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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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TRIANGLE EQUALITIES

K. R. S. Sastry

The lengths of the sides of a triangle are relatively prime integers and the sum of the lengths of the two sides equals the sum of the length of the base and the length of the median to the base. Do you suspect that the length of the median is either a square or twice a square?

A line segment between a vertex and a point on the opposite side is called a *cevian* of the triangle. A median, an angle bisector, an altitude are examples of cevians of a triangle. In what follows we shall see that the triangle equalities involving sides and cevians lead to pleasant surprises. We use the standard notation: a, b, c for the lengths of the sides opposite the vertices A, B, C; m_a, m_b, m_c for the lengths of the medians from the vertices A, B, C to the opposite sides. The following equation has appeared in *Crux* solutions now and then:

$$4m_a^2 = 2b^2 + 2c^2 - a^2. \tag{1}$$

Readers unfamiliar with it may consult a college geometry textbook for its derivation or may prefer to derive it as an application of the cosine rule.

THEOREM. Let ABC be a triangle with integer-valued sides and $\gcd(a, b, c) = 1$. If $b + c = a + m_a$ then m_a is either an even square or twice an odd square.

Proof. We restate the hypothesis as $b + c - a = m_a$, square both sides and use (1). This yields

$$4(b + c - a)^2 = 2b^2 + 2c^2 - a^2.$$

We now expand the above equation and rewrite it as a quadratic in a:

$$5a^2 - 8(b + c)a + 2b^2 + 8bc + 2c^2 = 0.$$

By the quadratic formula we obtain

$$a = \frac{1}{5}\left[4(b + c) \pm \sqrt{6b^2 - 8bc + 6c^2}\right].$$

It is easily seen that $6b^2 - 8bc + 6c^2 \geq (b + c)^2$. Hence if we take the positive square root in the above expression for a then $a \geq b + c$ and the lengths a, b, c do not form a non-degenerate triangle. So

$$a = \frac{1}{5}\left[4(b + c) - \sqrt{6b^2 - 8bc + 6c^2}\right]. \tag{2}$$

Since a is rational, $6b^2 - 8bc + 6c^2 = 2(3b^2 - 4bc + 3c^2)$ must be an even square, $4d^2$, say. Then $3b^2 - 4bc + 3c^2 = 2d^2$ may be expressed in the form $(3b - 2c)^2 + 5c^2 = 6d^2$. Let us put $3b - 2c = e$ so that we have to solve the equation

$$e^2 + 5c^2 = 6d^2.$$

Then

$$e^2 - d^2 = 5(d^2 - c^2),$$

or

$$\frac{e-d}{d-c} = \frac{5d+5c}{e+d} = \frac{u}{v},$$

where u, v are relatively prime positive integers. These yield the simultaneous equations

$$ve - (u+v)d = -uc \quad \text{and} \quad ue + (u-5v)d = 5vc$$

with the solution

$$\frac{e}{-u^2+10uv+5v^2} = \frac{c}{u^2+2uv-5v^2} = \frac{d}{u^2+5v^2} = k,$$

the proportionality constant. This gives

$$e = k(-u^2+10uv+5v^2), \quad c = k(u^2+2uv-5v^2), \quad d = k(u^2+5v^2).$$

From $e = 3b - 2c$ we determine

$$b = \frac{k}{3}(u^2+14uv-5v^2).$$

We now use (2) to obtain

$$a = \frac{2k}{3}(u^2+8uv-11v^2).$$

By taking $k = 3$, we can define *integer* lengths a', b', c' that are proportional to the lengths a, b, c respectively as follows:

$$\begin{aligned} a' &= 2(u^2+8uv-11v^2), & b' &= u^2+14uv-5v^2, \\ c' &= 3(u^2+2uv-5v^2), & m'_a &= b'+c'-a' = 2(u+v)^2. \end{aligned} \tag{3}$$

To form a triangle, the lengths a', b', c' should satisfy the six inequalities: $a' > 0$, $b' > 0$, $c' > 0$, $a' + b' > c'$, $b' + c' > a'$ and $c' + a' > b'$. For example, $a' > 0$ if $u^2 + 8uv - 11v^2 > 0$, that is, if $u > (3\sqrt{3} - 4)v$. A comparison of six such quadratic inequalities involving u and v imposes the constraints

$$u > 2v > v \geq 1$$

on the positive integers u and v.

If $\gcd(a', b', c') = 1$ then we take $a = a'$, $b = b'$, $c = c'$, and $m_a = m'_a$. If $\gcd(a', b', c') > 1$ then we divide the lengths a', b', c' and m'_a by this gcd in order to get the triangle with relatively prime sides a, b, c.

Now

$$\gcd(a', b', c') = \gcd(b'+c'-a', c') = \gcd(2(u+v)^2, 3(u+v)^2 - 18v^2). \tag{4}$$

First of all we note that $u + v$ and v are also relatively prime integers because u and v are. Hence from (4) it is evident that no prime $p \neq 2, 3$ can divide $\gcd(a', b', c')$.

(I) If $\gcd(a', b', c') = 1$ then (4) shows that $u + v$ must be odd. In this case

$$a = a',\ b = b',\ c = c', \quad \text{and} \quad m_a = m'_a = 2(u+v)^2 \quad \text{is twice an odd square.}$$

Now suppose that $\gcd(a', b', c') > 1$. We have already observed that the only primes that this gcd can contain are 2 and 3. Suppose 2 divides $\gcd(a', b', c')$. Then (4) shows that $u + v$ must be even. Therefore u and v must be both odd because they are relatively prime. Hence the highest power of 2 that this gcd can contain is just 2 because it has to divide 18 also.

Suppose now that 3 divides $\gcd(a', b', c')$. Then (4) shows that 3 divides $u + v$ but not v as these are relatively prime. Again from (4) it follows that the highest power of 3 that the gcd can contain is 9 because it has to divide 18 also. Hence $\gcd(a', b', c') = 9$ if $u + v$ is odd and $\gcd(a', b', c') = 18$ if $u + v$ is even. Summarising we have

(II) $\gcd(a', b', c') = 2$ implies $u + v$ is even and

$$(a, b, c) = \frac{1}{2}(a', b', c') \quad \text{and} \quad m_a = \frac{1}{2}m'_a = (u+v)^2, \quad \text{an even square.}$$

(III) $\gcd(a', b', c') = 18$ implies $u + v$ is even and

$$(a, b, c) = \frac{1}{18}(a', b', c') \quad \text{and} \quad m_a = \frac{1}{18}m'_a = \left(\frac{u+v}{3}\right)^2, \quad \text{an even square.}$$

(IV) $\gcd(a', b', c') = 9$ implies $u + v$ is odd and

$$(a, b, c) = \frac{1}{9}(a', b', c') \quad \text{and} \quad m_a = \frac{1}{9}m'_a = 2\left(\frac{u+v}{3}\right)^2, \quad \text{twice an odd square.}$$

□

We now use (3) for $1 \le v < 2v < u \le 10$ to prepare the table below to display all the four possibilities: $\gcd(a', b', c') = 1$, 2, 9 or 18.

u	v	a'	b'	c'	a	b	c	m_a
3	1	44	46	30	22	23	15	16
4	1	74	67	57	74	67	57	50
5	1	108	90	90	6	5	5	4
5	2	122	145	75	122	145	75	98
6	1	146	115	129	146	115	129	98
7	1	188	142	174	94	71	87	64
7	2	234	225	171	26	25	19	18
7	3	236	298	138	118	149	69	100
8	1	234	171	225	26	19	25	18
8	3	314	355	201	314	355	201	242

u	v	a'	b'	c'	a	b	c	m_a
9	1	284	202	282	142	101	141	100
9	2	362	313	291	362	313	291	242
9	4	386	505	219	386	505	219	338
10	1	338	235	345	338	235	345	242
10	3	482	475	345	482	475	345	338

CONCLUSION. In a similar spirit we can prove many interesting results having to do with other triangle cevians. The following (inexhaustive) list of problems, for example, contains some delightful surprises. The lengths of the sides are relatively prime integers.

1. Let w_a denote the length of the (internal) bisector of angle A of ΔABC. A formula for w_a is $w_a^2 = bc - a^2bc/(b+c)^2$. If $b + c = a + w_a$, derive the parametric representation

$$(a, b, c, w_a) = \Big((m+n)(m^2+mn+n^2), m(m^2+3mn+n^2),$$
$$n(m^2+3mn+n^2), 2mn(m+n)\Big)$$

where m and n are relatively prime natural numbers with $m \geq n$. It is easy to see that the semiperimeter of such triangles is $(m+n)^3$, a perfect cube. The Euler ϕ-function counts the number of natural numbers less than and prime to a given positive integer. For example, $\phi(6) = 2$ as 1 and 5 are the only natural numbers less than and prime to 6.

