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Mathematics—A Way of Looking at Life†

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The great physicist Willard Gibbs is said to have made only one speech in his life: he stood up at a university discussion on the teaching of languages and said: 'Mathematics is a language'—and then sat down. This statement by itself is not a very startling one; we are becoming accustomed to thinking of mathematics as the language in which the truths of mechanics, physics, and astronomy are most naturally and most efficiently expressed, but is it a language in which one can profitably talk about anything else? The thesis of this article is that mathematics is a universal language, of value to everyone, and with the aid of which it is possible to discuss in a fruitful way *at least some aspects* of almost every field of human experience and endeavour. Such a statement cannot be proved; one can only hope to persuade the reader of its truth by giving examples, and this I shall now proceed to do.

Let us begin with the past; historians argue about whether their subject is a science, but it is now becoming clear that some historical inquiries do quite naturally give rise to mathematical activity. In this country at the present time there is a great deal of interest in the changing structure of social groupings, in the drift from the country towards the towns, and in the gradual diffusion of knowledge about, and practice of, the principles of public health, and of family limitation, etc., during the last four centuries. Many questions in this area can in principle be answered by making a thorough use of the magnificent store of national and ecclesiastical archives which is still preserved and which is now at last being adequately indexed and otherwise cared for. But to use this mass of material in a systematic way is a task beyond the powers of the individual historian carrying out his calculations by hand; the speed and storage of a modern computing complex are required, and the skills of the mathematician must be married to those of the historian in order to ensure that the material is presented to the computer in an objective way, free from 'selection errors', that the data are stored in the manner best permitting efficient retrieval, and (most important of all) that the computer is programmed to sort, analyse, or organize the data in a way which is meaningful for the historian and

† An earlier version of this article was written for and published by the Romanian Society of Mathematics, by whose kind permission the present (English) version is republished here.

which extracts the maximum amount of relevant information bearing on the questions which are of interest to him.

If we push back from history to pre-history we encounter problems which call not merely for the computer as a device for speeding up and making relatively painless a tedious and lengthy sorting process, but also for the development of new sophisticated mathematical techniques before these problems can be handled at all. Thus my colleague Dr Roy Hodson recently introduced me to a series of some sixty Iron Age graves in a cemetery at Münsingen in Switzerland, which it was desired to arrange in something approaching the correct chronological order. This is normally done by the archaeologist using information of many different kinds, for example, if the graves lie one beneath the other and if there is no reason to fear that the vertical arrangement of the graves has been disturbed by natural or artificial earth movements, then it will be reasonable to assume that the deepest graves are the earliest, etc., and there is virtually no problem. But for the Münsingen cemetery (and in a number of similar examples) this simple method is not available because the graves all occur on the surface, at essentially the same depth beneath it. 'Carbon dating' may then help, but will often not be applicable and will in any case rarely possess the 'resolving power' necessary to establish the finer details of the chronology.

In such a case (as was first pointed out in 1899 by the egyptologist Flinders Petrie) we may still be able to recover the chronology of the graves by making use of the simple principle that *the more nearly contemporary graves will be expected to have the more similar contents*. Thus we must rearrange the 60 graves in such a way that those with similar contents occur near to one another in the sequence. (It is presumed that, before this operation is attempted, a classification of the contents—whether pottery, metal ornaments, tools, etc.—has been carried out, *varieties* being distinguished by the style of ornamentation, the measurements of various characteristic lengths, and the presence or absence of features judged not to be merely fortuitous.) The records of the excavation, wholly objective save for the intervention of the classification system just mentioned, can then be completely summarized by a table of double-entry in which each *row* is identified with a *grave*, and each *column* is identified with a *variety* (of pottery, etc.), the entry in the cell corresponding to a specified grave (row) and variety (column) being 1 if that variety was found in the grave, and 0 otherwise.

We must then try to interchange the rows of this table in such a way that, *as far as possible*, the 1's are bunched together in each separate column. I say, 'as far as possible', because it will be clear that a 'good' rearrangement of the rows for one column may be very unsuitable for some of the other columns, and so a compromise is called for.

It is a curious fact that the human brain seems to be able to achieve an approximate solution to this task even though a systematic approach to it is beyond the powers of a computer. To see this, let us first note that there are about 10^{80} different ways of rearranging 60 objects, and this is enormously greater than the number of steps which could be performed by the fastest possible computer (carrying out one step in the time it takes for light to travel through a distance equal to the radius of an atomic nucleus) even if the computer was run continuously for 10^9 years (the conventional 'age of the universe'). Thus a systematic scanning of all possible rearrangements is *utterly and absolutely impossible in principle*—and yet, after a fashion, archaeologists do it.

The explanation of this paradox seems to be that the archaeologist uses not merely the mathematical information contained in the 0–1 table, but also a vast amount of background information incorporated in his professional judgment; this will often include an explicit (or implicit, intuitive) feeling that some varieties supply a more pertinent criterion than others, a corpus of beliefs about what is and what is not a plausible developmental sequence for a cultural trait, and so on. Such information is not available to the computer; also archaeologists will occasionally not agree in their judgments on these matters. We may therefore say that the archaeologist performs his task by using *both* the 'objective' information which we hoped to present to the computer and *also* what we might call 'subjective' information which is personal to him, to the school of thought in which he works, or to the generation of archaeologists to which he belongs. Evidently, if a solution were possible using the 'objective' information alone, it would have a substantial but not over-riding value. In some respects it would be less valuable than the archaeologist's solution, for the quite simple reason that some ('subjective', but still important) information will have been wholly ignored. But against this we have the fact that an 'objective' solution, if one could be obtained, would be a very useful check on the archaeologist's procedures and presuppositions, and would represent an irreducible minimum of non-controversial fact with which all could agree and which all could then separately refine in accordance with such further special knowledge or beliefs as they might possess or hold.

During the last twelve months such an 'objective' method of analysis has been worked out in the Statistical Laboratory at Cambridge University, and applied to the Hodson data. This cannot, as we have explained above, involve a scrutiny by the machine of all possible rearrangements, and instead it takes a series of 'short cuts' of somewhat too technical a character to be explained here.† The final outcome was a computed chronology for the graves which turned out to place them more or less in the geographical order in which they occurred in the 'cemetery'—in this case roughly a 'linear' one. (The data were used as a test of the method for exactly this reason, but the computer—and the mathematician controlling it—*did not know beforehand what the geographical order was.*) It is of course very natural to suppose that in a nearly linear cemetery the order on the ground would be associated with the chronology, so that this agreement was the source of considerable satisfaction to all concerned, and an encouragement to pursue the method further.

I have written at some length about these applications of mathematics to history and allied fields because to many people it seems at first well nigh incredible

† The mathematical basis of the method was presented by the author at the Third Colloquium on Probability Theory held by the Romanian Academy of Sciences at Braşov, 2–7 September 1968.

that any such application could be made. But now I wish to turn to quite a different range of applications, contemporary in their relevance and very much concerned with the efficiency of our industries and indeed the comfort of our daily lives; I refer to what is called the 'theory of queues and waiting-times'.

Wherever an excess of demands compete for the services of a limited number of facilities, someone or something has to be 'disappointed'; according to the nature of the system this may mean having to wait, or having to retire with the demand unsatisfied. In the everyday world in which we live, work, and play we have inevitably to wait, whether for food, transport, information, or instructions vital to the next stage in our work, or the apartment or house in which we hope to live. When such waiting occurs on a large scale in a system with a semi-permanent structure there will usually be alternative modes in which the structure can be operated, and by choosing one of these carefully we may be able to cut down the *average* time spent in waiting, or the *average* number of persons who are waiting, to a very dramatic extent. There is a notable example in the field of medicine.

A medical clinic specializing in say children's ailments meets once a day and is normally capable of interviewing perhaps thirty patients. The easy, lazy, way of proceeding is to invite all the thirty patients to arrive together at the clinic at a fixed time, and then to interview them in some order—perhaps at random, or alphabetically. In this case most of the patients will have to wait a long time, the waiting-room will be very crowded, and the conditions for an accelerated spread of the latest children's disease will have been created by the very authority most concerned to prevent it.

