Mathematical Spectrum

Volume 4 1971/72 *Number 1*

A Magazine of *Contemporary Mathematics*

Published by *Oxford University Press*

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges, and universities. It is published by Oxford University Press on behalf of the Applied Probability Trust, a non-profit making organization established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 4 of *Mathematical Spectrum* (1971/72) will consist of two issues, the second of which will be published in the Spring of 1972.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

The Managing Editor, *Mathematical Spectrum*,
Hicks Building, The University, Sheffield S3 7RH.

Mathematics: The Language of Science[1]

J. B. HELLIWELL

University of Bradford

Pure and applied mathematics

It is an accepted fact that three of the most eminent names among those of the great mathematicians of the past are Archimedes, Newton and Gauss. One of the reasons why each towers so high above his contemporaries is that he acted not only as a source of inspiration in the abstract thinking associated with pure mathematics, but also established new fields of application of these abstract concepts to the physical world. Despite his remarkable achievements on the applied side—and what schoolchild is not aware of his jubilant shout of 'Eureka' from the bathroom—we know that Archimedes delighted more in other aspects of his subject, whilst Newton found his chief joy in the applications to which he could put his mathematical powers. On the other hand, Gauss found both facets of his work equally congenial and is reputed to have coined the descriptive phrase that mathematics is the 'queen and servant of science', a phrase which was taken by the mathematical historian, E. T. Bell, as the title of his well-known book.

The profession of mathematician is one that has, in recent years, been brought more and more into the public eye. It is becoming increasingly evident that the technological advancement and well-being of our society is closely dependent on an adequate supply of mathematicians. But let us be clear: society is not interested in such people because of what they are but rather because of what they can do. Further, the services of specialists in other scientific disciplines who are also well versed in the processes of mathematical abstraction and development are equally solicited. The essential requirement is that the scientist or technologist who wishes to investigate real-world problems must be able to formulate these before he is able to analyse and solve them, and for this work mathematics is almost always necessary. Many of the recent developments in the fields of economics and the biological sciences, for instance, have stemmed from the application of mathematical methods and have themselves given rise to new branches of mathematics; an important example of this is the growth of operational research. Particularly noteworthy is the interplay between application and the associated invention and development of new aspects of pure mathematics. It is significant that the professional body incorporated some seven years ago to embrace all mathematicians

[1] Based on an inaugural lecture given at the University of Bradford.

was founded with the aim of promoting the advancement of mathematics *and its applications*.

Mathematics is the language of abstract logical thought. Not only does the mathematician contemplate affairs in an abstract fashion; the ordinary man-in-the-street in his everyday life is often obliged to think in this way. If, for instance, he boards a bus with his family of a wife and two children and asks for two full and two half tickets, then he presupposes that the conductor and he will agree upon the price for these. He thus assumes that the conductor and he will pursue the same logical path in the process of computation, which presupposes that they both speak a common mathematical language. And presumably the calculation in question may be described as applied mathematics since the problem relates to real objects.

It is not always easy to decide whether a particular problem and its solution belong to the realm of pure mathematics or applied mathematics; nor, frequently, is it very profitable to do so. The following, probably apocryphal story, relating to a certain eminent geometer, may illustrate the point. A very tall slim man, he was in search of lodgings; having found rooms in every way suitable, he was inspecting the bed. The landlady looked at the professor, then at the bed and thought it would be too short. But he took his walking stick and laid it first across the bed and then along it and proclaimed, "That will be all right. I shall sleep diagonally." Was the geometer for that brief moment an applied mathematician?

Mathematics is a language and a literature

Our common language, English, may be used in many different ways; so also may the language of mathematicians. In the early Middle Ages, whilst only an oral tradition existed among virtually all people, the minstrel brought in his ballads news of the outer world, crudely portrayed but easily remembered. In later times, and in some crafts even to the present day, the techniques and expertise of the trade are handed down by similar oral means. Among the descriptive terms used are to be found those of elementary mathematics chiefly relating to calculations of length, area and volume. However, with the development of writing and printing the power of the written word has taken over from the spoken word, so that reference to earlier statements may be made with precision by future readers. In our libraries are to be found works of literature—belles-lettres, evocative prose, stirring novels with finely drawn characters and dramatic plots, and flowing verse, among other volumes.

In the world of mathematics, Euclid's geometry is familiar to us from our early schooldays. From basic postulates or axioms relating to mathematical entities such as points and lines we build up a set of theorems or laws concerning the structure of the mathematical system. Now one of the basic axioms of the Euclidean system concerns straight lines, which may be defined as the shortest distance between two points; the axiom states that only one such line passes through two points. Whilst this appears reasonable for points and lines restricted to lie on a plane surface it may not be so for spatial relations on the surface of a sphere. For on a sphere the shortest distance between two points is along the shorter arc of the great circle joining them, i.e., along the circle lying on the surface of the sphere having the same centre as it and passing through the two points. If the two points in question lie at the ends of a diameter of the sphere then there are many 'shortest distances' between the points, just as there are many lines of longitude joining the North and South poles on the Earth. Thus we may build up a mathematical structure for spatial relationships with the stated Euclidean postulate relaxed and replaced by alternatives. The elegance of the various geometries which result from our different assumptions is such that their detailed analysis merits placing on a par with the cherished library works in our mother tongue. In such a way we may build up a library of mathematics; belles-lettres from the field of mathematical logic, evocative prose from the realms of the complex variable, extensive fictional works from the world of mechanics, sonnets from algebra and the like. This is the world of the pure mathematician.

The art of modelling

It is, however, the applications of mathematics to so many human activities which has caused, throughout the ages, essentially hard-headed communities to support an elite class of mathematicians. Archimedes, who has been called the first scientific civil servant, was allowed to spend much of his time developing abstract theories because his master Hieron, the king of Sicily, knew that he was always available to bend his mind to practical problems and that some of his theories had direct application to the real world. Thus on one occasion the king had built a ship so heavy that it would not slip off the stocks. Upon applying to his chief scientist for advice, the latter devised a system of cogwheels and screws so that a launch could be effected single-handed. Archimedes, when faced with the real-life situation, stripped it down to its basic essentials and formulated these in mathematical language to set up what we would call a mathematical model. It is this process of construction and development of mathematical models to describe real-world phenomena which lies at the root of the importance of mathematics today.

At the outset, a scientist or technologist is faced with a problem in the real world for which he must find a solution. The problem may be connected with the flow of a gas, the trajectory of space vehicles, the economy, meteorology, bacteriology, and so on. Before it is possible to begin an analysis, information is needed of both a qualitative and quantitative nature which describes the phenomenon in question. When this information has been assembled it must be abstracted into mathematical form and consolidated in the model. If we are particularly fortunate it may be possible for us to build a model which is such an accurate representation of the actual state of affairs that we can succeed in making it reproduce quantitative aspects of the original information. Then we may reasonably expect to be able to use this model to predict future behaviour, as is done for example in the compilation of tidal charts. On the other hand, the real phenomena may be so complex that it is not possible to formulate a model of this nature, owing perhaps to an incomplete understanding of the physics incorporated in the qualitative information. Then we are obliged to form a simpler construction which satisfies a few of the requirements but may otherwise be very artificial; we hope that on analysing it and comparing its predictions with some of the neglected information, we may be led to a better understanding of the phenomena. A useful model of this sort yields fruitful suggestions for its own modification, and so leads to further analysis, both theoretical and experimental. A bad model would lead us into unproductive paths; but it is often difficult in the early stages of a large-scale study to distinguish between this and a good model. A particular skill which is necessary for successful model building is the capacity to recognize those physical characteristics of the situation which are essential. Thus if we are surveying a tract of earth and the area is of small dimensions then a model may be constructed on the assumption that the Earth is a plane; but for larger tracts it is vital to take account of the curvature of the Earth with a corresponding increase in the geometrical and trigonometrical complexity of the model. The aim is always to obtain a model which represents the physical situation as accurately as possible but which, above all, is simple enough to be amenable to mathematical analysis. If the simplification is overdone and some important item of information is ignored, further useful development is only possible when the model is reformulated. For instance a crude, but not useless, model of the flow of water predicts no drag on a body placed in a stream, a result which is at variance with experimental evidence. However, by including viscous effects one finds the drag theoretically predicted, but at the expense of a markedly more involved mathematical formulation. This sequential process of model-making, checking against experiment, reformulating the model, re-checking, is a vital feature of the situation. Mathematical modelling and experimentation go hand in hand and we have moved far from the attitude of mind prevalent among some mathematicians at the turn of the century. It is said that the celebrated Victorian physicist Clerk Maxwell invited the eminent Cambridge mathematician Todhunter to see some experiments in conical refraction. "No", replied Todhunter, "I have been teaching conical refraction in physical optics for many years and it might upset me." Maxwell then offered to show the experiments to Todhunter's pupils, but received the rebuff, "If a young man will not believe his tutor, a gentleman and often in Holy Orders, I fail to see what can be gained by practical demonstration."

A model and its developments

As an illustration of some of the above ideas let us now consider the particular field of gas dynamics, which is the subject dealing with the flow of compressible fluids. The physical state of the fluid in a very simple model in which all dissipative effects such as viscosity and thermal conductivity are neglected is described by the density, pressure, temperature, entropy and velocity (called collectively, the field variables) each of which in general varies from point to point and also from instant to instant. Represented symbolically by ρ, p, T, s, $\boldsymbol{v}$, $\boldsymbol{x}$, t respectively these quantities form the mathematical entities of the model, and the mathematical equations relating them are the field equations. These are none other than the basic postulates and have precisely the same status as axioms in geometry. The field equations are of two distinct types. They express either basic physical truths which are believed to be valid for all sorts of material, or else they describe particular properties of the material in question. The former are the laws of conservation of mass, momentum and energy together with a condition of thermodynamic irreversibility, whilst the latter form the constitutive equations. In addition to a statement of the governing equations, to complete the model for any particular flow configuration it is necessary to specify in what manner the fluid is bounded, and, if the motion is unsteady, what its state is at some initial instant and how it is to be disturbed.

A fundamental problem in elementary gas dynamics is to determine the flow pattern which develops in a long tube filled with gas at rest in a uniform state and closed at one end by a piston which at some instant is set in motion, pushing into the gas. It is found reasonable to neglect the effect of the confining sidewalls and to assume that conditions are uniform on any cross-section of the tube, so that all flow variables are functions only of the time and distance along the tube. Then if L is the initial length of the gas column and the time, t, is measured from the instant when the piston begins to move with velocity $U(t)$ from its position at the spatial origin it is necessary that:

(i) $p = p_0, \quad \rho = \rho_0, \quad v = 0; \qquad t < 0, \quad 0 \leqslant x \leqslant L,$

(ii) $v = 0; \qquad t \geqslant 0, \quad x = L,$

(iii) $v = U(t); \quad t \geqslant 0, \quad x = X(t).$

In the above statements the suffix zero denotes conditions in the undisturbed state (i.e., before the piston moves) and X, which is a function of t, is the location of the piston at any subsequent time. The conservation laws comprise the field equations:

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial s}{\partial t}+v\frac{\partial s}{\partial x} = 0,$$

and the irreversibility condition has been absorbed in an assumption of adiabatic conditions. In addition, for a so-called perfect gas with constant ratio γ of specific heats the constitutive equations may be written:

$$p/p_0 = (\rho/\rho_0)^{\gamma} \exp\{(s-s_0)/c_v\},$$
$$p = R\rho T,$$

where R is constant for a given gas and c_v is the specific heat at constant volume.

These five equations with associated conditions form the mathematical model of the problem and may be solved exactly. They give rise to solutions which we may interpret as waves of disturbance propagating in each direction along the tube. In this particular piston problem, in the early stages before reflexion from the closed end, compressive waves are continuously being sent down the tube to set the gas in motion. The speed of propagation of these waves increases with the gas velocity so that, as the piston accelerates, successive wavelets overtake each other and cause a violent disturbance termed a shock wave. This wave is such that, across a plane surface running ahead of the piston, sudden and considerable changes in the values of the field variables occur. This kind of behaviour is markedly different from that associated with the propagation of sound, electromagnetic signals and light and in contrast with the latter linear phenomena is governed by non-linear partial differential equations. Of course such a violent discontinuity as that presented by the mathematical model could not actually arise in nature, but in the formulation of the model all dissipative effects were ignored whereas one would expect these to become of increasing importance as the gradients of the field variables increase. Indeed if the model is modified to take account of viscosity and heat conductivity then it can be shown that, in agreement with observation, the changes in physical state take place within regions so narrow that the concept of mathematical discontinuity is perfectly satisfactory provided that we are only interested in the study of conditions on either side of the discontinuity.

As a consequence of the compressive action across a shock wave there occurs a considerable rise in the temperature of the gas. Indeed the high temperature behind a shock passing through combustible material can be used to initiate detonation, in the burnt gases of which a yet greater increase of pressure and temperature is achieved. However, ionization of a gas occurs at temperatures of the order of 10^4 degrees Kelvin so that it is quite possible for the products of combustion to be ionized; in this case, if a magnetic field exists, the model which we have developed becomes quite inadequate. For, from the time of Faraday it has been known that conducting material moving in a magnetic field experiences an electromotive force, and, if the configuration is suitable, electric currents develop. These currents themselves generate an induced magnetic field which perturbs the imposed external field and due to the interaction of the currents and magnetic field an electromagnetic force is produced which perturbs the original motion of the material. These effects are basic to the subject of magnetogasdynamics, which received its first major stimulus when astrophysicists became aware that the universe is chiefly composed of plasma, or gas in the ionized conducting state, and is also permeated by fairly strong magnetic fields. A second boost was given by the birth of atomic and nuclear power generators. For instance, the pumping by electromagnetic forces of liquid metal coolants in nuclear reactors is now well established, and controlled thermonuclear fusion can be achieved by using magnetogasdynamic effects to confine the extremely hot gases away from all walls. Even more recently, space research has begun to make demands on a similar mathematical model. Three problems in this field which are at present under investigation are: studies of the passage of space vehicles through the ionosphere; studies of the stream of high speed particles (known as the solar wind) ejected by the Sun and passing near the Earth; studies of the tenuous airstream past a body moving at very high speeds. In all these examples, the theoretical approach is of paramount importance to our hopes of increasing understanding.

