MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AT SCHOOLS
COLLEGES AND UNIVERSITIES

Volume 15 1982/83 Number 3

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 15 of *Mathematical Spectrum* consists of three issues, of which this is the third. The first was published in September 1982 and the second in January 1983.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Euler

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It is a remarkable fact that quite ordinary families can suddenly throw up a quite extraordinary mathematical genius. Isaac Newton was an example—and so was the much less well-known Leonhard Euler. So great was his talent and importance that in the field of mathematics the eighteenth century has been described as 'The Age of Euler'. This year, 1983, we mark the bicentenary of Euler's death, and it is appropriate to introduce this truly great man to readers who may be hardly aware that he even existed.

Leonhard Euler was born on 15 April 1707, and spent his childhood in the village of Riehen, near Basle, Switzerland. His forefathers had been mainly craftsmen, but his father, Paul, had achieved an academic education and became a Protestant minister. During his studies, Paul Euler had attended mathematics lectures given by the eminent professor at Basle University, Jakob Bernoulli. Interestingly, the Bernoulli family show that mathematical genius is sometimes inherited—no fewer than seven of them were outstanding mathematicians!

Leonhard Euler's father taught him elementary mathematics, and he then studied by himself for some time before receiving guidance from a private tutor. At the tender age of 13 years, he entered Basle University to study for a 'general arts degree' (Such an early age of entry was by no means as uncommon or surprising then as it appears to us today.) Being bored with his normal studies, he sought the additional challenge of mathematics in his spare time. Fortunately for the young Euler, Johann Bernoulli, who had succeeded his brother Jakob to the post of professor of mathematics at Basle in 1705, was on hand to give expert advice, and Euler was able to progress well with his private studies. In 1722, at the age of 15, Euler obtained his Bachelor of Arts degree, and in 1723 he followed this with his Master of Philosophy degree.

Complying with his father's wishes, he then joined the department of theology to study Hebrew, Greek and theology, but his heart was really elsewhere. Although he was to remain a committed Christian throughout his life, Euler wisely rejected the path to becoming a church minister, and turned to mathematical and scientific enlightenment. His investigations began in 1725, when he was just 18 years old.

Finding a job was always a big problem for an aspiring mathematician in those days. There were few universities, and those which did exist were small—Basle, for example had barely 100 students, and fewer than 20 'professors'. Having failed to secure a vacant post at Basle in 1727, probably because of his youth rather than through lack of merit, he took up a position in Russia at the newly formed St.

Petersburg Academy. It was at about this time that his work on exponential functions led him to introduce the familiar symbol e for the important transcendental number 2·7128 . . .

Although his new job really required him to study physiology, he was able to work in the mathematical field, in the company of eminent specialists in geometry, trigonometry, analysis, number theory, mechanics, astronomy, cartography, etc.

One function of the St. Petersburg Academy was the solving of surveying and other technological problems for the Russian government. This led Euler to apply his genius to a very wide range of problems—in map construction, shipbuilding, navigation, etc. Nevertheless, what to us would appear major scientific achievements were but minor entertaining interludes to Euler. By the year 1741, when he was 34 years old, he had published no fewer than 55 major mathematical papers, and had another 30 already written! It is doubtful whether any man before or since has had such a fertile mind. An indication of this is that he had so many brilliant ideas when a young man that sometimes it was only 20 years later that he had time to work out all the details for publication!

During this early period he discovered the amazing relationship, which bears his name, connecting e, π and i (his own invented symbol for $\sqrt{-1}$):

$$e^{i\pi} = -1,$$

and he investigated the Euler constant, represented by

$$\gamma = \lim_{k \to \infty} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k} - \log_e k \right\} \simeq 0{\cdot}5772.$$

This result can be very useful for estimating the sum of the finite series $\sum_{k=1}^{n} 1/k$, since his result showed that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \simeq \gamma + \log_e (n).$$

An application of this is in determining how many packets of chewing gum one must expect to buy in order to get a complete set of picture cards, if one is given free with each packet! It can be shown (you are invited to try) that the appropriate formula for n different cards in a set is:

$$\text{expected number} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).$$

Thus, for $n = 25$, an exact calculation of the above gives 95·4 (rounded to 3 significant figures), and the use of Euler's approximation gives 94·9, a surprisingly accurate result, with *much* less effort.

The political atmosphere in Russia became turbulent, and in 1740 Euler was pleased to accept an invitation to move to Berlin (then in the state of Prussia), to help reorganize the decaying Berlin Society of Sciences. He went in 1741, taking with him his wife Katharina, whom he had married in 1733, and their two sons. Such was Euler's energy that, after his move, he continued to work for the St. Petersburg

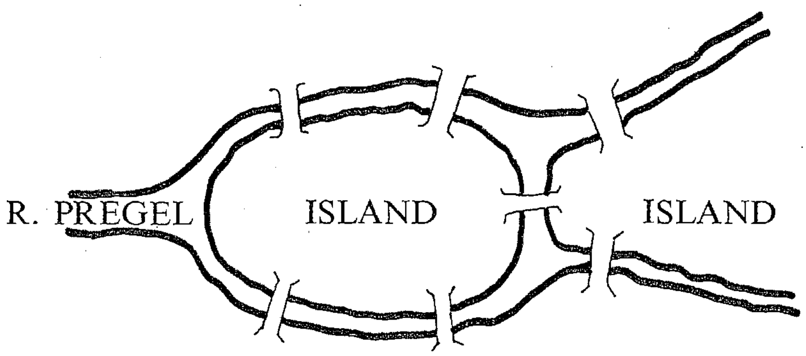

Figure 1.

Academy whilst simultaneously building up the Berlin equivalent! Just as he increased the reputation and activities of the Berlin academy, so did he build up the size of his family, with the addition of a third son and two daughters. It is recorded that a further eight children died in early infancy. This reminder that not all was as rosy as we might be led to believe is strengthened by noting that, as the result of a disease, Euler lost the sight of one eye in 1738, although it did not diminish his own vigour or output one jot.

Euler was the first to study the now major mathematical area of topology. One simple example of this is his solving of the problem of the seven bridges of Königsberg (Figure 1). The problem was to devise a route to enable one to walk over each bridge in turn without crossing any bridge twice. (The reader may like to investigate this.)

The general approach to this kind of traversability problem is to convert it (Figure 2) into a simple network, involving nodes (i.e. points), arcs (i.e. lines) and regions (i.e. areas) (Figure 3). This reduces to a network with one 5-node and three 3-nodes. Euler was able to show that any network with no more than two odd-nodes is traversable. (The reader may like to investigate this.)

Euler wrote mainly in Latin and French, but was conversant with other

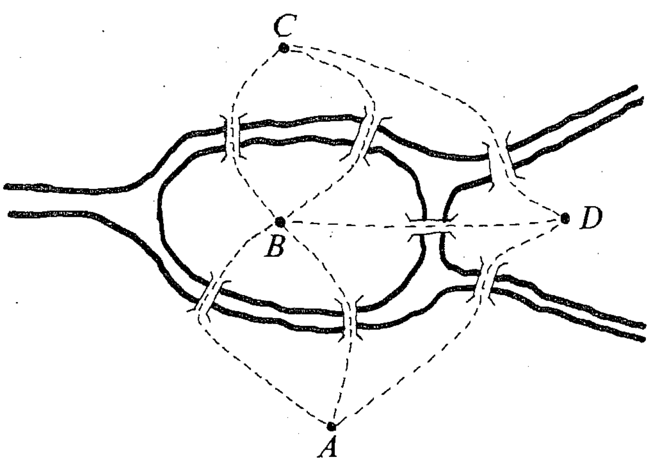

Figure 2.

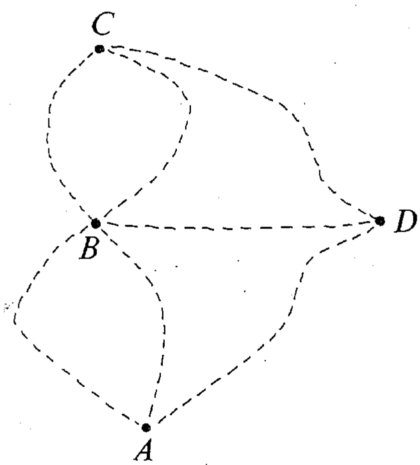

Figure 3.

languages too. This is illustrated by the fact that he translated into German the important English work on artillery, *New Principles of Gunnery*, by B. Robins. Euler added his own appendix on ballistics (the theory of firing projectiles) which was four times as long as the original book itself!

During his 25 years in Berlin, Euler published 275 books and papers, with a further hundred or so written. A list of the topics which he studied includes: the theory of toothed gears, hydraulic turbines, lunar and planetary motion (which involved developing methods of solving differential equations, which bear his name), optics, magnetism, electricity, hydrodynamics, the theory of calculus, etc.

At the age of 59, when most would have been thinking about a cosy retirement, Euler left Berlin to return to the St. Petersburg Academy to continue his work there. Soon after his arrival in 1766, he suffered almost complete loss of sight in his remaining eye. He could see only dim outlines of large objects, and so was quite unable to read or write. Even this major setback could not quell the flow of ideas and publications. He produced almost as many books and papers after his blindness as he did before! He used his sons and students as collaborators, and his valet as an amanuensis. He continued to produce work of the highest standard—often following up brilliant ideas formulated years earlier. Some of his major works in this period were: three volumes on optics (1769–71); three volumes on integral calculus (1768–70); two volumes dictated to his valet on algebra (1770); a 775-page volume on lunar motion (1772) the merits of which could not be fully appreciated for over 100 years; a manual for naval cadets (1773); and a very important work establishing the principles of insurance (1776).

Only in the last few years of his life did he slow down at all, and he died, quite suddenly, on 18 September 1783.

Leonhard Euler, the most prolific mathematician of all time, accomplished so much that his influence lasted a hundred years after his death. During his lifetime he published about 560 substantial articles and books. It took eighty years after his death to edit and publish the memoires he left behind. Collection and publication of his complete works, or *Omnia Opera*, has only recently been completed, and has been one of the most important undertakings in the history of science. The *Omnia Opera* consist of 29 volumes on mathematics, 31 volumes on mechanics, and 12 volumes on physics and other topics. In all, there are 72 massive volumes of original work, currently available, should you wish to buy the set, for 9880 DM (about £3000).

