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CONTENTS

A PROPERTY OF TRIANGLES INSCRIBED IN RECTANGLES

TOM M. APOSTOL
California Institute of Technology

In the course of a series of lectures he gave at the California Institute of Technology in the spring of 1977, H.S.M. Coxeter gave the following theorem as a problem:

THEOREM 1. *An equilateral triangle is inscribed in, and has a common vertex with, a rectangle (as shown in Figure* 1*). If A, B, C are the areas of the complementary right triangles, prove that $A+B=C$.*

This theorem was first proved in [1][1] (in which the proposer was unspecified) by W.Ch. Riemens and J.G.H. Pabon. They gave a simple trigonometric proof of the theorem and proved also that, if r denotes the ratio of the vertical to the horizontal side of the rectangle, then

$$\frac{\sqrt{3}}{2} < r < \frac{2\sqrt{3}}{3}.$$

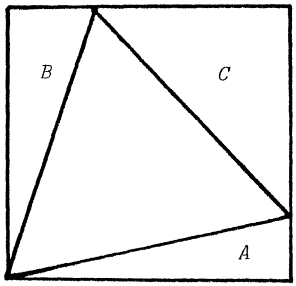

Figure 1

In a later issue of the same journal [2], a note by R. Kooistra gives a new proof of the theorem (due to D. Kruyswijk) by transformation geometry.

In solving this problem, the author was led to the following generalization:

THEOREM 2. *A triangle with angles α, β, γ is inscribed in, and has a common vertex with, a rectangle (as shown in Figure* 2*).*

(a) *If the right triangles opposite α, β, γ have areas A, B, C, respectively, then*

$$A\cot\alpha + B\cot\beta = C\cot\gamma. \tag{1}$$

(b) *If ρ denotes the ratio of the vertical to the horizontal side of the rectangle, then*

$$\frac{\sin\beta}{\sin\alpha}\cdot\sin\gamma \le \rho \le \frac{\sin\beta}{\sin\alpha}\cdot\frac{1}{\sin\gamma}. \tag{2}$$

Proof. (a) Let a, b, c denote the sides of the inscribed triangle opposite

[1]This reference and the next were obtained by the editor of EUREKA through the courtesy of O. Bottema and P.J. de Doelder.

the angles α, β, γ, respectively, and introduce the angles λ, μ, ν, as shown in Figure 2. Then we have

$$A = \frac{1}{2}(a\cos\lambda)(a\sin\lambda) = \frac{1}{4}a^2\sin 2\lambda,$$

and similarly

$$B = \frac{1}{4}b^2\sin 2\mu$$

and

$$C = \frac{1}{4}c^2\sin 2\nu.$$

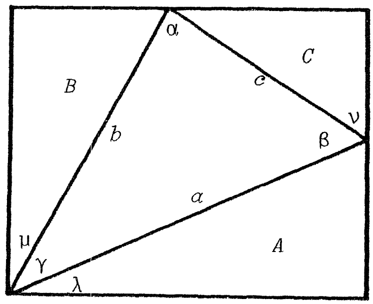

Figure 2

By the law of sines,

$$a = \frac{c\sin\alpha}{\sin\gamma} \quad \text{and} \quad b = \frac{c\sin\beta}{\sin\gamma};$$

therefore

$$A = \frac{1}{4}c^2\frac{\sin^2\alpha\ \sin 2\lambda}{\sin^2\gamma}, \quad B = \frac{1}{4}c^2\frac{\sin^2\beta\ \sin 2\mu}{\sin^2\gamma}, \quad C = \frac{1}{4}c^2\frac{\sin^2\gamma\ \sin 2\nu}{\sin^2\gamma}.$$

Hence

$$A\cot\alpha = \frac{1}{4}c^2\frac{\sin\alpha\ \cos\alpha\ \sin 2\lambda}{\sin^2\gamma} = \frac{1}{8}c^2\frac{\sin 2\alpha\ \sin 2\lambda}{\sin^2\gamma}$$

and similarly

$$B\cot\beta = \frac{1}{8}c^2\frac{\sin 2\beta\ \sin 2\mu}{\sin^2\gamma}, \quad C\cot\gamma = \frac{1}{8}c^2\frac{\sin 2\gamma\ \sin 2\nu}{\sin^2\gamma}.$$

Therefore, to prove (1) it suffices to verify the identity

$$\sin 2\alpha\ \sin 2\lambda + \sin 2\beta\ \sin 2\mu = \sin 2\gamma\ \sin 2\nu. \tag{3}$$

Since $\alpha+\beta+\gamma = \pi$, we have $2\alpha = 2\pi - 2\beta - 2\gamma$; hence $\sin 2\alpha = -\sin(2\beta+2\gamma)$ and

$$\sin 2\alpha\ \sin 2\lambda = -\sin 2\lambda\ \sin 2\beta\ \cos 2\gamma - \sin 2\lambda\ \cos 2\beta\ \sin 2\gamma. \tag{4}$$

The sum of the angles at the common vertex is $\mu+\gamma+\lambda = \frac{\pi}{2}$, so $2\mu = \pi - 2\gamma - 2\lambda$; hence $\sin 2\mu = \sin(2\gamma+2\lambda)$ and

$$\sin 2\beta\ \sin 2\mu = \sin 2\beta\ \sin 2\gamma\ \cos 2\lambda + \sin 2\beta\ \cos 2\gamma\ \sin 2\lambda. \tag{5}$$

Finally, adding β to the angles adjacent to it gives $\beta+\nu+\frac{\pi}{2}-\lambda = \pi$, so $2\nu = \pi - 2\beta + 2\lambda$; hence $\sin 2\nu = \sin(2\beta - 2\lambda)$ and

$$\sin 2\gamma\ \sin 2\nu = \sin 2\gamma\ \sin 2\beta\ \cos 2\lambda - \sin 2\gamma\ \cos 2\beta\ \sin 2\lambda. \tag{6}$$

The sum of the right members of (4) and (5) is the right member of (6); so this proves (3) and hence (1).

(b) Referring to Figure 2, we have

$$\rho = \frac{b\cos\mu}{a\cos\lambda} = \frac{b}{a}\,\frac{\cos\left(\frac{\pi}{2}-\gamma-\lambda\right)}{\cos\lambda} = \frac{b}{a}\left\{\frac{\cos\left(\frac{\pi}{2}-\gamma\right)\cos\lambda}{\cos\lambda} + \frac{\sin\left(\frac{\pi}{2}-\gamma\right)\sin\lambda}{\cos\lambda}\right\}$$

$$= \frac{b}{a}\,(\sin\gamma + \cos\gamma\ \tan\lambda).$$

Now $0 \le \lambda \le \frac{\pi}{2} - \gamma$; hence $0 \le \tan\lambda \le \tan(\frac{\pi}{2}-\gamma) = \cot\gamma$. Using this in the last equation for ρ, we obtain

$$\frac{b}{a}\sin\gamma \le \rho \le \frac{b}{a}(\sin\gamma + \cos\gamma\ \cot\gamma) = \frac{b}{a}\,\frac{1}{\sin\gamma}.$$

Now $\frac{b}{a} = \frac{\sin\beta}{\sin\alpha}$ by the law of sines, and (2) follows.

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1. W.Ch. Riemens and J.G.H. Pabon, Solution to Problem 1580, *Nieuw Tijdschrift voor Wiskunde*, 54 (1966-1967), 129-130.

2. R. Kooistra, A solution by transformation geometry (due to D. Kruyswijk) of Problem 1580*a*, *ibid.*, 55 (1967-1968), 125.

* * *

PRINCE RUPERT'S CUBES[1]

SAHIB RAM MANDAN

Indian Institute of Technology

1. *History.* In several well-known German works a note is found on the following problem: *To perforate a cube in such a way that a second cube of the same size may pass through the hole.* All writers agree in that this problem was first proposed by Prince Rupert, Count Palatine of the Rhine and Duke of Bavaria.

