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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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THE OLYMPIAD CORNER

No. 135

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with the problems of the 1991 Australian Mathematical Olympiad which was written February 12 and 13, 1991. Many thanks to Hans Lausch of Melbourne who sent them to Andy Liu, who then forwarded the contests to me.

1991 AUSTRALIAN MATHEMATICAL OLYMPIAD

Paper I — February 12, 1991 (Time: 4 hours)

1. Let $ABCD$ be a convex quadrilateral. Prove that if g is the greatest and h is the least of the distances AB, AC, AD, BC, BD, CD, then $g \geq h\sqrt{2}$.

2. Let M_n be the least common multiple of the numbers $1, 2, 3, \ldots, n$; e.g., $M_1 = 1$, $M_2 = 2$, $M_3 = 6$, $M_4 = 12$, $M_5 = 60$, $M_6 = 60$. For which positive integers n does $M_{n-1} = M_n$ hold? Prove your claim.

3. Let A, B, C be three points in the xy-plane and X, Y, Z the midpoints of the line segments AB, BC, AC respectively. Furthermore, let P be a point on the line BC so that $\angle CPZ = \angle YXZ$. Prove that AP and BC intersect in a right angle.

4. Prove that there is precisely one function f, that is defined for all non-zero reals, satisfying

(a) $f(x) = xf(1/x)$, for all non-zero reals x; and

(b) $f(x) + f(y) = 1 + f(x + y)$, for each pair (x, y) of non-zero reals where $x \neq -y$.

Paper II — February 13, 1991 (Time: 4 hours)

5. Let $P_1, P_2, \ldots, P_n$ be n different points in a given plane such that each triangle $P_iP_jP_k$ $(i \neq j \neq k \neq i)$ has an area not greater than 1. Prove that there exists a triangle Δ in this plane such that

(a) Δ has an area not greater than 4; and

(b) each of the points $P_1, P_2, \ldots, P_n$ lies in the interior or on the boundary of Δ.

6. For each positive integer n, let

$$f(n) = \frac{1}{\sqrt[3]{n^2+2n+1} + \sqrt[3]{n^2-1} + \sqrt[3]{n^2-2n+1}} .$$

Determine the value of $f(1) + f(3) + f(5) + \cdots + f(999997) + f(999999)$.

7. In triangle ABC, let M be the midpoint of BC, and let P and R be points on AB and AC respectively. Let Q be the intersection of AM and PR. Prove that if Q is the midpoint of PR, then PR is parallel to BC.

8. Find a sequence $a_0, a_1, a_2, \ldots$ whose elements are positive and such that $a_0 = 1$ and $a_n - a_{n+1} = a_{n+2}$ for $n = 0, 1, 2, \ldots$. Prove that there is only one such sequence.
[*Editor's note.* This problem is essentially the same as *Crux* 1378 [1989: 306].]

* * *

The next Olympiad we give was forwarded to us by Xiong Bin and Huang Tu Sheng of the Department of Mathematics, East China Normal University, Shanghai, China.

THE FIRST UNITED MATHEMATICS OLYMPIAD OF SENIOR NORMAL UNIVERSITY

Zhejiang, China

December 16, 1990 (Time: 3 hours)

1. Find all real solutions (x, y, z) satisfying the simultaneous equations

$$(x+2y)(x-2z) = 24$$
$$(y+2x)(y-2z) = -24$$
$$(z-2x)(z-2y) = -11.$$

2. The letters a, b, c, d, e have been written on the blackboard. Altogether 1990 letters have been written, but a given letter may have no, some or many occurrences. The following rule may be used to reduce the number of letters on the board by one.

Choose any two letters occurring on the board, erase each of them and write the letter which occurs at the intersection of the row determined by the first letter erased and the column for the second letter erased:

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	c	b	e	d
d	a	d	e	c	b
e	a	e	d	b	c

For example if c and e are erased, in order, we take the row for c and column for e and at the intersection find a d. So the c and e are replaced by d.

Does the last letter left on the board (after many repetitions of this operation) depend in any way on the order of the choices? Prove your conclusion.

3. The point P in the square $ABCD$ satisfies the following conditions:
(1) $PA < PB < PD < PC$
(2) $PA + PC = 2PB$
(3) $(PD)^2 = PB \times PC$.
Prove that such a point P exists, and is unique.

4. Partition the set $A = \{1, 2, 3, \ldots, n\}$ into two subsets I_1 and I_2, satisfying the following conditions:
(1) $I_1 \cap I_2 = \emptyset$
(2) $I_1 \cup I_2 = A$
(3) Any three numbers in I_1 (I_2 respectively), do not form an arithmetic progression.
What is the maximum value of n for which such a partition is possible?

5. AD', BE', CF' are the three medians of the non-obtuse triangle ABC with centroid G. They respectively intersect at D, E, F with the circumcircle of triangle ABC. Prove that $GD + GE + GF \geq 8R/3$.

6. Let n ($n \geq 2$) points be given in the plane. Let $0 < a \leq 1$. Prove that the points can always be covered by circles which don't intersect each other, and for which the sum of the diameters is less than $n - a + 1$, while the minimum distance between any two of the circles is at least a. (The minimum distance is the distance between the two closest points on the two circles.)

* * *

Last issue we gave the problems of the A.I.M.E. for 1992. As promised, we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A., 68588–0322.

1. 400	2. 502	3. 164	4. 062	5. 660
6. 156	7. 320	8. 819	9. 164	10. 572
11. 945	12. 792	13. 820	14. 094	15. 396

* * *

We now turn to solutions and comments of readers. First a correction: Lászlo Kőszegi writes pointing out that the 8 problems translated from *KVANT* used in the June 1991 issue of *Crux* were the problems of the *15th All Russian Mathematical Olympiad*, which is different from the 15th All Union Mathematical Olympiad which was actually held in 1981. Somehow we got the wrong Olympiad named!

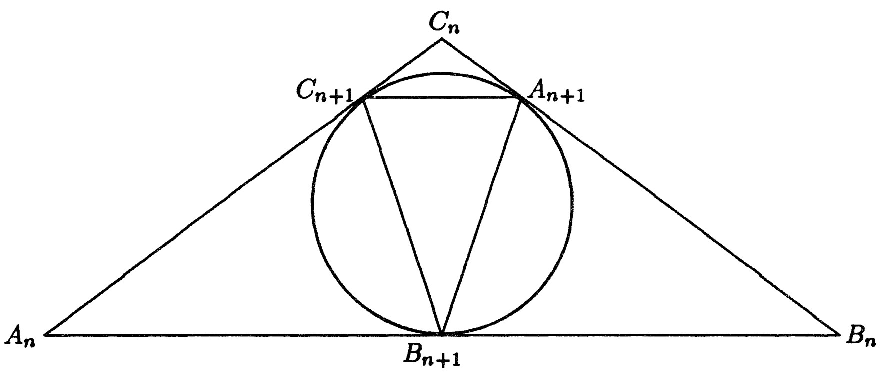
* * *

In the February number I issued a challenge to send in solutions to problems 2, 4 and 6 of the 25th Spanish Mathematical Olympiad in order to complete the set. One quick reader rose to the occasion. However, in the interim a solution to problem 2 was published the next month [1992: 67].

4. [1990: 226] *25th Spanish Mathematics Olympiad.*

The angles of triangle $A_1B_1C_1$ are given. For each natural number n, let A_{n+1}, B_{n+1}, C_{n+1} be the points of contact of the incircle of $\Delta A_nB_nC_n$ with the sides B_nC_n, C_nA_n, A_nB_n, respectively. Determine, in terms of n, the angles of the triangle $A_nB_nC_n$.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.



Let $\alpha_n, \beta_n, \gamma_n$ be the angles of the triangle $A_nB_nC_n$ at A_n, B_n, C_n, respectively. We claim

$$\alpha_{n+1} = 90^\circ - \frac{1}{2}\alpha_n, \quad \beta_{n+1} = 90^\circ - \frac{1}{2}\beta_n, \quad \gamma_{n+1} = 90^\circ - \frac{1}{2}\gamma_n.$$

To see this,

$$\angle A_nC_{n+1}B_{n+1} = \angle A_nB_{n+1}C_{n+1} = 90^\circ - \frac{1}{2}\alpha_n,$$
$$\angle C_nC_{n+1}A_{n+1} = \angle C_nA_{n+1}C_{n+1} = 90^\circ - \frac{1}{2}\beta_n,$$

and thus

$$\gamma_{n+1} = 180^\circ - \angle A_nC_{n+1}B_{n+1} - \angle C_nC_{n+1}A_{n+1} = \frac{1}{2}(\alpha_n + \beta_n) = 90^\circ - \frac{1}{2}\gamma_n.$$

The other equalities follow similarly. Thus

$$\begin{aligned}\alpha_n &= 90^\circ - \frac{1}{2}\alpha_{n-1} = 90^\circ - \frac{90^\circ}{2} + \frac{\alpha_{n-2}}{4} = 90^\circ - \frac{90^\circ}{2} + \frac{90^\circ}{4} - \frac{\alpha_{n-3}}{8} \\ &= 90^\circ\left[1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \left(-\frac{1}{2}\right)^{n-2}\right] + \left(-\frac{1}{2}\right)^{n-1}\alpha_1.\end{aligned}$$

This gives

$$\alpha_n = 90^\circ\left(\frac{1-(-1/2)^{n-1}}{3/2}\right) + \left(-\frac{1}{2}\right)^{n-1}\alpha_1,$$

$$\beta_n = 90^\circ\left(\frac{1-(-1/2)^{n-1}}{3/2}\right) + \left(-\frac{1}{2}\right)^{n-1}\beta_1,$$

$$\gamma_n = 90^\circ\left(\frac{1-(-1/2)^{n-1}}{3/2}\right) + \left(-\frac{1}{2}\right)^{n-1}\gamma_1.$$

Notice that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\gamma_n = 60^\circ$. Thus the triangles become more equilateral in the limit.

