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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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THE OLYMPIAD CORNER

No. 134

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first problems we present are from the American Invitational Mathematics Examination (A.I.M.E.) written April 2, 1992. The time allowed was three hours. The problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

1992 AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

1. Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

2. A positive integer is called "ascending" if, in its decimal representation, there are at least two digits and each digit is less than any digit to its right. How many ascending positive integers are there?

3. A tennis player computes her "win ratio" by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly .500. During the weekend she plays four matches, winning three and losing one. At the end of the weekend her win ratio is greater than .503. What is the largest number of matches that she could have won before the weekend began?

4. In Pascal's triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below.

Row 0:							1						
Row 1:						1		1					
Row 2:					1		2		1				
Row 3:				1		3		3		1			
Row 4:			1		4		6		4		1		
Row 5:		1		5		10		10		5		1	
Row 6:	1		6		15		20		15		6		1

In which row of Pascal's triangle do three consecutive entries occur that are in the ratio $3:4:5$?

5. Let S be the set of all rational numbers r, $0 < r < 1$, that have a repeating decimal expansion of the form

$$0.abcabcabc\ldots = 0.\overline{abc},$$

where the digits a, b, c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

6. For how many pairs of consecutive integers in $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

7. Faces ABC and BCD of tetrahedron $ABCD$ meet at an angle of 30°. The area of face ABC is 120, the area of face BCD is 80, and $BC = 10$. Find the volume of the tetrahedron.

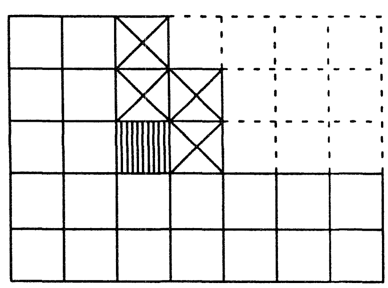
8. For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$, whose nth term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta\,(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

9. Trapezoid $ABCD$ has sides $AB = 92$, $BC = 50$, $CD = 19$, and $AD = 70$, with $\overline{AB}$ parallel to $\overline{CD}$. A circle with center P on $\overline{AB}$ is drawn tangent to $\overline{BC}$ and $\overline{AD}$. Given that $AP = m/n$, where m and n are relatively prime positive integers, find $m + n$.

10. Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1 inclusive. What is the integer that is nearest the area of A? (If $z = x + iy$ with x and y real, then $\overline{z} = x - iy$ is the conjugate of z.)

11. Lines ℓ_1 and ℓ_2 both pass through the origin and make first-quadrant angles of $\frac{\pi}{70}$ and $\frac{\pi}{54}$ radians, respectively, with the positive x-axis. For any line ℓ, the transformation $R(\ell)$ produces another line as follows: ℓ is reflected in ℓ_1, and the resulting line is then reflected in ℓ_2. Let $R^{(1)}(\ell) = R(\ell)$, and for integer $n \geq 2$ define $R^{(n)}(\ell) = R(R^{(n-1)}(\ell))$. Given that ℓ is the line $y = \frac{19}{92}x$, find the smallest positive integer m for which $R^{(m)}(\ell) = \ell$.

12. In a game of *Chomp*, two players alternately take "bites" from a 5-by-7 grid of unit squares. To take a bite, the player chooses one of the remaining squares, then removes ("eats") all squares found in the quadrant defined by the left edge (extended upward) and the lower edge (extended rightward) of the chosen square. For example, the bite determined by the shaded square in the diagram would remove the shaded square and the four squares marked by $\times$. (The squares with two or more dotted edges have been removed from the original board in previous moves.) The object of the game is to make one's opponent take the last bite. The diagram shows one of the many subsets of the set of 35 unit squares that can occur during games of Chomp. How many different subsets are there in all? Include the full board and the empty board in your count.



13. Triangle ABC has $AB = 9$ and $BC : CA = 40 : 41$. What is the largest area that this triangle can have?

14. In triangle ABC, A', B', and C' are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92,$$

find the value of

$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} .$$

15. Define a positive integer n to be a "factorial tail" if there is some positive integer m such that the base-ten representation of $m!$ ends with exactly n zeros. How many positive integers less than 1992 are *not* factorial tails?

* * *

The Olympiad problems we give this month are those of the *21st Austrian Mathematical Olympiad, 1990.* Many thanks to Walther Janous, Ursulinengymnasium, Innsbruck, Austria, for translating the problems. He points out that the distinction between 'natural' and 'integer' number in 2nd Round #1 and Final Round #4 was the cause of confusion and quite a few erroneous 'solutions'.

21st AUSTRIAN MATHEMATICAL OLYMPIAD

2nd Round (May 3, 1990): 4 hours

1. Prove: There exists no natural number n such that the total number of integer factors of n equals 1990 and the sum of the inverses $1/b$ of all natural number factors b of n equals 2.

2. Solve (in $\mathbf{R}$) the equation $\sqrt[3]{2x-7} + \sqrt[3]{3x-3} = \sqrt[3]{x-8} + \sqrt[3]{4x-2}$.

3. Let ABC be a triangle with E and D the feet of the altitudes to sides b and a, respectively. Let M be the point on AD such that $AD = DM$.

(a) Show there exists no acute-angled triangle ABC such that C, D, E, M lie on a circle.

(b) Determine all triangles ABC such that $CDEM$ do lie on a circle.

4. For natural numbers $k, n \geq 2$, determine the sum

$$S(k,n) = \left[\frac{2^{n+1}+1}{2^{n-1}+1}\right] + \left[\frac{3^{n+1}+1}{3^{n-1}+1}\right] + \cdots + \left[\frac{k^{n+1}+1}{k^{n-1}+1}\right]$$

where $[x]$ denotes the greatest integer $\leq x$.

Final Round

1st Day — May 30, 1990: 4.5 hours

1. Determine the number of all natural numbers n such that $1 \leq n \leq N = 1990^{1990}$ with $n^2 - 1$ and N relatively prime.

2. Show that for all natural numbers $n \geq 2$

$$\sqrt{2\sqrt[3]{3\sqrt[4]{4\ldots\sqrt[n]{n}}}} < 2.$$

3. In a convex quadrilateral $ABCD$ (all interior angles $< 180°$) let E be the intersection of the diagonals and F_1, F_2, and F the areas of ABE, CDE and $ABCD$, respectively. Show that

$$\sqrt{F_1} + \sqrt{F_2} \leq \sqrt{F}$$

and determine when equality holds.

2nd Day — May 31, 1990: 4.5 hours

4. For each integer $n \neq 0$ determine all functions $f : \mathbf{R}\backslash\{-3, 0\} \to \mathbf{R}$ such that

$$f(x+3) + f\left(-\frac{9}{x}\right) = \frac{(1-n)(x^2+3x-9)}{9n(x+3)} + \frac{2}{n}$$

for all $x \neq 0, -3$. Furthermore, for each fixed natural number n determine all integers x such that $f(x)$ is an integer.

5. Determine all rational numbers r such that all solutions of $rx^2+(r+1)x+(r-1) = 0$ are integers.

6. The convex pentagon $ABCDE$ has a circumcircle. The perpendicular distances of A from the lines through B and C, C and D, and D and E are a, b and c, respectively. Determine (as a function of a, b and c) the perpendicular distance of A from the diagonal BE.

* * *

We next move to solutions of problems posed in the 1991 volume of *Crux*.

2. [1991: 1] *1980 Celebration of Chinese New Year Contest.*

Let n be a positive integer. Is the greatest integer less than $(3+\sqrt{7})^n$ odd or even?

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

If $(3+\sqrt{7})^{2n} = a_n + b_n\sqrt{7}$ (a_n and b_n are integers), from the binomial formula $(3-\sqrt{7})^{2n} = a_n - b_n\sqrt{7}$. It follows that $(3+\sqrt{7})^{2n} + (3-\sqrt{7})^{2n} = 2a_n$ and $(3+\sqrt{7})^{2n+1} + (3-\sqrt{7})^{2n+1} = 2(3a_n + 7b_n)$. Since $0 < 3-\sqrt{7} < 1$, we have

$$[(3+\sqrt{7})^{2n}] = 2a_n - 1 \quad \text{and} \quad [(3+\sqrt{7})^{2n+1}] = 2(3a_n + 7b_n) - 1.$$

Thus $[(3+\sqrt{7})^n]$ is odd for all integers n.

4. [1991: 1] *1980 Celebration of Chinese New Year Contest.*
Denote by a_n the integer closest to $\sqrt{n}$. Determine

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{1980}}\,.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Shailesh Shirali, Rishi Valley School, Chittoor Dt., India. We give Shirali's solution.

Note that

$$\begin{aligned} a_n = k &\iff k-\frac{1}{2} < \sqrt{n} < k+\frac{1}{2} \\ &\iff \left(k-\frac{1}{2}\right)^2 < n < \left(k+\frac{1}{2}\right)^2 \\ &\iff k^2-k < n \le k^2+k. \end{aligned}$$

Therefore the cardinality of the set $\{n : a_n = k\}$ is just $k^2+k-(k^2-k) = 2k$. That is, there are $2k$ n's for which $a_n = k$. It follows that if $l^2 - l < m \le l^2 + l$ then

$$\sum_{n=1}^{m}\frac{1}{a_n} = \underbrace{2+\cdots+2}_{(l-1)\ 2\text{'s}} + \frac{m-(l^2-l)}{l} = l-1+\frac{m}{l}\,.$$

For $m = 1980$, $l = 44$, so the required sum equals $43+\frac{1980}{44} = 88$.

1. [1991: 2] *1981 Celebration of Chinese New Year Contest.*
What is the coefficient of x^2 when

$$(\ldots(((x-2)^2-2)^2-2)^2-\cdots-2)^2$$

is expanded and like terms are combined?

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let $P_n(x)$ be defined recursively as follows:

$$P_n(x) = (P_{n-1}(x)-2)^2, \quad P_1(x) = (x-2)^2.$$

Then the coefficient of x^2 in the polynomial is $P_n''(0)/2!$. Note that

$$P_n'(x) = 2(P_{n-1}(x)-2)P_{n-1}'(x)$$

and

$$P_n''(x) = 2(P_{n-1}'(x))^2 + 2(P_{n-1}(x)-2)P_{n-1}''(x).$$

Also $P_n(0) = 4$. Since $P_n'(0) = 2(P_{n-1}(0)-2)P_{n-1}'(0) = 4P_{n-1}'(0)$ and $P_1'(0) = -4$, we have $P_n'(0) = -4^n$. Also,

$$P_n''(0) = 2(P_{n-1}'(0))^2 + 2(P_{n-1}(0)-2)P_{n-1}''(0) = 2\cdot 4^{2n} + 4P_{n-1}''(0)$$

and $P_1''(0) = 2$. Solving the recurrence relation, we have

$$P_n''(0) = \frac{-61}{6}4^n + \frac{8}{3}4^{2n}.$$

The coefficient of x^2 in case the expansion results from n applications is thus $\frac{-61}{6}4^n + \frac{8}{3}4^{2n}$.

