Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- How Round is a Plane Figure?
- Tiling a Tromino

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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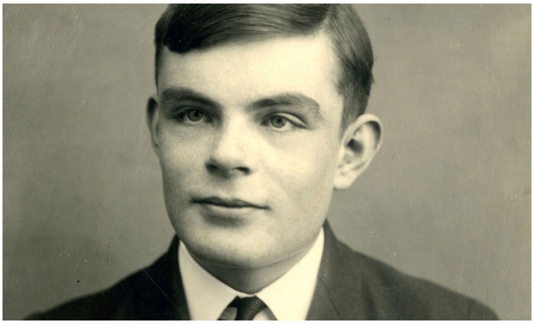
Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Breaking the code



Alan Turing

Mathematics seems to be featuring increasingly in the Arts these days, on TV, in films, and the theatre. At the time of writing, I have just been to see *Breaking the Code* at the Little Theatre in Leicester, UK. Although an amateur company, you would never know from the high standard of acting and production. It is a thought-provoking play on the life if Alan Turing, widely regarded as one of the greatest mathematicians not only that the UK has produced but internationally. He is called the father of the modern computer. His work at Bletchley Park in the Second World War in breaking the German Enigma Code, as well as leading to the development of the computer, is said to have shortened the war by up to two years, saving many lives, and was decisive in the war. The play concentrates more on his tortured personal life and the appalling way in which he was treated by the authorities because of his homosexuality, leading to his tragic suicide. It wasn't only the Enigma Code that he broke!

For a more cinematic view, readers may have seen Benedict Cumberbatch's portrayal of Alan Turing in the film *The Imitation Game*. A view of one whose mother worked at Bletchley Park in the war is that certain liberties were taken in the film. But that is cinema for you! My informant recently collected a posthumous award on behalf of her mother. She tells me that she knew nothing of her mother's work as part of the team at Bletchley Park, such was the code of secrecy almost to this day. This was a code that was not broken! Only recently has the crucial nature of the work at Bletchley Park come into the open. Alan Turing and other mathematicians, as well as many unsung heroes (and heroines!) like my informant's mother, were crucial in not only helping to defeat a tyrant but also in developing the computers that we all, for better and worse, rely on today.

Three sticks

This is the title of a new geometrical board game. To quote: 'Players are challenged to think creatively and use only three types of sticks to create geometrical shapes. In turn, players add two sticks to the board and earn points for the shapes created. With compelling game mechanics and exciting surprises, *Three Sticks* is a promising entertainer for both kids and adults'. See http://igg.me/at/threesticks.

A Serendipitous Encounter with a Bessel Function

MARTIN GRIFFITHS

In this article we describe an encounter with a mathematical object known as a *Bessel function*. Such functions often arise in advanced problems associated with, for example, heat conduction in cylindrical objects, modes of vibration of thin circular drums, and dynamics of floating bodies. The interesting point here though is that our meeting came about as a result of a student's question in a school mathematics classroom.

1. Introduction

Undergraduate students of engineering, physics, or mathematics often meet *Bessel functions* in connection with, for example, heat conduction in cylindrical objects, modes of vibration of thin circular drums, dynamics of floating bodies, and signal processing (see reference 1). These functions are also associated with several continuous probability distributions. By way of an example, a Bessel function appears in the definition of the *von Mises distribution* (see reference 2), which is a continuous probability distribution on the unit circle. The von Mises distribution has been used to investigate the seasonality of the onset of various diseases, including glaucoma and leukaemia (see reference 3).

In this article I describe my own recent encounter with a Bessel function. There are two reasons why this turned out to be interesting. First, it came about as a result of a student's question in a school mathematics classroom. Second, the student's question concerned a discrete probability distribution, as opposed to a continuous one.

2. Bessel functions

We provide here, for interested readers, a brief summary of the mathematical background to Bessel functions. There are in fact several varieties of these functions, although we shall be concerned only with the *Bessel function of the first kind* (see references 1 and 4). This occurs as a solution to

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0,$$

which is known as *Bessel's differential equation*. Note that both the variable x and the constant α may take on complex values.

The Bessel function of the first kind $J_\alpha(x)$ is given by the infinite series

$$J_\alpha(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k+\alpha}, \tag{1}$$

where $\Gamma(x)$ is the gamma function (see reference 5). This might appear a little intimidating at first glance. However, we shall only have cause to consider the function in the special case for which $\alpha = 0$. Furthermore, for x real and positive, $\Gamma(x)$ might be regarded as a continuous version of the factorial function. Indeed, when x is a nonnegative integer, we have

$$\Gamma(x+1) = x!.$$

The function $J_\alpha(x)$ exhibits oscillatory behaviour when x is real.

3. An introductory problem

In this section I describe the question and its extension that led me to stumble across a Bessel function. As part of our revision schedule, I recently gave the students in my statistics class a past examination paper on probability. One of the questions concerned the discrete random variables X and Y, representing the total number of cars owned by a randomly chosen household in Hamilton, Canada, and Hamilton, New Zealand, respectively. The mass functions of X and Y are given in tables 1 and 2, respectively. The question was: 'what is the probability that a randomly selected household in Hamilton, New Zealand, owns more cars than a randomly selected household in Hamilton, Canada?'

Assuming the random variables X and Y to be independent, the answer to the question is

$$\sum_{k=1}^{4} \mathrm{P}(Y = k)\, \mathrm{P}(X < k) \approx 0.37.$$

A student subsequently asked whether it would be possible for an alternative situation to arise in which, instead of being restricted to finitely many values, each of the random variables was able to take on infinitely many values. If so, he wondered, how would we go about obtaining a solution to the corresponding problem?

A natural way to extend the problem in the manner mooted by the student is by way of the Poisson distribution (see references 6 and 7). In particular, we consider here a probabilistic

Table 1 The mass function of X.

x	$\mathrm{P}(X = x)$
0	0.059
1	0.383
2	0.377
3	0.153
4	0.028

Table 2 The mass function of Y.

y	$\mathrm{P}(Y = y)$
0	0.09
1	0.298
2	0.423
3	0.119
4	0.07

scenario involving two telephone call centres, A and B. Suppose that each receives, on average, λ calls per minute throughout the working day. What is the probability that, during the course of a randomly chosen minute, A receives more calls than B?

4. The calculation of a probability

In order to answer the question posed above, we will assume (although in reality there are clearly a number of practical constraints here) that the number of calls arriving at A in one minute follows a Poisson distribution with mean λ, and similarly for B. Let T be a Poisson random variable with parameter λ, so that

$$\mathrm{P}(T = t) = \frac{\mathrm{e}^{-\lambda}\lambda^t}{t!},$$

for $t = 0, 1, 2, \ldots$. Suppose that T_1 and T_2, the number of calls received over a one-minute period by call centres A and B, respectively, are independently and identically distributed as T. We seek an expression for the probability $\mathrm{P}(T_1 > T_2)$.

First, note that

$$\mathrm{P}(T_1 = T_2) = \sum_{k=0}^{\infty}\left(\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\right)^2.$$

Then, since $\mathrm{P}(T_1 > T_2) = \mathrm{P}(T_2 > T_1)$ by symmetry, we have

$$\begin{aligned}\mathrm{P}(T_1 > T_2) &= \tfrac{1}{2}(1 - \mathrm{P}(T_1 = T_2)) \\ &= \frac{1}{2}\left(1 - \sum_{k=0}^{\infty}\left(\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\right)^2\right) \\ &= \frac{1}{2}\left(1 - \mathrm{e}^{-2\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(k!)^2}\right). \qquad (2)\end{aligned}$$

From the comments in Section 2, we know that $\Gamma(k+1) = k!$. Therefore,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(k!)^2} &= \sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(k+1)}\left(\frac{2\lambda}{2}\right)^{2k} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+1)}\left(\frac{2\lambda\mathrm{i}}{2}\right)^{2k} \\ &= J_0(2\lambda\mathrm{i}), \qquad (3)\end{aligned}$$

where (1) has been used, and i is the purely imaginary number such that $\mathrm{i}^2 = -1$. From (2) and (3) it then follows that

$$\mathrm{P}(T_1 > T_2) = \frac{1}{2}\left(1 - \frac{J_0(2\lambda\mathrm{i})}{\mathrm{e}^{2\lambda}}\right).$$

It is worth noting that

$$\lim_{\lambda\to\infty}\mathrm{P}(T_1 > T_2) = \tfrac{1}{2},$$

as may clearly be seen in figure 1.

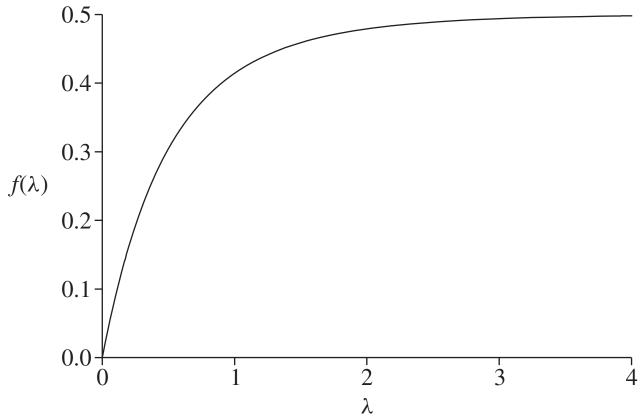


Figure 1 A graph showing how the probability $f(\lambda) = \mathrm{P}(T_1 > T_2)$ changes with λ.

5. Closing comments

This might make for an interesting mathematical diversion in the classroom or lecture theatre. Indeed, it is possible that undergraduate engineering, physics, and mathematics students who have seen these functions in other contexts will find the result derived here intriguing. This serves further to demonstrate the wonderfully interconnected nature of mathematics.

Bessel functions appear in the library of functions that may be implemented on MATHEMATICA®, and this is in fact how figure 1 was obtained. In section 2 it was mentioned that $J_\alpha(x)$ exhibits oscillatory behaviour when x is real. When the Bessel function of the first kind takes on a purely imaginary argument we observe exponential rather than oscillatory behaviour. Such a relationship is also observed between the trigonometric functions and their hyperbolic counterparts.

References

1 http://en.wikipedia.org/wiki/Bessel_function.
2 http://en.wikipedia.org/wiki/Von_Mises_distribution.
3 F. Gao *et al.*, On the application of the von Mises distribution and angular regression methods to investigate the seasonality of disease onset, *Statist. Medicine* **25** (2006), pp. 1593–1618.
4 R. Wrede and M. R. Spiegel, *Schaum's Outline of Advanced Calculus* (McGraw-Hill, New York, 2010), 3rd edn.
5 D. E. Knuth, *The Art of Computer Programming*, Vol. 1, *Fundamental Algorithms* (Addison-Wesley, Reading, MA, 1968).
6 G. Grimmett and D. Stirzaker, *Probability and Random Processes* (Oxford University Press, 2001), 3rd edn.
7 http://en.wikipedia.org/wiki/Poisson_distribution.

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A Gibonacci Puzzle with Dividends

THOMAS KOSHY

We develop an interesting pattern involving Fibonacci and Lucas numbers. We then extend it to Fibonacci, Lucas, Pell, and Pell–Lucas polynomials, and then to Pell and Pell–Lucas numbers.

Introduction

Fibonacci numbers F_n and Lucas numbers L_n satisfy the same second-order recurrence

$$x_{n+2} = x_{n+1} + x_n,$$

where $n \geq 1$. When $x_1 = 1 = x_2$, $x_n = F_n$; and when $x_1 = 1$ and $x_2 = 3$, $x_n = L_n$ (see reference 1). Pell numbers P_n and Pell–Lucas numbers Q_n, on the other hand, satisfy the recurrence

$$x_{n+2} = 2x_{n+1} + x_n,$$

where $n \geq 1$. When $x_1 = 1$ and $x_2 = 2$, $x_n = P_n$; and when $x_1 = 1$ and $x_2 = 3$, $x_n = Q_n$ (see references 1–3). They are all closely related, as we will see shortly. Table 1 shows the first ten Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

In *Pell's Equation* (see reference 2), the author presented a beautiful number pattern and invited the reader to predict the formula for the following pattern and to confirm it (see references 3 and 4):

$$3^4 - 5 \cdot 4^2 = 1,$$
$$7^4 - 24 \cdot 10^2 = 1,$$
$$17^4 - 145 \cdot 24^2 = 1,$$
$$41^4 - 840 \cdot 58^2 = 1,$$

Table 1 The first ten Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

n	F_n	L_n	P_n	Q_n
1	1	1	1	1
2	1	3	2	3
3	2	4	5	7
4	3	7	12	17
5	5	11	29	41
6	8	18	70	99
7	13	29	169	239
8	21	47	408	577
9	34	76	985	1 393
10	55	123	2 378	3 363

and so on. The pattern might not be obvious. But, using Pell and Pell–Lucas numbers, we can rewrite the pattern as follows:

$$3^4 - 1 \cdot 5 \cdot (2 \cdot 2)^2 = 1,$$
$$7^4 - 2 \cdot 12 \cdot (2 \cdot 5)^2 = 1,$$
$$17^4 - 5 \cdot 29 \cdot (2 \cdot 12)^2 = 1,$$
$$41^4 - 12 \cdot 70 \cdot (2 \cdot 29)^2 = 1,$$

and so on. Now the pattern is quite obvious: $Q_{n+1}^4 - \mathrm{P}_n\mathrm{P}_{n+2}(2\mathrm{P}_{n+1})^2 = 1$. We can establish this identity using 2×2 determinants (see references 3 and 4).

Our pursuit of a corresponding formula for Fibonacci and Lucas numbers took us to an interesting formula with a slightly different look:

$$\mathrm{L}_n^4 - (5\mathrm{F}_n^2)^2 - 40(-1)^n\mathrm{F}_n^2 = 16. \tag{1}$$

For example,

$$1^4 - 5^2 + 10 \cdot 2^2 = 16,$$
$$3^4 - 5^2 - 10 \cdot 2^2 = 16,$$
$$4^4 - 20^2 + 10 \cdot 4^2 = 16,$$
$$7^4 - 45^2 - 10 \cdot 6^2 = 16,$$

and so on.

The Gibonacci family

Interestingly, we can generalize identity (1). To this end, we introduce the family of *Gibonacci* (generalized Fibonacci) polynomials $\mathrm{g}_n(x)$, where $\mathrm{g}_{n+2}(x) = x\mathrm{g}_{n+1}(x) + \mathrm{g}_n(x)$. (Here, $\mathrm{g}_1(x) = a(x)$ and $\mathrm{g}_2(x) = b(x)$ are arbitrary.) It then follows that $\mathrm{g}_0(x) = \mathrm{g}_2(x) - x\mathrm{g}_1(x) = b(x) - xa(x)$. When $a(x) = 1$ and $b(x) = x$, $\mathrm{g}_n(x) = \mathrm{f}_n(x)$, the nth *Fibonacci polynomial*; and when $a(x) = x$ and $b(x) = x^2 + 2$, $\mathrm{g}_n(x) = \mathrm{l}_n(x)$, the nth *Lucas polynomial*. Table 2 gives the first six Fibonacci and Lucas polynomials. Clearly, $\mathrm{f}_0(x) = 0$, $\mathrm{l}_0(x) = 2$, $\mathrm{f}_n(1) = \mathrm{F}_n$, and $\mathrm{l}_n(1) = \mathrm{L}_n$.

