Crux Mathematicorum

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

EDITORIAL

Did you notice that we actually welcome repetition in songs? In fact, cognitive scientists claim that it is precisely the repetition that makes songs catchy as we happily sing/hum/holler along if we know what is coming. Furthermore, it is repetition that actually allows listeners to notice new things: already knowing the notes of the piano in the song lets us focus on the guitar sounds next time we listen and the drums the time after and so on.* While I think some students will agree that knowing what next step to take in a mathematical problem is somewhat satisfying, it is the second bit about discovering something new that I wish transferred to students doing math. So when solving a system of equations, I wish the students, knowing how to do it by elimination, would decide (by themselves!) to try the substitution method; when figuring out why Newton's method fails in a particular situation, they would (spontaneously!) choose to justify it graphically after figuring it out algebraically. Only deliberate practice of different techniques will result in understanding when to apply what tool.

As adults, we like sticking to what is familiar. I invite you to be more like a child once in a while and try a less comfortable and familiar approach to a problem. Sometimes the most elegant solution comes from an unexpected angle.

Kseniya Garaschuk

*For more details and nice graphics, check out TedEx video *Why we love repetition in music.*

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THE CONTEST CORNER

No. 56

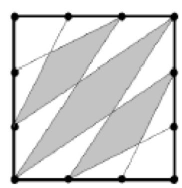
John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

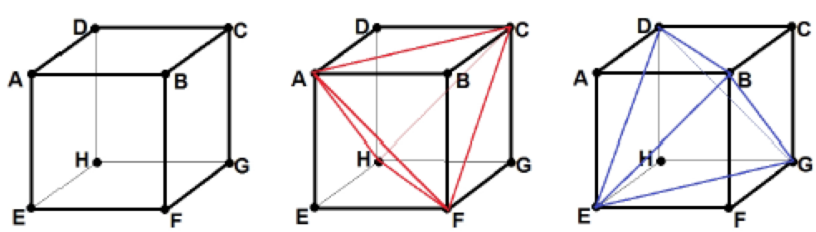
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er février 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC276. Le carré ci-dessous mesure 90×90. Chaque côté du carré est divisé en trois segments égaux par deux points et certains de ces points sont joints pour former d'autres segments. Déterminer l'aire de la région ombrée.



CC277. Soit A, B, C, D, E, F, G et H les sommets d'un cube $30 \times 30 \times 30$.

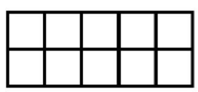
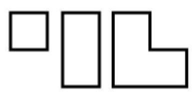
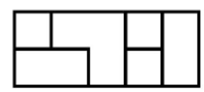


Les deux solides $ACFH$ et $BDEG$ sont des tétraèdres réguliers isométriques. Déterminer le volume de l'intersection de ces deux tétraèdres.

CC278. Deux entiers strictement positifs m et n forment une fraction $\frac{m}{n}$ dont la représentation décimale commence par 0,711 avec d'autres chiffres qui suivent. Déterminer la plus petite valeur possible de n.

CC279. On considère une grande quantité d'oeufs. Jeanne, Thomas et Raoul ont compté les oeufs, mais ils ont tous commis une erreur: dans la colonne des unités, la réponse de Jeanne diffère de la bonne réponse par 1; dans la colonne des dizaines, la réponse de Thomas diffère de la bonne réponse par 1 et dans la colonne des centaines, la réponse de Raoul diffère de la bonne réponse par 1. Sasha, José, Pierre et Maurice ont compté les oeufs et ont obtenu la bonne réponse. Lorque les sept réponses sont additionnées, on obtient un total de 3162. Combien y a-t-il d'oeufs?

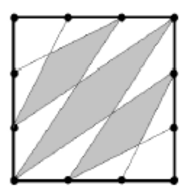
CC280. On considère une grille 2×5 de carrés. Il est possible de paver la grille en utilisant trois sortes de tuiles, soit des tuiles carrées 1×1, des tuiles formées de deux carrés 1×1 contigus et des tuiles en forme de L formées de trois carrés 1×1. La première figure suivante montre la grille, la deuxième montre les trois formes de tuiles et la troisième montre une façon de réussir le pavage.

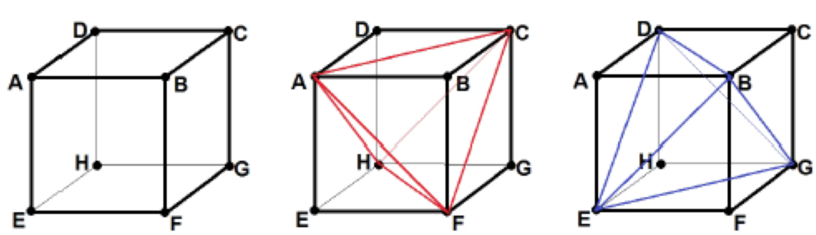
De combien de façons peut-on réussir le pavage?

. . . .

CC276. The figure below shows a 90×90 square with each side divided into three equal segments. Some of the endpoints of these segments are connected by straight lines. Find the area of the shaded region.



CC277. Let A, B, C, D, E, F, G and H be the eight vertices of a $30 \times 30 \times 30$ cube as shown.



The two figures $ACFH$ and $BDEG$ are congruent regular tetrahedra. Find the volume of the intersection of these two tetrahedra.

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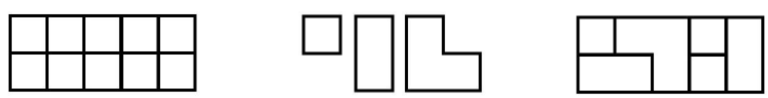
CC278. For positive integers m and n, the decimal representation for the fraction $\frac{m}{n}$ begins with 0.711 and is followed by other digits. Find the least possible value for n.

CC279. There is a pile of eggs. Joan counted the eggs, but her count was off by 1 in the 1's place. Tom counted the eggs, but his count was off by 1 in the 10's place. Raoul counted the eggs, but his count was off by 1 in the 100's place. Sasha, Jose, Peter, and Morris all counted the eggs and got the correct count. When these seven people added their counts together, the sum was 3162. How many eggs were in the pile?

CC280. You can tile a 2×5 grid of squares using any combination of three types of tiles: single unit squares, two side by side unit squares, and three unit squares in the shape of an L. The diagram below shows the grid, the available tile shapes, and one way to tile the grid.



In how many ways can the grid be tiled?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(6), p. 240–241.

CC226. In the table below we write all the different products of two distinct counting numbers between 1 and 100:

$$\begin{array}{ccccc} 1\cdot 2, & 1\cdot 3 & \dots & 1\cdot 99, & 1\cdot 100 \\ & 2\cdot 3 & \dots & 2\cdot 99, & 2\cdot 100 \\ & & \ddots & \vdots & \vdots \\ & & & & 99\cdot 100 \end{array}$$

Find the sum of all of these products.

Originally problem 46 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).

We received 17 solutions of which 16 were correct and complete. We present 2 solutions.

Solution 1, by Miguel Amengual Covas.

We add all entries of the given table by adding the entries of each column and then adding the column sums. We obtain

$$1\cdot 2 + (1+2)\cdot 3 + \ldots + (1+2+\ldots+98)\cdot 99 + (1+2+\ldots+99)\cdot 100.$$

Since each parenthesis consists of a sum of consecutive integers starting with 1, this expression may be rewritten as

$$\begin{aligned} &1\cdot 2 + \frac{2\,(1+2)}{2}\cdot 3 + \ldots + \frac{98\,(1+98)}{2}\cdot 99 + \frac{99\,(1+99)}{2}\cdot 100 \\ &= \frac{1}{2}\left(1\cdot 2^2 + 2\cdot 3^2 + \ldots + 98\cdot 99^2 + 99\cdot 100^2\right). \end{aligned}$$

Now we use the summation notation and write the last expression as

$$\frac{1}{2}\sum_{i=1}^{99} i\,(i+1)^2\,,$$

obtaining

$$\frac{1}{2}\sum_{i=1}^{99} i\,(i+1)^2 = \frac{1}{2}\sum_{i=1}^{99}\left(i^3+2i^2+i\right) = \frac{1}{2}\left(\sum_{i=1}^{99} i^3 + 2\sum_{i=1}^{99} i^2 + \sum_{i=1}^{99} i\right). \qquad (1)$$

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Applying the formulas

$$\sum_{i=1}^{n} i^3 = \left(\frac{n\,(n+1)}{2}\right)^2,$$
$$\sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6},$$
$$\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$$

with $n = 99$, we get

$$\sum_{i=1}^{99} i^3 = 24502500, \qquad \sum_{i=1}^{99} i^2 = 656700, \qquad \sum_{i=1}^{99} i = 4950.$$

When these results are substituted into (1), we get

$$\frac{1}{2}\sum_{i=1}^{99} i\,(i+1)^2 = 12582075.$$

Solution 2, by Ivko Dimitrić.

