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EUREKA

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All communications about the content of the magazine (articles, problems, solutions, book reviews, etc.) should be sent to the editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ont., K1S 1N4.

All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ont., K1S 0C5.

* * *

CONTENTS

AND NOW. . .A WORD FROM OUR SPONSOR

EUREKA had its origin, early in 1975, in the conviction of the Editor that those who teach mathematics have a professional obligation to *do* mathematics. The Secretary-Treasurer of the Carleton-Ottawa Mathematics Association and others agreed that a practical way to put such a conviction into effect was by the circulation of a problem-solving newsletter.

Algonquin College supported the project wholeheartedly, and COMA made it available to its membership and subsequently to the membership of the Ontario Association for Mathematics Education, of which it is a chapter.

The response — fanned by word of mouth — was most encouraging. Early in 1976, we had to make financial decisions for what had become, in effect, a new mathematical publication. It was decided to make EUREKA available free to all who asked, and to invite readers to contribute to a sustaining fund. We are grateful to the readers whose contributions enabled us to carry on. However, such a policy was necessarily interim, and sole reliance on a sustaining fund results in uncertainties which make it difficult to manage the project.

The Council of the Association has now decided on subscription rates which we hope readers will find acceptable. They will be given on the front page of each issue beginning with this one. Reduced rates are available to paid-up members of COMA and OAME.

With this issue, therefore, comes an invitation to subscribe to EUREKA for the calendar year 1977 (ten issues), and to obtain bound copies (with a reduced page size) of the combined Volumes 1 and 2 (1975-1976). Readers are requested to renew promptly — we are not computerized, nor do we have a paid staff to look after routine work.

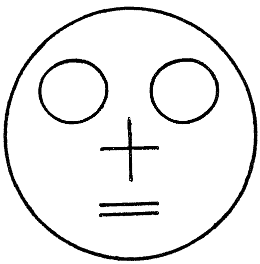
It is likely that the introduction of subscription rates will, at the outset, result in a reduced circulation for EUREKA, and the money received may not be sufficient to cover our fixed costs. So readers who value EUREKA are invited to recommend it to their libraries, and to their colleagues and students; and we do not discourage any of the generous impulses some readers may have of sending a contribution to the EUREKA Sustaining Fund over and above the regular subscription rate.

The best way to help EUREKA, however, is not with money, but with contributions to the magazine itself, with carefully-wrought articles, problems, solutions, and book reviews. EUREKA will only be as good as you want to make it.

F.G.B. MASKELL,
Secretary-Treasurer of COMA.

* * *

$0 + 0 = 0$



LEON BANKOFF,
Los Angeles, California.

ANOTHER PROOF OF THE BUTTERFLY THEOREM

DAN SOKOLOWSKY, Yellow Springs, Ohio

1. *Preliminary remarks.*

Mathematical lepidopterists are forever seeking new varieties of proofs of the Butterfly Theorem. The extensive bibliography given in EUREKA [1976: 2-5, 90-91] makes it clear that the number of known varieties is indeed very large. I offer here another specimen, a proof, which I believe to be new, based on the power of a point with respect to a circle.

It is sufficient for our purpose to define the power of a point only for points P interior to a circle K. For such a point P at distance d from the center, O, of circle K of radius R (see Figure 1), we define the *power*, P_K, *of* P *with respect to* K by

(a) $\mathrm{P}_K = R^2 - d^2$.

If P' is at distance d' from O, it then follows from (a) that

(b) $d = d' \iff \mathrm{P}_K = \mathrm{P}'_K$.

Also, if a chord of K through P is divided by P into segments of lengths a, b, we have

(c) $ab = (R+d)(R-d) = \mathrm{P}_K$.

Finally, if P, P', P" are distinct non-collinear points within K and O' is the circumcenter of ΔPP'P", then it follows directly from (b) that

(d) $\mathrm{O} \equiv \mathrm{O}' \iff \mathrm{P}_K = \mathrm{P}'_K = \mathrm{P}''_K$.

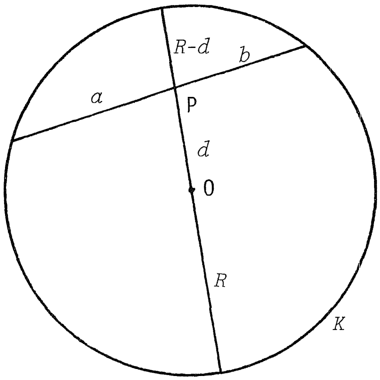


Figure 1

2. *THE BUTTERFLY THEOREM.*

Let C_1 *denote a circle, center* O*;* ST *a chord of* C_1 *with midpoint* E*;* AB*,* CD *chords of* C_1 *through* E. *Let* AD*,* BC *meet* ST *at* N*,* M*, respectively* (see Figure 2).

