MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Mathematical Spectrum Awards for Volume 16

Each year the editors offer two prizes to contributors who are still at school or are students in colleges or universities. A prize of £20 is for an article published in the magazine and another of £10 is for a letter or the solution of a problem.

For 1983/84 the winner of the £20 prize is John Hey for his article 'The proportions of quadratic equations with real and non-real roots'. The £10 prize has been awarded to Ruth Lawrence for her solutions to several problems. The editors have this year decided to award an additional prize of £5: this goes to Richard Hilditch for his solutions to problems.

The Best Viewing Distance for Highway Signboards

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At what distance does a highway signboard appear the largest? Does a square signboard appear larger than a rectangular one? How does the position of the signboard affect its apparent size? These are the questions analysed in this article.

A highway motorist may be interested in knowing the best viewing distance of a signboard, which is the distance at which the signboard appears the largest. The problem is one of calculating the solid angle subtended by the signboard at the observer's eye and finding the distance at which the solid angle is maximised.

In figure 1, the signboard is situated in the y-z plane of a right-handed coordinate system and the motorist is moving on the x-axis towards the origin. The signboard has dimensions $a \times b$ $(a > b)$, with its base at a height d from the eye level and its near side a distance c from the nearest approach of the observer. Although the figure corresponds to the continental system of driving (i.e. on the right), the same results will be obtained for the British system of driving on the left by replacing the limits c and $a+c$ by $-(a+c)$ and $-c$ respectively.

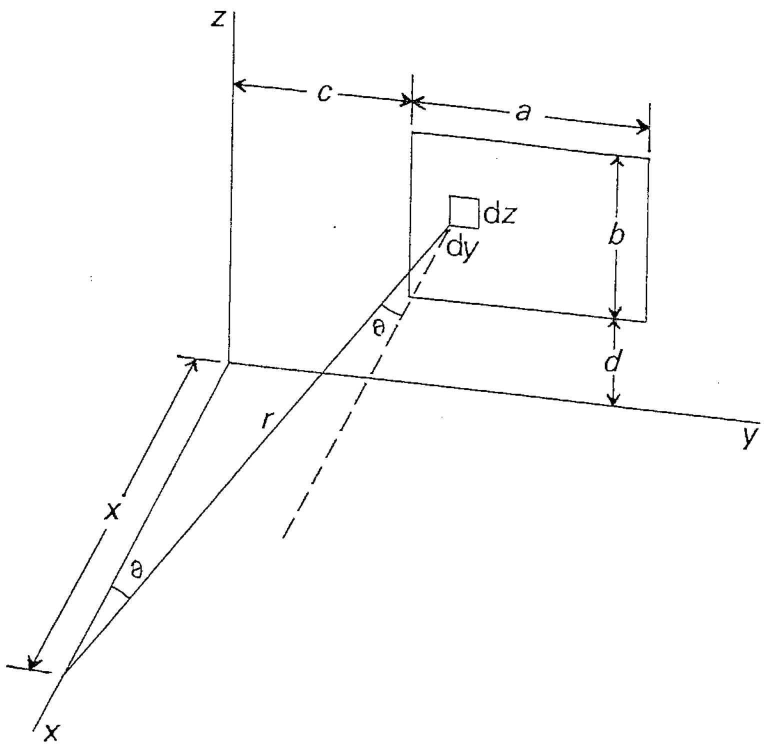

Figure 1. Geometry

Now the elementary solid angle $\mathrm{d}\Omega$ subtended by an elementary area $\mathrm{d}s = \mathrm{d}y\,\mathrm{d}z$ of the signboard is given by

$$\mathrm{d}\Omega = \frac{\mathrm{d}s\cos\theta}{r^2} = \frac{\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2}\,\frac{x}{\sqrt{x^2+y^2+z^2}}. \tag{1}$$

Hence, the total solid angle subtended by the entire signboard is

$$\Omega = \int \mathrm{d}\Omega = \int_{y=c}^{a+c}\int_{z=d}^{b+d} \frac{x}{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}y\,\mathrm{d}z. \tag{2}$$

On carrying out the integration, we obtain

$$\begin{aligned}\Omega = {} & \tan^{-1}\frac{(a+c)(b+d)}{x\sqrt{x^2+(b+d)^2+(a+c)^2}} + \tan^{-1}\frac{cd}{x\sqrt{x^2+d^2+c^2}} \\ & - \tan^{-1}\frac{c(b+d)}{x\sqrt{x^2+(b+d)^2+c^2}} - \tan^{-1}\frac{(a+c)d}{x\sqrt{x^2+d^2+(a+c)^2}}.\end{aligned} \tag{3}$$

A typical highway signboard is rectangular, having an area of around 200 square feet, and is situated around 30 feet from the highway and about 10 feet above ground. Calculated results for solid angles are shown in figure 2 for a few representative cases. A rectangular signboard with $a = 20$ feet and $b = 10$ feet and a square signboard having the same area are considered. Two cases are considered for each signboard: (a) where the signboards are

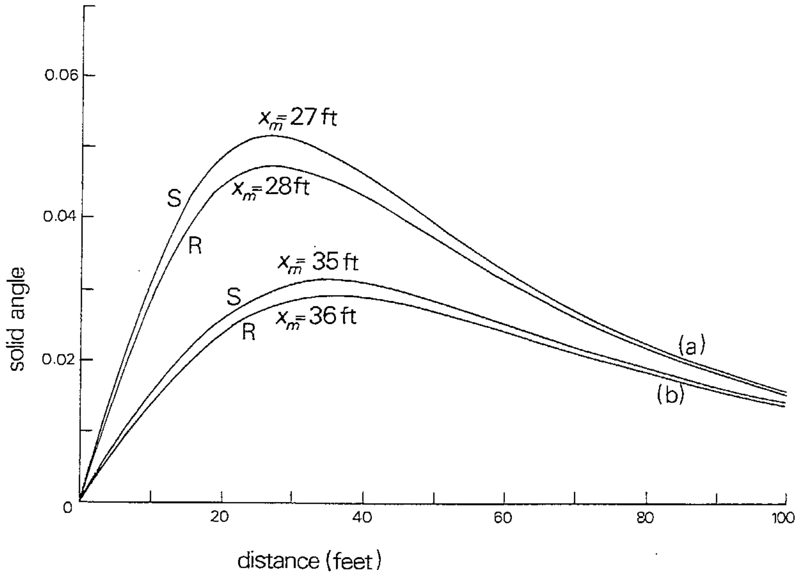

Figure 2. Solid angle Ω against distance x
S = square R = rectangular

situated such that $c = 30$ feet and $d = 5$ feet and (b) where $c = 40$ feet and $d = 10$ feet. The distances x_m where the signboards appear the largest are indicated in the figure.

The following observations are made.

(1) A square signboard appears substantially larger than a rectangular one (with a base greater than height) of the same area.

(2) The apparent size of a signboard decreases dramatically when it is situated further away from the road. For instance, in the above examples, the maximum apparent sizes for the square and rectangular signboards decrease by 40 and 39 per cent, respectively, when the signs are moved a mere 10 feet away and raised 5 feet. Conversely speaking, the maximum apparent sizes increased by 66 and 63 per cent respectively when they are moved 10 feet nearer and lowered 5 feet.

(3) The best viewing distance for the rectangular signboard is one foot longer than that for the square signboard.

(4) The best viewing distance increases when the signboards are situated further away from the road.

Can you write the integers 1 to 12 round the circumference of a circle in such a way that, for every three consecutive digits a, b, c, $b^2 - ac$ is divisible by 13?

Return of the Prisoner Paradox

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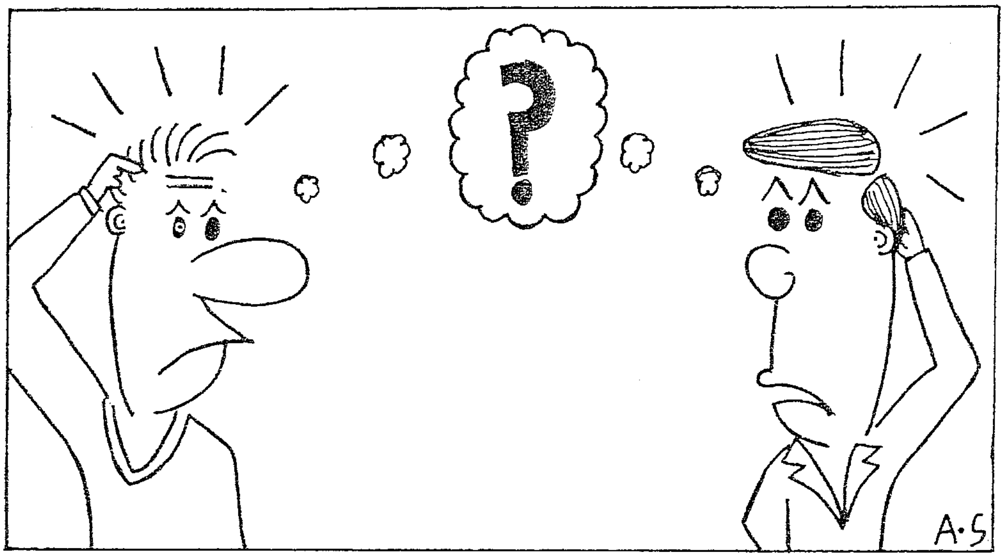

Readers of the monthly magazine *Scientific American* in May and June 1983 were faced with a tantalising offer, called the 'Luring Lottery'. This lottery, which is surely not a hoax, offered a prize of one million dollars divided by the total number of entries received. Each entrant could send in as many entries as desired and, to avoid proliferation of postcards, could do so by simply specifying on one postcard the number of entries to be considered.

A few moments reflection should convince you that *Scientific American* was not being totally crazy. Whilst it is theoretically possible that only one entry, a single one, would be received, getting a payoff of a whole million dollars, the chances are heavily in favour of a large number of entries coming in, many of them multiple entries, so that the actual payout would be a fraction of one dollar. (Even so, I have delayed writing this article until after the lottery deadline passed. Just in case!) The lottery appeared in an article by Douglas Hofstadter, the author of the best-selling book *Gödel, Escher, Bach*, and was intended to provide an example of the kinds of difficulty encountered in trying to maximise the outcome from a situation where other people are involved. The lottery itself was therefore little more than a curious piece of fun, and the final outcome will be of only passing interest. But the articles to which the lottery was related, in the May and June 1983 issues of *Scientific American*, concern a problem which has been around a long time and has decidedly ominous overtones.