6. [1990: 226] *25th Spanish Mathematics Olympiad.*

Several collections of natural numbers are written, such that: (a) in each collection, each one of the digits $0, 1, 2, \ldots, 9$ is used once; (b) no number in a collection has 0 as left-most digit; (c) the sum of the numbers in each collection is less than 75. How many such collections can be written in this way? (The order in which the numbers of a collection are written is immaterial.)

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

The following statements then follow:

1. No number in any collection can be a three or more digit number.
2. The collection with the smallest possible sum is $(2, 3, 4, 5, 6, 7, 8, 9, 10)$.
3. The sum of each collection is divisible by 9.
4. The only possible sums are 54, 63, and 72.
5. The "0" must be the rightmost digit of a two-digit number.

We may now enumerate the possibilities.

There are eight collections which use 10:

$$\begin{gathered}(10, 2, 3, 4, 5, 6, 7, 8, 9)\\(10, 23, 4, 5, 6, 7, 8, 9)\\(10, 24, 3, 5, 6, 7, 8, 9)\\\ldots\\(10, 29, 3, 4, 5, 6, 7, 8).\end{gathered}$$

There are eight more collections using 20:

$$\begin{gathered}(20, 1, 3, 4, 5, 6, 7, 8, 9)\\(20, 13, 4, 5, 6, 7, 8, 9)\\(20, 14, 3, 5, 6, 7, 8, 9)\\\ldots\\(20, 19, 3, 4, 5, 6, 7, 8).\end{gathered}$$

And there is one collection using 30:

$$(30, 1, 2, 4, 5, 6, 7, 8, 9).$$

This gives a total of 17 collections.

*　　*　　*

I received another envelope of solutions from M. Selby, University of Windsor, that arrived after the April number was finalized. It included solutions to five problems for which the solutions were discussed in that number, namely numbers 2 and 4 of the 1980 Chinese New Year Contest and numbers 1, 2 and 3 of the 1981 Chinese New Year Contest. The mail also included a package of solutions from Pavlos Maragoudakis, student, University of Athens, Greece. It included solutions to number 4 of the 1980 Chinese New Year Contest, and numbers 1, 2 and 3 of the 1981 Chinese New Year Contest. Two solutions to problem 1 of the 12th Austrian-Polish Mathematics Competition (discussed last issue [1992: 104]) were also in the mail. The solvers were Murray S. Klamkin, University of Alberta, and Selby.

* * *

Next we give solutions to the three problems of the Team Competition of the *12th Austrian-Polish Mathematics Competition* [1991: 66].

7. Functions $f_0, f_1, f_2, \ldots$ are recursively defined by

(1) $f_0(x) = x$, for $x \in \mathbf{R}$;

(2) $f_{2k+1}(x) = 3^{f_{2k}(x)}$, where $x \in \mathbf{R}$, $k = 0, 1, 2, \ldots$;

(3) $f_{2k}(x) = 2^{f_{2k-1}(x)}$, where $x \in \mathbf{R}$, $k = 1, 2, 3, \ldots$.

Determine (with proof) the greater one of the numbers $f_{10}(1)$ and $f_9(2)$.

Solutions by Margherita Barile, student, Genova, Italy; and Pavlos Maragoudakis, student, University of Athens, Greece. We give Maragoudakis's solution.

It is easy to check that $f_0(1) = 1$, $f_1(1) = 3$, $f_2(1) = 2^3 = 8$, $f_3(1) = 3^8 = 6561$ and $f_0(2) = 2$, $f_1(2) = 3^2 = 9$, $f_2(2) = 2^9 = 512$. We observe that $f_3(1) > 2f_2(2)$. Now

$$2^{f_3(1)} > 2^{2\cdot f_2(2)} = 4^{f_2(2)} > 3^{f_2(2)}$$

which implies $f_4(1) > f_3(2)$ and so

$$3^{f_4(1)} > 3^{f_3(2)}. \tag{1}$$

It is clear that if $k \in \mathbf{N}$ then $3^k > 2 \cdot 2^k \Leftrightarrow k \geq 2$. Also it is clear that $f_n(2) \geq 2$ for all $n \in \mathbf{N}$. Thus (1) implies that $3^{f_4(1)} > 2 \cdot 2^{f_3(2)}$ or $f_5(1) > 2f_4(2)$. In this manner one can prove, by induction, that

$$f_{2k+1}(1) > 2f_{2k}(2), \quad k = 1, 2, 3, \ldots ,$$
$$f_{2k}(1) > f_{2k-1}(2), \quad k = 2, 3, 4, \ldots .$$

Therefore $f_{10}(1) > f_9(2)$.

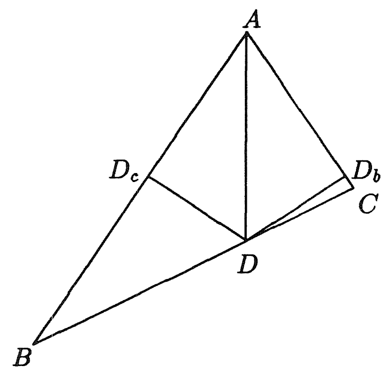
8. We are given an acute triangle ABC. For each point P of the interior or boundary of ABC let P_a, P_b, P_c be the orthogonal projections of P to the sides a, b and c, respectively. For such points we define the function

$$f(P) = \frac{\overline{AP_c} + \overline{BP_a} + \overline{CP_b}}{\overline{PP_a} + \overline{PP_b} + \overline{PP_c}} .$$

Show that $f(P)$ is constant if and only if ABC is an equilateral triangle.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and Pavlos Maragoudakis, student, University of Athens, Greece. The solution given is from Maragoudakis.

Consider the points D, E, F on sides BC, AC, AB respectively such that AD, BE, CF are the bisectors of the angles at A, B, C, respectively. Let $a = \overline{BC}$, $b = \overline{AC}$, and $c = \overline{AB}$. Then



$$f(D) = \frac{\overline{AD_c} + \overline{BD} + \overline{CD_b}}{\overline{DD_c} + \overline{DD_b}} .$$

We know that $\overline{AD_c} = \overline{AD_b}$, so $\overline{AD_c} + \overline{CD_b} = b$ and $\overline{DD_c} = \overline{DD_b}$, so

$$f(D) = \frac{b + \overline{BD}}{2\overline{DD_c}} .$$

Furthermore $\overline{DD_c} = \overline{BD} \cdot \sin B$ and $b/(2R) = \sin B$, where R is the circumradius. Thus

$$f(D) = \frac{1}{2\sin B} \cdot \frac{b + \overline{BD}}{\overline{BD}} = \frac{R}{b}\left(\frac{b}{\overline{BD}} + 1\right) .$$

But $\overline{BD}/\overline{DC} = c/b \Rightarrow \overline{BD} = ac/(b+c)$, and $\Delta = abc/(4R)$ where Δ is the area of ABC. Finally we obtain

$$f(D) = \frac{b^2 + bc + ac}{4\Delta} ,$$

and cyclically

$$f(E) = \frac{c^2 + ac + ab}{4\Delta}, \quad f(F) = \frac{a^2 + ab + bc}{4\Delta} .$$

If $f(P)$ is constant then

$$b^2 + bc + ac = c^2 + ac + ab = a^2 + ab + bc.$$

From this

$$b^2 - ab + (ab + bc + ac) = c^2 - bc + (ab + bc + ac) = a^2 - ac + (ab + bc + ca)$$

and $b^2 - ab = c^2 - bc = a^2 - ac = l$, say. Thus

$$\frac{l}{b} = b - a, \quad \frac{l}{c} = c - b, \quad \frac{l}{a} = a - c,$$

$$l\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = b - a + c - b + a - c = 0,$$

and so $l = 0$. Hence $b - a = c - b = 0$ and $a = b = c$.

Conversely if ABC is equilateral then

$$\overline{PP_a} + \overline{PP_b} + \overline{PP_c} = h_a = \frac{\sqrt{3}}{2}a \quad \text{and} \quad \overline{AP_c} + \overline{BP_a} + \overline{CP_b} = \frac{3}{2}a,$$

so $f(P) = \sqrt{3}$.

9. Determine the smallest odd natural number N such that N^2 is the sum of an odd number (> 1) of squares of adjacent natural numbers (> 0).

Solutions by Margherita Barile, student, Genova, Italy; and Pavlos Maragoudakis, student, University of Athens, Greece.

We are looking for a positive integer p such that

$$N^2 = \sum_{k=-p}^{p} (a+k)^2 = (2p+1)a^2 + 2\sum_{k=1}^{p} k^2 = (2p+1)a^2 + \frac{p(p+1)(2p+1)}{3} \tag{1}$$

with $a \in \mathbf{N}$, $a > p$ and N least possible. We consider the possible values of p.

$p = 1$. In this case (1) gives $N^2 = 3a^2+2$ whence $N^2 \equiv 2 \bmod 3$, which is impossible, since N^2 is 0 or 1 mod 3.

$p = 2$. From (1) $N^2 = 5a^2 + 10 = 5(a^2 + 2)$. Thus $5|N$ and so $5|a^2 + 2$. Thus $a^2 \equiv 3 \bmod 5$, which is impossible.

$p = 3$. Now $N^2 = 7a^2 + 28 = 7(a^2 + 4)$. This is again impossible, because it implies that $7|(a^2 + 4)$ so that $a^2 \equiv 3 \bmod 7$, an impossibility.

