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➢ Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

Mathematicorum

☆   ☆   ☆

AESCULAPIUS BOUND

Medicine is one thing,
Mathematics is another;
But mathematics can be medicine,
Most certainly—oh brother!
When a patient takes his medicine
In doses weighed just right,
The doctor will encourage him
To feel the outlook's bright.
So swallow down mathematics
In doses good and strong,
And then to higher learning
You soon will pass along ...
In medical establishments
Where healers wise consort,
Will blossom the accomplishments
Which labour here hath wrought.
And there we have no doubt
All ills you'll put to rout;
There'll be no lingering toxin
Near a product of ALGONQUIN!

MONICA MASKELL, Ottawa.
Written for 1968 *Christmas Exams*
for Medical Technology students

☆   ☆   ☆

Yet what are all such gaieties to me
  Whose thoughts are full of indices and surds?
$x^2 + 7x + 53$
  $= 11/3.$

LEWIS CARROLL

THE CELEBRATED BUTTERFLY PROBLEM

LÉO SAUVÉ, Algonquin College

1. *Introduction.*

The Butterfly Problem has a history that goes back at least to 1815 [1], and the interest it arouses is such that it keeps on reappearing in the literature [1 - 24]. In its simplest form it may be stated as follows:

Through the midpoint M *of chord* AB *of a circle (see Figure 1), two chords,* CD *and* EF, *are drawn.* ED *and* CF *intersect* AB *in* P *and* Q, *respectively. Prove that* PM = MQ.

The resemblance of the figure to the wings of a butterfly explains the name under which the problem has become known. The problem looks simple, but its proof turns out to be unexpectedly elusive. Coxeter and Greitzer [19] mention that one of the earliest solvers (1815) was W.G. Horner, discoverer of Horner's Method for approximating the roots of a polynomial equation. Many of the short proofs found over the years have been projective in nature, and most of the elementary proofs have been fairly long and complicated, involving in most cases the use of auxiliary lines. Indeed, Eves [23] says: "It is a real stickler if one is limited to the use of only high school geometry."

2. *A simple proof of the Butterfly Theorem.*

The elegant proof I give below appears to me to be the simplest of those that I have seen, in that it is fairly short, involves only high school geometry, and requires no auxiliary lines whatever. It was discovered by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y. [24].

In Figure 1, four pairs of equal angles are denoted by α, β, γ, δ, and MQ = x, PM = y, AM = MB = a, so that AQ = $a - x$ and PB = $a - y$. The notation K(RST) will be used to denote the area of $\triangle$RST.

Figure 1

From Figure 1, we see that

$$\frac{K(QCM)}{K(PEM)} \cdot \frac{K(PEM)}{K(QFM)} \cdot \frac{K(QFM)}{K(PDM)} \cdot \frac{K(PDM)}{K(QCM)} = 1.$$

Hence

$$\frac{CM \cdot CQ \cdot \sin\alpha}{EM \cdot EP \cdot \sin\alpha} \cdot \frac{EM \cdot MP \cdot \sin\gamma}{FM \cdot MQ \cdot \sin\gamma} \cdot \frac{FM \cdot FQ \cdot \sin\beta}{MD \cdot DP \cdot \sin\beta} \cdot \frac{MD \cdot MP \cdot \sin\delta}{CM \cdot MQ \cdot \sin\delta} = 1.$$
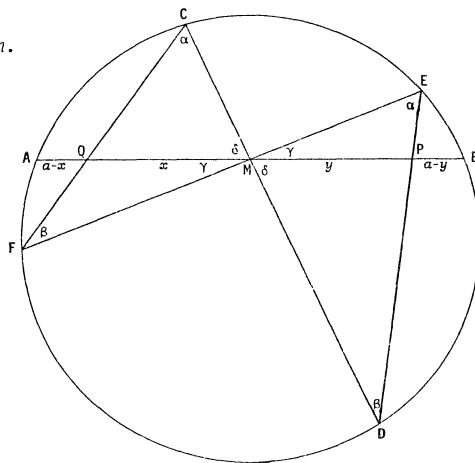
Upon cancellation, multiplication, and rearrangement, it follows that

$$CQ \cdot FQ \cdot (MP)^2 = EP \cdot DP \cdot (MQ)^2.$$

However, since $CQ \cdot FQ = AQ \cdot QB$ and $EP \cdot DP = BP \cdot AP$, it is true that

$$AQ \cdot QB \cdot (MP)^2 = BP \cdot AP \cdot (MQ)^2$$

or

$$(a^2 - x^2)y^2 = (a^2 - y^2)x^2.$$

Since x and y are positive, this equation implies $x = y$, from which it follows that $PM = MQ$.

3. *Extensions and Generalizations.*

(a) It has long been known that if CE and DF meet AB produced in R and S, then $RM = MS$ (Figure 2). This can be shown by a slight modification of existing proofs of the original Butterfly Problem.