A natural solution to this problem is to call the patients to the clinic at different times, instead of simultaneously, and to try to do this in such a way that (1) the patient never has to wait because the doctor is busy and (2) the doctor never has to wait because there are no patients ready for him. To achieve both (1) and (2) is impossible, firstly because consultations do not all last for the same length of time and their lengths cannot be *exactly* predicted, and secondly because the patients will not always arrive exactly when required (either because of unforeseen personal difficulties, or because of the irregularities of the transport facilities to the hospital). Thus we are immediately within the domain of the calculus of probabilities; more exactly we are concerned with what is now called a *stochastic process*. If the *degree of variability* of the random elements in the system (the length of the consultation time, and the inaccuracy with which appointments are kept) can be assessed numerically, then in principle (and sometimes in practice) a realistic appointments system can be worked out to the great benefit of all parties. This subject has been extensively studied during the last fifteen years in this country. At first the recommendations were greeted with some scepticism by the medical authorities, but now they are to a great extent an accepted part of medical practice.

Similar problems arise in the design of an airport, the organization of a transport system, the planning of a steel plant, and so on. The general principles are the same throughout, although from one application to the next the details will vary.

Now let us leave mankind and think about birds: how do they navigate? That birds are capable of astonishing navigational feats is known to everyone, but we have no clear understanding of how their task is carried out, although there is growing evidence that some sort of astronomical navigation involving the sun may play an important role here. Whatever 'observations' the bird 'makes', and whatever 'conclusions' it 'draws' from them about the most desirable direction for the next stage of its flight, it is certain that both will involve considerable possibilities of error, so that we should expect the plot of the flight-path of a navigating bird to have a small-scale random appearance, but a large-scale tendency to approach its home. Biologists have recently suggested specific models for such a combination of random and systematic motions which might describe very sketchily how the bird carries out its navigational task, and at this point the mathematician finds a role to play. He can study theoretically how such a 'fictitious bird' *would* behave, in order to make predictions based on the model, for subsequent comparison with the observations which show how real birds *do* behave, and he can also simulate such flight-paths within the computer (which will actually draw them out, if so requested) and compare these with the qualitative characteristics of flight-paths observed, for example, by following real birds with aeroplanes.

As a final example we return to the human world and to the problem of contagious epidemics which we touched upon very briefly earlier in this article. It is of the essence of a contagious disease that its spreading involves *both* the infectious persons *and* those who are at present free from the disease, but may catch it. In other words we have to deal with what mathematicians call a *non-linear* phenomenon; effects are not simply proportional to the numbers present, but also involve the products of these numbers. At the same time the transmission of such a disease depends on chance contacts between individuals and so probability is also involved —once again we have a 'stochastic process'. The mathematical study of such phenomena started in 1927 but has only been prosecuted with real vigour during the last fifteen years or so, when medical authorities began to perceive the possibilities of a mathematical approach in the general field of public health, and simultaneously a small group of mathematicians began an active search for such co-operation. This 'dialogue' between two apparently very different disciplines has now proceeded to the point at which the World Health Organization maintains a powerful group of mathematicians at its central office in Geneva in order that they should devote their whole energies to such matters and their practical exploitation for the benefit of mankind.

I have tried to indicate how wide is the range of subjects to which mathematics is now being applied, and I have succeeded (without loss of clarity, I hope) in doing so by using ordinary words; no symbols and no formulae. Symbols and formulae are *not* the stuff of mathematics, but rather an efficient conventional shorthand in which mathematical concepts can be expressed and examined. The mathematical *ideas* are already there, in our heads, and the decision to employ a professional mathematician in such a field is simply a recognition that a degree of complexity has been reached after which it would be foolish not to take full advantage of the technical aids available.

Once this fact is grasped, mathematics ceases to be an arcane mystery, and we recognize the mathematical approach as a natural part of our ordinary way of looking at the world. This is not quite the whole story, for we have made no mention of the 'pure' mathematician whose business it is to recognize the mathematical structure common to many apparently quite different everyday situations (for example, the different 'waiting' situations described earlier), and who then (say for the sake of economy of effort) studies the structure itself, without special reference to any particular instance of it. He is a specialist in structures, and is to be thought of like any other specialist. If you press hard enough, you will find in his environment the real situation which (perhaps very remotely) prompted his present abstract inquiries. Thus for me, mathematics, whether 'pure' or 'applied', is *a way of looking at life.*

The Laws of Thought†

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At the beginning of the twentieth century Bertrand Russell wrote, in a popular essay,‡ these famous words: 'Pure mathematics was discovered by Boole, in a work which he called the *Laws of Thought*.... His book was in fact concerned with formal logic, and this is the same thing as mathematics .... Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true .... Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.'

Boole's book *An Investigation of the Laws of Thought* was published in 1854. In 1954 the Royal Irish Academy celebrated the centenary of this event, and received from Bertrand Russell a letter in which he said: 'The remark that you quote from me to the effect that pure mathematics was discovered by Boole was of course not intended to be taken literally, but only as an emphatic statement of the importance of the subject which he inaugurated. This subject has now grown to

† Text of an address given to the Literary and Philosophical Society, University College, Swansea, on 9 October 1968.

‡ 'Recent work on the principles of mathematics', *International Monthly* **4**, 1901, 83–101; reprinted, with added footnotes, as Chapter V, 'Mathematics and the metaphysician', of his book *Mysticism and Logic* (Longmans, 1918; Pelican Books, 1953).

vast proportions and has developed in directions that would have surprised Boole, but the developments have made his importance continually more evident.' (Russell has explained elsewhere† that when he wrote the essay he was responding to an editorial request for something romantic.)

There are now few who would maintain that mathematics is exactly the same thing as formal logic. Nevertheless, Russell's epigrammatic definition still conveys an important truth about the nature of pure mathematics—though it perhaps does little to account for the aesthetic appeal of the subject, or for its practical utility. And Boole can still be seen as an outstanding figure among the many nineteenth-century pioneers of pure mathematics. I have no numerological excuse for talking about him this evening (I know of no relevant centenary), but it happens that today I am exactly as old as Boole was when he gave his last lecture.

Although he started a revolution in the study of logic, Boole seems to be regarded less highly by philosophers than by mathematicians: only a page or so is devoted to him in the eight volumes of the *Encyclopedia of Philosophy* (1967), as against a whole chapter—one of twenty-nine—in E. T. Bell's *Men of Mathematics* (1937). Bell's sketch of Boole's life and work is lively but unscholarly—his judgment as a biographer seems to have been clouded by his contempt for Victorian social attitudes. For the most part, Boole's own writings are easy to read, and there are several accounts of him by persons intimately connected with him; I would recommend particularly a book published in 1952 with the title *Studies in Logic and Probability*, which contains papers by and about Boole, edited, with a long introduction, by Rush Rhees. Also of interest is a special issue of the *Proceedings of the Royal Irish Academy* (Vol. 57, Section A, No. 6), published in 1955; this contains, among other items, the letter from Bertrand Russell that I have mentioned, a biographical article by Boole's grandson Sir Geoffrey Taylor (the distinguished mathematical physicist), an article by Rush Rhees on Boole as student and teacher, and one by an engineer of the Automatic Telephone Company explaining how 'Boolean algebra' can be used to analyse complicated systems of switches. (Such systems have proliferated greatly since 1954, with the rapid development of digital computers.)

George Boole was born in Lincoln, the son of a cobbler who apparently had some mathematical knowledge and who made optical instruments in his spare time. George had a simple elementary schooling, and as much help in the pursuit of learning as his father could give him, but he was largely self-educated. He studied the classics at first, and learned to read Latin, Greek, French, German, and Italian. At the age of sixteen he began his career as a teacher, at a school in Doncaster; he later had a school of his own in Lincoln. It was not until he was seventeen that he began the serious study of mathematics. Within a few years he was writing original papers, and corresponding with leading mathematicians of the time. His book *The Mathematical Analysis of Logic*—a precursor of the *Laws of Thought*—appeared in 1847, when he was 32; and by then he had published other mathematical work of

† Preface to *Mysticism and Logic*.

lasting value. In 1849 his standing was recognized by his election to the chair of mathematics at the new Queen's College (now University College) at Cork, which he occupied with distinction for the rest of his life.

In his fortieth year Boole married Mary Everest, a niece of his friend Ryall, the Professor of Greek at Cork. (Another of her uncles was George Everest, a native of Breconshire, who achieved great distinction as a surveyor and military engineer; he made the first accurate determination of the position and height of the Himalayan mountain that was named after him.) She was a woman of strong character who in later life wrote quite extensively on educational matters. Her collected works occupy four volumes. The first item in this collection has the title 'Home side of a scientific mind', and is an affectionate memoir of her husband (whom she survived by more than fifty years). Here she records that he took seven years to write the *Laws of Thought*, and that he wished he had taken fifteen; she also tells us that when, as a girl, she read part of it in manuscript (several years before their marriage) she did not fully understand it but 'got a vague sense of safety and comfort from finding it proved that all our thinking was under the shelter of algebraical laws, even when we were not conscious of using mathematical symbols'. (She was a clergyman's daughter, and had a proper respect for algebra.) Another item is her book *Lectures on the Logic of Arithmetic*, published in 1903 (the year of Russell's book *The Principles of Mathematics*, which began with the sentence: 'Pure Mathematics is the class of all propositions of the form "p implies q", where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants'). Mary Boole's book was addressed mainly to teachers, and it makes interesting reading today: she seems to have anticipated some of the ideas now being propagated by reformers of primary education. She and George had five children, all daughters. The second daughter was the mother of Sir Geoffrey Taylor; the third had some mathematical talent, and in late life collaborated with the distinguished geometer H. S. M. Coxeter (now of Toronto); the fourth became a professor of chemistry in London.