The polynomials $\mathrm{p}_n(x) = \mathrm{f}_n(2x)$ and $\mathrm{q}_n(x) = \mathrm{l}_n(2x)$ are the *Pell* and *Pell–Lucas polynomials*, respectively. Table 3 shows the first six Pell and Pell–Lucas polynomials. Then $\mathrm{p}_n(1) = \mathrm{f}_n(2) = \mathrm{P}_n$ and $\mathrm{q}_n(1) = \mathrm{l}_n(2) = 2\mathrm{Q}_n$.

Table 2 The first six Fibonacci and Lucas polynomials.

n	$\mathrm{f}_n(x)$	$\mathrm{l}_n(x)$
1	1	x
2	x	$x^2 + 2$
3	$x^2 + 1$	$x^3 + 3x$
4	$x^3 + 2x$	$x^4 + 4x^2 + 2$
5	$x^4 + 3x^2 + 1$	$x^5 + 5x^3 + 5x$
6	$x^5 + 4x^3 + 3x$	$x^6 + 6x^4 + 9x^2 + 2$

Table 3 The first six Pell and Pell–Lucas polynomials.

n	$\mathrm{p}_n(x)$	$\mathrm{q}_n(x)$
1	1	$2x$
2	$2x$	$4x^2+2$
3	$4x^2+1$	$8x^3+6x$
4	$8x^3+4x$	$16x^4+16x^2+2$
5	$16x^4+12x^2+1$	$32x^5+40x^3+10x$
6	$32x^5+32x^3+6x$	$64x^6+96x^4+36x^2+2$

In the interest of brevity, we will denote the polynomial $\mathrm{g}_n(x)$ by its name g_n, when there is no ambiguity.

Fibonacci and Lucas polynomials can also be defined explicitly by the Binet-like formulas as follows:

$$\mathrm{f}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{l}_n = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x) = (x+\Delta)/2$ and $\beta = \beta(x) = (x-\Delta)/2$ are the solutions of the equation $t^2 - xt - 1 = 0$ and $\Delta = \Delta(x) = \sqrt{x^2+4}$ (see reference 1).

Using the Binet-like formulas (or induction), we can show that

$$\mathrm{l}_n^2 - \Delta^2\mathrm{f}_n^2 = 4(-1)^n, \tag{2}$$

$$\mathrm{l}_n^2 + \Delta^2\mathrm{f}_n^2 = 2\mathrm{l}_{2n}, \tag{3}$$

$$\Delta^2\mathrm{f}_n^2 + 2(-1)^n = \mathrm{l}_{2n}. \tag{4}$$

A Gibonacci hybridity

We can now generalize (1) to Fibonacci and Lucas polynomials:

$$\mathrm{l}_n^4 - (\Delta^2\mathrm{f}_n^2)^2 - 8(-1)^n\Delta^2\mathrm{f}_n^2 = 16. \tag{5}$$

This can be established fairly quickly, as we now demonstrate.

Using identities (2), (3), and (4), we have

$$\mathrm{l}_n^4 - (\Delta^2\mathrm{f}_n^2)^2 = (\mathrm{l}_n^2 - \Delta^2\mathrm{f}_n^2)(\mathrm{l}_n^2 + \Delta^2\mathrm{f}_n^2) = [4(-1)^n](2\mathrm{l}_{2n}),$$
$$\mathrm{l}_n^4 - (\Delta^2\mathrm{f}_n^2)^2 - 8(-1)^n\Delta^2\mathrm{f}_n^2 = 8(-1)^n(\mathrm{l}_{2n} - \Delta^2\mathrm{f}_n^2) = 8(-1)^n \cdot 2(-1)^n = 16,$$

as desired.

Since $\Delta(1) = \sqrt{5}$, clearly, (1) follows from (5). Since $\mathrm{q}_n(x) = \mathrm{l}_n(2x)$ and $2\mathrm{Q}_n = \mathrm{q}_n(1)$, it follows from (5) that

$$\mathrm{q}_n^4 - 16(x^2+1)^2\mathrm{p}_n^4 - 32(x^2+1)(-1)^n\mathrm{p}_n^2 = 16,$$
$$\mathrm{Q}_n^4 - 4\mathrm{P}_n^4 - 4(-1)^n\mathrm{P}_n^2 = 1. \tag{6}$$

Equation (6) yields the following interesting pattern:

$$\begin{aligned}1^4 - 4 \cdot 1^4 + 2^2 &= 1,\\ 3^4 - 4 \cdot 2^4 - 4^2 &= 1,\\ 7^4 - 4 \cdot 5^4 + 10^2 &= 1,\\ 17^4 - 4 \cdot 12^4 - 24^2 &= 1,\end{aligned}$$

and so on. Using (2) and the fact that $\mathrm{f}_{2n} = \mathrm{f}_n \mathrm{l}_n$, we can rewrite (5) in a slightly different way. Since

$$\begin{aligned}\mathrm{l}_n^4 - (\Delta^2 \mathrm{f}_n^2)^2 - 8(-1)^n \Delta^2 \mathrm{f}_n^2 &= \mathrm{l}_n^4 - \Delta^2 \mathrm{f}_n^2[\Delta^2 \mathrm{f}_n^2 + 8(-1)^n]\\ &= \mathrm{l}_n^4 - \Delta^2 \mathrm{f}_n^2[\mathrm{l}_n^2 + 4(-1)^n]\\ &= \mathrm{l}_n^4 - \Delta^2 \mathrm{f}_{2n}^2 - 4(-1)^n \Delta^2 \mathrm{f}_n^2,\end{aligned}$$

it follows that

$$\mathrm{l}_n^4 - \Delta^2 \mathrm{f}_{2n}^2 - 4(-1)^n \Delta^2 \mathrm{f}_n^2 = 16. \tag{7}$$

For example,

$$\begin{aligned}\mathrm{l}_3^4 - \Delta^2 \mathrm{f}_6^2 + 4\Delta^2 \mathrm{f}_3^2 &= (x^3 + 3x)^4 - (x^2 + 4)(x^5 + 4x^3 + 3x)^2 + 4(x^2 + 4)(x^2 + 1)^2\\ &= 16.\end{aligned}$$

In particular, (7) yields

$$\begin{aligned}\mathrm{L}_n^4 - 5\mathrm{F}_{2n}^2 - 20(-1)^n \mathrm{F}_n^2 &= 16,\\ \mathrm{q}_n^4 - 4(x^2 + 1)\mathrm{p}_{2n}^2 - 16(x^2 + 1)(-1)^n \mathrm{p}_n^2 &= 16,\\ 2\mathrm{Q}_n^4 - \mathrm{P}_{2n}^2 - 4(-1)^n \mathrm{P}_n^2 &= 2.\end{aligned}$$

For example, $2\mathrm{Q}_5^4 - \mathrm{P}_{10}^2 + 4\mathrm{P}_5^2 = 2 \cdot 41^4 - 2378^2 + 4 \cdot 29^2 = 2$.

Acknowledgement The author would like to thank the Editor for his thoughtful suggestions which improved the quality of the exposition of the original version.

References

1 T. Koshy, *Fibonacci and Lucas Numbers with Applications* (John Wiley, New York, 2001).
2 E. J. Barbeau, *Pell's Equation* (Springer, New York, 2003).
3 T. Koshy, *Pell and Pell–Lucas Numbers with Applications* (Springer, New York, 2014).
4 T. Koshy, A Pell and Pell–Lucas hybridity, *J. Recreational Math.* **36** (2007), pp. 324–326.

Thomas Koshy *received his PhD in Algebraic Coding Theory from Boston University in 1971. He has authored several books, including 'Catalan Numbers with Applications' (Oxford), 'Triangular Arrays with Applications' (Oxford), and most recently 'Pell and Pell–Lucas Numbers with Applications' (Springer). He is a Professor Emeritus at Framingham State University.*

Constructing a Right-Angled Triangle from its Hypotenuse and Inradius

MICHEL BATAILLE

Given positive a and r subject to some restriction, we offer three different constructions of a right-angled triangle with hypotenuse a and inradius r.

Introduction

The starting point of this article is the observation that the inradius r of a right-angled triangle ABC with hypotenuse $BC = a$ and legs $CA = b$ and $AB = c$ obeys a very simple formula, namely

$$r = \frac{b + c - a}{2}. \tag{1}$$

To see this, let the incircle, with centre I, be tangent to the sides BC, CA, and AB at D, E, and F, respectively, and set $x = AE = AF$, $y = BF = BD$, and $z = CD = CE$ (see figure 1).

On the one hand, because of the right angle at vertex A, the quadrilateral $AEIF$ is a square; hence, $r = x$, and on the other hand, $2(x + y + z) = a + b + c$ and $y + z = a$. A short calculation then yields (1). This formula gives rise to the following question. If we consider all right-angled triangles with hypotenuse a, that is, inscribed in a circle with diameter a, what are the possible values of r? As a first approach, we will check the intuitive result that the inradius is maximal when the right-angled triangle is isosceles.

The maximal value of *r*

The inradius r and circumradius R of any triangle satisfy Euler's inequality $r \leq R/2$ (see reference 1 for a proof). If we limit ourselves to right-angled triangles with hypotenuse a, we can expect some stronger inequality in view of the particular formulas (1) and $R = a/2$.

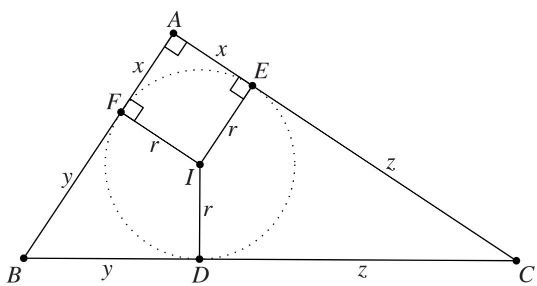

Figure 1

Indeed, using the inequality $2(b^2+c^2) \geq (b+c)^2$ (a straightforward consequence of $2(b^2+c^2)-(b+c)^2=(b-c)^2$) we obtain

$$r \leq \frac{\sqrt{2(b^2+c^2)}-a}{2} = \frac{\sqrt{2a^2}-a}{2} = \frac{a(\sqrt{2}-1)}{2} = R(\sqrt{2}-1),$$

with equality in the case of an isosceles right-angled triangle.

Thus, for any right-angled triangle with hypotenuse a, the inradius r satisfies

$$0 < r \leq \frac{a(\sqrt{2}-1)}{2}. \tag{2}$$

We will now go deeper and show that the interval $(0, a(\sqrt{2}-1)/2]$ is exactly the range of r. Specifically, given positive real numbers a, r subject to (2), we will construct a right-angled triangle ABC with hypotenuse $BC=a$ and inradius r. Three constructions are presented in the following section.

The constructions

In each of the three constructions that follow, we fix a line segment BC with length $a=2R$. We call δ the perpendicular bisector of BC and Γ the circle with radius R centered at the midpoint O of BC. Also given at the start is the length r such that $0<r<a(\sqrt{2}-1)/2$ (we discard $r=a(\sqrt{2}-1)/2$, the construction being immediate in that case).

Construction 1 The first idea coming to mind is likely to determine the legs b and c of the sought triangle. They are the positive solutions of the system formed by the equations $b^2+c^2=a^2$ and $b+c=a+2r$. Using $2bc=(b+c)^2-(b^2+c^2)$, we readily see that b and c are the numbers whose sum and product are $a+2r$ and $2r^2+2ar$, respectively. As such, they are the solutions to the quadratic equation $X^2-(a+2r)X+2r^2+2ar=0$. The restriction $2r<a(\sqrt{2}-1)$ ensures that the discriminant is positive and, assuming that $b>c$, we find

$$b = \frac{a+2r+\sqrt{a^2-4ar-4r^2}}{2} = r+R+\sqrt{R^2-2rR-r^2}$$

and

$$c = r+R-\sqrt{R^2-2rR-r^2}.$$

Constructing line segments of lengths b and c is not so difficult as it might appear. First, with the help of the circle with centre C and radius $R+r$, we obtain the point U of δ such that $CU=R+r$ (see figure 2). Because $r<R(\sqrt{2}-1)$, we have $OU^2=2rR+r^2<R^2$; hence, the parallel to BC through U certainly intersects the circle Γ, say at V on the same side of δ as B. Then $UV^2=OV^2-OU^2=R^2-2rR-r^2$. Thus, $b=CU+UV$ and $c=BU-UV$ and we can easily complete the construction of A providing a suitable triangle ABC (see figure 2).

Construction 2 Interestingly, once V has been obtained as in Construction 1, an alternative construction is possible. We construct the orthogonal projection D of V onto BC and the point I of the line segment DV such that $DI=r$ (see figure 3). The circle γ with centre I and radius r is tangent to BC at D. Let the tangents to γ from B and C, distinct from BC,

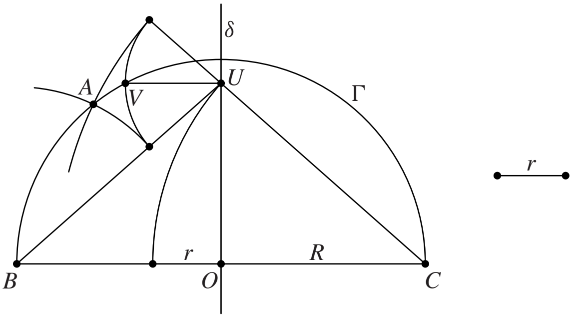

Figure 2

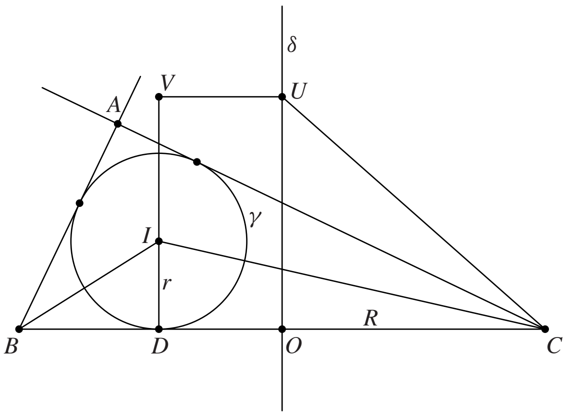

Figure 3

intersect at A and let $\theta = \angle BIC$. Using that twice the area of ΔBIC is equal to $IB \cdot IC \cdot \sin\theta$ as well as $2rR$, we have $\sin\theta = 2rR/IB \cdot IC$. The law of cosines yields

$$\begin{aligned} \cos\theta &= \frac{IB^2 + IC^2 - BC^2}{2IB \cdot IC} \\ &= \frac{r^2 + (R - OD)^2 + r^2 + (R + OD)^2 - 4R^2}{2IB \cdot IC} \\ &= \frac{r^2 - R^2 + OD^2}{IB \cdot IC}. \end{aligned}$$

Since $OD^2 = UV^2 = R^2 - 2rR - r^2$, we readily obtain $\cos\theta = -\sin\theta$; hence, $\theta = 135^\circ$. Thus,

$$\angle CBA + \angle BCA = 2(\angle CBI + \angle BCI) = 2(180^\circ - 135^\circ) = 90^\circ;$$

therefore, the triangle ABC has a right angle at A. Since, in addition, its inradius is r, this triangle ABC meets the requirements.

