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CONTENTS

A KNOCKOUT TOURNAMENT PROBLEM

CURTIS COOPER

A knockout tournament between 2^n players [2, 3] is conducted as follows:

In the first round the 2^n players are split into 2^{n-1} pairs who play each other. The 2^{n-1} winners proceed to the second round and play each other in pairs. The 2^{n-2} winners proceed to the third round, and the process repeats. The one winner of the nth round is declared the winner of the tournament.

A problem in [1] asks the question: What chance has a given player of winning a knockout tournament involving 2^n players?

This paper seeks to answer that question. To that end, we introduce some concepts. A *tournament* T_n is a set A_n of 2^n players, together with the results of the $2^n - 1$ matches between players according to the scheme described above. Let w be the winner of the tournament, w having beaten x in the final round. Then T_n can be thought of as two sub-tournaments, each having 2^{n-1} players, T_n^w won by w and T_n^x won by x, and a final match where w beats x.

Let M be the set of all possible tournaments T_n for a given set A_n of players. Let p_{ij} denote the probability that player a_i beats player a_j in a match. Also, let $p(T_n)$ be the probability that tournament T_n has its stated outcome (including all intermediate games). We see that

$$p(T_n) = \prod_B p_{ij},$$

where $p_{ij} \in B$ if and only if player a_i beats a_j in the tournament T_n.

Another convenient quantity for our discussion is s_n, defined by

$$s_1 = 1; \qquad s_n = \frac{1}{2}\binom{2^n}{2^{n-1}}(s_{n-1})^2, \qquad n \geq 2.$$

We then have, by mathematical induction,

$$s_n = (2^n - 1)!\cdot 2^{n+1-2^n}.$$

LEMMA 1. Let T_n be a tournament and T_n' the same tournament except that the result of the final match is reversed. Then

$$p(T_n) + p(T_n') = p(T_n^w)\cdot p(T_n^x).$$

Proof. This formula follows from the fact that $p_{ij} + p_{ji} = 1$.

THEOREM 1. We have

$$s_n = \sum_{T_n \in M} p(T_n).$$

Proof. We use mathematical induction on n. The case $n = 1$ is obvious. So assume the theorem is true for $n = k\text{-}1$, and let

$M' = \{T\colon T$ is a knockout tournament involving 2^{k-1} players from $A_k\}$.

Then we have

$$\sum_{T_k \in M} p(T_k) = \tfrac{1}{2} \sum_{T_k \in M} p(T_k^w)\cdot p(T_k^x), \text{ by Lemma 1}$$

$$= \tfrac{1}{2} s_{k-1} \sum_{T \in M'} p(T), \qquad \text{by induction hypothesis}$$

$$= \tfrac{1}{2} s_{k-1} \binom{2^k}{2^{k-1}} s_{k-1}, \text{ by induction hypothesis}$$

$$= s_k.$$

LEMMA 2. *The probability that T_n occurs, assuming that the contestants in each tournament are chosen at random, is $p(T_n)/s_n$.*

Proof. We again use induction. The case where $n = 1$ is trivial. So assume the formula true for $n = k\text{-}1$. Then the probability that T_k occurs is given by

$$\frac{2}{\binom{2^k}{2^{k-1}}} \cdot \frac{p(T_k^w)}{s_{k-1}} \cdot \frac{p(T_k^x)}{s_{k-1}} \cdot p_{wx} = \frac{p(T_k)}{s_k}. \quad \square$$

We are now ready to give the answer to the stated problem, which follows immediately from Lemma 2.

THEOREM 2. *The probability that player a_i wins a knockout tournament is given by*

$$\sum_{T_n \in M_i} p(T_n)/s_n,$$

where $M_i = \{T_n \in M\colon a_i$ *wins* $T_n\}$.

As an example of the computations involved in the final formula, we let $p_{ij} = \frac{1}{2}$ for all players in A_n and verify that our formula leads to the expected result, $1/2^n$. The number of elements in M_i is $(2^n - 1)!$ and, for each T_n,

$$p(T_n) = (\tfrac{1}{2})^{2^n - 1}.$$

Thus

$$\sum_{T_n \in M_i} \frac{p(T_n)}{s_n} = \frac{(\frac{1}{2})^{2^n-1}\cdot(2^n-1)!}{(2^n-1)!\cdot 2^{n+1-2^n}} = \frac{1}{2^n}.$$

Acknowledgment. The author would like to thank the referee for helpful suggestions which did much to improve the format and content of this paper.

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2. J.A. Hartigan, "Probabilistic Completion of a Knockout Tournament", *Annals of Mathematical Statistics*, 37 (1966) 495-503.

3. J.W. Moon, *Topics on Tournaments*, Holt, Rinehart & Winston, New York, 1968, pp. 47-49.

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* * *

POSTSCRIPT TO "AN INTERESTING RECURSIVE FUNCTION"

I found out, just too late for inclusion in my note published here last month [1982: 69], that the conjecture "the sequence will eventually enter a cycle containing 000...0*XX*" is invalid. Accordingly, the last paragraph of the note should be amended to read as follows:

Readers are invited to show that, if n_0 has any number of digits, the sequence will eventually enter a cycle. Can you describe a general component of this cycle? It is conjectured (but this is only a wild conjecture with little evidence to support it) that one can guarantee that $n_0 \to \ldots \to 000\ldots00$ for all k-digit n_0 only if k is a power of 2 .

RICHARD V. ANDREE
University of Oklahoma

* * *

THE PUZZLE CORNER

Puzzle No. 15: Heteronyms (2, 5; 3, 4; 7)

Professor Matrix is MY LAST
For mental mathematics.
Extracting roots? He's very fast.
At sight, he solves quadratics,
Does logarithms with MY FIRST,
And "crypts" as if they're all plain text;
But here is where he's not well versed —
To check his check book, he's MY NEXT.

ALAN WAYNE, Holiday, Florida

MAGIC SQUARES IN MINIMAL PRIMES

ALLAN WM. JOHNSON JR.

In a fifth-order magic square composed of distinct primes, the prime 2 cannot appear because the row containing 2 would have an even magic sum whereas any other row would have an odd magic sum. Thus a fifth-order prime magic square must be composed of odd primes only and its magic sum must be odd. Because the first twenty-five odd primes, 3, 5, ..., 101, sum to 1159, the smallest possible magic sum for a fifth-order prime magic square is 233, which is 1159/5 rounded up to the next odd integer. Here is a fifth-order prime magic square whose magic sum is 233, the smallest possible:

3	83	41	101	5
89	67	11	29	37
19	7	103	31	73
79	53	17	13	71
43	23	61	59	47

This magic square is very nearly composed of consecutive primes: it has each of the primes 3, 5, ..., 103 except 97.

The smallest group of twenty-five consecutive odd primes whose sum is divisible by 5 is 13, 17, ..., 113, and their sum is $1565 = 5 \cdot 313$. These twenty-five primes rearrange into the following magic square whose magic sum is 313, the smallest possible:

13	109	61	113	17
67	73	23	103	47
101	31	79	19	83
89	71	53	41	59
43	29	97	37	107

By chance, the magic sums 233 and 313 are both prime.

A sixth-order magic square composed of distinct primes cannot contain the prime 2 and its magic sum must be even. Because the thirty-six primes 3, 5, ..., 157 sum to 2582, a sixth-order prime magic square cannot have a magic sum less than 432, which is 2582/6 rounded up to the next even integer. The thirty-five primes 3, 5, ..., 151 with the prime 167 form a magic square with the smallest possible magic sum 432:

3	101	37	167	53	71
43	11	83	61	97	137
103	47	149	7	19	107
13	79	59	151	41	89
139	67	73	17	113	23
131	127	31	29	109	5

A sixth-order magic square composed of consecutive primes cannot have a magic sum less than 484, which is one-sixth the sum of the primes 7, 11, ..., 167. These thirty-six primes rearrange into the following magic square, which has the smallest possible magic sum 484:

11	107	47	167	73	79
53	41	59	17	151	163
109	83	157	31	67	37
23	113	71	137	43	97
149	13	61	29	131	101
139	127	89	103	19	7

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* * *

THE PUZZLE CORNER

Answer to Puzzle No. 11 [1982: 76]: Laconic (*L* a conic).