Suppose now $m + n$ is given. From the definition of the ϕ-function it follows that there are precisely $\phi(m+n)/2$ triangles all having semiperimeter $(m+n)^3$ in each of which $b + c = a + w_a$ holds. For example, if $m + n = 7$ then there are $\phi(7)/2 = 3$ triangles all with semiperimeter $7^3 = 343$ in each of which $b + c = a + w_a$ holds. Calculated from the expressions given above these are precisely

(i) $m = 6, \quad n = 1, \quad (a, b, c, w_a) = (301, 330, 55, 84)$,

(ii) $m = 5, \quad n = 2, \quad (a, b, c, w_a) = (273, 295, 118, 140)$,

(iii) $m = 4, \quad n = 3, \quad (a, b, c, w_a) = (259, 244, 183, 168)$.

If N is a positive integer then a formula for $\phi(N)$ can be found in a number theory textbook or in [1].

2. Determine triangles ABC in which $a + b = c + m_a$ holds.

3. In triangle ABC we have $w_a = b$. Prove that c is a cube.

4. Does there exist ΔABC in which $a + b = c + w_a$?

5. Investigate triangles ABC in which $w_b = 2m_a$.

6. Investigate triangles ABC in which $b + c = a + h_a$, where h_a is the altitude to the side BC.

7. Let D be a point on side BC of ΔABC such that $BD : DC = m : n$. Discuss the existence of ΔABC in which $b + c = a + AD$.

In the same vein we may go a step further:

8. Find a characterization of parallelograms in which longer side + shorter diagonal = shorter side + longer diagonal. Discuss the existence of such parallelograms in which both sides and both diagonals are natural numbers. Can such a parallelogram have the area also a natural number?

Acknowledgement. The author thanks the referee for the suggestions: in particular for the observation that m_a is either an *even* square or twice an *odd* square.

Reference:

[1] Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover, N.Y. (1966) 88–93.

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THE OLYMPIAD CORNER

No. 160

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

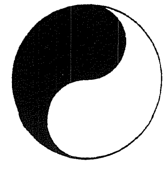
It has been some time since we gave a pre-Olympiad contest and so this number of the Corner will start with the problems of the 1989 United Kingdom Junior Mathematical Olympiad. My thanks to Andy Liu, The University of Alberta, and Tony Gardiner, The University of Birmingham, U.K., for sending me the questions.

1989 U.K. JUNIOR MATHEMATICAL OLYMPIAD

(2 hours)

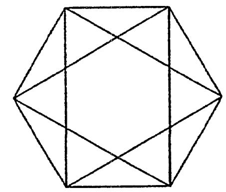
1. Find the smallest multiple of 9 which has no odd digits.

2. The two semicircles inside the big circle divide the big circle into two congruent pieces. Show how to cut the shaded and the unshaded regions exactly in half with a single straight cut. Explain why your method works.



3. How could you multiply 7,628,954,301 by 125 in your head while looking at the number?

4. Joining each vertex in the big regular hexagon to the next vertex but one produces a smaller hexagon inside. Prove that the smaller hexagon is regular. Find its edge length in terms of the edge length of the big hexagon.



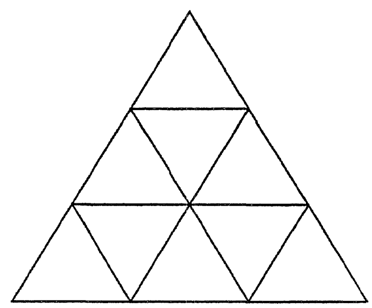
5. Find two perfect squares that differ by 105. How many different solutions are there?

6. A right angled triangle has legs of length a and b. A circle of radius r touches the two legs and has its centre on the hypotenuse. Show that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{r}\,.$$

7. Find all ordered pairs of numbers (x, y) whose sum $x+y$, product $x \cdot y$, and quotient x/y are all equal.

8. The diagram shows an equilateral triangle with sides of length 3 cut into nine equilateral triangles of side length 1. Place the numbers 1 to 9 in the nine small triangles so that the sum of the numbers inside any equilateral triangle with side length 2 is the same. What are the smallest and largest possible values of this sum?

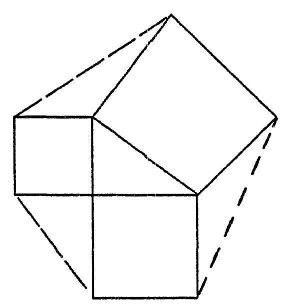


9. The three sides a, b, c of a triangle are such that

$$a^2+b^2+c^2=ab+bc+ca.$$

Does the triangle have to be equilateral?

10. Pythagoras' theorem for the right angled triangle ABC is often illustrated by a diagram like the one shown here (without the dotted lines). Putting in the three dotted lines shows that the whole figure can be enclosed in a hexagon. Find the area of this hexagon in terms of the edge lengths a, b, c of the original triangle.



11. (a) $5^2=4^2+3^2$. If we ignore multiples of this equation (such as $10^2=8^2+6^2$) what is the next square which can be written as a sum of two squares?

(b) $6^3=5^3+4^3+3^3$. If we ignore multiples of this equation (such as $12^3=10^3+8^3+6^3$) what is the next cube which can be written as a sum of three cubes?

12. Two straight cuts, one through each of two vertices of a triangle, divide this triangle into three smaller triangles and one quadrilateral.

(a) Is it possible for the areas of all four parts to be equal to one another?

(b) More generally, if three of the parts have area p while the fourth has area q what are the possible values of p/q?

13. Mr. and Mrs. A invite some other married couples to dinner. As they all meet, some pairs shake hands. Married couples do not shake hands with one another. Over coffee, while Mr. A is washing up, the rest discover that they all shook hands a different number of times.

(a) Suppose there were four couples altogether (including Mr. and Mrs. A). Can you say how many hands Mr. A must have shaken?

(b) What if there were five couples altogether? Can you say how many hands Mr. A must have shaken?

* * *

As an Olympiad contest we give the problems of the Nordic Mathematical Contest, which was written April 8, 1992. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, who collected the contest when he was Canadian Team leader at the I.M.O. in Moscow.

NORDIC MATHEMATICAL CONTEST

April 8, 1992 (Time: 4 hours)

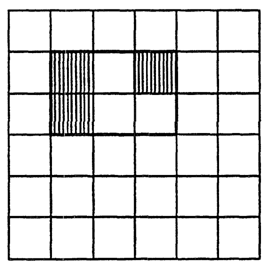
1. Determine all real numbers x, y, z greater than 1, satisfying the equation

$$x+y+z+\frac{3}{x-1}+\frac{3}{y-1}+\frac{3}{z-1}=2\left(\sqrt{x+2}+\sqrt{y+2}+\sqrt{z+2}\right).$$

2. Let n be an integer greater than 1 and let $a_1, a_2, \ldots, a_n$ be n different integers. Prove that the polynomial $f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)-1$ is *not divisible* by any polynomial of positive degree less than n and with integer coefficients and leading coefficient 1.

3. Prove that among all triangles whose incircle has radius equal to 1, the equilateral triangle has the shortest *perimeter*.

4. Peter has a great number of squares, some of them are black, some are white. Using these squares, Peter wants to construct a square, where the edge has length n, and with the following property: The four squares in the corners of an arbitrary subrectangle of the big square, must *never* have the same color. How large a square can Peter build?



$n = 6$

* * *

We now turn to solutions sent in by our readers to problems from the May 1993 number of the Corner and the 28th Spanish Mathematical Olympiad, First Round — November 22–23, 1991, Valladolid [1993: 131–132].

1. Let z be a complex number. Show that

$$\tan(\arg(z)) > \sqrt{2}-1 \quad\Rightarrow\quad \operatorname{Re}(z^2) < \operatorname{Im}(z^2) \quad\Rightarrow\quad \cot(\arg(z)) < 1+\sqrt{2}.$$

Is the converse true?

Solutions by Seung-Jin Bang, Seoul, Korea; by Beatriz Margolis, Paris, France; by Michael Selby, University of Windsor; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Selby's solution.

First, we assume $z \neq 0$ since $\arg(0)$ is not defined. Using polar form, we observe that $\operatorname{Re}(z^2) < \operatorname{Im}(z^2)$ if and only if $r^2\cos 2\theta < r^2 \sin 2\theta$, where $z = r(\cos\theta + i\sin\theta)$, $\theta = \arg(z)$.

Since $r \neq 0$, $\operatorname{Re}(z^2) < \operatorname{Im}(z^2)$ if and only if $\cos 2\theta < \sin 2\theta$. This is true just in case $\sin 2\theta - \cos 2\theta > 0$, or

$$\sin^2\theta - 2\sin\theta\cos\theta - \cos^2\theta > 0. \tag{1}$$

Suppose $\tan\theta > \sqrt{2}-1$. If $\cos\theta = 0$, (1) holds and the result is true. If $\cos\theta \neq 0$ (1) can be written as $\cos^2\theta(\tan^2\theta + 2\tan\theta - 1) > 0$ or

$$\cos^2\theta(\tan\theta - (\sqrt{2}-1))(\tan\theta - (-\sqrt{2}-1)) > 0. \tag{2}$$

Since $\tan\theta > \sqrt{2}-1 > -\sqrt{2}-1$, clearly (2) holds.

