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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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THE OLYMPIAD CORNER

No. 145

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The "pre-Olympiad" set of problems we give this month is the 1993 AIME, which is a regular feature of this column. The American Invitational Mathematics Examination, written April 1, 1993, and its problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America, and they may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

1993 AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1. How many even integers between 4000 and 7000 have four different digits?

2. During a recent campaign for office, a candidate made a tour of a country which we assume lies in a plane. On the first day of the tour he went east, on the second day he went north, on the third day west, on the fourth day south, on the fifth day east, etc. If the candidate went $n^2/2$ miles on the nth day of his tour, how many miles was he from his starting point at the end of the 40th day?

3. The table below displays some of the results of last summer's Frostbite Falls Fishing Festival, showing how many contestants caught n fish for various values of n.

n	0	1	2	3	$\cdots$	13	14	15
number of contestants who caught n fish	9	5	7	23	$\cdots$	5	2	1

In the newspaper story covering the event, it was reported that

a) the winner caught 15 fish;
b) those who caught 3 or more fish averaged 6 fish each;
c) those who caught 12 or fewer fish averaged 5 fish each.

What was the total number of fish caught during the festival?

4. How many ordered 4-tuples of integers (a, b, c, d) with $0 < a < b < c < d < 500$ satisfy $a + d = b + c$ and $bc - ad = 93$?

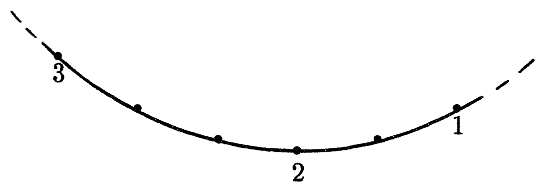
5. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

6. What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

7. Three numbers, a_1, a_2, a_3, are drawn randomly and without replacement from the set $\{1, 2, 3, \ldots, 1000\}$. Three other numbers, b_1, b_2, b_3, are then drawn randomly and without replacement from the remaining set of 997 numbers. Let p be the probability that, after a suitable rotation, a brick of dimensions $a_1 \times a_2 \times a_3$ can be enclosed in a box of dimensions $b_1 \times b_2 \times b_3$, with the sides of the brick parallel to the sides of the box. If p is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

8. Let S be a set with six elements. In how many different ways can one select two not necessarily distinct subsets of S so that the union of the two subsets is S? The order of selection does not matter; for example, the pair of subsets $\{a, c\}, \{b, c, d, e, f\}$ represents the same selection as the pair $\{b, c, d, e, f\}, \{a, c\}$.

9. Two thousand points are given on a circle. Label one of the points 1. From this point, count 2 points in the clockwise direction and label this point 2. From the point labeled 2, count 3 points in the clockwise direction and label this point 3. (See figure.) Continue this process until the labels $1, 2, 3, \ldots, 1993$ are all used. Some of the points on the circle will have more than one label and some points will not have a label. What is the smallest integer that labels the same point as 1993?



10. Euler's formula states that for a convex polyhedron with V vertices, E edges, and F faces, $V - E + F = 2$. A particular convex polyhedron has 32 faces, each of which is either a triangle or a pentagon. At each of its V vertices, T triangular faces and P pentagonal faces meet. What is the value of $100P + 10T + V$?

11. Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is m/n, where m and n are relatively prime positive integers. What are the last three digits of $m + n$?

12. The vertices of ΔABC are $A = (0, 0)$, $B = (0, 420)$, and $C = (560, 0)$. The six faces of a die are labeled with two A's, two B's, and two C's. Point $P_1 = (k, m)$ is chosen in the interior of ΔABC, and points $P_2, P_3, P_4, \ldots$ are generated by rolling the die repeatedly and applying the rule: If the die shows label L, where $L \in \{A, B, C\}$, and P_n is the most recently obtained point, then P_{n+1} is the midpoint of $\overline{P_n L}$. Given that $P_6 = (14, 92)$, what is $k + m$?

13. Jenny and Kenny are walking in the same direction, Kenny at 3 feet per second and Jenny at 1 foot per second, on parallel paths that are 200 feet apart. A tall circular building 100 feet in diameter is centered midway between the paths. At the instant when the building first blocks the line of sight between Jenny and Kenny, they are 200 feet apart. Let t be the amount of time, in seconds, before Jenny and Kenny can see each other again. If t is written as a fraction in lowest terms, what is the sum of the numerator and denominator?

14. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called *unstuck* if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a 6 by 8 rectangle, the smallest perimeter has the form $\sqrt{N}$, for a positive integer N. Find N.

15. Let $\overline{CH}$ be an altitude of ΔABC. Let F and S be the points where the circles inscribed in triangles ACH and BCH are tangent to $\overline{CH}$. If $AB = 1995$, $AC = 1994$, and $BC = 1993$, then RS can be expressed as m/n, where m and n are relatively prime positive integers. Find $m + n$.

* * *

The Olympiad for this month comes to us from F. Bellot, Valladolid, Spain. The problems are proposed by the Royal Spanish Mathematical Society.

28th SPANISH MATHEMATICAL OLYMPIAD

First Round — November 22–23, 1991, Valladolid

First Day

1. Let z be a complex number. Show that

$$\tan(\arg(z)) > \sqrt{2} - 1 \quad \Rightarrow \quad \mathrm{Re}\,(z^2) < \mathrm{Im}\,(z^2) \quad \Rightarrow \quad \cot(\arg(z)) < 1 + \sqrt{2}.$$

Is the converse true?

2. Let S be the set of straight lines which join a point from the set $A = \{(0, 1/a); a \in \mathbb{N}\}$ with a point from the set $B = \{(b+1, 0); b \in \mathbb{N}\}$. Show that a necessary and sufficient condition that the natural number m be composite is that the point $M = (m, -1)$ belong to a line from S. Determine the number of lines of S to which m belongs.

3. The abscissa of a point which moves in the positive part of the axis Ox is given by $x(t) = 5(t+1)^2 + a/(t+1)^5$, in which a is a positive constant. Find the minimum a such that $x(t) \geq 24$ for all $x \geq 0$.

4. Two equal and tangent half circles S_1 and S_2, diameters lying on the same straight line, are given. A common tangent line r is drawn to S_1 and S_2. A circle C_1, tangent to r, S_1 and S_2 is produced. Next, another circle C_2, tangent to C_1, S_2 and S_1 is drawn, and so on.

a) Find the radius r_n of C_n in terms of n and R, the radius of S_1.

b) Using the construction of this problem, show that the limit of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

as $n \longrightarrow \infty$, is 1.

Second Day

5. For each natural number n, let

$$(1+\sqrt{2})^{2n+1} = a_n + b_n\sqrt{2},$$

with a_n and b_n integers.

a) Show that a_n and b_n are odd, for all n.

b) Show that b_n is the hypotenuse of a right triangle with legs

$$\frac{a_n + (-1)^n}{2} \qquad \text{and} \qquad \frac{a_n - (-1)^n}{2}.$$

6. Two medicines, A and B, have been tested in two hospitals. In both hospitals, a better result was obtained with medicine A than with B; but when the results were combined it was found with astonishment that medicine B obtained better results than A. Is this possible, or is it due to a mistake in the calculations?

7. Let m be a natural number. Show that if $2^m + 1$ is prime, $2^m + 1 > 3$, then m is even.

8. Let ABC be any triangle. Two squares $BAEP$ and $ACDR$ are constructed externally to ABC. Let M and N be the midpoints of BC and ED, respectively. Show that $AM \perp ED$ and $AN \perp BC$.

* * *

To begin the solutions for this month we give readers' answers to the four problems of the *21st Austrian Mathematical Olympiad, 2nd Round* [1992: 99]. One of the solvers of the problems did not put a name on the page of solutions. While the handwriting is that of a regular solver, I am unable to confidently assert which one. My apologies to Anonymous since I forgot to check the page before discarding the envelope in which it arrived.

1. Prove: There exists no natural number n such that the total number of integer factors of n equals 1990 and the sum of the inverses $1/b$ of all natural number factors b of n equals 2.

Solutions by Anonymous; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Wang's solution.

Suppose there exists a natural number n satisfying the given conditions. Let $n = P_1^{a_1}P_2^{a_2}\dots P_k^{a_k}$ be the prime power decomposition of n, where $P_1 < P_2 < \cdots < P_k$ are primes and $a_i - 1 \geq 1$ for all i. Then it is well known that the number of positive divisors of n is $d(n) = \prod_{i=1}^{k}(1+a_i)$. Thus from the assumption that $2d(n) = 1990$ we get

$$\prod_{i=1}^{k}(1+a_i) = 995 = 5 \times 199. \tag{1}$$

From this we see that a_i is even for all i and hence $a_i = 2c_i$ where $c_i \geq 1$. Now by the second assumption we get

$$\prod_{i=1}^{k}\left(1+\frac{1}{P_i}\right)\left(1+\frac{1}{P_i^2}\right)\cdots\left(1+\frac{1}{P_i^{a_i}}\right) = 2 \tag{2}$$

since the left hand side, on expansion, clearly gives the sum of the reciprocals $1/b$ of all the positive divisors of n. From (2) we obtain

$$\prod_{i=1}^{k}(1+P_i)(1+P_i^2)\cdots(1+P_i^{2c_i}) = 2\prod_{i=1}^{k}P_i^{c_i(1+2c_i)}.$$

If P_i is odd for all i, then the left hand side of this expression is divisible by 4 while the right side is not, a contradiction. Thus $P_1 = 2$. If $k = 1$, then the left hand side is odd, a contradiction. Thus $k > 1$, and we see from (1), since both 5 and 199 are primes, that $k = 2$ and either $a_1 = 4, a_2 = 198$ or $a_1 = 198, a_2 = 4$. In either case

$$\prod_{i=1}^{2}\left(1+\frac{1}{P_i}\right)\left(1+\frac{1}{P_i^2}\right)\cdots\left(1+\frac{1}{P_i^{a_i}}\right) > \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{16}\right) = \frac{2295}{1024} > 2,$$

a contradiction. The proof is thus complete.

