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ROSS HONSBERGER

1. RAMANUJAN

The following letter, dated January 16, 1913, was received out of the blue by the prominent English mathematician G.H. Hardy:

"Dear Sir,

"I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only 20 pounds per annum. I am now about 23 years of age. [*He was actually 25—Ed.*] I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'....

"I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

"I remain, Dear Sir, Yours truly,

"S. RAMANUJAN."

To the letter were attached about 120 theorems, of which the 15 given below were part of a group selected by Hardy as "fairly representative":

$$(1.1)\quad 1 - \frac{3!}{(1!2!)^3}x^2 + \frac{6!}{(2!4!)^3}x^4 - \dots$$
$$= \left(1 + \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} + \dots\right)\left(1 - \frac{x}{(1!)^3} + \frac{x^2}{(2!)^3} - \dots\right).$$

$$(1.2)\quad 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots = \frac{2}{\pi}.$$

$$(1.3)\quad 1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1.5}{4.8}\right)^4 + 25\left(\frac{1.5.9}{4.8.12}\right)^4 + \dots = \frac{2^{\frac{3}{2}}}{\pi^{\frac{1}{2}}\{\Gamma(3/4)\}^2}.$$

(1.4) $$1 - 5\left(\frac{1}{2}\right)^5 + 9\left(\frac{1.3}{2.4}\right)^5 - 13\left(\frac{1.3.5}{2.4.6}\right)^5 + \ldots = \frac{2}{\{\Gamma(3/4)\}^4}.$$

(1.5) $$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\cdots dx$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a+\frac{1}{2})\,\Gamma(b+1)\,\Gamma(b-a+\frac{1}{2})}{\Gamma(a)\,\Gamma(b+\frac{1}{2})\,\Gamma(b-a+1)}.$$

(1.6) $$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots)}.$$

(1.7) If $\alpha\beta = \pi^2$, then

$$\alpha^{-1/4}\left(1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right) = \beta^{-1/4}\left(1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right).$$

(1.8) $$\int_0^a e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi} - \frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\ldots}.$$

(1.9) $$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx = \frac{1}{1+}\,\frac{1^2}{1+}\,\frac{1^2}{1+}\,\frac{2^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\,\frac{3^2}{1+\ldots}.$$

(1.10) If $u = \dfrac{x}{1+}\,\dfrac{x^5}{1+}\,\dfrac{x^{10}}{1+}\,\dfrac{x^{15}}{1+\ldots}$, $v = \dfrac{x^{\frac{1}{5}}}{1+}\,\dfrac{x}{1+}\,\dfrac{x^2}{1+}\,\dfrac{x^3}{1+\ldots}$,

then $$v^5 = u\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(1.11) $$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+\ldots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\}e^{2\pi/5}.$$

(1.12) $$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt{5}}}{1+}\,\frac{e^{-4\pi\sqrt{5}}}{1+\ldots} = \left[\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1\right\}}} - \frac{\sqrt{5}+1}{2}\right]e^{2\pi/\sqrt{5}}.$$

(1.13) If $F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1.3}{2.4}\right)^2 k^2 + \ldots$ and

$F(1-k) = \sqrt{(210)}F(k)$, then

$$k = (\sqrt{2} - 1)^4(2 - \sqrt{3})^2(\sqrt{7} - \sqrt{6})^4(8 - 3\sqrt{7})^2(\sqrt{10} - 3)^4$$
$$\times (4 - \sqrt{15})^4(\sqrt{15} - \sqrt{14})^2(6 - \sqrt{35})^2.$$

(1.14) The coefficient of x^n in $(1 - 2x + 2x^4 - 2x^9 + \ldots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right).$$

(1.15) The number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{(\log t)}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small compared with the previous integral.

Hardy commented as follows.

"I should like you to begin by trying to reconstruct the immediate reactions of an ordinary professional mathematician who receives a letter like this from an unknown Hindu clerk.

"The first question was whether I could recognise anything. I had proved things rather like (1.7) myself, and seemed vaguely familiar with (1.8). Actually (1.8) is classical; it is a formula of Laplace first proved properly by Jacobi; and (1.9) occurs in a paper published by Rogers in 1907. I thought that, as an expert in definite integrals, I could probably prove (1.5) and (1.6), and did so, though with a good deal more trouble than I had expected....

"The series formulae (1.1)-(1.4) I found much more intriguing, and it soon became obvious that Ramanujan must possess much more general theorems and was keeping a great deal up his sleeve. The second is a formula of Bauer well known in the theory of Legendre series, but the others are much harder than they look....

"The formulae (1.10)-(1.13) are on different level and obviously both difficult and deep. An expert in elliptic functions can see at once that (1.13) is derived somehow from the theory of 'complex multiplication', but (1.10)-(1.12) defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally...the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill...."

Hardy's invitation to come to Cambridge led to one of the most fruitful collaborations in the history of mathematics. Ramanujan spent 5 years with Hardy, returned home to India in 1919, and died of tuberculosis the following year at age 32. The story of Ramanujan is briefly told in [1], the source for the above notes;

[2] is a full biography and [3] and [4] contain many of his contributions to mathematics.

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2. *Ramanujan: The Man and the Mathematician*, S.R. Ranganathan, (Asia Publishing House).

3. *Collected Papers of Ramanujan*, G.H. Hardy, Aiyar, and Wilson.

4. *Ramanujan: 12 Lectures on his Life and Work*, G.H. Hardy.

2. HIGHLY COMPOSITE NUMBERS

In [3] there is a 60-page paper on what Ramanujan calls "highly composite numbers." In this introduction we shall establish only one simple theorem concerning the form such numbers must take.

As one would expect from the name, these numbers have "particularly many" divisors. Using $d(N)$ to denote the number of positive divisors of the positive integer N, a highly composite number is one which has a peculiarly large value of $d(N)$. To be precise, a positive integer N is highly composite if it has more divisors than any smaller positive integer:

$$N \text{ is } \textit{highly composite} \text{ if } d(N) > d(N') \text{ for all } N' < N.$$

In a sense, these are the opposite of the prime numbers, which (except for the number 1) have the minimum possible number of divisors (2). From Table I we can see that the first half-dozen highly composite numbers are 2, 4, 6, 12, 24, and 36.

Table I

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$d(N)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2

N	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
$d(N)$	6	4	4	2	8	3	4	4	6	2	8	2	6	4	4	4	9	2	4

In his paper, Ramanujan gave a list of the more than 100 highly composite numbers up to 6746328388800 (Table II). This represents a staggering amount of calculation, for he knew of no formula for generating them. For the sake of completeness, it should be noted that Ramanujan evidently missed one highly composite number, for 293318625600 (starred in the table) does not appear in his list.

