Crux Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 6, No. 10

December 1980

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Mathematics Department, the Ottawa Valley Education Liaison Council, and the University of Ottawa Mathematics Department are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $12.00. Back issues: $1.20 each. Bound volumes with index: Vols. 1&2 (combined), $12.00; Vols. 3,4,5, $12.00 each. Cheques and money orders, payable to CRUX MATHEMATICORUM (in US funds from outside Canada), should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

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CONTENTS

BICOMPLEX NUMBERS

HERBERT E. SALZER

Abstract. Iteration of the process by which complex numbers are defined as an algebra of ordered pairs of real numbers, by replacing those real numbers by ordered pairs of real numbers, leads to a commutative algebra with divisors of zero, of bicomplex numbers $a+ib+jc+kd$ where $i^2=j^2=-1$, $k^2=+1$, $ij=k$, etc. These bicomplex numbers have the mathematically disappointing property of being essentially equivalent to a direct sum of two complex numbers. Moreover, it is proven that any n-fold iteration of the ordered pair definition of complex numbers still gives a direct sum of now 2^n complex numbers. None of the other authors cited have stated this direct sum property. The multiplicative relations of i, j and k, when interpreted physically, are qualitatively suggestive of electrical, magnetic and gravitational fields and their interactions. Further problems and a speculation concerning these bicomplex analogues of physical fields are mentioned briefly.

I. *Definition and Properties.*

The system of ordinary complex numbers $A=a+ib$ is often presented as an algebra of ordered pairs of real numbers, where for $A=(a,b)$ and $B=(c,d)$ the sum $A+B$ is the ordered pair $(a+c,\ b+d)$ and the product AB is the ordered pair $(ac-bd,\ ad+bc)$. The numbers 0, 1 and $i=\sqrt{-1}$ correspond to (0,0), (1,0) and (0,1). Suppose that we iterate the ordered pairing by having in place of the real numbers a, b, c and d the complex numbers A, B, C and D given as the ordered pairs $A=(a,b)$, $B=(c,d)$, $C=(e,f)$ and $D=(g,h)$, and in the sum and product of $\mathbb{A}=(A,B)$ and $\mathbb{B}=(C,D)$ in terms of A, B, C and D we follow the rules given above for a, b, c and d. Thus

$$\mathbb{A}+\mathbb{B}=(A+C,\ B+D)=((a+e,\ b+f),\ (c+g,\ d+h)) \tag{1}$$

and

$$\begin{aligned}\mathbb{A}\mathbb{B}&=(AC-BD,\ AD+BC)\\ &=((ae-bf-cg+dh,\ af+be-ch-dg),\ (ag-bh+ce-df,\ ah+bg+cf+de)).\end{aligned} \tag{2}$$

We may drop the inner parentheses without ambiguity so that

$$\mathbb{A}=(A,B)=((a,b),\ (c,d))=(a,b,c,d). \tag{3}$$

We may also write

$$\mathbb{A}=(A,B)=A+jB=(a+ib)+j(c+id), \tag{4}$$

where j plays a role similar to that of i in ordinary complex numbers. $\mathbb{A}$ in any of these forms will be referred to as a *bicomplex number*. We have this correspondence:

$$0 = (0,0,0,0), \quad 1 = (1,0,0,0), \quad i = (0,1,0,0), \quad j = (0,0,1,0), \tag{5}$$

the last because $\mathbb{B} = (B,0)$, when multiplied by $(0,0,1,0) = ((0,0),\ (1,0)) = (0,1)$, becomes $(0,B)$ by the product law. Since the multiplication law gives

$$\begin{aligned} ij &= (0,1,0,0)(0,0,1,0) = ((0,1),\ (0,0))((0,0),\ (1,0)) = ((0,0),\ (0,1)) \\ &= (0,0,0,1), \end{aligned} \tag{6}$$

we denote $(0,0,0,1)$ by k. Also from the multiplication law any bicomplex number (a,b,c,d), where each real component a, b, c, d is expressible as $a = (a,0,0,0)$, $b = (b,0,0,0)$, etc., may be written as $a + ib + jc + kd$. Finally, from the multiplication law we have

$$i^2 = j^2 = -1, \quad k^2 = +1, \quad ij = ji = k, \quad ik = ki = -j, \quad jk = kj = -i. \tag{7}$$

The bicomplex numbers $\mathbb{A} = (A,B)$ have all the algebraic properties of ordinary complex numbers, including commutativity of multiplication, with the sole exception of having divisors of zero; that is, we can have $\mathbb{AB} = 0 = (0,0,0,0)$ with both $\mathbb{A} \neq 0$ and $\mathbb{B} \neq 0$. This occurs, for example, for $\mathbb{A} = (1,0,0,1)$ and $\mathbb{B} = (1,0,0,-1)$. In other words, bicomplex numbers satisfy only that part of the product law where $\mathbb{A}\cdot 0 = 0\cdot\mathbb{A} = 0$, but not the part where $\mathbb{AB} = 0$ implies that either $\mathbb{A} = 0$ or $\mathbb{B} = 0$.

II. *Other Authors.*

Bicomplex numbers are not new, having been known to W.R. Hamilton [1], who merely mentions the "bi-couple" $[q'] + i[q'']$, said to be "complanar with i," where

$$[q] = x + hy, \quad h^2 + 1 = 0, \quad hi = ih, \text{ and } h \neq \pm i\,. \tag{8}$$

Further on, Hamilton states that every equation of degree n has n^2 bicomplex roots. The most extensive treatment of bicomplex numbers is given by A. Seydler [2] who refers to Hamilton, saying that this "doppelt-complexen Form" is identical with a "class of quaternions." The writer recalls seeing also, some time ago, around 1935-1940, the rediscovery of bicomplex numbers by a Japanese mathematician who gave geometrical applications (reference not handy).

III. *Equivalence to Direct Sums, and a Generalization.*

At first glance, these bicomplex numbers might appear to be promising, since they bear a superficial resemblance to quaternions, and also the asset of commutativity of multiplication seems to counterbalance advantageously the liability of divisors of zero. But all hopes for something mathematically new are immediately dashed by the apparently fatal property that every bicomplex number may be written as a direct sum of two complex numbers since , from (7), we have

$$a+ib+jc+kd=\tfrac{1}{2}(1+k)[(a+d)+i(b-c)]+\tfrac{1}{2}(1-k)[(a-d)+i(b+c)], \tag{9}$$

where $\alpha_1 \equiv \frac{1}{2}(1+k)$ and $\alpha_2 \equiv \frac{1}{2}(1-k)$ satisfy

$$\alpha_1^2 = \alpha_1, \quad \alpha_2^2 = \alpha_2, \quad \text{and} \quad \alpha_1\alpha_2 = 0. \tag{10}$$

From (9) and (10), any mathematical property or function of A becomes simply that property or function of two separate complex numbers. For example, the property mentioned in Section II that every equation (allowing also bicomplex coefficients) of degree n has n^2 bicomplex roots is immediately apparent when each bicomplex coefficient is written as a direct sum of complex numbers, and the bicomplex variable X is written as the direct sum $\alpha_1X_1+\alpha_2X_2$, where X_1 and X_2 are complex, so that, from (10),

$$\mathsf{X}^m = \alpha_1X_1^m + \alpha_2X_2^m, \qquad m=0,1,\ldots,n.$$

Any equation is thus seen to be a direct sum of two complex equations of degree n in the complex variables X_1 and X_2 so that, by the pairing of the n roots from the separate equations in n^2 ways, we obtain all the bicomplex roots of the bicomplex equation. For an example involving some function, let $f(x)$ be any function represented by an infinite series with real or complex coefficients. Then for

$$\mathsf{A} = A + jB = \alpha_1A_1 + \alpha_2B_1$$

we have

$$f(\mathsf{A}) = \alpha_1f(A_1) + \alpha_2f(B_1). \tag{11}$$

To illustrate (11) more specifically, consider

$$\begin{aligned}
\exp(\mathsf{A}) &= \exp(\alpha_1A_1+\alpha_2B_1) = \exp(\alpha_1A_1)\cdot\exp(\alpha_2B_1)\\
&= \{1+\alpha_1(A_1+\tfrac{1}{2}A_1^2+\ldots)\}\cdot\{1+\alpha_2(B_1+\tfrac{1}{2}B_1^2+\ldots)\}\\
&= 1+\alpha_1(A_1+\tfrac{1}{2}A_1^2+\ldots)+\alpha_2(B_1+\tfrac{1}{2}B_1^2+\ldots)\\
&= \alpha_1+\alpha_2+\alpha_1(A_1+\tfrac{1}{2}A_1^2+\ldots)+\alpha_2(B_1+\tfrac{1}{2}B_1^2+\ldots)\\
&= \alpha_1(1+A_1+\tfrac{1}{2}A_1^2+\ldots)+\alpha_2(1+B_1+\tfrac{1}{2}B_2^2+\ldots)\\
&= \alpha_1\exp(A_1)+\alpha_2\exp(B_1).
\end{aligned} \tag{12}$$

The converse is also true (with the three reservations that follow concerning commutativity) except for an indeterminate idempotent factor of proportionality. Any direct sum of complex numbers $\alpha_1(a+ib)+\alpha_2(c+id)$, where α_1 and α_2 satisfy (10) without necessarily being $\frac{1}{2}(1\pm k)$, is equal to a bicomplex number multiplied by that proportionality factor. Thus

$$\alpha_1(a+ib)+\alpha_2(c+id) = \tfrac{1}{2}(\alpha_1+\alpha_2)(a+c)+\frac{i}{2}(\alpha_1+\alpha_2)(b+d)$$
$$+\frac{i}{2}(\alpha_1-\alpha_2)(b-d)+\tfrac{1}{2}(\alpha_2-\alpha_1)(-a+c). \tag{13}$$