Boole died before he was fifty. On a day in late November 1864, he walked from his home to the college (about two miles) in heavy rain, and lectured in his wet clothes. According to his wife, his health had been weakened by overwork—he was of course a professor—and he soon succumbed to a bronchial infection. He had received many scientific honours, and his personal qualities had attracted much esteem and affection.

Nowadays, a good approach to Boole's symbolic logic is by way of the abstract concept of a 'ring'. This is a powerful unifying concept in modern mathematics; it was implicit in the work of some of the nineteenth-century algebraists, but has been explicitly recognized and studied only in the last fifty years. (The German word 'Ring' was used in algebra more than 70 years ago, by Hilbert, but with a meaning considerably narrower than the modern one.) Consider first the system of integers that one meets in elementary arithmetic:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots.$$

Suppose that the symbols x and y represent arbitrary integers (allowing the possibility that $x = y$, meaning that x and y represent the same integer); then the composite symbols $x+y$ and $x \times y$ also represent integers (the sum and the product, respectively, of the integers x and y). We thus have a system which, as we say, is 'closed' with respect to 'operations' denoted by the symbols $+$ and $\times$ (and called 'addition' and 'multiplication' respectively). Moreover, the system is subject to certain 'laws' concerning these operations:

$$\left.\begin{aligned}(x+y)+z &= x+(y+z),\\ (x\times y)\times z &= x\times(y\times z)\end{aligned}\right\} \quad \text{(the \textit{associative laws}),}$$

$$\left.\begin{aligned}(x+y)\times z &= (x\times z)+(y\times z),\\ x\times(y+z) &= (x\times y)+(x\times z)\end{aligned}\right\} \quad \text{(the \textit{distributive laws}),}$$

$$x+y = y+x \quad \text{(the \textit{commutative law} for addition),}$$

and (the *law of subtraction*), for each choice of y and z there is, within the system, an x such that

$$x+y = z.$$

(The law of subtraction does not assert that, given y and z, there is only one x such that $x+y = z$; but from this law and the others concerning addition we can deduce that in fact there is only one such x, and this we denote by the symbol $z-y$.) Now we can conceive of an *arbitrary* collection of entities $x, y, \ldots$ closed with respect to a pair of hypothetical 'operations' which we might choose to denote by the symbols $+$ and $\times$: such a 'system' is called a *ring* if it is subject to the above laws; and in this case the operations $+$ and $\times$ are referred to as 'addition' and 'multiplication' respectively, because of their formal resemblance to the operations that have these names in arithmetic. The system of integers is thus an example of a ring. Other examples are the system of rational numbers, the system of real numbers, and the system of complex numbers. These number-systems have other features in common: in particular, each of them is a *commutative ring*, in that multiplication as well as addition is subject to the commutative law ($x \times y = y \times x$), and each of them has a *unit element*, 1, characterized by the property† that

$$1 \times x = x = x \times 1$$

for every x. There are, however, non-commutative rings (the system of quaternions is an example), and not every ring has a unit element (consider, for example, the system of even integers). As a matter of convenience, when one is working with a ring, it is usual to write xy for $x \times y$; and to write $2x$ for $x+x$, and x^2 for xx. In elementary algebra there are certain rules about the placing of brackets: these rules

† For each positive integer m, the system of residue classes modulo m, described in L. Mirsky's article in the preceding number of *Mathematical Spectrum*, is a commutative ring with unit element.

derive their validity from the associative and distributive laws, and are therefore applicable to any ring.

For none of the particular rings I have mentioned is it the case that

$$x^2 = x$$

for every x; but there are rings subject to this special law, and they are called *Boolean rings*. A Boolean ring is necessarily commutative, so that any proposition that is valid for commutative rings in general is valid in particular for any Boolean ring; the binomial theorem is such a proposition. But the algebra of Boolean rings is uncommonly simple: for instance, it is permissible to write

$$(x+y)^2 = x^2+y^2,$$

which, of course, is incorrect in the ordinary algebra of numbers (unless x or y is 0); another feature of Boolean rings is that $x-y$ is always the same as $x+y$.

To get an example of a Boolean ring, consider the class of all individuals of some particular kind—say, for the sake of illustration, the class of all animals (assuming that this class can be unambiguously defined). Let the letters $x, y,\ldots$ be used to denote subclasses of this given class, and let us adopt the following interpretation for the symbols $x+y$ and xy: whatever subclasses we choose to denote by x and y, $x+y$ shall denote the class of those individuals that belong either to the class x or to the class y but not to both, and xy shall denote the class of those individuals that belong both to x and to y. Thus if x denotes the class of horned animals, and y the class of sheep, then xy denotes the class of horned sheep; we could write this alternatively as yx (sheep that are horned). Provided that we admit the concept of an 'empty' or 'zero' class (having no members), we now have a system which is closed with respect to the operations of 'adding' and 'multiplying' classes; and it is a simple matter to verify that this system is a Boolean ring (we need logic to do this, of course, but there are no subtleties).

We can construct a Boolean ring in this way by considering the subclasses of any given class; in each case the ring has a unit element, namely the whole of the given class. For any such ring, the assertion that x is a subclass of y can be expressed by the equation

$$x = xy.$$

The symbol $1+x$ (or, equivalently, $x+1$) denotes the subclass consisting of those individuals that belong to the given class, 1, but not to the particular subclass x. To assert that x and y are disjoint (have no members in common) we can write

$$y = (1+x)y,$$

and by the standard rules about brackets, this equation can be written in the equivalent form

$$y = y+xy;$$

alternatively, we can write simply

$$xy = 0$$

with the understanding that the symbol 0 stands for the empty class.

Let us now suppose that we have chosen our unit class (which for Boole was 'the Universe') so that it includes the class of 'all that is mortal', the class of all men, and the class whose only member is Socrates; and let us denote these classes by x, y, and z respectively. Then the proposition 'all men are mortal' can be expressed by the equation

$$y = xy.$$

(I am not saying that this proposition is true, but merely that if we wish to assert it we can do so in this way.) Likewise, the proposition 'Socrates is a man' can be expressed by the equation

$$z = yz.$$

From these two equations we infer, by substitution from the first into the second, that

$$z = xyz.$$

(The symbol xyz has the same meaning as $(xy)z$ and $x(yz)$.) Multiplying both sides of this equation by x, we have

$$xz = x^2yz.$$

Thus far, we have proceeded as in ordinary algebra. But now we may note that, since we are working with a Boolean ring, x^2 is the same as x; so that

$$xz = xyz.$$

This, with our earlier equation $z = xyz$, implies that

$$z = xz.$$

Now this last equation expresses the proposition 'Socrates is mortal'. Thus we have passed from the premises of a *syllogism* to its conclusion by a process of algebraic manipulation.

These simple illustrations may give some idea of the way in which Boole brought algebra and logic together. However, I ought to point out, at the risk of being excessively technical, that in one important respect I have not followed Boole exactly: he assigned a meaning to the symbol $x+y$ only when the classes x and y were disjoint; his algebraic system was therefore not a ring, since it was not closed with respect to addition. Others, notably Augustus de Morgan, a contemporary of Boole who became the first president of the London Mathematical Society (which was founded a few weeks after Boole's death), wrote $x+y$ for what is now called the union of the classes x and y, consisting of all individuals that belong to either or both of these classes; this gave a closed system which, however, failed to be a ring since it was not subject to the law of subtraction. I suspect that these differing uses of the symbol $+$ have been the cause of some misunderstandings about Boolean algebra, and of some misuse of it. (Even Garrett Birkhoff, in his well-known and authoritative book *Lattice Theory*, 3rd edition, 1967, gives a definition of the term 'ring of sets' which is not equivalent to the standard definition.) If we write $x \vee y$ for the union of classes x and y, reserving for $x+y$ the interpretation that I have been using, then, clearly,

$$x \vee y = x+y+xy;$$

and if we write x' for $x+1$ (and y' for $y+1$) then

$$x+y = (x \vee y)(x' \vee y').$$

These two equations enable us to translate formulae between two notations currently used in Boolean algebra. (Boole wrote $1-x$ for x'; his rule about the meaning of $+$ forbade him to write $x+1$.) In the algebra of classes (or sets, as they are now more usually called by mathematicians), as distinct from general Boolean algebra, it is now customary to write $X \cup Y$, $X \cap Y$, and $X \Delta Y$ rather than, respectively, $x \vee y$, xy, and $x+y$; and when Boolean algebra is considered as a part of the subject known as 'lattice theory' (rather than as a part of the theory of rings), it is usual to write $x \wedge y$ instead of xy (though in another kind of algebra $\wedge$ denotes a certain non-associative operation). Recognition of the importance of the operation Δ, in the algebra of classes, is traceable to some early work of P. J. Daniell, a mathematician who, as Professor D. G. Kendall recently remarked (see Obituary, Norbert Wiener, *J. London Math. Soc.* **43**, 1968, 763–764), has been greatly underestimated; Daniell died in 1946—when, as a new assistant lecturer in his department at Sheffield, I had just begun to know him.

