Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

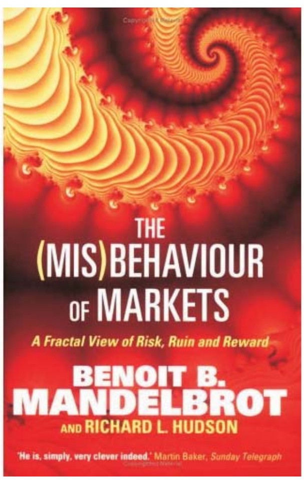
The (mis)Behaviour of Markets

> My life's work has been to develop a new mathematical tool to add to man's small survival kit. I call it fractal and multifractal geometry. It is the study of roughness, of the irregular and jagged. I coined its name in 1975.... Fractal geometry has come to be viewed as 'natural.' It is used today for an improbably diverse set of tasks: compressing digital images over the Internet, measuring metal fractures, analyzing brain waves in an EEG machine, designing ultra-small radio antennae, making better optical cables, and studying the anatomy of lung bronchia.

So wrote Benoit Mandelbrot, with his economist co-author Richard Hudson, in their fascinating and important book *The (mis)Behaviour of Markets* (pp. 116–117). Mandelbrot goes on to apply his fractal geometry to the study of financial markets and the rise and fall of stocks, shares, and currencies. You do not need to be a budding economist to find this fascinating. This book is written for a nonmathematical reader, so there is not a formula in sight in the main text; a few appear in the notes at the end. It is a very good read. The original hardback version appeared in 2004 and was the Financial Times' best business book of the year.

A prelude by Richard Hudson is a fascinating portrait of Mandelbrot, one of the most original mathematical minds of our age. Part I is a devastating critique of the mathematical models hitherto used in finance. To quote from later in the book (p. 247):

> The classic Random Walk model makes three essential claims. First is the so-called martingale condition: that your best guess of tomorrow's price is today's price. Second is a declaration of independence: that tomorrow's price is independent of past prices. Third is a statement of normality: that all the price changes taken together, from small to large, vary in accordance with the mild, bell-curve distribution. In my view, that is two claims too many.



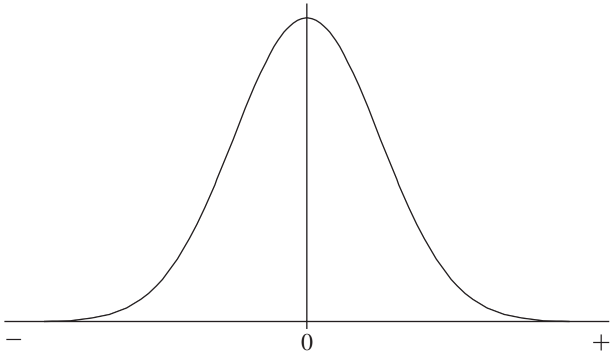


Figure 1

All students of statistics are familiar with the bell-shaped normal distribution curve (see figure 1). In the standard model, price changes of a stock, share, or currency conform to a normal distribution. To quote (p. 276):

> I believe the conventional models and their more recent 'fixes' ... are not merely wrong; they are dangerously wrong.

Financial markets are much more volatile than this model predicts.

Mandelbrot goes on to apply the ideas of fractal geometry to finance (p. 118):

> A fractal ... is a pattern or shape whose parts echo the whole.

Thus, the frond of a fern is a smaller version of the whole, each floret of a cauliflower is itself a cauliflower, and branches of a tree are complete smaller trees (p. 125). Examples are legion. So, says Mandelbrot (p. 239):

> The genius of fractal analysis is that the same risk factors, the same formulae apply to a day as to a year, an hour as to a month. Only the magnitude differs, not the proportions.

I am not an economist, but I found this plea for a new, more realistic mathematical model for financial markets both fascinating and convincing. I urge you to read this book. It may encourage the more adventurous student of mathematics, 'the bungee jumpers', into working in finance or to play the markets. Those, like me, who prefer to keep their feet firmly on the ground, will be tempted to keep their money under the mattress!

Reference

1 B. B. Mandelbrot and R. L. Hudson, *The (mis)Behaviour of Markets* (Profile Books, London, 2005).

Rational Approximation to Square Roots of Integers

M. A. KHAN

1. Introduction

A calculator that displays ten digits gives 1.414 213 562 for the square root of 2 which, when written as a fraction in lowest terms, is

$$\frac{707\,106\,781}{500\,000\,000}.$$

In this article we describe a method for finding rational approximations that are nearly as good but have smaller numbers in the numerator and denominator. The calculations are simple and do not require the use of calculus or infinite series. For example, the fraction

$$\frac{19\,601}{13\,860} = 1.414\,213\,564$$

agrees with $\sqrt{2}$ to eight decimal places.

Our algorithm essentially consists of first finding an initial solution (x_0, y_0) in positive integers by trial which satisfies the Diophantine equation

$$y^2 = nx^2 + 1. \tag{1}$$

Here, n is the positive integer whose square root is sought. Once the initial trial solution (x_0, y_0) is found, it can be used to produce a chain of other possible solutions

$$(x_1, y_1), (x_2, y_2), \ldots, (x_r, y_r), \ldots,$$

with increasing numerical values. Then, for any solution (x_r, y_r), we can write (1) in the following form:

$$n = \frac{y_r^2 - 1}{x_r^2}. \tag{2}$$

Now, if y_r^2 is much greater than 1 then $y_r^2 - 1$ can be approximated by y_r^2 and, with the aid of (2), we can now write

$$\sqrt{n} = \frac{y_r}{x_r}.$$

Thus, y_r/x_r is a rational approximation of $\sqrt{n}$.