$p = 4$. $N^2 = 9a^2+60 = 3(3a^2+20)$. This too is impossible, since $3|N^2 \Rightarrow 3|3a^2+20$, so $3|20$, a contradiction.

$p = 5$. $N^2 = 11a^2 + 110 = 11(a^2 + 10)$. This implies $a^2 \equiv 1 \bmod 11$. This occurs if and only if $a \equiv 1$ or $a \equiv 10 \bmod 11$. Since $a > 5$, we start by examining $a = 10$. This gives $11(a^2 + 10) = 11 \cdot 110 = 11^2 \cdot 10$, which is not a square. Continuing the trials:

a	$11(a^2+10)$
$11 + 1 = 12$	$11^2 \cdot 2 \cdot 7$
$11 + 10 = 21$	$11^2 \cdot 41$
$2 \cdot 11 + 1 = 23$	$11^2 \cdot 7^2$

Thus $N = 11 \cdot 7 = 77$ is a solution, with the required decomposition:

$$77^2 = 18^2 + 19^2 + 20^2 + 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2.$$

It is the smallest solution for $p \le 5$, and so in the following we restrict to $N < 77$.

$p = 6$. Then $N^2 = 13(a^2 + 14)$. Thus $N = 13x$ where $1 \le x \le 5$, since $N < 77$. So $a^2 = 13x^2 - 14$. Now $a \ge 7$ gives $x \ge 3$. But $x = 3$ gives $a^2 = 103$, $x = 4$ has $a^2 = 194$ and $x = 5$ yields $a^2 = 311$, which are not perfect squares.

$p = 7$. Then $n^2 = 5(3a^2 + 56)$ so $3a^2 + 1 \equiv 0 \bmod 5$, which is a contradiction as $a^2 \equiv 0$, 1, or 4 mod 5, and $3a^2 + 1 \equiv 1$, 3 or 4 mod 5.

$p = 8$. Then $N^2 = 17(a^2 + 24)$. Writing $N = 17x$ we get $x \le 4$. $x = 1, 2, 3, 4$ gives $a^2 = -7, 44, 109, 249$, which are not perfect squares.

$p = 9$. Then $N^2 = 19(a^2 + 30)$. Writing $N = 19x$ we have $x \le 4$ so $a^2 = 19x^2 - 30$ and checking $x = 1, 2, 3, 4$ we get $a^2 = -11, 46, 141, 274$, again impossible.

$p = 10$. Then $N^2 = 21a^2 + 770$. This implies that $7|N$, so $3a^2 + 110 \equiv 0 \bmod 7$. But $3a^2 + 5 \equiv 0 \bmod 7$ is impossible since $a^2 \equiv 0, 1, 2$ or $4 \bmod 7$.

$p = 11$. Then $N^2 = 23(a^2 + 44)$. Writing $N = 23x$ we get $x \le 3$. Now $x = 1, 2, 3$ give $a^2 = -21, 48, 163$, respectively, which is impossible.

$p = 12$. Then $N^2 = 25a^2 + 1300$. Now $a \ge p + 1$ so $N^2 \ge 25 \cdot 13^2 + 1300$, so $N^2 \ge 5525$. This makes $N \ge 75$. Since we seek $N < 77$, and since 5 divides N, we consider $N = 75$. This does not work since $a^2 = 273$, a contradiction.

$p \ge 13$. Since $a \ge p + 1$ we have $N^2 \ge 27 \cdot 14^2 + 1638 > 77^2$.

Thus 77 is the least solution.

* * *

That's all the space available this month. Send me your nice solutions and contests.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

From Zero to Infinity, by Constance Reid, published by the Mathematical Association of America, Washington, 1992, fourth edition, ISBN 0-88385-505-4, softcover, 168 pages, US$19 ($14 for MAA members). *Reviewed by Andy Liu, University of Alberta.*

This is an updated version of a classic by one of the best and best known mathematical biographers. It is in some sense a collection of twelve biographies, but of numbers! In each chapter, a particular number comes to life before our eyes as we trace through its history and development, and learn about concepts and problems associated with it. Here is the cast, with their primary roles and principal scenes:

0	additive identity	positional values
1	multiplicative identity	unique factorization
2	smallest meaningful base	binary arithmetic
3	first odd prime	Eratosthenes' sieve
4	square number	Pythagorean triples
5	pentagonal number	partition problems
6	first perfect number	Mersenne primes
7	relatively prime to 1–10	Fermat primes
8	cubic number	Waring's problem
9	largest digit	digital roots
e	base of natural logarithm	prime distribution
$\aleph_0$	infinite cardinal number	denumerable sets

This is a book one can read and enjoy from cover to cover in one sitting. There are short quizzes scattered throughout, with answers printed upside down immediately following. They serve as a reminder that one is reading a narrative account on serious mathematics.

After this, one is likely to return to the book again and again, with paper and pencil at hand, to explore further into doorways which have been bypassed on the initial tour. It is unfortunate that the book does not have a bibliography. There are no references other than those contained in occasional footnotes.

While the book is apparently not written as a text, it should make a very stimulating one, for instance in an introductory course for liberal arts and education students. An instructor would have to supplement it with an ample selection of examples and exercises, as well as notes to elaborate certain points. However, having the students excited about the subject matter is well worth the extra effort.

In the preface to this edition, the author gives the reader a glimpse of her mathematical life as well as that of her younger sister, the late Julia Robinson. Both serve as inspiring role models for female students with aspirations in mathematics, although their contributions are significant without reference to the gender issue. It is hoped that this book will help make gender in mathematics an irrelevant issue some day. It is highly recommended for all.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1992,** *although solutions received after that date will also be considered until the time when a solution is published.*

1741. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex cyclic quadrilateral, and P is an interior point of $ABCD$ such that $\angle BPC = \angle BAP + \angle PDC$. Let E, F and G be the feet of the perpendiculars from P to AB, AD and DC. Prove that ΔFEG is similar to ΔPBC.

1742. *Proposed by Murray S. Klamkin, University of Alberta.*

Let $1 \leq r < n$ be integers and $x_{r+1}, x_{r+2}, \ldots, x_n$ be given positive real numbers. Find positive $x_1, x_2, \ldots, x_r$ so as to minimize the sum

$$S = \sum \frac{x_i}{x_j}$$

taken over all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.

(This problem is due to Byron Calhoun, a high school student in McLean, Virginia. It appeared, with solution, in a science project of his.)

1743*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < \gamma < 180°$ be fixed. Consider the set $\Delta(\gamma)$ of all triangles ABC having angle γ at C, whose altitude through C meets AB in an interior point D such that the line through the incenters of ΔADC and ΔBCD meets the sides AC and BC in interior points E and F respectively. Prove or disprove that

$$\sup_{\Delta(\gamma)} \left(\frac{\text{area}(\Delta EFC)}{\text{area}(\Delta ABC)} \right) = \left(\frac{\cos(\gamma/2) - \sin(\gamma/2) + 1}{2\cos(\gamma/2)} \right)^2 .$$

(This would generalize problem 5 of the 1988 IMO [1988: 197].)

1744. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Find the number of points of intersection of the graphs of $y = a^x$ and $y = \log_a x$ for any $0 < a < 1$.

1745. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let $\mathcal{A} = A_0A_1A_2A_3$ be a square and X an arbitrary point in the plane of $\mathcal{A}$. Define the kth vertex (k an integer mod 4) of the quadrilaterals $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ as follows:

(i) B_k is the midpoint of XA_k;

(ii) $C_k = A_kB_{k+1} \cap A_{k+1}B_k$;

(iii) D_k is the centroid of the quadrilateral $B_kA_kA_{k+1}B_{k+1}$.

Prove that $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are squares whose centres lie on a line through the centre of $\mathcal{A}$.

1746. *Proposed by Richard K. Guy, University of Calgary, and Richard J. Nowakowski, Dalhousie University.*

(i) Find infinitely many pairs of integers a, b, with $1 < a < b$, so that ab exactly divides $a^2 + b^2 - 1$.

(ii) With a and b as in (i), what are the possible values of $(a^2 + b^2 - 1)/ab$?

1747. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$ABCDE$ is a convex pentagon in which each side is parallel to a diagonal. Two of its angles are right angles. Find the sum of the squares of the sines of the other three angles.

1748. *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

Several medieval arithmetic/algebra books give a problem with a fountain located between two towers so that pigeons of equal speeds can get from the tops of the towers to the fountain in equal times. The solution of this works out quite easily. However, in the *Trattato d'Aritmetica* attributed to Paolo dell'Abbaco, c. 1370, there are problems where a rope is strung between the two tower tops and a weight is hung from a ring on the rope, where the rope is just long enough for the weight to touch the ground. Solving this is a bit trickier than the previous problem. To make it even trickier, suppose the rope isn't long enough—suppose the towers are 1748 and 1992 *uncia* high, the rope is 2600 *uncia* long, and the two towers are 2400 *uncia* apart. How far above the ground does the weight hang?

1749. *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Let ABC be a triangle with external angle-bisectors w_a', w_b', w_c', inradius r, and circumradius R. Prove that

(i) $$\left(\sqrt{\frac{1}{w_a'}}+\sqrt{\frac{1}{w_b'}}+\sqrt{\frac{1}{w_c'}}\right)^2 < \frac{2}{r}\ ;$$

(ii) $$\left(\frac{1}{w_a'}+\frac{1}{w_b'}+\frac{1}{w_c'}\right)^2 < \frac{R}{3r^3}\ .$$

1750. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Pairs of numbers from the set $\{11, 12, \ldots, n\}$ are adjoined to each of the 45 different (unordered) pairs of numbers from the set $\{1, 2, \ldots, 10\}$, to obtain 45 4-element sets $A_1, A_2, \ldots, A_{45}$. Suppose that $|A_i \cap A_j| \le 2$ for all $i \ne j$. What is the smallest n possible?