Experience and language in partnership

The history of man is in many ways a history of technology. As each new technical development was exploited so man gained stature among his fellow creatures and power over his environment. Whilst one would not wish to suggest that our primeval ancestors used mathematical models to develop the stone axe, yet it is fair to claim for mathematics a very considerable share of the credit for the developing space programmes of the major nations. Clearly engineering and mathematics are now in partnership and the interplay between them can perhaps be best described in the following brief quotation from an unpublished paper of Richard von Mises, who until his death in 1953 was Gordon McKay Professor of Aerodynamics and Applied Mathematics at Harvard University. He wrote, "The leitmotif, the ever-recurring melody, is that two things are indispensable in any reasoning, in any description we shape of a segment of reality; to submit to experience and to face the language that is used, with unceasing logical criticism."

The Algebra of Genealogy

DAVID KENDALL

University of Cambridge

Recently Mr E. A. Leeb wrote as follows to my colleague Professor J. W. S. Cassels. "There is an old conundrum, 'Brothers and sisters have I none, yet this man's father is my father's son'. While it is possible to solve the problem by trial-and-error, I was wondering whether the relationships involved could be algebraically expressed. Or, to consider the broader question, can familial relationships be written mathematically?"

Professor Cassels showed the letter to me, and my approach to the problem is printed overleaf. Before turning the page, the reader may care to explore Mr Leeb's two questions for himself.

Let us denote individuals by $u, v, \ldots, u = v$ denoting identity. Let us define a mapping F from the set of all individuals into itself, by writing Fx for 'the father of x'. This is well defined (if we are evolutionists rather than fundamentalists), although Fx may, of course, not be *known* to be so describable. The equation $Fx = Fy$ asserts that x and y have the same father, and this situation can arise in *just three ways*:

(i) x and y may be identical;

(ii) x and y may be full sibs (i.e., have the same father *and mother*);

(iii) x and y may have the same father but different mothers.

To say that u is the son of v can be written, in our notation, as $Fu = v$, provided that the data tell us that v is male.

Now let w be the speaker, and z the person spoken of; we do not know at the outset that w and z are distinct. We are told that z is male, and we are told that his father, Fz, is 'my father's son'; i.e., that

$$F^2 z = Fw.$$

Thus $x = Fz$ and $y = w$ satisfy $Fx = Fy$, and we have already analysed that equation exhaustively. Case (i) gives us the traditional solution to the problem: $x = y$, so $Fz = w$, so *'I' am male and 'this man' is my son.* Case (ii) is excluded by the first line of the rhyme, for if x and y (that is, Fz and w) were full sibs, then 'I' would have a brother. We are left with case (iii), which on its simplest interpretation yields the following further and usually overlooked solution:

'I' may be male or female; my father had at least two wives (legally or not); I am his child by one wife and 'this man' is his grandson by another wife.

(The situation may be more complicated than this, because of the multiple relationships associated with more or less extreme forms of inbreeding such as the brother–sister marriages traditional in some of the royal families of the ancient world.)

I was surprised to get this second solution, for I have known the conundrum for some forty years and always believed the customary solution to be unique.

Mr Leeb's closing query is particularly timely, because genealogists are currently much concerned about the problem of reconstructing 'family trees' on a computer. Work of this sort is being done, for example, by the population geneticists in Pavia and in Cambridge (reference 1), by the historical demographers in Cambridge (reference 2), and by the genealogists in Salt Lake City. (See also Newcombe (reference 3).) It is clear that a trial or final 'tree' cannot be held in the computer, much less retrieved from it, unless its nodes are adequately labelled, and the non-trivial task of devising an efficient and readily comprehensible labelling, and writing machine programs in terms thereof, is very close to that proposed by Mr. Leeb.

References
1. M. H. Skolnick, A. Moroni, C. Cannings and L. L. Cavalli-Sforza, The reconstruction of genealogies from parish books. *Proceedings of the Anglo-Romanian Conference on Mathematics in the Archaeological and Historical Sciences* (Edinburgh University Press, 1971). Edited by F. R. Hodson, D. G. Kendall and P. Tautu.
2. R. S. Schofield, Population in the past: computer linking of vital records. *Bull. Inst. Math. Appl.* **6** (1970), 45–47.
3. H. B. Newcombe, Record linking: the design of efficient systems for linking records into individual and family histories. *Amer. J. Human. Genet.* **19** (1967), 335–359.

Gambling and Probability: Some Early Problems[1]

J. GANI

University of Sheffield

Students of chance, faced with a difficult problem in probability theory, may find it amusing to recall that the foundations of the subject, laid in the 16th and 17th centuries, were based on the frivolities of gambling. Today, gambling still provides minor motivation for the study of probability theory. Not long ago, Dr E. O. Thorp, an American mathematician, after studying the game of 'twenty one' was able to beat the banks in Las Vegas casinos, and accumulate a small fortune!

The origins of gambling are lost in ancient history. Early Egyptian tomb paintings depict board games played with sheep's heel-bones, known as astragals, which could fall on one of four faces, and served as imperfect dice. These astragals gradually evolved into regular six-sided cubic dice some time before the Christian era; such primitive dice have been found in many archaelogical excavations, including some in England dating from the Roman invasion. Both astragals and dice survived into the Middle Ages, when dicing games with rules very similar to those played today became firmly established.

The calculus of probabilities owes much to early gamblers; they were able to record from their very wide experience that certain throws of (one, two or more) astragals and dice were more likely to occur than others. Their practical knowledge greatly assisted the mathematicians who were attempting to formulate and solve the early problems of probability theory.

One of the earliest of these was Gerolamo Cardano (1501–1576), whose name is perhaps best known for his dispute with Tartaglia concerning the solution of

[1] This article formed part of a talk at the *Mathematics of Today* Conference for sixth formers, organised by the Department of Mathematics, University of Southampton, 14–16 April 1971.

the cubic equation. Cardano was a physician, astrologer, mathematician—and passionate gambler. In his *Liber de Ludo Aleae* (Book on Games of Chance) probably written around 1530, but not published until 1663 after his death, he laid the foundations of the calculus of probabilities. In it, he works out the probabilities of certain throws of two or more astragals and dice, as well as of certain card hands. He does this essentially by enumeration methods; although the results may now appear simple to us, they were a great achievement for his time.

This is how he describes the throws of two dice: "In the case of 2 dice, there are 6 throws with like faces, and 15 combinations with unlike faces, which when doubled gives 30, so that there are 36 throws in all, ..." He then argues, for example, that "the number of throws containing at least 1 ace is 11 out of the circuit of 36, or somewhat more than one quarter . . .". What he was considering can in modern terms be expressed more simply in terms of the sample space for throws of two dice: this is shown in Figure 1.

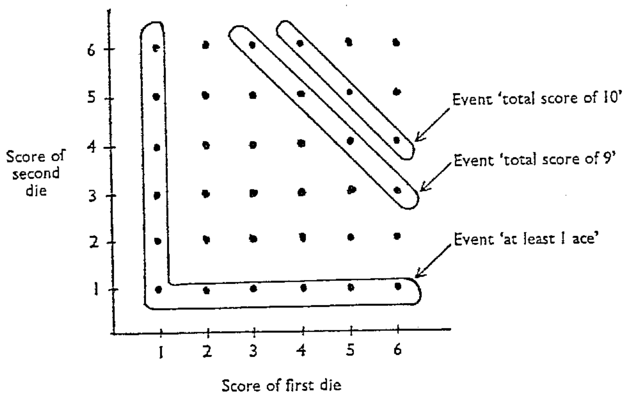


Figure 1. Sample space for throw of two dice.

Assuming independence of the two dice, each of whose faces will occur with probability 1/6, we can associate with each of the possible 36 points in the sample space a probability 1/36. Hence, simply counting points, Pr{at least 1 ace} = 11/36. We can also see directly from our sample space that

$$\Pr\{\text{total score of } 10\} = 3/36,$$

$$\Pr\{\text{total score of } 9\} = 4/36.$$

To the early gamblers, however, the problem seemed much more confusing. After all, 10 could be partitioned in two ways, either as $10 = 6+4$ or $10 = 5+5$; so likewise could 9 as $9 = 6+3$ or $9 = 5+4$. So why was there a difference in their chances of occurrence, as the gamblers had noted in practice? This is how Cardano resolved the problem: "The point 10 consists of (5, 5) and (6, 4), but the latter can occur in two ways, so that the whole number of ways of obtaining 10 will be 3/36 of the circuit . . . . Again in the case of 9, there are (5, 4) and (6, 3) both of which can occur in two ways, so that it will be 4/36 of the circuit." Thus, Cardano, by enumerative methods, had effectively discovered the equivalent of the sample space for the throw of two dice.

He was equally successful in resolving problems for throws of three dice. However, because of the late publication of his book, Galileo Galilei (1564–1642) published similar results independently around 1630 in his book *Considerazione sopra il Giuoco dei Dadi.* At that time Galileo was at the court in Florence, where he is reported to have been asked by the Grand Duke of Tuscany to explain the following paradox. Why, although there were an equal number of partitions into three different numbers of 9 and 10, did experienced gamblers state that the chance of throwing a 9 was less than that of throwing a 10. The six partitions of 9 and 10 are

$$9 = 1+2+6 = 1+3+5 = 1+4+4 = 2+2+5 = 2+3+4 = 3+3+3,$$
$$10 = 1+3+6 = 1+4+5 = 2+2+6 = 2+3+5 = 2+4+4 = 3+3+4.$$

Today, we would again draw the sample space for a throw of three dice, and note that for independent dice, we could associate with each of the 216 possible

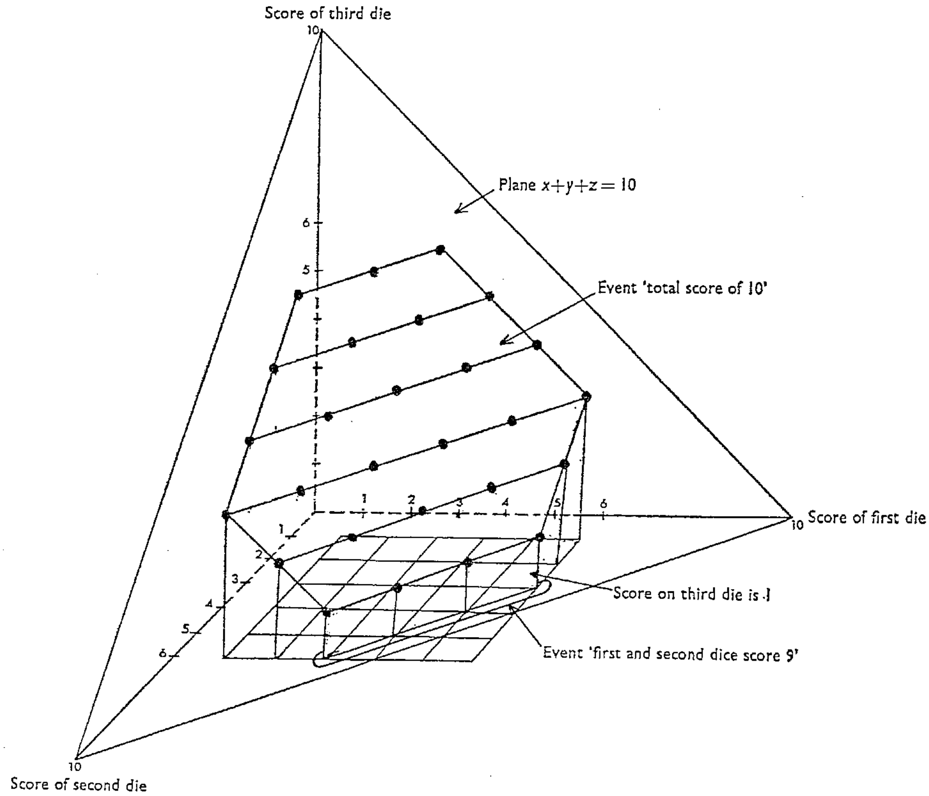


Figure 2. Sample space for throw of three dice. The heights of the points on the vertical planes, measured from the front plane backwards are 1, 2, 3, 4, 5 and 6 respectively, representing the score on the third die.

throws a probability 1/216. Hence if we counted the 27 points which lead to a total score of 10 we could assert that

$$\Pr\{\text{total score of } 10\} = 27/216.$$

A similar count would show that there are only 25 points leading to a 9 so that

$$\Pr\{\text{total score of } 9\} = 25/216.$$

Galileo obtained the same result by enumerating all possible throws, and noting the scores of the first, second and third dice, in all possible orders. He found the throws were:

$$\left.\begin{array}{r@{\;}l}
9 = & 1+2+6 \\
& 1+3+5 \\
& 1+4+4 \\
& 1+5+3 \\
& 1+6+2 \\
\\
9 = & 2+1+6 \\
& 2+2+5 \\
& 2+3+4 \\
& 2+4+3 \\
& 2+5+2 \\
& 2+6+1 \\
\\
9 = & 3+1+5 \\
& 3+2+4 \\
& 3+3+3 \\
& 3+4+2 \\
& 3+5+1 \\
\\
9 = & 4+1+4 \\
& 4+2+3 \\
& 4+3+2 \\
& 4+4+1 \\
\\
9 = & 5+1+3 \\
& 5+2+2 \\
& 5+3+1 \\
\\
9 = & 6+1+2 \\
& 6+2+1
\end{array}\right\} \text{25 possible throws}
\qquad
\left.\begin{array}{r@{\;}l}
10 = & 1+3+6 \\
& 1+4+5 \\
& 1+5+4 \\
& 1+6+3 \\
\\
10 = & 2+2+6 \\
& 2+3+5 \\
& 2+4+4 \\
& 2+5+3 \\
& 2+6+2 \\
\\
10 = & 3+1+6 \\
& 3+2+5 \\
& 3+3+4 \\
& 3+4+3 \\
& 3+5+2 \\
& 3+6+1 \\
\\
10 = & 4+1+5 \\
& 4+2+4 \\
& 4+3+3 \\
& 4+4+2 \\
& 4+5+1 \\
\\
10 = & 5+1+4 \\
& 5+2+3 \\
& 5+3+2 \\
& 5+4+1 \\
\\
10 = & 6+1+3 \\
& 6+2+2 \\
& 6+3+1
\end{array}\right\} \text{27 possible throws}$$

thus verifying that $\Pr\{9\} = 25/216$ and $\Pr\{10\} = 27/216$ and explaining the gamblers' practical odds.