Answer to Puzzle No. 12 [1982: 76]: "There goes the neighborhood."

Answer to Puzzle No. 13 [1982: 76]: SQUARE = 328509, CUBE = 1849, THREE = 67099 = 17·3947.

Answer to Puzzle No. 14 [1982: 76]: $\dfrac{78}{589}\ .$ 45942

THE OLYMPIAD CORNER: 34

M.S. KLAMKIN

Since I strongly believe that *Problem Proposing* is an important aspect of *Problem Solving*, I encourage students to come up with their own problems. The following problems were proposed by students at the 1981 U.S.A. Mathematical Olympiad Practice Session held at the U.S. Military Academy in West Point. For a previous set of such problems, see [1980: 210]. As usual, I solicit from all readers (especially high school students) elegant solutions for possible later publication in this column.

1. Sum the series

$$\sum_{i=1}^{\infty} \frac{36i^2+1}{(36i^2-1)^2}.$$

(Jack Brennen, Poolesville, Maryland.)

2. How many equations must be considered in order to maximize a given differentiable function $F(x_1,x_2,\ldots,x_n)$ in the region $0 \le x_i \le 1$, $i = 1,2,\ldots,n$, by differential calculus? (Gregg Patruno, Princeton University, Princeton, N.J.)

3. A quick proof that the rationality of p, q, and $\sqrt{p}+\sqrt{q}$ implies the rationality of $\sqrt{p}$ is furnished by the identity

$$2\sqrt{p} = \frac{(\sqrt{p}+\sqrt{q})^2+p-q}{\sqrt{p}+\sqrt{q}}.$$

Prove in similar fashion that if p, q, r, and $\sqrt{p}+\sqrt{q}+\sqrt{r}$ are rational, then so is $\sqrt{p}$. (Gregg Patruno, Princeton University, Princeton, N.J.)

4. If $a_i,b_i > 0$ for $i = 1,2,\ldots,n$ and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1,$$

prove that, for all $m > 1$,

$$\left\{\sum_{i=1}^{n} a_i^m\right\}^{m+1} \cdot \left\{\sum_{i=1}^{n} b_i^{-m}\right\}^{m-1} \ge 1.$$

(Brian Hunt, Montgomery Blair H.S., Silver Spring, Maryland.)

5. One solution of the Diophantine equation

$$7x^2 + 8x - 3 = y^2$$

is $(x,y) = (-3,6)$. Are there solutions for which x is positive? (Noam D. Elkies, Stuyvesant H.S., New York, N.Y.)

6. Let S_1, S_2, S_3, S_4 be four spheres in 3-space such that S_1, S_2, S_3 intersect at poihts A_4 and B_4; S_1, S_2, S_4 intersect at A_3 and B_3; S_1, S_3, S_4 intersect at A_2 and B_2; and, finally, S_2, S_3, S_4 intersect at A_1 and B_1. If P is any point distinct from A_i, B_i $(i = 1,2,3,4)$, prove that the circumcenters of triangles PA_1B_1, PA_2B_2, PA_3B_3, and PA_4B_4 are coplanar. (Noam D. Elkies, Stuyvesant H.S., New York, N.Y.)

7. Show how to construct a convex quadrilateral given the lengths of two opposite noncongruent sides and the diagonals of its Varignon parallelogram. (A *Varignon parallelogram* of a quadrilateral Q is formed by joining the midpoints of adjacent sides of Q.) (Scott Berkenblit, North Valley Stream, N.Y.)

*

I now give solutions by Andy Liu to a number of problems which appeared earlier in this column.

4. [1981: 42] Prove that $\Sigma\{1/(i_1 i_2 \dots i_k)\} = n$, where the summation is taken over all nonempty subsets $\{i_1, i_2, \dots, i_k\}$ of $\{1,2,\dots,n\}$.

Solution.

Our proof is by induction. Let S_n denote the given sum. Clearly $S_1 = 1$. Assume that $S_n = n$ for all $n \le k$ and consider the set $\{1,2,\dots,k+1\}$. Its nonempty subsets are of three types:

(i) those which are nonempty subsets of $\{1,2,\dots,k\}$;

(ii) those which are of type (i) with the element $k+1$ adjoined;

(iii) the singleton $k+1$.

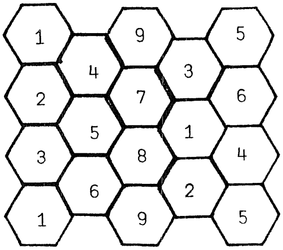
The induction hypothesis then yields

$$S_{k+1} = S_k + \frac{1}{k+1} S_k + \frac{1}{k+1} = k+1,$$

and so $S_n = n$ for all n.

8. [1981: 42] Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

Solution.



Tile the plane with "half-open" regular hexagons (interior plus north, north-east and south-east edges) of diameter 1 and color the hexagons in nine colors as shown in the figure. At least 1980/9 = 220 points must be in hexagons of the same color, and no two of these 220 points are in the same hexagon, as otherwise the distance between them would be less than 1. It is easy to check that, if two points are in different hexagons of the same color, the distance between them is at least $\sqrt{3}$, and the desired result follows.

1. [1981: 43] Find all functions $f:Q \to Q$ (where Q is the set of all rational numbers) satisfying the following two conditions:

(a) $f(1) = 2$;

(b) $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x,y \in Q$.

Solution.

Put $x = 1$ in (b); then, by (a), $f(y) = 2f(y)-f(y+1)+1$, or

$$f(y+1) = f(y) + 1 \tag{1}$$

for all y. Now, by induction,

$$f(y+n) = f(y) + n \tag{2}$$

for all y and all integers n. In particular, $f(0) = 1$ follows from (1), and so

$$f(n) = n + 1 \tag{3}$$

for all integers n.

Now let $r = p/q$ be rational. With $x = r$ and $y = q$ in (b), we get

$$f(p) = f(r)f(q) - f(q+r) + 1,$$

or, by (2) and (3),

$$p + 1 = (q+1)f(r) - (f(r)+q) + 1$$

and, finally,

$$f(r) = \frac{p+q}{q} = r + 1 \tag{4}$$

for all $r \in Q$.

Conversely, the function $f:Q \to Q$ defined by (4) certainly satisfies (a) and (b), and is thus the only such function.

3. [1981: 43] Let p be a prime number and n a positive integer. Prove that the following statements (a) and (b) are equivalent:

(a) None of the binomial coefficients $\binom{n}{k}$ for $k = 0,1,\ldots,n$ is divisible by p.

(b) n can be represented in the form $n = p^s q - 1$, where s and q are integers, $s \geq 0$, $0 < q < p$.

Solution.

We show that (a) $\Longrightarrow$ (b). Suppose that $p \nmid \binom{n}{k}$ for any k, and that n has no representation in the form $n = p^s q - 1$ with $q < p$. Since we can always write $n+1 = p^s q$ with $(p,q) = 1$, it follows that $q > p$. Consider the number $k = p^s(q-p)$. We have $0 < k < n$ and

$$\binom{n}{k} = \binom{n}{k-1}\cdot\frac{n-k+1}{k} = \binom{n}{k-1}\cdot\frac{p}{q-p}.$$

Since $(p,q) = 1$ implies that $(p,q-p) = 1$, it follows that $p \mid \binom{n}{k}$, a contradiction.

To prove that (b) $\Longrightarrow$ (a), suppose we have $n = p^s q - 1$ with $q < p$. We use induction to show that $p \nmid \binom{n}{k}$ for $k = 0,1,\ldots,n$. This is certainly true for $k = 0$ (and $k = n$). Suppose $p \nmid \binom{n}{k-1}$ and write $k = p^t r$ with $(p,r) = 1$. Since $q < p$ and $k < n$, we must have $t < s$. Now

$$\binom{n}{k} = \binom{n}{k-1}\cdot\frac{n-k+1}{k} = \binom{n}{k-1}\cdot\frac{p^{s-t}q-r}{r}.$$

If $p \mid \binom{n}{k}$, then either $p \mid \binom{n}{k-1}$ or $p \mid (p^{s-t}q - r)$. Since neither of these cases is possible, we must have $p \nmid \binom{n}{k}$.

