# Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- Why Twin Primes are Scarce
- *n*-Volumes
- Handcrafted Polyhedral Symmetry
- Dual Divisors

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# From the Editor

## War Stories from Applied Math

Projects are the in-thing nowadays. All teachers are familiar with the glaze that comes over the faces of their audience as they expound on the blackboard (or overhead projector or computer screen) what is to them a beautiful piece of mathematics. But get students working on a topic for themselves, discovering it, searching it out, writing it up, expounding it, and the students' interest comes alive. That is the theory. But does it work? And what sort of projects can be set? And is it possible to set problems which a firm wants solving rather than an exercise whose solution is already known? This volume, with its somewhat misleading title, addresses this issue.

Academics in a number of US institutions describe how they have set up links with firms, commercial, and academic institutions (usually their own!). Their 'clients' have proposed a problem of current interest to them, which has then been set to a group of undergraduate, not graduate, students. The results have been surprising. Not the failure you might expect. Work of use to the clients has usually been the outcome. And the students have been stimulated.

Various case-studies are given in the sort of problems which have been tackled successfully. One notable one is in the application of algorithms for finding the shortest routes. The sort of pitfalls into which such a project can fall are spelled out—hence the title. The conclusion is that such a project is not only possible but of value to both the students and the clients, who may even pay a fee! So, if you are in a position to plan a curriculum, how about having a go? You might want to read this volume first.

### Reference

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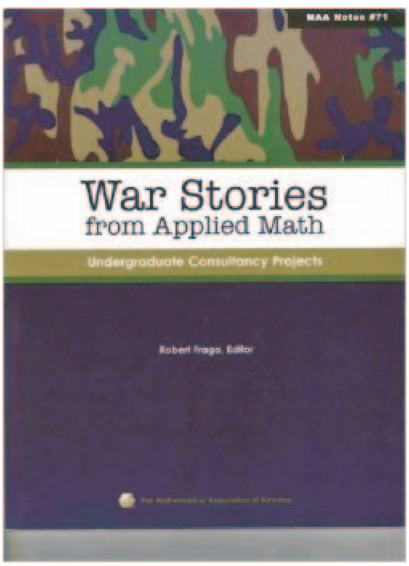

# Fibonacci Identities via Some Simple Expansions

MARTIN GRIFFITHS

In this article we show how two expansions involving the golden ratio lead to a quartet of related Fibonacci identities. What is intriguing here is that, despite the apparent simplicity of these expansions, the sequences that result enumerate relatively complex combinatorial objects.

## 1. Introduction

The inspiration for this article came whilst I was going through some straightforward problems involving the binomial theorem (see reference 1) with my students. At one point, I asked them to simplify the following numerical expression:

$$(1+\sqrt{2})^5 + (1-\sqrt{2})^5. \tag{1}$$

Here is a typical solution,

$$\begin{aligned}(1+\sqrt{2})^5 + (1-\sqrt{2})^5 &= \sum_{k=0}^{5}\binom{5}{k}(\sqrt{2})^k + \sum_{k=0}^{5}\binom{5}{k}(-\sqrt{2})^k \\ &= \sum_{k=0}^{5}\binom{5}{k}[(\sqrt{2})^k + (-\sqrt{2})^k] \\ &= 2\left[\binom{5}{0} + 2\binom{5}{2} + 2^2\binom{5}{4}\right] \\ &= 82.\end{aligned}$$

As my mind wandered, I considered replacing each $\sqrt{2}$ and each 5 in (1) with the golden ratio $\phi$ and $n$, respectively. It occurred to me that, because of the intimate relationship between the golden ratio and the Fibonacci numbers, this might lead to some interesting identities involving these numbers and the binomial coefficients. My explorations did in fact produce identities that do not appear on the authoritative Fibonacci-number website (see reference 2) but do give rise to sequences to be found in Sloane's *On-Line Encyclopedia of Integer Sequences* (http://oeis.org/).

By using some reasonably well-known properties of the golden ratio, we show in this article how two of these identities are derived from the expansion of $(1+\phi)^n + (1-\phi)^n$. The expansion of a related expression leads to two further identities. In the following section the Fibonacci numbers and the golden ratio are defined, and a number of results linking these numbers are given. We then go on to obtain our identities and look at some of the interesting properties of the resulting sequences.

## 2. Some preliminaries

The $n$th *Fibonacci number* $\mathrm{F}_n$ may be defined by way of the recurrence relation $\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}$ for $n \geq 2$, where $\mathrm{F}_0 = 0$ and $\mathrm{F}_1 = 1$ (see references 3, 4, and 5). This definition can be extended to negative subscripts. For example, $\mathrm{F}_{-1} = \mathrm{F}_1 - \mathrm{F}_0 = 1$, $\mathrm{F}_{-2} = \mathrm{F}_0 - \mathrm{F}_{-1} = -1$, and so on. In fact, it is straightforward to show that, for any $n \in \mathbb{N}$,

$$\mathrm{F}_{-n} = (-1)^{n+1}\mathrm{F}_n. \tag{2}$$

In order to obtain some of the results in this article, we will use various mathematical properties of the *golden ratio* $\phi$ (see references 2, 5, and 6), where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

First, it is easily checked that

$$1 + \phi = \phi^2 \tag{3}$$

and

$$1 - \phi = -\frac{1}{\phi}. \tag{4}$$

It is also true that

$$\phi^k = \mathrm{F}_k\phi + \mathrm{F}_{k-1}. \tag{5}$$

This result is given in references 2 and 5, and it is in fact a straightforward matter to prove it by induction using (3), noting, by virtue of (2), that it is also valid when $k$ is a negative integer.

In addition, we shall have cause to use the following simple lemma concerning irrational numbers in general.

**Lemma 1** *For any irrational number $\alpha$ and $m, n, r, s \in \mathbb{Q}$, $m\alpha + r = n\alpha + s$ if and only if $m = n$ and $r = s$.*

*Proof* Suppose that $m\alpha + r = n\alpha + s$. If $m = n$ then it is clear that $r = s$. Let us assume, therefore, that $m \neq n$. We then can write

$$\alpha = \frac{s-r}{m-n} \in \mathbb{Q}.$$

This, however, is a contradiction since $\alpha$ is irrational. It must thus be the case that $m\alpha + r = n\alpha + s$ implies $m = n$ and $r = s$. The converse is obviously true, thereby proving the lemma.

## 3. Our results

In this section we obtain two results that arise by expanding $(1+\phi)^n + (1-\phi)^n$ in different ways. First, we have

$$\begin{aligned}
(1+\phi)^n + (1-\phi)^n &= \sum_{k=0}^{n}\binom{n}{k}\phi^k + \sum_{k=0}^{n}\binom{n}{k}(-\phi)^k \\
&= \sum_{k=0}^{n}\binom{n}{k}\phi^k[1+(-1)^k] \\
&= 2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\phi^{2k},
\end{aligned} \tag{6}$$

where $\lfloor x \rfloor$ is the *floor function*, denoting the largest integer not exceeding $x$. From (2), (3), (4), and (5) it also follows that

$$\begin{aligned}(1+\phi)^n + (1-\phi)^n &= \phi^{2n} + \left(-\frac{1}{\phi}\right)^n \\ &= \phi^{2n} + (-1)^n\phi^{-n} \\ &= \phi \mathrm{F}_{2n} + \mathrm{F}_{2n-1} + (-1)^n(\phi\mathrm{F}_{-n} + \mathrm{F}_{-n-1}) \\ &= \phi \mathrm{F}_{2n} + \mathrm{F}_{2n-1} + (-1)^n((-1)^{n+1}\phi\mathrm{F}_n + (-1)^{n+2}\mathrm{F}_{n+1}) \\ &= \phi(\mathrm{F}_{2n} - \mathrm{F}_n) + \mathrm{F}_{2n-1} + \mathrm{F}_{n+1}. \end{aligned} \tag{7}$$

Therefore, on using (5), (6), and (7), we obtain

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(\phi\mathrm{F}_{2k} + \mathrm{F}_{2k-1}) = \phi(\mathrm{F}_{2n} - \mathrm{F}_n) + \mathrm{F}_{2n-1} + \mathrm{F}_{n+1}.$$

Specialising to $\alpha = \phi$ in Lemma 1 then leads to the following two results:

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\mathrm{F}_{2k} = \tfrac{1}{2}(\mathrm{F}_{2n} - \mathrm{F}_n) \tag{8}$$

and

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\mathrm{F}_{2k-1} = \tfrac{1}{2}(\mathrm{F}_{2n-1} + \mathrm{F}_{n+1}). \tag{9}$$

By considering next the expansion of $(1+\phi)^n - (1-\phi)^n$, readers might like also to show that

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2k+1}\mathrm{F}_{2k+1} = \tfrac{1}{2}(\mathrm{F}_{2n} + \mathrm{F}_n) \tag{10}$$

and

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2k+1}\mathrm{F}_{2k} = \tfrac{1}{2}(\mathrm{F}_{2n-1} - \mathrm{F}_{n+1}). \tag{11}$$

## 4. Some properties of our sequences

In this final section we consider a number of noteworthy properties of the sequences associated with the results obtained in section 3. To this end, let us denote the sequences with $n$th terms comprising the right-hand side of each of (8), (9), (10), and (11) by $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$, respectively. I found that each appears as a separate entry in the *On-Line Encyclopedia of Integer Sequences*.

An interesting initial point to make is that these four sequences satisfy the same recurrence relation, namely the one having the form

$$X_n = 4X_{n-1} - 3X_{n-2} - 2X_{n-3} + X_{n-4}. \tag{12}$$

They merely have different initial conditions, as follows:

$$(a_0, a_1, a_2, a_3) = (0, 0, 1, 3),$$
$$(b_0, b_1, b_2, b_3) = (1, 1, 2, 4),$$
$$(c_0, c_1, c_2, c_3) = (0, 1, 2, 5),$$
$$(d_0, d_1, d_2, d_3) = (0, 0, 0, 1).$$

Next, it is the case that $c_n$ and $d_n$ may each be expressed as Fibonacci convolutions. These are given by

$$c_n = \sum_{k=0}^{n} F_{2k-1} F_{n-k} \tag{13}$$

and

$$d_n = \sum_{k=0}^{n-1} F_{2k} F_{n-k-1}, \tag{14}$$

for $n \geq 1$. We provide a proof of (13) here, proceeding by induction on $n$. First,

$$\sum_{k=0}^{1} F_{2k-1} F_{1-k} = F_{-1}F_1 + F_1F_0 = 1 = c_1$$

and

$$\sum_{k=0}^{2} F_{2k-1} F_{2-k} = F_{-1}F_2 + F_1F_1 + F_3F_0 = 2 = c_2.$$

Now assume that (13) is true when $n = m - 1$ and $n = m$ for some $m \geq 2$. Then,

$$\begin{aligned}
\sum_{k=0}^{m+1} F_{2k-1} F_{(m+1)-k} &= \sum_{k=0}^{m} F_{2k-1}(F_{m-k} + F_{m-k-1}) + F_{2(m+1)-1}F_0 \\
&= \sum_{k=0}^{m} F_{2k-1}F_{m-k} + \sum_{k=0}^{m} F_{2k-1}F_{m-k-1} \\
&= \sum_{k=0}^{m} F_{2k-1}F_{m-k} + \sum_{k=0}^{m-1} F_{2k-1}F_{(m-1)-k} + F_{2m-1}F_{-1} \\
&= \tfrac{1}{2}(F_{2m} + F_m) + \tfrac{1}{2}(F_{2(m-1)} + F_{m-1}) + F_{2m-1},
\end{aligned}$$

on utilising the inductive hypothesis. This final expression does indeed simplify to

$$\tfrac{1}{2}(F_{2(m+1)} + F_{m+1}),$$

thereby proving (13). The interested reader might like to obtain (14) in a similar manner. Incidentally, the sequence $\{c_n\}$ crops up in knot theory.

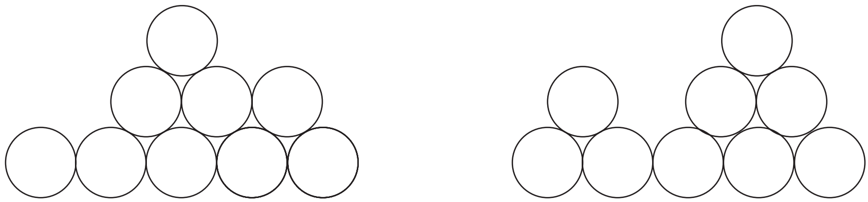

**Figure 1** These are both coin fountains, but only the one on the left is a block fountain.

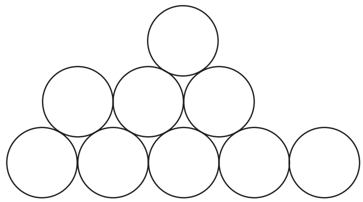

**Figure 2** A mirror image of the fountain on the left-hand side of figure 1.

The sequence $\{b_n\}$ enumerates a particular type of *coin fountain* (see reference 7). A coin fountain is an arrangement of coins in rows such that there are no gaps in the bottom row, while those in higher rows touch exactly two coins in the row below. A fountain possessing no 'troughs' is termed a *block* fountain. In figure 1 the fountain on the left is a block fountain whilst the one on the right is not. The term $b_n$ gives the number of distinct block fountains with exactly $n$ coins in the bottom row subject to the condition that fountains that are mirror images of one another are regarded as the same object. When obtaining $b_5$, for example, the fountain on the left in figure 1 and that in figure 2 only contribute a combined total of 1 to the evaluation of $b_5$.

We briefly outline here a proof of the fact that $b_n$ does indeed give the number of block fountains (subject to the mirror-image criteria stated in the previous paragraph) with $n$ coins in the bottom row. Notice that any such fountain may be represented by a sequence of letters from the set {U, D, H} where U, D, and H denote 'up', 'down', and 'horizontal', respectively. To take an example, the fountain in figure 2 would, by starting at the bottom left-hand corner and tracing a path from left to right along the top-most coins, give rise to the sequence UHUDDH. For block fountains in particular, any D appearing in a sequence will always be to the right of any U. Furthermore, on assigning 1 to each of U and D, and 2 to H, the total for any sequence arising from a block fountain with $n$ coins in the bottom row will always be $2(n-1)$.

We then need the fact that the number of ways of tiling a $1 \times n$ board with $1 \times 1$ squares and $1 \times 2$ dominoes is given by $\mathrm{F}_{n+1}$ (see reference 8). From the comments above it may be seen that, without any restrictions, the number of distinct sequences arising from block fountains with $n$ coins in the bottom row is $\mathrm{F}_{2n-1}$. In order to complete the proof, it needs to be shown that the number of such fountains possessing mirror symmetry is $\mathrm{F}_{n+1}$, from which the result does indeed follow.

In addition, we have the following formula,

$$a_n = \frac{2}{5}\sum_{k=0}^{4} \sin\left(\frac{2\pi k}{5}\right) \sin\left(\frac{4\pi k}{5}\right)\left(1 + 2\cos\left(\frac{\pi k}{5}\right)\right)^n. \tag{15}$$

As a first step in proving this, it is worth considering a regular pentagon with an edge length of one unit. Each of its diagonals has length $\phi$, and from this it is possible to show that

$$\cos\frac{\pi}{5} = \frac{\phi}{2}, \quad \cos\frac{2\pi}{5} = \frac{\phi-1}{2}, \ldots$$

and

$$\sin\frac{\pi}{5} = \frac{\sqrt{3-\phi}}{2}, \quad \sin\frac{2\pi}{5} = \frac{\sqrt{2+\phi}}{2}, \ldots.$$

It then makes a good challenge to complete the proof. There are at least two alternative ways of approaching this. First, the right-hand side of (15) might be manipulated in order to transform it into something that is recognisable as being equal to $a_n$. Second, it may be demonstrated that (15) satisfies the recurrence relation (12) along with the correct initial conditions.