2. [1991: 2] *1981 Celebration of Chinese New Year Contest.*
Prove that $1980^{1981^{1982}} + 1982^{1981^{1980}}$ is divisible by 1981^{1981}.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.
Let $a = 1981 = 7 \cdot 283$. From the binomial formula

$$(a-1)^{a^{a+1}} + (a+1)^{a^{a-1}} = \sum_{r=0}^{a-1} a^r \left\{ \binom{a^{a+1}}{r}(-1)^r + \binom{a^{a-1}}{r} \right\} + a^a L$$

for some integer L. It is well known that if p is a prime and p^s divides $r!$, then

$$s \le \left[\frac{r}{p}\right] + \left[\frac{r}{p^2}\right] + \cdots \le \left[\frac{r}{p-1}\right].$$

Since 7 and 283 are primes and $[r/6] \le r-1$, $[r/282] \le r-1$, $(r \ge 1)$, we have

$$\binom{a^{a+1}}{r} a^r = a^a M \quad \text{and} \quad \binom{a^{a-1}}{r} a^r = a^a N$$

for some integers M and N for $r \ge 1$. It follows that $(a-1)^{a^{a+1}} + (a+1)^{a^{a-1}}$ is divisible by a^a.

3. [1991: 2] *1981 Celebration of Chinese New Year Contest.*
Let $f(x) = x^{99} + x^{98} + x^{97} + \cdots + x^2 + x + 1$. Determine the remainder when $f(x^{100})$ is divided by $f(x)$.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.
Note that

$$f(x) = \frac{x^{100} - 1}{x - 1}.$$

If x_0 is a (complex) zero of $f(z)$ then x_0 is a simple zero of $f(z)$ because

$$f'(x_0) = \frac{100x_0^{99}(x_0 - 1) - (x_0^{100} - 1)}{(x_0 - 1)^2} = \frac{100}{x_0(x_0 - 1)} \ne 0.$$

Let $x_1, x_2, \ldots, x_{99}$ be all the distinct zeroes of $f(x)$, and let

$$r(x) = a_{98}x^{98} + a_{97}x^{97} + \cdots + a_1 x + a_0$$

be the remainder when $f(x^{100})$ is divided by $f(x)$. From $f(x_i^{100}) = 100 = r(x_i)$ for $i = 1, 2, \ldots, 99$ we have the following system of equations for $a_0, \ldots, a_{98}$:

$$\begin{aligned}
a_0 + a_1x_1 + &\cdots + a_{98}x_1^{98} = 100\\
a_0 + a_1x_2 + &\cdots + a_{98}x_2^{98} = 100\\
&\cdots\\
a_0 + a_1x_{99} + &\cdots + a_{98}x_{99}^{98} = 100.
\end{aligned}$$

Note that the determinant of the coefficient matrix is

$$\Delta = \begin{vmatrix} 1 & x_1 & \dots & x_1^{98} \\ 1 & x_2 & \dots & x_2^{98} \\ \vdots & & & \\ 1 & x_{99} & \dots & x_{99}^{98} \end{vmatrix} = \prod_{j<i} (x_i - x_j) \neq 0.$$

It follows from Cramer's rule that $a_i = 0$ $(i = 1, \dots, 98)$ and $a_0 = 100$. Thus $r(x) = 100$.

* * *

The only solution received to problems from the February 1991 number of *Crux* is the following.

Exercise IV. [1991: 34] *Composition de Mathématiques (France).*

Consider five points M_1, M_2, M_3, M_4, M situated on a circle C in the plane. Show that the product of the distances of M from the lines M_1M_2 and M_3M_4 equals the product of the distances from M to the lines M_1M_3 and M_2M_4. What can one deduce about $2n+1$ distinct points $M_1, \dots, M_{2n}, M$ situated on C?

Solution by Murray S. Klamkin, University of Alberta.

This is a known result which appears, for example, in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, N.Y., 1960, pp. 71–72.

Firstly, using the formula $abc = 4FR$ for a triangle with sides a, b, c, area F and circumradius R, it follows that for triangle MM_1M_2,

$$MM_1 \cdot MM_2 = h_{12}D$$

where h_{12} is the distance from M to M_1M_2 and D is the diameter of the circle, and similar results for the other segments. Thus

$$h_{12} \cdot h_{34} = h_{13} \cdot h_{24} = MM_1 \cdot MM_2 \cdot MM_3 \cdot MM_4/D^2 \quad (= \text{ also } h_{14} \cdot h_{23}).$$

For more general results with $2n+1$ points, just note the next two theorems (also in Johnson) which are obtained similarly.

Theorem. Let $M_1, M_2, \dots, M_{2n}, M$ be points on a circle and h_{ij} the distance from M to M_iM_j. Then if we form a product of n of the h_{ij}'s so that each subscript appears once and only once then all such products have the same value.

Theorem. If a polygon is inscribed in a circle, and a second polygon is circumscribed by drawing tangents to the circle at the vertices of the first, the product of the perpendiculars on the sides of the first, from a point on the circle, equals the products of the perpendiculars from the same point to the sides of the second.

* * *

For the next solutions we turn to problems from the March 1991 number of *Crux* and the problems of the *12th Austrian-Polish Mathematics Competition* [1991: 65–66].

1. Let $a_1, \dots, a_n$, $b_1, \dots, b_n$, $c_1, \dots, c_n$ be positive real numbers. Show that

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^3 \le \left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{k=1}^{n} b_k^3\right)\left(\sum_{k=1}^{n} c_k^3\right).$$

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.
We begin by noting that if x, y, z are positive real numbers, then

$$xyz = \sqrt[3]{x^3y^3z^3} \le \frac{x^3+y^3+z^3}{3}.$$

by the Arithmetic Mean-Geometric Mean Inequality.

Let

$$A^3 = \sum_{k=1}^{n} a_k^3, \quad B^3 = \sum_{k=1}^{n} b_k^3, \quad C^3 = \sum_{k=1}^{n} c_k^3,$$
$$x_1 = a_1/A,\ x_2 = a_2/A,\ \dots,\ x_n = a_n/A,$$
$$y_1 = b_1/B,\ y_2 = b_2/B,\ \dots,\ y_n = b_n/B,$$
$$z_1 = c_1/C,\ z_2 = c_2/C,\ \dots,\ z_n = c_n/C.$$

From the above,

$$a_1b_1c_1 = ABCx_1y_1z_1 \le \frac{ABC(x_1^3+y_1^3+z_1^3)}{3},$$
$$a_2b_2c_2 = ABCx_2y_2z_2 \le \frac{ABC(x_2^3+y_2^3+z_2^3)}{3},$$
$$\dots$$
$$a_nb_nc_n = ABCx_ny_nz_n \le \frac{ABC(x_n^3+y_n^3+z_n^3)}{3}.$$

Summing the above inequalities yields

$$\sum_{k=1}^{n} a_kb_kc_k \le ABC\left(\frac{\sum_{k=1}^n x_k^3 + \sum_{k=1}^n y_k^3 + \sum_{k=1}^n z_k^3}{3}\right).$$

But

$$\sum_{k=1}^{n} x_k^3 = \sum_{k=1}^{n}\left(\frac{a_k}{A}\right)^3 = \frac{\sum_{k=1}^n a_k^3}{A^3} = \frac{A^3}{A^3} = 1.$$

Similarly $\sum_{k=1}^n y_k^3 = 1 = \sum_{k=1}^n z_k^3$. Hence $\sum_{k=1}^n a_kb_kc_k \le ABC$. It follows that

$$\left(\sum_{k=1}^{n} a_kb_kc_k\right)^3 \le A^3B^3C^3 = \left(\sum_{k=1}^{n} a_k^3\right)\left(\sum_{k=1}^{n} b_k^3\right)\left(\sum_{k=1}^{n} c_k^3\right).$$

[*Editor's note.* Seung-Jin Bang, Seoul, Republic of Korea, points out that the problem is a special case of Hölder's inequality:

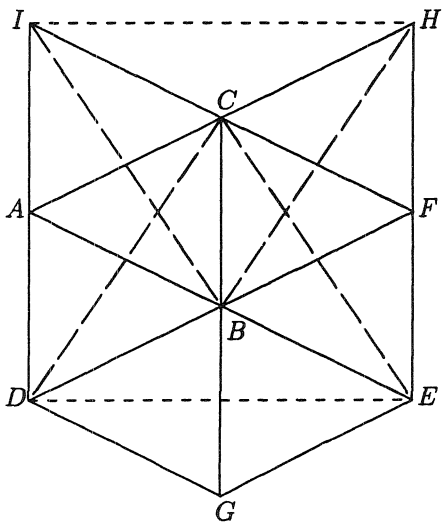
$$\sum_{i=1}^{m} a_{i1}^{q_1}a_{i2}^{q_2}a_{i3}^{q_3}\dots a_{in}^{q_n} < \left(\sum_{i=1}^{m} a_{i1}\right)^{q_1}\left(\sum_{i=1}^{m} a_{i2}\right)^{q_2}\dots\left(\sum_{i=1}^{m} a_{in}\right)^{q_n}$$

where $\sum_{i=1}^{n} q_i = 1$, $q_i \geq 0$, unless either $(a_{i1}, a_{i2}, \ldots, a_{in})$ $(i = 1, \ldots, m)$ are all proportional, or $(a_{i1}, \ldots, a_{in}) = (0, \ldots, 0)$ for some $i = 1, \ldots, m$. (Let $q_1 = q_2 = q_3 = 1/3$, $a_{k1} = a_k^3$, $a_{k2} = b_k^3$, $a_{k3} = c_k^3$). See Hardy, Littlewood, & Polya, *Inequalities*, 2nd Edition (1952), pp. 21–24.]

2. Each point of the plane ($\mathbf{R}^2$) is coloured by one of the two colours A and B. Show that there exists an equilateral triangle with monochromatic vertices.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Margherita Barile, student, Genova, Italy; and John Morvay, Springfield, Missouri. We give Barile's solution.