A harder problem which Cardano formulated, but was unable to solve was the following. If a gambler keeps throwing the dice until, say, a double six shows up, how should one bet on the number of throws which will be required to achieve this double six? Cardano reasoned roughly as follows: if in a single trial the probability of a double six is 1/36, then in n throws it will appear on average $n/36$ times. If this number is greater than $\frac{1}{2}$, then the odds will be in its favour, so that we would require $(n/36) > \frac{1}{2}$, or $n > 18$ throws (say 19).

This is in fact an incorrect solution. The correct answer was given by the French mathematician and philosopher Pascal (1623–1662) in his famous correspondence on probability with his fellow countryman Fermat (1601–1665). The Chevalier de Méré, a notorious gambler at court, had noticed that if a six were to be thrown with one die, the odds were in favour of obtaining it in 4 throws. If, however, two sixes were to be thrown with two dice, the odds did not appear in favour of doing so in 24 throws. De Méré appears to have reasoned that in the case of a single die, since 4 throws is to the 6 possible results in the same proportion as 24 throws is to the 36 possible results in the case of two dice, the two cases should have been equally favourable.

Pascal in attempting to resolve this problem, derived what is now referred to as the geometric, or sometimes, the Pascal probability distribution. His reasoning was as follows. The probability of obtaining a six in 4 independent throws is such that it can be obtained either in the first, the second, the third or the fourth throw. The probability of obtaining a six in the first throw will be 1/6; if a six is not thrown the first time, this event will have probability 5/6. Then the probability of obtaining a six for the first time in the second throw will clearly be $(5/6)\times(1/6)$, or the probability of not obtaining a six in the first throw multiplied by the probability of a six in the second throw. Similarly, for the probabilities of obtaining a six for the first time in the third and fourth throws, we derive the values $(5/6)^2\times(1/6)$ and $(5/6)^3\times(1/6)$. Since these results are mutually exclusive, the probability of obtaining a six in the first 4 throws will be their sum, and this is given by

$$\frac{1}{6}+\frac{1}{6}\left(\frac{5}{6}\right)+\frac{1}{6}\left(\frac{5}{6}\right)^2+\frac{1}{6}\left(\frac{5}{6}\right)^3=1-\left(\frac{5}{6}\right)^4=\frac{671}{1296}>\frac{1}{2}.$$

In exactly the same way, the result for the probability of throwing two sixes in 24 independent throws can be obtained. In this case, the probability of throwing two sixes in the first throw will be 1/36, while that of not throwing them will be 35/36. Thus for two sixes in the second throw the probability will be $(35/36)\times(1/36)$, and so on. In this way, we find that the probability of two sixes in 24 throws will be equal to $1-(35/36)^{24}$, which is approximately $625/1296<\frac{1}{2}$. It turns out, however, that the probability of two sixes in 25 throws is greater than $\frac{1}{2}$; thus Cardano's solution of 19 was clearly wrong. Although it may have appeared strange at first sight that the throwing of two sixes in 24 should be less favourable than the throwing of one six in 4, the mathematics indicates that the gambler's hunch of de Méré was quite correct.

Gambling is no longer our main source of problems in probability theory, but there are many practical chance events in queueing, telephone and car traffic, insurance, genetics, demography (the study of human populations) and the social sciences which have stimulated new research in this area. The field of probability remains as lively today as it was in Cardano's, Galileo's, Pascal's and Fermat's time: if anything, perhaps more so!

Suggestions for further reading

1. Oystein Ore, *Cardano, the Gambling Scholar* (Dover, New York, 1965).
2. F. N. David, *Games, Gods and Gambling* (Griffin, London, 1969).
3. D. Bergamini, *Mathematics* (Time-Life Books Pocket Edition, 1969).
4. C. Solomon, *Mathematics* (Paul Hamlyn, London, 1969).

5. E. O. Thorp, *Beat the Dealer: a Winning Strategy for the Game of Twenty One* (Random House, New York, 1962).
6. I. Todhunter, *A History of the Mathematical Theory of Probability* (Chelsea, New York, 1949.)

The Control of a Hovering VTOL Aircraft

E. HUNTLEY
University of Sheffield

1. Introduction

From even a casual glance at popular accounts of scientific developments it becomes apparent that the related subjects of automatic control and cybernetics are assuming ever-increasing importance in the modern world and that they hold challenging problems for the control engineer and for the mathematician. A vast number of physical, biological, economic and management systems operate under the influence of control mechanisms. These mechanisms differ enormously in their different realisations in the various fields and yet the analysis of all such systems can be tackled by the application of a common class of techniques coming under the heading of control theory. One of the principal ideas, common to many such systems, is the concept of negative feedback and one purpose of this article is to introduce and explain this concept and to present a simple example of its application. First, however, let us consider in general terms some examples of control systems from the various fields mentioned.

2 A survey of some situations involving control mechanisms

All modern transportation systems make extensive use of control systems and this of course is particularly true in the aerospace field. However, consider firstly the more mundane problem of controlling the speed of a car in a stream of traffic flowing along a motorway at speeds fluctuating slightly about 60 m.p.h. The driver knows from experience the desirable separation distance between his car and the next and he repeatedly checks this distance. In a simple analysis we might argue that he actuates his speed controls (accelerator + brake) in a manner which depends upon the difference between the actual separation distance and what he regards as a desirable separation. This difference we refer to as the error. The following functional block diagram would then illustrate the information flow.

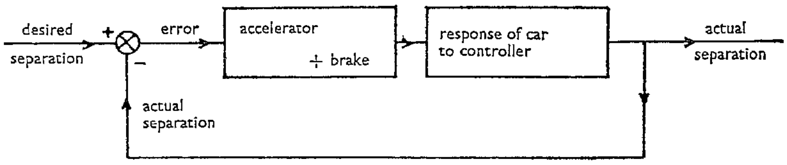


Figure 1

The accelerator + brake controller alters the speed of the car and therefore its distance from the car in front. At the summation point ⊗ the actual separation distance is subtracted from the desired separation distance. If the resultant error is positive the driver uses the accelerator and if negative he applies the brakes. In either case he attempts to reduce the error to zero and if he is successful the actual separation remains near to the desired value in spite of fluctuations in the speed of the car in front. This is an example of negative feedback. It is termed feedback because the system output is 'fed back' to a point where it can be compared with the input. It is 'negative' because it is subtracted from the input.

The overall car + driver system is then operating in the so-called closed-loop mode. Such a mode is inherently more stable and better able to combat disturbance than the alternative open-loop mode. In this example the open-loop mode is trivial and merely corresponds to driving at a constant pre-set speed. It would be acceptable only for stretches of the road when traffic was very light and/or speed fluctuations slight.

If the driver provided a control actuation signal proportional to the error or to the rate of change of error, he would be said to be operating in a linear manner. Unfortunately for the analyst he is much more sophisticated than that and the linearity assumption can only be made in certain circumstances. However one could quite feasibly replace the driver by an automatic system operating in this linear mode.

Now let us consider briefly examples of control mechanisms at work in biological systems. Any living organism lives and moves by virtue of a very large number of built-in control loops. The body thermostat for example, maintains body temperature remarkably accurately at 98·4°F over a wide range of environmental temperatures. The intricate muscles controlling the eye enable a young man to maintain the focus and alignment of his gaze on a pretty girl walking down the street. This is an example of 'tracking' clearly related to the problems of radar tracking of less attractive objects in the military sphere. Such problems concerning communication and control in biological systems come under the heading of cybernetics.

In the management sciences, the control of a continuously acting production-inventory system can be studied as a simple feedback mechanism. In this case the manager has to make decisions on rate of production of finished goods so as to meet the fluctuating market demand. He can maintain a store of finished goods to act as a buffer against sudden peak demands but this inventory incurs costs to the firm (e.g., capital costs for buildings and goods and wages for warehousemen). These costs must be weighed against other probable costs incurred if the factory should fail to produce goods on time. The aim therefore is to provide the manager with a strategy so that he makes the best decisions on rate of production and the inventory to be maintained. The optimal strategy turns out to involve a feedback control structure.

In a broader context the management and allocation of resources on a nation-wide basis is a research topic of considerable current interest to the economists.

There is clearly a need for better economic regulator devices which will eliminate the highly undesirable fluctuations apparent in the nation's economy.

Having mentioned some of the many fields of application of control mechanisms let us now examine in detail a mechanical system which can be fairly simply modelled.

3. The VTOL aircraft

A jet-lift VTOL aircraft such as the Hawker Harrier is able to take off and land vertically supported only by the deflected thrust of its jet engine. During these phases of operation the normal aerodynamic controls for pitch, roll and yaw are inoperative and have to be supplanted by small jet reaction controls at the extremities of the aircraft. When it is hovering the upward thrust balances the weight and the aircraft is effectively pivoted on a point, its centre of gravity. We now consider the pilot's problem of keeping the wings level in turbulent air. We restrict the analysis to the control of rolling motion about the horizontal longitudinal axis of the aircraft, defined as the line through the centre of gravity passing through the nose and tail. (A similar analysis would apply to the control of the pitching motion.)

Assume that the aircraft is initially trimmed with its wings level and at time $t = 0$ a disturbing moment $L(t)$ is applied about the longitudinal axis. Denote the consequent (small) roll angle displacement by ϕ. For an uncontrolled aircraft the equation of motion is simply

$$B\ddot{\phi} = L(t), \tag{1}$$

where B is the moment of inertia of the aircraft about the longitudinal axis. Equation (1) may be written alternatively in D operator notation as

$$D^2\phi = L(t)/B \quad \text{where} \quad D \equiv d/dt. \tag{2}$$

The solution of this differential equation for any $L(t)$ may be written symbolically as

$$\phi = \frac{1}{D^2}(L(t)/B),$$

the interpretation of this statement being that ϕ is the result of applying the operation $1/D^2$ to the input function $L(t)/B$. In this simple example $1/D^2$ merely involves integrating $L(t)/B$ twice but the D-notation provides a convenient way of dealing with more complicated operations, as we now see.

If we have a system described by the equation

$$P(D)\,\phi = L(t)/B,$$

where $P(D)$ is an nth degree polynomial in D then the solution is written as

$$\phi = \frac{1}{P(D)}(L(t)/B)$$

and we may call $1/P(D)$ the transfer operator of the system. The important point is that it entirely classifies the system and is independent of the form of the input $L(t)/B$. Returning to the example we can represent the situation expressed by equation (2) by a black box with transfer operator $1/D^2$, which operates upon $L(t)/B$ to produce ϕ (Figure 2). This black box representation is much exploited in analysing complicated systems.

With regard to representative behaviour of the system, if $L(t)/B$ is taken to be a unit step function (Figure 3), then the response is $\phi = t^2/2$ (Figure 4). Thus the uncontrolled aircraft would turn turtle and clearly must be controlled in some way.

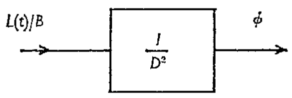


Figure 2

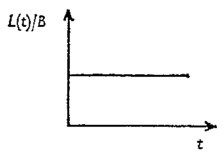


Figure 3

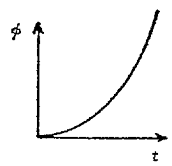


Figure 4

Suppose now that a controller (which may be the pilot or some mechanical device) gives a restoring moment at each instant proportional to the angle ϕ so that

$$B\ddot{\phi} = -k_1\phi + L(t) \quad \text{or} \quad (D^2 + k_1/B)\,\phi = L(t)/B \tag{3}$$

is the new equation of motion. The block diagram for this case (Figure 5) illustrates that the restoring moment gives rise to a negative feedback loop. If we are not interested in the actual internal structure we can enclose the system within the broken lines and obtain another black box, akin to that of Figure 2, with transfer operator $1/P(D) = 1/(D^2 + k_1/B)$, input $L(t)/B$ and output ϕ. In this way we can build up black boxes of increasing complexity.

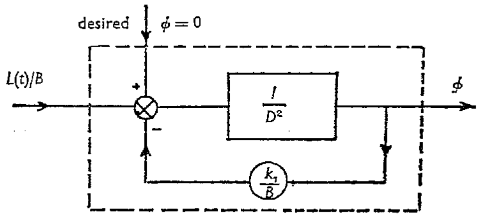


Figure 5

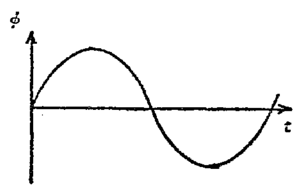


Figure 6

For this new system, the response to a unit step in rolling moment is given in Figure 6. The oscillation in roll angle is clearly an improvement on the divergence obtained in the uncontrolled case but it is still not satisfactory. Further improvement can be obtained by having the controller sense the rate of rotation $\dot{\phi}$ and provide a restoring moment proportional to $\dot{\phi}$. Then the equation of motion becomes

$$B\ddot{\phi} = -k_1\phi - k_2\dot{\phi} + L(t) \quad \text{or} \quad \left(D^2 + \frac{k_2}{B}D + \frac{k_1}{B}\right)\phi = L(t)/B.$$

The block diagram (Figure 7) now contains two feedback loops, one for ϕ and the other for $\dot{\phi}$. The overall transfer operator is $1/P(D) = 1/(D^2 + k_2\,D/B + k_1/B)$.

The response to a unit step input, shown in Figure 8, is a damped harmonic wave and thus we have achieved the desired result that the initial disturbance should die away with time. The rate of decay will depend upon the magnitudes of B, k_1 and k_2.

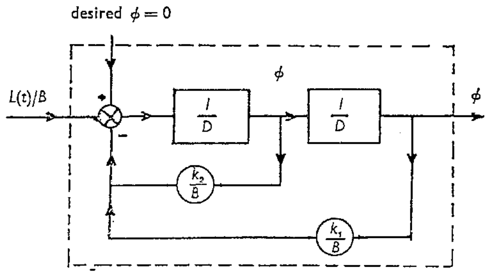


Figure 7

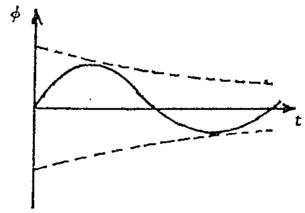


Figure 8

4. Concluding comments

In this article simple physical examples have been used to illustrate some of the more important concepts of control theory. The main points we have observed are:

(i) that control mechanisms occur in a very large number of widely differing contexts;

(ii) that in spite of the different external forms they take nevertheless many control problems have a very similar structure which allows one to treat them by the application of a comparatively small number of techniques;

(iii) that a central idea is the structural concept of negative feedback; and

(iv) that for the large class of problems which can be modelled by linear differential equations one very useful analytic device is the concept of a transfer operator. This linked with the visual assistance provided by a block diagram gives a very powerful technique for the analysis of systems of large dimensions formed from interacting sub-systems.