2. [1981: 44] The sequence $a_0, a_1, \ldots, a_n$ is defined by

$$a_0 = \tfrac{1}{2}, \quad a_{k+1} = a_k + (1/n)a_k^2, \quad k = 0,1,\ldots,n-1.$$

Prove that $1 - (1/n) < a_n < 1$.

Solution.

We show by induction that $a_k < n/(2n-k)$ for all $k \geq 1$. For $k = 1$, we have

$$a_1 = \frac{1}{2} + \frac{1}{4n} = \frac{2n+1}{4n} < \frac{n}{2n-1}.$$

Now, proceeding inductively,

$$a_{k+1} = a_k + \frac{a_k^2}{n} < \frac{n}{2n-k} + \frac{\left(\frac{n}{2n-k}\right)^2}{n} = \frac{n(2n-k+1)}{(2n-k)^2} < \frac{n}{2n-(k+1)}.$$

In particular, for any $n \geq 1$ we must have

$$a_n < \frac{n}{2n-n} = 1.$$

We now show inductively that $a_k > (n+1)/(2n+2-k)$ for $1 \le k \le n$. For $k = 1$, we have

$$a_1 = \frac{2n+1}{4n} > \frac{n+1}{2n+1} = \frac{n+1}{2n+2-1}.$$

With the induction assumption, we have, for $k \le n$,

$$a_{k+1} = a_k + \frac{a_k^2}{n} > \frac{n+1}{2n+2-k} + \frac{\left(\frac{n+1}{2n+2-k}\right)^2}{n} = \frac{(n+1)(2n^2+3n-nk+1)}{n(2n+2-k)^2} > \frac{n+1}{2n+2-(k+1)}.$$

It follows that, in particular,

$$a_n > \frac{n+1}{2n+2-n} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n}.$$

We conclude that, for all $n \ge 1$,

$$1 - \frac{1}{n} < a_n < 1.$$

3. [1981: 44] Consider the equation

$$x^n + 1 = y^{n+1},$$

where $n \ge 2$ is a natural number. Prove that no positive integer solutions (x,y) exist for which x and $n+1$ have no common factor.

Solution.

We assume that (x,y) is a solution such that $(x,n+1) = 1$ and find a contradiction. If we set $z = y-1$, then

$$x^n = y^{n+1} - 1 = (z+1)^{n+1} - 1 = zA, \tag{1}$$

where

$$A = z^n + (n+1)z^{n-1} + \ldots + \frac{(n+1)n}{2}z + (n+1).$$

If $p|z$ and $p|A$ for some prime p, then $p|x$ and $p|(n+1)$, contrary to the assumption $(x,n+1) = 1$. Hence $(z,A) = 1$, and it follows from (1) that A is a perfect nth power, say

$$A = \frac{(z+1)^{n+1} - 1}{z} = a^n,$$

where a is an integer. We have

$$z(z+1)^n < z(z+1)^n + (z+1)^n - 1 = (z+1)^{n+1} - 1 = za^n,$$

so $(z+1)^n < a^n$. On the other hand, since $n \ge 2$,

$$z\{(z+2)^n-(z+1)^n\} = z\{(z+2)-(z+1)\}\sum_{i=0}^{n-1}(z+2)^{n-1-i}(z+1)^i > zn(z+1)^{n-1} > (z+1)^n;$$

hence

$$z(z+2)^n > z(z+1)^n + (z+1)^n = (z+1)^{n+1} > (z+1)^{n+1} - 1 = za^n,$$

from which $a^n < (z+2)^n$. Thus $z+1 < a < z+2$, and this is the required contradiction.

6. [1981: 44] Find, with proof, the digit immediately to the left and the digit immediately to the right of the decimal point in the decimal expansion of the number

$$(\sqrt{2}+\sqrt{3})^{1980}.$$

Solution.

Since 1980 is even, binomial expansions show that

$$(\sqrt{3}+\sqrt{2})^{1980} = M + N\sqrt{6} \qquad \text{and} \qquad (\sqrt{3}-\sqrt{2})^{1980} = M - N\sqrt{6},$$

where

$$M = \sum_{i=0}^{990} \binom{1980}{2i} \cdot 3^{990-i} \cdot 2^{i} \qquad \text{and} \qquad N = \sum_{i=1}^{990} \binom{1980}{2i-1} \cdot 3^{990-i} \cdot 2^{i-1}.$$

Now

$$M^2 - 6N^2 = (M+N\sqrt{6})(M-N\sqrt{6}) = ((\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}))^{1980} = 1\ ;$$

hence

$$(\sqrt{3}+\sqrt{2})^{1980} = M + \sqrt{6N^2} = M + \sqrt{M^2-1} = (2M-1) + (1-(M-\sqrt{M^2-1})).$$

Let $t = M-\sqrt{M^2-1} > 0$, so that

$$(\sqrt{3}+\sqrt{2})^{1980} = (2M-1) + (1-t).$$

Since clearly $M > 5.05$, it follows that $0.2M > 1 + 0.01$ and

$$M^2 - 1 > M^2 - 0.2M + 0.01 = (M-0.1)^2,$$

and hence $\sqrt{M^2-1} > M-0.1$. Thus $t < 0.1$, the fractional part of $(\sqrt{3}+\sqrt{2})^{1980}$ is $1-t$, and the first decimal digit is 9.

The integral part of $(\sqrt{3}+\sqrt{2})^{1980}$ is $2M-1$. Since

$$M + N\sqrt{6} = (\sqrt{3}+\sqrt{2})^{1980} = (5+2\sqrt{6})^{990},$$

it follows that

$$M = \sum_{i=0}^{495} \binom{990}{2i} \cdot 5^{990-2i} \cdot 2^{2i} \cdot 6^{i},$$

whence

$$M \equiv 2^{990} \cdot 6^{495} \equiv 4 \pmod 5.$$

Thus $M = 5k+4$, $2M-1 = 10k+7$, and the last digit of the integral part is 7.

H-4. [1981: 114] Show that, for every natural number $n \geq 4$, there is always at least one integer between $n!$ and $(n+1)!$ which is divisible by n^3.

Solution.

Since $(n+1)! - n! = n \cdot n! > n^3$ for $n \geq 4$, there are at least n^3 integers between $n!$ and $(n+1)!$. Clearly one of them is divisible by n^3.

H-5. [1981: 114] Let "$\circ$" denote a binary operation on the integers such that

(i) $0 \circ a = a$ for all integers a;

(ii) $(a \circ b) \circ c = c \circ (ab) + (a \circ c) + (b \circ c) - 2c$ for all integers a, b, c.

Determine $8 \circ 9$.

Solution.

Put $a = 0$ in (ii). By (i) we have

$$b \circ c = (c \circ 0) + c + (b \circ c) - 2c,$$

from which $c \circ 0 = c$ for all c. Now put $c = 0$ in (ii). This yields

$$a \circ b = ab + a + b.$$

Thus $8 \circ 9 = 8 \cdot 9 + 8 + 9 = 89$.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

725. [1982: 78] *Correction.* In the statement of the problem, certain symbols A and B representing matrices were mistakenly underlined. The underlining should be ignored.

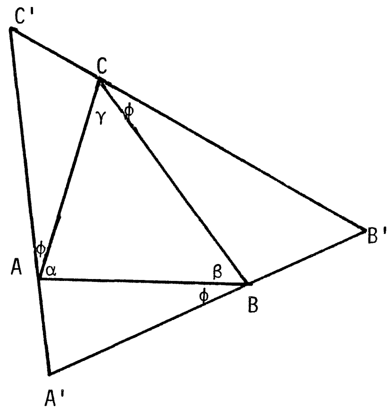
731. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Pity poor Stan. He built a better rat trap and, although he used only prime TRAPS, the world did not beat a path to his door: all he got was better rats. So, regretfully, we continue to urge:

$$\begin{array}{r} \text{TRAP} \\ \text{RATS} \\ \text{STAN} \\ \text{IN} \\ \text{RAT} \\ \hline \text{TRAPS} \end{array}.$$

732. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a fixed triangle ABC with angles α,β,γ and a variable circumscribed triange A'B'C' determined by an angle $\phi \in [0,\pi)$, as shown in the figure. It is easy to show that triangles ABC and A'B'C' are directly similar.