Let the two colours be black and white and let ABC be an equilateral triangle. We may assume, without loss of generality, that the vertices A and B are black. Suppose C is white and consider the figure where $ADGEFC$ is a regular hexagon (with B the centre). Now proceed as follows:



- ADB is equilateral. A, B are black; so suppose D is white.
- CDE is equilateral; C, D are white; so suppose E is black.
- BEF is equilateral; B, E are black; so suppose F is white.
- CFH is equilateral; C, F are white; so suppose H is black.
- BHI is equilateral; B, H are black; so suppose I is white.

Now IDF is equilateral and I, D, F are white, completing the proof.

3. Determine all natural numbers N (in decimal representation) satisfying the following properties:

(1) $N = (aabb)_{10}$, where $(aab)_{10}$ and $(abb)_{10}$ are primes.

(2) $N = P_1 \cdot P_2 \cdot P_3$, where P_k $(1 \leq k \leq 3)$ is a prime consisting of k (decimal) digits.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; Margherita Barile, student, Genova, Italy; and Stewart Metchette, Culver City, California. We give Bang's solution.

Since $(aab)_{10} = 110a + b$ is a prime, we obtain $b = 1, 3, 7$ or 9. From (1) we have $N = 11(100a + b)$. In a table of primes we obtain the pairs $((aab)_{10}, (aab)_{10})$ satisfying (1) as follows:

$$\begin{array}{llll} (223,233), & (227,277), & (331,311), & (443,433), \\ (449,499), & (557,577), & (773,733), & (881,811), \\ (887,877), & (991,911), & (997,977). & \end{array}$$

Corresponding to $100a + b$ are

$$\begin{array}{llll} 203 = 7 \times 29, & 207 = 9 \times 23, & 301 = 7 \times 43, & 403 = 13 \times 31, \\ 409 = \text{ prime }, & 507 = 3 \times 13^2, & 703 = 19 \times 37, & 801 = 3^2 \times 89, \\ 807 = 3 \times 269, & 901 = 17 \times 53, & 907 = \text{ prime} & \end{array}$$

Answer: $N = (8877)_{10} = 3 \times 11 \times 269$.

5. Let A be a vertex of a cube ω circumscribed about a sphere κ of radius 1. We consider lines g through A containing at least one point of κ. Let P be the point of $g \cap \kappa$ having minimal distance from A. Furthermore, $g \cap \omega$ is AQ. Determine the maximum value of $\overline{AP} \cdot \overline{AQ}$ and characterize the lines g yielding the maximum.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let the vertices of the given cube ω be $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, $(0,0,2)$, $(2,2,0)$, $(2,0,2)$, $(0,2,2)$, $(2,2,2)$ and let the equation of the given sphere κ be

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1.$$

We may assume that $A = (0,0,0)$, and the equation of the line g is $t(a,b,c)$ where t is real and $a^2 + b^2 + c^2 = 1$, $a, b, c \geq 0$. From $(ta-1)^2 + (tb-1)^2 + (tc-1)^2 = 1$ we have

$$P = (a + b + c - \sqrt{(a+b+c)^2 - 2}\,)(a,b,c).$$

Let $d = \max(a,b,c)$. Since $g \cap \omega = AQ$, we obtain $td = 2$ and

$$Q = \frac{2}{\max(a,b,c)}\,(a,b,c).$$

Note that

$$\begin{aligned} \overline{AP} \cdot \overline{AQ} &= 2 \cdot \frac{a+b+c - \sqrt{(a+b+c)^2 - 2}}{\max(a,b,c)} \\ &= \frac{4}{\max(a,b,c)(a+b+c+\sqrt{(a+b+c)^2-2}\,)}. \end{aligned}$$

We may assume that $a = \max(a,b,c)$. Then

$$f_a(b,c) = \overline{AP} \cdot \overline{AQ} = \frac{4}{a(a+b+c+\sqrt{(a+b+c)^2-2}\,)}$$

and $b^2 + c^2 = 1 - a^2$. Note that f_a is strictly decreasing as a function of $b + c$ and $\sqrt{1-a^2} \leq b + c$ holds for $b = 0$ or $c = 0$. So we may take $b = 0$ and $c = \sqrt{1-a^2}$. From $a = \max(a,b,c)$ we then have $a \geq \sqrt{1-a^2}$, so $a \geq 1/\sqrt{2}$. It follows that

$$\begin{aligned} f_a(b,c) &\leq \frac{4}{a(a+\sqrt{1-a^2}+\sqrt{(a+\sqrt{1-a^2}\,)^2-2}\,)} \\ &\leq \frac{4\sqrt{2}}{a+\sqrt{1-a^2}+\sqrt{(a+\sqrt{1-a^2}\,)^2-2}} \leq 4. \end{aligned}$$

The last inequality holds because

$$\frac{d}{da}\left(a+\sqrt{1-a^2}\,\right)=\frac{\sqrt{1-a^2}-a}{\sqrt{1-a^2}}<0$$

and $g(1/\sqrt{2})=4$ where

$$g(a)=\frac{4\sqrt{2}}{a+\sqrt{1-a^2}+\sqrt{(a+\sqrt{1-a^2}\,)^2-2}}\,.$$

It follows that $\overline{AP}\cdot\overline{AQ}\le 4$, with equality holding for $(a,b,c)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$, $(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$ or $(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. Thus $\overline{AP}\cdot\overline{AQ}\le 4$ with equality holding only for lines g such that $g\cap\kappa$ is one point.

6. We consider sequences $\{a_n : n\ge 1\}$ of squares of natural numbers (>0) such that for each n the difference $a_{n+1}-a_n$ is a prime or the square of a prime. Show that all such sequences are finite and determine the longest sequence $\{a_n : n\ge 1\}$.

Solution by Margherita Barile, student, Genova, Italy.

Given a sequence $\{a_n : n\ge 1\}$ of squares of positive integers, let

$$A=\sup\{p\in\mathbf{N} : a_n-a_{n-1}\text{ is a prime or the square of a prime for every } n\le p\}.$$

We prove that A is finite.

For each n let $a_n=b_n^2$, where $b_n>0$ is a natural number. Then if $n<A$,

$$a_{n+1}-a_n=b_{n+1}^2-b_n^2=(b_{n+1}+b_n)(b_{n+1}-b_n)=P_n^{\alpha_n}$$

where P_n is a prime and $\alpha_n=1$ or $\alpha_n=2$. Since $b_{n+1}+b_n=b_{n+1}-b_n$ implies $b_n=0$ and $a_n=0$, it must be the case that $b_{n+1}+b_n\ne b_{n+1}-b_n$. Therefore

$$b_{n+1}-b_n=1 \tag{1}$$

and

$$b_{n+1}+b_n=P_n^{\alpha_n}. \tag{2}$$

By (1) we have in particular that

$$b_2=b_1+1 \tag{3}$$

and

$$b_n=b_1+n\text{ for all } n \tag{4}$$

follows by induction. By (2) we have that

$$c_n=a_{n+1}-a_n=P_n^{\alpha_n}=2b_1+2n-1. \tag{5}$$

Thus the sequence $\{c_n : n\ge 1\}$ is defined recursively by

$$c_1=a_2-a_1,\quad c_{n+1}=c_n+2. \tag{6}$$

Let $C = \sum\{p \in \mathbf{N} : c_n$ is a prime or the square of a prime for every $n \leq p\}$. If C is finite, then $C = A + 1$ and A is finite. We prove that C is finite.

For $n = 1$ (5) yields $P_1^{\alpha_1} = 2b_1 + 1$. Suppose next that $A > n_0 = 2b_1 + 2$. By (5),

$$P_{n_0}^{\alpha n_0} = 2b_1 + 4b_1 + 3 = 6b_1 + 3 = 3P_1^{\alpha_1}.$$

This implies that $3|P_{n_0}$, i.e. $P_{n_0} = 3$, since P_{n_0} is a prime. Since $P_1^{\alpha_1} > 1$, it follows that $\alpha_0 = 2$, so that $6P_1 + 3 = 3^2 = 9$, implying $b_1 = 1$. Thus by (3) $b_2 = 2$ and $a_2 = b_2^2 = 4$, so that, by (6)

$c_1 = 4 - 1 = 3$	prime
$c_2 = 5$	prime
$c_3 = 7$	prime
$c_4 = 9 = 3^2$	square of a prime
$c_5 = 11$	prime
$c_6 = 13$	prime

Since $c_6 + 2 = 15$, which is neither a prime nor the square of a prime, $C = 6$ and therefore $A = 7$. In fact $a_1 = 1$, $a_2 = 4$, $a_3 = a_2 + c_2 = 4 + 5 = 9$, $a_4 = a_3 + c_3 = 9 + 7 = 16$, $a_5 = a_4 + c_4 = 16 + 9 = 25$, $a_6 = a_5 + c_5 = 25 + 11 = 36$ and $a_7 = a_6 + c_6 = 36 + 13 = 49$. We now prove that this is the sequence of maximal length A by showing that the corresponding sequence $\{c_n : n \geq 1\}$ is of maximal length C. If $C \geq 3$ then one of the numbers

$$c_1, \quad c_2 = c_1 + 2, \quad c_3 = c_1 + 4$$

is divisible by 3, and so is equal to 3 or 9. By (5), since $b_1 > 0$, we have $c_1 \geq 3$. Therefore, in the first case $c_1 = 3$, which yields the above example. Otherwise $c_n = 9$ for a certain $n \in \{1, 2, 3\}$. But then $c_{n+3} = c_n + 3 \cdot 2 = 15$, so that $C = n + 2 \leq 5 < 6$. This completes the proof.

* * *

That's all the space we have this issue. We'll continue with solutions for the Team Competition of the 12th Austrian-Polish Mathematics Competition next issue. The alert reader will note we did not give a solution to number 4. Meanwhile send me your national and regional Olympiads, as well as your nice solutions.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1992,** *although solutions received after that date will also be considered until the time when a solution is published.*

1731*. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Let P be a point within or on an isosceles right triangle and let c_1, c_2, c_3 be the lengths of the three concurrent cevians through P. Prove or disprove that c_1, c_2, c_3 form the sides of a nonobtuse triangle. [This problem was inspired by Murray Klamkin's problem 1631, solution this issue.]

1732. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $pal(n)$ be the nth palindromic number (i.e. $pal(1) = 1, \ldots, pal(9) = 9$, $pal(10) = 11$, $pal(11) = 22$, etc.). Determine the set of all exponents α such that

$$\sum_{n=1}^{\infty} \frac{1}{[pal(n)]^{\alpha}}$$

converges.