Variational Principles in Mathematical Physics

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1. Introduction

Variational principles are of great importance in mathematical physics. Almost exactly three hundred years ago Fermat showed that the laws of geometrical optics could be summarized by the variational principle named after him. About 1830 the great Irish mathematician Sir William Hamilton showed that the mechanics based on Newton's laws, known as classical mechanics, could also be derived from a variational principle. These two subjects are apparently otherwise unrelated. However, it turned out that there is in fact a subtle connection between them which was demonstrated in the present century by the experimental physicists investigating the behaviour of atomic particles. These experimental results, together with the tenuous connection between the principles of Fermat and Hamilton, enabled Erwin Schrödinger in 1926 to write down the basic equation of the mechanics which describes atomic systems known now as quantum mechanics.

Scientists are always looking for powerful concepts which link apparently different kinds of phenomena. Einstein's general relativity theory, which may be regarded as a theory of gravitation, leads to equations which are derivable from a variational principle. More recently, variational principles have been used in the formulation of quantum field theory, a branch of quantum mechanics which attempts to describe the interactions of the mesons ($\pi, \kappa, \rho, \omega, \ldots$) and the baryons ($n, p, \Lambda, \Sigma, \ldots$) commonly known as 'elementary' particles; although it is far from clear yet which, if any, of the host of new particles recently discovered in high-energy accelerators merits the description 'elementary'.

In the following sections we shall discuss the principles of Fermat and Hamilton; similar methods are applicable to the other topics mentioned above. First it will be convenient to discuss some mathematical ideas and state a few results.

2. Preliminary ideas and results

Let S be a set of real numbers and let x represent any member of S, a situation which we denote by the symbolism $x \in S$. With each $x \in S$ we associate a unique number denoted by $f(x)$, say. The correspondence defines a *function* f on S and $f(x)$ is the *value* of the function at the *point* x. (Members of sets are frequently referred to as *points* or *elements*.)

We can extend this notion by allowing S to consist of ordered pairs† (x, y) where x and y are real numbers. With each ordered pair $(x, y) \in S$ we associate a unique number denoted by $g(x, y)$, say. This correspondence defines a function g on S and $g(x, y)$ is the value of the function at the point (x, y).

For convenience we now assume that f is defined for all values of x and that it can be differentiated as often as we please.

Let

$$\delta f = f(x+\delta x) - f(x).$$

Then we know that

$$\delta f = f'(x)\,\delta x \tag{1}$$

to a *first* order in δx. We shall say that the point x is a *stationary* point of the function f if δf, the *first* order change in the value of f, is zero for arbitrary δx. From (1) we have

$$f'(x) = 0$$

at a stationary point of the function f (the converse also holds), this being the definition with which the reader is probably familiar.

Similarly we assume that g is defined for all pairs (x, y). If we keep y fixed and vary x we can calculate the *partial derivative* of g with respect to x in an obvious way. This derivative is denoted by $\partial g/\partial x$. In like fashion $\partial g/\partial y$ is obtained by differentiating g with respect to y when x is treated as a constant.

As an example, suppose that the function g is defined by

$$g(x, y) = 2x^3y^2 - 4(x+y^2)^2.$$

Then

$$\frac{\partial g}{\partial x} = 6x^2y^2 - 8(x+y^2),$$

† Two pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

and

$$\frac{\partial g}{\partial y} = 4x^3y - 16y(x+y^2).$$

Let us write

$$\delta g = g(x+\delta x, y+\delta y) - g(x, y).$$

A justification for the following expression for δg, which holds to a first order in δx and δy is given in the Appendix:

$$\delta g = \frac{\partial g}{\partial x}\,\delta x + \frac{\partial g}{\partial y}\,\delta y, \tag{2}$$

where the values of $\partial g/\partial x$ and $\partial g/\partial y$ are calculated at the point (x, y).

As an example consider the case where the function g is defined by

$$g(x, y) = x^2 + y^2.$$

Here,

$$\begin{aligned}\delta g &= (x+\delta x)^2 + (y+\delta y)^2 - (x^2+y^2)\\ &= 2x\,\delta x + 2y\,\delta y + (\delta x)^2 + (\delta y)^2\\ &= 2x\,\delta x + 2y\,\delta y\end{aligned}$$

to first order in δx and δy. But $\partial g/\partial x = 2x$ and $\partial g/\partial y = 2y$, so that (2) is clearly satisfied in this case.

3a. Functionals

In the previous section the notion of a function defined on a set of real numbers was introduced. By a slight abstraction we arrive at the idea of a *functional.*

Let S be a set of functions and suppose that with each $f \in S$ we associate a definite real number denoted by $H(f)$ say. This correspondence defines a *functional* H on S and $H(f)$ is the value of the functional at the *point* $f \in S$.

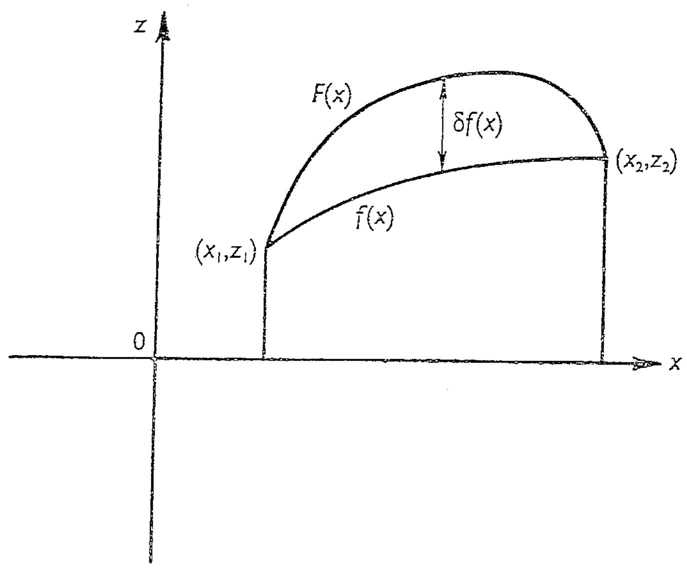

Figure 1

In Section 2 we defined what we meant by the stationary points of a function. In a similar fashion we now say that a point $f \in S$ is a stationary point of a functional H on S if δH is zero, to first order at the point f, where

$$\delta H = H(f+\delta f) - H(f).$$

This idea will be clearer if we think of $f+\delta f$ as a function, say F, which is 'close' to f,† and write

$$F(x) = f(x) + \delta f(x),$$

see Figure 1.

3b. A class of functionals and Euler's equation

Let S be the class of functions of a single variable whose graphs pass through two given points (x_1, z_1) and (x_2, z_2). The notation implies that if $f \in S$ then $z_1 = f(x_1)$ and $z_2 = f(x_2)$. For convenience let us denote df/dx by f_x. Consider the functional H on S defined by

$$H(f) = \int_{x_1}^{x_2} I(f(x), f_x(x), x)\, dx,$$

where the integrand can depend on f, f_x, x. The stationary points of H (following the notation and ideas of Section 3a) are obtained as follows.

