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CIRCLE PATTERNS ARISING FROM A SEVEN-CIRCLE PROBLEM

Hiroshi Okumura

In a recent paper [3], we discuss a certain five-circle problem and use it as a basis to display a periodic pattern of circles in the plane. In this paper, we consider a seven-circle problem and display periodic patterns of circles based on it.

PROBLEM 1: Let C_0 be a circle of radius r; C_a, C'_a circles of radii a touching C_0 internally at the end of a diameter of C_0; C_b, C'_b circles of radii b touching C_0 internally at the end of a diameter of C_0 and touching C_a and C'_a externally respectively; and C_c, C'_c circles of radii c touching C_0 internally at the end of a diameter of C_0 and touching C_b, C'_a and C'_b, C_a externally respectively. Suppose that the centres of $C_a, C_b, C_c, C'_a, C'_b, C'_c$ form vertices of a convex hexagon. Show that $r = a + b + c$.

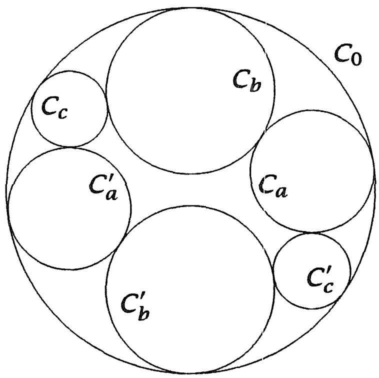

Figure 1

It is not appropriate to reproduce the long proof in [2] here, which is based on the Pythagorean theorem. Let us assume that three circles of radii a, b, c touch each other externally. Next, tessellate the plane, without gaps or overlaps, by copies of the triangle formed by the three centres and consider a vertex D of the tessellation (see Figure 2). There are six segments joining the neighboring vertices to D, which have length $a + b, b + c$ and $c + a$ in pairs. We denote these vertices by $C, C'; A, A'; B, B'$ and assume that A, B, C, A', B' and C' lie around D in this order. Then draw six circles $C_A, C_B, C_C, C_{A'}, C_{B'}$ and $C_{C'}$ of radii a, b, c, a, b, c with centres A, B, C, A', B', C' respectively. It is obvious that each of the six circles touches the two neighbors externally. Since the distance between D and one of the intersections of the line DA and C_A is $a + b + c$, C_A touches the circle of radius $a + b + c$ with centre D internally. Similarly the remaining five circles are tangent internally to the seventh circle with centre D. Therefore a solution of Problem 1 follows from the uniqueness of the figure.

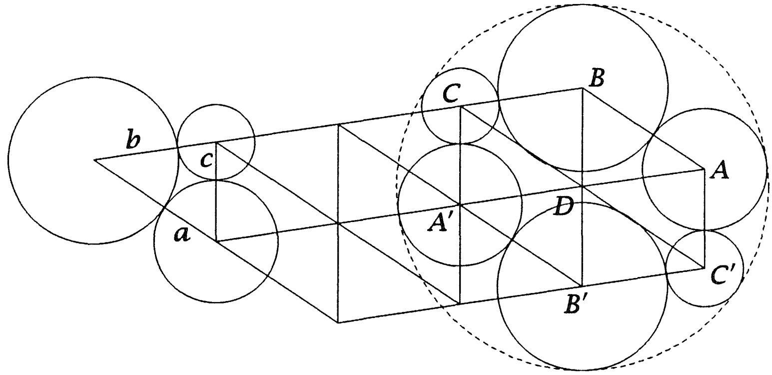

Figure 2

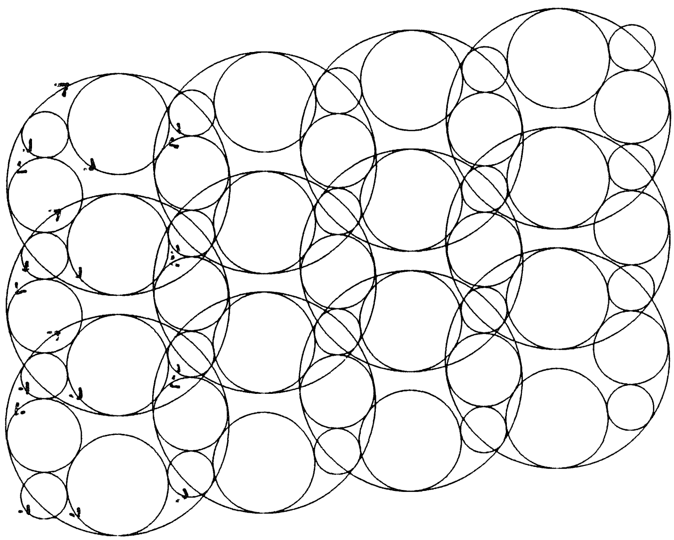

Figure 3

Let us denote the seventh circle by C_D. The above observation suggests that we can construct a pattern in the plane consisting of copies of C_A, C_B, C_C and C_D. Let S and T be the translations mapping A into A' and B into B' respectively. If we draw the images of C_A, C_B, C_C and C_D by $S^m T^n$ for all the integers m and n, we get a circle pattern in the plane (see Figure 3, the arrows will be explained later).

The centres of the circles form a triangular lattice in the pattern. The reader may think that the largest circles (the copies of C_D) play a special role among the others. But we will show that each of the circles plays exactly the same role if we ignore the relation "one circle contains another" or "one circle is contained in another". To see this fact we need to assign orientations to the circles. Such oriented circles are called *cycles*, and we describe the orientations by the arrows on the perimeters of the circles. We regard that the sign of the radius of the cycle is plus if the orientation is counterclockwise, otherwise minus. Two touching circles are said to *anti-touch* as cycles if they have the opposite orientations at the points of tangency. Let us assign orientations to the circles in our pattern so that each pair of touching circles anti-touch as cycles as in Figure 3. Then *for each cycle, there are six cycles anti-touching it. The six cycles fall into three pairs such that each of the pairs consists of two congruent cycles anti-touching the first at the end of a diameter.* Since the orientations of the largest cycles are opposite to the others, their radii have different signs from the others. Therefore the conclusion of Problem 1 can be restated as: *the sum of the four different radii of the cycles in the pattern is equal to zero.*

Returning to Figure 2, let us consider the case in which $A'BC$ is a right triangle with the right angle at C. Then AA' and BB' are perpendicular, and the figure consisting of $C_A, C_B, C_{A'}, C_{B'}$ and C_D is symmetric in the line AA'. This implies that the images of C_C and $C_{C'}$ by the reflection in the line AA' coincide with C_C^T and $C_{C'}^{T^{-1}}$ respectively. Therefore C_C^T and $C_{C'}^{T^{-1}}$ touch C_D internally. Similarly symmetry in the line BB' implies that C_A and $C_{B'}$ (and C_B and $C_{A'}$) also touch. If we draw our pattern in this situation, it is symmetric in the lines AA' and BB', and for each of the circles there are eight circles touching it (see Figure 4).

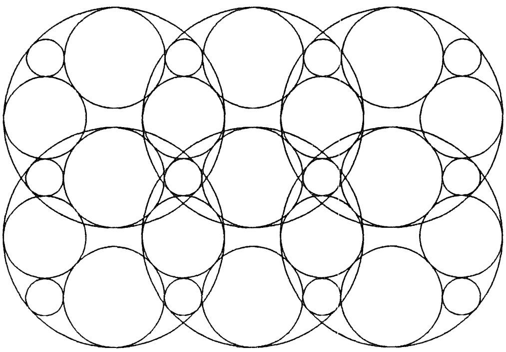

Figure 4

A related problem, with no solution given, appears in [1, p. 32].

Problem 2: Three circles touch each other externally, and another circle contains them and touches them, and the centres of the three small circles form a right triangle. Show that the radius of the largest circle is equal to the sum of the radii of the other three.

Let us again draw a tessellation, as in the solution to Problem 1, starting with the centres of the three smaller circles. Then we can see that there is a circle touching the small three, and its radius is equal to the sum of the three radii. Therefore the uniqueness of the figure gives an immediate solution of Problem 2. It is also easily seen that the sum of the sides of the right triangle is equal to a diameter of the largest circle. A problem stating this fact can also be found in [2].

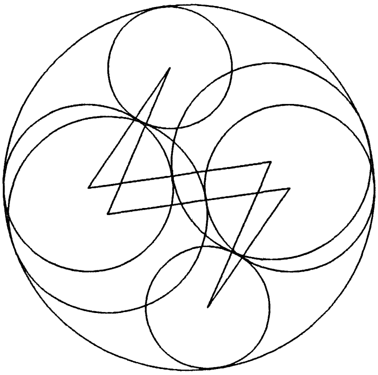

Figure 5

There are several things to be considered for the problems and our patterns. In Problem 1, the inner six circles touch the largest circle so that the six centres form a convex hexagon. But if the six centres form a non-convex hexagon, what can we say about the relationship between the radii of the circles (see Figure 5)? Is any sort of generalization of Problem 1 possible?

Every circle is always tangent to six others in our pattern. When every circle is tangent to exactly n other circles in our pattern, we may say that the pattern is of *n-type*. Since our pattern is symmetric in the centre of any circle, n is always even. In general our pattern is of 6-type, but as in Figure 4 it becomes of 8-type with an additional condition. Is it possible to construct patterns of 10-type, 12-type, 14-type, $\cdots$? It seems that Figure 6 suggests that a pattern of 10-type exists. If a pattern of n-type exists, find a necessary and sufficient condition under which our pattern is of n-type.

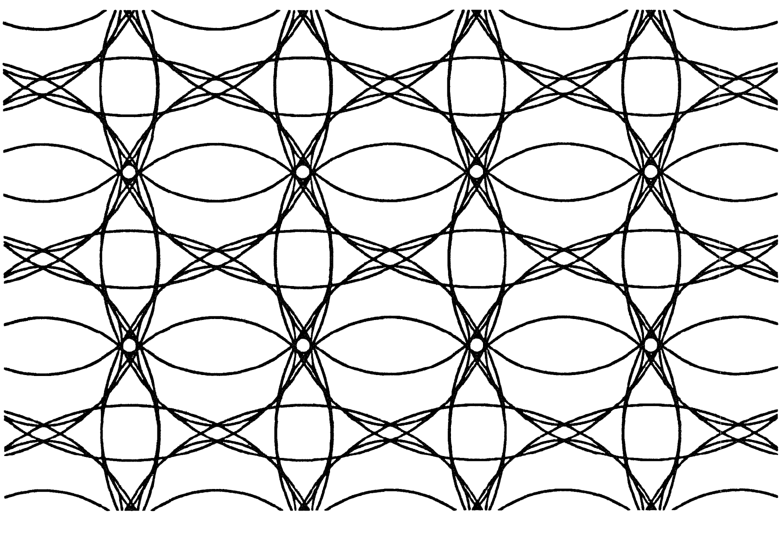

Figure 6

Acknowledgement: The author wishes to express his thanks to Y. Fujii for kindly providing a copy of [4], and to the referee for a lot of helpful comments.

References:

[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, Canada, 1989.

[2] A. Hirayama and J. Yamaki, *An addendum to the sangaku in south Miyagi*, private edition, 1967.

[3] H. Okumura, A Five-Circle Problem, *Crux Mathematicorum*, **20**(1994), pp. 121-126.

[4] Sakuma, *Sampō KigenShū* Vol. 2, 1877.

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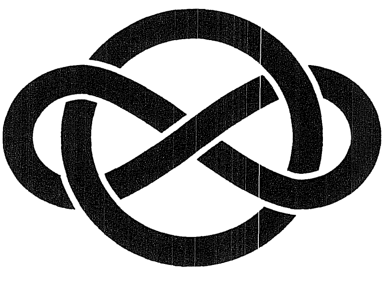
IMO95 PUZZLES



Above is the nifty logo for the 1995 IMO, held this past July in Canada, at York University. (The editor was lucky enough to attend as a problem captain.) Here are two little problems inspired by this logo, which were suggested at the IMO, and which readers might like to figure out.

1. (by Stan Rabinowitz) Consider the four intersections of the circle and the infinity sign. In general, there are $2^4 = 16$ possibilities for how they overlap (the actual logo shows one of these), because at each intersection either the circle or the infinity sign could be on top. In how many of these 16 possibilities are the circle and infinity sign **unlinked**?

2. (by Mike Dawes) Suppose you take a physical model of this logo, and manipulate it to turn the infinity sign into a circle. What does the circle turn into?

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THE SKOLIAD CORNER

No. 7

R. E. WOODROW

This month we feature a Calgary regional contest at the Junior High School level. The contest is run under the aegis of the Calgary Mathematical Association and is sponsored by The Faculties of Education and Science, The Department of Mathematics and Statistics of The University of Calgary, as well as the Mathematics Council of the Alberta Teachers' Association and the Calgary Catholic Board of Education.

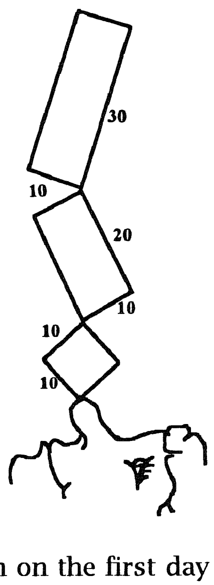
19th CALGARY JUNIOR MATHEMATICS CONTEST

May 1995 — Time: 90 minutes

PART A (5 marks each)

1. A cookie recipe calls for $\frac{1}{4}$ teaspoon of cinnamon and $\frac{1}{3}$ teaspoon of nutmeg (among other ingredients). Len accidentally puts in $\frac{1}{3}$ teaspoon of cinnamon and $\frac{1}{4}$ teaspoon of nutmeg. How much nutmeg (in teaspoons) should he add so that these two ingredients are in the correct proportion?

