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# Mathematicorum

# *CRUX MATHEMATICORUM*

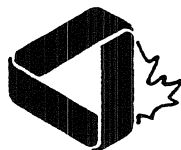
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# ON TILING AN $m \times m$ SQUARE WITH $m$ SQUARES

Stephen T. Ahearn and Charles H. Jepsen<sup>1</sup>

Algebraic identities can lead to interesting geometry problems. For example, consider the identity

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2,$$

which can be written as

$$1 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + \cdots + n \cdot n^2 = \left[ \frac{n(n+1)}{2} \right]^2.$$

The latter equation suggests the possibility of tiling a square of side length  $m = \frac{1}{2}n(n+1)$  with one unit square, two  $2 \times 2$  squares, three  $3 \times 3$  squares,  $\dots$ ,  $n$   $n \times n$  squares (a total of  $m$  squares in all). For what values of  $n$  does such a tiling exist?

Martin Gardner in [1], p. 301, attributes this problem to Robert T. Wainwright. Gardner notes that no solutions are possible for  $n = 2, 3, 4, 5$  and says that Wainwright has found no solutions for  $n = 6$  through 11 but has been unable to prove impossibility. Wainwright found a solution for  $n = 12$  (shown in Figure 1) and conjectured that this is the smallest square that can be so tiled.

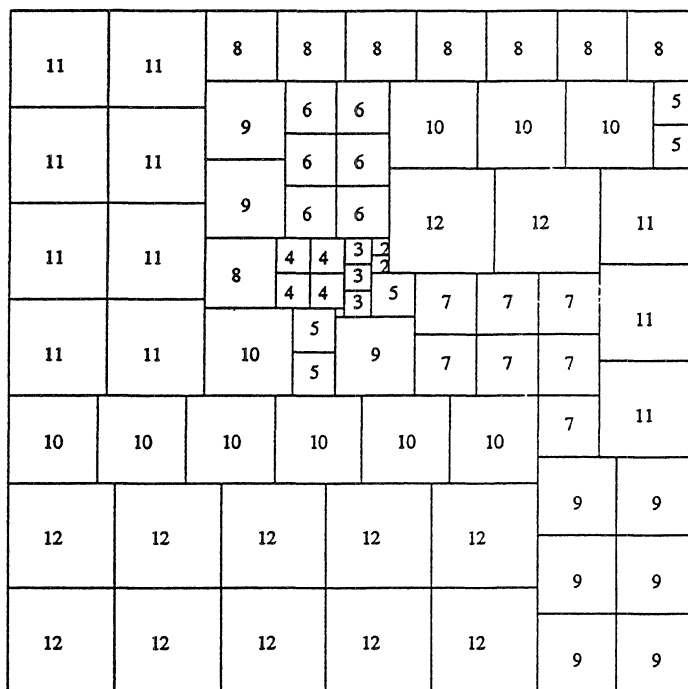


Figure 1

<sup>1</sup>This paper is the result of a 1992 summer student research project by the first author under the supervision of the second author.

In this note, we announce a collection of new results for this problem. Our first contribution is given in Figure 2: **a solution for  $n = 11$**  (found by the second author). In addition, we report that no solutions are possible for  $n = 6$ . Impossibility was proved (by the first author) by systematic exhaustion of all possible cases. The problem remains unsolved for  $n = 7$  through 10.

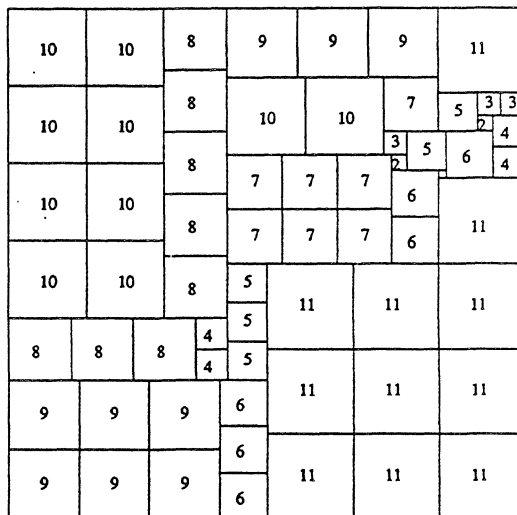


Figure 2

What about values of  $n$  larger than 12? It is easy to see that a solution for  $n$  even yields a solution for  $n + 1$ : given a tiling of a square of side length  $\frac{1}{2}n(n + 1)$ , adjoin a border of  $n + 1$  squares of side length  $n + 1$  along two adjacent edges. (Figure 3 shows the case where  $n + 1 = 13$ .) This technique will not work when  $n$  is odd; here one of the squares added along the border must be cut in half. However, we can still obtain a solution whenever the tiling of the square of side length  $\frac{1}{2}n(n + 1)$  contains four squares of side length  $\frac{1}{2}(n + 1)$  arranged in a square. Simply replace these four squares by a single square of side length  $n + 1$  and place the four squares along the border in the positions of the two half squares. (See Figure 4 for the case where  $n + 1 = 14$ .) Hence Wainwright's solution for  $n = 12$  yields solutions for 13, 14, and 15 (but not 16 and beyond).

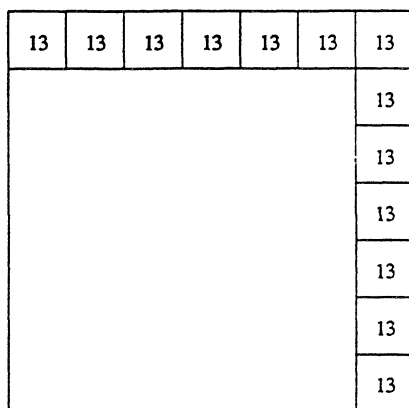


Figure 3

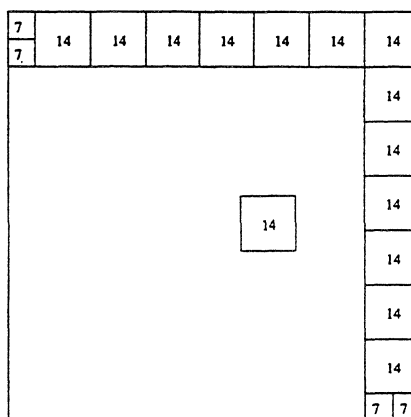


Figure 4

Figure 5 shows a solution for  $n = 16$  (found by the second author). Further, using the techniques described in the preceding paragraph, we get solutions for  $n = 17$  through 33. It appears likely that solutions exist for  $n \geq 34$  as well. However, these cases remain open for now.

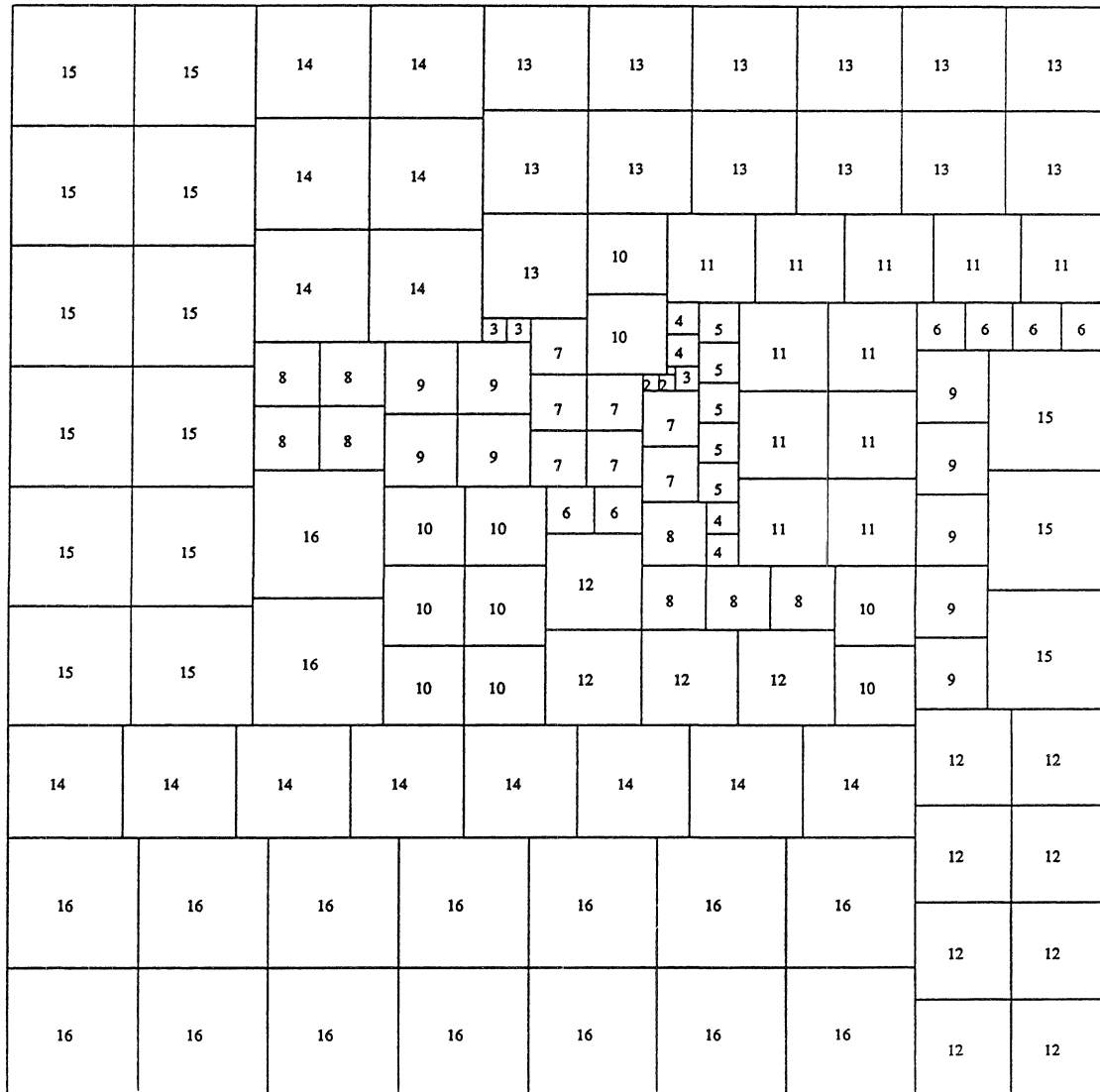


Figure 5

*Reference:*

- [1] Martin Gardner, *Fractal Music, Hypercards and More: Mathematical Recreations from Scientific American Magazine*, W. H. Freeman and Company, New York, 1992.

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# THE OLYMPIAD CORNER

No. 147

R.E. WOODROW

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We begin this number with the problems of the Irish Mathematical Olympiad 1991. My thanks to Georg Gunther, Team leader of the Canadian I.M.O. team, for collecting this problem set and forwarding it to me.