A simple extension of (2) leads to

$$I(F(x), F_x(x), x) - I(f(x), f_x(x), x) = \frac{\partial I}{\partial f}\delta f(x) + \frac{\partial I}{\partial f_x}\delta f_x(x).$$

The last term of this equation can be written as

$$\frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\delta f(x)\right) - \delta f(x)\frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)$$

since

$$\delta f_x(x) = \frac{d}{dx}\delta f(x).$$

Now

$$\delta f(x_1) = \delta f(x_2) = 0;$$

this follows at once, since $F \in S, f \in S$ and so $F(x_1) = z_1, f(x_1) = z_1$, hence

$$\delta f(x_1) = F(x_1) - f(x_1) = 0,$$

and similarly for the point x_2. So, finally

$$\delta H = \int_{x_1}^{x_2} \delta f(x) \left\{\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right)\right\} dx.$$

Since $\delta f(x)$ is arbitrary, δH will be zero to first order if and only if

$$\frac{\partial I}{\partial f} - \frac{d}{dx}\left(\frac{\partial I}{\partial f_x}\right) = 0. \qquad (3)$$

† This idea is only made really precise when a notion of *distance* in S is introduced, when the points of S are functions.

Thus the stationary points of the functional H are the functions satisfying the differential equation (3) which is known as Euler's equation.

As an application of the above result let us consider the case where

$$I(f(x), f_x(x), x) = (1+f_x^2(x))^{\frac{1}{2}}.$$

Then

$$H(f) = \int_{x_1}^{x_2} (1+f_x^2(x))^{\frac{1}{2}}\, dx = \int_{x_1}^{x_2} ds,$$

and H is just the length of arc from (x_1, z_1) to (x_2, z_2) determined by the graph of the function f passing through these points, ds being the usual symbol for an element of arc. Now

$$\frac{\partial I}{\partial f} = 0, \quad \frac{\partial I}{\partial f_x} = \frac{f_x(x)}{(1+f_x^2(x))^{\frac{1}{2}}}.$$

The stationary points of H are given by (3) which reduces to

$$\frac{d}{dx}\left\{\frac{f_x(x)}{(1+f_x^2(x))^{\frac{1}{2}}}\right\} = 0.$$

This equation implies that

$$\frac{f_x}{(1+f_x^2)^{\frac{1}{2}}} = \text{constant},$$

which in turn implies that f_x is a constant and so finally that

$$f(x) = ax+b,$$

where a and b are constants. Since $f \in S$ the constants a and b are determined by requiring the graph of f to pass through (x_1, z_1) and (x_2, z_2). The graph is of course a straight line.

4. Fermat's variational principle

We learn in geometrical optics that rays of light travel in straight lines *in vacuo*. If A and B are points on the path of a light ray, the path itself is given by

$$\delta \int_A^B ds = 0,$$

a result which follows from the investigation in Section 3b.

We could extend this idea to light rays in a general medium by postulating that the paths are given by the differential equations resulting from the requirement that

$$\delta \int_A^B \mu\, ds = 0,$$

where A and B, as before, are points on the path and μ is a function of position on the path of the ray. In the case where μ is a constant

$$\int_A^B \mu\, ds = \mu \int_A^B ds,$$

and so

$$\delta \int_A^B \mu \, ds = 0$$

reduces to

$$\delta \int_A^B ds = 0$$

and the paths are still straight lines.

If μ is not a constant the path will be a curve.

What happens at the boundary of two media where μ has a discontinuity? If ABC is the actual path, see Figure 2, one can show that

$$\delta \int_A^B \mu \, ds = 0$$

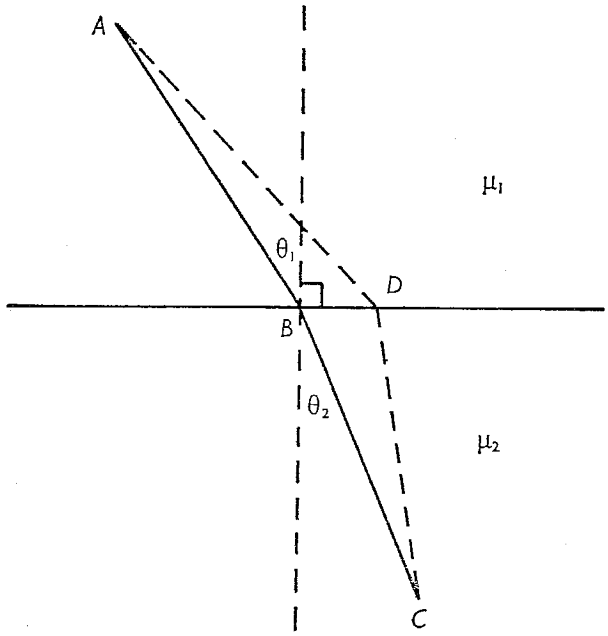

Figure 2

provided that

$$\mu_1 \sin \theta_1 = \mu_2 \sin \theta_2.$$

If we interpret μ as the refractive index of the medium, this is Snell's law of refraction. In fact we can *postulate* that the laws of geometrical optics are given by the principle

$$\delta \int_A^B \mu \, ds = 0,$$

where the symbols have their previously defined meanings. As well as the law of refraction all the other laws of geometrical optics are summarized by this single postulate which is known as Fermat's *variational principle* (the word *variational* is used to indicate that the paths are given by the *stationary* points of a functional).

5. A variational principle for classical mechanics

In order to express the ideas in this section as simply as possible let us consider a single particle of mass m whose position at time t is given by a single coordinate x (in other words the particle is moving in a straight line). Let the potential energy of the particle be represented by $V(x)$.

One of the great applied mathematicians who developed Newton's work was the Frenchman Joseph Louis Lagrange. A key function in the development of classical mechanics is named after him, the Lagrangian, it is denoted by L and for our case it is defined by

$$L(x, x_t, t) = T - V,$$

where T is the kinetic energy, in our case given by $\frac{1}{2}mx_t^2$. Lagrange's work was extended by Sir William Hamilton who showed that the equations of classical mechanics, based on Newton's laws, which determine the motion of a dynamical system once the initial conditions are prescribed, could be expressed in a very compact form. The result is summarized in the famous *variational principle* of Hamilton which states that the actual motion of a system is given by the stationary points of the *action functional*, namely

$$\int_{t_1}^{t_2} L\,dt.$$

In the case of the single particle which we are considering this statement becomes

$$\delta \int_{t_1}^{t_2} L\{x(t), x_t(t), t\}\,dt = 0;$$

see Figure 3. It follows from (3) that the function $x(t)$ which determines the motion

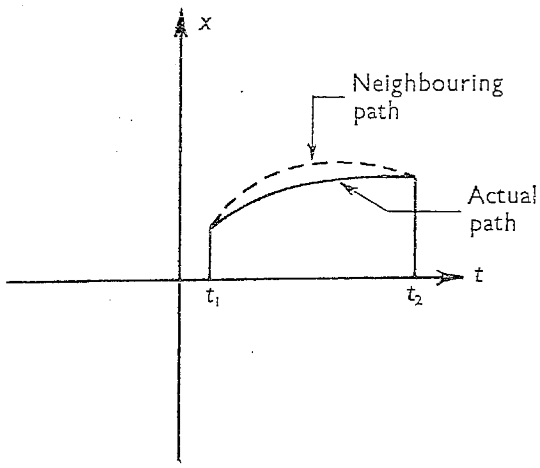

Figure 3

satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_t}\right) - \frac{\partial L}{\partial x} = 0, \tag{4}$$

an equation known as Lagrange's equation.

In cases where more than one coordinate is required to specify the motion there is a set of equations each similar to (4).

As an example let us take the case when

$$V(x) = \tfrac{1}{2}m\omega^2 x^2,$$

where ω is a constant. Then

$$L = \tfrac{1}{2}mx_t^2 - \tfrac{1}{2}m\omega^2 x^2,$$

$$\frac{\partial L}{\partial x_t} = mx_t,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_t}\right) = m\frac{d^2x}{dt^2},$$

$$\frac{\partial L}{\partial x} = -m\omega^2 x.$$

Equation (4) becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

which is the well-known equation describing a particle executing simple harmonic motion.

Some readers may not be familiar with the potential energy function V although they are probably familiar with a method of solution relying on a knowledge of the forces acting. However, in a physical investigation the formula giving the force is not known *a priori*, it must in the last analysis be found from experiment. In the Lagrangian formulation of dynamics one can consider that it is the function V which is determined by experimental means.

Finally it must be emphasized that Hamilton's principle has great generality and many complicated problems can be solved by applying the Lagrange equations which result from it.

Reference

R. W. Ditchburn, *Light* (Blackie, 1952), pp. 216–224.

Appendix

A more precise statement of the equation (1) is the following, which is true for the type of function in which we are interested,

$$f(x+\delta x) - f(x) = f_x(x+\theta\,\delta x)\,\delta x, \tag{5}$$

where $0 < \theta < 1$ and the subscript x denotes differentiation with respect to x. Then when the derivative is continuous we have

$$f(x+\delta x) - f(x) = f_x(x)\,\delta x$$

to first order in δx.