2. A juggler balances a 10cm by 10cm square, a 10cm by 20cm rectangle, and a 10cm by 30cm rectangle, all stacked up corner to corner, on the end of her nose. How far (in cm) above the end of her nose is the top of the top rectangle?



3. Theresa travels 200 km on the first day of her vacation. On each subsequent day she travels half the total distance she has travelled since her departure. How many km will she have travelled by the end of the fourth day?

4. What is the smallest whole number bigger than 1 which is simultaneously a perfect square, a perfect cube, and a perfect fourth power?

5. Karen's house number is greater than 150 and divides into 1995. What is the smallest possible value of her house number?

6. The fraction $\frac{2}{8}$ is equal to $\frac{1}{4}$, and when you add 1 to the numerator and denominator of $\frac{2}{8}$ you get $\frac{3}{9}$ which is equal to $\frac{1}{3}$. Find a fraction which is equal to $\frac{1}{8}$, so that when you add 1 to the numerator and denominator of your fraction you get a fraction which is equal to $\frac{1}{7}$.

7. The isosceles triangle ABC with $AB = AC$ is inscribed in a circle of radius 5cm. If the distance from the centre of the circle to the base BC is 4cm find the area of the triangle in square cm.

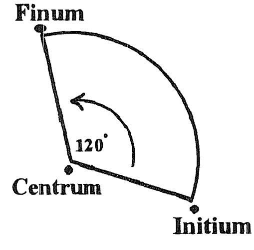
8. A trawler fishing off the nose of the Grand Banks observes a patrol boat approaching from the west. The patrol boat is 25 nautical miles away from the trawler when the trawler flees travelling east at 20 knots. If the top speed of the patrol boat is 25 knots, how long will it take the patrol boat to catch up to the trawler? [1 knot = 1 nautical mile per hour].

9. Bruce's television has channels 2 through 39. If he starts on channel 13 and surfs, pushing the channel-up button 417 times, on what channel is the television when he stops?

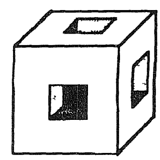
PART B (9 marks each)

1. Find all the whole numbers between 1 and 100 which can be written as a sum of integers constructed by using each of the digits 0 through 9 exactly once.
(Example: $90 = 0 + 1 + 52 + 3 + 4 + 6 + 7 + 8 + 9$ is one such number.)

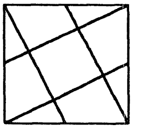
2. Two routes lead from Initium to Finum in a certain country. The Circular Route follows the arc of a circle of radius 100km centered at Centrum and passing through Initium and Finum. The Radial Route follows the radius from Initium to Centrum and then the radius from Centrum to Finum. The arc of the circular route measures 120°, and the speed limit on the Circular Route is 105 km/hr while that on the Radial Route is 100 km/hr. Which is the quicker route to take?



3. A wooden cube 9cm × 9cm × 9cm has three square holes drilled through it, each of which forms a 3cm × 3cm × 9cm tunnel through the centre of opposite faces. What is the total surface area of the exposed wood?



4. The large square has area 1. The inside lines join a vertex of the square to the midpoint of a side as shown. What is the area of the small central square?



5. A decimal number between 0 and 1 is called an "E-number" if all of its decimal digits are even, and is called an "O-number" if all of its decimal digits are odd. For instance .028 is an E-number and .1195 is an O-number, while .037 is neither an E-number nor an O-number, because 0 is even and 3 is odd.

(a) Suppose you want to find an E-number and an O-number so that the E-number is less than the O-number, but they are as close as possible. How close can they be?

(b) How close can they be if you want the O-number less than the E-number in (a)?

6. Alicia, Brad, Chris, Drew, Elsie and Farid went to a Stampede breakfast together and sat around a circular table eating pancakes. Alicia ate more pancakes than the person to her right, but fewer than Brad. Brad ate fewer pancakes than the person to his right, but more than Chris. Chris ate fewer pancakes than the person to her left, but more than Drew. Drew ate more pancakes than the person to his left, but fewer than Elsie. Farid wasn't hungry, and didn't eat any pancakes at all.

Find all the possible ways they could have been seated around the table.

* * * * *

Last issue we gave the Concours de Mathématiques from New Brunswick. As promised we now give the solutions.

1. E	**2.** D	**3.** B	**4.** D	**5.** C
6. B	**7.** D	**8.** D	**9.** E	**10.** C
11. C	**12.** E	**13.** D	**14.** E	**15.** C
16. C	**17.** E	**18.** A	**19.** C	**20.** C
21. A	**22.** C	**23.** D	**24.** C	**25.** E

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That completes our space for the Skoliad Corner this month. Send me your material, suggestions, criticisms and comments.

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THE OLYMPIAD CORNER

No. 167

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Because this number of *Crux* is longer than normal (and the remaining numbers a bit shorter) I give two Olympiad sets. The first is the 16th Austrian Polish Mathematics Competition held in Graz, Austria. Many thanks go to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, for translating the problems and sending them to me.

16th AUSTRIAN POLISH MATHEMATICS COMPETITION

First Day — June 30, 1993

Time: 4.5 hours (individual competition)

1. Determine all natural numbers $x, y \geq 1$ such that $2^x - 3^y = 7$.

2. We consider all tetrahedra $ABCD$ such that the sums of the three surface triangles ΔABD, ΔACD and ΔBCD is less or equal to 1. Determine (among the considered tetrahedra) the ones with maximum volumes.

3. Let the function f be defined as follows:

If $n = p^k > 1$ is a power of a prime number p, then $f(n) := n + 1$.

If $n = p_1^{k_1} \cdots p_r^{k_r}$ $(r > 1)$ is a product of powers of pairwise different prime numbers,then $f(n) := p_1^{k_1} + \cdots + p_r^{k_r}$.

For every $m > 1$ we construct the sequence $\{a_0, a_1, \ldots\}$ such that $a_0 = m$ and $a_{j+1} = f(a_j)$ for $j \geq 0$. We denote by $g(m)$ the smallest element of this sequence. Determine the value of $g(m)$ for all $m > 1$.

Second Day — July 1, 1993

Time: 4.5 hours (individual competition)

4. The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Let A and B be natural numbers such that A^{19} divides B^{93} and also B^{19} divides A^{93}. Prove: For all natural numbers $n \geq 1$ the number $(A^4 + B^8)^{F_{n+1}}$ is divisible by $(AB)^{F_n}$.

5. Determine all real solutions x, y, z of the system of equations:

$$\begin{aligned} x^3 + y &= 3x + 4 \\ 2y^3 + z &= 6y + 6 \\ 3z^3 + x &= 9z + 8 \end{aligned}$$

6. Show: For all real numbers $a, b \geq 0$ the following chain of inequalities is valid

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}\right)^3}$$

Also, for all three inequalities determine the cases of equality.

Third Day — July 2, 1993

Time: 4 hours (team competition)

7. We consider the sequence $\{a_n, n \geq 0\}$ defined by

$$a_0 = 0 \qquad \text{and} \qquad a_{n+1} = [\sqrt[3]{a_n + n}]^3$$

for $n \geq 0$. ($[x]$ is the greatest integer $\leq x$.)

a) Determine an explicit formula representing a_n as a function of n only.

b) Determine all n such that $a_n = n$.

8. Determine all real polynomials $P(x)$ such that there exists precisely one real polynomial $Q(x)$ satisfying the conditions $Q(0) = 0$ and $x + Q(y + P(x)) = y + Q(x + P(y))$ for all real numbers x and y. (Remark: For different polynomials $P(x)$ there may exist different polynomials $Q(x)$.)

9. Let ΔABC be equilateral. On side AB produced we choose a point P such that A lies between P and B. We now denote a as the length of sides of ΔABC; r_1 as the radius of incircle of ΔPAC; and r_2 as the exradius of ΔPBC with respect to side BC. Determine the sum $r_1 + r_2$ as a function of a alone.

* * * * *

The second contest comes from a nearby part of Europe — it is the VII Nordic Contest. My thanks go to Šefket Arslanagić, Berlin, Germany, for sending it to us.

VII NORDIC MATHEMATICAL CONTEST

March 17, 1993

1. Let F be an increasing real function defined for all real x, $0 \le x \le 1$, such that

i) $F\left(\frac{x}{3}\right) = \frac{F(x)}{2}$.

ii) $F(1-x) = 1 - F(x)$.

Find $F(173/1993)$ and $F(1/13)$.

2. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation

$$2r^3 - 7r - 3 = 0.$$

3. Find all solutions of the system of equations

$$\begin{cases} \text{i)} & s(x) + s(y) = x \\ \text{ii)} & x + y + s(z) = z \\ \text{iii)} & s(x) + s(y) + s(z) = y - 4 \end{cases}$$

where x, y and z are positive integers and $s(x)$, $s(y)$ and $s(z)$ denote the number of digits of the integers x, y and z respectively.

4. For a positive integer n let $T(n)$ be defined as the sum of the digits in the decimal representation of n.

a) Find a positive integer N such that $T(k \cdot N)$ is even for all integers k, $1 \le k \le 1992$, and $T(1993 \cdot N)$ is odd.

b) Show that there exists no positive integer N such that $T(k \cdot N)$ is even for all positive integers k.

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As promised last issue we give the "official" solutions of the 1995 Canadian Mathematical Olympiad. My thanks go to Edward T. H. Wang, Sir Wilfrid Laurier University and chairperson of the Canadian Mathematical Olympiad Committee, for mailing them to me.

1. Let $f(x) = \frac{9^x}{9^x + 3}$. Evaluate the sum

$$f\left(\frac{1}{1996}\right) + f\left(\frac{2}{1996}\right) + f\left(\frac{3}{1996}\right) + \cdots + f\left(\frac{1995}{1996}\right).$$

Solution. Note that

$$f(1-x) = \frac{9^{1-x}}{9^{1-x} + 3} = \frac{9}{9 + 3 \times 9^x} = \frac{3}{9^x + 3},$$

from which we get

$$f(x)+f(1-x)=\frac{9^x}{9^x+3}+\frac{3}{9^x+3}=1.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{1995} f\left(\frac{k}{1996}\right) &= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right)+f\left(\frac{1996-k}{1996}\right)\right]+f\left(\frac{998}{1996}\right)\\ &= \sum_{k=1}^{997}\left[f\left(\frac{k}{1996}\right)+f\left(1-\frac{k}{1996}\right)\right]+f\left(\frac{1}{2}\right)\\ &= 997+\frac{3}{3+3}=997\frac{1}{2}\,.\end{aligned}$$

2. Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}\,.$$

Solution I. We prove equivalently that $a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$. Due to complete symmetry in a, b, and c, we may assume, without loss of generality, that $a \geq b \geq c > 0$. Then $a-b \geq 0$, $b-c \geq 0$, $a-c \geq 0$ and $a/b \geq 1$, $b/c \geq 1$, $a/c \geq 1$. Therefore,

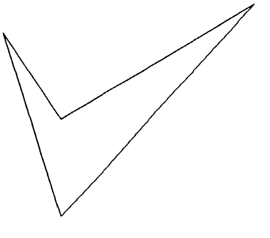
$$\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}}=\left(\frac{a}{b}\right)^{1-b}\left(\frac{b}{c}\right)^{b-c}\left(\frac{a}{c}\right)^{1-c} \geq 1.$$

Solution II. If we assign the weights a, b, c to the numbers a, b, c, respectively, then by the weighted geometric mean-harmonic mean inequality followed by the arithmetic mean-geometric mean inequality, we get:

$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a+b+c}{\frac{a}{a}+\frac{b}{b}+\frac{c}{c}}=\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

from which $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$ follows immediately.

3. Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees. (See Figure displayed.) Let C be a convex polygon having s sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b+(s-2)/2$.



Solution. For convenience, the interior angle in a boomerang which is greater than 180° will be called a "reflex angle".

Clearly there are b reflex angles, each occurring in a different boomerang and each with the corresponding vertex in the interior of C. Angles around these vertices add up to $2b\pi$. On the other hand, the sum of all the interior angles of C is $(s-2)\pi$ and the sum of the interior angles of all the q quadrilaterals is $2\pi q$.

Therefore, $2\pi q \geq 2b\pi + (s-2)\pi$ from which $q \geq b + (s-2)/2$ follows.

4. Let n be a fixed positive integer. Show that for any nonnegative integer k, the diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

Solution. Since

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

we see that when $k = 0$,

$$(x_1, x_2, \ldots, x_n; y) = \left(1, 2, \ldots, n; \frac{n(n+1)}{2}\right)$$

is a solution. To see that we can generate infinitely many solutions in general, set $c = (n(n+1))/2$ and notice that for all positive integers q, we have:

$$\begin{aligned}(c^k q^{3k+2})^3 &+ (2c^k q^{3k+2})^3 + \cdots + (nc^k q^{3k+2})^3 \\ &= c^{3k} q^{3(3k+2)} (1^3 + 2^3 + \cdots + n^3) \\ &= c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2 \\ &= c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}.\end{aligned}$$

That is, $(x_1, x_2, \ldots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \ldots, nc^k q^{3k+2}; cq^3)$ is a solution. This completes the proof.

5. Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq u \\ 1 - (\sqrt{ux} + \sqrt{(1-u)(1-x)})^2 & \text{if } u \leq x \leq 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows:

$$u_1 = f(1), \quad \text{and} \quad u_n = f(u_{n-1}) \quad \text{for all } n > 1.$$

Show that there exists a positive integer k for which $u_k = 0$.

Solution. Note first that $u_1 = 1 - u$. Since for all $x \in [u, 1]$, $u \leq x$ and $1 - x \leq 1 - u$ we have

$$\begin{aligned}1 &- (\sqrt{ux} + \sqrt{(1-u)(1-x)})^2 \\ &= 1 - ux - (1-u)(1-x) - 2\sqrt{ux(1-u)(1-x)} \\ &= u + x - 2ux - 2\sqrt{ux(1-u)(1-x)} \\ &\leq u + x - 2ux - 2u(1-x) = x - u.\end{aligned}$$

Therefore,

$$f(x) = 0 \quad \text{if} \quad 0 \le x \le u \tag{1}$$

and

$$f(x) \le x - u \quad \text{if} \quad u \le x \le 1. \tag{2}$$

From (2) we get $u_2 = f(u_1) - u = 1 - 2u$ if $u_1 \ge u$. An easy induction then yields $u_{n+1} = f(u_n) \le u_n - u \le 1 - (n+1)u$ if $u_i \ge u$ for all $i = 1, 2, \ldots, n$.

Thus for sufficiently large k, we must have $u_{k-1} < u$ and then $u_k = f(u_{k-1}) = 0$ by (1).

* * * * *

For the remainder of this number we turn the Corner over to Sam Maltby, formerly a student at Calgary, and now at Warwick in England. He gives a personal impression of participating in the IMO this summer in Canada. I hope that you will find his account as interesting as I did. Send me your nice problem sets and solutions.

The 1995 IMO

by

Sam Maltby

In the autumn of 1993, while languishing in the university division of the Sirius Cybernetics Corporation, I received an e-mail message from Bill Sands saying that Andy Liu was rounding up members for the Problem Selection Committee for the 1995 IMO (which would produce the shortlist of problems), and would I be interested in contributing a student's viewpoint? I got in touch with Andy and was put on the committee, but there was nothing to do for over a year since we couldn't select problems until they were submitted by the participating countries, and that wouldn't be until about three months before the contest. The only substantial movement in that time was that I was taken on as a co-ordinator for the contest itself, although even after the contest I cannot say why we were called co-ordinators, since our job was to mark the contest.

Finally, at the start of April 1995, the list of proposed problems had been compiled and was sent out to the thirteen committee members to look at before the selection meeting in June. In that time we were to attempt as many problems as we could, getting some idea as to their suitability for the Olympiad, writing up solutions to ones we solved, and alerting Andy whenever we recognised a problem from the literature. With over 100 problems to plough through in two months, there wasn't time to give every problem thorough treatment, and in fact I probably only gave about half of them enough time. Murphy's law coming into play, about half of those were known to someone else on the committee. The references were sometimes obscure; the two problems I had seen before were from training for the Olympiad in my high school days, and it seemed unlikely that any of today's competitors would have access to any Canadian training sets from eight years before. But then again, if Canada had trained with them, other countries might also, and we easily had enough problems to be able to err on the side of safety.

We weren't initially given the proposers' solutions, basically because we were supposed to try the problems ourselves, and if we could just look at the solutions then we might not give the problems a fair trial. In some cases the problem was extremely scary at first sight, but often there was a simple solution. In some of these cases, though, the proposer had sent in an equally scary solution, which could, in theory, have led to an easy problem showing up on the short list and rated as difficult.

The meeting took place in Edmonton on June 10 and 11. It was generally uneventful, with the procedure simply being to gauge the popularity of those problems which weren't previously known, selecting about forty of the most popular, then whittling that list down to twenty-eight problems (which we judged to be the perfect number for the short list), trying to achieve a variety of problems on different subjects and of varying difficulty. After that, we had to reword some of the problems so that all competitors would be likely to understand them, and write up the solutions in a way that adequately demonstrated the nature of the problem.

The meeting broke up, with the Edmonton members and Claude Laflamme (who was charged with translating the problems into French) forming the Problem Interpolation Group, which had about half a week to get the problems and solutions typed up for the International Jury, which would choose the six problems to be on the paper. We, of course, had to keep silent on which problems had made the short list, and in fact we had to destroy all material related to other problems since they might be resubmitted in future years. (This often happens. Andy recognised a few problems from the list for Hong Kong last year, and Murray Klamkin disqualified one problem with style, something like "This problem is being disqualified for the same reason I disqualified it in 1981, namely that it appeared on the 1975 USAMO.")

I headed to Toronto six weeks later, arriving on Day 1 of the contest. When I found out which problems had been on that day's paper, I thought the selection rather too easy. I wasn't counting on question 2, which we'd ranked as being the easiest one in its category. This was based on the theory that since it had a nice short solution which anyone used to fiddling with inequalities would have little trouble finding, it must rank as an easy problem. The obvious flaw in that reasoning was that not everyone knew how to manipulate inequalities. As a corollary, many competitors tried to use Lagrange multipliers, with only one complete success. This also caused massive headaches for the co-ordinators when it came to assigning part marks to such solutions, especially when an incomplete solution could go on for nine or ten pages.

In fact the first day's paper, while easier for the students, turned out to be harder for the markers. The first problem is a very easy geometry problem (I know because if it's geometry and I can solve it, then by definition it is very easy), but even before marking began, Ed Wang, the problem captain, had seven different solutions, which meant that the markers had to find a marking scheme to deal with them all. The third problem was straightforward, but it had the nuisance of requiring the markers to check that all cases had been dealt with.

By comparison, the second day's problems were substantially more difficult for the students, but for the markers they were more straightforward. There were numerous ways to do problem 4 (which I was marking), but they all had some common elements. Problems 5 and 6 each had only two methods of any merit.

Marking started the day after the contest finished. The co-ordinators were divided into six teams (one for each problem), each with a problem captain. Each team agreed in advance on a marking scheme (which had to be abandoned occasionally when a student came up with an unforeseen method of attack), and then split into pairs for the marking. The leader and deputy leader for each country would bring in their students' papers, with the co-ordinators looking at photocopies which were brought in before the marking started. Since there were 412 papers to go through for each problem and usually only three pairs of co-ordinators to look at them, it was impossible to go through each paper with a fine tooth comb, so in general the markers would get the leaders to give a general idea of what had been done, ask a few salient questions (did the student prove this, or merely take it for granted? etc.), and give a mark based on the scheme. If the leader thought the paper was worth more, some discussion took place, with the problem captain being brought in if the leaders and markers couldn't reach common ground. If that didn't work, the next stage of appeal was Ed Barbeau, and after that the International Jury. In the end nobody appealed to the Jury, although many threatened to.

It is one of the unfortunate (although unavoidable) facets of the IMO that a student's mark depends largely on the leader's ability to present a paper and, sometimes, bully the markers. The threat of appealing to the Jury is probably genuine in some cases, but sometimes it sounds like "We're going to make trouble if you don't give in." Sometimes the markers give in, sometimes not.

Another rather random factor in a student's mark is whether the co-ordinators can read the language the student has used. In my co-ordination team we had people who could, when necessary, read English, French, Italian, Spanish, German, and Russian, but that left numerous other languages, such as Chinese, which the country's leader would have to translate for us. Since verbatim translations were tedious, translations would more often take the form of:

"Here the student proves by induction that the general term has this form."

"And is the induction correct?"

"I believe so."

We had to take their word for it, as there was an average of five minutes allotted for each student on each problem. If the leader hadn't read the paper thoroughly, or had poor English, or missed a point in translation, it could cost the student, but on the other hand, as we couldn't notice minor details, it might also gain a point or two in places.

When the papers were marked was also important. We started marking at about ten o'clock on Friday morning and finished twelve hours later, with about two hours off for meals, then started at about nine o'clock on Saturday and kept it up until we were finished (which varied between about noon and three o'clock, depending on the problem). A team which had to be marked first thing on Friday was at a disadvantage because the markers were sharper and might have had time to look at the solutions before the leaders arrived, whereas ones which came in during the bustle of mid-afternoon were more likely to have the markers miss something, either through tiredness or from wanting to get back on schedule and moving too quickly. However, since each team had its marking spread out over the two days, it tended to balance out.

All three of these factors came into play midway through the session on Friday morning. A team came to my table and said, in a rather arrogant tone, "We believe these are all sevens." (Seven is the top mark for each question.) I didn't like their attitude (which continued unabated throughout the session), so I was extra careful not to let mistakes go by undetected. On the fourth paper I was rewarded by spotting an "iff" where an "only if" should have been, a minor mistake which cost the student a mark. If the leaders hadn't got my back up, or if it had been at a time when I was having trouble concentrating, or if I had been relying on a translation instead of looking at English, it is very doubtful whether I would have picked it up.

Their pushy disposition may have cost them later as well. When word gets around that a team is causing trouble for the markers, they may be met at other sessions by a pair specifically chosen as the least likely to give in. If it then becomes a question of the difference between a 3 and a 4, say, the markers may well opt for the 3 just to avoid giving in. (Of course, in some cases the markers may give in, either to avoid unpleasantness or to keep discussion short. It works both ways.)

Some teams quietly took whatever was offered, some argued hard for marks which they clearly felt were justified, some just seemed to argue for the sake of argument. By the lunch break, we all knew that it was possible to go from a very polite session where all the marks were fairly clearcut to a long, drawn-out, no-holds barred debate which would only be settled by bringing in the problem captain and sometimes Ed. The worst was yet to come, though.

My afternoon went fairly smoothly, all in all, with only the occasional minor dispute. Meanwhile, though, there was one stretch where the three other tables were occupied by the same teams for a full hour, where each was only supposed to have half an hour. Even when they left, some were going to come back later, when missing sheets were looked for or some other problem investigated. (It was not uncommon for a telling piece of scrap paper to wind up in the wrong room, or in one case for the student to have put the paper in the folder in such a way that the photocopying staff didn't see it. With around 10000 sheets of paper to photocopy, it's not surprising that there was an occasional glitch.) After they had left, the table in the far corner came in for the worst treatment yet. It was a session of approximately two and a half hours, with one marker loving to argue, and the coaches not having perfect English. I sat in briefly when one marker got too tired, and I spent five minutes in a cross-purposes argument over whether something was true, because the translation left something to be desired. Specifically, the statement given to me was that under certain conditions, including $x_i = x_j$, one of the following would hold: $x_i > 1 > x_j$, $x_i < 1 < x_j$, or $x_i = x_j = 1$. In fact, what the leader had translated as "x_i is greater than 1 and x_j is less than 1" was supposed to be "there is a 1 in the sequence between x_i and x_j." I had just sorted that out when the other marker returned and rescued me.

After that team had finally left, it was getting near supper, but that table decided to take on one more before the break. They should have known better. An hour later, they called for an intermission, and when they came back, the discussion resumed for another half hour.

Supper was a chance to discuss things with the other marking teams. The favourite topic seemed to be the qualifications of some of the leaders. Many had staged arguments which were quite clearly done with a lack of mathematical background. In our question, the winner had to be the leader who expected one of his students to receive a full seven for a solution which led to the answer of 1. Considering that this is marginally different from 2^{997} [the right answer], he didn't win the argument, but at least the student could have the satisfaction of knowing he'd predicted his own mark. More entertaining, though, were the stories of having to explain to the leaders that their students deserved more than they were asking for. I'd only had one of these come through, where the leader asked for a 3 or 4 and I overruled him and gave a 5. We were all quite impressed by the marker who had given 4 when only 1 was asked. Everyone, that is, except the marker who'd been confronted by, "This one's about a 4, this one's a 3, and the rest aren't worth anything." He decided to check the zeroes just in case he could raise one to a 1, and he discovered that one of them was actually worth the full seven marks.

In the evening, we were all getting a bit worn down, but the teams must have been as well because there weren't so many arguments. We had to switch our marking pairs around because some of us couldn't face another session immediately, and I ended the evening with one of the guys from the far corner. The relative calm of the evening did us some good, and when the Moroccans came in just when we were about to leave, we decided we could face one more team. My brain was now ready to make use of its accumulated knowledge from the day, and I discovered that after seeing so many papers go by, I could more or less discern simply from the arrangement of the mathematical expressions on a page which of the approaches a student was using. This was easier in French (which half the Moroccan team used) because I could also understand some of the words, but I could also manage it in Arabic (which the other half used), and this saved a bit of time because although the leader would willingly translate from French into English, when he tried translating from Arabic, he would occasionally use a more familiar French word than the English one. ("a_i is impair parce que if it was even... .")

The following day I snapped, along with a few others who succumbed to the pace. In my case, it was a "Bourbaki" solution which finished me. It nearly did my partner in as well, who threw up his hands and declared that there wasn't any point in checking all the little details because anyone who had mastered Bourbaki's style was sure to have all the details correct, even though the writing gave no clue as to where in the solution a detail might be. (It was also this partner who defeated an argument from one team leader that a certain point was so obvious it shouldn't need stating by saying, "One of the Bulgarians has been participating for four years. He now knows that there can never be too many details in an IMO solution. If you want to see where in his paper he proves that 1995 is odd, it's in there.") After Bourbaki I had to go for coffee, leaving the team leaders waiting with three more papers to grade.

