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Founding Editors: Léo Sauvé, Frederick G.B. Maskell.

Editor: G.W. Sands, Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada, T2N 1N4.

Managing Editor: Dr. Kenneth S. Williams, Canadian Mathematical Society, 577 King Edward Avenue, Ottawa, Ontario, Canada, K1N 6N5.

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CONTENTS

THE OLYMPIAD CORNER: 85

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month's column with Selected Problems from the *Bulgarian Spring Competition-Kazanlik* written March 31, 1985, for which we thank Ivan Tonov.

1. Find the values of the parameters a and b for which the system

$$|x + y| + |x - y| = 2a$$
$$y - b = |x - b|$$

has

(a) exactly 2 solutions;

(b) exactly 1 solution;

(c) no solutions.

(Grade 7)

2. Let x_1, x_2 and x_3, x_4 be the roots of the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ respectively where p, p' and q, q' are non-zero real numbers. Prove that if $x_1x_4 = x_2x_3$ then $\left[\frac{p}{p'}\right]^2 = \frac{q}{q'}$. (Grade 8)

3. Six different natural numbers less than 108 are given. Prove that it is possible to choose three of them, a, b and c, so that $a < bc$, $b < ca$ and $c < ab$. (Grade 8)

4. Find all positive values of the parameter a for which the common solutions of the inequalities $x^2 - 2x \leq a^2 - 1$ and $x^2 - 4x \leq -a - 2$ form an interval of length 1 on the real axis. (Grade 9)

5. The diagonals AC and BD of a trapezoid $ABCD$ intersect at the point O. Let the bases of this trapezoid be $AB = a$ and $CD = b$ and let $\angle AOD = \theta$. Prove that the circles with diameters AC and BD are tangent exactly when the area of $ABCD$ equals $ab\tan(\theta/2)$. (Grade 10)

6. Let the point M be chosen arbitrarily on the face ABC of the regular tetrahedron $ABCD$. Denote the orthogonal projections of M on the faces BCD, CAD and ABD by P, Q and T, respectively. Find the locus of

the centroid of the triangle PQT, as M describes the triangle ABC. (Grade 10)

7. Let $S_n = \sum_{k=0}^{n} \binom{3n}{3k}$. Prove that $\lim_{n\to\infty} (S_n)^{1/3n} = 2$. (Grade 11)

*   *   *

The next four problems we pose were also forwarded by Ivan Tonov and are selected from the *Bulgarian Spring Math. Competition-Gambol*, March 31, 1986. As always, we invite readers to submit elegant solutions, which we will consider for publication in future numbers of the Corner.

1. Let ABC be a triangle. Suppose that RST is an obtuse triangle, whose sides are equal to the medians of triangle ABC. Prove that at least one of the angles of ABC is less than 45°. (Grade 9)

2. A regular quadrilateral pyramid with base $ABCD$ and vertex Q is given. All edges of $ABCDQ$ are of length 1. Let K be the incircle of triangle BCQ. Find the shortest segment XY, where X is a point on K and Y a point on the segment BD. (Grade 10)

3. For any integer m, let $\tau(m)$ denote the number of positive integers which divide m. Prove that there exist infinitely many positive integers n such that $\tau(2^n - 1) > n$. (Grades 10, 11)

4. Prove that for every real number x the inequality

$$x^2 - x + 0.96 > \sin x$$

holds. (Grade 11)

*   *   *

As promised in last month's *Crux* we next give the numerical answers for the 1987 AIME. These problems and their solutions are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Detailed solutions, and additional copies of the problems may be obtained, for a nominal fee, from Professor Walter E. Mientka, CAMC Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588-0322.

Answers:

1.	300	2.	137	3.	182	4.	480
5.	588	6.	193	7.	070	8.	112
9.	033	10.	120	11.	486	12.	019
13.	931	14.	373	15.	462.		

We, of course, solicit new and original solutions which we may select for publication in future numbers of the Olympiad Corner.

* * *

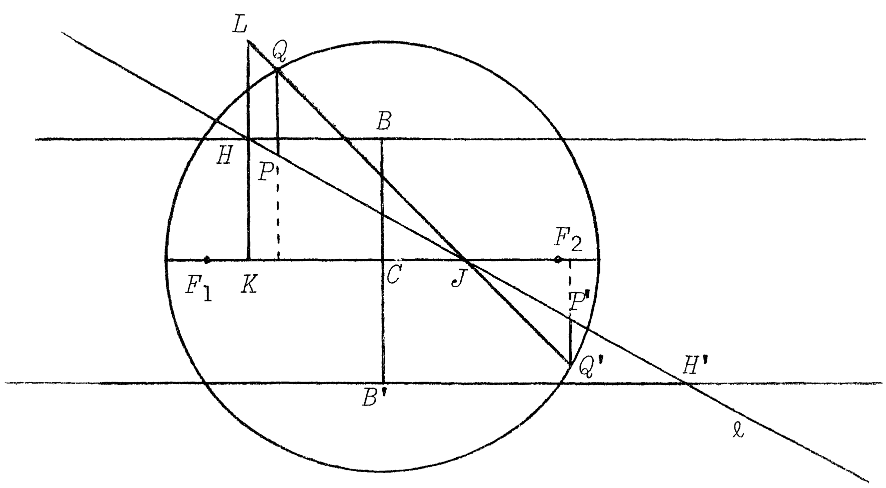
We next give two comments by Professor R.K. Guy, The University of Calgary, regarding two problems discussed earlier this year.

2. [1984: 74, 1987: 6] *West Point Proposals.*

Given the foci F_1, F_2 and the major axis of an ellipse, show how to construct with straightedge and compass the intersection of the ellipse with a given straight line ℓ.

Synthetic solution by R.K. Guy, The University of Calgary.

Given the foci F_1, F_2 and the major axis of length $2a$ of an ellipse, first with centres F_1, F_2 and radius a construct the two points B, B' of intersection of the corresponding circles. These are the ends of the minor axis. Let BB' cut F_1F_2 in C. Construct the parallels to F_1F_2 which pass through B and B'. Let the given line ℓ cut these at H and H', respectively, and let ℓ intersect F_1F_2 in J. Construct HK perpendicular to F_1F_2 with K on F_1F_2. Extend KH to L so that $KL = a$. Let LJ cut the circle centred at C with radius a in Q, Q'. Drop perpendiculars from Q, Q' to F_1F_2 and denote the points of intersection with ℓ by P, P', respectively. P and P' are the required points. The proof is by orthogonal projection in the ratio $b:a$.



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6. [1987: 71] *6th Brazilian Mathematical Olympiad.*

Figure 1 shows a board used in a game called "One Left". The game starts with one marker on each square of the board except for the central square which has no marker. Let A, B, C (or C, B, A) be three adjacent squares in a horizontal row or a vertical column. If A and B are occupied and C is not, then the marker at A may be jumped to C, and the marker at B removed (see Figure 2). Is it possible to end the game with one marker in the position shown in Figure 3?

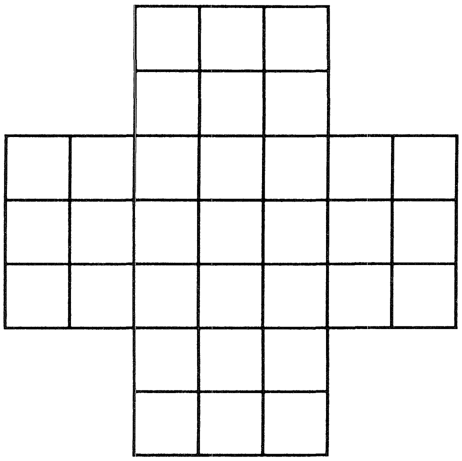


Figure 1

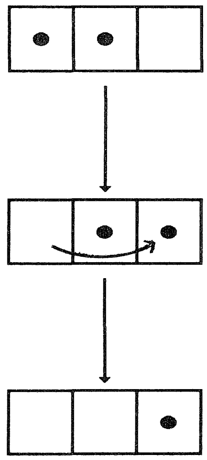


Figure 2

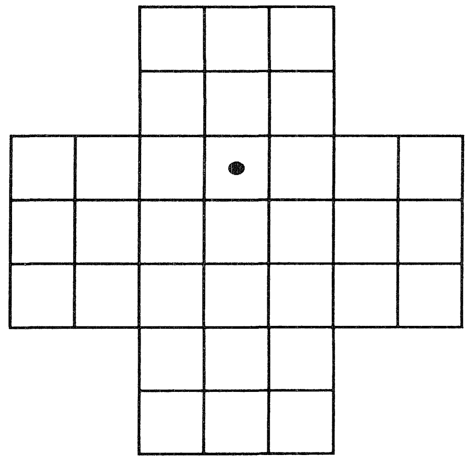


Figure 3

Comment by R.K. Guy, The University of Calgary.

No! This is essentially due to M. Reiss, Beitrage zur Theorie der Solitär-Spiels, *Crelle's J.* 54 (1857) 344-379.

See *Winning Ways*, p.705, the rule of two and the rule of three, and the "algebra" on pp.706-709, leading to the 16 Reiss classes.

Editor's note. *Winning Ways*, by E. Berlekamp, J. Conway and R.K. Guy, Academic Press, 1982, is an excellent source of information on the theory of games. We still solicit independent "nice" solutions to this problem for consideration for future columns.

* * *

We turn now to some solutions for problems posed in the October 1985 number of this column.

1. [1985: 237] *1981 Leningrad High School Olympiad (Third Round).*

Is there a sequence consisting of 1981 natural numbers whose sum is the cube of a natural number? (Grade 8)

Solution by John Morvay, Dallas, Texas.

More generally, for all positive integers n, k, $k \geq 2$ there exist arithmetic progressions with n terms and common difference $2 \cdot n^{k-2}$ whose sum is

n^k. Here the required sequence is

$$1981 + 3\cdot 1981 + 5\cdot 1981 + \ldots + 3961\cdot 1981.$$

2. [1985: 237] *1981 Leningrad High School Olympiad (Third Round).*

A square is partitioned into several rectangles whose sides are parallel to those of the square. For each rectangle, the ratio of the length of the smaller side to that of the longer side is calculated. Show that the sum of these ratios is not less than 1. (Grades 8, 9)

Solution by Su Chan, J.S. Woodsworth Secondary School, Nepean, Ontario.