Prince Rupert, Ruprecht or Robert (1619-82) is well known in the history of the world. He was son of Frederick V, Elector Palatine and surnamed the Winter King, because of his tragical ephemeral reign as king of Bohemia. His mother was Elizabeth, daughter of James I of England, so that he was a nephew of Charles I. He was educated in the Netherlands and from his will we learn that he possessed a house near Rhenen in that country. In the Civil War he was one of Charles's most loyal supporters and one of the bravest. Until after the battle of Naseby (1645), he was commander of the Royalist forces and, by his boldness and recklessness, the

[1]Reprinted from *Mathematics Teacher (India)*, 12 (1976), 32-34, by permission of the author and the editor.

dread of the parliamentary troops. Nor did he give up the struggle after Charles's defeat and execution. For several years, as a Naval Commander, he combated Cromwell's fleet. After the Restoration (1660) Rupert came back to England. By this time he had become quite an Englishman in life and thought and served his country again in the Anglo-Dutch wars. Prince Rupert took a great interest in scientific research. In afteryears he laboured at his own forge and in his own laboratory at Windsor Castle, of which he was appointed the Governor.

He invented an alloy, still called Prince's metal, and studied the curious properties of quickly cooled drops of glass ("glass tears", "Rupert drops").

In 1660 "The Royal Society of London for the Promotion of Natural Knowledge", with the proud device *Nullius in Verba*, By No One's Words (we swear), had been founded by Charles II. Prince Rupert was among the first elected Fellows.

Rupert's biographies tell us that he promulgated his discoveries and inventions in the *Philosophical Transactions of the Royal Society*. It is not sure whether he did so with the present problem. Schrek has not been able to find it in the *Transactions* of the years 1665 up to 1682, the year of Rupert's death. Having applied to the Hon. Secretary of the Royal Society Schrek learned that, in spite of much trouble, his staff had not been able to trace any information about this question.

The problem was taken up by no less an authority than John Wallis (1616-1703). In his *De Algebra Tractus* (*Opera Mathematica*, v. II, Oxoniae 1693, Caput CIX, p. 470-71) he tells us under the heading "Perforatio Cubi, alterum ipsi aequalem recipients":

"Rupert, Prince Palatine, staying in the palace of Charles II, King of England, a man of great wit and boldness, affirmed one day that it was quite possible (and having laid a wager undertook making) that, a hole being made in one of two equal cubes, the other might pass through; which I learn he has achieved. In which way this is done we shall show"; and he enunciates the problem in the following words:

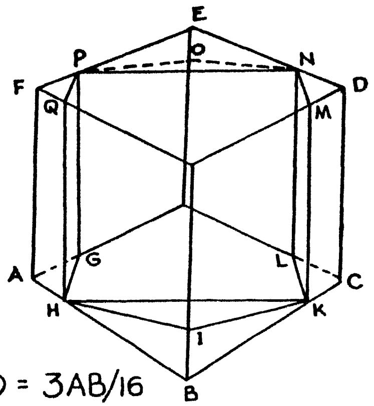
PROBLEM. To perforate one of two equal cubes in such a way that the intact cube may pass through.

Some time afterwards the same question was studied in another part of the British Isles. Gibson in his *History of the County and City of Cork* (v. II, p. 408) quotes an older similar work (Smith, *History of Cork*) in which the author mentions one Philip Ronayne who lived near Cork in the beginning of the 18th Century. The Ronaynes were an old Cork family with a very long history. We learn that Philip Ronayne had distinguished himself by several essays in the most sublime parts of Mathematics. He wrote a *Treatise of Algebra* in 2 books, the 2nd edition of which was printed in London in 1726. A copy of that edition is in the British Museum; it is very rare now. Smith says that it passed through several editions and that it was "much read and esteemed by all the philomaths of the present time".

Now Smith affirms emphatically that Philip Ronayne solved the problem of the interpenetrating cubes and that he demonstrated its possibility both geometrically and algebraically. Moreover, Smith declares that *he saw a model of the cubes* made by a Daniel Voster, a Dutchman, who kept a school in Cork. This scanty information is all we know about it (see H. Hennessy "Ronayne's Cubes", *Phil. Mag.* Series 5, v. 39, January-June 1895, pp. 183-87).

Half a century later we find the problem again in a work by the well-known Dutch mathematician, Jan Henri van Swinden (1746-1823), who refers to both Rupert and Ronayne. In modern times the problem reappears from time to time in mathematical journals.

So far the original problem. But there is another side to the question. We shall see that not only a cube of *equal* size, but a somewhat *larger* one may pass through a hole in the other. This being the case, the question arises: which is the maximum size of the first cube? It was another Dutch mathematician, Pieter Nieuwland (1764-94), who studied this problem. Born near Amsterdam as the son of a carpenter, he was distinguished by his precocity. His father, though a humble artisan, had some knowledge of Geometry and taught the boy the *Elements* of Euclid. An Amsterdam gentleman who, during his stay in the countryside, was struck by the talents of the child, enabled him to study Arts as well as Science; he was gifted for both and his knowledge of foreign languages, classical and modern, was amazing. Later on he applied himself especially to Mathematics, Physics and Astronomy, and became the beloved pupil of J.H. van Swinden; both were active members of "Felix Meritis", the then prominent Amsterdam Society "for promotion of Arts and Science". After having been Lecturer in Nautical Science in Amsterdam,



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PRINCE RUPERT'S CUBES

he was appointed Professor in Leyden University in 1793, but died a year later.

Van Swinden executed Nieuwland's will as far as Science was concerned. Among his scientific papers is the correct solution of our problem which Van Swinden inserted into his Geometry.

2. *Solution*. It can be proved that a cube whose side is *less than* $3\sqrt{2}/4$ = 1.06066... times that of another can be made to pass through the latter, and the channel has its maximum cross section in the position shown in the figure.

3. *Exercise*. With due devotion and care make this *delicate model*.

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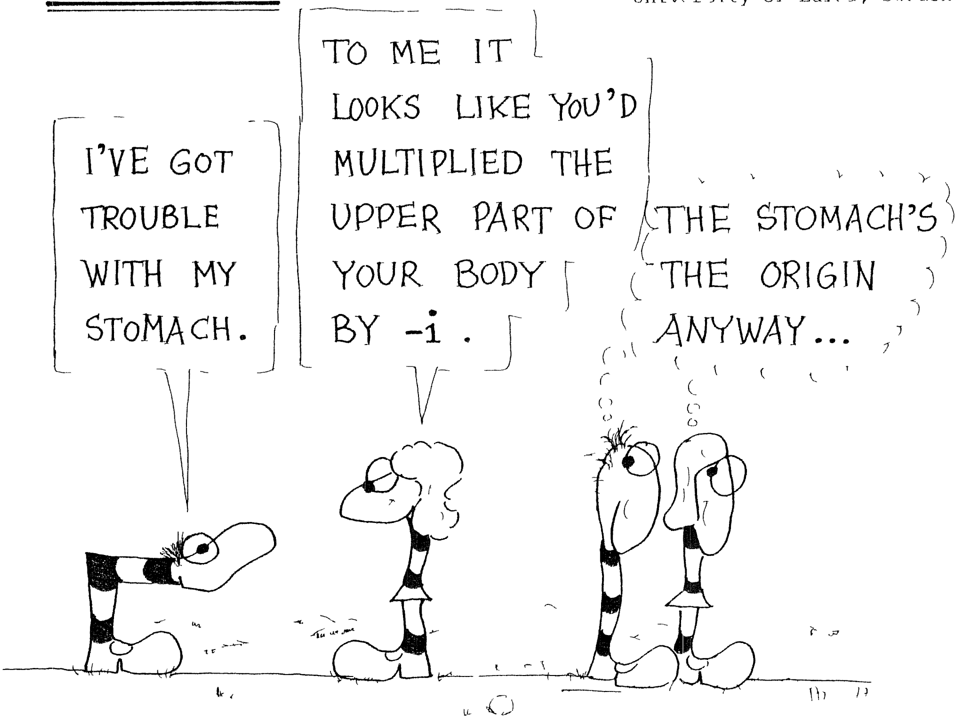
1. H.M. Cundy and A.P. Rollet: *Mathematical Models*. Oxford, 1961, 157 158.