Further reading

Mathematical Thinking in Behavioral Sciences (Readings from Scientific American), published in paperback by W. H. Freeman, 1968. This book contains a wealth of interesting and easily accessible contributions by famous scientists. In particular, N. Wiener on Cybernetics, A. Tustin on Feedback and R. Bellman on Control Theory.

Proving the Rule or First Steps in Rigour

D. B. SCOTT
University of Sussex

One of the greatest difficulties in learning mathematics is that it takes a certain amount of experience to learn to recognise correct methods of proof. The ideal view of mathematics is that it is all a sequence of steps each of which is a logical consequence of the previous ones. Although a convincing account of a subject one has mastered should be in that form, it is not usually the way most of us begin to learn. In the early stages it is not at all easy to grasp the principles of mathematical argument: in fact one usually learns through some kind of brain-washing treatment in which the procedures are gradually acquired through a judicious mixture of precept and example from our teachers. It takes the learner a very long time to be able to appreciate, criticise and evaluate techniques of proof.

In this naturally murky and confused situation there is one shining beacon, the old, and much misunderstood, principle that

'the exception proves the rule',

and this is the source of our title. Of course 'proves' is used here in the sense of 'tests'. An exception is the crucial test of any mathematical or logical proposition. So that, while we may be unsure as to what constitutes a proof of a mathematical statement, we can nevertheless be absolutely certain about some kinds of disproof. So the one thing on which all can agree at an early stage is the principle that if a proposition or a technique of proof purports to establish a result which is known to be false, then the proposition is false or the method of proof is either invalid or has been wrongly applied. (And if we can rule out the two latter alternatives we have the principle of '*reductio ad absurdum*' or of 'proof by contradiction'.)

Thus the notion of a counter-example is the first basic mathematical idea which can be universally agreed to be fully convincing. It is from this and not, as some people imagine, from the pontifications of long-haired (or even bald-headed) professors that our notions of rigour must be derived. The essential point is that rigour is the self-discipline we have to impose on ourselves when we find the sophistication of our grasp of a subject outrunning that of our techniques of proof. And it is this process, which depends on a growing mastery of our material, which explains why rigour is not a static concept, but one that is continually developing, and increasingly illuminating what we know.

The purpose of this article is to demonstrate the use of counter-examples in consolidating our grasp of mathematical ideas.

Let us begin with a well-known false proposition:

$$\boxed{1 = 2.}$$

We shall give two 'proofs' of this and show how the necessity of rejecting each of them illustrates how rigour is implanted.

First proof that $1 = 2$

Let $x = 1$.

$$\therefore \quad x = x^2, \qquad \therefore \quad x-1 = x^2-1,$$
$$(x-1) = (x-1)(x+1),$$
$$\therefore \quad \underline{1 = x+1,} \quad \text{i.e., } 1 = 2.$$

What has gone wrong here? Clearly the error, at the stage underlined, was in dividing by $(x-1)$ which is zero, and division by zero is not allowed, because *undefined.*

So the moral is that *we must avoid division by zero, or by anything which* MIGHT *be zero.* The only adequate remedy is to adopt a policy of *always* explaining why any expression by which we divide is not zero.

The second proof that $1 = 2$ is based on the highly popular method of 'backward argument'. The idea is to prove an identity by reducing it to something which clearly is true and then claiming that the original one was true also. The proof which follows was tried out on a tutorial group as a protest against their excessive use of the technique and to discover their reaction.

Second proof that $1 = 2$

Suppose $1 = 2$.

$$\underline{\therefore \quad 2 = 1.}$$

Hence (adding) $3 = 3$ which is true!

$$\therefore \quad \text{original hypothesis correct.}$$

The reactions to this were a bit disappointing. They could not see what was wrong with the argument—they just did not like the conclusion. But even that response was better than nothing!

The remedy in this case is clear. It is to make a point of using the implication sign $\Rightarrow$ or $\Leftarrow$ (or even $\Downarrow$ or $\Uparrow$ if that is required by the context) and the equivalence sign $\Leftrightarrow$ (or $\Updownarrow$) where we have two-way implication. If now we set out the previous argument with implication signs we see that *all* of them point downwards! There is not therefore an allowable path *upwards* from the conclusion $(3 = 3)$ to the hypothesis $(1 = 2)$. This incidentally draws attention to the valuable point that backward argument, while ungainly, is not *always* illegal: it is permissible so long as *all intermediate steps are reversible.*

Another simple counter-example arises from a question often given in Sussex to candidates for admission.

For what values of x is $1/(x-1) < 2/(x-2)$?

The usual solution goes as follows:

$$\text{‘}x-2<2(x-1)$$
$$\therefore \quad x > 0,$$

so the inequality holds for all positive x’.

At this stage we ask the customers what actually happens to the original inequality if $x = 1\frac{1}{2}$.

We then get $1/\frac{1}{2} < 2/-\frac{1}{2}$ or $2 < -4$, and after a lot of worrying over it they admit that it *is* incorrect. So then we ask what has gone wrong, usually without much result.

We enquire what we have actually done with the original inequality. In fact, we began by cross-multiplying, i.e., multiplying both sides by $(x-1)(x-2)$ and for $x = 1\frac{1}{2}$ this is equal to $-\frac{1}{4}$, and *multiplying by negative numbers reverses inequalities.* So the original step was illegitimate: we cannot multiply inequalities by negative numbers, or by quantities which *might* be negative, as this reverses the inequality.

A correct solution is as follows (an alternative procedure is to clear denominators by multiplying only by *squares* of denominators):

$$\frac{2}{x-2}-\frac{1}{x-1} > 0, \quad \text{i.e.,} \ \frac{x}{(x-1)(x-2)} > 0$$

$\therefore$ one, or all three, of x, $x-1$, $x-2$ must be positive.

So either $x > 2$ or $0 < x < 1$. (Our original solution wrongly included the range $1 \leqslant x \leqslant 2$ and it is obvious that there is something fishy at the ends of the interval as well as in the middle.)

Thus we have learnt a lot about rigour from these very simple errors.

* * * * *

Now no serious mathematics can be done without the principle of induction in some form. This can be stated in several ways, but a good and simple one, much used, is the following.

If T_n is a proposition depending on n, then

$$\left.\begin{array}{ll} \text{(a)} & T_n \Rightarrow T_{n+1} \text{ for } n \geqslant 1 \\ \text{and} & \\ \text{(b)} & T_1 \text{ is true} \end{array}\right\} \Rightarrow T_n \text{ is true for } n \geqslant 1.$$

This principle can be used to prove the (very reasonable) result that

$$1+2+ \ldots +n = \tfrac{1}{2}n(n+1).$$

We might equally try to prove that $1+2+ \ldots +n = \frac{1}{2}n(n+1)+126$. In this case (a) is valid but (b) is not.

A more sophisticated example is the following. We wish to prove the (rather surprising) proposition which follows:

'All the candidates taking any examination score the same number of marks'. To prove this it is enough to establish (as we do not envisage infinitely many candidates) for all positive integral n the following proposition T_n:

'Any n of the candidates taking the same examination have the same marks'.

We proceed as follows. First T_1 is obviously true. Now assume T_n and try to establish T_{n+1}. To this end we consider any set, say G_{n+1}, of $n+1$ candidates and try to deduce from T_n that they must all have the same marks.

Remove one candidate A from G so that we are left with n others. Then by the inductive hypothesis all candidates in $G-A$ have the same marks. Now replace A and extract a second candidate B and consider $G-B$. Again by our inductive hypothesis all these candidates have the same marks. Thus A and B each have the same marks as the remainder of G and hence the same as one another. Hence *all* members of G_{n+1} have the same marks, so we have established that $T_n \Rightarrow T_{n+1}$. Thus by the principle of induction we deduce that T_n is true for all values of n, so that our original unlikely proposition appears to be proved.

So assuming that we do not really believe that the proposition is true we have to explain what has gone wrong. Is the principle of induction incorrect, or have we applied it wrongly?

* * * * *

Another useful example, in a different field, is the following result known as 'Scott's Theorem on Determinants'[1] which asserts that

"All determinants are zero'.

Let us illustrate this by proving the result for a 3×3 determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We know that we can add or subtract one row from any other without affecting Δ. Assuming this we take the second row from the first, the third from the second, and the first from the third. Thus

$$\Delta = \begin{vmatrix} a_1-a_2 & b_1-b_2 & c_1-c_2 \\ a_2-a_3 & b_2-b_3 & c_2-c_3 \\ a_3-a_1 & b_3-b_1 & c_3-c_1 \end{vmatrix}.$$

Now add rows 2 and 3 to row 1 and we get a string of zeros along the top. Expanding by the first row we get $\Delta = 0$.

So are the theorems on determinants we have used incorrect, or if not what has gone wrong?

* * * * *

Now for some more subtle counter-examples, obtained by routine applications of the rule for integration by substitution.

[1] Rediscovered by successive generations of students.

This says that

$$\boxed{\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)dt, \quad \text{where} \quad a = g(\alpha), \quad b = g(\beta).}$$

This is simply

$$\int f(x)\,dx = \int f(x)\frac{dx}{dt}\,dt$$

written in a form allowing us to bring in the limits of integration.

We give three examples of this technique.

Example A.

$$\int_0^1 \frac{dx}{1+x^2}.$$

Let $x = \tan t$ so that $dx/dt = \sec^2 t$, and now evaluate the integral in two ways.

A(i) $\alpha = 0$, $\beta = \pi/4$.

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{\pi/4} \frac{1}{1+\tan^2 t}\sec^2 t\,dt = \pi/4.$$

A(ii) $\alpha = 0$, $\beta = 5\pi/4$.

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^{5\pi/4} \frac{1}{1+\tan^2 t}\sec^2 t\,dt = 5\pi/4.$$

Of course *both* these answers cannot be correct: indeed the second one is clearly wrong as the integrand is always $\leqslant 1$ so the integral cannot exceed 1. Naturally this does not prove that the first answer is right.

Example B.

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}}.$$

Let $x = \sin t$ so that $dx/dt = \cos t$.

B(i) $\alpha = 0$, $\beta = \pi/6$.

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}} = \int_0^{\pi/6} \frac{\cos t}{\sqrt{(1-\sin^2 t)}}\,dt = \int_0^{\pi/6} dt = \pi/6.$$

B(ii) $\alpha = 0$, $\beta = 5\pi/6$.

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}} = \int_0^{5\pi/6} \frac{\cos t}{\sqrt{(1-\sin^2 t)}}\,dt = \int_0^{5\pi/6} dt = 5\pi/6.$$

Again we have to ask whether the first answer is right, but we can easily see that the second one is wrong. For

$$\frac{1}{\sqrt{(1-x^2)}} \leqslant \frac{1}{\sqrt{(1-(\frac{1}{2})^2)}} \leqslant \frac{2}{\sqrt{3}}.$$

So the integral cannot exceed $1/\sqrt{3} = 2\sqrt{3}/6$, and as 5π is bigger than $2\sqrt{3}$ the second answer is inadmissible. (Note, however, that π is actually less than $2\sqrt{3}$ so the first answer is not inconceivable.)

Example C.

$$\int_0^1 \sqrt{(1-x^2)}\, dx.$$

A very well-known integral indeed. Again $x = \sin t$, $dx/dt = \cos t$.

C(i) $\alpha = 0$, $\beta = \pi/2$.

$$\int_0^1 \sqrt{(1-x^2)}\, dx = \int_0^{\pi/2} \sqrt{(1-\sin^2 t)} \cos t\, dt = \int_0^{\pi/2} \cos^2 t\, dt.$$

This can be calculated neatly by showing that

$$\int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} \sin^2 t\, dt$$

so that each is

$$\frac{1}{2}\int_0^{\pi/2} (\cos^2 t + \sin^2 t)\, dt = \frac{\pi}{4}.$$

We use a cruder technique because we need it later. Since

$$\cos^2 t = \tfrac{1}{2}(1+\cos 2t),$$

$$\int_0^1 \sqrt{(1-x^2)}\, dx = \tfrac{1}{2}[t + \tfrac{1}{2}\sin 2t]_0^{\pi/2} = \frac{\pi}{4}.$$

(A not implausible result.)

C(ii) $\alpha = \pi$, $\beta = \pi/2$.

$$\int_0^1 \sqrt{(1-x^2)}\, dx = \int_\pi^{\pi/2} \sqrt{(1-\sin^2 t)} \cos t\, dt = \int_\pi^{\pi/2} \cos^2 t\, dt = \tfrac{1}{2}[t + \tfrac{1}{2}\sin 2t]_\pi^{\pi/2} = -\frac{\pi}{4}.$$

A highly implausible answer because the integral should not be negative.

C(iii) $\alpha = 0$, $\beta = 5\pi/2$.

$$\int_0^1 \sqrt{(1-x^2)}\, dx = \int_0^{5\pi/2} \sqrt{(1-\sin^2 t)} \cos t\, dt = \int_0^{5\pi/2} \cos^2 t\, dt = \tfrac{1}{2}[t + \tfrac{1}{2}\sin 2t]_0^{5\pi/2} = \frac{5\pi}{4}.$$

Again this answer is wrong because the integrand (and hence the integral) cannot exceed 1.

All these answers have been obtained by routine use of a much employed formula. Having obtained so many wrong answers we have no option but to look more closely at the formula and make explicit the hidden assumptions on which it depends.

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt, \quad g(\alpha) = a, \quad g(\beta) = b.$$

Now what are the restrictions imposed on f and g? Clearly essential conditions are

$$\underline{f(x) \text{ is defined in } a \leqslant x \leqslant b \quad \text{and} \quad g(t) \text{ in } \alpha \leqslant t \leqslant \beta.}$$

However, even these minimal restrictions are violated in A(ii) since then $g(t) = \tan t$, the interval is $0 \leqslant t \leqslant 5\pi/4$ and $\tan t$ is *not* defined at $t = \pi/2$ (where $g(t)$ is undefined and wildly discontinuous).

