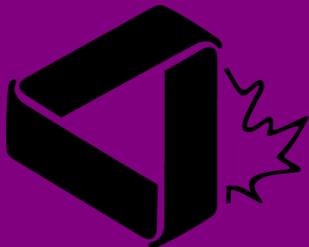


Mathematicorum

Crux

Published by the Canadian Mathematical Society.



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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 17 #6

June / juin

1991

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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1991 subscription rate for ten issues is \$ 17.50 for members of the Canadian Mathematical Society and \$35.00, for non-members. Back issues: \$3.50 each. Bound Volumes with index: volumes 1 & 2 (combined) and each of 3, 7, 8 & 9: \$10.00 (Volumes 4, 5, 6 & 10 are out-of-print). All prices are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

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Canadian Mathematical Society
577 King Edward
Ottawa, Ontario, Canada K1N 6N5

ACKNOWLEDGEMENTS

The support of the Department of Mathematics and Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

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Published by the Canadian Mathematical Society
Printed at Ottawa Laser Copy

ISSN 0705-0348

Second Class Mail Registration Number 5432

THE OLYMPIAD CORNER

No. 126

R.E. WOODROW

*All communications about this column should be sent to Professor R.E. Woodrow,
Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta,
Canada, T2N 1N4.*

This month we begin with the *Canadian Mathematics Olympiad* for 1991, which we reproduce with the permission of the Canadian Olympiad Committee of the Canadian Mathematical Society. My thanks to Ed Barbeau for sending me the contest along with the “official” solutions which will be given in the next issue.

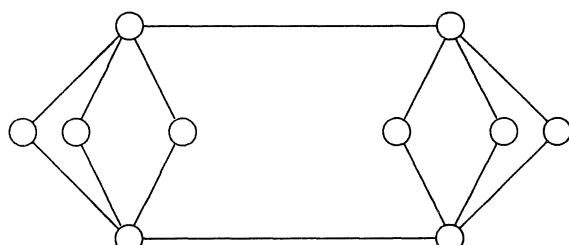
1991 CANADIAN MATHEMATICS OLYMPIAD

April 1991

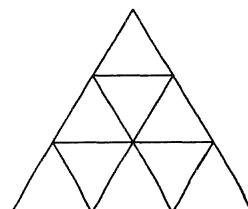
Time: 3 hours

1. Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.
2. Let n be a fixed positive integer. Find the sum of all positive integers with the following property: in base 2, it has exactly $2n$ digits consisting of n 1's and n 0's. (The first digit cannot be 0.)
3. Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C . Show that the midpoints of these chords lie on a circle.

4. Ten distinct numbers from the set $\{0, 1, 2, \dots, 13, 14\}$ are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.



5. In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for $f(n)$, the number of parallelograms, for a triangle of side length n .



* * *

The next set of problems are from the twentieth annual *United States of America Mathematical Olympiad*, written in April. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588-0322. As always we welcome your original "nice" solutions and generalizations.

20th USA MATHEMATICAL OLYMPIAD

April 23, 1991

Time: 3.5 hours

- 1.** In triangle ABC , angle A is twice angle B , angle C is obtuse, and the three side lengths a, b, c are integers. Determine, with proof, the minimum possible perimeter.
- 2.** For any nonempty set S of numbers, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S . Prove that

$$\sum \frac{\sigma(S)}{\pi(S)} = (n^2 + 2n) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)(n+1),$$

where the " \sum " denotes a sum involving all nonempty subsets S of $\{1, 2, 3, \dots, n\}$.

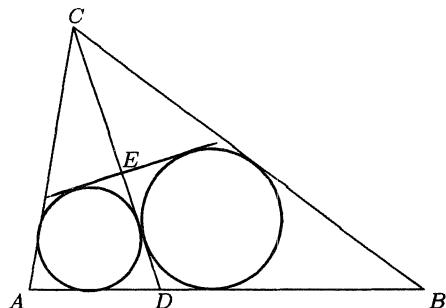
- 3.** Show that, for any fixed integer $n \geq 1$, the sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \dots \pmod{n}$$

is eventually constant. [The tower of exponents is defined by $a_1 = 2$, $a_{i+1} = 2^{a_i}$. Also, $a_i \pmod{n}$ means the remainder which results from dividing a_i by n .]

- 4.** Let $a = \frac{m^{m+1} + n^{n+1}}{m^m + n^n}$, where m and n are positive integers. Prove that $a^m + a^n \geq m^m + n^n$. [You may wish to analyze the ratio $\frac{a^N - N^N}{a - N}$ for real $a \geq 0$ and integer $N \geq 1$.]

- 5.** Let D be an arbitrary point on side AB of a given triangle ABC , and let E be the interior point where CD intersects the external, common tangent to the incircles of triangles ACD and BCD . As D assumes all positions between A and B , prove that point E traces the arc of a circle.



* * *

Next is a selection of problems from the *15th All Union Mathematical Olympiad – 10th Grade*. These appeared in *Kvant* and were translated by Hillel Gauchman and Duane Broline of Eastern Illinois University, Charleston. My thanks to them for sending these in.

15TH ALL UNION MATHEMATICAL OLYMPIAD–TENTH GRADE

First Day

1. Find natural numbers $a_1 < a_2 < \dots < a_{2n+1}$ which form an arithmetic sequence such that the product of all terms is the square of a natural number. (5 points)

2. The numbers $1, 2, \dots, n$ are written in some order around the circumference of a circle. Adjacent numbers may be interchanged provided the absolute value of their difference is larger than one. Prove in a finite number of such interchanges it is possible to rearrange the numbers in their natural order. (5 points)

3. Let ABC be a right triangle with right angle at C and select points D and E on sides AC and BC , respectively. Construct perpendiculars from C to each of DE , EA , AB , and BD . Prove that the feet of these perpendiculars are on a single circle. (10 points)

4. Let $a \geq 0$, $b \geq 0$, $c \geq 0$, and $a + b + c \leq 3$. Prove

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3}{2} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} .$$

(first inequality 3 points, second inequality 7 points)

Second Day

5. Prove, for any real number c , that the equation

$$x(x^2 - 1)(x^2 - 10) = c$$

cannot have five integer solutions. (10 points)

6. A rook is placed on the lower left square of a chessboard. On each move, the rook is permitted to jump one square either vertically or horizontally. Prove the rook may be moved so that it is on one square once, one square twice, \dots , one square 64 times, and

(a) so that the last move is to the lower left square,

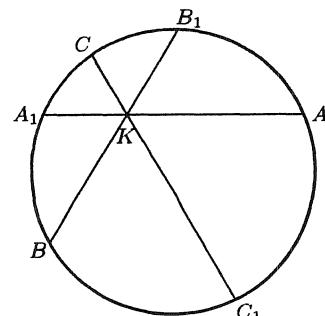
(b) so that the last move is to a square adjacent (along an edge) to the lower left square.

(part (a) 4 points, part (b) 4 points)

7. Three chords, AA_1 , BB_1 , CC_1 , to a circle meet at a point K where the angles B_1KA and AKC_1 are 60° , as shown. Prove that

$$KA + KB + KC = KA_1 + KB_1 + KC_1 .$$

(10 points)



8. Is it possible to put three regular tetrahedra, each having sides of length one, inside a unit cube so that the interiors of the tetrahedra do not intersect (the boundaries are allowed to touch)? (12 points)

* * *

We now turn to solutions of “archive” problems received from the readership. The problems are from the *1985 Spanish Olympiad* that appeared in the May 1986 number of the Corner.

4. [1986: 97] *1985 Spanish Mathematical Olympiad, 1st Round.*

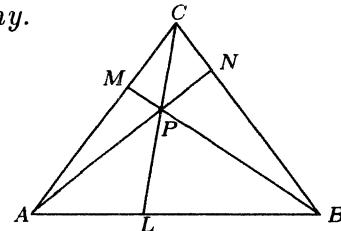
L and M are points on the sides AB and AC , respectively, of triangle ABC such that $\overline{AL} = 2\overline{AB}/5$ and $\overline{AM} = 3\overline{AC}/4$. If BM and CL intersect at P , and AP and BC intersect at N , determine $\overline{BN}/\overline{BC}$.

Solution by Hans Engelhardt, Gundelsheim, Germany.

From Ceva's theorem one has

$$\frac{\overline{AL}}{\overline{LB}} \cdot \frac{\overline{BN}}{\overline{NC}} \cdot \frac{\overline{CM}}{\overline{MA}} = 1$$

This gives $\overline{BN}/\overline{NC} = 9/2$ and so $\overline{BN}/\overline{BC} = 9/11$.



5. [1986: 97] *1985 Spanish Mathematical Olympiad, 1st Round.*

Determine all the real roots of $4x^4 + 16x^3 - 6x^2 - 40x + 25 = 0$.

Solution by Hans Engelhardt, Gundelsheim, Germany.

Dividing by x^2 we get $4x^2 + 16x - 6 - 40/x + 25/x^2 = 0$. The substitution $z = 2x - 5/x$ yields $z^2 + 8z + 14 = 0$. This gives solutions $z_1 = -4 + \sqrt{2}$ and $z_2 = -4 - \sqrt{2}$.

Case 1. $2x^2 - (-4 + \sqrt{2})x - 5 = 0$. Then

$$x = \frac{-4 + \sqrt{2} \pm \sqrt{58 - 8\sqrt{2}}}{4}.$$

Case 2. $2x^2 + (4 + \sqrt{2})x - 5 = 0$. Then

$$x = \frac{-4 - \sqrt{2} \pm \sqrt{58 + 8\sqrt{2}}}{4}.$$

This gives the four real roots.

2. [1986: 98] *1985 Spanish Mathematical Olympiad, 2nd Round.*

Let \mathbf{Z} be the set of integers and $\mathbf{Z} \times \mathbf{Z}$ be the set of ordered pairs of integers. On $\mathbf{Z} \times \mathbf{Z}$, define $(a, b) + (a', b') = (a + a', b + b')$ and $-(a, b) = (-a, -b)$. Determine if there exists a subset E of $\mathbf{Z} \times \mathbf{Z}$ satisfying:

- (i) Addition is closed in E ,
- (ii) E contains $(0, 0)$,
- (iii) For every $(a, b) \neq (0, 0)$, E contains exactly one of (a, b) and $-(a, b)$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Yes. Let

$$E = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : \text{ either } a \geq 0, b \geq 0 \text{ or } a < 0, b > 0\}$$

(i.e. E is the set of all lattice points in the first and second quadrants, together with those on the non-negative x -axis). It is easy to verify that E satisfies all three conditions.