Now, for a function g of two variables, we have

$$\begin{aligned}\delta g &= g(x+\delta x, y+\delta y) - g(x,y)\\ &= \{g(x+\delta x, y+\delta y) - g(x+\delta x, y)\}\\ &\quad + \{g(x+\delta x, y) - g(x,y)\}.\end{aligned}$$

An application of (5) shows that

$$\delta g = g_y(x+\delta x, y+\theta_1\,\delta y)\,\delta y + g_x(x+\theta_2\,\delta x, y),$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. If we assume that the partial derivatives of g with respect to x and y are continuous we see that

$$\delta g = \frac{\partial g}{\partial x}\,\delta x + \frac{\partial g}{\partial y}\,\delta y$$

to a first order in δx and δy. This is the result quoted in the text.

How Many Extra Components in a System?

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1. Introduction

Most cars have a spare wheel. Traditionally cricket teams have a reserve player. We have only one heart but two kidneys and if necessary could carry on with one kidney. All these are examples of extra components in a system. These extra components are rather like the premium of an insurance policy. The more you pay, the more security you buy. The question is, how many extra components should a system have?

Three ways of arranging for the extra components come readily to mind. If a car needs a new clutch, it has usually to be repaired in a garage. In this case the needed clutch is carried by the garage and is switched in after some delay. In the meantime, the car is inoperative. This is an example of the extra components being available outside the system and in this case the system may have to cease operation for an appreciable time while the extra components are switched in. For obvious reasons a plane in flight carries a number of extra components which can be switched in when needed. In other cases some extra components are in operation all the time, for example two kidneys.

2. A problem of reliability

Consider a system which consists of K subsystems or components. The system needs at least one of each of the K components and will fail if any one (or more) of the components is unsatisfactory. The system is rather like a circular chain with K links. If any of the links is broken, the chain is broken.

The system is such that we cannot be absolutely certain that it will behave in a satisfactory manner. We shall be content if the system has a low probability of failure over some specified time. In practice, operation in a satisfactory manner usually means that under some specified conditions the system should not fail before some specified time. For example, a type of radio valve is defined to be satisfactory if under some specified conditions of use it lasts for at least 1000 hours.

Let q_i, $i = 1, 2, \ldots, K$, be the probability that the ith type of component is satisfactory. That is, it does not fail during the specified time of operation of the system. Let $p_i = 1 - q_i$ be the probability that it fails. Let Q be the probability that the system is satisfactory. Q depends on q_i and is often called the *reliability* of the system. A simple case is that of independence and in this case, the behaviour of any component is completely independent of the rest of the components and Q is simply equal to the product of all the q_i values. For example, it is reasonable to suppose that the tyres and the battery of a car are independent. If now during a given journey the probabilities of a satisfactory performance from the battery and the tyres are 0·95 and 0·90 respectively, then the application of the product rule shows that the probability that *both* battery and tyres will perform satisfactorily is 0·855. For our system, in the simple case of independence, we have

$$Q = q_1\, q_2\, q_3 \ldots q_{k-1}\, q_k. \tag{1}$$

From a knowledge of q_i and the relationship between Q and q_i we can see if Q is sufficiently large. If it is not, we have a problem.

Suppose we wish to increase the reliability Q of a system by operating the system with $n_i + 1$ components of the ith type. One general formulation of the problem is that we want to find the values n_i such that *both* the following conditions hold:

(a) the system obeys certain restrictions; for example restrictions on total cost, weight or the maximum number of extra components; and

(b) Q is maximized.

In many practical problems the solution to this problem has to be obtained by using a computer.

3. A simple illustration

As an illustration of increasing Q through a suitable choice of n_i, consider a simple system with 3 components. We shall suppose that, apart from a desire to keep the number of extra components small, there are no special restrictions and the problem is to achieve a reliability of at least 0·95 at a minimum cost. Suppose Q has the form shown in equation (1). The cost per component is c_i and this cost together with values of q_i is shown below.

Component i	Cost c_i	q_i
1	4	0·9
2	1	0·8
3	2	0·7

The minimum possible cost of the system is 7 and this gives $Q = 0{\cdot}504$. Let us denote this minimum cost solution by (1, 1, 1) showing that there is one component of each type. Now Q is a product and a natural simplification is to work with $\log Q$. If n_i extra components are used so that the ith type failure occurs with a probability

$p_i^{(n_i+1)}$, we have

$$\log Q = \log(1-p_1^{n_1+1)}) + \log(1-p_2^{n_2+1}) + \log(1-p_3^{n_3+1}).$$

We add one extra component to (1, 1, 1) such that the increase in $\log Q$ per unit cost is maximum. We continue in this fashion until either Q is sufficiently large, or we come up against some restriction. To see which will be the first extra component we compare:

$$\frac{1}{c_1}[\log(1-p_1^2) - \log(1-p_1)] = 0{\cdot}0104;$$

$$\frac{1}{c_2}[\log(1-p_2^2) - \log(1-p_2)] = 0{\cdot}0792;$$

$$\frac{1}{c_3}[\log(1-p_3^2) - \log(1-p_3)] = 0{\cdot}0570.$$

Thus the next solution is $(1, 2, 1)$ giving $Q = 0{\cdot}605$ at a cost of 8.

For the second extra component we compare:

$$\frac{1}{c_1}[\log(1-p_1^2) - \log(1-p_1)] = 0{\cdot}0104;$$

$$\frac{1}{c_2}[\log(1-p_2^3) - \log(1-p_2^2)] = 0{\cdot}0142;$$

$$\frac{1}{c_3}[\log(1-p_3^2) - \log(1-p_3)] = 0{\cdot}0570.$$

Thus the next solution is $(1, 2, 2)$ giving $Q = 0{\cdot}786$ at a cost of 10.

Can you find the solution such that $Q \geqslant 0{\cdot}95$?

Problems and Solutions

Readers who have not yet reached the age of 20 on 1 April 1969 are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college, or university.

Problems

7. Evaluate the integral

$$\int_0^{\pi} \log \sin \theta \, d\theta.$$

8. Let ABC be a triangle such that $AC = BC$ and $A\hat{C}B = 90°$. Let P be a point in the plane of ABC such that P and C lie on opposite sides of AB. Show that, if PC bisects $A\hat{P}B$ and if $AP \neq BP$, then $A\hat{P}B = 90°$.

9. Let C denote the class of positive integers which, when written in the scale of 3, do not require the digit 2. Show that no three integers in C are in arithmetic progression.

10. Let a, b, c, d be real numbers, and write

$$f(x, y) = axy + bx + cy + d.$$

Let R be a rectangle with its sides parallel to the coordinate axes. Show that, if $f(x, y) \geqslant 0$ at the vertices of R, then $f(x, y) \geqslant 0$ throughout R.

Does this inference remain valid if the requirement that the sides of R should be parallel to the coordinate axes is dropped?

11. A problem on minimum cost for a specified reliability is given at the end of the article on page 58. Find the solution such that the reliability $Q \geqslant 0{\cdot}95$.

Solutions to Problems in Volume 1, Number 1

1. Let G be the centroid of the acute-angled triangle ABC of circumradius R. Show that

$$AG^2 + BG^2 + CG^2 \geqslant 2R^2.$$

Solution. Let L denote the midpoint of BC. Then

$$\tfrac{9}{4}AG^2 = AL^2 = c^2 + \tfrac{1}{4}a^2 - ca \cos B = \tfrac{1}{4}(2b^2 + 2c^2 - a^2).$$

Hence, by symmetry,

$$AG^2 + BG^2 + CG^2 = \tfrac{1}{3}(a^2 + b^2 + c^2).$$

Assuming that $a = \max(a, b, c)$, we have

$$a^2+b^2+c^2 \geqslant 2a^2 = 8R^2 \sin^2 A$$

and so

$$AG^2+BG^2+CG^2 \geqslant \tfrac{8}{3}R^2 \sin^2 A.$$

But $\frac{1}{3}\pi \leqslant A \leqslant \frac{1}{2}\pi$ and so $\sin^2 A \geqslant \frac{3}{4}$. Hence the assertion follows.

Also solved by R. P. Allen (Grammar School for Boys, Cambridge), H. E. Clarke (Attleborough, Norfolk), T. T. N. Woo (Ipswich School), Lynne Woodward (Manchester High School for Girls). Two of our correspondents pointed out that a sharper result, with $\frac{8}{3}R^2$ in place of $2R^2$, is in fact valid.

