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The Managing Editor, *Mathematical Spectrum*,
Hicks Building, The University, Sheffield S3 7RH.

Charles Babbage and his Computer*

J. HOWLETT
Atlas Computer Laboratory

Very few people, in England or elsewhere, knew anything about Charles Babbage until quite recently. He lived a most active and vigorous life through most of the 19th century, and he wrote and published many books and papers on an astonishingly wide variety of subjects. He was brilliantly clever. But his work was almost completely unknown until the late 1940's when some of the people working with what was then a very new device, the electronic digital computer, read his books and papers and found that he had thought out nearly all the principles of an automatic, general-purpose digital computer over 100 years ago. He had even made a detailed design for such a machine and had tried, unsuccessfully, to build it. In fact, he spent a great deal of his life, and money, on two machines, which he called the Difference Engine and the Analytical Engine respectively; it is the second which was the forerunner of the modern computer.

Babbage the man

Babbage was born in Devonshire, in South-West England, in 1791; his father was a banker and he lived the life of a rich, eccentric English gentleman who travelled widely and was on equal terms everywhere in the best society. He was already deeply interested in mathematics when he was quite young and went to Cambridge in 1810, taking a first-class degree in 1814. He published several mathematical papers from 1815 to 1822 and, with two distinguished friends, Herschel and Peacock, who had been undergraduates with him, worked hard to raise the standard of mathematics in England. In 1828, although he was not then connected with the university, he was elected to its most distinguished Lucasian Chair of Mathematics, which had been held by Newton. He resigned in 1839; in all the 11 years as Professor he never lived in Cambridge or gave a single lecture—probably a world record.

* This article is reprinted by permission of the University of Tübingen; it originally appeared in 350 *Jahre Rechenmaschinen*, a collection of papers given at a colloquium in November 1973 to mark the 350th anniversary of the construction of a calculating machine by Wilhelm Schickard.

Babbage's interest in calculating machines seems to date from his undergraduate days, around 1812–1813. He started to work on his first machine, the Difference Engine, in 1823 and was forming his first ideas on the much more complex Analytical Engine in 1833. Although he had many other interests, his life was dominated by these machine projects from then until his death in 1871.

Babbage was undoubtedly a most remarkable man. He was interested in, and curious about, practically everything, and especially about any kind of machinery. He had a very quick and agile mind, seemed able to understand everything down to the smallest technical detail and store a vast amount of information in his memory. At the same time he was thoroughly original and could always make a personal and fresh approach to any problem. All this comes out clearly in his autobiography, *Passages from the Life of a Philosopher*, which he published in 1864 and which was reprinted by Dawson of London in 1968. It is fascinating reading, though parts of it are tedious: one sees very clearly that whilst Babbage could be very good and stimulating company, he could also be very obsessive. The chapter headings show something of the scope of his interests; there are four dealing with his calculating machines, and from the whole set of thirty-six, here are a few:

IX Of the Mechanical Notation

XIV Recollections of Laplace, Biot, and Humboldt

XVIII Picking Locks and Deciphering

XXV Railways

XXIX Miracles

XXXIII The Author's Contribution to Human Knowledge

The book contains a list of his eighty publications, and the range is remarkable. Here are a few: Demonstration of a theorem relating to prime numbers; On the proportion of letters occurring in various languages, in a letter to M. Quetelet; *Economy of Manufactures and Machinery*, 8 volumes, 1832; On the statistics of light-houses, Compte Rendu des Travaux du Congrès Général, Bruxelles, Sept. 1853; On the action of ocean-currents in the formation of the strata of the earth.

Passages from the Life of a Philosopher is a very quotable book; I could easily use up all my space in quoting selections from it. Let me give just three, none of which concern Babbage's main work but which I think show several different aspects of his character very clearly.

First, the trained mind of the true scientist. Whilst he was staying in Naples in 1828 he decided to climb Vesuvius, which was then moderately active, to descend into the crater, and if possible to observe the movement of the lava in the active region. He gives, incidentally, a splendid picture of the rich English traveller: 'As I wished to see as much as possible, I made arrangements to economize my strength by using horses or mules to carry me wherever they could go. Where they could not carry me, as for instance, up the steep slope of the cone of ashes, I employed men to convey me in a chair.' His account, too long to quote, is quite dramatic. He descended into the main crater, an extensive oval plain about 500 feet below the top of the ridge, covered with a pattern of intersecting cracks in the rock and with a small subsidiary active crater near one part of the edge. He had with him a barometer, a sextant, several thermometers and a measuring tape, not to mention a flask of Irish whiskey.

He surveyed the area and timed the intervals between successive eruptions from the active crater, finding these to be reasonably constant at about 15 minutes. He recorded all his information in a note-book and made a detailed plan of action in which he would approach the edge of the crater along a definite line and could safely watch the interior for 6 minutes before he must retreat to escape the next eruption. It all worked out perfectly, and he had sufficient faith in his timings not to retreat when he saw a large bubble forming and growing in the red-hot liquid lava—it was not yet time for the next eruption.

The second quotation shows how he could not resist recording trivialities. In his chapter on 'Hints for Travellers' he writes at one point: 'One of the most useful accomplishments for a philosophical traveller with which I am acquainted, I learned from a workman, who taught me how to punch a hole in a sheet of glass without making a crack in it.' He goes on to say quite a lot about this: it seems to have been one of his favourite party tricks.

The third illustrates how he could immediately start to elaborate a new idea. In the chapter 'Further Contributions to Human Knowledge' he makes a short but penetrating study of games of skill in which he sees the possibility of applying his Analytical Engine (i.e., a computer) to game-playing. He analyses noughts-and-crosses (which he calls tit-tat-to) in detail, sees that there is always a winning strategy and 'I therefore easily sketched out a mechanism by which such an automaton could be guided'—i.e., he designed a machine to play the game. He then thought that he might turn the idea into an entertainment for which people would pay, and so make money to spend on building the Analytical Engine. He goes on: 'It occurred to me that if an automaton were made to play this game, it might be surrounded with such attractive circumstances that a very popular and profitable exhibition might be produced. I imagined that the machine might consist of the figures of two children playing against each other, accompanied by a lamb and a cock. That the child who won the game might clap his hands whilst the cock was crowing, after which, that child who was beaten might cry and wring his hands whilst the lamb began bleating.' Fortunately, a few inquiries showed him that there was no money to be made from this kind of display and he dropped the idea.

The Difference Engine

Babbage was led to design this machine by his thoughts on mathematical tables and on the great amount of human labour which went into their calculation. The theoretical basis of the machine is a few simple results from what we now call the Calculus of Finite Differences. First, if we evaluate a polynomial function at equal intervals of the argument, write down the differences between consecutive values, then the corresponding differences of this set of numbers, and so on, we arrive at some stage at a set of constant numbers. Thus if $f(x) = x^3+2x+1$ we have:

x	$f(x)$			
0	1			
		3		
1	4		6	
		9		6
2	13		12	
		21		6
3	34		18	
		39		6
4	73		24	
		63		6
5	136		30	—
		93	—	(6)
6	229	—	(36)	
	—	(129)		
7	(358)			

The theorem is that the nth differences of a polynomial of degree n are constant; here the polynomial is of degree 3 and the 3rd differences are all equal to 6. Having calculated the function for a few values, we can now use this knowledge to extend the table as far as we wish simply by successive additions—that is, no multiplications: the rule is simply that if $c = a-b$, then $a = b+c$. Thus having calculated $f(x) = x^3+2x+1$ and its differences as far as $x = 6$, as in the table, we get for the next line of differences ($x = 7$):

$$36 = 30+6, \quad 129 = 93+36, \quad \text{and} \quad (7)\colon 229+129 = 358,$$

and so on.

Further, by means of quite simple formulae (called Interpolation Formulae) we can use the table of differences to calculate values between the tabulated values, for example to form a whole new table at a finer interval, say 0·1 in x in place of the original interval of 1. And finally, if the function is not a polynomial—$\log x$, say—differencing will lead, not to a constant but to a set of values which are small in comparison with the function values, and we can use the interpolation formulae to calculate intermediate values which, whilst they are approximate in the strict sense, can be guaranteed not to be in error by more than some stated (and small) amount. This use of interpolation is not of any importance for simple functions like x^3+2x+1; but for complicated mathematical functions, or for tables used in navigation (such as the positions of the moon or certain stars at all times and dates), the direct calculation is most laborious and interpolation is well worth while, even, in many cases, with modern high-speed computers.

The purpose of the Difference Engine was to perform these operations: differencing a table, extending a table by additions of difference and interpolation, entering automatically. Babbage of course lived before any kind of electrical or electronic engineering was known and his machine could only be mechanical. The basic principle was to represent a single decimal digit (i.e., 0 to 9) by the position of a gear wheel which had ten teeth; and a number of, say, six digits (i.e., 0 to 999,999) by a group of such wheels—we may call such a group a *register.* It is clear that the action of the machine must consist of sequences of additions between registers and that there must be means for setting up cycles of operations (such as the sequence of additions which are required in going from the constant 3rd difference to the next value in the example) and for repeating these. Babbage foresaw every need and designed everything in the greatest of detail, including a means for automatically producing stereo-plates from which the results could be printed, so that no hand-copying should be needed. He obtained the support of the British government of his day and started to build the machine in 1823, expecting to finish it in three or four years. The project, however, proved far more difficult than Babbage or anyone else had expected. Mechanical engineering was then in a relatively primitive state; the precision which was necessary if the machine was to work satisfactorily was unobtainable, and Babbage had to start by designing and making tools with which the gears and other parts of his machine could be produced with the required accuracy. Also, and characteristically, he started with a plan for a machine on far too large a scale: it was to work with numbers with up to twenty decimal digits and with differences up to the 6th order. This strained the technology of the day beyond what it would bear.

Work on the building of the machine stopped in 1833. After many delays, discussions and argument the government finally withdrew its support in 1842, having spent £17,000 of public money on the project—probably equivalent to about £300,000 to-day. Babbage had spent about the same amount from his private fortune. Very little of the machine was actually built. There is a very small part in the Science Museum in London, a drawing of which appears in *Passages from the Life of a Philosopher*, with the following sad legend:

> 'It was commenced 1823
> This portion put together 1833
> The construction abandoned 1842
> This plate was printed June, 1853
> This portion was in the Exhibition 1862.'