## FOURTH IRISH MATHEMATICAL OLYMPIAD

May 4, 1991

Paper 1

**1.** Three points  $X$ ,  $Y$  and  $Z$  are given which are respectively, the circumcentre of triangle  $ABC$ , the midpoint of  $BC$ , and the foot of the altitude from  $B$  on  $AC$ . Show how to reconstruct the triangle  $ABC$ .

**2.** Find the polynomials  $f(x) = a_0 + a_1x + \cdots + a_nx^n$  satisfying the equation

$$f(x^2) = (f(x))^2$$

for all real numbers  $x$ .

**3.** Three operations  $f$ ,  $g$  and  $h$  are defined as follows:

$$f(n) = 10n \quad \text{if } n \text{ is a positive integer}$$

$$g(n) = 10n + 4 \quad \text{if } n \text{ is a positive integer}$$

$$h(n) = n/2 \quad \text{if } n \text{ is an even positive integer}$$

Prove that: starting from 4, every natural number can be constructed by performing a finite number of the operations  $f$ ,  $g$  and  $h$  in some order. [For example:

$$35 = h(f(h(g(h(h(4)))))). \quad ]$$

**4.** Eight politicians stranded on a desert island on January 1st, 1991 decided to establish a parliament. They decided on the following rules of attendance:

(1) There should always be at least one person present on each day.

(2) On no two days should the same subset attend.

(3) The members present on day  $N$  should include for each  $K < N$ , ( $K \geq 1$ ), at least one member who was present on day  $K$ .

For how many days can the parliament sit before one of the rules is broken?

**5.** Find *all* polynomials  $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$  with the following properties:

- (i) all the coefficients  $a_1, a_2, \dots, a_n$  belong to the set  $\{-1, 1\}$ ;
- (ii) all the roots of the equation  $f(x) = 0$  are real.

[Hint: First find the maximum possible value of  $n$ .]

### Paper 2

**1.** The sum of two consecutive squares can be a square ( $3^2 + 4^2 = 5^2$ ).

(a) Prove that the sum of  $m$  consecutive squares cannot be a square for the cases  $m = 3, 4, 5, 6$ .

(b) Find an example of eleven consecutive squares whose sum is a square.

**2.** Let

$$a_n = \frac{n^2 + 1}{\sqrt{n^4 + 4}} \quad \text{for } n = 1, 2, 3, \dots$$

and let  $b_n$  be the product  $a_1 a_2 \dots a_n$ . Prove that

$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2 + 1}}{\sqrt{n^2 + 2n + 2}}$$

and deduce that

$$\frac{1}{(n+1)^3} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$

for all positive integers  $n$ .

**3.** Let  $ABC$  be a triangle and  $L$  the line through  $C$  parallel to the side  $AB$ . Let the internal bisector of the angle at  $A$  meet the side  $BC$  at  $D$  and the line  $L$  at  $E$ , and let the internal bisector of the angle at  $B$  meet the side  $AC$  at  $F$  and the line  $L$  at  $G$ . If  $GF = DE$  prove that  $AC = BC$ .

**4.** Let  $P$  be the set of positive rational numbers and let  $f : P \rightarrow P$  such that

$$f(x) + f\left(\frac{1}{x}\right) = 1$$

$$\text{and} \quad f(2x) = 2f(f(x))$$

for all  $x \in P$ . Find, with proof, an explicit expression for  $f(x)$  for all  $x \in P$ .

**5.** Let  $\mathbb{Q}$  denote the set of rational numbers. A nonempty subset  $S$  of  $\mathbb{Q}$  has the following properties:

- (i) 0 is not in  $S$ ;
- (ii) for each  $s_1, s_2$  in  $S$ ,  $s_1/s_2$  is in  $S$  also;
- (iii) there exists a nonzero rational number  $q$  which is not in  $S$  and which has the property that every nonzero rational number not in  $S$  is of the form  $qs$  for some  $s$  in  $S$ .

Prove that if  $x$  is in  $S$ , then there exist elements  $y, z$  in  $S$  such that  $x = y + z$ .

\* \* \*



In the June number we gave the questions of the Canadian Mathematical Olympiad. Next we give the “official solutions”. My thanks to Edward Wang, Chairman of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society, for furnishing the problems and solutions.

### 1993 CANADIAN MATHEMATICAL OLYMPIAD

1. Determine a triangle whose three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

*Solution.*

Let the four consecutive integers be denoted by  $n - 2$ ,  $n - 1$ ,  $n$ , and  $n + 1$ , where  $n$  is a natural number with  $n \geq 3$ . Let  $h$  denote the length of the altitude which divides the corresponding side into two parts with integral lengths  $a$  and  $b$ . Then there are four possible configurations, illustrated as follows:

*Case (i):* (See Figure 1). In this case  $a + b = n$  and  $h = n - 2$ . From  $a^2 + h^2 = (n - 1)^2$  and  $b^2 + h^2 = (n + 1)^2$  we get  $(b - a)(b + a) = 4n$  which implies  $b - a = 4$  or  $b = a + 4$ . Hence  $n = 2a + 4$  from which we get

$$\begin{aligned} (2a + 2)^2 &= (n - 2)^2 = h^2 \\ &= (n - 1)^2 - a^2 \\ &= (2a + 3)^2 - a^2 \\ &= 3(a + 3)(a + 1). \end{aligned}$$

Therefore  $4(a + 1) = 3(a + 3)$  which yields  $a = 5$ . It then follows that  $b = 9$ ,  $n = 14$ . Thus we obtain the triangle as displayed in Figure 2.

*Case (ii):* (See Figure 3). In this case  $b^2 - a^2 = n^2 - (n - 1)^2 = 2n - 1$  implies

$$b - a = \frac{2n - 1}{n + 1} = 2 - \frac{3}{n + 1}$$

which is not an integer since  $n + 1 \geq 4$ . Thus this case is impossible.

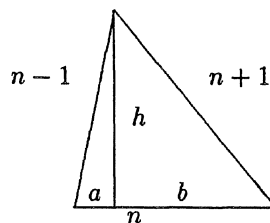


Figure 1

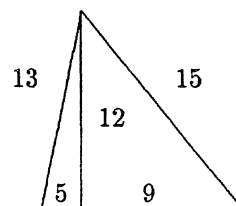


Figure 2

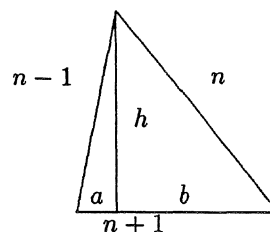


Figure 3

Case (iii) (See Figure 4). In this case  $b^2 - a^2 = (n+1)^2 - n^2 = 2n+1$  implies

$$\begin{aligned} b-a &= \frac{2n+1}{n-1} \\ &= 2 + \frac{3}{n-1}. \end{aligned}$$

Since  $n \geq 3$  we must have  $n = 4$ . Thus  $b-a = 3 = n-1 = b+a$  which yields  $a = 0$ , a contradiction.

Case (iv): (See Figure 5). In this case  $b^2 - a^2 = 2n+1$  implies

$$\begin{aligned} b-a &= \frac{2n+1}{n-2} \\ &= 2 + \frac{5}{n-2}. \end{aligned}$$

If  $n = 3$  then  $b-a = 7 > 1 = b+a$ , a contradiction.

If  $n = 7$  then  $b-a = 3$  and  $b+a = 5$  imply  $a = 1$  which is impossible since  $h = n-1 = 6$  and  $1^2 + b^2 \neq 7^2$ .

Therefore the configuration displayed in Figure 2 is the only such triangle.

**2.** Show that the number  $x$  is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence  $x, x+1, x+2, x+3, \dots$ .

*Solution.*

**Sufficiency:** Suppose  $x+a, x+b$  and  $x+c$  form a geometric progression, where  $a, b$ , and  $c$  are integers such that  $0 \leq a < b < c$ . Then from  $(x+a)(x+c) = (x+b)^2$  we get

$$(a+c)x + ac = 2bx + b^2 \quad \text{or} \quad (a+c-2b)x = b^2 - ac.$$

If  $a+c = 2b$  then  $b^2 = ac = a(2b-a)$  implies  $(a-b)^2 = 0$ , and thus  $a = b$ , a contradiction. (Variation:  $b^2 = ac \leq ((a+c)/2)^2 = b^2$  implies  $a = c$ , a contradiction.) Therefore,  $a+c-2b \neq 0$  and so  $x = (b^2 - ac)/(a+c-2b)$  is a rational number.

**Necessity:** Suppose  $x$  is a rational number. Choose a natural number  $n$  large enough so that  $x+n > 0$ . Let  $x+n = h/k$  where  $h$  and  $k$  are natural numbers. Consider the condition

$$\frac{h}{k} \left( \frac{h}{k} + c \right) = \left( \frac{h}{k} + b \right)^2$$

which simplifies to

$$c = \frac{k}{h} \left( \frac{2bh}{k} + b^2 \right) = 2b + \frac{b^2 k}{h}.$$

Choosing  $b = h$  and  $c = 2h + kh$  we conclude that

$$x+n = \frac{h}{k}, \quad x+n+b = \frac{h(1+k)}{k} \quad \text{and} \quad x+n+c = \frac{h}{k}(1+2k+k^2) = \frac{h}{k}(1+k)^2$$

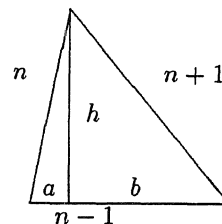


Figure 4

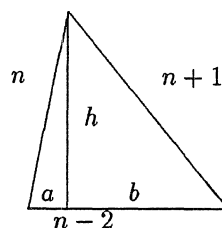


Figure 5

form a geometric progression.

**3.** In triangle  $ABC$ , the medians to the sides  $AB$  and  $AC$  are perpendicular. Prove that  $\cot B + \cot C \geq \frac{2}{3}$ .

*Solution 1.* Let  $BE$  and  $CF$  denote the medians to  $AC$  and  $AB$ , respectively. Let  $H$  denote the centroid of the triangle. (See the Figure.) Let  $HF = x$  and  $HE = y$ . Then  $CH = 2x$  and  $BH = 2y$ . Furthermore, let  $\Theta = \angle CBH$  and  $\Phi = \angle FBH$ . Then

$$\begin{aligned}\cot B &= \cot(\Theta + \Phi) = \frac{\cos \Theta \cos \Phi - \sin \Theta \sin \Phi}{\sin \Theta \cos \Phi + \cos \Theta \sin \Phi} \\ &= \frac{(2y/BC)(2y/BF) - (2x/BC)(x/BF)}{(2x/BC)(2y/BF) + (2y/BC)(x/BF)} \\ &= \frac{4y^2 - 2x^2}{6xy} = \frac{2y^2 - x^2}{3xy}.\end{aligned}$$

(Variation:

$$\cot(\Theta + \Phi) = \frac{\cot \Theta \cot \Phi - 1}{\cot \Theta + \cot \Phi} = \frac{(2y/2x)(2y/x) - 1}{(2y/2x) + (2y/x)} = \frac{2y^2 - x^2}{3xy}.)$$

Similarly,

$$\cot C = \frac{2x^2 - y^2}{3xy}.$$

Hence

$$\cot B + \cot C = \frac{x^2 + y^2}{3xy} \geq \frac{2xy}{3xy} = \frac{2}{3}$$

by the Arithmetic-Geometric-Mean Inequality.