As a final comment, we note that both $\{a_n\}$ and $\{d_n\}$ are associated with the enumeration of potential paths in certain types of restricted one-dimensional random walks.

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***Martin Griffiths*** *is currently Head of Mathematics at Christ's College in New Zealand. He has previously held academic posts in the UK as a Lecturer in Mathematics.*


**Reversing digits**

Select an odd number $n$ having at least two digits. Write the number in the reverse order to get $R(n)$. Then the numerical difference of $n$ and $R(n)$ is always divisible by 18. For example, if $n = 1257$ then $R(n) = 7521$ and $7521 - 1257 = 6264$, which is divisible by 18.


Integral University, Lucknow, India **Mohammed Faraz Khan**

# Why Twin Primes are Scarce

DES MACHALE and PETER MACHALE

We explore the possibility that one of the reasons why twin primes are so scarce is the number of stringent congruence conditions they must satisfy and list some of these conditions.

If $p$ and $p+2=q$ are both prime numbers, $p$ and $q$ are called *twin primes*. Examples are $(p,q)=(3,5),(5,7),(11,13),(17,19)$. It is one of the great unsolved problems of number theory to decide whether or not there are infinitely many pairs of twin primes. The general belief is that there are, but currently a proof is lacking. However, there is no doubt that there are appreciably fewer pairs of twin primes than primes themselves. For example, there are 168 primes, but only 35 pairs of twin primes, less than 1000. A good estimate for the number of primes less than $n$ is $n/\log n$, whereas there exists an absolute constant $c>0$ such that the number of twin primes less than $n$ is less than $cn/(\log n)^2$. In addition, there is the curious fact that $\sum_{n=1}^{\infty} 1/p_n$, where $p_n$ runs through the primes, is a divergent series, whereas the series $\sum_{t=1}^{\infty} 1/p_t$, where $p_t$ runs through the twin primes, $(=\frac{1}{3}+\frac{1}{5}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\cdots)$ actually converges to $1.902\,160\,582\,3\ldots$ (Brun's constant). This suggests that, even if there are infinitely many twin primes, they are pretty thinly dispersed.

Finally, and amazingly, there is a very beautiful necessary and sufficient congruence condition for $n$ and $n+2$ to be twin primes. It is $4[(n-1)!+1]+n\equiv 0 \pmod{n(n+2)}$, $n\geq 2$. It is derived from Wilson's theorem, namely that $n$ is a divisor of $(n-1)!+1$ if and only if $n$ is a prime, but the catch is that the factorial makes it computationally unfeasible.

In this article we explore the possibility that one of the reasons for the scarcity of twin primes is that their product $pq$ must satisfy a large number of congruences. For convenience, we choose $p>3$ and $q>5$ throughout, where $(p,q)$ is a pair of twin primes. Of course, since $p=2n-1$ and $q=2n+1$, $pq+1=4n^2$, an even perfect square.

**Theorem 1** *Let $p$ and $q$ be twin primes. Then* $pq+1\equiv 0 \pmod{36}$.

*Proof* Every integer greater than 4 is of the form $6n-1$, $6n$, $6n+1$, $6n+2$, $6n+3$, or $6n+4$, for some $n\in\mathbb{N}$. Now each of $6n$, $6n+2$, $6n+3$, and $6n+4$ is composite, so the only situation in which we can get twin primes is if $p=6n-1$ and $q=6n+1$ are both prime. If this is the case, then

$$pq+1=(6n-1)(6n+1)+1=36n^2\equiv 0 \pmod{36}.$$

Obviously this congruence does not hold for $p=3$, $q=5$.

**Theorem 2** *Let $p$ and $q$ be twin primes. Then* $pq\equiv -1 \pmod{16}$ *or* $pq\equiv 3 \pmod{16}$.

*Proof* This time we work modulo 4. Since $4n$ and $4n+2$ are both composite, twin primes can arise in only two ways.

(i) $p=4n-1$, $q=4n+1$. Then $pq=16n^2-1$, so $pq\equiv -1 \pmod{16}$. (For example, $p=59$, $q=61$.) We note that this congruence also applies to $p=3$, $q=5$.

(ii) $p=4n+1$, $q=4n+3$. Then $pq=16n^2+16n+3$, so $pq\equiv 3 \pmod{16}$. (For example, $p=17$, $q=19$.)

**Theorem 3** *If $p$ and $q$ are twin primes, then either $pq \equiv -1 \pmod{64}$ or $pq \equiv 15 \pmod{64}$ or $pq \equiv 3 \pmod{32}$.*

*Proof* Working modulo 8, we note that $8n$, $8n+2$, $8n+4$, and $8n+6$ are all composite, so twin primes can arise only in the following situations.

(i) $p = 8n-1$, $p = 8n+1$. Then $pq = 64n^2 - 1$ so $pq \equiv -1 \pmod{64}$. (For example, $p = 71$, $q = 73$.)

(ii) $p = 8n+1$, $p = 8n+3$. Then $pq = 64n^2 + 32n + 3$ so $pq \equiv 3 \pmod{32}$. (For example, $p = 17$, $q = 19$.)

(iii) $p = 8n+3$, $p = 8n+5$. Then $pq = 64n^2 + 64n + 15$ so $pq \equiv 15 \pmod{64}$. (For example, $p = 11$, $q = 13$.) This also holds for $p = 3$, $q = 5$.

(iv) $p = 8n+5$, $p = 8n+7$. Then $pq = 64n^2 + 96n + 35$ so $pq \equiv 3 \pmod{32}$ as in (ii). (For example, $p = 29$, $q = 31$.)

**Theorem 4** *If $p$ and $q$ are twin primes, then $pq \equiv -1 \pmod{81}$ or $pq \equiv 8 \pmod{27}$.*

*Proof* Working modulo 9, we see that the only possibilities are as follows.

(i) $p = 9n-1$, $q = 9n+1$. Then $pq = 81n^2 - 1$, so $pq \equiv -1 \pmod{81}$. (For example, $p = 17$, $q = 19$.)

(ii) $p = 9n+2$, $q = 9n+4$. Then $pq = 81n^2 + 54n + 8$, so $pq \equiv 8 \pmod{27}$. (For example, $p = 11$, $q = 13$.)

(iii) $p = 9n+5$, $q = 9n+7$. Then $pq = 81n^2 + 108n + 35$, so $pq \equiv 8 \pmod{27}$ as in (ii). (For example, $p = 41$, $q = 43$.)

We note that the number of cases to be dealt with is reduced considerably if we pick a modulus $t$ such that $\phi(t)$, the Euler $\phi$-function, is small. Since $\phi(10) = 4 = \phi(12)$, we finish by looking at the cases $t = 10$ and $t = 12$. Larger moduli give further results but these are more cumbersome and less useful. Also, as the modulus increases, we may have to exclude a further finite number of initial twin primes. In the next case for example, we must exclude $p = 5$, $q = 7$, as well as $p = 3$, $q = 5$.

**Theorem 5** *If $p$ and $q$ are twin primes, then $pq \equiv -1 \pmod{100}$ or $pq \equiv 3 \pmod{20}$.*

*Proof* Working modulo 10, the possibilities are as follows.

(i) $p = 10n-1$, $q = 10n+1$. Then $pq + 1 = 100n^2 \equiv 0 \pmod{100}$. (For example, $p = 59$, $q = 61$.)

(ii) $p = 10n+1$, $q = 10n+3$. Then $pq = 100n^2 + 40n + 3 \equiv 3 \pmod{20}$. (For example, $p = 71$, $q = 73$.)

(iii) $p = 10n+7$, $q = 10n+9$. Then $pq = 100n^2 + 160n + 63 \equiv 3 \pmod{20}$ as in (ii). (For example, $p = 107$, $q = 109$.)

**Theorem 6** *If $p$ and $q$ are twin primes, then $pq \equiv -1 \pmod{144}$ or $pq \equiv 35 \pmod{144}$.*

*Proof* Working modulo 12, there are just two cases to consider.

(i) $p = 12n - 1$, $q = 12n + 1$. Then $pq = 144n^2 - 1 \equiv -1 \pmod{144}$. (For example, $p = 11$, $q = 13$.)

(ii) $p = 12n + 5$, $q = 12n + 7$. Then $pq = 144n^2 + 144n + 35 \equiv 35 \pmod{144}$. (For example, $p = 29$, $q = 31$.)

At the suggestion of the Editor, we can combine all our results to give the following composite result.

**Theorem 7** *If $p$, $q$ are twin primes, then one of the following congruences must hold:*

$$\begin{gathered}
pq \equiv 8\,099 \pmod{129\,600},\\
pq \equiv 899 \pmod{43\,200},\\
pq \equiv 72\,899 \pmod{129\,600},\\
pq \equiv 22\,499 \pmod{43\,200},\\
pq \equiv 13\,283 \pmod{25\,920},\\
pq \equiv 6\,083 \pmod{8\,640},\\
pq \equiv 323 \pmod{25\,920},\\
pq \equiv 1\,763 \pmod{8\,640},\\
pq \equiv 129\,599 \pmod{129\,600} \quad \textit{or} \quad pq \equiv -1 \pmod{129\,600},\\
pq \equiv 32\,399 \pmod{129\,600},\\
pq \equiv 3\,599 \pmod{43\,200},\\
pq \equiv 5\,183 \pmod{25\,920},\\
pq \equiv 143 \pmod{8\,640},\\
pq \equiv 11\,663 \pmod{25\,920},\\
pq \equiv 14\,399 \pmod{43\,200},\\
pq \equiv 2\,303 \pmod{8\,640}.
\end{gathered}$$

These could all be made into congruences modulo 129 600.

We ask what is the relevance of all this in the search of a proof of the twin prime conjecture. It seems likely that much stronger results are true. For example, we are prepared to make the following conjecture.

**Conjecture 1** *There are infinitely many prime pairs of the form $10n - 1$, $10n + 1$.*

By the famous theorem of Dirichlet, there are infinitely many primes of the form $10n - 1$ and also infinitely many primes of the form $10n + 1$. Sometimes a specific stronger theorem is easier to prove than a general result because one has very strong information about the form of the numbers involved and this could very well be one of these cases. So, we ask, are there infinitely many twin pairs of the type $(29, 31)$, $(59, 61)$, $(149, 151)$, $(179, 181)$, $(239, 241)$, $(269, 271)$, $(419, 421)$, $(569, 571)$, $(659, 661), \ldots$?

Finally, we mention four interesting snippets.

1. Currently the largest known twin primes are $200\,366\,313 \cdot 2^{195\,000} \pm 1$, each with $58\,711$ decimal digits. They were discovered in 2007.

2. The following variant on Goldbach's conjecture has been proposed. With finitely many exceptions (35 integers beginning 2, 4, 94, 96, 98, 400, 402, 404, 514, 516, 518, ... and no others less than $10^9$), every even integer is the sum $a + b$, where each of $a$ and $b$ is a component of (not necessarily the same) twin prime pair. For example, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 5 + 7$, $14 = 7 + 7$, $16 = 3 + 13, \ldots$.

3. For $1 \leq x \leq 20$, $n = 30(2x - 27)(x - 5)$ implies that $n - 1$ and $n + 1$ are both prime.

4(i). If $p$, $2p - 1$, and $2p + 1$ are all prime, then $p = 2$ or 3.

4(ii). If $2^n - 1$ and $2^n + 1$ are twin primes, then $n = 2$.

We close with another conjecture.

**Conjecture 2** *If $n! - 1$ and $n! + 1$ are twin primes, then $n = 3$.*

This conjecture is based on numerical evidence and is true for $n < 103\,041$ (see reference 1).

**Reference**

1 N. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

***Des MacHale*** *is Emeritus Professor of Mathematics at University College Cork where he has taught for nearly forty years. His mathematical interests are in abstract algebra but he also works in number theory, geometry, combinatorics, and the history of mathematics. His other interests include humour, geology, and words.*

***Peter MacHale*** *is the Systems Manager in the Cork Constraint Computation Centre (4C), Computer Science Department, University College Cork. His interests are constraint programming and graph theory. His other interests include music, science fiction, and gaming.*

**$2^{57\,885\,161} - 1$**

This is the new largest known Mersenne prime number, discovered by Curtis Cooper of the University of Central Missouri, USA, on 25 January 2013. That makes it the 48th known Mersenne prime, and is part of GIMPS, the Great Internet Mersenne Prime Search, in which many people all over the world pool their computing resources to search for Mersenne primes. Their name goes back to the French monk Father Marin Mersenne (1588–1648), who made claims, not all correct, about when $2^p - 1$ is prime. For more details, see http://en.wikinews.org/wiki/Record_size_17.4_million-digit_prime_found.

(Sent in by Spiros Andriopoulos.)

# A Surprising Fact About Some *n*-Volumes?

MARTIN GRIFFITHS, ANTHONY BAKER
and CHRISTOPHER BROWN

In this article we consider the generalisation of certain shapes and solids to $n$ dimensions. During the course of our explorations we found a property of these generalised objects that might initially seem somewhat counterintuitive. We state and prove this property here, and make several comments in this regard.

## 1. Introduction

This article arose by way of a series of mathematical discussions that took place over a period of several weeks between two high-school students and a mathematics teacher. We were interested initially in the idea of extending the notion of regular polygons to $n$ dimensions. Regular polygons are two-dimensional objects, and there are infinitely many of them. It is well known, however, that when we try to extend the idea of a regular polygon to three dimensions, there turn out to be only five possibilities, namely the cube, the regular tetrahedron, regular octahedron, regular dodecahedron, and regular icosahedron. These are known as the *Platonic solids*.

What happens then when we attempt to extend things further still? It is in fact found that only three of the Platonic solids generalise to $n$ dimensions. These are the cube, the regular octahedron, and the regular tetrahedron.

The relative simplicity of the construction of an $n$-dimensional version of a unit cube means that this is usually the easiest of the three to consider initially. It can be built up in a recursive manner, starting with a zero-dimensional cube, which consists of a single point. A one-dimensional cube is obtained by joining together two zero-dimensional cubes via a line segment of length 1 unit. To construct a two-dimensional cube we place two one-dimensional cubes in a two-dimensional plane such that the corresponding vertices are at a distance of 1 unit from each other and the line segments joining these corresponding vertices are perpendicular to the edges of the one-dimensional cubes; this results in a unit square. Our familiar three-dimensional cube is obtained by placing two of these squares in three-dimensional space in such a way that the corresponding vertices are all at a distance of 1 unit from each other and the line segments joining these corresponding vertices are perpendicular to the edges of the square. Similarly, by carrying out this process with two cubes in four dimensions, we obtain a four-dimensional hypercube, which is commonly known as the *tesseract* (see http://en.wikipedia.org/wiki/Tesseract).

A two-dimensional representation of a tesseract is shown in figure 1. Note that it is impossible in such a representation to give the impression that all the edges are of equal length. See reference 1 for an animated clip showing how the tesseract may be constructed.