2. An algorithm for evaluating successively increasing values of (x_r, y_r)

Let (x_0, y_0) satisfy (1), so that

$$y_0^2 - nx_0^2 = 1 \quad \text{or} \quad (y_0 + \sqrt{n}\,x_0)(y_0 - \sqrt{n}\,x_0) = 1.$$

Suppose that (x, y) is also a solution to (1). Then we obtain

$$(y_0 + \sqrt{n}\,x_0)(y + \sqrt{n}\,x)(y_0 - \sqrt{n}\,x_0)(y - \sqrt{n}\,x) = 1,$$

so that

$$((y_0 y + n x_0 x) + \sqrt{n}(x_0 y + y_0 x))((y_0 y + n x_0 x) - \sqrt{n}(x_0 y + y_0 x)) = 1,$$

i.e.

$$(y_0 y + n x_0 x)^2 - n(x_0 y + y_0 x)^2 = 1;$$

$(y_0 y + n x_0 x, x_0 y + y_0 x)$ is also a solution to (1). We can denote this solution in matrix form as follows:

$$\begin{bmatrix} y_0 & n x_0 \\ x_0 & y_0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{or} \quad A \begin{bmatrix} y \\ x \end{bmatrix}.$$

Thus, we can obtain a sequence of solutions to (1), namely

$$\begin{bmatrix} y_r \\ x_r \end{bmatrix} = A^r \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \quad \text{for } r = 0, 1, 2, \ldots,$$

and y_r / x_r gives the rth rational approximation to $\sqrt{n}$.

We can obtain these approximations successively by means of the recurrence relation

$$\begin{bmatrix} y_r \\ x_r \end{bmatrix} = A \begin{bmatrix} y_{r-1} \\ x_{r-1} \end{bmatrix}, \tag{3}$$

so that

$$y_r = y_0 y_{r-1} + n x_0 x_{r-1}, \qquad x_r = x_0 y_{r-1} + y_0 x_{r-1}.$$

For example, if we choose $(x_0, y_0) = (2, 3)$ as an initial solution to $y^2 = 2x^2 + 1$, then we obtain

$$y_r = 3y_{r-1} + 4x_{r-1}, \qquad x_r = 2y_{r-1} + 3x_{r-1},$$

and the successive approximations of $\sqrt{2}$ for $r = 0, 1, 2, \ldots$ are

$$\sqrt{2} \simeq \tfrac{3}{2}, \tfrac{17}{12}, \tfrac{99}{70}, \tfrac{577}{408}, \tfrac{3\,363}{2\,378}, \tfrac{19\,601}{13\,860}, \ldots.$$

3. A comparison of the matrix method with Newton's method of iteration

According to Newton's formula, if u_i is some initial trial solution to $f(u) = 0$ then a better approximation is

$$u_{i+1} = u_i - \frac{f(u_i)}{f'(u_i)}.$$

The corresponding equation to determine $\sqrt{n}$ is $f(u) \equiv u^2 - n = 0$ so that, for some approximation u_i for the solution to $u^2 - n = 0$, a better approximation is

$$u_{i+1} = \frac{u_i^2 + n}{2u_i}, \qquad i = 0, 1, 2, \ldots. \tag{4}$$

Table 1 The matrix method and Newton's method ($\sqrt{3} = 1.732\,050\,808$ and $\sqrt{3} \approx y_r/x_r$).

Matrix method			Newton's method
r, i	y_r/x_r	y_r/x_r in decimals	u_i
1	$\frac{7}{4}$	1.75	$\frac{7}{4}$
2	$\frac{26}{15}$	1.733	$\frac{97}{56}$
3	$\frac{97}{56}$	1.732 1	$\frac{18\,817}{10\,864}$
4	$\frac{362}{209}$	1.732 05	$\cdots$
5	$\frac{1\,351}{780}$	1.732 051	$\cdots$
6	$\frac{5\,042}{2\,911}$	1.732 050 8	$\cdots$
7	$\frac{18\,817}{10\,864}$	1.732 050 81	$\cdots$

A comparative study of the iterations obtained by the two methods providing successive rational approximations to the value of $\sqrt{3}$ by using (3) and (4) respectively is shown in table 1. The initial values $x_0 = 1$ and $y_0 = 2$ for $n = 3$, which satisfy $y^2 = 3x^2 + 1$, can be used in (3), while, for the purpose of comparison, u_0 may be taken as 2 (corresponding to $u_0 = y_0/x_0$) in Newton's formula.

Table 1 shows that the third iteration in Newton's method corresponds to the seventh iteration in the matrix method, so the convergence is much faster in the former. However, this abrupt transition makes it difficult to choose the desired number of digits in the numerator and the denominator of the fractional representation of u_i. We note that x_r and y_r have no common factors because $y_r^2 - nx_r^2 = 1$.

