z) + (xy)^{-\lambda}(x-z)(y-z) \geq 0.$$

The case $x \geq z \geq y$ goes through in a similar way.

Note that if λ is an even integer, x, y, z can be any real numbers.

Next a comment about a solution from the February 1994 number.

6. [1994: 43; 1992: 297] *Vietnamese National Olympiad.*

Let x, y, z be positive real numbers with $x \geq y \geq z$. Prove that

$$\frac{x^2}{y} + \frac{y^2 z}{x} + \frac{z^2 x}{y} \geq x^2 + y^2 + z^2.$$

Comment by Murray S. Klamkin, University of Alberta. We give a more direct solution than the previous ones and which applies in many cases where the constraint conditions are $x \geq y \geq z$.

Let $z = a$, $y = a + b$, $x = a + b + c$ where $a > 0$, and $b, c \geq 0$. Substituting back in the inequality, multiplying by the least common denominator and combining the cubic terms, we get

$$(a + b + c)^3(ab + b^2) + a^3c(a + b + c) \geq (a + b)^3(ab + ac).$$

On inspection, for every term in the expansion of the right hand side there is a corresponding term on the left hand side, which establishes the inequality.

Amongst the solutions sent in were three solutions by Klamkin to problems 3, 4 and 7 of the 1992 Austrian-Polish Mathematics Competition. We discussed reader's solutions to these in the December number [1995: 336–340]. My apologies for not mentioning his solutions there. He also sent in a complete set of solutions to the 43rd Mathematical Olympiad in Poland, for which we discussed solutions to most of the problems in the February 1996 Corner [1996: 24–27]. One problem solution was not covered there and we next give his solution to fill that gap.

5. [1994: 130] *43rd Mathematical Olympiad in Poland.*

The rectangular $2n$-gon is the base of a regular pyramid with vertex S. A sphere passing through S cuts the lateral edges SA_i in the respective points B_i $(i = 1, 2, \ldots, 2n)$. Show that

$$\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}.$$

Solution by Murray S. Klamkin, University of Alberta. Let S be the origin $(0, 0, 0)$ of a rectangular coordinate system and let the coordinates of the vertices A_k of the regular $2n$-gon be given by $(r\cos\theta_k, r\sin\theta_k, a)$, $k = 1, 2, \ldots, 2n$ where $\theta_k = \pi k/n$. A general sphere through S is given by

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = h^2 + k^2 + l^2.$$

Since the parametric equation of the line SA_k is given by

$$x = tr\ \cos\theta_k, \quad y = tr\ \sin\theta_k, \quad z = ta,$$

its intersection with the sphere is given by

$$(tr\ \cos\theta_k - h)^2 + (tr\ \sin\theta_k - k)^2 + (ta - l)^2 = h^2 + k^2 + l^2.$$

Solving for t, $t = 0$ corresponding to point S and

$$t = \frac{(hr\cos\theta_k + kr\sin\theta_k + al)}{(r^2 + a^2)}.$$

Since $SB_k = t\sqrt{r^2 + a^2}$, the desired result will follow if

$$\sum \cos\theta_{2k-1} = \sum \cos\theta_{2k} \quad \text{and} \quad \sum \sin\theta_{2k-1} = \sum \sin\theta_{2k}$$

(where the sums are over $k = 1, 2, \dots, n$). Since in the plane

$$(\cos\theta_{2k-1}, \sin\theta_{2k-1}), \quad k = 1, 2, \dots, n,$$

are the vertices of a regular n-gon, both $\sum \cos\theta_{2k-1}$ and $\sum \sin\theta_{2k-1}$ vanish and the same for $\sum \cos\theta_{2k}$ and $\sum \sin\theta_{2k}$.

Next we give a comment and alternative solution to a problem discussed in the May 1994 Corner.

7. [1994: 133; 1993: 66-67] 14*th Austrian-Polish Mathematical Olympiad.*

For a given integer $n \geq 1$ determine the maximum of the function

$$f(x) = \frac{x + x^2 + \cdots x^{2n-1}}{(1 + x^n)^2}$$

over $x \in (0, \infty)$ and find all $x > 0$ for which this maximum is attained.

Comment by Murray S. Klamkin, University of Alberta. Here is a more compact solution than the previously published one. We show that the maximum value of $f(x)$ is attained for $x = 1$, by establishing the inequality

$$4(x + x^2 + \cdots x^{2n-1}) \leq (2n - 1)(1 + 2x^n + x^{2n}).$$

This is a consequence of the majorization inequality [1], i.e., if $F(x)$ is convex and the vector $(x_1, x_2, \dots, x_n)$ majorizes the vector $(y_1, y_2, \dots, y_n)$, then

$$F(y_1) + F(y_2) + \cdots + F(y_n) \leq F(x_1) + F(x_2) + \cdots + F(x_n).$$

Note that for a vector X of n components x_i in non-increasing order to majorize a vector Y of n components y_i in non-increasing order, we must have

$$\sum_{i=1}^{p} x_i \geq \sum_{i=1}^{p} y_i \quad \text{for} \quad p = 1, 2, \ldots, n-1 \quad \text{and} \quad \sum x_i = \sum y_i$$

and we write $X \succ Y$. Since x^t is convex in t, we need only show that $X = (2n, 2n, \ldots, 2n, n, n, \ldots, n, 0, 0, \ldots, 0) \succ Y = (2n-1, \ldots, 2n-1, 2n-2, \ldots, 2n-2, \ldots, 1, \ldots, 1)$ where for X there are $2n-1$ components of $2n$; $2(2n-1)$ components of n; $(2n-1)$ components of 0, while for Y there are 4 components of each of $(2n-1)$; $(2n-2)$; $\ldots$; 1.

It follows easily that X and Y have the same number of components and that the sums of their components are equal. As for the rest, actually all we really need show is that

$$\begin{aligned} &4[(2n-1) + (2n-2) + \cdots + (n+1)] \\ &\quad = 6n^2 - 12n < (2n-1)(2n) + (2n-3)(n) \\ &\quad = 6n^2 - 5n. \end{aligned}$$

Reference.

1. A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, NY, 1979.

Now we turn to some more solutions to problems proposed to the jury but not used at the IMO at Istanbul. Last number we gave solutions to some of these. My "found mail" includes another solution to 2 [1994: 216] by Murray Klamkin, University of Alberta, and solutions to 13 [1994: 241] by Bob Prielipp, University of Wisconsin-Oshkosh and by Cyrus C. Hsia, student, Woburn Collegiate Institute, Scarborough, Ontario.

1. [1994: 216] *Proposed by Brazil.*

Show that there exists a finite set $A \subset \mathbb{R}^2$ such that for every $X \in A$ there are points $Y_1, Y_2, \ldots, Y_{1993}$ in A such that the distance between X and Y_i is equal to 1, for every i.

Solution by Cyrus C. Hsia.

We will prove the following proposition P_n: There exists a finite set $A \subset \mathbb{R}^2$ such that for every point X in A there are n points $Y_1, Y_2, \ldots, Y_n$ in A such that the distance between X and Y_i is equal to 1, for every i, and n is an integer greater than 1.

(By mathematical induction on n). For the case $n = 2$ just take any two points a unit distance apart. For $n = 3$ just take any three points a unit distance apart from each other (i.e. any equilateral triangle of side 1 has vertices with this property). The proposition is satisfied in these two cases.

Suppose that the proposition P_n is true for $n = k$ points. Let there be L (finite) points that satisfy the proposition. Since there are a finite number of points in A_k there are a finite number of unit vectors formed between any two points of distance 1 unit apart. Now choose any unit vector, $\vec{v}$, different from any of the previous ones. Consider the set of $2L$ points formed by the original L points translated by $\vec{v}$ and the original L points. (No point can be translated onto another point by our choice of $\vec{v}$). Now by the induction hypothesis, every point in the original L points was a unit distance from k other points. But the translation produced two such sets with each point from one set a unit distance from its translated point in the other. Thus every point of the $2L$ points are at least a unit distance from $k+1$ other points, which means the proposition is true for $n = k+1$. Induction complete.

6. [1994: 217] *Proposed by Ireland.*

Let n, k be positive integers with $k \le n$ and let S be a set containing distinct real numbers. Let T be the set of all real numbers of the form $x_1 + x_2 + \cdots x_k$ where $x_1, x_2, \ldots, x_k$ are distinct elements of S. Prove that T contains at least $k(n-k)+1$ distinct elements.

Solution by Cyrus C. Hsia.

The problem should say: Let n, k be positive integers with $k \le n$ and let S be a set containing n distinct real numbers. Let T be the set of all real numbers of the form $x_1 + x_2 + \cdots + x_k$ where $x_1, x_2, \ldots, x_k$ are distinct elements of S. Prove that T contains at least $k(n-k)+1$ distinct elements.

WOLOG let $x_1 < x_2, \cdots < x_n$ since all x_i are distinct.

