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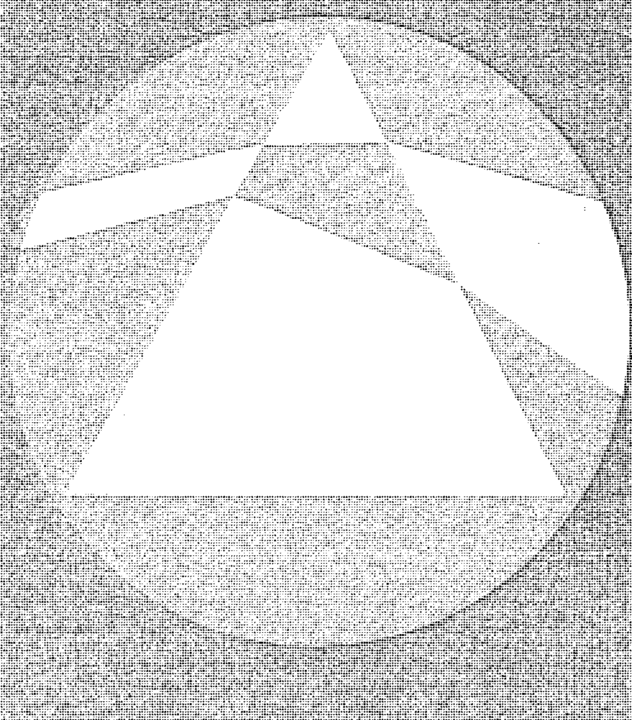

- Karl Weierstrass
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A magazine for students and teachers of mathematics in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Karl Theodor Wilhelm Weierstrass

ANNE C. BAKER

'The father of modern analysis'
1815–1897

Unlike most mathematicians Karl Weierstrass wrote the major part of his mathematical work after the age of forty, for his circumstances had previously prevented him from devoting sufficient time to his passion for mathematics. Once he had secured a university position he became a leading influence in the mathematics of his time. In his teaching and research alike he insisted on rigorous reasoning, refusing to accept the intuitive arguments which were still prevalent among many contemporary mathematicians, especially analysts. Numerous students from far and wide, many of whom Weierstrass initiated into research, went on to become professional mathematicians who transmitted his ideas and methods to the next generation. Thus Weierstrass's logical development of analysis, starting with the construction of the real number system, gradually won universal acceptance and has since been the basis of much modern development. For these reasons his sobriquet *'the father of modern analysis'* is apt on both the personal and ideological levels.

Karl Theodor Wilhelm Weierstrass was born first child of Wilhelm Weierstrass, secretary to the mayor of Ostenfelde, then part of Prussia, and Theodora Vonderforst on 31 October 1815. Afterwards there were three more children, Peter, Klara and Elise who, like Karl, never married. Their father was an intelligent, educated man with a knowledge of the arts and science, who had been converted from the Protestant faith and brought up the children in the Catholic tradition. Karl adhered to Catholicism throughout his life, but he acknowledged the cultural significance of the Reformation. Wilhelm Weierstrass took a number of administrative posts in Westphalia, mostly in taxation departments, before becoming treasurer at the main customs office in the ancient town of Paderborn. The family's frequent moves naturally meant somewhat unsettled school lives for the children. Theodora died when Karl was twelve and his father remarried a year later.

In 1829 Karl entered the Catholic Gymnasium (grammar school) in Paderborn and remained there until 1834, receiving many prizes for his studies in all his subjects except good penmanship. He had an interest in lyric poetry but never in painting, theatre or other arts; he had no talent or liking for music. At the age of fifteen he did some accounting work in order to supplement the family income.

Karl's mathematical gifts became evident in his teens when he began to read the research papers in Crelle's journal for pure and applied mathematics (which still flourishes). Although Wilhelm had a meticulous regard for detail which he may have passed on to Karl, it is notable that no other member of the Weierstrass family had any leaning towards mathematics. Peter Weierstrass recalled being the recipient of unsuccessful coaching attempts by his brother.

Despite Karl's early interest in mathematics he acceded to the wishes of his domineering father and in 1834 entered the University of Bonn in order to follow a course in public finance, economics and administration. This move was intended to prepare Karl for administrative posts in Prussia and proved to be a disastrous mistake. Karl's interest in mathematics did not wane and the conflict between duty and interest caused him mental and physical strain. After four years in which mathematics, fencing and drinking played more important parts in his life than his supposed course of study, Karl returned home to a very disappointed father without having attempted his examinations.

Karl Theodor Wilhelm Weierstrass

In 1839 Karl was sent, at the suggestion of a family friend, to the Theological and Philosophical Academy at Münster (now the Westfälische Wilhelms-Universität) where he was to prepare for a career as a secondary school teacher, able at last to study mathematics officially. Karl came under the influence of Cristof Gudermann, professor of mathematics, who soon recognised his rare talent. It is recorded that a course given by Gudermann initially drew thirteen students, but was down to a single one, Karl Weierstrass, after the first lecture. The topic was elliptic functions, which were studied intensively during much of the nineteenth century and, although they are far from simple, an indication of their origin and nature is fairly easily given.

Let $R(x, y)$ be a rational function of x and y, i.e. the quotient of two polynomials in x and y. Also let $f(x)$ be a polynomial in x and consider

$$u = \int_0^v R\big(t, \sqrt{f(t)}\big)\,dt = I(v), \tag{1}$$

say. It is known that, if f is a polynomial of degree less than or equal to 2, then the integral in equation (1) can be evaluated in terms of 'elementary' functions, i.e. rational, exponential and trigonometric functions and their inverses. However, no such evaluation is generally possible when f is a polynomial of degree greater than 2. If the degree of f is 3 or 4, equation (1) is called an *elliptic integral* because the arc length of an ellipse is given by an integral of this kind. (The arc length of the ellipse $x = a\cos\theta$, $y = b\sin\theta$, for $0 \leqslant \theta \leqslant 2\pi$, is

$$8\int_0^1 \frac{\sqrt{4a^2t^2 + b^2(1-t^2)^2}}{(1+t^2)^2}\,dt.\Big)$$

The mathematics now becomes much more interesting if the relationship $u = I(v)$ is inverted to $v = F(u)$, say. Then $F(u)$ is called an *elliptic function* and the final step is to extend this function to the whole complex plane. The complex function $F(z)$ has really remarkable properties, chief of which is that it is doubly periodic. This means that to each F there correspond complex numbers w_1 and w_2 such that w_1/w_2 is not real and

$$F(z + w_1) = F(z), \qquad F(z + w_2) = F(z)$$

for all z, so that the plane can be divided into parallelograms in all of which F has the same behaviour pattern. (See figure 1.)

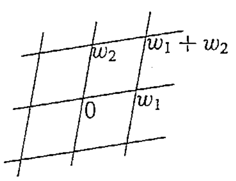

Figure 1

Weierstrass's dissertation on Gudermann's course on elliptic functions contained significant new material. Gudermann, greatly impressed, said that Weierstrass 'was of equal rank with the discoverers (N. H. Abel and C. G. J. Jacobi) who were crowned with glory'. However the work did not reach the outside world until much later and Karl Weierstrass went on, as planned, to become a certificated teacher. Nevertheless, Weierstrass ever after took every opportunity to acknowledge Gudermann's help and encouragement.

After a probationary year, Weierstrass taught at the Catholic gymnasium in Deutsch-Krone from 1842 to 1848, and then in a similar school in Braunsberg from 1848 to 1855. As well as mathematics and physics Weierstrass taught German, botany, geography, history, gymnastics and calligraphy, the one subject in which he himself did not shine at school. In his spare time he single-mindedly pursued mathematical research despite his isolation and lack of access to mathematical literature. He continued the work that he had begun in Münster, but now tackling Abelian functions which form a much larger class than the elliptic functions. *Abelian integrals* are defined like elliptic integrals except that the function f in the integral (1) is of a very general type which includes all polynomials. Inversion then yields *Abelian functions*, just as elliptic functions arise from the inversion of elliptic integrals. It is not clear why Weierstrass published these important results in the Braunsberg school prospectus of 1848 and not in a learned journal. Naturally enough, readers of this prospectus were most unlikely to understand Weierstrass's contribution which went unnoticed by the mathematical community. However, finally he sent his paper to Crelle's journal, where it appeared in 1854, causing a sensation in the mathematical world. The fact that it was written by an unknown schoolmaster was an additional source of wonder. It was the turning point in his career.

The first consequence of Weierstrass's emergence into the limelight was the conferment on him of an honorary doctorate by the University of Königsberg in eastern Prussia (now Kalingrad in Russia). Even more welcome was that, in 1855, the Prussian ministry of education gave Weierstrass a year's paid leave from the Braunsberg gymnasium to enable him to concentrate on his research. However, he never returned to school teaching. Instead, in 1856 he accepted a chair at the Industry Institute in Berlin and an associate professorship at the University of Berlin, having declined an offer of a special professorship tenable at any Austrian university he liked to choose. His ambition of holding a full professorship at the University of Berlin was not fulfilled until 1864.

Weierstrass lectured on a great variety of topics, including elliptic and Abelian functions, geometry and mechanics. However, his meticulous development of analysis, based on a rigorous construction of the real numbers, was perhaps his most enduring achievement. Eventually his lectures became models of exposition, but initially it was the exciting content of his courses that attracted audiences up to two hundred and fifty strong, including not only undergraduates but also graduate students and university lecturers. Many of the results that featured in Weierstrass's lectures were his original work. The four results that follow come from his course on analysis.

(i) If E is a set of real numbers, then the real number c is called a *limit point* of E if every interval $(c-h, c+h)$, however small a positive number h is, contains infinitely many points of E. For instance the set $\{1, 1/2, 1/3, 1/4, ...\}$ has the sole limit point 0, while the set $\{..., -2, -1, 0, 1, 2, ...\}$ has no limit point.

A very useful theorem, quite early in Weierstrass's course, is that *every infinite set which is also bounded, i.e. contained in some interval* (a, b), *has at least one limit point.* The result was formulated rather earlier by Bernard Bolzano (1781–1848), a Catholic priest and professor of philosophy at the University of Prague, whose profound writings on the foundations of analysis did not reach the outside world until well after his death. No attempted proof of his has been found, but in any case no rigorous proof by him was possible since he did not have available an adequate definition of real numbers. Nevertheless the result is known as the *Bolzano–Weierstrass theorem.*

(ii) Continuous functions feature prominently in analysis. Their definition in terms of limits is necessary for the rigorous derivation of their properties, but in an informal account it is usual to say that a function is continuous if its graph can be drawn without taking the pencil from the page. *Weierstrass's approximation theorem* states that *a function $f(x)$ which is continuous on an interval $a \leqslant x \leqslant b$ can be approximated arbitrarily closely by a suitable polynomial in that interval.* In other words, given any positive integer n, there is a polynomial $p_n(x)$ $(a \leqslant x \leqslant b)$ whose graph lies in the band between the curves $y = f(x) - 1/n$ and $y = f(x) + 1/n$ $(a \leqslant x \leqslant b)$. (See figure 2.) The theorem may not seem too surprising when described geometrically but Weierstrass's proof is not elementary and nor are any of the other proofs that have since been devised.

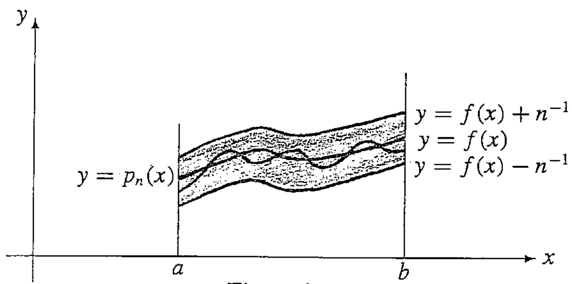

Figure 2

(iii) Weierstrass emphatically countered the unthinking assumption still made by many mathematicians of his day, that the order in which two operations are carried out is immaterial. He pointed out, like a few of his contemporaries, that, if a sequence of differentiable functions $f_n(x)$ converges for all x in an interval, then the relation

$$\lim_{n\to\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}f_n(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to\infty} f_n(x)\right) \tag{2}$$

need not hold. In fact $\lim_{n\to\infty} f_n(x)$ need not even be differentiable. A simple illustration is provided by the differentiable functions $f_n(x) = x^n$ for $0 \leqslant x \leqslant 1$ whose limit function $f(x)$ is such that $f(x) = 0$ for $0 \leqslant x < 1$ and $f(1) = 1$. Clearly f is not differentiable at 1.

Weierstrass also considered the integration of sequences of functions. The relationship analogous to (2), namely

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\mathrm{d}x, \tag{3}$$

equally fails to be true for some sequences $f_n(x)$ of integrable functions which converge on an interval. For instance, if $f_n(x) = nx\mathrm{e}^{-nx^2}$ for $0 \leqslant x \leqslant 1$, then $f_n(x) \to 0$ as $n \to \infty$ for $0 \leqslant x \leqslant 1$ so that

$$\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0,$$

but

$$\int_0^1 f_n(x)\mathrm{d}x = \left[-\tfrac{1}{2}\mathrm{e}^{-nx^2}\right]_0^1 = \tfrac{1}{2}(1 - \mathrm{e}^{-n})$$

and so

$$\lim_{n\to\infty}\int_0^1 f_n(x)\mathrm{d}x = \frac{1}{2}.$$

Weierstrass showed that (3) holds when ordinary convergence of the sequence $f_n(x)$ to $f(x)$ is replaced by a stronger concept, called *uniform convergence*, in which the maximum of $|f_n(x) - f(x)|$ for $a \leqslant x \leqslant b$ has to tend to 0. Actually, in (ii) above the convergence of $p_n(x)$ to $f(x)$ is uniform since $\max|p_n(x) - f(x)| \leqslant 1/n$. The conditions sufficient for (2) to hold are a little more complicated, namely ordinary convergence of the sequence $f_n(x)$ and the uniform convergence of $f_n'(x)$.

