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ON ϕ -PERFECT NUMBERS

MIRIAM HAUSMAN

If $\phi(n)$ denotes the Euler function and $\sigma(n)$ the sum of the divisors of n , then, except for $n = 1$, $\sigma(n)$ is greater than $\phi(n)$. Thus the question arises whether there exist any integers n for which $\phi(n)$ divides $\sigma(n)$. As a multiply-perfect number is one for which $\sigma(n) = kn$, we define a ϕ -perfect number to be one for which $\sigma(n) = k\phi(n)$, and consider conditions which lead to the existence of a finite number of integers n such that $\phi(n)$ divides $\sigma(n)$.

THEOREM.¹ *There exist at most a finite number of integers n such that $\phi(n)$ divides $\sigma(n)$ and*

- (i) *the number of distinct prime factors of n is fixed and*
- (ii) *the sum of the exponents of these prime factors is bounded.*

Proof. Note first that if $\sigma(n) = k\phi(n)$ and n has l distinct prime factors, then the possible values of k are bounded. For if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ and q_l is the l th prime, then, by Mertens's Theorem [1],

$$\frac{\sigma(n)}{\phi(n)} = k = \prod_{i=1}^l \frac{1 - \frac{1}{p_i^{\alpha_i+1}}}{\left\{1 - \frac{1}{p_i}\right\}^2} \leq \prod_{p \leq q_l} \left\{ \frac{1}{1 - \frac{1}{p}} \right\}^2 \leq c \log^2 l$$

for some constant c . Thus it suffices to show that, for each fixed k , there exist a finite number of solutions of

$$\sigma(n) = k\phi(n), \tag{1}$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ and $\alpha_1 + \alpha_2 + \dots + \alpha_l$ is bounded.

We assume that there are an infinite number of such solutions. Writing each of these in the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}, \text{ where } p_1^{\alpha_1} < p_2^{\alpha_2} < \dots < p_l^{\alpha_l},$$

we note that $p_l^{\alpha_l} \rightarrow \infty$ over this sequence. Hence from these n we can extract an infinite subsequence $\{n_j\}$ such that

$$n_j = p_1^{\alpha_1} \dots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1},j} \dots p_{l_j}^{\alpha_{l_j,j}},$$

¹David Chiang proved this result for square-free integers in his January 1980 project for the Westinghouse National Science Talent Search.

where

(I) the $p_i^{\alpha_i}$ are fixed independently of j and

(II) $\lim_{j \rightarrow \infty} p_{ij} = \infty$, $i = r+1, \dots, l$.

Although a priori (I) may well be empty, (II) is not empty.

Since the n_j are solutions of (1), we have

$$\frac{\sigma(n_j)}{\phi(n_j)} = \prod_I \prod_{II} \frac{p_{ij}^{\alpha_{ij}+1} - 1}{p_{ij}^{\alpha_{ij}-1} (p_{ij} - 1)^2} = \prod_I \prod_{II} \frac{1 - \frac{1}{p_{ij}^{\alpha_{ij}+1}}}{\left\{1 - \frac{1}{p_{ij}}\right\}^2} = k. \quad (2)$$

Taking the limit as $j \rightarrow \infty$, we get

$$\prod_I \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i-1} (p_i - 1)^2} = k. \quad (3)$$

If (I) is empty, then (3) yields $k = 1$, which in turn implies that $n = 1$. Thus we may assume that (I) is not empty. Dividing (2) by (3) yields

$$\frac{\sigma(\hat{n})}{\phi(\hat{n})} = 1,$$

where $\hat{n} = \prod_{II} p_i^{\alpha_i}$, which implies that $\hat{n} = 1$, contradicting the fact that II is not empty. This contradiction completes the proof of the theorem.

It might be reasonable to conjecture that the conclusion of the theorem holds even if one relinquishes the condition that the sum of the exponents be bounded. Proceeding as in the proof of the theorem, we assume that there exist an infinite number of integers n satisfying (1), each n having l distinct prime factors. Thus there is an infinite subsequence $\{n_j\}$ such that

$$n_j = p_1^{\alpha_1} \dots p_r^{\alpha_r} p_{r+1}^{\alpha_{r+1},j} \dots p_t^{\alpha_{tj}} p_{t+1}^{\alpha_{t+1},j} \dots p_l^{\alpha_{lj}},$$

where

(I) the $p_i^{\alpha_i}$ are fixed independently of j for $j = 1, \dots, r$;

(II) the p_i are fixed independently of j and $\lim_{j \rightarrow \infty} \alpha_{ij} = \infty$, $i = r+1, \dots, t$; and

(III) $\lim_{j \rightarrow \infty} p_{ij} = \infty$, $i = t+1, \dots, l$.

Since the n_j are solutions of (1), we have

$$\prod_I \prod_{II} \prod_{III} \frac{p_{ij}^{\alpha_{ij}+1} - 1}{p_{ij}^{\alpha_{ij}-1} (p_{ij} - 1)^2} = k. \quad (4)$$

Taking the limit as $j \rightarrow \infty$ yields

$$\prod_I \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \prod_{II} p_i^2 = k \prod_{II} (p_i - 1)^2 \prod_I p_i^{\alpha_i-1} (p_i - 1), \quad (5)$$

which does not give rise to a contradiction. Analogously dividing (4) by (5) gives

$$\prod_{II} \frac{p_i^{\alpha_i+1} - 1}{p_i^{\alpha_i+1}} \sigma \left(\prod_{III} p_i^{\alpha_i} \right) = 1,$$

which implies that the largest prime in (II) must divide $\sigma \left(\prod_{III} p_i^{\alpha_i} \right)$; and since there does not seem to be any apparent reason why this could not occur, this procedure fails to shed light on the conjecture.

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1. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Fourth Edition 1960 (or Fifth Edition 1979), p. 351.

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CONTRIBUTION REQUEST FOR "OPEN QUESTIONS IN MATHEMATICS"

A preliminary version of unsolved problems—containing mainly contributions by Academicians (and an occasional Nobel prize winner)—is now available.

The collection contains the contributors' "favorite problem" (excluding usually the well-known ones treated in other publications). Also included, whenever desirable is the problem's history, hints for solution, references, and a short biography of the contributor.

Additional contributions and comments will be considered for publication in the next edition. Manuscripts not exceeding three pages should be submitted, in duplicate and in final form for reproduction, to the editor

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MATHEMATICAL SWIFTIE

"It just doesn't add up," Tom said, nonplussed.

RICHARD A. GIBBS

AREA CHARACTERIZATIONS OF CURVES: II

ALFRED AEPPLI

Parabolas can be characterized by an area condition, as explained in [1], and [2] contains various area conditions leading to exponential spirals, certain closed curves, and circles. In this note, another characterization of exponential spirals and circles is given. It can be considered as a generalization of Proposition 1 in [2] and has been found because of Proposition 3 in [2], as indicated later on.

Let Γ be a C^1 curve $r = r(\phi) > 0$ in the plane (see Figure 1) and let

$$A_{\phi_1}^{\phi_2} = \frac{1}{2} \int_{\phi_1}^{\phi_2} r^2(\phi) d\phi$$

be the surface area of the finite domain bounded by Γ and the rays in directions ϕ_1 and ϕ_2 (shaded in Figure 2). Let $\psi = \psi(\alpha)$ be a C^1 function with $0 < \psi(\alpha) < \alpha$ if $\alpha > 0$.

PROPOSITION. (i) *If in the described situation*

$$A_{\phi_1}^{\phi_1 + \psi(\phi_2 - \phi_1)} = A_{\phi_1 + \psi(\phi_2 - \phi_1)}^{\phi_2} \quad (1)$$

for all ϕ_1, ϕ_2 , $\phi_1 < \phi_2$ (see Figure 3), then Γ is an exponential spiral $r(\phi) = r_0 e^{c\phi}$ for some constant c and some $r_0 = r(0) > 0$, and

$$\psi(\alpha) = \frac{1}{2\alpha} \log \frac{1}{2}(1 + e^{2c\alpha}).$$

(ii) *If in the described situation (1) holds, and if Γ is a closed curve (i.e., compact), then Γ is a circle $r(\phi) = r_0$ and $\psi(\alpha) = \alpha/2$.*

Proof. Fix $\phi_1 = \phi_0$ and consider $\phi_2 = \phi$ variable, with $\phi > \phi_0$. Then condition (1) says that