Then EN = EM.

Proof. It suffices to show that ON = OM.

Let the circle C_2, with center O and radius OE, meet AB, CD again at F, G, respectively, and let FG meet AD at R, BC at L. Then ST is tangent to C_2 at E and, by symmetry, AF = BE = m, CG = DE = n.

The following pairs of triangles are similar:

$$\Delta\,\mathrm{DEN} \sim \Delta\,\mathrm{BFL},\ \Delta\,\mathrm{BEM} \sim \Delta\,\mathrm{DGR},\ \Delta\,\mathrm{AFR} \sim \Delta\,\mathrm{CEM},\ \Delta\,\mathrm{CGL} \sim \Delta\,\mathrm{AEN}\,. \qquad (1)$$

The first similarity follows from $\angle EDN = \angle EBM$ and $\angle BFL = \angle MEG = \angle DEN$, and the others are proved in like fashion. Since, from the first similarity, $\angle DNE = \angle BLF$, we have $\angle ENR + \angle BLF = 180°$, and it follows that M, N, R, L lie on some circle C_3 of center, say, O'.

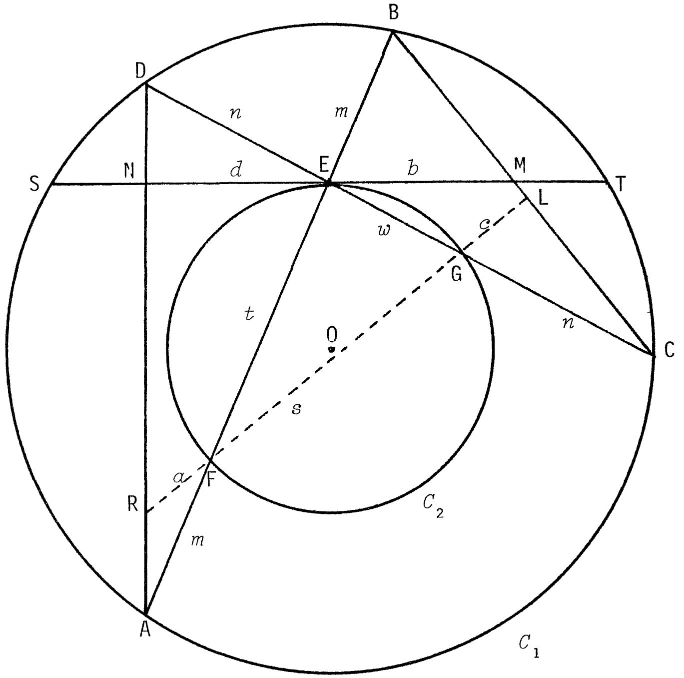


Figure 2

We show that $O \equiv O'$, and hence that $ON = OM$. Let $P_i(X)$ denote the power of a point X with respect to C_i, $i = 1,2,3$. By 1(c), $P_1(E) = m(m+t) = n(n+w)$, so that

$$\frac{n}{m+t} = \frac{m}{n+w}, \tag{2}$$

and

$$P_3(E) = bd,\ P_3(F) = a(c+s),\ P_3(G) = c(a+s).$$

Also, from the similarities (1),

$$\frac{d}{c+s} = \frac{n}{m+t},\quad \frac{b}{a+s} = \frac{m}{n+w},\quad \frac{a}{b} = \frac{m}{n+w},\quad \frac{c}{d} = \frac{n}{m+t}. \tag{3}$$

From (2) and (3), we now get

$$\frac{d}{c+s} = \frac{a}{b}, \quad bd = a(c+s), \quad P_3(E) = P_3(F),$$

and

$$\frac{b}{a+s} = \frac{c}{d}, \quad bd = c(a+s), \quad P_3(E) = P_3(G).$$

Thus $P_3(E) = P_3(F) = P\ (G)$, and $O \equiv O'$ follows from 1(d). Hence $ON = OM$ and the proof is complete.