We have therefore disposed rapidly of A(ii).

The restriction above is quite inadequate as it does not guarantee the existence of both the integrals in the formula, so that we have to impose at least sufficient conditions to ensure that the integrals exist. As we do not want to apply this formula in pathological cases it is not unreasonable to assume that the functions occurring are continuous.

Let us, then, start with the following assumptions and see how they serve us.

$$\boxed{\begin{array}{c} f(x) \text{ is continuous in } a \leqslant x \leqslant b. \\ g'(t) \text{ exists and is continuous in } \alpha \leqslant t \leqslant \beta. \end{array}}$$

Both these conditions are satisfied in cases B and C. Thus we obviously need some further restrictions.

The conditions of continuity given above were intended to cover the existence of both integrals. But do they in fact do so? Clearly they will if $f(g(t))$ is continuous. But of course this is a continuous function of a continuous function and is therefore continuous.

Unfortunately this is too slick. We only know that $f(x)$ is continuous in $a \leqslant x \leqslant b$ and we cannot guarantee that, if $\alpha \leqslant t \leqslant \beta$, $g(t)$ is in the range $[a, b]$. Indeed example B(ii) illustrates this. Here $1/\sqrt{(1-x^2)}$ is continuous in $0 \leqslant x \leqslant \frac{1}{2}$, but if $x = \sin t$ and $\alpha \leqslant t \leqslant \beta$ is $0 \leqslant t \leqslant 5\pi/6$, then for t in the range $\pi/6 \leqslant t \leqslant 5\pi/6$, x is *not* in the range $(0, \frac{1}{2})$ and indeed for $t = \pi/2$, *$f(g(t))$ is not even defined.*

So to get a continuous integrand in the second integral we must at least impose the following conditions.

$$\boxed{\begin{array}{c} g(t) \text{ has a continuous derivative in } \alpha \leqslant t \leqslant \beta \text{ and } a = g(\alpha),\ b = g(\beta). \\ f(x) \text{ is continuous in an interval of values of } x \text{ containing all values of} \\ g(t) \text{ with } t \text{ in } [\alpha, \beta]. \end{array}}$$

Note that these conditions rule out the awkward possibilities of A and B but they do not touch C.

We have already explained why the conditions of validity of the theorem must include the ones above. But do we need any others? The answer is *no*: the conditions above do enable us to prove the result.

Well then what about example C? We cannot ignore it: we must either regard it as disposing of our alleged proof of the result, or we have to sort it out in some other way. The solution of this problem is remarkably simple. We have assumed the elementary equation

$$\cos t = \sqrt{(1-\sin^2 t)}.$$

Unfortunately *this is obviously false*. It is true only in the first and fourth quadrants. But in the second and third quadrants the correct result is

$$\cos t = -\sqrt{(1-\sin^2 t)}.$$

With this emendation, the correct evaluation of the integrals in C(ii) and C(iii) gives the same answer as in C(i), so that there is no real trouble.

*   *   *   *   *

All this illustrates the point that investigating the conditions under which a well-known result actually holds is not purely an exercise of pedantry, but an important matter which is forced upon us as soon as our grasp of a subject becomes sufficiently firm for problems to arise. So the moral of what we have done is that rigour is not, as so many of us at first imagine, the natural enemy of the mathematical learner. It is rather a vital navigational aid which we really need as soon as we learn to recognise the reefs and shoals through which it can steer us.

Suggestions for further reading

D. B. Scott and S. R. Tims, *Mathematical Analysis: An Introduction* (Cambridge University Press, 1966).

Approximation to Square Roots

P. SHIU

University of Technology, Loughborough

1. Introduction

A rational number, or fraction, is a number of the form n/m where m, n are integers and $m > 0$. Clearly a rational number must satisfy an equation of the form

$$\alpha_0 x + \alpha_1 = 0, \tag{1}$$

where α_0, α_1 are integers; and conversely, the root of equation (1) must be a rational number. A number which is not rational is called an irrational number. If an irrational number is a root of an equation

$$\alpha_0 x^k + \alpha_1 x^{k-1} + \ldots + \alpha_{k-1} x + \alpha_k = 0, \tag{2}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are integers, we call it an irrational algebraic number. Irrational numbers which are not algebraic are called transcendental numbers. Examples of irrational algebraic numbers are $\sqrt{2}$, $\sqrt[3]{7}$ and $(1+\sqrt{5})/2$ for they satisfy respectively the equations $x^2-2=0$, $x^3-7=0$ and $x^2-x-1=0$. Examples of transcendental numbers are e, π, and $\log 2$, though it is difficult to prove that they actually satisfy no equation of the type such as (2).

2. Approximations

Although an irrational number α cannot be put in the form n/m, there are rational numbers which are arbitrarily close to it. We call such rational numbers rational approximations to α. Since the closeness can be arbitrary, we need a method to measure the accuracy of the approximation. It is clear that, in general, the more accurate the approximation is required, the larger the denominator must be. We therefore compare the degree of accuracy with the denominator of the rational approximation.

Let α be an irrational number. A rational number n/m such that

$$\left|\frac{n}{m}-\alpha\right| < \frac{1}{m^v}$$

$$\text{is called}\begin{cases}\text{a trivial approximation to } \alpha \text{ if } v=1,\\ \text{a reasonable approximation to } \alpha \text{ if } v=2,\\ \text{a good approximation to } \alpha \text{ if } v>2.\end{cases}$$

Clearly there are infinitely many trivial approximations to α since such approximations can easily be obtained by converting any decimal approximations to the corresponding rational numbers. For example, since $\sqrt{2} = 1{\cdot}414\ldots$, $\frac{141}{100}$ is a trivial approximation to $\sqrt{2}$ because

$$\left|\tfrac{141}{100}-\sqrt{2}\right| = 0{\cdot}004\ldots < 0{\cdot}01 = \tfrac{1}{100}.$$

But $\frac{141}{100}$ is obviously not a reasonable approximation. In fact, associated with each denominator m, there exists a trivial approximation to α, for we can always choose n such that

$$\left.\begin{aligned} &\frac{n}{m} < \alpha < \frac{n+1}{m}, \\ \text{so that}\qquad &\\ &0 < \alpha - \frac{n}{m} < \frac{1}{m}. \end{aligned}\right\} \tag{3}$$

It can be proved that there are always infinitely many reasonable approximations to any irrational number, and in particular to $\sqrt{N}$ where N is a square-free integer, that is an integer which is not divisible by any square other than 1. We shall extend (3) to show how to obtain reasonable approximations from trivial approximations by considering the means of the numbers n/m and $(n+1)/m$, and also how to generate infinitely many of them once one is obtained.

It is not too difficult to prove that $\sqrt{N}$ cannot have infinitely many good approximations; but whether irrational numbers in general have infinitely many good approximations is a difficult problem, which was solved only in 1955 by the British mathematician K. F. Roth who proved that there are at most a finite number of good approximations to any irrational algebraic number. It is interesting to note that the first number known to be transcendental was purposely constructed in 1851 by the French mathematician J. Liouville so that there are infinitely many abnormally good approximations, i.e., approximations with a large v. To determine whether a given number is transcendental or not is a really difficult problem, and one possible method is to study its rational approximations.

3. Application of means

The idea here is that if

$$\frac{n}{m} < \sqrt{N} < \frac{n+1}{m}$$

so that n/m and $(n+1)/m$ are approximations from below and from above respectively to $\sqrt{N}$, then their arithmetic mean and geometric mean may be 'better' approximations. But of course the arithmetic mean $(2n+1)/2m$ will have a larger denominator for the comparison of the reasonableness, and more important still, the geometric mean $\sqrt{[n(n+1)]}/m$ will not even be rational since $n(n+1)$ cannot be a square. This however turns out to be in our favour because we can therefore hope that $\sqrt{[n(n+1)]}/m$ may in fact be exactly $\sqrt{N}$. If it is, that is, if $n(n+1) = Nm^2$, then we should expect the arithmetic mean $(2n+1)/2m$, being only slightly bigger than the geometric mean, to be a reasonable approximation from above. This is indeed the case.

Theorem 1. If $n(n+1) = Nm^2$, then

$$0 < \frac{2n+1}{2m} - \sqrt{N} < \frac{1}{8\sqrt{N}m^2} < \frac{1}{(2m)^2},$$

and so $(2n+1)/2m$ is a reasonable approximation to $\sqrt{N}$ from above.

Proof. Let $n(n+1) = Nm^2$. Then $4n(n+1) = N(2m)^2$, i.e., $(2n+1)^2 - 1 = N(2m)^2$, and so

$$\left(\frac{2n+1}{2m}\right)^2 - N = \frac{1}{4m^2} > 0,$$

and therefore

$$0 < \frac{2n+1}{2m} - \sqrt{N} = \frac{1}{(2n+1)/2m+\sqrt{N}} \cdot \frac{1}{4m^2} < \frac{1}{2\sqrt{N}} \cdot \frac{1}{4m^2} = \frac{1}{8\sqrt{N}m^2} < \frac{1}{(2m)^2},$$

as required.

Theorem 2. Let N be a given square-free integer, and let

$$n(n+1) = Nm^2 \tag{4}$$

be satisfied by $m = m_0$, $n = n_0$. Then it is also satisfied by $m = m_1$, $n = n_1$, where

$$m_1 = 2m_0(2n_0+1), \quad n_1 = 4n_0(n_0+1).$$

Proof. Since $n_0(n_0+1) = Nm_0^2$, we have

$$\begin{aligned} n_1(n_1+1) &= 4n_0(n_0+1).[4n_0(n_0+1)+1] \\ &= 4Nm_0^2.(2n_0+1)^2 \\ &= N[2m_0(2n_0+1)]^2 \\ &= Nm_1^2, \end{aligned}$$

as required.

It is easy to see now that if equation (4) has a set of initial solutions m_0, n_0, then there are infinitely many sets of solutions m_k, n_k given by

$$\begin{array}{ll} m_1 = 2m_0(2n_0+1) & n_1 = 4n_0(n_0+1) \\ m_2 = 2m_1(2n_1+1) & n_2 = 4n_1(n_1+1) \\ \vdots & \vdots \\ m_k = 2m_{k-1}(2n_{k-1}+1) & n_k = 4n_{k-1}(n_{k-1}+1) \\ \vdots & \vdots \end{array}$$

Also by theorem 1 each set of solutions m_k, n_k gives rise to a reasonable approximation to $\sqrt{N}$ from above namely $(2n_k+1)/2m_k$.

4. An example

We shall illustrate the method with an example. Suppose we want to find reasonable approximations to $\sqrt{2}$ which give high degrees of accuracy.

Let $N = 2$. We therefore try to find m, n such that $n(n+1) = 2m^2$. This is easy—since $1.2 = 2.1^2$, $m_0 = n_0 = 1$ will do. We then calculate the following numbers:

$$\begin{array}{ll} m_0 = 1 & n_0 = 1 \\ m_1 = 2m_0(2n_0+1) = 6 & n_1 = 4n_0(n_0+1) = 8 \\ m_2 = 2m_1(2n_1+1) = 204 & n_2 = 4n_1(n_1+1) = 288 \\ m_3 = 2m_2(2n_2+1) = 235{,}416 & n_3 = 4n_2(n_2+1) = 332{,}928 \\ m_4 = 2m_3(2n_3+1) = 313{,}506{,}783{,}024 & n_4 = 4n_3(n_3+1) = 443{,}365{,}544{,}448 \\ \vdots & \vdots \end{array}$$

By theorem 1 we have

$$0 < \frac{2n_k+1}{2m_k} - \sqrt{2} < \frac{1}{8\sqrt{2m_k^2}}, \quad k = 0, 1, 2, \ldots.$$

Using $k = 4$, we have

$$0 < \frac{886{,}731{,}088{,}897}{627{,}013{,}566{,}048} - \sqrt{2} < 10^{-24},$$

giving a reasonable approximation to $\sqrt{2}$ correct to 24 decimal places.

5. Remarks

(1) It is natural to ask if we get a better result by using the harmonic mean[1] instead of the arithmetic mean. It turns out that the harmonic mean, though closer to the geometric mean than is the arithmetic mean does not always give a reasonable approximation, because $2/[m/n+m/(n+1)] = 2n(n+1)/m(2n+1)$ has a much larger denominator for the comparison of its own reasonableness. In fact, using the same idea as in theorem 1, we have

$$\left[\frac{2n(n+1)}{m(2n+1)}\right]^2 = \frac{4m^2N^2}{4m^2N+1} = N - \frac{1}{4m^2} + \rho,$$

where the remainder term ρ is of the order of magnitude $1/m^4$, which shows that $2n(n+1)/m(2n+1)$ is a 'close' approximation to $\sqrt{N}$ from below, but we cannot expect any reasonableness.

(2) We have not guaranteed the existence of an initial solution of equation (4) for the iteration, though actually equation (4) does always possess such an initial solution; but the proof is not easy.

(3) We may very likely ask whether

$$\frac{2n_{k+1}+1}{2m_{k+1}} - \sqrt{N} < \frac{2n_k+1}{2m_k} - \sqrt{N},$$

since it does not follow from our argument directly that successive approximation does improve the error. However, it is easy to show that the ratio of successive approximations is, in fact,

$$\frac{8n_k^2+8n_k+1}{8n_k^2+8n_k+2} < 1,$$

so that the sequence of approximations decreases steadily and tends to $\sqrt{N}$.

Suggestions for further reading

Younger students who want to know more about the number system would find helpful *Numbers Without End* by Professor C. Lanczos (Contemporary Science Paperback, 22. Oliver & Boyd, Edinburgh, 1968). Sixth formers who wish to follow up the topics of this article are recommended to try *Numbers: Rational and Irrational* by I. Niven (Random House, New York, 1961). University students should profit from reading *The Higher Arithmetic* by H. Davenport (Hutchinson's University Library, London, 1952), where there is a detailed solution to the question raised in Remark 2.

[1] The harmonic mean of two positive numbers a, b is defined to be $2ab/(a+b)$.