A suggestive upper triangular display of the products $i \cdot j$ in the ith row and jth column $(i < j)$ prompts us to consider also the missing part of the 100×100 symmetric square array, filling in the lower triangular part with products for which $i > j$ and diagonal with products $i \cdot i$ for $1 \le i, j \le 100$. Because of the symmetry of the completed array about the main diagonal $(i \cdot j = j \cdot i)$, the sum we are asked to find also appears as the sum of all the entries below the diagonal. Therefore, we find that sum by subtracting the sum of the diagonal entries from the sum of all the entries in the table and dividing the rest by two. The sum of all the entries in the square table equals

$$\sum_{j=1}^{100}\sum_{i=1}^{100} i \cdot j = \left(\sum_{i=1}^{100} i\right)\left(\sum_{j=1}^{100} j\right) = \left(\sum_{i=1}^{100} i\right)^2 = \left(\frac{100 \cdot 101}{2}\right)^2,$$

and we need also to recall the well-known summation formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then the sum we are asked to compute equals

$$\frac{1}{2}\left[\left(\sum_{i=1}^{100} i\right)^2 - \sum_{i=1}^{100} i^2\right] = \frac{1}{2}\left[\left(\frac{100 \cdot 101}{2}\right)^2 - \frac{100 \cdot 101 \cdot 201}{6}\right] = 12582075.$$

CC227. Suppose $\{a_1, a_2, \ldots\}$ is a geometric sequence of real numbers. The sum of the first n terms is S_n. If $S_{10} = 10$ and $S_{30} = 70$, determine the value of S_{40}.

Adapted from problem 36 from the Forty-second Annual Columbus State Invitational Mathematics Tournament (2016).

We received 14 correct solutions and one incorrect submission. We present the solution of Ivko Dimitrić.

Let r be the common ratio of the sequence. Then the n^{th} term is $a_n = a_1 r^{n-1}$ and the sum of the first n terms is

$$S_n = \frac{a_1(1-r^n)}{1-r}.$$

(Clearly, from the information given, $r \neq 1$). We are given

$$S_{10} = \frac{a_1(1-r^{10})}{1-r} = 10 \quad \text{and} \quad S_{30} = \frac{a_1(1-r^{30})}{1-r} = 70.$$

Dividing the two we have,

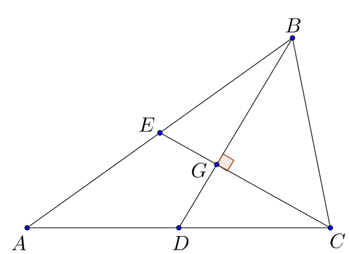
$$7 = \frac{S_{30}}{S_{10}} = \frac{1-(r^{10})^3}{1-r^{10}} = 1 + r^{10} + (r^{10})^2.$$

Setting $q = r^{10}$ we get a quadratic equation $q^2 + q + 1 = 7$ with roots $q = -3$ and $q = 2$. The first root must be discarded since it cannot equal an even power of a real number. Thus $q = r^{10} = 2$. Next we compute the ratio,

$$\frac{S_{40}}{S_{10}} = \frac{1-r^{40}}{1-r^{10}} = (1+r^{10})(1+r^{20}) = (1+2)(1+2^2) = 15.$$

Therefore $S_{40} = 15 \cdot S_{10} = 150$.

CC228. In the triangle ABC, $AB = 2\sqrt{13}, AC = \sqrt{73}$, E and D are the midpoints of AB and AC, respectively. Furthermore, BD is perpendicular to CE. Find the length of BC.



Originally problem 48 from the Forty-second Annual Columbus State Invitational Mathematics Tournament (2016).

We received 12 correct solutions and two incorrect submissions; we present two solutions.

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Solution 1, by David Manes.

Since G is the centroid of triangle ABC it splits each median into a 2:1 ratio. Let $|EG| = x$ and $|DG| = y$, which means $|GC| = 2x$ and $|GB| = 2y$. Applying the Pythagorean Theorem to each of the right triangles EGB, CGD and CBG we get

$$\begin{aligned} x^2 + 4y^2 &= (\sqrt{13})^2 && (1) \\ 4x^2 + y^2 &= \left(\frac{\sqrt{73}}{2}\right)^2 && (2) \\ 4x^2 + 4y^2 &= |BC|^2 && (3) \end{aligned}$$

Adding equations (1) and (2) and multiplying both sides of the result by $\frac{4}{5}$ we get

$$4x^2 + 4y^2 = 25,$$

which combined with (3) allows us to conclude that $|BC| = 5$.

Solution 2, by Titu Zvonaru.

Draw in the segment ED; since it joins two midpoints, $|ED| = |BC|/2$. The diagonals of quadrilateral $BCDE$ are perpendicular, and hence the sum of the squares of opposite sides is constant; that is,

$$\begin{aligned} &|BC|^2 + |ED|^2 = |BE|^2 + |CD|^2 \iff \\ &|BC|^2 + \frac{|BC|^2}{4} = \frac{52}{4} + \frac{73}{4} \iff \\ &5|BC|^2 = 125, \end{aligned}$$

whence we obtain $|BC| = 5$.

CC229. A store has objects that cost either 10, 25, 50, or 70 cents. If Sharon buys 40 objects and spends seven dollars, what is the largest quantity of the 50 cent items that could have been purchased?

Originally problem 37 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).

We received eight correct solutions. We present the solution of Hannes Geupel.

If 8 of the 40 objects cost each 50 cents, Sharon has to buy 32 objects that cost 10 cents to keep the price as small as possible. Altogether these 40 objects cost \$7.20. So Sharon has to buy less than 8 items that cost 50 cents.

If 7 of the 40 objects cost each 50 cents, she has to buy 33 objects that cost 10 cents to keep the price as small as possible. These 40 objects cost \$6.80 in total. So Sharon has to buy more expensive objects. The minimum price that can be added to this \$6.80 is 15 cents, by buying one 10 cent object less and one 25 cent object instead. Then the objects cost \$6.95. This is still not enough. So Sharon has to buy another more expensive object. The minimum price that can be added is 15 cents again, by buying one 10 cent object less and one 25 cent object instead. Then the 40 objects cost \$7.10. So Sharon has to buy less than 7 items that cost 50 cents.

If 6 of the 40 objects cost each 50 cents, it is possible to spend \$7 for all objects by buying 33 items that cost 10 cents, 0 that cost 25 cents, 6 that cost 50 cents, and 1 that costs 70 cents. Thus, 6 is the largest number of 50 cent items she could have bought.

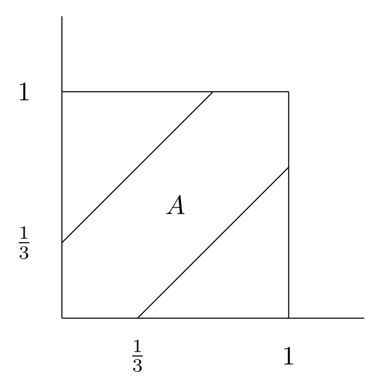
CC230. Two friends agree to meet at the library between 1:00 P.M. and 2:00 P.M. Each agrees to wait 20 minutes for the other. What is the probability that they will meet if their arrivals occur at random during the hour and if the arrival times are independent?

Originally problem 38 from the Thirty-fifth Annual Columbus State Invitational Mathematics Tournament (2009).

We received seven correct solutions. We present the solution of Ángel Plaza.

Let us use as sample space the unit square. Let X and Y be the two independent, randomly chosen times by person X and Y respectively from the one hour period.

These assumptions imply that every point in the unit square is equally likely, where the first coordinate represents the time when person X shows up and the second coordinate represents when person Y shows up. As 20 minutes is one third of the time between 1 pm and 2 pm, the required probability in this situation is the area of the the set of all points lying in region A in the figure below.



This can be seen as the unit square minus two triangles, giving the area of A as

$$1-2\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3}\right)=1-\frac{4}{9}=\frac{5}{9}.$$

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THE OLYMPIAD CORNER

No. 354

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er février 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC336. Déterminer toutes les fonctions f ($f : \mathbb{N} \to \mathbb{N}$) telles que

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2$$

pour tous entiers a et b ($a, b \in \mathbb{N}$).

OC337. Déterminer tous les polynômes $P(x)$ à coefficients entiers pour lesquels

$$P(P(n) + n)$$

est un nombre premier pour un nombre infini de valeurs entières de n.

OC338. Soit Γ le cercle exinscrit du triangle ABC opposé au sommet A (ce cercle est tangent au côté BC et aux prolongements des côtés AB et AC). Soit D le centre de Γ et E et F les points de contact de Γ avec les prolongements respectifs de AB et de AC. Soit J le point d'intersection des segments BD et EF. Démontrer que l'angle CJB est droit.

OC339. Soit n un entier strictement positif. Démontrer que

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n+1}}.$$

OC340. Soit k un entier fixe strictement positif. Alberto et Beralto jouent au jeu suivant: Étant donné un nombre initial N_0 ils effectuent l'opération suivante à tour de rôle en commençant par Alberto: le nombre n est remplacé par un nombre m tel que $m < n$ et que les représentations binaires (en base 2) de m et de n diffèrent l'une de l'autre pour exactement l chiffres consécutifs, l étant un entier quelconque ($1 \leq l \leq k$). Le joueur qui ne peut plus jouer est perdant. On dit qu'un entier non négatif t est un nombre gagnant si le joueur qui reçoit ce nombre a une stratégie gagnante, c'est-à-dire que ce joueur peut choisir les nombres suivants de manière à s'assurer une victoire, peu importe ce que fait son adversaire. Un nombre qui n'est pas gagnant est appelé un nombre perdant.

Démontrer que pour tout entier strictement positif N, le nombre de tous les nombres perdants non négatifs inférieurs à 2^N est égal à

$$2^{N-\lfloor \frac{\log(\min\{N,k\})}{\log 2} \rfloor}.$$

. . . .

OC336. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}$, we have

$$(f(a) + b)f(a + f(b)) = (a + f(b))^2.$$

OC337. Find all polynomials $P(x)$ with integer coefficients such that

$$P(P(n) + n)$$

is a prime number for infinitely many integers n.