Ironically, a working machine *was* made based on Babbage's design. A Swedish printer, P. G. Scheutz (1785–1873), read about Babbage's machine in 1834 and with his son Edward built a small prototype. Much later, in 1851, they got the support of the Swedish Academy and money to build a larger machine and in 1853, having had much help and advice from Babbage himself, they had completed a machine which worked with 14-digit numbers and 4th-order differences. The final realisation of Babbage's ideas for the Difference Engine came in the 1940's with the electro-mechanical accounting machines; these are still in use, but are being displaced by small computers. The National machine, for example, has been used very extensively for construction of tables in exactly the way which Babbage had described more than 100 years previously.

The Analytical Engine

In 1833, when he was in such difficulties over the building of his Difference Engine, Babbage had his great visionary idea: the fully automatic, completely general calculating machine which he called the Analytical Engine. In fact he never built more than a few fragments of this, but this does not detract from the greatness of his vision. It is unlikely that anyone could have built the machine with the technology of the day, as it would have been far more demanding than the Difference Engine. It is the concept which ensures his place in history. The Analytical Engine was to be able to perform any calculation, however long and complex. It was to consist of the following parts:

(i) A *store*, or set of registers in which numbers could be held.
(ii) A unit which could perform arithmetical operations on numbers put into it: Babbage called this the *mill*.
(iii) A unit which would take numbers from the store, cause the mill to operate on them and return the results to the store, all as required to carry out the successive steps in the calculation: we can call this the *control unit*.
(iv) Means for putting numbers and instructions defining the sequence of operations into the machine, and for getting out the results—*input/output units*, in fact.

This is almost exactly the description of a modern digital computer. Babbage also saw that:

(a) Any calculation can be broken down into a sequence of additions, subtractions, multiplications and divisions; so that the mill can be built so that it can do only these four basic operations,
(b) The course of a calculation may depend on the result obtained at some intermediate stage, for example solving an algebraic equation by a purely numerical process, and this cannot be predicted at the start; so the control unit must have some means for changing its sequence of instructions, according to the result obtained at some stage. In fact, Babbage saw that the decision to change course or not could always be made to depend on whether the number in a specified register was positive or negative.

This last is known now as 'conditional transfer of control' and is quite fundamental: no machine could be truly automatic without it, for everything would have to be foreseen at the start of the calculation. Babbage saw also that the same principle could be used to cause a cycle of operations to be repeated any chosen number of times.

Again the machine was to be entirely mechanical; as in the Difference Engine a register would be a group of toothed wheels and Babbage planned for 1000 registers, each made up of fifty wheels—that is, each could hold a number of fifty decimal digits. He was clearly aware of the need for high precision in long calculations, but fifty decimals is high even by present-day standards.

The most striking and original feature of the Analytical Engine is the concept of the control unit; nothing like this had ever been proposed before. Babbage thought it out, complete with a detailed mechanism, which was based on the principle of the Jacquard loom: it seems likely that he got the idea as a result of seeing one of these machines at work. In 1805 J. M. Jacquard invented a means for the control of a loom which enabled complex patterns to be woven automatically. The problem is to select, for each successive row of the material which is being woven, the warp threads which have to be raised before the shuttle goes across the material between the warp and the weft. This can be done well enough by hand, as it has for thousands of years, for plain material and simple patterns, but becomes almost impossible in the case of complicated patterns. Jacquard's idea was to make this selection by means of a card in which a hole could be punched in a position corresponding to the end of each of the rods which moved the warp threads: a rod could move through a hole if one was punched in its position, if there was no hole it could not. Thus any selection could be moved, and in order to produce any pattern in the cloth automatically, all that one had to do was decide in advance which threads had to be moved at each row, and punch a series of cards with holes in the corresponding positons. In modern terms, the pattern to be woven is coded by the pattern of holes in the cards. Jacquard's invention was an immediate and very great success and revolutionised the weaving industry.

Babbage saw that this concept of coding, although he never used the term, was of fundamental importance and could be used to control any mechanism; in particular, that it could control the sequence of operations in his Analytical Engine. To use the modern term again, the sequence of patterns of holes in the cards represented the program for the calculation. Again characteristically, Babbage worked it all out in great detail and produced engineers' drawings from which, in principle, the machine could have been built. Further, he foresaw the possibility for such things as 'library' packs of cards for frequently used programs or numerical tables (e.g., logarithms). He said explicitly that experience would be needed before one could decide whether it was better to get the value of a particular function from a table or to calculate it afresh by a library program whenever it was needed.

The most detailed available account of the Analytical Engine is one which was based on a series of lectures which he gave in Turin in 1840. L. F. Menabrea, an army officer who later became a General under Garibaldi, wrote a very full description of the principles of the machine, with examples of how it could be applied. This was later translated by a very remarkable woman, Ada Augusta, Countess of Lovelace, the daughter of the poet Byron; she had many accomplishments and in particular was a good mathematician, a most uncommon ability in a woman at that time. She obviously understood the principles of the machine very well indeed, probably as well as Babbage himself. She herself added very extensive and detailed notes to her translation of Menabrea's paper, which in total are longer than the paper itself; in one of these she gives what we should now call the program for computing the Bernoulli numbers. One of the most significant statements which she, or anyone else, made about the machine was that 'it can do only what we know how to order it to do'. Her translation and notes are reproduced in full in the book *Faster than Thought* by B. V. Bowden (now Lord Bowden), and also *Charles Babbage and his Calculating Engines* by E. and P. Morrison.

The more one reads about the Analytical Engine, the more one is struck by Babbage's foresight and by his grasp of both the broad concept and the fine details. But it remained only a concept. He made many attempts to get government support but never succeeded. He spent a great deal of his own time, effort and money in trying to build it, but made only a few fragments. His son, H. P. Babbage, succeeded in building a small part of the mill after Babbage had died, and this, like the part of the Difference Engine, is in the Science Museum in London. There is a tendency to blame the government of Babbage's day for being too timid and unimaginative to give him support. I think that is unfair: the ideas were too new for their importance to be appreciated at the time, and there was no obvious and pressing national need for the machine. Its construction was too much for the technology of the day, and even if this purely mechanical device had been built, it would almost certainly have proved too slow to be of real value. Babbage himself was predicting a time of one minute for a multiplication. Finally, Babbage was not an easy man to deal with: his mind was too active, and he was always producing new ideas. This had the disastrous result that he would abandon a piece of construction before it was finished, because he could see a better way to do it; to use the modern term once again, he did not realise the importance of freezing the design. He was undoubtedly embittered when he died in 1871; it was his misfortune that in so many ways he was a century ahead of his time.

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Isoperimetric Inequalities

DAVID BURGHES
University of Newcastle

According to legend, the city of Carthage was founded in 850 B.C. by Dido, a princess from the land of Tyre. Fleeing from the tyranny of her brother, she landed on the North African coast seeking land for a settlement. The local natives gave her a grudging concession, saying that she could occupy 'as much land as could be contained by the skin of an ox'.

Dido's ingenious solution involved cutting the ox's skin into extremely thin strips, which were then joined together to form one continuous closed strip. This strip could then encompass a much larger territory than the ox-skin area that the natives had intended. Her next problem was to choose the shape that her strip should take so as to maximise the area enclosed. Should she choose a square, or a circle, or perhaps a triangle with her fixed length of perimeter? Does in fact a change in perimeter shape change the enclosed area? Dido's solution was to choose a circle, but it was not until the nineteenth century that this result was rigorously proved to be correct.

We can formulate Dido's problem in precise mathematical terms using the following definition:

For a plane figure of area A with perimeter L, we define the *isoperimetric quotient* (I.Q.) *number* of the figure as

$$\text{I.Q.} = \frac{A}{\pi(L/2\pi)^2} = \frac{4\pi A}{L^2}. \tag{1}$$

This is a non-dimensional number, and, for a circle of radius a, the I.Q. number is 1. In Dido's problem, L is a given constant length, and so she requires the figure with maximum I.Q. number (if one exists!). The *isoperimetric theorem* states that, for any plane figure,

$$\text{I.Q.} \leqslant 1, \tag{2}$$

equality holding only in the case of a circle. Equation (2) is an example of an *isoperimetric inequality*. In the table below we list some simple figures with their I.Q. numbers. You can soon check these and evaluate many other I.Q. numbers.

Figure		I.Q.
Circle		1·00
Square		$\pi/4 \approx 0{\cdot}79$
Rectangle: sides in ratio	$1 : \lambda$	$\pi\lambda/(1+\lambda)^2$
	$\lambda = 2$	0·70
	$\lambda = 3$	0·59
	$\lambda = 9$	0·28
Semi-circle		0·75
Sector of circle:	angle γ (radians)	$2\pi\gamma/(2+\gamma)^2$
	$\gamma = 3\pi/2$	0·66
	$\gamma = 3\pi/4$	0·78
	$\gamma = \pi/2$	0·77
Triangle:	equilateral	$\pi/3\sqrt{3} \approx 0{\cdot}61$
	45°, 45°, 90°	0·54
	30°, 60°, 90°	0·49
Hexagon		0·91
Pentagon		0·86

We can make a number of conjectures from this table:

(i) For any rectangle,

$$\text{I.Q.} \leqslant \frac{\pi}{4}, \tag{3}$$

equality occurring only for a square.

(ii) For any triangle,

$$\text{I.Q.} \leqslant \frac{\pi}{3\sqrt{3}}, \tag{4}$$

equality occurring only for an equilateral triangle.

(iii) For sectors of circles, it is not at first obvious what angle gives a maximum I.Q. number. Using the general formula, I.Q. $= 2\pi\gamma/(2+\gamma)^2$ we can show that the I.Q. number has a maximum when $\gamma = 2$ radians. Thus for any sector of a circle,

$$\text{I.Q.} \leqslant \frac{\pi}{4}, \tag{5}$$

equality occurring only for the sector enclosing angle 2 radians.

In (3), (4) and (5) there are further examples of isoperimetric inequalities. The table also illustrates that increasing symmetry increases the I.Q. number. The circle for instance is symmetric about any diameter. This is the key to a major attempt at proving the isoperimetric theorem due to the Swiss mathematician Jacob Steiner

(1796–1863). He used 'symmetrisation' (see reference 1) in his attempted proof, but the German mathematician Karl Weierstrass (1815–1897) pointed out that Steiner's proof assumed the *existence* of a figure with maximum I.Q. number.

The question of existence is not a trivial question. For instance, consider the problem of finding the curve of minimum length which joins two points P and Q and leaves Q at right angles to the straight line PQ. Clearly, if one such curve is found, a second curve can also be found with less length!