TABLE II: HIGHLY COMPOSITE NUMBERS

$dd(N)$	$d(N)$	N
2	$2 = 2$	$2 = 2$
2	$3 = 3$	$4 = 2^2$
3	$4 = 2^2$	$6 = 2 \cdot 3$
4	$6 = 2 \cdot 3$	$12 = 2^2 \cdot 3$
4	$8 = 2^3$	$24 = 2^3 \cdot 3$
3	$9 = 3^2$	$36 = 2^2 \cdot 3^2$
4	$10 = 2 \cdot 5$	$48 = 2^4 \cdot 3$
6	$12 = 2^2 \cdot 3$	$60 = 2^2 \cdot 3 \cdot 5$
5	$16 = 2^4$	$120 = 2^3 \cdot 3 \cdot 5$
6	$18 = 2 \cdot 3^2$	$180 = 2^2 \cdot 3^2 \cdot 5$
6	$20 = 2^2 \cdot 5$	$240 = 2^4 \cdot 3 \cdot 5$
8	$24 = 2^3 \cdot 3$	$360 = 2^3 \cdot 3^2 \cdot 5$
8	$30 = 2 \cdot 3 \cdot 5$	$720 = 2^4 \cdot 3^2 \cdot 5$
6	$32 = 2^5$	$840 = 2^3 \cdot 3 \cdot 5 \cdot 7$
9	$36 = 2^2 \cdot 3^2$	$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$
8	$40 = 2^3 \cdot 5$	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$
10	$48 = 2^4 \cdot 3$	$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$
12	$60 = 2^2 \cdot 3 \cdot 5$	$5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$
7	$64 = 2^6$	$7560 = 2^3 \cdot 3^3 \cdot 5 \cdot 7$
12	$72 = 2^3 \cdot 3^2$	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$
10	$80 = 2^4 \cdot 5$	$15120 = 2^4 \cdot 3^3 \cdot 5 \cdot 7$
12	$84 = 2^2 \cdot 3 \cdot 7$	$20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
12	$90 = 2 \cdot 3^2 \cdot 5$	$25200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7$
12	$96 = 2^5 \cdot 3$	$27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
9	$100 = 2^2 \cdot 5^2$	$45360 = 2^4 \cdot 3^4 \cdot 5 \cdot 7$
12	$108 = 2^2 \cdot 3^3$	$50400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
16	$120 = 2^3 \cdot 3 \cdot 5$	$55440 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
8	$128 = 2^7$	$83160 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
15	$144 = 2^4 \cdot 3^2$	$110880 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
12	$160 = 2^5 \cdot 5$	$166320 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
16	$168 = 2^3 \cdot 3 \cdot 7$	$221760 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
18	$180 = 2^2 \cdot 3^2 \cdot 5$	$277200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
14	$192 = 2^6 \cdot 3$	$332640 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
12	$200 = 2^3 \cdot 5^2$	$498960 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$
16	$216 = 2^3 \cdot 3^3$	$554400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
12	$224 = 2^5 \cdot 7$	$665280 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
20	$240 = 2^4 \cdot 3 \cdot 5$	$720720 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
9	$256 = 2^8$	$1081080 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
18	$288 = 2^5 \cdot 3^2$	$1441440 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
14	$320 = 2^6 \cdot 5$	$2162160 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$336 = 2^4 \cdot 3 \cdot 7$	$2882880 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$360 = 2^3 \cdot 3^2 \cdot 5$	$3603600 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$384 = 2^7 \cdot 3$	$4324320 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
15	$400 = 2^4 \cdot 5^2$	$6486480 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
20	$432 = 2^4 \cdot 3^3$	$7207200 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
14	$448 = 2^6 \cdot 7$	$8648640 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
24	$480 = 2^5 \cdot 3 \cdot 5$	$10810800 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$504 = 2^3 \cdot 3^2 \cdot 7$	$14414400 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$

$dd(N)$	$d(N)$	N
10	$512 = 2^9$	$17297280 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
21	$576 = 2^6 \cdot 3^2$	$21621600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
24	$600 = 2^3 \cdot 3 \cdot 5^2$	$32432400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
16	$640 = 2^7 \cdot 5$	$36756720 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$672 = 2^5 \cdot 3 \cdot 7$	$43243200 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
30	$720 = 2^4 \cdot 3^2 \cdot 5$	$61261200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$768 = 2^8 \cdot 3$	$73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$800 = 2^5 \cdot 5^2$	$110270160 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$864 = 2^5 \cdot 3^3$	$122522400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
16	$896 = 2^7 \cdot 7$	$147026880 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$960 = 2^6 \cdot 3 \cdot 5$	$183783600 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1008 = 2^4 \cdot 3^2 \cdot 7$	$245044800 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
11	$1024 = 2^{10}$	$294053760 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
24	$1152 = 2^7 \cdot 3^2$	$367567200 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
30	$1200 = 2^4 \cdot 3 \cdot 5^2$	$551350800 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
18	$1280 = 2^8 \cdot 5$	$698377680 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
28	$1344 = 2^6 \cdot 3 \cdot 7$	$735134400 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
36	$1440 = 2^5 \cdot 3^2 \cdot 5$	$1102701600 = 2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
20	$1536 = 2^9 \cdot 3$	$1396755360 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
21	$1600 = 2^6 \cdot 5^2$	$2095133040 = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
40	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$	$2205403200 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$
28	$1728 = 2^6 \cdot 3^3$	$2327925600 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
18	$1792 = 2^8 \cdot 7$	$2793510720 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$1920 = 2^7 \cdot 3 \cdot 5$	$3491888400 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2016 = 2^5 \cdot 3^2 \cdot 7$	$4655851200 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
12	$2048 = 2^{11}$	$5587021440 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
27	$2304 = 2^8 \cdot 3^2$	$6983776800 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$2400 = 2^5 \cdot 3 \cdot 5^2$	$10475665200 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$2688 = 2^7 \cdot 3 \cdot 7$	$13967553600 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
42	$2880 = 2^6 \cdot 3^2 \cdot 5$	$20951330400 = 2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
22	$3072 = 2^{10} \cdot 3$	$27935107200 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
48	$3360 = 2^5 \cdot 3 \cdot 5 \cdot 7$	$41902660800 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
32	$3456 = 2^7 \cdot 3^3$	$48886437600 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
20	$3584 = 2^9 \cdot 7$	$64250746560 = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
45	$3600 = 2^4 \cdot 3^2 \cdot 5^2$	$73329656400 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$3840 = 2^8 \cdot 3 \cdot 5$	$80313433200 = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4032 = 2^6 \cdot 3^2 \cdot 7$	$97772875200 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
13	$4096 = 2^{12}$	$128501493120 = 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$4320 = 2^5 \cdot 3^3 \cdot 5$	$146659312800 = 2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
30	$4608 = 2^9 \cdot 3^2$	$160626866400 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
42	$4800 = 2^6 \cdot 3 \cdot 5^2$	$240940299600 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
60	$5040 = 7 \cdot 5 \cdot 3^2 \cdot 2^4$	$*293318625600 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
36	$5376 = 2^8 \cdot 3 \cdot 7$	$321253732800 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
48	$5760 = 2^7 \cdot 3^2 \cdot 5$	$481880599200 = 2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
24	$6144 = 2^{11} \cdot 3$	$642507465600 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$6720 = 2^6 \cdot 3 \cdot 5 \cdot 7$	$963761198400 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
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22	$7168 = 2^{10} \cdot 7$	$1606268664000 = 2^6 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
54	$7200 = 2^5 \cdot 3^2 \cdot 5^2$	$1686582097200 = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
40	$7680 = 2^9 \cdot 3 \cdot 5$	$1927522396800 = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

$dd(N)$	$d(N)$	N
48	$8064 = 2^7 \cdot 3^2 \cdot 7$	$2248776129600 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
14	$8192 = 2^{13}$	$3212537328000 = 2^7 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
56	$8640 = 2^6 \cdot 3^3 \cdot 5$	$3373164194400 = 2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
33	$9216 = 2^{10} \cdot 3^2$	$4497552259200 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$
72	$10080 = 2^5 \cdot 3^2 \cdot 5 \cdot 7$	$6746328388800 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

This brings to mind a penetrating remark made by Harold Edwards in the preface of his new book *Fermat's Last Theorem* (Springer-Verlag Graduate Texts Series, number 50):

> "..., as even a superficial glance at history shows, Kummer and the other great innovators in number theory did vast amounts of computation and gained much of their insight in this way. I deplore the fact that contemporary mathematical education tends to give students the idea that computation is demeaning drudgery to be avoided at all costs."

Now let us consider the following theorem on the structure of highly composite numbers:

THEOREM. If $N = 2^{a_2}3^{a_3}\ldots p^{a_p}$ *is the prime decomposition of the highly composite nunber* N, *then*

(i) *the primes* $2, 3, \ldots, p$ *form an unbroken sequence of consecutive primes as far as they go;*

(ii) *the exponents are nonincreasing:* $a_2 \geq a_3 \geq \ldots \geq a_p$;

(iii) *the final exponent* a_p *is always* 1, *except for the two cases* $N = 4 = 2^2$ *and* $N = 36 = 2^2 \cdot 3^2$, *when it is* 2.

