Welcome!

Welcome to ***CRUX MATHEMATICORUM with MATHEMATICAL MAYHEM***. We hope that those of you new to any part of our journal will enjoy all the material. We see this merger as beneficial to all our readers. We now offer a broader range of problems. High School students and teachers will find more material at that level. Please note that we will accept solutions to problems in the **MAYHEM** section only from students. These are the people that we wish to stimulate. When they find they are successful with the **MAYHEM** and **SKOLIAD** problems, we hope they will then find a further challenge with the problems in the **OLYMPIAD** corner, and, eventually, with the **PROBLEMS** section. Good luck, and we look forward to hearing from you.

You will note that the majority of the members of the Crux Editorial Board are continuing in their previous jobs. Sadly, Colin Bartholomew has taken early retirement from Memorial University, and I shall miss him. However, we are pleased to welcome Clayton Halfyard as Associate Editor. There will be a short profile of him elsewhere in this issue. We are also delighted to welcome Naoki Sato as Mayhem Editor and Cyrus Hsia as Assistant Mayhem Editor. These two Canadian IMO medallists bring with them a wealth of experience of problem solving, and many years of experience of writing for ***MAYHEM***.

There are some minor administrative changes: the table of contents (which is larger) is now on the back outside cover; and the ordering of the various sections has been changed. There is no change in the quantity of material that has been recently published in ***CRUX***. Whereas previously ***MAYHEM*** appeared five times per year, it now appears eight times per year. The annual quantity of material will be approximately the same.

Here are some details that will help you in cross-referencing previous material:

1. Please note that the volume number is consecutive for ***CRUX***.

2. Material occurring in previous volumes of ***CRUX*** and material occuring in ***CRUX with MAYHEM*** will be referenced as [year: page no].

   For example, the last page of the last issue of ***CRUX*** is [1996: 384], and the first page of ***CRUX with MAYHEM*** is [1997: 1].

3. Material occurring in previous volumes of ***MAYHEM*** will be referenced as [MAYHEM volume: issue, page no: year].

   For example, [MAYHEM 8: 5, 28: 1996] is the last page of the last issue of ***MAYHEM***.

Bruce Shawyer
Editor-in-Chief

Bienvenue!

Bienvenue au ***CRUX MATHEMATICORUM with MATHEMATICAL MAYHEM***. Nous espérons que ceux qui ne connaissent pas l'une des nouvelles parties de notre journal en apprécieront tout le contenu. À notre avis, la fusion de ces deux magazines sera tout à l'avantage de nos lecteurs. La gamme des problèmes publiés y est plus vaste qu'auparavant, et les élèves et les professeurs du secondaire y trouveront davantage de problèmes à leur niveau. Nous vous signalons au passage que nous accepterons uniquement les solutions aux problèmes de la section **MAYHEM** provenant des étudiants; après tout, c'est eux que nous voulons stimuler! Lorsqu'ils pourront réussir les problèmes des sections **MAYHEM** et **SKOLIAD**, nous espérons qu'ils s'attaqueront à ceux de la partie **OLYMPIAD**, et même, un peu plus tard, à ceux de la section **PROBLEMS**. Bonne chance et au plaisir de recevoir de vos nouvelles!

Vous aurez remarqué que la majorité des membres de l'équipe éditoriale du Crux sont demeurés en poste. Malheureusement, Colin Bartholomew a pris une retraite anticipée de l'Université Memorial; il me manquera. Nous sommes toutefois heureux d'accueillir Clayton Halfyard au poste de rédacteur adjoint. Vous trouverez son profil ailleurs dans ce numéro. Nous sommes également enchantés d'avoir parmi nous Naoki Sato et Cyrus Hsia, qui occupent respectivement les postes de rédacteur et de rédacteur adjoint du Mayhem. Ces deux médaillés canadiens de l'OIM apportent avec eux tout un bagage d'expertise en résolution de problèmes, et des années d'expérience à la rédaction du ***MAYHEM***.

Vous constaterez en outre quelques petits changements d'ordre administratif : la table des matières (élargie) est désormais en quatrième de couverture et les sections ne sont plus dans le même ordre. Le volume d'information du ***CRUX*** sera comparable à la quantité d'information qu'il contenait avant la fusion. Le ***MAYHEM***, qui paraissait cinq fois l'an, sera publié huit fois par année; ainsi, le volume annuel de contenu demeurera approximativement le même.

Voici quelques détails sur la façon de noter les renvois :

1. Les numéros de volume du ***CRUX*** continuent dans la même séquence.
2. On notera ainsi les renvois à des volumes «pré-fusion» du ***CRUX*** et aux volumes du ***CRUX with MAYHEM*** : [anné: no de page].
   Par exemple, la dernière page du dernier numéro du ***CRUX*** sera [1996: 384], et la première page du ***CRUX with MAYHEM*** sera [1997: 1].
3. On notera ainsi les renvois à des volumes «pré-fusion» du ***MAYHEM*** : [volume du MAYHEM: no, no de page: année].
   Par exemple, [MAYHEM 8: 5, 28: 1996] est la dernière page du dernier numéro du ***MAYHEM***.

Le rédacteur en chef
Bruce Shawyer

THE ACADEMY CORNER

No. 8

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In this issue, courtesy of Waldemar Pompe, student, University of Warsaw, Poland, we print an international contest paper for university students. Please send me your nice solutions.

INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS July 31 – August 1996, Plovdiv, Bulgaria

First day — August 2, 1996

1. Let for $j = 0, 1, \ldots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}.$$

   Calculate $\det A$ — the determinant of A.

2. Evaluate the integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\,\sin x}\,dx\,,$$

   where n is a natural number.

3. A linear operator A on a vector space V is called an *involution* if $A^2 = E$, where E is the identity operator on V.

   Let $\dim V = n < \infty$.

   (i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

   (ii) Find the maximal number of distinct pairwise commuting involutions on V.

4. Let $a_1 = 1,\ \ a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$.

Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$;

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

5. (i) Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every $x \in [0, 1]$.

Prove that

$$\lim_{n\to\infty} n \int_0^1 (1 + ax + bx^2)^n \, dx = \begin{cases} -1/a & \text{if } a < 0; \\ +\infty & \text{if } a \geq 0. \end{cases}$$

(ii) Let $f\colon [0, 1] \to [0, \infty)$ be a function with continuous second derivative and $f''(x) \leq 0$ for every $x \in [0, 1]$. Suppose that

$$L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, dx$$

exists and $0 < L < +\infty$.

Prove that f' has a constant sign and $L = \left(\min_{x\in[0,1]} |f'(x)| \right)^{-1}$.

6. *Upper content* of a subset E of the plane $\mathbb{R} \times \mathbb{R}$ is defined as

$$\mathcal{C}(E) = \inf \left\{ \sum_{i=1}^{n} \operatorname{diam}(E_i) \right\}$$

where the infimum is taken over all finite families of sets $E_1, E_2, \ldots, E_n$ in $\mathbb{R} \times \mathbb{R}$ such that $E \subset \bigcup_{i=1}^{n} E_i$.

Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\operatorname{length}(L)\}$$

such that L is a closed line segment onto which E can be contracted.

Show that

(i) $\mathcal{C}(L) = \operatorname{length}(L)$, if L is a closed line segment.

(ii) $\mathcal{C}(E) \geq \mathcal{K}(E)$.

(iii) equality in (ii) need not hold even if E is compact.

Hint: If $E = T \cup T'$ where T is the triangle with vertices $(-2, 2)$, $(2, 2)$ and $(0, 4)$, and T' is its reflection about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks:

All distances used in this problem are Euclidean.

Diameter of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x,y) \mid x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f\colon E \to F$ such that $\operatorname{dist}(f(x), f(y)) \le \operatorname{dist}(x,y)$ for all $x, y \in E$.

A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, that is such that $f(E) = F$.

Triangle is defined as the union of the three line segments joining its vertices, so that it does not contain the interior.

Problems 1 and 2 are worth 10 points, problems 3 and 4 are worth 15 points, problems 5 and 6 are worth 25 points.

You have 5 hours.

Please write the solutions on separate sheets of paper. Good luck!

THE OLYMPIAD CORNER

No. 179

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Another year has passed, and the first with Bruce Shawyer as Editor-in-Chief. He has made the transition pleasant and easy. Special thanks go to Joanne Longworth whose TEX skills have made changed fonts and formats easy to incorporate. Thanks also go to the many contributors to the two Corners including:

Miguel Amengual Covas	Cyrus C. Hsia	Dieter Ruoff
Sĕfket Arslanagić	Murray Klamkin	Toshio Seimiya
Mansur Boase	Derek Kisman	Michael Selby
Seung-Jin Bang	Ted Lewis	D.J. Smeenk
Christopher Bradley	Joseph Ling	Daryl Tingley
Francisco Bellot Rosado	Beatriz Margolis	Panos E. Tsaoussoglou
Paul Colucci	Stewart Metchette	Ravi Vakil
Hans Engelhaupt	Richard Nowakowski	Edward T.H. Wang
Tony Gardiner	Michael Nutt	Hoe Teck Wee
Solomon Golomb	Siu Taur Pang	Chris Wildhagen
Gareth Griffith	Bob Prielipp	Siming Zhan
Georg Gunther	Chandan Reddy	

Let me open with a major apology. In the November 1996 number of the corner we gave twelve of the problems proposed to the jury but not used at the 36th International Mathematical Olympiad held in Canada. Those familiar with the process of selection will know that the problems do not initiate with the host country. They come from proposers in other countries, and the responsibility of the host country selection committee is to refine and select from these submissions the official list of problems proposed to the jury. Over the years I have loosely referred to the problems proposed to the jury at the Xth International Olympiad in Y as the "Y-problems" for short. However, when that became part of a longer more official sounding sub-title "Canadian Problems for consideration by the International Jury," I should have seen that it read as if the original proposers are from Canada, thus insulting the creators. In retrospect I do not understand why that interpretation did not jump off the page. To the many creative non-Canadians who submitted problems for possible use at the 36th IMO my sincere apologies.

As an Olympiad Contest this issue we give the problems of the 44th Mathematical Olympiad from Latvia. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong for collecting the problems for me.

LATVIAN 44 MATHEMATICAL OLYMPIAD
Final Grade, 3rd Round
Riga, 1994

1. It is given that $\cos x = \cos y$ and $\sin x = -\sin y$. Prove that $\sin 1994x + \sin 1994y = 0$.

2. The plane is divided into unit squares in the standard way. Consider a pentagon with all its vertices at grid points.

(a) Prove that its area is not less than $3/2$.

(b) Prove that its area is not less than $5/2$, if it is given that the pentagon is convex.

3. It is given that $a > 0$, $b > 0$, $c > 0$, $a + b + c = abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

4. Solve the equation $1! + 2! + 3! + \cdots + n! = m^3$ in natural numbers.

5. There are 1994 employees in the office. Each of them knows 1600 of the others. Prove that we can find 6 employees, each of them knowing all 5 others.

1st SELECTION ROUND

1. It is given that x and y are positive integers and $3x^2 + x = 4y^2 + y$. Prove that $x - y$, $3x + 3y + 1$ and $4x + 4y + 1$ are squares of integers.

2. Is it possible to find 2^{1994} different pairs of natural numbers (a_i, b_i) such that the following 2 properties hold simultaneously:

(1) $\dfrac{1}{a_1b_1} + \dfrac{1}{a_2b_2} + \cdots + \dfrac{1}{a_{2^{1994}}b_{2^{1994}}} = 1$,

(2) $(a_1 + a_2 + \cdots + a_{2^{1994}}) + (b_1 + b_2 + \cdots + b_{2^{1994}}) = 3^{1995}$?

3. A circle with unit radius is given. A system of line segments is called a cover iff each line with a common point with the circle also has some common point with some of the segments of the system.

(a) Prove that the sum of the lengths of the segments of a cover is more than 3,

(b) Does there exist a cover with this sum less than 5?

4. A natural number is written on the blackboard. Two players move alternatively. The first player's move consists of replacing the number n on the blackboard by $n/2$, by $n/4$ or by $3n$ (first two choices are allowed only if they are natural numbers). The second player's move consists of replacing the number n on the blackboard by $n + 1$ or by $n - 1$. The first player wants the number 3 to appear on the blackboard (no matter who writes it down). Can he always achieve his aim?

5. Three equal circles intersect at the point O and also two by two at the points A, B, C. Let T be the triangle whose sides are common tangents of the circles; T contains all the circles inside itself. Prove that the area of T is not less than 9 times the area of ABC.

2nd SELECTION ROUND

1. It is given that $0 \le x_i \le 1$, $i = 1, 2, \ldots, n$. Find the maximum of the expression

$$\frac{x_1}{x_2x_3\ldots x_n + 1} + \frac{x_2}{x_1x_3x_4\ldots x_n + 1} + \cdots + \frac{x_n}{x_1x_2\ldots x_{n-1} + 1}.$$

2. There are $2n$ points on the circle dividing it into $2n$ equal arcs. We must draw n chords having these points as endpoints so that the lengths of all chords are different. Is it possible if:

(a) $n = 24$,

(b) $n = 1994$?

3. A triangle ABC is given. From the vertex B, n rays are constructed intersecting the side AC. For each of the $n+1$ triangles obtained, an incircle with radius r_i and excircle (which touches the side AC) with radius R_i is constructed. Prove that the expression

$$\frac{r_1r_2\ldots r_{n+1}}{R_1R_2\ldots R_{n+1}}$$

depends on neither n nor on which rays are constructed.

3rd SELECTION ROUND

1. A square is divided into n^2 cells. Into some cells "1" or "2" is written so that there is exactly one "1" and exactly one "2" in each row and in each column. We are allowed to interchange two rows or two columns; this is called a move. Prove that there is a sequence of moves such that after performing it "1"-s and "2"-s have interchanged their positions.

2. Let a_{ij} be integers, $|a_{ij}| < 100$. We know that the equation

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz = 0$$

has a solution $(1234, 3456, 5678)$. Prove that this equation also has a solution with x, y, z pairwise relatively prime which is not proportional to the given one.

3. Let $ABCD$ be an inscribed quadrilateral. Its diagonals intersect at O. Let the midpoints of AB and CD be U and V. Prove that the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent.

Next we give five Klamkin Quickies. My thanks go to Murray Klamkin, the University of Alberta, for supplying them to us. Next issue we will give the "quick" solutions along with another five of his special teasers.

FIVE KLAMKIN QUICKIES
October 21, 1996

1. For $x, y, z > 0$, prove that

(i) $1 + \dfrac{1}{(x+1)} \geq \left\{1 + \dfrac{1}{x(x+2)}\right\}^x$,

(ii) $[(x+y)(x+z)]^x[(y+z)(y+x)]^y[(z+x)(z+y)]^z \geq [4xy]^x[4yz]^y[4zx]^z$.

2. If $ABCD$ is a quadrilateral inscribed in a circle, prove that the four lines joining each vertex to the nine point centre of the triangle formed by the other three vertices are concurrent.

3. How many six digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square?

4. Let V_iW_i, $i = 1, 2, 3, 4$, denote four cevians of a tetrahedron $V_1V_2V_3V_4$ which are concurrent at an interior point P of the tetrahedron. Prove that

$$PW_1 + PW_2 + PW_3 + PW_4 \leq \max V_iW_i \leq \textbf{longest edge.}$$

5. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d, respectively.

We now turn to solutions from the readers to problems posed in the May 1995 number of the corner on the Sixth Irish Mathematical Olympiad, May 8, 1993 [1995: 151–152].

SIXTH IRISH MATHEMATICAL OLYMPIAD
May 8, 1993 — First Paper
(Time: 3 hours)

1. The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0, \qquad \beta^3 - 3\beta^2 + 5\beta + 11 = 0.$$

Find $\alpha + \beta$.

Solutions by Šefket Arslanagić, Berlin, Germany; by Beatriz Margolis, Paris, France; by Vedula N. Murty, Andhra University, Visakhapatnam, India; by D.J. Smeenk, Zaltbommel, the Netherlands; by Panos E. Tsaoussoglou, Athens, Greece; and comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Margolis's solution.