After putting calculus on a sound logical basis, Weierstrass gave a rigorous proof of the isoperimetric theorem—more than 2000 years after it was first conjectured. Isoperimetric inequalities have been used more recently for obtaining properties of solutions of partial differential equations. In reference 2 a large number of isoperimetric inequalities is developed, and reference 3 is a very readable introduction to inequalities, including isoperimetric inequalities.

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Statistics and Radiocarbon Dating

R. M. CLARK
Monash University, Australia

1. Background to the problem

It may surprise most readers to know that all living organisms (including people!) are radioactive. Yet this fact forms the basis of radiocarbon dating, a technique developed only in the last twenty-five years and now widely used by archaeologists to determine the age of organic matter dug up at archaeological sites.

Radiocarbon or carbon-14 is produced in the atmosphere by the action of cosmic rays at the rate of two radiocarbon atoms per square centimetre of the earth's surface per second or, equivalently, 7·5 kilograms globally per year. Although these isotopes of carbon are radioactive, they behave chemically like ordinary carbon, and so, for example, they combine with oxygen to produce carbon dioxide. This radioactive carbon dioxide is absorbed by plants during photosynthesis, and by animals which eat the plants, and in turn by *people* who eat the by-products of plants and animals. While a plant or animal is alive, any of its radiocarbon atoms which decay are presumed to be immediately replaced by 'fresh' radiocarbon from the atmosphere, and so the concentration of radiocarbon in all living matter should be the same as that in the atmosphere, currently 1 in 10^{12} relative to the most common isotope, carbon-12.

However, once an organism dies, its store of radiocarbon is not replaced but decays exponentially at a known rate. Thus, in principle, the age of any sample of organic matter (such as fragments of wood or charcoal from an archaeological site) can be determined simply by measuring the current concentration of radiocarbon in that sample.

The steady exponential decay of radiocarbon may be represented by the equation

$$A_m = A_i e^{-\lambda x}, \tag{1}$$

where A_i denotes the *initial* concentration of radiocarbon in the sample when it 'died' x years ago, and A_m the measured concentration of radiocarbon in the sample *now*. The decay constant λ is related to the *half-life* of 5,730 years. That is, half of the radiocarbon present in any sample decays away in 5,730 years; half of the remainder (or one quarter of the original amount) decays away in a further 5,730 years, and so on.

In practice, both x and A_i are unknown. The radiocarbon age, y, of the sample is determined by *assuming* that $A_i = A_0$, the concentration of radiocarbon in living material *now*, and solving the equation

$$A_m = A_0 e^{-\lambda y}$$

giving

$$y = \frac{1}{\lambda}\log[A_0/A_m]. \tag{2}$$

The quantities A_m and A_0 in (2) cannot be measured directly but are estimated indirectly by counting for several days the emitted beta-particles arising from the disintegration of carbon-14 atoms in the sample. This emission of particles is a random process, the number of particles in any fixed time-interval being a Poisson random variable with mean value proportional to the length of the time-interval and the concentration of radiocarbon in the sample. Thus, for this reason alone, every radiocarbon date is subject to unavoidable random errors of measurement. The resultant lack of precision due to these so-called 'counting errors' of any radiocarbon date is expressed in terms of its standard deviation, which may range from 40 to 120 years. There are also other sources of error of a more-or-less random nature which may affect radiocarbon dates, and so the real standard deviation of a given radiocarbon date may be greater than the figure reported by the laboratory concerned.

2. Correction of radiocarbon dates

Clearly, radiocarbon dating depends on several assumptions, and the validity of the method can only be checked by obtaining radiocarbon dates for a series of samples of *known* age, and then comparing the radiocarbon age and the known age of each sample. This was done, in the first instance, using samples of known age from certain archaeological sites in Europe and the Middle East. The agreement between radiocarbon dates and 'true' dates was not exact, but as close as could be expected, given that the radiocarbon dates were subject to inevitable random errors of measurement, and that the real age of some of the samples was not known exactly.

More accurate comparisons were later made using samples of wood from the giant *Sequoia* trees of California. The age of each such sample was known *exactly* simply by counting the annual growth rings of the tree from which the sample was taken. The results of this study confirmed that radiocarbon dates were sufficiently accurate for practical purposes, at least for samples not more than 1,300 year sold.

However, this comforting view of the radiocarbon dating method has been challenged in the past five years, following an extensive analysis of tree-ring-dated samples of bristlecone pine, *Pinus aristata*. These trees, which grow high in the White Mountains of California, live for up to 4,600 years, and it is possible to date fragments of wood which are even older than the oldest living tree, by 'cross-dating' the tree-rings (see reference 1). So far, more than 600 fragments of bristlecone pine, up to 8,200 years old, have been dated by tree-ring methods, with an error claimed to be no greater than 10 years.

A comparison of the radiocarbon dates and tree-ring dates of these 600 samples shows that there are major discrepancies between radiocarbon dates and tree-ring dates prior to 1500 B.C., the radiocarbon dates being systematically younger (i.e., more recent) by up to 700 years. Thus it is essential to correct, adjust or 'calibrate' radiocarbon dates, at least for material more than 3,000 years old.

In retrospect, it is clear that the assumption $A_i = A_0$ is not justified, and that the concentration of radiocarbon in living matter has not been constant during the past. The initial concentration, A_i, of radiocarbon in a sample x years old really depends, in some way, on x, and so it is advisable to replace A_i in Equation (1) by $A(x)$, which may be thought of as the global concentration of radiocarbon in all living matter x years ago.

Thus all samples of age x should have the same radiocarbon age $F(x)$, related to the radiocarbon concentration $A(x)$ by

$$\begin{aligned} F(x) &= \frac{1}{\lambda}\log\left[\frac{A_0}{A(x)\,e^{-\lambda x}}\right] \\ &= x+\frac{1}{\lambda}\log\left[\frac{A_0}{A(x)}\right]. \end{aligned} \tag{3}$$

This 'calibration function' F thus defines the theoretical relationship between radiocarbon dates and tree-ring or calendar dates.

3. The archaeologist's problem

Let us now consider the archaeologist's problem of finding the 'true' age, say x_0, of a particular sample from an archaeological site. The analysis of the bristlecone pine samples has shown that he can no longer assume that the radiocarbon age, y_0, of his sample is the same as the calendar age x_0. How then can he determine x_0?

In theory, the answer is simple. Clearly the number x_0 must satisfy the equation

$$y_0 = F(x_0). \tag{4}$$

In practice, there are some complications. Firstly, since we have no independent information concerning past variations in the radiocarbon concentration of the atmosphere, the function A in Equation (3) and hence F itself is unknown, and must somehow be estimated. Secondly, Equation (4) is the 'wrong way around', in the sense that it is x_0 which is unknown, not y_0. In other words, Equation (4) does not give us a direct expression for the unknown quantity x_0. Thirdly, the radiocarbon date y_0, like all radiocarbon dates, is not known exactly, but is subject to random errors of measurement. We will now consider each of these difficulties in turn.

(a) Clearly, F must be estimated using the data from the bristlecone pine samples. If we let x_i and y_i denote the tree-ring age and radiocarbon age respectively for the ith bristlecone pine sample, then these data must satisfy the n equations

$$y_i = F(x_i)+e_i, \quad i = 1, 2, \ldots, n, \tag{5}$$

where e_i denotes the measurement error associated with y_i, and n is the number of samples considered. (Here we have tacitly ignored any errors in the tree-ring dates, since any such errors are believed to be small.)

One way of estimating F from these equations is to plot the n points (x_i, y_i) on a graph and then draw a smooth freehand curve through these points. This was done in 1970 by Professor Suess of the University of California. His calibration curve (reference 2), extending back to 5,300 B.C., has been widely used by archaeologists; the consequent corrections to existing radiocarbon dates from archaeological sites have had a dramatic effect on the interpretation and understanding of European prehistory (see reference 2).

The trouble with this method is that it is ambiguous. If *you* were to draw a freehand curve through exactly the same data as Suess used, you would almost certainly produce a curve rather different from Professor Suess's.

Such ambiguity is avoided by using an explicit statistical procedure for estimating F from the available data. One such method is to *assume* that F can be represented, at least over a fairly short interval, by some low-degree polynomial, and then to estimate the coefficients of this polynomial by the Method of Least Squares. For example, we might assume that F is a cubic, so that Equations (5) become

$$y_i = B_0+B_1x_i+B_2x_i^2+B_3x_i^3+e_i, \quad i = 1, 2, \ldots, n. \tag{6}$$

The unknown coefficients B_0, B_1, B_2, B_3 are then determined in such a way that the sum of the squares of the e_i's in Equation (6) is minimised. This method has the added and important advantage that it is possible to state, by means of *confidence limits*, how close our resulting estimate of F can be expected to be to the theoretical function F defined in Equation (3).

(b) *If* F were a one-to-one function, it would have an inverse function F^{-1}, and so we could express the unknown x_0 in terms of the known quantity y_0 by means of the equivalent equation

$$x_0 = F^{-1}(y_0).$$

However, since the form of F is unknown, it would be unwise to assume that it is a one-to-one function.

In fact, the present evidence suggests that F is certainly not one-to-one. Suess's calibration curve is highly irregular, and contains numerous 'kinks' or 'wriggles' (like the curve in Figure 1) so that it is possible for several distinct tree-ring dates to correspond to the same radiocarbon date (as in Figure 1).