2. Solve (in $\mathbb{R}$) the equation $\sqrt[3]{2x-7} + \sqrt[3]{3x-3} = \sqrt[3]{x-8} + \sqrt[3]{4x-2}$.

Solutions by Anonymous; by Seung-Jin Bang, Albany, California; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Joseph Ling, The University of Calgary; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. The solutions were very similar and we give Ling's.

The solutions are -1, $-5/2$ and 2. We first establish a

Lemma. $\alpha, \beta, \gamma, \delta$ satisfy the system $\alpha + \beta = \gamma + \delta$ and $\alpha^3 + \beta^3 = \gamma^3 + \delta^3$ if and only if

(i) $\alpha = -\beta$ and $\gamma = -\delta$, or

(ii) $\{\alpha, \beta\} = \{\gamma, \delta\}$.

Proof. If $\alpha, \beta, \gamma, \delta$ satisfy the system, then

$$\begin{aligned} 0 = (\alpha+\beta)^3 - (\gamma+\delta)^3 &= \alpha^3 + \beta^3 - \gamma^3 - \delta^3 + 3\alpha\beta(\alpha+\beta) - 3\gamma\delta(\gamma+\delta) \\ &= 3\alpha\beta(\alpha+\beta) - 3\gamma\delta(\gamma+\delta) = 3(\alpha\beta - \gamma\delta)(\alpha+\beta). \end{aligned}$$

If $\alpha + \beta = \gamma + \delta = 0$ then we have (i). If $\alpha\beta - \gamma\delta = 0$, when we have both $\alpha + \beta = \gamma + \delta$ and $\alpha\beta = \gamma\delta$. Then $\{\alpha, \beta\} = \{\gamma, \delta\}$ since both are the set of roots of the same quadratic equation $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. This gives (ii). The converse of the lemma is trivial.

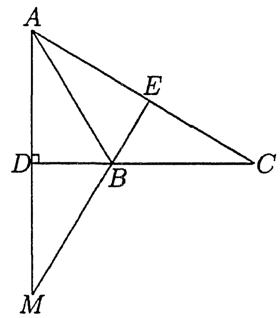
Apply the lemma to $\alpha = \sqrt[3]{2x-7}$, $\beta = \sqrt[3]{3x-3}$, $\gamma = \sqrt[3]{x-8}$ and $\delta = \sqrt[3]{4x-2}$, since we have $\alpha + \beta = \gamma + \delta$ by hypothesis, and $\alpha^3 + \beta^3 = 5x - 10 = \gamma^3 + \delta^3$. This gives $\alpha = -\beta$ and $\gamma = -\delta$ in which case $x = 2$, or $\{\alpha, \beta\} = \{\gamma, \delta\}$. Setting $\alpha = \gamma$, $\beta = \delta$, we get $x = -1$, and setting $\alpha = \delta$, $\beta = \gamma$ gives $x = -5/2$.

3. Let ABC be a triangle with E and D the feet of the altitudes to sides b and a, respectively. Let M be the point on AD such that $AD = DM$.

(a) Show there exists no acute-angled triangle ABC such that C, D, E, M lie on a circle.

(b) Determine all triangles ABC such that $CDEM$ do lie on a circle.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.



Let the angles at A, B, C be α, β, γ respectively. Then $AD = c\sin\beta$, $AE = c\cos\alpha$, $AC = b$, $AM = 2c\sin\beta$. If C, D, E, M are concyclic then $(AD)(AM) = (AE)(AC)$ giving $2c^2\sin^2\beta = bc\cos\alpha$. Thus $2c\sin\beta = 2R\cos\alpha$. From this $2\sin\beta\sin\gamma = \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma$. Then $\cos(\beta - \gamma) = 0$ and $\beta = \gamma + \pi/2$.

So (a) there is no acute angle triangle ABC such that C, D, E, M are concyclic.

(b) Those triangles for which C, D, E, M are concyclic have $\beta = \gamma + \pi/2$ and $\alpha = \pi/2 - 2\gamma$. Clearly $\angle EBC = \pi/2 - \gamma = \angle DBM$, so $\angle DMB = \gamma$. Now $\angle ABC = \pi/2 + \gamma$, so $\angle ABE = 2\gamma$, from which one deduces $\angle ABD = \pi/2 - \gamma$ also. Hence $\angle DAB = \gamma$ and $AB = BM$. In triangle ABC and MBC, BC is common, $AB = BM$, and $\angle ABC = \angle MBC = \pi/2 + \gamma$. So $\angle MCB = \gamma$ and triangle ACM is isosceles. B is the orthocentre of triangle AMC.

4. For natural numbers $k, n \geq 2$, determine the sum

$$S(k,n) = \left[\frac{2^{n+1}+1}{2^{n-1}+1}\right] + \left[\frac{3^{n+1}+1}{3^{n-1}+1}\right] + \cdots + \left[\frac{k^{n+1}+1}{k^{n-1}+1}\right]$$

where $[x]$ denotes the greatest integer $\leq x$.

Solutions by Seung-Jin Bang, Albany, California; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Joseph Ling, The University of Calgary; by Pavlos Maragoudakis, student, University of Athens, Greece; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

First consider the case with $n = 2$. Since

$$\frac{m^3+1}{m+1} = m^2 - m + 1,$$

we have

$$S(k,2) = \sum_{j=2}^{k}(j^2 - j + 1) = \frac{k(k+1)(2k+1)}{6} - \frac{k(k+1)}{2} + k - 1$$
$$= \frac{1}{3}k^3 + \frac{2}{3}k - 1 = (k-1)(k^2+k+3)/3.$$

Next suppose that $n \geq 3$. For $a \geq 2$ it is clear that

$$\frac{a^{n+1}+1}{a^{n-1}+1} < a^2.$$

Also

$$a^2 - 1 < \frac{a^{n+1}+1}{a^{n-1}+1}$$

just in case $a^{n+1} - a^{n-1} + a^2 - 1 < a^{n+1} + 1$. Equivalently, $a^2 < a^{n-1} + 2$ which is true. It follows that

$$\left\lfloor \frac{a^{n+1}+1}{a^{n-1}+1} \right\rfloor = a^2 - 1.$$

Therefore

$$S(k,n) = \sum_{a=2}^{k}(a^2-1) = \frac{k(k+1)(2k+1)}{6} - k$$
$$= \frac{k(2k+5)(k-1)}{6}.$$

* * *

Next are solutions from the readers to problems of the *21st Austrian Mathematical Olympiad, Final Round* [1992: 100].

1. Determine the number of all natural numbers n such that $1 \leq n \leq N = 1990^{1990}$ with $n^2 - 1$ and N relatively prime.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Joseph Ling, The University of Calgary. We use Ling's solution.

The answer is $1990^{1989} \cdot 589$.

The prime factorization of 1990 is $1990 = 2 \times 5 \times 199$. Hence $n^2 - 1$ is relatively prime to $N = 1990^{1990}$ if and only if $n^2 - 1 \not\equiv 0 \bmod 2$, mod 5, mod 199. That is $n \not\equiv 1 \bmod 2$, $n \not\equiv 1, 4 \bmod 5$ and $n \not\equiv 1, 198 \pmod{199}$. Let S be the set of all integers n with $1 \leq n \leq N$, and define the following subsets of S:

$$A = \{n : n \equiv \bmod 2\}, \quad B = \{n : n \equiv 1 \text{ or } 4 \bmod 5\}, \quad C = \{n : n \equiv 1 \text{ or } 198 \bmod 199\}.$$

Then we want to find $|S \backslash (A \cup B \cup C)|$. Now it is easily seen that

$$A \cap B = \{n : n \equiv 1 \text{ or } 9 \bmod 10\}, \quad B \cap C = \{n : n \equiv 1 \text{ or } 994 \bmod 995\},$$

$$A \cap C = \{n : n \equiv 1 \text{ or } 397 \bmod 398\}$$

and $A \cap B \cap C = \{n : n \equiv 1,\ 1989 \bmod 1990\}$. From this we get $|S| = N$, $|A| = N/2$, $|B| = 2N/5$, $|C| = 2N/199$, $A \cap B = 2N/10$, $|B \cap C| = 2N/995$, $|A \cap C| = 2N/398$, and $|A \cap B \cap C| = 2N/1990$. By the inclusion–exclusion principle

$$\begin{aligned} |S \backslash (A \cup B \cup C)| &= N - \frac{N}{2} - \frac{2N}{5} - \frac{2N}{199} + \frac{2N}{10} + \frac{2N}{995} + \frac{2N}{398} - \frac{2N}{1990} \\ &= \frac{589}{1990} N = 1990^{1989} \cdot 589. \end{aligned}$$

2. Show that for all natural numbers $n \geq 2$

$$\sqrt{2\sqrt[3]{3\sqrt[4]{4 \ldots \sqrt[n]{n}}}} < 2.$$

Solutions by Seung-Jin Bang, Albany, California; Christopher J. Bradley, Clifton College, Bristol, U.K.; Curtis Cooper, Central Missouri State University, Warrensburg; Murray S. Klamkin, University of Alberta; Andy Liu, University of Alberta; Pavlos Maragoudakis, student, University of Athens, Greece; Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give Cooper's solution, which illustrates one mode of attack.