(a) Find a formula for the ratio of similitude

$$\lambda = \lambda(\phi) = B'C'/BC.$$

(b) Find the maximal value λ_m of λ as ϕ varies in $[0,\pi)$, and show how to construct triangle A'B'C' when $\lambda = \lambda_m$.

(c) Prove that $\lambda_m \geq 2$, with equality just when triangle ABC is equilateral.

733*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

A triangle has sides a,b,c, and the medians of this triangle are used as sides of a new triangle. If r_m is the inradius of this new triangle, prove or disprove that

$$r_m \leq \frac{3abc}{4(a^2+b^2+c^2)},$$

with equality just when the original triangle is equilateral.

734. *Proposed by H. Kestelman, University College, London, England.*

The first $n-1$ columns of a real $n\times n$ matrix are given mutually orthogonal vectors of unit length. How can one choose the last column to ensure that the matrix is orthogonal and has determinant $+1$?

735*. *Proposed by S.C. Chan, Singapore.*

Solve the following problem, which is given without solution in Hall & Stevens, *A School Geometry*, Macmillan, London, 1944, page 310, Problem 50:

In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

736. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given is a regular n-gon $V_1V_2\ldots V_n$ inscribed in a unit circle. Show how to select, among the n vertices V_i, three vertices A,B,C such that

(a) The area of triangle ABC is a maximum;

(b) The perimeter of triangle ABC is a maximum.

737. *Proposed by Charles W. Trigg, San Diego, California.*

(a) Distribute the nine decimal nonzero digits so as to form six prime integers. In how many ways can this be done?

(b) Distribute the ten decimal digits so as to form six prime integers. In how many ways can this be done?

738*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Find, in terms of p,q,r, a formula for the area of a triangle whose vertices are the roots of

$$x^3 - px^2 + qx - r = 0$$

in the complex plane.

739. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Prove that, if the incentre I of a triangle is equidistant from the circumcentre O and the orthocentre H, then one angle of the triangle is 60°.

740. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Find all functions $y = y(x)$ which are defined and continuous for all $x > 0$ and satisfy $y + 1/y = x + 1/x$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

584. [1980: 283; 1981: 290; 1982: 16, 51] A solution was received from DAN SOKOLOWSKY, California State University at Los Angeles.

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606. [1981: 48; 1982: 24] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in Euclidean space R^n and let $\sigma_n' = A_0'A_1'\ldots A_n'$ be an n-simplex similar to and inscribed in σ_n, and labeled in such a way that

$$A_i' \in \sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n, \qquad i = 0,1,\ldots,n.$$

Prove that the ratio of similarity

$$\lambda \equiv A_i'A_j'/A_iA_j \geq 1/n.$$

II. *Solution by G.P. Henderson, Campbellcroft, Ontario.*

We will prove that $\lambda \geq 1/n$ even if we allow A_k' to be outside the simplex $A_0\ldots A_{k-1}A_{k+1}\ldots A_n$ (but in the $(n-1)$-space of these points). Let A_0 be the origin of a rectangular coordinate system and let A_k have coordinates b_{jk}. (The domain of free subscripts will always be $\{1,2,\ldots,n\}$ and all sums will be from 1 to n.)

The vector $\vec{A_0'A_k'}$ can be obtained from $\lambda\vec{A_0A_k}$ by an orthogonal transformation which corresponds to a rotation and possibly a reflection of σ_n. Let (c_{ij}) be the matrix of the transformation. Then the ith component of $\vec{A_0'A_k'}$ is $\lambda\sum_j c_{ij}b_{jk}$. Let the ith coordinate of A_0' be x_i. Then the ith coordinate of A_k' is

$$x_i + \lambda\sum_j c_{ij}b_{jk}. \tag{1}$$

The conditions that A_k' be in the $(n-1)$-space of $A_0\ldots A_{k-1}A_{k+1}\ldots A_n$ give us n linear equations for the x_i. Solving these and using the fact that A_0' is in the $(n-1)$-space $A_1\ldots A_n$ leads to the required condition on λ.

In these calculations, we need B_{jk}, the cofactor of b_{jk} in the determinant $D = |b_{jk}|$. These cofactors satisfy

$$\sum_k b_{ik}B_{jk} = \sum_k b_{ki}B_{kj} = D\delta_{ij}, \tag{2}$$

where δ_{ij} is the Kronecker delta ($= 1$ if $i = j$ and $= 0$ if $i \neq j$). Now it is clear that $A_0\ldots A_{k-1}A_k'A_{k+1}\ldots A_n$ are in an $(n-1)$-space if and only if a certain determinant of order $n+1$ is zero. Except for the last row, the determinant consists of the coordinates of the $n+1$ points, written as columns. The last row contains 1's. Since the coordinates of A_0 are all zero, we can delete the first column and the last row. This leaves a determinant of order n, the jth column of which is b_{ij} except for $j = k$. The kth column is given by (1). Expanding the determinant down the kth column yields

$$\sum_i (x_i + \lambda\sum_j c_{ij}b_{jk})B_{ik} = 0.$$

Multiplying by b_{mk}, summing with respect to k and using (2), we get

$$\sum_{i}\sum_{k} x_i B_{ik} b_{mk} = D\sum_{i} x_i \delta_{im} = Dx_m = -\lambda \sum_{i}\sum_{j}\sum_{k} c_{ij} B_{ik} b_{jk} b_{mk}.$$

Now $D \neq 0$ because the vectors $\vec{A_0A_k}$ are independent. Therefore

$$\begin{aligned} x_m &= -(\lambda/D)\sum_{i}\sum_{j}\sum_{k} c_{ij} B_{ik} b_{jk} b_{mk} \qquad (3)\\ &= \sum_{k} w_k b_{mk}, \end{aligned}$$

where

$$w_k = -(\lambda/D)\sum_{i}\sum_{j} c_{ij} B_{ik} b_{jk}.$$

Now A_0' will be in the $(n-1)$-space $A_1 \ldots A_n$ if and only if $\sum_{k} w_k = 1$, that is, if and only if

$$\sum_{i}\sum_{j}\sum_{k} c_{ij} B_{ik} b_{jk} = -D/\lambda.$$

Summing with respect to k, this reduces to

$$\sum_{i} c_{ii} = -1/\lambda.$$

Since $-1 \le c_{ij} \le 1$ and $\lambda > 0$, we have

$$0 < -\sum_{i} c_{ii} \le n,$$

that is, $\lambda \ge 1/n$.

If $\lambda = 1/n$, then $c_{ij} = -\delta_{ij}$, and using (1) and (3) we find that A_k' is the centroid of the points $A_0, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n$.

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624. [1981: 116] *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

ABC is a given triangle of area K, and PQR is the equilateral triangle of smallest area K_0 inscribed in triangle ABC, with P on BC, Q on CA, and R on AB.

(a) Find the ratio

$$\lambda = K/K_0 \equiv f(A,B,C)$$

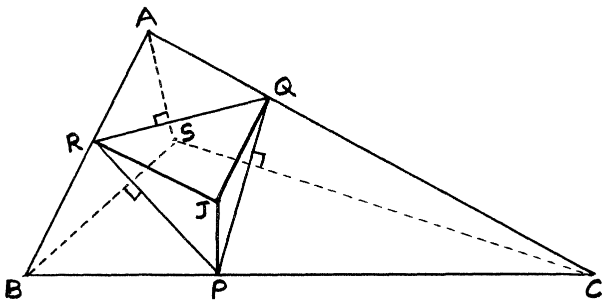
as a function of the angles of the given triangle.

(b) Prove that λ attains its minimum value when the given triangle ABC is equilateral.

(c) Give a Euclidean construction of triangle PQR for an arbitrary given triangle ABC.

Adapted from the solution of George Tsintsifas, Thessaloniki, Greece.

(a) It follows from the solution of Crux 623 [1982: 91] that PQR is the pedal triangle of J, one of the isodynamic points of triangle ABC (see figure).