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1640. [1991: 114] *Proposed by P. Penning, Delft, The Netherlands.*

Find

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n-1}\right).$$

I. *Solution by Kenneth S. Williams, Carleton University.*

It is well-known that for any positive integer n we have

$$1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\ . \tag{1}$$

Set

$$E_n=\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n-1}\ .$$

Then we have

$$\begin{aligned}1-\frac{1}{2}+\cdots-\frac{1}{4n} &= \frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{4n}\\ &= E_n+\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)\\ &= E_n+\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}\right),\end{aligned}$$

and letting $n \to \infty$ we obtain

$$\log 2 = \lim_{n\to\infty} E_n + \frac{1}{2}\log 2,$$

so

$$\lim_{n\to\infty} E_n = \frac{1}{2}\log 2.$$

[*Editor's Note.* As a reference for (1), we could give Ken Williams' own *Crux* article "Limits of students' limits" [1980: 204]! Thanks to solver Peter Hurthig for remembering this article.]

II. *Solution by Kee-Wai Lau, Hong Kong.*

For $x > 0$ the function $1/x$ is convex and decreasing. Hence

$$\begin{aligned}\ln\left(\frac{2n+1/2}{n+1/2}\right) &= \int_{n+1/2}^{2n+1/2}\frac{dx}{x}\\ &< \frac{1}{n+1/2}+\frac{1}{n+3/2}+\cdots+\frac{1}{2n-1/2}\\ &< \int_{n-1/2}^{2n-1/2}\frac{dx}{x} = \ln\left(\frac{2n-1/2}{n-1/2}\right).\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n-1}\right) = \frac{1}{2}\ln 2.$$

Also solved by H.L. ABBOTT, University of Alberta; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado (two solutions); IAN GOLDBERG, student, University of Toronto Schools; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; PETER HURTHIG, Columbia College, Burnaby, B.C.; ESTEBAN INDURAIN, Universidad Pública de Navarra, Pamplona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S.

KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; LEROY F. MEYERS, The Ohio State University; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; JEAN-MARIE MONIER, Lyon, France; CORY PYE, student, Memorial University of Newfoundland, St. John's; DANIEL REISZ, Vincelles, Champs-sur-Yonne, France; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; ROBERT E. SHAFER, Berkeley, California; D.J. SMEENK, Zaltbommel, The Netherlands (two solutions); UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one incorrect solution sent in.

*Bang, Heuver and Reisz note that an equivalent problem appears as #*13*, page* 50 *of Pólya and Szegö,* Problems and Theorems in Analysis I, *Springer-Verlag,* 1972.

Janous, Klamkin and Wang gave generalizations.

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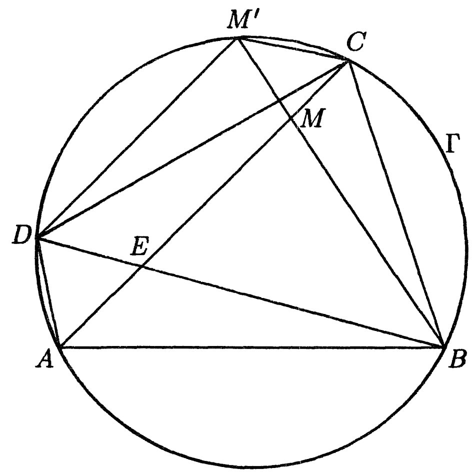
1641. [1991: 140] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Quadrilateral $ABCD$ is inscribed in circle Γ, with $AD < CD$. Diagonals AC and BD intersect in E, and M lies on EC so that $\angle CBM = \angle ACD$. Show that the circumcircle of ΔBME is tangent to Γ at B.

Solution by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.



Extend BM so that it intersects Γ in M'; then since

$$\angle CBM' = \angle ACD = \angle ABD$$

it follows that $CM' = AD$. From this we conclude that $AC \| DM'$. Hence $\Delta BDM'$ is similar to ΔBEM, and therefore the circumcircle of $\Delta BDM'$ is multiplied with a factor BE/BD with respect to B which is on Γ. It follows that the circumcircle of ΔBEM is tangent to Γ.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; ROLAND EDDY, Memorial University of Newfoundland; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Several solvers note that the condition $AD < CD$ isn't necessary.

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1642. [1991: 140] *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum value of

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)$$

subject to $yz+zx+xy=1$ and $x,y,z\geq 0$.

Solution by Jeff Higham, student, University of Toronto.
With all sums cyclic over x, y, z,

$$\begin{aligned}\sum x(1-y^2)(1-z^2) &= \sum x - \sum(xy^2+xz^2) + xyz\sum yz \\ &= \sum x - \sum(x^2y+x^2z) + xyz \\ &= \sum x - x\sum(1-yz) + xyz \\ &= (\sum xyz) + xyz = 4xyz.\end{aligned}$$

But

$$(xyz)^{2/3} \leq \frac{xy+yz+zx}{3} = \frac{1}{3}$$

by the A.M.–G.M. inequality, so

$$\sum x(1-y^2)(1-z^2) = 4xyz \leq \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9}\ .$$

Indeed when $x=y=z=1/\sqrt{3}$, the given expression has the value $4\sqrt{3}/9$, so this is the maximum value of the expression subject to the constraints.

Also solved by H.L. ABBOTT, University of Alberta; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incomplete solution was sent in.

Hurthig points out that the inequality is equivalent to the known fact that the maximum volume of a rectangular box of given surface area occurs for a cube.

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1643. [1991: 140] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Characterize all triangles ABC such that

$$\overline{AI_a} : \overline{BI_b} : \overline{CI_c} = \overline{BC} : \overline{CA} : \overline{AB},$$

where I_a, I_b, I_c are the excenters of ΔABC corresponding to A, B, C, respectively.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

On sait que

$$AI_a = \frac{s}{\cos A/2}\ , \quad BI_b = \frac{s}{\cos B/2}\ , \quad CI_c = \frac{s}{\cos C/2}\ ,$$

où s désigne le demi-périmètre du triangle ABC. Le relation donnée est donc équivalente à

$$a\cos\frac{A}{2} = b\cos\frac{B}{2} = c\cos\frac{C}{2}$$

ou encore à

$$\sin A\cos\frac{A}{2} = \sin B\cos\frac{B}{2} = \sin C\cos\frac{C}{2}\ . \tag{1}$$

1*er Cas:* $A = B$. Alors $C = 180° - 2A$, et la relation (1) devient

$$\sin A\cos\frac{A}{2} = \sin 2A\sin A$$

d'où

$$\cos\frac{A}{2} = \sin 2A.$$

Ainsi

$$\frac{A}{2} = 90° - 2A \quad \Leftrightarrow \quad A = 36°$$

ou

$$\frac{A}{2} = -90° + 2A \quad \Leftrightarrow \quad A = 60°.$$

On obtient ainsi

$$A = B = 36°,\ C = 108° \quad \text{ou} \quad A = B = C = 60°.$$

En faisant des hypothèses $B = C$ ou $C = A$, on obtiendrait les solutions $B = C = 36°$, $A = 108°$ ou $C = A = 36°$, $B = 108°$.

2*e Cas:* $A \neq B \neq C \neq A$. Alors la relation (1) s'écrit

$$\sin\frac{A}{2}\cos^2\frac{A}{2} = \sin\frac{B}{2}\cos^2\frac{B}{2} = \sin\frac{C}{2}\cos^2\frac{C}{2}\ ,$$

$$\sin\frac{A}{2} - \sin^3\frac{A}{2} = \sin\frac{B}{2} - \sin^3\frac{B}{2} = \sin\frac{C}{2} - \sin^3\frac{C}{2}\ ,$$

$$\begin{cases} \sin\frac{A}{2} - \sin\frac{B}{2} = \sin^3\frac{A}{2} - \sin^3\frac{B}{2} \\ \sin\frac{A}{2} - \sin\frac{C}{2} = \sin^3\frac{A}{2} - \sin^3\frac{C}{2}\ , \end{cases}$$

$$\begin{cases} 1 = \sin^2\frac{A}{2} + \sin\frac{A}{2}\sin\frac{B}{2} + \sin^2\frac{B}{2} \\ 1 = \sin^2\frac{A}{2} + \sin\frac{A}{2}\sin\frac{C}{2} + \sin^2\frac{C}{2}\ , \end{cases}$$

et en soustrayant ces deux égalités, on obtient

$$0 = \sin\frac{A}{2}\left(\sin\frac{B}{2} - \sin\frac{C}{2}\right) + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$$

d'où

$$0 = \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\ ,$$

ce qui est impossible.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer. Two incorrect solutions were received.

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1644*. [1991: 140] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $f : \mathbf{R} \to \mathbf{R}$ be continuous such that it attains both positive and negative values, and let $n \geq 2$ be an integer. Show that there exists a strictly increasing arithmetic sequence $a_1 < \cdots < a_n$ such that $f(a_1) + \cdots + f(a_n) = 0$.

Solution by Phil Reiss, student, University of Manitoba, Winnipeg.