Adventures in Ballistics, 1915–1918. I

J. E. LITTLEWOOD

Trinity College, Cambridge

1. I became a second lieutenant in the R.G.A. (7) soon after the outbreak of the 1914 war. About September 1915, I conceived a plan for dealing with zeppelins. This involved anti-aircraft range tables, and I had a short period of leave to work one out, which I did successfully, I felt, at least so far as targets not too far from the vertical were concerned. I visited the Ballistic office with my plan, which was immediately turned down as impracticable by Captain (later Major) Douglas, the regular officer responsible for all range tables. But he asked me: 'Can you make A.A. range tables?', a problem for which they had no glimmer of an idea. (It was possible to calculate any single trajectory by a step-by-step process, called the 'method of small arcs', but with computers in the far future a single small arc trajectory might take a week with the computing staff available. Since tables were needed for several guns, small arcs were quite impracticable.)

(It may be news to readers that a computer can now calculate a trajectory as fast as the shell traverses it.)

2. I must now describe my method. In order to find approximations in such a problem it is necessary to suppose that something, K say, is small (or, equivalently, large), otherwise theory has no grip on the subject.

A 'gun' is an ordered pair of numbers c, and V, the initial velocity. The air resistance at time t is $c\rho p(v)$, where v is the velocity at time t, $p(v)$ is given by a table, and $\rho(y)$ is the density of the air at the height y of the projectile. $\rho(y)$ is taken (pretty accurately) as e^{-ky}, which has approximately the value $\frac{1}{2}$ for $y = 15{,}000$ feet. A rough idea of $p(v)$ is that it is Kv^2 for $v < w$, the velocity of sound at ground level, and $3Kv^2$ for $v > w$. In practice $cp(V)$ is comparable with g and we shall assume this in stating O-terms.

3. Consider the trajectory Γ initially making an angle $\alpha = \frac{1}{2}\pi - \phi$ with the vertical, ϕ being the elevation of Γ.

For co-ordinates of P, (x, y) at time t, we take, in Figure 1, $\xi = OT$, $\eta = PT$, where PT is vertical.

Evidently ξ, η are even functions of α and so functions of $1 - \sin\phi$;

$$\xi = \sum_0 (-)^n (1-\sin\phi)^n X_n(t), \quad \eta = \sum_0 (-)^n (1-\sin\phi)^n Y_n(t).$$

My initial guess was that sufficient accuracy could be obtained by not going beyond $n = 2$, when the formulae become

$$\xi = X - X_1(1-\sin\phi), \quad \eta = Y - Y_1(1-\sin\phi). \tag{3.1}$$

We shall find ample justification for this later.

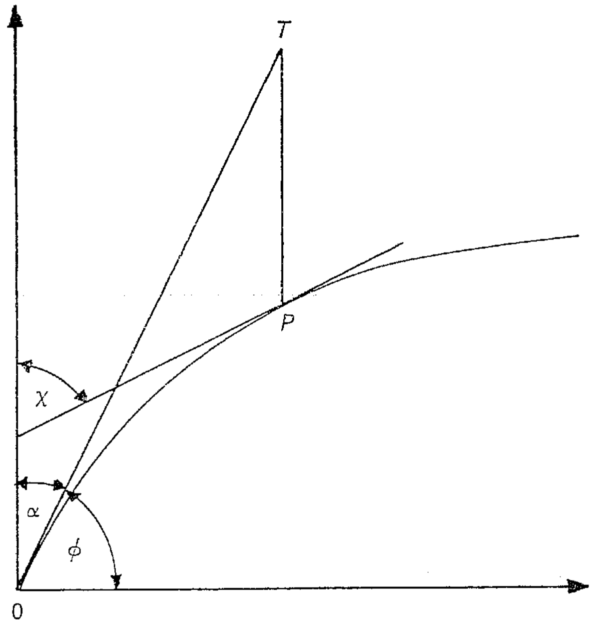


Figure 1

4. In the first place we can prove rigorously:

Theorem. Let $H(t)$ be the height and w the velocity at time t in the vertical Γ (calculated by a step-by-step process—not very exacting in this special case $\alpha = 0$). Let

$$\dot{Y} = w(t)\left[\int_0^t \exp\{g/w(t)\}\,dt - 1\right], \quad Y = \int_0^t \dot{Y}\,dt,$$
$$X = H + Y.$$

Then

$$\lim_{\alpha\to 0} \xi = X, \quad \lim_{\alpha\to 0} \eta = Y.$$

Let the tangent to Γ at P make an angle χ as shown in Figure 1. We have

$$x = \xi\sin\alpha, \quad y = \xi\cos\alpha - \eta, \quad \eta = x\cot\alpha - y,$$
$$\dot{x} = v\sin\chi, \quad \dot{y} = v\cos\chi,$$

and so

$$\dot{\eta} = v(\sin\chi\cot\alpha - \cos\chi) = v\sin(\chi-\alpha)/\sin\alpha. \tag{4.1}$$

Resolving normally at P we have

$$v\dot{\chi} = \left(\frac{ds}{dt}\right)^2 \frac{d\chi}{ds} = \frac{v^2}{\rho} = g\sin\chi,$$

so

$$\frac{\dot{\chi}}{\chi} = \frac{g}{v}\frac{\sin\chi}{\chi}, \quad \chi = \alpha \quad \text{at} \quad t = 0. \tag{4.2}$$

By (4.1),

$$\eta = \frac{v\sin(\chi-\alpha)}{\sin\alpha}.$$

As χ and thus α tends to 0,

$$\dot{Y} = \lim \dot{\eta} = \lim \int_0^t v\frac{\chi-\alpha}{\alpha}\,dt. \tag{4.3}$$

Now

$$\log(\chi/\alpha) = \log\chi - \log\alpha = \int_0^t (\dot{\chi}/\chi)\,dt$$

$$= \int_0^t \frac{g}{v}\frac{\sin\chi}{\chi}\,dt = J, \quad \text{say,}$$

$$\frac{\chi-\alpha}{\alpha} = \frac{\chi}{\alpha} - 1 = e^J - 1.$$

By this and (4.3)

$$\dot{Y} = \lim_{\alpha\to 0} \int_0^t v(e^J - 1)\,dt$$

$$= w\left[\int_0^t \exp\{g/w\}\,dt - 1\right],$$

the result of the theorem for $\dot{Y}$. It is evident that $X = H + Y$, and the proof is completed.

5. I went on to establish exact formulae for X_1, Y_1; these are double integrals involving X and Y. For reasons which will appear it is not necessary to give the details.

Douglas and I carried out the numerical calculations of X, Y, X_1, Y_1 for a test case. We then pushed the results to the limit of a small ϕ (α near $\frac{1}{2}\pi$); there was a reasonably accurate method for such 'direct fire' Γ. We both said: 'We cannot expect good agreement in this extreme case'; but to our surprise and delight the agreement was quite good. The obvious course was then to choose X_1, Y_1 so as to *force* agreement, a procedure which reduced the maximum error by a factor of $\frac{1}{4}$.

The ultimate procedure, to avoid the errors of the 'direct fire' formulae, was to calculate the $\phi = 30°$ Γ by 'small arcs' and make X_1, Y_1 fit this (X and Y are exact, remember).

6. Soon after our initial success there was a comic interlude. We were sent the results of two experimental observations of gun bursts of an actual gun; one of these differed violently with our formulae. Though we were confident, this was disconcerting, and any official communication had to receive a full answer.

(I digress to give an instance. A V.I.P. proposed to black out the Thames, a guide to zeppelins, with sawdust. Fortunately there was a two word answer: 'Sawdust sinks'.) For some time we could not find a convincing answer, but eventually I found that the perpendicular distance of the burst from the initial tangent of the Γ was considerably greater than it would be *in vacuo*, apart from the component of resistance!

7. With resistance μv and a homogeneous atmosphere it is not difficult to prove that

$$\xi = V\mu^{-1}(1-e^{-\mu t}) = X,$$
$$\eta = X-H.$$

Clearly our formulae hold exactly, with $X_1 = Y_1 = 0$.

(As a matter of historical fact I started with η perpendicular to ξ and changed because of this result.)

If we suppose a resistance μv and air density e^{-ky} a rather more elaborate discussion shows that the formulae are exact to the first power of k.

8. A numerical test. This was made for the gun $c^{-1} = 2{\cdot}8345$, $V = 2150$ f.s., and was carried out (in another department than ours) to 3 decimal figures of the unit foot! It is instructive to record sample values of X, Y, X_1, and Y_1.

TABLE 1

t (approx.)	11·5	18	25
X	17,895	25,492	33,501
Y	1,863	4,353	8,360
X_1	1,242	2,955	5,503
Y_1	121	421	1,001
X_1/X	0·069	0·115	0·164
Y_1/Y	0·065	0·096	0·130
$X_1/X-Y_1/Y$	0·004	0·009	0·034

The results were manipulated, on obvious lines, to provide values of y and t for given x for the Γ with $\phi = 45°$, for which errors would be near a maximum. In Table 2 (1) refers to the ξ, η formulae, (2) to the exact values.

TABLE 2

x	12,386	17,402	22,568
y (1)	10,563·3	13,179·5	14,489
y (2)	10,561·6	13,165·7	14,486
Error	+1·7	+13·8	+3
t (1)	11·64	14·82	24·57
t (2)	11·65	14·83	24·55
Error	−0·01	−0·01	+0·02

Similar results were found for two other guns for c's and V's differing considerably.

This degree of accuracy did come as a surprise. In what follows I develop some further theory throwing light on the point.

9. Recall that at the beginning of the story X_1, Y_1 were found by exact but complicated formulae, and that these gave very good agreement with the $\bar{\xi}$, $\bar{\eta}$ of the Γ with $\phi = 0$. After this the method—method 2, say—was to choose X_1, Y_1 so as to force exact agreement at $\phi = 0$.

If, however, we wish to find upper bounds of the form Kt^n for the errors in ξ, η *for all* $\phi \geqq 0$, method 2 cannot do this.

What we do is to expand ξ, η in powers of t, reject terms in $\sigma = 1-\sin\phi$ of higher powers than the first, and take for X_1, Y_1 the coefficients of $-\sigma$ in ξ, η. We find that this rejection creates an error $\leqslant K_1 t^4$ in ξ, and one $\leqslant K_2 t^5$ in η, for all $\phi \geqslant 0$. Definite values of $K_{1,2}$ *could* be found, but this would be very laborious, and not worth while in view of the numerical tests.

10. By a suitable choice of the units of length and time we can suppose $V = 1$, $g = 1$, and air resistance $= cp(v)\,e^{-ky}$.

We proceed to expand ξ and η in powers of t. (It is worth noting about errors that the numerical factors in the expansion of a function $f(t)$ are $\frac{1}{24}$ for t^4 and $\frac{1}{120}$ for t^5.)

Let the tangent at P of a trajectory Γ make an angle ψ with the horizontal. Initially $\psi = \phi$, the 'elevation'. Then

$$\dot{x} = v\cos\psi, \quad \dot{y} = v\sin\psi.$$

Resolving horizontally and normally we have

$$\dot{v} = -(cpe^{-ky}+\sin\psi),$$

$$v\dot{\psi} = \left(\frac{ds}{dt}\right)^2\frac{d\psi}{ds} = \frac{v^2}{\rho} = -\cos\psi.$$

Other basic formulae are

$$\begin{aligned} \ddot{x} &= -cpe^{-ky}\cos\psi, & \ddot{y} &= -cpe^{-ky}\sin\psi - 1; \\ \dot{\psi} &= -\cos\psi/v, & \dot{p} &= p'(v)\,(cpe^{-ky}+\sin\psi); \\ \xi &= x\sec\phi, & \eta &= x\tan\phi - y. \end{aligned} \tag{10.1}$$

These enable us in principle to expand ξ and η to any power of t.

X_1 and Y_1 are the coefficients of $\sin\phi$ in ξ and η.

The work is rather heavy, but in the end there are what seem extraordinary cancellations and simplifications. The relevant results are, for all ϕ of $0 \leqslant \phi \leqslant \pi$,

$$\left.\begin{aligned} \xi &= X - X(1-\sin\phi) + O(t^4), \\ \eta &= Y - Y_1(1-\sin\phi) + O(t^5), \end{aligned}\right\} \tag{10.2}$$

where results are (with $p = p(V)$, $p' = p'(V)$)

$$\left.\begin{aligned} X &= t + O(t^2), \quad Y = \tfrac{1}{2}t^2 + O(t^3); \\ X_1 &= \tfrac{1}{6}t^3\{c(p'-p)+kcp\} + O(t^4); \\ Y_1 &= \tfrac{1}{12}t^4\{c(p'-p)+kcp\} + O(t^5); \end{aligned}\right\} \tag{10.3}$$

X and Y are exact and do not depend on ϕ.

The curly brackets for X_1 and Y_1 in (10.3) being (astonishingly) the same, we have the vital result

$$\frac{X_1}{X}-\frac{Y_1}{Y}=O(t^3). \tag{10.4}$$

The most important practical error is in y for given x, or equivalently η for given ξ. Now we have, writing $1-\sin\phi=\sigma$,

$$\begin{aligned}\frac{\eta}{\xi}&=\frac{Y}{X}\frac{1-\sigma Y_1/Y+O(t^3)}{1-\sigma X_1/X+O(t^3)}\\&=\frac{Y}{X}\left[1+\sigma\left(\frac{X_1}{X}-\frac{Y_1}{Y}\right)+O\left\{\left(\frac{X_1}{X}\right)^2\right\}+O(t^3)\right],\end{aligned}$$

which by (10.4) is

$$\frac{Y}{X}\{1+O(t^3)\}=\frac{Y}{X}+O(t^4),$$

so that

$$\frac{\eta}{\xi}=\frac{Y}{X}+O(t^4), \tag{10.5}$$

a fourth power error.

11. Consider now method 2 (used in Tables 1 and 2). Let $(\bar{\xi},\bar{\eta})$ be the Γ with $\phi=0$, for which

$$\bar{\xi}=X-\bar{X}_1,\quad \bar{\eta}=Y-\bar{Y}_1. \tag{11.1}$$

$\bar{X}_1$, $\bar{Y}_1$ are exact (by small arc calculations). Now taking $\phi=0$ in (10.2) gives

$$|\bar{\xi}-(X-X_1)|\leqslant K_1t^4,\quad |\bar{\eta}-(Y-Y_1)|\leqslant K_2t^5,$$

and so, by (11.1),

$$|\bar{X}_1-X_1|\leqslant K_1t^4,\quad |\bar{Y}_1-Y_1|\leqslant K_2t^5.$$

Then, for all ϕ,

$$\begin{aligned}|\xi-(X-\sigma\bar{X}_1)|&\leqslant|\xi-(X-\sigma X_1)|+|\bar{X}_1-X|\\&\leqslant 2K_1t^4,\end{aligned}$$

and similarly for η. Thus the errors in ξ, η in method 2 are certainly not more than double those of (10.2).