For reasons connected with its original formulation, the problem is known as the 'Prisoner Paradox', and can be described as follows. (Warning. If you are reading this over breakfast and have a busy day ahead, postpone the rest until evening. This problem has proven abilities to wreck an entire day.) Imagine you and I are in the process of making some kind of deal with a third party, which involves each of us submitting to the third party a sealed 'bid'. Our bids are to be submitted simultaneously, and are to consist of either the single word 'alone' or else 'together'. If we both bid 'together', we get the contract jointly and each collect £1000. If one of us bids 'alone' and the other bids 'together', the former gets the contract and £2000, the latter gets nothing at all. If we both bid 'alone', neither of us gets the contract and we each collect £100 for our trouble. Before making the bid we are not able to communicate with each other in any way, though we do know who our adversary is and can investigate each other's character and previous behaviour.

So what should we do? If we both bid 'together', we collect £1000 each, which is not to be sniffed at. (After all, this is not the *Financial Times*.) But neither of us wants to be in the position of being left empty-handed because we have bid 'together' whilst the other bids 'alone'. So to avoid this it seems that we have to bid 'alone'. If the other does bid 'together', we collect the entire £2000 but, most likely, we both feel that we have to bid 'alone', which leaves us with a paltry £100 apiece. And we could have had £1000 each if only we had bid to cooperate. Which is where the thing becomes really tricky.

Since we are both mathematicians, we may assume (may we not) that we are both educated, intelligent and, above all, fair-minded individuals, who are all for cooperation. I know that you are like that and you know that I am like that. I know that you know that, and you know that I know that, etc. So I know that you know that it is better to bid 'together' and collect £1000 each rather than to walk away with only £100 each. And you know that I know that. So why don't we both bid 'together'? Well, because from my point of view this would seem a crazy thing to do. My bid cannot possibly have any effect upon your bid, so I lose nothing by bidding 'alone'. If you do bid 'together', then I collect £2000, which might upset you and destroy your faith in me, but I would be crying all the way to the bank. (It is assumed throughout that the only aim is to get the maximum monetary gain from this exercise. All questions of getting satisfaction from cooperating, even in the face of loss, are ruled out. Sorry.) But of course, I expect that you will reason in the same way and will also bid 'alone', so I will only get £100 out of it. And there is the rub. We both know that it would be best to cooperate and collect £1000 each, but we also know that our own bid cannot affect that of the other, so that we may just as well go for broke and bid 'alone', knowing full well that in all probability we shall end up collecting just £100 rather than the £1000 we could have had. Think about it. The more you look at it the worse it appears.

If you are still with this, it has probably become clear to you why mathematicians (and others) show interest in this problem. If, instead of being harmless (?) *Spectrum* readers, we were opposing superpowers (who are surely not *Spectrum* readers), and if the issue were whether or not to deploy a new weapons system, then the game assumes an altogether more serious nature. If we cooperate (i.e. if we both avoid deployment) we each save ourselves billions of pounds of expenditure which can be used to improve our environment (say). But if one of us fails to deploy whilst the other deploys, that one may no longer be in a position to improve the environment to his taste, and may even have no environment left at all. So, perhaps reluctantly, we both decide to deploy. And so it goes on.

In the form described above, the 'Prisoner Paradox' has tantalised logicians during the thirty years since it was first proposed. What sparked off the May 1983 article in *Scientific American* was the result of a fairly recent experiment conducted by Robert Axelrod, a political scientist at the University of Michigan. He had organised a sort of round-robin tournament based on the Prisoner Paradox. The idea here is that each competitor makes not just one bid but a whole series of bids, one after the other, and not just against one opponent but against a whole bank of them. In this way, each competitor can build up a file of information on each 'opponent' upon which to base future decisions. What is the best strategy to adopt in these circumstances?

For the first run of the tournament, Axelrod got fourteen assorted mathematical types each to submit computer programs which described decision strategies to be followed in the tournament, and then got the computer to let each strategy play against every other strategy 200 times, building up information about the opponents' strategies all the time. The detailed description of the outcome of this tournament makes interesting reading, but the most significant fact is that the 'winner' by far was the simplest strategy submitted. Called 'tit for tat', this strategy called for an initial bid of 'together', after which you do exactly what your opponent did last time.

For the second run of the tournament, Axelrod got 62 entries, including 'tit for tat'. Each entrant knew in advance that 'tit for tat' had won the first time round and would be in again. Despite the fact that all other strategies could try to take advantage of 'tit for tat', the outcome was the same. 'Tit for tat' was easily the winner. Mathematically at least the result was conclusive.

The best strategy is to try to cooperate and, when crossed, to avoid over-reaction. In fact, out of the fifteen most successful strategies, only one could be called a 'mean' one and, after 1000 rounds of the tournament, even that one had lost out. There is probably a lesson for us all in this.

Reference

D. R. Hofstadter, Metamagical themas, *Scientific American* **248** (1983) May, 14–21; June, 14–20.

The Alternate Couplet Square Sequence

1 2   3 4   7 10   17 24   41 58   99 140   ....

The terms of this sequence are given in pairs. The first pair is 1 2. From then on, the first term in each pair is the sum of the preceding two (so $3 = 1+2$, $7 = 3+4$ etc.) and the second term is the sum of the first term and the first term of the preceding pair (so $4 = 3+1$, $10 = 7+3$ etc.) Then:

1.
$$\begin{aligned} 2(1^2+1) &= 2^2 \\ 2(3^2-1) &= 4^2 \\ 2(7^2+1) &= 10^2 \\ 2(17^2-1) &= 24^2 \quad \text{etc.} \end{aligned}$$

2.
$$\begin{aligned} 2(1^2+3^2) &= 2^2+4^2 \\ 2(3^2+7^2) &= 4^2+10^2 \\ 2(7^2+17^2) &= 10^2+24^2 \\ 2(17^2+41^2) &= 24^2+58^2 \quad \text{etc.} \end{aligned}$$

3.
$$\begin{aligned} 2\times 3+1 &= 7 \\ 2\times 4+2 &= 10 \\ 2\times 7+3 &= 17 \\ 2\times 10+4 &= 24 \\ 2\times 17+7 &= 41 \\ 2\times 24+10 &= 58 \quad \text{etc.} \end{aligned}$$

4.
$$\begin{aligned} 2+4 &= 2\times 3 \\ 4+10 &= 2\times 7 \\ 10+24 &= 2\times 17 \\ 24+58 &= 2\times 41 \quad \text{etc.} \end{aligned}$$

Problem 1. Can you find general formulae for the first and second terms of each pair in the sequence?
Problem 2. Can you write down a general rule for each of 1–4 above?
Problem 3. Can you prove your rules?
Problem 4. Can you find any other properties that the sequence has?

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Recurring Decimals

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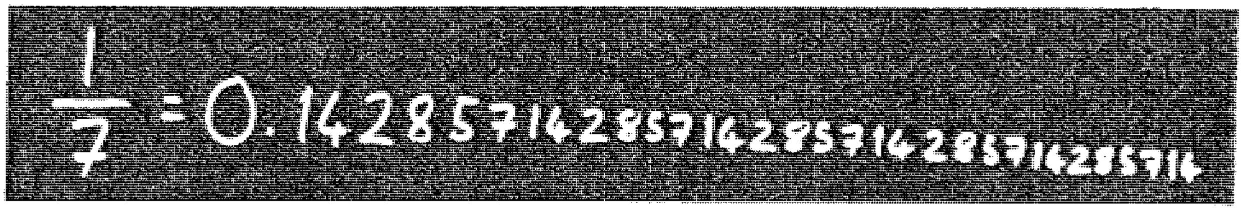

In the days before pocket calculators, most people regarded long division as something of a chore, so much so that, where possible, mnemonics, tricks and rules were encouraged to reduce the amount of paper work. Although many of our forebears learned these aids grudgingly, some of these tools provide refreshing examples for arithmetic on the pocket calculator. One problem is that of obtaining the complete cycle of digits in a recurring decimal of which only a portion can be accommodated in the calculator's register. In tackling this we first demonstrate one of the aids alluded to above for a simple case, and then extend it to a more complicated example.

Consider the reciprocal of 7, shown above, which we can write as

$$\tfrac{1}{7} = 0.\dot{1}4285\dot{7}.$$

Notice firstly that the sum of the digits in a cycle is 27 and the sum of the digits in 27 is 9, i.e. the so-called *digital root* of $\frac{1}{7}$ is 9. Secondly, there is an even number of digits in the cycle, specifically $7-1 = 6$. However, the most interesting property pertains to the relationship between the first and last trio of digits in the cycle. The digits can be written as

$$1\ 4\ 2\ 9{-}1\ 9{-}4\ 9{-}2.$$

Evidently a digit in the latter half-cycle is obtained by subtracting from 9 the corresponding digit in the first half-cycle. This subtraction will be called the 'nines-complement'. Apparently the rule embodied here is true for all recurring decimals with an even number of digits in their cycle. Perhaps someone could supply a proof. Now we see the reason for the digital root 9.

The rule fails for $\frac{1}{3} = 0.\dot{3}$, $\frac{1}{6} = 0.1\dot{6}$, $\frac{1}{27} = 0.\dot{0}3\dot{7}$ and $\frac{1}{4} = 0.25 = 0.25\dot{0}$ because these have an odd number of recurring digits. The rule is trivially true for numbers like $\frac{1}{11} = 0.\dot{0}\dot{9}$.

Now let us apply this rule to the problem of obtaining a complete cycle of recurring decimals of which the pocket calculator can accommodate only a portion. One more observation is useful. In finding the reciprocal of a natural number n by long division, the only allowable remainders if the reciprocal is to recur, other than with zeros, will be $1, 2, 3, \ldots, n-1$, i.e. the maximum number of different remainders possible will be $n-1$. This means that the maximum number of digits in a recurring decimal cycle cannot exceed $n-1$. It can be less than $n-1$. With $n = 13$, for instance, the reciprocal $\frac{1}{13} = 0.\dot{0}7692\dot{3}$ possesses only 6 of an allowable 12 digits in its cycle. Different sets occur in certain multiples of 13, but that is a different story.