(b) Klamkin [18] credits Cantab [3] for the following extension. If AOB is a diameter of the circle, $OM = ON$, and CD, EF are arbitrary chords through M and N (Figure 3), then $PO = OQ$ and $RO = OS$.

(c) Klamkin [18] extends (b) still further by showing that the diameter AOB can be replaced by an arbitrary chord. Thus, in Figure 4, if $OM = ON$, then $PO = OQ$ and $RO = OS$. Klamkin adds that since midpoints are invariant under an affine transformation, this result also holds for ellipses.

(d) Chakerian, Sallee, and Klamkin [21] showed that the Butterfly Property characterizes ellipses among ovals by proving the following theorem, which had already been conjectured by Klamkin alone in [18].

THEOREM. Let S be a closed, bounded, plane convex set with the following property: whenever M is the midpoint of a chord AB, and CD and EF are any two chords containing M, then $PM = MQ$ (as in Figure 1). Then S is an ellipse.

(e) Finally, Eves [23] extends the Butterfly Property to all proper conics by means of
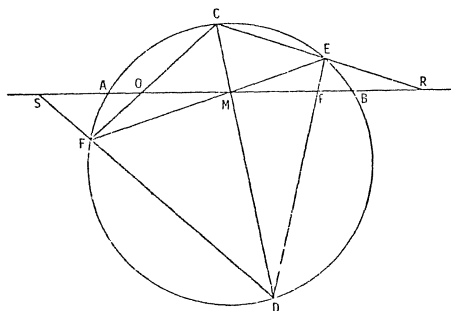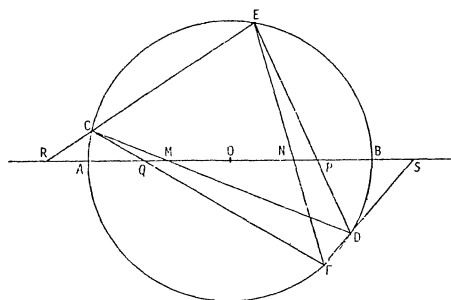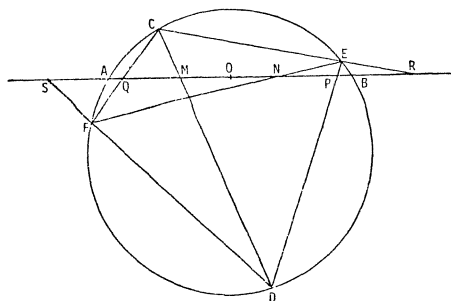

Figure 2

Figure 3

Figure 4

the following theorem:

THE GENERALIZED BUTTERFLY THEOREM. Let M (see Figure 5) be the midpoint of a chord AB of a proper conic c_1, let two other chords CD and EF be drawn through M, and let a conic c_2 through C, E, D, F cut the given chord in P and Q. Then M is the midpoint of PQ.
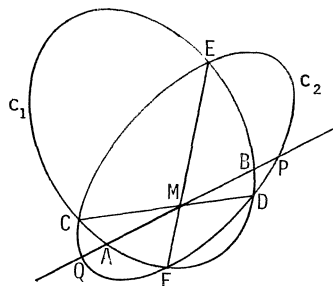
4. Acknowledgments.

I obtained reference 3 from [18]; references 14 and 17 from [19]; reference 20 from [21]; references 6, 19, 23, 24 are my own; all the remaining references are from [24].

Figure 5

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<div align="center">✻   ✻   ✻</div>

P R O B L E M S - - P R O B L È M E S

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 1.

For the problems given below, solutions, if available, will appear in EUREKA Vol.2, No.4, to be published around May 5, 1976. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than April 25, 1976.

101. *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que le cube de tout nombre rationnel est égal à la différence des carrés de deux nombres rationnels.

102. *Proposé par Léo Sauvé, Collège Algonquin.*

Si, dans un $\triangle ABC$, on a $a = 4$, $b = 5$, et $c = 6$, montrer que $C = 2A$.

103. *Proposed by H.G. Dworschak, Algonquin College.*

If $\dfrac{\cos\alpha}{\cos\beta} + \dfrac{\sin\alpha}{\sin\beta} = -1$, prove that

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1.$$

104. *Proposed by H.G. Dworschak, Algonquin College.*

Prove that $\sqrt[3]{5} - \sqrt[3]{3}$ is irrational.

105. *Proposed by Walter Bluger, Department of National Health and Welfare.*

INA BAIN declared once at a meeting
That she'd code her full name (without cheating),
    Then divide, so she reckoned,
    The first name by the second,
Thus obtaining five digits repeating.

106. *Proposed by Viktors Linis, University of Ottawa.*

Prove that, for any quadrilateral with sides a, b, c, d,

$$a^2 + b^2 + c^2 > \frac{1}{3}d^2.$$

107. *Proposed by Viktors Linis, University of Ottawa.*

For which integers m and n is the ratio $\frac{4m}{2m + 2n - mn}$ an integer?

108. *Proposed by Viktors Linis, University of Ottawa.*

Prove that, for all integers $n \geq 2$, $\sum\limits_{k=1}^{n} \frac{1}{k^2} > \frac{3n}{2n + 1}$.

