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Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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THE RECURRENCE RELATION $a_n = (An + B)a_{n-1} + (Cn + D)a_{n-2}$

Kenneth S. Williams

Suppose n objects are labelled $1, 2, \cdots, n$. A permutation of these objects in which object i is not placed in the ith place for any i is called a *derangement.* It is well-known (see for example [3, p. 204]) that the number D_n of derangements of n objects satisfies the recurrence relation

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2} \quad (n = 3, 4, \cdots)$$

with $D_1 = 0$, $D_2 = 1$. Less well-known is the fact that the number R_n of permutations of the n objects in which object $i+1$ is not positioned immediately to the right of object i for any i satisfies the recurrence relation

$$R_n = (n-1)R_{n-1} + (n-2)R_{n-2} \quad (n = 3, 4, \cdots)$$

with $R_1 = 1, R_2 = 1$, see for example [2, p. 115].

Both of these recurrence relations are of the type

$$a_n = (An + B)a_{n-1} + (Cn + D)a_{n-2} \quad (n = 3, 4, \cdots) \tag{1a}$$

with

$$a_1 = \alpha, \ a_2 = \beta \tag{1b}$$

for suitable real numbers A, B, C, D, α and β. If both A and C are zero the recurrence relation (1a) reduces to $a_n = Ba_{n-1} + Da_{n-2}$ whose solution is well-known (see for example [1, p. 250]) so we can exclude this possibility. Similarly if A and B are both zero or C and D are both zero, the recurrence relation is easy to solve, so we can exclude these possibilities as well. However, it is very difficult to solve the recurrence relation (1ab) in general. We identify four situations when a simple explicit solution can be given.

To solve the above recurrence relation for the number of derangements D_n, we usually rewrite the recurrence as

$$D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2}) \quad (n = 3, 4, \cdots)$$

so that

$$D_n - nD_{n-1} = (-1)^n(D_2 - 2D_1) = (-1)^n \quad (n = 2, 3, \cdots).$$

Thus

$$\frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!} \quad (n = 2, 3, \cdots)$$

and so

$$\frac{D_n}{n!} - \frac{D_1}{1!} = \sum_{k=2}^{n} \left(\frac{D_k}{k!} - \frac{D_{k-1}}{(k-1)!} \right) = \sum_{k=2}^{n} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots)$$

giving

$$D_n = n!\sum_{k=2}^{n}\frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots),$$

which is the well-known expression for D_n (see for example 2, p. 114], [3, p. 225]).

We now attempt to apply this technique to the general recurrence relation (1ab) by trying to express it in the form

$$a_n - (An + B + X)a_{n-1} = Y(a_{n-1} - (A(n-1) + B + X)a_{n-2}) \quad (n = 3, 4, \cdots) \tag{2}$$

for suitable constants X and Y. By comparing (2) with (1a) we see that this is possible if and only if

$$X = \frac{C}{A}, \quad Y = -\frac{C}{A}, \quad D = -C + \frac{BC}{A} + \frac{C^2}{A^2}.$$

From (2) with these values, we obtain

$$a_n - \left(An + B + \frac{C}{A}\right)a_{n-1} = \left(\frac{-C}{A}\right)^{n-2}\left(a_2 - \left(2A + B + \frac{C}{A}\right)a_1\right) \quad (n = 2, 3, \cdots).$$

Appealing to (1b) we see that

$$a_n - \left(An + B + \frac{C}{A}\right)a_{n-1} = \left(\frac{-C}{A}\right)^{n-2}\left(\beta - \left(2A + B + \frac{C}{A}\right)\alpha\right) \quad (n = 2, 3, \cdots). \tag{3}$$

We solve the two-term recurrence relation (3) by making use of the following result.

LEMMA: Let $f(n)$ and $g(n)$ be real-valued functions defined for $n = 2, 3, 4, \ldots$. If b_n $(n = 1, 2, 3, \cdots)$ satisfies the recurrence relation

$$b_n = f(n)b_{n-1} + g(n) \quad (n = 2, 3, \cdots)$$

then

$$b_n = b_1\prod_{i=2}^{n} f(i) + \sum_{k=2}^{n} g(k)\prod_{i=k+1}^{n} f(i) \quad (n = 1, 2, 3, \cdots).$$

Remark: We remark that the empty sum is always understood to be 0 and the empty product to be 1.

Proof: For $k = 1, 2, \cdots, n$ we set

$$p(n,k) = \prod_{i=k+1}^{n} f(i),$$

so that is particular we have $p(n,n) = 1$. Multiplying the relation

$$b_k - f(k)b_{k-1} = g(k) \quad (k = 2, 3, \cdots)$$

by $p(n,k)$, we obtain

$$p(n,k)b_k - p(n,k-1)b_{k-1} = p(n,k)g(k) \quad (k = 2, 3, \cdots).$$

Summing over $k = 2, \cdots, n$, we have

$$\sum_{k=2}^{n}(p(n,k)b_k - p(n,k-1)b_{k-1}) = \sum_{k=2}^{n} p(n,k)g(k),$$

so that

$$b_n - p(n,1)b_1 = \sum_{k=2}^{n} p(n,k)g(k),$$

that is

$$b_n = b_1 \prod_{i=2}^{n} f(i) + \sum_{k=2}^{n} g(k) \prod_{i=k+1}^{n} f(i) \quad (n = 1, 2, \cdots),$$

as asserted. $\square$

Applying the Lemma with

$$f(n) = An + B + \frac{C}{A} \quad \text{and} \quad g(n) = \left(\frac{-C}{A}\right)^{n-2} \left(\beta \left(2A + B + \frac{C}{A}\right) \alpha\right),$$

we obtain the following explicit formula for the solution a_n of the recurrence relation (1ab) when $D = -C + BC/A + C^2/A^2$.

THEOREM 1: Let A, B, C, D be real numbers such that

$$D = -C + \frac{BC}{A} + \frac{C^2}{A^2}.$$

Then the solution of the recurrence relation (1ab) is

$$a_n = \alpha \prod_{i=2}^{n} \left(Ai + \left(B + \frac{C}{A}\right)\right) + \left(\beta - \left(2A + B + \frac{C}{A}\right) \alpha\right) \sum_{k=2}^{n} \left(\frac{-C}{A}\right)^{k-2} \prod_{i=k+1}^{n} \left(Ai + \left(B + \frac{C}{A}\right)\right) \tag{4}$$

$(n = 1, 2, \cdots)$. $\square$

The recurrence relation for the number D_n of derangements is the special case of Theorem 1 with $A = 1$, $B = -1$, $C = 1$, $D = -1$, $\alpha = 0$, $\beta = 1$. With these values (4) gives

$$D_n = \sum_{k=2}^{n}(-1)^{k-2} \prod_{i=k+1}^{n} i = \sum_{k=2}^{n}(-1)^k \frac{n!}{k!} = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots).$$

The recurrence relation for R_n given above has $A = 1$, $B = -1$, $C = 1$, $D = -2$ and so does not satisfy the condition $D = -C + BC/A + C^2/A^2$. However, $b_n = nR_n$ is a solution of the recurrence relation $b_n = nb_{n-1} + nb_{n-2}$ which does satisfy this condition. Theorem 1 then gives an explicit expression for $R_n = b_n/n$.

More generally if we try to find real numbers X and Y such that

$$b_n = (Xn + Y)a_n \tag{5}$$

satisfies a recurrence relation of the type (1a) where A, B, C, D satisfy $D = -C + BC/A + C^2/A^2$, we can obtain the following result.

THEOREM 2: Let A, B, C, D be real numbers such that

$$C = A^2, \quad D = AB - A^2. \tag{6}$$

Then the solution of the recurrence relation (1ab) is

$$\begin{aligned} a_n = \frac{1}{An + (A+B)} \Bigg\{ & \alpha(2A+B) \prod_{i=2}^{n} (Ai + 2A + B) \\ & + ((3A+B)\beta - (2A+B)(4A+B)\alpha) \sum_{k=2}^{n} (-A)^{k-2} \prod_{i=k+1}^{n} (Ai + (2A+B)) \Bigg\} \end{aligned} \tag{7}$$

$(n = 1, 2, 3, \cdots)$. $\square$

The recurrence relation for the numbers R_n has $A = 1$, $B = -1$, $C = 1$, $D = -2$ so that (6) is satisfied. With these values of A, B, C, D and $\alpha = \beta = 1$, (7) gives

$$R_n = \frac{1}{n} \left\{ \prod_{i=2}^{n} (i+1) - \sum_{k=2}^{n} (-1)^{k-2} \prod_{i=k+1}^{n} (i+1) \right\} = \frac{(n+1)!}{n} \left\{ \frac{1}{2!} + \sum_{k=2}^{n} \frac{(-1)^{k-1}}{(k+1)!} \right\},$$

that is

$$R_n = \frac{(n+1)!}{n} \sum_{k=2}^{n+1} \frac{(-1)^k}{k!} \quad (n = 1, 2, \cdots).$$

Another method of trying to solve the recurrence relation (1a) is to try and express it in the form

$$a_n + Xa_{n-1} = (Yn + Z)(a_{n-1} + Xa_{n-2}) \quad (n = 3, 4, \cdots)$$

for suitable constants X, Y, Z. Again with some effort we may obtain the following result.