* * *

A BIBLIOGRAPHY OF THE STEINER-LEHMUS THEOREM

CHARLES W. TRIGG, Professor Emeritus, Los Angeles City College

The following references to the Steiner-Lehmus Theorem and related theorems will serve to augment the bibliography given by Sauvé in EUREKA [1976: 23-24].

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* * *

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 187.

For the problems given below, solutions, if available, will appear in EUREKA *Vol.* 3, *No.* 2, *to be published around Feb.* 15, 1977. *To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Feb.* 1, 1977.

181. *Proposed by Charles W. Trigg, San Diego, California.*

A polyhedron has one square face, two equilateral triangular faces attached to opposite sides of the square, and two isosceles trapezoidal faces, each with one edge equal to twice a side, e, of the square. What is the volume of this pentahedron in terms of a side of the square?

182. *Proposed by Charles W. Trigg, San Diego, California.*

A framework of uniform wire is congruent to the edges of the pentahedron in the previous problem. If the resistance of one side of the square is 1 ohm, what resistance does the framework offer when the longest edge is inserted in a circuit?

183. *Proposed by Viktors Linis, University of Ottawa.*

If $x+y=1$, show that

$$x^{m+1}\sum_{j=0}^{n} y^j C^j_{m+j} + y^{n+1}\sum_{i=0}^{m} x^i C^i_{n+i} = 1$$

holds for all $m, n = 0,1,2,\ldots$

This problem is taken from the list submitted for the 1975 Canadian Mathematical Olympiad (but not used on the actual exam).

184. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Si $I = \{x \in R \mid a \le x \le b\}$ et si la fonction $f : I \to I$ est continue, montrer que l'équation $f(x) = x$ admet au moins une solution dans I.

185. *Proposed by H.G. Dworschak, Algonquin College.*

Prove that, for any positive integer $n > 1$, the equation

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = n^2$$

has a rational root between 1 and 2.

186. *Proposed by Leroy F. Meyers, The Ohio State University.*

Let A, B, C, and D be the subsets of the plane R^2 having, respectively, both coordinates rational, both coordinates irrational, exactly one coordinate rational, and both coordinates or neither rational. Which of these sets is/are connected? (A subset of the plane is connected just when it cannot be expressed as the union of two disjoint nonempty sets neither of which contains a boundary point of the other.)

187. *Proposé par André Bourbeau, École Secondaire Garneau.*

Si $m = 2^n \cdot 3 \cdot p$, où n est un entier positif et p un nombre premier impair, trouver toutes les valeurs de m pour lesquelles $\sigma(m) = 3m$, $\sigma(m)$ étant la somme de tous les diviseurs de m.

188. *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Show that the only positive integer solution of the equation $a^b = b^a$, $a < b$, is $a = 2$, $b = 4$.

189. *Proposed by Kenneth S. Williams, Carleton University.*

If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

190. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Find all integral values of m for which the polynomial

$$P(x) = x^3 - mx^2 - mx - (m^2 + 1)$$

has an integral zero.

* * *

OMNIA VINCIT AMOR

The extract given below is reprinted accurately from a historical note in *James Cook Mathematical Notes No.* 3 (March 1976), edited by Professor Basil C. Rennie, James Cook University of North Queensland, Townsville, Australia.

> Magnetic Island, five miles North of Townsville, was discovered and named by Captain Cook in June, 1770, under the mistaken impression that the magnetic deviation changed suddenly near the island. I used to know Professor W.A. Osborne who loved there in his retirement after being Professor of Physiology at Melbourne University from 1904 to 1938.

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

151. [1976: 109] *Proposed by the editor.*

METAPHORS

I'm a riddle in nine syllables,
An elephant, a ponderous house,
A melon strolling on two tendrils.
O red fruit, ivory, fine timbers!
This loaf's big with its yeasty rising.
Money's new-minted in this fat purse.
I'm a means, a stage, a cow in calf.
I've eaten a bag of green apples,
Boarded the train there's no getting off.

SYLVIA PLATH (1932-1963)
From *Crossing the Water.*

Identify the speaker and thereby solve the riddle.

Solution by Gali Salvatore, Ottawa, Ont.

The interpretation of this riddle is so transparent that I'm sure the editor will be deluged with solutions. The phrases "a cow in calf" and "in nine syllables" make it as clear as clear can be that the speaker is a pregnant woman (possibly the author herself). The "riddle in nine syllables" refers to the sex of the unborn child, which takes nine months to determine. The expressions "ponderous house," "melon," "big loaf," and "fat purse," all refer to the woman's belly, which is distended as if she had "eaten a bag of green apples." The last line of the poem means, as the old saying goes, that there is no such thing as being "a little bit pregnant"; once you get on (or in), you're on for the whole ride. The "new-minted money". . .but why go on?