7. [1986: 98] 1985 Spanish Mathematical Olympiad, 2nd Round.

Determine the value of p such that the equation $x^5 - px - 1 = 0$ has two roots r and s which are the roots of an equation $x^2 - ax + b = 0$ where a and b are integers.

Solution by Hans Engelhardt, Gundelsheim, Germany, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

If p satisfies the condition

$$x^5 - px - 1 = (x^2 - ax + b)(x^3 + ux^2 + vx - 1/b),$$

comparing coefficients of x^4 we get $u - a = 0$, so $u = a$. From coefficients of x^3 and x^2 we then get, respectively,

$$b - a^2 + v = 0 \quad \text{and} \quad \frac{-1}{b} - av + ab = 0, \quad (1)$$

and from coefficients of x ,

$$\frac{a}{b} + bv = -p. \quad (2)$$

From (1),

$$\frac{-1}{b} - a(a^2 - b) + ab = 0.$$

Since a and b are integers it follows that b must be ± 1 .

Now, $b = -1$ gives $a^3 + 2a - 1 = 0$. Since $(-1)^3 + 2(-1) - 1 = -4$ and $1^3 + 2 \cdot 1 - 1 = 2$, this gives no rational and hence no integer solutions for a . The case $b = 1$ gives $a^3 - 2a + 1 = 0$ which gives $a = 1$. From (1) $v = 0$ and so from (2) $p = -1$.

8. [1986: 98] 1985 Spanish Mathematical Olympiad, 2nd Round.

A square matrix is “sum-magic” if the sum of all the elements on each row, column, and major diagonal is constant. Similarly, a square matrix is “product-magic” if the product of all the elements in each row, column, and major diagonal is constant. Determine if there exist 3×3 matrices of real numbers which are both “sum-magic” and “product-magic”.

Solution by Hans Engelhardt, Gundelsheim, Germany.

If the square is “sum-magic” then it must have the form

$$\begin{bmatrix} m-x & m+x+y & m-y \\ m+x-y & m & m-x+y \\ m+y & m-x-y & m+x \end{bmatrix}$$

where $3m$ is the constant value of the sums. [Editor's note. To see this, first observe that we can assume that the row sum is zero, hence $m = 0$, since subtracting the same amount from all entries gives another "sum-magic" matrix. We may then suppose that the matrix has the form

$$\begin{bmatrix} -x & x+y & -y \\ -a & a+b & -b \\ x+a & -(x+y+a+b) & y+b \end{bmatrix}.$$

Now from the diagonal sums,

$$a + 2b + y - x = 0 \quad \text{and} \quad 2a + b + x - y = 0.$$

Adding these two equations gives $a = -b$, so $a = y - x$ and $b = x - y$, from which the claim is immediate.]

If the square is "product-magic" then from the diagonals,

$$(m-x)m(m+x) = (m-y)m(m+y). \quad (1)$$

Case 1. $m = 0$. From the row and column products,

$$xy(x+y) = 0 = xy(x-y).$$

Solving, we obtain $x = 0$ or $y = 0$ (or both). This gives the two forms

$$\begin{bmatrix} r & -r & 0 \\ -r & 0 & r \\ 0 & r & -r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & s & -s \\ -s & 0 & s \\ s & -s & 0 \end{bmatrix}.$$

Case 2. $m \neq 0$. Then from (1), $x^2 = y^2$. If $x = y$, then from the product for the second row and column,

$$m(m^2 - 4x^2) = m^3$$

so $m^2 - 4x^2 = m^2$ and $x = 0$, giving all entries equal. If $x = -y$ we similarly obtain $m^3 = m(m^2 - 4x^2)$ and $x = 0$ giving all entries equal.

In summary, a "sum-magic" square that is also "product-magic" has all entries equal unless one of the diagonals contains only zeroes, and then the matrix is uniquely determined by an off-diagonal entry.

* * *

We now turn to solutions to the problems from the October 1989 number of the Corner. These were problems proposed to the jury, but not used, at the 30th IMO at Braunschweig, (then) West Germany [1989: 225–226]. We give the solutions we've received which differ from the "official" solutions published in the booklet *30th International Mathematical Olympiad* (see the review on [1991: 42]).

6. Proposed by Greece.

Let $g : \mathbf{C} \rightarrow \mathbf{C}$, $\omega \in \mathbf{C}$, $a \in \mathbf{C}$, with $\omega^3 = 1$ and $\omega \neq 1$. Show that there is one and only one function $f : \mathbf{C} \rightarrow \mathbf{C}$ such that

$$f(z) + f(\omega z + a) = g(z), \quad z \in \mathbf{C}.$$

Find the function f .

Solutions by George Evangelopoulos, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

If $f : \mathbf{C} \rightarrow \mathbf{C}$ satisfies

$$g(z) = f(z) + f(\omega z + a) \quad \text{for all } z \in \mathbf{C}, \quad (1)$$

then

$$g(\omega z + a) = f(\omega z + a) + f(\omega^2 z + \omega a + a) \quad \text{for all } z \in \mathbf{C}. \quad (2)$$

Since $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$, we have

$$\omega^2 z + \omega a + a = \omega^2 z + a(1 + \omega) = \omega^2(z - a) = \frac{z - a}{\omega},$$

and so

$$g\left(\frac{z - a}{\omega}\right) = f\left(\frac{z - a}{\omega}\right) + f\left(w \cdot \frac{z - a}{\omega} + a\right) = f(\omega^2 z + \omega a + a) + f(z). \quad (3)$$

Therefore (1) minus (2) plus (3) yields

$$g(z) - g(\omega z + a) + g\left(\frac{z - a}{\omega}\right) = 2f(z),$$

and so

$$f(z) = \frac{1}{2} \left\{ g(z) - g(\omega z + a) + g\left(\frac{z - a}{\omega}\right) \right\}.$$

On the other hand, it is easily checked that this choice of f satisfies (1) and hence it is the unique function with this property.

7. Proposed by Hungary.

Define the sequence $\{a_n\}_{n=1}^{\infty}$ of integers by

$$\sum_{d|n} a_d = 2^n.$$

Show that $n|a_n$. [Editor's note. Of course $x|y$ means that x divides y .]

Solution by Graham Denham, student, University of Alberta, and by Curtis Cooper, Central Missouri State University.

The proof is by induction on n . For a_1 , the result is obvious. Now assume that $i|a_i$, for all $i < n$. In order to show that $n|a_n$, it will suffice to show that $p^\alpha|a_n$, where p^α is any prime-power divisor of n . Say $n = p^\alpha k$, where k is not divisible by p . Then

$$\begin{aligned} 2^n &= \sum_{d|p^\alpha k} a_d = \sum_{d|p^{\alpha-1}k} a_d + \sum_{d|k} a_{p^\alpha d} \\ &= 2^{p^{\alpha-1}k} + \sum_{d|k, d < k} a_{p^\alpha d} + a_{p^\alpha k}. \end{aligned}$$

So,

$$a_{p^\alpha k} = 2^{p^\alpha k} - 2^{p^{\alpha-1}k} - \sum_{d|k, d < k} a_{p^\alpha d}.$$

Now, by hypothesis, each term $a_{p^\alpha d}$ is divisible by p^α , since $p^\alpha d < n$. Therefore,

$$\begin{aligned} a_n = a_{p^\alpha k} &\equiv (2^{p^\alpha k} - 2^{p^{\alpha-1}k}) \pmod{p^\alpha} \\ &\equiv 2^{p^{\alpha-1}k} (2^{p^{\alpha-1}k(p-1)} - 1) \pmod{p^\alpha}. \end{aligned}$$

If $p = 2$, then $2^{p^{\alpha-1}k} \equiv 0 \pmod{p^\alpha}$, since $p^{\alpha-1}k \geq \alpha$. For $p > 2$, we may use Euler's Theorem, giving

$$2^{\phi(p^\alpha)} \equiv 1 \pmod{p^\alpha},$$

that is,

$$2^{p^{\alpha-1}(p-1)} \equiv 1 \pmod{p^\alpha}.$$

So $2^{p^{\alpha-1}(p-1)k} \equiv 1 \pmod{p^\alpha}$, again giving $a_n \equiv 0 \pmod{p^\alpha}$. Hence $p^\alpha|a_n$, and as p was arbitrary, $n|a_n$, and the result follows by induction.

[*Editor's note.* This is the solution sent in by Graham Denham. The solution of Curtis Cooper differed by an initial application of Möbius inversion giving $a_n = \sum_{d|n} \mu(d)2^{n/d}$, where μ denotes the Möbius function. For an elegant combinatorial solution see the official solution, in which a_n is viewed as the number of nonrepeating 0–1 sequences.]

9. Proposed by Ireland.

Let a, b, c, d, m, n be positive integers such that

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 1989, \\ a + b + c + d &= m^2, \end{aligned}$$

and the largest of a, b, c, d is n^2 . Determine, with proof, the values of m and n .

Solution by Hans Engelhardt, Gundelsheim, Germany, and also by George Evangelopoulos, Athens, Greece.

Without loss of generality we may suppose $0 < a \leq b \leq c \leq d$. Let $S = a^2 + b^2 + c^2 + d^2$. Since $x^2 + y^2 \geq 2xy$ it follows that

$$3S \geq 2(ab + ac + ad + bc + bd + cd)$$

whence

$$4S \geq (a+b+c+d)^2.$$

Thus $4 \cdot 1989 \geq m^4$ and $m \leq 9$.