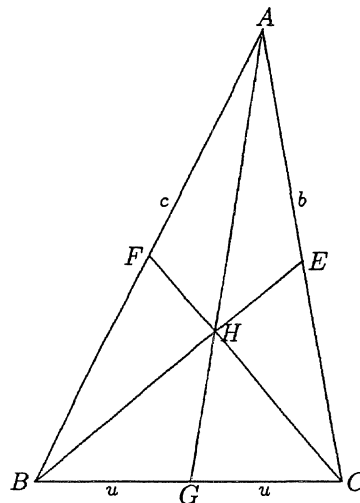
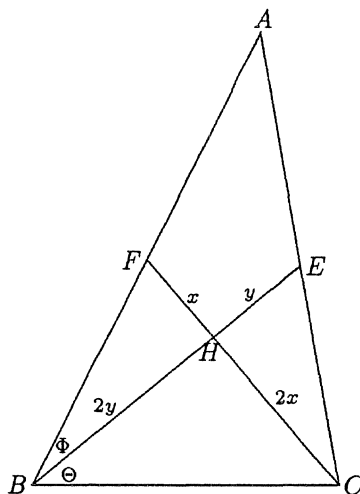
*Solution 2.* Let  $AG$  be the median to  $BC$ , and let  $BC = 2u$ . Furthermore, let  $AC = b$  and  $AB = c$ . (See the Figure.) Since  $\triangle BHC$  is a right triangle  $HG = BC/2 = u$  and thus  $AH = 2u$ . By the Law of Sines we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin A}{2u}$$

and thus

$$\sin B = \frac{b \sin A}{2u} \quad \text{and} \quad \sin C = \frac{c \sin A}{2u}.$$

Hence



$$\begin{aligned}\cot B + \cot C &= \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} = \frac{\sin(B+C)}{\sin B \sin C} = \frac{\sin A}{\sin B \sin C} \\ &= \frac{\sin A}{(bc \sin^2 A)/4u^2} = \frac{2u^2}{(bc \sin A)/2} = \frac{2u^2}{\Delta}\end{aligned}\quad (1)$$

where  $\Delta$  denotes the area of triangle  $ABC$ . Since the height of triangle  $ABC$  with base  $BC$  does not exceed  $AG$  we have

$$\Delta \leq \frac{1}{2}(2u)(3u) = 3u^2. \quad (2)$$

From (1) and (2) we get

$$\cot B + \cot C \geq \frac{2u^2}{3u^2} = \frac{2}{3}.$$

4. A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

*Solution 1.* Let  $b_i$  ( $g_i$ ) denote the number of boys (girls) from the  $i$ th school, and let  $n$  denote the total number of schools. If  $s$  ( $m$ ) denotes the number of singles (mixed singles) played, and for each  $i$  we let  $d_i = b_i - g_i$ , then we have

$$\begin{aligned}s &= \sum_{i < j} (b_i b_j + g_i g_j), & m &= \sum_{i < j} (b_i g_j + g_i b_j), \\ s - m &= \sum_{i < j} (b_i - g_i)(b_j - g_j) = \sum_{i < j} d_i d_j.\end{aligned}$$

The conditions stated in the problem can be written

$$(1) \left| \sum_i d_i \right| \leq 1 \quad \text{and} \quad (2) \left| \sum_{i < j} d_i d_j \right| \leq 1.$$

As

$$\sum_i d_i^2 = \left( \sum_i d_i \right)^2 - 2 \sum_{i < j} d_i d_j \leq 3$$

it follows that at most 3 values of  $d_i$  can be different from 0. Because the  $i$ th school has an odd number of players if and only if  $d_i$  is odd, at most 3 schools therefore can have an odd number of players.

If two schools have 1 girl and 0 boys, and one school has 0 girls and 1 boy, the conditions of the problem are satisfied, showing that it is possible to actually attain the upper bound of 3.

*Solution 2.* We show first that it suffices to consider the case in which all students of each school are of the same sex. Indeed, if some school has both a boy and a girl, then the number of singles matches played by this boy is the same as the number of mixed single matches played by the girl, and vice versa. It follows that deleting both the boy and the girl alters neither of the conditions of the problem.

Now, suppose that as above each school comprises either all girls or all boys, and that  $k$  schools have an odd number of students. Suppose there are, in all,  $B$  boys,  $G$  girls,  $S$  singles, and  $M$  mixed singles, with

$$|B - G| \leq 1 \quad \text{and} \quad |S - M| \leq 1.$$

Then  $M = BG$  and

$$\begin{aligned} -1 &\leq S - M \leq \left[ \frac{1}{2}B(B-1) + \frac{1}{2}G(G-1) \right] - BG \\ &= \frac{1}{2}(B-G)^2 - \frac{1}{2}(B+G) \leq \frac{1}{2}[1 - (B+G)]. \end{aligned}$$

It follows that  $-3 \leq -(B+G)$ , and so  $B+G \leq 3$ . Since each of the  $k$  schools has at least one student,  $k \leq B+G \leq 3$ , and there are at most 3 schools with an odd number of students.

The upper bound  $k = 3$  is attainable: let two schools have 1 girl and 0 boys, and let one school have 0 girls and 1 boy.

**5.** Let  $y_1, y_2, y_3, \dots$  be a sequence such that  $y_1 = 1$  and which, for  $k > 0$ , is defined by the relationship:

$$\begin{aligned} y_{2k} &= \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases} \\ y_{2k+1} &= \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Show that the sequence  $y_1, y_2, y_3, \dots$  takes on every positive integer value exactly once.

*Solution.* Calculating the first few values reveals that

$$\begin{aligned} y_1 &= 1 \\ y_2 &= 2y_1 + 1 = 3 \\ y_3 &= 2y_1 = 2 \\ y_4 &= 2y_2 = 6 \\ y_5 &= 2y_2 + 1 = 7 \\ y_6 &= 2y_3 + 1 = 5 \\ y_7 &= 2y_3 = 4 \text{ etc.} \end{aligned}$$

Note that

$$\{y_1\} = \{1\}, \quad \{y_2, y_3\} = \{2, 3\}, \quad \{y_4, y_5, y_6, y_7\} = \{4, 5, 6, 7\}.$$

Clearly we will be done if we can prove that

$$\{y_{2^{m-1}}, y_{2^{m-1}+1}, \dots, y_{2^m-1}\} = \{2^{m-1}, 2^{m-1} + 1, \dots, 2^m - 1\} \quad (*)$$

each as a set of  $2^{m-1}$  elements. In view of the observations made above we could take  $(*)$  as our induction hypothesis for some  $m \geq 1$ . Consider  $y_k$  where  $k = 2^m + d$ ,  $d = 0, 1, 2, \dots, 2^m - 1$ . Then

$$\left\lfloor \frac{k}{2} \right\rfloor \geq 2^{m-1} \quad \text{and} \quad \left\lfloor \frac{k}{2} \right\rfloor \leq \left\lfloor \frac{2^{m+1} - 1}{2} \right\rfloor = 2^m - 1.$$

Since  $k = 2(k/2) = 2[k/2]$  if  $k$  is even and  $k = 2((k-1)/2) + 1 = 2[k/2] + 1$  if  $k$  is odd we have, in either case,

$$2y_{[k/2]} \leq y_k \leq 2y_{[k/2]} + 1$$

and hence by  $(*)$  we get

$$\{y_{2^m}, y_{2^m+1}, \dots, y_{2^{m+1}-1}\} \subseteq \{2^m, 2^m + 1, \dots, 2^{m+1} - 1\}.$$

It remains to show that the map  $k \rightarrow y_k$  is one-to-one for  $2^m \leq k \leq 2^{m+1} - 1$ . Suppose

$$2^m \leq i, j \leq 2^{m+1} - 1 \quad \text{and} \quad y_i = y_j.$$

Then

$$i = 2k_i + \varepsilon_i \text{ and } j = 2k_j + \varepsilon_j \text{ where } \varepsilon_i, \varepsilon_j \in \{0, 1\} \text{ and } 2^{m-1} \leq k_i, k_j \leq 2^m - 1.$$

It is then clear from the sequence's defining relationship that  $2y_{k_i} = 2y_{k_j}$  which implies  $k_i = k_j$  by  $(*)$ . Furthermore,  $\varepsilon_i = \varepsilon_j$  or else  $|y_i - y_j| = 1$ , a contradiction. Thus  $i = j$  and the proof is complete.

\* \* \*

To finish this month's number we consider solutions to the 8th *Annual University of Michigan Undergraduate Mathematics Competition* [1992: 167].

**1.** Let  $f(x)$  be a continuous function defined on the closed interval  $[0, 1]$ , and suppose that  $f(0) = f(1)$ . Show that there is a real number  $a \in [0, 1/2]$  such that  $f(a) = f(a + 1/2)$ .

*Solutions by Seung-Jin Bang, Albany, California; by Tak Wing Lee, St. Francis of Assisi's College, and Wai Fun Lee, St. Mark's School, Hong Kong; by Beatriz Margolis, Paris, France; and by Robert Sealy, Mount Allison University, Sackville, New Brunswick.*

Define  $g(x) = f(x) - f(x + 1/2)$  on the closed interval  $[0, 1/2]$ . Then  $g(x)$  is continuous,  $g(0) = f(0) - f(1/2)$  and  $g(1/2) = f(1/2) - f(1) = f(1/2) - f(0) = -g(0)$ . If  $g(0)$  and  $g(1/2)$  do not equal 0, then they are opposite in signs. So by the intermediate value theorem, there is  $a \in [0, 1/2]$  such that  $g(a) = 0$ . Then  $f(a) = f(a + 1/2)$ .

**2.** Suppose that  $a_0, a_1, \dots, a_n$  are integers with  $a_n \neq 0$ , and let  $P(x) = a_0 + a_1x + \dots + a_nx^n$ . Suppose that  $x_0$  is a rational number such that  $P(x_0) = 0$ . Show that if  $1 \leq k \leq n$  then

$$a_kx_0 + a_{k+1}x_0^2 + \dots + a_nx_0^{n-k+1}$$

is an integer.