Now, assuming (1), $\sin\theta \neq 0$ and we can write (1) as $\sin^2\theta(1 + 2\cot\theta - \cot^2\theta) > 0$ or $(\cot^2\theta - 2\cot\theta - 1)\sin^2\theta < 0$. This is true if and only if $\cot^2\theta - 2\cot\theta - 1 < 0$ or $(\cot\theta - (1+\sqrt{2}))(\cot\theta - (1-\sqrt{2})) < 0$. This is true if and only if

$$1-\sqrt{2} < \cot\theta < 1+\sqrt{2}. \tag{3}$$

Hence $\cot\theta < 1+\sqrt{2}$.

To see that the converse fails, choose $z = -1+i$, $\cot\theta = -1 < 1+\sqrt{2}$, $z^2 = -2i$, $\operatorname{Re}(z^2) = 0$, $\operatorname{Im}(z^2) = -2$. Therefore $\operatorname{Re}(z^2) > \operatorname{Im}(z^2)$ and the converse is false. (We need $\cot\theta > 1-\sqrt{2}$ also from (3).)

2. Let S be the set of straight lines which join a point from the set $A = \{(0, 1/a); a \in \mathbb{N}\}$ with a point from the set $B = \{(b+1, 0); b \in \mathbb{N}\}$. Show that a necessary and sufficient condition that the natural number m be composite is that the point $M = (m, -1)$ belong to a line from S. Determine the number of lines of S to which m belongs.

Solutions by Michael Selby, University of Windsor; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Let $\mathbf{P} = (0, 1/a) \in A$ and $\mathbf{Q} = (c, 0) \in B$ where $c \geq 2$. Then the line l connecting $\mathbf{P}$ and $\mathbf{Q}$ has equation

$$\frac{y}{x-c} = \frac{-1/a}{c} \quad \text{or} \quad x + acy = c.$$

If $(m,-1)$ lies on l, then $m - ac = c$ or $m = (a+1)c$ which shows that m is composite since $a+1 \geq 2$ and $c \geq 2$. Conversely, if m is composite $m = dq$ where $d \geq 2$ and $q \geq 2$, then $(m,-1)$ lies on the line l^* connecting $(0, 1/(d-1))$ and $(1,0)$ since an equation of l^* is $x + (d-1)qy = q$ and $m - (d-1)q = dq - (d-1)q = q$.

Finally, combining the two parts of the above argument and noticing that for different values of d the equation $x + (d-1)qy = q$ represents different lines, we conclude that the number of lines of S to which M belongs is equal to the number of proper divisors of m besides 1; i.e., $\tau(m) - 2$ for all $m \geq 2$ and $\tau(1) = 0$ where $\tau(m)$ denotes the total number of (positive) divisors of m.

3. The abscissa of a point which moves in the positive part of the axis Ox is given by $x(t) = 5(t+1)^2 + a/(t+1)^5$, in which a is a positive constant. Find the minimum a such that $x(t) \geq 24$ for all $x \geq 0$.

Solution by Michael Selby, University of Windsor.

We want $5(t+1)^2 + a/((t+1)^5) \geq 24$ for all $t \geq 0$. This is true iff $a \geq 24(t+1)^5 - 5(t+1)^7$ for all $t \neq 0$. Hence the minimum a occurs at the maximum of $24(t+1)^5 - 5(t+1)^7$ for all $t \geq 0$.

Let $f(t) = 24(t+1)^5 - 5(t+1)^7$, $t \geq 0$. Then $f'(t) = 120(t+1)^4 - 35(t+1)^6$ and $f'(t) = 0 \Rightarrow t+1 = 0$ or $(t+1)^2 = 24/7$. Since we want $t \geq 0$, the only critical value occurs at $t_0 + 1 = \sqrt{24/7}$. This is clearly a maximum. Therefore, the minimum

$$a = 24(t_0+1)^5 - 5(t_0+1)^7 = \sqrt{\frac{24}{7}}\left(24\left(\frac{24}{7}\right)^2 - 5\left(\frac{24}{7}\right)^3\right) = \frac{2\cdot 24^3}{7^3}\sqrt{\frac{24}{7}}\,.$$

4. Two equal and tangent half circles S_1 and S_2, diameters lying on the same straight line, are given. A common tangent line r is drawn to S_1 and S_2. A circle C_1, tangent to r, S_1 and S_2 is produced. Next, another circle C_2, tangent to C_1, S_2 and S_1 is drawn, and so on.

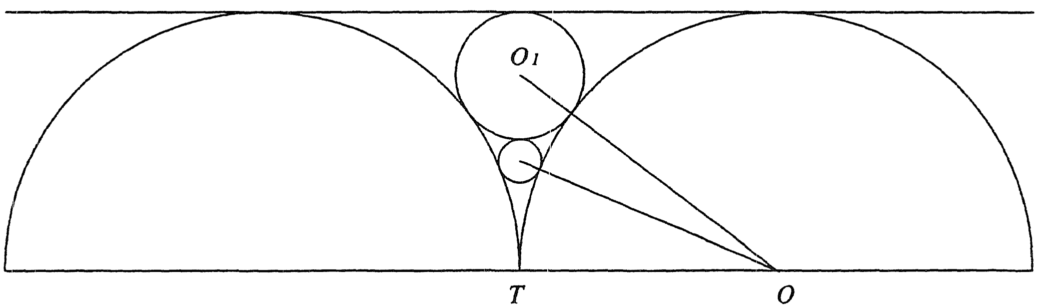
a) Find the radius r_n of C_n in terms of n and R, the radius of S_1.

b) Using the construction of this problem, show that the limit of

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

as $n \longrightarrow \infty$, is 1.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Michael Selby, University of Windsor. We use Amengual's solution.



(a) We claim that

$$r_n = \frac{R/2}{n(n+1)} \ .$$

Let T denote the point of contact of S_1 and S_2, O_n the centre of circle C_n and O the centre of one of S_1 or S_2. Note that $O_1T = R - r_1$, $TO = R$ and $O_1O = R + r_1$. The Pythagorean theorem applied to the right triangle O_1TO yields $(O_1T)^2 + (TO)^2 = (O_1O)^2$, i.e.

$$(R - r_1)^2 + R^2 = (R + r_1)^2$$

and

$$r_1 = \frac{R}{4} = \frac{R/2}{1 \cdot 2} \ .$$

Now suppose the assertion $r_k = \frac{R/2}{k(k+1)}$ is valid for $1 \le k \le n$. Then we find

$$\begin{aligned}
R - 2r_1 - 2r_2 - \cdots - 2r_n &= R - \frac{R}{1 \cdot 2} - \frac{R}{2 \cdot 3} - \cdots - \frac{R}{n(n+1)} \\
&= R\left[1 - \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}\right)\right] \\
&= R\left[1 - \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]\right] \\
&= \frac{R}{n+1} \ , \qquad \text{by telescoping.}
\end{aligned}$$

So we have

$$O_{n+1}T = R - 2r_1 - 2r_2 - \cdots - 2r_n - r_{n+1} = \frac{R}{n+1} - r_{n+1},$$

and from Pythagoras applied to $\Delta O_{n+1}TO$, $(O_{n+1}T)^2 + (TO)^2 = (O_{n+1}O)^2$

$$\left(\frac{R}{n+1} - r_{n+1}\right)^2 + R^2 = (R + r_{n+1})^2,$$

solving

$$r_{n+1} = \frac{R/2}{(n+1)(n+2)} \ ,$$

and the claim follows by induction.

(b) Note that

$$\begin{aligned}
R &= \lim_{n\to\infty} (2r_1 + 2r_2 + \cdots + 2r_n) = \lim_{n\to\infty} \left(\frac{R}{1 \cdot 2} + \frac{R}{2 \cdot 3} + \cdots + \frac{R}{n(n+1)}\right) \\
&= R \lim_{n\to\infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}\right) \ .
\end{aligned}$$

So

$$\lim_{n\to\infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} \cdots + \frac{1}{n(n+1)}\right) = 1.$$

5. For each natural number n, let

$$(1+\sqrt{2})^{2n+1} = a_n + b_n\sqrt{2},$$

with a_n and b_n integers.

a) Show that a_n and b_n are odd, for all n.

b) Show that b_n is the hypotenuse of a right triangle with legs

$$\frac{a_n + (-1)^n}{2} \qquad \text{and} \qquad \frac{a_n - (-1)^n}{2}.$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Seung-Jin Bang, Seoul, Korea; by Bob Prielipp, University of Wisconsin–Oshkosh; by Michael Selby, University of Windsor; by D. J. Smeenk, Zaltbommel, The Netherlands; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Prielipp's solution (which was typical).

(a) $a_1 b_1\sqrt{2} = (1+\sqrt{2})^3 = 7 + 5\sqrt{2}$ so $a_1 = 7$ and $b_1 = 5$.

