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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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THE OLYMPIAD CORNER

No. 131

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Here it is, the beginning of another year. We are getting more used to publishing with LaTeX and, after trying our own hand at typing, then the help of a graduate student, Dave Gunderson, who left to continue his studies, we have finally settled down a bit with the capable help of Joanne Longworth. It seems that pencil and paper are still my medium. Hopefully Volume 18 will see us getting ahead a bit on publication so that we appear in a more timely way.

It is also the time to thank those who have contributed problem sets, solutions and comments. Without the faithful readers this Corner would be vastly the poorer effort. Among the contributors whose work was mentioned last year were:

Mangho Ahaja	George Evagelopoulos	John Morvay
Seung-Jin Bang	Hidetosi Fukagawa	Bob Prielipp
Leon Bankoff	Hillel Gauchman	Michael Selby
Ed Barbeau	Richard Gibbs	Bruce Shawyer
Dieter Bennewitz	Georg Gunther	David Singmaster
Aage Bondeson	R.K. Guy	Florentin Smarandache
Duane Broline	Walther Janous	D.J. Smeenk
S.R. Cavior	O. Johnson	Don St. Jean
S.C. Chan	Leung Yu Kiang	L.J. Upton
Pak-Hong Cheung	Murray S. Klamkin	David Vaughan
Jason Colwell	Marcin E. Kuczma	G.R. Veldkamp
Curtis Cooper	Calvin Li	Edward T.H. Wang
Graham Denham	Andy Liu	Willie Yang
Nicos Diamantis	Jie Lou	Joseph Zaks
Hans Engelhaupt	Stewart Metchette	

Thank you all (and anyone I've left out by accident).

* * *

To warm up for the new year, here is a set of five of Murray S. Klamkin's "Quickies". They may look impossible but there is a short answer in each case. The answers will be given next issue.

FIVE KLAMKIN QUICKIES

1. Determine the extreme values of $r_1/h_1 + r_2/h_2 + r_3/h_3 + r_4/h_4$ where h_1, h_2, h_3, h_4 are the four altitudes of a given tetrahedron T and r_1, r_2, r_3, r_4 are the corresponding signed perpendicular distances from any point in the space of T to the faces.

2. Determine the minimum value of the product

$$P = (1 + x_1 + y_1)(1 + x_2 + y_2) \dots (1 + x_n + y_n)$$

where $x_i, y_i \geq 0$, and $x_1 x_2 \dots x_n = y_1 y_2 \dots y_n = a^n$.

3. Prove that if $F(x, y, z)$ is a concave function of x, y, z, then $\{F(x, y, z)\}^{-2}$ is a convex function of x, y, z.

4. If a, b, c are sides of a given triangle of perimeter p, determine the maximum values of

(i) $(a-b)^2 + (b-c)^2 + (c-a)^2$,

(ii) $|a-b| + |b-c| + |c-a|$,

(iii) $|a-b||b-c| + |b-c||c-a| + |c-a||a-b|$.

5. If A, B, C are three dihedral angles of a trihedral angle, show that $\sin A$, $\sin B$, $\sin C$ satisfy the triangle inequality.

* * *

The first contest we give this month is the XLI Mathematics Olympiad in Poland, Final Round. Thanks go to Marcin E. Kuczma of the University of Warsaw.

XLI MATHEMATICS OLYMPIAD

Poland, April 7–8, 1990

1. Determine all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ which satisfy

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

for all real x, y.

2. Let $n > 2$ be a natural number and let $x_1, \dots, x_n$ be positive real numbers. Prove:

$$\frac{x_1^2}{x_1^2 + x_2 x_3} + \frac{x_2^2}{x_2^2 + x_3 x_4} + \dots + \frac{x_n^2}{x_n^2 + x_1 x_2} \leq n - 1.$$

3. In a tournament of n participants, each pair has played exactly one game (no ties). Show that either

(i) the league splits into two (nonempty) subsets A, B so that each player of group A has beaten each player of group B, or

(ii) all players can be arranged into a cyclic sequence in which every player has beaten his successor.

4. A triangle of all sides ≥ 1 is placed in a square of side 1. Show that the centre of the square belongs to the triangle.

5. Given a sequence of positive integers (a_n) with $\lim_{n\to\infty}(n/a_n) = 0$, show that, for some k, there are not less than 1990 perfect squares between $a_1 + \dots + a_k$ and $a_1 + \dots + a_k + a_{k+1}$.

6. Prove that for every integer $n > 2$ the number

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{3k}$$

is divisible by 3.

*

The other set of Olympiad problems we give this month also came from the same region. These were sent to us both by Walther Janous, of Innsbruck, Austria, and by Marcin E. Kuczma of Warsaw, Poland. These are the problems of the 13th Austrian-Polish Mathematics Competition, written at Poznan, Poland, June 27–29, 1990.

13th AUSTRIAN-POLISH MATHEMATICS COMPETITION 1990

Individual Competition

First day: June 27, 1990 (Time: 4 1/2 hours)

1. Let A, B, $P_1, \ldots, P_6$ be eight distinct points in the plane, the P_i's all lying on the same side of line AB. Suppose the six triangles ABP_i $(1 \le i \le 6)$ are similar. Show that $P_1, \ldots, P_6$ lie on a circle.

2. Determine all triples (x, y, z) of positive integers such that

$$x^{(y^z)} \cdot y^{(z^x)} \cdot z^{(x^y)} = 1990^{1990} xyz.$$

3. Show that there are exactly two triples (x, y, z) of real numbers satisfying the system of equations

$$\begin{aligned} x + y^2 + z^4 &= 0 \\ y + z^2 + x^4 &= 0 \\ z + x^2 + y^4 &= 0. \end{aligned}$$

Second day: June 28 (Time: 4 1/2 hours)

4. For a given $n > 1$ consider the system of equations

$$\begin{aligned} x_1^4 + 14x_1x_2 + 1 &= y_1^4 \\ x_2^4 + 14x_2x_3 + 1 &= y_2^4 \\ &\cdots\cdots \\ x_{n-1}^4 + 14x_{n-1}x_n + 1 &= y_{n-1}^4 \\ x_n^4 + 14x_nx_1 + 1 &= y_n^4. \end{aligned}$$

Find all solutions $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with x_i $(1 \le i \le n)$ and y_i $(1 \le i \le n)$ being positive integers.

5. Given a natural number $n > 1$, let S_n denote the set of all permutations (one-to-one maps) $p : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$. For every permutation $p \in S_n$ write

$$F(p) = \sum_{k=1}^{n} |k - p(k)|.$$

Compute

$$M_n = \frac{1}{n!} \sum_{p \in S_n} F(p)$$

(summation spreading over all permutations $p \in S_n$).

6. Let $P(x)$ be a polynomial with integer coefficients. Suppose that the integers $x_1, x_2, \ldots, x_n$ ($n \geq 3$) satisfy the conditions

$$P(x_i) = x_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1, \qquad P(x_n) = x_1.$$

Show that $x_1 = x_3$.

Team Competition

Third day: June 29 (Time: 4 hours)

7. Given is a set S_n of domino pieces

$$[0|0],\ [0|1],\ \ldots,\ [0|n],\ [1|1],\ [1|2],\ \ldots,\ [n|n]$$

(one piece for each pair a, b with $0 \leq a \leq b \leq n$). By a chain we mean any sequence of successively arrayed pieces

$$[a_1|a_2][a_2|a_3]\ldots[a_{k-2}|a_{k-1}][a_{k-1}|a_k]$$

(e.g. $[0|5][5|5][5|1][1|2]$). The chain is closed when $a_k = a_1$.

(a) If n is even, prove that there exists a closed chain composed of all pieces.

(b) If n is odd, prove that every closed chain leaves at least $(n+1)/2$ pieces unused.

(c) Assume n is odd. How many sets A are there, each consisting of exactly $(n+1)/2$ pieces, such that all pieces in $S_n \backslash A$ can be arranged into a closed chain?

8. Let R be a 28 by 48 rectangle. We are concerned with dissections of R into congruent a by b rectangles ($a \neq b$, $a, b \in \mathbf{N}$) with sides parallel to the sides of R. For some (a, b) there exist several such dissections, for some (a, b) just one.

(a) Determine the sides a, b of the rectangle of smallest area such that the dissection of R into a by b rectangles is unique.

(b) Determine the sides a, b of the rectangle of largest area such that the dissection of R into a by b rectangles is not unique.

(Two dissections are considered to be different whenever they are not identical.)

9. Let $a_1, \ldots, a_n$ be integers such that every "partial" sum $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ ($1 \leq i_1 < i_2 < \ldots < i_k \leq n$) is different from zero. Prove that the set of positive integers can be partitioned into finitely many classes so that $a_1x_1 + \cdots + a_nx_n \neq 0$ whenever $x_1, \ldots, x_n$ belong to a common class.

*

Many thanks to both M. Kuczma and W. Janous for their continued support of the Corner. Walther Janous asks me to ask the readers their opinions of the continuously increasing difficulty of competition problems and the bad effect this can have on motivation of students. Is there a noticeable effect in other countries on the number of students taking part? In Austria apparently the numbers are decreasing, partly because students are "chased" by computers and their surroundings. Complaints about difficulty also are heard here in Canada and may affect participation, particularly away from certain centres of population with special academic programs. I would be interested in hearing from the readers on these points.

* * *

Before turning to solutions from the readers, a comment on the 1991 I.M.O.

6. [1991: 259] *32nd I.M.O.*

An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$. Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j| \cdot |i-j|^a \geq 1$$

for every pair of distinct non-negative integers i, j.

Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The problem is by far not new as an Olympiad problem. The case $a = 1$, i.e. $|x_i - x_j| > 1/|i-j|$, was a problem of the 12th All-Soviet Olympiad 1978 (U.S.S.R.), as can be seen from p. 70, problem 257* of N.B. Vasiliev and A.A. Egorov, *Problems of the All-Soviet Mathematical Olympiads* (Russian), Moscow, 1988.

* * *

While going through the solutions for this number of *Crux*, I found four solutions sent in by O. Johnson, student, King Edward's School, Birmingham, England, to problems 1, 2, 5 and 8 of the "All Union Mathematical Olympiad". These problems were discussed in the last two numbers. Fortunately, some of the remaining solutions that were included will be mentioned later in this number. My apologies!