Let the square be partitioned into n rectangles and let $a_1, a_2, \ldots, a_n$ be the lengths of the shorter sides of each rectangle and denote by $b_1, \ldots, b_n$ the lengths of the longer sides, respectively.

Then

$$\sum_{i=1}^{n} a_i b_i = x^2$$

where x is the length of a side of the original square.

Then

$$\begin{aligned}\sum_{i=1}^{n} \frac{a_i}{b_i} &= \sum_{i=1}^{n} \frac{a_i b_i}{b_i^2} \\ &\geq \sum_{i=1}^{n} \frac{a_i b_i}{x^2} \\ &= \frac{1}{x^2} \sum_{i=1}^{n} a_i b_i \\ &= \frac{x^2}{x^2} \\ &= 1.\end{aligned}$$

3. [1985: 237] *1981 Leningrad High School Olympiad (Third Round).*

Show that if $a^2 + ab + ac < 0$, then $b^2 > 4ac$. (Grade 8)

Solutions by Mangho Ahuja, Mathematics Department, Southeast Missouri State University (with generalization); and also by John Morvay, Dallas, Texas; Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin; and Elaine Tang, J.S. Woodsworth Secondary School, Nepean, Ontario.

Let $P(x) = a + bx + cx^2$. Now $P(0) = a$, $P(1) = a + b + c$. Since $a(a + b + c) < 0$, $P(0)$ and $P(1)$ have opposite signs and $P(x)$ has a real zero between 0 and 1.

We conclude that both roots of the quadratic equation $a + bx + cx^2 = 0$ are real, whence $b^2 \geq 4ac$. But $b^2 = 4ac$ would mean there are two equal roots between 0 and 1 and so $P(0)$ and $P(1)$ would have the same sign. Thus, $b^2 > 4ac$.

Generalization. If a, b, c, α, β are real numbers and

$$(a + b\alpha + c\alpha^2)(a + b\beta + c\beta^2) < 0$$

then $b^2 > 4ac$.

8. [1985: 238] *1981 Leningrad High School Olympiad (Third Round).*

The integers a, b, c, d, and A are such that

$$a^2 + A = b^2 \qquad \text{and} \qquad c^2 + A = d^2.$$

Show that the number

$$2(a + b)(c + d)(ac + bd - A)$$

is the square of a natural number. (Grades 9, 10)

Solution by Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin.

$$\begin{aligned}
2(a + b)(c + d)(ac + bd - A) &= (a + b)(c + d)(2ac + 2bd - 2A) \\
&= (a + b)(c + d)[(a - b)(c - d) + (a + b)(c + d) + (a - b)(a + b) \\
&\qquad + (c - d)(c + d)] \\
&= (a + b)(c + d)[((a - b) + (c + d))(a + b) + ((a - b) + (c + d))(c - d)] \\
&= (a + b)(c + d)[(a - b) + (c + d)][(a + b) + (c - d)] \\
&= (a + b)[(c + d) + (a - b)](c + d)[(a + b) + (c - d)] \\
&= [(a + b)(c + d) - A][(a + b)(c + d) - A].
\end{aligned}$$

The required result is now obvious.

* * *

The next problem discussed is from the 1983 Annual Greek High School Competition. George Evagelopoulos points out that solutions have appeared [pp.7-9, Volume 17, 1983, *Euclides B*]. *Euclides B* is a publication of the Greek Mathematical Society and appears 4 times a year. Nevertheless we solicit "nice" solutions for the Corner.

4. [1985: 240] *The 1983 Annual Greek High School Competition.*

A *lattice point* in a given coordinate plane is a point both of whose coordinates are integers.

(a) Given five lattice points $A_1, A_2, \ldots, A_5$ in the plane, show that the midpoint of at least one of the segments A_iA_j $(i \neq j)$ is a lattice point.

(b) Given are two lattice points $B(k,a)$ and $C(\ell,a)$ in the plane, with $k < \ell$. If A is a point in the plane such that $BC = CA = AB$, prove that A is not a lattice point.

Solution by John Morvay, Dallas, Texas.

(a) Call a subset of the integers *good* if its elements are all odd, or all even.

Let A_i have coordinates (x_i, y_i).

Notice that the midpoint of A_iA_j is a lattice point if and only if $\{x_i, x_j\}$ and $\{y_i, y_j\}$ are both good. By the pigeon-hole principle the set of first coordinates has a good subset of at least three members, say $\{x_1, x_2, x_3\}$, without loss of generality. The corresponding set of second coordinates $\{y_1, y_2, y_3\}$ has a good subset with at least two members, say $\{y_1, y_2\}$ without loss. Then the midpoint of A_1A_2 is a lattice point.

(b) Let $M\left[\frac{\ell + k}{2}, \alpha\right]$ be the midpoint of BC. Then A has coordinates $\left[\frac{\ell + k}{2}, \alpha + \sqrt{3}\frac{(\ell - k)}{2}\right]$ or $\left[\frac{\ell + k}{2}, \alpha - \sqrt{3}\frac{(\ell - k)}{2}\right]$. In any case the second coordinate is irrational!

*　　　　　*　　　　　*

Several problems from the October number remain without nice solutions in this column - in particular the problems from the 1984 Bulgarian Mathematical Olympiad. Here is a good chance to write up your solution and send it in for our column!

*　　　　　*　　　　　*

The next block of solutions we present are to the student proposals from the 1985 U.S.A.M.O. Training Session. Most of the solutions we have were generated by the student proposers, but problem 8 generated considerable interest, and lengthy solutions!

1. [1985: 270] *Proposed by Waldemar Horwat, Hoffman Estates, Illinois.*

$\mathbb{N}$ being the set of positive integers, determine all functions $f: \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$f(f(n)) = n + k$$

where k is a given odd positive integer.

Solution by the proposer.

The sequences $a, f^2(a), f^4(a), \ldots$ and $f(a), f^3(a), f^5(a), \ldots$ are both arithmetic progressions with common difference k. Moreover, they are disjoint, since

$$f^r(a) = f^s(a), \quad r \le s \implies a = f^{s-r}(a)$$
$$\implies a = f^{2(s-r)}(a) = a + (s - r)k$$
$$\implies r = s.$$

(The first step is valid because f^2, and hence f^r, is one to one.)

Therefore

$$a \not\equiv f(a) \bmod k. \tag{1}$$

An easy consequence of the functional equation is that

$$a \equiv b \bmod k \iff f(a) \equiv f(b) \bmod k.$$

From this we conclude that

$$\text{no two of } f(1),\ldots,f(k) \text{ are congruent modulo } k \tag{2}$$

and that

$$f(a) \equiv b \bmod k \implies f(b) \equiv a \bmod k. \tag{3}$$

Properties (1), (2) and (3) imply that in arithmetic modulo k, $f(1),\ldots,f(k)$ is a permutation of $1,\ldots,k$ whose cycles all have length 2. This is impossible when k is odd, so there are *no functions* with the desired properties.

2. [1985: 271] *Proposed by Bjorn Poonen, Winchester, Massachusetts.*

Is there a function f such that, for all real numbers x,

$$f(f(x)) = \sin 3x?$$

Solution by the proposer.

A "quickie". Yes: $f(x) = \begin{cases} -\sin 3x & \text{if } \sin 3x \ge 0 \\ -x & \text{if } \sin 3x < 0. \end{cases}$

3. [1985: 271] *Proposed by Bjorn Poonen, Winchester, Massachusetts.*

For a given positive integer n, prove that there are infinitely many positive integers m for which

$$\frac{(m!)^{n-1}}{(n!)^{m-1}}$$

is an integer.

Solution by the proposer.

If $n = 1$ the problem is trivial.

If $n > 1$, m can equal n^k, $k = 0,1,2,3,\ldots$, for then

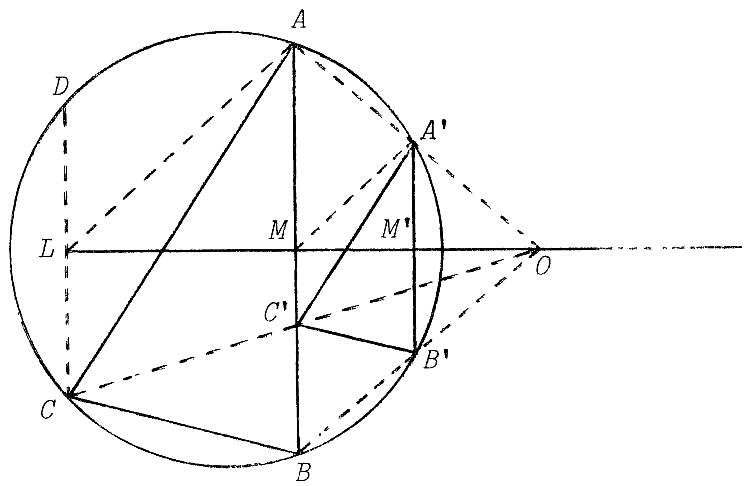
$$\frac{(m!)^{n-1}}{(n!)^{m-1}} = \left[\frac{(n^k)!}{(n!)^{n^{k-1}}(n!)^{n^{k-2}}\cdots(n!)^n\,(n!)}\right]^{n-1}$$

and the quantity in brackets counts the number of ways in which an n^k-element set can be partitioned hierarchically into n^{k-1}-element subsets, n^{k-2}-element subsets, and so on down to n-element subsets.

4. [1985: 271] *Proposed by Robert Adams, Annandale, Virginia.*

Starting with a given triangle ABC inscribed in a circle, show how to construct a directly homothetic triangle $A'B'C'$ with A', B' on the circle and C' on segment AB.

Solution by Steven Morrell, West Jordan, Utah.



Draw chord CD parallel to AB and hence to $A'B'$. The three respective midpoints L, M, M' of CD, AB and $A'B'$ are collinear and MM' passes through the homothetic center O of triangles ABC and $A'B'C'$, hence O lies on the constructible line LM.

The quadrilateral $A'B'C'M$ is homothetic to $ABCL$, so the line through M parallel to AL locates the point A' on the curve. Lines LM and AA' now determine O, and lines OB and OC locate B' and C'.