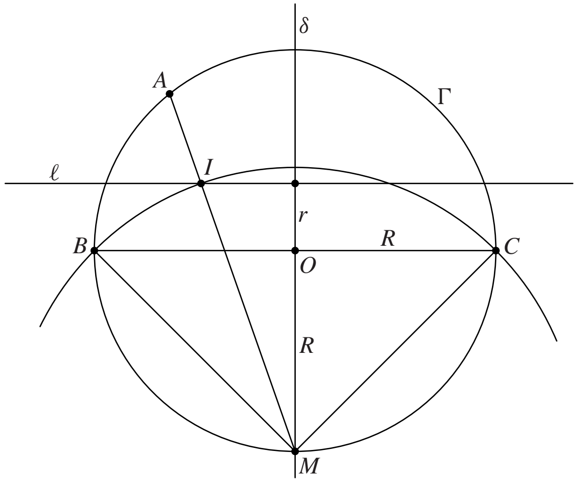

Figure 4

Construction 3 Our third construction is very quick, as it is achieved by drawing only two lines and a circle in addition to Γ and δ. First, we draw a line ℓ parallel to BC and at a distance r from BC. Let M be the point both on Γ and δ with M and ℓ on opposite sides of BC (see figure 4). Let I be one of the points of intersection of ℓ and the circle with centre M and radius $MB = MC = R\sqrt{2}$ (such a point I does exist because the distance $R + r$ from M to ℓ is less than the radius $R\sqrt{2}$). Then the line MI meets Γ again at A such that ΔABC is a solution.

To see why, first note that A is on Γ; hence, ΔABC has a right angle at A. Second, since M is the midpoint of the arc BC of Γ not containing A, AM is the internal bisector of $\angle BAC$. It is now sufficient to prove that the point I is the incentre of ΔABC. Denoting as usual each angle of ΔABC by its vertex, $\angle BMA$ and $\angle BCA$ subtend the same arc of Γ; hence, $\angle BMI = C$. Since $MB = MI$, it follows that $\angle MBI = \frac{1}{2}(180° - C)$. Observing that $\angle MBC = \angle MAC = A/2$, we deduce that $\angle IBC = \frac{1}{2}(180° - C) - A/2 = B/2$. Thus, BI bisects $\angle CBA$ and the conclusion follows.

Reference

1 M. Bataille, A trip from trig to triangle, *Math. Spectrum* **44** (2011/2012), pp. 19–23.

Michel Bataille *now retired, taught at an undergraduate level near Rouen in France. His main mathematical interests are geometry and problem solving, both as a solver and as a setter.*

Find the one positive integer less than 2016 to have more factors than 2016.

Jonny Griffiths

The Hypergeometric Distribution: an Application from Poker

JOHN C. B. COOPER

'People think poker's about statistics, odds, probability. But, if you play at the highest level, all the players know the odds off by heart, so that's not where the battle takes place. What separates the best from the rest is their ability to read others.' Jo Nesbo

Card games offer interesting and accessible opportunities to illustrate various aspects of probability. This article introduces the hypergeometric probability distribution and uses it to confirm the probability of occurrence of a flush in the card game poker. In addition, the results of a simple experiment to examine the expected versus actual frequencies of poker hands are reported.

Introduction

Poker is a family of card games played by both professionals and amateurs alike, with television and internet coverage of international poker tournaments attracting considerable worldwide interest. In the simplest variant of poker, each player is dealt a hand of five cards from a standard deck of 52. Thus, there are

$$\binom{52}{5} = 2\,598\,960$$

possible hands. These hands are in turn classified into ten possible groups, the values of which are inversely related to their probability of occurrence. A brief description of these groups and their probability of occurrence are listed in table 1.

Table 1

Hand	Description	Probability	Expected	Observed
royal flush	A, K, Q, J, 10 of same suit	0.15×10^{-5}	0	0
straight flush	five cards of same suit in rank order	0.14×10^{-4}	0	0
four of a kind	four cards of same rank plus one other	0.24×10^{-3}	0	0
full house	three cards of same rank plus two of another rank	0.001 4	1	2
flush	five cards of same suit	0.002 0	2	3
straight	five consecutive ranks	0.003 9	4	5
three of a kind	three cards of same rank plus two others	0.021 1	21	25
two pair	two sets of two cards of same rank plus one other	0.047 5	48	25
one pair	two cards of same rank plus three others	0.422 6	423	413
highest card	highest rank (Ace highest, two lowest)	0.501 2	501	527
		total 1.000 0	total 1 000	total 1 000

The following three hands are relatively rare and therefore valuable.

Royal flush This consists of the Ace, King, Queen, Jack, and 10, all of the same suit. Clearly, there can be only four ways of obtaining this combination so that the probability of occurrence is

$$\frac{4}{2\,598\,960} \approx 0.15 \times 10^{-5}.$$

Straight flush This consists of five successive cards of the same suit in rank order where, importantly, an Ace may count as either the highest or the lowest card. It is easy to see that there is a total of 40 ways of obtaining such a combination. Excluding the four possible royal flushes, which are themselves a special case of the straight flush, 36 remain so that the probability of occurrence is

$$\frac{36}{2\,598\,960} \approx 0.14 \times 10^{-4}.$$

Flush This consists of any five cards of the same suit. Given that there are $\binom{13}{5} = 1287$ ways of selecting five cards from any suit and $\binom{4}{1} = 4$ ways of selecting a suit, then a flush may be obtained in $\binom{13}{5}\binom{4}{1} = 5148$ ways. Again, excluding the four royal flushes and the 36 straight flushes, we obtain 5108 combinations with probability of occurrence equal to $5108/2\,598\,960 \approx 0.0020$. Interestingly, this probability may be confirmed by invoking the hypergeometric distribution.

Generalised hypergeometric distribution

Consider a nonhomogeneous population of N items consisting of m groups with n_1 items in the first group, n_2 items in the second group, ..., and n_m items in the mth group. The probability of selecting X items from N containing exactly $x_1 \leq n_1$ items from the first group, $x_2 \leq n_2$ items from the second group and, ..., $x_m \leq n_m$ items from the mth group at random, without replacement, and without ordering, may be obtained from the hypergeometric distribution. Its probability mass function can be written as

$$\Pr(x_1, x_2, \ldots, x_m) = \binom{n_1}{x_1}\binom{n_2}{x_2}\cdots\binom{n_m}{x_m} \Bigg/ \binom{N}{X},$$

where $x_i \geq 0$, $X = \sum_{i=1}^{m} x_i$, and $N = \sum_{i=1}^{m} n_i$.

Suppose that the nonhomogeneous population is a pack of $N = 52$ playing cards comprising the $m = 4$ suits (hearts, diamonds, spades, and clubs) each with $n = 13$ ranks ranging from Ace, 2, ..., King. Thus, in this case, $n_1 = n_2 = n_3 = n_4$. Since a flush is defined as a hand of cards containing five hearts or five diamonds or five spades or five clubs, then its probability of occurrence can be computed as

$$\begin{aligned}\Pr(\text{flush}) = \Pr(x_1 &= \text{five hearts}) + \Pr(x_2 = \text{five diamonds}) \\ &+ \Pr(x_3 = \text{five spades}) + \Pr(x_4 = \text{five clubs}),\end{aligned}$$

where

$$\begin{aligned}\Pr(x_1 = \text{five hearts}) &= \tbinom{13}{5}\tbinom{13}{0}\tbinom{13}{0}\tbinom{13}{0}/\tbinom{52}{5}, \\ \Pr(x_2 = \text{five diamonds}) &= \tbinom{13}{0}\tbinom{13}{5}\tbinom{13}{0}\tbinom{13}{0}/\tbinom{52}{5}, \\ \Pr(x_3 = \text{five spades}) &= \tbinom{13}{0}\tbinom{13}{0}\tbinom{13}{5}\tbinom{13}{0}/\tbinom{52}{5}, \\ \Pr(x_4 = \text{five clubs}) &= \tbinom{13}{0}\tbinom{13}{0}\tbinom{13}{0}\tbinom{13}{5}/\tbinom{52}{5}.\end{aligned}$$

Thus, the probability of *any* flush is equal to $\frac{5148}{2\,598\,960} \approx 0.001\,98$.

Excluding the four royal flushes and the 36 straight flushes, we obtain the marginally reduced probability of $0.001\,965 \approx 0.002\,0$, the same as above.

Expected versus actual frequency of poker hands

On an interminable return flight from the UK to Australia, the author dealt himself 1000 hands of five-card poker and recorded the results. Table 1 provides a brief description of all groups of poker hands, their probability of occurrence, together with the expected and actual frequencies. The derivation of these probabilities has been widely published elsewhere and will not be reproduced here. The interested reader is referred to reference 2, where the derivations are accessible and user-friendly.

Postscript

It is interesting to note that, with the exception of the hand two pair, the observed results are reasonably close to those expected. The author can offer no explanation as to why the observed frequency of the hand two pair is almost only half of that expected. One suggestion by an impartial reader is that there has been a data recording error. Admittedly, the aircraft environment was confined and restricted, jet lag was setting in, and the task itself was repetitive and tedious. The author would, however, reject that hypothesis. The task was broken down into manageable chunks, completed with the utmost care, and interspersed with other distractions.

References

1 J. Nesbo, *The Snowman* (Vintage, London, 2010), p. 282.

2 N. Rimmer, Counting poker hands and finding probabilities, available at www.math.upenn.edu/~rimmer/poker.pdf.

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Mathematical Spectrum Awards for Volume 47

Prizes have been awarded to the following student readers for contributions in Volume 47:

Avery Wilson

for the article ‘Pascal’s Triangle Modulo 3’;

Annanay Kapila

for the solution to Problem 47.2.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Compatible Group Operations

LOUIS RUBIN

Let X be a nonempty set, and suppose that $\bullet$ and $\odot$ are binary operations on X such that, for all a, b, $c \in X$, $(a \bullet b) \odot c = a \bullet (b \odot c)$ and $(a \odot b) \bullet c = a \odot (b \bullet c)$. Then we say that $\bullet$ and $\odot$ are *compatible*. In this article, we show that such operations arise on disjoint unions of isomorphic groups. In fact, given disjoint groups G and H, we show that G is isomorphic to H if and only if the group operations of G and H extend, respectively, to compatible group operations $\bullet$, $\odot$ on $G \cup H$.

Let X be a nonempty set, and suppose that $\bullet$ and $\odot$ are binary operations on X such that, for all $a, b, c \in X$, the following equations hold:

$$(a \bullet b) \odot c = a \bullet (b \odot c), \tag{1}$$

$$(a \odot b) \bullet c = a \odot (b \bullet c). \tag{2}$$

Then we refer to the operations $\bullet$ and $\odot$ as *compatible*. Evidently, any associative binary operation is compatible with itself. The following proposition supplies a less trivial example of compatibility.

Proposition 1 *Let G and H be disjoint groups. (We refer to two groups as 'disjoint' if their underlying sets have no elements in common.) Then $G \cong H$ if and only if the group operations of G and H extend, respectively, to compatible group operations $\bullet$ and $\odot$ on $G \cup H$.*

Proof Denote the group operations of G and H by $\cdot$ and $*$, respectively. First suppose that $G \cong H$, and let $f : G \to H$ be an isomorphism. Extend $\cdot$ to the binary operation $\circ_f$ on $G \cup H$ defined by

$$a \circ_f b = \begin{cases} a \cdot b & \text{for } a, b \in G, \\ f^{-1}(a * b) & \text{for } a, b \in H, \\ a * f(b) & \text{for } a \in H, b \in G, \\ f(a) * b & \text{for } a \in G, b \in H. \end{cases}$$

We claim that $(G \cup H, \circ_f)$ is a group. Indeed, if $\mathbb{Z}_2$ is the usual additive group of integers modulo 2, then

$$(G \cup H, \circ_f) \cong G \times \mathbb{Z}_2,$$

by virtue of the mapping

$$\phi : G \times \mathbb{Z}_2 \to G \cup H,$$

given by

$$\phi(g, i) = \begin{cases} g & \text{for } i = 0, \\ f(g) & \text{for } i = 1. \end{cases}$$

Since $G \times \mathbb{Z}_2$ is a group, $(G \cup H, \circ_f)$ must be as well. (The author initially proved that $(G \cup H, \circ_f)$ is a group directly from the group axioms. Amitai Yuval provided this more insightful argument via http://math.stackexchange.com/.) But $f : G \to H$ is an arbitrary group isomorphism, so it follows immediately that the operation $\circ_{f^{-1}}$ defined by

$$a \circ_{f^{-1}} b = \begin{cases} a * b & \text{for } a, b \in H, \\ f(a \cdot b) & \text{for } a, b \in G, \\ a \cdot f^{-1}(b) & \text{for } a \in G, b \in H, \\ f^{-1}(a) \cdot b & \text{for } a \in H, b \in G, \end{cases}$$

is a group operation on $G \cup H$. Now, taking $\bullet = \circ_f$ and $\odot = \circ_{f^{-1}}$, it is straightforward to check that (1) and (2) are satisfied for all $a, b, c \in G \cup H$. For instance, when $a \in G$ and $b, c \in H$, we have

$$\begin{aligned} (a \circ_f b) \circ_{f^{-1}} c &= (f(a) * b) \circ_{f^{-1}} c \\ &= (f(a) * b) * c \\ &= f(a) * (b * c) \\ &= f(a) * (b \circ_{f^{-1}} c) \\ &= a \circ_f (b \circ_{f^{-1}} c). \end{aligned}$$

We leave the remaining cases to the reader. (Note that (2) follows from (1) by replacing f with f^{-1}.) Hence, one direction of the proof is complete.

To prove the converse, suppose that $\cdot$ and $*$ extend, respectively, to compatible group operations $\bullet$ and $\odot$ on $G \cup H$. Let e_G and e_H denote, respectively, the identity elements of G and H. We claim that $g \bullet h \in H$ for all $g \in G$, $h \in H$. To see this, assume that $g \bullet h = g' \in G$, for some $g \in G$, $h \in H$. If $g^{-1} \in G$ is the inverse of g under $\cdot$, then g^{-1} is also the inverse of g in $(G \cup H, \bullet)$. Hence,

$$h = g^{-1} \bullet g' = g^{-1} \cdot g' \in G,$$

a contradiction, as G and H are disjoint. The claim follows, and we may similarly conclude that $h \odot g \in G$ for all $g \in G$, $h \in H$. Define $f : G \to H$ by

$$f(g) = g \bullet e_H.$$

We show that f is a group homomorphism from G into H. For $g, g' \in G$, we have

$$f(g \cdot g') = f(g \bullet g') = (g \bullet g') \bullet e_H = g \bullet (g' \bullet e_H) = g \bullet f(g') = g \bullet (e_H \odot f(g')),$$

where the last equality follows since e_H is the identity element of the group $(G \cup H, \odot)$. Yet $\bullet$ and $\odot$ are compatible, so

$$g \bullet (e_H \odot f(g')) = (g \bullet e_H) \odot f(g') = f(g) \odot f(g') = f(g) * f(g').$$

Hence, $f(g \cdot g') = f(g) * f(g')$, and it follows that f is a group homomorphism. But f is injective, for if $f(g) = f(g')$, then $g \bullet e_H = g' \bullet e_H$, which implies that $g = g'$. Moreover, f is surjective, since, for any $h \in H$, we have

$$h = h \odot e_H = h \odot (e_G \bullet e_H) = (h \odot e_G) \bullet e_H = f(h \odot e_G),$$

where $h \odot e_G \in G$. Thus, f is a group isomorphism, and the proof is complete.

