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Editor: Léo Sauvé, Algonquin College, 281 Echo Drive, Ottawa, Ontario, Canada K1S 1N3.

Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Nghi Chung.

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CONTENTS

THE EASY SOLUTION TO THE RUBIK'S CUBE(TM)

FRIEND H. KIERSTEAD, JR.

In the last few years at least a dozen books have appeared on the market claiming to have an easy solution to the Rubik's Cube(TM). However, with one exception the methods presented require in the worst case several hundred rotations to effect a solution and require that the solver memorize dozens of procedures of as many as 25 rotations each. The method presented here requires little more than 100 rotations and uses only nine procedures, the longest of which has only seven rotations.

A substantial portion of this paper will be devoted to definitions and the presentation of a notation. However, with these in place the remainder of the text will be much simplified. It will be assumed that the reader knows what a Rubik's Cube(TM) is and is familiar in a general way with what can be done with it. It is highly recommended that the text be read with the Cube in hand and that each procedure be performed as it is encountered.

The Cube appears to consist of 26 smaller cubes[1]. To distinguish between the small cubes and the large Cube, a capital C will always be used when referring to the large Cube. There are eight cubes with three faces showing (corner cubes), twelve with two faces showing (edge cubes), and six with only one face showing (face cubes).

Solving the Cube means getting it to the state in which each face is a solid color. The face cubes are attached to the core of the Cube so that they never change their relative positions, but merely rotate in place. Therefore solving the Cube requires moving the corner and edge cubes around so that their faces match the adjacent face cubes.

A cube can be in its correct position and still not match the colors of the adjacent face cubes. A corner cube has three-way symmetry and thus can occupy a given position with three different orientations, representing 120° rotations about an axis from the center of the Cube to the center of the cube. A 120° clockwise rotation about this axis will be denoted by a plus sign and a 120° counterclockwise rotation by a minus sign. Similarly, an edge cube has two-way symmetry and can occupy a given position in two different orientations representing a 180° rotation about a similar axis. Such a rotation will be denoted by a plus sign.

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[1] They are not actually cubes, as may be seen if the Cube is disassembled.

An *outer layer* is a set of nine cubes that share a face and rotate as a unit. There are six outer layers, one for each face. An *inner layer* is a set of four face cubes and four edge cubes, nested between two outer layers, that also rotate as a unit. There are three inner layers. When not otherwise qualified, the term *layer* refers to either an inner or an outer layer.

Usually when we need to designate a specific face (or layer) of the Cube, we need to do so with respect to the particular orientation of the Cube at the moment. Thus the six faces (and outer layers) of the cube are disignated u (up), d (down), r (right), l (left), f (front), and b (back). We designate a cube (or the position of a cube) by concatenating, in clockwise order, the designations of its faces. For example, the corner cube at upper right front (or the upper right front position) is called *urf*, *rfu*, or *fur*; the upper front edge cube is called *uf* or *fu*; and the front face cube is f.

Let x be any outer layer. We denote a 90° clockwise rotation of that layer by X, a 180° rotation (in either direction) by X^2, and a 90° counterclockwise rotation by X'. In all of these rotations, the face cubes remain fixed in position while four corner cubes and four edge cubes change position. Rotations of the inner layer adjacent to the x layer, in which four face cubes and four edge cubes change position while the eight corner cubes remain fixed, are called *slice rotations* and are denoted by X_s, X_s^2 , and X_s'. Note that $R_s = L_s'$ and $R_s^2 = L_s^2$.

A *procedure* is a series of rotations performed to achieve a specific effect and is denoted by a concatenation of the designators for the individual rotations. For example, $FRF'R'$ denotes clockwise rotations of the front and then the right-hand face, followed by counterclockwise rotations of the same faces. Parentheses and exponents may be used to indicate repetitions of a series of rotations. For instance, $(FRF'R')^2 = FRF'R'FRF'R'$. Parentheses without exponents will also be used to indicate significant groupings of rotations, but in this context they have no effect on the operations to be performed.

The execution of a procedure results in the cyclic permutation of groups of cubes; these permutations will be indicated by enumerating the cubes affected and enclosing each group in parentheses. For example,

$$F^2R_sF^2R_s' \qquad (df,\ uf,\ db)$$

indicates that the stated procedure results in the *df* edge cube moving to the *uf* position, the *uf* cube moving to the *db* position, the *db* cube moving to the *df* position, and all other cubes remaining in or returning to their original position. Now consider

FRF'R' (+*ufl*, *rfu*) (-*frd*, *drb*) (*uf*, *rf*, *rd*).

In this procedure the up face of *ufl* becomes the right face of *rfu*, but the right face of *rfu* becomes the front face of *ufl*, not the up face. Thus the cube has been rotated 120° clockwise in the process. This is the meaning of the plus sign preceding *ufl*. In the second group the front face of *frd* becomes the down face of *drb*, but the down face of *drb* becomes the down face of *frd*, and this 120° counterclockwise rotation is indicated by the minus sign.

For some procedures, particularly in the early stages of the solution, only one or two cubes are of interest, and the movements of the others can be ignored. In this situation the ellipsis is used:

BRB' (*frd*, *urf*, ...) ... .

At other times all permutations will be shown, but the ones of particular interest will be underlined.

In the course of the solution it is necessary to be able to distinguish certain physical faces or layers of the Cube regardless of the current orientation of the Cube. It would seem natural to use the colors of the faces for this prupose, but unfortunately the colors are not always arranged in the same way, even among Cubes from the same manufacturer. However, almost all Cubes have a white face, so the term *white* will be used to distinguish the layer or face containing the white face cube, if the Cube has one; otherwise any other arbitrarily chosen face cube. The face or layer opposite the white face or layer will be called the *anti-white* face or layer. The layer between these two will be called the *middle layer*. Also the term *white cube* will refer to any cube having a white face.

The solution of the Cube proceeds in six steps:

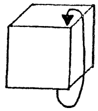
1. Position white corner cubes.
2. Position anti-white corner cubes.
3. Orient corner cubes.
4. Position and orient white edge cubes.
5. Position and orient remaining edge cubes.
6. Align layers.

Before proceeding to step 1, a couple of remarks are in order. First, in step 1 it is only necessary to get the white corner cubes in the correct location, not necessarily with the correct orientation, since the orientation can be accomplished in step 3. However, it will be easier for beginners if they are properly oriented, and the procedures given will accomplish this. Likewise it is not necessary to keep the three layers (white, middle, and anti-white) aligned with each other; to do so costs an extra rotation or two for each procedure and saves only the two rotations in step 6. However, beginners will find it easier to visualize the emerging solution and avoid mistakes if the alignment is restored after each procedure. In many of the procedures the last rotation serves only to restore the alignment of the layers and thus may be omitted. Such rotations will be enclosed in parentheses.

Step 1. Position white corner cubes.

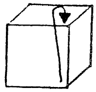
Turn the Cube so that the white face is up. If there is a corner cube with white on its top face, start with it; otherwise a 90° or 180° rotation of one of the vertical layers will bring a white corner cube to the up layer with its white face up. If you wish to keep layers aligned, rotate the top layer until the two vertical faces of the selected cube match the adjacent face cubes. Now locate one of the two white corner cubes that shares a color with the one just positioned. If it is in the up layer, move it to the down layer by rotating one of the vertical layers it shares, being careful not to dislodge the corner cube already positioned. If its white face is now down, rotate the Cube about a vertical axis until the intended position of the white cube is at *urf*, and then rotate the down layer until the white cube is at *dfr*. Then perform

$R'B'D^2BR$ $(dfr,\ urf)\ \ldots$. (1)



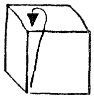
If the white face of the selected cube is not down, rotate the down layer until the selected cube is directly below its intended position and then rotate the Cube about a vertical axis until the selected cube is at either *frd* or *fdl* and its white face is in front. Then perform, respectively,

BRB' $(frd,\ urf,\ \ldots)\ \ldots$ (2a)



or

$B'L'B$ $(fdl,\ ufl,\ \ldots)\ \ldots$. (2b)



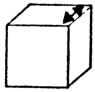
Note that these procedures are mirror images of each other. The rotations B and B' are used to "save" the cube at *ubr* or *ulb*, respectively, and may be omitted if that location does not already hold a correctly-positioned cube.