Thus when $n = 1$, a_n and b_n are both odd. Assume that a_k and b_k are both odd where k is a positive integer. Then

$$\begin{aligned} a_{k+1} + b_{k+1}\sqrt{2} &= (1+\sqrt{2})^{2k+3} = (1+\sqrt{2})^{2k+1}(1+\sqrt{2})^2 \\ &= (a_k + b_k\sqrt{2})(3+2\sqrt{2}) = (3a_k + 4b_k) + (2a_k + 3b_k)\sqrt{2} \end{aligned}$$

so $a_{k+1} = 3a_k + 4b_k$ and $b_{k+1} = 2a_k + 3b_k$. Hence if a_k and b_k are both odd then a_{k+1} and b_{k+1} are both odd.

Therefore a_n and b_n are both odd for each positive integer n by mathematical induction.

(b) $\left(\dfrac{a_n + (-1)^n}{2}\right)^2 + \left(\dfrac{a_n - (-1)^n}{2}\right)^2 = \dfrac{a_n^2 + 1}{2}$.

Since

$$a_n = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2} \qquad \text{and} \qquad b_n = \frac{(1+\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}}{2\sqrt{2}},$$

we have

$$\begin{aligned} \frac{a_n^2+1}{2} &= \frac{(1+\sqrt{2})^{4n+2} - 2 + (1-\sqrt{2})^{4n+2} + 4}{8} = \frac{(1+\sqrt{2})^{4n+2} + 2 + (1-\sqrt{2})^{4n+2}}{8} \\ &= \left[\frac{(1-\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}}{2\sqrt{2}}\right]^2 = b_n^2. \end{aligned}$$

Therefore b_n is the hypotenuse of a right triangle with legs $(a_n + (-1)^n)/2$ and $(a_n - (-1)^n)/2$.

6. Two medicines, A and B, have been tested in two hospitals. In both hospitals, a better result was obtained with medicine A than with B; but when the results were combined it was found with astonishment that medicine B obtained better results than A. Is this possible, or is it due to a mistake in the calculations?

Solution by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Clearly, any valid solution (with a logical argument) to this problem will have to depend on the interpretation of the phrase "a better result". Assuming it means that A has a higher *success ratio* than B (the number of tests conducted for A and B need not be the same), then the answer to the question is "Yes, it is entirely possible." For example, suppose the success ratios for A and B in one hospital are $3/4$ and $35/48$ respectively, while in the other hospital they are $17/24$ and $2/3$ respectively. Then $3/4 > 35/48$ and $17/24 > 2/3$ show that A produces better results than B. However, combining we get

$$\frac{3+17}{4+24} = \frac{5}{7} < \frac{37}{51} = \frac{35+2}{48+3}\,.$$

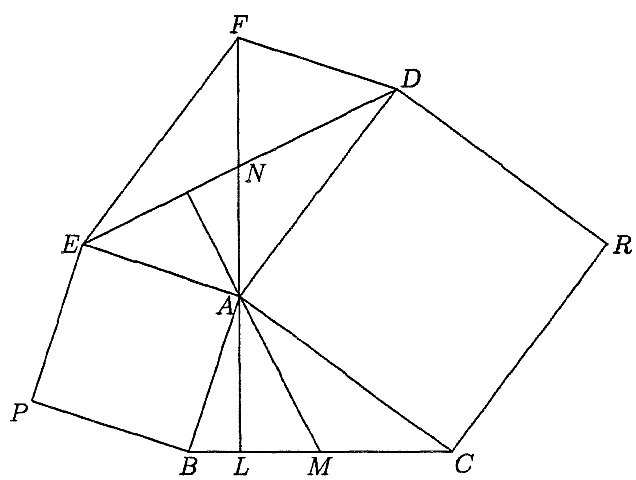
7. Let m be a natural number. Show that if 2^m+1 is prime, $2^m+1>3$, then m is even.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Seung-Jin Bang, Seoul, Korea; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Selby's solution.

Since $2^m+1>3$, $m>1$. Consider 2^m+1. Since $2\equiv -1 \bmod 3$, $2^m+1\equiv (-1)^m+1 \bmod 3$. If m is odd $(-1)^m=-1$ and $2^m+1\equiv 0 \pmod 3$. Hence, 2^m+1 cannot be prime since $m>1$. Therefore m must be even.

8. Let ABC be any triangle. Two squares $BAEP$ and $ACDR$ are constructed externally to ABC. Let M and N be the midpoints of BC and ED, respectively. Show that $AM \perp ED$ and $AN \perp BC$.

Solutions by Geoffrey A. Kandall, Hamden, Connecticut; by D. J. Smeenk, Zaltbommel, The Netherlands; and by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Smeenk's solution.



The problem should read "Two squares $BAEP$ and $ACRD$ are constructed externally to ABC... ."

Complete the parallelogram $EADF$. Then $AB = DF$, $AB \perp DF$, $AC = DA$, $AC \perp DA$ and $\angle BAC = \angle FDA$ whence ΔDFA and ΔABC are congruent.

Now since $DF \perp AB$ and $DA \perp AC$ it follows that $FA \perp BC$. But as FA is a diagonal of the parallelogram $EADF$ it passes through N whence $AN \perp BC$.

Interchanging the roles of ΔABC and ΔAED we conclude easily that the altitude through A in ΔAED passes through the midpoint M of AB whence $AM \perp ED$.

As a second solution we give Kandall's approach via vectors.

Let $\overrightarrow{AB} = \mathbf{b}$, $\overrightarrow{AC} = \mathbf{c}$, $\overrightarrow{AD} = \mathbf{d}$, $\overrightarrow{AE} = \mathbf{e}$. Then

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b}) = \frac{1}{2}(\mathbf{b} + \mathbf{c}).$$

Note that $\mathbf{b} \cdot \mathbf{e} = 0$, $\mathbf{c} \cdot \mathbf{d} = 0$, and $\mathbf{b} \cdot \mathbf{d} = |\mathbf{b}|\,|\mathbf{d}| \cos \angle BAD = |\mathbf{e}|\,|\mathbf{c}| \cos \angle EAC = \mathbf{e} \cdot \mathbf{c}$. Consequently

$$\overrightarrow{AM} \cdot \overrightarrow{ED} = \frac{1}{2}(\mathbf{b} + \mathbf{c})(\mathbf{d} - \mathbf{e}) = \frac{1}{2}(\mathbf{b} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{e} + \mathbf{c} \cdot \mathbf{d} - \mathbf{c} \cdot \mathbf{e}) = 0,$$

so $AM \perp ED$.

The second part of the problem is analogous to the first.

Editor's note. This result was mentioned by Jordi Dou in his solution of *Crux* 1493 [1991: 53].

* * *

Let me finish with an apology to Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, whose solution to Problem #4 of the 22nd Austrian Mathematical Olympiad [1993: 101; 1994: 219] was misfiled with the May solutions and not mentioned when those solutions were discussed.

That completes this number of the Corner. Send me your Olympiad and pre-Olympiad contests and your nice solutions.

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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

The Art of Problem Solving, Volume 2, by S. Lehoczky and R. Rusczyk. Published by Greater Testing Concepts, P. O. Box A-D, Stanford, CA 94309, 1994. Paperback, 390+ pages, solution manual 212 pages, without ISBN number, US $27 without solution manual or $35 with. (May be ordered in combination with Volume 1, which has been reviewed in *Crux* [1994: 135–136], US $47 without solution manuals or $60 with.) *Reviewed by Andy Liu.*

This volume consists of 27 chapters, 7 on algebra, 4 on geometry, 2 on analytic geometry, 2 on trigonometry, 2 on vectors, 5 on combinatorics, 3 on number theory, an introductory chapter on proof techniques and a final chapter of further problems. In all, there are 237 examples, 412 exercises and 509 problems. The companion manual contains solutions to all the exercises and problems.

Much that was said enthusiastically about Volume 1 applies here also. The only slight disappointment is that while the topics covered are more advanced, the treatment is at essentially the same level as that in Volume 1. Nevertheless, this set provides excellent preparation for introductory level mathematics competitions, and important stepping stones towards further study in solving problems of Olympiad calibre.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **July 1, 1995,** *although solutions received after that date will also be considered until the time when a solution is published.*

1991. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Ω is a fixed circle with center O. Let M be the foot of the perpendicular from O to a fixed line ℓ, and let P be a variable point on Ω. Let Γ be the circle with diameter PM, intersecting Ω and ℓ again at X and Y respectively. Prove that the line XY always passes through a fixed point.

1992. *Proposed by K.R.S. Sastry, Dodballapur, India.*
Find all pairs of cubic polynomials

$$x^3 + ax^2 + bx + c \quad \text{and} \quad x^3 + bx^2 + ax + c,$$

where a and b are positive integers and c is a nonzero integer, so that they have three integer roots each, exactly one of which is common.