Then consider the $k(n-k)+1$ increasing numbers

$$\begin{aligned}
& x_1 + x_2 + x_3 + \cdots + x_{k-1} + x_k < x_1 + x_2 + x_3 + \cdots \\
& \quad + x_{k-1} + x_{k+1} < \cdots < x_1 + x_2 + x_3 + \cdots + x_{k-1} + x_n && (n-k+1) \\[1ex]
& < x_1 + x_2 + x_3 + \cdots + x_{k-2} + x_k + x_n < x_1 + x_2 + x_3 + \cdots \\
& \quad + x_{k+1} + x_n < \cdots < x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n && (n-k) \\[1ex]
& < x_1 + x_2 + \cdots + x_{k-1} + x_{n-1} + x_n < x_1 + x_2 + \cdots + x_{k-1} \\
& \quad + x_{n-1} + x_n < \cdots < x_1 + x_2 + \cdots + x_{n-2} + x_{n-1} + x_n && (n-k) \\[1ex]
& \qquad \vdots && \vdots \\[1ex]
& < x_2 + x_{n-k+2} + \cdots + x_{n-1} + x_n < x_3 + x_{n-k+2} + \cdots \\
& \quad + x_{n-1} + x_n < \cdots < x_{n-k+1} + x_{n-k+2} + \cdots + x_{n-1} + x_n && (n-k)
\end{aligned}$$

There are at least $(n-k+1) + (k-1)(n-k) = k(n-k)+1$ distinct numbers.

9. [1994: 217] *Proposed by Poland.*

Let S_n be the number of sequences $(a_1, a_2, \ldots a_n)$, where $a_i \in \{0, 1\}$, in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ when $n \to \infty$.

Solution by Cyrus C. Hsia.

"No six consecutive blocks are equal" interpretation: There is no sequence with the consecutive numbers $0, 0, 0, 0, 0, 0$ or $1, 1, 1, 1, 1, 1$ anywhere.

Consider the blocks $0, 1$ and $1, 0$. If the sequences were made only with these then we cannot have six consecutive blocks equal. Let T_n be the number of such sequences for n even (and ending with 0 or 1 for n odd). For example

$$T_2 = 2\ \{0, 1 \text{ or } 1, 0\}$$
$$T_3 = 4\ \{0, 1, 0 \text{ or } 0, 1, 1 \text{ or } 1, 0, 0 \text{ or } 1, 0, 1\}$$

Thus $T_n = 2^{\lceil \frac{n}{2} \rceil}$. Now $T_n < S_n$ since any 1 in a T_n sequence can be changed to a 0 to form a new sequence in S_n which was not counted in T_n.

E.g. $1, 0, 0, 1, 0, 1$ in T_n changes to $1, 0, 0, 0, 0, 1$ which is in S_n.

Note: this change cannot produce six consecutive blocks equal.

Thus as $n \to \infty$, $T_n = 2^{\lceil \frac{n}{2} \rceil} \to \infty$. And since $S_n > T_n$, $S_n \to \infty$.

18. [1994: 242] *Proposed by the U.S.A.*

Prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

Solution by Cyrus C. Hsia.

Using the Cauchy-Schwartz-Buniakowski inequality, we have

$$\begin{aligned}&\left(\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c}\right)\\ &\quad \times(a(b+2c+3d) + b(c+2d+3a) + \cdots) \geq (a+b+c+d)^2\\ &\quad \Rightarrow S(4)(ab+a+ad+bc+bd+cd) \geq (a+b+c+d)^2,\end{aligned}$$

where

$$S = \left(\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c}\right).$$

Now $a^2 + b^2 \geq 2ab$ from the AM-GM inequality. Likewise for the other five pairs we have the same inequality. Adding all six pairs gives

$$\begin{aligned}3(a^2+b^2+c^2+d^2) &\geq 2(ab+ac+\cdots)\\ \Rightarrow 3(a+b+c+d)^2 &\geq 8(ab+ac+\cdots).\end{aligned}$$

Therefore

$$S \geq \frac{(a+b+c+d)^2}{4(ab+ac+\cdots)} \geq \frac{(\frac{8}{3}(ab+ac+\cdots)}{4(ab+ac+\cdots)} \geq \frac{2}{3},$$

as required.

Comment by Murray Klamkin, University of Alberta.

A generalization of this problem appeared recently in *School Science & Mathematics* as problem #4499. The generalization is the following:

Let n be a natural number greater than one. Show that, for all positive numbers $a_1, a_2, \dots, a_n$,

$$\sum \frac{a_i}{a_{i+1} + 2a_{i+2} + \cdots + (n-1)a_{i+n-1}} \geq \frac{2}{n-1},$$

with equality if and only if $a_1 = a_2 = \cdots a_n$. Here all the subscripts greater than n are reduced modulo n. For $n = 2$ and $n = 3$, the stated inequality reduces to

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2$$

and

$$\frac{a_1}{a_2 + 2a_3} + \frac{a_2}{a_3 + 2a_1} + \frac{a_3}{a_1 + 2a_2} \geq 1,$$

respectively. The special case $n = 4$ was proposed to the jury for the 34th International Mathematical Olympiad.

That's all the space I have this issue. Send me your nice solutions, and your regional and national Olympiads!

Mathematical Literacy

1. Which Victorian physicist characterised nonnumerical knowledge as "meagre and unsatisfactory".
2. Who said, in 1692, "There are very few things which we know; which are not capable of being reduc'd to a Mathematical Reasoning".
3. Which mathematician is reputed to have examined (and passed) Napoleon at the Ecole Militaire in 1785.
4. Which mathematician was said to have: "frequented low company, with whom he used to guzzle porter and gin".

THE ACADEMY CORNER

No. 2

Bruce Shawyer

All communications about this column should be sent to Professor Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In the first corner, I mentioned the APICS (Atlantic Provinces Council on the Sciences) annual undergraduate contest. This contest is held at the end of October each year during the Fall APICS Mathematics Meeting. As well as the competition, there is a regular mathematics meeting with presentations, mostly from mathematicians in the Atlantic Provinces. An effort is made to ensure that most of the talks are accessible to undergraduate mathematics students. In fact, a key talk is the annual W.J. Blundon Lecture. This is named in honour of Jack Blundon, who was Head of Department at Memorial University of Newfoundland for the 27 years leading up to 1976. Jack was a great support of *CRUX*, and long time subscribers will remember his many contributions to this journal.

APICS Mathematics Contest 1995

Time allowed: three hours

1. Given the functions $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$, with $g(g(x)) = x$ for every $x \in \mathbb{R}$, and α a real number such that $|\alpha| \neq 1$, prove that there exists exactly one function $f\colon \mathbb{R} \to \mathbb{R}$ such that

   $\alpha\, f(x) + f(g(x)) \;=\; h(x)$ for every $x \in \mathbb{R}$.

2. A solid fence encloses a square field with sides of length L. A cow is in a meadow surrounding the field for a large distance on all sides, and is tied to a rope of length R attached to a corner of the fence.

   What area of the meadow is available for the cow to use?

3. Find all solutions to

   $(x^2 + y)(x + y^2) \;=\; (x - y)^3$

   where x and y are integers different from zero.

4. For what positive integers n, is the n^{th} Catalan number, $\dfrac{\binom{2n}{n}}{n+1}$, odd?

5. N pairs of diametrically opposite points are chosen on a circle of radius 1. Every line segment joining two of the $2N$ points, whether in the same pair or not, is called a *diagonal*.

   Show that the sum of the squares of the lengths of the diagonals depends only on N, and find that value.

6. A finite pattern of checkers is placed on an infinite checkerboard, at most one checker to a square; this is *Generation* 0.

   Generation N is generated from *Generation* $(N - 1)$, for $N = 1, 2, 3, \ldots$, by the following process:

   if a cell has an odd number of immediate horizontal of vertical neighbours in *Generation* $(N - 1)$, it contains a checker in *Generation*N; otherwise it is vacant.

   Show that there exists an X such that *Generation*X consists of at least 1995 copies of the original pattern, each separated from the rest of the pattern by an empty region at least 1995 cells wide.

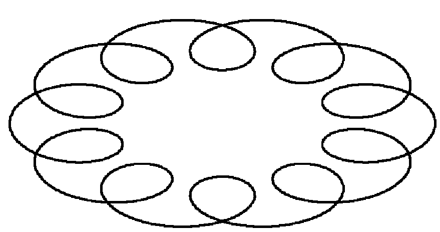
7. A and B play a game. First A chooses a sequence of three tosses of a coin, and tells it to B;

   then B chooses a difference sequence of three tosses and tells it to B.

   Then they throw a fair coin repeatedly until one sequence or the other shows up as three consecutive tosses.

   For instance, A might choose (head, tail, head); then B might choose (tail, head, tail). If the sequence of tosses is (head, tail, tail, head, tail), then B would win.

   If both players play rationally (make their best possible choice), what is the probability that A wins?