When extending shapes and solids to $n$ dimensions it is quite natural also to extend the concepts of area and volume to these higher dimensions. We term the $n$-dimensional volume of an $n$-dimensional object its $n$*-volume*, a notion that will be discussed in greater depth in

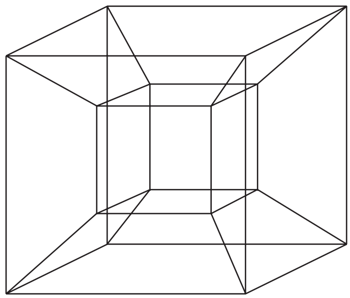

**Figure 1** A representation of a tesseract.

section 3. Our explorations led us eventually to consider the behaviour of the $n$-volumes of the rather less well-known $n$-dimensional octahedron and tetrahedron as $n$ tends to infinity. Although some of the results we obtain here will certainly not be new, our intention is to take a fresh look at these intricate objects by demonstrating an aspect of them that may seem somewhat counterintuitive initially.

## 2. The regular *n*-dimensional octahedron

With a view to extending the octahedron to $n$ dimensions, it is easiest first to define the 'normal' three-dimensional octahedron as the solid having six vertices with coordinates given by:

$$(1, 0, 0), \quad (-1, 0, 0), \quad (0, 1, 0), \quad (0, -1, 0), \quad (0, 0, 1), \quad (0, 0, -1).$$

Each vertex is connected, via an edge, to all other vertices except for its 'opposite'. For example, $(1, 0, 0)$ is connected to $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$ but not to $(-1, 0, 0)$. There are 12 edges and eight faces, each face being an equilateral triangle. A two-dimensional representation of an octahedron is shown in figure 2, where the dashed lines denote the hidden edges.

Let us now demonstrate how to construct an $n$-dimensional octahedron recursively for any $n \geq 2$. We first build a two-dimensional octahedron, as follows. It has four vertices, $A$, $B$,

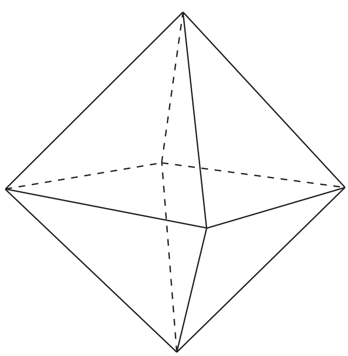

**Figure 2** A regular octahedron.

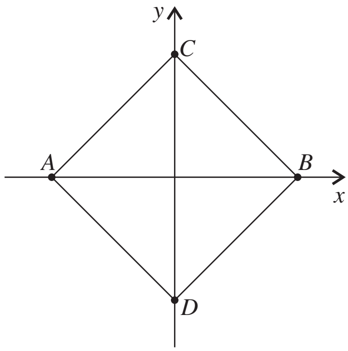


**Figure 3** A regular two-dimensional octahedron.

$C$, and $D$, which are positioned at $(-1, 0)$, $(1, 0)$, $(0, 1)$, and $(0, -1)$, respectively. Each vertex is then joined, via a line segment, to all other vertices except for its opposite. Thus, for example, there is an edge between $A$ and $C$ and one between $A$ and $D$, but not between $A$ and $B$. This results in the two-dimensional octahedron shown in figure 3. Note that the edge lengths are all $\sqrt{2}$ units, and that the regular two-dimensional octahedron is in fact a square.

In order next to obtain a three-dimensional octahedron, we extend the coordinate system so that it now has a $z$-axis which is perpendicular to both the $x$-axis and the $y$-axis. On considering figure 3, the positive portion of the $z$-axis comes out of the front of the page, while the negative part exits from the back. We now denote $(0, 0, 1)$ and $(0, 0, -1)$ by $E$ and $F$, respectively. Vertex $E$ is joined by line segments of length $\sqrt{2}$ units to each of $A$, $B$, $C$, and $D$, and similarly for the vertex at $F$. Note that $E$ and $F$ are not joined by a line segment, and that the resultant object is indeed a regular octahedron as depicted in figure 1, albeit somewhat different in size and orientation.

The four-dimensional octahedron is constructed by introducing a fourth axis, the $w$-axis, say, which is perpendicular to the $x$-, $y$-, and $z$-axes. Vertices $G$ and $H$ are positioned at $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$, respectively. They are subsequently joined by line segments of length $\sqrt{2}$ units to each of $A$, $B$, $C$, $D$, $E$, and $F$. As in the previous cases, the two new vertices are not joined to each other. The construction continues in this manner until we have reached the desired number of dimensions.

In general, we consider the $n$-dimensional octahedron with $2n$ vertices at

$$(\pm 1, 0, \dots, 0), (0, \pm 1, \dots, 0), \dots, (0, 0, \dots, \pm 1),$$

where each of these coordinates is an $n$-tuplet. For ease of notation we use $\mathrm{Oct}_n(a)$ to denote a regular $n$-dimensional octahedron with edge length $a$, so the octahedron given above is $\mathrm{Oct}_n(\sqrt{2})$.

## 3. The *n*-volume

Dealing first with an $n$-dimensional hypercube of edge length $a$ units, since all of the $n$ edges at any of its vertices are pairwise perpendicular, it has an $n$-volume equal to $a^n$ units$^n$. Thus, if $a = 1$, the numerical value of the $n$-volume is equal to 1 for all $n \in \mathbb{N}$, while if $a < 1$ or $a > 1$ it tends to zero or infinity, respectively, as $n$ tends to infinity.

The calculation of the $n$-volume of $\mathrm{Oct}_n(a)$ is not quite so straightforward. However, the construction given in section 2 provides us with a natural way to visualise the generalisation of its volume to $n$ dimensions. We start with the two-dimensional object, $\mathrm{Oct}_2(\sqrt{2})$, which is a square of area 2 units$^2$. We term this area the *two-volume* of $\mathrm{Oct}_2(\sqrt{2})$. Next, $\mathrm{Oct}_3(\sqrt{2})$ is constructed in three-dimensional space from $\mathrm{Oct}_2(\sqrt{2})$ essentially by gluing together two identical square-based pyramids each having $\mathrm{Oct}_2(\sqrt{2})$ as their base and a height of 1 unit. This gives us our familiar three-dimensional octahedron, having a three-volume that will be calculated in due course; note that in this special case the $n$-volume corresponds to the normal usage of the word 'volume'. In constructing $\mathrm{Oct}_4(\sqrt{2})$ in four-dimensional space from $\mathrm{Oct}_3(\sqrt{2})$, the latter acts as a three-dimensional 'base', and we glue two identical octahedron-based pyramids (each with a height of 1 unit once more) together. Note that, by dimensions, this process does indeed give us a four-dimensional object.

This process may be continued indefinitely, where at each stage $\mathrm{Oct}_{n-1}(\sqrt{2})$ provides the $(n-1)$-dimensional base on which $\mathrm{Oct}_n(\sqrt{2})$ may be built. The method of construction very much lends itself to the calculation of $V_n(a)$, the $n$-volume of $\mathrm{Oct}_n(a)$, in a recursive manner. In order to calculate the three-volume of $\mathrm{Oct}_3(a)$, we may integrate the square elemental slices that result as we pass from the base $\mathrm{Oct}_2(\sqrt{2})$ at height 0 units to the vertex at height 1 unit. This leads to the following integral:

$$\begin{aligned} V_3(\sqrt{2}) &= 2V_2(\sqrt{2})\int_0^1 (1-t)^2\,\mathrm{d}t \\ &= 4\int_0^1 (1-t)^2\,\mathrm{d}t \\ &= \tfrac{4}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned} V_4(\sqrt{2}) &= 2V_3(\sqrt{2})\int_0^1 (1-t)^3\,\mathrm{d}t \\ &= \tfrac{8}{3}\int_0^1 (1-t)^3\,\mathrm{d}t \\ &= \tfrac{2}{3}, \end{aligned}$$

and in general

$$V_n(\sqrt{2}) = 2V_{n-1}(\sqrt{2})\int_0^1 (1-t)^{n-1}\,\mathrm{d}t,$$

for $n \geq 3$, from which it is a straightforward matter to show, by induction, that

$$V_n(\sqrt{2}) = \frac{2^n}{n!}. \tag{1}$$

It is interesting to consider the numerical behaviour of $V_n(a)$ as $n$ tends to infinity. From (1) it follows that

$$V_n(a) = \frac{(a\sqrt{2})^n}{n!}.$$

Therefore,

$$\frac{V_{n+1}(a)}{V_n(a)} = \frac{(a\sqrt{2})^{n+1}}{(n+1)!}\,\frac{n!}{(a\sqrt{2})^n} = \frac{a\sqrt{2}}{n+1}.$$

Then, since

$$\frac{V_{n+1}(a)}{V_n(a)} < \frac{1}{2},$$

when $n > 2a\sqrt{2}$, it does in fact become clear that

$$\lim_{n\to\infty} V_n(a) = 0,$$

for any positive value of $a$.

This result means that no matter how large the (fixed) edge length $a$ of each of the elements of the set $\{\mathrm{Oct}_n(a) : n \in \mathbb{N}\}$, we would, for any given $\varepsilon > 0$, be able to find a value of $n$, $N$ say, such that the $n$-volume of $\mathrm{Oct}_n(a)$ was less than $\varepsilon$ units$^n$ for all $n > N$. To take a specific example, suppose that $a = 10$ m. Then, as is easily checked on a piece of mathematical software such as MATHEMATICA®, $V_n(a) < 1\,\mathrm{cm}^n$ for all $n > 3839$. The fact that the numerical value of $V_n(a)$ always tends to zero as $n$ tends to infinity might initially appear somewhat surprising, and this is indeed something that we discuss briefly in section 5.

## 4. The regular *n*-dimensional tetrahedron

The four vertices of a regular tetrahedron are mutually equidistant, and, noting that no set of more than four distinct vertices has this property in three dimensions, we see that this can in fact be used to extend its definition to $n$ dimensions. In two dimensions, for example, any equilateral triangle has three mutually equidistant vertices, but it is not possible for four or more distinct vertices to possess this property. For this reason we may regard an equilateral triangle as a two-dimensional tetrahedron. Similarly, a point and a line segment are zero-dimensional and one-dimensional tetrahedra, respectively. More generally, in $n$ dimensions any mutually equidistant set of $n + 1$ distinct points comprises the vertices of a regular $n$-dimensional tetrahedron.

Let us use $\mathrm{Tet}_n(a)$ to denote a regular $n$-dimensional tetrahedron with edge length $a$ units. With rather more effort than was the case with the $n$-dimensional octahedron, it can be shown that the $n$-volume of $\mathrm{Tet}_n(a)$ is given by

$$\frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}}.$$

We see from this that the $n$-volume of $\mathrm{Tet}_n(a)$ also tends to zero as $n$ tends to infinity.

## 5. Final comments

In the light of the potentially counterintuitive results given in the previous two sections, it is worth making a couple of further observations regarding $\mathrm{Oct}_n(a)$ and $V_n(a)$ as $n$ increases without limit.

Firstly, note that although $V_n(a)$ tends to zero as $n$ tends to infinity, it is not generally the case that the sequence $(V_n(a))$ is strictly decreasing. For example, the largest term in the sequence $(V_n(4))$ occurs when $n = 5$. It is true, however, that for any fixed $a > 0$, $(V_n(a))$ is an eventually-decreasing sequence in the sense that there exists some $K \in \mathbb{N}$ such that $V_{n+1}(a) < V_n(a)$ for all $n > K$.

Secondly, the result

$$\lim_{n\to\infty} V_n(a) = 0,$$

which is valid for any positive value of $a$, does not actually imply that $\mathrm{Oct}_n(a)$ is becoming small in the sense that it is vanishing to a point. Indeed, the length of each edge of $\mathrm{Oct}_n(\sqrt{2})$ remains at $\sqrt{2}$ units for any $n$.

It is possible to construct heuristic arguments in order to explain this limiting behaviour, although such arguments have the tendency to sound somewhat hand-waving. A key point is that the $n$-volume of an $n$-dimensional pyramid with a base 'area' of $A$ units$^{n-1}$ and a height of $h$ units is $Ah/n$ units$^n$. This, in conjunction with the fact that the recursive construction of both $\mathrm{Oct}_n(a)$ and $\mathrm{Tet}_n(a)$ involves the creation of $n$-dimensional pyramids, does indeed imply the long-term behaviour of the $n$-volumes.

As a final challenge for those interested in taking things even further, it is worth investigating the corresponding situation for $n$-dimensional spheres, also known as *hyperspheres* (see http://en.wikipedia.org/wiki/N-sphere and references 2 and 3).

### References

1 A. Eivy, Hypercube: the building blocks of four dimensional space, 2013, http://4d.shadowpuppet.net/.

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3 A. K. Jobbings, The volume of the $n$-ball – II, *Math. Gazette* **82** (1998), pp. 105–106.

***Martin Griffiths*** *is currently a mathematics teacher at Christ's College in New Zealand. He has previously held academic posts in the United Kingdom as a Lecturer in Mathematics.*

***Anthony Baker*** *is a student at Christ's College. He is interested in all fields of mathematics, but particularly combinatorics and probability.*

***Christopher Brown*** *is also a student of Christ's College. In 2012 he participated in the Junior Mathematics Competition of New Zealand, run by the University of Otago, and obtained first position in his age group.*

**Mathematical Competitive Game 2013–2014**

The Mathematical Competitive Game 2013–2014, organized jointly by the French Federation of Mathematical Games and Société de Calcul Mathématique SA, is now open. It is endowed with 2 000 Euros of prizes. The topic for this year is 'Checking an Industrial Process'.

See http://scmsa.eu/archives/SCM_FFJM_Competitive_Game_2013_2014.pdf for a complete description of the game.

Answers should be sent no later than 30 April 2014.

# Exploring the Fibonacci Sequence of Order Three

JAY L. SCHIFFMAN

The Fibonacci sequence of order three is the sequence of numbers 1, 3, 10, 33, 109, .... Each term in this sequence from the third term on is equal to three times the previous term plus the term two places before it. This article will explore ideas such as divisibility and periodicity as well as prime elements in this sequence. We conclude by furnishing a closed formula similar to the Binet formula for the Fibonacci sequence in addition to some palatable number tricks.

## 1. Introduction

The reader is undoubtedly familiar with the *Fibonacci sequence*, $F_n$, satisfying the recursion relation $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for $n \geq 3$. A fairly familiar companion sequence known as the *Pell sequence* or the *Fibonacci sequence of order two*, $A_n$, is defined using the recursion relation $A_1 = 1$, $A_2 = 2$, $A_n = 2A_{n-1} + A_{n-2}$. For the Fibonacci sequence of order three, $B_n$, we have $B_1 = 1$, $B_2 = 3$, $B_n = 3B_{n-1} + B_{n-2}$. This sequence will serve as our theme for this article. We begin by considering the sequence 1, 3, 10, 33, 109, 360, 1 189, 3 927, 12 970, ....

Table 1 generates the initial thirty integers in the sequence and their prime factorizations, enabling us to observe any prime outputs. We observe that four of the initial thirty outputs generated by our sequence produce primes; namely $B_2 = 3$, $B_5 = 109$, $B_{11} = 141\,481$, and $B_{17} = 183\,642\,229$. The only other prime output less than 100 is $B_{61} = 12\,435\,170\,468\,929\,559\,540\,592\,573\,684\,601$. Observe that the prime outputs correspond to prime indices. The converse is not true; e.g. consider $B_7 = 1189 = 29 \cdot 41$.

## 2. Some divisibility conjectures

Based on the analysis of table 1, the following conjectures can be formed.