OC338. Let Γ be the excircle of triangle ABC opposite to the vertex A (namely, the circle tangent to BC and to the extensions of the sides AB and AC from the points B and C). Let D be the center of Γ and E, F, respectively, the points in which Γ touches the extensions of AB and AC. Let J be the intersection between the segments BD and EF. Prove that $\angle CJB$ is a right angle.

OC339. Let n be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n+1}}.$$

OC340. Let k be a fixed positive integer. Alberto and Beralto play the following game: Given an initial number N_0 and starting with Alberto, they alternately do the following operation: change the number n for a number m such that $m < n$ and m and n differ, in their base-2 representation, in exactly l consecutive digits for some l such that $1 \le l \le k$. If someone can't play, they lose. We say a non-negative integer t is a winning number when the player who receives the number t has a winning strategy, that is, they can choose the next numbers in order to guarantee their own victory, regardless of the options of the other player. Otherwise, we call t a losing number.

Prove that, for every positive integer N, the total of non-negative losing integers less than 2^N is

$$2^{N-\lfloor \frac{\log(\min\{N,k\})}{\log 2} \rfloor}.$$

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OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(4), p. 149–150.

OC276. Let m and n be positive integers. If the number

$$k = \frac{(m+n)^2}{4m(m-n)^2+4}$$

is an integer, prove that k is a perfect square.

Originally problem 1 from day 1 of the 2015 Turkey Mathematical Olympiad.

No submitted solutions.

OC277. Find all real functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2+yf(x)) = xf(x+y)$.

Originally problem 3 of the 2015 India National Olympiad.

We received 3 correct submissions. We present the solution by Oliver Geupel.

The two functions $f(x) = 0$ and $f(x) = x$ are solutions to the problem. We prove that there are no other solutions.

Let f be any solution which is not identically 0. Putting $x = y = 0$, we see that $f(0) = 0$. Suppose a is a real number distinct from zero such that $f(a) = 0$. Then

$$f(a^2) = af(a+y)$$

for every $y \in \mathbb{R}$. Hence, f is a constant function, that is $f(x) = 0$, a contradiction to our hypothesis. Thus, $f(x) = 0$ holds only if $x = 0$.

Putting $y = -x$ in the given functional equation, we obtain

$$f(x^2 - xf(x)) = xf(0) = 0.$$

Consequently, for every $x \in \mathbb{R}$, $x^2 - xf(x) = 0$. It follows $f(x) = x$ for $x \neq 0$. The result follows.

OC278. Find all possible $\{x_1, x_2, \ldots, x_n\}$ permutations of $\{1, 2, \ldots, n\}$ so that when $1 \leq i \leq n-2$ then we have $x_i < x_{i+2}$ and when $1 \leq i \leq n-3$ then we have $x_i < x_{i+3}$. Here $n \geq 4$.

Originally problem 5 from day 2 of the 2015 Kazakhstan National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

Let f_j be the function of switching entry j and entry $j+1$ of a permutation of $\{1, 2, \cdots, n\}$, for all integers $1 \leq j \leq n-1$.

Define $(1,2,3)$, $(1,3,2)$, and $(2,1,3)$ to be the possible permutations if $n=3$.

It shall be proved, using mathematical induction, that for all integers $n \geq 3$, all possible permutations of $\{1,2,\cdots,n\}$ that satisfy the conditions are $(1,2,\cdots,n)$ and the permutations obtained by applying any of f_j for all integers $1 \leq j \leq n-1$, such that there does not exist $1 \leq m \leq n-2$ such that both f_m and f_{m+1} were applied.

If $n=3$, then the statement is true.

If $n=4$, then the possible permutations are $(1,2,3,4)$, $(1,3,2,4)$, $(2,1,3,4)$, $(1,2,4,3)$, and $(2,1,4,3)$, so the statement is true.

Assume that for some integer $m \geq 3$, if $n \in \{m, m+1\}$, then the statement is true. Let $n = m+2$. From the inequalities $x_j < x_{j+2}$ for all $1 \leq j \leq m$, element $m+2$ is either entry $m+2$ or entry $m+1$.

If element $m+2$ is entry $m+2$, then the inequalities $x_m < x_{m+2}$ and $x_{m-1} < x_{m+2}$ are satisfied, and the conditions on entries 1 to $m+1$ are equivalent to the case if $n = m+1$.

If element $m+2$ is entry $m+1$, then from the inequalities

$$x_m < x_{m+2}, \quad x_{m-1} < x_{m+2}, \quad x_j < x_{j+2}$$

for all integers $1 \leq j \leq m-2$, element $m+1$ is entry $m+2$. Therefore, the inequalities

$$x_{m-1} < x_{m+1}, \quad x_m < x_{m+2}, \quad x_{m-2} < x_{m+1}, \quad x_{m-1} < x_{m+2}$$

are satisfied, and the conditions on entries 1 to m are equivalent to the case if $n=m$, (so f_{m+1} is applied and not f_m).

Therefore for $n = m+2$, the statement is true and thus the claim is true by mathematical induction. Therefore all possible permutations are those as described at the beginning.

Editor's note. Can you count all the above permutations for a particular n? What do you notice? Can this lead to a different proof?

OC279. Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP \cdot CQ = AP \cdot AQ$$

Prove that I is tangent to the circumcircle of triangle BOC.

Originally problem 4 of the 2015 Canadian Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.

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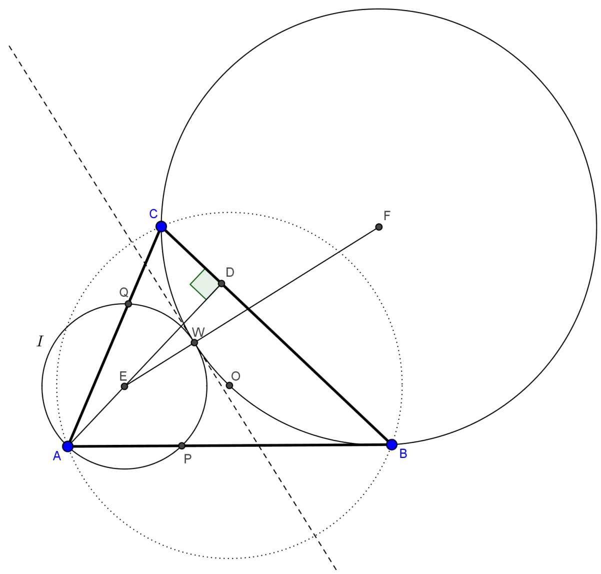

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC.

Equation of the altitude from A is

$$AD : S_B y - S_C z = 0.$$

Therefore a generic point on this line has coordinates

$$E(S_B - a^2 t : S_C t : S_B t)$$

where t is a parameter.

Now if we consider E the center of circle I we have the radius of this circle that is given from the distance between E to A

$$AE = \frac{aSt}{S_B},$$

so the equation of the generic circle I is

$$a^2yz + b^2zx + c^2xy - \frac{c^2S_B - 2S^2t}{S_B}y - \frac{b^2S_B - 2S^2t}{S_B}z = 0.$$

Points P and Q have coordinates

$$P = I \cap AB = (c^2S_B - 2S^2t : 2S^2t : 0), \qquad Q = I \cap AC = (b^2S_B - 2S^2t : 0 : 2S^2t),$$

so then we have

$$BP = \frac{c^2S_B - 2S^2t}{cS_B}, \qquad CQ = \frac{b^2S_B - 2S^2t}{bS_B}, \qquad AP = \frac{2S^2t}{cS_B}, \qquad AQ = \frac{2S^2t}{bS_B}.$$

Therefore, from the condition given, we obtain that

$$t = \frac{b^2c^2S_B}{2S^2(b^2+c^2)}, \quad \Rightarrow \quad I: \quad a^2yz + b^2zx + c^2xy - \frac{c^4}{b^2+c^2}y - \frac{b^4}{b^2+c^2}z = 0,$$
$$E(b^2S^2 + 2c^2S^2 - b^2S_B^2 : b^2c^2S_C : b^2c^2S_B).$$

Circumcircle of triangle BOC has to pass at the points

$$B(0,1,0), \quad O(a^2S_A : b^2S_B : c^2S_C) \quad \text{and} \quad C(0,0,1),$$

so it has equation

$$a^2yz + b^2zx + c^2xy - \frac{b^2c^2}{2S_A}x = 0.$$

So the center is the point $F\left(a^2(S_A^2 - S^2) : b^2(S^2 + S_AS_B) : c^2(S^2 + S_AS_C)\right)$.

Now the radical axis is the line given from the intersection between the two circles, that is

$$O_{BOC} \cap I = (b^4c^2 + b^2c^4)x - 2c^4S_Ay - 2b^4S_Az = 0.$$

We denote with W the intersection between the radical axis and the line EF connecting the circles' centers

$$W(2S_A : b^2 : c^2)$$

and the distances of this point from the centers are

$$EW = \frac{ab^2c^2}{2S(b^2+c^2)}, \qquad FW = \frac{ab^2c^2}{4S_AS}$$

that are equal to the radii of I and the circumcircle of triangle BOC, therefore the circles are tangent.

OC280. Let $g(n)$ be the greatest common divisor of n and 2015. Find the number of triples (a,b,c) which satisfy the following two conditions:

1. $a, b, c \in 1, 2, ..., 2015$;
2. $g(a), g(b), g(c), g(a+b), g(b+c), g(c+a), g(a+b+c)$ are pairwise distinct.

Originally problem 4 from day 1 of the 2015 China Girls Mathematics Olympiad.

