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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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ASIAN THOUGHTS

Andy Liu

The International Mathematical Olympiad (IMO) began as an East–European affair, but has become truly global in scope. The most marked increase in activity is occurring in Asia, with China hosting the event in 1990, Hong Kong in 1994, India in 1996, Taiwan in 1998 and South Korea in 2000.

I attended high school in Hong Kong, and have since maintained constant contact with mathematicians there. I had been the deputy leader of the USA IMO team from 1981 to 1984, and was a special guest of the organizing committee of the IMO in China. So when the competition came to Hong Kong, I was asked to lend a hand.

I set out from Edmonton on May 4, 1994, arriving at Hong Kong on May 5. I obtained my Taiwan visa on May 6, and flew to Taipei in the evening. I was invited to give a few talks to the Taiwan IMO team. I was very impressed by the fact that they had already been guaranteed acceptance into universities, on the basis of their achievements in mathematics competitions. They had been housed in the top high school in Taiwan since the Chinese New Year, and seemed to do little else but train.

I visited this school on May 7. I was greeted by the principal who showed a half-hour promotional video-tape about the school. It is for boys only, and has 33 classes of each of Grades 10, 11 and 12. Then I met the students, five boys and one girl, in the Resource Centre and had an informal discussion about problem solving. On May 8, which was a Sunday, I visited the National University of Taiwan, where some enrichment classes for high school students were taking place. I gave an impromptu short talk.

On May 9, Mr. W.H. Sun, the chief executive officer of Chiu Chang Mathematics Publishers, took me on a tour of the east coast of Taiwan. He is a mathematics graduate, and now a very successful businessman. His main source of income is supplying the schools with instructional material and stationery. Whatever profit he makes he channels back to his first love, mathematics. His company publishes nothing but mathematics books, primarily translations of outstanding titles from the West, but also works by local and mainland authors.

I had a lovely time, and we came back to Taipei on May 11. The next day saw the beginning of an intensive two-and-a-half days' training program, the second this year so far. The main topic was combinatorics, conducted by Prof. Y.N. Yeh of the Chinese Academy of Sciences, Taipei, and me, along with several other professors. Prof. Yeh and I were with the students for twenty hours. Seven hours were devoted to tests, which we graded. The rest of the time was allotted for lectures and problem-solving sessions. My own feeling is that this was a bit too much for the students, and they certainly tired out towards the end.

*

On May 16, I flew back to Hong Kong and began my work as a member of the Problem Selection Committee for the 35th IMO. The Committee was chaired by Prof. T.W. Leung of the Hong Kong Polytechnic. We stayed in the Chinese University of Hong

Kong for ten days. It was hard work for all concerned, going over the hundred plus problems submitted. However, it was also very enjoyable, with good problems and company, and we came up with a few alternative or improved solutions.

We received very late the problem proposals from Armenia, but in time for them to be considered. After working through them, we were astonished that one actually asks for the proof of the converse proposition in a problem submitted by Australia. We recommended this joint-problem, even though it deals only with straight lines and therefore yields to a sustained attack by the method of coordinates. It eventually became Problem 2 in the IMO. On the other hand, the proposals from Belarus did not arrive until late June, even though they were sent by surface mail in April. There was really not much we could do about them.

The first stage of work was concluded on May 26. A subcommittee of six stayed behind to compile and refine the work done so far. On May 31, the subcommittee was succeeded by the Problem Interpolation Group, which worked on the English wording of the problems and solutions, and the preparation of the book *Short-listed Problems for the 35th IMO.* Mr. K.K. Yeung, chief executive officer of Golden Cup Printing Co. Ltd., was responsible for the high physical quality of the finished product, at no charge to the IMO Committee.

The Committee was headed by Prof. K.Y. Chan, Chairman of the Department of Mathematics of the University of Hong Kong, and Prof. K.P. Shum of the Chinese University. The two provided outstanding leadership, but in very different ways. K.P., as a member of the IMO Site Committee, was instrumental in bringing the contest to Hong Kong. He epitomizes perpetual motion. Among other things, he was personally responsible for soliciting over HK$3,000,000 for the event. K.Y. is the unobtrusive helmsman who kept everything on the straight and narrow by making firm decisions at critical moments.

*

On June 30, Daniel van Vliet of Salisbury Composite High School, Sherwood Park, Alberta, joined me in Hong Kong. He was one of the two students whom I took to Beloretsk, Russia last year (see my article "A Mathematical Journey" in the January 1994 issue of *Crux*). Daniel had done a lot of computer work on a book which I translated from Chinese into English. Prof. Z.H. Qiu of the Chinese Academy of Sciences, Beijing, the author of the original Chinese version, invited Daniel to this magnificent metropolis.

We flew there on July 1, and met Prof. Peter Taylor, Head of the Mathematics Department of the University of Canberra and senior Vice-President of the International Mathematics Tournament of the Towns. Unfortunately, the President, Prof. Nikolay Konstantinov of the Independent University of Moscow, could not join us as arranged. So we held an informal discussion about China's intention in taking part in the Tournament, and asked Prof. Taylor to communicate our ideas to Prof. Konstantinov whom he would meet in Moscow later. Peter and I gave a talk each at the Academy.

The Chinese Mathematical Olympiad Committee hosted a welcoming dinner for Peter and me. Prof. Qiu ordered a dish specially for Daniel, spicy beef in broth. Daniel, who is proficient in using chopsticks, dug in. As it happened, the meat stuck together, and he picked up all but one piece. Everyone was deep in conversation and nobody noticed anything.

Eventually, Prof. Y.C. Xu got around to that dish. After fishing around for a little while, he asked, "Where is the beef?"

Prof. Qiu looked and found the last one. He declared triumphantly, "Here is a piece."

Prof. Xu looked again, and proclaimed, "There is no more."

It was then observed that they had unwittingly engaged in a mathematical collaboration, one establishing existence and the other proving uniqueness!

*

On July 8, Daniel and I returned to Hong Kong, at the same time as the IMO team leaders were arriving. Daniel continued to work on the book together with Mr. S.N. Suen, a high school mathematics teacher in Hong Kong. Mr. Suen was in charge of the guides for the IMO, and Daniel volunteered to serve as the guide to the Dutch team. So he moved to the camp where the students were to be housed, and I helped out in the Jury meeting.

The IMO was held in the Chinese University of Hong Kong. In retrospect, this was a mistake as the university administration did not support the event. They seemed to have the ridiculous notion that high school students were too lowly to be allowed on the hallowed grounds of their campus, which makes you wonder where their students came from. Apart from an appearance by the Chancellor at the opening ceremony, nobody, from the Vice-Chancellor down to the Chairman of the Department of Mathematics, met the contestants, many of whom did not come away with a favourable impression of the institution.

Only three colleagues of K.P. helped out, though they made up for the lack in number by their outstanding effort. Prof. Raymond Chan took on the onerous duty of running the office in the hotel where the leaders were housed. With over seventy countries present, he had to look after many people. His service was in constant demand, but his efficient handling of things was remarkable. Prof. C.W. Leung looked after the leaders when they were at the Chinese University, mainly arranging their meals and transportation. Prof. K.K. Au served as the leader of the Hong Kong team.

One of the reasons why the event was at the Chinese University was Sir Q.W. Lee, chairman of the Board of Governors. He was the main sponsor of the IMO, donating HK$2,000,000. From time to time, K.P. had to bother Sir Q.W. Lee to overcome obstruction from within. Most fortunately for us, the association of non-academic staffs, under the leadership of their president Mr. Y.M. Man, was on side and provided the IMO with much needed logistic support.

Speaking of looking after details, the highest praise goes to Prof. P.H. Cheung of the University of Hong Kong and Chairman of the Hong Kong IMO Committee since August, 1994. He did a lot of planning, such as the seating plans during the contest. He was in charge of the printing of the contest papers and putting those in the appropriate languages in the correct envelopes. There were so many other things he did that I can say the event would have had a much rougher ride without his many contributions.

The Jury meeting began on July 9. It did not run as smoothly as had been hoped, mainly because there was a shortage of combinatorics problems among those short-listed. It was really ironic that the Problem Interpolation Group had included seven of them, but five were found to have appeared in other sources or to be very similar to some that have. In the end, no combinatorics problems were selected. Prof. Jozsef Pelikan, the leader of the Hungarian team, was a great help to the local organizers, providing on important occasions multi-lingual translations and chairing two of the meetings.

I also had the occasion to chair one or two meetings. I kept reminding the assembly that only team leaders might participate in the discussion. Observers were not allowed to speak. I was bellowing this out at the top of my voice when it was pointed out that I was wearing a badge labelled "HKG 9", defining my status as an observer of the Hong Kong team!

*

The students were not welcome to stay in the Chinese University. In any case, the dormitories were not air-conditioned. They were in a holiday camp some distance away, and had to travel for the contest. On both July 13 and 14, one of the buses got held up for half an hour, thus delaying the contest for the same duration. Given the traffic conditions in Hong Kong, this was unavoidable.

The coordination process began in the evening of July 15. In recent years, the leaders had no input into the marking scheme, and on some occasions, they did not even know it while marking their students' scripts. At the request of the Jury, the Chief Coordinator, Prof. C.C. Yang, called a meeting to discuss this matter. C.C., President of the Hong Kong Mathematical Society since April, is from the Hong Kong University of Science and Technology. He was able to defuse the leaders' concerns, revising slightly the marking scheme for one of the questions in the process.

The beginning of the coordination was quite chaotic, mainly because the lateness of the dinner threw C.C.'s carefully planned schedule off balance. However, by the morning of July 16, it was running like clockwork. A few cases were referred to the Chief Coordinator. On the whole, most differences were resolved amicably. The only hitch was that two of the coordinating groups were not informed of the change in the marking scheme. Prof. Ron Turner-Smith, Assistant Chief Coordinator, had to sit up all night to double check scripts for that problem coordinated by those two groups.

Of their own accord, the leaders of the Czech and Moldavan teams informed Ron that some of their students appeared to have received extra marks due to the mix-up in the marking scheme. Ron did not deduct those marks, because he did not have the mandate to do so. He could only change the marks for those teams coordinated by those two coordinating groups, but not correct apparent mistakes made by other coordinators.