Acknowledgment I thank Dr Edward Letzter for his comments regarding Proposition 1.

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Lyness Cycles and their Invariant Curves

JONNY GRIFFITHS and MARTIN GRIFFITHS

Second-order recurrence relations are often associated with invariant curves. Although they are not particularly common, some order-2 recurrences exhibit periodic behaviour, as their terms repeat, and such recurrences possess invariant curves that are relatively easy to find. In this article we explore some of the properties of these recurrence relations; in particular, we examine the way in which the invariant curves for cycles of different periods can almost coincide, as they degenerate into three straight lines.

Introduction

Readers may well be familiar with the idea of a recurrence relation, and how this may be used to define infinite sequences. For example, the simple recurrence

$$u_{n+2} = u_{n+1} + u_n, \qquad u_1 = 1,\ u_2 = 1,$$

defines the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$. We call this an *order-2 recurrence*, as the next term is defined in terms of the previous two. Now consider the mapping

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x+y \end{pmatrix}.$$

If we put x and y equal to 1, and iterate T, it is easy to see how the Fibonacci sequence is generated (in the more general case, we can choose any starting values for x and y to produce a Fibonacci-type sequence).

Now consider the curve $(x^2 + xy - y^2)^2 = k^2$ (which could be regarded as the union of two conics). Let us choose a point $\binom{a}{b}$ and choose k so that this point is on the curve. Now $T\binom{a}{b}$ is also on the curve if and only if $(b^2 + b(a+b) - (a+b)^2)^2 = k^2$, or $(b^2 + ab + b^2 - a^2 - 2ab - b^2)^2 = k^2$, or $(a^2 + ab - b^2)^2 = k^2$, which is true. So, clearly $T^n\binom{a}{b}$ will be on the curve for all n, and this will be true for any starting point on the curve (see figure 1). We say that $(x^2 + xy - y^2)^2 = k^2$ is an invariant curve for the recurrence; as T is iterated, the points generated appear on each conic in turn.

The reader might justifiably point out that this curve appeared out of nowhere, and we would have to agree – invariant curves can be hard to find. But in certain rare cases, the order-2 recurrence of interest is periodic (its terms repeat in a cycle), and in such circumstances it is possible to find invariant curves with relative ease. These curves in a profound sense share the period of the generating recurrence – if the recurrence is period-5, then the invariant curve is period-5 too.

The invariant curves here are often elliptic (that is, cubic curves without singular points, like cusps or crossings), but they can sometimes become degenerate (cubic curves *with* singular

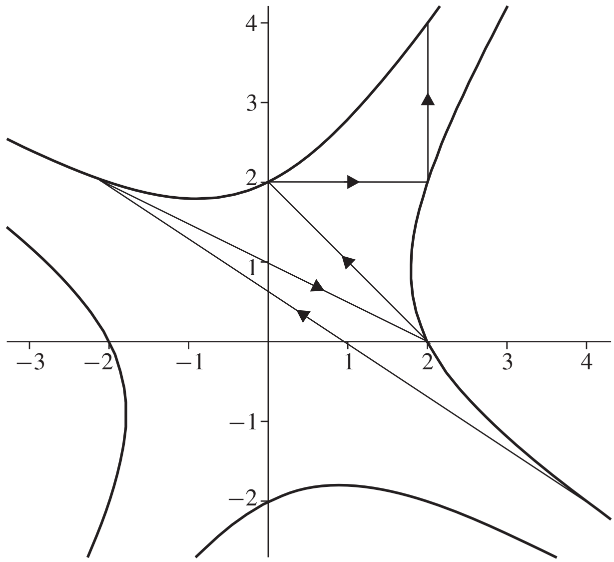

Figure 1 Fibonacci-type sequence $4, -2, 2, 0, 2, 2, \ldots$ with invariant curve $(x^2 + xy - y^2)^2 = 4^2$.

points, for example, if the equation factorises). Our aim here is to show how, in the situation for which the equations for a set of invariant curves come close to factorising into a degenerate set of three straight lines, it is possible for invariant curves of different periodicities to become as close to each other as we like.

An interesting recurrence

Consider the order-1 recurrence relation

$$u_{n+1} = \frac{3u_n + 5}{1 - u_n}.$$

If we put $u_1 = x$, then we can find subsequent terms in terms of x, and something pleasing happens. The sequence goes

$$x, \frac{3x+5}{1-x}, -\frac{x+5}{x+1}, \frac{x-5}{x+3}, x, \ldots. \tag{1}$$

So if $f(x) = (3x+5)/(1-x)$, then $f^4(x) = x$, and the recurrence returns to its start; we say it is period-4.

We note here that $f(x)$ is periodic not just for some freak value of x, but for all values of x except for the special cases $x = 1$, $x = -1$, and $x = -3$. This globally periodic behaviour is unusual among recurrence relations.

What periods are possible in the order-1 case?

A natural next question is as follows. If $f(x) = (ax+b)/(cx+d)$, where a, b, c, and d are rational, what periods are possible for the recurrence x, $f(x)$, $f^2(x), \ldots$? There is a lot of drudgery if we tackle this by hand, but a computer algebra package makes things much easier. It turns out that the only possible periods are 1, 2, 3, 4, and 6. See reference 1 for a proof that is accessible, but far from trivial.

At its heart is *Euler's totient function* $\phi(n)$ (see reference 2), the number of positive integers smaller than n that are coprime with n. For example, $\phi(8) = 4$, $\phi(9) = 6$, and $\phi(p) = p-1$ if p is prime. We may also note that $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, and $\phi(6) = 2$. It is also true that $\phi(n) \geq 3$ for $n > 6$. The proof mentioned above shows that a period-n cycle is impossible if $\phi(n) \geq 3$, which is why 5 misses out.

A related quartic equation

Multiplying the terms in the nonrepeating block in (1) together and equating this to a constant k turns out to be a helpful idea. We obtain

$$x\left(\frac{3x+5}{1-x}\right)\left(-\frac{x+5}{x+1}\right)\left(\frac{x-5}{x+3}\right) = k.$$

This is a quartic equation. What happens if we substitute $(3x+5)/(1-x)$ for x? We get

$$\left(\frac{3x+5}{1-x}\right)\left(\frac{3(3x+5)/(1-x)+5}{1-(3x+5)/(1-x)}\right)\left(-\frac{(3x+5)/(1-x)+5}{(3x+5)/(1-x)+1}\right)\left(\frac{(3x+5)/(1-x)-5}{(3x+5)/(1-x)+3}\right)$$
$$= k,$$

which, by periodicity, is

$$\left(\frac{3x+5}{1-x}\right)\left(-\frac{x+5}{x+1}\right)\left(\frac{x-5}{x+3}\right)x = k.$$

The equation is thus left unchanged by this substitution (it is invariant). It follows from this that if we know that one root is α, the other roots are given by $(3\alpha+5)/(1-\alpha)$, $-(\alpha+5)/(\alpha+1)$, and $(\alpha-5)/(\alpha+3)$.

The periodic order-2 recurrence

We have investigated an order-1 example; what happens for order-2 recurrences, like our Fibonacci example? Are there periodic examples here, and if so, which periods are possible? We turn now to one of the most famous examples in the literature:

$$u_{n+2} = \frac{u_{n+1}+1}{u_n},$$

for $n \geq 1$, where $u_1 = x$ and $u_2 = y$.

Iterating, we get the sequence

$$x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}, x, y, \ldots. \tag{2}$$

This is known as a *Lyness cycle* (see references 3–5), after Robert Cranford Lyness, the late mathematics teacher, inspector, and Olympiad organiser. The way the algebra simplifies halfway through is a joy. Again, this is globally periodic, as long as x and y are such that we never divide by zero.

What periods are possible in the order-2 case?

So what periods are possible here? If $f(x, y) = p(x, y)/q(x, y)$, where p and q are polynomials with rational coefficients, what periods are possible for $x, y, f(x, y), \ldots$? Certainly periods 2, 3, 4, 5, 6, 8, and 12 are possible. We have

- period-2: $x, y, x, y, \ldots,$
- period-3: $x, y, 1/xy, x, y, \ldots,$
- period-4: $x, y, -(x + 1)(y + 1)/y, -(xy + x + 1)/(x + 1), x, y, \ldots,$
- period-5: $x, y, (y + 1)/x, (x + y + 1)/xy, (x + 1)/y, x, y, \ldots,$
- period-6: $x, y, y/x, 1/x, 1/y, x/y, x, y, \ldots,$
- period-8: $x, y, (x - 1)/(x + 1), (y - 1)/(y + 1), -1/x, -1/y, (x + 1)/(1 - x), (y + 1)/(1 - y), x, y, \ldots,$
- period-12: $x, y, (2x-1)/(x+1), (2y-1)/(y+1), (x-1)/x, (y-1)/y, (x-2)/(2x-1), (y-2)/(2y-1), 1/(1-x), 1/(1-y), (x+1)/(2-x), (y+1)/(2-y), x, y, \ldots,$

as long as, once again, x and y are such that we never divide by zero. The examples for period-8 and period-12 might, however, be regarded as a little unsatisfactory; they are formed by interleaving two order-1 recurrences (we call these *pseudo-cycles*). Tsuda (see reference 6) has shown that cycles of the form

$$x, y, \frac{f(y) - g(y)x}{g(y) - h(y)x}, \ldots,$$

where f, g, and h are rational polynomials of degree at most 4, can only possess the periods 2, 3, 4, 5, and 6. (If we allow θ to be real, the recurrence $x, y, \theta y - x, \ldots$ can have any period for the right choice of θ (see reference 7).)

An invariant curve for the Lyness cycle

Now we return to (2). What happens if we multiply the terms of the recurrence together and equate the result to a constant k? We obtain

$$xy\left(\frac{y+1}{x}\right)\left(\frac{x+y+1}{xy}\right)\left(\frac{x+1}{y}\right) = k.$$

This gives us the cubic equation $(x + 1)(y + 1)(x + y + 1) = kxy$, an invariant curve for this Lyness cycle. Using the same logic used in the order-1 example, if we know that (x, y) is on the curve, then, by periodicity, (x, y), $(y, (y + 1)/x)$, $((y + 1)/x, (x + y + 1)/xy)$, $((x + y + 1)/xy, (x + 1)/y)$, and $((x + 1)/y, x)$ will also be on the curve. We can say the curve is globally period-5; this time starting from (almost) any point on the curve, the terms form a closed circuit (see figure 2).

Note also that we could arrive at an invariant curve for the cycle using any symmetric function in the terms; for example, if the cycle is period-5, and the terms are $x, y, p, q, r, x, y, \ldots$, then

$$xypqr = k, \qquad x + y + p + q + r = k, \qquad xy + yp + pq + qr + rx = k,$$

and so on, will all be invariant curves for the recurrence.

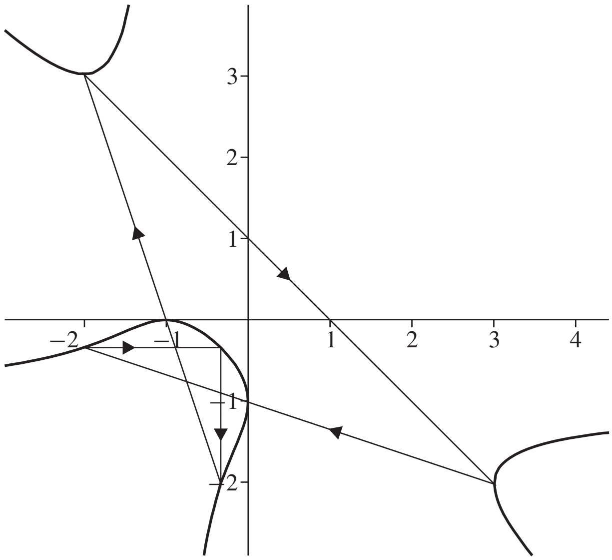

Figure 2 $(x+1)(y+1)(x+y+1) = kxy$ for $k = \frac{4}{3}$ starting from $(3, -2)$.

Different periods, but (almost) the same invariant curve

Now to the observation that we hope is original. We ask the following question: is it possible for an invariant curve to host cycles of different periods? The immediate answer would seem to be 'no'; if a curve is globally periodic period-5, then how can there be a point on the curve that is period-6? (For elliptic curve specialists, the period of the recurrence must match the torsion of the curve.) There is only one way out of this impasse; the invariant curve must collapse into three straight lines (become degenerate), and now, when it comes to periodicity, we might say that all bets are off.

So consider the following Lyness cycles:

$$\text{period-3:} \qquad x,\, y,\, \frac{2xy - 22x - 22y + 62}{xy - 2x - 2y + 22},\, x,\, y,\, \ldots,$$

$$\text{period-4:} \qquad x,\, y,\, \frac{xy - y^2 + y - 2}{xy - x - y},\, \frac{x^2 - xy + x + 2}{y - xy + x},\, x,\, y,\, \ldots,$$

$$\text{period-5:} \qquad x,\, y,\, \frac{5 - x - 3y}{x + 1},\, \frac{-xy + 8x + 8y - 10}{(x+1)(y+1)},\, \frac{5 - 3x - y}{y + 1},\, x,\, y,\, \ldots,$$

$$\text{period-6:} \qquad x,\, y,\, \frac{x - 3y + 1}{2x - 1},\, \frac{x + 4}{2x - 1},\, \frac{y + 4}{2y - 1},\, \frac{3x - y - 1}{1 - 2y},\, x,\, y,\, \ldots.$$

In each case adding the terms in a block and equating to k gives an invariant curve for the cycle. If we then put k equal to 3 (in the period-3 case), 5 (in the period-4 case), 4 (in the period-5 case), and 6 (in the period-6 case) then, in each instance, the curve (remarkably) simplifies to $(x-2)(y-2)(x+y-1) = 0$, or three straight lines (checking this *definitely* needs a computer algebra system!).

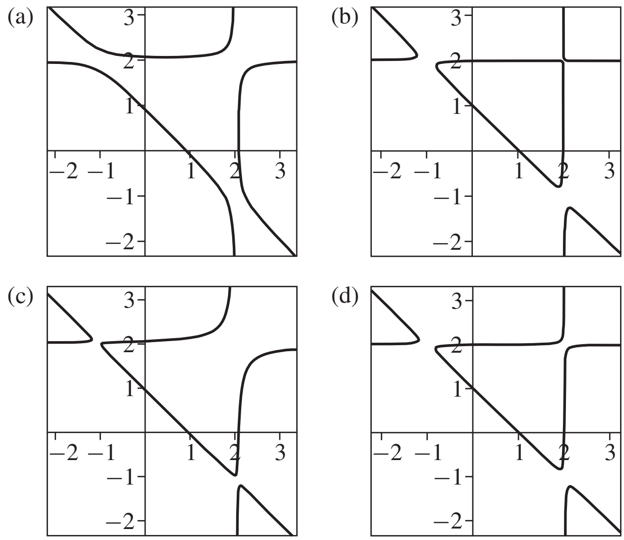

Figure 3 Invariant curves of (a) period-3, $k = 2.99$, (b) period-4, $k = 4.99$, (c) period-5, $k = 3.95$, and (d) period-6, $k = 5.99$, that approximate to three straight lines.