For $n = 17$ my 10-digit calculator gives

$$\tfrac{1}{17} = 0.058\,823\,529$$

with as yet no evident calculating pattern. This suggests that $\frac{1}{17}$ has its full complement of 16 digits. Nine of them appear above, but what of the other 7? The rule outlined above can be used, for the 'nines-complement' of the first digit 0 behind the decimal point is 9, which is the first digit in the second half-cycle of this 16-digit recurrence. The required digits are then determined from the scheme below.

$$\begin{array}{rr} & 99\,999\,999 \\ \text{subtract} & \underline{05\,882\,352} \\ & 0588\,235\,294\,117\,647 \end{array}$$

Thus

$$\tfrac{1}{17} = 0.\dot{0}58\,823\,529\,411\,764\,\dot{7}.$$

I found the following interesting number on my Oric 1 48 K.

$$3435 = 3^3 + 4^4 + 3^3 + 5^5$$

MALCOLM SMITHERS
Open University

Rotation by Quaternions

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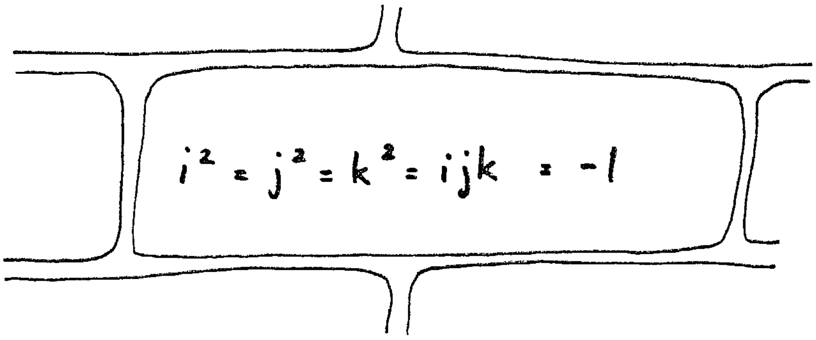

1. Introduction

Suppose that we have an object which is rotated about the origin. Often we need to know where a point which started out with coordinates (x, y) or (x, y, z) ends up, for example, when we wish to draw the object on a computer graphics terminal. In two dimensions, two different methods of doing the calculation are usually taught, one using complex numbers and the other using vectors and matrices.

However, if we now consider rotations in three dimensions, it is very unusual for students to be told about any way of performing the calculation other than by use of vectors and matrices. Yet several other methods exist — for an excellent review of the possibilities see reference 5 — and the matrix method is not always best. In this article I shall review the description of rotations in two dimensions using complex numbers and matrices. I shall then describe how quaternions may be used in three dimensions and compare them to the normal methods based on matrices, to show that at times less calculation is needed using quaternions.

2. Rotation in two dimensions using matrices

Let P be a point in some object, with coordinates (x, y), which, after rotation anticlockwise about the origin through an angle θ, ends up at the new position P' with coordinates (x', y') (see figure 1).

Because P is rotated about the origin, both P and P' are the same distance from the origin; let this distance be r. Let the angle between OP and the x-axis be α. Then we can see that

$$x = r\cos\alpha, \qquad y = r\sin\alpha, \tag{1}$$

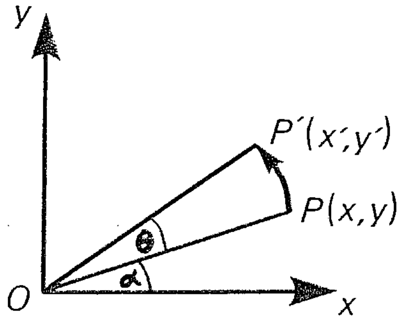

Figure 1

whereas

$$x' = r\cos(\alpha+\theta), \qquad y' = r\sin(\alpha+\theta), \tag{2}$$

which gives, after expanding the trigonometric functions of sums of angles and using (1),

$$x' = x\cos\theta - y\sin\theta, \qquad y' = y\cos\theta + x\sin\theta. \tag{3}$$

Equations (3) thus give the new coordinates of the point in terms of the old coordinates and the angle of rotation θ.

If we now wish to describe this rotation in terms of matrices, we can write the new and old coordinates as column vectors and multiply the old coordinate vector by the 2×2 matrix given below to give the new coordinate vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

These equations just put equations (3) into matrix form; the rotation is expressed as a 2×2 matrix.

3. Rotation in two dimensions using complex numbers

If we now think of the points P and P' as lying in the complex plane, the position of each of them can be represented by a complex number, z and z', respectively:

$$\begin{aligned} z &= x + \mathrm{i}y = r\cos\alpha + \mathrm{i}r\sin\alpha = r\mathrm{e}^{\mathrm{i}\alpha}, \\ z' &= x' + \mathrm{i}y' = r\cos(\alpha+\theta) + \mathrm{i}r\sin(\alpha+\theta) = r\mathrm{e}^{\mathrm{i}(\alpha+\theta)}, \end{aligned} \tag{5}$$

using the standard result for complex numbers that

$$\mathrm{e}^{\mathrm{i}\psi} = \cos\psi + \mathrm{i}\sin\psi.$$

Thus we can now see from (5) that

$$z' = z\mathrm{e}^{\mathrm{i}\theta}, \tag{6}$$

which is the formula we require to give us the new coordinates z' in terms of the old coordinates z and the angle of rotation θ. The rotation is thus expressed as the complex number

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{7}$$

4. Rotation in three dimensions using matrices

Rotations in three dimensions are, not surprisingly, more complicated than those in two dimensions. Now, as well as having to specify the angle turned through, we have to give the axis of rotation.

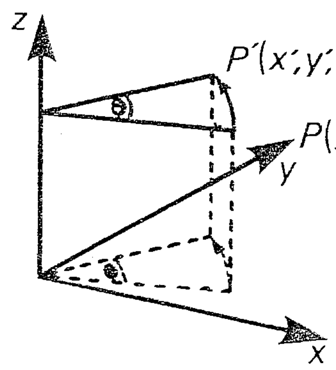

Figure 2

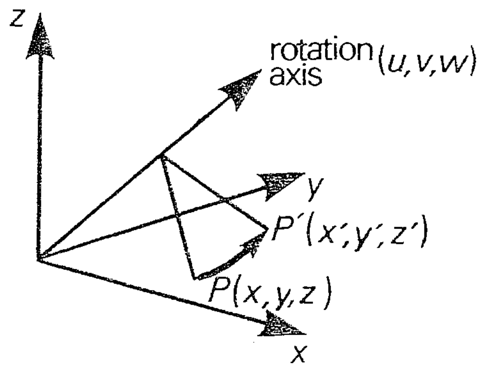

Figure 3

Let us first of all consider the simple case of a rotation about the z axis through an angle θ (see figure 2). Because we are rotating about the z axis, the heights of P and P' above the xy-plane are the same, and so

$$z = z'. \tag{8}$$

In fact, rotation is taking place in a plane parallel to the xy-plane and so the two-dimensional formulae (3) can be used to find the values of x' and y'. If we write these and equation (8) in matrix form, now using 3×1 column vectors for the coordinates and a 3×3 matrix to represent the rotation we have

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{9}$$

If we now consider a rotation about a more general axis, whose direction is given by the unit vector (u, v, w), again through an angle θ (see figure 3), the rotation can still be expressed using a 3×3 matrix, this time given in equation (10). A derivation of this result may be found in reference 3.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} =$$

$$\begin{pmatrix} u^2(1-\cos\theta)+\cos\theta & uv(1-\cos\theta)-w\sin\theta & uw(1-\cos\theta)+v\sin\theta \\ uv(1-\cos\theta)+w\sin\theta & v^2(1-\cos\theta)+\cos\theta & vw(1-\cos\theta)-u\sin\theta \\ uw(1-\cos\theta)-v\sin\theta & vw(1-\cos\theta)+u\sin\theta & w^2(1-\cos\theta)+\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{10}$$

5. Quaternions

Let us now take a break from looking at rotations to discover the properties of quaternions. They were invented by W. R. Hamilton after fifteen years of struggling to extend the idea of complex numbers in two dimensions to some equivalent for dealing with rotations in three dimensions. When he finally saw how to do it, while out walking, he was so pleased that he carved the formulae on a bridge. E. T. Bell's book *Men of Mathematics* (reference 1) gives a very readable account of Hamilton's life and a description of how he arrived at quaternions. Hamilton's own description of quaternions may be found in reference 4; Brand's book (reference 2) is somewhat easier to read and goes further than this article.