2. D.J.E. Schrek: "Prince Rupert's Problem and its extension by Pieter Nieuwland", *Scripta Mathematica*, 16 (1950), 73-80, 261-267.

* * *

FOOTIES

ANDREJS DUNKELS
University of Luleå, Sweden



Reminiscences

THE BIGGEST NUMBER

R. ROBINSON ROWE, Sacramento,[1] California

As my early school days introduced me to numbers, bigger numbers, and still bigger numbers—fascinatingly—I started naively to list them all! I filled reams of foolscap—way up into the thousands—and wondered what, if anything, came after thousands. I knew better than to ask my father—in divinity school[2] he had had to learn to read Latin, Greek, Semitic and Sanskrit, with no time for such mundane and profane things as numbers.

But my mother had graduated in engineering[3] and when I asked her what came after thousands, she told me about millions. I couldn't write them all, of course, but I did quite a few, thinking surely these were the biggest of all numbers. Hearing me say that, my mother corrected me—after millions come billions. Were those the biggest? No, after that came trillions! And so it went, as I wrote bigger and bigger numbers—meaningless and beyond my imagination but with digits stretched across the paper—up to decillions—finally realizing that there was no limit. As elusive as the pot of gold at the end of a rainbow, one could chase it but never reach it.

My interest in big numbers had eclipsed an interest in big words. In a clergyman's family circle, selected readings from the Bible were regular preludes to bedtime stories, but Psalms and Proverbs were less exciting than Little Red Riding-Hood and Cinderella. But when I began to have a serious interest in reading and turned to the Bible, lo and behold—there was a whole Book of Numbers!

Excited, I read the whole Book of Numbers at one sitting, skipping over the long words I didn't know (and could not care less about). On succeeding days, I listed all the numbers, chapter by chapter, and counted them—there were 504. Many were repeated, but there were 89 different numbers. There were no figures; all were spelled out. The most frequent was 'one' which occurred 143 times.[4] Following in order of frequency were: 2, 5, 7, 20, 10, 70 and 12; things were reckoned by scores more often than by dozens.

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[1]If R^3 is back in Sacramento, can winter be far behind? (Editor)

[2]Morgan Park Union Theological Seminary, now in University of Chicago, 1889.

[3]Ohio State University, 1892; Robinson Hall of Engineering is named for her father, Professor S.W. Robinson.

[4]Not counting the pronoun 'one', nor articles 'a' or 'an', implying 'one'.

I checked Moses' arithmetic; the big numbers in Chapters 1 and 2 had been correctly subtotalled and totalled, but in Chapter 3, after adding several times to make sure, he[5] goofed. He hadn't added all the 504 numbers, but I did.[6]

Meanwhile my Sunday School teacher had asked each of her class to recite a newly-memorized verse of the Bible. As a ruse to induce some extra reading, she specified that on succeeding Sundays our verses should begin with the letters A, B, C, and so on. I was quite ready the fifth Sunday, and solemnly recited:

> Even all they that were numbered were six hundred thousand and three thousand and five hundred and fifty. *Numbers* I:46.

My teacher blinked and gasped—and might have said something nasty but, after all, I was the Parson's son—so she let it pass. Nor did I tell her that that number, 603550, was the biggest number in Numbers. She was happier on the tenth Sunday when I recited John 11:35.[7]

In epilogue, this fascination for big numbers never ended. But it would be years before I would hear of the vigintillion, then the googol, then the googolplex.

* * *

Tinca Tinca Parley-voo

LETTER TO THE EDITOR

Dear Editor:

I thought there was something fishy about Tinca Tinca [1977: 190]. Perhaps the (s)tench got to me.

Since you have omitted the middle lines of the limerick in [1977: 169], I submit one possible completed version, in honor of one of your eminent contributors whose home town is mentioned in the first line.

There was a young girl of Topeka
For whom all mathematics seemed Greek. A
Professor in shrill key
Screamed, "Problem by Wilke!"
So she tried it and shouted, "Eureka!"

Might Edith Orr be the young girl referred to?

L.F. MEYERS,
The Ohio State University.

Commentaire de l'éditeur.

Si les effluves de Tinca Tinca sont parvenus jusqu'aux narines de M. Meyers, le poisson devait être dans une boîte à cloison non (é)tanche.

Edith Orr, qui baragouine un peu le français, répond ainsi à l'insinuation mal fondée de M. Meyers: "Jay nay souis pas le jeune fille de Topeka; jay souis oune canadienne pure laine."

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[5]Or the translators; I suppose the original scripture was in Hebraic or Semitic and passed through Sanskrit, Greek, Latin to my English version.

[6]The total was 6,014,994, if anyone cares.

[7]Look it up yourself.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

281. *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

Reconstituer les chiffres de l'addition décimale suivante:

HUIT + HUIT + HUIT = DOUZE + DOUZE.

282. *Proposed by Erwin Just and Sidney Penner, Bronx Community College.*

On a 6×6 board we place 3×1 trominoes (each tromino covering exactly three unit squares of the board) until no more trominoes can be accommodated. What is the maximum number of squares that can be left vacant?

283*. *Proposed by A.W. Goodman, University of South Florida.*

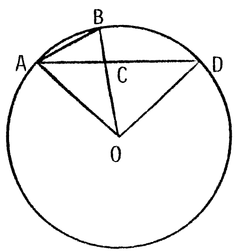
The function

$$y = -\frac{2x \ln x}{1 - x^2}$$

is increasing for $0 < x < 1$ and in fact y runs from 0 to 1 in this interval. Therefore an inverse function $x = g(y)$ exists. Can this inverse function be expressed in closed form and if so what is it? If it cannot be expressed in closed form, is there some nice series expression for $g(y)$? The series need not be a power series.

284. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Given a sector AOD of a circle (see figure), can a straightedge and compass construct the line OB so that AB = AC?



285. *Proposed by Robert S. Johnson, Montreal, Quebec.*

Using only the four digits 1, 7, 8, 9 (each exactly once) and four standard mathematical symbols (each at least once), construct an expression whose value is 109.

286. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

This problem generalizes *Mathematics Magazine* Problem 939 (proposed May, 1975; solution May, 1976, p. 151) and EUREKA Problem 204 (proposed January, 1977; solution May, 1977, p. 140).

Find, for positive integers $W \le L \le H$:

(a) the number of rectangular parallelepipeds (r.p.),

(b) the number of cubes,

(c) the number of different sizes of r.p.'s

imbedded in a $W \times L \times H$ r.p. made up of WLH unit cubes.

287. *Proposed by M.S. Klamkin, University of Alberta.*

Determine a real value of x satisfying

$$\sqrt{2ab + 2ax + 2bx - a^2 - b^2 - x^2} = \sqrt{ax - a^2} + \sqrt{bx - b^2}$$

if $x > a, b > 0$.

288. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Show how to construct (with compass and straightedge) a triangle given the circumcenter, the incenter and one vertex.

289. *Proposed by L.F. Meyers, The Ohio State University.*

Derive the laws of reflection and *refraction from the principle of least time without use of calculus or its equivalent. Specifically, let L be a straight line, and let A and B be points not on L. Let the speed of light on the side of L on which A lies be c_1, and let the speed of light on the other side of L be c_2. Characterize the points C on L for which the time taken for the route ACB is smallest, if

(a) A and B are on the same side of L (reflection);

*(b) A and B are on opposite sides of L (refraction).