1993. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*
$ABCD$ is a convex quadrilateral inscribed in a circle Γ. Assume that A, B and Γ are fixed and C, D are variable, so that the length of the segment CD is constant. X, Y are the points on the rays AC and BC respectively, such that $AX = AD$ and $BY = BD$. Prove that the distance between X and Y remains constant. [This problem was inspired by *Crux* 1902, which is solved in this issue.]

1994. *Proposed by N. Kildonan, Winnipeg, Manitoba.*
This problem marks the one and only time that the number of a *Crux* problem is equal to the year in which it is published. In particular this is the *first* time that

$$\text{a problem number is an integer multiple of its publication year.} \tag{1}$$

Assuming that *Crux* continues indefinitely to publish 10 problems per issue and 10 issues per year, will there be a *last* time (1) happens? If so, when will this occur?

1995. *Proposed by Jerzy Bednarczuk, Warszawa, Poland.*
Given two pyramids $SABCD$ and $TABCD$ ($S \neq T$) with a common base $ABCD$. The altitudes of their eight triangular faces, taken from the vertices S and T, are all equal to 1. Prove or disprove that the line ST is perpendicular to the plane containing points A, B, C, D.

1996. *Proposed by Murray S. Klamkin, University of Alberta.*
(a) Find positive integers a_1, a_2, a_3, a_4 so that

$$(1 + a_1\omega)(1 + a_2\omega)(1 + a_3\omega)(1 + a_4\omega)$$

is an integer, where ω is a complex cube root of unity.

(b)* Are there positive integers $a_1, a_2, a_3, a_4, a_5, a_6$ so that

$$(1 + a_1\omega)(1 + a_2\omega)(1 + a_3\omega)(1 + a_4\omega)(1 + a_5\omega)(1 + a_6\omega)$$

is an integer, where ω is a complex *fifth* root of unity?

1997. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
ABC is a triangle which is not equilateral, with circumcentre O and orthocentre H. Point K lies on OH so that O is the midpoint of HK. AK meets BC in X, and Y, Z are the feet of the perpendiculars from X onto the sides AC, AB respectively. Prove that AX, BY, CZ are concurrent or parallel.

1998. *Proposed by John Clyde, student, New Plymouth High School, New Plymouth, Idaho.*

Let $a = \sin 10°$, $b = \sin 50°$, $c = \sin 70°$. Prove that

$$\text{(i) } a + b = c, \qquad \text{(ii) } a^{-1} + b^{-1} = c^{-1} + 6, \qquad \text{(iii) } 8abc = 1.$$

1999. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let ABC be a (variable) isosceles triangle with constant sides $a = b$ and variable side c. Denote the median, angle bisector and altitude, measured from A to the opposite side, by m, w and h respectively. Find

$$\lim_{c \to a} \frac{m - h}{w - h} .$$

2000. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A 1000-element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to $1/5$?

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1901. [1994: 16] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous even function such that $f(0) = 0$ and $f(x + y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Must f be monotonic on $\mathbb{R}^+$?

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

No. The counterexample: $f(x) = |\sin x|$. $f(0) = 0$, f is continuous and even, and for $x, y \in \mathbb{R}$ we have

$$\begin{aligned} f(x+y) &= |\sin(x+y)| = |\sin x \cos y + \sin y \cos x| \\ &\le |\sin x||\cos y| + |\sin y||\cos x| \le |\sin x| + |\sin y| = f(x) + f(y). \end{aligned}$$

But f is not monotonic on $\mathbb{R}^+$.

Also solved by H. L. ABBOTT, University of Alberta; BILL CORRELL JR., student, Denison University, Granville, Ohio; KEITH EKBLAW, Walla Walla, Washington; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; PAVLOS KONSTADINIDIS, student, University of Arizona, Tucson; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; R.P. SEALY, Mount Allison University, Sackville, *New Brunswick; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, New York; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

Several of these solvers give the same counterexample as Pompe. Ekblaw comments that the answer to the problem is yes if we add that f must be differentiable.

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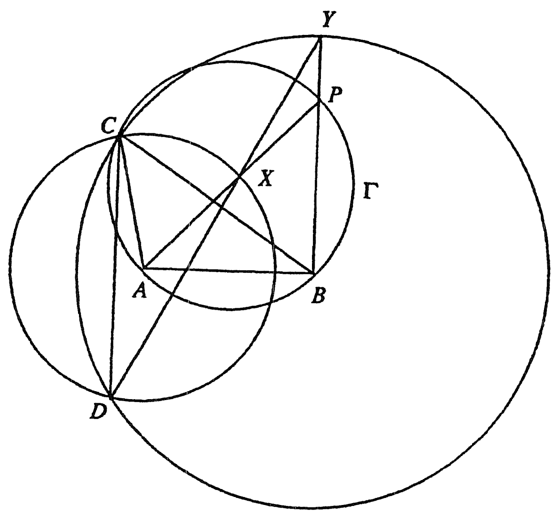
1902. [1994: 16] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcircle Γ. Let P be a variable point on the arc ACB of Γ, other than A, B, C. X and Y are points on the rays AP and BP respectively such that $AX = AC$ and $BY = BC$. Prove that the line XY always passes through a fixed point.

I. *Solution by Himadri Choudhury, student, Hunter High School, New York.*



Draw in the circle with center A and radius AC, and the circle with center B and radius BC. These two circles meet at the point D (in addition to C). Obviously X and Y lie on the circles centered at A and B respectively. We will show that, for all P on arc ACB, XY goes through D. First we note that $\angle XAC = 2\angle XDC$ and similarly $\angle YBC = 2\angle YDC$. Also $\angle PBC = \angle PAC$. Now $\angle PBC = \angle YBC$ and $\angle PAC = \angle XAC$ since B, P, Y and A, P, X are collinear. Combining this with our first three equalities, we get $\angle XDC = \angle YDC$. This implies that the points X, Y and D are collinear, which is what we set out to prove. [Note that points X and Y are on the same side of the line CD.—*Ed.*]

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Denote by D the mirror image of C across line AB. We will show that D is the fixed point in question. Look at the quadrilateral $APBD$. Let ℓ and m be the bisectors of its internal angles PAD and PBD, respectively. In view of the equality $\angle APB = \angle ACB = \angle ADB$, lines ℓ and m are parallel. By the definition of X and Y, $AX = AC = AD$ and $BY = BC = BD$. Therefore $DX \perp \ell$ and $DY \perp m$. And since $\ell \| m$, lines DX and DY coincide; the proof is complete (and case–independent!).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; JORDI DOU, Barcelona, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Bradley's and Dou's solutions are very similar to Solution I. *Bradley also notes that points P, C, X, Y are concyclic with centre lying on the circle Γ; readers may enjoy finding a proof.*

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1903. [1994: 16] *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let $n > 1$ be an integer. How many permutations $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ are there such that

$$1 \mid a_1 - a_2, \quad 2 \mid a_2 - a_3, \quad \ldots \quad , \quad n-1 \mid a_{n-1} - a_n?$$

I. *Solution by H. L. Abbott, University of Alberta.*

For $n = 2$ the only such permutations are $(1,2)$ and $(2,1)$, and for $n = 3$ the only such permutations are $(2,1,3)$ and $(2,3,1)$. Suppose then that $n \geq 4$. From $n-1 \mid a_{n-1} - a_n$ and $|a_{n-1} - a_n| \in \{1, 2, \ldots, n-1\}$, it follows that either (i) $a_{n-1} = 1$ and $a_n = n$, or (ii) $a_{n-1} = n$ and $a_n = 1$. Suppose that (i) holds. We claim that for $m = 0, 1, \ldots, \lfloor n/2 \rfloor - 1$,

$$a_{n-2m} = n - m \qquad \text{and} \qquad a_{n-2m-1} = m + 1.$$

This holds for $m = 0$. Suppose it holds for all $m = 0, 1, \ldots, k$ for some k with $0 \leq k < \lfloor n/2 \rfloor - 1$. Then from $n - 2k - 2 \mid a_{n-2k-2} - a_{n-2k-1}$, $a_{n-2k-1} = k + 1$, $n - 2k - 2 > 0$ and $a_{n-2k-2} \in \{k+2, k+3, \ldots, n-k-1\}$, we get $a_{n-2k-2} - (k+1) = n - 2k - 2$ and hence $a_{n-2k-2} = n - k - 1$. It follows that $a_{n-2k-3} \in \{k+2, k+3, \ldots, n-k-2\}$. From this and the conditions $n - 2k - 3 \mid a_{n-2k-3} - a_{n-2k-2}$, $a_{n-2k-2} = n - k - 1$ and $n - 2k - 3 > 0$, we find that $a_{n-2k-3} - (n - k - 1) = -(n - 2k - 3)$ and hence that $a_{n-2k-3} = k + 2$. This establishes the claim. Note that if n is odd, $a_1 = \lfloor n/2 \rfloor + 1$ is forced. Thus (i) leads to a single permutation with the desired property. An almost identical argument shows that (ii) also leads to a single permutation.