Proof. Parts (i) and (ii) are particularly easy to prove from a simple formula for $d(N)$ which we shall begin by deriving. Suppose

$$N = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$$

is the prime decomposition of a positive integer N. Then every divisor m of N must have a like prime decomposition

$$m = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n},$$

where each b_i is restricted to the range $\{0,1,2,\ldots,a_i\}$. Thus there are $a_1 + 1$ choices for the value of b_1, $a_2 + 1$ choices for b_2, and so on, implying that the number of divisors m which it is possible to construct is

$$d(N) = (a_1 + 1)(a_2 + 1)\ldots(a_n + 1).$$

Observe that $d(N)$ depends only on the collection of exponents, not on the primes themselves, and that the value of $d(N)$ is not changed by permuting the exponents over the same or a different set of prime bases.

(i) Suppose that some prime P is missing from the string of prime factors 2, 3,..., p of the highly composite number N:

$$N = 2^{a_2}3^{a_3}\ldots(\quad)\ldots p^{a_p},$$

where the empty parentheses remind us of the missing factor P^r. Dropping the final factor p^{a_p} and including the smaller P^{a_p} instead, we obtain the smaller number

$$N' = 2^{a_2}3^{a_3}\ldots(P^{a_p})\ldots$$

having the same exponents and therefore the same value of d:

$$N' < N \quad \text{and} \quad d(N') = d(N).$$

This contradicts the highly composite character of N, and part (i) follows.

(ii) Suppose the exponent of p_2 is greater than the exponent m of the smaller prime p_1. Then, for some positive integer n we have

$$N = 2^{a_2}3^{a_3}\ldots p_1^{\,m}\ldots p_2^{\,m+n}\ldots p^{a_p}.$$

Interchanging the exponents of p_1 and p_2, then, we obtain the contradiction of a smaller number

$$N' = 2^{a_2}3^{a_3}\ldots p_1^{\,m+n}\ldots p_2^{\,m}\ldots p^{a_p}$$

having $d(N') = d(N)$.

(iii) Before we are finished with part (iii), we shall need the following results from number theory. Suppose p_1, p_2, p_3, p_4, p_5 are consecutive in the sequence of primes; then

(a) $p_1^2 > p_2$ for all choices of p_1;
(b) $p_1^3 > p_4$ for all p_1;
(c) $p_2p_3 > p_1p_4$ for $5 \le p_3 \le 19$;
(d) $p_1p_2p_3 > p_4p_5$ for $p_3 \ge 11$.

There is no difficulty in verifying part (c) by direct observation of the few cases it describes, and the other parts follow easily from the very useful theorem known as Bertrand's Postulate. This declares the existence of a prime number between n and $2n$ for n a positive integer > 1. A complete account of Paul Erdős' marvellous elementary proof of this difficult theorem is given in one of my all-time favourite books—Wacław Sierpiński's *Theory of Numbers* (see the chapter on prime numbers). Erdős did this when he was 17 or 18 (1931); an auspicious beginning to a truly amazing creative career, which is still flourishing today.

By Bertrand's Postulate we have $p_{n+1} < 2p_n$, and repeated application gives

$$p_{n+1} < 2p_n < 4p_{n-1} < 8p_{n-2} < \dots .$$

Any of the above relations can be verified directly for the early primes and Bertrand's Postulate takes care of all cases beyond a very small minimum.

(a) $p_2 < 2p_1 < p_1^2$ for $p_1 \geq 3$;
(b) $p_4 < 2p_3 < 4p_2 < 8p_1 < p_1^3$ for $p_1 \geq 3$;
(d) $p_4 < 2p_3$ and $p_5 < 2p_4 < 4p_3 < 8p_2$; thus $p_4p_5 < 16p_2p_3 < p_1p_2p_3$ for $p_1 \geq 17$.

The first use we make of these results is to show that the final exponent cannot exceed 2. Suppose, to the contrary, that the final prime p_n has exponent $m+2$ where $m \geq 1$:

$$N = 2^{a_2}3^{a_3}\dots p_n^{m+2}.$$

By (a) we have $p_n^2 > p_{n+1}$, and so the number

$$N' = \frac{p_{n+1}}{p_n^2} \cdot N = 2^{a_2}3^{a_3}\dots p_n^m p_{n+1}$$

is less than N. Since N is highly composite, we have $d(N') < d(N)$, that is,

$$(a_2+1)(a_3+1)\dots(m+1)\cdot 2 < (a_2+1)(a_3+1)\dots(m+3),$$

so $(m+1)\cdot 2 < m+3$ and $m < 1$, a contradiction.

Next we similarly prove that the third last exponent cannot exceed 4. Suppose that the last three primes are p_1, p_2, and p_3, and that the exponent of p_1 is $4+m$ with $m \geq 1$:

$$N = 2^{a_2}3^{a_3}\dots p_1^{4+m}p_2^s p_3^t.$$

By (b) we have $p_1^3 > p_4$, implying that

$$N' = 2^{a_2}3^{a_3}\dots p_1^{m+1}p_2^s p_3^t p_4 < N,$$

which leads to $d(N') < d(N)$, that is,

$$(a_2+1)\ldots(m+2)(s+1)(t+1)\cdot 2 < (a_2+1)\ldots(5+m)(s+1)(t+1),$$

so $2(m+2) < 5+m$ and $m<1$, a contradiction.

Now let us consider highly composite numbers N having at least three prime divisors, and suppose that the last three primes in the prime decomposition of such an N are p_1, p_2, and p_3. We show that the final exponent is 1, first in the case $p_3 \le 19$, and then for the somewhat overlapping case of all $p_3 \ge 11$.

1) $\underline{p_3 \le 19.}$ We have seen that the final exponent cannot exceed 2. Suppose, then, that it is 2:

$$N = 2^{a_2}3^{a_3}\ldots p_1^r p_2^s p_3^2.$$

With two primes smaller than p_3, p_3 must be at least 5, and for $p_3 \le 19$ we obtain by (c) above that $p_2p_3 > p_1p_4$. Consequently,

$$N' = \frac{p_1p_4}{p_2p_3}\cdot N = 2^{a_2}3^{a_3}\ldots p_1^{r+1}p_2^{s-1}p_3p_4 < N,$$

and we have $d(N') < d(N)$, yielding

$$(r+2)\cdot s\cdot 2\cdot 2 < (r+1)(s+1)\cdot 3,$$

which reduces easily to $r(s-3)+5s < 3$. Now, since the exponents are nonincreasing, we have $r \ge s \ge 2$. Clearly for $s \ge 3$ the value of $r(s-3)+5s \ge 15$; and for $s=2$ we have $r>7$. However, the third last exponent cannot exceed 4, so we have a contradiction in all cases.

2) $\underline{p_3 \ge 11.}$ Again, suppose that the final exponent is 2:

$$N = 2^{a_2}3^{a_3}\ldots p_1^r p_2^s p_3^2.$$

By (d) we have

$$N' = \frac{p_4p_5}{p_1p_2p_3}\cdot N = 2^{a_2}3^{a_3}\ldots p_1^{r-1}p_2^{s-1}p_3p_4p_5 < N,$$

and $d(N') < d(N)$ yields $rs\cdot 2\cdot 2\cdot 2 < (r+1)(s+1)\cdot 3$, so

$$8 < \left(1+\frac{1}{r}\right)\left(1+\frac{1}{s}\right)\cdot 3.$$

Since $r \ge s \ge 2$, each of $1+1/r$, $1+1/s$ is less than or equal to $3/2$. Thus

$$\left(1+\frac{1}{r}\right)\left(1+\frac{1}{s}\right)\cdot 3 \le \frac{3}{2}\cdot\frac{3}{2}\cdot 3 = \frac{27}{4} < 8,$$

a contradiction. Therefore, if N has three or more prime divisors, the final exponent is 1.

There remain to be considered all N which have one or two prime divisors. Since the prime divisors form an unbroken string in the sequence of primes, we have only numbers of the form

$$N = 2^a 3^b.$$

From our earlier results we have $a \geq b$ and $b \leq 2$, and we are interested only in the exceptional times when $b \neq 1$.

For $b = 0$, we have $N = 2^a$ where $a \leq 2$. Thus $a = 2$ gives the only time the final exponent is not 1, and we get $N = 2^2 = 4$.