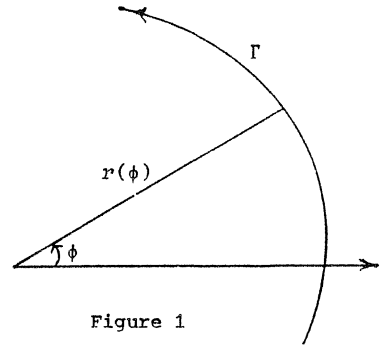


Figure 1

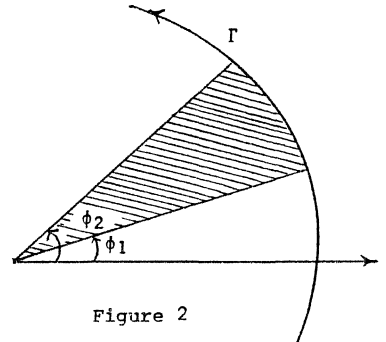


Figure 2

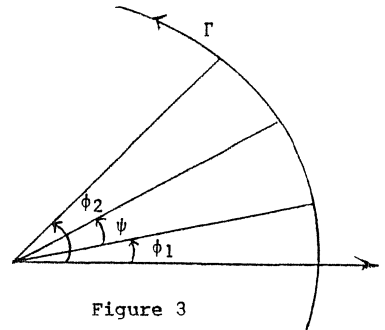


Figure 3

$$\int_{\phi_0}^{\phi_0+\psi(\phi-\phi_0)} r^2(\xi) d\xi = \int_{\phi_0+\psi(\phi-\phi_0)}^{\phi} r^2(\xi) d\xi \quad (2)$$

for all $\phi > \phi_0$. Differentiation of (2) with respect to ϕ yields

$$r^2(\phi_0+\psi(\phi-\phi_0)) \cdot \psi'(\phi-\phi_0) = r^2(\phi) - r^2(\phi_0+\psi(\phi-\phi_0)) \cdot \psi'(\phi-\phi_0),$$

that is,

$$2r^2(\phi_0+\psi(\phi-\phi_0)) \cdot \psi'(\phi-\phi_0) = r^2(\phi). \quad (3)$$

Set $\alpha = \phi - \phi_0$ and $\eta = \phi_0 + \psi(\alpha)$; then (3) becomes

$$2r^2(\eta) \cdot \psi'(\alpha) = r^2(\eta + \alpha - \psi(\alpha)) \quad (4)$$

and differentiation with respect to η gives

$$2r(\eta)r'(\eta)\psi'(\alpha) = r'(\eta + \alpha - \psi(\alpha)) \cdot r'(\eta + \alpha - \psi(\alpha)). \quad (5)$$

Now dividing (5) by (4) yields

$$\frac{r'(\eta)}{r(\eta)} = \frac{r'(\eta + \alpha - \psi(\alpha))}{r(\eta + \alpha - \psi(\alpha))}.$$

This we integrate over η from ζ_0 to ζ to obtain

$$\log \frac{r(\zeta)}{r(\zeta_0)} = \log \frac{r(\zeta + \alpha - \psi(\alpha))}{r(\zeta_0 + \alpha - \psi(\alpha))}.$$

from which

$$r(\zeta + (\alpha - \psi(\alpha))) = \frac{r(\zeta_0 + (\alpha - \psi(\alpha)))}{r(\zeta_0)} \cdot r(\zeta)$$

and, by iteration,

$$r(\zeta + n(\alpha - \psi(\alpha))) = \left\{ \frac{r(\zeta_0 + (\alpha - \psi(\alpha)))}{r(\zeta_0)} \right\}^n \cdot r(\zeta).$$

Therefore

$$\begin{aligned} r(\zeta + n(\alpha - \psi(\alpha))) &= \left\{ 1 + \frac{r(\zeta_0 + (\alpha - \psi(\alpha))) - r(\zeta_0)}{r(\zeta_0)} \right\}^n \cdot r(\zeta) \\ &= \left\{ 1 + \frac{1}{n} \cdot \frac{1}{r(\zeta_0)} \cdot \frac{r(\zeta_0 + (\alpha - \psi(\alpha))) - r(\zeta_0)}{\alpha - \psi(\alpha)} \cdot n(\alpha - \psi(\alpha)) \right\}^n \cdot r(\zeta). \end{aligned}$$

Now choose a sequence $\alpha_n \rightarrow 0$, $n = 1, 2, 3, \dots$, with $n(\alpha_n - \psi(\alpha_n)) \rightarrow \phi$ for $n \rightarrow \infty$ and for a fixed $\phi > 0$. Then $\alpha_n - \psi(\alpha_n) \rightarrow 0$ and

$$\frac{r(\zeta_0 + (\alpha_n - \psi(\alpha_n))) - r(\zeta_0)}{\alpha_n - \psi(\alpha_n)} \rightarrow r'(\zeta_0),$$

and since $\lim_{n \rightarrow \infty} (1 + \alpha_n/n)^n = e^a$ if $\lim_{n \rightarrow \infty} \alpha_n = a$, the expression for $r(\zeta + n(\alpha - \psi(\alpha)))$ becomes

$$r(\zeta+\phi) = r(\zeta) \cdot e^{(r'(\zeta_0)/r(\zeta_0)) \cdot \phi} = r(\zeta) \cdot e^{c\phi},$$

from which, when $\zeta = 0$,

$$r(\phi) = r_0 e^{c\phi} \quad (6)$$

for $r_0 = r(0)$ and some constant c .

To compute $\psi(\alpha)$, we set $x = \phi_1 + \psi(\phi_2 - \phi_1) = \phi_1 + \psi(\alpha)$. It then follows immediately from (1) and (6) that

$$\int_{\phi_1}^x e^{2c\phi} d\phi = \int_x^{\phi_2} e^{2c\phi} d\phi,$$

that is,

$$e^{2cx} = \frac{1}{2}(e^{2c\phi_1} + e^{2c\phi_2});$$

hence

$$\begin{aligned} x = \phi_1 + \psi(\alpha) &= \frac{1}{2c} \log \frac{1}{2}(e^{2c\phi_1} + e^{2c\phi_2}) \\ &= \frac{1}{2c} \log (\frac{1}{2}e^{2c\phi_1}(1 + e^{2c\alpha})), \end{aligned}$$

that is,

$$\psi(\alpha) = \frac{1}{2c} \log \frac{1}{2}(1 + e^{2c\alpha}).$$

This completes the proof of (i) in the Proposition.

Part (ii) is a consequence of (i): if Γ is compact, then $c = 0$, Γ is a circle, and $\psi(\alpha) = \alpha/2$.

Comment on the Proposition. The area condition (1) is "natural" in view of the problem solved in Proposition 3 of [2]. There the curves with

$$A_{P_1Q} = A_{QP_2}$$

are determined (see Figure 4; A_{P_1Q} and A_{QP_2} are the areas of triangles OP_1Q and OQP_2 , respectively). So I became interested in curves with

$$A_{\phi_1+\psi} = A_{\phi_1+\psi} \quad (7)$$

(equality between the areas of curvilinear triangles OP_1P and OPP_2 in Figure 4), where $\psi = \psi(\phi_1, \phi_2)$ is given by the construction

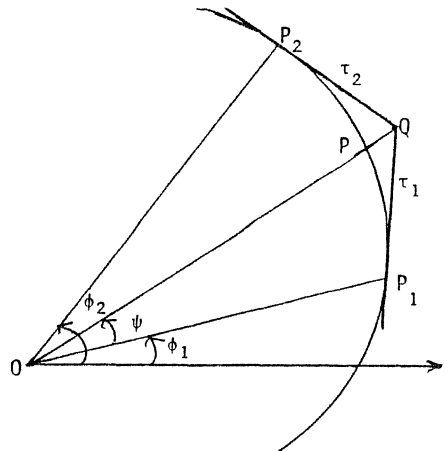


Figure 4

indicated in Figure 4: $\Psi = \angle P_1 O Q$, where Q is the intersection of the tangents τ_1 at P_1 and τ_2 at P_2 . What are those curves explicitly? If ψ is assumed to depend only on the difference $\phi_2 - \phi_1$, (7) leads to condition (1) and the Proposition above.

Remarks on a general version of (7).