Reference

An excellent short account of Euler's life and work may be found in Carl Boyer's *A History of Mathematics*, Chapter XXI. (Wiley, New York, 1968).

Would You Believe It?

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These are not tall stories. All the assertions are true, though you may think some of them are very hard to believe at first. The arguments range in difficulty and sophistication very considerably; and all that the results have in common is that most of us would find them to be surprising.

1. A buckled railway line

We begin with a very elementary problem. Figure 1 illustrates a piece of railway line which was originally 1 mile long. It is buckled in the sun and, as a result, expands

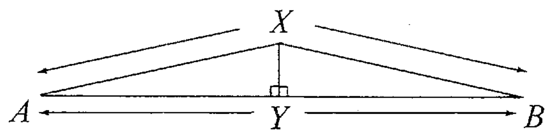


Figure 1

(uniformly) by 1 foot while the ends A, B remain fixed. For simplicity, we suppose that it becomes lifted from the ground at its middle point X as shown. At a guess, how high is X above the ground? Perhaps a few inches or maybe a yard? No, X is more than 50 feet above the ground!

If we measure in feet (remembering that 5280 feet = 1 mile), we have, in the notation of Figure 1,

$$\begin{aligned} XY^2 &= AX^2 - AY^2 \\ &= (2640{\cdot}5)^2 - 2640^2 \\ &= 2640{\cdot}25 \end{aligned}$$

and so

$$XY \simeq 51{\cdot}4.$$

2. Common birthdays

This problem and the next one are probabilistic. To set the scene, suppose you were to ask two dozen people in the street (chosen at random) the dates of their birthdays. Then it is more likely than not that you would find that two of them had birthdays on the same day. The assertion that this is so for such a small number of people is usually greeted with surprise. So let us supply a proof. However, to begin with, we should emphasize that all we are saying is that the probability of the occurrence of two birthdays falling on the same date exceeds $\frac{1}{2}$. Without too much inaccuracy, we shall suppose that there are 365 days in each year and that all dates are equally likely to be birthdays. Then the number of lists of n birthdays is 365^n, the number containing no repeated date is

$$365(365-1)(365-2)\cdots(365-n+1),$$

and so the probability of the occurrence of a repeated birthday for n people is

$$1-\frac{365(365-1)(365-2)\cdots(365-n+1)}{365^n}$$
$$=1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)=1-p_n \qquad \text{(say)}.$$

Now

$$\log_e p_n=\sum_{k=1}^{n-1}\log_e\left(1-\frac{k}{365}\right)$$

and, for n small compared with 365,

$$\log_e p_n \simeq \sum_{k=1}^{n-1}-\frac{k}{365}$$
$$=-\frac{1+2+\cdots+n-1}{365}$$
$$=-\frac{n(n-1)}{2\times 365}.$$

Thus, for $n=24$,

$$\log_e p_n \simeq -0{\cdot}76$$

whereas

$$\log_e \tfrac{1}{2} \simeq -0{\cdot}69.$$

Therefore, for $n=24$, it follows that $1-p_n>\frac{1}{2}$. (The critical number is 23 in fact, not 24, but since our calculations are approximate we should need to discuss the size of the error term to convince ourselves of the stronger result.†)

† In this connexion, we mention a recent article on the birthday problem by Susan Wilson (*Mathematical Spectrum*, Volume 13, Number 2), where a table of computed values for p_n is given.

3. The hats problem

Ten gentlemen attend a small party and leave their hats in the cloakroom. One thousand gentlemen attend a public reception and leave their hats in the cloakroom. On each occasion some confusion arises, and the hats are handed back to the guests at random. The probability, in each case, that no gentleman receives his own hat is about 0·37. Indeed, this probability to all intents and purposes is independent of the number of people involved.† This comes as something of a surprise; but a few fairly simple calculations will convince us.

Consider n individuals (with n hats); and denote by D_n the number of ways in which they can receive the wrong hats. This is equal to the number of permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$ such that $a_i \neq i$ for each i, i.e. the number of *derangements* of $(1, 2, \ldots, n)$. The probability that no man receives his own hat is then $D_n/n!$, which is what we wish to calculate. To this end, consider those derangements of $(1, 2, \ldots, n)$ in which the first position is occupied by the integer $k \neq 1$. The number of these in which the integer 1 occupies the kth position is evidently equal to D_{n-2}; and the number in which 1 does not occupy the kth position is D_{n-1} (since we may regard the kth position as a forbidden position not for k but for 1). Since k itself may take any one of the $n-1$ values $2, 3, \ldots, n$, we obtain the relation

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

provided $n \geq 3$. Also $D_1 = 0$ and $D_2 = 1$. Thus

$$\begin{aligned} D_n - nD_{n-1} &= -(D_{n-1} - (n-1)D_{n-2}) \\ &= (-1)^2(D_{n-2} - (n-2)D_{n-3}) \\ &= \cdots \\ &= (-1)^{n-2}(D_2 - 2D_1) \\ &= (-1)^{n-2} \\ &= (-1)^n. \end{aligned}$$

It is helpful to write this last equation in the form

$$\frac{D_n}{n!} = \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}.$$

Now if we examine this for the first few values of n we shall see the pattern at once:

$$\frac{D_3}{3!} = \frac{D_2}{2!} + \frac{(-1)^3}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

$$\frac{D_4}{4!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

† This problem was also considered by Harris S. Schultz in Volume 12, Number 2 of *Mathematical Spectrum*.

and so

$$\frac{D_n}{n!}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^n\frac{1}{n!}$$

$$=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+(-1)^n\frac{1}{n!}.$$

Further,

$$\left|\frac{D_n}{n!}-e^{-1}\right|=\frac{1}{(n+1)!}-\frac{1}{(n+2)!}+\frac{1}{(n+3)!}-\cdots$$

$$=\frac{1}{(n+1)!}-\left(\frac{1}{(n+2)!}-\frac{1}{(n+3)!}\right)-\cdots$$

$$<\frac{1}{(n+1)!}$$

and the right-hand side is small even for quite small values of n; for instance, for $n=5$ it is equal to 1/720. Therefore $D_n/n!\simeq e^{-1}\simeq 0{\cdot}37$.

4. Infinite collections

It is not difficult to find very surprising results when we venture into the infinite. Let us begin our discussion by getting our basic notions clearly defined. Two finite sets evidently have the same number of elements in them precisely when they can be put in one-to-one correspondence with each other. It seems entirely reasonable, therefore, to take this as the *definition* of 'having the same number of elements' for sets which are not necessarily finite. When we do so, however, we meet with some surprises. For instance, in the collection of all (positive) integers, the even integers form a proper subcollection; but, on the other hand, in view of the pairing

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & \cdots \end{array}$$

we are bound to admit that, according to our definition, there are just as many even integers as there are integers altogether. This is only the first of the surprises that await us, however. Surely there are more fractions than there are integers! Let us consider the fraction p/q, where p and q are positive integers, and associate with it the symbol (p,q). The diagram

$$\begin{array}{ccccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) & & \cdots \\
 & & \downarrow & & \uparrow & & \downarrow & & \\
(2,1) & \leftarrow & (2,2) & & (2,3) & & (2,4) & & \cdots \\
\downarrow & & & & \uparrow & & \downarrow & & \\
(3,1) & \rightarrow & (3,2) & \rightarrow & (3,3) & & (3,4) & & \cdots \\
 & & & & & & \downarrow & & \\
(4,1) & \leftarrow & (4,2) & \leftarrow & (4,3) & \leftarrow & (4,4) & & \cdots
\end{array}$$

indicates just how we can make a list of all these, and thus pair them off with the integers $1, 2, 3, \ldots$. If we delete symbols which represent the same fraction (for instance $(2, 4), (3, 6), \ldots$, which all represent the same fraction $\frac{1}{2}$) we still get a list; namely $1, \frac{1}{2}, 2, 3, \frac{3}{2}, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}$ and so on (in the ordinary notation for fractions). Therefore there are just as many integers as there are fractions. In contrast to all this, it is possible to show that the real numbers cannot be paired off with the integers, and thus that there are genuinely more real numbers than integers or fractions. Instead of looking into the proof of this last assertion, we turn to something a little different and very surprising. Let us consider the unit interval on a straight line. Its length is 1 unit. We shall proceed to show how to delete from it a sequence of subintervals whose lengths together add up to 1 unit and yet leave behind just as many points as there were in the original interval. It will make our calculations simpler (but is not significant in any other way) if we work with intervals which are 'closed' on the left and 'open' on the right, so that for instance our original interval includes 0 but does not include 1. We shall indicate this by writing this interval as

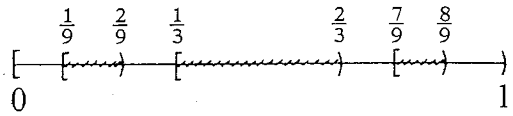


Figure 2

$[0, 1)$; with a similar notation for the deleted subintervals. First, we delete the 'middle third' of $[0, 1)$, namely $[\frac{1}{3}, \frac{2}{3})$; next, the middle third of each of the remaining two intervals, namely $[\frac{1}{9}, \frac{2}{9})$ and $[\frac{7}{9}, \frac{8}{9})$; and so on, always removing middle thirds. Let us denote by F the set of points which ultimately remain. The total length removed is evidently equal to

$$\begin{aligned}
&\tfrac{1}{3} + 2\cdot\tfrac{1}{9} + 4\cdot\tfrac{1}{27} + \cdots \\
&= \tfrac{1}{3} + \tfrac{2}{3^2} + \tfrac{2^2}{3^3} + \cdots \\
&= \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1.
\end{aligned}$$

Next, let us consider any point x of the original interval $[0, 1)$ and write x in binary form as $0 . b_1 b_2 b_3 \ldots$. Each b_i is then 0 or 1. (We avoid recurring 1's in order to make the representation unique.) Put $t_i = 2b_i$ for each i and regard $0 . t_1 t_2 t_3 \ldots$ as the representation of a real number $f(x)$ to base 3. Then each t_i is 0 or 2. Now, since $t_1 \neq 1, f(x) \notin [\frac{1}{3}, \frac{2}{3})$;† and since $t_2 \neq 1, f(x) \notin [\frac{1}{9}, \frac{2}{9})$ or to $[\frac{7}{9}, \frac{8}{9})$; and so on. Thus $f(x)$ does not belong to any of the deleted intervals, and so $f(x) \in F$. We have therefore defined a mapping f: $[0, 1) \to F$. Under this mapping, distinct points of $[0, 1)$ clearly go into distinct points of F; and hence there are as many points in F as in the whole of $[0, 1)$.