109. *Proposed by Léo Sauvé, Algonquin College.*

(a) Prove that rational points (i.e. both coordinates rational) are dense on any circle with rational centre and rational radius.

(b) Prove that if the radius is irrational the circle *may* have infinitely many rational points.

(c) Prove that if even one coordinate of the centre is irrational, the circle has at most two rational points.

110. *Proposed by H.G. Dworschak, Algonquin College.*

(a) Let AB and PR be two chords of a circle intersecting at Q. If A, B, and P are kept fixed, characterize geometrically the position of R for which the length of QR is maximal. (See figure).

(b) Give a Euclidean construction for the point R which maximizes the length of QR, or show that no such construction is possible.

*   *   *
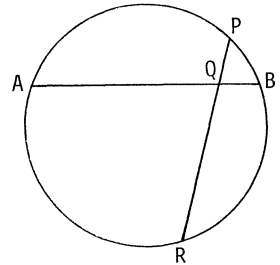
THE DEVIL A MATHEMATICIAN WOULD BE

The Devil said to Daniel Webster, "Set me a task I can't carry out, and I'll give you anything in the world you ask for."

Daniel Webster: "Fair enough. Prove that for n greater than 2, the equation $a^n + b^n = c^n$ has no non-trivial solution in integers."

They agreed on a three-day period for the labor, and the Devil disappeared.

At the end of the three days the Devil presented himself, haggard, jumpy, biting his lip. Daniel Webster said to him, "Well—how did you do at my task? Did you prove the theorem?"

"Eh? No . . . no, I haven't proved it."

"Then I can have whatever I ask for? Money? The Presidency?"

"What? Oh, *that*—of course. But listen! If we could just prove the following two lemmas—"

—A.J. LOHWATER[1]

[1]Quoted in *The Mathematical Magpie* by Clifton Fadiman, Simon and Schuster, 1962.

S O L U T I O N S

39. [1975; 26, 64] *Proposé par Maurice Poirier, École Secondaire Garneau.*

On donne un point P à l'intérieur d'un triangle équilatéral ABC tel que les longueurs des segments PA, PB, PC sont 3, 4, et 5 unités respectivement. Calculer l'aire du △ABC.

III. *Solution by Richard Popplewell, Architectural Technology student, Algonquin College.*

As in solution II, suppose we have given an equilateral △ABC, the length of whose side is X, and an interior point P such that PA = a, PB = b, and PC = c, as shown in the figure. In finding the area of △ABC in terms of a, b, and c, the solver in II relies heavily on Problem 38 [1975; 26, 63], a theorem which few people could be expected to know. I show here that the same result can be arrived at, and even more simply, using only well-known theorems of elementary geometry.
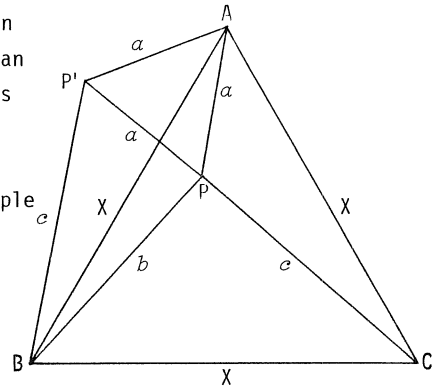
Rotate △APC about A through an angle of 60°, so that AC coincides with AB and P occupies new position P'. △APP' is clearly equilateral, so that PP' = a, and its area is $\frac{\sqrt{3}}{4}a^2$. The area K of △PP'B, whose sides are a, b, c, is easily found by Heron's formula. Then

$$\triangle APB + \triangle APC = \triangle APP' + \triangle PP'B = \frac{\sqrt{3}}{4}a^2 + K.$$

Similar rotations about B and C will achieve corresponding results, so that altogether we have

$$2\triangle ABC = \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) + 3K$$

and finally

$$\triangle ABC = \frac{\sqrt{3}}{8}(a^2 + b^2 + c^2) + \frac{3}{2}K,$$

where K = $\sqrt{s(s-a)(s-b)(s-c)}$ and s = $\frac{1}{2}(a+b+c)$.

51. [1975; 48, 86] *Late solution: André Bourbeau, École Secondaire Garneau.*

53. [1975; 48, 88] *Late solution: André Bourbeau.*

55. [1975; 48, 89] *Late solution: André Bourbeau.*

62. [1975; 56, 99] *Late solution: André Bourbeau.*

65. [1975; 57, 100] *Late solution: André Bourbeau.*

71. [1975; 71] *Proposed by Léo Sauvé, Algonquin College.*

If ten sheep jump over a fence in ten seconds, how many would jump over the fence in ten minutes?

Solution by the proposer.

It is not the ten sheep which make up the ten seconds, nor the ten jumps, but *the nine intervals of time between jumps*, each of which therefore lasts 10/9 of a second. In 10 minutes, or 600 seconds, there are 540 such intervals of time, and therefore 541 sheep jump over the fence in 10 minutes.