THEOREM 3: Let A, B, C, D be real numbers such that

$$D = \frac{BC}{A} + \frac{C^2}{A^2}. \tag{8}$$

Then the solution of the recurrence relation (1ab) is

$$a_n = \alpha \left(\frac{-C}{A} \right)^{n-1} + \left(\beta + \frac{C}{A} \alpha \right) \sum_{k=2}^{n} \left(\frac{-C}{A} \right)^{n-k} \prod_{i=3}^{k} \left(Ai + B + \frac{C}{A} \right) \quad (n = 1, 2, \cdots). \quad \square$$

The recurrence relation

$$\begin{cases} a_n = (2n+1)a_{n-1} + (4n+6)a_{n-2} & (n = 3, 4, \cdots), \\ a_1 = 1, a_2 = 2, \end{cases}$$

satisfies the condition (8) of Theorem 3. Hence

$$a_n = (-2)^{n-1} + 4\sum_{k=2}^{n}(-2)^{n-k}\prod_{i=3}^{k}(2i+3) \quad (n = 1, 2, \cdots).$$

Expressing $\prod_{i=3}^{k}(2i+3)$ in the form

$$\frac{(2k+3)!}{105 \cdot 2^{k+1}(k+1)!} \quad (k = 2, 3, \cdots),$$

we obtain

$$a_n = (-1)^n 2^{n-1}\left\{\frac{4}{105}\sum_{k=2}^{n}\frac{(-1)^k(2k+3)!}{2^{2k}(k+1)!} - 1\right\} \quad (n = 1, 2, \cdots).$$

Finally, if we consider the recurrence relation (1ab) with $A = 0$, $D = -C$, $\alpha = B$, $\beta = B^2 + C$, that is

$$\begin{cases} a_n = Ba_{n-1} + C(n-1)a_{n-2} & (n = 3, 4, \cdots), \\ a_1 = B, a_2 = B^2 + C, \end{cases}$$

and apply the standard generating function method, we obtain the following result.

THEOREM 4: The solution of the recurrence relation

$$\begin{cases} a_n = Ba_{n-1} + C(n-1)a_{n-2} & (n = 2, 3, \cdots) \\ a_0 = 1, a_1 = B, \end{cases} \tag{9}$$

is given by

$$a_n = n!\sum_{s=0}^{[n/2]}\frac{B^{n-2s}C^s}{2^s(n-2s)!s!} \quad (n = 0, 1, 2, \cdots). \qquad \square$$

As a concluding example we consider the number S_n of $n \times n$ symmetric $(0,1)$ matrices all of whose columns sum to 1. When $n = 1$ there is exactly one such matrix, namely [1], so that $S_1 = 1$. When $n = 2$ there are exactly two such matrices, namely,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $S_2 = 2$. When $n = 3$ there are exactly four such matrices, namely,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $S_3 = 4$. It is also easy to show that

$$S_n = S_{n-1} + (n-1)S_{n-2} \quad (n = 3, 4, \cdots). \tag{10}$$

If we set $S_0 = 1$ the recurrence relation (10) is (9) with $B = C = 1$. Thus, by Theorem 4, we deduce

$$S_n = n! \sum_{s=0}^{[n/2]} \frac{1}{2^s (n-2s)! s!} \quad (n = 0, 1, \cdots).$$

References:

[1] Norman L. Biggs, *Discrete Mathematics*, Oxford University Press (1985).

[2] Kenneth P. Bogart, *Introductory Combinatorics*, Pitman Publishing Inc. (1983).

[3] Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall Inc. (1984).

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THE OLYMPIAD CORNER

No. 154

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

As an Olympiad set for this issue we give the problems of the 15th Austrian–Polish Mathematics Competition held in Nowy Sácz (Poland), June 22–24, 1992. Many thanks go to Walther Janous, Ursulinengymnasium, Innsbruck, Austria; to Marcin E. Kuczma, Warszawa, Poland; and to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, who separately sent me copies of the contest.

1992 AUSTRIAN–POLISH MATHEMATICS COMPETITION

Individual Competition, First Day (Time: 4.5 hours)

1. For natural $n \geq 1$, let $s(n)$ denote the sum of all positive divisors of n. Prove that for every integer $n > 1$ the product $s(n-1)s(n)s(n+1)$ is even.

2. Each point on the boundary of a square has to be coloured in one colour. Consider all right triangles with three vertices on the boundary of the square. Determine the least number of colours for which there exists a colouring such that no such triangle has all its vertices of equal colour.

3. Prove that for all positive real numbers a, b, c the following inequality holds:

$$2\sqrt{bc+ca+ab} \leq \sqrt{3}\sqrt[3]{(b+c)(c+a)(a+b)}.$$

Individual Competition, Second Day (Time: 4.5 hours)

4. Let k be a natural number and u, v be real numbers. Set

$$P(x) = (x-u^k)(x-uv)(x-v^k) = x^3 + ax^2 + bx + c.$$

(a) For $k=2$ prove: If a, b, c are rational, then the product uv is rational.

(b) Is that also true for $k=3$?

5. Given a circle k of center M and radius r, let AB be a fixed diameter of k and let K be a fixed point of segment AM. Denote by t the line tangent to k at A. For any chord CD (other than AB) passing through K construct P and Q as the points of intersection of lines BC and BD with t. Prove that the product $AP \cdot AQ$ remains constant as the chord CD varies.

6. Let $\mathbf{Z}$ denote the set of all integers. Consider a function $f : \mathbf{Z} \to \mathbf{Z}$ with the properties:

$$\begin{aligned} f(92+x) &= f(92-x) \\ f(19\cdot 92+x) &= f(19\cdot 92-x) \quad (19\cdot 92 = 1748) \\ f(1992+x) &= f(1992-x) \end{aligned}$$

for all $x \in \mathbf{Z}$. Is it possible that all positive divisors of 92 occur as values of f?

Team Competition (Time: 4 hours)

7. We are considering triangles ABC in space.

(a) What conditions must be fulfilled by the angles α, β, γ of triangle ABC in order that there exists a point P in space such that $\angle APB$, $\angle BPC$, $\angle CPA$ are right angles?

(b) Let d be the maximum distance among PA, PB, PC and let h be the longest altitude of triangle ABC. Show that $(\sqrt{6}/3)h \leq d \leq h$.

8. Let $n \geq 3$ be a fixed integer. Real numbers $a_1, \ldots, a_n$ (different from zero) satisfy the equations

$$\begin{aligned} \frac{-a_1-a_2+a_3+\cdots+a_n}{a_1} &= \frac{a_1-a_2-a_3+a_4+\cdots+a_n}{a_2} = \cdots \\ &= \frac{a_1+\cdots+a_{n-2}-a_{n-1}-a_n}{a_{n-1}} = \frac{-a_1+a_2+\cdots+a_{n-1}-a_n}{a_n}. \end{aligned}$$

What values can be taken (for the given n) by the product

$$\frac{a_2+\cdots+a_n}{a_1}\cdot\frac{a_1+a_3+\cdots+a_n}{a_2}\cdots\frac{a_1+\cdots+a_{n-1}}{a_n}\ ?$$

9. Let $n > 1$ be an integer. We are considering words composed of n characters A and n characters B. A word $X_1 \ldots X_{2n}$ is said to belong to $R(n)$ if no initial segment $X_1 \ldots X_k$ $(1 \le k < 2n)$ consists of equally many A's and B's. A word $X_1 \ldots X_{2n}$ is said to belong to $S(n)$ if exactly one initial segment $X_1 \ldots X_k$ $(1 \le k < 2n)$ consists of equally many A's and B's. Let $r(n)$ and $s(n)$ be the cardinalities of $R(n)$ and $S(n)$. Calculate the ratio $s(n)/r(n)$.

* * *

We now turn to solutions sent in by our readers, starting with a simpler solution to a problem discussed in the last issue.

4. [1993: 4; 1994: 67] *1991–92 A.H.S.M.C. Part II.*

Suppose x, y and z are real numbers which satisfy the equation $ax + by + cz = 0$, where a, b and c are given positive numbers.

(a) Prove that $x^2 + y^2 + z^2 \ge 2xy + 2yz + 2xz$.

(b) Determine when equality holds in (a).

Comment by Murray S. Klamkin, University of Alberta.

A simpler solution is given by the following:

As before by replacing z by $-(ax + by)/c$, we have to show that

$$(a+c)^2 + 2(a+c)(b+c)xy + (b+c)^2y^2 \ge 4c^2xy$$

or

$$\{(a+c)x + (b+c)y\}^2 \ge 4c^2xy.$$

This immediately follows from the A.M.–G.M. inequality, i.e.,

$$\{(a+c)x + (b+c)y\}^2 \ge 4(a+c)(b+c)xy \ge 4c^2xy.$$

For equality, it is now immediate that $x = y = 0$ and then $z = 0$.

* * *

Next come solutions to problems from the February 1993 number of the Corner with solutions to problems of the *XXV Soviet Mathematical Olympiad, 11th Form* [1993: 36–37].

11.1 For any nonnegative integer n the number a_{n+1} is obtained from a_n by the following rule: if the last digit of a_n does not exceed 5, then this digit is removed and the remaining sequence of digits forms a decimal representation of a_{n+1} (if a_{n+1} contains no digits the process stops). Otherwise $a_{n+1} = 9a_n$. Can a_0 be chosen so that this process is infinite?

(A. Azamov, S. Konjagin)

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Joseph Ling, The University of Calgary; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's presentation (although they all used the same idea).