But $(a+b+c+d)^2 > a^2 + b^2 + c^2 + d^2$ so $m^4 > 1989$ and $m > 6$. Since $a^2 + b^2 + c^2 + d^2$ is odd, so is $m^2 = a + b + c + d$, thus $m = 7$ or $m = 9$. Suppose for a contradiction that $m = 7$. Now $(49-d)^2 = (a+b+c)^2 > a^2 + b^2 + c^2 = 1989 - d^2$. Thus $d^2 - 49d + 206 > 0$. It follows that $d > 44$ or $d \leq 4$. However, $d < 45$, since $45^2 = 2025$. Also, $d \leq 4$ implies $a^2 + b^2 + c^2 + d^2 \leq 64 < 1989$.

It follows that $m = 9$. Now $n^2 = d > 16$, since $d \leq 16$ implies $a+b+c+d \leq 64 < 81$. As before $d \leq 44$ so $n^2 = 25$ or 36 . If $d = n^2 = 25$ then let $a = 25-p$, $b = 25-q$, $c = 25-r$, with $p, q, r \geq 0$. Furthermore $a+b+c = 56$ implies $p+q+r = 19$. $(25-p)^2 + (25-q)^2 + (25-r)^2 = 1364$ implies $p^2 + q^2 + r^2 = 439$. Now $(p+q+r)^2 > p^2 + q^2 + r^2$ gives a contradiction.

Thus the only possibility is that $n = 6$, and there is a solution with $a = 12$, $b = 15$, $c = 18$, $d = 36$. Thus $m = 9$ and $n = 6$.

11. Proposed by Mongolia.

Seven points are given in the plane. They are to be joined by a minimal number of segments such that at least two of any three points are joined. How many segments has such a figure? Give an example.

Generalization and solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We use the terminology of graph theory. For each positive integer n , let ℓ_n denote the minimum number of edges that a graph on n vertices must have so that among any three vertices at least two are joined by an edge. Then we claim

$$\ell_n = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

To see this it is useful to consider m_n , the *maximum* number of edges that a graph on n vertices can have, but contain *no* triangle. We first show that $\ell_n + m_n = \binom{n}{2}$. Let G be a graph on n vertices with ℓ_n edges so that there are no “empty triangles”. Then the complementary graph \overline{G} , obtained by interchanging edges and non-edges, is a graph without triangles, so $\binom{n}{2} - \ell_n \leq m_n$. Similarly one argues that $\ell_n \leq \binom{n}{2} - m_n$.

Now by Turán’s theorem $m_n = \lfloor n^2/4 \rfloor$ (c.f. Ex. 30, p. 68 of *Combinatorial Problems and Exercises* by L. Lovász). Thus $\ell_n = \binom{n}{2} - \lfloor \frac{n^2}{4} \rfloor$. By checking separately n even and n odd, one obtains $\ell_n = \lfloor (n-1)^2/4 \rfloor$ in all cases. This situation is realized by the disjoint union of two complete graphs on $(n-1)/2$ and $(n+1)/2$ points respectively if n is odd, and by two copies of the complete graph $K_{n/2}$ if n is even.

* * *

This completes the correct solutions received for problems from the October 1989 number and the space we have this month. Send me your problems and nice solutions!

* * * *

BOOK REVIEW

Mathematical Olympiad in China, edited by W. Ouyang, published by Hunan Education Publishing House, Changsha, 1990, ISBN 7-5355-1152-X, hardcover, 312+ pages. *Reviewed by Andy Liu, University of Alberta.*

This book was put together by the Chinese Mathematical Olympiad Committee as a gift to the Leaders and Deputy Leaders of the teams participating in the 31st International Mathematical Olympiad in Beijing. It consists of four parts.

The first part consists of an essay sketching the history and development of the Olympiad movement in China, climaxing in the 1990 IMO. The second part consists of eight articles on mathematics that arose from the Olympiads. Some are translations of articles of exceptional merit that were published previously in Chinese, while others are contributed especially for this volume.

Here is a problem considered in the article “A conjecture concerning six points in a square” by L. Yang and J. Zhang, translated from the Chinese version published in 1980. Everyone knows that 9 points in a square of area 1 are sufficient to guarantee that 3 of them will determine a triangle of area at most $1/4$. Actually, 9 points are not necessary. What is the minimum?

The answer is 6. A key lemma states that among 4 points determining a convex quadrilateral in a triangle of area 1, 3 will determine a triangle of area at most $1/4$. The simple proof is an elegant combination of Euclidean geometry and the Arithmetic-Mean Geometric-Mean Inequality.

Another easy corollary of this lemma is that 5 points in a triangle of area 1 are sufficient to guarantee that 3 of them will determine a triangle of area at most $1/4$. Here, it is easy to see that 5 points are also necessary.

The third part of the book consists of three articles on the art and science of problem-proposing and problem-solving. The fourth part consists of the problems and solutions of the first five annual Mathematical Winter Camps in China.

A very limited number of extra copies of this book have been printed. The price at US \$5.80 (plus 25% overseas postage) is a true bargain. It may be ordered with prepayment from:

Hunan Education Publishing House,
1 Dongfen Road, Changsha, Hunan,
410005, People's Republic of China.

* * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1992, although solutions received after that date will also be considered until the time when a solution is published.

1651. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and A_1, B_1, C_1 the common points of the inscribed circle with the sides BC, CA, AB , respectively. We denote the length of the arc B_1C_1 (not containing A_1) of the incircle by S_a , and similarly define S_b and S_c . Prove that

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{9\sqrt{3}}{\pi}.$$

1652. *Proposed by Murray S. Klamkin, University of Alberta.*

Given fixed constants $a, b, c > 0$ and $m > 1$, find all positive values of x, y, z which minimize

$$\frac{x^m + y^m + z^m + a^m + b^m + c^m}{6} - \left(\frac{x + y + z + a + b + c}{6} \right)^m.$$

1653. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be the intersection of the diagonals AC, BD of a quadrangle $ABCD$, and let M, N be the midpoints of AB, CD , respectively. Let l, m, n be the lines through P, M, N perpendicular to AD, BD, AC , respectively. Prove that if l, m, n are concurrent, then A, B, C, D are concyclic.

1654*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\sum \frac{x}{x + \sqrt{(x+y)(x+z)}} \leq 1,$$

where the sum is cyclic over x, y, z , and determine when equality holds.

1655. *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be a trapezoid with $AD \parallel BC$. M, N, P, Q, O are the midpoints of AB, CD, AC, BD, MN , respectively. Circles m, n, p, q all pass through O , and are tangent to AB at M , to CD at N , to AC at P , and to BD at Q , respectively. Prove that the centres of m, n, p, q are collinear.

1656. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

Given a triangle ABC , we take variable points P on segment AB and Q on segment AC . CP meets BQ in T . Where should P and Q be located so that the area of ΔPQT is maximized?

1657. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Pythagoras the eternal traveller reached Brahmapura on one of his travels. "Dear Pythagoras", told the townspeople of Brahmapura, "there on the top of that tall vertical mountain 99 *Brahmis* away resides Brahmagupta, who exhibits unmatched skills in both travel through the air and mathematics". "Aha", exclaimed Pythagoras, "I must have a chat with him before leaving." Pythagoras made his way directly to the foot of the mountain. No sooner had he reached it than he found himself in a comforting magic spell that flew him effortlessly to the summit where Brahmagupta received him. The two exchanged greetings and ideas. "You will have noticed that the foot and summit of this mountain forms with the town a triangle that is Pythagorean [an integer-sided right triangle—*Ed.*]", Brahmagupta remarked. "I can rise high vertically from the summit and then proceed diagonally to reach the town thus making yet another Pythagorean triangle and in the process equalling the distance covered by you from the town to the summit." Tell me, dear *Crux* problem solver, how high above the mountain did Brahmagupta rise?

1658. *Proposed by Avinoam Freedman, Teaneck, New Jersey.*

Let P be a point inside circle O and let three rays from P making angles of 120° at P meet O at A, B, C . Show that the power of P with respect to O is the product of the arithmetic and harmonic means of PA, PB and PC .

1659*. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

For any integer $n > 1$, prove or disprove that the largest coefficient in the expansion of

$$(1 + 2x + 3x^2 + 4x^3)^n$$

is the coefficient of x^{2n} .

1660. *Proposed by Isao Ashiba, Tokyo, Japan.*

Construct equilateral triangles $A'BC, B'CA, C'AB$ exterior to triangle ABC , and take points P, Q, R on AA', BB', CC' , respectively, such that

$$\frac{AP}{AA'} + \frac{BQ}{BB'} + \frac{CR}{CC'} = 1.$$

Prove that ΔPQR is equilateral.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

93. [1975: 97; 1976: 45, 111; 1984: 293] *Proposed by H.G. Dworshak, Algonquin College.*

Is there a convex polyhedron having exactly seven edges?

V. Solution by William Pippin and James Underwood, students, The Ohio State University.

Since the vertices and edges of a polyhedron form the vertices and edges of a simple graph in which each vertex has degree at least 3, the following result provides an immediate negative answer.

A simple graph in which each vertex has degree at least 3 cannot have exactly 7 edges.

Proof. Suppose that the graph has V vertices. Then the number of edges must be at least $3V/2$, since each vertex is incident to at least 3 edges and each edge has 2 ends. But if $V \geq 5$, then $3V/2 \geq 15/2 > 7$. On the other hand, if $V \leq 4$ then the graph cannot have more than $\binom{4}{2} = 6$ edges.

* * * *

964*. [1984: 217; 1985: 296] *Proposed by R.B. Killgrove, Alhambra, California.*

Let T be the image of the Euler ϕ -function, that is,

$$T = \{\phi(n) : n = 1, 2, 3, \dots\}.$$

Prove or disprove that T is a *Dirichlet set*, as defined in the proposer's article "Elementary Dirichlet sets" [1984: 206–209, esp. p. 206 and last paragraph p. 209].

Solution by Sam Maltby, student, University of Calgary.

We need to show that if S is an arithmetic sequence,

$$S = \{a, a + d, a + 2d, a + 3d, \dots\}$$

with a and d positive integers, then $|S \cap T| \in \{0, 1, \infty\}$. We in fact prove: *if $S \cap T$ contains an element greater than or equal to d , then it contains infinitely many elements.*

Assume $t \in S \cap T$ with $t \geq d$. Since $t \in T$,

$$t = \phi(m) = p_1^{\alpha_1}(p_1 - 1)p_2^{\alpha_2}(p_2 - 1)\dots p_r^{\alpha_r}(p_r - 1) \quad (1)$$

where $m = p_1^{\alpha_1+1}p_2^{\alpha_2+1}\dots p_r^{\alpha_r+1}$, the p_i 's are distinct primes and each $\alpha_i \geq 0$.