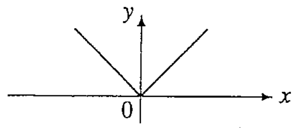

Figure 3

(iv) If a function is differentiable at a point, then it is continuous at that point. However, continuity does not imply differentiability, as is demonstrated by the function $|x|$ which is everywhere continuous but not differentiable at 0. (See figure 3.) Moreover it is easy to sketch the graph of a continuous function which is not differentiable at a given finite number of points by making corners at those points. In fact, it is even quite simple to construct a function which is continuous on an interval but is not differentiable at an infinite number of points. For instance the function whose graph is sketched in figure 4 is continuous for $0 \leqslant x \leqslant 2$, but is not differentiable at $1, 1/2, 1/3, 1/4, \ldots$ and at 0.

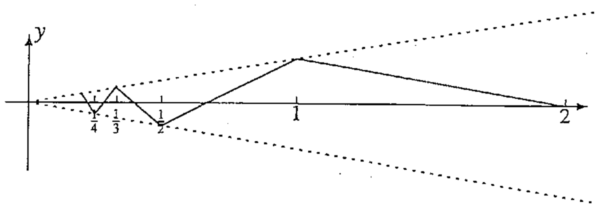

Figure 4

The question naturally arises as to how densely the points where a continuous function is not differentiable can be packed. The completely unexpected answer given by Weierstrass in his lectures in 1861, but not published until 1874, is that there are functions that are *everywhere continuous*, yet *nowhere differentiable.* His example of such a function is

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{4}$$

where $0 < a < 1, ab > 1 + \frac{3}{2}\pi$. It is impossible to draw the graph of this function for whose existence geometrical intuition cannot prepare us. The ripples in the mathematical world made by the discovery of this function reached far beyond the original problem it was designed to solve, for it strikingly illustrates the fact that in mathematics there is no substitute for rigorous proof. Although the function (4) took mathematicians entirely by surprise, it was much later found that Bolzano had constructed a function with the same properties in about 1830. However, Bolzano had not proved the continuity of his function satisfactorily.

Weierstrass had a strong constitution and sense of purpose. Without these he would not have been able to persist with his self-imposed regime during his school-teaching period when, after a hard day's teaching, he would spend much of the night on his research. However, eventually his body rebelled and, starting in 1850, he had a long period when he would suffer hour-long attacks of vertigo which made activity of any kind impossible. Weierstrass's move to the higher education system unfortunately did not bring any reduction of his work load. In December 1861, after a very productive year in which, notably, he was instrumental in setting up the first seminar in Germany devoted exclusively to mathematics, he had a nervous breakdown. He recovered in about a year and thereafter he lectured sitting down, with a student writing the required text on the blackboard. For the rest of his life he suffered recurring bouts of bronchitis and phlebitis, but his determination kept him teaching and pursuing his research.

Sonya Kovalevsky

When well, Karl Weierstrass was lively and cheerful, enjoying the company of colleagues and students alike. He even managed to be on good terms with two notoriously difficult members of the mathematics faculty in Berlin, the geometer Jacob Steiner (1796–1863) and the algebraist and number theorist Leopold Kronecker (1823–1891). After the former's death Weierstrass fulfilled a promise to him by giving, for several years, a course on geometry, though the subject matter held no charms for him. However, relations with Kronecker deteriorated sharply when the latter attacked Georg Cantor (1845–1918) and his revolutionary work on infinite sets, for Weierstrass was a strong supporter of Cantor on both personal and mathematical grounds. (See Hazel Perfect's biographies of Kronecker and Cantor in *Mathematical Spectrum* 1991/2, Volume 24, Number 1, pages 1–7, and 1994/5, Volume 27, Number 2, pages 25–28, respectively.)

A great many of Weierstrass's devoted students went on to become distinguished mathematicians, having benefited from his lectures, his generous guidance in the choice of research topics and the stimulus provided by the seminar he had founded. Without doubt the most remarkable among these students was Sonya Kovalevsky.

Born in 1850, the daughter of a Russian artillery general, Sonya was introduced to mathematics early in life when her room was papered with the lithographed pages of a calculus text used by her father in his school days. When she was seventeen she was given more orthodox mathematical instruction in the form of private lessons from a professor at the St. Petersberg Naval Academy, who soon recognised her brilliance. As a woman, she was unable to attend a Russian university. In 1868 she therefore contracted a marriage of convenience, later to become a love match, with Vladimir Kovalevsky, a young paleontologist who, like herself, wished to study abroad. The couple first enrolled in the University of Heidelberg, but after two years they separated. Vladimir went to Jena to work for a doctorate while Sonya travelled to Berlin, hoping to attend Weierstrass's lectures. Weierstrass, impressed by her enthusiasm, the ability she showed in the solution of problems set by him and her references from Heidelberg, appealed unsuccessfully to Berlin University to waive its ban on women students. He therefore taught her privately for four years and, by the end of this period, she had produced three outstanding pieces of research, one on differential equations, another on Abelian integrals, the third on Saturn's rings. Owing to Weierstrass's influence Sonya was allowed to combine these into a doctoral dissertation for the University of Göttingen, and the degree was awarded in 1874. However, even Weierstrass was unable to secure a lectureship for her anywhere in Europe. The Kovalevskys therefore returned to Russia. Much to Weierstrass's regret, Sonya's mathematical activities now stopped temporarily because she channelled her mental energy into highly acclaimed literary work and also into the movement for the emancipation of women. In 1883 Sonya suffered the devastating blow of the suicide of Vladimir who had unwittingly become involved in shady business dealings. Fortunately by then she was once more active mathematically, and her connection with Weierstrass came to her aid. In 1884 the Swedish mathematician Gustav Mittag-Leffler, a former student of Weierstrass, succeeded in persuading the University of Stockholm to appoint her as a lecturer. Five years later she was promoted to professor. In Stockholm she flourished mathematically working in collaboration with many of the leading mathematicians of the time and keeping in close touch with Weierstrass. Her famous paper 'On the rotation of a solid body about a fixed point' won the French Academy's Bordin prize, hailed by the judges as being of exceptional merit. She died in 1891, at the height of her powers, of influenza and pneumonia.

Weierstrass and Sonya Kovalevsky corresponded fairly regularly from the time they met. Sadly Weierstrass burned all Sonya's letters after her death. However his letters survived and the relationship between them is beautifully encapsulated in a letter of 1873 in which he recalled their having 'dreamed and been enraptured of so many riddles that remain for us to solve, on finite and infinite spaces, on the stability of the world system, and on all the other major problems of the mathematics and the physics of the future'.

Since Weierstrass never rushed into print, much of his work reached the mathematical public second or third hand, in the transcripts of his lectures or in textbooks that used these transcripts. The resulting versions of his ideas were not always up to his high standard. In 1887 he therefore decided to embark, with the help of many former students, on the publication of all his work. The first two volumes appeared in 1894 and 1895, respectively, but the next five volumes were published between 1902 and 1927, after his death. Though three more volumes had been planned, it is highly unlikely that they will ever materialise.

On his 70th birthday Karl Weierstrass was honoured at a gathering of many of his former students who were now spread over much of Europe. His 80th birthday was the occasion of even greater celebration. Between these events Weierstrass gradually reduced his lecturing commitments but continued to enjoy the company of students when he was in reasonable health. He was looked after by his two sisters and he died peacefully on 19 February 1897, in his eighty second year.

Anne Baker *taught mathematics at the University College of Swansea and at Sheffield Hallam University. Her main mathematical interest has been in the interaction between analysis and algebra.*

The Daughter's Dilemma

DAVID SHARPE

The following investigation was posed to a class of 15-year olds.

> An emperor wishes to choose a husband for his daughter from her suitors. He decides to arrange them in a circle and go round executing every alternate person until one is left, who will marry his daughter. Where should the daughter position her favourite suitor to ensure that he is the one she will marry?

Readers may like to try a few cases. For example, with sixteen suitors it is number 16 who survives, with seventeen it is number 2 and with eighteen number 4. (We count from the first person executed and assume that his daughter knows at which point the executions begin. Perhaps we should also assume that the unfortunate suitors do not know the strategy adopted by the emperor to choose a son-in-law!)

It is clear after a few trials that the best number of suitors from the daughter's point of view is a power of 2. Take 16 as an example, and number the suitors from 1 to 16 using binary notation:

1 10 11 100 101 110 111 1000 1001 1010 1011
1100 1101 1110 1111 10000.

For typographical reasons, we write them in a list rather than in a circle. The order of execution is now

1 11 101 111 1001 1011 1101 1111 10 110
1010 1110 100 1100 1000,

leaving 10000, i.e. 16, to marry the emperor's daughter. Notice that the order of executions is, first, those ending in 1 (i.e. the odd numbers), then those ending in 10, then those ending in 100, then the one ending in 1000, leaving 10000, i.e. 16. This pattern works for all powers of 2. Thus, if the number of suitors is a power of 2, the daughter places her favourite suitor last in the circle, i.e. immediately before the first to be executed.

She will need her wits about her if the number of suitors is not a power of 2. Suppose it is n, that the largest power of 2 smaller than n is 2^k and that n exceeds 2^k by r. The first r executions are to suitors $1, 3, 5, \ldots, 2r-1$. This leaves 2^k suitors, as in the previous situation. The first to be executed is now $2r+1$, so his immediate predecessor, namely $2r$, marries the daughter. So the daughter places her intended in position $2r$. For example, when $n = 17$ he is in position 2, when $n = 18$ in position 4. For 300 suitors, $2^k = 256$, $r = 44$, so he is in position 88. We can only hope that the daughter is familiar with binary arithmetic, or else is a reader of *Mathematical Spectrum*!

Readers may like to consider other strategies that the emperor might adopt.

David Sharpe *is editor of 'Mathematical Spectrum' and was presented with this problem, posed to pupils at Ryde High School, whilst holidaying on the Isle of Wight.*

Spirals Galore

P. GLAISTER

This article contains a mathematical perspective on some intriguing spirals.

Most readers will be familiar with the Koch snowflake. This is where an equilateral triangle has three similar, but smaller, triangles constructed on each of its edges, and which are exterior to the original triangle. Each of these triangles has two similar, but again smaller, triangles constructed on its exterior edges. This process is repeated indefinitely with two triangles being added to each of the previously constructed triangles, and where the aspect ratio is 1 : 3 in each case. It is well known that both the area and the perimeter of the resulting shape can be determined using geometric progressions, and that the corresponding case with squares produces a different result. Readers may like to investigate. In this article we pose a related problem on the area and length of spirals generated within triangles and squares, and which also involves the summing of an infinite geometric progression. The results are extended to regular polygons, and variations of the problem for different polygons suggest further investigation. If nothing else, the patterns produced are very appealing!

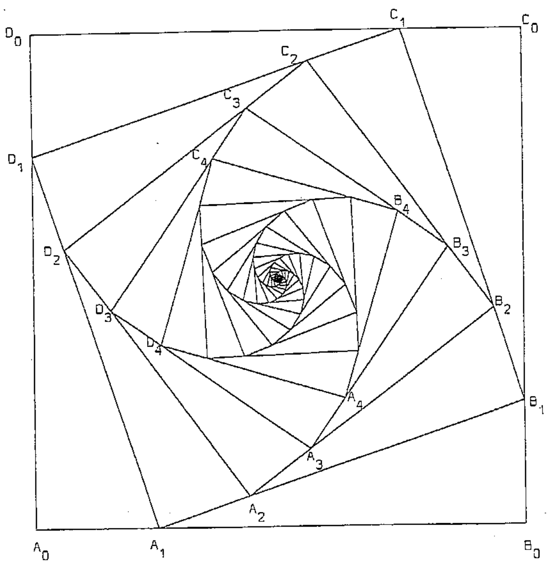

Figure 1

Consider the square $A_0B_0C_0D_0$ shown in figure 1. The vertices of a second square $A_1B_1C_1D_1$ divide the sides A_0B_0, B_0C_0, C_0D_0 and D_0A_0 in the ratio $p : q$. If this process is repeated indefinitely, then a number of spiral patterns are formed. The one of interest here is that formed by the triangles $A_1B_0B_1$, $A_2B_1B_2$, $A_3B_2B_3$, etc., as shown in figure 2, and the problem is to determine the area of this spiral.

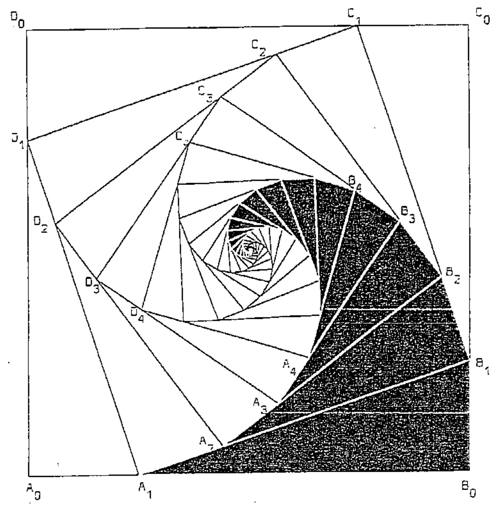

Figure 2

First, by applying Pythagoras' theorem to triangle $A_1B_0B_1$, we have

$$(A_1B_1)^2 = (A_1B_0)^2 + (B_0B_1)^2, \qquad (1)$$

and, if we denote the side of square $A_0B_0C_0D_0$ by a, then

$$\frac{A_0A_1}{A_1B_0} = \frac{p}{q}, \qquad A_0A_1 + A_1B_0 = a,$$

and thus

$$A_1B_0 = \frac{aq}{p+q}, \qquad B_0B_1 = A_0A_1 = \frac{ap}{p+q}.$$

Substituting these into (1) gives

$$(A_1B_1)^2 = \frac{a^2(p^2+q^2)}{(p+q)^2}. \qquad (2)$$

Also, the area of triangle $A_1B_0B_1$ is

$$\tfrac{1}{2}(A_1B_0)(B_0B_1) = \frac{a^2pq/2}{(p+q)^2}.$$

Similar results hold for all other triangles in the spiral, i.e.

$$(A_{i+1}B_{i+1})^2 = \frac{(A_iB_i)^2(p^2+q^2)}{(p+q)^2} \qquad (3)$$

and the area of triangle $A_{i+1}B_iB_{i+1}$ is

$$\frac{(A_iB_i)^2pq/2}{(p+q)^2},$$

both starting with $i = 0$ with $A_0B_0 = a$.