Conclusion

Maybe this is nothing more than a curiosity. But it does show that we can find invariant curves for Lyness cycles which possess different periods, and yet are arbitrarily close to each other (see figure 3). We hope that this fact is worth noting.

References

1 P. Cull, M. Flahive and R. Robson, *Difference Equations: From Rabbits to Chaos* (Springer, New York, 2005).
2 D. M. Burton, *Elementary Number Theory* (McGraw-Hill, New York, 1998), 4th edn.
3 R. C. Lyness, 1581, Cycles, *Math. Gazette* **26** (1942), p. 62.
4 R. C. Lyness, 1847, Cycles, *Math. Gazette* **29** (1945), pp. 231–233.
5 R. C. Lyness, 2952, Cycles, *Math. Gazette* **45** (1961), pp. 207–209.
6 T. Tsuda, Integrable mappings via rational elliptic surfaces, *J. Phys. A* **37** (2004), pp. 2721–2730.
7 J. Griffiths, Periodic recurrence relations of the type $x, y, y^k/x, \ldots$, *Math. Spectrum* **37** (2004/2005), pp. 61–66.

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The authors are not related (!)

How Round is a Plane Figure?

ARJUN TAN

The obscure isoperimetric quotient, henceforth called the roundness coefficient, is proposed as a measure for the roundness of a plane figure. The roundness coefficients of several classes of regular and irregular figures are calculated. It is found that convex boundary sections of a figure generally enhance the roundness coefficient and straight boundary sections do not. Concave boundary sections are extremely detrimental to the roundness coefficient. Also, elongation of a figure diminishes the roundness coefficient.

1. Introduction

On a wintry day in January 1986, the Space Shuttle *Challenger* exploded shortly after takeoff. During the following Congressional Investigation of the tragedy, Physics Nobel Laureate Richard Feynman was vocally critical about some of the practices followed by the United States Space Agency NASA. In a particular aspect, attention was focused on whether the solid rocket boosters became slightly out-of-round during their transit by rail before they were packed with propellants and whether some propellant could consequently have escaped through the O-rings causing the explosion. It was revealed that roundness was assured when three diameters measured in three directions 120 degrees apart matched. To this, Feynman produced his now famous figure with three equal diameters which was obviously not round (see figure 1, which is taken from reference 1, p. 122). Unfortunately, however, Feynman did not provide an answer as to how the roundness of a plane figure could be defined or measured.

In this article, we show that there actually exists a quantity called the *isoperimetric quotient* which could be taken as a measure of the roundness of a plane figure. It is defined by

$$Q \equiv \frac{4\pi A}{L^2},$$

where A is the area of the plane figure and L is its perimeter (see reference 2, p. 23). For a circle $Q = 1$, and it can be proved that $Q < 1$ for any other closed figure (see reference 3,

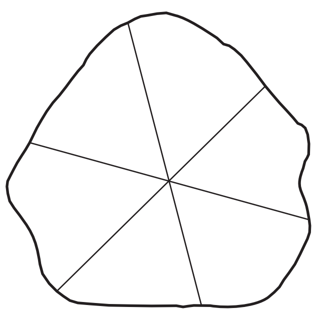

Figure 1 A plane figure with three equal diameters at $120°$ apart (see reference 1, p. 122).

pp. 168–189). In this article, we shall simply call Q the *roundness coefficient* and denote it by R. We calculate the roundness coefficients for several classes of geometrical figures to illustrate how R depends on the shape. Finally, we estimate the roundness coefficient of Feynman's figure graphically to find how much it departed from unity.

2. Roundness coefficient of a regular *n*-sided polygon

For a regular n-sided polygon of side a, we have $A = na^2/4\tan(\pi/n)$ and $L = na$; hence,

$$R = \frac{\pi}{n\tan(\pi/n)}.$$

We now provide a few examples.

- For an equilateral triangle, $R = \sqrt{3}\pi/9 \approx 0.605$.
- For a square, $R = \pi/4 \approx 0.785$.
- For a regular pentagon, $R = \sqrt{5(5+\sqrt{5})}\pi/25 \approx 0.865$.
- For a regular hexagon, $R = \sqrt{3}\pi/6 \approx 0.907$.
- For a regular octagon, $R = \sqrt{5+2\sqrt{5}}\,\pi/8 \approx 0.948$.
- For a regular decagon, $R = \sqrt{5+2\sqrt{5}}\,\pi/10 \approx 0.967$.

As n tends to ∞, R converges to unity and the polygon tends to a circle.

3. Roundness coefficient of a rectangle

For a rectangle of length a and width b, $A = ab$ and $L = 2(a+b)$; hence,

$$R = \frac{\pi ab}{(a+b)^2}. \tag{1}$$

If the rectangle is a square, then $a = b$ and $R = \pi/4 \approx 0.785$, a result obtained earlier.

If the length of the rectangle is twice its width, then $a = 2b$ and $R = 2\pi/9 \approx 0.698$. If $a = \phi b$, where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the *golden ratio*, then $R = 2\pi(1+\sqrt{5})/(3+\sqrt{5})^2 \approx 0.742$. As $b/a \to 0$, by (1), $R = (\pi b/a)/(1+b/a)^2 \to 0$.

4. Roundness coefficient of an ellipse

The ellipse is one category of the conic sections which occurs frequently in everyday life. For an ellipse, the semi-major axis a and semi-minor axis b are related by

$$b = a\sqrt{1-e^2},$$

where e is the eccentricity of the ellipse. The area of the ellipse is $A = \pi ab$. However, there is no exact closed-form expression for its perimeter. A simple approximate expression is given by

$$L \approx \pi\sqrt{2(a^2 + b^2)}$$

(see reference 4, p. 870). Then

$$R \approx \frac{2\sqrt{1-e^2}}{2-e^2}.$$

In the limiting case of $e = 0$, the ellipse is a circle and $R = 1$.

For an ellipse with $a = 2b$, we have $e = \sqrt{3}/2$, and $R = 0.8$.

A more accurate alternative expression for L was given by Ramanujan (see reference 4, p. 870),

$$L \approx \pi[3(a+b) - \sqrt{(a+3b)(3a+b)}],$$

which gives $R \approx 0.841$ in the latter case.

5. Roundness coefficients for some convex figures

Here we consider three convex figures with the longer dimension twice the shorter one.

5.1. Semi-circle

For a semi-circle of radius a, $A = \frac{1}{2}\pi a^2$ and $L = (\pi + 2)a$; hence, $R = 2\pi^2/(\pi+2)^2 \approx 0.747$.

5.2. A lens shape

A lens shape having a length equal to twice its width is constructed out of two circular arcs of the same radius and centres situated on opposite sides. Figure 2 shows one half of this pattern between the circular arc ACB with centre at O and the chord AB, which is the segment $ACBA$. The width of this segment is d so that its length is $4d$. By the chord theorem of a circle, we have $4d^2 = d(2a-d)$, where a is the radius of the circular arc; hence, $d = 0.4a$. If 2α is the angle subtended by the arc at O (see figure 2), then $\alpha = \sin^{-1}\frac{4}{5} \approx 0.927295\text{rad} \approx 53.1301°$.

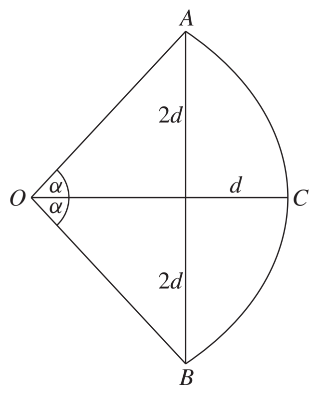

Figure 2 Segment $ACBA$ between circular arc ACB and chord AB. The area of the segment is equal to the area of sector $OACB$ minus area of triangle OAB.

The length of the arc ACB is $s = 2\alpha a \approx 1.85459a$. Furthermore, the area of the segment $ACBA$ is $\Delta = \Delta_1 - \Delta_2$, where Δ_1 is the area of the sector $OACB$ and Δ_2 is the area of the triangle OAB. We find that $\Delta_1 = \alpha a^2 \approx 0.927295a^2$ and $\Delta_2 = \frac{1}{2}(0.6a)(1.6a) \approx 0.48a^2$; hence, $\Delta \approx 0.447295a^2$.

For the whole lens shape, $A = 2\Delta \approx 0.89459$ and $L = 2s \approx 3.70918a$; hence, $R \approx 0.817$.

5.3. A crescent shape

A crescent shape is constructed out of a semi-circular arc of radius a and another circular arc such that the mid-section has a width of $a/2$. The radius of the latter arc as found by the chord theorem is $1.25a$. The angle subtended by this arc at its centre is equal to $2\sin^{-1}\frac{4}{5} \approx 106.2602°$.

The length of the inner arc is calculated to be $2.318238a$, giving the perimeter of the crescent as $L \approx \pi a + 2.318238a \approx 5.45983a$. The area of the crescent is the difference between those of the two circular segments which is equal to $A \approx \frac{1}{2}\pi a^2 - 0.447295a^2 = 1.1235a^2$. The roundness coefficient of the crescent is $R \approx 0.474$.

6. Roundness coefficients of miscellaneous figures

Here we consider two figures constructed entirely out of straight lines.

6.1. Isosceles triangle with base twice the height

Consider an isosceles triangle with base $2a$ and height a. We easily get $A = a^2$, $L = 2(\sqrt{2}+1)a$, and $R = \pi/(\sqrt{2}+1)^2) \approx 0.539$.

6.2. Diamond-shaped figure

Consider a diamond-shaped figure with its longest diameter ($2a$) twice the shortest diameter (a). We have $A = a^2$, $L = 2\sqrt{5}a$, and $R = \pi/5 \approx 0.628$.

7. Roundness coefficients of irregular plane figures

For an irregular plane figure such as Feynman's figure, the roundness coefficient cannot be calculated analytically, but it can be done numerically by employing graphical methods. The area can be calculated by drawing the figure on graph paper and counting the number of squares which fall within the figure completely and adding to that the squares with at least half of their areas within that figure. The perimeter is calculated by approximating the boundary with small linear segments and measuring their lengths with a ruler and adding them. The roundness coefficient of figure 1 was calculated in this fashion. On an inch graph paper with smallest length unit of 0.1 inches, we got $L \approx 144$ units, $A \approx 1431$ square units, and $R \approx 0.867$, a seemingly high figure!

8. Summary and conclusions

For a comparative analysis, we gather the results for seven elongated figures which fit completely into a 2×1 rectangle in figure 3 in order of descending roundness coefficients. Some observations can be made from the results. Firstly, the top row consists of figures having at least one convex section on its boundary. Evidently, the existence of a rectilinear section of the boundary in part (c) may well have reduced its roundness coefficient. Thus, we might infer

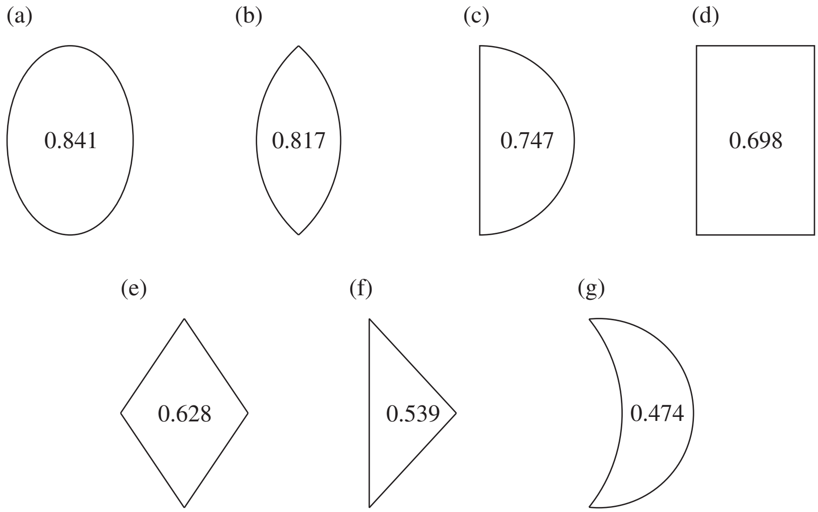

Figure 3 Seven plane figures in the order of descending roundness coefficient: (a) ellipse, (b) lens shape, (c) semi-circle, (d) rectangle, (e) diamond-shaped figure, (f) isosceles triangle, and (g) crescent shape.

that convex boundary sections generally enhance the roundness coefficient whereas rectilinear boundary sections do not. It may also be conjectured here that amongst all figures within the boundary of a rectangle, the ellipse touching the rectangle will have the highest roundness coefficient. Secondly, the middle row consists of figures made entirely from straight lines. Here, the roundness coefficient was the smallest for the figure with the smallest number of sides as was the case with regular polygons. Thirdly, the bottom row consists of the sole figure with a concave boundary section. This figure has the lowest roundness coefficient amongst the seven figures. Evidently, the concave boundary section had a deleterious effect on the roundness coefficient. Finally, Feynman's figure had a higher roundness coefficient (0.867) than all the figures just considered, even as it is very roughly an irregular triangular figure with corners rounded. Recall that the equilateral triangle had the lowest roundness coefficient amongst all the regular polygons. The reason for this is traced to the fact that all the seven parts of figure 3 are elongated ones whereas Feynman's figure is basically a rotund one. It is concluded that elongation diminishes the roundness coefficient of a figure.

References

1 R. P. Feynman, *What do You Care What Other People Think?* (Norton, New York, 1988).

2 H. T. Croft, K. J. Falconer and R. K. Guy, *Unsolved Problems in Geometry* (Springer, New York, 1991).

3 G. Pólya, *Mathematics and Plausible Reasoning*, Vol. 1, *Induction and Analogy in Mathematics* (Princeton University Press, 1954).

4 E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics* (Chapman & Hall/CRC, Boca Raton, FL, 2003).

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Can a Tromino be Tiled with Unit Trominoes?

C. REBECCA THOMAS, SIÂN K. JONES
and STEPHANIE PERKINS

Polominoes are well known due to their use in the game *Tetris*, in which shapes made from four squares called *tetrominoes* are arranged within a game area. Polominoes can be constructed using any number of squares. In this article *trominoes*, which consist of three squares in an L-shape formation, are examined. We determine whether these can be used to fill larger L-shaped formations.

1. Introduction

A *tromino* (also known as a *triomino*) is a geometric shape formed by three squares. Trominoes can be either I-shaped or L-shaped and can be rotated in any orientation. In this article only the L-shaped tromino is used and throughout we use the word tromino to mean an L-shaped tromino. The possible orientations of the tromino are given in figure 1.

Larger L-shapes can also be formed as shown in figure 2 and the aim of this article is to show that such shapes can be completely filled with copies of the unit trominoes given in

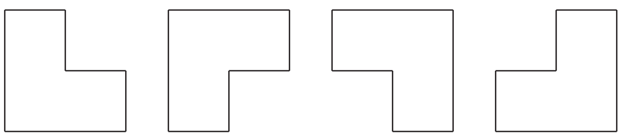

Figure 1 A unit tromino and its rotations.