Since f is continuous and $f(x)$ attains both positive and negative values, there is an open interval over which $f(x)$ is positive and an open interval over which $f(x)$ is negative. Let $f(x)$ be positive for $x \in (p, p+k)$ $(k > 0)$ and negative for $x \in (q, q+k)$. Then, for integer $n \geq 2$,

$$f\left(p + \frac{k}{n+1}\right) + f\left(p + \frac{2k}{n+1}\right) + \cdots + f\left(p + \frac{nk}{n+1}\right) > 0,$$

$$f\left(q + \frac{k}{n+1}\right) + f\left(q + \frac{2k}{n+1}\right) + \cdots + f\left(q + \frac{nk}{n+1}\right) < 0.$$

Let

$$g(x) = f\left(x + \frac{k}{n+1}\right) + f\left(x + \frac{2k}{n+1}\right) + \cdots + f\left(x + \frac{nk}{n+1}\right)\ .$$

Then $g(p) > 0$, $g(q) < 0$, and g is continuous, so the intermediate value theorem tells us that $g(r) = 0$ for some r between p and q. For $a_i = r + ik/(n+1)$, $i = 1, 2, \ldots, n$, we then have

$$f(a_1) + f(a_2) + \cdots + f(a_n) = 0,$$

with the a_i's forming a strictly increasing arithmetic sequence as required.

Also solved by H.L. ABBOTT, University of Alberta; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; LEROY F. MEYERS, The Ohio State University; JEAN-MARIE MONIER, Lyon, France; CARLOS JOSÉ PÉREZ JIMÉNEZ, student, Logroño, Spain; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The proposer points out that a similar result holds for geometric sequences provided that f attains both positive and negative values on $(0, \infty)$ (proof similar and left to the reader!).

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1645. [1991: 140] *Proposed by J. Chris Fisher, University of Regina.*

Let P_1, P_2, P_3 be arbitrary points on the sides A_2A_3, A_3A_1, A_1A_2, respectively, of a triangle $A_1A_2A_3$. Let B_1 be the intersection of the perpendicular bisectors of A_1P_2 and A_1P_3, and analogously define B_2 and B_3. Prove that $\Delta B_1B_2B_3$ is similar to $\Delta A_1A_2A_3$.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

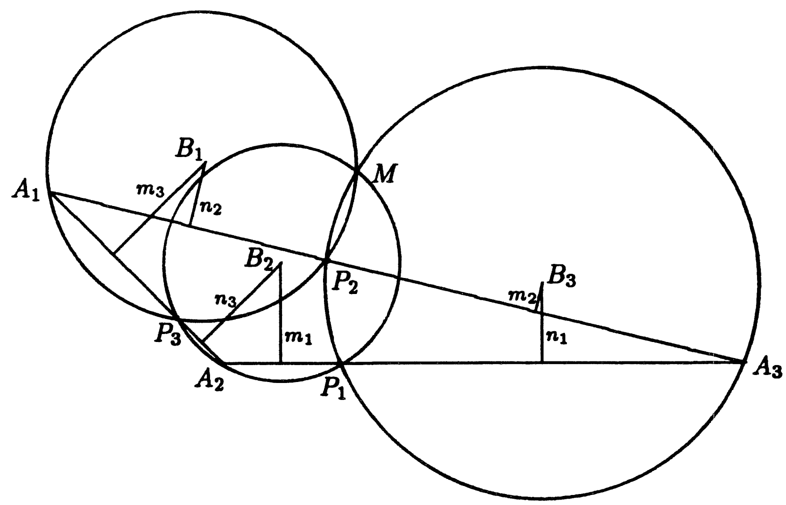

Figure 1

The points B_1, B_2, B_3 are the centers of the three so-called *Miquel circles* ($A_1P_2P_3$, etc.) for the triad $P_1P_2P_3$ with respect to $\Delta A_1A_2A_3$. These three circles pass through a common point M. [This result is the special case of Miquel's theorem known as the *pivot theorem*, which is discussed below.—*Ed.*] Consider the Miquel circle of center B_1 (namely, $A_1P_2P_3$). We have, comparing the inscribed angle to the angle at the center,

$$\angle P_3A_1P_2 = \frac{1}{2}\angle P_3B_1P_2 = \frac{1}{2}(\angle P_3B_1M - \angle P_2B_1M).$$

But $\angle P_3B_1M = 2\angle B_2B_1M$, and $\angle MB_1P_2 = 2\angle MB_1B_3$, because B_2B_1 and B_1B_3 are the perpendicular bisectors of P_3M and MP_2, respectively. Therefore

$$\angle A_2A_1A_3 = \angle P_3A_1P_2 = \frac{1}{2}\cdot 2(\angle B_2B_1M - \angle B_3B_1M) = \angle B_2B_1B_3.$$

Likewise $\angle A_1A_3A_2 = \angle B_1B_3B_2$, so the triangles are similar as required.

This problem is not new. It can be found in [2, p. 161] and [3, p. 134].

[*Comments contributed by the proposer.* Two solvers, Kuczma and Bluskov, point out that care is needed in applying theorems involving cyclic quadrangles. They suggest checking separate cases for when the Miquel point M is interior to, exterior to, or on a side of $\Delta A_1A_2A_3$. So in Figure 1, $\angle P_1A_2P_3 = \pi - \frac{1}{2}\angle P_3B_2P_1$. Alternatively, the notion of *directed angle* greatly simplifies the discussion [3, paragraphs 16–19]: The directed angle $\angle P_1B_2P_3$ (from P_1B_2 to B_2P_3) is defined to be *that angle through which the entire line P_1B_2 must be rotated about B_2 in the positive direction in order to coincide with B_2P_3.* As an example of how this definition works, note that four points A, B, C, D lie on a circle if and only if the directed angles $\angle ABC$ and $\angle ADC$ are equal. More to the point, the directed angles $\angle A_1A_3A_2$ and $\angle B_1B_3B_2$ of Figure 1 are equal, so Bellot's proof is valid without the need for any special cases. In [1, section 8.7], Berger provides a careful treatment of the subtleties involving directed angles, adding that it is worth the trouble because of the quick and elegant proofs that the notion allows.

A. Miquel, whose theorem dates from 1833, is not to be confused with another solver, Miguel Amengual Covas! The latter supplied the following short proof of his near-namesake's theorem (which has been modified by the use of directed angles). Define M to be the second point of intersection of the circles centered at B_2 and B_3 (see Figure 1). We wish to show that $\angle A_1P_3M = \angle A_1P_2M$ as directed angles. But $\angle A_1P_3M = \angle A_2P_1M$ since both equal $\angle A_2P_3M$ — remember, these are directed angles. Similarly, $\angle A_2P_1M = \angle A_3P_2M$ since both equal $\angle A_3P_1M$. Finally, since both equal the positive angle through which the line A_1A_3 must be rotated about P_2 to coincide with P_2M, $\angle A_3P_1M = \angle A_1P_2M$ as desired.]

References:

[1] M. Berger, *Geometry I*, Springer, N.Y., 1980.

[2] Davis, *Modern College Geometry*, Addison-Wesley, 1949.

[3] R. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960.

II. *Solution by the proposer.*

For this solution, let the subscripts be mod 3 integers (e.g., $A_3 = A_0$) and, as in Figure 1, define n_{j+1} to be the perpendicular bisector of A_jP_{j+1} and m_{j-1} of A_jP_{j-1}. The result follows from two simple applications of transformation geometry, both more readily proved than stated.

(i) *The product of rotations about the points B_j and B_{j-1} through the respective angles $2\alpha_j$ and $2\alpha_{j-1}$ is a rotation through the angle $-2(\pi - \alpha_j\alpha_{j-1})$ about the unique point B_{j+1} for* *which* $\angle B_j$ $(= \angle B_{j+1}B_jB_{j-1}) = \alpha_j$ *and* $\angle B_{j-1}$ $(= \angle B_jB_{j-1}B_{j+1}) = \alpha_{j-1}$. Proof: Write the rotations as the product of reflections in the sides B_jB_{j+1} and B_jB_{j-1} (in that order).

(ii) *The product of three translations along the directed sides* A_jA_{j-1} *of* $\Delta A_1A_2A_3$ *(through the lengths of these sides) is the identity.* Proof: Use vector addition to describe the translations.

To prove that the given triangles are similar, write each translation T_j as the product $T_j = \nu_j\mu_j$ of reflections in the lines n_j and m_j respectively. Note that since these mirrors are perpendicular to the sides of $\Delta A_1A_2A_3$ the rotation $\mu_j\nu_{j-1}$ is a rotation about B_j through twice $\angle A_j$. Using (ii) along with properties of reflections, the identity

$$I = \nu_2 I \nu_2 = \nu_2(T_2T_1T_0)\nu_2 = \nu_2(\nu_2\mu_2)(\nu_1\mu_1)(\nu_0\mu_0)\nu_2 = (\mu_2\nu_1)(\mu_1\nu_0)(\mu_0\nu_2).$$

This last product represents the identity as a product of rotations about the vertices of $\Delta B_1B_2B_3$. (i) says that for any triangle $\Delta B_1B_2B_3$ and rotations R_j about the vertices B_j, if $R_2R_1R_0$ is the identity, then the angle of rotation must for each j be twice $\angle B_j$; consequently, $\angle A_j = \angle B_j$ as desired.

Comment. Since the pivot theorem follows immediately from the similarity of the two given triangles (for example, the argument used in solution I can be reversed), an independent proof of this result provides yet another proof of the pivot theorem.

