Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

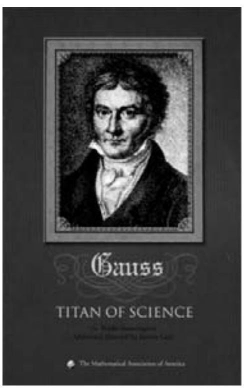
Gauss: Titan of Science

The example of a master in a particular field can either be hugely inspiring or discouraging, as any aspiring sportsperson or musician can testify. Carl Friedrich Gauss is described as 'Titan of Science' in the title of a biography, by G. Waldo Dunnington. Gauss is quoted as having said that there have only been three epoch-making mathematicians in all history (up to then!): Archimedes, Newton and Eisenstein, one of his pupils. Dunnington adds that 'history has ... given Gauss himself the position he gave Eisenstein'. He quotes the French mathematician Laplace who was asked who was the foremost mathematician in Germany. Laplace supplied a name. 'Why not Gauss?' his questioner asked. 'Oh, Gauss is the greatest mathematician in the world', replied Laplace.

Gauss was born on 30 April 1777 in Brunswick, and lived through tumultuous times in Europe (Napoleon and all that), yet his life seemed a haven of calm. He joined the staff of the University of Göttingen in 1807, where he remained for the rest of his life. He travelled little, and died on 23 February 1855. During his life he was honoured by many learned societies throughout the world. The centenary of his death was commemorated by the issue of a postage stamp by Deutsche Bundespost.

Gauss did foundational work in mathematics, physics and astronomy, both theoretical and practical. To quote from Dunnington's biography 'Gauss used to say that he was entirely a mathematician.... He called mathematics the queen of the sciences, and the theory of numbers the queen of mathematics, saying that she often condescended to serve astronomy and other sciences, but that under all circumstances top rank belonged to her. Gauss regarded mathematics as the principal means of educating the human mind'.

Gauss was a somewhat grudging teacher, 'I have a true aversion to teaching' he wrote in 1802. Again, in 1810, 'This winter I am teaching two courses for three listeners, of whom one is only moderately prepared, one scarcely moderately prepared and the third lacks preparation as well as ability. Those are the onera of a mathematical profession'. In 1808, he wrote 'It is not knowledge but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment'. On being greeted by a student in the street who was the worse for wear, Gauss replied, smiling, 'my young friend, I wish that science would intoxicate you as



much as our good Göttingen beer'. On gifted students, he wrote in 1808 'Instruction is very purposeless for such individuals who do not want merely to collect a mass of knowledge, but are mainly interested in exercising their own powers. One doesn't need to grasp such a one by the hand and lead him to the goal, but only from time to time to give him suggestions, in order that he may reach out himself in the shortest way'.

In case any teacher or prospective teacher should feel like giving up, even Gauss had a teacher! Inevitably, the famous story of J. G. Büttner, a teacher in 1784 at St. Katharine's Volksschule, is included in Dunnington's biography. Büttner set his young pupils the task of adding up the numbers 1 to 100. Gauss almost immediately put his slate on the table on which he had written a single number, 5050. Many of the answers of the other pupils were wrong, and were 'at once rectified by the whip'. How did Gauss arrive at his answer so quickly?

These quotes are all from a biography of Gauss, which was first published in 1955. The author, G. Waldo Dunnington, was an American professor of German. It has the feeling of having been written for a former generation, but there is useful additional mathematical material in this new edition by Jeremy Gray and Fritz-Egbert Dohse. You can read Gauss' mathematical diary; even his will and a list of the books that he borrowed from the library. There is no mention of Harry Potter!

If Gauss is at the summit of mathematical achievement, most of the rest of us have to be content to be on the nursery slopes or the plain. But don't despair. Would there have been a Gauss without a Büttner, even with his whip?

Reference

1 G. W. Dunnington, with additional material by J. Gray, *Gauss: Titan of Science* (MAA, Washington, DC, 2003).

Continuous squares

$$\begin{aligned} 49 &= 7^2 \quad \text{and} \quad 4 = 2^2, \quad 9 = 3^2, \\ 169 &= 13^2 \quad \text{and} \quad 16 = 4^2, \quad 9 = 3^2, \\ 1681 &= 41^2 \quad \text{and} \quad 16 = 4^2, \quad 81 = 9^2. \end{aligned}$$

Are there others?

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Ptolemy versus Copernicus

THOMAS J. OSLER and JOSEPH DIACO

1. Introduction

Before recorded history, it was observed that the stars formed fixed patterns in the sky. These became our constellations. The ancients watched the Sun, Moon and five of the brightest stars (the planets) move through the fixed stars in a great band called the zodiac that circles the heavens. The familiar twelve signs of astrology are the twelve constellations that comprise this zodiac.

The path that the Sun takes through the zodiac is called the *ecliptic*. Imagine a map of the entire sky, much like a map of the world, only it shows the stars and constellations. Figure 1 is such a map (without the stars plotted). Here, the ecliptic is a sinusoidal curve. The Moon and planets move in paths that are very close to the ecliptic. To identify an object's position on the ecliptic, astronomers measure its distance in degrees from the vernal equinox, the point where the Sun crosses the equator on the first day of spring. We denote this important angle by the variable θ.

About 150AD, Claudius Ptolemy published his *Almagest* (see references 1 and 2), which was the bible of astronomy for around 1500 years. In his cosmology, the Earth was at the centre of the universe. In Section 2, we will use a simplified version of Ptolemy's cosmos to obtain equations for the position θ of a planet. In 1543, Nicolaus Copernicus, a Polish priest, published his *De Revolutionibus* (see references 3 and 4) in which he placed the Sun at the centre. In Section 3, we use a simplified version of the Copernican view to derive equations for θ. Remarkably, we get the same equations that we obtained from Ptolemy's geocentric universe, we discuss this in Section 4. This analysis requires nothing more than familiarity with trigonometry, in particular, it does not use calculus.

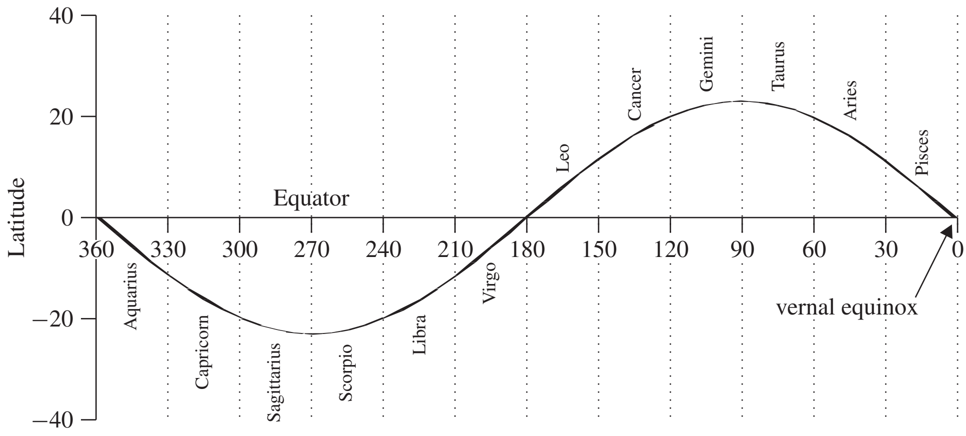

Figure 1 The ecliptic and constellations of the zodiac as they appear on a celestial map, showing latitudes from −40 to 40 degrees.

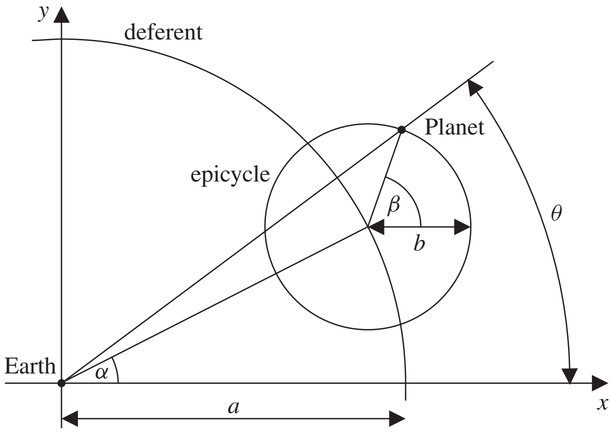

Figure 2 A simplified Ptolemaic system.

2. The simplified Ptolemaic view

Figure 2 shows the essential features of the Ptolemaic system on the ecliptic plane. The planet is on the rim of a small circle of radius b called the epicycle. This circle rotates with constant angular velocity ω_{E}. The angle β is given by

$$\beta = \omega_{\mathrm{E}} t + \gamma_{\mathrm{E}}, \tag{1}$$

where t is time and γ_{E} is the value of β when $t = 0$. The centre of the epicycle, in turn, moves uniformly on the rim of a large circle of radius a called the *deferent*. The angle α is given by

$$\alpha = \omega_{\mathrm{P}} t + \gamma_{\mathrm{P}}. \tag{2}$$

The deferent rotates with constant angular velocity ω_{P}, and γ_{P} is the value of α when $t = 0$.

From figure 2, it is easy to see that the x and y coordinates of the planet can be found by using

$$x = a\cos(\omega_{\mathrm{P}} t + \gamma_{\mathrm{P}}) + b\cos(\omega_{\mathrm{E}} t + \gamma_{\mathrm{E}}), \tag{3}$$

$$y = a\sin(\omega_{\mathrm{P}} t + \gamma_{\mathrm{P}}) + b\sin(\omega_{\mathrm{E}} t + \gamma_{\mathrm{E}}). \tag{4}$$

Since the Earth is at the origin, the position of the planet is seen as the angle

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{5}$$

3. The simplified Copernican view

In the Copernican view, the Sun is at the centre of coordinates as shown in figure 3. The Earth is orbiting the Sun on a circle of radius b, while the planet is on a circle of radius a. The angles α and β are again given by (1) and (2).

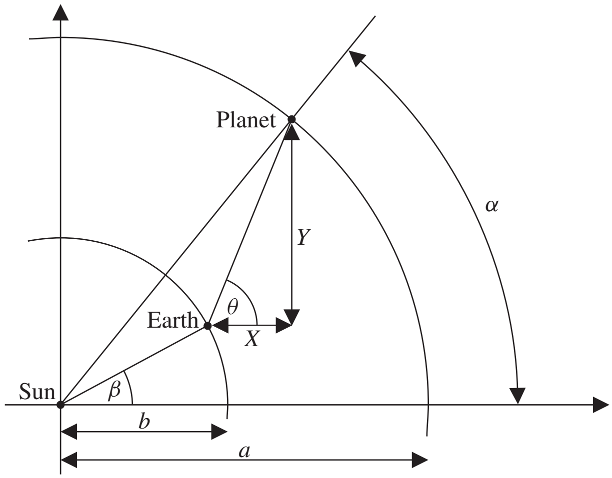

Figure 3 A simplified Copernican system.

The values of X and Y are given by

$$X = a\cos(\omega_{\mathrm{P}}t + \gamma_{\mathrm{P}}) - b\cos(\omega_{\mathrm{E}}t + \gamma_{\mathrm{E}}), \tag{6}$$

$$Y = a\sin(\omega_{\mathrm{P}}t + \gamma_{\mathrm{P}}) - b\sin(\omega_{\mathrm{E}}t + \gamma_{\mathrm{E}}). \tag{7}$$

The angle at which the planet is observed in the heavens from an observer on the Earth is given by

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right). \tag{8}$$

4. Ptolemy versus Copernicus

Historically, the Copernican hypothesis was very controversial. Early supporters of the theory faced the ridicule of those who believed that a moving Earth would result in everything flying off into the air. Worse, there were religious objections to moving man from the central location of God's creation. Johannes Kepler was excommunicated from the Lutheran church, and Galileo was tried by the Inquisition for supporting the Copernican view.

However, from a mathematical point of view, the two theories are equivalent. Compare the Ptolemaic equations (3)–(5) with their equivalent Copernican formulas (6)–(8). The only difference is the minus signs in (6) and (7). If we replace γ_{E} by $\gamma_{\mathrm{E}} + \pi$ in (6) and (7), then the minus signs disappear, and the Ptolemaic and Copernican formulas for the position of a planet become identical. The addition of π simply changes the reference angle from the positive x direction to the negative x direction. Thus, from a mathematical point of view, there is no difference in the Ptolemaic and the Copernican systems.

This means that the geocentric and the heliocentric descriptions of planetary motion are equally accurate (or inaccurate) for calculating the positions of the planets as observed from the Earth. (We say inaccurate because Kepler showed that the ellipse, and not the circle, was needed to describe the motion with better precision.)

In figures 2 and 3 we have described the situation for a *superior planet*, that is a planet rotating outside the orbit of the Earth. We could easily have done the same for an *inferior planet*, i.e. a planet rotating inside the orbit of the Earth. Look at the two circles in the Ptolemaic system (figure 2), and compare them to the two circles in the Copernican system (figure 3). Notice that the deferent circle becomes the orbit of the planet, while the epicyclic circle becomes the orbit of the Earth. One outstanding advantage of the Copernican system is that it enabled astronomers to determine the relative size of the solar system. That is, assuming the radius of the orbit of the Earth to be one, they could easily determine the size of the orbits of the other planets. This measure of the size of the known solar system was impossible for Ptolemy. You can read the actual calculations of Copernicus in reference 4 for Mercury (p. 287), Venus (p. 277), Mars (p. 275), Jupiter (p. 266) and Saturn (p. 255).

5. Final remarks

The original systems of Ptolemy and Copernicus are more involved than the simplified versions presented above. For a good description of the original systems (see reference 5). There are many interesting discussions of the work of Ptolemy and Copernicus in references 6 and 7. The remarkable book by Price (see reference 8) discussed attempts to build mechanical calculators to implement the theories of Ptolemy during the Middle Ages.

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More Odd Abundant Sequences

JAY L. SCHIFFMAN and CHRISTOPHER S. SIMONS

An article by the first author (see reference 1) contained a discussion of the arithmetic sequence $945 + 630n$. This sequence appeared to be both curious and fascinating in the sense that odd abundant numbers were obtained for each of the initial 52 whole number inputs. During a colloquium presentation by the first author, the second author discovered that the sequence $3465 + 2310n$ generates odd abundant number outputs for the initial 193 whole number inputs. Energized by these discoveries, we pondered as to whether such sequences were rare at all. These two sequences turn out to be but the first of an infinite family of sequences that produce long initial odd abundant values. The goal of this article is to examine the existence of such arithmetic sequences.

