Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 4, No. 3

March 1978

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education
Publié par le Collège Algonquin

(32 issues of this journal, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name EUREKA.)

CRUX MATHEMATICORUM is published monthly (except July and August). The yearly subscription rate for ten issues is $8.00 in Canadian or U.S. dollars ($1.50 extra for delivery by first-class mail). Back issues: $1.00 each. Bound volumes: Vol. 1-2 (combined), $10.00; Vol. 3, $10.00. Cheques or money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, permission to reprint, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Nancy Makila.

*　　　*　　　*

CONTENTS

*　　　*　　　*

For Vol. 4 (1978), the support of Algonquin College and of the Samuel Beatty Fund is gratefully acknowledged.

THE POWER OF THE POWER CONCEPT

DAN PEDOE

If we have a circle with normalized equation

$$C(X,Y) \equiv X^2 + Y^2 + 2pX + 2qY + r = 0,$$

the *power* of the point $P(x,y)$ with regard to this circle is the number $C(x,y)$. If a line through P intersects the circle in the points Q and R, then

$$C(x,y) = \overline{PQ} \cdot \overline{PR}.$$

If we have two circles C_1 and C_2, the locus of a point P, which moves so that the ratio of its powers with regard to the two given circles is a given number k, is given by the equation

$$C_1(X,Y) - kC_2(X,Y) = 0,$$

and this is a member of the coaxal system determined by the given circles. All this is treated in Chapter III of my book [1].

Dan Sokolowsky, Antioch College, raised the following question:

If C_1 and C_2 are circles intersecting in a point U*, and a chord through* U *intersects C_1 again in* V *and C_2 again in* W*, what is the locus of the midpoint* P *of the segment* VW?

The power of P with regard to C_1 is $\overline{PU} \cdot \overline{PV}$, and the power of P with regard to C_2 is $\overline{PU} \cdot \overline{PW}$; and if P is the midpoint of VW the ratio of these powers is

$$\frac{\overline{PU} \cdot \overline{PV}}{\overline{PU} \cdot \overline{PW}} = \frac{\overline{PV}}{\overline{PW}} = -1,$$

and therefore P moves on a circle through the intersections of the two given circles. Its centre bisects the join of the centres of the given circles.

If the segment VW is divided in a given ratio, the same method shows that the dividing point P moves on a circle of the coaxal system determined by C_1 and C_2, the centre of this circle dividing the join of the centres of C_1 and C_2 in the given ratio.

I have not seen this theorem before, and the nearest approach to the method I use is given in some books to prove the theorem:

QR *is a chord of a circle C which subtends a right angle at a given point* L. *Show that the midpoint* P *of* QR *moves on a circle of the coaxal system determined by C and the point-circle* L.

The proof, which is indicated in Exercise 27.3, p. 108, of my book [1], is to observe that the power of P with regard to the circle C is $-(PL)^2$, since $\overline{PQ} = -\overline{PR} = |PL|$. The power of P with regard to the point-circle L is $(PL)^2$, and so the theorem follows.

REFERENCE

1. Dan Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970.

School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455.

* * *

Reminiscences

APRÈS LA TRIGONOMÉTRIE, QUOI?

R. ROBINSON ROWE

In my high school, the mathematics curriculum ended with trigonometry in the 7th semester. Having mastered that, with nothing offered in the last semester, I presumed I now knew it all. There wasn't any more. Mother corrected me; there was more, much more in college. Until then, I had had no thought of college and at first the idea was frightening, but my concern was relieved when my counselor told me the minimum age for matriculating at the university[1] was 17. Being only 15, with a birthday in August, I would have to wait a year.

Then came a fortuitous sequence. There would be a Prom the last week of high school. When a committee decreed it would be formal, I was dismayed. Formal wear in 1912 was white tie and tails—and I was just a boy in short pants. Mother mentioned that casually in a letter to her sister, and back came a box with a note, "Here is Otto's[2] dress suit he wore at Harvard and outgrew; maybe it can be altered in time." It was—and for my first date I escorted a lovely classmate to the Prom. Then, having counted me out, the committee rescinded its decree—and I was the only one there in tails! Funny, I didn't feel conspicuous; everybody was out of step but me.

But the incident fostered an idea that grew and grew—now that I was outfitted for Harvard, I wanted to go there. I talked again with my counselor. There was no minimum age for matriculation after all, but my diploma wasn't enough: I would have

---

[1] University of Michigan, where my grandfather graduated in Engineering in 1863.
[2] Otto FitzAlan Hakes, Harvard, 1901.

to pass an entrance examination. Also, I had had only two years of language and would need two more. The only way you can get two more years of language in one year is to add 3rd and 4th year Latin to your two years in that language.

So I had an easy postgraduate year at high school: Cicero and Vergil together are easier than one at a time. I also opted for a third subject—Music Appreciation—and was pleasantly surprised to learn some more mathematics: binary arithmetic, harmonious ratios, and the remarkable tempered scale.[3]

There were other rewards in this extra year. My classmates that had dubbed me "Hayseed" were gone, and while that nickname wasn't intended to be complimentary, and was as acceptable as those of my contemporaries, like Red, Slim, Shrimp, Puke, Shorty, Skinny, Peewee, Squint, Jerk, Runt, Dizzy, etc., I was now seated with my sister's Class of 1913 where she called me "Bob", which I have been ever since.

And figures. In the country school, they were numerals; in geometry, triangles and circles; in trigonometry, sines and logarithms. Now I found others—girls. Perhaps two factors helped: (1) girls in the Class of 1913 and in my Latin classes were more nearly my own age; (2) they were at an age when their figures were becoming more interesting. I was no Lothario, but I capitalized on a gimmick. In the winter of 1912-13, boys squired their dates to dances and parties in the family Model T, Overland, Cartercar, Winton or whatever. On cold nights, they drained the radiator—there was no antifreeze. *Après le bal*, while their dates waited inside, they refilled their radiators, with warm water if available, then got out in front and cranked—and cranked—and cranked. But my generous mother regularly lent me her Detroit Electric, so me and my gal, with a pal and his date, could just waltz right out and drive away. Besides, it was enclosed like a modern sedan, while gas cars were soft-topped with, at best, flapping side curtains.

Also in this younger class was a very attractive and talented musician whom I admired—more and more as I learned to appreciate good music. When I went to college, she would go abroad to study,[4] graduate, then coach for concert work in Boston, where we renewed our friendship, which ripened into marriage.

Finally, I was ready and took the Harvard Entrance Examination, in June 1913, in Chicago—four 3-hour tests on four successive mornings. With afternoons to kill, I learned how to ride the El for hours on one nickel: get off at one station, down one stair, under the tracks, up another stair, board another El to the Loop. The

---

[3]E.g., if $r^{12} = 2$, then $r^4 = 1.260$, $r^5 = 1.335$, $r^7 = 1.498$, near the ratios 5/4, 4/3, 3/2 of the major third, perfect fourth, and perfect fifth.

[4]Edythe Evelyn Reily, Conservatoire Royal de Bruxelles, Violoncelle, 1914.

acme of low finance and nickel-pinching!

2701 Third Avenue, Sacramento, California 95818.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

321. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

For some we loved, the loveliest and the best
That from his Vintage rolling Time has prest,
Have drunk their Cup a Round or two before,
And one by one crept silently to rest.

OMAR KHAYYAM.

ONE × ONE = BYGONE

Regard the above equality as an arithmetical multiplication in the decimal system. Each digit has been replaced by one and only one letter. Different digits have been replaced by different letters. Restore the digits.

322. *Proposed by Harry Sitomer, Huntington, N.Y.*

In parallelogram ABCD, $\angle A$ is acute and $AB = 5$. Point E is on AD with $AE = 4$ and $BE = 3$. A line through B, perpendicular to CD, intersects CD at F. If $BF = 5$, find EF.