III. *Combination of solutions by P. Penning, Delft, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

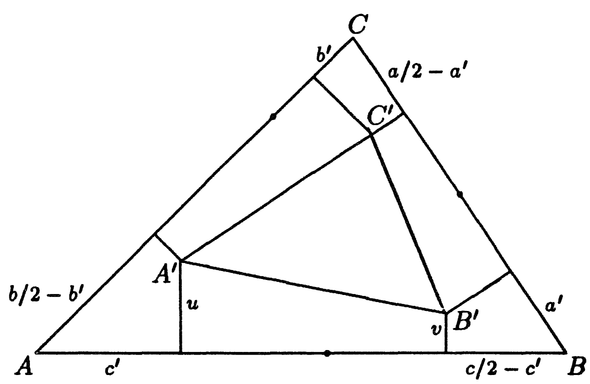

Figure 2

Avoiding subscripts, we change the notation to that of Figure 2. [So B_1 becomes A', etc.] To prove that $\Delta ABC \sim \Delta A'B'C'$ we show that the angles each side of the primed triangle (extended if necessary) makes with the respective side of ΔABC are equal. This will follow at once from the equality of the tangents of these angles. Letting u and v be the respective *directed distances* of A' and B' to AB, the tangent of the angle that $A'B'$ makes with AB is $t = 2(u - v)/c$, (where, as usual, c is the length of the side AB). By trigonometry,

$$u = \frac{\left(\frac{1}{2}b - b'\right) - c'\cos A}{\sin A} \quad \text{and} \quad v = \frac{a' - \left(\frac{1}{2}c - c'\right)\cos B}{\sin B}.$$

For the first equality, drop a perpendicular to AC from the point where u meets AB; this gives a right triangle whose hypotenuse is c' and base is $((b/2) - b') - u\sin A$; similarly for the second. Thus our tangent t equals

$$\frac{b\sin B - 2b'\sin B - 2c'\cos A\sin B - 2a'\sin A + c\cos B\sin A - 2c'\cos B\sin A}{c\sin A\sin B}.$$

The terms with c' can be combined to get $-2c'\sin(A+B) = -2c'\sin C$. Reversing the process with $\sin B$, one gets $b\sin B = b(\sin A\cos C + \sin C\cos A)$. Using the law of sines ($a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, with R the circumradius of ΔABC), t becomes

$$t = \frac{R\sum(\cos A\sin B\sin C) - (a'\sin A + b'\sin B + c'\sin C)}{R\sin A\sin B\sin C},$$

where $\sum$ is taken over the cyclic permutations of A, B, C. Since this last expression is symmetric in all six parameters, the three tangents will be equal as claimed.

The same proof gives $\Delta ABC \sim \Delta A'B'C'$ if the edges of the primed triangle are defined by the rule that the lengths of their projections are a fixed fraction of the sides of the unprimed triangle. (In the original problem the fraction is $1/2$.)

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; ANDY LIU, University of Alberta; JEAN-MARIE MONIER, Lyon, France; and TOSHIO SEIMIYA, Kawasaki, Japan.

Most solvers gave solutions which were much like Solution I, *and most of these also included a proof of the pivot theorem.*

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1646. [1991: 140] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Find all positive integers n such that the polynomial

$$(a-b)^{2n}(a+b-c) + (b-c)^{2n}(b+c-a) + (c-a)^{2n}(c+a-b)$$

has $a^2 + b^2 + c^2 - ab - bc - ca$ as a factor.

Solution by Murray S. Klamkin, University of Alberta.

Letting $a = y+z$, $b = z+x$, $c = x+y$, we equivalently want

$$P \equiv 2x(y-z)^{2n} + 2y(z-x)^{2n} + 2z(x-y)^{2n}$$

to be divisible by

$$\begin{aligned}(y+z)^2 + (z+x)^2 + (x+y)^2 - (y+z)(z+x) - (z+x)(x+y) - (x+y)(y+z)\\ = x^2 + y^2 + z^2 - yz - zx - xy\\ = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),\end{aligned}$$

where ω is a primitive cube root of unity (so $\omega^2 + \omega = -1$). Thus P must vanish for $x = -\omega y - \omega^2 z$, for any y and z, i.e.

$$\begin{aligned} 0 &= (-\omega y - \omega^2 z)(y-z)^{2n} + y(z + \omega y + \omega^2 z)^{2n} + z(-\omega y - \omega^2 z - y)^{2n} \\ &= (-\omega y - \omega^2 z)(y-z)^{2n} + y\omega^{2n}(y-z)^{2n} + z\omega^{4n}(y-z)^{2n} \\ &= [(\omega^{2n} - \omega)y + (\omega^{4n} - \omega^2)z](y-z)^{2n}. \end{aligned}$$

Hence we must have $\omega^{2n} = \omega$, and thus $n = 3m + 2$ for $m = 0, 1, 2, \ldots$.

Also solved by ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; JEAN-MARIE MONIER, Lyon, France; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was also one correct solution without proof, and one incorrect solution, sent in.

Kuczma notes that the exponents $2n$ in the question could have been replaced by n, i.e., allowed to be odd, in which case the result is that $n \equiv 1 \bmod 3$. The above proof can easily be modified to show this.

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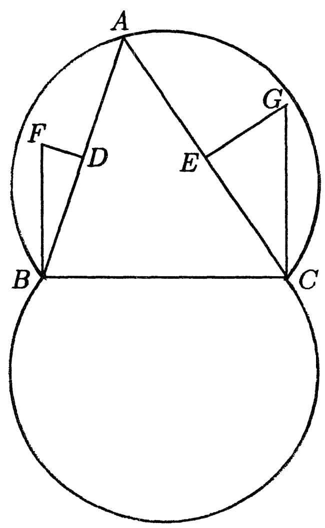
1647. [1991: 141] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

B and C are fixed points and A a variable point such that $\angle BAC$ is a fixed value. D and E are the midpoints of AB and AC respectively, and F and G are such that $FD \perp AB, GE \perp AC$, and FB and GC are perpendicular to BC. Show that $|BF| \cdot |CG|$ is independent of the location of A.

Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

Let $\angle A$ be fixed and let $\angle B = \beta$, $\angle C = \gamma$. Then A belongs to the union of two parts of congruent circles as shown in the picture. Let R be the radius of these circles. Since $\angle FBD$ is equal to $90° - \beta$ or $\beta - 90°$ we get



$$\begin{aligned} FB &= \frac{BD}{\cos \angle FBD} = \frac{BD}{\sin \beta} = \frac{AB}{2 \sin \beta} \\ &= \frac{2R \sin \gamma}{2 \sin \beta} = \frac{R \sin \gamma}{\sin \beta} . \end{aligned}$$

Analogously,

$$GC = \frac{R \sin \beta}{\sin \gamma} .$$

Thus $BF \cdot CG = R^2$, which completes the proof.

Also solved (usually the same way) by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; ANDY LIU, University of Alberta; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1648. [1991: 141] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Evaluate $\lim_{n\to\infty}(u_n/\sqrt{n})$, where $\{u_n\}$ is defined by $u_0 = u_1 = u_2 = 1$ and

$$u_{n+3} = u_{n+2} + \frac{u_n}{2n+6}, \quad n = 0, 1, \dots .$$

Solution by Harvey L. Abbott, University of Alberta.

We use the following proposition which is a special case of a theorem of Darboux.

(P) *Let g be regular in a region of the complex plane containing the disk $\{z : |z| \le 1\}$ and let $f(z) = g(z)/(1-z)^\alpha$, $\alpha > 0$. Then if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ one has, as $n \to \infty$,*

$$a_n = (1 + o(1))\frac{g(1)\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)} . \quad \square$$

Now let $f(z) = \sum_{n=0}^{\infty} u_n z^n$. Since $0 < u_n \le n$ for $n \ge 1$, the series for f converges for $|z| < 1$. From the given recurrence relation we then get

$$\sum_{n=0}^{\infty} u_{n+3} z^{n+3} = z\sum_{n=0}^{\infty} u_{n+2} z^{n+2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{u_n z^{n+3}}{n+3} . \tag{1}$$

Suppose, for the moment, that z is real. It follows from (1) that for $-1 < z < 1$

$$f(z) - u_0 - u_1 z - u_2 z^2 = z(f(z) - u_0 - u_1 z) + \frac{1}{2}\int_0^z t^2 f(t)dt.$$

Since $u_0 = u_1 = u_2 = 1$, this gives

$$(1-z)f(z) - 1 - \frac{1}{2}\int_0^z t^2 f(t)dt = 0. \tag{2}$$

On differentiating (2) we get the differential equation

$$(1-z)f'(z) - f(z) - \frac{1}{2}z^2 f(z) = 0, \quad f(0) = 1. \tag{3}$$

The solution of (3) is given by

$$f(z) = \frac{e^{-(z^2+2z)/4}}{(1-z)^{3/2}} \ . \tag{4}$$

By analytic continuation, (4) defines f as a regular function in the complex plane, apart from the singularity at $z = 1$. f clearly satisfies the hypotheses of proposition (P), with $g(z) = e^{-(z^2+2z)/4}$ and $\alpha = 3/2$. It follows from (P) that, as $n \to \infty$,

$$u_n = (1+o(1))\frac{e^{-3/4}\Gamma(n+3/2)}{n!\,\Gamma(3/2)} \ .$$

From $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$ and the asymptotic formula for the Γ-function we now get

$$u_n = (1+o(1))\frac{2\sqrt{n}}{e^{3/4}\sqrt{\pi}} \ .$$

Hence

$$\lim_{n\to\infty}\frac{u_n}{\sqrt{n}} = \frac{2}{e^{3/4}\sqrt{\pi}} = 0.53301\cdots \ .$$

Remark. For other applications of the method of Darboux and a more general formulation of proposition (P) see the paper of E.A. Bender, Asymptotic methods in enumeration, *SIAM Review* 16 (1974), pages 485–515, especially section 6.

Also solved by SEUNG-JIN BANG, Seoul, Korea; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Two other solutions giving numerical estimates were submitted.

The problem was taken from a paper of Robin Robinson, "A new absolute geometrical constant?", Amer. Math. Monthly 58 (1951) 462–469, *where it was unsolved.*

The editor had hoped there would be some elementary approach to this problem, but it seems that this was a futile wish! Considering the intended audience of Crux, *problems requiring solutions of this depth, however interesting, are inappropriate. The editor will try to edit them more carefully in the future.*

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1649*. [1991: 141] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Prove or disprove that

$$\sum \cot\frac{\alpha}{2} \ - \ 2\sum \cot\alpha \ge \sqrt{3},$$

where the sums are cyclic over the angles α, β, γ of a triangle.