If in (13) we denote $\alpha_1+\alpha_2$ by $\mathbb{1}$, since we have $\mathbb{1}^2=\mathbb{1}$ we see that $\alpha_2-\alpha_1$ should correspond to $k\mathbb{1}$ for some k satisfying $k^2=+1$ because $(\alpha_2-\alpha_1)^2$ is also $\mathbb{1}$. Then $i(\alpha_1+\alpha_2)=i\mathbb{1}$ and $i(\alpha_1-\alpha_2)=-ik\mathbb{1}=j\mathbb{1}$, where the last equation defines j as $-ik$. In (13) we have already assumed tacitly that i commutes with α_1 and α_2. One further assumption, that k also commutes with α_1 and α_2 (we do not require that $ik=ki$), suffices for

$$\alpha_1(a+ib)+\alpha_2(c+id) = \mathbb{1}\{\tfrac{1}{2}(a+c)+\tfrac{1}{2}i(b+d)+\tfrac{1}{2}j(b-d)+\tfrac{1}{2}k(-a+c)\}. \tag{13'}$$

However, we do require that $ik=ki$ in order that i, j and k satisfy all the relations in (7). Without making any of the preceding three assumptions on commutativity, we can still assert that

$$\alpha_1(a+ib)+\alpha_2(c+id) = \tfrac{1}{2}\alpha(a+c)+\tfrac{1}{2}\beta(b+d)+\tfrac{1}{2}\gamma(b-d)+\tfrac{1}{2}\delta(-a+c), \tag{13''}$$

where $\alpha^2=\delta^2=\alpha$ and $\beta^2=\gamma^2=-\alpha$.

Oddly enough, neither Hamilton, Seydler nor any other writer has mentioned, or even appeared to have been aware of, this direct sum property. Thus in reading Seydler's article one gets the impression that he believes bicomplex numbers to have a significance on a par with that of quaternions. Of course, at the present time anyone would be immediately suspicious of something like bicomplex numbers being purportedly new, in view of Frobenius' theorem that the only finite associative real algebras with principal unit and product law are the real, complex and quaternion algebras [3]. If we include commutativity of multiplication, we are left with only the real and complex algebras.

We might anticipate failure in trying to find something new by iterating the definition of ordered pairs of ordered pairs once more, in the hope that the direct sum property might not continue into that algebra of "quatrocomplex" numbers having eight real components. We can show even more by induction, namely, that the n-fold iterate, $n\geq 1$, of the definition of ordered pairs gives a number that is the direct sum of 2^n complex numbers. This was just shown for $n=1$. Assume that the m-fold iterate is the direct sum

$$\sum_{r=1}^{2^m} \alpha_r(a_r+ib_r).$$

Then the $(m+1)$-fold iterate gives a number that is algebraically equivalent to

$$\sum_{r=1}^{2^m} \alpha_r(a_r+ib_r) + J\sum_{r=1}^{2^m} \alpha_r(c_r+id_r),$$

where J is a new unit differing from the 2^{m+1} units $1, i, j, k, \ldots$ which occur in the m-fold iterate and where $J^2=-1$. Also, J is easily seen to commute with all previous units which, from the hypothesis of the induction, are already known to commute with one another. In particular, $Ji=iJ=K$, so that $K^2=+1$ and $J=-iK=-Ki$. The preceding sum may then be written as

$$\sum_{r=1}^{2^m} \alpha_r\{\tfrac{1}{2}(1+K)[(a_r+d_r)+i(b_r-c_r)]+\tfrac{1}{2}(1-K)[(a_r-d_r)+i(b_r+c_r)]\},$$

which is seen to be a direct sum of the 2^{m+1} complex numbers

$$(a_r+d_r)+i(b_r-c_r) \quad\text{and}\quad (a_r-d_r)+i(b_r+c_r), \qquad r=1,2,\ldots,2^m,$$

because in taking the product of any two such expressions, which is a sum of terms $\alpha_r\{\ldots\}\cdot\alpha_s\{\ldots\}$, the cross products where $r\neq s$ vanish since then $\alpha_r\alpha_s=0$, and when $r=s$ there are no cross products from the complex numbers inside the corresponding braces since $\{\tfrac{1}{2}(1+K)\}\cdot\{\tfrac{1}{2}(1-K)\}=0$. The product will therefore be of exactly the same form as either factor since

$$\{\tfrac{1}{2}(1+K)\}^2=\tfrac{1}{2}(1+K),\quad \{\tfrac{1}{2}(1-K)\}^2=\tfrac{1}{2}(1-K),\quad\text{and}\quad \alpha_r^2=\alpha_r,\quad r=1,2,\ldots,2^m.$$

IV. *Some Physical Field Interpretations.*

Having established the mildly interesting property that continued iteration of the ordered pair definition produces nothing algebraically new, only direct sums of complex numbers, it appears that there should be no further *mathematical* interest in bicomplex numbers. However, from a *physical* standpoint there is something suggestive in the multiplication of the units i, j and k, if we do not view the parallels between the mathematical and physical properties in too literal a fashion. Even though the physics of today must take into account all kinds of new forces and particles, as well as quantum theory, we shall here revert to those comfortable earlier days when the electrical, magnetic and gravitational forces were the only types that were recognized besides mechanical forces. Then $i^2=j^2=-1$ and $k^2=+1$ suggest making i and j correspond to static electrical and magnetic fields, since like charges and poles rēpel each other, while k might correspond to a gravitational field, since all material particles attract one another. It is convenient to imagine that an action-reaction at a point, when consisting of a pull reacting to a pull, which is thus an attraction, could be separated in space to become a field of mutual attraction at a distance, whereas a push reacting to a push, which is now a repulsion, could be separated in space to become a field of mutual repulsion at a distance. The real number 1 or -1 might correspond to a doublet of action and reaction, constituting either an attraction or a repulsion at the same point, while the i, j and k fields of attraction and repulsion might be thought of as the electrical, magnetic and gravitational factors of this action-reaction doublet, but acting at a distance. In other words, we may have action and reaction separated in space and assuming the character of mutually interacting i, j or k fields. This is strictly a classical physics picture, since instant action at a distance is ruled out in relativity physics.

To avoid confusion with real numbers as scalar multipliers of i, j and k, it will probably be necessary, without change in the algebraic properties, to employ the form given in (13") for the physical representation of a bicomplex number, which is

$$\alpha a + \beta b + \gamma c + \delta d, \quad \text{where } \alpha^2 = \delta^2 = \alpha, \quad \beta^2 = \gamma^2 = -\alpha,$$

so that α will denote the unfactored doublet of action and reaction, which is consistent with its being idempotent, and real numbers will be just positive or negative multipliers of α, β, γ or δ. If we are constantly aware of this distinction, we could continue to employ in the further physical discussion the more familiar 1, i, j and k instead of the more accurately descriptive α, β, γ and δ.

We might notice that this bicomplex number representation of physical force fields leaves no provision for antigravity or gravitational repulsion, since -1 [$-\alpha$] has only $\pm i$ and $\pm j$ [$\pm\beta$ and $\pm\gamma$] for square roots, there being no physically meaningful distinction between $(-i)^2 = -1$ and $i^2 = -1$, etc.

We might be tempted to try the assumption of four types of the most familiar classical fields being represented by the $\alpha a + \beta b + \gamma c + \delta d$ form, where β, γ and δ are for electrical, magnetic and gravitational fields, while α would denote a mechanical field acting only on contact. But that has three difficulties offhand, namely,

1) $\alpha^2 = \alpha$ would equate the action-reaction doublet to its field component;

2) it provides only for a pull-pull contact action-reaction; and

3) having two types of fields that act upon masses, tending to accelerate them, goes against the spirit of Einstein's principle of equivalence of uniform gravitational and mechanical acceleration fields.

All action and reaction at the same point are obviously instantaneous, from both classical as well as relativistic viewpoints. Some further extension of the static i, j and k fields may cover action and reaction at a distance, which are equal and opposite, but delayed (i.e., separated in time as well as in space), so that it would involve possibly something like a finite time of propagation of force or momentum of a mechanical or electromagnetic nature, and some fundamental velocity like c = velocity of light. This speculation will not be pursued here.

Incidentally, we should not be misled into thinking that, because a bicomplex number is a direct sum of complex numbers, a representation of the electrical, magnetic and gravitational fields is available from two separate and noninteracting combinations of an action-reaction doublet with a repulsion field. The reason is that the α_1 and α_2 factors in the direct sum, when they are $\frac{1}{2}(1 \pm k)$, involve the k field of attraction, and have the added difficult property of not being physically homogeneous or easy to interpret. Furthermore, for α_1 and α_2 not $\frac{1}{2}(1 \pm k)$ but satisfying (10), we would still be bringing in those two arbitrary and unexplained quantities having physical significance and constrained to meet the three conditions of (10).