A geometric solution (no trigonometry) is desired.

323. *Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y., and M.S. Klamkin, University of Alberta.*

If $xyz = (1-x)(1-y)(1-z)$ where $0 \le x,y,z \le 1$, show that

$$x(1-z)+y(1-x)+z(1-y) \ge 3/4.$$

324. *Proposed by Gali Salvatore, Ottawa, Ontario.*

In the determinant

$$\Delta = \begin{vmatrix} 6 & a & 6 & b \\ c & 8 & d & 2 \\ 1 & e & 5 & f \\ g & 1 & h & 1 \end{vmatrix}$$

replace the letters $a,b,\dots,h$ by eight different digits so as to make the value of the determinant a multiple of the prime 757.

325. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

It is well-known that if you put two pins (thumb-tacks) in a drawing board and a loop of string around them you can draw an ellipse by pulling the string tight with a pencil. Now suppose that instead of the two pins you use an ellipse cut out from plywood. Will the pencil in the loop of string trace out another ellipse?

326. *Proposed by Harry D. Ruderman, Hunter College, New York.*

If the members of the set

$$S = \{2^x3^y \mid x, y \text{ are nonnegative integers}\}$$

are arranged in increasing order, we get the sequence beginning

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, \dots .$$

(a) What is the position of 2^a3^b in the sequence in terms of a and b?

(b)* What is the nth term of the sequence in terms of n?

327. *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Soit p_n le $n$ième nombre premier. Pour quelle(s) valeur(s) de n le nombre $p_n^2 + 2$ est-il premier?

328. *Proposed by Charles W. Trigg, San Diego, California.*

$2k(k+1)$ dominoes, each $2" \times 1"$, can be arranged to form a square with an empty $1" \times 1"$ space in the center.

(a) Show that for all k there is an arrangement such that no straight line can divide the ensemble into two parts without cutting a domino.

(b) Is it always possible to arrange the dominoes so that the ensemble can be separated into two parts by a straight line that cuts no domino?

329. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

"The product of the ages of my three children is less than 100," said

Bill, "but even if I told you the exact product, and even told you the sum of their ages, you still couldn't figure out each child's age."

"I would have trouble if different ages are very close," said John as he looked at the children, "but tell me the product anyway."

Bill told him, and John confidently told each child his age.

If *you* can now also tell the three ages, what are they?

330*. *Proposed by M.S. Klamkin, University of Alberta.*

It is known that if any one of the following three conditions holds for a given tetrahedron, then the four faces of the tetrahedron are mutually congruent (i.e., the tetrahedron is isosceles):

1. The perimeters of the four faces are mutually equal.
2. The areas of the four faces are mutually equal.
3. The circumcircles of the four faces are mutually congruent.

Does the condition that the incircles of the four faces be mutually congruent, also, imply that the tetrahedron be isosceles?

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

261. [1977: 189] *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

Identifier les chiffres de l'addition décimale que voici:

UN + DEUX + DEUX + DEUX + DEUX = NEUF

I. *Solution by Kenneth M. Wilke, Washburn University, Topeka.*

```
  UN
DEUX
DEUX
DEUX
DEUX
----
NEUF
```

It will be easier to follow the reasoning if the coded numbers are written in a column, as shown on the right. Starting from the right, let c_i denote the carry from the ith column to the next column on the left. It is clear that we must have $0 \le c_i \le 4$ for $i = 1, 2, 3$. The last three columns yield the equations given in the table on the following page together with their compatible solutions that yield no duplication of digits (bearing in mind that D can only be 1 or 2). Line (b) must be rejected, since the unused digits contain no value of X consistent with $c_1 = 4$; and line (a) gives immediately X = 6 and F = 9, producing the unique solution

$$25 + 1326 + 1326 + 1326 + 1326 = 5329.$$

	$4U + c_1 = 10c_2$			$3E + c_2 = 10c_3$			$4D + c_3 = N$		
	c_1	c_2	U	c_2	c_3	E	c_3	D	N
	0	2	5	2	2	6			
(a)	2	1	2	1	1	3	1	1	5
	2	3	7	3	3	9			
	4	2	4	2	2	6			
(b)	4	4	9	4	1	2	1	1	5

II. *Comment by Charles W. Trigg, San Diego, California.*

This type of cryptarithm, which has been designated a "doubly true addition" (see [1], [2]), was introduced to the *American Mathematical Monthly* [3] in the form of

FORTY + TEN + TEN = SIXTY

by our proposer, Professor Wayne. Previously, he had published a number of them, such as

SEVEN + SEVEN + SIX = TWENTY,

in *The Cryptogram*, a publication of the American Cryptogram Association. Since then, Wayne and his imitators have produced many of these cryptarithms.

The term "cryptarithmie" was applied to arithmetical restoration puzzles by MINOS (S. Vatriquant) in *Sphinx*, Vol. 1 (1931), p. 50.

"A charming cryptarithm should (1) make sense in the given letters as well as the solved digits, (2) involve all the digits, (3) have a unique solution, and (4) be such that it can be broken by logic, without recourse to trial and error." [4]

III. *"Solution" by Y.*

I was surprised to find only one solution in base ten:

25 + 1326 + 1326 + 1326 + 1326 = 5329.

There is no solution in base six; the smallest base for which there is a solution is seven:

36 + 1432 + 1432 + 1432 + 1432 = 6430.

IV. *"Solution" by Z.*

68 + 2964 + 2964 + 2964 + 2964 = 8960

UN + DEUX + DEUX + DEUX + DEUX = NEUF

and a translation

ONE + TWO + TWO + TWO + TWO = NINE

541 + 875 + 875 + 875 + 875 = 4041

Also solved by CLAYTON W. DODGE, University of Maine at Orono; LEIGH JANES, Rocky Hill, Connecticut; the following students in the class of JACK LeSAGE, Eastview Secondary School, Barrie, Ontario (independently): M.W. McCUAIG, SUSAN McARTHUR, HEATHER STECKLEY, BEV WALLBANK, HILARY COTTER, ANDY SCHWENDIMAN, BRUCE MURRELL, ALEX MILLS, HARRY BINNENDYK, IAIN BEATON, ALBERTA SIMPSON, and JIM ROBB; MYOUNG HEE AN, Long Island City H.S., N.Y.; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; CHARLES W. TRIGG, San Diego, California (solution as well); and the proposer.

Editor's comment.

Trigg's comment II makes us realize that we are fortunate enough to have here a doubly true cryptarithmic addition produced by the originator of the *genre* (and more are to come).

I would like to take this opportunity to make some remarks about cryptarithmic problems. A knowledge of elementary arithmetic is all that is needed to understand and solve most of them. Perhaps for this reason, some people turn up their nose at these problems, feeling it beneath their dignity to solve them. Others solve them by blundering about, trying this and that until they arrive at an answer; but they would be hard put to describe in a coherent fashion how the answer was obtained. A well-made cryptarithm will have an elegant solution. But it takes ability of a high order to

1) devise a logical attack upon the problem that forces out all the answers with a minimum of calculation;

2) describe the method of solution in a form suitable for publication: in grammatical, properly punctuated, complete sentences in the language of discourse (English or French), not just a handful of numbers and other symbols sprinkled on a sheet of paper.

The elementary nature of these problems understandably makes them favourites among the younger readers, but even the best of us can benefit from the challenge of *finding* and *writing* an elegant solution.