* * *

We next turn to solutions for the problems of the *1989 Indian Mathematical Olympiad* given in the May 1990 number of the Corner [1990: 133].

1. Prove that the polynomial

$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$

cannot be expressed as a product

$$f(x) = p(x)q(x)$$

where $p(x)$, $q(x)$ are both polynomials with integral coefficients and with degree < 4.

Solutions by J. Lou, student, Halifax West High School, Halifax, N.S., and by M. Selby, University of Windsor.

If $f(x) = p(x)q(x)$ there are essentially two cases to consider.

Case 1: $p(x) = x + a$, $q(x) = x^3 + bx^2 + cx + d$. Then $a + b = 26$, $ab + c = 52$, $ac + d = 78$ and $ad = 1989$. Now if $13 \mid a$, we get $13 \mid b$, so $13 \mid c$, $13 \mid d$ and $13^2 \mid ad$, but this is impossible since $13^2 \not\mid 1989$. But $13 \not\mid a$ gives $13 \not\mid b$ and $13 \not\mid c$, $13 \not\mid d$ and then $13 \not\mid ad$, contradicting that $13 \mid 1989$.

Case 2: $p(x) = x^2 + ax + b$, $q(x) = x^2 + cx + d$. Now $a + c = 26$, $ac + b + d = 52$, $ad + bc = 78$, and $bd = 1989$. Now $13 \mid 1989$ so ad or bc is divisible by 13. Since $13 \mid 78$ both are divisible by 13. Hence $13^2 \mid abcd$. Now $13 \mid ac$, else $13^2 \mid 1989$. Then $13 \mid b + d$, and so 13 divides both b and d, a contradiction, as before.

[*Editor's Note.* Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, points out that one is just applying Eisenstein's Irreducibility Criterion since 13 is a prime which divides 26, 52, 78 and 1989, while 13^2 does not divide 1989.]

2. Let a, b, c, d be any four real numbers, not all equal to zero. Prove that the roots of the polynomial

$$f(x) = x^6 + ax^3 + bx^2 + cx + d$$

cannot all be real.

Solutions by J. Lou, student, Halifax West High School; and also by Michael Selby, University of Windsor.

Suppose $f(x)$ has six real roots $x_1, x_2, \ldots, x_6$. Then since the coefficients of x^5 and x^4 are zero

$$\sum_{i=1}^{6} x_i = 0 \quad \text{and} \quad \sum_{1 \le i < j \le 6} x_i x_j = 0.$$

Then

$$\sum_{i=1}^{6} x_i^2 = \left(\sum_{i=1}^{6} x_i\right)^2 - 2 \sum_{1 \le i < j \le 6} x_i x_j = 0.$$

Since the roots are real we have $x_i = 0$ for $1 \le i \le 6$. This gives $f(x) = x^6$ so $a = b = c = d = 0$, which is impossible.

[*Editor's Note.* Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, gave a solution based on Descartes' rule of signs. Considering the sequence of coefficients 1, a, b, c, d for $f(x)$ and 1, $-a$, b, $-c$, d for $f(-x)$, he notes that a change of sign for one corresponds to the sign remaining unchanged for the other. Thus the total number of positive and negative roots cannot exceed 4. He then analyses the multiplicity of the root $x = 0$. M. Selby also gave a second argument via the calculus. If $f(x)$ has six real roots then $f'(x)$ has five real roots, $f''(x)$ has four, and $f'''(x) = 120x^3 + a$ must have three. This gives $a = 0$. Then $f''(x) = 30x^4 + b$ gives $b = 0$, and then $c = 0$. Finally $f(x) = x^6 + d$ cannot have six real roots unless $a = b = c = d = 0$.]

3. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots, 541, 551\}$ having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

Solutions by O. Johnson, student, King Edward's School, Birmingham, England; J. Lou, student, Halifax West High School; Bob Prielipp, University of Wisconsin–Oshkosh; Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Consider the following family of 28 subsets which partition the original set:

$$\{\{1, 551\}, \{11, 541\}, \ldots, \{271, 281\}\}.$$

If A contains at least 29 elements, then at least two come from the same subset in the family, by the pigeonhole principle. These two integers sum to 552. Thus, A can contain no more than 28 elements.

4. Determine, with proof, all the positive integers n for which (i) n is not the square of any integer, and (ii) $[\sqrt{n}]^3$ divides n^2. (Notation: $[x]$ denotes the largest integer that is less than or equal to x.)

Solutions by J. Lou, student, Halifax West High School, and by Michael Selby, University of Windsor. (We give the solution by Lou.)

Write $n = k^2 + m$, where $0 < m < 2k + 1$, and set the greatest common divisor $(k, m) = p$. Then $[\sqrt{n}]^3 = k^3$, $n^2 = k^4 + 2k^2m + m^2$. Now $k^3 \mid k^4$, so from (ii) $k^3 \mid 2k^2m + m^2$. This gives that $k^2p \mid m^2$ since $k^2p \mid k^3$, so $k^2p \mid 2k^2m + m^2$ and $k^2p \mid 2k^2m$. Now write $k = pp_1$. Then we have that $p_1 = 1$ since $(k, m) = p$. So we have $m = k$ or $m = 2k$. If $m = k$, then $k^3 \mid k^2$, and $k = 1$. If $m = 2k$, then k divides 4, giving $k = 1, 2$, or 4. This gives the solutions 2, 3, 8, 24 to the problem, and these are easily checked.

5. Let a, b, c denote the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

must lie between the limits 3/2 and 2. Can equality hold at either limit?

Solutions by George Evagelopoulos, Athens, Greece; by J. Lou, student, Halifax West High School; Bob Prielipp, University of Wisconsin–Oshkosh; Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Now

$$\begin{aligned}
\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + 3 &= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \\
&= \frac{1}{2}[(a+b) + (b+c) + (a+c)]\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \\
&\geq 9/2
\end{aligned}$$

by the Arithmetic–Harmonic Mean inequality, and so

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{9}{2}-3=\frac{3}{2}\,.$$

Equality can hold if $a=b=c$ (the triangle is equilateral). For the other limit, note that since a, b, c denote the sides of a non-degenerate triangle, $a<b+c$. Therefore

$$\frac{a}{b+c}-\frac{2a}{a+b+c}=\frac{a(a-b-c)}{(b+c)(a+b+c)}<0,$$

and

$$\frac{a}{b+c}<\frac{2a}{a+b+c}\,.$$

Similarly, we have

$$\frac{b}{c+a}<\frac{2b}{a+b+c} \quad \text{and} \quad \frac{c}{a+b}<\frac{2c}{a+b+c}\,.$$

Note that equality cannot hold at this end.

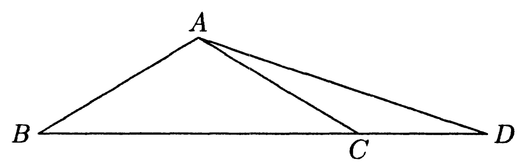
6. Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which

$$|AD|=\sqrt{|BD|\cdot|CD|},$$

where $|BD|$ refers to the (positive) distance between B and D.

Solution by J. Lou, student, Halifax West High School.

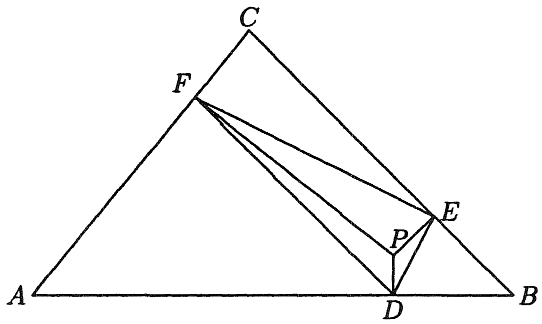
Referring to the figure, consider first D on the right side of BC. Then $|AD|/|BD| = |CD|/|AD|$ and $\angle ADC = \angle BDA$, so we have that triangles ADC and BDA are similar. Hence $\angle CAD = \angle ABD$, and there is only one solution. D can also lie to the left of BC, but this similarly gives one solution.

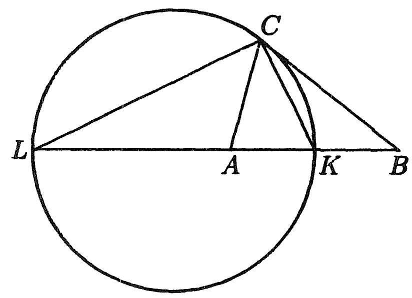


It is easy to see that no solutions exist between B and C. Because $\angle A > \angle B + \angle C$, there are points D_1 and D_2 with D_1 left of D_2 such that $|BD_1| = |AD_1|$ and $|AD_2| = |CD_2|$. Only if $D_1 = D_2$ could $|AD|/|DC| = |BD|/|AD|$ for some D.

7. Let ABC be an arbitrary acute angled triangle. For any point P lying within this triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all positions of the point P for which the triangle DEF is isosceles. For which P will the triangle DEF become equilateral?

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.





With α the angle at A, β the angle at B and γ the angle at C we have $DF = AP\sin\alpha$, since AP is a diameter of the circumcircle of $ADPF$, and the central angle of chord DF is 2α. Also $DE = BP\sin\beta$ and $EF = CP\sin\gamma$. Now $DF = DE$ implies $AP\sin\alpha = BP\sin\beta$ and thus

$$AP : BP \;=\; \sin\beta : \sin\alpha \;=\; b : a \tag{1}$$

The locus of the points P satisfying (1) is a circle of Apollonius, with diameter KL, K being the point of intersection of the angle bisector of $\angle ACB$ with side AB, and L being the point of intersection of the exterior bisector of $\angle ACB$ with AB. Since P must lie within triangle ABC, the locus asked for is the union of the arc of this circle within the triangle, and the two arcs arising from the assumption that $ED = EF$ and $FE = FD$. These three arcs intersect at a point Q that yields an equilateral triangle.

* * *

We now give solutions to two of the problems of the 38th Bulgarian Mathematics Olympiad. I solicit solutions to the other four problems from that set.

