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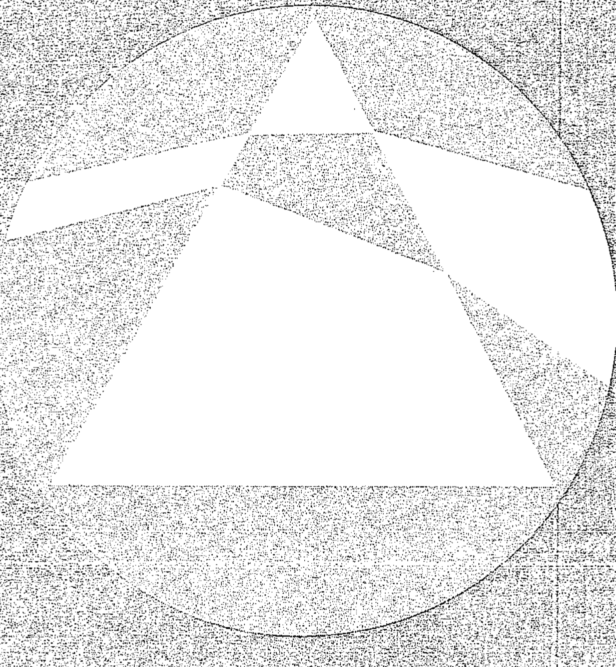

- Paul Erdős
- From Pascal to groups
- From building blocks to entropy

A magazine for students and teachers of mathematics
in schools, colleges and universities

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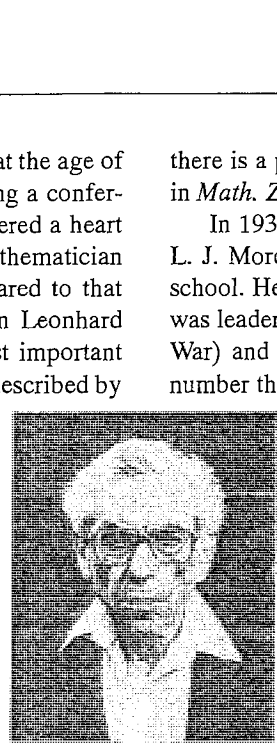
Paul Erdős, 1913–1996

ROGER COOK

Paul Erdős died on Friday, 20 September 1996, at the age of 83. Still mathematically active, he was attending a conference on combinatorics in Warsaw when he suffered a heart attack. He was probably the most prolific mathematician of the twentieth century, with an output compared to that of the eighteenth century Swiss mathematician Leonhard Euler. (Euler is generally accepted as the most important mathematician of the eighteenth century and is described by the *Oxford Concise Dictionary of Mathematics* as 'beyond comparison, the most prolific of famous mathematicians'.) Erdős published over 1500 articles on mathematics, mainly in number theory and combinatorics but touching on other areas such as probability, analysis, set theory and geometry. To comprehend the significance of this output you have to understand that most mathematicians cease publishing original work by the time they are 40, and a lifetime output of, say, fifty papers is regarded as a quite respectable contribution to the field.



His parents were mathematics teachers in Budapest. His birth, on 26 March 1913, was accompanied by tragedy. His two sisters, aged 3 and 5, died whilst his mother was in hospital, falling victims to a scarlet fever epidemic. After the outbreak of the First World War in 1914 his father was conscripted into the Austro–Hungarian army, captured by the Russians and was not released until November 1920. In these circumstances it is easy to understand that Paul's mother may have been overprotective of her son. He admitted that he had never 'learned' to boil a kettle of water, and did not butter his first slice of bread until he was 21. What he could do though, even at a very early age, was mathematics. By the age of 3 he could multiply two three-digit numbers in his head.

He went to the University in Budapest as a teenager and left after four years with a doctorate in mathematics. His thesis was concerned with Bertrand's postulate; for every integer $x > 1$, there is a prime number between x and $2x$. This was first proved in the mid-nineteenth century, following the work of Chebyshev on the distribution of prime numbers. For large integers it is an easy consequence of the Prime Number Theorem, proved at the end of the nineteenth century, which states that the number of prime numbers up to x is closely approximated by $x/\log x$. However Erdős came up with a new and simple proof which provided his first publication (*Acta Sci. Math. Szeged* 1932). His thesis, prepared in 1932 under the supervision of Fejer, extended the result to primes in certain arithmetic progressions; between x and $2x$ there is a prime congruent to 3 mod 4, say. This appeared in *Math. Zeitschrift* in 1935.

In 1934 he went to the University of Manchester where L. J. Mordell was building up an excellent number theory school. Here he met, among others, Harold Davenport (who was leader of British number theory after the Second World War) and L. K. Hua (who returned to China to build up number theory there). Erdős stayed in Manchester for four years and it was the last time he ever stayed so long in one place. By 1938 it was clear that it would be foolhardy for him, a Jew, to remain in Hungary. He had returned home for the summer but left hurriedly as the international situation deteriorated. It was to be the last time he saw his father. He left to take up a fellowship in the United States at the Institute of Advanced Studies in Princeton, New Jersey.

After a couple of years, however, he took up a singular lifestyle, holding no permanent position at any university, but adopting instead an itinerant regime, constantly moving on from conference to conference, from mathematics department to mathematics department. He had no permanent home, travelling around with two suitcases. One contained his clothes, the other his current mathematical research. He believed that private property was a nuisance and although in later life he owned an apartment in Budapest, it was usually available for visiting mathematicians, and friends took care of his finances for him.

It was not unknown for him to arrive in a town and knock on a friend's door with the greeting 'My brain is open'. This greeting, from a slightly built grey haired man, standing about 5 ft 6in tall, casually dressed and carrying two well-worn suitcases, may sound like a visitation by the house-guest from hell; but Erdős was always welcomed by colleagues, who were eager to house him and feed him so that they could work with him. He was obsessed by mathematics, working up to nineteen hours a day fuelled mainly by coffee and, in later life, by amphetamines and antidepressants. Erdős defined a mathematician as 'a machine for turning coffee into theorems'. After a couple of weeks Paul would move on to collaborate with another colleague, and we can begin to understand his phenomenal output; it represents a steady output of a paper roughly every two weeks throughout a working lifetime of sixty years.

This workstyle also generated a new concept, the Erdős number. No mathematician in history has been such a social mathematician. He wrote joint papers with over 450 different coauthors; they are given Erdős number 1. Anyone who has written a paper with one of these, but not with Erdős himself, is given Erdős number 2. Basically the Erdős number of a mathematician is the minimal number of collaborative joint papers to form a path to Erdős. Because much of his work has a probabilistic flavour it is not just mathematicians who have Erdős numbers, but also physicists, economists and biologists. The Erdős Number Project, maintained by Jerrold W. Grossman at the website `http://www.oakland.edu/~grossman`, keeps track of this extended web of collaborations. Over 4500 people, including Albert Einstein, have Erdős number 2.

In combinatorics a graph is a finite collection of points, called vertices, and lines joining certain of the points, called edges. For example, a system of roads (the edges) joining towns (the vertices) is a graph. Another example would be to take the vertices as mathematicians, and join two vertices when the mathematicians have written a joint paper. A mathematician's Erdős number is the number of edges you need to traverse in this graph to get from the mathematician to Erdős.

Paul Erdős was a problem-solving mathematician rather than a builder of abstract theories. He liked to provide short solutions, showing deep insight, to simply stated problems. Although mathematics started out by solving problems, theory-building has become the more dominant style in recent years. Perhaps the most prestigious prizes in mathematics are the Fields Medals, often described as the Nobel prizes of mathematics. These are usually awarded to theory-builders rather than problem-solvers. Nevertheless, most recipients of Fields Medals in recent years have an Erdős number, and usually a fairly low one. (It's an interesting point whether Andrew Wiles's proof of Fermat's Last Theorem — see *Mathematical Spectrum* Volume 26 Number 3 (1993/4) pages 65–73 — should be regarded as problem-solving or theory-building. Perhaps it shows that the two styles are not too widely separated; there is little point in building a theory if there are no worthwhile problems that it solves.)

In a group of six people at a party, either there are three who know both of the others in their subset or three none of whom knows either of the other two in their subset. This simple problem is the basis for Ramsey Theory, named after Frank Plumpton Ramsey — brother of a former Archbishop of Canterbury. Unfortunately Frank Ramsey died in 1930 at the age of 27, so never developed the concept. The idea was taken up by Erdős and Szekeres (another Hungarian mathematician) in a 1935 paper which became a major influence on the development of this subject, so that Erdős is regarded as the father of modern Ramsey theory.

The *chromatic number* of a graph is the smallest number of colours required to colour its vertices, one colour to each vertex, so that two vertices which are joined by an edge are assigned different colours. On a map we can replace each country by a vertex, and join two vertices by an edge when the corresponding countries have a common border. Thus the chromatic number corresponds to the number of colours we need to colour the map in the usual way.

In number theory many of the papers Erdős wrote have a probabilistic flavour and have heavily influenced the development of probabilistic number theory, and the same is true of his work in graph theory. For example, in a classic paper he showed that there exist graphs with large chromatic number and large girth (i.e. the shortest cycle in the graph is large) by counting the graphs on n vertices and showing that most of them have the required properties. Results like this have been studied closely because of their applications to communications networks.

In the late 1940s Erdős and a Norwegian mathematician, Atle Selberg, were both working on the problem of finding an elementary proof of the Prime Number Theorem. ('Elementary' here has a technical meaning; the proof should not use complex analysis.) They could combine their results to provide a proof and were going to publish back-to-back articles in a journal, each explaining how the contributions fitted together. However Selberg went ahead and published his results separately. In 1948 Selberg won the Fields Medal for his work. Erdős rated his own contribution as one of the three most important pieces of mathematics that he produced. The other two were showing that additive functions (i.e. functions for which $f(mn) = f(m) + f(n)$) are essentially logarithmic, and that the number of divisors of a positive integer has a certain probability distribution.

Although he never received a Fields Medal, Erdős did receive the prestigious Wolf prize, worth $50,000. He had little use for the money himself, and donated $30,000 to endow a postdoctoral fellowship. Throughout his life he offered cash prizes for solutions to mathematical problems he posed, with values ranging from $10 to $25,000 — see his letter in *Mathematical Spectrum* Volume 27 Number 2 (1994/5) pages 43–44. The largest prize that was ever claimed was for $1000, when Szemeredi showed that reasonably dense subsequences of the positive integers must contain arbitrarily long arithmetic progressions. Erdős gave away most of his money as prizes or to deserving causes. He was a fellow of the Royal Society, and a fellow of the national academies of countries on three different continents.

I am grateful to Aleksandar Ivic for his comments on an earlier draft of this article. □

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From Pascal to Groups

GUIDO LASTERS and DAVID SHARPE

A geometrical problem

Consider the following problem. *Given a circle C and a straight line ℓ not intersecting the circle, construct a quadrilateral inscribed in C whose opposite sides meet on ℓ.* This is illustrated in figure 1. A few attempts should soon convince you that this is easier said than done. By 'construct' we understand this to be in the classical sense, by means of a straight edge and compass (and pencil!) alone.

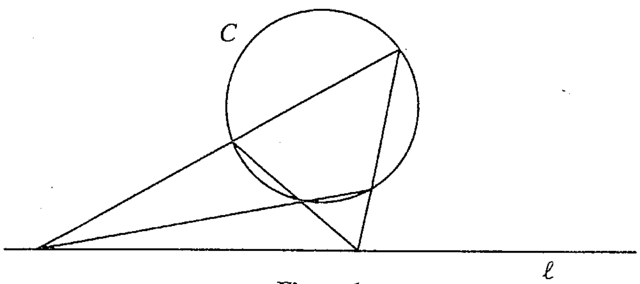

Figure 1

We first describe a construction that works; it is illustrated in figure 2. Choose a point E on the circle and construct the tangent to the circle to meet ℓ at X_1 (say). (If this tangent is parallel to ℓ then X_1 is taken as a point at infinity, but in any case E can be chosen so that the tangent is not parallel to ℓ.) That such a construction is possible using only a straight edge and a compass is shown in the appendix. Construct the other point A_2 on the circle such that $X_1A_2 = X_1E$; this is the point at which the other tangent to the circle through X_1 touches the circle.

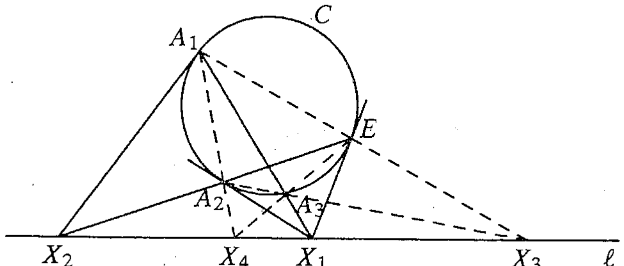

Figure 2

Let the line EA_2 produced meet ℓ at X_2. Construct one of the tangents from X_2 to the circle, meeting the circle at A_1. (See the appendix.) Let the line A_1X_1 cut the circle again at A_3. We claim that the quadrilateral $EA_1A_2A_3$ has the required properties, so that A_1E and A_2A_3 produced meet on ℓ, as do A_1A_2 and A_3E produced. These lines are shown as dashed lines in figure 2. We need to show that X_3 and X_4 in the diagram lie on ℓ.

An accurately-drawn diagram may convince us that this is so, but of course it does not constitute a proof, for which we turn to a theorem of Pascal which applies to any conic. A conic is any curve which is obtained by intersecting a circular cone by a plane (figure 3).