In arriving at this answer, of course, certain simplifying assumptions, which are implicit in the solution, have had to be made to render the problem determinate.

Also solved by R. Elichuk, Sir John A. Macdonald High School; F.G.B. Maskell, Algonquin College; and four other readers who gave 600 *as the answer.*

Editor's comments.

1. Elichuk's solutions (there were four different ones) were more in the nature of poking gentle fun at the proposer, who seems to be an old acquaintance of his, for proposing what appeared to be a frivolous problem.

It would be more correct to say that Elichuk found four solutions but gave only three, for he writes: "I did arrive at one marvelous conclusion—which also was undeniably correct—however, my supply of foolscap ran out before I was able to record it for posterity."

Elichuk should know that in mathematics it is customary, since Fermat, to run out of margin, not foolscap.

2. The proposer, in an internal communication, has assured the editor that, far from being frivolous, he had a serious didactic intent in proposing this problem. He wanted to show, as is clear from his solution, how easy it is even for trained people to arrive at the wrong answer (600 sheep in this case) by thoughtlessly equating apples and oranges.

3. Maskell, apparently with his tongue firmly lodged in his cheek, assumed that the intervals of time between jumps were, not constant as did the proposer, but formed a normal distribution. He then gave a detailed statistical analysis, complete with confidence limits, to arrive at the probable number of sheep that would jump in ten minutes.

It still remains to be determined whether this statistical method of counting sheep is as effective in curing in-som-nia..as..the..u--su--al...zZzz....zzZZzzz.....

72. ⌈1975; 71] *Proposé par Léo Sauvé, Collège Algonquin.*

Déterminer le couple (p,q) sachant que p et q

(a) sont *les* racines de l'équation $x^2 + px + q = 0$;

(b) sont chacun racine de l'équation $x^2 + px + q = 0$.

Solution by H.G. Dworschak, Algonquin College.

(a) If p and q are *the* roots of the equation, then

$$p + q = -p,$$
$$pq = q,$$

and the solutions of this system are $(0,0)$ and $(1,-2)$.

(b) If each of the numbers p and q is a solution, then each must satisfy the equation, so that

$$p^2 + p^2 + q = 0,$$
$$q^2 + pq + q = 0,$$

and the solutions of this system are $(0,0)$, $(1,-2)$, and $(-\frac{1}{2}, -\frac{1}{2})$.

Also solved by R. Elichuk, Sir John A. Macdonald High School; F.G.B. Maskell, Algonquin College; G.D. Kaye, Department of National Defence; and the proposer.

Editor's comment.

Not all of these solutions were correct in all respects. Some solvers had difficulty interpreting the subtle difference between (a) and (b).

73. [1975; 71] *Proposed by Viktors Linis, University of Ottawa.*

Is there a polyhedron with exactly ten pentagons as faces?

*Solution by F.G.B. Maskell, Algonquin College.*

Ten pentagons have 50 sides and 50 interior angles. Therefore a polyhedron of ten pentagonal faces (F), if it exists, has 25 edges (E) and no more than 16 vertices (V), since each vertex of the polyhedron requires at least three face angles. Therefore

$$V - E + F = V - 25 + 10 = V - 15 \leq 16 - 15 = 1,$$

contrary to Euler's formula $V - E + F = 2$.

Coxeter [1] gives an excellent version of Euler's proof based on the Schlegel diagram for a polyhedron, which is what you would see if you put your eye close enough to one face of the polyhedron to see all the other faces through it.

The Schlegel diagram for the dodecahedron is shown here: $F = 12$, $E = 30$, and $V = 20$; all faces are pentagons, and there are three face angles at each vertex. No polyhedron with fewer than 12 pentagonal faces is possible.
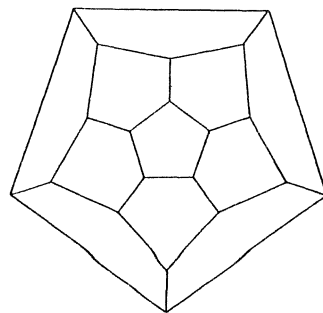
Also solved by H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; and Léo Sauvé, Collège Algonquin.

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74. [1975; 71] *Proposed by Viktors Linis, University of Ottawa.*

Prove that if the sides a, b, c of a triangle satisfy $a^2 + b^2 = kc^2$, then $k > \frac{1}{2}$.

Solution by André Bourbeau, École Secondaire Garneau.

By a well-known theorem of Pappus [1,2], in any triangle ABC,

$$\overline{BC}^2 + \overline{CA}^2 = 2\overline{AM}^2 + 2\overline{CM}^2,$$

where M is the midpoint of AB; thus

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2\overline{CM}^2.$$

From the hypothesis, we therefore have

$$kc^2 = \frac{c^2}{2} + 2\overline{CM}^2$$

and

$$k = \frac{1}{2} + \frac{2\overline{CM}^2}{c^2} > \frac{1}{2}.$$

Also solved by H.G. Dworschak, Algonquin College; R. Elichuk, Sir John A. Macdonald High School; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; Léo Sauvé, Collège Algonquin; and B. Vanbrugghe, Université de Moncton.