1. [1990: 134] *38th Bulgarian Mathematical Olympiad.*

Let p and q be prime numbers for which the number

$$\sqrt{p^2 + 7pq + q^2} + \sqrt{p^2 + 14pq + q^2}$$

is an integer. Prove that $p = q$.

Solutions by O. Johnson, student, King Edward's School, Birmingham, England, and by Stewart Metchette, Culver City, California. (We give Johnson's solution.)

Lemma. If x, y and $(\sqrt{x} + \sqrt{y})$ are integers then both $\sqrt{x}$ and $\sqrt{y}$ are integers as well.

Proof. Now $\sqrt{y} = (\sqrt{x} + \sqrt{y}) - \sqrt{x}$ so

$$y = (\sqrt{x} + \sqrt{y})^2 - 2\sqrt{x}(\sqrt{x} + \sqrt{y}) + x$$

and

$$\sqrt{x} = \frac{(\sqrt{x} + \sqrt{y})^2 + x - y}{2(\sqrt{x} + \sqrt{y})} \in \mathbf{Q}.$$

But for the square root of an integer to be rational, the integer must be a perfect square, and so $\sqrt{x}$ is an integer. It follows immediately that $\sqrt{y}$ is an integer as well.

Now we turn to the question at hand. By the lemma, since $\sqrt{p^2+7pq+q^2}+\sqrt{p^2+14pq+q^2}$ is an integer, both $p^2+7pq+q^2$ and $p^2+14pq+q^2$ must be perfect squares. Without loss of generality $p \geq q$. Then we can write

$$p^2+14pq+q^2=(4p-m)^2 \quad \text{where} \quad 0 \leq m \leq 4p \tag{1}$$

and

$$p^2+7pq+q^2=(3p-n)^2 \quad \text{where} \quad 0 \leq n \leq 3p. \tag{2}$$

Then subtracting,

$$7pq=(7p-m-n)(p-m+n). \tag{3}$$

Since $7, p, q$ are prime we have that $7pq$ factorizes into $7k$ and l, where $7k$ is either $7p-m-n$ or $p-m+n$ and l is the other factor in (3), and $k, l \in \{1, p, q, pq\}$ with $kl=pq$. Then

$$7k+l=(7p-m-n)+(p-m+n)=8p-2m$$

and

$$7k-l=\pm[(7p-m-n)-(p-m+n)]=\pm(6p-2n).$$

This gives two cases.

Case 1: $14k=(8p-2m)+(6p-2n)=14p-2(m+n)$. Then $m+n=7j$ say. From (3) $pq=(p-j)(p-m+n)$. Since p and q are prime either

(i) $j=0$ and $p-m+n=q$,
(ii) $m-n=0$ and $p-j=q$,
(iii) $p-j=1$ and $p-m+n=pq$, or
(iv) $p-j=pq$ and $p-m+n=1$.

In subcase (i) $p-m+n=q$, and $m+n=0$ gives $p-2m=q$. But from (1)

$$p^2+14p(p-2m)+(p-2m)^2=(4p-m)^2,$$

$$3m(m-8p)=3m^2-24mp=0,$$

so $m=0$ or $8p$. But $0 \leq m \leq 4p$ so $m=0$, $n=0$, and $p=q$.

In subcase (ii) $p-m-n=q$, $m-n=0$ and $p-2m=q$, and we finish as in (i).

In subcase (iii) $p-j=1$ and $p-m+n=pq$. Now $pq>p$ so $n \geq m$. Thus $m+n \leq 6p$ and so $7=7p-7j=7p-(m+n) \geq p$. It is easy to check for p a prime less than or equal to 7 that unless $p=q$ the expression is not an integer.

In subcase (iv) $p-j=pq$ and $p-m+n=1$. This is impossible as $j \geq 0$ so $p-j \leq p$.

Case 2: $14k=(8p-2m)-(6p-2n)=2(p-m+n)$. Thus $7k=p-m+n$, $l=7p-m-n$. Once again there are four subcases.

(i): $k=p$ and $l=q$. Then $q=(p-m+n)-m-n=p-2m$, which reduces to case 1.

(ii): $k = q$ and $l = p$. Then $p = p - m - n$ with $m, n \geq 0$ gives $m = n = 0$. Substitution and solution using (1) and (2) gives $p = q$.

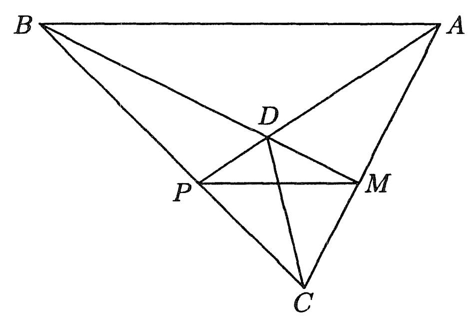
(iii): $k = 1$ and $l = pq$. Now $pq = p - (m + n)$ is impossible as $pq > p$.

(iv): $k = pq$ and $l = 1$. From $7pq = p - m + n$ we get a contradiction since $p - m + n \leq 4p < 7pq$. This completes the proof.

3. [1990: 134] *38th Bulgarian Mathematics Olympiad.*

A triangle ABC is given. A line parallel to the base AB intersects the sides AC and BC respectively at the inner points M and P. Let D be the point of intersection of the lines AP and BM. Prove that the line joining the orthocentres of the triangles ADM and BDP is perpendicular to the line CD.

Solutions by O. Johnson, student, King Edward's School, Birmingham, England and by D.J. Smeenk, Zaltbommel, The Netherlands. (We give Johnson's "vector solution".)



Set the origin at C and let $\mathbf{A}$ be the vector $\overrightarrow{CA}$, $\mathbf{B} = \overrightarrow{CB}$, and $\mathbf{D} = \overrightarrow{CD}$.

Now as P lies between C and B there is k with $0 < k < 1$ such that $\overrightarrow{CP} = k\mathbf{B}$. Since $PM \| AB$, $\overrightarrow{CM} = k\mathbf{A}$. Now D is the intersection of AP and BM, so there are λ and μ with $0 < \lambda, \mu < 1$ and

$$\mathbf{A} + \lambda(k\mathbf{B} - \mathbf{A}) = \mathbf{D} = \mathbf{B} + \mu(k\mathbf{A} - \mathbf{B}).$$

Hence

$$\mathbf{D} = \frac{k}{k+1}(\mathbf{A} + \mathbf{B})$$

and $\lambda = \mu = \frac{1}{k+1}$. Now at X, the orthocentre of ΔADM, and with $\mathbf{X} = \overrightarrow{CX}$,

$$(\mathbf{X} - \mathbf{A}) \cdot (\mathbf{B} - k\mathbf{A}) = \mathbf{B} \cdot \mathbf{X} - k\mathbf{A} \cdot \mathbf{X} - \mathbf{A} \cdot \mathbf{B} + k\mathbf{A} \cdot \mathbf{A} = 0$$

and

$$(\mathbf{X} - k\mathbf{A}) \cdot (\mathbf{A} - k\mathbf{B}) = \mathbf{A} \cdot \mathbf{X} - k\mathbf{B} \cdot \mathbf{X} + k^2\mathbf{A} \cdot \mathbf{B} - k\mathbf{A} \cdot \mathbf{A} = 0.$$

Thus

$$(1 - k)\mathbf{X} \cdot (\mathbf{A} + \mathbf{B}) + (k^2 - 1)\mathbf{A} \cdot \mathbf{B} = 0.$$

As $k \neq 1$,

$$\mathbf{X} \cdot (\mathbf{A} + \mathbf{B}) + (k + 1)\mathbf{A} \cdot \mathbf{B} = 0. \tag{1}$$

Similarly for the orthocentre Y of ΔBDP, and with $\mathbf{Y} = \overrightarrow{CY}$, we have

$$\mathbf{Y} \cdot (\mathbf{A} + \mathbf{B}) + (k + 1)\mathbf{A} \cdot \mathbf{B} = 0. \tag{2}$$

From (1) and (2), $(\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{A} + \mathbf{B}) = 0$. Thus $XY \perp CD$, as desired.

*　　　*　　　*

These are all the solutions on file for problems up to the May 1990 issue. The challenge is out to solve the rest! As the June issue gave the Canadian and USA Olympiads for 1990, and we use only the official or "very nice" solutions, we will take up with the September 1990 number of the Corner next issue. Send me your nice solutions and contest problem sets.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1992,** *although solutions received after that date will also be considered until the time when a solution is published.*

1701*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

If ABC is a triangle, prove or disprove that

$$R \geq 4\max\left\{\frac{h_a\cos A}{1+8\cos^2 A}, \frac{h_b\cos B}{1+8\cos^2 B}, \frac{h_c\cos C}{1+8\cos^2 C}\right\},$$

where h_a, h_b, h_c are the altitudes of the triangle and R is its circumradius.

1702. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABDE$ and $BCFG$ are squares described externally upon the sides of an acute triangle ABC with $\overline{AB} < \overline{BC}$. Let M and N be the midpoints of BC and AC, respectively, and let S be the intersection of BN and GM. Suppose that M, C, S, N are concyclic. Prove that $\overline{MD} = \overline{MG}$.

1703. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum and minimum values of

$$x^2 + y^2 + z^2 + \lambda xyz,$$

where $x + y + z = 1$, $x, y, z \geq 0$, and λ is a given constant.

1704. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Two chords of a circle (neither a diameter) intersect at right angles inside the circle, forming four regions. A circle is inscribed in each region. The radii of the four circles are r, s, t, u in cyclic order. Show that

$$(r-s+t-u)\left(\frac{1}{r}-\frac{1}{s}+\frac{1}{t}-\frac{1}{u}\right)=\frac{(rt-su)^2}{rstu}\ .$$

1705. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $n \geq 2$ and $b_0 \in [2, 2n-1]$ be integers, and consider the recurrence

$$b_{i+1}=\begin{cases} 2b_i-1 & \text{if } b_i \leq n, \\ 2b_i-2n & \text{if } b_i > n. \end{cases}$$

Let $p = p(b_0, n)$ be the smallest positive integer such that $b_p = b_0$.

