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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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PRIME PYRAMIDS

Richard K. Guy

Margaret J. Kenney of Boston College, in the *Student Math Notes* enclosed with the Nov. 1986 NCTM News Bulletin, proposed the following **prime pyramid** as a classroom activity.

$$\begin{array}{ccccccccccccc}
 & & & & & & * & & & & & & \\
 & & & & & 1 & & 2 & & & & & \\
 & & & & 1 & & 2 & & 3 & & & & \\
 & & & 1 & & 2 & & 3 & & 4 & & & \\
 & & 1 & & 4 & & 3 & & 2 & & 5 & & \\
 & 1 & & 4 & & 3 & & 2 & & 5 & & 6 & \\
1 & & - & & - & & - & & - & & - & & 7
\end{array}$$

Row n contains the numbers $1, 2, \ldots, n$. It begins with 1 and ends with n. The sum of two consecutive entries in a row is prime.

There are various questions we may ask.

1. Can the pyramid be continued?
2. How many ways are there of arranging the numbers in row n? (Check that the arrangements shown above are unique.)
3. Can you find a pattern that will serve for an infinity of rows?
4. Will there ever be a row that cannot be completed?

The $n-1$ sums add to

$$1 + 2(2 + 3 + \cdots + (n-1)) + n = n^2 - 1,$$

so the average sum of two consecutive members is $n+1$. The probability that a number of this size is prime is about $1/\ln n$. So, if we wrote the numbers down at random, our chance of success would only be about

$$\left(\frac{1}{\ln n}\right)^{n-1}$$

However, there are $(n-2)!$ ways in which the numbers may be arranged and when we estimate

$$\left[1 - \left(\frac{1}{\ln n}\right)^{n-1}\right]^{(n-2)!}$$

(which is our chance of failure if we try every order) we find that we're pretty certain to succeed. Of course, this isn't a proof, but if you start answering question 2, you find

$n =$	1	2	3	4	5	6	7	8	9	10	11	12
# of solutions	*	1	1	1	1	1	2	4	8	28	84	216

and it seems likely that the answer grows super-exponentially, at a factorial-type rate.

Now we could answer question 3 if we knew that there were infinitely many twin primes. If n and $n+2$ are both prime, then

$$1 \quad (n-1) \quad 3 \quad (n-3) \quad 5 \quad \ldots \quad (n-2) \quad 2 \quad n$$

will work.

Here is a more convincing algorithm which it may be possible to formalize into a rigorous proof. It's clear that we have to arrange the numbers alternately odd and even. This reduces our number of arrangements from $(n-2)!$ to

$$\left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lfloor \frac{n-2}{2} \right\rfloor !$$

but this is still a pretty huge number.

Assume for the moment that n is odd. You can see how the argument is easily adapted for n even. Take $n = 11$ for an example. Consider all 4! ways of arranging the odd numbers 3, 5, 7, 9 between 1 and 11, leaving spaces for the even numbers. Then we can fill in the even numbers, subject only to the constraints that

the number	2	4	6	8	10
mustn't sit next to	7	5 or 11	3 or 9	1 or 7	5 or 11

But this is almost exactly Lucas's classical *problème des ménages*, to seat couples round a table with no-one sitting next to their own spouse. The difference is that some even numbers have as many as two spouses, but the number of legal arrangements is still a substantial fraction of the $((n-1)/2)!$ arrangements, and this grows rapidly. For $n = 11$ you could be unlucky. You can't seat the even numbers among the arrangements

$$1\,.\,3\,.\,9\,.\,5\,.\,7\,.\,11, \qquad 1\,.\,7\,.\,3\,.\,5\,.\,9\,.\,11,$$
$$1\,.\,7\,.\,9\,.\,5\,.\,3\,.\,11, \qquad 1\,.\,9\,.\,3\,.\,5\,.\,7\,.\,11,$$

but the other 20 all allow solutions, some of them,

$$1\,.\,3\,.\,9\,.\,7\,.\,5\,.\,11, \qquad 1\,.\,7\,.\,3\,.\,9\,.\,5\,.\,11,$$
$$1\,.\,7\,.\,9\,.\,3\,.\,5\,.\,11, \qquad 1\,.\,7\,.\,9\,.\,3\,.\,5\,.\,11,$$

as many as eight.

What is going on? The constraints above are only needed to avoid the divisor 3. The odd and even arrangement avoids the divisor 2. And since our sums are at most

$11 + 10 < 25$ we don't have to worry about divisor 5. For larger n we need watch out only for divisors $\leq \sqrt{2n-1}$ and the number of constraints is less than

$$\frac{n}{6} + \frac{n}{10} + \frac{n}{14} + \cdots$$

where the denominators are twice the odd primes less than $\sqrt{2n}$. Unfortunately this series, even when added only thus far, is divergent. Its sum is presumably $cn \ln\ln\sqrt{2n}$, where c is something like 0.43, though one needs a much more powerful telescope than the one I've used. A better estimate, possibly good enough to formalize into a proof, could be obtained by using the inclusion-exclusion principle.

Although there is no necessary connexion between one row and another, note that any row can probably be used as the initial segment of a later row, provided we leave ourselves enough room to manœuvre. We couldn't extend 1 6 5 2 3 4 7 to a row of length 8, 9 or 10, but it can be the beginning of

1 6 5 2 3 4 7 10 9 8 11 12 or 1 6 5 2 3 4 7 12 11 8 9 10 13,

not the beginning of rows of length 14 or 15, but

1 6 5 2 3 4 7 12 11 8 15 14 9 10 13 16
1 6 5 2 3 4 7 12 11 8 9 10 13 16 15 14 17
1 6 5 2 3 4 7 10 9 8 11 12 17 14 15 16 13 18
1 6 5 2 3 4 7 10 9 8 11 18 13 16 15 14 17 12 19

and probably any longer row. Note how bits of the 'twin prime' construction can be patched together.

Has anyone any better answers?

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* * * * *

Editor's note. *Crux* readers are encouraged to check out the long and interesting interview with Professor Guy, complete with photos, which graces a recent issue of the *College Mathematics Journal* (Volume 24, Number 2, March 1993). Look for Richard and Louise mountaineering on the cover!

* * * * *

THE OLYMPIAD CORNER

No. 144

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first set of problems we give are "pre-Olympiad" even though the title seems to say otherwise. They are the 9th form problems of the XXV Soviet Mathematical Olympiad, written at Smolensk, Russia, April 17–24, 1991. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, leader of the Canadian I.M.O. team, for collecting and forwarding these to me.

XXV SOVIET MATHEMATICAL OLYMPIAD

9th Form

1. Find all integer solutions of the system

$$xz - 2yt = 3, \qquad xt + yz = 1.$$

(Ju. Nesterenko)

2. On the blackboard are written n numbers. One may erase any pair of them, say a and b, and write down the number $(a+b)/4$ to replace them. After this procedure is repeated $n-1$ times, only one number remains on the blackboard. Prove that given that the n numbers at the beginning are all 1, the last number will not be less than $1/n$. (B. Berlov)

3. There are four straight lines, each two of them intersecting, and no three of them having a common point. Each line is divided into four pieces with two finite length segments among them. The total number of segments is eight. Is it possible that the lengths of these segments are equal to:

(a) $1, 2, 3, 4, 5, 6, 7, 8$?

(b) pairwise different natural numbers? (A. Berzinsh)

4. A ticket for a lottery is a card which has 50 empty cells in a line. Each participant writes down in the cells the numbers $1, 2, \ldots, 50$ without repetitions. The organizer of the lottery has his own card with the numbers written on it according to the rule. A ticket wins if at least one number in it coincides with the number in the corresponding cell of the organizer's card. What is the least possible number of cards which the participant must fill to guarantee a win? (A. Berzinsh)

* * *

For an Olympiad set this issue we give the problems of the 22nd Austrian Mathematical Olympiad (2nd Round). Many thanks to Walter Janous, Ursulinengymnasium, Innsbruck, Austria, for sending them to me.

22nd AUSTRIAN MATHEMATICAL OLYMPIAD 1991

2nd Round — May 7, 1991 (Time allowed: 4 hours)

1. Let a, b be rational numbers such that $\sqrt[3]{a}+\sqrt[3]{b}$ is a rational number $c \neq 0$. Show that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

2. Determine all real solutions of the equation

$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14}=0.$$

3. Determine the number of all square numbers contained in the sequence $\{a_0, a_1, a_2, \ldots\}$ where $a_0 = 91$ and $a_{n+1} = 10a_n + (-1)^n$, $n \geq 0$.

4. Let A, B be two points on a circle k of radius r such that $\overline{AB} = c$.

(i) Give a construction of all triangles ABC having k as their circumcircle and such that one of the medians s_a or s_b (through vertices A or B, resp.) is of given length d.

(ii) How have r, c and d to be chosen such that ΔABC is uniquely determined?

Final Round, 1st Day — June 11, 1991 (Time allowed: 4 hours)

1. We are given a convex solid K in $\mathbb{R}^3$ (i.e. for any two points of K the segment joining them belongs to K), and two parallel planes ε_1 and ε_2 with mutual distance 1 both tangent to K. Let ε be parallel to ε_1 and ε_2, between ε_1 and ε_2 and at distance d_1 from ε_1. Determine the values of d_1 such that the part of K between ε_1 and ε has (a) at least, (b) at most half of the volume of K?

2. Determine all functions $f : \mathbb{Z} \setminus \{0\} \to \mathbb{Q}$ satisfying the functional equation

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}.$$

3. (a) Show that $91 \mid n^{37} - n$ for all $n \in \mathbb{N}$.

(b) Determine the greatest integer k such that $k \mid n^{37} - n$ for all $n \in \mathbb{N}$.

Final Round, 2nd Day — June 12, 1991 (Time allowed: 4 hours)

4. The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 0$ and $a_{2k+1} = a_k + a_{k+1}$, $a_{2k+2} = 2a_{k+1}$, $k \geq 1$. Determine a_m, where $m = 2^{19} + 91$.

5. Show that for all natural numbers $n > 1$ the inequality

$$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$$

is valid.

6. Determine the number of numbers $(a_9 \ldots a_0)_{10}$ which have no initial zeros and do not contain the block of digits 1991, when written in decimal notation.