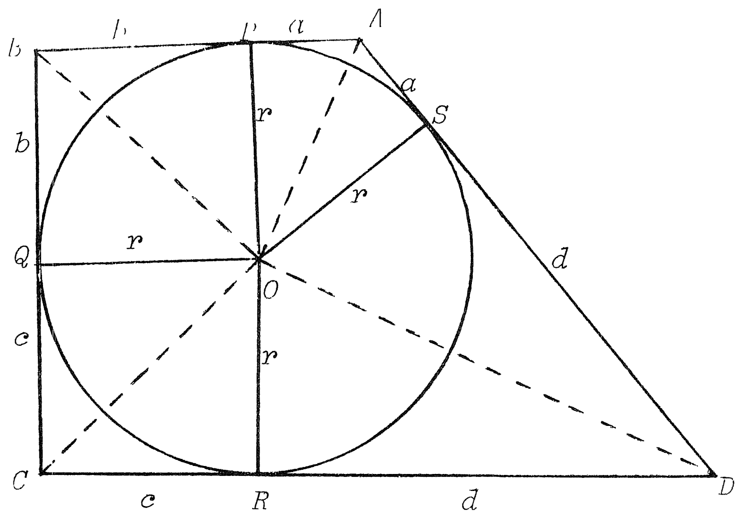
Comment by Gregg Patruno.

The proof above remains valid with the circle replaced by any conic section.

5. [1985: 271] *Proposed by Peter Yu, Richmond, California.*

A circle is tangent to the sides AB, BC, CD, and DA of a quadrilateral at points P, Q, R, and S, respectively. If $AP = 1$, $BQ = 2$, $CR = 2$ and $DS = 4$, determine the area of $ABCD$.

Solution by Gregg Patruno.



Let O denote the centre of the circle. Let θ_1 be $\angle AOB$ and let θ_2 equal $\angle DOC$. Let $AP = a = AS$, $BQ = b = BP$, $CR = c = CQ$ and $DS = d = DR$. Now

$$\tan\theta_1 = \frac{\frac{a}{r} + \frac{b}{r}}{1 - \frac{a}{r}\cdot\frac{b}{r}} = \frac{(a + b)r}{r^2 - ab}$$

and

$$\tan\theta_2 = \frac{\frac{c}{r} + \frac{d}{r}}{1 - \frac{c}{r}\cdot\frac{d}{r}} = \frac{(c + d)r}{r^2 - cd}.$$

Now

$$\theta_1 + \theta_2 = \pi \implies \tan\theta_1 + \tan\theta_2 = 0$$

$$\implies r^2 = \frac{abc + bcd + cda + dab}{a + b + c + d}.$$

Hence the area

$$K = rs = r(a + b + c + d) = \sqrt{(a + b + c + d)(abc + bcd + cda + dab)}.$$

For the problem in question

$$(a,b,c,d) = (1,2,2,4)$$

so the area is $18 = \sqrt{(1 + 2 + 2 + 4)(4 + 16 + 8 + 8)}$.

6. [1985: 271] *Proposed by Patrick Brown, Fairfax, Virginia.*

In a given quadrilateral $ABCD$, line AC is a nonperpendicular bisector of BD. If the bisectors of angles B and D intersect on AC, prove that they meet at right angles.

Solution by the proposer.

A "quickie". Suppose that the bisectors of the angles at B and D intersect at P on AC. Denote $\beta = \frac{1}{2}\angle B$ and $\mu = \angle BPA$. Now by the law of sines

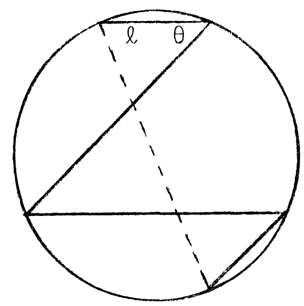
$$\frac{AB}{\sin\mu} = \frac{AP}{\sin\beta} \quad \text{and} \quad \frac{BC}{\sin(180^\circ - \mu)} = \frac{PC}{\sin\beta}$$

giving $\frac{AB}{BC} = \frac{AP}{PC}$. Similarly $\frac{AD}{DC} = \frac{AP}{PC}$. Thus B, D and P all lie on a circle of Apollonius with diameter PP' a segment of the line through AC. Since this diameter is not a perpendicular bisector of BD, (but *is* a bisector), the chord BD must itself be a diameter of the circle; and it is then immediate that $\angle BPD$ is a right angle.

Editor's remark. One difficulty here is that the examples cannot be convex, as a corollary of this proof.

7. [1985: 271] *Proposed by Zinkoo Han, Brooklyn, N.Y.*

A zigzag path across a unit circle consists of n chords lying in two directions. If the initial and terminal points of the path are ends of a diameter and the first chord has length ℓ, find, in terms of ℓ and n, the angle θ between the first and second chords. (The figure illustrates the case $n = 4$.)



$n = 4$

Solution by the proposer.

A "quickie". By adding the appropriate arcs, we find

$$\frac{n}{2}(2\theta) = \pi \qquad \text{for } n \text{ even}$$

and

$$\frac{n-1}{2}(2\theta) = 2\cos^{-1}\left[\frac{\ell}{2}\right] \qquad \text{for } n \text{ odd.}$$

So

$$\theta(\ell,n) = \begin{cases} \dfrac{\pi}{n} & \text{for } n \text{ even} \\ \dfrac{2}{n-1}\cos^{-1}\left[\dfrac{\ell}{2}\right] & \text{for } n \text{ odd.} \end{cases}$$

*   *   *

We leave the solutions to numbers 8 & 9 of this set for the June number of the Corner. We have no recorded solution to problem 10, for which we would welcome your submissions.

*   *   *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

1241. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.* (Dedicated to Murray S. Klamkin.)

A quadrilateral $ABCD$ and a triangle EFG are inscribed in a circle Γ. For an arbitrary point X of Γ, $s(X)$ denotes the sum of the distances from A, B, C, and D to the tangent to Γ at X. Prove that if $s(E) = s(F) = s(G)$, then $ABCD$ is a rectangle.

1242. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

The following problem appears in a book on matrix analysis: "Show that $\sum_{i,j=1}^{n} a_{ij}x_ix_j$ is positive definite if

$$\sum_i a_{ii}x_i^2 + \sum_{i\neq j} |a_{ij}|x_ix_j$$

is positive definite."

Give a counterexample!

1243. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and M an interior point with barycentric coordinates $(\lambda_1,\lambda_2,\lambda_3)$. The distances of M from the vertices A, B, C are x_1, x_2, x_3 and the circumradii of the triangles MBC, MCA, MAB, ABC are R_1, R_2, R_3, R. Show that

$$\lambda_1R_1 + \lambda_2R_2 + \lambda_3R_3 \geq R \geq \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3.$$

1244. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all triangles with integral sides a, b, c and area A such that a, b, c, and A form an arithmetic progression.

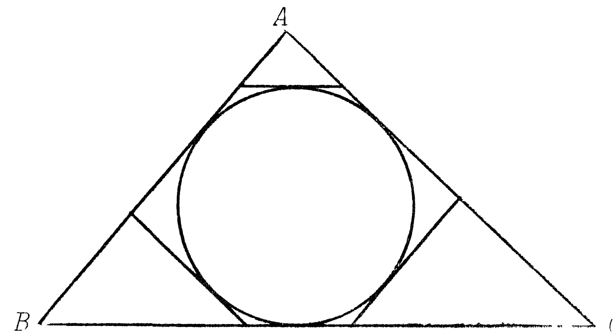
<u>1245</u>. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(Dedicated to Léo Sauvé.)

Let ABC be a triangle, and let $\mathcal{H}$ be the hexagon created by drawing tangents to the incircle of ABC parallel to the sides of ABC. Prove that

$$\text{perimeter}(\mathcal{H}) \le \frac{2}{3}\,\text{perimeter}(ABC).$$

When does equality occur?



<u>1246</u>. *Proposed by Lanny Semenko, Erewhon, Alberta.*

Canadians may notice (in the summer only, of course) that 28° C and 82° F are almost, but not exactly, the same temperature, where 82 is 28 written in reverse order. Find positive integers M and N such that M° C = N° F, and M and N in decimal are reverses of each other, or prove that no such M and N exist.

<u>1247</u>. *Proposed by Robert E. Shafer, Berkeley, California.*

Prove that for $0 \le \phi < \theta \le \pi/2$,

$$\cos^2\frac{\phi}{2}\log\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\log\sin^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\log\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\log\sin^2\frac{\theta}{2} < \frac{3}{4}[\sin^{4/3}\theta - \sin^{4/3}\phi].$$

<u>1248</u>. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Suppose that m and n are positive integers and that the decimal expansion of the rational number m/n has a repetend of 4356. Prove that n is divisible by 101.

<u>1249</u>*. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

Prove the triangle inequalities

(a) $\quad \Sigma \sin^4 A \le 2 - \frac{1}{2}\left[\frac{r}{R}\right]^2 - 3\left[\frac{r}{R}\right]^4 \le 2 - 5\left[\frac{r}{R}\right]^4$

(b) $\quad \Sigma \sin^2 2A \ge 6\left[\frac{r}{R}\right]^2 + 12\left[\frac{r}{R}\right]^4 \ge 36\left[\frac{r}{R}\right]^4$

(c) $\quad \Sigma \sin 2B \sin 2C \le 5\left[\frac{r}{R}\right]^2 + 8\left[\frac{r}{R}\right]^3 \le 9\left[\frac{r}{R}\right]^2$

where the sums are cyclic over the angles A, B, C of a triangle, and r, R are the inradius and circumradius respectively.

<u>1250</u>. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

We have a regular octahedron with vertices $A_1,A_2,\ldots,A_6$. Let P be a point and let $n_1,n_2,\ldots,n_8$ be the distances from P to the eight faces of the octahedron. Let

$$S_1 = \sum_{i=1}^{6} \overline{PA_i}^2 , \qquad S_2 = \sum_{j=1}^{8} n_j^2 .$$

Prove that S_1/S_2 is independent of P.