Recall that an *abundant number* is one in which the sum of all its divisors, including itself, exceeds twice the number in question. The smallest odd abundant number is 945. A good introduction to odd abundant numbers is given in reference 1. A computer program in conjunction with MATHEMATICA® (see reference 2) enabled us to produce values for sequences with initial value less than three billion, see table 1. These sequences are obtained from so-called *seed values* a. These seed values are odd deficient numbers that are very close to being perfect. For example, if $\sigma(a)$ denotes the sum of the divisors of a, then

$$\begin{aligned}\sigma(1155) &= \sigma(3 \times 5 \times 7 \times 11)\\ &= \sigma(3) \times \sigma(5) \times \sigma(7) \times \sigma(11)\\ &= (3+1) \times (5+1) \times (7+1) \times (11+1)\\ &= 2304\\ &< 2310\\ &= 2 \times 1155.\end{aligned}$$

We recall that if $\sigma(a)/a > 2$ then n is abundant, while if $\sigma(a)/a = 2$ then a is perfect, and if $\sigma(a)/a < 2$ then a is deficient. Now

$$\frac{\sigma(1155)}{1155} = \frac{2304}{1155} = 1.994\,81,$$

which is just under 2. In order to get abundant numbers from the deficient seed values, we note that $\sigma(ka) > \sigma(a)$, whenever $k > 1$. So we choose our sequence to be $f(n) = (3 + 2n)a = 3a + 2an$, starting with $n = 0$. In general, the closer a is to being abundant the more likely the initial terms will be abundant. The *first failure point* for such a sequence is the smallest positive integer n for which the sequence $f(n) = 3a + 2an$ yields a deficient output. For example, if we start with seed value $a = 1155$, we get the sequence $f(n) = 3465 + 2310n$. Then

$$f(193) = 449\,295, \qquad \frac{\sigma(449\,295)}{449\,295} = 1.999\,93,$$

and so $f(193)$ is deficient, while for all smaller values of n, $f(n)$ is abundant. Since $f(n) = (3 + 2n)a$ and any multiple of an abundant number is itself abundant, the first failure point can

Table 1 Record holders.

Seed value a	Factorization of seed value	First failure point	Arithmetic sequence $3a+2an$
315	$3^2 \times 5 \times 7$	52	$945 + 630n$
1 155	$3 \times 5 \times 7 \times 11$	193	$3\,465 + 2\,310n$
40 365	$3^3 \times 5 \times 13 \times 23$	452	$121\,095 + 80\,730n$
55 335	$3 \times 5 \times 7 \times 17 \times 31$	710	$166\,005 + 110\,670n$
106 425	$3^2 \times 5^2 \times 11 \times 43$	1 613	$319\,275 + 212\,850n$
629 145	$3^2 \times 5 \times 11 \times 31 \times 41$	2 062	$1\,887\,435 + 1\,258\,290n$
702 405	$3^3 \times 5 \times 11^2 \times 43$	2 128	$2\,107\,215 + 1\,404\,810n$
730 125	$3^2 \times 5^3 \times 11 \times 59$	8 113	$2\,190\,375 + 1\,460\,250n$
1 805 475	$3 \times 5^2 \times 7 \times 19 \times 181$	25 795	$5\,416\,425 + 3\,610\,950n$
13 800 465	$3^2 \times 5 \times 7 \times 193 \times 227$	85 190	$41\,401\,395 + 27\,600\,930n$
16 029 405	$3^2 \times 5 \times 7 \times 151 \times 337$	86 185	$48\,088\,215 + 32\,058\,810n$
16 286 445	$3^2 \times 5 \times 7 \times 149 \times 347$	180 962	$48\,859\,335 + 32\,572\,890n$
21 003 885	$3^2 \times 5 \times 7 \times 131 \times 509$	233 387	$63\,011\,655 + 42\,007\,770n$
32 062 485	$3 \times 5 \times 7 \times 13 \times 83 \times 283$	763 402	$96\,187\,455 + 64\,124\,970n$
132 701 205	$3 \times 5 \times 7 \times 13 \times 67 \times 1451$	3 159 554	$398\,103\,615 + 265\,402\,410n$
594 397 485	$3^2 \times 5 \times 11 \times 29 \times 47 \times 881$	6 604 424	$1\,783\,192\,455 + 1\,188\,794\,970n$
815 634 435	$3 \times 5 \times 7 \times 11 \times 547 \times 1291$	135 939 073	$2\,446\,903\,305 + 1\,631\,268\,870n$

only occur when $3+2n$ is prime. Indeed, in our example, $3+2\times 193 = 389$ is prime. Table 1 lists those seed values a along with their sequences $f(n) = 3a+2an$ whose first failure point is greater than that of any such sequence with a smaller seed value.

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Sums of squares and cubes

$$3^2 + 4^2 = 5^2 \quad \text{and} \quad 3^3 + 4^3 + 5^3 = 6^3.$$

Are there any other triples (x, y, z) of natural numbers such that $x^2 + y^2 = z^2$ and $x^3 + y^3 + z^3$ is a perfect cube, apart from multiples of $(3, 4, 5)$?

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Pick's Theorem and Greatest Common Divisors

M. A. NYBLOM

A good example of a result from elementary mathematics which can readily be comprehended by the layman but whose proof is rather challenging is Pick's theorem. This result, which was published in 1899 (see reference 1), concerns the study of simple lattice polygons, that is polygons whose boundaries consist of sequences of connecting nonintersecting straight-line segments such that the coordinates of their vertices are integers. To picture these lattice polygons better, imagine a board on which nails are inserted one unit apart in a square grid pattern. On such a board, commonly referred to as a geoboard, an elastic band can be stretched over various nail heads to produce all manner of polygonal shapes. Pick's simple geometric theorem allows the area of any polygon embedded on a geoboard to be determined in the following manner. Count up the number of points on the boundary of the polygon where the elastic band touches a nail, and divide the resulting number by two. Add to this the number of points in the interior of the polygon, that is inside but not touching the elastic band, and subtract one. This gives the area of the polygonal figure outlined by the elastic band.

Theorem 1 (Pick's theorem) *The area of a simple lattice polygon S is given by either of the following equations:*

$$A(S) = I + \tfrac{1}{2}B - 1 = V - \tfrac{1}{2}B - 1, \tag{1}$$

where I, B and V are the number of interior lattice points of S, the number of boundary lattice points of S and the total number of lattice points of S respectively. (A lattice point is a point with integer-valued coordinates.)

As testament to its enduring interest and appeal, Pick's theorem has been subject over the years to a number of extensions, which include lattices other than the usual square lattice (see references 2 and 3), more general polygons (see reference 4) and higher-degree polyhedra (see references 5 and 6). In this article, we return to Pick's original theorem and show how (1) can be used to derive an explicit formula for the greatest common divisor of two positive integers. In particular, we present the following result.

Theorem 2 *For any $a, b \in \mathbb{N}$, the greatest common divisor of a and b, denoted by (a, b), is given by*

$$(a, b) = 2\sum_{i=1}^{b-1}\left\lfloor \frac{a}{b}i \right\rfloor + a + b - ab, \tag{2}$$

where $\lfloor\cdot\rfloor$ is the integer-part function.

Before proving Theorem 2, it will be instructive for the reader to consider the following example of calculating $(57, 3)$ using (2).

Example 1 Take $a = 57$ and $b = 6$. Then the right-hand side of (2) is equal to

$$\begin{aligned} 2&\left(\left\lfloor \frac{57 \times 1}{6} \right\rfloor + \left\lfloor \frac{57 \times 2}{6} \right\rfloor + \left\lfloor \frac{57 \times 3}{6} \right\rfloor + \left\lfloor \frac{57 \times 4}{6} \right\rfloor + \left\lfloor \frac{57 \times 5}{6} \right\rfloor\right) + 57 + 6 - 57 \times 6 \\ &= 2(9 + 19 + 28 + 38 + 47) + 63 - 342 \\ &= 3, \end{aligned}$$

which is the correct value of $(57, 6)$.

This example illustrates that (2) is quite effective in calculating the greatest common divisor for small integers, but this is not generally the case for larger integers due to the increased number of terms involved in the summation. Thus, (2) cannot be viewed as a viable alternative to the use of, say, Euclid's algorithm in determining greatest common divisors. We now establish Theorem 2 via Pick's theorem.

Proof of Theorem 2 Consider the simple lattice polygon S consisting of three vertices at the lattice points $(0, 0)$, $(b, 0)$ and (b, a). Furthermore, consider the set of all lattice points of S inside and on the boundary of S. To extract the required expression in (2), we shall need to evaluate both sides of the second formula in (1), namely $A(S) = V - \frac{1}{2}B - 1$. Clearly, the area $A(S)$ of S is equal to $\frac{1}{2}ab$. To determine V, suppose that n is an integer with $0 \le n \le b$. Then the number of lattice points (n, m) such that $0 < m \le (a/b)n$ is equal to $\lfloor (a/b)n \rfloor$. Thus, by counting the remaining lattice point $(n, 0)$ for each n, we see that

$$V = \sum_{i=0}^{b} \left\{ \left\lfloor \frac{a}{b} i \right\rfloor + 1 \right\} = \sum_{i=1}^{b-1} \left\lfloor \frac{a}{b} i \right\rfloor + (a + b + 1). \tag{3}$$

Turning to the evaluation of B, we observe that $B = |B_1| + |B_2| + |B_3|$, where

$$\begin{aligned} B_1 &= \{(x, 0) \in \mathbb{Z}^2 : \ 0 < x \le b\}, \\ B_2 &= \{(b, y) \in \mathbb{Z}^2 : \ 0 < y < a\}, \\ B_3 &= \left\{(x, y) \in \mathbb{Z}^2 : \ 0 \le x \le b, \ y = \frac{a}{b} x\right\}. \end{aligned}$$

Clearly, $|B_1| = b$ and $|B_2| = a - 1$, while $|B_3|$ is equal to the number of integer solutions (x, y) of the linear equation $by - ax = 0$ with $0 \le x \le b$. If $d = (a, b)$, the highest common factor of a and b, then we claim that all integer solutions (x, y) of $by - ax = 0$ are of the form $((b/d)t, (a/d)t)$, where $t \in \mathbb{Z}$. To establish this, consider the following argument. If we suppose that (x', y') is an arbitrary integer solution of $by - ax = 0$, then $(b/d)y' = (a/d)x'$ where b/d and a/d are relatively prime integers, as $d = (a, b)$. Hence a/d divides $(b/d)y'$ with a/d and b/d relatively prime, so that a/d must divide y', say $y' = (a/d)t$ for some $t \in \mathbb{Z}$, whence also $x' = (b/d)t$, as required.

Consequently, $(x, y) = ((b/d)t, (a/d)t)$, for $t = 0, 1, \ldots, d$, are the only integer solutions of $by - ax = 0$ with $0 \le x \le b$. Thus, $((b/d)t, (a/d)t)$, for $t = 0, 1, \ldots, d$, are the only lattice points in B_3, and so $|B_3| = d + 1$. Hence

$$B = a + b + d. \tag{4}$$

Finally, substituting (3) and (4) into (1), we find that

$$\frac{1}{2}ab = \sum_{i=1}^{b-1}\left\lfloor \frac{a}{b}i \right\rfloor + (a+b+1) - \frac{1}{2}(a+b+d) - 1,$$

which upon solving for d yields (2), as required.

To conclude, we show how (2) can be used to deduce a number of identities involving the integer-part function. The first of these is a well-known identity which appeared as a problem in reference 7 (p. 186). The reader may see a marked similarity between the identity

$$\sum_{i=1}^{(p-1)/2}\left\lfloor \frac{q}{p}i \right\rfloor + \sum_{i=1}^{(q-1)/2}\left\lfloor \frac{p}{q}i \right\rfloor = \frac{(p-1)}{2}\frac{(q-1)}{2},$$

which is used in proving the Law of Quadratic Reciprocity for odd primes p and q, and the identity (5).

Corollary 1 *If a and b are two relatively prime integers, then*

$$\sum_{i=1}^{b-1}\left\lfloor \frac{a}{b}i \right\rfloor = \frac{(a-1)(b-1)}{2}. \tag{5}$$

Proof After setting $(a,b)=1$ in (2), observe that $1-(a+b)+ab=(a-1)(b-1)$.

If we interchange the roles of the integers a and b in (2), we can deduce the following curious identity from the equality $(a,b)=(b,a)$.

Corollary 2 *If a and b are positive integers, then*

$$\sum_{i=1}^{b-1}\left\lfloor \frac{a}{b}i \right\rfloor = \sum_{i=1}^{a-1}\left\lfloor \frac{b}{a}i \right\rfloor.$$

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Equicentric Patterns

P. GLAISTER

Some time ago, my attention was drawn to a tiled pattern which looked something like the one shown in figure 1. It wasn't until much later that I recalled the individual design, which is shown in figure 2. Apart from the optical illusion of the exterior straight lines appearing curved, I was intrigued by the nature of this pattern and investigated further.

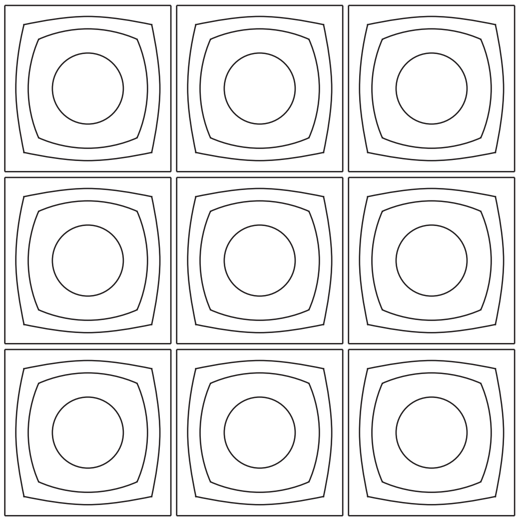

Figure 1

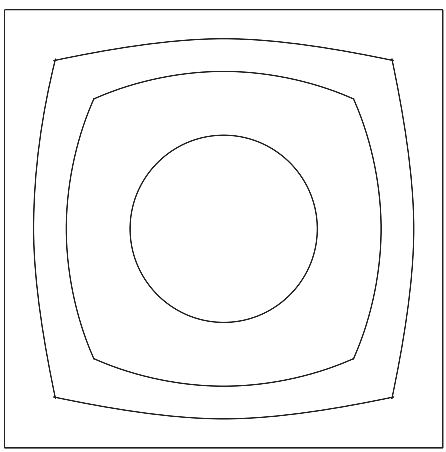

Figure 2

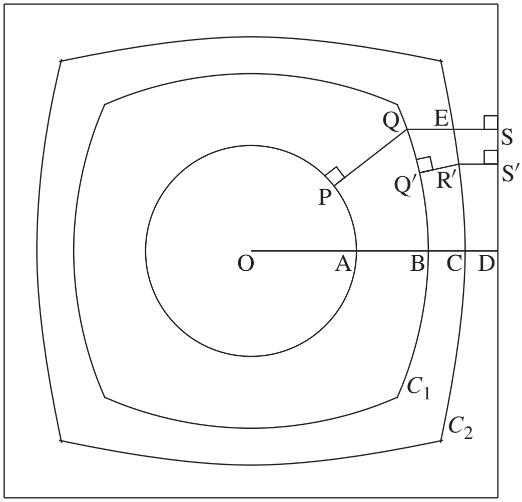

Figure 3

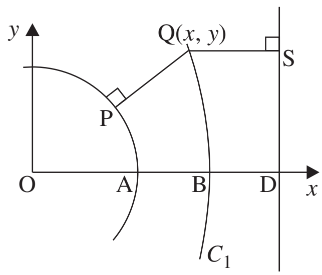

Figure 4

We see from figure 3 that the basic design comprises a circle inside a square and the point B on the curve C_1 is half way between points A and D. Moreover, as we move along C_1 each point is equidistant from the circle and the square, i.e. the shortest distance from a typical point Q to the circle is QP, which is equal to the shortest distance QS to the square, in each case meeting at right angles. So, in a sense, when the remaining parts of C_1 are drawn in, this curve, together with the square and circle, are equally spaced and similar to concentric circles, and so we refer to these as *equicentric*.