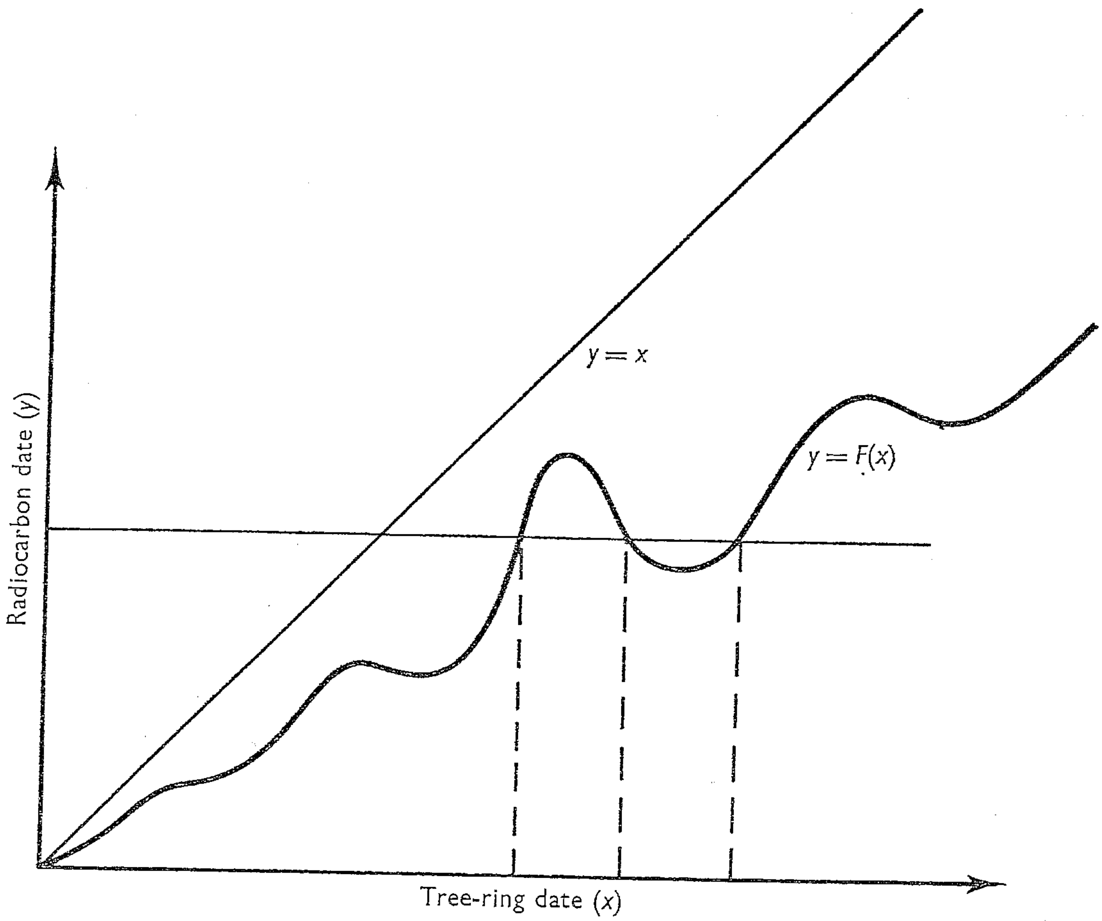

Figure 1

Thus if the radiocarbon date (y_0) of the archaeologist's sample corresponds to a kinky part of the function F, he has no alternative but to conclude that the calendar age of his sample could be any of several possible dates. However, in certain cases, it may be possible to eliminate some of these alternative dates using other archaeological evidence.

(c) Since y_0 is not an exact number but in fact an observation on a random variable with known standard deviation, we should try to find the likely *range* of x_0. This may be done in terms of confidence limits. However, the situation is complicated by the fact that F is estimated from the observations $y_1, y_2, \ldots, y_n$, each of which contains random errors of measurement, and so our estimate of F must be subject to random fluctuations to some extent. More advanced students may like to see how these complications can be resolved in the special case when F is assumed to be a straight line (reference 3).

4. Validity of the calibration

Since, as already mentioned, the archaeological implications of the tree-ring calibration of radiocarbon dates are so dramatic, it is desirable, if possible, to verify this correction of radiocarbon dates. Indeed, the validity of the bristlecone pine results has been questioned recently on both geophysical and archaeological grounds. It has been suggested that carbon-14 may actually be produced in living bristlecone pine trees by the action of cosmic rays at the high altitude at which these trees grow. Also, it is possible that, because of the very narrow rings of the bristlecone pine (typically 0·1 mm), the inner rings of a tree may be enriched in radiocarbon by sap from the outer (younger) rings. In either case, the radiocarbon content of the bristlecone pine samples would be abnormally high, and the corresponding radiocarbon dates would be too young (recent), as we actually find.

At present, the historical calendar of ancient Egypt provides the only independent chronology for testing the bristlecone pine calibration curve. In a recent paper (reference 4), statistical methods were used to compare the radiocarbon dates of the bristlecone pine samples and a series of samples from archaeological sites in Egypt, assuming that over the time-period considered, namely 3100–1800 B.C., the calibration function F could be represented by either a polynomial or a continuous piecewise linear curve. This analysis confirmed the general trend of the bristlecone pine calibration curve, but this result should be regarded as provisional to some extent, because of the uncertainties regarding the absolute dates of the Egyptian samples concerned.

Concluding remarks

The main trend of the calibration function F in Equation (3) now appears to be well established, at least up to 8,000 years ago, but there has been much argument concerning the magnitude, the location and even the existence of the kinks in Suess's calibration curve. Unfortunately, there is not enough space here to describe how, in principle, statistical methods can help to resolve these questions also (see reference 5 for further details).

Although one should not overlook the technical difficulties of obtaining accurate radiocarbon dates, the subsequent correction of a given radiocarbon date is essentially a statistical problem. In this article, I have tried to show *why* and *how* the statistician is able to help the archaeologist to solve his most urgent problem, namely that of accurately dating objects from archaeological sites.

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More about Lions and other Animals

PETER A. RADO
Royal Holloway College, London
and
RICHARD RADO
University of Reading

It was shown in reference 3 how a man can save himself from an enraged lion with whom he is compelled to share a cage. In the present note we consider some analogues and extensions. Developments in a different direction were pursued by H. T. Croft (reference 2). In each case there will be a man M and one or more lions L, or other animals. $M(t), L(t)$ denote the positions of M and L, respectively, at time t. The whole operation starts at the time $t = 0$. By saying that M (and similarly L, etc) has *maximum speed* w, all we mean is that

$$\overline{M(t_1)\,M(t_2)} \leqslant w(t_2 - t_1)$$

for $t_1 < t_2$, where $\overline{AB}$ denotes the distance between the points A and B. We use rectangular cartesian coordinates x, y.

1. Many lions in an infinite desert Π

(i) *M not inside the region dominated by the lions $L_1, \ldots, L_n$.*

Let $M(0)$, $L_1(0), \ldots, L_n(0)$ be points in a plane Π. Suppose that man and lions have the same maximum speed v. Denote by C_0 the *convex hull* of the points $L_i(0)$. This means that C_0 is the set of all points of the form $\sum_{i=1}^{n} t_i L_i(0)$, where the points $L_i(0)$ are treated as vectors and the t_i are arbitrary non-negative numbers such that $\sum_{i=1}^{n} t_i = 1$. In fact C_0 is either a convex polygon, plus its interior, whose vertices are some or all of the points $L_i(0)$, or else C_0 is a straight segment, which we consider as a degenerate polygon.

Now suppose that $M(0)$ is not an interior point of C_0, which is only relevant if C_0 is non-degenerate. Furthermore, let $M(0) \neq L_i(0)$ for all i. Then the man can save himself as follows. By a general theorem on convex sets, which is intuitive for convex polygons, there exists a straight line l through $M(0)$ such that all points $L_i(0)$

lie in one and the same of the two half-planes bounded by l. We may assume that $M(0) = (0, 0)$, that l is the line $x = 0$, and that all points $L_i(0)$ lie in the half-plane $x \leqslant 0$. Now let M move at his maximum speed v along the positive x-axis, so that $M(t) = (vt, 0)$ for $t \geqslant 0$. Then (see Figure 1)

$$\overline{M(0)\,M(t)} < \overline{L_i(0)\,M(t)} \quad \text{for } t \geqslant 0 \text{ and all } i,$$

so that M is safe for ever and ever.

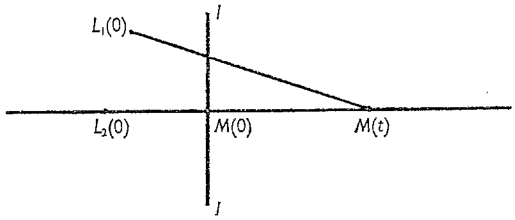

Figure 1

(ii) *$M, L_1, \ldots, L_n$ at arbitrary points in Π.*

Let M and L_i have maximum speeds w and v_i, respectively, and let $M(0) \neq L_i(0)$ for all i. Then M can save himself provided that $w \geqslant v_i$ for each i, and

$$\sum_{i=1}^{n} \sin^{-1}(v_i/w) < \pi. \tag{1}$$

Proof. According to [1], the locus of all points P in Π such that

$$\overline{PL_i(0)}/\overline{PM(0)} = v_i/w$$

is a circle C_i (see Figure 2) which degenerates into a straight line when $v_i = w$. Let the line $M(0)\,T_i$ touch the circle C_i at the point T_i. It is not very difficult to

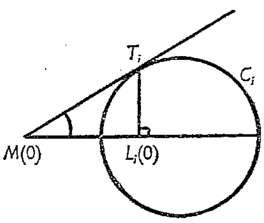

Figure 2

prove that the angle $M(0)L_i(0)\,T_i$ is $\frac{1}{2}\pi$. (See problem 7.7.) Therefore, if θ_i is the angle $L_i(0)\,M(0)\,T_i$,

$$\sin\theta_i = \overline{T_i L_i(0)}/\overline{T_i M(0)} = v_i/w.$$

So the circle C_i subtends at $M(0)$ the angle $2\theta_i = 2\sin^{-1}(v_i/w)$, and (1) gives $\sum_{i=1}^{n} 2\theta_i < 2\pi$. It follows that there is a half-line h through $M(0)$ which lies outside each of the circles C_i; and if the man moves along h at his top speed w he will be safe for ever and ever. To see this we first observe that, for each i, the region outside C_i is the locus of all points P such that

$$\overline{PL_i(0)} > (v_i/w)\,\overline{PM(0)}. \tag{2}$$

Now assume (contrary to what we wish to prove) that L_i catches M at time t_0, so that $L_i(t_0) = M(t_0) = X$, say. Denote by $\widetilde{L_i(0)\,X}$ the length of the path of L_i from $t = 0$ to $t = t_0$ (which need not be straight). Then we have from (2), since X must be on h (and so outside C_i),

$$\overline{M(0)\,X}/\overline{L_i(0)\,X} < w/v_i = wt_0/v_i\,t_0$$

$$\leqslant \overline{M(0)\,X}/\widetilde{L_i(0)\,X} \leqslant \overline{M(0)\,X}/\overline{L_i(0)\,X}.$$

In fact, we have an inequality of the form $x < x$, which is impossible. This shows that L_i never coincides with M.

(iii) *Three lions having the same maximum speed v.*

It is worth discussing this case separately, although, of course, it is a special case of (i) or (ii); for it gives rise to what seems to be an interesting problem.

