Crux Mathematicorum

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IN THIS ISSUE / DANS CE NUMÉRO

YEAR-END FINALE

Year 2018 has been a real rollercoaster for ***Crux***: we started out hopeful to secure funding for the journal going forward, our hopes dwindled as the year went on, the situation looked gloomy in October and we were saved by the bell in November. In 2018, we are saying goodbye to the physical copies of the little purple book. As of 2019, ***Crux*** will be available online as an open access journal. I know that many of you will miss receiving hardcopies of ***Crux*** in the mail (I write this as I stare at my bookshelf with the last 11 volumes of ***Crux*** on it). But electronic format will allow us to reach wider audiences, include more materials and experiment with dynamic content – all at no subscription costs. After all, we can all share an electronic bookshelf: you can find all issues of ***Crux*** from 1975 to the present day at `www.math.ca/crux`.

Before you turn this page, physically or electronically, I offer to you the very first page of this journal from 1975, then known as Eureka. Its message still stands today.

Kseniya Garaschuk

EUREKA

No. 1

March 1975

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MATHEMATICS, ANYONE?

Have you come across an interesting problem lately? Do you have a comment to make about matters mathematical? Let your friends and colleagues know about it, through this magazine.

THE CONTEST CORNER

No. 70

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

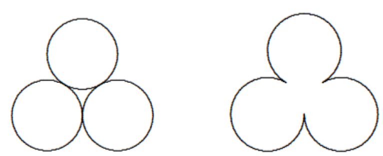
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

CC346. Une fourmi avance le long de l'axe x au rythme constant d'un unité par seconde. Son point de départ est $x = 0$ et son trajet est d'un unité en avant, puis deux unités en arrière, puis trois unités en avant, etc. Combien de fois la fourmi passe un point $x = 10$ dans les cinq premières minutes de son trajet ?

CC347. Trouver la somme de toutes les fractions p/q entre 0 et 1 qui ont un dénominateur de 100 lorsqu'elles sont réduites au maximum.

CC348. Trois cercles de même rayon r sont mutuellement tangents tel que montré dans la figure de gauche. Les arcs du milieu sont retirés de façon à faire un trèfle (figure de droite). Déterminer la longueur exacte du trèfle exprimée selon r.



CC349. Soit O le centre d'un triangle équilatéral ABC (c'est-à-dire l'unique point équidistant de chaque sommet). Un point P quelconque est choisi à l'intérieur du $\triangle ABC$. Quelle est la probabilité que P soit plus près que O que de n'importe quel des points A, B ou C.

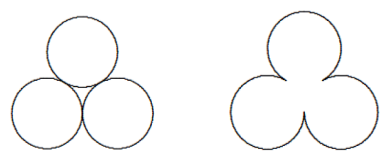
CC350. Un ami vous propose le jeu de devinettes suivant : il choisit un entier entre 1 et 100 inclusivement et vous tentez de deviner ce nombre. Il vous dit pour chaque réponse incorrecte si votre réponse est au dessus ou en dessous du nombre qu'il a choisi, mais vous ne pouvez faire qu'une seule tentative plus grande. Vous gagnez la partir si vous trouvez le nombre et vous perdez au moment où vous faite une deuxième tentative plus élevée que le nombre choisi. Quel est le nombre minimal de tentatives qui peut vous garantir la victoire ?

. . . .

CC346. An ant paces along the x-axis at a constant rate of one unit per second. He begins at $x = 0$ and his path takes him one unit forward, then two back, then three forward, etc. How many times does the ant step on the point $x = 10$ in the first five minutes of his walk?

CC347. Find the sum of all fractions p/q between 0 and 1 that have denominator 100 when expressed in lowest terms.

CC348. Three circles with the same radius r are mutually tangent as shown on the left figure. The arcs in the middle are removed, making a trefoil (right figure). Determine the exact length of the trefoil in terms of r.



CC349. Let O be the centre of equilateral triangle ABC (i.e. the unique point equidistant from each vertex). Another point P is selected uniformly at random in the interior of $\triangle ABC$. Find the probability that P is closer to O than it is to any of A, B or C.

CC350. A friend proposes the following guessing game: He chooses an integer between 1 and 100, inclusive, and you repeatedly try to guess his number. He tells you whether each incorrect guess is higher or lower than his chosen number, but you are allowed at most one high guess overall. You win the game when you guess his number correctly. You lose the game the instant you make a second high guess. What is the minimum number of guesses in which you can guarantee you will win the game?

. . . .

Note from the Editor. Problems CC 336, 337 and 338 in Volume 44 (8) are duplicates of problems CC308, 309 and 307, respectively, in Volume 44 (2). As Contest Corner section will be replaced with a new section in Volume 45, we will not be replacing these problems with new ones. We apologize for the inconvenience and will not be accepting solutions to the later re-print of the problems.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(10), p. 420–423.

CC296. Find the number of positive integers k with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

Originally problem 22 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 8 submissions, of which 4 were correct and complete and others contain a variety of counting mistakes. We present two of the solutions.

Solution 1, by Henry Ricardo.

Consider the number $k = \overline{abcde}$, where $10000 \leq k \leq 99999$. If digit c is to be the average of digits a and e, then $a + e$ must be even, so that both a and e are even or both a and e are odd.

Case 1: Both a and e are even. Since a can't equal zero, we have 4 choices for a and 5 choices for e. Any values for a and e determine c uniquely. After selecting a, e, and c, we have 10 choices for b and 10 choices for d, giving us $4 \times 10 \times 1 \times 10 \times 5 = 2000$ numbers satisfying our condition.

Case 2: Both a and e are odd. Now we have 5 choices for a, after which there are 5 ways to choose e. With a, e, and therefore c determined, we have 10 choices for b and 10 choices for d. Thus we have $5 \times 10 \times 1 \times 10 \times 5 = 2500$ numbers satisfying our condition.

This analysis yields a total of 4500 positive integers k with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

Solution 2, by Doddy Kastanya.

Each integer of interest will have 5 digits and can be represented by $AxByC$, where $B = \frac{1}{2}(A + C)$, where A could be any integer from 1 through 9 and C could be any integer between 0 and 9. There are 45 feasible triples (A, B, C) as indicated by the shaded cells in the table on the next page.

The other two digits of the integer (i.e., x and y) are independent of the values of A, B, or C. Since there are 10 possible values of x and 10 possible values of y, there are 100 different permutations of x and y. Therefore, there are $45 \times 100 = 4500$ positive integers k with $10000 \leq k \leq 99999$ such that the middle digit is the average of the first and fifth digits.

A	C									
	0	1	2	3	4	5	6	7	8	9
1		1		2		3		4		5
2	1		2		3		4		5	
3		2		3		4		5		6
4	2		3		4		5		6	
5		3		4		5		6		7
6	3		4		5		6		7	
7		4		5		6		7		8
8	4		5		6		7		8	
9		5		6		7		8		9

CC297. Six black checkers are placed on squares of a 6 by 6 checkerboard in the positions shown in Figure 1 and are left in place. A white checker begins on the square at the lower left corner of the board (marked A in Figure 1) and follows a path from square to square across the board, ending in the upper right corner of the board (marked B). How many different paths are there from A to B if at each step the white checker can move one square to the right, one square up or one square diagonally upward to the right and may not pass though any square occupied by a black checker? One such path is shown in Figure 2.

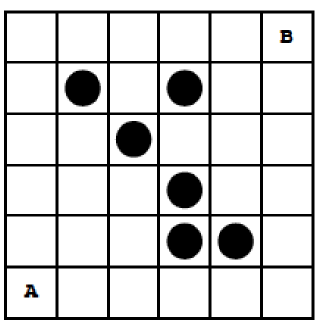


Figure 1

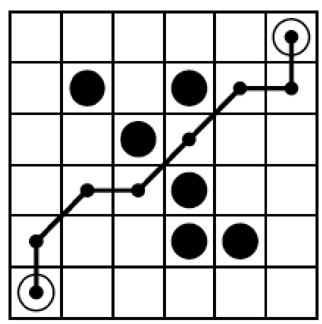


Figure 2

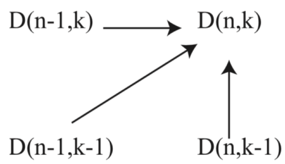
Originally problem 27 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 4 submissions, of which 2 were correct and complete. We present the solution by Carlos Moreno and Ángel Plaza.

Let us assign integer coordinates to the cells in our checkerboard, beginning with $A = (0, 0)$, so $B = (5, 5)$. Let $D(n, k)$ be the number of different paths from A to the cell (n, k). The following recurrence relation holds:

$$D(n,k) = \begin{cases} 0, & \text{if there is a black checker in it. Otherwise,} \\ 1, & \text{if } n = 0 \text{ or } k = 0 \\ D(n-1,k) + D(n-1,k-1) + D(n,k-1) & \text{otherwise.} \end{cases}$$

Using the recurrence relation in our problem gives the numbers in each cell as in the following figure:



1	2	9	16	42	122
1	●	7	●	26	54
1	7	●	13	13	15
1	5	13	●	0	2
1	3	5	●	●	2
1	1	1	1	1	1

Therefore, there are 122 different paths from A to B.

Note: this problem may be solved using the Delannoy numbers.

CC298. Let ABC be the triangle with vertices $(0,0),(4,0)$ and $(2,3)$. Find the coordinates of the point P that is equidistant from A, B and C.