The final Jury meeting was held in the evening of July 17, at the beautiful campus of the four-year-old University of Science and Technology. C.C. was the gracious host and provided a sumptuous buffet dinner. It would have been ideal to hold the IMO here, with ultra-modern facilities from classrooms to dormitories. A large number of C.C.'s colleagues served as coordinators. Prof. T.M. Ko served as the deputy leader of the Hong Kong team.

It was felt that the paper was on the easy side, and all six members from the American team were among the twenty-two who obtained perfect scores. However, it seems reasonable that a contestant should get about 50% of the overall score to get a medal, and this took 19 points out of 42 this year. Problem 1 turned out to be the second hardest among the six, while Problem 3 was the second easiest. No special prizes were awarded. Although there were some nice solutions, too many contestants used the same approach.

*

On the last two days, the leaders finally got some rest and did some sightseeing. The farewell dinner was in the evening of July 19, funded by the Pui Kiu Alumni Association. Another book I translated, the "Chinese Mathematical Olympiad", was published by Chiu Chang of Taiwan. Copies were donated to the local organizers who used them as gifts to the team leaders and deputy leaders. It was also printed by Golden Cup. Mr. K.K. Yeung, a Pui Kiu alumnus himself, was on hand to present them.

Daniel rejoined me on July 20. We had budgeted five extra days to spend on our own, but it rained cats and dogs every day. Parts of Hong Kong were flooded, and there were a few mud-slides. Fortunately, the IMO was already over, as otherwise the camp would have closed and we would have had to relocate almost 400 students at very short notice. Daniel and I spent some time in a book exhibit, a computer mall (three floors, with over one hundred shops), a boat cruise (including dinner on a floating restaurant), and a few trips to the cinemas (including a Japanese comedy).

We came back to Canada on July 25. For me, it had been a hectic three months, but wonderful too, meeting old friends and making new acquaintances. It was also a most valuable experience seeing the IMO from the inside. I faced some situations in which things seemed to be tottering on the brink of disaster. Yet, looking back over the five times I had been to an IMO before, the same situations certainly existed there too. But since I had been an outsider then, they did not even register on my mind at the time. All is well that ends well!

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THE OLYMPIAD CORNER

No. 158

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Elsewhere in this issue of *Crux* you will find the very interesting account of Andy Liu's participation in the I.M.O. at Hong Kong in July. I am still waiting for information from my sources with the Canadian Olympiad team before preparing my own report on this year's Olympiad. Normally, too, we give the problems posed for last year's I.M.O. before giving the annual report. So this number of the Corner begins with problems proposed to the jury but not used at the 34th International Mathematical Olympiad at Istanbul, Turkey.

PROBLEMS PROPOSED AT ISTANBUL

1. *Proposed by Brazil.* Show that there exists a finite set $A \subset \mathbb{R}^2$ such that for every $X \in A$ there are points $Y_1, Y_2, \ldots, Y_{1993}$ in A such that the distance between X and Y_i is equal to 1, for every i.

2. *Proposed by Canada.* Let triangle ABC be such that its circumradius $R = 1$. Let r be the inradius of ABC and let p be the inradius of the orthic triangle $A'B'C'$ of triangle ABC. Prove that $p \leq 1 - 1/3(1+r)^2$.

3. *Proposed by Spain.* Consider the triangle ABC, its circumcircle k of center O and radius R, and its incircle of center I and radius r. Another circle k_c is tangent to the sides CA, CB at D, E, respectively, and it is internally tangent to k. Show that the incenter I is the midpoint of DE.

4. *Proposed by Georgia.* Let a, b, c be given integers $a > 0$, $ac - b^2 = P = P_1 \ldots P_n$ where $P_1, \ldots, P_n$ are (*distinct*) prime numbers. Let $M(n)$ denote the number of pairs of integers (x, y) for which

$$ax^2 + 2bxy + cy^2 = n.$$

Prove that $M(n)$ is finite and $M(n) = M(P^k \cdot n)$ for every integer $k \geq 0$.

5. *Proposed by India.* (a) Show that the set $\mathbb{Q}^+$ of all positive rationals can be partitioned into three disjoint subsets A, B, C satisfying the following conditions:

$$BA = B; \quad B^2 = C; \quad BC = A;$$

where HK stands for the set $\{hk : h \in H,\ k \in K\}$ for any two subsets H, K of $\mathbb{Q}^+$ and H^2 stands for HH.

(b) Show that all positive rational cubes are in A for such a partition of $\mathbb{Q}^+$.

(c) Find such a partition $\mathbb{Q}^+ = A \cup B \cup C$ with the property that for no positive integer $n \leq 34$, both n and $n+1$ are in A, that is,

$$\min\{n \in \mathbb{N} : n \in A,\ n+1 \in A\} > 34.$$

6. *Proposed by Ireland.* Let n, k be positive integers with $k \leq n$ and let S be a set containing distinct real numbers. Let T be the set of all real numbers of the form $x_1 + x_2 + \cdots + x_k$ where $x_1, x_2, \ldots, x_k$ are distinct elements of S. Prove that T contains at least $k(n-k)+1$ distinct elements.

7. *Proposed by Israel.* The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides, and S is the area of ABC, prove that

$$DE \geq 2\sqrt{2}\,S \cdot \{a^2 + b^2 + c^2 + 4\sqrt{3}\,S\}^{-1/2}.$$

8. *Proposed by Macau.* Let $n \in \mathbb{N}$, $n \geq 2$ and $A_0 = (a_{01}, a_{02}, \ldots, a_{0m})$ be any n-tuple of natural numbers such that $0 \leq a_{0i} \leq i-1$, for $i = 1, \ldots, n$. n-tuples $A_1 = (a_{11}, a_{12}, \ldots, a_{1n})$, $A_2 = (a_{21}, a_{22}, \ldots, a_{2n}), \ldots$ are defined by: $A_{i+1j} = \text{Card}\ \{a_{i,\ell} \mid 1 \leq \ell \leq j-1,\ a_{i,\ell} \geq a_{ij}\}$, for $i \in \mathbb{N}$ and $j = 1, \ldots, n$. Prove that there exists $k \in \mathbb{N}$, such that $A_{k+2} = A_k$.

9. *Proposed by Poland.* Let S_n be the number of sequences $(a_1, a_2, \ldots, a_n)$, where $a_i \in \{0, 1\}$, in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ when $n \to \infty$.

10. *Proposed by Romania.* Let a, b, n be positive integers, $b > 1$ and $b^n - 1 | a$. Show that the representation of the number a in the base b contains at least n digits different from zero.

* * *

In the June number we began solutions to the problems of the 22nd Austrian Mathematical Olympiad 1991 and promised to give the rest "next month". Since *Crux* doesn't appear in the summer, readers will have applied the word month liberally, but space did not permit their treatment in the September number either, so we begin them this number with apologies that 1 appears to equal 3. Let me also apologize to Christopher J. Bradley, Clifton College, Bristol, U.K., who submitted solutions to problems 1 and 2 of Part 2 of the contest but which got misfiled with the Final Round solutions. Here are readers' solutions to problems of the *Final Round* of the 22*nd Austrian Mathematical Olympiad* 1991 [1993: 101].

2. Determine all functions $f : \mathbb{Z} \setminus \{0\} \to \mathbb{Q}$ satisfying the functional equation

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}\,.$$

Solutions by Seung-Jin Bang, Albany, California; by Joseph Ling, The University of Calgary; by Waldemar Pompe, student, University of Warsaw, Poland; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. The solutions were very much alike. We use Ling's write-up.

We prove that f is a constant function. We first note that

1. $f(1) = f\left(\frac{1+2}{3}\right) = \frac{f(1)+f(2)}{2} \Rightarrow f(1) = f(2)$;

2. $f(2) = f\left(\frac{3+3}{3}\right) = \frac{f(3)+f(3)}{2} = f(3)$.

Next, we suppose that for some $k \geq 1$, $f(1) = f(2) = \cdots = f(3k)$. Then, we have

3. $f(2) = f(k+1) = f\left(\frac{3k+1+2}{3}\right) = \frac{f(3k+1)+f(2)}{2} \Rightarrow f(3k+1) = f(2)$;

4. $f(1) = f(k+1) = f\left(\frac{3k+2+1}{3}\right) = \frac{f(3k+2)+f(1)}{2} \Rightarrow f(3k+2) = f(1)$;

5. $f(3) = f(k+2) = f\left(\frac{3k+3+3}{3}\right) = \frac{f(3k+3)+f(3)}{2} \Rightarrow f(3k+3) = f(3)$.

Hence $f(1) = f(2) = f(3) = \cdots = f(3k) = f(3k+1) = f(3k+2) = f(3k+3)$.

This induction argument shows that f is constant on the positive integers. Finally, if x is a negative integer, then $-x+3$ is positive, and so

$$f(1) = f\left(\frac{x+(-x+3)}{3}\right) = \frac{f(x)+f(-x+3)}{2} = \frac{f(x)+f(1)}{2} \Rightarrow f(x) = f(1).$$

Therefore f is a constant.

It is obvious that all constant (rational valued) functions satisfy the given equation.

3. (a) Show that $91 \mid n^{37} - n$ for all $n \in \mathbb{N}$.
(b) Determine the greatest integer k such that $k \mid n^{37} - n$ for all $n \in \mathbb{N}$.

Solutions to part (a) *only were sent in by Seung-Jin Bang, Albany, California; and by John Morvay, Springfield, Missouri; solutions to both parts were submitted by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; Waldemar Pompe, student, University of Warsaw, Poland; by Michael Selby, University of Windsor; by P. Tsaussoglou, Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We give the solution by Bulman–Fleming and Wang.*

(a) By Fermat's little theorem $n^7 \equiv n \bmod 7$ for all $n \in \mathbb{N}$ and thus

$$n^{37} = (n^7)^5 \cdot n^2 \equiv n^5 \cdot n^2 = n^7 \equiv n \bmod 7,$$

showing that $7 | n^{37} - n$. Similarly

$$n^{13} \equiv n \bmod 13 \Rightarrow n^{26} \equiv n^2 \Rightarrow n^{37} \equiv n^{11} n^2 \equiv n^{13} \equiv n \bmod 13,$$

showing that $13 | n^{37} - n$. Since $91 = 7 \times 13$, the result follows.