Continuing the physical interpretation, we can go further on the basis of i, j and k alone, by a little stretch of the meaning of multiplication, and bring in something special to these bicomplex numbers that is suggestive of both Ampère's law and Faraday's law of electromagnetic induction, both of which constitute the essential basis of Maxwell's equations. Even though this parallel is only qualitative, since no units are specified or even hinted at, there are three interesting and also confirming physical aspects. One is that the independence of only two of the three force fields under consideration corresponds to the algebraic independence of only two of the i, j and k components. The second is that, from $ij = ji$ in (7) the simplest instance of the commutativity of multiplication, we find a reason to expect the opposite signs in the following two of the four Maxwell field equations *in vacuo*, which are the differential forms of Ampère's and Faraday's laws,

$$\operatorname{curl} \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \qquad \operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{14}$$

where the vector $\vec{E}$ = electric field, $\vec{H}$ = magnetic field, $\vec{D}$ = electric displacement and $\vec{B}$ = magnetic induction. The third is that all the discussion given below of $ij = k$ which interprets k as a mechanical force, in conjunction with the gravitational field designation of k, fits in nicely with Einstein's principle of equivalence of a uniform gravitational field and a uniform acceleration field that is produced by a mechanical force. Only the second of these confirmatory aspects needs further explanation (cf. paragraph after next).

In view of the facts that $i^2 = j^2 = -1$, $k^2 = +1$ are applied to two like *static* fields next to each other, we shall interpret $ij = k$ in an arbitrarily different physical manner. To bring in some geometry, consider i, j and k as an orthogonal configuration formed by the middle finger, thumb and forefinger of the right hand, the middle finger, for i, pointing left; the thumb, for j, pointing up; and the forefinger, for k, pointing out from the body. Now if we still interpret j as a static magnetic field, and adopt as an arbitrary convention the rule that whatever points to the left is to be considered as *moving* uniformly, so that in this case we may now regard i as a moving electric field or a current, according to a well-known law (that is an equivalent statement of Ampère's law), there is a mechanical force whose action upon any mass that is connected with i may be considered as equivalent to that of a gravitational or k field and which does point out from the body. Just as the gravitational field shows no real existence by itself, being detected through its interaction with some mass, or expressed otherwise, a k field is detected only by something that produces another k field, likewise the k field generated electromagnetically according to this interpretation of ij, which is a mechanical force field, shows its reality by interacting with an inertial field that is inherent in any mass connected with i. At this point, we should justify the consistency of having $i^2 = j^2 = -1$, $k^2 = +1$ (which are real numbers understood to be the earlier suggested α), for a force of repulsion or attraction that would be exerted on any mass attached to a charge, pole or particle, with having $ij = k$ also denote such a force. They are different because, as shown by the real number, in the former case we have an action-reaction doublet, while k in the latter case shows just a mechanical action with no reaction unless there is that attached inertial mass whose acceleration produces the reaction.

Next, suppose we rotate that right hand 90° counterclockwise about the forefinger as axis, so that now the thumb points to the left and the middle finger points downward. The commutativity in $ij = ji = k$ gives just one possible value for k so long as those fingers do not change their orientation with respect to each other. In any orientation of the hand there will always be just one k, but we are associating different physical phenomena with different hand orientations (for no *a priori* reason, just to fit the physical facts). If we did not have this commutativity we should expect some physical difference to correspond to the algebraic difference between ij and ji, which is more than is needed since the Ampère and Faraday laws follow only from the convention given above applied to different hand orientations. In the new orientation, the thumb pointing left represents now a moving magnetic or

j field which is crossing the electrostatic or i field that is directed in an *opposite sense* (middle finger now pointing down) of that static j field in the previous orientation, and producing a mechanical force or k field in the *same outward direction* as before. This is a qualitative statement of Faraday's law of electromagnetic induction that happens to be consistent with the difference of signs in the right members of (14).

V. *Further Physical Problems and a Speculation.*

To summarize so far whatever has been suggested physically, in a qualitative way, by all but the last two of the equations in (7), we have action and reaction, the behavior of three outstanding force fields (electrostatic, magnetostatic, gravitational), the equivalence of mechanical force fields with gravitational fields, Ampère's law of electric currents producing magnetic fields, and Faraday's law of electromagnetic induction, the last two being mutually consistent with respect to sign.

For use in three dimensions, all the preceding could be extended to where we have vector instead of scalar coefficients of the 1, i, j and k (really α, β, γ and δ, cf. IV, second paragraph), so long as we do not confuse the x, y and z components of these position vector coefficients with the i, j and k when employed with orientation in the product $ij = k$.

On a more sober note, we should be aware that these suggested physical analogues of bicomplex numbers present conceptual problems and tend to arouse further speculations having even less firmer bases than those offered for the interpretation of the i, j and k relations. For example, here are just three problems in connection with the further development of the i, j and k approach to fields:

1) There is the question of units and relative magnitudes and the tremendous disparity between the mechanical forces involved in electrical or magnetic attraction and the much smaller gravitational attraction which is not shown by k alone.

2) There is the question of the velocities involved when i or j in $ij = k$ are interpreted as moving fields.

3) The most fundamental question is whether this interpretation of i, j and k can be made conceptually compatible with the tremendously complicated present-day outlook on all the physical forces and fields.

We conclude by showing how this i, j, k system gives rise to an interesting speculation concerning some possibly new predictable electromechanical effects. Consider now $ki = -j$ or $kj = -i$ in connection with a third orientation of the right hand which is obtained from the first orientation by rotating 90° counterclockwise about the thumb, so that the forefinger pointing right now denotes a moving k field which may represent either a moving mechanical force or gravitational field that can also be regarded as equivalent to an acceleration that is moving uniformly along some direction (but this is not the same as a nonuniform acceleration at a point). The middle finger pointing toward the body denotes the electrostatic or i field in the negative direction, while the thumb still pointing up denotes the magnetostatic field in the positive direction, which is consistent with the minus signs above. This suggests the possibility of a physical effect when a moving mechanical force or gravitational field cuts a static electric [magnetic] field at right angles, which would appear as a static magnetic [electric] field which is at right angles to the plane of the force and electric [magnetic] field. This effect should be sought for by producing very rapidly moving forces or accelerations in extremely powerful electric [magnetic] fields.

We might thus say, in ending, that these bicomplex numbers are basically a disappointment as a mathematical entity and only suggestive as a representation of physical entities, and yet present an overall picture that resists dismissal as totally uninteresting!

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* * *

IT TAKES ALL KINDS...

Graduating from West Point in 1841, he [Nathaniel Lyon] had taken into the army a strong detestation for higher mathematics—the calculus, he insisted, "lies outside the bounds of reason" and had doubtless been invented by someone of disordered imagination... [p.36]

Into St. Louis toward the end of August came a plump balding young regular army officer, Major John M. Schofield, who had been Lyon's chief of staff; a serious-minded officer deeply interested in physics, whose spare-time pursuit it was to try to "work out the mathematical interpretation of all the phenomena of physical science, including electricity and magnetism,"... [p. 74]

BRUCE CATTON, *This Hallowed Ground*, Pocket Books, New York, 1961.

* * *

THE OLYMPIAD CORNER: 20

MURRAY S. KLAMKIN

I wish to thank all readers who have taken the trouble to send me comments on or solutions to problems published earlier in this column. Although lack of space does not permit me to publish or even to acknowledge here all contributions received, I shall from time to time publish selected ones, starting with the few given below.

1-1. [1979: 44] If a, b, c, d are positive integers such that $ab = cd$, prove that $a^2 + b^2 + c^2 + d^2$ is never a prime number. (From a West German Olympiad.)

Comment by F. David Hammer, Los Gatos, California.

The conclusion holds for sums of all kth powers, not only for sums of squares. For since a, b, c, d must have the form

$$a = pm, \quad b = qn, \quad c = mn, \quad d = pq,$$

then we have

$$a^k + b^k + c^k + d^k = (m^k + q^k)(p^k + n^k), \qquad k = 1,2,3,\ldots .$$

*

1. [1979: 259] Find all triangles ABC for which

$$\mathrm{AB} + \mathrm{AC} = 2 \text{ cm} \quad \text{and} \quad \mathrm{AD} + \mathrm{BC} = \sqrt{5} \text{ cm},$$

where AD is the altitude through A, meeting BC at right angles in D.

Comment by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.

I solve here the analogous 3-dimensional problem suggested by Professor Klamkin [1979: 260]:

Determine all tetrahedra P-ABC *such that*

$$\mathrm{PA} + \mathrm{PB} + \mathrm{PC} = 3 \quad \text{and} \quad \mathrm{QA} + \mathrm{QB} + \mathrm{QC} + \mathrm{QP} = \sqrt{10},$$

where PQ *is an altitude of* P-ABC.

Let $\mathrm{QA} = x$, $\mathrm{QB} = y$, $\mathrm{QC} = z$, and $\mathrm{QP} = h$; then

$$\mathrm{PA} = \sqrt{x^2+h^2}, \quad \mathrm{PB} = \sqrt{y^2+h^2}, \quad \mathrm{PC} = \sqrt{z^2+h^2},$$

and

$$\sqrt{x^2+h^2} + \sqrt{y^2+h^2} + \sqrt{z^2+h^2} = \frac{3}{\sqrt{10}}(x + y + z + h). \tag{1}$$

Since $(x-3h)^2 \geq 0$, we have $\sqrt{x^2+h^2} \geq (3x+h)/\sqrt{10}$, with equality if and only if $x = 3h$, and two similar inequalities involving y and z. It now follows that (1) is satisfied if and only if

$$x = y = z = 3h \qquad \text{or} \qquad x = y = z = \frac{3}{\sqrt{10}}, \quad h = \frac{1}{\sqrt{10}}.$$

This leads to an infinite class of tetrahedra obtained by starting out with the altitude PQ $= 1/\sqrt{10}$ and then constructing three arbitrary congruent segments QA, QB, QC, each of length $3/\sqrt{10}$ and each perpendicular to PQ.

*

12-3. [1980: 149] If Alice and Bob toss 11 and 9 fair coins, respectively, show that the probability that Alice gets more heads than Bob is

$$\frac{1}{2} + \frac{1}{2^{21}}\binom{20}{10} \approx 0.588.$$

Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.