We will prove more than required, namely that for each integer $m \geq 2$, if $n \geq m$ then

$$\sqrt[m]{m \sqrt[m+1]{(m+1) \ldots \sqrt[n]{n}}} < 2,$$

by reverse induction (that is, first for $m = n$ and then down to $m = 2$). Clearly $\sqrt[n]{n} < 2$. For $m < n$ assume inductively that

$$\sqrt[m+1]{(m+1) \ldots \sqrt[n]{n}} < 2.$$

Then

$$\sqrt[m]{m \sqrt[m+1]{(m+1) \ldots \sqrt[n]{n}}} < \sqrt[m]{m \cdot 2} \leq 2.$$

The result follows immediately setting $m = 2$.

To illustrate the other style of solution we give Klamkin's solution (with finer estimates).

If P denotes the left hand side, then

$$\ln P = \frac{\ln 2}{2!} + \frac{\ln 3}{3!} + \cdots + \frac{\ln n}{n!}.$$

Since $(\ln x)/x$ is decreasing for $x \geq 3$,

$$\begin{aligned} \ln P &< \frac{\ln 2}{2!} + \frac{\ln 3}{3} \left\{ \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \right\} \\ &= \frac{\ln 2}{2!} + \frac{\ln 3}{3}(e-2). \end{aligned}$$

Hence $P < \sqrt{2} \cdot 3^{(e-2)/3} \sim 1.8397 < 2$.

For a lower bound

$$\ln P > \frac{\ln 2}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \frac{\ln 2}{2!} + e - \frac{5}{2} .$$

Hence

$$P > \sqrt{2} \cdot e^{(e-5/2)} \sim 1.1423.$$

3. In a convex quadrilateral $ABCD$ (all interior angles $< 180°$) let E be the intersection of the diagonals and F_1, F_2, and F the areas of ABE, CDE and $ABCD$, respectively. Show that

$$\sqrt{F_1} + \sqrt{F_2} \leq \sqrt{F}$$

and determine when equality holds.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Murray S. Klamkin, University of Alberta; and by Joseph Ling, The University of Calgary. We use Klamkin's solution.

Let $AE = x$, $EC = y$, $BE = u$, $ED = v$ and $\angle AEB = \theta$. Then $2F_1 = xu\sin\theta$, $2F_2 = vy\sin\theta$ and $F = (x+y)(u+v)\sin\theta$. We now want to show that

$$\sqrt{xu} + \sqrt{yv} \leq \sqrt{(x+y)(u+v)} .$$

This follows immediately from Cauchy's inequality. Moreover, there is equality if and only if $x/u = y/v$, and this is equivalent to the quadrilateral being a trapezoid (i.e. AB is parallel to CD).

4. For each integer $n \neq 0$ determine all functions $f : \mathbb{R}\backslash\{-3, 0\} \to \mathbb{R}$ such that

$$f(x+3) + f\left(-\frac{9}{x}\right) = \frac{(1-n)(x^2+3x-9)}{9n(x+3)} + \frac{2}{n}$$

for all $x \neq 0, -3$. Furthermore, for each fixed natural number n determine all integers x such that $f(x)$ is an integer.

Solution by Joseph Ling, The University of Calgary.

We claim that $f(x) = (n+x-1)/nx = 1/n + 1/x - 1/nx$, $x \neq 0, -3$. Furthermore, if $n = 1$ or 4, then $f(x)$ is an integer just in case $x = 1$; and if n is a natural number other than 1 or 4, then $f(x)$ is an integer if and only if $x = 1$ or $1 - n$.

Suppose that f satisfies the given condition. Define $g(x) = f(x) - 1/n - 1/x + 1/nx$, $x \neq 0, -3$. A direct calculation shows that $g(x+2) + g(-9/x) = 0$. Define $h(x) = g(x+3)$, $x \neq -3, -6$. Then $h(x) + h(-3 - 9/x) = 0$, i.e. $h(x) = -h(\phi(x))$, where $\phi(x) = -3 - 9/x$.

Now, a direct computation shows that $\phi(\phi(x)) = -9/(x+3)$ and $\phi(\phi(\phi(x))) = x$. Therefore we have $h(x) = -h(\phi(x)) = h(\phi(\phi(x))) = -h(\phi(\phi(\phi(x)))) = -h(x)$, and so $2h(x) = 0$. This forces $h(x) = 0$ and $g(x) = 0$. [*Editor's note.* There is a technical difficulty, because $x = 0 \notin \operatorname{dom}\phi(x)$ as well as with $x = 3$ (since $\phi(3) = -6 \notin \operatorname{dom} h$). However this problem was inherent with f since $f(-9/x)$ is not defined when $x = 3$.]

Therefore $f(x) = (n+x-1)/nx$. If $f(x) = k$ is an integer then $n+x-1 = knx$ and so $x = (n-1)/(kn-1)$. Since n is a positive integer, x is an integer only when $k = 0$ or 1. The corresponding values for x are $1-n$ and 1. These are solutions unless the domain of f rejects $1-n$, i.e. if $n = 1$ or 4. For those two values of n, $f(x)$ is an integer if and only if $x = 1$.

5. Determine all rational numbers r such that all solutions of $rx^2+(r+1)x+(r-1) = 0$ are integers.

Solutions by Joseph Ling, The University of Calgary; by Pavlos Maragoudakis, student, University of Athens, Greece; and by Michael Selby, University of Windsor.

If $r = 0$ then the equation becomes $x - 1 = 0$, so $x = 1$.

If $r \neq 0$, then let x_1, x_2 be the roots of the quadratic equation, $x_1 \leq x_2$. Then $x_1 + x_2 = -(r+1)/r$, and $x_1x_2 = (r-1)/r$. So

$$x_1x_2 - x_1 - x_2 = \frac{r-1}{r} + \frac{r+1}{r} = 2$$

$$(x_1 - 1)(x_2 - 1) = 3$$

and we have $(x_1 - 1 = 1, x_2 - 1 = 3)$ or $(x_1 - 1 = -3, x_2 - 1 = -1)$.

If $x_1 = 2$, and $x_2 = 4$ then $x_1 \cdot x_2 = 8$ so $(r-1)/r = 8$ and $r = -1/7$.

If $x_1 = 0$ and $x_2 = -2$, then $x_1 \cdot x_2 = 0$, so $r = 1$.

Thus the possibilities for r are $0, 1, -1/7$.

6. The convex pentagon $ABCDE$ has a circumcircle. The perpendicular distances of A from the lines through B and C, C and D, and D and E are a, b and c, respectively. Determine (as a function of a, b and c) the perpendicular distance of A from the diagonal BE.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Joseph Ling, The University of Calgary. We give Bradley's answer.

Let R be the circumradius. Consider the area of triangle ABC. It can be expressed in two ways as

$$\frac{(AB)(BC)(CA)}{4R} = \frac{a}{2}BC.$$

Thus $a = (AB)(AC)/2R$. Similarly $b = (AC)(AD)/2R$ and $x = (AD)(AE)/2R$. It follows that the perpendicular distance P from A to BE is given by

$$P = \frac{(AB)(AE)}{2R} = \frac{ac}{b}\ .$$

* * *

That completes the solutions for problems from the April 1992 number of *Crux*, and makes a logical place to finish this month's column. This is the season of the Olympiads. Please send me your national and regional contests. Don't forget your nice solutions to problems from the Corner as well!

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Cariboo College High School Mathematics Contest Problems, 1973–1992, edited by Jim Totten, solution manual edited by Jim Totten and Leonard Janke, published privately (no ISBN) by Cariboo College, Kamloops, Canada, 1992. Softcover, 193+ pages, solution manual coil-bound, 246+ pages, \$14.95 apiece, plus \$1.50 postage and handling, and GST for Canadian orders. (Order from The Bookstore, University College of the Cariboo, Box 3010, Kamloops, British Columbia, V2C 5N3, Canada.) *Reviewed by Andy Liu.*

The Cariboo Contest consists of two papers and two rounds. The Junior Paper is for students in Grades 8 to 10, and the Senior Paper for those in Grades 11 and 12. The Preliminary Round consists exclusively of multiple-choice questions. The number of questions was 25 in 1973. This was reduced to 20 starting from 1974, and further to 15 starting from 1981. The Final Round consists of 10 multiple-choice questions and 5 full-solution questions, except in two years. In 1973 when the contest began, there were no multiple-choice questions but 10 full-solution questions. In 1979, there were 10 multiple-choice questions and 6 full-solution questions.

There is considerable overlap between the Junior Paper and the Senior Paper. This is clearly indicated in the book. Up to and including 1982, the Final Rounds of the two papers are identical, and in 1977, so are their Preliminary Rounds. The duplicate papers are not reprinted, but other common questions are retained within the respective papers.

Among the over 1000 problems, one can expect all sorts: routine exercises and challenging problems, favourite classics and original compositions, formal statements and anecdotal presentations, and so on. The collection is a good reference, especially for people making up or writing competitions.

The cover of the book features this geometry problem. Squares $ABFE$, $BCHG$ and $CADJ$ are constructed outside triangle ABC. If the total area of the first two squares is half that of the hexagon $DEFGHJ$, compute angle ABC. The solution is given on the cover of the solution manual.

*   *   *   *   *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1841. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral with vertex angles A, B, C, D, and O is the intersection of the diagonals AC and BD. Show that $ABCD$ is a parallelogram if and only if $OA\sin A = OC\sin C$ and $OB\sin B = OD\sin D$.