(b) We claim that $k = 2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37 = 1919190$. First of all the prime power decompositions of $2^{37}-2$ and $3^{37}-3$ are

$$\begin{aligned} 2^{37}-2 &= 2(2^{18}-1)(2^{18}+1) \\ &= 2\cdot 3^3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73\cdot 109 \end{aligned}$$

and

$$\begin{aligned} 3^{37}-3 &= 3(3^{18}-1)(3^{18}+1) \\ &= 2^4\cdot 3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37\cdot 73\cdot 757\cdot 530713 \end{aligned}$$

As in (a), we use Fermat's little theorem repeatedly:

$$\begin{aligned} &n^2 \equiv n \bmod 2 \Rightarrow n^9 \equiv n \bmod 2 \text{ for all } q\in\mathbb{N} \Rightarrow n^{37}\equiv n \bmod 2. \\ &n^3 \equiv n \bmod 3 \Rightarrow n^{36}\equiv n^{12}\equiv n^4\equiv n^2 \Rightarrow n^{37}\equiv n^3\equiv n \bmod 3. \\ &n^5\equiv n \bmod 5 \Rightarrow n^{35}\equiv n^7\equiv n^3 \bmod 5 \Rightarrow n^{37}\equiv n^5\equiv n \bmod 5. \\ &n^{37}\equiv n \bmod 37 \text{ by the theorem itself.} \end{aligned}$$

In view of the result in (a), it remains to show that it is not true that $73|n^{37}-n$ for all $n\in\mathbb{N}$. Note that $5^{37}-5 = 5(5^{18}-1)(5^{18}+1)$. Since $5^6 = 15625 \equiv 3 \bmod 73$, we have $5^{18}-1\equiv 26$ and $5^{18}+1\equiv 28 \bmod 73$ and thus $5^{37}\not\equiv 5 \bmod 73$. This completes the proof.

4. The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 0$ and $a_{2k+1} = a_k + a_{k+1}$, $a_{2k+2} = 2a_{k+1}$, $k\geq 1$. Determine a_m, where $m = 2^{19}+91$.

Solutions by Seung-Jin Bang, Albany, California; and by Chris Wildhagen, Rotterdam, The Netherlands. We use Wildhagen's solution.

We show by induction on $k\geq 1$ that for $n = 2^k+j$, where $0\leq j\leq 2^k$ we have

$$a_n = \begin{cases} j & \text{if } 0\leq j\leq 2^{k-1} \\ 2^k - j & \text{if } 2^{k-1}\leq j\leq 2^k. \end{cases}$$

Now $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 0$ so that the statement holds for $k = 1$.

If $0\leq t\leq 2^{k+1}$ and $n = 2^{k+1}+t$ then if $0\leq t\leq 2^k$ we have

$$a_{2^{k+1}+t} = \begin{cases} 2a_{2^k+t/2} = 2(\frac{t}{2}) = t, & \text{t even} \\ a_{2^k+[t/2]} + a_{2^k+[t/2]+1} = [t/2]+[t/2]+1 = t, & \text{t odd.} \end{cases}$$

Similarly if $2^k\leq t\leq 2^{k+1}$

$$a_{2^{k+1}+t} = \begin{cases} 2a_{2^k+t/2} = 2(2^k - t/2) = 2^{k+1}-t, & \text{t even} \\ a_{2^k+[t/2]} + a_{2^k+[t/2]+1} = 2^k - [t/2] + 2^k - [t/2] - 1 = 2^{k+1}-t, & \text{t odd.} \end{cases}$$

Thus $a_{2^{19}+91} = 91$.

5. Show that for all natural numbers $n>1$ the inequality

$$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$$

is valid.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

We claim
$$x-1<\left(\frac{1+x^x}{x+1}\right)^{1/(x-1)}<x \quad \text{for} \quad x>1.$$
In fact, for $x>1$
$$x^x+x^{x-1}>x^x+1$$
i.e.,
$$x^{x-1}(1+x)>x^x+1$$
whence
$$x^{x-1}>\frac{x^x+1}{x+1}$$
and
$$x>\left(\frac{1+x^x}{x+1}\right)^{1/(x-1)}.$$
And, for $x\geq 2$
$$(x^2-1)(x-1)^{x-2}<(x^2-1)x^{x-2}=x^x-x^{x-2}<1+x^x.$$
Thus $(x-1)^{x-1}(x+1)<1+x^x$ from which
$$x-1<\left(\frac{1+x^x}{x+1}\right)^{1/(x-1)}.$$
It follows that if n is an integer greater than 1
$$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{1/n}>\left(\frac{1+n^n}{n+1}\right)^{1/(n-1)}$$
begin separated by the integer n. The required result follows.

6. Determine the number of numbers $(a_9\dots a_0)_{10}$ which have no initial zeros and do not contain the block of digits 1991, when written in decimal notation.

Solutions by Edward T.H. Wang and Siming Zhan, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands.

Let n denote the number asked for, and let S denote the set of numbers $(a_9\dots a_0)_{10}$ with no initial zeros. Let $B\subset S$ be the subset of those numbers which contain one or more blocks of 1991. For any number in B label the positions of its digits from left to right so a_9 is in the first position, a_8 is in the second position, etc.

For each $i=1,2,\dots,7$ define
$$B_i=\{x\in B:\ \text{the leading one in } \textit{some} \text{ block of } 1991 \text{ in } x \text{ is in the } i\text{th position}\}.$$

Then by simple counting we find that

$$|B_1| = 10^6, \quad |B_i| = 9 \times 10^5 \text{ for all } i = 2, 3, \ldots, 7;$$

$$|B_1 \cap B_4| = 10^3,$$

$$|B_1 \cap B_5| = |B_1 \cap B_6| = |B_1 \cap B_7| = 10^2$$

$$|B_2 \cap B_5| = |B_3 \cap B_6| = |B_4 \cap B_7| = 9 \times 10^2$$

$$|B_2 \cap B_6| = |B_2 \cap B_7| = |B_3 \cap B_7| = 9 \times 10$$

$$|B_1 \cap B_4 \cap B_7| = 1$$

and all the other intersections of two or more B_i's are empty. Hence by the Inclusion–Exclusion Principle, we get

$$\begin{aligned} m &= |S| - \sum |B_i| + \sum |B_i \cap B_j| - \sum |B_i \cap B_j \cap B_k| \\ &= 9 \times 10^9 - 10^6 - 6 \times 9 \times 10^5 + 10^3 + 3 \times 10^2 + 3 \times 9 \times 10^2 + 3 \times 9 \times 10 - 1 \\ &= 8,993,604,269. \end{aligned}$$

* * *

In the June number of the Corner we gave solutions to problems from the 22nd Austrian Mathematical Olympiad. Michael Selby, University of Windsor, writes that it is possible to generate a solution to problem 2 [1994: 156] using Maple using six commands.

* * *

In May, we gave solutions to most of the problems of the 14th Austrian–Polish Mathematics Competition [1993: 66–67], [1994: 130–134]. An astute reader noticed the problems for which we didn't have solutions and supplied the answers.

4. Let $P(x)$ be a real polynomial with $P(x) \geq 0$ for $x \in [0,1]$. Prove that there exist polynomials $P_i(x)$ $(i = 0, 1, 2)$ with $P_i(x) \geq 0$ for all real x and such that $P(x) = P_0(x) + xP_1(x) + (1-x)P_2(x)$.

Solution by Roger W. Lee, White Plains, New York.

Induct on the degree of P. If $P(x) = ax + b$ for constants a and b, then choose

$$(P_0, P_1, P_2) = \begin{cases} (b, a, 0) & \text{if } a \geq 0 \\ (a+b, 0, -a) & \text{if } a < 0. \end{cases}$$

Consider P of degree $k \geq 2$ and assume the claim true for any polynomial of degree less than k. It suffices to assume that

$$\min_{x \in [0,1]} P(x) = 0,$$

because for any constant $c > 0$, a satisfactory decomposition for $P(x) + c$ follows easily from a satisfactory decomposition of $P(x)$.

Case 1. $P(r) = 0$ for some $r \in (0,1)$. Then r is a root of multiplicity at least two and we may write $P(x) = (x-r)^2 Q(x)$ for some real polynomial Q. Since $Q(x) \geq 0$ for $x \in [0,1]$, we have

$$Q(x) = Q_0(x) + xQ_1(x) + (1-x)Q_2(x) \text{ for some } Q_i(x) \geq 0$$

from the induction hypothesis. Let

$$(P_0, P_1, P_2) = ((x-r)^2 Q_0, (x-r)^2 Q_1, (x-r)^2 Q_2).$$

Case 2. If Case 1 does not obtain then $P(0) = 0$ or $P(1) = 0$. If $P(0) = 0$ then by the induction hypothesis,

$$P(x) = x[Q_0(x) + xQ_1(x) + (1-x)Q_2(x)] \text{ for some } Q_i(x) \geq 0.$$

Rearranging,

$$P(x) = x^2 Q_1(x) + xQ_0(x) + [x(1-x)^2 + (1-x)x^2]Q_2(x).$$

So let

$$(P_0, P_1, P_2) = (x^2 Q_1, Q_0 + (1-x)^2 Q_2, x^2 Q_2).$$

The proof for $P(1) = 0$ is similar.

8. Consider the system of simultaneous congruences

$$xy \equiv -1 \pmod{z}, \quad yz \equiv 1 \pmod{x}, \quad zx \equiv 1 \pmod{y}.$$

Find the number of triples (x, y, z) of distinct positive integers satisfying the system and such that one of x, y, z equals 19.

Solution by Roger W. Lee, White Plains, New York.