BOOK REVIEWS

Edited by ANDY LIU

e *The Story of a Number* by Eli Maor.
Published by Princeton University Press, Princeton NJ, 1994,
ISBN 0-691-03390-0, xiv+223 pages, US$24.95.
Reviewed by **Richard Guy**, *University of Calgary.*

As the author says in his Preface:

> The story of π has been extensively told, no doubt because its history goes back to ancient times, but also because much of it can be grasped without a knowledge of advanced mathematics. Perhaps no book did better than Petr Beckmann's *A History of* π, a model of popular yet clear and precise exposition. The number e fared less well.

It continues to fare less well. Many of us are searching for ways to bring mathematics to a much broader audience. This book is a good opportunity missed.

There are too many formulas, and they appear too early; even in the Preface there are many. Moreover, they are rarely displayed (only one out of more than a dozen in the Preface), which makes the text hard to read — particularly for the layman.

Several chapters are no more than excerpts from standard calculus texts. Chap. 4, Limits. Chap. 10, $dy/dx = y$, Chap. 12, Catenary. Chaps. 13 & 14, Complex Numbers; even including a treatment of the Cauchy-Riemann and Laplace equations.

This is a history book, but the history is often invented, either consciously or unconsciously (Pythagoras experimenting with strings, bells and glasses of water; Archimedes using parabolic mirrors to set the Roman fleet ablaze; Descartes watching a fly on the ceiling; 'quotations' of Johann and Daniel Bernoulli which are figments of E. T. Bell's fertile imagination), or just plain wrong.

Pascal's dates (1623-1662) are given, and an illustration of "his" triangle from a work published nearly a century before Pascal was born, and another from a Japanese work more than a century after he died. The author, elsewhere, mentions that it appeared in 1544 in Michael Stifel's *Arithmetica integra*; all of which leaves the reader bewildered. The triangle was known to the Japanese some centuries before, to the Chinese some centuries before that, and to Omar Khayyam before that.

After saying that it is unlikely that a member of the Bach family met one of the Bernoullis, the author gives a conversation between J. S. Bach and Johann Bernoulli. This piece of fiction distracts the reader from the real topic, which is the mathematics of musical scales. No connection with e is mentioned.

One of the various examples that is given of the occurrence of the exponential function, or of its inverse, the natural logarithm, is the Weber-Fechner law, which purports to measure the human response to physical stimuli. This is probably also regarded as a piece of fiction by most modern scientists. It has far less plausibility than the gas laws, say.

There are many missed opportunities: more examples of damping, the distribution of temperature and pressure in the earth's atmosphere, alternating currents, radioactive decay and carbon dating; and why not mention the distribution of the prime numbers — they are as relevant and probably of more interest to the person-in-the-street than many of the topics covered.

Let us look at some of the examples that are given.

Newton's law of cooling says that the temperature (difference) is proportional to its derivative. There is no mention of its eclipse by the work of Dulong & Petit and the Stefan-Boltzmann law [**1**], presumably because they have little connexion with e.

The parachutist whose air resistance is proportional to his velocity; in practice the resistance is at least quadratic and probably more complicated [**2**].

Yet another example is the occurrence of the logarithmic spiral in art and in nature, without any hint of the controversial aspects of the subject. There is the oft-quoted and illustrated capitulum of a sunflower; though the current wisdom [**3**] is that Fermat's spiral, $r = a\sqrt{\theta}$, which does not involve e, provides a more realistic model. Incidentally, this spiral is essentially identical to the orbit of a non-relativistic charged particle in a cyclotron; there are area, (resp. energy), considerations which make this the 'right' spiral for these two applications.

Population growth gets only a one-sentence mention, whereas there are important mathematical and social lessons to be learnt from this topic. The author is not afraid of quoting whole pages of calculus, so why not the following? There are many good examples of exponential growth, or decay, in everyday life. But none of them is perfect. Look at a perturbation of the equation for the exponential function, a particular case of Bernoulli's equation:

$$\frac{dp}{dt} = kp - \epsilon p^2$$

which, after multiplying by e^{kt}/p^2, we may write as

$$\frac{d}{dt}\left(\frac{e^{kt}}{p}\right) = \epsilon e^{kt}$$

(Yes! Integrating factors and even characteristic equations occur on pp. 104–105.) Integrate and divide by e^{kt}:

$$\frac{1}{p} = Ce^{-kt} + \frac{\epsilon}{k}$$

As the time tends to infinity, the population tends to k/ϵ. If ϵ is small, this is large, but at least it is finite, and there is some hope for our planet. What is the ϵp^2 term? It is roughly proportional to the number of pairs of people, and the term represents the effect of competition. But what if ϵ is negative? What if we substitute cooperation for competition? As we approach the finite time $(1/k)\ln(Ck/(-\epsilon))$ the population tends to infinity! No wonder that capitalism is more successful than communism.

There is a good deal of irrelevant padding, presumably in an attempt to make the book 'popular': the cycloid, the lemniscate, Euler's formula for polyhedra, the Newton-Leibniz controversy, and even Fermat's Last Theorem, though Andrew Wiles has a wrong given name in the Index.

There are misprints: 'Appolonius' (p. 66; he does not appear in the Index), 'audibile' (p. 112); — no mathematics book is free of misprints. But there are worse than misprints: — 'each additional term brings us closer to the limit (this is not so with a series whose terms have alternating signs)' (p. 36); 'can be thought of as the sum of infinitely many small triangles' (p. 54) and Note 1 on p. 93 might lead many readers to infer that a continuous function has a derivative.

Let us hope that the book will spark the interest of non-mathematicians, but the fear is that it will confirm their suspicions that we mathematicians just juggle symbols which the rest of the world has little hope of comprehending.

REFERENCES

1. Charles H. Draper, *Heat and the Principles of Thermodynamics*, Blackie & Son, London and Glasgow, 1893, pp. 221–222.

2. Horace Lamb, *Dynamics*, Cambridge Univ. Press, 1914, p. 290.

3. Helmut Vogel, A better way to construct the sunflower head, *Math. Biosciences*, **44** (1979), pp. 179–189.

PROBLEMS

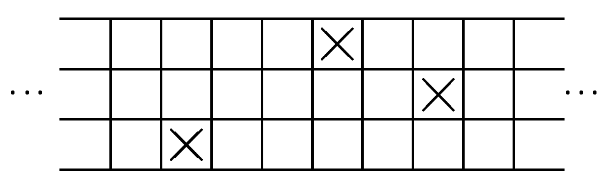
Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 November 1996**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX, *preferably in* LaTeX2e). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2125 *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each. The lockers in the top row are numbered 1 to N, the middle row $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right. Ann, Beth, and Carol are three friends whose lockers are located as follows:



By the way, the three girls are not only friends, but also next-door neighbours, with Ann's, Beth's, and Carol's houses next to each other (in that order) on the same street. So the girls are intrigued when they notice that Beth's house number divides into all three of their locker numbers. What is Beth's house number?

2126 *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

At Lake West Collegiate, the lockers are in a long rectangular array, with three rows of N lockers each, where N is some positive integer between 400 and 450. The lockers in the top row were originally numbered 1 to N, the middle row $N+1$ to $2N$, and the bottom row $2N+1$ to $3N$, all from left to right. However, one evening the school administration changed around the locker numbers so that the first column on the left is now numbered 1 to 3, the next column 4 to 6, and so forth, all from top to bottom. Three friends, whose lockers are located one in each row, come in the next morning to discover that each of them now has the locker number that used to belong to one of the others! What are (were) their locker numbers, assuming that all are three-digit numbers?

2127 *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with circumcentre O, and D is a point on the minor arc AC of the circumcircle ($D \neq A, C$). Let P be a point on the side AB such that $\angle ADP = \angle OBC$, and let Q be a point on the side BC such that $\angle CDQ = \angle OBA$. Prove that $\angle DPQ = \angle DOC$ and $\angle DQP = \angle DOA$.

2128 *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square. Let P and Q be interior points on the sides BC and CD respectively, and let E and F be the intersections of PQ with AB and AD respectively. Prove that

$$\pi \leq \angle PAQ + \angle ECF < \frac{5\pi}{4}.$$

2129* *Proposed by Stanley Rabinowitz, Westford, Massachusetts, USA.*

For $n > 1$ and $i = \sqrt{-1}$, prove or disprove that

$$\frac{1}{4i} \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{4n} i^k \tan\left(\frac{k\pi}{4n}\right)$$

is an integer.

2130 *Proposed by D. J. Smeenk, Zaltbommel, the Netherlands.*

A and B are fixed points, and ℓ is a fixed line passing through A. C is a variable point on ℓ, staying on one side of A. The incircle of ΔABC touches BC at D and AC at E. Show that line DE passes through a fixed point.

2131 *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Find all positive integers $n > 1$ such that there exists a cyclic permutation of $(1, 1, 2, 2, \ldots, n, n)$ satisfying:

(i) no two adjacent terms of the permutation (including the last and first term) are equal; and

(ii) no block of n consecutive terms consists of n distinct integers.