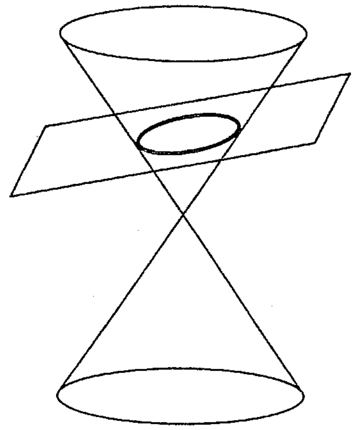

Figure 3

For example, ellipses, hyperbolas and parabolas can be obtained in this way. A circle is a special case of an ellipse, obtained when the cross-section is taken at right angles to the axis of the cone. Let A_1, A_2, A_3 and B_1, B_2, B_3 be six points on a conic (see figure 4). Denote by C_1 the point of intersection of A_2B_3 and A_3B_2, by C_2 the point of intersection of A_3B_1 and A_1B_3 and by C_3 the point of intersection of A_1B_2 and A_2B_1. Pascal's Theorem is that the points C_1, C_2 and C_3 are collinear. (See references [1] or [2], for example.) It is not required that all the points $A_1, A_2, A_3, B_1, B_2, B_3$ are distinct. For example, if $A_1 = B_2$ then the chord A_1B_2 becomes the tangent to the conic at this point. Also, parallel lines are taken to meet at infinity.

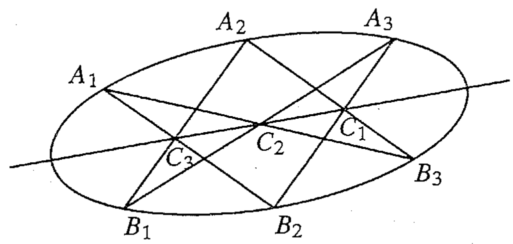

Figure 4

We return now to figure 2. Consider the points A_1, E, A_3 and A_2, A_1, E of the circle. By Pascal's Theorem, the points X_1, X_3, X_2 are collinear, so X_3 lies on ℓ. Consider also the points A_1, A_3, A_2 and E, A_2, A_1. Pascal's Theorem now shows that the points X_1, X_2, X_4 are collinear, so X_4 lies on ℓ. This justifies our construction.

The construction of a group

Readers may have come across the notion of an abstract group. This is a set G on which is defined a binary operation, so that, given elements a, b of G, there is defined an element of G, usually written as ab and called the *product* of a and b. Three simple axioms have to be satisfied.

(1) The binary operation is associative, i.e. $a(bc) = (ab)c$ for all a, b, c in G.

(2) There is a *neutral element* e in G satisfying the property that $ea = ae = a$ for all a in G.

(3) Every element a of G has an *inverse* element a^{-1} satisfying the condition $a^{-1}a = aa^{-1} = e$.

If one further property is satisfied, namely

(4) the binary operation is commutative, i.e. $ab = ba$ for all a, b in G,

then the group is said to be an *Abelian group* (after the Norwegian mathematician Niels Abel).

A beautiful example of an Abelian group arises if, as in the previous section, we start with a conic C and a straight line ℓ not meeting C. The elements of the group are the points of the conic. Choose any point of the conic to be the neutral element e. The binary operation is shown in figure 5.

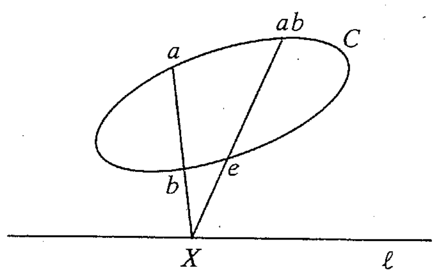

Figure 5

Given points a and b of C, construct the straight line through a and b to meet ℓ in X (say). The point where the straight line through X and e meets the conic again is the point ab. If $a = b$, then ab is taken to be the tangent to C at a, and if Xe is tangential to C then $ab = e$. Also, parallel lines are taken to meet at infinity as in the previous section. Finally, note that $ab = cd$ if and only if the line joining a, b and the line joining c, d intersect on ℓ.

We need to verify the axioms for an Abelian group. Axiom 4, that $ab = ba$ for all a, b is clear, and axioms 2 and 3 are easily seen; they are illustrated in figures 6 and 7.

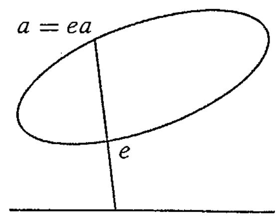

Figure 6

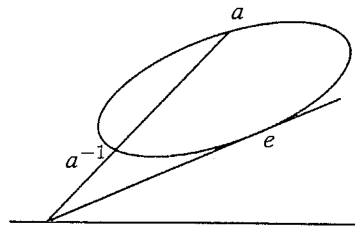

Figure 7

The problem axiom is axiom 1. This is surprising because, for most groups that occur in mathematics, the associative law seems to be self-evident.

Take points a, b, c of C. Figure 8 shows the points ab and bc. Consider the triples of points a, c, e and ab, bc, b of C. Now the straight lines through a, b and through e, ab respectively meet on ℓ, as do those through c, b and through e, bc. It follows from Pascal's Theorem that the straight lines through a, bc and through ab, c also meet on ℓ. This means that $a(bc) = (ab)c$. Thus we do have an Abelian group.

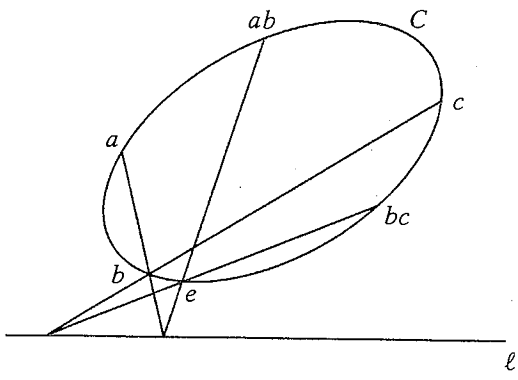

Figure 8

We now return to our original problem, namely, given a circle C and a straight line ℓ not passing through C, to construct, using ruler and compass only, a quadrilateral inscribed in C with opposite sides meeting on ℓ. We shall examine the construction of this quadrilateral given in the previous section in the light of our group.

Choose a point e on C; this is to be the neutral element of the group. Construct the tangent at e to meet ℓ in X_1, and construct the other point b on the circle so that $X_1e = X_1b$; this is the point at which the other tangent to the circle through X_1 touches the circle. Let the line eb meet ℓ at X_2 and construct a tangent from X_2 touching C at a (say). Then, in our group, $b = a^2$. Draw the line through a

and $b(= a^2)$ to meet ℓ at X_4. Then the line joining X_4 to e meets C at $ab = a^3$. Since $b^2 = e$, $a^4 = e$. We thus have the cyclic group $\{e, a, a^2, a^3\}$. Now the remaining two sides of the quadrilateral e, a, a^2, a^3 join a, e and a^2, a^3 respectively. Since $a^2 \times a^3 = a^5 = a = ae$, this means that these sides meet on ℓ.

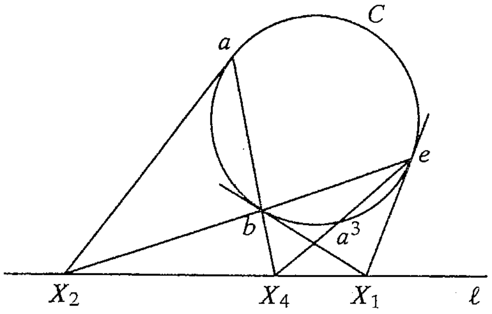

Figure 9

Thus the proof of our construction can be expressed in group-theoretic language. However, the associative law of the group is intricately bound up with the classical theorem of Pascal. The moral seems to be that abstract algebra does not render classical geometry obsolete but merely masks its use!

Appendix

We describe in this appendix two 'ruler and compass' constructions that we have used.

1. *The construction of the tangent to a circle at a point E of the circle.*

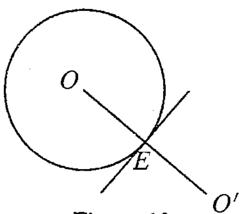

Figure 10

Draw the straight line through E and centre O, and mark the point O' on OE produced so that $OE = EO'$. Bisect OO'. The perpendicular bisector of OO' is the tangent to the circle at E. See figure 10. (If the centre of the circle is not known, it can be located by drawing two non-parallel chords and finding the point where their perpendicular bisectors meet. This is the centre of the circle.)

2. *The construction of a tangent to a circle through a given point X outside the circle.*

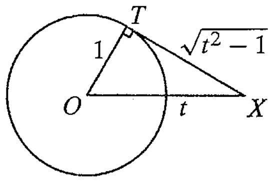

Figure 11

We denote the centre of the circle by O, let its radius be 1 unit and put $OX = t$, so that $t > 1$. If T denotes a point where a tangent to the circle through X touches the circle, then $XT = \sqrt{t^2 - 1}$ (figure 11), and we can mark T using compasses provided that we can construct the distance $\sqrt{t^2 - 1}$. Figure 12 shows how to construct t^2 since, by similar triangles, $x/t = t/1$, so $x = t^2$. We can now construct $t^2 - 1$.

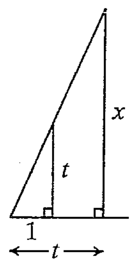

Figure 12

Finally, figure 13 shows how to constuct $\sqrt{r}$ from r. Mark a length $r + 1$ along the x-axis as shown and construct the semi-circle on this diameter. Then the distance OY shown is $\sqrt{r}$.

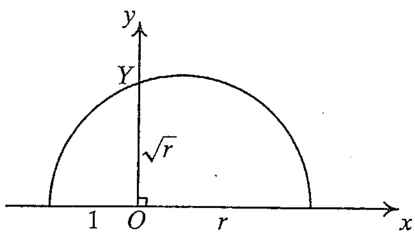

Figure 13

References

1. C. W. O'Hara and D. R. Ward, *Projective Geometry* (OUP, Oxford 1937) page 107 — a purely goemetrical approach.
2. D. M. Y. Somerville, *Analytical Conics* (Bell, London, 1956) page 235 – a more algebraic approach. □

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Find a million consecutive natural numbers none of which is prime.

How many palindromic primes are there with an even number of digits?

Optimal Angle of Projection from a Height

ALI VAHDATI

In a desperate attempt to improve his performance in the shot-put event, the author finally turned to the art of mathematics for inspiration. The result? Well, he solved the mathematical problem — but his shot-put still has a long way to go!

Being a keen discus and shot-put thrower, I have always been interested to find the optimum angle of projection for a person of my height. When I was a sixth-form student myself I had come across a lengthy analytical method of solving this problem, which I will describe later. So I thought that it would be a good idea to set this problem to my lower-sixth students. With a little guidance they all managed to do the following.

Suppose the projectile is projected with velocity v and angle θ from the horizontal at a height h above the ground. From figure 1, at time t after the projection,

$$x = vt\cos\theta,$$
$$y = vt\sin\theta - \tfrac{1}{2}gt^2,$$

and when the projectile hits the ground,

$$-h = \frac{xv\sin\theta}{v\cos\theta} - \frac{gx^2}{2v^2\cos^2\theta},$$

i.e.

$$x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} + h = 0,$$

or

$$xv^2\sin 2\theta - gx^2 + 2hv^2\cos^2\theta = 0. \qquad (1)$$

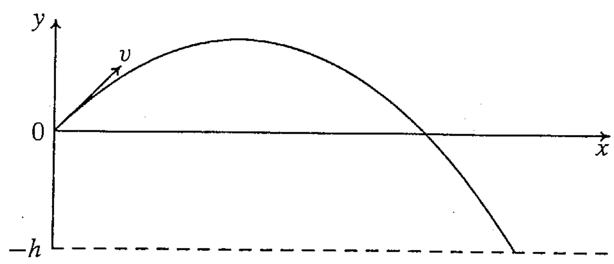

Figure 1

The class did not like the idea of maximising x by using calculus, so I introduced them to a spreadsheet in order to give them some help in obtaining numerical solutions for particular heights and velocities. Their approach was to fix the velocity and the height and then use (1) to find the distance travelled horizontally for a range of angles and see which angle gave the greatest distance. The spreadsheet that they used is shown at the end of this article.

Now to get back to analytical methods of solving this problem. There are three ways of doing this.

Method 1. Using calculus

Keeping v and h fixed and varying θ we differentiate (1) with respect to θ:

$$v^2\sin 2\theta\frac{\mathrm{d}x}{\mathrm{d}\theta} + 2v^2x\cos 2\theta - 2gx\frac{\mathrm{d}x}{\mathrm{d}\theta} - 2hv^2\sin 2\theta = 0. \qquad (2)$$

For a maximum we need $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$. This gives

$$2v^2x\cos 2\theta - 2hv^2\sin 2\theta = 0,$$

so that $x = h\tan 2\theta$. Substituting this in (1) we have

$$gh\tan^2 2\theta = v^2\left(\frac{\sin^2 2\theta}{\cos 2\theta} + 1 + \cos 2\theta\right)$$

which simplifies to

$$gh = \frac{v^2\cos 2\theta}{1 - \cos 2\theta},$$

and we can use this to find an expression for $\cos 2\theta$, namely

$$\cos 2\theta = \frac{gh}{v^2 + gh}.$$

If we put $v^2/g = R$, we obtain

$$\cos 2\theta = \frac{h}{R + h}. \qquad (3)$$

Now we have to show that (3) really gives a maximum. Differentiating (2) with respect to θ we have, when $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$,

$$(v^2\sin 2\theta - 2gx)\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2} = 4hv^2\cos 2\theta + 4xv^2\sin 2\theta. \qquad (4)$$

But, when $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$, $x = h\tan 2\theta$ and $\cos 2\theta = h/(R+h)$. Hence the right-hand side of (4) is positive and the coefficient of $\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2}$ is

$$\begin{aligned} v^2\sin 2\theta - 2gh\tan 2\theta &= \sin 2\theta\left(v^2 - \frac{2gh}{\cos 2\theta}\right) \\ &= \sin 2\theta[v^2 - 2g(R+h)] \\ &= \sin 2\theta\left[v^2 - 2g\left(\frac{v^2}{g} + h\right)\right] \\ &= \sin 2\theta(-v^2 - 2gh) \\ &< 0. \end{aligned}$$

Hence $\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2} < 0$.

Method 2. Limiting parabola

The limiting parabola marks the boundary of all the possible positions the projectile can reach with the same speed using different angles of projection. First we need to show that the envelope of the parabolas is indeed a parabola.