If the selected white cube is already in the up layer but in the wrong location or orientation, it can be moved to the down layer by rotating the Cube until it is at *urf* and then performing (2a). Then (1), (2a), or (2b) can be used, as explained above, to return it to the top in its correct position and orientation.

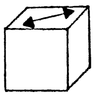
Continue in this manner until all four white corner cubes have been placed. If the procedures have been done correctly, there will be a white *X* on the up face, and the colors of the two white corner cubes on each vertical face will match the face cube. Practice these procedures until you can do them readily, because if you make a mistake in Step 2 (or later), you may have to repeat Step 1.

Step 2. Position anti-white corner cubes.

Now turn the Cube so that the white face is down and examine the anti-white corner cubes. You will have to imagine that each is rotated in place until the anti-white face is up, and you may, if you wish, rotate the up layer to align the colors on the vertical faces with the vertical face cubes. There is one chance in six that all of the anti-white corner cubes are in their correct positions, or may be placed there by rotating the up layer. If so, you are finished with Step 2. There are four chances in six that two adjacent corner cubes must be interchanged. If so, rotate the Cube until they are at *urf* and *rub* and perform

$F'UFURU'R'$ (-$\underline{urf}$, $\underline{rub}$) (+*ulb*) (*fr*, *lu*, *ru*, *ub*). (3) 

Finally there is one chance in six that two non-adjacent corner cubes must be interchanged. If so, rotate the Cube so that they are at *ufl* and *ubr* and perform

$UFURU'R'F'$ (-$\underline{ufl}$, $\underline{ubr}$) (+*ulb*) (*fu*, *lu*, *ru*, *ub*). (4) 

Note that (4) is the same as (3) except that the F' at the beginning has been moved to the end.

Step 3. Orient corner cubes.

If the procedures of Step 1 were performed as described, the white corner cubes are already correctly oriented. This case will be covered in detail. If some of the white corner cubes still need to be rotated, the same general methods can be used, and it is possible to rotate both white and anti-white corner cubes at the same time. The details will be left to the ingenuity of the reader.

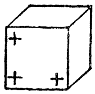
Turn the Cube so that the anti-white face is to the front and examine the orientations of the four anti-white corner cubes. There are four possible cases:

1. All corner cubes are correctly oriented (probability 1/27): Go to Step 4.
2. One corner cube is correctly oriented (probability 8/27): Go to Step 3b.
3. Two corner cubes are correctly oriented (probability 12/27): One corner cube needs to be rotated clockwise, the other counterclockwise. Rotate the Cube about a front-back axis until the cube that needs to be rotated counterclockwise is at *fur* and go to Step 3a.
4. No corner cubes are correctly oriented (probability 6/27): Two corner cubes need to be rotated clockwise and two counterclockwise. Rotate the Cube about a front-back axis until one of the cubes that needs to be rotated clockwise is at *fur* and continue with Step 3a.

Step 3a.

Now perform

$$U'F^2UFU'FU(F^2) \qquad (+\underline{ufl})\ (+\underline{fdl})\ (+\underline{frd})\ (fl,\ fr,\ fd). \qquad (5a)$$

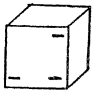


Note that f always rotates clockwise, while u alternates between counterclockwise and clockwise rotations. Note also that the last rotation serves merely to align the layers and may therefore be omitted.

Step 3b.

At this stage one of the anti-white corner cubes is properly oriented and the other three all need to be rotated in the same direction. If they need to be rotated clockwise, rotate the Cube about a front-back axis until the corner that is correctly oriented is at *fur* and perform (5a). Otherwise position the correctly oriented corner cube to *ufl* and perform

$$UF^2U'F'UF'U'(F^2) \qquad (-\underline{fur})\ (-\underline{frd})\ (-\underline{fdl})\ (fr,\ fl,\ fd), \qquad (5b)$$



which is the mirror image of (5a).

Procedures (13), (15), and (16) in the Appendix may also be used for cases 3 and 4.

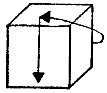
If all has gone correctly, all of the corner cubes should now be correctly positioned and oriented, and if the layers are aligned each face should show an X in the color of its face cube. Steps 4, 5, and 6 will not disturb the corners unless a serious error is made.

Step 4. Position and orient white edge cubes.

Turn the Cube so that the white face is up. Step 4 will have to be performed four times, once for each white edge cube.

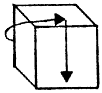
Select one of the white edge cubes. If it is in the top layer but not in its correct position or orientation, rotate the Cube about a vertical axis until the selected cube is at *fu* and move it out of the up layer by

$$FU_sF'(U_s') \qquad (\underline{rb}, \underline{fu}, \underline{df}, lf, fr). \qquad (6a)$$



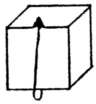
If the selected cube is in the middle layer, rotate the Cube about a vertical axis until the white face of the selected cube is to the back, i.e., the cube is at *rb* or *lb*. Now rotate the up layer until its desired position is at *fu*. If the cube is at *rb*, perform (6a); if it is at *lb*, perform

$$F'U_s'F(U_s) \qquad (\underline{lb}, \underline{fu}, df, rf, fl). \qquad (6b)$$



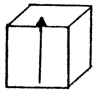
If the selected cube is in the down layer, rotate the Cube about a vertical axis until it is at *fd* and then rotate the up layer until its desired position is at *fu*. If the white face of the selected cube is on the down face, perform

$$R_s'D^2R_s(D^2) \qquad (\underline{fd}, \underline{fu}, bd); \qquad (7)$$



if it is in the front face, perform

$$FU_s'F'(U_s) \qquad (\underline{fd}, \underline{uf}, rf, fl, lb). \qquad (8)$$



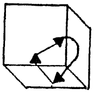
After each of these procedures the layers may be realigned, if desired, by rotating the top layer as necessary to nullify the preparatory rotation used to get the desired position to *fu*.

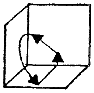
Procedures (17)-(20) in the Appendix may also be used in Step 4.

Step 4 may be performed immediately after Step 1, since Steps 2 and 3 do not disturb the white layer. This variation has the advantage of providing one complete layer as a guide for Steps 2 and 3. However, a mistake in Step 2 or Step 3 will probably mean that Step 4 must be repeated, whereas a mistake in Step 4 is not likely to disturb the corner cubes.

Step 5. Position and orient remaining edge cubes.