The total area of the spiral is therefore given by the following sum:

area of triangle $A_1B_0B_1$ + area of triangle $A_2B_1B_2$ + area of triangle $A_3B_2B_3 + \dots$

$$= \frac{a^2pq/2}{(p+q)^2} + \frac{(A_1B_1)^2 pq/2}{(p+q)^2} + \frac{(A_2B_2)^2 pq/2}{(p+q)^2} + \dots$$

$$= \frac{pq/2}{(p+q)^2}[a^2 + (A_1B_1)^2 + (A_2B_2)^2 + \dots]$$

$$= \frac{pq/2}{(p+q)^2}\left[a^2 + \frac{a^2(p^2+q^2)}{(p+q)^2} + \frac{(A_1B_1)^2(p^2+q^2)}{(p+q)^2} + \dots\right]$$

$$= \frac{pq/2}{(p+q)^2}\left[a^2 + \frac{a^2(p^2+q^2)}{(p+q)^2} + (A_0B_0)^2\left(\frac{(p^2+q^2)}{(p+q)^2}\right)^2 + \dots\right]$$

$$= \frac{a^2pq/2}{(p+q)^2}(1 + r + r^2 + \dots),$$

where

$$r = \frac{p^2+q^2}{(p+q)^2}.$$

Thus the area is the sum of an infinite geometric progression with common ratio

$$\begin{aligned} r &= \frac{p^2+q^2}{(p+q)^2} \\ &< \frac{p^2+q^2+2pq}{(p+q)^2} \\ &= 1, \end{aligned}$$

since $pq > 0$, and hence is convergent. The area is therefore

$$\begin{aligned} \frac{a^2pq/2}{(p+q)^2}\frac{1}{1-r} &= \frac{a^2pq/2}{(p+q)^2}\frac{1}{1-\dfrac{p^2+q^2}{(p+q)^2}} \\ &= \tfrac{1}{2}a^2\frac{pq}{(p+q)^2-(p^2+q^2)} \\ &= \tfrac{1}{2}a^2\frac{pq}{2pq} \\ &= \tfrac{1}{4}a^2, \end{aligned}$$

which, incidentally, is independent of the ratio $p : q$, i.e. the division of the sides. Thus the area of the spiral in figure 2, for which $p : q = 1 : 3$, is the same as that in figure 3, where $p : q = 1 : 1$.

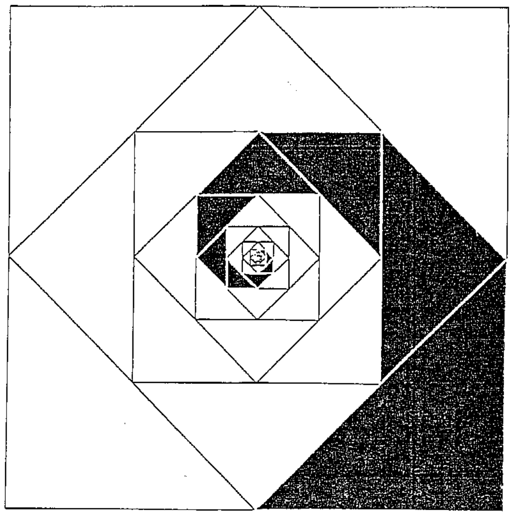

Figure 3

Consider now the length of the inner spiral $A_0A_1A_2\dots$ (which is clearly equal to that of the outer spiral $B_0B_1B_2\dots$). Since $A_0A_1 = ap/(p+q)$, and more generally, $A_iA_{i+1} = A_iB_i p/(p+q)$, and in view of (2) and (3), this length is equal to

$$\begin{aligned} A_0A_1 &+ A_1A_2 + A_2A_3 + \dots \\ &= \frac{ap}{p+q} + \frac{A_1B_1p}{p+q} + \frac{A_2B_2p}{p+q} + \dots \\ &= \frac{ap}{p+q} + \frac{a\sqrt{p^2+q^2}}{p+q}\frac{p}{p+q} + \frac{A_1B_1\sqrt{p^2+q^2}}{p+q}\frac{p}{p+q} + \dots \\ &= \frac{ap}{p+q} + \frac{a\sqrt{p^2+q^2}}{p+q}\frac{p}{p+q} + \frac{a\sqrt{p^2+q^2}}{p+q}\frac{\sqrt{p^2+q^2}}{p+q}\frac{p}{p+q} + \dots \\ &= \frac{ap}{p+q}(1 + s + s^2 + \dots), \end{aligned}$$

where $s = \sqrt{p^2+q^2}/(p+q) = \sqrt{r}$ is the square root of the previous common ratio. Thus the progression is convergent and the sum is

$$\begin{aligned} \frac{ap}{p+q}\frac{1}{1-s} &= \frac{ap}{p+q}\left(1 - \frac{\sqrt{p^2+q^2}}{p+q}\right)^{-1} \\ &= \frac{ap}{p+q-\sqrt{p^2+q^2}}. \end{aligned}$$

In general, for a regular polygon with n sides, $n \geqslant 3$, and a corresponding lettering, then

$$(A_{i+1}B_{i+1})^2 = \frac{p^2+q^2+2pq\cos\theta}{(p+q)^2}(A_iB_i)^2,$$

and the area of the triangle $A_{i+1}B_iB_{i+1}$ is

$$\frac{\frac{1}{2}pq\sin\theta(A_iB_i)^2}{(p+q)^2},$$

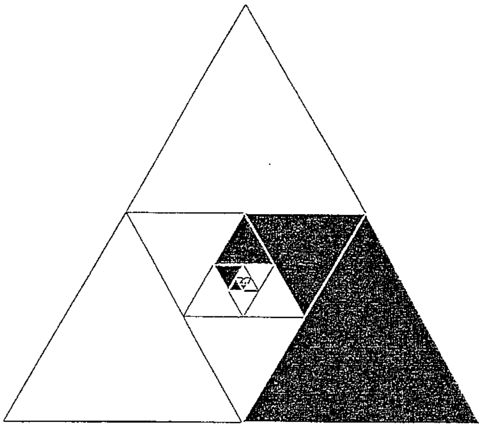

Figure 4

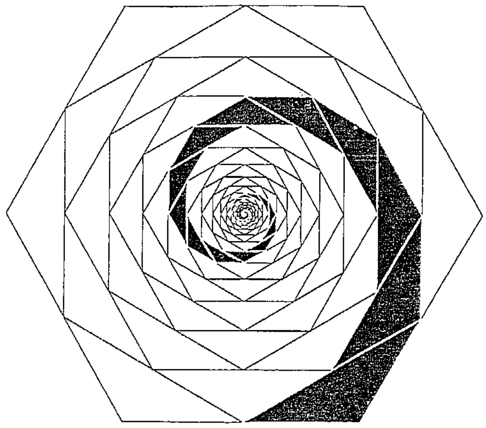

Figure 6

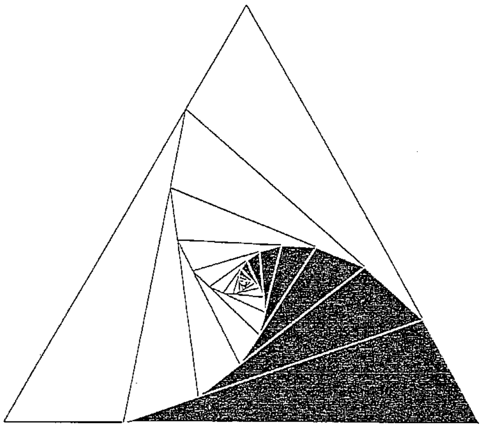

Figure 5

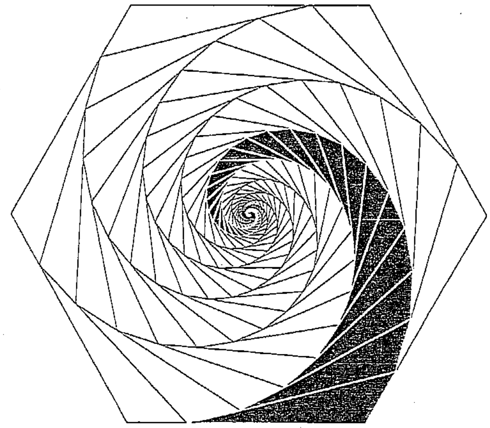

Figure 7

where $A_0B_0 = a$ and $\theta = 360°/n$ is the exterior angle; the area of the spiral is

$$\frac{a^2}{4}\frac{\sin\theta}{1-\cos\theta},$$

again independent of the ratio $p : q$. The corresponding formula for the length of the spiral $A_0A_1A_2\ldots$ is

$$\frac{ap}{p+q-\sqrt{p^2+q^2+2pq\cos\theta}}.$$

Interesting cases are for midpoint division, i.e. where A_1, A_2, etc. are midpoints of the respective sides, so that $p = q = 1$. In this case, the length is $a/(4\sin^2(\theta/4))$, and this value for a triangle, square and hexagon is a, $a/(2-\sqrt{2})$ and $a/(2-\sqrt{3})$, respectively.

Figures 4 and 5 show the spirals for a triangle with $p : q = 1 : 1$ and $p : q = 1 : 3$. Another interesting case is that of a hexagon, as shown in figures 6 and 7. Readers may also like to consider what happens as the number of sides n increases, maybe indefinitely.

The area results could be found by noting that n such spirals form the complete polygon, which of course explains why the area is independent of the ratio $p : q$. □

Paul Glaister *lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, perturbation methods as well as mathematics and science education. His main leisure interests are his two children and, when they allow him, watching cricket.*

The largest prime

The new largest known prime number is $2^{1,257,787} - 1$, discovered in 1996 by Paul Gage and David Slowinski at the Cray Research Unit in USA using a Cray T94 supercomputer.

An attempt is being made to beat this record by linking PCs all over the world. Maybe you would like to join in. If so, visit George Woltman's web site at

`http://ourworld.compuserve.com/homepages/justforfun/prime.html`

A Shorthand Notation for a Class of Integrals

ASTRID BAUMANN

A typical application of repeated integration by parts is to evaluate integrals of the type $\int x^n f(x)\,\mathrm{d}x$, $(n \in \mathbb{N})$. Many recurrence formulae are given in standard integral tables; for example:

$$\int x^n \mathrm{e}^x\,\mathrm{d}x = x^n\mathrm{e}^x - n\int x^{n-1}\mathrm{e}^x\,\mathrm{d}x.$$

Repeated application of this formula yields

$$\begin{aligned}\int x^n \mathrm{e}^x\,\mathrm{d}x &= x^n\mathrm{e}^x - n\left(x^{n-1}\mathrm{e}^x - (n-1)\int x^{n-2}\mathrm{e}^x\,\mathrm{d}x\right)\\ &= x^n\mathrm{e}^x - nx^{n-1}\mathrm{e}^x + n(n-1)\int x^{n-2}\mathrm{e}^x\,\mathrm{d}x\\ &= x^n\mathrm{e}^x - nx^{n-1}\mathrm{e}^x + n(n-1)x^{n-2}\mathrm{e}^x\\ &\quad - n(n-1)(n-2)x^{n-3}\mathrm{e}^x + \ldots\\ &\quad + (-1)^n n! x^0\mathrm{e}^x.\end{aligned}$$

Writing this result in the form

$$\int x^n\mathrm{e}^x\,\mathrm{d}x = (-1)^n n!\mathrm{e}^x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!}\right),$$

we notice that the truncated series expansion of the function

$$\mathrm{e}^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

occurs. The notation

$$f_n(x) \quad \text{or} \quad [f(x)]_n = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$$

for the truncated Maclaurin series expansion of the function f now leads to the following shorthand notation for the result of the integration:

$$\int x^n\mathrm{e}^x\,\mathrm{d}x = (-1)^n n!\mathrm{e}^x[\mathrm{e}^{-x}]_n \quad (+C). \qquad (1)$$

This formula can be generalized by the substitution $x \to ax$:

$$\int x^n\mathrm{e}^{ax}\,\mathrm{d}x = (-1)^n\frac{n!}{a^{n+1}}\mathrm{e}^{ax}[\mathrm{e}^{-ax}]_n \quad (+C). \qquad (2)$$

To familiarize the reader with the new notation we note that, for instance,

$$\begin{aligned}\cos_7 x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!},\\ \cos_0 x &= 1\\ \tan_7^{-1} z &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}.\end{aligned}$$

This article presents a collection of formulae for integrals of the type $\int \big(f(x)\big)^n g(x)\,\mathrm{d}x$, which can be obtained similarly by repeated integration by parts. The short notation introduced above reveals the structure of the indefinite integral and its connection with the relevant series expansion. This should be of interest even to students who use computer algebra programs to do the integration work. Since the short notation is new, existing tables of integrals (e.g. reference 1) naturally do not use it.

The compact notation of an indefinite integral has an interesting feature. If the subscript denoting a truncated Maclaurin series is omitted, for example if (2), namely

$$(-1)^n\frac{n!}{a^{n+1}}\mathrm{e}^{ax}[\mathrm{e}^{-ax}]_n,$$

is replaced by

$$(-1)^n\frac{n!}{a^{n+1}}\mathrm{e}^{ax}\mathrm{e}^{-ax},$$

the result is a constant, in this case $(-1)^n(n!/a^{n+1})$. However, so far there is no proof that this phenomenon occurs universally.