Perpendiculars from vertices A,B,C upon QR,RP,PQ, respectively, are concurrent [1, p. 270] in S, one of the isogonic centres of triangle ABC [2, p. 296], which is isogonally conjugate to J. Now we have [2, p. 220]

$$(\mathrm{SA} + \mathrm{SB} + \mathrm{SC})^2 = \tfrac{1}{2}(a^2+b^2+c^2) + 2\sqrt{3}\,K, \tag{1}$$

where a = BC, etc. (This relationship holds even if one angle of triangle ABC is greater than 120°, provided SA,SB,SC are considered as directed segments.) With brackets denoting (signed) area, we have

$$\begin{aligned} K &= [\mathrm{ARSQ}] + [\mathrm{BPSR}] + [\mathrm{CQSP}] \\ &= \tfrac{1}{2}l(\mathrm{SA} + \mathrm{SB} + \mathrm{SC}), \end{aligned} \tag{2}$$

where l is the side length of triangle PQR. With $K_0 = l^2\sqrt{3}/4$, we get from (1) and (2)

$$\frac{K}{K_0} = 2 + \frac{a^2+b^2+c^2}{2\sqrt{3}K}. \tag{3}$$

Since

$$\frac{a^2+b^2+c^2}{4K} = \cot\omega = \cot A + \cot B + \cot C, \tag{4}$$

where ω is the Brocard angle of triangle ABC [2, p. 266], we obtain from (3)

$$\lambda = \frac{K}{K_0} = 2 + \frac{2}{\sqrt{3}}(\cot A + \cot B + \cot C). \tag{5}$$

(b) Since $\cot A + \cot B + \cot C \geq \sqrt{3}$ [3, p. 28], we have

$$\lambda \geq 4, \tag{6}$$

with equality if and only if triangle ABC is equilateral. We can also obtain (6) directly from (3) by using $a^2+b^2+c^2 \geq 4K\sqrt{3}$ [3, p. 42].

(c) With triangle PQR identified as in the first sentence of part (a) of this solution, the construction is trivial. Draw two of the Apollonius circles of triangle ABC. They intersect in two points, one of which is J. Then drop perpendiculars from J to the sides of triangle ABC. *Voilà*.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; VADIM V. MUZYCHENKO, Moscow, U.S.S.R; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

Dou obtained formula (5) in a way that was not much different form our featured solution. All other solvers used calculus, and after (in some cases) acres of calculations they arrived at formulas different from (and much more complicated than) our formula (5). We chickened out, charitably assumed that their formulas were all equivalent to (5), and credited them with a correct solution.

A few years ago, in a comment about the Cosine Circle [1978: 82], we derived the beautifully symmetric identity

$$\frac{a\sin A + b\sin B + c\sin C}{a\cos A + b\cos B + c\cos C} = \cot A + \cot B + \cot C \tag{7}$$

from the fact that each side is equal to R/ρ, where R is the circumradius and ρ the radius of the Cosine Circle, and we mentioned at the time that we had never come across (7) before. Well, now we have. In looking up references for our present problem, we came across the identity [2, p. 267]

$$\frac{a\sin A + b\sin B + c\sin C}{a\cos A + b\cos B + c\cos C} = \cot\omega ,$$

which, in view of (4), is the same as (7).

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1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952.

2. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960.

3. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publishing Co., Groningen, 1968.

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625. [1981: 116] *Proposed by Gali Salvatore, Perkins, Québec.*

(a) Let R denote the real field and let P be a polynomial in $R[x]$. Prove that if there are *positive definite* polynomials $Q_1, Q_2 \in R[x]$ such that $P = Q_1 - Q_2$, then there are infinitely many such pairs (Q_1, Q_2).

(b) Exhibit one such pair (Q_1,Q_2) for the polynomial P defined by

$$P(x) = a_0 + a_1x + \ldots + a_nx^n, \qquad a_i \in R.$$

Solution by Leroy F. Meyers, The Ohio State University.

I interpret "Q is a positive definite polynomial" to mean "$Q(x) > 0$ for all real numbers x".

(a) This part is trivial. Suppose (Q_1,Q_2) is a solution. Then, since

$$P = Q_1 - Q_2 = (Q_1+S) - (Q_2+S),$$

the pair $(Q_1+S,\ Q_2+S)$ is also a solution for any positive definite polynomial S.

(b) For the given polynomial P, set

$$Q_1(x) = 1 + \tfrac{1}{4}\sum_{k=0}^{n}(x^k+a_k)^2 \qquad\text{and}\qquad Q_2(x) = 1 + \tfrac{1}{4}\sum_{k=0}^{n}(x^k-a_k)^2.$$

Now Q_1 and Q_2 are both positive definite, each being a sum of squares with at least one positive term, and $Q_1 - Q_2 = P$. So the pair (Q_1,Q_2) constitutes one solution.

Also solved by HIPPOLYTE CHARLES, Waterloo, Québec; M.S. KLAMKIN, University of Alberta; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Editor's comment.

The interpretation of "positive definite" given in our featured solution is the usual one. If $Q(x) \geq 0$ for all real x, then Q is usually called *positive semi-definite* [1]. So pairs like

$$Q_1(x) = \{\tfrac{1}{2}(P(x)+1)\}^2, \qquad Q_2(x) = \{\tfrac{1}{2}(P(x)-1)\}^2,$$

given by one solver, do not constitute a solution to part (b) if $P(x) \equiv 1$ or -1. For the same reason, one of the other answers given was invalid if $P(x) \equiv 0$. But we did not, on such weak semantic grounds, call these solutions incorrect.

A related problem [2] states that "any polynomial with real coefficients can be written as a difference of two real monotone increasing polynomials."

REFERENCES

1. James/James, *Mathematics Dictionary*, Fourth Edition, Van Nostrand Reinhold, New York, 1976, p. 154.

2. K.L. Yocom, Solution to Problem 813 (proposed by L. Carlitz and R.A. Scoville), *Mathematics Magazine*, 45 (1972) 233.

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626. [1981: 116] *Proposed by A. Liu, University of Alberta.*

A (v,b,r,k,λ)-configuration on a set with v elements is a collection of b k-subsets such that

(i) each element appears in exactly r of the k-subsets;

(ii) each pair of elements appears in exactly λ of the k-subsets.

Prove that $k^r \geq v^\lambda$ and determine the value of b when equality holds.

Solution by the proposer.

Clearly $v \geq k$ and $r \geq \lambda$. Since each element appears in r sets and appears with every other element in λ sets, we have $r(k-1) = \lambda(v-1)$. Now

$$rk = r(k-1) + r = \lambda(v-1) + r = \lambda v + (r-\lambda).$$

When rk is partitioned into r summands, their product is maximal when all the terms are equal. Hence

$$k^r \geq v^\lambda \cdot 1^{r-\lambda} = v^\lambda.$$

Equality holds when $v = k$ and $r = \lambda$, and it follows that then $b = 1$.

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627. [1981: 116] *Proposed by F. David Hammer, Santa Cruz, California.*

Consider the double inequality

$$6 < 3^{\sqrt{3}} < 7.$$

Using *only* the elementary properties of exponents and inequalities (no calculator, computer, table of logarithms, or estimate of $\sqrt{3}$ may be used), prove that the first inequality implies the second.

I. *Solution de Hippolyte Charles, Waterloo, Québec.*

Posons p: $6 < 3^{\sqrt{3}}$ et q: $3^{\sqrt{3}} < 7$. Nous avons à démontrer que l'implication $p \Longrightarrow q$ est vraie. Il suffit pour cela de démontrer que la proposition q est vraie, car alors l'implication $p \Longrightarrow q$ sera vraie quelle que soit la valeur logique de p. Or l'implication

$$(3^{\sqrt{3}})^4 = 3^{\sqrt{48}} < 3^{\sqrt{49}} = 3^7 = 2187 < 2401 = 7^4 \quad \Longrightarrow \quad 3^{\sqrt{3}} < 7$$

est vraie, et son hypothèse aussi. Donc q est vraie.