The poem itself is a beautiful one, but the real riddle in this problem is why the editor thought it appropriate to be published in a mathematical journal such as this one.

Also solved by the proposer and by nobody else.

Editor's comment.

The editor had his umbrella open, but the deluge never came.

Be it said in defence of the editor's judgment, there are in the poem certain numerical patterns which may justify (barely) its inclusion in this journal. The title contains nine letters, and the poem itself consists of nine lines, each of which has nine syllables. This internal evidence confirms beyond the shadow of a doubt that the above solver's interpretation is the correct one.

152. [1976: 109] *Proposé par Jacques Marion, Université d'Ottawa.*

Si $a > e$, montrer que l'équation $e^z = az^m$ possède m solutions à l'intérieur du cercle $|z| = 1$.

Solution by Kenneth S. Williams, Carleton University.

As $a > e$, on $|z| = 1$ we have

$$|-az^m| = a > e \geq e^x = |e^z|,$$

and so, by Rouché's Theorem, $-az^m$ and $-az^m + e^z$ have the same number of zeros inside $|z| = 1$. But $-az^m$ has m zeros inside $|z| = 1$; hence the equation $e^z = az^m$ has m solutions inside $|z| = 1$.

This problem appears in many texts on complex variable theory.

Also solved by F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment.

The problem can be found, for example, on p. 136 in the Schaum's Outline Series *Complex Variables*, by Murray R. Spiegel; and on p. 328 in *Basic Complex Analysis*, by Jerrold E. Marsden, W.H. Freeman and Co., 1973. Furthermore, the second reference contains a solution similar to the one given above.

153. [1976: 110] *Proposé par Bernard Vanbrugghe, Université de Moncton.*

Montrer que les seuls entiers positifs qui vérifient l'équation

$$ab = a + b$$

sont $a = b = 2$.

I. *Solution by R. Robinson Rowe, Sacramento, California.*

Without loss of generality, assume $a \leq b$. Division of $ab = a + b$ by b gives $a = \frac{a}{b} + 1$, which is between 1 and 2 and so not an integer if $a < b$. Hence $a = b$, whence $a^2 = 2a$ and $a = 2 = b$ is the only solution.

II. *Solution by Kenneth S. Williams, Carleton University.*

Since $a | ab - a$, $ab = a + b$ implies $a | b$, and similarly $b | a$. Thus $a = b$, $a^2 = 2a$, and $a = b = 2$.

III. *Second solution by Kenneth S. Williams, Carleton University.*

Without loss of generality, assume $a \geq b$. If $b \geq 3$, then

$$ab \geq 3a > 2a \geq a + b = ab,$$

a contradiction; hence $b \leq 2$. Since $b = 1$ is clearly impossible, we have $b = 2$ and $a = 2$.

IV. *Solution by Charles W. Trigg, San Diego, California.*

If $ab = a + b$, then $a = b/(b - 1) = 1 + 1/(b - 1)$. Since $1/(b - 1)$ is an integer, $b = 2 = a$.

V. *Solution by Clayton W. Dodge, University of Maine at Orono.*

From $ab = a+b$, we obtain $a = b/(b-1)$, and hence $b-1|b$. Since $gcd(b-1,b) = 1$, we have $b-1 = 1$ and $b=2=a$.

Also solved by HIPPOLYTE CHARLES, *Waterloo, Québec;* G.D. KAYE, *Department of National Defence;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.;* KENNETH S. WILLIAMS, *Carleton University (third solution); and the proposer.*

Editor's comment.

This is the simplest case of the more general problem of finding n positive integers whose sum equals their product. There is at least one solution for every $n \geq 2$. For $n=3$, the unique solution (disregarding order) is $\{1,2,3\}$. For $n=4$, the unique solution is $\{1,1,2,4\}$. For $n=5,\ldots$ ah, but that would be telling. If you want to know the answer for $n=5$, do your own think and solve Problem 172.

E.P. Starke [1] has verified that, for $n \leq 232$, the solution is unique only when n = 2, 3, 4, 6, 24, 114, and 174. Solvers of Problem 172 should therefore note that there is more than one solution for $n=5$.