The second set of curves, C_2, are equidistant from C_1 and the square, i.e. C_1, C_2 and the square are also equicentric. This means that C is half-way between B and D. Similarly, $\mathrm{Q'R'} = \mathrm{R'S'}$ and the two lines Q′R′ and R′S′ meet C_1 and the square at right angles. To determine the mathematical nature of these two sets of equicentric curves, we assume for simplicity that the square has side 14 and the radius of the circle is 3, so that $\mathrm{OD} = 7$, $\mathrm{AD} = 4$ and, hence, $\mathrm{AB} = 2$ and $\mathrm{BC} = \mathrm{CD} = 1$.

From figure 4, we can see that a typical point Q on C_1 satisfies the equidistant relation

$$\mathrm{PQ} = \mathrm{QS}, \tag{1}$$

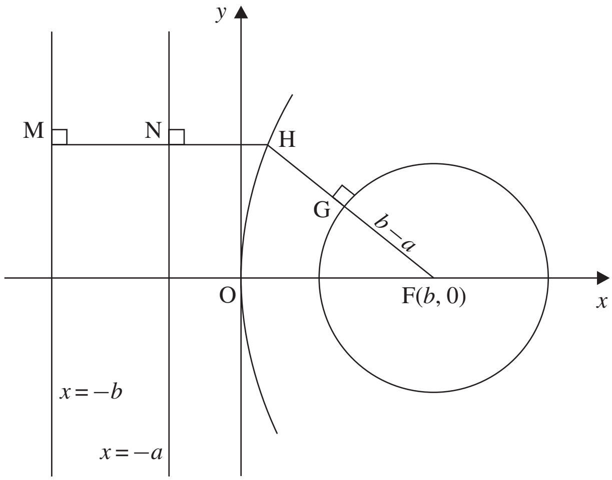

Figure 5

so, with x and y axes shown with origin O, (1) can be written as

$$7 - x = \sqrt{x^2 + y^2} - 3,$$

which can be simplified to

$$y^2 = 100 - 20x, \tag{2}$$

i.e. C_1 is part of a parabola. (The right-hand part of C_1 in figure 3 is given by (2) for $x \in [10(\sqrt{2} - 1), 5]$.)

To see why this is so, we need only consider a simpler example of the locus of points equidistant from a line $x = -a$ and a circle centre $\mathrm{F}(b, 0)$, radius $b - a$, where $b \geq a > 0$, as shown in figure 5. Here, the equidistant property GH = HN can be rewritten as FH − FG = HM − NM, i.e. FH − $(b - a)$ = HM − $(b - a)$ and, thus, FH = HM (the standard focus-directrix property of a parabola as the locus of points equidistant from the focus $\mathrm{F}(b, 0)$ and directrix $x = -b$, the cartesian equation for which is $y^2 = 4bx$).

Hence, the parabola can be thought of as points equidistant from a circle and a straight line, which is the case for C_1, because the line joining F to H through G in figure 5 (and O to Q through P in figures 3 and 4) necessarily meets the circle at right angles, and the distance from the centre of a circle to its circumference is obviously fixed.

Now, we consider the curve C_2, the locus of all points equidistant from the square and the parabola C_1. This is a less straightforward calculation than the previous one. In this case, with $\mathrm{R}'(x, y)$ a typical point on C_2 as shown in figure 3, we have $\mathrm{Q}'\mathrm{R}' = \mathrm{R}'\mathrm{S}'$, i.e.

$$\sqrt{\left(x - \left(5 - \tfrac{1}{20}t^2\right)\right)^2 + (y - t)^2} = 7 - x, \tag{3}$$

where $\mathrm{Q}'(5 - \frac{1}{20}t^2, t)$, for some t, denotes the point on C_1 (with cartesian equation $y^2 = 100 - 20x$ from (2)) which is closest to R'. Furthermore, the slope of the perpendicular to C_1 is given by

$$-\frac{1}{\mathrm{d}y/\mathrm{d}x} = -\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{\mathrm{d}}{\mathrm{d}y}\left(5 - \frac{1}{20}y^2\right) = \frac{1}{10}y,$$

which has a value of $\frac{1}{10}t$ at Q′. Thus, for the line Q′R′,

$$\frac{y-t}{x-(5-\frac{1}{20}t^2)} = \frac{t}{10}. \tag{4}$$

Equations (3) and (4) represent parametric equations of the locus C_2.

Simplifying by rearranging (4),

$$y-t = \frac{t}{10}\big(x-\big(5-\tfrac{1}{20}t^2\big)\big), \tag{5}$$

and substituting into (3), we obtain

$$\sqrt{\big(1+\tfrac{1}{100}t^2\big)\big(x-\big(5-\tfrac{1}{20}t^2\big)\big)^2} = 7-x,$$

i.e.

$$\pm\big(x-\big(5-\tfrac{1}{20}t^2\big)\big)\sqrt{1+\tfrac{1}{100}t^2} = 7-x. \tag{6}$$

Rearranging (6) to isolate x, and then substituting this into (5), we obtain

$$x = \frac{7 \pm \big(5-\frac{1}{20}t^2\big)\sqrt{1+\frac{1}{100}t^2}}{1 \pm \sqrt{1+\frac{1}{100}t^2}}, \tag{7}$$

$$y = \frac{\frac{6}{5}t + \frac{1}{200}t^3 \pm t\sqrt{1+\frac{1}{100}t^2}}{1 \pm \sqrt{1+\frac{1}{100}t^2}}, \tag{8}$$

with $t \in [0, 4.6849]$, and the positive sign corresponds to that part of C_2 between B and E as shown in figure 3. The remaining parts of the curve can be determined by successive reflections in the lines $y=0$, $x=0$ and $y=x$, i.e. replacing y by $-y$, x by $-x$ and interchanging x and y. (The limits on the parameter t can be found by noting that the point E is where $y=x$, and then equating the expressions in (7) and (8), and solving for the appropriate positive value of t.)

Variations on this theme include curves equicentric to circles. For example, the three circles shown in figure 6 are clearly concentric, with the circle C equidistant from the exterior and

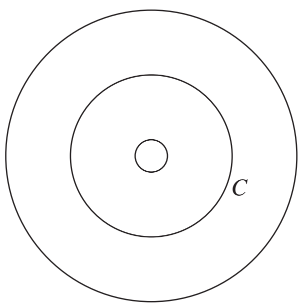

Figure 6

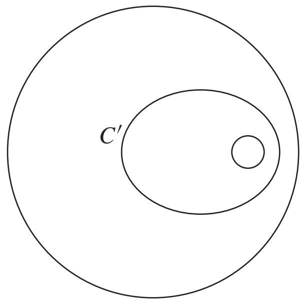

Figure 7

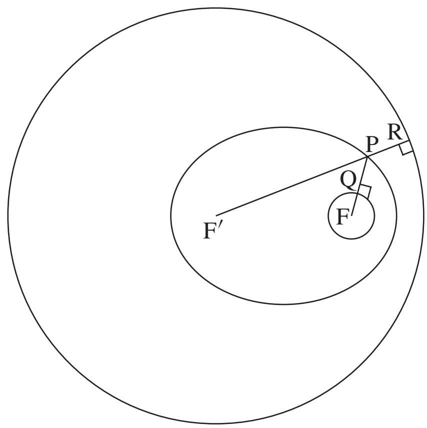

Figure 8

interior circles. If we displace the interior circle, however, then the equicentric curve C' is shown in figure 7 and looks elliptical.

In fact, from figure 8 with F and F$'$ denoting the centres and a and b denoting the radii of the smaller and larger circles respectively, and using the fact that FQ is perpendicular to the smaller circle at Q and F$'$R (passing through P) is perpendicular to the larger circle at R, then the equidistant property $\mathrm{QP} = \mathrm{PR}$ can be written as $\mathrm{FP} - \mathrm{FQ} = \mathrm{F'R} - \mathrm{F'P}$. Now, since $\mathrm{FQ} = a$ and $\mathrm{F'R} = b$, both fixed, then this relation can be written as $\mathrm{FP} - a = b - \mathrm{F'P}$, i.e. $\mathrm{FP} + \mathrm{F'P} = a + b$, a constant, which is a well-known property of an ellipse with foci at F and F$'$ and semi-major axis $\frac{1}{2}(a + b)$. Thus the equicentric curve to the 'off-centre' circles is indeed an ellipse.

One further variation on the original problem is to have a rectangular tile and to determine the nature of the equicentric curves when the interior circular shape is replaced by an ellipse.

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Partitioning Fibonacci Sets

PHILIP MAYNARD

Among other problems, the following two problems were posed and solved in reference 1. Firstly, for which natural numbers n is it possible to find disjoint nonempty sets A and B such that $\{1, 2, \ldots, n\} = A \cup B$ and the sum of the elements of A is equal to that of B? It is not too hard to show that the answer is precisely those n for which $4 \mid n$ or $4 \mid (n+1)$. Secondly, it was shown that, for all $n \geq 4$, it is possible to delete an element of $\{1, 2, \ldots, n\}$ so that the resulting set can be partitioned into two disjoint nonempty subsets, so that the sum of the elements in both subsets are equal. We are interested in proving similar conclusions for a set related to the Fibonacci numbers. A *Fibonacci-type* sequence is any sequence $F := [f_1, f_2, \ldots, f_k, \ldots]$, where f_1 and f_2 are natural numbers and $f_k = f_{k-1} + f_{k-2}$ for all $k \geq 3$. For each integer n we define

$$F_n := [f_1, f_2, f_3, f_4, \ldots, f_n].$$

We begin with the following simple lemma.

Lemma 1 *For any natural number n, we have*

$$\sum_{i=1}^{n} f_i = f_{n+2} - f_2.$$

Proof The proof is by induction; the case when $n = 1$ is trivially true. Thus, assume that $\sum_{i=1}^{t-1} f_i = f_{t+1} - f_2$. Then

$$\begin{aligned}\sum_{i=1}^{t} f_i &= \sum_{i=1}^{t-1} f_i + f_t \\ &= f_{t+1} - f_2 + f_t \\ &= f_{t+2} - f_2.\end{aligned}$$

This completes the induction and thus the proof.

For any finite sequence M we shall say that M is *separable* if M can be partitioned into two subsequences N_1 and N_2 such that the sum of the members of N_1 is equal to the sum of the members of N_2. Also, for any finite sequence M, we shall say that M is *near-separable* if there exists an element $m \in M$ such that $M \setminus m$ is separable. Here $M \setminus m$ is the sequence obtained from M by deleting the element m from M.

Assume that $F = [f_1, f_2, \ldots]$ is a Fibonacci-type sequence with f_1 and f_2 even. Let 2^x be the highest integer power of 2 that divides both f_1 and f_2. Then $g_1 = f_1 2^{-x}$ and $g_2 = f_2 2^{-x}$ are both integers, at least one of which is odd. Also, if $g_k = g_{k-1} + g_{k-2}$ for all $k \geq 3$ then

$$F = [f_1, f_2, \ldots] = 2^x [g_1, g_2, \ldots]$$

and, certainly, F_n is separable if and only if $[g_1, g_2, \ldots, g_n]$ is separable. So in this way we can restrict our analysis to Fibonacci-type sequences whose first terms are not both even.

Theorem 1 *Let $F = [f_1, f_2, \ldots]$ be a Fibonacci-type sequence. Let $n \geq 2$. Then the following conditions hold.*

(i) *If f_1 is odd and f_2 is even then F_n is separable if and only if $n \equiv 0 \mod 3$.*

(ii) *If f_1, f_2 are odd and $f_1 = f_2$ then F_n is separable if and only if $n \equiv 0, 2 \mod 3$.*

(iii) *If f_1, f_2 are odd and $f_1 \neq f_2$ then F_n is separable if and only if $n \equiv 0 \mod 3$.*

(iv) *If f_1 is even, f_2 is odd and $f_1 = 2f_2$ then F_n is separable if and only if $n \equiv 0, 1 \mod 3$.*

(v) *If f_1 is even, f_2 is odd and $f_1 \neq 2f_2$ then F_n is separable if and only if $n \equiv 0 \mod 3$.*

To prove this theorem, we first need the following fact.

Fact 1 For any f_1, f_2, if $n \equiv 0 \mod 3$ then F_n is separable. Now, using $f_1 + f_2 = f_3$, $f_4 + f_5 = f_6$ etc., with $n = 3k$, we obtain

$$(f_1 + f_2) + (f_4 + f_5) + \cdots + (f_{3k-2} + f_{3k-1}) = f_3 + f_6 + \cdots + f_{3k}.$$

Proof of Theorem 1 (i) If F_n is separable then $\sum_{i=1}^{n} f_i$ is even, so by Lemma 1 we have $f_{n+2} - f_2$ is even. Since f_2 is even, f_{n+2} must be even. For these f_1, f_2, we have that f_k is even if and only if $k \equiv 2 \mod 3$. Hence, if F_n is separable then $n \equiv 0 \mod 3$. Conversely, by Fact 1, if $n \equiv 0 \mod 3$ then F_n is separable.

(ii) If F_n is separable then $f_{n+2} - f_2$ is even, i.e. f_{n+2} is odd. However, f_k is odd if and only if $k \equiv 1, 2 \mod 3$. Hence, if F_n is separable then $n \equiv 0, 2 \mod 3$. Conversely, if $n \equiv 0 \mod 3$ then F_n is separable by Fact 1. Assume now that $n \equiv 2 \mod 3$, say $n = 3k+2$ for some integer k. Since $f_1 = f_2$ and $f_3 + f_4 = f_5, \ldots, f_{3k} + f_{3k+1} = f_{3k+2}$ we conclude that F_n is separable.

(iii) As in (ii), if F_n is separable then $n \equiv 0, 2 \mod 3$. We shall show that $n \equiv 2 \mod 3$ leads to a contradiction. Thus let $n \equiv 2 \mod 3$. Let $t \geq 5$ be the smallest integer such that $t \equiv 2 \mod 3$, and F_t is separable into subsequences A and B. Now if, say, $f_t \in A$ and $f_{t-1}, f_{t-2} \in B$, this would imply that F_{t-3} is separable, which contradicts the minimalism of t. Hence, $f_t, f_{t-1} \in A$ or $f_t, f_{t-2} \in A$. However, for $t \geq 5$, we have

$$\begin{aligned}
f_t + f_{t-2} - (f_{t-1} + f_{t-3} + f_{t-4} + \cdots + f_2 + f_1) &= f_t + f_{t-2} - f_{t-1} - (f_{t-1} - f_2) \\
&> f_t - 2f_{t-1} + f_{t-2} \\
&= (f_t - f_{t-1}) - (f_{t-1} - f_{t-2}) \\
&= f_{t-2} - f_{t-3} \\
&> 0.
\end{aligned}$$

So $f_t, f_{t-2} \in A$ is impossible. Since $f_t + f_{t-1} > f_t + f_{t-2}$ it is also impossible for $f_t, f_{t-1} \in A$. Conversely, if $n \equiv 0 \mod 3$ then F_n is separable by Fact 1.

(iv) If F_n is separable then f_{n+2} is odd, which implies that $n \equiv 0, 1 \mod 3$. Conversely, if $n \equiv 0 \mod 3$ then F_n is separable by Fact 1. Assume now that $n \equiv 1 \mod 3$, say $n = 3k + 1$. Since F_4 is separable ($f_2 + f_4 = 2f_2 + f_3 = f_1 + f_3$) and $f_5 + f_6 = f_7, \ldots, f_{3k-1} + f_{3k} = f_{3k+1}$, we conclude that F_n is separable.