4. Evaluation of the error involved in approximating $\sqrt{n}$ by y_r/x_r

The *relative error* δ_r in the calculated value $(\sqrt{n})_{\mathrm{c}} = y_r/x_r$ with respect to the true value $(\sqrt{n})_{\mathrm{t}} = (\sqrt{y_r^2 - 1})/x_r$ is given by

$$\delta_r = \frac{(\sqrt{n})_{\mathrm{c}} - (\sqrt{n})_{\mathrm{t}}}{(\sqrt{n})_{\mathrm{t}}} = \frac{y_r}{\sqrt{y_r^2 - 1}} - 1 = \left(1 - \frac{1}{y_r^2}\right)^{-1/2} - 1 \simeq 1 + \frac{1}{2y_r^2} - 1 = \frac{1}{2y_r^2},$$

for a large value of y_r, which represents an error of less than one part in 10^7 if y_r contains at least four digits.

References

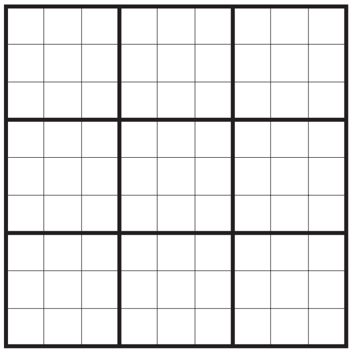
1 S. Goldberg, *Introduction to Difference Equations* (Dover, New York, 1986).
2 G. Strang, *Calculus* (Wellesley-Cambridge Press, Wellesley, MA, 1991).

The author retired from government service as a Deputy Director in 1994 after serving the Indian Railways for over 30 years as an Engineer.

Mathematics of Sudoku II

ED RUSSELL and FRAZER JARVIS

We recall that the idea of the Sudoku puzzle is extremely simple; the solver is faced with a 9×9 grid, divided into nine 3×3 blocks.



In some of these boxes, the setter puts some of the digits $1, \ldots, 9$; the aim of the solver is to complete the grid by filling in a digit in every box in such a way that each row, each column, and each 3×3 box contains each of the digits $1, \ldots, 9$ exactly once.

In an earlier article by Bertram Felgenhauer and Frazer Jarvis (see reference 1), the number of different Sudoku grids was calculated to be

$$N_0 = 6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \times 10^{21}.$$

However, distinct solutions were treated as different solutions in this calculation, even when one could be transformed to another by some symmetry. You may feel that two grids should not be counted differently if, say, the second is just the first rotated by 90 degrees. Similarly, a grid can be reflected to get another valid grid – should these be counted differently? Also, all the entries can be re-labelled (exchanging 1s and 2s, for example) to give another grid. In this article, we want to explain how to refine the counting method of reference 1 to find how many *essentially different* grids there are, if we allow various possible symmetries.

Let us firstly study all the possible symmetries of a grid. That is, we want to find all the operations that we can make on a general Sudoku grid which preserve the property that every row, column, and box contains each of the digits $1, \ldots, 9$ exactly once.

As well as re-labelling, reflecting, and rotating, there are a few other operations that can be performed. In order to define these, we need to define some terminology. We say that a *block* refers to one of the 3×3 sub-squares within the 9×9 grid with a boundary of bold lines. A *stack* consists of three blocks in a vertical 9×3 arrangement; a *band* consists of three blocks in a horizontal 3×9 arrangement.

The following is a complete list of operations that we can perform:

1. re-label (i.e. permute) the nine digits,

2. permute the three stacks,

3. permute the three bands,

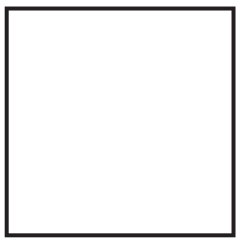
4. permute the three columns within a stack,

5. permute the three rows within a band,

6. any reflection or rotation.

In this article, we explain how to work out the number of grids which are ‘essentially different’, where we regard two grids as essentially the same if there is a sequence of the above operations which transforms one of the grids to another.

We start by focusing on operations 2–6, which are operations that are done on the grid squares, not on the *labels* within those squares. In a sense, we are going to think of these operations as *symmetries* of Sudoku puzzles, in the same way that a cube, for example, has symmetries (we can rotate a cube in several ways to put it back into the same position). To describe the method, and the solution, we need to discuss some *group theory*.



Symmetries of geometrical objects form what is known as a *group*; this means that if we are given two symmetries, we can do one and then the other, and the result is itself a symmetry. For example, if we take the above square frame of wire, then we have the following eight symmetries:

- rot_0 (rotation by 0 degrees (in other words, do nothing)),
- rot_1 (rotate by 90 degrees clockwise),
- rot_2 (rotate by 180 degrees),
- rot_3 (rotate by 90 degrees anticlockwise),
- ref_0 (reflect in the horizontal axis),
- ref_1 (reflect in the top-right–bottom-left diagonal axis),
- ref_2 (reflect in the vertical axis),
- ref_3 (reflect in the top-left–bottom-right diagonal axis).

You might like to check that doing rot_1 and then ref_2, say, has exactly the same effect as doing ref_3. If we start with our square $\square$, we apply first rot_1 to get to $\mathrm{rot}_1(\square)$, and then apply ref_2 to that to get $\mathrm{ref}_2(\mathrm{rot}_1(\square))$, this is the same as $\mathrm{ref}_3(\square)$. We just write $\mathrm{ref}_2\mathrm{rot}_1 = \mathrm{ref}_3$. Whichever pair of symmetries you take, doing one then the other will have the same effect as one of the eight symmetries on the list.