One of the reasons that it took Hamilton so long to invent quaternions is that they need to have four components, rather than the three that might be thought necessary for three dimensions. A quaternion may be written as

$$q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d, \tag{11}$$

where a, b, c and d are real numbers. Two quaternions are equal if and only if their corresponding components (i.e. a, b, c, d) are equal. Addition and subtraction are defined in the obvious manner, so that, with obvious notation,

$$q_1 \pm q_2 = (a_1 \pm a_2) + \mathrm{i}(b_1 \pm b_2) + \mathrm{j}(c_1 \pm c_2) + \mathrm{k}(d_1 \pm d_2). \tag{12}$$

The rules for multiplication are a little more complicated. We have

$$\begin{aligned} &\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \\ &\mathrm{ij} = \mathrm{k}, \qquad \mathrm{jk} = \mathrm{i}, \qquad \mathrm{ki} = \mathrm{j}, \\ &\mathrm{ji} = -\mathrm{k}, \qquad \mathrm{kj} = -\mathrm{i}, \qquad \mathrm{ik} = -\mathrm{j}. \end{aligned} \tag{13}$$

Note that quaternions are more like matrices than complex numbers in their multiplication rules in so far that multiplication is non-commutative (i.e. the order of the two quaternions in a product matters, and in general $q_1 q_2 \neq q_2 q_1$). Using the definitions in equations (13) we obtain for the product of two quaternions

$$
\begin{aligned}
q_1 q_2 &= (a_1+\mathrm{i}b_1+\mathrm{j}c_1+\mathrm{k}d_1)(a_2+\mathrm{i}b_2+\mathrm{j}c_2+\mathrm{k}d_2) \\
&= (a_1a_2-b_1b_2-c_1c_2-d_1d_2)+\mathrm{i}(a_1b_2+b_1a_2+c_1d_2-d_1c_2) \\
&\quad +\mathrm{j}(a_1c_2-b_1d_2+c_1a_2+d_1b_2)+\mathrm{k}(a_1d_2+b_1c_2-c_1b_2+d_1a_2).
\end{aligned} \tag{14}
$$

Because multiplication is non-commutative, division is not defined. However, just as in the case of the matrices, we can define an inverse, which takes the place of division by a quaternion. If we define the modulus $|q|$ of the quaternion q by

$$|q| = \sqrt{a^2+b^2+c^2+d^2}, \tag{15}$$

then the inverse of q, written as q^{-1}, is given by

$$q^{-1} = \frac{a-\mathrm{i}b-\mathrm{j}c-\mathrm{k}d}{|q|^2}. \tag{16}$$

It is easy to check, using (14), (15) and (16) that

$$q^{-1}q = qq^{-1} = 1. \tag{17}$$

6. Rotation in three dimensions using quaternions

Just as we can use complex numbers to represent points in two dimensions, so we can use quaternions to represent points in three dimensions; the point P (x, y, z) is represented by the quaternion

$$q = \mathrm{i}x+\mathrm{j}y+\mathrm{k}z. \tag{18}$$

Note that the real ('1') component of this quaternion is zero.

How are rotations in three dimensions calculated using quaternions? It can be shown that, if we wish to rotate the point P through an angle θ about the axis (u, v, w) as before, the rotation is represented by the quaternion

$$r = \cos\tfrac{1}{2}\theta+\mathrm{i}u\sin\tfrac{1}{2}\theta+\mathrm{j}v\sin\tfrac{1}{2}\theta+\mathrm{k}w\sin\tfrac{1}{2}\theta. \tag{19}$$

If the old position of the point is given by the quaternion q, then the new position of the point is given by q', where q' is calculated using

$$q' = rqr^{-1}. \tag{20}$$

This is the formula corresponding to (4), (6) and (10), telling us how to do the calculation in two dimensions using matrices and complex numbers, and in three dimensions using matrices, respectively. Note, however, that this formula is different from the other three in one particular respect: in each of those cases we had only to multiply the old coordinates by *one* quantity (a matrix or a complex number) to obtain the new coordinates, whereas here we have to multiply the old coordinates by *two* separate quaternions (or, at least, one quaternion and its inverse) to obtain the answer.

I shall not give the boring proof of equations (19) and (20) here, but readers may be interested to see the result below which is needed along the way. Suppose we represent two vectors $\boldsymbol{v}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{v}_2 = (x_2, y_2, z_2)$ by the quaternions q_1 and q_2 in the manner given in (18). Then, if we form the quaternion product $q_1 q_2$, we obtain the following:

$$\begin{aligned} q_1 q_2 &= (\mathrm{i}x_1 + \mathrm{j}y_1 + \mathrm{k}z_1)(\mathrm{i}x_2 + \mathrm{j}y_2 + \mathrm{k}z_2) \\ &= -(x_1x_2 + y_1y_2 + z_1z_2) + \mathrm{i}(y_1z_2 - z_1y_2) + \mathrm{j}(z_1x_2 - x_1z_2) + \mathrm{k}(x_1y_2 - y_1x_2). \end{aligned} \tag{21}$$

Let us now write this answer in terms of the two original vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The real part of $q_1 q_2$ is $-\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$, whereas the other (i.e. i, j, k) parts of $q_1 q_2$ represent the vectors $\boldsymbol{v}_1 \times \boldsymbol{v}_2$. Thus the quaternion product of two vectors contains both the scalar and vector products (dot product and cross product) of these two vectors.

7. A quaternion revival?

The theory of quaternions was invented by Hamilton over a century ago, with one of its main objects being the calculation of rotations in three dimensions. At a later stage Gibbs invented the dot- and cross-product notation now in use for vectors. To cut a long story short, after much heated argument between the supporters of vector methods and those in favour of quaternions, we have reached the position we are in now, where vectors are widely taught in schools and universities, and quaternions are mentioned only as a curiosity, if at all. Perhaps the reasons for this at the time were valid, although I suspect that the argument got to such a stage that neither side was capable of admitting that some of the other's ideas might be useful. However, in recent years the use of computer graphics and computational geometry has become widespread, leading to a need for the representation of transformations such as rotation by computers. In some of the problems of current interest, it is certainly no longer true that vector methods are always better than quaternions.

Let us consider two simple examples. Often we wish to compute the effect of two successive rotations. When using vectors and matrices, this must be done by multiplying two 3×3 matrices together, which when performed in the usual way requires a total of 27 multiplications and 18 additions. If we represent the rotations by quaternions, however, their combination needs only a total of 16 multiplications and 12 additions or subtractions. Thus, in this case, the use of quaternions involves less work.

The second example is even more straightforward. A 3×3 matrix used to represent a rotation needs storage space for 9 numbers, whereas if we use a quaternion, only 4 numbers are required, so saving memory space in the computer.

Thus, with the requirements of today's calculations, performed on computers, differing so much from those of the days of Hamilton and Gibbs, perhaps the time is ripe for quaternions to be dusted down and given a second chance.

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2. L. Brand, *Vector and Tensor Analysis* (Wiley, New York, 1947).
3. I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture* (Ellis Horwood, Chichester, 1979).
4. W. R. Hamilton, *Elements of Quaternions* (Cambridge University Press, 1899).
5. J. Rooney, A survey of representations of spatial rotation about a fixed point. *Environment and Planning* B 4 (1977), 185-210.

The Physics of Boomerangs

ROBERT J. O. REID, *University of Leeds*

Robert J. O. Reid is a Senior Lecturer in Physics at the University of Leeds. His research work is in high-energy cosmic radiation, but boomerangs have been a hobby for more than ten years.

Different types and shapes of boomerang

'I'm right smittled wi'it'—the verdict of a Yorkshire woman when, after a few attempts, her boomerang returned, hovered briefly and descended nearby. The word 'smittle' was new to me, but her obvious delight defined it nicely.

So it is with all age groups; both the sophisticated and the simple are fascinated by the flight of an object which can be fashioned in half an hour from a sheet of 10-mm plywood. Some feel that it is a pity to reduce such an elegant flight to mere physics, but for me it has not lessened the magic; on the contrary, understanding admits of subtle variations in the flight. My only regret is that I was presented with the explanation in the same excellent article (reference 1) which, for the first time, gave a clear description of boomerangs and of how to throw them. Thus I will never know if, given the description first, I could have worked it out for myself!

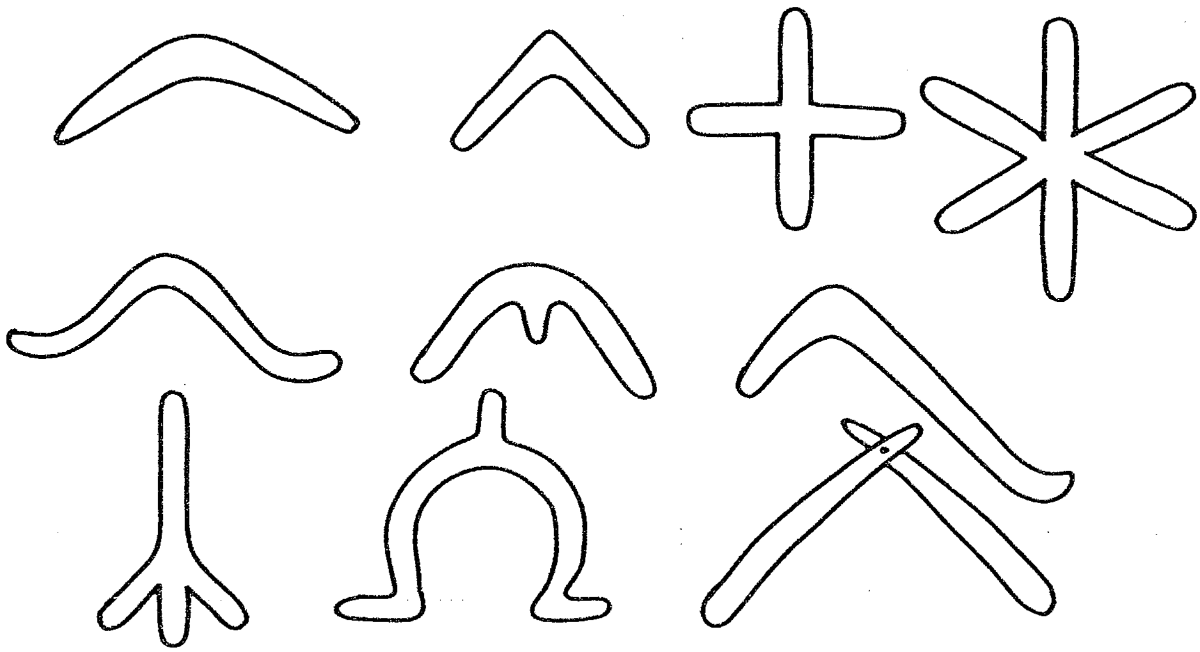


Figure 1. The common and essential feature is the cross-sectional profile which is shown in figure 2.