Originally problem 36 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 9 solutions. We present 2 solutions, slightly modified by the editor.

Solution 1, by Catherine Doan.

Let (x, y) be the coordinates of P. Let P_A, P_B, P_C denote the distances from point P to the points A, B, C, respectively. As $P_A = P_B$, it follows that $x = 2$. As $P_A = P_C$ and $x = 2$, it follows that $4 + y^2 = (y-3)^2$, so $y = \frac{5}{6}$. Therefore the coordinates of P are $\left(2, \frac{5}{6}\right)$.

Solution 2, by Ivko Dimitrić.

Since the point P is the circumcenter of an isosceles $\triangle ABC$, we use the well known formula for the circumradius R of a triangle in terms of its area $[ABC]$ and side lengths a, b, c to get

$$R = \frac{abc}{4[ABC]} = \frac{4 \cdot \sqrt{2^2+3^2}^{\,2}}{4 \cdot \frac{1}{2} \cdot 4 \cdot 3} = \frac{13}{6}.$$

The y-coordinate of P is $3 - \frac{13}{6} = \frac{5}{6}$. Thus the coordinates of P are $\left(2, \frac{5}{6}\right)$.

CC299. Find the area of the region bounded by the graphs of

$$\begin{cases} x + y + |x| = 10, \\ x + y - |x| = -8. \end{cases}$$

Originally problem 24 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

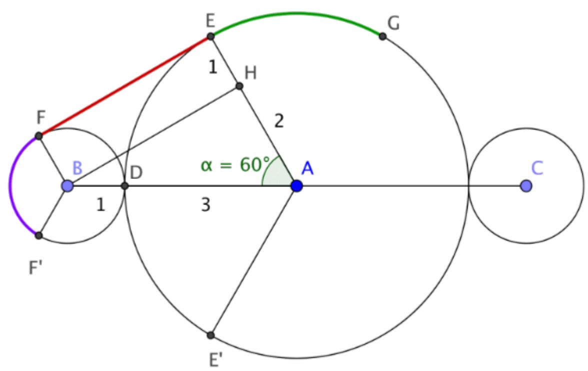
We received 8 correct solutions and 1 incorrect submission. We present the solution by Kathleen Lewis.

The function $x + y + |x| = 10$ is equal to $y = 10$ if $x < 0$ and $y = 10 - 2x$ when $x \geq 0$. The function $x + y - |x| = -8$ is equal to $y = -8$ if $x \geq 0$ and $y = -2x - 8$ if $x < 0$. The functions are equal when $|x| = 9$, so $x = \pm 9$. The enclosed area is a parallelogram with base length 9 and height 18. Therefore the area is 162.

CC300. Three poles with circular cross sections are to be bound together with a wire. The radii of the circular cross sections are 1, 3 and 1 inches. The centers of the circles are on the same straight line as indicated in the sketch. If the length of the wire is written in the form $a\sqrt{3} + b\pi$, where a and b are rational numbers, find $a + b$. Assume that the wire has negligible thickness.

Originally problem 41 from the University of Vermont, 56th Annual High School Prize Examination, 2013.

We received 5 solutions. We present the solution by Catherine Doan.



Let A, B, C be the centers of the three circles, i.e. the three circular cross sections. Let E, F be the tangent points of the wires to the circles with A and B centers as shown in the above diagram. We have $EF \perp AE$, $EF \perp BF$. Let H be on EA such that $BH \parallel EH$.

Due to ratio of side to hypotenuse being $1 : 2$, triangle ABH is a 30-60-90 right triangle, hence $EF = BH = 2\sqrt{3}$. Since $\angle FBF' = 120°$, the length of FF' arc (shown in purple) is $(2\pi \times 1)/3 = 2\pi/3$. Similarly because $\angle EAG = 60°$, the length of EG arc (shown in green) is $(2\pi \times 3)/6 = \pi$. Due to symmetry, the total length of the wire is:

$$2 \times \frac{2\pi}{3} + 2 \times \pi + 4 \times 2\sqrt{3} = 8\sqrt{3} + \frac{10}{3}\pi.$$

The answer is: $a = 8$, $b = 10/3$, and $a + b = 8 + 10/3 = 34/3$.

THE OLYMPIAD CORNER

No. 368

Alessandro Ventullo

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2019**.

La rédaction souhaite remercier Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

OC406. Soit D le point à l'intérieur du triangle ABC tel que $BD = CD$ et $\angle BDC = 120°$. Soit E un point à l'extérieur du triangle ABC tel que $AE = CE$, $\angle AEC = 60°$ et les points B et E soient dans les différents demi-plans par rapport à AC. Montrer que $\angle AFD = 90°$, où F est le point milieu du segment BE.

OC407. Le triangle acutangle isocèle ABC ($AB = AC$) est inscrit dans un cercle de centre O. Les rayons BO et CO interseptent les côtés AC et AB aux points B' et C', respectivement. Une droite l est parallèle au segment AC et passe par le point C'. Montrer que la droite l est tangente au cercle circonscrit ω du triangle $B'OC$.

OC408. Est-ce qu'il existe une suite infinie $a_1, a_2, a_3, \ldots$ d'entiers positifs telle que la somme de deux termes distincts de la suite est copremière avec la somme de n'importe quels trois termes distincts de cette suite ?

OC409.

(a) Donner un exemple de fonction continue $f : [0, \infty) \to \mathbb{R}$ telle que

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1$$

et $f(x)/x$ n'a pas de limite lorsque $x \to \infty$.

(b) Soit $f : [0, \infty) \to \mathbb{R}$ une fonction croissante telle que

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\ dt = 1.$$

Montrer que $f(x)/x$ possède une limite lorsque $x \to \infty$ et déterminer cette limite.

OC410. Soit $a_0, a_1, \ldots, a_{10}$ des entiers tels que $a_0 + a_1 + \cdots + a_{10} = 11$. Quel est le nombre maximal de solutions entières distinctes à l'équation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$

. . . .

OC406. Let D be a point inside the triangle ABC such that $BD = CD$ and $\angle BDC = 120^\circ$. Let E be a point outside the triangle ABC such that $AE = CE$, $\angle AEC = 60^\circ$ and points B and E are in different half-planes with respect to AC. Prove that $\angle AFD = 90^\circ$, where F is the midpoint of the segment BE.

OC407. The acute isosceles triangle ABC ($AB = AC$) is inscribed in a circle with center O. The rays BO and CO intersect the sides AC and AB in the points B' and C', respectively. A line l parallel to the line AC passes through point C'. Prove that the line l is tangent to the circumcircle ω of the triangle $B'OC$.

OC408. Does there exist an infinite increasing sequence $a_1, a_2, a_3, \ldots$ of positive integers such that the sum of any two distinct terms of the sequence is coprime with the sum of any three distinct terms of the sequence?

OC409.

(a) Give an example of a continuous function $f : [0, \infty) \to \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\, dt = 1$$

and $f(x)/x$ has no limit as $x \to \infty$.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be an increasing function such that

$$\lim_{x\to\infty} \frac{1}{x^2} \int_0^x f(t)\, dt = 1.$$

Prove that $f(x)/x$ has a limit as $x \to \infty$ and determine this limit.

OC410. Let $a_0, a_1, \ldots, a_{10}$ be integers such that $a_0 + a_1 + \cdots + a_{10} = 11$. Find the maximum number of distinct integer solutions to the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{10} x^{10} = 1.$$

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017: 43(8), p. 336–337.

OC346. Two real number sequences are given, one arithmetic $(a_n)_{n\in\mathbb{N}}$ and another geometric $(g_n)_{n\in\mathbb{N}}$, neither of them constant. These sequences satisfy $a_1 = g_1 \neq 0$, $a_2 = g_2$ and $a_{10} = g_3$. Prove that, for every positive integer p, there is a positive integer m, such that $g_p = a_m$.

Originally 2016 Spain Mathematical Olympiad Day 1, Problem 1.

We received 10 solutions. We present the solution by C. R. Pranesachar.

Assume that d is the common difference of the arithmetic sequence $(a_n)_{n\in\mathbb{N}}$ and r is the common ratio of the geometric sequence $(g_n)_{n\in\mathbb{N}}$. Then

$$a_2 = a_1 + d = g_2 \quad \text{and} \quad a_{10} = a_1 + 9d = g_3.$$

Since $g_2^2 = g_1 g_3$, we get $(a_1+d)^2 = a_1(a_1+9d)$. This gives $d = 7a_1$, as $d \neq 0$ (the sequence $(a_n)_{n\in\mathbb{N}}$ is nonconstant). Furthermore,

$$r = \frac{g_2}{g_1} = \frac{a_1+d}{a_1} = \frac{8a_1}{a_1} = 8.$$

Hence, if $p \in \mathbb{N}$, then

$$\begin{aligned} g_p &= a_1 r^{p-1} = a_1 \cdot (1+7)^{p-1} \\ &= a_1(1+7k) \\ &= a_1 + kd \end{aligned}$$

for some $k \in \mathbb{N}$. Taking $m = k+1$, we have

$$g_p = a_1 + (m-1)d = a_m,$$

as desired. This completes the proof.

OC347. Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

Originally 2016 Canadian Mathematical Olympiad, Problem 2.

We received 4 solutions. We present the solution by Mohammed Aassila.