Define $f(x) = x^3 - 3x^2 + 5x$. We show that if $f(\alpha) + f(\beta) = 6$, then $\alpha + \beta = 2$. Since $f(x) = (x-1)^3 + 2(x-1) + 3$, we have

$$\begin{aligned} f(\alpha) - 3 &= (\alpha-1)^3 + 2(\alpha - 1) \\ f(\beta) - 3 &= (\beta-1)^3 + 2(\beta - 1) \end{aligned}$$

Adding gives

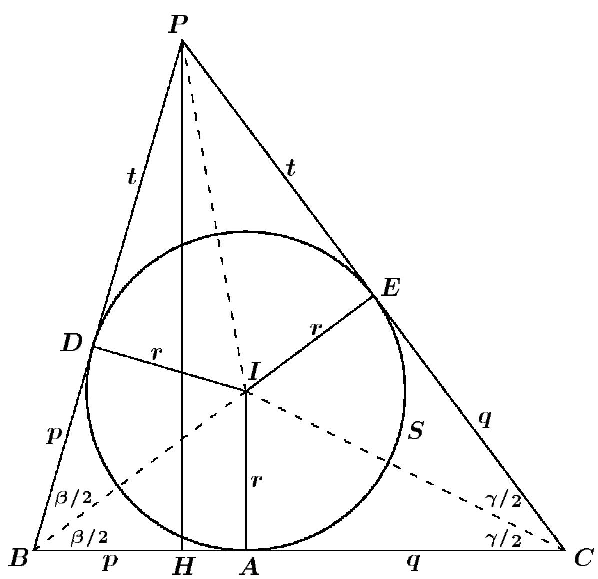
$$\begin{aligned} 0 &= (\alpha-1)^3 + (\beta-1)^3 + 2(\alpha+\beta-2) \\ &= (\alpha+\beta-2)[(\alpha-1)^2 + (\alpha-1)(\beta-1) + (\beta-1)^2 + 2] \end{aligned}$$

and, since the second factor is positive, we obtain the result. (See Olympiad Corner 142 and its solution.)

[*Wang's Comment:*] The problem is strikingly similar to problem 11.2 of the XXV Soviet Mathematical Olympiad, 11th Form [1993: 37]. The method used by Bradley given in the published solution [1994: 99] works for the present problem as well and, in fact, yields the same answer: $\alpha + \beta = 2$.

3. The line l is tangent to the circle S at the point A; B and C are points on l on opposite sides of A and the other tangents from B, C to S intersect at a point P. If B, C vary along l in such a way that the product $|AB| \cdot |AC|$ is constant, find the locus of P.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by D.J. Smeenk, Zaltbommel, the Netherlands. We use Smeenk's solution.



Let S be the incircle $s(I, 2)$ of $\triangle BCP$. We denote $\angle PBA = \beta$, $\angle PCA = \gamma$

$$\overline{AB} = p\overline{AC} = q$$

with $pq = k^2$, a constant.

Let S touch BP and CP at D and E respectively. For $\triangle PEI$ we have $\angle EIP = \frac{1}{2}(\beta+\gamma)$. Thus

$$t = r\tan\frac{1}{2}(\beta+\gamma) = \frac{(p+q)r^2}{pq-r^2}.$$

The semiperimeter of $\triangle BCP$ is

$$p+q+t = p+q+\frac{(p+q)r^2}{pq-r^2} = \frac{pq(p+q)}{pq-r^2}.$$

The area, F, of $\triangle BCP$ is

$$r\frac{pq(p+q)}{pq-r^2} = \frac{1}{2}(p+q)PH,$$

where PH is the altitude to BC. It follows immediately that

$$PH = \frac{2pqr}{pq-r^2} = \frac{2k^2r}{k^2-r}.$$

So the locus of P is a line parallel to BC.

4. Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $1/2^n$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ where the α_i denote the n real roots of f, $i = 1, 2, \ldots, n$. Then from $|f(0)| = f(1)$ we get $\prod(1-\alpha_i) = \prod \alpha_i$. (All products are over $i = 1, 2, \ldots, n$.) Using the Arithmetic Mean–Geometric Mean Inequality we then get

$$\left(\prod \alpha_i\right)^2 = \prod \alpha_i(1-\alpha_i) \;\leq\; \prod\left(\frac{\alpha_i + (1-\alpha_i)}{2}\right)^2 \;=\; \frac{1}{2^{2n}}$$

from which $\prod \alpha_i \leq \frac{1}{2^n}$ follows. Equality holds if and only if $\alpha_i = \frac{1}{2}$ for all $i = 1, 2, \ldots, n$.

May 8, 1993 — Second Paper

(Time: 3 hours)

3. For non-negative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty}\binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$$

for all integers n, r with $1 \leq r \leq n$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We use a combinatorial argument to establish the obviously equivalent identity

$$\sum_{d=1}^{k} \binom{n-r+1}{d}\binom{r-1}{r-d} = \binom{n}{r} \qquad (*)$$

where $k = \min\{r, n-r+1\}$. It clearly suffices to demonstrate that the left hand side of $(*)$ counts the number of ways of selecting r objects from n distinct objects (without replacements). Let $|S_2| = r-1$. For each fixed $d = 1, 2, \ldots, k$, any selection of d objects from S_1 $(S \backslash S_2)$ together with any selection of $r-d$ objections from S_2 would yield a selection of r objects from S. The total number of such selections is $\binom{n-r+1}{d}\binom{r-1}{r-d}$. Conversely, each selection of r objects from S clearly must arise in this manner. Summing over $d = 1, 2, \ldots,$ $(*)$ follows.

4. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Solutions by Šefket Arslanagić, Berlin, Germany; and by Vedula N. Murty, Andhra University, Visakhapatnam, India. We give Murty's solution.

We use mathematical induction. Let

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{(2k-1)}.$$

$S_1(x) = \sin x > 0$ for $x \in (0, \pi)$. Thus the proposed inequality is true for $n = 1$. Let $S_r(x) > 0$ for $r = 1, 2, \ldots, n-1$. We will deduce that $S_n(x) > 0$ for $x \in (0, \pi)$. Suppose that $S_n(x_0) \leq 0$ for some $x_0 \in (0, \pi)$, and that $S_n(x)$ attains its minimum at $x = x_0$. Hence $\frac{d}{dx}[S_n(x)]_{x=x_0} = 0$. That is

$$S_n'(x_0) = \sum_{k=1}^{n} \cos((2k-1)x_0) = 0,$$

so that

$$\begin{aligned} 2\sin x_0 S_n'(x_0) &= \sum_{k=1}^{n} 2\cos((2k-1)x_0)\sin x_0 \\ &= \sum_{k=1}^{n} [\sin(2kx_0) - \sin((2k-2)x_0)] \\ &= \sin 2nx_0. \end{aligned}$$

Thus $S_n'(x_0) = \frac{\sin 2nx_0}{2\sin x_0} = 0$ implying $\sin 2nx_0 = 0$. Hence

$$x_0 \in \left\{\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{(2n-1)}{2n}\right\}.$$

It is easily verified that at each of these values $S_n(x_0) > 0$, a contradiction. Hence $S_n(x) > 0$ for $x \in (0, \pi)$.

Editor's Note: Both solutions used the calculus. Does anyone have a more elementary solution?

We complete this number of the Corner with solutions by our readers to problems of the 1992 Dutch Mathematical Olympiad, Second Round given in the June 1995 number of the Corner [1995; 192–193].

1992 DUTCH MATHEMATICAL OLYMPIAD
Second Round
September 18, 1992

1. Four dice are thrown. What is the chance that the product of the numbers equals 36?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There are four different kinds of outcomes in which the product is 36: each of $\{1,1,6,6\}$ and $\{2,2,3,3\}$ can occur in $\frac{4!}{2!2!} = 6$ ways; $\{1,4,3,3\}$ can occur in $\frac{4!}{2!} = 12$ ways; and $\{1,2,3,6\}$ can occur in $4! = 24$ ways. Hence the probability that the product equals 36 is $\frac{48}{6^4} = \frac{1}{27}$.

2. In the fraction and its decimal notation (with period of length 4) every letter represents a digit. Different letters denote different digits. The numerator and denominator are mutually prime. Determine the value of the fraction:

$$\frac{ADA}{KOK} = .SNELSNELSNELSNEL\dots$$

[*Note*. ADA KOK is a famous Dutch swimmer. She won gold in the 1968 Olympic Games in Mexico. SNEL is Dutch for FAST.]

Solution by the Editor.

Let $x = \frac{ADA}{KOK} = .\overline{SNEL}$. Then $10^4 x = SNEL.SNELSNEL\dots$ and $10^4 x - x = SNEL$. So

$$x = \frac{SNEL}{9999} = \frac{SNEL}{11 \times 909} = \frac{SNEL}{33 \times 303} = \frac{SNEL}{99 \times 101}.$$

Taking $KOK = 909$ we obtain $SNEL = 11 \times ADA$, and $A = L$, which is impossible.

Taking $KOK = 303$, we obtain $SNEL = 33 \times ADA$, so $3A < 10$, as the product has four digits and $3A = L$. Because $S \neq L \neq 0$, $3 \leq 3A < 9$, giving $A = 1$ or $A = 2$. Now $A = 1$ gives $L = 3 = K$, which is impossible, so $A = 2$. This gives $L = 6$, and $D \geq 2$, so there is a carry. This gives $D \geq 4$, as $A = 2$, $K = 3$.

For $D = 4$, $\frac{ADA}{KOK} = .SNEL$ is $\frac{242}{303} = \overline{.7986}$, a solution.

For $D = 5$, $ADA = 252$ is not coprime to $KOK = 303$.

For $D = 6$, $SNEL = 8646$ and $N = L$.

For $D = 7$, $SNEL = 8976$ and $D = E$.

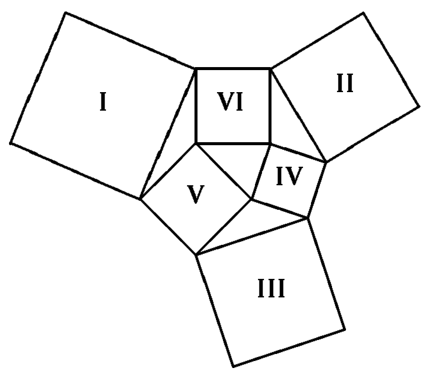
For $D = 8$, $SNEL = 9036$ and $O = N$.

For $D = 9$, $SNEL = 9636$ and $N = L$.

Taking $KOK = 101$ gives $SNEL = 99 \times ADA$ forcing $A = 1$ for $SNEL$ to have four digits, but then $A = K$.

Thus the only solution is $\frac{242}{303} = \overline{.7986}$.

3. The vertices of six squares coincide in such a way that they enclose triangles; see the picture. Prove that the sum of the areas of the three outer squares (I, II and III) equals three times the sum of the areas of the three inner squares (IV, V and VI).



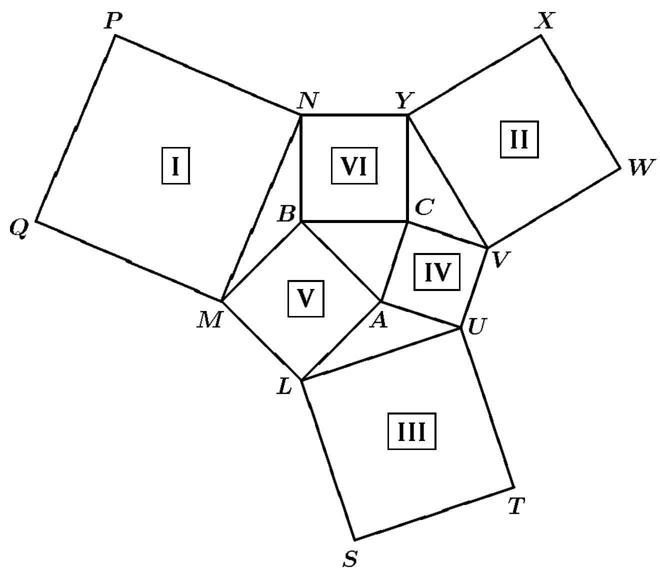
Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Vedula N. Murty, Andhra University, Visakhapatnam, India. We give Murty's solution.

Let the figure on the next page be labelled as shown:

Let $MN = x_1$, $LU = x_3$, $UY = x_2$, $AC = x_4$, $AB = x_5$, $BC = x_6$, $\angle MBN = \alpha$, $\angle LAU = \beta$, $\angle VCY = \gamma$, $\angle BAC = A$, $\angle ACB = C$ and $\angle ABC = B$.

Then we have $\alpha + \beta = \pi$, $\beta + A = \pi$, $\gamma + C = \pi$

$$\left.\begin{aligned} x_1^2 &= x_6^2 + x_5^2 - 2x_5x_6\cos\alpha \\ x_2^2 &= x_4^2 + x_6^2 - 2x_4x_6\cos\gamma \\ x_3^2 &= x_4^2 + x_5^2 - 2x_4x_5\cos\beta \end{aligned}\right\}\ldots \qquad (1)$$



$$\left.\begin{aligned} x_4^2 &= x_5^2 + x_6^2 - 2x_5x_6\cos B \\ x_4^2 &= x_4^2 + x_6^2 - 2x_4x_6\cos C \\ x_6^2 &= x_4^2 + x_5^2 - 2x_4x_6\cos A \end{aligned}\right\}\ldots \qquad (2)$$

From (2), we have

$$\begin{aligned} x_4^2 + x_5^2 + x_6^2 &= 2x_4x_5\cos A + 2x_5x_6\cos B + 2x_4x_6\cos C \\ &= -2x_4x_5\cos\beta - 2x_5x_6\cos\alpha - 2x_4x_6\cos\gamma \quad \ldots (3) \end{aligned}$$

From (1), we have

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 2(x_4^2 + x_5^2 + x_6^2) - 2x_5x_6\cos\alpha \\ &\quad -2x_4x_5\cos\beta - 2x_4x_6\cos\gamma \end{aligned}$$

using (3)

$$\begin{aligned} &= 2(x_4^2 + x_5^2 + x_6^2) + x_4^2 + x_5^2 + x_6^2 \\ &= 3(x_4^2 + x_5^2 + x_6^2). \end{aligned}$$

That is, Area of $(I + II + III) = 3$ Area of $(IV + V + VI)$.

4. For every positive integer n, $n?$ is defined as follows:

$$n? = \begin{cases} 1 & for\ n = 1 \\ \frac{n}{(n-1)?} & for\ n \geq 2 \end{cases}$$

Prove $\sqrt{1992} < 1992? < \frac{4}{3}\sqrt{1992}$.

Solutions by Vedula N. Murty, Andhra University, Visakhapatnam, India; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution and his remark.

Using the more convenient notation $f(n)$ for $n?$, we show that in general

$$\sqrt{n+1} < f(n) < \frac{4}{3}\sqrt{n} \qquad (*)$$

for all **even** $n \geq 6$. In particular, for $n = 1992$, we would get $\sqrt{1993} < f(1992) < \frac{4}{3}\sqrt{1992}$.

First note that $f(n) = \frac{n}{f(n-1)} = \frac{n}{n-1} f(n-2)$ for all $n \geq 3$. If $N = 2k$ where $k \geq 2$, then multiplying $f(2q) = \frac{2q}{2q-1} f(2q-2)$ for $q = 2, 3, \ldots, k$, we get

$$\begin{aligned} f(2k) &= \frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2k}{2k-1}\cdot f(2) \\ &= \left(\frac{2}{1}\right)\cdot\left(\frac{4}{3}\right)\cdot\left(\frac{6}{5}\right)\cdots\left(\frac{2k}{2k-1}\right) > \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\cdots\left(\frac{2k+1}{2k}\right). \end{aligned}$$

Hence

$$(f(2k))^2 > \frac{2\cdot 4\cdot 6\cdots 2k}{1\cdot 3\cdot 5\cdots(2k-1)}\cdot\frac{3\cdot 5\cdot 7\cdots(2k+1)}{2\cdot 4\cdot 6\cdots 2k} = 2k+1,$$

from which it follows that

$$f(n) = f(2k) > \sqrt{2k+1} = \sqrt{n+1}. \qquad (1)$$

On the other hand, for $k \geq 3$ we have

$$\begin{aligned} 2(2k) &= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{2k-2}{2k-1}\right)\cdot 2k \\ &< \left(\frac{2}{3}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\cdots\left(\frac{2k-1}{2k}\right) 2k. \end{aligned}$$

Hence

$$\begin{aligned} (f(2k))^2 &< \left(\frac{2}{3}\right)^2\cdot\frac{4\cdot 6\cdots(2k-2)}{5\cdot 7\cdots(2k-1)}\cdot\frac{5\cdot 7\cdots(2k-1)}{6\cdot 8\cdots 2k}\cdot(2k)^2 \\ &= \left(\frac{2}{3}\right)^2\cdot 4\cdot 2k, \end{aligned}$$

from which it follows that

$$f(n) = f(2k) < \frac{4}{3}\sqrt{2k} = \frac{4}{3}\sqrt{n}. \qquad (2)$$

The result follows from (1) and (2).