As in the original solution [1980: 149], we suppose, more generally, that Alice and Bob toss $m+n$ and n coins, respectively, and denote by $P(m,n)$ the probability that Alice gets more heads than Bob. By the symmetry between heads and tails, we get

$P(m,n) =$ (Probability that no. of Alice's heads - no. of Bob's tails > 0)

$=$ (Probability that no. of Alice's heads $- n +$ no. of Bob's heads > 0)

$=$ Probability of $n+1$ or more heads in $m+2n$ tosses

$$= \frac{1}{2^{m+2n}} \sum_{k=n+1}^{m+n} \binom{m+2n}{k}. \tag{1}$$

For small positive values of m, the calculation is facilitated by noting that

$$2^{m+2n}P(m,n) = \sum_{j=0}^{m+n-1}\binom{m+2n}{j} = 2^{m+2n} - \sum_{j=m+n}^{m+2n}\binom{m+2n}{j}, \tag{2}$$

and thus, combining (1) and (2), we have

$$P(m,n) = \frac{1}{2} + \frac{1}{2^{m+2n+1}} \sum_{j=n+1}^{m+n-1}\binom{m+2n}{j}, \qquad \text{if } m > 0, \tag{3}$$

and

$$P(0,n) = \frac{1}{2} - \frac{1}{2^{2n+1}}\binom{2n}{n}.$$

In particular, (3) gives $P(2,9) = \dfrac{1}{2} + \dfrac{1}{2^{21}}\dbinom{20}{10} \approx 0.588.$

*

J-1. [1980: 145] Prove that $\left(\frac{\sin x}{x}\right)^3 \geq \cos x$; $0 < x \leq \frac{\pi}{2}$.

Solution by P.R. Beesack, Carleton University, Ottawa.

The analysis that follows is valid for all x in the interval $(0,\pi/2]$. Since

$$1 > \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots > 1 - \frac{x^2}{6} > 0 ,$$

we have

$$F(x) \equiv \left(\frac{\sin x}{x}\right)^3 - \cos x \geq G(x) \equiv \left(1 - \frac{x^2}{6}\right)^3 - \cos x.$$

Since, furthermore, $G(0) = 0$ and

$$G'(x) = \sin x - x\left(1 - \frac{x^2}{6}\right)^2 = x\left\{\frac{\sin x}{x} - \left(1 - \frac{x^2}{6}\right)^2\right\} > 0,$$

it follows that $F(x) \geq G(x) > 0$.

Comment by M.S.K.

This problem appears with extensions in Mitrinović's *Analytic Inequalities*, Springer-Verlag, 1970, pp. 238-239, where it is proved by showing that if

$$H(x) = x - \sin x\,(\cos x)^{-1/3},$$

then $H''(x) < 0$, $H'(x) < H'(0) = 0$, and $H(x) < H(0) = 0$.

I had earlier proposed (in *Pi Mu Epsilon Journal*, 4 (1966) 182) the following inequality:

$$\left(\frac{\sin x}{x}\right)^m \geq \cos x, \qquad 0 < x \leq \frac{\pi}{2}, \quad 0 \leq m \leq 3.$$

*

J-2. [1980: 145] Solve in rational numbers x, y, z, and t:

$$(x + y\sqrt{2})^2 + (z + t\sqrt{2})^2 = 5 + 4\sqrt{2}.$$

Solution by P.R. Beesack, Carleton University, Ottawa.

We use an indirect proof to show there are no solutions. Suppose a solution exists. Expanding the squares, we must have

$$xy + zt = 2 \quad \text{and} \quad x^2 + z^2 + 2(y^2 + t^2) = 5. \tag{1}$$

By Cauchy's inequality,

$$4 = (xy + zt)^2 \leq (x^2 + z^2)(y^2 + t^2) = mn,$$

where $m + 2n = 5$. Since $m,n > 0$ and $m = 5-2n$, we have

$$(5 - 2n)n \geq 4 \quad \text{or} \quad 2n^2 - 5n + 4 \leq 0.$$

This gives a contradiction since

$$2n^2 - 5n + 4 = 2(n - \frac{5}{4})^2 + \frac{7}{8} > 0.$$

There would be rational solutions if the 5 in (1) were replaced by 6; for example,

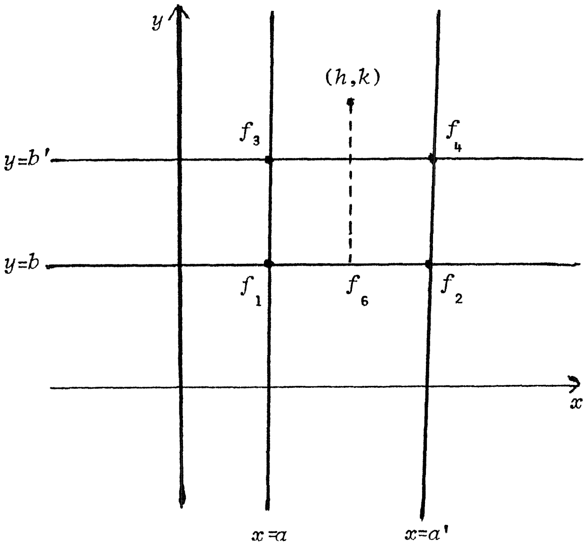
$$(x,y,z,t) = (0,0,2,1).$$

*

J-3. [1980: 145] A function $f(x,y)$ of two variables takes on at least 3 values. For some fixed numbers a and b, we have $f(a,y) \neq$ constant, $f(x,b) \neq$ constant. Prove that there exist numbers p,q,r,s such that $f(p,q)$, $f(r,q)$, $f(p,s)$ are three pairwise-distinct values of $f(x,y)$.

Solution by K.S. Murray, Brooklyn, N.Y.

Equivalently, we have to prove the existence of three points (x_i,y_i) which are vertices of a right triangle with legs parallel to the axes and such that the function values at these three points are all distinct.



Let $f_1 = f(a,b)$. Then there exist points (a',b) and (a,b') such that

$$f_2 = f(a',b) \neq f_1 \quad \text{and} \quad f_3 = f(a,b') \neq f_1,$$

as indicated in the figure. If $f_2 \neq f_3$, we are done. So we assume $f_2 = f_3$.

Consider the point (a',b') and let $f_4 = f(a',b')$. If f_4 is distinct from f_1 and f_2, we are done; if not, there is another point (h,k) such that $f_5 \equiv f(h,k)$ is distinct from f_1 and f_2. If (h,k) lies on one of the four lines $x = a,a'$ or $y = b,b'$ we are done. If not, consider its projection onto the line $y = b$ (i.e., the point (h,b)). Then, regardless of the value of $f_6 \equiv f(h,b)$ we are done.

*

J-4. [1980: 146] Let $ab = 4$, $c^2 + 4d^2 = 4$. Prove the inequality

$$(a-c)^2 + (b-d)^2 \geq 1.6.$$

Solution by Dieter K. Ross in James Cook Mathematical Notes 2 (*September* 1980) 122-123, *reprinted by permission of the editor of JCMN.*

Clearly what is needed is an expression for the square of the shortest distance between the hyperbola $xy = 4$ and the ellipse $x^2 + 4y^2 = 4$. The line representing this shortest distance is normal to both curves and this leads to the relation $ac = 4bd$.

Since the points (a,b) and (c,d) can be taken to lie in the first quadrant of the Cartesian plane it follows by simple algebra that

$$b = \frac{4}{a}, \qquad c = \frac{16}{\sqrt{a^4 + 64}}, \qquad d = \frac{a^2}{\sqrt{a^4 + 64}},$$

and that

$$(a-c)^2 + (b-d)^2 = \left(a - \frac{16}{\sqrt{a^4+64}}\right)^2 + \left(\frac{4}{a} - \frac{a^2}{\sqrt{a^4+64}}\right)^2.$$

This expression has its least value when a is the positive root of the equation

$$12a^3 = (a^4 - 16)\sqrt{a^4 + 64}$$

The appropriate root is $a = 2.390978...$ which leads to the final inequality

$$(a-c)^2 + (b-d)^2 \geq 1.774796... \;.$$

*

J-5. [1980: 146] Let ABCD be a tetrahedron with $DB \perp DC$ such that the perpendicular to the plane ABC coming through the vertex D intersects the plane of the triangle ABC at the orthocenter of this triangle. Prove that

$$(|AB| + |BC| + |AC|)^2 \leq 6(|AD|^2 + |BD|^2 + |CD|^2).$$

For which tetrahedra does the equality take place?

Comment.

This problem previously appeared in the Twelfth International Mathematical

Olympiad (1970). It can be found, along with several solutions, in Samuel L. Greitzer, *International Mathematical Olympiads 1959-1977*, Mathematical Association of America, Washington, D.C., 1978, pp. 12, 127-129.

*

J-6. [1980: 146] What is more: $\sqrt[3]{60}$ or $2+\sqrt[3]{7}$?

Solution by D. Cross, University of Exeter, England. (Similar solution by I. Pressman, Carleton University, Ottawa.)

Since the nth root function is concave for $n>1$, we have, whenever $x>y>0$,

$$\sqrt[n]{x+y}+\sqrt[n]{x-y} < 2\sqrt[n]{\tfrac{1}{2}\{(x+y)+(x-y)\}}. \tag{1}$$

With $n=3$, $x=7.5$, and $y=0.5$, (1) becomes

$$2+\sqrt[3]{7}<\sqrt[3]{60}.$$

Comment by M.S.K.