(i) Given a sufficiently nice curve Γ , $r = r(\phi) > 0$, there is a unique function $\psi = \psi(\phi_1, \phi_2)$ such that

$$A_{\phi_1}^{\phi_1 + \psi(\phi_1, \phi_2)} = A_{\phi_1 + \psi(\phi_1, \phi_2)}^{\phi_2}. \quad (7)$$

(ii) The function $\psi = \psi(\phi_1, \phi_2)$ determined by (7) has the following properties:

$$0 < \psi(\phi_1, \phi_2) < \phi_2 - \phi_1 \text{ if } \phi_1 < \phi_2,$$

$$0 > \psi(\phi_1, \phi_2) > \phi_2 - \phi_1 \text{ if } \phi_1 > \phi_2,$$

$$\psi(\phi, \phi) \equiv 0.$$

(iii) Partial differentiation of (7) with respect to ϕ_1 and ϕ_2 gives (with $\psi_i = \partial\psi/\partial\phi_i$ for $i = 1, 2$)

$$2r^2(\phi_1 + \psi(\phi_1, \phi_2)) \cdot (1 + \psi_1(\phi_1, \phi_2)) = r^2(\phi_1) \quad (8)$$

and

$$2r^2(\phi_1 + \psi(\phi_1, \phi_2)) \cdot \psi_2(\phi_1, \phi_2) = r^2(\phi_2). \quad (9)$$

Now (8) and (9) imply

$$r^2(\phi_2) = \frac{\psi_2(\phi_1, \phi_2)}{1 + \psi_1(\phi_1, \phi_2)} \cdot r^2(\phi_1); \quad (10)$$

in particular, for $\phi_1 = 0$ and $\phi_2 = \phi$,

$$r^2(\phi) = \frac{\psi_2(0, \phi)}{1 + \psi_1(0, \phi)} \cdot r^2(0). \quad (10')$$

This shows that, given a C^1 function $\psi = \psi(\phi_1, \phi_2)$, there is at most one curve Γ , $r = r(\phi)$, with (7), starting at $r(0)$ as long as $1 + \psi_1(0, \phi) \neq 0$.

(iv) Further properties of ψ :

(a) Relations (8) and (9) with $\psi(\phi, \phi) \equiv 0$ imply that

$$\psi_1(\phi, \phi) \equiv -\frac{1}{2} \quad \text{and} \quad \psi_2(\phi, \phi) \equiv \frac{1}{2}.$$

(b) Set

$$\Phi(\phi_1, \phi_2) = \frac{\psi_2(\phi_1, \phi_2)}{1 + \psi_1(\phi_1, \phi_2)}.$$

Application of (10) for (ϕ_1, ϕ_2) , (ϕ_2, ϕ_3) , (ϕ_1, ϕ_3) produces

$$\Phi(\phi_1, \phi_2) \cdot \Phi(\phi_2, \phi_3) = \Phi(\phi_1, \phi_3). \quad (11)$$

(c) Relations (9) and (10') imply that

$$\frac{2\psi_2(0, \phi_1 + \psi(\phi_1, \phi_2)) \cdot \psi_2(\phi_1, \phi_2)}{1 + \psi_1(0, \phi_1 + \psi(\phi_1, \phi_2))} = \frac{\psi_2(0, \phi_2)}{1 + \psi_1(0, \phi_1)};$$

hence for ϕ_0 instead of 0,

$$2\psi_2(\phi_0, \phi_1 + \psi(\phi_1, \phi_2)) \cdot \psi_2(\phi_1, \phi_2) \cdot (1 + \psi_1(\phi_0, \phi_2)) = \psi_2(\phi_0, \phi_2) \cdot (1 + \psi_1(\phi_0, \phi_1 + \psi(\phi_1, \phi_2))), \quad (12)$$

that is,

$$2\Phi(\phi_0, \phi_1 + \psi(\phi_1, \phi_2)) \cdot \psi_2(\phi_1, \phi_2) = \Phi(\phi_0, \phi_2).$$

Also, (8) and (10') for ϕ_0 instead of 0 imply that

$$\frac{2\psi_2(\phi_0, \phi_1 + \psi(\phi_1, \phi_2)) \cdot (1 + \psi_1(\phi_1, \phi_2))}{1 + \psi_1(\phi_0, \phi_1 + \psi(\phi_1, \phi_2))} = \frac{\psi_2(\phi_0, \phi_1)}{1 + \psi_1(\phi_0, \phi_1)},$$

that is,

$$2\Phi(\phi_0, \phi_1 + \psi(\phi_1, \phi_2)) \cdot (1 + \psi_1(\phi_1, \phi_2)) = \Phi(\phi_0, \phi_1),$$

which is also a consequence of (11) and (12).

(v) The calculations show: if a C^1 function $\psi = \psi(\phi_1, \phi_2)$ is given with $\psi(\phi, \phi) \equiv 0$, (11) and (12), and given $r(0)$, then condition (7) is fulfilled for $r(\phi)$ defined by (10').

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2. Alfred Aeppli, "Area Characterizations of Curves:I", this journal, 7 (1981) 34.

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THE OLYMPIAD CORNER: 25

M.S. KLAMKIN

I give this month four different problem sets, most of which are from recent competitions. I would be happy to receive from readers and to consider for possible publication in this column copies of other recent competitions.

*

I give first the problems posed at the Thirteenth Canadian Mathematics Olympiad, which took place on 6 May 1981. The questions were prepared by the Olympiad Committee of the Canadian Mathematical Society, consisting of G. Butler (Chairman), M. Klamkin, G. Labelle, G. Lord, J. Schaer, J. Wilker, and E. Williams. I hope to be able to give

next month solutions edited from those prepared by the Olympiad Committee.

THIRTEENTH CANADIAN MATHEMATICS OLYMPIAD

6 May 1981 - 3 hours

1. For any real number t , denote by $[t]$ the greatest integer which is less than or equal to t . For example: $[8] = 8$, $[\pi] = 3$ and $[-5/2] = -3$. Show that the following equation has no real solution:

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345.$$

2. Given a circle of radius r and a tangent line l to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to l with Q on l . Determine the maximum of the area of triangle PQR .

3. Given a finite collection of lines in a plane P , show that it is possible to draw an arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, etc.) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

4. $P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P(Q(x)) \equiv Q(P(x))$ for all real numbers x . If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P(P(x)) = Q(Q(x))$ also has no real solution.

5. Eleven theatrical groups participated in a festival. Each day, some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival, each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last?

*

I now give the problems posed at the Tenth U.S.A. Mathematical Olympiad, which took place on 5 May 1981. The problems will appear with solutions (along with those of the Twenty-second International Mathematical Olympiad (1981), which has not yet taken place) in a booklet, *Olympiads for 1981*, to be compiled by Samuel L. Greitzer. The booklet will be obtainable later this year (50¢ per copy) from

Dr. Walter E. Mientka,
Executive Director,
MAA Committee on H.S. Contests,
917 Oldfather Hall,
University of Nebraska,
Lincoln, Nebraska 68588.

TENTH U.S.A. MATHEMATICAL OLYMPIAD

5 May 1981 - 3½ hours

1. The measure of a given angle is $180^\circ/n$ where n is a positive integer not divisible by 3. Prove that the angle can be trisected by Euclidean means (straightedge and compass).
2. Every pair of communities in a county are linked directly by exactly one mode of transportation: bus, train or airplane. All three modes of transportation are used in the county with no community being serviced by all three modes and no three communities being linked pairwise by the same mode. Determine the maximum number of communities in this county.
3. If A , B and C are the measures of the angles of a triangle, prove that
$$-2 \leq \sin 3A + \sin 3B + \sin 3C \leq 3\sqrt{3}/2$$
and determine when equality holds.
4. The sum of the measures of all the face angles of a given convex polyhedral angle is equal to the sum of the measures of all its dihedral angles. Prove that the polyhedral angle is a trihedral angle.

Note: A convex polyhedral angle may be formed by drawing rays from an exterior point to all points of a convex polygon.
5. If x is a positive real number and n is a positive integer, prove that

$$[nx] \geq \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \dots + \frac{[nx]}{n},$$

where $[t]$ denotes the greatest integer less than or equal to t . For example, $[\pi] = 3$ and $[\sqrt{2}] = 1$.

*

Next comes the 1981 Alberta High School Prize Examination in Mathematics, which took place on 11 March 1981. This competition is sponsored by the Canadian Mathematical Society and administered by the Mathematics Department of the University of Alberta. It is in two parts. Part I (not given here) consists of 20 questions with multiple choice answers to be done in 60 minutes. Students are then allowed 110 minutes to do the 5 problems of Part II, which are given below. Solutions to these 5 problems will be given here in the next issue. All questions were weighted equally and all were to be answered. Extra credit was given for particularly elegant solutions as well as for nontrivial generalizations with proof.

1981 ALBERTA HIGH SCHOOL
PRIZE EXAMINATION IN MATHEMATICS
PART II

11 March 1981 - 110 minutes

1. Show that the two equations

$$x^4 - x^3 + x^2 + 2x - 6 = 0$$

and

$$x^4 + x^3 + 3x^2 + 4x + 6 = 0$$

have a pair of complex roots in common.

2. Trevor wrote down a four-digit number x , transferred the right-most digit to the extreme left to obtain a smaller four-digit number y , and then added the two numbers together to obtain a four-digit number z . The next day he was unable to find his calculations but remembered that the last three digits of z were 179. What was x ?

[A four-digit number does not start with zero.]

3. A baseball league is made up of 20 teams. Each team plays at least once and there are no tie games. A team's average is defined to be its number of wins divided by its total number of games played.