We prove that there are eleven 10-tuples that are solutions of the system:

$$\left(4, \frac{4}{3}, \frac{4}{3} \ldots, \frac{4}{3}\right) \quad \text{(and all its permutations) and} \quad \left(\frac{8}{5}, \frac{8}{5}, \ldots, \frac{8}{5}\right).$$

Let $C = \sum_{i=1}^{10} v_i^2$. The term v_i is a solution to the equation

$$\frac{6}{C}x^2 - x + 1 = 0.$$

Therefore, $v_i \in \{a, b\}$. Assume that there are u terms equal to a and v terms equal to b. So, $u + v = 10$ and adding all the equations, we get

$$ua + vb = 16. \tag{1}$$

By Vieta's formulas we have

$$a + b = ab = \frac{C}{6} = \frac{ua^2 + vb^2}{6}.$$

Plugging $b = \dfrac{a}{a-1}$ and $v = 10 - u$ into (**??**), we get

$$ua^2 - 2a(u+3) + 16 = 0.$$

The discriminant of this equation in a is $\Delta_u = (u+3)^2 - 16u = (u-1)(u-9)$. So, $u \le 1$ or $u \ge 9$. Assume without loss of generality that $u \ge 9$.

(i) If $u = 9$, we have $9a + b = 16$ and $a + b = ab = \dfrac{9a^2 + b^2}{6}$. The last equation gives $(3a - 2b)^2 = 0$, i.e. $3a = 2b$, which gives the solution $\left(4, \frac{4}{3}, \frac{4}{3}, \dots, \frac{4}{3}\right)$ (and all its permutations).

(ii) If $u = 10$, we have $a = 1 + \dfrac{6a^2}{10a^2} = 1 + \dfrac{6}{10} = \dfrac{8}{5}$, so we have the solution $\left(\frac{8}{5}, \frac{8}{5}, \dots, \frac{8}{5}\right)$.

OC348. Triangle ABC is an acute isosceles triangle ($AB = AC$) and CD one altitude. Circle $C_2(C, CD)$ meets AC at K, AC produced at Z and circle $C_1(B, BD)$ at E. Line DZ meets circle (C_1) at M. Show that:

a) $\widehat{ZDE} = 45^\circ$.

b) Points E, M, K lie on a line.

c) $BM \parallel EC$.

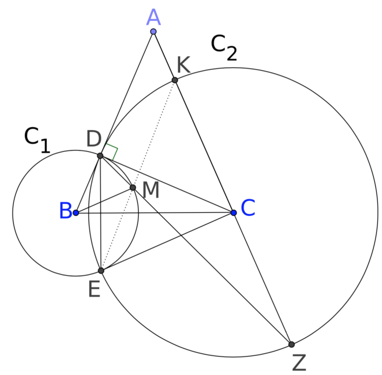
Originally 2016 Greece National Olympiad, Problem 3.

We received 7 solutions. We present the solution by Oliver Geupel.

Since CD is an altitude in triangle ABC, we have $\angle ACD = 90^\circ - \angle A$ and $\angle DCB = 90^\circ - \angle C$. Because D and E are symmetric with respect to the axis BC, we have that $\angle BCE = \angle DCB$. Moreover, $\angle B = \angle C$. Hence,

$$\angle ECZ = 180^\circ - (\angle ACD + \angle DCB + \angle BCE) = \angle A + \angle B + \angle C - 90^\circ = 90^\circ.$$

Since D, E, Z lie on a circle with centre C, we obtain $\angle ZDE = \angle ECZ/2 = 45°$, which is the result a).



Because D, E, K, Z lie on a circle with centre C, we have

$$\angle KED = \angle KZD = \frac{1}{2}\angle KCD = \frac{1}{2}\angle ACD = \frac{1}{2}(90° - \angle A) = \angle C - 45°.$$

D, E, M lie on a circle with centre B. Thus,

$$\angle DME = 180° - \frac{1}{2}\angle EBD = 180° - \angle C.$$

Therefore $\angle MED = 180° - \angle EDM - \angle DME = \angle C - 45° = \angle KED$, so that E, M, K lie on a line, which proves result b).

Since CE is tangent to C_1, we have $\angle CEB = 90°$. Moreover, because D, E, M lie on a circle with centre B, we have $\angle EBM = 2\angle EDM = 90°$. Consequently, $BM \parallel EC$. This completes the proof of result c).

OC349. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x) - x) = f(x)f(y) + 2x$$

for all $x,\ y \in \mathbb{R}$.

Originally 2016 Japan Mathematical Olympiad Finals, Problem 4.

We received 5 solutions. We present the solution by Oliver Geupel.

It is straightforward to verify that the two functions $x \mapsto -2x$ and $x \mapsto 1 - x$ are solutions of the problem. We prove that there are no further solutions.

Suppose that f is such a solution. Let $P(x, y)$ denote the assertion that the given functional equation holds for x and y. From $P(0, 0)$, we easily obtain $f(0) \in \{0, 1\}$. If $f(0) = 0$, then $P(-x, 0)$ yields $f(x) = -2x$ for all $x \in \mathbb{R}$.

It remains to consider the case $f(0) = 1$. From $P(x, 0)$, we get

$$f(-x) = f(x) + 2x \qquad \text{(assertion } Q(x)\text{)}.$$

It follows:

$$\begin{aligned} f(yf(x)-x) &= f(x)f(y)+2x && \text{by } P(x,y)\\ &= f(x)\cdot(f(-y)-2y)+2x && \text{by } Q(y)\\ &= f(x)f(-y)+2x-2(yf(x)+x)+2x && \\ &= f(-yf(x)-x)-2(yf(x)+x)+2x && \text{by } P(x,-y)\\ &= f(yf(x)+x)+2x && \text{by } Q(yf(x)+x). \end{aligned}$$

Therefore, we see that for all $x, y \in \mathbb{R}$ such that $f(x) \neq 0$,

$$f(y-x) = f(y+x)+2x \qquad (\text{assertion } R(x,y)).$$

Let a be a real number with the property that $f(a) = 0$. Then, $a \neq 0$ and, by $Q(a)$, $f(-a) = f(a) + 2a = 2a \neq 0$, so that $R(-a, y)$ and thus $R(a, y)$ does hold. As a consequence, $R(x, y)$ is satisfied in fact for *all* $x, y \in \mathbb{R}$. We deduce $f(y-x)+y-x = f(y+x)+y+x$, that is, $f(x)+x$ is a constant for all x. Taking account of $f(0) = 1$, we conclude $f(x) = 1-x$ for all $x \in \mathbb{R}$. This completes the proof that no further solutions exist.

OC350. Two players, A (first player) and B, take alternate turns in playing a game using 2016 chips as follows: the player whose turn it is, must remove s chips from the remaining pile of chips, where $s \in \{2, 4, 5\}$. No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Which of the two players has a winning strategy?

Originally 2016 Philippines Mathematical Olympiad, Problem 4.

We received 2 solutions and we present both of them.

Solution 1, by Ivko Dimitrić.

A player whose turn it is can win if that player is presented at each turn with a pile of chips whose number is between $7k+2$ and $7k+6$ for some integer k and leaves a reduced pile (after removing a suitable number of chips) with number of chips in it equal to $7k$ or $7k+1$ for the next player. Namely, if the number of chips from that range is $7k+2$ or $7k+3$, the player removes 2 chips. If the number of chips is $7k+4$ or $7k+5$, the player removes 4 chips and if the number is $7k+6$ the player is to remove 5 chips. That way, the reduced pile now containing $7k$ or $7k+1$ chips is left to the next player at that player's turn. However many chips $s \in \{2, 4, 5\}$ the next player removes, the new pile that remains would have at least $7k-5 = 7(k-1)+2$ and at most $(7k+1)-2 = 7(k-1)+6$ chips, leaving it in the desirable range for the player who plays subsequently.

Since $2016 = 288 \cdot 7$ is a multiple of 7 at the beginning of the game, however many chips $s \in \{2, 4, 5\}$ player A removes, it would leave a new pile whose number is in the range $287 \cdot 7 + 2 = 2011$ to $287 \cdot 7 + 6 = 2015$ for player B. Then player B removes a certain number of chips following the strategy as explained above so that the pile left after B makes the move has between $287 \cdot 7$ and $287 \cdot 7 + 1$ chips at player A's turn. Whatever move the player A next does, the new reduced pile left will have a number of chips between $286 \cdot 7 + 2$ and $286 \cdot 7 + 6$ at B's turn. Consequently, in order to win, following the strategy, the player B ensures that the number of chips left after B's turn is between $285 \cdot 7$ and $285 \cdot 7 + 1$, and so on, resulting in B leaving always a number of chips between $7k$ and $7k + 1$, so that after A's subsequent move B has a pile numbering between $7(k-1)+2$ and $7(k-1)+6$ chips at B's disposal. Thus, eventually, as k decreases to 1, player B, who follows the strategy, will leave the pile reduced to only 7 or 8 chips. If that number is 7 and player A next takes 2, 4 or 5 chips (leaving a new pile of 5, 3 or 2 chips, respectively), player B will remove 4, 2 or 2 chips, respectively, so that A cannot make the next move. Likewise, if the number of chips remaining is 8 and player A removes 2, 4 or 5 chips (leaving the reduced pile of 6, 4 or 3 chips) then player B removes 5, 4 or 2 chips, respectively, so that A cannot continue the play according to the rules and player B wins.