I gave this problem at the last Olympiad training session at Annapolis and apparently many of the students had difficulty with it.

A purely algebraic solution follows from the equivalence

$$a^3-b^3-c^3>3abc \iff a>b+c, \tag{2}$$

which itself follows from the factorization

$$a^3-b^3-c^3-3abc = \tfrac{1}{2}(a-b-c)\{(b-c)^2+(c+a)^2+(a+b)^2\}.$$

With $a=\sqrt[3]{60}$, $b=2$, and $c=\sqrt[3]{7}$, the left member of (2) becomes $45>6\sqrt[3]{420}$, which is equivalent to $225>224$. Hence, from the right member of (2),

$$\sqrt[3]{60}>2+\sqrt[3]{7}.$$

*

In the Olympiad Corner: 15 [1980: 145], I listed ten "Jewish" problems, and solutions to the first six are given above. Solutions to the remaining four are solicited. I give below a further list of eleven "Jewish" problems. Readers, particularly high school students, are encouraged to send me their solutions, some of which may then be published in a subsequent issue.

The first nine problems (J-11 to J-19) are given by Melvyn B. Nathanson in Grigori Freiman's book *It Seems I Am A Jew*, A Samizdat Essay on Soviet Mathematics (Southern Illinois University Press, 1980, pp. 95-96). The last two are from a letter of Professor Boris M. Schein quoted in the *Notices of the A.M.S.* 27 (June 1980) 347. The last problem, J-21, is correct as given here. Professor Schéin's letter had incorrectly given the last word as "zeros" instead of "nines."

J-11. Which is larger, $\sqrt[3]{413}$ or $6+\sqrt[3]{3}$?

J-12. Solve the equation $x^y = y^x$ in positive integers.

J-13. Prove that a convex polygon of area 1 contains a triangle of area $\frac{1}{4}$.

J-14. Show that the following statement is false: *A convex polyhedron of volume* 1 *contains a tetrahedron of volume* 1/8.

A natural estimate is not $(1/2)^3 = 1/8$ but $(1/3)^3 = 1/27$. Obtain this estimate and try to prove it.

J-15. Does there exist an infinite family of pairwise noncongruent right triangles such that the lengths of the sides are natural numbers and the lengths of the two short sides differ by 1?

J-16. Construct a straight line that divides into two equal parts both the area and the perimeter of a given triangle.

J-17. Prove the inequality $(\sin x)^{-2} \geq x^{-2} + 1 - 4/\pi^2$ for $0 < x < \pi/2$.

J-18. Let ABCDE be a convex pentagon with the property that each of the five triangles ABC, BCD, CDE, DEA, and EAB has area 1. Find the area of pentagon ABCDE.

J-19. Six points are given, one on each edge of a tetrahedron of volume 1, none of them being a vertex. Consider the four tetrahedra formed as follows: Choose one vertex of the original tetrahedron and let the remaining vertices be the three given points that lie on the three edges incident with the chosen vertex. Prove that at least one of these four tetrahedra has volume not exceeding 1/8.

J-20. Let $\{x\}$ denote the fractional part of x. Find

$$\lim_{n\to\infty} \{(2+\sqrt{3})^n\}.$$

J-21. Prove that the first thousand digits after the decimal point in the decimal expansion of $(6+\sqrt{35})^{1979}$ are nines.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

MAMA-THEMATICS

Madame Galois, reminiscing about her son: "Evariste just loved numbers, but his own number came up too soon."

CHARLES W. TRIGG

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

587. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma = A_0A_1 \ldots A_n$ be an n-simplex in R^n. A straight line cuts the $(n-1)$-dimensional faces

$$\sigma_i \equiv A_0A_1 \ldots A_{i-1}A_{i+1} \ldots A_n, \qquad i = 0, 1, \ldots n$$

in the points B_i. If M_i is the midpoint of the straight line segment A_iB_i, show that all the points M_i lie in the same $(n-1)$-dimensional plane.

588. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given is a triangle ABC with internal angle bisectors t_a, t_b, t_c and medians m_a, m_b, m_c to sides a, b, c, respectively. If

$$m_a \cap t_b = P, \quad m_b \cap t_c = Q, \quad m_c \cap t_a = R,$$

and L, M, N are the midpoints of the sides a, b, c, respectively, prove that

$$\frac{AP}{PL} \cdot \frac{BQ}{QM} \cdot \frac{CR}{RN} = 8.$$

589. *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

In a triangle ABC with semiperimeter s, sides of lengths a, b, c, and medians of lengths m_a, m_b, m_c, prove that:

(a) There exists a triangle with sides of lengths $a(s-a)$, $b(s-b)$, $c(s-c)$.

(b) $(m_a/a)^2 + (m_b/b)^2 + (m_c/c)^2 \geq 9/4$, with equality if and only if the triangle is equilateral.

590. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find all real solutions of the equation $[x^3] - 3[x^2] + 2[x] = 0$, where the brackets denote the greatest integer function.

591. *Proposed by Charles W. Trigg, San Diego, California.*

December is a good month to solve the cryptarithm

$$\text{SEER} + \text{SEER} + \text{SEER} = \text{MAGI},$$

which memorializes the start of a historical event. The three wise men, Pythagoreans all, insisted that AG be twice the square of a prime. Find the unique solution.

592. *Proposed by Leroy F. Meyers, The Ohio State University.*

(a) Given a segment AB of length l, and a rusty compass of fixed opening r, show how to find a point C such that the length of AC is the mean proportional between r and l, by use of the rusty compass only, if $\frac{1}{4}l \le r \le l$ but $r \ne \frac{1}{2}l$.

(b) Show that the construction is impossible if $r = \frac{1}{2}l$.

(c)* Is the construction possible if $r < \frac{1}{4}l$ or $r > l$?

(This problem was inspired by Dan Pedoe's Problem 492.)

593. *Proposed by Andy Liu, University of Alberta.*

Grandpa is 100 years old and his memory is fading. He remembers that last year — or was it the year before that? — there was a big birthday party in his honor, each guest giving him a number of beads equal to his age. The total number of beads was a five-digit number, $x67y2$, but to his chagrin he cannot recall what x and y stand for. How many guests were at the party?

594*. *Proposed by John Veness, Cremorne, N.S.W., Australia.*

Let N be a natural number which is not a perfect cube. Investigate the existence, nature, and number of solutions of either or both of the Diophantine equations

$$x^3 - Ny^3 = \pm 1$$

in positive integers x and y.

595. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let $f(x,y) = a^2\cos x \cos y + a(\sin x + \sin y) + 1$. Prove that

$$f(\beta,\gamma) = 0 \text{ and } f(\gamma,\alpha) = 0 \implies f(\alpha,\beta) = 0.$$

* * *

THE PEN IS MIGHTIER THAN THE SURD

Don't talk to me of your Archimedes' lever. He was an absent-minded person with a mathematical imagination. Mathematics commands all my respect, but I have no use for engines. Give me the right word and the right accent and I will move the world.

JOSEPH CONRAD, in the
preface to *A Personal Record*.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

342. [1978: 133,297] *Proposed by James Gary Propp, Great Neck, N.Y.*

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets $S_1, S_2, \ldots, S_n$ as follows: for each positive integer m, we have $m \in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m \in S_n$ for all sufficiently large m. (If $n = 2$, this is essentially equivalent to Problem 226 [1977: 205].)

Partial solution by Dan Sokolowsky, Antioch College and Sam LaMacchia, The Ohio State University (jointly).

We prove that the statement is true for all odd $n \geq 3$ but draw no conclusion for even n (except to recall that, as noted in the proposal, the theorem holds for $n = 2$). We shall need the following three lemmas:

LEMMA 1. *From some term on, the terms of S_n form an infinite arithmetic progression.*

LEMMA 2. *If d is the common difference of this arithmetic progression, then $d|n$ and*

LEMMA 3. $d = 1$ *or* 2.

For odd n, Lemmas 2 and 3 imply $d = 1$, and the required result then follows from Lemma 1. From now on, we assume $n > 2$.

Proof of Lemma 1. For any $k = 1,2,\ldots,n$, we call the sum of p elements of S_k a *p-sum of* S_k and denote it by $\sigma_k(p)$. It follows from the hypothesis that all $\sigma_k(n) \in S_n$; in particular, all $\sigma_n(n) \in S_n$. Since at least one S_k is infinite, for this k there are infinitely many distinct n-sums $\sigma_k(n) \in S_n$, and hence S_n is an infinite set. Now let

$$d = \min|x - y|, \qquad x,y \in S_n, \qquad x \neq y.$$

Then $d = \beta - \alpha$ for some $\alpha,\beta \in S_n$. Let α, β, t, γ_i, $i = 1,2,\ldots,n-2$ be $n+1$ distinct members of S_n and form the sums

$$p = \alpha + \sum_{i=1}^{n-2}\gamma_i, \qquad q = \beta + \sum_{i=1}^{n-2}\gamma_i.$$

Then p and q are $(n-1)$-sums of S_n and $q - p = \beta - \alpha = d$. Since $t \in S_n$, each of the sums $p + t$ and $q + t$ is a $\sigma_n(n)$, and hence each belongs to S_n. In general, if

$$a = mp + wq + t \in S_n,$$

then

$$p + a = (m+1)p + wq + t \in S_n$$

and

$$q + a = mp + (w+1)q + t \in S_n.$$

Thus

$$U = \{mp + wq + t; \quad m,w = 0,1,2,\ldots\} \subset S_n.$$

If, for $s = 0,1,2,\ldots$, we let

$$U_s = \{mp + wq + t; \quad m + w = s; \quad m,w = 0,1,\ldots,s\},$$

then

$$U_s \subset U \subset S_n, \qquad s = 0,1,2,\ldots .$$

For each $s = 0,1,2,\ldots$, the smallest term, L_s, and the largest term, M_s, of U_s are given by

$$L_s = sp + t, \quad M_s = sq + t.$$

Moreover, for $r = 0,1,\ldots,s$,

$$L_s + rd = (sp + t) + r(q - p) = (s - r)p + rq + t \in U_s,$$

and hence

$$U_s = \{L_s + rd; \quad r = 0,1,\ldots,s\},$$

that is, the terms of U_s form an arithmetic progression with common difference d. From this and the defining property of d, it follows that

$$x \in S_n \text{ and } L_s \le x \le M_s \implies x \in U_s. \tag{1}$$

We now show that

$$L_{s+1} \le M_s + d \tag{2}$$

implies that

(I) the terms of $U_s \cup U_{s+1}$ form an arithmetic progression with common difference d.