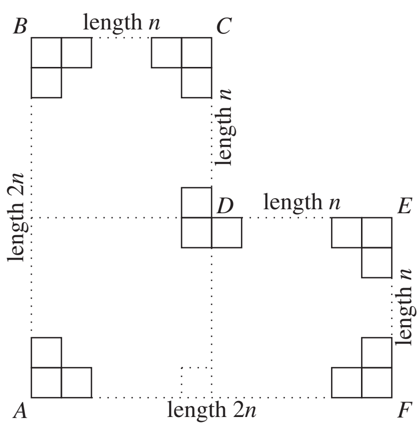

Figure 2 A tromino of size n.

figure 1. A regular L-shaped tromino can be considered to comprise three $n \times n$ squares in the arrangement given by the dotted lines in figure 2.

The unit trominoes in figure 1 are denoted by L_1 and the larger L-shaped tromino, in figure 2, of size n is denoted by L_n. The aim of this article is to show that L_n for any integer n can be tiled completely with unit trominoes (L_1).

2. Tiling L_n for n a multiple of 2 or 3

For the cases where n is a multiple of 2 or 3 it can be shown explicitly that a tiling exists. The smallest case L_2 is given in figure 3(a) using four copies of L_1. L_3 is given in figure 3(b) and uses five copies of L_1 and one L_2 (which is itself composed of four copies of L_1).

In figure 4(a) it is demonstrated that the same arrangement as L_2 can be used to tile L_n when n is a multiple of 2 ($n = 2k$, for k an integer) using four copies of L_k. Similarly in figure 4(b) the same arrangement as L_3 can be used to tile L_n when n is a multiple of 3 ($n = 3k$, for k an integer) using five copies of L_k and one copy of L_{2k}. Therefore, if there exists a tiling of a tromino L_k, of size k, then there exists a tiling of a tromino L_{2k}, of size $2k$, and there exists a tiling of a tromino L_{3k} of size $3k$. Hence, any L_n for $n = 2^x 3^y m$ can be tiled using unit trominoes (L_1) if L_m can be tiled using unit trominoes. Therefore, it is sufficient to prove that L_m can be tiled by L_1 when m is not a multiple of 2 or 3.

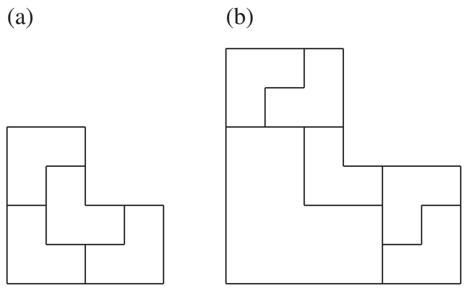

Figure 3 Tilings of (a) L_2 and (b) L_3.

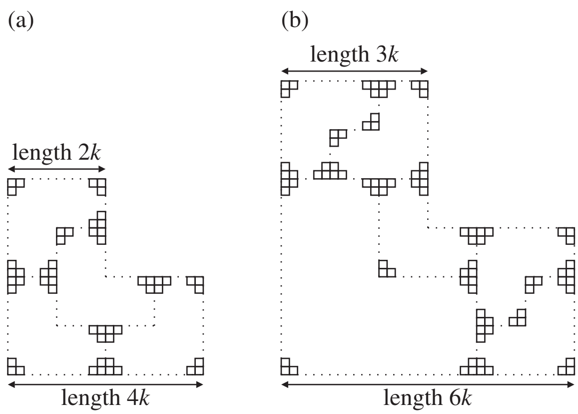

Figure 4 A general (a) L_{2k} and (b) L_{3k}.

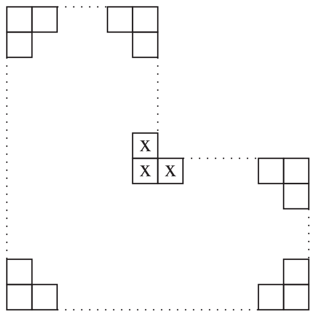

Figure 5 A tromino constructed from deficient squares.

3. Tiling of L_n, for $n \geq 5$

Consider L_n in figure 5. First tile the squares marked 'x' using an L_1. The remaining tromino consists of three $n \times n$ squares with a corner square removed, called *deficient squares*. To give a tiling for a tromino for L_n, for $n \geq 5$ and n not a multiple of 2 or 3, it suffices to show that an $n \times n$ deficient square (for $n \geq 5$ and not a multiple of 2 or 3) can be tiled with multiple copies of L_1.

4. Tiling a deficient $n \times n$ square

In reference 1 it was shown that a deficient square of dimension $n \times n$ can be tiled with unit trominoes if $n \geq 5$ and 3 does not divide n. Both n odd and n even were considered, and with the removed square located anywhere within the $n \times n$ square. However, only the case of n odd and the corner square removed is of interest in this article and is summarized in this section.

In reference 1 an explicit tiling of a deficient 5×5 square was given; we provide an example of this in figure 6(a). In reference 1 a 2×3 configuration (as shown in figure 6(b)) was also given, which was used to show that a $2i \times 3j$, or $3i \times 2j$, board can be tiled exactly using unit trominoes (using an $i \times j$ arrangement of 2×3, respectively 3×2 configurations).

Using the 5×5 deficient square and the 2×3 and 3×2 configurations, the 7×7 deficient square can now be tiled (see figure 7(a)).

A $n \times n$ deficient square for $n \geq 11$ and n odd is given in figure 7(b) and comprises of four shapes. This is a rearranged version of figure 9 from reference 1 using the same shapes

(a) (b)

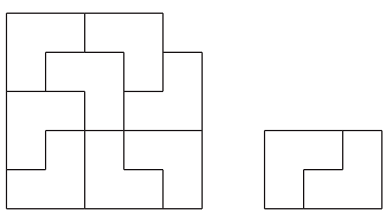

Figure 6 (a) A 5×5 deficient board tiled using trominoes and (b) a 2×3 configuration.

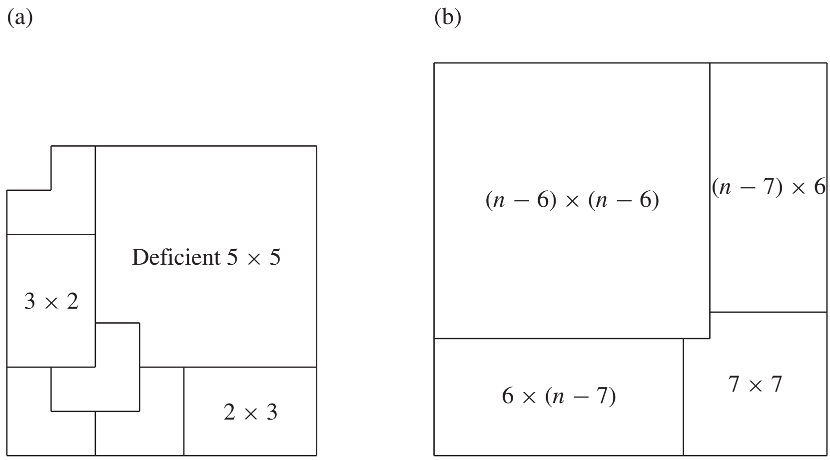

Figure 7 (a) A 7×7 deficient square and (b) a $n \times n$ (for n odd) deficient square.

but with a corner square removed. Consider each of these shapes in turn.

- A $(n-7) \times 6$ (and hence also $6 \times (n-7)$) rectangle can be tiled using the configurations given in figure 6(b).
- A 7×7 deficient square is given in figure 7(a).
- A $(n-6) \times (n-6)$ deficient square; the first case $(n-6) \times (n-6) = 5 \times 5$ is given in figure 6(a) and $(n-6) \times (n-6) = 7 \times 7$ is given in figure 7(a). Therefore, since an 11×11 deficient square can be created when $(n-6) \times (n-6) = 5 \times 5$ and 13×13 is created using $(n-6) \times (n-6) = 7 \times 7$. Then by an inductive argument all remaining cases for $n = 5 + 6k$, $7 + 6k$, i.e. n odd and not a multiple of three, can be generated.

5. A tiling of L_n

In section 2 it was demonstrated that L_n can be tiled using unit trominoes (L_1) for n a multiple of 2 or 3. The results of reference 1 demonstrate that an $n \times n$ deficient square can be tiled using unit trominoes for $n \geq 5$, n odd and where 3 does not divide n. Since the deficient square can be chosen to be a corner square then three such squares plus one copy of L_1 can be arranged as shown in figure 5 to construct L_n where $n \geq 5$, n odd and where 3 does not divide n. Hence, it has been shown that there exists a tiling of L_n, for any integer n using unit trominoes.

Reference

1 I.-P. Chu and R. Johnsonbaugh, Tiling deficient boards with trominoes, *Math. Magazine* **59** (1986), pp. 34–40.

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Exploring The Curiously Fascinating Integer Sequence 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 1234567891, 12345678912, 123456789123, . . .

JAY L. SCHIFFMAN

This article considers the integer sequence 1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, 1234567891, 12345678912, 123456789123, ... Our goal is to examine the structure of the sequence by exploring divisibility patterns including securing prime outputs and determining the highest power of two that is a possible factor of any term in the sequence. Using MATHEMATICA®, I was able to obtain the complete prime factorizations for the initial 108 terms in the sequence. The deployment of modular arithmetic will enable us to secure recurring prime factors from complete groupings such as 123456789, 123456789123456789, 123456789123456789123456789, ... We conclude by suggesting future directions for companion sequences that serve to furnish additional stimulating research. Such possibilities include extensions, the sequence reversal, and examining the sequence and its reversals in different number bases such as hexadecimal and duodecimal (base twelve).

1. Introduction

Our initial goal is to determine any early prime outputs in the sequence. Observe that the first term 1 is not prime by definition while the terms 12, 1234, 123456, and 12345678 are even, and the term 12345 is divisible by 5. It is well known that any integer is divisible by 3 or 9 if and only if the integer obtained by forming the digital sum is divisible by 3 or 9. As a consequence, the terms 12, 123, 12345, 123456, 12345678, and 123456789 are divisible by 3. Meanwhile the term 123456789 is divisible by 9. If we append the digits 12, 123, 12345, 123456, 12345678, and 123456789 to the integer 123456789, then divisibility by 3 is preserved. In addition, appending the digits 123456789 to the integer 123456789 preserves divisibility by 9. Clearly, 12345678912345 is divisible by 5 while 12345678912, 1234567891234, 123456789123456, and 12345678912345678 are divisible by 2. In order to view divisibility patterns, it is helpful to factor the integers in question.

2. Factorization of the integers

I was able to obtain the complete factorizations for the initial 108 terms using MATHEMATICA. This enables patterns to be viewed and conjectures to be formed based on the analysis of these patterns, and then conjectures can be affirmed with formal proofs. Due to space limitations, we will focus on the factorizations of the first thirty-six terms in table 1.

An analysis of table 1 illustrates that the 10th and 28th terms in the sequence are prime; namely 1234567891 and 1234567891234567891234567891. In addition, the 70th term

$$1234567891234567891234567891234567891234567891234567891234567891234567$$

is likewise prime. Based on the structure of the sequence with regards to its digits, no term of the sequence is divisible by 10 (for there is no 0) and no term is divisible by 5^n, for $n \geq 2$. In the following section, we likewise prove that no term in the sequence is divisible by 2^n, for $n \geq 8$.

3. Some divisibility theorems

We denote the kth term of the sequence by $S(k)$.

Theorem 1 *$S(k)$ is even if and only if $k \equiv 2, 4, 6,$ or $8 \pmod 9$.*

Theorem 2 *$S(k)$ is divisible by 3 if and only if $k \equiv 0, 2, 3, 5, 6,$ or $8 \pmod 9$. More simply, $S(k)$ is divisible by 3 if and only if $k \not\equiv 1 \pmod 3$.*

Theorem 3 *$S(k)$ is divisible by 4 if and only if $k \equiv 2$ or $6 \pmod 9$.*

Theorem 4 *$S(k)$ is divisible by 5 if and only if $k \equiv 5 \pmod 9$.*

Theorem 5 *$S(k)$ is divisible by 6 if and only if $k \equiv 2, 6,$ or $8 \pmod 9$.*

Theorem 6 *$S(k)$ is divisible by 7 if and only if $k \equiv 0$ or $11 \pmod{18}$.*

Theorem 7 *$S(k)$ is divisible by 8 if and only if $k \equiv 2$ or $6 \pmod 9$ and $k \neq 2$.*

Theorem 8 *$S(k)$ is divisible by 9 if and only if $k \equiv 0$ or $8 \pmod 9$.*

Theorem 9 *$S(k)$ is divisible by 11 if and only if $k \equiv 0 \pmod{18}$.*

Theorem 10 *$S(k)$ is divisible by 12 if and only if $k \equiv 2$ or $6 \pmod 9$.*

Theorem 11 *$S(k)$ is divisible by 13, 19, 33, 39, 57, 63, 77, 91, 99, and 117 if and only if $k \equiv 0 \pmod{18}$.*

Theorem 12 *$S(k)$ is divisible by 14, 28, and 56 if and only if $k \equiv 0$ or $17 \pmod{27}$.*

Theorem 13 *$S(k)$ is divisible by 15 if and only if $k \equiv 5 \pmod 9$.*

Theorem 14 *$S(k)$ is divisible by 16, 24, 32, 48, 64, and 96 if and only if $k \equiv 2$ or $6 \pmod 9$, starting with the sixth and eleventh terms.*

Theorem 15 *$S(k)$ is divisible by 18 if and only if $k \equiv 8 \pmod 9$.*

Theorem 16 *$S(k)$ is divisible by 24 if and only if $k \equiv 2$ or $6 \pmod 9$, starting with the sixth term.*

Theorem 17 *$S(k)$ is divisible by 27 if and only $k \equiv 0$ or 17 (mod 27).*

Theorem 18 *$S(k)$ is divisible by 54 if and only $k \equiv 17$ (mod 27).*

Table 1 Factorization of the initial 36 terms in the sequence.