- (a) If each team played the same number of games, show that the sum of the averages of all the teams is 10.
(b) If each team did not play the same number of games, show that the sum of the averages of all the teams must be at least 1 and at most 19.

4. A farmer owns a fenced yard in the shape of a square 30 metres by 30 metres. He wishes to divide the yard into three parts of equal area, using 50 metres length of fencing. Find two different (i.e., noncongruent) ways that he can do this.

5. Do *either* part (a) *or* part (b).

- (a) If $P'Q'R'$ is the parallel projection of a triangle PQR onto any plane, prove that the volumes of the two tetrahedra $P'Q'R'P$ and $PQRP'$ are the same.
(b) Prove that if $x \geq 0$, then

$$\left(\frac{x+1}{n+1}\right)^{n+1} \geq \left(\frac{x}{n}\right)^n, \quad n = 1, 2, 3, \dots$$

In Olympiad Corners 15 [1980: 145] and 20 [1980: 316], I listed 21 so-called "Jewish" problems. Here I extend the list with 16 additional "Jewish" problems. I am grateful to Boris M. Schein for sending me these problems in Russian and to M.L. Glasser for translating them into English. As usual, I invite all readers to send me solutions to these problems (secondary school students should include the name of their school, its location, and their grade). I shall from time to time publish some of the more elegant solutions received.

Selected problems from the oral mathematics examination, Mechomat, Moscow State University, 1979.

- J-22, Can a spatial figure have exactly six axes of symmetry?
- J-23, Three given circles, O_1 , O_2 , O_3 , intersect pairwise: O_1 and O_2 at points A and B, O_2 and O_3 at points C and D, and O_3 and O_1 at points E and F. Prove that the straight lines AB, CD, and EF intersect at a point.
- J-24, A point P is selected in the base BCD of a given tetrahedron A-BCD (not necessarily regular) and lines are drawn through it parallel to the edges AB, AC, AD, intersecting the faces of the tetrahedron in other points U, V, W. Find the point P of base BCD for which the volume of tetrahedron P-UVW is a maximum.
- J-25, In a convex quadrilateral ABCD, the sides AB and CD are congruent and the midpoints of diagonals AC and BD are distinct. Prove that the straight line through these two midpoints makes unequal angles with AB and CD.
- J-26, In a given triangle inscribe a rectangle having given diagonals. Carry out a complete investigation.
- J-27, Solve the inequality $2xy \ln(x/y) < x^2 - y^2$.
- J-28, The lengths of the sides of a convex quadrilateral are, in order, a , b , c , d and its area is S . Prove that $2S \leq ac + bd$.
- J-29, The base ABC of a pyramid P-ABC is an equilateral triangle. If the angles PAB, PBC, and PCA are all congruent, prove that P-ABC is regular.
- J-30, Construct a convex quadrilateral given its angles and diagonals. Give a complete discussion.
- J-31, Solve $x(3y - 5) = y^2 + 1$ in integers.
- J-32, What conditions must be satisfied by the coefficients u, v, w of the polynomial

$$x^3 - ux^2 + vx - w$$

in order that line segments whose lengths are roots of the polynomial can form a triangle.

J-33, A straight line CD and two points A and B not on the line are given. Locate the point M on this line such that $\angle AMC = 2\angle BMD$.

J-34, ABC is a triangle of perimeter p . The tangent to its incircle which is parallel to BC meets AB in E and AC in F. Among all triangles of perimeter p , is there one for which EF is of maximum length?

J-35, Given are three disjoint noncongruent spheres.

(a) Show that, for any two of the spheres, their common external tangents all intersect in a point.

(b) The three spheres pairwise determine three points as in (a). Prove that these points are collinear.

J-36, A point O lies in the base ABC of a tetrahedron P-ABC. Prove that the sum of the angles formed by the line OP and the edges PA, PB, and PC is less than the sum of the face angles at vertex P and greater than half this sum.

J-37, If a, b, c, d are, in order, the sides of a convex quadrilateral and S is its area, prove that

$$S \leq \left\{ \left(\frac{a+b}{2} \right) \left(\frac{b+c}{2} \right) \left(\frac{c+d}{2} \right) \left(\frac{d+a}{2} \right) \right\}^{\frac{1}{2}}.$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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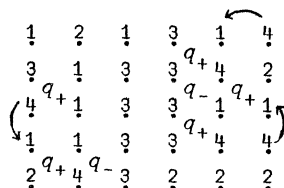
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ANOTHER LATTICE POINT THEOREM

HARRY D. RUDERMAN

This note is an addendum to my article "A Lattice Point Assignment Theorem" published in this journal last month [1981: 98], to which readers should refer for notation and terminology. We relax the restrictions on the mapping $f: L \rightarrow E$ so that only restriction (i) holds: that 4 may not follow 1 on the boundary.

Let T be the set of all small squares having 1, 4, and at least one of 2, 3 among their vertex assignments. Let Q_+ be the subset of T consisting of all squares q_+ in which 1 follows 4 but 4 does not follow 1, and Q_- the subset of T consisting of all squares q_- in which 4 follows 1 but 1 does not follow 4. Then we have the



THEOREM. If $|Q_+|$ and $|Q_-|$ denote the number of squares in Q_+ and Q_- , respectively, and J is the number of jumps, then

$$J + |Q_-| = |Q_+|.$$

For the mapping $f: L \rightarrow E$ shown in the figure, for example, we have $J = 3$, $|Q_-| = 2$, $|Q_+| = 5$, and $3 + 2 = 5$.

Proof. As in the proof of the original theorem, we sum in two different ways the assignments given to all the directed segments. Each square in Q_+ contributes 5 to the sum, each square in Q_- contributes -5, and each jump contributes 5, so we have

$$5|Q_+| - 5|Q_-| = 5J,$$

and the desired result follows.

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PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

633. *Proposed by Janis Aldins; James S. Kline, M.D.; and Stan Wagon, Smith College, Northampton, Massachusetts (jointly).*

It follows from the Wallace-Bolyai-Gerwien Theorem of the early nineteenth century that *any triangle may be cut up into pieces which may be rearranged using only translations and rotations to form the mirror image of the given triangle.* This problem once appeared in a Moscow Mathematical Olympiad (see V.G. Boltianskii, *Hilbert's Third Problem*, Winston, Washington, 1978, p.70, where a three-cut solution is given).

Show that such a dissection may be effected with only two straight cuts.

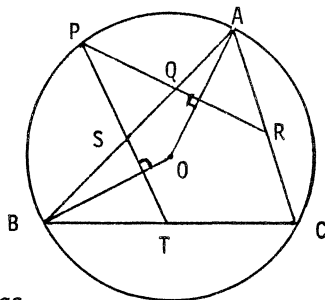
634. *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Le nombre $2701 = 37 \cdot 73$ n'est pas premier. Montrer, néanmoins, que $3^{2701} \equiv 3 \pmod{2701}$.

635. Proposed by Dan Sokolowsky, Antioch College,
Yellow Springs, Ohio.

In the adjoining figure, O is the circumcenter of triangle ABC , and $PQR \perp OA$, $PST \perp OB$. Prove that

$$PQ = QR \iff PS = ST.$$



636.* Proposed by Ferrell Wheeler, student,
Texas A & M University, College Station, Texas.

For an n -point planar graph G_n consisting of the points A_1, A_2, \dots, A_n , let $N(i)$ be the number of points A_j , $j \neq i$, such that the line $A_i A_j$ divides the plane into two half-planes both containing the same number of points of G_n . The spread of G_n is then defined by

$$\sigma(G_n) \equiv \sum_{i=1}^n N(i).$$

(a) For what values of k does there exist an n -point graph G_n such that $\sigma(G_n) = k$?

(b) In particular, prove or disprove that $\sigma(G_6) \neq 8$ for any 6-point graph G_6 .

637. Proposed by Jayanta Bhattacharya, Midnapur, West Bengal, India.

Given $a, b, c > 0$, $0 < A, B, C < \pi$, and

$$a = b \cos C + c \cos B,$$

$$b = c \cos A + a \cos C,$$

$$c = a \cos B + b \cos A,$$

prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{and} \quad A + B + C = \pi.$$

638.* Proposed by S.C. Chan, Singapore.

A fly moves in a straight line on a coordinate axis. Starting at the origin, during each one-second interval it moves either a unit distance in the positive direction or, with equal probability, a unit distance in the negative direction.

(a) Obtain the mean and variance of its distance from the origin after t seconds.

(b) The fly is trapped if it reaches a point 6 units from the origin in the positive direction. What is the probability that it will be trapped within 8 seconds?

639. Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.