Editor's comments by Edward Wang. From the solution by Abbott, one easily finds that the two permutations are given by

$$(m, m+1, m-1, m+2, \ldots, 2, 2m-1, 1, 2m)$$

and

$$(m+1, m, m+2, m-1, \ldots, 2m-1, 2, 2m, 1)$$

when $n = 2m$ is even, and by

$$(m+1, m, m+2, m-1, \ldots, 2, 2m, 1, 2m+1)$$

and

$$(m+1, m+2, m, m+3, \ldots, 2, 2m, 2m+1, 1)$$

when $n = 2m+1$ is odd. This was pointed out explicitly by solvers Bradley, Hess and Wildhagen.

For (ii), one could also use the proposer's nice observation that if $a_1, \ldots, a_n$ is a permutation satisfying the condition of the problem, so is $n+1-a_1, \ldots, n+1-a_n$.

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

Clearly, the last two positions (a_{n-1} and a_n) must be occupied by 1 and n. Thus a feasible permutation (in short, an FP) ends either with n or with 1; accordingly, we will call it an FP of type (i) or type (ii).

If $(a_1, \ldots, a_n)$ is an FP of type (i) (i.e., $a_n = n$, $a_{n-1} = 1$), then $(a_1, \ldots, a_{n-1})$ is an FP of length $n-1$ of type (ii), and conversely.

If $(a_1, \ldots, a_n)$ is an FP of type (ii) (i.e., $a_n = 1$, $a_{n-1} = n$), then $(a_1 - 1, \ldots, a_{n-1} - 1)$ is an FP of length $n-1$ of type (i), and conversely.

These statements establish a bijection between FP's of length n and FP's of length $n-1$. Thus the number of FP's is the same for every n. For $n = 2$ there are two FP's. Hence there are two FP's for every n.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; TIM CROSS, Wolverley High School, Kidderminster, U. K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THOMAS KLAUSNER, student, Technische Universität Wien, Vienna, Austria; KEE-WAI LAU, Hong Kong; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE, student, Memorial University of Newfoundland, St. John's; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer.

Besides Abbott and Kuczma, only Lau, Perz and Yiu gave precise proofs, using induction. All the other solvers simply "deduced" the answer by pointing out the fact that after the two choices for values of a_{n-1} and a_n are made the choice for values of other a_i's, at each step, working from right to left, is forced.

* * * * *

1904. [1994: 16] *Proposed by Kee-Wai Lau, Hong Kong.*

If m_a, m_b, m_c are the medians of a triangle with sides a, b, c, prove that

$$m_a(bc - a^2) + m_b(ca - b^2) + m_c(ab - c^2) \geq 0.$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

We will show that

$$bm_c + cm_b \geq 2am_a, \tag{1}$$

equality holding (in a non-degenerate triangle) if and only if $2a^2 = b^2 + c^2$. Multiplying (1) by a and adding the resulting inequality and its two cyclic counterparts we then obtain exactly the proposed inequality (turning into equality only when $2a^2 = b^2 + c^2$, $2b^2 = c^2 + a^2$, $2c^2 = a^2 + b^2$, i.e., when $a = b = c$).

Define

$$P = (2bm_c)^2 = b^2(2a^2 + 2b^2 - c^2), \qquad Q = (2cm_b)^2 = c^2(2a^2 + 2c^2 - b^2),$$
$$R = (4am_a)^2 = 4a^2(2b^2 + 2c^2 - a^2),$$
$$X = 2\sqrt{PQ} + (P + Q - R) = (\sqrt{P} + \sqrt{Q} + \sqrt{R})(\sqrt{P} + \sqrt{Q} - \sqrt{R}),$$
$$Y = 2\sqrt{PQ} - (P + Q - R).$$

The claimed inequality (1) is equivalent to $\sqrt{P} + \sqrt{Q} \geq \sqrt{R}$, i.e., to $X \geq 0$.

The product XY factorizes as follows:

$$\begin{aligned} XY &= 4PQ - (P + Q - R)^2 \\ &= 4(a + b + c)(b + c - a)(c + a - b)(a + b - c)(2a^2 - b^2 - c^2)^2 \\ &= 64\,F^2(2a^2 - b^2 - c^2)^2, \end{aligned} \tag{2}$$

where $F = \text{Area}(\Delta ABC)$ (the verification is simple though tedious; it is not difficult to check that each of

$$a^2 = (b + c)^2, \quad a^2 = (b - c)^2, \quad 2a^2 = b^2 + c^2$$

implies that $XY = 0$, which yields most of the above factorization). Thus $XY \geq 0$ — strictly unless $2a^2 = b^2 + c^2$.

Now, if $P + Q - R \geq 0$ then $X > 0$ by the definition of X. And if $P + Q - R < 0$ then $Y > 0$ by the definition of Y, so the inequality $XY \geq 0$ entails $X \geq 0$ also in this case.

Remarks: (i) It is seen from (2) that XY is zero also for triangles degenerating to segments. Direct examination of those cases shows that equality holds in (1) for every "triangle" $(a, a + d, d)$ or $(a, d, a + d)$, where $0 \leq d \leq a$. The *original* inequality, however, is strict unless $a = b = c$ or $\min(a, b, c) = 0$.

(ii) Let A', B', C' be the midpoints of the sides BC, CA, AB of a triangle ABC and let

$$\varphi = \angle ABB', \qquad \psi = \angle ACC', \qquad \omega = \angle AA'C.$$

From

$$F = cm_b \sin\varphi = bm_c \sin\psi = (am_a/2)\sin\omega$$

we obtain the following restatement of (1):

$$\frac{1}{\sin\varphi} + \frac{1}{\sin\psi} \geq \frac{4}{\sin\omega}$$

— another item to add to the collection of triangle inequalities.

Also solved by the proposer. Two incorrect solutions were sent in.
The editor would like to see an easier way to obtain equality (2).

* * * * *

1905. [1994: 17] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Find all real solutions of the equation

$$\sqrt{x_1^2+x_2^2+\cdots+x_n^2}=\sqrt[3]{x_1^3+x_2^3+\cdots+x_n^3}.$$

Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.

Suppose there exists a positive number k such that

$$x_1^2+x_2^2+\cdots+x_n^2=k^2 \quad \text{and} \quad x_1^3+x_2^3+\cdots+x_n^3=k^3$$

thereby providing a possible solution; then

$$(x_1-k)x_1^2+(x_2-k)x_2^2+\cdots+(x_n-k)x_n^2=0.$$

Now either all the terms are zero or one of the terms at least is positive. Suppose $(x_i-k)x_i^2>0$; then $x_i>k$, contradicting $\sum x_i^2=k^2$. Hence all solutions arise from when the terms are all zero, and are thus of the form

$$x_j=k>0 \quad \text{and} \quad x_i=0 \text{ for all } i\neq j$$

for some $j\in\{1,2,\ldots,n\}$, or else all $x_i=0$. [Note that from $\sum x_i^2=k^2$, at most one x_i can equal k.—*Ed.*]

Comment by the editor.

Seung-Jin Bang, Seoul, Korea, and Václav Konečný, Ferris State University, Big Rapids, Michigan, point out that this problem is contained in problem 2 of the fourth Nordic Mathematical Olympiad, and that two solutions of that problem appear on [1992: 42–43]. (There is a small typo in the last line of solution II [1992: 43].) In some ways, Bradley's solution is simpler than either of these.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; CORY PYE, student, Memorial University of Newfoundland, St. John's; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; ASHISH K. SINGH, student, Kanpur, India; N. T. TIN, Hong Kong; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Five incorrect solutions were received.

A generalization of this problem to

$$\sqrt[p]{x_1^p+x_2^p+\cdots+x_n^p}=\sqrt[q]{x_1^q+x_2^q+\cdots+x_n^q}, \tag{1}$$

where p and q are positive real numbers and the x_i are nonnegative, is given on [1992: 42] *(the solutions are again that at most one x_i can be nonzero). Janous wonders if, for p and q positive odd integers and x_i arbitrary reals, the only solutions of* (1) *are when, under some renumbering of the x_i's,*

$$x_1=-x_2, \quad x_3=-x_4, \quad x_5=-x_6, \quad etc.,$$

with x_n arbitrary when n is odd. Does anyone out there know?

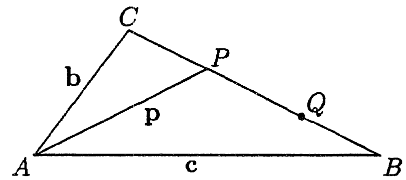
* * * * *

1906. [1994: 17] *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*

Let AP bisect angle A of triangle ABC, with P on BC. Let Q be the point on segment BC such that $BQ = CP$. Prove that $(AQ)^2 = (AP)^2 + (b-c)^2$.

Solution by Tim Cross, Wolverley High School, Kidderminster, U. K.