Solution by Stephen D. Hnidei, student, University of Windsor.

The given inequality is true. Using the double-angle formula,

$$\begin{aligned}\sum \cot\frac{\alpha}{2} - 2\sum\cot\alpha &= \sum\cot\frac{\alpha}{2} - 2\sum\frac{\cot^2(\alpha/2)-1}{2\cot(\alpha/2)} \\ &= \sum\tan\frac{\alpha}{2} \ge \sqrt{3}\ ,\end{aligned}$$

by item 2.33, p. 27, of Bottema et al, *Geometric Inequalities.*

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; JEFF HIGHAM, student, University of Toronto; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; BOB PRIELIPP, University of Wisconsin–Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; and D.J. SMEENK, Zaltbommel, The Netherlands. A second solution was sent in by Hnidei.

Several solutions were similar to the above, with some of the solvers proving the inequality $\sum \tan(\alpha/2) \geq \sqrt{3}$ directly using Jensen's inequality. Other solvers reduced the problem to the known inequality $r + 4R \geq s\sqrt{3}$, which is also in Bottema.

* * * * *

1650. [1991: 141] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Find all real numbers α for which the equality

$$[\sqrt{n+\alpha} + \sqrt{n}\,] = [\sqrt{4n+1}\,]$$

holds for every positive integer n. Here [ ] denotes the greatest integer function. (This problem was inspired by problem 5 of the 1987 Canadian Mathematics Olympiad [1987: 214].)

Solution by H.L. Abbott, University of Alberta.

The solution is

$$9 - 6\sqrt{2} \leq \alpha \leq 2.$$

Each positive integer n determines a unique non-negative integer t such that either

(a) $t(t+1) \leq n \leq t^2 + 2t$

or

(b) $t^2 + 2t + 1 \leq n < (t+1)(t+2)$.

Suppose (a) holds for a given n and set $n = t(t+1) + u$, $0 \leq u \leq t$. Then

$$4t^2 + 4t + 1 \leq 4n + 1 \leq 4t^2 + 8t + 1$$

so that

$$(2t+1)^2 \leq 4n + 1 < (2t+2)^2$$

from which it follows that $[\sqrt{4n+1}\,] = 2t + 1$. The condition $[\sqrt{n+\alpha} + \sqrt{n}\,] = [\sqrt{4n+1}\,]$ then becomes

$$2t + 1 \leq \sqrt{t(t+1) + u + \alpha} + \sqrt{t(t+1) + u} < 2t + 2.$$

This inequality is easily seen to be equivalent to

$$(2t+1)^2 - 2(2t+1)\sqrt{t(t+1)+u} \leq \alpha < (2t+2)^2 - 2(2t+2)\sqrt{t(t+1)+u}.$$

The lower bound on α is maximal when $u = 0$ and the upper bound is minimal when $u = t$. We must therefore have for all $t \geq 1$,

$$\alpha \geq (2t+1)^2 - 2(2t+1)\sqrt{t(t+1)} := f(t),$$

$$\alpha < (2t+2)^2 - 2(2t+2)\sqrt{t^2+2t} := g(t).$$

If case (b) holds, one finds by a similar argument that for all $t \geq 0$,

$$\alpha \geq (2t+2)^2 - 2(2t+2)(t+1) = 0,$$

$$\alpha < (2t+3)^2 - 2(2t+3)\sqrt{t^2+3t+1} := h(t).$$

It is easy to verify, by elementary calculus, that f is decreasing on $[1,\infty)$ so that $\alpha \geq f(1) = 9 - 6\sqrt{2} > 0$. Also, g and h are decreasing on $[0,\infty)$, and $g(t) \to 2$ and $h(t) \to 5/2$ as $t \to \infty$. Thus $\alpha \leq g(\infty) = 2$. Thus α satisfies $9 - 6\sqrt{2} \leq \alpha \leq 2$.

Also solved by DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; C. WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Two incorrect solutions were also sent in.

A similar problem (prove that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}]$$

for all positive integers n) occurred in the 1948 *Putnam contest. See pp.* 26 *and* 257 *of* The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1938–1964, *A.M. Gleason, R.E. Greenwood, and L.M. Kelly, MAA,* 1980. *Can this problem be generalized in the same way that* 1650 *generalized the CMO problem? Is there a "super-generalization" of both problems?*

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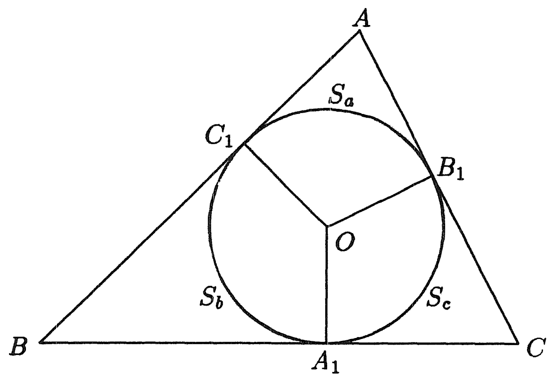
1651. [1991: 171] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and A_1, B_1, C_1 the common points of the inscribed circle with the sides BC, CA, AB, respectively. We denote the length of the arc B_1C_1 (not containing A_1) of the incircle by S_a, and similarly define S_b and S_c. Prove that

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{9\sqrt{3}}{\pi}.$$

Solution by Pavlos Maragoudakis, student, University of Athens, Greece.



Without loss of generality we may suppose $a \geq b \geq c$. Let O be the centre of the inscribed circle, and put $\omega_a = \angle B_1OC_1$, $\omega_b = \angle A_1OC_1$, $\omega_c = \angle A_1OB_1$. We have

$$A + \omega_a = \pi, \text{ etc.},$$

and so

$$a \geq b \geq c \Rightarrow \omega_a \leq \omega_b \leq \omega_c.$$

But $S_a = r\omega_a$, $S_b = r\omega_b$, $S_c = r\omega_c$, where r is the inradius, therefore $S_a \leq S_b \leq S_c$, or

$$\frac{1}{S_a} \geq \frac{1}{S_b} \geq \frac{1}{S_c}.$$

By Chebyshev's inequality,

$$3\left(\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c}\right) \geq (a+b+c)\left(\frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c}\right).$$

But also

$$(S_a + S_b + S_c)\left(\frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c}\right) \geq 3^2,$$

so

$$\begin{aligned} 3\left(\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c}\right) \cdot 2\pi r &= 3\left(\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c}\right)(S_a + S_b + S_c) \\ &\geq (a+b+c)\left(\frac{1}{S_a} + \frac{1}{S_b} + \frac{1}{S_c}\right)(S_a + S_b + S_c) \\ &\geq 18s, \end{aligned}$$

where s is the semiperimeter, so

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{3s}{\pi r}.$$

Now it is enough to prove that $s \geq 3\sqrt{3}\,r$, which is true (e.g., item 5.11, of Bottema et al, *Geometric Inequalities*).

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia, and the proposer. One incorrect solution was received.

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1652. [1991: 171] *Proposed by Murray S. Klamkin, University of Alberta.*

Given fixed constants $a, b, c > 0$ and $m > 1$, find all positive values of x, y, z which minimize

$$\frac{x^m + y^m + z^m + a^m + b^m + c^m}{6} - \left(\frac{x+y+z+a+b+c}{6}\right)^m .$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

Denote

$$A = \frac{a+b+c}{3}, \quad M = \sqrt[m]{\frac{a^m + b^m + c^m}{3}}, \quad X = \frac{x+y+z}{3} .$$

Using (twice) the strict convexity of $t \mapsto t^m$ we get, with sums cyclic in a, b, c or x, y, z,

$$\begin{aligned}
\frac{1}{6}\left(\sum x^m + \sum a^m\right) - \left[\frac{1}{6}\left(\sum x + \sum a\right)\right]^m
&= \frac{1}{2}\left(\frac{1}{3}\sum x^m + \frac{1}{3}\sum a^m\right) - \left(\frac{1}{2}(X+A)\right)^m \\
&\geq \frac{1}{2}\left[\left(\frac{1}{3}\sum x\right)^m + \left(\frac{1}{3}\sum a^m\right)\right] - \left(\frac{1}{2}(X+A)\right)^m \\
&\geq \frac{1}{2}(X^m + M^m) - \frac{1}{2}(X^m + A^m) = \frac{1}{2}(M^m - A^m).
\end{aligned}$$

This is the desired minimum value, attained only if $x = y = z$ and $X = A$; i.e., if

$$x = y = z = \frac{a+b+c}{3} .$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer. One incorrect solution was sent in.

Janous and the proposer both point out that, for any positive integers k, l, real $m > 1$, and real $a_1, \ldots, a_l > 0$, the minimum value of

$$\frac{x_1^m + \cdots + x_k^m + a_1^m + \cdots + a_l^m}{k+l} - \left(\frac{x_1 + \cdots + x_k + a_1 + \cdots + a_l}{k+l}\right)^m$$

occurs for $x_1 = \cdots = x_k = (a_1 + \cdots + a_l)/l$, as can be proved as above.

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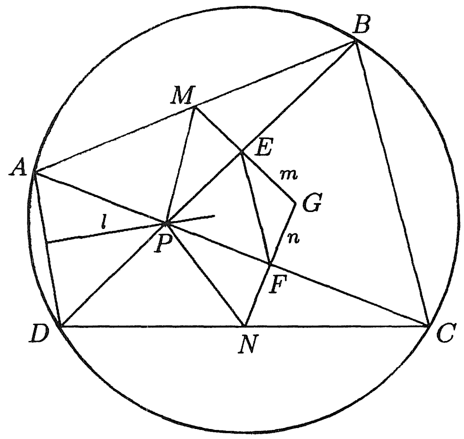
1653. [1991: 171] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be the intersection of the diagonals AC, BD of a quadrangle $ABCD$, and let M, N be the midpoints of AB, CD, respectively. Let l, m, n be the lines through P, M, N perpendicular to AD, BD, AC, respectively. Prove that if l, m, n are concurrent, then A, B, C, D are concyclic.