For $b = 2$, we have $N = 2^a 3^2$ where $a \geq 2$. Suppose $a = 2 + n$ where $n \geq 0$. Since $2 \cdot 3 > 5$, we have

$$N' = 2^{1+n} \cdot 3 \cdot 5 < 2^{2+n} 3^2 = N,$$

and then $d(N') < d(N)$ gives $(2+n) \cdot 2 \cdot 2 < (3+n) \cdot 3$. Thus $n < 1$, implying that $n = 0$ and $N = 2^2 3^2 = 36$.

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* * *

THE OLYMPIAD CORNER: 1

MURRAY S. KLAMKIN

On behalf of the Canadian Mathematical Olympiad Committee, the U.S.A. Mathematical Olympiad Committee, and the U.S.A. Annual High School Mathematics Examination Committee, I would like to thank the editor for giving us access to the pages of this journal.

This column will provide, on a continuing basis, information about various mathematical contests taking place in Canada, the U.S.A., and internationally. It will also provide from time to time practice sets of problems on which interested students can test and sharpen their mathematical skills and thereby possibly qualify to participate in some Olympiad. "Official" solutions to these problems will be published in a succeeding issue. Teachers are encouraged to go into these solutions and possible extensions more thoroughly with their interested students. They should impress upon the students the importance of having *clear*, *concise*, and *complete* solutions since *good presentation counts* (in both the Canadian and U.S.A. Mathematical Olympiads). This includes *legibility*, *good English*, and *mathematical clarity*. Also, extra credit is given in both Olympiads for particularly elegant solutions as well as for nontrivial generalizations with proof.

I will be glad to receive communications on these problems from students (or their teachers) who have come up with an elegant solution different from the "official" one, and/or a nice extension with proof. Some of these may then be published in this section.

To start the ball rolling, I give below three Practice Sets. "Official" solutions will start appearing in the next issue.

PRACTICE SET 1 (3 hours)

1-1. If a, b, c, d are positive integers such that $ab = cd$, prove that $a^2 + b^2 + c^2 + d^2$ is never a prime number.

1-2. If two circles pass through the vertex and a point on the bisector of an angle, prove that they intercept equal segments on the sides of the angle.

1-3. (a) If a, b, $c \geq 0$ and $(1+a)(1+b)(1+c) = 8$, prove that $abc \leq 1$.
(b) If a, b, $c \geq 1$, prove that $4(abc+1) \geq (1+a)(1+b)(1+c)$.

PRACTICE SET 2 (3 hours)

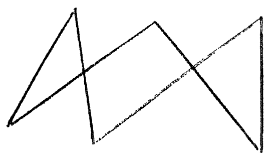
2-1. It is easy to see that there exists an infinite family of ellipses which can be inscribed in a given square. Prove, however, that only *one* ellipse can be inscribed in a given regular pentagon.

2-2. Determine all pairs of rational numbers (x,y) such that $x^3 + y^3 = x^2 + y^2$.

2-3. Three unequal disjoint circles are given on a large (planar) card. If the centers of the circles are collinear, show that it is always possible to fold the card along two straight lines such that the three circles lie on a common sphere.

PRACTICE SET 3 (3 hours)

3-1. Does there exist a polygon of 17 sides such that some straight line intersects each of its sides in some point other than a vertex of the polygon? Note that the polygon need not be convex nor simple, e.g.,



3-2. Prove that from any row of n integers one may always select a block of adjacent integers whose sum is divisible by n.

3-3. If

$$\frac{a}{bc - a^2} + \frac{b}{ca - b^2} + \frac{c}{ab - c^2} = 0,$$

prove that also

$$\frac{a}{(bc - a^2)^2} + \frac{b}{(ca - b^2)^2} + \frac{c}{(ab - c^2)^2} = 0.$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

401. *Proposed by Herman Nyon, Paramaribo, Surinam.*

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 HAPPY
   NEW
  YEAR
 -----
*1979
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In this decimal alphametic, replace the eight distinct letters and the asterisk by nine distinct *nonzero* digits. Since years come in cycles of seven (seven lean years, seven fat years), YEAR should be divisible by 7.

I hope this will turn out to be a doubly-true alphametic for all readers of this journal.

402. *Proposed by the late R. Robinson Rowe, Sacramento, California.*

An army with an initial strength of A men is exactly decimated each day of a 5-day battle and reinforced each night with R men from the reserve pool of P men, winding up on the morning of the 6th day with 60% of its initial strength. At least how large must the initial strength have been if

(a) R was a constant number each day;

(b) R was exactly half the men available in the dwindling pool?

403*. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $Z^+ = \{0,1,2,\ldots\}$ and set

$$\begin{aligned}
A_1 &= \{3m^2+6mn+3n^2+2m+3n+1 \ : \ m,n \in Z^+\},\\
A_2 &= \{3m^2+6mn+3n^2+4m+5n+2 \ : \ m,n \in Z^+\},\\
A_3 &= \{3m^2+6mn+3n^2+5m+6n+3 \ : \ m,n \in Z^+\},\\
A_4 &= \{3m^2+6mn+3n^2+6m+7n+4 \ : \ m,n \in Z^+\},\\
A_5 &= \{3m^2+6mn+3n^2+7m+8n+5 \ : \ m,n \in Z^+\},\\
A_6 &= \{3m^2+6mn+3n^2+9m+10n+8 \ : \ m,n \in Z^+\},
\end{aligned}$$

so that

$$\begin{aligned}
A_1 &= \{1, 6, 7, 17, 18, 19, 34, 35, 36, 37, 57, 58, 59, 60, 61,\ldots\},\\
A_2 &= \{2, 9, 10, 22, 23, 24, 41, 42, 43, 44,\ldots\},\\
A_3 &= \{3, 11, 12, 25, 26, 27, 45, 46, 47, 48,\ldots\},\\
A_4 &= \{4, 13, 14, 28, 29, 30, 49, 50, 51, 52,\ldots\},\\
A_5 &= \{5, 15, 16, 31, 32, 33, 53, 54, 55, 56,\ldots\},\\
A_6 &= \{8, 20, 21, 38, 39, 40, 62, 63, 64, 65,\ldots\}.
\end{aligned}$$

Prove or disprove that

(a) the elements of A_i are all distinct for $1 \le i \le 6$;

(b) $A_i \cap A_j = \emptyset$ for $1 \le i < j \le 6$;

(c) $\{0\} \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = Z^+$.

404. *Proposed by A. Liu, University of Alberta.*

Let A be a set of n distinct positive numbers. Prove that

(a) the number of distinct sums of subsets of A is at least $\frac{1}{2}n(n+1)+1$;

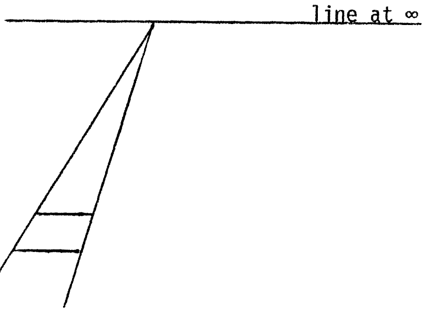
(b) the number of distinct subsets of A with sum equal to half the sum of A is at most $2^n/(n+1)$.

405. *Proposed by Viktors Linis, University of Ottawa.*

A circle of radius 16 contains 650 points. Prove that there exists an annulus of inner radius 2 and outer radius 3 which contains at least 10 of the given points.

406. *Proposed by W.A. McWorter Jr., The Ohio State University.*

The figure shows an unfinished perspective drawing of a railroad track with two ties drawn parallel to the line at ∞. Can the remaining ties be drawn, assuming that the actual track has equally spaced ties?



407. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

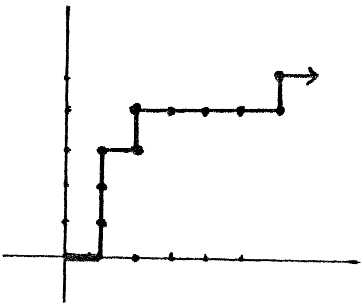
There are decimal integers whose representation in some number base $B = 2, 3, 4, \ldots$ consists of three nonzero digits whose cubes sum to the integer. For example,

$$43_{10} = 223_4 = 2^3 + 2^3 + 3^3,$$
$$134_{10} = 251_7 = 2^3 + 5^3 + 1^3,$$
$$433_{10} = 661_8 = 6^3 + 6^3 + 1^3.$$

Prove that infinitely many such integers exist.