Let $M(0) \neq L_i(0)$ for $1 \leqslant i \leqslant 3$. If $M(0)$ is not inside the triangle $L_1(0)\,L_2(0)\,L_3(0)$, then, according to (i), the man can save himself provided that his maximum speed w satisfies $w \geqslant v$. Now let $M(0)$ be an interior point of this triangle. Then the strategy of (ii) will save the man if

$$3\sin^{-1}(v/w) < \pi,$$

i.e.,

$$\frac{w}{v} > \frac{2}{\sqrt{3}},$$

or

$$w > (1{\cdot}154700\ldots)\,v.$$

We have not been able to make a complete analysis of the case when

$$1 \leqslant \frac{w}{v} \leqslant \frac{2}{\sqrt{3}}.$$

(iv) *One lion, and his maximum speed exceeds that of the man.*

Let M and L have maximum speeds w, v respectively, and let $0 < w < v$. Then (α) the man can move in such a way that he is safe at least until the time

$$t^* = \overline{L(0)\,M(0)}\,(v-w)^{-1};$$

and (β) there is a strategy for the lion such that, whatever the man does, he is caught at or before the time t^*.

Proof of (α). Let the man choose, not unnaturally, to run at his top speed w along a straight line l through $M(0)$ and $L(0)$ in a direction away from $L(0)$ (see Figure 3). Define the *mock-lion* $L'(t)$ to be the projection of $L(t)$ onto l. The man

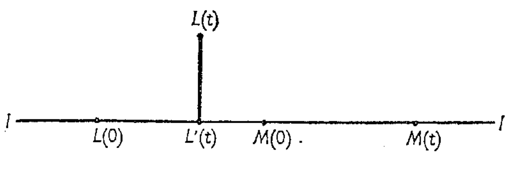

Figure 3

observes that, whatever the lion does, the mock-lion has at most the speed v, and thus his own speed relative to $L'(t)$ is at most $v-w$. He also reflects that he cannot be caught by the real lion unless the mock-lion closes in on him at the same time. So he concludes that he will be safe until at least the time $t = t^*$.

Proof of (β). The authors wish to acknowledge gratefully that the inspiration for this proof was entirely derived from reference 4. We shall define a sequence t_i such that $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots$. Let, for some $i \geqslant 0$, the numbers $t_0, \ldots, t_i$ be already

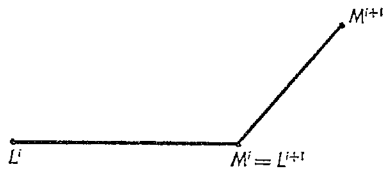

Figure 4

defined and let $0 = t_0 \leqslant \ldots \leqslant t_i$. Put $M^i = M(t_i)$ and $L^i = L(t_i)$. We now define t_{i+1} and a strategy for the lion. The lion moves at top speed from L^i to M^i, putting $L^{i+1} = M^i$. Whilst this is happening, the man moves from M^i to some point M^{i+1} of his choice (see Figure 4). Let t_{i+1} be the time at which the lion arrives at L^{i+1}. Now we have

$$\overline{L^iM^i} = v(t_{i+1} - t_i)$$

and

$$\overline{L^{i+1}M^{i+1}} \leqslant w(t_{i+1} - t_i) = \frac{w}{v}\overline{L^iM^i}.$$

Hence

$$\overline{L^nM^n} \leqslant \left(\frac{w}{v}\right)^n \overline{L^0M^0} \quad \text{for } n \geqslant 0$$

and so

$$\lim_{n\to\infty} \overline{L^nM^n} = 0.$$

Furthermore,

$$0 \leqslant t_{i+1} - t_i = \frac{1}{v}\overline{L^iM^i} \leqslant \frac{1}{v}\left(\frac{w}{v}\right)^i \overline{L^0M^0}.$$

On summing for $i = 0, 1, \ldots, n-1$ we find, since $t_0 = 0$, that

$$t_n = t_n - t_0 \leqslant \frac{1}{v}\overline{L^0M^0}\sum_{i=0}^{n-1}\left(\frac{w}{v}\right)^i \leqslant \frac{1}{v}\overline{L^0M^0}\left(1-\frac{w}{v}\right)^{-1} = \overline{L^0M^0}(v-w)^{-1} = t^*.$$

This shows that t_n converges to a limit between 0 and t^* inclusive, all of which can be interpreted as meaning that the lion will get his man at or before time t^*.

2. A hawk in a cage

Let H and M be points in a cage C having a level floor F. Suppose that M can move on F and H can move in C, each with maximum speed v. Then M can save himself from being caught by H by adopting the strategy described below. We assume, of course, that $M(0) \neq H(0)$.

Step 1. The man chooses M_1, an interior point of F, such that the whole straight segment from $M(0)$ to M_1 lies in F and, moreover, $\overline{M(0)M_1} < \frac{1}{2}\overline{M(0)H(0)}$. This is, for instance, always possible if C is a cylinder or a hemisphere or, more generally, whenever C is convex. The man moves at top speed in a straight line from $M(0)$ to M_1. The hawk cannot catch him during this manœuvre and will move in the same time from $H(0)$ to H_1.

Step 2. The man chooses a straight segment S having one end-point at M_1 and lying entirely in F. Moreover, the length of S is to be less than $\frac{1}{2}\overline{M_1H_1}$. This step takes no time at all, of course.

Now the strategy involves an alternative depending on the hawk's movements. We introduce the *mock-hawk* H' which we define to be the projection of the real hawk H onto the plane of the floor F. The projection of H_1 is called H_1', and similarly for $H_2, H_3, \ldots$.

Case 1. $H_1' \neq M_1$. In this case the man ceases to worry about the hawk and instead considers the mock-hawk as his enemy. He notes with relief that the hawk can only peck at him provided the mock-hawk gets its beak into him at the same time. Also, the mock-hawk cannot move faster than the real hawk and hence not faster than he himself at top speed. Thus, by employing the strategy defined in reference 3 he can for ever remain safe from H' and therefore also from H.

Case 2. $H_1' = M_1$. Then the man moves at top speed along the whole length of the segment S, from M_1 to M_2, say, whilst H moves from H_1 to H_2, say. According to the choice of S he will be safe from H whilst doing this. If $H_2' = M_2$, he moves back again along S getting to M_3 $(= M_1)$, whilst the hawk flies to H_3. The man observes with satisfaction that, if $H_2' = M_2$, then the hawk must have flown at top speed horizontally from H_1 to H_2, so that $\overline{M_1H_1'} = \overline{M_2H_2'}$. The man continues this procedure, moving to and fro along S as long as the hawk keeps shadowing him, which means as long as $H_i' = M_i$.

There are only two possibilities:

(i) We have $H_i' = M_i$ for each $i = 1, 2, 3, \ldots$. In this case the total length of the path which the man describes is infinite, and the hawk always remains at the fixed positive distance $\overline{M_1H_1}$ from the man.

(ii) There is at least $i \geqslant 1$ such that $H_i' \neq M_i$, so that while the man moved from M_{i-1} to M_i the hawk ceased flying horizontally. Then the man uses from now on the strategy of reference 3 against H', as explained in Case 1, and he is again safe from H.

We wish to thank the Editor for his helpful comments and suggestions.

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Some New Applications of the Unit Step Function

J. D. GRIFFITHS
University of Wales Institute of Science and Technology

The (Heaviside) Unit Step Function is defined for real t by

$$H(t) = 0, \quad t < 0,$$
$$= 1, \quad t \geqslant 0.$$

This function has well-known applications in mechanics and engineering, notably in the representation of various periodic functions. For example, the square wave pulse shown in Figure 1 may be represented as a combination of step functions thus:

$$V(t) = H(t) - H(t-1) + H(t-2) - H(t-3) + \dots .$$

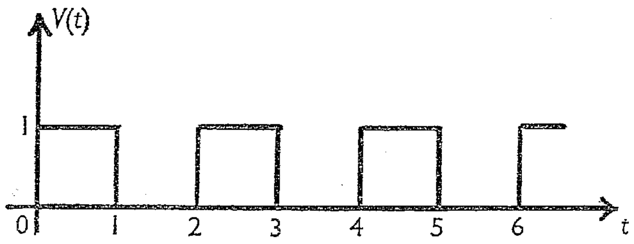


Figure 1

However, the function has applications in optimization and curve fitting which are less well known, and yet of considerable mathematical interest.

Let us look first at the problem of choosing that value of the integer k which minimizes $\sum_{i=1}^{n} |k - x_i|$, where the x_i are given integers. This problem has many applications, e.g., in the siting of a warehouse to supply customers where total distance travelled is to be minimized. Assume, without loss of generality, that the x_i are arranged in ascending order with $x_1 \leqslant x_2 \leqslant \dots \leqslant x_n$.

Clearly, if $k \geqslant x_i$ then $|k - x_i| = k - x_i$, and if $k < x_i$ then $|k - x_i| = x_i - k$. Writing this in terms of the unit step function we have

$$|k - x_i| = (k - x_i) H(k - x_i) + (x_i - k)\{1 - H(k - x_i)\}$$
$$= 2(k - x_i) H(k - x_i) - (k - x_i).$$

Then

$$d(k) = \sum_{i=1}^{n} |k - x_i|$$

$$= 2\sum_{i=1}^{n} (k - x_i)\,H(k - x_i) + \sum_{i=1}^{n} x_i - nk.$$

For minimum $d(k)$ we require

$$d(k) \leqslant d(k+1) \quad \text{and} \quad d(k) \leqslant d(k-1).$$

Now

$$d(k) - d(k+1) = 2\sum_{i=1}^{n} \{(k - x_i)\,H(k - x_i) - (k+1-x_i)\,H(k+1-x_i)\} + n.$$

A little thought shows that

$$\{(k - x_i)\,H(k - x_i) - (k+1-x_i)\,H(k+1-x_i)\} = -H(k - x_i).$$

Hence

$$d(k) - d(k+1) = -2\sum_{i=1}^{n} H(k - x_i) + n,$$

and $d(k) - d(k+1) \leqslant 0$ implies

$$\sum_{i=1}^{n} H(k - x_i) \geqslant \tfrac{1}{2}n.$$

Since the x_i are arranged in ascending order, we have:

$$k \geqslant \text{median value of the } x_i.$$

(The median of a set of numbers is the middle-most value when the numbers are arranged in ascending (or descending) order. For example, the median of the set of values 7, 11, 12, 14, 17 is 12. When the set contains an even number of values the median is taken as the average of the two middle values. For example, the median of the set of values 7, 11, 12, 14, 17, 18 is $(12+14)/2$, i.e., 13.)

Similarly, $d(k) - d(k-1) \leqslant 0$ implies

$$\sum_{i=1}^{n} H(k - 1 - x_i) \leqslant \tfrac{1}{2}n.$$

Hence,

$$k \leqslant \text{median value of the } x_i.$$

Thus,

$$k = \text{median value of the } x_i.$$

(If n is even, this result may need modification as all integral values between the two middle values will be optimal.)