2. Let $f(x)$ be a polynomial of degree n with real coefficients and such that $f(x) \geqslant 0$ for all real x. Show that

$$f(x)+f'(x)+f''(x)+\ldots+f^{(n)}(x) \geqslant 0$$

for all real x.

Solution by H. E. Clarke (Attleborough, Norfolk)

Write $f(x) = a_0 x^n+\ldots+a_n$. Since $f(x) \geqslant 0$ for all x, it follows that $a_0 > 0$ and that n is even. Write

$$F(x) = f(x)+f'(x)+\ldots+f^{(n)}(x).$$

Then $F(x) = a_0 x^n+\ldots$ and so the minimum value of $F(x)$ is finite. Let it be attained at x_0, so that $F'(x_0) = 0$. But $F'(x) = F(x)-f(x)$ and so $F(x_0) = f(x_0) \geqslant 0$. Hence $F(x) \geqslant F(x_0) = 0$ for all x.

Alternative solution. We have, for all x,

$$0 \leqslant \int_0^\infty e^{-u}f(x+u)\,du = \int_0^\infty \left(e^{-u}\sum_{k=0}^{n}\frac{u^k}{k!}f^{(k)}(x)\right)du$$

$$= \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x)\int_0^\infty e^{-u}u^k\,du = \sum_{k=0}^{n}f^{(k)}(x).$$

3. Let $z_1,\ldots, z_n$ be complex numbers in a sector of the complex plane with the origin as its vertex and of angle θ, where $0 \leqslant \theta < \pi$. Establish the inequality

$$\left|\sum_{k=1}^{n} z_k\right| \geqslant \cos\tfrac{1}{2}\theta \sum_{k=1}^{n} |z_k|.$$

Solution. Let the given sector be specified by the inequalities

$$\alpha-\tfrac{1}{2}\theta \leqslant \arg z \leqslant \alpha+\tfrac{1}{2}\theta.$$

Then

$$\cos(\arg z-\alpha) \geqslant \cos\tfrac{1}{2}\theta \quad (1 \leqslant k \leqslant n),$$

and so

$$\left|\sum_{k=1}^{n} z_k\right| = \left|e^{-i\alpha}\sum_{k=1}^{n} z_k\right| \geqslant \sum_{k=1}^{n}\mathscr{R}(e^{-i\alpha}z_k)$$

$$= \sum_{k=1}^{n}|z_k|\cos(\arg z_k-\alpha) \geqslant \cos\tfrac{1}{2}\theta\sum_{k=1}^{n}|z_k|.$$

Also solved by H. E. Clarke (Attleborough, Norfolk).

4. Show that the set of all positive integers can be partitioned into two subsets neither of which contains an infinite arithmetic progression.

Solution by H. E. Clarke (Attleborough, Norfolk)

We partition the positive integers into two sets, say A and B as follows: 1 is placed in A; 2 and 3 in B; 4, 5, 6 in A; 7, 8, 9, 10 in B; and so on. Now let d be an arbitrary positive integer. Then neither A nor B contains an infinite arithmetic progression with common difference d since each set has gaps of length greater than d.

Also solved by H. Blake (Rugby School).

5. Three boxes A, B, and C contain respectively 4 red balls; 1 white and 3 red balls; 2 white and 2 red balls. The boxes are shuffled around: one is put on your right hand, one in the centre, and the last on your left. You are now asked to pick one ball from whichever box you prefer: this turns out to be red. Which of the boxes A, B, or C is the ball likely to have come from?

Solution. The probability of picking balls, red or white, from box A is $\frac{1}{3}$; likewise for boxes B and C. The probabilities of picking a red ball (event R), given that the box A, B, or C from which it is drawn is known, are

$$\Pr\{R \mid A\} = 1, \quad \Pr\{R \mid B\} = \tfrac{3}{4}, \quad \Pr\{R \mid C\} = \tfrac{1}{2}.$$

Hence,

$$\Pr\{RA\} = \tfrac{1}{3}, \quad \Pr\{RB\} = \tfrac{1}{4}, \quad \Pr\{RC\} = \tfrac{1}{6}.$$

It follows from Bayes' theorem that the probability that the red ball drawn was from A is

$$\Pr\{A \mid R\} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}+\frac{1}{6}} = \tfrac{4}{9}.$$

Similarly we find

$$\Pr\{B \mid R\} = \tfrac{1}{3}, \quad \Pr\{C \mid R\} = \tfrac{2}{9}.$$

Also solved by M. G. Eastwood (Altrincham Grammar School for Boys, Altrincham, Cheshire), A. Mingay (Grammar School for Boys, Cambridge), and H. Blake (Rugby School).

6. The probability of a student's passing an examination is p, where $0 < p < 1$. It is assumed that his efforts in consecutive examinations are independent. Find, to 2 significant figures, the minimum value of p such that the probability of a student's passing in not more than 10 attempts is at least $\frac{1}{2}$.

Solution by A. Mingay (Grammar School for Boys, Cambridge).

The probability of a student's failing all 10 examination attempts is $(1-p)^{10}$. Thus the probability that the student will pass in 10 or fewer attempts is $1-(1-p)^{10}$. Since this is to be greater than 0·5, then

$$1-(1-p)^{10} > 0{\cdot}5, \quad \text{or} \quad (1-p)^{10} < 0{\cdot}5.$$

Taking logarithms to base 10,

$$10\log(1-p) < 0{\cdot}3010, \quad \text{or} \quad \log(1-p) < 0{\cdot}0301.$$

It follows that $1-p < 0{\cdot}9330$, whence $p > 0{\cdot}067$. The minimum p required is therefore 0·067.

Also solved by H. Blake (Rugby School).

Letters to the Editor

Dear Editor,

The Dainton† Committee's proposal of 'mathematics for all' has been criticized on the grounds that there are insufficient mathematics teachers. A programme has recently been suggested for the retraining of mathematics teachers;‡ while I agree that this would be desirable, I fail to see how it would solve the problem of providing *more* mathematics teaching.

It seems to me that the aim behind this Dainton proposal is that everyone should have a better understanding of mathematics than has previously been considered necessary; that it should no longer be fashionable for sympathetic arts graduates to say to children, 'I was never any good at maths either'; that we should dispel the myth that only those with a somewhat peculiar intellect can understand or enjoy mathematics; in fact that mathematics should become an integral part of our total culture. I will not write 'that numeracy be considered as important as literacy'; the word 'numeracy' implies that mathematics means only familiarity with number. Mathematics includes very much more.

If everyone should endeavour to understand mathematics better, then surely teachers should be among the first to make such an attempt. I mean *all* teachers—not only mathematics teachers.

Although I am primarily a scientist I confess I always liked mathematics at school. I particularly disliked history lessons, and I was bored by French. (This is not intended as a criticism of history or French, nor of their teachers: merely a statement of personal experience.)

One day, shortly after I had begun teaching mathematics, a child asked me: 'Why are there 360 degrees in a full revolution' rather than some other, perhaps more convenient or more logical number. I confessed I did not know. I asked the Head of Department, and apart from the clue that the ancient Babylonians used radix sixty in counting, he confessed that he did not know either. He suggested I refer to a history of mathematics in the library which I dutifully read. I have avidly read every book on the history of mathematics I could lay my hands on ever since, and can now claim to be interested in 'pure' history.

Could not my experience suggest a parallel course to history (and other) teachers? Would history teachers not become interested in mathematics through reading its history, and French teachers through reading Descartes and Laplace? Is it not possible that they could then teach mathematics to future specialists in their own subject, if they themselves felt a genuine enthusiasm for it. 'Yes, I like history best too, but mathematics is great fun and helps in understanding history' is more likely to be successful as an attitude than the specialist's 'I don't expect you to be as competent in mathematics as I am, but everyone's got to learn mathematics'.

I am not suggesting that all teachers are automatically capable of teaching mathematics unaided. But if they could be trained in mathematics, they could surely help.

There are obvious advantages in science, economics, sociology and geography teachers' teaching the applications of mathematics in their own particular subject.

† *Inquiry into the Flow of Candidates in Science and Technology into Higher Education*. Cmnd 3514, HMSO (1968). One of the Dainton Report's recommendations is a broader sixth-form curriculum, which would normally include the teaching of mathematics to all pupils until they leave school.

‡ D. J. Roaf, 'Crisis in Maths', *Times Educational Supplement*, p. 691 (4 October 1968).

Many of them would undoubtedly benefit from a better understanding of mathematics, if they could receive more mathematical instruction.

How are they all to be taught mathematics? Dr Roaf has suggested filling the 'empty places' in universities. What of those in the colleges of further education? Many have been compelled to discontinue courses because of the Department of Education and Science's policy of economy following the Pilkington Report.† Many of their staff may feel that their talents are not now being used to the full.