Solution 2, by Missouri State University Problem Solving Group.

More generally, if the game starts with n chips, we claim that B has a winning strategy if $n \equiv 0$ or $1 \bmod 7$ and A has a winning strategy otherwise. We proceed by induction on n. If $n = 0$ or 1, A clearly loses. If $n = 2, 4$ or 5 A picks up all the chips and wins. If $n = 3$, A picks up 2 chips and B loses. If $n = 6$, A picks up 5 chips and B loses. Now suppose $n > 6$ and the result holds for all $k < n$. Note that if at any point in the game, removing 2,4, or 5 chips always results in a number of chips where A has a winning strategy at the beginning of the game , then the player whose turn is next has a winning strategy. If at any point in the game, some choice of removing 2, 4, or 5 chips results in a number of chips where B has a winning strategy at the beginning of the game, then the player whose turn it is has a winning strategy.

Suppose $n \equiv 0 \bmod 7$. Removing 2, 4 or 5 chips leaves k chips where $k \equiv 2, 3$ or $5 \bmod 7$. By induction, these all yield a winning strategy for A, so by discussion above, since A is starting, this gives a winning strategy for player B.

Suppose $n \equiv 1 \bmod 7$. Removing 2,4, or 5 chips leaves k chips where $k \equiv 3, 4,$ or $6 \bmod 7$. By induction, these all yield a winning strategy for A, so again B has a winning strategy.

Suppose $n \equiv 2 \bmod 7$. If A removes 2 chips we have $k \equiv 0 \bmod 7$ chips remaining. By induction, this number gives a winning strategy for B, so the original number of chips gives a winning strategy for A.

Proceeding in a similar manner if $n \equiv 3 \bmod 7$, A removes 2 chips, if $n \equiv 4 \bmod 7$, A removes 4 chips, if $n \equiv 5 \bmod 7$, A removes 5 chips, and if $n \equiv 6 \bmod 7$, A removes 5 chips. In each case the number of remaining chips is $k \equiv 0$ or $1 \bmod 7$. Consequently A has a winning strategy in each of these cases.

If the initial number of chips is $2016 \equiv 0 \bmod 7$, then B has a winning strategy.

PROBLEM SOLVING 101

No. 7

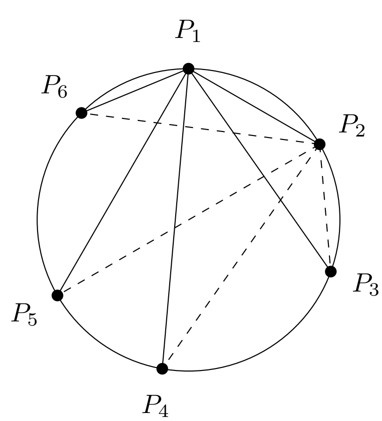
Shawn Godin

This month, we will look at problems 6 to 10 from the course C & O 380 that I took back in 1986. See [2017: 43(4), p. 151 - 153] and [2018: 44(4), p. 157 - 159] for more about this course and the problems.

#6. Prove that n^2+3n+5 is never divisible by 121 for any natural number n.
#7. Of 5 points inside a square of unit side, show that some pair is less than $\frac{\sqrt{2}}{2}$ units apart.
#8. On a circle n points are selected and chords joining them in pairs are drawn. If no three of these chords are concurrent inside the circle, how many points of intersection are there in the circle?
#9. Four points, A, B, C, D are given on a straight line. Show how to construct a pair of parallel lines through A and B, and another pair of parallel lines through C and D, so that these pairs of parallel lines intersect in the vertices of a square.
#10. Into how may regions do n great circles, no three concurrent, divide the surface of a sphere? (A great circle has the same centre as the sphere itself).

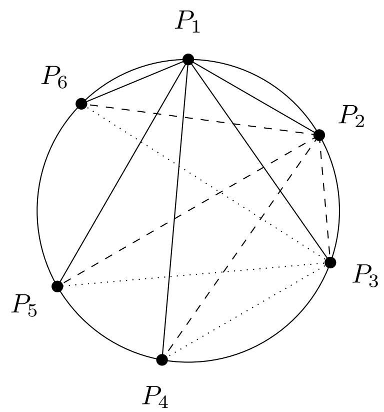
In this issue we will look at problem #8. We will solve the problem in two ways: the long way and the short way. Hopefully we will see that a little insight usually goes a long way!

We will look at the case where $n = 6$ and generalize our findings. First, we will label our points P_1 through P_6 as in the diagram. From point P_1 we will draw 5 chords, from point P_2 we will draw 4 chords, and so on.



When drawing chords from point P_2, we notice that chord P_2P_3 doesn't intersect any chords from P_1, while P_2P_4 intersects 1 (namely the chord from P_1 to P_3), P_2P_5 intersects 2 (the chords to P_3 and P_4), and P_2P_6 intersects 3 (the chords to P_3, P_4 and P_5). So drawing the chords from P_2 introduces $1 + 2 + 3 = 6$ points.

When drawing chords from point P_3, we notice that chord P_3P_4 doesn't intersect any chords from P_1 or P_2, while P_3P_5 intersects 1 each from P_1 and P_2, and P_3P_6 intersects 2 from each. So drawing the chords from P_3 introduces $2(1 + 2) = 6$ points.



Finally, drawing P_4P_5 introduces no new points but P_4P_6 introduces 1 point each from chords from P_1, P_2 and P_3 for a total of $3(1) = 3$ points.

Generalizing the process we see that if we have n points, there will be $n-1$ chords from P_1. The chords from P_2 to P_4 through to P_n will cross the chords from P_1 at

$$\sum_{i=1}^{n-3} i$$

points. The chords from P_3 to P_5 through to P_n will cross chords from both P_1 and P_2 to any particular point, hence the number of new points introduced is

$$2\sum_{i=1}^{n-4} i.$$

In general, when $2 \leq r \leq n-2$ and $r+2 \leq s \leq n-1$, the chord P_rP_s will intersect (inside the circle) each chord P_iP_j with i running from 1 to $r-1$ and j strictly between r and s; the number of crossings produced by the $n-r$ chords initiated at P_r is therefore

$$(r-1)\sum_{i=1}^{n-r-1} i.$$

Thus the total number of crossings is

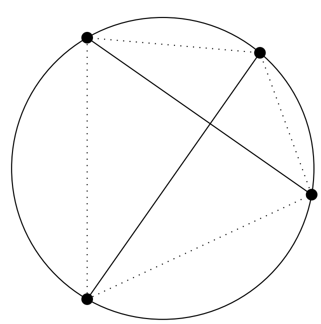
$$\sum_{i=1}^{n-3} i + 2\sum_{i=1}^{n-4} i + 3\sum_{i=1}^{n-5} i + \cdots + (n-3)(1) = \sum_{i=1}^{n-3}(n-2-i)\sum_{j=1}^{i} j$$

Simplifying yields

$$\begin{aligned}
&\sum_{i=1}^{n-3}(n-2-i)\sum_{j=1}^{i} j\\
&= \frac{1}{2}\sum_{i=1}^{n-3}(n-2-i)(i^2+i)\\
&= \frac{1}{2}\left[(n-3)\sum_{i=1}^{n-3} i^2 + (n-2)\sum_{i=1}^{n-3} i - \sum_{i=1}^{n-3} i^3\right]\\
&= \frac{1}{2}\left[(n-3)\frac{(n-3)(n-2)(2n-5)}{6} + (n-2)\frac{(n-3)(n-2)}{2} - \frac{(n-3)^2(n-2)^2}{4}\right]\\
&= \frac{(n-2)(n-3)}{24}\left[2(n-3)(2n-5) + 6(n-2) - 3(n-3)(n-2)\right]\\
&= \frac{(n-2)(n-3)}{24}\left[n^2-n\right]\\
&= \frac{n(n-1)(n-2)(n-3)}{24}
\end{aligned}$$

That was a little bit fun, but quite a bit of algebraic manipulation.

It would be nice to get at things a little easier. If we start using some counting techniques, we have $\binom{n}{2}$ chords in total. It might be nice if we could just pair the chords up to count the intersections, but we see that not all chords cross other chords, while others cross many. We need a way that we can identify a point of crossing. If we examine two crossing chords that identify a point, we notice that these two chords are associated with 4 distinct points on the circle. On top of that, those four points also produce four other chords that do not intersect inside the circle.



Hence, it seems that each collection of 4 points on the circumference, upon drawing in all the associated chords, produces one point of intersection inside the circle. Thus, the total number of interior crossing points is

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$$

which we had found earlier with much more grunt work.

The second solution gives us much more insight into how the chords and points of intersection are related to each other. Even if you may not have considered the second solution, now that you have seen it a similar technique may be used in future problems. Also, the problem suggests the value in reflecting and reexamining a problem after we have arrived at a solution. The algebraic technique leads to an expression that, when put in factored form, suggests $\binom{n}{4}$. Recognizing this, it is worth going back to figure out why. Making a habit of reflecting on a solution will help you make connections between mathematical concepts and strengthen your problem solving.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4391. *Proposé par Leonard Giugiuc et Oai Thanh Dao.*

Soit ABC un triangle équilatéral et soit W un point à l'intérieur de ABC. Une ligne l_1 passant par W intersecte les segments BC et AB en D et P, respectivement. De façon similaire, une ligne l_2 passant par W intersecte AC et BC en E et M, et une ligne l_3 passant par W intersecte AB et AC en F et N. Si $\angle DWE = \angle EWF = \angle FWD = 120°$, démontrer que les triangles DEF et MNP sont similaires.