If $x + y + z = 0$, prove that

$$\frac{x^5 + y^5 + z^5}{5} = \frac{x^3 + y^3 + z^3}{3} \cdot \frac{x^2 + y^2 + z^2}{2}.$$

640, *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let O be an interior point of an oval (compact convex set) in the real Euclidean plane R^2 , and let $\chi(\phi)$ be the length of the "chord" of the oval through O making an angle ϕ with some fixed ray OX . If K is the area of the oval, prove that

$$K \geq \frac{1}{2} \int_0^{\pi/2} \chi(\phi) \chi(\phi + \pi/2) d\phi.$$

641, *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Deduce three different solutions of the decimal multiplication

$$\begin{array}{r} \text{TWO} \\ \text{SIX} \\ \hline \text{TWELVE} \end{array}$$

from these doubly-true clues:

(a) In the first solution, the sum $\text{TWO} + \text{SIX}$ is a cube and both $\text{SIX} - 3$ and $\text{SIX} + 5$ are prime numbers.

(b) In the second solution, $\text{TWO} + 9$ is a prime number and both $\text{TWO} + 7$ and $4 \cdot \text{SIX} + \text{TWO}/2$ are square numbers.

(c) In the third solution, $3 \cdot \text{SIX} + 2 \cdot \text{TWO} + 3$ is a square number and SIX equals a power of 2 multiplied by a power of 3.

642*, *Proposed by Charles W. Trigg, San Diego, California.*

In the convenient notation $a_k = aaaa$, the subscript indicates the number of consecutive like digits. Thus $2_2 6_0 9_3 4 = 229994$.

The base ten number $10_k 30_k 1$ is known to be a palindromic prime for $k = 0, 1, 2, 3$. Is it composite for any $k > 3$, and if so what are its factors?

643, *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For $i = 1, 2, 3$, a given triangle has vertices A_i , interior angles α_i , and sides a_i . Segment $A_i D_i$, which terminates in α_i , bisects angle α_i ; m_i is the perpendicular bisector of $A_i D_i$; and $E_i = \alpha_i \cap m_i$. Prove that

(a) the three points E_i are collinear;

(b) the three segments $E_i A_i$ are tangent to the circumcircle of the triangle;

(c) if p_i is the length of $E_i A_i$, and if $a_1 \leq a_2 \leq a_3$, then $(1/p_3) + (1/p_1) = 1/p_2$.

644, *Proposed by Jack Garfunkel, Flushing, N.Y.*

If I is the incenter of triangle ABC and lines AI , BI , CI meet the circumcircle of the triangle again in D , E , F , respectively, prove that

$$\frac{AI}{ID} + \frac{BI}{IE} + \frac{CI}{IF} \geq 3.$$

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

543, [1980: 153] Proposed by George Taintsifas, Thessaloniki, Greece.

Let F_1, F_2, \dots, F_n be ovals (compact convex bodies) in the real Euclidean plane R^2 . We suppose that, for all i and j ,

$$\Phi_{ij} \equiv F_i \cap F_j \neq \emptyset.$$

Prove that, if each three of the Φ 's are intersected by a straight line, then the F_i 's have a common point, that is,

$$\bigcap_{i=1}^n F_i \neq \emptyset.$$

Solution by the proposer.

The theorem is vacuously true if $n = 1$ or 2 , so we assume $n \geq 3$. Let F_a, F_b, F_c be three distinct ovals of the n -family, and let l be a line intersecting the three nonempty sets $\Phi_{bc}, \Phi_{ca}, \Phi_{ab}$, with

$$A \in \Phi_{bc} \cap l, \quad B \in \Phi_{ca} \cap l, \quad C \in \Phi_{ab} \cap l.$$

It follows from convexity that segment $BC \subset F_a$, and similarly $CA \subset F_b$ and $AB \subset F_c$. We can assume that B is between A and C on l (relabeling the F 's if necessary).

From convexity again, $B \in F_b$ and so $B \in F_a \cap F_b \cap F_c \neq \emptyset$. Since each three of the ovals have a common point, the desired result,

$$\bigcap_{i=1}^n F_i \neq \emptyset,$$

follows from Helly's Theorem. \square

To arrive at a generalization, we first prove the

LEMMA. Let $F_i, i = 1, 2, \dots, n+1$ be compact convex bodies in Euclidean space R^n . We assume that

$$\Phi_k \equiv \bigcap_{\substack{i=1 \\ i \neq k}}^{n+1} F_i \neq \emptyset, \quad k = 1, 2, \dots, n+1.$$

If there is an $(n-1)$ -plane $\pi \subset R^n$ such that $\pi \cap \Phi_k \neq \emptyset$ for $k = 1, 2, \dots, n+1$, then

$$\bigcap_{i=1}^{n+1} F_i \neq \emptyset.$$

Proof. For $k = 1, 2, \dots, n+1$, let A_k be a point in the nonempty set $\pi \cap \Phi_k$.

Since the $n + 1$ points A_k all lie in the $(n-1)$ -dimensional Euclidean space π , Radon's Theorem assures us that they can be partitioned into two subsets whose convex hulls intersect, say

$$S_1 = \text{convex hull of } \{A_{i_1}, \dots, A_{i_r}\}$$

and

$$S_2 = \text{convex hull of } \{A_{i_{r+1}}, \dots, A_{i_{n+1}}\},$$

with $P \in S_1 \cap S_2 \neq \emptyset$. But then

$$P \in S_1 \subset (F_{i_{r+1}} \cap \dots \cap F_{i_{n+1}})$$

and

$$P \in S_2 \subset (F_{i_1} \cap \dots \cap F_{i_r}),$$

and finally

$$P \in \bigcap_{i=1}^{n+1} F_i \neq \emptyset. \quad \square$$

Our problem is now easily generalized to the

THEOREM. Let F_i , $i = 1, 2, \dots, m$, be compact convex bodies in R^n . We assume that, whenever $1 \leq i_1 < i_2 < \dots < i_n \leq m$, we have

$$F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_n} \neq \emptyset.$$

If, for each set of $n+1$ distinct F 's, there is an $(n-1)$ -plane which intersects each F in the set, then

$$\bigcap_{i=1}^m F_i \neq \emptyset.$$

Proof. The theorem is vacuously true if $m \leq n$, so we assume $m \geq n + 1$. It follows from the lemma that any $n+1$ distinct F 's have a common point; hence, by Helly's Theorem,

$$\bigcap_{i=1}^m F_i \neq \emptyset.$$

Also solved by M.S. KLAMKIN, University of Alberta.

Editor's comment.

Klamkin mentioned two related results [1]:

1. If a family of ovals is such that each two of its members have a common point, then through each point of the plane there is a line that intersects all the ovals of the family.
2. If a family of ovals is such that each two of its members have a common point, then for each line in the plane there is a parallel line that intersects all the ovals of the family.

REFERENCE

1. H. Hadwiger, H. Debrunner, and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York, 1964, pp. 7-9.

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544, [1980: 153] Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.

Prove that, in any triangle ABC,

$$2\Sigma \sin \frac{B}{2} \sin \frac{C}{2} \leq \Sigma \sin \frac{A}{2} \quad (1)$$

(where the sums are cyclic over A,B,C), with equality if and only if the triangle is equilateral.

I. Solution by George Tsintsifas, Thessaloniki, Greece.

We may assume that the triangle has been labeled so that B and C are the two angles "on the same side" of 60° , that is, that $A \leq 60^\circ \leq B, C$ or $B, C \leq 60^\circ \leq A$. The desired result (1) then follows from the double inequality

$$2\Sigma \sin \frac{B}{2} \sin \frac{C}{2} \leq 1 - \sin \frac{A}{2} + 2 \sin \frac{C}{2} \sin \frac{A}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \leq \Sigma \sin \frac{A}{2} \quad (2)$$

which we proceed to establish. The first inequality in (2) results from

$$2 \sin \frac{B}{2} \sin \frac{C}{2} = \cos \frac{B-C}{2} - \cos \frac{B+C}{2} \leq 1 - \cos \frac{B+C}{2} = 1 - \sin \frac{A}{2} \quad (3)$$

and the second is equivalent to

$$(2 \sin \frac{A}{2} - 1)(\sin \frac{B}{2} + \sin \frac{C}{2} - 1) \leq 0 \quad (4)$$

which is true since it holds both when $A \leq 60^\circ \leq B, C$ and when $B, C \leq 60^\circ \leq A$.

Suppose equality holds in (1); then it holds throughout in (2), and hence in (3) and (4). Now we get $B=C$ from (3) and $A = 60^\circ$ from (4), so the triangle is equilateral. The converse is trivial.