Another example: for an *odd* number $n \in \mathbb{N}$ we have the formula

$$\int\frac{\sin^n x}{\cos x}\,\mathrm{d}x = -\ln|\cos x| + \left[\tfrac{1}{2}\ln_n(1-z^2)\right]_{z=\sin x} \qquad (3)$$

with truncated series expansion

$$\tfrac{1}{2}\ln_n(1-z^2) = -\frac{z^2}{2} - \frac{z^4}{4} - \frac{z^6}{6} - \frac{z^8}{8} - \cdots - \frac{z^{n-1}}{n-1}.$$

Instead of the right-hand side of equation (3) now calculate the term

$$-\ln|\cos x| + \left[\tfrac{1}{2}\ln(1-z^2)\right]_{z=\sin x}$$

(without index n) to find that it is 0. Since $\ln_n(1-z^2) \to \ln(1-z^2)$, it follows that

$$\lim_{n\to\infty}\int_a^b \frac{\sin^n x}{\cos x}\,\mathrm{d}x = 0, \qquad a, b \in (-\pi/2, \pi/2).$$

It is convenient to introduce another new notation. For every natural number m we write

$$m!! = m(m-2)!! \qquad \text{with} \quad 0!! = (-1)!! = 1.$$

Table of integrals

$\sinh x, \cosh x$	n even	n odd
$\int x^n \sinh x dx$	$n!(\cosh x\ \cosh_n x - \sinh x\ \sinh_n x)$	$n!(\cosh x\ \sinh_n x - \sinh x\ \cosh_n x)$
$\int x^n \cosh x dx$	$n!(\sinh x\ \cosh_n x - \cosh x\ \sinh_n x)$	$n!(\sinh x\ \sinh_n x - \cosh x\ \cosh_n x)$
$\int \tanh^n x dx$	$x - [\tanh_n^{-1} z]_{z=\tanh x}$	$\ln(\cosh x) + \left[\frac{1}{2}\ln_n(1-z^2)\right]_{z=\tanh x}$
$\int \coth^n x dx$	$x - [\tanh_n^{-1} z]_{z=\coth x}$	$\ln\lvert\sinh x\rvert + \left[\frac{1}{2}\ln_n(1-z^2)\right]_{z=\coth x}$
$\sin x, \cos x$	n even	n odd
$\int x^n \sin x dx$	$(-1)^{1+\frac{1}{2}n} n!(\cos x\ \cos_n x + \sin x\ \sin_n x)$	$(-1)^{(n-1)/2} n!(\sin x\ \cos_n x - \cos x\ \sin_n x)$
$\int x^n \cos x dx$	$(-1)^{n/2} n!(\sin x\ \cos_n x - \cos x\ \sin_n x)$	$(-1)^{(n-1)/2} n!(\cos x\ \cos_n x + \sin x\ \sin_n x)$
$\int \tan^n x dx$	$(-1)^{n/2}\big(x - [\tan_n^{-1} z]_{z=\tan x}\big)$	$(-1)^{(n+1)/2}\big(\ln\lvert\cos x\rvert + [\frac{1}{2}\ln_n(1+z^2)]_{z=\tan x}\big)$
$\int \cot^n x dx$	$(-1)^{n/2}\big(x + [\tan_n^{-1} z]_{z=\cot x}\big)$	$(-1)^{(n-1)/2}\big(\ln\lvert\sin x\rvert + [\frac{1}{2}\ln_n(1+z^2)]_{z=\cot x}\big)$
$\int \sin^n x dx$	$\frac{(n-1)!!}{n!!}\left(x - \cos x \times \left[\frac{\sin^{-1} z}{\sqrt{1-z^2}}\right]_{n\lvert z=\sin x}\right)$	$-\frac{(n-1)!!}{n!!}\cos x \times \left[\frac{1}{\sqrt{1-z^2}}\right]_{n\lvert z=\sin x}$
$\int \cos^n x dx$	$\frac{(n-1)!!}{n!!}\left(x + \sin x \times \left[\frac{\sin^{-1} z}{\sqrt{1-z^2}}\right]_{n\lvert z=\cos x}\right)$	$\frac{(n-1)!!}{n!!}\sin x \times \left[\frac{1}{\sqrt{1-z^2}}\right]_{n\lvert z=\cos x}$

The complete table with shorthand notation for 70 different integrals, which includes most of the current standard integrals, is available from the author, address Kiefernweg 15, D 61169 Friedberg, Germany.

Examples:

$6!! = 2 \times 4 \times 6, \qquad 7!! = 1 \times 3 \times 5 \times 7, \qquad 1!! = 1.$

In the table of integrals above we assume the well-known Maclaurin series for e^x, $\cos x$, $\sin x$, $\cosh x$, $\sinh x$, $\tan^{-1} x$ and $\tanh^{-1} x$. We also use the following expansions, all with radius of convergence 1:

$$\tfrac{1}{2}\ln(1+x^2) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \dots, \tag{4}$$

$$\tfrac{1}{2}\ln(1-x^2) = -\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{6} - \frac{x^8}{8} - \dots, \tag{5}$$

and

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{3!!}{4!!}x^4 - \frac{5!!}{6!!}x^6 + \frac{7!!}{8!!}x^8 - \dots, \tag{6}$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3!!}{4!!}x^4 + \frac{5!!}{6!!}x^6 + \frac{7!!}{8!!}x^8 + \dots, \tag{7}$$

and corresponding to (6) and (7):

$$\frac{\sinh^{-1} x}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{4!!}{5!!}x^5 - \frac{6!!}{7!!}x^7 + \frac{8!!}{9!!}x^9 - \dots, \tag{8}$$

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{4!!}{5!!}x^5 + \frac{6!!}{7!!}x^7 + \frac{8!!}{9!!}x^9 + \dots. \tag{9}$$

It may be noted that the functions on the left in (8) and (9) are best obtained as follows. For instance, if the sum of the series in (9) is denoted by $f(x)$, then $f(0) = 0$ and

$$f'(x) = 1 + xf(x) + x^2 f'(x),$$

and this differential equation has solution

$$f(x) = \sin^{-1} x/\sqrt{1-x^2}.$$

The differential equation yielding (8) is

$$f'(x) = 1 - xf(x) - x^2 f'(x).$$

Integration examples and applications

Example (a). To calculate the integral $\int \tan^{10} x\,dx$ we use the entry

$$\int \tan^n x\,dx = (-1)^{n/2}\big(x - [\tan_n^{-1} z]_{z=\tan x}\big) \qquad (n \text{ even})$$

in the table. So

$$\int \tan^{10} x\,dx = -\left(x - \left[z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9}\right]_{z=\tan x}\right)$$
$$= -x + \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} \quad (+C).$$

Again note that, if the suffix in $\tan_n^{-1} z$ is omitted, the right-hand side of the above equation reduces to a constant, namely 0.

We now use the technique of evaluating an integral in two different ways to derive some summation formulae for binomial coefficients.

Example (b). By the table, for an odd number $n = 2m+1$,

$$\int \cos^n t \mathrm{d}t = \frac{(n-1)!!}{n!!} \sin t \times \left[\frac{1}{\sqrt{1-z^2}}\right]_{n \mid z=\cos t} + C_1. \tag{10}$$

On the other hand, the substitution $x = \sin t$ gives

$$\begin{aligned}\int \cos^{2m+1} t \mathrm{d}t &= \int (1-\sin^2 t)^m \cos t \mathrm{d}t = \int (1-x^2)^m \,\mathrm{d}x \\ &= \int \left[1 - \binom{m}{1}x^2 + \binom{m}{2}x^4 - \binom{m}{3}x^6 \right. \\ &\quad \left. + \cdots + (-1)^m \binom{m}{m} x^{2m}\right] \mathrm{d}x \qquad (11) \\ &= x - \frac{1}{3}\binom{m}{1}x^3 + \frac{1}{5}\binom{m}{2}x^5 - \frac{1}{7}\binom{m}{3}x^7 \\ &\quad + \cdots + (-1)^m \frac{1}{2m+1}\binom{m}{m}x^{2m+1} + C_2,\end{aligned}$$

where $x = \sin t$. But the right-hand sides of (10) and (11) represent the same function and so, putting $t = 0$, we obtain $C_1 = C_2$. Hence

$$\begin{aligned}&\sin t - \frac{1}{3}\binom{m}{1}\sin^3 t + \frac{1}{5}\binom{m}{2}\sin^5 t - \frac{1}{7}\binom{m}{3}\sin^7 t \\ &\quad + \cdots + (-1)^m \frac{1}{2m+1}\binom{m}{m}\sin^{2m+1} t \\ &= \frac{(2m)!!}{(2m+1)!!}\sin t\left(1 + \frac{1}{2}\cos^2 t + \frac{3!!}{4!!}\cos^4 t \right. \\ &\quad \left. + \cdots + \frac{(2m-1)!!}{(2m)!!}\cos^{2m} t\right).\end{aligned}$$

Evaluating this equation for $t = \pi/2$ we finally obtain

$$\begin{aligned}&\binom{m}{0} - \frac{1}{3}\binom{m}{1} + \frac{1}{5}\binom{m}{2} - \frac{1}{7}\binom{m}{3} + \ldots \\ &\quad + (-1)^m \frac{1}{2m+1}\binom{m}{m} = \frac{(2m)!!}{(2m+1)!!}. \qquad (12)\end{aligned}$$

The same method will yield other summation formulae involving binomial coefficients of *one single row* of Pascal's triangle.

The method of repeated integration by parts also leads to summation formulae for binomial coefficients of *different rows* of Pascal's triangle, as is shown by the following example, the detailed working of which is omitted.

Example (c). For a function f with convergent series expansion in an open interval containing 0, we calculate $\int \left(f(x)/x^{n+1}\right) \mathrm{d}x$ in two different ways:

(A) integrate by parts n times and then replace $f(x)$ and its derivatives by their Maclaurin series;

(B) replace $f(x)$ by its Maclaurin series expansion *before* evaluating the integral term by term.

Comparing the coefficients of the powers of x obtained in (A) and (B) we now have

$$\frac{\binom{s}{0}}{\binom{n}{0}} + \frac{\binom{s}{1}}{\binom{n}{1}} + \frac{\binom{s}{2}}{\binom{n}{2}} + \cdots + \frac{\binom{s}{s}}{\binom{n}{s}} = \frac{n+1}{n+1-s} \tag{13}$$

for all $s, n \in \mathbb{N}_0$ with $0 \leqslant s \leqslant n$, and

$$\frac{\binom{s}{0}}{\binom{n}{0}} + \frac{\binom{s}{1}}{\binom{n}{1}} + \frac{\binom{s}{2}}{\binom{n}{2}} + \cdots + \frac{\binom{s}{n}}{\binom{n}{n}} = \frac{1}{s-(n+1)} \times \left[s\binom{s-1}{n} - (n+1)\right] \tag{14}$$

for all $s \in \mathbb{N}$ with $s \geqslant n+2$. The case $s = n+1$ not covered by (14) can be treated separately. The following identity is easily proved:

$$\frac{\binom{n+1}{0}}{\binom{n}{0}} + \frac{\binom{n+1}{1}}{\binom{n}{1}} + \frac{\binom{n+1}{2}}{\binom{n}{2}} + \cdots + \frac{\binom{n+1}{n}}{\binom{n}{n}} = (n+1)\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n+1}\right). \tag{15}$$

The identities (13)–(15) cover all cases of summations of quotients of the corresponding binomial coefficients of two different rows of Pascal's triangle.

The analogous summation formula for *products* of corresponding binomial coefficients is

$$\binom{n}{0}\binom{s}{0} + \binom{n}{1}\binom{s}{1} + \binom{n}{2}\binom{s}{2} + \cdots + \binom{n}{s}\binom{s}{s} = \binom{n+s}{s} \tag{16}$$

for all $n, s \in \mathbb{N}_0$ with $n \geqslant s$; it can be obtained by calculating $(\mathrm{d}^n/\mathrm{d}x^n)\left(x^n f(x)\right)$ in two different ways using the series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

(The case $n = s$ is treated in reference 2, page 6, number 32.)

References

1. W. Gröbner and N. Hofreiter, *Integraltafel*, vol. 1, (Springer, Vienna, 1957).
2. G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, vol. 1, (Springer, Berlin, 1972).

□

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Scaling into Pythagoras' Theorem

BRIAN R. STONEBRIDGE

According to fairly reliable sources, the year 2000 is the 2500th anniversary of the death of Pythagoras of Alexandria, and 2001 is the 2600th anniversary of his birth. His theorem has possibly been proved more times than any other theorem, yet even after almost $10^4/4$ years new approaches appear.

Introduction

There are very many proofs of Pythagoras' Theorem (reference 1). The most attractive of them usually rely upon geometrical relationships between sides of similar triangles (reference 2) or upon the fact that areas of similar triangles are proportional to the squares of corresponding sides.

However, the following proof indicates that the sophistication of 'similarity' or 'area' is not really necessary if one is prepared to accept the idea of 'scaling' a triangle, that is, redrawing it with the angles unchanged and lengths of the sides multiplied by a constant, s, say.

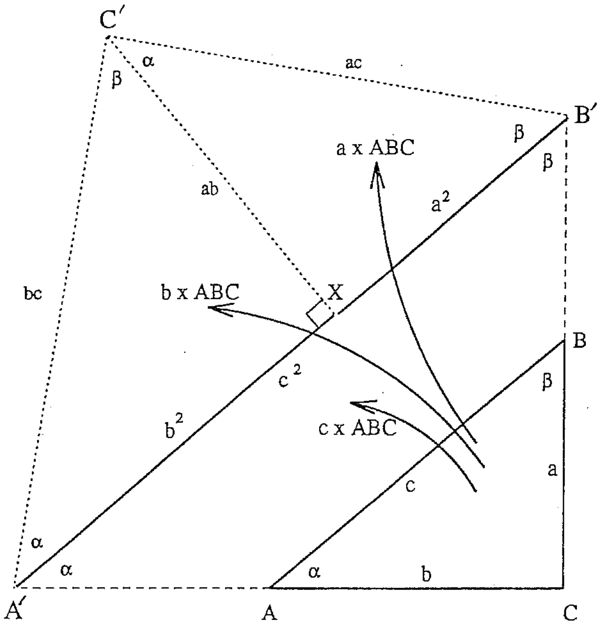

Figure 1. Constructions for a proof of Pythagoras' Theorem *by inspection*.