290. *Proposed by R. Robinson Rowe, Sacramento, California.*

Find a 9-digit integer A representing the area of a triangle of which the three sides are consecutive integers.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

189. [1976: 194; 1977: 74, 193] *Proposed by Kenneth S. Williams, Carleton University.*

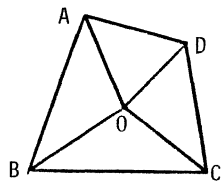
If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

V. *Comment by Basil Rennie, James Cook University of North Queensland, Australia.*

This problem was shown by the editor [1977: 75] to be a trivial consequence of a theorem of Léon Anne. Two proofs of this theorem were presented by the editor in [1977: 183]. One was the now old-fashioned original proof by Léon Anne (1879), and the other was an elegant up-to-the-minute vectorial proof by Victor Milenkovic and Paul Weiss.

Here is how the theorem might have been proved earlier this century:

THEOREM (Léon Anne). Let ABCD *be a quadrilateral. Then the line joining the midpoints of the diagonals* AC *and* BD *is the locus of the points* O *such that the sum of the areas of* Δ*s* OAD, OBC *is equal to the sum of the areas of* Δ*s* OAB, OCD.



Proof. Cartesian coordinates having been introduced in the plane, the area of each of the triangles is a linear function of the coordinates x and y of the permissible point O, and so the locus of O is a straight line. Because any triangle is bisected by a median, the locus contains the midpoints of the two diagonals AC and BD. If the midpoints coincide, the quadrilateral is a parallelogram.

Editor's comment.

There is something almost magical about this proof. The theorem of Léon Anne is not an obvious one; yet here is Rennie giving a valid proof with a nonchalant flick of the wrist and no calculations whatsoever!

* * *

233. [1977: 104] *Proposed by Viktors Linis, University of Ottawa.*

The three points (1), (2), (3) lie in this order on an axis, and the distances [1,2] = a and [2,3] = b are given. Points (4) and (5) lie on one side of the axis, and the distance [4,5] = $2c > 0$ and the angles (415) = v_1, (425) = v_2,

(435) = v_3 are also known. Determine the position of the points (1), (2), (3) relative to (4) and (5).

Gauss gave a solution to this problem which was found in a book on navigation [*Handbuch der Schiffahrtskunde* von C. Rümker, 1850, p. 76].

Editor's comment.

No solutions were received for this problem. The proposer submitted a photocopy of Gauss's own solution, as it was reported in the Rümker reference given in the proposal. I give below a faithful, nearly literal, translation of Gauss's solution. The translation was made by H.G. Dworschak of Algonquin College, who also added footnotes to try to clarify a few obscure points.

The solution begins with a restatement of the problem in Gauss's own notation, which differs slightly from that in our proposal. The solution itself is a beautiful one, but it will naturally appear quaint and archaic to modern readers.

I hope someone will undertake to rewrite Gauss's solution in a clear, uncluttered way using modern mathematical idiom (and including, one hopes, notational and other improvements). The goal is to have a model of elegant mathematical exposition for today's readers. I will gladly publish the best of those I receive.

Copies of the original German text are available from the editor on request.

Solution by Carl Friedrich Gauss.

Three points (1), (2), (3) lie on a straight line (*I*) at given distances from one another, A from (1) to (2) and B from (2) to (3). The angles θ, θ', θ'' between two other points (4), (5), at a distance $2c$ from each other, are measured.[1] The positions of the first three points are required, relative to the last two. In order to avoid all ambiguity, I assume that all three angles are measured in the same increasing sense from (4) to (5), that a positive sense is chosen on line (*I*) (so that, if one chose for some reason not to denote the point between the other two by (2), then A and B would have opposite signs), and c is assumed to be positive.

I choose as the axis of abscissas the straight line (*II*) which meets (4),(5) at its midpoint (6) at right angles, and consider the abscissas from (6) as positive on that side of (4),(5) where the angle[2] from (4) to (5) is less than 180°, that is, on the right when the angles increase from left to right; the ordinates are positive in the sense from (6) to (5). On (*II*) I denote (1*), (2*), (3*) the points with abscissas

$$c \cdot \text{cotang}\,\theta = n - a, \qquad c \cdot \text{cotang}\,\theta' = n, \qquad c \cdot \text{cotang}\,\theta'' = n + b;$$

[1] From points of observation (1), (2), (3).

[2] As observed from a point of positive abscissa.

they are the centres of three circles that pass respectively through (1), (2), (3) and simultaneously all three circles pass through (4) and (5). The radii of these circles are

$$\frac{c}{\sin\theta} = \sqrt{(cc+(n-a)^2)}, \quad \frac{c}{\sin\theta'} = \sqrt{(cc+nn)}, \quad \frac{c}{\sin\theta''} = \sqrt{(cc+(n+b)^2)},$$

or, if one sets $\frac{c}{\sin\theta'} = r$, then the other two become

$$\sqrt{(rr-2an+aa)} \quad \text{and} \quad \sqrt{(rr+2bn+bb)}.$$

Finally, let (7) be the point of intersection of (I) and (II), T and t the respective distances between points (2), (2*) and (7), ϕ the angle between the respective arms of those lines, measured of course from (I) to (II) in the chosen positive sense.[3] Hence the abscissa of (7) $= n-t$, and consequently the coordinates of the three points of observation are

$$\begin{array}{lll} (1) & n-t+(T-A)\cos\phi, & (T-A)\sin\phi; \\ (2) & n-t+T\cos\phi, & T\sin\phi; \\ (3) & n-t+(T+B)\cos\phi, & (T+B)\sin\phi. \end{array}$$

We can solve for the three unknown quantities t, T, ϕ from the following equations, where for brevity x has been written for $\cos\phi$:

$$tt+TT-2tTx = rr, \qquad [1]$$
$$(t-a)^2+(T-A)^2-2(t-a)(T-A)x = rr-2an+aa,$$
$$(t+b)^2+(T+B)^2-2(t+b)(T+B)x = rr+2bn+bb.$$

Instead of the last two, I use the following which are obtained by subtracting them from the first:

$$2at+2AT-2an-AA = (2At+2aT-2aA)x, \qquad [2]$$
$$2bt+2BT-2bn+BB = (2Bt+2bT+2bB)x. \qquad [3]$$

From $-\frac{1}{2}b[2]+\frac{1}{2}a[3]$ and $\frac{1}{2}B[2]-\frac{1}{2}A[3]$, using the abbreviations

$$aB-bA = \lambda, \quad ab(A+B) = f, \quad \tfrac{1}{2}AB(A+B)+\lambda n = g,$$
$$AB(a+b) = F, \quad \tfrac{1}{2}aBB+\tfrac{1}{2}bAA = G,$$

we can write

$$\lambda T+G = \lambda(t+f)x, \qquad \lambda t-g = (\lambda T-F)x \quad \text{or} \quad \lambda(t-Tx) = g-Fx$$

and consequently

$$\lambda t = \frac{+g-(F+G)x+fxx}{1-xx}, \qquad [4]$$

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[3] ϕ is thus the angle between (I) and (II), measured positively from the ray of (I) that contains (2) to the ray of (II) that contains (2*).

$$\lambda T = \frac{-G + (f+g)x - Fxx}{1 - xx}. \qquad [5]$$

Equation [1] in the form $(t - Tx)^2 + TT(1 - xx) = rr$ gives

$$\lambda\lambda rr - (g - Fx)^2 = \lambda\lambda TT(1 - xx). \qquad [6]$$

If we substitute the value of λT from [5], the following cubic equation results:

$$\begin{aligned} 2fFx^3 &- (ff + 2fg + FF + 2FG + \lambda\lambda rr)xx \\ &+ (2fG + 2gF + 2gG)x + \lambda\lambda rr - gg - GG = 0. \end{aligned} \qquad [7]$$

After x has been determined, the coordinates of point (2) can be found from the above expressions which, on substituting for $t - Tx$ and T the above given values, have the following form[4]

$$n - \frac{g - Fx}{\lambda}, \qquad \frac{-G + (f+g)x - Fxx}{\lambda\sqrt{(1 - xx)}},$$

and then the coordinates of (1) and (3) can be found by adding respectively $-Ax$, $-A\sqrt{(1 - xx)}$ and $+Bx$, $+B\sqrt{(1 - xx)}$ to those of (2).

