Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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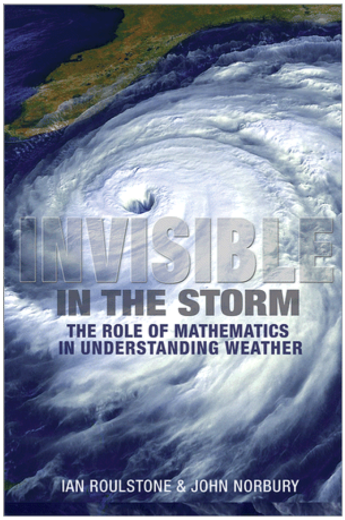
Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Invisible in the Storm



We think nothing of checking the weather report each day. For some it can be vitally important. But how did it all happen? Our ancestors relied on folklore: ‘Red sky at night, shepherd’s delight, red sky in the morning, shepherd’s warning’. Now it is isobars, high and low pressure, and the jet stream. *Invisible in the Storm* charts the role of mathematics in understanding the weather.

At the end of the nineteenth century, the Norwegian Vilhelm Bjerknes saw how to use measurements of air pressure and density to predict average winds, and proposed his fundamental circulation theorem. He went on to found the Bergen School. The problem was a practical one. Such was the complexity of the calculations needed when data was received from monitoring stations that, by the time the weather report was issued it was already out of date. It would be 50 years before computers came on the scene and made it possible to do the calculations in a realistic time. Happily, Bjerknes lived just long enough to see the dawn of the computer age.

It was the great French mathematician Henri Poincaré who pointed out at the time the fundamental dilemma facing the weather forecaster, the problem of unpredictability or chaos. To quote:

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that universe at each succeeding moment. But … it may happen that small differences in the initial conditions produce great ones in the final phenomena.

This possibility of chaos was propounded by Edward Lorenz in 1972 in his ‘butterfly effect’. He is reported to have said that the flapping of a butterfly’s wings in Brazil might cause a hurricane over Texas. So the forecaster needs to adjust the initial conditions slightly and see where they lead. If they lead to broadly similar situations, such as the path of a hurricane, the forecaster can be reasonably confident that the forecast is reliable. But if the outcomes are very different, then the forecast may be totally unreliable.

So the laws which Bjerknes began to apply to the weather need to be augmented by those of probability to take into account unpredictability, and we get weather reports that there is a 10% chance of rain. In addition, new geometries such as symplectic geometry, in which areas but not shapes remain invariant, come into play. So there is a ready place for the mathematician in the world of meteorology.

The authors of this well-written and well-illustrated book, in which technicalities are confined to boxes to avoid intimidating the general reader, well summarize their account as follows:

> To figure out where and when the weather will change its mood, we need mathematics: we need to appeal to what lies invisible amid the beauty, the power and the enigma of the weather. The laws of physics must be encoded in a computer model. The latest observations and measurements need to be assimilated in just the right way, and the equations of motion, heat, and moisture need to be integrated so that the overarching conservation principles are respected. When this is all done, we would like to estimate how much faith can be placed on the forecast. Even further, we would like to evaluate Earth's future climate .... Our message is that the smart use of mathematics gets the best out of these programs and helps us to see the ever-changing weather perhaps emerging from the mists of chaos.

In 1924, fishermen successfully lobbied to keep open Bjerknes' Bergen Institute, which faced closure (government cuts!). Weather reporting was the best thing the state had done for them, they said. Of all the scientific recognition he had received, this gladdened Bjerknes the most.

Reference

1 I. Roulstone and J. Norbury, *Invisible in the Storm: The Role of Mathematics in Understanding Weather* (Princeton University Press, 2013).

Correction

In Volume 47, Number 2, p. 63, in the article 'A brief study of abundant numbers not divisible by any of the first n primes' by Jay L. Schiffman, a section was repeated in error. We apologize for this error. The corrected version reads as follows.

> A second easily proven result shows that any positive integer that is the product of two distinct primes p and q is deficient save for $p = 2$ and $q = 3$ which yields the initial perfect number 6. For primes p, q with $p < q$, we have $2pq - \sigma(pq) = pq - 1 - p - q = (p-1)(q-1) - 2 > 0$, except for $p = 2$, $q = 3$, when it is 0.

A note from the Editor

Submissions to *Mathematical Spectrum* are currently running below the level needed, so we are requesting articles and other contributions suitable for our intended student readership so that we can maintain the present publication of three issues per year.

Hikorski Triples

JONNY GRIFFITHS

A new triple of natural numbers is introduced, (a, b, $(ab + 1)/(a + b)$). These *Hikorski triples* arise from a simple yet evocative mathematical situation, and there is a link with the world of *periodic recurrence relations* (or *Lyness cycles*). When the terms of such cycles are multiplied or added, they can give rise to *elliptic curves*.

Introduction

Mathematics contains many positive integer triples. The most famous are the *Pythagorean triples* (where (a, b, c) satisfies $a^2 + b^2 = c^2$, e.g. $(3, 4, 5)$), but there are *amicable triples* (where (a, b, c) satisfies $\sigma(a) = \sigma(b) = \sigma(c) = a + b + c$, where $\sigma(x)$ is the sum of the positive divisors of x including x (an example is $(1\,980, 2\,016, 2\,556)$), *Markov number triples* (where (a, b, c) satisfies $a^2 + b^2 + c^2 = 3abc$, e.g. (5, 13, 194)), triples that arise from the Euler brick problem ((a, b, c) where $a^2 + b^2$, $b^2 + c^2$, and $c^2 + a^2$ are all squares, e.g. (44, 117, 240)), and so on. These examples of triples each arise from a simple mathematical situation; indeed, you might argue that the simpler the motivating situation, the more profound the significance of the triple.

Hikorski triples

We define here a new triple of positive integers, called the *Hikorski triple* (HT). The name is explained in reference 1 (p. 200). A triple of natural numbers (a, b, c), with $a \geq b \geq c$, is an HT if and only if $c = (ab + 1)/(a + b)$. HTs arise naturally from the following task (see *Mathematical Spectrum*, Volume 35, Number 3, p. 65). This simple situation arose out of writing a GCSE worksheet on equations for my students; ten years later, it provided all the raw material for my MSc by Research.

> Put the numbers in the bag into the circles in any way you choose (no repeats!) – see figure 1. Solve the resulting equation. What solutions are possible as you choose different orders? What happens if you vary the numbers in the bag, keeping them as distinct integers?

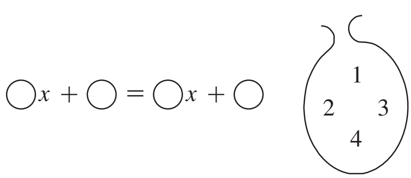


Figure 1 Numbers in a bag.

The reader is invited to:

1. show that if S is the set of solutions for a given quartet in the bag, the maximum size for S is 12,

2. show that if x is in S, then $-x$, $1/x$, and $-1/x$ are also in S,

3. show that S can contain at most three positive integers, and find an example of this.

Let us now show that if S does contain three positive integers, they form an HT. The solution to $ax + b = cx + d$ is $x = (d - b)/(a - c)$. Thus, if we replace a, b, c, d in the bag with $a + k$, $b + k$, $c + k$, $d + k$, we will get the same solutions. Similarly, replacing a, b, c, d with ka, kb, kc, kd ($k \neq 0$) gives the same solutions. Thus, without loss of generality, we can replace our numbers in the bag with $0 < 1 < i < j$, where i and j are rational, without changing the solutions. Here, suppose that $j - i < 1$. Then 0, -1, $-i$, $-j$ give the same solutions, as do j, $j-1$, $j-i$, 0 (adding j), as do $j/(j-i)$, $(j-1)/(j-i)$, 1, 0, (dividing by $j - i$). Now note that

$$\frac{j}{j-i} - \frac{j-1}{j-i} = \frac{1}{j-i} > 1.$$

So we can also insist without loss of generality that the numbers in our bag are $0 < 1 < i < j$, where $j - i > 1$. So the 12 solutions possible with this standardised bag are

$$\frac{j}{i-1}, \frac{-j}{i-1}, \frac{i-1}{j}, \frac{1-i}{j}, \frac{j-i}{1}, \frac{i-j}{1}, \frac{1}{j-i}, \frac{1}{i-j}, \frac{j-1}{i}, \frac{1-j}{i}, \frac{i}{j-1}, \frac{i}{1-j}.$$

Of these, only three can be positive integers, $p = j/(i-1)$, $q = j - i$, and $r = (j-1)/i$. Eliminating i and j gives us

$$r = \frac{pq+1}{p+q}.$$

Notice that $(n, 1, 1)$ is always an HT for n a positive integer, as given by the bag $\{2n, n+1, n-1, 0\}$, for example, and is called a *trivial HT*. So, in general, there are 12 solutions to our numbers-in-the-bag equation, which fall into three quartets:

$$\left\{p, -p, \frac{1}{p}, -\frac{1}{p}\right\}, \left\{q, -q, \frac{1}{q}, -\frac{1}{q}\right\}, \left\{\frac{pq+1}{p+q}, -\frac{pq+1}{p+q}, \frac{p+q}{pq+1}, -\frac{p+q}{pq+1}\right\}.$$

What resonances do the four functions in the third quartet have for us? We might think of the functions tanh and coth:

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{\tanh(x)\tanh(y) + 1}, \qquad \coth(x+y) = \frac{\coth(x)\coth(y) + 1}{\coth(x) + \coth(y)}.$$

Relativity comes to mind. If nothing can travel faster than the speed of light, how do we add speeds? For example, if someone is moving at three-quarters the speed of light down a train that is itself moving at three-quarters the speed of light, what is the person's resultant speed? If we say that the speed of light is 1, we can add parallel speeds relativistically as follows: define '$\odot$' as

$$a \odot b = \frac{a+b}{1+ab};$$

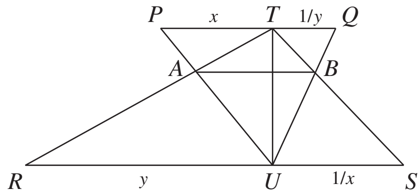


Figure 2 Constructing $(xy+1)/(x+y)$.

thus, $\frac{3}{4} \odot \frac{3}{4} = \frac{24}{25}$. The reader is invited to check that the set of numbers in the interval $(-1, 1)$ in $\mathbb{R}$ together with the binary operation $\odot$ form a group. Further exercises for the reader are as follows.

1. Show that if in figure 2 the lines PQ and RS are horizontal and TU is vertical, then the line AB is horizontal and of length $(xy+1)/(x+y)$.

2. Explore a link with the world of recurrence relations. If we write the recurrence relation $u_{n+1} = f(u_{n-1}, u_n)$ with $u_0 = x$, $u_1 = y$ as $x, y, f(x, y), \ldots$, then show that, if $x + y$ is nonzero and $x \neq \pm 1$, $y \neq \pm 1$, the recurrence $x, y, -(xy+1)/(x+y), \ldots$ is periodic, with period 3.

How many HTs are there? They are relatively numerous; discounting the trivial HTs, there are 2252 HTs with all terms less than 1000 (the same count for Pythagorean triples, including both primitive and nonprimitive triples, yields 1034). A question arises; given n, in how many HTs does n appear? Counting up, we find that the number 11 features in 16 distinct HTs, while the number 12 appears in only four. Let $h(n) =$ 'number of HTs in which n is an element', and note that

$$(n, 1, 1), \qquad (n^2 - n - 1, n, n - 1), \qquad (n^2 + n - 1, n + 1, n), \qquad (2n + 1, 2n - 1, n)$$

are always distinct HTs for $n > 2$. So we have $h(1) = \infty$, $h(2) = 2$ (2 only occurs in $(2, 1, 1)$ and $(5, 3, 2)$), and $h(n) \geq 4$ for $n \geq 3$. Now, $11^2 - 1 = 120$, which has 16 positive divisors, while $12^2 - 1 = 143 = 11 \times 13$, which has only four positive divisors. It is easy to conjecture that $h(n) = d(n^2 - 1)$, where $d(n) =$ 'the number of positive divisors of n'.

Theorem 1 *The number of HTs in which n appears is $h(n) = d(n^2 - 1)$ for $n > 1$.*

Proof The theorem is clearly true for $n = 2$. Now we have that $d(k)$ is odd if and only if k is a square, and $n^2 - 1$ cannot be a square if $n > 1$, so $d(n^2 - 1) = 2j$ for some j. So there exist j pairs of integers (u, v) with $u > v$, so that $uv = n^2 - 1$, where $j \geq 2$ since 1, $n - 1$, $n + 1$, and $n^2 - 1$ all divide $n^2 - 1$ for $n > 2$. How many HTs contain n? Firstly, it is easy to show n cannot appear twice in an HT unless $n = 1$. Now consider the case where n is the smallest member of the HT $(n + u, n + v, n)$, where $u > v > 0$. We have

$$\frac{(n+u)(n+v)+1}{2n+u+v} = n \qquad \Longleftrightarrow \qquad uv = n^2 - 1.$$

There are exactly j pairs for (u, v) that satisfy this equation, so there are j HTs with n as the smallest member. Now consider the case where n is not the smallest member of the HT, so we may write the HT as $(u-n, n, n-v)$, where $n-v$ is the smallest element (we cannot be sure how $u-n$ and n are ordered), with $1 \leq v < n$, $u > n$. From the HT definition, we have

$$\frac{(u-n)n+1}{u} = n-v \qquad \Longleftrightarrow \qquad uv = n^2-1.$$

We know that there are precisely j values for v between (and including) 1 and $n-1$ that divide n^2-1, and each gives a value for u between (and including) $n+1$ and n^2-1 that divides n^2-1, and so $u-n$ is between (and including) 1 and n^2-n-1. We need to check that $u-n \geq n-v$ in every case, which is true if and only if $u+v \geq 2n$. But $u+v \geq 2\sqrt{uv}$ by the AM–GM inequality, and so $u+v \geq 2\sqrt{n^2-1}$. Now,

$$2n > 2\sqrt{n^2-1} > 2n-1 \qquad \Longleftrightarrow \qquad 4n^2 > 4n^2-4 > 4n^2-4n+1,$$

which is true, and so since $u+v$ is an integer, $u+v \geq 2n$ for $n > 1$. Thus, we always have exactly $2j$ HTs for a given $n > 1$, and so $h(n) = d(n^2-1)$.