II. *Solution by the proposer.*

[With p and q as in solution I], the truth of the implication $p \Longrightarrow q$ follows by transitivity from the following chain of implications, each one of which is true:

$$
\begin{aligned}
6 < 3^{\sqrt{3}} &\Longrightarrow 18 < 3^{\sqrt{3}+1} \\
&\Longrightarrow 16 = 4^{(\sqrt{3}+1)(\sqrt{3}-1)} < 3^{\sqrt{3}+1} \\
&\Longrightarrow 4^{\sqrt{3}-1} < 3 \\
&\Longrightarrow 4^{\sqrt{3}} < 12 \\
&\Longrightarrow 4^{\sqrt{3}} < \frac{49}{4} \\
&\Longrightarrow 4^{\sqrt{3}+1} < 49 = 7^{(\sqrt{3}+1)(\sqrt{3}-1)} \\
&\Longrightarrow 4 < 7^{\sqrt{3}-1} \\
&\Longrightarrow 28 < 7^{\sqrt{3}} \\
&\Longrightarrow 27 = 3^3 < 7^{\sqrt{3}} \\
&\Longrightarrow 3^{\sqrt{3}} < 7.
\end{aligned}
$$

Also solved by PAUL R. BEESACK, Carleton University, Ottawa; CLAYTON W. DODGE, University of Maine at Orono; RICHARD H. HUDSON, University of South Carolina, Columbia, S.C., and KENNETH S. WILLIAMS, Carleton University, Ottawa (jointly); VIKTORS LINIS, University of Ottawa; and V.N. MURTY, Pennsylvania State University, Capitol Campus (two solutions).

Editor's comment.

With any given propositions p and q, the truth of the implication $p \Longrightarrow q$ can be proved in three ways:

(i) Prove that p is false; the truth value of q then does not matter.

(ii) Prove that q is true; the truth value of p then does not matter.

(iii) Give a chain of true implications, $p \Longrightarrow \ldots \Longrightarrow q$; the truth values of p *and* q then do not matter.

Note that it is *never* necessary to prove that p is true.

With p and q as given in solution I, method (i) is not available to us, because it can be shown (in various "legal" or "illegal" ways) that p is true. Our first solver used method (ii), and our second solver used method (iii).

Several solvers apparently did not read the problem carefully, because they seemed unaware that the problem was more a logical exercise than an arithmetical one. They proved independently that p and q were both true, and then contentedly awaited to receive their laurels. (One even said that he did so "without logical contortions", which shows that he has completely missed the point of the problem.) Nevertheless, having proved q true, they have accidentally stumbled upon a technically correct solution. *Caveat lector*!

628. [1981: 117] *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Given a triangle ABC with sides a,b,c, let T_a, T_b, T_c denote the angle bisectors extended to the circumcircle of the triangle. If R and r are the circum- and in-radii of the triangle, prove that

$$T_a + T_b + T_c \le 5R + 2r,$$

with equality just when the triangle is equilateral.

I. *Solution by S.C. Chan, Singapore, and the proposer (independently).*

Let I be the incentre and $T_a = AA'$, etc. According to Johnson [1], we have $IA' = 2R\sin(A/2)$, etc. With the inequalities

$$\Sigma AI \le 2(R+r) \qquad \text{and} \qquad \Sigma \sin(A/2) \le \frac{3}{2} \tag{1}$$

from the Bottema Bible [2, pp. 103, 20], we obtain

$$T_a+T_b+T_c = \Sigma AI + \Sigma IA' \le 2(R+r) + 3R = 5R + 2r.$$

With each of the inequalities we have used, equality occurs just when the triangle is equilateral.

II. *Solution by Jack Garfunkel, Flushing, N.Y.*

Since

$$T_a = 2R\cos\tfrac{1}{2}(B-C), \text{ etc.} \qquad \text{and} \qquad r = 4R\Pi\sin\tfrac{1}{2}A, \tag{2}$$

the proposed inequality is equivalent to

$$\Sigma\cos\tfrac{1}{2}(B-C) \le \frac{5}{2} + 4\Pi\sin\tfrac{1}{2}A. \tag{3}$$

It is known [3] that

$$\Sigma\cos\tfrac{1}{2}(B-C) \le \Sigma\cos A + \Sigma\sin\tfrac{1}{2}A;$$

hence (3) follows from the sharper inequality

$$\Sigma\cos A + \Sigma\sin\tfrac{1}{2}A \le \frac{5}{2} + 4\Pi\sin\tfrac{1}{2}A. \tag{4}$$

To establish (4), we simply note that the well-known identity

$$\Sigma\cos A = 1 + 4\Pi\sin\tfrac{1}{2}A$$

reduces (4) to

$$1 + \Sigma\sin\tfrac{1}{2}A \le \frac{5}{2},$$

which, in turn, is equivalent to the known [second inequality in (1)].

III. *Comment by J.T. Groenman, Arnhem, The Netherlands.*

[With the first relation in (2)] and the inequality of Crux 613 [1982: 55, 67], we get

$$T_a+T_b+T_c \geq \frac{4R}{\sqrt{3}}\Sigma \sin A = \frac{4s}{\sqrt{3}},$$

where s is the semiperimeter; and this, with the inequality of this problem, gives

$$\frac{4s}{\sqrt{3}} \leq 5R + 2r. \qquad (5)$$

We now attempt a bit of exegesis. Reverently citing chapter and verse from the Bottema Bible [2], from $2r \leq R$ [Ch. 5, v. 1] and (5), we obtain

$$2s \leq \frac{\sqrt{3}}{2}(5R+2r) \leq 3\sqrt{3}R,$$

which sharpens the inequality $2s \leq 3\sqrt{3}R$ [Ch. 5, v. 3].

Also solved by JAYANTA BHATTACHARYA, Midnapur, West Bengal, India; W.J. BLUNDON, Memorial University of Newfoundland; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

Editor's comment.

Klamkin noted that

$$\Sigma \cos \tfrac{1}{2}(B-C) \leq \frac{5}{2} + \frac{r}{R}$$

is equivalent to the proposed inequality. This is obvious from the first relation in (2).

Calculus is rarely, if ever, the most appropriate tool to use in establishing triangle inequalities. Three of the above solvers used calculus. They showed, after extensive calculations involving first and second partial derivatives, that a certain function $f(A,B,C)$ attains a (relative) extremum when $A = B = C$. But it is not clear that in all cases they succeeded in proving that this extremum is *absolute* for all triangles ABC.

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1. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960, p. 185.

2. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

3. Leon Bankoff, Solution of Problem E 2298 (proposed by Anders Bager), *American Mathematical Monthly*, 79 (1972) 520.

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629. [1981: 117] *Proposed by S.C. Chan, Singapore.*

For which constants λ and m does the infinite series

$$\frac{1^3\lambda}{1!}e^{-m} + \frac{2^3\lambda^2}{2!}e^{-2m} + \frac{3^3\lambda^3}{3!}e^{-3m} + \frac{4^3\lambda^4}{4!}e^{-4m} + \ldots$$

converge, and to what sum?

Solution by W.J. Blundon, Memorial University of Newfoundland.

Let $z = \lambda e^{-m}$. For any nonnegative integer k, the series

$$S_k(z) = \sum_{n=1}^{\infty} \frac{n^k z^n}{n!}$$

converges absolutely for all complex z by the ratio test (hence for all λ and m), and we have to evaluate $S_3(z)$. Since

$$n^3 = n + 3n(n-1) + n(n-1)(n-2),$$

so that, for $n \geq 3$,

$$\frac{n^3}{n!} = \frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!},$$

we have

$$\begin{aligned} S_3(z) &= z + 4z^2 + \sum_{n=3}^{\infty}\left\{\frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!}\right\}z^n \\ &= z + 4z^2 + z\sum_{n=2}^{\infty}\frac{z^n}{n!} + 3z^2\sum_{n=1}^{\infty}\frac{z^n}{n!} + z^3\sum_{n=0}^{\infty}\frac{z^n}{n!} \\ &= z + 4z^2 + z(e^z-1-z) + 3z^2(e^z-1) + z^3e^z \\ &= z(1+3z+z^2)e^z. \end{aligned}$$

In terms of λ and m, we therefore have, for all complex λ and m,

$$S_3(\lambda,m) = \lambda e^{-m}(1 + 3\lambda e^{-m} + \lambda^2 e^{-2m})\exp(\lambda e^{-m}).$$

The sum $S_k(\lambda,m)$ can be evaluated just as easily for any nonnegative integer k.