When I got back, things didn't improve much. Lying in wait two papers on was the Banach-Tarski proof. That was the only way I could think of describing it at the time. It was essentially two different solutions interlaced; if you pulled all the bits apart and stuck them back together in a different way, you would have two solutions. Of course, it took a while to realise that there was actually a solution buried there; it looked more like random correct statements leading nowhere, but we spent so long trying to figure out what it was worth that the pattern emerged.

I was almost finished later when Thailand came through. Most of the solutions were nice, but one was astounding for its lack of clarity. Of course, it didn't help that it was written in Chinese, or something which might as well have been for all I could make of it, but the leader diligently translated as best she could, continually apologising for how much of a mess it was. I wasn't even at the worst part when I sweetly enquired whether she would mind bringing in the student when we'd finished so that I could shoot him. She replied, "We already did. We spent three hours last night going over this, and this morning I kicked him for it." I asked her to kick him for me as well, and she assured me she would.

When it was all over, though, we all sat back and relaxed, congratulating ourselves on what we thought was a fairly good job, discussing the strangest of the incidents, which teams were nice, and who had received the nicest tokens from team leaders. (Some teams passed out souvenirs of their countries to the markers, generally pins or stickers, and always after the marking so that we couldn't think it was a bribe. Maybe it's different in other countries, but I think it would take quite a bit more than a sticker to bribe a Canadian.) We then waited for the results to come in, which they finally did at the Jury Meeting. That was accompanied by brief chats with leaders and other markers, discussing what everyone thought of the problems and our marking. There were no fistfights, so I guess we did all right.

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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

[*Preamble by the Reviews Editor.* Although most of the books we review are in English, occasionally we come across books of exceptional merit in other languages. The following is an example. If the review generates sufficient interest, perhaps some publisher may translate the book into English. At the moment, only a Chinese edition exists. For ordering information, please write to the Reviews Editor.]

A New Approach to Plane Geometry (in Chinese), by J. Z. Zhang. Published in 1992 by the Sichuan Educational Publishers. ISBN 7-5408-1611-2, paperback, 434+ pages. *Reviewed by Andy Liu.*

This book promotes using area as a basic tool for solving problems in plane geometry. We use $[F]$ to denote the area of the figure F. Chapter One sets the scene and contains the following result.

Common-height Lemma. *If A, B, C and D lie on a line not passing through P, then*

$$\frac{[PAB]}{[QAB]} = \frac{AB}{CD}\,.$$

This follows easily from the formula for the area of a triangle.

Chapter Two contains the two principal results of the book.

Common-side Theorem. *Triangles PAB and QAB share the common side AB. If PQ intersects AB at M, then*

$$\frac{[PAB]}{[QAB]} = \frac{PM}{QM}\,.$$

Common-angle Theorem. *If angles ABC and $A'B'C'$ are either equal or supplementary, then*

$$\frac{[ABC]}{[A'B'C']} = \frac{AB \cdot BC}{A'B' \cdot B'C'}\,.$$

The first follows from the above lemma, and the second can be derived from the first. The rest of the chapter contains numerous applications of these two results, including proofs of the theorems of Ceva, Menelaus and Routh. We give a sample here.

Problem. If K, L and M are points on the sides BC, CA and AB of triangle ABC respectively, prove that the minimum of $[ALM]$, $[BMK]$ and $[CKL]$ is at most $[ABC]/4$.

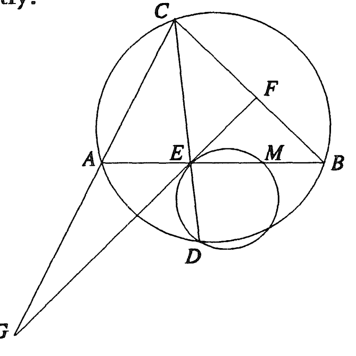
Solution: Let $AM = rAB$, $BK = sBC$ and $CL = tCA$, where r, s and t are real numbers between 0 and 1. From the Common-angle Theorem,

$$\frac{[ALM]}{[ABC]} \cdot \frac{[BMK]}{[ABC]} \cdot \frac{[CKL]}{[ABC]} = \frac{AM \cdot AL}{AB \cdot AC} \cdot \frac{BK \cdot BM}{BC \cdot BA} \cdot \frac{CL \cdot CK}{CA \cdot CB}$$
$$= r(1-t) \cdot s(1-r) \cdot t(1-s)$$
$$= r(1-r) \cdot s(1-s) \cdot t(1-t) \leq \left(\frac{1}{4}\right)^3.$$

Hence the minimum of $[ALM], [BMK]$ and $[CKL]$ is at most $[ABC]/4$. □

The remaining three chapters apply the method of area to geometric computations, geometric constructions and the study of circles, regular polygons and other geometric figures with a certain degree of symmetry. We give an example which does not involve area explicitly.

Problem. Two chords AB and CD of a circle intersect at the point E. M is an arbitrary point on BE. The tangent at E to the circumcircle of triangle MED intersects CB at F and the extension of CA at G. Prove that $EG/EF = MA/MB$.



Solution: By the Common-height Lemma,

$$\frac{EB}{EA} = \frac{[CBE]}{[ACE]} = \frac{[FCE] + [FEB]}{[GCE] - [AGE]}.$$

Note that $\angle CEF = \angle DEG = \angle EMD$ and $\angle GEA = \angle FEB = \angle EDM$. Dividing the numerator and denominator by $[MED]$ and applying the Common-angle Theorem, we have

$$\frac{EB}{EA} = \frac{\frac{EC \cdot EF}{MD \cdot EM} + \frac{EB \cdot EF}{MD \cdot ED}}{\frac{EC \cdot EG}{MD \cdot EM} - \frac{EA \cdot EG}{MD \cdot ED}} = \frac{EF(EC \cdot ED + EB \cdot EM)}{EG(EC \cdot ED - EA \cdot EM)}.$$

Since $EC \cdot ED = EA \cdot EB$, this reduces to

$$\frac{EB}{EA} = \frac{EF \cdot EB(EA + EM)}{EG \cdot EA(EB - EM)} = \frac{EF \cdot EB \cdot MA}{EG \cdot EA \cdot MB},$$

from which the desired result follows. □

The book presents an elegant and systematic treatment of elementary geometry that is both refreshing and easily accessible. The unifying theme of area makes the approach to various problems very easy and natural. It is highly recommended.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1996**, *although solutions received after that date will also be considered until the time when a solution is published.*

2061. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with centroid G, and P is a variable interior point of ABC. Let D, E, F be points on sides BC, CA, AB respectively such that $PD \parallel AG$, $PE \parallel BG$ and $PF \parallel CG$. Prove that $[PAF] + [PBD] + [PCE]$ is constant, where $[XYZ]$ denotes the area of triangle XYZ.

2062. *Proposed by K. R. S. Sastry, Dodballapur, India.*

Find a positive integer n so that both the continued roots

$$\sqrt{1995 + \sqrt{n + \sqrt{1995 + \sqrt{n + \cdots}}}} \quad \text{and} \quad \sqrt{n + \sqrt{1995 + \sqrt{n + \sqrt{1995 + \cdots}}}}$$

converge to positive integers.

2063. *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Triangle ABC has a right angle at C.

(a) Prove that the three ellipses having foci at two vertices of the given triangle, while passing through the third, all share a common point.

(b) Prove that the principal vertices of the ellipses of part (a) (i.e., the points where an ellipse meets the axis through its foci) form two pairs of collinear triples.

2064. *Proposed by Murray S. Klamkin, University of Alberta.*

Show that

$$3\max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

for arbitrary positive real numbers a, b, c.

2065. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find a monic polynomial $f(x)$ of lowest degree and with integer coefficients such that $f(n)$ is divisible by 1995 for all integers n.

2066. *Proposed by John Magill, Brighton, England.*

The inhabitants of Rigel III use, in their arithmetic, the same operations of addition, subtraction, multiplication and division, with the same rules of manipulation, as are used by Earth. However, instead of working with base 10, as is common on Earth, the people of Rigel III use a different base, greater than 2 and less than 10.

```
     BC
AB)CBC
   AB
   BDC
   BDC
   ----
```

This is the solution to one of their long division problems, which I have copied from a school book. I have substituted letters for the notation originally used. Each of the letters represents a different digit, the same digit wherever it appears.

As the answer to this puzzle, substitute the correct numbers for the letters and state the base of the arithmetic of Rigel III.

2067. *Proposed by Moshe Stupel and Victor Oxman, Pedagogical Religious Girls' College "Shanan", Haifa, Israel.*

Triangle ABC is inscribed in a circle Γ. Let AA_1, BB_1 and CC_1 be the bisectors of angles A, B and C, with A_1, B_1 and C_1 on Γ. Prove that the perimeter of the triangle is equal to

$$AA_1 \cos\frac{A}{2} + BB_1 \cos\frac{B}{2} + CC_1 \cos\frac{C}{2}\,.$$

2068. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Find all real solutions of the equation

$$\sqrt{17 + 8x - 2x^2} + \sqrt{4 + 12x - 3x^2} = x^2 - 4x + 13.$$

2069. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

M is a variable point of side BC of triangle ABC. A line through M intersects the lines AB in K and AC in L so that M is the midpoint of segment KL. Point K' is such that $ALKK'$ is a parallelogram. Determine the locus of K' as M moves on segment BC.

2070. *Proposed by Joaquín Gómez Rey, I. B. Luis Buñuel, Alcorcón, Madrid, Spain.*

For which positive integers n is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}$$

odd?

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1754*. [1992: 175; 1993: 151; 1994: 196] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be positive integers such that $2 \le k < n$, and let $x_1, \ldots, x_n$ be nonnegative real numbers satisfying $\sum_{i=1}^{n} x_i = 1$. Prove or disprove that

$$\sum x_1 x_2 \ldots x_k \le \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1, x_2, \ldots, x_n$. [The case $k = 2$ is known — see inequality (1) in the solution of *Crux* 1662 [1992: 188].]

II. *Partial solution by Marcin E. Kuczma, Warszawa, Poland.*

If $k \le 4$, the bound $1/n^{k-1}$ in the two cases missing in Pompe's partial solution [1994: 196] can be proved by multivariable calculus as follows.

$k = 3$, $n = 5$. The continuous function

$$F(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$$

attains its maximum value at some point of the compact simplex

$$\Delta = \left\{(x_1, x_2, x_3, x_4, x_5):\ x_i \ge 0,\ \ \sum x_i = 1\right\}.$$

At any local extremum point inside Δ (i.e., point with all $x_i > 0$), all the partial derivatives of F are equal, according to the Lagrange multiplier theorem. The equality between

$$\frac{\partial F}{\partial x_1} = x_4x_5 + x_5x_2 + x_2x_3 \qquad \text{and} \qquad \frac{\partial F}{\partial x_4} = x_2x_3 + x_3x_5 + x_5x_1$$

implies (in view of $x_5 > 0$)

$$x_1 + x_3 = x_2 + x_4.$$

Thus, by cyclicity, all sums $x_i + x_{i+2}$ are equal, hence all the x_i's must be equal. So $(1/5, \ldots, 1/5)$ is the only possible extremum point inside Δ.

Considering the face $x_5 = 0$ of Δ, we have

$$F(x_1, x_2, x_3, x_4, 0) = (x_1 + x_4)x_2x_3 \le \left(\frac{(x_1 + x_4) + x_2 + x_3}{3}\right)^3 = \frac{1}{27}.$$

Therefore $F(1/5, \ldots, 1/5) = 1/5^2 = 1/25$ is the maximum value of F on Δ.

$k = 4$, $n = 6$. The method is the same, the details are more involved. Now we are maximizing

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2x_3x_4 + \cdots \quad [\text{cycl}] \quad \cdots + x_6x_1x_2x_3$$

on

$$\Delta = \left\{(x_1, x_2, x_3, x_4, x_5, x_6)\colon\ x_i \ge 0,\ \ \sum x_i = 1\right\}.$$

Suppose $(x_1, \ldots, x_6)$ is a local extremum point with $x_i > 0$, $i = 1, \ldots, 6$. The equality between

$$\frac{\partial F}{\partial x_3} = x_6x_1x_2 + x_1x_2x_4 + x_2x_4x_5 + x_4x_5x_6 \tag{1}$$

and

$$\frac{\partial F}{\partial x_5} = x_2x_3x_4 + x_3x_4x_6 + x_4x_6x_1 + x_6x_1x_2 \tag{2}$$

implies (in view of $x_4 > 0$)

$$x_1x_2 + x_2x_5 + x_5x_6 = x_2x_3 + x_3x_6 + x_6x_1.$$

By cyclicity, this can be produced to

$$\cdots = x_3x_4 + x_4x_1 + x_1x_2 = x_4x_5 + x_5x_2 + x_2x_3.$$

Look at the beginning and the end of this chain of equalities; the term x_2x_5 cancels and we obtain

$$x_1x_2 - x_4x_5 = x_2x_3 - x_5x_6.$$

Again, this produces to

$$\cdots = x_3x_4 - x_6x_1 = x_4x_5 - x_1x_2.$$

Consequently, all these differences must be equal to zero, and we get

$$x_1x_2 = x_4x_5, \qquad x_2x_3 = x_5x_6, \qquad x_3x_4 = x_6x_1.$$

Thus the ratios $x_1/x_4, x_5/x_2, x_3/x_6$ are equal. Denoting their common value by q and inserting $x_1 = qx_4$, $x_5 = qx_2$, $x_3 = qx_6$ into the equality $\partial F/\partial x_3 = \partial F/\partial x_5$ (see (1) and (2)), we obtain

$$q(2x_2x_4x_6 + x_2x_4^2 + x_2^2x_4) = q(2x_2x_4x_6 + x_4x_6^2 + x_4^2x_6).$$

Hence $x_2x_4 + x_2^2 = x_6^2 + x_4x_6$, i.e.,

$$(x_2 - x_6)(x_2 + x_4 + x_6) = 0,$$

showing that $x_2 = x_6$. By cyclicity,

$$x_2 = x_4 = x_6 =: c, \qquad x_1 = x_3 = x_5 = qc,$$

which inserted into (1) and, say, into the formula expressing $\partial F/\partial x_2$ gives us

$$\frac{\partial F}{\partial x_3} = 4qc^3, \qquad \frac{\partial F}{\partial x_2} = 4q^2c^3.$$

These derivatives have to be equal. Thus, finally, $q = 1$ and $x_i = c$ for all i. So $(1/6, \ldots, 1/6)$ is the only possible extremum point inside Δ.