Proof

Figure 1 shows a proof which involves a scaling ($\times c$) of triangle ABC, to give A′B′C, reflection of this to give A′B′C′, and the construction of a perpendicular, C′X to simplify identification of the other scaled triangles ($\times a$ and $\times b$). The proof then is self-explanatory, by considering the length

$$\mathrm{A'B'} = \boxed{c^2 = a^2 + b^2} = \mathrm{A'X} + \mathrm{XB'}.$$

When several scalings are used in this way, the process is sometimes called a 'spray' operation in computer graphics; see figure 2. This may be helpful as a description.

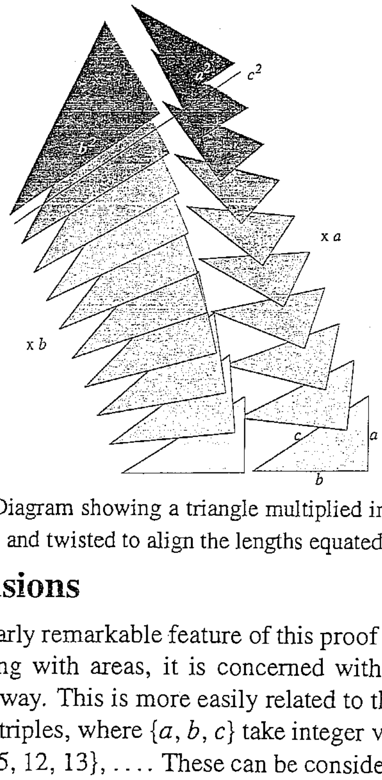

Figure 2. Diagram showing a triangle multiplied in turn by a and b and twisted to align the lengths equated.

Conclusions

A particularly remarkable feature of this proof is that, rather than dealing with areas, it is concerned with *lengths* in a very clear way. This is more easily related to the sets of Pythagorean triples, where $\{a, b, c\}$ take integer values such as $\{3, 4, 5\}, \{5, 12, 13\}, \ldots$. These can be considered as purely numerical relationships.

Various equivalent proofs are possible, e.g. the figure may be folded along A′B′, so making C′ coincident with C. This reduces the number of construction lines but does not simplify the proof.

Note to analysts

I readily concede that this proof implicitly allows the more recent notion of a *real* number, s, rather than the *commensurable* numbers of the time. This merely makes the theorem, as conceived, a special case of its current interpretation.

References

1. Euclid, *Elements* i. 47.
2. H. S. M. Coxeter, *Introduction to Geometry*, 2nd edition, (Wiley, New York, 1969). □

In common with most mathematicians, ***Brian Stonebridge*** *gave up Euclidean geometry at an early age, but he now believes that advances in computer graphics and the possibility of animated proofs will enable it to regain a prominent place in its own right and as a model for representations of logic, inference and proof.*

A Famous Theorem and Another Theorem

BRIAN D. BUNDAY

Perhaps the most famous and widely known theorem in mathematics is the theorem of Pythagoras: *in a right angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.* Even public figures admit to knowledge of this result, in contrast to their general reticence whenever the subject of mathematics gets a mention.

There are of course numerous ways of proving this theorem. For completeness, and for reasons that will soon become apparent, a vector proof is given.

For any triangle ABC (figure 1) it is well known that the sum of the vectors representing the three sides taken in one sense is the zero vector.

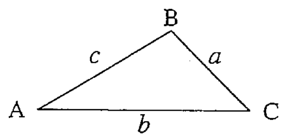

Figure 1

Thus

$$\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{0}. \tag{1}$$

Hence

$$\mathbf{AB} = -(\mathbf{BC} + \mathbf{CA}) \tag{2}$$

so that $\mathbf{AB} \cdot \mathbf{AB} = (\mathbf{BC} + \mathbf{CA}) \cdot (\mathbf{BC} + \mathbf{CA})$ on equating the scalar products. Therefore,

$$\mathrm{AB}^2 = \mathrm{BC}^2 + \mathrm{CA}^2 + 2\mathbf{BC} \cdot \mathbf{CA} \tag{3}$$

and, if the angle at C is a right angle, $\mathbf{BC} \cdot \mathbf{CA} = 0$ so that

$$\mathrm{AB}^2 = \mathrm{BC}^2 + \mathrm{CA}^2 \quad \text{or} \quad c^2 = a^2 + b^2. \tag{4}$$

A result that is not so well known is that there is an analogous theorem for a 'right angled tetrahedron', i.e. a tetrahedron formed by slicing off a vertex from a cuboid (see figure 2). For such a tetrahedron we have the following result: *the square of the area of the face opposite the vertex common to the three right angled faces is equal to the sum of the squares of the areas of the other three faces.*

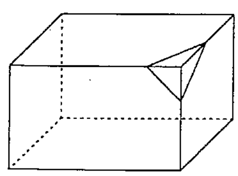

Figure 2

A simple proof using vector methods, and analogous to our earlier proof of Pythagoras' Theorem, is easy to construct. We need to remember that the vector area of the triangle defined by two intersecting vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\frac{1}{2}\boldsymbol{a} \wedge \boldsymbol{b}$. The magnitude of this vector is the area of the triangle, and its direction is perpendicular to the plane of the triangle in a sense given by a right-hand corkscrew.

Now, corresponding to (1), it is well known that the sum of the vector areas for the faces of a closed polyhedron is zero. For the tetrahedron OXYZ defined by the three edges (figure 3), $\mathbf{OX} = \boldsymbol{x}$, $\mathbf{OY} = \boldsymbol{y}$, and $\mathbf{OZ} = \boldsymbol{z}$, so that $\mathbf{XY} = \boldsymbol{y} - \boldsymbol{x}$ and $\mathbf{XZ} = \boldsymbol{z} - \boldsymbol{x}$ and the identity

$$(\boldsymbol{z} - \boldsymbol{x}) \wedge (\boldsymbol{y} - \boldsymbol{x}) + \boldsymbol{y} \wedge \boldsymbol{z} + \boldsymbol{z} \wedge \boldsymbol{x} + \boldsymbol{x} \wedge \boldsymbol{y} = \mathbf{0}$$

establishes this result.

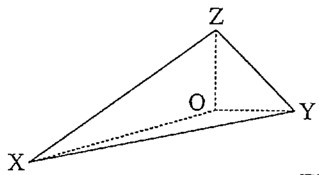

Figure 3

If we denote the corresponding vector areas, directed inwards, by $\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ then, corresponding to (2), we have

$$\boldsymbol{A} = -(\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}). \tag{5}$$

On squaring (forming the scalar product), we see that

$$A^2 = X^2 + Y^2 + Z^2 + 2\boldsymbol{Y} \cdot \boldsymbol{Z} + 2\boldsymbol{Z} \cdot \boldsymbol{X} + 2\boldsymbol{X} \cdot \boldsymbol{Y}. \tag{6}$$

If the tetrahedron is 'right angled', then $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ are mutually perpendicular and all the scalar products are zero, which yields the required result.

Of course, Pythagoras' Theorem can be regarded as a special case of the cosine rule for a triangle; indeed our result (3), which is always true, shows this, for in general $\mathbf{BC} \cdot \mathbf{CA} = -ab\cos C$ in the usual notation, so that, for any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{7}$$

Equation (6) shows that there is also a corresponding cosine formula for any tetrahedron. We note that the angle between a pair of area vectors is the supplement of the angle between the corresponding planes, i.e. the two angles add up to 180°. Then from (6) we can state that, for any tetrahedron, *the square of the area of one face is equal to the sum of the squares of the areas of the other three faces minus twice the sum of the products of the areas of the other faces two at a time and the cosine of the angle between them.*

Reference

C. E. Weatherburn, *Elementary Vector Analysis*, (Bell, London, 1958) pages 45, 55 for scalar and vector products. □

Brian Bunday *is Head of the Department of Mathematics at the University of Bradford. He finds relaxation in gardening (besides an acre of garden he grows his own vegetables and fruit on an allotment), chess and music.*

Designing Tiles

TAKAKAZU YAMAMOTO, TOMOHIRO OHTA and KAZUAKI KITAHARA

This article describes how to design your own tiles or wallpaper pattern; all you need is a computer.

Let L be a set of lattice points in $\mathbb{R}^2$; for example, set $L = \{x_{m,n} = (m, n) | m, n \in \mathbb{Z}\}$ (see figure 1).

With each point $x_{m,n}$ we associate its tile $T_{m,n}$ such that

$$T_{m,n} = \{x | x \in \mathbb{R}^2, r(x, x_{m,n}) \leqslant r(x, x_{p,q}) \; \forall p, q \in \mathbb{Z}\}, \quad (1)$$

where $r(x, y)$ denotes the usual distance between the points x and y. Each $T_{m,n}$ is the territory which consists of the points nearest to $x_{m,n}$ among all $x_{p,q}$ $(p, q \in \mathbb{Z})$. Each $T_{m,n}$ forms a square with side 1 (see figure 2).

We now replace $r(x, y)$ by the function $d(x, y)$, where

$$d(x, y) = r(x, y)(|\cos(\theta + 1) + \cos(2\theta + 1) + \cos(3\theta + 1)| + 1),$$

and θ is the argument of $y - x$ regarded as a complex number. (Note that $d(x, y)$ is not a distance function in the usual sense. For one thing, $d(x, y) = d(y, x)$ does not hold.) What do the tiles $T_{m,n}$ look like when, in (1), we replace $r(x, y)$ by $d(x, y)$? The design, obtained by means of a QBasic program (on an IBM PC) is shown in figure 3.

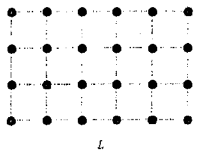

Figure 1

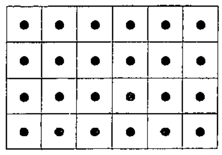

Figure 2

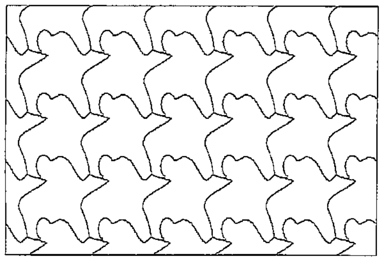

Figure 3

Below is a sample QBasic program for generating tiles; with the aid of a computer one can clearly obtain many other tilings simply by varying the distance function.

```
100 DIM Dx%(49), Dy%(49), Dc%(49)
110 CONST pi! = 3.141593
120 SCREEN 12
130 CLS
140
150 i% = 0
160 FOR y% = 1 TO 7
170  FOR x% = 1 TO 7
180   i% = i% + 1
190   Dx%(i%) = x% * 80
200   Dy%(i%) = (y% - 1) * 80
210   Dc%(i%) = (y% MOD 2) * 2
             + (x% MOD 2) + 1
220  NEXT x%
230 NEXT y%
240
250 FOR y% = 80 TO 400
              - 5 STEP 5
260  FOR x% = 160 TO 480
              - 5 STEP 5
270   MinDist! = dist!(x%,
              y%, Dx%(1),
              Dy%(1),
              Dc%(1))
280   col% = Dc%(1)
290   FOR i% = 2 TO 49
300     IF dist!(x%, y%, Dx%(i%),
            Dy%(i%), Dc%(i%))
                      < MinDist!
        THEN
310      MinDist! = dist!(x%, y%,
               Dx%(i%), Dy%(i%),
                       Dc%(i%))
320      col% = Dc%(i%)
330     END IF
340    NEXT i%
350    LINE (x%, y%) - STEP(5, 5),
                    col% + 8, BF
360  NEXT x%
370 NEXT y%
380
390 END
400
410 FUNCTION dist! (x1%, y1%,
            x2%, y2%, col%)
420  x! = x1% - x2%
430  y! = y1% - y2%
440  R! = SQR(x! * x! + y! * y!)
450
460  IF x! > 0 THEN
470   IF y! >= 0 THEN
480    arg! = ATN(y! / x!)
490    ELSE
500     arg! = ATN(y! / x!)
                  + 2 * pi!
510    END IF
520  ELSEIF x! < 0 THEN
530   arg! = ATN(y! / x!) + pi!
540  ELSE
550   IF y! >= 0 THEN
560    arg! = pi! / 2
570   ELSE
580    arg! = pi! * 3 / 2
590   END IF
600  END IF
610
620 DO WHILE arg! > 2 * pi!
630  arg! = arg! - 2 * pi!
640 LOOP
650 DO WHILE arg! < 0
660  arg! = arg! + 2 * pi!
670 LOOP
680
690 dist! = R! * (ABS
        (COS(arg!+1)
       + COS(2*arg!+1)
       + COS(3*arg!+1)) +1)
700 END FUNCTION
```

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Mathematics in the Classroom

Simulation

One of the optional modules in our new A-level syllabus is called *Decision and Discrete Mathematics* and this area of mathematics has certainly become very popular in recent times as computer technology has advanced. One area that it covers is simulation, which provides ample opportunities for some game-like activities to grab the attention of most students. The purpose of these exercises is to illustrate the ideas of probability and modelling, and to gain some idea of what will happen in a real situation. This is achieved if we can find a mathematical model that is probabilistically the same as our real situation, and then observe the model over as many simulations as we need to get a clear idea of what is happening.

The computer is a powerful tool when it comes to simulations, but much can also be achieved with pencil, paper, dice, coins and random numbers. Indeed, computer programs written to carry out simulations are more likely to be successful if a few have first been carried out by hand to test the logic.

The following examples provide classic situations where answers can be achieved by simulations as well as probabilistic analysis.