Formally, a group is a set G with a product rule such that

1. if g_1, g_2 are both in G, then $g_1 g_2 \in G$,
2. there is an element $e \in G$ such that $eg = ge = g$ for all $g \in G$ (so $e = \mathrm{rot}_0$ in the example above),
3. every element $g \in G$ has an inverse $g' \in G$, so that $gg' = g'g = e$,
4. $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for any three elements g_1, g_2, and g_3 in G.

Here is a question that could be asked. Colour the edges of the square wire frame red and blue, how many essentially different ways are there to do this (where two colourings count as the same if there is a symmetry turning one into the other)? It turns out that there is a mathematical theorem known as Burnside's lemma which allows you to work out this sort of thing (in fact, although Burnside re-discovered it around 1900, it seems to have first been noticed by Cauchy around 1845). We address this question in the following way.

> Write down all the possible symmetries, as we did above. For each symmetry, count the number of colourings which are fixed by the symmetry. The answer to the question is the average of these numbers.

Let us explain how this works in the example of the coloured square wire frame.

- The symmetry rot_0 is going to fix everything; there are $2^4 = 16$ colourings of the square (each of the four edges can be coloured in two ways).
- The symmetry rot_1 will only fix two of the colourings, where each edge is red, or each edge is blue. (If a colouring is fixed by rotating it by 90 degrees, then each edge must be coloured the same way as its neighbour.)
- The symmetry rot_2 will fix four of the colourings, the ones where the opposite edges are coloured in the same way. So the top and bottom edges are either both red or both blue, and the left and right edges are either both red or both blue.
- The symmetry rot_3 fixes two colourings, just as rot_1 does.
- The symmetry ref_0 fixes eight colourings, those where the top and bottom edges are coloured the same.
- The symmetry ref_1 fixes four colourings, those where the top and right edges and the bottom and left edges are coloured the same.
- The symmetry ref_2 fixes eight colourings, where the left and right edges are coloured the same.
- The symmetry ref_3 fixes four colourings, those where the top and left edges and the bottom and right edges are coloured the same.

Then the average of these is

$$\tfrac{1}{8}(16 + 2 + 4 + 2 + 8 + 4 + 8 + 4) = \tfrac{48}{8} = 6,$$

and Burnside's lemma tells us that this is exactly the answer to the original question of how many essentially different colourings there are.

We are trying to do the same to count the number of essentially different Sudoku grids. For simplicity, our group G will consist of all the symmetries built up from operations 2–6. These symmetries can be thought of as being operations permuting the 81 squares of the grid, and there is a computer program available, called GAP (see reference 2), which allows us to work with groups arising by permuting sets.

To take into account the re-labelling of operation 1, we are going to regard two grids as *equivalent* if one can be transformed into the other by re-labelling. Then our symmetry group G is acting on grids; as in the example above, we want to count the number of grids which are fixed *up to equivalence* by a given symmetry, i.e. the grids which are transformed by the symmetry into an equivalent grid.

For example, the following grid is equivalent to itself under a quarter turn, where the re-labelling takes $1 \to 3 \to 9 \to 7 \to 1$, $2 \to 6 \to 8 \to 4 \to 2$, and 5 is fixed.

1	2	4	5	6	7	8	9	3
3	7	8	2	9	4	5	1	6
6	5	9	8	3	1	7	4	2
9	8	7	1	2	3	4	6	5
2	3	1	4	5	6	9	7	8
5	4	6	7	8	9	3	2	1
8	6	3	9	7	2	1	5	4
4	9	5	6	1	8	2	3	7
7	1	2	3	4	5	6	8	9

It turns out that the group G generated by the symmetries rot_1, rot_2, rot_3, ref_0, and ref_1 has 3 359 232 symmetries. To count the number of fixed points for all these would take an immense amount of time. But just as the rotations by 90 degrees and by 270 degrees gave the same answer above, it turns out that there are 275 'conjugacy classes' (i.e. any two elements in the same class will have the same number of fixed grids), so if we choose one symmetry from each class, we only have to see what happens for these 275 symmetries (any symmetry will have the same number of fixed grids as one of these 275).

But for many symmetries, it is easy to see that there are no grids which are transformed to equivalent grids. For example, let us think about reflecting a Sudoku grid in a horizontal axis. Notice that all the numbers in the middle (fifth) row are unchanged. Since each of the nine digits occurs in this row, we see that there can be no re-labelling; the grid must be transformed under the relection not just to an equivalent grid, but to *the same* grid. However, this is clearly impossible – this would mean, for example, that the two entries in the first column of rows 4 and 6 would have to be the same, but they are in the same box (B4), and two identical numbers in the same box is not allowed.

In fact, for all but 27 of these 275 symmetries, it can be shown that there are no grids at all transformed to equivalent grids. Using brute-force methods like those in reference 1, the first author computed the number of elements transformed to equivalent grids by each of the remaining 27 symmetries. For example, it turns out that there are $9! \times 13\,056$ Sudoku grids which are equivalent to themselves under a quarter turn, like the example above, i.e. 13 056 essentially different grids equivalent to themselves under a quarter turn.

Taking the average over all of the 3 359 232 group elements, we find that the number of essentially different Sudoku grids is 5 472 730 538.

If we just allow re-labelling, rotations, and permutations (but not reflections), the corresponding group H has order $6^8 = 1\,679\,616$ and has 484 conjugacy classes. Burnside's lemma gives 10 945 437 157 essentially different grids with these symmetries. (Both of these numbers have subsequently been confirmed by Kjell Fredrik Pettersen.)