**Conjecture 1** *Every third term is divisible by 2, 5, and 10.*

**Conjecture 2** *Every second term is divisible by 3.*

**Conjecture 3** *Every sixth term is divisible by 4, 6, 8, and 9.*

**Conjecture 4** *Every eighth term is divisible by 7.*

**Conjecture 5** *Every fourth term is divisible by 11.*

These conjectures are readily proven using modular arithmetic. Consider, for example, divisibility by 5. The sequence modulo 5 is given by **1**, **3**, **0**, **3**, **4**, **0**, **4**, **2**, **0**, **2**, **1**, **0**, 1, 3, ..., and hence repeats after the twelfth term. It follows that every third term is congruent to 0 (mod 5) and so is divisible by 5. Similarly the sequence modulo 7 is **1**, **3**, **3**, **5**, **4**, **3**, **6**, **0**, **6**, **4**, **4**, **2**, **3**, **4**, **1**, **0**, 1, 3, ..., so it repeats after the sixteenth term. Thus every eighth term is congruent to 0 (mod 7) and so is divisible by 7.

**Table 1**

| Term | Value | Prime factorization |
|---|---:|:---:|
| 1st | 1 | 1 |
| 2nd | 3 | **3 (prime)** |
| 3rd | 10 | $2 \cdot 5$ |
| 4th | 33 | $3 \cdot 11$ |
| 5th | 109 | **109 (prime)** |
| 6th | 360 | $2^3 \cdot 3^2 \cdot 5$ |
| 7th | 1 189 | $29 \cdot 41$ |
| 8th | 3 927 | $3 \cdot 7 \cdot 11 \cdot 17$ |
| 9th | 12 970 | $2 \cdot 5 \cdot 1\,297$ |
| 10th | 42 837 | $3 \cdot 109 \cdot 131$ |
| 11th | 141 481 | **141 481 (prime)** |
| 12th | 467 280 | $2^4 \cdot 3^2 \cdot 5 \cdot 11 \cdot 59$ |
| 13th | 1 543 321 | $13 \cdot 118\,717$ |
| 14th | 5 097 243 | $3 \cdot 29 \cdot 41 \cdot 1\,429$ |
| 15th | 16 835 050 | $2 \cdot 5^2 \cdot 109 \cdot 3\,089$ |
| 16th | 55 602 393 | $3 \cdot 7 \cdot 11 \cdot 17 \cdot 14\,159$ |
| 17th | 183 642 229 | **183 642 229 (prime)** |
| 18th | 606 529 080 | $2^3 \cdot 3^3 \cdot 5 \cdot 433 \cdot 1\,297$ |
| 19th | 2 003 229 469 | $37 \cdot 54\,141\,337$ |
| 20th | 6 616 217 487 | $3 \cdot 11 \cdot 19 \cdot 109 \cdot 131 \cdot 739$ |
| 21st | 21 851 881 930 | $2 \cdot 5 \cdot 29 \cdot 41 \cdot 1\,837\,837$ |
| 22nd | 72 171 863 277 | $3 \cdot 23 \cdot 7\,393 \cdot 141\,481$ |
| 23rd | 238 367 471 761 | $137 \cdot 1\,739\,908\,553$ |
| 24th | 787 274 278 560 | $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 59 \cdot 7\,079$ |
| 25th | 2 600 190 307 441 | $109 \cdot 7\,949 \cdot 3\,001\,001$ |
| 26th | 8 587 845 200 883 | $3 \cdot 13 \cdot 53 \cdot 79 \cdot 443 \cdot 118\,717$ |
| 27th | 28 363 725 910 090 | $2 \cdot 5 \cdot 1\,297 \cdot 5\,237 \cdot 417\,581$ |
| 28th | 93 679 022 931 153 | $3 \cdot 11 \cdot 29 \cdot 41 \cdot 1\,429 \cdot 1\,670\,761$ |
| 29th | 309 400 794 703 549 | $233 \cdot 1\,327\,900\,406\,453$ |
| 30th | 1 021 881 407 041 800 | $2^3 \cdot 3^2 \cdot 5^2 \cdot 61 \cdot 109 \cdot 131 \cdot 211 \cdot 3\,089$ |

For integers $n$ from 1 to 150, table 2 lists the period of divisibility by $n$ and the length of the repeating block modulo $n$.

In order to secure the length of the repeating block modulo $n$ for composite indices $n$, the algebra of residue classes and the least common multiple (LCM) are both key. For example, the period of the sequence modulo 10 is 12. The period of the sequence modulo 2 is 3 (the sequence of remainders is $\mathbf{1}, \mathbf{1}, \mathbf{0}, 1, 1, 0, \ldots$) and the period of the sequence modulo 5 is 12 (the sequence of remainders is $\mathbf{1}, \mathbf{3}, \mathbf{0}, \mathbf{3}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{1}, \mathbf{0}, 1, 3, \ldots$). Now the LCM of 3 and 12 is 12. Indeed, we can examine the sequence of remainders in the Fibonacci sequence of order three modulo 10, this sequence is $\mathbf{1}, \mathbf{3}, \mathbf{0}, \mathbf{3}, \mathbf{9}, \mathbf{0}, \mathbf{9}, \mathbf{7}, \mathbf{0}, \mathbf{7}, \mathbf{1}, \mathbf{0}, 1, 3, \ldots$, which is of length 12. It should be observed that when we come to consecutive terms 1 and 0, we know that we have completed a cycle based on the recursion relation in the sequence.

A similar procedure can be used for the period of divisibility by a composite integer. Consider, for example, divisibility by 90. First note that $90 = 2 \cdot 3^2 \cdot 5$. The period of

**Table 2** Divisibility and periodicity table for the Fibonacci sequence of order three for the initial 150 counting integers.

| Integer $n$ | Period of divisibility | Length of repeating block modulo $n$ | Integer $n$ | Period of divisibility | Length of repeating block modulo $n$ | Integer $n$ | Period of divisibility | Length of repeating block modulo $n$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 51 | 8 | 16 | 101 | 50 | 50 |
| 2 | 3 | 3 | 52 | 78 | 156 | 102 | 24 | 48 |
| 3 | 2 | 2 | 53 | 26 | 26 | 103 | 102 | 102 |
| 4 | 6 | 6 | 54 | 18 | 18 | 104 | 78 | 156 |
| 5 | 3 | 12 | 55 | 12 | 24 | 105 | 24 | 48 |
| 6 | 6 | 6 | 56 | 24 | 48 | 106 | 78 | 78 |
| 7 | 8 | 16 | 57 | 20 | 40 | 107 | 106 | 106 |
| 8 | 6 | 12 | 58 | 21 | 84 | 108 | 18 | 18 |
| 9 | 6 | 6 | 59 | 12 | 24 | 109 | 5 | 20 |
| 10 | 3 | 12 | 60 | 6 | 12 | 110 | 12 | 24 |
| 11 | 4 | 8 | 61 | 30 | 30 | 111 | 38 | 76 |
| 12 | 6 | 6 | 62 | 96 | 192 | 112 | 24 | 48 |
| 13 | 13 | 52 | 63 | 24 | 48 | 113 | 56 | 112 |
| 14 | 24 | 48 | 64 | 48 | 96 | 114 | 60 | 120 |
| 15 | 6 | 12 | 65 | 39 | 156 | 115 | 66 | 132 |
| 16 | 12 | 24 | 66 | 12 | 24 | 116 | 42 | 84 |
| 17 | 8 | 16 | 67 | 68 | 136 | 117 | 78 | 156 |
| 18 | 6 | 6 | 68 | 24 | 48 | 118 | 12 | 24 |
| 19 | 20 | 40 | 69 | 22 | 22 | 119 | 8 | 16 |
| 20 | 6 | 12 | 70 | 24 | 48 | 120 | 6 | 12 |
| 21 | 8 | 16 | 71 | 72 | 144 | 121 | 44 | 88 |
| 22 | 12 | 24 | 72 | 6 | 12 | 122 | 30 | 30 |
| 23 | 22 | 22 | 73 | 37 | 148 | 123 | 14 | 28 |
| 24 | 6 | 12 | 74 | 57 | 228 | 124 | 96 | 192 |
| 25 | 15 | 60 | 75 | 30 | 60 | 125 | 75 | 300 |
| 26 | 39 | 156 | 76 | 60 | 120 | 126 | 24 | 48 |
| 27 | 18 | 18 | 77 | 8 | 16 | 127 | 126 | 126 |
| 28 | 24 | 48 | 78 | 78 | 156 | 128 | 96 | 192 |
| 29 | 7 | 28 | 79 | 26 | 26 | 129 | 42 | 42 |
| 30 | 6 | 12 | 80 | 12 | 24 | 130 | 39 | 156 |
| 31 | 32 | 64 | 81 | 54 | 54 | 131 | 10 | 10 |
| 32 | 24 | 48 | 82 | 21 | 84 | 132 | 12 | 24 |
| 33 | 4 | 8 | 83 | 84 | 168 | 133 | 40 | 80 |
| 34 | 24 | 48 | 84 | 24 | 48 | 134 | 204 | 408 |
| 35 | 24 | 48 | 85 | 24 | 48 | 135 | 18 | 36 |
| 36 | 6 | 6 | 86 | 42 | 42 | 136 | 24 | 48 |
| 37 | 19 | 76 | 87 | 14 | 28 | 137 | 23 | 92 |
| 38 | 60 | 120 | 88 | 12 | 24 | 138 | 66 | 66 |
| 39 | 26 | 52 | 89 | 45 | 180 | 139 | 138 | 138 |
| 40 | 6 | 12 | 90 | 6 | 12 | 140 | 24 | 48 |
| 41 | 7 | 28 | 91 | 104 | 208 | 141 | 48 | 96 |
| 42 | 24 | 48 | 92 | 66 | 66 | 142 | 72 | 144 |
| 43 | 42 | 42 | 93 | 32 | 64 | 143 | 52 | 104 |
| 44 | 12 | 24 | 94 | 48 | 96 | 144 | 12 | 24 |
| 45 | 6 | 12 | 95 | 60 | 120 | 145 | 21 | 84 |
| 46 | 66 | 66 | 96 | 24 | 48 | 146 | 111 | 444 |
| 47 | 48 | 96 | 97 | 49 | 196 | 147 | 56 | 112 |
| 48 | 12 | 24 | 98 | 168 | 336 | 148 | 114 | 228 |
| 49 | 56 | 112 | 99 | 12 | 24 | 149 | 75 | 300 |
| 50 | 15 | 60 | 100 | 30 | 60 | 150 | 30 | 60 |

divisibility for 2 is 3, that for $9 = 3^2$ is 6, and that for 5 is 3. The LCM of 3, 6, and 3 is 6. Hence the period of divisibility by 90 is 6. We note that $B_6 = 360 = 4 \cdot 90$. Based on table 2, we observed that the length of the repeating block modulo $n$ either coincides with the period of divisibility by $n$, or is twice or four times that value, and all repeating blocks for $n \geq 3$ are of even length. We extended the table to $n = 1\,000$ and observed the truth of the above two conjectures as well. The reader is invited to determine if these conjectures are true in general.

## 3. A mathematical recreation: some palatable number tricks

Fibonacci enthusiasts are familiar with the fun fact that if we select any ten consecutive members in the standard Fibonacci (or any Fibonacci-like) sequence, form their sum, and divide this sum by 11, the quotient will always coincide with the seventh term in the sequence. (A Fibonacci-like sequence is any sequence that satisfies the recurrence relation in the Fibonacci sequence where the initial two terms can assume any prescribed values. For example, the Lucas sequence begins with the terms 1 and 3 and thereafter satisfies the identical recurrence relation given in the Fibonacci sequence.) In a similar fashion, the sum of any six consecutive terms is divisible by 4 and the quotient coincides with the fifth term in the sequence while the sum of any fourteen consecutive terms is divisible by 29 and the quotient coincides with the ninth term in the given sequence. We explore a trio of palatable number tricks dealing with our Fibonacci sequence of order three. In table 3, we give the terms of the sequence, the value of each term, and the cumulative sum. Let the initial two terms be $x$ and $y$, respectively.

In our first observation, we note that the sum of six consecutive terms is $48x + 156y$. The greatest common divisor (GCD) of 48 and 156 is 12 and

$$\frac{48x + 156y}{12} = 4x + 13y = (x + 3y) + (3x + 10y).$$

Thus the sum of any six consecutive terms is divisible by 12 and the quotient is the sum of the third and fourth terms in the sequence.

As a second excursion, the sum of any eight consecutive terms in this sequence is $517x + 1705y$. The GCD of 517 and 1 705 is 11 and $(517x+1705y)/11 = 47x+155y$. It is interesting to observe that

$$\frac{517x + 1705y}{11} = 47x + 155y = (x + 3y) + (3x + 10y) + (10x + 33y) + (33x + 109y).$$

Thus the sum of any eight consecutive terms is divisible by 11 and the quotient is the sum of the third, fourth, fifth, and sixth terms in the sequence.

**Table 3**

| Term number in the sequence | Value | Cumulative sum |
|---|---|---|
| 1 | $x$ | $x$ |
| 2 | $y$ | $x + y$ |
| 3 | $x + 3y$ | $2x + 4y$ |
| 4 | $3x + 10y$ | $5x + 14y$ |
| 5 | $10x + 33y$ | $15x + 47y$ |
| 6 | $33x + 109y$ | $48x + 156y$ |
| 7 | $109x + 360y$ | $157x + 516y$ |
| 8 | $360x + 1189y$ | $517x + 1705y$ |
| 9 | $1189x + 3927y$ | $1706x + 5632y$ |
| 10 | $3927x + 12970y$ | $5633x + 18602y$ |

For our final number trick, the sum of any ten consecutive terms in this sequence is $5\,633x + 18\,602y$. The GCD of 5 633 and 18 602 is 131 and

$$\frac{5\,633x + 18\,602y}{131} = 43x + 142y = (10x + 33y) + (33x + 109y).$$

Thus the sum of any ten consecutive terms is divisible by 131 and the quotient is the sum of the fifth and sixth terms in the sequence. We leave it to the reader to generate additional palatable number tricks with this sequence.

## 4. Exploring the ratios of consecutive terms in the sequence and finding an explicit closed formula

We consider the ratios of successive terms in the sequence in table 4. Note that the sequence formed in the first column is increasing while the sequence in the second column is decreasing. Both sequences appear to be converging to the same number! This irrational number appears to play a role analogous to that of the golden ratio in the standard Fibonaccci sequence. In fact, the sequence of successive iterations is approaching the irrational number

$$\frac{3+\sqrt{13}}{2} \approx 3.302\,775\,637\,73.$$

The aforementioned irrational number represents the positive solution to the quadratic equation $x^2 = 3x + 1$ in much the same manner that

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\,033\,988\,75$$

is the solution to the quadratic equation $x^2 = x + 1$ in the standard Fibonacci sequence.