Without loss of generality take $b < c$. Let A be the origin and put $\overrightarrow{AB} = \mathbf{c}$ and $\overrightarrow{AC} = \mathbf{b}$ (so that $AB = |\mathbf{c}| = c$ and $AC = |\mathbf{b}| = b$). Then



$$\mathbf{p} = \mathbf{b} + \lambda(\mathbf{c} - \mathbf{b}) = (1-\lambda)\mathbf{b} + \lambda\mathbf{c}$$

for some scalar parameter λ, and

$$\mathbf{q} = \lambda\mathbf{b} + (1-\lambda)\mathbf{c}$$

since $BQ = CP$. By the Angle Bisector Theorem,

$$\frac{\lambda}{1-\lambda} = \frac{CP}{PB} = \frac{CA}{AB} = \frac{b}{c} \quad \Longrightarrow \quad \lambda = \frac{b}{b+c}\,.$$

Now

$$(AQ)^2 = \mathbf{q}\cdot\mathbf{q} = \lambda^2 b^2 + (1-\lambda)^2 c^2 + 2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{c}$$

and

$$(AP)^2 = \mathbf{p}\cdot\mathbf{p} = (1-\lambda)^2 b^2 + \lambda^2 c^2 + 2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{c}.$$

Subtracting,

$$(AQ)^2 - (AP)^2 = (1-2\lambda)(c^2-b^2) = \left(1 - \frac{2b}{b+c}\right)(c^2-b^2) = \frac{c-b}{b+c}(c^2-b^2) = (c-b)^2,$$

as required.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; BILL CORRELL JR., student, Denison University, Granville, Ohio; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C.; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; GEOFFREY A. KANDALL, Hamden, Connecticut; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong: DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, *Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; ASHISH KUMAR SINGH, student, Kanpur, India; D. J. SMEENK, Zaltbommel, The Netherlands; N. T. TIN, Hong Kong; PANOS E. TSAOUSSOGLOU, Athens, Greece; ALBERT W. WALKER, Toronto, Ontario (two solutions); CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

Hess and Klamkin sent in solutions similar to Cross's. In fact, Klamkin gave a generalization.

Walker states the similar result: if AP is the external *bisector of angle A with P on line BC, and Q is on line BC so that $\overrightarrow{BP} = -\overrightarrow{CQ}$, then $(AQ)^2 = (AP)^2 + (b+c)^2$. Readers may enjoy working on it. Walker also congratulates the proposer for a neat result with a variety of proofs.*

* * * * *

1908. [1994: 17] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

In triangle ABC the feet of the perpendiculars from A, B, C onto BC, CA, AB are denoted by D, E, F respectively. H is the orthocentre. The triangle is such that all of $AH - HD$, $BH - HE$, $CH - HF$ are positive. K is an internal point of ABC and L, M, N are the feet of the perpendiculars from K onto BC, CA, AB respectively. Prove that AL, BM, CN are concurrent if $KL : KM : KN$ is equal to

(i) $AH - HD : BH - HE : CH - HF$; (ii) $\dfrac{1}{AH-HD} : \dfrac{1}{BH-HE} : \dfrac{1}{CH-HF}$.

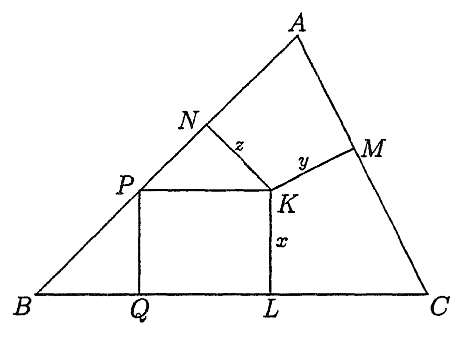
Solution by D. J. Smeenk, Zaltbommel, The Netherlands.

We denote the lengths of KL, KM and KN by x, y and z respectively. Let $P \in AB$ such that $KP \| CB$, and $Q \in BC$ such that $PQ \perp BC$. It is easy to verify that

$$PK = \frac{z}{\sin\beta}, \qquad BQ = x\cot\beta$$

(α, β, γ are the angles of ΔABC). So

$$BL = \frac{x\cos\beta + z}{\sin\beta},$$



and similarly

$$CL = \frac{x\cos\gamma + y}{\sin\gamma}, \qquad CM = \frac{y\cos\gamma + x}{\sin\gamma},$$

$$AM = \frac{y\cos\alpha + z}{\sin\alpha}, \qquad AN = \frac{z\cos\alpha + y}{\sin\alpha}, \qquad BN = \frac{z\cos\beta + x}{\sin\beta}.$$

Now AL, BM and CN are concurrent if and only if

$$BL \cdot CM \cdot AN = CL \cdot BN \cdot AM$$

[Ceva's theorem], or

$$(x\cos\beta + z)(y\cos\gamma + x)(z\cos\alpha + y) = (x\cos\gamma + y)(z\cos\beta + x)(y\cos\alpha + z),$$

which becomes

$$x(z^2 - y^2)(\cos\alpha - \cos\beta\cos\gamma) + y(x^2 - z^2)(\cos\beta - \cos\gamma\cos\alpha) + z(y^2 - x^2)(\cos\gamma - \cos\alpha\cos\beta) = 0. \quad (1)$$

We denote

$$\frac{AH - HD}{2R} = \cos\alpha - \cos\beta\cos\gamma = p,$$
$$\frac{BH - HE}{2R} = \cos\beta - \cos\gamma\cos\alpha = q,$$
$$\frac{CH - HF}{2R} = \cos\gamma - \cos\alpha\cos\beta = r$$

[e.g., see p. 163 of R. A. Johnson, *Advanced Euclidean Geometry.—Ed.*]. With these we rewrite (1):

$$px(z^2 - y^2) + qy(x^2 - z^2) + rz(y^2 - x^2) = 0. \quad (2)$$

Condition (i) is $x : y : z = p : q : r$, or $p = \lambda x$, $q = \lambda y$, $r = \lambda z$, with $\lambda > 0$. Then (2) becomes

$$\lambda[x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)] = 0,$$

and that holds indeed!

Condition (ii) is $x : y : z = p^{-1} : q^{-1} : r^{-1}$, or $px = qy = rz = \mu > 0$. Then (2) becomes

$$\mu[(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)] = 0,$$

and that holds as well.

We remark that the conditions mentioned are sufficient, not necessary.

Also solved by FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

*   *   *   *   *

1909. [1994: 17] *Proposed by Charles R. Diminnie, Saint Bonaventure University, Saint Bonaventure, New York.*

Solve the recurrence

$$p_0 = 1, \quad p_{n+1} = 5p_n(5p_n^4 - 5p_n^2 + 1)$$

for p_n in terms of n. [This problem was inspired by *Crux* 1809 [1994: 19].]

Solution by Achilleas Sinefakopoulos, student, University of Athens, Greece.

Put $f_0(x) = 2$, $f_1(x) = x$ and $f_n(x) = xf_{n-1}(x) - f_{n-2}(x)$ for real x and $n \geq 2$. We first solve the recurrence

$$p_0 = 1, \qquad p_{n+1} = \frac{1}{\sqrt{5}} f_m(\sqrt{5}\,p_n), \qquad m \text{ odd } \geq 3. \tag{1}$$

An easy induction yields

$$f_n\left(y + \frac{1}{y}\right) = y^n + \frac{1}{y^n} \tag{2}$$

for $n \geq 0$ and $y \neq 0$. Also, observe that since m is odd we have, by Binet's formula,

$$\sqrt{5}\,F(m^n) = \left(\frac{1+\sqrt{5}}{2}\right)^{m^n} - \left(\frac{1-\sqrt{5}}{2}\right)^{m^n} = \left(\frac{1+\sqrt{5}}{2}\right)^{m^n} + \left(\frac{2}{1+\sqrt{5}}\right)^{m^n}, \tag{3}$$

where $F(n)$ is the nth Fibonacci number ($F(1) = F(2) = 1$). Now we prove by induction that $p_n = F(m^n)$ for $n \geq 0$. Indeed, the case $n = 0$ is trivial, so assume that $p_n = F(m^n)$ for some $n \geq 0$. Then

$$\begin{aligned} p_{n+1} &= \frac{1}{\sqrt{5}} f_m(\sqrt{5}\,p_n) = \frac{1}{\sqrt{5}} f_m(\sqrt{5}F(m^n)) \\ &= \frac{1}{\sqrt{5}} f_m \left[\left(\frac{1+\sqrt{5}}{2}\right)^{m^n} + \left(\frac{2}{1+\sqrt{5}}\right)^{m^n}\right] \qquad \text{by (3)} \\ &= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{m^{n+1}} + \left(\frac{2}{1+\sqrt{5}}\right)^{m^{n+1}}\right] \qquad \text{by (2)} \\ &= F(m^{n+1}). \end{aligned}$$

This completes the proof that $p_n = F(m^n)$ for all $n \geq 0$.