Remark: Using similar arguments, upper and lower bounds for $f(n)$ when n is odd can also be easily derived. In fact, if we set $P = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}$ (usually denoted by $\frac{(2k-1)!!}{(2k)!!}$) then various upper and lower bounds for P abound in the literature; for example, it is known that

$$\frac{1}{2}\sqrt{\frac{5}{4k+1}} \leq P \leq \frac{1}{2}\sqrt{\frac{3}{2k+1}}$$

and

$$\frac{1}{\sqrt{(n+\frac{1}{2})\pi}} < P \leq \frac{1}{\sqrt{n\pi}}.$$

(Compare, for example, §3.1.16 on p. 192 of *Analytic Inequalities* by D.S. Mitronović.)

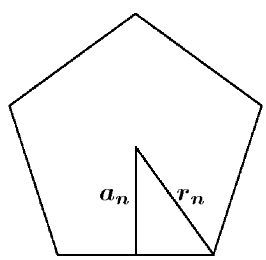
Clearly, each pair of these double inequalities would yield corresponding upper and lower bounds for the function $f(n)$ considered in the given problem.

5. We consider regular n-gons with a fixed circumference 4. We call the distance from the centre of such a n-gon to a vertex r_n and the distance from the centre to an edge a_n.

a) Determine a_4, r_4, a_8, r_8.

b) Give an appropriate interpretation for a_2 and r_2.

c) Prove: $a_{2n} = \frac{1}{2}(a_n + r_n)$ and $r_{2n} = \sqrt{a_{2n} r_n}$.



Let $u_0, u_1, u_2, u_3, \ldots$ be defined as follows:

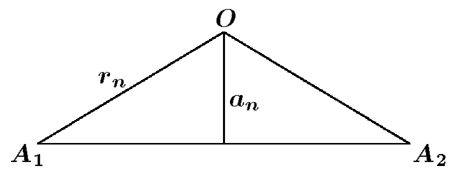
$$u_0 = 0,\ u_1 = 1;\ u_n = \frac{1}{2}(u_{n-2} + u_{n-1}) \text{ for } n \text{ even and}$$

$$u_n = \sqrt{u_{n-2}\cdot u_{n-1}} \text{ for } n \text{ odd.}$$

d) Determine: $\lim_{n\to\infty} u_n$.

Solution by Vedula N. Murty, Andhra University, Visakhapatnam, India.

Let O be the centre of the regular n-gon. Let A_1A_2 denote one side of the regular n-gon



Then we have $\angle A_1OA_2 = \frac{2\pi}{n}$, $\angle OA_1A_2 = \angle OA_2A_1 = \frac{\pi}{2} - \frac{\pi}{n}$. Thus

$$\begin{aligned}|\overrightarrow{A_1A_2}| &= \sqrt{r_n^2 + r_n^2 - 2r_n^2 \cos\frac{2\pi}{n}} \\ &= \sqrt{2r_n^2(1-\cos\frac{2\pi}{n})} \\ &= \sqrt{4r_n^2 \sin^2\frac{\pi}{n}} = 2r_n \sin\frac{\pi}{n}.\end{aligned}$$

The circumference of the regular n-gon is $2nr_n \sin\frac{\pi}{n} = 4$ whence

$$r_n = \frac{2}{n\sin\frac{\pi}{n}},$$

$$a_n = r_n \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = r_n \cos\frac{\pi}{n} = \frac{2}{n}\cot\frac{\pi}{n}.$$

In particular

$$r_4 = \frac{1}{2}\frac{1}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{2}, \qquad a_4 = \frac{2}{4}\cot\frac{\pi}{4} = \frac{1}{2},$$

$$r_8 = \frac{2}{8\sin\frac{\pi}{8}} = \frac{1}{4\sin\frac{\pi}{8}}.$$

Now, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 1 - 2\sin^2\frac{\pi}{8}$ gives

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}},$$

so

$$r_8 = \frac{1}{4}\frac{2}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\cdot\frac{1}{\sqrt{2-\sqrt{2}}},$$

and

$$a_8 = r_8\cos\frac{\pi}{8} = \frac{1}{4}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \frac{1}{4}\frac{1}{2-\sqrt{2}}\sqrt{2},$$

since $\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$.

For (b), $r_2 = 1$, $a_2 = 0$ as the 2-gon is a straight line with O lying at the middle of A_1 and A_2.

For (c), we have

$$\begin{aligned} a_n + r_n &= r_n\left(1+\cos\frac{\pi}{n}\right) = 2r_n\cos^2\frac{\pi}{2n} \\ &= \frac{4}{n\sin\frac{\pi}{n}}\cos^2\frac{\pi}{2n} \\ &= \frac{4}{2n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\cos^2\frac{\pi}{2n} \;=\; \frac{2}{n}\cot\frac{\pi}{2n}. \end{aligned}$$

Thus $\frac{1}{2}(a_n + r_n) = \frac{1}{n}\cot(\frac{\pi}{2n}) = a_{2n}$, and

$$a_{2n}r_n = \frac{1}{n}\,\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}\cdot\frac{2}{n\sin\frac{\pi}{n}} = \frac{1}{n^2}\,\frac{\cos\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{1}{n^2\sin^2\frac{\pi}{2n}},$$

so $\sqrt{a_{2n}r_n} = \frac{1}{n\sin\frac{\pi}{2n}} = r_{2n}$.

For (d), note $u_0 = 0$, $u_1 = 1$, and $u_2 = \frac{1}{2}$. For $n \geq 2$ we have that u_n is either the arithmetic or geometric mean of u_{n-1} and u_{n-2} and in either case lies between them. It is also easy to show by induction that $u_0, u_2, u_4, \ldots$ form an increasing sequence, and $u_1, u_3, u_5, \ldots$ form a decreasing sequence with $u_{2l} \leq u_{2s+1}$ for all $l, s \geq 0$. Let $\lim_{k\to\infty} u_{2k} = P$ and $\lim_{k\to\infty} u_{2k+1} = I$. Then $P \leq I$. We also have from $u_{2n} = \frac{1}{2}(u_{2n-1} + u_{2n-2})$ that $P = \frac{1}{2}(I + P)$ so that $I = P$ and $\lim_{n\to\infty} u_n$ exists. Let $\lim_{n\to\infty} u_n = L$.

With $a_2 = 0$ and $r_2 = 1$, let $\overline{u}_{2k} = a_{2^{k+1}}$ and $\overline{u}_{2k+1} = r_{2^{k+1}}$, for $k = 0, 1, 2, \ldots$. From (c), $\overline{u}_0 = a_{2^1} = a_2 = 0$ and $\overline{u}_1 = r_{2^1} = r_2 = 1$. Also for $n = 2k+2$, $\overline{u}_{2k+2} = a_{2^{k+1+1}} = a_{2\cdot 2^{k+1}} = \frac{1}{2}(a_{2^{k+1}} + b_{2^{k+1}}) = \frac{1}{2}(\overline{u}_{2k} + \overline{u}_{2k+1})$; that is $\overline{u}_n = \frac{1}{2}(\overline{u}_{n-2} + \overline{u}_{n-1})$ and for $n = 2k+3$

$$\begin{aligned} \overline{u}_{2k+3} &= \overline{u}_{2(k+1)+1} = r_{2^{k+1+1}} = r_{2(2^{k+1})} \\ &= \sqrt{a_{2(2^{k+1})}\cdot r_{2^{k+1}}} = \sqrt{a_{2^{k+1+1}}\cdot r_{2^{k+1}}} \\ &= \sqrt{\overline{u}_{2(k+1)}\cdot\overline{u}_{2k+1}} \end{aligned}$$

so $\overline{u}_n = \sqrt{\overline{u}_{n-1}\cdot\overline{u}_{n-2}}$. Thus u_n and $\overline{u}_n$ satisfy the same recurrence and it follows that $L = \lim_{k\to\infty} a_{2^{k+1}} = \lim_{k\to\infty} r_{2^{k+1}}$. Now, from the solution to (c),

$$r_n = \frac{2}{n\sin\frac{\pi}{n}} = \frac{2}{\pi}\,\frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}},$$

so $\lim_{n\to\infty} r_n = \frac{2}{\pi}$ since $\frac{\pi}{n} \to 0$. Therefore $\lim_{n\to\infty} u_n = \frac{2}{\pi}$.

That completes the column for this issue. Olympiad season is approaching. Send me your contests and nice solutions.

BOOK REVIEWS

Edited by ANDY LIU

Shaking Hands in Corner Brook and other Math Problems,
edited by Peter Booth, Bruce Shawyer and John Grant McLoughlin,
published by the Waterloo Mathematics Foundation, Waterloo, 1995,
153 pages, paperback, ISBN 0-921418-31-0.
Reviewed by ***Robert Geretschläger*** and ***Gottfried Perz***.

- One representative from each of six regions met in Corner Brook, Newfoundland, to discuss math problems. Each of these delegates shook hands with each other delegate. How many handshakes were there?

This is the first of many problems that can be found in this collection of problems from the Newfoundland and Labrador Teachers' Association (NLTA) Senior Mathematics League, which conveniently offers an explanation as to the whereabouts of Corner Brook.

The NLTA Math League was started in 1987, and has since developed into a very interesting competition at the regional level. A number of aspects make this competition different from most math competitions. First of all, it is purely a team competition, with four students from each participating school comprising a team. There is no individual ranking, and so students are motivated to work together at finding solutions. For each of ten questions posed, a team can receive five points for a correct team answer. If the members of a team cannot agree on the correct answer, they can submit individual answers, for which their team can get one point each, if correct. Finally, there is a relay question, made up of four parts. In the relay, each part yields an answer, which is necessary to be able to solve the next part (much as in the American Regions Mathematics League (ARML), which may be better known to many readers). The relay section can yield a maximum of 15 points (made up of five points for the solution and extra points for solving the problems in a short time), for a possible total of 65 points. If teams end up with the same point sum, a tie breaker question is posed.

The concept behind this competition is geared to fostering cooperative problem solving, something that is generally ignored in olympiad-style competitions. The level of difficulty of the problems posed is adequate to the time allowed (usually from 3 to 10 minutes per question) and the intentions of the competition, and ranges from fairly easy to pre-olympiad level. The book is divided into sections covering regular questions, relay questions, tie breakers and solutions. The problems are in a random order, and no indication is given of which questions were posed at which competition. Perhaps at least one example of ten specific questions posed at one competition, and the order they were posed in, might have been of interest.

Here are a few problems to whet your appetite:

- How many three digit numbers include at least one seven but have no zeros?
- Al, Betty, Charles, Darlene and Elaine play a game in which each is either a frog or a moose. A frog's statement is always false while a moose's statement is always true.

  Al says that Betty is a moose.
  Charles says that Darlene is a frog.
  Elaine says that Al is not a frog.
  Betty says that Charles is not a moose.
  Darlene says that Elaine and Al are different kinds of animals.

  How many frogs are there?
- Triangle ABC is isosceles, with $\angle ABC = \angle ACB$. There are points D, E and F on BC, CA and AB, respectively, that form an equilateral triangle. Given that $\angle AFE = x^\circ$ and $\angle CED = y^\circ$, calculate $\angle BDF$ in terms of x and y.

The book has a very pleasing layout, with the cover showing the densest packing of seven circles in an equilateral triangle. The solutions are nicely presented, and in several cases, alternate solutions are given, occasionally labeled the "routine way" and the "subtle" or "smart way".

Shaking Hands in Corner Brook should be of interest to anyone involved with high school mathematics, either in competitions, or simply seeking enrichment material for the interested student.

Copies of the above reviewed book may be obtained from:

Canadian Mathematics Competition
Faculty of Mathematics, University of Waterloo
Waterloo, Ontario, Canada. N2L 3G1

The cost is $12 in Canadian funds (plus 7% GST for shipping to Canadian addresses). Cheques or money orders in Canadian funds should be made payable to: Canadian Mathematics Competition. All profits from the sale of this book are for the Newfoundland Mathematics Prizes Fund.

Teachers interested in providing a lively and stimulating high school mathematics competition for their students may be interested in participating in a NLTA Senior Mathematics League in their own area. Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario, a regular contributor to ***CRUX***, is already a participant. Meetings can be held within individual schools, or between teams from more than one school.

Sample games and other information on how the NLTA Senior Mathematics League is organised may be obtained free from:

Dr. Peter Booth
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland, Canada. A1C 5S7

Tel: int+ 709–737–8786
Fax: int+ 709–737–3010
email: pbooth@fermat.math.mun.ca

Schools that participate on a regular basis will be sent questions and detailed solutions five times per year (October, November, February, March and May). There is an annual fee of $50 (Canadian funds) for each group of schools participating. Cheques or money orders should be made payable to Newfoundland Mathematics Prizes Fund.

A Probabilistic Approach to Determinants with Integer Entries

Theodore Chronis

It is well known that the probability for an integer number to be odd is equal to the probability for the number to be even. What about determinants? What is the probability for a rectangular matrix with integer entries to have odd determinant? More generally, if m is a natural number, what is the probability for which $\det A \equiv m_i \,\mathbf{mod}\, m$, where m_i is chosen from the set $\{0, 1, 2, \dots, m-1\}$?

I have the following problem to propose; I hope you will find it interesting.

Let A be an $n \times n$ matrix whose elements are integers. What is the probability the determinant of A is an odd number?

Solution:

Let $A = [a_{ij}]$, $i, j = 1, \dots, n$. It is obvious that

$$\det A \equiv \det\left([a_{ij} \,\mathbf{mod}\, 2]\right) \,\mathbf{mod}\, 2.$$

So the problem is to find the probability that the determinant of an $n \times n$ matrix with elements from the set $\{0, 1\}$ is an odd number. Let A_n be an $n \times n$ matrix with elements from the set $\{0, 1\}$. Let also $N(\det A_n)$ be the number of odd $n \times n$ determinants, and $P(\det A_n)$ be the corresponding probability. Let

$$f(K = \{k_1, k_2, \dots, k_n\}) = N\left(\begin{vmatrix} k_1 & k_2 & \dots & \dots & k_n \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{vmatrix}\right),$$

where $k_1, k_2, \dots, k_n \in \{0, 1\}$ are fixed.

Then $N(\det A_n) = \sum(f(K))$, where the sum is calculated for all the $2^n - 1$ possible permutations $K = \{k_1, k_2 \dots, k_n\}$ with $k_i \in \{0, 1\}$, $i = 1, \dots, n$ and $k_1, k_2, \dots, k_n$ not all zero. (Note that $f(0, 0, \dots, 0) = 0$.)

Lemma: $f(K) = 2^{n-1}N(\det A_{n-1})$, where $k_1 + k_2 + \ldots + k_n \neq 0$.