We prove by induction on $k \geq 0$ that the process terminates if started with k. $P(0)$ is obviously true. Suppose that the process terminates when started at k' for each $0 \leq k' < k$ and let $a_0 = k$. Let c be the last digit of a_0. We distinguish two cases:

I. $0 \leq c \leq 5$. Then $a_1 = (a_0 - c)/10 < k$.

II. $6 \leq c \leq 9$. Then $a_1 = 9a_0$. The crucial observation is that the last digit of a_1 is less than 5, hence $a_2 < 9a_0/10 < k$. Therefore $a_1 < k$ or $a_2 < k$, after which an appeal to the induction hypothesis finishes the proof.

11.2 The reals α and β are such that

$$\alpha^3 - 3\alpha^2 + 5\alpha = 1, \quad \beta^3 - 3\beta^2 + 5\beta = 5.$$

Find $\alpha + \beta$. (B. Kukushkin)

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Tim Cross, Wolverley High School, Kidderminster, U.K.; by George Evagelopoulos, Athens, Greece; by Joseph Ling, The University of Calgary; by Beatriz Margolis, Paris, France; by Bob Prielipp, University of Wisconsin–Oshkosh; by Henry J. Ricardo, Tappan, N.Y.; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We first give Bradley's solution, which exemplifies a common approach.

Let $x = \alpha - 1$ and $y = \beta - 1$. Then the two given equations become

$$x^3 + 2x + 2 = 0$$

$$y^3 + 2y - 2 = 0.$$

Adding gives $x^3 + y^3 + 2(x + y) = 0$ and so $(x + y)(x^2 - xy + y^2 + 2) = 0$. Now for real x, y, $x^2 - xy + y^2 \geq 0$ so $x^2 - xy + y^2 + 2 > 0$.

Hence the only real x, y obey $x + y = 0$. This gives $\alpha + \beta = 2$, for real α, β.

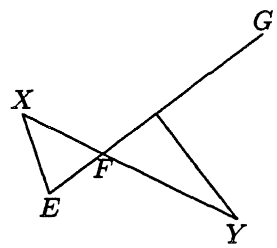
[*Editor's note.* Several other solutions used the calculus to give the existence of the solution and to deduce the result from applications of monotonicity of $x^3 - 3x^2 + 5x$. Another method was to use the Cardano method to produce an exact solution. The following novel elementary method comes from Cross.]

Consider the curve $y = x^3 - 3x^2 + 5x - 3 = (x-1)^3 + 2(x-1) = (x-1)[(x-1)^2 + 2]$ which has a single real root at $x = 1$. Moreover the function is strictly increasing since $z^3 + 2z$ is monotonic. Now the graph of $y = x^3 - 3x^2 + 5x - 1$ and $y = x^3 - 3x^2 + 5x - 5$ are translates of this graph parallel to the y axis by $+2$ and -2 respectively. They have each a unique x-intercept, and since the function $z^3 + 2z$ is odd, the intercepts are equally distant from $x = 1$, that is $1 - \alpha = \beta - 1$ or $\alpha + \beta = 2$.

11.3 There are points A, B, C, D, E on a sphere, such that the chords AB and CD intersect at a point F and A, C, F are equidistant from the point E. Prove that BD and EF are perpendicular. (B. Chinik, I. Sergeev)

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by George Evagelopoulos, Athens, Greece. We use the solution by Evagelopoulos.

Let G be the point of intersection between the semiline EF and the sphere and let XY be either of the segments AB or CD. We will prove that Y (and hence the segment BD) lies on the plane perpendicular to the line segment FG passing through the midpoint. Indeed $XEYG$ lies on a circle which is the intersection of the sphere and the plane of triangle XEF. The triangles XEF and GYF are similar. But since $XE = FE$, then $GY = FY$ which means that Y lies on the plane perpendicular to FG and through the midpoint, as required. But then BD and EF are perpendicular, as required.



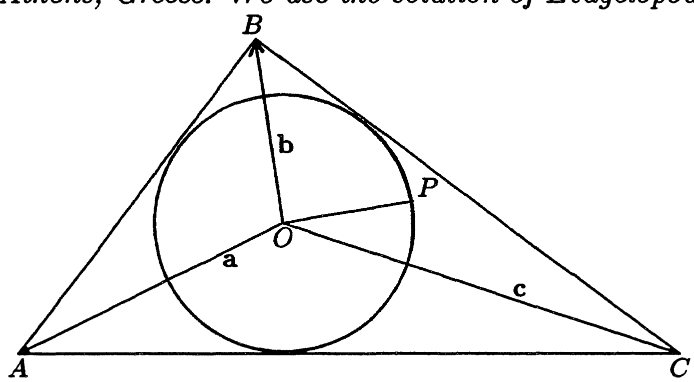
11.4 Does there exist a set containing:

(a) four noncollinear vectors such that the sum of each possible pair is perpendicular to the sum of the other two?

(b) 91 nonzero vectors with the sum of any 19 vectors perpendicular to the sum of all others?

All vectors are supposed to belong to the same plane. (D. Fomin)

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by George Evagelopoulos, Athens, Greece. We use the solution of Evagelopoulos.



(a) The answer to the first question is yes. In the equilateral triangle ABC, where O is the centre of the inscribed circle and P a point on the circle, vectors $\overrightarrow{PA}$, $\overrightarrow{PB}$, $\overrightarrow{PC}$ and $\overrightarrow{PO}$ form the desired tetrad, only if vector $\overrightarrow{PO} = \mathbf{r}$ is not parallel to the others.

Indeed, for example

$$(\overrightarrow{PA}+\overrightarrow{PB})(\overrightarrow{PC}+\overrightarrow{PO}) = (\mathbf{a}+\mathbf{b}+2\mathbf{r})\cdot(\mathbf{c}+2\mathbf{r}) = (2\mathbf{r}-\mathbf{c})(2\mathbf{r}+\mathbf{c})+4\mathbf{r}^2-\mathbf{c}^2 = 0.$$

(b) The answer to the second question is no. We assume that the collection of 91 vectors asked for exists and derive a contradiction. So if $\overrightarrow{OS}$ is their sum and $\overrightarrow{OX}$ is the sum of any 19 of the vectors, then the point X lies on the circle σ with diameter OS. We will prove that it is not possible to choose *five* vectors α_1, α_2, α_3, ℓ_1, ℓ_2 from the collection such that α_1, α_2, and α_3 are distinct and $\ell_1 \neq \ell_2$. Indeed, if such a choice were possible we could choose 17 vectors from the set, set $\mathbf{q}$ to be the sum, and form 6 vectors $\overrightarrow{OX_{ij}} = \mathbf{q}+\alpha_i+\ell_j$ $(i=1,2,3,\;\; j=1,2)$ with the terminal vertices X_{ij} on the circle σ. But then 3 equal vectors $\overrightarrow{X_{i1}X_{i2}} = \overrightarrow{OX_{i2}} - \overrightarrow{OX_{i1}} = \ell_2-\ell_1$ will form 3 chords which will not coincide, which is impossible.

From this we conclude that there are not more than 5 distinct vectors in the collection. Furthermore, there are not distinct vectors x, y, z, t, because otherwise one of them, say t, would be repeated twice and we could select vectors $\alpha_1=t$, $\alpha_2=x$, $\alpha_3=y$, $\ell_1=z$, $\ell_2=t$. For the same reason if there are only 3 different kinds, then one of these kinds has more than one representative. There must be either two or three kinds of vector since they cannot all be equal. Let $\mathbf{x}$ be the kind with the largest number, m, of repetitions.

Suppose first $m \le 89$. Now we can find vectors $\mathbf{y}$, $\mathbf{z}$ distinct from $\mathbf{x}$ and form the sums

$$\overrightarrow{OY}_1 = 18\mathbf{x}+\mathbf{y} \qquad \overrightarrow{OY}_2 = 18\mathbf{x}+\mathbf{z},$$
$$\overrightarrow{OY}_3 = 17\mathbf{x}+\mathbf{y}+\mathbf{z}, \qquad \overrightarrow{OY}_4 = 19\mathbf{x}.$$

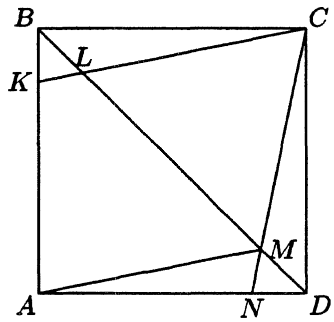
From the equation $\frac{1}{2}(\overrightarrow{OY}_1+\overrightarrow{OY}_2) = \frac{1}{2}(\overrightarrow{OY}_3+\overrightarrow{OY}_4)$ we conclude that the midpoints of the chords Y_1Y_2 and Y_3Y_4 coincide. However the chords do not coincide, because $Y_3 \neq Y_1$ and $Y_3 \neq Y_2$. We conclude that they are diameters of the circle σ. Then $\mathbf{y}\neq\mathbf{z}$, $m=89$, and $\overrightarrow{OY}_1+\overrightarrow{OY}_2=\overrightarrow{OS}$, namely $36\mathbf{x}+\mathbf{y}+\mathbf{z} = 89\mathbf{x}+\mathbf{y}+\mathbf{z}$, which is impossible.