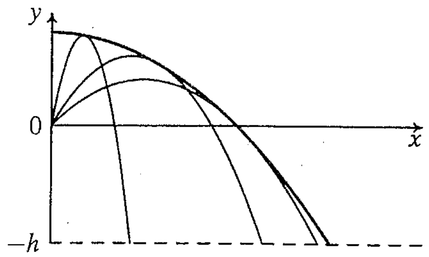

Figure 2

If the point (x, y) can be reached by projection from the origin at angle θ with the horizontal and with velocity v, then, as before,

$$x = vt\cos\theta, \qquad y = vt\sin\theta - \tfrac{1}{2}gt^2.$$

Hence

$$\frac{gx^2}{2v^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2v^2} + y = 0. \tag{5}$$

This quadratic in $\tan\theta$ has distinct real roots, equal (real) roots or non-real roots according as

$$x^2 - \frac{2gx^2}{v^2}\left(\frac{gx^2}{2v^2} + y\right) \gtreqless 0,$$

or

$$\frac{2g}{v^2}\left(\frac{gx^2}{2v^2} + y\right) \lesseqgtr 1.$$

Thus the boundary between the points that can be reached and those that cannot is the curve

$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2},$$

which is a parabola. The furthest point $(x, -h)$ that can be reached from the origin therefore satisfies the equation

$$-h = \frac{v^2}{2g} - \frac{gx^2}{2v^2},$$

so that

$$\begin{aligned} x^2 &= \frac{2v^2}{g}\left(\frac{v^2}{2g} + h\right) \\ &= R^2 + 2Rh, \end{aligned}$$

where $R = v^2/g$, as in method 1. Moreover, the angle θ corresponding to this point is then obtained from (5) when the equation has a double root in $\tan\theta$, and the double root is

$$\tan\theta = \frac{v^2}{gx},$$

i.e.

$$\tan\theta = \sqrt{\frac{R}{R+2h}}. \tag{6}$$

It is not difficult to show that (3) and (6) are equivalent.

Method 3. Projection down an inclined plane

To solve the problem by this method we first need to find an expression for the range down an inclined plane.

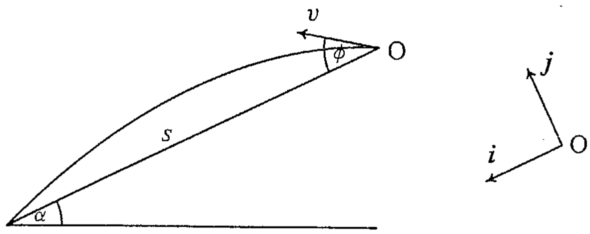

Figure 3

Suppose that the projectile is projected with velocity v and angle of projection ϕ from an inclined plane that is at an angle α to the horizontal. From figure 3, at time t after projection the position vector $\boldsymbol{r}$ of the projectile is given by

$$\boldsymbol{r} = \begin{pmatrix} vt\cos\phi + \frac{1}{2}gt^2\sin\alpha \\ vt\sin\phi - \frac{1}{2}gt^2\cos\alpha \end{pmatrix}.$$

When the projectile hits the plane again

$$vt\sin\phi - \tfrac{1}{2}gt^2\cos\alpha = 0,$$

i.e.

$$t = \frac{2v\sin\phi}{g\cos\alpha}.$$

For this value of t, the displacement down the inclined plane is

$$s = \frac{2v^2\sin\phi\cos\phi}{g\cos\alpha} + \frac{g}{2}\left(\frac{4v^2\sin^2\phi}{g^2\cos^2\alpha}\right)\sin\alpha,$$

which simplifies to

$$\begin{aligned} s &= \frac{2v^2\sin\phi\cos(\phi-\alpha)}{g\cos^2\alpha} \\ &= \frac{v^2}{g\cos^2\alpha}\big(\sin(2\phi-\alpha) + \sin\alpha\big). \end{aligned}$$

Now, for a fixed α, s is maximal when $\sin(2\phi - \alpha) = 1$, i.e.

$$\phi = 45 + \tfrac{1}{2}\alpha; \tag{7}$$

and s_α, the maximum range down the plane, is given by

$$\begin{aligned} s_\alpha &= \frac{v^2(1+\sin\alpha)}{g\cos^2\alpha} \\ &= \frac{v^2}{g(1-\sin\alpha)}. \end{aligned} \tag{8}$$

Note that, for $0 < \alpha < 90$, s_α increases as α increases.

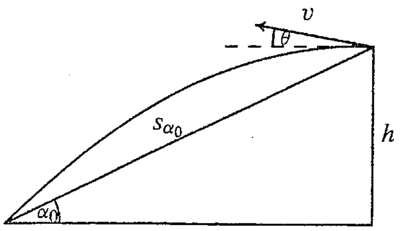

Figure 4

Now (see figure 4) define α_0 by the equation

$$\sin\alpha_0 = \frac{h}{s_{\alpha_0}}. \tag{9}$$

This is possible since, by (8), equation (9) is equivalent to

$$\frac{v^2\sin\alpha_0}{g(1-\sin\alpha_0)} = h,$$

i.e. to

$$\sin\alpha_0 = \frac{h}{(v^2/g)+h},$$

or to

$$\sin\alpha_0 = \frac{h}{R+h},$$

if, as in methods 1 and 2 above, $R = v^2/g$.

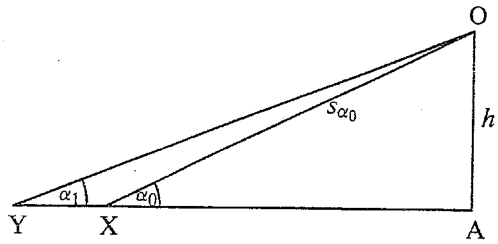

Figure 5

We wish to show (see figure 5) that $\mathrm{AX} = s_{\alpha_0}\cos\alpha_0$ is the maximum horizontal range of the projectile when it is propelled from O, a height h above the ground. Suppose, then, that the projectile can reach the point Y on AX produced, so that $\mathrm{OY} > s_{\alpha_0}$. However, if $\angle\mathrm{OYA}$ is α_1, then $\alpha_1 < \alpha_0$ and, since s_α increases with α,

$$\mathrm{OY} \leqslant s_{\alpha_1} < s_{\alpha_0},$$

so there is a contradiction. Hence AX is indeed the maximum horizontal range.

Finally, taking as previously θ to be the angle of projection with the horizontal, we have $\theta = \phi - \alpha$. Hence, by equation (7), for maximum horizontal range, $\theta = 45 - \frac{1}{2}\alpha_0$ and so

$$\cos 2\theta = \sin\alpha_0 = \frac{h}{R+h},$$

which agrees with equation (3).

The spreadsheet

The spreadsheet reproduced in figure 6 is set up to solve the quadratic equation (1) in x. Here h and v are fixed and x is found for θ ranging from 29° to 50°. I have used the quadratic formula with

$$a = -g, \qquad b = v^2\sin 2\theta, \qquad c = 2hv^2\cos^2\theta.$$

Also notice that, since $a < 0$ and $b, c > 0$, there is just one positive root which is given by the negative sign in the formula. The fixed references to A2, B2 and C2 make the spreadsheet easily adaptable to solve the same problem for different velocities and different heights.

gravity	speed of projection	height projected from	
-9.8	15	2	
angle in degrees	b	c	distance thrown
+A6+1	`($B$2)^2*@SIN(2*A6*@PI/180)`	`2*$C$2*(@COS(A6*@PI/180))^2*($B$2)^2`	`(-B6-@SQRT((B6*B6-(4*$A$2)*D6)))/(2*$A$2)`
50.00	221.58	371.86	24.18
49.00	222.81	387.37	24.36
48.00	223.77	402.96	24.51
47.00	224.45	418.61	24.64
46.00	224.86	434.30	24.74
45.00	225.00	450.00	24.81
44.00	224.86	465.70	24.86
43.00	224.45	481.39	24.88
42.00	223.77	497.04	24.87
41.00	222.81	512.63	24.84
40.00	221.58	528.14	24.78
39.00	220.08	543.56	24.70
38.00	218.32	558.86	24.60
37.00	216.28	574.04	24.46
36.00	213.99	589.06	24.31
35.00	211.43	603.91	24.13
34.00	208.62	618.57	23.93
33.00	205.55	633.03	23.70
32.00	202.23	647.27	23.45
31.00	198.66	661.26	23.18
30.00	194.86	675.00	22.89
29.00	190.81	688.46	22.58

Figure 6

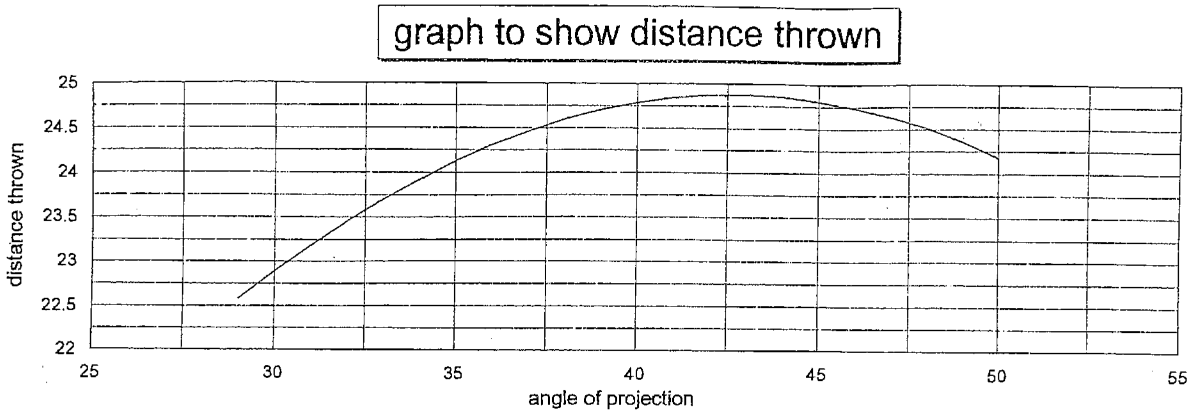

Figure 7

We can check whether our analytical method gives the same result as the numerical one. If we put $h = 2$ and $v = 15$ in (3) we get $\theta = 42.7°$, which agrees with figure 7.

The obvious question is: how does this relate to the real-life situation? Although the theory suggests that the best angle of projection for a world-class thrower (who is about 2 metres tall and throws the shot about 21 metres) is about 42°, experimental data shows otherwise. Through analysis of film clips, it is found that most world-class shot-putters project the shot at angles between 38°–40°. This can be explained by one particular oversimplification of our model. We assumed that the shot-putter can attain the same maximum speed at all angles of projection, but in fact he can generate more speed at lower angles of projection. Hence a better angle is sacrificed to superior speed. □

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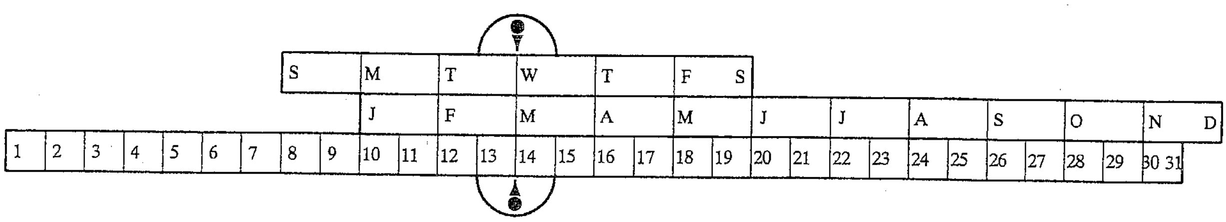
Braintwister

2: Moment of madness

I have an unusual calendar. Three uniform rulers (of the same thickness, depth and material) slide through a frame as shown:



The 12 cm ruler has marks for Sunday–Saturday at 2 cm intervals, the 22 cm ruler has marks for January–December at 2 cm intervals, and the 30 cm ruler has marks for 1–31 at 1 cm intervals. For example, the picture shows how the calendar should look on Wednesday, March 14.

The calendar should be anchored to the wall by two central screws in the frame, as shown. In a moment of madness I only used the upper screw and the calendar can swing freely about that point. As a consequence the calendar is rarely horizontal. In fact, it will only hang horizontally on one day in 1997.

Which day?

(The solution will be published next time.)

VICTOR BRYANT

Laplace Transforms and Differential Equations

RUSSELL EULER

In reference [1], Laplace transforms were used to solve a non-homogeneous, linear differential equation with constant coefficients. The non-homogeneous terms were differentiable functions. The purpose of this paper is to use Laplace transforms to solve differential equations such as $y' + ay = f(t)$ or $y'' + ay' + by = f(t)$, where $f(t)$ is a *piecewise* continuous function. If f has a jump discontinuity at $t = u$ and standard methods are used to solve the differential equation, the equation has to be solved separately on the intervals $(-\infty, u]$ and $[u, \infty)$. This can be tedious, whereas the method employed in this article yields the solution without consideration of these intervals separately.

The *unit step function* is denoted by $U(t)$ and defined by

$$U(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqslant 0. \end{cases}$$

As a result,

$$U(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geqslant a. \end{cases}$$

The function $U(t-a)$ has a jump discontinuity at $t = a$. These step functions can be used to express a piecewise continuous function in terms of a single formula.

If

$$f(t) = \begin{cases} l(t) & \text{for } t < a, \\ r(t) & \text{for } t \geqslant a, \end{cases}$$

then it is easy to see that $f(t) = l(t) + U(t-a)[r(t) - l(t)]$. If $t < a$, then $U(t-a) = 0$ and so $f(t) = l(t)$, while if $t \geqslant a$, then $U(t-a) = 1$ and so

$$f(t) = l(t) + [r(t) - l(t)] = r(t).$$

This can easily be extended to functions such as

$$f(t) = \begin{cases} p(t) & \text{if } a \leqslant t < b, \\ q(t) & \text{if } b \leqslant t < c, \\ r(t) & \text{if } c \leqslant t < d. \end{cases}$$

Now

$$f(t) = p(t) + U(t-b)[q(t) - p(t)] + U(t-c)[r(t) - q(t)].$$

The following result is useful when solving differential equations of the type mentioned at the beginning of this article.