1842. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For given $\lambda > 1$ and $x_1 \in (0,1)$, the sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_{n+1} = \lambda^{x_n} - 1, \quad n \geq 1.$$

Determine the set of all λ's such that $\sum_{n=1}^{\infty} x_n$ converges for every starting value $x_1 \in (0,1)$.

1843. *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D.M. Milošević, Pranjani, Yugoslavia.*

Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.

(i) Prove that

$$\sum \frac{a}{2A(s-a)} \geq \frac{9}{\pi}\ .$$

(ii)* It is obvious that also

$$\sum \frac{1}{A} \geq \frac{9}{\pi}\ .$$

Do these two summations compare in general?

1844. *Proposed by Jordi Dou, Barcelona, Spain.*

Let Ω be the circumcircle of triangle ABC, and let Γ be the parabola tangent to AB at B and to AC at C. Construct an equilateral triangle XYZ so that X lies on AB, Y on AC and Z on Ω, and XY is tangent to Γ.

1845. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K..*

Suppose that x_1, x_2, x_3, x_4, x_5 are real numbers satisfying $x_1 < x_2 < x_3 < x_4 < x_5$ and

$$\sum_i x_i = 10, \quad \sum_{i<j} x_i x_j = 35, \quad \sum_{i<j<k} x_i x_j x_k = 50, \quad \sum_{i<j<k<l} x_i x_j x_k x_l = 25.$$

Prove that

$$\frac{5+\sqrt{5}}{2} < x_5 < 4.$$

1846. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Consider the three excircles of a given triangle ABC. Let $A'B'C'$ be the triangle containing these three circles and whose sides are each tangent to two of the circles. Prove that $[A'B'C'] \geq 25[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

1847. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

The points $(0,0) = O$, $(a,0)$, $(0,b)$, (a,b) are the corners of an $a \times b$ rectangle. For a point Z in the interior of the rectangle, draw the vertical and horizontal lines through Z, and let them meet the sides of the rectangle at points P, P' and Q, Q' respectively. Define $X = PQ' \cap P'Q$ and $Y = PQ \cap P'Q'$. Prove that the vector whose x- and y-coordinates are the slopes of the lines OX and OY, respectively, is orthogonal to the vector whose coordinates are the slopes of lines ZX and ZY.

1848. *Proposed by Neven Jurić, Zagreb, Croatia.*

Suppose that $x_1, x_2, \ldots, x_n$ are integers in $\{1, 2, \ldots, n\}$ such that

$$x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2} \quad \text{and} \quad x_1 x_2 \ldots x_n = n!\,.$$

Must $x_1, \ldots, x_n$ be a permutation of $1, 2, \ldots, n$?

1849. *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*

Let three points P, Q, R be on the sides BC, CA, AB, respectively, of a triangle ABC, such that they cut the perimeter of ΔABC into three equal parts; i.e. $QA + AR = RB + BP = PC + CQ$.

(a) Prove that

$$RP \cdot PQ + PQ \cdot QR + QR \cdot RP \geq \frac{1}{12}(a+b+c)^2.$$

(b)* Prove or disprove that the circumradius of ΔPQR is at least half the circumradius of ΔABC.

1850. *Proposed by Esteban Indurain, Universidad Pública de Navarra, Pamplona, Spain.*

Given a square in the plane, divide it into nine congruent smaller squares by lines parallel to the sides, and remove the interior of the cross-shaped central region; four isolated squares remain. Do the same with each of these squares; sixteen isolated squares remain. Do the same with each of these squares, and continue this process indefinitely. There are some points of the original square that still remain. Prove that any line parallel to a diagonal of the original square, and intersecting the original square, must pass through one of these remaining points.

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1636*. [1991: 114; 1992: 123] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the set of all real exponents r such that

$$d_r(x,y) = \frac{|x-y|}{(x+y)^r}$$

satisfies the triangle inequality

$$d_r(x,y) + d_r(y,z) \ge d_r(x,z) \quad \text{for all } x,y,z > 0$$

(and thus induces a metric on $\mathbf{R}^+$—see *Crux* 1449, esp. [1990: 224]).

II. *Comment by Murray S. Klamkin, University of Alberta.*

In an editorial note [1992: 125], the problem of whether or not

$$d'_r(x,y) = \frac{|x-y|}{(|x|+|y|)^r}$$

satisfies the triangle inequality for $x, y \in \mathbb{R} - \{0\}$, $r \in [0,1] \cup \{-1\}$, was raised. Here we show it satisfies the triangle inequality (and so is a metric) for $r \in [0,1]$.

Since $d'_r(x,y) = d'_r(-x,-y)$, the case $x,y,z \le 0$ is equivalent to the case $x,y,z \ge 0$ which is *Crux* 1636 and was taken care of previously. Since the case $x, y \ge 0$, $z \le 0$ is equivalent to the case $x \ge 0$, $y,z \le 0$, we need only check when

$$\frac{x-y}{(x+y)^r}, \quad \frac{y+z}{(y+z)^r}, \quad \frac{x+z}{(x+z)^r} \tag{1}$$

satisfy the triangle inequality for $x \ge y \ge 0$ and $z \ge 0$.

For $r = -1$, (1) becomes

$$x^2 - y^2, \quad (y+z)^2, \quad (x+z)^2,$$

and since $x > y$ implies

$$(x+z)^2 > (x^2 - y^2) + (y+z)^2$$

just by multiplying out, $d'_{-1}(x,y)$ is not a metric. For $r = 0$ or 1, $d'_r(x,y)$ is obviously a metric. We are now left with the case $0 < r < 1$. Since

$$\frac{x-y}{(x+y)^r} \le \frac{x}{x^r},$$

it follows that the largest of the three numbers in (1) is $(x+z)/(x+z)^r$, so we need only show that

$$\frac{x-y}{(x+y)^r} \ge \frac{x+z}{(x+z)^r} - \frac{y+z}{(y+z)^r} = (x+z)^{1-r} - (y+z)^{1-r}.$$

Since the right hand side of this inequality is decreasing in z, it suffices to show that

$$\frac{x-y}{(x+y)^r} \geq x^{1-r} - y^{1-r}$$

or equivalently (putting $y = x\lambda$, $s = 1-r$) that

$$\frac{1-\lambda}{1+\lambda} \geq \frac{1-\lambda^s}{(1+\lambda)^s}\,, \quad 0 < s < 1, \quad 0 < \lambda < 1 \tag{2}$$

(the cases for $\lambda = 0$ and 1 are trivial).

We now show that (2) is actually valid for $0 \leq s \leq 1$ and $s \geq 2$, and that for $1 < s < 2$ the inequality goes the other way. Let

$$F(s) = \ln\left(\frac{1-\lambda^s}{(1+\lambda)^s}\right).$$

It follows from

$$F''(s) = -\frac{\lambda^s(\ln\lambda)^2}{(1-\lambda^s)^2} < 0$$

that $F(s)$ is concave for $s \geq 0$. Since

$$F(1) = F(2) = \ln\left(\frac{1-\lambda}{1+\lambda}\right),$$

the rest follows.

Editor's note. Would anyone care to tackle when the triangle inequality holds for

$$d_{r,s}(x,y) = \frac{|x-y|}{(|x|^s + |y|^s)^r}\ ?$$

* * * * *

1749. [1992: 140] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Let ABC be a triangle with external angle-bisectors w_a', w_b', w_c', inradius r, and circumradius R. Prove that

(i) $\left(\sqrt{\dfrac{1}{w_a'}} + \sqrt{\dfrac{1}{w_b'}} + \sqrt{\dfrac{1}{w_c'}}\right)^2 < \dfrac{2}{r}$;

(ii) $\left(\dfrac{1}{w_a'} + \dfrac{1}{w_b'} + \dfrac{1}{w_c'}\right)^2 < \dfrac{R}{3r^3}$.

I. *Solution to part* (i) *by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

From

$$w_a \cos\frac{|B-C|}{2} = h_a = c\sin B = 2R\sin C\sin B, \qquad \text{etc.},$$

where h_a, h_b, h_c are the altitudes, and

$$w_a = w_a' \tan\frac{|B-C|}{2}\,, \qquad \text{etc.},$$

we know

$$w_a' = \frac{2R\sin B\sin C}{\sin(|B-C|/2)}\,, \qquad \text{etc.}$$

So the proposed inequality becomes

$$\left(\sum\sqrt{\frac{r\sin(|B-C|/2)}{2R\sin B\sin C}}\right)^2 < 2.$$

With the identities

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \qquad \text{and} \qquad \sin B\sin C = 4\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\,,$$

it becomes

$$\sum\sqrt{\frac{\sin(A/2)\sin(|B-C|/2)}{\cos(B/2)\cos(C/2)}} < 2. \tag{1}$$

Assume (without loss of generality) that $A \ge B \ge C$.

Case (a): $\tan(A/2)\tan(B/2) \le 4\sqrt{3}/9$. Then

$$\frac{\sin(A/2)}{\cos(B/2)\cos(C/2)}\cdot\left|\sin\left(\frac{B}{2}-\frac{C}{2}\right)\right| = \cos\frac{A}{2}\tan\frac{A}{2}\left|\tan\frac{B}{2}-\tan\frac{C}{2}\right|,$$

and by Cauchy's inequality and the inequality $\sum\cos(A/2) \le 3\sqrt{3}/2$ ([1], item 2.27) we have

$$\begin{aligned}
\left(\sum\sqrt{\frac{\sin(A/2)\sin(|B-C|/2)}{\cos(B/2)\cos(C/2)}}\right)^2 &= \left(\sum\sqrt{\cos\frac{A}{2}\tan\frac{A}{2}\left|\tan\frac{B}{2}-\tan\frac{C}{2}\right|}\right)^2 \\
&\le \left(\sum\cos\frac{A}{2}\right)\left(\sum\tan\frac{A}{2}\left|\tan\frac{B}{2}-\tan\frac{C}{2}\right|\right) \\
&\le \frac{3\sqrt{3}}{2}\left(2\tan\frac{A}{2}\tan\frac{B}{2} - 2\tan\frac{B}{2}\tan\frac{C}{2}\right) \\
&< 3\sqrt{3}\tan\frac{A}{2}\tan\frac{B}{2} \le 4,
\end{aligned}$$

so (1) follows.