4392. *Proposé par Leonard Giugiuc et Kadir Altintas.*

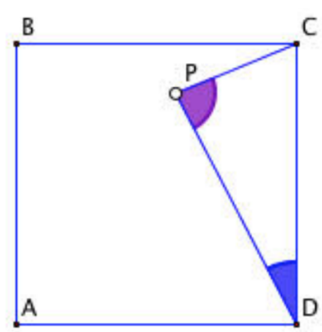
Soit M un point à l'intérieur du triangle ABC dont les côtés sont $BC = a$, $CA = b$ et $AB = c$. Si $MA = x, MB = y$ et $MC = z$, démontrer que

$$\sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} + \sqrt{(c+x-y)(c-x+y)} = \sqrt{3}(x+y+z),$$

alors ABC est équilatéral.

4393. *Proposé par Ruben Dario Auqui et Leonard Giugiuc.*

Soit $ABCD$ un carré. Déterminer le lieu géométrique des points P à l'intérieur de $ABCD$ tels que $\cot\angle CPD + \cot\angle CDP = 2$.



4394. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle acutangle où $M \in BC$, $BM \equiv MC$, $E \in AB$, $F \in AC$, $\measuredangle BEM \equiv \measuredangle CFM = 90°$. Les tangentes au cercle circonscrit du $\triangle MEF$, en E et F, intersectent en X. Si $XM \cap EF = \{Y\}$, démontrer que $YB = YC$.

4395. *Proposé par Michel Bataille.*

Soit $ABCD$ un tétraèdre et soient $A1, B1, C1, A2, B2, C2$ les mipoints de BC, CA, AB, DA, DB, DC, respectivement. Démontrer que

$$(\overrightarrow{DA} \cdot \overrightarrow{BC})A_1A_2^2 + (\overrightarrow{DB} \cdot \overrightarrow{CA})B_1B_2^2 + (\overrightarrow{DC} \cdot \overrightarrow{AB})C_1C_2^2 = 0,$$

où $\overrightarrow{X} \cdot \overrightarrow{Y}$ dénote le produit scalaire des vecteurs $\overrightarrow{X}$ and $\overrightarrow{Y}$.

4396. *Proposé par David Lowry-Duda.*

Démontrer qu'il existe une bijection $f : \mathbb{N} \mapsto \mathbb{N}$ telle que la série $\sum_{n=1}^{\infty} \dfrac{1}{n + f(n)}$ converge, ou démontrer qu'aucune telle bijection existe.

4397. *Proposé par George Stoica.*

Soit $n \in \mathbb{N}$ et $k \in \{0, 1, \ldots, 2^n\}$. Démontrer qu'il existe $k' \in \{0, 1, \ldots, 2^{n+1}\}$ tel que

$$\left|\sin \frac{k'\pi}{2^{n+2}} - \frac{k}{2^n}\right| \leq \frac{1}{2^n}.$$

4398. *Proposé par Daniel Sitaru.*

Démontrer que pour $n \in \mathbb{N}^*$, l'inégalité qui suit est valide:

$$\frac{1}{2n-1} + \int_0^1 \sin^2(x^n)dx \geq \frac{2}{n}(1 - \cos 1).$$

4399. *Proposé par Lacin Can Atis.*

Soit $ABCDE$ un pentagone. Démontrer que

$$|AB||EC||ED| + |BC||EA||ED| + |CD||EA||EB| \geq |AD||EB||EC|.$$

Quand l'égalité tient-elle ?

4400. *Proposé par Daniel Sitaru.*

Démontrer que l'inégalité qui suit tient pour tout triangle ABC:

$$\sum_{cyc} \frac{\sin(\frac{\pi}{3} - \frac{A}{2})\sin(\frac{\pi}{3} - \frac{B}{2})}{\cos(\frac{C-A}{2})\cos(\frac{C-B}{2})} < 1.$$

. . . .

4391. *Proposed by Leonard Giugiuc and Oai Thanh Dao.*

Let ABC be an equilateral triangle and let W be a point inside ABC. A line l_1 through W intersects the segments BC and AB in D and P, respectively. Similarly, a line l_2 through W intersects AC and BC in E and M, and a line l_3 through W intersects AB and AC in F and N. If

$$\angle DWE = \angle EWF = \angle FWD = 120^\circ,$$

show that the triangles DEF and MNP are similar.

4392. *Proposed by Leonard Giugiuc and Kadir Altintas.*

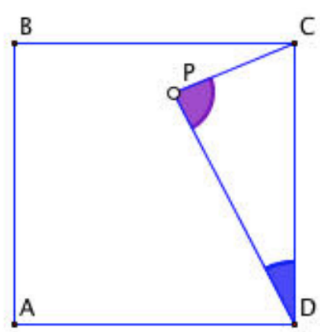
Let M be an interior point of a triangle ABC with sides $BC = a$, $CA = b$ and $AB = c$. If $MA = x, MB = y$ and $MC = z$, then prove that if

$$\sqrt{(a+y-z)(a-y+z)} + \sqrt{(b+z-x)(b-z+x)} + \sqrt{(c+x-y)(c-x+y)} = \sqrt{3}(x+y+z),$$

then ABC is equilateral.

4393. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let $ABCD$ be a square. Find the locus of points P inside $ABCD$ such that $\cot\angle CPD + \cot\angle CDP = 2$.



4394. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and $M \in BC$, $BM \equiv MC$, $E \in AB$, $F \in AC$, $\measuredangle BEM \equiv \measuredangle CFM = 90^\circ$. The two tangents at the points E and F to the circumcircle of $\triangle MEF$ intersect at the point X. If $XM \cap EF = \{Y\}$, show that $YB = YC$.

4395. *Proposed by Michel Bataille.*

Let $ABCD$ be a tetrahedron and let $A_1, B_1, C_1, A_2, B_2, C_2$ be the midpoints of BC, CA, AB, DA, DB, DC, respectively. Prove that

$$(\overrightarrow{DA} \cdot \overrightarrow{BC})A_1A_2^2 + (\overrightarrow{DB} \cdot \overrightarrow{CA})B_1B_2^2 + (\overrightarrow{DC} \cdot \overrightarrow{AB})C_1C_2^2 = 0,$$

where $\overrightarrow{X} \cdot \overrightarrow{Y}$ denotes the dot product of the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$.

4396. *Proposed by David Lowry-Duda.*

Show that there is a bijection $f : \mathbb{N} \mapsto \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} \dfrac{1}{n + f(n)}$ converges or show that no such bijection exists.

4397. *Proposed by George Stoica.*

Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n\}$. Show that there exists $k' \in \{0, 1, \ldots, 2^{n+1}\}$ such that

$$\left| \sin \frac{k'\pi}{2^{n+2}} - \frac{k}{2^n} \right| \leq \frac{1}{2^n}.$$

4398. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}^*$, we have

$$\frac{1}{2n-1} + \int_0^1 \sin^2(x^n)dx \geq \frac{2}{n}(1 - \cos 1).$$

4399. *Proposed by Lacin Can Atis.*

Let $ABCDE$ be a pentagon. Prove that

$$|AB||EC||ED| + |BC||EA||ED| + |CD||EA||EB| \geq |AD||EB||EC|.$$

When does equality hold?

4400. *Proposed by Daniel Sitaru.*

Prove that in any triangle ABC, the following relationship holds:

$$\sum_{cyc} \frac{\sin(\frac{\pi}{3} - \frac{A}{2}) \sin(\frac{\pi}{3} - \frac{B}{2})}{\cos(\frac{C-A}{2}) \cos(\frac{C-B}{2})} < 1.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(10), p. 444–448.

An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

4291. *Proposed by George Stoica.*

i) Find the number of permutations of n distinct integers from the set of integers $1, 2, \dots, N$ so that no two integers in a permutation are consecutive.

ii) Find the number of permutations of $n+1$ distinct integers from the set of integers $1, 2, \dots, N$ so that no two of the first n integers in a permutation are consecutive, but the $(n+1)^{th}$ is consecutive with one of the first n.

We received one solution. We present the solution by Kathleen Lewis, slightly edited.

(i) First, we need to choose the set of n non-consecutive integers to be permuted. For this we create a bijection between sets of n integers from $\{1, \dots, N-n+1\}$ and sets of n non-consecutive integers from $\{1, \dots, N\}$. Suppose we have a set of n integers from $\{1, \dots, N-n+1\}$. Put the integers in increasing order and add 0 to the first number, 1 to the second number, and so on, up to adding $n-1$ to the last number. This gives us a set of n non-consecutive numbers from $\{1, \dots, N\}$. In reverse, given such a set of non-consecutive numbers, we can subtract $0, 1, \dots, n-1$ respectively to obtain n distinct numbers from $\{1, \dots, N\}$. Therefore there are $\binom{N-n+1}{n}$ such sets. Once the set is chosen, there are $n!$ ways to permute it. Altogether we obtain

$$\binom{N-n+1}{n} \cdot n! = \frac{(N-n+1)!}{(N-2n+1)!}$$

permutations.