It is worth noting that if $n-1$ and $n+1$ are both prime (that is, they form a prime pair), then n^2-1 has four positive divisors, 1, $n-1$, $n+1$, and $(n-1)(n+1)$, so that $h(n^2-1) = 4$. Are there infinitely many n so that $h(n) = 4$?

The uniqueness conjecture for HTs

If we add the elements of an HT, the sum may well be shared with other HTs. Indeed, we can prove that the number of HTs whose three elements add to k is unbounded as $k \to \infty$. Multiplying the elements of an HT, however, proves to be more intriguing. Of course, the trivial HTs $(n, 1, 1)$ can multiply to any whole number. What if we exclude these? A computer program enables a reasonably swift check on the first 250 000 nontrivial HTs, and multiplying their elements gives distinct products. The first few HTs with their products are given in table 1. And so we have the following uniqueness conjecture that remains open as I type, and which is offered as a challenge to the reader.

Table 1 Early HT products.

a	b	c	abc
5	3	2	30
7	5	3	105
11	4	3	132
9	7	4	252
19	5	4	380
11	9	5	495
13	8	5	520
17	7	5	595
13	11	6	858
29	6	5	870
15	13	7	1 365

Uniqueness conjecture *If* (a, b, c) *and* (p, q, r) *are nontrivial HTs with* $abc = pqr$, *then* $(a, b, c) = (p, q, r)$.

Here are two HTs, (a, b, c) and (p, q, r), where abc is close to pqr (a counterexample to the idea that such products are necessarily far apart): $(1957, 1955, 978)$ is an HT, and $1957 \times 1955 \times 978 = 3\,741\,764\,430$, while $(7897, 719, 659)$ is also an HT, and $7897 \times 719 \times 659 = 3\,741\,764\,437$.

The uniqueness conjecture initially looks straightforward to prove, but if it holds, the consensus of experienced observers is that a proof may well be difficult. There is coincidentally also a uniqueness conjecture for Markov triples (see reference 1, p. 175), which also remains unresolved, despite much attention.

The $-(xy + 1)/(x + y)$ cycle

Given our interest in HTs, the period-3 recurrence

$$x, y, -\frac{xy+1}{x+y}, x, y \dots$$

deserves special consideration. (A recurrence that is periodic in general for any starting terms is known as a *Lyness cycle*. R. C. Lyness was a mathematician and school-teacher who first discussed such recurrences (see reference 1, p. 22).) Multiplying the three terms and equating this to $-k$ gives the curve

$$xy\left(-\frac{xy+1}{x+y}\right) = -k \quad \text{or} \quad x^2y^2 + xy - kx - ky = 0. \tag{1}$$

This is an example of an *elliptic curve*, which is essentially a curve of the form $y^2 = P(x)$, where $P(x)$ is a polynomial of degree 3 that does not factorise, and which has no singularities, that is, crossing points or cusps. Some quartic curves reduce to elliptic ones. Using the transformation

$$x = \frac{X}{4k}, \qquad y = \frac{2kY - 2kX + 8k^3}{X^2}, \qquad X = 4kx, \qquad Y = 8kx^2y + 4kx - 4k^2,$$

our curve becomes

$$Y^2 = X^3 + X^2 - 8k^2X + 16k^4.$$

This is now in *Weierstrass form*, a standard representation for elliptic curves.

Integer point implications

Elliptic curves are vital in many areas of mathematics. When Hardy and Wright's masterpiece *An Introduction to the Theory of Numbers* reached its sixth edition (see reference 2) just a single chapter was deemed to be an essential addition to the original, one on elliptic curves. One of the many remarkable properties of these curves is that the points they contain form a group under a certain simple addition law. They also form the basis for much of our most effective cryptography. Elliptic curve theorems include Siegel's, proved in 1929, which tells us that there are only finitely many integer points on any elliptic curve. What, then, will be the integer points on (1) if k is $30 = 5 \times 3 \times 2$? (The triple $(5, 3, 2)$ is an HT.) First note that, given the periodic nature of our recurrence, the expression $xy(-(xy+1)/(x+y))$ is invariant under the transformation

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -(xy+1)/(x+y) \end{pmatrix}.$$

Since $(5, 3, 2)$ is an HT, integer points will therefore include $(5, 3)$, $(3, -2)$, and $(-2, 5)$. Their reflections in $y = x$, $(3, 5)$, $(-2, 3)$, and $(5, -2)$ can also be added. We also have $(30, 1)$, $(1, -1)$, $(-1, 30)$, $(30, -1)$, $(-1, 1)$, and $(1, 30)$ from the trivial HT, a total of 12 integer points. But if the UC is true, this will be all, and we will therefore have a family of elliptic curves (as we vary k to give HT products) that each have 12 integer points (that are easily found), and only 12. The unlikelihood of this situation casts doubt on the plausibility of the uniqueness conjecture, but if the uniqueness conjecture turns out to be true, then the implications are exciting.

References

1 J. Griffiths, Lyness cycles, elliptic curves, and Hikorski triples (MSc Thesis, University of East Anglia, 2012, available at http://www.s253053503.websitehome.co.uk/jg-msc-uea/index.html).

2 G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford University Press, 2008), 6th edn.

Jonny Griffiths *teaches at Paston College in Norfolk, where he has been for the last twenty-three years. He has also taught at St Dominic's Sixth Form College in Harrow-on-the-Hill. He has studied maths at Cambridge University, with the Open University, and at the University of East Anglia. Possible claims to fame include being an ex-member of Harvey and the Wallbangers, a popular band in the 1980s, and playing the character Stringfellow on the children's television programme Playdays.*

Mathematical Spectrum **Awards for Volume 46**

Prizes have been awarded to the following student readers for contributions in Volume 46:

Sudharaka Palamakumbura

for the article 'The Mathematics of Origami';

Anthony Baker and Christopher Brown

for the article (written with their teacher, Martin Griffiths) 'A Surprising Fact About Some *n*-Volumes?'.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Lottery Perception

JORGE ALMEIDA

Data from two complementary sources can inform our choices when betting on lottery numbers. Firstly, records of past winning combinations that signal whether or not lottery results are obtained through random processes. Secondly, the combinations that people bet on. It may be the case that there is, in general, no reason to question the fairness of lottery draws. In contrast, people seem to express preferences for certain classes of combinations. Therefore, as argued here, only data of the second kind would be relevant for choosing lottery numbers or, perhaps more interestingly, for understanding people's behaviour.

'Perception of randomness' is a subject of research in Experimental Psychology (see, for example, reference 1). One of the beliefs shared by many of those working on this theme is that, when assessing outcomes of repeated tosses of a fair coin, people tend to underestimate probabilities of runs of consecutive heads (H) or tails (T) and, conversely, overestimate probabilities of alternations (e.g. HTTHTHHHTHT has one run of three, one run of two, six runs of one, and seven alternations). For example, given 21 tosses of a fair coin, the most frequent class of outcomes will be that with 10 alternations. However, people tend to judge as more likely the class of sequences with 12 to 14 alternations. Of course, work in this field has to bring together two disciplines as it requires confronting people's beliefs (Psychology) and calculating probabilities of sequences or classes of sequences (Mathematics). One such calculation was presented by McPherson and Hodson (see reference 2), who addressed the problem of the occurrence of runs of consecutive numbers in lotteries in which people bet on six out of 49 numbers.

McPherson and Hodson (reference 2) estimated the probabilities of drawing combinations of all unattached numbers or combinations consisting of possible types of runs of the six numbers and then compared their results to data on past winning combinations. No significant deviation from expectation, on the assumption of randomness, was found, attesting to the fairness of the lottery draw. For example, out of 1028 lottery draws, sequences with a run of three, a run of two, and a single have occurred five times, very close to the predicted value of 5.84, and sequences with 'all unattached' numbers (six singles) which are expected to occur 519 times have occurred 555 times. Of course, given that the probability of occurrence of a run of six is only $\frac{44}{13983816}$ and that the number of draws in the dataset (1 028) is very small compared to the number of possible lottery results (13 983 816), we expect runs of six never to have occurred and, indeed, so it is. (Note that the probability that a run of six does not occur in 1 028 draws is given by $(1 - \frac{44}{13983816})^{1028} = 0.996\,77$. In other words, in out of one thousand sets each of 1028 draws, we expect runs of six to occur in only under four of the sets.)

A parallel can be readily established between alternations in repeated coin tosses and alternations in the lottery by considering that, in the string of 49 numbers, the six chosen numbers in a combination are Hs and the remaining numbers are Ts (or ones and zeros). For example, a combination such as {2, 7, 11, 12, 13, 18} can be expressed as

{THTTTTHTTTHHHTTTTHT, followed by all Ts up to position 49}.

Given that an alternation is a transition from H to T or T to H, this combination would have eight alternations. Using the equation of McPherson and Hodson for relating numbers of combinations to run structure it can be determined that the average number of alternations in the $\frac{6}{49}$ lottery is nearly 10.53 (for a range of 1 to 12 alternations). Combinations with a run of six have only one alternation if at the ends of the 1–49 range or two alternations otherwise, that is, far below the 10.53 average. Therefore, according to Psychologists' views, most people would be inclined to believe that it is obviously a bad idea to bet on runs of six. That this belief is illusory, if founded on probabilities of occurrence of lottery combinations, was acknowledged by the French Mathematician Pierre Simon de Laplace (1749–1827), a founder of Probability Theory. Quoting from Hawking's commentary on Laplace's work in the compilation *God Created the Integers* (reference 3):

> '…when playing a lottery in which six numbers get picked out of fifty…many people would avoid playing the set {01, 02, 03, 04, 05, 06} supposing that that set of six numbers shows much more regularity than the recent winning set of numbers {06, 13, 15, 15, 32, 36} [one of the 15s is certainly a lapse, my comment]. But, Laplace would argue, a thorough analysis of the lottery process would reveal the independence of each number's selection. Thus, the set {01, 02, 03, 04, 05, 06} shows neither more nor less regularity than the set {06, 13, 15, 15, 32, 36}….'

To account for the illusion brought to light by Laplace it is perhaps useful to view people's perception of probability as a two-step process. First, people inevitably partition the set of all possible outcomes according to given equivalence relations such as alternations/run-lengths. Second, following the partition, the illusion arises through some mechanism leading to confounding probabilities of subsets resulting from the partition, and probabilities of particular elements of subsets. For example, McPherson and Hodson (reference 2) calculated that the probability of the class of runs of six is only $\frac{44}{13983816}$, whereas the probability of the class 'all unattached' is $\frac{7059052}{13983816}$, that is, over 50%. Thus, one would rather bet on 'all unattached' than on a run of six. But this is an illusion since if a lottery draw yields an 'all unattached', highly probable, result, the probability that a particular 'all unattached' ticket is the winner is only $\frac{1}{7059052}$, whereas if a draw gives an improbable run of six, the probability that a particular ticket with a six-run wins is a relatively high $\frac{1}{44}$. Hence, the probability of a particular element of the 'all unattached' class is

$$\frac{7059052}{13983816} \times \frac{1}{7059052} = \frac{1}{13983816},$$

which is exactly the same as the probability of a particular element of the six-run class ($\frac{44}{13983816} \times \frac{1}{44} = \frac{1}{13983816}$). Of course, all this is saying is that any two outcomes are equally likely. The somewhat convoluted way of arriving at such equivalence serves, however, the purpose of establishing a parallel between calculation of probabilities and the proposed two-step process of probability perception. Should people see in a single step that any two particular results out of the 14 million are equally likely, there would be no problem of probability perception.

It therefore seems that it is not such a bad idea to bet on a run of six. Neither, at first sight, would it be a good idea. Consider, however, the problem of betting in the hope of becoming not just a lottery winner but the sole lottery winner, that is, one who, in the very unlikely event of winning, does not have to share the prize with other contenders. Under these conditions, suppose that all but one of the players follow the advice 'have your numbers all unattached'. Would the odd player be acting smartly? Clearly, if there is only one player betting on the subset 'runs of six', this player is guaranteed to be a sole winner in the very unlikely event of winning. Laplace himself acknowledged this in following through the idea mentioned above.

Quoting again from Hawking's comment

> '…Given the fact that a lottery payout depends on the number of winners, the very fact that many people would avoid playing $\{01, 02, 03, 04, 05, 06\}$ is, in fact, a very good reason to play it. There would be less likelihood of anyone else having the same winning set of numbers….'

One problem with Laplace's advice is that it is self-defeating. A combination perceived as unpopular, i.e. a good combination, may attract a crowd of Laplace's followers, thus turning bad. For example, lottery organizers do not systematically divulge numbers on players' preferences, but rumour has it that thousands of people bet on the combination $\{1, 2, 3, 4, 5, 6\}$ (see, for example, references 4 and 5). Notwithstanding the veracity of this claim, analysis of what little information lottery organizers release (e.g. numbers of winners for jackpots and other categories) indicates that unlike lottery draws which seem to be random, people do indeed make biased choices (see reference 4). This can be explained, as in the two-step process of probability perception proposed above, if people tend to confound probabilities of subsets resulting from partitions and probabilities of particular elements of subsets (e.g. perceiving a particular element of the six-run class as less likely than a particular element of the all unattached class). In this view, betting randomly amounts to not making such confusion. One sure way of not making such confusion is in turn to partition the set of all possible combinations (say, a set of size C) into C subsets of a single element each, as recognized, for example, by Haigh (reference 4). It is hard to see how most people would make their choices according to a partition into nearly 14 million subsets, if not with widespread and systematic use of devices for generating random numbers, hence the inevitability for people to base their decisions on partitions into 'manageable' numbers of subsets.