Since to each cosine belong two values of the angle or, what is the same, since the radical $\sqrt{(1 - xx)}$ can be taken either positively or negatively, each admissible root of the equation gives two solutions, that is, two positions of the points (1), (2), (3) symmetric with respect to line (II), which is in any case obvious. In the case that +1 or -1 is a root of equation [7], then the above formula for the ordinates cannot be used in any case because then the numerator and denominator become zero, and one has to use instead the following after [6]:

$$\sqrt{\left\{rr - \left(\frac{g \mp F}{\lambda}\right)^2\right\}}.$$

Even here for one root there are two solutions, namely because of the symmetric positions of the points (1), (2), (3) with respect to (II) on opposite sides at equal distances; for $x = +1$ the positive sense on (I) is the same as on (II); for $x = -1$, opposed. Only the single case, without regard to the sign of $\lambda r = g - F$ (for $x = +1$) or $= g + F$ (for $x = -1$) must be considered separately, inasmuch as then both symmetric positions of (I) coincide with (II).

One must exclude from the roots of equation [7] not only the imaginary ones but also the real ones outside the limits -1 and +1, and the roots +1 or -1 themselves, when λr regardless of sign is respectively smaller than $g - F$ or $g + F$. Moreover, it can be proved that at least one of the three roots must always be excluded and that

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[4] That is, first determine x ($= \cos\phi$) from [7], then find $t - Tx$ and T from [4] and [5], and substitute in the coordinates of (2) given earlier.

in any case there can never be more than four different solutions with real coordinates. To be exact, however, a *singular* solution represents an exception to the above statement, since in that case no root is excluded. This singular solution is the one already mentioned where, for $x = \pm 1$, the ordinates $= 0$ and where (as can easily be proved) the required root is counted twice, i.e. where the expression to the left of the equality sign in equation [7] contains the factor $(x \mp 1)^2$; the equation then has only two unequal roots of which the second may certainly be admissible. The conclusion itself, however, remains valid with the exception of the above special case.

Finally, it must be noted that among the real solutions there may be some physically impossible ones. Namely, the locus of point (1) is not the entire circle with centre (1*)[5] through (4) and (5); rather it is only an arc of the circle that lies on the positive side of (4),(5) when θ is less than 180°, and that lies on the negative side when θ is a reflex angle; the same goes for the other two circles. This physical constraint is, however, not taken into account in our solution. Consequently, among the various real solutions, only those are acceptable where the values of the abscissas of the three points (1), (2), (3) all have respectively the same sign as that of the sines of θ, θ', θ''.

I must not omit to mention that, for the singular cases, the given general solution either loses its applicability completely or else requires several modifications, but I restrict myself to only an indication of the most important points:

I. If one of the measured angles = 0 or 180°, then it would be advantageous to name that point as (1) or (3), even if it should lie between the other two points. If one were to choose the latter, then all parts of the general solution remain valid, as long as one considers b as increasing without limit and one keeps it in the calculations as a mere symbol, which thereafter drops off by itself from all results. The angle cannot be 0 or 180° for more than one point, because in that case all three points lie in the line (4),(5) and the problem becomes indeterminate.

II. If two of the measured angles are equal, two of the points (1*), (2*), (3*) coalesce, or one of the quantities a, b, $a+b$ becomes $= 0$, from which also $fF = 0$; in this case the cubic equation reduces to a quadratic. In any case it is easily seen that the vanishing of the first coefficient of the cubic equation occurs only

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[5]The German text here has (1) instead of (1*); this is clearly a misprint.

in the case of equality of two angles.

III. The given solution is not applicable when A, B are proportional to a, b, from which $\lambda = 0$. In this case, the cubic equation is used incorrectly and contains one root not applicable to the given conditions and the correct root twice. It is clear that then the two combinations through which from [2] and [3] the equations [4] and [5] were found are *not different*; these equations become therefore identical, and each gives $x = A/2a = B/2b$. Obviously, one of the equations [2], [3] must be used again, and then a combination with [1] yields: $cx = (t-n)\sqrt{(1-xx)}$. From this it becomes clear that the line (I) passes through either point (4) or (5) and that in fact for this singular case there are always four solutions, by drawing through (4) or (5) one of the straight lines whose angle with the axis of abscissas has as its cosine $A/2a = B/2b$. The reality of this solution depends on this quantity not being greater than 1, for which the four solutions reduce themselves to two.

* * *

234. [1977: 104, 154] (Corrected) *Proposed by Viktors Linis, University of Ottawa.*

If $\sin\frac{2^n\pi}{13} = \pm\sin\frac{\pi}{13}$, prove that

$$\cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{4\pi}{13}\cdots\cos\frac{2^{n-1}\pi}{13} = \pm\frac{1}{2^n}.$$

Gauss's remark: inspect a polygon!

Solution by Gali Salvatore, Ottawa, Ontario.

If

$$P_n = \cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{4\pi}{13}\cdots\cos\frac{2^{n-1}\pi}{13},$$

a trivial induction proof shows that, for $n = 1, 2, \ldots$,

$$2^n\sin\frac{\pi}{13}\cdot P_n = \sin\frac{2^n\pi}{13};$$

hence, for any n for which the hypothesis holds, we have

$$2^n\sin\frac{\pi}{13}\cdot P_n = \pm\sin\frac{\pi}{13},$$

and so $P_n = \pm\frac{1}{2^n}$, as required.

But for which n does the hypothesis hold?

Since 13 is a prime, Fermat's Theorem gives $2^{12} \equiv 1 \pmod{13}$; hence for every nonnegative integer m and $k = 0, 1, \ldots, 5$, we have

$$2^6 \equiv \pm 1, \qquad 2^{6m} \equiv \pm 1, \qquad 2^{6m+k} \equiv \pm 2^k \pmod{13},$$

and so

$$\sin\frac{2^{6m+k}\pi}{13} = \pm\sin\frac{2^k\pi}{13}.$$

It is easy to verify that the equation

$$\sin\frac{2^k\pi}{13} = \pm\sin\frac{\pi}{13}$$

is true for $k=0$ and false for $k = 1, 2, \ldots, 5$. Hence the hypothesis

$$\sin\frac{2^n\pi}{13} = \pm\sin\frac{\pi}{13}$$

holds if and only if n is a multiple of 6.

Solutions and/or comments were also submitted by NICOMEDES ALONSO III and SANDA ILIESCU, both of Long Island City High School, N.Y. (jointly); CLAYTON W. DODGE, University of Maine at Orono; RICK KUZNIAK, Upper Canada College, Toronto; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; JOHN I. NASSAR, Muhlenberg College, Allentown, Pennsylvania; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

Proceeding essentially as in our featured solution, Maskell proved the following generalization:

$$\cos\frac{\pi}{2m+1}\cos\frac{2\pi}{2m+1}\cos\frac{4\pi}{2m+1}\cdots\cos\frac{2^{n-1}\pi}{2m+1} = \pm\frac{1}{2^n}$$

holds if and only if $2m+1$ *is an odd prime and* n *is a multiple of* m.

I don't know what Gauss's own solution was. I think I'll ask the old boy as soon as he returns from inspecting that polygon.

* * *

235. [1977: 105] *Proposed by Viktors Linis, University of Ottawa.*

Prove Gauss's *Theorema Elegantissimum:* If

$$f(x) = 1+\frac{1}{2}\cdot\frac{1}{2}xx+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}x^6+\ldots,$$

show that

$$\sin\phi\, f(\sin\phi)\, f'(\cos\phi) + \cos\phi\, f(\cos\phi)\, f'(\sin\phi) = \frac{2}{\pi\sin\phi\cos\phi}.$$

(Gauss actually wrote xx, but feel free to write x^2 if you prefer.)