**Table 4**

| | |
|---|---|
| $\frac{B_2}{B_1} = \frac{3}{1} = 3$ | $\frac{B_3}{B_2} = \frac{10}{3} = 3.\overline{3}$ |
| $\frac{B_4}{B_3} = \frac{33}{10} = 3.3$ | $\frac{B_5}{B_4} = \frac{109}{33} = 3.\overline{30}$ |
| $\frac{B_6}{B_5} = \frac{360}{109} = 3.302\,752\,293\,58$ | $\frac{B_7}{B_6} = \frac{1\,189}{360} = 3.302\,\overline{7}$ |
| $\frac{B_8}{B_7} = \frac{3\,927}{1\,189} = 3.302\,775\,441\,55$ | $\frac{B_9}{B_8} = \frac{12\,970}{3\,927} = 3.302\,775\,655\,72$ |
| $\frac{B_{10}}{B_9} = \frac{42\,837}{12\,970} = 3.302\,775\,636\,08$ | $\frac{B_{11}}{B_{10}} = \frac{141\,481}{42\,837} = 3.302\,775\,637\,88$ |
| $\frac{B_{12}}{B_{11}} = \frac{467\,280}{141\,481} = 3.302\,775\,637\,72$ | $\frac{B_{13}}{B_{12}} = \frac{1\,543\,321}{467\,280} = 3.302\,775\,637\,73$ |
| $\frac{B_{14}}{B_{13}} = \frac{5\,097\,243}{1\,543\,321} = 3.302\,775\,637\,73$ | $\frac{B_{15}}{B_{14}} = \frac{16\,835\,050}{5\,097\,243} = 3.302\,775\,637\,73$ |

Suppose that we let

$$\alpha = \frac{3+\sqrt{13}}{2} \quad \text{and} \quad \beta = \frac{3-\sqrt{13}}{2}$$

be the two roots of the quadratic equation $x^2 - 3x - 1 = 0$. Then $\alpha + \beta = 3$ and $\alpha\beta = -1$. Utilizing the recurrence relation, we obtain

$$\mathrm{B}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \left(\left(\frac{3+\sqrt{13}}{2}\right)^n - \left(\frac{3-\sqrt{13}}{2}\right)^n\right) \Big/ \sqrt{13}$$

as our Binet-like formula.

In addition, observe the following:

$$\begin{aligned} \alpha^2 &= 3\alpha + 1, \\ \alpha^3 = \alpha\alpha^2 = \alpha(3\alpha+1) &= 3\alpha^2 + \alpha = 3(3\alpha+1) + \alpha = 10\alpha + 3, \\ \alpha^4 = \alpha\alpha^3 = \alpha(10\alpha+3) &= 10\alpha^2 + 3\alpha = 10(3\alpha+1) + 3\alpha = 33\alpha + 10. \end{aligned}$$

In general, $\alpha^n = \mathrm{B}_n\alpha + \mathrm{B}_{n-1}$.

## 5. Concluding remarks

Recursive sequences are appropriate for obtaining new and illuminating insights into dynamic mathematics. Additionally, such sequences illustrate the essence of the nature of mathematics as the science of patterns. This article incorporates discrete mathematical ideas entailing patterns and modular arithmetic with the aid of technology such as MATHEMATICA®. The reader is cordially invited to partake further of this and other Fibonacci-like sequences, exploring and discovering a world of possibilities in the process.

### Acknowledgments

The author would like to thank his colleague, Marcus Wright, and the Editor for their helpful suggestions and comments which served to improve this article.

#### References

1 Wolfram MathWorld, http://mathworld.wolfram.com/.
2 The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

***Jay L. Schiffman*** *has taught mathematics at Rowan University for the past twenty years. His research interests are in number theory, discrete mathematics, and the interface of mathematics with technology in the classroom as manifested via the use of graphing calculators, computer algebra systems, and mathematical websites. In addition, he enjoys travelling across the USA to conduct workshops and present papers at mathematical conferences.*

# Handcrafted Polyhedral Symmetry

RABE VON RANDOW

We analyse the geometric properties of a polyhedral lampshade.

Always on the lookout for novel representations of polyhedrons, I was really elated when I quite fortuitously came across this supremely beautiful lampshade (figure 1). Quite apart from its captivating, haunting beauty, it embodies the realization of a polyhedron that one certainly does not come across very often, and it achieves this in such a basically simple way: by joining 30 identical, essentially rhombic pieces of stiff paper at their corners (figure 2). It is crafted by a small firm which, in their own words,

> is based in Freiburg at the foot of the Black Forest. That's also the place where the lamps are handmade, produced using only exclusive paper. Having remained under the ownership of one family for three generations, our clear passion is paper (www.lampen-oberkirch.de).

They make only this one lampshade, in three different sizes, and these are only available wholesale.

I will now describe its polyhedral nature. For the following facts I have consulted perhaps the best-known source, namely the book *Mathematical Models* by Cundy and Rollett (see

**Figure 1**

**Figure 2**

reference 1). We shall be working with the five platonic (regular) polyhedrons and several of the 13 archimedean (semi-regular) polyhedrons (ignoring the two infinite families of prisms and antiprisms). The defining properties of the archimedean polyhedrons are given below. We shall also need to know the duals of these polyhedrons, which are obtained as follows. Let $P$ be one of these polyhedrons. Then $P$ is inscribable in a sphere $S$ with all its vertices lying on $S$. If we replace each vertex of $P$ by the tangent plane to $S$ at this vertex, then all these tangent planes form a new polyhedron, namely the dual $P'$ of $P$. Note that the number of faces (vertices) of $P'$ equals the number of vertices (faces) of $P$, and both have the same number of edges.

The duals of the platonic polyhedrons do not produce any new polyhedrons; the tetrahedron is self-dual, the cube and the octahedron are duals of one another, as are the dodecahedron and the icosahedron. The archimedean polyhedrons have the following defining properties:

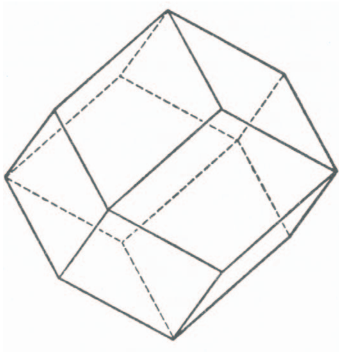
- every face is a regular polygon but the faces are not all of the same kind, and
- all the vertices are the same with respect to the way the faces are arranged around them.

In the archimedean duals, these properties are clearly replaced by the following ones:

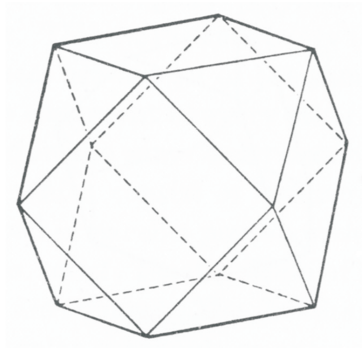
- every vertex is regular (i.e. all the faces incident with it have the same angle there and are arranged symmetrically around it) but the vertices are not all of the same kind, and
- all the faces are congruent.

Thus, the archimedean duals are 13 new solids. Only two of them have rhombic faces, firstly, the *rhombic dodecahedron* (RD) (see figure 3a) which is the dual of the *cuboctahedron* (CO) (figure 3b), and secondly, the *rhombic triacontahedron* (RT) (figure 3c) which is the dual of the *icosidodecahedron* (ID) (figure 3d). Of course, the (rhombic) faces of an RD have different angles from those of an RT.

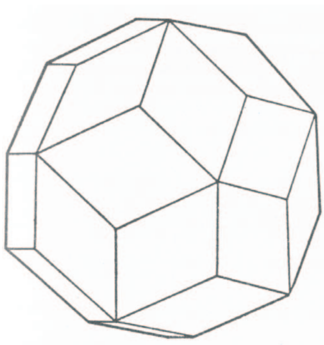
From figure 3 we can see that a CO has 12 vertices, each of degree 4 (i.e. each vertex has four edges incident with it), 24 equal edges, eight triangular faces, and six square faces, while



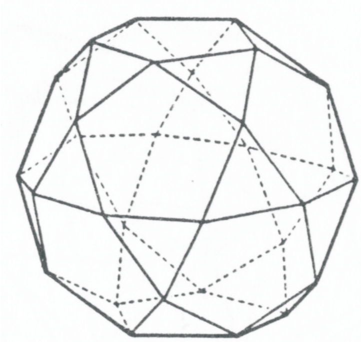
(a) Rhombic dodecahedron (RD)



(b) Cuboctahedron (CO)



(c) Rhombic triacontahedron (RT)



(d) Icosidodecahedron (ID)

**Figure 3**

an RD has 12 rhombic faces, 24 equal edges, eight vertices of degree 3, and six vertices of degree 4. Likewise, an ID has 30 vertices, each of degree 4, 60 equal edges, 20 triangular faces, and 12 pentagonal faces, while an RT has 30 rhombic faces, 60 equal edges, 20 vertices of degree 3, and 12 vertices of degree 5.

The above duality correspondences also yield the following beautiful geometric results.

1. The 12 short diagonals of the faces of an RD form a cube, and the 12 long ones form an octahedron (the dual of the cube). In other words, the vertices of the cube are the eight vertices of degree 3 of the RD, and the vertices of the octahedron are the six vertices of degree 4 of the RD.

2. Similarly, the 30 short diagonals of the faces of an RT form a dodecahedron, and the 30 long ones form an icosahedron (the dual of the dodecahedron). In other words, the vertices of the dodecahedron are the 20 vertices of degree 3 of the RT, and the vertices of the icosahedron are the 12 vertices of degree 5 of the RT.

One cannot but marvel at the many relationships between these constructs, reflecting the duality relations between the platonic polyhedrons. In this connection one question arises naturally. Is there a 'rhombic solid' that yields two self-dual tetrahedrons in this manner? Yes, there is—it is the cube, which is not strictly archimedean as all its faces are the same, however, its face-diagonals clearly form two tetrahedrons.

The secret of the lampshade is now out of the bag; it is an RT. The 30 identical units from which it is constructed are the rhombic faces of an RT (see figure 3c), and the number of faces (three or five) meeting at a vertex can be seen clearly in the lovely three- or five-petal flowers where the faces are joined in their corners (figure 2). Moreover, with some concentration, one can see that the 20 three-petal flowers are the vertices of a dodecahedron and the 12 five-petal flowers are the vertices of an icosahedron.

For the sake of completeness, one must remark that one can clearly build a similar model based on an RD, but this will require only 12 (instead of 30) identical rhombic pieces of paper. As it has only 14 (instead of 32) vertices, its corners will be much more peaked and the entire structure will be much less spherical.

### Reference

1 H. M. Cundy and A. P. Rollett, *Mathematical Models* (Oxford University Press, London, 1951).

***Rabe von Randow*** *studied mathematics, physics, and chemistry at Auckland University, New Zealand, graduating with an MSc in mathematics. He then did his PhD with Professor F. Hirzebruch in Bonn, Germany. After university teaching posts in Dunedin, New Zealand; Tucson, Arizona; Durham, England; and Cologne, Germany; he returned to the University of Bonn, where he remained until his retirement.*

## ***Mathematical Spectrum*** Awards for Volume 45

Prizes have been awarded to the following student readers for contributions in Volume 45:

**Daniel Gould**

for the article 'The Arithmetic Mean-Geometric Mean Inequality';

**Dean G. Hathout**

for the article 'A Stochastic Random-Walk Analysis of the Sport of Squash'.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

# An Elliptic Approach to Constructing the Hyperbola

MICHEL BATAILLE

A well-known construction of the ellipse leads, with surprising ease, to various constructions of the hyperbola.

## 1. Introduction

Confronted with the construction of the hyperbola $\mathcal{H} : x^2/a^2 - y^2/b^2 = 1$, we cannot fail to recall the case of the ellipse $\mathcal{E} : x^2/a^2 + y^2/b^2 = 1$ and would expect a short sequence of simple steps, just as in the following well-known construction of $\mathcal{E}$.

**Construction (c)** Draw the circles $\mathcal{C}_a : x^2 + y^2 = a^2$ and $\mathcal{C}_b : x^2 + y^2 = b^2$ with centre $O(0, 0)$ and radii $a$ and $b$, respectively. If a ray originating at $O$ intersects $\mathcal{C}_a$ at $M$ and $\mathcal{C}_b$ at $N$, the parallels to the $x$-axis through $N$ and to the $y$-axis through $M$ meet at a point $E$ of $\mathcal{E}$ (see figure 1 where $A(a, 0)$ and $B(0, b)$). This directly follows from the fact that $E$ has the same abscissa as $M$ and the same ordinate as $N$, that is $E(a\cos\theta, b\sin\theta)$, where $\theta$ denotes the angle from the $x$-axis to the ray $(ONM)$.

The purpose of this note is to exploit construction (c) and offer constructions of $\mathcal{H}$ in the same vein. In section 2, we work in the wake of construction (c) itself and present two extensions leading to a point on $\mathcal{H}$ by just adding a few lines. In a sense, the rotation of the ray $(ONM)$ around $O$ will produce both $\mathcal{H}$ and $\mathcal{E}$ in one sweep! In section 3, we try to mimic construction (c) and finish, in section 4, with a very simple construction in the same spirit.

## 2. Extending construction (c)

In figure 1, add the tangent $t$ to $\mathcal{C}_a$ at $A$, intersecting $OE$ at $P$. Then, the parallel to the $x$-axis through $P$ and the line $A'E$, where $A'(-a, 0)$, meet at a point $H$ of the hyperbola $\mathcal{H}$ (see figure 2).

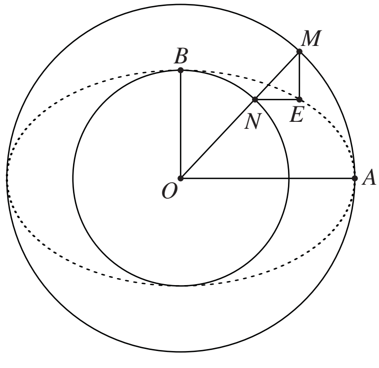



**Figure 1**

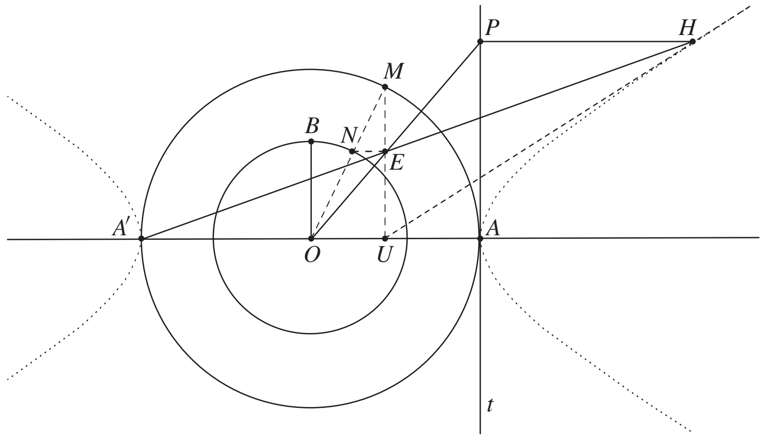



**Figure 2**

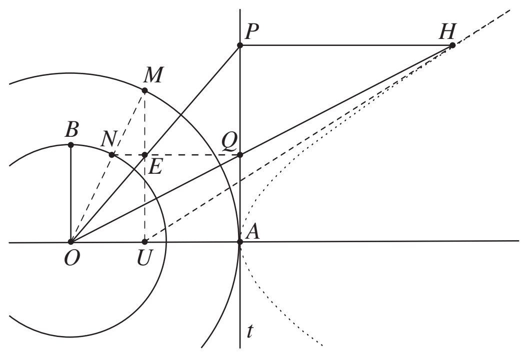



**Figure 3**

The proof is straightforward. We readily obtain the equation of the line $A'E$ as $y = (b/a)(x+a)\tan(\theta/2)$ and $P(a, b\tan\theta)$ (note that $AP/UE = OA/OU$ where $U$ is the orthogonal projection of $E$ onto the $x$-axis). Clearly, the ordinate of $H$ is $y_H = b\tan\theta$ and a simple calculation gives its abscissa:

$$x_H = a\left(\frac{\tan\theta}{\tan(\theta/2)} - 1\right) = a\left(\frac{2\cos^2(\theta/2) - \cos\theta}{\cos\theta}\right) = \frac{a}{\cos\theta}.$$

Since $x_H^2/a^2 - y_H^2/b^2 = 1$, the point $H$ is on the hyperbola $\mathcal{H}$.