If $\mathcal{L}[f(t)]$ exists for $s > c$ and if $a > 0$, then

$$\mathcal{L}[f(t-a)U(t-a)] = e^{-as}\mathcal{L}[f(t)]$$

for $s > c$. The proof of this is straightforward:

$$\begin{aligned} \mathcal{L}[f(t-a)U(t-a)] &= \int_0^\infty e^{-st} f(t-a)U(t-a)\mathrm{d}t \\ &= \int_a^\infty e^{-st} f(t-a)\mathrm{d}t \\ &= \int_0^\infty e^{-s(u+a)} f(u)\mathrm{d}u \\ &= e^{-sa}\mathcal{L}[f(t)]. \end{aligned}$$

This can also be stated as

$$\mathcal{L}^{-1}\big[e^{-sa}\mathcal{L}[f(t)]\big] = f(t-a)U(t-a) = f(t)U(t)|_{t\to t-a} \tag{1}$$

for $a > 0$.

Before solving a specific initial value problem we consider the formula $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$ when f' has a jump discontinuity at the point $a \geqslant 0$. So suppose that f has a left and a right derivative at a and f' is continuous on $[0, a]$ and on $[a, \infty)$, where $f'(a)$ is interpreted as the left derivative in $[0, a]$ and as the right derivative in $[a, \infty)$. Assuming also that $\lim_{t\to\infty}[e^{-st} f(t)] = 0$ for $s > c$, we have by integration by parts in $[0, a]$ and in $[a, \infty)$,

$$\begin{aligned} \mathcal{L}[f'(t)] &= \int_0^\infty e^{-st} f'(t)\mathrm{d}t \\ &= \int_0^a e^{-st} f'(t)\mathrm{d}t + \lim_{R\to\infty}\int_a^R e^{-st} f'(t)\mathrm{d}t \\ &= [e^{-st} f(t)]|_0^a + s\int_0^a e^{-st} f(t)\mathrm{d}t \\ &\quad + \lim_{R\to\infty}\left([e^{-st} f(t)]|_a^R + s\int_a^R e^{-st} f(t)\mathrm{d}t\right) \\ &= e^{-sa} f(a) - f(0) + s\int_0^a e^{-st} f(t)\mathrm{d}t - e^{-sa} f(a) \\ &\quad + s\int_a^\infty e^{-st} f(t)\mathrm{d}t \\ &= s\int_0^\infty e^{-st} f(t)\mathrm{d}t - f(0) \\ &= s\mathcal{L}[f(t)] - f(0). \end{aligned}$$

As an example, we solve the initial value problem

$$y' + y = \begin{cases} t & \text{if } 0 \leqslant t < 4, \\ 1 & \text{if } 4 \leqslant t, \end{cases}$$
$$y(0) = 1.$$

This equation can be written as

$$\begin{aligned} y' + y &= t + (1-t)U(t-4) \\ &= t - (t-4)U(t-4) - 3U(t-4). \end{aligned}$$

Taking the Laplace transform of both sides of this equation we have

$$\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[t] - \mathcal{L}[(t-4)U(t-4)] - 3\mathcal{L}[U(t-4)].$$

Therefore

$$s\mathcal{L}[y] - 1 + \mathcal{L}[y] = \frac{1}{s^2} - \mathrm{e}^{-4s}\mathcal{L}[t] - 3\mathrm{e}^{-4s}\mathcal{L}[1]$$

and so

$$(s+1)\mathcal{L}[y] = 1 + \frac{1}{s^2} - \mathrm{e}^{-4s}\left(\frac{1}{s^2} + \frac{3}{s}\right).$$

Hence

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{s^2(s+1)} - \mathrm{e}^{-4s}\left(\frac{3s+1}{s^2(s+1)}\right)$$

and, by partial fractions,

$$\mathcal{L}[y] = \frac{2}{s+1} - \frac{1}{s} + \frac{1}{s^2} - \mathrm{e}^{-4s}\left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right).$$

So, by (1),

$$\begin{aligned} y &= 2\mathrm{e}^{-t} - 1 - t - \mathcal{L}^{-1}\left[\mathrm{e}^{-4s}\left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right)\right] \\ &= 2\mathrm{e}^{-t} - 1 - t - \left[U(t)\mathcal{L}^{-1}\left(\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s+1}\right)\right]\Bigg|_{t\to t-4} \\ &= 2\mathrm{e}^{-t} - 1 - t - [U(t)(2 + t - 2\mathrm{e}^{-t})]|_{t\to t-4} \\ &= 2\mathrm{e}^{-t} - 1 - t - U(t-4)(t - 2 - 2\mathrm{e}^{-t+4}). \end{aligned}$$

The next differential equation could occur in electrical engineering. Solve the initial value problem

$$y'' + 5y' + 6y = \begin{cases} \sin 2t & \text{if } 0 \leqslant t \leqslant \frac{1}{2}\pi, \\ 0 & \text{if } t \geqslant \frac{1}{2}\pi, \end{cases}$$
$$y(0) = y'(0) = 0.$$

This equation can be written as

$$\begin{aligned} y'' + 5y' + 6y &= \sin 2t - U(t - \tfrac{1}{2}\pi)\sin 2t \\ &= \sin 2t + U(t - \tfrac{1}{2}\pi)\sin[2(t - \tfrac{1}{2}\pi)]. \end{aligned}$$

Therefore

$$s^2\mathcal{L}[y] + 5s\mathcal{L}[y] + 6\mathcal{L}[y] = \frac{2}{s^2+4} + \mathrm{e}^{-\pi s/2}\mathcal{L}[\sin 2t],$$

or

$$(s+2)(s+3)\mathcal{L}[y] = \frac{2}{s^2+4} + \frac{2\mathrm{e}^{-\pi s/2}}{s^2+4}.$$

Hence

$$\mathcal{L}[y] = \frac{2}{(s+2)(s+3)(s^2+4)} + \frac{2\mathrm{e}^{-\pi s/2}}{(s+2)(s+3)(s^2+4)}$$

and, by use of partial fractions,

$$\begin{aligned} \mathcal{L}[y] = {} & \frac{\frac{1}{4}}{s+2} - \frac{\frac{2}{13}}{s+3} + \frac{-\frac{5}{52}s + \frac{1}{26}}{s^2+4} \\ & + \mathrm{e}^{-\pi s/2}\left(\frac{\frac{1}{4}}{s+2} - \frac{\frac{2}{13}}{s+3} + \frac{-\frac{5}{52}s + \frac{1}{26}}{s^2+4}\right). \end{aligned}$$

Taking inverse Laplace transforms we now obtain

$$\begin{aligned} y = {} & \tfrac{1}{4}\mathrm{e}^{-2t} - \tfrac{2}{13}\mathrm{e}^{-3t} - \tfrac{5}{52}\cos 2t + \tfrac{1}{52}\sin 2t \\ & + \left[U(t)\left(\tfrac{1}{4}\mathrm{e}^{-2t} - \tfrac{2}{13}\mathrm{e}^{-3t} - \tfrac{5}{52}\cos 2t \right.\right. \\ & \left.\left. + \tfrac{1}{52}\sin 2t\right)\right]\Bigg|_{t\to t-\frac{1}{2}\pi} \end{aligned}$$

and so

$$\begin{aligned} y = {} & \tfrac{1}{4}\mathrm{e}^{-2t} - \tfrac{2}{13}\mathrm{e}^{-3t} - \tfrac{5}{52}\cos 2t + \tfrac{1}{52}\sin 2t \\ & + U(t - \tfrac{1}{2}\pi)\left(\tfrac{1}{4}\mathrm{e}^{-2t+\pi} - \tfrac{2}{13}\mathrm{e}^{-3t+\frac{3}{2}\pi} \right. \\ & \left. + \tfrac{5}{52}\cos 2t - \tfrac{1}{52}\sin 2t\right). \end{aligned}$$

As these examples illustrate, Laplace transforms can provide a technique for solving some differential equations in which the non-homogeneous terms are merely piecewise continuous functions.

Reference

1. B. H. Denton, Laplace transforms in theory and practice, *Mathematical Spectrum*, Volume **27**, Number 1, (1994/95) pp. 8–10. □

***Russell Euler** is a professor in the Department of Mathematics and Statistics at Northwest Missouri State University. He is an avid problem solver/proposer, and has been successful in passing his enthusiasm on to some of his students.*

From Building Blocks to Entropy

MERVYN HUGHES

Beginning with some fairly simple mathematics, namely the number of ways of arranging children's coloured building blocks, we then use the result to find the number of ways that atoms can share energy (i.e. entropy).

I was recently watching my children playing with a set of coloured building blocks when I came across the following problem. It appeared to be the same problem that occurred in an A-level physics class, although it is a mathematical one. The standard colours for the building blocks are blue, green, red and yellow. We can place one on top of another to form a tower. Three examples are given in figure 1.

B	B	B
R	R	G
G	G	R
Y	G	G
G	Y	Y

Figure 1

My son said that these arrangements were the same but clearly they are not. What he meant to say was that each tower had one blue, one red, one yellow and two greens. We decided to call these *similar* towers. He had an ample supply of each colour and I could easily see that the total number of different towers is simply 4^5. If instead of a tower of 5 blocks we have n blocks, the total number is 4^n. So, in general, if we have m colours and n blocks, the total number is m^n.

My son continued playing with 5 blocks and varied the number of greens, blues, reds and yellows. He wanted to know how many *non-similar* arrangements of 5 blocks there were if we can have at most 4 colours. The answer did not seem easy. We decided to start with something simple.

Problem 1. Given 4 colours, how many non-similar arrangements are there?

By considering just 2 colours and 5 blocks and using a systematic approach, we listed all possible towers in figure 2.

B	B	B	B	B	G
B	B	B	B	G	G
B	B	B	G	G	G
B	B	G	G	G	G
B	G	G	G	G	G

Figure 2

The answer is simply 6. In fact with 2 colours and n blocks, it is not hard to see that the answer is $n+1$. When a third colour is introduced, the problem is trickier. A systematic counting approach, using just one colour, then two colours and then finally listing all towers with three colours, gives the answer as 21. When a fourth colour is introduced, the problem is even harder. The answer is 56. Surely there must be an easier way. Can we find an easier way and can we generalise to n blocks?

I decided to leave my young son playing with his building blocks and returned to the blackboard.

Problem 2. Given 4 colours, how many non-similar arrangements of n blocks are there?

Let $A_{m,n}$ be the answer for m colours and n blocks. A simple counting argument gives $A_{4,2} = 10$, $A_{4,3} = 20$ and $A_{4,4} = 35$.

Main problem. Given m colours, how many non-similar arrangements of n blocks are there?

Upon investigation, using a systematic counting approach and building upon previous results, we found that

$$A_{m,n} = \sum_{i=0}^{m-1} \binom{n-3+i}{i} \binom{m-i+l}{2},$$

which is a sum of products of binomial coefficients. This can be evaluated to give a simple binomial coefficient, bearing out the results above. Since the answer seems fairly simple, we seek a simpler proof.

Alternative proof. Let us return to the original problem with the 5 building blocks of 4 colours. We can have as many blues, greens, reds and yellows as we like, so long as the total number is 5:

X X X X X where X can be B, G, R or Y.

This time the blocks are laid in a row. If we insert vertical bars to separate the colours we have: X X | X || X X. This stands for 2 Blues, 1 Green, 0 Reds and 2 Yellows. Remember that the positions of the colours do not matter, only the number. Thus the answer to our problem is equal to the total number of ways of arranging the 5 X's and the 3 vertical bars, i.e. ${}^8C_3 = 56$.

For the main problem above, we will need $m-1$ vertical bars to separate the m colours, giving the very simple answer $A_{m,n} = {}^{m+n-1}C_{m-1}$. This is, in fact, a well-known result from occupancy theory.

Application in entropy

Readers in physics or chemistry may have met the following problem in the topic of entropy. The electrons in any atom are associated with particular energy levels. These energy levels may be represented by a series of horizontal lines (states). There are only certain permitted energy states since the energy is quantized; it cannot vary continuously but can only take discrete values. For a fixed total energy (units of energy are called 'quanta') shared between more than one atom, there will be several different ways of sharing the energy. The 'entropy' of the system can be defined as $S = k \ln g$, where k is the Boltzmann constant and g is the number of ways the atoms can share the total energy.

Example 1. There are three different ways that 2 atoms can share 2 quanta.

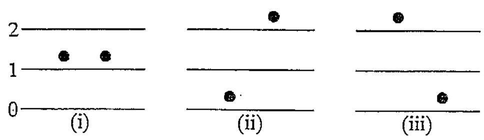


Figure 3

Example 2. There are four different ways that 2 atoms can share 3 quanta.

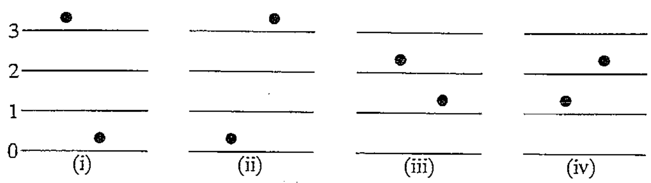


Figure 4

Given a number of atoms (say m) and a total energy of, say, n quanta, how many different ways are there of sharing the total energy among all the m atoms? For small values of m and n it is easy to count, as above. The number of ways of sharing a total energy of 4 among two particles is 5. For larger numbers, even as small as $m = 5$, $n = 5$, the number of ways is too many to write down, but the various ways can be generated with a simple computer program.

However, the problem is the same as arranging the building blocks, as we shall see. Think of each atom as having a different colour; thus the number of atoms becomes the number of different colours. The total energy, n, can then be considered to be the number of blocks in a tower. For example 1 above, we have two atoms, so let them be coloured blue and green. Then, in figure 3, (i) represents 1 blue and 1 green, (ii) represents 0 blues and 2 greens, (iii) represents 2 blues and 0 greens.