(ii) [*Note that a number k is consecutive with both $k-1$ and $k+1$.*] We consider the set of all permutations of $n+1$ distinct integers from $\{1, \dots, N\}$ so that no two of the first n integers are consecutive. If we place no restriction on the last integer, then we have $\frac{(N-n+1)!}{(N-2n+1)!}$ choices for the first n integers of the permutation (by the first part) and $N-n$ for the last integer, in total $\frac{(N-n+1)!}{(N-2n+1)!}(N-n)$. From these we subtract the number of permutations where the last integer is not consecutive with any of the first n integers either; by part (i) there are $\frac{(N-n)!}{(N-2n-1)!}$ such permutations. We obtain

$$\frac{(N-n+1)!}{(N-2n+1)!}(N-n) - \frac{(N-n)!}{(N-2n-1)!}.$$

4292. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and let $A_1 \in BC, B_1 \in CA, C_1 \in AB$ be the feet of its altitudes. Suppose further that X, Y, Z are the incenters of triangles AC_1B_1, BA_1C_1 and CB_1A_1, respectively. Show that the given triangle is equilateral if and only if $\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ} = \overrightarrow{0}$.

All ten of the submissions that we received were correct; we feature the solution by the AN-anduud Problem Solving Group.

($\Rightarrow$) If ΔABC is equilateral, then the line segments AX, BY, CZ have the same length and are perpendicular to the opposite sides, which easily implies

$$\overrightarrow{AX} + \overrightarrow{BY} + \overrightarrow{CZ} = \overrightarrow{0}. \tag{1}$$

($\Leftarrow$) For the interesting part of the theorem we assume that (1) holds and will prove that the altitudes must coincide with the medians.

First we recall that triangles AB_1C_1 and ABC are oppositely similar. [For a proof, note that AC_1HB_1 is cyclic so that $\angle AB_1C_1 = \angle AHC_1 = \angle CBA$, etc.] From the right triangle ACC_1 we see that the ratio of similarity is

$$\frac{AC_1}{AC} = \cos A.$$

Similarly, $\Delta A_1BC_1 \sim \Delta ABC$ so that

$$\frac{BC_1}{BC} = \cos B.$$

Because X is the incenter of ΔAB_1C_1, it follows that the incenter I of ΔABC satisfies

$$\overrightarrow{AX} = \cos A \cdot \overrightarrow{AI}.$$

Similarly,

$$\overrightarrow{BY} = \cos B \cdot \overrightarrow{BI} \quad \text{and} \quad \overrightarrow{CZ} = \cos C \cdot \overrightarrow{CI}.$$

From (1), we get

$$(\cos A) \cdot \overrightarrow{AI} + (\cos B) \cdot \overrightarrow{BI} + (\cos C) \cdot \overrightarrow{CI} = \overrightarrow{0}.$$

This equation tells us that I is the center of mass of triangle ABC with weights $\cos A, \cos B, \cos C$ placed at the vertices. Consequently, the intersection D of AB with CI satisfies

$$\frac{AD}{BD} = \frac{\cos B}{\cos A}. \tag{2}$$

On the other hand, because D is the foot of the angle bisector from C we have

$$\frac{AD}{BD} = \frac{AC}{BC}. \tag{3}$$

Equations (2) and (3) imply that

$$(\cos A) \cdot AC = (\cos B) \cdot BC.$$

But we have seen that

$$\cos A = \frac{AC_1}{AC} \quad \text{and} \quad \cos B = \frac{BC_1}{BC};$$

hence, we get $AC_1 = C_1B$. Similarly, $AB_1 = B_1C$ and $CA_1 = A_1B$. Consequently, in triangle ABC the altitudes coincide with the medians, whence ΔABC is equilateral.

Editor's comment. Apparently, the given triangle was assumed to be acute in order to simplify the diagram. None of the submitted solutions referred to the restriction.

4293. *Proposed by Eugen Ionascu.*

Let ϕ be the golden ratio. Prove that there exist infinitely many $0-1$ sequences $(x_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} \frac{x_n}{\phi^n} = 1.$$

We received 8 submissions, of which 7 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

We claim that, for any $k \geq 1$, we have $\sum\limits_{i=1}^{k} \frac{1}{\phi^{2i-1}} + \frac{1}{\phi^{2k}} = 1$; the result will follow. We proceed by induction on k.

It is known that $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$ (can also be easily verified from $\phi = \frac{1+\sqrt{5}}{2}$), so the equality holds for $k = 1$. If the result holds for $k \geq 1$ then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{\phi^{2i-1}} + \frac{1}{\phi^{2(k+1)}} &= \sum_{i=1}^{k} \frac{1}{\phi^{2i-1}} + \frac{1}{\phi^{2k+1}} + \frac{1}{\phi^{2(k+1)}} \\
&= \sum_{i=1}^{k} \frac{1}{\phi^{2i-1}} + \frac{1}{\phi^{2k}} \left(\frac{1}{\phi} + \frac{1}{\phi^2} \right) \\
&= \sum_{i=1}^{k} \frac{1}{\phi^{2i-1}} + \frac{1}{\phi^{2k}} \\
&= 1,
\end{aligned}$$

where the last equality holds by the induction hypothesis. The claimed equality follows by the principle of induction. Therefore, for any fixed $k \geq 1$, the sequence (x_n) defined by $x_n = 1$ for $n = 1, 3, \ldots, 2k-1$ and $n = 2k$, and $x_n = 0$ otherwise satisfies the condition $\sum\limits_{n=1}^{\infty} \frac{x_n}{\phi^n} = 1$.

4294. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be an isosceles triangle with $AB = AC$. Let D be a point on the side AC such that $CD = 2DA$. Let P be a point on BD such that $PA \perp PC$. Prove that

$$\frac{BP}{PD} = \frac{3BC^2}{4AC^2}.$$

We received 12 solutions, all correct, and present two of them.

Solution 1, by C.R. Pranesachar.

Let $BC = a$ and $AB = AC = b$ be the side lengths of triangle ABC, and let $BD = d$. By the Law of Cosines applied to triangle ABD one has

$$\begin{aligned} BD^2 &= AB^2 + AD^2 - 2 \cdot AB \cdot AD \cdot \cos A \\ &= b^2 + \frac{b^2}{9} - 2 \cdot b \cdot \frac{b}{3} \cdot \left(\frac{2b^2 - a^2}{2b^2}\right) \\ &= \frac{1}{9}(4b^2 + 3a^2), \end{aligned}$$

giving

$$9d^2 - 4b^2 = 3a^2. \qquad (1)$$

Since $\angle APC = 90^\circ$, the circle Γ with AC as the diameter passes through P. If M is the midpoint of BC, then $\angle AMC = 90^\circ$, and so Γ also passes through M. Draw the chord AQ parallel to MC, so that quadrilateral $AQCM$ is a rectangle inscribed in Γ. So $AQ = MC = \frac{1}{2}BC$.

We shall now show that B, D, Q are collinear: Note that triangles DAQ and DCB are similar (because $\angle DAQ = \angle DCB$ and $\frac{AD}{CD} = \frac{AQ}{CB} = \frac{1}{2}$). So $\angle BDC = \angle QDA$, which implies that B, D, Q are collinear; moreover, $d = BD = 2DQ$. By the theorem of intersecting chords, we have

$$PD = \frac{AD \cdot DC}{DQ} = \frac{\frac{b}{3} \cdot \frac{2b}{3}}{\frac{d}{2}} = \frac{4b^2}{9d}.$$

It follows, with the help of (1), that

$$BP = BD - PD = d - \frac{4b^2}{9d} = \frac{9d^2 - 4b^2}{9d} = \frac{3a^2}{9d}.$$

Finally,

$$\frac{BP}{PD} = \frac{3a^2/9d}{4b^2/9d} = \frac{3a^2}{4b^2},$$

as desired.

Solution 2, by Roy Barbara.

We introduce Cartesian coordinates with origin the midpoint of BC; for $m > 0$ define

$$B(-3, 0), \quad C(3, 0), \quad \text{and} \quad A(0, 3m).$$

We then get $D(1, 2m)$, and we set $P(p, q)$. Note that $p < 0$ (since $p \geq 0$ makes $\angle APC$ obtuse). From $BC^2 = 36$ and $AC^2 = 9(1 + m^2)$ we get $\frac{3BC^2}{4AC^2} = \frac{3}{1+m^2}$. Hence, the problem is reduced to proving that

$$\frac{BP}{PD} = \frac{3}{1 + m^2}.$$

The equation of line BD is $y = \frac{m}{2}x + \frac{3m}{2}$. Hence,

$$q = \frac{m}{2}p + \frac{3m}{2}. \qquad (2)$$

From $AP^2 + PC^2 = AC^2$ we easily calculate

$$p^2 + q^2 = 3p + 3mq. \qquad (3)$$

Inserting in (3) the value of q given by (2) yields

$$(m^2 + 4)p^2 - 12p - 9m^2 = 0,$$

whose negative root is $p = \dfrac{-3m^2}{m^2 + 4}$.

Let P' and D' be the respective projections of P and D onto the x-axis. We then have

$$\frac{BP}{PD} = \frac{BP'}{P'D'} = \frac{p - (-3)}{1 - p} = \frac{\left(\frac{-3m^2}{m^2+4}\right) - (-3)}{1 - \left(\frac{-3m^2}{m^2+4}\right)} = \frac{3}{1 + m^2},$$

as desired.