A brief solution is naturally preferable to a long one, provided compactness is not achieved by leaving out steps necessary to the understanding. Some "solvers" achieve the ultimate in compactness: they chicken out completely by sending in *only* an answer. What must the editor do to convince those readers that we are not in the business of finding *answers*? Answers can be found by the million at the back of textbooks. We are in the business of finding *solutions*, that is, *methods* leading to an answer. The word "solution" is frequently used as a synonym for "answer" (e.g.

a problem has a unique solution), but any reader who would like to see the expression "solved by" followed by his name and his submission should not take it to mean "answered by".

In "solution" III, Y supplies only the correct answer to our problem. His unsupported statements about base six and base seven happen to be true, although he could have added that the answer is also unique in base seven. (The editor may decide to publish without proof statements such as these about problems related to the one under consideration. But first he has to know if they are true. So it would be a kindness to him to supply proofs or references along with the statements.)

In "solution" IV, Z also gives only an answer to our problem. His answer is wrong, but that may be because his French is poor. So he tries to recoup by giving an answer to an English translation of our problem. *This* answer is correct but incomplete, for the translated version has exactly *thirty-five* distinct answers, which makes it a very, *very* bad cryptarithm (*traduttori traditori*).

REFERENCES

1. C.W. Trigg, A Doubly True Addition (Solution of Problem E 1461), *American Mathematical Monthly*, 68 (December 1961) 1006.

2. William L. Schaaf, *A Bibliography of Recreational Mathematics*, National Council of Teachers of Mathematics, Reston, Virginia; Vol. 3, 1973, Glossary, p. 131.

3. Alan Wayne, proposer, Problem E 751, *American Mathematical Monthly*, 54 (January 1947) 38; solution by A. Chulick, *ibid.* 54 (August 1947) 413.

4. Howard Eves, Note on a cryptarithm, *American Mathematical Monthly*, 54 (August 1947) 413.

* * *

262. [1977: 189] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

In *Challenging Problems in Algebra* 2, by Alfred S. Posamentier and Charles T. Salkind, Macmillan, New York, 1970, p. 14, occurs the following problem (originally proposed in a New York City Junior Contest on April 9, 1965):

Find the real values of x such that $3^{2x^2-7x+3} = 4^{x^2-x-6}$.

In their solution, the authors "prove" that 3 is the only solution. Show this to be incorrect, and find all solutions of this equation.

Solution by Myoung Hee An, Long Island City H.S., N.Y.

The given equation can be written

$$3^{(x-3)(2x-1)} = 4^{(x-3)(x+2)},$$

so of course $x=3$ is a solution. A number $x\neq 3$ is a solution if and only if

$$3^{2x-1} = 4^{x+2} = 2^{2x+4},$$

which is equivalent to the linear equation

$$(2x-1)\log 3 = (2x+4)\log 2,$$

where logarithms are taken to any convenient base. This yields

$$x = \frac{\log 3 + 4\log 2}{2(\log 3 - \log 2)}, \qquad (1)$$

which is the remaining solution. To approximate (1), we can use, say, base 10 or base e for the logarithms, and obtain

$$x = 4.773\ 778\ 228\ldots .$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; LEIGH JANES, Rocky Hill, Connecticut; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; BRUCE McCOLL, St. Lawrence College, Kingston, Ontario; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; H.L. RIDGE, University of Toronto; R. ROBINSON ROWE, Sacramento, California; HARRY D. RUDERMAN, Hunter College, New York; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer. A comment was received from ROBERT S. JOHNSON, Montréal, Québec.

* * *

263. [1977: 189] *Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Ten friends, identified by the digits 0, 1,..., 9, form a lunch club. Each day *four* of them meet and have lunch together. Describe minimal sets of lunches *ijkl* such that

(a) every two of the friends lunch together an equal number of times;

(b) every three of them lunch together just once;

(c) every four of them lunch together just once.

Solution by the proposer (revised by the editor).

(a) It is not possible for every two of the friends to lunch together exactly once. For, in such a scheme, each person would eat exactly three meals: one with three of his nine friends, one with three of the remaining six, and one with the last three. Thus there would be 30 meals in all, which is impossible since the total number of meals must be divisible by four.

In any scheme in which every two of the friends lunch together exactly twice, it is clear from the preceding paragraph that each person would eat at least six meals, making a total of at least 60 meals spread over at least 15 daily four-meal lunches. So a 15-lunch scheme, if such exists, is certainly minimal.

The adjoining two-part table gives two such 15-lunch schemes, in each of which every two of the friends lunch together exactly twice.

(b) There are 120 combinations of the ten friends taken three at a time, and every four-meal daily lunch accounts for four of these combinations. Hence any scheme in which every three of the friends lunch together just once, if such exists, must consist of 30 daily lunches.

The two parts of the table mentioned earlier are complementary in the sense that, taken *together*, they form a 30-lunch scheme in which every three of the friends lunch together just once.

(c) This part is trivial. Just list the 210 combinations of the ten objects 0, 1,..., 9 taken four at a time.

0	1	2	3	4	5	6	7	8	9
0	1	2	3						
0	1			4			7		
0		2			5			8	
0			3			6			9
0				4	5	6			
0							7	8	9
	1	2				6			9
	1		3		5			8	
	1			4				8	9
	1				5	6	7		
		2	3	4			7		
		2		4		6		8	
		2			5		7		9
			3	4	5				9
			3			6	7	8	

0	1				5				9
0	1					6		8	
0		2		4					9
0		2				6	7		
0			3	4				8	
0			3		5		7		
	1	2		4	5				
	1	2					7	8	
	1		3	4		6			
	1		3				7		9
		2	3		5	6			
		2	3					8	9
				4	5		7	8	
				4		6	7		9
					5	6		8	9

Partial solutions were received from ROBERT S. JOHNSON, Montréal, Québec; and R. ROBINSON ROWE, Sacramento, California.

* * *

264. [1977: 189] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Find a formula that gives the number of digits in the nth Fibonacci number explicitly in terms of n.

Solution by Clayton W. Dodge, University of Maine at Orono.

The Fibonacci sequence (F_n) is defined by

$$F_0 = 0,\ F_1 = 1; \qquad F_n = F_{n-1} + F_{n-2}, \qquad n \ge 2.$$

The following formula, due to Binet [2], can be found in nearly every book on number theory:

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}. \tag{1}$$

So one formula for d_n, the number of digits in F_n, is obviously

$$d_n = [1 + \log F_n],$$

where the brackets denote the greatest integer function and the logarithm is to base 10. But a somewhat simpler formula can be derived.

Since F_n is an integer and, for $n \ge 1$,

$$\frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right|^n \le \frac{1}{\sqrt{5}}\left|\frac{1-\sqrt{5}}{2}\right| < 0.28,$$

it follows that F_n is the integer nearest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

We will show that, for $n > 1$,

$$d_n = [1 + n\log\frac{1+\sqrt{5}}{2} - \frac{1}{2}\log 5]. \tag{2}$$

It is clear from the above discussion that (2) fails only when n is odd and $F_n = 10^k$ for some nonnegative integer k, because then $\log F_n = k$ but $n\log(1+\sqrt{5})/2 < k$ since the second term in (1) is positive. These conditions occur for $n = 1$, for example.

We complete the proof of (2) by showing that F_n is never a positive integral power of 10. It is known (see [1], for example) that $F_n | F_m$ if and only if $n | m$; hence since $F_3 = 2$ and $F_5 = 5$, we have

$$2 | F_n \iff 3 | n \quad \text{and} \quad 5 | F_n \iff 5 | n.$$

Therefore

$$10 | F_n \iff 15 | n \iff F_{15} | F_n.$$

Since $F_{15} = 610$, it follows that F_n can never be a positive integral power of 10, and the proof is complete.