Run length is but one of diverse criteria that people use for partitioning the set of nearly 14 million combinations in $\frac{6}{49}$ lotteries. For example, people may avoid numbers at the upper end of the range 1–49 (especially numbers greater than 31 which cannot be birthdates) or may even respond to the particular layout of the numbers in the ticket (see reference 5). We may hope that research combining Probability Theory and Experimental Psychology will reveal with increasing accuracy how people make decisions when confronted with uncertainty. But rather than expecting this to produce a guide for action at betting counters, it is perhaps more interesting to expect deeper insight into the workings of the human mind.

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The Euler Conic via Pascal

GUIDO LASTERS and DAVID SHARPE

This article shows how the Euler conic or nine-point conic of a triangle can be deduced from Pascal's theorem for conics.

Readers may be familiar with the *Euler circle* of a triangle ABC, also called its *nine-point circle*. Denote by A', B', C' the midpoints of BC, CA, and AB, respectively. Denote by H the orthocentre of the triangle and let P_1 be the intersection of AH (produced) and BC, P_2 be the intersection of BH and CA, and P_3 be the intersection of CH and AB. Thus, P_1, P_2, P_3 are the feet of the perpendiculars from the vertices to the opposite sides. Denote by A'', B'', C'' the midpoints of AH, BH, and CH, respectively. Then the nine points

$$A',\ B',\ C',\ P_1,\ P_2,\ P_3,\ A'',\ B'',\ C''$$

lie on a circle, called the *Euler circle* or the *nine-point circle* of the triangle. This is illustrated in figure 1.

Suppose that, instead of the orthocentre H, we use any point P in the plane of the triangle and construct the same nine points. Now these points all lie on a conic – see figure 2. In this article we show how this follows from the well-known Pascal's theorem for conics.

We need two facts. Firstly, given any five distinct points, there is a unique conic passing through them. Or, to put it another way, a conic is determined by any five of its points. The second fact we need is that, if six points in the plane are the vertices of a hexagon with opposite sides parallel, then these six points lie on a conic. This follows from Pascal's theorem or, more precisely, from a converse of Pascal's theorem. Consider six distinct points

$$A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3,$$

which lie on a conic. Denote the intersection of A_2B_3 and A_3B_2 by C_1, that of A_3B_1 and A_1B_3 by C_2, and that of A_1B_2 and A_2B_1 by C_3. Then C_1, C_2, C_3 are collinear – see

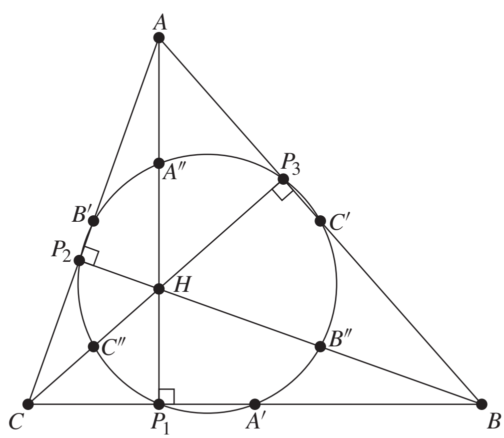

Figure 1 The nine-point circle.

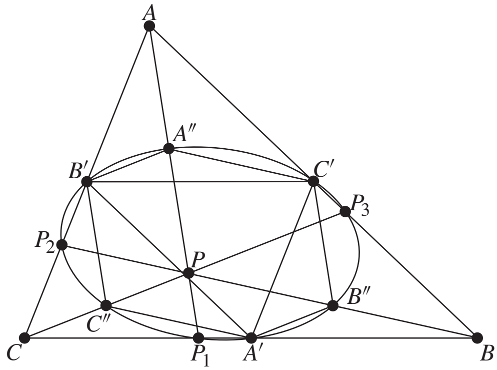

Figure 2 The nine-point conic.

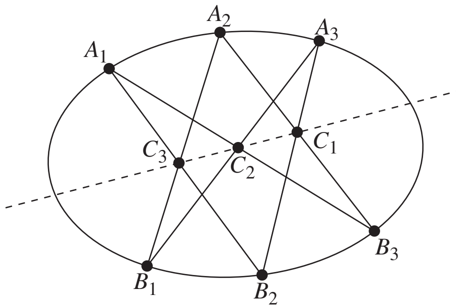

Figure 3 Pascal's theorem.

figure 3. When two straight lines are parallel, we say conventionally that they meet on the line at infinity. The converse result to Pascal's theorem that we need is that, starting with six points A_1 to B_3, construct C_1, C_2, C_3 as before. If C_1, C_2, C_3 are collinear, and this includes the line at infinity, then the six points A_1 to B_3 lie on a conic. Now, suppose that we have a hexagon $A_1B_2A_3B_1A_2B_3$ in which opposite sides are parallel. Then A_1B_2 and A_2B_1 meet on the line at infinity, as do A_2B_3 and A_3B_2, as do A_3B_1 and A_1B_3. Hence, the six vertices of the hexagon lie on a conic.

We now consider figure 2. Because A' is the midpoint of BC and B'' is the midpoint of PB, $A'B''$ and PC are parallel. Likewise, $A''B'$ and PC are also parallel. Hence, $A'B''$ and $A''B'$ are parallel. By the same token, $B'C''$ and $B''C'$ are parallel, as are $C'A''$ and $C''A'$. Hence, the six points

$$A',\ B',\ C',\ A'',\ B'',\ C''$$

lie on a conic.

Now consider the hexagon $P_1A'B''C'B'A''$. Since B' and C' are the midpoints of AC and AB, $B'C'$ is parallel to BC and so to P_1A'. Since B'' is the midpoint of PB and C' is the midpoint of AB, $B''C'$ is parallel to PA and so to P_1A''. We already know that $A'B''$ and $B'A''$ are parallel. Hence, P_1 lies on the conic determined by A', B', C', A'', and B'', on which C'' also lies. By a cyclic interchange of letters, the same argument gives that P_2 and P_3

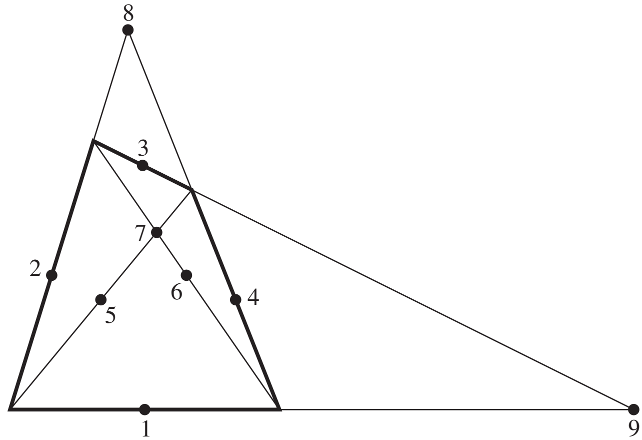

Figure 4

also lie on this conic. Thus, all nine points lie on the conic, aptly called the *nine-point conic* or *Euler conic*.

An alternative way of describing this result is that, given any quadrilateral in the plane, the midpoints of its sides, the midpoints of its diagonals, the point of intersection of its diagonals, and the points where opposite sides extended meet, nine points in all, lie on a conic (see figure 4). This follows by taking one of the vertices of the quadrilateral as the point P and the triangle formed by the other three as the triangle. Readers may like to sketch a conic passing through these nine points, evidently a hyperbola in figure 4. To obtain an ellipse, readers could start with a nonconvex quadrilateral.

Camilla Jordan has designed two interactive GeoGebra files that illustrate beautifully what is happening. One shows the Euler conic and allows users to move the point P at will. When P is inside the triangle, the Euler conic is an ellipse; when P is outside the triangle, it is a hyperbola; when P lies on a side of the triangle, the conic degenerates to a pair of parallel straight lines. The other shows a hexagon with opposite sides parallel and its circumscribed conic. These can be seen on the *Mathematical Spectrum* website, http://ms.appliedprobability.org/.

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We are sorry report the death of **Douglas Quadling**, who has been a long-time member of the advisory board of *Mathematical Spectrum* as well as a contributor. He has served the Mathematical community as teacher and educator over many years, and will be greatly missed. We extend our sympathy to his family and friends.

Looking For a Good Time to Bet

LAURENT SERLET

Suppose that the cards in a well-shuffled deck of cards are turned up one after another. At any time – but once only – you may bet that the next card to be turned up will be black. If it is correct you win 1$, if not you lose 1$. If you never bet you lose 1$. Is there a good strategy to play that game and make it profitable? Discussing this will be a good opportunity to see an example of a random walk and introduce the notion of a martingale.

1. Let's play a game!

To play this game we need an ordinary deck of cards, for instance with $N = 52$ cards. In fact, we will only look at the colours of the cards, black for half of them and red for the other half. First the cards are properly shuffled. Then the cards are turned up one after another in front of you. At any time – but once only – you may bet that the next card to be turned up will be black. If it is correct you win 1$, if not you lose 1$. If you never bet you lose 1$. Naturally, the question is:

is there a way to make money playing this game?

Of course the gain is random so 'making money' must be understood as choosing a strategy which ensures that the mean gain is strictly positive, i.e. the probability to win 1$ minus the probability to lose 1$ is strictly positive. If it is the case, the most famous theorem of probability theory, called Kolmogorov's law of large numbers, implies that if you play the game a large number of times, the average gain per play will be close to this mean, hence strictly positive.

Let us examine two naive strategies. First betting at a fixed time, for instance at the very beginning, before the first card is revealed – that will at least save time. In this case black or red have the same probability of occurring so the average gain is 0. This is also true for betting at any other fixed time, in particular on the last card. Note that in this case the colour of this last card is known for sure but it happens to be black only with a probability of $\frac{1}{2}$! However, you also have the right to bet at a *random* time depending on the colours of the cards previously revealed. Also, there are random times such that the next card revealed has a greater chance of being black than red, which implies a strictly positive (conditional) mean gain. Typically it happens if there is a deficit of black cards among the cards revealed up to that time. More precisely, suppose that, among the k first cards turned up ($1 \leq k \leq N-1$), there are b black ones and r red ones ($k = b + r$). Then, betting at this time, your mean gain will be

$$\frac{r-b}{N-k}. \tag{1}$$

Indeed, in the specified situation, $N/2 - b$ black cards and $N/2 - r$ red ones are left to be drawn, so the probability of a black card being drawn is

$$\frac{N/2-b}{N/2-b+N/2-r} = \frac{1}{2}\left(1 + \frac{r-b}{N-k}\right). \tag{2}$$

Similarly, the probability of a red one is

$$\frac{1}{2}\left(1 - \frac{r-b}{N-k}\right), \tag{3}$$

and (1) follows.

We can see from this formula that the mean gain is proportional to the deficit of black cards and inversely proportional to the number of cards still to be drawn, so we could be tempted to wait for a large deficit of black cards or a low number of remaining cards. But the longer you wait the less probability there is that this situation will occur and remember that in the case of non-betting there is a 1$ penalty.

To go further we need to formalize the problem a little more. In view of (1), let us introduce, for $0 \leq k \leq N$, the random variable S_k defined as the number of red cards minus the number of black cards, among the k first cards drawn. In probability theory, $(S_k)_{0 \leq k \leq N}$ is called an unbiased random walk on $\mathbb{Z}$ starting from 0 and conditioned on $S_N = 0$. Many explicit computations can be performed on this object by combinatorial techniques as we will see in the next section.

For the moment let us give a general formula for the mean gain generated by a strategy. Let us say that a strategy is associated with a random time τ if it consists of betting after τ cards have been revealed. Thus, τ is a random variable taking its values in $\{0, 1, \ldots, N-1\}$ and we add the possibility that τ takes the value $+\infty$ to mean that the player never bets. We restrict our analysis to the strategies that we call *admissible*, in the sense that, for every $k \in \{0, 1, \ldots, N-1\}$, the event $\tau = k$ only depends on the values $S_1, \ldots, S_k$. In other words, we assume that the player has no psychic power to anticipate the future. For instance, the strategy of waiting for a deficit of q black cards drawn ($q \geq 1$) is denoted by τ_q and given by

$$\tau_q = \min\{k \in \{1, \ldots, N-1\}, S_k = q\}$$

with the convention $\tau_q = +\infty$ if $S_k < q$ for all $k \in \{0, 1, \ldots, N\}$. In probabilistic terms, τ_q is the hitting time of q for the random walk $(S_k)_{0 \leq k \leq N}$. Here, 'hitting time' means 'first hitting time' but we could possibly consider the strategy associated to the second or third hitting time, and so on.

With the notation above, it follows from (1) that the mean gain of the admissible strategy associated with τ is

$$\mathrm{MG}(\tau) = \mathrm{E}\left[\sum_{k=0}^{N-1} \frac{S_k}{N-k} \mathbf{1}_{\{\tau=k\}}\right] - \mathrm{P}(\tau = +\infty). \tag{4}$$

In this formula, we denote by $\mathbf{1}_{\{\tau=k\}}$ the variable equal to 1 if the event $\tau = k$ is realized and 0 if not; P is our notation for probability and E is the associated expectation (mean).

The problem we want to solve is to find the admissible strategies τ which maximize $\mathrm{MG}(\tau)$ given by (4) and determine if the maximum is strictly positive. Probability theory has the ideal tool to solve the problem completely as we will explain in section 3. However, this argument could seem rather abstract so we first perform the explicit computation with a natural candidate to be a good strategy. This strategy is simply τ_1 which consists of betting as soon as red cards have been turned up more often than black ones. This will be an opportunity to see the techniques needed to do such a computation and the difficulty to conclude this way. By contrast, it will hopefully convince the reader of the power of the abstract argument developed in section 3.