On the face $x_6 = 0$ of Δ,

$$\begin{aligned} F(x_1, x_2, x_3, x_4, x_5, 0) &= (x_1 + x_5)x_2x_3x_4 \\ &\le \left(\frac{(x_1 + x_5) + x_2 + x_3 + x_4}{4}\right)^4 = \frac{1}{256}. \end{aligned}$$

Therefore $F(1/6, \ldots, 1/6) = 1/6^3 = 1/216$ is the maximum value of F on Δ.

Remark. It seems that this method does not give much hope for a quick generalization. Already the calculations one comes across in trying to handle the case of $n = 7$, $k = 5$ are rather disgusting. The only case where the situation is really clear is that of $k = n - 1$, the cyclic sum coinciding with the full symmetric form, maximized when all x_i's are equal, on account of the Geometric Mean-Harmonic Mean Inequality.

[*Editor's note.* This completes the solution of the problem when $k \le 4$. For $k \ge 5$, the problem is still mostly unsolved. For example, for $k = 5$ the answer is not yet known for $n = 7$ or 8, or in fact for any $n > 5$ satisfying $n \equiv 2$ or 3 mod 5.]

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1969. [1994: 195] *Proposed by Jerzy Bednarczuk, Warszawa, Poland.*

We have two parallelepipeds whose twelve faces are all congruent rhombi. Must these parallelepipeds be congruent?

Solution by Richard I. Hess, Rancho Palos Verdes, California.

The parallelepipeds need not be congruent! For example, consider a stick cube with hinged vertices. Pull or push two diagonally opposite vertices directly toward or away from each other to arbitrarily squash or elongate the cube into a parallelepiped with congruent rhombic faces.

More mathematically, consider unit vectors **A**, **B** and **C** such that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} = \cos\theta$$

where $0 < \theta < 120°$. Then the vertices are

$$\mathbf{0}, \quad \mathbf{A}, \quad \mathbf{B}, \quad \mathbf{C}, \quad \mathbf{A}+\mathbf{B}, \quad \mathbf{A}+\mathbf{C}, \quad \mathbf{B}+\mathbf{C}, \quad \mathbf{A}+\mathbf{B}+\mathbf{C}.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; and the proposer.

This problem (and the later Crux 1995*) were forwarded to the editor by the proposer's student Waldemar Pompe, who informs us that the proposer's nickname is "Proof", pronounced "prof"!*

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1970. [1994: 195] *Proposed by Susan Gyd, Nose Hill, Alberta.*

Find an isosceles triangle and a rectangle, both with integer sides, which have the same area and the same perimeter.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton.

Let $2b, c, c$ be the lengths of the sides of the isosceles triangle, and h, k those of the rectangle. Clearly,

(i) $b = h+k-c$ [since the perimeters $2b+2c$ and $2h+2k$ of the triangle and rectangle are equal—*Ed.*], and therefore b is an integer;

(ii) the height a of the isosceles triangle should be rational [since the areas ab and hk of the triangle and rectangle are equal] and therefore an integer [since $a^2 = c^2 - b^2$ is an integer]; consequently,

(iii) the isosceles triangle is made up of two congruent Pythagorean triangles;

(iv) we may assume that the sides of the Pythagorean triangle are relatively prime. There are integers m and n, relatively prime, and of different parity, such that either

(a) $c = m^2 + n^2$, $b = m^2 - n^2$, $a = 2mn$, or

(b) $c = m^2 + n^2$, $b = 2mn$, $a = m^2 - n^2$.

In each case, the area of the isosceles triangle is $2mn(m^2 - n^2)$.

In (a), the perimeter is $4m^2$. This means

$$h + k = 2m^2, \quad hk = 2mn(m^2 - n^2).$$

From this, $(h-k)^2 = 4[m^4 - 2mn(m^2 - n^2)]$. It is easy to see that $m = l^2$, l odd, or $m = 2l^2$. For $m < 10000$, the only possibilities are (i) and (iii) in the table below.

In (b), the perimeter is $2(h+k) = 2(c+b) = 2(m+n)^2$. Again, $hk = 2mn(m^2 - n^2)$. It follows that $(h-k)^2 = (m+n)^4 - 8mn(m^2 - n^2)$. From this, $m+n$ must be a square. For $m < 10000$, the only solutions are (ii), (iv), (v) in the table below.

Isosceles triangles and rectangles with equal areas and perimeters

	Parameters		Triangle		Rectangle	
	m	n	c	$2b$	h	k
(i)	2	1	5	6	6	2
(ii)	7	2	53	56	60	21
(iii)	49	44	4337	930	4340	462
(iv)	532	429	467065	912912	51832	871689
(v)	7645	2964	67231321	90639120	26004616	86546265

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, New York; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D. J. SMEENK, Zaltbommel, The Netherlands; DAVID VELLA, Skidmore College, Saratoga Springs, New York; and the proposer. There were also two incorrect solutions sent in.

Most solvers found only the smallest solution (the 6, 5, 5 *triangle and the* 2 *by* 6 *rectangle), although many of them also remarked that one may multiply all sides by a constant to obtain other solutions. Other solvers found some or all of the solutions given by Yiu. Guy in fact found one more, involving even larger numbers, and has shown that there are infinitely many. He has also written an article ("My favorite elliptic curve: a tale of two types of triangle", to appear shortly in the* American Math. Monthly*), in which he solves this problem and an unrelated one which reduce to the same elliptic curve.*

Interestingly, there is (according to Guy) **no** *example of a right triangle and a rectangle with integer sides and the same area and same perimeter.*

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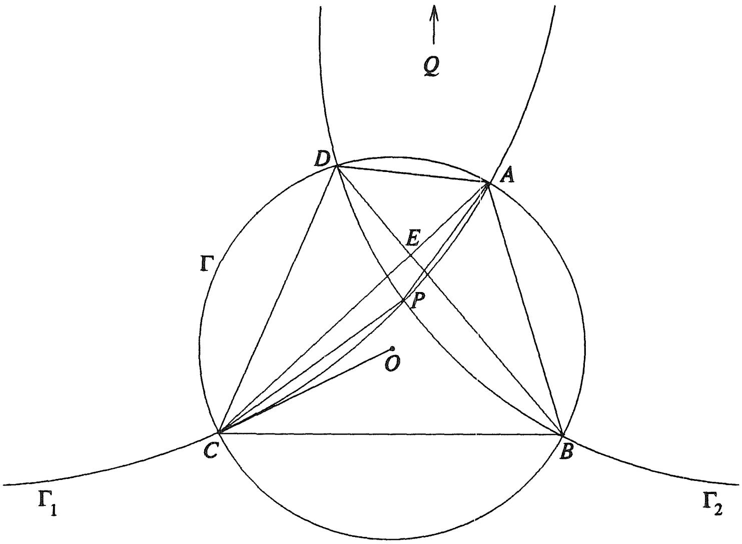
1971. [1994: 225] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O, and E is the intersection of diagonals AC and BD. Let P be an interior point of $ABCD$ such that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90°.$$

Prove that O, P and E are collinear.

I. *Solution by D. J. Smeenk, Zaltbommel, The Netherlands.*



Let the circumcircle of $ABCD$ be Γ. We denote $\angle DAB = \alpha$ and $\angle ABC = \beta$. Then

$$\angle APC = \angle PAB + \angle PCB + \angle ABC = \frac{\pi}{2} + \beta.$$

The locus of the point P_1 with the property that $\angle AP_1C = \pi/2 + \beta$ is (an arc of) a circle Γ_1 passing through A and C and touching OC at C. [*Editor's note.* Since

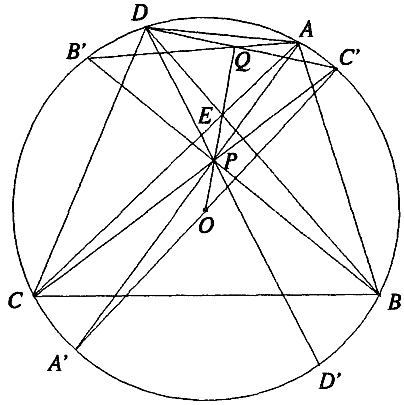
$$\angle APC + \angle ACO = \frac{\pi}{2} + \beta + \frac{\pi}{2} - \beta = \pi,$$

OC must be tangent to Γ_1 at C.] In the same way, $\angle BPD = \pi/2 + \alpha$, and the locus of the point P_2 with the property that $\angle BP_2D = \pi/2 + \alpha$ is (an arc of) a circle Γ_2 passing through B and D and touching OB at B. Γ_1 and Γ_2 intersect in P and Q (see the figure) but only P has the property that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = \frac{\pi}{2}. \tag{1}$$

Now we'll show that O, E and P are collinear. Consider the circles Γ, Γ_1 and Γ_2. The radical axis of Γ and Γ_1 is AC. The radical axis of Γ and Γ_2 is BD. So E is the radical point of Γ, Γ_1 and Γ_2 [and thus E lies on the radical axis of Γ_1 and Γ_2]. O is a point of this axis as well; OC is tangent to Γ_1 at C, and OB is tangent to Γ_2 at B, and $OC = OB$. In other words, OE is the radical axis of Γ_1 and Γ_2, and the intersection points P and Q of these circles are lying on OE. Only P satisfies (1).

II. *Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge.*



Extend AP, BP, CP, DP to meet the circle in points A', B', C', D'. Then $\angle PAB + \angle PCB = 90°$ implies that

$$\frac{\operatorname{arc} A'B}{2} + \frac{\operatorname{arc} BC'}{2} = 90°,$$

so $\operatorname{arc} A'C' = 180°$. Therefore $A'C'$ is a diameter of the circle, and similarly so is $B'D'$. Thus $A'C' \cap B'D' = O$. Let

$$AB' \cap DC' = Q.$$

Using Pascal's Theorem on the hexagon $BB'ACC'D$ gives that

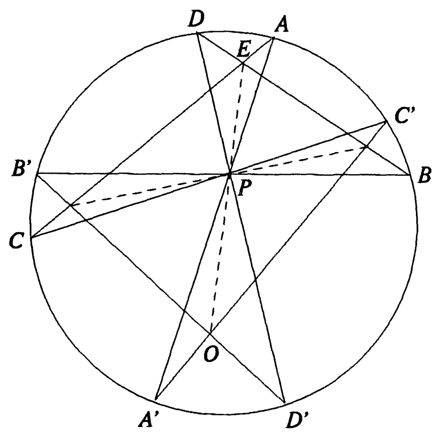
$$BB' \cap CC' = P, \quad BD \cap AC = E, \quad AB' \cap C'D = Q$$

are collinear, while using it on the hexagon $AA'C'DD'B'$ gives that

$$D'B' \cap A'C' = O, \quad D'D \cap AA' = P, \quad C'D \cap AB' = Q$$

are collinear. Since P, E, Q are collinear and O, P, Q are collinear, it follows that O, P, E are collinear.

Also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; and the proposer.



Konečný points out that (to borrow Ardila's notation) the problem reduces to two intersecting "butterflies" $ACC'A'$ and $BDD'B'$, inscribed in the same circle and with the same intersection point P. In this situation it is known that the other points of intersection of the butterflies are collinear with P as in the diagram. A proof (courtesy of Chris Fisher): transform the circle to an ellipse so that P projects to its centre, and by symmetry the result follows. For more about butterflies, see Léo Sauvé's article on [1976: 2-5].

Question: is the condition $AC \neq BD$ needed?

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1972. [1994: 225] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Define a sequence $a_0, a_1, a_2, \ldots$ of nonnegative integers by: $a_0 = 0$ and

$$a_{2n} = 3a_n, \qquad a_{2n+1} = 3a_n + 1 \qquad \text{for } n = 0, 1, 2, \ldots .$$

(a) Characterize all nonnegative integers n so that there is exactly **one** pair (k, l) satisfying

$$k > l \qquad \text{and} \qquad a_k + a_l = n. \tag{1}$$

(b) For each n, let $f(n)$ be the number of pairs (k, l) satisfying (1). Find

$$\max_{n < 3^{1972}} f(n).$$

Combination of solutions by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario, and Pavlos B. Konstadinidis, student, University of Arizona, Tucson.