Here is a variant of this construction: instead of $A'E$, draw the line $NE$ which intersects $t$ at $Q$. Then the line $OQ$ and the parallel to the $x$-axis through $P$ meet at $H$ (see figure 3). The verification is easy and the details are left to the reader.

Note that all the points of $\mathcal{H}$ are obtained when $M$ traverses the circle $\mathcal{C}_a$ (if $OM$ is perpendicular to $OA$, then $H$ is at infinity on an asymptote of $\mathcal{H}$). As an added bonus, the tangent $T_H$ to $\mathcal{H}$ at $H$ is just the line $UH$ (see figures 2 and 3). The proof is simple: $U(a\cos\theta, 0)$ satisfies the equation of $T_H$, $(x/a^2)(a/\cos\theta) - (y/b^2)b\tan\theta = 1$. The reader

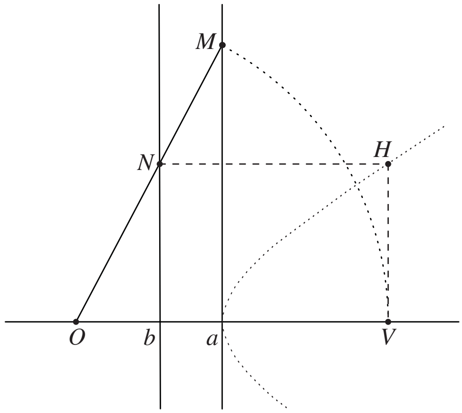



**Figure 4**

may enjoy observing (and proving) that symmetrically, the tangent to $\mathcal{E}$ at $E$ passes through the orthogonal projection of $H$ onto the $x$-axis.

## 3. Trying to mimic construction (c)

We would like to keep the gist of construction (c): the desired point $H$ should borrow its coordinates from the points of intersection of a turning ray with two previously drawn curves (the circles $\mathcal{C}_a$ and $\mathcal{C}_b$ in construction (c)). What curves could replace these circles? Of course, $\mathcal{C}'_a : x^2 - y^2 = a^2$ and $\mathcal{C}'_b : x^2 - y^2 = b^2$ instantly jump to mind. But $\mathcal{C}'_a$ and $\mathcal{C}'_b$ are already hyperbolas (rectangular hyperbolas) so this does not seem promising and we set the idea aside. Instead, let us focus on the coordinates $(a/\cos\theta, b\tan\theta)$ we want for the point $H$. After a moment of reflection, we realize that the lines $L_a : x = a$ and $L_b : x = b$ can do the job! Consider figure 4 where we call $M$ and $N$ the points of intersection of the turning ray with $L_a$ and $L_b$, respectively. If $V$ is a point of the $x$-axis such that $OV = OM$, the parallels to the $x$-axis through $N$ and to the $y$-axis through $V$ intersect at a point $H$ of the hyperbola $\mathcal{H}$ (see figure 4). It is that simple! (The proof is obvious since $OM = a/\cos\theta$ and $y_N = b\tan\theta$.)

When $\theta$ takes all possible values in the interval $(-\pi/2, \pi/2)$ (and with $V$ always of positive abscissa), the construction provides all points of one of the two branches of $\mathcal{H}$.

## 4. A pleasant surprise

An unexpected property will arise from simply remarking that the construction of figure 4 remains valid and is even simpler when $a = b$ (in that case, $L_a = L_b$ and $N = M$; see figure 5). Thus, points on a rectangular hyperbola are particularly easy to obtain and this rings a bell, reminding us of our earlier rejected attempt to use $\mathcal{C}'_a$ and $\mathcal{C}'_b$. In figure 6, which prolongs figure 4, we have constructed $M'$ on $\mathcal{C}'_a$ from the point $M$ and similarly $N'$ on $\mathcal{C}'_b$ from $N$. The pleasant surprise is that $O$, $M'$, $N'$ are collinear! Indeed, it is easily checked that they are on the line $y = (\sin\theta)x$. In addition, $H$ borrows its coordinates from the points of intersection of the ray $(OM'N')$ with the rectangular hyperbolas $\mathcal{C}'_a$, $\mathcal{C}'_b$ just as $E$ does with $(OMN)$ and the circles $\mathcal{C}_a$, $\mathcal{C}_b$ in construction (c). In a hidden way, the construction achieved on figure 4 is the perfect counterpart of our initial construction of the ellipse!

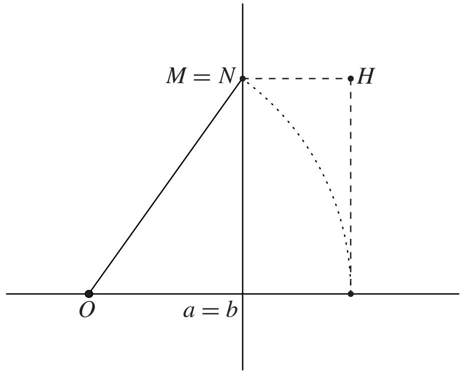



**Figure 5**

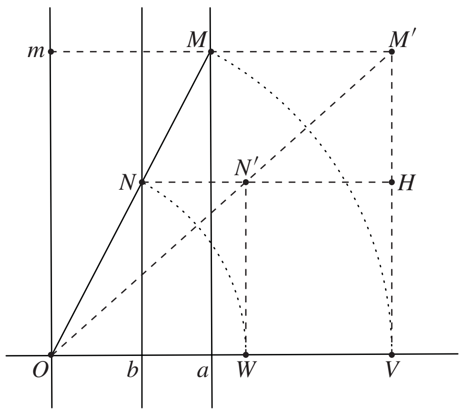



**Figure 6**

A word about the tangent to $\mathcal{H}$ at $H$: it is just the perpendicular through $H$ to the line $mW$ where $m$ is the projection of $M$ onto the $y$-axis and $W$ is to $N$ as $V$ to $M$ (see figure 6). For other constructions of the hyperbola, we refer the reader to two articles that have appeared before in *Mathematical Spectrum* (see references 1 and 2).

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***Michel Bataille*** *teaches at undergraduate level near Rouen in France. His main mathematical interests are geometry and problem solving, both as a solver and as a setter.*

# Dual Divisors

ROGER WEBSTER and GARETH WILLIAMS

We show that 1, 3, 9, 11, 33, and 99 are the only positive integers with the property that divisibility by them is *preserved* under reversal of digits in the dividend.

## 1. Introduction

Nonpalindromic positive integers in decimal form $ab \ldots cd$ $(a \neq 0)$ that divide their reverses $dc \ldots ba$ are called *reverse divisors*. There are infinitely many of these, the first being $1089$, and they are all identified in reference 1. In section 3 of that article, Roger Webster noted that 1, 3, 9, 11, 33, 99 have the property that whenever they divide a positive integer, they *also* divide its reverse. Here we show that *these* are the only such positive integers.

## 2. Basics

Throughout the discussion, a *number* $n$ will mean a positive integer $ab \ldots cd$ $(a \neq 0)$ in *decimal* form, with *reverse* $dc \ldots ba$ denoted by $n^R$. Such a number is called a *dual divisor* if it has the property that whenever it divides a number, it *also* divides the reverse of the number. Thus our objective is to prove that 1, 3, 9, 11, 33, and 99 are the *only* dual divisors.

**Theorem 1** *The numbers* $1,\ 3,\ 9,\ 11,\ 33$, *and* $99$ *are dual divisors.*

*Proof* Clearly, 1 is a dual divisor. That 3, 9, 11 *are*, follows from the facts that a number is divisible by 3 (or 9) if and only if the sum of its digits is divisible by 3 (or 9), and is divisible by 11 if and only if the *alternating* sum of its digits is divisible by 11. Since 3 (or 9) is *coprime* to 11, a number is divisible by 33 (or 99) if and only if it is both divisible by 3 (or 9) and 11. It follows that a number is divisible by 33 (or 99) if and only if its reverse is, showing that 33 and 99 are dual divisors. See reference 2.

Before embarking on the intricate proof that 1, 3, 9, 11, 33, and 99 are the only dual divisors, we pause for breath. Many of our arguments involve showing that certain numbers are *not* dual divisors. To prove that a number $n$ is *not* a dual divisor, it suffices to find a number $k$ such that $(k \times n)^R$ is *not* divisible by $n$. By way of an interesting example, to show that the reverse divisor $109\,989$ is *not* a dual divisor, we note that $(911 \times 109\,989)^R$ is *not* divisible by $109\,989$, whereas for *all* $k < 911$, $(k \times 109\,989)^R$ *is*.

A word of caution. Our results are stated concisely, and our arguments, whilst elementary, are both formal and detailed. To help readers unaccustomed to such situations, we have included more discussion in some proofs than is necessary for the more seasoned reader. For reasons of length, and to avoid repeating similar arguments, we do not include statements and proofs of *everything* here, preferring to make a full version of this article available, with *every* detail included, online (see reference 3).

**Lemma 1** *Every number has a multiple whose first digit is* $1$.

*Proof* Let $n$ be a $d$-digit number, so $n < 10^d$. Let $r$ be the *smallest* number such that $r \times n > 10^d$, so $(r-1) \times n \leq 10^d$. Then $10^d < r \times n < 2 \times 10^d$, whence the first digit of $r \times n$ is 1.

**Corollary 1** *No dual divisor is even or ends in* $5$.

*Proof* Let $n$ be a number that is even or ends in 5. Then by lemma 1, there is a number $k$ such that $(k \times n)^R = \dots 1$, which is *not* divisible by $n$. Thus $n$ divides $k \times n$ but not $(k \times n)^R$, and $n$ is *not* a dual divisor.

**Lemma 2** *No number whose first and final digits are* $7$ *is a dual divisor.*

*Proof* Let $n = 7 \dots 7$ be a $d$-digit $(d \geq 1)$ number whose first and final digits are 7. Then the $(d+1)$-digit number $m = (2 \times n)^R = 4 \dots 1$ is *not* divisible by $n$, for if it were, the quotient would end in 3, but $3 \times n < m$ and $13 \times n > m$. Thus $n$ divides $2 \times n$ but not $(2 \times n)^R$, and so $n$ is *not* a dual divisor.

We leave it to the reader to complete the details as to why $3 \times n < m$ and $13 \times n > m$. To build confidence, here is a hint: in the proof of lemma 2, $n$ must be less than $80 \dots 00$ ($d$ digits), so what may be deduced about $3 \times n$? Similarly, $n$ must be greater than $70 \dots 00$ ($d$ digits), so what of $13 \times n$?

**Theorem 2** *A dual divisor is either a reverse divisor or a palindrome beginning with a* $1$, $3$, *or* $9$.

*Proof* Suppose that the number $n$ is a dual divisor. Since $n$ divides itself, it must divide $n^R$. If $n^R \neq n$, then $n$ is a reverse divisor. If $n^R = n$, then $n$ is a palindrome, which begins with a 1, 3, or 9 by corollary 1 and lemma 2.

In the remainder of this article, we progressively eliminate numbers of the forms specified in theorem 2 from being dual divisors, until only the six dual divisors given in theorem 1 remain.

## 3. Reverse divisors and dual divisors

We now prove that *no* reverse divisor is a dual divisor. Our analysis relies upon properties of reverse divisors established in reference 1. Reverse divisors *either* have quotient 4 *or* quotient 9, which means that, in the former case, the reverse is *four* times the original number, and in the latter, *nine* times. We also use the facts that reverse divisors with quotient 4 are *even*, and that reverse divisors with quotient 9 have the form $10 \dots 89$. For each nonnegative integer $r$, we denote by $0_r$ the string of $r$ consecutive 0s.

**Theorem 3** *No reverse divisor is a dual divisor.*

*Proof* Reverse divisors with quotient 4 are even, and so by corollary 1 *cannot* be dual divisors. Let, then, $n = 10A89$ be a $d$-digit $(d \geq 4)$ reverse divisor with quotient 9, where $A$ is a string of $(d-4)$ digits. We establish the theorem by proving $(k \times n)^R$ is not divisible by $n$ for $k = 10^{d-1} + 1$. Consider the number

$$m = ((10^{d-1} + 1) \times n)^R = (10A890_{d-1} + 10A89)^R = 98A^R009A^R01.$$

Next, remembering that $n$ is a reverse divisor with quotient 9, we observe that

$$9 \times 10^{d-1} \times n = 10^{d-1} \times n^R = 98A^R010_{d-1}.$$

Since the number $(9 \times 10^{d-1} \times n) - m = 10^{d-1} - 9A^R01$ has $(d-2)$ digits, it cannot be divisible by $n$, whence neither can $m$ be. Thus $n$ is *not* a dual divisor.

## 4. Palindromes of the form 1...1 and dual divisors

Here we show that there are *no* palindromic dual divisors with *three* or more digits which begin with a 1.

**Lemma 3** *No palindrome of the form* $1a\dots 1$ $(a \neq 1)$ *is a dual divisor.*

Our proof considers each of the *nine* possible cases for $a$. The case $a = 0$ (a good warm up for the proof of lemma 4, below) is the hardest, and could be left until *after* the reader has mastered the other cases.

*Proof of lemma 3* Let $n$ be a $d$-digit ($d \geq 3$) palindrome of the form $1a\dots 1$ ($a \neq 1$).

Case $a = 0$. If $n = 101$, then $m = (109 \times n)^R = 90\,011$ is not divisible by $n$. If $n = 10A01$, where $A$ is a palindromic string of $(d-4)$ digits ($d \geq 4$), then $9 \times n = 9B09$, for some string $B$ of $(d-3)$ digits. We now establish this case by proving $(k \times n)^R$ is not divisible by $n$ for $k = 9 \times 10^{d-1} + 1$. Consider the number

$$m = ((9 \times 10^{d-1} + 1) \times n)^R = (9B090_{d-1} + 10A01)^R = 10A^R001B^R9 = 10A001B^R9.$$

Then the number $(10^{d-1} \times n) - m = 10^{d-1} - 1B^R9$ has $(d-1)$ digits, and so is not divisible by $n$. Thus neither is $m$, and $n$ is *not* a dual divisor.

Case $a = 2$. We have $m = (5 \times n)^R = 5\dots 6$ is not divisible by $n$, for if it were, it would be $6n$, but $6n > m$. Thus $n$ is *not* a dual divisor.

Case $a = 3$. We have $m = (4 \times n)^R = 4\dots 5$ is not divisible by $n$, for if it were, it would be $5n$, but $5n > m$. Thus $n$ is *not* a dual divisor.

Case $a = 4$. We have that the inequalities $2n < (3 \times n)^R = 3\dots 4 < 3n$ show that $n$ is *not* a dual divisor.

Cases $a \geq 5$. We have that the inequalities $n < (2 \times n)^R = 2\dots 3 < 2n$ show that $n$ is *not* a dual divisor.