Case 1: $z = 19$. The congruences imply

$$19y = xa + 1, \qquad 19x = yb + 1$$

for positive integers a, b. So $361y = (yb+1)a + 19$, implying that $(361 - ab)y = a + 19$ (and that $ab < 361$). Similarly $(361 - ab)x = b + 19$. So

$$ab = [(361 - ab)y - 19][(361 - ab)x - 19]$$

which justifies the middle inequality in:

$$(361 - ab) - 19 \leq \min((361 - ab)y - 19, (361 - ab)x - 19) \leq \sqrt{ab} < 19.$$

Therefore $ab > 323$. From $xa \equiv -1 \equiv yb \pmod{19}$, we have $-1 \equiv (-xy)(ab) \equiv ab \pmod{19}$. So $ab = 341$ or 360.

If $ab = 341$ then $20x = b + 19$, so $b = 11$ or $b = 31$ would not yield an integral x. The remaining possibilities $b = 1$ and $b = 341$ yield the solutions $(x, y) = (18, 1)$ and $(1, 18)$.

If $ab = 360$ then $x = b + 19$ and $y = a + 19$. The possible solutions are then the 24 different (x, y) such that $b = x - 19$ and $a = y - 19$ are positive integers whose product is 360. All 24 pairs are in fact solutions since they satisfy $xy \equiv ab \equiv -2 \pmod{19}$ and

$$19y = 19a + 361 = a(19 + b) + 1 = ax + 1 \equiv 1 \pmod{x}$$

and similarly $19x \equiv 1 \pmod{y}$. Therefore we have 26 solutions where $z = 19$.

Case 2: $y = 19$. The congruences imply

$$19z = xa + 1, \qquad 19x = zc - 1$$

for positive integers a, c. Substitute as in Case 1 to find $(361 - ac)x = c - 19$ and $(361 - ac)z = 19 - a$. So

$$ac = [(361 - ac)x + 19][19 - (361 - ac)z].$$

If both right-hand side factors are negative then both $361 - ac < 0$ and $361 - ac > 0$, which is impossible. Hence both factors are positive, implying $361 - ac > -19$ and $361 - ac < 19$, so $342 < ac < 380$. From $-xa \equiv 1 \equiv zc \pmod{19}$, we have $-1 \equiv (xa)(ac) \equiv ac \pmod{19}$. So $ac = 379$ or 360.

If $ac = 379$ then $c - 19 < 0$ and $19 - a < 0$, so $a = 379$ and $c = 1$, yielding the solution $(x, z) = (1, 20)$.

If $ac = 360$ then $x = c - 19$ and $z = 19 - a$ with $a < 19$ and $c > 19$. The possible solutions are then the 12 different (x, z) such that $c = x + 19 > 19$ and $a = 19 - z < 19$ are positive integers whose product is 360. All 12 pairs are in fact solutions since they satisfy $xz \equiv -ac \equiv 1 \pmod{19}$ and

$$19x = 19c - 361 = c(19 - a) - 1 = cz - 1 \equiv -1 \pmod{z};$$

$$19z = 361 - 19a = a(c - 19) + 1 = ax + 1 \equiv 1 \pmod{x}.$$

Therefore we have 13 solutions where $y = 19$.

Case 3: $x = 19$. This case is symmetric to Case 2, and yields 13 more solutions.

Since the cases do not overlap, the total number of solutions is $26 + 13 + 13 = 52$.

9. Let $A = \{1, 2, \ldots, n\}$ with n a positive even integer. Suppose $g : A \to A$ is a function with $g(k) \neq k$, $g(g(k)) = k$ for $k \in A$. How many functions $f : A \to A$ are there such that $f(k) \neq g(k)$ and $f(f(f(k))) = g(k)$ for $k \in A$?

Solution by Roger W. Lee, White Plains, New York.

Clearly g is the product of $n/2$ disjoint transpositions. Also each f must be a permutation, and is hence expressible as the product of disjoint cycles. Since f^6 is the identity and $f^3 = g$, each cycle takes the form $(a_1, a_2, a_3, g(a_1), g(a_2), g(a_3))$ These six elements are distinct because $g(k) \neq k \Rightarrow f(f(f(k))) \neq k \Rightarrow f(k) \neq k$; also $f(k) \neq g(k) \Rightarrow f(k) \neq f(f(f(k))) \Rightarrow f(f(k)) \neq k$.

If 6 does not divide n then no such f exist. If 6 divides n then the number of such f is the product of x and $y^{n/6}$; where x is the number of partitions of $n/2$ distinct objects

(g's transpositions) into $n/6$ parts (f's cycles) each of size 3; and where, given three pairs of elements, y is the number of distinct 6-cycles, using all six of these elements, such that the ith element and the $(i+3)$th element of the cycle come from the same pair.

Now $y = 8$ (arbitrarily fix the 6-cycle's first element, leaving 4 choices for the second, and 2 choices for the third), and

$$x = \frac{(n/2)!}{(n/6)!(3!)^{n/6}}$$

so the answer whenever $6|n$ is

$$\frac{(n/2)!}{(n/6)!}\left(\frac{4}{3}\right)^{n/6}.$$

* * *

That is all the room we have this month. Send me your nice solutions and pre–Olympiad and Olympiad material.

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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

The Asian Pacific Mathematics Olympiad: The First Five Years of a Regional Competition, 1989–1993. Edited by Hans Lausch, published jointly in 1994 by Academia de la Investigacion Cientifica, A. C., Mexico, and the Australian Mathematics Trust in its Mathematics Competition Enrichment Series. ISBN 968-7428-00-7, 106+ pages, softcover, US \$15 in North America, Australian \$20 elsewhere. *Reviewed by Andy Liu.*

This book contains a very informative account of the conception and inception of the contest by Prof. Peter O'Halloran, the prime mover of the APMO. The reviewer was privileged to have travelled by car with Peter from Sydney to Canberra in early May, 1989. It was clear that the new contest was uppermost in his mind, and our conversation revealed that much careful thought had gone into this venture. The meticulous care with which Peter developed the contest is amply reflected in the detailed procedures and regulations, including a yearly timetable. This is most valuable for people interested in getting similar contests started.

The main part of the book consists of the twenty-five problems posed, and their solutions, with several alternatives given on many occasions. People involved either in administration or problem-proposing are fully acknowledged. Many of the latter provided useful insights into how they came up with their problems. The book concludes with detailed statistical data and analysis. Hans, who chaired the Problem Committee for the five years, has done a wonderful job in that as well as editing this highly recommended volume.

Here is a sample problem from the book. Show that for every integer n greater than or equal to 6, there exists a convex hexagon which can be dissected into exactly n congruent triangles. Three constructions are given, including a beautiful one by a South Korean contestant.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1995**, *although solutions received after that date will also be considered until the time when a solution is published.*

1971. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O, and E is the intersection of diagonals AC and BD. Let P be an interior point of $ABCD$ such that

$$\angle PAB + \angle PCB = \angle PBC + \angle PDC = 90^\circ.$$

Prove that O, P and E are collinear.

1972. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Define a sequence $a_0, a_1, a_2, \ldots$ of nonnegative integers by: $a_0 = 0$ and

$$a_{2n} = 3a_n, \qquad a_{2n+1} = 3a_n + 1 \qquad \text{for } n = 0, 1, 2, \ldots .$$

(a) Characterize all nonnegative integers n so that there is exactly **one** pair (k, l) satisfying

$$k > l \qquad \text{and} \qquad a_k + a_l = n. \tag{1}$$

(b) For each n, let $f(n)$ be the number of pairs (k, l) satisfying (1). Find

$$\max_{n < 3^{1972}} f(n).$$

1973. *Proposed by K. R. S. Sastry, Dodballapur, India.*

Triangle ABC is inscribed in a circle. The chord AD bisects $\angle BAC$. Assume that $AB = \sqrt{2}\,BC = \sqrt{2}\,AD$. Determine the angles of ΔABC.

1974. *Proposed by Neven Jurić, Zagreb, Croatia.*

Prove or disprove that

$$\sqrt{5+\sqrt{21}}+\sqrt{8+\sqrt{55}}=\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}\,.$$

1975*. *Proposed by Murray S. Klamkin, University of Alberta.*

Let $s(x)$ be the side of an equilateral triangle inscribed in the ellipse $x^2/a^2+y^2/b^2 = 1$ (where $a > b$) having one vertex with the abscissa x. Prove or disprove that $s(x)$ is a monotonic function of x in the interval $[0,a]$.

1976. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

If a, b and c are positive numbers, prove that

$$\frac{a(3a-b)}{c(a+b)}+\frac{b(3b-c)}{a(b+c)}+\frac{c(3c-a)}{b(c+a)}\le\frac{a^3+b^3+c^3}{abc}\,.$$

1977. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcenter O. Let ℓ be the line through O parallel to BC, and let P be a variable point on ℓ. The projections of P onto BC, CA and AB are Q, R and S respectively. Show that the circle passing through Q, R and S passes through a fixed point, independent of P. [This is not a new problem. A reference will be given when a solution is published.]

1978. *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

A fair coin is tossed repeatedly till it shows up heads for the first time. Let n be the number of coin tosses required for this. We then choose at random one of the n integers 1 to n. Find the probability that the chosen integer is 1.

1979. *Proposed by Edward Kitchen, Santa Monica, California.*

Let P be a convex pentagon which is affinely regular, that is, each diagonal is parallel to a side. Let P^* be the convex pentagon inside P formed by the diagonals of P. Using each of two consecutive sides of P^* as base, construct outwards two $36°$–$72°$–$72°$ isosceles triangles, and using the segment between their two summits as base, erect a third such triangle inwards. Prove that the third summit coincides with a vertex of P.

1980. *Proposed by István Bech and Niels Bejlegaard, Stavanger, Norway.*

Find all sets of four points in the plane so that the sum of the distances from each of the points to the other three is a constant.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1825. [1993: 77; 1994: 56] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose that the real polynomial $x^4 + ax^3 + bx^2 + cx + d$ has four positive roots. Prove that $abc \geq a^2d + 5c^2$.

III. *Comment by Ji Chen, Ningbo University, China.*

We show that for $n \geq 2$ and $a_1, a_2, \ldots, a_n \geq 0$ with $\sum_{i=1}^n a_i = S$,

$$\frac{1}{n-1}\sum_{i=1}^n a_i^{n-2}(S - a_i) \geq \prod_{i=1}^n a_i \sum_{i=1}^n \frac{1}{a_i}\,, \tag{1}$$

which was suggested by Walther Janous in his published solution [1994: 57].