*Solutions by Seung-Jin Bang, Albany, California; and by Pak Kuen Lee, St. Paul's College, Hong Kong. We give Lee's solution.*

We will prove the assertion by induction on  $k$ . The case  $k = 1$  is true because  $a_1x_0 + \dots + a_nx_0^n = -a_0$ . Suppose the assertion is true for  $k = l$ , ( $1 \leq l < n$ ), say  $a_lx_0 + \dots + a_nx_0^{n-l+1} = c_0$ . Let  $x_0 = p/q$  with  $p, q$  integers,  $q \neq 0$ ,  $(p, q) = 1$ . Then

$$p(a_lq^{n-l} + a_{l+1}pq^{n-l+1} + \dots + a_n p^{n-l}) = c_0q^{n-l+1}.$$

Since  $(p, q) = 1$ ,  $p$  divides  $c_0$ . So  $c_0 = pc_1$  for some integer  $c_1$ . Then

$$a_{l+1}x_0 + a_{l+2}x_0^2 + \dots + a_nx_0^{n-l} = \frac{c_0}{x_0} - a_l = qc_1 - a_l$$

is an integer. This completes the induction.

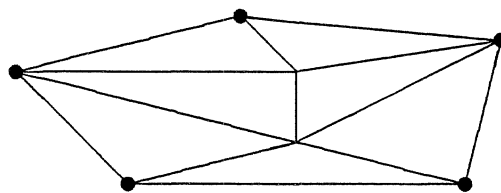
**3.** Let  $A = [a_{ij}]$  be an  $n \times n$  matrix all of whose entries are  $\pm 1$ . Suppose that the various columns of  $A$  are orthogonal to each other in the sense that if  $1 \leq j < k \leq n$  then  $\sum_{i=1}^n a_{ij}a_{ik} = 0$ . Let  $S$  denote the difference between the number of  $+1$ 's in the matrix and the number of  $-1$ 's. Show that  $|S| \leq n^{3/2}$ .

*Solution by the 1992-3 Hong Kong I.M.O. trainees.*

We have  $S = \sum_{i=1}^n \sum_{j=1}^n a_{ij}$ . By the Cauchy-Schwarz inequality

$$\begin{aligned} |S| &\leq \sqrt{\sum_{i=1}^n 1^2} \sqrt{\sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \right)^2} = \sqrt{n} \sqrt{n^2 + 2 \sum_{1 \leq j < k \leq n} \sum_{i=1}^n a_{ij}a_{ik}} \\ &= n^{3/2}. \end{aligned}$$

**4.** Suppose that a polygon with  $n$  vertices is decomposed into a union of finitely many non-overlapping triangles. Show that the number of interior edges is at least  $n - 3$ . In the example depicted,  $n = 5$  and there are 8 interior edges.




*Solution by Kwong Shing Lin, Tsuen Wan Government Secondary School, Hong Kong.*

Consider the process of removing triangles from the decomposition. At any stage, the term "boundary" will refer to the polygon consisting of the exterior edges. Suppose the decomposition of the original polygon with  $n$  vertices has  $n$  exterior edges. Each time we remove a triangle with at least one edge on the boundary. After removal, at least one

exterior edge is taken away and at least one interior edge becomes an exterior edge. After  $k$  removals, the boundary has at least  $n - k$  edges and there are at most  $m - k$  interior edges. When one triangle remains we have  $n - k \leq 3$  and  $0 \leq m - k$ , so  $m \geq n - 3$ .

Alternatively, suppose there are  $n_i$  interior vertices and  $e_i$  interior edges. Then for the entire graph, there are  $v = n + n_i$  vertices,  $e = n + e_i$  edges and  $f = 1/3(n + 2e_i) + 1$  faces (because each interior edge is shared by two triangles and there is one unbounded face). Applying Euler's formula  $v - e + f = 2$  and  $n_i \geq 0$ , we get  $e_i \geq n - 3$ .

[*Editor's note:* Joel Brenner points out that one cannot allow edges to cross, as a "butterfly"  shows.]

**5.** Let  $\mathcal{P}$  denote a convex polygon in the plane whose centroid is  $C$ . For each edge  $e$  of  $\mathcal{P}$  let  $l_e$  denote the line through  $e$ , and let  $C_e$  denote the point on  $l_e$  such that the line through  $C_e$  and  $C$  is perpendicular to  $l_e$ . Show that there is an edge  $e$  of  $\mathcal{P}$  such that  $C_e$  lies on the edge  $e$ , not just on the line through  $e$ .

*Solution by the 1992–3 Hong Kong I.M.O. Trainees.*

By convexity,  $C$  is inside the polygon. Let  $e$  be an edge such that  $C$  is closest to  $l_e$ . Through  $C$  draw a line  $l$  parallel to  $l_e$ . Let  $v$  be the vertex of the polygon on  $e$  that is closer to  $C_e$ ,  $e'$  be the other edge of the polygon containing  $v$ , and  $w$  be the foot of the perpendicular to  $l$  from  $v$ . Now  $C$  and  $e$  must lie on the opposite side of the line through  $v$  and  $w$ . By the convexity of the polygon the angle made by  $e'$  and the ray from  $v$  to  $w$  is less than or equal to  $90^\circ$ . Then  $C$  would be closer to  $e'$  than  $e$ , a contradiction.

**6.** In a group of  $2n$  people, each person has at least  $n$  friends. Show that it is possible to seat these people around a circular table in such a way that each pair of neighbors are friends. (We assume that if  $B$  is a friend of  $A$  then  $A$  is a friend of  $B$ .)

*Comment and solution by the 1992–3 Hong Kong I.M.O. Trainees.*

Identify each person with a vertex of a graph. For each pair of friends, draw an edge connecting the corresponding pair of vertices. Then the degree of each vertex is at least  $n$ . By Dirac's theorem on Hamiltonian graphs (cf. Gary Chartrand and Linda Lesniak, *Graphs & Digraphs*, second edition, Wadsworth/Cole, 1986, p. 187) this graph has a Hamiltonian cycle. Such a cycle corresponds to a desired seating arrangement.

\* \* \*

That completes this month's number of the Corner. Send me your nice solutions, Olympiad contests, and pre-Olympiad material.

\* \* \* \* \*



## BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

*The Wohascum County Problem Book*, by G. T. Gilbert, M. Krusemeyer and L. Larson. Published by the Mathematical Association of America, Washington, 1993. ISBN 0-88385-316-7, paperbound, 244 pages, US\$26.00. *Reviewed by Andy Liu.*

This is Volume 14 in the MAA's Dolciani series, which has gone paperbound since Volume 10. It contains 130 problems at the senior high school and undergraduate level. The authors state that the problems are all original, apart from some variations on classic themes. The title refers to a fictitious county in Minnesota (woebegone are those who try to find it on a map). Many of the problems depict country life there. For easy reference, every problem is followed by the number of the page on which its solution appears. Multiple solutions are often presented. There are two useful appendices. The first gives the prerequisite (and implicitly some hints) for the problems. The second classifies the problems by topics. It is a well-written book which students preparing for contests will find particularly useful.

In the following sample problem, the first solution given in the book is a gem by Jason Colwell, now 15 and entering the third year Honours Mathematics program at the University of Alberta. "Let  $C$  be a circle with centre  $O$ , and  $Q$  a point inside  $C$  different from  $O$ . Where should a point  $P$  be located on the circumference of  $C$  to maximize angle  $OPQ$ ?"

\* \* \* \* \*

## PROBLEMS

*Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (\*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.*

*To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1994, although solutions received after that date will also be considered until the time when a solution is published.*

**1861.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let  $f : \mathbf{R}^+ \rightarrow \mathbf{R}$  be an increasing and concave function from the positive real numbers to the reals. Prove that if  $0 < x \leq y \leq z$  and  $n$  is a positive integer then

$$(z^n - x^n)f(y) \geq (z^n - y^n)f(x) + (y^n - x^n)f(z).$$

**1862.** *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABC$  is an isosceles triangle with  $AB = AC$  and  $\angle A = 120^\circ$ . Let  $P, Q$  be points on sides  $AB, AC$  respectively so that  $PQ$  is tangent to the incircle of  $\triangle ABC$ . Prove that  $BP \cdot CQ$  is equal to twice the area of quadrilateral  $PBCQ$ .

**1863.** *Proposed by Murray S. Klamkin, University of Alberta.*

Are there any integer solutions of the equation

$$(x + y + z)^5 = 80xyz(x^2 + y^2 + z^2)$$

such that none of  $x, y, z$  are 0?

**1864.** *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Consider the three excircles of a given triangle  $ABC$ . Let  $P$  be the radius of the circle containing and internally tangent to these three circles. Prove that  $P \geq 7r$ , where  $r$  is the inradius of  $\triangle ABC$ .

**1865.** *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Find an integer-sided right-angled triangle with sides  $x^2 - 1, y^2 - 1, z^2 - 1$  where  $x, y, z$  are integers.

**1866.** *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Perpendicular chords  $AC, BD$  of a given circle intersect in point  $J$ . Let  $P, Q, R$  be the orthogonal projections of  $J$  onto segments  $AB, BC, CD$  respectively and let  $N$  be the midpoint of  $AD$ . Prove that  $N, P, Q, R$  are concyclic.

**1867.** *Proposed by N. Kildonan, Winnipeg, Manitoba.*

(a) Celebrate Canadian Confederation (July 1, 1867) by finding a 6-digit number CANADA (C, A, N, D distinct decimal digits with  $C \neq 0$ ) which is divisible by 1867.

(b) Prove (not using a computer) that there is no CANADA divisible by 1887.

**1868.** *Proposed by De-jun Zhao, Chengtun High School, Xingchang, China.*

Let  $n \geq 3$ ,  $a_1 > a_2 > \cdots > a_n > 0$ , and  $p > q > 0$ . Show that

$$a_1^p a_2^q + a_2^p a_3^q + \cdots + a_{n-1}^p a_n^q + a_n^p a_1^q > a_1^q a_2^p + a_2^q a_3^p + \cdots + a_{n-1}^q a_n^p + a_n^q a_1^p.$$

**1869.** *Proposed by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

For every positive integer  $n$ , let  $a_n$  be the biggest odd factor of  $n$ . Calculate the sum of the series

$$\frac{a_1}{1^3} + \frac{a_2}{2^3} + \frac{a_3}{3^3} + \cdots.$$

**1870\***. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In any convex pentagon  $ABCDE$  prove or disprove that

$$AC \cdot BD + BD \cdot CE + CE \cdot DA + DA \cdot EB + EB \cdot AC \\ > AB \cdot CD + BC \cdot DE + CD \cdot EA + DE \cdot AB + EA \cdot BC.$$

(Note: the first sum involves diagonals, the second sum involves sides.)

\* \* \* \* \*

## SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**1510\***. [1990: 20; 1991: 91; 1993: 50] *Proposed by Jack Garfunkel, Flushing, New York.*

$P$  is any point inside a triangle  $ABC$ . Lines  $PA, PB, PC$  are drawn and angles  $PAC, PBA, PCB$  are denoted by  $\alpha, \beta, \gamma$  respectively. Prove or disprove that

$$\cot \alpha + \cot \beta + \cot \gamma \geq \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2},$$

with equality when  $P$  is the incenter of  $\triangle ABC$ .