Allowing only re-labelling and rotations, so that the group just has four elements (rotation by 0, 90, 180, and 270 degrees), the same method gives 4 595 805 644 052 864 essentially different grids. If we allow reflections as well, so that the symmetry group has eight elements (the same as the symmetry group of the wire frame above), it turns out that we get 2 297 902 829 591 040 essentially different grids.

The programs and data are stored at http://www.afjarvis.staff.shef.ac.uk/sudoku/.

References

1 B. Felgenhauer and F. Jarvis, Mathematics of Sudoku I, *Math. Spectrum* **39** (2006/2007), pp. 15–22.

2 The GAP Group, GAP – Groups, Algorithms, Programming, Version 4.4, 2005, available at http://www.gap-system.org.

Ed Russell *is a government scientist with interests in mathematics and computing. Outside of work he has a penchant for wine and cycling, though not at the same time.*

Frazer Jarvis *lectures in pure mathematics, with research interests in algebraic number theory. Outside mathematics, he is a keen pianist and sings with the Sheffield Philharmonic Chorus.*

Different bases

Show that

$$(0.\overline{101})_{3/2} = \tfrac{26}{19},$$
$$(0.\overline{031})_{4/3} = (3.\overline{3520})_6 = \tfrac{135}{37}.$$

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Bob Bertuello

Integral Triangles and Diophantine Equations

KONSTANTINE ZELATOR

1. Introduction

Our aim in this article is two-fold. The first goal is to establish, by using a simple method, a special family of triangles with integer side lengths and integer area. This is accomplished in Section 5. By *integral triangles* we simply mean triangles with integer side lengths. On the other hand, triangles with rational side lengths and rational area are referred to as *Heron triangles*. In reference 1, 25 printed pages of condensed information on Heron triangles are offered. As it turns out, if a Heron triangle is an integral triangle, its area must, in fact, be an integer which is a multiple of 6 (see reference 2 for details).

The second purpose of this article is to generate, in a plain manner, a family of integral triangles with an internal angle-bisector which has rational length; see Section 6. In Section 7, we present two numerical examples. In Section 2, we state the well-known result *Heron's formula* for the area of a triangle and we derive a formula for the length of an internal angle-bisector. In Section 3, we state the parametric formulae describing the solutions to the Diophantine equation $x^2 + y^2 + z^2 = t^2$, which is used in Section 5. In Section 4, we offer a very short derivation of a two-parameter family of solutions to the Diophantine equation $2(u^2 + v^2) = w^2 + s^2$, which we make use of in Section 6.

2. Two formulae from geometry

We firstly discuss Heron's formula. According to reference 3, Heron's proof can be found in Proposition 1.8 of his work *Metrica* (circa 100BC–100AD). This manuscript had been lost for centuries until a fragment was discovered in 1894 and a complete copy found in 1896. In reference 4, derivations of formulae for the area, the three internal (and the three) external angle-bisectors, as well as for the three medians can be found. We now state Heron's formula (see reference 5).

Heron's formula We have

$$\text{Area} = \mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

where a, b, and c are the lengths of the sides of the triangle and $s = (a + b + c)/2$ is the semi-perimeter of the triangle.

Next, we derive a formula for the length d of the internal bisector AD (see figure 1). We have

$$\text{Area } ABD + \text{Area } ADC = \text{Area } ABC,$$

$$\frac{1}{2}cd \sin\frac{A}{2} + \frac{1}{2}bd \sin\frac{A}{2} = \frac{1}{2}bc \sin A,$$

$$d(b+c)\sin\frac{A}{2} = 2bc \sin\frac{A}{2}\cos\frac{A}{2},$$

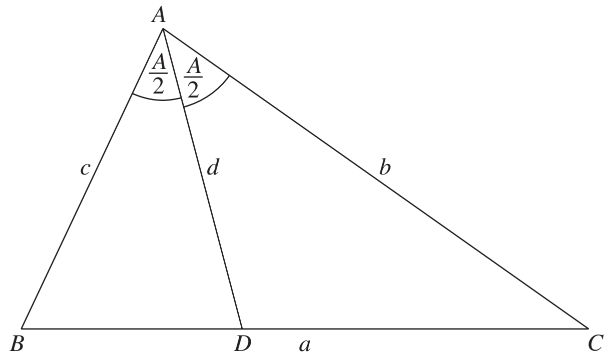

Figure 1

and

$$\begin{aligned}
d(b+c) &= 2bc\sqrt{\frac{1+\cos A}{2}} \\
&= bc\sqrt{2\left(1+\frac{b^2+c^2-a^2}{2bc}\right)} \\
&= \sqrt{bc((b+c)^2-a^2)} \\
&= \sqrt{bc((2s-a)^2-a^2)} \\
&= \sqrt{4bcs(s-a)}.
\end{aligned}$$

Hence,

$$d = \frac{2\sqrt{bcs(s-a)}}{b+c}. \tag{2}$$

3. All the solutions to the Diophantine equation $x^2 + y^2 + z^2 = t^2$

The complete parametric description of the entire solution set to

$$x^2 + y^2 + z^2 = t^2 \tag{3}$$

can be found in reference 6. The derivation is short and uncomplicated. Observe that a congruence modulo 4 in (3) easily shows that, in any solution (x, y, z, t), at least two of x, y, and z must be even integers. This is because if all three were odd then

$$x^2 \equiv y^2 \equiv z^2 \equiv 1 (\mathrm{mod}\, 4) \quad \text{and} \quad t^2 \equiv 3 (\mathrm{mod}\, 4),$$

which is impossible. If two of x, y, and z were odd and the third even, then $t^2 \equiv 2 (\mathrm{mod}\, 4)$, which is impossible. The general solution to (3) in positive integers x, y, z, and t, with x and y even, is

$$x = 2r, \qquad y = 2t, \qquad z = \frac{r^2 + t^2 - R^2}{R}, \qquad t = \frac{r^2 + t^2 + R^2}{R},$$

where r and t can be any positive integers and R is a positive integer divisor of $r^2 + t^2$.