Let $\mathbb{N}$ denote the set of nonnegative integers, and let $A = \{a_i\}_{i \geq 0}$. For $n, N \in \mathbb{N}$, $N \geq 2$, let's write $n = (d_m, d_{m-1}, \ldots, d_1, d_0)_N$ for the base N representation of n, that is, if $n = \sum_{j=0}^{m} d_j N^j$ with $d_j \in \{0, 1, \ldots, N-1\}$ for all $j \in \{0, 1, \ldots, m\}$.

Lemma. For every $i \in \mathbb{N}$,

$$\text{if} \quad i = (b_m, b_{m-1}, \ldots, b_1, b_0)_2 \quad \text{then} \quad a_i = (b_m, b_{m-1}, \ldots, b_1, b_0)_3.$$

Proof. By induction on i. It's true for $i = 0$. Supposing the result valid for i, let's prove the result for $2i$ and $2i+1$. Writing $i = (b_m, b_{m-1}, \ldots, b_1, b_0)_2$, we have that

$$2i = (b_m, b_{m-1}, \ldots, b_1, b_0, 0)_2 \quad \text{and} \quad 2i + 1 = (b_m, b_{m-1}, \ldots, b_1, b_0, 1)_2,$$

and by assumption

$$a_i = (b_m, b_{m-1}, \ldots, b_1, b_0)_3,$$

which implies that

$$a_{2i} = 3a_i = (b_m, b_{m-1}, \ldots, b_1, b_0, 0)_3$$

and

$$a_{2i+1} = 3a_i + 1 = (b_m, b_{m-1}, \ldots, b_1, b_0, 1)_3,$$

as claimed. $\square$

Therefore, we have proved that *A is the sequence of nonnegative integers that have only digits 0 and 1 in their base 3 representation.*

We're now ready for part (a).

(a) Note that there are no carries in the addition of the base 3 values of any a_k and a_l, for the maximum sum in each column is $1 + 1 = 2$.

For $a_k + a_l = n$, note that the digits 0 and 2 in n have "unique splits", i.e., if n contains 0 then a_k and a_l both have 0 in the corresponding digit, and if n contains 2 then a_k and a_l both have 1 in the corresponding digit. However, if n contains 1 then the corresponding digit in one of a_k or a_l is 1 and the other 0.

If n contains only one 1 then the split $n = a_k + a_l$ is unique such that a_k is bigger than a_l. However, if there is more than one 1 in n then we have $n = \_\_\_\_1\_\_\_\_1\_\_\_\_$, and so

$$\left\{\begin{array}{l} a_k = \_\_\_\_1\_\_\_\_0\_\_\_\_ \\ a_l = \_\_\_\_0\_\_\_\_1\_\_\_\_ \end{array}\right\} \text{ or } \left\{\begin{array}{l} a_k = \_\_\_\_1\_\_\_\_1\_\_\_\_ \\ a_l = \_\_\_\_0\_\_\_\_0\_\_\_\_ \end{array}\right\}.$$

Both of these situations are possible whether or not there are other digits.

Finally, if n does not contain any 1's then the split into a_k and a_l is unique and further it is the same, so that $k = l$, which is not possible if $k > l$. This means that the nonnegative integers n satisfying part (a) **are exactly those whose base 3 representation contains exactly one 1.**

(b) Again consider n in base 3. As shown before, if any of the digits are 0 or 2 then they must be split in exactly one way. Thus it's clear that

$$\max_{n<3^{1972}} f(n) = f(t),$$

where t is the integer in $\{0, 1, \ldots, 3^{1972} - 1\}$ that has the greatest number of digits 1 in its base 3 representation, namely

$$t = 3^{1971} + 3^{1970} + \cdots + 1 \quad \left(= (\overbrace{1, 1, \ldots, 1}^{1972})_3\right).$$

There are 2^{1972} possibilities for the 1972 digits of a_k to be 0 or 1, and each digit of a_l is the opposite value of the digit of a_k. This means that there are

$$2^{1972}/2 = 2^{1971}$$

pairs (a_k, a_l) with $k > l$ such that $a_k + a_l = t$. Thus

$$\max_{n<3^{1972}} f(n) = f(t) = 2^{1971}.$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales; CHRIS WILDHAGEN, Rotterdam, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. One incorrect solution was sent in.

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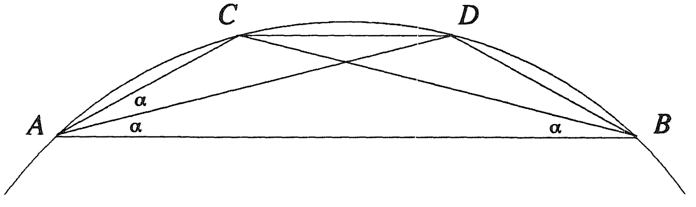
1973. [1994: 226] *Proposed by K. R. S. Sastry, Dodballapur, India.*

Triangle ABC is inscribed in a circle. The chord AD bisects $\angle BAC$. Assume that $AB = \sqrt{2}\,BC = \sqrt{2}\,AD$. Determine the angles of ΔABC.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.



Since $BC = AD$, the cyclic quadrilateral $ABDC$ must be an isosceles trapezoid. So either $AC = BD$ or $AB = CD$. The diagonal of an isosceles trapezoid is greater than its arm, while here $AB = \sqrt{2}\,BC$, so the latter case is not possible. Therefore $AC = BD$ (see the figure). D is the midpoint of the arc BC, so that $BD = CD$. Thus $AC = BD = CD = x$. Let $a = BC$ ($= AD$). Then $AB = a\sqrt{2}$. Using the Ptolemy theorem on $ABCD$ we get $AC \cdot BD + AB \cdot CD = AD \cdot BC$, or $x^2 + xa\sqrt{2} - a^2 = 0$, which gives

$$\frac{x}{a} = \frac{\sqrt{6}-\sqrt{2}}{2}.$$

Let $\alpha = \angle ABC$. Clearly $\angle CAB = 2\angle ABC = 2\alpha$. Using the law of sines on ΔABC we obtain

$$2\cos\alpha = \frac{\sin 2\alpha}{\sin\alpha} = \frac{a}{x} = \frac{2}{\sqrt{6}-\sqrt{2}},$$

which gives $\cos\alpha = (\sqrt{6}+\sqrt{2})/4$. Hence

$$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{8+2\sqrt{12}}{8} - 1 = \frac{\sqrt{3}}{2},$$

or equivalently $2\alpha = 30°$. Therefore the angles of the triangle ABC are

$$15°, \qquad 30° \qquad \text{and} \qquad 135°.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CHENG YU, student, Southwest Missouri State University, Springfield; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain;

TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; L. J. UPTON, Mississauga, Ontario; JOHN VLACHAKIS, Athens, Greece; and the proposer. There were three incorrect solutions sent in.

Engelhaupt notes that the points A, C, D, B form four consecutive vertices of a regular dodecagon.

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1974. [1994: 226] *Proposed by Neven Jurić, Zagreb, Croatia.*

Prove or disprove that

$$\sqrt{5+\sqrt{21}}+\sqrt{8+\sqrt{55}}=\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}.$$

Solution by Aaron Tievsky, student, Thomas W. Pyle Middle School, Bethesda, Maryland.

In order to simplify the radicals, the radicands should be forced to equal square numbers (e.g., $7+\sqrt{33}$ should be a square of some number). Numbers whose squares have a rational and radical part are usually in the form $a+b$. So let

$$\sqrt{7+\sqrt{33}}=a+b=\sqrt{(a+b)^2}=\sqrt{a^2+b^2+2ab},$$

and set

$$a^2+b^2=7 \quad \text{and} \quad 2ab=\sqrt{33}, \quad \text{i.e. } b=\frac{\sqrt{33}}{2a}.$$

Thus

$$a^2+\left(\frac{\sqrt{33}}{2a}\right)^2=7$$

which multiplying by $4a^2$ gives

$$(2a^2-3)(2a^2-11)=4a^4+33-28a^2=0.$$

So $2a^2=3$ or $2a^2=11$, i.e.

$$a=\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}, \quad b=\frac{\sqrt{33}}{\sqrt{6}}=\frac{\sqrt{22}}{2}$$

or

$$a=\sqrt{\frac{11}{2}}=\frac{\sqrt{22}}{2}, \quad b=\frac{\sqrt{6}}{2},$$

and so

$$\sqrt{7+\sqrt{33}}=a+b=\frac{\sqrt{6}+\sqrt{22}}{2}.$$

Using the same process for the other radicals, we get

$$\sqrt{6+\sqrt{35}}=\frac{\sqrt{10}+\sqrt{14}}{2},$$

so

$$\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}=\frac{\sqrt{6}+\sqrt{22}+\sqrt{10}+\sqrt{14}}{2}\ ; \tag{1}$$

and

$$\sqrt{8+\sqrt{55}}=\frac{\sqrt{10}+\sqrt{22}}{2}\ , \quad \sqrt{5+\sqrt{21}}=\frac{\sqrt{6}+\sqrt{14}}{2}\ ,$$

so

$$\sqrt{8+\sqrt{55}}+\sqrt{5+\sqrt{21}}=\frac{\sqrt{10}+\sqrt{22}+\sqrt{6}+\sqrt{14}}{2}\ . \tag{2}$$

From (1) and (2),

$$\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}=\sqrt{8+\sqrt{55}}+\sqrt{5+\sqrt{21}}\ .$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; LEON BANKOFF, Beverly Hills, California; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CHENG YU, student, Southwest Missouri State University, Springfield; TIM CROSS, Wolverley High School, Kidderminster, U. K.; C. R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SOLOMON W. GOLOMB, University of Southern California, Los Angeles; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; LARRY HOEHN, Austin Peay State University, Clarksville, Tennessee; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; I. KLIMANN, student, Université de Paris 6, France; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; J. A. MCCALLUM, Medicine Hat, Alberta; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Ytre Laksevaag, Norway; VICTOR OXMAN, Haifa University, Israel; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Spain; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; JOHN VLACHAKIS, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; CHRISTIAN WOLINSKI, Halifax, Nova Scotia; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer.

This problem is identical to problem Q601 *in the journal* Parabola *(Vol.* 20, 1984, *p.* 26*), and is listed on page* 61 *of Stanley Rabinowitz's* Index to Mathematical Problems 1980-1984. *(And will the editor ever learn to consult this book before using a problem?)*

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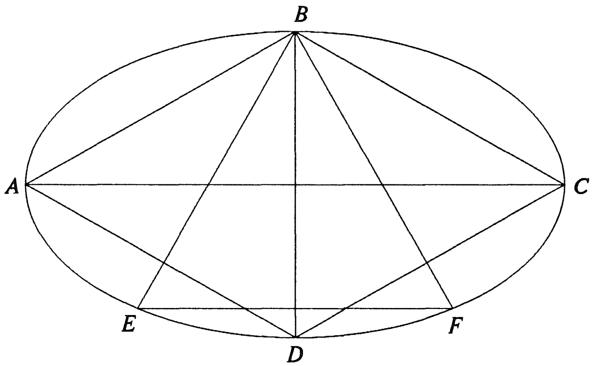
1975*. [1994: 226] *Proposed by Murray S. Klamkin, University of Alberta.*

Let $s(x)$ be the side of an equilateral triangle inscribed in the ellipse $x^2/a^2+y^2/b^2 = 1$ (where $a > b$) having one vertex with the abscissa x. Prove or disprove that $s(x)$ is a monotonic function of x in the interval $[0, a]$.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

Consider the ellipse

$$\frac{x^2}{3} + y^2 = 1 \qquad \text{(i.e., } a = \sqrt{3},\ \ b = 1).$$



We have

$$A(-\sqrt{3},0), \quad B(0,1), \quad C(\sqrt{3},0), \quad D(0,-1),$$
$$E\left(-\frac{3}{5}\sqrt{3}, -\frac{4}{5}\right), \quad F\left(\frac{3}{5}\sqrt{3}, -\frac{4}{5}\right).$$

There are three equilateral triangles with vertex B: triangles ABD and CBD with side 2, and triangle BEF with side $6\sqrt{3}/5 \approx 2.078$. I understand the problem in the following sense: $s(x)$ is the side of the equilateral triangle inscribed in the ellipse such that x is the maximal abscissa of the three vertices of the equilateral triangle. This implies

$$s(0) = 2, \quad s\left(\frac{3\sqrt{3}}{5}\right) \approx 2.078, \quad s(\sqrt{3}) = 2.$$

Thus $s(x)$ is **not** a monotonic function in the interval $[0, a]$.

[*Editor's note.* Unfortunately, as Engelhaupt and the other solvers of this problem realized, the function $s(x)$ is not precisely defined for every x. In particular, in the above example it is not clear from the problem whether $s(0) = 2$ or $6\sqrt{3}/5$. Engelhaupt's interpretation is as good as any and certainly shows $s(x)$ is not monotonic; but even if we prefer $s(0) = 6\sqrt{3}/5$, the values $s(3\sqrt{3}/5) = 6\sqrt{3}/5$ and $s(\sqrt{3}) = 2$ (which are not in question) would still imply that $s(x)$ is not monotonic, granting only that $s(x)$ is not constant for x between 0 and $3\sqrt{3}/5$.]