Finally, let $\mathbf{x}$ be found $m=90$ times and y once. Then denoting $\overrightarrow{OZ}_1 = 18\mathbf{x}+\mathbf{y}$, we obtain that chords $\overrightarrow{OZ}_2 = 19\mathbf{x}$ and $\overrightarrow{SZ}_1 = -72\mathbf{x}$ are parallel to the vector $\mathbf{x}$, and as a result symmetrical to the midpoint of the diameter OS, which is impossible as well.

11.5 On the sides AB and AD of the square $ABCD$ points K, N respectively are given so that $AK\cdot AN = 2BD\cdot DN$. The lines CK and CN intersect the diagonal BD at the points L and M. Prove that the points K, L, M, N, A are concyclic. (D. Tereshin)

Correction and solutions by Seung-Jin Bang, Albany, California; and by George Evagelopoulos, Athens, Greece. We give Evagelopoulos' solution.

The condition should be $AK \cdot AN = 2BK \cdot DN$.



First we will prove that $\angle BKC + \angle DNC = 3\pi/4$. Indeed, considering the side of the square as a unit we denote $a = BK = \cot\angle BKC$, $b = DN = \cot\angle DNC$ and we obtain $(1-a)(1-b) = 2ab$ from which $\tan(\angle BKC + \angle DNC) = (a+b)/(ab-1) = -1 = \tan(3\pi/4)$. Thus $\angle BKC + \angle DNC = 3\pi/4$.

Now we have

$$\angle BLK = \frac{3\pi}{4} - \angle BKL = \angle DNC = \angle BCM = \angle BAM,$$

so $\angle KLM + \angle KAM = \pi$, and the points A, K, L, M lie on a circle. Similarly one can show that A, N, M, L lie on a circle (the same one). Thus the points A, K, L, M, N are concyclic.

11.6 There are 100 mutually conflicting countries on the planet Xenon. For maintaining peace military alliances were established. No one alliance contains more than 50 countries and each two countries both belong to at least one alliance. What is the least possible number of alliances? What would this number be under the additional condition that no pair of alliances jointly contains more than 80 countries? (D. Flaas)

Solution by George Evagelopoulos, Athens, Greece.

Each country must belong to no less than three alliances. (Otherwise, a country would be a member in the same alliance with at most $49 + 49 = 98$ of the remaining 99 countries.) For this reason the number of alliances cannot be less than $3 \cdot 100/50 = 6$.

(a) It would be enough to divide the 100 countries into 4 groups of 25 members and form alliances of pairs of groups with all the six possible ways.

(b) It is enough to divide all the countries into 10 groups α_1, α_2, α_3, α_4, α_5, β_1, β_2, β_3, β_4, β_5 consisting of 10 countries each and form alliances.

11.7 The $2n$ distinct reals $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are given. A table of 100×100 cells is filled so that in a cell, that is the intersection of the ith row with the jth column, the number $a_i + b_j$ is written down. Given that the products of all numbers in each column are equal to each other, prove that the products of all numbers in each row are equal to each other as well. (D. Fomin)

Solutions by Seung-Jin Bang, Albany, California; by George Evagelopoulos, Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Bang's solution.

We don't have to assume that $a_1, \dots, a_n$, $b_1, \dots, b_n$ are real numbers and $a_1, \dots, a_n$ are distinct. Let k be the product of all numbers in the first column. Since $b_1, \dots, b_n$ are distinct, from the given condition, $b_1, b_2, \dots, b_n$ are all solutions of the equation $(x + a_1)(x + a_2)\dots(x + a_n) - k = 0$. The relations between roots and coefficients imply

$$\begin{aligned} b_1 + \cdots + b_n &= -(a_1 + \cdots + a_n) \\ b_1 b_2 + \cdots &= a_1 a_2 + \cdots \\ &\cdots \\ b_1 b_2 \ldots b_{n-1} + \cdots &= (-1)^{n-1}(a_1 a_2 \ldots a_{n-1} + \cdots) \\ b_1 \ldots b_n &= (-1)^n (a_1 a_2 \ldots a_n - k). \end{aligned}$$

It follows that $a_1, a_2, \ldots, a_n$ are all solutions of the equation $(x+b_1)(x+b_2)\ldots(x+b_n) + (-1)^n k = 0$ and the products of all numbers in each row are equal to $(-1)^{n-1}k$. This completes the proof of an extended version.

11.8 The numbers $x_1, \ldots, x_{1991}$ are such that

$$|x_1 - x_2| + \cdots + |x_{1990} - x_{1991}| = 1991.$$

What is the greatest possible value of the expression $|y_1 - y_2| + \cdots + |y_{1990} - y_{1991}|$ where $y_k = (x_1 + \cdots + x_k)/k$? (A. Kachurovski)

Solutions by Seung-Jin Bang, Albany, California; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We show that if $x_1, x_2, \ldots, x_n$ are n numbers such that $|x_1 - x_2| + |x_2 - x_3| + \cdots + |x_{n-1} - x_n| = n$, $n \geq 2$ and $y_k = \frac{1}{k}(x_1 + x_2 + \cdots + x_k)$, $k = 1, 2, \ldots, n$, then $|y_1 - y_2| + |y_2 - y_3| + \cdots + |y_{n-1} - y_n| \leq n - 1$ with equality if and only if $x_1 = c \pm n$ and $x_k = c$ for all $k = 2, 3, \ldots, n$, where c is a constant. For each $k = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} |y_k - y_{k+1}| &= \frac{1}{k(k+1)}|x_1 + x_2 + \cdots + x_k - kx_{k+1}| \\ &= \frac{1}{k(k+1)}|(x_1 - x_2) + 2(x_2 - x_3) + \cdots + k(x_k - x_{k+1})| \\ &\leq \frac{1}{k(k+1)}(|x_1 - x_2| + 2|x_2 - x_3| + \cdots + k|x_k - x_{k+1}|). \end{aligned}$$

Summing we have

$$\begin{aligned} \sum_{k=1}^{n-1} |y_k - y_{k+1}| &\leq \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \sum_{i=1}^{k} i|x_i - x_{i+1}| \\ &= |x_1 - x_2| \sum_{k=1}^{n-1} \frac{1}{k(k+1)} + 2|x_2 - x_3| \sum_{k=2}^{n-1} \frac{1}{k(k+1)} + \cdots \\ &\quad + (n-1)|x_{n-1} - x_n| \sum_{k=n-1}^{n-1} \frac{1}{k(k+1)}. \end{aligned}$$

By telescoping, we have, for each $q = 1, 2, \ldots, n-1$

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \frac{1}{q} - \frac{1}{n}.$$

Combining we have

$$\sum_{k=1}^{n-1}|y_k-y_{k+1}| \le \left(1-\frac{1}{n}\right)|x_1-x_2| + 2\left(\frac{1}{2}-\frac{1}{n}\right)|x_2-x_3|$$
$$+\cdots+(n-1)\left(\frac{1}{n-1}-\frac{1}{n}\right)|x_{n-1}-x_n|$$
$$=\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)|x_k-x_{k+1}| \le \left(1-\frac{1}{n}\right)\sum_{k=1}^{n-1}|x_k-x_{k+1}| = n-1.$$

If equality holds then from the last inequality we see that $x_k - x_{k+1} = 0$ for $k = 2, 3, \ldots, n$ and thus $x_1 = c \pm n$, and $x_2 = x_3 = \cdots = x_n = c$ for some c.

Then it is easy to verify that $y_k = c \pm n/k$ and thus $|y_k - y_{k+1}| = n/(k(k+1))$ from which it follows that

$$\sum_{k=1}^{n-1}|y_k-y_{k+1}| = n\left(1-\frac{1}{n}\right) = n-1.$$

* * *

That is all for this number. Send me your nice solutions, Olympiads, and pre-Olympiad materials.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Memorabilia Mathematica — The Philomath's Quotation Book, by Robert Edouard Moritz. Published by the Mathematical Association of America, Washington, D.C., 1993. viii+410 pages, paperback, ISBN 0-88385-321-3, US $24 (MAA members $19).

Out of the Mouths of Mathematicians — A Quotation Book for Philomaths, by Rosemary Schmalz. Published by the Mathematical Association of America, Washington, D.C., 1993. x+294 pages, paperback, ISBN 0-88385-509-7, US $29 (MAA members $23).

Both reviewed by Murray S. Klamkin, University of Alberta.

The first book is a reprint of a 1914 edition consisting of 1140 anecdotes, aphorisms and passages by famous mathematicians, scientists, and writers. Particularly apropos is the following quotation by Emerson near the front of the book: "A great man quotes bravely, and will not draw on his invention when his memory serves him with a word as good." It pleased me to see this quotation since I have used it many times.

A good short description of the book appears on the back cover: "Grouped in twenty-one chapters, the quotations deal with such topics as the definitions and objects of mathematics; the teaching of mathematics; mathematics as a language or as a fine art; the relationship of mathematics to philosophy, to logic, or to science; the nature of mathematics; and the value of mathematics. Other sections contain passages referring to specific subjects in the field such as arithmetic, algebra, geometry, calculus, and modern mathematics. Of special interest is the extensive amount of material on great mathematicians which provides irreplaceable glimpses into the lives and personalities of mathematical giants."