For any $k \geq 2$ [by the A.M.–G.M. inequality, the power mean inequality, and the Newton–Maclaurin inequality],

$$\begin{aligned}
\sum_{i=1}^n a_i^{k-1}(S - a_i) &= \sum_{\substack{i,j=1\\ j\neq i}}^n a_i^{k-1}a_j = \sum_{1\leq i<j\leq n} (a_i^{k-1}a_j + a_j^{k-1}a_i) \\
&\geq 2\sum_{i<j}(a_ia_j)^{k/2} = \frac{n(n-1)}{\binom{n}{2}}\sum_{i<j}(a_ia_j)^{k/2} \\
&\geq n(n-1)\left(\frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j\right)^{k/2} \geq \frac{n(n-1)}{\binom{n}{k}}\sum_{1\leq i_1<\cdots<i_k\leq n} a_{i_1}\ldots a_{i_k}.
\end{aligned}$$

When $k = n - 1$, this is (1).

*　　*　　*　　*　　*

1877. [1993: 235] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $B_1, B_2, \ldots, B_b$ be k-element subsets of $\{1, 2, \ldots, n\}$ such that $|B_i \cap B_j| \leq 1$ for all $i \neq j$. Show that

$$b \leq \left[\frac{n}{k}\left[\frac{n-1}{k-1}\right]\right],$$

where $[x]$ denotes the greatest integer $\leq x$.

Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.

Define the $b \times n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } j \in B_i, \\ 0 & \text{if } j \notin B_i. \end{cases}$$

We prove that $bk \leq n\left[\frac{n-1}{k-1}\right]$. Assume to the contrary that $bk > n\left[\frac{n-1}{k-1}\right]$. As A has b rows and each row has exactly k ones (because of the cardinality of the B_i), the matrix

A has exactly bk ones. Then by the pigeonhole principle, there is a column, without loss of generality the first column, which has more than $\left[\frac{n-1}{k-1}\right]$ ones. Then, without loss of generality, we may assume that

$$a_{11} = a_{21} = \cdots = a_{t1} = 1 \qquad \text{for } t = \left[\frac{n-1}{k-1}\right] + 1.$$

As $|B_1 \cap B_i| \le 1$ for $i = 2, 3, \ldots, b$, at most one of a_{1i}, a_{2i}, …, a_{ti} can also be a one, thus

$$\begin{gathered} a_{12} + a_{22} + \cdots + a_{t2} \le 1, \\ a_{13} + a_{23} + \cdots + a_{t3} \le 1, \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{tn} \le 1, \end{gathered}$$

and so summing up these $n-1$ inequalities column by column gives

$$(k-1) + (k-1) + \cdots + (k-1) \le n - 1,$$

because each row of A contains exactly k ones, one of which is a_{i1} for $i = 1, 2, \ldots, t$. Therefore $t(k-1) \le n-1$ and so

$$\left[\frac{n-1}{k-1}\right] + 1 = t \le \frac{n-1}{k-1}$$

which is clearly a contradiction.

Therefore $bk \le n\left[\frac{n-1}{k-1}\right]$, and thus $b \le \frac{n}{k}\left[\frac{n-1}{k-1}\right]$, and as b is an integer, we must have

$$b \le \left[\frac{n}{k}\left[\frac{n-1}{k-1}\right]\right]$$

as we wished to prove.

Also solved by HARVEY L. ABBOTT, University of Alberta; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer.

The proposer actually solved the more general problem where $|B_i \cap B_j| \le t$ for all $i \ne j$ (t fixed). Readers may like to calculate the resulting upper bound for b.

* * * * *

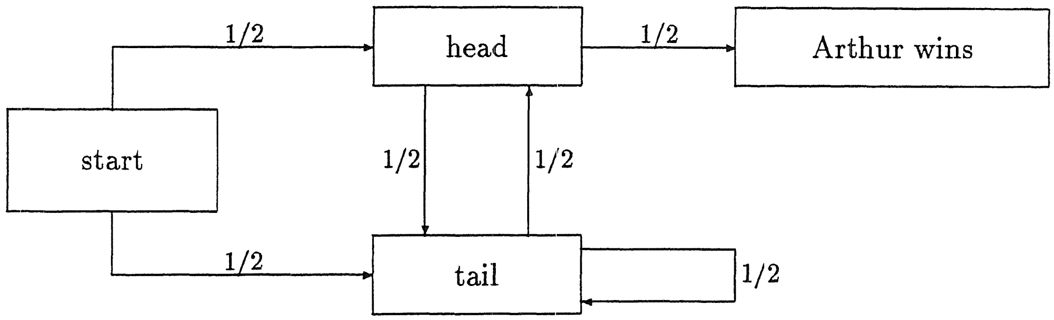
1882. [1993: 264] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Arthur tosses a fair coin until he obtains two heads in succession. Betty tosses another fair coin until she obtains a head and a tail in succession, with the head coming immediately prior to the tail.

(i) What is the average number of tosses each has to make?

(ii) What is the probability that Betty makes *fewer* tosses than Arthur (rather than the same number or more than Arthur)?

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.
In this solution I use absorbing Markov chains (e.g., see [1]).
(i) For Arthur, we have the following diagram:



[*Editor's comment.* For example, the arrow from "start" to "head" indicates that, with probability 1/2, Arthur's first toss is a head; the arrow from "head" to "tail" indicates that, with probability 1/2, Arthur's next toss after throwing a head will be a tail. The arrow from "head" to "Arthur wins" denotes the outcome that Arthur's next toss after throwing a head is another head, which means he wins.]

Let

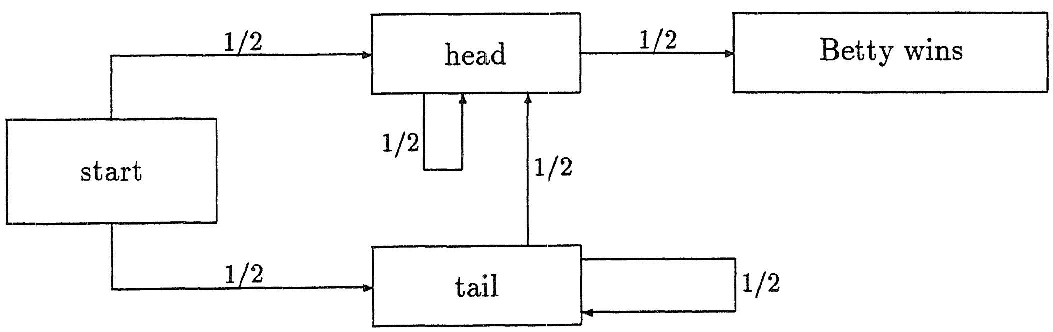
$$\begin{array}{ll} \text{start} = \text{state } 0, & \text{tail} = \text{state } 1, \\ \text{head} = \text{state } 2, & \text{Arthur wins} = \text{state } 3, \end{array}$$

and let m_i be the average number of tosses until Arthur wins if he starts from state i, for $i = 0, 1, 2, 3$. Then clearly $m_3 = 0$, and

$$\left.\begin{array}{l} m_0 = \frac{1}{2}m_1 + \frac{1}{2}m_2 + 1 \\ m_1 = \frac{1}{2}m_1 + \frac{1}{2}m_2 + 1 \\ m_2 = \frac{1}{2}m_1 + \frac{1}{2}m_3 + 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} m_0 = 6 \\ m_1 = 6 \\ m_2 = 4. \end{array}\right.$$

Thus the average number of tosses Arthur has to make is $m_0 = \mathbf{6}$.

For Betty, we have the diagram



Similarly, let

$$\begin{array}{ll} \text{start} = \text{state } 0, & \text{tail} = \text{state } 1, \\ \text{head} = \text{state } 2, & \text{Betty wins} = \text{state } 3, \end{array}$$

and let m_i be the average number of tosses until Betty wins if she starts from state i, for $i = 0, 1, 2, 3$. Then $m_3 = 0$, and

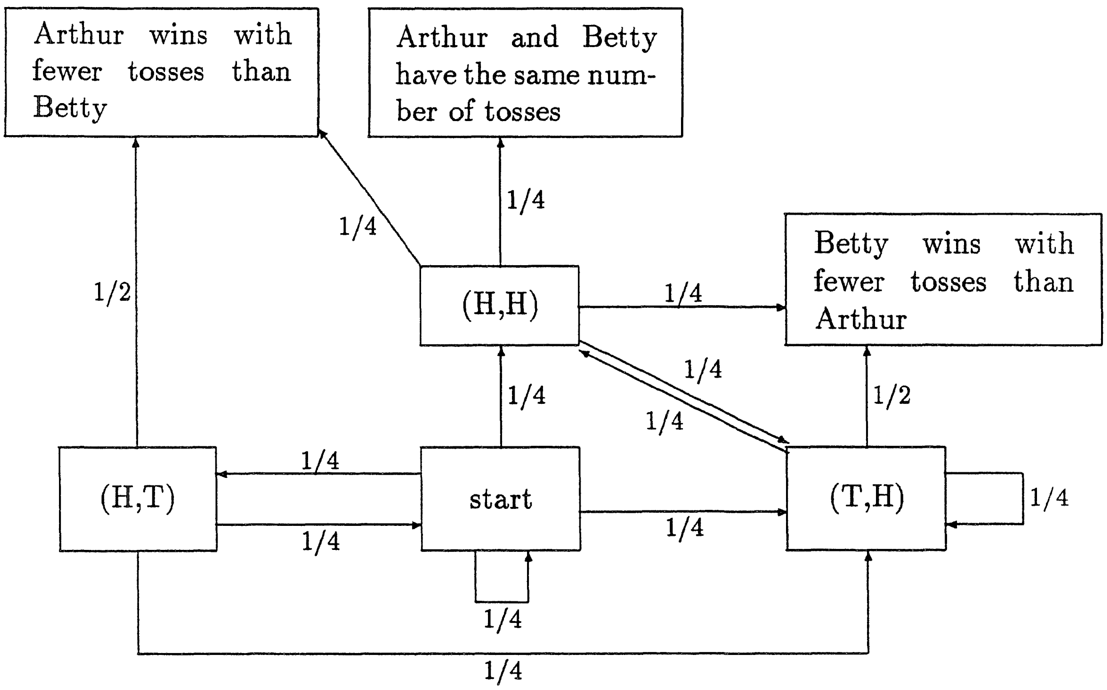
$$\left.\begin{aligned} m_0 &= \tfrac{1}{2}m_1 + \tfrac{1}{2}m_2 + 1 \\ m_1 &= \tfrac{1}{2}m_1 + \tfrac{1}{2}m_2 + 1 \\ m_2 &= \tfrac{1}{2}m_2 + \tfrac{1}{2}m_3 + 1 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} m_0 &= 4 \\ m_1 &= 4 \\ m_2 &= 2. \end{aligned}\right.$$

Thus the average number of tosses Betty has to make is $m_0 = \mathbf{4}$.