408*. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A *zigzag* is an infinite connected path in a Cartesian plane formed by starting at the origin and moving successively one unit right or up (see figure). Prove or disprove that for every zigzag and for every positive integer k, there exist (at least) k collinear lattice points on the zigzag.



(This problem was given to me by a classmate at City College of New York in 1971-72. Its origin is unknown to me.)

409. *Proposed by L.F. Meyers, The Ohio State University.*

In a certain bingo game for children, each move consists in rolling (actually, "popping") two dice. One of the dice is marked with the symbols B, I, N, G, O, and *, and the other die is marked with 1, 2, 3, 4, 5, and 6. A disadvantage of this form of bingo, in comparison with the adult form of the game, is that a combination (such as B3) may appear repeatedly. What is the expected number of the move at which the first repetition occurs in each of these cases:

(a) all 36 combinations (B1 through *6) are considered to be different (and equally likely)?

(b) all 36 combinations (B1 through *6) are considered to be equally likely, but the 6 combinations containing * are considered to be the same?

410.* *Proposed by James Gary Propp, student, Harvard College, Cambridge, Massachusetts.*

Are there only finitely many powers of 2 that have no zeros in their decimal expansions?

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

250. [1977: 132; 1978: 39] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

(a) Find all pairs (m,n) of positive integers such that

$$|3^m - 2^n| = 1.$$

(b) If $|3^m - 2^n| \neq 1$, is there always a prime between 3^m and 2^n?

IV. *Partial solution of part (b) by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

It is well known that

$$\lim_{m,n\to\infty} |3^m - 2^n| = \infty.$$

This is a consequence of the Thue-Siegel Theorem (see Landau, *Vorlesungen*). Even more than this was proved by Pillai [2]. He showed that for every δ, $0<\delta<1$, for all sufficiently large m and n, we have

$$|3^m - 2^n| > (\mathrm{Max}\ \{3^m, 2^n\})^{1-\delta}.$$

From the above it follows that if $3^m > 2^n$, for sufficiently large m and n we have

$$3^m - 2^n > (2^n)^{1-\delta}.$$

It is known (see, e.g., Huxley [1]) that, for x large, there is always a prime between x and $x+x^{2/3}$. So, using $\delta = 1/3$ in the above, we conclude that for large m and n there is a prime between 2^n and 3^m. The case where $2^n > 3^m$ may be treated similarly.

REFERENCES

1. M. Huxley, "On the difference between consecutive primes", *Invent. Math.*, 15 (1972) 164-170.

2. S.S. Pillai, "On the inequality $0 < a^x - b^y \le n$", *Journal of the Indian Mathematical Society*, 19 (1931) 1-11.

* * *

321, 322, 323, 325, 327, 332. [1978: 65, 66, 100, 252, 254, 255, 258, 260, 266] The following name was inadvertently omitted from the lists of solvers of these problems: ROBERT S. JOHNSON, Montréal, Québec.

326. [1978: 66, 259] *Proposed by Harry D. Ruderman, Hunter College, New York.*

If the members of the set

$$S = \{2^x 3^y \mid x,y \text{ are nonnegative integers}\}$$

are arranged in increasing order, we get the sequence beginning

$$1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 16,\ 18,\ldots\ .$$

(a) What is the position of $2^a 3^b$ in the sequence in terms of a and b?

(b) What is the nth term of the sequence in terms of n?

I. *Partial solution by the proposer.*

(a) Given a and b, let

$$S_i = \{(i,y) \mid 2^i 3^y \le 2^a 3^b,\ y \in \{0,1,2,\ldots\}\}, \tag{1}$$

$$C_i = \text{the cardinality of } S_i,$$

and

$$M = [a + b\log_2 3],$$

where the square brackets denote the greatest integer function. M is the largest value of i such that $2^i \le 2^a 3^b$. Since the inequality in (1) is equivalent to

$$y \le (a-i)\log_3 2 + b,$$

we have, remembering that y can be zero,

$$C_i = [(a-i)\log_3 2] + b + 1.$$

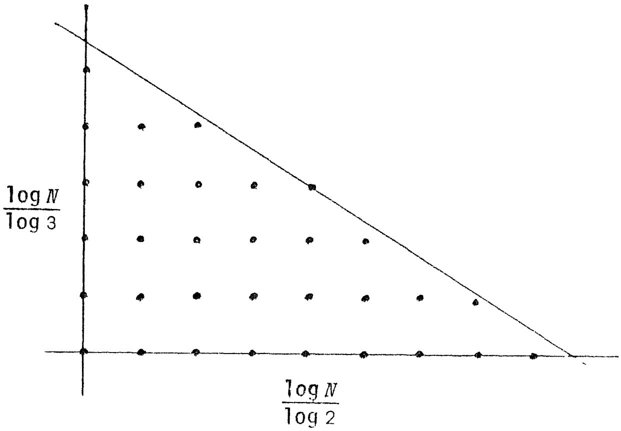
The rank n of $2^a 3^b$ in the sequence is thus given by

$$n = n(a,b) = \sum_{i=0}^{M} C_i = (M+1)(b+1) + \sum_{i=0}^{M} [(a-i)\log_3 2].$$

II. *Partial solution by Viktors Linis, University of Ottawa, and the editor (jointly).*

In part (a), given a and b it is not hard to find the rank n of 2^a3^b in the sequence by a formula involving a summation. But such a formula is just an efficient way of counting the terms of the sequence, and as such it amounts to little more than a restatement of the problem. Furthermore, it offers little hope that it can be used to gain information about the more difficult part (b).

We offer instead for part (a) a formula which will yield, after a single calculation, a value n^* which is either the exact rank n or a very close approximation to it, and this is sufficient for many purposes. Our formula will have the added advantage of furnishing us with much insight into part (b).



(a) Let L be the set of lattice points in the closed first quadrant of a coordinate plane. The mapping $\mu : L \to S$ defined by $\mu(x,y) = 2^x3^y$ is clearly a bijection. Hence, if $N = 2^a3^b$, where a and b are fixed, the number n of elements of S such that

$$1 \le 2^x3^y \le N \tag{2}$$

(i.e. the rank of 2^a3^b in the sequence) is the same as the number of lattice points whose coordinates satsify (2) or its equivalent

$$0 \le x\log 2 + y\log 3 \le \log N,$$

where it is clear that the logarithms may be taken to any convenient base. These lattice points are all those inside and on the boundary of a triangular region in the plane (see the figure, which is drawn for the case $(a,b) = (4,3)$). The figure suggests that their number n must be very close to

$$n^* = \left[\frac{1}{2}\left(\frac{\log N}{\log 2} + 1\right)\left(\frac{\log N}{\log 3} + 1\right)\right] + 1, \tag{3}$$

where the square brackets denote the greatest integer function. If we set

$$\lambda = \frac{\log 3}{\log 2} = 1.5849625007\ldots,$$

then (3) can be written

$$n^* = n^*(a,b) = [\tfrac{1}{2}(a+b\lambda+1)(a/\lambda+b+1)]+1, \tag{4}$$

and this is the promised formula for part (a), whose accuracy must now be assessed.

The first 309 terms t_n of the sequence were first determined, from $t_1 = 1$ to $t_{309} = 2^{30}$, and the value of n^* for each was then calculated from (4). In nearly 90% of the cases, (4) gave the correct rank n, and when it did not the calculated rank n^* differed from the true one by only *one unit*. To give as much information as possible to readers who may wish to work further on this problem, we list all the cases where (4) failed to give the correct rank n.