Also solved by PAUL R. BEESACK, Carleton University, Ottawa; J.E. CHANCE, Pan American University, Edinburg, Texas; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; M.S. KLAMKIN, University of Alberta; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and KENNETH S. WILLIAMS, Carleton University, Ottawa. A partial solution was received from BIKASH K. GHOSH, Bombay, India.

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630. [1981: 117] *Proposed by Charles W. Trigg, San Diego, California.*

Partition the palindrome 2662 into two integers, one of which is divisible by 29 and the other by 43.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

We have to find integer pairs (x,y) such that $29x + 43y = 2662$. Since 29 and 43 are relatively prime, we use the Euclidean algorithm to find integers a and b such that $29a + 43b = 1$, obtaining

$$29\cdot 3 + 43\cdot(-2) = 1,$$

from which

$$29\cdot 7986 + 43\cdot(-5324) = 2662,$$

and all solutions are given by

$$(x, y) = (7986-43t, -5324+29t)$$

for any integer t.

If (as probably intended by the proposer) positive integer solutions are desired, then we must have

$$183.6 \approx \frac{5324}{29} < t < \frac{7986}{43} \approx 185.7,$$

and setting $t = 184$ and 185 yields the solutions

$$(x, y) = (74, 12) \text{ and } (31, 41).$$

The required partitions are therefore

$$29\cdot 74 + 43\cdot 12 = 2146 + 516 = 2662,$$
$$29\cdot 31 + 43\cdot 41 = 899 + 1763 = 2662.$$

Also solved by JAYANTA BHATTACHARYA, Midnapur, West Bengal, India; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; SIDNEY KRAVITZ, Dover, New Jersey; J.A. McCALLUM, Medicine Hat, Alberta; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; ALAN WAYNE, Holiday, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Partial solutions (one answer only) were received from PAUL R. BEESACK, Carleton University, Ottawa; W.J. BLUNDON, Memorial University of Newfoundland; BIKASH K. GHOSH, Bombay, India; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; and ROBERT S. JOHNSON, Montréal, Québec.

Editor's comment.

Most of our partial solvers "proved" that their answer was unique.

The proposer noted that in the first partition the digit sums of the two integers are the consecutive 12 and 13, and that in the second partition the palindrome is the sum of the products of consecutive twin primes. One expected no less from a Count (formerly Prince) of Digit Delvers.

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631. [1981: 117] *Proposed by Sidney Kravitz, Dover, New Jersey.*

Richard Bedford BENNETT (1870-1947) and Sir Wilfrid LAURIER (1841-1919) are former Canadian prime ministers; and Pierre-Elliott TRUDEAU is, of course, the current one. They have buried their political differences and find themselves united in the following alphametic, which you are asked to solve:

$$\begin{array}{r} \text{BENNETT} \\ \underline{\text{TRUDEAU}} \\ \text{LAURIER} \end{array}.$$

Solution by Clayton W. Dodge, University of Maine at Orono.

We denote the carries into the various columns by Greek letters, thus:

$$\begin{array}{l} \alpha\beta\gamma\delta\theta\lambda \\ \text{BENNETT} \\ \underline{\text{TRUDEAU}} \\ \text{LAURIER} \end{array}.$$

Since $\gamma+N+U = U+10\beta$, we have either

(a) $\beta = \gamma = 1$ and $N = 9$, so $\delta = 0$, $D = R+1$, and $E \le 4$;

or else

(b) $\beta = \gamma = N = 0$, so $\delta = 1$, $D+1 = R$, and $E \ge 5$.

In either case, from the 10- and 10^5-columns, we have

$$\lambda+T+\beta+R = 10(\alpha+\theta),$$

so $\alpha+\theta = 1$ and $T+R = 8$, 9, or 10.

In case (a), we have $T+R = 8$ or 9, and testing all the possibilities yields no solution. In case (b), we have $T+R = 9$ or 10, and testing the various possibilities produces the unique solution

$$\begin{array}{r} 1800855 \\ \underline{5493829} \\ 7294684 \end{array}.$$

Also solved by J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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632. [1981: 117] *Proposed by Leroy F. Meyers, The Ohio State University.*

Let ABCD be a plane quadrilateral in Euclidean 3-dimensional space. Find a simple formula for the area of ABCD.

I. *Solution by the proposer.*

Using only the information given, viz., the four vertices A,B,C,D (assumed given in the order of their occurrence along the perimeter of the quadrilateral), the area (or its absolute value if signed areas are used) is given by

$$|\vec{AC} \times \vec{BD}|/2, \tag{1}$$

a formula that contains only 11 symbols, including the arrows. For if lines are drawn through A and C parallel to BD, and through B and D parallel to AC, then these lines enclose a parallelogram whose area is twice that of the given quadrilateral, but is also $|\vec{AC} \times \vec{BD}|$.

Formula (1) is valid for every quadrilateral ABCD, convex or not, simple or not, degenerate or not. In particular, if we set D = C, we obtain $|\vec{AC} \times \vec{BC}|/2$, the well-known formula for the area of a triangle ABC.

II. *Suggested by a comment of M.S. Klamkin, University of Alberta.*

Using only the information given, the formula

$$[ABC] + [CDA], \tag{2}$$

where the brackets denote signed area, is just as "simple" as (1), since it also contains 11 symbols; and it has the added advantage of giving the area [ABCD] in sign as well as in absolute value. However, (2) is of limited usefulness by itself since it is merely the *definition* of the signed area [ABCD].

Also solved by CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

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633. [1981: 145] *Proposed by Janis Aldins; James S. Kline, M.D.; and Stan Wagon, Smith College, Northampton, Massachusetts (jointly).*

It follows from the Wallace-Bolyai-Gerwien Theorem of the early nineteenth century that *any triangle may be cut up into pieces which may be rearranged using only translations and rotations to form the mirror image of the given triangle.* This problem once appeared in a Moscow Mathematical Olympiad (see V.G. Boltianskii, *Hilbert's Third Problem,* Winston, Washington, 1978, p.70, where a three-cut solution is given).

Show that such a dissection may be effected with only two straight cuts.

I. *Solution by Benji Fisher, student, Bronx High School of Science, Bronx, N.Y.*

Given a triangle ABC, at least one altitude, say AD, falls within the triangle (see Figure 1). If M and N are the midpoints of AB and AC, respectively, then the two straight cuts DM and DN suffice. For if we rotate triangles DBM and DCN about M and N, respectively, through 180°, we obtain triangle DEF, the mirror image of triangle ABC.

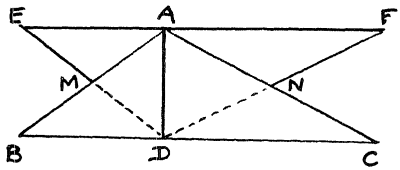

Figure 1

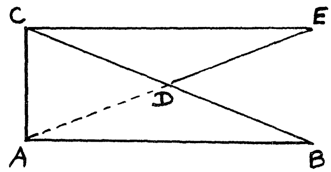

Figure 2

If triangle ABC is right-angled, say at A, as shown in Figure 2, then one straight cut along the median AD suffices. For a 180° rotation of triangle ABD about D produces triangle CEA, the mirror image of triangle ABC.

The proof in each case is obvious.

II. *Comment by Joe Konhauser, Macalester College, Saint Paul, Minnesota.*

A solution with two straight cuts [similar to the above] was given in 1971 by Engel [1].

Also solved by JORDI DOU, Barcelona, Spain; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposers.

Editor's comment.

All solutions received were the same, but we decided to let the junior member of the team carry the flag. The solution was (re)discovered and proposed here by a carpenter, a physician, and a mathematician.

REFERENCE

1. Arthur Engel, "Geometrical Activities for the Upper Elementary School", *Educational Studies in Mathematics*, 3 (June 1971) 353-394, esp. p. 385.