4295. *Proposed by Khang Nguyen Thanh.*

Let a, b and c be distinct non-zero real numbers such that $\frac{b}{a} + \frac{c}{b} + \frac{a}{c} = 3$. Find the maximum value of

$$P = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

We received five correct solutions and six incorrect solutions. Some solvers did not take note that a, b, c were to be distinct, and some just established an upper bound. Kee-Wai Lau, Digby Smith, and the proposer followed the approach of Solution 1, below.

Solution 1.

Without loss of generality, suppose that $c = 1$. The given condition is

$$b^2 + a(a - 3)b + a = 0.$$

The values of a for which there exist real b satisfying the equation are given by $a^2(a - 3)^2 - 4a = a(a - 1)^2(a - 4) \geq 0$. The case $a = 1$ corresponds to $a = b = c$, which is excluded. Hence, either $a < 0$ or $a \geq 4$.

We have that

$$\begin{aligned} P &= a\left(\frac{1}{b}+\frac{b}{a}\right)+\frac{1}{a} = a\left(\frac{1}{b}+\frac{b}{a}+a\right)+\left(\frac{1}{a}-a^2\right) \\ &= 3a+\frac{1}{a}-a^2 = \frac{1+3a^2-a^3}{a} \\ &= \frac{(4-a)(1+2a)^2}{4a}-\frac{15}{4} \le -\frac{15}{4}, \end{aligned}$$

since $(4-a)/a \le 0$. Since $P = -15/4$ when $(a,b,c) = (4,-2,1)$, the required maximum is $-15/4$.

Solution 2, by Leonard Giugiuc.

Let $(u,v,w) = (b/a, c/b, a/c)$. Then $u+v+w = 3$ and $uvw = 1$, so that u, v, w are roots of the cubic $f(x) = x^3-3x^2+Px-1$ where $P = uv+vw+wu$ is the quantity to be maximized.

The possible values of P are those for which $f(x)$ has three real roots, counting multiplicity. When $P = 1$, then $f(x) = (x-1)^3$ which corresponds to $u = v = w = 1$ and the situation we have excluded. Otherwise, the possible values of P are those for which the equation $g(x) = P$ has three solutions, not all equal (counting multiplicity), where $g(x) = -x^2+3x+x^{-1}$. Since $g'(x) = -x^{-2}(x-1)^2(2x+1)$, we see that $g(x)$ increases on $(-\infty,-1/2)$ to a local maximum $g(-1/2) = -15/4$, and decreases on $(-1/2,0)$ as well as on $(0,\infty)$. A graphical analysis indicates that $P \le -15/4$ when there are three roots. The maximum is attained when $(u,v,w) = (-1/2,-1/2,4)$ which occurs for example when $a:b:c = 4:-2:1$.

4296. *Proposed by Marius Drăgan.*

Prove that the following inequality holds for every triangle ABC:

$$5\sum_{cyclic}\tan^4\frac{A}{2}\tan^4\frac{B}{2} - 4\sum_{cyclic}\tan^5\frac{A}{2}\tan^5\frac{B}{2} \ge \frac{11}{81}.$$

We received eleven submissions, all correct. We present the solution by Nghia Doan, modified slightly by the editor.

Let

$$x = \tan\frac{A}{2}\tan\frac{B}{2}, \quad y = \tan\frac{B}{2}\tan\frac{C}{2}, \quad z = \tan\frac{C}{2}\tan\frac{A}{2}.$$

Then $x,y,z > 0$ and it is well known that

$$x+y+z = 1 \tag{1}$$

Note that for any u,v,w, if we set $a = u+v+w$, $b = uv+vw+wu$, and $c = uvw$, then

$$\begin{aligned}
&u^4 + v^4 + w^4 \\
&= (u^2 + v^2 + w^2)^2 - 2(u^2v^2 + v^2w^2 + w^2u^2) \\
&= \left((u + v + w)^2 - 2(uv + vw + wu)\right)^2 - 2\left((uv + vw + wu)^2 - 2uvw(u + v + w)\right) \\
&= (a^2 - 2b)^2 - 2(b^2 - 2ac) = a^4 - 4a^2b + 2b^2 + 4ac. \qquad (2)
\end{aligned}$$

Next,

$$\begin{aligned}
u^3 + v^3 + w^3 &= (u + v + w)(u^2 + v^2 + w^2 - uv - vw - wu) + 3uvw \\
&= a(a^2 - 3b) + 3c = a^3 - 3ab + 3c, \qquad (3)
\end{aligned}$$

so

$$u^5 + v^5 + w^5 = (u + v + w)(u^4 + v^4 + w^4) - \sum_{cyc}\left(u^2v + uv^2\right). \qquad (4)$$

Note that

$$\begin{aligned}
\sum_{cyc}(u^4v + uv^4) &= \sum_{cyc}\left(uv(u^3 + v^3 + w^3) - uvw^3\right) \\
&= (uv + vw + wu)(u^3 + v^3 + w^3) - uvw(u^2 + v^2 + w^2) \\
&= b(a^3 - 3ab + 3c) - c(a^2 - 2b) = a^3b - 3ab^2 - a^2c + 5bc. \qquad (5)
\end{aligned}$$

From (2) – (5) we then have

$$\begin{aligned}
u^5 + v^5 + w^5 &= a(a^4 - 4a^2b + 2b^2 + 4ac) - (a^3b - 3ab^2 - a^2c + 5bc) \\
&= a^5 - 5a^3b + 5ab^2 + 5a^2c - 5bc. \qquad (6)
\end{aligned}$$

If we set $u = x$, $v = y$, and $w = z$, then $a = 1$ so we obtain from (2) and (6) that

$$x^4 + y^4 + z^4 = 1 - 4b + 2b^2 + 4c \quad \text{and} \quad x^5 + y^5 + z^5 = 1 - 5b + 5b^2 + 5c - 5bc,$$

so

$$5(x^4 + y^4 + z^4) - 4(x^5 + y^5 + z^5) = 1 - 10b^2 + 20bc.$$

Thus, the given inequality is equivalent to

$$1 - 10b^2 + 20bc \geq \frac{11}{81}, \quad \text{or} \quad b(b - 2c) \leq \frac{7}{81}. \qquad (7)$$

Now,

$$x^3 + y^3 - (x^2y + xy^2) = x^2(x - y) + y^2(y - x) = (x + y)(x - y)^2 \geq 0,$$

so $x^3 + y^3 \geq x^2y + xy^2$. Adding up with two other similar inequalities, we get

$$2(x^3 + y^3 + z^3) \geq \sum_{cyc}\left(x^2y + xy^2\right) \qquad (8)$$

Note that

$$(x+y+z)^3 = x^3+y^3+z^3+3\sum_{cyc}\left(x^2y+xy^2\right)+6xyz, \tag{9}$$

and

$$(x+y+z)(xy+yz+zx) = \sum_{cyc}\left(x^2y+xy^2\right)+3xyz. \tag{10}$$

From (8) – (10) we then have

$$\begin{aligned}2(x+y+z)^3+9xyz &= 2(x^3+y^3+z^3)+6\sum_{cyc}\left(x^2y+xy^2\right)+21xyz\\ &\geq 7\sum_{cyc}\left(x^2y+xy^2\right)+21xyz = 7(x+y+z)(xy+yz+zx).\end{aligned}$$

That is, $2+9c \geq 7b$, or $b \leq \frac{2+9c}{7}$, or

$$b(b-2c) \leq \left(\frac{2+9c}{7}\right)\left(\frac{2-5c}{7}\right) = \frac{1}{49}(-45c^2+8c+4).$$

Since $-45c^2+8c+4$ attains its maximum at $c=\frac{4}{45} > \frac{1}{27}$, and

$$0 < c < \left(\frac{x+y+z}{3}\right)^3 = \frac{1}{27},$$

it follows that

$$\begin{aligned}b(b-2c) &\leq \frac{1}{49}\left(-45\left(\frac{1}{27}\right)^2+8\left(\frac{1}{27}\right)+4\right)\\ &= \frac{1}{49}\left(\frac{-45+216+2916}{27^2}\right) = \frac{3087}{49\cdot 27^2} = \frac{7}{81},\end{aligned}$$

proving (7) and completing the proof.

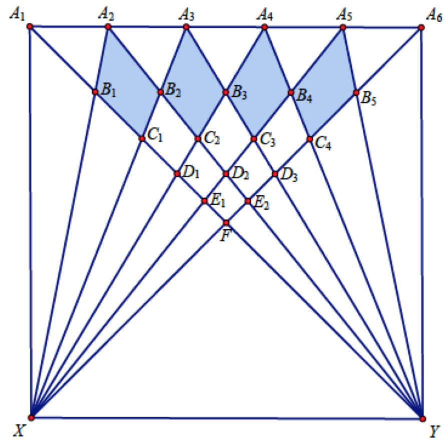
Editor's comments. From the proof, it is clear that in general, if $x, y, z \geq 0$ such that $x+y+z=1$, then

$$5(x^4+y^4+z^4)-4(x^5+y^5+z^5) \geq \frac{11}{81}$$

with equality if and only if $x=y=z=\frac{1}{3}$.