Everything is now in place except the four anti-white edge cubes and the four middle edge cubes. To complete the solution of the Cube, only one procedure and its mirror image will be used:

$F'U_s'FR_sFR_s'F'(U_s)$ $(fd,\ fr,\ db)$, (9a) 

$FU_sF'R_sF'R_s'F(U_s')$ $(fd,\ fl,\ db)$. (9b) 

As with many of the previous procedures, the last rotation of each of these is required only to keep the layers aligned. These routines effect a cyclic permutation of two anti-white edge cubes and one middle edge cube. Usually at least five applications of one or the other are required, but never more than eight. With one exception an edge cube that is in the proper layer (middle or anti-white) but in the wrong position or orientation must be moved to the other layer before it is moved into its proper position. The one exception is an anti-white edge cube that is in the position opposite its correct position (e.g., *bd* vs. *df*) with its anti-white face on a vertical face. In fact the only way to get two opposing anti-white edge cubes correctly positioned is to bring the first one into the opposite position and then bring its mate into the same position. Just how this works will be shown in the detailed instructions below.

One should always try to do two things at the same time if possible; for example, move an incorrectly positioned or oriented anti-white cube into the middle layer with the same procedure that moves another anti-white cube out of the middle layer.

The following steps will always effect a solution, but may not always be the most efficient route: (a) move two opposing anti-white cubes into their correct positions; (b) move a third anti-white cube into the anti-white slice; and (c) position all four middle edge cubes.

Step 5a.

Select two anti-white edge cubes that when correctly positioned will occupy opposing positions. If either is now in the anti-white layer, move it to the middle layer (any position) as follows: Rotate the Cube about a vertical axis until the selected cube is at *fd* and apply either (9a) or (9b).

Now rotate the Cube about a vertical axis until one of the selected cubes is at *fl* or *fr* with the anti-white face to left or right. Then rotate the down layer until its final (correct) position is at *fd*. If the selected cube is at *fr*, perform (9a); otherwise perform (9b). The cube should now be in the position opposite its correct position, ready to pop home with the next procedure. Now rotate the Cube about a vertical axis until the second selected cube is at *fl* or *fr* with its anti-white face at *front*. Then rotate the down layer until its correct position is at *db*. If the selected cube is at *fr*, perform (9a); otherwise perform (9b). The two selected cubes should now be in their correct positions and be correctly oriented.

Step 5b.

Select one of the two remaining anti-white edge cubes. If it is in its correct final position and orientation, or if it is in the opposite position with the anti-white color on its vertical face, Step 5b is completed. Otherwise move it to the middle layer if necessary as in the first paragraph of Step 5a, and then return it to the anti-white slice by either of the methods used in the second paragraph of Step 5a.

Step 5c.

Look at the fourth edge cube in the anti-white layer. If it is the fourth anti-white edge cube, move it to the middle layer as in the first paragraph of Step 5a. There is now a middle edge cube in the anti-white layer. Rotate the down layer until that cube is directly below the face cube of the same color as its vertical face, and rotate the Cube about a vertical axis until these cubes are at *f* and *fd*. The correct position for the cube now at *fd* will be either *fl* or *fr*. If it is *fr*, perform (9a); otherwise apply (9b).

Continue in this manner until all middle edge cubes are correctly positioned and oriented. During this process, the third anti-white edge cube will shuttle back and forth between its correct position and the position opposite. If the fourth anti-white edge cube returns to the down layer before all four middle edge cubes are correctly positioned and oriented, return it immediately to the middle layer, displacing one of the middle edge cubes that is not yet correctly positioned and oriented. When finally the fourth middle edge cube is correctly positioned and oriented, the third and fourth anti-white edge cubes will be also.

To maintain alignment of the layers, each procedure should be followed by a rotation of the down layer.

Procedures (21) and (22) in the Appendix may also be used in Step 5.

Step 6. Align layers.

If alignment of the layers has been done after each procedure, the solution is now complete. If not, at most one rotation of the white layer and one rotation of the anti-white layer will complete the solution.

Concluding remarks.

The number of rotations required in the worst case (assuming the layers are not kept in alignment) is approximately[1] 120, apportioned as follows: Step 1, 15; Step 2, 7; Step 3, 14; Step 4, 24; Step 5, 58; and Step 6, 2. However, the worst case rarely occurs; ordinarily the method requires less than 100 rotations, and this can easily be reduced to 75-80 by judicious use of some of the procedures in the Appendix, particularly 15, 16, 21, and 22.

Some Rubik's Cubes have appeared on the market, usually as an advertising gimmick, that have pictures rather than solid colors on the faces. With one of these Cubes, the solution given above may leave some of the face cubes rotated with respect to the edge and corner cubes. Procedures (23)-(27) in the Appendix may then be used to rotate the face cubes without disrupting the edge and corner cubes. For these procedures a plus sign indicates a 90° clockwise rotation, a minus sign indicates a 90° counterclockwise rotation, and two plus signs indicate a 180° rotation.

For those who would like to know more about the Rubik's Cube(TM), particularly the group-theoretic implications, [1] and [2] are highly recommended. Bandelow's book presents a method very similar to that given above and requires about the same number of moves, although it uses a larger collection of procedures. Frey and Singmaster's book presents a quite different (and lengthier) method, but the group theory is easier to understand, particularly for readers who are not comfortable with the terse notation of set theory.

Acknowledgements.

The procedures listed in the Appendix have come from many sources; some are original and others have been invented independently by many authors. In particular I would like to give credit to David Singmaster, who started it all, Christoph Bandelow, and James Angevine.

REFERENCES

1. Christoph Bandelow, *Inside Rubik's Cube and Beyond*, Birkhauser, Boston, 1982.

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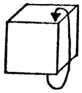
[1] It is not easy to determine the worst case, particularly for Steps 1 and 5.

2. Alexander H. Frey, Jr., and David Singmaster, *Handbook of Cubic Math*, Enslow Publishers, Hillside, New Jersey, 1982.

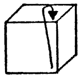
Appendix.

This is not intended to be an exhaustive catalog of procedures; there are probably thousands of procedures that can be found in the many books about the Rubik's Cube(TM). Listed here are those that have been found useful with the method presented, and a few that are simply curiosities. More extensive catalogs may be found in [1] and [2]. For convenience the procedures given in the text are repeated, though without their mirror-image counterparts.

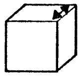
$R'B'D^2BR$ $(\underline{dfr}, \underline{urf})\ (dlf, rbd)\ (fr, dl)\ (fd, dr)$ (1)



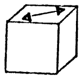
BRB' $(\underline{frd}, \underline{urf}, bdr, ldb)\ (fr, ur, bd, dr)$ (2a)



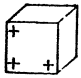
$F'UFURU'R'$ $(-\underline{urf}, \underline{rub})\ (+ulb)\ (fr, ur, bd, dr)$ (3)



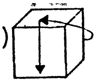
$UFURU'R'F'$ $(-\underline{ufl}, \underline{ubr})\ (+ulb)\ (uf, ul, ur, bu)$ (4)



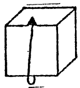
$U'F^2UFU'FUF^2$ $(+\underline{ful})\ (+\underline{fdl})\ (+\underline{frd})\ (fl, fr, fd)$ (5a)



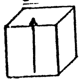
$FU_sF'U'_s$ $(\underline{rb}, \underline{fu}, df, lf, fr)$ (6a)



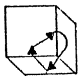
$R'_sD^2R_sD^2$ $(\underline{fd}, \underline{fu}, bd)$ (7)



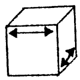
$FU'_sF'U_s$ $(\underline{fd}, \underline{uf}, rf, fl, lb)$ (8)



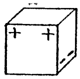
$F'U'_sFR_sFR'_sF'U_s$ (fd, fr, db) (9a)



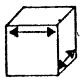
$FRF'R'$ $(+ufl, rfu)\ (-frd, drb)\ (uf, rf, rd)$ (10)

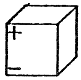


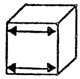
$(FRF'R')^2$ $(+ufl)\ (+rfu)\ (-frd)\ (-drb)\ (uf, rd, rf)$ (11)