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

<u>478</u>. [1979: 229; 1980: 219; 1985: 189] *Proposed by Murray S. Klamkin, University of Alberta.*

Consider the following theorem:

If the circumcircles of the four faces of a tetrahedron are mutually congruent, then the circumcenter O of the tetrahedron and its incenter I coincide.

An editor's comment following *Crux* 330 [1978: 264] claims that the proof of this theorem is "easy". Prove it.

III. *Further comment by the proposer and Andy Liu, University of Alberta.*

Our previous comment [1985: 189] raised the question of the completeness of the Altshiller-Court proof showing that if the circumcenter and incenter of a tetrahedron coincide then the tetrahedron must be isosceles. In response to this comment, the then editor (Léo Sauvé) gave a different proof using a theorem of Bang which may be harder to prove than the result in question. Here we resolve the Altshiller-Court proof, and we *now* believe that the missing step was so apparent to him that he did not bother to include it (this is often done by many of us). The questionable statement was "Now equal chords subtend equal angles in a circle" since one had to rule out possible supplementary angles. These are ruled out by noting that the points of tangency of the insphere with the faces of the tetrahedron must all be interior points. Since these points are also the circumcenters of the four faces, the faces are all acute triangles. Nevertheless, problem 478 is still open since the circumcenter could possibly coincide with one of the many excenters of the tetrahedron rather than with the incenter.

* * *

1039. [1985: 122] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given are three collinear points O, P, H (in that order) such that $OH < 3OP$. Construct a triangle ABC with circumcentre O and orthocentre H and such that AP is the internal bisector of angle A. How many such triangles are possible?

Solution by Dan Sokolowsky, Williamsburg, Virginia.

Let $A \neq P$ be an arbitrary point such that

$$\frac{AH}{AO} = \frac{PH}{PO}\,. \tag{1}$$

Draw the circle K with center O and radius $R = OA$. Let AH meet K again at S. (If $AH \perp AO$, let $S = A$.)

If $S \neq H$, draw the perpendicular bisector L of HS which (as we prove later) meets K in two distinct points; label them B, C in either order. If $S = H$, label H also as B (or C), and let C (or B) denote the point at which the perpendicular to AH at H meets K again.

In all cases the resulting triangle ABC obviously has circumcenter O, and, as we now prove, meets the remaining requirements:

(a) AP is the internal bisector of $\angle A$;

(b) H is the orthocenter of ΔABC.

In particular we note that, since there are infinitely many points A satisfying (1), there are infinitely many triangles ABC meeting the given requirements.

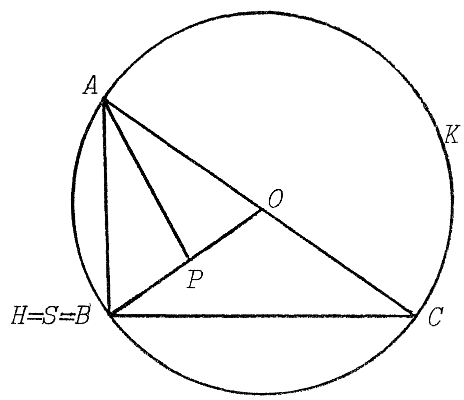


Figure 1

We consider the three possible cases:

(i) H is on K;

(ii) H is interior to K;

(iii) H is exterior to K.

In case (i) (Figure 1), (a) is implied by (1), and (b) is obvious.

In case (ii) (Figure 2), it is obvious that L meets K in two distinct points B, C. Here (1) implies that

$$\angle PAH = \angle PAO. \tag{2}$$

Let M be the midpoint of HS. Then, since $\angle AMB = 90^\circ$, $\angle BAM = 90^\circ - \angle B$. In

ΔAOC, $\angle AOC = 2\angle B$, and since $OA = OC$,

$$\angle OAC = 90^\circ - \frac{1}{2}\angle AOC = 90^\circ - \angle B = \angle BAM.$$

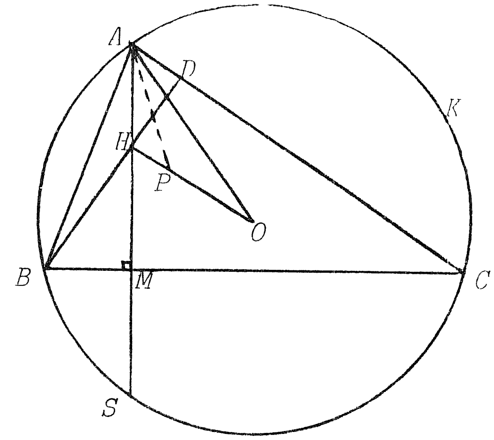


Figure 2

Hence by (2),

$$\angle BAP = \angle BAM + \angle PAM = \angle OAC + \angle PAO = \angle CAP,$$

which implies (a).

Let BH meet AC at D. Since $AH \perp BC$, to prove (b) it suffices to show $BC \perp AC$. Since BM is the perpendicular bisector of HS, $\Delta BMH \equiv \Delta BMS$, and hence

$$\angle HBC = \angle CBS = \angle CAS.$$

From triangles BMH and AHD, we now have

$$\angle ADH = \angle BMH = 90^\circ,$$

and (b) follows.

For case (iii) there are three possible subcases, namely (α) $A - S - H$, (β) $S - A - H$, (γ) $S = A$. We provide the analysis for (α) (those for (β) and (γ) are similar).

Note first that $OH < 3OP$ implies $PH < 2PO$, so by (1)

$$AH < 2AO = 2R. \qquad (3)$$

Let M denote the midpoint of HS, Q the foot of the perpendicular from O to AS, and T on MC such that $OQMT$ is a rectangle. Then by (3),

$$OT = QM = QS + SM$$
$$= \frac{1}{2}AS + \frac{1}{2}SH = \frac{1}{2}AH < R,$$

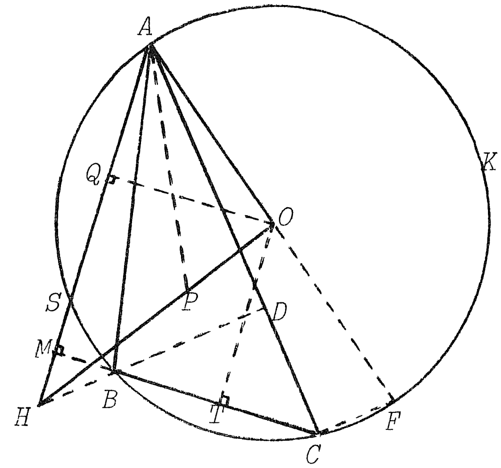


Figure 3

so T is interior to K. Since H is exterior to K, so is M, and hence MT meets K in two distinct points B, C. Let AF be a diameter of K; then $\angle ACF = 90^\circ = \angle AMB$ and $\angle AFC = 180^\circ - \angle ABC = \angle ABM$, so $\angle CAF = \angle MAB$. Thus by (2), $\angle BAP = \angle CAP$, proving (a).

To prove (b), let BH meet AC at D. Since MT is the perpendicular bisector of HS, $BH = BS$, so

$$\angle MHB = \angle MSB = 180^\circ - \angle ASB = \angle ACB.$$

We also have $\angle MBH = \angle DBC$, so $\angle BDC = \angle BMH = 90^\circ$, which implies (b).

Remark. Given $O - P - H$, the above argument shows that the condition $OH < 3OP$ is sufficient to guarantee the existence of a triangle ABC meeting the desired requirements. It is also necessary. The proof is similar in all cases; we illustrate it for the case (iii) (α) above.

Thus, in Figure 3, let O be the circumcenter of a triangle ABC satisfying (a) and (b). Let BH meet AC at D, BC meet AH at M, $OT \perp BC$, $OQ \perp AB$, $R = OA$. Since (b) holds, $\angle BMH = 90^\circ = \angle BDC$ and thus $\angle BHM = \angle BCD$. Since also

$$\angle BCD = 180^\circ - \angle ASB = \angle BSH,$$

$\angle BHM = \angle BSH$ and hence $HM = SM = \frac{1}{2}HS$. Then

$$OT = MQ = MS + SQ = \frac{1}{2}HA.$$

Since $OT < R = OA$,

$$\frac{AH}{AO} < 2. \tag{4}$$

Since (a) holds, $\angle PAB = \angle PAC$, and also

$$\angle ABM = 180^\circ - \angle ABC = \angle AFC.$$

Hence $\angle BAM = \angle CAF$ and so $\angle PAH = \angle PAO$. Thus AP bisects $\angle OAH$, and, using (4),

$$\frac{AH}{AO} = \frac{PH}{PO} < 2,$$

which, with $O - P - H$, implies $OH < 3OP$.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; and the proposer.

Meyers assumed that the point P was the intersection of the bisector of ∠A with BC. He constructed the triangle under this assumption, but showed that its construction is impossible in general when the points O, P, H are not collinear, thus settling Problem 79 of William Wernick, Triangle Constructions with three located points, Mathematics Magazine 55 *(1982) 227-230.*

Dou gave the following properties of triangles ABC satisfying the given conditions: (i) the vertices A all lie on a fixed circle φ (this follows from (1) above); (ii) the sides AB and AC intersect φ in fixed points M and N; (iii) the loci of vertices B and C are circles of equal radii.

Finally, this problem is an extension of, and was suggested by, Crux 818 *[1984: 159].*

* * *

<u>1096</u>. [1985: 325] *Proposed by M.S. Klamkin, University of Alberta.*

Determine the maximum and minimum values of

$$S \equiv \cos A/4 \cos B/4 \cos C/4 + \sin A/4 \sin B/4 \sin C/4,$$

where A, B, C are the angles of a triangle. (No calculus, please!)