This is certainly true if $M_s < L_{s+1}$, for then

$$0 < L_{s+1} - M_s \le d \implies L_{s+1} = M_s + d$$

and (I) holds. The only alternative is $L_{s+1} \le M_s$ for which (2) certainly holds; and then $L_{s+1} \in U_s$, a consequence of $L_s < L_{s+1} \le M_s$ and (1), implies that (I) holds.

If we substitute in (2) the expressions given above for L_{s+1}, M_s, and d in terms of p, q, and t, we obtain the equivalent condition

$$(s + 1)(q - p) \ge p. \tag{3}$$

Let s_0 be the smallest value of s satisfying (3) and hence (2); then (I) holds for all $s \geq s_0$. This implies that the terms of

$$A = \bigcup_{s \geq s_0} U_s$$

form an infinite arithmetic progression with common difference d; and $A \subset S_n$ since $U_s \subset S_n$ for all s.

The smallest term, L, of A is clearly the smallest term L_{s_0} of U_{s_0}, so $L = s_0 p + t$, and the terms of A are those of the arithmetic progression

$$L + kd, \quad k = 0,1,2,\ldots .$$

Finally, it follows from the defining property of d that, if $x \in S_n$ and $x \geq L$, then $x \in A$.

Proof of Lemma 2. For the numbers d and L defined in the proof of Lemma 1, let $R = \{0,1,\ldots,d-1\}$ be the set of least residues modulo d, and suppose $L \equiv r \pmod d$ with $r \in R$. We have in effect shown earlier that

$$x \in S_n \text{ and } x \geq L \implies x \equiv L \equiv r \pmod d. \tag{4}$$

Let $r' \in R$. Since there are infinitely many integers congruent to r' modulo d, there is a k such that S_k contains infinitely many such integers from which we can select n integers x_i, $i = 1,2,\ldots,n$. Now the sum

$$x = \sum_{i=1}^{n} x_i \equiv nr' \pmod d \tag{5}$$

is an n-sum of S_k, and so $x \in S_n$. If we assume (as we can) that $x_1 \geq L$, then also $x \geq L$ and

$$nr' \equiv r \pmod d \tag{6}$$

follows from (4) and (5) for any $r' \in R$. We can assume $d > 1$, so that $0,1 \in R$. For $r' = 0$, (6) implies

$$r \equiv 0 \pmod d; \tag{7}$$

and for $r' = 1$, (6) and (7) imply $n \equiv 0 \pmod d$, that is, $d|n$. $\square$

The proof of Lemma 3 will require the following results:

(II) If $r' \in R$ and $r' \neq 0$, then $(n-1)r' \not\equiv 0 \pmod d$.

Proof. For $nr' \equiv 0 \pmod d$ since $d|n$, and this together with $r' \neq 0$ gives $nr' - r' = (n-1)r' \not\equiv 0 \pmod d$.

(III) $d|x$ for all $x \in S_n$.

Proof. That this is true when $x \in S_n$ and $x \geq L$ follows from (4) and (7). So suppose $x \in S_n$ and $x < L$. Select numbers

$$x_i \in S_n, \quad x_i \geq L, \quad i = 1,2,\ldots,n-1$$

and form the sum

$$\sigma = x + \sum_{i=1}^{n-1} x_i.$$

Now $\sigma \in S_n$ since it is an n-sum of S_n, and $\sigma > L$. Since $\sigma > L$ and $x_i \geq L$, it follows that $d|\sigma$, $d|x_i$, and hence $d|x$.

(IV) If some S_k contains at least $n+1$ elements, then $x \equiv y \pmod d$ for all $x,y \in S_k$.

Proof. Suppose x, y, z_i, $i = 1,2,\ldots,n-1$, are all in S_k and let

$$z = \sum_{i=1}^{n-1} z_i.$$

Then $x + z \in S_n$ and $y + z \in S_n$, each being an n-sum of S_n; hence, by (III), $d|x+z$ and $d|y+z$, and so $d|x-y$.

Proof of Lemma 3. It suffices to show that $d < 3$, so suppose $d \geq 3$, so that $1,2 \in R$. There is an S_j [an S_k] which contains infinitely many terms $x \equiv 1 \pmod d$ [$y \equiv 2 \pmod d$] from which we select x_i, $i = 1,2,\ldots,n-1$ [y_i, $i = 1,2,\ldots,n-1$]. If

$$a = \sum_{i=1}^{n-1} x_i \quad \text{and} \quad b = \sum_{i=1}^{n-1} y_i,$$

then $a \equiv n-1 \pmod d$ and $b \equiv 2(n-1) \pmod d$, so $a,b \not\equiv 0 \pmod d$ by (II) and $a,b \notin S_n$ by (III). Since a,b are $(n-1)$-sums which are not in S_n, each is in S_{n-1}, and S_{n-1} is an infinite set since infinitely many sums like a and b can be formed. Now, from (IV)

$$a \equiv n-1 \equiv 2(n-1) \equiv b \pmod d,$$

which implies $a \equiv n-1 \equiv 0 \pmod d$, contradicting the earlier $a \not\equiv 0 \pmod d$.

We conclude that $d < 3$ and $d = 1$ or 2.

The same partial result was established by HAROLD N. SHAPIRO, Courant Institute, New York University; and a less inclusive partial result was proved by ANDY LIU, University of Alberta.

Editor's comment.

The editor hopes that some reader can settle the question for even $n > 2$. Ideal would be a simpler proof valid for all n.

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495. [1979: 292] A solution to this problem will appear shortly, possibly in the next issue.

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496. [1979: 292] *Proposed by E.J. Barbeau, University of Toronto.*

Solve the Diophantine equation

$$(x+1)^k - x^k - (x-1)^k = (y+1)^k - y^k - (y-1)^k$$

for $k = 2,3,4$ and $x \neq y$.

Solution by Jan van de Craats, Leiden University, The Netherlands.

When $k = 2$ and $x \neq y$, the given equation is equivalent to

$$x + y = 4,$$

and all integer solutions are given by

$$(x,y) = (2+t,\ 2-t), \qquad t = \pm1, \pm2, \pm3, \ldots\ .$$

When $k = 3$ and $x \neq y$, the given equation is equivalent to

$$x^2 + xy + y^2 = 6(x+y), \tag{1}$$

from which it is clear that x and y are both even. The substitution

$$x = u + v, \quad y = u - v, \quad v \neq 0, \tag{2}$$

with u and v integers, is therefore allowable and it makes (1) equivalent to

$$3(u-2)^2 = 12 - v^2,$$

for which the only solutions are easily found to be $v = \pm3$ and $u-2 = \pm1$, from which the required solutions are

$$(x,y) = (6,0), \quad (0,6), \quad (4,-2), \quad \text{or} \quad (-2,4).$$

When $k = 4$ and $x \neq y$, the given equation is equivalent to

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1),$$

from which it is clear that x and y have the same parity. The substitution (2) is therefore again allowable, and it yields

$$(u-2)(u^2 + v^2) = 2(2u^2 + 1). \tag{3}$$

Since $v \neq 0$, we have

$$0 < 2(2u^2 + 1) < 2(2u^2 + 2v^2) = 4(u^2 + v^2);$$

hence, from (3), $0 < u-2 < 4$. Only $u = 5$ gives integral values for v, $v = \pm3$, from which we get the required solutions

$$(x,y) = (8,2) \quad \text{or} \quad (2,8).$$

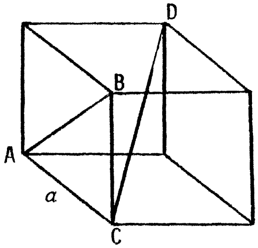
Also solved by ACADIA UNIVERSITY PROBLEM-SOLVING GROUP, Wolfville, Nova Scotia; W.J. BLUNDON, Memorial University of Newfoundland; G.C. GIRI, Midnapore College, West

Bengal, India; ALLAN WM. JOHNSON JR., Washington, D.C.; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP and JOHN OMAN, The University of Wisconsin-Oshkosh (jointly); SANJIB KUMAR ROY, Research Scholar, Indian Institute of Technology, Kharagpur, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

497. [1979: 293] *Proposed by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

Given is a cube of edge length a with diagonal CD, face diagonal AB, and edge CB, as shown in the figure. Points P and Q start at the same time from A and C, respectively, move at constant rates along AB and CD, respectively, and reach B and D, respectively, at the same time. Find the area of the surface swept out by segment PQ.



Solution by G. Ramanaiah, Perarignar Anna University of Technology, Madras, India.