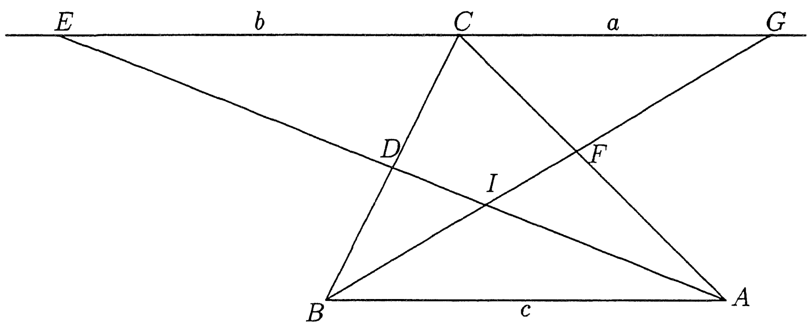
* * *

First we give an alternate solution to one given in the January number.

8. [1991: 197; 1993: 8] *Proposed by Ireland.*

Let ABC be a triangle and ℓ the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line ℓ at E. Let the internal bisector of the angle at B meet the side AC at F and the line ℓ at G. If $GF = DE$ prove that $AC = BC$.

Alternate solution by Geoffrey A. Kandall, Hamsden, Connecticut.



Let $AD = t_a$, $BF = t_b$. Using similar triangles, we obtain

$$\frac{GF}{t_b} = \frac{a}{c}, \qquad \frac{DE}{t_a} = \frac{b}{c}.$$

Therefore $GF = DE$ iff $at_b = bt_a$. So the problem really is to show that $at_b = bt_a \Rightarrow a = b$.

Suppose $a \neq b$, say $a < b$. Then it is well known that $t_b < t_a$, and hence $at_b < bt_a$, a contradiction.

Even if one does not use the "well-known" fact, it is possible to give an elementary proof of it from $at_b = bt_a$. Suppose $at_b = bt_a$. Suppose $a \neq b$, say $a < b$. Then $\angle IAB < \angle IBA$, hence $BI < AI$. Since $BI/IF = (a+c)/b$ and $AI/ID = (b+c)/a$, it follows that

$$\frac{t_b}{IF} = \frac{a+b+c}{b}, \qquad \frac{t_a}{ID} = \frac{a+b+c}{a}$$

and from $BI < AI$ this gives

$$\frac{a+c}{a+b+c}t_b < \frac{b+c}{a+b+c}t_b, \quad \text{i.e.} \quad (a+c)t_b < (b+c)t_a.$$

Since $at_b = bt_a$ it follows that $ct_b < ct_a$, and $t_b < t_a$.

* * *

Next a typo spotted by an ever vigilant reader, Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. Problem 4 of the Singapore Interschool Mathematics Competition, Part A [1993: 65] should replace $a + b + 1$ by $a + b + c = 1$, to read

4. Let a, b, c be real numbers satisfying $a+b+c=1$, $a^2+b^2+c^2=1/2$. Find the maximum value of c.

* * *

Shortly after the March number of the Corner was off to the printers, the mail brought a package of solutions to the five Selection Questions For the 1990 Irish I.M.O. Team, as well as to problem 1 of the 1988 Chinese Olympiad Training Camp. Thanks to Michael Selby, University of Windsor, for the solutions.

* * *

The next solutions we give deal with responses to the problems *More Selection Questions For the 1990 Irish I.M.O. Team* [1992: 65].

1. Let ABC be a right-angled triangle with right angle at A. Let X be the foot of the perpendicular from A to BC and Y the midpoint of XC. Let AB be extended to D so that $AB = BD$. Prove that DX is perpendicular to AY.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Dieter Bennewitz, Koblenz, Germany; by C. Bradley, Clifton College, Bristol, United Kingdom; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Pavlos Maragoudakis, student, University of Athens, Greece. Most solvers used coordinates in one form or another. We use the (slightly) more geometric proof of Maragoudakis.

Let BC be extended to X' so that $BX = BX'$. Then AX' and XD are parallel and of equal length. It is enough to prove that triangle YAX' is right-angled at A. Let $BC = a$, $AC = b$, $AB = c$. We have $AX = h_a = (BC)/a$. Also, by similar triangles, $b^2 = XC \cdot a$. Therefore $XC = b^2/a$, and $XY = b^2/2a$. Now $c^2 = XB \cdot a$ so $XB = c^2/a$.

From Pythagoras,

$$AY^2 = AX^2 + XY^2 = \frac{b^2c^2}{a^2} + \frac{b^4}{4a^2}\ . \tag{1}$$

Because XB is the median of triangle AXD,

$$AX^2 + XD^2 = 2XB^2 + \frac{AD^2}{2}\ ,$$

$$XD^2 = \frac{2c^4}{a^2} + 2c^2 - \frac{b^2c^2}{a^2}\ ,$$

and

$$(AX')^2 = \frac{2c^4}{a^2} + 2c^2 - \frac{b^2c^2}{a^2}\ . \tag{2}$$

We also have

$$(X'Y)^2 = (XY + 2XB)^2 = \left(\frac{b^2}{2a} + \frac{2c^2}{a}\right)^2$$

which gives

$$X'Y^2 = \frac{b^4}{4a^2} + \frac{4c^4}{a^2} + \frac{2b^2c^2}{a^2}\ . \tag{3}$$

Finally $\angle YAX' = 90°$ iff $(X'Y)^2 = AY^2 + (AX')^2$ which by (1), (2) and (3) is equivalent to

$$\frac{2c^4}{a^2} + \frac{2b^2c^2}{a^2} = 2c^2,$$

i.e. $c^2 + b^2 = a^2$, which is true.

2. The real number x satisfies all the inequalities

$$2^k < x^k + x^{k+1} < 2^{k+1}$$

for $k = 1, 2, \ldots, n$. What is the greatest possible value of n?

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Pavlos Maragoudakis, student, University of Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Bradley's solution.

We cannot have $n \geq 4$ because $2 < x + x^2 < 4$ and $16 < x^4 + x^5 < 32$ would mean $x^3 > 4$ or $x > 2^{2/3}$. But $2^{2/3} + 2^{4/3} > 4$, by the Arithmetic Mean – Geometric Mean inequality, so we have a contradiction.

On the other hand $n = 3$ is possible, for example $x = 3/2$ will do, since

$$2 < \frac{3}{2} + \left(\frac{3}{2}\right)^2 = \frac{15}{4} < 4, \quad 4 < \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 = \frac{45}{8} < 8,$$

and

$$8 < \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^4 = \frac{135}{16} < 16.$$

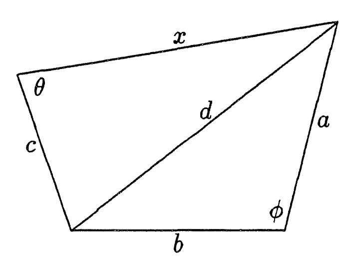
3. Three sides of a quadrilateral are given, of lengths a, b, c, respectively. If the area of the quadrilateral is as large as possible prove that the length x of the remaining side satisfies the equation

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

We have

$$d^2 = a^2 + b^2 - 2ab\cos\varphi = c^2 + x^2 - 2cx\cos\theta \qquad (1)$$



and $2\Delta = ab\sin\varphi + cx\sin\theta$, where Δ is the area of the quadrilateral. It is perfectly possible, but cumbersome, to prove purely algebraically that the maximum area occurs when the quadrilateral is cyclic. Using Lagrange multipliers is shorter, using three variables θ, φ, x and the constraint (1). One obtains

$$(cx\cos\theta, ab\cos\varphi, c\sin\theta) = \lambda(2cx\sin\theta, -2ab\sin\varphi, 2x - 2c\cos\theta).$$

So, $2\lambda = \cot\theta = -\cot\varphi$, showing that $\theta + \phi = 180°$. Also $2\lambda(x - c\cos\theta) = c\sin\theta$, from which $x\cos\theta = c$, and $\cos\varphi = -c/x$. Substituting back in (1) gives

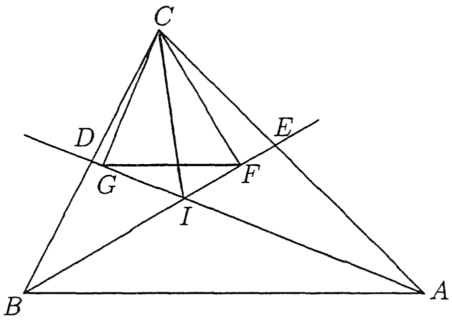
$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

4. Let ABC be a triangle and let the internal bisectors of the angles at A and B meet the sides BC and AC at D and E, respectively. Let CF and CG be the perpendiculars from C to the lines BE and AD, respectively. Prove that the line FG is parallel to AB.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Pavlos Maragoudakis, student, University of Athens, Greece. We use the solution of Maragoudakis.

Let I denote the incentre of the triangle, i.e. the point of intersection of AD and BE. The quadrilateral $IFCG$ can be inscribed in a circle since $\angle IFC + \angle IGC = 180°$. So

$$\angle IFG = \angle ICG. \qquad (1)$$



But

$$\begin{aligned}\angle ICG &= \angle ACG - \angle ICA \\ &= 90° - \angle GAC - \angle ICA = 90° - \frac{1}{2}\angle A - \frac{1}{2}\angle C. \qquad (2)\end{aligned}$$

From (1), (2) we have $\angle IFG = \frac{1}{2}\angle B = \angle ABF$. Thus AB is parallel to FG as required.

5. Let $n = 2k - 1$ where $k \geq 6$ is an integer. Let T be the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where x_i is 0 or 1 ($i = 1, 2, \ldots, n$). For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in T, let $d(\mathbf{x}, \mathbf{y})$ denote the number of integers j with $1 \leq j \leq n$ such that $x_j \neq y_j$. (In particular, $d(\mathbf{x}, \mathbf{x}) = 0$). Suppose that there exists a subset S of T with 2^k elements which has the following property: given any element $\mathbf{x}$ in T, there is a unique element $\mathbf{y}$ in S with $d(\mathbf{x}, \mathbf{y}) \leq 3$. Prove that $n = 23$.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

It is not clear whether such a set S exists, but if it does, then it clearly partitions T into disjoint subsets T_i such that the elements of T_i are linked to the element $\mathbf{x}$ in S (uniquely) by the specification that the distance from any element in T_i to $\mathbf{x}$ is less than or equal to three. Furthermore these subsets will each have the same number of elements, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$. The conditions of the problem require this to be equal to $2^{(n-1)/2}$. This is only true if $n = 23$.

* * *

Next come solutions to some of the problems from the *Second, Third, and Fourth Tests of the 1988 Chinese Olympiad Training Camp* [1992: 66,67].

2. *Second Test.* For a finite sequence A of 0's and 1's, let $f(A)$ denote the sequence obtained from A by replacing each 1 by 0, 1, and each 0 by 1, 0; e.g., $f((1,0,0,1)) =$

$(0,1,1,0,1,0,0,1)$. Let $f^n(A)$ denote the nth iterate of f on A. Determine the number in $f^n((1))$ of two consecutive terms which are $0,0$.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

Let a_n be the number in $f^n((1))$ of two consecutive terms which are $0,0$ and b_n the number in $f^n((1))$ of two consecutive terms that are $1,1$. The following facts about the set $f^n((1))$ are easy to establish by induction, and the proofs are omitted.