The collector's problem

If a cereal manufacturer is giving away a set of eight model cars, one in each packet that is purchased, how many packets would you expect to buy in order to achieve a full set? In addition, if there were two collectors in one family, how many packets would then be expected to be purchased in order to achieve two full sets?

This situation can be modelled quite easily with a set of random number tables. Starting at a random point in the tables and discarding any 0's or 9's that occur, we continue until the numbers 1–8 have all occurred and the total number of random numbers used to achieve this has been noted. This is repeated many times, and the average of the total number of random numbers used to achieve all eight outcomes is calculated to give one an idea of the investment in cereal packets needed to acquire the complete set of cars.

Probabilistically, this can be solved by recourse to the geometric distribution. This probability distribution involves trials with only two possible outcomes ('I get a car that I want' or 'I get a duplicate' in this case), usually called success and failure. If X is the random variable that denotes the number of trials needed to obtain a success, where the probability of a success is p, can you show that $E(X)$, i.e. the mean value, or expected value, of X is $1/p$? Then can you use this result to show that you must expect to buy between 21 and 22 packets of cereal (21.743 to be precise) in order to obtain a full set of the cars? This is left as an exercise for the reader.

As soon as we add in the complication of a second collector in the same family, then probabilistically the problem becomes much more tricky. But simulation still works. After 5000 simulations (with a computer program this time), a group of my students had arrived at an average of 34.6 packets needing to be purchased. Along the way, we observed a tremendous spread of observations. Not only had we detected as few as 19, but also as many as 102 packets needing to be purchased before two complete sets had been obtained. Mathematically this problem is challenging and I owe my solution to Dr David Grey (personal communication) who showed that

$$E(N) = \int_0^\infty \left[1 - (1 - \mathrm{e}^{-u} - u\mathrm{e}^{-u})^8\right] \mathrm{d}u$$

where N is the number needed to achieve two complete sets. After a tedious expansion (again left to the reader) the answer emerges as 34.88 so our simulation had worked very well. David Grey also pointed out that, using inclusion–exclusion arguments, and observing only the dominant term, you could deduce that

$$\Pr(N > n) \sim n(\tfrac{7}{8})^{n-1} \quad \text{as } n \to \infty$$

and hence

$$\Pr(N > 100) = 0.0002$$

which means that in 5000 simulations we should expect one over 100!

Sharing out the Christmas pudding (reference 1).

Suppose a Christmas pudding is shared among six people so that all have equal portions. Hidden in the pudding are six coins. What are the chances that everyone will get a coin? How many, on average, will get no coin? How many people will get more than one coin?

Using a die, a circle cut into six equal sectors numbered 1–6 to represent the six portions of pudding, and six counters to represent the coins, this situation can be readily simulated in the classroom. The die is rolled six times with a counter placed each time on the sector of the circle corresponding to the outcome of the roll. Once the six counters have all been placed, the number of empty sectors is recorded and also the number of sectors with more than one counter. This single simulation can be repeated many times and estimates found for the three quantities defined above.

The theoretical model underlying this simulation relies on the binomial distribution. The number of coins, X, in any portion of pudding will have a binomial distribution with $n = 6$ and $p = 1/6$. Hence $\Pr(X = 0) = (\frac{5}{6})^6 = 0.3349$.

Therefore we expect $6 \times 0.3349 = 2.009$ portions to have no coins.

Can you now show that $\Pr(X > 1) = 0.2632$ and so arrive at the expected number of portions containing more than one coin as 1.579?

Can you verify that the probability that all six portions contain one coin apiece is 0.0154? This is not very likely, so you would expect to carry out about 100 simulations before this outcome occurs.

Waiting at the doctor's surgery (reference 2).

You can test an appointments system by a simulation in the following way. Suppose that the doctor makes appointments at 10-minute intervals between 9.00am and 11.00am. We now need a model for the consultation times. An unsophisticated version of this might describe consultation times as being 5 minutes, 10 minutes or 15 minutes in the ratios 3:5:2. Suppose we use random digits and associate 0, 1, 2 with a consultation time of 5 minutes, the occurrence of 3, 4, 5, 6, 7 with a consultation time of 10 minutes and finally, 8, 9 with a consultation time of 15 minutes. We then run the simulation for the 18 appointments, recording the outcomes as in table 1.

On such a system you can soon see why lengthy queues build up in the doctor's waiting room even though the appointment times seem quite reasonable. At the end of the simulation you can calculate the amount of time that the doctor is idle and the average waiting time of the patient.

What refinements can you suggest for the model for consultation times? Run a simulation using your refinements to see what happens. An ideal system is one in which the doctor's idle time is low and so is the patient's waiting time. Can you suggest a model that will achieve both these aims?

Table 1

Random number	App. time	Consult. time	Start	End	Doctor idle	Patient wait
6	9.00	10	9.00	9.10	0	0
9	9.10	15	9.10	9.25	0	0
8	9.20	15	9.25	9.40	0	5
5	9.30	10	9.40	9.50	0	10
⋮	⋮	⋮	⋮	⋮	⋮	⋮

If you have any favourite simulations that illustrate ideas that perhaps can or cannot be treated mathematically within the confines of your syllabus, please write and let us know.

Carol Nixon

References

1. F. R. Watson, *A Simple Introduction to Simulation*, (Keele Mathematical Education Publications, 1980).

2. P. Sprent, *Statistics in Action*, (Penguin, London, 1977) □

Computer Column

Genetic algorithms

Huxley once wrote that 'six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum'. We can quickly see that Huxley's statement is nonsense, even if we replace six monkeys with millions of computers.

Let's be less ambitious than Huxley and ask if random typing can produce just the complete works of Shakespeare. My edition of the complete works is 1230 pages long; assuming 500 words per page, and five characters per word, the volume contains about 3×10^6 characters. There are about 60 characters from which to choose (A–Z, a–z, and a few punctuation marks). The probability of obtaining a correct string of three million characters is therefore $P = 1/60^{3\times 10^6} \approx 10^{5.334.000}$. Roughly $10^{5.334.000}$ trials are needed for an even chance of producing the complete works.

Modern PCs run at around 100 MHz, so let's assume that a computer can make 10^8 trials every second. Furthermore, assume that *every single particle* in the Universe is a computer (physicists guess that there are something like 10^{80} subatomic particles). Finally, assume that every one of these subatomic computers has been at the task since the start of the Universe (something like 10^{17} seconds, according to astronomers). Could this universe full of computers, typing at random since the beginning of time, reproduce the works of Shakespeare? No. They could have made only 10^{105} trials, which is vanishingly small compared to $10^{5.334.000}$.

Instead of the complete works, what are our chances of typing just the word SHAKESPEARE at random? If we restrict the character set to A–Z, there are $1/26^{11}$ combinations: about 3.67×10^{15} trials are needed. A computer could do this, but it would take a long time. Is there any way that a random process can perform better? Here is one method.

Step 1. Generate a random string 11 characters long. Call this the 'reference' string.

Step 2. Generate another random string of 11 characters. Call this the 'new' string.

Step 3. Check the first character in each string. Is the character from the new string closer to the letter S than that from the reference string? If yes, write it to the first position of the reference string. If no, leave the reference string unchanged. Repeat for the second character in each string (comparing with the letter H), then the third (comparing with A) and so on. This produces a modified reference string.

Step 4. Is the reference string SHAKESPEARE? If yes, we have succeeded. If no, go to step 2.

My Basic program, based on the above ideas, began with the string URLAKAMABZS and after 20 steps had reached SKBDKPMEBTL. This may look little better, but with each step the program gets closer to the required answer.

Try writing your own program. (Any language will do, and you can no doubt improve on this sketchy algorithm.) Typically how many trials are required before the reference string converges to SHAKESPEARE? You may be surprised!

This is a *very* crude example of a genetic algorithm (GA). A GA has a goal function and a set of candidate solutions; you 'mate' two candidates to produce 'offspring', which replace the least-fit candidates. The candidates 'evolve' to the goal, in a manner reminiscent of the way in which living beings evolve. GAs are widely used — from designing the shape of aeroplane wings, to picking investment strategies. Perhaps you can think of more uses?

Stephen Webb

Letters to the Editor

Dear Editor,

A note on a sequence free from powers

Let $p_1 = 2, p_2 = 3, \ldots$ be the sequence of primes and let

$$T_n = (p_1 p_2 \ldots p_n) + 1$$

so that $T_1 = 3, T_2 = 7 \ldots$. The first composite value of T_n is

$$T_7 = 510511 = 19 \times 26869.$$

The sequence T_n is frequently used to show that there exist infinitely many primes. Recently Prakash (*Mathematical Spectrum* Volume 22 Number 3 pages 92–3) proved the following result.

Theorem 1. For each $n \geqslant 2$, T_n

(i) is never a square or higher power of any natural number;

(ii) has at most $n-1$ prime factors, counting multiplicities.

It is straightforward to improve the second result.

Theorem 2. For each $n \geqslant 4$, T_n has at most $n-3$ prime factors, counting multiplicities.

We need a lemma.

Lemma. For $n \geqslant 6$,

$$T_n < (p_{n+1})^{n-2}.$$

Proof. The proof is by induction on n. Since

$$30031 = T_6 < p_7^4 = 83521,$$

the result is true for $n = 6$. Now, assuming the validity for n, we get

$$\begin{aligned} T_{n+1} &= T_n p_{n+1} - (p_{n+1} - 1) < T_n p_{n+1} \\ &< (p_{n+1})^{n-2} p_{n+1} = (p_{n+1})^{n-1} < p_{n+2}^{n-1}. \end{aligned}$$

Thus the result is true for $n+1$, completing the induction.

Proof of Theorem 2. Clearly, any prime divisor of T_n satisfies $p \geqslant p_{n+1}$. If T_n had $n-2$ (or more) prime divisors (when counted with multiplicity) then $T_n \geqslant (p_{n+1})^{n-2}$, contradicting the lemma.

We define the sequence D_n by

$$D_n = T_n - 2 = p_1 p_2 \ldots p_n - 1$$

so that $D_1 = 1$, $D_2 = 5$, $D_3 = 29$ and the first composite value is $D_4 = 209 = 11 \times 19$. There are analogous results for the sequence D_n.

Theorem 3. (i) For $n \geqslant 3$, D_n has at most $n-2$ prime factors, when counted with multiplicity;

(ii) for $n \geqslant 2$, D_n is never the square or higher power of a natural number.

Proof. (i) Since $D_n < T_n$, for $n \geqslant 6$ this follows in the same way that we proved the result for T_n. The cases $n = 3, 4, 5$ are checked individually. Note that it is the case $n = 4$ that requires $n-2$ rather than $n-3$ prime factors.

(ii) Suppose that $D_n = N^k$ for some natural numbers N and $k > 1$. If $k = 2$ we have

$$N^2 = D_n = 2 \times 3 \times \ldots p_n - 1 \equiv -1 \pmod 3,$$

which is not possible. Note that this also shows the impossibility whenever k is even.

Now we may suppose that $k > 2$, and further that k is an odd prime. (If D_n is a power it will be a power to a prime exponent.) From part (i) we see that $k < n$ so k must be one of $p_2, \ldots p_n$. By Fermat's Theorem

$$-1 \equiv D_n \equiv N^k \equiv N \pmod k$$

so $N = mk - 1$ for some natural number m. Then

$$\begin{aligned} p_1 \ldots p_n &= D_n + 1 \\ &= (mk-1)^k + 1 \\ &= (mk)^k - k(mk)^{k-1} + \cdots + k(mk), \end{aligned}$$

which is divisible by k^2, which is clearly not possible.

Yours sincerely,

A. A. K. MAJUMDAR
(Jahangirnagar University,
Bangladesh)

Dear Editor,

Difficult questions

Readers may be able to help me solve the following questions.

1. How many positive integers less than 10^{10} have exactly one digit equal to a square and have the sum of their digits equal to a square? For example, 592 satisfies both conditions as 9 is a square and $5+9+2 = 16$, a square.
2. How many positive integers less than 10^{10} have all their digits equal to a prime and have the sum of their digits equal to a prime? For example, 23572 satisfies both conditions since 2, 3, 5 and 7 are all primes and $2+3+5+7+2 = 19$, a prime.

Yours sincerely,

KAMLESH GAYA
(204 Antelme Avenue,
Quatre-Bornes, Mauritius)

Dear Editor,

On a type of arithmetic function

I should like to introduce a type of elementary arithmetic function which has very interesting results.

The square case. A natural number A is written in its decimal form as

$$\begin{aligned} A &= a_n a_{n-1} \dots a_1 \\ &= a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + \dots + a_1, \end{aligned}$$

and take

$$f(A) = a_n^2 + a_{n-1}^2 + \dots + a_1^2.$$

Starting with a natural number A, we consider the sequence

$$A,\ f(A),\ f(f(A)),\ f(f(f(A))), \dots.$$

For example, when $A = 23$ the sequence is

$$23, 13, 10, 1, 1, 1, \dots.$$

Question. When does this sequence converge?

First we observe that if $A \geqslant 1000$ then $n \geqslant 4$, and

$$A \geqslant 10^{n-1} > 81n \geqslant f(A),$$

so the sequence will decrease until it reaches a value smaller than 1000. For $100 \leqslant A \leqslant 999$,

$$\begin{aligned} A &= 100a_3 + 10a_2 + a_1 \\ &\geqslant a_3^2 + a_2^2 + 91a_3 + a_1 > a_3^2 + a_2^2 + a_1^2 = f(A), \end{aligned}$$

so the sequence decreases until it reaches a value smaller than 100.

Thus it is sufficient to investigate what happens to the numbers $A < 100$. We find that either

(i) the numbers 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97 all start sequences which converge to 1; or

(ii) all other natural numbers $A < 100$ start sequences which end up in the 8-element cycle 89, 145, 42, 20, 4, 16, 37, 58.

It is an interesting result, I think. It also shows that $A = 1$ is the only fixed point of the sequences, i.e. the only solution of $f(A) = A$. However, a question remains. What is the mathematical meaning of the cyclic number set? Does it have any connection with group theory and, if so, what sort of group is involved? I still have no idea about it.