Example 3. In the case $m = 4$, $n = 5$, figure 5 represents blue with 3 quanta, green with 1 quantum, red with 1 quantum and yellow with 0 quanta.

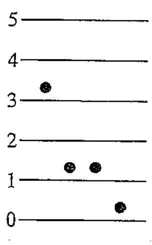


Figure 5

Thus, returning to the entropy problem: how many ways can we arrange m atoms with total energy n quanta? The answer is ${}^{m+n-1}C_{m-1}$. For various values of m and n we can calculate the number of ways. This is shown in table 1.

No. of atoms	Total energy	No. of ways
2	4	5
3	3	10
10	10	92378
100	100	8×10^{59}

Table 1

When energy is added to a substance, the number of ways of arranging the quanta among the atoms increases. If $W = W(m, n)$ is the number of ways in which n quanta can be distributed among m atoms, then the addition of one more quantum increases it to $W' = W(m, n+1)$, and

$$\frac{W'}{W} = \frac{{}^{m+n}C_{m-1}}{{}^{m+n-1}C_{m-1}} = \frac{(m+n)!/(n+1)!(m-1)!}{(m+n-1)!/(m-1)!n!} = \frac{m+n}{n+1}.$$

For a given number of atoms, m, and quanta, n, we can calculate the probability $P(q)$ of any particular atom having any particular number of quanta, q say. Clearly this is zero for $q > n$ and can be calculated for other values:

$$P(q) = \frac{W(m-1, n-q)}{W(m, n)} = \frac{(m-1)n!(m+n-q-2)!}{(m+n-1)!(n-q)!}.$$

The expected number of atoms having a particular number of quanta is displayed in the histogram (see figure 6).

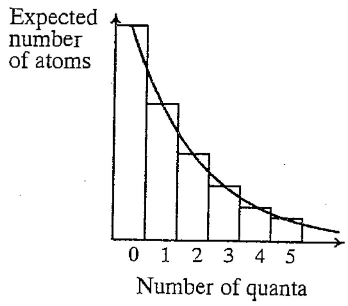


Figure 6

When m and n are large, we need a computer to calculate the values. For $q \ll m, n$ the distribution approximates to the exponential curve $P(q) = P(0)\exp(-aq)$ (the Boltzmann distribution), where $a = \ln((m+n)/n) \cong \ln(W'/W)$ and is inversely proportional to the 'temperature' — the lower the value, the higher the temperature.

Further reading

1. T. Duncan, *Advanced Physics*, (Murray, London, 1975).
2. P. Whelan and M. Hodgson, *Essential Principles of Physics*, (Murray, London, 1971).
3. *Advanced Nuffield Physics*, (Longman, London, 1971). □

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Remainderless Division of Positive Integers

JOHN MACNEILL

This is not about division involving rounding or truncation.

What is 14 divided by 5?

Fourteen floral table decorations have been prepared for the guests' tables at a wedding. There are five long tables for the guests. Consider the number of floral decorations per table.

Rational-number division gives $14/5 = 2.8$ floral decorations per table. We reject this answer since the decorations are artistic unities, to be kept whole.

Integer division gives $14/5 = 2$, with remainder 4. However we would not wish to be wasteful and leave four floral decorations unused.

We can have a whole number of decorations on each table with no decorations left over only by relaxing the condition that every table must have the same number of decorations. Instead, the condition would be that the allocation of decorations to tables must be, in some sense, as even as possible. If the tables are parallel to each other, somewhat like | | | | |, then the natural allocation of decorations to tables is $(3, 3, 2, 3, 3)$.

Using a double slash to represent this type of division gives

$$14//5 = (3, 3, 2, 3, 3).$$

$r//s$ may be thought of as the allocation of r indistinguishable objects to s pigeon-holes in a row so that the distribution of objects is as even as possible; in essence, a partition of the positive integer r into s integral parts which are as nearly equal as possible and where the order of the parts matters in making the distribution as even as possible.

Naturally $24//8 = (3, 3, 3, 3, 3, 3, 3, 3)$, which we may abbreviate to 33333333; this style of abbreviation is used throughout this article, although a problem would arise if a number of objects explicitly more than nine is assigned to a particular pigeon-hole. It also would seem clear that $26//8 = 33433433$, where the fours are in positions three and six to make the distribution as even as possible.

However, it is far from obvious what 'as even as possible' means in general. For instance there are six credible candidates for the value of $5//10$:

0101010101	1010101010
0101011010	0101101010
1010010101	1010100101.

Both 0101010101 and 1010101010 pleasingly avoid having consecutive zeros or consecutive ones, whereas each of 0101011010, 0101101010, 1010010101 and 1010100101 has a 'centre of mass' that is more centrally placed. The appropriate value for $5//10$ must depend on the application.

So there are six possible definitions of $r//s$, exemplified by the six candidate values of $5//10$. Algorithms 1 and 2 (see below) in their various forms cover all six cases.

Let $r//s = (c[1], c[2], ..., c[s])$.

Algorithm 1: to find $r//s$. (Exemplified by $5//10 = 1010101010$.)

```
01 set j to 0;
02 while j<s
03  begin
04  add 1 to j;
05  set c[j] to 0;
06  end;
07 set i to s;
08 set j to 1;
09 while i<2*r*s
10  begin
11  while i>2*j*r
12   begin
13   add 1 to j
14   end;
15  add 1 to c[j];
16  add 2*s to i;
17  end;
```

As is usual in such an algorithm, an asterisk is used as a multiplication sign. Algorithm 1 may be understood as follows. Think of a line segment of length $2rs$ divided into s equal line segments ('pigeon-holes') each of length $2r$ and also divided into r equal line segments (each with an 'object' at its midpoint) each of length $2s$. The variable i is used to step through the objects (see lines 07, 09 and 16) and the variable j is used to step through the pigeon-holes (see lines 08, 11 and 13), with the assignment of an object to a pigeon-hole occurring in line 15; lines 01 to 08 of algorithm 1 give initial values to $c[j]$, i and j. Algorithm 1 has a leftward bias which gives

$$26//8 = 34333433,$$

not 33433433.

What happens in algorithm 1 if an object coincides with a boundary of a pigeon-hole? The object may be assigned (as above) to the pigeon-hole of which the boundary is the upper boundary. Alternatively, the object may be assigned to the pigeon-hole of which the boundary is the lower boundary, which is achieved by modifying algorithm 1, replacing $>$ in line 11 with $\geqslant$; using this rightward-biased form of algorithm 1,

$$5//10 = 0101010101$$

and

$$26//8 = 33433343.$$

Algorithm 2: to find $r//s$. (Exemplified by $5//10 = 0101101010$.)

```
01 function E(r,s);
02  begin
03  if r>s then set E(r,s) to E(r-s,s) + 111...1;
04  if r=s then set E(r,s) to 111...1;
05  if r<s then
06   begin
07   if r=1 and s=2 then set E(r,s) to 01
08   else
09    begin
10    if s<2*r then set E(r,s) to h(E(r,s-r+1))
11    else set E(r,s) to g(E(s-r,r+1))
12    end;
13   end;
14  end:
```

Algorithm 2 defines $E(r, s)$ [$= r//s$] recursively and in terms of the functions g and h (lines 11 and 10) which are defined as follows. The function g maps $a_1 a_2 a_3 \ldots a_k$ to a_1 zeros followed by a one followed by a_2 zeros followed by a one followed by a_3 zeros followed by a one ... followed by a_k zeros; for example, $g(232) = 001000100$. The function h maps $a_1 a_2 a_3 \ldots a_k$ to a_1 ones followed by a zero followed by a_2 ones followed by a zero followed by a_3 ones followed by a zero ... followed by a_k ones; for example, $h(232) = 110111011$.

Here are some explanatory notes for algorithm 2.

(a) If there are more objects than pigeon-holes, then start by putting one object into each pigeon-hole (line 03).

(b) If the number of objects equals the number of pigeon-holes, then simply put one object in each pigeon-hole (line 04).

(c) If there are two pigeon-holes and one object, then put the object in the second pigeon-hole (line 07).

(d) If, for instance, there are 17 objects and 22 pigeon-holes, then five pigeon-holes will end up empty. These five empty pigeon-holes may be thought of as separating the 17 non-empty pigeon-holes into six groups. So the problem reduces to finding how to allocate 17 objects to six pigeon-holes (line 10).

(e) If, for instance, there are five objects and 22 pigeon-holes, then five pigeon-holes will end up non-empty. These five non-empty pigeon-holes may be thought of as separating the 17 empty pigeon-holes into six groups. So the problem reduces to finding how to allocate 17 objects to six pigeon-holes (line 11).

The question of whether algorithm 2 really does define a value for $E(r, s)$ for all positive integers r and s is left to the reader.

There is arbitrariness in algorithm 2 when the number of pigeon-holes is exactly double the number of objects. For one object and two pigeon-holes, the object is put in the second pigeon-hole (line 07); it could equally well be put in the first. For other cases where the number of pigeon-holes is exactly double the number of objects, the condition $s < 2r$ in line 10 causes the allocation of objects to pigeon-holes to be made in line 11 using the function g; the condition in line 10 could equally well be $s \leqslant 2r$, causing the allocation of objects to pigeon-holes to be made in line 10 using the function h. So there are four forms of algorithm 2 (01 or 10 in line 07; $<$ or $\leqslant$ in line 10) corresponding to the values 0101011010, 0101101010, 1010010101 and 1010100101 for $E(5, 10)$. Each form of algorithm 2 gives $26//8 = 33433433$.

The material in this article was motivated by an investigation into linear interpolation in integers, undertaken in the hope that the outcome would be of use in computer graphics programming. □

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Mathematics in the Classroom

Olympic statistics

In July 1996, one hundred years after the first modern Olympic Games had been held in Athens, the Games took place in Atlanta, Georgia and provided us with a wealth of statistics. Prior to their start, there was much speculation as to whether Linford Christie, the Olympic 100 m champion, would be able to successfully defend his title. Caught up in this excitement, some of my students decided to look at the times taken by the winners of some of the Men's 100 m over the hundred years and try to predict the time for the 1996 winner. The data that they started with (published in *The Guardian*, 26 July 1996) are given in table 1.

We presumed that greater accuracy in time-measuring equipment was responsible for the move from 1 to 2 decimal place accuracy in 1960, but perhaps it was brought about by the need to separate the times of the runners even further.

Table 1. Men's 100 m Olympic champions.

Year	Winner	Country	Time (s)
1896	Francis Lane	US	12.2
1924	Harold Abrahams	GB	10.6
1936	Jesse Owens	US	10.3
1948	Harrison Dillard	US	10.3
1960	Armin Hary	Germany	10.32
1968	James Hines	US	9.95
1972	Valeriy Borzov	USSR	10.14
1980	Alan Wells	GB	10.25
1984	Carl Lewis	US	9.95
1992	Linford Christie	GB	9.96

A scatter plot of time (x) against year (y) for these data (see figure 1) suggests that a linear relationship might prove satisfactory over the time period given, and could even be extrapolated for an estimate for 1996. The product moment correlation coefficient, r, between x and y was calculated as -0.85, giving r^2, a measure of tightness of the fit of a regression line through the data, as 0.72.

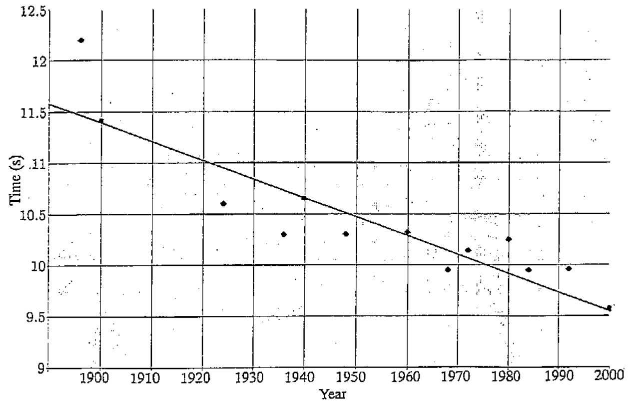

Figure 1. Times of the winners of the Men's 100 m at the Olympic Games. Points for regression line: ($x = 1900, y = 11.45$), ($x = 1940, y = 10.70$), ($x = 2000, y = 9.57$).

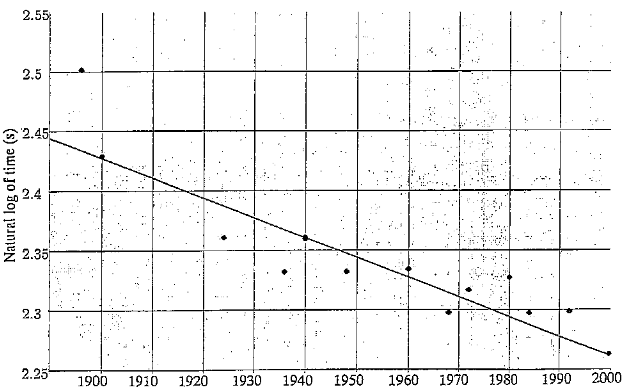

Figure 2. Times of the winners of the Men's 100 m: $\ln y$ against x. Points for regression line: ($x = 1900, y = 2.436$), ($x = 1940, y = 2.367$), ($x = 2000, y = 2.264$).

This value of r seemed encouraging. After all, we were not expecting a perfect relationship between x and y as we have ignored many other influential variables. For example, the Games take place in varying locations at different altitudes and in different weather conditions, all of which could influence the time of the winner. The time of 10.25 s in 1980 seems unexpectedly slow, but that year there were absences from the Moscow Games as a protest against the Soviet invasion of Afghanistan, and this could have resulted in a changed field of runners. We have not accounted for such factors as these. So this model seemed reasonable. Hence a regression line equation was obtained ($y = 47.20 - 0.02x$) which yielded a prediction of 9.64 s for 1996.