Also solved by LEIGH JANES, Rocky Hill, Connecticut; LEROY F. MEYERS, The Ohio State University (partial solution); R. ROBINSON ROWE, Sacramento, California (partial solution); and KENNETH M. WILKE, Washburn University, Topeka, Kansas. A comment was received from ROBERT S. JOHNSON, Montréal, Québec.

REFERENCES

1. David M. Burton, *Elementary Number Theory*, Allyn and Bacon, Boston, 1976, p. 292.

2. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. I, p.394.

* * *

265. [1977: 190] *Proposed by David Wheeler, Concordia University, Montréal, Québec.*

A game involves tossing a coin n times. What is the probability that two heads will turn up in succession somewhere in the sequence of throws?

I. *Solution by Bob Prielipp, The University of Wisconsin-Oshkosh.*

We are looking for $1-p_n$, where p_n is the probability that two consecutive heads will not appear in n tosses of a coin. If $n \geq 2$, the event "n trials produce no sequence HH" can occur in either of two mutually exclusive ways:

i) the trials begin with a T and the next $n-1$ tosses produce no sequence HH, and the probability of this event is $\frac{1}{2}p_{n-1}$; or

ii) the trials begin with HT and the next $n-2$ tosses produce no sequence HH, and the probability here is $\frac{1}{4}p_{n-2}$.

Thus p_n satisfies

$$p_0 = 1, \quad p_1 = 1; \qquad p_n = \tfrac{1}{2}p_{n-1} + \tfrac{1}{4}p_{n-2}, \quad n \geq 2. \tag{1}$$

We claim that

$$p_n = 2^{-n}F_{n+2}, \tag{2}$$

where (F_n) is the Fibonacci sequence defined by

$$F_0 = 0, \quad F_1 = 1; \qquad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2. \tag{3}$$

Relation (2) clearly holds for $n = 0$ and $n = 1$. Suppose it holds for $n-1$ and $n-2$, where $n \geq 2$; then, from (1) and (3), we have

$$p_n = \frac{1}{2}\cdot 2^{-(n-1)}F_{n+1} + \frac{1}{4}\cdot 2^{-(n-2)}F_n = 2^{-n}(F_{n+1}+F_n) = 2^{-n}F_{n+2},$$

and the induction is complete.

The required probability is thus

$$1 - p_n = 1 - 2^{-n}F_{n+2}. \tag{4}$$

II. *Comment by Robert S. Johnson, Montréal, Québec.*

[It follows from (4) that] a game of as few as 5 tosses favours the person who bets he'll get two heads in succession.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec (solution as well); BRUCE McCOLL, St. Lawrence College, Kingston, Ontario; LEROY F. MEYERS, The Ohio State University; H.L. RIDGE, University of Toronto; R. ROBINSON ROWE, Sacramento, California; and the proposer. MURRAY S. KLAMKIN, University of Alberta, sent in a comment; and one incorrect solution was received.

Editor's comment.

Klamkin mentioned that this problem (proposed by Donald J. Persico and Henry C. Friedman) was solved by J. Pullen in *SIAM Review*, Vol. 6, No. 3 (July 1964), pp. 313-314. This reference contains (in addition to a proof essentially the same as our own) several references to related problems.

*　　　　*　　　　*

266. [1977: 190] *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Let d_n be the first digit in the decimal representation of $n!$, so that

$$d_0 = 1,\ d_1 = 1,\ d_2 = 2,\ d_3 = 6,\ d_4 = 2, \ldots .$$

Find expressions for d_n and $\Sigma_{i=0}^{n} d_i$.

Partial solution by Gali Salvatore, Ottawa, Ontario.

The formula

$$d_n = \left[\frac{n!}{10^{[\log (n!)]}}\right], \tag{1}$$

where the brackets denote the greatest integer function and the logarithm is to base 10, is clearly true for all $n = 0, 1, 2, \ldots$; but it is singularly unhelpful, since to find the first digit of $n!$ requires knowledge of $n!$ itself! This disadvantage may be obviated by using Stirling's formula [1]:

$$n! = e^{-n}n^{n+\frac{1}{2}}\sqrt{2\pi}\left\{1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + O(n^{-5})\right\}.$$

We can try replacing $n!$ in (1) by

$$S(n) = e^{-n}n^{n+\frac{1}{2}}\sqrt{2\pi}\left\{1 + \frac{1}{12n} + \frac{1}{288n^2}\right\}$$

to get

$$d_n = \left[\frac{S(n)}{10^{[\log S(n)]}}\right]. \qquad (2)$$

It will be found that (2) gives the correct value of d_n for $n = 1, 2, \ldots, 10$. For $n = 10$, the relative error in using $S(n)$ for $n!$, which is

$$\frac{|10! - S(10)|}{10!} \approx 2.674 \times 10^{-6},$$

is already so small that the first digit of $n!$ is not likely to be affected for any $n > 10$. However, a proof of this conjecture is still needed.

REFERENCE

1. L.B.W. Jolley, *Summation of Series*, Dover, New York, 1961, p. 186.

* * *

267. [1977: 190] *Proposed by John Veness, Cremorne, N.S.W., Australia.*

Some products, like $56 = 7 \cdot 8$ and $17820 = 36 \cdot 495$, exhibit consecutive digits without repetition. Find more (if possible, all) such products $c = a \cdot b$ which exhibit without repetition four, five,..., ten consecutive digits.

Combined list of all the acceptable answers submitted. The list of solvers is given at the end, each name being followed by the number of acceptable answers submitted. An acceptable answer was judged to be one that contains a sequence of consecutive digits in the cyclic order 0, 1, 2,..., 9, 0, 1, 2,..., *in a product of two factors, with no leading zeros.*

12 = 3 • 4	630 = 15 • 42	2718 = 3 • 906
56 = 7 • 8	702 = 18 • 39	2718 = 6 • 453
		3672 = 8 • 459
52 = 4 • 13	1036 = 4 • 259	3708 = 9 • 412
	1368 = 24 • 57	3712 = 58 • 64
162 = 3 • 54	1746 = 3 • 582	3752 = 8 • 469
342 = 6 • 57	1768 = 34 • 52	4296 = 8 • 537
364 = 7 • 52	1827 = 3 • 609	4632 = 8 • 579
658 = 7 • 94	1863 = 9 • 207	4736 = 8 • 592
756 = 9 • 84	1920 = 5 • 384	4876 = 53 • 92
760 = 8 • 95	2106 = 39 • 54	5392 = 8 • 674
	2146 = 37 • 58	5394 = 62 • 87
360 = 15 • 24	2190 = 5 • 438	5432 = 8 • 679

$5742 = 9 \cdot 638$
$5823 = 9 \cdot 647$
$5936 = 8 \cdot 742$
$6318 = 9 \cdot 702$
$6352 = 8 \cdot 794$
$7456 = 8 \cdot 932$
$7506 = 9 \cdot 834$
$7524 = 9 \cdot 836$
$7536 = 8 \cdot 942$
$7624 = 8 \cdot 953$
$7632 = 8 \cdot 954$

$3096 = 2 \cdot 1548$
$3690 = 2 \cdot 1845$
$4396 = 28 \cdot 157$
$4827 = 3 \cdot 1609$
$5082 = 3 \cdot 1694$
$5346 = 18 \cdot 297$
$5346 = 27 \cdot 198$
$5427 = 3 \cdot 1809$
$5724 = 3 \cdot 1908$
$5796 = 12 \cdot 483$