1733. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcenter O and such that $\angle A > 90°$ and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN with side AC. Suppose that $AD = (AB + AC)/2$. Find $\angle A$.

1734. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the minimum value of

$$\sqrt{(1-ax)^2+(ay)^2+(az)^2}+\sqrt{(1-by)^2+(bz)^2+(bx)^2}+\sqrt{(1-cz)^2+(cx)^2+(cy)^2}$$

for all real values of a, b, c, x, y, z.

1735. *Proposed by P. Penning, Delft, The Netherlands.*

In a conic (ellipse or hyperbola) with centre O, chords AB have the property that all triangles OAB have the same area. Find the locus of the midpoint of AB.

1736. *Proposed by Jordi Dou, Barcelona, Spain.*

Let A', B', C' be the feet of the altitudes of ΔABC, $K = AA' \cap B'C'$, and L, M the intersections of AB, AC with the perpendicular bisector of $A'K$. Prove that A, A', L, M are concyclic.

1737. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ describe the Fibonacci sequence. A point of the plane is a *Fibonacci point* if both of its coordinates are Fibonacci numbers. In the *Fibonacci Quarterly* Vol. **28** (1990) pp. 22–27, Clark Kimberling calls a hyperbola (with axes parallel to the coordinate axes) a *Fibonacci hyperbola* if it contains an infinity of Fibonacci points.

(a) Consider the sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Fibonacci points (listed by distance from the origin) in the first quadrant which lie on the Fibonacci hyperbola $x^2 + xy - y^2 + 1 = 0$. Prove that the area of the triangle $\phi_{n-1}\phi_n\phi_{n+1}$ is constant over all positive integers n.

(b)* What about for other Fibonacci hyperbolas?

1738. *Proposed by Shiko Iwata, Gifu, Japan.*

There are four points A_1, A_2, A_3, A_4 in a plane. Let H_i be the orthocenter of the triangle formed by excluding A_i from these four points. Show that the areas of the quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are equal.

1739. *Proposed by Richard K. Guy, University of Calgary.*

Express 19 as the sum of two cubes of positive rational numbers in two different ways.

1740. *Proposed by Dan Pedoe, Minneapolis, Minnesota.* (Dedicated in memoriam to Joseph Konhauser.)

In triangle ABC the points N, L, M, in that order on AC, are respectively the foot of the perpendicular from B onto AC, the intersection with AC of the bisector of $\angle ABC$, and the midpoint of AC. The angles ABN, NBL, LBM and MBC are all equal. Determine the angles of ΔABC. [Some comments on the origin of this proposal will be given when a solution is published.]

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1626. [1991: 79] *Proposed by P. Penning, Delft, The Netherlands.*

Determine the average number of throws of a standard die required to obtain each face of the die at least once.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

This is a particular case of a famous problem from probability theory, sometimes called the "collector's problem" (see, e.g., pp. 174–175 of W. Feller, *Introduction to Probability Theory and its Applications*, Vol. I, John Wiley, New York–London, 1950), with the result for an n-sided die:

$$\text{Expected number of throws } = n\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}+1\right);$$

i.e., for $n=6$, the expected number of throws is 14.7.

Also solved by MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; TOM LEINSTER, New College, Oxford; CHRIS WILDHAGEN, Rotterdam, The Netherlands (two solutions); and the proposer.

Klamkin also referred to Feller's book, but p. 211 *of the* 1961 *edition. He also mentions the paper "The double dixie cup problem" by D.J. Newman and L. Shepp, in the* American Math. Monthly **67** (1960), *pp.* 58–61, *which considers the problem if each number is to be obtained m times. For a more recent reference, see "Majorization and the birthday inequality" by M.L. Clevenson and W. Watkins, in* Mathematics Magazine **64** (1991), *pp.* 183–188, *especially p.* 187.

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1628*. [1991: 79] *Proposed by Remy van de Ven, student, University of Sydney, Sydney, Australia.*

Prove that

$$(1-r)^k\sum_{i=1}^{\infty}\binom{k+i-1}{i}r^i\left(\frac{1}{k^2}+\frac{1}{(k+1)^2}+\cdots+\frac{1}{(k+i-1)^2}\right)=\sum_{i=1}^{\infty}\frac{r^i}{i^2\binom{k+i-1}{i}},$$

where k is a positive integer.

Solution by G.P. Henderson, Campbellcroft, Ontario.

It is not necessary that k be an integer. The relation is true provided k is not zero or a negative integer.

The coefficient of r^n on the left side is

$$\begin{aligned}C_n&=\sum_{s=0}^{n-1}\sum_{t=0}^{n-s-1}(-1)^s\binom{k}{s}\binom{k+n-s-1}{n-s}\frac{1}{(k+t)^2}\\&=\sum_{t=0}^{n-1}\sum_{s=0}^{n-t-1}(-1)^s\binom{k}{s}\binom{k+n-s-1}{n-s}\frac{1}{(k+t)^2}\\&=\sum_{t=0}^{n-1}\left[\binom{k+n-1}{n}+\sum_{s=1}^{n-t-1}(-1)^s\binom{k}{s}\binom{k+n-s-1}{n-s}\right]\frac{1}{(k+t)^2}\,. \qquad (1)\end{aligned}$$

We have

$$\binom{k}{s}\binom{k+n-s-1}{n-s} = \frac{k(k+n-s-1)(k+n-s-2)\dots(k-s+1)}{s!(n-s)!}$$
$$= \left[\frac{(k-s)(n-s)}{n} + \frac{(k+n-s)s}{n}\right] \frac{(k+n-s-1)\dots(k-s+1)}{s!(n-s)!}$$
$$= \binom{n-1}{s}\binom{k+n-s-1}{n} + \binom{n-1}{s-1}\binom{k+n-s}{n}.$$

Therefore in (1), the inner sum is

$$\sum_{s=1}^{n-t-1}\left[(-1)^s\binom{n-1}{s}\binom{k+n-s-1}{n} - (-1)^{s-1}\binom{n-1}{s-1}\binom{k+n-s}{n}\right]$$
$$= (-1)^{n-t-1}\binom{n-1}{n-t-1}\binom{k+t}{n} - \binom{k+n-1}{n},$$

and

$$C_n = \sum_{t=0}^{n-1}(-1)^{n-t-1}\binom{n-1}{n-t-1}\binom{k+t}{n}\frac{1}{(k+t)^2}.$$

$\binom{k}{n}\frac{1}{k}$ is a polynomial of degree $n-1$ in k and its constant term is $(-1)^{n-1}/n$. Hence

$$\binom{k}{n}\frac{1}{k^2} = P(k) + \frac{(-1)^{n-1}}{nk}$$

where P is a polynomial of degree $n-2$ if $n \geq 2$ and is zero if $n = 1$. Then

$$C_n = \sum_{t=0}^{n-1}(-1)^{n-t-1}\binom{n-1}{n-t-1}\left[P(k+t) + \frac{(-1)^{n-1}}{n}\frac{1}{k+t}\right]$$
$$= \Delta^{n-1}P(k) + \frac{(-1)^{n-1}}{n}\Delta^{n-1}\frac{1}{k}.$$

[Here

$$\Delta f(x) = f(x+1) - f(x), \quad \Delta^m f(x) = \Delta(\Delta^{m-1}f(x)), \quad m = 2,3,\dots,$$

are the *forward differences.—Ed.*] The first part of this is zero because P is either zero or a polynomial of degree $n-2$. The second part is

$$\frac{1}{n^2\binom{k+n-1}{n}}$$

which is the coefficient of r^n on the right side.

The proposer found the identity (used without proof) in an old paper "The negative binomial distribution", by R.A. Fisher, in The Annals of Eugenics **11** (1941), *pp.* 182–187.

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1629. [1991: 79] *Proposed by Rossen Ivanov, student, St. Kliment Ohridsky University, Sofia, Bulgaria.*

In a tetrahedron x and v, y and u, z and t are pairs of opposite edges, and the distances between the midpoints of each pair are respectively l, m, n. The tetrahedron has surface area S, circumradius R, and inradius r. Prove that, for any real number λ with $0 \leq \lambda \leq 1$,

$$x^{2\lambda}v^{2\lambda}l^2 + y^{2\lambda}u^{2\lambda}m^2 + z^{2\lambda}t^{2\lambda}n^2 \geq \left(\frac{\sqrt{3}}{4}\right)^{1-\lambda} (2S)^{1+\lambda}(Rr)^{\lambda}.$$

Solution by the proposer.

We use the inequality of Neuberg–Pedoe:

$$\sum a^2(-a'^2 + b'^2 + c'^2) \geq 16FF',$$

where a, b, c and a', b', c' are the sides of triangles with areas F and F', respectively, and the sums here and below are cyclic (e.g., see p. 355 of [1]). If S_1, S_2, S_3 and S_4 are the areas of the faces of the tetrahedron determined by edges v, y, z; x, u, z; x, y, t; and v, u, t respectively, then we have four inequalities:

$$\sum a^2(-v^2 + y^2 + z^2) \geq 16S_1F,$$

$$\sum a^2(-x^2 + u^2 + z^2) \geq 16S_2F,$$

$$\sum a^2(-x^2 + y^2 + t^2) \geq 16S_3F,$$

$$\sum a^2(-v^2 + u^2 + t^2) \geq 16S_4F.$$

If we sum up these four inequalities, we get

$$2\sum a^2(-x^2 - v^2 + y^2 + u^2 + z^2 + t^2) \geq 16SF.$$

But the bimedian connecting the midpoints of edges with lengths x and v has length l, so

$$4l^2 = -x^2 - v^2 + y^2 + u^2 + z^2 + t^2, \quad \text{etc.}$$

(e.g., see the remark after 1.3 on p. 547 of [1]). Therefore

$$8\sum a^2l^2 \geq 16SF.$$

Now we will use Oppenheim's result: if a, b, c are the sides of a triangle with area F, and if $0 < \lambda \leq 1$ is a real number, then a triangle exists with sides a^λ, b^λ, c^λ and area

$$F_\lambda \geq \left(\frac{\sqrt{3}}{4}\right)^{1-\lambda} F^\lambda$$

(see p. 104 of [1]). This gives

$$8\sum a^{2\lambda}l^2 \geq 16S\left(\frac{\sqrt{3}}{4}\right)^{1-\lambda} F^\lambda.$$

Furthermore, according to Crelle's theorem (item 1.6, p. 549 and (4), p. 555 of [1]), we can let

$$a = xv, \quad b = yu, \quad c = zt, \quad F = 2SRr,$$

from which the proposed inequality follows.