(a,b)	n	n^*
(3,5)	47	48
(8,2)	50	49
(4,5)	55	56
(3,6)	60	61
(10,2)	67	66
(4,6)	69	70
(6,5)	73	74
(6,6)	89	90
(20,1)	165	166
(14,5)	170	171
(22,0)	171	172
(0,14)	175	174

(a,b)	n	n^*
(23,0)	186	187
(14,6)	194	195
(22,1)	195	196
(2,14)	206	205
(23,1)	211	212
(20,3)	214	215
(25,0)	218	219
(25,1)	245	246
(22,3)	248	249
(19,5)	251	252
(8,12)	254	253
(5,14)	257	256

(a,b)	n	n^*
(2,16)	260	259
(26,1)	263	264
(23,3)	266	267
(20,5)	269	270
(19,6)	280	281
(8,13)	283	282
(21,5)	288	289
(10,12)	291	290
(20,6)	299	300
(25,3)	304	305
(22,5)	307	308

We have just seen that $|n(a,b) - n^*(a,b)| \leq 1$ at least for the first few hundred terms of the sequence, but it is unlikely that this inequality holds for *all* terms. More information is needed about the behaviour of this difference as $a,b \to \infty$.

It might be possible to refine formula (4) to increase its range of validity, but it is in its present form that it furnishes valuable insights into part (b), as we shall now see.

(b) Here, given the positive integer n, we must determine nonnegative integers $a(n)$ and $b(n)$ such that $N = 2^a 3^b$ is the nth term of the sequence. The bijection $\mu : L \to S$ assures us that for every n the pair $(a(n), b(n))$ exists and is unique. What we will do is to put the given n in the left side of (4) and solve for a and b. The resulting $N^* = 2^a 3^b$ should then be either the required N or a term relatively close to it in the sequence. It is clear from (4) that $n = 1$ if and only if $(a,b) = (0,0)$ and $N^* = N = 1$. We now assume $n \geq 2$.

With $u = a + b\lambda$, (4) becomes

$$n = [\tfrac{1}{2}(u+1)(u/\lambda+1)] + 1, \tag{5}$$

from which we get

$$n - 1 < \tfrac{1}{2}(u+1)(u/\lambda+1) < n,$$

the inequalities being strict because λ is irrational. Separately, the two inequalities are equivalent to

$$u^2 + (1+\lambda)u - (2n-3)\lambda > 0 \tag{6}$$

and

$$u^2 + (1+\lambda)u - (2n-1)\lambda < 0. \tag{7}$$

The corresponding quadratic equations each have a unique positive root, those for (6) and (7) being respectively

$$u_1 = \frac{-(1+\lambda) + \sqrt{(1+\lambda)^2 + 4(2n-3)\lambda}}{2} \tag{8}$$

and

$$u_2 = \frac{-(1+\lambda) + \sqrt{(1+\lambda)^2 + 4(2n-1)\lambda}}{2}. \tag{9}$$

Here we have $u_1 < u_2$ and the solutions $u = a + b\lambda$ of (5) must satisfy $u_1 < u < u_2$; hence

$$u_1 - b\lambda < a < u_2 - b\lambda. \tag{10}$$

Now for $n \geq 2$,

$$\begin{aligned}
(u_2 - b\lambda) - (u_1 - b\lambda) &= u_2 - u_1 \\
&= \frac{4\lambda}{\sqrt{(1+\lambda)^2 + 4(2n-1)\lambda} + \sqrt{(1+\lambda)^2 + 4(2n-3)\lambda}} \\
&\leq \frac{4\lambda}{\sqrt{(1+\lambda)^2 + 12\lambda} + \sqrt{(1+\lambda)^2 + 4\lambda}} \\
&= 0.7305\ldots \\
&< 1;
\end{aligned}$$

hence the nonnegative integers $a(n)$ and $b(n)$ must satisfy

$$a = [u_2 - b\lambda] = [u_1 - b\lambda] + 1 = [u_1 + 1 - b\lambda].$$

It now turns out that, while we lack an explicit formula for N^*, we have the next best thing: an efficient algorithm to find N^* given $n \geq 2$. The work can conveniently be carried out as follows: given n, we first calculate u_1 and u_2 from

(8) and (9); we then determine by trial and success all the nonnegative integers $b(n)$ which satisfy

$$\left[\lambda\left(\frac{u_2}{\lambda} - b\right)\right] = \left[\lambda\left(\frac{u_1+1}{\lambda} - b\right)\right], \tag{11}$$

the common value of the two sides being the corresponding $a(n)$. Each pair (a,b) then gives a value of $N^* = 2^a 3^b$. The number of values of b to be tried is finite, since $b < u_2/\lambda$ by (10).

When n is not too large, certain tentative conclusions can be drawn. In most cases, only one value of b will satisfy (11), and then $N^* = N = 2^a 3^b$ is uniquely determined. For example, if $n = 32$ we have

$$\frac{u_2}{\lambda} = 5.5417\ldots, \qquad \frac{u_1+1}{\lambda} = 6.0725\ldots, \qquad b = 3, \qquad a = 4,$$

and so $N^* = N = 2^4 3^3$. This was the case illustrated in our figure. Occasionally, for a given n there will be no value of b that satisfies (11). This occurs, for example, when $n = 47$. There should then be two values of N^* for $n = 46$ (of which we take the larger for N) or for $n = 48$ (of which we take the smaller for N). What actually happens here is that $n = 48$ yields $N^* = 2^{11}$ or $2^3 3^5$, and we have no difficulty in identifying the latter as the correct 47th term, the former being the 48th term.

As a final flourish, we try to find the millionth term of the sequence. For $n = 1000000$, we find

$$\frac{u_2}{\lambda} = 1122.509751\ldots, \qquad \frac{u_1+1}{\lambda} = 1123.140119\ldots,$$

and we try in (11) all values of b such that $0 \le b \le 1122$. Two values of b are found to be satisfactory, and they yield

$$N_1^* = 2^{1161}3^{390} = 3.742090\ldots \times 10^{535}$$

and

$$N_2^* = 2^{107}3^{1055} = 3,742254\ldots \times 10^{535}.$$

One of these numbers *may* be the millionth term of the sequence, and the other either the 999999th or the 1000001st. But here n is pretty large, so all we can say for sure is that the two values of N^* are relatively close to the millionth term.

Partial solutions were also received from H.G. DWORSCHAK, Algonquin College, Ottawa; and ROBERT S. JOHNSON, Montréal, Québec.

Editor's comment.

I announced in the November issue [1978: 259] that discussion of this problem, whose turn had come, was postponed until the present January issue. The reason was that one of our readers, Michael Urciuoli, a Senior at Benjamin N. Cardozo High School, Bayside, N.Y., had selected this difficult problem as his topic for a research paper he was writing for the Westinghouse National Science Talent Search. The deadline for submitting these research papers was December 15, 1978, and Urciuoli's former mathematics teacher, Steven R. Conrad, asked me if I would mind waiting until after that date to discuss the problem in print.

A preliminary selection of the 40 best papers was made (out of approximately 1000 submitted), and the judges released the names of the 40 finalists in mid-January. Only 6 of the finalists had selected a mathematical topic. One of them was Michael Urciuoli.

I sincerely hope I shall have more good news to report when the winners of the competition are announced, sometime in March.

* * *

338. [1978: 101, 290] Late solution: ANDY LIU, University of Alberta.

344. [1978: 133] *Proposed by Viktors Linis, University of Ottawa.*

Given is a set S of n positive real numbers. With each nonempty subset P of S, we associate the number

$$\sigma(P) = \text{sum of all its elements.}$$

Show that set $\{\sigma(P) \mid P \subseteq S\}$ *can be partitioned into* n *subsets such that in each subset the ratio of the largest element to the smallest is at most* 2.

Solution by Leroy F. Meyers, The Ohio State University.

Let the n numbers in S, which are not assumed to be all distinct, be given, in nondecreasing order, by

$$0 < x_1 \le x_2 \le \ldots \le x_n,$$

and set

$$y_0 = 0, \qquad y_k = x_1 + \ldots + x_k, \quad 1 \le k \le n.$$

The set

$$X = \{\sigma(P) \mid P \subseteq S\}$$

contains $2^n - 1$ positive numbers, not necessarily all distinct. We define the class of subsets of X,

$$Y = \{A_1, A_2, \ldots, A_n\},$$

as follows: for $1 \le k \le n$,

$$\sigma(P) \in A_k \iff y_{k-1} < \sigma(P) \le y_k.$$

Each $\sigma(P) \in X$ belongs to exactly one of the A_k since $0 < \sigma(P) \le y_n$; and each A_k is nonempty since y_k is a maximal element of A_k. Thus Y is a partition of X into n subsets.