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154. [1976: 110, 159] (Corrected) *Proposed by Kenneth S. Williams, Carleton University.*

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \leq r \leq p_n - 1)$ in this list, say $r = p_1^{a_1}\ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1}\ldots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

1	2	3	4	5	6	7	8	9	10	11	12
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	3	5	9	7	15	11	27	25	21	13	45

We observe that $\ell = 17 = p_7$.

Solution by the editor.

It is easy to verify that the conjecture is true for the first few small values of n. We will show that it holds for all n.

Let $L = \{1,2,\ldots,p_n - 1\}$ and $L' = L \cup \{p_n\}$. For every $x = p_1^{\alpha_1}\ldots p_{n-1}^{\alpha_{n-1}} \in L$, let

$f(x) = p_2^{\alpha_1}\ldots p_n^{\alpha_{n-1}}$ and let $f(p_n) = p_{n+1}$. The function $f: L' \to N$ is clearly injective; hence if we set $M = f(L)$, then $p_{n+1} = f(p_n) \notin M$. It is clear from the definition of f that M contains only odd numbers. Indeed, every odd number from 1 to p_n is in M. For

$$1 = 3^0 = f(2^0) = f(1),$$

so that $1 \in M$; and if $y = p_2^{\alpha_1}\ldots p_{n-1}^{\alpha_{n-2}}$ is any of the remaining odd numbers of L, then, for $x = p_1^{\alpha_1}\ldots p_{n-2}^{\alpha_{n-2}}$, we have $x < y$, so that $x \in L$, and $y = f(x) \in M$; and finally $p_n = f(p_{n-1}) \in M$.

Let ℓ be the smallest odd integer not in M; then $p_n < \ell \leq p_{n+1}$, since $p_{n+1} \notin M$. We will show that $\ell = p_{n+1}$. Suppose $\ell < p_{n+1}$; then ℓ must be composite, say $\ell = p_2^{\alpha_1}\ldots p_n^{\alpha_{n-1}}$, where $\alpha_1 + \ldots + \alpha_{n-1} \geq 2$, for otherwise ℓ would be prime. If $\ell - 2$ is prime, it must equal p_n, in which case, for $\lambda = p_1^{\alpha_1}\ldots p_{n-1}^{\alpha_{n-1}}$, we must have $\lambda < \ell - 2$ (this is a consequence of $\alpha_1 + \ldots + \alpha_{n-1} \geq 2$), so that $\lambda \in L$ and $\ell = f(\lambda) \in M$, a contradiction. If $\ell - 2$ is composite, it must be the largest composite in M by the choice of ℓ, so that $\ell - 2 \leq p_n - 2$, whence $\ell \leq p_n$ and $\ell \in M$, again a contradiction. Thus $\ell = p_{n+1}$ and the proof is complete.

Comment.

Queried as to the source of this problem, the proposer said that he had heard of it a few years ago from a graduate student, but could not identify the source further. He added that to the best of his knowledge — and he has consulted several number theorists about it — the problem was still unsolved. So the proof given above, which unexpectedly turns out to be quite elementary, may well be the first in existence.

It would be interesting to be able to attach a name to this fascinating conjecture, which has now been promoted to the rank of a full theorem. So if any reader can furnish a clue as to the source of this problem, as well as to other proofs of it if any, I would appreciate being informed of it.

155. [1976: 110] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside N.Y., and Ira Ewen, James Madison H.S., Brooklyn, N.Y.*

A plane is *tessellated* by regular hexagons when the plane is the union of congruent regular hexagonal closed regions which have disjoint interiors. A *lattice point* of this tessellation is any vertex of any of the hexagons.

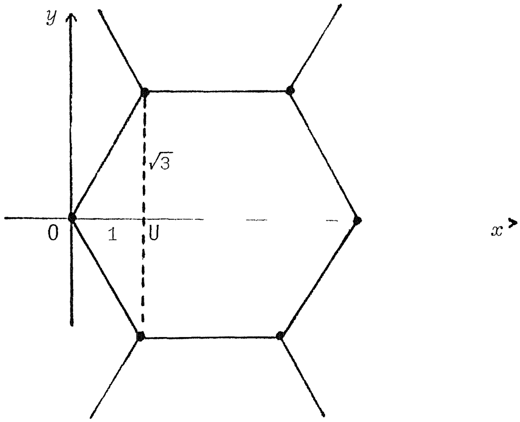
Prove that no four lattice points of a regular hexagonal tessellation of a plane can be the vertices of a regular 4-gon (square).