Generalizations are easily made to solve $\min \sum_i w_i |k - x_i|$, where the w_i are some suitable set of weights.

As a further application consider the following well-known problem in stock control.

Demands on a stock occur as follows: Monday, 9; Tuesday, 2; Wednesday, 17; Thursday, 4; Friday, 2; Saturday, 14; Sunday, 1. (This demand pattern is cyclic with a period of one week.) New stock is delivered at the start of each day in constant amounts. Any shortages which occur on a particular day are made up from the next day's delivery of new stock, but shortages cost twice as much per day as holding stock. How much stock should be on hand on Monday morning?

Since the total demand per week is 49, it is clear that 7 units of new stock must be delivered daily. Suppose there is amount k in stock on Monday. The following table shows the situation each day.

Day	Demand	Stock in hand Before demand	Stock in hand After demand
Monday	9	k	$k-9$
Tuesday	2	$k-2$	$k-4$
Wednesday	17	$k+3$	$k-14$
Thursday	4	$k-7$	$k-11$
Friday	2	$k-4$	$k-6$
Saturday	14	$k+1$	$k-13$
Sunday	1	$k-6$	$k-7$

The entries in the last column of the table could be positive, zero or negative, depending on the value of k. Consider, for example, the entry $k-9$. If $k > 9$ this represents a positive stock; if $k < 9$ this represents a shortage of size $9-k$. Hence this entry represents a positive stock of $(k-9)\,H(k-9)$ and a shortage of $(9-k)\{1-H(k-9)\}$.

If $C(k)$ represents the total cost per week with initial stock k, then

$$\begin{aligned}C(k) = {} & [(k-9)\,H(k-9)+(k-4)\,H(k-4)+\ldots+(k-7)\,H(k-7)] \\ & +2[(9-k)\{1-H(k-9)\}+(4-k)\{1-H(k-4)\} \\ & \quad +\ldots+(7-k)\{1-H(k-7)\}] \\ = {} & 3[(k-9)\,H(k-9)+(k-4)\,H(k-4)+\ldots+(k-7)\,H(k-7)]-2(7k-64).\end{aligned}$$

For minimum cost we require $C(k) \leqslant C(k+1)$ and $C(k) \leqslant C(k-1)$.

$$C(k)-C(k+1) = -3[H(k-9)+H(k-4)+\ldots+H(k-7)]+14.$$

$C(k)-C(k+1) \leqslant 0$ implies (on ordering the unit step functions)

$$\begin{aligned}& H(k-4)+H(k-6)+H(k-7)+H(k-9) \\ & \qquad +H(k-11)+H(k-13)+H(k-14) \geqslant \tfrac{14}{3}.\end{aligned}$$

Thus, $k \geqslant 11$. Similarly, $C(k)-C(k-1) \leqslant 0$ gives

$$\begin{aligned}& H(k-5)+H(k-7)+H(k-8)+H(k-10) \\ & \qquad +H(k-12)+H(k-14)+H(k-15) \leqslant \tfrac{14}{3}.\end{aligned}$$

Thus, $k \leqslant 11$. Hence the required value of k is 11.

Again generalizations of this problem are possible, including the case where continuous variables are allowed.

As a final application we consider an example of statistical curve fitting. Suppose it is required to fit the line $y = mx$ to a set of n observed pairs $\{x_i, y_i\}$. Assuming that no errors are present in the x_i values, the traditional method would be to minimize the sum of the squares of the deviations of observed y_i from the ordinates of the line at corresponding values of x_i. The resulting line is known as 'the line of best fit' in the least squares sense; the required value of m is that which minimizes $R(m)$, where

$$R(m) = \sum_{i=1}^{n} (mx_i - y_i)^2.$$

Thus,

$$\frac{dR}{dm} = 2\sum_{i=1}^{n} x_i(mx_i - y_i).$$

Equating this expression to zero yields the least squares estimate of m to be $\sum_{i=1}^{n} x_i y_i / \sum_{i=1}^{n} x_i^2$. One of the main reasons for using this least squares approach is that mathematically it is easy to obtain the above estimate of m in explicit form. This explicit form is very easy to program economically on an electronic computer, which adds to its attractiveness.

However, one might question the use of the least squares principle in defining the 'best' fitting line on other grounds. Essentially, from an intuitive point of view, one would wish a line of best fit to be 'as near as possible', in some sense, to the set of observations $\{x_i, y_i\}$. The most obvious interpretation of 'as near as possible' is to minimize the sum of the vertical distances of the points from the line (regardless of sign). That is, it is required to minimize $S(m)$ with respect to the continuous variable m, where

$$S(m) = \sum_{i=1}^{n} |mx_i - y_i|.$$

Using the unit step function, $S(m)$ may be expressed in the form

$$S(m) = 2\sum_{i=1}^{n} (mx_i - y_i)\, H(mx_i - y_i) - \sum_{i=1}^{n} (mx_i - y_i).$$

From a geometrical point of view $S(m)$ would have a form similar to that shown in Figure 2.

We wish to find that value, $\hat{m}$, of m such that $dS/dm \leqslant 0$ for $m < \hat{m}$, and $dS/dm \geqslant 0$ for $m > \hat{m}$. Now, $dS/dm = 2\sum_{i=1}^{n} x_i H(mx_i - y_i) - \sum_{i=1}^{n} x_i$, using the fact that $dH(t)/dt = 0$ for all $t \neq 0$, as is clear from consideration of a graph of $H(t)$.

Hence,

$$\frac{dS}{dm} \leqslant 0 \quad \text{when} \quad \sum_{i=1}^{n} x_i H(mx_i - y_i) \leqslant \frac{1}{2}\sum_{i=1}^{n} x_i. \tag{1}$$

and

$$\frac{dS}{dm} \geqslant 0 \quad \text{when} \quad \sum_{i=1}^{n} x_i H(mx_i - y_i) \geqslant \frac{1}{2}\sum_{i=1}^{n} x_i. \tag{2}$$

Although the inequalities (1) and (2) do not yield an explicit formula for the required value $\hat{m}$, in practice they provide this value with less burden of computation than in the least squares case. Consider the following example where it is required to fit $y = mx$ to the set of observed values

$$(1,1), \quad (\tfrac{3}{2},3), \quad (2,4), \quad (\tfrac{5}{2},\tfrac{7}{2}), \quad (3,4).$$

Using result (1), and ordering the step functions, we have

$$H(m-1)+3H(3m-4)+\tfrac{5}{2}H(\tfrac{5}{2}m-\tfrac{7}{2})+\tfrac{3}{2}H(\tfrac{3}{2}m-3)+2H(2m-4) \leqslant 5.$$

Hence, $m < \frac{7}{5}$. Similarly result (2) gives $m \geqslant \frac{7}{5}$. Thus, the required value of $\hat{m}$ is $\frac{7}{5}$.

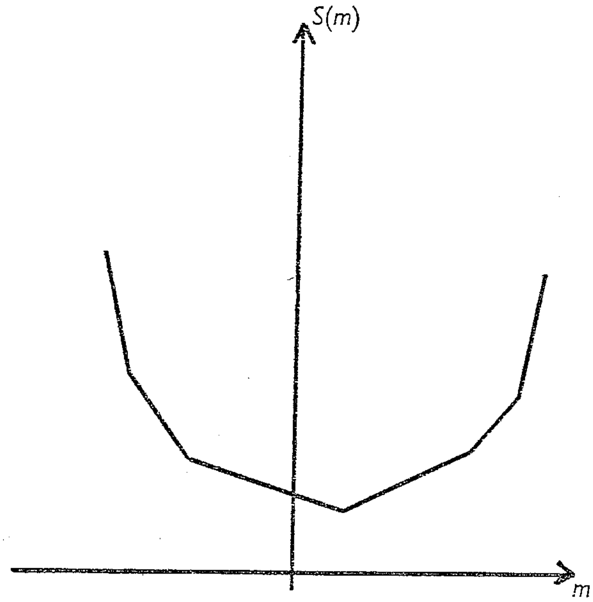


Figure 2

The least squares estimate of m for the above data is 1·52. Figure 3 shows the original data together with the least squares and minimum modulus lines.

Comparing results (1) and (2) with the least squares estimate of m we see that the minimum modulus estimate is the *median* of the set $\{y_i/x_i\}$ weighted with x_i, while the least squares estimate is the *mean* of the set $\{y_i/x_i\}$ weighted with x_i^2.

It is interesting to note that gross errors in extreme values of y_i can affect the least squares estimare of m substantially, whilst having no effect on the minimum modulus estimate. For example, if the point $(2,4)$ in the above example were mistakenly written as $(2,14)$ then the least squares estimate of m is 2·41, but the minimum modulus estimate remains unchanged. On reflection this result is not surprising since the least squares estimate is a mean value and uses all the data; hence any change in the y_i must produce a change in m. However, the minimum modulus estimate is a median value and hence is not affected by changes in extreme values of y_i.

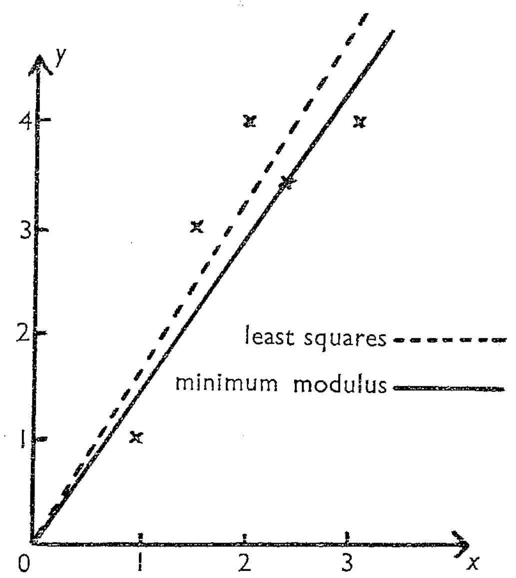


Figure 3

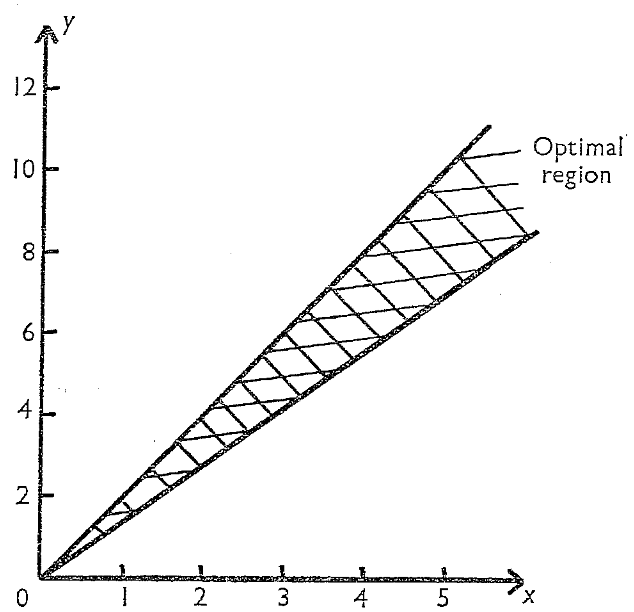


Figure 4

A further point of interest concerns the fact that it is not always possible to find a unique line of best fit using the minimum modulus principle. For example, if we take the set of points

$$(1,1), \quad (\tfrac{3}{2},3), \quad (\tfrac{5}{2},\tfrac{7}{5}), \quad (3,4), \quad (5,12),$$

then the minimum modulus line through the origin can be any line such that $\frac{7}{5} \leqslant m \leqslant 2$ (see Figure 4). This non-uniqueness property can be useful in practical problems since some physical criteria can often be employed to choose the best line within the optimal region.