Now if we can find infinitely many elements of T congruent to a modulo d , we are done. Since $t \equiv a \pmod{d}$, it is sufficient to find these elements congruent to $t \pmod{d}$.

Let

$$d = kp_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}, \beta_i \geq 0 \text{ for all } i, (k, p_i) = 1 \text{ for all } i.$$

We cannot have $\beta_i > \alpha_i$ for all i , for then we would have

$$d \geq p_1^{\beta_1} \cdots p_r^{\beta_r} \geq p_1^{\alpha_1+1} \cdots p_r^{\alpha_r+1} > t.$$

So $\beta_i \leq \alpha_i$ for some i , say $\beta_r \leq \alpha_r$.

Put $x = tp_r^{n\phi(d)}$, where n is any positive integer; then I claim that

$$x \equiv t \pmod{d}$$

for all $n = 1, 2, \dots$. Putting

$$d = k' p_r^{\beta_r} \quad (\text{i.e., } k' = kp_1^{\beta_1} \cdots p_{r-1}^{\beta_{r-1}}), \quad (2)$$

we get from (1) that $t \equiv 0 \pmod{p_r^{\beta_r}}$, so

$$x \equiv t \equiv 0 \pmod{p_r^{\beta_r}}.$$

Also by (2) and Euler's theorem,

$$x = t \cdot p_r^{\phi(k') \cdot n\phi(p_r^{\beta_r})} \equiv t \cdot 1^{n\phi(p_r^{\beta_r})} \pmod{k'} \equiv t \pmod{k'},$$

so since $(k', p_r^{\beta_r}) = 1$ we have $x \equiv t \pmod{d}$, as claimed. Thus $x \in S$ for all n .

Also, x is of the form (1), so $x \in T$. Thus T is an elementary Dirichlet set.

Note incidentally that the sequence $S = \{1, 3, 5, 7, \dots\}$, with and without its first element, shows that $|S \cap T|$ may equal 0 or 1.

* * * *

1415. [1989: 47; 1990: 114] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Given the system of differential equations

$$\begin{aligned}\dot{x}_1 &= -(c_{12} + c_{13})x_1 + c_{12}x_2 + c_{13}x_3 \\ \dot{x}_2 &= c_{21}x_1 - (c_{21} + c_{23})x_2 + c_{23}x_3 \\ \dot{x}_3 &= c_{31}x_1 + c_{32}x_2 - (c_{31} + c_{32})x_3,\end{aligned}$$

where the c 's are positive constants, show that $\lim_{t \rightarrow \infty} x_i(t)$ is a weighted average, independent of i , of the initial values $x_1(0)$, $x_2(0)$, $x_3(0)$.

II. Comment by the editor.

It has been pointed out by John Lindsey, Northern Illinois University, Dekalb, that an “explanation” offered by the editor for one point in the published proof was incorrect! On [1990: 115], in an “editor’s note” appears the equation

$$x_i(t) = x_i(0) + f_i t e^{m_i t}.$$

This equation, and the rest of the editor’s note, should be ignored! Instead one can substitute the following argument, obtained by consultation with the solution’s author, Kee-Wai Lau.

From equation (2) on [1990: 115],

$$x_i(t) = K + e_i e^{m_1 t} + f_i t e^{m_1 t}$$

so that

$$\dot{x}_i(t) = e_i m_1 e^{m_1 t} + f_i e^{m_1 t} + f_i m_1 t e^{m_1 t}.$$

Thus from $\dot{x}_i(0) = 0$ we get $e_i m_1 + f_i = 0$, and since $e_1 = e_2 = e_3$ it follows that $f_1 = f_2 = f_3$. Hence $x_1(t) = x_2(t) = x_3(t)$, and we deduce from the original system that $\dot{x}_1(t) = \dot{x}_2(t) = \dot{x}_3(t) = 0$. Thus $x_1(t) = x_2(t) = x_3(t) = \text{constant}$, as claimed. Apologies to Professor Lau.

Lindsey also contributed a generalization of this problem to n variables.

* * * *

1425. [1989: 73; 1990: 146] *Proposed by Jordi Dou, Barcelona, Spain.*

Let D be the midpoint of side BC of the equilateral triangle ABC and ω a circle through D tangent to AB , cutting AC in points B_1 and B_2 . Prove that the two circles, distinct from ω , which pass through D and are tangent to AB , and which respectively pass through B_1 and B_2 , have a point in common on AC .

III. *Comment by Chris Fisher, University of Regina.*

On [1990: 148] the editor asked for another solution to this problem. Ironically *Crux* has already published several as solutions to *Crux* 975 [1985: 328–331]. Note that the figure in the comment by Michal Szurek and me [1985: 330] is the same as the one Jordi Dou used on [1990: 148]. If you invert that figure you get (according to Dou) *Crux* 1425; if you project it (as explained in our comment) you get *Crux* 975. In other words, both problems are specializations of the classical theorem known as Poncelet's Porism.

A further comment on this problem was received from P. PENNING, Delft, The Netherlands.

* * * *

1529. [1990: 75] *Proposed by Jordi Dou, Barcelona, Spain.*

Given points A, B, C on line l and A', B', C' on line l' , construct the points $P \neq l \cap l'$ such that the three angles APA', BPB', CPC' have the same pair of bisectors.

Solution by the proposer.

Let π be the projectivity of l onto l' defined by $\pi(A) = A', \pi(B) = B', \pi(C) = C'$. Let ω be the conic tangent to l, l', AA', BB' and CC' . We put $l \cap l' = S = T', \pi(S) = S'$, and $\pi^{-1}(T') = T$; note that S' and T are the points where l' and l meet ω .

For a point X , let π_X denote the projectivity of the pencil of lines through X that takes XA, XB, XC to XA', XB', XC' . We seek the position of $X = P$ for which π_P is induced by a reflection of the plane; in other words, π_P is an involution whose fixed lines are orthogonal. For π_X to be an involution it is necessary and sufficient that X be on the line $S'T$ (because $\pi_X(XS) = XS'$ must coincide with $\pi_X^{-1}(XT') = XT$).

The fixed lines of π_X are the tangents from X to ω . In order that these be perpendicular it is necessary and sufficient that X be on the circle μ (of Monge) that is the locus of points from which the tangents to ω form a right angle.

Therefore the required points P are the points of intersection of $S'T$ with μ .

Construction of P . Since $S'T$ is the axis of π it must contain $D = AC' \cap A'C$ and $E = AB' \cap A'B$ (as well as $BC' \cap B'C$). But $S'T$ is also the polar of S with respect to ω ; thus if M is the midpoint of $S'T$, then SM passes through the centre O of ω (which is also the centre of μ). Analogously, letting $F = AA' \cap BB'$, $G = A'B \cap FS$ and $R = GT \cap AA'$, then RT is the polar of A . Letting N be the midpoint of RT , AN passes through O . Thus $O = SM \cap AN$. It remains to find a point on the circle μ . Let A'', B'', T'' be the orthogonal projections of A', B', T' on l (note $T'' = T'$). Let U and U' be the fixed points of the projectivity $A, B, T \rightarrow A'', B'', T''$. Then the perpendiculars to l through U, U' are tangent to ω and therefore U and U' lie on μ . (The fixed points of a projectivity are easily found; for example, project A, B, T, A'', B'', T'' onto any convenient circle α from a point V of α . Let a, b, t, a'', b'', t'' be these projections onto α . The line through points $at'' \cap a''t$ and $bt'' \cap b''t$ cuts α in u and u' , which project back to give us U and U' .)

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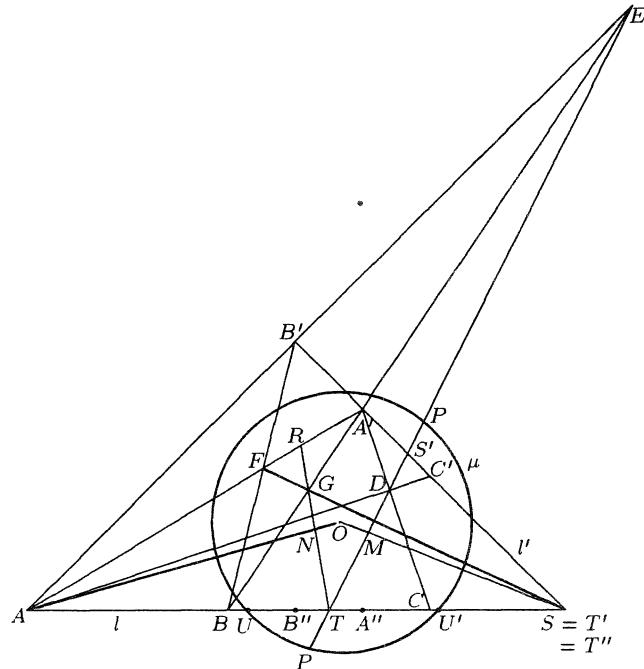
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1532. [1990: 109] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine all (possibly degenerate) triangles ABC such that

$$(1 + \cos B)(1 + \cos C)(1 - \cos A) = 2 \cos A \cos B \cos C.$$

Solution by Kee-Wai Lau, Hong Kong.

We show that the equality of the problem holds if and only if $A = 0$ and $B = C = \pi/2$.