2132 *Proposed by Šefket Arslanagić, Berlin, Germany.* Let n be an even number and z a complex number.
Prove that the polynomial $P(z) = (z+1)^n - z^n - n$ is not divisible by $z^2 + z + n$.

2133 *Proposed by K. R. S. Sastry, Dodballapur, India.*

Similar non-square rectangles are placed outwardly on the sides of a parallelogram π. Prove that the centres of these rectangles also form a non-square rectangle if and only if π is a non-square rhombus.

2134* *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let $\{x_n\}$ be an increasing sequence of positive integers such that the sequence $\{x_{n+1} - x_n\}$ is bounded. Prove or disprove that, for each integer $m \geq 3$, there exist positive integers $k_1 < k_2 < \ldots < k_m$, such that $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ are in arithmetic progression.

2135 *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let n be a positive integer. Find the value of the sum

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{(k+1)!(n-k)!(n-2k)!}.$$

2136 *Proposed by G. P. Henderson, Campbellcroft, Ontario.*

Let a, b, c be the lengths of the sides of a triangle. Given the values of $p = \sum a$ and $q = \sum ab$, prove that $r = abc$ can be estimated with an error of at most $r/26$.

2137 *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{2R}{R+2}$, (ii) T lies on the line segment joining the centres of the circumcircle and the incircle of ΔABC.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2015. [1995: 53 and 129 (Corrected)] *Proposed by Shi-Chang Chi and Ji Chen, Ningbo University, China.*

Prove that

$$\Big(\sin(A)+\sin(B)+\sin(C)\Big)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right) \;\geq\; \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

Editor's comment on the featured solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia, Canada. [1996: 47]

There is a slight and very subtle flaw in the published solution. To correct this, all that is required is to replace the open domain S by a **closed** domain.

The error is a very natural one, and has been made in the past by many others. We refer readers to *Mathematics Magazine*, **58** (1985), pp. 146–150, for several examples illustrating this subtle point.

2025. [1995: 90] *Proposed by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

(a) An equilateral triangle ABC is drawn on a sheet of paper. Prove that you can repeatedly fold the paper along the lines containing the sides of the triangle, so that the entire sheet of paper has been folded into a wad with the original triangle as its boundary. More precisely, let f_a be the function from the plane of the sheet of paper to itself defined by

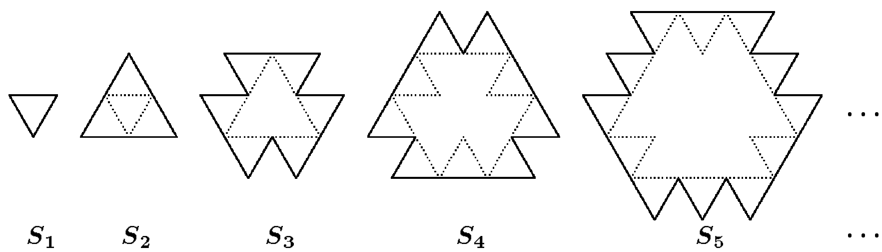
$$f_a(x) = \begin{cases} x & \text{if, } x \text{ is on the same side of } BC \text{ as } A \text{ is,} \\ \text{the reflection of } x \text{ about line } BC, & \text{otherwise,} \end{cases}$$

(f_a describes the result of folding the paper along the line BC), and analogously define f_b and f_c. Prove that there is a finite sequence $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$, with each $f_{i_j} = f_a$, f_b or f_c, such that $f_{i_n}(\ldots(f_{i_2}(f_{i_1}(x)))\ldots)$ lies in or on the triangle for every point x on the paper.

(b)* Is the result true for arbitrary triangles ABC?

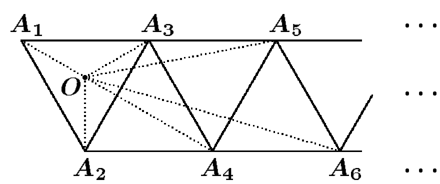
Solution to (a) *by Catherine Shevlin, Wallsend, England.*

We shall show that it is possible by reversing the problem: we start with the triangle ABC and unfold three copies of it along the lines containing the edges of the original triangle, to create a larger shape. This process of unfolding shapes is repeated as illustrated in the diagram. We see how the plane is covered by the sequence of shapes, $\{S_k\}$. The areas of the shapes increase: the area of S_k is $1 + 3 + 6 + \ldots + 3(k-1) = \dfrac{3k^2 - 3k + 2}{2} \to \infty$ as $k \to \infty$.



We now show that the minimum distance from a point on the boundary of the shape to the centre increases without bound.

Let O be the centre of the original triangle. Then, proceeding away from triangle ABC in each of the six principal directions is a sequence of triangles, as illustrated below. For convenience, we rename triangle ABC as $A_1A_2A_3$.

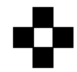


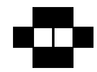
The distance OA_k is the minimum distance of the k^{th} repetition from O. This is easy to calculate in terms of the side of the original triangle. It is clear that $OA_k \to \infty$ as $k \to \infty$.

It is easy to see that the area covered includes a circle of radius OA_k. Hence, the area covered by the repetitions tends to infinity, and so it will cover any finite area. Thus the sheet of paper will be covered by the foldings.

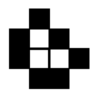
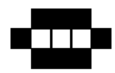
Part (a) *was also solved by the proposer. One incorrect attempt was received. No-one sent in anything on part* (b)*, so this problem remains open.*

2026. [1995: 90] *Proposed by Hiroshi Kotera, Nara City, Japan.*

One white square is surrounded by four black squares: 

Two white squares are surrounded by six black squares: 

Three white squares are surrounded by seven or eight black squares:

What is the largest possible number of white squares surrounded by n black squares? [According to the proposer, this problem was on the entrance examination of the junior high school where he teaches!]

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

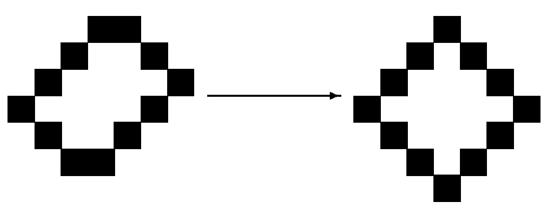
The largest possible number of white squares is

$$\begin{cases} 2k^2 - 2k + 1 & \text{if } n = 4k, \\ 2k^2 - k & \text{if } n = 4k + 1, \\ 2k^2 & \text{if } n = 4k + 2, \\ 2k^2 + k & \text{if } n = 4k + 3. \end{cases}$$

Consider an arrangement of black squares surrounding some white region.

Three black squares that are horizontally or vertically adjacent can be changed as follows to increase the number of white squares surrounded by one as shown below.

Suppose there are two pairs of black squares that are horizontally adjacent. Then we can shift all the black squares between these pairs down as shown and still keep [at least] the same number of white squares surrounded, as shown below.



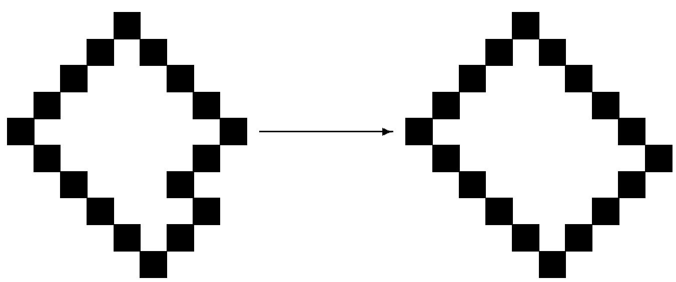
[Similarly we can assume there are not two pairs of vertically adjacent black squares.] Thus the maximum number of white squares surrounded can be obtained when there are either one horizontal and one vertical pair of adjacent black squares or when there are no horizontally or vertically adjacent black squares. [For example, it is impossible to have one horizontal pair and no vertical pair of adjacent black squares: just consider the usual chessboard colouring of the squares. — *Ed.*]

Label a set of diagonally adjacent squares as shown below.

1		1		1		1		1		1		1		1	
	2		2		2		2		2		2		2		2
1		1		1		1		1		1		1		1	
	2		2		2		2		2		2		2		2
1		1		1		1		1		1		1		1	
	2		2		2		2		2		2		2		2
1		1		1		1		1		1		1		1	
	2		2		2		2		2		2		2		2

Then if all the black squares are diagonally adjacent, we see that the squares alternate from type 1 to type 2 and that there must be an even number of them. Thus, if there is an odd number of black squares, we have two black squares that are vertically adjacent and two black squares that are horizontally adjacent.