† $\notin$ means 'does not belong to'. Thus it is not true that $\frac{1}{3} \leq f(x) < \frac{2}{3}$.

5. Turning a line segment

In 1917 the Japanese mathematician Kakeya proposed the following problem. Find the region with least area within which it is possible to turn a line segment of length 1 unit continuously through a complete revolution. Certainly the area of this region will be less than or equal to $\frac{1}{4}\pi$, which is the area of a disc of diameter 1 unit. It was quite soon shown to be, in fact, less than or equal to $\frac{1}{8}\pi$, which is the area inside a deltoid inscribed in a circle of diameter $\frac{3}{2}$ units. Since, indeed, in Figure 3, the

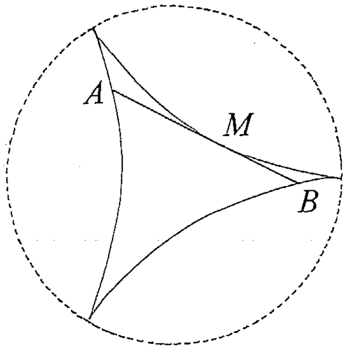


Figure 3

intercept AB on the tangent at M is equal to 1 unit for each position of M, it is readily seen just how to rotate a unit line segment through 360° entirely inside the deltoid. But this was far from being the end of the story. The answer to Kakeya's question is that a unit line segment can be turned through 360° *inside a region of arbitrarily small area*. It will take a little time for us to justify this truly surprising assertion, for although the original argument has been considerably simplified over the years, yet it is by no means easy. It depends on a basic lemma, and we shall reserve the proof of this for an appendix (to be read by those of you who want more than a sketch of the argument).

Lemma. Let ABC be a triangle with base AB; divide AB into 2^n equal subintervals, and join each point of subdivision to C to form 2^n small triangles inside ABC. For a suitable choice of n, it is possible to slide these small triangles along the base AB so that they overlap each other to such an extent that the total area which they cover in their new positions is arbitrarily small.

Now consider a circle with centre O and radius 1 unit. Describe an equilateral triangle ABC about this circle as shown in Figure 4 and join AO, BO, CO. Consider separately each of the triangles OBC, OCA, OAB with bases BC, CA, AB and, as in the lemma, divide each of them into 2^n small triangles by means of equally spaced points of subdivision of their bases. Figure 4 shows the subdivision of OBC into 2^3 small triangles. According to the lemma, given any positive number ε, however small, we may choose n so that these small triangles when translated parallel to the bases BC, CA, AB cover a total area not exceeding (say) $\frac{1}{2}\varepsilon$. Our circle is divided by small triangles into sectors (one such sector is shown shaded in the figure), and we may consider translations of the triangles as translations of the sectors. Let us denote by U the figure formed by all the sectors in their final positions. Then the area

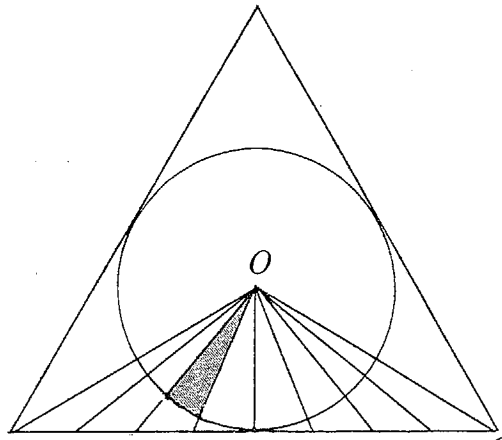


Figure 4

of U is certainly less than $\frac{1}{2}\varepsilon$. We are going to show how to enlarge U to a figure V whose area does not exceed ε and within which a unit line segment may be turned continuously through 360°. To this end, let us look first at two adjacent small sectors in their new positions. (It does not matter whether they correspond to small triangles in the same triangle, say OBC, in which case they will both have been translated in the same direction, or in neighbouring ones, in which case they will have been translated in different directions.) No directions will have been changed since the sectors have been translated only, *not* rotated. As in Figure 5, denote these two sectors by $O'PQ$, $O''RS$, where $O'Q$ and $O''R$ are parallel. Take two radii $O'X$, $O''Y$, one in each sector, which make a small angle θ. Let $O'X$, $O''Y$ (perhaps produced) meet in O_1 and form a sector $O_1X_1Y_1$ at O_1 of radius 1 unit. Evidently $O'P$ can be moved continuously to $O''S$ entirely within the shaded region. To be specific: rotate $O'P$ to $O'X$, slide $O'X$ to O_1X_1, rotate O_1X_1 to O_1Y_1, slide O_1Y_1 to $O''Y$, and finally rotate $O''Y$ to $O''S$. By considering all adjacent pairs of segments, let us augment U by means of all these small sectors $O_1X_1Y_1$ each with centre angle θ, to form V. A unit line segment can evidently be turned through a complete revolution inside V. But the total area added to U by this procedure is equal to $3\,.\,2^n(\theta/2)$; so it only remains to choose θ to be less than or equal to $\varepsilon/3\,.\,2^n$ to make this total area added to U to be less than or equal to $\frac{1}{2}\varepsilon$. Thus V has area not exceeding ε, and we can indeed turn our line segment through a complete revolution inside a region of arbitrarily small area.

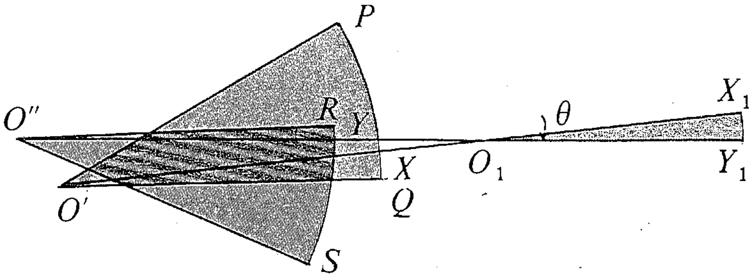


Figure 5

Appendix

We supply a proof of the lemma stated on p. 74, omitting details of routine (albeit tedious!) elementary geometry. Let us denote by S the area of the triangle ABC and by c the length of AB. Now there is an even number of small triangles inside ABC, namely 2^n, and we begin by sliding the first, third, fifth (and so on) of these along AB through a small distance $cx/2^n$ towards B. To fix our attention, we shall look at the first four triangles to the right of A; so let us denote the first four points of subdivision (read from A) by X, Y, Z, W and indicate, as in Figure 6, the new positions of the triangles ACX and YCZ by corresponding dashed letters. Let

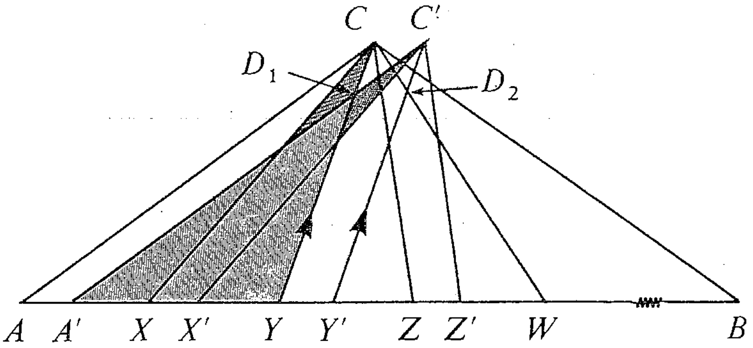


Figure 6

$A'C'$ meet YC in D_1 and $Y'C'$ meet WC in D_2. Now the total area covered by $A'C'X'$ and XCY is equal to the area of the (shaded) triangle $A'D_1Y$ together with that of two tiny (darker shaded) triangles. We may calculate this first area to be $S(1-\frac{1}{2}x)^2/2^{n-1}$, and the sum of the areas of the two tiny triangles to be $Sx^2/2^n$. Now, by a consideration of appropriate ratios, it is easily checked that YD_1 and $Y'D_2$ are equal in length; they are also parallel and so, by a further slide, the triangles $A'D_1Y$ and $Y'D_2W$ may clearly be 'fitted together'. This happens all the way along, and we form a new triangle from the 2^{n-1} triangles $A'D_1Y$, $Y'D_2W, \dots$ whose area is $S(1-\frac{1}{2}x)^2$, and in addition we have 2^{n-1} pairs of tiny triangles partly outside this triangle with total area at most $\frac{1}{2}Sx^2$. We repeat the whole procedure on the new triangle with its subdivision into 2^{n-1} small triangles, and go on repeating it. A moment's consideration will confirm that all the sliding involved is induced by sliding the original 2^n triangles ACX, XCY, $YCZ, \dots$ in ABC. After $n-1$ repetitions, the total area covered by these triangles in their new positions will be at most equal to

$$S\underbrace{(1-\tfrac{1}{2}x)^2\cdots(1-\tfrac{1}{2}x)^2}_{n}+\tfrac{1}{2}Sx^2+\tfrac{1}{2}S(1-\tfrac{1}{2}x)^2x^2+\cdots+\tfrac{1}{2}S(1-\tfrac{1}{2}x)^{2n-2}x^2$$

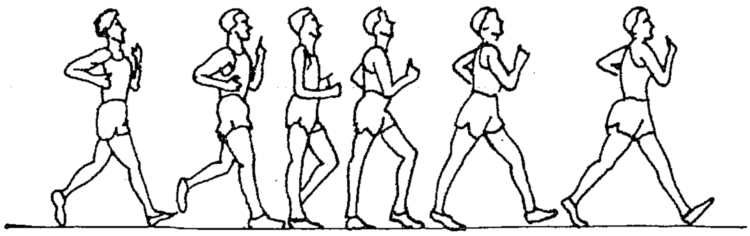
$$<S(1-\tfrac{1}{2}x)^{2n}+Sx^2/2(1-(1-\tfrac{1}{2}x)^2).$$

Finally, given any positive number δ, however small, we may *first* choose x small enough to make the second term above less than $\delta/2$ and *then* choose n large enough to make the first term less than $\delta/2$. The whole area is then less than δ.