III. *Comment by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

We know from Kuczma's solution on [1991: 91] that the proposed inequality is false. However, we can prove that

$$\cot \alpha + \cot \beta + \cot \gamma \geq \frac{1}{2} \left( \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \right). \quad (1)$$

From [1991: 92] the minimum value of the sum  $\sum \cot \alpha$  is

$$\sum \cot A + 3(\sin A \sin B \sin C)^{-1/3}.$$

Thus, by the known relations

$$\prod \sin A = \frac{sr}{2R^2}, \quad \sum \cot A = \frac{s^2 - r^2 - 4Rr}{2sr}, \quad \sum \cot \frac{A}{2} = \frac{s}{r}$$

(see items (29), (69) and (89), pages 55–60 of [2]), where  $s, R, r$  are the semiperimeter, circumradius, and inradius of the triangle, (1) will follow from

$$\frac{s^2 - r^2 - 4Rr}{2sr} + 3 \left( \frac{2R^2}{sr} \right)^{1/3} \geq \frac{s}{2r}$$

which simplifies to

$$\frac{(6sr)^3 \cdot 2R^2}{sr} \geq r^3(4R + r)^3$$

or

$$3s^2 \geq \frac{r(4R+r)^3}{9 \cdot 16R^2}. \quad (2)$$

Now (2) follows by using

$$3s^2 \geq 3r(16R - 5r) \geq 9r(4R + r) \quad (3)$$

(the first inequality is item 5.8 of [1], the second reduces to  $R \geq 2r$ ); for from (3) we need only prove  $81 \cdot 16R^2 \geq (4R + r)^2$ , i.e.  $36R \geq 4R + r$ , which is obvious.

*References:*

- [1] O. Bottema et al, *Geometric Inequalities*.
- [2] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

\* \* \* \* \*

**1768.** [1992: 207] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let  $S$  be any point in the plane of triangle  $ABC$  different from a vertex.  $S_1, S_2, S_3$  are the feet of the respective perpendiculars from  $S$  to  $BC, CA, AB$ , and  $l_1, l_2, l_3$  are the respective perpendiculars from  $A, B, C$  to the lines  $S_2S_3, S_3S_1, S_1S_2$ .

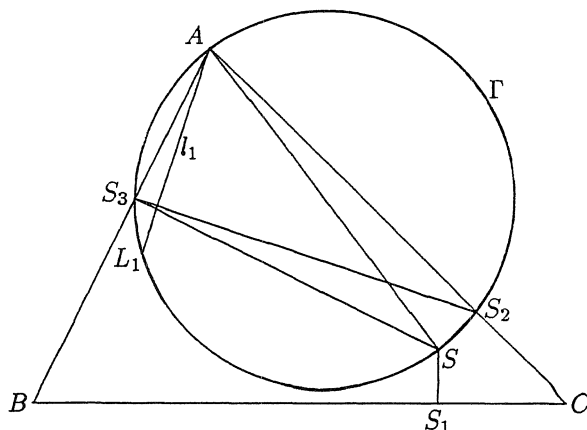
- (a) Show that  $l_1, l_2, l_3$  are concurrent, and
- (b) determine the locus of their common point as  $S$  moves along the Euler line of triangle  $ABC$ .

*I. Solution by Jordan Tabov, Sofia, Bulgaria.*

(a) This part follows immediately from a theorem of Steiner: *If the perpendiculars from the vertices of a triangle  $\Delta_1$  to the sides of a triangle  $\Delta_2$  are concurrent, then the perpendiculars from the vertices of  $\Delta_2$  to the sides of  $\Delta_1$  are concurrent.* [cf. Pedoe's discussion of orthologic triangles in *Geometry, A Comprehensive Course*, §8.3. The result of part (a) also appears as Theorem 237 in R.A. Johnson, *Advanced Euclidean Geometry*, p. 156.]

We shall present another proof, which gives a result necessary for part (b).

Consider the circle  $\Gamma$  with  $AS$  as diameter. Since  $\angle SS_2A = \angle SS_3A = 90^\circ$ , then  $S_2$  and  $S_3$  lie on  $\Gamma$ . Denote by  $L_1$  the point (other than  $A$ ) where  $l_1$  meets  $\Gamma$ . Since  $SS_2 \perp CA$  and  $l_1 \perp S_2S_3$ , then  $\angle S_2AL_1 = \angle SS_2S_3$ . But  $\angle SS_2S_3 = \angle SAS_3$  because they subtend the same arc  $SS_3$ ; hence  $\angle S_2AL_1 = \angle SAS_3$ , which means that the lines  $l_1$  and  $AS$  are isogonal conjugates. Consequently,  $l_1$  passes through the point  $j(S)$ , the isogonal conjugate of  $S$ . Similarly  $l_2$  and  $l_3$  pass through  $j(S)$ , and the  $l_i$  are concurrent as desired.



(b) When  $S$  describes the Euler line  $\varepsilon$ , the points  $j(S)$  describe  $j(\varepsilon)$ , the set of isogonal conjugates of the points of  $\varepsilon$ .

Introduce homogeneous trilinear coordinates with  $\triangle ABC$  as the reference triangle. Then the equation of  $\varepsilon$  is

$$x \sin 2A \sin(B - C) + y \sin 2B \sin(C - A) + z \sin 2C \sin(A - B) = 0.$$

Since  $j$  takes  $(x, y, z)$  to  $(1/x, 1/y, 1/z)$ , the equation of  $j(\varepsilon)$  is

$$yz \sin 2A \sin(B - C) + zx \sin 2B \sin(C - A) + xy \sin 2C \sin(A - B) = 0. \quad (1)$$

This is an equation of a conic passing through the five points  $A, B, C, O, H$ , where  $O$  and  $H$  are the circumcentre and the orthocentre of  $\triangle ABC$ . (Note that  $O$  and  $H$  lie on the Euler line and that they are isogonally conjugate.) Since no three of these five points are collinear,  $j(\varepsilon)$  is nondegenerate; since the convex hull of  $A, B, C$  and  $H$  is a triangle,  $j(\varepsilon)$  is a hyperbola. Thus, the desired locus is the hyperbola (1) through the points  $A, B, C, O, H$ .

II. *Comment by Roland H. Eddy, Memorial University of Newfoundland, St. John's.*

It is known, see [1], that as  $S$  moves along a straight line, the locus of  $j(S)$  is a conic circumscribing the reference triangle. This conic is an ellipse, a parabola, or a hyperbola according as the line intersects the circumcircle of  $ABC$  in none, one, or two points.

Here are some other line and circumscribing conic pairs related by isogonal conjugation. For more details, see [2] or [3].

Line	Circumscribing Conic
Line at infinity	Circumcircle
Lemoine line (line of centres of the circles of Apollonius)	Steiner's circumscribed ellipse (minimum circumscribing ellipse)
Brocard axis (radical axis of the circles of Apollonius)	Kiepert's hyperbola (through, among others, the centroid, the orthocentre, and the Fermat point of $\triangle ABC$ )
Line connecting the incentre and the circumcentre	Feuerbach's hyperbola

#### References:

- [1] H. Eves and C. Kimberling, Problem E2990, solution by O.P. Lossers, *Amer. Math. Monthly* **93** (1986) 132–133.
- [2] R. Eddy and R. Fritsch, The Kiepert conics: A comprehensive lesson in the geometry of the triangle, *Math. Magazine* (to appear).
- [3] R. Eddy and J.B. Wilker, Plane mappings of isogonal/isotomic type, *Soochow J. Math.* **18** (1992) 135–158.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland, St. John's; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

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**1771\***. [1992: 237] Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let  $a, b, c$  be the sides of a triangle and  $u, v, w$  be non-negative real numbers such that  $u + v + w = 1$ . Prove that

$$\sum abc - s \sum vwa \geq 3Rr,$$

where  $s, R, r$  are the semiperimeter, circumradius and inradius of the triangle, and the sums are cyclic.

*Solution by G.P. Henderson, Campbellcroft, Ontario.*

The conditions  $u, v, w \geq 0$  are not needed. Set

$$F = \sum abc - s \sum vwa - 3Rr$$

and replace  $Rr$  by  $abc/4s$ . In  $(u, v, w)$ -space,  $F = 0$  is a quadric with centre  $(a/2s, b/2s, c/2s)$ . Translating the origin to this point, set

$$u = x + \frac{a}{2s}, \quad v = y + \frac{b}{2s}, \quad w = z + \frac{c}{2s},$$

where  $x + y + z = 0$  (since  $u + v + w = 1$ ). Then

$$\begin{aligned} F &= \sum \left(x + \frac{a}{2s}\right) bc - s \sum \left(y + \frac{b}{2s}\right) \left(z + \frac{c}{2s}\right) a - \frac{3abc}{4s} \\ &= \sum xbc + \frac{3abc}{2s} - s \sum ayz - \sum xbc - \frac{3abc}{4s} - \frac{3abc}{4s} = -s \sum ayz \end{aligned}$$

and

$$\begin{aligned} \frac{2F}{s} &= -2 \sum ayz = \sum a[y^2 + z^2 - (y + z)^2] = \sum a(y^2 + z^2 - x^2) \\ &= \sum x^2(b + c - a) \geq 0, \end{aligned}$$

so  $F \geq 0$ . For a non-degenerate triangle there is equality if and only if  $x = y = z = 0$ , i.e.,  $(u, v, w) = (a/2s, b/2s, c/2s)$ .

Also solved by MURRAY S. KLAMKIN, University of Alberta.

\* \* \* \* \*

**1772.** [1992: 237] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

The equation  $x^3 + ax^2 + (a^2 - 6)x + (8 - a^2) = 0$  has only positive roots. Find all possible values of  $a$ .

*Solution by Kee-Wai Lau, Hong Kong.*

We show that  $a = -3$  is the only solution.

It is known that the roots of the equation  $x^3 + ax^2 + bx + c = 0$  are real if and only if

$$a^2b^2 - 4a^3c + 18abc - 4b^3 - 27c^2 \geq 0$$

[e.g., see problem 6.1.9, page 194 of Ed Barbeau's *Polynomials*, Springer-Verlag, 1989]. For  $b = a^2 - 6$  and  $c = 8 - a^2$ , this inequality is equivalent to

$$\begin{aligned} 0 &\leq (a^2 - 6)^2[a^2 - 4(a^2 - 6)] - (8 - a^2)[4a^3 - 18a(a^2 - 6) + 27(8 - a^2)] \\ &= (a^2 - 6)^2 \cdot 3(8 - a^2) + (8 - a^2)(14a^3 + 27a^2 - 108a - 216) \\ &= (8 - a^2)(3a^4 + 14a^3 - 9a^2 - 108a - 108) = (8 - a^2)(a + 3)^2(3a^2 - 4a - 12) \end{aligned}$$

or

$$(a + 3)^2(a^2 - 8)(3a^2 - 4a - 12) \leq 0. \quad (1)$$

By Descartes' rule of signs all the roots are positive only if

$$a < 0, \quad a^2 - 6 > 0, \quad \text{and} \quad 8 - a^2 < 0. \quad (2)$$

[Alternatively, the sum of the roots is  $-a$  and their product is  $-(8 - a^2)$ , so both  $-a$  and  $a^2 - 8$  must be positive.] From (2) we have  $a < -\sqrt{8}$ . Note that  $3a^2 - 4a - 12 > 0$  for  $a < -\sqrt{8}$  [e.g., since the quadratic is positive at  $a = -2$  and negative at  $a = 0$ ], so by (1) we obtain  $a = -3$ . In fact for  $a = -3$  the original equation becomes  $(x - 1)^3 = 0$ , with  $x = 1$  as a triple root.