(i) $f^n((1))$ is a sequence of 2^n digits, all zero or one.

(ii) The last digit is always 1.

(iii) The first digit alternates $0,1,0,1,\ldots$, with the first digit 1 for n even.

(iv) The first 2^{n-1} digits in $f^{(n)}((1))$ are the complements with respect to 1 of the last 2^{n-1} digits.

These facts taken together, imply that

$$b_n = a_{n-1} + b_{n-1}, \quad n \geq 2 \tag{1}$$

$$a_{2n+1} = a_{2n} + b_{2n}, \quad n \geq 1 \tag{2}$$

and

$$a_{2n+2} = a_{2n+1} + b_{2n+1} + 1, \quad n \geq 0 \tag{3}$$

The extra 1 creeps into a_{2n} because of a "010" created in the centre, which occurs only on even rows.

From these equations one may eliminate b_n to obtain

$$a_{2n+2} = 2a_{2n+1} + 1, \quad n \geq 0, \tag{4}$$

and

$$a_{2n+1} = 2a_{2n} - 1, \quad n \geq 1. \tag{5}$$

Also, for the start $a_1 = 0$ and $a_2 = 1$. Observe that there is a different recurrence relation for odd and even n, so readjusting to take account of this, from (4) and (5), one has

$$a_{2n+1} = 4a_{2n-1} + 1, \quad n \geq 1 \tag{6}$$

and

$$a_{2n+2} = 4a_{2n} - 1, \quad n \geq 1. \tag{7}$$

The solution of these difference equations provides the answer to the problem:

$$a_{2n+1} = \frac{1}{3}(4^n - 1) \quad \text{and} \quad a_{2n} = \frac{1}{6}(4^n + 2).$$

[*Editor's note*: this problem is the same as *Crux* 1790 [1992: 275].]

3. *Second Test.* A mathematics teacher wants her two intelligent students S and P to derive the exact value of a 2-digit natural number n by revealing the number of positive divisors of n to S and the sum of the digits of n to P. A brief conversation between S and P goes as follows:

P: I can not determine n.
S: I can't either but I know whether n is even or not.
P: Now I know what n is.
S: So do I now.

Suppose both students are honest and have perfect logical reasoning for whatever they say. Determine n and justify your answer.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Pavlos Maragoudakis, student, University of Athens, Greece.

First we make a table which shows the number of positive divisors of all 2-digit natural numbers:

2	3	4	5	6	7	8	9	10	12
11, 13	25	10, 14, 15	16	12, 18	64	24	36	48	60
17, 19	49	21, 22, 26	81	20, 28		30		80	72
23, 29		27, 33, 34		32, 44		40			84
31, 37		35, 38, 39		45, 50		42			90
41, 43		46, 51, 55		52, 63		54			96
47, 53		57, 58, 62		68, 76		56			
59, 61		65, 69, 74		92, 98		66			
67, 71		75, 77, 82		99		70			
73, 79		85, 86, 87				78			
83, 89		93, 94, 95				88			
91, 97									

P can not determine n, so n is neither 10 nor 99, and the sum of the digits lies between 2 and 17 inclusive. S can't either, so, from the table n is neither 36 nor 64. But he knows whether n is even or not. So n can have only 2, 3, 8, 10 or 12 divisors because from the table these are the columns which have only even or only odd number entries.

Then P says that he knows n. This means that among the 2-digit numbers with 2, 3, 8, 10 or 12 divisors there is exactly one with the sum of its digits equal to the sum of the digits of n. We therefore make a second table which shows the sum of the digits of the numbers with 2, 3, 8, 10 or 12 divisors:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
11	30	13	23	42	25	17	54	19	29	48	67	59	78	79	89
		31	41	60	43	53	72	37	47	66	49		96	88	
		40		24	61	71	90	73	56	84				97	
					70			91	83						

From this table S can see that n is one of the numbers 11, 59, 89, 30. Then he says he also knows n. This means that among 11, 59, 89, 30 there is exactly one with its number of divisors the same as for n. The numbers 11, 59, 89 have 2 divisors, so n must be the number 30.

1. *Third Test.* Suppose the inscribed circle of $\Delta A_1A_2A_3$ touches the sides A_2A_3, A_3A_1 and A_1A_2 at T_1, T_2 and T_3 respectively. From the midpoints M_1, M_2 and M_3 of A_2A_3, A_3A_1 and A_1A_2, draw lines perpendicular to T_2T_3, T_3T_1, and T_1T_2 respectively. Prove that these three perpendicular lines are concurrent at a point P and determine the location of P.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

We make use of the following known lemma, which can be called the distance lemma. Given its usefulness in problems in geometry it is surprising it is not more widely known.

Lemma. Given any triangle ABC with sides a, b, c, then it is possible to choose an origin O such that with $\overrightarrow{OA} = \mathbf{x}$, $\overrightarrow{OB} = \mathbf{y}$, and $\overrightarrow{OC} = \mathbf{z}$ and areal coordinates (α, β, γ), any point P in the plane of the triangle has $\overrightarrow{OP} = \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$ with $\alpha + \beta + \gamma = 1$. Furthermore O can be chosen so that if $P_1(\alpha_1, \beta_1, \gamma_1)$ and $P_2(\alpha_1, \beta_2, \gamma_2)$ are given then

$$(\overrightarrow{P_1P_2})^2 = (\alpha_1-\alpha_2)^2\left(\frac{b^2+c^2-a^2}{2}\right)+(\beta_1-\beta_2)^2\left(\frac{c^2+a^2-b^2}{2}\right)+(\gamma_1-\gamma_2)^2\left(\frac{a^2+b^2-c^2}{2}\right).$$

Let O be as given in the lemma. The point T_1 has then areal coordinates

$$\left(0, \frac{a+b-c}{2a}, \frac{a-b+c}{2a}\right)$$

since $T_1A_3 = s-c$ and $T_1A_2 = s-b$. T_2 and T_3 may similarly be represented as

$$\left(\frac{a+b-c}{2b}, 0, \frac{-a+b+c}{2b}\right) \quad \text{and} \quad \left(\frac{a-b+c}{2c}, \frac{-a+b+c}{2c}, 0\right)$$

respectively. M_1, M_2, M_3 have coordinates $(0,1/2,1/2)$, $(1/2,0,1/2)$ and $(1/2,1/2,0)$ respectively. We have

$$\overrightarrow{T_2T_3} = \frac{b+c-a}{2bc}(c-b, b, -c).$$

Consider now the point K with areal coordinates

$$\frac{1}{2(a+b+c)}(b+c, c+a, a+b).$$

We have

$$\overrightarrow{M_1K} = \frac{1}{2(a+b+c)}(b+c, -b, -c).$$

From the distance lemma $\overrightarrow{M_1K}$ and $\overrightarrow{T_2T_3}$ are perpendicular since

$$\overrightarrow{M_1K}\cdot\overrightarrow{T_2T_3} = \frac{b+c-a}{8bc(a+b+c)}[(c^2-b^2)(b^2+c^2-a^2)-b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)] = 0.$$

It follows that the perpendiculars from M_1 to T_2T_3, M_2 to T_3T_1 and M_3 to T_1T_2 are concurrent at K. It is easily verified that K is the centre of mass of a uniform wire in the shape of triangle $A_1A_2A_3$ and it lies on the line joining the incentre to the centroid of the triangle. It is also the radical centre of the three excircles.

2. *Third Test.* Consider a quadrilateral $ABCD$ inscribed in a circle. Suppose we fix A and C and move B and D along the arcs AC and CA in the clockwise direction in such a way that $BC = CD$. Let M denote the point of intersection of AC and BD. Find the locus of the circumcenter of triangle AMB.

Comment and solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

There appears to be something mildly wrong with the wording of the question, for in order to maintain $BC = CD$ then if D moves clockwise along CA then B must move anticlockwise along AC. However, I shall answer this amended question rather than suppose the question should read $AB = CD$, which is rather a different question altogether.

Set up a coordinate system with O the centre of the circle, and having $A(\cos\alpha, \sin\alpha)$, $C(1, 0)$ where α is fixed. Let B have coordinates $(\cos\theta, \sin\theta)$. Then for $BC = CD$ we have $D(\cos\theta, -\sin\theta)$. We require the locus of the circumcentre of triangle AMB as θ varies, and shall prove it is a **circle**. The equation of AC is $x\sin\alpha + y(1-\cos\alpha) = \sin\alpha$. The equation of BD is $x = \cos\theta$. So the coordinates of M are

$$\left(\cos\theta\ ,\ \frac{\sin\alpha(1-\cos\theta)}{1-\cos\alpha}\right).$$

Let (X, Y) be the coordinates of the circumcentre S of triangle ABM. The condition $SA = SB$ gives $X\cos\alpha + Y\sin\alpha = X\cos\theta + Y\sin\theta$, and the condition $SB = SM$ gives, after some algebra,

$$Y\sin\frac{\alpha}{2} = \sin\frac{\theta}{2}\sin\frac{\alpha+\theta}{2}$$

giving the coordinates as

$$\left(\sin\frac{\theta}{2}\cos\frac{\alpha+\theta}{2}\csc\frac{\alpha}{2}\ ,\ \sin\frac{\theta}{2}\sin\frac{\alpha+\theta}{2}\csc\frac{\alpha}{2}\right)$$

or

$$2X\sin\frac{\alpha}{2} = \sin\left(\frac{\alpha}{2}+\theta\right) - \sin\frac{\alpha}{2}\ ,$$
$$2Y\sin\frac{\alpha}{2} = -\cos\left(\frac{\alpha}{2}+\theta\right) + \cos\frac{\alpha}{2}\ .$$

So the locus of S is

$$\left(2X\sin\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2 + \left(2Y\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)^2 = 1$$

which is a part of a circle centred at $(-1/2, 1/2\cot(\alpha/2))$, of radius $1/2\csc(\alpha/2)$ and passing through O, the centre of the given circle.

1. *Fourth Test.* Suppose $x_1, x_2, \ldots, x_n$ are positive reals with sum equal to 1. Determine the minimum value of the function

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1}{1 + x_2 + \cdots + x_n} + \frac{x_2}{1 + x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n}{1 + x_1 + x_2 + \cdots + x_{n-1}}$$

and find all n-tuples $(x_1, x_2, \ldots, x_n)$ which yield this minimum value.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Murray S. Klamkin, University of Alberta. We give Klamkin's solution and generalization.