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75. [1975; 71] *Proposed by R. Duff Butterill, Ottawa Board of Education.*

M is the midpoint of chord AB of the circle with centre C shown in the figure below. Prove that RS > MN.
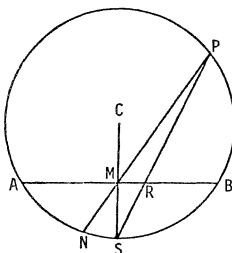
I. *Solution by Walter Bluger, Department of National Health and Welfare.*

Join ND and NS, where D is the extremity of the diameter through S. Two pairs of equal angles are denoted by α and β in Figure 1, and $x = \frac{\pi}{2} - \beta$, $y = \frac{\pi}{2} - \alpha$. From $\triangle$'s MNS and MSR, we have

$$\frac{MN}{MS} = \frac{\sin y}{\sin x} = \frac{\cos \alpha}{\sin x} \quad \text{and} \quad \frac{MS}{RS} = \sin x,$$

so that

$$\frac{MN}{RS} = \frac{MN}{MS} \cdot \frac{MS}{RS} = \cos \alpha \le 1,$$

and so $RS \ge MN$, equality being attained only when P is at D.

    II. *Solution by Léo Sauvé, Algonquin College*.

    In $\triangle$'s TMN and MRS in Figure 1, the angles at N and S are equal and $TM = MR$ by the Butterfly Theorem (see p. 2 in this issue). Hence these two triangles have equal circumcircles (in any triangle, the diameter of the circumcircle is the ratio of a side to the sine of the opposite angle). In these circumcircles RS is a diameter and MN is a chord, from which $RS > MN$ follows. If P is at D, the triangles collapse and RS coincides with MN.
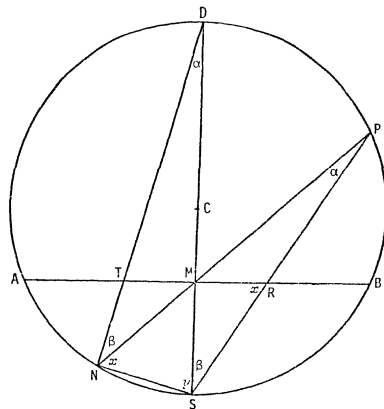
Figure 1

    Also solved by G.D. Kaye, Department of National Defence; and the proposer.

    Editor's comments.

    1. There is a curious history behind the publication of this problem. I will give the facts as they appear chronologically from the editor's viewpoint.

    In early summer 1975, the proposer submitted the problem to be published in EUREKA at the editor's discretion. He told me he had heard of this problem a few weeks earlier from Professor Ross Honsberger of Waterloo University.

    I decided to publish the problem in the October 1975 issue of EUREKA, which came out around October 15.

    When he saw the problem in EUREKA, University of Ottawa professor Viktors Linis informed me that it had already been proposed in the May 1975 issue of the *Ontario Secondary School Mathematics Bulletin* under the name of that roving ambassador *extraordinaire* of mathematics, Paul Erdös, of the Hungarian Academy of Science. Linis added that two solutions to the problem (both different from the solutions given here) had been published in the September 1975 issue of the *Bulletin*. One of these solutions was attributed to the proposer, Paul Erdös.

    !low the plot thickens. On December 10, 1975, I received my copy of the September 1975 issue of *Mathematics Magazine*, whose delivery had been delayed more than two months by the Canadian postal strike. The problem appeared there again, on p. 238, jointly proposed by Paul Erdös and Waterloo University professor M. S. Klamkin. Solutions to the problem have not yet been published in the *Mathematics Magazine*. They will probably appear in the September 1976 issue or thereabout. A copy of this issue of EUREKA will be sent to the problem editor of the *Mathematics Magazine* for his information.

2. Historical matters aside, there are several features of this problem which are fascinating.

Although it is short and elegant, solution II cannot really be said to be elementary, since the Butterfly Property is not normally part of a high school geometry curriculum.

Solution I, on the other hand, which is equally short and elegant, is completely elementary. It has the added advantage of not merely showing that $\frac{MN}{RS} \leq 1$, but of giving the exact value of the ratio: $\frac{MN}{RS} = \cos \alpha$.

It is natural to wonder, with reference to Figure 2, whether, as G ranges over the arc ASB, segment FG attains its maximum length when G is at S; and if not, whether the position of G which maximizes the length of FG can be characterized geometrically. H. G. Dworschak has answered this question (as far as I know, this is a new result). He invites you to share in the joy of his discovery in Problem 110 of this issue.
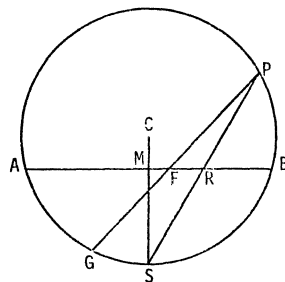

Figure 2

76. [1975; 71] *Proposed by H.G. Dworschak, Algonquin College.*

What is the remainder when 23^{23} is divided by 53?