I. *Solution by George Tsintsifas, Thessaloniki, Greece.*

We use the following well-known identities:

$$\sin x + \sin y + \sin z - \sin(x+y+z) \;=\; 4\sin\frac{x+y}{2}\sin\frac{y+z}{2}\sin\frac{z+x}{2},$$

$$\cos x + \cos y + \cos z + \cos(x+y+z) \;=\; 4\cos\frac{x+y}{2}\cos\frac{y+z}{2}\cos\frac{z+x}{2}.$$

We put

$$x = 45^\circ - A/2,$$
$$y = 45^\circ - B/2,$$
$$z = 45^\circ - C/2.$$

Then $\frac{x+y}{2} = 45^\circ - \frac{A+B}{4} = \frac{C}{4}$, etc., so we will have

$$\Sigma \sin(45^\circ - A/2) - \sqrt{2}/2 \;=\; 4\,\Pi \sin A/4 \tag{1}$$

$$\Sigma \cos(45^\circ - A/2) + \sqrt{2}/2 \;=\; 4\,\Pi \cos A/4, \tag{2}$$

where the sums and products are cyclic over A, B, C. But we easily see

$$\sin(45^\circ - A/2) + \cos(45^\circ - A/2) \;=\; \sqrt{2}\cos A/2.$$

Therefore adding (1) and (2) we have

$$4S \;=\; \sqrt{2}(\cos A/2 + \cos B/2 + \cos C/2), \tag{3}$$

where

$$S \;=\; \cos A/4 \cos B/4 \cos C/4 + \sin A/4 \sin B/4 \sin C/4.$$

It is known (2.27 of Bottema, Djordjevic, Janic, Mitrinovic, Vasic, *Geometric Inequalities*) that

$$2 \;<\; \cos A/2 + \cos B/2 + \cos C/2 \;\le\; \frac{3\sqrt{3}}{2} \tag{4}$$

and that these bounds are attained (by the degenerate triangle $A = B = 0^\circ$, $C = 180^\circ$ and by the equilateral triangle, respectively). Thus from (3) and (4) follows

$$\frac{\sqrt{2}}{2} < S \le \frac{3\sqrt{6}}{8}.$$

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We deal more generally with

$$S(\lambda) := \Pi \cos \lambda A + \Pi \sin \lambda A$$

where $0 < \lambda \le 1/2$ and the products are cyclic over A, B, C. It is easily checked that the functions $f(t) = \log \sin \lambda t$ and $g(t) = \log \cos \lambda t$ are concave on $[0,\pi]$. Thus

$$\Sigma f(A) \le 3f(\pi/3), \qquad \Sigma g(A) \le 3g(\pi/3)$$

and finally

$$S(\lambda) \le \sin^3\frac{\lambda\pi}{3} + \cos^3\frac{\lambda\pi}{3}. \tag{1}$$

On the other hand, by pp.22-23 of D.S. Mitrinovic, *Analytic Inequalities*, $\Sigma f(A) > 2f(0) + f(\pi)$ and $\Sigma g(A) > 2g(0) + g(\pi)$, and therefore

$$S(\lambda) > \cos \lambda\pi \tag{2}$$

(the bound being approached by triangles with, say, $A,B \longrightarrow 0$ and $C \longrightarrow \pi$). Putting $\lambda = 1/4$ in (1) and (2), we get

$$\frac{\sqrt{2}}{2} < S \le \frac{3\sqrt{6}}{8} .$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; and the proposer. The upper bound only was found by SVETOSLAV BILCHEV, Russe, Bulgaria and by LEROY F. MEYERS, The Ohio State University, Columbus, Ohio.

* * *

<u>1097</u>. [1985: 325; 1987: 135] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let AD, BE, CF be the angle bisectors and AM, BN, CP the medians of a triangle ABC. Prove that

$$\overrightarrow{AD}\cdot\overrightarrow{AM} + \overrightarrow{BE}\cdot\overrightarrow{BN} + \overrightarrow{CF}\cdot\overrightarrow{CP} = s^2,$$

where s is the semiperimeter.

II. *Generalization by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

We have [1987: 136] that

$$\overrightarrow{AD}\cdot\overrightarrow{AM} = s(s - a)$$
$$\overrightarrow{BE}\cdot\overrightarrow{BN} = s(s - b)$$
$$\overrightarrow{CF}\cdot\overrightarrow{CP} = s(s - c).$$

Thus for $n \ge 1$, using the power mean inequality,

$$\Sigma(\overrightarrow{AD}\cdot\overrightarrow{AM})^n = s^n\,\Sigma(s - a)^n = 3s^n[\Sigma(s - a)^n]/3$$
$$\ge 3s^n\left[\frac{\Sigma(s - a)}{3}\right]^n = 3s^n\left[\frac{s}{3}\right]^n = \frac{s^{2n}}{3^{n-1}} .$$

If $n < 1$, the inequality is reversed.

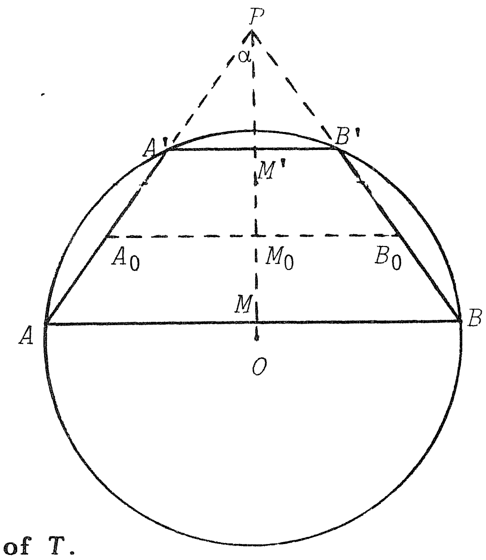
* * *

<u>1098</u>. [1985: 326] *Proposed by Jordi Dou, Barcelona, Spain.*

Characterize all trapezoids for which the circumscribed ellipse of minimal area is a circle.

Solution by the proposer.

Clearly we need only consider trapezoids having circumcircles, namely isosceles trapezoids. Let T be an isosceles trapezoid with parallel sides AB,



$A'B'$. Let M, M' be the midpoints of AB, $A'B'$, respectively, and let A_0, M_0, B_0 be the midpoints of AA', MM', BB', respectively. Let P be the intersection of AA' and BB' (and MM'), and put

$$k = \frac{\overline{A'B'}}{\overline{AB}}\ , \quad h = \overline{MM'}\ , \quad \alpha = \angle APM\ ,$$

where we assume $\overline{A'B'} < \overline{AB}$ so that $0 < k < 1$. Finally, let O be the centre and R the radius of the circumcircle of T.

Consider k, h fixed (so that P is fixed) and α varying, and denote the trapezoid by T_α. Letting E_α be the ellipse of minimal area circumscribing T_α, we have, by considering the affinity (orthogonal projection) with axis PM, that $[E_\alpha]/[T_\alpha]$ is constant, where $[X]$ denotes the area of figure X. Letting C_α be the circumcircle of T_α, it follows that $C_\alpha = E_\alpha$ for that value of α minimizing $[C_\alpha]/[T_\alpha]$.

Letting $m = \overline{M_0P}$, we have

$$k = \frac{\overline{A'B'}}{\overline{AB}} = \frac{\overline{M'P}}{\overline{MP}} = \frac{m - h/2}{m + h/2}$$

and thus

$$m = \frac{h}{2}\cdot\frac{1 + k}{1 - k}\ .$$

Put

$$\rho = \frac{2m}{h} = \frac{1 + k}{1 - k}\ ; \tag{1}$$

then from

$$\overline{M_0A_0} = m \tan\alpha$$

we obtain

$$R^2 = (\overline{A_0A})^2 + (\overline{OA_0})^2 = \frac{h^2}{4\cos^2\alpha} + \frac{m^2\tan^2\alpha}{\cos^2\alpha}$$

$$= \frac{h^2}{4}\left[\frac{\cos^2\alpha + \rho^2\sin^2\alpha}{\cos^4\alpha}\right].$$

Hence

$$[C_\alpha] = \frac{\pi h^2}{4}\left[\frac{\cos^2\alpha + \rho^2\sin^2\alpha}{\cos^4\alpha}\right]$$

and

$$[T_\alpha] \;=\; 2h\cdot\overline{M_0A_0} \;=\; 2hm \tan\alpha.$$

Thus

$$\frac{[C_\alpha]}{[T_\alpha]} = \frac{\pi h}{8m}\,\varphi(\alpha)$$

where

$$\varphi(\alpha) \;=\; \frac{\cos^2\alpha + \rho^2\sin^2\alpha}{\sin\alpha\,\cos^3\alpha},$$

and so we wish to minimize $\varphi(\alpha)$, $0 < \alpha < 90^\circ$. Now

$$\varphi'(\alpha) = \frac{\sin\alpha\cos^3\alpha(-2\sin\alpha\cos\alpha+2\rho^2\sin\alpha\cos\alpha) - (\cos^2\alpha+\rho^2\sin^2\alpha)(\cos^4\alpha-3\sin^2\alpha\cos^2\alpha)}{\sin^2\alpha\,\cos^6\alpha}$$

$$= \frac{2\sin^2\alpha\,\cos^2\alpha(\rho^2 - 1) - \cos^4\alpha + (3 - \rho^2)\sin^2\alpha\,\cos^2\alpha + 3\rho^2\sin^4\alpha}{\sin^2\alpha\,\cos^4\alpha}$$

$$= \frac{3\rho^2\tan^4\alpha + (\rho^2 + 1)\tan^2\alpha - 1}{\sin^2\alpha}.$$

It follows that $\varphi(\alpha)$, and thus $[C_\alpha]/[T_\alpha]$, is minimized for α satisfying

$$3\rho^2\tan^4\alpha + (\rho^2 + 1)\tan^2\alpha - 1 \;=\; 0. \tag{2}$$

Putting $\overline{AM} = b$ so that $\overline{A'M'} = kb$, we have

$$\tan\alpha = \frac{b - kb}{h}. \tag{3}$$

From (1) and (3), (2) becomes

$$\frac{3(1 + k)^2}{(1 - k)^2}\cdot\frac{b^4}{h^4}(1 - k)^4 + \left[\frac{(1 + k)^2}{(1 - k)^2} + 1\right]\frac{b^2}{h^2}(1 - k)^2 - 1 \;=\; 0$$

$$3(1 - k^2)^2\left(\frac{b}{h}\right)^4 + 2(1 + k^2)\left(\frac{b}{h}\right)^2 - 1 \;=\; 0$$

or

$$\left(\frac{h}{b}\right)^4 - 2(1 + k^2)\left(\frac{h}{b}\right)^2 - 3(1 - k^2)^2 \;=\; 0,$$

whence

$$\left(\frac{h}{b}\right)^2 = \frac{2(1 + k^2) + \sqrt{4(1 + k^2)^2 + 12(1 - k^2)^2}}{2}$$

and thus

$$\frac{h}{b} = \sqrt{1 + k^2 + \sqrt{1 + 2k^2 + k^4 + 3 - 6k^2 + 3k^4}}$$

$$= \sqrt{1 + k^2 + 2\sqrt{1 - k^2 + k^4}}\,. \tag{4}$$

Therefore the required trapezoids are those for which the height $\overline{MM'} = h$, the base $\overline{AB} = 2b$, and the ratio $k = \overline{A'B'}/\overline{AB}$ $(0 < k < 1)$ satisfy (4). In addition, letting $k \longrightarrow 1$ we obtain from (4) that $h/b \longrightarrow 2$; thus the square is a further solution. Finally, as $k \longrightarrow 0$, we see that $h/b \longrightarrow \sqrt{3}$, which from (3) says that $\alpha \longrightarrow 30^\circ$, yielding the equilateral triangle as another solution.