The cubic case. Now we write

$$g(A) = a_n^3 + a_{n-1}^3 + a_{n-2}^3 + \dots + a_1^3$$

and consider the sequence $A, g(A), g(g(A)), \dots$. For example,

$$g(1234) = 100,\ g(100) = 1,\ g(1) = 1,\ \dots.$$

I found the following results, which are proved in the same way as results about f.

(i) For $A \geqslant 10000$, then $n \geqslant 5$ and

$$A \geqslant 10^{n-1} > 729n \geqslant g(A)$$

so the sequence decreases until it reaches a value smaller than 10000.

(ii) For $2000 \leqslant A \leqslant 9999$,

$$\begin{aligned} A &= 1000a_4 + 100a_3 + 10a_2 + a_1 \\ &\geqslant a_4^3 + 919a_4 + a_3^3 + 19a_3 + a_2^3 - 71a_2 + a_1^3 - 80a_1 \\ &\geqslant a_4^3 + a_3^3 + a_2^3 + a_1^3 + 919 \times 2 - 71 \times 9 - 80 \times 9 \\ &> a_4^3 + a_3^3 + a_2^3 + a_1^3, \end{aligned}$$

so the sequence will decrease until it reaches a value smaller than 2000.

Thus it is sufficient to investigate the cases $A < 2000$ and we find that the sequence ends up either

(a) at one of 5 fixed points of the mapping, 1, 153, 370, 371 and 407; or

(b) in one of 4 cycles $\{136, 244\}$, $\{919, 1459\}$, $\{55, 250, 133\}$, $\{160, 217, 352\}$.

We see that g has only 5 fixed points. Do the fixed points and cycles have any mathematical meaning? I have no idea. Readers may like to investigate the analogous results for higher powers and for bases other than 10. If anyone is interested in this topic I should like to hear your ideas.

Yours sincerely,

KENICHIRO KASHIHARA
(Kamitsuruma 4-13-15, Sagamihara, Kanagawa 228, Japan)

Dear Editor,

Paradox in probability

In Volume 28 Number 3 page 62, Basil Rennie sets out a paradox that is expressed in terms of gamblers tossing silver dollars. One of them comments that the likeness of George Washington on the face of the coin is admirable (which, of course, gives away the information that the coin he or she is looking at is likely to be heads up).

I believe that Washington has appeared only on the 'quarter' or 25-cent piece, among coins. (He is, of course, on the dollar bill, but that would not work well for tossing!)

Yours sincerely,

BRIAN TREADWAY
(Department of Mathematics, The University of Iowa)

Dear Editor,

The domino problem

In the latest issue of *Mathematical Spectrum*, Volume 29 Number 1 page 19, you publish a further letter from Alastair Summers concerning the 'domino problem'. At the end he gives his result for the 6×14 case, and in his third paragraph he asks readers to apply his methods to find the number of ways of covering the chessboard.

Both these numbers have already been found using the formula of P. W. Kasteleyn, which was reported in Robin Chapman's letter in the previous issue of *Mathematical Spectrum*, Volume 28 Number 3 page 68. Indeed, the total for the 8×8 chessboard of $12{,}988{,}816 = 3604^2$ was reported in Mr Chapman's letter. The total was apparently first explicitly calculated by James Propp in 1992.

Earlier this year Mr Chapman sent me a print-out showing all the totals for boards up to 20×20, calculated by using the computer algebra system Maple. This also includes the 6×14 case mentioned above, confirming Mr Summers' result.

I published many of these results, those up to 72 squares, in an article entitled 'Dominizing the Chessboard' in *The Games and Puzzles Journal*, which I started up again at the end of May after a break of seven years.

Yours sincerely,

GEORGE JELLISS
(63 Eversfield Place,
St Leonards on Sea,
East Sussex, TN37 6DB)

Old Prime's Crossnumber Puzzle

1	2	3	4	5	6	7	8	9	10
11				12			13		
14			15			16			
17			18		19		20		
21	22	23				24		25	
26				27			28		
29			30			31			
32				33					

Clues—Across

1. A multiple of this number is obtained by removing the first digit and placing it after the last digit.
7. The year in the twentieth century when Easter is earliest.
11. Divisible by 7, 11, 13.
12. Multiple of 30 Down.
13. When added to 16 Across is equal to the sum of 23 Down and 25 Down.
14. See 26 Across.
15. A multiple of 9.
16. See 13 Across.
17. This number has the same first and last digits.
18. A multiple of 3.
19. Ten times 31 Across plus five times 13 Across.
21. Factorial 9.
24. Multiple of 28 Across.
26. Sum of 3 Down and 14 Across.
27. See 8 Down.
28. See 24 Across.
29. See 4 Down.
31. See 19 Across.
32. Equal to 22 Down.
33. $100\,000 \times \pi$ to the nearest integer.

Clues—Down

1. The cube of a prime number.
2. A multiple of 17 Across.
3. A multiple of 7.
4. Sum of twice 21 Across and 29 Across.
5. See 10 Down.
6. This number is equal to the sum of the cubes of its digits.
7. A cube number.
8. The sum of 15 Across and 27 Across.
9. See 20 Down.
10. A multiple of 5 Down.
19. A square number.
20. Ten times 9 Down plus 1.
22. Equal to 32 Across.
23. See 13 Across.
25. See 13 Across.
30. A factor of 12 Across.

JAN SUTO

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

29.5 Prove that a triangle can be triangulated into n similar triangles for every $n \geqslant 6$.

(Submitted by Mansur Boase, St Paul's School, London)

29.6 Find all prime numbers p for which $2p-1$ and $2p+1$ are both prime.

(Submitted by Des MacHale, University College, Cork)

29.7 A goat is tethered by a rope to a point on the circumference of a circular field of radius r. The goat can reach exactly half the area of the field. How long is the rope?

(Submitted by Kamlesh Gaya, Mauritius)

29.8 Using compasses only, find the centre of a given circle.

(Submitted by Eddie Kent)

Solutions to Problems in Volume 28 Number 3

28.9 A polynomial function of degree n is such that $p(x) \geqslant 0$ for all x. Prove that

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geqslant 0$$

for all x.

Solution by Adrian Sanders, King's College School, Wimbledon

Write

$$f(x) = p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x).$$

Since $p(x) \geqslant 0$ for all x, n must be even and the leading coefficient of $p(x)$ is positive. So f is also of even degree and has a positive leading coefficient. Thus, if $f(x)$ takes its least value at $x = X$, then $f'(X) = 0$. Now,

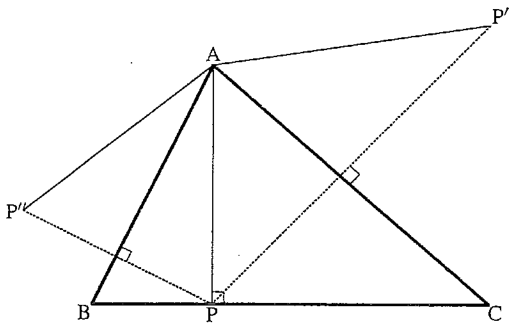
$$f(x) = f'(x) + p(x).$$

Hence $f(X) = p(X) \geqslant 0$. Hence $f(x) \geqslant f(X) \geqslant 0$ for all x.

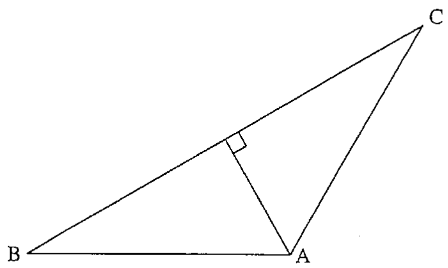
Also solved by Toby Gee (The John of Gaunt School, Trowbridge).

28.10 Let P be a point on one of the sides of triangle ABC, not equal to A, B or C. Reflect P across each of the other two sides to P′ and P″. What is the minimum value of the radius of the circumscribed circle of triangle PP′P″ as P travels round the perimeter of triangle ABC?

Solution by Andrew Lobb, St Olave's Grammar School, Orpington



When a point X is reflected in a line YZ to X′, then YZ is the perpendicular bisector of XX′. Also, to find the circumcentre of a triangle, we find the point where the perpendicular bisectors of two sides meet. Thus, when P is on side BC of triangle ABC, the circumcentre of the circle through P, P′ and P″ is A and the circumradius is AP; and similarly when P is on sides CA or AB. Thus, when the triangle is acute-angled, the minimum circumradius is the smallest of the altitudes of the triangle.

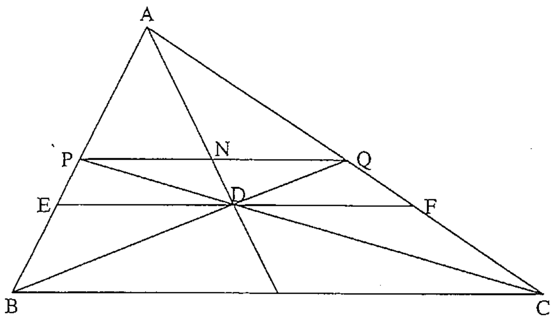


Now consider an obtuse-angled triangle, with the obtuse angle at A. When P is on BC, the minimum circumradius is the altitude from A. When P is on CA, the circumradius is always greater than AB (P cannot be at A); when P is on AB, the circumradius is always greater than AC. Both AB and AC are greater than the altitude from A, so the minimum circumradius is the altitude from A, which is again the smallest altitude of the triangle.

Also solved by Adrian Sanders, Can Minh (University of California, Berkeley), Paul Russell (St Bride's High School, East Kilbride), Mansur Boase, Toby Gee.

28.11 Let P and Q be points on the sides AB and AC of triangle ABC, respectively, and let D be the point of intersection of BQ and CP. Prove that AD bisects PQ if and only if PQ is parallel to BC.

Solution by Junji Inaba, William Hulme's Grammar School, Manchester



Suppose that PQ is parallel to BC. Draw the line through D parallel to PQ to meet AB and AC in E and F respectively. Denote the distance between PQ and BC by H and the distance between EF and BC by h. Since triangles PQB, EDB and PQC, DFC are similar,

$$\frac{\mathrm{PQ}}{\mathrm{ED}} = \frac{\mathrm{PB}}{\mathrm{EB}} = \frac{H}{h}$$

and

$$\frac{\mathrm{PQ}}{\mathrm{DF}} = \frac{\mathrm{QC}}{\mathrm{FC}} = \frac{H}{h}$$

so ED = DF. Since triangles APN, AED and ANQ, ADF are similar,

$$\frac{\mathrm{PN}}{\mathrm{ED}} = \frac{\mathrm{AN}}{\mathrm{AD}} = \frac{\mathrm{NQ}}{\mathrm{DF}}.$$

It follows that PN = NQ.

Now suppose that AD bisects PQ and draw the line EF parallel to PQ as before. This time, let H, h be the respective distances of B from PQ and EF and H', h' the respective distances of C from PQ and EF. Then, as before,

$$\frac{\mathrm{PQ}}{\mathrm{ED}} = \frac{H}{h}$$

and

$$\frac{\mathrm{PQ}}{\mathrm{DF}} = \frac{H'}{h'}.$$

Also as before,

$$\frac{\mathrm{PN}}{\mathrm{ED}} = \frac{\mathrm{NQ}}{\mathrm{DF}}.$$

Since PN = NQ it follows that ED = DF. Hence

$$\frac{H}{h} = \frac{H'}{h'}.$$

Since PQ is parallel to EF,

$$H - h = H' - h'.$$

Hence

$$Hh' - hh' = H'h' - h'^2$$

so

$$H'h - hh' = H'h' - h'^2$$

i.e.

$$(H' - h')(h - h') = 0.$$

Since $H' \neq h'$, it follows that $h = h'$, whence also $H = H'$. Hence BC is parallel to PQ.

Also solved by Adrian Sanders, Can Minh, Mansur Boase and Toby Gee.

28.12 Prove that no term of the Fibonacci sequence ends in 1996.

Andrew Lobb, Adrian Sanders and Paul Russell all gave the same solution.

A number that ends in 1996 is congruent to 1996 mod 10000 and so to 4 (mod 8). The Fibonacci sequence mod 8 is

$$1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0$$

and then repeats, so no term in it is congruent to 4 (mod 8).

Also solved by Junji Inaba and Mansur Boase.

Mathematical Spectrum Awards for Volume 28

Prizes have been awarded to the following student readers for contributions published in Volume 28:

- **James Horth** for his article 'Graphic calculators and the Mandelbrot set' (pages 16–18);
- **Filip Sajdak** for his article 'The Roseberry conjecture' (page 33);
- **Toby Gee** for other contributions.

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Show that all positive integers can be written as $5k - 3l$, where k and l are positive integers.

KAMLESH GAYA
(Mauritius)

Each square of an $n \times n$ chessboard is occupied by a piece. Show that each piece can be moved horizontally or vertically to an adjacent square so that each square is again occupied by a piece if and only if n is even.

Reviews

She Does Math! Edited by MARLA PARKER. The Mathematical Association of America, Washington, DC, 1995. Pp. 272. Paperback \$24.00 (ISBN 0-88385-702-2).

In many ways this is an interesting book. It is partly biographical, showing how various women have become interested in certain aspects of mathematics, how it has become relevant in their professional lives and how they have coped with family responsibilities and a career. I certainly found the potted biographies eminently readable and some of the attitudes displayed by families and contemporaries are still encountered today. Biographies such as these may help some schoolgirls to realise that they are not alone in meeting opposition when they contemplate a scientific training.

The 'problem' pages at the end of each biography make interesting reading and hopefully may fire the imagination of some young people. At certain stages it is easy to find routine practice of mathematical techniques boring and repetitive, and to wonder where it is leading and whether it has any relevance to life outside the classroom. The problems give a glimpse of some of the possible uses.