**Lemma 4** *Let* $n$ *be a* $d$*-digit* ($d \geq 5$) *palindrome of the form* $n = 11A11$, *where* $A$ *is a* $(d-4)$*-digit palindromic string containing a digit other than* 1. *Then* $n$ *is not a dual divisor.*

*Proof* We consider the *two* possible cases for the number of digits of $9 \times A$. We establish *both* cases by proving $(k \times n)^R$ is not divisible by $n$ for $k = 9 \times 10^{d-1} + 1$.

Case $9 \times A$ has $(d-4)$ digits. Since $A$ contains a digit other than 1, the number $9 \times A$ contains a digit other than 9. Thus $B = (9 \times A) + 1$ contains $(d-4)$ digits. Consider the number

$$\begin{aligned} m &= ((9 \times 10^{d-1} + 1) \times n)^R \\ &= (99(9 \times A)990_{d-1} + 11A11)^R \\ &= 11A^R100B^R99 \\ &= 11A100B^R99. \end{aligned}$$

Then the number $(10^{d-1} \times n) - m = 10^{d-1} - B^R99$ has $(d-1)$ digits, and so is not divisible by $n$. Thus neither is $m$, and $n$ is *not* a dual divisor.

Case $9 \times A$ has $(d-3)$ digits. We have $9 \times n = 10C99$, where both $C$ and $D = C+1$ have $(d-3)$ digits. Consider the number

$$m = ((9 \times 10^{d-1} + 1) \times n)^R = (10C990_{d-1} + 11A11)^R = 11A^R100D^R01 = 11A100D^R01.$$

Then the number $(10^d \times n) - m = 10^d - D^R01$ is a $d$-digit number ending in 9. But no $d$-digit number ending in 9 can be divisible by $n$, for $9 \times n$ has $(d+1)$ digits, and $l \times n$ $(l < 9)$ does not end in 9. Thus $(10^d \times n) - m$, and hence $m$, is not divisible by $n$, and $n$ is *not* a dual divisor.

**Lemma 5** *Let $n$ be a $d$-digit ($d \geq 3$) number containing only 1s. Then $n$ is not a dual divisor.*

*Proof* The $(d+1)$-digit number $m = (19 \times n)^R = 90\ldots12$ is not divisible by $n$, since the smallest $l$ for which $l \times n$ ends in 12 is 92, but $92 \times n > m$. Thus $n$ is *not* a dual divisor.

**Summary 1** *No palindrome beginning with a 1, having three or more digits, is a dual divisor.*

As a postscript, all the results in this section have analogues for palindromes beginning with a 3 or 9. Their statements and proofs, given in reference 3, require modification in their detail, but not in their style.

Theorems 1, 2, and 3 combine with summary 1, and its direct analogues for palindromes beginning with a 3 or 9, to bring us to our goal.

**Conclusion 1** *There are precisely six dual divisors: 1, 3, 9, 11, 33, and 99.*

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***Roger Webster*** *has lectured at Sheffield University for forty-nine years and is the author of 'Convexity' (Oxford University Press, 1994). He has written many general interest articles, including the very first to appear in 'Mathematical Spectrum' in 1968.*

***Gareth Williams*** *is a staff tutor in mathematics at The Open University. He is a pure mathematician specializing in topology, and is in great demand as a popularizer of mathematics.*


**Maximum value of a determinant**

Consider a $3 \times 3$ determinant having nine cells, where the entry of each cell is either 1 or 0. What is the maximum possible value of this determinant?


Integral University, Lucknow, India **Mohammed Faraz Khan**

# Rational Approximation of Square Roots by Recurrence Relations

HO-HON LEUNG

This article describes a method of finding rational approximations to square roots by certain recurrence relations.

## 1. Introduction

Historically it was an interesting problem to compute square roots. For example, Newton's method and the Bakhshali approximation are two methods that were known in the past, see references 1 and 2 respectively. In particular, for the square root of 2, that the Pell numbers can be used for approximation has been known since ancient times, see reference 3. Nowadays there are many iterative methods of computing square roots in numerical analysis, see references 4 and 5 for examples. In this article we describe an elementary method of finding rational approximations by certain recurrence relations.

## 2. Main results

For any positive integer $q$ which is not a perfect square, let $p$ be the closest integer to $\sqrt{q}$. Define a recurrence relation as follows:

$$a_n = 2pa_{n-1} + (q - p^2)a_{n-2}, \tag{1}$$

for $n = 2, 3, \dots$, where $a_0$ and $a_1$ are integers chosen arbitrarily. We look for solutions of the form $a_n = r^n$, where $r$ is a nonzero constant. Note that $a_n = r^n$ is a solution of the recurrence relation (1) if and only if

$$r^n = 2pr^{n-1} + (q - p^2)r^{n-2}.$$

Dividing both sides by $r^{n-2}$, we obtain the *characteristic equation* of the recurrence relation

$$r^2 - 2pr + p^2 - q = 0.$$

The roots of this equation are

$$r_1 = p + \sqrt{q}, \qquad r_2 = p - \sqrt{q}.$$

(In the literature they are called the *characteristic roots* of the recurrence relation.) The general solution of the recurrence relation (1) is

$$a_n = k_1(p + \sqrt{q})^n + k_2(p - \sqrt{q})^n, \tag{2}$$

where $k_1$ and $k_2$ are constants that depend on the values of $a_0$ and $a_1$. Since $|p - \sqrt{q}| < \frac{1}{2} < 1$,

$$\lim_{n \to \infty} (p - \sqrt{q})^n = 0.$$

This means that, for large values of $n$, the $(p+\sqrt{q})^n$ term dominates the expression of $a_n$. Hence, we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=p+\sqrt{q}. \tag{3}$$

It implies that the sequence of rational numbers $b_n$ defined by

$$b_n=\frac{a_{n+1}}{a_n}-p=\frac{a_{n+1}-pa_n}{a_n} \tag{4}$$

approaches $\sqrt{q}$ as $n$ tends to infinity.

**Remark 1** Note that (3) holds true regardless of the values of $a_0$ and $a_1$. So $b_n$ defined by (4) can be used to approximate $\sqrt{q}$ regardless of the values of $b_0$ and $b_1$.

**Example 1** To approximate $\sqrt{69}$, $q=69$ and $p=8$. Then the recurrence relation (1) becomes

$$a_n=16a_{n-1}+5a_{n-2}.$$

We pick $a_0=0$ and $a_1=1$. Then we have $a_2=16a_1+5a_0=16$, $a_3=16a_2+5a_1=261$, and $a_4=16a_3+5a_2=4256$. Hence,

$$b_2=\frac{a_3-8a_2}{a_2}=\frac{133}{16}=8.312\,5,$$
$$b_3=\frac{a_4-8a_3}{a_3}=\frac{2\,168}{261}=8.306\,513\,410\,0\ldots.$$

We have done the computation up to $b_3$ only, yet it is a close approximation to $\sqrt{69}=8.306\,623\,862\,9\ldots$.

**Remark 2** Note that $p$ is chosen such that it is the integer closest to $\sqrt{q}$. If $p<\sqrt{q}<p+1$ and $p+1$ is chosen instead of $p$, then we still have $|(p+1)-\sqrt{q}|<1$ and hence $\lim_{n\to\infty}[(p+1)-\sqrt{q}]^n=0$. So if we define an alternative recurrence relation by

$$a_n=2(p+1)a_{n-1}+(q-(p+1)^2)a_{n-2},$$

then we can define $b_n$ by

$$b_n=\frac{a_{n+1}}{a_n}-(p+1)$$

and use it to approximate $\sqrt{q}$. But the rate of convergence in this case may not be as fast as the rate of convergence by using recurrence relation (1) and the definition of $b_n$ by (4) because $|p-\sqrt{q}|<\frac{1}{2}<|(p+1)-\sqrt{q}|$.

**Remark 3** The rate of convergence of $b_n$ to $\sqrt{q}$ is governed by the rate of convergence of $a_{n+1}/a_n$ to $p+\sqrt{q}$. Since

$$\frac{a_{n+1}}{a_n}=\frac{k_1(p+\sqrt{q})^{n+1}+k_2(p-\sqrt{q})^{n+1}}{k_1(p+\sqrt{q})^n+k_2(p-\sqrt{q})^n},$$

our approximation works at its best for extremely small values of $p-\sqrt{q}$ (as $a_{n+1}/a_n$ converges to $p+\sqrt{q}$ faster if $p-\sqrt{q}$ is smaller), that is, when $\sqrt{q}$ is extremely close to an integer.

**Remark 4** The constants $k_1$ and $k_2$ in the general solution (2) of the relation (1) are determined by $a_0$ and $a_1$. By (2),

$$a_0 = k_1 + k_2,$$
$$a_1 = p(k_1 + k_2) + \sqrt{q}(k_1 - k_2).$$

If $a_0 = 2$ and $a_1 = pa_0 = 2p$, then $\sqrt{q}(k_1 - k_2) = 0$ and hence $k_1 - k_2 = 0$. So we have

$$k_1 = k_2 = \frac{a_0}{2} = 1.$$

The general solution of the relation (1) becomes

$$a_n = (p + \sqrt{q})^n + (p - \sqrt{q})^n$$

and $a_n$ is the integer closest to

$$(p + \sqrt{q})^n$$

since $|p - \sqrt{q}| < \frac{1}{2}$.

More generally this method can be applied to the square root of $q$ when $q$ is a rational number. Let

$$q = \frac{c}{d},$$

where $c$ and $d$ are positive integers. Let $p$ be the integer closest to $\sqrt{c/d}$. Define a recurrence relation as follows:

$$a_n = 2pa_{n-1} + \left(\frac{c}{d} - p^2\right)a_{n-2}, \tag{5}$$

for $n = 2, 3, \ldots$, where $a_0$ and $a_1$ are arbitrarily chosen. Then the rational sequence $b_n$ defined by

$$b_n = \frac{a_{n+1} - pa_n}{a_n}$$

can be used to approximate $\sqrt{c/d}$.

**Example 2** To approximate $\sqrt{10/3}$, $q = \frac{10}{3}$ and $p = 2$. Then the recurrence relation (5) becomes

$$a_n = 4a_{n-1} + \left(\tfrac{10}{3} - 4\right)a_{n-2} = 4a_{n-1} - \tfrac{2}{3}a_{n-2}.$$

We choose $a_0 = 0$ and $a_1 = 1$. Then $a_2 = 4$, $a_3 = \frac{46}{3}$, and $a_4 = \frac{176}{3}$. Hence,

$$b_2 = \frac{\frac{46}{3} - 2(4)}{4} = \frac{11}{6} = 1.833\,33\ldots,$$
$$b_3 = \frac{\frac{176}{3} - 2(\frac{46}{3})}{\frac{46}{3}} = \frac{42}{23} = 1.826\,086\,956\,5\ldots;$$

$b_3$ is a fairly close approximation to $\sqrt{10/3} = 1.825\,741\,858\,4\ldots$.

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*Born on the 24th of September 1984,* ***Ho Hon Leung*** *is currently an Assistant Professor of Mathematics at the Canadian University of Dubai, UAE. He graduated from Cornell University with a PhD in Mathematics in 2011. His mathematical interests are primarily in geometry and topology. He is also interested in recreational problems in geometry and number theory.*

# Letters to the Editor

Dear Editor,

$$x^n + y^n = z^n \pm 1 \text{ when } n > 2$$

By Fermat's Last Theorem, the equation $x^n + y^n = z^n$ has no solution in positive integers when $n > 2$. However, the equations $x^3 + y^3 = z^3 \pm 1$ *do* have such solutions. I have found the identity

$$(9k^4)^3 + (1 - 9k^3)^3 + (3k - 9k^4)^3 = 1.$$

This gives solutions to the equation $x^3 + y^3 = z^3 - 1$, for example

$$\begin{aligned} 8^3 + 6^3 &= 9^3 - 1 \quad (\text{from } k = 1), \\ 71^3 + 138^3 &= 144^3 - 1 \quad (\text{from } k = 2), \\ 242^3 + 720^3 &= 729^3 - 1 \quad (\text{from } k = 3). \end{aligned}$$

I found other solutions by computer which do not come from this identity, specifically

$$\begin{aligned} 135^3 + 138^3 &= 172^3 - 1, \\ 372^3 + 426^3 &= 505^3 - 1, \\ 426^3 + 486^3 &= 577^3 - 1, \\ 566^3 + 823^3 &= 904^3 - 1. \end{aligned}$$

The identity gives solutions to the equation $x^3 + y^3 = z^3 + 1$, for example

$$\begin{aligned} 9^3 + 10^3 &= 12^3 + 1 \quad (\text{from } k = -1), \\ 73^3 + 144^3 &= 150^3 + 1 \quad (\text{from } k = -2), \\ 244^3 + 729^3 &= 738^3 + 1 \quad (\text{from } k = -3). \end{aligned}$$

I found other solutions by computer which do not come from the identity, specifically

$$64^3 + 94^3 = 103^3 + 1,$$
$$135^3 + 235^3 = 249^3 + 1,$$
$$334^3 + 438^3 = 495^3 + 1.$$

Can readers find solutions of the equations $x^n + y^n = z^n \pm 1$ when $n > 3$?

I have found another identity:

$$(1 + 6k^3)^3 + (1 - 6k^3)^3 + (-6k^3)^3 = 2,$$

which will provide solutions to the equations $x^3 + y^3 = z^3 \pm 2$ in positive integers.

Yours sincerely,
**Abbas Rouholamini**
(Sirjan
Iran)

Dear Editor,

*A geometric appearance of relativity*

The well-known formula for the relativistic sum of two speeds $a$ and $b$ is

$$\frac{a+b}{1+(ab)/c^2},$$

where $c$ is the speed of light. This formula makes a geometrical appearance as follows. Parameterize the points on the unit circle as shown in figure 1. (The point $(0, 1)$ can be given the parameter $\infty$.) Then the point on the circle with parameter $a$ has Cartesian coordinates

$$\left(\frac{2a}{a^2+1}, \frac{a^2-1}{a^2+1}\right).$$

Given points with parameters $a, b$ on the unit circle, the point with parameter $a + b$ can be found as shown in figure 2. First draw the straight line through $a$ and $b$ to meet the line $y = 1$ in the point $P$, which will have coordinates $(2/(a + b), 1)$. (If $a = b$, draw the tangent to the

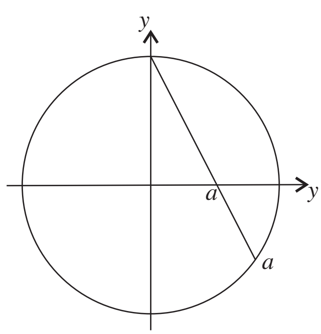


**Figure 1**

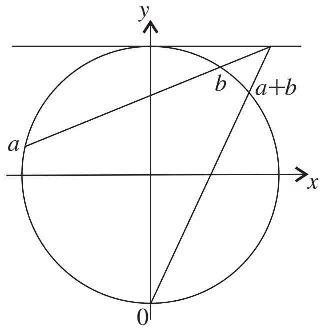


**Figure 2**

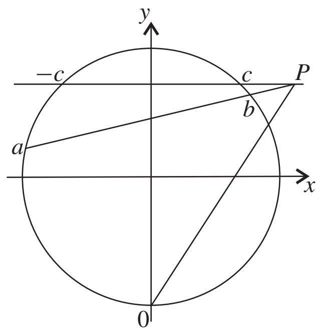


**Figure 3**

circle at $a$.) The straight line through the points with parameters $0$ and $a+b$ will also pass through $P$, so join $P$ to the point $(0,-1)$ to meet the circle again at the point $a+b$.