With C as the origin of a rectangular Cartesian coordinate system, we have

$$C = (0,0,0),\quad A = (0,a,0),\quad B = (0,0,a),\quad D = (a,a,a).$$

Let the points P and Q start at time $t=0$ and reach their destination at $t=1$. Noting that the speeds of P and Q are $a\sqrt{2}$ and $a\sqrt{3}$, respectively, we find that at time t $(0 \le t \le 1)$ the positions of P and Q are given by

$$P = a(0,1-t,t) \quad\text{and}\quad Q = at(1,1,1).$$

Let P occupy P' and Q occupy Q' at time $t+\delta t$, where δt is a small increment in t; then

$$P' = a(0,1-t-\delta t,t+\delta t) \quad\text{and}\quad Q' = a(t+\delta t)(1,1,1).$$

The area of quadrilateral PQQ'P' can be easily computed as

$$\delta S = \frac{a^2}{2}\sqrt{24t^2-20t+5}\,\delta t + o((\delta t)^2)\,.$$

Thus

$$\frac{dS}{dt} = \frac{a^2}{2}\sqrt{24t^2-20t+5}$$

and the required area is

$$\begin{aligned} S &= \frac{a^2}{2}\int_0^1 \sqrt{24t^2-20t+5}\,dt \\ &= \frac{a^2}{48}\left\{21 + 5\sqrt{5} + \frac{5}{\sqrt{6}}\ln\left(\frac{7+3\sqrt{6}}{\sqrt{30}-5}\right)\right\} \\ &\approx 0.815a^2. \end{aligned}$$

Also solved by JOHN IM, Grade 12 student, Toronto French School, Toronto, Ontario; and partially solved by DAVE DIXON, Haliburton Highlands Secondary School, Haliburton, Ontario.

* * *

498. [1979: 293] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let $a_i(t)$, $i = 1,2,3$ be given functions whose Wronskian, $w(t)$, never vanishes. Let

$$u(t) = \sqrt{\Sigma a_i^2} \quad \text{and} \quad v(t) = (\Sigma a_i^2)(\Sigma a_i'^2) - (\Sigma a_i a_i')^2.$$

Prove that the general solution of the system

$$x_1'/a_1 = x_2'/a_2 = x_3'/a_3 \tag{1}$$

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \tag{2}$$

can be expressed in terms of

$$\int \frac{uw}{v}\,dt,$$

no other quadratures being required.

Solution and comment by the proposer.

We shall for convenience use capital letters to denote vectors; thus $A = (a_1,a_2,a_3)$, $X = (x_1,x_2,x_3)$, and so on. In particular, $O = (0,0,0)$.

The condition $w \neq 0$ implies $u \neq 0$, and we can then define the unit vector $B = A/u$ parallel to A. In terms of B, (1) and (2) become

$$X' \times B = O \tag{3}$$

and

$$X \cdot B = 0. \tag{4}$$

Since B, B', and $B\times B'$ are mutually orthogonal, and $w \neq 0$ implies $B' \neq O$ and $B\times B' \neq O$, we can use these three vectors as a basis. From (4), we have

$$X = pB' + q(B\times B'),$$

where p and q are scalar functions to be determined. Differentiating, we get

$$X' = p'B' + pB'' + q'(B\times B') + q(B\times B'')$$

which, by (3), is perpendicular to both B' and $B\times B'$. Now the scalar products $X'\cdot B'$ and $X'\cdot(B\times B')$ result in

$$\left.\begin{aligned} p'|B'|^2 &= -pB'\cdot B'' + q[B,B',B''] \\ q'|B'|^2 &= -qB'\cdot B'' - p[B,B',B''] \end{aligned}\right\} \tag{5}$$

where $[B,B',B'']$ is the scalar triple product $(B\times B')\cdot B''$. Noting that $2B'\cdot B'' = d|B'|^2/dt$, the form of these equations suggests the substitutions

$$p = r/|B'|, \quad q = s/|B'|.$$

Equations (5) now become

$$r' = cs, \qquad s' = cr,$$

where $c = [B,B',B'']/|B'|^2$; hence

$$r = k\sin\Theta, \quad s = k\cos\Theta,$$

where k is an arbitrary constant and

$$\Theta = \int c\,dt .$$

To express the solution in terms of A, it will be helpful to note that

$$\begin{aligned} B &= A/u, \\ B' &= -Au'/u^2 + A'/u, \\ B'' &= (\quad * \quad)A + (\quad * \quad)A' + A''/u, \end{aligned}$$

where the terms not written out are irrelevant; hence

$$[B,B',B''] = [A,A',A'']/u^3 = w/u^3$$

and, since $|A|^2 = u^2$ and $A\cdot A' = uu'$,

$$\begin{aligned} |B'|^2 &= |A|^2u'^2/u^4 - 2A\cdot A'u'/u^3 + |A'|^2/u^2 \\ &= u'^2/u^2 - 2u'^2/u^2 + |A'|^2/u^2 \\ &= (|A'|^2 - u'^2)/u^2 \\ &= v/u^4 . \end{aligned}$$

We can now write the solution in the form

$$X = (k/\sqrt{v})\{(uA' - u'A)\sin\Theta + (A\times A')\cos\Theta\},$$

where

$$\Theta = \int \frac{uw}{v}\,dt .$$

Every solution is of this form. On the other hand, it is easy to check that this actually is a solution. □

These equations occur in the theory of parallel curves. Two curves in Euclidean n-space are said to be parallel if there is a 1-1 correspondence between their points such that the tangents at corresponding points are parallel and the join of corresponding points is perpendicular to these tangents.

Let $R(t)$ be the position vector of a point on a given curve C and let $R+X$ be the corresponding point on a parallel curve D. Then $R'=A$ is in the direction of the tangent at R, and (1) and (2) are the conditions for parallelism (for n=3).

It is known [2] that finding D is equivalent to the problem of determining a curve in $(n-1)$-space in terms of its curvatures. For $n=3$, this can be done by means of a single quadrature. The proposed problem amounts to verifying this directly.

Parallel curves in 3-space were studied by P.J. da Cunha [1]. He reduced (1) and (2) to a Riccati equation in the variable x_1/x_3. The equation was solved by J.N. Machado in an unpublished communication. Da Cunha gives the solution but does not tell us how Machado obtained it.

REFERENCES

1. Pedro José da Cunha, "Du parallélisme dans l'espace euclidéen [*sic*]", *Portugaliae Math.*, 2(1941) 177-246.

2. G.P. Henderson, "Parallel Curves", *Canadian Journal of Mathematics* 6 (1953) 99-107.

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499. [1979: 293] *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

A certain polyhedron has all its edges of unit length. An ant travels along the edges and, at each vertex it reaches, chooses at random a new edge along which to travel (each edge at a vertex being equally likely to be chosen). The expected (mean) length of a return trip from one vertex back to it is 6 for some vertices and 7.5 for the other vertices.

Calculate the volume of the polyhedron.

Solution by Friend H. Kierstead, Cuyahoga Falls, Ohio.

Let T be the expected trip length. Rowe [1] has shown that

$$T = \Sigma J_i/J_1, \tag{1}$$

where J_i is the number of edges meeting at the ith vertex, the subscript 1 refers to the starting vertex, and the summation is over all the vertices of the polyhedron. In the present case, we have two types of vertices, say a vertices with J_a edges and b vertices with J_b edges. Then we have, from (1),

$$aJ_a + bJ_b = 6J_a = 7.5J_b. \tag{2}$$

From the equation on the right in (2), we have immediately

$$J_a = 5d, \qquad J_b = 4d,$$

where d is a positive integer. Substituting these values into the equation on the left in (2), we find only one solution in positive integers a and b: $a=2$ and $b=5$. With $d=1$, we have two vertices with five edges and five vertices with four edges:

our polyhedron is then a decahedron consisting of two pentagonal pyramids back to back. If $d > 1$, Euler's equation will show that each face of the polyhedron has, on the average, fewer than three edges, an impossibility. Therefore the decahedron is the required polyhedron.

For a regular pentagon with unit sides, it is known (see, e.g., [2]) that

$$R = \tfrac{1}{2}\csc 36^\circ, \quad K = (5/4)\cot 36^\circ,$$

where R is the circumradius and K is the area. The altitude of one of the pyramids is

$$h = \sqrt{1 - R^2} = \sqrt{1 - \tfrac{1}{4}\csc^2 36^\circ}.$$

Hence the required volume is

$$V = (2/3)Kh = (5/6)\cot 36^\circ \sqrt{1 - \tfrac{1}{4}\csc^2 36^\circ}$$
$$= (1/12)\sqrt{30 + 10\sqrt{5}} \approx 0.603006.$$

Also solved by the proposer. In addition, one incorrect solution was received.

Editor's comment.

Clearly, the point of this problem was simply to identify the polyhedron in question, and asking point-blank for its volume was nothing but a piece of bravura on the part of our Spanish proposer. *Olé!*

See [3] for a related problem.

REFERENCES

1. R. Robinson Rowe, "Roundtripping a Chessboard", *Journal of Recreational Mathematics*, 4 (1971) 265-267.

2. Samuel M. Selby, Ed., *Standard Mathematical Tables*, 18th Edition, The Chemical Rubber Co., Cleveland, 1964, p. 11.

3. F.P. Callahan (proposer), Problem E 1897, *American Mathematical Monthly*, 74 (1967) 1008-1010.

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500. [1979: 293] *Proposed by H.S.M. Coxeter, University of Toronto.*

Let 1, 2, 3, 4 be four mutually tangent spheres with six distinct points of contact 12, 13, ..., 34. Let 0 and 5 be the two spheres that touch all the first four. Prove that the five "consecutive" points of contact 01, 12, 23, 34, 45 all lie on a sphere (or possibly a plane).