II. Comment by M.S. Klamkin, University of Alberta.

The desired result (1) follows from (and is greatly strengthened by) the following chain of inequalities, in each of which equality holds just when the triangle is equilateral:

$$2\Sigma \sin \frac{B}{2} \sin \frac{C}{2} \leq \sqrt{\Sigma \sin B \sin C} \leq \Sigma \cos A \leq \Sigma \sin \frac{A}{2}. \quad (5)$$

The last inequality in (5),

$$\Sigma \cos A \leq \Sigma \sin \frac{A}{2},$$

was given and proved in 1971 by Bager [1, p.15]. It was proposed again a couple of years later by Bankoff and proved in essentially the same way by Starke [4]. That the same proof was given twice independently is not surprising, for it is a simple and practically inevitable proof:

$$2\Sigma \cos A = \Sigma(\cos B + \cos C) = 2\Sigma \sin \frac{A}{2} \cos \frac{B-C}{2} \leq 2\Sigma \sin \frac{A}{2}.$$

The inequality involving the first and third members in (5),

$$2\Sigma \sin \frac{B}{2} \sin \frac{C}{2} \leq \Sigma \cos A,$$

is easily shown to be equivalent to

$$(\Sigma \sin \frac{A}{2})^2 \leq \Sigma \cos^2 \frac{A}{2}, \quad (6)$$

an inequality proposed by Thébault and proved by Bankoff [2]. Inequality (6) and the same proof were repeated later in Bottema et al. [3, p. 32] as well as in Bager [1].

The second inequality in (5) is equivalent to

$$\Sigma \sin B \sin C \leq (\Sigma \cos A)^2. \quad (7)$$

This was proved algebraically by Carlitz and used to solve *A.M.M. Problem E 1573* [5]: *Prove that the arithmetic mean of the angle bisectors of a triangle T never exceeds the sum of the distances of the circumcenter from the three sides of T, with equality if and only if T is equilateral.*

Carlitz first obtained (7) in the equivalent form (in the usual notation)

$$4(R+r)^2 \geq bc + ca + ab. \quad (7a)$$

Since $bc + ca + ab = s^2 + 4Rr + r^2$, this is equivalent to the known inequality

$$4R^2 + 4Rr + 3r^2 \geq s^2 \quad (7b)$$

which had previously been proved by Steinig [3, p. 50] by showing that it was also equivalent to the distance between the incenter and orthocenter of a triangle being nonnegative, that is,

$$IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C \geq 0. \quad (7c)$$

Other proofs of (7) or its above equivalent forms were given by Bager [1, p. 16] and Blundon [3, p. 51]. Still other equivalent (and nicer) forms of (7) are given by Bager [1, p. 20]:

$$\Pi(1 - \cos A) \geq \Pi \cos A, \quad (7d)$$

$$\Pi(\csc \frac{A}{2} - 1) \geq 1. \quad (7e)$$

We have only left to prove the first inequality in (5), which is equivalent to

$$4(\Sigma \sin \frac{B}{2} \sin \frac{C}{2})^2 - \Sigma \sin B \sin C \leq 0. \quad (8)$$

First observe that, from standard identities,

$$\begin{aligned} 4(\Sigma \sin \frac{B}{2} \sin \frac{C}{2})^2 &= \Sigma 4 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} + (2\Sigma \sin \frac{A}{2})(4\Pi \sin \frac{A}{2}) \\ &= \Sigma(1 - \cos B)(1 - \cos C) + (2\Sigma \sin \frac{A}{2})(\Sigma \cos A - 1) \end{aligned}$$

and then that

$$\begin{aligned} \Sigma(1 - \cos B)(1 - \cos C) - \Sigma \sin B \sin C &= 3 + \Sigma \cos(B+C) - \Sigma(\cos B + \cos C) \\ &= 3 - 3\Sigma \cos A. \end{aligned}$$

Now (8) is seen to be equivalent to

$$(\Sigma \cos A - 1)(2\Sigma \sin \frac{A}{2} - 3) \leq 0. \quad (8a)$$

It is known [3, pp. 20,22] that $\Sigma \cos A > 1$ and $\Sigma \sin \frac{A}{2} \leq 3/2$, with equality just when the triangle is equilateral. This establishes (8a) and hence (8).

Now we show that the proposed inequality (1) is equivalent to *A.M.M. Problem S 23* proposed by Garfunkel and Bankoff [6], [for which a solution has not yet been published at the time this is written, although one is due soon (Editor)]: *Prove that the sum of the distances from the incenter of a triangle ABC to the vertices does not exceed half the sum of the internal angle bisectors, each extended to its intersection with the circumcircle of triangle ABC (see figure).*

Corresponding to vertex A we have, by the power of a point formula,

$$AI \cdot ID = R^2 - OI^2 = 2Rr;$$

also

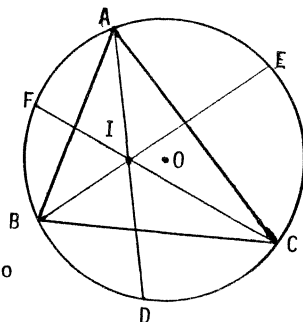
$$AI = r / \sin \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}.$$

Thus $ID = 2R \sin \frac{A}{2}$, with similar formulas corresponding to vertices B and C. Then

$$AI + BI + CI \leq \frac{1}{2}(AD+BE+CF) \quad (1a)$$

is equivalent to

$$4R\Sigma \sin \frac{B}{2} \sin \frac{C}{2} \leq 2R\Sigma \sin \frac{B}{2} \sin \frac{C}{2} + R\Sigma \sin \frac{A}{2},$$



and so equivalent to (1).

Finally, in a recent letter T. Sekiguchi (University of Arkansas) noted that (1a) can be shown to be equivalent to

$$2(\cot \alpha + \cot \beta + \cot \gamma) \leq \tan \alpha + \tan \beta + \tan \gamma, \quad (1b)$$

where α, β, γ are the direction angles of an arbitrary interior ray from the origin in the positive octant of a rectangular coordinate system; and also equivalent to

$$2\left\{\sqrt{\frac{x}{y+z}} + \sqrt{\frac{y}{z+x}} + \sqrt{\frac{z}{x+y}}\right\} \leq \sqrt{\frac{y+z}{x}} + \sqrt{\frac{z+x}{y}} + \sqrt{\frac{x+y}{z}} \quad (1c)$$

for $x, y, z > 0$.

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

Satyanarayana showed that the proposed inequality (1) is also equivalent to (see figure)

$$AD + BE + CF \leq P, \quad (1d)$$

where P is the perimeter of hexagon AFBDC E, and to (usual notation)

$$2\left\{\sqrt{\frac{s-a}{a}} + \sqrt{\frac{s-b}{b}} + \sqrt{\frac{s-c}{c}}\right\} \leq \sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}}. \quad (1e)$$

The equivalence of (1c) and (1e) is easily seen by putting

$$y+z = a, \quad z+x = b, \quad x+y = c.$$

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2. Leon Bankoff, Solution to Problem E 1272 (proposed by Victor Thébault), *American Mathematical Monthly*, 67 (1960) 693-694.
3. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.
4. E.P. Starke, Solution to Problem 860 (proposed by Leon Bankoff), *Mathematics Magazine*, 47 (1974) 50-51.
5. Leonard Carlitz, Solution to Problem E 1573 (proposed by Franz Leuenberger), *American Mathematical Monthly*, 71 (1964) 93-94.
6. Jack Garfunkel and Leon Bankoff, proposers of Problem S 23, *American Mathematical Monthly*, 86 (1979) 863.

545, [1980: 153] *Proposed by Jack Garfunkel, Flushing, New York.*

Given three concentric circles, construct an equilateral triangle having one vertex on each circle.

Comment by Clayton W. Dodge, University of Maine at Orono.

If there is a solution triangle, then it may be rotated about the common center, thus showing that any point on any of the three circles can serve as a vertex of a solution triangle. If we identify solutions obtainable from one another by a rotation about the common center, then it suffices to find all solutions corresponding to a fixed point O on one of the circles. If the other two circles are called C_1 and C_2 , then the problem becomes a special case of the general problem cited by Howard Eves in his solution of Crux 463 [1980: 163]:

Given a point O and two curves C_1 and C_2 , to locate a triangle OP_1P_2 , where P_1 is on C_1 and P_2 is on C_2 , directly similar to a given triangle $O'P_1'P_2'$.

The simple construction then given by Eves in the earlier problem applies equally well to the present problem and shows that there may be as many as four distinct solutions.

Solutions were received from JOHN BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; JAMES BOWE and BILL JUNKIN, Erskine College, Due West, South Carolina (jointly); J.T. GROENMAN, Arnhem, The Netherlands; B.M. SALER, Aurora High School, Aurora, Ontario; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. Comments were received from M.S. KLAMKIN, University of Alberta; JAN VAN DE CRAATS, Leiden University, The Netherlands; and STEPHEN WISMATH, University of Lethbridge.

Editor's comment.