Integer	Prime factorization
1	1
12	$2^2 \cdot 3$
123	$3 \cdot 41$
1234	$2 \cdot 617$
12345	$3 \cdot 5 \cdot 823$
123456	$2^6 \cdot 3 \cdot 643$
1234567	$127 \cdot 9721$
12345678	$2 \cdot 3^2 \cdot 47 \cdot 14593$
123456789	$3^2 \cdot 3607 \cdot 3803$
1234567891	1234567891 (prime)
12345678912	$2^6 \cdot 3 \cdot 7 \cdot 9185773$
123456789123	$3 \cdot 12049 \cdot 3415409$
1234567891234	$2 \cdot 617283945617$
12345678912345	$3 \cdot 5 \cdot 43 \cdot 2371 \cdot 8072791$
123456789123456	$2^7 \cdot 3 \cdot 10303 \cdot 31204703$
1234567891234567	$47 \cdot 167 \cdot 167953 \cdot 936511$
12345678912345678	$2 \cdot 3^3 \cdot 228623683561957$
123456789123456789	$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 3607 \cdot 3803 \cdot 52579$
1234567891234567891	$31 \cdot 241 \cdot 1019 \cdot 162166841159$
12345678912345678912	$2^6 \cdot 3 \cdot 313 \cdot 35129 \cdot 5847949643$
123456789123456789123	$3 \cdot 829 \cdot 2423 \cdot 20487349591123$
1234567891234567891234	$2 \cdot 28513 \cdot 21649210732553009$
12345678912345678912345	$3 \cdot 5 \cdot 17 \cdot 7523 \cdot 67777 \cdot 94951404589$
123456789123456789123456	$2^7 \cdot 3 \cdot 43 \cdot 7476791976953536163$
1234567891234567891234567	$224234251 \cdot 5505706134227317$
12345678912345678912345678	$2 \cdot 3^2 \cdot 4812168377 \cdot 142528481331623$
123456789123456789123456789	$3^3 \cdot 757 \cdot 3607 \cdot 3803 \cdot 440334654777631$
1234567891234567891234567891	1234567891234567891234567891 (prime)
12345678912345678912345678912	$2^6 \cdot 3 \cdot 7 \cdot 17 \cdot 4111 \cdot 1314375062637858481 79$
123456789123456789123456789123	$3 \cdot 585341 \cdot 6348673 \cdot 11073931153575637$
1234567891234567891234567891234	$2 \cdot 17 \cdot 133673 \cdot 569451119 \cdot 477019260610423$
12345678912345678912345678912345	$3 \cdot 5 \cdot 823045260823045260823045260823$
123456789123456789123456789123456	$2^7 \cdot 3 \cdot 3221 \cdot 21592663$ $\cdot 1934404501 \cdot 2389678783$
1234567891234567891234567891234567	$31 \cdot 39824770684986061007566706168857$
12345678912345678912345678912345678	$2 \cdot 3^2 \cdot 17 \cdot 8663 \cdot 36919$ $\cdot 126146529730509013060079$
123456789123456789123456789123456789	$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 101 \cdot 3607$ $\cdot 3803 \cdot 9901 \cdot 52579 \cdot 999999000001$

Theorem 19 *$S(k)$ is divisible by 128 if and only $k \equiv 6 \pmod 9$, starting with the fifteenth term.*

In order to prove these theorems, we appeal to modular arithmetic. For example, let us consider a test for divisibility of an integer by 13. We note the following:

$$\begin{aligned}
1 &\equiv 1 \pmod{13},\\
10 &\equiv 10 \pmod{13} \equiv -3 \pmod{13},\\
10^2 &\equiv 9 \pmod{13} \equiv -4 \pmod{13},\\
10^3 &\equiv 12 \pmod{13} \equiv -1 \pmod{13},\\
10^4 &\equiv 3 \pmod{13},\\
10^5 &\equiv 4 \pmod{13},\\
10^6 &\equiv 1 \pmod{13},\\
N &= a_0 \cdot 10^0 + a_1 \cdot 10^1 + a_2 \cdot 10^2 + a_3 \cdot 10^3 + a_4 \cdot 10^4 + a_5 \cdot 10^5 + a_6 \cdot 10^6 + \cdots\\
&\equiv a_0 - 3 \cdot a_1 - 4 \cdot a_2 - 1 \cdot a_3 + 3 \cdot a_4 + 4 \cdot a_5 + 1 \cdot a_6 + \cdots \pmod{13}.
\end{aligned}$$

To summarize, the test for whether an integer is divisible by 13 entails the following. Proceeding from right to left, examine the integer obtained by taking the units digit minus three times the tens digit minus four times the hundreds digit minus the thousands digit plus three times the ten thousands digit plus four times the hundred thousands digit plus the millions digit, and so on. If the result is congruent to 0 modulo 13, then the integer is divisible by 13. The sequence of multipliers $(1, -3, -4, -1, 3, 4, \ldots)$ repeats in that pattern. It is useful to view this on a six hour analogue clock which the reader is invited to partake of.

To cite an example,

$$S(18) = 123456789123456789$$

is divisible by 13, since

$$\begin{aligned}
&1 \cdot 9 - 3 \cdot 8 - 4 \cdot 7 - 1 \cdot 6 + 3 \cdot 5 + 4 \cdot 4 + 1 \cdot 3 - 3 \cdot 2 - 4 \cdot 1 - 1 \cdot 9 + 3 \cdot 8 + 4 \cdot 7\\
&\quad + 1 \cdot 6 - 3 \cdot 5 - 4 \cdot 4 - 1 \cdot 3 + 3 \cdot 2 + 4 \cdot 1\\
&\qquad = 9 - 24 - 28 - 6 + 15 + 16 + 3 - 6 - 4 - 9 + 24 + 28 + 6 - 15 - 16\\
&\qquad\quad - 3 + 6 + 4\\
&\qquad = 0\\
&\qquad \equiv 0 \pmod{13}.
\end{aligned}$$

We next test an integer for divisibility by 27. Observe the following:

$$\begin{aligned}
1 &\equiv 1 \pmod{27},\\
10 &\equiv 10 \pmod{27},\\
10^2 &\equiv 19 \pmod{27} \equiv -8 \pmod{27},\\
10^3 &\equiv 1 \pmod{27},\\
N &= a_0 \cdot 10^0 + a_1 \cdot 10^1 + a_2 \cdot 10^2 + a_3 \cdot 10^3 + \cdots\\
&\equiv a_0 + 10 \cdot a_1 - 8 \cdot a_2 + 1 \cdot a_3 + \cdots \pmod{27}.
\end{aligned}$$

To cite an example, observe that

$$S(17) = 12345678912345678 \quad \text{and} \quad S(27) = 123456789123456789123456789$$

are each divisible by 27.

For $S(17)$, note that

$$\begin{aligned}
&1 \cdot 8 + 10 \cdot 7 - 8 \cdot 6 + 1 \cdot 5 + 10 \cdot 4 - 8 \cdot 3 + 1 \cdot 2 + 10 \cdot 1 - 8 \cdot 9 + 1 \cdot 8 + 10 \cdot 7 - 8 \cdot 6 \\
&\quad + 1 \cdot 5 + 10 \cdot 4 - 8 \cdot 3 + 1 \cdot 2 + 10 \cdot 1 \\
&\qquad = 8 + 70 - 48 + 5 + 40 - 24 + 2 + 10 - 72 + 8 + 70 - 48 + 5 + 40 - 24 + 2 \\
&\qquad\quad + 10 \\
&\qquad = 54 \\
&\qquad \equiv 0 \pmod{27}.
\end{aligned}$$

For $S(27)$, observe that

$$\begin{aligned}
&1 \cdot 9 + 10 \cdot 8 - 8 \cdot 7 + 1 \cdot 6 + 10 \cdot 5 - 8 \cdot 4 + 1 \cdot 3 + 10 \cdot 2 - 8 \cdot 1 + 1 \cdot 9 + 10 \cdot 8 - 8 \cdot 7 \\
&\quad + 1 \cdot 6 + 10 \cdot 5 - 8 \cdot 4 + 1 \cdot 3 + 10 \cdot 2 - 8 \cdot 1 + 1 \cdot 9 + 10 \cdot 8 - 8 \cdot 7 + 1 \cdot 6 + 10 \cdot 5 \\
&\quad - 8 \cdot 4 + 1 \cdot 3 + 10 \cdot 2 - 8 \cdot 1 \\
&\qquad = 9 + 80 - 56 + 6 + 50 - 32 + 3 + 20 - 8 + 9 + 80 - 56 + 6 + 50 - 32 + 3 \\
&\qquad\quad + 20 - 8 + 9 + 80 - 56 + 6 + 50 - 32 + 3 + 20 - 8 \\
&\qquad = 216 \\
&\qquad \equiv 0 \pmod{27}.
\end{aligned}$$

These two computations aid in establishing theorem 17.

If we examine the structure of the sequence, the highest power of two that is possible as a factor of any term is 2^7. An integer is divisible by 2^n if and only if the integer consisting of the last n digits is divisible by 2^n. Note that an integer is divisible by two if and only if the units digit is even. In short, $S(k)$ is even if and only if $k \equiv 2$, 4, 6, or $8 \pmod 9$. In contrast, $S(k)$ is odd if and only if $k \equiv 0$, 1, 3, 5, or $7 \pmod 9$. Table 2 furnishes the relevant information regarding divisibility of the sequence by powers of two. No integers are divisible by any higher power of 2, since the number consisting of the last eight digits of the integer must be divisible by 256 for the entire integer to be divisible by 256. The only candidate is 89123456 which is not divisible by 256.

Table 2

$S(k)$ is divisible by	Possible last digit(s)
2	2, 4, 6, 8
2^2	12, 56
2^3	456, 912
2^4	3456, 8912
2^5	23456, 78912
2^6	123456, 678912
2^7	9123456
2^k, for $k \geq 8$	no combination works

4. Next steps

This article served to delve into a fascinating integer sequence as we toured around the clock with the aid of modular arithmetic. Companion sequences such as

$$1, 12, 123, 1234, 12345, 123456, 1234561, 12345612,$$
$$123456123, 1234561234, 12345612345, 123456123456, \ldots,$$

and their reversals, are curious and fascinating to explore. Another direction to pursue is how a different number base such as bases twelve and sixteen would affect the outcomes alluded to here.

Jay L. Schiffman *has taught mathematics at Rowan University, Glassboro, NJ, USA, for the past twenty years. His interests include number theory, discrete mathematics, mathematics education, and the interface of mathematics with technology. He enjoys travel around the USA, especially to conferences to disseminate his research.*

Letters to the Editor

Dear Editor,

A Fibonacci curiosity

I enjoyed the recursive proof that Thomas Koshy and Zhenguang Gao gave in their article ‘A Fibonacci curiosity’ (see Volume 48, Number 1, pp. 13–15) to show that

$$\sum_{k\geq 0,\ k\text{ even}} \binom{n-k}{k} = \sum_{k\geq 0,\ k\text{ odd}} \binom{n-k}{k}$$

holds if and only if $n \equiv 2 \pmod 3$.

Here I observe that Chebyshev polynomials of the second kind may be used to give an alternative proof. These are characterised by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

which generates the polynomials

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

With $\theta = 2\pi/3$ so that $x = \cos\theta = -\frac{1}{2}$, we obtain

$$(-1)^n U_n\left(-\tfrac{1}{2}\right) = \sum_{k\geq 0,\ k\text{ even}} \binom{n-k}{k} - \sum_{k\geq 0,\ k\text{ odd}} \binom{n-k}{k},$$

where

$$U_n\left(-\tfrac{1}{2}\right) = \frac{\sin(2\pi/3)(n+1)}{\sin(2\pi/3)}.$$

But $\sin(2\pi/3)(n+1) = 0$ if and only if $n \equiv 2 \pmod 3$.

It is also worth noting that, on setting $x = \cos\theta$ and using Euler's formula,

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sqrt{1-x^2}} = \frac{1}{2\mathrm{i}\sqrt{1-x^2}}[(x+\mathrm{i}\sqrt{1-x^2})^{n+1} - (x-\mathrm{i}\sqrt{1-x^2})^{n+1}],$$

so that

$$U_n\left(\frac{1}{2}\mathrm{i}\right) = \frac{\mathrm{i}^n}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],$$

while, from the earlier formula for $U_n(x)$,

$$U_n\left(\frac{1}{2}\mathrm{i}\right) = \mathrm{i}^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

We thus deduce, from Binet's formula, that

$$\mathrm{i}^{-n} U_n\left(\frac{1}{2}\mathrm{i}\right) = F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

Yours sincerely,

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Dear Editor,

Apollonius' theorem and Pythagorean triples

I am currently investigating aspects of Apollonius' theorem, which addresses certain features of the median lines constructed within a scalene triangle with sides of length (a, b, c). It can be shown that these three median lines can never be simultaneously rational, and hence are not a secondary source of Pythagorean triples.

Given that $0 < a < b < c$ and $(a + b) > c$, the dimension of the three median lines are defined as m_1, m_2, m_3, where $m_1 < m_2 < m_3$. Then Apollonius' theorem states that:

(i) $m_1^2 = \frac{1}{2}(a^2 + b^2 - \frac{1}{2}c^2)$,

(ii) $m_2^2 = \frac{1}{2}(a^2 + c^2 - \frac{1}{2}b^2)$,

(iii) $m_3^2 = \frac{1}{2}(b^2 + c^2 - \frac{1}{2}a^2)$.

These equations can be used to define $(m_1^2 + m_2^2) \pm m_3^2$, leading to

(iv) $\sum m_i^2 = \frac{3}{4}(a^2 + b^2 + c^2)$, where $5a^2 = b^2 + c^2$,

(v) $(m_1^2 + m_2^2) = m_3^2$.

Where solutions are known to the Diophantine equation at (v), it can be shown that $m_3 = 1\frac{1}{2}a$ in all cases. Actual solutions seem to be rare and thinly spread, with only the following five such cases being identified so far:

1. $(a, b, c) = (13, 19, 22)$, $m_1^2 = 144$, $m_2^2 = 236\frac{1}{4}$, $m_3^2 = 380\frac{1}{4}$,
2. $(a, b, c) = (17, 22, 31)$, $m_1^2 = 146\frac{1}{4}$, $m_2^2 = 504$, $m_3^2 = 650\frac{1}{4}$,
3. $(a, b, c) = (25, 38, 41)$, $m_1^2 = 614\frac{1}{4}$, $m_2^2 = 792$, $m_3^2 = 1\,406\frac{1}{4}$,
4. $(a, b, c) = (37, 58, 59)$, $m_1^2 = 1\,496\frac{1}{4}$, $m_2^2 = 1\,584$, $m_3^2 = 3\,080\frac{1}{4}$,
5. $(a, b, c) = (53, 62, 101)$, $m_1^2 = 776\frac{1}{4}$, $m_2^2 = 5\,544$, $m_3^2 = 6\,320\frac{1}{4}$.

Readers can confirm that $m_1^2 + m_2^2 = m_3^2$ and $m_3 = 1\frac{1}{2}a$ in all cases. The first of these solutions is an example of where two of the three median lines are actually rational (note that $m_1^2 = 144$, then $m_1 = 12$, $m_3 = 1\frac{1}{2}a = 19\frac{1}{2}$); m_2 is therefore irrational. It would be interesting to know if any of your readers can add any further solutions to this very short list, in which case, please send details to the address below.

Yours sincerely,

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Dear Editor,

The inequality $\frac{9}{4} \geq \sin^2 A + \sin^2 B + \sin^2 C > 2$ for acute-angled triangles

A proof of these inequalities first appeared in reference 1 (see reference 2). In this letter we present a new proof.

First, we can easily deduce that there is always an angle greater than or equal to $\pi/3$. Suppose that

$$A \geq \frac{\pi}{3}. \tag{1}$$

In a similar way we can see that there is always a second angle greater than or equal to $\pi/4$. Suppose that

$$B \geq \frac{\pi}{4}. \tag{2}$$

The function $f(x) = \sin^2 x$ has $f''(x) = 2(\cos^2 x - \sin^2 x) \leq 0$ on the interval $[\pi/4, \pi/2)$. Hence, this function is concave downwards on this interval, so it satisfies Jensen's inequality. Since $A, B \in [\pi/4, \pi/2)$ we get

$$\frac{f(A) + f(B)}{2} \leq f\left(\frac{A + B}{2}\right).$$

Thus,

$$\frac{\sin^2 A + \sin^2 B}{2} \leq \sin^2\left(\frac{A + B}{2}\right) = \sin^2\left(\frac{\pi - C}{2}\right) = \cos^2\left(\frac{C}{2}\right),$$

so that

$$\sin^2 A+\sin^2 B+\sin^2 C \le 2\cos^2\left(\frac{C}{2}\right)+\sin^2 C = 1+\cos C+\sin^2 C = \cos^2 C+\cos C+2.$$

The maximum value of the last quadratic form is obtained when $\cos C = \frac{1}{2}$ and it is

$$-\left(\tfrac{1}{2}\right)^2+\tfrac{1}{2}+2=\tfrac{9}{4}.$$

Hence,

$$\sin^2 A+\sin^2 B+\sin^2 C \le \tfrac{9}{4}.$$

Note that there is equality for an equilateral triangle.