Reference:

[1] D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

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1630. [1991: 79] *Proposed by Isao Ashiba, Tokyo, Japan.*

Maximize

$$a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n}$$

over all permutations $a_1, a_2, \ldots, a_{2n}$ of the set $\{1, 2, \ldots, 2n\}$.

I. *Solution by Pavlos Maragoudakis, student, University of Athens, Greece.*

Since

$$\begin{aligned} a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n} &= \frac{1}{2}\left(\sum_{i=1}^{2n} a_i^2 - \sum_{i=1}^{n}(a_{2i} - a_{2i-1})^2\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{2n} i^2 - \sum_{i=1}^{n}(a_{2i} - a_{2i-1})^2\right), \end{aligned}$$

the above sum is maximized when $\sum_{i=1}^{n}(a_{2i} - a_{2i-1})^2$ is minimized. But

$$\sum_{i=1}^{n}(a_{2i} - a_{2i-1})^2 \geq n,$$

with equality when $|a_{2i} - a_{2i-1}| = 1$, $i = 1, 2, \ldots, n$, for example when $a_i = i$ for all i. Therefore we obtain the maximization of $a_1a_2 + \cdots + a_{2n-1}a_{2n}$ when $a_i = i$ for all i.

II. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

We claim that, given any set S of $2n$ real numbers, $a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n}$ is maximized over all permutations of S by pairing the largest two numbers, then the remaining largest two numbers, and so on down to the smallest two numbers. For consider any four elements

$$a, \quad a + \Delta_1, \quad a + \Delta_1 + \Delta_2, \quad a + \Delta_1 + \Delta_2 + \Delta_3$$

of S, where Δ_1, Δ_2, $\Delta_3 \geq 0$, and let

$$\begin{aligned} M &= (a + \Delta_1 + \Delta_2 + \Delta_3)(a + \Delta_1 + \Delta_2) + a(a + \Delta_1), \\ L_1 &= (a + \Delta_1 + \Delta_2 + \Delta_3)(a + \Delta_1) + a(a + \Delta_1 + \Delta_2), \\ L_2 &= (a + \Delta_1 + \Delta_2 + \Delta_3)a + (a + \Delta_1 + \Delta_2)(a + \Delta_1). \end{aligned}$$

Then

$$M - L_1 = \Delta_2(\Delta_1 + \Delta_2 + \Delta_3) \geq 0 \quad \text{and} \quad M - L_2 = (\Delta_1 + \Delta_2)(\Delta_2 + \Delta_3) \geq 0,$$

so M is the best to pick. A permutation claiming to be maximizing in a way other than above could be improved by finding the largest a_i deviating from the rule and switching the permutation to obey the rule. [So if a_i were not paired off with its mate a_j according to the above rule, suppose a_i were paired with a_k and a_j with a_l, where $a_i > a_j > a_k, a_l$; then $a_i a_k + a_j a_l$ is increased by switching to $a_i a_j + a_k a_l$. —*Ed.*]

Also solved by H.L. ABBOTT, University of Alberta; MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; PETER HURTHIG, Columbia College, Burnaby, B.C.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; TOM LEINSTER, New College, Oxford; ANDY LIU, University of Alberta; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; DAVID G. POOLE, Trent University, Peterborough, Ontario; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The solutions of Janous, Leinster, and the proposer were similar to Solution I. *Several solvers obtained the more general result in Solution* II. *Several more noted that the minimum of $a_1a_2 + \cdots + a_{2n-1}a_{2n}$ occurs by pairing up the smallest and largest numbers, next smallest and next largest, etc.; this can be seen as in Hess's proof, since $L_1 - L_2 = \Delta_1\Delta_3 \geq 0$, so $\min\{M, L_1, L_2\} = L_2$.*

Monier refers to exercise 1.3.33, *p.* 5 *of his book* Analyse, *Tome* 1, *Dunad Université,* 1991. *Other readers note a similar problem on* [1986: 23–24], *and the related result (*§10.2, *p.* 261 *of Hardy, Littlewood and Pólya,* Inequalities*) that if $\{a_k\}$ and $\{b_k\}$ are given except in arrangement, then $\sum_{k=1}^{n} a_k b_k$ is greatest when $\{a_k\}$ and $\{b_k\}$ are monotonic in the same sense and least when they are monotonic in the opposite sense.*

Janous and Klamkin would like to see a proof (or disproof) that the maximum of

$$a_1a_2 \ldots a_k + a_{k+1}a_{k+2} \ldots a_{2k} + \cdots + a_{(n-1)k+1} \ldots a_{nk}$$

over all permutations of $\{1, 2, \ldots, nk\}$ (or more generally any nk real numbers) occurs for $a_i = i$ for all i.

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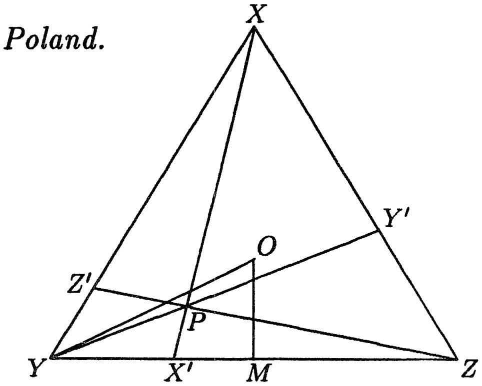
1631*. [1991: 113] *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)

Let P be a point within or on an equilateral triangle and let c_1, c_2, c_3 be the lengths of the three concurrent cevians through P. Determine the largest constant λ such that $c_1^\lambda, c_2^\lambda, c_3^\lambda$ are the sides of a triangle for any P.

Solution by Marcin E. Kuczma, Warszawa, Poland.



The largest such λ is $\ln 4/\ln(9/8) \approx 11.77$.

Let XYZ be the triangle, of side length 1, and XX', YY', ZZ' be the three cevians. Assume without loss of generality that $P \in \Delta OMY$ where O is the center of XYZ and M is the midpoint of YZ. Then $ZZ' \geq XX'$ and $ZZ' \geq YY'$. Thus, given an exponent $\lambda > 0$, the requirement of the problem is fulfilled if and only if

$$\Phi_\lambda(P) := (XX')^\lambda + (YY')^\lambda - (ZZ')^\lambda \geq 0 \tag{1}$$

for $P \in \Delta OMY$.

If P lies on side MY then clearly $ZZ' = YY' = 1$ and $XX' \geq \sqrt{3}/2$, so

$$\Phi_\lambda(P) \geq (3/4)^{\lambda/2}. \tag{2}$$

For every other position of P in ΔOMY, let

$$x = \frac{ZX'}{X'Y} \geq 1, \quad y = \frac{XY'}{Y'Z} \geq 1, \quad z = \frac{YZ'}{Z'X} = \frac{1}{xy} \leq 1$$

(since $xyz = 1$ by Ceva's theorem), and put

$$f(x) = 1 - \frac{x}{(x+1)^2} = f\left(\frac{1}{x}\right) .$$

Then we have

$$X'Y = \frac{1}{1+x} ,$$

so

$$XX' = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} - \frac{1}{1+x}\right)^2} = \sqrt{1 - \frac{1}{1+x} + \frac{1}{(1+x)^2}} = \sqrt{f(x)} ,$$

and likewise

$$YY' = \sqrt{f(y)}, \quad ZZ' = \sqrt{f(z)} .$$

Hence

$$\Phi_\lambda(P) = f(x)^{\lambda/2} + f(y)^{\lambda/2} - f(z)^{\lambda/2} = f(x)^{\lambda/2} + f(y)^{\lambda/2} - f(xy)^{\lambda/2}. \tag{3}$$

Consider the point P_0 determined by the cevian ratios $x = y = 2 + \sqrt{3} =: x_0$ (so $z = 1/x_0^2 = 7 - 4\sqrt{3}$). (This point minimizes the ratio $\operatorname{med}(c_i)/\max(c_i)$ asked for in *Crux* 1621 [1992: 88].) We have $f(x_0^2) = \frac{9}{8} f(x_0)$ and therefore

$$\Phi_\lambda(P_0) = f(x_0)^{\lambda/2}(2 - (9/8)^{\lambda/2}).$$

This number is nonnegative if and only if

$$\lambda \leq \frac{2\ln 2}{2\ln 3 - 3\ln 2} =: \lambda_0 \quad (\approx 11.77) \tag{4}$$

Hence (4) is a necessary condition for (1) to hold.

We will show that it is also sufficient, so that $\lambda = \lambda_0$ is actually the threshold value for the problem in question. Write for convenience $p = \lambda_0/2$ and denote the right side of (3) with $\lambda = \lambda_0$ by $F(x,y)$. So we claim

$$F(x,y) := f(x)^p + f(y)^p - f(xy)^p \geq 0 \quad \text{for } x, y \geq 1. \tag{5}$$

This is a continuous function, with boundary values $F(x,1) = F(1,y) = f(1)^p = (3/4)^p$; its limit behavior at infinity can be estimated from (2):

$$\liminf_{x^2+y^2 \to \infty} F(x,y) \geq \left(\frac{3}{4}\right)^p.$$

Since F vanishes at (x_0, x_0), we see that $F(x,y)$ must attain its minimum value at some point (x,y) with $1 < x, y < \infty$. At this point,

$$\frac{\partial F}{\partial x} = p(f(x)^{p-1}f'(x) - yf(xy)^{p-1}f'(xy)) = 0, \quad \frac{\partial F}{\partial y} = 0.$$

Writing

$$g(x) = xf(x)^{p-1}f'(x), \tag{6}$$

we have that

$$g(x) = g(y) = g(xy) \tag{7}$$

at the extreme point (x,y).

Let $t = (x + 1/x)/2 > 1$. Then $2t + 2 = (x+1)^2/x$, so

$$f(x) = 1 - \frac{x}{(x+1)^2} = 1 - \frac{1}{2t+2} = \frac{1}{2} \cdot \frac{2t+1}{t+1},$$

$$xf'(x) = \frac{x(x-1)}{(x+1)^3} = \left(\frac{x}{(x+1)^2}\right)^2 \left(x - \frac{1}{x}\right) = \frac{1}{2} \cdot \frac{\sqrt{t^2-1}}{(t+1)^2},$$

and hence by (6)

$$g(x) = \frac{(2t+1)^{p-1}(t^2-1)^{1/2}}{2^p(t+1)^{p+1}} =: h(t).$$

Now

$$\begin{aligned} \text{sign } h'(t) &= \text{sign } [2(p-1)(t+1)(t^2-1) - (p+1)(2t+1)(t^2-1) + t(2t+1)(t+1)] \\ &= \text{sign } [2(p-1)(t^2-1) - (p+1)(2t+1)(t-1) + t(2t+1)] \\ &= -\text{sign } [2t^2 - (p+2)t + (p-3)]. \end{aligned}$$

The last expression in square brackets is a quadratic trinomial with one root in $(0,1)$ and the other in $(1,\infty)$. Thus, as t varies from 1 to infinity, $h(t)$ first increases and then decreases. The same must be the behavior of $g(x)$ because the substitution $t = (x+1/x)/2$ is strictly monotonic in $(1,\infty)$. So $(1,\infty)$ is the union of two intervals I and J, I to the left of J, such that g increases in I and decreases in J. Consequently, (7) forces $x = y \in I$, $x^2 \in J$, and $g(x) = g(x^2)$. The last equation can have at most one solution in $(1, \infty)$, in view of the monotonicity relations just established. So it will be enough to check that $x = x_0$ *is* a solution.