Solution by D.E. Fisher, Algonquin College.

Since $23^2 = 529 \equiv -1 \pmod{53}$, we have
$$23^{23} = 23[(23)^2]^{11} \equiv 23(-1) = -23 \equiv 30 \pmod{53},$$
and so the required remainder is 30.

Also solved by André Bourbeau, École Secondaire Garneau; G.D. Kaye, Department of National Defence; Léo Sauvé, Collège Algonquin; Frank Stoyles, Algonquin College; and the proposer. One incorrect solution was received.

77. [1975; 71] *Proposed by H.G. Dworschak, Algonquin College.*

Let A_n, G_n, and H_n denote the arithmetic, geometric, and harmonic means of the n positive integers $n+1$, $n+2$, ..., $n+n$. Evaluate
$$\lim_{n\to\infty} \frac{A_n}{n}, \ \lim_{n\to\infty} \frac{G_n}{n}, \ \lim_{n\to\infty} \frac{H_n}{n}.$$

Solution de Léo Sauvé, Collège Algonquin.

Si l'on pose
$$f(x) = 1 + x, \quad g(x) = \ln(1+x), \quad h(x) = \frac{1}{1+x},$$
on obtient
$$\frac{A_n}{n} = \frac{1}{n}\sum_{k=1}^{n}\left(1 + \frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right), \tag{1}$$

$$\ln \frac{G_n}{n} = \frac{1}{n} \sum_{k=1}^{n} \ln(1 + \frac{k}{n}) = \frac{1}{n} \sum_{k=1}^{n} g(\frac{k}{n}), \qquad (2)$$

$$\frac{n}{H_n} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k}{n}} = \frac{1}{n} \sum_{k=1}^{n} h(\frac{k}{n}). \qquad (3)$$

Les limites quand $n \to \infty$ des membres gauches de (1)—(3) sont donc les valeurs moyennes des fonctions f, g, et h dans l'intervalle [0,1], c'est-à-dire

$$\lim_{n \to \infty} \frac{A_n}{n} = \int_0^1 f(x)\, dx = \frac{3}{2},$$

$$\lim_{n \to \infty} \ln \frac{G_n}{n} = \int_0^1 g(x)\, dx = 2 \ln 2 - 1 = \ln \frac{4}{e},$$

$$\lim_{n \to \infty} \frac{n}{H_n} = \int_0^1 h(x)\, dx = \ln 2.$$

On conclut que les limites demandées sont $\frac{3}{2}$, $\frac{4}{e}$, et $\frac{1}{\ln 2}$.

Also solved by F.G.B. Maskell, Algonquin College; and the proposer.

78. [1975; 72] *Proposed by Jacques Sauvé, student, University of Ottawa.*

There is a well-known formula for the sum of all the combinations of n objects: $\sum_{r=0}^{n} C(n,r) = 2^n$. But is there a simple formula for the sum of all the permutations $\sum_{r=0}^{n} P(n,r)$?

The need for such a formula arose in a study of reliability in systems engineering.

Solution by Léo Sauvé, Algonquin College.

The solution given here is essentially the same as that given by P. B. Johnson in [4]. We let P_n denote the required sum of permutations and arbitrarily assign $P_0 = 1$. If $n \geq 1$, we have

$$n!e = \sum_{r=0}^{\infty} \frac{n!}{r!} = \sum_{r=0}^{n} \frac{n!}{r!} + \sum_{r=n+1}^{\infty} \frac{n!}{r!} = \sum_{r=0}^{n} \frac{n!}{(n-r)!} + \sum_{r=n+1}^{\infty} \frac{n!}{r!}$$

$$= P_n + R_n,$$

where

$$0 < R_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots < \sum_{r=1}^{\infty} \frac{1}{(n+1)^r} = \frac{1}{n} \leq 1. \qquad (1)$$

Since P_n is surely an integer and $0 < R_n < 1$, it follows that $P_n = [n!e]$, where the bracket denotes the greatest integer function.

Also solved by G.D. Kaye, Department of National Defence; and Viktors Linis, University of Ottawa.

Editor's comments.

1. This result goes back at least to 1939 and probably much earlier, since it appears in [1] just as one problem among many, without solution or special comment. It reappeared in 1955 in [2], as a problem proposed by David Freidman who was then a student at McGill University in Montreal. The solution to Freidman's proposal, which appeared in [4], is the one paraphrased above. The problem then resurfaced in [5,6,7,11]. It is interesting to note that [1,5,11] give the result as

$$P_n = \text{the integer } nearest \text{ to } n!e,$$

which is not true if $n = 1$. But the two forms are equivalent if $n > 1$ since then $R_n < \frac{1}{2}$ by (1).