(v) As in (iv), if F_n is separable then $n \equiv 0, 1 \mod 3$. We shall show that $n \equiv 1 \mod 3$ leads to a contradiction. Thus let $n \equiv 1 \mod 3$. Let t be the smallest integer such that $t \equiv 1 \mod 3$, and F_t is separable into subsequences A and B. Now if, say, $f_t \in A$ and $f_{t-1}, f_{t-2} \in B$, this would imply that F_{t-3} is separable, which contradicts the minimalism of t. Hence, $f_t, f_{t-1} \in A$ or $f_t, f_{t-2} \in A$. However, as in (iii), this is impossible unless $t < 5$, i.e. $t = 4$. We can check that $t = 4$ implies that $f_1 = 2f_2$, i.e. a contradiction. Conversely, if $n \equiv 0 \mod 3$ then F_n is separable by Fact 1.

Theorem 2 *Let $F = [f_1, f_2, \ldots]$ be a Fibonacci-type sequence. Assume that $n \geq 3$. Then the following conditions hold.*

(i) *If f_1 is odd and f_2 is even then F_n is near-separable if and only if $n \equiv 1, 2 \mod 3$.*

(ii) *If f_1, f_2 are odd and $f_1 = f_2$ then F_n is near-separable for all n.*

(iii) *If f_1, f_2 are odd and $f_1 \neq f_2$ then F_n is near-separable if and only if $n \equiv 1, 2 \mod 3$.*

(iv) *If f_1 is even, f_2 is odd and $f_1 = 2f_2$ then F_n is near-separable for all $n \neq 3$.*

(v) *If f_1 is even, f_2 is odd and $f_1 \neq 2f_2$ then F_n is near-separable if and only if $n \equiv 1, 2 \mod 3$.*

To begin the proof of this theorem, we need the following fact.

Fact 2 For any f_1, f_2, if $n \equiv 1, 2 \mod 3$ then F_n is near-separable. Firstly, if $n \equiv 1 \mod 3$ then removing f_n from F_n leaves F_{n-1}, which is separable by Fact 1. Next, assume that $n \equiv 2 \mod 3$, say $n = 3k + 2$. We show that F_5 is near-separable. We obtain $F_5 = [f_1, f_2, f_1 + f_2, f_1 + 2f_2, 2f_1 + 3f_2]$. Now, removing $f_1 + f_2$ from F_5 leaves a separable sequence, since $f_1 + f_2 + (f_1 + 2f_2) = (2f_1 + 3f_2)$. Since also $f_6 + f_7 = f_8, \ldots, f_{3k} + f_{3k+1} = f_{3k+2}$, we conclude that F_n is near-separable.

Proof of Theorem 2 (i) All we need to show is that F_n is not near-separable if $n \equiv 0 \mod 3$. It is easy to see that F_3 is not near-separable. Assume that F_n is near-separable for some $n \equiv 0 \mod 3$. Let $t \geq 6$ be the smallest integer such that $t \equiv 0 \mod 3$ and $F_t \setminus x$ is separable into subsequences A and B for some $x \in F_t$. Now $x \neq f_t$, since otherwise we would have that F_{t-1} is separable, which contradicts Theorem 1(i). We show that $x = f_{t-1}$ or $x = f_{t-2}$, since, if not, as in the argument in Theorem 1(iii), we have $f_{t-1}, f_{t-2} \in A$ and $f_t \in B$, and then F_{t-3} is near-separable, which contradicts the minimalism of t. It is not difficult to show that $\sum_{i=1}^{t} f_i$ is even, f_{t-1} is even and f_{t-2} is odd, (since $t \equiv 0 \mod 3$, f_1 is odd and f_2 is even). Hence, $x = f_{t-1}$. But then $\sum_{i=1}^{t-2} f_i = f_t - f_2 < f_t$, i.e. a contradiction.

(ii) Let $n \equiv 0 \mod 3$, say $n = 3k$. Then F_n is near-separable since $F_n \setminus f_3$ is separable. This is since $f_1 = f_2, f_4 + f_5 = f_6, \ldots, f_{3k-2} + f_{3k-1} = f_{3k}$.

(iii) Assume that F_n is near-separable and $n \equiv 0 \mod 3$. It is easy to see that F_3 is not near-separable. Let $t \geq 6$ be the smallest integer such that $t \equiv 0 \mod 3$ and $F_t \setminus x$ is separable into subsequences A and B for some $x \in F_t$. Now, as in (i), by the minimalism of t we have $x = f_{t-1}$ or $x = f_{t-2}$. However, for $t \equiv 0 \mod 3$, it follows that $\sum_{i=1}^{t} f_i$ is even and both f_{t-1} and f_{t-2} are odd, i.e. a contradiction.

(iv) Let $n \equiv 0 \mod 3$, say $n = 3k$ with $k \geq 2$. We show that F_n is near-separable by showing that $F_n \setminus f_4$ is separable. We obtain

$$F_6 = [f_1, f_2, f_1 + f_2, f_1 + 2f_2, 2f_1 + 3f_2, 3f_1 + 5f_2].$$

Now, removing $(f_1 + 2f_2)$ leaves a separable sequence, since $f_1 + (f_1 + f_2) + (2f_1 + 3f_2) = f_2 + (3f_1 + 5f_2)$. Since also $f_7 + f_8 = f_9, \dots, f_{3k-2} + f_{3k-1} = f_{3k}$, we are done. (It can also be checked that F_3 is not near-separable.)

(v) Assume that F_n is near-separable and $n \equiv 0 \mod 3$. Let $t \geq 6$ be the smallest integer such that $t \equiv 0 \mod 3$ and $F_t \setminus x$ is separable into subsequences A and B for some $x \in F_t$. Now by the minimalism of t we have $x = f_{t-1}$ or $x = f_{t-2}$. It is not difficult to show that $\sum_{i=1}^{t} f_i$ is even, f_{t-1} is odd and f_{t-2} is even, so $x = f_{t-2}$. As previously, we cannot have $f_t, f_{t-1} \in A$, say, so take $f_t \in A$ and $f_{t-1} \in B$. We now show that $f_{t-3}, f_{t-4} \in B$, if $t \geq 7$. Now, if f_{t-3} or $f_{t-4} \in A$ and $t \geq 7$ then

$$\begin{aligned}
\sum_{f \in A} f - \sum_{g \in B} g &\geq f_t - f_{t-1} + f_{t-4} - f_{t-3} - \sum_{i=1}^{t-5} f_i \\
&= f_{t-2} - f_{t-5} - f_{t-3} + f_2 \\
&> f_{t-2} - f_{t-4} - f_{t-3} + f_2 \\
&= f_2 \\
&> 0.
\end{aligned}$$

Note that $f_{t-4} > f_{t-5}$ since $t \geq 7$. But now, since $f_t = f_{t-1} + f_{t-3} + f_{t-4}$, it follows that F_{t-5} is separable, which contradicts Theorem 1(v). It can be checked by hand that F_3 and F_6 are not near-separable.

In general it is not possible to partition F_n into more than two subsequences the sums of whose respective elements are equal. This is since we claim that in F_{k+3} the element f_{k+3} is more than a third of the sum of all the elements. This claim follows on by establishing that $2f_{k+3} > \sum_{i=1}^{k+2} f_i$, for any $k \geq 1$, and this is since

$$\sum_{i=1}^{k+2} f_i = f_{k+4} - f_2 = f_{k+2} + f_{k+3} - f_2 < 2f_{k+3} - f_2.$$

Finally we note that $F_n = [1, 1, 2, 3, 5, 8, \dots, f_n]$ *acts* like the set $\{1, 2, \dots, n\}$, in so far as both are near-separable for any n, and if $2 \mid \sum_{i=1}^{n} f_i$ then F_n is separable. This is comparable to $\{1, 2, \dots, n\}$ (if $2 \mid n(n+1)/2$ then $\{1, 2, \dots, n\}$ is separable).

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Philip Maynard *has interests in combinatorics and permutation group theory. However, he also has a keen interest in recreational mathematics. He is currently a research associate in the School of Mathematics at the University of East Anglia, Norwich.*

Progression of Scores and Averages in Test Cricket

A. TAN, R. RAMACHANDRAN and W. SHENG

1. Introduction

The game of cricket is unsurpassed in terms of the magnitudes of scores generated. For example, the innings scores in test cricket ranged from a high of 952 (for 6 *wickets*) to a low of only 26. In traditional cricket, each team (consisting of 11 players) bat for up to two innings. The outcome of the game is decided by the aggregate runs scored from two completed innings. Each innings is completed after the fall of 10 wickets (*all out*). Occasionally, one or more players are not able to bat through injury (*retired hurt* or *absent hurt*) or illness (*retired ill*). In such a case, the innings is still recorded as all out. An innings can also be terminated through a *declaration* at any time, in which case the number of runs scored and wickets fallen are both specified (see reference 1). Among many facets of the game, the main struggle in cricket is between the *batsmen* and the *bowlers* (cf. reference 2). Whereas batsmen try to score as many runs as they can, the bowlers try to dismiss the batsmen (i.e. capture their *wickets*) by conceding as few runs as possible.

A *run* is scored when the batsmen successfully cross over the length of the pitch after the ball has been struck by the bat. A boundary (4 runs) is awarded if the struck ball reaches the boundary or perimeter of the field. An over-boundary (6 runs) is awarded if the struck ball crosses the boundary airborne, i.e. without touching the ground. Runs scored this way are credited to the batsman and charged to the bowler. There are several ways in which a batsman can be dismissed: *bowled* (when the ball hits the wicket and dislodges the bails after the batsman has been beaten), *caught* (when a fielder, the wicketkeeper or the bowler catches the ball as it leaves the bat airborne), *leg before wicket* (when the batsman prevents the ball from hitting the wicket with his leg pads), *stumped* (when the batsman is outside the batting crease and the wicketkeeper removes the bails), or *hit wicket* (when the batsman dislodges the bails with his bat or body). The bowler is credited with the wicket in these modes of dismissals.

Besides the runs scored by the batsmen, runs are also added through *extras*, which comprise *byes* (when the batsman misses the ball and the wicketkeeper fails to gather it), *leg-byes* (when the batsman deflects the ball with a part of his body, usually the leg pads), *no-balls* (when a bowler fails to bowl a legal delivery, usually by overstepping the bowling crease), and *wides* (when the ball is delivered beyond the reach of the batsman), (see reference 1). The extras are neither credited to the batsman nor charged to the bowler. Thus, the total runs scored by all the batsmen is exactly equal to the total runs conceded by all the bowlers. However, the total number of dismissals of all batsmen is not exactly equal to the total number of wickets captured by all bowlers. Batsmen may be dismissed by *run outs* (when they fail to complete the runs) and other rare forms of dismissals (such as *obstructing the field, hitting the ball twice*, etc.) where the bowler is not credited with the dismissals (see reference 1).

The *batting average* (α) of a batsman is the total number of runs scored divided by the number of times dismissed. The *bowling average* (β) of a bowler is the total number of runs conceded divided by the number of wickets captured by the bowler. This article examines the yearly progression of scores and averages in test cricket since the first test was played between England and Australia in 1876 up to and including the year 2000. The scores and averages are calculated from all test matches played everywhere in a year. Each year contains two seasons: the summer season (in England) and the winter season (elsewhere). The test scoreboards are readily available from reference 3. Upon completion of the calculations, trend studies of the scores and averages are made, which shed considerable light on the progression and evolution of the great game of cricket.

2. Average innings scores

Firstly, we study the progression of innings scores in test cricket. For each year, the *total number of runs* scored in all test matches (R) and the *total number of wickets fallen* (W) are summed. (Note that the number of wickets in a completed innings (all out) is taken as 10 even when the innings contained batsmen who were retired hurt, absent hurt or absent ill.) The ratio R/W is the average runs scored per wicket for the year. This quantity, when multiplied by 10, gives the *Average Completed Innings* (*ACI*) for that year as follows:

$$\text{ACI} = \frac{R}{W} \times 10.$$

Figure 1 shows the ACI of each year from 1876 up to 2000. Amidst considerable scatter of the data points, two distinct trends emerge from the figure. In the pre-World War I (WWI) era,

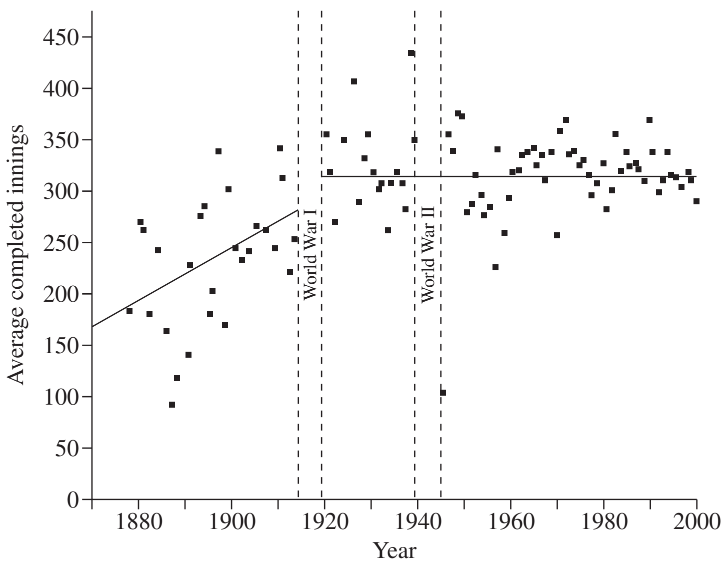

Figure 1 Progression of average completed innings (10 wickets) in test cricket.

the trend is one of steady increase in the ACI, whereas in the post-WWI era, the ACI remained nearly horizontal. The straight lines in these two periods are the least-squares fit straight lines (LSLs) through the data points of the respective periods given by the general equation of the straight line

$$y = mx + p,$$

where x is the year, m the slope and p the y-intercept. The constants m and p are calculated from the normal equations

$$m \sum x + pn = \sum y$$

and

$$m \sum x^2 + p \sum x = \sum xy,$$

where n is the number of data points and the summations are from 1 to n.

During the first period, the ACI increased from 181 in 1876 to 277 in 1913 along the LSL, which gives an increase of 53% in 37 years. The reason for this dramatic increase is clear. In the early days of test cricket, pitch preparation was poor and batting scores were low. As the pitch preparation and batting techniques gradually improved, so did batting scores. The bat became so dominant over the ball that the duration of test matches had to be lengthened in order to obtain a result, first from 3 to 4 days and then to the standard 5 days (or 30 hours). (Timeless tests were played in Australia during the early days, but they were normally concluded in 3 or 4 days.)

After WWI, the struggle between the bat and the ball reached a stalemate and the ACI remained remarkably constant. As the number of tests increased, the scatter of the data points diminished, but the LSL remained horizontal with ACI values at both ends equal to 318. During this period, test cricket witnessed important changes. Newer teams (West Indies, New Zealand, India, Pakistan, Sri Lanka and Zimbabwe) attained test status. The front foot rule was changed to curb the growing menace from the fast bowlers. The introduction of the helmet also gave the batsmen a protective shield against bouncers. In addition, the covering of pitches brought an end to sticky wickets that had given the bowlers a great advantage. But all these changes did little to alter the ACI. The introduction of weaker teams produced smaller scores by these teams, which were counterbalanced by the bigger scores produced against them. The advantages gained by the batsmen from the protective helmet were offset by the video replays from which the bowlers could spot additional weaknesses of the batsmen.