(a) Find $p(2, 2^k)$ and $p(2, 2^k + 1)$ for all $k \geq 1$.

(b) Prove that $p(b_0, n) \mid p(2, n)$.

1706. *Proposed by Jordi Dou, Barcelona, Spain.*

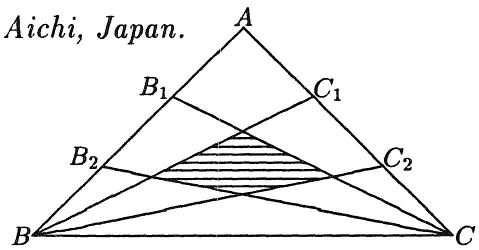
Given four lines in general position in a plane, and a point P in the plane, a pair of lines through P will usually cut off a segment from each of the four given lines. Construct such a pair of lines so that the midpoints of the four segments are collinear.

1707. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

What is the largest integer m for which an $m \times m$ square can be cut up into 7 rectangles whose dimensions are $1, 2, \ldots, 14$ in some order?

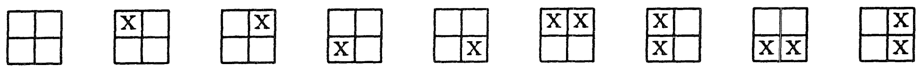
1708. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

ABC is a triangle of area 1, and B_1, B_2 and C_1, C_2 are the points of trisection of edges AB and AC respectively. Find the area of the quadrilateral formed by the four lines CB_1, CB_2, BC_1, BC_2.



1709. *Proposed by Bill Sands, University of Calgary.*

Find the number of ways to choose cells from a $2 \times n$ "chessboard" so that no two chosen cells are next to each other **diagonally** (one way is to choose no cells at all). For example, for $n = 2$ the number of ways is 9, namely



1710. *Proposed by P. Penning, Delft, The Netherlands.*

A tetrahedron $TABC$ of volume 1 has top T and base an equilateral triangle ABC. The projection T' of T onto the base is the centre of ABC. Point I is the midpoint of TT'. A congruent tetrahedron $T'A'B'C'$ is generated by reflecting the original one through I (so $\overrightarrow{AI}=\overrightarrow{IA'}$, etc.). Find the volume that the two tetrahedra have in common.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

993. [1984: 318; 1986: 56] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let P be the product of the $n+1$ positive real numbers $x_1, x_2, \ldots, x_{n+1}$. Find a lower bound (as good as possible) for P if the x_i satisfy

(a) $\displaystyle\sum_{i=1}^{n+1} \frac{1}{1+x_i} = 1;$

(b)* $\displaystyle\sum_{i=1}^{n+1} \frac{a_i}{b_i+x_i} = 1,$ where the a_i and b_i are given positive real numbers.

II. *Solution to part* (b) *by Sam Maltby, student, University of Calgary.*

We give the best lower bound for P in the case $n=1$. For $n>1$ we can only find a lower bound for P which is not sharp.

As observed by Klamkin [1986: 58] we may assume $b_i = 1$ for all i. Also, if $\sum_{i=1}^{n+1} a_i \le 1$ then there is no solution, since

$$\sum_{i=1}^{n+1} \frac{a_i}{1+x_i} < \sum_{i=1}^{n+1} a_i \le 1.$$

So we assume $\sum_{i=1}^{n+1} a_i > 1$.

If some $a_j < 1$ then, as Klamkin also noted, setting x_j arbitrarily close to 0 makes P close to 0, and 0 is the best lower bound. So we assume $a_i \ge 1$ for all i.

Now suppose $n=1$.

Case (i): $a_1 = 1$. Then

$$\frac{1}{1+x_1} + \frac{a_2}{1+x_2} = 1 \implies P = x_1x_2 = a_2 + (a_2-1)x_1.$$

By setting x_1 arbitrarily close to 0, we get P arbitrarily close to but not less than a_2, so a_2 is the best lower bound. Similarly, if $a_2 = 1$ then a_1 is the best lower bound.

Case (ii): $a_1 > 1$ and $a_2 > 1$. Then

$$x_1 > a_1 - 1 \quad \text{and} \quad x_2 > a_2 - 1 \tag{1}$$

since we must have $a_i/(1+x_i) < 1$ for each i. Furthermore, $x_1 = 2a_1 - 1$ and $x_2 = 2a_2 - 1$ give $\sum_{i=1}^{2} a_i/(1+x_i) = 1$, so the lower bound for P is at most $(2a_1-1)(2a_2-1)$. If

$$x_1 \ge \frac{(2a_1-1)(2a_2-1)}{a_2-1} \quad \text{or} \quad x_2 \ge \frac{(2a_1-1)(2a_2-1)}{a_1-1}, \tag{2}$$

then, by (1), $x_1x_2 > (2a_1-1)(2a_2-1)$; thus any pairs (x_1, x_2) satisfying (2) are essentially irrelevant to the lower bound of P, so we may assume

$$x_1 < \frac{(2a_1-1)(2a_2-1)}{a_2-1} \quad \text{and} \quad x_2 < \frac{(2a_1-1)(2a_2-1)}{a_1-1}\,.$$

Also, if $x_1 \leq 4(a_1-1)/3$ then

$$\frac{a_2}{1+x_2} = 1 - \frac{a_1}{1+x_1} \leq 1 - \frac{a_1}{1+4(a_1-1)/3} = \frac{a_1-1}{4a_1-1}\,,$$

so

$$x_1x_2 > (a_1-1)\left(\frac{a_2(4a_1-1)}{a_1-1} - 1\right) = 4a_1a_2 - a_2 - a_1 + 1 > (2a_1-1)(2a_2-1).$$

Thus pairs (x_1, x_2) with $x_1 \leq 4(a_1-1)/3$ (or likewise $x_2 \leq 4(a_2-1)/3$) are irrelevant to the lower bound of P.

We have therefore restricted consideration to the surface

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} = 1$$

with

$$\frac{4(a_1-1)}{3} \leq x_1 \leq \frac{(2a_1-1)(2a_2-1)}{a_2-1}\,, \quad \frac{4(a_2-1)}{3} \leq x_2 \leq \frac{(2a_1-1)(2a_2-1)}{a_1-1}\,,$$

and we know that the lower bound of $P = x_1x_2$ does not occur on the boundary, so it occurs in the interior. Thus we can use Lagrange multipliers.

Now

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} = 1 \implies x_1x_2 + x_1(1-a_2) + x_2(1-a_1) + 1 - a_1 - a_2 = 0, \qquad (3)$$

so we may use the latter as our definition of the surface and get

$$x_2 = \frac{\partial P}{\partial x_1} = \lambda \frac{\partial}{\partial x_1}(x_1x_2 + x_1(1-a_2) + x_2(1-a_1) + 1 - a_1 - a_2) = \lambda(x_2 + 1 - a_2)$$

and so

$$x_2 = \frac{\lambda(a_2-1)}{1-\lambda} \quad \text{and similarly} \quad x_1 = \frac{\lambda(a_1-1)}{1-\lambda}\,.$$

This gives us

$$x_2 = \left(\frac{a_2-1}{a_1-1}\right)x_1.$$

Putting this in (3), we get

$$(a_2-1)x_1^2 + 2(1-a_2)(a_1-1)x_1 + (1-a_1-a_2)(a_1-1) = 0,$$

or

$$\begin{aligned} x_1 &= a_1 - 1 + \frac{\sqrt{(a_2-1)^2(a_1-1)^2 + (a_2-1)(a_1-1)(a_2+a_1-1)}}{a_2-1} \\ &= a_1 - 1 + \sqrt{\frac{a_1 a_2 (a_1-1)}{a_2-1}} \end{aligned} \tag{4}$$

(since $x_1 > 0$), and so

$$x_2 = \left(\frac{a_2-1}{a_1-1}\right) x_1 = a_2 - 1 + \sqrt{\frac{a_1 a_2 (a_2-1)}{a_1-1}}. \tag{5}$$

Thus the best lower bound for $P = x_1 x_2$ in this case is

$$\begin{aligned} &\left(a_1 - 1 + \sqrt{\frac{a_1 a_2 (a_1-1)}{a_2-1}}\right)\left(a_2 - 1 + \sqrt{\frac{a_1 a_2 (a_2-1)}{a_1-1}}\right) \\ &\qquad = 2a_1 a_2 - a_1 - a_2 + 1 + 2\sqrt{a_1 a_2 (a_1-1)(a_2-1)}. \end{aligned}$$

[This can easily be shown to be $\leq (2a_1-1)(2a_2-1)$.] Note that if $a_1 = 1$ (or $a_2 = 1$), this bound reduces to a_2 (or a_1), as given in Case (i).

Now suppose $n > 1$ and assume $a_1 \leq a_2 \leq \ldots \leq a_{n+1}$. It can be shown as above (using Lagrange multipliers) that P has a unique minimum value which is attained when

$$x_i = \frac{a_i(1+x_1)^2 - 2a_1 x_1 + \sqrt{a_i^2(1+x_1)^4 - 4a_i a_1 x_1 (1+x_1)^2}}{2a_1 x_1},$$

for $i = 2, 3, \ldots, n+1$; putting these into

$$\sum_{i=1}^{n+1} \frac{a_i}{1+x_i} = 1$$

gives, after simplifying,

$$\frac{a_1}{1+x_1} + \sum_{i=2}^{n+1} \frac{a_i(1+x_1) - \sqrt{a_i^2(1+x_1)^2 - 4a_i a_1 x_1}}{2(1+x_1)} = 1,$$

so the minimum value of P can be found if this can be solved for x_1. But, this cannot be done generally.