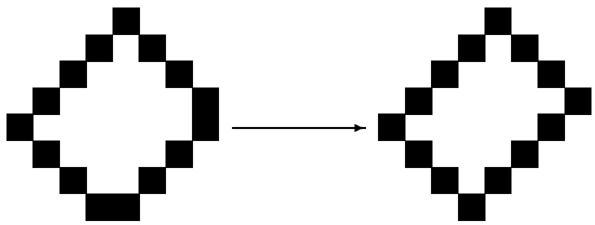
Connect the centres of adjacent squares. Then for a maximal number of white squares surrounded, this polygon must be convex, since if it is not convex, we can increase the number of white squares surrounded, as shown below.



Thus the black squares must form a rectangular formation, as in the second part of the figure above, if the number of black squares is even. If there are $2a + 2b$ black squares, $a + 1$ along two opposite sides and $b + 1$ along the other two sides, the number of white squares enclosed can be found to be $ab + (a-1)(b-1) = 2ab - a - b + 1$, so the maximum value is $2k^2 - 2k + 1$ if there are $4k$ black squares and $2k^2$ if there are $4k + 2$ black squares, for each $k > 0$.

[*Editor's note*. With $2a+2b = 4k$ the number of white squares is $2ab-2k+1$, which is maximized when $a = b = k$; when $2a + 2b = 4k + 2$ the number of white squares is $2ab - 2k$, which is maximized when $\{a, b\} = \{k, k + 1\}$.]

If the number of black squares is odd, there are two squares that are horizontally adjacent and two squares that are vertically adjacent. We can remove one of these squares and shift the others to form a closed figure as shown below.



This second figure must be a rectangle, so we can add on at most $a - 1$ squares, where $a + 1$ is the number of squares in one of the largest sides of the rectangle. Therefore if there are $4k + 1$ black squares, we can surround $2k^2 - k$ white squares; with $4k + 3$ black squares, we can surround $2k^2 + k$ white squares. [*Editor's note*. Here are some details. Letting the resulting black rectangle have $a+1$ squares on one side and $b+1$ on the other, where $a \geq b$, there are $2a + 2b$ black squares surrounding $2ab - a - b + 1$ white squares as before. Thus in the original figure there are $2a + 2b + 1$ black squares surrounding $2ab - b$ white squares. Now if the number of black squares is $4k+1$, then $a+b = 2k$, and $2ab-b$ is maximized when $a = b = k$; thus the number of white squares surrounded is $2k^2 - k$. Similarly, if the number of black squares is $4k + 3$, then $a + b = 2k + 1$, and $2ab - b$ is maximized when $a = k + 1$, $b = k$; and this time the number of white squares surrounded is $2(k + 1)k - k = 2k^2 + k$.]

Also solved by HAYO AHLBURG, Benidorm, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; R. DANIEL HURWITZ, Skidmore College, Saratoga Springs, New York; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KOJI SHODA, Nemuro City, Japan; UNIVERSITY OF ARIZONA PROBLEM SOLVING LAB, Tucson; and the proposer. One incorrect and one incomplete solution were also received.

Bosley and Janous actually made similar minor errors at the end of their solutions. Bosley's has been corrected in the above writeup.

2029***.** [1995: 91] *Proposed by Jun-hua Huang, the Middle School Attached To Hunan Normal University, Changsha, China.*

ABC is a triangle with area F and internal angle bisectors w_a, w_b, w_c. Prove or disprove that

$$w_b w_c + w_c w_a + w_a w_b \geq 3\sqrt{3}F.$$

Solution by Kee-Wai Lau, Hong Kong.

The inequality is true.

Denote, as usual, the semi-perimeter, inradius and circumradius by s, r and R respectively. In the following, sums and products are cyclic over A, B, C, and/or over a, b, c, as is appropriate.

We need the following identities:

$$\begin{aligned}
w_a &= \frac{2ab}{b+c}\cos(A/2), \quad \text{etc.}, & (1)\\
F &= \tfrac{1}{2}ab\sin(C), \quad \text{etc.}, & (2)\\
\sum \sin^2(A) &= \frac{s^2-4Rr-r^2}{2R^2}, & (3)\\
\sum \sin(A) &= \frac{s}{R}, & (4)\\
\sum \sin(A)\sin(B) &= \frac{s^2+4Rr+r^2}{4R^2}, & (5)\\
\prod \sin(A) &= \frac{sr}{2R^2}, & (6)\\
\sum \cos(A) &= \frac{R+r}{R}, & (7)\\
\sum \cos(A)\cos(B) &= \frac{r^2+s^2-4R^2}{4R^2}, & (8)\\
\prod \cos(A) &= \frac{s^2-4R^2-4Rr-r^2}{4R^2}, & (9)\\
\prod \cos\left(\frac{A-B}{2}\right) &= \frac{s^2+2Rr+r^2}{8R^2}. & (10)
\end{aligned}$$

We also need the inequality:

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2. \qquad (11)$$

Identities (1) and (2) are well-known. Identities (3) through (9) can be found on pages 55–56 of [1]. Identity (10) can be obtained from (5) and (8) by rewriting $\cos\left(\frac{A-B}{2}\right)$ as $\frac{1}{4}\left(1+\sum\sin(A)\sin(B)+\sum\cos(A)\cos(B)\right)$.

Inequality (11) is due to J.C. Gerretsen, and can be found on page 45 of [1].

From (1) and (2), we have

$$\begin{aligned}\frac{\sum w_a w_b}{F} &= 8\sum\left(\frac{c^2\cos(A/2)\cos(B/2)}{(c+a)(c+b)\sin(C)}\right)\\ &= \sum\left(\frac{2\sin(C)}{\cos((B-C)/2)\cos((A-C)/2)}\right).\end{aligned}$$

Thus, it is sufficient to prove that

$$\sum\sin(A)\cos((B-C)/2)\geq\frac{3\sqrt{3}}{2}\prod\cos((B-C)/2). \tag{12}$$

By squaring both sides, we see that (12) is equivalent to

$$\begin{aligned}&4\sum\sin^2(A)\cos^2((B-C)/2)\\ &\quad+8\sum\sin(A)\sin(B)\cos((B-C)/2)\cos((A-C)/2)\\ &\quad\quad\geq 27\prod\cos^2((B-C)/2).\end{aligned} \tag{13}$$

To prove (13), it suffices to show that

$$\begin{aligned}&4\sum\sin^2(A)\cos^2((B-C)/2)\\ &\quad+8\prod\cos((B-C)/2)\sum\sin(A)\sin(B)\\ &\quad\quad-27\prod\cos^2((B-C)/2)\geq 0.\end{aligned} \tag{14}$$

Since

$$\begin{aligned}&4\sum\sin^2(A)\cos^2((B-C/2))\\ &= 2\Big(\sum\sin^2(A)+\sum\cos(B)\cos(C)\\ &\quad-\prod\cos(A)\sum\cos(A)+\prod\sin(A)\sum\sin(A)\Big),\end{aligned}$$

we use (3) – (10), to obtain that (14) is equivalent to

$$11s^4+(22r^2-20rR-64R^2)s^2-148r^2R^2-20r^3R+11r^4\leq 0. \tag{15}$$

Let $x=s^2$ and denote the left side of (15) by $f(x)$. Then $f(x)$ is a convex function of x. In view of (11), in order to prove (15), it is sufficient to show both

$$f(16Rr-5r^2)\leq 0, \tag{16}$$
$$f(4R^2+4Rr+3r^2)\leq 0. \tag{17}$$

Now

$$f(16Rr-5r^2)=-4r(R-2r)(256R^2-155rR+22r^2),$$

and

$$f(4R^2+4Rr+3r^2)=-4(R-2r)(20R^3+36rR^2+45r^2R+22r^3).$$

Since $R\geq 2r$, both (16) and (17) hold, and the desired inequality is proved.

Reference

[1.] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

No other solutions were received.

2030. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

For which complex numbers s does the polynomial $z^3 - sz^2 + \overline{s}z - 1$ possess exactly three distinct zeros having modulus 1?

I. *Combination of solutions by F.J. Flanigan, San Jose State University, and the late John B. Wilker, University of Toronto.*

Denote by S the set of such complex numbers s. We offer two parametrizations of S (neither injective), the second of these yielding a description of S as the interior of a certain curvilinear triangle (*hypocycloid* or *deltoid*) inscribed in the disc $|z| \leq 3$.

Since the three zeros have modulus 1 and product equal to unity, they may be written as $e^{i\alpha}$, $e^{i\beta}$, $e^{-i(\alpha+\beta)}$ with $0 \leq \alpha, \beta < 2\pi$. It follows that

$$s = e^{i\alpha} + e^{i\beta} + e^{-i(\alpha+\beta)}. \tag{1}$$

Moreover, the three zeros will be distinct if and only if $\alpha \neq \beta$ and neither $2\alpha + \beta$ nor $2\beta + \alpha$ is an integer multiple of 2π. (We note in passing that the fact the coefficient of z in the cubic is $\overline{s}$ does not impose a further restriction.)