Notice that for $m = 3$ we have *Crux* 1809, and for $m = 5$ we have *Crux* 1909. [*Editor's note.* Specifically, we get:

$$f_2(x) = xf_1(x) - f_0(x) = x^2 - 2,$$

$$f_3(x) = xf_2(x) - f_1(x) = x(x^2 - 2) - x = x^3 - 3x,$$

and thus, putting $m = 3$ in the recurrence (1),

$$p_{n+1} = \frac{1}{\sqrt{5}} f_3(\sqrt{5}p_n) = \frac{1}{\sqrt{5}}\left[(\sqrt{5}p_n)^3 - 3\sqrt{5}p_n\right] = 5p_n^3 - 3p_n,$$

the recurrence in *Crux* 1809, with solution $p_n = F(3^n)$ given above;

$$f_4(x) = xf_3(x) - f_2(x) = x(x^3 - 3x) - (x^2 - 2) = x^4 - 4x^2 + 2,$$

$$f_5(x) = xf_4(x) - f_3(x) = x(x^4 - 4x^2 + 2) - (x^3 - 3x) = x^5 - 5x^3 + 5x,$$

and so, putting $m = 5$ into (1),

$$p_{n+1} = \frac{1}{\sqrt{5}} f_5(\sqrt{5}p_n) = \frac{1}{\sqrt{5}}\left[(\sqrt{5}p_n)^5 - 5(\sqrt{5}p_n)^3 + 5\sqrt{5}p_n\right] = 25p_n^5 - 25p_n^3 + 5p_n,$$

the given recurrence, with solution

$$p_n = F(5^n).$$

We could continue to obtain further recurrences. For instance one finds that

$$p_{n+1} = 125p_n^7 - 175p_n^5 + 70p_n^3 - 7p_n$$

($p_0 = 1$) has solutions $p_n = F(7^n)$.]

Also solved by H. L. ABBOTT, University of Alberta; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SEUNG-JIN BANG, Seoul, Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; HIMADRI CHOUDHURY, student, Hunter High School, New York; BILL CORRELL JR., student, Denison University, Granville, Ohio; TIM CROSS, Wolverley High School, Kidderminster, U. K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Klamkin also gives instructions for obtaining a recurrence with solution $p_n = F(m^n)$ for odd m. Further, he states that the recurrences

$$p_{n+1} = p_n^2 - 2, \quad p_1 = 3$$

$$p_{n+1} = p_n^4 - 4p_n^2 + 2, \quad p_1 = 7$$

$$p_{n+1} = p_n^6 - 6p_n^4 + 9p_n^2 - 2, \quad p_1 = 18$$

have solutions, respectively: $p_n = L(2^n)$, $p_n = L(4^n)$, and $p_n = L(6^n)$, where $L(n)$ is the nth Lucas number, defined by $L(1) = 1$, $L(2) = 3$, $L(n) = L(n-1) + L(n-2)$. Note the connection between these polynomials and the polynomials $f_2(x)$ and $f_4(x)$ above!

*   *   *   *   *

YEAR-END WRAPUP

With the end of another year it is time again to record various comments, late solutions, etc. which have come in to the *Crux* office during the past 12 months.

939 [1985: 224, 251]. Murray S. Klamkin, University of Alberta, points out a small error in the proposer's solution of this problem, namely that the inequality $\angle MBC > \angle MCB$ is needed to obtain the inequality $BB' < CC'$ given on [1985: 225]. The effect of this is that the shaded area in Figure 2 on [1985: 225] should be bounded on the right by the perpendicular bisector of BC. As the symmedian, angle bisector and median from vertex A all still lie in this smaller shaded area, the rest of the proof should not be affected.

1217 [1988: 123]. For some reason lost in the mists of time, after printing two solutions to this problem the editor failed to list the other solvers; moreover, none of these deserving readers ever informed him of this omission! For the record then, the solvers were Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany; the late J. T. Groenman, Arnhem, The Netherlands; and the proposer (Niels Bejlegaard).

1463 [1990: 280]. Murray S. Klamkin, University of Alberta, points out that more general sums are known and can be found in E. R. Hanson, *A Table of Series and Products*, Prentice Hall, 1975, p. 256.

1559 [1991: 248]. Juan-Bosco Romero Márquez, Universidad de Valladolid, Spain, notes that the analogous problem for a general cubic appears on pp. 128–140 of J. Rivaud, *Ejercicios de Algebra*, Madrid, 1968, and a further generalization to arbitrary degree appears on pp. 85–86 of (the Spanish edition of) D. Faddeev and I. Sominski, *Problemas en Algebra Superior*, Mir, Moscow, 1971.

1598 [1992: 27; 1993: 51]. Another proof of Murray Klamkin's conjectured inequality (5) on [1992: 29] was independently sent in by Ji Chen, Ningbo University, China.

1773 [1993: 208]. In answer to the editor's question about which other properties the class of triangles in this problem possesses, D. J. Smeenk, Zaltbommel, The Netherlands, found some. However, the editor has also since learned that this class of triangles has popped up numerous times before! As well as occurring in *Crux* 1751 [1993: 148], it is also the subject of *Crux* problems 388 [1979: 201], 659 [1982: 215] and 660 [1982: 216], and the earlier problem 758 of *Mathematics Magazine*, solution on pages 285–286 of the 1970 volume.

1777 [1993: 214]. Regarding the editor's question "for any relatively prime a and b is there always a prime $\equiv a \bmod b$ between $n/2$ and n for sufficiently large n?", Pieter Moree, of Leiden University but visiting at Princeton, says that it is clear from the Prime Number Theorem for arithmetic progressions that the answer is yes. He also refers to his article "Bertrand's postulate for primes in arithmetic progressions", *Computers Math. Applic.* **26**, No. 5 (1993) pp. 35–43, which contains more information.

Regarding the article "The Ochoa curve" by Richard Guy in the March 1990 issue [1990: 65–69], Richard reports that R. J. Stroeker and B. M. M. de Weger ("On elliptic diophantine equations that defy Thue's method (the case of the Ochoa curve)", Report

9437/B, Econometric Institute, Erasmus University, Rotterdam, 1994) have proved that the curve is indeed of rank 4, as suspected, and that there are just 23 integer points on it, for which

$$(X,Y) = (247,3528),\ (499,3276),\ (751,14112) \quad \text{and} \quad (-761,504)$$

form a basis. Their paper will appear in the journal *Experimental Mathematics* next year.

In answer to Bob Prielipp's query, noted last December [1993: 305], about possible misprints in Viktors Linis's article "Gauss and Easter Dates" [1977: 102–103], Chris Fisher writes that in the formula for e on [1977: 103] the a should have been a d, i.e. the formula should read

$$e \equiv 2b + 4c + 6d + n \pmod 7.$$

Interestingly, this equation is *correct* in my bound copy of the 1977 volume, but not in Chris's or Bob's! The error must have been corrected in some later printing of this volume.

Late solutions were received from Hayo Ahlburg, Benidorm, Spain (**1805** and **1810**(a)); Francisco Bellot Rosado, I. B. Emilio Ferrari, Valladolid, Spain (**1833**); Shawn Godin, St. Joseph Scollard Hall S. S., North Bay, Ontario (**1879**); and Walther Janous, Ursulinengymnasium, Innsbruck, Austria (**1820**, **1848**, **1850**, **1854**, **1862**, **1865**, **1866**, **1867**(a), **1868**, **1869**, **1870**, **1874** and **1879**). Many of Janous's late submissions appear to have been caused by slow mail delivery!

Many thanks to the following people for their assistance to the editor and other members of the Editorial Board during 1994, in giving advice regarding problems, articles, and solutions: *ED BARBEAU, LEN BOS, PAK-HONG CHEUNG, ROLAND EDDY, PETER EHLERS, DOUG FARENICK, BRUCE GILLIGAN, WALTHER JANOUS, STEVE KIRKLAND, MURRAY KLAMKIN, JOANNE MCDONALD, RICHARD MCINTOSH, DIETER RUOFF, JONATHAN SCHAER, HARLEY WESTON, SIMING ZHAN.* The editor would also like to thank *JAN CERNY, MIKE DOOB, CLAUDE LAFLAMME, MIKE LOGOZAR*, and *ARUNAS SALKAUSKAS* for their efforts in 1994 to improve the computer production of *Crux*. Hopefully readers will see the results soon!

As usual, special thanks are due to the members of the *Crux* Editorial Board for their constant contributions to the quality of our journal, and to *JOANNE LONGWORTH* for her work at odd hours, usually lunch hours, evenings and weekends, putting *Crux* into LaTeX.

Finally, to you the readers, a big thank-you coupled with a wish that you have a

send in problems send in problems send in problems send in problems send in problems

HAPPY NEW YEAR!

send in problems send in problems send in problems send in problems send in problems

* * * * *

ARTICLES AND NOTES

PROPOSALS AND SOLUTIONS

PROPOSERS AND SOLVERS

The numbers refer to the pages in which the corresponding name appears with a problem proposal, a solution or a comment.

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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