To find a general lower bound for P, note that it follows from the case $n = 1$, especially (4) and (5), that if $a_1 \leq a_2$ are given positive reals and x_1, x_2 positive variables satisfying

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} = C,$$

a given positive constant, then

$$\frac{a_1}{1+x_1} \geq \frac{a_2}{1+x_2}$$

for x_1, x_2 minimizing x_1x_2. Therefore in the present case ($n > 1$, $a_1 \leq a_2 \leq \ldots \leq a_{n+1}$), when the minimum value of P is attained we must have

$$\frac{a_1}{1+x_1} \geq \frac{a_2}{1+x_2} \geq \cdots \geq \frac{a_{n+1}}{1+x_{n+1}} .$$

Since these fractions sum to 1,

$$\frac{a_1}{1+x_1} \leq 1, \quad \frac{a_2}{1+x_2} \leq \frac{1}{2}, \quad \frac{a_3}{1+x_3} \leq \frac{1}{3}, \quad \text{etc.}$$

Thus $x_i \geq ia_i - 1$ for all i, and a lower bound for P is

$$\prod_{i=1}^{n+1}(ia_i - 1).$$

Incidentally, the published solution to part (a) [1986: 56] stated that $P \geq n^{n+1}$, with equality if and only if all x_i's were equal. This is the only condition for equality when $n > 1$, but not when $n = 1$, since $P = x_1x_2$ equals 1 for *any* positive x_1, x_2 satisfying

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} = 1.$$

* * * * *

1548*. [1990: 144; 1991: 219] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a_1, a_2 be given positive constants and define a sequence $a_3, a_4, a_5, \ldots$ by

$$a_n = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}, \quad n > 2.$$

Show that $\lim_{n\to\infty} a_n$ exists and find this limit.

II. *Further comment by the editor.*

As mentioned on [1991: 219], this problem is equivalent to problem E3388 of the *American Math. Monthly.* The solution of the *Monthly* problem has recently appeared (see pp. 69–70 of the January 1992 issue), and is vaguely similar to, but shorter than, solutions earlier received for *Crux* 1548. Thus no additional solution will be printed here, for now anyway. All that remains is to list those solvers of *Crux* 1548, namely: *EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; G.P. HENDERSON, Campbellcroft, Ontario; MARCIN E. KUCZMA, Warszawa, Poland;* and *REX WESTBROOK, University of Calgary.* Henderson gave a generalization. Additionally, Kuczma, Walther Janous, and John H. Lindsey alerted the editor about *Monthly* problem E3388 (Janous and Kuczma were two of the three – independent – proposers of E3388!), and one reader sent in the correct limit without proof. And by the way, the limit is $\sqrt{2}$.

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1589. [1990: 268] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that, for any natural number n,

$$\sqrt[n]{n!} + \sqrt[n+2]{(n+2)!} < 2 \cdot \sqrt[n+1]{(n+1)!}\ .$$

Combination of solutions by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Let

$$f(x) = (\Gamma(x+1))^{1/x}.$$

Because $f''(x) < 0$ for $x \geq 7$ [see *Editor's note* below!], f is concave for $x \geq 7$. Then everything follows by Jensen's inequality:

$$\begin{aligned} f(x+1) &= f\left(\frac{x+(x+2)}{2}\right) > \frac{f(x)+f(x+2)}{2} \\ &\Longrightarrow 2[\Gamma(x+2)]^{\frac{1}{x+1}} > [\Gamma(x+1)]^{\frac{1}{x}} + [\Gamma(x+3)]^{\frac{1}{x+2}}. \end{aligned}$$

If $x = n \geq 7$, the required inequality comes from $\Gamma(n+1) = n!$. The remaining values $n = 1, 2, \ldots, 6$ are checked directly.

[*Editor's note.* The statement that $f(x)$ is concave was given by both Janous and the proposer (they differed on when the concavity took effect; Janous said $x \geq 20$, the proposer $x \geq 7$). Neither supplied the editor with a reference. The editor has since asked several colleagues and experts about this, and none of them could recall the result or find a reference either. However, one of them made calculations suggesting that the concavity should hold for *all* x (≥ 1?). The editor would very much like some reader to clear this up. Most likely, more on this problem will be forthcoming.]

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and MARCIN E. KUCZMA, Warszawa, Poland.

*   *   *   *   *

1590. [1990: 268] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Four familiar circles in the plane of a scalene triangle are the incircle, circumcircle, nine-point circle, and the Spieker circle. Let I, O, F, S be their respective centers. Prove that the lines IO and FS are parallel.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Let the triangle be ABC, with centroid G, and let D, E, F be the midpoints of BC, CA, AB respectively. Triangles ABC and DEF are homothetic with centre G and ratio $2:1$. The Spieker circle is the incircle of ΔDEF, so IS passes through G and $|IG| = 2|GS|$. The nine-point circle is the circumcircle of ΔDEF, so OF passes through G and $|OG| = 2|GF|$. Also G lies on the segments IS and OF. It follows easily that $IO \| FS$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

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1591. [1990: 298] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on the side BC of a triangle ABC. Suppose that $\overline{AC} = \overline{BD}$, $\angle ADC = 30°$, and $\angle ACB = 48°$. Calculate angle B.

Solution by Dag Jonsson, Uppsala, Sweden.

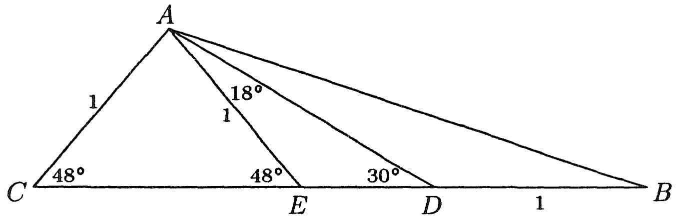

Figure 1

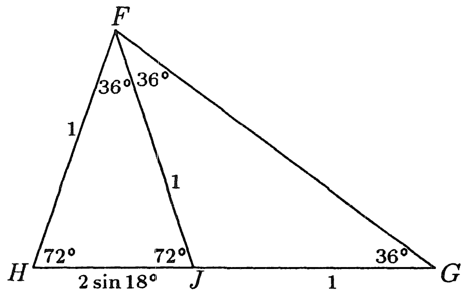

Figure 2

Let $AC = 1$. E is a point on CD such that $AE = 1$. Then $\angle EAD = 48° - 30° = 18°$ (see Figure 1). The sinus theorem applied to the triangle AED gives

$$ED = \frac{\sin 18°}{\sin 30°} = 2\sin 18°.$$

The triangles AED and BEA have an angle, E, in common. We will show that the triangles are similar by showing that $BE/AE = AE/DE$ or

$$1 + 2\sin 18° = \frac{1}{2\sin 18°}\ . \tag{1}$$

But (1) is obvious from Figure 2 (here ΔFGH is similar to ΔHFJ). Thus

$$\angle ABD = \angle EAD = 18°.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SAM BAETHGE, Science Academy, Austin, Texas; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and M^A. ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; PETER HURTHIG, Columbia College, Burnaby, B.C. ; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands;

D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One other reader calculated only the value of $\tan B$.

Many solutions used the relation $\sin 18^\circ = (\sqrt{5}-1)/4$ *(e.g., obtainable from* (1) *above). Lau's solution, which was similar in some ways to Jonsson's, concluded that*

$$ED \cdot EB = \frac{\sqrt{5}-1}{2} \cdot \frac{\sqrt{5}+1}{2} = 1 = (EA)^2,$$

therefore EA *is tangent to the circumcircle of* ΔABD *at* A, *and so* $\angle B = \angle DAE = 18^\circ$.

Baethge notes that with $\angle ADC = 30^\circ$ *and varying* $\angle C$, *the following integer pairs seem to be produced for* (B, C):

$$(15,30),\ (18,48),\ (20,80),\ (20,100),\ (18,132).$$

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1592. [1990: 298] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

If P is a monic polynomial of degree $n > 1$, having n negative roots (counting multiplicities), show that

$$P'(0)P(1) \ge 2n^2 P(0),$$

and find conditions for equality.

Solution by David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We prove the stronger result that

$$P'(0)P(1) \ge \frac{n^{n+1}}{(n-1)^{n-1}}\, P(0), \tag{1}$$

with equality if and only if the n roots of P are all equal to $-1/(n-1)$.

Let

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

and let the roots be $-r_i$ where $r_i > 0$ for $i = 1, 2, \ldots, n$. Then

$$P(x) = \prod_{i=1}^{n}(x + r_i) \quad \Longrightarrow \quad P(1) = \prod_{i=1}^{n}(1 + r_i).$$

Since $P'(0) = a_1$ and $P(0) = a_0$, the inequality to be proved is equivalent to

$$a_1 P(1) \ge \frac{n^{n+1}}{(n-1)^{n-1}}\, a_0.$$

Since

$$a_0 = \prod_{i=1}^{n} r_i \quad \text{and} \quad a_1 = \sum_{i=1}^{n}\left(\frac{\prod_{j=1}^{n} r_j}{r_i}\right),$$

we have

$$\frac{a_1 P(1)}{a_0} = \left(\sum_{i=1}^{n} \frac{1}{r_i}\right)\left(\prod_{i=1}^{n} (1+r_i)\right) .$$

Now it is known (cf. solution to Problem 425, *College Math. Journal* 22(2) (1991) p. 171) that

$$\left(\sum_{i=1}^{n} \frac{1}{r_i}\right)\left(\prod_{i=1}^{n} (1+r_i)\right)$$

has an absolute minimum value of $n^{n+1}/(n-1)^{n-1}$ attained uniquely when $r_i = 1/(n-1)$ for all $i = 1, 2, \ldots, n$, and (1) follows.

Since

$$\frac{n^{n+1}}{(n-1)^{n-1}} = n^2\left(1+\frac{1}{n-1}\right)^{n-1} ,$$

and since

$$2 \leq \left(1+\frac{1}{n-1}\right)^{n-1} < e$$

with equality holding if and only if $n = 2$, we conclude that $P'(0)P(1) > 2n^2P(0)$ for all $n > 2$. If $n = 2$, then equality holds if and only if $r_1 = r_2 = 1$, i.e., $P(x) = (x+1)^2$.

Comment. This problem, with very minor changes in statement and notation, actually appeared in *Crux* before. See problem 2 of the 1986 Austrian–Polish Mathematical Competition [1986: 231].