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1099. [1985: 326] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find a solution to each of the Diophantine equations

$$x^2 - 726y^2 = 3 \qquad \text{and} \qquad x^2 - 363y^2 = -2$$

in which the integers x and y are both greater than 500.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

With the substitution $z = 11y$ the two equations become

$$x^2 - 6z^2 = 3 \qquad \text{and} \qquad x^2 - 3z^2 = -2.$$

These are Pell equations, which can be solved by standard methods. For the first we obtain

$$\begin{aligned} x_1 &= 3, & z_1 &= 1; \\ x_2 &= 27, & z_2 &= 11; \\ x_i &= 10x_{i-1} - x_{i-2}, & z_i &= 10z_{i-1} - z_{i-2}. \end{aligned} \tag{1}$$

To solve the original equation we require solutions for z that are divisible by 11. From the recursion relation (1) it is easy to show that

$$z_{i+3} \equiv z_i \pmod{11}.$$

Therefore since $11|z_2$, $11|z_{3k-1}$ for all positive values of k. In particular we find $z_5 = 10681$, whence $y_5 = 971$ and $x_5 = 26163$. Thus (26163,971) is the first of an infinite set of solutions (x,y) that meet the conditions of the problem statement.

Similarly for the second equation we obtain

$$\begin{aligned} x_1 &= 1, & z_1 &= 1; \\ x_2 &= 5, & z_2 &= 3; \\ x_3 &= 19, & z_3 &= 11; \\ x_i &= 4x_{i-1} - x_{i-2}, & z_i &= 4z_{i-1} - z_{i-2}; \\ z_{i+5} &\equiv z_i \pmod{11}; \end{aligned}$$

and thus $11|z_{5k-3}$ for all positive k. Hence $z_8 = 7953$, $y_8 = 723$, $x_8 = 13775$, and the smallest solution is (13775,723).

Also solved by J.G. FLATMAN, *Timmins, Ontario;* RICHARD I. HESS, *Rancho Palos Verdes, California;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* SIDNEY KRAVITZ, *Dover, New Jersey;* SUSIE LANIER *and* DAVID STONE, *Georgia Southern College, Statesboro, Georgia;* STEWART METCHETTE, *Culver City,* *California; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; KENNETH M. WILKE, Topeka, Kansas; and the proposer.*

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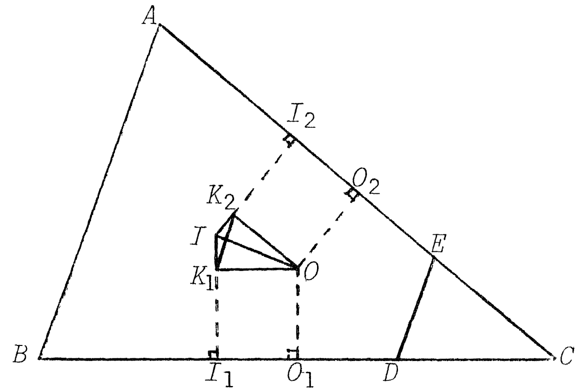
1100. [1985: 326] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with $C = 30°$, circumcentre O and incentre I. Points D and E are chosen on BC and AC, respectively, such that $BD = AE = AB$. Prove that $DE = OI$ and $DE \perp OI$.

I. *Solution by J.T. Groenman, Arnhem, The Netherlands.*



Denote the sides and semiperimeter of ΔABC by a, b, c, s as usual.

Let I_1, O_1 on BC and I_2, O_2 on AC be the feet of the perpendiculars from I, O, respectively. Let K_1, K_2 be the feet of the perpendiculars from O to II_1, II_2, respectively. Then

$$K_1O = I_1O_1 = \left\{\begin{matrix} \frac{1}{2}a - (s - b) & \text{if} & b \geq c \\ \\ \frac{1}{2}a - (s - c) & \text{if} & c \geq b \end{matrix}\right\} = \frac{1}{2}|b - c|$$

and in the same way we have

$$K_2O = \frac{1}{2}|a - c|.$$

Moreover $K_2O \parallel AC$ and $K_1O \parallel BC$, so

$$\angle K_2OK_1 = \angle DCE.$$

Thus since

$$CE = |b - AE| = |b - AB| = |b - c|$$

and

$$DC = |a - c|,$$

triangles K_2K_1O and DEC are similar. In particular,

$$\angle K_1K_2O = \angle CDE.$$

Now as $\angle IK_1O = \angle IK_2O = 90°$, OK_2IK_1 is cyclic, and thus

$$\angle K_1IO = \angle K_1K_2O = \angle CDE.$$

Since $IK_1 \perp DC$, we conclude that $IO \perp DE$. We remark that the assumption $C = 30°$ has not yet been used.

Now

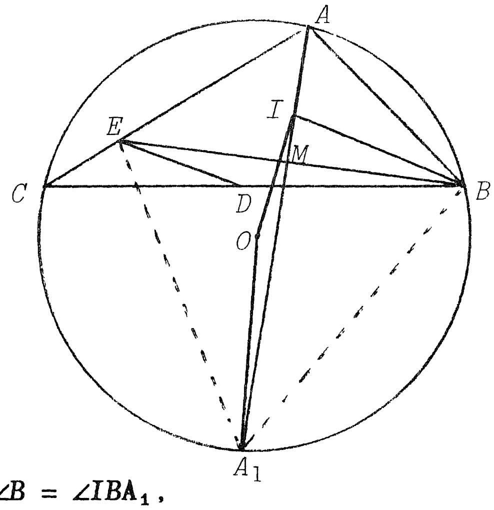
$$\sin \angle CDE = \frac{K_1O}{IO} = \frac{|b - c|}{2(IO)},$$

and so

$$IO = \frac{1}{2}\frac{|b - c|}{\sin \angle CDE} = \frac{1}{2}\frac{CE}{\sin \angle CDE} = \frac{1}{2}\frac{DE}{\sin C} = DE,$$

using $C = 30^\circ$. I suspect without doubt that the result holds when $C = 150^\circ$ as well.

II. *Solution by H. Fukagawa, Yokosuka High School, Tokai-city, Aichi, Japan.*



Let AI meet the circumcircle again at A_1, and let M be the intersection of AA_1 and BE. Then from $AB = AE$ and $\angle BAA_1 = \angle CAA_1$ we have that $\Delta BAA_1 \equiv \Delta EAA_1$, and thus $A_1E = A_1B$ and $AM \perp BE$. Since

$\angle EA_1A = \angle AA_1B = \angle ACB = 30^\circ$,

ΔBEA_1 is equilateral. Moreover, since

$$\angle BIM = \frac{1}{2}\angle A + \frac{1}{2}\angle B = \angle IBA_1,$$

we have

$$IA_1 = BA_1 = BE. \tag{1}$$

From $BC \perp A_1O$ and $BE \perp A_1M$, it follows that

$$\angle MA_1O = \angle EBD. \tag{2}$$

Further,

$$\angle AOB = 2\angle ACB = 60^\circ,$$

so

$$DB = AB = OA_1. \tag{3}$$

Now (1), (2), and (3) imply

$$\Delta OIA_1 \equiv \Delta EDB.$$

We conclude that $ED = IO$, and since $AM \perp BE$, $ED \perp IO$.

Also solved by BENO ARBEL, *Tel Aviv University, Tel Aviv, Israel;* R.H. EDDY, *Memorial University, St. John's, Newfoundland;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

* * *

<u>1101</u>. [1986: 11] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

This problem is the dual of *Crux* 941 [1985: 227]. Independently solve each of the following alphametics in base ten:

$$6 \cdot \text{FLOCK} = \text{GEESE},$$
$$7 \cdot \text{FLOCK} = \text{GEESE},$$
$$8 \cdot \text{FLOCK} = \text{GEESE}.$$

Solution.

$$6 \cdot 15367 = 92202,$$
$$7 \cdot 10948 = 76636,$$
$$8 \cdot 12356 = 98848.$$

Found by JOHN FLATMAN, Timmins, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. MCCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Valencia Community College, Orlando, Florida; J. SUCK, Essen, Federal Republic of Germany; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, Federal Republic of Germany; and the proposer.

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<u>1102</u>. [1986: 11] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in n-dimensional Euclidean space. Let M be an interior point of σ_n whose barycentric coordinates are $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and, for $i = 0, 1, \ldots, n$, let p_i be its distance from the $(n - 1)$-face

$$\sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n.$$

Prove that $\lambda_0p_0 + \lambda_1p_1 + \ldots + \lambda_np_n \ge r$, where r is the inradius of σ_n.

Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

Let F_i denote the volume of the $(n - 1)$-dimensional face opposite A_i and V the volume of σ_n. Then

$$\lambda_i = \frac{p_iF_i}{nV}.$$

Since also $\sum_{i=0}^{n} p_iF_i = nV$, it follows by Hölder's inequality that for any $m > 0$,

$$\left[\sum_{i=0}^{n} \lambda_i p_i^m\right]^{\frac{1}{m+1}} \left[\sum_{i=0}^{n} F_i\right]^{\frac{m}{m+1}} = \left[\sum_{i=0}^{n} \frac{p_i^{m+1}F_i}{nV}\right]^{\frac{1}{m+1}} \left[\sum_{i=0}^{n} F_i\right]^{\frac{m}{m+1}}$$

$$\ge \sum_{i=0}^{n} p_iF_i/(nV)^{\frac{1}{m+1}}$$

$$= (nV)^{\frac{m}{m+1}} = \left[r \sum_{i=0}^{n} F_i \right]^{\frac{m}{m+1}}$$

which implies

$$\sum_{i=0}^{n} \lambda_i p_i^m \geq r^m .$$

Putting $m = 1$ gives the desired result. There is equality if and only if p_i is constant, i.e. M is the incenter.

Also solved by WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

*   *   *

<u>1103</u>. [1986: 11] *Proposed by Roger Izard, Dallas, Texas.*

Three concurrent cevians through the vertices A, B, C of a triangle meet the lines BC, CA, AB in D, E, F, respectively, and the internal bisector of angle A meets BC in V. If A, F, D, V, E are all concyclic, prove that $AD \perp BC$.

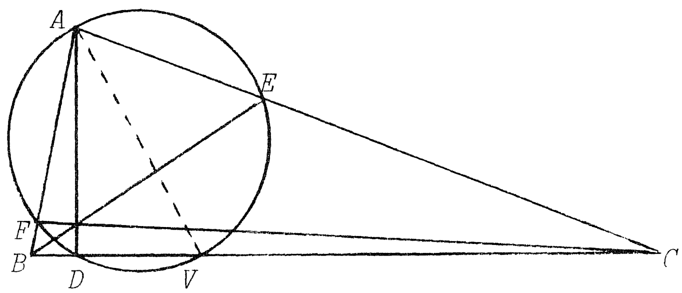
Solution by R.H. Eddy, Memorial University, St. John's, Newfoundland.

Letting AC = b and AB = c, the equations

$$FB \cdot c = BV \cdot BD$$

and

$$CE \cdot b = VC \cdot DC$$



are obtained immediately by applying the tangent-secant relationship at vertices B and C respectively. Consequently,

$$\frac{BD}{DC} = \frac{FB \cdot c \cdot VC}{CE \cdot b \cdot BV} .$$

But

$$\frac{BV}{VC} = \frac{c}{b} ,$$

hence

$$\frac{BD}{DC} = \frac{FB}{CE}$$

and so $AF = EA$ from Ceva's theorem. Since also $\angle FAV = \angle VAC$, AV is a diameter of the given circle, and so $\angle VDA = 90^\circ$.

Also solved by JORDI DOU, *Barcelona, Spain;* J.T. GROENMAN, *Arnhem, The Netherlands;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the* *proposer.*

*   *   *

1104*. [1086: 12] *Proposed by D.S. Mitrinovic, University of Belgrade, Yugoslavia.*

As a by-product, Giuseppina Masotti Biggiogero (*Ist. Lombardo, Sci. Lett. Rend. Cl. Sci. Math. Nat.*, (3) 14 (83) (1950) 735-752) obtained the following result about a square matrix (a_{ij}) of order $m + 1$:

If

$$a_{ij} = \begin{cases} c, & \text{if } i = j, \\ m - j + 1, & \text{if } i = j + 1, \\ -j + 1, & \text{if } i = j - 1, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$\det(a_{ij}) = \begin{cases} c(c^2 + 2^2)(c^2 + 4^2) \ldots (c^2 + m^2), & \text{if } m \text{ is even}, \\ (c^2 + 1^2)(c^2 + 3^2) \ldots (c^2 + m^2), & \text{if } m \text{ is odd}. \end{cases}$$

Prove this assertion.

Solution by Len Bos, University of Calgary, Calgary, Alberta.

Let

$$A = A_m(c) = \begin{bmatrix} c & -1 & 0 & . & . & . & . & 0 \\ m & c & -2 & 0 & . & . & . & 0 \\ 0 & m-1 & c & -3 & 0 & . & . & 0 \\ \vdots & & & \ddots & & & & \vdots \\ & & & & & & c & -m \\ 0 & . & . & . & . & . & 1 & c \end{bmatrix}$$

be the given matrix and let

$$J = \begin{bmatrix} i & & & & \\ & 1 & & 0 & \\ & & -i & & \\ & 0 & & -1 & \\ & & & & \ddots \end{bmatrix}$$

be the $m \times m$ diagonal matrix $J_{kk} = i(-i)^{k-1}$. Then $J_{kk}^{-1} = -i(i)^{k-1}$ and so

$$JAi(J)^{-1} = \begin{bmatrix} ic & 1 & 0 & . & . & . & . & . & 0 \\ m & ic & 2 & 0 & . & . & . & . & 0 \\ 0 & m-1 & ic & 3 & 0 & . & . & . & 0 \\ \vdots & & & \ddots & & & & & \vdots \\ & & & & & & & ic & m \\ 0 & . & . & . & . & . & 0 & 1 & ic \end{bmatrix}.$$

Hence, letting

$$D_m(x) = \begin{vmatrix} x & 1 & & & & \\ m & x & 2 & & 0 & \\ & m-1 & x & & & \\ & & & \ddots & & \\ & 0 & & & x & m \\ & & & & 1 & x \end{vmatrix},$$

the required determinant will be

$$|A| = \frac{1}{i^{m+1}} D_m(ic).$$

$D_m(x)$ is a known determinant (e.g. problem 399 of I.V. Proskuryakov, *Problems in Linear Algebra*). We claim that

$$D_m(x) = \begin{cases} (x^2 - 1^2)(x^2 - 3^2) \ldots (x^2 - m^2), & m \text{ odd}, \\ x(x^2 - 2^2) \quad . \quad . \quad . \quad (x^2 - m^2), & m \text{ even}. \end{cases} \tag{1}$$

The proof is by induction. If $m = 1$, then

$$D_m(x) = \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = x^2 - 1,$$

and if $m = 2$, then

$$D_m(x) = \begin{vmatrix} x & 1 & 0 \\ 2 & x & 2 \\ 0 & 1 & x \end{vmatrix} = x(x^2 - 4).$$

Assume (1) holds for $m - 1$. To each row of $D_m(x)$ add all the following rows to obtain

$$D_m(x) = \begin{vmatrix} x+m & x+m & \ldots & \ldots & \ldots & \ldots & x+m \\ m & x+m-1 & x+m & \ldots & \ldots & \ldots & x+m \\ 0 & m-1 & x+m-2 & x+m & \ldots & & x+m \\ \vdots & & & \ddots & & & \vdots \\ 0 & \ldots & \ldots & \ldots & 2 & x+1 & x+m \\ 0 & \ldots & \ldots & \ldots & 0 & 1 & x \end{vmatrix}.$$

Now subtract from each column the preceding column. We obtain

$$D_m(x) = \begin{vmatrix} x+m & 0 & 0 & 0 & & & \\ m & x-1 & 1 & 0 & & 0 & \\ 0 & m-1 & x-1 & 2 & & & \\ & & m-2 & x-1 & & & \\ & & & m-3 & \ddots & & \\ & 0 & & & & x-1 & m-1 \\ & & & & & 1 & x-1 \end{vmatrix}.$$

Hence

$$D_m(x) = (x + m)D_{m-1}(x - 1).$$

Therefore if m is even, $m - 1$ is odd and

$$\begin{aligned} D_m(x) &= (x + m)[((x - 1)^2 - 1^2)((x - 1)^2 - 3^2) \ldots ((x - 1)^2 - (m - 1)^2)] \\ &= (x + m)[x(x - 2)(x + 2)(x - 4)(x + 4)(x - 6) \ldots (x + m - 2)(x - m)] \\ &= x(x^2 - 2^2)(x^2 - 4^2) \ldots (x^2 - m^2). \end{aligned}$$

Similarly, if m is odd,

$$D_m(x) = (x + m)[(x - 1)((x - 1)^2 - 2^2) \ldots ((x - 1)^2 - (m - 1)^2)]$$
$$= (x + m)[(x - 1)(x + 1)(x - 3) \ldots (x + m - 2)(x - m)]$$
$$= (x^2 - 1^2)(x^2 - 3^2) \ldots (x^2 - m^2).$$

This completes the proof of (1). Hence

$$|A_m| = \frac{1}{i^{m+1}}\begin{cases} (ic)((ic)^2 - 2^2)((ic)^2 - 4^2) \ldots ((ic)^2 - m^2)\ , & m \text{ even} \\ ((ic)^2 - 1^2)((ic)^2 - 3^2) \ldots ((ic)^2 - m^2)\ , & m \text{ odd} \end{cases}$$

$$= \frac{1}{i^{m+1}}\begin{cases} i(-1)^{m/2}(c^2 + 2^2)(c^2 + 4^2) \ldots (c^2 + m^2)\ , & m \text{ even} \\ (-1)^{(m+1)/2}(c^2 + 1^2)(c^2 + 3^2) \ldots (c^2 + m^2)\ , & m \text{ odd} \end{cases}$$

and the result follows.

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1105*. [1986: 12] *Proposed by László Csirmaz, Mathematical Institute of the Hungarian Academy of Sciences.*

The Extraterrestrials' Secret Invisible Submarine (ESIS) carries a deadly weapon through the Grid Ocean ($\mathbb{Z} \times \mathbb{Z}$). You, as the Captain of the Most Advanced Space Command, are the only person in the universe who can save (wo)mankind by destroying the ESIS. Your Secret Agent reported that the command computer of the enemy's submarine was programmed to travel at a uniform speed along a straight line so that the ESIS must be in a grid point at every full hour, and only then is its energy shield weak enough to blow it up. You have infinite supplies, but only countably many rockets. The rockets can reach any grid point within an hour. Your equipment, however, allows one launching per hour. Well, Captain, could you save me?

Salvation by Duane Broline, University of Evansville, Evansville, Indiana.

Yes, I can. I can even save you if you wander into Grid Hyperspace ($\mathbb{Z}^n$). But if you travel to higher dimensions, you're on your own.

We assume that the Secret Agent is able to tell us the time, according to our watches, that the ESIS is at a grid point. Let $\mathbf{V} = \mathbb{Z}^n$ and let addition and scalar multiplication in $\mathbf{V}$ be defined in the usual way. Choose an arbitrary time that the ESIS is at a grid point to be Zero Hour.