I. *Solution by M.S. Klamkin, University of Alberta.*

It is a known result (see, e.g., [2]) that $f(x)$ is proportional to the complete elliptic integral of the first kind, i.e.,

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2\sin^2\theta}} = \frac{2}{\pi}K(x), \text{ say.}$$

Also, it is known and easy to show that

$$K'(k) = \frac{E(k) - k'^2K(k)}{kk'^2},$$

where $E(k)$ is the complete elliptic integral of the second kind and $k'^2 + k^2 = 1$. Consequently, the given identity reduces to

$$K(k)E(k') + K(k')E(k) - K(k)K(k') = \frac{\pi}{2}, \tag{1}$$

where $k = \sin\phi$. However, (1) is a known identity due to Legendre (see, e.g., [1] or [3]).

II. *Comment by the proposer.*

The given series is a special case of the (Gauss) hypergeometrical series (see [4], p. 45)

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}, \qquad |z| < 1,$$

whose derivative is ([4], p. 69, Ex. 1)

$$\frac{d}{dz}F(a,b,c,z) = \frac{ab}{c}F(a+1,\ b+1,\ c+1,\ z).$$

In fact ([4], p. 71, Ex. 19)

$$f(x) = F(\tfrac{1}{2},\ \tfrac{1}{2},\ 1,\ x^2),$$

and a solution to the problem could be worked out on this basis.

However, Gauss's own solution is based on his earlier work (of which he was always very proud!) on the arithmetico-geometric mean. His sketchy solution contained no more than is given below. Interested readers may wish to fill in the yawning gaps.

Gauss uses as his main tool the following result:

if

$$x = \sin\phi = \frac{b}{a}, \qquad \sqrt{1-x^2} = \cos\phi = \frac{c}{a},$$

then

$$f(\sin\phi) = \frac{a}{M(a,c)} \qquad \text{and} \qquad f(\cos\phi) = \frac{a}{M(a,b)},$$

where $M(s,t)$ is the arithmetico-geometric mean of s and t. Then he applies the identity

$$\frac{b^2\, d\log\dfrac{a}{M(a,b)}}{d\log\dfrac{c}{a}} - \frac{c^2\, d\log\dfrac{a}{M(a,c)}}{d\log\dfrac{a}{b}} = \frac{2}{\pi} M(a,b)\, M(a,c)$$

to obtain the required relation.

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4. Earl D. Rainville, *Special Functions*, Chelsea Pub. Co., Bronx, N.Y., 1971.

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236. [1977: 105] *Proposed by Viktors Linis, University of Ottawa.*

Solve the cryptarithmic subtraction:

$$\begin{array}{r} \text{GAUSS} \\ -\ \text{DIED} \\ \hline 1855 \end{array}\ .$$

Solution by Nicomedes Alonso III, Long Island City High School, New York.

First rewrite the problem as an addition:

$$\begin{array}{r} \text{DIED} \\ +\ 1855 \\ \hline \text{GAUSS} \end{array}\ .$$

It is clear that $G = 1$ and $D = 8$ (since $D = 9$ implies $A = G = 1$); hence $S = 3$ and $E = 7$. Applying the remaining digits to I shows that only $I = 5$, 6 yield no contradiction, giving $(I,U) = (5,4)$ or $(6,5)$, with $A = 0$ in both cases.

Thus there are exactly two solutions:

$$\begin{array}{r} 10433 \\ -\ 8578 \\ \hline 1855 \end{array} \quad \text{and} \quad \begin{array}{r} 10533 \\ -\ 8678 \\ \hline 1855 \end{array}\ .$$

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; SANDER A. COHEN, Grade 6, D. Roy Kennedy School, and DAVID COHEN, Grade 10, Nepean High School, both of Ottawa (jointly); CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montreal, Quebec; JUDY LYNCH, Statesboro, Georgia; RAMA KRISHNA MANDAN, Hindustan Mineral Products, Bombay, India; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; and KENNETH M. WILKE, Washburn University, Topeka, Kansas. One incorrect solution was received.

Editor's comment.

Of the above twelve solvers, five found only one of the two answers to the problem, and six sent in *only* an answer. Thus I find it necessary to remind readers that the name of this game is *solutions*, not *answers*; and that a solution consists of an argument (even a brief one) leading to an answer. Readers presumably send in solutions in the hope that the editor will single out theirs as the most worthy of publication (fame is the spur). Sending in only an answer is the surest way of remaining among the also-rans.

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237. [1977: 105] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

It is a well-known theorem (due to Gauss and F. Lucas) that if $f(z)$ is a polynomial with complex coefficients, then the zeros of the derivative $f'(z)$ all lie in the convex hull of the zeros of $f(z)$.

Prove or disprove the following converse: Suppose a closed set E in the complex plane has the property that if a polynomial has all its zeros in E then the derivative also has all its zeros in E; then E is convex.

Solution by Clayton W. Dodge, University of Maine at Orono.

If the set E contains at most one point, then E is closed and convex, and the theorem holds. Otherwise, let z_1 and z_2 be two distinct points in E. We show that, given a rational r with $0 < r < 1$, there is a polynomial f such that

$$f(z_1) = f(z_2) = 0 \qquad \text{and} \qquad f'(rz_2 + (1-r)z_1) = 0.$$

To that end, take $r = \frac{m}{m+n}$, where m and n are positive integers, and let

$$f(z) = (z - z_1)^m (z - z_2)^n,$$

so that

$$f'(z) = (z - z_1)^{m-1}(z - z_2)^{n-1}[(m+n)z - (mz_2 + nz_1)].$$

Then we have

$$f(z_1) = f(z_2) = 0 \qquad \text{and} \qquad f'\left(\frac{mz_2 + nz_1}{m+n}\right) = f'(rz_2 + (1-r)z_1) = 0.$$

Thus E contains the dense subset of the segment between z_1 and z_2 consisting of all points of the form $rz_2 + (1-r)z_1$ with r rational. Since E is closed, it must contain *all* points of the segment, and so E is convex.

The word "closed" cannot be deleted from the hypothesis. For otherwise the set E of all algebraic numbers would satisfy the hypothesis; yet in this case E is not convex since no polynomial with algebraic coefficients can have transcendental zeros.

Also solved, in much the same way, by LEROY F. MEYERS, The Ohio State University; and the proposer.

Editor's comment.

The theorem of Gauss-Lucas mentioned in the proposal is an extension to the complex plane of the following well-known corollary to Rolle's Theorem: *any interval of the real axis containing all the zeros of a given polynomial P also contains all the critical points of P.*

Much fascinating information about the zeros of polynomials and their derivatives can be found in the references given below. (References [2]-[5] were obtained from [1].)

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238. [1977: 105] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution to this cryptarithm having both 1 and 7 represented among the letters:

$$\begin{array}{r} \text{CARL} \\ +\,1777 \\ \hline \text{GAUSS} \end{array}.$$

Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.

Clearly $G=1$ and $A=0$; then $C=9$ since there is no carry-over from the preceding column. There is a carry-over from the first column on the right to the next (otherwise $L=R$); hence

$$L+7 = 10+S, \quad 1+R+7 = 10+S, \quad 1+7 = U.$$

Thus $U=8$, $L=3+S$, and $R=2+S$.

We know that 7 is one of the letters. If $S=7$, then $R=9=C$; if $R=7$, then

$S = 5$ and $L = 8 = U$. Therefore $L = 7$, from which $S = 4$ and $R = 6$ follow.

The unique solution is thus

$$\begin{array}{r} 9067 \\ +\,1777 \\ \hline 10844 \end{array},$$

and it may be of interest to note that 9067 and 1777 are both primes.