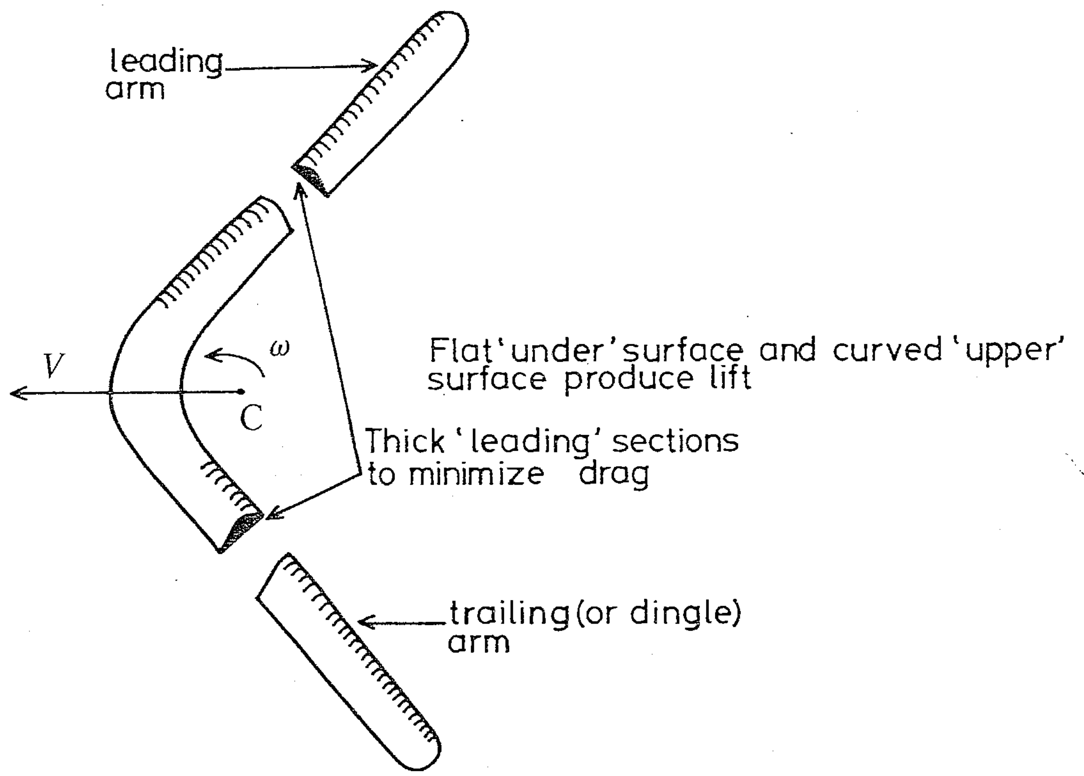


Figure 2. A boomerang *thrown with its plane vertical*, having a horizontal forward velocity V and rotating around its centre of gravity C with angular velocity ω. Cut-away sections show cross-sectional profiles.

The first thing to recognise is the distinction between returning boomerangs and hunting boomerangs. The latter are heavy weapons, have complicated cross-sections and are thrown in a totally different way from the returning variety. The mass is typically 500 grammes, the forward velocity 100 kilometres per hour and the rotation rate 10 revolutions per second, so only the suicidal would wish for a good return! Their purpose is to fly along a predetermined, almost linear, path for distances of about 200 metres and deliver a killing or maiming blow to the animal being hunted. Returning boomerangs are of simpler cross-section, are thrown with the plane of the boomerang vertical, and the less massive ones can be caught on return without risk of serious injury. Their purpose is solely for fun and the pleasure of perfecting a skill; they are to hunting boomerangs what show-jumping is to hunting on horseback. They come in a wide range of shapes and sizes; the selection of shapes shown in figure 1 all fly well and, given wind-free conditions, will return to within a few feet of the thrower. Many sources state that a boomerang can be made to return only by judicious warping of the wings. This is wrong. It is true that some boomerangs with a small degree of twist and/or bend in the wings return satisfactorily, but both the performance and stability of such devices are unduly sensitive to minute changes to the degree of warp/bend. For this reason I prefer to use plane boomerangs and go to some trouble in the selection and treatment of the wood to ensure that it does not subsequently warp.

The flight path of a returning boomerang

Now to the flight itself. Assume, as in figure 3, a large flat field and zero wind. The thrower launches the boomerang from point (a), *with its plane vertical*, with a forward horizontal velocity of some 100 kilometres per hour and, by checking his hand just on release, also imparts as much 'forward' spin as he can muster. Almost immediately the boomerang begins to veer to the left (a right-handed thrower and a right-handed boomerang!). Gradually the plane of the boomerang begins to tilt away from the vertical and, as it does so, there is a gain in height [point (b), figure 3]. By the time the boomerang has reached point (c) its forward velocity has been slowed appreciably by drag and its plane has become almost horizontal, so that the aerodynamic forces are now almost entirely devoted to overcoming gravity, and it hovers gently to the grass. To the thrower the flight has seemed timeless; forced to make a quantitative estimate he will feel sure it lasted more than a minute. A stopwatch, however, would have registered about 10 seconds.

The path when projected on to the ground (the dotted line in figure 3) approximates closely to a circle. Variations in design can result in quite different trajectories. For example, by increasing the lift on the leading arm, figure 2, the boomerang can be made to lie down much earlier so that, by point (b) of figure 4, its plane is almost horizontal and it has gained significantly in height—it then swoops down excitingly to point (c) where, having tilted beyond the horizontal plane, the curvature of the path reverses and it reaches the thrower via a figure of 8. Alternatively, reducing the lift of

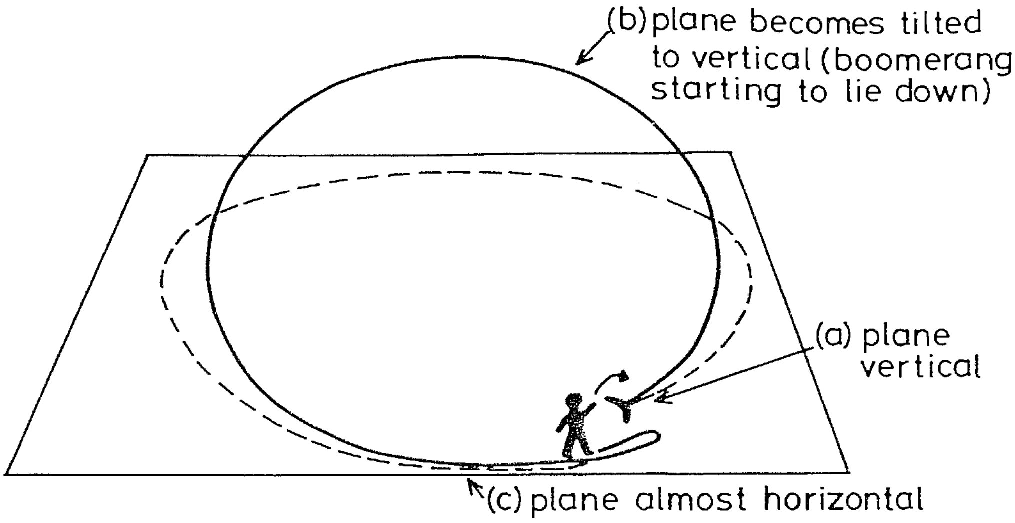


Figure 3

the leading arm (or increasing that of the trailing arm) delays 'lie down' and the boomerang will cheerfully pass the thrower and begin a second and even a third circular orbit. For a given boomerang, variations in the throw, that is, in the forward velocity or angle of the boomerang's plane to the vertical, give different orbits. Whereas these differences appear and are spectacular, measurement will show that, for a flight which ends in a return, the maximum distance from the thrower alters very little. In this sense the path of a boomerang is built-in and is independent of the throw.

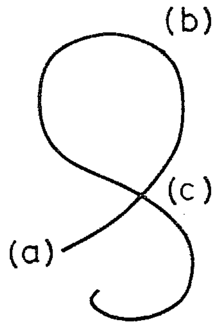


Figure 4

For my home-made boomerangs, orbits range in diameter from 3 metres to 70 metres, masses from 10 grammes to 350 grammes and times of flight from 2 to 15 seconds. The world record is held by an American, a throw of 117 metres followed by a full return.

Analysis of idealised flight path

At first sight it might seem necessary to take into account the various instantaneous forces acting at time t and thus calculate the changes in position and altitude at time $t+\Delta t$, so that the new forces can be determined, thus proceeding in small steps to derive the detailed orbit. Such approaches have been followed with success, but their very detail obscures the basic mechanism. The good scientist seeks instead to model what is happening in as simple a way as possible so as to understand the phenomenon he is studying. We shall therefore choose to consider the simple geometry of a crossed boomerang, assume zero aerodynamic drag, ignore gravity and determine the conditions for a perfectly circular orbit. Such a boomerang would not stop after one orbit but would be condemned to retrace its path endlessly. Ignoring drag is of course wrong, but drag can be minimised by having a well-streamlined cross-section. A practical boomerang will slow down in both forward velocity and spin. As to gravity, the first consideration is to design the boomerang so that the ratio of aerodynamic to gravitational forces is large; better still, it is possible to use a small portion of the aerodynamic lift to counterbalance gravity and in practice to arrange that, as the boomerang slows down, it also lies down, so that an increasing component of the decreasing lift is devoted to keeping it aloft.

The term 'drag' force and 'lift' force have been introduced rather loosely; we now need to explore fluid flow around an obstacle, understand how it can produce both types of force and, in those cases where lift does result, find out how it depends on the fluid and its velocity and on the shape of the obstacle. Drag is familiar to everyone who has ridden a bicycle or stood outside in a strong wind; it is due in part to the viscous flow and in part to the loss of momentum of the oncoming air. With boomerangs the latter component, referred to as the inertial force, dominates and, for a fluid of density ρ approaching an obstacle with velocity v, the inertial force F_D is proportional to ρv^2. (The change of momentum per second is mv and m, the mass arriving per second, is proportional to ρv.) Thus the drag force $F_D = C_D A' \rho v^2$, where A' is the area normal to the direction of the oncoming fluid and C_D, the drag coefficient, depends strongly on whether the flow remains streamlined or becomes turbulent in passing over the obstacle. For example, the drag force exerted on a wire of circular cross-section, figure 5 (a), can be reduced by a factor of almost 100 by the addition of material which leads the flow smoothly over the trailing edge, figure 5 (b), and thus avoids the wide low-pressure turbulent wake seen in figure 5 (a). In figure 5 (b) the pressure above the obstacle is less than the pressure upstream but, being exactly balanced by the pressure below, provides no vertical lifting force. If we remove the lower half of the streamlined obstacle, the pressure above the remaining aerofoil, figure 5 (c), is now less than the pressure below it and a net vertical upward force results; by Newton's third law an equal and opposite force must act on the fluid which is deflected downwards as shown. To summarise: figure 5 (a) is characterised by turbulent flow, wide low-pressure wake, high drag force and no lift; figure 5 (b) is characterised by streamlined flow, low drag and no lift; figure 5 (c) is characterised by streamlined flow, low drag and lift provided by lack of symmetry in pressure above and below the aerofoil.