From (1) we see that $|s| \leq 3$ with equality if and only if $\alpha = \beta = -(\alpha + \beta) \mod 2\pi$.

We improve (1) by noting that $e^{i\alpha} + e^{i\beta} = e^{i(\alpha+\beta)/2}(e^{i(\alpha-\beta)/2} + e^{-i(\alpha-\beta)/2}) = 2\cos\delta e^{i\mu}$ where $\mu = (\alpha+\beta)/2$ and $\delta = (\alpha - \beta)/2$. Thus we have

$$s = 2\cos\delta e^{i\mu} + e^{-2i\mu}. \tag{2}$$

Equation (2) enables us to visualize the parameter set S as follows. Fix μ and let $\cos\delta$ vary through its full range: $-1 \leq \cos\delta \leq 1$. Then the complex numbers $2\cos\delta e^{i\mu}$ lie on a line segment of length 4 centred at the origin in the direction of the vector $e^{i\mu}$. Thus the points s for this fixed μ lie on a line segment of length 4 joining $P = e^{-2i\mu} + 2e^{i\mu}$ to $P' = e^{-2i\mu} - 2e^{i\mu}$. This segment is centred at the point $e^{-2i\mu}$ and makes an angle μ with the x-axis. The set S is the union of all these (overlapping) segments PP'.

As μ varies, the path of either endpoint of the segment is the curvilinear triangle with vertices at $3, 3e^{\pi i/3}$, and $3e^{-\pi i/3}$, namely,

$$z(t) = 2e^{it} + e^{-2it}$$

(where $P = z(\mu)$ and $P' = z(\mu + \pi)$). One verifies easily that this curve is a hypocycloid with three cusps: it is the locus of a point P fixed to the circumference of a circle (whose diameter is PP') that is rolling clockwise around the inside of the circle $|z| = 3$ as μ runs from 0 to 2π. This curve is called a deltoid because its shape resembles the Greek letter Δ ([2], pp 73–79).

Define $P'' = 2\cos(3\mu)e^{i\mu} + e^{-2i\mu}$. Note that P'' lies on PP', and since

$$P'' = (e^{3i\mu} + e^{-3i\mu})e^{i\mu} + e^{-2i\mu} = e^{4i\mu} + 2e^{-2i\mu} = z(-2\mu),$$

it lies on the deltoid as well. Moreover, P'' is the only point of the deltoid in the interior of the segment PP' (as is easily verified, or see [2], p. 75), so that PP' is tangent to the deltoid at P''. It follows that all points inside the deltoid are in S (which is the union of these diameters). Note further that P (the point where $\delta = 0$) corresponds to $\alpha = \beta$, P' (where $\delta = \pi$) to $\alpha = \beta + 2\pi$, and P'' (where $\delta = 3\mu$) to $\alpha = -2\beta$; consequently, the values of s on the deltoid correspond to multiple zeros of the given cubic ($e^{i\alpha} = e^{i\beta}$ or $e^{i\beta} = e^{-i(\alpha+\beta)}$). Since the proposal excludes multiple zeros we conclude that S is precisely the interior of the region bounded by the deltoid.

Editor's comment (by Chris Fisher). It is certainly clear (from (2)) that all points between P and P' on the segment PP' lie in S. Many of those whose solutions described S geometrically concluded that the interior of the segment joining the points P and P' automatically lies inside the deltoid, since P and P' lie on the boundary. However, further argument seems to be required: since S is not a convex region we have no guarantee against an interval of points on PP' that (like P'') lie outside the interior of the deltoid.

II. *Solution by Kurt Fink and Jawad Sadek, Northwest Missouri State University.*

Let $P(z) = z^3 - sz^2 + \overline{s}z - 1$ and let $P'(z) = 3z^2 - 2sz + \overline{s}$ be its derivative. A result of A. Cohn, specialized to $P(z)$ (see [3], p. 206, Exercise 3), states that the zeros of $P(z)$ lie on the unit circle and are simple if and only if the zeros of $P'(z)$ lie in $|z| < 1$ or, equivalently, the zeros of the polynomial $\overline{s}z^2 - 2sz + 3$ lie in $|z| > 1$. An application of Theorem 6.8b on p. 493 of [1], shows that (we omit the elementary calculations) this occurs if and only if $|s| < 3$ and $|s|^4 + 18|s|^2 - 8\Re(s^3) - 27 < 0$. If $s = x + iy$, the condition becomes

$$(x^2+y^2)^2+18(x^2+y^2)-8(x^3-3xy^2)-27<0.$$

This is the interior of a deltoid whose closure is contained in the disk of radius 3 centred at the origin; it touches the boundary circle at the points $(3,0)$, $(-3/2, 3\sqrt{3}/2)$, and $(-3/2, -3\sqrt{3}/2)$. The closed deltoid region contains the unit circle and boundaries of these curves meet at $(-1,0)$, $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$.

References

1. Peter Henrici, *Applied and Computational Complex Analysis*, Vol. 1, Wiley, New York, 1974.

2. E.H. Lockwood, *A Book of Curves*, Cambridge Univ. Press, Cambridge, 1963.

3. Morris Marden, *Geometry of Polynomials*, Amer. Math. Society, Princeton, 1985.

Also solved by ED BARBEAU, University of Toronto, Toronto, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Janous showed that if the polynomial

$$p(z)=z^n-sz^{n-1}+\cdots+(-1)^n$$

has n zeros of modulus 1, then $s=\sum_{k=1}^{n} e^{i\lambda_k}$, where $\lambda_1,\ldots,\lambda_n\in[0,2\pi)$ and $\lambda_1+\cdots+\lambda_n = 0 \pmod{2\pi}$. Furthermore, if a_k is the coefficient of z^k then, necessarily, $(-1)^n\,\overline{a}_k$ is the coefficient of z^{n-k}. The proposer outlined an argument that if the polynomial

$$p(z)=z^n-sz^{n-1}+\overline{s}z-1$$

has n zeros of modulus 1, then s belongs to the interior of the region bounded by the hypocycloid

$$z(t)=\frac{n-1}{n-2}\,e^{it}+\frac{1}{n-2}\,e^{-i(n-1)t}.$$

2031. [1995: 129] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that α, β, γ are acute angles such that

$$\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}+\frac{\sin(\beta-\gamma)}{\sin(\beta+\gamma)}+\frac{\sin(\gamma-\alpha)}{\sin(\gamma+\alpha)}=0.$$

Prove that at least two of α, β, γ are equal.

All solvers had the same idea, so we present a composite solution.

By dividing the first term, top and bottom, by $\cos\alpha\cos\beta$, and the other two terms similarly, the given condition is equivalent to

$$\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}+\frac{\tan\beta-\tan\gamma}{\tan\beta+\tan\gamma}+\frac{\tan\gamma-\tan\alpha}{\tan\gamma+\tan\alpha}=0.$$

By multiplying by the common denominator, this reduces to

$$(\tan\alpha-\tan\beta)(\tan\beta-\tan\gamma)(\tan\gamma-\tan\alpha)=0.$$

Hence, at least two of $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are equal, and since α, β, γ are acute angles, at least two of them must be equal.

Solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; THEODORE CHRONIS, student, Aristotle University of Thessalonika, Thessalonika, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, Haifa University, Haifa, Israel; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D. J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2032. [1995: 129] *Proposed by Tim Cross, Wolverley High School, Kidderminster, UK.*

Prove that, for nonnegative real numbers x, y and z,

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\geq\sqrt{6(x+y+z)}\,.$$

When does equality hold?

This problem attracted 25 correct solutions. Six solvers (marked † in the list of solvers) gave the generalization highlighted below. So we present a composite solution based on the submissions of several solvers.

We prove the generalization: for nonnegative real numbers $\{x_k\}$,

$$\sum_{k=1}^{n} \sqrt{x_k^2 + 1} \geq \sqrt{2n \sum_{k=1}^{n} x_k}.$$

From $(x_k x_j - 1)^2 \geq 0$, we get

$$x_k^2 x_j^2 + x_k^2 + x_j^2 + 1 \geq x_k^2 + 2x_k x_j + x_j^2.$$

Hence

$$(x_k^2 + 1)(x_j^2 + 1) \geq (x_k + x_j)^2.$$

Taking the square root, and since the x_k are nonnegative, we have

$$2\sqrt{(x_k^2 + 1)}\sqrt{(x_j^2 + 1)} \geq 2(x_k + x_j).$$

Hence

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \sqrt{(x_k^2 + 1)}\sqrt{(x_j^2 + 1)} \geq 2(n-1) \sum_{k=1}^{n} x_k.$$

We now add the nonnegative quantities $(x_k - 1)^2$, to get

$$\sum_{k=1}^{n} (x_k - 1)^2 + \sum_{k=1}^{n} \sum_{j=1}^{n} \sqrt{(x_k^2 + 1)}\sqrt{(x_j^2 + 1)} \geq 2(n-1) \sum_{k=1}^{n} x_k.$$

Thus

$$\sum_{k=1}^{n} x_k^2 \; + \; n \sum_{k=1}^{n} \sum_{j=1}^{n} \sqrt{(x_k^2 + 1)}\sqrt{(x_j^2 + 1)} \geq 2n \sum_{k=1}^{n} x_k.$$

But the left side is the square of

$$\sum_{k=1}^{n} \sqrt{(x_k^2 + 1)},$$

and so the result follows.