$5796 = 42 \cdot 138$
$6158 = 2 \cdot 3079$
$6174 = 3 \cdot 2058$
$6390 = 5 \cdot 1278$
$6920 = 5 \cdot 1384$
$6952 = 4 \cdot 1738$
$6970 = 2 \cdot 3485$
$7096 = 2 \cdot 3548$
$7236 = 4 \cdot 1809$
$7254 = 39 \cdot 186$
$7632 = 4 \cdot 1908$
$7632 = 48 \cdot 159$
$7690 = 2 \cdot 3845$
$7852 = 4 \cdot 1963$
$7854 = 6 \cdot 1309$
$8156 = 4 \cdot 2039$
$8316 = 4 \cdot 2079$
$9036 = 2 \cdot 4518$
$9076 = 2 \cdot 4538$
$9230 = 5 \cdot 1846$
$9320 = 5 \cdot 1864$
$9360 = 5 \cdot 1872$
$9370 = 2 \cdot 4685$

$9630 = 2 \cdot 4815$
$9670 = 2 \cdot 4835$
$9706 = 2 \cdot 4853$
$9730 = 2 \cdot 4865$

$15628 = 4 \cdot 3907$
$15678 = 39 \cdot 402$
$16038 = 27 \cdot 594$
$16038 = 54 \cdot 297$
$17082 = 3 \cdot 5694$
$17820 = 36 \cdot 495$
$17820 = 45 \cdot 396$
$20457 = 3 \cdot 6819$
$20754 = 3 \cdot 6918$
$21658 = 7 \cdot 3094$
$24507 = 3 \cdot 8169$
$27504 = 3 \cdot 9168$
$28156 = 4 \cdot 7039$
$28651 = 7 \cdot 4093$
$34902 = 6 \cdot 5817$
$65128 = 7 \cdot 9304$
$65821 = 7 \cdot 9403$

Solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono (87); ROBERT S. JOHNSON, Montréal, Québec (7); HERMAN NYON, Paramaribo, Surinam (30); R. ROBINSON ROWE, Sacramento, California (2); KENNETH M. WILKE, Washburn University, Topeka, Kansas (10); and the proposer (24).

Editor's comment.

Purely by accident, there happen to be exactly 100 answers in the above list. And the list is obviously far from complete. I hope we can finish the job now that we have started it. I will gladly publish any further results discovered by readers.

Observe the interesting factorizations:

$2718 = 3 \cdot 906$
$2718 = 6 \cdot 453$

$7632 = 4 \cdot 1908$
$7632 = 8 \cdot 954$
$7632 = 48 \cdot 159$

$5346 = 18 \cdot 297$
$5346 = 27 \cdot 198$

$5796 = 12 \cdot 483$	$16038 = 27 \cdot 594$	$17820 = 36 \cdot 495$
$5796 = 42 \cdot 138$	$16038 = 54 \cdot 297$	$17820 = 45 \cdot 396$

Dodge observed two consecutive integers which produce solutions with the same multiplier. They are:

$$8 \cdot 953 = 7624 \quad \text{and} \quad 8 \cdot 954 = 7632.$$

Nyon unearthed in *Nouvelles Annales de Mathématiques*, 1911, p. 46, the fact that Thié solved the equation

$$\overline{ab} \times \overline{cde} = \overline{fghi},$$

where the letters represent nine distinct nonzero digits. He proved there are exactly seven solutions, which appear in our list.

Anyone want to try this problem in base sixteen? If you do, *don't* send the answers to the editor.

* * *

268. [1977: 190] *Proposed by Gali Salvatore, Ottawa, Ont.*

Show that in ΔABC, with $a \geq b \geq c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3\left(\tan\frac{C}{2} + \tan\frac{A}{2}\right). \tag{1}$$

I. *Composite of the solutions submitted independently by Leon Bankoff, Los Angeles, California; and Charles W. Trigg, San Diego, California.*

From the half-angle formulas of trigonometry, we have, in the usual notation,

$$c + a = 2b \iff \tan\frac{C}{2}\tan\frac{A}{2} = \frac{s-b}{s} = \frac{c+a-b}{c+a+b} = \frac{1}{3}. \tag{2}$$

Now the well-known identity $\Sigma \tan\frac{B}{2}\tan\frac{C}{2} = 1$ shows that (2) is equivalent to

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = \frac{2}{3},$$

which is just another way of writing the formula in the proposal.

II. *Solution by the proposer.*

The relation $c + a = 2b$ holds if and only if B lies on an ellipse with foci at C and A and eccentricity $e = \frac{1}{2}$. By Eustice's characterization of an ellipse [1976: 132], this in turn is equivalent to

$$\tan\frac{C}{2}\tan\frac{A}{2} = \frac{1-e}{1+e} = \frac{1}{3}.$$

[The rest of the proof is as in solution I.]

III. *Comment by Charles W. Trigg, San Diego, California.*

In triangles where $c+a = 2b$, some other relationships that follow easily are:[1]

1) $\cos C + \cos A = 4\sin^2 \frac{B}{2}$.

2) $a\cos C - c\cos A = 2(a-c)$.

3) $ca = 6Rr$ (where R, r = circumradius, inradius).

4) $\cos A = \frac{4c-3b}{2c}$.

5) The intermediate angle B is less than $60°$, except when the triangle is equilateral.

6) GI is parallel to the intermediate side b (where G, I = centroid, incenter).

7) In the special case where $A = C+90°$, then $a:b:c::(\sqrt{7}+1):\sqrt{7}:(\sqrt{7}-1)$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; BRUCE McCOLL, St. Lawrence College, Kingston, Ontario; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; KENNETH M. WILKE, Washburn University, Topeka, Kansas.

Editor's comment.

Bankoff also submitted the following two results, which are striking when seen in juxtaposition. He says he saw them out of the cornea of his eye.

i) If $c+a = 2b$, then $\cot\frac{C}{2} + \cot\frac{A}{2} = 2\cot\frac{B}{2}$;

ii) If $c^2+a^2 = 2b^2$, then $\cot C + \cot A = 2\cot B$.

The first follows easily from (1), and the second is Property 3 in [1978: 14].

* * *

269. [1977: 190] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Let $<\sqrt{10}>$ denote the fractional part of $\sqrt{10}$. Prove that for any positive integer n there exists an integer I_n such that

$$(<\sqrt{10}>)^n = \sqrt{I_n+1} - \sqrt{I_n}.$$

I. *Solution by Lindsay Reynolds, student in the class of H.L. Ridge, University of Toronto.*

We will show that, for every nonnegative integer n, there exist nonnegative integers a_n, b_n verifying

$$a_n^2 - 10b_n^2 = (-1)^n \qquad (1)$$

[1]Trigg submitted proofs for all, and copies are available from the editor on request.

such that

$$(\langle\sqrt{10}\rangle)^n = (-1)^n(a_n - b_n\sqrt{10}). \tag{2}$$

The required result will then follow by taking

$$I_n = a_n^2, \quad \text{if } n \text{ is odd;}$$

$$I_n = 10b_n^2, \quad \text{if } n \text{ is even.}$$

For $n = 0$, we can choose $a_0 = 1$, $b_0 = 0$; these satisfy both (1) and (2). For $n \geq 0$, assume the existence of a_n, b_n satisfying (1) and (2). Then

$$\begin{aligned}(\langle\sqrt{10}\rangle)^{n+1} &= (\langle\sqrt{10}\rangle)(\langle\sqrt{10}\rangle)^n \\ &= (-3+\sqrt{10})\cdot(-1)^n(a_n - b_n\sqrt{10}) \\ &= (-1)^{n+1}\{(3a_n + 10b_n) - (a_n + 3b_n)\sqrt{10}\}.\end{aligned}$$

Thus (2) will be satisfied if we choose

$$a_{n+1} = 3a_n + 10b_n, \qquad b_{n+1} = a_n + 3b_n;$$

and since

$$a_{n+1}^2 - 10b_{n+1}^2 = (3a_n + 10b_n)^2 - 10(a_n + 3b_n)^2 = -(a_n^2 - 10b_n^2) = (-1)^{n+1},$$

our choice satisfies (1) as well as (2), and the induction is complete.