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DUANE M. BROLINE, Eastern Illinois University, Charleston; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Klamkin gave the stronger result (1) *as well, but proved that*

$$\left(\sum_{i=1}^{n} \frac{1}{r_i}\right)\left(\prod_{i=1}^{n}(1+r_i)\right) \geq \frac{n^{n+1}}{(n-1)^{n-1}} \tag{2}$$

(the College Math. Journal *problem) rather nicely as follows: by the A.M.–G.M. inequality,*

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_n} \geq n(r_1 r_2 \ldots r_n)^{-1/n}$$

and

$$1 + r_i = r_i + \overbrace{\frac{1}{n-1}+\cdots+\frac{1}{n-1}}^{n-1 \text{ terms}} \geq n r_i^{1/n}(n-1)^{(1-n)/n},$$

hence (2).

Bang also pointed out the College Math. Journal *problem, and Janous also noted the earlier appearance of (almost) the same problem in the Austrian–Polish contest.*

*　　*　　*　　*　　*

1593. [1990: 298] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let the rigid triangle ABC move in the Cartesian plane such that B moves along the y axis and C moves along the x axis. Then it is well known that A will describe an ellipse $\mathcal{E}$. Find all points A' of the plane such that the ellipse described by A' when $\Delta A'BC$ moves as above is congruent to $\mathcal{E}$.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

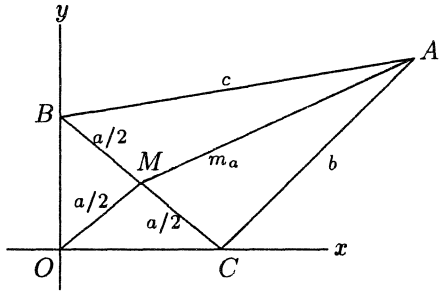

Figure 1

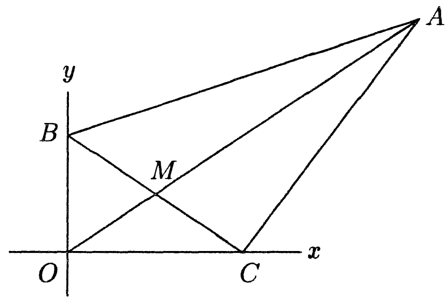

Figure 2

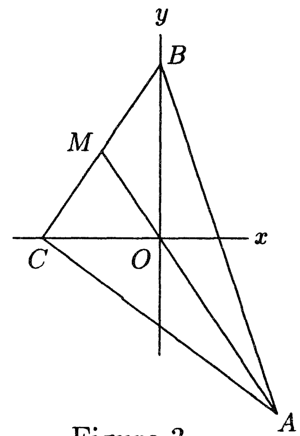

Figure 3

Le point A décrit une ellipse de centre O (Figure 1). A est le plus éloigné de O lorsque O, M, A sont alignés et dans cet ordre (Figure 2); A est alors une extrémité du grand axe de l'ellipse. A est le plus proche de O lorsque M, O, A sont alignés et dans cet ordre (Figure 3); A est alors une extrémité du petit axe de l'ellipse. Les longueurs des axes sont donc $2m_a + a$ et $2m_a - a$. L'ellipse décrite par A' est isométrique à celle décrite par A si et seulement si ses axes ont pour longueurs $2m_a + a$ et $2m_a - a$. Dès lors le lieu de A' est le cercle de centre M et de rayon $m_a = |AM|$.

Also solved by JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; and the proposer.

*The solutions of Pedoe and Penning were the same as the above. Pedoe referred to the diagram in his paper "The ellipse as a hypotrochoid" (*Math. Magazine **48** (1975) 228–230*).*

Dou and Pedoe note that a similar result holds if the lines along which B and C move are not at right angles.

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1594. [1990: 298] *Proposed by Murray S. Klamkin, University of Alberta.*

Express

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1x_2x_3x_4$$

as a sum of squares of rational functions with real coefficients. (By the A.M.-G.M. inequality, this polynomial is nonnegative for all real values of its variables, and so by a theorem of Hilbert it can be so expressed.)

Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

$$\begin{aligned} x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1x_2x_3x_4 &= x_1^4 + x_2^4 - 2x_1^2x_2^2 + x_3^4 + x_4^4 - 2x_3^2x_4^2 \\ &\quad +2x_1^2x_2^2 + 2x_3^2x_4^2 - 4x_1x_2x_3x_4 \\ &= (x_1^2 - x_2^2)^2 + (x_3^2 - x_4^2)^2 + (\sqrt{2}(x_1x_2 - x_3x_4))^2. \end{aligned}$$

Also solved (usually the same way) by SEUNG-JIN BANG, Seoul, Republic of Korea; DUANE M. BROLINE, Eastern Illinois University, Charleston; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

As Bang points out, it is known more generally that for any even n,

$$a_1^n + a_2^n + \cdots + a_n^n - na_1a_2\cdots a_n$$

can be expressed as a sum of squares; see p. 55, section 2.23 of Hardy, Littlewood, and Pólya, Inequalities, *Cambridge Univ. Press.*

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1595. [1990: 298] *Proposed by Isao Ashiba, Tokyo, Japan.*

P is a variable point on the circumcircle of a triangle ABC. Show that

$$PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C$$

is constant.

Solution by Murray S. Klamkin, University of Alberta.

This problem is a special case of the more general result [1]: *if A, B, C are any three fixed points and P is any point on a circle whose center is the circumcenter O of ΔABC, then*

$$S \equiv (AP)^2[BOC] + (BP)^2[COA] + (CP)^2[AOB]$$

is constant for all positions of P on the circle (here, e.g., $[BOC]$ denotes the area of ΔBOC).

We give a slightly less computational proof than in Hobson. Let $\mathbf{X}$ denote the vector from origin O to any point X. Since the barycentric coordinates of O itself are $[BOC]/[ABC]$, $[COA]/[ABC]$, $[AOB]/[ABC]$, the sum

$$\mathbf{A}[BOC] + \mathbf{B}[COA] + \mathbf{C}[AOB]$$

must be $\mathbf{0}$. Since

$$(AP)^2 = (\mathbf{A} - \mathbf{P})^2 = \mathbf{A}^2 + \mathbf{P}^2 - 2\mathbf{A}\cdot\mathbf{P}, \text{ etc.,}$$

and $\mathbf{P}^2$ is constant, the only part of S which could vary with P is

$$-2(\mathbf{A}\cdot\mathbf{P}[BOC] + \mathbf{B}\cdot\mathbf{P}[COA] + \mathbf{C}\cdot\mathbf{P}[AOB]) = -2\mathbf{P}\cdot(\mathbf{A}[BOC] + \mathbf{B}[COA] + \mathbf{C}[AOB]) = 0.$$

Also noted in [1] are the following three particular cases:
(a) This case corresponds to the given problem above.
(b) $(PA)^2 \sin A + (PB)^2 \sin B + (PC)^2 \sin C$ is constant if P lies on the incircle.
(c) $(PA)^2 \sin A \cos(B-C) + (PB)^2 \sin B \cos(C-A) + (PC)^2 \sin C \cos(A-B)$ is constant if P lies on the nine point circle.

Reference:

[1] E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, pp. 210–211.

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; LJUBOMIR LJUBENOV, Stara Zagora, Bulgaria; P. PENNING, Delft, The Netherlands; K.R.S. SASTRY, Addis Ababa, Ethiopia; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

The reference to Hobson was also supplied by Bellot Rosado. Several solvers pointed out that the constant of the problem comes out to be 4 times the area of the triangle.

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1596. [1990: 298] *Proposed by Mark Kisin, student, Monash University, Clayton, Australia.*

Given an integer $n \geq 1$, what is the least integer $f(n)$ such that, from any collection of $f(n)$ integers, one can always choose n of them so that their sum is divisible by n?

Comment by the editor.

The result that $f(n) = 2n - 1$ is a known one, as several readers point out. In fact MURRAY S. KLAMKIN, University of Alberta, notes that the result has already appeared in *Crux*, having been mentioned in the solution of problem 3 of the 1981 West German Math. Olympiad, Second Round [1986: 43]. There, Klamkin (then editor of the Olympiad Corner) gave the original reference for the result, which is due to Erdős, Ginzburg and Ziv [2]. He also mentioned two further proofs which appeared in *The Mathematical Intelligencer* (vol. 1 (1979) p. 250 and vol. 2 (1980) p. 106), and a generalization in the paper [4]. The Erdős–Ginzburg–Ziv reference was also given by MARCIN E. KUCZMA, Warszawa, Poland; SYDNEY BULMAN-FLEMING and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and CHRIS WILDHAGEN, Rotterdam, The Netherlands, the latter noting the additional recent paper [1]. KEE-WAI LAU, Hong Kong, mentions the reference [5]. The editor apologizes for not sufficiently researching this problem before using it!

The proposer's solution contains no references, but apparently the proposer discovered some time after having submitted the problem that it had been solved before. As

Bulman-Fleming and Wang point out, a preprint [3] by the proposer, dated a full year after the problem was submitted to *Crux*, contains the result and credits it to Erdős, Ginzburg and Ziv.

The proposer was led to the problem by a special case (n a power of 2) which was used in the Correspondence Course to select the 1990 Australian I.M.O. team. This is the same special case as occurs in the 1981 West German Olympiad.

One other reader sent in the correct value of $f(n)$ without proof and without references. Most of the above mention that $f(n) = 2n - 1$ is best possible, as can be seen from the collection of $2n - 2$ integers consisting of $n - 1$ 0's and $n - 1$ 1's.

References:

[1] C. Bailey and R.B. Richter, Sum zero (mod n), size n subsets of integers, *American Math. Monthly* **96** (1989) 240–242.

[2] P. Erdős, A. Ginzburg and A. Ziv, Theorem in additive number theory, *Bull. Research Council Israel* **10**F (1961) 41–43.

[3] M. Kisin, The number of zero sums modulo m in a sequence of length n, preprint (Monash University, paper no. 125, June 1991).

[4] H.B. Mann and J.E. Olson, Sum of sets in the elementary Abelian group of type (p, p), *J. Combinatorial Theory* **2** (1967) 275–284.

[5] Zun Shan, On a conjecture of elementary number theory, *Advances in Mathematics (Beijing)* **12** (1983) 299–301. (Chinese)

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1597. [1990: 298] *Proposed by Jordi Dou, Barcelona, Spain.*

Given three lines a_1, a_2, a_3 and a point S, find the line s through S so that the lines s_1, s_2, s_3 symmetric to s with respect to a_1, a_2, a_3, respectively, are concurrent.