2. Is the ‘natural strategy’ profitable?

We want to compute $\mathrm{MG}(\tau_1)$ using (4). We first remark that, on the event $\tau_1 = k$, we have $S_k = 1$ so (4) simplifies to

$$\mathrm{MG}(\tau_1) = \left(\sum_{k=0}^{N-1} \frac{1}{N-k}\,\mathrm{P}(\tau_1 = k)\right) - \mathrm{P}(\tau_1 = +\infty). \tag{5}$$

We are left with computing the probabilities appearing in this formula which amounts to computing the law of τ_1. To state the results we need to introduce the usual binomial coefficients,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

when n is a nonnegative integer and $k \in \{0, \ldots, n\}$; moreover, when these requirements are not satisfied, the binomial coefficient above is set equal to 0. We claim that

$$\mathrm{P}(\tau_1 = +\infty) = 1 - \binom{N}{N/2+1} \Big/ \binom{N}{N/2} \tag{6}$$

and, for every $k \in \{1, \ldots, N-1\}$,

$$\mathrm{P}(\tau_1 = k) = \frac{1}{k}\binom{k}{(k+1)/2}\binom{N-k}{(N-k-1)/2} \Big/ \binom{N}{N/2}. \tag{7}$$

The proof is given in section 5. Substituting these values in (5) and putting $N = 2n$, we get

$$\mathrm{MG}(\tau_1) = \frac{1}{2}\sum_{i=0}^{n-1} \frac{1}{(2n-2i-1)(i+1)}\binom{2i}{i}\binom{2n-2i}{n-i} \Big/ \binom{2n}{n} - \frac{1}{n+1}.$$

Remember that we want to determine if this expression is strictly positive. *A priori* it is not clear but computing the value for $n = 1, 2, 3$, we always find 0. In fact, the value is 0 for every $n \geq 1$. This will be a consequence of the arguments given later in this article, but very similar identities exist in the literature, see for instance equation (1.34) in reference 1, which was attributed to Bruckman.

This result is disappointing because the strategy associated with τ_1, which seemed rather sensible, is not better than the strategy of betting on the first card. Both strategies are not profitable. However, both are fair in the sense that the mean gain is 0. We could ask about the strategy associated with τ_q for $q > 1$. The computations are rather similar but it is even harder to see the sign of $\mathrm{MG}(\tau_q)$.

Because of the rather negative result of this 'experimental' approach, we feel the need of a more abstract probabilistic tool to settle the problem. The next section is going to present the *ad hoc* concept.

3. Let's use a martingale

In gambling vocabulary, a martingale is a technique which enables one to win. In probability theory a martingale is simply a sequence of random variables which satisfy a certain property concerning the (conditional) means. See reference 2 for the historical aspects and, for instance, references 3 or 4 for an account of the full theory. The paradox is that a martingale of probability theory can often be used to show that a 'gambler's martingale' does not exist. In our setting, a martingale $(M_k)_{0\le k\le N}$ is defined as a sequence of random variables such that, for every $i \in \{0, \dots, N-1\}$ and for every event B depending only on the values of $S_1, \dots, S_i$, we have

$$\mathrm{E}[M_i \mathbf{1}_B] = \mathrm{E}[M_{i+1}\mathbf{1}_B]. \tag{8}$$

In our problem, it happens that the sequence

$$(M_k)_{0\le k\le N} = \left(\frac{S_k}{N-k}\right)_{0\le k\le N} \tag{9}$$

is a martingale. Let us justify this fact. Equation (8) amounts to showing that, for all integers $s_1, \dots, s_i$, the mean of

$$M_{i+1} = \frac{S_{i+1}}{N-i-1}$$

conditionally on $S_1 = s_1, \dots, S_i = s_i$ is $s_i/(N-i)$. But as seen in (2) and (3), conditionally on $S_1 = s_1, \dots, S_i = s_i$, we have $S_{i+1} = s_i - 1$ with probability

$$\frac{1}{2}\left(1 + \frac{s_i}{N-i}\right)$$

and $S_{i+1} = s_i + 1$ with probability

$$\frac{1}{2}\left(1 - \frac{s_i}{N-i}\right),$$

so the conditional mean of S_{i+1} is

$$(s_i - 1)\frac{1}{2}\left(1 + \frac{s_i}{N-i}\right) + (s_i + 1)\frac{1}{2}\left(1 - \frac{s_i}{N-i}\right) = \frac{N-(i+1)}{N-i}s_i.$$

Simply dividing by $N-(i+1)$, we obtain the martingale property. From this property we deduce the following result.

Theorem 1 *The mean gain associated with an admissible strategy τ is*

$$\mathrm{MG}(\tau) = -2\,\mathrm{P}(S_{N-1} = 1; \tau = +\infty). \tag{10}$$

Proof Let $k \in \{1, \dots, N-1\}$. If we iterate (8) for the martingale given in (9), letting i take the values $k, k+1, \dots, N-2$ and $B = \{\tau = k\}$, we obtain

$$\mathrm{E}\left[\frac{S_k}{N-k}\mathbf{1}_{\{\tau=k\}}\right] = \mathrm{E}[S_{N-1}\mathbf{1}_{\{\tau=k\}}].$$

Substituting this expression in (4), we get that the mean gain is

$$\mathrm{MG}(\tau) = \mathrm{E}\left[S_{N-1}\left(\sum_{k=0}^{N-1}\mathbf{1}_{\{\tau=k\}}\right)\right] - \mathrm{P}(\tau = +\infty). \tag{11}$$

But

$$\sum_{k=0}^{N-1}\mathbf{1}_{\{\tau=k\}} = 1 - \mathbf{1}_{\{\tau=+\infty\}},$$

and it is clear by symmetry that $\mathrm{E}(S_{N-1}) = 0$. Substituting in (11), we obtain

$$\mathrm{MG}(\tau) = -\mathrm{E}[(1 + S_{N-1})\mathbf{1}_{\{\tau=+\infty\}}],$$

and (10) is just another formulation. The proof is complete.

Let us now discuss the consequences of theorem 1. The most important is that the mean gain cannot be strictly positive. There is no way to make money with this game. Conversely, any strategy associated with τ such that $\mathrm{P}(\tau = +\infty) = 0$, i.e. such that the player always bets at some time, is fair. It also confirms that the strategy associated with τ_1 is fair because, for this strategy, the event $\tau_1 = +\infty$ (no bet) automatically implies $S_{N-1} = -1$. So this theorem shows that no strategy can do better than the strategy of betting on the first card drawn. More sophisticated strategies like those associated with τ_q for $q > 1$ can in fact be even worse since, as we will see in (12), below, the mean gain is strictly negative. Conversely, it is astonishing that certain strategies which sound stupid are in fact fair and by consequence among the best ones. For instance, suppose that you bet at the first time when black cards have turned up q times more than red ones ($q \geq 1$) or on the last card if this never happens. It does not seem to make sense but in fact this strategy is as clever as the best ones. In fact, (10) shows that losing money is caused by non-betting even when you know for sure that the last card drawn will be black, which is effectively stupid behaviour!

4. Further questions and remarks

Some new questions arise if the problem is changed or extended. In particular, what happens if the penalty for non-betting is cancelled? Of course, profitable strategies arise. Starting again from (11), we obtain that the mean gain of an admissible strategy associated with τ is now

$$\begin{aligned}\widetilde{\mathrm{MG}}(\tau) &= \mathrm{E}[S_{N-1}\mathbf{1}_{\{\tau<+\infty\}}] \\ &= \mathrm{P}(S_{N-1} = 1, \tau < +\infty) - \mathrm{P}(S_{N-1} = -1, \tau < +\infty).\end{aligned}$$

The strategy which maximizes this quantity is $\tilde{\tau}$, defined by $\tilde{\tau} = N-1$ if $S_{N-1} = 1$ and $+\infty$ otherwise. This is the 'last minute' strategy which consists of waiting for the last draw and in this case $\widetilde{\mathrm{MG}}(\tilde{\tau}) = \frac{1}{2}$. In this case too the best strategy is not a clever one.

We can also imagine several generalizations of this game with bets on the suit, but note that the modelling will then require us to consider a random walk in dimension bigger than 1, which certainly complicates the arguments.

We end this section by giving a few additional results that the reader can treat as application exercises.

(i) Show that

$$\mathrm{MG}(\tau_q) = -2\left(\binom{N-1}{N/2} - \binom{N-1}{N/2+q-1}\right) \Big/ \binom{N}{N/2}, \tag{12}$$

and, using the well-known monotonicity properties of binomial coefficients, show that, for $q > 1$, the mean gain $\mathrm{MG}(\tau_q)$ is strictly negative.

(ii) Combine (12) with (4) to deduce a, not so obvious, combinatorial identity: for N an even integer and $q \geq 1$,

$$\sum_{k=q}^{N-q} \frac{1}{k(N-k)} \binom{k}{(k+q)/2} \binom{N-k}{(N-k-q)/2} = \frac{2}{qN} \binom{N}{N/2+q},$$

which is a generalization of equation (1.34) in reference 1.

(iii) With the notation $N = 2n$, show that

$$\widetilde{\mathrm{MG}}(\tau_q) = \frac{q}{n} \frac{(n!)^2}{(n+q)!\,(n-q)!}.$$

Check that this quantity reaches a maximum for q_0 being the least integer q such that $2q(q+1) > n$. In particular, when n is large, $q_0 \approx \sqrt{n/2}$ and $\widetilde{\mathrm{MG}}(\tau_{q_0})$ is, up to multiplicative constants, of order $n^{-1/2}$, which is much lower than the last minute strategy.

5. Computation of a law

This section is devoted to the computation of the law of τ_1 that we announced in (6) and (7). Since it requires no more effort, we will compute the law of τ_q for $q \geq 1$. It uses well-known combinatorial techniques on the set of trajectories of random walks, the main argument being the so-called *reflection principle*. Chapter III of reference 5 is a nice, classical reference on the techniques we use in this section. Note that τ_q takes in fact its values in $\{q, \ldots, N-q\} \cup \{+\infty\}$ and we will moreover see that τ_q may only take integer values having the same parity as q.

Let us denote by $\mathcal{S}(N, a, b)$ the set of all finite sequences of integers $(s_k)_{0 \leq k \leq N}$ of length $N+1$ with increments $+1$ or -1, starting from $s_0 = a$ and arriving at $s_N = b$. Such a sequence is easily represented by a graph in the plane. A sequence belonging to $\mathcal{S}(N, a, b)$ is entirely determined by specifying which increments are $+1$ and the number of those increments equal to $+1$ is exactly $(N+b-a)/2$ in order to meet the requirements on s_0 and s_N. Hence, the cardinal (number of elements) of $\mathcal{S}(N, a, b)$ is

$$\#[\mathcal{S}(N, a, b)] = \binom{N}{(N+b-a)/2}. \tag{13}$$

The random sequence $(S_k)_{0\le k\le N}$ takes its values in $\mathcal{S}(N,0,0)$ and, moreover, the distribution of this random sequence is the uniform probability on $\mathcal{S}(N,0,0)$. As a consequence, computing the desired probabilities will consist essentially of determining the cardinal of the specified events. For instance, let us start with $\mathrm{P}(\tau_q = +\infty)$ and, for convenience, we prefer to compute the probability $\mathrm{P}(\tau_q < +\infty)$ of the complementary event. It is the cardinal of the subset of $\mathcal{S}(N,0,0)$ formed by the sequences that reach q, divided by the cardinal of $\mathcal{S}(N,0,0)$. But there is a one-to-one correspondence between sequences in $\mathcal{S}(N,0,0)$ that reach q and sequences in $\mathcal{S}(N,0,2q)$. The correspondence consists of applying a reflection with respect to level q from the hitting time of q up to time N. Therefore, this argument is called the reflection principle. By (13), it follows that

$$\mathrm{P}(\tau_q < +\infty) = \frac{\#[\mathcal{S}(N,0,2q)]}{\#[\mathcal{S}(N,0,0)]} = \binom{N}{N/2-q} \Big/ \binom{N}{N/2};$$

hence, (6) holds. Now we pass to $\mathrm{P}(\tau_q = k)$. Decomposing into the part before time k and the part after time k, we get

$$\mathrm{P}(\tau_q = k) = \frac{\#[\mathcal{S}(k,0,q;<q)] \times \#[\mathcal{S}(N-k,q,0)]}{\#[\mathcal{S}(N,0,0)]}, \tag{14}$$

where $\mathcal{S}(k,0,q;<q)$ is the subset of $\mathcal{S}(k,0,q)$ consisting of sequences that reach level q for the first time at time k. This set has the same cardinal as the set of sequences in $\mathcal{S}(k-1,0,q-1)$ which do not reach level q. Another application of the reflection principle shows that the latter set has the same cardinal as

$$\#[\mathcal{S}(k-1,0,q-1)] - \#[\mathcal{S}(k-1,0,q+1)] = \frac{q}{k}\binom{k}{(k+q)/2}.$$

Substituting in (14) and expressing the other terms using (13), we get the final expression of the law of τ_q,

$$\mathrm{P}(\tau_q = k) = \frac{q}{k}\binom{k}{(k+q)/2}\binom{N-k}{(N-k-q)/2} \Big/ \binom{N}{N/2},$$

and (7) is a particular case.

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Catalan-Related Polynomials

MARTIN GRIFFITHS

A well-known combinatorial interpretation of the Catalan numbers involves a simple voting scenario. In this article we generalise the situation somewhat, and show how this leads, in a natural way, to an infinite family of polynomials where each of these polynomials has a combinatorial interpretation associated with the Catalan numbers.

1. Introduction

The sequence of Catalan numbers (see references 1–3) was named after the Belgian mathematician Eugène Charles Catalan (1814–1894) (see reference 4). The numbers in this sequence arise from a well-known combinatorial scenario associated with voting, as we now explain. Suppose that an election is taking place, and that only two candidates are involved. Since I am currently living in New Zealand, and the two main political parties here are Labour and the Nationals, let us assume that one candidate represents the Labour Party and the other represents the National Party.