This theorem may be called the 4-*gon conclusion.*

(This problem was originally written for the 1976 New York State Math League Meet, held on May 1, 1976.)

Solution by Gali Salvatore, Ottawa, Ontario.

Introduce a rectangular coordinate system in the tessellated plane with the origin at one of the lattice points and the unit point on the x-axis at U as shown in the figure. Then every lattice point has coordinates of the form $(x, y\sqrt{3})$, where x and y are integers.



Suppose the lattice points $A(a, r\sqrt{3})$, $B(b, s\sqrt{3})$, $C(c, t\sqrt{3})$, and $D(d, u\sqrt{3})$ are, in counterclockwise order, the vertices of a square. By the usual "cross-multiplication" method of analytic geometry, the area of the square is

$$[(a-c)(s-u)+(b-d)(t-r)]\frac{\sqrt{3}}{2},$$

an irrational number. On the other hand, the area of the square is also

$$|AB|^2 = (a-b)^2 + 3(r-s)^2,$$

which is an integer. With this contradiction, the *4-gon conclusion* is indeed a foregone conclusion.

If, in addition, the centres of all the hexagons are considered as lattice points, thus inducing a tessellation of the plane by equilateral triangles, then the above proof, without any change, shows that the *4-gon conclusion* holds as well for this augmented set of lattice points.

Also solved by G.D. KAYE, *Department of National Defence; and the proposers.*

156. [1976: 110] *Proposé par l'éditeur.*

Déterminer tous les entiers n pour lesquels l'implication suivante est vraie:

Pour tous les réels a, b, c non nuls et de somme non nulle,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \Rightarrow \frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$

Solution de Leroy F. Meyers, The Ohio State University.

Un calcul facile montre que l'hypothèse équivaut à

$$(b+c)(c+a)(a+b) = 0,$$

et donc à

$$b = -c \quad \text{ou} \quad c = -a \quad \text{ou} \quad a = -b. \qquad (1)$$

Si n est un entier impair (plus généralement, si $n = p/q$, où p et q sont des entiers impairs), (1) entraîne

$$b^n = -c^n \quad \text{ou} \quad c^n = -a^n \quad \text{ou} \quad a^n = -b^n,$$

ce qui équivaut à

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n}\ . \tag{2}$$

Si n est un entier pair (plus généralement, si $n = p/q$, où p est pair et q est impair), tous les termes de (2) sont positifs et il n'y a jamais égalité puisque le membre gauche est toujours supérieur au membre droit. On ne peut envisager la possibilité $n = p/q$, où p est impair et q est pair, sans sortir du domaine des réels.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle;* KENNETH S. WILLIAMS, *Carleton University; and the proposer.*

Editor's comment.

Meyers found in [1] the following Russian variant of our problem: prove that for arbitrary odd n

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c} \Rightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^n = \frac{1}{a^n + b^n + c^n} = \frac{1}{(a+b+c)^n}\ .$$

I myself found the problem years ago in a now-forgotten textbook. Since publishing it here, however, I have rediscovered it in Burkhill and Cundy [2] and in Hall and Knight [3], and it probably appears in dozens of other books.

REFERENCES

1. V.G. Boltjanskii and N.Ja. Vilenkin, *Simmetrija v algebre* (Symmetry in Algebra), Moscow, 1967, pp. 78, 248.

2. Burkhill and Cundy, *Mathematical Scholarship Problems*, Cambridge University Press 1962, p. 34.

3. Hall and Knight, *Higher Algebra*, Macmillan, London, 1891, p. 517.

157. [1976: 110] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

In base fifty, the integer x is represented by the numeral CC and x^3 is represented by the numeral $ABBA$. If $C > 0$, express all possible values of B in base ten.

Solution by G.D. Kaye, Department of National Defence.

We have (in base fifty)

$$x = CC = C \cdot 11, \quad x^3 = C^3 \cdot 1331 = ABBA.$$

The last equation only holds for values of C^3 for which the multiplication $C^3 \cdot 1331$ can be effected without a carry-over since 1331 is already in the form *XYYX*. Hence we must have $B = 3C^3 <$ fifty, which holds only for $C = 1$ or 2, giving, in base ten, $B = 3$ or 24. The given numbers are, in base fifty,

$$11^3 = 1331 \quad \text{and} \quad 22^3 = 8\,\overline{24}\,\overline{24}\,8\,.$$

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle;* R. ROBINSON ROWE, *Sacramento, California;* CHARLES W. TRIGG, *San Diego, California; and the proposer.*

Editor's comment.