From (1) and (2) we can deduce that

$$\tan A\tan B \ge \sqrt{3}\cdot 1 > \tfrac{3}{2},$$

so that

$$2\sin A\sin B > 3\cos A\cos B$$

or

$$2\sin A\sin B\cos A\cos B > \cos^2 A\cos^2 B+\cos^2 A\cos^2 B+\cos^2 A\cos^2 B.$$

Thus,

$$2\sin A\sin B\cos A\cos B+\sin^2 A\cos^2 B+\sin^2 B\cos^2 A > \cos^2 A+\cos^2 B+\cos^2 A\cos^2 B.$$

Hence,

$$(\sin A\cos B+\sin B\cos A)^2=\sin^2(A+B) > \cos^2 A+\cos^2 B.$$

Since $\sin C=\sin(A+B)$, we get

$$\sin^2 A+\sin^2 B+\sin^2 C > \sin^2 A+\cos^2 A+\sin^2 B+\cos^2 B+1+1=2.$$

Therefore,

$$\sin^2 A+\sin^2 B+\sin^2 C > 2. \tag{3}$$

A more complicated proof of (3) exists in reference 3, p. 89.

Finally, we note that as the triangle tends to a right-angled triangle, inequality (3) tends to an equality. From (3) we can easily deduce that $\sin A+\sin B+\sin C > 2$, for any acute-angled triangle ABC.

References

1 R. Kooistra, Nieuw tijdschr, *Wisk*. **45** (1957/1958), pp. 108–115.

2 O. Bottema *et al.*, *Geometric Inequalities* (Wolters-Noordhoff, Groningen, 1969).

3 Z. Cvetkovski, *Inequalities* (Springer, Berlin, 2012).

Yours sincerely,

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Dear Editor,

Dividing by $x^2 - x - 1$

The division of the polynomial x^n $(n \geq 2)$ by $x^2 - x - 1$ gives

$$x^n = (x^2 - x - 1)(F_1 x^{n-2} + F_2 x^{n-3} + \cdots + F_{n-1}) + (F_n x + F_{n-1}),$$

where F_r denotes the r th Fibonacci number. It follows, for example, that

$$\phi^n = F_n \phi + F_{n-1},$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, and

$$a^n \equiv F_n a + F_{n-1} \pmod{a^2 - a - 1}$$

for any integer $a \geq 2$.

More generally, for any polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n^2,$$

we have

$$f(x) = (x^2 - x - 1)g(x) + (ax + b).$$

If the roots of $x^2 - x - 1$ are denoted by α and β, then

$$f(\alpha) = a\alpha + b, \qquad f(\beta) = a\beta + b,$$

which gives

$$a = \frac{f(\alpha) - f(\beta)}{\alpha - \beta}, \qquad b = \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta},$$

and the remainder when $f(x)$ is divided by $x^2 - x - 1$ is

$$\begin{aligned}
&\frac{f(\alpha) - f(\beta)}{\alpha - \beta} x + \frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta} \\
&\quad = \frac{(a_0 + a_1\alpha + \cdots + a_n\alpha^n) - (a_0 + a_1\beta + \cdots + a_n\beta^n)}{\alpha - \beta} x \\
&\qquad + \frac{\alpha(a_0 + a_1\beta + \cdots + a_n\beta^n) - \beta(a_0 + a_1\alpha + \cdots + a_n\alpha^n)}{\alpha - \beta} \\
&\quad = (a_0F_0 + a_1F_1 + a_2F_2 + \cdots + a_nF_n)x + (a_0F_{-1} + a_1F_0 + a_2F_1 + \cdots + a_nF_{n-1})
\end{aligned}$$

since $\alpha\beta = -1$ and

$$F_r = \frac{\alpha^r - \beta^r}{\alpha - \beta} \qquad \text{(Binet's formula)},$$

with $F_0 = 0$ and $F_{-1} = 1$.

Yours sincerely,

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Dear Editor,

Euler's infinite product generates series

In my previous letter 'An application of Euler's infinite product for the sine function' (see Volume 47, Number 3, pp. 133–135), I presented Euler's identity:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots .$$

This can be written as

$$\sin x = x\left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots . \tag{1}$$

Therefore,

$$\sin x = x - x^3\left(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \frac{1}{(4\pi)^2} + \cdots\right) + \sum_{i=2}^{\infty} a_{2i+1}x^{2i+1}. \tag{2}$$

The Maclaurin series expansion for the sine function is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots . \tag{3}$$

Combining (2) and (3) we get

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \frac{1}{(4\pi)^2} + \cdots .$$

Therefore,

$$\frac{\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

When $x \in (0, \pi)$ we can take the logarithms of both sides of (1) and we obtain

$$\ln(\sin x) = \ln x + \ln\left(1 - \frac{x}{\pi}\right) + \ln\left(1 + \frac{x}{\pi}\right) + \ln\left(1 - \frac{x}{2\pi}\right) + \ln\left(1 + \frac{x}{2\pi}\right) + \cdots .$$

Differentiating both sides of this we take

$$\frac{\cos x}{\sin x} = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \cdots . \tag{4}$$

Substituting $x = \pi/4$ in (4) we obtain

$$1 = \frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \frac{4}{7\pi} + \frac{4}{9\pi} - \cdots .$$

Hence,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Substituting $x = \pi/2$ in (1) we have

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{(2\cdot 2)^2}\right)\left(1 - \frac{1}{(2\cdot 3)^2}\right)\left(1 - \frac{1}{(2\cdot 4)^2}\right)\cdots$$

or

$$\frac{2}{\pi} = \prod_{k=1}^{\infty} \frac{(2k)^2 - 1}{(2k)^2} = \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k)}.$$

The last identity is the Wallis formula.

References

1 http://mathworld.wolfram.com/WallisFormula.html.

2 P. Levrie, Euler's wonderful insight, *Mathematical Intelligencer* **34** (2012), 1pp. Available at http://www.springerlink.com/content/24x7362687555751/fulltext.pdf.

3 C. J. Sangwin, An infinite series of surprises, School of Mathematics and Statistics, University of Birmingham, 2001. Available at http://plus.maths.org/issue19/features/infseries/2pdf/index.html/op.pdf.

Yours sincerely,

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Eleia
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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

48.5 A regular octagon is enclosed in a square as in figure 1. What is the angle α?

(Submitted by M. N. Deshpande, Nagpur, India)

48.6 What is the smallest area of a triangle in the coordinate plane whose vertices have integer coefficients?

(Submitted by Fionntan Roukema, University of Sheffield, UK)

48.7 Distinct points A, B, C are collinear in the plane, ABK is an anticlockwise triangle, BCL is an anticlockwise triangle similar to ABR, and KLM is an clockwise triangle similar to the other two. Show that M lies on the straight line through A, B, C.

(Submitted by Guido Lasters, Tienen, Belgium)

48.8 Points M, N, P lie on the sides BC, CA, AB of triangle ABC respectively such that $AB + BM = AC + CM$, $BA + AN = BC + CN$, and $CA + AP = CB + BP$. Show that area $\triangle MNP \leq \frac{1}{4}$area $\triangle ABC$.

(Submitted by Marcel Chirita, Bucharest, Romania)

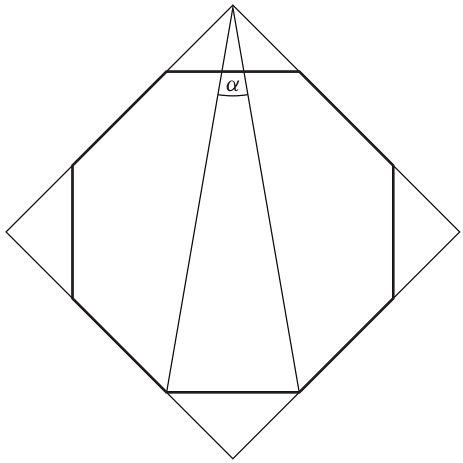

Figure 1

Solutions to Problems in Volume 47 Number 3

47.9 Solve the equation $e^x = x^{3x}$ for $x > 0$.

Solution by Henry Ricardo (New York Math Circle, USA), Lucía Ma Li (IES Isabel de España, Las Palmas de Gran Canaria), Ángel Plaza (Universidad de Las Palmas de Gran Canaria), Annanay Kapila (Year 12, Nottingham High School, UK), and Martin Egozcue (Norte Construcciones S.A., Punta del Este, Uruguay), independently.

We have

$$\begin{aligned} e^x = x^{3x} \quad (x > 0) \quad &\iff \quad x = 3x \ln x \\ &\iff \quad \ln x = \tfrac{1}{3} \\ &\iff \quad x = e^{1/3}. \end{aligned}$$

47.10 Let a, b, c, respectively, be the length, breadth, and height of a rectangular box. Prove that

$$\sqrt{2} < \frac{\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2}}{\sqrt{a^2+b^2+c^2}} \le \sqrt{6}.$$

Solution by Annanay Kapila

Put

$$x = \sqrt{a^2+b^2}, \qquad y = \sqrt{b^2+c^2}, \qquad z = \sqrt{c^2+a^2}.$$

Then the inequalities become

$$\begin{aligned} &1 < \frac{x+y+z}{\sqrt{x^2+y^2+z^2}} \le \sqrt{3} \\ \iff \quad &\sqrt{x^2+y^2+z^2} < x+y+z \le \sqrt{3(x^2+y^2+z^2)} \\ \iff \quad &x^2+y^2+z^2 < x^2+y^2+z^2+2(xy+xz+yz) \le 3(x^2+y^2+z^2) \\ \iff \quad &0 < 2(xy+xz+yz) \le 2(x^2+y^2+z^2). \end{aligned}$$

The first inequality is true because $x, y, z > 0$. For the second inequality,

$$2(x^2 + y^2 + z^2) - 2(xy + xz + yz) = (x - y)^2 + (x - z)^2 + (y - z)^2 \geq 0,$$

with equality if and only if $x = y = z$.

Also solved by Ángel Plaza and Henry Ricardo.

47.11 For positive real numbers $a_1, a_2, \ldots, a_n$ $(n \geq 1)$, prove that the following inequality holds:

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} \geq (a_1 a_2 \cdots a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

When does equality occur?

Solution by Annanay Kapila

Let $n > 1$. We use the inequality of arithmetic and geometric means on all but one member of the set $\{a_1^{n-1}, a_2^{n-1}, \ldots, a_n^{n-1}\}$, each time omitting a different member:

$$\begin{aligned}
a_2^{n-1} + \cdots + a_n^{n-1} &\geq (n-1)a_2 \cdots a_n,\\
a_1^{n-1} + a_3^{n-1} + \cdots + a_n^{n-1} &\geq (n-1)a_1 a_3 \cdots a_n,\\
&\vdots\\
a_1^{n-1} + a_2^{n-1} + \cdots + a_{n-1}^{n-1} &\geq (n-1)a_1 \cdots a_{n-1}.
\end{aligned}$$

If we sum these inequalities, we have

$$(n-1)(a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1}) \geq (n-1)a_1 a_2 \cdots a_n\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

If we divide by $n-1$, we obtain the result, which is obvious when $n = 1$. When $n = 2$, the result is true for all a_1, a_2. When $n > 2$, there is equality if and only if $a_1^{n-1} = a_2^{n-1} = \cdots = a_n^{n-1}$, i.e. if and only if $a_1 = a_2 = \cdots = a_n$.

Also solved by Lucía Ma Li, Ángel Plaza, and Henry Ricardo.

47.12 The point P lies inside the tetrahedron $A_1A_2A_3A_4$. The distance of A_i from the opposite face is denoted by h_i and d_i denotes the distance of P from that face. Prove that

$$\frac{d_1}{h_1} + \frac{d_2}{h_2} + \frac{d_3}{h_3} + \frac{d_4}{h_4} = 1.$$

Solution adapted from that given by Annanay Kapila.

Denote the areas of the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, and $A_1A_2A_3$ by S_1, S_2, S_3, and S_4 respectively and denote the volume of the tetrahedron by V. Then

$$V = \tfrac{1}{3}h_1S_1 = \tfrac{1}{3}h_2S_2 = \tfrac{1}{3}h_3S_3 = \tfrac{1}{3}h_4S_4.$$

The tetrahedron can be divided into four tetrahedra $PA_2A_3A_4$, $PA_1A_3A_4$, $PA_1A_2A_4$, and $PA_1A_2A_3$, with respective volumes V_1, V_2, V_3, and V_4. Then

$$V_1 = \tfrac{1}{3}d_1S_1, \qquad V_2 = \tfrac{1}{3}d_2S_2, \qquad V_3 = \tfrac{1}{3}d_3S_3, \qquad V_4 = \tfrac{1}{3}d_4S_4,$$

so that

$$\frac{V_1}{V} = \frac{d_1}{h_1}, \qquad \frac{V_2}{V} = \frac{d_2}{h_2}, \qquad \frac{V_3}{V} = \frac{d_3}{h_3}, \qquad \frac{V_4}{V} = \frac{d_4}{h_4}.$$

But $V = V_1 + V_2 + V_3 + V_4$. Hence,

$$\frac{d_1}{h_1} + \frac{d_2}{h_2} + \frac{d_3}{h_3} + \frac{d_4}{h_4} = 1.$$

Reviews

The Proof and the Pudding: What Mathematicians, Cooks, and You Have in Common. By Jim Henle. Princeton University Press, 2015. Hardback, 176 pages, £18.95 (ISBN 9780691164861).

The book is split into 25 chapters, each varying in length from one to 11 pages. Each chapter has a theme related to both mathematicians and cooks where key connections are detailed via examples. As the title may suggest, the book details the qualities mathematicians and cooks both have. The chapters cover mathematics that some may not have thought of, though much of the content concentrates on problems with a strong mathematical element coming from logical ideas.

There are many recipes for a keen cook to try, as well as problems to finish solving on your own. Not only this but there are various analogues between mathematicians and cooks, which I had not considered. One example is that both have choices to make, now it is easy to see that cooks have choices in their recipes, and people eating have choices too, but maybe it is less easy to see that mathematicians have a choice. There are many more examples detailed in the book.

Overall I enjoyed the book, and would recommend to anyone who has studied high school mathematics and enjoyed it. Admittedly, I have not tried any of the recipes yet but I plan to soon. However, I have solved many of the problems throughout the book, and found them to be fun.

University of Sheffield **Sarah Browne**

2016

Our annual puzzle is to express as many positive integers as possible using each of the digits of the year once in order, using only the operations of addition, subtraction, multiplication, division, factorials, square roots, and concatenation. For example,

$$1 = -(2 + 0!)! + 1 + 6$$

Your editor managed to reach 27 with a couple of gaps whilst doodling in a boring meeting.

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