Since

$$2 \cos A \cos B \cos C = (1 + \cos B)(1 + \cos C)(1 - \cos A) \geq 0 ,$$

so $0 \leq A, B, C \leq \pi/2$. Clearly $A \neq \pi/2$, and if $A = 0$ then $B = C = \pi/2$. In what follows we assume that $0 < A < \pi/2$ and $0 \leq B, C < \pi/2$. For $0 \leq x < \pi/2$, let $f(x) = \ln(\sec x + 1)$. We have

$$f''(x) = \frac{1 + \cos x - \cos^2 x}{(1 + \cos x) \cos^2 x} > 0$$

and so $f(x)$ is convex. Hence

$$\begin{aligned} (\sec B + 1)(\sec C + 1)(\sec A - 1) &\geq \left(\sec \frac{B+C}{2} + 1\right)^2 (\sec A - 1) \\ &= \left(\csc \frac{A}{2} + 1\right)^2 (\sec A - 1) \\ &> \csc^2 \frac{A}{2} (\sec A - 1) \\ &= 2 \sec A \geq 2. \end{aligned}$$

Therefore

$$(1 + \cos B)(1 + \cos C)(1 - \cos A) > 2 \cos A \cos B \cos C.$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer.

Festraets-Hamoir and the proposer solved the problem by showing that the given equality is equivalent to the orthocentre of the triangle lying on the excircle to side a , which can be seen geometrically to happen only for the degenerate $(0^\circ, 90^\circ, 90^\circ)$ triangle.

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1533. [1990: 109] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

For any integers $n \geq k \geq 0$, $n \geq 1$, denote by $p(n, k)$ the probability that a randomly chosen permutation of $\{1, 2, \dots, n\}$ has exactly k fixed points, and let

$$P(n) = p(n, 0)p(n, 1) \dots p(n, n).$$

Prove that

$$P(n) \leq \exp(-2^n n!).$$

Solution.

Since $p(n, n-1) = 0$, $P(n) = 0$.

Solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer, who claims he didn't invent the problem, but that it was "circulating".

This problem was intended as an "April Fool's" joke by the editor, and, who knows, might have been better appreciated by the readers if the April 1990 Crux had been available anywhere near the beginning of April! Some future spring, when Crux is on schedule, the editor may be moved to try again.

* * * *

1534. [1990: 109] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Triangle $H_1H_2H_3$ is formed by joining the feet of the altitudes of an acute triangle $A_1A_2A_3$. Prove that

$$\frac{s}{r} \leq \frac{s'}{r'},$$

where s, s' and r, r' are the semiperimeters and inradii of $A_1A_2A_3$ and $H_1H_2H_3$ respectively.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

This will be a solution "from the books". First we observe that if S is the area of the triangle ABC , then $S = sr$, $s' = S/R$ and $r' = 2R \cos A_1 \cos A_2 \cos A_3$ (see p. 191 of Johnson, *Advanced Euclidean Geometry*). Then the proposed inequality can be written

$$\frac{s}{r} \leq \frac{sr}{2R^2 \cos A_1 \cos A_2 \cos A_3},$$

or equivalently

$$\cos A_1 \cos A_2 \cos A_3 \leq \frac{r^2}{2R^2},$$

and this last is an inequality of W.J. Blundon (see problem E1925 of the *American Math. Monthly*, solution on pp. 196-197 of the February 1968 issue).

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

Several solvers reduced the problem of one of the equivalent inequalities

$$(1 - \cos A_1)(1 - \cos A_2)(1 - \cos A_3) \geq \cos A_1 \cos A_2 \cos A_3,$$

which is contained in Crux 836 [1984: 228], or

$$4R^2 + 4Rr + 3r^2 \geq s^2$$

which is item 5.8 of Bottema et al, Geometric Inequalities. See the proof of Crux 544 [1981: 150-153] for yet more equivalent forms. Also see the proof of Crux 1539, this issue!

Janous used item 35, p. 246 of Mitrinović et al, Recent Advances in Geometric Inequalities, to refine the given inequality:

$$\frac{s'}{r'} = \frac{F}{2R^2 \cos A_1 \cos A_2 \cos A_3} = \tan A_1 \tan A_2 \tan A_3 \geq \frac{27R^2}{4F} = \frac{27R^2}{4s^2} \cdot \frac{s}{r},$$

this being a refinement because $3\sqrt{3}R \geq 2s$ (item 5.3 of Bottema et al).

Klamkin considers replacing $H_1H_2H_3$ by the pedal triangle of a point P , and asks for which P the inequality still holds. He notes that it holds if P lies on the circumcircle of $A_1A_2A_3$.

1535. [1990: 109] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let P be a variable point inside an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Through P draw two chords with slopes b/a and $-b/a$ respectively. The point P divides these two chords into four pieces of lengths d_1, d_2, d_3, d_4 . Prove that $d_1^2 + d_2^2 + d_3^2 + d_4^2$ is independent of the location of P and in fact has the value $2(a^2 + b^2)$.

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

Let $x' = (b/a)x$. Then the ellipse becomes the circle $x'^2 + y^2 = b^2$ when plotted in x', y axes. Moreover the chords become at angles of 45° to the axes.

Rotate the circle so that the given chords are parallel to the axes. Denote the resulting coordinates of P by (r, s) and the lengths of the pieces of chords through P by d'_1, d'_2, d'_3, d'_4 . Then it's clear that

$$d'_1 = \sqrt{b^2 - s^2} - r \quad \text{and} \quad d'_3 = \sqrt{b^2 - s^2} + r,$$

so

$$d'^2_1 + d'^2_3 = 2b^2 - 2s^2 + 2r^2.$$

Similarly

$$d'_2 = \sqrt{b^2 - r^2} - s \quad \text{and} \quad d'_4 = \sqrt{b^2 - r^2} + s,$$

so

$$d'^2_2 + d'^2_4 = 2b^2 - 2r^2 + 2s^2.$$

Thus

$$d'^2_1 + d'^2_2 + d'^2_3 + d'^2_4 = 4b^2, \tag{1}$$

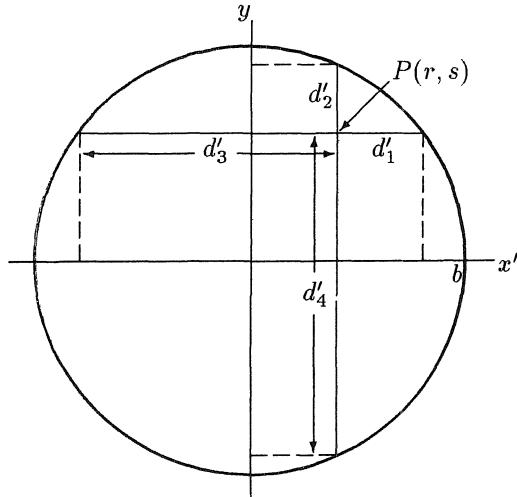
independent of the placement of P .

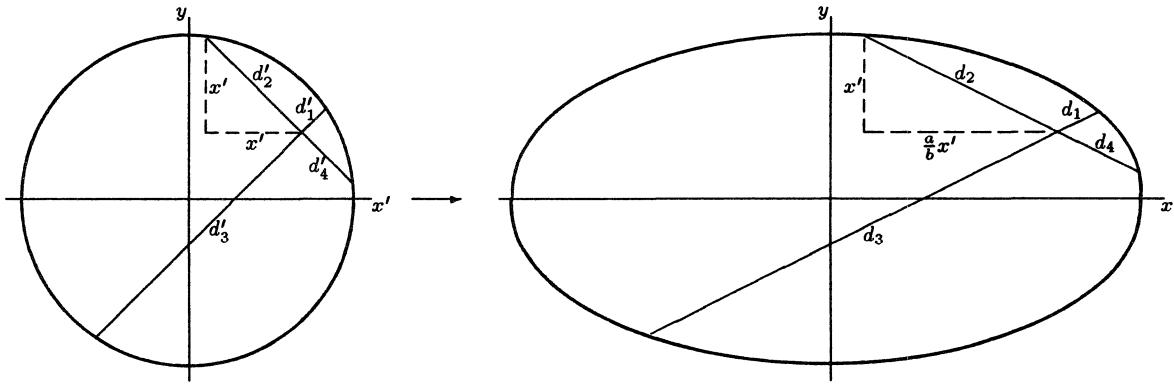
In going from (x', y) -coordinates to (x, y) -coordinates as shown on the next page, we have

$$d_i^2 = \left(1 + \frac{a^2}{b^2}\right) x'^2 = \left(1 + \frac{a^2}{b^2}\right) \cdot \frac{1}{2} d'^2_i,$$

and so by (1),

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 = \frac{1}{2} \left(1 + \frac{a^2}{b^2}\right) 4b^2 = 2(a^2 + b^2).$$





II. *Solution par Jacques Choné, Lycée Blaise Pascal, Clermont-Ferrand, France.*

Soit (α, β) les coordonnées de P , avec $\alpha^2/a^2 + \beta^2/b^2 - 1 \leq 0$. Un vecteur unitaire des cordes considérées a pour coordonnées:

$$\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{\varepsilon b}{\sqrt{a^2 + b^2}} \right), \quad \varepsilon \in \{-1, 1\}.$$

Les points des cordes considérées ont pour coordonnées:

$$\left(\alpha + \frac{\lambda a}{\sqrt{a^2 + b^2}}, \beta + \frac{\lambda \varepsilon b}{\sqrt{a^2 + b^2}} \right), \quad \lambda \text{ réel.}$$

Les réels d_i ($i = 1, \dots, 4$) sont les valeurs absolues des solutions de l'équation

$$\frac{1}{a^2} \left(\alpha + \frac{\lambda a}{\sqrt{a^2 + b^2}} \right)^2 + \frac{1}{b^2} \left(\beta + \frac{\lambda \varepsilon b}{\sqrt{a^2 + b^2}} \right)^2 = 1,$$

c'est à dire de

$$\frac{2}{a^2 + b^2} \lambda^2 + \frac{2\lambda}{\sqrt{a^2 + b^2}} \left(\frac{\alpha}{a} + \frac{\varepsilon \beta}{b} \right) + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 = 0.$$

La somme des carrés des solutions de l'équation $Ax^2 + Bx + C = 0$ étant $(-B/A)^2 - 2C/A$, on obtient

$$\begin{aligned} d_1^2 + d_2^2 + d_3^2 + d_4^2 &= \left[\left(\frac{\alpha}{a} + \frac{\beta}{b} \right)^2 + \left(\frac{\alpha}{a} - \frac{\beta}{b} \right)^2 \right] (a^2 + b^2) - 2 \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 \right) (a^2 + b^2) \\ &= 2(a^2 + b^2), \end{aligned}$$

ce qui est le résultat désiré.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona,

Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Klamkin and Kuczma note that equation (1), the special case for circles, has already appeared in Crux on [1989: 293].