For x_0 we have $f(x_0^2)/f(x_0) = 9/8$ and $(9/8)^p = 2$. Hence by (6) (and by $f'(x) = (x-1)(x+1)^{-3}$)

$$\frac{g(x_0^2)}{g(x_0)} = x_0 \left(\frac{f(x_0^2)}{f(x_0)}\right)^{p-1} \frac{(x_0+1)^4}{(x_0^2+1)^3} = (2+\sqrt{3}) \cdot \frac{16}{9} \cdot \frac{(3+\sqrt{3})^4}{(8+4\sqrt{3})^3} = 1,$$

as needed.

All this taken into account, we infer that $F(x, y)$ is minimized at (x_0, x_0). So its minimum value is 0 and (5) is settled. Therefore (1) holds for $\lambda = \lambda_0$, hence for every positive $\lambda \le \lambda_0$. The lengths of the three cevians, raised to power λ, satisfy the triangle inequality for every P if and only if $\lambda \le \lambda_0$, where λ_0 is the constant defined in (4).

Also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; and P. PENNING, Delft, The Netherlands.

As both Kuczma and Penning observe, the points giving solutions to problems 1621 *and* 1631 *coincide! Is there a simple proof that this nice "coincidence" had to be?*

Penning also calculates that the minimum value of λ satisfying the problem is $\lambda = -\ln 4/\ln(4/3) \approx -4.82$.

Konečný's work on this problem led him to propose problem 1731, *this issue.*

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1632. [1991: 113] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find all x and y which are rational multiples of π (with $0 < x < y < \pi/2$) such that $\tan x + \tan y = 2$.

Solution by Diane and Roy Dowling, University of Manitoba, Winnipeg.

The only solution is $x = \pi/12$, $y = \pi/3$.

A result of Gordan [1] says that for u, v and w rational multiples of π with $0 < u \le v \le w < \pi$, the equation

$$\cos u + \cos v + \cos w + 1 = 0$$

has only the two solutions

$$u = \frac{2\pi}{5}, \quad v = \frac{2\pi}{3}, \quad w = \frac{4\pi}{5} \quad \text{and} \quad u = \frac{\pi}{2}, \ v = \frac{2\pi}{3}, \ w = \frac{2\pi}{3}.$$

It follows that, when θ and ϕ are rational multiples of π between 0 and π, *the equation* $\cos\theta + 2\cos\phi + 1 = 0$ *has only one solution:* $\theta = \pi/2$ and $\phi = 2\pi/3$.

Note that if $\tan x + \tan y = 2$ then

$$\sin(y+x) = 2\cos y \cos x = \cos(y+x) + \cos(y-x),$$

so

$$2 - 2\sin(2y + 2x) = 2[\sin(y + x) - \cos(y + x)]^2 = 2\cos^2(y - x) = 1 + \cos(2y - 2x),$$

so

$$\begin{aligned}\cos(\pi - 2y + 2x) + 2\cos\left(\frac{3\pi}{2} - 2y - 2x\right) + 1 &= -\cos(2y - 2x) - 2\sin(2y + 2x) + 1 \\ &= 0. \qquad (1)\end{aligned}$$

Suppose now that x and y satisfy the conditions of the problem; then $0 < y - x < \pi/2$, so

$$0 < \pi - 2y + 2x < \pi.$$

Also

$$\tan(y + x) = \frac{\tan y + \tan x}{1 - \tan y \tan x} = \frac{2}{1 - (2 - \tan x)\tan x} = \frac{2}{(1 - \tan x)^2} > 2,$$

since $0 < \tan x < 2$, and from $y + x < \pi$ we easily see that

$$0 < \frac{3\pi}{2} - 2y - 2x < \pi.$$

Thus, applying the above result, it follows from (1) that

$$\pi - 2y + 2x = \frac{\pi}{2} \quad \text{and} \quad \frac{3\pi}{2} - 2y - 2x = \frac{2\pi}{3}\,.$$

Solving these last two equations we find $x = \pi/12$ and $y = \pi/3$.

Reference:

[1] P. Gordan, *Mathematische Annalen*, Vol. 12 (1877), p. 35.

Also solved by the proposer. Two other readers sent in the correct answer, one without proof, and one without decipherable proof.

The proposer used a result of Conway and Jones, in Acta Arithmetica 30 (1976) *pp. 229–240, to prove more generally that the only rational solution of* $\tan x\pi + \tan y\pi = z$ *with* $0 < x < y < 1/2$ *is the one above, i.e.,* $x = 1/12$, $y = 1/3$, $z = 2$. *The problem was suggested by* Crux 892 [1985: 57].

Walther Janous found the problem very familiar (and gave the correct solution without proof), but could not supply a reference.

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1633. [1991: 113] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

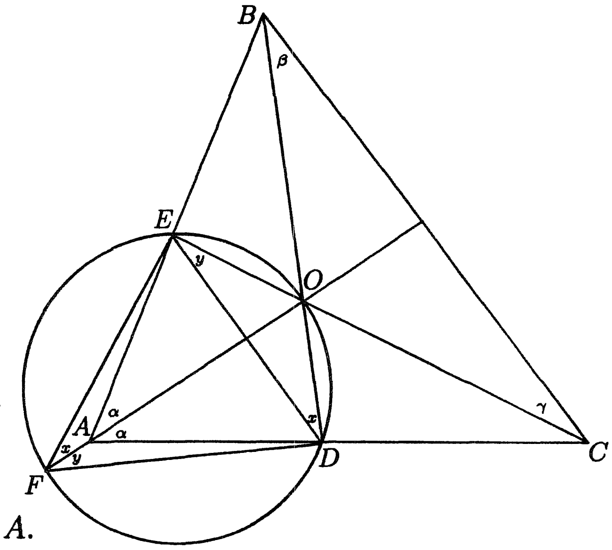
In triangle ABC, the internal bisectors of $\angle B$ and $\angle C$ meet AC and AB at D and E, respectively. We put $\angle BDE = x$, $\angle CED = y$. Prove that if $\angle A > 60°$ then $\cos 2x + \cos 2y > 1$.

Solution by Dag Jonsson, Uppsala, Sweden (slightly altered by the editor).

Let $\angle A = 2\alpha$, $\angle B = 2\beta$, $\angle C = 2\gamma$, and let the bisectors intersect at O. We will show that

$$\sin^2 x + \sin^2 y < \frac{1}{2},$$

which is equivalent to $\cos 2x + \cos 2y > 1$.



Case 1. $30° < \alpha \leq 45°$. Then the circumcircle of ΔEOD intersects the extension of the bisector AO at a point F beyond A, since $x = \angle EDO = \angle EFO$ and similarly $y = \angle DFO$ so that

$$\angle F = x + y = \beta + \gamma = 90° - \alpha < 2\alpha = \angle A.$$

Thus $x < \alpha$ and $y < \alpha$. Assume without loss of generality that $x \geq y$. Then

$$90° - 2\alpha < y \leq x < \alpha$$

(from $x + y = 90° - \alpha$) and $x \geq (90° - \alpha)/2$. Now for given s, $0 < s < 90°$, the function

$$f(x) = \sin^2 x + \sin^2(s - x)$$

is increasing for $s/2 \leq x \leq 45°$, since

$$f'(x) = 2\sin x \cos x - 2\sin(s - x)\cos(s - x) = \sin 2x - \sin 2(s - x) \geq 0.$$

Therefore

$$\sin^2 x + \sin^2 y = \sin^2 x + \sin^2(90° - \alpha - x) < \sin^2 \alpha + \sin^2(90° - 2\alpha).$$

But letting

$$g(\alpha) = \sin^2 \alpha + \sin^2(90° - 2\alpha),$$

we have

$$\begin{aligned} g'(\alpha) &= 2\sin\alpha\cos\alpha - 4\sin(90° - 2\alpha)\cos(90° - 2\alpha) \\ &= \sin 2\alpha - 2\sin 4\alpha = \sin 2\alpha(1 - 4\cos 2\alpha), \end{aligned}$$

so g decreases and then increases on $30° < \alpha < 45°$. Also

$$g(30°) = \frac{1}{2} = g(45°),$$

so

$$\sin^2 x + \sin^2 y < g(\alpha) \leq \frac{1}{2}.$$

Case 2. $45° < \alpha < 90°$. Then $0 < x + y < 45°$, giving (since the function $\sin^2 x$ is increasing and convex on $0 < x < 90°$)

$$\sin^2 x + \sin^2 y < \sin^2(x+y) < \sin^2 45° = \frac{1}{2} .$$

Also solved by KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and the proposer.

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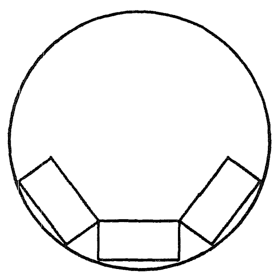
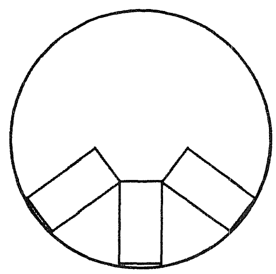
1634. [1991: 113] *Proposed by F.F. Nab, Tunnel Mountain, Alberta.*

A cafeteria at a university has round tables (of various sizes) and rectangular trays (all the same size). Diners place their trays of food on the table in one of two ways, depending on whether the short or long sides of the trays point toward the centre of the table:

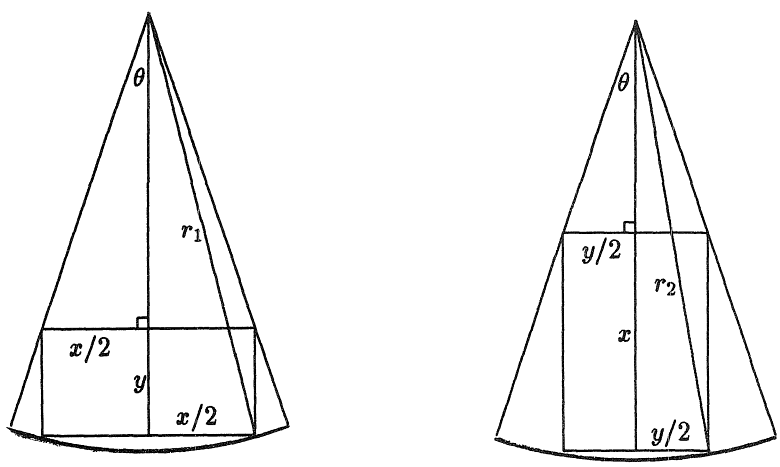


or



Moreover, at the same table everybody aligns their trays the same way. Suppose n mathematics students come in to eat together. How should they align their trays so that the table needed is as small as possible?