The sum can be rewritten as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{x_i}{2 - x_i} = -n + \sum_{i=1}^{n} \frac{2}{2 - x_i} .$$

Applying Cauchy's inequality

$$\sum_{i=1}^{n} (2 - x_i) \cdot \sum_{i=1}^{n} \frac{2}{2 - x_i} \geq \left(\sum_{i=1}^{n} \sqrt{2} \right)^2 = 2n^2 .$$

Since $\sum_{i=1}^{n} (2 - x_i) = 2n - 1$,

$$\min f(x_1, x_2, \ldots, x_n) = \frac{2n^2}{2n - 1} - n = \frac{n}{2n - 1}$$

and with equality just in case each $x_i = 1/n$.

Generalization. Let $G(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} F(x_i/(a - x_i))$ where the x_i's are nonnegative and with sum s and F is a non-increasing convex function over the range of $x_i/(a - x_i)$ where $a > s$. Since $x/(a - x)$ is a convex function in $[0, a)$, $F(x/(x - a))$ is also convex. Hence by the majorization inequality (see [1988: 120] or [1]),

$$F\left(\frac{s}{a - s}\right) + (n - 1)F(0) \geq G(x_1, x_2, \ldots, x_n) \geq \frac{n}{as - 1} .$$

Here the maximum value is achieved for the n-tuple $(s, 0, 0, \ldots, 0)$ and the minimum value is achieved for the n-tuple $(1/s, 1/s, \ldots, 1/s)$.

Reference:

[1] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

2. *Fourth Test.* (a) Prove that there exist positive real numbers λ such that $[\lambda^n]$ and n have the same parity for all positive integers n. ($[\cdot]$ denotes the greatest integral part function.)

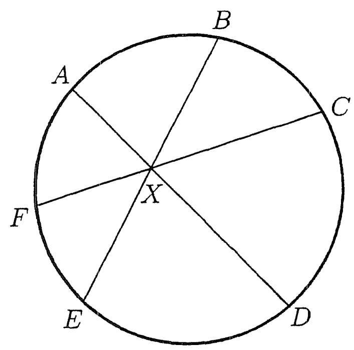
(b) Find one such number λ.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

Consider the sequence $7, 57, 427, \ldots$ defined by $u_1 = 7$, $u_2 = 57$, and $u_{n+2} = 7u_{n+1} + 4u_n$, $n \geq 1$. First observe that with u_1, u_2 odd it follows that u_n is odd for all positive integral n. Secondly by standard methods on recurrence relations, it is readily shown that $u_n = \lambda^n + \mu^n$ where λ, μ are the roots of the quadratic equation $x^2 - 6x - 4 = 0$. We take $\lambda = \frac{1}{2}(7 + \sqrt{65})$ and $\mu = \frac{1}{2}(7 - \sqrt{65})$. The claim is that λ satisfies the conditions of the problem, namely $[\lambda^n] - n \equiv 0 \bmod 2$, for all positive integral n. The reason for this is that μ is small and negative, that is $-1 < \mu < 0$, from which it follows that μ^n lies between -1 and 0 when n is odd, and between 0 and 1 when n is even. Since u_n is odd for all n, it follows that $[\lambda^n]$ is even when n is even, and odd when n is odd. Part (a) of the problem asks for the existence of such numbers (plural) and the same construction holds with different sequences. For example, $\lambda = \frac{1}{3}(3 + \sqrt{17})$ is another case.

3. *Fourth Test.* Suppose convex hexagon $ABCDEF$ can be inscribed in a circle. Prove that a necessary and sufficient condition for the three diagonals AD, BE and CF to be concurrent is $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.



Suppose first that AD, BE, and CF are concurrent at X. Now triangles ABX and EDX are similar with $\angle BAX = \angle DEX$, etc. So $AB/DE = BX/XD$. Similarly $CD/AF = XD/FX$ and $EF/BC = FX/BX$. It follows immediately that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Suppose next that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Let AD and BE meet at X. Produce CX to meet the circle again at F'. We want to show that F and F' coincide. By the first part

$$AB \cdot CD \cdot EF' = BC \cdot DE \cdot F'A.$$

Hence $EF/EF' = FA/F'A.(*)$

Now in the case with F' nearer to A than F, we would have $F'A < FA$ and $EF < EF'$ and so $(F'A)(EF) < (FA)(EF')$ and $(*)$ could not hold. It follows that F' must lie on arc FE. Similarly it must lie on the arc FA. Hence F' and F coincide.

* * *

That concludes the file of solutions to problems from the March 1992 number as well as this month's Corner. Send me your Olympiad and pre-Olympiad contests as well as your nice solutions.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1831*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be any real numbers and let λ be an *odd* positive integer. Prove or disprove that

$$x(x+y)^\lambda + y(y+z)^\lambda + z(z+x)^\lambda \geq 0.$$

1832. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

An old unsolved problem is: "is it possible that a box can have its sides, face diagonals, and space diagonal all of integer lengths, i.e., are there positive integers a, b, c such that

$$a^2+b^2, \quad b^2+c^2, \quad c^2+a^2, \quad \text{and} \quad a^2+b^2+c^2$$

are all perfect squares?" What if we replace the squares by triangular numbers? For n a positive integer, let $t_n = n(n+1)/2$ be the nth triangular number.

(a) Find positive integers a, b, c such that

$$t_a+t_b, \quad t_b+t_c, \quad t_c+t_a$$

are all triangular numbers.

(b)* Is there such a solution so that $t_a+t_b+t_c$ is also a triangular number?

1833. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

E and F are points on sides BC and AD, respectively, of a quadrilateral $ABCD$. Let $P = AE \cap BF$ and $Q = CF \cap DE$. Prove that E and F divide BC and AD (or BC and DA) in the same ratio if and only if

$$\frac{[FPQ]}{[EDA]} = \frac{[EQP]}{[FBC]},$$

where $[XYZ]$ denotes the area of triangle XYZ.

1834. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Given positive numbers A, G and H, show that they are respectively the arithmetic, geometric and harmonic means of some three positive numbers x, y, z if and only if

$$\frac{A^3}{G^3} + \frac{G^3}{H^3} + 1 \leq \frac{3}{4}\left(1 + \frac{A}{H}\right)^2 .$$

1835. *Proposed by Joaquín Gómez Rey, I.B. Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\binom{kn-1}{n-1}$$

for $n = 1, 2, 3, \ldots$.

1836. *Proposed by Jisho Kotani, Akita, Japan.*

Let $ABCD$ be a quadrilateral inscribed in a circle Γ, and let $AC \cap BD = P$. Assume that the center of Γ does not lie on AC or BD. Draw circles with diameters AB, BC, CD, DA, and let the areas of the moon-shaped regions inside these circles and outside Γ be F_1, F_2, F_3, F_4. M_1, M_2, M_3, M_4 are the midpoints of the sides of $ABCD$, and H_1, H_2, H_3, H_4 are the feet of the perpendiculars from P to the sides of $ABCD$. Prove that, if $F_1 + F_2 + F_3 + F_4 = \text{area}(ABCD)$, then M_1, M_2, M_3, M_4, H_1, H_2, H_3, H_4 are concyclic.

1837. *Proposed by Andy Liu, University of Alberta.*

A function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be *strictly log-convex* if

$$f(x_1)f(x_2) \geq \left(f\left(\frac{x_1 + x_2}{2}\right)\right)^2$$

for all x_1, $x_2 \in \mathbb{R}$, with equality if and only if $x_1 = x_2$. f is said to be *strictly log-concave* if the inequality is reversed.

(a) Prove that if f and g are strictly log-convex functions, then so is $f + g$.

(b)* Does the same conclusion hold for strictly log-concave functions?

1838. *Proposed by Stoyan Kapralov and Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Find all sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ of positive integers such that

$$a_1 + a_2 + \cdots + a_n = 26, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = 62, \quad a_1^3 + a_2^3 + \cdots + a_n^3 = 164.$$

1839. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Notice that

$$122 = 11^2 + 1 = 12^2 - 22,$$

i.e., the (base 10) integer $N = 122$ can be partitioned into two parts (1 and 22), so that the first part is the difference between N and the greatest square less than N, and the second part is the difference between N and the least square greater than N. Find another positive integer with this property.

1840. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Let ΔABC be an acute triangle with area F and circumcenter O. The distances from O to BC, CA, AB are denoted d_a, d_b, d_c respectively. $\Delta A_1B_1C_1$ (with sides a_1, b_1, c_1) is inscribed in ΔABC, with $A_1 \in BC$ etc. Prove that

$$d_a a_1 + d_b b_1 + d_c c_1 \geq F.$$

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1621*. [1991: 78; 1992: 88] *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)

Let P be a point within or on an equilateral triangle and let $c_1 \leq c_2 \leq c_3$ be the lengths of the three concurrent cevians through P. Determine the minimum value of c_2/c_3 over all P.

II. *Comment by the proposer.*

In Baron's solution [1992: 89], he uses calculus to determine the minimum of

$$\frac{(1+3s^2)^2}{1+9s^2} .$$

More elementarily, by letting $1 + 9s^2 = t$, we find that

$$\frac{(1+3s^2)^2}{1+9s^2} = \frac{(t+2)^2}{9t} = \frac{1}{9}\left(t + \frac{4}{t} + 4\right) = \frac{1}{9}\left(8 + \left(\sqrt{t} - \frac{2}{\sqrt{t}}\right)^2\right) \geq \frac{8}{9} ,$$

with equality when $t = 2$, i.e. $s = 1/3$.

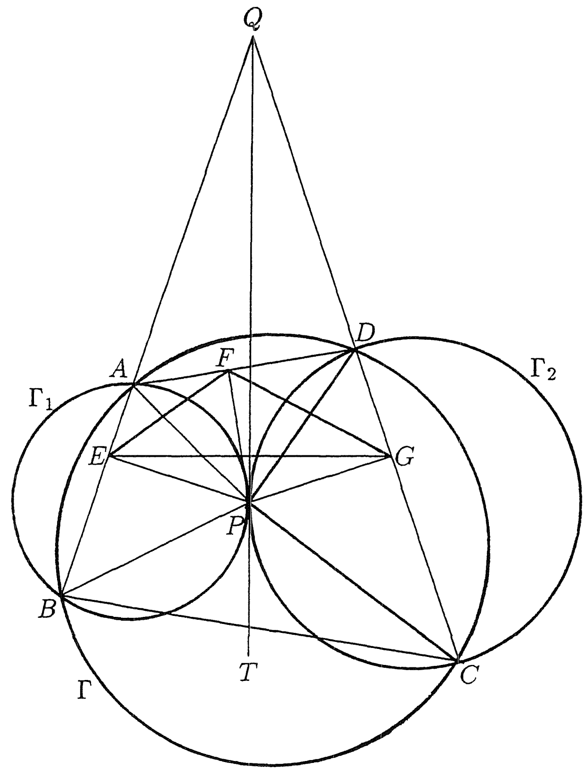
* * * * *

1741. [1992: 138] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex cyclic quadrilateral, and P is an interior point of $ABCD$ such that $\angle BPC = \angle BAP + \angle PDC$. Let E, F and G be the feet of the perpendiculars from P to AB, AD and DC. Prove that ΔFEG is similar to ΔPBC.