* * * *

1536*. [1990: 109] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

This being problem 1536 in volume 16 of *Crux*, note that the numbers 16 and 1536 have the following nice property:

$$\begin{array}{r}
 \begin{array}{c} 96 \\ 16 \overline{) 1536} \\ 144 \\ \hline 96 \\ 96 \\ \hline 0 \end{array}
 \end{array}
 \quad \left(\begin{array}{l} \text{in Europe: } 1536 : 16 = 96 \\ \qquad \qquad \qquad 96 \\ \qquad \qquad \qquad 0 \end{array} \right).$$

That is, 1536 is exactly divisible by 16, and upon dividing 16 into 1536 via “long division” the last nonzero remainder in the display is equal to the quotient $1536/16 = 96$. Assuming *Crux* continues to publish 100 problems each year, what will be the next *Crux* volume and problem numbers to have the same property?

Solution by Kenneth M. Wilke, Topeka, Kansas.

Since there are exactly 100 problems published annually we have

$$P = 100(v - 1) + p \quad (1)$$

where p denotes the last two digits of the problem number P and v is the volume number. Both v and p are integers with $v > 0$ and $1 \leq p \leq 99$. [This omits the last problem in each volume, whose number ends in 00, but it is easy to see such problems can never yield a solution.] Letting q be the quotient resulting from the exact division of P by v , we get from (1)

$$q = \frac{P}{v} = 100 - \frac{100 - p}{v}. \quad (2)$$

(2) guarantees that $v \leq 100$ since q cannot be an integer for $v > 100$.

By the conditions of the problem, q is also the last nonzero remainder appearing in the division process, so we have $q \leq 9v$, or by (2)

$$9v^2 - 100v + 100 - p \geq 0.$$

Hence

$$v \leq \frac{100 - \sqrt{6400 + 36p}}{18} \quad \text{or} \quad v \geq \frac{100 + \sqrt{6400 + 36p}}{18}.$$

Taking $p = 0$ provides the useful bounds $v \leq 10/9$ or $v \geq 10$.

We first take $v \geq 10$. Then by (2) $q = 90 + k$ for some integer k such that $0 \leq k \leq 9$. But since q is the last nonzero remainder and k is the units digit of q , we must have $k = q/v$. Hence $90 + k = q = kv$ or

$$\frac{90}{v-1} = k. \quad (3)$$

Since $0 \leq k \leq 9$, (3) yields $(v, k) = (11, 9), (15, 6), (19, 5), (31, 3), (46, 2)$ and $(91, 1)$. Substituting $q = 90 + k$ into (2) and rearranging yields

$$p = 100 - (10 - k)v. \quad (4)$$

Using each of the above possibilities for (v, k) , we discard the last three since they yield $p < 0$. Hence we have the following solutions:

$$\begin{aligned} v = 11, \quad k = 9, \quad p = 89, \quad P = \mathbf{1089}; \\ v = 15, \quad k = 6, \quad p = 36, \quad P = \mathbf{1536}; \\ v = 19, \quad k = 5, \quad p = 5, \quad P = \mathbf{1805}. \end{aligned}$$

It remains to consider $v \leq 10/9$ or $v = 1$. Here we have nine trivial solutions $(v, k, p) = (1, k, k)$ where $1 \leq k \leq 9$. They correspond to the first nine problems published in *Eureka* (*Crux's* predecessor).

Hence the next and last problem number satisfying the conditions of the problem will be *Crux* 1805 appearing in volume 19.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; and K.R.S. SASTRY, Addis Ababa, Ethiopia. One incorrect solution was received.

Readers may like to contribute an appropriate problem to be numbered 1805, to carry on the “tradition” newly established by Janous! The editor suggests that such a problem (i) involve the number 1805 and possibly the volume number (19) or even the year (which will be 1993), and (ii) have exactly one integer solution larger than 1805. (Not too large!)

* * * *

1537. [1990: 110] Proposed by Isao Ashiba, Tokyo, Japan.

ABC is a right triangle with right angle at A . Construct the squares $ABDE$ and $ACFG$ exterior to ΔABC , and let P and Q be the points of intersection of CD and AB , and of BF and AC , respectively. Show that $AP = AQ$.

I. Solution by O. Johnson, student, King Edward's School, Birmingham, England.

Let $AB = AE = p$ and $AC = AG = q$.

Then $\triangle CAP$ is similar to $\triangle CED$, so

$$\frac{AP}{q} = \frac{AP}{AC} = \frac{DE}{EC} = \frac{p}{p+q},$$

and therefore

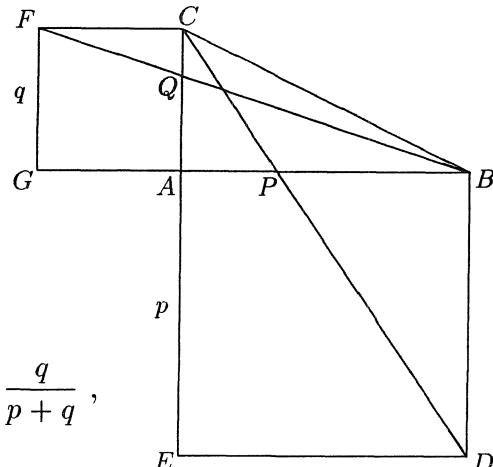
$$AP = \frac{pq}{p+q}.$$

Also $\triangle BAQ$ is similar to $\triangle BGF$, so

$$\frac{AQ}{p} = \frac{AQ}{AB} = \frac{FG}{GB} = \frac{q}{p+q},$$

and therefore

$$AQ = \frac{pq}{p+q} = AP.$$



II. Comment by Hidetosi Fukagawa, Aichi, Japan.

Several interesting facts about the figure of this problem are shown in Shiko Iwata, *Encyclopedia of Geometry* Vol. 3 (1976), published in Japanese. They include, as well as this problem itself,

- (i) $BP : CQ = (AB)^2 : (AC)^2$;
- (ii) if DE meets GF in M , then $AM \perp BC$ and the three lines AM, DC, BF meet in a point H ;
- (iii) $1/AH = 1/AL + 1/BC$, where AM meets BC in L ;
- (iv) $BD' = CF'$, where D' and F' are the feet of the perpendiculars from D and F to BC ;
- (v) $DD' + FF' = BC$.

[Fukagawa then gave proofs of (i)–(v) and *Crux* 1537. —Ed.]

Also solved by WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; J. CHONÉ, Lycée Blaise Pascal, Clermont-Ferrand, France; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANTONIO LUIZ SANTOS, Rio de Janeiro, Brazil; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; H.J. MICHEL WIJERS, Eindhoven, The Netherlands; KA-PING YEE, student, St. John's-Ravenscourt School, Winnipeg, Manitoba; and the proposer.

Heuver found the problem with solution on pp. 83–84 of Coxeter and Greitzer, Geometry Revisited, L.W. Singer, New York.

McCallum notes that another problem, rather familiar, has the same answer $pq/(p + q)$ as was obtained for AP and AQ above: namely, given two vertical posts of heights p and q on a flat plain, and wires connecting the top of each post with the bottom of the other, find the height of the intersection of the wires. He wonders if there is a nice way to turn each problem into the other.

* * * *

1538. [1990: 110] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Find all functions $y = f(x)$ with the property that the line through any two points $(p, f(p)), (q, f(q))$ on the curve intersects the y -axis at the point $(0, -pq)$.

Solution by Ka-Ping Yee, student, St. John's-Ravenscourt School, Winnipeg, Manitoba.

All functions $y = f(x) = x(c + x)$, where c is some constant, have this property, and no other functions do. We can show this as follows.

If a line passes through the points $(p, f(p))$ and $(q, f(q))$, then

$$f(p) = mp + b, \quad f(q) = mq + b.$$

Eliminating the slope m and solving for the y -intercept b we get

$$b = \frac{pf(q) - qf(p)}{p - q} = -pq \quad (\text{as given}),$$

which implies

$$pf(q) - qf(p) = pq(q - p)$$

and so

$$\frac{f(q)}{q} - q = \frac{f(p)}{p} - p.$$

Since this is true for all [nonzero] real p and q , $f(x)/x - x$ has a constant value for all $x \neq 0$. If we let this constant value be c , then $f(x) = x(c + x)$.

[*Editor's note.* To this solution should be added two points, missed by many solvers: (i) from the statement of the problem, $f(0)$ must be 0, and the functions $f(x) = x(c + x)$ all have this property, thus $f(x) = x(c + x)$ for all x (not just $x \neq 0$); (ii) all functions $f(x) = x(c + x)$ do in fact satisfy the problem.]

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; J. CHONÉ, Lycée Blaise Pascal, Clermont-Ferrand, France; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D. HNIDEI and ROBERT PIDGEON, students, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland;

KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID C. VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer. (Three of these solutions assumed that the required functions were differentiable, but we'll let it go this time.)

The problem is the converse of a result proved in Herb Holden's article "Chords of the parabola", Two Year College Math. Journal, June 1982, pp. 186–190, and was suggested by that article.

* * * *

1539*. [1990: 110] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

If α, β, γ are the angles, s the semiperimeter, R the circumradius and r the inradius of a triangle, prove or disprove that

$$\sum \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \leq \left(\frac{2R - r}{s} \right)^2,$$

where the sum is cyclic.

Solution by Stephen D. Hnidei, student, University of British Columbia.

Starting with

$$\tan \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \tan \frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}, \quad (1)$$

where a, b, c are the sides opposite the angles α, β, γ respectively, the given inequality becomes

$$(s-a)^2 + (s-b)^2 + (s-c)^2 \leq (2R-r)^2,$$

or equivalently

$$a^2 + b^2 + c^2 \leq (2R-r)^2 + s^2. \quad (2)$$

Using the fact that

$$\frac{a^2 + b^2 + c^2}{2} + r(4R+r) = s^2, \quad (3)$$

(2) becomes

$$a^2 + b^2 + c^2 \leq 8R^2 + 4r^2.$$

This is inequality 5.14, page 53 of Bottema et al, *Geometric Inequalities*.