The following generating function for P_n is given by Riordan [5] and Comtet [11]:

$$\frac{e^t}{1-t} = \sum_{n=0}^{\infty} \frac{P_n}{n!} t^n.$$

2. Some useful by-products can be derived from the solution given above.

(a) The irrationality of e is an immediate consequence. For if e were rational there would be a positive integer n large enough to make $n!e$ integral. This is impossible, since it implies $R_n = 0$, contrary to (1).

(b) For $x > 0$ let (x) represent $x - [x]$, the fractional part of x. If α is irrational, we know from Kronecker's Theorem that the values of the sequence $\{(n\alpha)\}$, $n = 1,2,3,\ldots$, are dense in the open interval $(0,1)$ so that, in particular, the sequence does not converge. Hence, by analogy, there is little reason to expect the sequence $\{(n!\alpha)\}$, $n = 1,2,3,\ldots$, to exhibit any kind of regularity, and still less reason to expect that it will converge. But surprisingly, for $\alpha = e$, the sequence $\{(n!e)\}$, $n = 1,2,3,\ldots$, converges since $(n!e) = R_n \longrightarrow 0$ by (1).

3. The proposer mentioned that the formula for P_n may be useful in systems engineering. Here are two other applications of it:

(a) Suppose N points are given in general position in space, and all pairs of points are connected by a segment. Each segment is then coloured with one of n different colours. If N is large enough but n is fixed, this chromatic graph will surely contain monochromatic triangles (all three sides of the same colour). For fixed n, what minimal value of N will ensure the presence of at least one monochromatic triangle? A question equivalent to this one was asked by Arthur Engel in [8], and a solution by J.W. Ellis was published in [9], but the question had already been answered by R. E. Greenwood and A. M. Gleason in [3]. The minimal value of N turns out to be

$$[n!e] + 1 = P_n + 1.$$

(b) It has long been known that the number of arithmetical operations occurring in the expansion of an nth order determinant by minors and cofactors is of the order of $n!$, but until recently the exact number was unknown. In 1969 A. J. Wise showed in [10] that the exact number is

$$[n!e] - 2 = P_n - 2.$$

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10. A. J. Wise, The Number of Arithmetic Operations Required to Evaluate an n by n Determinant, *The Mathematical Gazette*, May, 1969, pp. 174-176.

11. Louis Comtet, *Analyse Combinatoire*, Presses Universitaires de France, 1970, Tome I, p. 90.

79. [1975; 72] *Proposed by John Thomas, University of Ottawa.*

Show that, for $x > 0$,

$$\left| \int_x^{x+1} \sin(t^2)\, dt \right| < \frac{2}{x^2}.$$

Solution (with a correction!) by Viktors Linis, University of Ottawa.

If we let

$$f(x) = \int_x^{x+1} \sin t^2\, dt = \int_x^{x+1} \frac{1}{2t} \cdot \sin t^2 \cdot 2t\, dt,$$

then integration by parts yields

$$f(x) = -\frac{\cos t^2}{2t}\Big|_x^{x+1} - \frac{1}{2}\int_x^{x+1}\frac{\cos t^2}{t^2}dt = -\frac{\cos(x+1)^2}{2(x+1)} + \frac{\cos x^2}{2x} - \frac{1}{2}\int_x^{x+1}\frac{\cos t^2}{t^2}dt;$$

hence

$$\left| f(x) \right| \leq \left| -\frac{\cos (x+1)^2}{2(x+1)} \right| + \left| \frac{\cos x^2}{2x} \right| + \frac{1}{2}\left| \int_x^{x+1} \frac{\cos t^2}{t^2}\, dt \right|$$

$$\leq \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2}\int_x^{x+1}\frac{dt}{t^2} = \frac{1}{2(x+1)} + \frac{1}{2x} + \frac{1}{2}\left(-\frac{1}{x+1} + \frac{1}{x}\right)$$

$$= \frac{1}{x}.$$

Also solved by Léo Sauvé, Algonquin College.

Editor's comment.

The problem as proposed is incorrect. It implies, for example, that $\left| f(4) \right| < \frac{1}{8} = 0.125$, whereas the use of Simpson's Rule with 20 intervals yields $\left| f(4) \right| \approx 0.219$. The proposer probably intended the inequality to read

$$\left| f(x) \right| < \frac{2}{x}, \qquad x > 0. \tag{1}$$

Indeed, (1) occurs as a problem in Rudin [1], with a hint to solve it by using the second mean value theorem. The solution above shows that, using nothing more complicated than integration by parts, the upper bound in (1) can be improved to

$$\left| f(x) \right| \leq \frac{1}{x}, \qquad x > 0. \tag{2}$$

In a remark following his solution, Linis outlined an argument showing that, for large x, $\left| f(x) \right|$ is of order $\frac{1}{x}$, so that no substantial improvement of the upper bound in (2) can be expected. This was confirmed and sharpened to a fine edge by Gillis and Shimshoni in [2]. They showed that the upper bound in (2) is the best possible by proving that