Solution by Leroy F. Meyers, The Ohio State University.



Let the trays have dimensions x and y, with $x > y$. (The orientation is irrelevant for square trays.) Let r_1 be the radius of the table needed when the short sides of the trays point towards the centre of the table, and r_2 the radius needed when the long sides point towards the centre. Each tray is inscribed in a circular sector with central angle $2\pi/n$. Let $\theta = \pi/n$. Then from the diagrams it is obvious that

$$r_1^2 = \left(\frac{x}{2}\right)^2 + \left(y + \frac{x}{2}\cot\theta\right)^2, \quad r_2^2 = \left(\frac{y}{2}\right)^2 + \left(x + \frac{y}{2}\cot\theta\right)^2.$$

Subtraction yields

$$r_1^2 - r_2^2 = \frac{x^2 - y^2}{4}(\cot^2\theta - 3).$$

Then

$$r_1 > r_2 \Leftrightarrow \cot^2\theta > 3 \Leftrightarrow \theta < \frac{\pi}{6} \Leftrightarrow n > 6.$$

Hence to obtain the smallest table, point the short sides toward the centre if $n \leq 5$, but point the long sides toward the centre if $n \geq 7$; there is no difference if $n = 6$. Note that the criterion depends only on the number of diners, and not on the relative dimensions of the tray. It may be interesting to investigate what happens if some of the trays are placed one way and some the other, or if only one corner of each tray is at the table edge.

Also solved by MARGHERITA BARILE, student, University of Essen, Germany; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; IAN GOLDBERG, student, University of Toronto Schools; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer. One incorrect solution was sent in.

Meyers and Kierstead suggest the possibility that an even smaller table may be enough in some cases, if the trays are not all placed the same way. However no readers seem to have looked into this. Can anyone find any values of n (and proportions of trays?) for which it is better to alternate the two ways of placing trays?

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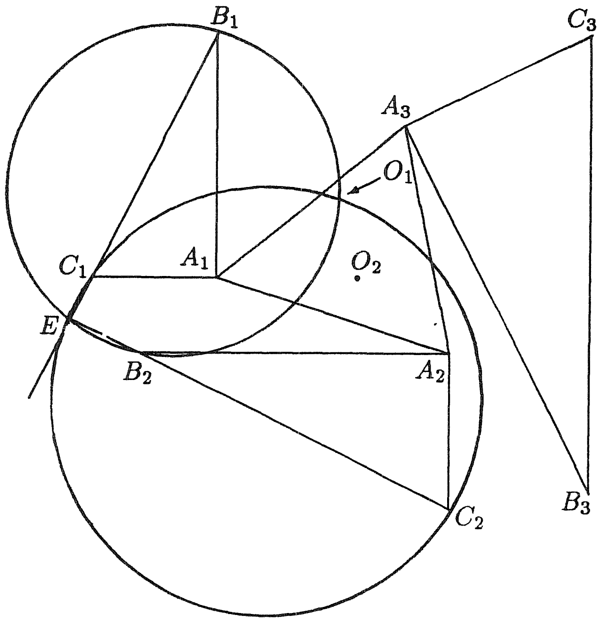
1635. [1991: 114] *Proposed by Jordi Dou, Barcelona, Spain.*

Given points $B_1, C_1, B_2, C_2, B_3, C_3$ in the plane, construct an equilateral triangle $A_1A_2A_3$ so that the triangles $A_1B_1C_1$, $A_2B_2C_2$ and $A_3B_3C_3$ are directly similar.



Solution by the proposer.

Consider the direct similarity S_1 so that $S_1(B_1C_1) = B_2C_2$. Its centre O_1 is the intersection (other than E) of the circles EB_1B_2 and EC_1C_2, where $E = B_1C_1 \cap B_2C_2$. [Because in the figure $\angle EC_1O_1 = \angle EC_2O_1$ and $\angle EB_1O_1 = \angle O_1B_2C_2$.— *Ed.*] Analogously S_2 denotes the similarity $S_2(B_2C_2) = B_3C_3$ of centre O_2. Since $A_1B_1C_1$ is similar to $A_2B_2C_2$, we will have $A_2 = S_1(A_1)$ and analogously $A_3 = S_2(A_2)$. Conversely, if $S_1(A_1) = A_2$ and $S_2(A_2) = A_3$, $A_1B_1C_1$ is similar to $A_2B_2C_2$ and $A_2B_2C_2$ is similar to $A_3B_3C_3$. Therefore the problem reduces to constructing an equilateral triangle $A_1A_2A_3$ such that $S_1(A_1) = A_2$ and $S_2(A_2) = A_3$.

It holds that $O_1A_1A_2$ is similar to $O_1B_1B_2$ (and to any $O_1X_1X_2$, where $X_2 = S_1(X_1)$), and $O_2A_2A_3$ is similar to $O_2B_2B_3$. Let $A_1'A_2'A_3'$ be any equilateral triangle. Construct O_1' and O_2' such that $O_1'A_1'A_2'$ is similar to $O_1B_1B_2$, and $O_2'A_2'A_3'$ to $O_2B_2B_3$. Then triangles $O_1O_2A_1$ and $O_1O_2A_2$ are similar to $O_1'O_2'A_1'$ and $O_1'O_2'A_2'$, respectively. Knowing O_1 and O_2, we can construct A_1 and A_2.

Also solved by TOSHIO SEIMIYA, Kawasaki, Japan.

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1636*. [1991: 114] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the set of all real exponents r such that

$$d_r(x,y) = \frac{|x-y|}{(x+y)^r}$$

satisfies the triangle inequality

$$d_r(x,y) + d_r(y,z) \geq d_r(x,z) \quad \text{for all } x,y,z > 0$$

(and thus induces a metric on $\mathbf{R}^+$—see *Crux* 1449, esp. [1990: 224]).

Solution by Murray S. Klamkin, University of Alberta.

We will show that the only valid exponents are $r \in [0,1]$ and $r = -1$.

Assume that $x \geq y \geq z$ and consider all three permuted versions of the triangle inequality. If $r = -1$, then $d_r(x,y)$, $d_r(y,z)$, $d_r(x,z)$ become $x^2 - y^2$, $y^2 - z^2$, $x^2 - z^2$ which obviously satisfy the condition since the first two sum to the third.

Suppose next that $0 \leq r \leq 1$. Of the three numbers $d_r(x,y)$, $d_r(y,z)$, $d_r(x,z)$, we first show that $d_r(x,z)$ is not less than the other two. Clearly

$$d_r(x,z) = \frac{x-z}{(x+z)^r} \geq \frac{x-y}{(x+y)^r} = d_r(x,y).$$

To show that $d_r(x,z) \geq d_r(y,z)$, it suffices to show that

$$F(t) \equiv \frac{t-z}{(t+z)^r}$$

is increasing for $t \in [z,x]$, and this follows from

$$F'(t) = \frac{(t+z)^r - (t-z)r(t+z)^{r-1}}{(t+z)^{2r}} = \frac{2rz + (1-r)(t+z)}{(t+z)^{r+1}} \geq 0$$

for $0 \leq r \leq 1$. Thus, to show that r satisfies the problem, it now suffices to prove that

$$\frac{x-y}{(x+y)^r} + \frac{y-z}{(y+z)^r} \geq \frac{x-z}{(x+z)^r}\ . \tag{1}$$

Since t^{-r} is convex for $t > 0$ and all $r \geq 0$, we have by Jensen's inequality that, for any $t_1, t_2 > 0$,

$$w_1 t_1^{-r} + w_2 t_2^{-r} \geq (w_1 t_1 + w_2 t_2)^{-r},$$

where $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$. Now letting

$$w_1 = \frac{x-y}{x-z}\ , \quad w_2 = \frac{y-z}{x-z}\ , \quad t_1 = x+y, \quad t_2 = y+z,$$

we obtain (1).

We now show that no other value of r will work. For $r > 1$, let $z = 1$ and $y = 2$; then for sufficiently large x,

$$\frac{y-z}{(y+z)^r} = \frac{1}{3^r} > \frac{x-2}{(x+2)^r} + \frac{x-1}{(x+1)^r} = \frac{x-y}{(x+y)^r} + \frac{x-z}{(x+z)^r}\ .$$

For $-1 < r < 0$, t^{-r} is concave so that, by the above argument, the inequality in (1) goes the other way. Finally for $r < -1$, by again choosing $z = 1$, $y = 2$, we can make

$$\frac{x-y}{(x+y)^r} = (x-2)(x+2)^{-r} > 3^{-r} + (x-1)(x+1)^{-r} = \frac{y-z}{(y+z)^r} + \frac{x-z}{(x+z)^r}$$

by choosing x sufficiently large.

Comment. The following related result appears (in a more general form) in [1]: *the function*

$$d(x,y) = \frac{|x-y|}{(|x|^p + |y|^p)^{1/p}}$$

satisfies the triangle inequality, i.e., is a metric on $\mathbf{R}$, *for every real* $p \geq 1$. (In fact this result still holds if x and y are vectors in $\mathbf{R}^n$ and $|\cdot|$ becomes length.) An open problem noted in [1] is whether $d(x,y)$ satisfies the triangle inequality for every $p \geq 1/2$. We could prove that the triangle inequality holds if $p = 1/2$ and fails if $p = 1/4$.

For $p = 1$, the above result is due to Schattschneider [2]; see also [1980: 248].

References:

[1] M.S. Klamkin and A. Meir, Ptolemy's inequality, chordal metric, multiplicative metric, *Pacific J. Math.* 10 (1982) 389–392.