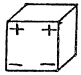


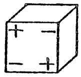
$(FRF'R')^3$ $(ufl, fur)\ (frd, bdr)$ (12)

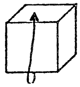


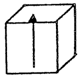
$(FRF'R')^2(DR'D'R)^2$ $(+\underline{ufl})\ (-\underline{fdl})\ (uf,\ fd,\ rf)$ (13) 

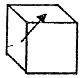
$D(FRF'R')D'$ $(+\underline{ufl},\ \underline{rfu})\ (-\underline{lfd},\ \underline{dfr})\ (uf,\ rf,\ fd)$ (14) 

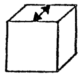
$D(FRF'R')^2D'$ $(+\underline{ufl})\ (+\underline{rfu})\ (-\underline{lfd})\ (-\underline{dfr})\ (uf,\ fd,\ rf)$ (15) 

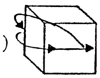
$R^2F^2RF^2R^2F$ $(+\underline{flu})\ (-\underline{fur})\ (-\underline{fdl})\ (+\underline{frd})\ (fu,\ fr,\ ur)\ (fl,\ dr,\ fd)$ (16) 

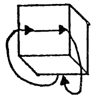
$F^2R_sF^2R'_s$ $(\underline{df},\ \underline{uf},\ db)$ (17) 

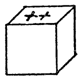
$R_sFR'_sF'$ $(\underline{df},\ \underline{fu},\ fl,\ rf,\ bd)$ (18) 

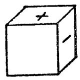
$FR_sF'R'_s$ $(\underline{fl},\ \underline{fu},\ df,\ bd,\ rf)$ (19) 

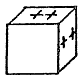
$R_s^2D^2R_s^2D^2$ $(uf,\ ub)\ (df,\ db)$ (20) 

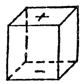
$F^2U'_sF^2U_s$ $(\underline{fl},\ \underline{fr},\ bl)$ (21) 

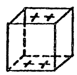
$FR_sF^2R'_sF$ $(fr,\ bd,\ fl)$ (22) 

$(URLU^2R'L')^2$ $(++u)$ (23) 

$R'_sU_sR_sU'R'_sU'_sR_sU$ $(+u)\ (-r)$ (24) 

$R'_sU_sR_sU^2R'_sU'_sR_sU^2$ $(++u)\ (++r)$ (25) 

$R'_sU_s^2R_sU'R'_sU_s^2R_sU$ $(+u)\ (-d)$ (26) 

$R'_sU_s^2R_sU^2R'_sU_s^2R_sU^2$ $(++u)\ (++d)$ (27) 

2504 Fifteenth Street, Cuyahoga Falls, Ohio 44223.

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THE OLYMPIAD CORNER: 50

M.S. KLAMKIN

I shall give later solutions to a number of problems proposed here earlier. But first I present the problems set at the second and third rounds of the 1980 Leningrad High School Olympiad. (I am grateful to Alex Merkurjev and to Larry Glasser for the transmittal and translation, respectively, of these problems.) As usual, I solicit elegant solutions from all readers, but particularly from high school students. Since each problem is identified as being intended for students of grades 8, 9, or 10, I look forward to an increased response from good high school students of any grade.

1980 LENINGRAD HIGH SCHOOL OLYMPIAD

Second Round

1. An arithmetic progression contains an odd number of terms. The sum of the terms in the even-numbered positions equals the sum of the terms in the odd-numbered positions. Find the sum of all the terms in the progression. (Grade 8)

2. A trapezoid has perpendicular diagonals and altitude 4. Find the area of the trapezoid if one diagonal has length 5. (Grades 8, 9, 10)

3. Find all natural numbers n with unit digit 4 which are such that the sum of the squares of the digits in n is not less than n. (Grades 8, 9)

4. Find all primes p such that $2p^4 - p^2 + 16$ is a perfect square. (Grade 8)

5. The function f satisfies $f(0) = 1$ and, for any natural number n,

$$1 + f(0) + f(1) + f(2) + \ldots + f(n-1) = f(n).$$

Find $f^2(0) + f^2(1) + \ldots + f^2(n)$. (Grades 9, 10)

6. Construct the common tangents to the parabolas

$$y = -x^2 + 2x \qquad \text{and} \qquad y = x^2 + 2.5.$$

Show that the points of tangency are the vertices of a parallelogram. (Grade 9)

7. Show that $\sqrt[3]{2 + \frac{10}{3\sqrt{3}}} + \sqrt[3]{2 - \frac{10}{3\sqrt{3}}} = 2.$ (Grade 10)

8. The point O is the midpoint of diagonal AC_1 of the cube $ABCDA_1B_1C_1D_1$, and M is the midpoint of segment OC_1. Through M we draw all possible segments limited by the surface of the cube which are bisected by the point M.

What set is formed on the surface of the cube by the endpoints of these segments? (Grade 10)

9. Find the primes p and q if it is known that the equation $x^4 - px^3 + q = 0$ has an integer root. (Grade 9)

10. From a point A outside a circle of radius R two tangents AB and AC are drawn, where B and C are the points of tangency. Let BC $= a$. Show that

$$4R^2 = r^2 + r_0^2 + \tfrac{1}{2}a^2,$$

where r is the inradius of triangle ABC and r_0 is its circumradius. (Grade 9)

11. If m and n are natural numbers such that $m/n < \sqrt{2}$, show that

$$\frac{m}{n} < \sqrt{2}\left(1 - \frac{1}{4n^2}\right).$$

(Grade 10)

12. Prove that the equation

$$\sin(\sin x + x^2 + 1) + (\sin x + x^2 + 1)^2 = x - 1$$

has no real root. (Grade 10)

Third Round

1. If $a, b, c, d > 0$ and $a+b+c+d = 1$, prove that

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6.$$

(Grade 8)

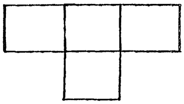
2. On the sides AC and BC of a triangle ABC, points M and K, respectively, are chosen such that

$$\mathrm{BK}\cdot\mathrm{AB} = \mathrm{IB}^2 \quad \text{and} \quad \mathrm{AM}\cdot\mathrm{AB} = \mathrm{IA}^2,$$

where I is the incenter of the triangle. Prove that the points M, I, and K are collinear. (Grade 8)

3. Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, the first player has won 10 games and the second has won 21. How many games has the third player won? (Grade 8)

4. Is it possible to arrange the natural numbers from 1 to 64 on an 8×8 checkerboard in such a way that the sum of the numbers in any figure of the form shown on the right is divisible by 5? (The figure can be placed on the board with any orientation.) (Grade 8)



5. A convex quadrilateral is divided by its diagonals into four triangles. The sum of the squares of the areas of the triangles adjacent to opposite sides is the same. Show that at least one of the diagonals is bisected by the point of intersection. (Grade 8)

6. For natural numbers m and n, what is the smallest value of n such that in its decimal representation the fraction m/n has the sequence ...501... after the decimal point? (Grades 8, 9)

7. Among nine coins are two counterfeits. The real coins weigh 10 g and the false ones weigh 11 g. How can the false coins be identified in five weighings on a single-pan balance if the counterweights are in units of 1 g? (Grade 8)

8. A triangle ABC has sides a,b,c in the usual order. If angle A is twice angle B, show that $a^2 = b(b+c)$. (Grade 9)

9. Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, it turns out that the first player has won 10 games, the second 15, and the third 17. Who lost the second game? (Grade 9)

10. Show that if, for any value of x in the interval $[0,1]$, the inequality $|ax^2+bx+c| \leq 1$ is satisfied, then $|a|+|b|+|c| \leq 1$. (Grade 9)