*Also solved by HARVEY L. ABBOTT, University of Alberta; F.J. FLANIGAN, San Jose State University, San Jose, California; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; SHALESH SHIRALI, Rishi Valley School, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. C. FESTRAETS-HAMOI, Brussels, Belgium, sent in a nearly-correct solution. There were also four incorrect solutions submitted.*

*The handy factorized inequality (1) was also found by Shirali; but in his solution it fell out, after "substantial simplification", from the necessary condition that the given equation must be negative at its relative minimum  $x = (\sqrt{18 - 2a^2} - a)/3$ .*

\* \* \* \*

**1773.** [1992: 237] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABC$  is a triangle inscribed in a circle with radius  $R$ . Let  $L$ ,  $M$  and  $N$  be the midpoints of the arcs  $AB$ ,  $BC$  and  $CA$ , not containing  $C$ ,  $A$  and  $B$ , respectively. Let  $E$  and  $F$  be the feet of the perpendiculars from  $M$  to  $AB$  and  $AC$ , respectively. Suppose that  $\overline{AB} \neq \overline{AC}$  and  $\overline{LE} = \overline{NF}$ . Prove that  $\overline{NF} = R$ .

*Solution by Jordi Dou, Barcelona, Spain.*

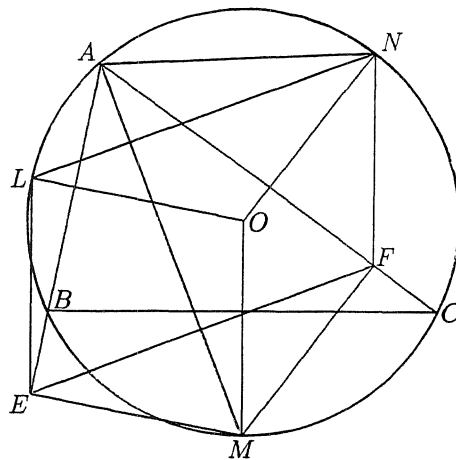
It holds that  $LN \perp AM$  because

$$\text{arc } LA + \text{arc } NM = \text{arc } AN + \text{arc } ML$$

[and therefore

$$\angle ANL + \angle NAM = \frac{1}{2}(\text{arc } AL + \text{arc } NM) = \frac{\pi}{2}$$

—Ed.]. Also  $EF \perp AM$  [because  $MA$  bisects  $\angle A$ , so  $AE = AF$ ]. Therefore  $LN \parallel EF$ . If  $EL = FN$ , then  $EFNL$  is either a parallelogram or an isosceles trapezoid. The perpendicular bisector  $AM$  of the line  $EF$  does not coincide with that of the line  $LN$ , because  $AB \neq AC$  and hence  $AL \neq AN$ . Thus  $EFNL$  is a parallelogram. Let  $O$  be the circumcentre of  $\triangle ABC$ . Triangles  $LON$  and  $EMF$  have parallel sides [ $LO \parallel EM$  because both are perpendicular to  $AB$ , and similarly  $ON \parallel MF$ ], and  $LN = EF$ , so these triangles are congruent, and  $LE$ ,  $OM$  and  $NF$  are equal and parallel.



Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; L.J. HUT, Groningen, The Netherlands; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The proposer and Dou had very similar solutions.

Dou, and in fact most solvers, also determined that the condition in the problem is equivalent to  $\cos B + \cos C = 1$ . As pointed out by Penning, this is the same condition satisfied by the triangles in the proposer's earlier problem Crux 1751 [1993: 148]! What other properties does this interesting class of triangles possess?

\* \* \* \* \*

**1774\***. [1992: 237] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the smallest  $\lambda \geq 0$  such that

$$2(x^3 + y^3 + z^3) + 3xyz \geq (x^\lambda + y^\lambda + z^\lambda)(x^{3-\lambda} + y^{3-\lambda} + z^{3-\lambda})$$

for all non-negative  $x, y, z$ .

*Solution by G.P. Henderson, Campbellcroft, Ontario.*

We will prove that the minimum  $\lambda$  is 1. The left side exceeds the right by

$$F_\lambda(x, y, z) = \sum x^3 + 3xyz - \sum x^\lambda y^{3-\lambda} - \sum x^{3-\lambda} y^\lambda$$

where the sums are cyclic. Setting  $\lambda = 1$ , we get

$$\begin{aligned} F_1 &= \sum x^3 + 3xyz - \sum xy^2 - \sum x^2y \\ &= x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y). \end{aligned} \tag{1}$$



Let the notation be chosen so that  $x \geq y \geq z$ . If we replace  $x - z$  in the first term by  $y - z$ ,  $F_1$  can only decrease. This gives

$$F_1 \geq (y - z)[(x - y)^2 + z(x - z)] \geq 0.$$

For  $1 \leq \lambda \leq 2$ , set

$$\begin{aligned} G &= \sum xy^2 + \sum x^2y - \sum x^\lambda y^{3-\lambda} - \sum x^{3-\lambda} y^\lambda \\ &= \sum xy(x^{\lambda-1} - y^{\lambda-1})(x^{2-\lambda} - y^{2-\lambda}). \end{aligned}$$

Each of the three terms is nonnegative. Therefore  $G \geq 0$  and

$$F_\lambda = F_1 + G \geq 0.$$

For  $0 \leq \lambda < 1$  we will show that  $F_\lambda$  takes on negative values. Consider

$$F_\lambda(x, 1, 1) = x^3 + 3x - 2x^\lambda - 2x^{3-\lambda}.$$

When  $x$  is small, the dominant terms are the second and third. If  $x$  is chosen so that these terms cancel, the last term will dominate and  $F_\lambda$  will be negative. Thus set  $x = (2/3)^{1/(1-\lambda)}$ . Then

$$F_\lambda = \left(\frac{2}{3}\right)^{3/(1-\lambda)} \left[1 - 2\left(\frac{3}{2}\right)^{\lambda/(1-\lambda)}\right] < 0.$$

*Also solved by MANUEL BENITO MUÑOZ, I.B. Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

*The proposer proved that the inequality is valid for  $\lambda = 1$ ; he first obtained the expression (1) above, then noted that its being positive is just the case  $n = 1$  of Schur's inequality.*

\* \* \* \* \*

**1775.** [1992: 237] *Proposed by P. Penning, Delft, The Netherlands.*

Find the radius of the smallest sphere (in three-dimensional space) which is tangent to the three lines  $y = 1, z = -1$ ;  $z = 1, x = -1$ ;  $x = 1, y = -1$ ; and whose centre does *not* lie on the line  $x = y = z$ .

*Solution by Kee-Wai Lau, Hong Kong.*

We show that the required radius is  $5\sqrt{2}$ .

Let the equation of the sphere be  $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ . Since the sphere is tangent to the lines  $y = 1, z = -1$ ;  $z = 1, x = -1$ ;  $x = 1, y = -1$ , we have

$$(1 - b)^2 + (1 + c)^2 = r^2, \tag{1}$$

$$(1 - c)^2 + (1 + a)^2 = r^2, \tag{2}$$

$$(1 - a)^2 + (1 + b)^2 = r^2. \quad (3)$$

[For example, to get equation (1) note that the point where the sphere touches the line  $y = 1, z = -1$  must be  $(a, 1, -1)$ .—*Ed.*] From (1) and (2) we obtain

$$4c + b^2 - 2b - a^2 - 2a = 0, \quad (4)$$

and from (1) and (3) we obtain

$$4b - c^2 - 2c + a^2 - 2a = 0. \quad (5)$$

Let  $s = (b - c)/2$  and  $t = (b + c)/2$ , so that  $4st = b^2 - c^2$  and  $2s^2 + 2t^2 = b^2 + c^2$ . By adding (4) and (5) we obtain  $4st + 4t - 4a = 0$  or

$$t + st = a, \quad (6)$$

and by subtracting (5) from (4) we obtain  $2s^2 + 2t^2 - 12s - 2a^2 = 0$  or

$$s^2 + t^2 - 6s = a^2. \quad (7)$$

If  $s = 0$  then by (6) we have  $a = t = b = c$ , which we avoid. For  $s \neq 0$  we obtain from (6) and (7) that

$$s^2 - 6s + \frac{a^2}{(1 + s)^2} = a^2,$$

so

$$s(s + 2)a^2 = [(1 + s)^2 - 1]a^2 = s(s - 6)(1 + s)^2$$

or

$$a^2 = \frac{(s - 6)(s + 1)^2}{s + 2}, \quad (8)$$

where  $s < -2$  or  $s \geq 6$  (since  $a^2 \geq 0$ ). Now by (1), (7) and (8) we have

$$\begin{aligned} r^2 &= 2 + b^2 + c^2 + 2(c - b) = 2(1 + s^2 + t^2 - 2s) \\ &= 2 \left( 1 - 2s + 6s + \frac{(s - 6)(s + 1)^2}{s + 2} \right) = 2 \left( 1 + 4s + 1 - 6s + s^2 - \frac{8}{s + 2} \right) \\ &= 2 \left( 2 + s^2 - 2s - \frac{8}{s + 2} \right). \end{aligned}$$

By differentiation it is easy to check that  $r^2$  attains its minimum value of 50 when  $s = -3$  or 6. [*Editor's note.* In fact

$$\frac{d}{ds} \left( \frac{r^2}{2} \right) = 2s - 2 + \frac{8}{(s + 2)^2},$$

and it is clear that this is positive, i.e.  $r^2$  is increasing, for  $s \geq 6$ , while it easily turns out that in the interval  $s < -2$  there is a relative minimum at  $s = -3$ .]

*Also solved by JORDI DOU, Barcelona, Spain; and the proposer.*

\* \* \* \* \*

**1776.** [1992: 238] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Given  $0 < x_0 < 1$ , the sequence  $x_0, x_1, \dots$  is defined by

$$x_{n+1} = \frac{3}{4} - \frac{3}{2} \left| x_n - \frac{1}{2} \right|$$

for  $n \geq 0$ . It is easy to see that  $0 < x_n < 1$  for all  $n$ . Find the smallest closed interval  $J$  in  $[0, 1]$  so that  $x_n \in J$  for all sufficiently large  $n$ .

*Solution by Shailesh Shirali, Rishi Valley School, India.*

We show that  $J = [3/8, 3/4]$ .