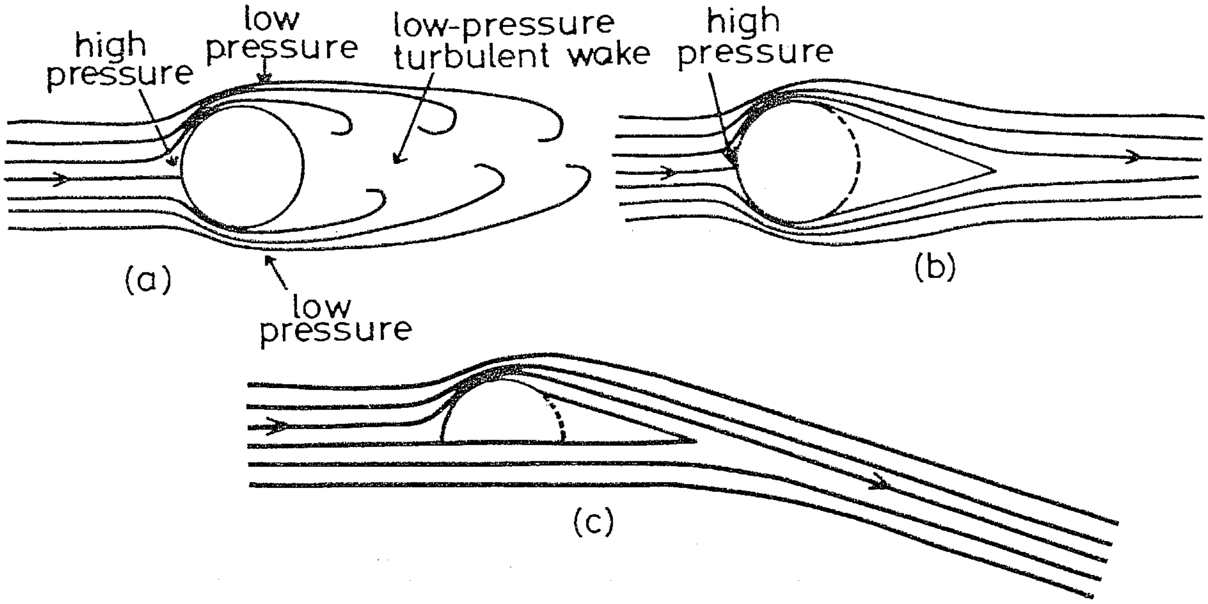


Figure 5

To find the magnitude of the lift force, consider the simpler situation illustrated in figure 6 in which an incompressible fluid flows through a constriction in a horizontal tube, the pressure being directly monitored by the level of the liquid in the vertical tubes. Provided viscous losses are low and the flow remains streamlined, the pressure at (b) will be less than that at (a) and (c). That this should be so is in accordance with Bernoulli's principle and is a direct consequence of energy conservation. If the fluid is incompressible, the volume rate of flow must be the same at (a), (b) and (c), so that the fluid velocity and hence its kinetic energy per unit volume, $\frac{1}{2}\rho v^2$, is higher at (b) than at (a) and (c). This increase in kinetic energy has to be supplied from the existing reservoir of potential energy, which in this case is the pressure P, the potential energy per unit volume. Hence for any streamline $\frac{1}{2}\rho v^2 + P =$ constant, and changes in pressure and velocity are directly linked.

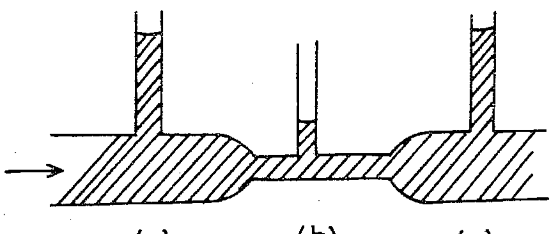


Figure 6

For air flow over an aerofoil, it is possible to ensure that the flow remains streamlined; viscous forces play an important role in determining the flow pattern but are much smaller than inertial forces. More surprisingly, we can treat air as an incompressible fluid provided that we are considering velocities less than the velocity of sound. We can therefore express the lifting force acting on a surface of area A as $F_L = C_L A\rho v^2$, where C_L is a constant of proportionality known as the *lift coefficient* and depends on both the cross-sectional profile and on the angle at which it is presented to the oncoming air (the angle of incidence or angle of attack).

Since the aerodynamic lift is horizontal when the boomerang is in the vertical plane, the recipe for a circular orbit is that this force should be constant in magnitude and always remain perpendicular to the instantaneous motion. This will be true if and only if the angle of incidence of the wing to the oncoming air remains constant, i.e. if and only if the boomerang rotates slowly (precesses) about a vertical axis through its centre of mass. Thus we need a precessional velocity Ω so that in time t

$$\alpha = \Omega t = \frac{Vt}{R},$$

that is,

$$\Omega = \frac{V}{R}. \tag{1}$$

Without this constant rate of turning, the lift force alone would not give a returning boomerang; after initial deflection to the left the angle of incidence would decrease, resulting in reduced lift, and the path would gradually straighten out again. Equally, if the rate of turning is faster than that prescribed by equation (1), the angle of incidence will increase rapidly until the stall angle of the aerofoil is reached (usually less than 15°), terminating the flight abruptly.

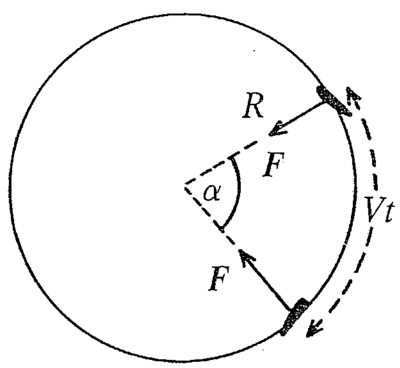


Figure 7. Idealised circular path of boomerang as seen from above.

The remaining physics is to understand why the plane of the boomerang precesses and how to arrange that it does so at the correct rate. Consider the

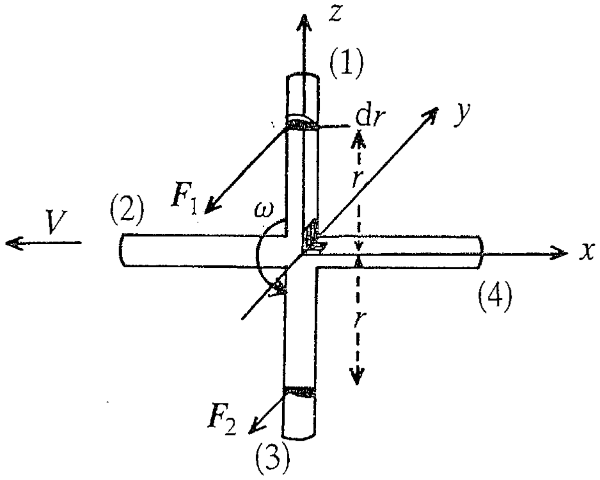


Figure 8

boomerang of figure 8 at the instant when one pair of arms is in the directly up/down position [arms (1) and (3) of figure 8] and focus attention on two small elements of length dr of these arms which are at distance r from the centre of mass. The instantaneous velocity of the element on the uppermost arm is $v_1 = V+\omega r$, whereas that of the corresponding lower arm is $v_2 = V-\omega r$. The forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ on the two elements are

$$F_1 = C_\mathrm{L} a\,\mathrm{d}r\rho(V+\omega r)^2 \quad \text{and} \quad F_2 + C_\mathrm{L} a\,\mathrm{d}r\rho(V-\omega r)^2,$$

where a is the width of each arm. These forces produce a torque $\Delta\tau$ about the horizontal axis in figure 8 which is given by

$$\Delta\tau = (F_1 - F_2)r = C_\mathrm{L} a\,\mathrm{d}r\rho[(V+\omega r)^2 - (V-\omega r)^2] = 4C_\mathrm{L}\rho a V\omega r^2\,\mathrm{d}r, \quad (2)$$

and the total torque τ about the horizontal axis by all the elements of the two arms is

$$\tau = 4C_\mathrm{L}\rho a V\omega \int_0^l r^2\,\mathrm{d}r = \tfrac{4}{3}C_\mathrm{L}\rho a V\omega l^3, \quad (3)$$

where l is the length of each arm.

The exact rotational analogue of Newton's second law tells us that the action of the torque is to cause a rate of change $\mathrm{d}L/\mathrm{d}t$ of the angular momentum L in the direction of the torque, so that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \tau. \quad (4)$$

The angular momentum L at time t is $I\omega$, where I is the moment of inertia of the boomerang about the axis of spin. For elements of length dr, $\Delta I = A\,\mathrm{d}r\rho' r^2$, where A is the cross-sectional area of an arm which is made from material of density ρ'. Thus, for the complete boomerang about the spin axis,

$$I = \tfrac{4}{3}\rho' A l^3. \quad (5)$$

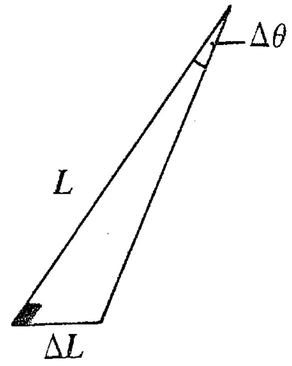


Figure 9

The vector diagram figure 9 describes the situation. The original angular momentum vector $\boldsymbol{L}$ is along the negative y-axis and, in a time Δt, the resultant torque $\boldsymbol{\tau}$ causes a change in angular momentum ΔL which is in the positive x-direction. The addition of these two vectors causes the angular momentum vector to swing around in the x–y plane† by an angle $\Delta\theta = \Delta L/L$ in the time Δt.