Proof. It is well known that a determinant remains unchanged if from the elements of one of its columns we substract the corresponding elements of another column. It is also obvious that the same is true for $N(\det A_n)$. Additionally, a determinant just changes sign if we interchange two columns, while $N(\det A_n)$ remains unchanged. So

$$N\left(\begin{vmatrix} k_1 & k_2 & .. & .. & k_n \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{vmatrix}\right) = N\left(\begin{vmatrix} 1 & 0 & .. & .. & 0 \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{vmatrix}\right)$$

$$\iff f(K) = 2^{n-1}N(\det A_{n-1}).$$

Hence $N(\det A_n) = 2^{n-1}N(\det A_{n-1})(2^n - 1)$.
Of course $N(\det A_1) = 1$ and so

$$N(\det A_n) = 2^{n(n-1)/2}\prod_{i=1}^{n}(2^i - 1).$$

Finally, $P(\det A_n) = \frac{N(\det A_n)}{2^{n^2}} \iff P(\det A_n) = \prod_{i=1}^{n}(1 - 2^{-i})$.

Note: The infinite sequence $\prod_{i=1}^{n}(1 - 2^{-i}),\ n = 1, 2, \ldots$ is decreasing and bounded below by 0, so $\lim_{n\to\infty} P(\det A_n)$ exists. Using **Mathematica**, we found that

$$\lim_{n\to\infty} P(\det A_n) \cong 0.2887880950866024212788997219290\ldots.$$

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THE SKOLIAD CORNER

No. 19

R.E. Woodrow

The problem set we give in this issue comes to us with our thanks from Tony Gardiner of the UK Mathematics Foundation, School of Mathematics, University of Birmingham. The Nat West Junior Mathematical Challenge was written Tuesday, April 26, 1994 by about 105,000 students. Students from England and Wales must be in school year 8 or below. The use of calculators, calendars, rulers, and measuring instruments was forbidden.

1994 NAT WEST UK JUNIOR MATHEMATICAL CHALLENGE

Tuesday, April 26, 1994 — Time: 1 hour

1. $38 + 47 + 56 + 65 + 74 + 83 + 92$ equals

A. 425 **B.** 435 **C.** 445 **D.** 456 **E.** 465.

2. What is the largest possible number of people at a party if no two of them have birthdays in the same month?

A. 11 **B.** 12 **C.** 13 **D.** 23 **E.** 334.

3. I have $500 in 5p coins. How many 5p coins is that?

A. 100 **B.** 500 **C.** 1000 **D.** 2500 **E.** 10000.

4. What was the precise date exactly sixty days ago today? (No calendars!)

A. Friday 25th February **B.** Saturday 26th February

C. Friday 26th February **D.** Saturday 27th February

E. Tuesday 26th February.

5. You have to find a route from A to B moving horizontally and vertically only, from one square to an adjacent square. Each time you enter a square you add the number in that square to your total. What is the lowest possible total score for a route from A to B?

3	9	B
8	5	6
9	11	7
A	8	10

A. 28 **B.** 29 **C.** 30 **D.** 31 **E.** 34.

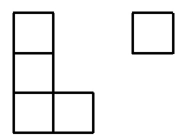
6. On a clock face, how big is the angle between the lines joining the centre to the 2 O'clock and the 7 O'clock marks?

A. 160° **B.** 150° **C.** 140° **D.** 130° **E.** 120°.

7. Gill is just six and boasts that she can count up to 100. However, she often mixes up nineteen and ninety, and so jumps straight from nineteen to ninety one. How many numbers does she miss out when she does this?

A. 70 **B.** 71 **C.** 72 **D.** 78 **E.** 89.

8. In how many ways can you join the two shapes shown here to make a figure with a line of symmetry?



A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4.

9. If you divide 98765432 by 8, which non-zero digit does not appear in your answer?

A. 2 **B.** 4 **C.** 6 **D.** 8 **E.** 9.

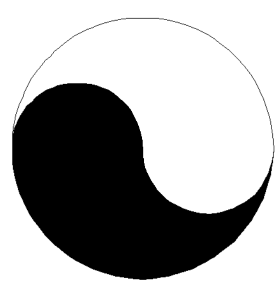
10. How many numbers between 20 and 30 (inclusive) cannot be written as a multiple of 5, or as a multiple of 7, or as the sum of a multiple of 5 and a multiple of 7?

A. 1 **B.** 2 **C.** 3 **D.** 5 **E.** 6.

11. Four children are arguing over a broken toy. Alex says Barbara broke it. Barbara says Claire broke it. Claire and David say they do not know who broke it. Only the guilty child was lying. Who broke the toy?

A. Alex **B.** Barbara **C.** Claire **D.** David **E.** can't be sure.

12. The diagram is made up of one circle and two semicircles. Which of the three regions — the black region, the white region and the large circle — has the longest perimeter?



A. the black region **B.** the circle **C.** the white region
D. black and white are equal and longest
E. all three perimeters are equal.

13. From my house the church spire is in the direction NNE. If I face in this direction and then turn anticlockwise through 135° I can see the Town Hall clock. In which direction am I then facing?

A. WSW **B.** due West **C.** SW **D.** due South **E.** SSE.

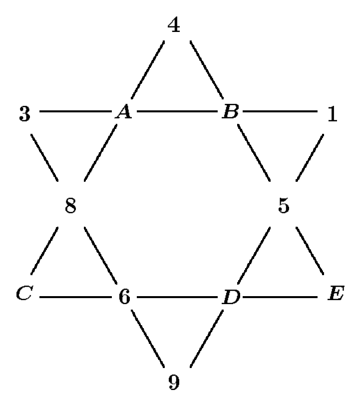
14. Samantha bought seven super strawberry swizzles and ten tongue twisting toffees for \$1.43. Sharanpal bought five super strawberry swizzles and ten tongue twisting toffees for \$1.25. How much is one tongue twisting toffee?

A. 7p **B.** 8p **C.** 9p **D.** 10p **E.** 18p.

15. I am forty eight years, forty eight months, forty eight weeks, forty eight days and forty eight hours old. How old am I?

A. 48 **B.** 50 **C.** 51 **D.** 52 **E.** 53.

16. The numbers 1 to 12 are to be placed so that the sum of the four numbers in each of the six rows is the same. Where must the 7 go?



A. at A **B.** at B **C.** at C **D.** at D **E.** at E

17. Three hedgehogs — Roland, Spike and Percival — have a leaf collecting race. Roland collects twice as many as Percival, who collects one and a half times as many as Spike. (Spike is moulting, and so has fewer prickles for her to stick the leaves onto.) Between them they collect 198 leaves. How many did Spike manage to collect?

A. 18 **B.** 22 **C.** 36 **D.** 44 **E.** 66.

18. The four digits $1, 2, 3, 4$ are writtin in increasing order. You must insert one plus sign and one minus sign between the 1 and the 2, or between the 2 and the 3, or between the 3 and the 4, to produce expressions with different answers. For example,

$$1 - 23 + 4 \quad \text{gives the answer} - 18.$$

How many different *positive* answers can be obtained in this way?

A. 2 **B.** 3 **C.** 4 **D.** 5 **E.** 6.

19. Roger Rabbit has twice as many sisters as brothers. His sister Raquel notices that $2/5$ of her brothers and sisters are boys. How many Rabbit children are there in the family?

A. 2 **B.** 4 **C.** 8 **D.** 16 **E.** 32.

20. The population of a new town in 1990 was $10,000$. It has since doubled every year. If it kept on doubling every year for ten years, what would its population be in the year 2000?

A. $100,000$ **B.** $200,000$ **C.** $1,000,000$ **D.** $2,000,000$ **E.** $10,000,000$.

21. $LMNO$ is a square. P is a point inside the square such that NOP is an equilateral triangle. How big is the angle PMN?

A. $75°$ **B.** $70°$ **C.** $60°$ **D.** $45°$ **E.** $30°$.

22. If the perimeter of a rectangle is $16x + 18$ and its width is $2x + 6$, what is its length?

A. $18x + 24$ **B.** $7x + 6$ **C.** $12x + 6$ **D.** $6x + 3$ **E.** $14x + 12$.

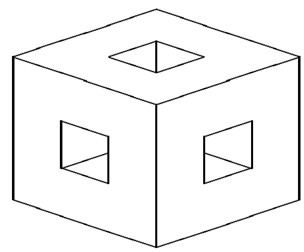
23. In a group of fifty girls each one is either blonde or brunette and is either blue-eyed or brown-eyed. Fourteen are blue-eyed blondes, thirty one are brunettes and eighteen are brown-eyed. How many are brown-eyed brunettes?

A. 5 **B.** 7 **C.** 9 **D.** 13 **E.** 18.

24. A bottle of *Jungle Monster Crush (JMC)* makes enough drink to fill sixty glasses when it is diluted in the ratio 1 part *Crush* to 4 parts water. How many glasses of drink would a bottle of *JMC* make if it is diluted in the ratio 1 part *Crush* to 5 parts water?

A. 48 **B.** 60 **C.** 72 **D.** 75 **E.** 80.

25. A 3 by 3 by 3 cube has three holes, each with a 1 by 1 cross section running from the centre of each face to the centre of the opposite face. What is the total surface area of the resulting solid?



A. 24 **B.** 48 **C.** 72 **D.** 78 **E.** 84.

That completes the Skoliad Corner for this issue. Send me your contests, suggestions, and recommendations to improve this feature.

Introducing the new Associate Editor-in-Chief

For those of you who do not know Clayton, here is a short profile:

Born:	Haystack, Newfoundland [1]
Educated	Haystack School School, Newfoundland Thornlea School, Newfoundland Memorial University of Newfoundland Queen's University, Kingston, Ontario
Employment	Random South School Board, Trinity Bay, Newfoundland Memorial University of Newfoundland
Mathematical Interests	Mathematical Education Commutative Algebra

[1] *You may have difficulty finding Haystack on a map of Newfoundland — it is not lost — it no longer exists*!

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 1A1. The electronic address is

`mayhem@math.toronto.edu`

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Editorial

It gives me great pleasure to unveil the premiere issue of "***Crux Mathematicorum with Mathematical Mayhem***". This merger has been in the works for quite some time, and it has finally been successfully realized.

For the benefit of those who have not heard of ***Mayhem***, I will provide a brief description. ***Mayhem*** was founded in 1988 by two high school students Ravi Vakil and Patrick Surry, who wished to establish a journal specifically oriented towards students, and totally operated by students. Although the journal has been passed down through many hands, and though it has not always been easy, this mandate has always been resolutely upheld; it has made ***Mayhem*** a unique and exceptional journal. And rest assured, we will still be running our share of "***Crux with Mayhem***".

Our features include articles, olympiads, and a problems section. The material is generally focused towards contests and olympiads, and how to prepare for them, and the topics range from high school mathematics to undergraduate material. We cannot emphasize enough that we are a journal dedicated to mathematics students. I myself am a fourth-year student at the University of Toronto, and Cyrus Hsia (the ***Mayhem*** Assistant Editor) is a third-year student, also at the University of Toronto.

Our fearless staff also consists of undergraduate and high school students.

This brings me to my next point. After considerable discussion, "***Mayhem***" has decided to restrict itself to publishing solutions only from students. The rationale behind this move is that the ***Crux*** problems already draw many solutions, and if the same people were to respond to our problems, which are considerably easier, it would simply overwhelm the section. We know that there are many non-students who have contributed to the problems sections over the years, who have our full gratitude, and hope they understand our position. We are, however, prepared to make exceptions in, well, exceptional cases.

However, we warmly welcome submissions for articles and problems from all people. Back issues are available; the information is inside the back cover. Any correspondence about ***Mayhem*** should be sent to ***Mathematical Mayhem***, c/o Naoki Sato, Department of Mathematics, University of Toronto, M5S 1A1, or at the e-mail address `<mayhem@math.toronto.edu>`. Subscriptions, however, should be sent to the Canadian Mathematical Society offices, as mentioned on the inside back cover.

Well, I think that's about it. For people who have subscribed to ***Mayhem***, welcome back, I know it's been a long wait. I look forward to working with Bruce Shawyer on our new project (I think he will bring a certain "discipline" to ***Mayhem***, but we will resist it as much as possible. Don't tell him I said that.) Here's to a new year and a new era for ***Mayhem***.

Naoki Sato
Mayhem Editor

Shreds and Slices

Positive Matrices and Positive Eigenvalues

Theorem. An $n \times n$ matrix M with positive real entries has at least one positive eigenvalue.

Proof. Let $S = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \geq 0, x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$; that is, S is the portion of the unit sphere in $\mathbb{R}^n$ with all coordinates non-negative.

Define the map $f : S \to S$ by $f(\vec{v}) = \frac{M\vec{v}}{|M\vec{v}|}$. Since M has all positive real entries, for any $\vec{v} \in S$, $M\vec{v}$ also has all non-negative coordinates, so f is well-defined, and does indeed map S into S.

Note that S is a closed, simply connected set. Then by Brouwer's Fixed Point Theorem, there is a fixed point of f; that is, for some $\vec{v} \in S$, $f(\vec{v}) = \vec{v} = \frac{M\vec{v}}{|M\vec{v}|} \Rightarrow M\vec{v} = \lambda\vec{v}$, for some positive value λ (since λ cannot be zero), namely $\lambda = |M\vec{v}|$. This λ is a positive eigenvalue of M.

Newton's Relations

Given n reals $a_1, a_2, \ldots, a_n$, let S_k be the sum of the products of the a_i taken k at a time, and let $P_k = a_1^k + a_2^k + \cdots + a_n^k$. Consider the generating functions

$$\begin{aligned} & S_0 + S_1x + S_2x^2 + \cdots + S_nx^n \\ & \quad = 1 + (a_1 + a_2 + \cdots + a_n)x + (a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n)x^2 \\ & \qquad + \cdots + a_1a_2\cdots a_nx^n \\ & \quad = (1 + a_1x)(1 + a_2x)\cdots(1 + a_nx) \end{aligned}$$

and

$$\begin{aligned} & P_0 - P_1x + P_2x^2 - \cdots \\ & \quad = (1 - a_1x + a_1^2x^2 - \cdots) + (1 - a_2x + a_2^2x^2 - \cdots) \\ & \qquad + \cdots + (1 - a_nx + a_n^2x^2 - \cdots) \\ & \quad = \frac{1}{1 + a_1x} + \frac{1}{1 + a_2x} + \cdots + \frac{1}{1 + a_nx}. \end{aligned}$$

Their product is

$$\begin{aligned} & (n - P_1x + P_2x^2 - \cdots)(1 + S_1x + S_2x^2 + \cdots) \\ & \quad = \ (1 + a_1x)(1 + a_2x)\cdots(1 + a_nx) \\ & \qquad \times \left(\frac{1}{1 + a_1x} + \frac{1}{1 + a_2x} + \cdots + \frac{1}{1 + a_nx}\right), \end{aligned}$$

since $P_0 = n$ and $S_0 = 1$.

We claim the expression is equal to $n + (n-1)S_1x + (n-2)S_2x^2 + \cdots + S_{n-1}x^{n-1}$. To see this, consider the coefficient of x^k. Since the expression is symmetric, the coefficient is some multiple of S_k. How many times does the term $a_1a_2\cdots a_kx^k$ appear? It must have appeared in the product $(1 + a_1x)(1 + a_2x)\cdots(1 + a_nx)$ as $a_1a_2\cdots a_ka_l$, where $k < l \leq n$, before having the term a_l divided out. There are $n - k$ choices for l, and hence the coefficient is $(n-k)S_k$.

Hence,

$$\begin{aligned} & (n - P_1x + P_2x^2 - \cdots)(1 + S_1x + S_2x^2 + \cdots) \\ & \quad = \ n + (n-1)S_1x + (n-2)S_2x^2 + \cdots + S_{n-1}x^{n-1}, \end{aligned}$$

and equating coefficients:

$$\begin{aligned} nS_1 - P_1 &= (n-1)S_1 \\ nS_2 - S_1P_1 + P_2 &= (n-2)S_2 \\ nS_3 - S_2P_1 + S_1P_2 - P_3 &= (n-3)S_3 \\ &\cdots \\ nS_{n-1} - S_{n-2}P_1 + S_{n-3}P_2 - \cdots + (-1)^{n-1}P_{n-1} &= S_{n-1}, \end{aligned}$$

or

$$\begin{aligned}
P_1 - S_1 &= 0 \\
P_2 - S_1P_1 + 2S_2 &= 0 \\
P_3 - S_1P_2 + S_2P_1 - 3S_3 &= 0 \\
&\cdots \\
P_{n-1} - S_1P_{n-2} + S_2P_{n-3} - \cdots + (-1)^{n-1}(n-1)S_{n-1} &= 0, \\
&\quad \text{and} \\
P_m - S_1P_{m-1} + S_2P_{m-2} - \cdots + (-1)^n P_{m-n}S_n &= 0 \\
&\quad \text{for } m \geq n.
\end{aligned}$$

The last equation is the well-known recursion sequence for the P_i, and the previous equations (known as Newton's relations) can help pin down the values of $P_1, P_2, \ldots, P_{n-1}$, or vice-versa.