11. Find two different natural numbers whose arithmetic and geometric means are two-digit numbers one of which is obtained from the other by interchanging the digits. (Grade 9)

12. We shall call a segment in a convex quadrilateral a *midline* if it joins the midpoints of opposite sides. Show that if the sum of the midlines of a quadrilateral is equal to its semiperimeter, then the quadrilateral is a parallelogram. (Grades 9, 10)

13. Are there real numbers a and b such that the function $f(x) = ax + b$ satisfies the inequality

$$\{f(x)\}^2 - \cos x \cdot f(x) < \tfrac{1}{4}\sin^2 x$$

for all $x \in [0,2\pi]$? (Grade 10)

14. Three people are playing table tennis, and after every match the loser gives up his place to the person not playing. At the end, it turns out that the first player has won 10 games, the second 12, and the third 14. How many games did each person play? (Grade 10)

15. How many different numbers appears in the sequence

$$\left[\frac{1^2}{1980}\right], \left[\frac{2^2}{1980}\right], \ldots, \left[\frac{1980^2}{1980}\right],$$

where the square brackets denote the greatest integer function? (Grade 10)

16. At the intersections of an $n\times n$ square array real numbers are located. We decide to write in the places of any two numbers their arithmetic mean. Find all natural numbers n for which, from any initial arrangement of these numbers in the array, this operation can end up with a single number appearing everywhere in the array. (Grade 10)

17. Show that any two points on the surface of a regular tetrahedron of edge 1 cm can be joined by a broken line passing along the surface of the tetrahedron whose length does not exceed $2/\sqrt{3}$ cm. (Grade 10)

18. Two spiders sit along the sides of a convex polygon. Simultaneously they begin to run along the polygon in the same sense and with the same speed. For what initial arrangement of the spiders will the shortest distance between them during the motion be the greatest? (Grade 9)

*

3. [1981: 15] For any natural number p, consider the equation $1/x + 1/y = 1/p$. We are looking for solutions (x,y) of this equation in natural numbers (solutions (x,y) and (y,x) being considered distinct unless $x = y$). Show that the equation has exactly 3 solutions if p is prime, and more than 3 solutions if p is composite.

Solution.

The given equation is equivalent to $(x-p)(y-p) = p^2$. So if p is prime, then there are exactly three possibilities:

$$(x-p,\ \ y-p) = (1,\ p^2),\quad (p,\ p),\quad \text{or}\quad (p^2,\ 1).$$

The three solutions are therefore

$$(x,\ y) = (p+1,\ p+p^2),\quad (2p,\ 2p),\quad \text{and}\quad (p+p^2,\ p+1).$$

These are also solutions if p is composite. But then there are more solutions, for if $p^2 = mn$, then we have also

$$(x,\ y) = (p+m,\ p+n).$$

*

7. [1981: 15] Given is a polynomial with integral coefficients. For three integral values of the variable, it takes on the value 2. Show that for no integral value of the variable can it take on the value 3.

Solution by Dan Sokolowsky, California State University at Los Angeles.

Let $Q(x)$ be the given polynomial. Then $P(x) \equiv Q(x) - 2$ vanishes for three distinct integral values of x, say $x = n_1, n_2, n_3$. Thus

$$P(x) = (x-n_1)(x-n_2)(x-n_3)R(x),$$

where $R(x)$ is a polynomial with integral coefficients. Then for any integer n we have

$$P(n) = (n-n_1)(n-n_2)(n-n_3)R(n),$$

which is a product of four integers at least three of which are distinct. Thus $P(n) \neq 1$, and $Q(n) \neq 3$, for any integer n.

*

14. [1981: 16] If $n \geq 2$ is a given natural number, show that there exists a natural number k such that

$$\frac{k + \sqrt{k^2-4}}{2} = \left(\frac{n + \sqrt{n^2-4}}{2}\right)^5. \tag{1}$$

Solution.

Suppose there exists a number k satisfying (1). Rationalizing the numerators and inverting the results yield

$$\frac{k - \sqrt{k^2-4}}{2} = \left(\frac{n - \sqrt{n^2-4}}{2}\right)^5, \tag{2}$$

and

$$k = \left(\frac{n + \sqrt{n^2-4}}{2}\right)^5 + \left(\frac{n - \sqrt{n^2-4}}{2}\right)^5 \tag{3}$$

follows from (1) and (2). It is easy to see that this value of k is a natural number for any $n \geq 2$. Conversely, if the natural number k is given by (3), then $k \geq 2$ and

$$\begin{aligned} k^2 - 4 &= \left(\frac{n + \sqrt{n^2-4}}{2}\right)^{10} + \left(\frac{n - \sqrt{n^2-4}}{2}\right)^{10} - 2 \\ &= \left\{\left(\frac{n + \sqrt{n^2-4}}{2}\right)^5 - \left(\frac{n - \sqrt{n^2-4}}{2}\right)^5\right\}^2, \end{aligned}$$

so that (1) is satisfied.

*

16. [1981: 17] The bisectors of the face angles of a trihedral angle are drawn. Show that the angles between these bisectors (taken in pairs) are either all acute, all right, or all obtuse.

Solution.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ denote unit vectors along the edges of the trihedral angle, all with origin at the vertex. Then the face angle bisectors lie along $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$, and $\vec{a}+\vec{b}$. The desired result now follows immediately from

$$(\vec{b}+\vec{c})\cdot(\vec{c}+\vec{a}) = (\vec{c}+\vec{a})\cdot(\vec{a}+\vec{b}) = (\vec{a}+\vec{b})\cdot(\vec{b}+\vec{c}) = 1 + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{a}\cdot\vec{b}.$$

*

1. [1981: 45] Determine the range of the function f defined for all real x by

$$f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x+1}.$$

Solution.

The function f is defined, continuous, and differentiable for all real x, it is an odd function, and $f(0) = 0$; so it suffices to investigate its behavior for $x > 0$.

It is clear from

$$f'(x) = \frac{1}{2}\left\{\frac{2x+1}{\sqrt{x^2+x+1}} - \frac{2x-1}{\sqrt{x^2-x+1}}\right\}$$

that f is increasing for all $x \in [0,\frac{1}{2}]$; and it is also increasing for all $x \in [\frac{1}{2},\infty)$, since

$$(2x+1)^2(x^2-x+1) - (2x-1)^2(x^2+x+1) = 6x > 0.$$

Rewriting the functional relation in the form

$$f(x) = \frac{2x}{\sqrt{x^2+x+1} + \sqrt{x^2-x+1}}$$

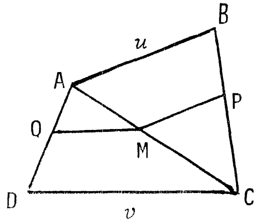
shows that $\lim_{x\to+\infty} f(x) = 1$. The required range is therefore the open interval $(-1, 1)$. The graph of f looks very much like that of $y = \tanh x$.

*

2. [1981: 45] Let ABCD be an arbitrary convex quadrilateral and M a point on the diagonal AC. The parallel to AB [resp. DC] through M intersects BC in P [AD in Q].

(a) Prove that

$$MP^2 + MQ^2 \ge \frac{AB^2 \cdot DC^2}{AB^2 + DC^2}.$$



When does equality hold?

(b) Determine the locus of the midpoint of PQ as M ranges over the segment AC.

Solution.