4. A family of solutions to the Diophantine equation $2(u^2+v^2) = w^2+s^2$

A family of solutions to

$$2(u^2 + v^2) = w^2 + s^2 \tag{4}$$

is given by

$$u = k - m, \qquad v = k + m, \qquad w = 2k, \qquad s = 2m, \tag{5}$$

where k and m can be any positive integers with $k > m$. These are among the solutions to (4), and are by no means the entire set of solutions.

5. A family of integral Heron triangles

By using $s = (a + b + c)/2$, we can rewrite (1) as

$$(4\text{A})^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c). \tag{6}$$

To try to find all integral Heron triangles would be an extremely complex problem, we would have to consider the greatest common divisors of each two of the four factors on the right-hand side of (6). Instead, our approach here is a simple one. Below, we find a specific family of solutions among the entire set of solutions. We consider the case where each of the four factors is a perfect square, i.e.

$$-a + b + c = n_1^2, \qquad a - b + c = n_2^2, \qquad a + b - c = n_3^2, \qquad a + b + c = n_4^2.$$

Then we have

$$a = \frac{n_2^2 + n_3^2}{2}, \qquad b = \frac{n_1^2 + n_3^2}{2}, \qquad c = \frac{n_1^2 + n_2^2}{2} \tag{7}$$

and

$$n_1^2 + n_2^2 + n_3^2 = n_4^2, \tag{8}$$

so that (n_1, n_2, n_3, n_4) is a positive integer solution to the Diophantine equation (3). Since a, b, and c are integers, (7) requires that n_1, n_2, and n_3 must have the same parity, i.e. either they are all odd integers or they are all even integers. But since (n_1, n_2, n_3, n_4) satisfies (3), at least two of n_1, n_2, and n_3 must be even. Hence, they must all be even and, by (8), so must n_4. We set $n_i = 2f_i$, for $i = 1, 2, 3, 4$. Thus, $f_1^2 + f_2^2 + f_3^2 = f_4^2$ and, from (6), $\text{A} = f_1 f_2 f_3 f_4$. Therefore, a family of integral Heron triangles is given by

$$a = 2(f_2^2 + f_3^2), \qquad b = 2(f_1^2 + f_3^2), \qquad c = 2(f_1^2 + f_2^2),$$

where the positive integers f_1, f_2, f_3, and f_4 satisfy $f_1^2 + f_2^2 + f_3^2 = f_4^2$. In other words, (f_1, f_2, f_3, f_4) is a solution to the Diophantine equation (3). Futhermore, the area is given by $\text{A} = f_1 f_2 f_3 f_4$.

6. A family of integral triangles with a rational integral angle-bisector length

If we take $b = \beta^2$, $c = \gamma^2$, $\beta, \gamma \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers), and require that $a + b + c = \lambda^2$ and $b + c - a = \mu^2$ for some $\lambda, \mu \in \mathbb{Z}^+$, then we end up with $a = \lambda^2 - \beta^2 - \gamma^2$, and the integers β, γ, λ, and μ must satisfy $2(\beta^2 + \gamma^2) = \lambda^2 + \mu^2$.

Therefore, $(\beta, \gamma, \lambda, \mu)$ must be a solution to (4); and, by (5), we may take $\beta = k - m$, $\gamma = k + m$, $\lambda = 2k$, and $\mu = 2m$, for some $k, m \in \mathbb{Z}^+$ with $k > m$. We obtain the following family:

$$\begin{aligned} &a = 2(k^2 - m^2), \quad b = (k-m)^2, \\ &c = (k+m)^2, \qquad d = \frac{2km(k^2 - m^2)}{k^2 + m^2}, \qquad k > m,\ k, m \in \mathbb{Z}^+, \end{aligned} \tag{9}$$

where d is obtained using (2). Note that, as we can easily check, the two triangle inequalities $b + c > a$ and $c + a > b$ are satisfied. However, the triangle inequality $a + b > c$ is not satisfied for all k and m with $k > m$. Indeed,

$$a + b > c \iff k^2 - 2km - m^2 > 0 \iff \left(\frac{k}{m}\right)^2 - 2\frac{k}{m} - 1 > 0. \tag{10}$$

The roots of the quadratic function $f(x) = x^2 - 2x - 1$ are the real numbers $1 + \sqrt{2}$ and $1 - \sqrt{2}$. Since $k, m \in \mathbb{Z}^+$, in particular $k/m > 0$, we can see that we must have

$$\frac{k}{m} > 1 + \sqrt{2} \approx 2.414, \tag{11}$$

for (10) to be satisfied.