Problem. If

$$\begin{aligned}
x + y + z &= 1, \\
x^2 + y^2 + z^2 &= 2, \\
x^3 + y^3 + z^3 &= 3,
\end{aligned}$$

determine the value of $x^4 + y^4 + z^4$.

Solution. Newton's relations become

$$\begin{aligned}
&P_1 - S_1 = 1 - S_1 = 0, \\
&P_2 - S_1P_1 + 2S_2 = 2 - S_1 + 2S_2 = 0, \\
&P_3 - S_1P_2 + S_2P_1 - 3S_3 = 3 - 2S_1 + S_2 - 3S_3 = 0,
\end{aligned}$$

which imply that $S_1 = 1$, $S_2 = -1/2$, and $S_3 = 1/6$. Also, $P_4 - 3S_1 + 2S_2 - S_3 = 0 \Rightarrow P_4 = 25/6$.

Here is the last problem of the 1995 Japan Mathematical Olympiad, Final Round.

Problem. Let $1 \leq k \leq n$ be positive integers. Let $a_1, a_2, \ldots, a_k$ be complex numbers satisfying

$$\begin{aligned}
a_1 + a_2 + \cdots + a_k &= n \\
a_1^2 + a_2^2 + \cdots + a_k^2 &= n \\
&\cdots \\
a_1^k + a_2^k + \cdots + a_k^k &= n
\end{aligned}$$

Show that $(x+a_1)(x+a_2)\cdots(x+a_n) = x^k + \binom{n}{1}x^{k-1} + \binom{n}{2}x^{k-2} + \cdots + \binom{n}{k}$.

Solution. Given $P_1 = P_2 = \cdots = P_k = n$, we must find $S_1, S_2, \ldots, S_k$. We will prove that $S_m = \binom{n}{m}$ by induction. Clearly $S_1 = P_1 = n = \binom{n}{1}$.

Now, for some m, assume $S_1 = \binom{n}{1}$, $S_2 = \binom{n}{2}$, ... , $S_{m-1} = \binom{n}{m-1}$. Then by the equations above, $P_m - S_1 P_{m-1} + S_2 P_{m-2} - \cdots + (-1)^{m-1} S_{m-1} P_1 + m(-1)^m S_m = 0$, or

$$n - n\binom{n}{1} + n\binom{n}{2} - \cdots + (-1)^{m-1} n \binom{n}{m-1} + m(-1)^m S_m = 0$$

so that

$$\frac{m}{n} S_m = \binom{n}{m-1} - \binom{n}{m-2} + \binom{n}{m-3} - \cdots = \binom{n-1}{m-1},$$

and further,

$$S_m = \frac{n}{m}\binom{n-1}{m-1} = \binom{n}{m}.$$

So by induction, we are done.

Mathematically Correct Sayings

[The following shred/slice appeared in the newsgroup `rec.humor.funny`.]

After applying some simple algebra to some trite phrases and cliches, a new understanding can be reached of the secret to wealth and success. Here it goes.

Knowledge is Power,
Time is Money,
and as everyone knows, Power is Work divided by Time.

So, substituting algebraic equations for these time worn bits of wisdom, we get:

$$K = P \quad (1)$$
$$T = M \quad (2)$$
$$P = W/T \quad (3)$$

Now, do a few simple substitutions. Put W/T in for P in equation (1), which yields:

$$K = W/T \quad (4)$$

Put M in for T into equation (4), which yields:

$$K = W/M \quad (5)$$

Now we've got something. Expanding back into English, we get: Knowledge equals Work divided by Money.

What this MEANS is that:

1. The More You Know, the More Work You Do, and
2. The More You Know, the Less Money You Make.

Solving for Money, we get:

$$M = W/K \tag{6}$$

Money equals Work divided by Knowledge.

From equation (6) we see that Money approaches infinity as Knowledge approaches 0, regardless of the Work done.

What THIS MEANS is: The More you Make, the Less you Know.

Solving for Work, we get

$$W = M \times K \tag{7}$$

Work equals Money times Knowledge

From equation (7) we see that Work approaches 0 as Knowledge approaches 0.

What THIS MEANS is: The stupid rich do little or no work.

Working out the socioeconomic implications of this breakthrough is left as an exercise for the reader.

Contest Dates

Here are some upcoming (or in some cases, already past) contest dates to mark on your calendar.

Contest	Grade	Date
Gauss	Grades 7 & 8	Wednesday, May 14, 1997
Pascal	Grade 9	Wednesday, February 19, 1997
Cayley	Grade 10	Wednesday, February 19, 1997
Fermat	Grade 11	Wednesday, February 19, 1997
Euclid	Grade 12	Tuesday, April 15, 1997
Descartes	Grades 12 & 13	Wednesday, April 16, 1997
CIMC	Grades 10 & 11	Wednesday, April 16, 1997
AJHSME	Grades 7 & 8	Thursday, November 21, 1996
AHSME	High School	Thursday, February 13, 1997
AIME	High School	Thursday, March 20, 1997
USAMO	High School	Thursday, May 1, 1997
AJHSME	Grades 7 & 8	Thursday, November 20, 1997
COMC	High School	Wednesday, November 27, 1996
CMO	High School	Wednesday, March 26, 1997
APMO	High School	March, 1997
IMO	High School	July 18 – 31, 1997

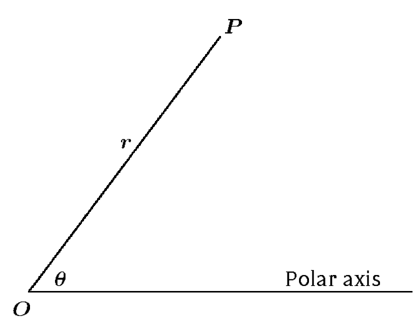
A Journey to the Pole — Part I

Miguel Carrión Álvarez
student, Universidad Complutense de Madrid
Madrid, Spain

For those of us who can not seem to get a strong grip on synthetic geometry, analytic geometry comes in handy. Even though polar coordinates can be superior to rectangular coordinates in some situations, they are systematically ignored by instructors and students alike. The purpose of this series is to introduce their uses with the idea that, as is always happening in mathematics, with a little ingenuity, the concepts central to polar coordinates can be applied elsewhere. This first article uses polar coordinates in elementary geometry.

Definition

In polar coordinates, the position of a point P is determined by the distance r from a point O called the *pole* and the angle θ between OP and a semi-infinite line called the *polar axis*. By convention, the polar axis is taken to be the positive x-axis, and the transformation from polar to cartesian coordinates is given by $x = r\cos\theta$, $y = r\sin\theta$. The inverse change of coordinates is not so straightforward; The obvious expressions are $r = \sqrt{x^2+y^2}$, $\theta = \arctan(\frac{y}{x})$, but the equation for θ is not single-valued, even if θ is restricted to $[0, 2\pi)$, and θ is undefined at the origin. Fortunately, we need not worry about this: when handling curves in polar coordinates, the change from rectangular to polar coordinates is of little use, and it is convenient to allow r and θ to take on all real values. With this provision, a point can be referred to by an infinite set of coordinate pairs: $(r, \theta) = ((-1)^n r, \theta + n\pi)$. Unless you want to do multiple integrals, this is not a problem, but rather something to exploit!



Polar Curves

Polar curves are usually written in the form $r = r(\theta)$, and unlike curves of the form $y = y(x)$, they can be closed and need not be simple (they can intersect themselves). Implicit curves of the form $f(r, \theta) = 0$ can be even more general. From a cartesian equation, the substitution $x = r\cos\theta$, $y = r\sin\theta$ yields a polar expression. Throughout this article, I have tried to avoid this whenever possible, and it turns out that it is always possible.

Some symmetries of the curves can be detected by checking the functions $r(\theta)$ or $f(r, \theta)$ above for the following simple properties (this is not a complete list):

- The curve is symmetric about the pole if $r(\theta) = r(\theta + \pi)$ or $f(-r, \theta) = f(r, \theta)$
- The curve has n-fold symmetry about the pole if $r(\theta) = r(\theta + \frac{2\pi}{n})$
- The curve is symmetric about the polar axis if $r(\theta) = r(-\theta)$
- The curve is symmetric about a line at an angle ϕ to the polar axis if $r(\theta) = r(2\phi - \theta)$

The following transformations are also useful:

- Any curve can be rotated through ϕ by substituting $\theta - \phi$ for θ
- The x- and y-axes can be permuted by substituting $\frac{\pi}{2} - \theta$ for θ

Example 1. The equation of a circle of radius a centered at the origin is $r = a$.

Example 2. The equation of a line passing through the origin at an angle ϕ to the polar axis is $\theta = \phi$.

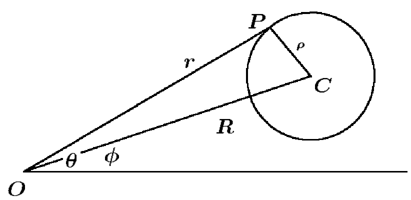
Exercise 1. Find the equation of a line at an angle ϕ to the polar axis passing at a distance d to the pole.

Exercise 2. Identify the curve $r = 2a\cos\theta$.

The Cosine Law

More often than not, when working in polar coordinates, one uses nothing but trigonometry, and the cosine law is the starting point of many derivations. If you think about it, it comes closest to being a 'vector addition rule' to use if you need to translate a curve, although this is best done in rectangular coordinates. I will not give a translation rule, because it is cumbersome and is of little use. Instead, I will use the cosine law to derive the equation of a circle of radius ρ centered at (R, ϕ) (see figure). Applying the cosine law to side ρ of $\triangle OCP$, we have

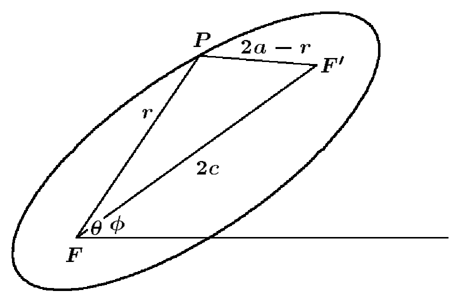
$$\begin{aligned} \rho^2 &= R^2 + r^2 - 2Rr\cos(\theta - \phi) \\ &= [r - R\cos(\theta - \phi)]^2 + R^2 - R^2\cos^2(\theta - \phi) \\ \Longrightarrow & \quad [r - R\cos(\theta - \phi)]^2 \; = \; \rho^2 - R^2\sin^2(\theta - \phi). \end{aligned}$$



From this equation, it is evident that if $R \geq \rho$, then the curve is defined for a limited range of θ given by $-\frac{\rho}{R} \leq \sin(\theta-\phi) \leq \frac{\rho}{R}$, as we would expect when the origin lies outside the circle. The squared length of the tangent from O, when $\sin(\theta-\phi) = \frac{\rho}{R}$, is $P = R^2\cos^2(\theta-\phi) = R^2(1-\frac{\rho^2}{R^2}) = R^2-\rho^2$; this is called the *potence* of the origin w.r.t. the circle. Note that this formula is correct, even when $R < \rho$ and $\sin\theta = \frac{\rho}{R}$ has no solution. Incidentally, the solution to Exercise 2 can be obtained easily by noting that if the origin is on the circle, then $R = \rho$, and

$$r - R\cos(\theta-\phi) = R\cos(\theta-\phi) \Longrightarrow r = 2R\cos(\theta-\phi).$$

Example 3. Polar equation of the ellipse with one focus at the origin and the main axis at an angle ϕ to the polar axis. The cosine law applied to side $F'P$ of triangle $FF'P$ gives



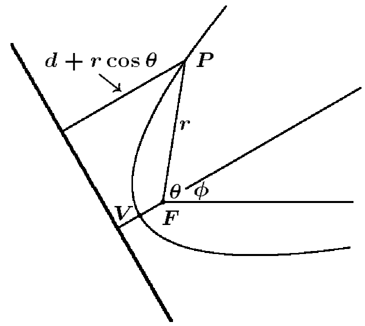
$$\begin{aligned}
(2a-r)^2 \quad &= \quad r^2+4c^2-4rc\cos(\theta-\phi) \\
\Longrightarrow \quad & a^2-ar \ = \ c^2-rc\cos(\theta-\phi) \\
\Longrightarrow \quad & r[a-c\cos(\theta-\phi)] \ = \ a^2-c^2 \\
\Longrightarrow \quad & r \ = \ \frac{b^2/a}{1-e\cos(\theta-\phi)}.
\end{aligned}$$

A Catalogue of Important Curves

The following curves are all important in their own right, but since their polar expressions are particularly simple, they make good examples of the use of polar coordinates.

Conic sections

We already have the equation for the ellipse. The polar equation of the parabola is even easier to derive. In the figure, we have the focus at the origin, the axis at an angle ϕ to the polar axis and a distance d from the focus to the directrix. From the figure on the previous page, we have

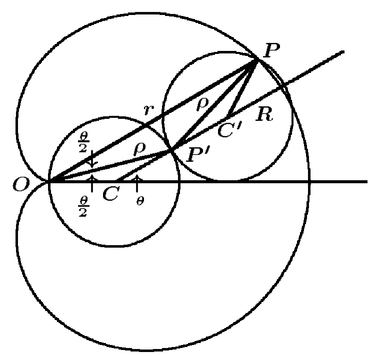


$$r = d + r\cos(\theta - \phi) \Longrightarrow r = \frac{d}{1 - \cos(\theta - \phi)}.$$

Exercise 3. In a similar way, derive the equation for the hyperbola, noting how both branches are handled. Hence, deduce that the general equation of the conic is $r = \frac{de}{1 - e\cos(\theta - \phi)}$, where e is the eccentricity. This equation can be obtained immediately from the definition of a conic as the locus of the points whose distances to a line (called the *directrix*) and a point (called the *focus*) are at a constant ratio e.

The Cardioid

The cardioid is the trajectory of a point on a circle that rolls on another circle of the same radius. So defined, it is a special case of the epicycloid, which is the curve described by a point on a circle rolling on another circle with no restriction on the radii; the general equation of the epicycloid is best expressed in parametric form.



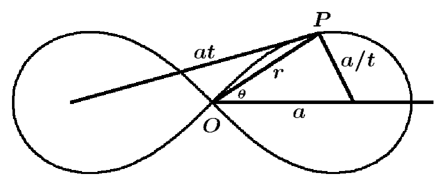
In the figure, the two circles have radius R. The condition that the one centered at C' rolls on the one centered at C implies that triangles OCP' and $PC'P'$ must be congruent. In triangles OCP' and $PC'P'$, $\rho = 2R\cos(\theta/2)$. Similarly, in triangle $OP'P$, $r = 2\rho\cos(\theta/2)$. Putting all together, we have

$$r = 4R\cos^2(\theta/2) \Longrightarrow r = 2R(1 + \cos\theta).$$

The cardioid is also a special case of *Pascal's Limaçon*, of the equation $r = b + a\cos\theta$. The cusp at O becomes a loop if $b < a$, and a smooth indentation if $b > a$. The limaçon can be defined as the locus of the feet of perpendiculars dropped from the origin to tangent lines to a circle. The radius of the circle is b and the distance from O to its centre is a.

The Lemniscate

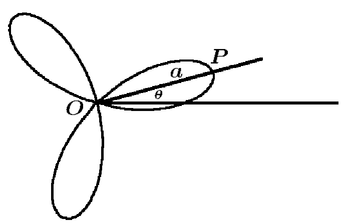
The lemniscate is the locus of the points such that the product of their distances to two points $2a$ apart is a^2. In the figure, the cosine law gives



$$\begin{aligned} a^2t^2 &= r^2 + a^2 + 2ar\cos\theta, \quad a^2/t^2 = r^2 + a^2 - 2ar\cos\theta \\ \Longrightarrow\ a^4 &= (r^2+a^2)^2 - 4a^2r^2\cos^2\theta \\ \Longrightarrow\ r^4 &= 2a^2r^2(2\cos^2\theta - 1) \\ \Longrightarrow\ r^2 &= 2a^2\cos(2\theta). \end{aligned}$$

The lemniscate is a special case of *Cassini's ovals*, which are defined in the same way, but with no restriction on the product of the distances.