(ii) Now suppose Arthur and Betty throw their dice at the same time. Let (H,T) mean that Arthur has a head and Betty has a tail, etc., and let

start = state 0, (H,H) = state 1, (T,H) = state 2, (H,T) = state 3,
Arthur wins with fewer tosses than Betty = state 4,
Arthur and Betty have the same number of tosses = state 5,
Betty wins with fewer tosses than Arthur = state 6.

Then we have the diagram



Let p_i be the probability that Betty wins with fewer tosses than Arthur if she starts from state i, for $0 \le i \le 6$. Then $p_4 = p_5 = 0$, $p_6 = 1$, and

$$\left.\begin{aligned} p_0 &= \tfrac{1}{4}p_0 + \tfrac{1}{4}p_1 + \tfrac{1}{4}p_2 + \tfrac{1}{4}p_3 \\ p_1 &= \tfrac{1}{4}p_2 + \tfrac{1}{4} \\ p_2 &= \tfrac{1}{4}p_1 + \tfrac{1}{4}p_2 + \tfrac{1}{2} \\ p_3 &= \tfrac{1}{4}p_0 + \tfrac{1}{4}p_2 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} p_0 &= 65/121 \\ p_1 &= 5/11 \\ p_2 &= 9/11 \\ p_3 &= 41/121. \end{aligned}\right.$$

Therefore Betty makes fewer tosses than Arthur with probability 65/121.

With the same method one gets

$$\text{Prob(Arthur and Betty make the same number of tosses)} = 17/121,$$

$$\text{Prob(Arthur makes fewer tosses than Betty)} = 39/121.$$

Reference:

[1] Arthur Engel, *Wahrscheinlichkeitsrechnung und Statistik*, Band 2, Stuttgart, 1978.

Also solved by JORDI DOU, Barcelona, Spain; KEITH EKBLAW, Walla Walla, Washington; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia, solved part (i), *but didn't get a final answer for part* (ii). *Two other readers sent in incomplete or incorrect solutions.*

Engelhaupt actually sent in a second solution, using infinite series (and Fibonacci numbers), which was the method employed by every other solver.

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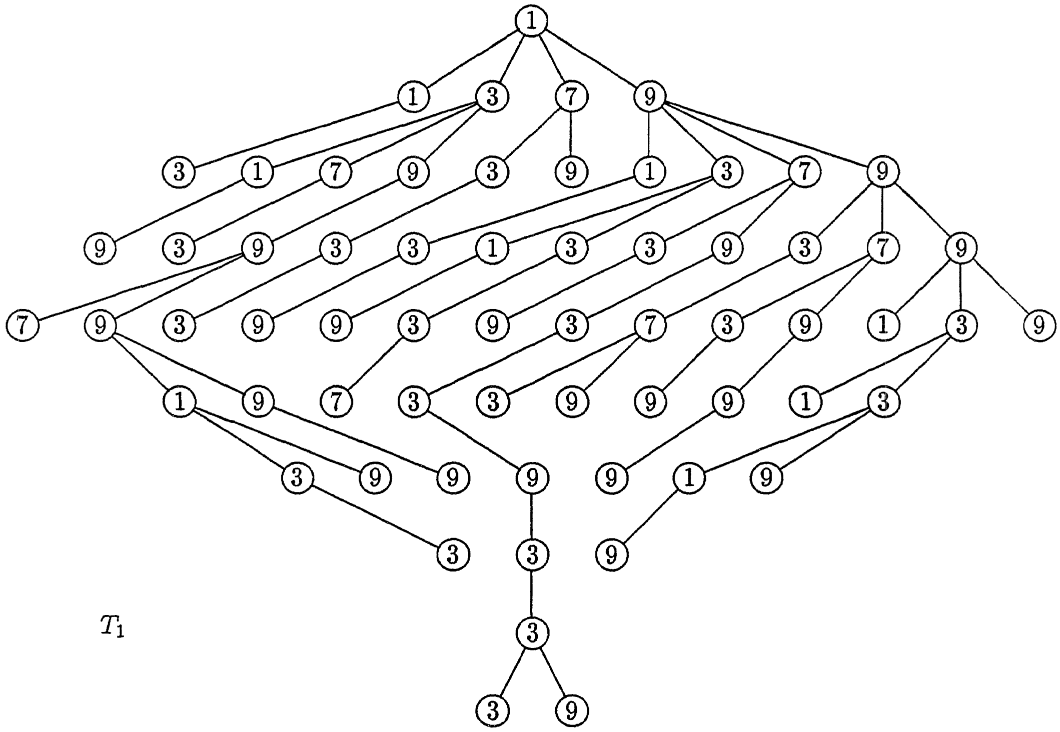
1884. [1993: 264] *Proposed by Ian Affleck, student, University of Regina.*

(a) Let $v(n)$ be the set of integers that result from "truncations" of the positive integer n; for example $v(135) = \{1,3,5,13,35,135\}$. Call n a *v-prime* if every number in $v(n)$ is a prime or 1, so that 173 is v-prime for example. Find all v-primes.

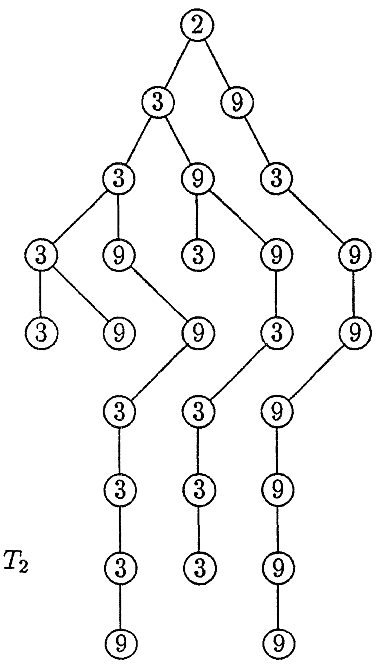
(b)* Let $t(n)$ be the set of integers that result from a *single* truncation of n; for example, $t(1806) = \{1,18,180,1806,806,6\}$. Define *t-prime* analogously to v-prime. How many t-primes are there?

Solution by H.L. Abbott, University of Alberta.

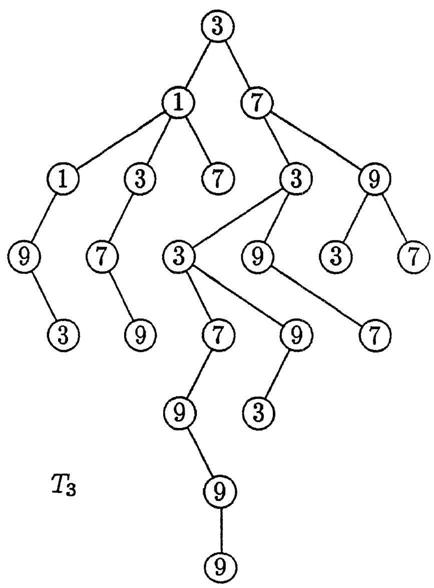
If n is a positive integer, denote by $s(n)$ the set of integers that result from right truncations. For example, $s(1739) = \{1739,173,17,1\}$. Call n an *s-prime* if every number in $s(n)$ is a prime or 1. Every v-prime is a t-prime and every t-prime is an s-prime. We show that there are finitely many s-primes and determine all of them. The argument is entirely computational. We carry out the computations as follows. The first digit of an s-prime must be one of $1,2,3,5,7$ and each digit beyond the first must be one of $1,3,7,9$. For $a \in \{1,2,3,5,7\}$ construct a rooted tree T_a and assign to each vertex a label from $\{1,2,3,5,7,9\}$ via the following scheme: The root is labelled a and after the vertices in level k have been chosen and labelled, select a vertex v in level k with label x and let $a,b,c,\ldots,x$ be the labels attached to the consecutive vertices of the unique chain from the root to v. For $y \in \{1,3,5,7\}$, if the $(k+1)$-digit number $abc\ldots xy$ is a prime place a vertex labelled y in the $(k+1)$st level and join this vertex to v. If $abc\ldots xy$ is composite for all $y \in \{1,3,7,9\}$ then v is a vertex of degree 1 in T_a. This is done for every vertex in the kth level and the $(k+1)$st level is thus determined. It happens that each T_a is finite. The trees are shown in the diagram.