Combination of solutions by Christopher J. Bradley, Clifton College, Bristol, U.K., and the proposer.

We denote the circumcircles of $ABCD$, ΔPAB and ΔPCD by Γ, Γ_1 and Γ_2 respectively. Draw a ray PT in $\angle BPC$ such that $\angle BPT = \angle BAP$; then we get



$$\begin{aligned}\angle TPC &= \angle BPC - \angle BPT \\ &= \angle BPC - \angle BAP = \angle PDC.\end{aligned}$$

Therefore PT is tangent to both Γ_1 and Γ_2. As AB and CD are the common chords of Γ_1, Γ and Γ_2, Γ respectively, AB, CD and PT are concurrent at the radical center Q of Γ_1, Γ_2 and Γ. As PQ is a common tangent to Γ_1 and Γ_2 at P, we get

$$\angle QPA = \angle PBA. \tag{1}$$

Since $\angle AEP = \angle AFP = 90°$, A, E, P, F are concyclic and we have

$$\angle FEP = \angle FAP = \angle DAP. \tag{2}$$

Similarly, F, D, G, P are concyclic, and

$$\angle BPC = \angle BAP + \angle PDC = \angle EFP + \angle PFG = \angle EFG. \tag{3}$$

Since $\angle PEQ = \angle PGQ = 90°$, Q, E, P, G are concyclic and we have

$$\angle GEP = \angle GQP = \angle DQP. \tag{4}$$

From (2) and (4), and $\angle DAP + \angle QPA = \angle QDA + \angle DQP$, we get

$$\angle FEG = \angle FEP - \angle GEP = \angle DAP - \angle DQP = \angle QDA - \angle QPA. \tag{5}$$

As A, B, C, D are concyclic, we have $\angle QDA = \angle QBC$. Then we obtain from (1) and (5) that

$$\angle FEG = \angle QBC - \angle PBA = \angle PBC. \tag{6}$$

By (3) and (6), ΔFEG is similar to ΔPBC.

When Q is a point on AB produced beyond B, the proof is similar. And when $AB \| CD$, we get $PT \| AB \| CD$ and then E, P, G are collinear, so we have $\Delta PAD \equiv \Delta PBC$. From $\Delta FEG \sim \Delta PAD$ we get $\Delta FEG \sim \Delta PBC$.

Also solved by D.J. SMEENK, Zaltbommel, The Netherlands.

Smeenk in fact notes that the locus of point P is an arc of the circle with centre Q which is orthogonal to Γ.

* * * * *

1742. [1992: 139] *Proposed by Murray S. Klamkin, University of Alberta.*

Let $1 \le r < n$ be integers and $x_{r+1}, x_{r+2}, \ldots, x_n$ be given positive real numbers. Find positive $x_1, x_2, \ldots, x_r$ so as to minimize the sum

$$S = \sum \frac{x_i}{x_j}$$

taken over all $i, j \in \{1, 2, \ldots, n\}$ with $i \ne j$.

(This problem is due to Byron Calhoun, a high school student in McLean, Virginia. It appeared, with solution, in a science project of his.)

Solution by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.

Let

$$x = x_1 + x_2 + \cdots + x_r, \quad y = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_r},$$

$$a = x_{r+1} + x_{r+2} + \cdots + x_n, \quad b = \frac{1}{x_{r+1}} + \frac{1}{x_{r+2}} + \cdots + \frac{1}{x_n}.$$

Then

$$\begin{aligned}
S = \sum_{i\ne j} \frac{x_i}{x_j} &= \sum_{1\le i\ne j\le r} \frac{x_i}{x_j} + \sum_{r<i\ne j\le n} \frac{x_i}{x_j} + \sum_{\substack{1\le i\le r\\ r<j\le n}} \frac{x_i}{x_j} + \sum_{\substack{r<i\le n\\ 1\le j\le r}} \frac{x_i}{x_j} \\
&= x_1\left(y - \frac{1}{x_1}\right) + x_2\left(y - \frac{1}{x_2}\right) + \cdots + x_r\left(y - \frac{1}{x_r}\right) \\
&\quad + x_{r+1}\left(b - \frac{1}{x_{r+1}}\right) + x_{r+2}\left(b - \frac{1}{x_{r+2}}\right) + \cdots + x_n\left(b - \frac{1}{x_n}\right) \\
&\quad + (x_1 + x_2 + \cdots + x_r)b + \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_r}\right)a \\
&= xy - r + ab - (n-r) + xb + ya = (x+a)(y+b) - n \\
&\ge (\sqrt{xy} + \sqrt{ab})^2 - n \qquad \text{[true by Cauchy's inequality]} \\
&\ge (r + \sqrt{ab})^2 - n
\end{aligned}$$

[equivalent to $\sum_{i=1}^r x_i \cdot \sum_{i=1}^r 1/x_i \ge r^2$, which is again true by Cauchy's inequality], which is independent of $x_1, \ldots, x_r$.

Equality holds in the first inequality if and only if $x/y = a/b$, and in the second inequality if and only if $x_1 = x_2 = \cdots = x_r$ $(= x/r = r/y)$. So S attains its minimum value if and only if

$$x_1 = x_2 = \cdots = x_r = \sqrt{\frac{a}{b}}\,.$$

Also solved by SEUNG-JIN BANG, University of California, Berkeley; C. J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; IGNASI MUNDET I RIERA, Barcelona, Catalunya, Spain; and the proposer.

The proposer gave a generalization to the sum $\sum x_i^p/x_j^q$, where $p, q > 0$.

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1743*. [1992: 139] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < \gamma < 180°$ be fixed. Consider the set $\Delta(\gamma)$ of all triangles ABC having angle γ at C, whose altitude through C meets AB in an interior point D such that the line through the incenters of ΔADC and ΔBCD meets the sides AC and BC in interior points E and F respectively. Prove or disprove that

$$\sup_{\Delta(\gamma)}\left(\frac{\text{area}(\Delta EFC)}{\text{area}(\Delta ABC)}\right) = \left(\frac{\cos(\gamma/2)-\sin(\gamma/2)+1}{2\cos(\gamma/2)}\right)^2.$$

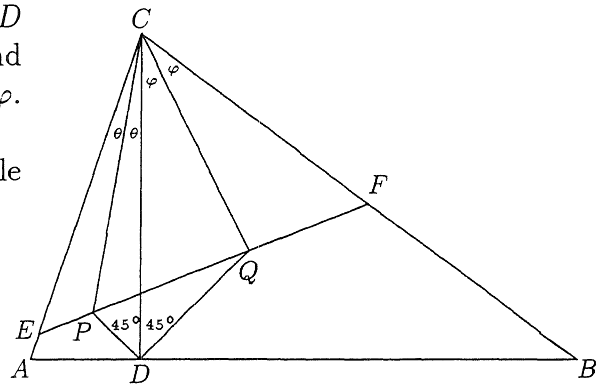
(This would generalize problem 5 of the 1988 IMO [1988: 197].)

Solution by Toshio Seimiya, Kawasaki, Japan.

We shall prove that the given relation does hold.

Let the incenters of ΔADC and ΔBCD be P and Q, and put $\gamma/2 = \varepsilon$, $CD = h$, and $\angle ACP = \angle PCD = \theta$, $\angle DCQ = \angle QCB = \varphi$. Then $\theta + \varphi = \varepsilon$, where $0° < \varepsilon < 90°$.

Let $[XYZ]$ denote the area of triangle XYZ. Because



$$[CEP] + [CPQ] = [CEQ],$$

we have

$$\frac{CE\cdot CP\sin\theta}{2} + \frac{CP\cdot CQ\sin\varepsilon}{2} = \frac{CE\cdot CQ\sin(\varepsilon+\theta)}{2},$$

which implies

$$\frac{\sin\varepsilon}{CE} = \frac{\sin(\varepsilon+\theta)}{CP} - \frac{\sin\theta}{CQ}. \tag{1}$$

By the law of sines for ΔCPD,

$$\frac{h}{\sin(45°+\theta)} = \frac{CP}{\sin 45°} = \sqrt{2}\,CP,$$

thus we get

$$\frac{1}{CP} = \frac{\sqrt{2}\sin(45°+\theta)}{h} \quad\text{and similarly}\quad \frac{1}{CQ} = \frac{\sqrt{2}\sin(45°+\varphi)}{h}. \tag{2}$$

From (1) and (2) we have

$$\frac{\sin\varepsilon}{CE} = \frac{\sqrt{2}\sin(\varepsilon+\theta)\sin(45°+\theta)}{h} - \frac{\sqrt{2}\,\sin\theta\sin(45°+\varphi)}{h}.$$

Therefore

$$\begin{aligned}\frac{\sqrt{2}\,h\sin\varepsilon}{CE} &= 2\sin(\varepsilon+\theta)\sin(45^\circ+\theta) - 2\sin\theta\sin(45^\circ+\varphi)\\ &= [\cos(45^\circ-\varepsilon) - \cos(45^\circ+\varepsilon+2\theta)] - [\cos(45^\circ+\varphi-\theta) - \cos(45^\circ+\theta+\varphi)]\\ &= [\cos(45^\circ-\varepsilon) + \cos(45^\circ+\varepsilon)] - [\cos(45^\circ+\varepsilon+2\theta) + \cos(45^\circ+\varphi-\theta)]\\ &= 2\cos 45^\circ\cos\varepsilon - 2\cos(45^\circ+\varepsilon)\cos 2\theta\\ &= \sqrt{2}\,\cos\varepsilon - \sqrt{2}(\cos\varepsilon - \sin\varepsilon)\cos 2\theta.\end{aligned}$$

Thus we have

$$CE = \frac{h\sin\varepsilon}{\cos\varepsilon + (\sin\varepsilon - \cos\varepsilon)\cos 2\theta}\,. \tag{3}$$

Similarly we get

$$CF = \frac{h\sin\varepsilon}{\cos\varepsilon + (\sin\varepsilon - \cos\varepsilon)\cos 2\varphi}\,. \tag{4}$$