Walking or Running? When Does Lifting Occur?

E. A. TROWBRIDGE, *University of Sheffield*

Tony Trowbridge is a Lecturer in Applied and Computational Mathematics at the University of Sheffield. His main research interests are in the application of mechanics to problems in sport and the application of continuum theories to the biomedical sciences. The present article was written in support of a controversial suggestion by the National Event Coach in Race Walking. But Tony tells us that, if he wishes to travel quickly, he would rather run than walk!



Introduction

Many readers will have observed what appears to be the strange and awkward style adopted by athletes in walking races. As we shall see, this style is necessary in order to move at a fast walking speed and still stay within the rules applied to walking races.

The major rule of race walking states that a walker must have at least one foot in contact with the ground at all times. This means that in a single step the supporting (back) foot must not leave the ground until the swinging (leading) foot has touched the ground. If both feet are off the floor simultaneously the walker is running; in the jargon of race walking he is said to be 'lifting'. If a walker is seen to be 'lifting' by a judge he is first warned and then, if he continues to break the walking rule, he is disqualified.

In the last two or three years the world records in race walking have been revised radically. Slow-motion film of some of the leading walkers has shown that, at the speeds achieved, many of the walkers cannot maintain contact with the ground. The human eye operates like a camera taking pictures at 24 frames per second, whereas the slow-motion film operates at many more frames per second. As a consequence it is sometimes very difficult for a judge to decide whether or not a walker is in fact running. Only the slow-motion replay makes it clear. Since it is not practical to film all the walkers for the total duration of a race some coaches and officials have suggested that the rules of walking should be revised. (See *Athletics Weekly*, 11 October 1980.) This article develops a simple mathematical model which predicts a relationship between leg length, stride length and maximum speed of travel without 'lifting'. It provides quantitative theoretical evidence that there is a limit to the speed which can be achieved by a walker without running. Furthermore, it points to some aspects of walking which can lead to a greater speed without 'lifting'.

A mathematical model of race walking

For an initial mathematical model the inertia of the legs is neglected compared with that of the total inertia of the athlete. This means that mechanically the athlete can be considered as a single mass M (the mass of the athlete) supported by weightless rods.

The mass is assumed to move with average speed $\bar{v}$, and is constrained to move on the arc of a circle during one stride; in other words, the supporting leg is assumed to be straight throughout the stride. Half a stride can be represented pictorially as in Figure 1.

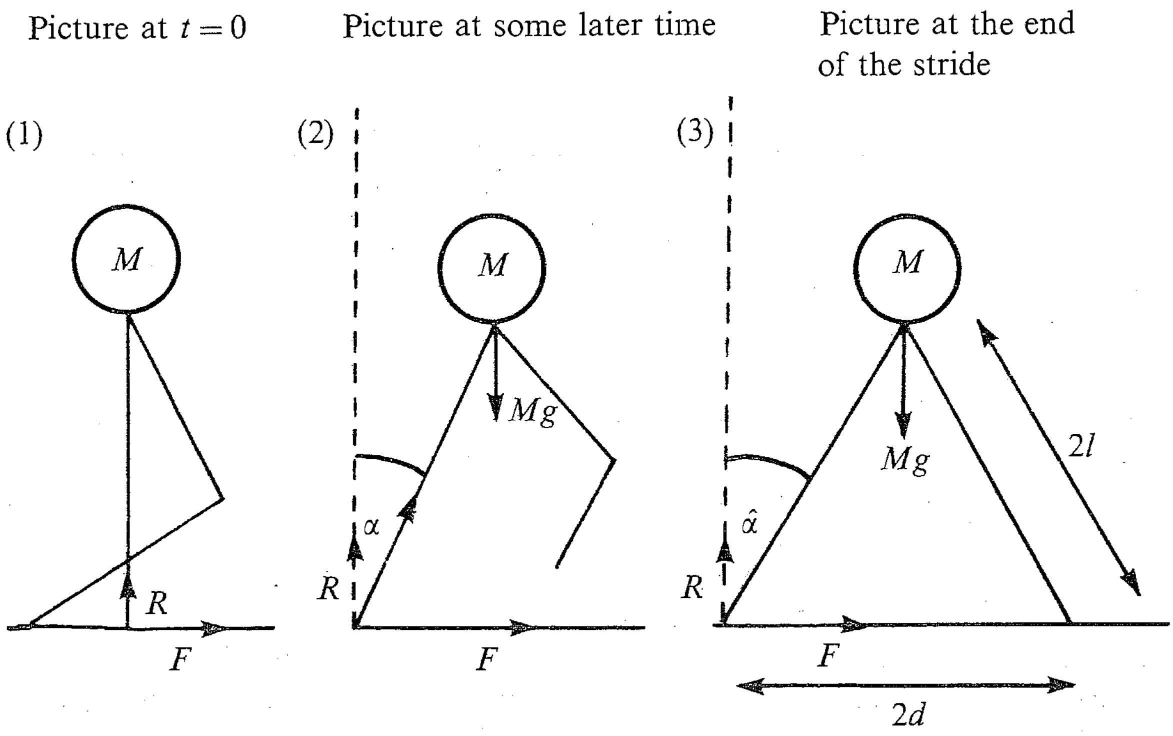


Figure 1. Schematic representation of half a stride in walking.

The legs play no part in the equations of motion; they just provide a restraint to the curve upon which the mass centre travels. The important question to be answered, which is related to 'lifting', involves the magnitude of R, the reaction force acting on the supporting foot. If the back leg breaks contact with the ground before the front foot reaches the ground, then $R = 0$ and 'lifting' occurs. For correct race walking the above picture (3) occurs, at some angle $\alpha = \hat{\alpha}$, before $R = 0$.

Although some error is introduced by neglecting the inertia of the legs, the general principles are the same, and this model should give some estimate of maximum speeds which can be achieved before walking becomes running.

If the leg length of the athlete is $2l$, then the equations of motion at any time t are as follows:

Vertical equation of motion

$$R - Mg = M\frac{d^2}{dt^2}[2l\cos\alpha]. \tag{1}$$

Horizontal equation of motion

$$F = M\frac{d^2}{dt^2}[2l\sin\alpha]. \tag{2}$$

Rotational equation of motion about the point of ground contact

$$4Ml^2\frac{d^2\alpha}{dt^2} = 2Mgl\sin\alpha. \tag{3}$$

Multiplying equation (3) by $\dot{\alpha} = (d\alpha/dt)$ and then integrating with respect to time we have

$$\dot{\alpha}^2 = \frac{g}{l}[1 - \cos\alpha] + \dot{\alpha}_0^2, \tag{4}$$

where

$$\dot{\alpha} = \dot{\alpha}_0 \qquad \text{when } \alpha = 0.$$

By substituting equations (4) and (3) into equation (1) and using the result

$$\frac{d^2}{dt^2}(\cos\alpha) = -\sin\alpha\,\ddot{\alpha} - \cos\alpha\,\dot{\alpha}^2$$

we obtain

$$R - Mg = Mg\left[\cos\alpha\left\{3\cos\alpha - 2\left(1 + \frac{l\dot{\alpha}_0^2}{g}\right)\right\} - 1\right]. \tag{5}$$

So $R = 0$ when

$$\cos\alpha\left[3\cos\alpha - 2\left(1 + \frac{l\dot{\alpha}_0^2}{g}\right)\right] = 0.$$

Since the mass M moves on the arc of a circle with an average speed of $\bar{v}$ it would seem sensible to relate the angular speed $\dot{\alpha}_0$, when the mass centre passes over the supporting foot, to $\bar{v}$ by

$$\dot{\alpha}_0 = \frac{\bar{v}}{2l},$$

so that

$$3\cos\alpha - \left(2 + \frac{\bar{v}^2}{2lg}\right) = 0. \tag{6}$$

Consequently, for a given speed $\bar{v}$, $R = 0$ when

$$\bar{v}^2 = 2lg[3\cos\alpha - 2]. \tag{7}$$

If the front foot just touches down as the back foot is about to leave the ground, when $R=0$,

$$\alpha = \hat{\alpha} \qquad \text{and} \qquad \cos\hat{\alpha} = \frac{\sqrt{4l^2 - d^2}}{2l}$$

(see Picture 3 in Figure 1). The maximum speed which can be achieved without 'lifting' taking place is thus given by

$$\bar{v}_{\max}^2 = lg[3\sqrt{4-r^2} - 4],$$

where

$$r = \frac{2d}{2l} = \frac{\text{stride length}}{\text{leg length}}.$$

A walker's stride length and leg length now determine the maximum speed at which he can travel. If we take a typical leg length of 1 metre and a typical stride length of $\frac{4}{3}$ metre then

$$\bar{v}_{\max} = 1{\cdot}52\,\text{m/sec}.$$

This maximum speed is much lower than those achieved in race walking. Data supplied by Julian Hopkins, National Event Coach for Race Walking, suggests that race walkers travel at about 4·13 m/sec.

There are two ways that the walker can improve on the maximum speed of 1·52 m/sec. Either he can reduce his stride length and as a consequence increase his frequency of step, or he can increase his effective leg length by horizontal hip extension which tends to flatten the top of the classic 'walker's triangle'. (See Picture 3 in Figure 1 which depicts this triangle.)

Increase in effective leg length

In an effort to move at a fast speed while keeping within the laws of race walking the hip is moved so that the upper end of the front or leading leg is displaced from the upper end of the back leg. The attempt to provide this hip extension leads to the apparently strange walking action adopted by top-class walkers. Although at first glance the action appears awkward, further investigation reveals a smoothness of

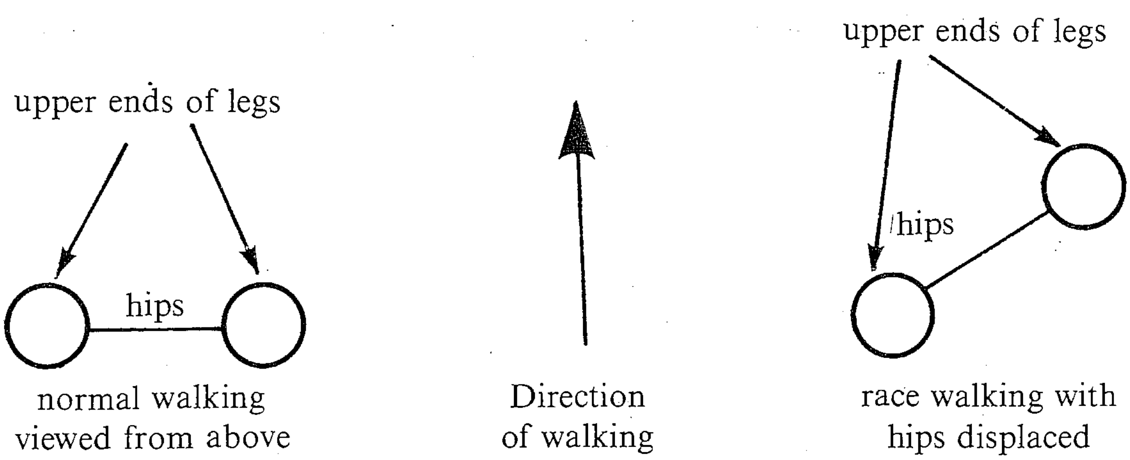


Figure 2. A comparison of normal walking and race walking.