II. *Comment by W.J. Blundon, Memorial University of Newfoundland.*

Assuming that, for $n \geq 0$, the existence of an I_n such that

$$(\langle\sqrt{10}\rangle)^n = (\sqrt{10} - 3)^n = \sqrt{I_n + 1} - \sqrt{I_n} \tag{3}$$

has been demonstrated [as in solution I or otherwise], we will find a simple recurrence relation that will generate all I_n beyond $I_0 = 0$ and $I_1 = 9$, which are obvious from (3).

Taking reciprocals in (3) gives

$$(\sqrt{10} + 3)^n = \sqrt{I_n + 1} + \sqrt{I_n}\,; \tag{4}$$

and adding (3) and (4) and squaring the result yield

$$4I_n + 2 = \theta^n + \theta^{-n},$$

where $\theta = (\sqrt{10} - 3)^2 = 19 - 6\sqrt{10}$ (and hence $\theta^{-1} = 19 + 6\sqrt{10}$). If $n \geq 1$, we now have

$$38(4I_n + 2) = (\theta + \theta^{-1})(\theta^n + \theta^{-n})$$

$$= (\theta^{n+1} + \theta^{-n-1}) + (\theta^{n-1} + \theta^{-n+1})$$

$$= (4I_{n+1} + 2) + (4I_{n-1} + 2),$$

which reduces to

$$I_{n+1} = 38I_n - I_{n-1} + 18.$$

Values of I_n for small n are set forth in the following table.

n	I_n
0	0
1	9
2	360
3	13689
4	519840
5	19740249
6	749609640

III. *Comment by M.S. Klamkin, University of Alberta.*

More generally, given any positive integer a, we can prove the existence of a sequence of nonnegative integers $I_0, I_1, I_2, \ldots$ such that

$$(\sqrt{a+1} - \sqrt{a})^n = \sqrt{I_n + 1} - \sqrt{I_n} \qquad n = 0,1,2,\ldots\ .$$

Assuming the existence of such a sequence [the proof of existence could proceed as in solution I, for example, where $a = 9$], we find an explicit expression for I_n.

[Proceeding as in comment II], we find

$$4I_n + 2 = (\sqrt{a+1} + \sqrt{a})^{2n} + (\sqrt{a+1} - \sqrt{a})^{2n}$$

$$= 2\{(a+1)^n + \binom{2n}{2}(a+1)^{n-1}a + \ldots + \binom{2n}{2n}a^n\},$$

from which we get

$$I_n = \frac{1}{2}\left\{\sum_{r=0}^{n} \binom{2n}{2r}(a+1)^{n-r}a^r - 1\right\}.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; H.G. DWORSCHAK, Algonquin College, Ottawa; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; LEIGH JANES, Rocky Hill, Connecticut; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; H.L. RIDGE, University of Toronto and his students Y. de BRUYN and LOIS THOMPSON (independently); R. ROBINSON ROWE, Sacramento, California; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

Editor's comment.

Thompson proved the following generalization:

If R is a positive integer, $[\sqrt{R}]$ the integral part of its square root, and $R-[\sqrt{R}]^2=k$, then there exists a sequence (I_n) of nonnegative integers such that

$$(<\sqrt{R}>)^n = \sqrt{I_n+k^n} - \sqrt{I_n}, \qquad n=0,1,2,\ldots .$$

Proceeding as in comment II, we can find the following recurrence relation for I_n:

$$I_0=0, \quad I_1=R-k; \quad I_{n+1} = 2(2R-k)I_n - k^2 I_{n-1} + 2(R-k)k^n, \quad n\ge 1.$$

And the method of comment III yields the following explicit expression for I_n:

$$I_n = \frac{1}{2}\left\{\sum_{r=0}^{n} \binom{2n}{2r} R^{n-r}(R-k)^r - k^n\right\}, \quad n=0,1,2,\ldots .$$

* * *

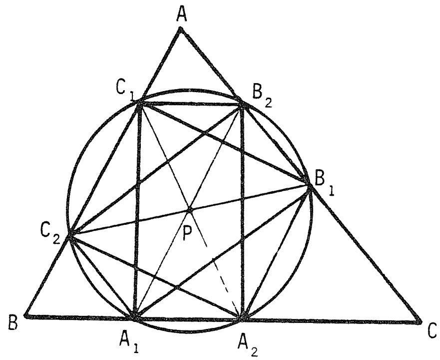
270. [1977: 190] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Call a *chord* of a triangle a segment with endpoints on the sides. Show that for every acute-angled triangle there is a unique point P through which pass three *equal* chords each of which is bisected by P.

Solutions and/or comments were received from LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; M.S. KLAMKIN, University of Alberta; SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; BRUCE McCOLL, St. Lawrence College, Kingston, Ontario; HERMAN NYON, Paramaribo, Surinam; and the proposer. One incorrect solution was received.

Editor's comment.

If two equal chords of a triangle are each bisected by a point P, then their four extremities are the vertices of a rectangle with centre P, one side lying in a side of the given triangle, with the remaining two vertices on the other two sides. Thus, if the point P of the proposal exists, it must be the centre of three such rectangles, whose diagonals are the three chords taken in pairs. The six extremities of the three chords will then lie on a circle with



centre P (see figure).

It is well-known (by few people) that such a circle exists and is unique for *every* triangle, even if obtuse-angled (provided in this case that chords are allowed to terminate on a side produced). I was informed of this fact by Bankoff, Klamkin, Mandan, and McColl. The circle was discovered in 1873 by Lemoine [11]. It is called the *Cosine Circle* or the *Second Lemoine Circle*. Its centre P is called the *Cosine Centre* or *Lemoine Point* (it is, in fact, the symmedian point of the triangle and is usually denoted by K in the literature). The three equal chords through P are antiparallels to the sides of the given triangle with respect to the opposite angles. The reason for the name *Cosine Circle* is that the segments cut off on the sides of the given triangle by the equal chords are proportional to the cosines of the opposite angles; in fact,

$$\frac{A_1A_2}{\cos A} = \frac{B_1B_2}{\cos B} = \frac{C_1C_2}{\cos C} = 2\rho,$$

where ρ is the radius of the Cosine Circle. (By analogy, the circumcircle could be called the Sine Circle.)

For proofs and more information on the Second Lemoine Circle, readers are invited to look into some of the references given below. (And while they are at it, they might look up as well the *First* Lemoine Circle, to lessen the chance that *it* might later turn up as a problem proposal.) All references except [7], [11], and [12] were sent in by Bankoff; and [5] or [8] were submitted also by Klamkin, Mandan, and McColl.

Blundon derived the following formulas for the radius of the Cosine Circle (in the usual notation where R, r, s = circumradius, inradius, semiperimeter):

$$\rho = \frac{R}{\cot A + \cot B + \cot C} = \frac{2Rrs}{s^2 - 4Rr - r^2} = \frac{abc}{a^2 + b^2 + c^2}.$$

From these, it is not too difficult to derive the following beautifully symmetric identity, in which each side equals R/ρ. It looks as if it should be known, although I don't recall seeing it anywhere before:

$$\frac{a\sin A + b\sin B + c\sin C}{a\cos A + b\cos B + c\cos C} = \cot A + \cot B + \cot C.$$

So we have in this proposal another instance of the frequently occurring phenomenon of rediscovery in mathematics. Our proposer will probably not be the last to rediscover this beautiful theorem. And he was not the first: Klamkin wrote that he submitted an equivalent problem to the *American Mathematical Monthly* about

25 years ago. The problem was returned to him, with information about the Cosine Circle, by the then Problem Editor, Howard Eves.