Suppose, instead of using the line $y=1$, we use the line through the points with parameters $c$ and $-c$ (figure 3). The straight line through $a$ and $b$ will meet this line in the point $P$ with coordinates

$$\left(2\Big/\frac{a+b}{(1+(ab)/c^2)}\times\frac{c^2}{c^2+1},\frac{c^2-1}{c^2+1}\right).$$

(Note that this tends to the previous point $P$ as $c\to\infty$.) The straight line through $(0,-1)$, with parameter $O$, and the point with parameter

$$\frac{a+b}{1+(ab/c^2)}$$

will also pass through $P$, so the straight line joining $P$ to $(0,-1)$ will meet the unit circle again at the point whose parameter is the relativistic sum of $a$ and $b$.

Yours sincerely,

**Guido Lasters**
(Ganzendries 245
Tienen, 3300
Belgium)

Dear Editor,

*Was the scheduling of the final rounds of the IPL 2012 fair?*

In their article in Volume 46, Number 1, pp. 30–31, J. Shiwalkar and M. N. Deshpande suggested that the schedule for the final rounds of the Indian Premier League in 2012 was not fair. The top four teams, A, B, C, and D, qualify. First A plays B and C plays D. The winner of the first match proceeds to the final and the loser of the second match is eliminated. The loser of the first match then plays the winner of the second match with the winner also proceeding to the final. Assuming that each team has an equal chance of winning each game (which may not be the case) gives the probability that A or B will win the final as $\frac{3}{8}$ and the probability that C or D will win the final as $\frac{1}{8}$. The suggestion is made that this is unfair. But teams finishing in a higher position should be rewarded by having a higher probability than those in a lower position. However, this does seem a little extreme.

A better system which I believe was once used by the Australian Football League is the following. The top five teams, A, B, C, D, and E, qualify. In Round 1, Match 1 is between B and C and Match 2 is between D and E. The loser of the latter is eliminated. In Round 2, Match 1 is between A and the winner of Round 1 Match 1 and Match 2 is between the loser of Round 1 Match 1 and the winner of Round 1 Match 2. Again the loser of the latter is eliminated. The winner of Round 2 Match 1 proceeds to the final and there is a further game between the loser of Round 2 Match 1 and the winner of Round 2 Match 2 to determine the other finalist.

Thus to proceed to the final, teams D and E must win three consecutive games, teams B and C must win two games out of a possible three and team A must win one game out of a possible two. Every possible combination of teams can contest the final except that D cannot play E. There are five teams involved instead of four which is good for maintaining interest throughout the season and a total of six games instead of four which is good for revenue. The probabilities of winning the tournament are $\frac{3}{8}$ for A, $\frac{1}{4}$ for B and C, and $\frac{1}{16}$ for D and E. Thus A has a better chance than any other team which seems fair and the teams finishing 4th and 5th have a much reduced chance than the three teams above them.

Yours sincerely,

**Terry S. Griggs**
(The Open University)

Dear Editor,

*Decimal expansion of fractions*

Readers will probably be familiar with the technique of expressing a fraction as a decimal. Take $\frac{7}{17}$ as an example. The procedure is to multiply the numerator by 10, divide 70 by 17 to give quotient 4 and remainder 2. The decimal expansion thus begins 0.4. This is then repeated with 2, and so on. Since the possible remainders are 1 to 16, there must be a repetition at least after 16 steps, so the decimal repeats. In fact, the infinite decimal expansion of $\frac{7}{17}$ is

$$0.\dot{4}11\,764\,705\,882\,352\,\dot{9},$$

where the dots denote the repeating block. This can be wearing on patience, especially with a large denominator, with a propensity for errors to arise.

This procedure is pre-calculator. These days, if one is asked to express $\frac{7}{17}$ as a decimal, the universal reaction is to reach for a calculator and write, using a standard 10-digit calculator,

$$\tfrac{7}{17} = 0.411\,764\,705,$$

which is only correct to its eigth decimal place.

Can we use our calculator to obtain the correct decimal representation without going through the pre-calculator procedure with which we began?

The answer is 'yes'. At each step, instead of multiplying by 10, we multiply by a power of 10. We know that the decimal must repeat by the 17th stage, and we cannot use $10^9$ because of the limits of the calculator, so we cannot do it in two steps, but we can do it in three if we multiply by $10^6$ $(3 \times 6 = 18)$. So, divide $7 \times 10^6$ by 17 using the calculator to obtain the quotient 411 764 and remainder $7 \times 10^6 - 411\,764 \times 17 = 12$. For the second step, divide $12 \times 10^6$ by 17 to give the quotient 705 882 and remainder $12 \times 10^6 - 705\,882 \times 17 = 6$. In the third step, divide $6 \times 10^6$ by 17 to give the quotient 352 941. Thus, the decimal expansion of $\frac{7}{17}$ is

$$0.411\,764\,705\,882\,352\,941$$

to 18 decimal places. But it must have repeated by the 17th stage. It has not repeated before that, so 41 must start the repeating block, and

$$\tfrac{7}{17} = 0.\dot{4}11\,764\,705\,882\,352\,\dot{9},$$

which our pre-calculator ancestors had already worked out!

**Reference**

1 K. S. Leung, Converting a fraction into a decimal by hand calculation, *Math. Spectrum* **45** (2012/2013), pp. 34–37.

Yours sincerely,

**Bob Bertuello**
(12 Pinewood Road
Midsomer Norton BA3 2RG
UK)

Dear Editor,

*Goldbach $\Longrightarrow$ Bertrand*

The *Goldbach conjecture* states that every even integer greater than 2 is the sum of two primes. The weaker version of *Bertrand's postulate* is that, for every integer $n > 1$, there is a prime number strictly between $n$ and $2n$. (The stronger version has $2n$ replaced by $2n - 2$, with $n > 3$.) Bertrand's postulate was first proved by Tchebychef in 1852; the Goldbach conjecture is unproven up to now. I have noticed that the Goldbach conjecture implies the weak version of Bertrand's postulate. Let $n > 1$ be an integer. Then by Goldbach,

$$2n + 2 = p_1 + p_2,$$

for some primes $p_1$, $p_2$, which must be at least 3 and so both less than $2n$, but one of them must be greater than $n$.

Yours sincerely,

**Guido Lasters**
(Ganzendries 245
Tienen, 3300
Belgium)

Dear Editor,

*A reciprocal sum of Fibonacci products*

Sums of the type $\sum_{i=1}^{n}(1/\mathrm{F}_i)$ have been discussed in *The Fibonacci Quarterly* (see reference 1), but it appears that the results of such summation cannot be put in a closed form. However, it is interesting to note that the following sum of reciprocals of Fibonacci products can be represented by a closed formula:

$$\sum_{i=1}^{n}\frac{1}{\mathrm{F}_{2i}\mathrm{F}_{2i+2}}=\frac{\mathrm{F}_{2n}}{\mathrm{F}_{2n+2}}.$$

We use induction to prove this. We note that the formula is true for $n=1$ as

$$\frac{1}{\mathrm{F}_2\mathrm{F}_4}=\frac{1}{1\times 3}=\frac{\mathrm{F}_2}{\mathrm{F}_4}.$$

Now assume that the result is true for some $n$. To prove that it is true for $n+1$, we need to show that

$$\frac{\mathrm{F}_{2n}}{\mathrm{F}_{2n+2}}+\frac{1}{\mathrm{F}_{2n+2}\mathrm{F}_{2n+4}}=\frac{\mathrm{F}_{2n+2}}{\mathrm{F}_{2n+4}},$$

which simplifies to

$$\mathrm{F}_{2n}\mathrm{F}_{2n+4}+1=\mathrm{F}_{2n+2}^2.$$

Binet's formula for $\mathrm{F}_n$ is $\mathrm{F}_n=(\alpha^n-\beta^n)/\sqrt{5}$, where $\alpha=(1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2$, so that $\alpha\beta=-1$. Hence

$$\begin{aligned}\mathrm{F}_{2n}\mathrm{F}_{2n+4}+1&=\frac{(\alpha^{2n}-\beta^{2n})(\alpha^{2n+4}-\beta^{2n+4})}{5}+1\\&=\frac{\alpha^{4n+4}+\beta^{4n+4}-(\alpha^4+\beta^4)}{5}+1.\end{aligned}$$

Now $\alpha^4+\beta^4=[(\alpha+\beta)^2-2\alpha\beta]^2-2\alpha^2\beta^2=7$, so that

$$\begin{aligned}\mathrm{F}_{2n}\mathrm{F}_{2n+4}+1&=\frac{\alpha^{4n+4}+\beta^{4n+4}-7}{5}+1\\&=\frac{\alpha^{4n+4}-2+\beta^{4n+4}}{5}\\&=\frac{(\alpha^{2n+2}-\beta^{2n+2})^2}{5}\\&=\mathrm{F}_{2n+2}^2,\end{aligned}$$

as required.

**Reference**

1 R. S. Melham and A. G. Shannon, On reciprocal sums of Chebyshev related sequences, *Fibonacci Quart.* **33** (1995), pp. 194–202.

Yours sincerely,

**M. A. Khan**
(Lucknow
India)

# Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

## Problems

**46.5** Find all isosceles triangles such that the length of each side is numerically equal to the square of the secant of the angle opposite.

(Submitted by Herb Bailey, Rose-Hulman Institute of Technology, USA, and William Gosnell, Amherst, MA, USA)

**46.6** A sheet of paper is 11 cm $\times$ 8 cm. It is folded so that the bottom right-hand corner touches the left-hand edge to form a crease of length $c$ cm. What is the minimum value of $c$?

(Submitted by Bob Bertuello, Midsomer Norton, UK)

**46.7** If four fair six-sided dice are thrown, what is the probability that the numbers add up to a prime?

(Submitted by Chris Caldwell, University of Tennessee at Martin, USA)

**46.8** If $\boldsymbol{A}$, $\boldsymbol{B}$ are $3 \times 3$ matrices such that $\det \boldsymbol{A} = \det \boldsymbol{B} = 3$ and $\det(\boldsymbol{A} + \boldsymbol{B}) = -1$, how are $\det(3\boldsymbol{A} + 2\boldsymbol{B})$ and $\det(2\boldsymbol{A} + 3\boldsymbol{B})$ related?

(Submitted by Marcel Chirita, Bucharest, Romania)

## Solutions to Problems in Volume 45 Number 3

**45.9** If $a + b > 0$, find the minimum value of $a\sin^4 x + b\cos^4 x$.

*Solution* by Abbas Rouholamini Gugheri, who proposed the problem

Suppose that $a, b \geq 0$. From

$$(a\sin^2 x - b\cos^2 x)^2 \geq 0,$$

we have

$$a^2\sin^4 x + b^2\cos^4 x - 2ab\sin^2 x\cos^2 x \geq 0.$$

But

$$1 = (\sin^2 x + \cos^2 x)^2 = \sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x,$$

so that

$$a^2\sin^4 x + b^2\cos^4 x + ab(\sin^4 x + \cos^4 x) \geq ab,$$

whence

$$(a+b)(a\sin^4 x + b\cos^4 x) \ge ab.$$

Thus

$$a\sin^4 x + b\cos^4 x \ge \frac{ab}{a+b},$$

since $a+b > 0$. Equality occurs when $a\sin^2 x = b\cos^2 x$, i.e. when $\tan x = \pm\sqrt{b/a}$. When $a > 0$, $b < 0$, $a\sin^4 x + b\cos^4 x \ge b$ and equality occurs when $x = 0$. When $a < 0$, $b > 0$, $a\sin^4 x + b\cos^4 x \ge a$ and equality occurs when $x = \pi/2$.

Also solved by Subramanyam Durbha (Comunnity College of Philadelphia, USA) and Spiros Andriopoulos (Third High School of Amaliada, Eleia, Greece).

**45.10** A known inequality for natural logarithms is $\ln x \le x - 1$ for all $x > 0$. Prove the following refinement of this inequality:

$$\ln x \le \frac{2}{3}\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{1}{3x} + 1 \le \sqrt{x} - \frac{1}{\sqrt{x}} \le x - 1$$

for all $x \ge 1$.

*Solution*

Put

$$f(x) = \frac{2}{3}\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{1}{3x} + 1 - \ln x \qquad (x \ge 1).$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{3\sqrt{x}} + \frac{1}{x\sqrt{x}} - \frac{1}{3x^2} - \frac{1}{x} \\ &= \frac{1}{3x^2}(\sqrt{x}-1)^3 \\ &\ge 0 \quad \text{for } x \ge 1, \end{aligned}$$

and $f(1) = 0$. Hence $f(x) \ge 0$ for all $x \ge 1$. Next,

$$\begin{aligned} \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) - \left(\frac{2}{3}\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{1}{3x} + 1\right) &= \frac{1}{3}\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{1}{3x} - 1 \\ &= \frac{1}{3x}(\sqrt{x}-1)^3 \\ &\ge 0, \quad \text{for all } x \ge 1. \end{aligned}$$

Finally,

$$\begin{aligned} (x-1) - \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) &= \frac{1}{\sqrt{x}}(x-1)(\sqrt{x}-1) \\ &\ge 0, \quad \text{for all } x \ge 1. \end{aligned}$$

**45.11** Let $T_n = \frac{1}{2}n(n+1)$ be the $n$th triangular number and $P_n = \frac{1}{2}n(3n-1)$ be the $n$th pentagonal number. Prove that among $T_m + P_n$, where $m$, $n$ are positive integers, there are infinitely many different cubes.

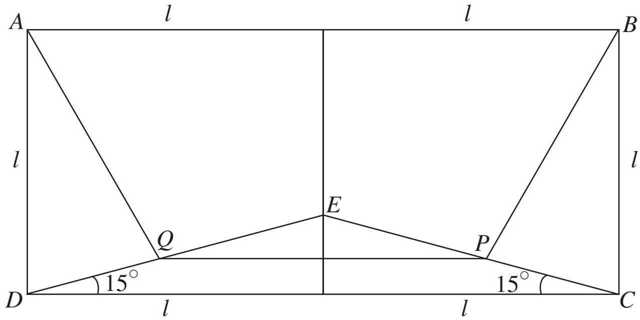


**Figure 1**

*Solution* by Kuntal Chatterjee

Since $T_n + P_n = 2n^2$, we can put $m = n = 2a^3$ for any natural number $a$.

**45.12** In figure 1, $ABCD$ is a rectangle and $P$, $Q$ are the midpoints of the straight lines $CE$, $DE$. What is the area of the quadrilateral $ABPQ$?

*Solution* by Spiros Andriopoulos

The altitudes of triangles $EPQ$ and $ECD$ are $(l/2)\tan 15°$ and $l\tan 15°$, respectively, so the distance between the parallel lines $PQ$ and $CD$ is $(l/2)\tan 15°$. Hence

$$\begin{aligned}\text{area } ABPQ &= \frac{1}{2}(AB + PQ)\left(l - \frac{l}{2}\tan 15°\right)\\ &= \tfrac{1}{2}(2l + l)l\big(1 - \tfrac{1}{2}\tan 15°\big)\\ &= \frac{3l^2}{4}(2 - \tan 15°).\end{aligned}$$

Now

$$\begin{aligned}\tan 15° &= \tan(60° - 45°)\\ &= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}\\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3}}\\ &= \frac{(\sqrt{3} - 1)^2}{2}\\ &= 2 - \sqrt{3},\end{aligned}$$

and

$$\text{area } ABPQ = \frac{3l^2}{4}\sqrt{3}.$$

Also solved by Tom Peak, Hills Road Sixth Form College, Cambridge, UK.

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