7. Two numerical examples

Example 1 For the first family, the triangle with smallest value of $f_1 f_2 f_3$ is obtained for $f_1 = f_2 = 2$ and $f_3 = 1$ and, thus, with $f_4 = 3$. The corresponding triangle has $a = 10$, $b = 10$, $c = 16$, and $\mathrm{A} = 48$.

Example 2 For the second family, using (9) and (11) we see that the triangle with the smallest value of k/m and with $km \leq 10$ is obtained for $k = 5$ and $m = 2$. The corresponding triangle has $a = 42$, $b = 9$, $c = 49$, and $d = 2 \times 21 \times 5 \times 2/29 = \frac{420}{29} = 14.482\,758$.

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80% Incan

JONATHAN SMITH

The other day a friend of mine declared that he was 80% Incan. He meant ‘three-quarters Incan’, but it made me wonder whether it is actually possible for him to be 80% Incan and, if so, what his family tree would look like.

If you go back n generations, you have 2^n direct ancestors of that generation (2 parents, 4 grandparents, and so on). It follows that each ancestor in that generation accounts for $1/2^n$ of your nationality.

In my friend’s case, we need to go back far enough so that the number of (pure) Incan ancestors of that generation will account for 80% of the total ancestors of that generation. In other words, we are looking for an integer solution to

$$\frac{a}{2^n} = 0.8,$$

where n generations back we have a pure Incan ancestors. We can simplify this to

$$5a = 2^{n+2},$$

which has no integer solutions because 2^{n+2} is not divisible by 5. Therefore, it is impossible to be 80% Incan.

However, suppose that you have a family tradition whereby for two generations your family marry pure Incans, and then for two generations marry non-Incans. The origins of this tradition are lost in the mists of time (so we can assume that it has gone on for ever). If so, the degree to which you are Incan, S, is given by

$$S = \left(\underset{\text{one parent}}{\frac{1}{2^1}} + \underset{\text{one grandparent}}{\frac{1}{2^2}}\right) \underset{\text{skip two generations}}{+} \left(\frac{1}{2^5} + \frac{1}{2^6}\right) + \left(\frac{1}{2^9} + \frac{1}{2^{10}}\right) + \cdots .$$

Therefore

$$\frac{S}{2^4} = \left(\frac{1}{2^5} + \frac{1}{2^6}\right) + \left(\frac{1}{2^9} + \frac{1}{2^{10}}\right) + \cdots .$$

Subtracting, we obtain

$$S - \frac{S}{2^4} = \frac{1}{2^1} + \frac{1}{2^2},$$

so that

$$S = 0.8.$$

So maybe it is possible to be 80% Incan after all!

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A Matrix Inequality with Functional Applications

STUART SIMONS

1. Introduction

In this article, we state and prove a new inequality concerning a rectangular matrix with all nonnegative elements. We use this in Section 3 to formulate an inequality concerning polynomials and power series with nonnegative coefficients, while in Section 4 this latter result is applied to determining the global minimum of a function of several variables with a particular structure.

2. Matrix inequality

Theorem 1 *Given a rectangular $N \times M$ matrix with real nonnegative elements a_{pq}, $1 \le p \le N$, $1 \le q \le M$, the following inequality holds:*

$$\begin{aligned}(a_{11}+a_{12}+\cdots+a_{1M})^{1/N}(a_{21}+a_{22}+\cdots+a_{2M})^{1/N}\cdots(a_{N1}+a_{N2}+\cdots+a_{NM})^{1/N}\\ \ge (a_{11}a_{21}\cdots a_{N1})^{1/N}+(a_{12}a_{22}\cdots a_{N2})^{1/N}+\cdots+(a_{1M}a_{2M}\cdots a_{NM})^{1/N}. \qquad (1)\end{aligned}$$

That is, the product of the Nth roots of all the row sums is greater than or equal to the sum of the Nth roots of all the column products.

Proof Consider

$$P=(a_{11}+a_{12}+\cdots+a_{1M})(a_{21}+a_{22}+\cdots+a_{2M})\cdots(a_{N1}+a_{N2}+\cdots+a_{NM}). \qquad (2)$$

Clearly, P can be expressed as the sum of M^N terms, each of which is the product of N of the elements a_{pq}, one taken from each of the bracketed expressions. Of these M^N terms, let $Q(n_1, n_2, \ldots, n_M)$ be the sum of those terms formed from the product of

$$\begin{array}{ll} n_1 & \text{elements of the form } a_{p1},\\ n_2 & \text{elements of the form } a_{p2},\\ \vdots & \quad\vdots \qquad\qquad \vdots\\ n_M & \text{elements of the form } a_{pM},\end{array}$$

where $n_1+n_2+\cdots+n_M=N$. Then we obtain

$$P=\sum_{\substack{n_1,n_2,\ldots,n_M\\(n_1+n_2+\cdots+n_M=N)}} Q(n_1,n_2,\ldots,n_M). \qquad (3)$$