The election takes place, and it turns out that an even number of votes were cast, say $2n$ for some $n \in \mathbb{N}$. Furthermore, the outcome was a tie, with each candidate receiving n votes. We will assume that the chronological order in which the votes were cast is recorded, and that no two votes were cast simultaneously. An interesting question that may then be asked is as follows.

> How many possible voting orders are there such that the Labour candidate had, at any stage throughout the day of the election, received at least as many votes as the National candidate?

The answer to this is indeed the nth Catalan number, denoted by C_n, and given by the simple formula

$$\mathrm{C}_n = \frac{1}{n+1}\binom{2n}{n}.$$

A proof of this is given in references 2 and 3.

By way of a manageable example, let us suppose that there were a total of only eight votes cast, so that $n = 4$. Reading from left to right, here are the following possibilities for the order of votes cast such that the above conditions are satisfied, using L and N to represent a Labour and a National voter, respectively:

LLLLNNNN,	LLNLLNNN,	LLNNLNLN,	LNLNLLNN,
LLLNLNNN,	LLNLNLNN,	LNLLLNNN,	LNLNLNLN,
LLLNNLNN,	LLNLNNLN,	LNLLNLNN,	
LLLNNNLN,	LLNNLLNN,	LNLLNNLN.	

This confirms that

$$\mathrm{C}_4 = 14 = \frac{1}{5}\binom{8}{4}.$$

Table 1 The first ten Catalan numbers.

n	C_n
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1 430
9	4 862
10	16 796

The first few terms of the sequence of Catalan numbers are given in table 1. One of the fascinating things about these numbers is that they do in fact possess numerous combinatorial interpretations, and it is not always immediately obvious that the various interpretations will lead to the same sequence. For example, C_n also gives the number of distinct binary trees on n nodes (see reference 2).

In this article we extend the scenario to the situation where, although we are still enumerating the possible voting orders for which the Labour candidate is never lagging behind, the number of Labour and National votes are no longer necessarily equal. We show that this leads to an infinite family of polynomials, and go on to obtain a general formula for the members of this family.

2. Some initial results

Suppose that there are n voters in total, k of which are National, with the remaining $n-k$ being Labour. Let $f_k(n)$ denote, for $n \geq 2k$, the number of possible voting orders such that at any point in the day, the Labour Party has at least as many votes as the National Party. We shall term any of the voting orders enumerated by $f_k(n)$ an *allowable pattern*. First, it is very straightforward to determine $f_1(n)$ for $n \geq 2$. For example, if $n = 5$ and $k = 1$ we have the following possible four voting orders:

LLLLN, LLLNL, LLNLL, LNLLL.

In general, the single vote for the National Party can occupy any of the positions from 2 to n, and so we have $f_1(n) = n - 1$.

To make further progress, use may be made of the fact that $f_k(n)$ obeys the following recurrence relation:

$$f_k(n) = f_k(n-1) + f_{k-1}(n-1), \tag{1}$$

with 'boundary' conditions $f_1(n) = n - 1$ for all $n \geq 2$ and $f_k(2k) = C_k$ for all $k \geq 1$. In order to see that this recurrence holds in general, note that $f_k(n-1)$ counts the number of allowable patterns enumerated by $f_k(n)$ for which the last vote is L while $f_{k-1}(n-1)$ counts those for which the last vote is N.

We now show how to obtain $f_2(n)$ using (1) in conjunction with the boundary conditions. To this end, we have, for $n \geq 4$,

$$\begin{aligned}
f_2(n) &= f_2(n-1) + (n-2) \\
&= f_2(n-2) + (n-3) + (n-2) \\
&= f_2(n-3) + (n-4) + (n-3) + (n-2) \\
&\ \ \vdots \\
&= f_2(4) + 3 + 4 + \cdots + (n-2) \\
&= \mathrm{C}_2 + \frac{(n-2)(n-1)}{2} - 1 - 2 \\
&= \frac{n(n-3)}{2}.
\end{aligned}$$

Similarly, for $n \geq 6$, it is the case that

$$f_3(n) = f_3(6) + \sum_{j=1}^{n-6} \frac{(n-j)(n-j-3)}{2} = \frac{n(n-1)(n-5)}{6},$$

where we have used the summation results

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2} \quad \text{and} \quad \sum_{j=1}^{m} j^2 = \frac{m(m+1)(2m+1)}{6}.$$

The following recurrence holds in general:

$$f_k(n) = f_k(2k) + \sum_{j=1}^{n-2k} f_{k-1}(n-j). \tag{2}$$

3. A general formula

In section 2 we established a recurrence relation (2) that allows us to find, for any fixed $k \in \mathbb{N}$, the polynomial $f_k(n)$. However, if k is large, this could take a considerable amount of time and effort. Let us now concentrate, therefore, on obtaining an overall general formula for $f_k(n)$. In fact, we prove by induction on n that

$$f_k(n) = \frac{n-2k+1}{n-k+1}\binom{n}{k}, \tag{3}$$

for all $k \geq 1$ and $n \geq 2k$.

First, we have

$$f_k(2k) = \mathrm{C}_k = \frac{1}{k+1}\binom{2k}{k} = \frac{2k-2k+1}{2k-k+1}\binom{2k}{k},$$

so (3) is, for any $k \in \mathbb{N}$, true when $n = 2k$. Now suppose that, for any $k \in \mathbb{N}$,

$$f_k(n-1) = \frac{(n-1)-2k+1}{(n-1)-k+1}\binom{n-1}{k},$$

Table 2 Evaluations of $f_n(k)$ for $1 \le k \le 4$ and $2 \le n \le 8$.

k	$f_k(2)$	$f_k(3)$	$f_k(4)$	$f_k(5)$	$f_k(6)$	$f_k(7)$	$f_k(8)$
1	1	2	3	4	5	6	7
2			2	5	9	14	20
3					5	14	28
4							14

for some $n > 2k$. Then, by way of (1) and the inductive hypothesis, we obtain

$$\begin{aligned}
f_k(n) &= f_k(n-1) + f_{k-1}(n-1) \\
&= \frac{(n-1)-2k+1}{(n-1)-k+1}\binom{n-1}{k} + \frac{(n-1)-2(k-1)+1}{(n-1)-(k-1)+1}\binom{n-1}{k-1} \\
&= \frac{n-2k}{n-k}\,\frac{(n-1)!}{k!\,(n-k-1)!} + \frac{n-2k+2}{n-k+1}\,\frac{(n-1)!}{(k-1)!\,(n-k)!} \\
&= \frac{n-2k}{n-k}\,\frac{n-k}{n}\binom{n}{k} + \frac{n-2k+2}{n-k+1}\,\frac{k}{n}\binom{n}{k} \\
&= \frac{1}{n(n-k+1)}\binom{n}{k}[(n-2k)(n-k+1) + k(n-2k+2)] \\
&= \frac{n-2k+1}{n-k+1}\binom{n}{k},
\end{aligned}$$

which we can simplify to complete the proof.

We have therefore been able to show here that there is an infinite family of polynomials arising from the Catalan numbers in a natural way, such that each of these polynomials has a combinatorial interpretation associated with these numbers. For example,

$$f_5(n) = \frac{n-9}{n-4}\binom{n}{5} = \frac{1}{120}(n^5 - 15n^4 + 65n^3 - 105n^2 + 54n)$$

enumerates the possible voting orders for which there is a total of $n \ge 10$ votes, exactly five of which are for the National Party, such that the Labour candidate is never behind at any stage during the day. The numerical values of $f_k(n)$ for low values of k and n appear in table 2, noting that the Catalan numbers occupy the first nonzero entry in each row.

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Fibonacci Polynomials, Lucas Polynomials, and Operators

THOMAS KOSHY

Using differential calculus, we establish a close link between Fibonacci polynomials $f_n(x)$ and Lucas polynomials $l_n(x)$. We then employ integral calculus to express $l_n(x)$ in terms of $f_n(x)$. We use a bit of operator theory to confirm the well-known Lucas-like formula for f_{n+1} and then use this formula, coupled with integral calculus, to establish the Lucas-like formula for $l_n(x)$.

Introduction

Fibonacci polynomials $f_n(x)$ and Lucas polynomials $l_n(x)$ satisfy the second-order *gibonacci* (generalized Fibonacci) recurrence $g_{n+2}(x) = xg_{n+1}(x) + g_n(x)$, where $g_1(x)$ and $g_2(x)$ are arbitrary polynomials and $n \geq 1$ (see reference 1). When $g_1(x) = 1$ and $g_2(x) = x$, $g_n(x) = f_n(x)$; when $g_1(x) = x$ and $g_2(x) = x^2 + 2$, $g_n(x) = l_n(x)$. Clearly, $f_n(1) = F_n$, the nth Fibonacci number, and $l_n(1) = L_n$, the nth Lucas number. (In the interest of brevity, we will denote $g_n(x)$ by g_n when doing so creates no confusion.) Table 1 gives the first six Fibonacci and Lucas polynomials.

Fibonacci and Lucas polynomials can be defined explicitly by the Binet-like formulas

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x) = (x + D)/2$ and $\beta = \beta(x) = (x - D)/2$ are the solutions of the equation $t^2 - xt - 1 = 0$, and $D = D(x) = \sqrt{x^2 + 4}$ (see reference 1).

Next we will study two close links between f_n and l_n.

Links Between Fibonacci polynomials and Lucas polynomials

To begin with, $D' = x/D$, $\alpha' = \alpha/D$, and $\beta' = -\beta/D$, where the prime denotes differentiation with respect to x. By Binet's formula, we then have

$$l_n = \alpha^n + \beta^n$$

Table 1 The first six Fibonacci and Lucas polynomials.

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	$x^2 + 2$
3	$x^2 + 1$	$x^3 + 3x$
4	$x^3 + 2x$	$x^4 + 4x^2 + 2$
5	$x^4 + 3x^2 + 1$	$x^5 + 5x^3 + 5x$
6	$x^5 + 4x^3 + 3x$	$x^6 + 6x^4 + 9x^2 + 2$

and

$$\mathrm{l}_n' = n\alpha^{n-1}\frac{\alpha}{D} - n\beta^{n-1}\frac{\beta}{D} = n\mathrm{f}_n. \tag{1}$$

For example, $\mathrm{l}_6' = 6(x^5 + 4x^3 + 3x) = 6\mathrm{f}_6$, as can be seen from table 1. Equation (1) implies that we can recover l_n from f_n using integral calculus, i.e.

$$\begin{aligned}
\int_0^x \mathrm{l}_n'(y)\,\mathrm{d}y &= n\int_0^x \mathrm{f}_n(y)\,\mathrm{d}y,\\
\mathrm{l}_n(x) - \kappa_n &= n\int_0^x \mathrm{f}_n(y)\,\mathrm{d}y,\\
\mathrm{l}_n(x) &= \kappa_n + n\int_0^x \mathrm{f}_n(y)\,\mathrm{d}y, \qquad (2)
\end{aligned}$$

where

$$\kappa_n = \mathrm{l}_n(0) = 1 + (-1)^n = \begin{cases} 0 & \text{if } n \text{ is odd,}\\ 2 & \text{otherwise.}\end{cases}$$

Alternate explicit formulas

Both f_n and l_n can also be defined explicitly by the Lucas-like formulas

$$\mathrm{f}_{n+1} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} x^{n-2k}, \tag{3}$$

$$\mathrm{l}_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}, \tag{4}$$

respectively, where $\lfloor\lambda\rfloor$ denotes the *floor* of the real number λ, i.e. the greatest integer less than or equal to λ. Both can be confirmed using induction and some messy algebra (see reference 1). However, we can establish both in a short and elegant way, by employing a little bit of operator theory (see reference 2). To this end, let

$$S_n = S_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k} x^{n-2k}.$$

To prove that (3) holds, we must show that $S_n = \mathrm{f}_{n+1}$.

Let $\Delta S_n = S_{n+1} - xS_n$. With the convention that $\binom{n}{r} = 0$ when $r > n$, we then have

$$\begin{aligned}
\Delta S_n &= \sum_{k\ge 0}\left[\binom{n-k+1}{k} - \binom{n-k}{k}\right] x^{n-2k+1}\\
&= \sum_{k\ge 1}\binom{n-k}{k-1} x^{n-2k+1}\\
&= \sum_{k\ge 0}\binom{n-k-1}{k} x^{n-2k-1}
\end{aligned}$$

and

$$\begin{aligned}
\Delta^2 S_n &= \Delta(\Delta S_n)\\
&= \Delta S_{n+1} - x\Delta S_n\\
&= \sum_{k\geq 0}\left[\binom{n-k}{k} - \binom{n-k-1}{k}\right]x^{n-2k}\\
&= \sum_{k\geq 1}\binom{n-k-1}{k-1}x^{n-2k}.
\end{aligned}$$

Consequently,

$$\Delta S_{n+1} = \Delta^2 S_n + x\Delta S_n = \sum_{k\geq 1}\left[\binom{n-k-1}{k-1} + \binom{n-k-1}{k}\right]x^{n-2k} + \binom{n-1}{0}x^n.$$

That is, $S_{n+2} - xS_{n+1} = \sum_{k\geq 0}\binom{n-k}{k}x^{n-2k} = S_n$. Thus, S_n satisfies the gibonacci recurrence.

This, coupled with the initial conditions $S_0 = 1 = \mathrm{f}_1$ and $S_1 = x = \mathrm{f}_2$, implies that $S_n = \mathrm{f}_{n+1}$, as desired.

Equation (4) follows by a similar argument.

Interestingly, we can recover (4) from (3) by using (2) as follows:

$$\begin{aligned}
\mathrm{l}_n &= \kappa_n + n\int_0^x \mathrm{f}_n(y)\,\mathrm{d}y\\
&= \kappa_n + n\int_0^x \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-k-1}{k}y^{n-2k-1}\,\mathrm{d}y\\
&= \kappa_n + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{n}{n-2k}\binom{n-k-1}{k}x^{n-2k}\\
&= \kappa_n + \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}\\
&= \sum_{k=0}^{\lfloor n/2\rfloor}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k},
\end{aligned}$$

as required.