The Rose



Loosely related to the lemniscate are the roses, of equation

$$r = a\cos[n(\theta - \phi)]$$

with integer n. For odd n, the curve has n 'leaves' and it is traced completely when θ varies from 0 to π. For even n, the curve has $2n$ 'leaves', and it is traced completely only when θ varies from 0 to 2π (see the figure).

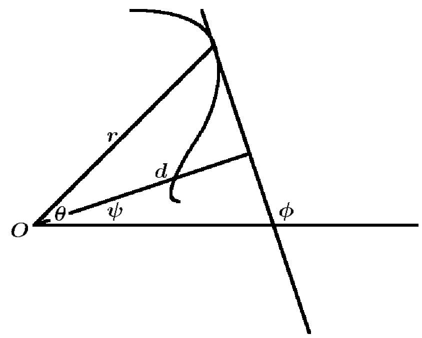
More general curves can be obtained if n is rational or irrational. In the first case there is an integer number of overlapping lobes and the curve is closed, but in the latter case the curve never closes, and in fact it is dense in the disc $r \leq a$.

The Spirals

Polar coordinates are particularly suited to spirals. The two most famous are the Archimedean Spiral $r = a\theta$, which is the trajectory of a point whose angular and radial velocities are proportional, and the Logarithmic Spiral $r = e^{a\theta}$. In fact, any continuous monotonic function that goes to infinity as θ goes to infinity defined in a semi-infinite interval will give rise to a spiral, like $r = a\ln\theta$. A related feature of polar curves is the *limit cycle*, occurring when $r(\theta)$ has a finite limit r_0 at infinity. In that case, the curve winds around the origin infinitely many times, approaching the circle $r = r_0$. Recognizing a limit cycle makes it easier to sketch a polar curve.

Straight Lines

We finish by deriving the equation of a straight line not passing through the origin. From the figure, we have $d = r\cos(\theta - \psi)$, where d is the distance from the line to the origin and ψ is the direction of the closest point. An alternative form is $d = r\sin(\theta - \phi)$, where ϕ is the direction of the line and $0 > \phi - \theta > \pi$ (see the figure).



Additional Problems

Problem 1. Considering $r(\theta_0 + \alpha)$ and $r(\theta_0 - \alpha)$, derive an expression for a secant line to a conic. Passing to the limit $\alpha \to 0$, write an equation for the tangent line at θ_0.

Problem 2. PQ is a chord through the focus F of a conic, and the tangents at P, Q meet at T. Prove that T lies on the directrix corresponding to F and that $FT \perp PQ$.

Problem 3. Let s be the tangent line at the vertex of a parabola and t be the tangent at P. If r and s meet at Q, prove that FQ bisects the angle between FP and the axis of the parabola, and that $FQ \perp s$.

IMO Report

Richard Hoshino
student, University of Waterloo
Waterloo, Ontario.

After ten days of intensive training at the Fields Institute in Toronto, the 1996 Canadian IMO team travelled to Mumbai, India to participate in the 37th International Mathematical Olympiad. For the first time in our team's history, every team member brought home a medal, with three silver and three bronze.

This year's team members were: Sabin "**Get me a donut**" Cautis, Adrian "**Da Chef**" Chan, Byung Kyu "**Spring Roll**" Chun, Richard "**YES! WE'VE GOT BAGELS!**" Hoshino, Derek "**Leggo my Eggo**" Kisman, and Soroosh "**Mr. Bean**" Yazdani. Our team leaders were J.P. "**Radishes**" Grossman and Ravi "**Oli**" Vakil (no, he's not Italian). Special thanks go out to our coaches, Naoki "**Dr. Cow**" Sato and Georg "**Where's my Ethanol**" Gunther.

This year's paper was one of the most difficult ever, and thus, the cutoffs for medals were among the lowest in history, 28 for gold, 20 for silver and 12 for bronze. Only one student, a Romanian, received a perfect score of 42. Our team's scores were as follows:

CAN 1	Sabin Cautis	13	Bronze Medal
CAN 2	Adrian Chan	14	Bronze Medal
CAN 3	Byung Kyu Chun	18	Bronze Medal
CAN 4	Richard Hoshino	22	Silver Medal
CAN 5	Derek Kisman	22	Silver Medal
CAN 6	Soroosh Yazdani	22	Silver Medal

Some weird coincidences: all the silver medallists got the exact same score, are graduating and are headed to the University of Waterloo in September, and all the bronze medallists are eligible to return to Argentina for next year's IMO. Overall, Canada finished sixteenth out of seventy-five countries, one of our highest rankings ever. A main reason for our success was our combined team score of 36 out of 42 on question #6, a problem that many countries answered very poorly (in fact, only two countries had more points on that problem than we did, and they both got 37 out of 42). Unfortunately, question #2 was answered very poorly by Canada, with only one 7, even though the problem was created by our own team leader, J.P. Grossman.

We all owe special thanks to Dr. Graham Wright of the Canadian Mathematical Society, Dr. Bruce Shawyer of Memorial University and Dr. Richard Nowakowski of Dalhousie University for their hard work and organization in making our trip possible and Dr. Ed Barbeau of the University of Toronto for all his commitment and dedication to training all the IMO team hopefuls with his year-long correspondence program.

Overall, the experience was memorable for all of us, although we could have done without the cockroaches in our rooms. Best of luck to all the students who will be working hard to make the 1997 IMO team, which will be held in Chapadmalal, Argentina near Mar del Plata.

Mayhem Problems

A new year brings new changes and new problem editors. *Cyrus Hsia* now takes over the helm as *Mayhem Advanced Problems Editor*, with *Richard Hoshino* filling his spot as the *Mayhem High School Problems Editor*, and veteran *Ravi Vakil* maintains his post as *Mayhem Challenge Board Problems Editor*. Note that all correspondence should be sent to the appropriate editor — see the relevant section.

In this issue, you will find only problems — the next issue will feature only solutions. We intend to have problems and solutions in alternate issues.

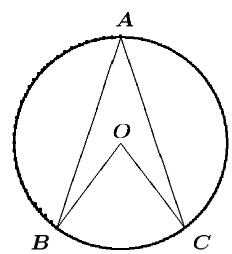
We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions be submitted by 1 June 1997, for publication in the issue 5 months ahead; that is, issue 6. We also request that **only students** submit solutions (see editorial), but we will consider particularly elegant or insightful solutions for others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H217. Let a_1, a_2, a_3, a_4, a_5 be a five-term geometric sequence satisfying the inequality $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 100$, where each term is an integer. How many of these five-term geometric sequences are there? (For example, the sequence 3, 6, 12, 24, 48 is a sequence of this type).

H218. A Star Trek logo is inscribed inside a circle with centre O and radius 1, as shown. Points A, B, and C are selected on the circle so that $AB = AC$ and arc BC is minor (that is $ABOC$ is not a convex quadrilateral). The area of figure $ABOC$ is equal to $\sin m°$, where $0 < m < 90$ and m is an integer. Furthermore, the length of arc AB (shaded as shown) is equal to $a\pi/b$, where a and b are relatively prime integers. Let $p = a + b + m$.



(i) If $p = 360$, and m is composite, determine all possible values for m.

(ii) If m and p are both prime, determine the value of p.

H219. Consider the infinite sum

$$S = \frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \cdots,$$

where the sequence $\{a_n\}$ is defined by $a_0 = a_1 = 1$, and the recurrence relation $a_n = 20a_{n-1} + 12a_{n-2}$ for all positive integers $n \geq 2$. If $\sqrt{S}$ can be expressed in the form $\frac{a}{\sqrt{b}}$, where a and b are relatively prime positive integers, determine the ordered pair (a, b).

H220. Let S be the sum of the elements of the set

$$\{1, 2, 3, \dots, (2p)^n - 1\}.$$

Let T be the sum of the elements of this set whose representation in base $2p$ consists only of digits from 0 to $p-1$.

Prove that $2n \times \frac{T}{S} = (p-1)/(2p-1)$.

Advanced Problems

Editor: Cyrus Hsia, 21 Van Allen Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A193. If $f(x,y)$ is a convex function in x for each fixed y, and a convex function in y for each fixed x, is $f(x,y)$ necessarily a convex function in x and y?

A194. Let H be the orthocentre (point where the altitudes meet) of a triangle ABC. Show that if $AH : BH : CH = BC : CA : AB$, then the triangle is equilateral.

A195. Compute $\tan 20° \tan 40° \tan 60° \tan 80°$.

A196. Show that $r^2 + r_a^2 + r_b^2 + r_c^2 \geq 4K$, where r, r_a, r_b, r_c, and K are the inradius, exradii, and area respectively of a triangle ABC.

Challenge Board Problems

Editor: Ravi Vakil, Department of Mathematics, One Oxford Street, Cambridge, MA, USA. 02138-2901 <ravi@math.harvard.edu>

C70. Prove that the group of automorphisms of the dodecahedron is S_5, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is A_5.

C71. Let L_1, L_2, L_3, L_4 be four general lines in the plane. Let p_{ij} be the intersection of lines L_i and L_j. Prove that the circumcircles of the four triangles $p_{12}p_{23}p_{31}$, $p_{23}p_{34}p_{42}$, $p_{34}p_{41}p_{13}$, $p_{41}p_{12}p_{24}$ are concurrent.

C72. A finite group G acts on a finite set X transitively. (In other words, for any $x, y \in X$, there is a $g \in G$ with $g \cdot x = y$.) Prove that there is an element of G whose action on X has no fixed points.

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) *after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 September 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2201. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Let L, M, N be the midpoints of DB, BC, CA respectively. Suppose that AL, OM, DN are concurrent. Show that

$$\text{either} \quad AD \parallel BC \qquad \text{or} \qquad [ABCD] = 2[OBC],$$

where $[\mathcal{F}]$ denotes the area of figure $\mathcal{F}$.

2202. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose than $n \geq 3$. Let $A_1, \ldots, A_n$ be a convex n-gon (as usual with interior angles $A_1, \ldots, A_n$).

Determine the greatest constant C_n such that

$$\sum_{k=1}^{n} \frac{1}{A_k} \geq C_n \sum_{k=1}^{n} \frac{1}{\pi - A_k}.$$

Determine when equality occurs.

2203. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $ABCD$ be a quadrilateral with incircle $\mathcal{I}$. Denote by P, Q, R and S, the points of tangency of sides AB, BC, CD and DA, respectively with $\mathcal{I}$.

Determine all possible values of $\angle(PR, QS)$ such that $ABCD$ is cyclic.

2204. *Proposed by Šefket Arslanagić, Berlin, Germany.*

For triangle ABC such that $R(a+b) = c\sqrt{ab}$, prove that

$$r < \frac{3}{10}a.$$

Here, a, b, c, R, and r are the three sides, the circumradius and the inradius of $\triangle ABC$.

2205. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Find the least positive integer n such that the expression

$$\sin^{n+2} A \ \sin^{n+1} B \ \sin^{n} C$$

has a maximum which is a rational number (where A, B, C are the angles of a variable triangle).

2206. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Let a and b denote distinct positive real numbers.

(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}\left(a^p b^{1-p} + a^{1-p} b^p\right) < 4p(1-p)\sqrt{ab} + (1 - 4p(1-p))\,\frac{a+b}{2}.$$

(b) Use (a) to deduce Pólya's Inequality:

$$\frac{a-b}{\log a - \log b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right).$$

Note: "log" is, of course, the natural logarithm.

2207. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Let p be a prime. Find all solutions in positive integers of the equation:

$$\frac{2}{a} + \frac{3}{b} = \frac{5}{p}.$$

2208. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

1. Find a set of positive integers $\{x, y, z, a, b, c, k\}$ such that

$$\begin{aligned} y^2z^2 &= a^2+k^2 \\ z^2x^2 &= b^2+k^2 \\ x^2y^2 &= c^2+k^2 \end{aligned}$$

2. Show how to obtain an infinite number of distinct sets of positive integers satisfying these equations.

2209. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Let $ABCD$ be a cyclic quadrilateral having perpendicular diagonals crossing at P. Project P onto the sides of the quadrilateral.

1. Prove that the quadrilateral obtained by joining these four projections is inscribable and circumscribable.

2. Prove that the circle which passes through these four projections also passes through the mid-points of the sides of the given quadrilateral.

2210*. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Given $a_0 = 1$, the sequence $\{a_n\}$ $(n = 1,\ 2,\ \ldots)$ is given recursively by

$$\binom{n}{n}a_n - \binom{n}{n-1}a_{n-1} + \binom{n}{n-2}a_{n-2} - \ldots \pm \binom{n}{\lfloor\frac{n}{2}\rfloor}a_{\lfloor\frac{n}{2}\rfloor} = 0.$$

Which terms have value 0?

2211. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

Several people go to a pizza restaurant. Each person who is "hungry" wants to eat either 6 or 7 slices of pizza. Everyone else wants to eat only 2 or 3 slices of pizza each. Each pizza in the restaurant has 12 slices.

It turns out that four pizzas are not sufficient to satisfy everyone, but that with five pizzas, there would be some pizza left over.

How many people went to the restaurant, and how many of these were "hungry"?

2212. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $S = \{1, 2, \ldots, n\}$ where $n \geq 3$.

(a) In how many ways can three integers x, y, z (not necessarily distinct) be chosen from S such that $x + y = z$? (Note that $x + y = z$ and $y + x = z$ are considered to be the same solution.)

(b) What is the answer to (a) if x, y, z must be distinct?

2213. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

A generalization of problem 2095 [1995: 344, 1996: 373].

Suppose that the function $f(u)$ has a second derivative in the interval (a, b), and that $f(u) \geq 0$ for all $u \in (a, b)$. Prove that

1. $(y - z)f(x) + (z - x)f(y) + (x - y)f(z) > 0$ for all $x, y, z \in (a, b)$, $z < y < x$

   if and only if $f''(u) > 0$ for all $u \in (a, b)$;

2. $(y - z)f(x) + (z - x)f(y) + (x - y)f(z) = 0$ for all $x, y, z \in (a, b)$, $z < y < x$

   if and only if $f(u)$ is a linear function on (a, b).

Correction

2137. [1996: 124, 317]*Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Three circles of (equal) radius t pass through a point T, and are each inside triangle ABC and tangent to two of its sides. Prove that:

(i) $t = \dfrac{rR}{R + r}$,
[**NB**: r instead of 2]

(ii) T lies on the line segment joining the centres of the circumcircle and the incircle of $\triangle ABC$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2101. [1996: 33] *Proposed by Ji Chen, Ningbo University, China.*

Let a, b, c be the sides and A, B, C the angles of a triangle. Prove that for any $k \leq 1$,

$$\sum \frac{a^k}{A} \geq \frac{3}{\pi} \sum a^k,$$

where the sums are cyclic. [The case $k = 1$ is known; see item 4.11, page 170 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*.]

I. *Solution by Hoe Teck Wee, Singapore.*

Let $f(x) = \dfrac{\sin^k x}{x}$ for $0 < x < \pi$. Then, for $0 < k \leq 1$, we have

$$f'(x) = \frac{(kx \cos x - \sin x) \sin^{k-1} x}{x^2}.$$

For $x \geq \pi/2$, we have $\cos x \leq 0$, so that $kx \cos x - \sin x \leq 0$.

For $x < \pi/2$, we have $\cos x > 0$ and $\tan x > x \geq kx$, so that $kx \cos x - \sin x \leq 0$.

Therefore $f'(x) = \dfrac{(kx \cos x - \sin x) \sin^{k-1} x}{x^2} \leq 0$.