To give the flavour of the book, the following are some of the short quotations which have appealed to me:

> **216.** In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure. — *Hermann Hankel*
>
> **813.** Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different. — *Goethe*
>
> **985.** I never come across one of Laplace's "*Thus it plainly appears*" without feeling sure I have hours of hard work before me to fill up the chasm and find out and show *how* it plainly appears. — *N. Bowditch*
>
> **1528.** In questions of science the authority of a thousand is not worth the humble reasoning of a single individual. — *Galileo*
>
> **1636.** Integral numbers are the fountainhead of all mathematics. — *H. Minkowski*
>
> **1706.** Algebra is but written geometry and geometry is but figured algebra. — *Sophie Germain*
>
> **1913.** The calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word. — *W. F. Osgood*

The second book is a companion volume to the first book; it picks up where Moritz left off and is a compilation of 727 quotations from 292 contributors, almost all of whom are twentieth century mathematicians. The author notes that her "primary objective is more in keeping with Moritz's second objective: to compile a volume that can be used by writers to emphasize or substantiate a point, by teachers to encourage and amuse their students, and by all readers, and in particular young readers, to get the flavor of mathematics and to whet their appetites and make them eager to learn more about it." By reading this book one can fairly easily get a sense of the "story" of twentieth century mathematics and similarly for the Moritz book for the earlier period of mathematics. As before, to give a flavour of the Schmalz book, I include the following short quotations:

> **p. 10.** Mathematical activity — like all of Gaul — may be divided into three areas: Education, Research and Applications ... much of the strength of the mathematical fabric comes from the interaction among these three. — *Henry O. Pollak*

p. 26. We can see that the development of mathematics is a process of conflict among the many contrasting elements: the concrete and the abstract, the particular and the general, the formal and the material, the finite and the infinite, the discrete and the continuous, and so forth. — *A. D. Aleksandrov*

p. 38. There appears a *fundamental principle* which can serve to characterize all possible geometries .... Given any group of transformations in space which includes the principal group as a subgroup, then the invariant theory of this group gives a definite kind of geometry, and every possible geometry can be obtained this way. — *Felix Klein*

p. 52. If logic is the hygiene of the mathematician, it is not his source of food; the great problems furnish the daily bread on which he thrives. — *André Weil*

p. 55. Usually in mathematics one has to choose between saying more and more about less and less on one hand, and saying less and less about more and more on the other. — *N. G. deBruijn*

p. 75. Proofs really aren't there to convince you that something is true — they're there to show you why it is true. — *Andrew Gleason*

p. 78. The value of a problem is not so much in coming up with the answer as in the ideas and attempted ideas it forces on the would-be solver. — *I. N. Herstein*

p. 91. Most of the best work starts in hopeless muddle and floundering, sustained on the "smell" that something is there. — *John E. Littlewood*

p. 160. In the quest for simplification, mathematics stands to computer science as diamond mining to coal mining. The former is a search for gems. ... The latter is permanently involved with bulldozing large masses of ore — extremely useful bulk material. — *Jacob T. Schwartz*

p. 174. One never knew where [Paul] Erdős was, not even the country. However, one could be sure that during the year that Erdős was everywhere. He was the nearest thing to an ergodic particle that a human being could be. — *Richard Bellman*

Both books have chapter headings and indexes which can be very helpful in tracking down various quotations of a given author or a given type. There are also citations for all the quotations, all the ones in the Schmalz book being complete. The sampling of quotations given above are only some of the very short ones which appealed to me. There are also as well much longer ones. I am now very glad to have these two books since they will help to buttress future papers with appropriate quotations and their citations, and additionally, as Schmalz notes, to give me a better picture of twentieth century mathematics and its leading mathematicians.

Finally, it would be very interesting to be able to give a liberal arts mathematics course based on these two books. One would select various quotations, e.g. by A. D. Aleksandrov or Felix Klein (above), and examine the quotations in some depth.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1931. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

M is the midpoint of side BC of a triangle ABC, and Γ is the circle with diameter AM. D and E are the other intersections of Γ with AB and AC respectively. Let P be the point such that PD and PE are tangent to Γ. Prove that $PB = PC$.

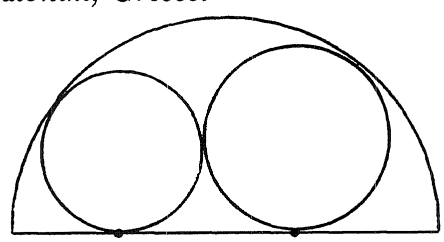
1932*. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Trivially, if $N = 1, 10, 100, \ldots$ then in each case the initial digits of N^3 are the digits of N. A nontrivial example is $N = 32$, since $32^3 = 32768$. Find another positive integer N with this property.

1933. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$$R_1 + R_2 \leq 2(\sqrt{2} - 1).$$



1934. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

A cone of radius 1 metre and height h metres is lowered point first at a constant rate of 1 metre per second into a tall cylinder of radius R (> 1) metres which is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?

1935. *Proposed by Murray S. Klamkin, University of Alberta.*

Given an ellipse which is not a circle, prove or disprove that the locus of the mid-points of sufficiently small constant length chords is another ellipse.

1936. *Proposed by Bolian Liu, South China Normal University, Guangzhou, China.*

Ten players participate in a ping-pong tournament, in which every two players play against each other exactly once. If player i beats player j, player j beats player k, and player k beats player i, then the set $\{i, j, k\}$ is called a *triangle.* Let w_i and l_i denote the number of games won and lost, respectively, by the ith player. Suppose that, whenever i beats j, then $l_i + w_j \geq 8$. Prove that there are exactly 40 triangles in the tournament.

1937. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcenter O, orthocenter H, and altitudes AD, BE and CF (with D on BC, etc.). Suppose $OH \parallel AC$.

(a) Show that EF, FD and DE are in arithmetic progression.

(b) Determine the possible values of angle B.

1938. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Find the exact value of

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cdots\cos\left(\frac{16\pi}{17}\right).$$

1939. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Let ABC be an acute-angled triangle with circumcentre O, incentre I and orthocentre H. Let AI, BI, CI meet BC, CA, AB respectively in U, V, W, and let AH, BH, CH meet BC, CA, AB respectively in D, E, F. Prove that O is an interior point of triangle UVW if and only if I is an interior point of triangle DEF.

1940. *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

*   *   *   *   *

SOLUTIONS

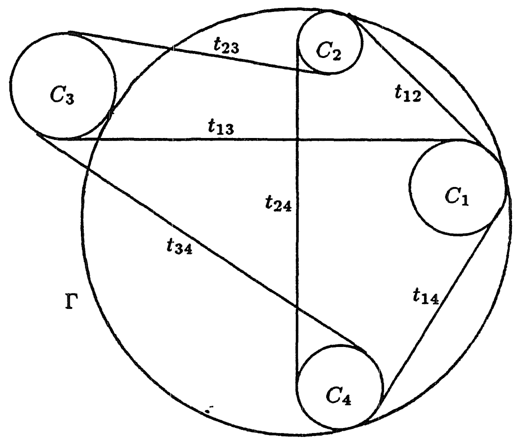
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1726. [1992: 75; 1993: 60] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

Circular arc AC lies inside rectangle $ABCD$, dividing it into two parts with inscribed circles of radii r_1 and r_2. Show that $r_1 + r_2 = 2r$, where r is the inradius of the right-angled triangle ADC.

II. *Solution and comment by Toshio Seimiya, Kawasaki, Japan.*

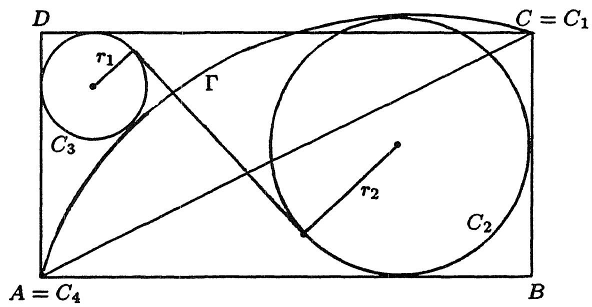
In the first figure, Γ is tangent to four circles C_1, C_2, C_3, C_4 (in that order), and t_{ij} is the length of the common tangents of C_i and C_j, for each $i \neq j$. Then by Casey's theorem,



$$t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24}.$$

[For example, this nice generalization of Ptolemy's theorem can be found as Theorem 172, page 121 of R.A. Johnson, *Advanced Euclidean Geometry.—Ed.*]

Now let Γ be the circle containing arc AC, and let C_3 and C_2 be the inscribed circles with radii r_1 and r_2 respectively. We regard points C and A as circles C_1 and C_4 with radii 0. Put $AB = a$ and $BC = b$, and let t_{ij} be the length of the common tangents of C_i and C_j. Then we get



$$t_{12} = b - r_2, \quad t_{13} = a - r_1, \quad t_{24} = a - r_2,$$

$$t_{34} = b - r_1, \quad t_{14} = AC = \sqrt{a^2 + b^2},$$

and

$$t_{23} = \sqrt{[a - (r_1 + r_2)]^2 + [b - (r_1 + r_2)]^2 - (r_1 + r_2)^2}.$$

By Casey's theorem we get $t_{14}t_{23} = t_{13}t_{24} - t_{12}t_{34}$, i.e.

$$\begin{aligned} &\sqrt{a^2 + b^2}\,\sqrt{(a^2 + b^2) - 2(a + b)(r_1 + r_2) + (r_1 + r_2)^2} \\ &\qquad = (a - r_1)(a - r_2) - (b - r_2)(b - r_1) = (a^2 - b^2) - (a - b)(r_1 + r_2). \quad (1) \end{aligned}$$