Instead of the sequence  $\{x_n\}$ , we shall work with the sequence  $\{v_n\}$ , where  $v_n = 3 - 4x_n$ . Then  $-1 < v_0 < 3$  and

$$v_{n+1} = \frac{3}{2} |1 - v_n|$$

[since the recurrence for the  $x$ 's can be written  $3 - 4x_{n+1} = \frac{3}{2} |(1 - (3 - 4x_n))|$ ].

Let  $f$  denote the function from the closed interval  $[-1, 3]$  into itself defined by  $f(v) = \frac{3}{2} |1 - v|$ . For closed intervals  $I$  within  $[-1, 3]$ , let  $I$  be said to have *property P* if  $f(I)$  is a subset of  $I$ , and to have *property Q* if for each  $v_0$  lying in the open interval  $(-1, 3)$  there exists a positive integer  $n$  (depending on  $v_0$ ) for which  $v_n$  lies in  $I$ . We require the smallest closed interval  $I$  that has both the properties  $P$  and  $Q$ . We shall show that this is the interval  $[0, 3/2]$ .

To begin with, let us locate the closed intervals that have property  $P$ . Let  $I = [a, b]$  where  $-1 \leq a \leq b \leq 3$ .

*Case 1:*  $a \leq b \leq 1$ . For  $f(I) \subseteq I$ , the inequalities to be solved are

$$a \leq \frac{3}{2}(1 - b) \leq \frac{3}{2}(1 - a) \leq b$$

which simplifies to

$$3 \leq 3a + 2b \leq 2a + 3b \leq 3$$

or  $a = b = 3/5$ , so the only such interval is  $[3/5, 3/5]$ .

*Case 2:*  $a \leq 1 \leq b$  and  $a + b \leq 2$ . Here  $f(a) = \frac{3}{2}(1 - a)$ ,  $f(1) = 0$ , and  $f(b) = \frac{3}{2}(b - 1)$ , and so  $f[a, 1] = [0, \frac{3}{2}(1 - a)]$  and  $f[1, b] = [0, \frac{3}{2}(b - 1)]$ . Also  $1 - a \geq b - 1$ , so  $f[a, b] = [0, \frac{3}{2}(1 - a)]$ , and the inequalities to be solved are

$$a \leq 0 \leq \frac{3}{2}(1 - a) \leq b,$$

i.e.  $a \leq 0$  and  $3a + 2b \geq 3$  (and  $a + b \leq 2$ ). This set of inequalities defines a triangular region in the  $a, b$  plane, with vertices at  $(-1, 3)$ ,  $(0, 2)$  and  $(0, 3/2)$  as shown in Figure 1. Any closed interval  $[a, b]$ , where the point  $(a, b)$  lies inside or on the boundary of this region, satisfies property  $P$ .

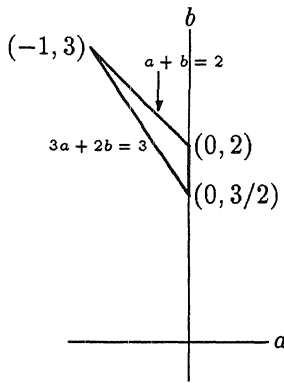


Figure 1

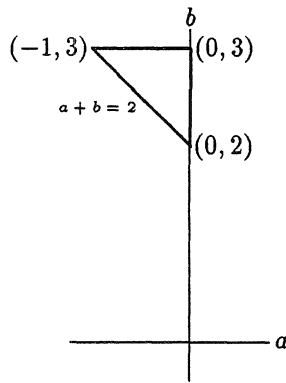


Figure 2

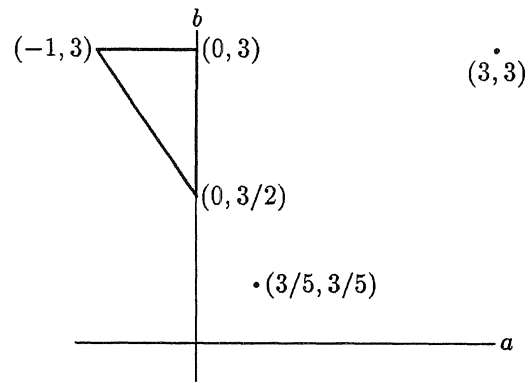


Figure 3

Case 3:  $a \leq 1 \leq b$  and  $a + b \geq 2$ . In this case  $b - 1 \geq 1 - a$ , so the inequality string becomes

$$a \leq 0 \leq \frac{3}{2}(b - 1) \leq b,$$

i.e.  $a \leq 0$  and  $1 \leq b \leq 3$  (and  $a + b \geq 2$ ). Once again we obtain a triangular region in the  $a, b$  plane, this time with vertices  $(-1, 3)$ ,  $(0, 2)$  and  $(0, 3)$ . (Figure 2)

Case 4:  $1 \leq a \leq b$ . For this situation we must have

$$a \leq \frac{3}{2}(a - 1) \leq \frac{3}{2}(b - 1) \leq b,$$

or  $3 \leq a \leq b \leq 3$ , so the only solution to this is  $a = b = 3$ ; i.e., the only interval with property  $P$  in this situation is  $[3, 3]$ .

Bringing together the above results, we see that the closed intervals that have property  $P$  are precisely the following:  $[3/5, 3/5]$ ,  $[3, 3]$ , and all  $[a, b]$  for which the point  $(a, b)$  in the  $a, b$  plane falls within the closed triangular region whose vertices are  $(-1, 3)$ ,  $(0, 3)$  and  $(0, 3/2)$ . (Figure 3)

We must now check for property  $Q$ . Obviously  $[3/5, 3/5]$  and  $[3, 3]$  do not have property  $Q$ . [Editor's note. Shirali's reason for this was slightly flawed. However, note that all  $v_n$ 's are rational if and only if  $v_0$  is rational, so choosing  $v_0$  to be any irrational shows that neither of these one-point intervals (or similarly any other one-point interval) can have property  $Q$ .] We shall show that  $[0, 3/2]$  has property  $Q$ , which will immediately identify it as the interval we seek, as it also happens to be set-theoretically minimal amongst all the remaining intervals that have property  $P$ .

Note that  $f(v) \geq 0$  for all  $v$  in  $[-1, 3]$ , which means that  $v_n \geq 0$  for all  $n \geq 1$ , irrespective of the choice of  $v_0$ .

Next, if  $v_n > 3/2$ , then  $v_{n+1} = f(v_n) = \frac{3}{2}(v_n - 1)$  and so

$$3 - v_{n+1} = 3 - \frac{3}{2}v_n + \frac{3}{2} = \frac{3}{2}(3 - v_n),$$

which shows that there is a sort of "repulsion" away from the point  $v = 3$ . It immediately follows that for some  $n$ ,  $v_n$  must belong to  $[0, 3/2]$ .

Thus for the  $v$ -sequence, the required interval is  $[0, 3/2]$ . For the given  $x$ -sequence, this translates into the interval  $[3/8, 3/4]$ .

*Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; and the proposer. None of these proved the minimality of  $J = [3/8, 3/4]$  quite as convincingly as Shirali, though.*

\* \* \* \* \*

**1777.** [1992: 238] *Proposed by Guy N. Bloodsucker, Stirling Pound, Banffshire, Scotland.*

Find all positive integer solutions of  $x^2 + y^2 = n!$ .

*Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*

There are two solutions:

$$1^2 + 1^2 = 2! \quad \text{and} \quad 24^2 + 12^2 = 6!.$$

First check that these are the only solutions for  $n \leq 6$ . Then note the following results:

(i) for any positive integer  $a$ , if a prime  $p$  with  $p \equiv 3 \pmod{4}$  divides into  $a$ , but  $p^2$  doesn't divide into  $a$ , then  $a \neq x^2 + y^2$  for any integers  $x, y$ ;

(ii) between  $n/2$  and  $n$  exists at least one prime  $p$  with  $p \equiv 3 \pmod{4}$ , for any  $n > 6$ .

These imply that there are no solutions for  $n > 6$  [since the prime  $p$  in (ii) will divide exactly once into  $n!$ ].

*Editor's note.* All solvers of this problem gave this solution. Result (i) above is pretty familiar, and is available in any number theory text that includes the characterization of which integers are the sums of two squares. But result (ii) is much harder to locate, and to prove, it seems! It is due to R. Breusch [1] in 1932, and is mentioned in the recent and more accessible paper [5]. Breusch proves in fact that the same thing holds (with varying lower bounds on  $n$ ) for primes  $\equiv 1 \pmod{4}$ ,  $\equiv 1 \pmod{3}$  and  $\equiv 2 \pmod{3}$ . Slightly stronger results are in [2] (due to Erdős) and also in [4]. In fact Erdős may have solved this problem long ago! See [3]. (For the number theory experts in the audience: is there a more general result known? That is, for any relatively prime  $a$  and  $b$  is there always a prime  $\equiv a \pmod{b}$  between  $n/2$  and  $n$  for sufficiently large  $n$ ?)

Both the proposer and the editor had hoped that this problem would have an easier solution! Apologies are offered for the too-tough result necessary to solve this problem.

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- [2] P. Erdős, Über die Primzahlen gewisser arithmetischer Reihen, *Math. Zeitschrift* **39** (1935), 473–491. (*Z. für M.* 10, p. 293)

- [3] P. Erdős and R. Obláth, Über diophantische Gleichungen der Form  $n! = x^p \pm y^p$  und  $n! \pm m! = x^p$ , *Acta Litt. Sci. Szeged* **8** (1937), 241–255. (*Z. für M.* 17, p. 4)
- [4] K. Molsen, Zur Verallgemeinerung des Bertrandschen Postulates, *Deutsche Math.* **6** (1941), 248–256. (MR 8, 197)
- [5] P. Schumer and J. Steinig, On the incongruence of consecutive fourth powers, *Elemente der Math.* **43** (1988), no. 5, 145–149.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer.

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**1778.** [1992: 238] Proposed by Marcin E. Kuczma, Warszawa, Poland.

Find all real numbers  $x_1, x_2, \dots, x_{1778}$  satisfying the system of equations

$$3 + 2x_{i+1} = 3|x_i - 1| - |x_i|, \quad i = 1, 2, \dots, 1778 \quad (x_{1779} = x_1).$$

*Solution by Chris Wildhagen, Rotterdam, The Netherlands.*

Let

$$F(x) = \frac{3}{2}|x - 1| - \frac{1}{2}|x| - \frac{3}{2} = \begin{cases} -x & \text{if } x \leq 0, \\ -2x & \text{if } 0 \leq x \leq 1, \\ x - 3 & \text{if } x \geq 1. \end{cases}$$

For all  $x_1 \in \mathbb{R}$  we define (inductively) the sequence  $(x_i)_{i \geq 1}$  by  $x_{i+1} = F(x_i)$ ,  $i \geq 1$ . This extends the finite sequence as given by the problem. With the notation  $x \xrightarrow{l} y$  we mean that  $l$  successive applications of  $F$  map  $x$  onto  $y$ ;  $x \xrightarrow{1} y$  is simplified to  $x \rightarrow y$ . Further, let  $\mathcal{J} = [-2, -1] \cup [1, 2]$ .