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634. [1981: 145] *Proposé par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Le nombre 2701 = 37·73 n'est pas premier. Montrer, néanmoins, que $3^{2701} \equiv 3 \pmod{2701}$.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

All congruences being modulo 2701, we have

$$3^{36} = 19683^4 \equiv 776^4 = 602176^2 \equiv (-147)^2 = 21609 \equiv 1;$$

hence

$$3^{2701} = 3\cdot(3^{36})^{75} \equiv 3\cdot 1^{75} = 3.$$

II. *Solution by W.J. Blundon, Memorial University of Newfoundland.*

Since 37 and 73 are primes, Fermat's Theorem gives

$$3^{37} \equiv 3 \pmod{37} \qquad \text{and} \qquad 3^{73} \equiv 3 \pmod{73};$$

hence

$$3^{2701} = (3^{37})^{73} \equiv 3^{73} = 3^{37}\cdot 3^{36} \equiv 3\cdot 1 = 3 \pmod{37} \tag{1}$$

and

$$3^{2701} = (3^{73})^{37} \equiv 3^{37} = (3^{6})^{6}\cdot 3 \equiv (-1)^{6}\cdot 3 = 3 \pmod{73}. \tag{2}$$

Now, from (1), (2), and the Chinese Remainder Theorem,

$$3^{2701} \equiv 3 \pmod{2701}.$$

III. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

Let a be an integer and p a positive integer such that $(a,p) = 1$. Fermat's Theorem states that *if p is a prime, then $a^{p-1} \equiv 1 \pmod p$.* The example of this problem, if true, shows that the converse of Fermat's Theorem is not true: if $(a,n) = 1$, then $a^{n-1} \equiv 1 \pmod n$ may hold for some composite positive integers n. Our problem will follow from the following

THEOREM. *Let p and q be primes with $q > p$, and let a be an integer such that $(a,p) = (a,q) = 1$. If there exist a positive integer k and an integer b such that $q - 1 = k(p-1)$ and $a \equiv b^k \pmod q$, then*

$$a^{pq-1} \equiv 1 \pmod{pq}, \tag{3}$$

and conversely.

With $p = 37$, $q = 73$, and $a = 3$, the integers $k = 2$ and $b = \pm 21$ satisfy the hypothesis of the theorem; hence

$$3^{2700} \equiv 1 \text{ and } 3^{2701} \equiv 3 \pmod{2701}.$$

Proof of the theorem. By Fermat's Theorem, the hypothesis yields

$$a^{pq-1} = (a^{p})^{q-1}\cdot a^{p-1} \equiv 1\cdot a^{p-1} \equiv (b^{k})^{p-1} = b^{q-1} \equiv 1 \pmod q \tag{4}$$

and

$$a^{pq-1} = (a^{q})^{p-1}\cdot a^{q-1} \equiv 1\cdot a^{q-1} = (a^{k})^{p-1} \equiv 1 \pmod p. \tag{5}$$

Now (3) follows from (4), (5), and the Chinese Remainder Theorem.

To prove the converse, we assume (3), which implies that

$$a^{pq-1} \equiv 1 \pmod p \text{ and } \pmod q. \tag{6}$$

If $q - 1 = k(p-1)+r$, $0 \le r < p - 1$, then we have

$$a^{pq-1} = (a^q)^{p-1} \cdot a^{q-1} \equiv 1 \cdot a^{q-1} = (a^k)^{p-1} \cdot a^r \equiv a^r \pmod{p}.$$

Now $a^r \equiv 1 \pmod{p}$ follows from (6), so $r = 0$ and $q - 1 = k(p-1)$. We also have

$$a^{pq-1} = (a^p)^{q-1} \cdot a^{p-1} \equiv a^{p-1} \pmod{q};$$

and since $a^{p-1} \equiv 1 \pmod{q}$ from (6), there is an integer b such that

$$a^{p-1} \equiv 1 \equiv b^{q-1} = (b^k)^{p-1} \pmod{q},$$

and so $a \equiv b^k \pmod{q}$. □

There is an extensive literature on the converse of Fermat's Theorem. References [1]-[5], from which many others can be obtained, are given below.

Our theorem and the specific example of the problem show that for certain composite positive integers n there may exist integers a such that

$$a^{n-1} \equiv 1 \pmod{n}. \tag{7}$$

Much more remarkable is the fact that there are composite positive integers n such that (7) holds for all integers a that are relatively prime to n (and hence $a^n \equiv a \pmod{n}$ for *all* a). Such numbers n are said to have the *Fermat property* and are called *F numbers* [4] or *Carmichael numbers* [5]. It is not known if there are infinitely many Carmichael numbers. The first few are 561, 1105, 1729, 2465 [5]. All Carmichael numbers are odd and have at least three different prime factors [4] (so $2701 = 37 \cdot 73$ is *not* a Carmichael number). A necessary and sufficient condition for a positive integer n to have the Fermat property is that

$$n \equiv 1 \pmod{\lambda(n)},$$

where $\lambda(n)$ is the smallest positive integer N such $a^N \equiv 1 \pmod{n}$ for every a relatively prime to n [4]. The number $\lambda(n)$ exists for every n, and $\lambda(n) \le \phi(n)$, where ϕ is the Euler function.

Also solved by JORDI DOU, Barcelona, Spain; BENJI FISHER, student, Bronx High School of Science, Bronx, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer. In addition, three incorrect solutions were received.

Editor's comment.

One of our incorrect solvers, call him Mr. A, used as a key step in his solution the false congruence $3^{18} \equiv 1 \pmod{73}$. Another one, Mr. B, applied Fermat's Theorem to congruences involving *fractional* exponents, thus: if $(a,73) = 1$, then

$$a^{108} = (a^{73-1})^{3/2} \equiv 1 \pmod{73}.$$

Suppose this is true. With $a = 5$ we get $(5^6)^{18} \equiv 1 \pmod{73}$. As $5^6 \equiv 3 \pmod{73}$, we therefore get again $3^{18} \equiv 1 \pmod{73}$. Mr. B, meet Mr. A. Our third incorect[1] solver, who will not be identified even by a letter, apparently did not put on his glasses before reading the problem. His solution ran essentially thus: Since 2701 is prime and $(3,2701) = 1$, we have $3^{2701} \equiv 3 \pmod{2701}$ by Fermat's Theorem. Next question?

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2. Jack Chernick, "On Fermat's Simple Theorem", *ibid.*, 45 (April 1939) 269-274.

3. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol.I, pp. 92-95.

4. Oystein Ore, *Number Theory and its History*, McGraw-Hill, New York, 1948, Chapter 14: The Converse of Fermat's Theorem.

5. Daniel Shanks, *Solved and Unsolved Problems in Number Theory*, Second Edition, Chelsea, New York, 1978, pp. 115-118.

* * *

NOT EUCLID ALONE
HAS LOOKED ON BEAUTY BARE

[*Page 16*] From my childhood to my university days, mathematics and science were my principal interests, and chess my main hobby. I was particularly fascinated by geometry, algebra, and physics because I was convinced — much as the Pythagoreans and the alchemists had been — that these disciplines contained the clue to the mystery of existence. ...

It is difficult to convey a child's delight and excitement in penetrating the mysteries of the Pythagorean triangle, or of Kepler's laws of planetary movement, or of Planck's theory of quanta. [*Some kid, huh?*]

[*Page 17*] The idea that infinity would remain an unsolved riddle was unbearable. The more so as I had learned that a finite quantity like the earth — or like myself reclining on it — shrank to zero when divided by an infinite quantity. So, mathematically, if space was infinite, the earth was zero and I was zero and one's life-span was zero, and a year and a century were zero. It made no sense, there was a miscalculation somewhere, and the answer to the riddle was obviously to be found by reading more books about gravity, electricity, astronomy, and higher mathematics. [*Much higher.*]

[*Page 49*] Chapter title: Mount of Olivers [*sic*] to Montparnasse.

ARTHUR KOESTLER, in *Bricks to Babel*
Hutchinson, London, 1980
[*Better title: Bricks Without Straws?*]

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[1] This is an autological adjective [1975: 55] not included in the solution of Problem 61 [1975: 98].