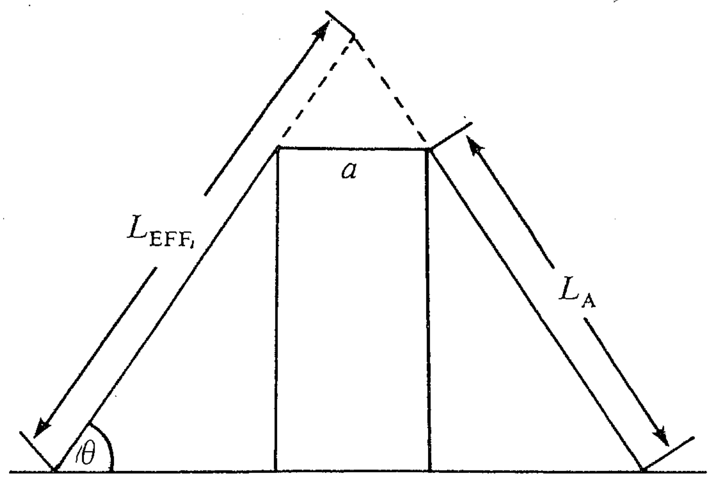


Figure 3

movement not apparent in normal walking. Expressed in two dimensions the hip displacement appears as a triangle with a flattened top (Figure 3). If the actual leg length is L_A, the effective leg length L_{EFF}, the angle θ of the leg at double contact (both legs on the ground), and the hip extension a, then

$$\text{stride length} = 2L_A \cos\theta + a.$$

For example, with a maximum speed without lifting of 14·88 km/h (4·13 m/sec) and stride length of 1·23 m, the effective leg length must be 2·07 m. (This figure for maximum speed was supplied by Julian Hopkins, National Event Coach for Race Walking.)

A walker with a leg length of 2 m would be approximately 4 m tall, which is obviously a physical impossibility. Such speeds may be achieved only with tremendous hip extensions for legs which are physically 0·92 m long. In fact for legs of such a physical length the hip extension will be $a = 0{\cdot}68$ m, according to the mechanics presented here. If such a large hip extension is necessary it casts doubts on the legality of the walking action at these high speeds. In fact it is probable that both feet are momentarily off the ground simultaneously, and the walker is 'running'. As stated previously, slow-motion film seems to support this view in many cases.

Conclusion

The simple model presented here shows how simple mechanical ideas can be employed to understand why race walkers adopt what appears to be an ungainly walking action. It has been shown that it is virtually impossible to walk at racing speeds without considerable hip extension. This hip extension effectively lengthens the physical length of the leg and enables the walker to remain in contact with the ground at all times. The model also supports the evidence, gleaned from slow-motion video recordings, that, even with substantial hip extension, race walking speeds can be achieved only by running.

The interested reader may like to improve the model by including the inertial properties of the legs and introducing some muscular control into the system.

Coupon Collecting by Computer

DERMOT ROAF, *Exeter College, Oxford*

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1. The problem

When my son James was studying statistics in the Lower Sixth, he was set a practical project on the 'Coupon Collecting Problem'. The problem is easily described: a collector accumulates m different coupons, each of which is equally likely to be found in a cereal packet. The collector buys packets until a complete set of m coupons has been obtained. What is the probability that n packets have to be bought for a complete set, and what is the average value of n?

In James's project, there were 10 coupons to be collected, and it was suggested that a table of random digits should be used instead of purchasing, at great expense, lots of cereal packets (which, anyway, might not be randomly mixed). James was told to collect 200 complete sets. I was at that time lecturing to first-year Oxford students on elementary probability, including the theory of coupon collecting, and became interested in the project.

James saw that 200 sets would require a very large number of random numbers (actually about 6000), and suggested using our Sinclair ZX-80 computer to simulate the problem. I thought this would make for an interesting comparison with the theoretical distribution of n which I had been deriving for my lecture audience. So James wrote a program to do this; anyone reading this article is also urged to try writing a program for this problem.

When James got his first 200 sets, he noticed that the number of trials varied between 12 and 83, with an average of about 30, but the program took a long time to run. He realised that a larger number of sets would provide a better comparison with the theory, so he decided to collect more sets. To do this he had to rewrite the program to speed up the test of whether the set was complete. Eventually, if a set required n random numbers, these could be processed by carrying out $(64 + 3n)$ BASIC instructions. Can you beat this? On average $(n = 29)$ each set took 0·7 seconds on my college's Commodore Pet and 0·4 seconds on the Sinclair ZX-80.

The theory gives a probability of $10!/10^{10}$ of collecting a complete set in the first 10 trials, i.e. about 4×10^{-4}, so with 1000 trials one would not expect to get one. In fact, James did not succeed, but a selection of his results is shown in column A of the table. So we then collected 10000 sets—but still did not obtain a complete set in only ten packets. But, with this number of sets, a surprising thing happened. The number of sets for any given value of n was either 0 or greater than 17 (see column B of the table). Can you explain this?

Table of sets (of ten coupons) collected in exactly n packets for selected values of n

Theoretical		Observed frequencies				
n	$P(n)$	A	B	C	D	E
1–9	0	0	0	0	0	0
10	0·0004	0	0	2	1	0
11	0·0016	0	0	7	4	5
12	0·0042	12	109	17	6	6
15	0·0186	12	129	74	33	23
20	0·0415	53	544	166	100	44
25	0·0438	52	489	175	75	42
30	0·0343	28	289	148	62	28
40	0·0150	10	90	60	19	18
50	0·0055	5	54	23	6	5
60	0·0020	0	0	13	8	4
61	0·0018	2	18	10	0	1
62	0·0016	0	0	13	1	1
63	0·0014	2	18	8	2	1
64	0·0013	1	18	7	3	2
65	0·0012	2	18	8	3	3
80	0·0002	0	0	1	0	2
Total	1·0000	1000	10000	4000	2000	1000

A ZX-80 one-digit numbers
B ZX-80 one-digit numbers
C ZX-80 four-digit numbers
D Commodore Pet one-digit numbers
E ZX-81 one-digit numbers.

2. The theory

The theory looks more complicated than it really is. Let us start by considering the expectation (or average) of the number of packets needed. Suppose we consider first the simple problem of repeating a trial until a success is obtained. If the probability of a success and a failure at each (independent) trial are p and $q\,(=1-p)$ respectively, the probability of success in exactly n trials is $q^{n-1}p$ (since there are $(n-1)$ failures followed by success). Thus the expectation of n is

$$\sum_{n=1}^{\infty} nq^{n-1}p = p(1-q)^{-2} = 1/p.$$

Now suppose that, at some point, the collector has exactly r of the m different coupons. Then the probability of obtaining a different coupon in the next packet is $p=(m-r)/m$, so he will need on average to buy $m/(m-r)$ packets. Thus, to get his first coupon, he buys $m/m=1$ packet; for the second an average of $m/(m-1)$ packets, for the rth $m/(m-r)$ on average, and for his last coupon m packets on average, so the total is

$$m\left[\frac{1}{m}+\frac{1}{m-1}+\cdots+\frac{1}{2}+1\right].$$

If $m = 10$ this gives 29·29.

But what of the individual probabilities? Clearly, for $n < m$ the probability is 0, and if $n = m$ the probability is $m!/m^m$. (Of the m^m possible sets of m packets, exactly $m!$ contain m different coupons.) But, for values of n much larger than m, a more complicated theory is required. (You may like to work out a proof that, for $n = m + 1$, the probability is $\frac{1}{2}(m-1)m!/m^m$.)

It is well known that

$$P(A \cup B) = P(A) + p(B) - P(A \cap B).$$

Here $P(A)$ means the probability of an event A, $P(A \cup B)$ is the probability of A or B or both and $P(A \cap B)$ is the probability of both A and B. This formula can be extended to give

$$P(A \cup B \cup C) = P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C].$$

Now

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$

so

$$P(A \cup B \cup C)$$
$$= P(A) + P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C).$$

By induction, we can readily prove that

$$P(A_1 \cup A_2 \cup \cdots \cup A_m) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots$$
$$+ (-1)^{m-1} P(A_1 \cap A_2 \cdots \cap A_m).$$

If A_i is the event that coupon i does not occur in the first n packets, then $P(A_i) = [(m-1)/m]^n$, $P(A_i \cap A_j) = [(m-2)/m]^n$ and $P(A_1 \cap A_2 \cdots A_m) = [0/m]^n$ (0 for $n > 0$ but 1 for $n = 0$). Now $P(A_1 \cup A_2 \cup \cdots \cup A_m)$, the probability that at least one coupon is missing in the first n packets, is $1 - Q(n)$, where $Q(n)$ is the probability of a complete set in the first n packets, namely

$$Q(n) = 1 - \sum[(m-1)/m]^n + \sum[(m-2)/m]^n - \cdots + (-1)^r \sum[(m-r)/m]^n + \cdots$$
$$= 1 - m[(m-1)/m]^n + \tfrac{1}{2}m(m-1)[(m-2)/m]^n - \cdots$$
$$+ (-1)^r \binom{m}{r}[(m-r)/m]^n + \cdots.$$

So the probability of completing a set in exactly n packets is

$$P(n) = Q(n) - Q(n-1)$$
$$= m\{[(m-1)/m]^{n-1} - [(m-1)/m]^n\} + \cdots$$
$$- (-1)^r \binom{m}{r}\{[(m-r)/m]^{n-1} - [(m-r)/m]^n\} + \cdots$$

$$= m[(m-1)/m]^{n-1}[1-(m-1)/m] + \cdots$$

$$+ (-1)^{r-1}\binom{m}{r}[(m-r)/m]^{n-1}[1-(m-r)/m] + \cdots$$

$$= [(m-1)/m]^{n-1} + \cdots + (-1)^{r-1}\binom{m-1}{r-1}[(m-r)/m]^{n-1} + \cdots.$$

If you write a program to calculate $P(n)$, you can easily check whether it is right, because for $n < m$ $P(n)$ should be 0, and $P(m)$ should be $m!/m^m$. Of course one does not actually get zeros because of rounding errors.