Without loss of generality, we may assume that $A \leq B \leq C$. For $0 < k \leq 1$, we have that $f(x)$ is a non-increasing function, so that $f(A) \geq f(B) \geq f(C)$. Thus, by Tchebyshev's inequality, we have

$$\left(\frac{\sin^k A}{A} + \frac{\sin^k B}{B} + \frac{\sin^k C}{C} \right) (A+B+C) \geq 3 \left(\sin^k A + \sin^k B + \sin^k C \right).$$

By the Sine Rule, we have $a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$, where R is the circumradius of $\triangle ABC$. Multiply the inequality by $(2R)^k$ and substitute $A + B + C = \pi$ to get

$$\sum \frac{a^k}{A} \;\geq\; \frac{3}{\pi} \sum a^k. \tag{1}$$

For $k \leq 0$, we have

$$a \leq b \leq c \Longrightarrow a^k \geq b^k \geq c^k \Longrightarrow \frac{a^k}{A} \geq \frac{b^k}{B} \geq \frac{c^k}{C}.$$

Thus, by using Tchebyshev's inequality again, (??) holds for $k \leq 0$.

In conclusion, (??) holds for any $k \leq 1$.

II. *Solution by Kee-Wai Lau, Hong Kong.* (Edited)

We let, without loss of generality, $a \geq b \geq c$, so that $A \geq B \geq C$. Next, put $f(k) = \left(a^k/A + b^k/B + c^k/C\right)\left(a^k + b^k + c^k\right)^{-1}$, then

$$f'(k) = -\sum \frac{a^k b^k (A-B)(\log a - \log b)}{AB}\left(\sum a^k\right)^{-2} \leq 0,$$

where the sums are cyclic. Since we are given that $f(1) \geq 3/n$, it follows that $f(k) \geq 3/n$ for $k \leq 1$.

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. One incorrect solution was received.

Janous establishes the following generalization.

Let k and ℓ be real numbers. Then:

1.
$$\sum \frac{a^k}{A^\ell} \geq \left(\frac{3}{\pi}\right)^\ell \sum a^k,$$

for the cases

- $0 \leq k \leq \ell \leq 1$,
- $k \leq 0 \leq \ell$,
- $k \geq 0$ and $\ell \leq -1$.

2.
$$\sum \frac{a^k}{A^\ell} \leq \left(\frac{3}{\pi}\right)^\ell \sum a^k,$$

where

$$k \leq 0 \quad \text{and} \quad -1 \leq \ell \leq 0.$$

2102. [1996: 33] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. Let P and Q be the feet of the perpendiculars from A to BI and CI respectively. Prove that

$$\frac{AP}{BI} + \frac{AQ}{CI} = \cot\frac{A}{2}.$$

Solution by Panos E. Tsaoussoglou, Athens, Greece, and by eight others!

In $\triangle APB$, we have $\sin(B/2) = \dfrac{AP}{AB}$,
in $\triangle AQC$, we have $\sin(C/2) = \dfrac{AQ}{AC}$,
in $\triangle ABI$, we have $\dfrac{BI}{\sin(A/2)} = \dfrac{AB}{\cos C/2}$,
in $\triangle ACI$. we have $\dfrac{CI}{\sin(A/2)} = \dfrac{AC}{\cos B/2}$,
so that

$$\begin{aligned}\frac{AP}{BI} + \frac{AQ}{CI} &= \frac{\sin(B/2)\ \cos(C/2)}{\sin(A/2)} + \frac{\sin C/2\ \cos B/2}{\sin A/2} \\ &= \frac{\sin(B/2 + C/2)}{\sin(A/2)} = \frac{\cos(A/2)}{\sin(A/2)} = \cot(A/2).\end{aligned}$$

Editor's comment. Our featured solution is based on the property: $\angle AIB = \pi - A/2 - B/2 = \pi/2 + C/2$, so that $\sin(\angle AIB) = \cos(C/2)$. Other solvers used the equally efficient relations: $BI = r/\sin(B/2)$, or $BI = 4R\sin(A/2)\sin(C/2)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (two solutions); FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; MELITIS D. VASILIOU. Elefsis, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

2103. [1996: 33] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle. Let D be the point on side BC produced beyond B such that $BD = BA$, and let M be the mid-point of AC. The bisector of $\angle ABC$ meets DM at P. Prove that $\angle BAP = \angle ACB$.

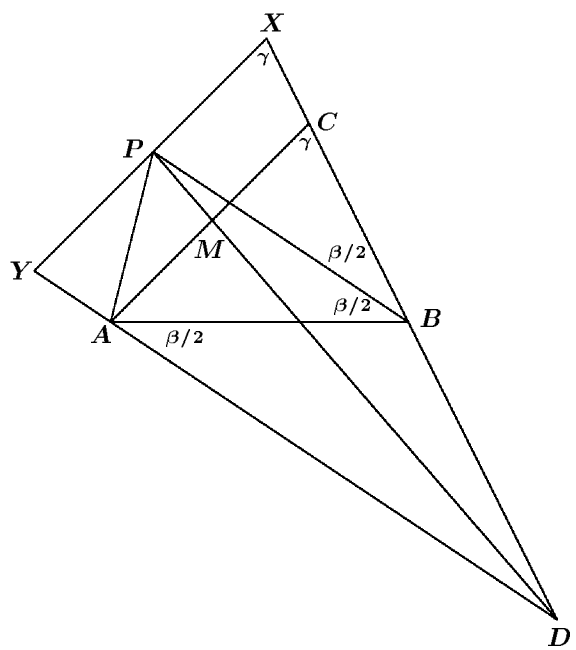
Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.

Let PX be parallel to AC with X lying on the line BC. Let Y be the intersection of PX with AD. P is the midpoint of XY because M is the mid-point of AC.

Then B is the mid-point of DX [PB is parallel to AD since $2\angle DAB = \angle DAB + \angle BDA = \angle ABC = 2\angle PBA$].

Hence $BX = BD = AB$. Triangle BPA is congruent to triangle BPX [$PB = PB$; $AB = XB$; $\angle ABP = \angle XBP$.]

Therefore, $\angle BAP = \angle BXP = \angle BCA$ [$PX \| AC$].



Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; TIM CROSS, King Edward's School, Birmingham, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MITKO

CHRISTOV KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; KEE-WAI LAU, Hong Kong; VASILIOU MELETIS, Elelsis, Greece; P. PENNING, Delft, the Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; HOE TECK WEE, Singapore; and the proposer.

2104. [1996: 34] *Proposed by K.R.S. Sastry, Dodballapur, India.*
In how many ways can 111 be written as a sum of three integers in geometric progression?

Solution by Zun Shan, Nanjing Normal University, Nanjing, China.

The answer is seventeen or sixteen depending on whether we allow the common ratio of the G.P. (geometric progression) to be zero or not.

Suppose $111 = a + ar + ar^2$ where a is an integer and r is a rational number. If $r = 0$, then we get the trivial solution

$$111 = 111 + 0 + 0. \tag{1}$$

Suppose $r = \frac{n}{m} \neq 0$ where m and n are nonzero integers. Without loss of generality, we may also assume that $m > 0$ and $(m, n) = 1$. Since the reverse of the G.P. a, ar, ar^2 is another G.P. ar^2, ar, a, we may also assume that $|r| \geq 1$ and so $0 < m \leq |n|$.

From $a(1 + r + r^2) = 111$ we get $a(m^2 + mn + n^2) = 111m^2$. Since clearly $(m^2 + mn + n^2, m^2) = 1$ we have $m^2|a$. Letting $a = km^2$ where k is an integer we then get $k(m^2 + mn + n^2) = 111$ which implies $k|111$. Since $m^2 + mn + n^2 > 0$ and $111 = 3 \times 37$, we have $k = 1, 3, 37$, or 111. Note that $m^2 + mn + n^2 = m^2 + |n|(\pm m + |n|) \geq m^2$.

Case [1] If $k = 1$, then $m^2 + mn + n^2 = 111 \Longrightarrow m^2 \leq 111 \Longrightarrow m \leq 10$. When $m = 1$, $a = 1$ and from $n^2 + n = 110$ we get $n = 10, -11$. Thus $r = 10$, -11 and we obtain the solutions:

$$111 = 1 + 10 + 100 = 1 - 11 + 121. \tag{2}$$

For $2 \leq m \leq 9$ it is easily checked that the resulting quadratic equation in n has no integer solutions.

When $m = 10$, $a = 100$ and from $n^2 + 10n = 11$ we get $n = 1$, -11. Since $m \leq |n|$, $n = -11$ and $r = -\frac{11}{10}$ yielding the solution:

$$111 = 100 - 110 + 121. \tag{3}$$

Case [2] If $k = 3$, then $m^2 + mn + n^2 = 37 \Longrightarrow m^2 \leq 37 \Longrightarrow m \leq 6$. Quick checkings reveal that there are no solutions for $m = 1$, 2, 5, 6.

When $m = 3$, $a = 27$ and from $n^2 + 3n = 28$ we get $n = 4$, -7. Thus $r = \frac{4}{3}, -\frac{7}{3}$ yielding the solutions:

$$111 = 27 + 36 + 48 = 27 - 63 + 147. \tag{4}$$

When $m = 4$, $a = 48$ and from $n^2 + 4n = 21$ we get $n = 3, -7$.

Since $m \le |n|$, $n = -7$ and $r = -\frac{7}{4}$ yielding the solution:

$$111 = 48 - 84 + 147. \tag{5}$$

Case [3] If $k = 37$, then $m^2 + mn + n^2 = 3 \Longrightarrow m^2 \le 3 \Longrightarrow m = 1 \Longrightarrow a = 37$ and from $n^2 + n = 2$ we get $n = 1, -2$. Thus $r = 1$ or -2 yielding the solutions:

$$111 = 37 + 37 + 37 = 37 - 74 + 148. \tag{6}$$

Case [4] If $k = 111$, then $m^2 + mn + n^2 = 1 \Longrightarrow m^2 \le 1 \Longrightarrow m = 1 \Longrightarrow a = 111$, and from $n^2 + n = 0$ we get $n = -1$ as $n \ne 0$. Thus $r = -1$ and we get the solution

$$111 = 111 - 111 + 111. \tag{7}$$

Reversing the summand in $(2) - (7)$ and noting that two of them are "symmetric", we obtain seventeen solutions in all:

$$\begin{array}{rlllll}
111 & = 111 + 0 + 0 & = 1 + 10 + 100 & = 100 + 10 + 1 \\
 & = 1 - 11 + 121 & = 121 - 11 + 1 & = 100 - 110 + 121 \\
 & = 121 - 110 + 100 & = 27 + 36 + 48 & = 48 + 36 + 27 \\
 & = 27 - 63 + 147 & = 147 - 63 + 27 & = 48 - 84 + 147 \\
 & = 147 - 84 + 48 & = 37 + 37 + 37 & = 37 - 74 + 148 \\
 & = 148 - 74 + 37 & = 111 - 111 + 111. &
\end{array}$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain, (who assumed $r \ne 0$ and found sixteen solutions); F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; (both of them found all seventeen solutions). There were also two incomplete and twenty-three incorrect solutions submitted!

Among these twenty-three, thirteen submissions claimed six solutions; six submissions claimed five solutions; two submissions claimed six or nine solutions; one submission claimed two solutions, and one submission claimed one solution only. Most of the errors were the result of assuming by mistake that $a(1 + r + r^2) = 111 \Longrightarrow 1 + r + r^2$ must be an integer.

2105. [1996: 34] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK*

Find all values of λ for which the inequality

$$2(x^3 + y^3 + z^3) + 3(1 + 3\lambda)xyz \ge (1 + \lambda)(x + y + z)(yz + zx + xy)$$

holds for all positive real numbers x, y, z.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

On setting $x = y = 1$ and $z = 0$, we obtain $4 \geq (1+\lambda)2$ and thus find that λ must be ≤ 1. We now show that the inequality holds for all $\lambda \leq 1$.

First, if $\lambda = 1$, the inequality reduces to

$$x^3 + y^3 + z^3 + 6xyz \geq (x+y+z)(yz+zx+xy),$$

which is equivalent to the special case $n = 1$ of the known Schur inequality

$$x^n(x-y)(x-z) + y^n(y-z)(y-x) + z^n(z-x)(z-y) \geq 0,$$

true for all real n, and which has come up many times in this journal. The rest will follow by showing that for all $\lambda < 1$,

$$\begin{aligned} &(1+\lambda)(x+y+z)(yz+zx+xy) - 3(1+3\lambda)xyz \\ \leq\ &(1+1)(x+y+z)(yz+zx+xy) - 3(1+3)xyz. \end{aligned} \tag{1}$$

[*Editor's note*: rewrite the original inequality as

$$2(x^3+y^3+z^3) \geq (1+\lambda)(x+y+z)(yz+zx+xy) - 3(1+3\lambda)xyz;$$

then (1) says that the right hand side is largest when $\lambda = 1$, so doing the case $\lambda = 1$ would be enough.] But (1) can be written

$$(1-\lambda)[(x+y+z)(yz+zx+xy) - 9xyz] \geq 0$$

which [after cancelling the positive factor $1-\lambda$] is a known elementary inequality, equivalent to Cauchy's inequality

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and HOE TECK WEE, Singapore. There were three incorrect solutions sent in.

2106. [1996: 34] *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A quadrilateral has sides a, b, c, d (in that order) and area F. Prove that

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F.$$

When does equality hold?

Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.

Let $ABCD$ be the quadrilateral with $AB = a$, $BC = b$, $CD = c$, and $DA = d$. We can assume, without loss of generality, that $AC = 1$. Therefore, we can locate the quadrilateral in a system of Cartesian coordinates where

$$A = (0,0), \quad B = (p,q), \quad C = (1,0), \quad D = (r,s).$$

We assume that $ABCD$ is simple so that its area is well-defined. If $ABCD$ is not convex we can make it convex and keep the side lengths the same while increasing the area. This means that we will be done if we can show that the result is true for convex quadrilaterals. It's also clear from this that if the result is true for convex quadrilaterals, then equality cannot hold for non-convex quadrilaterals. Therefore, assume $q < 0$ and $s > 0$. Now note that

$$\begin{aligned} 2a^2 + 5b^2 &= 2(p^2+q^2) + 5((p-1)^2+q^2) \\ &= 7p^2 - 10p + 5 + 7q^2 \\ &= 7\left(p - \tfrac{5}{7}\right)^2 - \tfrac{25}{7} + 5 + 7q^2 \\ 2a^2 + 5b^2 &\geq 7q^2 + \tfrac{10}{7}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} 8c^2 - d^2 &= 8((r-1)^2+s^2) - (r^2+s^2) \\ &= 7r^2 - 16r + 8 + 7s^2 \\ &= 7\left(r - \tfrac{8}{7}\right)^2 - \tfrac{64}{7} + 8 + 7s^2 \\ 8c^2 - d^2 &\geq 7s^2 - \tfrac{8}{7}. \end{aligned} \tag{3}$$

Combining (??) and (??), we get

$$\begin{aligned} &2a^2 + 5b^2 + 8c^2 - d^2 \\ &\quad\geq 7q^2 + 7s^2 + \tfrac{2}{7} \\ &\quad= \left(7q^2 + \tfrac{1}{7}\right) + \left(7s^2 + \tfrac{1}{7}\right) \\ &\quad= \left(7\left(|q| - \tfrac{1}{7}\right)^2 + 2|q|\right) + \left(7\left(|s| - \tfrac{1}{7}\right)^2 + 2|s|\right) \\ &\quad\geq 2(|q| + |s|) \\ &\quad= 4(\overline{ABC}) + 4(\overline{CDA}). \end{aligned} \tag{4}$$

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F,$$

as we wished to prove (where $\overline{ABC}$ and $\overline{CDA}$ refer to the areas of the two triangles ABC and CDA respectively). For equality to hold (when $A = (0,0)$ and $C = (1,0)$), it must hold in steps (??), (??), and (??). Therefore $p = \frac{5}{7}$, $r = \frac{8}{7}$, $q = -\frac{1}{7}$, and $s = \frac{1}{7}$. Thus, in general, equality holds if and only if $ABCD$ is directly similar to quadrilateral $A_0B_0C_0D_0$, where

$$A_0 = (0,0), \quad B_0 = \left(\tfrac{5}{7}, -\tfrac{1}{7}\right), \quad C_0 = (1,0), \quad D_0 = \left(\tfrac{8}{7}, \tfrac{1}{7}\right).$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer.