Actually, since the errors in ξ, η are 0 at $\phi=0$ and $\phi=\frac{1}{2}\pi$, we should expect them to be smaller by a factor like $\max|\sin\phi(1-\sin\phi)|=\frac{1}{4}$ for $0\leqslant\phi\leqslant\frac{1}{2}\pi$. But a rigorous proof would be very heavy.

12. Digression. A range table was once asked for, quickly, a gun in an aeroplane, to fire at any elevation from 0° to 180°: it did not need to be very accurate. We interpolated between the vertically upward and downward trajectories. But that was not all. For the downward Γ we took air density at $\rho = 1/(1-ky)$, y the downward distance. By a coincidence between constants (and a little faking) the downward Γ was in 'simple harmonic motion'. (The downward end of the motion is at $y = 1/k$, $\rho = \infty$.) How this comes about is explained in the Ballistic section of my book *A Mathematician's Miscellany.*

13. Further developments. Let S be the 'angle of sight', the angle OP makes with the horizontal; let $\phi = E+S$; E is called the 'tangent elevation'. From the triangle OPT we have

$$\sin E \sec S = \eta/\xi.$$

η/ξ is Y/X when $S = \phi = \frac{1}{2}\pi$ and is $\sin\phi_0$ when $S = 0$, where ϕ_0 is the ϕ for a 'flat' Γ with time of flight t. The linear function of $\sin S$ agreeing with these values is

$$\sin\phi_0 - \left(\sin\phi_0 - \frac{Y}{X}\right)\sin S.$$

If now we write

$$\frac{\eta}{\xi} = \sin E \sec S = \sin\phi_0 - \left(\sin\phi_0 - \frac{Y}{X}\right)\sin S + \varepsilon \tag{13.1}$$

$\beta = 1-\sin\phi$, $\beta_0 = 1-\sin\phi_0$, the equations

$$\eta/\xi = \{Y - Y_1(1-\sin\phi)\}/\{X - X_1(1-\sin\phi)\}$$

for $\phi = \phi$ and ϕ_0, give for the ε of (13.1),

$$\frac{\varepsilon}{\sin\phi_0} = \left(\frac{X_1}{X} - \frac{Y_1}{Y}\right)\frac{X}{X-\beta X_1}\cdot\frac{Y}{Y-\beta Y_1}Q, \tag{13.2}$$

$$Q = \beta\sin\phi_0 - 2\beta_0 \sin\tfrac{1}{2}E\cos(S+\tfrac{1}{2}E) - \tfrac{1}{2}\beta_0 X_1/X.$$

Now $\sin\phi_0$ is first order in t, $X_1/X - Y_1/Y$ is second order, E, X_1/X, and so Q are first order. Thus ε is fourth order. We therefore have, from (13.1),

$$\sin E = \{\sin\phi_0 - (\sin\phi_0 - Y/X)\sin S\}\cos S + O(t^4). \tag{13.3}$$

14. Digression. Since the second term in curly brackets in (13.3) is $O(t^2)$ we have

$$\sin E = \sin\phi_0 \cos S + O(t^2).$$

The approximate formulae $\sin E = \sin\phi_0\cos S$ was inferred from actual fire by Captain Dunkley before the above analysis was done. While short of (13.3) by two orders it is fairly accurate.

15. If (13.3) is to be used for making range tables we need a second formula, giving $R = OP$. Now we have, exactly,

$$R\cos S = \xi\cos\phi.$$

Hence by $\xi = X - X_1(1-\sin\phi)$,

$$R = \{X - X_1(1-\sin\phi)\}\cos\phi\sec S,$$

where $\phi = E+S$, and E is given by (13.1).

16. When this research was completed I was asked to write up the results, without proofs for official use. I said that I could not possibly do this in an office, and it was agreed that I should live where I liked, visiting the office once a week to report. My people were intelligently liberal, and I think my free-lance position to the end of the war (remaining a second lieutenant—rank was not important) was unique. (A worker in another research office told me that if he had to solve a serious problem he had to 'go sick', without pay.) I gave some time to the problem of really accurate formulae for 'direct fire', i.e., for Γ with a target at ground level, The results of this will be the subject of paper II.

Problems and Solutions

Readers who have not yet reached the age of 20 on 1 October 1971 are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

4.1. Show that $\min\{m^{1/n}, n^{1/m}\} \leqslant 3^{1/3}$ for all positive integers m, n.

4.2. Teams $T_1, T_2, \ldots, T_n$ take part in a tournament in which every team plays every other team just once. One point is awarded for each win, and it is assumed that there are no draws. Denoting by $s_1, s_2, \ldots, s_n$ the total scores of $T_1, T_2, \ldots, T_n$ respectively, show that, for $1 \leqslant k \leqslant n$,

$$s_1 + s_2 + \ldots + s_k \leqslant nk - \tfrac{1}{2}k(k+1).$$

4.3. A family has n children, where $n > 1$. Let A be the event that the family has at most one girl and B the event that not every child is of the same sex. Determine the value of n for which A and B are independent events.

4.4. A train leaves the station punctually. After having travelled for 8 miles, the driver looks at his watch and sees that the hour hand is directly over the minute hand. The average speed over the 8 miles is 33 miles per hour. At what time did the train leave the station?

Solutions to Problems in Volume 3, Number 2

3.4. Let u_n denote the nth Fibonacci number. (See the article by R. J. Webster.) Show that (i) when m divides n, then u_m divides u_n, (ii) when $n \geqslant 8$, then u_n has not more than $n/4$ digits in its decimal expansion.

Solution by A. H. Rodgers (Gonville and Caius College, Cambridge)

(i) First prove, by induction on k, that

$$u_{k+m} = u_k\, u_{m+1} + u_{k-1}\, u_m$$

for all integers k, m such that $k \geqslant 2$, $m \geqslant 1$. We now show that u_m divides u_{pm} for all integers $p, m \geqslant 1$. This is obvious when $m = 1$ because $u_1 = 1$, so we suppose that $m > 1$. We use induction on p, the result being obvious when $p = 1$. Now let $p > 1$ and put $k = (p-1)\,m$ in the above. Then

$$u_{pm} = u_{(p-1)m}\, u_{m+1} + u_{(p-1)m-1}\, u_m.$$

From this it follows that, if u_m divides $u_{(p-1)m}$, then it also divides u_{pm}. This completes the induction.

(ii) We have

$$u_{k+m} = u_k\, u_{m+1} + u_{k-1}\, u_m < u_k(u_{m+1} + u_m) = u_k\, u_{m+2}.$$

This is true for $k, m \geqslant 1$. Thus

$$u_{k+4} < u_k\, u_6 = 8u_k < 10u_k.$$

Hence, for all integers $q \geqslant 2$,

$$u_{4q+3} < 10u_{4(q-1)+3} < \dots < 10^{q-2}\, u_{11} = 10^{q-2}.89 < 10^q.$$

Let $n \geqslant 8$ be an integer, and let q be the largest integer which does not exceed $n/4$. Then $n/4 < q+1$ so that $n \leqslant 4q+3$ and

$$u_n \leqslant u_{4q+3} < 10^q.$$

Thus u_n has not more than q digits in its decimal expansion, and hence not more than $n/4$.

L. B. Raphael points out that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\sqrt{5}}{2}.$$

(See Problem 3.1.)

3.5. There are n sisters who live in n different suburbs of a town. Each wishes to hear daily the news of all the others. They agree that, to reduce their overall telephone account, each will pass on all the news she knows at every telephone call made to, or received from, another sister. Show that, for $n > 3$, all the news can be circulated in $2n-4$ telephone calls.

Solution by L. B. Raphael (Quarry Bank Comprehensive School, Liverpool)

We use induction on n. Consider the case $n = 4$, and label the sisters A, B, C, D. The news can be circulated by the following calls in the order given:

A rings B, $\quad C$ rings D, $\quad A$ rings C, $\quad B$ rings D.

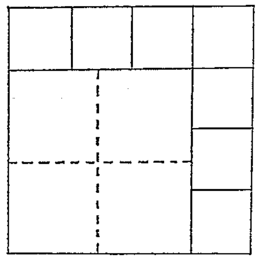
Now let $n > 4$, consider n sisters, and suppose that $n-1$ sisters can circulate their news in $2n-6$ calls. Pick out one of the sisters, labelled X. First, X phones any one of the other sisters, then the other sisters circulate their news and X's in $2n-6$ calls, then one of the other sisters phones X. This gives $2n-4$ calls in all. This completes the induction.

Also solved by A. H. Rodgers (Gonville and Caius College, Cambridge) and M. K. Ashbrook (Wyggeston Boys' School, Leicester).

3.6. Show that, for all integers $n \geqslant 6$, a square can be subdivided into n non-overlapping squares.

Solution by A. H. Rodgers

Let $k \geqslant 2$ be an integer. The square can be subdivided into $2k$ non-overlapping squares as indicated in the diagram (which shows the case $k = 4$).



By subdividing one of these squares into four non-overlapping squares, we obtain a subdivision into $2k+3$ squares. Thus the square can be subdivided into n squares for all integers $n \geqslant 6$.

Also solved by L. B. Raphael (Quarry Bank Comprehensive School, Liverpool), C. Goldthorpe (Leeds Grammar School) and M. K. Ashbrook (Wyggeston Boys' School, Leicester).

3.7. Let $z_1, z_2, \ldots, z_n$ be any complex numbers. Show that there exists a unique complex number z which minimizes the sum

$$\sum_{k=1}^{n} |z-z_k|^2.$$

Solution by A. H. Rodgers

$$\begin{aligned}\sum_{k=1}^{n} |z-z_k|^2 &= \sum_{k=1}^{n} (z-z_k)(\bar{z}-\bar{z}_k) \\ &= nz\bar{z} - z\sum_{k=1}^{n} \bar{z}_k - \bar{z}\sum_{k=1}^{n} z_k + \sum_{k=1}^{n} z_k \bar{z}_k \\ &= n\left(z-\frac{1}{n}\sum_{k=1}^{n} z_k\right)\left(\bar{z}-\frac{1}{n}\sum_{1=k}^{n} \bar{z}_k\right) + \sum_{k=1}^{n} z_k\bar{z}_k - \frac{1}{n}\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \bar{z}_k\right) \\ &= n\left| z-\frac{1}{n}\sum_{k=1}^{n} z_k\right|^2 + \sum_{k=1}^{n} |z_k|^2 - \frac{1}{n}\left|\sum_{k=1}^{n} z_k\right|^2,\end{aligned}$$

and this takes a minimum value when and only when

$$z = \frac{1}{n}\sum_{k=1}^{n} z_k.$$

A. H. Rodgers points out that this establishes the inequality

$$\left|\sum_{k=1}^{n} z_k\right|^2 \leqslant n\sum_{k=1}^{n} |z_k|^2.$$

Book Reviews

Functions of a Complex Variable, I and II. By D. O. TALL. Routledge and Kegan Paul Ltd, London, 1970. Pp. 72, 80. £0·45, paperback; £0·90, cloth.

Despite some serious omissions this book is quite a useful addition to the literature, if only because of its readability. It is best regarded as a text for service courses or as set reading preparatory to a 'serious' course. A good point is the attempt to explain, motivate and warn about points which more advanced texts usually take for granted.

In one or two cases the terminology and notation is non-standard; for example the definition of connectedness refers to what is usually called arc-wise connectedness—admittedly without harm here—and the use of log z and Log z is the reverse of the standard one.

The elementary work and the introduction to line integration are readable and the examples on calculating line integrals through parametrization will help students at a point where they often encounter difficulty.

Cauchy's theorem is proved for triangles (in an appendix), extended to star-domains and the general result is sketched. Personally I have never found this approach entirely satisfactory and would prefer a proof along the lines given in Titchmarsh's or Knopp's *Theory of Functions*, with indication of where appeal is made to intuition. This would satisfy the needs of an accessory course and the serious student will eventually need a more detailed text in any case.

A chapter is devoted to conformal mapping but without any examples. This would be much improved by including a brief study of some concrete mappings such as the bilinear, exponential, logarithmic and trigonometric ones. As it is the reader is left without any idea of how to actually map say a half-strip onto a half-plane.

A further unaccountable omission is the maximum principle. Personally I would also like to see Schwarz's lemma included, together with some application.

Laurent's theorem could have been handled better by simplifying the notation and integrating the infinite, rather than the finite geometric series for $(\zeta - z)^{-1}$. The residue theory is well exemplified including some summation of series. General methods are given for calculating the true improper integrals $\int_{-\infty}^{\infty} f(x)\, dx$ and $\int_{-\infty}^{\infty} e^{imx} f(x)\, dx$ by integrating around rectangular contours. A place might have been found for Jordan's lemma because of its convenience.

Finally, minor annoyances stem from inadequate proof reading (e.g., "We remark that the notation of extension to an analytic function does guarantee that such an extension exists, . . "!) and the frequent omission of the relative pronoun 'that', which in the reviewer's case at least meant that sentences often had to be read through twice.

University College of Wales, Aberystwyth K. W. LUCAS

Computation and Theory in Ordinary Differential Equations. By JAMES W. DANIEL and RAMON E. MOORE. W. H. Freeman and Company Ltd, Reading, 1970. Pp. 172. £3·10.

In this book the authors try to provide motivation for the seemingly dry subject of computational techniques. By considering vector fields, they make use of our geometrical insight to develop the theory of differential equations. Theorems are usually stated without proof (otherwise the text would have been unbearably long) but adequate references are given. Counter-examples are used liberally to demonstrate the necessity of hypotheses (a feature often lacking, sadly, in many text-books).

The first chapter shows the equivalence between differential equations and vector fields. The formulation is fairly abstract, dealing in n dimensions; but they use our geometrical insight to solve the initial value problem in one and two dimensions. There is an illuminating section which shows why there must always be unstable weather somewhere (but not why it always seems to occur in England!). The third chapter is devoted to the boundary value problem. To remind us of the limits of the theory, examples are cited which have no, many and even infinitely many solutions. These three chapters, forming the first part, lay the theoretical foundations for differential equations.