But the students, who had been learning about fitting curves to data, wondered if an exponential curve might provide a better fit. After all, the human body has its limitations and reductions in time cannot go on indefinitely; at some point there must be a levelling out of these winning times. So their second attempt was to fit a curve of the type $y = ae^{bx}$ which led to a plot of $\ln y$ against x to test the relationship $\ln y = \ln a + bx$ (see figure 2).

We were surprised to find that this second scatter looked much the same as the plot of the raw data. What could we discern from this? Was it simply the result of scaling? Fitting a regression line to these data gave $\ln a = 5.70$ (the intercept) and $b = -1.72 \times 10^{-3}$ (the gradient). The correlation coefficient between $\ln y$ and x was -0.86, and $r^2 = 0.74$, hardly changed from the linear model. Use of this exponential curve gave an estimate of 9.69s for 1996.

Still not satisfied that this was the best they could do, my students then set about seeing if a curve of the type $y = ax^n$ could provide a more convincing fit to the data. This led to a plot of $\ln y$ against $\ln x$ testing the relationship $\ln y = \ln a + n \ln x$ (see figure 3). Again, a very similar looking scatterplot and again, $r = -0.86$. Another regression line gave $\ln a = 27.78$ (the intercept) and $n = -3.36$ (the gradient). So assuming a relationship of the form $y = ax^n$ yielded an estimate for 1996 of 9.69 s, exactly the same as the estimate from the exponential relationship.

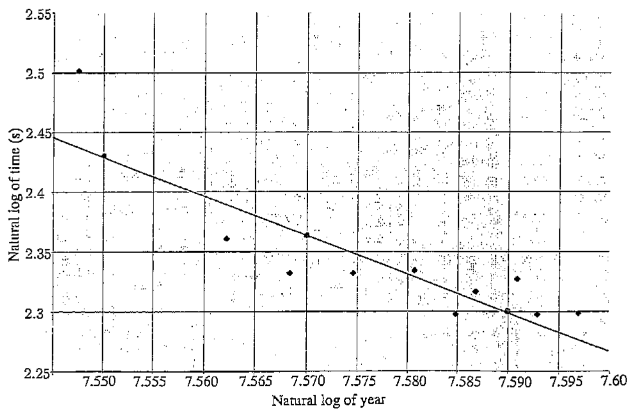

Figure 3. Times of the winners of the Men's 100 m: $\ln y$ against $\ln x$. Points for regression line: ($x = 7.55, y = 2.436$), ($x = 7.57, y = 2.369$), ($x = 7.59, y = 2.301$).

All these estimates seemed very low — could they be achievable? Records would need to be broken! Even Ben Johnson's time in 1988 (when he was dismissed days later after failure of a drugs test) was as high as 9.79 s. We knew that Atlanta's scorching heat suited sprinters (although it was the dread of the long distance runners), but there was nothing for it but to wait for the big race and see what happened.

Well, the first disappointment was the failure of Linford Christie to reach the final. The second was the failure of the winning athlete to reach our estimated time. The winner was Donovan Bailey of Canada whose record-breaking time was 9.84 s. But we had a satisfactory explanation for this as we knew that we had some unexplained variation in our models and hence none of them was a perfect fit to the data. Even so, it was agreed that it had been an interesting exercise from which we had learned much; we might even try to estimate the winning time in 2000! Any offers for a better model?

Carol Nixon

Letters to the Editor

Dear Editor,

Large prime triplets

It is easy to see that, in any set of three numbers $\{a, b, c\}$ with $3 < a < b < c < a+6$, at least one of the three must be composite. On the other hand, there is no reason why we cannot have three consecutive primes $\{p, q, p+6\}$. Such groupings clearly exist and they are known as *prime triplets*. The first few are $\{5, 7, 11\}$, $\{7, 11, 13\}$, $\{11, 13, 17\}$, $\{13, 17, 19\}$, $\{17, 19, 23\}$, $\{37, 41, 43\} \ldots$, and, just as with prime twins, which are pairs of prime numbers $\{p, p+2\}$, it is widely believed but unproven that the sequence of prime triplets goes on for ever. This is the *prime triplet conjecture*, first investigated by Hardy and Littlewood (see reference 1).

In 1984, Yates (reference 2) described any prime of at least 1000 decimal digits as 'titanic', although it has to be said that ordinary personal computers are now quite capable of generating such large primes with relatively little effort. Nevertheless, it is a rare occasion to find three titanic primes coming close together in the manner described above. So I thought it might be of some interest to try to find examples of what could be called 'titanic triplets'. On 4 December 1996, I succeeded — my computer found the triplet of 1083-digit primes

$$\begin{aligned} &437850590(2^{3567} - 2^{1189}) - 6 \times 2^{1189} - 5, \\ &437850590(2^{3567} - 2^{1189}) - 6 \times 2^{1189} - 1, \\ &437850590(2^{3567} - 2^{1189}) - 6 \times 2^{1189} + 1. \end{aligned} \tag{1}$$

According to Chris Caldwell's 'Titanic Primes' database, located on the World Wide Web at

`http://www.utm.edu/research/primes/largest.html` ,

it seems to be the only known example of a prime triplet where the individual primes have more than a thousand digits each.

The numbers I searched have the form

$$N-5, \qquad N-1, \qquad N+1, \tag{2}$$

where $N = m(2^{3\times 1189} - 2^{1189}) - 6\times 2^{1189}$ and $0 < m < 2^{31}$. The exponent 1189 was chosen because $2^{1189}+1$ has been completely factorized, and $2^{1189}+1$ divides $N-6$. We use this information when we have to prove that $N-5$ is prime. For, when $m = 437850590$, it turns out that, together with the divisor 3×2^{1189} of N, we easily obtain sufficient prime factors of both $N-6$ and N to verify the primality of the three numbers $N-5$, $N-1$ and $N+1$ by the methods of Brillhart, Lehmer and Selfridge (reference 3).

To find the triplet (1), I first used a sieving procedure to prepare a list of m's for which the corresponding $(N-5)(N-1)(N+1)$ is not divisible by any prime up to $2^{31}-1$. Then each $N-1$ was checked for probable primality with the *Fermat test*:

$$2^X \equiv 2 \pmod{X}. \tag{3}$$

The Fermat–Euler theorem asserts that *if X is prime then (3) holds*. However, and this is extremely useful for anybody who is interested in searching for large primes, it is known that the converse is in some sense almost always true; a large number X chosen more or less at random is likely to be prime if it satisfies (3). Rather than use (3) directly, we compute

$$2^{m2^{3\times 1189}} \pmod{N-1}$$

and compare with

$$2^{(m+6)2^{1189}+2} \pmod{N-1},$$

which can be obtained along the way at virtually no extra cost.

Reduction modulo $N-1$ is very fast because we can take advantage of the special form of $N-1$ and avoid doing most of the long divisions. The performance of the Fermat test is therefore dominated by the repeated squaring during the computation of

$$2^{m2^{3\times 1189}} \pmod{N-1}.$$

We use an efficient squaring algorithm — similar to the Schönhage–Strassen method — based on a certain integer transform operating on a vector obtained by chopping the number up into 60-bit digits. It is essentially the same as the algorithm described in reference 4.

With both these enhancements in place — fast reduction and fast squaring — the Fermat test takes about 3.7 seconds on a 120 MHz Pentium. For comparison, the time increases to about 21 seconds if we use the usual, brute-force method of squaring and long division for the reductions.

I would like to thank Oliver Atkin for showing me a general method (of which (2) is a special case) for devising patterns of three or four numbers such that the primality of each number can be readily established. I am also grateful to Harvey Dubner for providing independent verification of (1).

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1. G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes, *Acta Math.* **44**, pp. 1–70 (1922).
2. S. Yates, Sinkers of the Titanics, *J. Rec. Math.* **17**, pp. 268–274 (1984).
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4. T. Forbes, A large pair of twin primes, *Math. Comp.* **66**, pp. 451–455 (1997).

Yours sincerely,

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(22 St Albans Road,
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Surrey KT2 5HQ)

Dear Editor,

Fibonacci numbers

Martin Sandford conjectured in Volume 28 Number 2 page 43 that $F_n + 1$ and $F_n - 1$ were both composite for $n \geqslant 7$. More generally it can be proved that:

$F_n + F_k$ and $F_n - F_k$ are composite if $n + k$ is even and $n \geqslant 6 + k$ for some k.

Define L_n to be the nth Lucas number with $L_0 = 2, L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$. The proof of the above then follows from the identities

$$F_{2n+k'} + (-1)^{n+1} F_{k'} = F_n L_{n+k'},$$
$$F_{2n+k'} + (-1)^n F_{k'} = F_{n+k'} L_n,$$

which can be proved using the Binet forms of F_n and L_n, namely $F_n = (x^n - y^n)/(x - y)$ and $L_n = x^n + y^n$, where $x = \frac{1}{2}(1 + \sqrt{5})$ and $y = \frac{1}{2}(1 - \sqrt{5})$. $F_n L_{n+k}$ and $F_{n+k} L_n$ are composite since $2n + k' \geqslant 6 + k'$ gives $n \geqslant 3$ so $F_n > 1$ and $L_n > 1$.

Yours sincerely,
MANSUR BOASE
(St Paul's School, London)

Dear Editor,

Fibonacci numbers

With respect to Martin Sandford's letter in Volume 28 Number 2, it can be proved fairly easily (from Binet's formula for F_n, for example) that

$$(F_{m+1} + F_{m-1})F_n = \begin{cases} F_{m+n} + (-1)^{n+1} F_{m-n} & \text{if } m \geqslant n, \\ F_{m+n} + (-1)^m F_{n-m} & \text{if } n > m. \end{cases}$$

If we replace (m, n) by $(n+1, n-1)$, $(n-1, n+1)$, $(n+1, n)$, $(n, n+1)$ respectively, we obtain

$$F_{2n} + (-1)^n = (F_{n+2} + F_n)F_{n-1},$$
$$F_{2n} + (-1)^{n-1} = (F_n + F_{n-2})F_{n+1},$$
$$F_{2n+1} + (-1)^{n+1} = (F_{n+2} + F_n)F_n,$$
$$F_{2n+1} + (-1)^n = (F_{n+1} + F_{n-1})F_{n+1}.$$

This finds specific Fibonacci factors greater than 1 for all $F_n \pm 1$ when $n \geqslant 7$.

Yours sincerely,
TOBY GEE
(The John of Gaunt School, Trowbridge)

Dear Editor,

The Collatz problem

In case the dates of the references provided by Michael Mudge, *Mathematical Spectrum*, 1996/97, Volume 29, Number 1, page 20 give a different impression, it should perhaps be remarked that the Collatz problem is still very much alive in terms of current research. For example, Z. Franco and C. Pomerance, in *Mathematics of Computation*, 1995, Volume 64, pages 1333–1336, take up the paper by Crandall mentioned by Mudge, by looking at the conjecture that for any odd integer q greater than 3 there is some positive integer which under iteration in the '$qx + 1$ problem' never reaches 1. They show that, in the sense of asymptotic density, this holds for almost all odd integers q greater than 3.

D. Applegate and J. C. Lagarias, in *Mathematics of Computation*, 1995, Volume 64, pages 411–438, look specifically at the '$3x + 1$ problem', that is the *Collatz* problem, again referring to the work of Crandall. They are concerned with such issues as the number of integers whose kth iterate is some specified number not divisible by 3, or the number of integers in absolute value less than some prescribed amount which under iteration reach some specified integer not divisible by 3.

All these papers are highly technical and give further references to the technical literature. An accessible, general reference, although by now somewhat dated, is the survey by J. C. Lagarias, in *The American Mathematical Monthly*, 1985, Volume 92, pages 3–23. As it happens, Sir Bryan Thwaites, who promoted interest in the question from the early 1950s onwards, mentions it again in a retrospective piece in the centenary issue of *The Mathematical Gazette*, 1996, Volume 80, pages 35–36 and 420.

Yours sincerely,
D. G. ROGERS
(Halewood Cottage, The Green, Croxley Green, WD3 3HT)

Dear Editor,

Problems 28.4 and 28.5

These problems asked for all solutions in integers of the equations $2^n + n^2 = m^2$ and $3^n + n^3 = m^3$; the former has the only solution $n = 6$, $m = 10$ and the latter has no solution. Can Minh asked whether there are any other solutions of the equation $p^n + n^p = m^p$, where p is a prime number. More generally, we show here that the only solution of the equation $p^k = m^p - n^p$, where m, n, k are natural numbers and p is a prime, is given by $2^6 = 10^2 - 6^2$.

Write $m = p^a m'$ and $n = p^b n'$, where $p \nmid m'$ and $p \nmid n'$. Then, if $a > b$,

$$p^{k-bp} = (p^{a-b} m')^p - (n')^p.$$

Since $p \nmid n'$, the only possibility is that $k = bp$. But, in that case, we have two pth powers differing by 1, which is impossible. A similar argument shows that $a \not< b$. Hence $a = b$. Put $K = k - ap$. Then, rewriting m' and n' as m and n, we have $p^K = m^p - n^p$, where $p \nmid m$ and $p \nmid n$. Thus

$$p^k = (m - n)(m^{p-1} + m^{p-2} n + \cdots + n^{p-1}),$$

so $m - n = p^c$ and $m^{p-1} + m^{p-2} n + \cdots + n^{p-1} = p^d$ for some c and d with $d \geqslant 1$. Suppose that $c \geqslant 2$. Then

$$p^d \equiv n^{p-1} + n^{p-2} n + \cdots + n^{p-1} \equiv p n^{p-1} \pmod{p^2}$$

so $p^d \not\equiv 0 \pmod{p^2}$. Hence $d = 1$, so $m = n = 1$, which is not a solution of $p^K = m^p - n^p$. Hence $c = 0$ or 1.