Dear Editor,

The divergence of the harmonic series

In Volume 7, Number 1 (pages 9–12) Captain N. A. Draim gave three separate proofs of the divergence of the series

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\dots \tag{1}$$

and asked whether readers knew of any others. The first of these proofs can be simplified in the following way.

Suppose that the series (1) converges and has sum S, i.e., that if

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n},$$

then $S_n \to S$ as $n \to \infty$. Since the sequence S_n increases, we have

$$S_n < S_{2n} < S \quad \text{for all } n; \tag{2}$$

and, since $S_n \to S$, there exists an integer N such that

$$S - S_n < \tfrac{1}{4} \quad \text{for all } n \geqslant N. \tag{3}$$

Now (2) and (3) together imply that

$$S_{2n} - S_n < \tfrac{1}{4} \quad \text{for all } n \geqslant N. \tag{4}$$

However, for *every* integer n,

$$S_{2n} - S_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n} > n.\frac{1}{2n} = \frac{1}{2}.$$

This contradicts (4), and therefore the original supposition that the harmonic series converged must have been false. Thus the series diverges.

Yours sincerely,

P. A. P. MORAN
(Australian National University)

* * *

Dear Editor,

Fibonacci numbers

In Volume 3 of *Mathematical Spectrum*, on pages 51–55, there was an interesting article by R. J. Webster on Fibonacci numbers. The definition of these numbers u_n is there given in the customary form: u_1 and u_2 are taken to be 1 and $u_3, u_4, \dots$ are defined by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n \quad (n \geqslant 1).$$

It is shown that, for all n,

$$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}. \tag{1}$$

Spectrum readers may be interested to note an alternative compact expression for u_n; it is

$$u_n = \frac{\sin n\theta}{i^{n-1}\sin\theta}, \tag{2}$$

where θ is any one of the infinitely many complex numbers (differing from one another by integral multiples of 2π) such that

$$\cos\theta = \tfrac{1}{2}i. \tag{3}$$

For suppose that θ satisfies (3). Then $e^{i\theta}+e^{-i\theta} = i$, or

$$e^{2i\theta} - i\,e^{i\theta} + 1 = 0;$$

and the solution of this quadratic equation in $e^{i\theta}$ is

$$e^{i\theta} = i\frac{1\pm\sqrt{5}}{2},$$

so that

$$e^{-i\theta} = i\frac{1\mp\sqrt{5}}{2}.$$

Substitution of either of these pairs of values of $e^{i\theta}$ and $e^{-i\theta}$ in

$$\frac{\sin n\theta}{i^{n-1}\sin\theta} = \frac{(e^{in\theta}-e^{-in\theta})/2i}{i^{n-1}(e^{i\theta}-e^{-i\theta})/2i}$$

$$= \frac{(e^{i\theta})^n-(e^{-i\theta})^n}{i^{n-1}(e^{i\theta}-e^{-i\theta})}$$

immediately leads to Expression (1).

Various relations between Fibonacci numbers can be deduced from Identity (2). For instance readers may like to show that

$$u_{2n} = u_{n+1}^2 - u_{n-1}^2 \quad \text{and} \quad u_{2n+1} = u_{n+1}^2 + u_n^2$$

for all $n \geqslant 1$.

Yours sincerely,

A. B. Patel
(V. S. Patel College of Arts and Science,
Billimora, India)

* * *

Problem submitted by a Reader

Dear Editor,

The following problem may be of interest to your readers.

'In a non-viscous liquid, what is the motion of a small bubble of gas (of constant size) rising from the bottom?'

This problem arises from the contemplation of a glass of soda water or champagne.

Yours sincerely,

W. H. Carter (Major)
(69 Viceroy Court, Lord Street,
Southport, Lancs. PR8 1PW)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

7.7. Distinct points L and M are given in the plane, and k is a real number such that $0 < k < 1$. Then the locus of all points X in the plane such that $LX/MX = k$ is a circle (Apollonius' Circle). A tangent is drawn through M to touch the circle at T. Show that the angle TLM is a right angle.

7.8. Use the identity

$$\frac{4}{1+t^2} = 4-4t^2+5t^4-4t^5+t^6-\frac{t^4(1-t)^4}{1+t^2}$$

to show that

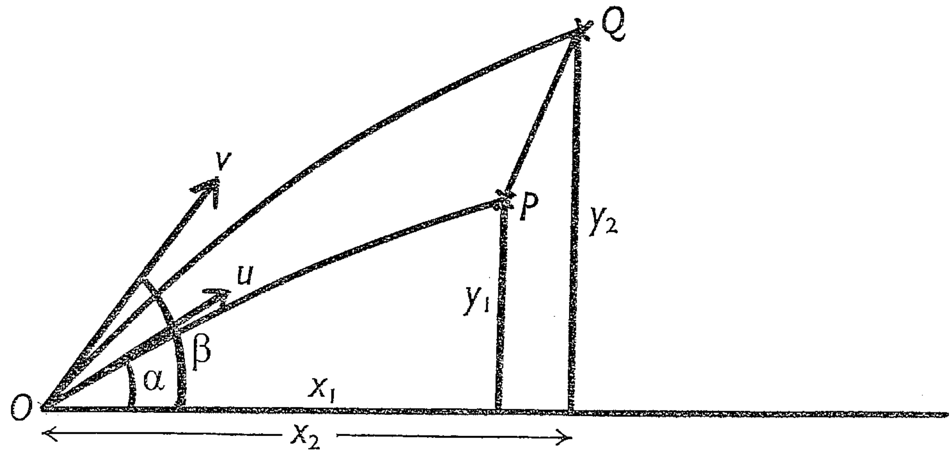
$$\tfrac{22}{7}-\tfrac{1}{1260} > \pi > \tfrac{22}{7}-\tfrac{1}{630}.$$

7.9. (Submitted by B. G. Eke, University of Sheffield.) Let a be a positive integer and let b, c be integers. Suppose that ax^2+bx+c has two distinct roots in the range $0 < x < 1$. Show that $a \geqslant 5$ and find such a quadratic with $a = 5$.

Solutions to Problems in Volume 7, Number 1

7.1. Two projectiles are fired from a point O at the same time. Describe how the direction and length of the straight line joining the projectiles vary with time during the subsequent flight. (Air resistance can be neglected.)

Solution



Suppose that the initial velocities of the projectiles are u, v inclined at angles α, β to the horizontal, and suppose that the projectiles have reached points P, Q respectively at time t. Then, with the notation shown in the diagram,

$$x_1 = ut\cos\alpha, \quad y_1 = ut\sin\alpha - \tfrac{1}{2}gt^2,$$
$$x_2 = vt\cos\beta, \quad y_2 = vt\sin\beta - \tfrac{1}{2}gt^2,$$

so

$$\begin{aligned} PQ &= \sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}} \\ &= t\sqrt{\{(v\cos\beta-u\cos\alpha)^2+(v\sin\beta-u\sin\alpha)^2\}}, \end{aligned}$$

and this is proportional to t. The angle γ of inclination of PQ to the horizontal is given by

$$\begin{aligned} \tan\gamma &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{v\sin\beta-u\sin\alpha}{v\cos\beta-u\cos\alpha}, \end{aligned}$$

and this is independent of time.

7.2. Let $A_1 A_2 \ldots A_n$ be a regular plane polygon with centre O, and let P be a point in the plane outside the circumcircle of the polygon. Compare the geometric mean of the lengths $A_r P$ $(1 \leqslant r \leqslant n)$ with the length OP in the following two cases: (i) when OP passes through a vertex of the polygon; (ii) when OP bisects a side of the polygon.

Solution by M. Ram Murty and V. Kumar Murty (Carleton University, Ottawa)

Let the radius of the circumcircle of the polygon be a. Consider in the complex plane the circle of radius a centred at O. If z is an arbitrary point outside the circle and $w = \exp(2\pi i/n)$, then we are asked to compare $|z|$ and

$$\{|z-aw||z-aw^2|\ldots|z-aw^n|\}^{1/n} = \{|z^n-a^n|\}^{1/n}.$$

In case (i), we may suppose that z is on the positive real axis, so that z is real and greater than a.

In this case

$$\{|z^n-a^n|\}^{1/n} < z = |z|.$$

In case (ii), we may suppose that $\arg z = \pi/n$. Hence, if

$$z = r\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right),$$

we have

$$\{|z^n-a^n|\}^{1/n} = \{r^n+a^n\}^{1/n} > r = |z|.$$

7.3. The real numbers $a_1, a_2, a_3, \ldots$ are positive, less than 1 and such that

$$a_n < \tfrac{1}{2}(a_{n-1}+a_{n+1})$$

for $n = 2, 3, \ldots$. Show that a_n tends to a limit as n tends to infinity.

Solution by M. Ram Murty and V. Kumar Murty

If the sequence is monotone decreasing, then, because it is bounded below, it has a limit. Suppose, therefore, that it is not monotone decreasing. This means that, for some r, $a_{r+1} > a_r$. But we can now show by induction on n that, for all $n \geqslant r$, $a_{n+1} > a_n$. This is certainly true when $n = r$. Assume that $a_{m+1} > a_m$ for some $m \geqslant r$. Then

$$a_{m+2} > 2a_{m+1}-a_m = a_{m+1}+(a_{m+1}-a_m) > a_{m+1}.$$

It follows, as asserted, that $a_{n+1} > a_n$ for all $n \geqslant r$, so the sequence is monotone increasing from its nth term on. Since it is bounded above, it must therefore have a limit.