Squaring both sides, we get

$$(a^2 + b^2)[(a^2 + b^2) - 2(a + b)(r_1 + r_2) + (r_1 + r_2)^2] = [(a^2 - b^2) - (a - b)(r_1 + r_2)]^2. \quad (2)$$

(This is assuming that $a \geq b$. When $a < b$ the right side of (1) becomes

$$(b^2 - a^2) - (b - a)(r_1 + r_2),$$

and we still have (2).) Therefore

$$4a^2b^2 - 2(a+b)(r_1+r_2)[a^2+b^2-(a-b)^2] + [a^2+b^2-(a-b)^2](r_1+r_2)^2 = 0.$$

Hence we obtain

$$(r_1+r_2)^2 - 2(a+b)(r_1+r_2) + 2ab = 0,$$

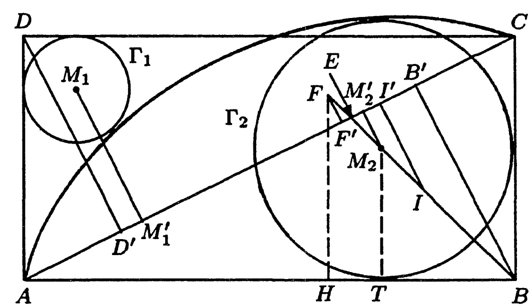
and therefore

$$r_1 + r_2 = \frac{2(a+b) - \sqrt{4(a+b)^2 - 8ab}}{2} = a + b - \sqrt{a^2+b^2} = 2r.$$

This finishes the solution of the problem.

P. Penning's conjecture [1993: 60], that one of the common internal tangents to the inscribed circles Γ_1 and Γ_2 is parallel to AC, is correct. We now give a proof.

In the figure, let M_1 and M_2 be the centers of inscribed circles Γ_1 and Γ_2 with radii r_1 and r_2 respectively. Let I be the incenter of ΔABC; then B, I, and M_2 are collinear. Let E be the intersection of BI and AC, and let F be the point on BI produced beyond I so that $BI = IF$. Because $BI > IE$, I and F lie on opposite sides of AC. Let B', D', F', I' be the feet of the perpendiculars from B, D, F, I to AC. As I is the midpoint of BF, we get



$$BB' - FF' = 2II' = 2r = r_1 + r_2.$$

Because $BB' = DD'$, we have

$$DD' - FF' = r_1 + r_2. \tag{3}$$

Let H and T be the feet of the perpendiculars from F and M_2 to AB. Then we have $FH = 2r$ and $M_2T = r_2$, so

$$DM_1 = \sqrt{2}r_1 = \sqrt{2}(2r - r_2) = \sqrt{2}(FH - M_2T) = FM_2.$$

Letting M_1' and M_2' be the feet of the perpendiculars from M_1 and M_2 to AC, we get from (4) and $DM_1 \| FM_2$ that

$$DD' - M_1M_1' = FF' + M_2'M_2.$$

Therefore from (3)

$$M_1M_1' + M_2'M_2 = r_1 + r_2,$$

and hence we obtain $M_1M_1' - r_1 = r_2 - M_2'M_2$. This implies the result.

[*Editor's note.* P. Penning, Delft, The Netherlands, has also since sent in a proof of the original problem.]

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1802. [1993: 15, 274] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that, for any real numbers x and y,

$$x^4 + y^4 + (x^2+1)(y^2+1) \geq x^3(1+y) + y^3(1+x) + x + y,$$

and determine when equality holds.

II. *Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

For any real x, y, z, by the A.M.–G.M. inequality,

$$\frac{x^4+x^2z^2}{2} \geq x^3z, \qquad \frac{x^4+x^2y^2}{2} \geq x^3y,$$
$$\frac{y^4+y^2x^2}{2} \geq y^3x, \qquad \frac{y^4+y^2z^2}{2} \geq y^3z,$$
$$\frac{z^4+z^2x^2}{2} \geq z^3x, \qquad \frac{z^4+z^2y^2}{2} \geq z^3y.$$

Adding these inequalities and factoring, we obtain

$$x^4 + y^4 + (x^2+z^2)(y^2+z^2) \geq x^3(z+y) + y^3(z+x) + z^3(x+y)$$

[equality holding if and only if $x = y = z$]. Setting $z = 1$ we obtain the desired inequality.

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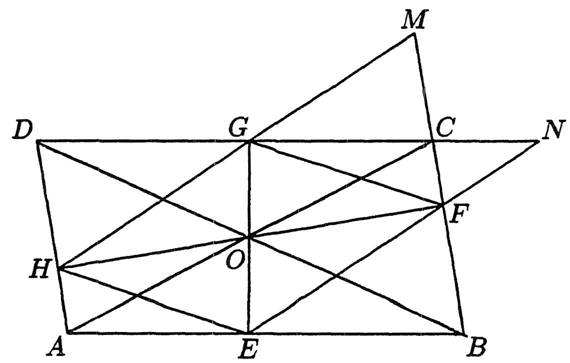
1841. [1993: 140] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral with vertex angles A, B, C, D, and O is the intersection of the diagonals AC and BD. Show that $ABCD$ is a parallelogram if and only if $OA \sin A = OC \sin C$ and $OB \sin B = OD \sin D$.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

When $ABCD$ is a parallelogram we have $OA = OC$, $OB = OD$, and $\sin A = \sin B = \sin C = \sin D$. Therefore $OA \sin A = OC \sin C$ and $OB \sin B = OD \sin D$.

Now for the reverse. Suppose $OA \sin A = OC \sin C$ and $OB \sin B = OD \sin D$. Let E, F, G, H be the projections of O onto AB, BC, CD, DA respectively. Then



$$\angle AEO = \angle AHO = \frac{\pi}{2}, \qquad (1)$$

so $AEOH$ is inscribable in a circle and

$$HE = OA \sin A.$$

In a similar way $FG = OC \sin C$, and so $HE = FG$. Similarly $EF = GH$, and we conclude that $EFGH$ is a parallelogram and thus $EF \| HG$ and $EH \| FG$. Let HG intersect BC in M, and EF intersect DC in N. Denote $\angle FNC = \angle CGM = \angle HGD = \varphi$. As $HOGD$ is inscribable in a circle we have

$$\angle DOH = \angle DGH = \varphi. \tag{2}$$

In ΔFNC we find $\angle CFN = C - \varphi$, so it follows that

$$\angle BOE = \angle BFE = C - \varphi. \tag{3}$$

Now $\angle DOH + \angle HOE + \angle EOB = \pi$, and by (1) $\angle HOE = \pi - A$, so by (2) and (3) we have $\varphi + \pi - A + C - \varphi = \pi$ and thus $A = C$. In a similar way $B = D$, so $ABCD$ is a parallelogram.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; and the proposer. There was one incomplete solution sent in.

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1842. [1993: 140] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For given $\lambda > 1$ and $x_1 \in (0,1)$, the sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_{n+1} = \lambda^{x_n} - 1, \quad n \geq 1.$$

Determine the set of all λ's such that $\sum_{n=1}^{\infty} x_n$ converges for every starting value $x_1 \in (0,1)$.

Solution by Marcin E. Kuczma, Warszawa, Poland.

The answer is: $1 < \lambda \leq 2$.

Let $f(x) = \lambda^x - 1$, so that $x_{n+1} = f(x_n)$. Note that f is strictly increasing and strictly convex.

Case (i): $\lambda > 2$. The tangent to the graph of f at $(1, f(1))$ has slope $f'(1) = \lambda \ln \lambda > 2 \ln 2 > 1$. The graph lies above that tangent. Since $f(1) = \lambda - 1 > 1$, it follows that the inequality $f(x) > x$ holds in an interval (a, ∞) for some a between 0 and 1. Choose $x_1 \in (a, 1)$; then

$$x_1 < f(x_1) = x_2 < f(x_2) = x_3 < \cdots ,$$

which implies divergence of $\sum_{n=1}^{\infty} x_n$.

Case (ii): $\lambda \leq 2$. As $f(0) = 0$ and $f(1) = \lambda - 1 \leq 1$, we get (by strict convexity) $f(x) < x$ in $(0,1)$. Choose $x_1 \in (0,1)$ arbitrarily; then

$$x_1 > f(x_1) = x_2 > f(x_2) = x_3 > \cdots .$$

Let $f(x_1)/x_1 = c < 1$. All x_n's lie in $(0, x_1]$, and for $x \in (0, x_1]$, $f(x) \leq cx$ (by convexity again). Hence $x_{n+1} \leq c x_n$ for all n. By induction, $x_n \leq c^{n-1} x_1 < c^{n-1}$. Thus the sum $\sum_{n=1}^{\infty} x_n$ is dominated by the geometric series with ratio $c < 1$, and so converges.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1843. [1993: 140] *Proposed by Šefket Arslanagić, Nyborg, Denmark, and D.M. Milošević, Pranjani, Yugoslavia.*

Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{a}{2A(s-a)} \geq \frac{9}{\pi} .$$

(ii)* It is obvious that also

$$\sum \frac{1}{A} \geq \frac{9}{\pi} .$$

Do these two summations compare in general?

Solution to both parts by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.