3. Average percentage of extras

In this section, the *total number of extras* from all tests (E) are summed for each year. That number is divided by R and multiplied by 100 to yield the *Average Percentage of Extras* (*APEs*) for the year, i.e.

$$\mathrm{APE} = \frac{E}{R} \times 100.$$

The APEs are plotted for each year in figure 2. Once again, the data shows interesting trends in the two periods separated by WWI. In the pre-WWI period, there was considerable scatter of

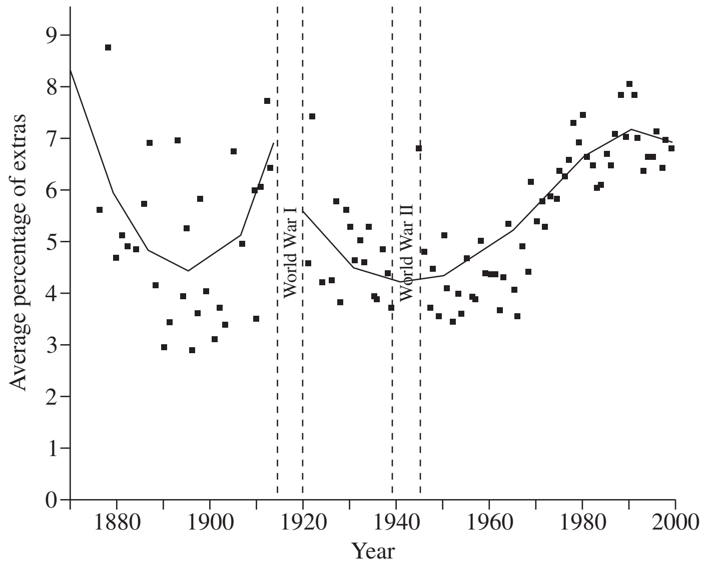

Figure 2 Progression of average percentage of extras in test cricket.

the data because of the smaller number of matches played. Yet, a parabolic trend is discernible from the data, which could be fitted by a quadratic function of x as follows:

$$y = ax^2 + bx + c.$$

The constants a, b and c are determined by the following normal equations:

$$\begin{aligned} a\sum x^2 + b\sum x + cn &= \sum y, \\ a\sum x^3 + b\sum x^2 + c\sum x &= \sum xy, \\ a\sum x^4 + b\sum x^3 + c\sum x^2 &= \sum x^2 y. \end{aligned}$$

Solving these equations simultaneously, we obtain $a = 0.006\,95$, $b = -26.36$ and $c = 24\,977$.

In the post-WWI period, a definite cubic trend in the APE data is evident. This trend can be fitted by a cubic polynomial given by

$$y = dx^3 + ex^2 + fx + g.$$

The constants d, e, f and g are determined by the following normal equations:

$$\begin{aligned} d\sum x^3 + e\sum x^2 + f\sum x + gn &= \sum y, \\ d\sum x^4 + e\sum x^3 + f\sum x^2 + g\sum x &= \sum xy, \\ d\sum x^5 + e\sum x^4 + f\sum x^3 + g\sum x^2 &= \sum x^2 y, \\ d\sum x^6 + e\sum x^5 + f\sum x^4 + g\sum x^3 &= \sum x^3 y. \end{aligned}$$

Solving these equations simultaneously, we obtain $d = -3.028 \times 10^{-5}$, $e = 0.179\,06$, $f = -352.89$ and $g = 231\,775$.

The values of the APE ranged from a high of 8 in the earliest years to a low of 4 around 1950. In the earliest years, the ball bounced unpredictably and thus produced a bumper crop of extras (mainly byes) for the batting side. That percentage rapidly diminished as both the pitches and wicket keeping techniques improved. There was a small rise on either side of WWI, which was produced by a rise of fast bowlers in that period. The APE reached its minimum of 4 around 1950, as in that period there was a dearth of fast bowlers everywhere except in Australia. Also, in the same period, matting wickets were used in India and the West Indies, which, with their predictable bounce, limited the number of byes. In the post-World War II period from 1960 onwards, there was a tremendous surge in the number of extras (mainly byes and no-balls). This period coincided with the dominance of West Indian cricket with their innovative 4-pronged pace attack which was quickly adopted by other nations. There was a slight but perceptible decline in the APE from 1995 which, coincidentally, marked the end of the West Indian dominance and also a re-emergence of spin bowling elsewhere.

4. Mean batting and bowling averages

Now, we calculate the *Mean Batting Average* (*MBA*) and the *Mean Bowling Average* (*MbA*) of all players for each year. The total number of runs scored by all batsmen is $R - E$. If W_1 is the total number of batsmen dismissed, then the MBA is given by

$$\text{MBA} = \frac{R - E}{W_1}. \tag{1}$$

It should be noted that W_1 is slightly smaller than W. If k is the total number of batsmen who were retired hurt, absent hurt or absent ill in a completed innings (all out), then

$$W_1 = W - k.$$

Similarly, if W_2 is the total number of wickets captured by the bowlers, then the MbA is given by

$$\text{MbA} = \frac{R - E}{W_2}. \tag{2}$$

Again, W_2 is slightly smaller than W_1 by a number l, which is mainly the number of run outs, but also includes other unusual modes of dismissals (such as, hitting the ball twice, obstructing the field, etc.) where the bowler is not credited with the wicket, i.e.

$$W_2 = W_1 - l. \tag{3}$$

Since $W_2 < W_1$ (unless l is zero), we have (see (1)–(3)) $\text{MbA} > \text{MBA}$. The MBA and MbA for each year are plotted in figure 3. Both curves are similar to those in figure 1, with two distinct trends separated by WWI. During the pre-WWI period, the MBA steadily increased from 17.24 in 1876 to 26.38 in 1913 along the LSL. During the post-WWI period, the MBA decreased very slightly from 30.79 in 1919 to 29.69 in 2000. This decline, even though small, is discernible to the naked eye. It is interesting to point out that even as the MBA decreased

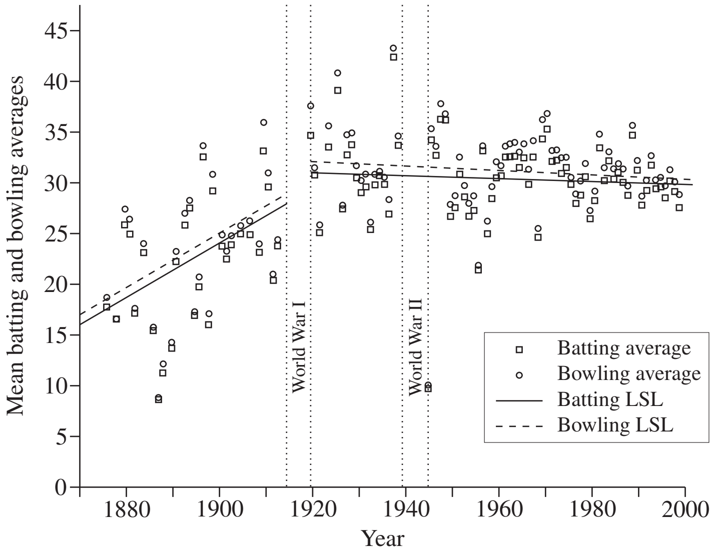

Figure 3 Progression of mean batting and bowling averages in test cricket.

by one run, the extras increased by just the amount to render the ACI constant throughout the post-WWI period (cf. figure 1). The pre-WWI MbA LSL was generally parallel with the MBA LSL, at about one unit above the latter. In the post-WWI period, the MbA LSL decreased slightly from 31.96 in 1919 to 30.76 in 2000 with a slightly steeper slope than that of the MBA LSL. The slight convergence of the MBA and MbA lines indicates a narrowing of the difference between W_1 and W_2. Since the latter difference is mainly the number of run outs, this implies that the running between the wickets had generally improved in the latter part of the twentieth century, thanks perhaps to video re-plays of the matches.

Batsmen and bowlers are most frequently judged by their career batting and bowling averages respectively. In view of the previous discussion, it is no wonder that the greatest bowling averages (with a qualification of 100 wickets) were registered in the pre-WWI era (from reference 3): Lohman ($\beta = 10.75$), Barnes (16.43), Turner (16.53), Peel (16.98), Briggs (17.75) and Blythe (18.63). Bowlers from the post-WWI era never had any chance of attaining these averages. The greatest batting averages, (among batsmen who scored over 2000 runs and retired by the year 2000) on the other hand, were mostly recorded in the earlier part of the post-WWI era where the MBAs were the highest: Bradman ($\alpha = 99.94$), Pollock (60.97), Headly (60.83), Sutcliffe (60.73), Barrington (58.67), Weekes (58.61), Hammond (58.45), Sobers (57.78), Hobbs (56.94), Walcott (56.68) and Hutton (56.67).

The greatness of a batsman or a bowler should be judged by his average relative to the MBA or MbA of his time. For example, Bradman ($\alpha = 99.94$) is mathematically equivalent to over three average batsmen of his time (MBA about 30.7). Similarly, Pollock, Headly and Sutcliffe all having averages of over 60, were equivalent to two average batsmen. But, Hobbs, generally acknowledged as the greatest batsman of his time, began his test career in 1907, when the MBA was generally lower. Thus, his average of 56.94 is actually worth more in terms of the post-WWI MBA. Similar consideration should be also given to W. G. Grace, generally acknowledged to be the greatest batsman of the nineteenth century.

5. All-rounders and wicketkeeper-batsmen

While a batsman is judged by his batting average and a bowler by his bowling average, there has been no mention of any quantity in the literature to measure the effectiveness of an all-rounder. The reason for this is clear: whereas α is a linear quantity, β is a reciprocal quantity, and the two are not amenable to addition. We can circumvent this problem by noting that the reciprocal of β is a linear quantity, and therefore the product of α and $1/\beta$ can be proposed as a measure of the all-round capability (both batting and bowling) of a player as follows:

$$\gamma = \frac{\alpha}{\beta}.$$

Note that γ is not dependent on the MBA or MbA, and therefore can be used to judge all-rounders of all times. When we compute γ for the greatest all-rounders (who had scored over 2000 runs, captured 100 wickets and retired by the year 2000), three players emerge well above all others: Gary Sobers ($\alpha = 57.78$, $\beta = 34.03$, $\gamma = 1.69$), Imran Khan ($37.69, 22.81, 1.65$) and Keith Miller ($36.97, 22.97, 1.60$). Indeed, Sobers is universally accepted as the greatest all-rounder of all time; and before him, Keith Miller was generally regarded as the greatest all-rounder. Since Imran Khan came to the scene after Gary Sobers, such accolades evaded him.

Just as an all-rounder can add to the depth of batting and bowling, it is a luxury to have a wicketkeeper who can also bat well. However, we find that among the leading wicketkeepers (who had over 100 dismissals and who retired before the year 2000), only three had batting averages above the MBA: Parks ($\alpha = 32.16$), Knott (32.75) and Dujon (31.94). Thus, true wicketkeeper-batsmen have been rare in test cricket.

6. The winning strategy

In the post-WWI period, the ACI has been constant at 318. For a team to win consistently, they must score well above that figure and, at the same time, hold their opposition to under that figure. The Australian team under Bradman in 1948 not only had great depth in batting (with Bradman alone equal to three average batsmen) but also three menacing pace bowlers in Lindwall ($\beta = 23.03$), Miller (22.97) and Johnston (23.91), who could consistently limit the opposition's innings total to well under 250. They easily conquered all before them with their lethal bowling combination.

It is bowling more than batting that wins matches. Great batting alone can only ensure a draw and not victory. For example, the victorious West Indies teams of 1978–1993 owed more of their success to a battery of fast bowlers than anything else. While they had a dependable opening pair in Greenidge and Haynes, they had only one batsman (Richards) with a batting average of over 50. A succession of fearsome fast bowlers helped them achieve victory after victory, sometimes from seemingly impossible situations. The list is as long as it is fearsome: Roberts ($\beta = 25.61$), Holding (23.68), Garner (20.97), Croft (23.30), Marshall (20.94), Walsh (24.44), Ambrose (20.99) and Bishop (24.27). Altogether, they captured 2296 test wickets at a price of only 22.81 runs each. They were ably supported by Patterson (30.91), W. Benjamin (27.01) and K. Benjamin (30.27), in what was definitely the greatest parade of fast bowlers ever produced in cricket history.

7. Summary and conclusions

In summary, the statistical analyses of scores and averages in test cricket shed considerable light on the progression and evolution of the game. The scores and averages steadily increased from the inception of test cricket up to WWI. In the post-WWI period, the mean batting and bowling averages declined slightly, but the extras increased sufficiently to render the ACI constant. During this period, the winning formula for a team is to score well above the ACI and at the same time to hold the opposition to under that figure twice in the game.

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3 http://usa.cricinfo.com/db/STATS/BY_CALENDAR/

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Pascal's triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{array}$$

$$1 = 11^0, \qquad 11 = 11^1, \qquad 121 = 11^2, \qquad 1\,331 = 11^3, \qquad 14\,641 = 11^4.$$

Why?

Queen's College,
Cambridge, UK

Jonathan Smith

But I Only Want to Half Differentiate!

LINDA ETTLIN and PAUL BELCHER

1. Introduction

We define the operator symbol D as follows:

$$D[f(x)] = \frac{\mathrm{d}f}{\mathrm{d}x} = f'(x), \qquad D^n[f(x)] = \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = f^{(n)}(x).$$

We will attempt to make a sensible definition and provide a clearer meaning for D^n even when $n \notin \mathbb{N}$. (We are using the International Baccalaureate notation, where the set $\mathbb{N}$ includes 0. If we wish to exclude 0 then we will write $\mathbb{N}^*$. We will also use the convention that an empty product is 1.)

We first consider $D^n[x^m]$, and start on familiar ground with $n \in \mathbb{N}$.

2. The case $n \in \mathbb{N}$

Let $m \in \mathbb{N}$. Then, for $m \geq n$, we obtain

$$D^n[x^m] = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}, \tag{1}$$

which gives the 'factorial formula'

$$D^n[x^m] = \frac{m!}{(m-n)!}x^{m-n}. \tag{2}$$

Equation (1) also gives the 'product formula'

$$D^n[x^m] = \left(\prod_{i=0}^{n-1}(m-i)\right)x^{m-n}. \tag{3}$$

Still letting $m \in \mathbb{N}$, we obtain $D^n[x^m] = D^{n-m}(D^m[x^m]) = D^{n-m}[m!] = 0$, for $m < n$. So, in this case, (2) does not apply but (3) does, as there will be a zero somewhere in the product.

Let $m \in \mathbb{Z}^-$ (i.e. m is a negative integer). Then we obtain

$$D^n[x^m] = (-1)^n \frac{(n-m-1)!}{(-m-1)!}x^{m-n}.$$

In this case, (2) does not apply but (3) does.

Let $m \in \mathbb{R} \setminus \mathbb{Z}$ (i.e. m is a real number that is not an integer). Then we obtain

$$D^n[x^m] = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}.$$

Here, again, (2) does not apply but (3) does.

3. The case $n \in \mathbb{Z}^-$

We now consider the case where n is a negative integer. It seems natural to consider D^{-1} as representing integration, and D^{-n}, for $n \in \mathbb{N}$, as integrating n times. We will apply the convention of ignoring all constants of integration (as many of us erroneously do anyway!).