Also solved by NICOMEDES ALONSO III, Long Island City High School, New York; LOUIS CAIROLI, Kansas State University, Manhattan, Kansas; SANDER A. COHEN, Grade 6, D. Roy Kennedy School, and DAVID COHEN, Grade 10, Nepean High School, both of Ottawa (jointly); ROBERT S. JOHNSON, Montreal, Quebec; JUDY LYNCH, Statesboro, Georgia; SURESH KUMAR MANDAN, Mazagon Docks, Bombay, India; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, New York; R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

Six of the above solvers submitted only an answer unsupported by any argument. Printing their "solutions" would save a lot of space but would be mathematically unproductive.

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239. [1977: 105] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve this addition cryptarithm. There is a unique solution in which each of the date digits 1, 5, 7, 8 is represented by a letter:

$$\begin{array}{r} \text{CARL} \\ 1777 \\ 1855 \\ \hline \text{GAUSS} \end{array}.$$

Solution by Louis H. Cairoli, Kansas State University, Manhattan, Kansas.

It is immediately clear that

$$G = 1, \quad C = 7,\ 8 \text{ or } 9, \quad A = 0,\ 2 \text{ or } 3.$$

Now $(C,A) = (9,0)$ is clearly impossible and it is tedious but easy to verify that $(C,A) = (7,2)$, $(7,3)$, $(8,0)$, $(8,2)$, $(8,3)$ and $(9,3)$ lead to impossible carry-overs or other contradictions. Furthermore, $(C,A) = (7,0)$ leads to a unique solution in which no letter represents 8.

This leaves only $(C,A) = (9,2)$, which requires $U = 8$. Applying to S the unused digits leads to contradictions for $S = 0, 3, 4, 5$; and $S = 6$ leads to a unique solution in which no letter represents 5 or 7. This leaves $S = 7$, from which $L = 5$ and $R = 4$, and we have the unique satisfactory solution

$$\begin{array}{r} 9245 \\ 1777 \\ \underline{1855} \\ 12877 \end{array}.$$

Also solved by NICOMEDES ALONSO III, Long Island City High School, New York; SANDER A. COHEN, Grade 6, D. Roy Kennedy School, and DAVID COHEN, Grade 10, Nepean High School, both of Ottawa (jointly); ROBERT S. JOHNSON, Montreal, Quebec; HARISH CHANDRA MANDAN, Kharagpur, India; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer. One incorrect solution was received.

Editor's comment. Six of the above solvers sent in only an answer.

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240. [1977: 105] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution for this base ten cryptarithm:

$$\begin{array}{r} \text{CARL} \\ \underline{\times \quad \text{F}} \\ \text{GAUSS} \end{array}.$$

I. *Solution by W_1, W_2, W_3, W_4, W_5 (independently).*

...so the unique solution is

$$\begin{array}{r} 7483 \\ \underline{\times \quad 2} \\ 14966 \end{array}.$$

II. *Solution by X_1, X_2 (independently).*

...so the unique solution is

$$\begin{array}{r} 7196 \\ \underline{\times \quad 3} \\ 21588 \end{array}.$$

III. *Solution by Y.*

...so the only two solutions are

$$\begin{array}{r} 7483 \\ \underline{\times \quad 2} \\ 14966 \end{array} \quad \text{and} \quad \begin{array}{r} 7196 \\ \underline{\times \quad 3} \\ 21588 \end{array}.$$

IV. *Solution by Z_1, Z_2, Z_3, Z_4 (independently).*

...so the unique solution is

$$\begin{array}{r} 5097 \\ \underline{\times \quad 4} \\ 20388 \end{array}.$$

Solutions were submitted by NICOMEDES ALONSO III, Long Island City High School, New York; SANDER A. COHEN, Grade 6, D. Roy Kennedy School, and DAVID COHEN, Grade 10, Nepean High School, both of Ottawa (jointly); ROBERT S. JOHNSON, Montreal, Quebec; JUDY LYNCH, Statesboro, Georgia; SUSHILA DEVI MANDAN, Indian Institute of Technology, Kharagpur, India; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

The proposer will report to the principal and receive ten lashes with a limp sine curve for misleading his classmates by claiming his problem had a unique solution.

All solutions submitted were incomplete, none having all three of the known answers. And there may well be other answers still to be found. But finding them all, or proving there are no others, may require an excessive amount of calculation if done by hand, although I'm sure even a baby computer could gurgle out all the answers in a few moments. I hope someone with access to a computer will undertake this chore and let me know the result so that I can later give readers a final report on this game of the name.

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241. [1977: 130] *Proposed by John J. McNamee, Executive Director, Canadian Mathematical Congress.*

Solve the base ten cryptarithm:

$$(HE)(EH) = WHEW.$$

I. *Solution by Charles W. Trigg, San Diego, California.*

We have

$$(HE)(EH) = WHEW = WooW + 10(HE);$$

hence

$$(HE)(EH - 10) = W(1001) = W(7)(11)(13).$$

Since $H \neq E$, $EH - 10$ is a multiple of 11 with $E - 1 = H$, and so HE is a multiple of 13 having consecutive digits, namely $6(13) = 78$.

The unique reconstruction of the equation is

$$(78)(87) = 6786.$$

No numerological nut would fail to find cosmic significance in the following facts: the sum of the digits on the right is 3^3, the difference of the factors on the left is 3^2, and the number of letters to be identified is 3.

II. *Comment by Herman Nyon, Paramaribo, Surinam.*

There are solutions to this cryptarithm in some bases other than ten. I have made a straightforward analysis and am able to report that in base b, $4 \le b \le 30$, $b \ne 10$, there is a solution (unique in each case) only for b = 11, 20, 29. These solutions are:

b	HE × EH = WHEW	Equivalent in base ten
11	34 × 43 = 1341	37 × 47 = 1739
20	67 × 76 = 2672	127 × 146 = 18542
29	9(10) × (10)9 = 39(10)3	271 × 299 = 81029

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montreal, Quebec; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; HARRY L. NELSON, Livermore, California; HERMAN NYON, Paramaribo, Surinam (solution as well); SIDNEY PENNER, Bronx Community College, New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Editor's comment.

Only a numerological *nut* would have failed to predict as well that the solution would appear in EUREKA Vol. 3, No. 3^2. E.P.B. Umbugio would never have overlooked this. But numerology has its limitations: this is shown by the fact that the cryptarithm has no solution in bases 3, 3^2, or 3^3.

Wilke pointed out that this cryptarithm is isomorphic to Problem 18 in [1], which also appeared as Problem 280 in [2].

This problem can now be laid to rest in the same grave as its isomorphic twin (HA)(AH) = THAT.

REFERENCES.

1. James F. Hurley, *Litton's Problematical Recreations*, Van Nostrand Reinhold, 1971, p. 266.

2. *The Pentagon*, Vol. 36, No. 1 (Fall 1976), p. 35.

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242. [1977: 130] *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

Give a geometrical construction for determining the focus of a parabola when two tangents and their points of contact are given.

I. *Solution by Dan Pedoe, University of Minnesota.*

This is Example 12 on page 37 of [6], and is dependent on the two previous Examples, which are:

10. If the tangents (to a parabola with focus S) at P, P' meet in T, the triangles SPT, STP' are similar.

11. If T is any point on the tangent at P, the circle SPT touches the second tangent from T to the parabola.

Hence, given the tangents TP and TP', the focus S can be determined from the following

Construction. Construct the circle through T and P which touches TP' at T, and the circle through T and P' which touches TP at T (see Figure 1). The other intersection of the two circles is the focus S.

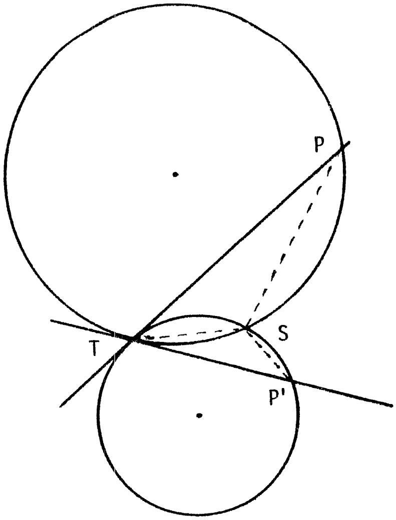

Figure 1

II. *Solution by R. Robinson Rowe, Sacramento, California.*

Construction. The given tangents being TP and TP' (see Figure 2), join PP' and bisect it at M. Join TM. The focus is then the point S determined by the arms of the angles TPS and TP'S equal respectively to angles PTM and P'TM.