Since the ESIS travels in a straight line and at a constant speed, an ESIS is uniquely determined by its position at Zero Hour and its position one hour later. Thus, it is possible to name the submarine by elements of $\mathbf{V} \times \mathbf{V}$

such that ESIS($\mathbf{a},\mathbf{b}$) is at grid point $\mathbf{a}$ at Zero Hour and at $\mathbf{b}$ at Zero Hour + 1. This ESIS is at grid point $\mathbf{a} + k(\mathbf{b} - \mathbf{a})$ after k hours.

Let $\mathbf{f}\colon \mathbb{N} \longrightarrow \mathbf{V} \times \mathbf{V}$ be a bijection where $\mathbb{N}$ is the set of natural numbers. If $k \in \mathbb{N}$, define $\mathbf{f}_1(k)$ and $\mathbf{f}_2(k)$ by $\mathbf{f}(k) = (\mathbf{f}_1(k),\mathbf{f}_2(k))$. The launching instructions are: at Zero Hour + $(k - 1)$ hours, $k = 1,2,3,\ldots$, launch a rocket so that it hits the grid point $\mathbf{f}_1(k) + k(\mathbf{f}_2(k) - \mathbf{f}_1(k))$ at Zero Hour + k hours. This rocket destroys the ESIS $\mathbf{f}(k)$.

If you have to wander into $\mathbb{Z}^{\aleph_0}$, you might examine the "CUBE WARS" defense popular in this country which is based upon a bijection from $\mathbb{N}$ to $\mathbb{Z}^{\aleph_0}$.

*   *   *

__1106__. [1986: 12] *Proposed by Jack Garfunkel, Flushing, N.Y.*

The directly similar triangles ABC and DEC are both right-angled at C. Prove that

(a) $AD \perp BE$;

(b) AD/BE equals the ratio of similitude of the two triangles.

Solution by Dan Pedoe, Minneapolis, Minnesota.

The problem can be generalized (the triangles need not be right-angled), but (b) is incorrectly stated. The generalized statement is:

The directly similar triangles ABC and DEC have angle Ω at C. Prove that

(a) AD makes angle Ω with BE;

(b) $AD/BE = CA/CB = CD/CE$.

Proof: With C as origin, using complex numbers, let a, b, d, e be the respective affixes of the points A, B, D, E. Then if the orientations of the two given triangles are both positive, $b = k\omega a$ and $e = k\omega d$ where

$$\frac{CB}{CA} = \frac{CE}{CD} = k, \qquad \omega = \cos\Omega + i\sin\Omega.$$

Hence

$$e - b = k\omega(d - a),$$

which shows that AD makes an angle Ω with BE, and that

$$\frac{AD}{BE} = \frac{1}{k} = \frac{CA}{CB} = \frac{CD}{CE}\ .$$

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J. SUCK, Essen, Federal Republic of Germany; and the proposer.

All solvers solved the corrected part (b), even the proposer (although he stated the problem incorrectly). Most solvers noted the above generalization.

* * *

1107. [1986: 12] *Proposed by Jordi Dou, Barcelona, Spain.*

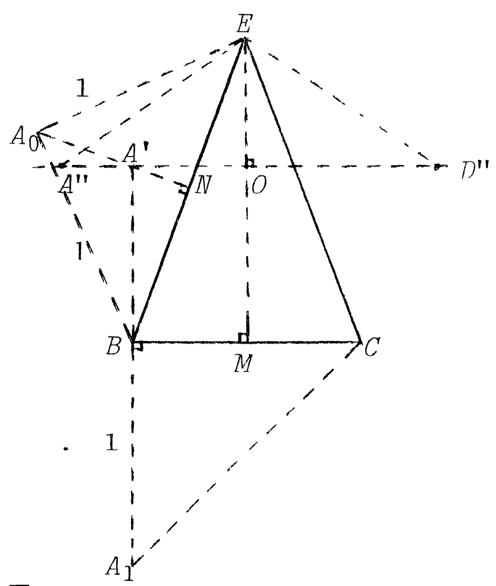
An equilateral skew pentagon has angles A, B, C, D all equal to 90°. Give a Euclidean construction for, and calculate the measure of, angle E, given that it is not 60°. (See Problem 1 [1985: 240].)

Solution by the proposer.

Let $AB = 1$. Then $EB = AC = BD = CE = \sqrt{2}$. Consider the tetrahedra $AEBC$ and $DECB$ with common face EBC. The faces EBC, AEC, and EBD of the two tetrahedra are congruent isosceles triangles, and the faces EAB, ABC, BCD, and CDE are congruent isosceles right triangles.

If the two tetrahedra are on the same side of EBC, then the tetrahedra are symmetric with respect to the plane perpendicular to BC through its midpoint M (and the point E). The vertices A, B, C, D will form a plane square, so $AD = 1$ and $\angle E$ will be 60°.

Thus the tetrahedra must be on opposite sides of EBC, and they will be congruent and symmetric with respect to the axis EM. The diagonal AD is perpendicular to EM and cuts it, say at O. Rotate ΔAEB about the line EB, keeping EB fixed, until A falls on a point A_0 in the plane of, and outside, ΔEBC. Similarly rotate ΔABC about BC so that A falls on a point A_1 in the plane of, and outside, ΔEBC. The projection A' of A on the plane of EBC is the intersection of A_1B and A_0N, where N is the midpoint of EB. The projection O of A on EM is the same as the projection of A' on EM.



Since $EA = ED = 1$ and $AO \perp EO$, the required angle E will satisfy

$\cos E/2 = EO/EA = EO.$

Since $\Delta A'BN \sim \Delta BEM$ and $EM = \sqrt{2 - 1/4} = \sqrt{7}/2$,

$$\frac{A'B}{\sqrt{2}/2} = \frac{A'B}{BN} = \frac{EB}{EM} = \frac{\sqrt{2}}{\sqrt{7}/2},$$

so $A'B = 2/\sqrt{7}$. Thus

$$\cos E/2 = EO = EM - OM = EM - A'B = \frac{\sqrt{7}}{2} - \frac{2}{\sqrt{7}} = \frac{3}{2\sqrt{7}},$$

and $E \approx 111^\circ$.

Angle E can easily be constructed by constructing points E'' and D'' on $A'O$ so that $A''E = D''E = 1$. Then $\angle E = \angle A''ED''$, since A and D lie symmetrically placed (with respect to O) on opposite sides of the plane of EBC.

* * *

1108. [1986: 12] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve the equation $4 \sec x + 3 \csc x = 10$ for $0 < x < 360^\circ$.

I. *Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

More generally, consider the equation

$$a \sec x + b \csc x = 2\sqrt{a^2 + b^2} \tag{1}$$

where $a,b > 0$. Letting

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos A, \qquad 0 < A < 90^\circ, \tag{2}$$

(1) becomes

$$\frac{\cos A}{\cos x} + \frac{\sin A}{\sin x} = 2,$$

$$\sin x \cos A + \cos x \sin A = 2 \sin x \cos x,$$

and finally

$$\sin(x + A) = \sin 2x.$$

Thus the solutions in the given range are

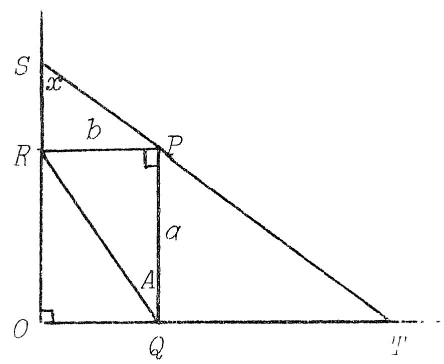
$$\begin{aligned} x + A &= 2x &&\text{or}&& x = A\,, \\ 2x &= 180^\circ - (x + A) &&\text{or}&& x = 60^\circ - A/3\,, \\ 2x &= 540^\circ - (x + A) &&\text{or}&& x = 180^\circ - A/3\,, \\ 2x &= 900^\circ - (x + A) &&\text{or}&& x = 300^\circ - A/3\,, \end{aligned}$$

with A defined by (2). For the given problem, $a = 4$ and $b = 3$ so

$$A = \cos^{-1}(4/5) \approx 36.87^\circ.$$

Note that of the above four solutions, the first two will be acute, and are distinct unless $A = 45^\circ$.

Geometrically, the problem is equivalent to finding chords ST through P equal to twice RQ (see diagram). One obvious solution is when $ST \parallel RQ$. The above observation says that there is always a second solution except when $A = 45^\circ$.



II. *"Solution" by N. Withheld.*

Differentiate

$$4 \sec x + 3 \csc x = 10$$

with respect to x. Then

$$\frac{4 \sin x}{\cos^2 x} - \frac{3 \cos x}{\sin^2 x} = 0$$

$$4 \tan^3 x = 3$$

$$\tan x = 0.9086$$

$$x = 42^\circ 15' \quad \text{or} \quad 222^\circ 15'.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; FRANCISCO BELLOT, Valladolid, Spain; DUANE BROLINE, University of Evansville, Evansville, Indiana; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, New York; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; M.A. SELBY, University of Windsor, Windsor, Ontario; W.R. UTZ, Rolla, Missouri; and the proposer. There was one incorrect solution (besides II above!) submitted.

* * *

1110.* [1986: 13] *Proposed by M.S. Klamkin, University of Alberta.*

How many different polynomials $P(x_1,x_2,\ldots,x_m)$ of degree n are there for which the coefficients of all the terms are 0's or 1's and

$$P(x_1,x_2,\ldots,x_m) = 1 \qquad \text{whenever} \qquad x_1 + x_2 + \ldots + x_m = 1?$$

Editor's comment. There has been only one response to this problem, and it was incorrect. It did note the easy case $m = 1$, where $P(x) = x^n$ is the only solution for each n. Equally easy is the case $n = 1$ which has unique solution $P(x_1,x_2,\ldots,x_m) = x_1 + x_2 + \ldots + x_m$ for each m. I wouldn't like to see this problem abandoned at this point. Can anyone find all such polynomials for any other values of m and/or n? Can anyone show that there is *at least* one such polynomial for each m and n?

* * *

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