We have $CA\cos 2\theta = h = CB\cos 2\varphi$, therefore we obtain from (3) and (4)

$$\frac{[EFC]}{[ABC]} = \frac{CE\cdot CF}{CA\cdot CB} = \frac{\sin^2\varepsilon\cos 2\theta\cos 2\varphi}{[\cos\varepsilon + (\sin\varepsilon-\cos\varepsilon)\cos 2\theta][\cos\varepsilon + (\sin\varepsilon-\cos\varepsilon)\cos 2\varphi]}\,, \tag{5}$$

and note

$$\begin{aligned}\cos 2\theta\cos 2\varphi &= \frac{1}{2}[\cos(2\theta+2\varphi) + \cos(2\theta-2\varphi)] = \frac{1}{2}[\cos 2\varepsilon + \cos 2(\theta-\varphi)]\\ &= \frac{1}{2}[1 - 2\sin^2\varepsilon + 2\cos^2(\theta-\varphi) - 1] = \cos^2(\theta-\varphi) - \sin^2\varepsilon.\end{aligned} \tag{6}$$

Case I: $\varepsilon = 45^\circ$. Then $\sin\varepsilon - \cos\varepsilon = 0$, and therefore we get from (5) and (6)

$$\frac{[EFC]}{[ABC]} = \frac{\sin^2\varepsilon[\cos^2(\theta-\varphi) - \sin^2\varepsilon]}{\cos^2\varepsilon} = \cos^2(\theta-\varphi) - \frac{1}{2} \le \frac{1}{2}$$

(equality holding when $\theta = \varphi$). Also

$$\left(\frac{\cos(\gamma/2) - \sin(\gamma/2) + 1}{2\cos(\gamma/2)}\right)^2 = \frac{1}{4\cos^2 45^\circ} = \frac{1}{2}\,,$$

and hence the given relation holds. [*Editor's note*: this case is the IMO problem mentioned above.]

When $\varepsilon \ne 45^\circ$, we have $\sin\varepsilon - \cos\varepsilon \ne 0$. We put

$$a = \frac{\cos\varepsilon}{\sin\varepsilon - \cos\varepsilon}\,, \qquad b = a^2 - \sin^2\varepsilon\,;$$

then (5) becomes

$$\frac{[EFC]}{[ABC]} = \frac{\sin^2\varepsilon}{(\sin\varepsilon-\cos\varepsilon)^2}\cdot\frac{\cos^2(\theta-\varphi) - \sin^2\varepsilon}{(a+\cos 2\theta)(a+\cos 2\varphi)}\,,$$

where by (6)

$$\begin{aligned}(a+\cos 2\theta)(a+\cos 2\varphi) &= a^2 + a(\cos 2\theta + \cos 2\varphi) + \cos 2\theta \cos 2\varphi \\ &= a^2 + 2a\cos(\theta+\varphi)\cos(\theta-\varphi) + \cos^2(\theta-\varphi) - \sin^2\varepsilon \\ &= a^2 + 2a\cos\varepsilon\cos(\theta-\varphi) + \cos^2(\theta-\varphi) - \sin^2\varepsilon \\ &= \cos^2(\theta-\varphi) + 2a\cos\varepsilon\cos(\theta-\varphi) + b.\end{aligned}$$

Therefore

$$\frac{[EFC]}{[ABC]} = \frac{\sin^2\varepsilon}{(\sin\varepsilon-\cos\varepsilon)^2}\cdot\frac{\cos^2(\theta-\varphi)-\sin^2\varepsilon}{\cos^2(\theta-\varphi)+2a\cos\varepsilon\cos(\theta-\varphi)+b}\,. \tag{7}$$

Put $x=\cos(\theta-\varphi)$, where $\cos\varepsilon \le x \le 1$. We consider

$$f(x) = \frac{x^2-\sin^2\varepsilon}{x^2+2a\cos\varepsilon\cdot x + b}\,,$$

for which

$$\begin{aligned}f'(x) &= \frac{2x(x^2+2a\cos\varepsilon\cdot x+b) - 2(x+a\cos\varepsilon)(x^2-\sin^2\varepsilon)}{(x^2+2a\cos\varepsilon\cdot x+b)^2} \\ &= \frac{2[a\cos\varepsilon\cdot x^2 + (b+\sin^2\varepsilon)x + a\sin^2\varepsilon\cos\varepsilon]}{(x^2+2a\cos\varepsilon\cdot x+b)^2} \\ &= \frac{2a\cos\varepsilon}{(x^2+2a\cos\varepsilon\cdot x+b)^2}\left(x^2+\frac{x}{\sin\varepsilon-\cos\varepsilon}+\sin^2\varepsilon\right).\end{aligned} \tag{8}$$

Case II: $\varepsilon > 45°$. Then $\sin\varepsilon-\cos\varepsilon > 0$, so we have $f'(x) > 0$. Hence $f(x)$ is increasing in $\cos\varepsilon \le x \le 1$ and has maximum at $x=1$, i.e., when $\theta=\varphi$. Therefore by (7) the maximum value of $[EFC]/[ABC]$ is

$$\begin{aligned}&\frac{\sin^2\varepsilon}{(\sin\varepsilon-\cos\varepsilon)^2}\cdot\frac{1-\sin^2\varepsilon}{1+2a\cos\varepsilon+b} = \frac{\sin^2\varepsilon\cos^2\varepsilon}{(\sin\varepsilon-\cos\varepsilon)^2(1+2a\cos\varepsilon+a^2-\sin^2\varepsilon)} \\ &= \frac{\sin^2\varepsilon\cos^2\varepsilon}{(\sin\varepsilon-\cos\varepsilon)^2(a+\cos\varepsilon)^2} = \frac{\sin^2\varepsilon\cos^2\varepsilon}{[\cos\varepsilon+(\sin\varepsilon-\cos\varepsilon)\cos\varepsilon]^2} \\ &= \left(\frac{\sin\varepsilon}{1+\sin\varepsilon-\cos\varepsilon}\right)^2\cdot\left(\frac{1-\sin\varepsilon+\cos\varepsilon}{1-\sin\varepsilon+\cos\varepsilon}\right)^2 = \left(\frac{\sin\varepsilon(1-\sin\varepsilon+\cos\varepsilon)}{2\sin\varepsilon\cos\varepsilon}\right)^2 \\ &= \left(\frac{\cos(\gamma/2)-\sin(\gamma/2)+1}{2\cos(\gamma/2)}\right)^2,\end{aligned}$$

as claimed.

Case III: $45° > \varepsilon > 0°$. Then $\sin\varepsilon-\cos\varepsilon < 0$. We put

$$g(x) = x^2 + \frac{x}{\sin\varepsilon-\cos\varepsilon} + \sin^2\varepsilon. \qquad (\cos\varepsilon \le x \le 1)$$

Then
$$g(\cos\varepsilon)=\cos^2\varepsilon+\frac{\cos\varepsilon}{\sin\varepsilon-\cos\varepsilon}+\sin^2\varepsilon=\frac{\sin\varepsilon}{\sin\varepsilon-\cos\varepsilon}<0,$$
and
$$g(1)=1+\frac{1}{\sin\varepsilon-\cos\varepsilon}+\sin^2\varepsilon=\frac{\sin\varepsilon-\cos\varepsilon+1+\sin^2\varepsilon(\sin\varepsilon-\cos\varepsilon)}{\sin\varepsilon-\cos\varepsilon}$$
where
$$\begin{aligned}\sin\varepsilon-\cos\varepsilon+1+\sin^2\varepsilon(\sin\varepsilon-\cos\varepsilon)&>\sin^2\varepsilon-\cos\varepsilon+1+\sin^3\varepsilon-\sin^2\varepsilon\cos\varepsilon\\&=(1-\cos\varepsilon)(1+\sin^2\varepsilon)+\sin^3\varepsilon>0,\end{aligned}$$

therefore $g(1)<0$. Because $g(x)$ is a convex function, we have $g(x)<0$ for $\cos\varepsilon\le x\le 1$. Since $a<0$, we get from (8) that $f'(x)>0$. Hence $f(x)$ is increasing in $\cos\varepsilon\le x\le 1$ and has maximum at $x=1$, i.e., when $\theta=\varphi$. Therefore as in Case II the maximum value of $[EFC]/[ABC]$ is again
$$\left(\frac{\cos(\gamma/2)-\sin(\gamma/2)+1}{2\cos(\gamma/2)}\right)^2.$$

[*Editor's note:* in all cases, the given maximum occurs when $CA=CB$, i.e., when $EF\|AB$.]

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1744. [1992: 139] *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Find the number of points of intersection of the graphs of $y=a^x$ and $y=\log_a x$ for any $0<a<1$.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

First observe that
$$y=\log_a x \quad\Leftrightarrow\quad x=a^y,$$
so clearly $y=a^x$, $y=\log_a x$ has at least one solution with $x=y$, for all a satisfying $0<a<1$. In fact when $a=1/c^c$ $(c>1)$ we have a value of a satisfying $0<a<1$, and since the function $1/c^c$ decreases continuously from 1 as c increases from 1, and tends to 0 as $c\to\infty$, every a corresponds to just one value c. Since $(1/c^c)^{1/c}=1/c$, there is thus only one solution of the type $x=y$, namely $x=y=1/c$ when $a=1/c^c$.

Now note that $x_1^{x_2}=x_2^{x_1}$ has solutions with $x_1\ne x_2$, the totality of which may be identified as follows. Consider the graph of
$$Y=\frac{\ln X}{X}$$
(capitals used here so as not to cause confusion with the x,y of the question). For any value k of Y in the interval $0<Y<1/e$ draw a line parallel to the X-axis. This meets the graph in exactly 2 points (x_1,k) and (x_2,k) satisfying
$$k=\frac{\ln x_1}{x_1}=\frac{\ln x_2}{x_2},$$

for which $x_1^{x_2} = x_2^{x_1}$. The value $k = 1/e$ is excluded because $Y = 1/e$ is tangent to $Y = (\ln X)/X$ at $X = e$ where the curve has its (maximum) turning point. Furthermore, for any pair of such values x_1, x_2 we have $x_1^{x_2} = x_2^{x_1} > e^e$, and conversely, if $c > e^e$ unique x_1, x_2 exist with $x_1^{x_2} = x_2^{x_1} = c$.