The second part of the book deals with methods of approximation. Rather than convey the minute details, the authors try to concentrate on the philosophy behind a particular method by giving an illustrative example. At the end of each chapter they give a highly personal (but indispensable) guide to facilitate the choice of method. The final part is devoted to computer-techniques. Good planning and careful thought are necessary to ensure greater accuracy and avoid needless wastage of time. The authors discuss global (applicable generally) and local (which depend on the different portions of the solution space) transformations; and also give helpful hints for improvements after some computing has been done.

The book is attractively laid-out with many diagrams. There are good, clear introductions to each part stating, precisely, what the authors intend to do. The exercises intermingle with the text but no answers are provided. Unfortunately the book requires quite a lot of sophistication: it does assume prior knowledge of differential equations and numerical analysis. There is a nice blend between geometrical insight and computational techniques; but one still feels that the motivation to use this book would come from the desire to solve a problem rather than for art's sake. Thus it may be more useful as a source book rather than for learning. But it probably would not be a useful addition to an average school library as only the very brightest students would be able to understand it.

University of Nottingham RAMESH KAPADIA

Probability. By PETER WHITTLE. Library of University Mathematics, Penguin Education, Harmondsworth, Middlesex, 1970. Pp. 239. £1·50.

Like Feller's famous book *An Introduction to Probability Theory and its Applications*, Volume I, this book treats probability theory, not as some preamble to, or prerequisite for, Statistics, but as a subject in its own right, with increasingly important and varied applications independent of that particular application called 'Statistics'.

From remarks in the Preface and on the cover, one is led to understand that the book covers a one-year introductory course in probability and that it is suitable for self study. Let us be clear that this is a book for the second- or third-year university student and constantly demands the levels of concentration and of mathematical maturity which such a student might have. Early in the text, the reader is expected to know the definitions and properties of absolutely convergent integrals and positive semi-definite matrices. In an exercise in Chapter 1, he meets 'a continuous infinity of dimensions'.

The book's 'quantum-mechanical' approach, with its emphasis on what is *observable* in an experiment, and with an axiom system taking '*expectation* (value on average) of a randomly determined number' instead of 'probability of an event' as the primitive concept, has much to commend it. There is an interesting chapter on *quantum mechanics* itself in which the Uncertainty Principle is derived from a similar axiom scheme for average values. Quite a lot of matrix theory is used in this chapter.

Inventory policy (how best to control level of stocks in the face of uncertain future demand) and *guidance theory* (how best to control a missile in the presence of unpredictable perturbing forces): these are parts of the theory of *stochastic dynamic programming*. How to encode a message for communication in the most efficient way is a problem in *information theory*. (It has nothing to do with secret codes.) The book's treatment of these newer branches of probability theory is highly condensed but the reader can gain an idea of the flavour of subjects not discussed in other books at this level.

A sixth-former would find the book extremely difficult to read. If he is keen to learn what probability is really about, he can browse in Feller's book with the same profit and delight as the research worker; there are easier books than Feller's but none is comparable. Perhaps he can appreciate something of the new approach and new applications in Professor Whittle's book from his teacher; my first-year university students should certainly benefit from my having read it.

University College, Swansea — D. WILLIAMS

A Programmed Vector Algebra. By K. L. GARDNER. Oxford University Press, 1968. Pp. 207. £1·75.

We enjoyed this book for two reasons; first, because it gave us a firm basis to work on, and secondly because it was refreshing to read owing to its unusual form of presentation. Basic ideas are developed by means of programmed instruction. The reader is invited to answer some short 'basis-building' questions on a topic which has previously been explained to him. If the reader feels that he can master the topic presented, he then continues either with more questions or with information on a new subject; however, if the reader finds himself in difficulties over answering the questions set, he then re-reads the information on that given topic, and works his way through the questions until he has satisfied himself that he is in complete control of the information he has been given. In this however lies a great danger; the reader must take care that he does not pass too quickly over a subject he does not fully understand. If he does so, the whole point of the book will have been lost. At the end of each chapter there are summaries written in the form of flow diagrams, and often, worked examples in the topic discussed.

The points which the book explained well were vector and scalar products; unfortunately, however, the author preferred to develop this concept under the treatment of vectors in the i, j, k form. We feel that the total concept would have been more easily grasped had scalar products $\mathbf{a}\cdot\mathbf{b}$ been expressed as $ab \cos \theta$; a similar treatment should have applied to vector products.

The book does have its weak points. Some of the flow diagrams are trivial, and more important, there is a lack of questions such as might be set in A-level or similar examinations. We feel that exercises should be placed at the end of every chapter to give the reader a chance of tackling questions which will not only test his understanding, but also allow him to use his imagination in discovering wider applications for vectors.

Since the book lacks those essentials, we would recommend that it should be used only for private studies or for revision purposes. We would stress, however, that for these two applications, it would be very useful; at a price of £1·75, it is a book which we would recommend all Mathematics libraries to obtain.

Maths VIII, St Paul's School, London, S.W.13 — T. E. COPE

P. J. LINGS

Modern Applied Mathematics: Probability, Statistics and Operational Research. By J. C. TURNER. English Universities Press Ltd, London, 1970. Pp. 502. £2·25, paperback; £3·45, cloth.

In the intense and sometimes acrimonious debate about the content and style of mathematics teaching in schools which has taken place during the last decade, and will no doubt continue for the next, the position of Applied Mathematics has been rather overlooked. It is refreshing therefore, to find a book, aimed at sixth-formers amongst others, which includes applications of mathematics which are genuinely modern in the historic sense.

The book is in three parts: Ancillary Modern Mathematics, Probability and Statistics, Applied Probability and Operational Research. Part 1 deals in an elementary, descriptive way with sets, matrices, inequalities and graphs. Part 2 (nearly half the book) is devoted to a careful but lively survey of elementary statistics which emphasises both the probabilistic basis of the subject and its applications to many aspects of everyday life. Part 3 presents 'quantitative common-sense' applied to Markov chains, queues, networks, optimisation problems and generally uses the ideas and techniques developed in Part 1. In general, these ideas are seen to be relevant although the introduction of sets and the associated notation sometimes appears a little contrived: not everyone will agree with the author that 'set theory is a delightful subject in itself'.

Another particular criticism one could make concerns the lack of an adequate definition of function although the term is widely used. These however are relatively minor matters and do little to diminish the overall appeal of the book. The style and presentation, containing as it does a good supply of illustrative examples, make this a book which one can confidently recommend to sixth-form mathematicians and (perhaps more importantly) their teachers and future employers.

University of Nottingham D. S. HALE

The Search for Pattern. By W. W. SAWYER. Penguin Books, Harmondsworth, Middlesex, 1970. Pp. 349. £0·40.

I liked this book when I read it; it seems to be aimed at people like me who have just completed a course of the approximate standard of Additional Mathematics and who are interested in mathematics.

The first chapters of the book are relatively basic and deal with simple subjects like linear functions, while the latter part discusses more advanced topics such as the binomial theorem, the remainder theorem and statistics. For those who have the necessary grounding, the way in which the author treats these topics is very interesting; it is also useful as background reading to supplement their knowledge. The author's ideas provide some interesting outlines of ways in which certain topics might be approached by schoolmasters.

However, the author states in his Preface that this book might be helpful to those doing advanced work in other subjects and who need some knowledge of mathematics. I hardly think he achieves this aim. Anyone who had never dealt with any of the topics in this book before would, in my opinion, almost certainly need a more detailed treatment; conversely this book would hardly be of much use to a person who had covered the subjects the author deals with already, except as a refresher course.

But Professor Sawyer has certainly written an interesting book. It is suitable for any enthusiast who has reached a standard equivalent to Additional Mathematics, and I would recommend it wholeheartedly for a school library.

Lower 8th, St Paul's School, London, S.W.13 R. STONE

Choice of Careers No. 109: Mathematical, Statistical and Computer Work. Central Youth Employment Executive, H.M.S.O., 1970. Pp. 40. £0·11.

This booklet aims to outline various aspects of careers which are open to people who are studying, or have qualifications in mathematical subjects. Sections are devoted to career prospects in Research and Development, Operational Research, Statistics, Computer Work, Education and Actuarial Work. In each case, a brief idea is given of what the work involves, the qualifications required to pursue such work (and how to obtain them), and the prospects and salaries that the associated jobs are likely to command. Each section concludes with suggestions as to where further information can be obtained.

On the whole, a simply-written, condensed but fairly comprehensive survey is given of the opportunities available to young would-be-mathematicians. As is to be expected in such a short account of a wide range of topics, some aspects are only touched upon. However, the booklet recognises its own limitations and provides a useful list of sources of further information. Certainly this is a booklet which anyone seeking a career in mathematics should read to gain a preliminary insight into the various possible openings, and at the price of 11p, it should be within the means of most sixth-formers and students.

University of Sheffield CAROL NIXON

Letter to the Editor

Dear Editor,

I have looked with particular interest at the two articles on sample surveys in No. 2 of Volume 3 of *Mathematical Spectrum*. I think these illustrate some of the difficulties all teachers encounter when trying to present a complex subject at an elementary level. From personal experience of attempting to teach survey theory and practice, I want to emphasize the great importance of adherence to a consistent and widely accepted terminology. The reader who, at his first attempt with a new subject, is encouraged to use words in a manner peculiar to one author may encounter much confusion later. I criticize these two articles with some reluctance, because I know how easily I can fall into the same error, but I think the issue is important for all elementary writing on science, and especially in a journal designed for students.

Dr Rao writes of 'multi-phase sampling' when he refers to sampling on successive occasions. In fact two-phase sampling is an accepted synonym for double-sampling, a method in which an auxiliary variate is measured in a large sample and both it and the variate under study are measured on a smaller sample. Professor Moore writes extensively about the 'precision' of estimation, and uses the word largely in an acceptable colloquial sense. However, he implies that precision is measured by a standard error; statistical usage is surely clear that a quantitative measure of precision is always the reciprocal of a variance or some minor modification of this. Dr Rao has compressed an immense amount of information into five pages. Inevitably, he is a little uncritical in this condensed presentation. A reader new to the subject might wonder why systematic sampling is not the method of choice in every investigation; it is introduced as an alternative to random sampling, with no indication that it has hazards or limitation. Indeed, he later gives the impression that systematic sampling is peculiarly appropriate to the sampling of trees, whereas farm crops should be sampled at random. The term 'crop cutting experiments' was at one time used in his sense, especially in India; fortunately it is not in general currency, because it causes confusion between the very different aims of sample survey and controlled experimentation.

Yours sincerely,

D. J. Finney
(University of Edinburgh).

Notes on Contributors

J. B. Helliwell is Professor of Engineering Mathematics at Bradford University. Before going to Bradford he worked in two of the principal Institutes of Science and Technology in this country, namely, Manchester and Glasgow; both Institutes have now become Universities. He has worked mainly in the field of fluid mechanics and gas dynamics.

David Kendall FRS is Professor of Mathematical Statistics in the University of Cambridge and a Fellow of Churchill College. He has wide interests in probability and statistics from theoretical problems to such practical applications as the spread of epidemics and rumours, the mechanism of bird navigation, and statistical studies in archaeology.

J. Gani is Professor of Statistics and Head of the Department of Probability and Statistics in the University of Sheffield. He has taught in universities in this country as well as in Australia and the U.S.A., where he has made some mathematical films for schools. His main interest is the application of probability to problems in biology.

E. Huntley is a Lecturer in the Department of Applied Mathematics and Computing Science in the University of Sheffield. He worked for several years in the Flight Dynamics Division of the Royal Aircraft Establishment, Bedford. His publications are mainly concerned with problems of control systems analysis and aircraft dynamics.

D. B. Scott, a graduate of Cambridge and of London, has taught in the University of London and (briefly) in the University of Aberdeen, and has been Professor of Mathematics in the University of Sussex since its foundation. A specialist in algebraic geometry, he is also deeply interested in problems of mathematical education and he has addressed many meetings of both teachers and pupils. He is the author, jointly with the late S. R. Tims, of *Mathematical Analysis: An Introduction* (C.U.P. 1966).

Peter Shiu Man-Kit is a Lecturer in Mathematics at Loughbourough University of Technology. He is also a member of the Centre for the Advancement of Mathematical Education in Technology (CAMET). His interests are in analysis and in mathematical education.

J. E. Littlewood, Fellow and Copley medallist of The Royal Society, De Morgan medallist of the London Mathematical Society, honorary doctor or member of many universities and academies, is the outstanding mathematical analyst of his generation. Born in 1885, he has been a Fellow of Trinity College, Cambridge, since 1908 and Rouse Ball Professor of Mathematics from 1928 to 1950. Littlewood's papers in analysis and number theory, of which over a hundred were written in collaboration with the late G. H. Hardy, have a striking power which to mere mortals seems nothing short of miraculous. Now, at the age of 86, Littlewood retains unimpaired his zest for mathematics; and the quality of his delight in mathematical activity may be gleaned from his charming booklet *A Mathematician's Miscellany* (Methuen, London, 1953).

Contents

J. B. HELLIWELL	1	Mathematics: the language of science
DAVID KENDALL	7	The algebra of genealogy
J. GANI	9	Gambling and probability: some early problems
E. HUNTLEY	14	The control of a hovering VTOL aircraft
D. B. SCOTT	19	Proving the rule or first steps in rigour
P. SHIU	26	Approximation to square roots
J. E. LITTLEWOOD	31	Adventures in ballistics, 1915–1918. I
	38	Problems and Solutions
	41	Book Reviews
	46	Letter to the Editor
	47	Notes on Contributors

PRICES (*postage included*)

	Individual annual subscription (1 volume of 2 issues)	Bulk annual subscriptions (5 or more copies of 1 volume of 2 issues)	Single issues
U.K. only	£0·40	£0·35	£0·23
Australia	$A1.45	$A1.25	$A0.80
U.S.A.	$1.60	$1.40	$0.90

Enquiries about rates, subscriptions, and advertisements should be directed to:

Oxford University Press, Education Department,
Walton Street, Oxford,

or to Oxford University Press Branches Overseas.

Printed in England by John Wright & Sons Ltd., Bristol