T_1

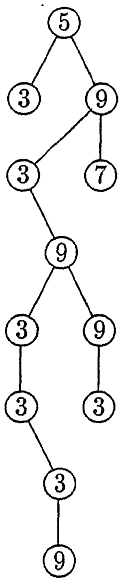


T_2

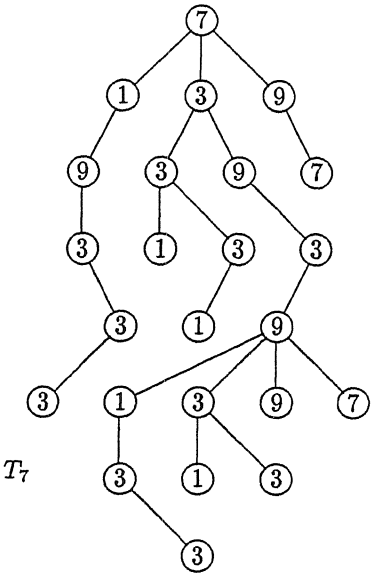


T_3

T_5



T_7



There is a natural one–one correspondence between the set of s-primes with first digit a and the set of chains in T_a emanating from the root. There are 245 s-primes. The largest is 1979339339. The t-primes are

$$1, 2, 3, 5, 7, 11, 13, 17, 23, 31, 37, 53, 71, 73, 113, 131, 137, 173, 197, 311,$$
$$313, 317, 373, 797, 1373, 1997, 3137, 3797, 7331, 73331, 739397.$$

The v-primes are

$$1, 2, 3, 5, 7, 11, 13, 17, 23, 31, 37, 53, 71, 73, 113,$$
$$131, 137, 173, 311, 313, 317, 373, 1373, 3137.$$

Both parts also solved by MANUEL BENITO MUÑOZ, I.B. Sagasta, Logroño, Spain; H. ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; and DAVID E. MANES, State University of New York, Oneonta. Part (a) *only solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. (Actually, Engelhaupt missed one t-prime, Ashbacher missed one v-prime, and Janous and Wildhagen each missed two v-primes.) There were also two more seriously incomplete solutions sent in.*

Hess points out Chris Caldwell's article "Truncatable primes", in the Journal of Recreational Mathematics **19** (1987) 30–33*, which contains related results and deals briefly with Abbott's s-primes (Caldwell calls them right-truncatable primes).*

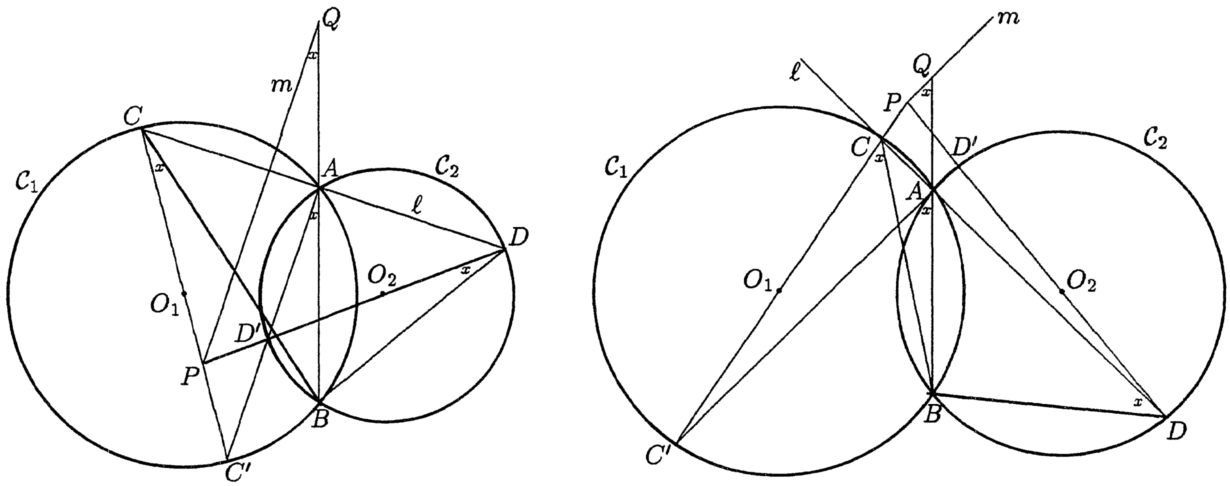
* * * * *

1885. [1993: 265] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Circles $\mathcal{C}_1$ and $\mathcal{C}_2$, with centres O_1 and O_2, intersect in A and B. A line ℓ through A intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ for the second time in C and D respectively. CO_1 and DO_2 intersect in P, and the line m through P perpendicular to CD intersects AB in Q. Show that P, D, Q, C and B are concyclic.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

[*Editor's comment by Chris Fisher.* Pompe's submitted solution discusses different cases (as in the accompanying figures). Those different cases can be avoided by using *directed angles*: the *directed angle from a line ℓ to a line ℓ'* is that angle through which ℓ must be rotated in the positive direction in order to become parallel to ℓ'. See, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, paragraphs 16–19. In particular, four points C', B, A, C are concyclic if and only if the directed angles $\angle C'CB$ and $\angle C'AB$ are equal (regardless of their relative positions on the circle).]



The construction is undefined when $AC \perp AB$, so assume that the chosen line through A is not perpendicular to AB. Let CC' and DD' be diameters of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Then $\angle DAD' = 90°$. Therefore $\angle CAD' = 90° = \angle CAC'$, which implies that A, C', D' are collinear. Since $PQ \| AC'$ (because $PQ \perp CD$ and $AC' \perp CD$) we obtain

$$\angle PCB = \angle C'CB = \angle C'AB = \angle PQB,$$

so that P, B, Q, C are concyclic. Moreover,

$$\angle D'AB = \angle D'DB = \angle PDB$$

so that P, B, Q, D are concyclic. Hence P, B, Q, C and D are concyclic, as desired.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain, with a second solution by Bellot alone; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HIMADRI CHOUDHURY, student, Hunter H.S., New York; JORDI DOU, Barcelona, *Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.*

All solvers produced a variant of the featured solution except for Bellot and López, whose solution utilized coordinates. Only Pompe made explicit reference to the complications arising from the different relative positions of the relevant points.

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1886. [1993: 265] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all integers $n > 1$ such that $\{1!, 2!, \ldots, n!\}$ is a complete set of residues modulo n. (This problem was inspired by problem 1424 in the June 1993 issue of *Mathematics Magazine.*)

Solution by David E. Manes, State University of New York, Oneonta.

If $n = \mathbf{2}$ or $\mathbf{3}$, then $\{1!, 2!, \ldots, n!\}$ is a complete set of residues modulo n by direct computation. There are no other values of n that work.

If $n > 3$ is a prime, then by Wilson's Theorem, $(n-1)! \equiv -1 \bmod n$ from which it follows that $(n-2)! \equiv 1 \bmod n$. On the other hand, if n is composite and $n = r \cdot s$ for some integers r and s with $0 < s < r < n$, then $r! \equiv 0 \equiv n! \bmod n$. Moreover, if $n = k^2$ for some $k > 2$, then $(2k)! \equiv 0 \equiv n! \bmod n$, while for $n = 4$, $2! \equiv 3! \bmod 4$. For each of these cases, it is then clear that $\{1!, 2!, \ldots, n!\}$ cannot be a complete set of residues modulo n.

Editor's comments by E. T. H. Wang. Interestingly, Wilson's Theorem is used by all solvers. However, not everyone's argument is completely valid though all obtained the correct answer. Several solvers claim that if $n \geq 4$ is composite or if $n = a^2$, then $(n-1)! \equiv 0 \bmod n$. Another solver claims that if $n > 3$ is composite, then the residue 0 will occur more than once. Clearly $n = 4$ provides a counterexample to all these statements. One solver considers only the three cases $n = ab$ $(1 < a < b < n)$, $n = p$ and $n = p^2$ where $p > 3$ is a prime, therefore leaving out the cases $n = 4, 9$. Two other solvers claim without proof that if $n > 4$ is composite, then clearly $(n-1)! \equiv 0 \bmod n$. In view of the inaccurate statements quoted above, this fact may not be that "clear" after all.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; TIM CROSS, Wolverley High School, Kidderminster, U.K.; HUGH EDGAR, San Jose State University, San Jose, California; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, John Dewey H. S., Brooklyn, N. Y.; RICHARD I. HESS, Rancho Palos Verdes, California; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; NICK LORD, Tonbridge School, Kent, England; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, *University of Warsaw, Poland; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; LAWRENCE SOMER, Catholic University of America, Washington, D.C.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.*

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1887. [1993: 265] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

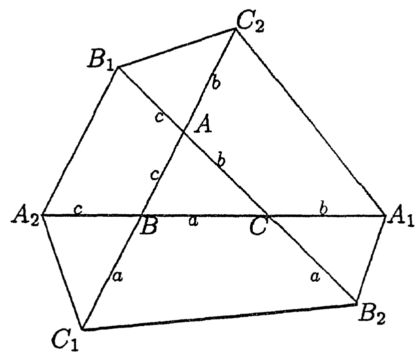
Given an acute triangle ABC, form the hexagon $A_1C_2B_1A_2C_1B_2$ as shown, where

$$BC = BC_1 = CB_2,$$

$$CA = CA_1 = AC_2,$$

and

$$AB = AB_1 = BA_2.$$



Prove that the area of the hexagon is at least 13 times the area of ΔABC, with equality when ABC is equilateral.

Solution by Himadri Choudhury, student, Hunter H.S., New York.