Let $m \in \mathbb{N}$. Then we obtain

$$D^n[x^m] = \frac{1}{(m+1)(m+2)\cdots(m-n)}x^{m-n} = \frac{m!}{(m-n)!}x^{m-n} = \frac{x^{m-n}}{\prod_{i=1}^{-n}(m+i)}.$$

So (2) applies but the product formula (3) does not, although there is an inverted product formula.

Let $m \in \mathbb{Z}^-$, with $m \le n-1$. Then we obtain the 'inverted product formula'

$$D^n[x^m] = \frac{1}{(m+1)(m+2)\cdots(m-n)}x^{m-n} = \frac{x^{m-n}}{\prod_{i=1}^{-n}(m+i)}. \tag{4}$$

Note that neither (2) nor (3) apply in this case.

Let $m \in \mathbb{Z}^-$, with $m > n-1$. Now, we run into the fact that x^{-1} integrates to $\ln|x|$,

$$D^n[x^m] = \frac{1}{(m+1)(m+2)\cdots(-2)(-1)}D^{n-m-1}[x^{-1}].$$

A proof by induction on $r \in \mathbb{N}^*$, using integration by parts, gives

$$D^{-r}[x^{-1}] = \frac{x^{r-1}}{(r-1)!}\left(\ln|x| - \sum_{i=1}^{r-1}\frac{1}{i}\right).$$

Thus,

$$D^n[x^m] = \frac{x^{m-n}}{(\prod_{i=1}^{-m-1}(m+i))(m-n)!}\left(\ln|x| - \sum_{i=1}^{m-n}\frac{1}{i}\right). \tag{5}$$

So, quite definitely, neither (2) nor (3) apply.

Let $m \in \mathbb{R} \setminus \mathbb{Z}$. Then we obtain

$$D^n[x^m] = \frac{1}{(m+1)(m+2)\cdots(m-n)}x^{m-n}.$$

In this case, neither (2) nor (3) apply.

4. The case $n \in \mathbb{R}$

To extend to $n \in \mathbb{R} \setminus \mathbb{Z}$, we would like to use a generalised version of (2). A generalisation of the factorial function is the gamma function, which is defined as follows:

$$\Gamma(r) = \int_0^\infty t^{r-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for } r \in \mathbb{R},\ r > 0.$$

Integration by parts gives $\Gamma(r+1)r\Gamma(r)$. As $\Gamma(1) = 1$, if $n \in \mathbb{N}$, then $n! = \Gamma(n+1)$. Thus (2) becomes

$$D^n[x^m] = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.$$

For $r < 0$, r not an integer, the gamma function is defined recursively by

$$\Gamma(r) = \frac{\Gamma(r+1)}{r}. \tag{6}$$

The graph of the gamma function is shown in figure 1.

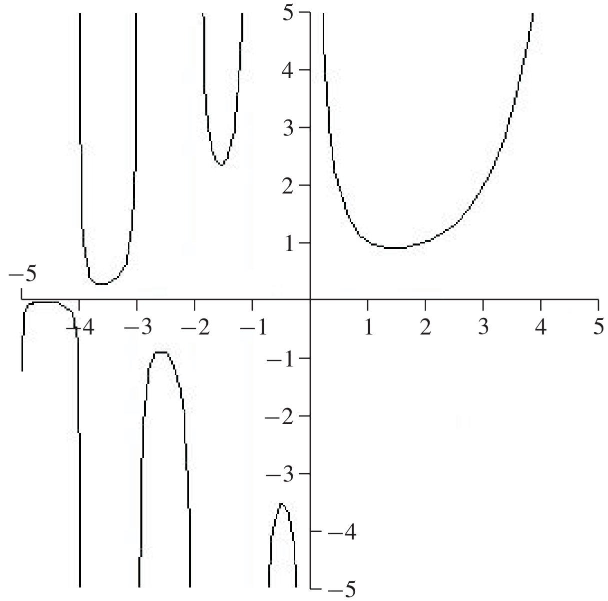


Figure 1 The gamma function

From (6), we obtain

$$\Gamma(r+1) = r\Gamma(r), \qquad r \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{0\}). \tag{7}$$

One disadvantage of the gamma function is that it is undefined for zero and negative integers.

We wish to define the 'gamma formula' as follows:

$$D^n[x^m] = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \quad \text{for } m, n \in \mathbb{R},\ m \notin \mathbb{Z}^-,\ m-n \notin \mathbb{Z}^-. \tag{8}$$

So, for example, with $m = 1$ and $n = \frac{1}{2}$, we obtain

$$D^{1/2}[x] = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{1/2} = \frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})} x^{1/2} = \frac{2}{\sqrt{\pi}} x^{1/2}.$$

Here, making the substitution $t = \frac{1}{2}x^2$, we obtain

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-1/2} \mathrm{e}^{-t}\, \mathrm{d}t = \int_0^\infty \sqrt{2} \mathrm{e}^{-x^2/2}\, \mathrm{d}x.$$

From the normal distribution, we know that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, \mathrm{d}x = 1.$$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \cdot \frac{1}{2} \cdot \sqrt{2\pi} = \sqrt{\pi}$.

In the previous work, (3) could only be applied if $n \in \mathbb{N}$. If $n \in \mathbb{N}$, $m \notin \mathbb{Z}^-$, and $m - n \notin \mathbb{Z}^-$ (so if $m \in \mathbb{N}$ then $m \geq n$) then repeated use of (7) gives

$$\Gamma(m+1) = (m)(m-1)\cdots(m-n+1)\Gamma(m-n+1),$$

and thus (8) agrees with (3). Note that, for $n \in \mathbb{N}$, if $m \in \mathbb{Z}$ and $m < n$ then, at present, (3) applies but (8) does not.

We now define D^n to have the following properties:

$$D^n[f(x) + g(x)] = D^n[f(x)] + D^n[g(x)] \quad \text{and} \quad D^n[kf(x)] = kD^n[f(x)],$$

for $k \in \mathbb{R}$. So, for example,

$$D^{1/2}[3x + 5x^{1/2}] = \frac{6}{\sqrt{\pi}}x^{1/2} + \frac{5}{2}\sqrt{\pi}.$$

We would like our gamma formula (8) to have the following property:

$$D^{a+b}[f(x)] = D^a[D^b[f(x)]]. \tag{9}$$

In fact,

$$D^{a+b}[x^m] = \frac{\Gamma(m+1)}{\Gamma(m-a-b+1)}x^{m-a-b},$$

provided that $m \notin \mathbb{Z}^-$ and $m - a - b \notin \mathbb{Z}^-$, and

$$D^a[D^b[x^m]] = D^a\left[\frac{\Gamma(m+1)}{\Gamma(m-b+1)}x^{m-b}\right] = \frac{\Gamma(m+1)}{\Gamma(m-b+1)}\frac{\Gamma(m-b+1)}{\Gamma(m-b-a+1)}x^{m-b-a},$$

provided that $m \notin \mathbb{Z}^-$, $m - b \notin \mathbb{Z}^-$ and $m - a - b \notin \mathbb{Z}^-$. So, we do have agreement when these quantities exist.

We now wish to extend our definition of the gamma formula (8) as follows:

$$D^n[x^m] = 0 \quad \text{for } m \notin \mathbb{Z}^- \text{ and } m - n \in \mathbb{Z}^-. \tag{10}$$

The rationale behind this is that, in this case, $\Gamma(m+1)/\Gamma(m-n+1)$ can be thought of as a finite quantity divided by an infinite quantity. This also assists in making (9) hold, since, for example, we require $D^{1/2}[D^{1/2}[x^0]] = D[x^0] = 0$. Using (10) we have, for example,

$$D^{1/2}[x^{-1/2}] = 0 \quad \text{and} \quad D^6[x^3] = 0.$$

If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $m < n$, then (10) gives $D^n[x^m] = 0$, which agrees with (3).

To apply D^n to functions other than those constructed from linear combinations of powers of x, we can consider the Maclaurin expansion of a function. For example,

$$\begin{aligned} D^{1/2}[\cos x] &= D^{1/2}\left[\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}}{(2i)!}\right] \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(2i+1)x^{2i-1/2}}{(2i)!\,\Gamma(2i+\frac{1}{2})} \\ &= \frac{x^{-1/2}}{\Gamma(\frac{1}{2})}\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}2^{2i}}{\prod_{r=0}^{2i-1} 2r+1} \\ &= \frac{x^{-1/2}}{\sqrt{\pi}}\sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}2^{4i}(2i)!}{(4i)!}. \end{aligned}$$

Similar results can be found for $D^{1/2}[\sin x]$ and $D^{1/2}[\mathrm{e}^x]$.

More generally, let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ be a power series. If $n \notin \mathbb{N}^*$, we then obtain

$$D^n[p(x)] = \sum_{i=0}^{\infty} \frac{a_i\Gamma(i+1)x^{i-n}}{\Gamma(i+1-n)} = \frac{x^{-n}}{\Gamma(1-n)} \sum_{i=0}^{\infty} \frac{a_i i!\, x^j}{\prod_{r=1}^{i}(r-n)}.$$

Summarising our search, so far, for a definition of $D^n[x^m]$, all the cases where $m \notin \mathbb{Z}^-$ are dealt with by (8) and (10). So we now have to consider $m \in \mathbb{Z}^-$ for the two cases $m-n \in \mathbb{Z}^-$ and $m-n \notin \mathbb{Z}^-$.

Case 1 ($m - n \in \mathbb{Z}^-$). We have that $m \in \mathbb{Z}^-$ and $m-n \in \mathbb{Z}^-$ implies $n \in \mathbb{Z}$ and $n > m$, so we will divide this case into two.

Case 1a ($m \in \mathbb{Z}^-$ and $n \in \mathbb{N}$). An example of this case is $D^7[x^{-3}]$. We have previously discussed this case, and we know that (3) applies. So we define

$$D^n[x^m] = \left(\prod_{i=0}^{n-1}(m-i)\right)x^{m-n} \quad \text{for } m \in \mathbb{Z}^- \text{ and } n \in \mathbb{N}. \tag{11}$$

Case 1b ($m \in \mathbb{Z}^-$, $n \in \mathbb{Z}^-$ and $n > m$). An example of this case is $D^{-2}[x^{-3}]$. Again, we have previously discussed this case, and we know that (4) applies. So we define

$$D^n[x^m] = \frac{x^{m-n}}{\prod_{i=1}^{-n}(m+i)} \quad \text{for } m \in \mathbb{Z}^-,\ n \in \mathbb{Z}^- \text{ and } n > m. \tag{12}$$

If we were to suppose (rather wickedly) that $\Gamma(0)$ existed and that (7) always holds, except for $r = 0$, then definitions (11) and (12) would follow from the major definition (8) upon cancelling $\Gamma(0)$, top and bottom.

Case 2 ($m - n \notin \mathbb{Z}^-$). We have that $m \in \mathbb{Z}^-$ and $m-n \notin \mathbb{Z}^-$. We will also divide this case into two.

Case 2a ($m - n \in \mathbb{N}$). We have that $m \in \mathbb{Z}^-$ and $m-n \in \mathbb{N}$, so $m \geq n$ and, thus, $n \in \mathbb{Z}^-$. An example of this case is $D^{-7}[x^{-3}]$. Again, we have already discussed this case, and we know that (5) applies. So we define

$$D^n[x^m] = \frac{x^{m-n}}{(\prod_{i=1}^{-m-1}(m+i))(m-n)!}\left(\ln|x| - \sum_{i=1}^{m-n}\frac{1}{i}\right) \quad \text{for } m \in \mathbb{Z}^- \text{ and } m \geq n.$$

Case 2b ($m - n \notin \mathbb{Z}$). We have that $m \in \mathbb{Z}^-$ and $m-n \notin \mathbb{Z}$, so $n \notin \mathbb{Z}$. An example of this case is $D^{1/2}[x^{-7}]$. Here we have a problem. We could reasonably expect $D^n[x^m]$ to be of the form $K(n,m)x^{m-n}$ for some constant $K(n,m)$. Now, $D[x^m] = mx^{m-1}$ but, by (9) and (10),

$$D[x^m] = D^{1-n}[D^n[x^m]] = D^{1-n}[K(n,m)x^{m-n}] = 0.$$

Sadly, we do not seem to be able to formulate a satisfactory definition for $D^n[x^m]$ in this case.

We already had an existing problem in desiring (9) to be true, since, for example,

$$D^{-7}[D^5[x^2]] = D^{-7}[0] = 0,$$

but

$$D^{-2}[x^2] = \tfrac{1}{12}x^4.$$

This is partly, but not entirely, due to the fact that we ignored constants of integration.

Our extension of differentiation is also lacking a product rule and a function of a function rule. We feel that it is time to quit and certainly not look at $n \in \mathbb{C}$.

*This article is an amended version of the Mathematics Extended Essay that **Linda Ettlin**, a 17 year old Swiss student at Atlantic College, produced as part of her assessment for the International Baccalaureate Diploma. **Paul Belcher** was her extended essay tutor.*

Mathematics in the Classroom

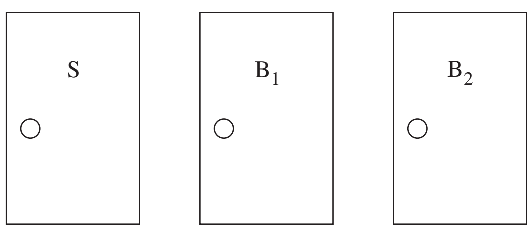
Game show problem

In a game show, the contestant is offered three doors. Behind two of the doors are booby prizes but behind one door is a star prize. The contestant chooses, but does not open, one door. The game show host, who knows which door the star prize is behind, then opens one of the two doors not selected by the contestant and reveals a booby prize. The contestant is then offered a chance to change the door of his choice. Is his best strategy to change, or stay with his original choice, or doesn't it matter?

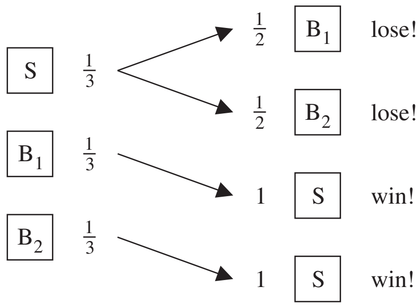
Solution

Initial thoughts seem to suggest that it does not matter – the contestant appears to have a 50–50 chance of winning once the host has opened a door, effectively leaving a choice between the two remaining closed doors. But further consideration shows that this is not in fact the case.

Consider the three doors behind one of which is a star prize (S), and behind each of the other two there are booby prizes (B_1 and B_2)



A probability tree diagram is the best way of solving such problems to do with probability: the second branches show which door the game show host chooses to leave closed following the first choice of the contestant. If the contestant decides to change his or her choice, we obtain the following possible outcomes.



If the contestant initially chooses the star prize door S, then the host will therefore open a booby prize door, either B_1 or B_2. If the contestant then changes his or her choice to the remaining door (B_2 or B_1 respectively), then the contestant would lose in both cases.

If the contestant initially chooses the booby prize door B_1, then the host can only open the remaining booby prize door B_2, and if the contestant then changes his or her choice to the remaining door (S) then the contestant would win.