Also solved by G. P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; and VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan. Henderson and Konečný gave substantial details of the behaviour of $s(x)$.

* * * * *

1976. [1994: 226] *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

If a, b and c are positive numbers, prove that

$$\frac{a(3a-b)}{c(a+b)} + \frac{b(3b-c)}{a(b+c)} + \frac{c(3c-a)}{b(c+a)} \le \frac{a^3+b^3+c^3}{abc}.$$

Solution by I. Klimann, student, Université de Paris 6, France.

On remarque que

$$\begin{aligned}\frac{a^2}{bc} - \frac{a(3a-b)}{c(a+b)} &= \frac{a}{bc}\left[\frac{a(a+b)-b(3a-b)}{a+b}\right] \\ &= \frac{a}{bc}\left[\frac{a^2-2ab+b^2}{a+b}\right] = \frac{a(a-b)^2}{bc(a+b)} \ge 0\end{aligned}$$

avec égalité si et seulement si $a = b$. Par permutation circulaire, on obtient donc l'inégalité souhaitée, avec égalité si et seulement si $a = b = c$.

Also solved (often the same way) by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CHENG YU, student, Southwest Missouri State University, Springfield; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, New Mexico Highlands University, Las Vegas; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; VICTOR OXMAN, Haifa University, Israel; WALDEMAR POMPE, student, University of Warsaw, Poland; ALEX POPOVICI, Hanau, Germany; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D. J. SMEENK, Zaltbommel, The Netherlands; PANOS TSAOUSSOGLOU, Athens, Greece; JOHN VLACHAKIS, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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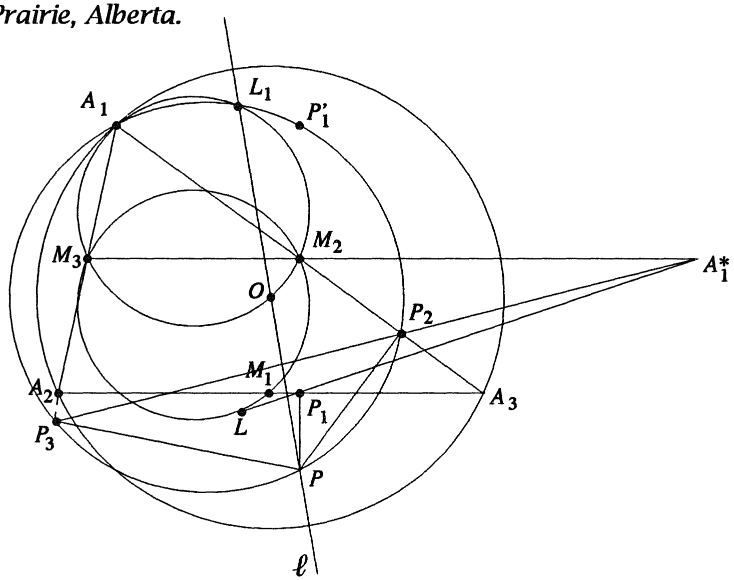
1977. [1994: 226] *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcenter O. Let ℓ be the line through O parallel to BC, and let P be a variable point on ℓ. The projections of P onto BC, CA and AB are Q, R and S respectively. Show that the circle passing through Q, R and S passes through a fixed point, independent of P. [This is not a new problem. A reference will be given when a solution is published.]

Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.



We give a solution, with slightly different notation, taken from [1]. [Replace ABC by $A_1A_2A_3$, and Q, R, S by P_1, P_2, P_3 — *Ed.*] The requirement that the line through O is parallel to BC has been dropped.

Let us see where the pedal circle of a point P meets the nine-point circle. Let M_1, M_2, M_3 be the midpoints of the sides of $\Delta A_1A_2A_3$. The intersection of the lines P_2P_3 and M_2M_3 shall be A_1^*, and A_2^*, A_3^* are defined analogously. We intend to show that the three lines $A_i^*P_i$ are concurrent in a point L of the nine-point circle. Construct the circle $A_1M_2M_3$. It will contain O which, parenthetically, is the orthocenter of $\Delta M_1M_2M_3$, and is diametrically opposite to A_1. Let PO meet this circle again in L_1. The points L_1, P_2, P_3 are vertices of three right triangles on A_1P as common hypotenuse and so are concyclic with A_1 and P. This circle will also contain P'_1, the reflection of P_1 in M_2M_3.

Moreover, the points A_1^*, L_1, P'_1 are collinear. For L_1 lies on the circles $A_1M_2M_3$ and $A_1P_2P_3$; hence the feet of the perpendiculars from L_1 to the four lines A_1M_2, A_1M_3, M_2M_3, P_2P_3 are collinear (by the theorem of Simson), so that L_1 lies on the circle $A_1^*M_3P_3$. Thus

$$\angle A_1^*L_1P_3 = \angle A_1^*M_3P_3.$$

The pentagon $A_1PP_3P_1'L_1$ is inscriptable, as we have just seen, and

$$\angle P_3L_1P_1' = \angle P_3A_1P_1' = \pi - \angle P_3PP_1' = \angle A_1^*M_3P_3,$$

the sides being perpendicular to each other ($P_3P \perp P_3M_3$ and $PP_1' \perp A_1^*M_3$), so

$$\angle A_1^*L_1P_3 = \angle P_3L_1P_1'.$$

Hence A_1^*, L_1, P_1' are collinear. [*Editor's note.* With other diagrams there are slight changes to the argument.]

Now let the reflection of L_1 in M_2M_3 be L. It lies on the line $A_1^*P_1$ and also on the nine-point circle. Also

$$(A_1^*L)(A_1^*P_1) = (A_1^*L_1)(A_1^*P_1') = (A_1^*P_2)(A_1^*P_3)$$

[so that L lies on the circle $P_1P_2P_3$]. Hence L is the intersection of the nine-point circle and pedal circles. If P moves along a fixed line through O the points L_i, L remain fixed. In the book [1] it is referred to as Theorem 68: "If a point move along a fixed line through the center of the circumscribed circle, its pedal circle will contain a fixed point of the nine-point circle." Coolidge refers to Fontené in *Nouvelles Annales de Mathématiques*, 1905.

Reference:

[1] J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Chelsea Publishing Company, Bronx, New York, 1971, pp. 51–52.

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. Bellot and Seimiya also drop the condition that ℓ is parallel to BC.

The proposer's original problem (including the condition $\ell \parallel BC$) asked to show that the required fixed point is the midpoint of AH (H the orthocentre of ABC), which of course lies on the nine-point circle of ABC. The proposer found the problem in the Journal de Mathématiques Élémentaires, 1911.

Bellot finds Fontené's theorem as Theorem 402, *p.* 245 *of R. A. Johnson's* Advanced Euclidean Geometry. *He also notes the more general theorem of T. Lemoyne (*Nouvelles Annales de Mathématiques, *Sept.* 1904, *p.* 400*), namely: for a fixed triangle and fixed line, there is a point P such that the power of P with respect to any of the pedal circles of the points on the line is the same.*

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1978. [1994: 226] *Proposed by Joaquín Gómez Rey, I. B. Luis Buñuel, Alcorcón, Madrid, Spain.*

A fair coin is tossed repeatedly till it shows up heads for the first time. Let n be the number of coin tosses required for this. We then choose at random one of the n integers 1 to n. Find the probability that the chosen integer is 1.

Solution by Robert P. Sealy, Mount Allison University, Sackville, New Brunswick.

The answer is $\ln 2$ (≈ 0.69314).

Define the random variable X to be the number of tosses of a fair coin until a head first appears. Then X is a geometric random variable with

$$P(X = n) = \left(\frac{1}{2}\right)^n.$$

The probability that the integer 1 is chosen is given by the formula

$$\sum_{n=1}^{\infty} P(X = n)\frac{1}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{n} = \ln 2.$$

To see the latter equality, write

$$f(x) = \sum_{n=1}^{\infty} x^n \frac{1}{n}, \qquad -1 < x < 1.$$

Then

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}.$$

Therefore $f(x) = -\ln|1-x|$. In particular, $f(1/2) = \ln 2$.

The problem can be generalized to show that the probability $p(m)$ that any positive integer m is chosen is given by the formula

$$\sum_{n=m}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{n},$$

which is a nice way of showing that

$$\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \left(\frac{1}{2}\right)^n \frac{1}{n} = 1.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHARLES ASHBACHER, Cedar Rapids, Iowa; C. J. BRADLEY, Clifton College, Bristol, U. K.; PAUL COLUCCI, student, University of Illinois; CURTIS COOPER, Central Missouri State University, Warrensburg; JORDI DOU, Barcelona, Spain; KEITH EKBLAW, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; SOLOMON W. GOLOMB, University of Southern California, Los Angeles; RICHARD I. HESS, Rancho Palos Verdes, California; DAG JONSSON, Uppsala, Sweden; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; KATHLEEN E. LEWIS, SUNY, Oswego, New York; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Ytre Laksevaag, Norway; P. PENNING, Delft, *The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Three incorrect solutions were sent in.*

Most solvers used this approach or a variation. Lewis also shows that if the probability of a head is p and of a tail is $q = 1 - p$, then the answer becomes $-(p/q)\ln(p)$.

* * * * *

1979. [1994: 226] *Proposed by Edward Kitchen, Santa Monica, California.*

Let P be a convex pentagon which is affinely regular, that is, each diagonal is parallel to a side. Let P^* be the convex pentagon inside P formed by the diagonals of P. Using each of two consecutive sides of P^* as base, construct outwards two 36°-72°-72° isosceles triangles, and using the segment between their two summits as base, erect a third such triangle inwards. Prove that the third summit coincides with a vertex of P.

Solution by Toshio Seimiya, Kawasaki, Japan.

The vertices of P and P^* are labeled $ABCDE$ and $A'B'C'D'E'$ as in the figure, and we assume that $AB \| CE$, etc. Because $ABA'E$ and $AE'DE$ are parallelograms, we get $BA' = AE = E'D$, so

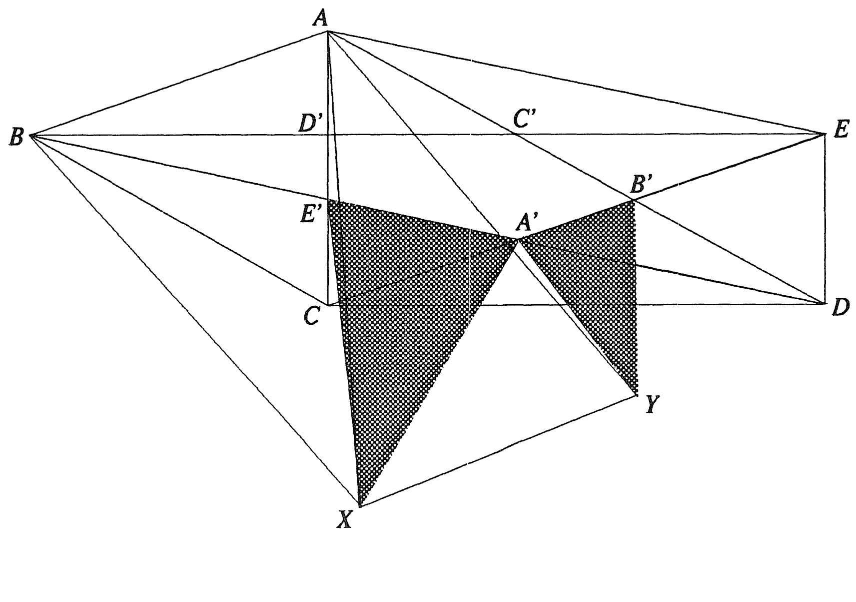
$$BE' = A'D.$$

Similarly we have

$$CA' = B'E.$$

Since $AB \| CE$, $AE \| BD$, and $CD \| EB$, we get

$$\frac{BE'}{E'A'} = \frac{AE'}{E'C} = \frac{EA'}{A'C} = \frac{BA'}{A'D} = \frac{BA'}{BE'}\ .$$



Therefore the line segment BA' is divided according to the golden section; that is,

$$\frac{BE'}{E'A'} = \frac{A'D}{E'A'} = \mu, \quad \text{where } \mu = \frac{1+\sqrt{5}}{2} \text{ (the *golden ratio*).}$$

Similarly we have

$$\frac{CA'}{A'B'} = \frac{B'E}{A'B'} = \mu.$$

[*Editor's comment by Chris Fisher.* The results obtained to this point could have been justified by simply stating that $ABCDE$ is the image of a regular pentagon under an affine transformation (which preserves ratios of segments along parallel lines). However, the details are useful for fixing the notation and, moreover, readers might appreciate seeing an affine result proved by affine methods. We now return to Seimiya's argument.]

We may assume without loss of generality that the consecutive sides of P^* mentioned in the problem are $A'E'$ and $A'B'$. We construct outwards two 36°-72°-72° isosceles triangles $XA'E'$ and $YB'A'$, and we shall prove that ΔAXY is also a 36°-72°-72° triangle.