Klamkin and van de Craats noted that our problem is given and solved in Yaglom [1]. The proposer showed that the problem is equivalent to that of constructing an equilateral triangle given the distances a, b, c of its vertices from a given point O , a problem already considered in this journal (Problem 39 [1975: 64]). Bowe and Junkin showed that, if a, b, c are the radii of the concentric circles, with $a \geq b, c$, then there is no solution if $b + c < a$, two solutions (a pair of congruent triangles) if $b + c = a$, and four solutions (two pairs of congruent triangles) if $b + c > a$. Furthermore, they showed that the length X of a side of any solution triangle (the existence of which implies that a, b, c can form a triangle of area, say, K) is the positive root of one of the equations

$$2X^2 = a^2 + b^2 + c^2 \pm 4\sqrt{3}K,$$

a formula which had already been obtained by Dworschak [1975: 65].

REFERENCE

1. I.M. Yaglom, *Geometric Transformations I*, N.M.L. No. 8, Mathematical Association of America, 1962, pp. 31, 90.

546, [1980: 154] *Proposed by John A. Winterink, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico.*

Prove the validity of the following simple method for finding the center of a conic, which is not given in most current texts:

For the central conic

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad ab - h^2 \neq 0,$$

the center is the intersection of the lines

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 0.$$

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

By Taylor's formula, we have

$$\phi(x+\alpha, y+\beta) \equiv \phi(\alpha, \beta) + \left\{ x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right\} + \frac{1}{2} \left\{ x^2 \frac{\partial^2 \phi}{\partial x^2} + 2xy \frac{\partial^2 \phi}{\partial x \partial y} + y^2 \frac{\partial^2 \phi}{\partial y^2} \right\}, \quad (1)$$

where all the partial derivatives are evaluated at (α, β) . It follows that $\phi(x, y) = 0$ represents a central conic with centre (α, β) if and only if (α, β) is the only pair that makes the coefficients of x and y in (1) simultaneously vanish. Hence the centre of the conic $\phi(x, y) = 0$ is the intersection (α, β) of the two lines

$$\frac{\partial \phi}{\partial x} \equiv 2(ax + hy + g) = 0,$$

$$\frac{\partial \phi}{\partial y} \equiv 2(hx + by + f) = 0,$$

which exists and is unique since $ab - h^2 \neq 0$. \square

The extension to n -dimensional conicoids is immediate. For $n = 3$, for example,

$$\phi(x, y, z,) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

is a central conicoid with centre (α, β, γ) if and only if (α, β, γ) is the unique intersection of the three planes

$$\frac{\partial \phi}{\partial x} \equiv 2(ax + hy + gz + u) = 0,$$

$$\frac{\partial \phi}{\partial y} \equiv 2(hx + by + fz + v) = 0,$$

$$\frac{\partial \phi}{\partial z} \equiv 2(gx + fy + cz + w) = 0,$$

the existence and uniqueness of this intersection being guaranteed by the condition

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0.$$

Also solved by JOHN BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; SAHIB RAM MANDAN, Bombay, India (3 solutions); V.N. MURTY, Pennsylvania State University, Capitol Campus; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer. Comments were received from W.J. BLUNDON, Memorial University of Newfoundland; and M.S. KLAMKIN, University of Alberta.

Editor's comment.

Most solvers showed that the centre of the conic is the unique solution (α, β) of the system

$$ax + hy + g = 0,$$

$$hx + by + f = 0.$$

They did this in a variety of ways, among which were:

1. showing that the terms in x and y vanish when the coordinate axes are translated so that (α, β) is the new origin, as is done in most current texts;
2. showing that (α, β) is the point which bisects every chord of the conic that passes through it;
3. showing that (α, β) is the pole of the line at infinity.

And so far not a peep about partial derivatives! Then, apparently nudged by the proposal, they became aware that hey, the left members of those two equations are just $\frac{1}{2}\partial\phi/\partial x$ and $\frac{1}{2}\partial\phi/\partial y$. So then they dragged the poor protesting partial derivatives into their solutions, after the answer was at hand and they were no longer needed.

Our featured solution shows that the partial derivatives can be made to play in this problem a role that is essential and not fortuitous. Which is not to say that this method should be used in teaching the present-day watered-down residue of the beautiful theory of analytical conics. As Klamkin noted in his comment, the more popular method of translation of axes is more elementary and better motivated, even if it involves a bit more algebra.

The partial derivative method discussed here has already been mentioned by our proposer in a letter to the editor of *The Mathematics Teacher*, 73 (March 1980) 167, 186.

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547. [1980: 154] *Proposed by Charles W. Trigg, San Diego, California.*

There are powers 2^k whose decimal digits sum to k ; for example, $2^5 = 32$ and $3 + 2 = 5$. Find another.

I. Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

We show that, for $1 \leq k < 106$, the only solutions are for $k = 5$, as noted in the proposal, and for $k = 70$, since 70 is the digital sum of

$$2^{70} = 1\,180\,591\,620\,717\,411\,303\,424.$$

A necessary (but not sufficient) condition for the decimal digits of 2^k to sum to k is that $2^k \equiv k \pmod{9}$. Since $2^6 \equiv 1 \pmod{9}$, we have, modulo 9, for an arbitrary nonnegative integer j ,

$$\begin{aligned} 2^{6j} &\equiv 1, & 2^{6j+1} &\equiv 2, & 2^{6j+2} &\equiv 4, \\ 2^{6j+3} &\equiv 8, & 2^{6j+4} &\equiv 7, & 2^{6j+5} &\equiv 5. \end{aligned}$$

Now the congruences modulo 9

$$6j \equiv 1, \quad 6j+1 \equiv 2, \quad 6j+2 \equiv 4, \quad 6j+3 \equiv 8$$

have no solution (since $ax \equiv b \pmod{m}$ has no solution when $(a, m) \nmid b$). So solutions to our problem can only occur when k is of the form $6j+4$ or $6j+5$. Since

$$6j+4 \equiv 7 \pmod{9} \iff 2j \equiv 1 \pmod{3} \iff j \equiv 2 \pmod{3}$$

and

$$6j+5 \equiv 5 \pmod{9} \iff j \equiv 0 \pmod{3},$$

we must have

$$k = 6j + 4 = 6(3u + 2) + 4 = 18u + 16$$

or

$$k = 6j + 5 = 6(3v) + 5 = 18v + 5.$$

So we need look for solutions only in the sets

$$\{18u + 16 \mid u = 0, 1, 2, \dots\} = \{16, 34, 52, 70, 88, 106, \dots\}$$

and

$$\{18v + 5 \mid v = 0, 1, 2, \dots\} = \{5, 23, 41, 59, 77, 95, 113, \dots\}.$$

That $k = 5$ and $k = 70$ are the only solutions with $k < 106$ can now be verified, for example, in the *CRC Standard Mathematical Tables*, all recent editions of which contain a table of powers of 2 up to 2^{101} .

II. *Solution by Harry L. Nelson, Livermore, California.*

I did a computer search for all positive integers k such that the base- b representation of 2^k has

(i) no more than one thousand digits and

(ii) a digital sum of k

and this for all bases b with $2 \leq b \leq 100$. The favorable results are tabulated below. The omitted bases are those for which there is no solution satisfying (i) and (ii). (The values of b and k in the table are in decimal notation.) The table shows,

in particular, that if there is a solution in base ten other than $k = 5$ and $k = 70$, then 2^k has more than one thousand digits and $k > 999/\log_{10} 2 > 3318$.

b	k	b	k	b	k
2	1	9	8, 136	32	8
3	2, 4	10	5, 70	59	6
5	4, 12, 16, 32, 52, 96	11	14	63	8
6	3, 14, 16, 34, 76, 77, 83, 94, 163, 174, 203, 303, 363, 374, 496, 543, 723, 794, 1457, 1543, 2176, 2537	13	4	64	16
		14	10	71	16
		15	16	85	16
		19	34		
		22	11		
7	4, 16, 22, 34, 52, 88, 94, 244, 250, 388, 454, 628, 820, 1222, 1234	28	5		
		30	6		
		31	16		

Later I put the computer through its paces again. Some of the interesting results that turned up are:

the (decimal) digital sum of the base-6 representation of 2^{2834} is 2834,	
...	base-6 ... 2^{8257} is 8257,
...	base-4 ... 3^{18} is 18,
...	base-4 ... 3^{24} is 24,
...	base-4 ... 3^{66} is 66,
...	base-5 ... 3^{23} is 23,
...	base-9 ... 6^{136} is 136.

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; SIDNEY KRAVITZ, Dover, New Jersey; HERMAN NYON, Paramaribo, Surinam; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Johnson noted that the property of 2^{70} discussed here had already been observed by Madachy [1].

REFERENCE

1. Joseph S. Madachy, *Mathematics on Vacation*, Scribner's, New York, 1966, p.168.

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548, [1980: 154] Proposed by M.S. Klamkin, University of Alberta.

If three equal cevians of a triangle divide the sides in the same ratio and same sense, must the triangle be equilateral?