For any subset A_k in the partition, let σ_k be a minimal element of A_k. If $x_k > y_{k-1}$, then x_k is a minimal element of A_k, and so

$$2\sigma_k = 2x_k > y_{k-1} + x_k = y_k.$$

On the other hand, if $x_k \le y_{k-1}$, then $y_k - y_{k-1} = x_k \le y_{k-1} < \sigma_k$, and so

$$y_k = y_{k-1} + x_k \le 2y_{k-1} < 2\sigma_k.$$

In either case we have $y_k/\sigma_k < 2$, as desired.

Note that our conclusion is stronger than that in the proposal, for we have shown "strictly less than" rather than "at most". We have also shown, as required, that for any set S of n numbers, distinct or not, a partition of X into n subsets A_k is sufficient. If the stronger property in our conclusion is to hold for *all* sets S of n numbers, a partition of X into n subsets A_k is also necessary, for if

$$S = \{2, 2^2, \ldots, 2^n\},$$

then every element of S is an element of X and no two of them may belong to the same A_k.

Also solved by PAUL R. BEESACK, Carleton University, Ottawa; and DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio.

Editor's comment.

The proposer found this problem in the Russian journal *Kvant* (No. 7, 1977, Problem M 453). He also submitted a translation of a solution by D. Bernshtein which appeared in *Kvant* No. 5, 1978, p. 26. Bernshtein proves nothing more than is contained in the proposal: "sufficiency" and "at most".

* * *

345. [1978: 134] *Proposed by Charles W. Trigg, San Diego, California.*

It has been shown [*Pi Mu Epsilon Journal*, 5 (Spring 1971) 209] that when the nine nonzero digits are distributed in a square array so that no column, row, or unbroken diagonal has its digits in order of magnitude, the central digit must always be odd.

(a) Can such a distribution be made for every odd central digit?

(b) Do any such distributions exist in which odd and even digits alternate around the perimeter of the array?

Composite of the solutions submitted by Clayton W. Dodge, University of Maine at Orono; and the proposer.

If an array that meets the conditions exists, it is clear that its complement (formed by subtracting each element from 10) also meets the conditions.

(a) The answer is YES. With a central 9, distribute 1, 2, 3, 4 in any order at the corners (or midpoints) of the array and 5, 6, 7, 8 in any order at the midpoints (or corners). The complement then has a central 1. For example, we have

$$\begin{matrix} 1 & 6 & 3 \\ 8 & 9 & 5 \\ 2 & 7 & 4 \end{matrix} \quad \text{and its complement} \quad \begin{matrix} 9 & 4 & 7 \\ 2 & 1 & 5. \\ 8 & 3 & 6 \end{matrix}$$

Distributions with other odd central digits are

$$\begin{matrix} 5 & 3 & 9 \\ 1 & 7 & 2, \\ 8 & 4 & 6 \end{matrix} \quad \text{its complement} \quad \begin{matrix} 5 & 7 & 1 \\ 9 & 3 & 8, \\ 2 & 6 & 4 \end{matrix} \quad \text{and} \quad \begin{matrix} 7 & 3 & 8 \\ 1 & 5 & 2. \\ 6 & 4 & 9 \end{matrix}$$

(b) The answer is again YES, although here not all odd digits can be central. The three arrays below show that 5 and 1 (and hence 9 by complementation) can be the central digit in an array with alternating perimeter:

$$\begin{matrix} 3 & 6 & 5 \\ 8 & 1 & 2, \\ 7 & 4 & 9 \end{matrix} \qquad \begin{matrix} 4 & 5 & 2 \\ 7 & 1 & 9, \\ 6 & 3 & 8 \end{matrix} \qquad \begin{matrix} 2 & 1 & 6 \\ 9 & 5 & 7. \\ 8 & 3 & 4 \end{matrix}$$

Note that for central 1 (and 9) we can have either the odd digits or the even digits in the corners. But with a central 5 the even digits only can be in the corners, for otherwise 3 would be in the corner opposite to 1, and hence 9 would be in a corner adjacent to 1, which is impossible for the row or column 1 x 9 would be in order of magnitude.

For a central 3, we must have 1 and 2 in diametrically opposite positions. But if odd and even digits alternate around the perimeter, then diametrically opposite digits must have the same parity. So an alternating perimeter is not possible with a central 3 (or with a central 7, by complementation).

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; ROBERT S. JOHNSON, Montréal, Québec; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; BOB PRIELIPP, The University of Wisconsin-Oshkosh; and KENNETH M. WILKE, Topeka, Kansas.

* * *

346. [1978: 134] *Proposed by Leroy F. Meyers, The Ohio State University.*

It has been conjectured by Erdős that every rational number of the form $4/n$, where n is an integer greater than 1, can be expressed as the sum of three or fewer unit fractions (reciprocals of positive integers, also called Egyptian fractions), not necessarily distinct. As a partial verification of the conjecture, show that at least 23/24 of such numbers have the required expansions.

I. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

First a few words about the conjecture itself. It is known (see Stewart [8, p. 199, Theorem 1]) that for $n > 4$ every fraction $4/n$ can be represented as the sum of at most four distinct unit fractions. For $n = 1$, it is clear that more than four distinct unit fractions are required. For n = 2, 3, 4, we have

$$\begin{aligned} \frac{4}{n} &= \frac{4}{2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}, \\ \frac{4}{n} &= \frac{4}{3} = \frac{1}{1} + \frac{1}{3} = \frac{1}{1} + \frac{1}{4} + \frac{1}{12}, \qquad (1) \\ \frac{4}{n} &= \frac{4}{4} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \end{aligned}$$

Thus the stated property actually holds for all $n \geq 2$. Observe that for $n = 3$ and 4 three distinct unit fractions suffice. According to [8, p. 203], what Erdős conjectured was that *for $n \geq 3$ every fraction $4/n$ can be represented as the sum of exactly three distinct unit fractions.* Note that if a representation with fewer than three distinct unit fractions exists, it can be increased to three by using the identity

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)},$$

since the unit fractions on the right are distinct when $x > 1$. This, in fact, is what we did in (1). So it is a matter of indifference whether the conjecture reads "exactly three" or "at most three". We shall assume that it reads "at most three distinct unit fractions".

We will prove that the conjecture holds for more than 139/140 of the integers $n \geq 3$. This result is not new. It appears as an exercise in [8, p. 207, Exercise

28.14]. A proof of the same result can be found in Mordell [6]. But Mordell's is more in the nature of an existence proof, with few actual expansions given.

If n is even, say $n = 2m$ where $m > 1$, the proof is easy, for then we have

$$\frac{4}{n} = \frac{2}{m} = \frac{1}{m} + \frac{1}{m} = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m(m+1)},$$

and we can restrict our further attention to odd $n > 3$.

Let $n = 4q + r$, where $q \geq 1$ and $r = 1$ or 3. The case $r = 3$ is quickly disposed of, for we have

$$\frac{4}{n} = \frac{4}{4q+3} = \frac{1}{q+1} + \frac{1}{(q+1)(4q+3)}.$$

The case $r = 1$ is more troublesome. Here we have at first

$$\frac{4}{n} = \frac{4}{4q+1} = \frac{1}{q+1} + \frac{3}{(q+1)(4q+1)} \tag{2}$$

$$= \frac{1}{q+1} + \frac{2}{(q+1)(4q+1)} + \frac{1}{(q+1)(4q+1)} \tag{3}$$

$$= \frac{1}{q+2} + \frac{6}{(q+2)(4q+1)} + \frac{1}{(q+2)(4q+1)}. \tag{4}$$

If $q \equiv 1$, 3, or $5 \pmod 6$, then q is odd, say $q = 2h+1$ where $h \geq 0$, and we have from (3)

$$\frac{4}{n} = \frac{4}{8h+5} = \frac{1}{2(h+1)} + \frac{1}{(h+1)(8h+5)} + \frac{1}{2(h+1)(8h+5)}.$$

If $q \equiv 2 \pmod 6$, then $q = 6i + 2$ where $i \geq 0$, and we have from (2)

$$\frac{4}{n} = \frac{4}{24i+9} = \frac{1}{3(2i+1)} + \frac{1}{(2i+1)(24i+9)}.$$

If $q \equiv 4 \pmod 6$, then $q = 6j + 4$ where $j \geq 0$, and we have from (4)

$$\frac{4}{n} = \frac{4}{24j+17} = \frac{1}{6(j+1)} + \frac{1}{(j+1)(24j+17)} + \frac{1}{6(j+1)(24j+17)}.$$

So far we have shown that the conjecture is true for all $n \geq 3$ except possibly when $n = 4q+1$ and $q \equiv 0 \pmod 6$, that is, when $n = 24k+1$ for some positive integer k. Thus the conjecture is true for at least 23/24 of the integers $n \geq 3$, as required by the proposal.