In terms of the original problem, this implies that for $0 < a < 1/e^e$ there are two more solutions of the simultaneous equations $y = a^x$ and $x = a^y$. For taking such an a we can then find corresponding x_1, x_2 with $x_1^{x_2} = x_2^{x_1} = 1/a$ and set either $x = 1/x_1$, $y = 1/x_2$ or $x = 1/x_2$, $y = 1/x_1$, and then we have

$$\frac{1}{x_1} = \left(\frac{1}{x_1^{x_2}}\right)^{1/x_2} \quad \text{and} \quad \frac{1}{x_2} = \left(\frac{1}{x_2^{x_1}}\right)^{1/x_1}, \quad \text{with} \quad a = \frac{1}{x_1^{x_2}} = \frac{1}{x_2^{x_1}}.$$

It is now claimed that *all* solutions of the equations $y = a^x$, $x = a^y$ derive from one of the above two methods. For if the equations hold then $\ln y = x \ln a$ and $\ln x = y \ln a$ so $x \ln x = y \ln y$. Now put $x = 1/x_1$ and $y = 1/x_2$; then

$$\frac{1}{x_1} \ln\left(\frac{1}{x_1}\right) = \frac{1}{x_2} \ln\left(\frac{1}{x_2}\right) \quad \Rightarrow \quad x_1 \ln x_2 = x_2 \ln x_1 \quad \Rightarrow \quad x_2^{x_1} = x_1^{x_2}.$$

So for $1 > a \geq 1/e^e$ there is one solution and for $0 < a < 1/e^e$ there are three. (I suppose some might say that $a = 1/e^e$ leads to a triple root.)

Also solved by H.L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer. Four incorrect solutions were sent in.

Ashbacher points out that the problem appeared as problem 4334 *in* School Science and Mathematics *(proposed by its editor Richard A. Gibbs), with solution on pages* 292–293 *of the May/June* 1992 *issue.*

The equation $x^y = y^x$ has been frequently studied. See for instance Mathematics Magazine, *Vol.* 63 (1990) 30–33, *or* Mathematical Gazette, *Vol.* 71 (1987) 131–135.

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1745. [1992: 139] *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let $\mathcal{A} = A_0A_1A_2A_3$ be a square and X an arbitrary point in the plane of $\mathcal{A}$. Define the kth vertex (k an integer mod 4) of the quadrilaterals $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ as follows:

(i) B_k is the midpoint of XA_k;

(ii) $C_k = A_kB_{k+1} \cap A_{k+1}B_k$;

(iii) D_k is the centroid of the quadrilateral $B_kA_kA_{k+1}B_{k+1}$.

Prove that $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are squares whose centres lie on a line through the centre of $\mathcal{A}$.

I. *Comment by the editor (in this case Chris Fisher).*

Before looking at the solution, note that the definition of the square $\mathcal{D}$ is ambiguous since, as pointed out in the solution to *Crux* 1527 [1991: 150–151], there is no universally approved meaning of *centroid.* Among the seven submitted solutions, one was independent of the choice of centroid, three interpreted D_k to be the centroid of equal masses placed at the vertices of the quadrangle $B_kA_kA_{k+1}B_{k+1}$, while the remaining three took D_k to be the centroid of the quadrangular lamina. Of course, another interpretation seems possible for $\mathcal{D}$: D_k might be the centroid of the boundary of the quadrangle; no solver dealt with this possibility, which would produce a counterexample to the claims made for $\mathcal{D}$ because this centroid is not an affine invariant. A recent discussion of the three types of centroids accompanies the solution to E3283 in the *Amer. Math. Monthly* **97** (1990) 849–850.

II. *Combination of solutions by Jordi Dou, Barcelona, Spain; by Dan Pedoe, Minneapolis, Minnesota; and by D.J. Smeenk, Zaltbommel, The Netherlands.*

The solution is based on two observations:

(1) the image of a square under a dilatation having centre X is a square, with the centres of the two squares collinear with X; and

(2) if $\mathcal{M}$ is the square formed by the midpoints of the sides of $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{M}$ have the same centre.

The dilatation $D(X, 1/2)$ — with centre X and ratio of magnification $1/2$ — takes $\mathcal{A}$ to $\mathcal{B}$, while $D(X, 2/3)$ takes $\mathcal{M}$ to $\mathcal{C}$. In the vertex case $D(X, 3/4)$ takes $\mathcal{M}$ to $\mathcal{D}$, while the dilatation $D(X, 7/9)$ works for the lamina. [Details of the calculation can be deduced from the cited *Crux* problem.] So, whichever of the two affine choices is made for $\mathcal{D}$, the centres of $\mathcal{B}$ and of $\mathcal{A}$, of $\mathcal{C}$ and of $\mathcal{M}$, and of $\mathcal{D}$ and of $\mathcal{M}$ are collinear with X, so that the centres of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ lie on a line through X as desired.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The proposer added various generalizations to his problem. In fact, one sees easily that the featured solution invites a frenzied orgy of generalization:

> *For $n \geq 3$ and $\lambda > 0$, let $\mathcal{A} = A_0A_1 \ldots A_{n-1}$ be any convex n-gon, k be an integer* mod n, *and the point B_k divide XA_k in the ratio $\lambda : 1$. If $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined as in the original proposal, then they are similar to $\mathcal{A}$ and the centroids of their vertex sets lie on a line through the centroid of the vertex set of $\mathcal{A}$.*

The dilatations required to verify the generalization have the ratios λ for $\mathcal{B}$, $2\lambda/(1+\lambda)$ for $\mathcal{C}$, and either $(\lambda+1)/2$ for $\mathcal{D}$ (when D_k is the centroid of the vertex set of the quadrangle) or $2(\lambda^2+\lambda+1)/(3(\lambda+1))$ (when D_k is the centroid of the quadrangular lamina).

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1746. [1992: 139] *Proposed by Richard K. Guy, University of Calgary, and Richard J. Nowakowski, Dalhousie University.*

(i) Find infinitely many pairs of integers a, b, with $1 < a < b$, so that ab exactly divides $a^2 + b^2 - 1$.

(ii) With a and b as in (i), what are the possible values of $(a^2 + b^2 - 1)/ab$?

Solution by Richard McIntosh, University of Regina.

Let a and b be integers such that $1 \leq a < b$. We claim that ab divides $a^2 + b^2 - 1$ if and only if a and b are consecutive terms in the sequence

$$u_0 = 1, \quad u_1 = k, \quad \ldots, \quad u_{n+1} = \frac{u_n^2 - 1}{u_{n-1}}, \quad \ldots,$$

where $k > 1$ is an integer. Moreover, $(a^2 + b^2 - 1)/ab = k$ and thus can take any integer value greater than 1.

Proof. Let $k > 1$ be any integer. Since

$$\frac{u_n^2 + u_{n+1}^2 - 1}{u_n u_{n+1}} = \frac{u_n^2 u_{n-1}^2 + (u_n^2 - 1)^2 - u_{n-1}^2}{u_n(u_n^2 - 1)u_{n-1}} = \frac{u_{n-1}^2 + u_n^2 - 1}{u_{n-1}u_n},$$

it follows by induction that if a and b are any two consecutive terms in the sequence then

$$\frac{a^2 + b^2 - 1}{ab} = \frac{u_0^2 + u_1^2 - 1}{u_0 u_1} = \frac{1 + k^2 - 1}{1 \cdot k} = k.$$

All pairs (a, b) with $1 \leq a < b$ such that ab divides $a^2 + b^2 - 1$ arise in this fashion. If $a = 1$ then put $k = b$ and we are done. Now suppose that $1 < a < b$ and ab divides $a^2 + b^2 - 1$. Let $c = (a^2 - 1)/b$ so that c is an integer, $1 \leq c < a$, and

$$\frac{c^2 + a^2 - 1}{ca} = \frac{(a^2 - 1)^2 + (a^2 - 1)b^2}{(a^2 - 1)ab} = \frac{a^2 + b^2 - 1}{ab}.$$

Hence the pair (a, b) gives rise to the smaller pair (c, a) satisfying the same divisibility criterion. This process can be continued until we obtain the pair $(1, k)$, where $k = (a^2 + b^2 - 1)/ab$. Therefore $(a, b) = (u_n, u_{n+1})$ for some n. [*Editor's comment.* Note that the equation $c = (a^2 - 1)/b$ says that the sequence $b, a, c, \ldots, k, 1$ generated by this process is just a sequence of u_n's in reverse order.]

Also solved by H.L. ABBOTT, University of Alberta; HAYO AHLBURG, Benidorm, Alicante, Spain; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; MARGHERITA BARILE, student, Universität Essen, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; RANDY HO, student, University of Arizona, Tucson; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; DAVID E. MANES, State University of New York, Oneonta; KAAREN MAY, student, St. John's, Newfoundland; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro; P. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposers. Part* (i) *only (probably due to misunderstanding part* (ii)*) solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; BEATRIZ MARGOLIS, Paris, France; W.R. UTZ, University of Missouri, Columbia; and KENNETH M. WILKE, Topeka, Kansas.*

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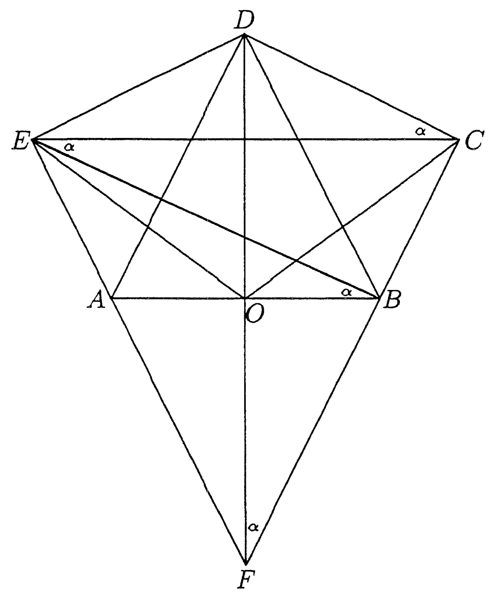
1747. [1992: 139] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$ABCDE$ is a convex pentagon in which each side is parallel to a diagonal. Two of its angles are right angles. Find the sum of the squares of the sines of the other three angles.

Solution by Jordi Dou, Barcelona, Spain.



The right angles can not be at consecutive vertices, so we can suppose that $\angle C = \angle E = 90°$. Put $F = BC \cap AE$, and let O be the midpoint of DF. We may put $OC = OD = OE = OF = 1$ [since $CDEF$ is cyclic with diameter DF]. We have $AD \| FC$, $AB \| CE$, $BE \| CD$, $BD \| AE$. [Note that $BDAF$ is a parallelogram, so O is the midpoint of AB.—*Ed.*] Put

$$\alpha = \angle DFC = \angle ABE = \angle BEC = \angle ECD.$$

[*Editor's note.* Here Dou seems to use symmetry of the figure beyond what has been established. All would be okay if we knew that $DF \perp AB$, i.e., $AD = BD$. This can be proved as follows. From $AE \| BD$ and $AD \| BC$ we know $\angle EAD = \angle DBC$, and since $\angle E = \angle C = 90°$ we have $\Delta EAD \sim \Delta CBD$. If $AE < BC$ then also $DE < DC$ and, since $EC \| AB$,

$$90° = \angle AEC + \angle DEC > \angle BCE + \angle DCE = 90°,$$

a contradiction. Similarly $AE > BC$ is impossible, so $AE = BC$, thus $\Delta EAD \equiv \Delta CBD$ and $AD = BD$. Or is it easier than this?]