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Some Remarks on the Pell Equation

W. J. A. COLMAN

In this article we determine simple explicit formulae for the i th solution of the famous Pell equation when the fundamental solution has been determined by the standard method. This enables us to count the number of solutions not greater than any arbitrary integer.

Introduction

The Pell equation $x^2 - my^2 = 1$, where m is a positive nonsquare integer, is to be solved in integers x and y. It is known (Lagrange) that it always has an infinity of integral solutions. The fundamental solution (x_1, y_1), with minimum $y_1 > 0$, can be found from the continued fraction expansion of $\sqrt{m}$ and is explained in standard textbooks. From this solution all subsequent solutions can be found. We shall determine some formulae that can simplify the calculations assuming that we have already determined the fundamental solution.

To illustrate this method we will consider a numerical example. Suppose that we have $x^2 - 5y^2 = 1$. The fundamental solution is $x_1 = 9$ and $y_1 = 4$. But for arbitrary m the fundamental solution can be large and not findable by means of a computer search. For example, if $m = 313$ then $x_1 = 2(126\,862\,368)^2 + 1$ and $y_1 = 2(7\,170\,685)(126\,862\,368)$, which was given by Brouncker and Wallis in about 1658 in answer to a challenge from continental mathematicians. All subsequent solutions for our example are given implicitly by

$$x_i + 5y_i = (9 + 4\sqrt{5})^i, \quad \text{for } i \geq 1.$$

The first six solutions are given in table 1.

We get the next solution by multiplying the previous one by $9 + 4\sqrt{5}$. For example, $(161 + 72\sqrt{5})(9 + 4\sqrt{5}) = 2889 + 1292\sqrt{5}$. We can of course determine any numerical solution by expanding the right-hand side by the binomial theorem. For example,

$$\begin{aligned} x_{10} + \sqrt{5}y_{10} &= (9 + 4\sqrt{5})^{10} \\ &= 9^{10} + 10 \cdot 9^9 \cdot 4\sqrt{5} + 45 \cdot 9^8 \cdot 4^2 \cdot 5 + \cdots + 4^{10} \cdot 5^5 \\ &= 1\,730\,726\,404\,001 + 774\,004\,377\,960\sqrt{5}. \end{aligned}$$

But what is the i th solution?

Table 1

i	x_i	y_i
1	9	4
2	161	72
3	2 889	1 292
4	51 841	23 184
5	930 249	4 160 202
6	16 692 641	7 465 176

An alternative procedure

The classical method for determining subsequent solutions relies entirely upon integers and the square root of m which never has to be evaluated. Therefore, there are never any problems with rounding errors. We shall now introduce a method based on real numbers which necessarily involves truncated decimals. We have

$$x_i + \sqrt{m}y_i = (x_1 + \sqrt{m}y_1)^i,$$
$$x_i - \sqrt{m}y_i = \frac{1}{(x_1 + \sqrt{m}y_1)^i}, \quad \text{as } x_i^2 - my_i^2 = 1.$$

Adding gives

$$\begin{aligned} x_i &= \frac{(x_1 + \sqrt{m}y_1)^i}{2} + \frac{1}{2(x_1 + \sqrt{m}y_1)^i} \\ &= \left[\frac{(x_1 + \sqrt{m}y_1)^i}{2}\right] + \left\{\frac{(x_1 + \sqrt{m}y_1)^i}{2}\right\} + \frac{1}{2(x_1 + \sqrt{m}y_1)^i}, \end{aligned}$$

where $[\cdot]$ is the integer part and $\{\cdot\}$ is the fractional part. Now,

$$0 < \left\{\frac{(x_1 + \sqrt{m}y_1)^i}{2}\right\} + \frac{1}{2(x_1 + \sqrt{m}y_1)^i} < 2.$$

But x_i is an integer so that this expression must be one, which gives

$$x_i = \left[\frac{(x_1 + \sqrt{m}y_1)^i}{2}\right] + 1, \quad \text{for } i \geq 1.$$

Similarly, we have

$$y_i = \left[\frac{(x_1 + \sqrt{m}y_1)^i}{2\sqrt{m}}\right] + \left\{\frac{(x_1 + \sqrt{m}y_1)^i}{2\sqrt{m}}\right\} - \frac{1}{2\sqrt{m}(x_1 + \sqrt{m}y_1)^i},$$

where now the fractional part lies between -1 and 1 and so must be zero as y_i is an integer. This gives

$$y_i = \left[\frac{(x_1 + \sqrt{m}y_1)^i}{2\sqrt{m}}\right].$$

So we have x_i and y_i given explicitly, but there is a catch which becomes apparent when evaluating these real numbers on a computer. To see why, consider determining x_{10} from the previous example. We have

$$x_{10} = \left[\frac{(9 + 4\sqrt{5})^{10}}{2}\right] + 1 = 1\,730\,726\,404\,001,$$

which we know is the correct answer. However, premature rounding will give the wrong answer. We have

$$y_{10} = \left[\frac{(9 + 4\sqrt{5})^{10}}{2\sqrt{5}}\right] = 774\,004\,377\,960.$$

Alternatively, we can just accept that there might be an error in the last digit, due to rounding, which we can easily check, as it is obvious that $1\,730\,726\,404\,002^2 - 5(774\,004\,377\,960)^2 \neq 1$.

Counting the number of solutions

The main reason for determining these explicit formulae for x_i and y_i is that it enables us to count the number of solutions less than or equal to some given number. For example, how many solutions of $x^2 - my^2 = 1$ are there with $x \le L$, where L is any positive integer? We can churn them out and see but if L is large this is not much fun. In any case is there a formula? We have

$$x_i = \frac{(x_1 + \sqrt{m}y_1)^i}{2} + \frac{1}{2(x_1 + \sqrt{m}y_1)^i} \le L.$$

This gives a quadratic inequality in $(x_1 + \sqrt{m}y_1)^i$. Solving gives

$$L - \sqrt{L^2 - 1} \le (x_1 + \sqrt{m}y_1)^i \le L + \sqrt{L^2 - 1},$$

giving

$$\frac{\ln(L - \sqrt{L^2 - 1})}{\ln(x_1 + \sqrt{m}y_1)} \le i \le \frac{\ln(L + \sqrt{L^2 - 1})}{\ln(x_1 + \sqrt{m}y_1)}.$$

This gives the number of solutions with $x_i \le L$ to be

$$\left\lfloor \frac{\ln(L + \sqrt{L^2 - 1})}{\ln(x_1 + \sqrt{m}y_1)} \right\rfloor.$$

If we put $L = x_1$ this gives 1. In exactly the same way, the number of solutions with $y \le L$ is

$$\left\lfloor \frac{\ln(\sqrt{m}L + \sqrt{mL^2 + 1})}{\ln(x_1 + \sqrt{m}y_1)} \right\rfloor,$$

which gives 1 if $L = y_1$.

We can do more complicated counts such as the number of solutions for which $x_i + y_i \le L$, which gives

$$\left\lfloor \ln\left(\frac{\sqrt{m}L + \sqrt{mL^2 - m + 1}}{1 + \sqrt{m}} \right) \Big/ \ln(x_1 + \sqrt{m}y_1) \right\rfloor.$$

It is an interesting bit of algebra to show that this number is 1 if $L = x_1 + y_1$.

So how many integer solutions of $x^2 - 5y^2 = 1$ are there with $x + y \le 10^{20}$? The answer is

$$\left\lfloor \ln\left(\frac{\sqrt{5} \cdot 10^{20} + \sqrt{5 \cdot 10^{40} - 4}}{1 + \sqrt{5}} \right) \Big/ \ln(9 + 4\sqrt{5}) \right\rfloor = 16.$$

We can verify this by using the explicit expressions for x_i and y_i, with $i = 16$, to show that $x_{16} + y_{16} = 8.362\,11\ldots \times 10^{19}$ and with $i = 17$ we get $x_{17} + y_{17} = 1.500\,52\ldots \times 10^{21}$.

W. J. A. Colman *taught at the University of East London and is now retired. He first used a computer in the early 1960s at the old Woolwich Polytechnic, where it filled a large room and had by today's standards an archaic programming language, although at the time it did seem like the beginning of a new age, which of course it was.*

New Trigonometric Identities From Old

DES MACHALE and PETER MACHALE

We derive new trigonometric identities from existing ones by means of the substitution $a = 2R \sin A$.

One of the most basic results in plane trigonometry is that if ABC is a triangle then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the radius of the circumcircle of the triangle. In this note we examine what happens if we make the substitution

$$a = 2R \sin A, \qquad b = 2R \sin B, \qquad c = 2R \sin C$$

in some known trigonometric formulae. We use the usual notation: $2s = a + b + c$, Δ is the area of the triangle, and r is the radius of the incircle.

Theorem 1 *In any triangle,* $\sin A + \sin B > \sin C$.

Proof The triangle inequality states that $a + b > c$. Thus,

$$2R \sin A + 2R \sin B > 2R \sin C.$$

By cancellation, $\sin A + \sin B > \sin C$, as claimed.

Theorem 2 $2R^2 \sin A \sin B \sin C = \Delta$.

Proof We have

$$2R^2 \sin A \sin B \sin C = \tfrac{1}{2}(2R \sin A)(2R \sin B) \sin C = \tfrac{1}{2} ab \sin C = \Delta.$$

We can also easily derive known trigonometric identities in a novel way using this technique.

Theorem 3 *If B and C are two angles of a triangle, then*

$$\sin(B + C) = \sin B \cos C + \cos B \sin C.$$

Proof We start with the easy-to-prove identity $a = b \cos C + c \cos B$. Then

$$2R \sin A = 2R \sin B \cos C + 2R \sin C \cos B.$$

Thus,

$$\sin B \cos C + \cos B \sin C = \sin A = \sin(180^\circ - A) = \sin(B + C),$$

as claimed.

Theorem 4 $R = s/(\sin A + \sin B + \sin C)$ *and* $Rr = \Delta/(\sin A + \sin B + \sin C)$.

Proof Adding the equations $2R\sin A = a$, $2R\sin B = b$, and $2R\sin C = c$, we get

$$2R(\sin A + \sin B + \sin C) = a + b + c = 2s.$$

So,

$$R = \frac{s}{\sin A + \sin B + \sin C}.$$

Now,

$$\Delta = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = rs.$$

So,

$$Rr = \frac{s}{\sin A + \sin B + \sin C}\,\frac{\Delta}{s} = \frac{\Delta}{\sin A + \sin B + \sin C}.$$

Theorem 5 *In any triangle,* $2\cos A\sin B\sin C = \sin^2 B + \sin^2 C - \sin^2 A$.

Proof We start with the identity

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and make our substitution to get

$$\cos A = \frac{4R^2\sin^2 B + 4R^2\sin^2 C - 4R^2\sin^2 A}{2(2R\sin B)(2R\sin C)} = \frac{\sin^2 B + \sin^2 C - \sin^2 A}{2\sin B\sin C},$$

from which the desired result follows. (The right-hand side is of course $\frac{1}{2}(1 + \cos 2A - \cos 2B - \cos 2C)$.)

The trick is to find expressions where by cancellation R vanishes from the discussion like a catalyst in chemistry. We note that our substitution can also be used to give simple proofs of relatively difficult trigonometric identities as the following result shows.

Theorem 6 *In any triangle,*

$$(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0.$$

Proof Using $\cot x = \cos x/\sin x$ and $\sin x = x/2R$, the expression becomes

$$\begin{aligned}
&\frac{(b^2 - c^2)\cos A(2R)}{a} + \frac{(c^2 - a^2)\cos B(2R)}{b} + \frac{(a^2 - b^2)\cos C(2R)}{c} \\
&\quad = 2R\Bigg[\frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2abc} + \frac{(c^2 - a^2)(a^2 + c^2 - b^2)}{2abc} \\
&\qquad\quad + \frac{(a^2 - b^2)(a^2 + b^2 - c^2)}{2abc}\Bigg] \\
&\quad = \frac{R}{abc}[(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) \\
&\qquad\quad + (a^2 - b^2)(a^2 + b^2 - c^2)] \\
&\quad = \frac{R}{abc}[b^4 - c^4 - a^2(b^2 - c^2) + c^4 - a^4 - b^2(c^2 - a^2) + a^4 - b^4 - c^2(a^2 - b^2)] \\
&\quad = 0.
\end{aligned}$$

Theorem 7 *In any triangle,*

$$\cos^2\frac{A}{2}=\frac{(\sin B+\sin C)^2-\sin^2 A}{4\sin B\sin C}\quad \textit{and}\quad \sin^2\frac{A}{2}=\frac{\sin^2 A-(\sin B-\sin C)^2}{4\sin B\sin C}.$$

Proof We start with

$$\cos^2\frac{A}{2}=\frac{s(s-a)}{bc}\quad\text{and}\quad\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc},$$

which follow from the well-known formula $a^2=b^2+c^2-2bc\cos A$. By theorem 4, $s=R(\sin A+\sin B+\sin C)$ and $s-a=R(\sin B+\sin C-\sin A)$, with similar expressions for $s-b$ and $s-c$. Then we have

$$\frac{s(s-a)}{bc}=\frac{R^2(\sin A+\sin B+\sin C)(\sin B+\sin C-\sin A)}{4R^2\sin B\sin C},$$

from which the desired expression follows easily. The proof of the second part of the theorem is similar.

Theorem 8 *In any triangle ABC,*

$$a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C.$$

Proof We have

$$\begin{aligned}a\cos A+b\cos B+c\cos C&=2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C\\&=2R\sin A\cos A+R\sin 2B+R\sin 2C.\end{aligned}$$

Now,

$$\sin A\cos A=-\sin A\cos(B+C)$$

and

$$\begin{aligned}\sin 2B+\sin 2C&=2\sin(B+C)\cos(B-C)\\&=2\sin A\cos(B-C),\end{aligned}$$

since $A+B+C=180°$. So our expression becomes

$$R[2\sin A(\cos(B-C)-\cos(B+C))]=4R\sin A\sin B\sin C.$$

(Note that, by theorem 2, both expressions are equal to $2\Delta/R=8\Delta^2/abc$. See reference 1 for details.)