Although I like the book in many ways, I find the title jars somewhat; 'She Does Math!' seems to suggest that it is particularly strange for a female to take mathematics. I feel that, in the modern world, this incredulity is out of place, and a mathematician should be judged on the quality of his or her scientific work and not on gender or appearance. Admittedly the admission that one is a mathematician is a great conversation stopper at any party!

Much as I like the biographies in the book I would have preferred a volume along the same lines which included both men and women scientists.

University of Sheffield MARY HART

From Polynomials to Sums of Squares By T. JACKSON. IOP Publishing, Bristol, 1995. Pp. xii+184. Paperback £17.50 (ISBN 0-750303298).

The great joy of this volume is its accompanying computer disk. Using the software you can, without mental effort, do arithmetic in polynomials with rational coefficients or integers modulo a prime p, or in the integers of quadratic number fields $\mathbb{Q}(\sqrt{d})$ for various values of d. You can evaluate Legendre symbols and watch the program using the Law of Quadratic Reciprocity, and can express natural numbers as sums of squares, or in the form $x^2 + ky^2 + z^2 + t^2$ for $k = 1, 2, 3$ and 5. The danger, as the author points out, is that readers may leave everything to the computer!

Readers who enjoyed the author's previous volume *From Number Theory to Secret Codes*, also with an accompanying disk, will not be disappointed. The text is informally but rigorously presented. Perhaps it gets carried away in the later stages, and it may be difficult for university students meeting this material for the first time. This volume may best be used as a supplement for one of the more conventional texts; it should give great pleasure.

One niggle. The statement of Eisenstein's Irreducibility Criterion on page 26 is incorrect; the conditions do not give that the polynomial is irreducible in $\mathbb{Z}[x]$, only in $\mathbb{Q}[x]$ — try $2x + 6$ with $p = 3$.

University of Sheffield DAVID SHARPE

100% Mathematical Proof By R. GARNIER AND J. TAYLOR. Wiley, Chichester, 1996. Pp. 326. Paperback £16.95 (ISBN 0-471-96199-X).

This book seeks to explain the idea of mathematical proof and 'to describe how proofs may be discovered and communicated'. It begins by covering propositional and predicate logic. This is followed by chapters on various types of proof: direct proof, existence proof, etc.

Throughout the book, there are detailed explanations which are, for the most part, very clear. However, the explanations seem to become rather long-winded in certain places, particularly in the sections on propositional and predicate logic. There are also several examples and sets of exercises. Another useful feature of the book is that detailed hints and solutions to many of the exercises are provided at the back

The descriptions of the various types of proof seem to suggest that it is aimed at those who are fairly unfamiliar with mathematical proof. However, I felt that several sections, particularly the formal proofs given to justify each method of proof, might seem rather daunting to such a student.

I also spotted a flaw in one of the proofs. The authors state the result that, if a prime p is a divisor of the product of two integers a and b, then it is a divisor of either a or of b, and prove this using the Fundamental Theorem of Arithmetic. In the following chapter, the Fundamental Theorem of Arithmetic is proved using this result.

Overall, this book could be useful as an introduction to mathematical proof, although I felt it was perhaps over-long in parts.

Student, East Kilbride, Glasgow PAUL RUSSELL

Groups and Symmetry By DAVID W. FARMER. American Mathematical Society, Providence, RI, 1995. Pp. 102. Paperback \$19.00 (ISBN 0-8218-0450-2).

Knots and Surfaces By DAVID W. FARMER AND THEODORE B. STANFORD. American Mathematical Society, Providence, RI, 1995. Pp. 101. Paperback \$19.00 (ISBN 0-8218-0451-2).

Expositions of mathematics, whether in books or in the lecture theatre, go something like this: Definition/Remark/Definition/Theorem/Example/Theorem etc. The reader is soon gazing through the window at the free world outside and the student is falling asleep (the lecture room has no windows). Little or no insight is given into how it all arose. It is like walking through a picture gallery, impressive perhaps but uninvolving.

These companion volumes attempt to take the reader into the artist's studio; more than that, to make the reader into the artist. Few of us are Eulers (or Rembrandts), so much of the time we shall fail, but we can be stimulated even in failure to discover something else that we were not looking for.

So there is not a theorem in sight, at least formally presented. There are lots of tasks for readers to attempt and no answers, so you cannot cheat and will sometimes fail in your task. Nor will you be able to check whether you have got it right, although, if you have, you will know in your bones! Readers are discovering their own mathematics.

The volume on *Groups and Symmetry* covers such topics as symmetries of plane figures, strip patterns and wall patterns; the volume on *Knots and Surfaces* covers graphs drawn in the plane and on spheres, toruses, double toruses etc, and knots, the sort that boy scouts and fishermen tie.

Sixth formers and undergraduates will be introduced to a whole new area which they may never have considered to be mathematics. Beware, these volumes may change your life! One small point: Sir William Hamilton was Irish, not English (*Knots and Surfaces*, page 16).

University of Sheffield DAVID SHARPE

A Cascade of Numbers. By R. P. BURN AND A. CHETWYND. Arnold, London, 1995. Pp. viii+148. Paperback £9.99 (ISBN 0-340-65251-9).

Perhaps you are a student tired of sitting in a lecture theatre listening — or not listening — to an incomprehensible, second-rate mathematical exposition. Or perhaps you are a lecturer tired of giving your all to a sea of blank faces. Then this book might be for you. Those who know Bob Burn's *A Pathway into Number Theory* will be familiar with his style, here used in collaboration with Amanda Chetwynd. You learn by doing. A glance at the contents page numbers will show you what I mean. Each topic is listed twice. The first time you try it yourself, under the guidance of the authors. The second half of the book is a re-run in which the authors tell you what you should have done.

The topics are those which occur in any course in elementary number theory up to quadratic reciprocity, but no continued fractions (which were included in the earlier text).

Certainly any student who has been through such a course of number theory will find this an invigorating refresher course. It would be interesting to know whether the authors or others have used this self-help approach for first timers, and with what results. In these days of declining staffing and increasing class sizes, it could be a winner!

University of Sheffield DAVID SHARPE

Other books received

Mechanics 5 and 6. By P. BRYDEN AND C. PAVELIN. Hodder and Stoughton, London, 1996. Pp. 312. Paperback £14.99 (ISBN 0-340-57860-2).

This attractively produced volume is part of a well-known series, covering the last two mechanics components of the MEI Structured Mathematics syllabus. It is suitable for any A-level Further Mathematics course or for a first-year university mechanics course. Vector methods are used throughout and application to problems in the real world is emphasized.

Pure Mathematics 4. By TERRY HEARD AND DAVID MARTIN. Hodder & Stoughton, London, 1996. Pp. 172. Paperback £8.25 (ISBN 0340578610).

This is the latest in a series of books covering the MEI syllabus. The material is part of the Further Mathematics course, covering such topics as proof, curve sketching, complex numbers, matrices and vectors. The material is presented in an attractive, reader-friendly way.

Mastering Advanced Pure Mathematics. By G. BUCKWELL. Macmillan, Basingstoke, 1996. Pp. x+381. Paperback £9.50 (ISBN 0-333-62049-6).

This text covers topics in the common core of Advanced Level syllabuses. Its attractive presentation makes it suitable for students to use on their own as well as for use in the classroom.

Mastering Electronic and Electrical Calculations. By NOEL M. MORRIS. Macmillan, Basingstoke, 1996. Pp. xxiii+452. Paperback £8.99 (ISBN 0-333-63345-8).

This book is designed for students of electronic and electrical engineering at schools and in the first year of degree courses.

Mathematical Modelling Skills. By D. EDWARDS AND M. HAMSON. Macmillan, Basingstoke, 1996. Pp. xi+164. Paperback £10.99 (ISBN 0-333-59595-5).

This volume is designed to help students to develop skills in using mathematics to model problems in many different areas. The level is suitable for students at school and in the first year of an undergraduate degree course.

Calculus and ODEs. By D. PEARSON. Arnold, London, 1995. Pp. 227. Paperback £8.99 (ISBN 0-340-62530-9).

Ordinary Differential Equations. By W. COX. Arnold, London, 1995. Pp. 222. Paperback £8.99 (ISBN 0-340-63203-8).

Statistics. By A. D. MAYER AND A. M. SYKES. Arnold, London, 1996. Pp. x+176. Paperback £8.99 (ISBN 0-340-63194-5).

These three volumes are in the well-known Modular Mathematics series of introductory texts for beginning university students.

Analysis By P. E. KOPP. Arnold, London, 1996. Pp. 188. Paperback £8.99 (ISBN 0-340-64596-2).

This volume, covering a first course in analysis, is an attractive addition to 'Modular Mathematics' — a series of introductory texts for undergraduates. It may need to be augmented by further exercises. A pity about the false argument purporting to prove that there are infinitely many prime numbers in the introductory section pointing out the need for rigorous proof!

Discrete Mathematics. By A. CHETWYND AND P. DIGGLE. Arnold, London, 1995. Pp. 209. Paperback £8.99 (ISBN 0-340-61047-6).

This volume is designed to introduce beginning undergraduates to a variety of topics under the general heading of Discrete Mathematics. The chapters cover Logic, Sets, Relations and Functions, Combinatorics, Probability, Graphs. The approach is student-friendly and the presentation attractive. But be careful when you come to Example 24 on page 180!

Linear Algebra By T. LAWSON. Wiley, Chichester, 1996. Pp. 408. Hardback £19.99 (ISBN 0-471-30897-8).

A textbook of linear algebra with applications to many areas of mathematics, suitable for first and second year undergraduates. Computer 'lab texts' based on MATLAB, Maple and Mathematica, as well as a Student Solutions Manual, are also available.

Carefree Calculations for Healthcare Students. By D. COBEN AND E. ATERE-ROBERTS. Macmillan, Basingstoke, 1996. Pp. xiii+108. Paperback £8.99 (ISBN 0-333-61530-1).

This is a volume designed to help nurses and healthcare students with the mathematical calculations they meet in their work. It is interesting for students of mathematics to see how widely mathematics is applied, but the mathematics in this volume is at too elementary a level for most of our readers.

Introduction to Graph Theory. By ROBIN J. WILSON. Addison Wesley Longman, Harlow, 1996. Pp. viii+171. Paperback £14.99 (ISBN 0-582-24993-7).

The fourth edition of a well-tried and well-used introduction to graph theory.

Symmetric Bends. How to Join Two Lengths of Cord. By ROGER E. MILES. World Scientific, Singapore, 1995. Pp. xii+163. Hardback (ISBN 981-02-21940).

A bend is a knot that securely joins two lengths of cord. This volume includes 60 symmetric bends, well illustrated by photographs and diagrams. 'For all knot lovers' is the dedication, but it will primarily interest the specialist.

Vita Mathematica. By RONALD CALINGER. The Mathematical Association of America, Washington, DC, 1996. Pp. xii+350. Paperback $34.95 (ISBN 0-88385-097-4).

The contention motivating this volume is that mathematics should be taught from a historical perspective at all levels. It will be of interest primarily to teachers of mathematics at university level.

Calculus: the Dynamics of Change. By A. WAYNE ROBERTS. The Mathematical Association of America, Washington DC, 1996. Pp. 175. Paperback $34.95 (ISBN 0-88385-098-2).

A report from a committee in the USA, the CRAFTY Committee (Calculus Reform and the First Two Years!), on what an undergraduate calculus course should contain.

Models that Work. By ALAN TUCKER. The Mathematical Association of America, Washington DC, 1995. Pp. 88. Paperback $24.00 (ISBN 0-88385-096-6).

This is a report into mathematical education at undergraduate level in the USA. A team visited a cross-section of higher education institutions to find 'Models that Work'.

Leningrad Mathematical Olympiads 1987–1991. By DMITRY FOMIN AND ALEXEY KIRICHENKO. MathPro Press, Westford MA, 1996. Pp. xix+197. Paperback $24.00 (ISBN 0-9626401-4-X).

ARML–NYSML Contests 1989–1994. By LAWRENCE ZIMMERMAN AND GILBERT KESSLER. MathPro Press, Westford MA, 1996. Pp. xviii+189. Paperback $19.95 (ISBN 0-9626401-6-6).

These two volumes contain problems and solutions from annual mathematics competitions for high school students in USA and Russia.

From Erdös to Kiev. By R. HONSBERGER. The Mathematical Association of America, Washington, 1995. Pp. xii+250. Paperback $31.00 (ISBN 0-88385-324-8).

A volume of problems and discussion of their solutions by this well-known author.

General Relativity By G. S. HALL AND J. R. PULHAM. IOP, Bristol, 1996. Pp. 422. Paperback £25.00 (ISBN 0750304197).

This volume contains the courses given at the 46th Scottish Universities Summer School in Physics held at Aberdeen in 1995, attended by postgraduates and postdoctoral students. For the specialist in the subject. □

Braintwister

1: No need for Mystic Meg

I have a brilliant way of choosing my lottery numbers (though it has not brought me any luck yet!). I know that all the blank balls weigh the same and that the weight of paint used for any particular digit is the same whichever ball it is on. However, some digits use more paint than others and consequently some of the numbered balls 01–49 are heavier than others.

It turns out that no ball numbered more than 25 is lighter than any ball numbered less than 25. In fact, most of the balls weigh a different amount from ball 25.

Since heavier balls *obviously* are more likely to drop to the bottom of the drum, I always choose my six numbers from amongst the heaviest.

How many do I have to choose from?

(The solution will be published next time.)

VICTOR BRYANT

Pythagorean 4-tuples

Find positive integers a, b, c, d such that

$$a^2 + b^2 + c^2 = d^2.$$

GUIDO LASTERS
(Tienen, Belgium)

The 1997 puzzle

Our annual puzzle is to express the numbers 1 to 100 with the digits of the year in order using only the operations of $+, -, \times, \div, \sqrt{\ }, !$, brackets and concatenation (e.g. putting the digits 1 and 9 together to make 19). Powers are not allowed. Thus, for example,

$$1 = 1 \times (-\sqrt{9} - \sqrt{9} + 7).$$

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