Similarly, if the contestant initially chooses the booby prize door B_2, then the host can only open the remaining booby prize door B_1, and if the contestant then changes his or her choice to the remaining door (S) then the contestant would win.

Working out the probability of overall winning or losing *if a change of choice is made* by the contestant, will allow us to find out if this is the best strategy, or whether staying with the original choice is best, or if it really does not matter which strategy is adopted.

So, if a change of door is made, we have the following probabilities of the events described in the above tree:

$$\left.\begin{array}{l}\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}\\[2ex] \frac{1}{3}\times\frac{1}{2}=\frac{1}{6}\end{array}\right\}\ \frac{1}{3}\qquad\text{lose,}$$

$$\left.\begin{array}{l}\frac{1}{3}\times 1=\frac{1}{3}\\[2ex] \frac{1}{3}\times 1=\frac{1}{3}\end{array}\right\}\ \frac{2}{3}\qquad\text{win,}$$

i.e., given that changing choices is the strategy to be employed, a lose can only be achieved by choosing the star prize door first, which has a probability of $\frac{1}{3}$.

It follows that, given that changing choices is the strategy to be employed, a win will be achieved if either B_1 or B_2 is the first choice, which has a probability of $\frac{2}{3}$.

This shows that the contestant's best strategy is to change and not to stay with the original choice, because he or she will then have twice the chance of winning than of losing! A somewhat surprising result!

Student, Solihull Sixth Form College, **Jen-chin Pang**

Letters to the Editor

Dear Editor,

Fibonacci and Lucas numbers

Homer W. Austin, in his recent article in Volume 37, Number 2, pp. 67–72, provided much of interest, but his Theorem 1 may be proved much more simply using the basic results

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \qquad L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = \frac{1}{2}(1-\sqrt{5})$. (Austin deploys this to prove his result (3).) We have

$$\begin{aligned} L_m F_{n+m} &= (\alpha^m + \beta^m)\frac{1}{\sqrt{5}}(\alpha^{n+m} - \beta^{n+m}) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2m+n} - \beta^{2m+n} + (\alpha\beta)^m(\alpha^n - \beta^n)) \\ &= F_{2m+n} + (-1)^m F_n, \end{aligned}$$

since $\alpha\beta = -1$. Hence

$$F_{2m+n} = L_m F_{m+n} + (-1)^{m+1} F_n,$$

which is Austin's Theorem 1.

In fact, Theorem 1 is a special case of the following more general result: if the sequence $f_n = \lambda a^n + \mu b^n$ (where λ, μ, a, b are constants) and $l_m = a^m + b^m$, then $l_m f_{n+m} = f_{2m+n} + (ab)^m f_n$. (The proof is similar to the one given above.)

Yours sincerely,

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Dear Editor,

Periodic recurrence relations

I enjoyed Jonny Griffiths' article on *Periodic recurrence relations of the type* $x, y, y^k/x, \ldots$ in *Mathematical Spectrum*, Volume 37, Number 2. A slight shift in perspective from algebra towards trigonometry enables many of the loose ends to be tied up.

The recurrence relation $v_n = kv_{n-1} - v_{n-2}$, with $v_0 = 0$ and $v_1 = 1$, has auxiliary equation $\lambda^2 - k\lambda + 1 = 0$ with roots $k/2 \pm \sqrt{k^2/4 - 1}$. Solving this in the usual way gives the following solutions.

Case 1 If $k = 2$ then $v_n = n$, and if $k = -2$ then $v_n = n(-1)^{n+1}$.

Case 2 If $|k| > 2$ then

$$v_n = \frac{1}{\sqrt{k^2 - 4}}\left[\left(\frac{k}{2} + \sqrt{\frac{k^2}{4} - 1}\right)^n - \left(\frac{k}{2} - \sqrt{\frac{k^2}{4} - 1}\right)^n\right].$$

(Thus, if $|k| \geq 2$ then (v_n) is unbounded and hence divergent.)

Case 3 If $|k| < 2$ then $k = 2\cos\theta$ for some $0 < \theta < \pi$, and the roots of the auxiliary equation are $\cos\theta \pm \mathrm{i}\sin\theta$, so that $v_n = \sin n\theta / \sin\theta$.

Note that the formula

$$v_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-1-r}{r} k^{n-2r-1}$$

quoted in Griffiths' article thus corresponds to the following trigonometric identity:

$$\sin n\theta = \sin\theta \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-r-1}{r} 2^{n-2r-1} \cos^{n-2r-1}\theta,$$

where $\lfloor\cdot\rfloor$ denotes the integer-part function.

Certainly, (v_n) is bounded with $|v_n| \le \operatorname{cosec}\theta$.

If $v_m = 0$ then $\theta = r\pi/m$ for some r, $1 \le r \le m-1$, and

$$v_{m+1} = \frac{\sin((m+1)r\pi/m)}{\sin(r\pi/m)} = (-1)^r.$$

Thus, if r is even then (v_n) is periodic (with period dividing m), and if r is odd then $v_{2m} = 0$ and $v_{2m+1} = 1$, so that (v_n) has period $2m$ (cf table 3 of Griffiths' article).

Thought of as a polynomial in k, $v_m(k) = 0$ has $m-1$ distinct nonzero roots given by $k = 2\cos(r\pi/m)$, $1 \le r \le m-1$. (Since, if m divides m', then the roots of $v_m(k) = 0$ appear among those of $v_{m'}(k) = 0$, and we confirm Griffiths' observation that, as polynomials in k, v_m divides $v_{m'}$, whenever m divides m'.)

The periodic sequences arise from $k = 2\cos(2\pi s/m)$, $1 \le s < m/2$, and it is a standard result that, as illustrated in Griffiths' table 4, exactly $\frac{1}{2}\phi(m)$ of them – corresponding to s and m coprime – have period m, where $\phi(m)$ denotes Euler's totient. For example, if $m = 15$ and $\phi(15) = 8$ then the four entries $(1.827, 1.338, -0.209, -1.956)$ in Griffiths' table 4 are revealed to be

$$2\cos\left(\frac{2\pi}{15}\right), \qquad 2\cos\left(\frac{4\pi}{15}\right), \qquad 2\cos\left(\frac{8\pi}{15}\right), \qquad 2\cos\left(\frac{14\pi}{15}\right).$$

Finally, it is worth recording that the bounded sequence

$$v_n = \frac{\sin n\theta}{\sin\theta}, \qquad 0 < \theta < \pi,$$

is never convergent. For the trigonometric identities

$$v_{n+1} - v_{n-1} = 2\cos n\theta,$$
$$\cos(n-1)\theta - \cos(n+1)\theta = 2\sin\theta\sin n\theta,$$

show that, if $v_n \to l$ then $\cos n\theta \to 0$ and $\sin n\theta \to 0$, contradicting the fact that $\sin^2 n\theta + \cos^2 n\theta = 1$.

Yours sincerely,

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Dear Editor,

Diophantine equations

With reference to the problems posed in Muneer Jebreel's letter *Solutions to Diophantine equations* (*Math. Spectrum*, Volume 37, Number 3), I assume that we are only interested in nonzero solutions, otherwise the problems are trivial. An infinite number of solutions may be found to all three equations as shown below. I illustrate the method for the second equation only. It is very similar for the other two. Indeed, such a method will work for a wider set of Diophantine equations of the type

$$\pm x^{mn-1} \pm y^{mn-1} \pm \cdots = k^n,$$

for example $x^8 + y^8 - z^8 = k^3$.

Now consider $x^3 + y^3 = k^2$. Select any nonnegative integers a and b. Let $a^3 + b^3 = cj^2$, where c and j are integers. This can always be done since, if necessary, we may take $j = 1$. Then $x = ca$, $y = cb$ and $k = c^2 j$ yield a solution of $x^3 + y^3 = k^2$. In fact, if $a = 3$, $b = 1$, $c = 7$, $j = 2$, then we obtain the solution $21^3 + 7^3 = 98^2$ given by Jebreel.

Allowing for the fact that c may be equal to 1, it follows that all possible solutions may be found by this method (this applies for all three equations).

I believe that the simplest solutions to Jebreel's three equations are

$$1^3 + 2^3 + 3^3 = 6^2, \qquad 1^3 + 2^3 = 3^2, \qquad 6^5 - 2^5 = 88^2.$$

The first two equations are simply illustrations of the well-known formula for summing the cubes of the first n natural numbers, i.e.

$$\sum_{r=1}^{n} r^3 = \left(\frac{r(r+1)}{2}\right)^2,$$

with $n = 2$ and 3 respectively. The solution to the third equation arises from noting that $3^5 - 1^5 = 2 \cdot 11^2$, and then using a similar method to that demonstrated above. I leave one further problem for readers. Is there a solution to the third equation where x, y and k have no common factor?

Yours sincerely,

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Dear Editor,

Square roots by subtraction

Reading Frazer Jarvis' article *Square roots by subtraction* in Volume 37, Number 3, I recalled a process which may be of interest to readers.

I was taught this process as a junior school pupil in the early 1930s, simply as a routine process with no explanation. Later, at university, I was set the task of justifying the method. Then, in the late 1940s while working at a research establishment when computers were a rarity, I used the same process on an electric calculator. It proved to be very efficient.

Fundamentally the process resembles that in Jarvis' article, being essentially a subtraction process.

Example

$$
\sqrt{234.5} = \begin{array}{r|cccccccccc}
 & 1 & & 5. & & 3 & & 1 & & 3 & \\
\hline
 & 2 & 3 & 4. & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 \times 1 & 1 & & & & & & & & & \\
 & 1 & 3 & 4 & & & & & & & \\
25 \times 5 & 1 & 2 & 5 & & & & & & & \\
 & & & 9 & 5 & 0 & & & & & \\
303 \times 3 & & & 9 & 0 & 9 & & & & & \\
 & & & & 4 & 1 & 0 & 0 & & & \\
3061 \times 1 & & & & 3 & 0 & 6 & 1 & & & \\
 & & & & 1 & 0 & 3 & 9 & 0 & 0 & \\
30623 \times 3 & & & & & 9 & 1 & 8 & 6 & 9 & \\
 & & & & & 1 & 2 & 0 & 3 & 1 & 0
\end{array}
$$

- Pair off the digits from the decimal point in both directions.
- Starting with the leftmost pair (in this case a single digit) subtract the greatest possible square number (in this case 1) and record it to the left and in the answer line at the top.
- Copy down the next pair of digits (to give 134). Now double the answer so far in the top line, multiply it by 10 and add a single digit x such that multiplication by x gives the largest number not exceeding this number ($(2 \times 1 \times 10 + 5) \times 5$). Record the multiplier (5) in the top line and subtract.
- Repeat the above step until sufficient accuracy is attained. The numbers in the top line give an approximation to the required square root.

As a clue to a justification of the process, note that

$$234.5 = 100 + 134.5 = 10 \times 10 + 2 \times 10 \times 5 + 5 \times 5 + 9.5 = 15 \times 15 + 9.5$$

and

$$9.5 = 3 \times 3 + 0.5 = 2 \times 15 \times 0.3 + 0.3 \times 0.3 + 0.4100,$$

which is in line with the above calculation, giving

$$234.5 = 15.3 \times 15.3 + 0.4100.$$

Yours sincerely,

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[Peter Shannon, of Boroughmuir High School, Edinburgh, has written describing a similar method – Ed.]

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

38.1 Determine all natural numbers n and all permutations $(p_1, \ldots, p_n)$ of $(1, \ldots, n)$ such that $|k - p_k| = 1$ for $k = 1, \ldots, n$.

(Submitted by H. A. Shah Ali, Tehran, Iran)

38.2 The Fibonacci and Lucas sequences are defined by

$$\mathrm{F}_0 = 0,\ \mathrm{F}_1 = 1, \qquad \mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2} \quad \text{for } n \geq 2$$

and

$$\mathrm{L}_0 = 2,\ \mathrm{L}_1 = 1, \qquad \mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2} \quad \text{for } n \geq 2$$

respectively. Prove that

(i) $\mathrm{F}_{3n} + (-1)^n \mathrm{F}_n = \mathrm{L}_n \mathrm{F}_{2n}$,

(ii) $\mathrm{L}_{3n}\mathrm{F}_{3n} + \mathrm{F}_n \mathrm{L}_n = \mathrm{F}_{2n}\mathrm{L}_{2n}^2$,

for all $n \geq 0$.

(Submitted by Mihály Bencze, Brasov, Romania)

38.3 Sum the finite series

$$\sum_{r=0}^{n} \frac{(n+r)!\,(-1)^{n-r}}{(n-r)!\,r!\,r!}.$$

(Submitted by J. A. Scott, Chippenham)

38.4 Prove that, for all real numbers x, y, z,

$$\sqrt{x^2 + xy + y^2} + \sqrt{x^2 + xz + z^2} \geq \sqrt{y^2 + yz + z^2}.$$

(Submitted by Abbas Roohol Aminy, 'New Meditation and Young Society', Sirjan, Iran)

Solutions to Problems in Volume 37 Number 2

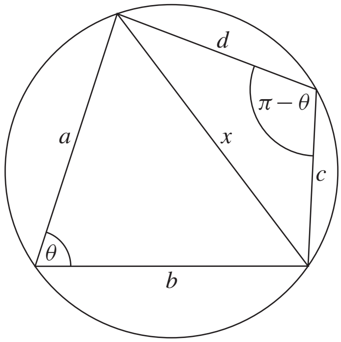
37.5 Prove Brahmagupta's formula for the area Δ of a cyclic quadrilateral with sides a, b, c, d, namely

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where

$$s = \tfrac{1}{2}(a+b+c+d).$$

Solution



From the figure,

$$x^2 = a^2 + b^2 - 2ab\cos\theta = c^2 + d^2 + 2cd\cos\theta,$$

so that

$$\cos\theta = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

The area of the quadrilateral is

$$\begin{aligned}
&\tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\theta \\
&\quad = \tfrac{1}{2}(ab+cd)\sqrt{1-\left(\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right)^2} \\
&\quad = \tfrac{1}{4}\sqrt{4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2} \\
&\quad = \tfrac{1}{4}\sqrt{(2(ab+cd)+a^2+b^2-c^2-d^2)(2(ab+cd)-a^2-b^2+c^2+d^2)} \\
&\quad = \tfrac{1}{4}\sqrt{((a+b)^2-(c-d)^2)((c+d)^2-(a-b)^2)} \\
&\quad = \tfrac{1}{4}\sqrt{(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)} \\
&\quad = \tfrac{1}{4}\sqrt{(2s-2d)(2s-2c)(2s-2b)(2s-2a)} \\
&\quad = \sqrt{(s-a)(s-b)(s-c)(s-d)}.
\end{aligned}$$

Mihály Bencze pointed out that Heron's formula for the area of a triangle, namely

$$\text{Area } \Delta\text{ABC} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$, follows by putting $s = 0$, since every triangle can be inscribed in a circle.