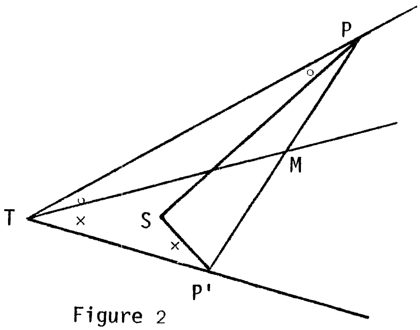

Figure 2

Proof. The line TM, connecting the intersection of two tangents to the midpoint of the chord of their contacts, is a diameter of the parabola. Diameters of a parabola are reflected by the curve (and/or its tangent) through the focus. The angles of incidence of the diametric rays at P and P' are equal to those between the tangents and diameter TM, since all diameters are parallel. The reflected rays are then PS and P'S, intersecting at focus S.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; JOSEPH D.E. KONHAUSER, Macalester College, St. Paul, Minnesota; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India (two solutions); F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; BASIL C. RENNIE, James Cook University of North Queensland, Australia. One incorrect solution was received.

Editor's comment.

Most of the other solutions received were essentially equivalent to the two given above.

The references given below, about this problem and related matters, were all sent in by readers, for which the editor is grateful. Solutions accompanied by references, including photocopies when convenient, greatly assist the editor in his task.

REFERENCES

1. E.H. Askwith, *A Course of Pure Geometry*, Hindustani Press, Patna, 1956, pp. 132-133.

2. H.F. Baker, *Principles of Geometry II*, Cambridge University Press, 1954, p. 81, Ex. 1.

3. __________, *An Introduction to Plane Geometry*, Chelsea, 1971, pp. 137-138.

4. L. Cremona, *Elements of Projective Geometry*, Dover, 1960, pp. 246, 265.

5. S.L. Loney, *Coordinate Geometry*, Macmillan, 1962, pp. 220-221.

6. F.S. Macaulay, *Geometrical Conics*, Cambridge University Press, 1921.

7. W.F. Osgood and W.C. Graustein, *Plane and Solid Analytic Geometry*, Macmillan, 1921, p. 100, Problem 26.

8. G. Salmon, *A Treatise on Conic Sections*, Chelsea, (no date) pp. 205-206, 212.

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243. [1977: 130] *Proposé par Hippolyte Charles, Waterloo, Québec.*

a) Trouver des conditions nécessaires et suffisantes pour que le plus grand commun diviseur des entiers positifs a et b, $a > b$, soit égal à leur différence.

b) *Application.* Trouver toutes les paires d'entiers positifs dont le plus grand commun diviseur est égal à leur différence et dont le plus petit commun multiple est 180.

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

a) Let $(a,b) = d$, with $a = dr$ and $b = ds$. If $d = a - b = d(r - s)$, then $r - s = 1$, so $r = s + 1$ and

$$a = d(s+1), \qquad b = ds. \tag{1}$$

Conversely, if a and b satisfy (1) for some positive integers s and d, then $a - b = d$ and

$$(a,b) = (d(s+1),\ ds) = d(s+1,\ s) = d = a - b.$$

Thus a necessary and sufficient condition is that a and b be the same multiple of two consecutive integers.

b) If

$$180 = [a,b] = \frac{ab}{d} = ds(s+1),$$

the only values of s for which $s(s+1)|180$ are $s = 1, 2, 3, 4, 5, 9$, from which we get the table

s	1	2	3	4	5	9
d	90	30	15	9	6	2
a	180	90	60	45	36	20
b	90	60	45	36	30	18

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas (partial solution); DOUG DILLON, Brockville, Ontario (partial solution); CLAYTON W. DODGE, University of Maine at Orono; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam (partial solution); BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; DAVID R. STONE, University of Kentucky, Lexington, Kentucky; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

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THE MANAGING EDITOR TAKES A BOW

F.G.B. MASKELL, Managing Editor

Readers who like to begin at the beginning will notice some changes this month in the management of EUREKA, as indicated on the front page of this issue: the Secretary-Treasurer of COMA is no longer directly involved; instead EUREKA now has a Managing Editor—and the price of a subscription is going up.

The two changes are not related—you will not be paying more for EUREKA in order to put a managing editor on salary. So with the Editor's agreement I can now reveal that the former Secretary-Treasurer of COMA did not do all that good a job in looking after EUREKA, and he was persuaded to hand over the commitments of his office to a colleague on the Executive—Professor Rosaline Séguin of the University of Ottawa—and to concentrate his attention on the administration of the Association's problem-solving monthly. Thus he became the Managing Editor, and a EUREKA bureaucracy has begun to take shape (we hope it is only a beginning).

In the first two years of its existence, EUREKA was sent free to all who asked for it—and to many who didn't—but we had no means of knowing how many welcomed each month's challenge. We found out in January 1977, when we first started to charge for subscriptions: our circulation dropped from about 400 to under 100. Since then, however, fanned almost exclusively by word of mouth, our circulation has tripled to about 275 interested paid-up subscribers in Australia, Canada, Denmark, Greece, India, Israel, The Netherlands, Surinam, Sweden, The United Kingdom, and the United States.

Since the combined population of these countries is about 750,000,000, there is still some slack to be picked up (not to speak of other countries). So, on the basis of growth rate, EUREKA has proved to be a success as a publishing venture (in the professional sense; financially there have been no profits, no losses, no pay for editors or contributors), and we can confidently conclude that there is a place under the sun for a monthly of this kind.

In 1975 - 1976, EUREKA averaged 17 pages an issue, and our 1977 subscription rates were set on that basis. In 1977, however, the issues averaged 30 pages each and this fact, together with the increase in postal rates, obliged us to raise the subscription price for 1978 from \$6 to \$8 (everyone having the option of receiving it by first-class mail for an additional \$1.50). Even at that price, of course, we would need a far larger subscription list to cover all our costs. As a service to the cause of mathematical education, Algonquin College has been very generous in supporting our operations so that we have managed so far to stay out of the red, but there is no reason why it should carry the burden alone: we would gladly clasp more institutional sponsors to our corporate bosom. Some individual subscribers may wish to obey a generous impulse by making a contribution to the EUREKA Sustaining Fund.

The bound combined Vol. 1-2 (1975 - 1976) will continue to be available at \$10. Early in the new year, a bound edition of Vol. 3 (1977), reduced in size and provided with a full index, will also be available at \$10. But for those who order the bound Vol. 3 along with their 1978 subscription we will, as we did last year, reduce the price to \$4, making \$12 in all for a 1978 subscription and a bound copy of Vol. 3.

Subscription renewal forms for 1978 will be mailed shortly to those whose subscription ends with the December 1977 issue. The Managing Editor hopes that readers will fill them out and return them promptly, if possible by return mail.

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SYMMETRY

Symmetry was an ideal of eighteenth-century architecture and horticulture to which Mr. Craik rigidly adhered. Robert Adam designed his new mansion house, which replaced the old one in 1755; and ten years earlier, after the great wall about the garden was erected, Mr. Craik insisted, against John Paul's objections, on building two round stone summerhouses, since one alone would have been unsymmetrical. The gardener, however, had his revenge. He caught a man stealing fruit, locked him up in one of the summerhouses; then clapped John Paul Jr. in the other and sent for Mr. Craik. The owner, astonished at seeing little John peering out of a window, inquired what he was doing there; to which John Paul Sr. replied, "I just put him in for the sake of symmetry!"

From Chapter I in *John Paul Jones*, by Samuel Eliot Morison, Atlantic Monthly Press, 1959.