I. *Solution by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Let $I(i) = i'$, $i = 0,1,\ldots,5$, where I is an inversion in a sphere with centre 12. Then 1' and 2' are parallel planes; and 3', 4', 0', 5' are equal spheres (sandwiched between and) tangent to 1' and 2'. In addition, 3' is tangent to 0', 4', 5'; 4' is tangent to 0', 3', 5'; and 0' and 5' are each tangent to 3' and 4'. The points 0'1', 1'4', 2'5', 2'3' are the vertices of a parallelogram. Its centre is 3'4', and 0'3' and 4'5' are the midpoints of its sides 0'1'-2'3' and 1'4'-2'5', respectively. Thus the points

0'1', 1'2', 2'3', 3'4', 4'5', 0'3', 1'4', 2'5'

all lie in a plane, and hence the eight points

01, 12, 23, 34, 45, 03, 14, 25

lie on a sphere (or possibly a plane).

II. *Solution by the proposer.*

Since this is a theorem of inversive geometry, no generality is lost by regarding the spheres 0 and 5 as being concentric, so that the spheres 1, 2, 3, 4 are all of the same size, with centres (1,-1,-1), (-1,1,-1), (-1,-1,1), (1,1,1). Now these spheres are

$$x^2 + y^2 + z^2 \pm 2x \pm 2y \pm 2z + 1 = 0$$

with an odd number of minus signs;

12 is $(0,0,-1)$, 23 is $(-1,0,0)$, 34 is $(0,0,1)$;

and the concentric spheres 5 and 0 are

$$x^2 + y^2 + z^2 = (\sqrt{3} \pm \sqrt{2})^2.$$

Thus

01 is $(1-\sqrt{2/3},\ -1+\sqrt{2/3},\ -1+\sqrt{2/3})$,
45 is $(1+\sqrt{2/3},\ 1+\sqrt{2/3},\ 1+\sqrt{2/3})$,

and the five points in question all lie on the sphere

$$x^2 + y^2 + z^2 - 2\sqrt{6}y - 1 = 0$$

or

$$x^2 + (y-\sqrt{6})^2 + z^2 = 7.$$

Since

03 is $(-1+\sqrt{2/3},\ -1+\sqrt{2/3},\ 1-\sqrt{2/3})$,
14 is $(1,0,0)$,
25 is $(-1-\sqrt{2/3},\ 1+\sqrt{2/3},\ -1-\sqrt{2/3})$,

these three points also lie on the same sphere.

In fact, the fourteen points of contact lie by eights on six spheres of radius $\sqrt{7}$ with their centres at the vertices of a regular octahedron of edge $2\sqrt{3}$.

It follows that, in an infinite sequence of spheres such that every five consecutive members are mutually tangent [1], all the "consecutive" points of contact 01, 12, 23,... lie on a sphere (or possibly a plane). Moreover, the points of contact 03, 14, 25, ... (where the numbers differ by 3) lie on the same sphere (or plane).

Also solved by HOWARD EVES, University of Maine; F.G.B. MASKELL, Algonquin College, Ottawa; J.F. RIGBY, University College, Cardiff, Wales; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer (second solution).

Editor's comment.

Van de Craats noted a related problem in [2]. Rigby proved the following more general theorem:

In Euclidean $(2n+1)$*-space, let* $S_1, S_2, \ldots, S_{2n+2}$ *be* $2n+2$ $2n$*-spheres all touching one another, and let* S_0, S_{2n+3} *be the two* $2n$*-spheres that touch each of* $S_1, S_2, \ldots, S_{2n+2}$*. Let* $P_{ij} = P_{ji}$ *denote the point of contact of* S_i *and* S_j*. Then the* $(n+1)(n+3)$ *points* P_{ij} *(where* $0 \le i < j \le 2n+3$*,* $i+j$ *is odd, and* $(i,j) \ne (0,2n+3)$*) lie on a* $2n$*-sphere.*

If we put $n=1$ in this theorem, we see that our problem naturally involves eight points (as noted in both of our featured solutions), not merely the five points mentioned in the proposal.

REFERENCES

1. H.S.M. Coxeter, "Loxodromic sequences of tangent spheres," *Aequationes Mathematicae*, 1 (1968) 118.

2. ________, *Introduction to Geometry*, Second Edition, Wiley, 1969, p. 92, Ex.5.

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501. [1980: 15] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

For the second time in less than a year, Canadians are undergoing the throes of a national election. In the alphametic

```
        N O
  F O O L'S
  A L O N E
        A T
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O T T A W A
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you won't be sure about the FOOLS (until after the election), but what's the unique value of OTTAWA?

Solution by Kenneth M. Wilke, Topeka, Kansas.

The reputation of the proposer as a prolific and careful creator of alphametics (he even coined the word *alphametic* in 1955) forces me to assume that the prime in

L' is not a mere flyspeck to be ignored but is meant to distinguish L' from L. Furthermore, the assumption $L' \neq L$ ensures that the alphametic involves ten different symbols, which makes it an ideal alphametic, and nothing less than the ideal is expected from this proposer.

It is immediately clear that $O = 1$. The possibility $A = 2$ is easily eliminated, so we have $3 \le A \le 5$. Since there is no carry from the third column to the fourth (counting from the right), we have $T = L + 1$, or else $T = 0$ and $L = 9$. We consider these two cases separately.

If $T = L + 1$, then $F + A = 10 + T$, so $A > T > L > 1$. (Note that this eliminates $A = 3$.) The only possibilities are given by

$$(A,T,L,F,S{+}E) = (4,3,2,9,10),\ (5,4,3,9,10),\ \text{or}\ (5,3,2,8,11).$$

(Note in the proposal that S and E are interchangeable so only the sum $S + E$ need be considered.) The first two possibilities are easily eliminated, the first because no satisfactory pair {S,E} exists, and the second because no satisfactory triple {N,L',W} exists. The third possibility yields uniquely $\{S,E\} = \{4,7\}$ and $(N,L',W) = (9,6,0)$. Thus we have only the two solutions

$$\begin{array}{r} 91 \\ 81164 \\ 52197 \\ \underline{53} \\ 133505 \end{array} \quad \text{and} \quad \begin{array}{r} 91 \\ 81167 \\ 52194 \\ \underline{53} \\ 133505 \end{array},$$

which are identical except for the interchange of 4 and 7.

If $T = 0$ and $L = 9$, then $F + A = 9$ and $S + E = 9 + A$. Remembering that $3 \le A \le 5$, we have the possibilities

$$(A,F,S{+}E) = (5,4,14),\ (4,5,13),\ \text{or}\ (3,6,12).$$

It is easy to verify [we omit the details] that the first two possibilities yield no solutions, and the third yields only the two solutions

$$\begin{array}{r} 21 \\ 61174 \\ 39128 \\ \underline{30} \\ 100353 \end{array} \quad \text{and} \quad \begin{array}{r} 21 \\ 61178 \\ 39124 \\ \underline{30} \\ 100353 \end{array},$$

which are identical except for the interchange of 4 and 8. □

It thus appears that, as the proposer predicted, there are several FOOL'S (probably most of them were elected), and even ALONE is not alone. But the proposer may be surprised to find that there are two different OTTAWA's. Apart from the lovely Canadian city (which I visited a couple of years ago) which is known throughout the world as the seat of *Crux Mathematicorum*, there is a town of the same name in my own state of Kansas, about 60 kilometers south-east of my home town of Topeka.

Also solved by ALLAN WM. JOHNSON JR., Washington, D.C.; POONAM MANDAN and RAMA KRISHNA MANDAN, Bombay, India (independently); J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; DONVAL R. SIMPSON, Fairbanks, Alaska; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; and the proposer.

Editor's comment.

All the other solvers (including the proposer, who had written "Don't omit the apostrophe" in the margin of his proposal) assumed that L' = L, so this was certainly not an unreasonable assumption to make. They all came up with the solutions

$$\begin{array}{r} 21 \\ 61194 \\ 39128 \\ \underline{30} \\ 100373 \end{array} \quad \text{and} \quad \begin{array}{r} 21 \\ 61198 \\ 39124 \\ \underline{30} \\ 100373 \end{array},$$

which are identical except for the interchange of 4 and 8, and do provide a unique value for OTTAWA. But is it Ottawa, Can. or Ottawa, Kan.?

* * *

CROOKS WHAT?

For the benefit of recent subscribers, we reprint the following from an earlier issue [1978: 90]:

Crux mathematicorum is an idiomatic Latin phrase meaning: a puzzle or problem for mathematicians. The phrase appears in the Foreign Words and Phrases Supplement of *The New Century Dictionary*, D. Appleton - Century Co., 1946, Vol. 2, p. 2438. It also appears in *Websters New International Dictionary*, Second Unabridged Edition, G. & C. Merriam Co., 1959, Vol. 1, p. 637.

* * *

MATHEMATICAL SWIFTIES

"Those two figures do have the same shape," Tom agreed similarly.

"That set of axioms is consistent and independent," Tom proclaimed categorically.

"We have determined all the zeros of $x^2 = \tan x$," Tom whispered exhaustedly.

"What was the average of the numbers turned up by the dice?" Tom asked expectantly.

"The right ideals of that set of 2×2 matrices consists of all matrices whose columns are scalar multiples of a fixed nonzero vector," Tom remarked properly.

"The determinant of the given matrix is zero," Tom argued singularly.

"Are the solutions really nonunique?" Tom asked repeatedly.

"I do not want solutions which are the ratio of two integers," Tom shouted irrationally.

M.S. KLAMKIN

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