First note that

$$\Delta ABC \cong \Delta AB_1C_2 \cong \Delta A_2BC_1 \cong \Delta A_1B_2C. \tag{1}$$

If the altitude is constant then the ratio of the areas of two triangles is equal to the ratio of the lengths of their respective bases. Thus, letting $[X]$ denote the area of figure X,

$$\frac{[BAA_1]}{[ABC]} = \frac{a+b}{a} \quad \text{and} \quad \frac{[A_1BC_2]}{[BAA_1]} = \frac{b+c}{c},$$

so

$$\frac{[A_1BC_2]}{[ABC]} = \frac{(a+b)(b+c)}{ac}.$$

Similarly

$$\frac{[A_2B_1C]}{[ABC]} = \frac{(b+c)(a+c)}{ab} \quad \text{and} \quad \frac{[AB_2C_1]}{[ABC]} = \frac{(a+b)(a+c)}{bc}.$$

Note that by (1),

$$[A_1B_2C_1A_2B_1C_2] = [A_1BC_2] + [A_2B_1C] + [AB_2C_1] + [ABC],$$

so

$$\begin{aligned}\frac{[A_1B_2C_1A_2B_1C_2]}{[ABC]} &= 1 + \frac{(a+b)(b+c)}{ac} + \frac{(b+c)(a+c)}{ab} + \frac{(a+b)(a+c)}{bc} \\ &= 4 + \left(\frac{b}{a}+\frac{a}{b}\right) + \left(\frac{b}{c}+\frac{c}{b}\right) + \left(\frac{c}{a}+\frac{a}{c}\right) + \left(\frac{b^2}{ac}+\frac{c^2}{ab}+\frac{a^2}{bc}\right). \end{aligned} \tag{2}$$

By the A.M.–G.M. inequality we know that

$$\frac{b}{a}+\frac{a}{b}\geq 2, \text{ etc.}$$

with equality when $a=b$, etc., and

$$\frac{b^2}{ac}+\frac{c^2}{ab}+\frac{a^2}{bc}\geq 3\sqrt[3]{\frac{b^2}{ac}\cdot\frac{c^2}{ab}\cdot\frac{a^2}{bc}}=3\sqrt[3]{1}=3$$

with equality when $a=b=c$. Adding them all together, we have from (2) that

$$\frac{[A_1B_2C_1A_2B_1C_2]}{[ABC]}\geq 4+2+2+2+3=13$$

with equality when $a=b=c$, i.e., ABC is equilateral.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; TIM CROSS, Wolverley High School, Kidderminster, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; LARRY HOEHN, Austin Peay State University, Clarksville, Tennessee; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; NICK LORD, Yateley, Surrey, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, and SIMING ZHAN, University of Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Hoehn, Janous and Seimiya remark that it is not necessary to assume that ΔABC *is acute, as the above proof shows.*

Some solvers point out the rather similar problem 6 *of the* 1992 *Iberoamerican Olympiad* [1993 : 287]. *Is there a common generalization of these two problems?*

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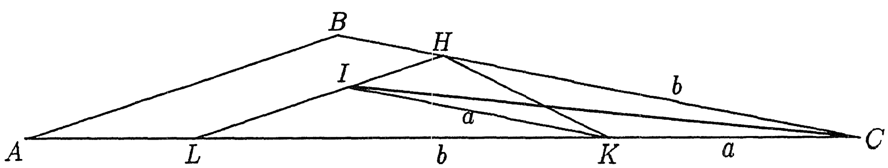
1889. [1993: 265] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

I is the incenter of ΔABC. IH parallel to AB meets BC at H, and IK parallel to BC meets AC at K. Assume that $C=12°$ and that the quadrilateral $ABHK$ is cyclic. Find angles A and B.

Solution by Dag Jonsson, Uppsala, Sweden.



For notation see the figure. Let $\alpha = \angle A$ giving $\angle B = 168° - \alpha$. We successively get $\angle HKL = \alpha + 12°$ (since $\angle B + \angle HKL = 180°$), then $\angle IKL = \angle C = 12°$ (parallel lines), and consequently $\angle HKI = \alpha$. Thus $\angle KHC = \alpha$ (since $IK \| HC$), and also $\angle ILK = \alpha$ (since $LI \| AB$). IC is the bisector of angle C, i.e. $\angle ICK = 6°$, and therefore

$$\angle KIC = \angle IKL - \angle ICK = 12° - 6° = 6°.$$

Thus the triangle IKC is isosceles with $IK = KC = a$, say. We see that the triangles HKC and LIK are congruent and are similar to the triangles LHC and ABC. Comparing ΔLIK with ΔLHC, where $HC = LK = b$ say, we get

$$\frac{b}{a} = \frac{a+b}{b} ;$$

i.e., putting $x = b/a$ we have $x = x^{-1} + 1$ which gives $x = (1+\sqrt{5})/2$ (the other root, $(1-\sqrt{5})/2$, is negative). The sinus theorem applied to ΔLIK gives

$$\frac{1}{\sin\alpha} = \frac{1+\sqrt{5}}{2\sin(168° - \alpha)} ,$$

i.e.

$$\sin 30° \sin\alpha = \frac{1}{2}\sin\alpha = \frac{1}{\sqrt{5}+1}\sin(12° + \alpha) = \sin 18° \sin(12° + \alpha).$$

Obviously $\alpha = 18°$ is a solution. For $0° < \alpha < 168°$ the solution is unique, since $\sin\alpha / \sin(12° + \alpha)$ is an increasing function, the derivative $\sin 12° / \sin^2(12° + \alpha)$ being positive [alternatively, just observe that

$$\frac{\sin(12° + \alpha)}{\sin\alpha} = \sin 12° \cot\alpha + \cos 12°$$

is decreasing.—*Ed.*].

So the answer is: the other angles of triangle ABC are $A = \mathbf{18°}$ and $B = \mathbf{150°}$.

[*Editor's note.* From the above proof, the result remains true if I is chosen to be any point on the bisector of angle C.]

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; KEE-WAI LAU, Hong Kong; PAUL PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Seimiya lists some other choices of angle C yielding integer values for A and B, namely

$$\begin{array}{llllll} C = 24°, & A = 30°, & B = 126°; & C = 36°, & A = 36°, & B = 108°; \\ C = 72°, & A = 36°, & B = 72°; & C = 96°, & A = 30°, & B = 54°; \\ C = 132°, & A = 18°, & B = 30°. & & & \end{array}$$

*   *   *   *   *

1890. [1993: 265; 1994: 47] *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let n be a positive integer and let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad g(x) = \frac{k}{a_n}x^n + \frac{k}{a_{n-1}}x^{n-1} + \cdots + \frac{k}{a_1}x + \frac{k}{a_0},$$

where k and the a_i's are positive real numbers. Prove that

$$f(g(1))g(f(1)) \geq 4k.$$

When does equality hold?

Correction and solution by Waldemar Pompe, student, University of Warsaw, Poland.

The inequality is correct for $k \geq 1/4$, but can fail for $k < 1/4$.

We first find a counterexample for $0 < k < 1/4$. Let $n = 1$ and $a_0 = a_1 = a$. Then

$$f(x) = ax + a, \quad g(x) = \frac{k}{a}x + \frac{k}{a}, \quad f(1) = 2a, \quad g(1) = \frac{2k}{a},$$

and so

$$f(g(1)) = a \cdot \frac{2k}{a} + a = 2k + a, \qquad g(f(1)) = \frac{k}{a} \cdot 2a + \frac{k}{a} = 2k + \frac{k}{a}.$$

We want to find a positive a such that $f(g(1))\,g(f(1)) < 4k$, or

$$(2k + a)\left(2 + \frac{1}{a}\right) < 4,$$

or

$$F(a) := 2a^2 + (4k - 3)a + 2k < 0.$$

F is a quadratic polynomial with discriminant

$$\Delta = (4k - 3)^2 - 16k = (1 - 4k)(9 - 4k) > 0$$

for $k < 1/4$, so F has two distinct roots. These roots are both positive, because $k > 0$ and $4k - 3 < 0$ for $k < 1/4$. Therefore it is enough to choose any a lying between the roots of F, and we have constructed a counterexample for any $0 < k < 1/4$.

Now assume $k \geq 1/4$. We shall prove that the desired inequality is valid.

We have

$$f(1) = a_n + a_{n-1} + \cdots + a_0, \qquad g(1) = \frac{k}{a_n} + \frac{k}{a_{n-1}} + \cdots + \frac{k}{a_0},$$

and so

$$f(g(1)) = a_n\left(\frac{k}{a_n} + \cdots + \frac{k}{a_0}\right)^n + \cdots + a_1\left(\frac{k}{a_n} + \cdots + \frac{k}{a_0}\right) + a_0,$$

$$g(f(1)) = \frac{k}{a_n}(a_n + \cdots + a_0)^n + \cdots + \frac{k}{a_1}(a_n + \cdots + a_0) + \frac{k}{a_0}.$$

Therefore, since the a_i's and k are positive,

$$f(g(1))\,g(f(1)) \geq \left(a_1\left(\frac{k}{a_1} + \frac{k}{a_0}\right) + a_0\right)\left(\frac{k}{a_1}(a_1 + a_0) + \frac{k}{a_0}\right)$$

with equality if and only if $n = 1$. Since $(a_0^{-1} + a_1^{-1})(a_0 + a_1) \geq 4$, with equality if and only if $a_0 = a_1$, we have

$$\begin{aligned}
&\left(a_1\left(\frac{k}{a_1} + \frac{k}{a_0}\right) + a_0\right)\left(\frac{k}{a_1}(a_1 + a_0) + \frac{k}{a_0}\right) \\
&\qquad = k\left[\left(\frac{k}{a_1} + \frac{k}{a_0}\right)(a_1 + a_0) + \frac{a_1}{a_0}\left(\frac{k}{a_1} + \frac{k}{a_0}\right) + \frac{a_0}{a_1}(a_1 + a_0) + 1\right] \\
&\qquad \geq k\left(4k + 2\sqrt{\left(\frac{k}{a_1} + \frac{k}{a_0}\right)(a_1 + a_0)} + 1\right) \\
&\qquad \geq k(4k + 2\sqrt{4k} + 1) \geq k(1 + 2 + 1) = 4k.
\end{aligned}$$

Equality holds if and only if $n = 1$, $k = 1/4$, $a_0 = a_1 = 1/2$.

Counterexamples to the original and/or first-time-corrected versions of this problem were also sent in by RICHARD I. HESS, Rancho Palos Verdes, California; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; and ALEX LEE, student, Choate Rosemary Hall, Wallingford, Connecticut. The proposer's original proof was correct for $k \geq 1/4$ but contained a hidden error for $k < 1/4$. He has since sent in counterexamples for all $k < 1/4$.

Konečný proves that when $k = 1$ (as in the editor's original incorrect version of this problem) one gets $f(g(1))g(f(1)) \geq 9$, and this is best possible (equality when $n = 1$, $a_1 = a_0 = 1$).

The editor thanks the above readers for alerting him to the mistakes in this problem!

* * * * *

A MESSAGE OF THANKS

Francisco Bellot Rosado reports that he has received **two** copies of Altshiller Court's *College Geometry* in answer to his enquiry in the June issue [1994: 180]. He wishes to thank very much all those who responded. The Editor is not surprised at the generosity of *Crux* readers toward Professor Bellot, who has been a longtime *Crux* supporter.

Ken Williams of Carleton University suggests that *Crux* may be an ideal forum through which other readers may search for hard-to-find mathematics books, and the Editor agrees. Any requests?

* * * * *

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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