Solution by Malcolm A. Smith, Georgia Southern College, Statesboro, Georgia.

The answer is YES. Let ABC be a triangle with equal cevians

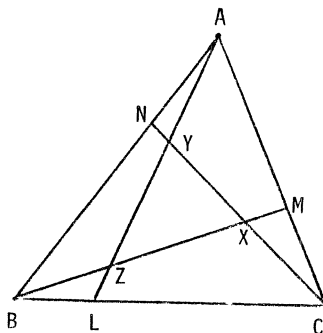
$$AL = BM = CN = l$$

such that

$$\frac{BL}{LC} = \frac{CM}{MA} = \frac{AN}{NB} = k \notin \{0, -1\},$$

and let the cevian intersections X, Y, Z be as shown in the figure. (Note that all segments are directed.)

Applying Menelaus' Theorem to triangle ABL and cevian CN, we get



$$\frac{AN}{NB} \cdot \frac{BC}{CL} \cdot \frac{LY}{YA} = -1.$$

Since $AN/NB = k$, $BC/CL = -(k+1)$, and $LY/YA = (l/AY) - 1$, we find that

$$AY = \frac{lk(k+1)}{k^2+k+1},$$

and the same result is obtained for BZ and CX by cyclic interchange. Thus

$$AY = BZ = CX. \quad (1)$$

Now we apply Menelaus' Theorem to triangle ALC and cevian BM, obtaining

$$\frac{AZ}{ZL} \cdot \frac{LB}{BC} \cdot \frac{CM}{MA} = -1.$$

Since $CM/MA = k$, $LB/BC = -k/(k+1)$, and $ZL/AZ = (l/AZ) - 1$, we find that

$$AZ = \frac{l(k+1)}{k^2+k+1},$$

and the same result is obtained for BX and CY by cyclic interchange. Thus

$$AZ = BX = CY. \quad (2)$$

Now $YZ = ZX = XY$ follows from (1) and (2), so triangle XYZ is equilateral.

Finally, triangles BCX, CAY, and ABZ are congruent (SAS), so $BC = CA = AB$ and triangle ABC is equilateral.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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549. [1980: 154] *Proposed by Ferrell Wheeler, [now at] Texas A & M University, College Station, Texas.*

The centers of eight congruent spheres of radius r are the vertices of a cube of edge length r .

(a) Find the volume of the intersection of the eight spheres.

(b) Into how many parts do these spheres divide the cube?

I. *Solution of part (a) by Bengt Månsson, Lund, Sweden.*

The required volume is proportional to r^3 , say it is Vr^3 . We will assume $r = 1$ and calculate V .

Introduce a Cartesian coordinate system with origin at the centre of the cube and axes parallel to its edges. Then one of the spheres has centre $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$; its surface S has the equation

$$(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 + (z + \frac{1}{2})^2 = 1;$$

and, together with the coordinate planes, it bounds a part of the first octant with volume $V/8$. Since S intersects the xy -plane along a circle of radius $\sqrt{3}/2$, we have

$$V/8 = \iint_{\substack{(x+\frac{1}{2})^2 + (y+\frac{1}{2})^2 \leq 3/4 \\ x, y \geq 0}} \{-\frac{1}{2} + \sqrt{1 - (x+\frac{1}{2})^2 - (y+\frac{1}{2})^2}\} dx dy.$$

With polar coordinates (ρ, ϕ) defined by

$$x + \frac{1}{2} = \rho \cos \phi, \quad y + \frac{1}{2} = \rho \sin \phi,$$

the integral reduces to

$$V/8 = 2 \int_{\arcsin(1/\sqrt{3})}^{\pi/4} d\phi \left\{ \int_{1/(2 \sin \phi)}^{\sqrt{3}/2} d\rho (-\frac{1}{2} + \sqrt{1 - \rho^2}) \rho \right\}.$$

A primitive for the ρ -integrand is easily found, with the help of which the integral becomes

$$V/8 = 2 \int_{\arcsin(1/\sqrt{3})}^{\pi/4} \left\{ -\frac{11}{48} + \frac{1}{16 \sin^2 \phi} + \frac{1}{3} \left(1 - \frac{1}{4 \sin^2 \phi}\right)^{3/2} \right\} d\phi.$$

The difficulty here is the integral of the last term which is, however, elementary. By means of the successive transformations

$$u = \sin \phi, \quad v = u^2, \quad w = \sqrt{\frac{4v-1}{1-v}},$$

we find

$$\int_{\arcsin(1/\sqrt{3})}^{\pi/4} \left(1 - \frac{1}{4 \sin^2 \phi}\right)^{3/2} d\phi = \frac{9}{8} \int_{1/\sqrt{2}}^{\sqrt{2}} \frac{w^4}{(4+w^2)(1+w^2)^2} dw,$$

which can be calculated using the partial fraction expansion

$$\frac{w^4}{(4+w^2)(1+w^2)^2} = \frac{16}{9} \cdot \frac{1}{4+w^2} - \frac{11}{18} \cdot \frac{1}{1+w^2} - \frac{1}{6} \cdot \frac{w^2-1}{(1+w^2)^2}.$$

It turns out that

$$\int_{\arcsin(1/\sqrt{3})}^{\pi/4} \left(1 - \frac{1}{4 \sin^2 \phi}\right)^{3/2} d\phi = \arctan(\sqrt{2}/5) - \frac{11}{16} \arctan(1/2\sqrt{2}).$$

The rest of the expression for $V/8$ is easily evaluated, and we finally obtain

$$V = 9 \arctan(\sqrt{2}/5) - 11\pi/12 + \sqrt{2} - 1 \approx 0.0152.$$

II. *Adapted from the solution of part (b) by John T. Barsby, St. John's-Ravenscourt School, Winnipeg, Manitoba.*

By drawing cross sections of the cube parallel to the base at various levels [8 detailed figures are omitted (Editor)], I determined that the spheres divide the top and bottom halves of the cube each into 54 regions. Since 13 regions extend into both halves, the required number of regions is

$$2 \cdot 54 - 13 = 95.$$

Editor's comment.

Barsby also solved part (a). He did not evaluate exactly his complicated integral for V but calculated instead the following bounds,

$$0.0139 < V < 0.0160,$$

and this agrees with the exact value found in solution I.

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550. [1980: 154] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Prove that

$$\int_0^{\frac{\pi}{2}} \cos(\cos \theta) \cdot \cosh(\sin \theta) d\theta = \frac{\pi}{2}.$$

Solution by M.S. Klamkin, University of Alberta.

The key to this problem is that the integrand

$$f(\theta) \equiv \cos(\cos \theta) \cosh(\sin \theta) = \cos(\cos \theta) \cos(i \sin \theta)$$

is the real part of

$$\cos(\cos \theta + i \sin \theta) = \cos e^{i\theta}.$$

Since $\cos e^{i\theta} + \cos e^{-i\theta} = 2f(\theta)$, we have to show equivalently that

$$\int_0^{\pi/2} (\cos e^{i\theta} + \cos e^{-i\theta}) d\theta = \pi$$

or that

$$\int_{-\pi/2}^{\pi/2} \cos e^{i\theta} d\theta = \pi. \quad (1)$$

Since $\cos e^{i(\pi+\theta)} = \cos e^{i\pi} e^{i\theta} = \cos e^{i\theta}$, (1) is equivalent to

$$\int_{\pi/2}^{3\pi/2} \cos e^{i\theta} d\theta = \pi. \quad (2)$$

From (1) and (2), our problem is therefore equivalent to

$$\int_0^{2\pi} \cos e^{i\theta} d\theta = 2\pi$$

or, if we put $z = e^{i\theta}$, to

$$\int_{|z|=1} \frac{\cos z}{z} dz = 2\pi i. \quad (3)$$

Now (3) is true by the residue theorem, since the integrand has a simply pole at $z = 0$ with residue 1, so the proposed equality is established.

Also solved by BENGT MÅNSSON, Lund, Sweden; V.N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; M. SELBY, University of Windsor; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

Editor's comment.

Some solvers started from (3), seemingly pulling it out of a hat, and were then able to proceed expeditiously to the answer. But the approach taken in our featured solution seems better motivated.

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SQUARES ON PARADE

$$\begin{aligned} 21^4 + 42^4 + 63^4 &= 4^2 + 11^2 + 76^2 + 4365^2 \\ &= 9^2 + 2916^2 + 3249^2 \\ &= 84^2 + 2541^2 + 3549^2 \\ &= 147^2 + 2352^2 + 3675^2 \\ &= 567^2 + 1575^2 + 4032^2 \\ &= 867^2 + 1200^2 + 4107^2 \\ &= 1029^2 + 1029^2 + 4116^2 \\ &= 3087^2 + 3087^2 \end{aligned}$$

DONALD CROSS, University of Exeter.