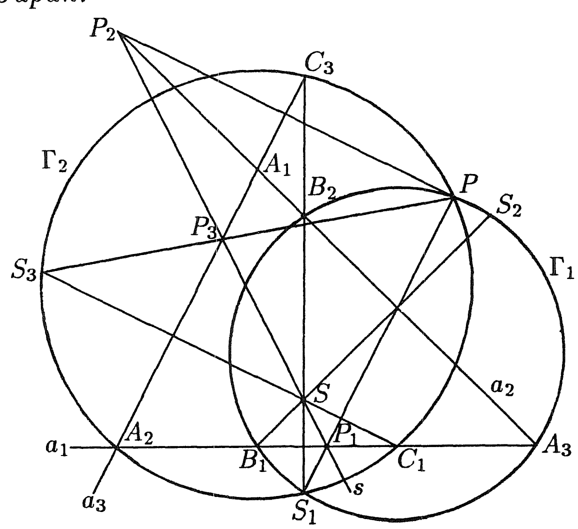
I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Let S_1, S_2, S_3 be the points symmetric to S with respect to a_1, a_2, a_3, respectively, and let

$$a_2 \cap a_3 = A_1, a_3 \cap a_1 = A_2, a_1 \cap a_2 = A_3,$$

$$SS_2 \cap a_1 = B_1, \;\; SS_1 \cap a_2 = B_2,$$

$$SS_3 \cap a_1 = C_1, \;\; SS_1 \cap a_3 = C_3.$$



Let Γ_1 be the conic which passes through the five points A_3, S_1, S_2, B_1, B_2, and let Γ_2 be the conic which passes through the five points A_2, S_1, S_3, C_1, C_3. (*Comment:* it may easily be proved that Γ_1 and Γ_2 are in fact circles.) Let

P be the intersection of Γ_1 with Γ_2, other than S_1, and let P_1 be the intersection of S_1P with a_1.

We claim that SP_1 is the line s which we are looking for. To prove this, let $PS_2 \cap a_2 = P_2$ and $PS_3 \cap a_3 = P_3$. As the six points $P, S_1, B_2, A_3, B_1, S_2$ lie on a conic Γ_1, by Pascal's theorem the points

$$P_1 = PS_1 \cap A_3B_1, \ S = S_1B_2 \cap B_1S_2, \ P_2 = PS_2 \cap B_2A_3$$

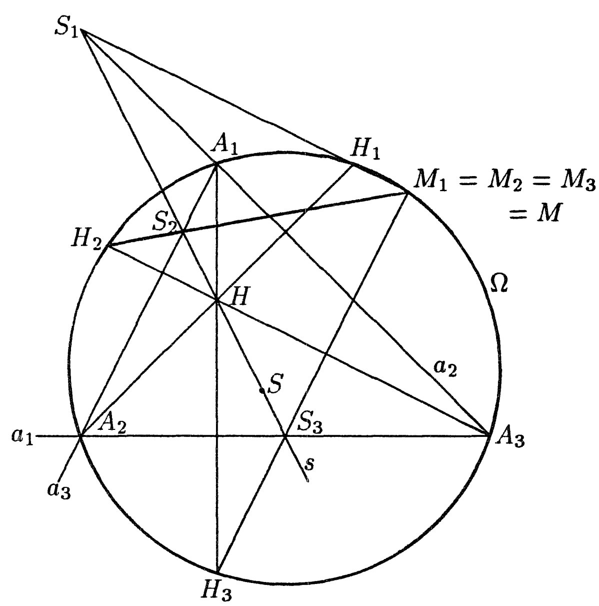
are collinear. Because the six points $P, S_1, C_3, A_2, C_1, S_3$ lie on a conic Γ_2, by Pascal's theorem the points

$$P_1 = PS_1 \cap A_2C_1, \ S = S_1C_3 \cap C_1S_3, \ P_3 = PS_3 \cap C_3A_2$$

are collinear. Therefore P_1, P_2, P_3, S lie on the line s. Lines P_1S_1, P_2S_2, P_3S_3 are s_1, s_2, s_3, and they concur at P.

II. *Solution by the proposer.*

We assume that the given lines form the sides of a triangle $A_1A_2A_3$ (where $A_1 = a_2 \cap a_3$, etc.). Let Ω be the circumcircle of this triangle and let H be its orthocentre. Put $s = SH$ and let S_1, S_2, S_3 be the intersections of s with a_1, a_2, a_3, and H_1, H_2, H_3 the intersections of the altitudes of $\Delta A_1A_2A_3$ with Ω. We know that H_1, H_2, H_3 are symmetric to H with respect to a_1, a_2, a_3. Thus we have $s_1 = S_1H_1$, $s_2 = S_2H_2$, $s_3 = S_3H_3$. Let M_1 be the other intersection of s_1 with Ω, and analogously define M_2 and M_3. Then we have



$$\angle A_2H_2M_2 = \angle H_2HS_2 = 90^\circ - \angle A_1S_2S_3$$

and

$$\angle A_3H_3M_3 = 90^\circ - \angle A_1S_3S_2.$$

Thus

$$\angle A_2A_1M_2 + \angle A_3A_1M_3 = \angle A_2H_2M_2 + \angle A_3H_3M_3 = \angle A_1.$$

Therefore M_2 and M_3 coincide at a point M on Ω. Analogously the points M_1 and M_2 coincide at M. Therefore SH is the required line.

Comment. This solution proves the following corollary: *the directrix of any parabola tangent to the three sides of a triangle passes through the orthocentre. The focus M is on the circumcircle and the tangent through the vertex is the Simson line of M with respect to* *the triangle.* [*Editor's note.* To get these results one uses the following reflection property of the parabola: *the image of the directrix of a parabola under reflection in any tangent passes through the focus.* A quite different yet common approach to these results is through projective geometry, as can be found in 17.4.3.5, page 231, of Marcel Berger, *Geometry II*, Springer-Verlag, New York, 1987.]

Also solved by P. PENNING, Delft, The Netherlands.

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1598*. [1990: 299] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$. Determine the maximum constant $C = C(\lambda)$ such that for all non-negative real numbers x_1, x_2 there holds

$$x_1^2 + x_2^2 + \lambda x_1 x_2 \geq C(x_1 + x_2)^2.$$

I. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

For $\lambda \geq 2$,

$$x_1^2 + x_2^2 + \lambda x_1 x_2 \geq x_1^2 + x_2^2 + 2x_1x_2 = (x_1 + x_2)^2,$$

with equality for $x_1x_2 = 0$. For $0 < \lambda < 2$,

$$\begin{aligned} x_1^2 + x_2^2 + \lambda x_1 x_2 &= (x_1 + x_2)^2 - (2 - \lambda)x_1x_2 \\ &\geq (x_1 + x_2)^2 - (2 - \lambda)\left(\frac{x_1 + x_2}{2}\right)^2 \\ &= \frac{2 + \lambda}{4}(x_1 + x_2)^2, \end{aligned}$$

with equality for $x_1 = x_2$. Thus

$$C(\lambda) = \begin{cases} 1 & \text{for } \lambda \geq 2, \\ (2 + \lambda)/4 & \text{for } 0 < \lambda < 2. \end{cases}$$

II. *Solution by Murray S. Klamkin, University of Alberta.*

More generally let us consider finding the least $M(\lambda)$ and the greatest $m(\lambda)$ such that

$$M(\lambda)(x_1 + \cdots + x_n)^n \geq x_1^n + \cdots + x_n^n + \lambda x_1 \cdots x_n \geq m(\lambda)(x_1 + \cdots + x_n)^n \qquad (1)$$

for all non-negative $x_1, x_2, \ldots, x_n$. By considering the case $x_1 = \cdots = x_n = 1/n$, we see that

$$M(\lambda) \geq \frac{n + \lambda}{n^n} \geq m(\lambda);$$

from the case $x_1 = 1$, $x_2 = \cdots = x_n = 0$ we see

$$M(\lambda) \geq 1 \geq m(\lambda);$$

and from the case $x_1 = \cdots = x_{n-1} = 1/(n-1)$, $x_n = 0$ we see

$$M(\lambda) \geq (n-1)^{1-n} \geq m(\lambda).$$

Therefore we must have

$$m \leq \min\left\{(n-1)^{1-n},\ \frac{n+\lambda}{n^n}\right\}\ , \quad M \geq \max\left\{1,\ \frac{n+\lambda}{n^n}\right\}\ .$$

We now show that, for all $n \geq 2$,

$$M = \max\left\{1,\ \frac{n+\lambda}{n^n}\right\}\ , \tag{2}$$

and that for $n = 2$ and 3,

$$m = \min\left\{(n-1)^{1-n},\ \frac{n+\lambda}{n^n}\right\}\ . \tag{3}$$

It is left as an open problem to show whether (3) is also valid for $n > 3$.

Proof of (2). If $1 \geq (n+\lambda)/n^n$, we must show that

$$(x_1 + \cdots + x_n)^n \geq x_1^n + \cdots + x_n^n + \lambda x_1 \ldots x_n,$$

where $\lambda \leq n^n - n$. This will follow by showing that

$$(x_1 + \cdots + x_n)^n \geq x_1^n + \cdots + x_n^n + (n^n - n)x_1 \ldots x_n. \tag{4}$$

Also, if $1 \leq (n+\lambda)n^n$ then we must show that

$$\frac{n+\lambda}{n^n}(x_1 + \cdots + x_n)^n \geq x_1^n + \cdots + x_n^n + \lambda x_1 \ldots x_n$$

or

$$\lambda\left[\left(\frac{x_1 + \cdots + x_n}{n}\right)^n - x_1 \ldots x_n\right] + n\left(\frac{x_1 + \cdots + x_n}{n}\right)^n \geq x_1^n + \cdots + x_n^n,$$

for $\lambda \geq n^n - n$. Since the term $[\ \cdots\ ]$ is nonnegative (by the A.M.–G.M. inequality), this will follow by showing that

$$(n^n - n)\left[\left(\frac{x_1 + \cdots + x_n}{n}\right)^n - x_1 \ldots x_n\right] + n\left(\frac{x_1 + \cdots + x_n}{n}\right)^n \geq x_1^n + \cdots + x_n^n,$$

which reduces to (4) again. Thus we need only prove (4). The number of terms in

$$(x_1 + \cdots + x_n)^n - (x_1^n + \cdots + x_n^n)$$

after expanding out (but not simplifying) is precisely $n^n - n$, and they are all positive, so that (4) will follow from the A.M.–G.M. inequality if the product of these $n^n - n$ terms is

$$(x_1 \ldots x_n)^{n^n - n}.$$

But this is clear by symmetry, since each of the $n^n - n$ terms is a product of some n x_i's.