$$\limsup_{x \to \infty} \left| x\, f(x) \right| = 1.$$

The integrals

$$C(x) = \int_0^x \cos t^2\, dt, \qquad S(x) = \int_0^x \sin t^2\, dt$$

are called *Fresnel integrals*, after AUGUSTIN FRESNEL (1788 - 1827), French physicist and mathematician. They are very useful in geometrical optics, especially in the theory of diffraction [3]. They have been extensively studied and their values tabulated [4]. In terms of the Fresnel sine integral, (2) can be written

$$\left| S(x+1) - S(x) \right| \leq \frac{1}{x}, \qquad x > 0.$$

<div align="center">REFERENCES</div>

1. Walter Rudin, *Principles of Mathematical Analysis*, 2nd Edition, McGraw-Hill, 1964, p. 129.

2. Solution to Problem E2211, *American Mathematical Monthly*, Vol. 77, 1970, p. 1108.

3. Arnold Summerfeld, *Optics*, Academic Press, 1964.

4. *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964.

80. [1975; 72] *Proposé par Jacques Marion, Université d'Ottawa.*

Existe-t-il une suite d'entiers $\{a_n\}$ telle que $\lim\limits_{n\to\infty} a_n = \infty$ et que la suite $\{\sin a_n x\}$ converge pour tout $x \in [0, 2\pi]$?

I. *Comment by H.G. Dworschak, Algonquin College.*

There is no such sequence. This problem can be found, with a solution, in Rudin [1]. The solution given there runs as follows:

Suppose there is such a sequence. In that case we must have

$$\lim_{n\to\infty} (\sin a_n x - \sin a_{n+1} x) = 0 \qquad (0 \le x \le 2\pi);$$

hence

$$\lim_{n\to\infty} (\sin a_n x - \sin a_{n+1} x)^2 = 0 \qquad (0 \le x \le 2\pi). \qquad (1)$$

By Lebesgue's theorem concerning integration of boundedly convergent sequences [2], (1) implies

$$\lim_{n\to\infty} \int_0^{2\pi} (\sin a_n x - \sin a_{n+1} x)^2 \, dx = 0. \qquad (2)$$

But a simple calculation shows that

$$\int_0^{2\pi} (\sin a_n x - \sin a_{n+1} x)^2 \, dx = 2\pi,$$

which contradicts (2).

II. *Comment by Léo Sauvé, Algonquin College.*

A stronger result is given as a problem (with a hint but no solution) in Rudin [3]: the set of all x at which the sequence $\{\sin a_n x\}$ converges must have measure zero.

REFERENCES

1. Walter Rudin, *Principles of Mathematical Analysis*, 2nd Edition, McGraw-Hill, 1964, p. 143.

2. *ibid.*, p. 246.

3. *ibid.*, p. 258.

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Re Popular Misconception No. 2 [1975; 69], George W. Maskell, Huddersfield, England, suggests:

One man's meat is another man's *poisson*.

REPORT ON PRIMES I

VIKTORS LINIS, University of Ottawa

Prime numbers continue to fascinate and frustrate most number-theoreticians and other number addicts. A recent report on *What's new on the prime number front* is given by the Swedish mathematician Hans Riesel [*Nordisk Matem. Tidskrift*, 23 - 1, pp. 5 - 14, Oslo 1975]. Since the article is in Swedish and the periodical is not easily available, here are some highlights for EUREKA readers.

1. The largest known twin primes are $76 \cdot 3^{139} \pm 1$ (69 digits); they were discovered by Williams and Zarnke.

2. The Fermat number $F_n = 2^{2^n} + 1$ is composite for $n = 7$. This was shown by Morehead in 1905, but its factors were unknown until, in 1970, Morrison and Brillhart found them to be:

$$F_7 = 59{,}649{,}589{,}127{,}497{,}217 \cdot 5{,}704{,}689{,}200{,}685{,}129{,}054{,}721.$$

It is also known that F_{1945} is composite with one factor $5 \cdot 2^{1947} + 1$, which has approximately 10^{585} digits.

3. Fancy primes of the form $\dfrac{10^p - 1}{9}$ (p prime):

$$\frac{10^2 - 1}{9} = 11,$$

$$\frac{10^{19} - 1}{9} = 1111111111111111111 \quad (19 \text{ digits}),$$

and

$$\frac{10^{23} - 1}{9} = 11111111111111111111111 \quad (23 \text{ digits})$$

are the only ones for $p \le 353$ (Brillhart).

Similarly the numbers $\dfrac{10^p + 1}{11}$ are primes for $p = 5, 7, 19, 31, 53, 67$; these look like

$$\frac{10^{19} + 1}{11} = 909090909090909091 \quad (18 \text{ digits}).$$

4. The number $\pi(x)$ of primes $\le x$ has been computed recently by Bohman for $x = 10^{13}$:

$$\pi(10^{13}) = 346{,}065{,}535{,}898.$$

It took over $4\frac{1}{2}$ hours on a very fast computer (Univac 1108) to do this.

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