No submitted solutions.

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Two Famous Formulas (Part II)

V. Vavilov and A. Ustinov

Part I of this article appears in ***Crux*** *43(2).*

The first part of this article discussed *Pick's formula* for calculating the area of a lattice polygon. Here, we will discuss *Euler's formula* for polyhedra and the connection between these two formulas.

We define a *polygonal map* to be a partition of a simple plane polygon into non-overlapping simple polygons that connect along entire sides (more precisely, any two of the partition's polygons that touch may do so either at a vertex or along a common side). A polygonal map is a special case of a *planar graph*, a graph that can be drawn on the plane without its edges crossing (that is, its edges intersect only at their endpoints). For connected planar graphs, and in particular for polygonal maps, we have the famous *Euler's formula*

$$V - E + F = 1,$$

where V is the number of vertices of the graph, E is the number of its edges and F is the number of its faces (not counting the outer face). See Figure 10 for an example. [Ed.: In Western literature the outer face of the planar graph is normally included in the count so that the right-hand side of the formula equals 2.]

The word "map" emphasizes the fact that Euler's formula holds also for topological partitions, where the sides of the regions are allowed to be curves (and not just straight lines) as long as the neighbouring regions share only a common side or a common vertex (see Figure 11).

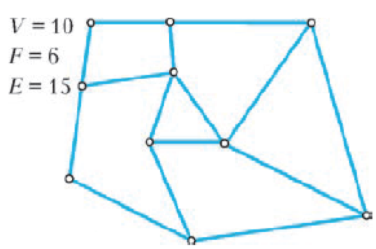

Figure 10

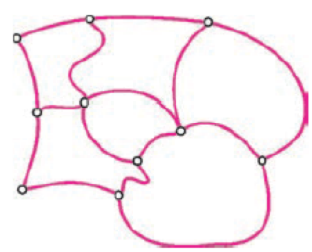

Figure 11

When all the polygons of the partition are triangles, the polygonal map is called a *triangulation*. Clearly, we can triangulate a given simple polygon in many different ways. However, for N triangles in a triangulation (not necessarily on a lattice), we always have that

$$N = 2N_i + N_e - 2,$$

where N_i is the number of vertices of the graph that lie inside the polygon and N_e is the number of its boundary vertices. (We proved this in Part I for lattice polygons.)

Exercise 3. Prove this formula for planar graph triangulations.

We also have that
$$E = 3N_i + 2N_e - 3,$$
where E is the total number of edges in the triangulation. Indeed, there are N_e edges that form the border of the polygon and every one of those edges appears in exactly one triangle. Therefore, there are $E - N_e$ other edges and each of those, being an inside edge, appears in exactly two triangles. In total, we have $3N$ triangle sides. Hence,
$$N = \frac{2(E - N_e) + N_e}{3} = \frac{2E - N_e}{3}.$$
Therefore,
$$E = \frac{3N + N_e}{2} = \frac{3(2N_i + N_e - 2) - N_e}{2} = 3N_i + 2N_e - 3,$$
as desired.

Theorem 3 (L. Euler) *For any polygonal map, we have*
$$V - E + F = 1.$$

Proof. Given a polygonal map P, introduce a new vertex inside each of the partition polygons. Connect these points (using either lines or curves) to the vertices of the corresponding polygon (see Figure 12) to form its triangulation.

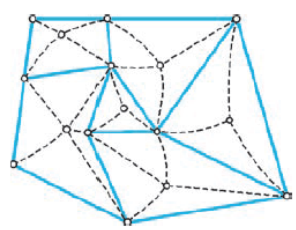

Figure 12

Now note that the formula $E' = 3N_i + 2N_e - 3$ holds for the number of edges E' of the newly created curved triangulation R of the polygon P. For this triangulation, we clearly have $N_i + N_e = V + F$. By construction, one edge of each triangle in R belongs to the original map P. Each of the newly drawn edges belongs to two triangles in R. Therefore, double the number of the new edges equals double the number of triangles. So for the number E' of all edges in R, we have $E' = E + N$, where N is the number of triangles in R. Now using the equation $N = \frac{2E' - N_e}{3}$ derived above, we have
$$V + F - E = (N_i + N_e) - (E' - N) = N_i + N_e - \frac{3N_i + 3N_e - 3}{3} = 1,$$
as desired. $\square$

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Exercise 4. Using Euler's formula, derive the equation $N = 2N_i + N_e - 2$.

Exercise 5. Prove Euler's formula without using the equation $N = 2N_i + N_e - 2$.

These two exercises together with the final section of Part I show that Pick's and Euler's formulas are equivalent – each can be derived from the other. They can also be generalized to maps with polygonal holes (see Figure 13) as follows:

Theorem 4 *For any simple lattice polygon P with n holes, we have*

$$[P] = N_i + \frac{N_e}{2} - 1 + n,$$

where $[P]$ denotes the area of P, N_i is the number of lattice points interior to P and exterior to all the holes, while N_e is the number of lattice points on the boundary of P and the boundaries of all the holes.

Theorem 5 *For any polygonal map with n holes, we have*

$$V - E + F = 1 - n.$$

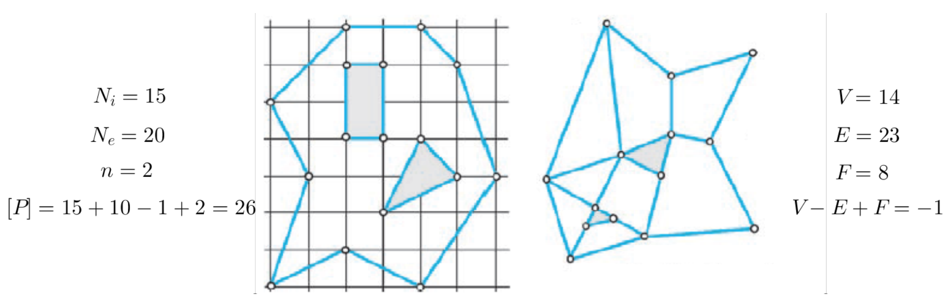

Figure 13

The proofs of these two generalized theorems are left as exercises for the reader.

Exercises.

6. Three frogs sit on the vertices of a lattice square and play leapfrog: each frog can jump over another frog and land on the point on the other side symmetrical to its original position (see Figure 14). Can any of the frogs end up on the fourth vertex of the original square? If they start on a lattice triangle, are there any points which frogs can never land on?

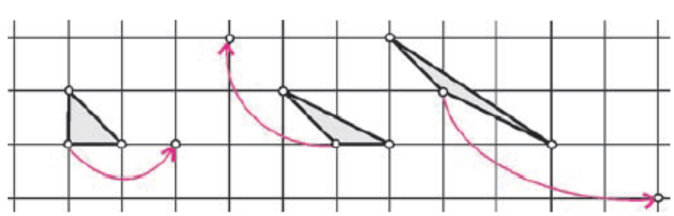

Figure 14

7. Consider a lattice triangle with no lattice points on its sides other than its vertices. Prove that if such a triangle contains exactly one lattice point in its interior, then this point is the point of intersection of its medians.

8. Consider a convex lattice n-gon with no lattice points on its sides other than its vertices and with no lattice points in its interior. Prove that $n \leq 4$.

9. Consider any two lattice points A and B such that there are no other lattice points on the line segment AB. Prove that there exists a lattice point C so that the triangle ABC is primitive (that is, contains no lattice points, except its vertices, inside or on its perimeter). If $|AB| = d$, find the distance from C to AB.

10. On a lattice, mark n ($n \geq 3$) lattice points so that any three of them form a triangle whose medians do not intersect at a lattice point. Find the largest value of n for which this is possible.

11. A king goes for a tour around an 8×8 chessboard; it visits every square exactly once before returning to the square where it started. A zigzag line that connects the centers of the squares in the order which the king visited them is non-intersecting. What is the maximum area of the polygon which this line borders?

12. Prove that an $n \times n$ square arbitrarily placed on a lattice will cover no more than $(n + 1)^2$ lattice points.

13. For two situations in Figure 15, calculate the area of the shaded parallelograms if the sides of the parallelogram $ABCD$ are divided by the given lattice into n and m equal parts as indicated.

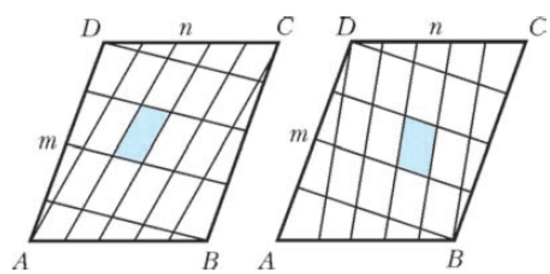

Figure 15

14. Each side of a triangle ABC is divided into three equal parts and one of the points on each side is connected to the opposite vertex as in Figure 16. Find the area of the shaded triangle in terms of the area of ABC.

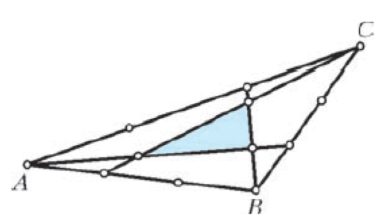

Figure 16

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15. Midpoints of the sides of a square are connected as shown in Figure 17. Find the area of the shaded octagon in terms of the area of the original square.