(a) Let AB $= u$ and DC $= v$, as shown in the figure, and set t = AM/AC. From similar triangles, we have

$$\frac{\text{MP}}{u} = 1-t \qquad \text{and} \qquad \frac{\text{QM}}{v} = t.$$

Thus

$$\text{MP}^2 + \text{MQ}^2 = u^2(1-t)^2 + v^2t^2 = (u^2+v^2)\left(t - \frac{u^2}{u^2+v^2}\right)^2 + \frac{u^2v^2}{u^2+v^2}$$
$$\geq \frac{u^2v^2}{u^2+v^2} = \frac{\text{AB}^2 \cdot \text{DC}^2}{\text{AB}^2 + \text{DC}^2},$$

with equality if and only if $t = u^2/(u^2+v^2)$.

Note that the above solution remains valid exactly as written if M is any point on the *line* AC, provided directed segments be used.

(b) We assume that M is any point on the *line* AC, and set $\vec{\text{CA}} = \vec{a}$, $\vec{\text{CB}} = \vec{b}$, $\vec{\text{CD}} = \vec{d}$. With t as in part (a), we have

$$\vec{\text{CP}} = (1-t)\vec{b} \qquad \text{and} \qquad \vec{\text{CQ}} = \vec{\text{CM}} + \vec{\text{MQ}} = (1-t)\vec{a} + t\vec{d}.$$

If R is the midpoint of PQ, then

$$\vec{\text{CR}} = \tfrac{1}{2}(\vec{\text{CP}} + \vec{\text{CQ}}) = \tfrac{1}{2}\{(1-t)(\vec{a}+\vec{b}) + t\vec{d}\}.$$

This shows that the locus of R is a straight line. If M is restricted to the *segment* AC, then the locus of R is a segment whose endpoints, corresponding to $t = 0$ and $t = 1$, are the midpoints of AB and DC, respectively.

*

4. [1981: 45] Find all real triples (x,y,z) such that

$$\sqrt{x} + \sqrt{y-1} + \sqrt{z-2} = \tfrac{1}{2}(x+y+z).$$

Solution.

Suppose the real triple (x,y,z) is a solution to the given equation. Then there exist $a,b,c \geq 0$ such that $x = a^2$, $y = b^2+1$, $z = c^2+2$, and

$$a + b + c = \tfrac{1}{2}(a^2 + b^2 + c^2 + 3). \tag{1}$$

Since (1) is equivalent to $(a-1)^2 + (b-1)^2 + (c-1)^2 = 0$, we must have $a = b = c = 1$ and

$$(x,\ y,\ z) = (1,\ 2,\ 3). \tag{2}$$

It is easily verified that, conversely, (2) is a solution to the given equation. Solution (2) is therefore unique.

*

1, [1981: 45] Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of real numbers and $\phi: A \to A$ a bijective map. Suppose that $a_1 < a_2 < \ldots < a_n$ and that

$$a_1 + \phi(a_1) < a_2 + \phi(a_2) < \ldots < a_n + \phi(a_n). \tag{1}$$

Show that ϕ coincides with the identical map of the set A. Is this result still true if A is replaced by the set of all integers?

Solution by Andy Liu, University of Alberta.

We first show that $\phi(a_n) = a_n$. If not, then $\phi(a_i) = a_n$ for some $i < n$. Now for at least one $j > i$, we must have $\phi(a_j) \le a_i$, and so

$$a_j + \phi(a_j) \le a_j + a_i \le a_i + a_n = a_i + \phi(a_i),$$

contradicting (1). The same reasoning shows that

$$\phi(a_{n-1}) = a_{n-1},\ \phi(a_{n-2}) = a_{n-2},\ \ldots,\ \phi(a_2) = a_2,$$

and $\phi(a_1) = a_1$ follows, so ϕ is indeed the identity map.

The bijective map defined by $\phi(n) = n+1$ shows that the result is not true if A is replaced by the set of all integers.

*

2, [1981: 45] Find an integer $k \ge 1$ such that the expression

$$f(k,x) \equiv \sin kx \cdot \sin^k x + \cos kx \cdot \cos^k x - \cos^k 2x$$

does not depend on x.

Solution.

Since $f(1,x) = 2\sin^2 x$ and $f(2,x) = \sin^2 x(\sin 2x + \cos 2x)$, neither of which is independent of x, we must have $k \ge 3$. With $C \equiv \cos x$ and $S \equiv \sin x$, we have

$$\begin{aligned} f(3,x) &= (3S-4S^3)S^3 + (4C^3-3C)C^3 - (1-2S^2)^3 \\ &= (3S-4S^3)S^3 + (1-4S^2)(1-S^2)^2 - (1-2S^2)^3 \\ &\equiv 0, \end{aligned}$$

so $k = 3$ is the smallest satisfactory value.

*

3, [1981: 46] Given is a convex polyhedron with $n \ge 5$ (blank) faces and exactly three edges emanating from each vertex. Two persons play the following game: each player in turn signs his name on one of the (remaining) blank faces. To win, a player must sign his name on three faces with a common vertex. Show that there is a winning strategy for the first player.

Solution by Andy Liu, University of Alberta.

All vertices are of degree 3. If every face were a triangle, then the polyhedron would be a tetrahedron, which has only 4 faces, whereas the given polyhedron has at least 5 faces. Hence one of the faces of the polyhedron has at least 4 adjacent faces. The first player should sign this face. Now, regardless of what his opponent does on his first play, there remain three consecutive unsigned faces adjacent to the first signed face. If the first player now signs the middle one, he is assured of a victory on his third move.

*

4. [1981: 47] Let $P(z)$ be a polynomial with complex coefficients. Prove that

$$(P\circ P\circ \ldots \circ P)(z) - z$$

is divisible by $P(z) - z$, where the composition $\circ$ is taken any finite number of times.

Solution.

Let

$$P_0(z) = z \quad \text{and} \quad P_{k+1}(z) = P(P_k(z)), \qquad k = 0,1,2,\ldots \;.$$

We show by induction that

$$P(z) - z \mid P_n(z) - z, \qquad n = 0,1,2,\ldots \;. \tag{1}$$

Now (1) clearly holds for $n = 0$. Suppose it holds for some $n \geq 0$ and consider

$$P_{n+1}(z) - z = \{P(P_n(z)) - P_n(z)\} + \{P_n(z) - z\}. \tag{2}$$

Since $Q(x) - Q(y)$ is always divisible by $x - y$ for any polynomial Q, it follows that $P_n(z) - z$ divides (2). From this and the induction assumption, we conclude that

$$P(z) - z \mid P_{n+1}(z) - z,$$

and the induction is complete.

*

1. [1982: 70] Let S be the sum of the greatest odd divisors of each of the numbers $1,2,3,\ldots,2^n$. Prove that $3S = 4^n + 2$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Let $A = \{1,2,3,\ldots,2^n\}$. For $k = 0,1,\ldots,n$, let A_k be the set of greatest odd divisors of the numbers in A which are divisible by 2^k but not by 2^{k+1}, and let S_k be the sum of the numbers in A_k. It is clear that $A_n = \{1\}$, and a straight-forward proof by induction shows that

$$A_k = \{1,\ 3,\ 5,\ \ldots,\ (2^{n-k}-1)\},\quad k = 0,1,\ldots,n-1.$$

Since $S_n = 1$ and, for $k = 0,1,\ldots,n-1$,

$$S_k = 1 + 3 + 5 + \ldots + (2^{n-k}-1) = (2^{n-k-1})^2 = 4^{n-k-1},$$

we have

$$S = \sum_{k=0}^{n} S_k = \sum_{k=0}^{n-1} S_k + S_n = \frac{4^n-1}{3} + 1 = \frac{4^n+2}{3},$$

and therefore $3S = 4^n + 2$, as required.