To reach our goal of 139/140, we will consider the numbers $n = 24k+1$ according to the residue classes of k modulo 5 and then modulo 7. A representation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{5}$$

with $1 \leq x < y < z$ requires

$$\frac{1}{x} < \frac{4}{n} < \frac{3}{x} \qquad \text{or} \qquad \frac{n}{4} < x < \frac{3n}{4},$$

so for a given n the number of possible values of x is finite. Having selected one such possible x, we try to find distinct positive integers y and z such that

$$\frac{4x-n}{nx} = \frac{4}{n} - \frac{1}{x} = \frac{1}{y} + \frac{1}{z}.$$

If we can find distinct, positive, relatively prime integers P and Q, with $P > Q$, such that $P|nx$, $Q|nx$, and $P + Q = w(4x - n)$, then we can set $y = wnx/P$, $z = wnx/Q$, and the desired representation is then given by (5).

For $n = 24k + 1$ with $k = 5u + s$, $u \geq 0$, a satisfactory representation is found for $s = 1, 3, 4$, and the results are:

$$\frac{4}{n} = \frac{4}{120u+25} = \frac{1}{30u+7} + \frac{1}{2(24u+5)(30u+7)} + \frac{1}{10(24u+5)(30u+7)},$$

$$\frac{4}{n} = \frac{4}{120u+73} = \frac{1}{10(3u+2)} + \frac{1}{2(3u+2)(120u+73)} + \frac{1}{5(3u+2)(120u+73)},$$

$$\frac{4}{n} = \frac{4}{120u+97} = \frac{1}{5(6u+5)} + \frac{1}{2(6u+5)(120u+97)} + \frac{1}{10(6u+5)(120u+97)}.$$

For $n = 24k + 1$ with $k = 7v + t$, $v \geq 0$, we are successful with $t = 2, 3, 4, 6$, getting the results:

$$\frac{4}{n} = \frac{4}{168v+49} = \frac{1}{14(3v+1)} + \frac{1}{14(3v+1)(24v+7)},$$

$$\frac{4}{n} = \frac{4}{168v+73} = \frac{1}{2(21v+10)} + \frac{1}{3(2v+1)(168v+73)} + \frac{1}{6(2v+1)(21v+10)(168v+73)},$$

$$\frac{4}{n} = \frac{4}{168v+97} = \frac{1}{2(21v+13)} + \frac{1}{2(3v+2)(168v+97)} + \frac{1}{2(3v+2)(21v+13)(168v+97)},$$

$$\frac{4}{n} = \frac{4}{168v+145} = \frac{1}{2(21v+19)} + \frac{1}{6(8v+7)(21v+19)} + \frac{1}{3(8v+7)(21v+19)(168v+145)}.$$

By the Chinese Remainder Theorem, the unsuccessful cases,

$$k \equiv 0 \text{ or } 2 \pmod 5 \qquad \text{and} \qquad k \equiv 0, 1, \text{ or } 5 \pmod 7,$$

are those for which

$$k \equiv 0, 5, 7, 12, 15, \text{ or } 22 \pmod{35},$$

that is,

$$n = 24k + 1 \equiv 1, 121, 169, 289, 361, 529 \pmod{840}, \tag{6}$$

and so the conjecture holds for at least 139/140 of the integers $n \geq 3$.

We can go a quarter-inch further, to justify our claim of "more than 139/140". The conjecture need be verified only when n is a prime, since

$$\frac{4}{c} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \implies \frac{4}{cd} = \frac{1}{xd} + \frac{1}{yd} + \frac{1}{zd}.$$

Since the conjecture is known to hold for n = 11, 13, 17, 19, 23, it must hold for their squares 121, 169, 289, 361, 529. The first few primes in the set (6) are 1009, 1129, 1201, 1801, 2521, 2689, 3049. Since

$$\frac{4}{1009} = \frac{1}{260} + \frac{1}{10090} + \frac{1}{52468}, \qquad \frac{4}{1129} = \frac{1}{290} + \frac{1}{11290} + \frac{1}{163705},$$

$$\frac{4}{1201} = \frac{1}{306} + \frac{1}{21618} + \frac{1}{61251}, \qquad \frac{4}{1801} = \frac{1}{456} + \frac{1}{43224} + \frac{1}{205314},$$

$$\frac{4}{2521} = \frac{1}{638} + \frac{1}{55462} + \frac{1}{804199}, \qquad \frac{4}{2689} = \frac{1}{676} + \frac{1}{139828} + \frac{1}{908882},$$

the conjecture certainly holds, and we know how to find an expansion, for all integers n such that $3 \le n \le 3048$. More expansions for individual primes could easily be found, but there hardly seems to be much point to it since Yamamoto [9] verified by computer that the conjecture holds for all n such that $3 \le n \le 10^7$.

II. *Comment by the proposer.*

The problem has an extensive literature. I have not seen Erdős's paper [5], which seems to be the source of the problem. Bernstein [1] mentions that he first heard of the problem in a lecture given by Erdős which mentioned the relation of this problem to another unsolved problem. There are extensive bibliographies in the papers by Bleicher [2], Rav [7], and, for related problems, in Campbell [3,4]. Yamamoto's verification [9] was done on an OKITAC 5090H computer at Kyushu University. Rav's paper [7] concludes with the following

THEOREM. The equation

$$\frac{m}{n} = \frac{1}{x_1} + \ldots + \frac{1}{x_k},$$

with given relatively prime positive integers m and n, is solvable (for fixed k) for $x_1, \ldots, x_k$ if and only if there exist positive integers M and N and positive relatively prime integers $N_1, \ldots, N_k$, each dividing N, such that

$$\frac{m}{n} = \frac{M}{N} \qquad \text{and} \qquad M \mid N_1 + \ldots + N_k.$$

Also solved by P.R. BEESACK, Carleton University, Ottawa; STEVE CURRAN for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of

Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; BOB PRIELIPP, The University of Wisconsin-Oshkosh; BASIL C. RENNIE, James Cook University of North Queensland, Australia; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

In solution I, the ratio 139/140 ($\approx$ 99.29%) was attained by considering, for $n = 24k+1$, residue classes of k modulo 5 and 7. There is, of course, no reason to stop there: one can consider residue classes of k modulo 11, 13... . Maskell, in fact, did just that. But he was only able to verify that the conjecture holds for two residue classes of k modulo 11 and for three residue classes modulo 13, thus increasing the ratio of validity to about 99.55%. It seems unlikely that the conjecture will ever be completely laid to rest in this piecemeal fashion.

Now that some readers have sharpened their teeth on this conjecture of Erdős, they might like to bite into an even more startling conjecture due to Sierpiński. It is known (see [8, p. 199, Theorem 1]) that for $n > 5$ every fraction $5/n$ can be represented as the sum of at most five distinct unit fractions. Sierpiński conjectured that for $n \geq 3$ every such fraction can be represented as the sum of at most *three* distinct unit fractions. At present it is known that this conjecture holds for 278459/278460 $\approx$ 99.99964% of integers $n \geq 3$ and that it is correct for all n in the range $3 \leq n \leq 1057438801$. For details see [8, pp. 203-206].

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