[Editor's note. (1) can be proved from the formulas

$$\tan \frac{\alpha}{2} = \frac{r}{s-a}, \text{ etc. and } r^2 s = (s-a)(s-b)(s-c).$$

(3) is proved on page 52 of Mitrinović et al, *Recent Advances in Geometric Inequalities*.]

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; JACK GARFUNKEL, Flushing, N.Y.; WALther JANOUS, Ursulinengymnasium, Innsbruck, Austria;

MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Penn State University at Harrisburg; and BOB PRIELIPP, University of Wisconsin-Oshkosh.

Most solvers reduced the given inequality to item 5.8, page 50 of Bottema et al, Geometric Inequalities. Thus this problem and Crux 1534 (solution this issue) are equivalent!

* * * *

1540. [1990: 110] *Proposed by Len Bos and Bill Sands, University of Calgary.*

For k a positive odd integer, define a sequence $\langle a_n \rangle$ by: $a_0 = 1$ and, for $n > 0$,

$$a_n = \begin{cases} a_{n-1} + k & \text{if } a_{n-1} \text{ is odd ,} \\ a_{n-1}/2 & \text{if } a_{n-1} \text{ is even ,} \end{cases}$$

and let $f(k)$ be the smallest $n > 0$ such that $a_n = 1$. Find all k such that $f(k) = k$.

Solution by Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain.

We will see that $f(k) = k$ if and only if $k = 3^q$ with $q \geq 1$.

First we look at some particular cases. For $k = 15$, the sequence is:

$$\langle a_n \rangle : 1, 16, 8, 4, 2, 1, \dots \quad \text{and } f(15) = 5.$$

For $k = 13$, we get

$$\langle a_n \rangle : 1, 14, 7, 20, 10, 5, 18, 9, 22, 11, 24, 12, 6, 3, 16, 8, 4, 2, 1, \dots \quad \text{and } f(13) = 18.$$

For $k = 9$, we get

$$\langle a_n \rangle : 1, 10, 5, 14, 7, 16, 8, 4, 2, 1, \dots \quad \text{and } f(9) = 9.$$

We can work better with the “auxiliary” sequence $\langle t_n \rangle$ defined by: $t_0 = 1$, and for $n > 0$

$$t_n = \begin{cases} (t_{n-1} + k)/2 & \text{if } t_{n-1} \text{ is odd,} \\ t_{n-1}/2 & \text{if } t_{n-1} \text{ is even;} \end{cases}$$

and let $g(k)$ be also the smallest $n > 0$ such that $t_n = 1$. For the same values of k , we have

$$(k = 15) \quad \langle t_n \rangle : 1, 8, 4, 2, 1, \dots \quad \text{and } g(15) = 4;$$

$$(k = 13) \quad \langle t_n \rangle : 1, 7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1, \dots \quad \text{and } g(13) = 12;$$

$$(k = 9) \quad \langle t_n \rangle : 1, 5, 7, 8, 4, 2, 1, \dots \quad \text{and } g(9) = 6.$$

Evidently, each odd t_i ($i \geq 0$) shortens by one term the length of sequence $\langle a_n \rangle$ in comparison to sequence $\langle t_n \rangle$, so:

$$\begin{aligned} f(15) &= g(15) + 1 = g(15) + \text{card}\{1\}, \\ f(13) &= g(13) + 6 = g(13) + \text{card}\{1, 7, 5, 9, 11, 3\}, \\ f(9) &= g(9) + 3 = g(9) + \text{card}\{1, 5, 7\}. \end{aligned}$$

In general, once we prove that $g(k)$ is finite for every positive odd integer k , then we shall have that

$$f(k) = g(k) + \text{card}\{\text{odd } t_i : 0 \leq i \leq g(k)\}. \quad (1)$$

We shall show that the coincidence between the sequence $\langle t_n \rangle$ (reversed) and the sequence of residues of the 2^n 's modulo k , for example,

$$\begin{aligned} & \langle 2^n \text{ modulo } 15 \rangle : 1, 2, 4, 8, 1, \dots, \\ & \langle 2^n \text{ modulo } 13 \rangle : 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, \dots, \\ & \langle 2^n \text{ modulo } 9 \rangle : 1, 2, 4, 8, 7, 5, 1, \dots \end{aligned}$$

is not accidental. In fact we can prove that, for every odd $k > 1$ (already excluding the case $k = 1$, for which $f(1) = 2$), we have:

- (a) $t_n < k$ for all n ;
- (b) $2^n t_n \equiv 1 \pmod{k}$ for all n ; and so
- (c) if h is a positive integer such that $2^h \equiv 1 \pmod{k}$, then $t_n \equiv 2^{h-n} \pmod{k}$ for $n \leq h$, and so the period of the sequence $\langle t_n \rangle$ is

$$g(k) = \min\{h > 0 : 2^h \equiv 1 \pmod{k}\} (\text{: the order of } 2 \pmod{k}).$$

Proof of (a). By induction. Note that

$$t_0 = 1 < k, \quad t_1 = \frac{1+k}{2} < k.$$

If we suppose $t_{n-1} < k$ and we put $t_{n-1} \equiv \varepsilon \pmod{2}$ ($\varepsilon = 0$ or 1), then

$$t_n = \frac{t_{n-1}}{2} + \varepsilon \cdot \frac{k}{2} < \frac{k}{2} + \frac{k}{2} = k.$$

Proof of (b). By induction. Note that $t_0 = 1$ and $2t_1 = 1 + k \equiv 1 \pmod{k}$. If we suppose that $2^{n-1}t_{n-1} \equiv 1 \pmod{k}$, then we have (as above)

$$\begin{aligned} 2^n t_n &= 2^n \left(\frac{t_{n-1}}{2} + \varepsilon \cdot \frac{k}{2} \right) = 2^{n-1}t_{n-1} + k2^{n-1}\varepsilon \\ &\equiv 2^{n-1}t_{n-1} \pmod{k} \equiv 1 \pmod{k}. \end{aligned}$$

Proof of (c). According to (b), $2^n t_n \equiv 1 \pmod{k}$, and (as k is odd) by division $t_n \equiv 2^{h-n} \pmod{k}$ for every $n \leq h$. By this and (a), the terms t_1, t_2, \dots, t_h of the sequence $\langle t_n \rangle$ are, in opposite order, just the same as the terms $2^0 \pmod{k}, 2^1 \pmod{k}, \dots, 2^{h-1} \pmod{k}$ of the sequence $\langle 2^n \pmod{k} \rangle$, for every positive h such that $2^h \equiv 1 \pmod{k}$. The periods of $\langle t_n \rangle$ and $\langle 2^n \pmod{k} \rangle$ are therefore equal.

Remembering (1) above, we have now that sequence $\langle a_n \rangle$ is pure periodic for all odd k , with period $f(k) = \ell + s$ with

$$\ell = \text{the order of } 2 \pmod{k},$$

$$s = \text{card}\{\text{odd residues of } 2^i \pmod{k}, 0 < i \leq \ell\}.$$

Finally we answer the problem, that is, we prove

$$f(k) = k \text{ if and only if } k = 3^q \text{ with } q \geq 1.$$

(\Rightarrow) It is known that ℓ divides the Euler phi-function $\varphi(k)$, because $2^{\varphi(k)} \equiv 1 \pmod{k}$ (k and 2 are relatively prime) and ℓ is the smallest exponent which satisfies the congruence. On the other hand, evidently $s < \ell$. If $\ell \neq \varphi(k)$ then $\ell \leq \varphi(k)/2 < k/2$ and so

$$f(k) = \ell + s < \frac{k}{2} + \frac{k}{2} = k,$$

a contradiction. Thus $\ell = \varphi(k)$. In this case the sequence $\langle 2^i \pmod{k} : 1 \leq i \leq \ell \rangle$ contains all the positive numbers less than k and prime to k . If $m < k$ is prime to k , then $k - m$ is prime to k too and has opposite parity to m since k is odd; therefore

$$s = \frac{\varphi(k)}{2} = \text{card}\{\text{all positive odd integers } < k \text{ and prime to } k\}.$$

So we have by the hypothesis

$$k = f(k) = \ell + s = \varphi(k) + \frac{\varphi(k)}{2} = \frac{3}{2}\varphi(k),$$

or

$$\frac{2k}{3} = \varphi(k) = k \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots,$$

where p_1, p_2, \dots are the distinct prime factors of k . Thus

$$\frac{2}{3} = \prod_{\substack{p|k \\ p \text{ prime}}} \frac{p-1}{p}$$

and consequently the only prime factor of k can be $p = 3$. Therefore $k = 3^q$ with $q \geq 1$.

(\Leftarrow) There remains to prove that

(*) for every $k = 3^q$ with $q \geq 1$, the order of 2 modulo k is $\ell = \varphi(k) = 2 \cdot 3^{q-1}$.

It then follows that $s = \varphi(k)/2 = 3^{q-1}$ and so

$$f(k) = \ell + s = 2 \cdot 3^{q-1} + 3^{q-1} = 3^q = k,$$

as claimed.