As $XE'/E'A' = \mu$, we get $BE' = XE' = XA'$, so $\angle E'BX = \frac{1}{2}\angle A'E'X = 36°$. Hence $\Delta BA'X$ is a 36°-72°-72° triangle. Similarly, $YA' = CA'$. We put $\angle BA'C = \theta$, so $\angle ABX = \angle ABA' + \angle A'BX = \theta + 36°$. Because

$$\begin{aligned} \angle XA'Y &= 360° - [\angle XA'B + \angle BA'E + \angle EA'Y] \\ &= 360° - [72° + (180° - \theta) + 72°] = \theta + 36°, \end{aligned}$$

we have

$$\angle ABX = \angle XA'Y. \tag{1}$$

Because

$$\frac{BX}{A'X} = \mu \quad \text{and} \quad \frac{BA}{A'Y} = \frac{BA}{A'C} = \frac{BE'}{E'A'} = \mu,$$

we get

$$\frac{BX}{A'X} = \frac{BA}{A'Y}\,. \tag{2}$$

From (1) and (2) we conclude that $\Delta XBA \sim \Delta XA'Y$, whence follows $\Delta XBA' \sim \Delta XAY$. Therefore ΔAXY is a 36°-72°-72° triangle as desired.

Hence, if we erect a 36°-72°-72° triangle on XY inwards, then the third vertex coincides with A.

Also solved by JORDI DOU, Barcelona, Spain; and the proposer.

The proposer included some examples of related results that could be dealt with by the same machinery (involving complex numbers) that he developed for his solution. Among them are Napoleon's theorem (the centers of equilateral triangles erected externally on the sides of any triangle form the vertices of an equilateral triangle), and Neuberg's theorem (see Crux 540 [1981: 127]*).*

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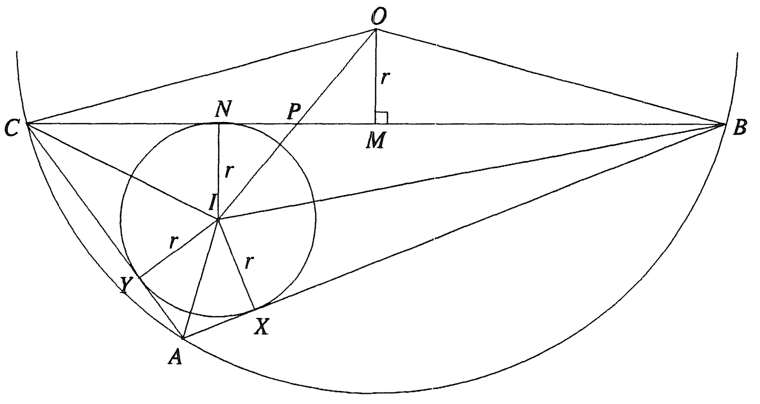
1981. [1994: 250] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an obtuse triangle with $\angle A > 90°$. Let I and O be the incenter and circumcenter of ΔABC. Suppose that $[IBC] = [OBC]$, where $[XYZ]$ denotes the area of triangle XYZ. Prove that

$$[IAB] + [IOC] = [ICA] + [IBO].$$

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.



Since $[IBC] = [OBC]$ we have $OM = IN = r$, the inradius of ΔABC. As usual put $AC = b$ and $AB = c$. Then from the figure,

$$c + CN = BX + XA + CN = BN + YA + CY = BN + b,$$

so

$$[IAB] + [ICN] = \frac{1}{2}rc + \frac{1}{2}r(CN) = \frac{1}{2}rb + \frac{1}{2}r(BN) = [ICA] + [IBN].$$

Thus

$$\begin{aligned}[IAB] + [IOC] &= [IAB] + [ICN] + [COM] \\ &= [ICA] + [IBN] + [BOM] = [ICA] + [IBO].\end{aligned}$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SAM BAETHGE, Science Academy, Austin, Texas; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; DONNY CHEUNG, student, St. John's-Ravenscourt School, Winnipeg, Manitoba; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, *Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; JOHN VLACHAKIS, Athens, Greece; and the proposer.*

In Konečný's diagram, extend MO past O to meet the circumcircle at D; then Konečný observes that I lies on the circle with centre D and radius DC. Bradley and Penning note that the triangle with angles $108°, 36°, 36°$ *satisfies the conditions of the problem.*

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1982. [1994: 250] *Proposed by Tim Cross, Wolverley High School, Kidderminster, U. K..*

Determine all sequences $a_1 \le a_2 \le \cdots \le a_n$ of positive real numbers such that

$$\sum_{i=1}^{n} a_i = 96, \quad \sum_{i=1}^{n} a_i^2 = 144 \quad \text{and} \quad \sum_{i=1}^{n} a_i^3 = 216.$$

I. *Solution by Ashish Kr. Singh, Kanpur, India.*

According to Cauchy's inequality, we have

$$\begin{aligned} 96 \times 216 &= (a_1 + a_2 + \cdots + a_n)(a_1^3 + a_2^3 + \cdots + a_n^3) \\ &= \left((a_1^{1/2})^2 + (a_2^{1/2})^2 + \cdots + (a_n^{1/2})^2\right)\left((a_1^{3/2})^2 + \cdots + (a_n^{3/2})^2\right) \\ &\ge (a_1^{1/2}a_1^{3/2} + a_2^{1/2}a_2^{3/2} + \cdots + a_n^{1/2}a_n^{3/2})^2 \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2)^2 = 144^2, \end{aligned}$$

where equality holds if and only if $a_i^{3/2} = \lambda a_i^{1/2}$ for all $1 \le i \le n$. We can see that $144^2 = 96 \times 216$, thus $a_i = \lambda$ for all $1 \le i \le n$, which implies

$$\lambda n = 96, \qquad \lambda^2 n = 144, \qquad \lambda^3 n = 216.$$

The first two equations imply $n = 96^2/144 = 64$. This gives $\lambda = 96/64 = 3/2$, and this (λ, n) satisfies $\lambda^3 n = 216$ too. Hence the only solution is

$$(a_i) = \Big(\overbrace{\frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2}}^{64 \text{ times}}\Big).$$

II. *Solution by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let

$$a_i = \frac{3}{2}b_i, \qquad i = 1, 2, \ldots, n.$$

Then we must find all sequences $0 < b_1 \le b_2 \le \cdots \le b_n$ such that

$$\sum_{i=1}^{n} b_i = \frac{2}{3}(96) = 64, \qquad \sum_{i=1}^{n} b_i^2 = \frac{4}{9}(144) = 64, \qquad \sum_{i=1}^{n} b_i^3 = \frac{8}{27}(216) = 64.$$

We can see that

$$\sum_{i=1}^{n} b_i(b_i - 1)^2 = \sum(b_i^3 - 2b_i^2 + b_i) = 64 - 2 \cdot 64 + 64 = 0.$$

It is obvious that since $b_i > 0$ it must be that $b_i = 1$ for all i, and thus $n = 64$. So the only sequence is $a_i = 3/2$, $i = 1, 2, \dots, 64$.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; PAUL COLUCCI, student, University of Illinois; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; F. J. FLANIGAN, San Jose State University, San Jose, California; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; JAMSHID KHOLDI, New York, N.Y.; KEE-WAI LAU, Hong Kong; KATHLEEN E. LEWIS, SUNY, Oswego; VICTOR OXMAN, Haifa University, Israel; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; JOHN VLACHAKIS, Athens, Greece; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. A further reader gave the correct solution without proving that it is unique. There were also two incorrect solutions sent in.

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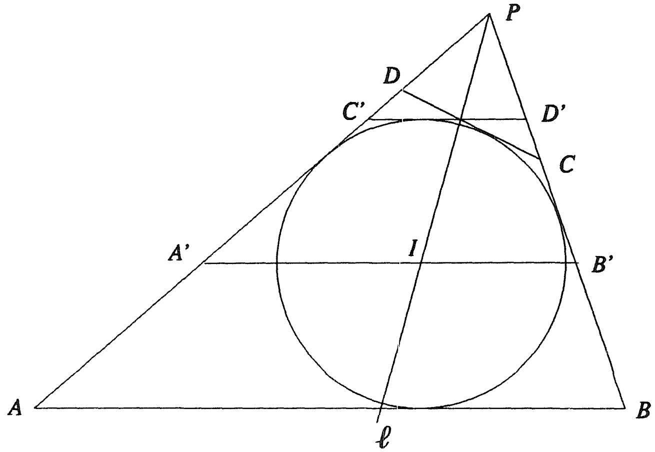
1983. [1994: 250] *Proposed by K. R. S. Sastry, Dodballapur, India.*

A convex quadrilateral $ABCD$ has an inscribed circle with center I and also has a circumscribed circle. Let the line parallel to AB through I meet AD in A' and BC in B'. Prove that the length of $A'B'$ is a quarter of the perimeter of $ABCD$.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.



Let ℓ be the angle bisector of $\angle APB$, where $P = AD \cap BC$. (If AD is parallel to BC then let ℓ be a line parallel to AD and BC passing through I.) Let C' and D' be the symmetric points to C and D respectively with respect to the line ℓ. Since $ABCD$ is cyclic *(i.e. has a circumscribed circle)*, $\angle ABC = \angle CDC' = \angle C'D'C$, so $C'D'$ and AB are parallel. Consequently, $ABD'C'$ is a trapezoid. Because of the symmetry $C'D'$ is tangent to the incircle of $ABCD$, and so this circle is also inscribed in the trapezoid $ABD'C'$. Hence $A'B' = \frac{1}{2}(AB + C'D')$. Since $ABCD$ has the inscribed circle, we obtain

$$A'B' = \frac{1}{2}(AB + C'D') = \frac{1}{2}(AB + CD) = \frac{1}{4}(AB + BC + CD + DA),$$

so we are done.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; PAUL PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CRISTÓBAL SÁNCHEZ-RUBIO, I. B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; ASHISH KR. SINGH, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; JOHN VLACHAKIS, Athens, Greece; and the proposer.

The solutions of Sánchez-Rubio and of Vlachakis are similar to Pompe's solution.

Bellot, Janous and the proposer note that, from the problem, the point I has the interesting property that all four parallels to the sides of $ABCD$ through I have the same length (1/4 of the perimeter). (Here the "parallel to AB" is the segment $A'B'$ as defined in the problem, and the parallels to the other sides are defined analogously. Notice that, for example, the parallel to BC through I in the above diagram does not lie completely inside the quadrilateral.) Janous then proposes the following related problem: determine all convex quadrilaterals $ABCD$ such that there exists at least one interior point P so that the four parallels to sides AB, BC, CD and DA through P have equal lengths.

In the same spirit, which convex quadrilaterals have an interior point P so that the sum of the four parallels to the sides through P is equal to the perimeter of the quadrilateral? (It works for trapezoids!)

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1984. [1994: 250] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

Find an integer $n > 1$ so that there exist n consecutive integer squares having an average of n^2.

Combination of solutions by Charles R. Diminnie, St. Bonaventure University, St. Bonaventure, New York, and Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

We seek positive integers k and n, with $n > 1$, such that

$$\begin{aligned} n^2 &= \frac{1}{n}\sum_{i=1}^{n}(k+i)^2 = \frac{1}{n}\left(\sum_{i=1}^{n}k^2 + 2k\sum_{i=1}^{n}i + \sum_{i=1}^{n}i^2\right) \\ &= \frac{1}{n}\left(nk^2 + kn(n+1) + \frac{n}{6}(n+1)(2n+1)\right) \\ &= k^2 + (n+1)k + \frac{(n+1)(2n+1)}{6}\,. \end{aligned} \tag{1}$$

Note that n is odd [since otherwise the fraction in (1) is not an integer — *Ed.*]. Complete the square in (1) by putting

$$k = a - \frac{n+1}{2}\,,$$

where k is an integer; then (1) becomes

$$\begin{aligned} n^2 &= a^2 - \left(\frac{n+1}{2}\right)^2 + \frac{(n+1)(2n+1)}{6} \\ &= a^2 + \frac{n+1}{12}[-3(n+1) + 2(2n+1)] \\ &= a^2 + \frac{(n+1)(n-1)}{12} = a^2 + \frac{n^2-1}{12}\,, \end{aligned}$$

or $12a^2 = 11n^2 + 1$. Putting $n = a + t$ this becomes

$$12a^2 = 11a^2 + 22at + 11t^2 + 1$$

or

$$(a - 11t)^2 - 132t^2 = 1.$$

With $x = a - 11t$ and $y = 2t$ one gets the Pell equation

$$x^2 - 33y^2 = 1. \tag{2}$$

The smallest positive integer solution is $y = 4$, $x = 23$, which implies $t = 2$, $a = 45$, $n = 47$, $k = 21$, and thus a solution to the problem is

$$\frac{22^2 + 23^2 + \cdots + 68^2}{47} = 47^2.$$

From the theory of Pell equations (and putting $x_1 = 23$, $y_1 = 4$), there are infinitely many positive integer solutions (x_i, y_i) of (2), given by

$$x_i + y_i\sqrt{33} = (23 + 4\sqrt{33})^i,$$

or alternately by

$$x_{i+1} = 23x_i + 4 \cdot 33y_i, \quad y_{i+1} = 23y_i + 4x_i.$$

The first four solutions and the corresponding values of k and n are

x	y	k	n
23	4	21	47
1057	184	988	2161
48599	8460	45449	99359
2234497	388976	2089688	4568353

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, Wolverley High School, Kidderminster, U. K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, New York; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D. J. SMEENK, Zaltbommel, The Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro; JOHN VLACHAKIS, Athens, Greece; C. WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incomplete solution was sent in.

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