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 49: Alphametic

Take a few moments and ponder this, please.
It's a novel idea called WAR
AND.
PEACE
Either solution provides the key
For the value of the digit E.

Puzzle No. 50: Alphametic

Got stumped? Can't see the forest for the trees?
I'll give you the option of WAR
OR.
PEACE
To the missing digits we may letters assign,
If NIECE and MAN be relatively prime.

HANS HAVERMANN, Weston, Ontario

Answer to Puzzle No. 47 [1983: 277]: Ell-square.

Answer to Puzzle No. 48 [1983: 277]: Tetrameter, terameter.

The context from Ezra Pound's *The Cantos*:

Here did they rites, Perimedes and Eurylochus,
And drawing sword from my hip
I dug the ell-square pitkin; ...

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

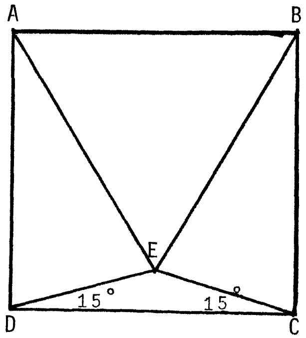
To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

891. *Proposed by Charles W. Trigg, San Diego, California.*

After the dog	R A C E
	A P A W
needed	C A R E
using unguent from the	E W E R.

Each letter in the square array uniquely represents a decimal digit. Identify the digits so that, when they replace the letters, each column and row will be a square integer.

892. *Proposed by Stan Wagon, Smith College, Northampton, Massachusetts.*



ABCD is a square and ECD an isosceles triangle with base angles 15°, as shown in the figure. Prove that $\angle AEB = 60°$ (and therefore triangle AEB is equilateral).

This problem is *very* well known, but all the published solutions use trigonometry and/or auxiliary lines. What is required here is a simple proof without trigonometry or any auxiliary lines (or circles).

893. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let C be the centre of the ellipse

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1 + 2a_{23}x_2 + a_{33} = 0,$$

and let $P_i(x_{1i}, x_{2i})$, $i = 1,2$, be two points on the ellipse. Find the area of the smaller of the regions bounded by CP_1, CP_2, and the ellipse.

894*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

(a) Find necessary and sufficient conditions on the complex numbers a,b,ω so that the roots of

$$z^2 + 2az + b = 0 \quad \text{and} \quad z - \omega = 0$$

shall be collinear in the complex plane.

(b) Find necessary and sufficient conditions on the complex numbers a,b,c,d so that the roots of

$$z^2 + 2az + b = 0 \quad \text{and} \quad z^2 + 2cz + d = 0$$

shall all be collinear in the complex plane.

895. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c in the usual order and circumcircle Γ. A line l through C meets the segment AB in D, Γ again in E, and the perpendicular bisector of AB in F. Assume that $c = 3b$.

(a) Construct the line l for which the length of DE is maximal.

(b) If DE has maximal length, prove that DF = FE.

(c) If DE has maximal length and also CD = DF, find a in terms of b and the measure of angle A.

896. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the inequalities

$$\Sigma \sin^2\frac{A}{2} \geq 1 - \frac{1}{4}\Pi \cos\frac{B-C}{2} \geq \frac{3}{4},$$

where the sum and product are cyclic over the angles A,B,C of a triangle. The inequality between the second and third members is obvious, and that between the first and third members is well known. Prove the sharper inequality between the first two members.

897. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $\lambda > \mu$ and $a \geq b \geq c > 0$, prove that

$$b^{2\lambda}c^{2\mu} + c^{2\lambda}a^{2\mu} + a^{2\lambda}b^{2\mu} \geq (bc)^{\lambda+\mu} + (ca)^{\lambda+\mu} + (ab)^{\lambda+\mu},$$

with equality just when $a = b = c$.

898*. *Proposed by S.C. Chan, Singapore.*

A fair coin is tossed n times. Let T_n be the number of times in the n tosses that a tail is followed by a head. Find (a) the expectation of T_n, (b) the variance of T_n.

899. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Let $\{a_i\}$ and $\{b_i\}$, $i = 1,2,\ldots,n$, be two sequences of real numbers with the a_i all positive. Prove that

$$\sum_{i\neq j} a_i b_j = 0 \implies \sum_{i\neq j} b_i b_j \le 0.$$

900. *Proposed by W.R. Utz, University of Texas at Austin.*

Show that there are an infinite number of sets of three integers in arithmetic progression such that the sum of the square of the first, twice the square of the second, and three times the square of the third is a square.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

773. [1982: 245] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Nine third-order magic squares can be combined into a ninth-order magic square. It is known (L.S. Frierson and W.S. Andrews, *Magic Squares and Cubes*, Dover, New York, 1960, p. 132) that such a ninth-order magic square can be constructed with 5 degrees of freedom, that is, with the numbers in each of the 81 cells being linear combinations of the same 5 independent variables. Prove that such a ninth-order magic square can also be constructed with 21 degrees of freedom.

Solution by the proposer.

Let the third-order magic squares T_i have the magic sums S_i, $i = 1,2,\ldots,9$, and let S be the magic sum of the ninth-order magic square shown in Figure 1. It follows that the third-order square in Figure 2 is magic with magic sum S.

T_1	T_2	T_3
T_4	T_5	T_6
T_7	T_8	T_9

Figure 1

S_1	S_2	S_3
S_4	S_5	S_6
S_7	S_8	S_9

Figure 2

$p+m$	$p-m-n$	$p+n$
$p-m+n$	p	$p+m-n$
$p-n$	$p+m+n$	$p-m$

Figure 3

Now every third-order magic square can be written in terms of three independent variables p,m,n, as shown in Figure 3; hence the magic sums S_i can be expressed in terms of p,m,n by equating the numbers in corresponding cells of

Figures 2 and 3. Thus $S_1 = p+m$, $S_2 = p-m-n$, $S_3 = p+n$, $S_4 = p-m+n$, etc. Since the magic sum of every third-order magic square is thrice its center number, the magic square T_i must have $\frac{1}{3}S_i$ in its center cell, and $S = 3S_5 = 3p$. In accordance with Figure 3, each magic square T_i has the form of Figure 4. Thus the numbers in the cells of T_i are linear combinations of the five independent variables p, m, n, m_i, and n_i. Hence the numbers in the 81 cells of the ninth-order magic square of Figure 1 can all be expressed as linear combinations of the 21 independent variables

$\frac{1}{3}S_i+m_i$	$\frac{1}{3}S_i-m_i-n_i$	$\frac{1}{3}S_i+n_i$
$\frac{1}{3}S_i-m_i+n_i$	$\frac{1}{3}S_i$	$\frac{1}{3}S_i+m_i-n_i$
$\frac{1}{3}S_i-n_i$	$\frac{1}{3}S_i+m_i+n_i$	$\frac{1}{3}S_i-m_i$

Figure 4

$$p,\ m,\ n,\ m_1,\ m_2,\ \ldots,\ m_9,\ n_1,\ n_2,\ \ldots,\ n_9. \quad \Box$$

As an interesting application, we set

$p = 3057 \quad m = 126 \quad n = 900$

$m_1 = 30$	$n_1 = 1008$
$m_2 = 186$	$n_2 = 420$
$m_3 = 90$	$n_3 = 1212$
$m_4 = 330$	$n_4 = 636$
$m_5 = 90$	$n_5 = 792$
$m_6 = 60$	$n_6 = 672$
$m_7 = 120$	$n_7 = 588$
$m_8 = 210$	$n_8 = 978$
$m_9 = 36$	$n_9 = 510$

and obtain the ninth-order magic square of Figure 5, which is composed of 81 distinct primes.