Dodge observed that the largest base b for which the problem has exactly n solutions is $b = 3(n+1)^3$.

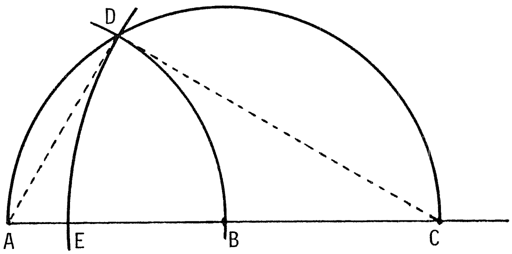
158. [1976: 111] *Proposed by André Bourbeau, École Secondaire Garneau.*

Devise a Euclidean construction to divide a given line segment into two parts such that the sum of the squares on the whole segment and on one of its parts is equal to twice the square on the other part.

Solution by Charles W. Trigg, San Diego, California.

Analysis. If the length of the given segment is y and that of the first part is x, then $x^2 + y^2 = 2(y-x)^2$ or $x^2 - 4xy + y^2 = 0$. Since $x < y$, the applicable solution of this equation is $x = y(2-\sqrt{3})$.

Construction. Extend the given segment AB. With radius BA and B as center describe semicircle AC. With the same radius and A as center describe an arc cutting the semicircle at D. With C as center and radius CD describe an arc cutting AB at E, the desired dividing point. (This procedure requires drawing 1 line, making 2 compass settings, and striking 3 arcs — a total of 6 operations.)



Proof. $\angle$ADC is a right angle, being inscribed in a semicircle. AD = AB = AC/2. Hence CA:AD:CD::2:1:$\sqrt{3}$, so AE = $(2-\sqrt{3})$AB, and EB = $(\sqrt{3}-1)$AB. Thus

$$1^2 + (2-\sqrt{3})^2 = 8 - 4\sqrt{3} = 2(\sqrt{3}-1)^2,$$

and so $\overline{AB}^2 + \overline{AE}^2 = 2\overline{EB}^2$.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence;* ANDRÉ LADOUCEUR, *École Secondaire De La Salle;* R. ROBINSON ROWE, *Sacramento, California;* BERNARD VANBRUGGHE, *Université de Moncton; and the proposer.*

Editor's comment.

The admirable low-energy construction given above should be a lesson for us all, in these energy-conscious times. A Euclidean construction problem should be judged by its efficiency, that is, by the number of Euclidean operations involved — drawing a line, setting a compass, striking an arc — since accuracy with the ruler and compass is inversely proportional to the number of operations performed. Why use a keg of nails in a construction when half a dozen will do a better job? The elegance of the attendant proof is a secondary matter, although it will be found that the more elegant proof frequently comes along with the more efficient construction.

The other solvers, energy wasters all, submitted constructions requiring from 13 to more than 30 Euclidean operations.

159. [1976: 111] *Proposed by Viktors Linis, University of Ottawa.*

Show that

$$x! + y! = z!$$

has only one solution in positive integers, and that

$$(x!)(y!) = z!$$

has infinitely many for $x > 1$, $y > 1$, $z > 1$.

Solution by Clayton W. Dodge, University of Maine at Orono.

Clearly $1! + 1! = 2!$. For $n > 2$, $n! > 2(n-1)!$, establishing the desired uniqueness. The second equation has the obvious solutions given, for all positive integers n, by

$$(n! - 1)!(n!) = (n!)!.$$

For example, when $n = 3$ we get $(5!)(3!) = 6!$ These values get large rapidly; when $n = 4$, $(n!)!$ is approximately 6.2×10^{23}.

Also solved by G.D. KAYE, *Department of National Defence;* R. ROBINSON ROWE, *Sacramento, California; and* KENNETH S. WILLIAMS, *Carleton University.*

Editor's comment.

All solvers gave the same solutions to the second part of the problem.