Case (b): $\tan(A/2)\tan(B/2) > 4\sqrt{3}/9$. Let $k = \sqrt{1-4\sqrt{3}/9}$ and $\theta = (A-B)/4$. Then

$$\sin\frac{A}{2} + \sin\frac{B}{2} = 2\sin\frac{A+B}{4}\cos\frac{A-B}{4} < \sqrt{2}\cos\theta,$$

$$2\sin\frac{A}{2}\sin\frac{B}{2} = \cos\frac{A-B}{2} - \cos\frac{A+B}{2} < \cos 2\theta,$$

$$\sqrt{\sin\frac{A}{2}} + \sqrt{\sin\frac{B}{2}} \le \sqrt{2\left(\sin\frac{A}{2} + \sin\frac{B}{2}\right)} < \sqrt{2\sqrt{2}\cos\theta},$$

and

$$\frac{\sin(A/2)}{\cos(B/2)\cos(C/2)} = \frac{\cos(B/2+C/2)}{\cos(B/2)\cos(C/2)} = 1 - \tan\frac{B}{2}\tan\frac{C}{2},$$

so

$$\begin{aligned}
\left(\sum\sqrt{\frac{\sin(A/2)\sin(|B-C|/2)}{\cos(B/2)\cos(C/2)}}\right)^2 &= \left(\sum\sqrt{\left(1-\tan\frac{B}{2}\tan\frac{C}{2}\right)\left|\sin\frac{B-C}{2}\right|}\right)^2 \\
&< \left(\sqrt{\sin\frac{B}{2}} + \sqrt{\sin\frac{A}{2}} + k\sqrt{\sin\frac{A-B}{2}}\right)^2 \\
&= k^2\sin\frac{A-B}{2} + \sin\frac{A}{2} + \sin\frac{B}{2} \\
&\qquad + 2k\sqrt{\sin\frac{A-B}{2}}\left(\sqrt{\sin\frac{A}{2}} + \sqrt{\sin\frac{B}{2}}\right) + 2\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}} \\
&\le k^2\sin 2\theta + \sqrt{2}\cos\theta + 2k\sqrt{\sin 2\theta\cdot 2\sqrt{2}\cos\theta} + \sqrt{2\cos 2\theta} = f(\theta).
\end{aligned}$$

Now

$$\begin{aligned}
f'(\theta) &= 2k^2\cos 2\theta - \sqrt{2}\sin\theta + \frac{k\sqrt{2\sqrt{2}}(2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta)}{\sqrt{\sin 2\theta\cos\theta}} - \frac{\sqrt{2}\sin 2\theta}{\sqrt{\cos 2\theta}} \\
&= 2k^2\cos 2\theta - \sqrt{2}\sin\theta + \frac{k\sqrt{2\sqrt{2}}(2\cos^3\theta - 4\sin^2\theta\cos\theta)}{\cos\theta\sqrt{2\sin\theta}} - \frac{\sqrt{2}\sin 2\theta}{\sqrt{\cos 2\theta}} \\
&= 2k^2\cos 2\theta - \sqrt{2}\sin\theta + \frac{2k\sqrt[4]{2}\cos^2\theta}{\sqrt{\sin\theta}} - 4k\sqrt[4]{2}\sin^{3/2}\theta - \frac{\sqrt{2}\sin 2\theta}{\sqrt{\cos 2\theta}}\ .
\end{aligned}$$

It is easy to verify that $f''(\theta) < 0$ for $0 < \theta < \pi/4$ [in fact each term of $f''(\theta)$ is negative]. Since $f'(\theta) = 0$ at $\theta_\circ \approx 20.516°$, the maximum value of $f(\theta)$ is $f(\theta_\circ) \approx 3.9693498 < 4$. Thus (1) follows.

Reference:

[1] O. Bottema et al, *Geometric Inequalities.*

[*Editor's note.* Huang also solved part (ii).]

II. *Solution to part* (ii) *by the proposer.*

By Cauchy's inequality, we have

$$\left(\sum \frac{1}{w'_a}\right)^2 = \left(\sum \frac{h_a}{w'_a} \cdot \frac{1}{h_a}\right)^2 \leq \left(\sum \frac{h_a^2}{w'^2_a}\right)\left(\sum \frac{1}{h_a^2}\right). \tag{2}$$

Then, since

$$\sum \frac{1}{h_a^2} \leq \frac{R}{6r^3}$$

(see [1], page 202, IX.9.12) and

$$\sum \frac{h_a^2}{w'^2_a} = \sum \sin^2 \frac{B-C}{2} < \sin^2 \frac{B+C}{2} + 2\sin^2 \frac{A}{2} = \cos^2 \frac{A}{2} + 2\sin^2 \frac{A}{2} = 1 + \sin^2 \frac{A}{2} < 2,$$

(2) implies

$$\left(\sum \frac{1}{w'_a}\right)^2 < \frac{R}{3r^3} .$$

Reference:

[1] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, Dordrecht, 1989.

III. *Solution to part* (ii) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Using

$$w'_a = \frac{h_a}{\sin(|B-C|/2)} = \frac{2F}{a\sin(|B-C|/2)} = \frac{2rs}{a\sin(|B-C|/2)},$$

(ii) reads equivalently

$$\left(\sum a \sin \frac{|B-C|}{2}\right)^2 < \frac{4Rs^2}{3r} . \tag{3}$$

The Cauchy–Schwarz inequality yields

$$\left(\sum a \sin \frac{|B-C|}{2}\right)^2 \leq \left(\sum a^2\right)\left(\sum \sin^2 \frac{B-C}{2}\right). \tag{4}$$

We now replace the sums on the right hand side of (4) by their R, r, s expressions; i.e.,

$$\sum a^2 = 2(s^2 - r^2 - 4Rr)$$

(see [2], page 52, (5)) and

$$\begin{aligned}\sum \sin^2 \frac{B-C}{2} &= 3 - \sum \cos^2 \frac{B-C}{2} = 3 - \frac{3}{4} - \frac{1}{4}\left[\left(4\prod \cos \frac{A}{2}\right)^2 + \left(1 + 4\prod \sin \frac{A}{2}\right)^2\right] \\ &= \frac{1}{4}\left[9 - \frac{s^2}{R^2} - \left(1 + \frac{r}{R}\right)^2\right] = \frac{1}{4R^2}(8R^2 - s^2 - r^2 - 2Rr)\end{aligned}$$

(see [2], p. 95, proof of (16) and p. 57, (52) and (56)), and hence (4) becomes

$$\left(\sum a\sin\frac{|B-C|}{2}\right)^2 \le \frac{1}{2R^2}(s^2-r^2-4Rr)(8R^2-s^2-r^2-2Rr). \tag{5}$$

Now

$$s^2 \ge 16Rr-5r^2$$

([1], item 5.8). Hence the right-hand expression of (5) is dominated by

$$\frac{1}{2R^2}s^2(8R^2-18Rr+4r^2) = \frac{s^2}{R^2}(4R^2-9Rr+2r^2).$$

Using this with (3) and (5), it is enough to prove

$$\frac{s^2}{R^2}(4R^2-9Rr+2r^2) \le \frac{4Rs^2}{3r}\ .$$

We now show how to improve this inequality. Consider

$$\frac{s^2}{R^2}(4R^2-9Rr+2r^2) \le \frac{4\lambda Rs^2}{r}\ ; \tag{6}$$

we want to find the smallest value of λ for which (6) holds. Putting $t = r/R \le 1/2$ we get for (6):

$$f(t) := 4t-9t^2+2t^3 \le 4\lambda.$$

Now $f'(t) = 4-18t+6t^2$ equals zero for

$$t = \frac{9-\sqrt{57}}{6}\ \left(<\frac{1}{2}\right),$$

so the maximum value of $f(t)$ for $0<t<1/2$ is

$$f\left(\frac{9-\sqrt{57}}{6}\right) = \frac{19\sqrt{57}-135}{18}$$

and the optimal value of λ is

$$\frac{19\sqrt{57}-135}{72} \approx 0.1173 < \frac{1}{8}\ .$$

In terms of the original part (ii), this finally leads to

$$\left(\sum\frac{1}{w'_a}\right)^2 < \frac{R}{r^3}\left(\frac{19\sqrt{57}-135}{72}\right)\left[<\frac{R}{8r^3}\right].$$

References:
[1] O. Bottema et al, *Geometric Inequalities.*
[2] Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities.*

One other reader misinterpreted "external angle bisectors" as being the sides of the triangle formed by the external angle bisectors, rather than as the segments from A to BC, etc., taken along the external angle bisectors, but proved the given inequalities nevertheless!