Proof of (3) *for* $n = 2$ *and* 3. Similar to the above, this will follow (for any n) from the single inequality

$$x_1^n + \cdots + x_n^n + \left(\frac{n^n}{(n-1)^{n-1}} - n\right) x_1 \ldots x_n \geq \frac{(x_1 + \cdots + x_n)^n}{(n-1)^{n-1}} . \tag{5}$$

For $n = 2$, (5) reduces to an identity (this solves the original problem). For $n = 3$, (5) reduces to

$$4(x_1^3 + x_2^3 + x_3^3) + 15x_1x_2x_3 \geq (x_1 + x_2 + x_3)^3,$$

which is equivalent to the special case $r = 1$ of Schur's inequality (e.g., see [1991: 50])

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; BEATRIZ MARGOLIS, Paris, France; JEAN-MARIE MONIER, Lyon, France; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Spain; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Margolis points out (as is clear from solution I*) that the condition* $\lambda > 0$ *is not needed.*

The proposer's original problem (submitted without solution) was more generally to find the greatest $C = C(\lambda, n)$ *such that*

$$x_1^n + \cdots + x_n^n + \lambda x_1 \cdots x_n \geq C(x_1 + \cdots + x_n)^n,$$

i.e., the right hand inequality of (1)*! However, the editor (wishing to improve the chances of receiving solutions) printed only a special case, to which the proposer then submitted a solution. Can anyone settle* (5) *for all* $n > 3$*, and thus complete the solution of* (1)*?*

For a related problem, see Crux 1703, *this issue.*

*   *   *   *   *

1599. [1990: 299] *Proposed by Milen N. Naydenov, Varna, Bulgaria.*

A convex quadrilateral with sides a, b, c, d has both an incircle and a circumcircle. Its circumradius is R and its area F. Prove that

$$abc + abd + acd + bcd \leq 2\sqrt{F}(F + 2R^2).$$

Solution by C. Festraets-Hamoir, Brussels, Belgium.

On sait que, pour un quadrilatère à la fois inscriptible et circonscriptible, on a

$$s = a + c = b + d, \quad F = \sqrt{abcd}, \tag{1}$$

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}\,. \tag{2}$$

Ainsi

$$\begin{aligned} abc + bcd + cda + dab &= ac(b+d) + bd(a+c) \\ &= \frac{ac(b+d)^2 + bd(a+c)^2}{s} \\ &= \frac{acb^2 + acd^2 + bda^2 + bdc^2 + 4abcd}{s} \\ &= \frac{(ab+cd)(ad+bc) + 4abcd}{s} \\ &= \frac{(ab+cd)(ad+bc)(ac+bd)}{s(ac+bd)} + \frac{4abcd}{s}\,. \end{aligned}$$

Aussi

$$s = \frac{1}{2}(a+b+c+d) \geq 2\sqrt[4]{abcd}$$

et

$$ac + bd \geq 2\sqrt{abcd}$$

[A.M.–G.M. inequality], donc

$$\begin{aligned} abc + bcd + cda + dab &\leq \frac{16R^2F^2}{2\sqrt[4]{abcd}\cdot 2\sqrt{abcd}} + \frac{4F^2}{2\sqrt[4]{abcd}} \\ &\leq \frac{4R^2F^2}{\sqrt{F}\,F} + \frac{2F^2}{\sqrt{F}} = 2\sqrt{F}(2R^2+F). \end{aligned}$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

The proposer's original problem was much longer, somewhat like his earlier related problem Crux 1203 [1988: 91]. *Equations* (1) *are mentioned on* [1988: 91], *and* (2) *can be obtained from information given there. Here is Kuczma's derivation of* (2). *Letting* e *and* f *be the diagonals, we have*

$$4RF = abe + cde \quad \textit{and} \quad 4RF = adf + bcf,$$

since, e.g., the triangle with sides a, b, e *has circumradius* R *and thus area* $abe/(4R)$. *Therefore*

$$16R^2abcd = 16R^2F^2 = (ab+cd)(ad+bc)ef,$$

and (2) *follows from* $ac + bd = ef$, *which is also given on* [1988: 91].

*　　*　　*　　*　　*

1600. [1990: 299] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, and Wan-Di Wei, Sichuan University, Chengdu, China.*

Find the number of unordered triples $\{a, b, c\}$ of positive integers such that $\mathrm{lcm}(a, b, c) = 1600$.

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Note that $1600 = 2^6 5^2$. We allow that among the numbers a, b, c for which $\mathrm{lcm}(a, b, c) = 1600$, there are duplicates. Let

$$\begin{aligned} V &= \{0,1,2,3,4,5,6\} \times \{0,1,2\} \\ &= \{(0,0),(0,1),(0,2),(1,0),(1,1),\ldots,(6,2)\}. \end{aligned}$$

With each element (k, l) of V there corresponds a unique divisor of 1600, namely $2^k 5^l$. The required number N of unordered triples $\{a, b, c\}$ such that $\mathrm{lcm}(a, b, c) = 1600$ is equal to the number of ways to choose with replacement three elements $(a_1, a_2), (b_1, b_2)$ and (c_1, c_2) from V such that $6 \in \{a_1, b_1, c_1\}$ and $2 \in \{a_2, b_2, c_2\}$. By an inclusion-exclusion argument we deduce

$$\begin{aligned} N &= \binom{7 \cdot 3 + 3 - 1}{3} - \binom{6 \cdot 3 + 3 - 1}{3} - \binom{7 \cdot 2 + 3 - 1}{3} + \binom{6 \cdot 2 + 3 - 1}{3} \\ &= \binom{23}{3} - \binom{20}{3} - \binom{16}{3} + \binom{14}{3} = 1771 - 1140 - 560 + 364 = 435. \end{aligned}$$

[For example, $\binom{7\cdot3+3-1}{3}$ is the number of ways of choosing $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ from V with replacement and without conditions; $\binom{6\cdot3+3-1}{3}$ is the number if $6 \notin \{a_1, b_1, c_1\}$; etc.]

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; and the proposers. There were five incorrect answers sent in, the most common mistake being that ordered rather than unordered triples were counted.

Some solvers (including the proposers) gave generalizations; for example, Janous calculates that for primes p, q and nonnegative integers α, β, the number of unordered triples $\{a, b, c\}$ *such that* $\mathrm{lcm}(a, b, c) = p^\alpha q^\beta$ *is*

$$\alpha^2\beta^2 + \frac{1}{2}\alpha\beta(\alpha + \beta + 2) + \frac{1}{2}(\alpha + 1)(\beta + 1)(\alpha\beta + \alpha + \beta + 2).$$

* * * * *

UNSOLVED PROBLEMS IN *CRUX*

In the summer of 1990, SAM MALTBY, an undergraduate student at the University of Calgary, was "hired" to make a list of all (numbered) *Crux* problems which were still partly or completely unsolved, and to solve whichever of them he could. As readers may have noticed lately, he did manage to settle a few, and his solutions (sometimes quite long!) have been appearing in *Crux* over the last year. With the publication of the last of these in this issue, the list of remaining unsolved problems (plus a few more recent ones to bring the list up to date) can now be presented. It could be said that in most cases there are very good reasons why these problems are still unsolved! Some are famous or near-famous uncrackable nuts, others simply have no nice solution. Problems which Maltby considers undoable for either reason are marked with a dagger (†): readers beware! In some cases footnotes have been added. Most of the earlier problems are number theory; most of the later, geometry. **Not** listed are any generalizations or related problems that may have been raised in connection with *Crux* problems. Readers are on their own in finding those!

133.† [1976: 144, 221][1]
154.† [1976: 197, 225; 1977: 20, 108, 191]
250.† [1978: 39; 1979: 17]
266. [1978: 75]
283.† [1978: 115, 195][2]
342. [1980: 319]
343. [1978: 297]
355(c),(d). [1979: 78, 168]
410.† [1979: 296]
434(b). [1980: 59]
473. [1980: 197]
490.† [1980: 288]
494(b). [1980: 296]
527(b).† [1981: 88][3]
533(b). [1981: 118]
592(c). [1981: 310]
609. [1982: 27]
648(b). [1982: 180]
714. [1983: 58]
757.† [1983: 218][4]
804. [1984: 120]
844(b). [1984: 264]
857(c).† [1984: 304; 1985: 20, 84]
860.† [1984: 308]
906. [1985: 92]
909.† [1985: 94]
928. [1985: 159]
942. [1985: 228][5]
972(b),(c). [1985: 326]
976.† [1986: 145; 1987: 16]
1062(b). [1987: 17]
1066. [1987: 24][6]
1077. [1987: 93]
1086. [1987: 100]
1110. [1987: 170; 1988: 13]
1180(b). [1988: 24]
1225. [1988: 206]
1338.† [1989: 179]
1357. [1989: 243]
1363. [1989: 250][7]
1464(c). [1990: 282]
1492(b). [1991: 50]
1495(b). [1991: 54]
1580. [1991: 308]
1581. [1991: 308]
1587. [1991: 314][8]

[1]The infamous Collatz $3x + 1$ problem!

[2]Given is a solution as a limit of elementary functions; a closed solution may be impossible.

[3]Maltby "solved" this, obtaining an expression with four nested summations which the editor has deemed unprintable.

[4]Related to **592**(c).

[5]Rumoured to be solved (see [1985: 229]). No evidence yet.

[6]One case left.

[7]Is Erdős's $25 prize still unclaimed?

[8]Actually, there is no footnote 8. This is just to fill up an annoying space that LaTeXwould otherwise insist on leaving above footnote 5.

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