You should also check that $\sum_1^\infty P(n) = 1$; one cannot actually sum an infinite number of terms, but it is easy to see that, for large n, $P(n) \to [(m-1)/m]^n$, so you should check that, say, $\sum_1^{100} p(n) + \sum_{101}^{\infty} [(m-1)/m]^n = 1$.

3. Discussion

But why did the ZX-80 give such odd results when we collected 10000 sets? The answer is that computers are too well behaved to be able to produce random numbers accurately. They actually produce 'pseudo-random numbers', and the ZX-80 has only 32768 pseudo-random numbers, which come up in the same order. So this will only let us collect about 1000 sets before we repeat! Other computers have similarly finite cycles of pseudo-random numbers. (How could you find out what the cycle is?)

To overcome this, I asked the ZX-80 to produce four-digit pseudo-random numbers and then used every digit, so I was able to collect 4000 sets before the pseudo-random numbers repeated. The results are shown in column C. I also asked the Commodore Pet and a Sinclair ZX-81 to collect sets of coupons with the results shown in columns D and E.

It is obvious that, for my selected values of n, the agreement is not very close. You should now try to repeat this project—can you do any better? Are your computer's pseudo-random numbers more random than the ZX-80's?

Mental Wizardry

To find the square of any two-digit number ending in 5 (say 65), take the first digit (6) and multiply it by one more than itself (7) and adjoin 25 to the product (42) and get the required square (4225).

Can you explain why this trick works?

Mathematical Models in Geography and Planning: Urban Retail Structure as an Example

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The author is Professor of Urban and Regional Geography in Leeds. He is currently interested in dynamical urban models and applications of catastrophe theory and bifurcation theory.

1. Mathematical models in geography and planning

Geography was long considered a descriptive subject and its theoretical content was essentially verbal. In the last twenty years, a new element has been introduced by the use of quantitative methods, first with the application of statistical techniques and secondly with the construction of mathematical models. The range of application is wide. In this article, I shall take one example which demonstrates how one quite simple model can have interesting properties and which also shows how this kind of analysis can be used as a basis for planning.

2. The Huff–Lakshmanan–Hansen model

Consider a city divided exclusively and exhaustively into discrete zones which are numbered consecutively from 1 to N, say. Let i and j be typical zones. Consider the flows of money from residents to shops, from zone to zone. Thus, S_{ij} can be taken as the flow of cash from zone i to zone j and an aim of model building will be to 'explain' this set of variables in terms of other variables. These can be taken as e_i, the per capita expenditure of zone i; P_i, the population of zone i; W_j, the attractiveness of shops in zone j—usually taken as proportional to centre size—and c_{ij}, the cost of travel from i to j. These variables, and a zoning system, are shown in Figure 1.

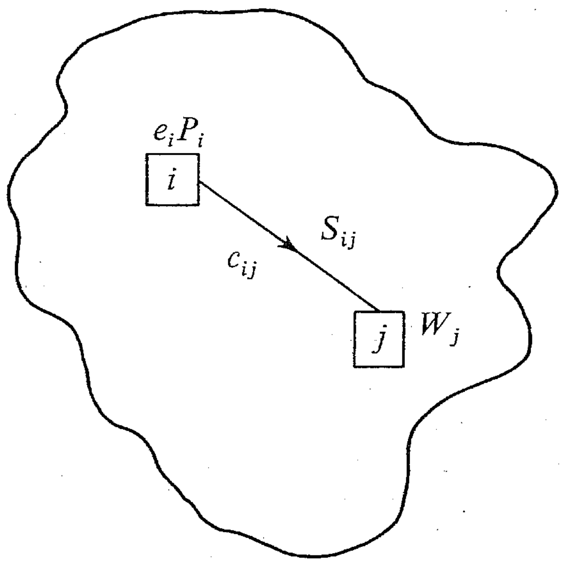

Figure 1.

The model which relates these variables was produced independently by Huff [2] and Lakshmanan and Hansen [3] in the mid 1960s. A simple derivation can be given as follows. The flows S_{ij} can be assumed to be proportional to three sets of explanatory variables:

$S_{ij} \propto e_i P_i$	(total expenditure by residents of zone i)
$S_{ij} \propto W_j^\alpha$	(attractiveness of shops in zone j, raised to a power)
$S_{ij} \propto f(c_{ij})$	(where f is some decreasing function of a measure of travel cost, c_{ij}).

Then,

$$S_{ij} = A_i e_i P_i W_j^\alpha e^{-\beta c_{ij}} \tag{1}$$

where A_i is a set of proportionality factors, indexed by i so that we can ensure that

$$\sum_j S_{ij} = e_i P_i \tag{2}$$

for each i; that is, we ensure that all available money is spent somewhere. We have taken f to be a negative exponential function for definiteness and thus introduced β as a second parameter. (Other functions can be used: the exact form for particular places is a matter of empirical testing.)

The set of flows $\{S_{ij}\}$ are written out in full in Figure 2, and the row and column sums are also shown. The row sums are $e_i P_i$ and the column sums are D_j, and we are thus defining D_j, which is the total revenue attracted to shops in j. We note, therefore, that

$$D_j = \sum_i S_{ij} \tag{3}$$

and that this will be predicted by the model which predicts S_{ij}.

Equation (1), of course, holds for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, and so there are N^2 flows in all. These flows are predicted in terms of the spatial distributions of spending power ($e_i P_i$), of shops (W_j) and the travel costs (c_{ij}).

$$\begin{array}{ccccc|c}
 & & & & & \text{Row sums} \\
S_{11} & S_{12} & S_{13} & \cdots & S_{1N} & e_1 P_1 \\
S_{21} & S_{22} & S_{23} & \cdots & S_{2N} & e_2 P_2 \\
S_{31} & S_{32} & S_{33} & \cdots & S_{3N} & e_3 P_3 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
S_{N1} & S_{N2} & S_{N3} & \cdots & S_{NN} & e_N P_N \\
D_1 & D_2 & D_3 & \cdots & D_N &
\end{array}$$

Figure 2. The matrix of retail trade flows.

The nature of the model can be made more explicit if we solve for A_i in (1) by substituting for S_{ij} from (1) into (2). This gives

$$A_i = 1 \Big/ \sum_k W_k^{\alpha} e^{-\beta c_{ik}}. \tag{4}$$

The main model equation, (1), can therefore be written

$$S_{ij} = e_i P_i \frac{W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}}. \tag{5}$$

It is now easy to see that the denominator, $\sum_k W_k^{\alpha} e^{-\beta c_{ik}}$, in relation to the S_{ij} flow, measures the 'competition' to j of other centres: if there are many other nearby centres, it will be large, and $W_j^{\alpha} e^{-\beta c_{ij}} / \sum_k W_k^{\alpha} e^{-\beta c_{ik}}$ will be small (and hence so will S_{ij}); and vice versa.

Our simple assumptions have therefore generated a model with nice plausible properties which works quite well. It is perhaps useful to add that there are alternative and better derivations—as an entropy-maximising model (Wilson, reference 5) or as a random-utility model (Williams, reference 4)—but the above ideas will suffice for now. The model is also improved if different types of retail goods are represented, but this is another complication we can neglect without affecting the presentation of the main ideas here.

3. Uses in planning

The same broad type of model can be used to represent residential location patterns, other service usages and all types of transport flows. All have uses in city planning, but here we focus again on the retail flows model to illustrate the principles.

The set of shopping centre 'sizes' $\{W_j\}, j = 1, 2, \ldots, N$, could be considered by a local authority as constituting a trial 'plan' for retailing. One planning application of the model is to compute the revenues D_j and use these to assess the viability of each centre. A second is to compute

$$X_i = A_i^{-1} = \sum_k W_k^{\alpha} e^{-\beta c_{ik}} \tag{6}$$

which is a measure of the *accessibility* for residents of zone i to shops in general. Note that the planner with a fixed amount of shopping floorspace to allocate may face competing objectives: improving viability may involve decreasing peripheral accessibility, i.e. profitability may demand the development of large shopping centres near to concentrations of population, rather than smaller ones designed to provide easy access for the rest of the population.

Essentially, what the model offers is the possibility of experimenting with plans and their impacts, using the computer, so that the best solutions can then be implemented in the real world.

4. Towards dynamical models

Suppose we now consider our model to describe a retail system operating on a free-market basis. What sort of centre-structure, represented as the $\{W_j\}$ pattern, would evolve?

Entrepreneurs at j receive revenue D_j if they provide a centre of size W_j. Let K be the unit annual cost so that KW_j is the cost figure comparable with D_j. A positive value for $D_j - KW_j$ is a profit, and vice versa. Thus, we can assume a reasonable hypothesis for describing the development of a centre is

$$\frac{dW_j}{dt} = \dot{W}_j = \varepsilon(D_j - KW_j) \tag{7}$$

for a suitable constant ε which measures entrepreneurial response per unit time period. Empirical evidence sometimes shows that the path to equilibrium differs from that generated by this equation and is slower initially and faster in the middle. This is called logistic growth. It can be modelled by adding a factor W_j:

$$\dot{W}_j = \varepsilon(D_j - KW_j)W_j. \tag{8}$$

At present, these equations are deceptively simple. If we substitute for S_{ij} from (5) into (3), and then for D_j from (3) into (8), we get

$$\dot{W}_j = \varepsilon\left[\sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} - KW_j\right] W_j. \tag{9}$$

This shows that we have a system of non-linear simultaneous equations of some complexity.