1091	23	2069	863	71	1097	1409	17	2531
2039	1061	83	911	677	443	2441	1319	197
53	2099	1031	257	1283	491	107	2621	1229
1607	311	1913	1109	137	1811	821	29	1433
1583	1277	971	1721	1019	317	1373	761	149
641	2243	947	227	1901	929	89	1493	701
839	11	1307	1571	173	2339	1013	431	1487
1187	719	251	2129	1361	593	1451	977	503
131	1427	599	383	2549	1151	467	1523	941

Figure 5

* * *

775. [1982: 246] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

(a) For $n > 2$, let the $2n$ points A_1, A_2, ..., A_{2n}, in general position in the plane (i.e., no three collinear), be such that, for every line A_iA_j there is a line $A_kA_l \perp A_iA_j$. Prove or disprove that the $2n$ points A_i must be the vertices of a regular polygon.

(b) Conjecture and, if possible, prove an analogous result when an odd number of points are given (perhaps in 3-dimensional space).

Solution by Branko Grünbaum, University of Washington.

This problem is a delight, but the conjecture is hopelessly wrong, and the problem of characterizing all the solutions appears to be very hard (if it has any reasonable solution at all). The following lines are meant to justify this assertion.

To shorten the statements, let us call *orthozygotic* a finite set S of points provided it has the Tsintsifas property, that is, any line determined by S is intersected by some line perpendicular to it which is also determined by S. Then:

For any set S of n points in general position in the plane, there exists an orthozygotic set T in general position such that T contains S and consists of at most $4n$ points; moreover, T can be chosen in such a way that the diameter of T is less than twice the diameter of S.

All this follows very simply from the observation that for any set of squares, of arbitrary sizes and orientations but with the same center, the set of their vertices is orthozygotic.

There are several other ways to generate orthozygotic sets, and some yield sets with an odd number of points (see (ii) and (iii) below). For example:

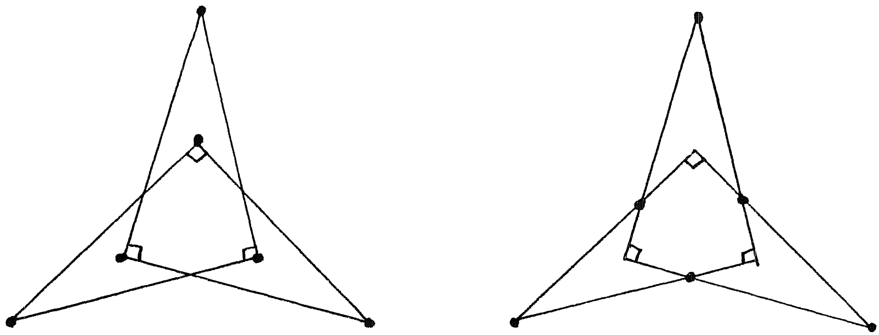
(i) For each $n \geq 1$, the set of vertices of any regular $(2n+1)$-gon together with the center of the polygon.

(ii) For any $n \geq 2$, the set of $3n$ of the vertices of the regular $4n$-gon obtained by omitting every fourth vertex of the $4n$-gon. Many subsets of the set of vertices of even-sided regular polygons are orthozygotic; for example, omitting a single vertex works for all $2n$-gons with $n \geq 3$.

(iii) If S is an orthozygotic set consisting of concyclic points, the union of any family of sets obtained by rotations of S about the common center is also orthozygotic.

(iv) Two 6-point orthozygotic sets that do not fit any of the above classes are illustrated below. (The outer vertices are those of an equilateral triangle, and not all the lines determined by the point sets are drawn in the figures.)

There certainly are many other possibilities.



Also solved by G.P. HENDERSON, Campbellcroft, Ontario; and ROBERT C. LYNESS, Southwold, Suffolk, England.

Editor's comment.

Henderson gave as counterexamples a 2-parameter family of 5-point orthozygotic sets, and a 4-parameter family of 6-point sets. He also gave a particularly simple 5-point set not in his 2-parameter family. It consists of the points with rectangular coordinates

$$(0,\ 1),\ (1,\ 1),\ (1,\ -1),\ (0,\ -1),\ \text{and}\ (2,\ 0).$$

*　　　*　　　*

776. [1982: 246] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Ann watched in amazement as Sam made out the check. "I said two mugs and three plates," she reminded him.

Sam nodded. "That's right. There's no quantity discount, so it's $4.05 for the five pieces."

"But you multiplied the two amounts instead of adding," Ann protested.

"Sure I did, lady," replied the old man. "But it made no difference to the total."

He was right! So what were the two prices?

Solution by Kenneth S. Williams, Carleton University, Ottawa.

Let m and p be the price (in dollars) of a mug and a plate, respectively. Then

$$2m + 3p = 2m\cdot 3p = 4.05,$$

and so $\{2m,3p\}$ are the roots of the quadratic $x^2 - 4.05x + 4.05 = 0$, that is, $\{2m,3p\} = \{1.80,2.25\}$. As there is no quantity discount, we cannot have $2m = 2.25$. Hence $2m = 1.80$, $3p = 2.25$, and the prices of a mug and a plate were 90 cents and 75 cents, respectively.

Also solved by ELWYN ADAMS, Gainesville, Florida; SAM BAETHGE, Southwest High School, San Antonio, Texas; the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; DONALD C. FULLER, Gainesville Junior College, Gainesville, Georgia; J.T. GROENMAN, Arnhem, The Netherlands; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DONALD P. SKOW, Edinburg, Texas; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One incorrect solution was received.

Editor's comment.

Several of the above solvers ignored the restriction "no quantity discount" and came up with two answers. Our incorrect solver came up with only one answer— the wrong one.

Dou commented: "Este problema es simpático, me recuerda los tiempos del bachillerato."

* * *

778. [1982: 246] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with incenter I, the lines AI, BI, CI meeting its circumcircle again in D, E, F, respectively. If S is the sum and P the product of the numbers

$$\frac{\mathrm{ID}}{\mathrm{AI}}, \quad \frac{\mathrm{IE}}{\mathrm{BI}}, \quad \frac{\mathrm{IF}}{\mathrm{CI}},$$

prove that $4P - S = 1$.

Solution de Hippolyte Charles, Waterloo, Québec.

Comme dans la solution I de Crux 644 [1982: 154], on trouve

$$\frac{\mathrm{ID}}{\mathrm{AI}} = \frac{a}{2(s-a)}, \quad \frac{\mathrm{IE}}{\mathrm{BI}} = \frac{b}{2(s-b)}, \quad \frac{\mathrm{IF}}{\mathrm{CI}} = \frac{c}{2(s-c)}.$$

Si l'on pose $x = 2(s-a)$, $y = 2(s-b)$, $z = 2(s-c)$, de sorte que $y+z = 2a$, $z+x = 2b$, $x+y = 2c$, l'équation proposée est équivalente à

$$(y+z)(z+x)(x+y) - \{yz(y+z) + zx(z+x) + xy(x+y)\} = 2xyz.$$

Or le membre gauche de cette équation est un polynôme homogène et symétrique du troisième degré en x,y,z qui s'annule pour $x = 0$. Ce membre égale donc $kxyz$, où k est une constante. Enfin, pour $x = y = z = 1$ on trouve $k = 2$, et le tour est joué.

Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; W.J. BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; the COPS of Ottawa; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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TITLES OF ARTICLES AND NOTES

PROBLEMS AND SOLUTIONS

The numbers refer to the pages in which the corresponding name appears with a problem proposal, a solution, or a comment.