If $c = 0$ then

$$p^K = (n+1)^p - n^p \equiv (n+1) - n \equiv 1 \pmod{p}$$

so $K = 0$. But $m^p - n^p = 1$ is not possible. If $c = 1$ then

$$p^K = (n+p)^p - n^p \equiv p^2 n^{p-1} \pmod{p^3} \qquad \text{if } p \geqslant 3.$$

Since $p \nmid n$, this gives $K \leqslant 2$. But

$$p^K = (n+p)^p - n^p \geqslant p^p,$$

so that $K \geqslant p$. Thus $p \geqslant 3$ is not possible, so $p = 2$. Then $2^K = (n+2)^2 - n^2 = 4(n+1)$, so that $K \geqslant 3$ and $n = 2^{K-2} - 1$. Put $K - 2 = N$. Then $n = 2^N - 1$ and $m = 2^N + 1$. If we now multiply through by 2^a we see that every solution of the equation $p^k + n^p = m^p$ is of the form

$$(p, k, m, n) = \big(2, N + 2(a+1), 2^a(2^N + 1), 2^a(2^N - 1)\big),$$

where $N \geqslant 1$ and $a \geqslant 0$, and it is easy to verify that all such are solutions, so this gives the complete solution.

In particular, the only solution of $p^n + n^p = m^p$ is $2^6 + 6^2 = 10^2$.

Yours sincerely,
TOBY GEE

Dear Editor,

Problem 28.8

This problem was set in Volume 28 Number 2; it was to prove that, for a prime number p,

$$\prod_{n=1}^{p-1} (1 + pn^{-1}) \equiv 1 \pmod{p}.$$

In fact, it is possible to prove the stronger result that

$$\prod_{n=1}^{p-1} (1 + pn^{-1}) \equiv 1 \pmod{p^3}$$

when p is a prime number greater than 3.

Yours sincerely,
TOBY GEE

Dear Editor,

The daughter's dilemma
(Volume 29 Number 2 page 29)

This is a special case of the standard 'Chiefs and Indians' (n in a circle, counted out m at a time) problem, and is interesting because there are some explicit solutions. 'Chiefs and Indians' — more properly known as the Josephus Problem (see Rouse Ball, page 32 of my 1947 edition) — was the theme of Mike Mudge's *Numbers Count* column in the April 1996 issue of *Personal Computer World*, and his December 1996 column reports that he was sent results for up to 5000 Indians and jumps of 2 to 20. If any of your readers are interested I can put them in touch with Mike Mudge.

Yours sincerely,
ALAN D. COX
(Pen-y-Maes, Ostrey Hill,
St Clears, Dyfed, SA33 4AJ)

(See also 'The Josephus Problem' by I. M. Richards, *Mathematical Spectrum* Volume 24 No. 4 pp. 97–104. – ED.)

Dear Editor,

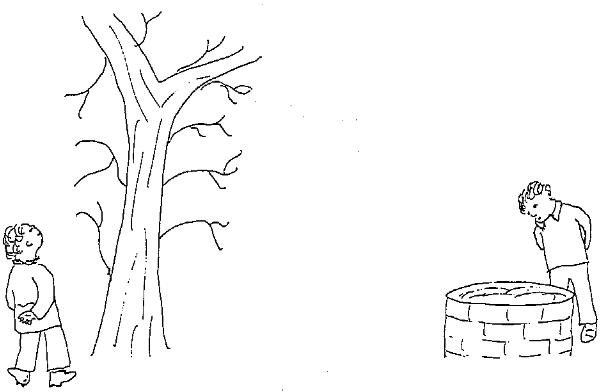
Letter from Kamlesh Gaya
(Volume 29 Number 2 page 41)

I am not sure exactly what Mr Gaya is seeking. I doubt whether a simple analytical solution to either of his problems is possible, but it is simple to write BASIC programs to solve them. QBASIC with LONG INTEGERs will handle up to 10^9 but you need either UBASIC or, for example, *Mathematica* to go to 10^{10}.

Yours sincerely,
ALAN D. COX

PS The program in the article 'Designing Tiles' (Volume 29 Number 2 page 38) is not true QBASIC, which abhors line numbers — it is more like GW BASIC.

How would you estimate ... the height of a tree?
the depth of a well?

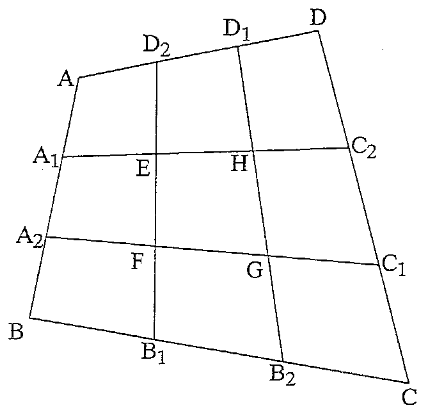


Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

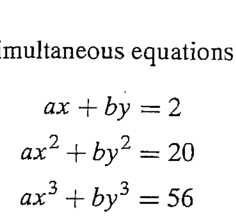
Problems

29.9 In the figure, $DD_1 = D_1D_2 = D_2A$, $AA_1 = A_1A_2 = A_2B$, $BB_1 = B_1B_2 = B_2C$, $CC_1 = C_1C_2 = C_2D$. Determine the area of the quadrilateral EFGH in terms of the area of ABCD.



(Submitted by Zhao Yueyang, Millfield School)

29.10 A piece of wire of length ℓ is bent into the shape of a sector of a circle. Find the maximum area of the sector.



(Submitted by Des MacHale, University College, Cork)

29.11 Solve the simultaneous equations

$$\begin{aligned} ax + by &= 2 \\ ax^2 + by^2 &= 20 \\ ax^3 + by^3 &= 56 \\ ax^4 + by^4 &= 272. \end{aligned}$$

(Submitted by Pak-San Man, Winchester College)

29.12 Simplify

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C},$$

where A, B, C are the angles of a triangle, and find all non-zero real numbers α for which

$$\frac{\sin \alpha A + \sin \alpha B + \sin \alpha C}{\sin A \sin B \sin C},$$

is independent of A, B and C.

(Submitted by Mansur Boase, St Paul's School, London)

Solutions to Problems in Volume 29 Number 1

29.1 Prove that

$$\sin A \sin B(2\cos C - 1) + \sin B \sin C(2\cos A - 1) + \sin C \sin A(2\cos B - 1) \geqslant 0,$$

where A, B and C are the angles of a triangle.

Solution independently by Pak-San Man (Winchester College) and Andrew Lobb (St Olave's Grammar School, Orpington)

With obvious notation

$$\begin{aligned} \sin A \sin B(2\cos C - 1) &= \frac{a}{2R}\frac{b}{2R}\left(2\frac{a^2+b^2-c^2}{2ab} - 1\right) \\ &= \frac{1}{4R^2}(a^2+b^2-c^2-ab), \end{aligned}$$

so the given expression is equal to

$$\begin{aligned} &\frac{1}{4R^2}\big[(a^2+b^2-c^2-ab) + (b^2+c^2-a^2-bc) \\ &\qquad + (c^2+a^2-b^2-ca)\big] \\ &= \frac{1}{4R^2}(a^2+b^2+c^2-ab-bc-ca) \\ &= \frac{1}{8R^2}\big[(a-b)^2+(b-c)^2+(c-a)^2\big] \geqslant 0 \end{aligned}$$

(with equality if and only if $a = b = c$, i.e. if and only if the triangle is equilateral).

Also solved by Konstantin Ardakov (Christ Church, Oxford), Junji Inaba (Trinity College, Cambridge), John Kelly (Scoil Uí Chonaill, Cahersiveen, Co. Kerry, Ireland), Deborah King (Haberdashers' Aske's School for Girls, Elstree), Maurice Bowler (Scoil Uí Chonaill, Cahersiveen, Co. Kerry, Ireland), Can Minh (University of Southern California, Los Angeles), Scott Brown (Auburn University, Alabama).

29.2 There are n sheep in a field, numbered 1 to n and some integer $m > 1$ is given such that $m^2 \leqslant n$. It is required to separate the sheep into two groups such that (1) no sheep has number m times the number of a sheep in the same group, and (2) no sheep has number the sum of the numbers of two sheep in its group. For which values of m, n is this possible?

Solution by Andrew Lobb

We first show that $n < 9$. Suppose that $n \geqslant 9$, denote by N the group containing sheep 9 and by N' the other group. There are four possibilities.

(1) $1 \in N$ and $2 \in N$ ($\in$ means 'belongs to'). Then $3, 7, 8 \in N'$ by condition (2) so $4, 5 \in N$ so $9 \in N'$, a contradiction.

(2) $1 \in N$ and $2 \in N'$. Then $8 \in N'$ so $6 \in N$ so $5, 7 \in N'$ so $2 \in N$, a contradiction.

(3) $1 \in N'$ and $2 \in N$. Then $7 \in N'$ so $6, 8 \in N$ so $2 \in N'$, a contradiction.

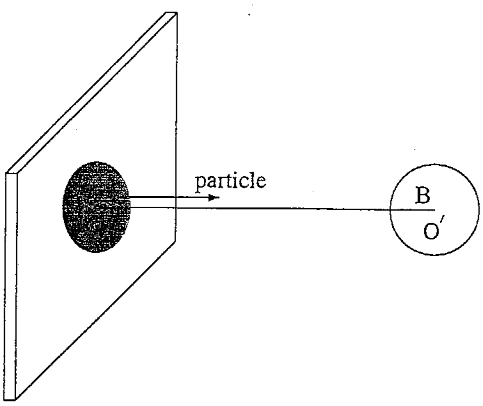
(4) $1 \in N'$ and $2 \in N'$. Then $3 \in N$ so $6 \in N'$ so $4, 5 \in N$ so $9 \in N'$, a contradiction.

We now have $9 > n \geqslant m^2 > 1$, so $m = 2$. Suppose that $n \geqslant 5$ and denote the group containing sheep 1 by O and the other group by O'. Then $2 \in O'$ by condition (1) so $4 \in O$ so $3, 5 \in O'$ by condition (2) so $2 \in O$, a contradiction. Hence $n < 5$. But now $4 \leqslant m^2 \leqslant n < 5$ so $n = 4$.

Thus $m = 2$ and $n = 4$; a solution of the problem for these values is $\{1, 4\}$ and $\{2, 3\}$.

Also solved by Michelle Allen (Trinity College, Cambridge), Konstantin Ardakov.

29.3 A very small spherical particle is fired horizontally in the direction OO′ from the plane circular firing area C of radius 1 — see the diagram. The line OO′ is at right angles to C and O is the centre of C. The particle is fired at a fixed sphere B, with centre O′, also of radius 1, and rebounds from B in a perfectly elastic manner. The particle is equally likely to be fired from anywhere in C. Show that all directions of rebound from B are equally likely. (This was proved by James Clerk Maxwell in his paper 'Illustration of the Dynamical Theory of Gases', 1859).



Solution by D. Forfar, who submitted the problem.

The area of a spherical cap A of a sphere radius 1, as shown in figure 1, is

$$\int_0^\alpha 2\pi \sin\theta \, d\theta = 2\pi(1 - \cos\alpha) = 4\pi \sin^2 \tfrac{1}{2}\alpha.$$

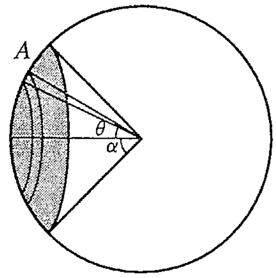


Figure 1

Let the directions of rebound of the particle be represented by unit vectors. The ends of these vectors form a sphere of radius 1. We consider the directions of rebound ending on a spherical cap A as in figure 1. Figure 2 shows that the particles rebounding in directions represented by A must have been fired from A'. The probability of being fired from A' is the area of A' divided by the area of C, i.e. it is

$$\frac{\pi \sin^2 \frac{1}{2}\alpha}{\pi} = \sin^2 \tfrac{1}{2}\alpha.$$

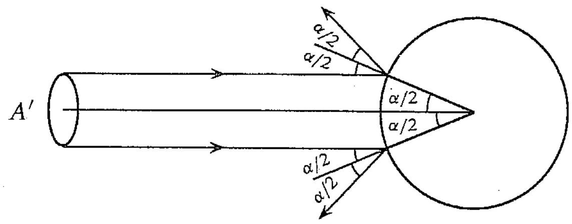


Figure 2

Hence the probability that the direction of rebound is through A is

$$\sin^2 \tfrac{1}{2}\alpha = \frac{\text{area of cap } A}{4\pi} = \frac{\text{area of cap } A}{\text{area of unit sphere}}.$$

Hence all directions of rebound are equally probable.

29.4 Let a, b, c be the lengths of the sides of a non-degenerate triangle. Prove that

$$\frac{3}{2} \leqslant \frac{\tanh a}{\tanh b + \tanh c} + \frac{\tanh b}{\tanh c + \tanh a} + \frac{\tanh c}{\tanh a + \tanh b} < 2.$$

Solution by Can Minh

Denote the given sum by S. Since tanh is an increasing function and $0 < a < b + c$,

$$\begin{aligned} 0 < \tanh a &< \tanh(b+c) \\ &= \frac{\tanh b + \tanh c}{1 + \tanh b \tanh c} \\ &< \tanh b + \tanh c. \end{aligned}$$

Hence

$$\tanh a + \tanh b + \tanh c < 2(\tanh b + \tanh c),$$

so

$$\frac{\tanh a}{\tanh b + \tanh c} < \frac{2\tanh a}{\tanh a + \tanh b + \tanh c}.$$

Hence

$$S < \frac{2\tanh a + 2\tanh b + 2\tanh c}{\tanh a + \tanh b + \tanh c} = 2.$$

Next,

$$\begin{aligned} &[(\tanh b + \tanh c) + (\tanh c + \tanh a) + (\tanh a + \tanh b)] \\ &\times \left[\frac{1}{\tanh b + \tanh c} + \frac{1}{\tanh c + \tanh a} + \frac{1}{\tanh a + \tanh b}\right] \\ &= 2S + 6. \end{aligned}$$

By use of the arithmetic mean $\geqslant$ geometric mean inequality, this gives

$$\begin{aligned} 2S + 6 &\geqslant 3 \\ &\times \left[(\tanh b + \tanh c)(\tanh c + \tanh a)(\tanh a + \tanh b)\right]^{1/3} \\ &\times \frac{3}{\left[(\tanh b + \tanh c)(\tanh c + \tanh a)(\tanh a + \tanh b)\right]^{1/3}} \\ &= 9, \end{aligned}$$

so $S \geqslant \frac{3}{2}$.