In their (otherwise correct) solutions of part (i)*, both Janous and the proposer used the inequality*

$$\sum \sin\frac{|B-C|}{2} < 2.$$

However this inequality is **false***, the sum seeming to reach its peak (of about* 2.1*) when* $A \approx 157°$*,* $B \approx 23°$*,* $C = 0°$*. (The inequality is true, and so are their solutions of part* (i)*, for triangles with no angle greater than* 120°*, say.) A shorter proof of part* (i) *would be desirable.*

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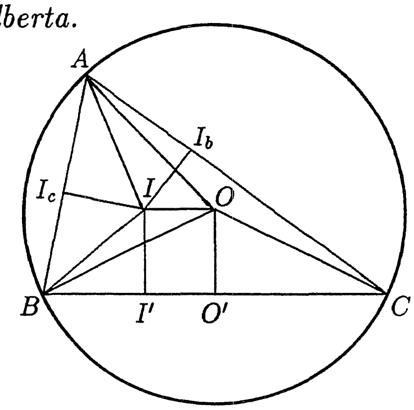
1751. [1992: 175] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with incenter I and circumcenter O. Suppose that $AB < AC$ and $IO = \frac{1}{2}(AC - AB)$. Prove that

$$\text{area}(\Delta IAO) = \frac{1}{2}[\text{area}(\Delta BAO) - \text{area}(\Delta CAO)].$$

Combination of solutions by Jordi Dou, Barcelona, Spain, and John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.



Let a, b, c be the sides, s the semi-perimeter, and r the inradius of ΔABC. Let I_b, I_c be the projections of I onto AC and AB respectively, and let $I'O'$ be the projection of IO onto BC. Then

$$\begin{aligned} 2IO &= AC - AB = I_bC - I_cB \\ &= I'C - I'B = 2I'O', \end{aligned}$$

so $IO \| BC$. Now

$$\text{area}(\Delta ABI + \Delta BCI + \Delta CAI) = \text{area}(\Delta ABO + BCO + CAO),$$

so since $II' = OO'$ implies that area $(\Delta BCI) = \text{area}(\Delta BCO)$, we have

$$\text{area}(\Delta ABO + \Delta CAO) = \text{area}(\Delta ABI + \Delta ACI) = \frac{1}{2}(rc + rb).$$

From this and

$$IO = I'O' = O'B - I'B = \frac{a}{2} - \frac{a+c-b}{2} = \frac{b-c}{2}$$

it follows that

$$\text{area}(\Delta BAO) = \text{area}(\Delta ABI + \Delta OBI + \Delta IAO) = \frac{rc}{2} + \frac{r}{2}\left(\frac{b-c}{2}\right) + \text{area}(\Delta IAO)$$
$$= \frac{r(c+b)}{4} + \text{area}(\Delta IAO) = \frac{1}{2}\text{area}(\Delta BAO + \Delta CAO) + \text{area}(\Delta IAO).$$

Hence

$$\text{area}(\Delta IAO) = \frac{1}{2}[\text{area}(\Delta BAO) - \text{area}(\Delta CAO)].$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; C. FESTRAETS-HAMOIR, Brussels, Belgium; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The nice fact $IO \| BC$ given above was also observed by Smeenk and the proposer. Another consequence of the property $2IO = AC - AB$, noted by several solvers, is that $\cos B + \cos C = 1$*; readers may enjoy proving this for themselves.*

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1752*. [1992: 175] *Proposed by Murray S. Klamkin, University of Alberta.*

If A and B are positive integers and p is a prime such that $p \mid A$, $p^2 \nmid A$ and $p^2 \mid B$, then the arithmetic progression

$$A, A+B, A+2B, A+3B, \ldots$$

contains no terms which are perfect powers (squares, cubes, etc.). Are there any infinite non-constant arithmetic progressions of positive integers, with no term a perfect power, which are *not* of this form?

I. *Solution by Margherita Barile, student, Universität Essen, Germany.*

The answer is affirmative. Let p be a prime, $p > 2$. Since $1^2 \equiv (p-1)^2 \equiv 1 \bmod p$, there is r, $0 < r < p$, such that $a^2 \not\equiv r \bmod p$ for all $a = 0, \ldots, p-1$. Let $A = p^2 r^p$, $B = p^3$. Then, for all i

$$A + iB = p^2 r^p + ip^3 = p^2(r^p + ip).$$

The greatest power of p dividing $A + iB$ is p^2. Hence, if $A + iB$ is a perfect power, it is a square. Then there is an integer s such that

$$r^p + ip = s^2.$$

But then by Fermat

$$s^2 \equiv r^p \equiv r \bmod p,$$

which provides a contradiction. Thus the arithmetic progression $A, A+B, A+2B, \ldots$ contains no perfect powers. But it is not of the given form. In fact, our proof shows how to construct infinitely many such arithmetic progressions. We give one example. For $p = 3$, one gets $r = 2$; then $A = 72$, and $B = 27$.

II. *Solution by Leroy F. Meyers, The Ohio State University.*

For each nonnegative integer m let $x_m = A + mB$.

The proof of the first statement of the problem is trivial. (In fact, it may not have been intended to be part of the problem.) Suppose that p is a prime and that $p \mid A$, $p^2 \nmid A$, and $p^2 \mid B$. Then $x_m \equiv A \not\equiv 0 \bmod p^2$, so that $p \mid x_m$ and $p^2 \nmid x_m$. But if x_m is a perfect kth power, then p^k must divide x_m, which is impossible if $k > 1$.

Two cases of the converse are considered.

If no prime dividing A occurs to a higher power in B than in A, then A is relatively prime to $C = B/\gcd(A, B)$. By Euler's theorem we have

$$A^{\phi(C)} \equiv 1 \bmod C,$$

so that

$$A^{1+\phi(C)} \equiv A \bmod AC.$$

but $AC = AB/\gcd(A, B) = \operatorname{lcm}(A, B)$, and so $B \mid AC$. Hence

$$A^{1+\phi(C)} \equiv A \bmod B,$$

and $A^{1+\phi(C)}$ is the required perfect power in the arithmetic sequence.

However, if A is divisible by a prime which occurs to a higher power in B, then there may be no perfect power in the sequence. For example, let $A = 12 = 2^2 \cdot 3$ and $B = 16 = 2^4$. If

$$12 + 16m = s^k \quad \text{for some positive integer } k,$$

then s must be even, in which case $12 + 16m$ must be divisible by 2^k, which is impossible if $k > 2$. However, if $k = 2$ and $s = 2t$, then

$$3 + 4m = t^2,$$

which also is impossible, since a perfect square must be congruent to either 0 or 1 modulo 4. Thus there can be a nonconstant perfect-power-free arithmetic sequence not of the specified form.

Also solved by CHRIS WILDHAGEN, Rotterdam, The Netherlands, whose counter-example was similar to Meyers'.

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1753. [1992: 175] *Proposed by Jordi Dou, Barcelona, Spain.*

Given points P_i, $i = 1, 2, 3, 4$, and line ℓ in the plane, find a pair of points X, Y so that points $X_i = XP_i \cap \ell$ and $Y_i = YP_i \cap \ell$ are symmetric on ℓ (i.e., the midpoints of X_iY_i coincide). Also prove that for all such X, Y the lines XY pass through a fixed point.

Solution by Toshio Seimiya, Kawasaki, Japan.

Let P_∞ be the point at infinity on ℓ, and let Γ be the conic through the five points P_1, P_2, P_3, P_4, and P_∞. (Γ is a hyperbola or parabola.) [*Editorial remarks by Chris Fisher:* (1) *to the nonprojectively inclined*: One can assume, with little loss of generality, that ℓ is the line $y = 1$ and that the P_i lie on the hyperbola $xy = 1$; (2) *to the projectively inclined*: Γ might also be degenerate, in which case the solver's argument goes through with only minor changes.] Let M be the point other than P_∞ where ℓ intersects Γ. We take two points X_1, Y_1 on ℓ such that M is the midpoint of X_1Y_1, and let X, Y be the second intersections of X_1P_1, Y_1P_1 with Γ. *Then X, Y are the points we are looking for.*

Proof. Let

$$X_i = XP_i \cap \ell \quad \text{and} \quad Y_i = YP_i \cap \ell, \quad i = 2, 3, 4.$$

Then, using standard arguments involving cross ratios and projectivities (as, for example, in Chapter 14 of Coxeter's *Introduction to Geometry*, or in any projective geometry text),

$$(P_1X, P_1Y; P_1M, P_1P_\infty) = (P_1X_1, P_1Y_1; P_1M, P_1P_\infty) = (X_1, Y_1; M, P_\infty) = -1,$$

and for $i = 2, 3, 4$,

$$\begin{aligned}(X_i, Y_i; M, P_\infty) &= (P_iX_i, P_iY_i; P_iM, P_iP_\infty) = (P_iX, P_iY; P_iM, P_iP_\infty) \\ &= (P_1X, P_1Y; P_1M, P_1P_\infty) = -1.\end{aligned}$$

Thus the midpoints of X_iY_i coincide with M.

Let T be the point on XY such that MT is tangent to Γ, and let $S = XY \cap \ell$. Then,

$$(X, Y; T, S) = (MX, MY; MT, MP_\infty) = (P_1X, P_1Y; P_1M, P_1P_\infty) = -1.$$

Therefore X, Y, T, S form a harmonic set, and $MS = \ell$ is the polar of T with respect to Γ. This implies that for any choice of X_1 and Y_1, XY passes through a fixed point (the pole of ℓ).

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; and the proposer.

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1754*. [1992: 175] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n and k be positive integers such that $2 \le k < n$, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n x_i = 1$. Prove or disprove that

$$\sum x_1x_2\ldots x_k \le \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1, x_2, \ldots, x_n$. [The case $k = 2$ is known — see inequality (1) in the solution of *Crux* 1662 [1992: 188].]

Comment by the Editor.

There were no solutions submitted for this problem. $k = 3$, anyone?