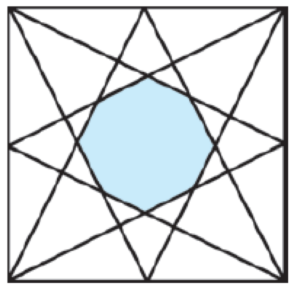

Figure 17

16. Let $f(P)$ be the function defined on all simple lattice polygons P as follows:

$$f(P) = aN_i(P) + bN_e(P) + c,$$

where a, b and c are some constants. Suppose also that $f(P) = f(P_1) + f(P_2)$ if the polygon P is divided into simple lattice polygons P_1 and P_2 by some zigzag line connecting lattice points in the interior of P. Prove that $b = a/2$ and $c = -a$.

17. Prove that for any simple lattice polygon P we have

$$2[P] = N(2P) - 2N(P) + 1,$$

where $N(P)$ denotes the total number of lattice points in the interior and on the boundary of P and $2P$ denotes a polygon that was obtained from P by a stretch with a scale factor 2 with respect to the origin.

18. Find one thousand points in the plane, no three of them collinear, so that the distance between any two of them is irrational and the area of a triangle formed by any three of them is rational.

This article appeared in Russian in Kvant, 2008(2), p. 11–15. It has been translated and adapted with permission.

UNSOLVED CRUX PROBLEMS

*As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Chris Fisher published a list of unsolved problems from **Crux** [2010 : 545, 547]. Below is a sample of two of these unsolved problems. Please email your solutions to* `crux-editors@cms.math.ca`.

2735$\star$. *Proposed by Richard I. Hess, Rancho Palos Verdes, California, USA.* [2002: 179; 2003: 190]

Given three Pythagorean triangles with the same hypotenuse, is it possible that the area of one triangle is equal to the sum of the areas of the other two triangles?

2950$\star$. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* [2004: 231, 234; 2005: 256]

Let ABC be a triangle whose largest angle does not exceed $2\pi/3$. For $\lambda, \mu \in \mathbb{R}$, consider the inequalities of the form

$$\cos\left(\frac{A}{2}\right)\cdot\cos\left(\frac{B}{2}\right)\cdot\cos\left(\frac{C}{2}\right) \geq \lambda + \mu\cdot\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right).$$

a) Prove that $\lambda_{\max} \geq \frac{2\sqrt{3}-1}{8}$.

b) Prove or disprove that $\lambda = \frac{2\sqrt{3}-1}{8}$ and $\mu = 1+\sqrt{3}$ yield the best inequality in the sense that λ cannot be increased. Determine also the cases of equality.

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The *pqr* Method: Part II

Steven Chow, Howard Halim and Victor Rong

*Part I of this article appears in **Crux** 43(5).*

The first part of this article discussed the *pqr* Lemma. In this part, we will examine its extension. For completeness, we re-state the *pqr* Lemma here:

The *pqr* Lemma. For three complex numbers a, b and c, let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$, and define

$$T(p,\, q,\, r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 = (a - b)^2(b - c)^2(c - a)^2.$$

When we fix two of p, q, r such that there exist triples $(p,\, q,\, r)$ satisfying $p,\, q,\, r \geq 0$ and $T(p,\, q,\, r) \geq 0$, the unfixed variable obtains its maximum and minimum values when two of a, b, c are equal. There is one exception – when r is the unfixed variable, its minimum value occurs when either two of a, b, c are equal, or one of them is equal to 0.

Special Conditions

Consider the following problem:

Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$a + b + c \geq 2 + \sqrt{abc(4 - a - b - c)}.$$

Here, the condition $a^2 + b^2 + c^2 + abc = 4$ is equivalent to $p^2 - 2q + r = 4$. However, fixing two of p, q, r would fix the third, so we cannot apply the *pqr* lemma directly. Fortunately, we can extend the *pqr* method to deal with conditions like these.

The *pqr* Lemma For Special Conditions. Let the variables a, b, c obey a symmetric condition that can be written as $G(p,\, q,\, r) = 0$, where G is a continuous function. Let (x, y, z) be a permutation of $(p,\, q,\, r)$. Fix some value $z \geq 0$ such that there exist triples $(p,\, q,\, r)$ satisfying $p,\, q,\, r \geq 0$, $T(p,\, q,\, r) \geq 0$, and $G(p,\, q,\, r) = 0$. Assume that the condition G is equivalent to $y = f(x)$ (while z is fixed), where a set of valid values of x is bounded, and f is continuous over that set. If $z = r$, x obtains its maximum and minimum values when two of a, b, c are equal. If $z \neq r$, x obtains its maximum and minimum values when either two of a, b, c are equal, or one of them equals 0.

We will prove this for $(x, y, z) = (q, p, r)$, because the proofs for the others are similar.

Proof. When r is fixed, each of the inequalities $T(f(q),\, q,\, r) = T(p,\, q,\, r) \geq 0$ and $f(q) = p \geq 0$ defines a union of several intervals and rays. The set of valid values of q is the intersection of those sets, so it is a union of several intervals (it cannot contain rays because we are assuming that this set is bounded). At the endpoints of those intervals, either $T(p, q, r) = 0$ or $p = 0$. The maximum and minimum values of q must be at an endpoint. Therefore q attains its maximum and minimum values when two of a, b, c are equal. $\square$

Now, we can use this to solve the problem above.

Solution. The inequality

$$a + b + c \geq 2 + \sqrt{abc(4 - a - b - c)}$$

is equivalent to $p \geq 2 + \sqrt{r(4 - p)}$, and the condition is equivalent to

$$q = \frac{p^2 - 4 + r}{2}.$$

If we fix r, q is a continuous function of p. Let $f(p) = p - 2 - \sqrt{r(4 - p)}$. Then, the inequality can be written as $f(p) \geq 0$. Since $f(p)$ is monotonic, it suffices to prove the inequality for the minimum value of p. From the above lemma, this occurs when two of the variables are equal.

WLOG assume that $a = b$. The condition becomes

$$2a^2 + c^2 + a^2c = 4, \quad \text{or} \quad (c + 2)(c + a^2 - 2) = 0,$$

so $c = 2 - a^2$. Substituting this into the inequality, we get $a^4(a - 1)^2 \geq 0$. Since it is true for the minimum value of p, it is true for all values of p, and we are done.

Below is an additional example to further show the usefulness of the *pqr* method:

Example. Let a, b, c be non-negative real numbers such that

$$a^2 + b^2 + c^2 = ab + bc + ca + (abc - 1)^2.$$

Prove that

$$ab + bc + ca + 3 \geq 2(a + b + c).$$

Solution. The condition is equivalent to $q = \frac{p^2-(r-1)^2}{3}$. Plugging in this value of q into the desired inequality yields

$$\frac{p^2 - (r - 1)^2}{3} + 3 - 2p \geq 0.$$

When p is fixed, this is a concave function in terms of r, and q is a continuous function of r. Therefore, it is only necessary to consider the extreme values of r. By the *pqr* lemma for special conditions, r takes an extreme value when WLOG $a = 0$ or $a = b$.

If $a = 0$, then the condition is $b^2 + c^2 = bc + 1$, which is equivalent to

$$(b + c)^2 = 3bc + 1.$$

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By the AM-GM inequality,

$$(b+c)^2 = 3bc + 1 \leq \frac{3(b+c)^2}{4} + 1.$$

From this inequality, $b + c \leq 2$. The desired inequality is

$$bc + 3 \geq 2(b+c).$$

By the condition, $bc = \frac{(b+c)^2-1}{3}$, so it is enough to prove that

$$\frac{(b+c)^2 - 1}{3} + 3 \geq 2(b+c).$$

This inequality is equivalent to $(2 - b - c)(4 - b - c) \geq 0$, which is true since $b + c \leq 2$.

If $a = b$, then the desired inequality is equivalent to $a^2 + 2ac + 3 \geq 4a + 2c$. The condition becomes

$$(a-c)^2 = (a^2c - 1)^2,$$

which means that $a - c = a^2c - 1$ or $a - c = 1 - a^2c$. If $a - c = a^2c - 1$, then

$$c = \frac{a+1}{a^2+1}.$$

Plugging this into the inequality yields

$$a^2 + 2a\left(\frac{a+1}{a^2+1}\right) + 3 \geq 4a + 2\left(\frac{a+1}{a^2+1}\right).$$

This is equivalent to $(a-1)^4 \geq 0$, which is clearly true.

If $a - c = 1 - a^2c$, then this is equivalent to $(a-1)(ac+c+1) = 0$. Since a and c are non-negative, $ac + c + 1$ cannot be 0, so a must equal 1 and c can be any non-negative real number. Plugging this into the inequality yields $2c+4 \geq 2c+4$, which always holds true.

Since we have proved the inequality for when $a = 0$ and $a = b$, we are done.

Problems with Special Conditions

The following problems may be solved using the *pqr* lemma for special conditions.

Problem 5. Let a, b, c be non-negative real numbers such that

$$(a+b)(b+c)(c+a) = 8.$$

Prove that

$$(a+b+c)^3 + 5abc \geq 32.$$

Problem 6. Let a, b, c be non-negative real numbers such that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10.$$

Prove that

$$\frac{9}{8}\le\frac{a^2+b^2+c^2}{ab+bc+ca}\le\frac{6}{5}.$$

Problem 7. Let a, b, c be positive real numbers such that

$$2(a+b+c)=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2.$$

Prove that

$$\frac{a^2}{2+bc}+\frac{b^2}{2+ca}+\frac{c^2}{2+ab}\ge 1.$$

Acknowledgements

We would like to thank our instructors at the Summer Conference: A. Doledenok, M. Fadin, A. Menshchikov, and A. Semchankau, for their excellent support during the conference. Also, huge thanks to Professor Kumar Murty and the Department of Mathematics at the University of Toronto for the generous financial support which allowed us to participate in this unique international learning experience. Last but not least, many thanks to Olga Zaitseva and Professor Victor Ivrii for running the Tournament of Towns program in Toronto, making all the travel arrangements, and overseeing us in Russia, otherwise none of this would have been possible.