I. *Solution by Botond Kőszegi, student, Fazekas Mihály Gimnázium, Budapest, Hungary.*

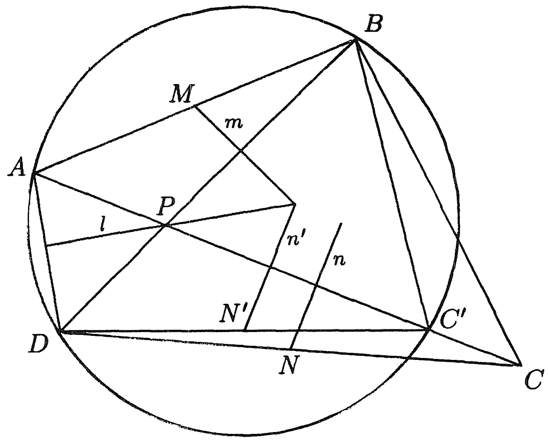


We will first prove the *converse*; that is, if A, B, C, D are concyclic, then lines l, m, n are concurrent. Let the feet of perpendiculars from M to BD and from N to AC be E and F, respectively, and let the intersection of lines m and n be G. We must prove that line l passes through G.

Since $ABCD$ is cyclic, $\Delta APB \sim \Delta DPC$. PM and PN are corresponding medians in these triangles, so $\Delta PNC \sim \Delta PMB$. Again, ME and NF are corresponding altitudes in the two triangles, so $PE/EB = PF/FC$. Thus EF and BC are parallel. It follows immediately that $\angle PEF = \angle PBC = \angle PAD$. Quadrangle $PFGE$ has two opposite right angles, so it is cyclic. Therefore

$$\angle GPF = \angle FEG = 90^\circ - \angle PEF = 90^\circ - \angle PAD.$$

Let's look at line l now. It is perpendicular to AD, so it makes an angle of $90^\circ - \angle PAD$ with line PF. Since line PG makes exactly the same angle with PF, line l must be the same as line PG. Hence l passes through G.



It is left to prove that if A, B, C, D are not concyclic, then l, m, n are not concurrent. Take an arbitrary non-cyclic quadrilateral $ABCD$, and draw the circumcircle of ΔABD. Obviously, this doesn't pass through C. Let the other intersection of the circle and line AC be C', and let the midpoint of DC' be N'. As we have just proved, the lines m, n' and l through M, N' and P perpendicular to BD, AC' and AD, respectively, are concurrent, because $ABC'D$ is cyclic. The line n through N perpendicular to AC is parallel to n'. Also, obviously these two lines are not the same, so they cannot both go through the intersection of lines l and m. Thus (since l, m and n' are concurrent), l, m and n can't possibly be concurrent.

Therefore we have proved that A, B, C, D are concyclic if and only if l, m, n are concurrent.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Let x, y be the positive scalars such that

$$\overrightarrow{PC} = -x\,\overrightarrow{PA}, \quad \overrightarrow{PB} = -y\,\overrightarrow{PD}\,.$$

Supposing l, m, n are concurrent at Q and denoting $\overrightarrow{PA} = \mathbf{u}$, $\overrightarrow{PD} = \mathbf{v}$, $2\,\overrightarrow{PQ} = \mathbf{w}$, we have

$$2\overrightarrow{PM} = \overrightarrow{PA} + \overrightarrow{PB} = \mathbf{u} - y\mathbf{v}, \qquad 2\overrightarrow{PN} = \overrightarrow{PC} + \overrightarrow{PD} = \mathbf{v} - x\mathbf{u},$$
$$2\overrightarrow{QM} = 2\overrightarrow{PM} - 2\overrightarrow{PQ} = \mathbf{u} - y\mathbf{v} - \mathbf{w}, \qquad 2\overrightarrow{QN} = 2\overrightarrow{PN} - 2\overrightarrow{PQ} = \mathbf{v} - x\mathbf{u} - \mathbf{w}.$$

By the condition of the problem, the vectors $\overrightarrow{PQ}$, $\overrightarrow{QM}$, $\overrightarrow{QN}$ are perpendicular to $\overrightarrow{AD}$, $\overrightarrow{PD}$, $\overrightarrow{PA}$, respectively (with the agreement that the zero vector is perpendicular to any vector). Equating to zero the corresponding scalar products:

$$0 = (\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w},$$
$$0 = (\mathbf{u} - y\mathbf{v} - \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - y|\mathbf{v}|^2,$$
$$0 = (\mathbf{v} - x\mathbf{u} - \mathbf{w}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w} - x|\mathbf{u}|^2,$$

we get $x|\mathbf{u}|^2 = y|\mathbf{v}|^2$, i.e., $PA \cdot PC = PB \cdot PD$. This is just the condition of concyclicity.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

The converse was also shown by Bang.

* * * * *

1654*. [1991: 171] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\sum \frac{x}{x + \sqrt{(x+y)(x+z)}} \le 1,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

Since

$$\sqrt{(x+y)(x+z)} \le \frac{(x+y)+(x+z)}{2} = x + \frac{y+z}{2}\,, \quad \text{etc.}$$

(arithmetic-geometric inequality), we get

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} = \frac{x(\sqrt{(x+y)(x+z)}-x)}{(x+y)(x+z)-x^2} \le \frac{x(y+z)/2}{xy+yz+zx},$$

so

$$\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} \le \frac{x(y+z)/2+y(z+x)/2+z(x+y)/2}{xy+yz+zx} = 1,$$

with equality for $x+y=x+z=y+z$, i.e., $x=y=z$.

Also solved by H.L. ABBOTT, University of Alberta; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; M. S. KLAMKIN, University of Alberta; PETER KOMJATH, Simon Fraser University, Vancouver; BOTOND KŐSZEGI, student, Fazekas Mihály Gimnázium, Budapest, Hungary; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; BEATRIZ MARGOLIS, Paris, France; and the proposer.

Kuczma notes that the inequality remains true provided that, say, $x, y > 0$ *and* $z > -xy/(x+y)$.

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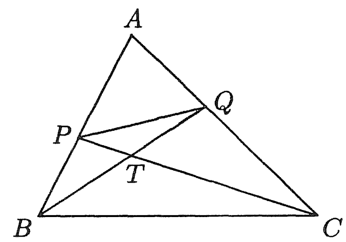
1656. [1991: 172] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

Given a triangle ABC, we take variable points P on segment AB and Q on segment AC. CP meets BQ in T. Where should P and Q be located so that the area of ΔPQT is maximized?

Solution by Kee-Wai Lau, Hong Kong.

We show that the area of ΔPQT is maximized if and only if

$$\frac{AP}{AB} = \frac{AQ}{AC} = \frac{\sqrt{5}-1}{2}.$$



Let $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$, $\overrightarrow{AP} = s\mathbf{b}$ and $\overrightarrow{AQ} = t\mathbf{c}$, where $0 < s, t < 1$. Then $\overrightarrow{PC} = \mathbf{c} - s\mathbf{b}$ and $\overrightarrow{QB} = \mathbf{b} - t\mathbf{c}$. Let $\overrightarrow{BT} = n\overrightarrow{BQ}$ and $\overrightarrow{PT} = m\overrightarrow{PC}$. Since $\overrightarrow{PT} + \overrightarrow{BP} = \overrightarrow{BT}$,

$$m(\mathbf{c} - s\mathbf{b}) - (1-s)\mathbf{b} = n(t\mathbf{c} - \mathbf{b})$$

or

$$(n-1+s(1-m))\mathbf{b} + (m-nt)\mathbf{c} = \mathbf{0}.$$

It follows that

$$n-1+s(1-m) = m-nt = 0,$$

from which we obtain

$$m = \frac{t(1-s)}{1-st}, \quad n = \frac{1-s}{1-st}.$$

Thus

$$\overrightarrow{TP} = \frac{t(1-s)}{1-st}\,(s\mathbf{b}-\mathbf{c})$$

and by symmetry

$$\overrightarrow{TQ} = \frac{s(1-t)}{1-st}\,(t\mathbf{c}-\mathbf{b}).$$

Now

$$\text{area }(\Delta PQT) = \frac{1}{2}\,|\overrightarrow{TP}\times\overrightarrow{TQ}\,| = \frac{1}{2}\,f(s,t)\,|\mathbf{b}\times\mathbf{c}|,$$

where

$$f(s,t) = \frac{st(1-s-t+st)}{1-st}\ .$$

Since $s+t \geq 2\sqrt{st}$,

$$f(s,t) \leq \frac{st(1-2\sqrt{st}+st)}{1-st} = \frac{st(1-\sqrt{st})}{1+\sqrt{st}}\ ,$$

with equality when $s=t$. By differentiation it is easy to check that for $0<x<1$ the function $x^2(1-x)/(1+x)$ attains its maximum when $x^2+x-1=0$ or $x=(\sqrt{5}-1)/2$. Thus $f(s,t)$, and so the area of ΔPQT, attains its maximum if and only if

$$s = t = \frac{\sqrt{5}-1}{2}\ .$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; JOSÉ YUSTY PITA, Madrid, and FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; and the proposer.

Yusty and Bellot wonder if there is a purely geometric solution. (Perhaps using some already known property of the Golden Ratio?)

The problem was taken from the 1879 Japanese book Kyokusu Taisei.

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