They had *editors* in those days.

REFERENCES

1. Early History of the Symmedian Point, *Proceedings of the Edinburgh Mathematical Society*, XI (1892-1893) 92.

2. Symmedians of a triangle and their Concomitant Circles, *Proceedings of the Edinburgh Mathematical Society*, XIV (1896) 37-103.

3. John Casey, *Géométrie Élémentaire Récente*, Gauthier-Villars, Paris, 1890, p. 20.

4. ________, *A Sequel to Euclid*, Hodges, Figgis & Co., Sixth Edition, Dublin, 1892, p. 192.

5. Nathan Altshiller Court, *College Geometry*, Second Edition, Barnes and Noble, New York, 1952, pp. 258-259 (First Edition, 1925, p. 234).

6. David R. Davis, *Modern College Geometry*, Addison-Wesley, Cambridge, Mass., 1949, p. 170.

7. F.G.-M., *Exercices de Géométrie*, Mame et Fils, Tours, Quatrième édition, 1907, pp. 1099, 1137.

8. Roger A. Johnson, *Modern Geometry*, Houghton Mifflin, New York, 1929 (reissued as *Advanced Euclidean Geometry* by Dover, New York, 1960), pp. 271-272.

9. R. Lachlan, *Modern Pure Geometry*, Macmillan, 1893, pp. 77-78.

10. Trajan Lalesco, *La Géométrie du Triangle*, Vuibert, Paris, 1952, p. 42.

11. Émile Lemoine, *Nouvelles Annales de Mathématiques*, 1873, 2e série, t. XII, p. 365.

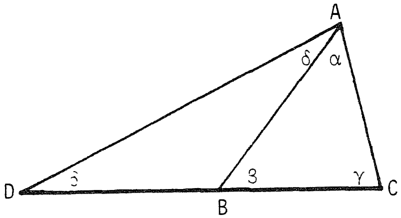
12. Rev. J.B. Lock, *A Treatise on Higher Trigonometry*, Macmillan, 1899, pp. 144-145.

13. Wm. J. McClelland, *Geometry of the Circle*, Macmillan, 1891, p. 74.

* * *

271. [1977: 226] *Proposed by Shmuel Avital, I.I.T. Technion, Haifa, Israel.*

Find all possible triangles ABC which have the property that one can draw a line AD, outside the triangular region, on the same side of AC as AB, which meets CB (extended) in D so that triangles ABD and ACD will be isosceles.



Solution by W.J. Blundon, Memorial University of Newfoundland.

Let the angles of ΔABC have measures α, β, γ. Nine cases are to be considered, according as to which pair of sides are to be equal in ΔABD and which pair in ΔACD. Only six of these cases will yield solutions.

Suppose ABC is a permissible triangle such that $AB = BD$, and let δ be the measure of $\angle DAB$ (see figure). If $AD = CD$, then $\gamma < 90°$ and

$$\gamma = \alpha + \delta, \qquad \beta = 2\delta, \qquad \alpha + \beta + \gamma = 180°.$$

Eliminating β and δ from these equations yields $\alpha = 3(\gamma - 60°)$. Thus

$$\alpha = 3(\gamma - 60°), \qquad 60° < \gamma < 90° \tag{1}$$

is necessary for the existence of a ΔABC with $AB = BD$ and $AD = CD$. Conversely, it is easy to verify that such a triangle can be constructed when (1) holds; hence (1) is necessary and sufficient. A similar analysis would show that for $AB = BD$ and $AC = AD$ necessary and sufficient conditions are

$$\alpha = 3(60° - \gamma), \qquad 0 < \gamma < 60°,$$

while $AB = BD$ and $AC = CD$ is impossible.

Proceeding similarly, we find that a ΔABC exists for which $AB = AD$ and ΔACD is isosceles if and only if

$$\alpha = \frac{3}{2}(60° - \gamma), \quad 0 < \gamma < 60° \quad \text{or} \quad \alpha = 3(60° - \gamma), \quad 45° < \gamma < 60°,$$

the lower bound in the second case being determined by the measure of $\angle DAB$, namely, $4(\gamma - 45°)$; and that one exists for which $AD = BD$ and ΔACD is isosceles if and only if

$$\alpha = \frac{3}{4}(60° - \gamma), \quad 0 < \gamma < 60° \quad \text{or} \quad \alpha = \frac{3}{2}(60° - \gamma), \quad 0 < \gamma < 60°.$$

Summarizing, we have for ΔABC:

(a) no solution for $\gamma = 60°$ or for $90° \le \gamma < 180°$;

(b) for each γ with $60° < \gamma < 90°$, a unique solution $\alpha = 3(\gamma - 60°)$;

(c) for each γ with $0° < \gamma < 60°$, three distinct solutions:

$$\alpha = 3(60° - \gamma), \qquad \alpha = \frac{3}{2}(60° - \gamma), \qquad \alpha = \frac{3}{4}(60° - \gamma).$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; LEIGH JANES, Rocky Hill, Connecticut; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; VIKTORS LINIS, University of Ottawa; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; and DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio.

* * *

272. [1977: 226] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Perhaps by coincidence, the following problem occurs in three different books (to be revealed when a solution is published here):

Solve the system

$$z^x = y^{2x}$$

$$2^z = 2(4)^x$$

$$x + y + z = 16.$$

Perhaps also by coincidence (?), the same incomplete answer is given in all three sources.

Give a complete solution of the system.

Solution by David R. Stone, University of Kentucky, Lexington.

I am inclined to guess that the three mysterious sources for this problem, to be revealed by the editor, are fairly elementary books. If so, the exponential nature of some of the equations leads me to suspect that the complete solution sought by the proposer consists of real triples only. In any case, I will so assume.

We can try $x = 0$ in the first equation, provided the resulting values of y and z are nonzero. Since the last two equations then yield $z = 1$ and $y = 15$, one possible solution is $(0,15,1)$.

We now assume $x \neq 0$. The first equation now gives $z = \pm y^2$ and the second $z = 2x + 1$. Substituting $x = \frac{1}{2}(z - 1)$ and then $z = y^2$ in the third leads to the quadratic

$$3y^2 + 2y - 33 = (3y + 11)(y - 3) = 0,$$

so $y = 3$ or $-11/3$, and we have the possible solutions $(4,3,9)$ and $(56/9,-11/3,121/9)$.

Substituting $x = \frac{1}{2}(z - 1)$ and then $z = -y^2$ in the third equation yields the quadratic $3y^2 - 2y + 33 = 0$, which has no real roots.

The three triples

$$(0,15,1), \qquad (4,3,9), \qquad (\tfrac{56}{9},-\tfrac{11}{3},\tfrac{121}{9}),$$

each of which is easily verified to be a solution of the given system, constitute the complete solution set.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; H.G. DWORSCHAK, Algonquin College, Ottawa; ROBERT S. JOHNSON, Montréal, Québec; JACK LeSAGE, Eastview Secondary School, Barrie, Ontario; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Partial solutions were received from LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; MICHAEL W. ECKER, City University of N.Y.; and R. ROBINSON ROWE, Sacramento, California.

Editor's comment.

The proposer found this problem in [1]-[3]. In each case, as our featured solver surmised, the solutions sought consisted of real triples; and in each case one solution was omitted, the most obvious one: (0,15,1). The proposer also reported that this problem occurred, with an incomplete "official" answer, in a New York City Interscholastic Mathematics League Contest in the 1940's. Another coincidence!

All but two of our "other solvers" (also [1] and [3]) assumed that if the real triple (x,y,z) satisfies the first equation and $x \neq 0$, then necessarily $z = y^2$. This is incorrect: for example, we get $z = -y^2$ from the triple $(2,1,-1)$. It is true that $z = -y^2$ yields no additional solutions, but the possibility must be investigated before one can be sure of having found all the solutions.