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er février 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4251. *Proposé par Paolo Perfetti.*

Soit $8x = y^2$, $z = 0$ l'équation d'une parabole dans le plan (x, y) de $\mathbb{R}^3$. Soit C le cône formé par le sommet $P = (0, 0, 4)$ et les segments allant de P aux points de la parabole. Soit S la sphère d'équation $x^2 + y^2 + z^2 - 4z = 0$.

a) Calculer la surface de la portion de C se trouvant à l'intérieur de S.

b) Calculer la surface de la portion de S se trouvant à l'intérieur de C.

4252. *Proposé par Leonard Giugiuc et Marian Cucoanes.*

Soit $ABCD$ un tétraèdre avec $\angle BAC = \angle CAD = \angle DAB = 60°$. Dénoter par R_a, R_b et R_c les rayons des cercles circonscrits des triangles BAC, CAD et DAB, respectivement. Démontrer que

$$R_a + R_b + R_c \geq \sqrt{AB^2 + AC^2 + BC^2}.$$

4253. *Proposé par Titu Zvonaru.*

Soit ABC un triangle tel que $A = 90°$ et $45° < C < 60°$. Soit M le mi point de BC. La perpendiculaire de C vers AM intersecte AB en D. Sur AC, choisissons un point E et soit K l'intersection des lignes CD et BE. Si $BK = 2AE$, démontrer que le triangle CEK est isocèle.

4254. *Proposé par George Apostolopoulos.*

Soit ABC un triangle. Démontrer que

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{\text{cyc}}\frac{(\sin A + \sin B)^2}{\sin C} \leq \frac{3\sqrt{3}}{4}.$$

4255. *Proposé par Michel Bataille.*

Soit ABC un triangle isocèle tel que $AB = AC$ et soit P un point sur son cercle circonscrit tel que $P \neq A$. La réflexion par rapport à AP du cercle avec diamètre

AB intersecte le cercle avec diamètre AP en A et Q. Démontrer que AQ et QC sont perpendiculaires.

4256. *Proposé par Daniel Sitaru.*

Soient $a, b, c \in \mathbb{R}$ tels que $a + b + c = 1$. Démontrer que

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 4.$$

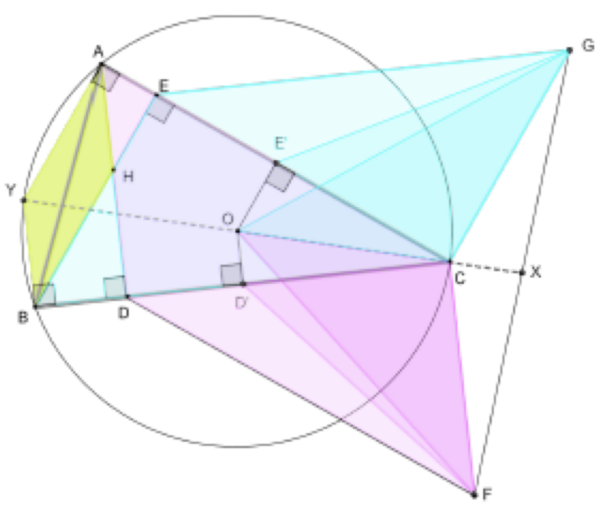
4257. *Proposé par Leonard Giugiuc et Dan Stefan Marinescu.*

Calculer la limite

$$\lim_{n\to\infty} \left(\frac{\sqrt[n(n+1)]{1! \cdot 2! \cdot \ldots n!}}{\sqrt{n}} \right).$$

4258. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle aigu avec cercle circonscrit O et orthocentre H, où $D \in BC, AD \perp BC, E \in AC$ puis $BE \perp AC$. Définissons F et G comme étant les quatrièmes sommets des parallélogrammes $CADF$ et $CBEG$. Si X est le mi point de FG et Y est le point où XC intersecte de nouveau le cercle circonscrit, démontrer que $AHBY$ est un parallélogramme.



4259. *Proposé par Mihály Bencze.*

Démontrer que

$$\prod_{k=1}^{n} \left(\frac{\sum_{p=1}^{k} \frac{1}{2p-1}}{\sum_{p=1}^{k} \frac{1}{p}} \right) \geq \frac{n+1}{2^n}.$$

4260. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Soient $a_i, i = 1, \ldots, 6$ des nombres positifs tels que $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 = 3$ et $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 9$. Démontrer que $a_1 a_2 a_3 a_4 a_5 a_6 \leq 1$.

. . . .

4251. *Proposed by Paolo Perfetti.*

Let $8x = y^2$ and $z = 0$ be the equations of a parabola in the (x, y) plane of $\mathbb{R}^3$. Let C be the cone of vertex in $P = (0, 0, 4)$ and generate the segments from P to the points of the parabola. Let S be the sphere of equation $x^2 + y^2 + z^2 - 4z = 0$.

a) Calculate the area of that portion of C inside S.

b) Calculate the area of that portion of S inside C.

4252. *Proposed by Leonard Giugiuc and Marian Cucoanes.*

Let $ABCD$ be a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 60^\circ$. Denote by R_a, R_b, R_c the circumradii of the triangles BAC, CAD and DAB, respectively. Prove that

$$R_a + R_b + R_c \geq \sqrt{AB^2 + AC^2 + BC^2}.$$

4253. *Proposed by Titu Zvonaru.*

Let ABC be a triangle with $A = 90^\circ$ and $45^\circ < C < 60^\circ$. Let M be the midpoint of BC. The perpendicular from C to AM intersects the leg AB at D. On the side AC we take a point E and let K be the intersection of the lines CD and BE. If $BK = 2AE$, then prove that the triangle CEK is isosceles.

4254. *Proposed by George Apostolopoulos.*

Let ABC be a triangle. Prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{\text{cyc}}\frac{(\sin A + \sin B)^2}{\sin C} \leq \frac{3\sqrt{3}}{4}.$$

4255. *Proposed by Michel Bataille.*

Let ABC be an isosceles triangle with $AB = AC$ and P a point of its circumcircle (with $P \neq A$). The reflection about AP of the circle with diameter AB intersects the circle with diameter AP at A and Q. Prove that AQ and QC are perpendicular.

4256. *Proposed by Daniel Sitaru.*

Let $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Prove that

$$\frac{e^b - e^a}{b - a} + \frac{e^c - e^b}{c - b} + \frac{e^a - e^c}{a - c} > 4.$$

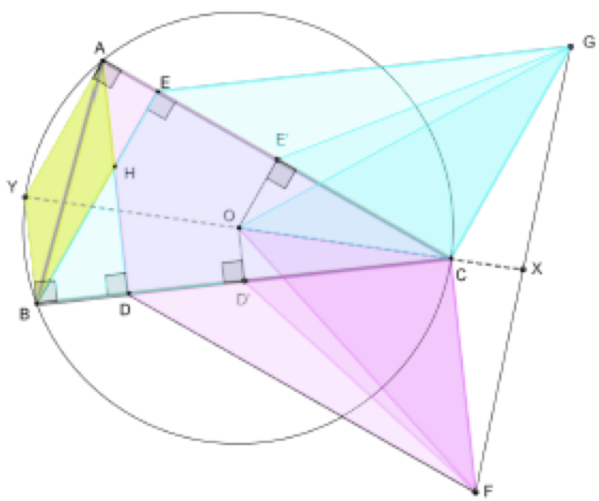
4257. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Calculate the following limit

$$\lim_{n\to\infty}\left(\frac{\sqrt[n(n+1)]{1!\cdot 2!\cdot \ldots n!}}{\sqrt{n}}\right).$$

4258. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle O, orthocentre H, $D \in BC, AD \perp BC, E \in AC, BE \perp AC$. Define points F and G to be the fourth vertices of parallelograms $CADF$ and $CBEG$. If X is the midpoint of FG, and Y is the point where XC intersects the circumcircle again, prove that $AHBY$ is a parallelogram.



4259. *Proposed by Mihály Bencze.*

Prove that

$$\prod_{k=1}^{n}\left(\frac{\sum_{p=1}^{k}\frac{1}{2p-1}}{\sum_{p=1}^{k}\frac{1}{p}}\right) \geq \frac{n+1}{2^n}.$$

4260. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let $a_i, i = 1, \ldots, 6$ be positive numbers such that $a_1 + a_2 + a_3 - a_4 - a_5 - a_6 = 3$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 9$. Prove that $a_1a_2a_3a_4a_5a_6 \leq 1$.

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SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(6), p. 267–270.

4151. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let n be an integer with $n \geq 2$. Find the real numbers t such that

$$(a_1a_2\dots a_n)^t + (b_1b_2\dots b_n)^t + (c_1c_2\dots c_n)^t \leq 1$$

for all $a_i, b_i, c_i > 0$ with $a_i + b_i + c_i = 1$, $i = 1, 2, \dots, n$.

Four correct solutions were received; all essentially followed the same approach.

First observe that when a_i, b_i, c_i are all equal to $1/3$, the left side of the inequality equals 3^{1-nt}, and this exceeds 1 when $t < 1/n$.