Could such staff not run evening courses in mathematics for local teachers? I believe many teachers would welcome these courses and prefer them to a year away from home. It would be a worth-while experiment, and there is little to lose if it should fail!

Yours sincerely,
VERA HARTLEY
(Burnley, Lancs)

* * *

Dear Editor,

With reference to Mr Richard Morton's letter in Volume 1 Number 1, The Association of Teachers of Mathematics (A.T.M.) has recently published a pamphlet called *Mathematical Films and Filmstrips*, which is available from their office at Vine Street Chambers, Nelson, Lancs (price 3/–).

This 1968 edition of the list of films and filmstrips available in England contains details of their content, duration, distributors, and hire charges.

It is possible that the film medium itself may suggest some form of mathematics which we have not examined in detail. Instead of using a film as a record of a classroom situation or a convenient way of showing animated diagrams, there may be a kind of mathematics that can best be expressed through the film medium.

For example, a dynamic demonstration of Pythagoras' theorem might give rise to a new approach to a proof—and thus to an examination of possible new axioms and the stretching of the concepts involved.

Demonstrations such as this have been used by G. Papy in his series *Mathématique Moderne*, and by the A.T.M. in *Notes on Mathematics in Primary Schools*.

The possibilities of animated diagrams and models in mathematics have yet to be fully explored.

Yours sincerely,
LEO F. ROGERS
(Digby Stuart College of the Sacred Heart, London).

† *Committee on Technical College Resources Report on the Size of Classes and Approval of Further Education Courses*. Department of Education and Science (1966). Some of the recommendations of this Report, such as minimum initial enrolments for courses and the avoidance of duplicate courses at neighbouring colleges, have resulted in a reduction of previously available courses.

Fundamentals of Probability Theory and Mathematical Statistics. By V. E. GMURMAN. Translated from the Russian by Scripta Technica Ltd. Iliffe Books Ltd, London, 1968. Pp. viii + 249. 50*s*.

At a time when the study of probability and statistics is becoming more common, both in schools and in undergraduate university courses, it is to be expected that elementary books on these subjects will be published with increasing frequency. The present book, originally written for engineering and economics students at technical institutes in the U.S.S.R., has been translated by Scripta Technica (edited by I. I. Berenblut) for the use of English-speaking students. The text, though somewhat stilted in parts, is perfectly intelligible and can be read by young mathematicians completing advanced school mathematics courses, or starting probability and statistics courses at college.

The book begins with an excellent introduction on probability theory; this is slightly marred by the final paragraph which gives a misleadingly inflated impression of the Soviet contribution to the subject. This is followed by Part I which consists of five chapters on random events. In these, the basic concepts and theorems of the calculus of probabilities are outlined. Numerous examples are given in the text, and each chapter concludes with problems to which the answers are provided.

Part II of the book consists of eight chapters on random variables, covering such topics as expectation and variance. The notations $M(X)$ and $D(X)$ for these differ from the usual ones $\mathscr{E}(X)$ and $\mathscr{V}(X)$ common in the literature in English on the subject. Next, Chebyshev's inequality and the law of large numbers are outlined. The main properties of distribution functions and probability density functions, including the normal density function, are discussed. There follows a long chapter on bivariate distributions including the bivariate normal.

Part III on the elements of mathematical statistics consists of four chapters on sampling, the estimation of parameters, and quality control. Confidence intervals for estimators and Student's distribution (without proof) are discussed. The joint estimation of multiple parameters of a distribution is outlined, and the estimation of the correlation coefficient illustrated. Finally some principles of quality control are given. Four tables, including the normal density function and integral, a table of Student's t, and a five-page index complete the book.

This is a book which could certainly be ordered for the school or college library. Some of its defects are possibly due to the difficulties of a translated text; the balance between probability and statistics is uneven, the latter topic being somewhat neglected. One of its main virtues is the very large number of worked examples and problems. The printing (from varitype) is reasonably clear, and there are few misprints, but the price of 50*s*. appears excessive for what the book has to offer.

University of Sheffield J. GANI

Finite Sets: Theory, Counting, and Applications. By LAURENCE P. MAHER JR. Merrill, Columbus, Ohio, 1968. Pp. xii + 110. 18*s*. 6*d*.

The author of this book tells us about Caspar Queeze, who has the misfortune to become involved in a duel with the infamous Baron Gunner von Piercecarcass. They agree to meet sometime between 5 a.m. and 6 a.m. Caspar Queeze hopes to satisfy honour and at the same time save his life by arriving between 5 a.m. and 5.15 a.m., waiting for *three* minutes, and then leaving quickly if the Baron has not yet arrived. Unknown to Caspar, the Baron is also reluctant to fight the duel, so he decides to arrive sometime between 5 a.m. and 5.15 a.m., wait for *two* minutes and then disappear if Caspar Queeze has not arrived. The question is: What is the probability that the two will meet? The author gives the answer.

This whetted my appetite to read on. Then I came to this statement: 'Every set has at least one member'. The author does not believe in the empty set. And I do. I meet it every day. It is the set of all elephants that have had a lift in my car. I am a keen 'empty-setter'. If you need convincing about the need for the empty set, read this book. You will see the complications that arise if the empty set is not around.

There are lots of illustrations and problems, although some will be unintelligible to a rather staid British reader like myself. There are problems about Snaggle toothpaste and 'rubber baby buggy bumpers'. And British readers may have to contact the American Embassy before answering the question: 'How many first-string teams can a baseball manager select from five pitchers, two catchers, seven infielders and five outfielders?'

Mathematics is a beautiful subject; it should give pleasure to those who study it. But your reviewer gained little pleasure from this book, except for the problem about Caspar Queeze: that was fun.

University of Sheffield D. W. SHARPE

Some Exercises in Pure Mathematics with Expository Comments. By J. D. WESTON and H. J. GODWIN. Cambridge University Press, 1968. Pp. vii+136. 13*s*.

During the last twenty years there have been many changes in the teaching of school mathematics; and even greater changes are proposed by the originators of a number of projects for the reform of mathematical curricula. This book is, in the first place, intended for those who follow a new A-level syllabus that has been adopted by several schools in Swansea. The syllabus is designed to eliminate the gap which exists between the traditional type of school mathematics and the material presented to most first-year university students specializing in mathematics. It includes some set theory, elementary number theory, the notion of a field, and a rigorous introduction to analysis (as opposed to the more usual intuitive treatment of the calculus). Accordingly, many of the exercises in this collection are on topics which are not currently taught in most schools. However, other exercises are suitable for and will thoroughly test the understanding of all sixth-form mathematicians. Here are some examples:

1. Prove that, if a, b, c are real numbers such that

$$ax^2+2bx+c \geqslant 0$$

for every real number x, then $b^2 \leqslant ac$.

2. Evaluate

$$\int_1^2 [x^2]\, dx,$$

where $[a]$ denotes the greatest integer less than or equal to the real number a.

3. Let the functions f and g be defined by the equations

$$f(x) = |x|, \quad g(x) = x\,|x|,$$

where x is any real number. Prove that f is not differentiable at 0 and that g is everywhere differentiable.

Exercises 1–50 are provided not only with solutions, but also with an ample and most illuminating commentary. For the next fifty exercises there are hints and more sparing comments; and for the remaining one hundred the reader is on his own. This last group includes the sixty questions set in the A-level examinations of 1966 and 1967.

Although most sixth formers will find a substantial portion of this book relatively unfamiliar, they will gain from it an excellent impression of what first-year university mathematics is about. If they then decide to embark on such a course, the book will prove an invaluable companion in the early stages of their university career.

University of Sheffield H. BURKILL

Notes on Contributors

David Kendall FRS is Professor of Mathematical Statistics in the University of Cambridge and a Fellow of Churchill College. Previously, he was a Fellow of Magdalen College, Oxford. His interests in probability and statistics have led him to investigate a wide range of practical problems, such as the spread of epidemics and rumours, the mechanism of bird navigation, and the mathematical problems arising in connection with archaeological and historical studies.

H. C. Rae is a lecturer in Applied Mathematics at Bedford College, London University. He has been a contributor to the College's annual Easter School for Sixth Form Mathematicians. His research interest is in the field of elementary particles.

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J. D. Weston was originally trained as an electrical engineer, but his interests soon shifted to mathematical analysis and he is now a professor of Pure Mathematics in the University of Wales, at Swansea. He is the author, together with H. J. Godwin, of *Some Exercises in Pure Mathematics* (reviewed elsewhere in this issue).

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