The AAS congruence condition on triangles implies that a formula must exist for the area of a triangle in terms of a side and the trigonometric functions of two angles. Indeed, a rather beautiful such formula does exist, but unfortunately it is not well known. We hope our next result will remedy this situation.

Theorem 9 *For any triangle,*

$$\Delta=\frac{a^2}{2(\cot B+\cot C)}.$$

Proof We start with $\Delta = \frac{1}{2}ab\sin C$. So,

$$\Delta = \tfrac{1}{2}a(2R\sin B)\sin C = \tfrac{1}{2}a\frac{a}{\sin A}\sin B\sin C = \frac{a^2\sin B\sin C}{2(\sin B\cos C + \cos B\sin C)},$$

by theorem 3. Dividing above and below by the nonzero expression $\sin B\sin C$ gives the result.

In the bad old days before calculators (or were they good old days?), for surveying purposes, it was very important to be able to use log tables for calculation, so trigonometric formulae tended to consist of multiplications only to facilitate computation. The following result, now almost forgotten, was frequently used.

Theorem 10 *In any triangle,*

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

Proof The standard proof is found in most old trigonometry books, but the point is that it depends on

$$\begin{aligned}\frac{b-c}{b+c} &= \frac{2R(b-c)}{2R(b+c)}\\ &= \frac{\sin B - \sin C}{\sin B + \sin C}\\ &= \frac{2\cos((B+C)/2)\sin((B-C)/2)}{2\sin((B+C)/2)\cos((B-C)/2)}\\ &= \tan\frac{A}{2}\tan\frac{B-C}{2}.\end{aligned}$$

Once $\tan((B-C)/2)$ was found logarithmically, since $\tan((B+C)/2)$ is known, B and C could then be found and $b/\sin B = c/\sin C = a/\sin A$ could be used to solve the triangle fully in the SAS case.

We observe that the neglected function, the cotangent, has made several welcome appearances in this article. It is, in fact, our favourite trigonometric function and is defined for all angles of a triangle, with none of the difficulties associated with the tangent, which behaves very badly with respect to right angles. It is only fitting that we finish with a very nice expression for the cotangent, which deserves to be better known.

Theorem 11 *In any triangle,*

$$\cot A = \frac{b^2+c^2-a^2}{4\Delta}.$$

Proof We have

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{b^2+c^2-a^2}{4\Delta}.$$

This gives at once that

$$\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}.$$

We observe that this gives $\tan A = 4\Delta/(b^2+c^2-a^2)$ for $A \neq 90°$. Is the exceptional case a new proof of the theorem of Pythagoras, $a^2 = b^2+c^2$? Discuss!

Reference

1 H. S. Hall and S. R. Knight, *Elementary Trigonometry* (MacMillan, London, 1893).

Des MacHale *is Emeritus Professor of Mathematics at University College Cork where he taught for nearly forty years. His mathematical interests are in abstract algebra but he also works in number theory, geometry, combinatorics, and the history of mathematics. His other interests include humour, geology, and words.*

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Letters to the Editor

Dear Editor,

The equation $x^3 + 1 = y^2$

Catalan conjectured that the only solution of this equation in positive integers is $x = 2$ and $y = 3$. We consider some consequences of this conjecture. We put

$$S_n = 1^3 + 2^3 + \cdots + n^3, \qquad A_n = 1^3 + 3^3 + \cdots + (2n-1)^3,$$
$$B_n = 2^3 + 4^3 + \cdots + (2n)^3, \qquad C_n = 1^3 + 2^3 + \cdots + n^3 + (n-1)^3 + \cdots + 2^3 + 1^3,$$

and ask for which values of n are these perfect cubes?

Write $S_n = k^3$. Then

$$\left(\tfrac{1}{2}n(n+1)\right)^2 = k^3,$$

so that

$$\tfrac{1}{2}n(n+1) = r^3$$

for some positive integer r; whence,

$$n^2 + n - 2r^3 = 0.$$

Hence,

$$n = \frac{-1 + \sqrt{1 + 8r^3}}{2}$$

and $1 + 8r^3$ must be a perfect square. Catalan's conjecture gives that $r = 1$, so that n must be 1. Since $B_n = 2^3 S_n$, it follows that, for B_n to be a perfect cube, n must be 1.

Now write $A_n = k^3$. Then

$$\left(\tfrac{1}{2}2n(2n+1)\right)^2 - 2^3\left(\tfrac{1}{2}n(n+1)\right)^2 = k^3;$$

whence, $n^2(2n^2-1)=k^3$, $n^2=(1+\sqrt{1+8k^3})/4$, and $1+8k^3$ must be a perfect square. Catalan's conjecture gives that k must be 1, so that $n=1$.

Lastly, write $C_n=k^3$. Then

$$\left(\tfrac{1}{2}n(n+1)\right)^2+\left(\tfrac{1}{2}(n-1)n\right)^2=k^3,$$

i.e.

$$\tfrac{1}{2}n^2(n^2+1)=k^3.$$

Hence,

$$n^4+n^2-2k^3=0$$

and Catalan's conjecture gives as before that k must be 1, so that $n=1$.

Yours sincerely,

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Dear Editor,

An application of Euler's infinite product for the sine function

In 1735 the great mathematician Leonhard Euler discovered that every nth degree polynomial $p(x)$ whose roots are $a_1, a_2, \ldots, a_n$ and whose constant term is 1 can be factored as a product of the form

$$p(x)=\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\cdots\left(1-\frac{x}{a_n}\right). \tag{1}$$

In the next step Euler applied this algebraic result to the sine function using the Taylor series expansion

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots.$$

The last identity can be written as

$$\frac{\sin x}{x}=p(x), \quad \text{where } p(x)=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots.$$

For $x\neq 0$, as we can see, the constant term of $p(x)$ is 1. Now Euler was ready to make a big step. He applied his algebraic result (1) to the infinite polynomial $p(x)$. The function $\sin x/x$ is zero when $x=\pm k\pi$, where $k=1,2,\ldots$.

Therefore, said Euler,

$$\frac{\sin x}{x}=p(x)=\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1-\frac{x}{-3\pi}\right)\cdots,$$

so that

$$\frac{\sin x}{x}=\left(1-\frac{x^2}{1^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots. \tag{2}$$

This admittedly nonrigorous argument led to the beautiful and surprising result (2), which proved to be very useful, as we now illustrate.

In 1969 Raymond Redheffer proposed the following interesting inequality as a new problem proposal for *The American Mathematical Monthly*: prove that

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad \text{for all real numbers } x \neq 0. \tag{3}$$

The above inequality can be written equivalently in the form

$$\frac{\sin \pi x}{\pi x} \geq \frac{1 - x^2}{1 + x^2}, \quad \text{for all real numbers } x \neq 0. \tag{4}$$

Today, (3) is known as *Redheffer's inequality*. The proof of (3) was based on (2).

We can distinguish three cases: (i) $0 < x < 1$, (ii) $x \geq 1$, and (iii) $x < 0$.

For case (i), we work as follows. Replacing x by πx in (2) we get

$$\frac{\sin \pi x}{\pi x} = \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right) \cdots = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right). \tag{5}$$

It is enough to prove that

$$\frac{1 + x^2}{1 - x^2}\,\frac{\sin \pi x}{\pi x} \geq 1.$$

Let

$$P_n = \prod_{k=2}^{n}\left(1 - \frac{x^2}{k^2}\right), \quad \text{with } n \geq 2.$$

From (5) we can see that $\sin \pi x/\pi x = (1 - x^2)\lim_{n\to\infty} P_n$. Therefore,

$$\frac{1 + x^2}{1 - x^2}\,\frac{\sin \pi x}{\pi x} = (1 + x^2)\lim_{n\to\infty} P_n. \tag{6}$$

Since $P_{k+1} = P_k(1 - x^2/(k+1)^2)$, using simple induction we can prove that

$$(1 + x^2)P_n > 1 + \frac{x^2}{n}, \quad \text{for all } n \geq 2. \tag{7}$$

Combining (6) and (7) we have

$$\frac{1 + x^2}{1 - x^2}\,\frac{\sin \pi x}{\pi x} > 1,$$

i.e. the desired inequality (4).

For case (ii), we work as follows. For $x = 1$ we have equality in (4). For $x > 1$ we have

$$\frac{1 - x^2}{1 + x^2} - \frac{\sin \pi x}{\pi x} = \frac{1 - x^2}{1 + x^2} - \frac{\sin(\pi - \pi x)}{\pi x} = \frac{1 - x^2}{1 + x^2} + \frac{\sin(\pi x - \pi)}{\pi x - \pi}\,\frac{x - 1}{x}. \tag{8}$$

But $\sin x < x$ for all $x > 0$. Hence,

$$\frac{\sin(\pi x - \pi)}{\pi x - \pi} < 1.$$

Considering (8) and the last inequality we get

$$\frac{1-x^2}{1+x^2}-\frac{\sin \pi x}{\pi x}<\frac{1-x^2}{1+x^2}+\frac{x-1}{x}=-\frac{(x-1)^2}{(1+x^2)x}<0,$$

or $(1-x^2)/(1+x^2)<\sin \pi x/\pi x$, i.e. the desired inequality (4).

For case (iii), using cases (i) and (ii) we get

$$\frac{\sin x}{x}=\frac{\sin(-x)}{-x}\geq\frac{\pi^2-x^2}{\pi^2+x^2}.$$

Professor Raymond Redheffer passed away in May 2005 at the age of 84.

References

1 M. Kline, Euler and infinite series, *Math. Magazine* **56** (1983), pp. 307–314.

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5 J. P. Williams, A delightful inequality (solution to Problem 5642), *Amer. Math. Monthly* **76** (1969), pp. 1153–1154.

6 http://www.universityofcalifornia.edu/senate/inmemoriam/raymondredheffer.htm

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Yours sincerely,

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Dear Editor,

Sudoku, magic squares, circles, and triangles

I have constructed the following which readers may find interesting.

(i) A sudoku pattern in which reflections in the centre add up to 10 (see figure 1).

3	6	9	7	1	5	4	8	2
1	4	7	8	2	6	5	9	3
2	5	8	9	3	4	6	7	1
5	9	3	1	4	7	8	2	6
6	7	1	2	5	8	9	3	4
4	8	2	3	6	9	7	1	5
9	3	4	6	7	1	2	5	8
7	1	5	4	8	2	3	6	9
8	2	6	5	9	3	1	4	7

Figure 1

70	81	2	13	24	35	37	48	59
36	38	49	60	71	73	3	14	25
74	4	15	26	28	39	50	61	72
40	51	62	64	75	5	16	27	29
6	17	19	30	41	52	63	65	76
53	55	66	77	7	18	20	31	42
10	21	32	43	54	56	67	78	8
57	68	79	9	11	22	33	44	46
23	34	45	47	58	69	80	1	12

Figure 2

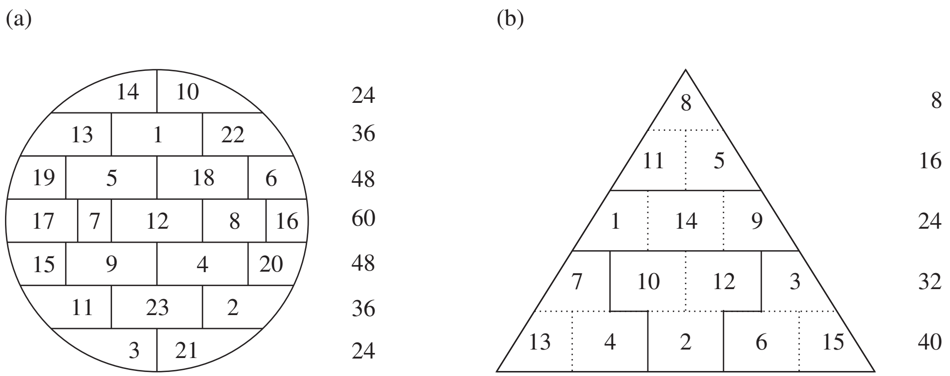

Figure 3

(ii) A 9×9 magic square using the numbers 1 to 81 in which reflections in the centre add up to 82 (see figure 2).

(iii) The numbers 1 to 23 arranged inside a circle as shown in figure 3(a) with row-sums increasing and decreasing multiples of 12.

(iv) The numbers 1 to 15 arranged in a triangle as shown in figure 3(b) with row-sums increasing multiples of 8 and the numbers in each of the five subdivisions adding up to 24.

Yours sincerely,

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Corrected reprint of Book Review 'Origins of Mathematical Words: A Comprehensive Dictionary of Latin, Greek, and Arabic Roots' by Roger Webster

We would like to apologize for the errors which appeared in this book review (see Volume 47, Number 2, pp. 94–95). Here is the corrected review, in full.

Origins of Mathematical Words: A Comprehensive Dictionary of Latin, Greek, and Arabic Roots. By A. Lo Bello. The Johns Hopkins University Press, Baltimore, MD, 2013. Paperback, 350 pages, £32.00 (ISBN 978-1-4214-1098-2).

Origins of Mathematical Words is a discursive etymological dictionary of mathematical terms with roots in Greek, Latin, or Arabic, the languages in which the larger-than-life author, Professor Anthony Lo Bello, has expertise. He provides Greek, Latin, and Arabic text in its original form to enhance each explanation and avoid doubtful consequences of transliteration. The book is about words, mathematical words, how they were constructed, their history, and how they are used today. A typical entry begins as follows.

> **conchoid** The Greek noun $\kappa o \gamma \chi \eta$ means *mussel* or *shell*, and $\epsilon \iota \delta o \varsigma$ means *shape*. The conchoid is therefore a *shell-shaped curve*. It is a curve first studied by Nicomedes (*circa* 225 B.C.), which, if allowed, permits the trisection of an arbitrary angle.