4297. *Proposed by Arsalan Wares.*

Suppose polygon A_1A_6YX is a square. Points A_2, A_3, A_4 and A_5 divide side A_1A_6 into five equal parts. Point X is connected to points A_2, A_3, A_4, A_5 and A_6, and point Y is connected to points A_1, A_2, A_3, A_4 and A_5. Points B_i, C_i, D_i, E_i and F are points of intersections of line segments shown in the figure (on the next page).



Find the ratio of the sum of the areas of the shaded quadrilaterals (namely, quadrilaterals $B_1A_2B_2C_1$, $B_2A_3B_3C_2$, $B_3A_4B_4C_3$ and $B_4A_5B_5C_4$) to the area of square A_1A_6YX.

We received 11 submissions; all were correct and most were similar to our featured solution, which is a composite of almost identical solutions from Chudamani P. Anilkumar and Nghia Doan.

Let us assume (without loss of generality) that the side length of the square A_1A_6YX is 1. For $1 \leq j \leq 5$ the triangles $A_jB_jA_{j+1}$ and YB_jX are similar with

$$\frac{A_jA_{j+1}}{XY} = \frac{1}{5}$$

as their ratio of magnification. Of course, their altitudes are also in the ratio $1:5$ and, since the altitudes of each corresponding pair sum to 1, the altitude of $\Delta A_jB_jA_{j+1}$ must be $\frac{1}{6}$. Consequently, for each j the area of the triangle $A_jB_jA_{j+1}$ equals

$$\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{60}.$$

Similarly, for $1 \leq j \leq 5$,

$$\Delta A_jC_jA_{j+2} \sim \Delta YC_jX \quad \text{with} \quad \frac{A_jA_{j+2}}{XY} = \frac{2}{5}$$

as the ratio of magnification. Arguing as before, we get that the area of triangle $A_jC_jA_{j+2}$ equals

$$\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{2}{7} = \frac{2}{35}.$$

Hence the area of each of the four quadrilaterals $A_jB_jC_{j-1}B_{j-1}$ is $\frac{2}{35} - 2 \cdot \frac{1}{60} = \frac{1}{42}$, so that the desired ratio is

$$\frac{4 \cdot \frac{1}{42}}{1} = \frac{2}{21}.$$

Editor's comment. Several solvers observed that the same argument works when the side of the square opposite XY is partitioned into n equal segments. More generally, Konečný began with a trapezoid whose base XY has length b while the parallel side has length a and is partitioned into n equal segments A_jA_{j+1} of length $\frac{a}{n}$. An argument as in our featured solution shows that if the trapezoid has unit area, then a quadrilateral $A_jB_jC_{j-1}B_{j-1}$ has area

$$\frac{2a^2b}{(nb+2a)(nb+a)(a+b)}.$$

This reduces to our problem when $a = b$ and $n = 5$.

4298. *Proposed by Daniel Sitaru.*

Compute:

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n+k}{2+\sin(n+k)+(n+k)^2}.$$

We received 11 correct solutions. We present the solution by Missouri State University Problem Solving Group.

Define

$$f(n,k) = \frac{n+k}{2+\sin(n+k)+(n+k)^2} \text{ and } g(n,k) = \frac{1}{n+k}.$$

Since $1 \le 2+\sin(n+k) \le 3$, then for $1 \le k \le n$, we have

$$\begin{aligned} |g(n,k) - f(n,k)| &= \frac{2+\sin(n+k)}{(n+k)(2+\sin(n+k)+(n+k)^2)} \\ &\le \frac{3}{(n+k)(1+(n+k)^2)} \\ &\le \frac{3}{n^3}. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left| \sum_{k=1}^{n} g(n,k) - f(n,k) \right| \le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{3}{n^3} = \lim_{n\to\infty} \frac{3}{n^2} = 0.$$

In particular, we now have

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} f(n,k) = \lim_{n\to\infty} \sum_{k=1}^{n} g(n,k) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

Let $h(x) = \frac{1}{x}$. Since h is continuous on $[1,2]$, it is integrable on $[1,2]$. Therefore

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n\left(1+\frac{k}{n}\right)} = \lim_{n\to\infty} \sum_{k=1}^{n} h\left(1+\frac{k}{n}\right)\left(\frac{1}{n}\right) = \int_1^2 h(x)\,dx = \ln 2.$$

4299. *Proposed by Michel Bataille.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(xy + f(x+y)\right) = f\left(f(xy)\right) + x + y$$

for all $x, y \in \mathbb{R}$.

We received eight correct solutions and two incorrect solutions. We present two of the solutions here.

Solution 1, by Charles and David Diminnie (jointly).

It is easily verified that $f(x) = x + k$ satisfies the equation for each k. We show that there are no other solutions.

Setting $y = 0$ yields $f(f(x)) = x + a$ for each real x, where $a = f(f(0))$. Hence

$$f(xy + f(x+y)) = xy + x + y + a = f(f(xy + x + y)).$$

Since f is one-one and onto, we get

$$xy + f(x+y) = f(xy + x + y).$$

Let $(x, y) = (t+1, -1)$. Then $-(t+1) + f(t) = f(-1)$, whence $f(t) = t + k$ with $k = f(-1) + 1$.

Solution 2, by Missouri State University Problem Solving Group.

As in the previous solution, we find that $f(f(x)) = x + f(b)$ where $b = f(0)$. Let $y = -x$. Then

$$f(-x^2 + b) = f(f(-x^2)) \quad \text{for all} \quad x.$$

Since f is one-one, $-x^2 + b = f(-x^2)$. Hence, for $z \leq 0$, $f(z) = z + b$.

Let $z > 0$. The quadratic equation $t^2 - zt - z$ has two real solutions x and y for which $x + y = z = -xy$. Therefore

$$f(-z + f(z)) = f(f(-z)) + z = -z + f(b) + z = f(b)$$

so that, applying the inverse of f, we get $f(z) = z + b$.

Editor's comment. As a variant on the second solution, we could set

$$(x, y) = (t, -t(t+1)^{-1}),$$

whereupon $xy + x + y = 0$. Feeding this into the equation and applying the inverse of f leads to

$$f(t^2(t+1)^{-1}) = t^2(t+1)^{-1} + f(0).$$

However, the function $t^2(t+1)^{-1}$ does not assume values in the interval $(-4, 0)$ and a separate argument is needed to finish the job.

4300$\star$. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers with $a+b+c = ab+bc+ca > 0$. Prove or disprove that

$$\sqrt{24ab+25}+\sqrt{24bc+25}+\sqrt{24ca+25} \geq 21.$$

We received 3 solutions. We present the solution by Shuborno Das.

Let $f(x) = \sqrt{24x^2+25}$. Then we have

$$f'(x) = \frac{24x}{\sqrt{24x^2+25}},$$

and

$$f''(x) = \frac{600}{(24x^2+25)^{\frac{3}{2}}}.$$

Therefore $f''(x) > 0$ for all $x \geq 0$, which implies that $f(x)$ is convex for $x \geq 0$.

From Jensen's inequality,

$$f(a)+f(b)+f(c) \geq 3f(\frac{a+b+c}{3}),$$

which implies

$$\sqrt{24ab+25}+\sqrt{24bc+25}+\sqrt{24ca+25} \geq 3\sqrt{24\left(\frac{a+b+c}{3}\right)^2+25}.$$

Hence we need

$$3\sqrt{24\left(\frac{a+b+c}{3}\right)^2+25} \geq 21 \leftrightarrow 24\left(\frac{a+b+c}{3}\right)^2+25 \geq 49,$$

or

$$\left(\frac{a+b+c}{3}\right)^2 \geq 1 \leftrightarrow a+b+c \geq 3.$$

We have assumed that $a+b+c = ab+bc+ca$, and from the AM-GM inequality we have $a^2+b^2+c^2 \geq ab+ac+bc$. Thus

$$\begin{aligned} a+b+c = (a+b+c)\left(\frac{a+b+c}{ab+bc+ca}\right) &= \frac{(a+b+c)^2}{ab+bc+ca} \\ &= \frac{a^2+b^2+c^2+2(ab+bc+ca)}{ab+bc+ca} = \frac{a^2+b^2+c^2}{ab+bc+ca}+2 \\ &\geq \frac{ab+bc+ca}{ab+bc+ca}+2 = 3, \end{aligned}$$

which completes the proof.

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Solvers and proposers appearing in issues 9 and 10

(Bold font indicates featured solution.)

Proposers

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AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia : **4282**, 4285, **4286**, 4287, 4288, **4292**, 4293, 4294, 4297, 4298, 4299

Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA : 4288

Marian Cucoanes, Marasesti, and Marius Dragan, Bucharest, Romania : 4286

Charles Diminnie, Angelo State University, and David Diminnie, Texas Instruments Inc, Dallas, TX, USA : **4299**

Missouri State University Problem Solving Group, Springfield, MO, USA : CC292, CC295, CC296, CC298, **4283**, 4285, 4288, **4293**, 4294, **4298**, **4299**, OC346, **OC350**

Julio Mohnsam and Luiz Eduardo Richardt de Azevedo, Pelotas-RS, Brazil: CC292

Carlos Moreno, Colegio Teresiano, and Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain : **CC297**, CC300

Miguel Ochoa Sanchez, Peru, and Leonard Giugiuc, Romania : 4294

George Stoica, Saint John, New Brunswick, Canada and Eugen J. Ionascu, Columbus State University, USA: 4293

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