KONEČNÝ makes the observation that the quadrilateral is cyclic and that $\angle ABD = 135°$. The proposer makes the early observation that the maximum area for a quadrilateral with fixed sides occurs when it is cyclic and uses properties of cyclic quadrilaterals in the proof. He also generalizes the result to an inequality which JANOUS uses in his proof and which appears in the Addenda to the Monograph "Recent Advances in Geometric Inequalities" *by D. S. Mitrinović et al. in I. Journal of Ningbo University 4, No. 2 (Dec. 1991), 79–145.*

2107. [1996: 34] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC is not isosceles nor equilateral, and has sides a, b, c. D_1 and E_1 are points of BA and CA or their productions so that $BD_1 = CE_1 = a$. D_2 and E_2 are points of CB and AB or their productions so that $CD_2 = AE_2 = b$. Show that $D_1E_1 \parallel D_2E_2$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let S be the intersection of AB and D_2E_1.

[*Editor's comment by Chris Fisher*. Even though there seem to be two choices for each D_i and E_i, no solver had any trouble choosing the positions that make the result correct; furthermore, it must have been "obvious" to eveyone but me that AB is not parallel to D_2E_1, so that S exists. Alas, perhaps I need stronger glasses.]

Then CS is the bisector of $\angle ACB$, since $CE_1 = CB$ and $CA = CD_2$. Therefore

$$\frac{D_1S}{SE_2} = \frac{BD_1 - BS}{AE_2 - AS} = a = BSb - AS = \frac{a}{b},$$

since $\dfrac{BS}{AS} = \dfrac{a}{b}$. It then follows that $D_1E_1 \| D_2E_2$, since

$$\frac{E_1S}{SD_2} = \frac{CE_1}{CD_2} = \frac{a}{b} = \frac{D_1S}{SE_2}.$$

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; JOEL SCHLOSBERG, student, Hunter

College High School, New York NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; MELITIS D. VASILIOU. Elefsis, Greece; and the proposer.

Janous adds the observation that D_1E_1 and D_2E_2 are not only parallel, but their lengths are in the ratios $a : b$ (as is clear from the featured solution).

2108. [1996: 34] *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

Prove that

$$\frac{a+b+c}{3} \leq \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}},$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria, (modified slightly by the editor).

By the arithmetic-geometric mean inequality we have

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 6\sqrt[6]{a^6b^6c^6} = 6abc,$$

which implies

$$\begin{aligned} &9(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc) \\ &\quad\geq\ 8(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc), \end{aligned}$$

or

$$\begin{aligned} &9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca) \\ &\quad=\ 4(a+b+c)(a(b+c) + b(c+a) + c(a+b)). \end{aligned}$$

Using the arithmetic-geometric mean inequality again, we then have

$$\begin{aligned} &\frac{3}{4}(a+b)(b+c)(c+a) \\ &\quad\geq\ (a+b+c)\cdot\frac{a(b+c)+b(c+a)+c(a+b)}{3} \\ &\quad\geq\ (a+b+c)\sqrt{abc(a+b)(b+c)(c+a)} \qquad (1) \end{aligned}$$

From (1) it follows immediately that

$$\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}} \geq \frac{a+b+c}{3}.$$

Clearly, equality holds if $a = b = c$. [Ed. In fact, if equality holds, then from (1) we have $a(b+c) = b(c+a) = c(a+b)$. The first equality implies

$a = b$ and the second one implies $b = c$. Thus, equality holds in the given inequality if and only if $a = b = c$. This was observed by about half of the solvers.]

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous commented that upon the transformation

$$a \to \frac{1}{a},\ b \to \frac{1}{b},\ c \to \frac{1}{c},$$

the given inequality can be shown to be equivalent to

$$\sqrt{\frac{ab+bc+ca}{3}} \le \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}$$

which is known in the literature as Carlson's inequality (cf. eg. P.S. Bullen, D.S. Mitrinovic and P.M. Vasic, "Means and Their Inequalities", *Dordrechf, 1988. An anonymous reader commented that in this form, the inequality was problem 3 of the 1992 Austrian-Polish Mathematics Competition and has appeared in Crux before (see* [1994:97; 1995:336-337]). *Several solvers showed that the given inequality is equivalent to various other trigonometric inequalities involving a triangle XYZ, for example*

$$\cos\left(\frac{X}{2}\right)\cos\left(\frac{Y}{2}\right)\cos\left(\frac{Z}{2}\right) \le \frac{3\sqrt{3}}{8}$$

or

$$\sin X + \sin Y + \sin Z \le \frac{3\sqrt{3}}{2},$$

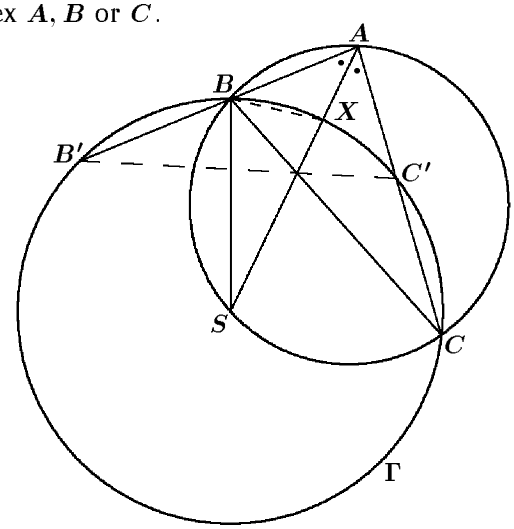
etc. These inequalities can be found in "Geometric Inequalities" *by O. Bottema et al.*

2109. [1996: 34] *Proposed by Victor Oxman, Haifa, Israel.*

In the plane are given a triangle and a circle passing through two of the vertices of the triangle and also through the incentre of the triangle. (The incentre and the centre of the circle are not given.) Construct, using only an unmarked ruler, the incentre.

Solution by P. Penning, Delft, the Netherlands.

Let the triangle be ABC, and Γ the circle passing through B, C and the incentre. The angles of the triangle are denoted by the symbol for the corresponding vertex A, B or C.



ANALYSIS:

Let point S be the intersection of the **circumcircle** and the angular bisector through A. The arcs SB and SC of the circumcircle are now equal and so are the chords SB and SC. Introduce X on AS such that $SX = SB = SC$.

$\angle BSA = \angle BCA = C$;

$\triangle XBS$ is isosceles with $SX = SB$, so $\angle XBS = 90^\circ - C/2$.

$\angle CBS = \angle CAS = A/2$; $\angle XBC = 90^\circ - C/2 - A/2 = B/2$.

So BX is the angular bisector at vertex B, and X must be the incentre. The circle Γ apparently has the point S as centre. [*It does, see Roger A. Johnson,* Modern Geometry, *292*].

[*Editor's note: If either AB or AC is tangent to Γ, then they both are and $AB = AC$. Suppose AB is tangent to Γ. Then $\angle SBA = \frac{\pi}{2}$, so $\angle BSA + \angle SAB = \frac{\pi}{2}$. Since $SC = SB$, $\angle BCS = \angle SBC = \angle SAC = \angle SAB$. Therefore, $\angle ACS = \angle ACB + \angle BCS = \angle BSA + \angle SAB = \frac{\pi}{2}$ and AC is tangent to Γ. In addition, tangents to a circle from an exterior point are equal, so $AB = AC$.*]

So if $AB \neq AC$, neither line is tangent to Γ. Let B' and C' be the other intersections of AB, respectively AC, with Γ. There is mirror-symmetry with respect to the line AS : Γ remains Γ; AB reflects into AC and AC reflects into AB; B and C' are mirror-images and so are C and B'. The side BC becomes $B'C'$; as a consequence they must intersect on the mirror-line AS.

CONSTRUCTION:

Find the other two intersections, B' and C', of AB and AC with the circle Γ. The intersection of BC and $B'C'$ is M. The incentre is the intersection of AM and Γ.

COMMENT:

The construction fails if ABC is isosceles, with $AB = AC$. In that case Γ touches both AB and AC in B and C respectively. M is now the midpoint of BC, but that cannot be found with unmarked ruler.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; and the proposer.

2110. [1996: 35] *Proposed by Jordi Dou, Barcelona, Spain.*

Let S be the curved *Reuleaux* triangle whose sides AB, BC and CA are arcs of unit circles centred at C, A and B respectively. Choose at random (and uniformly) a point M in the interior and let $C(M)$ be a chord of S for which M is the midpoint. Find the length ℓ such that the probability that $C(M) > \ell$ is $1/2$.

Solution by the proposer.

Let Σ be the locus of the mid-point M of segments of constant length σ, whose ends S_1 and S_2 move on the boundary of S. For the points M inside Σ the chords bisected by M are greater than σ.

$(\star)$ The area contained between S and Σ is $\frac{\pi}{4}\sigma^2$. It is sufficient to show that

$$\frac{\pi}{4}\ell^2 = \frac{[S]}{2}.$$

Since $[S] = \frac{\pi}{2} - \frac{2\sqrt{3}}{4}$, we will have

$$\ell = \left(\frac{\pi - \sqrt{3}}{\pi}\right)^{\frac{1}{2}} \simeq 0.67.$$

Brief proof of the assertion $(\star)$ (**Special Case of Holditch's Theorem**)

The ends S_1, S_2 of the chord of constant length move along the contour of the closed curve S. The mid-point M describes the curve Σ.

Let $S_1 = (x_1, y_1)$, $S_2 = (x_2, y_2)$, and $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. Suppose that x_1, y_1, x_2, y_2 are functions of t such that for $t_0 \le t \le t_1$, we have S_1, S_2 describing S.

$$\begin{aligned} [S] &= \int_{t_0}^{t_1} y_1\,dx_1 &= \int_{t_0}^{t_1} y_2\,dx_2 = \int_{t_0}^{t_1} \frac{1}{2}(y_1\,dx_1 + y_2\,dx_2) \\ [\Sigma] &= \int_{t_0}^{t_1} y\,dx &= \int_{t_0}^{t_1} \frac{1}{4}(y_1+y_2)(dx_1+dx_2). \end{aligned}$$

Then

$$[S] - [\Sigma] = \frac{1}{4}\int_{t_0}^{t_1} (y_2 - y_1)(dx_2 - dx_1).$$

Substitute $X = x_2 - x_1$, $Y = y_2 - y_1$, giving

$$[S] - [\Sigma] = \frac{1}{4}\int_{t_0}^{t_1} Y\,dX = \frac{\pi}{4}\sigma^2,$$

since (X, Y) describes a circle of radius σ.

2111. [1996: 35] *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Does there exist a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers) satisfying the three conditions:

(i) $f(1996) = 1$;

(ii) for all primes p, every prime occurs in the sequence

$f(p)$, $f(2p)$, $f(3p)$, $\ldots$, $f(kp)$, $\ldots$ infinitely often; and

(iii) $f(f(n)) = 1$ for all $n \in \mathbb{N}$?

I. *Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Yes, a function does exist that satisfies the three conditions. It is:

$$f(x) = \begin{cases} p_i & \text{if condition } * \text{ holds,} \\ 1 & \text{otherwise,} \end{cases}$$

where condition $*$ is: if the prime factorization of x is $x = \prod p_i^{e_i}$, there exists a power e_i such that $e_i > 2$ and $e_i > e_j$ for all $j \neq i$.

For example, 109850 has condition $*$, since $109850 = 2 \times 5^2 \times 13^3$ and the power of 13 is bigger than 2 and bigger than all other powers in the factorization; thus $f(109850) = 13$.

Now

- f satisfies condition (i) since $f(1996) = f(2^2 \times 499) = 1$;
- f satisfies condition (ii) because for any two primes p and q, $f(x_i) = q$ for every $x_i = q^i p$, $i = 3, 4, \ldots$;
- f satisfies condition (iii) since for all n either $f(n) = 1$ or $f(n) = p_i$ for some prime p_i, and in either case $f(f(n)) = 1$.

II. *Solution by Chris Wildhagen, Rotterdam, the Netherlands.*

For each $n \in \mathbb{N}$ let q_n be the nth prime and $b(n)$ be the number of 1's in the binary representation of n. Define $f : \mathbb{N} \to \mathbb{N}$ as follows:

$$f(m) = \begin{cases} q_{b(n)} & \text{if } m = p^n \text{ with } p \text{ prime and } n \geq 2, \\ 1 & \text{else.} \end{cases}$$

Clearly f satisfies the three given conditions.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; KEITH EKBLAW, Walla Walla, Washington, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; DAVID E. MANES, State University of New York, Oneonta, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposer. There were three incorrect solutions sent in.

Most solvers gave a variation of Solution I.

2112. [1996: 35] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find a four-digit base-ten number $abcd$ (with $a \neq 0$) which is equal to $a^a + b^b + c^c + d^d$.

Solution by Cyrus Hsia, student, University of Toronto, Toronto, Ontario (modified slightly by the editor).

We first stipulate that $0^0 = 1$. Let $m = abcd$, $s = a^a + b^b + c^c + d^d$ and assume that $m = s$. Clearly, $10^3 \leq m < 10^4$. If $x \geq 6$ for any $x \in \{a, b, c, d\}$ then $s \geq 6^6 > 10^4$ which is a contradiction. So $a, b, c, d \leq 5$.

If $x < 5$ for all $x \in \{a, b, c, d\}$, then $s \leq 4 \times 4^4 = 1024$ and furthermore $a = b = c = d = 4$ is the only combination for which $s \geq 10^3$. However, in this case, $s = 1024 \neq 4444 = m$. Hence $x = 5$ for some $x \in \{a, b, c, d\}$. We cannot have more than one 5 or else $s \geq 2 \times 5^5 = 6250$ would imply that some digit of m is at least 6. Hence, we have exactly one 5.

Since $s > 5^5 = 3125$ and $s \leq 5^5 + 3 \times 4^4 = 3893 < 4000$, we must have $a = 3$. Thus, $s = 5^5 + 3^3 + x^x + y^y = 3152 + x^x + y^y$ where $x, y \in \{a, b, c, d\}$. Without loss of generality, we may assume that $0 \leq y \leq x \leq 4$.

If $x = 0$, then $y = 0$ and $s = 3154$ which has no 0 among its digits.

If $x = 1$, then $y = 0, 1$ and $s = 3154$ while m has no 4 among its digits.

If $x = 2$, then $s = 3156 + y^y$ and it is easily verified that s has no 2 among its digits for $y = 0, 1, 2$.

If $x = 3$, then $s = 3179 + y^y$ and it is easily verified that s has no 5 among its digits for $y = 0, 1, 2, 3$.

If $x = 4$, then $s = 3408 + y^y$ and, again, it is readily checked that s has no 5 among its digits when $y = 0, 1, 2, 4$. However, when $y = 3$, $s = 3435$ which is clearly a solution.

To summarize, $m = 3435$ is the only solution.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; TIM CROSS, King Edward's School, Birmingham, England; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; LUKE LAMOTHE, student, St. Joseph Scollard Hall S.S., North Bay, Ontario, KATHLEEN E. LEWIS, SUNY Oswego. Oswego, NY, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; JOHN GRANT McLOUGHLIN, Okanagan University College, Kelowna British Columbia; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Of the twenty-six solvers (including the proposer), nine of them only gave the answer 3435. *About half of all the solvers claimed, with or without proof, that* 3435 *is the only solution. Chronis, Hess, and Janous found the answer by computer search. Hess remarked that no other solutions were found for the present problem and the corresponding problem on* 5*–digit integers. Janous investigated the corresponding* n*–digit problem of finding all* n*–digit integers* $a_{n-1}a_{n-2}\ldots a_1a_0$ *which equal* $\sum\limits_{k=0}^{n-1} a_k^{a_k}$. *He showed that a necessary condition is* $n \leq 10$. *For* $n > 1$, *he conducted an extensive, but not exhaustive, computer search, which revealed no solutions other than the one found by all the solvers! He made a guess that* 1 *and* 3435 *are the only integers with the desired property. Can any reader prove or disprove this?*