The proof of (*) is by induction on q . For $k = 3$, $\varphi(k) = 2$, and we have $2^2 \equiv 1 \pmod{3}$ but $2^1 \not\equiv 1 \pmod{3}$. Moreover $2^2 = 1 + 1 \cdot 3$ where the second 1 is prime to 3. Suppose now that $2 \cdot 3^{q-1}$ is the order of 2 modulo 3^q , and also that

$$2^{2 \cdot 3^{q-1}} = 1 + d \cdot 3^q$$

where d is prime to 3. Then we have

$$\begin{aligned} 2^{2 \cdot 3^q} &= (1 + d \cdot 3^q)^3 = 1 + 3 \cdot 3^q d + 3 \cdot 3^{2q} d^2 + 3^{3q} d^3 \\ &= 1 + 3^{q+1}(d + 3c) \quad \text{for some integer } c \\ &= 1 + 3^{q+1}d', \end{aligned}$$

where d' is prime to 3, since d is. Thus $2^{2 \cdot 3^q} \equiv 1 \pmod{3^{q+1}}$ (which of course follows by Euler's theorem) and moreover, if $2^h \equiv 1 \pmod{3^{q+1}}$ with $h < 2 \cdot 3^q$, then in particular we will have $2^h \equiv 1 \pmod{3^q}$, so h must be a multiple of $2 \cdot 3^{q-1}$, that is, $h = 2 \cdot 3^{q-1}$ or $h = 2^2 \cdot 3^{q-1}$. But

$$2^{2 \cdot 3^{q-1}} = 1 + d \cdot 3^q \not\equiv 1 \pmod{3^{q+1}}$$

since d is prime to 3, and (as $2^2 = 2 \cdot 2$)

$$\begin{aligned} 2^{2 \cdot 3^{q-1}} &= (2^{2 \cdot 3^{q-1}})^2 = (1 + 3^q d)^2 \\ &= 1 + 2 \cdot 3^q d + 3^{2q} d^2 \equiv 1 + 2 \cdot 3^q d \pmod{3^{q+1}} \\ &\not\equiv 1 \pmod{3^{q+1}} \end{aligned}$$

since d is prime to 3. Thus the order of 2 modulo 3^{q+1} is $2 \cdot 3^q$, and (*) is proved.

Also solved (the same way) by C. FESTRAETS-HAMOIR, Brussels, Belgium; MARCIN E. KUCZMA, Warszawa, Poland; and the proposers. Partial solutions were sent in by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; and by P. PENNING, Delft, The Netherlands (they each got one of the directions of the above “if and only if” statement). Two other readers submitted the correct answer with no proof.

As Engelhaupt points out, the result () in the above proof is known. For example, see Theorem 8.10, page 261 of K.H. Rosen, Elementary Number Theory and its Applications (2nd edition), Addison Wesley, 1988.*

* * * *

1541. [1990: 143] Proposed by J.T. Groenman, Arnhem, The Netherlands.

I_1 is the excenter of $\Delta A_1 A_2 A_3$ corresponding to side $A_2 A_3$. P is a point in the plane of the triangle, and $A_1 P$ intersects $A_2 A_3$ in P_1 . I_2, P_2, I_3, P_3 are analogously defined. Prove that the lines $I_1 P_1, I_2 P_2, I_3 P_3$ are concurrent.

Solution by R.H. Eddy, Memorial University of Newfoundland.

More generally, let $\Delta Q_1 Q_2 Q_3$ be circumscribed about $\Delta A_1 A_2 A_3$ such that $A_1 Q_1 \cap A_2 Q_2 \cap A_3 Q_3 = Q(q_1, q_2, q_3)$; then the lines $P_1 Q_1, P_2 Q_2, P_3 Q_3$ are concurrent.

We use trilinear coordinates taken with respect to $\Delta A_1 A_2 A_3$. If P has coordinates (p_1, p_2, p_3) then those for P_1, P_2, P_3 are $(0, p_2, p_3), (p_1, 0, p_3), (p_1, p_2, 0)$ respectively. Since the coordinates for Q_1, Q_2, Q_3 are $(-q_1, q_2, q_3), (q_1, -q_2, q_3), (q_1, q_2, -q_3)$, the line coordinates of $P_1 Q_1, P_2 Q_2, P_3 Q_3$ are given respectively by the rows of the determinant

$$\left| \begin{array}{ccc} p_2 q_3 - p_3 q_2 & -p_3 q_1 & p_2 q_1 \\ p_3 q_2 & p_3 q_1 - p_1 q_3 & -p_1 q_2 \\ -p_2 q_3 & p_1 q_3 & p_1 q_2 - p_2 q_1 \end{array} \right|$$

which is seen, by adding the rows together, to have the value zero. Thus the lines are concurrent.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Aichi, Japan; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; D. SOKOLOWSKY, Williamsburg, Virginia; E. SZEKERES, Turramurra, Australia; and the proposer.

The generalization given by Eddy was also found by Rausen, who later located it in Aubert et Papelier, Exercices de Géométrie Analytique, Vol. 1, 10th ed., Paris, 1957, problem 52, p. 35. Rausen also points out that the lines I_1P_1 , I_2P_2 , I_3P_3 may be parallel rather than concurrent: just pick P_1 on line A_2A_3 and P_2 on A_3A_1 such that lines I_1P_1 and I_2P_2 are parallel, then define P to be $A_1P_1 \cap A_2P_2$.

* * * *

1543. [1990: 143] Proposed by George Tsintsifas, Thessaloniki, Greece.

Show that the circumradius of a triangle is at least four times the inradius of the pedal triangle of any interior point.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let R_1, R_2, R_3 and r_1, r_2, r_3 be the distances of the interior point P from the vertices and sides, respectively, of triangle $A_1A_2A_3$ having sides a_1, a_2, a_3 . The sides of the pedal triangle are given by

$$R_i \sin A_i = \frac{a_i R_i}{2R}, \quad i = 1, 2, 3$$

(e.g. [1], p. 296), and its area is given by

$$F_{\text{ped}} = \frac{1}{2} \sum r_2 r_3 \sin A_1 = \frac{1}{4R} \sum a_1 r_2 r_3$$

where the sums are cyclic. (This can be seen by the trigonometric area formula and the fact that r_2 and r_3 make the angle $\pi - A_1$, etc.) Therefore the claimed inequality $R \geq 4r_{\text{ped}}$ reads $R s_{\text{ped}} \geq 4F_{\text{ped}}$ or

$$R \sum a_1 R_1 \geq 4 \sum a_1 r_2 r_3. \tag{1}$$

Now it is known ([2], item 12.19) that

$$\sum a_1 R_1 \geq 2 \sum a_1 r_1 = 4F.$$

Therefore (1) would follow from $RF \geq \sum a_1 r_2 r_3$, i.e.,

$$a_1 a_2 a_3 \geq 4 \sum a_1 r_2 r_3.$$

But this inequality can either be found in [1] (p. 333, item 7.1 or p. 339, item 7.28) or in *Crux* ([1990: 64] or [1987: 260]).

References:

- [1] Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer, Dordrecht, 1989.
[2] Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968.

Also solved by JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; MURRAY S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * *

1544. [1990: 143] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*
One root of $x^3 + ax + b = 0$ is λ times the difference of the other two roots ($|\lambda| \neq 1$).
Find this root as a simple rational function of a , b and λ .

Solution by Jean-Marie Monier, Lyon, France.

Letting the roots be x_1, x_2, x_3 , we have

$$x_1^2 = [\lambda(x_2 - x_3)]^2 = \lambda^2[(x_2 + x_3)^2 - 4x_2x_3] = \lambda^2 \left(x_1^2 + \frac{4b}{x_1} \right)$$

(since $x_1 + x_2 + x_3 = 0$ and $x_1x_2x_3 = -b$), hence $(1 - \lambda^2)x_1^3 = 4\lambda^2b$ or

$$x_1^3 = \frac{4\lambda^2b}{1 - \lambda^2} .$$

Thus

$$ax_1 + b = -x_1^3 = \frac{4\lambda^2b}{\lambda^2 - 1} ,$$

so

$$x_1 = \frac{1}{a} \left(\frac{4\lambda^2b}{\lambda^2 - 1} - b \right) = \frac{b(3\lambda^2 + 1)}{a(\lambda^2 - 1)} .$$

[*Editor's note.* Monier's solution actually contained a small error, but in other respects was the nicest one received, so the error has been corrected in the above write-up.]

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATHEW ENGLANDER, student, University of Waterloo; M. MERCEDES SÁNCHEZ BENITO, Madrid, and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D. HNIDEI, student, University of British Columbia; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward's School, Birmingham, England; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; BEATRIZ MARGOLIS, Paris, France; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

As was pointed out by several solvers, the solution is valid only for $a \neq 0$.

* * * *

1545. [1990: 144] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A sphere is said to be inscribed into the skeleton of a convex polyhedron if it is tangent to all the edges of the polyhedron. Given a convex polyhedron P and a point O inside it, suppose a sphere can be inscribed into the skeleton of each pyramid spanned by O and a face of P .

(a) Prove that if every vertex of P is the endpoint of exactly three edges then there exists a sphere inscribed into the skeleton of P .

(b) Is this true without the assumption stated in (a)?

Solution by the proposer.

(a) Label the faces F_1, \dots, F_n in such a way that F_1 and F_2 be adjacent and every successive F_k ($3 \leq k \leq n$) be adjacent to some two previous faces. [For instance, F_3 would be the third face meeting at one of the endpoints of the common edge of F_1 and F_2 .—Ed.] Let P_j be the pyramid of base F_j and vertex O , let D_j be the disc inscribed in F_j (it does exist) and let ℓ_j be the line through the center of D_j perpendicular to the plane of F_j . The two spheres inscribed into the skeletons of P_1 and P_2 touch the common edge of F_1 and F_2 in the same point (they both contain the incircle of triangle $P_1 \cap P_2$). The two planes passing through that point and through line ℓ_1 , resp. ℓ_2 , are perpendicular to the edge, hence they are identical. Thus ℓ_1 and ℓ_2 intersect.

Face F_3 is adjacent to F_1 and F_2 . The same argument shows that ℓ_3 cuts ℓ_1 and ℓ_2 . Since ℓ_1, ℓ_2, ℓ_3 do not lie in the same plane, we infer that they have exactly one point in common.

By induction, each successive line ℓ_4, \dots, ℓ_n passes through this point, which is therefore equally distant from all edges of P [because it is equidistant from all the edges of any one face]; the claim results.

(b) The assumption given in (a) is indeed essential. As an example consider the following construction. ABC is an equilateral triangle of side a , O is its center. Points D and E lie symmetrically with respect to plane ABC , at a distance h from it; O is the midpoint of DE . Polyhedron P is defined as the union of pyramids $ABCD$ and $ABCE$. A sphere can be inscribed into the skeleton of any one of the six pyramids spanned by O and the faces of P if and only if $h = (\sqrt{3} - 1)a/4$ (verification is routine and therefore omitted). A sphere inscribed into the skeleton of P exists if and only if $h = a/3$.

There was one incorrect solution sent in, which failed to take into account the condition in part (a).

* * * *

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