Then follows a formal definition of the curve, some general comments, and Pascal's generalization of it. A few entries demand an understanding of Greek, Latin, and Arabic grammar, which most readers, including this reviewer, do not have.

This book is not your common-or-garden reference work, where the author seldom graces its pages; unlike here, where the reader is constantly confronted by him, sitting in judgement on a raft of issues, expressing his views in no uncertain terms. He deplores the use of *macaronic* words, those formed from different languages; an infallible sign of a defective education, we are told! Of many examples, he offers **septagon** (a learned mistake for **heptagon**, combining Latin and Greek roots), **nonagon** (an absurd word, used by the unlearned for *enneagon*), and **cohomology** (a bad word, a corruption of the first two letters of a Latin prefix added to a word of Greek origin). Other examples of his red rags are: acronyms; the use of letters to name mathematical objects, e.g. **CW complex**; the regrettable abbreviation **math** (no less silly, than it is natural); and the noncapitalization of adjectives formed from proper names, such as **Abelian** (for otherwise they look ridiculous). Noah Webster is censured for the crime of changing many English spellings in his American dictionary of 1806, thereby severing words from their roots. No one escapes criticism, even the *Oxford English Dictionary* receives a slap on the wrist for including *incommutative* (not a good word, a cautious fellow may call it *rare*, a frank one, *wrong*).

The author's authoritarian exposition can at times be intimidating, so it is comforting to know that even he, like Homer, sometimes nods. For example, the last sentence of the entry for **abundant numbers** should be in that for **amicable numbers**; and the entry for **nonagon** refers to a nonexistent one for *enneagon*. How ironic that a book that is so particular about mathematical vocabulary should pay so little attention to mathematical typography, resulting in such mathematical monstrosities as

$$//x + y// \leq //x// + //y// \quad \text{and} \quad //ax// \leq /a/ \, //x//$$

in the entry for **pseudonorm**. Whilst the author deplores the ludicrous mixing of upper- and lower-case letters in the word LaTeX, a quality typesetting system, had he employed it, would have given the book the fully professional appearance that it deserves.

This sophisticated, one-of-a-kind dictionary, based on decades of painstaking research by its charismatic author, will delight mathematicians and word lovers alike. It is a treasure chest of rare gems, a godsend to all teachers of the subject and its history. But how to unearth these nuggets, tucked away in the **a** to **z** format of a dictionary? My own solution, to read the book from cover to cover, struck gold immediately, with the feisty first entry **a-**, covering almost three pages – the totality of *all* entries beginning **w**, **x**, **y**, or **z** fills just over a page! Lo Bello's scholarship, combined with his refreshingly personal exposition, brings to mind Briggs' remark on reading Napier's *Mirifici Logarithmorum Canonis Descriptio* in 1614, 'I never saw a book which pleased me better'.

University of Sheffield **Roger Webster**

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

47.9 Solve the equation $e^x = x^{3x}$ for $x > 0$.

(Submitted by Sidney Edwards)

47.10 Let a, b, c, respectively, be the length, breadth, and height of a rectangular box. Prove that

$$\sqrt{2} < \frac{\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}}{\sqrt{a^2+b^2+c^2}} \le \sqrt{6}.$$

(Submitted by Prithwijit De, Mumbai, India)

47.11 For positive real numbers $a_1, a_2, \ldots, a_n$ $(n \ge 1)$, prove that the following inequality holds:

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} \ge (a_1 a_2 \cdots a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

When does equality occur?

(Submitted by J. A. Scott, Chippenham, UK)

47.12 The point P lies inside the tetrahedron $A_1A_2A_3A_4$. The distance of A_i from the opposite face is denoted by h_i and d_i denotes the distance of P from that face. Prove that

$$\frac{d_1}{h_1}+\frac{d_2}{h_2}+\frac{d_3}{h_3}+\frac{d_4}{h_4}=1.$$

(Submitted by Zhang Yun, Xi An City, China)

Solutions to Problems in Volume 47 Number 1

47.1 Let f be a continuous positive function on the interval $[0, 1]$. Prove that

(i)

$$\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x=\frac{1}{2},$$

(ii)

$$\int_0^1 \frac{f(x)}{f(1-x)}\,\mathrm{d}x\geq 1,$$

(iii) if $f''(x)\geq 0$ for all $x\in[0,1]$ then

$$\int_0^1 f(x)\,\mathrm{d}x\geq f(\tfrac{1}{2}),$$

and if $f''(x)\leq 0$ for all $x\in[0,1]$ then the inequality is reversed.

Solution by Henry Ricardo (New York Math Circle) and Angel Plaza (Universidad de Las Palmas de Gran Canaria), independently

For part (i), the substitution $x=1-y$ gives

$$I=\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x=-\int_1^0 \frac{f(1-y)}{f(1-y)+f(y)}\,\mathrm{d}y=\int_0^1 \frac{f(1-x)}{f(x)+f(1-x)}\,\mathrm{d}x,$$

so that

$$2I=\int_0^1 \frac{f(x)+f(1-x)}{f(x)+f(1-x)}\,\mathrm{d}x=1.$$

For part (ii), we have

$$J=\int_0^1 \frac{f(x)}{f(1-x)}\,\mathrm{d}x=-\int_1^0 \frac{f(1-y)}{f(y)}\,\mathrm{d}y=\int_0^1 \frac{f(1-x)}{f(x)}\,\mathrm{d}x$$

and

$$\frac{f(x)}{f(1-x)}+\frac{f(1-x)}{f(x)}\geq 2,$$

so that

$$2J=\int_0^1\left(\frac{f(x)}{f(1-x)}+\frac{f(1-x)}{f(x)}\right)\mathrm{d}x\geq\int_0^1 2\,\mathrm{d}x=2.$$

Part (iii) follows from the Hermite–Hadamard inequality, namely

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geq f\left(\frac{a+b}{2}\right)$$

for a convex function f, with the inequality reversed for a concave function. Alternatively, as suggested by Spiros P. Andriopoulos, who proposed the problem,

$$\frac{f(x)+f(1-x)}{2} \geq f\left(\frac{1}{2}\right),$$

because $f''(x) \geq 0$ for all $x \in [0, 1]$, so that

$$\frac{f(x)}{f(x)+f(1-x)} \leq \frac{f(x)}{2f(\frac{1}{2})}$$

and

$$\frac{1}{2} = \int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x \leq \frac{1}{2f(\frac{1}{2})}\int_0^1 f(x)\,\mathrm{d}x,$$

which gives the result. If $f''(x) \leq 0$ for $x \in [0, 1]$, the inequalities are reversed.

47.2 A circle of radius n touches the x-axis and the line $y = nx$. Determine the locus of the centre of the circle as n varies.

Solution by Annanay Kapila (Year 12, Nottingham High School, UK)

First take the circle to lie in the first quadrant. The straight line joining the origin to the centre of the circle bisects the angle 2θ that the line $y = nx$ makes with the x-axis, so the coordinates of the centre are (a, n), where $\tan\theta = n/a$ and $\tan 2\theta = n$. Thus,

$$\frac{2\tan\theta}{1-\tan^2\theta} = n;$$

whence,

$$n\tan^2\theta + 2\tan\theta - n = 0$$

and

$$\tan\theta = \frac{-1+\sqrt{1+n^2}}{n}.$$

Thus,

$$\frac{n}{a} = \tan\theta = \frac{-1+\sqrt{1+n^2}}{n} \quad \text{and} \quad a = \frac{n^2}{-1+\sqrt{1+n^2}} = 1+\sqrt{1+n^2}.$$

Hence, the centre (a, n) lies on the curve with equation

$$x = 1+\sqrt{1+y^2},$$

so the locus is that part of the rectangular hyperbola

$$(x-1)^2 - y^2 = 1$$

in the fist quadrant.

If the circle lies in the second quadrant, the centre will be at $(-a, n)$, where

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{n}{a} \quad \text{and} \quad \tan 2\theta = n.$$

This time

$$\frac{n}{a} = \cot\theta = \frac{n}{-1+\sqrt{1+n^2}},$$

so that $a = -1 + \sqrt{1+n^2}$ and the centre $(-a, n)$ lies on the curve with equation

$$x = 1 - \sqrt{1+y^2}.$$

Hence, the locus will be that part of the same rectangular hyperbola which lies in the second quadrant.

The analysis when the circle lies in the third and fourth quadrants is similar. Now the locus of the centre is that part of the rectangular hyperbola

$$(x+1)^2 - y^2 = 1$$

which lies in the third and fourth quadrants.

Also solved by Angel Plaza for the first quadrant.

47.3 For positive real numbers a, b, c, prove that

$$(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9,$$

and generalize.

Solution by Sprios P. Andriopoulos, Third High School of Amaliada, Eleia, Greece

We prove that

$$(a_1^2+\cdots+a_n^2)\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right) \geq (a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) \geq n^2,$$

for positive real numbers $a_1, \ldots, a_n$. First note that

$$x + \frac{1}{x} \geq 2, \quad \text{for all } x > 0,$$

and $y^2 - y - 2 \geq 0$ when $y \geq 2$. Hence,

$$\left(x+\frac{1}{x}\right)^2 - \left(x+\frac{1}{x}\right) - 2 \geq 0, \quad \text{for all } x > 0;$$

whence,

$$x^2 + \frac{1}{x^2} \geq x + \frac{1}{x}.$$

Thus, for $i \neq j$,

$$\frac{a_i^2}{a_j^2} + \frac{a_j^2}{a_i^2} \geq \frac{a_i}{a_j} + \frac{a_j}{a_i};$$

whence,

$$\begin{aligned}(a_1 + \cdots + a_n^2)\left(\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}\right) &= n + \sum_{i<j}\left(\frac{a_i^2}{a_j^2} + \frac{a_j^2}{a_i^2}\right) \\ &\geq n + \sum_{i<j}\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) \\ &= (a_1 + \cdots + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \\ &\geq n(a_1 \cdots a_n)^{1/n} n\left(\frac{a}{a_1 \cdots a_n}\right)^{1/n} \\ &= n^2,\end{aligned}$$

by the arithmetic mean–geometric mean inequality.

Annanay Kapila pointed out that

$$(a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0,$$

so that

$$2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc);$$

whence,

$$a^2 + b^2 + c^2 \geq ab + ac + bc.$$

Hence,

$$\begin{aligned}(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) &\geq (ab + ac + bc)\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) \\ &= abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\frac{1}{abc}(a + b + c) \\ &= (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).\end{aligned}$$

Also solved by Angel Plaza.

47.4 Prove that a number which is both triangular and oblong is divisible by 6 and has units digit 0 or 6.

Solution

Let

$$n = \tfrac{1}{2}a(a+1) = b(b+1)$$

for some positive integers a, b. Then n is even. Also, $n = b(b+1) \equiv 0 \times 1$ or 1×2 or $2 \times 3 \pmod 3$, i.e. $n \equiv 0$ or $2 \pmod 3$. Suppose that $n \equiv 2 \pmod 3$. Then $a(a+1) \equiv 2 \times 2 \equiv 1 \pmod 3$, which is impossible. Hence, $n \equiv 0 \pmod 3$. So n is divisible by 2 and 3, and hence by 6.

Also, $n = b(b+1) \equiv 0 \times 1$ or 1×2 or … or $9 \times 10 \pmod{10}$, i.e. $n \equiv 0, 2,$ or $6 \pmod{10}$. Suppose that $n \equiv 2 \pmod{10}$. Then $a(a+1) \equiv 2 \times 2 \equiv 4 \pmod{10}$, which is impossible. Hence, $n \equiv 0$ or $6 \pmod{10}$, i.e. its units digit is 0 or 6.

Reviews

The First Twenty-Five Years of the Superbrain. By D. Early and D. MacHale. United Kingdom Mathematics Trust, Leeds, 2014. Paperback, 230 pages, £18.00 (ISBN 978-1906001223).

Keeping undergraduate mathematicians excited can be a difficult task. For thirty years now, University College Cork has had an exciting in-house solution to this problem: the annual Superbrain competition. This is the brainchild of Des MacHale, who is best known in some circles for the book *Comic Sections: A Contribution to the Literature of Mathematical Humour*.

The book under review documents all the problems and solutions from the first twenty-five years of Superbrain competitions. In its preparation, Professor MacHale was ably helped by Diarmuid Early, a past champion of three Superbrains.

The easier problems have a habit of being quirky and surprising, but some are written with an eye on practicality. Examples include, ‘Given the coordinates of three distinct points A, B, C in the plane, give ten essentially different methods of deciding whether or not A, B, and C lie on the same line’, and ‘What is the minimum number of different calendars it is necessary to have to cover every year?’.

The harder problems frequently showcase techniques for evaluating special cases in algebra, calculus, or coordinate geometry, and as such can be instructive. The solutions are generous, and worth reading: some comment on applications, or other points of interest arising from the problems.

The syllabus required to understand all the problems rarely strays beyond Further Maths A-level, and as a result many problems will be as suitable for bored sixth-formers as bored undergraduates. I shall be looking forward to a book describing the next twenty-five years.

United Kingdom Mathematics Trust **James Cranch**

50 Visions of Mathematics. Edited by S. Parc. Oxford University Press, 2014. Hardback, 208 pages, £24.99 (ISBN 978-0198701811).

This book marks the 50th anniversary of the founding of the Institute of Mathematics and its Applications. The book consists of 50 articles, each two to six pages in length, written by 50 different authors with a forward by Dara O’Briain. The articles are entertaining and accessible to mathematicians and nonmathematicians alike. From ants walking towards one another, to mathematical references in the Simpsons, via mathematician’s biographies and the tower of Hanoi, this book comprises a motley collection of enjoyable mathematical stories.

The length and depth of each article makes it an ideal source for quick fixes of mathematical entertainment. If you’re mathematically minded then you should enjoy dipping in and out of this book.

University of Sheffield **Fionntan Roukema**

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