Thus the precessional velocity

$$\Omega = \frac{\Delta\theta}{\Delta t} = \frac{1}{L}\frac{\Delta L}{\Delta\tau} = \frac{\tau}{L} = \frac{\frac{4}{3}C_{\mathrm{L}}\rho a\omega V l^3}{\frac{4}{3}\rho' A\omega l^3} = C_{\mathrm{L}}\frac{\rho a}{\rho' A}V. \tag{6}$$

From equation (1), $\Omega = V/R$ for a circular orbit and we obtain finally

$$R = \frac{V}{\Omega} = \frac{\rho' A}{C_{\mathrm{L}}\rho a}. \tag{7}$$

It is worthwhile to interpret equation (7). It predicts that the radius of the orbit of a boomerang is independent of (a) the angular spin velocity ω (because both torque and angular momentum change in direct proportion to ω), (b) the length of the arms (because the moment of inertia and torque both increase as l^2), (c) the number of arms n (because the moment of inertia and torque both increase as n) and (d) the forward velocity V. [Equation (6) shows that the rate of turning increases linearly with V, which is just what is required to maintain the same orbit.]

The radius of the orbit does depend on (e) the air density ρ (record throws are easier at high altitudes!), (f) the density of material ρ' (dense boomerangs travel further), (g) the lift coefficient C_{L} (i.e. the cross-sectional profile), (h) the ratio A/a, i.e. the cross-sectional area of the boomerang arm divided by the width of the boomerang arm. Note, however, that it is not

†This explanation of gyroscopic precession, while formally correct, is unhelpful to many. Why does a torque about one axis produce motion about a perpendicular axis? For a very good discussion see reference 2.

possible to change A/a without altering the profile geometry and hence C_L. So, for an indoor boomerang, I use light balsa wood and a profile giving high lift. This has a diameter of about 3 metres. For long-distance throws, high-density material is coupled with a thin profile. The use of sticky tape to attach a coin near the extremity of each wing adds about 10 metres to the throw (and is a sure hedge against inflation!), as the moment of inertia is increased with no change in the torque.

The progressive lying down, necessary to keep gravity at bay, comes from another precessional motion which requires for its production a torque about a vertical axis. It can be provided in the conventional boomerang of figure 2 by giving the leading wing a higher lift profile than that of the trailing wing. That the cross boomerang of figure 8 also lies down requires a different explanation, and is one of the many more difficult topics tackled by F. Hess in his Ph.D. thesis†, completed some seven years after his original article in *Scientific American* (reference 1). In figure 8, arm (4) is moving through air which has already been disturbed, whereas arm (2) is in 'new' air and for this reason experiences higher lift.

Reading about boomerangs is a poor substitute for flying them. Good ready-made boomerangs are not available in the shops, and those brought back from Australia were made for the mantlepiece! Mr M. J. Hanson, PO Box 1, Cumnock, Ayrshire, Scotland, offers an excellent selection of home-made and imported boomerangs at prices which range from £3 to £11. (N.B. You must specify whether you are a left-handed or right-handed thrower.) However, you'll get much more enjoyment if you make your own and, if you send me a large stamped addressed envelope, I will provide full-scale tracings with details of profiles, together with hints on construction and throwing. Finally you may wish to join the British Boomerang Society (J. Jordan, 10 The Spinney, Wolverhampton, WV5 9ER), or take part in the biannual workshop/competition, organised by the Horniman Museum, London Road, Forest Hill, London, SE23 3PQ.

Reference

1. Felix Hess, The aerodynamics of boomerangs, *Scientific American* **219** (1968) November, 124–136.
2. J. McCaughan, *Physics Education* **17** (1982), 133–138.

†This thesis, entitled 'Boomerangs Aerodynamics and Motion', is highly mathematical in some topics. It contains an extensive bibliography tracing the origins of boomerangs through to all continents of the world and as far back in time as 7000 BC, as well as descriptive material and excellent stereoscopic photographs of actual and computer-simulated flights.

Computer Column

MIKE PIFF

In Volume 16 Number 3 I asked for an interpretation of, and possible improvement on, a program a reader had sent us. The only significant attempt at improvement was from Mike Day of Manchester University.

The program found all numbers $n \leqslant M$ which have the maximum number of divisors. Thus, if d_n = the number of divisors of n, the program prints out only those n such that $d_n = \max_{1 \leqslant i \leqslant M} d_i$.

Mike Day's improvements, which were accompanied by voluminous notes and timings, were as follows.

(a) In finding d_n, you need only check possible divisors d between 2 and int(sqrt(n)), the case $d = 1$ needing no checking!

(b) If n is odd, only odd divisors need be checked.

(c) $\max_{1 \leqslant i \leqslant M} d_i = \max_{\frac{1}{2}(M+1) \leqslant i \leqslant M} d_i$.

All small improvements, but quite significant when they are put together. Try timing your own program incorporating these ideas. Another idea is to keep a list in an array or file of the numbers which make d_n a maximum. When you start, you don't know what that maximum is, but whenever a 'new' maximum is found, you just start your list afresh. Then just print out the list!

Try working out

$$1 \times 142857,$$
$$2 \times 142857,$$
$$3 \times 142857,$$
$$4 \times 142857,$$
$$5 \times 142857,$$
$$6 \times 142857.$$

What do you notice? Why does it work?

Letter to the Editor

Dear Editor,

Relating to a proof of Pythagoras' theorem

In his terse proof of the theorem of Pythagoras in Volume 17 Number 1, John MacNeill draws a triangle ABC, right-angled at A, and from A draws the altitude AD to BC. He uses the similarity of the triangles BCA, ACD and BAD.

It occurs to me that the triple similarity property of the three right-angled triangles has an interesting converse which I had not previously encountered. Its enunciation is as follows; the reader can easily establish the result.

Let ABC be any triangle and suppose that D is any point in BC. Suppose the triangles BCA, ACD and BAD are similar, with this correspondence of lettering. Then A must be a right angle and AD must be an altitude to BC.

Yours sincerely,

F. CHORLTON
(The University of Aston in Birmingham)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

17.4. (Submitted by John MacNeill, The Royal Wolverhampton School)
Determine $(9+4\sqrt{5})^{1/3}$ as a quadratic surd (i.e. in the form $a+b\sqrt{c}$, where a, b and c are rational numbers).

17.5. (Submitted by Alan Maclean, Dunfermline)
Prove that, when p and q are prime numbers greater than 5, then p^4-q^4 is always divisible by 10.

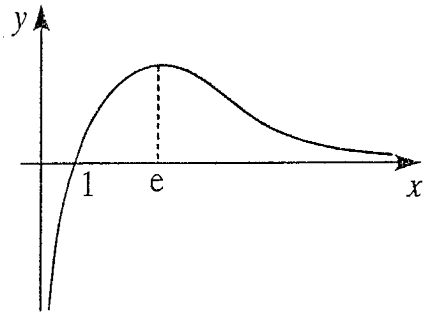
17.6. (Submitted by Péter Ivády, Budapest)
Prove that, for $0 \leqslant x \leqslant \frac{1}{2}\pi$,

$$\tanh x \geqslant \sin(\sin x) \geqslant \sin x \cos x,$$

and that equality occurs if and only if $x = 0$.

Solutions to Problems in Volume 16, Number 3

16.7. Determine all distinct pairs of positive integers m and n for which $m^n = n^m$.
Solution by Richard Dobbs (Winchester College)



We have $m^n = n^m$ if and only if

$$n \log m = m \log n$$

or

$$\frac{\log m}{m} = \frac{\log n}{n}.$$

Now consider the function

$$f(x) = \frac{\log x}{x};$$

we want $f(m) = f(n)$. We have

$$f'(x) = \frac{1 - \log x}{x^2},$$

and the function has a turning point when $\log x = 1$, i.e. when $x = \mathrm{e}$. The graph has the form shown in the figure. For there to be distinct positive integers m and n such that $f(m) = f(n)$, one of these, say m, must be in the range $1 < m < \mathrm{e}$, so $m = 2$. It is clear now that $n = 4$, so the only such distinct pair of positive integers is $2, 4$.
Also solved by Ruth Lawrence (St. Hugh's College, Oxford).

16.8. Does there exist a positive integer such that we obtain its (a) 57th, (b) 58th part when we delete its first digit?
Solution by Ruth Lawrence (St. Hugh's College, Oxford)
Consider a positive integer n with first digit a. Then n may be written as

$$n = a \times 10^r + k,$$

where k is the integer obtained from n by deleting its first digit. If $k = \frac{1}{57}n$, then

$$56k = a \times 10^r.$$

Now $(7, 10^r) = 1$, so $7|a$. Since $1 \leqslant a \leqslant 9$, this means that $a = 7$. Thus $8k = 10^r$, whence $r \geqslant 3$ and

$$n = 7 \times 10^r + \tfrac{1}{8} \times 10^r = 7125 \times 10^{r-3}.$$

Hence the numbers satisfying (a) are 7125 times powers of ten. If $k = \frac{1}{58}n$, we obtain

$$57k = a\times 10^r.$$

Since 57 is relatively prime to 10, this means that $57|a$, which is impossible because $1 \leqslant a \leqslant 9$. Hence there is no positive integer satisfying (b).

Also solved by Mike Day (University of Manchester) and Malcolm Smithers (Open University).

16.9. Let triangle ABC have circumcentre O, centroid G, orthocentre H and incentre I. Then it is known that O, G and H are collinear (Euler's line) with $OG/GH = \frac{1}{2}$.

But let W be the 'centroid of the perimeter' of the triangle ABC, i.e. W is the centre of mass of a uniform wire in the shape of the perimeter of the triangle. Then prove that W, G and I are collinear with $WG/GI = \frac{1}{2}$.

Solution by Ruth Lawrence (St Hugh's College, Oxford)

Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the position vectors of A, B and C from the origin G, let $\boldsymbol{x}$ be the position vector of W, put $BC = \alpha$, $CA = \beta$ and $AB = \gamma$, and denote the length of the perimeter of the triangle by s. Then

$$s\boldsymbol{x} = \alpha\times\tfrac{1}{2}(\boldsymbol{b}+\boldsymbol{c})+\beta\times\tfrac{1}{2}(\boldsymbol{c}+\boldsymbol{a})+\gamma\times\tfrac{1}{2}(\boldsymbol{a}+\boldsymbol{b}),$$

since G is the centre of mass of particles placed at the midpoints of the sides of the triangle whose masses are proportional to the lengths of those sides.