REFERENCES

1. Steven J. Bryant, George E. Graham, and Kenneth G. Wiley, *Nonroutine Problems*, McGraw-Hill, 1965, pp. 4, 31-32.

2. H.S. Hall and S.R. Knight, *Elementary Algebra for Schools*, Macmillan, London, 1885, p. 486, Problem 234(2). (This book was revised several times and reprinted 63 times between 1885 and 1960, and the incomplete answer to our problem is still in the 1960 reprint!)

3. Charles T. Salkind, *The Contest Problem Book*, The Mathematical Association of America, 1961, pp. 23, 98.

*　　　*　　　*

273. [1977: 226] *Proposed by M.S. Klamkin, University of Alberta.*

Prove that

$$\lim_{n\to 0} \int_c^\infty \frac{(x+a)^{n-1}}{(x+b)^{n+1}}\,dx = \int_c^\infty \frac{(x+a)^{-1}}{x+b}\,dx, \qquad (a,b,c > 0),$$

without interchanging the limit with the integral.

Solution by Paul J. Campbell for the Beloit College Solvers, Beloit, Wisconsin.

The theorem holds if $a = b$, since then each side is equal to the convergent integral

$$\int_c^\infty (x+a)^{-2}\,dx.$$

We now assume $a \neq b$.

The integral on the right can be evaluated by partial fractions and is found to converge to

$$\frac{1}{b-a}\ln\frac{c+b}{c+a}. \tag{1}$$

By means of the relation

$$\frac{d}{dx}\frac{(x+a)^n}{(x+b)^n} = n(b-a)\frac{(x+a)^{n-1}}{(x+b)^{n+1}},$$

we find that the integral on the left converges to

$$\frac{1}{n(b-a)}\left\{1-\left(\frac{c+a}{c+b}\right)^n\right\} \tag{2}$$

whenever $n \neq 0$ but becomes indeterminate when $n = 0$. But then L'Hôpital's Rule shows that as $n \to 0$ the value of (2) approaches (1), and the proof is complete.

Also solved by VIKTORS LINIS, University of Ottawa; HIPPOLYTE CHARLES, Waterloo, Québec; and the proposer.

* * *

274. [1977: 226] *Proposed by Charles W. Trigg, San Diego, California.*

Find triangular numbers of the form $abcdef$ such that

$$abc = 2def.$$

Solution by W.J. Blundon, Memorial University of Newfoundland.

If def represents the number m, then abc must represent the number $2m$ and $abcdef$ represents the number

$$2001m = T_n = \tfrac{1}{2}n(n+1),$$

where T_n is the nth triangular number. Since $2001 = 3 \cdot 23 \cdot 29$, the congruence

$$n(n+1) \equiv 0 \pmod{2001} \tag{1}$$

implies

$$n(n+1) \equiv 0 \pmod{3}, \quad n(n+1) \equiv 0 \pmod{23}, \quad n(n+1) \equiv 0 \pmod{29}. \tag{2}$$

Since all moduli are primes, each congruence in (2) has exactly two solutions, $n \equiv 0$ and $n \equiv -1$, and it follows from an application of the Chinese Remainder Theorem (see Theorem 5.28 in [1], for example) that (1) has exactly $2 \cdot 2 \cdot 2 = 8$ solutions. These are

$$n \equiv 0,\ -1,\ 1449,\ 1334,\ 1218,\ 782,\ 666,\ 551 \pmod{2001}.$$

The first three are inadmissible since T_{2001}, T_{2000}, and T_{1449} all have more than six digits, so the complete solution to our problem is given by the remaining five, as follows:

n	T_n
1334	890445
1218	742371
782	306153
666	222111
551	152076

If the proposer intended the digits of *abcdef* to be all distinct, then $T_{551} = 152076$ is the unique solution.

Also solved by BOB PRIELIPP, The University of Wisconsin-Oshkosh; and H.L. RIDGE, University of Toronto, and his students LOIS THOMPSON and DIETRICH BURBULLA (independently).

Incomplete solutions were submitted by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas and the proposer.

REFERENCE

1. Tom M Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976, pp. 118-119.

* * *

OUR JOURNAL HAS A NEW NAME

When this journal was in the planning stage a few years ago, our choice of a name for it was limited by several self-imposed restrictions. The journal was to be a Canadian bilingual (English and French) publication devoted primarily to problem solving, and we wanted a name that would reflect these realities. The name we finally chose, EUREKA, was unoriginal but perfectly appropriate, since its allusion to problem solving was unmistakable, and the name was neither English nor French but readily understandable in both languages.

It may be evidence of inexcusable ignorance but is nevertheless a fact that we were well into our first year of publication before we discovered the existence of another mathematical journal with the same name: *Eureka*, The Archimedeans' Journal, published once a year by the Cambridge University Mathematical Society, Junior Branch of the Mathematical Association in England. This publication had started in 1939. But publications come and go, and we wanted to find out if it was still being published. Enquiries were made in England, both at Cambridge University and at the Mathematical Association; these remained unanswered. Several other people we asked (including some university librarians) thought it had not been published for the last 25 years or so;and a search of other journals revealed no reference to it beyond the early 1950's. So we thought it was no longer being published and decided not to change the name of our own publication, feeling that the gap of an entire generation between the two would be enough to dispel most of the confusion.

The present issue of our journal, Vol. 4, No. 3, was already half completed when one of our new subscribers provided, at this editor's request, clear evidence that the Cambridge *Eureka* was still very much alive. It is still being published once a year and the current issue, for the academic year 1977-1978, is No. 40. We made an instantaneous decision to prevent further confusion by changing the name of our journal immediately. This will temporarily cause confusion for *us*, but at least the confusion will be removed from the doorstep of our sister publication in England.

We would have liked to seek the help of our readers in the selection of a new name, but this would have entailed a delay of several months while their suggestions were being winnowed down to one. So the editor gave the matter much thought, consulted a few close friends and advisers, and finally settled on the name you have just seen on the front page of this issue: *CRUX MATHEMATICORUM* (with new ISSN 0705 - 0348).

Crux mathematicorum is an idiomatic Latin phrase meaning: a puzzle or problem for mathematicians. The phrase appears in the Foreign Words and Phrases Supplement of *The New Century Dictionary*, D. Appleton - Century Co., 1946, Vol. 2, p. 2438. It also appears in *Webster's New International Dictionary*, Second Unabridged Edition, G. & C. Merriam Co., 1959, Vol. 1, p. 637. Webster's *Third* Unabridged Edition (1971) gives for *Crux*: a puzzling, confusing, or difficult problem; but it has dropped the phrase *Crux mathematicorum*. Perhaps our use of the phrase will reactivate it and it may reappear in Webster's *Fourth* Unabridged Edition, whenever that is to be published.

It may be argued that, since our journal publishes many problems and solutions in every issue, the plural form *Cruces Mathematicorum* might have been more appropriate. But one does not lightly tamper with an idiom, even a Latin one; and besides, it was felt that the plural *cruces* was somewhat lacking in *panache*.

So *alea jacta est*, as we Romans say. We hope you will like the flavour of our new name, if not right away at least after gingerly tasting it for a few days. In any case, the editor would be pleased to receive your reactions to it.

Readers who do not wish the name *Eureka* to depart from their consciousness might wish to subscribe to *Eureka*, The Archimedeans' Journal. For subscription information, write to: The Editor, *Eureka*, The Arts School, Benet St., Cambridge, England.

THE EDITOR.

* * *

En effeuillant la marguerite

Andrejs Dunkels,
Université de Luleå, Suède.