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1755. [1992: 176] *Proposed by Dan Pedoe, Minneapolis, Minnesota, and J. Chris Fisher and Robert E. Jamison, Clemson University, Clemson, South Carolina.*

(a) If $T_1 = \Delta A_1B_1C_1$ is a triangle inscribed in $T = \Delta ABC$, and each side of T_1 is one-half the corresponding side of T, prove that T_1 is the medial triangle of T (the triangle whose vertices are the midpoints of the sides of T).

(b) Define $\Delta A_1B_1C_1$ to be *properly inscribed* in ΔABC if the two triangles have the same orientation and A_1, B_1, C_1 are on the *lines* BC, CA, AB respectively. Prove that if $\Delta A_1B_1C_1$ is one such triangle, then there is exactly one other properly inscribed triangle congruent to it, with one exception. The exception occurs when the given inscribed triangle is the *pedal triangle* of some point M (i.e., when its vertices are the feet of the perpendiculars from M onto the respective sides of ΔABC); in this case there is no other inscribed triangle congruent to $\Delta A_1B_1C_1$.

Solution by the proposers.

We first look at part (b). Let $\Delta A_1B_1C_1$ be "properly inscribed" in ΔABC. Then the circles AB_1C_1, BC_1A_1, and CA_1B_1 have a point M in common, the *Miquel point of* $\Delta A_1B_1C_1$. This is described in detail in [1]. The proof, as with other proofs from Johnson that we shall assume, depends on the theorem that the opposite angles of a quadrangle inscribed in a circle sum to π. The Miquel point is sometimes referred to as the *pivot point* of the inscribed triangle, for good reasons as we shall see.

From the circle theorem just quoted we see that

$$\angle MA_1C = \angle MB_1A = \angle MC_1B.$$

It also follows that

$$\angle BMC = \angle BAC + \angle B_1A_1C_1,$$

with similar results for $\angle CMA$ and $\angle AMB$. (This is Theorem 186 in [1]; its proof, given by Johnson, makes a nice exercise.) Hence, *if the angles of* $\Delta A_1B_1C_1$ *are given, the point* M *is uniquely determined.*

Using the point M as a pivot, imagine the lines MA_1, MB_1, and MC_1 rotated, as if they formed a rigid structure, about the point M to a position where the given lines, extended if necessary, intersect BC, CA, and AB respectively in points A_2, B_2, C_2. Then since $\angle B_1MC_1 = \angle B_2MC_2$ $(= \pi - \angle A)$, with similar results for $\angle C_1MA_1$ and $\angle A_1MB_1$, the points CA_2MB_2 lie on a circle, as do AB_2MC_2 and BC_2MA_2, so that M is also the Miquel point of $\Delta A_2B_2C_2$. Our final theorem is that $\Delta A_2B_2C_2$ is similar to $\Delta A_1B_1C_1$. This follows from the theorem that $\angle B_1A_1C_1$ and $\angle B_2A_2C_2$ are both equal to $\angle MCA + \angle MBA$, and similarly for the angles on CA and on AB. The term "pivot" is therefore fully justified.

If we drop perpendiculars from M to the sides of ΔABC, obtaining $\Delta A_0B_0C_0$ (the *pedal* triangle of M), then a rotation about M through an angle α, say, will produce

$\Delta A_1B_1C_1$. This will be similar to $\Delta A_0B_0C_0$; in fact, $A_0B_0 = (\cos\alpha)A_1B_1$; and if $\Delta A_0B_0C_0$ does not coincide with our first inscribed triangle, then it will be smaller. Of course, a rotation of the pedal triangle through an angle $-\alpha$ will produce a triangle congruent to $\Delta A_1B_1C_1$, and there are no others. Thus, it is only when $\alpha = 0$ (so that $A_1 = A_0$, $B_1 = B_0$, $C_1 = C_0$) that there is no other properly inscribed triangle that is congruent to $\Delta A_1B_1C_1$. This completes the discussion of part (b).

For part (a), the midpoints of the sides of ΔABC form a triangle whose Miquel point is the circumcentre of the given triangle; that is, the medial triangle is the pedal triangle $A_0B_0C_0$ of the circumcentre. (Of course, the medial triangle has sides just half those of the given ΔABC.) As we have just seen, this pedal triangle is smaller than any similar triangle properly inscribed in ΔABC, so the claim in (a) follows immediately.

Reference:

[1] Roger A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, paragraphs 184–190.

Also solved by JORDI DOU, Barcelona, Spain. P. PENNING, Delft, The Netherlands, solved a representative special case. ROBIN HUR, Albuquerque, New Mexico, and MURRAY S. KLAMKIN, University of Alberta, comment that the "pedal simplex" of a simplex in n dimensions is easily seen to be the smallest among those similar to it that are "properly inscribed" in the given simplex.

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1757. [1992: 176] *Proposed by Avinoam Freedman, Teaneck, N.J.*

Let $A_1A_2A_3$ be an acute triangle with sides a_1, a_2, a_3 and area F, and let $\Delta B_1B_2B_3$ (with sides b_1, b_2, b_3) be inscribed in $\Delta A_1A_2A_3$ with $B_1 \in A_2A_3$, etc. Show that for any $x_1, x_2, x_3 > 0$,

$$(x_1a_1^2 + x_2a_2^2 + x_3a_3^2)(x_1b_1^2 + x_2b_2^2 + x_3b_3^2) \geq 4F^2(x_2x_3 + x_3x_1 + x_1x_2).$$

Solution by Murray S. Klamkin, University of Alberta.

The special case when $x_1 = 1/b_1$, $x_2 = 1/b_2$, $x_3 = 1/b_3$ yields the inequality

$$b_2b_3a_1^2 + b_3b_1a_2^2 + b_1b_2a_3^2 \geq 4F^2, \tag{1}$$

the proof of which was problem E3154 in the *American Math. Monthly*, proposed by George Tsintsifas. For the proof here, we employ the known results about the Miquel point (see p. 131 of [1]) which were used in the proof of E3154 by Jiro Fukuta (see pp. 659–660 of the 1988 *Monthly*).

The circumcircles of $A_1B_2B_3$, $A_2B_3B_1$, $A_3B_1B_2$ are concurrent in the Miquel point denoted by P. Let the radii of these circles and the circumcircle of $A_1A_2A_3$ be x, y, z and R, respectively. We then have

$$a_1 = 2R\sin A_1, \quad a_2 = 2R\sin A_2, \quad a_3 = 2R\sin A_3,$$

$$b_1 = 2x\sin A_1, \quad b_2 = 2y\sin A_2, \quad b_3 = 2z\sin A_3,$$

and

$$F = 2R^2 \sin A_1 \sin A_2 \sin A_3.$$

Since

$$2x \geq PA_1 = R_1, \quad 2y \geq PA_2 = R_2, \quad 2z \geq PA_3 = R_3,$$

it suffices to show that

$$(x_1 \sin^2 A_1 + x_2 \sin^2 A_2 + x_3 \sin^2 A_3)(x_1 R_1^2 \sin^2 A_1 + x_2 R_2^2 \sin^2 A_2 + x_3 R_3^2 \sin^2 A_3) \\ \geq 4R^2 \sin^2 A_1 \sin^2 A_2 \sin^2 A_3 \, (x_2x_3 + x_3x_1 + x_1x_2).$$

The latter inequality is just a variation of the known polar moment of inertia inequality [1989: 28], i.e.,

$$(x_1 + x_2 + x_3)(x_1 R_1^2 + x_2 R_2^2 + x_3 R_3^2) \geq x_2x_3a_1^2 + x_3x_1a_2^2 + x_1x_2a_3^2$$

(just change x_i into $x_i \sin^2 A_i$ and replace a_i by $2R \sin A_i$).

Other special cases of the given inequality: letting $x_i = 1/a_i$, we get

$$a_2a_3b_1^2 + a_3a_1b_2^2 + a_1a_2b_3^2 \geq 4F^2$$

[compare with (1)!]; and letting $x_i = 1$, we get

$$b_1^2 + b_2^2 + b_3^2 \geq \frac{12F^2}{a_1^2 + a_2^2 + a_3^2}$$

(see item 1.6, p. 344 of [2]). This corresponds to the known theorem (p. 217 of [1]) that, of all triangles inscribed in a given triangle, the one for which the sum of the squares of the sides is a minimum is the pedal triangle of the symmedian point.

References:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960.

[2] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

Also solved by the proposer.

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1759. [1992: 176] *Proposed by Isao Ashiba, Tokyo, Japan.*

A is a fixed point on a circle, and P and Q are variable points on the circle so that $AP + PQ$ equals the diameter of the circle. Find P and Q so that the area of ΔAPQ is as large as possible.

I. *Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

The maximum area of ΔAPQ occurs when $AP = PQ = R$ (the radius of the circle), $A \neq Q$; the corresponding area is $R\sqrt{3}/4$.

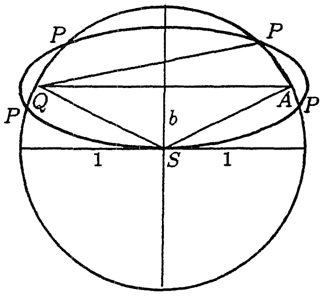

Figure 1

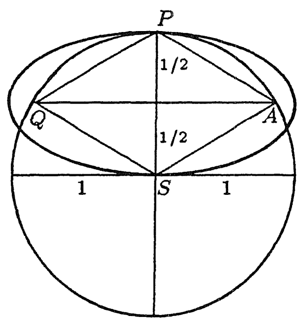

Figure 2

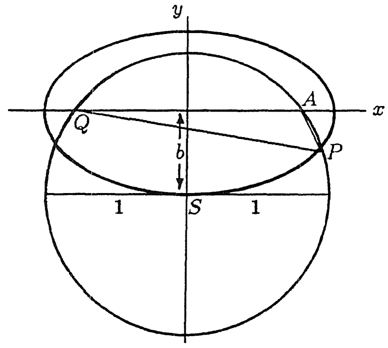

Figure 3

Consider the unit circle ($R = 1$), and for given A, P, Q let b be the distance of line AQ from the center S of the circle. Then P must lie on the ellipse with foci A and Q, semiaxes 1 and b and which passes through S (Figure 1), since $AP + PQ = 2 = AS + SQ$. For $0 < b \le 1/2$, area$(APQ) \le$ area$(ASQ) \le \sqrt{3}/4$ [note this is true whether P lies above AQ or between AQ and S], with equality only if $b = 1/2$ (Figure 2); thus for $b \le 1/2$ the solution as stated above is clearly true.