*Claim 1.* If  $x_1 = 3k$ ,  $k \in \mathbb{Z}$ , then  $x_i = 0$  for  $i \geq |k| + 2$ .

*Proof.*

$$\begin{aligned} k > 0 &\Rightarrow 3k \rightarrow 3(k-1) \rightarrow 3(k-2) \rightarrow \cdots \rightarrow 0, \text{ hence } 3k \xrightarrow{k} 0. \\ k = 0 &\Rightarrow x_1 = 0 \rightarrow 0. \\ k < 0 &\Rightarrow 3k \rightarrow -3k \xrightarrow{-k} 0. \quad \square \end{aligned}$$

*Claim 2.* Let  $x_1 \in \mathcal{J}$ . Then the sequence  $(x_i)_{i \geq 1}$  has period

$$\begin{cases} 4, & \text{if } x_1 \neq \pm 3/2; \\ 2, & \text{if } x_1 = \pm 3/2. \end{cases}$$

*Proof.*

$$x_1 \in [1, 2] \Rightarrow x_1 \rightarrow x_1 - 3 \rightarrow -x_1 + 3 \rightarrow -x_1 \rightarrow x_1,$$

hence if  $x_1 \neq -x_1 + 3$ , i.e.  $x_1 \neq 3/2$ , then  $(x_i)$  has period 4, else the period is 2.

$$x_1 \in [-2, -1] \Rightarrow x_1 \rightarrow -x_1 \rightarrow -x_1 - 3 \rightarrow x_1 + 3 \rightarrow x_1,$$

hence if  $x_1 \neq -x_1 - 3$ , i.e.  $x_1 \neq -3/2$ , then  $(x_i)$  has period 4, else the period is 2.  $\square$

*Claim 3.* Let  $x_1 \in \mathbb{R} \setminus \{3k | k \in \mathbb{Z}\}$ . Then  $x_i \in \mathcal{J}$  for  $i$  sufficiently large.

*Proof.* We distinguish 5 cases.

- (i)  $x_1 \in \mathcal{J}$ . Then  $x_i \in \mathcal{J}$  for all  $i \geq 1$ , as can easily be seen from the proof of Claim 2.
- (ii)  $x_1 \in (0, 1)$ . Choose  $n \in \mathbb{N}$  minimal such that  $2^n x_1 \geq 1$ . Then

$$x_1 \xrightarrow{2} 2x_1 \xrightarrow{2} \cdots \xrightarrow{2} 2^n x_1 \in [1, 2).$$

Now apply case (i).

- (iii)  $x_1 \in (-1, 0)$ .  $x_1 \rightarrow -x_1 \in (0, 1)$ ; see case (ii).

- (iv)  $x_1 > 2$ . Choose  $l \in \mathbb{N}$  minimal such that  $x_1 - 3l \leq 2$ . Then

$$x_1 \xrightarrow{l} x_1 - 3l \in (-1, 2] \setminus \{0\};$$

now see cases (i), (ii) and (iii).

- (v)  $x_1 < -2$ . Then  $x_1 \rightarrow -x_1 \in (2, \infty)$ ; see case (iv).  $\square$

Let us call  $x_1$  *good* if  $x_1 \xrightarrow{1778} x_{1779} = x_1$ . 0 is good since  $0 \rightarrow 0$ . The numbers  $3k$  with  $k \in \mathbb{Z} \setminus \{0\}$  are not good, as follows easily from Claim 1. Now take any  $x_1 \in \mathbb{R} \setminus \{3k | k \in \mathbb{Z}\}$ . By Claim 3,  $x_1$  is not good if  $x_1 \notin \mathcal{J}$ . Finally let  $x_1 \in \mathcal{J}$ . If  $x_1$  is good then the period of the sequence  $(x_i)$  is a divisor of 1778. Since  $1778 \equiv 2 \pmod{4}$  we deduce, using Claim 2, that  $x_1$  is good  $\Leftrightarrow x_1 = \pm 3/2$ . Summing up, the only good numbers are 0,  $3/2$  and  $-3/2$ . The iterates of  $x_1$  are in each case:

$x_1$	$\rightarrow$	$x_2$	$\rightarrow$	$x_3$	$\rightarrow$	$\cdots$	$\rightarrow$	$x_{1778}$
0	$\rightarrow$	0	$\rightarrow$	0	$\rightarrow$	$\cdots$	$\rightarrow$	0
$\frac{3}{2}$	$\rightarrow$	$-\frac{3}{2}$	$\rightarrow$	$\frac{3}{2}$	$\rightarrow$	$\cdots$	$\rightarrow$	$-\frac{3}{2}$
$-\frac{3}{2}$	$\rightarrow$	$\frac{3}{2}$	$\rightarrow$	$-\frac{3}{2}$	$\rightarrow$	$\cdots$	$\rightarrow$	$\frac{3}{2}$

*Also solved by HARVEY L. ABBOTT, University of Alberta; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer. SEUNG-JIN BANG, Albany, California, missed only the trivial solution  $x_i = 0$  for all  $i$ . There were two incorrect solutions sent in.*

\* \* \* \* \*

**1779.** [1992: 238] *Proposed by Christopher Bradley, Clifton College, Bristol, U.K.*

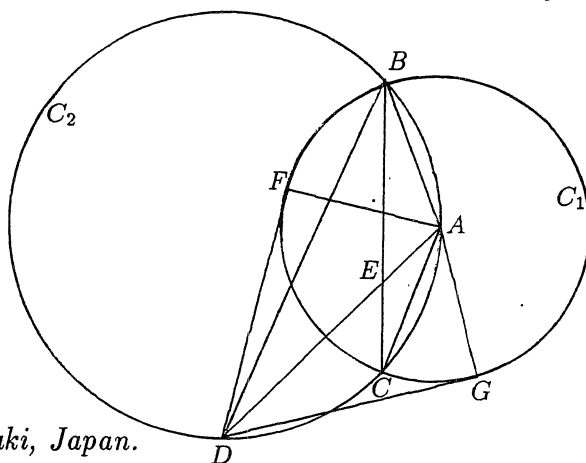
Two circles  $C_1$  and  $C_2$  are given with the centre  $A$  of circle  $C_1$  lying on  $C_2$ .  $BC$  is the common chord. The chord  $AD$  of  $C_2$  meets  $BC$  at  $E$ . From  $D$  lines  $DF$  and  $DG$  are drawn tangent to  $C_1$  at  $F$  and  $G$ . Prove that  $E, F, G$  are collinear.

I. *Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*

I consider inversion with respect to the circle  $C_1$ . Reflected through  $C_1$ ,

- the circle  $C_2$  becomes the line  $BC$ ;
- the circle  $AFDG$  becomes the line  $FG$ ;
- the point  $D$  becomes the point  $E$ .

Since  $D$  lies on the circles  $D_1$  and  $AFDG$  and the line  $AD$ ,  $E$  will lie on the lines  $BC$ ,  $FG$  and  $AD$ .



II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Since  $B, A, C, D$  are concyclic, and  $AB = AC$ ,

$$\angle ADB = \angle ACB = \angle ABE.$$

Therefore  $[\triangle ABE \sim \triangle ADB$ , so  $AB/AD = AE/AB$ , and thus]

$$(AF)^2 = (AB)^2 = AE \cdot AD.$$

Furthermore  $\angle AFD = 90^\circ$ , so we get  $FE \perp AD$  [because, for example,

$$\frac{AE}{AF} = \frac{AF}{AD} = \cos \angle FAD$$

forces  $\angle FEA = 90^\circ$ ]. Similarly we have  $GE \perp AD$ , hence  $E, F, G$  are collinear.

[*Editor's Note.* Seimiya also gave a second proof, related to solution I, but simply using the fact that the three common chords of the circles  $C_1$ ,  $C_2$  and  $AFDG$  (taken in pairs) must concur.]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SEUNG-JIN BANG, Albany, California; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTAETS-HAMMOIR, Brussels, Belgium; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Several solutions were like Solution I, or related to it. A couple were similar to Solution II.

The proposer found the problem, attributed to J. Wolstenholme, in *Mathematical Problems*, Macmillan and Co., 1891.



\* \* \* \*

**1780.** [1992: 238] *Proposed by Jordan Stoyanov, Queen's University, Kingston, Ontario.*

Prove that, for any natural number  $n$  and real numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$ ,

$$(1 - \sin^2 \alpha_1 \sin^2 \alpha_2 \dots \sin^2 \alpha_n)^n + (1 - \cos^2 \alpha_1 \cos^2 \alpha_2 \dots \cos^2 \alpha_n)^n \geq 1.$$

*Combination of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and Murray S. Klamkin, University of Alberta.*

If we let  $\sin^2 \alpha_i = p_i$  and  $\cos^2 \alpha_i = q_i$ , then the inequality becomes

$$(1 - p_1 p_2 \dots p_n)^n + (1 - q_1 q_2 \dots q_n)^n \geq 1, \quad (1)$$

where the  $p$ 's and  $q$ 's satisfy  $0 \leq p_i, q_i \leq 1$  and  $p_i + q_i = 1$ . A related inequality problem is #5 of the 1984 Bulgarian Mathematical Olympiad, i.e., prove that

$$(1 - p_1 p_2 \dots p_n)^m + (1 - q_1^m)(1 - q_2^m) \dots (1 - q_n^m) \geq 1 \quad (2)$$

where  $m, n$  are natural numbers and again  $0 \leq p_i, q_i \leq 1$  and  $p_i + q_i = 1$  for all  $i$ . On [1991: 38] it is noted that (2) is a special case of the more general inequality

$$\prod_{i=1}^n \left( 1 - \prod_{j=1}^m p_{ij} \right) + \prod_{j=1}^m \left( 1 - \prod_{i=1}^n q_{ij} \right) \geq 1 \quad (3)$$

where  $p_{ij} + q_{ij} = 1$ ,  $0 \leq p_{ij} \leq 1$ , and  $m, n$  are natural numbers. We now show that (2) for  $m = n$  is a stronger inequality than (1), i.e.,

$$(1 - q_1 q_2 \dots q_n)^n \geq (1 - q_1^n)(1 - q_2^n) \dots (1 - q_n^n) \quad (4)$$

and so (1) will follow. The proof of (4) uses the A.M.-G.M. inequality twice:

$$\left[ \prod_{i=1}^n (1 - q_i^n) \right]^{1/n} \leq \frac{1}{n} \sum_{i=1}^n (1 - q_i^n) = 1 - \frac{1}{n} \sum_{i=1}^n q_i^n \leq 1 - \left( \prod_{i=1}^n q_i^n \right)^{1/n} = 1 - \prod_{i=1}^n q_i.$$

There is equality in (1) if and only if the  $p_i$ 's are all 0 or all 1.

*Also solved by SEUNG-JIN BANG, Albany, California; KEE-WAI LAU, Hong Kong; and the proposer. A further reader gave inequality (3) without showing how it solved the problem.*

\* \* \* \*

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