[2] Doris J. Schattschneider, A multiplicative metric, *Mathematics Magazine* 49 (1976) 203–205.

Also solved by MARCIN E. KUCZMA, Warszawa, Poland.

A further solution, due to H. HAMETNER, appears in the article "Eine Metrik auf $\mathbf{R}^+$*",* Wissenschaftliche Nachrichten 85 *(Jänner* 1991*)* 31–32. *The problem had appeared in the same journal (Vol.* 83, *April* 1990, *pp.* 29–30*) in an earlier article (of the same title) by the proposer.*

The proposer would like to know whether the function

$$d'_r(x,y) = \frac{|x-y|}{(|x|+|y|)^r}\,, \quad x,y \in \mathbf{R}-\{0\},$$

satisfies the triangle inequality for any $r \in [0,1] \cup \{-1\}$. *A similar problem, with* x *and* y *in* $\mathbf{R}^2 - \{(0,0)\}$, *appeared in Hametner's article.*

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1637. [1991: 114] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

Comment by the Editor.

A solution to this problem, by Ian Goldberg, has already been published as part of his solution to *Crux* 1611 [1992: 62]. Other solvers of the present problem are listed below. Two other readers suggest that the bound in this problem could be improved to $9\sqrt{3}/\pi$, with equality holding when ABC is equilateral (one of them also guesses that the result should be true for all triangles). However, the editor feels that the only proof received of this (probably true) claim is incomplete, and so prefers to wait and see whether a better proof will be forthcoming.

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; VEDULA N. MURTY, Penn State University at Harrisburg; BOB PRIELIPP, University of Wisconsin–Oshkosh; and the proposer.

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1638. [1991: 114] *Proposed by Juan C. Candeal, Universidad de Zaragoza, and Esteban Indurain, Universidad Pública de Navarra, Pamplona, Spain.*

Find all continuous functions $f : (0, \infty) \to (0, \infty)$ satisfying the following two conditions:

(i) f is not one-to-one;

(ii) if $f(x) = f(y)$ then $f(tx) = f(ty)$ for every $t > 0$.

I. *Solution by Robert B. Israel, University of British Columbia.*

All such functions are constant.

Consider any two points u, v with $0 < u < v < \infty$. I claim $f(u) = f(v)$. Since f is not one-to-one, there exist a, b with $0 < a < b < \infty$ and $f(a) = f(b)$. Therefore f has a local maximum or local minimum at some $c \in (a, b)$. Suppose it is a local maximum (the proof for a local minimum is similar), i.e., there is $\epsilon > 0$ such that $f(x) \le f(c)$ whenever $|x - c| < \epsilon$. Take $\alpha = (v/u)^{1/n} > 1$ where n is so large that $|\alpha c - c| < \epsilon$ and $|\alpha^{-1} c - c| < \epsilon$. If $g(x) = f(x) - f(\alpha x)$, we have $g(c) \ge 0$ and $g(\alpha^{-1} c) \le 0$, so by continuity $g(d) = 0$ for some d between c and $\alpha^{-1} c$. Since $f(d) = f(\alpha d)$, we must have $f(x) = f(\alpha x)$ for all x, and therefore

$$f(u) = f(\alpha u) = f(\alpha^2 u) = \cdots = f(\alpha^n u) = f(v).$$

II. *Solution by Leroy F. Meyers, The Ohio State University.*

The only continuous functions satisfying the conditions are constants. Let

$$g(u) = f(e^u) - f(1) \quad \text{for } u \in (-\infty, \infty).$$

Then g is continuous on $(-\infty, \infty)$, $g(0) = 0$, and

(i′) g is not one-to-one,

(ii′) if $g(u) = g(v)$, then $g(w + u) = g(w + v)$ for every $w \in (-\infty, \infty)$.

To see (ii′), if u, v, and w are in $(-\infty, \infty)$ and $g(u) = g(v)$, then $f(e^u) = f(e^v)$ and

$$g(w + u) + f(1) = f(e^{w+u}) = f(e^w e^u) = f(e^w e^v) = f(e^{w+v}) = g(w + v) + f(1).$$

Suppose that $a < b$ and $g(a) = g(b)$. Let $c = b - a$. Then

$$g(u) = g((u - a) + a) = g((u - a) + b) = g(u + c)$$

for $u \in (-\infty, \infty)$, so that g has period c. Suppose that f is not constant. Then g is not constant on $(-\infty, \infty)$ and, by periodicity, g is nonconstant, as well as continuous, on $[0, c]$. Hence g achieves a minimum at some point $m \in (0, c]$ and a maximum at some point $M \in (m, m + c)$. By continuity, there is a positive real number δ such that $g(u) > g(m)$ for all u in $(M - \delta, M + \delta)$ [we select M so that g is not constant on either of the intervals $(M - \delta, M]$ and $[M, M + \delta)$—*Ed.*]. For s sufficiently close to, but less than, $g(M)$, there are u and v such that $M - \delta < u < M < v < M + \delta$ and $g(u) = g(v) = s$. But by the argument at the beginning of this paragraph, g has period $v - u$. Hence g must have the value $g(m)$ for some number in the interval (u, v), which contradicts the definition of δ. Hence f cannot be nonconstant.

III. *"Solution" by N. Withheld.*

We will show that the only solutions to this problem are constant functions.

First we show that any solution has exactly one value taken more than once. For, by (i), assume $f(x_1) = f(y_1)$ where $0 < x_1 < y_1$. Then by (ii), with $t_1 = x_1/y_1 < 1$, we get $f(x_1) = f(t_1 y_1) = f(t_1 x_1)$ and thus $f(x_1) = f(t_1^k x_1)$ for all $k \in \mathbf{N}$. Hence

$$f(x_1) = \lim_{k\to\infty} f(t_1^k x_1) = \lim_{a\to 0^+} f(a).$$

If $f(x_2) = f(y_2)$ where $0 < x_2 < y_2$, we would similarly obtain

$$f(x_2) = \lim_{a\to 0^+} f(a) = f(x_1),$$

and therefore there is exactly one value—call it λ—that f can assume more than once.

Let us show that, in fact, $f(x) = \lambda$ for all $x > 0$. Assume $f(a) \neq \lambda$ for some $a > 0$. By (i) and the first part of our proof, we can find $y > x > 0$ such that $f(x) = f(y) = \lambda$. By (ii), we can moreover assume $x < a < y$; for choosing $t \in (a/y, a/x)$, we get $f(tx) = f(ty)$ (therefore $= \lambda$) and $tx < a < ty$. Now by continuity, for any fixed number μ strictly between λ and $f(a)$, f assumes the value μ at least once in (x, a) and at least once in (a, y), i.e., at least twice in (x, y). This contradicts the first part of the proof.

Also solved by H.L. ABBOTT, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; JEAN-MARIE MONIER, Lyon, France; CARLOS JOSE PEREZ JIMENEZ, student, Logroño, Spain; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposers.

"Solution" III *was actually derived from two separate submissions containing the same subtle error, which the reader may enjoy discovering.*

Colleague Len Bos points out that the result is not true if conditions (i) *and* (ii) *are replaced by the single condition:*

there exist $a \neq b$ such that $f(ta) = f(tb)$ for all $t > 0$;

for example, consider the nonconstant function $f(x) = \sin(\ln x)$ *with* $a = 1$ *and* $b = e^{2\pi}$.

The problem was taken from the first phase (Navarre) of the 1990 *Spanish Mathematical Olympiad.*

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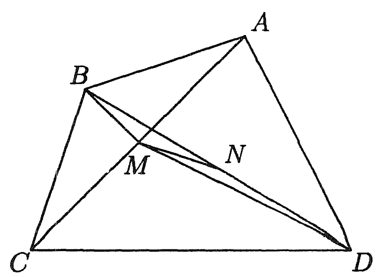
1639. [1991: 114] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

$ABCD$ is a convex cyclic quadrilateral. Prove that

$$(AB + CD)^2 + (AD + BC)^2 \geq (AC + BD)^2.$$

Solution by Toshio Seimiya, Kawasaki, Japan.



Let M and N be the midpoints of AC and BD respectively; then from

$$\begin{aligned} AB^2 + BC^2 &= AM^2 + BM^2 - 2AM \cdot BM \cos \angle BMA \\ &\quad + BM^2 + CM^2 - 2BM \cdot CM \cos \angle BMC \\ &= 2(BM^2 + AM^2), \quad \text{etc.} \end{aligned}$$

we get

$$\begin{aligned}AB^2+BC^2+CD^2+DA^2 &= 2(BM^2+AM^2)+2(DM^2+AM^2)\\ &= 2(BM^2+DM^2)+4AM^2\\ &= 4(BN^2+MN^2)+4AM^2\\ &= AC^2+BD^2+4MN^2. \qquad (1)\end{aligned}$$

Therefore we have

$$AB^2+BC^2+CD^2+DA^2 \geq AC^2+BD^2. \qquad (2)$$

In a general quadrilateral, it is well known that

$$AB\cdot CD+AD\cdot BC \geq AC\cdot BD \qquad (3)$$

(e.g., p. 63 of R.A. Johnson, *Advanced Euclidean Geometry*). From (2) and (3) we obtain

$$\begin{aligned}&(AB+CD)^2+(AD+BC)^2-(AC+BD)^2\\ &\qquad = (AB^2+BC^2+CD^2+DA^2-AC^2-BD^2)\\ &\qquad\quad + 2(AB\cdot CD+AD\cdot BC-AC\cdot BD) \geq 0.\end{aligned}$$

Hence we have the required inequality. As shown in the proof, the condition that $ABCD$ be cyclic is unnecessary.

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; IAN GOLDBERG, student, University of Toronto Schools; JEFF HIGHAM, student, University of Toronto; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; R.S. ODONKOR, International Secondary School, Agege, Nigeria; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Several readers observe that equality holds if and only if $ABCD$ is a rectangle, as can be seen from the above proof.

Janous and Klamkin also observe that the result holds for arbitrary quadrilaterals. Janous points out that (1) *was known to Euler (e.g., see p.* 547, *remark following item* 1.3, *of Mitrinović et al,* Recent Advances in Geometric Inequalities*). Klamkin remarks that the inequality of the problem is a special case of the following: if a, a'; b, b'; and c, c' are the lengths of the three pairs of opposite edges of an arbitrary tetrahedron, then $a+a'$, $b+b'$, $c+c'$ are the sides of an acute triangle. See* [1979: 130–131] *or pp.* 69–70 *of Klamkin's* USA Mathematical Olympiads 1972–1986, *MAA,* 1988.

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