It holds that

$$\angle A = \angle B = 90° + \alpha, \quad \angle D = 180° - 2\alpha; \tag{1}$$

also $\angle COD = 2\alpha$, and so $CE = 2\sin 2\alpha$. From this and

$$CB = FC - FB = 2\cos\alpha - \frac{1}{\cos\alpha} = \frac{\cos 2\alpha}{\cos\alpha}$$

we have

$$\sin\alpha = \frac{CB}{CE} = \frac{\cos 2\alpha}{\cos\alpha \cdot 2\sin 2\alpha}\,,$$

or

$$\cos 2\alpha = \sin^2 2\alpha. \tag{2}$$

From this,

$$2\cos^2\alpha - 1 = 4\sin^2\alpha\cos^2\alpha = 4\cos^2\alpha - 4\cos^4\alpha,$$
$$4\cos^4\alpha - 2\cos^2\alpha - 1 = 0,$$

so

$$\cos^2\alpha = \frac{1+\sqrt{5}}{4}\,. \tag{3}$$

Now using (1), (2) and (3),

$$\begin{aligned}\sin^2 A + \sin^2 B + \sin^2 D &= 2\cos^2\alpha + \sin^2 2\alpha = 2\cos^2\alpha + \cos 2\alpha \\ &= 4\cos^2\alpha - 1 = \sqrt{5}.\end{aligned}$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RANDY HO, student, University of Arizona, Tucson; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

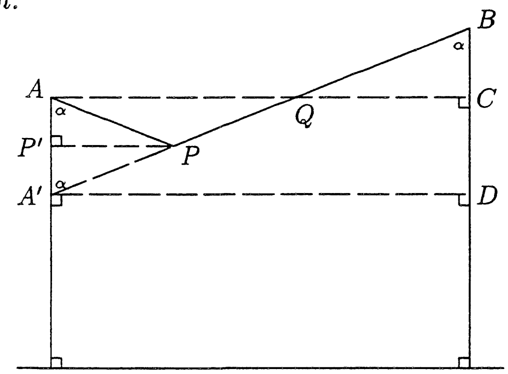
The proposer also wonders about the maximum number of right angles possible in a convex n-gon $A_1A_2\ldots A_n$ in which $A_iA_j \| A_kA_l$ whenever $i+j \equiv k+l \bmod n$ (an "affinely regular" n-gon — the case $n=5$ is the above situation).

*　　*　　*　　*　　*

1748. [1992: 140] *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

Several medieval arithmetic/algebra books give a problem with a fountain located between two towers so that pigeons of equal speeds can get from the tops of the towers to the fountain in equal times. The solution of this works out quite easily. However, in the *Trattato d'Aritmetica* attributed to Paolo dell'Abbaco, c. 1370, there are problems where a rope is strung between the two tower tops and a weight is hung from a ring on the rope, where the rope is just long enough for the weight to touch the ground. Solving this is a bit trickier than the previous problem. To make it even trickier, suppose the rope isn't long enough—suppose the towers are 1748 and 1992 *uncia* high, the rope is 2600 *uncia* long, and the two towers are 2400 *uncia* apart. How far above the ground does the weight hang?

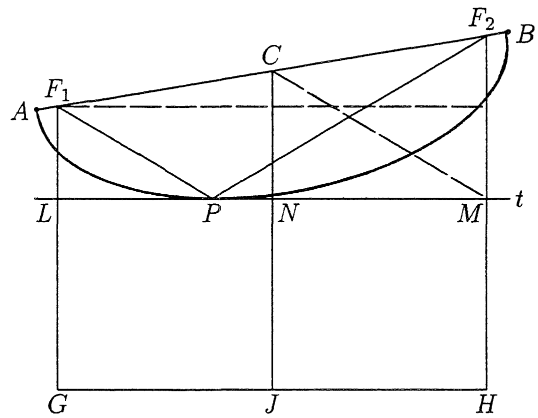
I. *Solution by Dag Jonsson, Uppsala, Sweden.*



For notation see the figure. The rope forms the same angle α at A and B (for the weight P does not change its position if the rope is fixed at Q). Thus, if the line BP is extended till A' we have $|AP| = |A'P|$ and $|A'B| = 2600$. Since $|A'D| = 2400$ we get $|BD| = \sqrt{2600^2 - 2400^2} = 1000$. Now $|BC| = 1992 - 1748 = 244$, $|CD| = 100 - 244 = 756$, and $|AP'| = |P'A'| = 756/2 = 378$. Thus the weight hangs $1748 - 378 = 1370$ *uncia* above the ground.

II. *Solution by Hayo Ahlburg, Benidorm, Alicante, Spain.*

This problem was inspired by work done hundreds of years ago. Its solution is supplied by work done thousands of years ago.



The size of ring and weight are considered to be negligible. Sliding along the tight rope which is tied to the two tower tops (focal points F_1 and F_2), the weight describes part of an ellipse. It comes to rest at the lowest point P, where the ellipse has a horizontal tangent t. The rope length $F_1P + PF_2$ is equal to the large axis AB of the ellipse,

$$F_1P + PF_2 = AB = 2600 \text{ \textit{uncia}}. \tag{1}$$

Let the center of the ellipse (midpoint of F_1F_2 and of AB) be C. L and M are the points where t intersects with the vertical towers F_1G and F_2H (which are normals to the horizontal tangent t) and N is their midpoint. G, H, and their midpoint J are assumed at the same (horizontal) ground level, although this is not explicitly stated in the problem, and you never know about those medieval towers. From $GH = 2400$ *uncia*, we have

$$GJ = HJ = LN = MN = \frac{GH}{2} = 1200 \text{ \textit{uncia}}. \tag{2}$$

With $F_1G = 1748$ *uncia* and $F_2H = 1992$ *uncia* we furthermore have

$$CJ = \frac{F_1G + F_2H}{2} = 1870 \text{ \textit{uncia}}. \tag{3}$$

Now let us cast our minds back 2200 years, to the great Apollonios. To him, the figure was quite familiar, and he would have been happy to point out that

$$CL = CM = CA = CB = \frac{AB}{2} = 1300\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota.^{1} \tag{4}$$

[1]As he called *uncia* (inches) in his day.

He had proved that A, L, M and B are points of the ellipse's circumcircle [1].

Using (1) to (4), we now simply have

$$CN = \sqrt{CM^2 - MN^2} = \sqrt{\left(\frac{AB}{2}\right)^2 - \left(\frac{GH}{2}\right)^2} = 500\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota \tag{5}$$

and

$$NJ = CJ - CN = 1370\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota. \tag{6}$$

This is also the elevation of P above ground.

If P were touching ground, the line $LPNM$ would already signify ground level, and the solution would stop at (5). Thanks to Apollonios — and to Pythagoras, of course — solution (5) is not tricky, and (6) is not trickier, but "elementary, dear Watson".

By the way, since $\angle LPF_1 = \angle MPF_2$ [2], we have

$$LP : PM = F_1L : F_2M = 378 : 622.$$

With $LP + PM = 2400\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota$, this gives

$$LP = 907.2\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota \quad \text{and} \quad PM = 1492.8\ \delta\acute{\alpha}\kappa\tau\upsilon\lambda o\iota.$$

References:

[1] It is a pleasure to honor this great mathematician whose works are so rarely quoted today: *'Απολλωνίου Περγαίου κωνικῶν γ', ν'* (Apollonios of Pergae, c. 220 B.C., *Conics III*, 50).

[2] *'Απολλωνίου Περγαίου κωνικῶν γ', μη'* (*Conics III*, 48).

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; P. TSAOUSSOGLOU, Athens, Greece; and the proposer. One incorrect solution was sent in.

* * * * *

1750. [1992: 140] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Pairs of numbers from the set $\{11, 12, \ldots, n\}$ are adjoined to each of the 45 different (unordered) pairs of numbers from the set $\{1, 2, \ldots, 10\}$, to obtain 45 4-element sets $A_1, A_2, \ldots, A_{45}$. Suppose that $|A_i \cap A_j| \leq 2$ for all $i \neq j$. What is the smallest n possible?

Solution by Randy Ho, student, University of Arizona, Tucson.

The smallest n possible is $n = 15$.

$n = 14$ *will not work.* There are only 6 pairs of numbers from $\{11, 12, 13, 14\}$. Consider these 7 pairs from $\{1, 2, \ldots, 10\}$:

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}.$$

By the Pigeonhole Principle, there are two pairs $\{1, i\}$ and $\{1, j\}$ that are assigned the same pair from $\{11, 12, 13, 14\}$ and so the intersection of the resulting 4-element sets has 3 elements.

$n = 15$ *will work.* See the following chart for assigning pairs.

	2	3	4	5	6	7	8	9	10
1	b	c	d	e	f	g	h	i	j
2		d	e	f	g	h	i	j	a
3			f	g	h	i	j	a	b
4				h	i	j	a	b	c
5					j	a	b	c	d
6						b	c	d	e
7							d	e	f
8								f	g
9									h

where

$a = 11, 12$
$b = 11, 13$
$c = 12, 13$
$d = 11, 14$
$e = 12, 14$
$f = 13, 14$
$g = 11, 15$
$h = 12, 15$
$i = 13, 15$
$j = 14, 15$

For example, the pair $\{5, 9\}$ gets assigned to c, or $\{12, 13\}$, to form the 4-element set $\{5, 9, 12, 13\}$.

[*Editor's note.* Ho's rule here is simply to assign to each pair $\{x, y\}$ from $\{1, 2, \ldots, 10\}$ one of the 10 pairs from $\{11, 12, 13, 14, 15\}$ depending on what $x + y$ is modulo 10. Thus it is clear that no two pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ with an element in common can be assigned the same pair, so the resulting 4-element sets must have at most two elements in common.]

Also solved by H.L. ABBOTT, University of Alberta; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; STOYAN N. KAPRALOV, Technical University, Gabrovo, Bulgaria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposer.

Abbott generalized the result by replacing $\{1, 2, \ldots, 10\}$ *by* $\{1, 2, \ldots, m\}$ *and* $\{11, 12, \ldots, n\}$ *by* $\{m+1, m+2, \ldots, m+s\}$. *The minimum value of* s *(depending on* m*) can be found as above.*

As pointed out by both Kapralov and the proposer, the problem is related to the construction of "balanced tournament designs".

* * * * *

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