Vector Methods. By R. J. COLE. Van Nostrand Reinhold Company Ltd, London, 1974. Pp. 117. £1·50, paperback; £3·50, hardback.

Dr Cole's book is an introduction to the methods of vector analysis. It takes the reader fairly briskly to Green's theorem, Stokes' theorem and Gauss' divergence theorem with a final chapter on orthogonal curvilinear coordinates. Written with the 'average student' in mind, it would make a very good companion to a course of lectures but would not, I think, be satisfactory as a first introduction on its own.

The main strength of the book lies in the clear and uncluttered way the material is presented. The strategy is rather similar to that of the Schaum Outlines series—books which have been regarded as rather 'infra dig.' by lecturers in the past, but which have always been appreciated by those on the receiving end. The fundamental results and properties in each chapter are gathered together for easy reference and they are followed by a set of 'notes' explaining the results and indicating ways in which they might be proved. The notes are themselves easily identifiable and so can be missed out on first reading, second reading, or both, depending on the reader's view of such things. In turn the notes are followed by well-chosen, if familiar, applications to geometry, mechanics and physics. The whole process is rounded off with some worked examples and exercises, the latter being chosen less sadistically than is sometimes the case in 'introductory' books. Many students will find the layout ideal for reference and revision.

The author of an introductory textbook has a responsibility to make his subject sensible and to avoid the temptation of giving the impression that sufficient explanation of an idea is there when it is not. This fake impression can only lead to the student either deciding he is not clever enough to cope or trying to condition himself to give correct answers without knowing why. The first decision the author of an introduction to the analysis or algebra of vectors has to make is where to start. Are vectors the elements of a vector space, n-tuples of numbers, equivalence classes of directed line segments, objects having magnitude and direction, or what? Dr Cole is very muddling at this point, for his axioms are certainly not axioms and his properties do not follow from them. Notation, for example $a+(-b) = a-b$, is muddled up with theorems. This concern for being 'sensible' makes me worry about the use of differentials—mnemonic devices which usually suggest the right results; but here, as in many books on vector analysis, they are treated with too much respect. There is a danger also that in reducing the frills to a minimum an author cuts out too much. The consistent choice of a normal to an orientable surface makes sense if it is carefully explained, but is reduced to nonsense in this account. There are one or two other curious lapses—the notation for the work done by a force in moving a particle along a path must include the path itself and not just the end points.

The printing is clear and the usual diagrams are well drawn. Most of the few misprints are in using the wrong type—particularly in the case of Greek letters. Some readers anxious to get on with the subject may grumble less than I do. We might agree, however, that it would have been helpful to include a list of books on the same subject, of about the same difficulty, some giving a more rigorous account and some covering more advanced methods.

Keele University

P. R. BAXANDALL

Finite Mathematics. By HUGH G. CAMPBELL and ROBERT SPENCER. Collier Macmillan Publishers, London, 1974. Pp. 326. £4·50.

This book, dealing as it does with sets, logic, probability, matrices, linear programming and games, will inevitably be compared with the classic *Introduction to Finite Mathematics* by Kemeny, Snell and Thompson.

The level is about the same, giving good background material to a sixth-form course and providing good resource material for teachers at other levels. However, I could not help thinking that Kemeny, Snell and Thompson win most of the comparisons because the illustrations, examples, and particularly the exercises, are so much more interesting. Nevertheless the layout of this book is better than that of Kemeny, Snell and Thompson and trouble is taken to give signposts to the reader about the knowledge required for particular sections and to highlight the key ideas involved.

University of Durham HUGH NEILL

The Geometry of Environment. By LIONEL MARCH and PHILIP STEADMAN. Methuen & Co. Ltd, London, 1974. Pp. 360. £2·90.

I enjoyed this book enormously. It is written by two architects who are aiming to give teachers and to sixth formers some idea of how mathematics, and in particular modern mathematics, is used in their profession. The book is well written and imaginatively illustrated from the worlds of art, architecture and design.

I was very impressed by the whole book but two chapters stood out for me. The first chaper on mappings was so much more lively than many that one reads; and a later chapter on planar graphs (networks) and how they can be applied to the layout of rooms in a building gave unexpected applications of something which tends to be left high and dry in the standard textbooks.

At £2·90 for 360 pages the book is quite excellent value and should find at least one place in every school.

University of Durham HUGH NEILL

The Art of Problem Solving. By S. MOSES. Transworld Student Library, London, 1974. Pp. 183. £0·85.

This is a book that develops Pólya's ideas on problem solving, but it is in no way a duplicate of *How to Solve It* by Pólya. Problems are separated into various types, and strategies are suggested for gaining a solution. Each method is amply illustrated with examples in which notes indicate why a given step was taken. Examples in the main require a knowledge of advanced level mathematics.

Problems are analysed under the following chapter headings.

Abilities. The abilities required in problem solving are discussed.

Strategies. Methods of attack are considered with possible lines of thought being developed. These are illustrated with flow diagrams. The reader gains a clear picture of mathematics in action. The failure of a given strategy is examined with new lines of attack suggested in such a case.

The Inductive Process. Many examples are given illustrating the various ways in which answers can be induced from the observed data.

Methods of Proof. Various types of proof are considered under the main headings of Direct, Indirect and Incomplete Proofs. The author discusses necessary and sufficient conditions, but then fails to give a set of sufficient conditions in defining a group; see example 9, page 106.

Problems and Extensions. An excellent chapter on how, having obtained a solution, the question can be varied to give further extension of the problems. This develops a strong insight into an area of problem solving which, although much neglected among students, is worth developing.

The book contains 100 problems, including many famous and stimulating ones. These are designed to give practice in developing the ideas in the book. Answers and hints are given.

I found this a stimulating book to read and to work with. Pencil and paper are a must. But it is a pity that the examples are of advanced level standard and above. A book of this nature and importance, dealing with basic methods, deserves to be available to the widest possible readership. I am sure some easier examples could have been found which would have been just as instructive. I strongly recommend this book for staff in schools and first-year college students. The methods given would form a sound introduction to a sixth-form course—if easier examples were given.

Sunderland Education Authority
The Esplanade, Sunderland

N. T. WALKER

The next three books are in the *Problem Solvers* series, which provide very brief summaries of relevant theory, a number of fully solved problems and a list of further problems for the student.

Stochastic Processes. By RODNEY COLEMAN. George Allen & Unwin Ltd, London, 1974. Pp. 93. £4·50, hardback; £1·95, paperback.

After an introduction on relevant probability results, this book contains sections on random walk, Markov chains, Poisson, Markov, non-Markov and diffusion processes. To cover all this in a short book has required a very compact layout and unless a student has considerable familiarity with the type of argument used, he will find the presentation of some solutions hard going. The earlier book of the same type by Takács has less extensive coverage with slightly more space so that its solutions are easier to follow. However, the present book will serve as a useful adjunct to courses on stochastic processes given to mathematics students or to mathematically competent students of other disciplines, although the price seems very high.

University of Durham

D. M. GREIG

Fluid Mechanics. By J. WILLIAMS. George Allen & Unwin Ltd, London, 1974. Pp. 107. £3·75, hardback; £1·50, paperback.

Although intended for first- and second-year students of pure or applied science, the problems solved in this book seem unlikely to be encountered in quantity outside an honours mathematics course; they are very much those occurring in classical hydrodynamics. After discussing the basic concepts, the author deals extensively with two-dimensional motion, including complex variable and conformal transformations. However, I failed to see the point of the brief mention of surface waves with two pieces of theory disguised as problems. There is a further chapter on axisymmetric flow. A small collection of exercises for the reader is provided at the end of each chapter. The existence of the velocity potential which is discussed at some length could have been listed with the useful results in vector analysis at the end. This book could usefully supplement a very theoretical course on fluids; it is a pity that it was not produced more cheaply.

University of Durham

D. H. WILSON

Groups. By D. A. R. WALLACE. George Allen & Unwin Ltd, London, 1974. Pp. 104. £3·75, hardback; £1·50, paperback.

I was very disappointed in this book which is simply a set of theorems and a few examples all thrown together and called problems. There is no motivation given and no reader would increase his insight into group theory as a result of reading it. Perhaps an undergraduate would find it useful as a source of solutions to problems set, but I imagine the Schaum series would do that better and in a more constructive way.

University of Durham H. NEILL

Notes on Contributors

Jack Howlett, one of the Editors of *Mathematical Spectrum*, is Director of the Atlas Computer Laboratory, an organisation under the Science Research Council which provides computing services to research workers in all British universities.

David Burghes was first an undergraduate, later a postgraduate student, and finally a Lecturer in Applied Mathematics at the University of Sheffield; but three years ago he moved to the University of Newcastle. His principal field of research is the subject with the formidable appellation of magnetohydrodynamics. He has a growing interest in mathematical education.

R. M. Clark is a Lecturer in Mathematics at Monash University in Victoria, Australia. He graduated at the University of Melbourne and has also held posts in the Operations Research Section of Imperial Chemical Industries Ltd, Melbourne, and at the University of Sheffield. His current research interest is applied statistics, with emphasis on applications in archaeology.

Richard Rado (who will be known to readers of *Mathematical Spectrum* from his article in Volume 6) retired about three years ago from the Chair of Pure Mathematics in the University of Reading, but one cannot imagine him retiring from the active pursuit of mathematics and mathematical research. He is a familiar figure at conferences and seminars the world over, and he continues to pour out a stream of papers on an exceptionally wide range of topics. Combinatorial mathematics in all its forms lies at the heart of his interests, although he is equally at home in algebra, analysis, geometry and the theory of numbers. In 1972, Richard Rado was awarded the Senior Berwick Prize by the London Mathematical Society.

Peter Rado is the son of Richard Rado and, not surprisingly, he keeps the flag flying. A graduate of Cambridge University, he is now a Lecturer in Mathematics at Royal Holloway College (University of London). His dominant interest is in various aspects of graph theory and combinatorics.

Jeff Griffiths is Lecturer in Mathematics at the University of Wales Institute of Science and Technology, Cardiff. After graduating in Mathematics at University College, Cardiff, he spent a number of years in the Operational Research Department of the Steel Company of Wales. His Ph.D. thesis was concerned with the applications of stochastic processes to road traffic queues, and his main current interests are in the techniques of operational research with particular reference to probabilistic model building.

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