On the other hand, taking note that each product lies between 0 and 1, when $t \geq 1/n$,

$$\begin{aligned}\left(\prod_{i=1}^n a_i\right)^t + \left(\prod_{i=1}^n b_i\right)^t + \left(\prod_{i=1}^n c_i\right)^t &\leq \left(\prod_{i=1}^n a_i\right)^{1/n} + \left(\prod_{i=1}^n b_i\right)^{1/n} + \left(\prod_{i=1}^n c_i\right)^{1/n}\\ &\leq \frac{1}{n}\left(\sum_{i=1}^n a_i + \sum_{i=1}^n b_i + \sum_{i=1}^n c_i\right)\\ &= \frac{1}{n}\sum_{i=1}^n (a_i + b_i + c_i) = 1.\end{aligned}$$

Hence the inequality holds when $t \in [1/n, \infty)$.

Editor's Comment. The inequality also follows from the generalized Hölder inequality

$$\sum_{j=1}^m \left(\prod_{i=1}^n u_{ij}\right) \leq \prod_{i=1}^n \left(\sum_{j=1}^m u_{ij}^n\right)^{1/n}$$

applied to $m = 3$ and $(u_{i1}, u_{i2}, u_{i3}) = (a_i^{1/n}, b_i^{1/n}, c_i^{1/n})$.

4152. *Proposed by Daniel Sitaru.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$\ln(1+a)^{\ln(1+b)^{\ln(1+c)}} \leq \ln^3(1 + \sqrt[3]{abc}).$$

The following solution is due, independently, to Ali Adnan, Leonard Giugiuc and the proposer. No other correct solutions and one incorrect solution were received.

The left side of the inequality should be bracketed

$$\ln\left[(1+a)^{\ln\left[(1+b)^{\ln(1+c)}\right]}\right]$$

so that it is equal to $\ln(1+a)\ln(1+b)\ln(1+c)$.

Let $f(x)=\ln(\ln(1+e^x))$. Then

$$f''(x)=e^x[(1+e^x)\ln(1+e^x)]^{-2}[\ln(1+e^x)-e^x]<0,$$

so that $f(x)$ is concave. By Jensen's theorem, for each x,y,z,

$$f(x)+f(y)+f(z)\le 3f\left(\frac{x+y+z}{3}\right).$$

Setting $(x,y,z)=(\ln a,\ln b,\ln c)$ yields that

$$\ln[\ln(1+a)\ln(1+b)\ln(1+c)]\le 3\ln\left[\ln\left(1+e^{(\ln abc)/3}\right)\right].$$

Exponentiating yields

$$\ln(1+a)\ln(1+b)\ln(1+c)\le \ln^3(1+\sqrt[3]{abc})$$

as desired.

4153. *Proposed by Michel Bataille.*

Let $ABCDEFG$ be a regular heptagon inscribed in a circle with radius r. Prove that

$$\frac{1}{AB^3\cdot BD}-\frac{1}{BD^3\cdot DG}+\frac{1}{DG^3\cdot GA}=\frac{1}{r^4}.$$

We received 11 submissions, 10 of which were correct solutions. The eleventh offered an approximate numerical verification. We present the solution by Titu Zvonaru, modified superficially by the editor.

We may assume (without loss of generality) that $r=1$ and denote $a=AB, b=BD(=AC)$, and $c=DG(=AD)$. With this notation the relation to be proved reduces to

$$b^2c^3-c^2a^3+a^2b^3=a^3b^3c^3. \tag{1}$$

Applying Ptolemy's theorem to quadrilateral $ACDE$ we obtain

$$ab+ac=bc, \tag{2}$$

and to $ACDF$ we obtain

$$b^2+ab=c^2. \tag{3}$$

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Using these relations we can reduce the left-hand side of (1) to

$$\begin{aligned}
&(bc)^2c - c^2a^3 + a^2b^3\\
=_{(2)}\quad & c(ab+ac)^2 - c^2a^3 + a^2b^3 = a^2(b^2c + 2bc^2 + c^3 - ac^2 + b^3)\\
=_{(3),(2)}\ & a^2(b^2c + 2bc^2 + c(b^2+ab) - c(bc-ab) + b^3) = a^2b(2bc + c^2 + 2ac + b^2)\\
=_{(3),(2)}\ & a^2b(2bc + (b^2+ab) + 2(bc-ab) + b^2\\
=\quad & a^2b^2(-a + 2b + 4c).
\end{aligned}$$

It follows that we must prove that

$$-a + 2b + 4c = abc^3. \tag{4}$$

By the Law of Sines we have $a = 2\sin\frac{\pi}{7}$, $b = 2\sin\frac{2\pi}{7}$, and $c = 2\sin\frac{4\pi}{7}$, so that the left-hand side of (4) becomes

$$2(-\sin\frac{\pi}{7} + 2\sin\frac{2\pi}{7} + 4\sin\frac{4\pi}{7}) = 2(-\sin\frac{\pi}{7} + 4\sin\frac{\pi}{7}\cos\frac{\pi}{7} + 16\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}),$$

while the right-hand side becomes

$$\begin{aligned}
32\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin^3\frac{4\pi}{7} &= 8\sin\frac{\pi}{7}\left(2\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)\left(2\sin^2\frac{4\pi}{7}\right)\\
&= 8\sin\frac{\pi}{7}\left(\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7}\right)\left(1 - \cos\frac{8\pi}{7}\right).
\end{aligned}$$

Consequently, (4) is equivalent to

$$-1 + 4\cos\frac{\pi}{7} + 16\cos\frac{\pi}{7}\cos\frac{2\pi}{7} = 4\left(\cos\frac{2\pi}{7} + \cos\frac{\pi}{7}\right)\left(1 + \cos\frac{\pi}{7}\right). \tag{5}$$

Setting $x = \cos\frac{\pi}{7}$ turns (5) into $-1 + 4x + 16x(2x^2 - 1) = 4(2x^2 - 1 + x)(1 + x)$, or

$$8x^3 - 4x^2 - 4x + 1 = 0. \tag{6}$$

Finally, since $\cos\frac{3\pi}{7} = -\cos\frac{4\pi}{7}$, we deduce that $4x^3 - 3x = -(8x^4 - 8x^2 + 1)$, so that $\cos\frac{\pi}{7}$ is a zero of

$$8x^4 + 4x^3 - 8x^2 - 3x + 1 = (x+1)(8x^3 - 4x^2 - 4x + 1) = 0.$$

Hence, equation (6) holds and the argument is complete.

Editor's Comment. Several solvers reduced the problem to equation (6), which they said is "well known." Indeed, a version of the cubic was established on page 57 of the proposer's recent article in issue 2 of *Crux*, "About the Side and Diagonals of the Regular Heptagon" [2017: 55-60]. Other solutions made use of the cubic $x^3 - \frac{\sqrt{7}}{2}x^2 + \frac{\sqrt{7}}{8} = 0$, whose zeros are $-\sin\frac{\pi}{7}, \sin\frac{2\pi}{7}$, and $\sin\frac{4\pi}{7}$, an equation that can be found implicitly on the pages of Bataille's article. In fact, the problem itself is solved there on page 59.

4154. *Proposed by Leonard Giugiuc.*

Find a 3×3 matrix X with integer coefficients such that

$$X^4 = 3\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

We received five correct solutions and will feature the one by Michel Bataille.

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

An easy calculation shows that $A^2 = 3A$, hence it is sufficient to find $X \in \mathcal{M}_3(\mathbb{Z})$ such that $X^2 = A$.

A classical method of diagonalization gives $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Now, let

$$Y = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$Y^2 = \begin{bmatrix} a^2 + bc & b(a+d) & 0 \\ c(a+d) & d^2 + bc & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we have $Y^2 = D$ if we take $a = 1, d = -1, b = 2$ and $c = 1$. With this choice, we obtain

$$PYP^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

and since

$$(PYP^{-1})^2 = PY^2P^{-1} = PDP^{-1} = A,$$

the matrix

$$X = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is a solution.

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4155. *Proposed by Mihaela Berindeanu.*

Show that in any triangle ABC with side lengths a, b, c and semi-perimeter p we have:

$$\sqrt{\frac{2(p-a)}{c}} + \sqrt{\frac{2(p-b)}{a}} + \sqrt{\frac{2(p-c)}{b}} \geq \frac{p^2}{a^2+b^2+c^2-p^2}.$$

We received seven submissions all of which were correct. We present the solution by Titu Zvonaru.

Let $a = y+z, b = z+x$ and $c = x+y$ for $x, y, z \geq 0$ (this is commonly known as Ravi substitution.) Then $p = x+y+z$ and

$$a^2+b^2+c^2-p^2 = (x+y)^2+(y+z)^2+(z+x)^2-(x+y+z)^2 = x^2+y^2+z^2.$$

Using the AM-GM Inequality, the Cauchy-Schwarz Inequality and the fact that $xy+yz+zx \leq x^2+y^2+z^2$, we have

$$\begin{aligned}
\sqrt{\frac{2(p-a)}{c}} + \sqrt{\frac{2(p-b)}{a}} + \sqrt{\frac{2(p-c)}{b}} &= \sqrt{\frac{2x}{x+y}} + \sqrt{\frac{2y}{y+z}} + \sqrt{\frac{2z}{z+x}} \\
&= \frac{2x}{\sqrt{2x(x+y)}} + \frac{2y}{\sqrt{2y(y+z)}} + \frac{2z}{\sqrt{2z(z+x)}} \\
&\geq \frac{4x}{3x+y} + \frac{4y}{3y+z} + \frac{4z}{3z+x} \\
&= \frac{4x^2}{3x^2+xy} + \frac{4y^2}{3y^2+yz} + \frac{4z^2}{3z^2+zx} \\
&\geq \frac{4(x+y+z)^2}{3(x^2+y^2+z^2)+xy+yz+zx} \\
&\geq \frac{(x+y+z)^2}{x^2+y^2+z^2} \\
&= \frac{p^2}{a^2+b^2+c^2-p^2}
\end{aligned}$$

and the proof is complete.