If $1/2 < b \le 1$ and P is between S and AQ as in Figure 3, we shall show now that the area of ΔAPQ is again less than $\sqrt{3}/4$, which will complete the proof. Let the x and y axes correspond to the axes of the ellipse as in Figure 3; then the equations of the ellipse and the circle are respectively

$$b^2x^2 + y^2 = b^2 \quad \text{and} \quad b^2x^2 + b^2(y+b)^2 = b^2.$$

Subtracting the equations we get

$$0 = b^2(y+b)^2 - y^2 = [b(y+b) - y][b(y+b) + y] = [y(b-1) + b^2][y(b+1) + b^2],$$

and thus the y coordinate of P is $-b^2/(b+1)$. Therefore

$$\text{area}(APQ) = \frac{1}{2} \cdot 2\sqrt{1-b^2} \cdot \frac{b^2}{b+1} = b \cdot \frac{b}{1+b} \cdot \sqrt{1-b^2} < 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\,.$$

II. *Solution by P. Penning, Delft, The Netherlands.*

Let A, P, Q be the angles of ΔAPQ, and the circumradius unity. Then $PQ = 2\sin A$ and $AP = 2\sin Q$, so $AP + PQ =$ diameter $= 2$ means

$$\sin A + \sin Q = 1. \tag{1}$$

Introduce

$$u = \frac{Q+A}{2}\,, \qquad v = \frac{Q-A}{2}\,;$$

then the constraint (1) reads

$$2 \sin u \cos v = 1. \tag{2}$$

Note that $u \geq 30°$. Now the area S of APQ satisfies

$$\begin{aligned} S &= 2\sin A \sin Q \sin P = 2\sin(u-v)\sin(u+v)\sin 2u \\ &= 4\sin u \cos u(\cos^2 v - \cos^2 u) = \frac{\cos u}{\sin u} - 4\cos^3 u \sin u \end{aligned}$$

by (2). The extremes are easily found by equating the derivative to zero:

$$\frac{dS}{du} = -\frac{1}{\sin^2 u} + 12\cos^2 u \sin^2 u - 4\cos^4 u = -\frac{4\cos^4 u \sin^2 u - 12\cos^2 u \sin^4 u + 1}{\sin^2 u} = 0.$$

Putting $m = 2\sin^2 u$, we must solve

$$\begin{aligned} 0 &= 4\left(1-\frac{m}{2}\right)^2 \cdot \frac{m}{2} - 12\left(1-\frac{m}{2}\right)\left(\frac{m^2}{4}\right) + 1 = 2m^3 - 5m^2 + 2m + 1 \\ &= (m-1)(2m^2 - 3m - 1). \end{aligned}$$

Since $m > 0$, the only two roots are:

$$m = 1 \Rightarrow u = v = 45° \Rightarrow A = 0°,\ Q = 90° \Rightarrow S = 0 \qquad \text{(a minimum)}$$

and

$$\begin{aligned} m = \frac{3+\sqrt{17}}{4} \quad &\Rightarrow u \approx 70.666°,\ v \approx 58° \Rightarrow A \approx 12.6634°,\ Q \approx 128.6683° \\ &\Rightarrow S \approx 0.21389 \qquad \text{(local maximum).} \end{aligned}$$

At the boundary:

$$u = 30°,\ v = 0° \Rightarrow A = Q = 30°,\ P = 120° \Rightarrow S = \sqrt{3}/4 \qquad \text{(absolute maximum).}$$

So the area is maximal if A, P, Q are consecutive vertices of the regular hexagon inscribed in the circle.

III. *Solution by Murray S. Klamkin, University of Alberta.*

More generally, we consider the problem of maximizing the area of an inscribed triangle in a given circle given the sum of two of its sides. If the angles subtended at the center by the two sides are $2x$ and $2y$, then we want to maximize

$$S = \sin 2x + \sin 2y - \sin(2x+2y)$$

where $\sin x + \sin y = 2a$ and a is a given constant in $(0,1)$ (here $S = 2\Delta/R^2$ where Δ is the area and R is the radius of the circle). Equivalently, putting $u = \sin x$ and $v = \sin y$,

$$\frac{S}{4} = \sin x \sin y \sin(x+y) = u\sqrt{u^2v^2 - u^2v^4} + v\sqrt{u^2v^2 - u^4v^2}\,.$$

By the weighted power mean inequality,

$$\frac{S}{8a} = \frac{u\sqrt{u^2v^2 - u^2v^4} + v\sqrt{u^2v^2 - u^4v^2}}{u+v}$$
$$\le \sqrt{\frac{u(u^2v^2 - u^2v^4) + v(u^2v^2 - u^4v^2)}{u+v}} = \sqrt{u^2v^2 - u^3v^3}\,,$$

with equality if and only if $u = v$. The function $t^2 - t^3$ has a maximum value when $t = 2/3$ and is increasing for $0 \le t \le 2/3$. Since $u + v = 2a$, $uv \le a^2$. Hence if $a^2 \le 2/3$, S will take on its maximum value when uv is as large as possible (namely $uv = a^2$), and

$$\max \Delta = 4R^2a^3\sqrt{1-a^2}\,.$$

For the proposed problem, $a = 1/2$, so $\max\Delta = R^2\sqrt{3}/4$, the maximum area triangle corresponding to an isosceles triangle whose equal sides are two successive sides of a regular hexagon.

It is conjectured that even for $2/3 \le a^2 \le 1$ the area will be maximized when the triangle is isosceles.

More generally, it is conjectured that the maximum area inscribed $(n+1)$-gon in a given circle given the sum of the lengths of n sides occurs when all these sides are equal. This is equivalent to maximizing

$$S = \sin 2x_1 + \sin 2x_2 + \cdots + \sin 2x_n - \sin(2x_1 + 2x_2 + \cdots + 2x_n)$$

subject to

$$\sin x_1 + \sin x_2 + \cdots + \sin x_n = \text{ constant.}$$

The proof or disproof is left as an open problem.

Also solved by H.L. ABBOTT, University of Alberta; C.J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, The Netherlands; P.E. TSAOUSSOGLOU, Athens, Greece; and the proposer. Two incorrect solutions were sent in.

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1760. [1992: 177] *Proposed by Ray Killgrove, San Diego State University.*

Find all positive integers B so that $(111)_B = (aabbcc)_6$, where a, b, c represent distinct base 6 digits, $a \ne 0$.

Solution by Kaaren May, student, St. John's, Newfoundland.

The given equation is equivalent to

$$B^2 + B + 1 = 6^4(6+1)a + 6^2(6+1)b + (6+1)c. \tag{1}$$

Therefore $B^2 + B + 1 \equiv 0 \bmod 7$, giving $B = 7n + 2$ or $B = 7n + 4$ for some integer n. Thus we obtain from (1)

$$(7n+2)^2 + (7n+2) + 1 = 6^4 \cdot 7a + 6^2 \cdot 7b + 7c$$

or

$$(7n+4)^2 + (7n+4) + 1 = 6^4 \cdot 7a + 6^2 \cdot 7b + 7c,$$

that is,

$$7n^2 + 5n + 1 = 1296a + 36b + c \tag{2}$$

or

$$7n^2 + 9n + 3 = 1296a + 36b + c, \tag{3}$$

respectively. Since $0 < a \leq 5$ and $0 \leq b, c \leq 5$, we can take a to be a constant and then solve (2) and (3) for all positive integer values of n on the interval $[1296a, 1296a + 185]$. Given the above restrictions, it is then possible to solve for b and c.

		(2)				(3)			
a	Interval	n	$7n^2+5n+1$	$36b+c$	Solution	n	$7n^2+9n+3$	$36b+c$	Solution
1	[1296,1481]	14	1443	147	$b=4, c=3$; $B=$**100**	13	1303	7	none
2	[2592,2777]	19	2623	31	none	19	2701	109	$b=3, c=1$; $B=$**137**
3	[3888,4073]	none				23	3913	25	none
4	[5184,5369]	27	5239	55	none	27	5349	165	none
5	[6480, 6665]	none				30	6573	93	none

Therefore, the only solutions are

$$(111)_{100} = (114433)_6 \qquad \text{and} \qquad (111)_{137} = (223311)_6.$$

Also solved by H.L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KIM LEE LIAN, Messiah College, Grantham, Pennsylvania; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. A partial solution was sent in by JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China.

Actually, the proposer's original problem had some additional conditions which eliminated one of the two solutions given above.

Janous wonders for which bases λ *the relation* $(111)_B = (aabbcc)_\lambda$ *has a solution. Besides* $\lambda = 6$, *he finds two:* $\lambda = 18$ *(*$B = 2006$, $a = 2$, $b = 6$, $c = 1$, *plus two more solutions) and* $\lambda = 20$ *(*$B = 2608$, $a = 2$, $b = 10$, $c = 13$*).*

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Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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