Madachy says in [1] that there are any number of examples of factorials which are the products of other factorials. He gives as examples

$$(4!)(23!) = 24!$$

$$(2!)(4!)(47!) = 48!$$

$$(2!)(3!)(4!)(287!) = 288!$$

He adds that only four cases are known of factorials which are the products of factorials of numbers in arithmetic progression; these are

$$(1!)(2!) = 2!$$
$$(6!)(7!) = 10!$$
$$(1!)(3!)(5!) = 6!$$
$$(1!)(3!)(5!)(7!) = 10!$$

If zero factorials are allowed, the following can be added to the list, making a total of six:

$$(0!)(1!) = 1!$$
$$(0!)(1!)(2!) = 2!$$

REFERENCE

1. Joseph S. Madachy, *Mathematics on Vacation*, Scribner's, 1966, p. 174.

160. [1976: 111] *Proposed by Viktors Linis, University of Ottawa.*

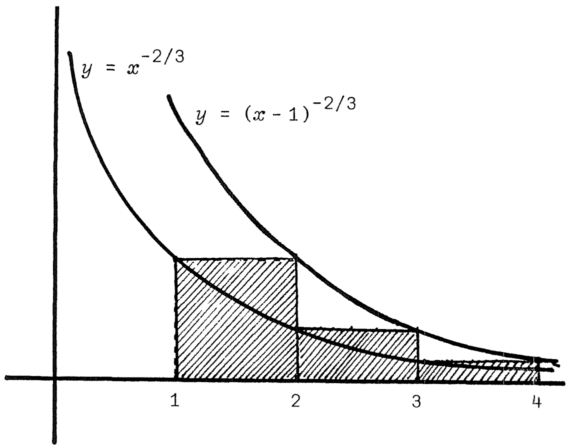
Find the integral part of

$$\sum_{n=1}^{10^9} n^{-2/3}.$$

This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

Solution d'André Ladouceur, École Secondaire De La Salle.

Soit S la somme donnée. S est la somme des aires des 10^9 rectangles dont les trois premiers paraissent sur la figure ci-jointe. On a donc



$$\int_1^{10^9+1} x^{-2/3}\,dx < S < 1 + \int_2^{10^9+1} (x-1)^{-2/3}\,dx,$$

d'où

$$2997 < 3\sqrt[3]{10^9+1} - 3 < S < 2998\,,$$

de sorte que la partie entière de S est 2997.

Also solved by HIPPOLYTE CHARLES, *Waterloo, Québec;* CLAYTON W. DODGE, *University of Maine at Orono; and* R. ROBINSON ROWE, *Sacramento, California.*

Editor's comment.

Rowe was more interested in finding a good approximation to the sum rather than merely its integral part. He noted that

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} x^{-2/3}\,dx = 3(\sqrt[3]{n+\tfrac{1}{2}} - \sqrt[3]{n-\tfrac{1}{2}}) \tag{1}$$

approximates $n^{-2/3}$, the error being very small for large values of n. This gave him the first approximation

$$S \approx \sum_{n=1}^{10^9} I_n = \int_{\frac{1}{2}}^{10^9+\frac{1}{2}} x^{-2/3}\,dx. \tag{2}$$

A careful error analysis, involving in part the expansion of (1) by the binomial theorem, enabled him to find the approximate error $E_n = I_n - n^{-2/3}$ for each term of the summation in (2). After a page or so of computing frenzy, he is able to announce with a flourish to an expectant world that $S \approx 2997.552\,419\,70\ldots$

* * *

THE WITLESS WONDER[1]

Who will fill the humour gap?
Who will spell the wit?
The Editor — he's fastened tight
Mathematizing day and night!
The "subs. man" has a similar slate. . .
We all can sense their sorry state.
The "spares" are those who should be light
And lend the humour — oh, that they might!
But muses are involved you know. . .
Their bounty's of uncertain flow!

MONICA MASKELL

TWO CLERIHEWS

Archimedes of Syracuse
To get into the news
Called out "Ευρηκα"
And became the first streaker.

J.C.W. DE LA BERE[2]

James Cook, Australia-bound
In 1774, on a hunch,
Stopped at the Sandwich Islands
For lunch.

ANN ONYMOUS[3]

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[1] *A propos* of the editor's comment [1976: 186] that EUREKA is sadly lacking in wit and humour.

[2] This clerihew first appeared in the *Australian Mathematical Society Gazette*, Vol. 1, No. 3 (Dec. 1974). It was sent to the editor by Professor Basil C. Rennie, James Cook University of North Queensland, Townsville, Australia.

[3] This clerihew was written by the well-known Miss Onymous, at the editor's request, to thank Professor Rennie for sending the first one.