Equality holds when $x_k = 1$ for all k such that $1 \leq k \leq n$.

Solved by †ŠEFKET ARSLANAGIĆ, Berlin, Germany; †NIELS BEJLEGAARD, Stavanger, Norway; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; †MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; SABIN CAUTIS, student, Earl Haig Secondary School, North York Ontario; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; †THEODORE CHRONIS, student, Aristotle University of Thessalonika, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; †WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; †VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; SAI C. KWOK, Boulder, Colorado, USA; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, Haifa, Israel; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; SCIENCE ACADEMY PROBLEM SOLVERS, Austin, Texas, USA; †HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Four incorrect solutions were received. They all used calculus. Those solvers are referred to the comment on the solution to problem 2015, earlier in this issue.

2033. [1995: 129] *Proposed by K. R. S. Sastry, Dodballapur, India.*

The sides AB, BC, CD, DA of a convex quadrilateral $ABCD$ are extended in that order to the points P, Q, R, S such that $BP = CQ = DR = AS$. If $PQRS$ is a square, prove that $ABCD$ is also a square.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario.

Suppose that $ABCD$ is not a rectangle. Then it must contain an interior angle greater than $\frac{\pi}{2}$. Without loss of generality, let $\angle A > \frac{\pi}{2}$.

Since $PQRS$ is a square, $PQ = QR = RS = SP$. Rotate QP about P clockwise until Q coincides with S. Let B' be the new position of B. Now

$$\begin{aligned} \angle B'PA &= \angle B'PS + \angle SPA \\ &= \angle BPQ + \angle SPA \quad \text{(from rotation)} \\ &= \angle SPQ = \frac{\pi}{2}. \end{aligned}$$

(Editor's note: Since $PQRS$ is a square, this is just a rotation through $\frac{3\pi}{2}$ radians and the fact that $\angle B'PA = \frac{\pi}{2}$ follows directly.)

Since $\angle A > \frac{\pi}{2}$, $\angle SAP < \frac{\pi}{2}$ which implies that $PB' = AS > h$, where h is an altitude from S to AP. Then

$$\frac{\pi}{2} > \angle SB'P = \angle QBP,$$

so $\angle ABC > \frac{\pi}{2}$.

Similar arguments show that $\angle BCD > \frac{\pi}{2}$ and $\angle CDA > \frac{\pi}{2}$. Therefore, the sum of the interior angles of $ABCD > 2\pi$. Impossible!

Therefore, $ABCD$ is a rectangle.

Since $\angle SAP = \angle PBQ = \angle QCR = \angle RDS = \frac{\pi}{2}$, the triangles ASP, BPQ, CQR, DRS are congruent and $AB = BC = CD = DA$. Therefore, $ABCD$ is a square.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incorrect solution. The proposer notes that his starting point was the analogous theorem for triangles.

Bradley notes that the problem was set in the Second Selection examination for the 36th IMO in Bucharest in April 1995 and is attributed to L. PANAITOPOL. He also notes that two solutions are given, one trigonometrical, which is correct, the other, a pure solution, which is in error.

Finally, both Bradley and Smeenk note that a simple modification of the proof works in the case of convex polygons of any number of sides.

2034. [1995: 130, 157] *Proposed by Murray S. Klamkin and M. V. Subbarao, University of Alberta.*

(a) Find all sequences $p_1 < p_2 < \cdots < p_n$ of distinct prime numbers such that

$$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_2}\right)\cdots\left(1+\frac{1}{p_n}\right)$$

is an integer.

(b) Can

$$\left(1+\frac{1}{a_1^2}\right)\left(1+\frac{1}{a_2^2}\right)\cdots\left(1+\frac{1}{a_n^2}\right)$$

be an integer, where $a_1, a_2, \ldots$ are distinct integers greater than 1?

Solution to (a), by Heinz-Jürgen Seiffert, Berlin, Germany.

If the considered product is an integer, then $p_n|(p_i+1)$ for some $i \in \{1, 2, \ldots, n-1\}$. Since $p_i + 1 \le p_n$, it then follows that $p_n = p_i + 1$, which implies $p_i = 2$ and $p_n = 3$. Thus, $n = 2$, $p_1 = 2$, $p_2 = 3$. This is indeed a solution since $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) = 2$.

Solution to (b). In the following, we present four different solutions submitted by eight solvers and the proposers. In all of them, p denotes the given product, and it is shown that $1 < p < 2$ and thus, p cannot be an integer. Clearly one may assume, without loss of generality, that $1 < a_1 < a_2 < \ldots < a_n$.

Solution I, by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA; Kee-Wai Lau, Hong Kong; and Kathleen E. Lewis, SUNY, Oswego, New York.

Since $1 + x < e^x$ for $x > 0$, we have

$$\begin{aligned} 1 < p &< \exp\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \ldots + \frac{1}{a_n^2}\right) \\ &< \exp\left(\frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \\ &= \exp\left(\frac{\pi^2}{6} - 1\right) \approx 1.90586 < 2. \end{aligned}$$

Solution II, by Toby Gee, student, the John of Gaunt School, Trowbridge, England.

Since $a_1 \geq 2$, we have

$$\begin{aligned} 1 < p &\leq \prod_{k=2}^{n+1}\left(1 + \frac{1}{k^2}\right) = \prod_{k=2}^{n+1}\frac{k^2+1}{k^2} < \prod_{k=2}^{n+1}\frac{k^2}{k^2-1} \\ &= \left(\prod_{k=2}^{n+1}\frac{-k}{k+1}\right)\left(\prod_{k=2}^{n+1}\frac{k}{k-1}\right) = \frac{2(n+1)}{n+2} < 2. \end{aligned}$$

Solution III, by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Václav Konečný, Ferris State University, Big Rapids, Michigan, USA; Heinz-Jürgen Seiffert, Berlin, Germany; and the proposers.

$$1 < p < \prod_{t=2}^{\infty}\left(1 + \frac{1}{t^2}\right) = \frac{1}{2}\prod_{t=1}^{\infty}\left(1 + \frac{1}{t^2}\right) = \frac{\sinh \pi}{2\pi} \approx 1.838 < 2.$$

Solution IV, Richard I. Hess, Rancho Palos Verdes, California, USA.

Let $p_\infty = \prod_{k=2}^{\infty}\left(1 + \frac{1}{h^2}\right)$. Then clearly $1 < p < p_\infty = Q \times R$ where $Q = \prod_{k=2}^{10}\left(1 + \frac{1}{h^2}\right) \approx 1.6714$ and $R = \prod_{k=11}^{\infty}\left(1 + \frac{1}{k^2}\right)$.

Now, $\ln R = \sum_{k=11}^{\infty}\ln\left(1 + \frac{1}{k^2}\right) < \int_{10}^{\infty}\ln\left(1 + \frac{1}{x^2}\right)dx < \int_{10}^{\infty}\frac{dx}{x^2} = \frac{1}{10}$.

(Ed: This is because $f(x) = \ln\left(1 + \frac{1}{x^2}\right)$ is strictly decreasing on $(0, \infty)$ and $\ln(1+t) < t$ for $t > 0$).

Therefore, $R < e^{0.1} \approx 1.1052$, and so $p_\infty < (1.68)(1.11) = 1.8648 < 2$.

Part (a) *was also solved by SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; SHAWN GODIN, St. Joseph Scollard Hall,*
North Bay, Ontario; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; and ASHISH KR. SINGH, student, Kanpur, India.

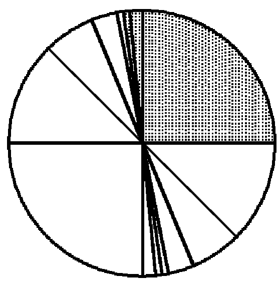
2036. [1995: 130] *Proposed by Victor Oxman, Haifa University, Israel.*

You are given a circle cut out of paper, and a pair of scissors. Show how, by cutting only along folds, to cut from the circle a figure which has area between 27% and 28% of the area of the circle.

Solution. The response to this problem was 10 different solutions in the range $[27.2535\ldots, 27.8834\ldots]$. Except for the proposer's solution, all involved straight line cuts. The proposer's solution involves cutting along four curved lines, outlined by parts of the circle that have been folded inwards.

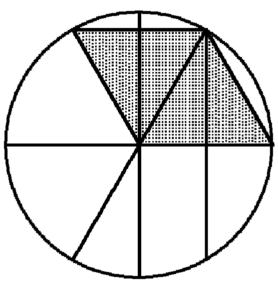
The minimum number of (single) folds was four – Hurthig wondered if a prize was available for the least!