37.6 Let $x_1, \ldots, x_n$ be real numbers such that

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n.$$

Prove that

$$\cos^{-1}\frac{x_1}{x_n} \leq \cos^{-1}\frac{x_1}{x_2} + \cos^{-1}\frac{x_2}{x_3} + \cdots + \cos^{-1}\frac{x_{n-1}}{x_n}.$$

Solution by Mihály Bencze, who proposed the problem

We use induction on n. The result is clear when $n = 2$. For $n = 3$,

$$\begin{aligned}\cos\left(\cos^{-1}\frac{x_1}{x_2} + \cos^{-1}\frac{x_2}{x_3}\right) &= \frac{x_1}{x_2}\frac{x_2}{x_3} - \sqrt{\left(1 - \frac{x_1^2}{x_2^2}\right)\left(1 - \frac{x_2^2}{x_3^2}\right)} \\ &\leq \frac{x_1}{x_3},\end{aligned}$$

so that

$$\cos^{-1}\frac{x_1}{x_2} + \cos^{-1}\frac{x_2}{x_3} \leq \cos^{-1}\frac{x_1}{x_3}.$$

Now, let $n > 3$ and assume the result for $n - 1$. Then we obtain

$$\begin{aligned}&\cos^{-1}\frac{x_1}{x_2} + \cdots + \cos^{-1}\frac{x_{n-2}}{x_{n-1}} + \cos^{-1}\frac{x_{n-1}}{x_n} \\ &\quad \geq \cos^{-1}\frac{x_1}{x_{n-1}} + \cos^{-1}\frac{x_{n-1}}{x_n} \quad \text{(by the inductive hypothesis)} \\ &\quad \geq \cos^{-1}\frac{x_1}{x_n} \quad \text{(by the case } n = 3\text{)}.\end{aligned}$$

This proves the inductive step.

37.7 Let x be a real number. Determine

$$\lim_{n\to\infty}\frac{[x] + [2x] + \cdots + [nx]}{n^2},$$

where $[\cdot]$ denotes the integer-part function.

Solution by Henry Ricardo, Medgar Evers College, New York

For a positive integer k,

$$kx - 1 < [kx] \leq kx,$$

so that

$$\sum_{k=1}^{n}(kx - 1) \leq \sum_{k=1}^{n}[kx] \leq \sum_{k=1}^{n}kx,$$

i.e.

$$\frac{n(n+1)}{2}x - n \leq \sum_{k=1}^{n}[kx] \leq \frac{n(n+1)}{2}x,$$

whence

$$\frac{n+1}{2n}x - \frac{1}{n} \leq \frac{1}{n^2}\sum_{k=1}^{n}[kx] \leq \frac{n+1}{2n}x.$$

The *squeeze principle* [also called the *sandwich rule* – Ed.] now gives

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}[kx]}{n^2} = \frac{x}{2}.$$

Mihály Bencze has sent us the generalization

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \sum_{k=1}^{n}[k^p x] = \frac{x}{p+1}.$$

Readers may like to try to prove this.

37.8 A set of $n \geq 6$ consecutive integers is partitioned into two subsets A and B, both with at least three elements. Prove that there exist a_1, a_2 in A and b_1, b_2 in B such that $a_1 - a_2 = b_1 - b_2 \neq 0$.

Solution by H. A. Shah Ali, who proposed the problem

First consider m integers such that $x_1 < x_2 < \cdots < x_m$. Then

$$x_2 - x_1 < x_3 - x_1 < \cdots < x_m - x_1,$$

so that there are at least $m - 1$ different integers among $x_i - x_j$, for $i > j$. Suppose that there are exactly $m - 1$ different integers among the $x_i - x_j$. Consider $x_i - x_{i-1}$. It cannot be less than $x_2 - x_1$, otherwise there would be at least m distinct integers among the $x_i - x_j$. If $x_i - x_{i-1} > x_2 - x_1$ then $x_i - x_2 > x_{i-1} - x_1$, so that

$$x_i - x_1 > x_i - x_2 > x_{i-1} - x_1,$$

and again there will be at least m distinct integers among the $x_i - x_j$. Hence $x_i - x_{i-1} = x_2 - x_1$ for all i, so the sequence $x_1, \ldots, x_m$ is an arithmetic progression.

Now, consider the problem posed. If one of the sets, say A, has exactly $|A| - 1$ differences, then its elements will form an arithmetic progression. If the common difference is 1, then A consists of $|A|$ consecutive integers and B consists of the remaining ones. Since $|B| \geq 3$, two of the integers in B will be consecutive, and the result is established. If the common difference is $d \geq 2$, then let x and $x + d$ be the first two terms in A. Then $x + 1, x + d + 1 \in B$, and the result again follows.

It remains to consider the case when A and B have at least $|A|$ and $|B|$ differences respectively. If there is no difference common to A and B, then the set as a whole will have at least $|A| + |B| = n$ differences. But it cannot, because it consists of n consecutive integers.

$$x^{\circ}\text{centigrade} = x^{\circ}\text{fahrenheit}.$$

What is x?

Reviews

Resources for the Study of Real Analysis. By Robert L. Brabenec. MAA, Washington, DC, 2004. Paperback, 240 pages, \$48.95 (ISBN 0-88385-737-5).

This book is for an undergraduate student taking a course in analysis, it would not replace their lecture notes or their recommended text, but would provide excellent additional material. A student who enjoys doing mathematics for its own sake, and wants to explore the subject further would get most out of this book. Only a conscientious and talented student would complete all the exercises given in the book. I would also recommend this book to anyone teaching analysis. It is a book to be dipped into and come back to again, rather than reading it all at once.

Section 1 gives an outline of a traditional course in calculus followed by some review items. A list is perhaps not the most exciting way to start a book and thus, for me, this would have been better at the end.

Section 2 gives 34 analysis problems divided into ‘Basic’, ‘Supplementary’ and ‘Enrichment’. I particularly enjoyed ‘properties of even and odd functions’ in the Basic section, ‘a function with derivatives of all orders, but no Maclaurin series’ in the Supplementary section and ‘subseries of the harmonic series’, which introduced me to gap sequences, in the Enrichment section.

Section 3 gives ten essays divided into ‘History and Biography’, ‘New Looks at Calculus Content’ and ‘General Topics for Analysis’. The historical ones are chronological, and well worth reading due to their insightfulness.

Section 4 gives five readings by other authors, several dealing with the relationships between famous mathematicians. There is also an extensive Bibliography that is annotated and allows you to follow up on points raised.

The whole book is well written and the mathematical derivations are clear. Reading the book gives me the strong impression that Brabenec’s students, the author teaches at Wheaton College, are lucky to have such an organised teacher who wishes to pass his enjoyment in the subject on to others.

Atlantic College **Paul Belcher**

Closepacks and Quasi-Closepacks. By Patrick Taylor. Nattygrafix, Ipswich, 2004. Paperback, 72 pages, £6.00 (ISBN 0-9516701-6-6).

This is the fifth volume by this author dealing with polyhedra.

Mathematical Delights. By Ross Honsberger. MAA, Washington, DC, 2004. Paperback, 264 pages, \$39.50 (ISBN 0-88385-334-5).

The book is, at first, a little daunting as it is full of very complicated sums and has an equally complicated layout. Most of the questions were beyond me with my current level of GCSE mathematics, but the solutions were still interesting to look at, giving a good insight into much more complex step-by-step answers, and how you go about setting them out – a valuable experience.

The layout differs from a more traditional question-and-answer type book. Rather than having all the answers separated into a section at the back of the book, the writer has followed each question directly with the solution. This is something that I should like to see more of, as it means you don't have to constantly flick between pages to find the right answer. However, perhaps the answer should be kept slightly more distant from the question to prevent the temptation of looking at the answer prematurely. The book would be perfect if it clearly stated where questions end and answers begin, as on numerous occasions I was thoroughly confused by a question, not realising that I had strayed into reading the answer.

All in all I think that this book would be very beneficial for anyone with an interest in mathematics, even if you don't understand advanced mathematics, or aren't very good at it, as it covers a very large spectrum of questions ranging from pure algebra to locus-type problems and logic questions. All this put together offers opportunities for widening the knowledge and understanding of the reader, a must for anyone in need of a challenge!

Student, Solihull School **Matthew Hull**

Mathematical Adventures for Students and Amateurs. Edited by David F. Hayes and Tatiana Shubin. MAA, Washington, DC, 2004. Paperback, 304 pages, $37.50 (ISBN 0-88385-548-8).

This book delves further into mathematics than my brain could ever have conceived possible before reading it. Focussing on questions in life that have no obvious links with mathematics, it sets about proving them through calculations. I found a lot of the book difficult to understand after the first read, but then tried to focus on some of the aspects that I had covered in my school work and found that a vast quantity of the calculations used some of the methods I had encountered, but had taken them to the next level.

A section of the book that I found particularly interesting (and sometimes amusing) was the section entitled 'Alice in Numberland' which approached mathematics in an exciting and fun way. It also highlighted how simple logic and common sense can prove vital when approaching seemingly hard questions.

This book opened my eyes to the wonders of mathematics, and maybe, in a few more years, I will be able to appreciate and understand a lot more of the content.

Student, Solihull School **Oliver Talbot**

Core Mathematics 2. By Greg Atwood, Alistair Macpherson, Bronwen Moran, Joe Petran, Geoff Staley and Dave Wilkins. Heinemann, Oxford, 2004. Paperback, 224 pages, £10.99 (ISBN 0-435-51098-3).

This book is designed to meet the new specifications for AS/A-level mathematics introduced in 2004. It covers the second pure mathematics module, which forms one third of the AS-level syllabus, and is endorsed by the Edexcel exam board.

The first chapter starts with basic knowledge required for the module, such as algebra and functions. It is clear that a student will require a sound knowledge of these topics, as they are presented and consolidated at GCSE level. Throughout the book, topics range from equation-related work such as binomial expansions to topics linked with graphs, with a recommendation that everything relating to coordinate geometry should be started from the basics. This enables the student to focus on developing and building a topic in a way that allows relationships with other topics to be very evident.

The layout of the text is very clear and presentation is lively. Diagrams are plentiful, and these illustrations are used along with detailed notes to ensure that the reader understands each individual step. Exercises at the end of each topic test understanding and build confidence in using the techniques that have been explored. Answers and hints are provided to aid self-marking; it is acknowledged that learning from mistakes is an excellent way to progress. Summaries at the end of each topic are short and straightforward providing good revision material when a topic is completed.

Colour coding is used in an interesting way – each topic has its own colour providing a good aid to memory. My only reservation about this book relates to a lack of really challenging problems for when the routine ones have been completed. Otherwise it will prove an invaluable support for anyone studying for an AS-level in mathematics.

Student, Solihull Sixth Form College **Jen-chin Pang**

Advancing Mathematics for AQA: Statistics 1. By Roger Williamson, Gill Buque, Jim Miller and Chris Worth. Heinemann, Oxford, 2004 (2nd edn). Paperback, 210 pages, £8.99 (ISBN 0-435-51338-9).

This is the second edition of a familiar book that has been revised to meet the new specifications for AS/A-level mathematics introduced in 2004. Having used the first edition of this book, which I found extremely beneficial, it has to be said that this new edition does not disappoint; it contains exactly what is required for the opening statistics module of the new AQA syllabus.

I feel that anyone who has taken GCSE Statistics, and found it as riveting as I did, will want to use this book as it will continue to aid him or her in broadening their understanding of statistics, which is almost a life-skill these days. Hence it has an appeal beyond the confines of those students studying for an AS-level in mathematics, regardless of the exam board.

The book is clearly divided into chapters, enabling the reader to distinguish different topics easily. Each chapter presents a brief introduction, usefully familiarising the reader with the topic in question. Explanations are enhanced by diagrams. These are clearly set out amongst the text in an uncongested way, providing pages that are visually very appealing and consequently easy to follow. The language used in the book is straightforward, allowing topics which could prove challenging to be accessed in a more user-friendly way.

Having very successfully used the first edition myself, I would recommend this book to all students studying the syllabus as essential reading. It also deserves a place in the college or school library for those students who want to expand their statistical understanding for use in other disciplines.

Student, Solihull Sixth Form College **Jason Kawa**

Advancing Maths for AQA: Mechanics 2. By Ted Graham and Aidan Burrows. Heinemann, Oxford, 2nd edn., 2004. Paperback, 154 pages, £9.99 (ISBN 0-435-51337-0).

The authors are senior members of the examining team, and have prepared the textbooks in this series specifically to support students in studying the course. Every topic has detailed and straightforward explanations detailing techniques used, and key definitions are highlighted. Methods are illustrated through step-by-step solutions to worked examples which use clear diagrams and also interpret the final answers. Questions are well structured and call for the use of clearly labelled diagrams, so developing good habits in the reader.

After each topic, key points are summarised with the formulae used, and definitions with page references enable easy revision. A test-yourself section of typical examination-style questions relates the situation to everyday life. Practice papers are also given, which helpfully show mark and time allocations.

A well-motivated student will enjoy the style of this book, which aims to ensure that basic methods and techniques will be understood. It will provide good support for any student working on his or her own.

Student, Solihull Sixth Form College **Jen-chin Pang**

Advancing Maths for AQA: Decision Maths 1. By Sam Boardman, Ted Graham, David Pearson and Roger Williamson. Heinemann, Oxford, 2nd edn., 2004. Paperback, 215 pages, £8.99 (ISBN 0-435-51335-4).

New specifications for AS/A-level mathematics were introduced in 2004, the first changes since Curriculum 2000 started four years earlier. These have generally increased, at the expense of applied mathematics, the proportion of pure mathematics contained in the syllabus and, consequently, redistributed some of the material contained in the various strands.

As a result, the decision mathematics module has expanded, and the book covers minimum connectors, shortest path, Chinese postman and travelling salesman problems, along with graph theory, matchings, sorting algorithms and linear programming. The actual presentation of the material remains in the format of the first edition, it is clearly laid out with well-labelled diagrams and marginal notes to aid the exposition.

Solihull Sixth Form College **Carol Nixon**

Advancing Maths for AQA: Core Maths 1. By Greg Atwood, Alistair Macpherson, Bronwen Moran, Joe Petran, and Dave Wilkins. Heinemann, Oxford, 2004. Paperback, 224 pages, £10.99 (ISBN 0-435-51097-5).

This book is aimed at students studying the core mathematics 1 module for the new 2004 specifications, and is divided into eight colour-coded topics. The book begins with the basics of algebra and functions and progresses towards differentiation and integration. Students are led through the topics well, each chapter being divided into sub-sections that have worked examples and plenty of exercises – with answers included. Guidance through the examples is provided in helpful yellow boxes and the hints in pink boxes are also useful, whilst the summary at the end of each topic acts as a good revision aid.

The exercise questions and mixed exercises, most of which are past exam questions, are substantial if not a little excessive in places. Yellow boxes pop up frequently, this may be distracting to some as well as helpful to others. The single exam-style paper at the end of the book provides limited whole-paper practice. However, as an A-level textbook this is very helpful indeed and I am sure that students using it will benefit substantially.

Student, Solihull Sixth Form College **Carsten Ghedia**

Advancing Maths for AQA: Core Maths 4. By Greg Atwood, Alistair Macpherson, Bronwen Moran, Joe Petran, Keith Pledger and Dave Wilkins. Heinemann, Oxford, 2004. Paperback, 140 pages, £10.99 (ISBN 0-435-51100-9).

Written in the same style as *Core Maths 1*, this book covers the content of the final pure mathematics module in the Edexcel A-level course. As well as further differentiation and integration, it covers partial fractions, coordinate geometry, binomial expansions and vectors in a student-friendly and succinct way.

Student, Solihull Sixth Form College **Carsten Ghedia**

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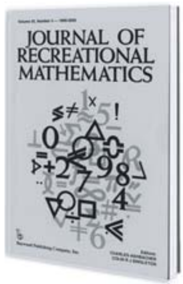

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