First note that the function $f(x) = x \cot x$ is decreasing for $0 \leq x \leq \pi/2$. This is because

$$f'(x) = \cot x - x \csc^2 x = \frac{2 \sin x \cos x - 2x}{2 \sin^2 x} = \frac{\sin 2x - 2x}{2 \sin^2 x} \leq 0$$

for $0 \leq x \leq \pi/2$.

Now we have that

$$s-a > s-b \quad \Leftrightarrow \quad b > a \quad \Leftrightarrow \quad \frac{B}{2} > \frac{A}{2} \quad \Leftrightarrow \quad \frac{B/2}{\tan(B/2)} < \frac{A/2}{\tan(A/2)} ,$$

and [since $\tan(A/2) = r/(s-a)$, etc., where r is the inradius] this is equivalent to

$$B(s-b) < A(s-a) \quad \Leftrightarrow \quad \frac{1}{A(s-a)} < \frac{1}{B(s-b)} .$$

Therefore

$$\{s-a, s-b, s-c\} \qquad \text{and} \qquad \left\{ \frac{1}{A(s-a)} , \frac{1}{B(s-b)} , \frac{1}{C(s-c)} \right\}$$

have inverse order, so by Chebyshev's inequality,

$$\frac{1}{3} \sum (s-a) \cdot \frac{1}{3} \sum \frac{1}{A(s-a)} \geq \frac{1}{3} \sum \frac{1}{A} .$$

Thus
$$\sum \frac{a}{A(s-a)} + \sum \frac{1}{A} = s\sum \frac{1}{A(s-a)} \geq 3\sum \frac{1}{A},$$
and so
$$\sum \frac{a}{2A(s-a)} \geq \sum \frac{1}{A} \quad \left(\geq \frac{9}{\pi}\right). \tag{1}$$

Both parts also solved (via a longer method) by MARCIN E. KUCZMA, Warszawa, Poland. Part (i) *only solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; KEE-WAI LAU, Hong Kong; and the proposer.*

Fernández gave a lengthy proof of inequality (1) *in the special case when the triangle is isosceles.*

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1844. [1993: 140] *Proposed by Jordi Dou, Barcelona, Spain.*

Let Ω be the circumcircle of triangle ABC, and let Γ be the parabola tangent to AB at B and to AC at C. Construct an equilateral triangle XYZ so that X lies on AB, Y on AC and Z on Ω, and XY is tangent to Γ.

Solution by Toshio Seimiya, Kawasaki, Japan.

Let F be the focus of Γ, and let XY be tangent to Γ, where $X \in AB$ and $Y \in AC$. As AB, AC and XY are tangent to a parabola, the feet of perpendiculars from the focus F to AB, AC and XY are collinear. Hence, by the converse of Simson's theorem, F lies on the circumcircle of ΔAXY. Therefore
$$\angle FXY = \angle FAY = \angle FAC \tag{1}$$
and
$$\angle FYX = \angle FAX = \angle FAB. \tag{2}$$
In particular, when XY coincides with BA or AC, (1) and (2) become
$$\angle FBA = \angle FAC \tag{3}$$
and
$$\angle FCA = \angle FAB. \tag{4}$$
From (3) the circumcircle of ΔFAB is tangent to AC at A, and from (4) the circumcircle of ΔFAC is tangent to AB at A. Consequently, the focus F of Γ can be constructed as the second intersection of the circle which passes through B that is tangent to AC at A, and the circle which passes through C that is tangent to AB at A. [*Editor's comment.* The above results have appeared before in *Crux*, for example [1992: 26–27]; they can also be found in standard references.]

Construct the three equilateral triangles BAD, ACE and XYZ (either all clockwise or all counterclockwise). From (1)–(4) we have
$$\Delta FBA \sim \Delta FXY \sim \Delta FAC.$$

Therefore $BD : XZ : AE = BA : XY : AC = BF : XF : AF$, and $\angle FBD = \angle FXZ = \angle FAE$. Hence we have $\Delta FBD \sim \Delta FXZ \sim \Delta FAE$. Because B, X, A are collinear, D, Z, E are also collinear. If X varies on AB, then Z varies on DE. In other words, DE is the locus of Z. Let Z be a point of intersection of DE with Ω (if one exists), and take X and Y such that $\Delta FXZ \sim \Delta FBD$ and $\Delta FYZ \sim \Delta FAD$. Then X, Y, Z are the points we are looking for.

Also solved by P. PENNING, Delft, The Netherlands; and the proposer.

Penning noted that there could be no solution or as many as four, depending on the number of intersections of DE with Ω in the clockwise and counterclockwise cases. He added that the problem could be modified to allow an arbitrary but fixed shape for triangle XYZ.

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1845. [1993: 140] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K..*

Suppose that x_1, x_2, x_3, x_4, x_5 are real numbers satisfying $x_1 < x_2 < x_3 < x_4 < x_5$ and

$$\sum_i x_i = 10, \quad \sum_{i<j} x_i x_j = 35, \quad \sum_{i<j<k} x_i x_j x_k = 50, \quad \sum_{i<j<k<l} x_i x_j x_k x_l = 25.$$

Prove that

$$\frac{5+\sqrt{5}}{2} < x_5 < 4.$$

Solution by Ed Barbeau, University of Toronto.

Let

$$p(t) = t^5 - 10t^4 + 35t^3 - 50t^2 + 25t.$$

Then

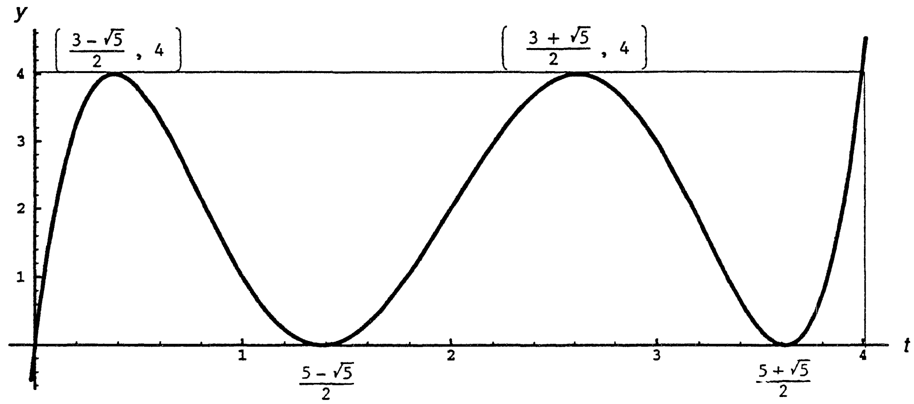
$$p(t) = t(t^2 - 5t + 5)^2 = (t-4)(t^2 - 3t + 1)^2 + 4.$$

From this, clearly

$$\begin{array}{ll} p(t) < 0 & \text{for } t < 0, \\ 0 \le p(t) \le 4 & \text{for } 0 \le t \le 4, \\ p(t) > 4 & \text{for } t > 4, \end{array}$$

and $p(t)$ is monotonely increasing for $t < 0$ and $t > 4$. [Also, $p(t) = 0$ when $t = 0$ and

$t = (5 \pm \sqrt{5})/2$, and $p(t) = 4$ when $t = 4$ and $t = (3 \pm \sqrt{5})/2$.] Thus the graph of $p(t)$ is



Now the numbers x_1, x_2, x_3, x_4, x_5 are zeros of a polynomial of the form $p(t) - k$, where the equation $p(t) = k$ has five real roots. From the graph we note that $0 < k < 4$, and that the largest root x_5 of this equation satisfies

$$\frac{5+\sqrt{5}}{2} < x_5 < 4.$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; TIM CROSS, Wolverley High School, Kidderminster, U.K.; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; F. J. FLANIGAN, San Jose State University, San Jose, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; FRANCISCO LUIS ROCHA PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

As several solvers pointed out, and can easily be seen from the above proof, the numbers x_i satisfy

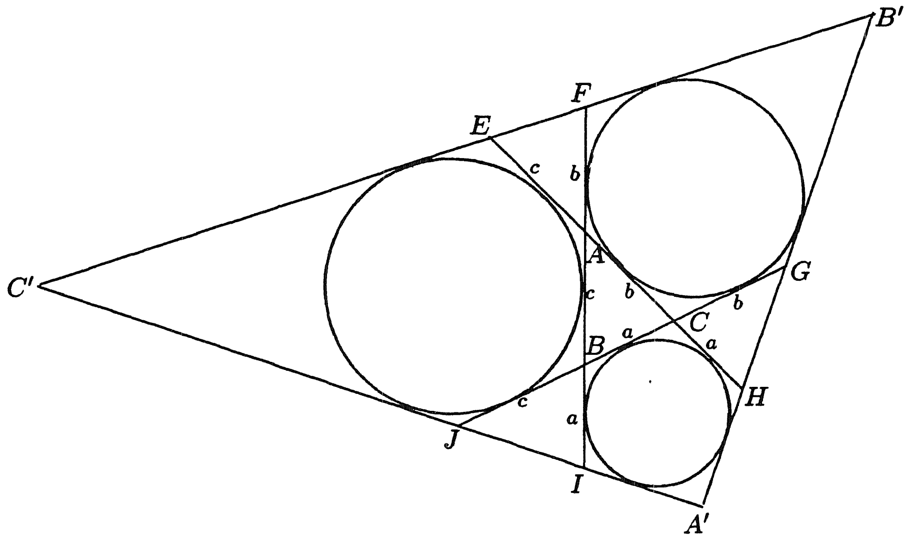
$$0 < x_1 < \frac{3-\sqrt{5}}{2} < x_2 < \frac{5-\sqrt{5}}{2} < x_3 < \frac{3+\sqrt{5}}{2} < x_4 < \frac{5+\sqrt{5}}{2} < x_5 < 4.$$

*   *   *   *   *

1846. [1993: 140] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Consider the three excircles of a given triangle ABC. Let $A'B'C'$ be the triangle containing these three circles and whose sides are each tangent to two of the circles. Prove that $[A'B'C'] \geq 25[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

Solution by Francisco Luis Rocha Pimentel, Fortaleza, Brazil.