It turns out that, not only can progress be made with the analysis of solutions, but also some very interesting results emerge. First, we can use (9) to analyse the existence and stability of non-zero equilibrium solutions. These are obviously the solutions of

$$\dot{W}_j = 0, \tag{10}$$

which implies

$$\sum_i \frac{e_i P_i W_j^{\alpha} e^{-\beta c_{ij}}}{\sum_k W_k^{\alpha} e^{-\beta c_{ik}}} = KW_j. \tag{11}$$

There are values of the parameters α, β and K at which, for a particular zone j, a non-zero equilibrium value jumps to 0; and vice versa. Such a set of values is critical. Consider (α, β, K) to define a point in a 3-dimensional space, the 'parameter space'. There is a surface in this space of critical values. On one side of this surface, a non-zero stable equilibrium value exists; on the other, not. This helps us to build a picture of the spatial distribution of the elements of urban structure, in this case W_j, and to interpret it. Changes can be interpreted in terms of catastrophe theory. (See Harris and Wilson (reference 1) and Wilson and Clarke (reference 7) for more details.)

If the system is not in equilibrium, either because it has been disturbed or because there is no stable equilibrium state, the equations (9) have to be solved directly. Again, however, the notion of 'criticality' is interesting. It turns out that there are critical values of the parameter ε at which the basic solution of (9), for a zone, changes from stable-equilibrium to two-cycle periodic, then to higher cycles, and finally to chaotic oscillation followed by divergence. When there is a change in the nature of a solution at critical parameter values, this is known as a bifurcation. The study of such behaviour is known as 'bifurcation theory'. (See Wilson (reference 6) for more details.)

These results imply that a theory of the evolution of urban structure can be developed and that the analysis of critical parameter values of models is an important element of this.

5. Concluding comments

In this short article, it has only been possible to use one main model as an illustration, and from one branch of geography. There are many other models within urban and regional geography, and in other branches too, involving resources, physical and ecological systems. Many of these have applications in planning. So while the subject matter of geography in many ways remains unchanged, new weapons have been added to the armoury for attacking a wide range of interesting theoretical problems. This has led to models which are now widely applied. Many geographers have been obliged to learn some mathematics to keep up with and contribute to these developments. It may now also be appropriate for mathematicians to see aspects of geography as a new field of applied mathematics.

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Letter to the Editor

Dear Editor,

Roots of polynomials

At the end of his article in Volume 15 Number 1, D. W. Sharpe asked who Lucas was and how his name came to be attached, with Gauss's, to the theorem proved in the article. In fact, Gauss proved a result in potential theory from which the French mathematician François-Edouard-Anatole Lucas proved the result on roots of polynomials in 1874. Full details may be found in the book *The Geometry of the Zeros of a Polynomial in a Complex Variable* by Morris Marden, American Mathematics Surveys No. III, American Mathematical Society (1949). Readers may also be interested in the entry on Lucas in the *Dictionary of Scientific Biography*, ed. C. C. Gillispie, Volume VIII.

Yours sincerely,
JOSEPH HAMMER
(Department of Mathematics, The University of Sydney)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and home address and also the postal address of your school, college or university.

Problems

15.7. (From the Hungarian Olympiad 1981) The points A, B, C are non-collinear and are such that $AB^2 \geq AC^2 + BC^2$. Prove that $CD^2 \leq AD^2 + BD^2$ for every point D in the plane in which A, B, C lie. Does this relation hold if D is not in this plane?

15.8. Show that, for every positive integer n and every positive odd integer k, $1 + 2 + \cdots + n$ divides $1^k + 2^k + \cdots + n^k$.

15.9. (Submitted by A. J. Douglas and G. T. Vickers, University of Sheffield) Let $x_1, x_2, \ldots, x_n$ be real numbers such that $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \leq \begin{cases} \dfrac{1}{4} & \text{when } n \text{ is even} \\[2ex] \dfrac{1}{4} - \dfrac{1}{4n^2} & \text{when } n \text{ is odd.} \end{cases}$$

Discuss when equality occurs.

Solutions to Problems in Volume 15, Number 1

15.1. Determine the exact value of $\sin 18^\circ$.

Solution by Richard Hilditch (Sidney Sussex College, Cambridge)
Let $\theta = 18^\circ$. Then $5\theta = 90^\circ$ and $\cos 3\theta = \cos(90^\circ - 2\theta) = \sin 2\theta$. Now

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta)\\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta\\ &= (1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta\\ &= (1 - 4\sin^2\theta)\cos\theta,\end{aligned}$$

so that $1 - 4\sin^2\theta = 2\sin\theta$. Hence $\sin\theta = (\surd 5 - 1)/4$ since $\sin\theta > 0$.

Also solved by Vincent Un (Ipswich School), Ruth Lawrence (Huddersfield), Ian Wright (Winchester College), Richard Conyers (Batley Grammar School).

This problem was sent to us by M. G. Sykes, who was at Huddersfield New College at the time, but is now at Magdalene College, Cambridge. He also supplied the following results:

$$\sin 15^\circ = \frac{\surd 3 - 1}{2\surd 2}, \qquad \sin 54^\circ = \frac{\surd 5 + 1}{4}, \qquad \sin 75^\circ = \frac{\surd 3 + 1}{2\surd 2},$$

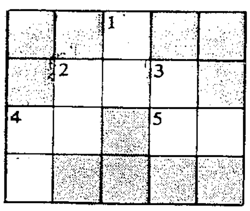
$$\sin 24^\circ = \tfrac{1}{8}[\surd 3 + \surd 15 - \surd(10 - 2\surd 5)].$$

He wrote that he had found that the sine of any angle which is an integer multiple of 3° can be expressed in surds (i.e. square roots). This amounts to saying that these angles are constructible, i.e. they can be constructed using a straight edge and compass only. He asked which integer-valued angles are constructible. M. G. Sykes has now brought to our attention a recent article in the *Mathematical Gazette*, Volume 66 (1982) 144–145, by Desmond MacHale, 'Constructing integer angles', which shows that an angle measuring an integer number k° is constructible if and only if k is a multiple of 3.

We can indicate why this is so. We know that 18° is constructible. Also 30° is constructible, so 15° is constructible (simply bisect 30°). Hence $3^\circ = 18^\circ - 15^\circ$ is constructible. Hence $3n^\circ$ is constructible for any positive integer n. It is a famous result following from Galois's work that 20° is not constructible. Hence $1^\circ, 2^\circ$ are not constructible (otherwise $18^\circ + 2^\circ = 20^\circ$ would be). Thus $(3n+1)^\circ$, $(3n+2)^\circ$ cannot be constructible for any non-negative integer n.

15.2. Each square of the crossnumber is to be filled with one of the digits $0, 1, 2, \ldots, 9$ (no number beginning with 0) so that the following triples of numbers are in arithmetical progression:

2 down,	4 across,	2 across
2 down,	3 down,	4 across
1 down,	3 down,	4 down
4 down,	5 across,	2 across



Solution by Michael Day (Northgate High School, Ipswich).
Suppose that the elements are denoted as shown:

		[1] a		
	[2] b	c	[3] d	
[4] e	f		[5] g	h
i				

The clues give us the following equations:

$$(10b+f)+(100b+10c+d)=2(10e+f), \tag{1}$$

$$(10b+f)+(10e+f)=2(10d+g), \tag{2}$$

$$(10a+c)+(10e+i)=2(10d+g), \tag{3}$$

$$(10e+i)+(100b+10c+d)=2(10g+h). \tag{4}$$

In (1), the right-hand side cannot exceed 198. It follows that $b=1$. Equation (1) now gives

$$f-d=110+10c-20e,$$

so that $f-d$ is divisible by 10. Since $f-d<10$, this means that $f=d$. Equation (1) now reduces to

$$2e-c=11. \tag{5}$$

Thus $e \geq 6$. Equation (2) reduces to

$$5e+5=9d+g. \tag{6}$$

Equations (5), (6) now give the following possibilities:

e	c	d	g
6	1	3	8
7	3	4	4
8	5	4	9
9	7	5	5

From (3), (4), $i+c$ and $i+d$ are both even, so that i, c, d have the same parity (i.e. they are all even or all odd). This excludes the possibilities $e=7$, $e=8$. Consider $e=9$. Then $g=5$ and the right-hand side of (4) cannot exceed 118, which the left-hand side clearly does. Hence we must have $e=6$, $c=1$, $d=f=3$, $g=8$, $b=1$. Equations (3), (4) now reduce to

$$10a+i=15$$
$$i+13=2h.$$

Since $a \neq 0$, this means that $a=1$, $i=5$, $h=9$. This gives the one and only solution of the crossnumber, since it is easy to verify that the numbers obtained do indeed satisfy the conditions.

Also solved by M. G. Sykes (Magdalene College, Cambridge), Richard Hilditch (Sidney Sussex College, Cambridge), Ruth Lawrence (Huddersfield).

15.3. The symbol f denotes a complex polynomial of positive degree, and z is a root of the derivative f' but not a root of f. If the maximum of the moduli of the roots of f is r, show that $|z| < r$.

Solution

This problem was proposed as an exercise on the use of the Gauss–Lucas theorem, which was the topic of the article 'Roots of Polynomials' in Volume 15 Number 1. Denote by C the circular disc, centre the origin, radius r. Then C is a convex set which contains the roots of f. By the Gauss–Lucas theorem, C will also contain the roots of f'. Suppose that $|z| = r$. Then C with the single boundary point z removed is again a convex set which contains the roots of f and so also the roots of f'. But this is not true because it does not contain z. Hence $|z| < r$, as required.

Also solved by Richard Hilditch (Sidney Sussex College, Cambridge), Ruth Lawrence (Huddersfield).

Book Reviews

Elementary Linear Algebra with Applications. By ADIL YAQUB and HAL. G. MOORE. Addison-Wesley Publishing Co., London, 1980. Pp. xiii + 369. £11·00.

Introductory Linear Algebra, with Applications, 2nd edn. By B. KOLMAN. Collier Macmillan, West Drayton, 1980. Pp. xvii + 535. £11·75.

Modern Algebra: A Natural Approach, with Applications. By C. F. GARDINER. John Wiley and Sons Ltd, Chichester, 1981. Pp. 288. £12·50 hardback; £5·90 paperback.

In the past twenty years there have been innumerable books on linear algebra at the elementary introductory level—presumably all their authors have good and valid reasons for adding to the seemingly saturated market. In more recent times it has become more fashionable to add 'with applications' to the title. This is the case with these three books—Gardiner, however, has been even more adventurous, having added 'a natural approach with applications' to 'modern algebra'!