Also solved by Konstantin Ardakov.

Reviews

Which Way did the Bicycle Go? By JOSEPH D.E. KONHAUSER, DAN VELLEMAN AND STAN WAGON. Mathematical Association of America, Washington DC, 1996. Pp. xv+235. Paperback $24.95 (ISBN 0-88385-325-6).

This book is a collection of 191 of the best problems from the Konhauser collection, named after the late Jo Konhauser (1924–1992), who, for 25 years, posted a Problem of the Week at Macalester College. This tradition is now being continued by Stan Wagon, a Macalester mathematician. The third author, Dan Velleman, teaches maths at Amherst College.

The title refers to the first problem in the collection in which Sherlock Holmes made a mistake in determining which direction a bicycle was travelling by studying its tracks in a patch of mud. This sets the tone for the rest of the problems in the book, which are intriguing, and readers are invited to use their deductive skills to solve them. They are divided into six sections, each with several subsections: plane geometry, number theory, algebra, combinatorics and graph theory, three-dimensional geometry and miscellaneous. Complete solutions to all the problems are included in addition to similar problems and topics which the reader is invited to investigate. In a few of the problems, some lines of *Mathematica* code are given to show how computers may be used to help solve a problem. There is a very detailed reference section at the end, and a combined author and subject index, mentioning everything from 'Bézier's curve' to 'leopard's teeth'. Furthermore, each problem has a name and there is a complete list of all these names in the contents, from 'Abe Lincoln's Somersaults' to 'A Puzzling Reflexicon'.

The problems require little prior mathematical knowledge and are æsthetically pleasing. This could explain why half of them are about geometry. They are well illustrated with large, clearly labelled diagrams. What is special about these problems is that the standard problem-solving techniques will often fail and the reader is forced to find his own method. The solutions are elegant, and usually there is a 'neat approach', as in Problem 15:

> Suppose $\triangle$ABC is isosceles with AB = AC and $\angle$BAC $= 20°$. P is on AB such that AP = BC. Determine $\angle$ACP.

Some of these problems have interesting origins. The late Samuel Greitzer intensely disliked 'new math' and would argue against including problems about liars and truthtellers, Venn diagrams, and non-base-10 arithmetic in maths contests. As a joke, his friend Stanley Rabinowitz invented Problem 172, which makes use of all three concepts. Problem 78, not known until 1977 when it was discovered by Paul Erdős and George Szekeres, asks the reader to prove the simple fact that any two entries, not equal to one, in the same row of Pascal's triangle have a common factor. The solution is a one-liner!

There are also some delicious pizza problems for readers to dig their teeth into. If a pizza is sliced into eight 45° wedges meeting at a point other than the centre of the pizza and two people eat alternate wedges, will they necessarily get equal amounts of pizza? This question was sent to the great problem solver, Professor Murray Klamkin, who interpreted it slightly differently and arrived at a beautiful variation, which readers can discover for themselves, demonstrating the truth of James Joyce's words: 'A man of genius makes no mistakes. His errors are volitional and are the portals of discovery.'

I have enjoyed reading this book and solving some of the problems in it. I strongly recommend it to all those who enjoy solving problems and to every school and college library. I hope we will not have to wait another 25 years for the sequel.

St Paul's School, London MANSUR BOASE

Julia. A Life in Mathematics. By CONSTANCE REID. Mathematical Association of America, Washington DC, 1996. Pp. xii+123. Hardback $27.00 (ISBN 0-88385-520-8).

In 1900, at the second International Congress of Mathematicians, held in Paris, David Hilbert propounded 23 mathematical problems. They range from those on axiomatics and the foundations of mathematics to more technical ones on differential equations and the calculus of variations. One or two were solved quite soon, others over the decades since then, and some defy solution today. Solved or unsolved, they continue to influence the directions of mathematical research. This book tells the story of how Hilbert's tenth problem was solved. The problem concerns Diophantine equations. These are polynomial equations with integer coefficients; and the question posed was essentially: does there exist a universal algorithm for deciding whether a given Diophantine equation can be solved (in integers) or not? In 1970 the question was answered, in the negative, by a young Russian mathematician Yuri Matijasevich.

Julia is the personal story of Julia Bowman Robinson (1919–1985), compiled by her sister Constance Reid, herself well known as the author of excellent biographies of Hilbert and Courant. Though Hilbert's tenth problem was ultimately solved by Matijasevich, a considerable part of the solution depended upon fundamental ideas in the work of Julia Robinson, whose main mathematical interests lay on the borderline of number theory and symbolic logic. Her crucial contribution to the solution of this famous problem established her reputation in the mathematical world, particularly in America. She became the first woman mathematician to be elected to the National Academy of Sciences, and was the first woman president of the American Mathematical Society.

Consisting of four previously published articles (the longest one by Constance Reid, and written in the form of an autobiography as though by Julia herself, and one of the others by Yuri Matijasevich), this book, partly descriptive and partly technical and containing many photographs, has been produced as a tribute to Julia. Its purpose has been to make available to readers, not only mathematicians but also teachers and students, something about her mathematics and her career.

In reading the book we get glimpses at times of the excitement of mathematical discovery as well as of its frustrations. But there is restraint apparent in the writing, for Julia Robinson had never wished for a biography. In her own words: 'I would prefer to be remembered . . . simply for the theorems I have proved and the problems I have solved'. As readers, we could perhaps have wished to be told a little more.

University of Sheffield HAZEL PERFECT

Mathematical Mysteries: The Beauty and Magic of Numbers. By CALVIN C. CLAWSON. Plenum, New York, 1996. Pp. x+313. Hardback $27.95 (ISBN 0-306-45404-1).

A thriller whose subject is mathematics? Impossible! Yet that is what the author has attempted to write. It is a book about mathematics, in particular number theory, and those who invented (or discovered?) it, rather than a book of mathematics. Your reviewer read it over Christmas, and it really was as good as a good whodunit. Here is the search for large prime numbers and how to use them to design an unbreakable code. Or the mystery and tragedy of the great Indian mathematician Ramanujan and the beautiful results he found; a sample is the formula

$$\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\left(1+\frac{1}{7^4}\right)\cdots=\frac{105}{\pi^4},$$

where there is a bracket for each prime number. Or the Goldbach conjecture that every even number greater than 2 is the sum of two primes; is it true or not?

It has to be said that some readers may feel rather patronized at times. At other times the same readers may feel out of their depth; the description of the continuum hypothesis and Gödel's incompleteness theorem may well leave readers bewildered.

But, if not everything is equally successful, young mathematicians should be stimulated and excited by the book as a whole. Beware; it may well start you on a voyage of discovery that could change your life!

Strongly recommended for all college and school libraries.

University of Sheffield DAVID SHARPE

A Primer of Real Functions. By RALPH P. BOAS, JR. (Revised and updated by HAROLD P. BOAS). The Mathematical Association of America, Washington, DC, 1996. Pp. xiv+305. Hardback $32.95 (ISBN 0-88385-029-X).

This is the fourth edition of a classic, written in a gracious style as a journey through the theory of real functions rather than as a textbook. To the original two chapters on sets and functions has been added a third on integration, the Lebesgue and Stieltjes integrals, written from a draft left by Ralph Boas on his death. The first edition was dedicated 'To my epsilons'. This revision has been undertaken by one of them, and is recommended to all who understand the joke.

University of Sheffield DAVID SHARPE

Introduction to the Design and Analysis of Experiments. By G. M. CLARKE AND R. E. KEMPSON. Arnold, London, 1997. Pp. vii + 344. Paperback £19.99 (ISBN 0-340-64555-5).

The authors have met their aim of producing a text that is mainly at undergraduate level but with a few postgraduate topics included. On the whole this is a careful, traditional presentation of the standard aspects of the principles of design and the analysis of data from, for example, orthogonal designs, factorial experiments, response surfaces, balanced incomplete block and general non-orthogonal designs. There is an unbiased appraisal of the more modern Taguchi approach, and illustrations of various analyses using the SAS procedure PROC GLM. Many examples are worked through and useful exercises provided at the end of each chapter.

University of Sheffield — IAN DUNSMORE

Introduction to Probability Theory, with Contemporary Applications. By LESTER L. HELMS. W. H. Freeman, New York, 1996. Pp. x + 351. Hardback £24.95 (ISBN 0-7167-3023-5).

This introductory text in probability theory is suitable for either an advanced undergraduate or a beginning graduate course. The author suggests that roughly the first half of the book could provide material for a course lasting one semester, while the entire book would be more than adequate for a two semester course.

The emphasis has been on providing a book 'written for students'. Some readers may find the explanations a little too lengthy, but the author points out that he has 'endeavored to err on the side of readability'. The material selected combines a reasonable range of topics with a very clear level of exposition. The eight chapters cover:

1. Classical Probability
2. Axioms of Probability
3. Random Variables
4. Expectation
5. Stochastic Processes
6. Continuous Random Variables
7. Expectation Revisited
8. Continuous Parameter Markov Processes

There are many exercises, and their solutions are included in pages 300–345 of the book. A helpful index is provided in conclusion.

This is a text from which students should be able to learn the elements of probability theory with relative facility. The author's relaxed style, the variety of examples, and the thorough treatment of the topics selected will commend the book to students and teachers alike.

Australian National University — JOE GANI

Statistically Speaking: A Dictionary of Quotations. By CARL C. GAITHER AND ALMA E. CALVAZOS-GAITHER. Institute of Physics Publishing, Bristol, 1996. Pp. xii+420. Hardback £19.95 (ISBN 0-7503-0401-4).

This book collects together a wealth of quotations relating to statistics, including topics such as error, dice and probability, but also more distant notions such as recurrence, reason and truth. There are 288 pages of such quotations plus, and it is this for which the authors should be warmly thanked, 132 pages of index which allow one to trace author within subject and subject within author, and full reference. It cannot be said that the book is full of wit and warmth, nor of sound and fury. 'Statistics' usually invites apprehension rather than affection, but certainly it will be a useful tool for those who wish to quote.

University of Sheffield — CHRIS CANNINGS

Other books received

National Engineering Mathematics. Volume 3. By J. C. YATES. Macmillan, Basingstoke, 1997. Pp. xiii+537. Paperback £16.99 (ISBN 0-333-54853-1).

This volume covers complex numbers, further differentiation and integration, series, numerical methods, differential equations, Laplace transforms, Fourier series and probability distributions.

Solution to Braintwister 1

(No need for Mystic Meg)

Answer. There are 12 heaviest balls.

Solution. In what follows all the equalities and inequalities refer to weights:

$$13 \leqslant 41 = 14 \leqslant 31 = 13 \text{ and } 13 \leqslant 32 = 23 \leqslant 31 = 13 \text{ and } 03 \leqslant 32 = 23 \leqslant 30 = 03.$$

Hence $3 = 4$, $1 = 2$ and $0 = 2$. It follows that $16 \leqslant 41 = 14 \leqslant 26 = 16$ and so $4 = 6$ (and similarly $4 = 7 = 8 = 9$). Hence, in increasing order of weight, we have

$$0 = 1 = 2 \text{ and } 5 \text{ and } 3 = 4 = 6 = 7 = 8 = 9,$$

with not all three batches equal. For most of the numbers to weigh a different amount from ball 25 we need 5 to weigh less than 3 etc. Then the heaviest balls use both digits from 3, 4, 6, 7, 8, 9 and are

$$33, 34, 36, 37, 38, 39, 43, 44, 46, 47, 48, 49.$$

VICTOR BRYANT

LONDON MATHEMATICAL SOCIETY

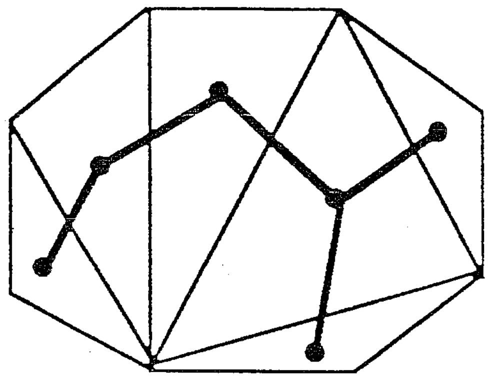
1997 POPULAR LECTURES

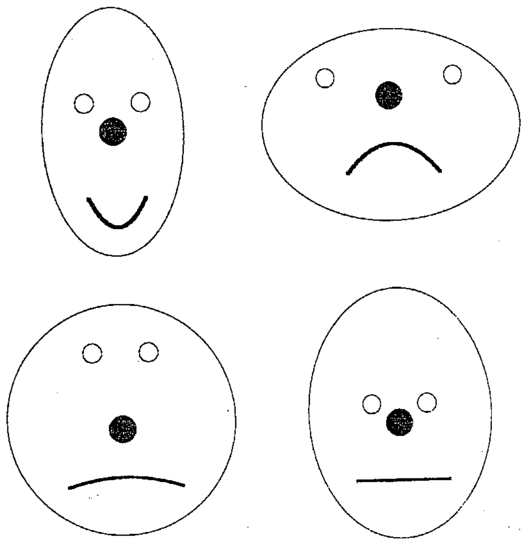
Edinburgh University - Tuesday 17 June
Birmingham University - Monday 23 June
Imperial College - Friday 4 July

Professor Mike Atkinson

Staying ahead of the Opposition

"Counting votes, partitioning a polygon into triangles, walking in a city whose roads only run North-South or East-West are all activities linked by the ubiquitous Catalan numbers"





Professor John Kent

How to Study Random Shapes

"Developed during the past 10-15 years, the new subject of shape analysis provides us with methods of comparing and averaging geometric objects, without regard for their location size and orientation."

For further information contact Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL (e-mail: lms@lms.ac.uk).
Admission is free.

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