In the figure we have

$$\Delta AEF \equiv \Delta GHC \equiv \Delta JBI \equiv \Delta ABC. \tag{1}$$

For by symmetry,

$$\begin{aligned} \angle C'IF &= \angle C = \angle C'FI, \\ \angle B'EH &= \angle B = \angle B'HE, \\ \angle A'JG &= \angle A = \angle A'GJ, \end{aligned} \tag{2}$$

and also

$$BI = BC = CH \ (= a), \quad FA = AC = CG \ (= b), \quad EA = AB = BJ \ (= c).$$

Thus by (1)

$$\begin{aligned} [A'B'C'] &= [C'FI] + [B'EH] + [A'GJ] - [AEF] - [CGH] - [BIJ] + [ABC] \\ &= [C'FI] + [B'EH] + [A'GJ] - 2[ABC]. \end{aligned} \tag{3}$$

Since $\angle C' = 180° - 2\angle C$ by (2), in $\Delta C'FI$ we have

$$\frac{C'I}{\sin C} = \frac{FI}{\sin C'} = \frac{2s}{\sin 2C}$$

where s is the semiperimeter of ΔABC, so

$$C'I = \frac{2s\sin C}{\sin 2C} = \frac{s}{\cos C} \ .$$

Then

$$[C'FI] = \frac{1}{2}C'I \cdot C'F \sin C' = \frac{1}{2}\frac{s}{\cos C} \cdot \frac{s}{\cos C} \sin 2C = s^2 \tan C.$$

In the same way,

$$[B'EH] = s^2 \tan B \qquad \text{and} \qquad [A'GJ] = s^2 \tan A.$$

Substituting in (3), we obtain

$$\begin{aligned} [A'B'C'] &= s^2(\tan A + \tan B + \tan C) - 2[ABC] \\ &\geq 3\sqrt{3}\,[ABC] \cdot 3\sqrt{3} - 2[ABC] = 25[ABC], \end{aligned}$$

where we used the inequalities

$$s^2 \geq 3\sqrt{3}\,[ABC] \qquad \text{and} \qquad \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

which are items 4.2 and 2.30 respectively of O. Bottema et al, *Geometric Inequalities.*

Also solved (usually the same way) by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

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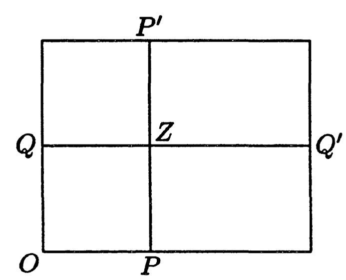
1847. [1993: 141] *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

The points $(0,0) = O$, $(a,0)$, $(0,b)$, (a,b) are the corners of an $a \times b$ rectangle. For a point Z in the interior of the rectangle, draw the vertical and horizontal lines through Z, and let them meet the sides of the rectangle at points P, P' and Q, Q' respectively. Define $X = PQ' \cap P'Q$ and $Y = PQ \cap P'Q'$. Prove that the vector whose x- and y-coordinates are the slopes of the lines OX and OY, respectively, is orthogonal to the vector whose coordinates are the slopes of lines ZX and ZY.

Comment by the editor.



Unfortunately, this problem is **false**. For example, it is clear from the picture that the slopes of lines OX and ZX will be *positive*, while the slopes of OY and ZY will be *negative*. Thus it is impossible that the mentioned vectors will be orthogonal. The proposer's "proof" contained at least one algebraic error.

The editor apologizes for this problem and for the time some readers (see below) spent on it. The problem may be curiously appropriate in one way however: as Chris Fisher has pointed out, 1847 was the year in which Heinrich Hoffman published his *Struwwelpeter*, a book (likely familiar to all our German-speaking readers) in which children are cautioned about what happens when they go astray!

Counterexamples found by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

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1848. [1993: 141] *Proposed by Neven Jurić, Zagreb, Croatia.*

Suppose that $x_1, x_2, \ldots, x_n$ are integers in $\{1, 2, \ldots, n\}$ such that

$$x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2} \quad \text{and} \quad x_1 x_2 \ldots x_n = n!\,.$$

Must $x_1, \ldots, x_n$ be a permutation of $1, 2, \ldots, n$?

Solution by R.P. Sealy, Mount Allison University, Sackville, New Brunswick.

The answer is **no** for $n \geq 9$ and **yes** for $n < 9$. In fact for $n \geq 9$, one can replace the factors 8, 6 and 3 in the expression for $n!$ by the factors 9, 4 and 4; then since

$$8 + 6 + 3 = 9 + 4 + 4 \quad \text{and} \quad 8 \cdot 6 \cdot 3 = 9 \cdot 4 \cdot 4,$$

this leaves the number of factors, the product of the factors and the sum of the factors all unchanged.

[*Editor's note.* Sealy then went on to show that there are no such examples for $n \leq 8$.]

Also solved by H.L. ABBOTT, University of Alberta; IAN AFFLECK, student, University of Regina; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; IGNOTUS, Maputo, Mozambique; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, N.Y.; A. N. 'T WOORD, Eindhoven University of Technology, Eindhoven, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario, and ROY WESTWICK, University of British Columbia; and the proposer. Two other readers sent in examples which were invalid because some x_i was greater than n.

About half the solvers found the example with $n = 9$, and most of these noted that it is minimal.

Abbott asks: for any real $\lambda > 0$ and for n large enough, is there a sequence $x_1, x_2, \ldots, x_n$ of integers between 1 and n, satisfying $\sum x_i = n(n+1)/2$ and $\prod x_i = n!$, and containing at most λn different numbers?

The proposer would like to see an example of a sequence $x_1, x_2, \ldots, x_n$, not a permutation of $\{1, 2, \ldots, n\}$, satisfying the conditions of the problem and in addition $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n i^2$.

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1850. [1993: 141] *Proposed by Esteban Indurain, Universidad Pública de Navarra, Pamplona, Spain.*

Given a square in the plane, divide it into nine congruent smaller squares by lines parallel to the sides, and remove the interior of the cross-shaped central region; four isolated squares remain. Do the same with each of these squares; sixteen isolated squares remain. Do the same with each of these squares, and continue this process indefinitely. There are some points of the original square that still remain. Prove that any line parallel to a diagonal of the original square, and intersecting the original square, must pass through one of these remaining points.

I. *Solution by Fernando Antonio Amaral Pimentel, student, Fortaleza, Brazil.*

One can easily verify that any line parallel to one of the diagonals that intersects the original square intersects one of the four remaining squares, too. Let ℓ be a line parallel to a diagonal, which intersects the original square. Let S_1 be the original square and S_2 the square, of the four squares remaining from S_1, which is intersected by ℓ. Do the same with S_2 to define S_3. Continue this process indefinitely to define a sequence (S_n) of "nested squares" such that $\ell \cap S_n \neq \emptyset$ for all positive integers n. Let $\ell_n = \ell \cap S_n$. Then (ℓ_n) is a sequence of (not empty) nested closed intervals, which implies $\bigcap_{n=1}^{\infty} \ell_n \neq \emptyset$. Let P be a point in $\bigcap_{n=1}^{\infty} \ell_n$. For any n, since $P \in \ell_n$ then $P \in S_n$ too. So $P \in \bigcap_{n=1}^{\infty} S_n$ and must be a point of the original square that still remains, and ℓ obviously passes through P.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

What remains is exactly the Cartesian product $C \times C$ of the Cantor set $C \subset I = [0,1]$ by itself (we assume that $I \times I$ is the original square). Consider the line $x + y = a$, a a constant, $0 \leq a \leq 2$. To say that this line meets $C \times C$ is just as much as to assert that a is representable as the sum of two numbers from C. The Cantor set consists of those numbers from $[0,1]$ which admit a base 3 representation $(0.c_1c_2c_3\cdots)_3$ with each c_i equal to either 0 or 2 (possibly with 2's from some point on). Let $a/2 = (0.a_1a_2a_3\cdots)_3$, $a_i \in \{0,1,2\}$, $i = 1,2,3\ldots.$ Write each a_i as the sum $a_i = u_i + v_i$ with $u_i, v_i \in \{0,1\}$ (this is unique if $a_i = 0$ or $a_i = 2$ and non-unique if $a_i = 1$; we may agree that, in the latter case, $u_i = 1$ and $v_i = 0$). Let $x_i = 2u_i$, $y_i = 2v_i$. So $x_i, y_i \in \{0,2\}$ and

$$a = 2(0.a_1a_2\cdots)_3 = 2[(0.u_1u_2\cdots)_3 + (0.v_1v_2\cdots)_3] = (0.x_1x_2\cdots)_3 + (0.y_1y_2\cdots)_3,$$

the sum of two elements of C, as wished.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The solvers were about equally split between the above two proofs. But neither solution is as elementary as the proposer had hoped for.

* * * * *

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