In this review, we shall concentrate on the applications side of the books and attempt to determine how they differ from other elementary introductions to linear algebra. In fact, as far as basic theory is concerned, the books by Yaqub and Moore and by Kolman cover roughly the same ground as the usual linear algebra book, but their books differ vastly in length—approximately 360 and 530 pages, respectively. To the reviewer's taste both books are far too long with much space wasted on trivial and undemanding examples. What about applications? Yaqub and Moore have only least square fit (8 pages), quadric surfaces and differential equations (12 pages) and a final chapter of about 30 pages on other applications to directed graphs, communications networks, regular absorbing Markov chains. This is all interesting and done well—but hardly good value for the title of the book. Kolman is far more ambitious. His book is split into three parts. Part 1 is on introductory linear algebra. Part 2 is on applications to linear programming, geometry, graph theory, theory of games, least squares, linear economic problems, Markov chains, the Fibonacci sequence and differential equations. Part 3 is on numerical linear algebra and finally an interesting appendix on the computer in linear algebra. With almost half the book devoted to these subjects this is far more ambitious and interesting—in fact, this book lives up to its title very well.

Gardiner's book is quite different. It is an introduction to algebra at the first year level and contains sections on logic, sets, functions and relations, numbers (including complex numbers)—a long chapter on matrices, determinants, vectors and linear transformations—two shorter chapters on 'applications' to geometry—a final short chapter on factorisation and euclidean domains (including an application to number theory, namely Fermat's theorem).

This is a good introductory book which can be recommended. But the reviewer saw nothing which would justify either 'natural' or 'applications' in the title of the book. The printing is also very unattractive in parts, the layout poor and the large brackets used around matrices and permutations are ugly.

All the books contain solutions. Gardiner's solutions are possibly given in too much detail—the lazy or hurried student will simply go to the back of the book and copy out the solutions rather than sweat it out.

The University College of Wales
Aberystwyth

A. O. MORRIS

Precalculus Mathematics in a Nutshell. By GEORGE F. SIMMONS. William Kaufmann, Inc. 1981. Pp. 119. £3·50 paperback.

If you are about to leave school for college or university and will be studying mathematics there, either for its own sake or as a part of your course, then this might well be a book for you. It will serve as a concise revision text to help you consolidate *some* of the mathematics you have learned over the last six or seven years. But this description makes it sound dull, and it is anything but dull; though the author makes only modest claims: 'If this book can occasionally ease the pain and smooth the learning process... it will have done its job'.

This is an American book and, in America, calculus is usually studied rather later than in Britain, and the three subjects covered are basic geometry, algebra and trigonometry; but you should find it very useful nevertheless. There is no nit-picking, but there is no glossing over of difficulties either; and all the time we read we feel confidence in the author. He knows what to include and what to emphasise, and throughout he states basic principles clearly and without fuss.

A surprising amount of material has been covered in an unhurried way in a very small book, and what the author says in his introduction to Chapter 3 applies to the whole text: 'At least, the student will have no difficulty here in separating the wheat from the chaff—for there is no chaff'.

I have described this book as a revision text, but you are almost certain to learn something new as well. My one complaint is of the poor-quality paper on which the book is printed and the lack of care in trimming some of the pages. But it is well worth buying.

University of Sheffield

HAZEL PERFECT

Mathematical Puzzles. By STEPHEN AINLEY. Bell and Hyman Ltd, London, 1982. Pp. 149. £3·95 paperback.

This is one of the best books of puzzles I have seen. It is suitable for a wide range of readers with an ability in mathematics from O level upwards.

It contains over 100 original puzzles which are interestingly presented. Many topics are covered, including ones involving shapes or patterns in two and three dimensions, as well as numerical ones, such as: 'Out of the whole numbers from 1 to 50 inclusive, make a list of as many as possible in such a way that no number in your list exactly divides any other'.

The solutions are very clear and occasionally use results which leave you wondering why they are not better known. For example, everyone knows the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, but how many know the corresponding formula for the area of any cyclic quadrilateral?

If you enjoy being challenged, then I can only recommend that you buy this book, and hope that you find it as stimulating as I did myself.

J. PORTEOUS

A Pathway into Number Theory. By R. P. BURN. Cambridge University Press, 1982. Pp. 257. £18·00 hardback; £7·50 paperback.

Devotees of number theory will give this book an enthusiastic welcome. Those who are as yet unconvinced of the wealth of interest in this branch of mathematics (neglected in this country), but who wish to be tempted, will find their appetites whetted from the start. Provided they are willing to make a considerable effort to cooperate with the author in the subsequent pages, they will find their efforts rewarded amply, and finally join the ranks of the number theory enthusiasts.

The structure of this book is unusual in that there is no textual exposition of the usual type. Rather, the exposition is entirely carried out by carefully graded questions for the reader, arranged in a 'linear' development. Each chapter is supplemented by notes and answers, followed by historical notes. A student could 'cheat' and turn this book into a straightforward text by jumping from each question to its solution and back again, to and fro. But there is strong motivation for carrying out the author's plan, which is fully explained in the preface.

The wealth of material, the variety and scope of the questions, are truly remarkable. The book develops the subject by proposing no fewer than 817 questions, many of them very substantial, ranging over 11 chapters. This is not only a scholarly work, it is also a work of teaching. The ample index at the end provides quick access to particular items for the mathematician who is not a number theorist, but who has occasional need to refer to its results and methods. There is also an annotated comprehensive bibliography and a useful final summary.

No book can be perfect. For all its thoroughness, I could find no mention of the fact that $\phi(n)$ (Euler's function) is always even, nor that $\phi(p) = p - 1$ for a prime p. There are some objectionable lapses of notation; perhaps the worst is to write a·2 for 2a, with an elevated dot to denote multiplication. Important results do not get enough emphasis, being 'buried' among the questions. For example, Fermat's theorem appears obscurely as Question 19 on page 55. Finally, in view of the fact that number theory is very much a minority pursuit, the price may prevent large sales among students, which would be a pity.

Royal Grammar School, Newcastle upon Tyne F. J. BUDDEN

An Introduction to Mechanics and Modelling. By D. G. MEDLEY. Heinemann Educational Books Ltd, London, 1982. Pp. xii + 340. £7·50.

In recent years, many universities have set out to revitalise their first-year courses in applied mathematics. One result of this has been the introduction of applications of mathematics in areas other than mechanics. Nevertheless, the traditional material of a mechanics course, in one form or another, still constitutes a major part of most first-year university courses. Here, as in the newer applications, attempts have been made to place more emphasis on the 'modelling' aspects of the subject. Dr Medley's book is very much in line with this trend, concentrating as it does on making explicit the assumptions underlying the models employed in Newtonian mechanics. The spirit of the book is expressed in the author's hope that 'no mechanics student who reads this book will emerge from his first/second year mechanics course feeling that he has been required to perform tricks whose underlying logic remains a mystery to him'.

This is a very laudable aim, and the book represents an adventurous attempt to achieve it. The task is by no means an easy one since, for most students at least, understanding mechanics is much harder than learning how to do it. For this reason I suspect that the book is probably more suited to those who have already learnt some of the tricks, for whom it should certainly provide fresh insight and a better understanding of the principles involved.

Nevertheless, there is much here too for those students who encounter mechanics for the first time at university level. In particular, Chapter 4 ('Rudimentary ideas of force and mass; . . .') provides a helpful and original introduction to the notion of a force and its various manifestations, with simple, clear illustrations of concepts which are often taken for granted or left unexplained in many more conventional texts.

The book is arranged in three main sections—Part I: 'Modelling of space by vectors, and an introduction to modelling in the real world'; Part II: 'Understanding the motion of real bodies'; Part III: 'Energy modelling in elementary mechanics'. Part I begins with a treatment of vector algebra in which the usual basic results are covered with particular reference, wherever possible, to the requirements of mechanics, and concludes with the excellent introduction to the concepts of force and mass, referred to above. These preliminary ideas are further expounded and developed, in a rather more rigorous fashion, in Part II, which embodies the main core material of mechanics, and in Part III, which deals specifically with energy modelling. Most of the standard topics usually encountered in a first-year mechanics course are covered, including two-dimensional rigid body motion. (A sequel volume is proposed to cover more advanced topics, some of which may be found in some first-year courses.) A good number of exercises is provided throughout the book, including many of a straightforward nature and others designed to cause the student to think carefully about the ideas involved.

Overall, this is a refreshing and readable book which would be a worthwhile acquisition for any first-year undergraduate student looking for something more than just a 'how to do it' book. At pre-university level it could also provide thought-provoking background reading for good A-level students, and would be a useful addition to the school library. Teachers too could find it a useful source of new ideas.

University of Sheffield A. M. DOWNS

Studies in Mathematics Education, Volume 1 (1980). Pp. 129. Volume 2 (1981). Pp. x + 179. Edited by ROBERT MORRIS. UNESCO, Paris.

Mathematical education is not a subject that usually enters into sixth-form studies. Nevertheless these two volumes from UNESCO are worth considering by teachers and pupils. We tend to take for granted our view of mathematics and what should be taught in schools. Volume 1 of these studies broadens horizons by giving an international viewpoint. There are articles on mathematics teaching in Hungary, Indonesia, Japan, the Philippines, the Soviet Union and Tanzania and they show the effect of different national characteristics and priorities on the approach to and content of mathematics teaching. Volume 2 looks in more detail at the goals of mathematical education. Of particular interest to British readers are the contributions of Margaret Brown and Ruth Rees looking at goals as a reflection of the needs of the learner and at links with commerce and industry respectively. Drawing on examples from the project 'Concepts in Secondary Mathematics and Science' Margaret Brown suggests that our mathematics teaching tends too often to ignore the needs of the child as they are perceived by him in the classroom. Ruth Rees reports on various attempts in England to forge links between school mathematics and employment. Other essays look at goals as reflecting the needs of society, goals for rural development, goals reflecting production requirements and at how to make curriculum change more rational and systematic.

The books are more for the teacher than the pupil, but both could gain from viewing their courses in the broader contexts detailed in these essays.

University of Sheffield P. HOLMES

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ISSN 0025–5653

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Published by the Applied Probability Trust
Printed in England by Galliard (Printers) Ltd, Great Yarmouth