Denote by J the point on WG produced such that $WG/GJ = \frac{1}{2}$ and denote by $\boldsymbol{y}$ the position vector of J from G. We must show that $J = I$. Now

$$\boldsymbol{y} = -2\boldsymbol{x} = -\frac{1}{s}\{(\beta+\gamma)\boldsymbol{a}+(\gamma+\alpha)\boldsymbol{b}+(\alpha+\beta)\boldsymbol{c}\}.$$

Thus

$$\begin{aligned}
\overrightarrow{AJ} &= \boldsymbol{y}-\boldsymbol{a}\\
&= -\frac{1}{s}\{(\beta+\gamma+s)\boldsymbol{a}+(\gamma+\alpha)\boldsymbol{b}+(\alpha+\beta)\boldsymbol{c}\}\\
&= -\frac{1}{s}\{\alpha(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})+2(\beta+\gamma)\boldsymbol{a}+\gamma\boldsymbol{b}+\beta\boldsymbol{c}\}\\
&= -\frac{1}{s}\{2(\beta+\gamma)\boldsymbol{a}+\gamma\boldsymbol{b}+\beta\boldsymbol{c}\} \quad (\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \boldsymbol{0} \text{ because } G \text{ is the origin})\\
&= -\frac{1}{s}\{(\beta+\gamma)\boldsymbol{a}-\beta\boldsymbol{b}-\gamma\boldsymbol{c}+(\beta+\gamma)(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})\}\\
&= -\frac{1}{s}\{(\beta+\gamma)\boldsymbol{a}-\beta\boldsymbol{b}-\gamma\boldsymbol{c}\}\\
&= \frac{1}{s}\{\beta(\boldsymbol{b}-\boldsymbol{a})+\gamma(\boldsymbol{c}-\boldsymbol{a})\}\\
&= \frac{1}{s}(\beta\overrightarrow{AB}+\gamma\overrightarrow{AC}).
\end{aligned}$$

Thus

$$\overrightarrow{AJ}\cdot\overrightarrow{AB} = \frac{1}{s}(\beta AB^2+\gamma\overrightarrow{AB}\cdot\overrightarrow{AC}) = \frac{\gamma}{s}(\beta\gamma+\overrightarrow{AB}\cdot\overrightarrow{AC}),$$

$$\overrightarrow{AJ}\cdot\overrightarrow{AC} = \frac{1}{s}(\beta\overrightarrow{AB}\cdot\overrightarrow{AC}+\gamma AC^2) = \frac{\beta}{s}(\overrightarrow{AB}\cdot\overrightarrow{AC}+\beta\gamma),$$

so

$$\frac{1}{AB}(\overrightarrow{AJ}\cdot\overrightarrow{AB}) = \frac{1}{AC}(\overrightarrow{AJ}\cdot\overrightarrow{AC}).$$

This means that AJ is equally inclined to AB and AC. Hence AJ bisects the angle BAC. Similarly, BJ bisects the angle CBA and CJ bisects the angle ACB. Hence J is the incentre of triangle ABC, so $I = J$, as required.

Book Reviews

Teach Yourself Modern Mathematics. By L. C. PASCOE. Hodder & Stoughton, Sevenoaks, 1984. (Revised issue of **Teach Yourself New Mathematics**, 1970). Pp. 235. £2·95.

If I were asked to recommend a book to someone who wished to acquaint himself with those aspects and topics in mathematics which have become prominent in the last 25 years, there are many texts I would recommend before this one. So far as the overall structure of this book is concerned, it is difficult to find much justification for the claim made in the introduction that 'the subject matter is developed logically...'. Indeed, the chapters are almost in watertight compartments, and there is little attempt to draw any of the subject matter together, little cross-reference between chapters, and many missed opportunities for cross-fertilization. The desultory treatment is apparent from the outset: definitions, results and algorithms are pulled out of a hat rather than led up to, justified and illustrated; proofs are frequently little more than checks or verifications; many of the explanations are poor, while some confuse rather than clarify. Many examples of objectionable notation could be quoted; for instance, on page 81 'x div y' is used to mean $y|x$ or $x \equiv 0 \pmod{y}$; on page 181 $\varnothing$ is used for the null matrix. The most successful chapters are 7 and 8 on inequalities (although the linear programming examples chosen are artificial) and 10 and 11, the last-mentioned being particularly good.

There are some serious omissions from some of the treatment: in Chapter 3 (on vectors) the important section formula $\boldsymbol{a}t+\boldsymbol{b}(1-t)$ is not mentioned, in Chapter 12 matrices are motivated only by simultaneous equations and by shopping lists, no mention being made of transformations, while in Chapter 13 (on probability) no reference is made to probability trees, the use of which would have saved a lot of verbiage. There are several instances of imbalance: far too heavy emphasis is placed on scales of notation and tests for factors of whole numbers. In the case of the former there is a long string of pointless exercises, e.g. 'solve $111x+1010 = 101\,101$ (base 2)' on page 67. On page 8 it is stated that the method of square roots by iteration 'does not supersede the usual process ... in books of arithmetic'. Surely in these days of electronic aids (much vaunted by the author) one would hope the contrary.

Finally, there is no index and no book list.

Formerly Head of the Mathematics Department
The Royal Grammar School
Newcastle upon Tyne

F. J. BUDDEN

Statistics for Advanced Level Mathematics. By I. G. EVANS. Hodder & Stoughton, Sevenoaks, 1984. Pp. x+307. £5·95.

It is not often that *Mathematical Spectrum* publishes a review of a traditional-style textbook. Nevertheless, many students, particularly in statistics, do like to have their own book to help them work for a particular examination. To choose the right book you need to know whether (a) it covers the syllabus, (b) it is accurate, (c) it is clear enough to follow without expert help, (d) it is of the right standard and (e) it is the right price.

Gwyn Evans's book is written for the many syllabuses in statistics that form part of an A-level mathematics with statistics course. It covers all the ground of some of these courses and most of the ground of all of them. As befits the work of an experienced examiner and teacher, the book is well laid out, clear and accurate. The student of average and above ability should be able to work through it, though occasionally some of the more technical jargon might be found difficult. Since the syllabuses for which the book is written concentrate on the theoretical and mathematical side of the statistics, so does the book. You will not find suggestions for practical work or projects, nor any encouragement to do them. For this you will have to go elsewhere.

And is the price right? Well, at £5.95 for 307 pages of text with a large number of examination questions, you will have to decide for yourself.

University of Sheffield

PETER HOLMES

Dual Models. By MAGNUS J. WENNINGER. Cambridge University Press, 1984. Pp. xii+156. £12·50.

If you have enjoyed making models of the 75 uniform polyhedra and some of the stellated polyhedra described in the author's earlier book *Polyhedron Models* (Cambridge University Press, 1971), you will want to try your hand at the dual models described in his new book. Two polyhedra are said to be *dual* if each is the polar reciprocal of the other with respect to a suitable sphere, so that among the regular polyhedra the cube and octahedron are dual, and so are the icosahedron and dodecahedron.

The author follows the same pattern in his new book as he did in his earlier book; he gives detailed instructions for constructing his models and a brief account of the underlying mathematical theory. He also gives photographs of each model, but, as he says, these are no substitute for making the models yourself.

University of Leicester

ERIC PRIMROSE

The Mathematical Experience. By PHILIP J. DAVIS and REUBEN HERSH. Penguin Books Ltd, London, 1983. Pp. xix + 440. £5·95 (paperback).

The authors explain that this book 'is not intended to present a systematic, self-contained discussion of a specific corpus of mathematical material' but 'rather to capture the inexhaustible variety presented by the mathematical experience.'

It is difficult in a short review to convey the variety of, topics discussed in over sixty short chapters; nevertheless the authors' success in achieving their aim can perhaps be hinted by mentioning a few in each of the eight sections in which they are grouped. As an introduction, The Mathematical Landscape (Section 1) attempts to answer such questions as: What is mathematics?, Where is mathematics?, How much mathematics is now known? Next Section 2, entitled Varieties of Mathematical Experience, includes a sketch of the ideal mathematician, and a physicist's view of mathematics. The following three groups of essays concern Outer Issues (for example, why mathematics works, mathematical models, the utility of mathematics), Inner Issues (such as symbols, generalisation, proof, infinity, abstraction, the Chinese remainder theorem), and Selected Topics in Mathematics (the prime number theorem, Fourier analysis, non-Euclidean geometry, etc.). Section 6, concerning Learning and Teaching, covers topics as diverse as the confessions of a preparatory school teacher and Pólya's craft of discovery. Section 7, From Certainty to Fallibility, includes a discussion of foundations found and lost, and the philosophical plight of the working mathematician, while the concluding section, Mathematical Reality, asks Why should I believe a computer? and discusses several topics including four-dimensional intuition and true facts about imaginary objects.

Most chapters can be read independently; a number will be readily comprehended by the layman, although some require a more specialised background and are therefore more suited to the professional. Interest ranges from the history of mathematics to modern developments, from the mathematics of everyday living to the abstract concepts which form the stock in trade of the researcher.

Almost every chapter concludes with suggestions for further reading listed in an extensive bibliography. There is a useful glossary and the book is extremely well illustrated with portraits, diagrams and sketches. It would be a valuable acquisition for a school library.

University of Sheffield MAVIS HITCHCOCK

Mathematical Magic Show. By MARTIN GARDNER. Viking Penguin Inc., 1984. Pp. 284. £9·95. (First published in USA, 1977.)

Martin Gardner's Sixth Book of Mathematical Diversions and **Martin Gardner's New Mathematical Diversions**. Both reprinted by the University of Chicago Press, 1984. Pp. 262, 253 respectively. Both paperback, £6·75.

If they are not already there, some of Martin Gardner's books of mathematical diversions and games (originally articles in the *Scientific American*) would make a welcome addition to your school library shelves—or to your own. The three listed above have recently been reprinted. An earlier edition of the third one was reviewed by a fifth-form student in Volume 5, page 35.

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