4156. *Proposed by José Luis Díaz–Barrero.*

Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{n} \sum_{k=1}^{n} \frac{\left(\sqrt{x_k} + \sqrt{x_{k+1}}\right)^4}{x_k + x_{k+1}} \geq 8,$$

where the subscripts are taken modulo n.

We received 13 submissions, all of which were correct. We present the solution by Michel Bataille.

Let S denote the left-hand side of the proposed inequality. Note that if $a, b > 0$, then
$$(\sqrt{a}+\sqrt{b})^4 = (a+b+2\sqrt{ab})^2 = (a+b)^2 + 4ab + 4(a+b)\sqrt{ab},$$
so
$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b} = \left((a+b) + \frac{4ab}{a+b} + 4\sqrt{ab}\right) \geq 2\sqrt{4ab} + 4\sqrt{ab} = 8\sqrt{ab}$$
by the AM-GM Inequality. Therefore, applying the AM-GM Inequality one more time, we then have
$$S \geq \frac{1}{n}\sum_{k=1}^{n} 8\sqrt{x_k x_{k+1}} \geq \frac{8}{n}\left(n\left(\prod_{k=1}^{n}\sqrt{x_k x_{k+1}}\right)^{1/n}\right) = 8\left(\prod_{k=1}^{n} x_k\right)^{1/n} = 8$$
completing the proof.

4157. *Proposed by Michael Rozenberg, Leonard Giugiuc and Daniel Sitaru.*

Let a, b and c be positive real numbers such that
$$a+b+c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = k$$
for some positive real number $k \geq 3$. Find the minimum value of $ab+bc+ca$ in terms of k.

We received four correct solutions and five incorrect ones. We present the solution by Kee-Wai Lau.

We show that the minimum value of $ab+bc+ca$ equals
$$\frac{k^4 + 18k^2 - 27 - (k-3)(k+3)\sqrt{(k-3)(k-1)(k+1)(k+3)}}{8k^2},$$
which we denote by $f(k)$.

Let $p = a+b+c$, $q = ab+bc+ca$, and $r = abc$, so that
$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 = (a-b)^2(b-c)^2(c-a)^2 \geq 0.$$
From the given equalities, we have $p = k$ and $r = \frac{q}{k}$, so
$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 = -\frac{q[4k^2q^2 + (27-18k^2-k^4)q + 4k^4]}{k^2}.$$
It follows that
$$4k^2q^2 + (27-18k^2-k^4)q + 4k^4 \leq 0$$
so that $q \geq f(k)$. By taking
$$a = \frac{k^2 - 3 + \sqrt{(k-3)(k-1)(k+1)(k+3)}}{2k}$$

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and

$$b = c = \frac{k^2 + 3 - \sqrt{(k-3)(k-1)(k+1)(k+3)}}{4k},$$

we obtain $ab + bc + ca = f(k)$, proving our claim for the minimum value.

4158. *Proposed by George Apostolopoulos.*

Let m_a, m_b and m_c be the lengths of medians of a triangle ABC with inradius r. Prove that

$$\frac{m_a + m_b + m_c}{\sin^2 A + \sin^2 B + \sin^2 C} \geq 4r.$$

We received 15 correct solutions. We present the solution by Adnan Ali.

Clearly, the inequality is equivalent to

$$m_a + m_b + m_c \geq r\left((2\sin A)^2 + (2\sin B)^2 + (2\sin C)^2\right) = r \cdot \frac{a^2 + b^2 + c^2}{R^2},$$

where the equality is due to the Sine Law (and the notations have their usual meanings). Then from the well-known inequalities (see D.S. Mitrinovic, J.E. Pecaric and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, pp. 219, 11.15)

$$a^2 + b^2 + c^2 \leq 9R^2$$

and

$$m_a + m_b + m_c \geq 9r.$$

the problem follows. Equality occurs if and only if ABC is equilateral.

4159. *Proposed by Michel Bataille.*

Prove that

$$\cosh x + \cosh y + \cosh(x+y) \leq 1 + 2\sqrt{\cosh x \cosh y \cosh(x+y)}$$

for any real numbers x, y. For which pairs (x, y) does equality hold?

We received four correct solutions and two incorrect submissions. We present the solution by Dan Daniel.

Let $a = e^x$ and $b = e^y$. Note that $a, b > 0$. Then

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\left(a + \frac{1}{a}\right) = \frac{a^2+1}{2a},\\
\cosh(y) &= \frac{b^2+1}{2b},\\
\cosh(x+y) &= \frac{(ab)^2+1}{2ab}.
\end{aligned}$$

The given inequality (once we make these substitutions and multiply by $2ab > 0$ to get rid of fractions) is equivalent to

$$a^2b + ab^2 + a + b + a^2b^2 + 1 \le 2ab + \sqrt{2(a^2+1)(b^2+1)(a^2b^2+1)} \iff$$
$$(ab+1)(a+b) + (ab-1)^2 \le \sqrt{2(a^2+1)(b^2+1)(a^2b^2+1)}. \qquad (1)$$

Applying the Cauchy-Schwarz inequality to the $\mathbb{R}^2$ vectors $(ab+1, ab-1)$ and $(a+b, ab-1)$ we get

$$\begin{aligned}(ab+1)(a+b) + (ab-1)^2 &\le \sqrt{\left((a+b)^2 + (ab-1)^2\right)\left((ab+1)^2 + (ab-1)^2\right)}\\ &= \sqrt{(a^2+b^2+a^2b^2+1)\cdot 2(a^2b^2+1)}\\ &= \sqrt{2(a^2+1)(b^2+1)(a^2b^2+1)},\end{aligned}$$

which is the same as (1) and hence concludes the proof of the inequality.

For the Cauchy-Schwarz inequality, equality holds if and only if the two vectors are linearly dependent. This leads us to consider two cases:

- $ab - 1 = 0$, in which case the two vectors are dependent regardless of the first coordinate. We have $ab = 1 \Leftrightarrow e^{x+y} = 1 \Leftrightarrow x + y = 0$.
- $ab - 1 \neq 0$, and the two vectors are dependent if and only if $\frac{a+b}{ab+1} = \frac{ab-1}{ab-1}$. This gives us $ab - a - b + 1 = 0 \Leftrightarrow (a-1)(b-1) = 0 \Leftrightarrow a = 1$ or $b = 1$, which translates to $x = 0$ or $y = 0$.

Therefore, equality holds for $(x, y) \in \{(t, -t), (t, 0), (0, t) : t \in \mathbb{R}\}$.

4160. *Proposed by Leonard Giugiuc and Marian Cucoanes.*

Let ABC be a triangle with circumradius R, inradius r and incenter I. Let D, E and F be the circumcenters of the triangles IBC, ICA and IAB, respectively. Prove that

$$\frac{Area(DEF)}{Area(ABC)} = \frac{R}{2r}.$$

We received 15 solutions, all correct, and feature the solution by Václav Konečný, modified by the editor.

Our argument depends on a theorem that can be found in college geometry textbooks such as Roger A. Johnson's *Advanced Euclidean Geometry* (Dover, 1960, Theorem 292, p. 185):

> *The point D that is equidistant from I, B, and C is the point where the circumcircle of triangle ABC intersects the interior bisector of angle A and the perpendicular bisector of the side BC.*

A proof is easier than looking up a reference; for example, the figure indicates how to show (for the midpoint D of the arc BC) that $DB = DI$ in triangle BDI by showing that the angle at D equals $\angle C$ and at B equals $\frac{\angle B}{2} + \frac{\angle A}{2}$, so that the angle at I must also equal $\frac{\angle B}{2} + \frac{\angle A}{2}$.

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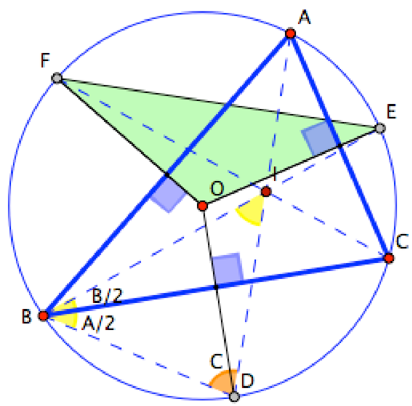

All we need here is that D, E, F (as defined in our problem to be centers of their respective circumcircles) are points of the circumcircle of ΔABC where it meets the perpendicular bisectors OD, OE, OF of the sides of ΔABC. Thus, in the shaded triangle FOE for example, $OE = OF = R$ and $\angle FOE = 180° - A$, whence

$$[FOE] = \frac{R^2}{2}\sin A$$

is the area of triangle FOE. Similarly for triangles DOE and FOD, so that

$$[DEF] = [FOE] + [DOF] + [EOD] = \frac{1}{2}R^2(\sin A + \sin B + \sin C).$$

Of course,

$$[ABC] = rs = r\frac{a+b+c}{2} = r(R\sin A + R\sin B + R\sin C).$$

Consequently, $\frac{[DEF]}{[ABC]} = \frac{R}{2r}$, as desired.

Editor's Comments. Most solvers reduced the problem to a known formula such as

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$

which can be found as Formula 293b on page 186 of the Johnson book cited above.

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

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