Rather than give the details of any one, we shall present ten diagrams that illustrate the ingenuity of our readership, and leave you, the reader, to do the exact calculation for each.

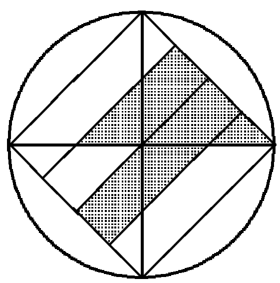


Gee, Godwin, Hsai

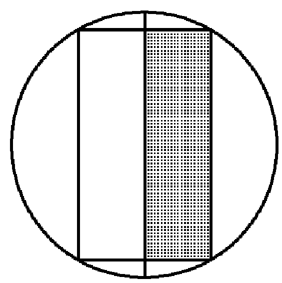
In the above diagram, each line bisects an angle formed previously, until an angle of $35\pi/64$ is generated.



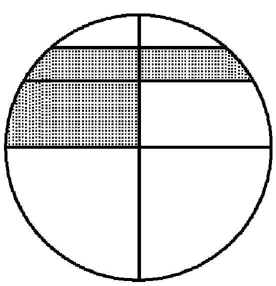
Bosley



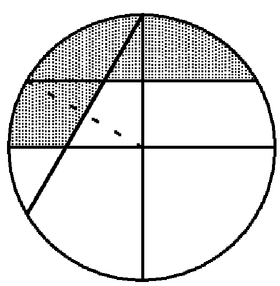
Godwin



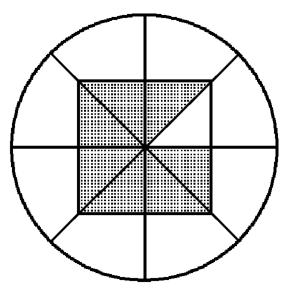
Grant, Hurthig



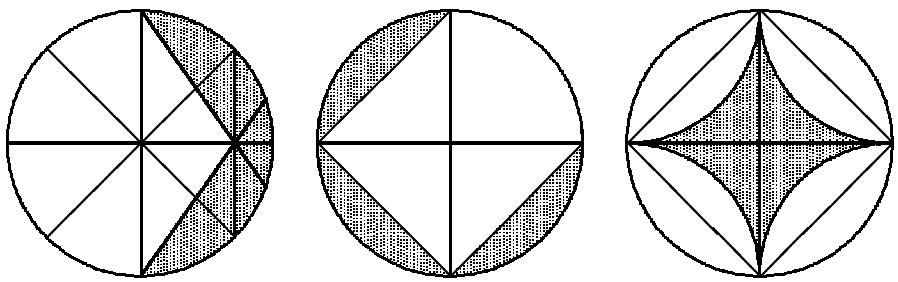
Hess



Hurthig



Jackson



Perz Tsaoussoglou The proposer

Solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; TOBY GEE; student, the John of Gaunt School, Trowbridge, England; DOUGLASS GRANT, University College of Cape Breton; Sydney, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales, New Mexico, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2037. [1995: 130] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

The lengths of the base and a slant side of an isosceles triangle are integers without common divisors. If the lengths of the angle bisectors of the triangle are all rational numbers, show that the length of the slant side is an odd perfect square.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

Let ΔABC be isosceles, with $b = c$, and let the angle bisectors be $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$. It is given that a and c are relatively prime integers. Using Stewart's Theorem, for example, one can show that the length of the angle bisector to $\overline{AB}$ (and hence to $\overline{AC}$) is

$$CF = \frac{\sqrt{ac(a+b+c)(a+b-c)}}{a+b} = \frac{a\sqrt{c(2c+a)}}{a+c}\,.$$

[This formula can also be derived from the formula for the angle bisector given in *CRUX* on [1995: 321]. — *Ed.*] Also [by the Pythagorean Theorem]

$$AD = \frac{\sqrt{4c^2-a^2}}{2}\,.$$

Since CF and AD are both rational, we must have, for some positive integers u and v,

$$c(2c+a) = u^2 \qquad \text{and} \qquad 4c^2 - a^2 = v^2.$$

Since $\gcd(a, c) = 1$, it follows that $\gcd(a, 2c+a) = 1$. Hence both c and $2c + a$ are perfect squares. Now, if c is even, then a, being relatively prime to c, is odd. From the equation $4c^2 - a^2 = v^2$ we see that v too is odd. Thus $a^2 + v^2 \equiv 2 \bmod 4$. But $4c^2 \equiv 0 \bmod 4$, a contradiction. Thus c is an odd perfect square.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; RICHARD K. GUY, University of Calgary, Calgary, Alberta; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. Four slightly incorrect solutions were also received, most of these assuming, without proof, that the altitude from the apex of the triangle is an integer.

Amengual notes that the perimeter of the triangle is also a perfect square. This is contained in the above proof, since $2c + a$ is the perimeter.

The proposer gave several examples of triangles satisfying the given property. The smallest has base 14 and slant sides 25.

2038. [1995: 130] *Proposed by Neven Jurić, Zagreb, Croatia.*

Show that, for any positive integers m and n, there is a positive integer k so that

$$\left(\sqrt{m} + \sqrt{m-1}\right)^n = \sqrt{k} + \sqrt{k-1}.$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain. We prove a more general result:

Let $a \geq b$ be non-negative real numbers, and let $n \in \mathbb{N}$. Let

$$p = \frac{(a+b)^2 - (a-b)^n}{2}.$$

Then

$$p^2 + \left(a^2 - b^2\right)^n = \frac{1}{4}\left((a+b)^n + (a-b)^n\right)^2,$$

giving

$$\sqrt{p^2 + (a^2 - b^2)^n} = \frac{1}{2}\left((a+b)^n + (a-b)^n\right) = (a+b)^n - p.$$

This given

$$\sqrt{p^2 + (a^2 - b^2)^n} + \sqrt{p^2} = (a+b)^n. \qquad (1)$$

In (1), set $a = \sqrt{s}$, $b = \sqrt{s-k}$, where $s, k \in \mathbb{N}$, $s \geq k$, so that p^2 is a positive integer, and we have

$$\left(\sqrt{s} + \sqrt{s-k}\right)^n = \sqrt{p^2} + \sqrt{p^2 - k^2}. \qquad (2)$$

Letting $k = 1$ in (2), gives the result of the original problem here:

$$\left(\sqrt{s} + \sqrt{s-1}\right)^n = \sqrt{p^2} + \sqrt{p^2 - 1}. \qquad (3)$$

If we let $s = 2$ in (3), we get

$$\left(\sqrt{2} + 1\right)^n = \sqrt{p^2} + \sqrt{p^2 - 1},$$

which was a problem proposed by Hong Kong (but not used) at the 1989 IMO at Braunschweig, Germany. This version also appears as Elementary Problem E 950 (proposed by W.R. Ransom, Tufts College) in the American Mathematical Monthly **58** (1951) p. 566, on which we have based our solution.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; TOBY GEE, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; VICTOR OXMAN, Haifa, Israel; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; SCIENCE ACADEMY PROBLEM SOLVERS; Austin, Texas, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. One incorrect submission was received.

Chan states that the problem is in the 1980 Romanian Mathematical Olympiad. Perz observes that the special case $m = 10$ occurs in **CRUX** [1977: 190]*, and, along with Hsia, notes that the case $m = 2$ was in the 1994 Canadian Mathematical Olympiad (see* **CRUX** [1994: 151], [1994: 186])

2039*. [1995: 130] Proposed by Dong Zhou, Fudan University, Shanghai, China, and Ji Chen, Ningbo University, China.

Prove or disprove that

$$\frac{\sin A}{B} + \frac{\sin B}{C} + \frac{\sin C}{A} \geq \frac{9\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle. [Compare with *CRUX* 1216 [1988: 120] and this issue!]

Solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia.

In problem 2015 [1995: 53 and 129 (Corrected), 1996: 47], it was shown that

$$K = (\sin A + \sin B + \sin C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angle of a triangle measured in radians, and that the lower bound is attained in the equilateral triangle.

Let

$$\begin{aligned} F &= \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}, \\ G &= \frac{\sin A}{B} + \frac{\sin B}{C} + \frac{\sin C}{A}, \\ H &= \frac{\sin A}{C} + \frac{\sin B}{A} + \frac{\sin C}{B}. \end{aligned}$$

Then $K = F + G + H$. By problem 1216 [1987: 53, 1988: 120], we have $2 < F \leq \frac{9\sqrt{3}}{2\pi}$, so that we have $G + H \geq \frac{18\sqrt{3}}{2\pi}$.

But G can be obtained from H by a permutation of symbols. Thus each of G and H has the same minimum, which will be at least $\frac{9\sqrt{3}}{2\pi}$.

But, in the equilateral triangle, G (and H) achieves this value, so the required inequality is true and sharp.

No other submission was received.