Let $(N; n_1, n_2, \ldots, n_M)$ be the number of terms contributing to the sum $Q(n_1, n_2, \ldots, n_M)$ and, following the same approach as in the standard proof of the multinominal theorem, we can then readily show that

$$(N; n_1, n_2, \ldots, n_M) = \frac{N!}{n_1!\, n_2! \cdots n_M!}. \tag{4}$$

We now use the result that the arithmetic mean of all the terms contributing to the sum $Q(n_1, n_2, \ldots, n_M)$ is greater than or equal to their geometric mean; this yields

$$Q(n_1, n_2, \ldots, n_M) \geq (N; n_1, n_2, \ldots, n_M)[R(n_1, n_2, \ldots, n_M)]^{1/(N; n_1, n_2, \ldots, n_M)}, \tag{5}$$

where $R(n_1, n_2, \ldots, n_M)$ is the product of all the terms which contribute to the sum $Q(n_1, n_2, \ldots, n_M)$. Now, $R(n_1, n_2, \ldots, n_M)$ will be a product of powers of all the matrix elements and, for a specified pair of values p and q, the element a_{pq} appearing in $R(n_1, n_2, \ldots, n_M)$ will be raised to a power equal to the number of terms involving a_{pq} which contribute to the sum $Q(n_1, n_2, \ldots, n_M)$. In the product which defines each such term there will be a further $N-1$ matrix elements since a_{pq} has already been specified. Of these, the number of elements of the form a_{r1} (for arbitrary r) will be n_1, the number of elements of the form a_{r2} will be n_2, and so on (except for elements of the form a_{rq} whose number will be $n_q - 1$, since one element (a_{pq}) of this form has already been specified). Thus, the number of terms involving a_{pq} which contribute to the sum $Q(n_1, n_2, \ldots, n_M)$ will be $(N-1; n_1, n_2, \ldots, n_q - 1, \ldots, n_M)$; this will therefore be the power to which a_{pq} is raised in the expression $R(n_1, n_2, \ldots, n_M)$. Hence,

$$R(n_1, n_2, \ldots, n_M) = \prod_{q=1}^{M} \prod_{p=1}^{N} a_{pq}^{(N-1; n_1, n_2, \ldots, n_q - 1, \ldots, n_M)};$$

it then follows from (5) that

$$\begin{aligned} &Q(n_1, n_2, \ldots, n_M) \\ &\quad \geq (N; n_1, n_2, \ldots, n_M) \prod_{q=1}^{M} \prod_{p=1}^{N} a_{pq}^{[(N-1; n_1, n_2, \ldots, n_q - 1, \ldots, n_M)/(N; n_1, n_2, \ldots, n_q, \ldots, n_M)]} \\ &\quad = (N; n_1, n_2, \ldots, n_M) \prod_{q=1}^{M} \left(\prod_{p=1}^{N} a_{pq} \right)^{n_q/N}, \end{aligned}$$

on making use of (4). Equation (3) then yields

$$\begin{aligned} P \geq &\sum_{\substack{n_1, n_2, \ldots, n_M \\ (n_1 + n_2 + \cdots + n_M = N)}} (N; n_1, n_2, \ldots, n_M)[(a_{11}a_{21} \cdots a_{N1})^{1/N}]^{n_1} \\ &\qquad \times [(a_{12}a_{22} \cdots a_{N2})^{1/N}]^{n_2} \cdots [(a_{1M}a_{2M} \cdots a_{NM})^{1/N}]^{n_M} \\ &= [(a_{11}a_{21} \cdots a_{N1})^{1/N} + (a_{12}a_{22} \cdots a_{N2})^{1/N} + \cdots + (a_{1M}a_{2M} \cdots a_{NM})^{1/N}]^N, \end{aligned}$$

by the multinomial theorem. Finally, on substituting for P from (2) and taking the Nth root of each side of this relation, we obtain (1). We note that the equality sign holds in (1) when a_{pq} is of the form $a_{pq} = \alpha_p \beta_q$ for arbitrary α_p and β_q.

3. Function inequality

In this section and in Section 4 we let $f(x)$ be any real function of x of the form $f(x) = \sum_{t=0}^{T} c_t x^t$, with $c_t \geq 0$; additionally we consider only $x \geq 0$. The specified value of T may be finite (when f is a polynomial) or infinite (when f is a power series). In the latter case we suppose that the value of x is such that the series $f(x)$ is convergent. Such series (with $c_t \geq 0$) include many standard functions, for example, $\exp x$, $\cosh x$, $\sinh x$, $-\ln(1-x)$, and $(1-x)^{-n}$, $n > 0$. We can then easily prove the following result.

Theorem 2 *Given a set of* N *values* $x_1, x_2, \ldots, x_N$, *we obtain*

$$[f(x_1)f(x_2)\cdots f(x_N)]^{1/N} \geq f([x_1x_2\cdots x_N]^{1/N}), \tag{6}$$

with equality when $x_1 = x_2 = \cdots = x_N$. *That is, for a given set of values of the argument, the geometric mean of the corresponding values of the function is greater than or equal to the value of the function for the geometric mean of the given values of the argument.*

Proof Theorem 1 can be expressed in the following form:

$$\prod_{p=1}^{N}\left(\sum_{q=1}^{M} a_{pq}\right)^{1/N} \geq \sum_{q=1}^{M}\left(\prod_{p=1}^{N} a_{pq}\right)^{1/N}. \tag{7}$$

We now let $a_{pq} = c_{q-1}x_p^{q-1}$ (≥ 0) and $M = T+1$. On substituting $t = q-1$ in the appropriate summations in (7), the latter yields

$$\prod_{p=1}^{N}\left(\sum_{t=0}^{T} c_t x_p^t\right)^{1/N} \geq \sum_{t=0}^{T}\left(\prod_{p=1}^{N} c_t x_p^t\right)^{1/N} = \sum_{t=0}^{T} c_t([x_1x_2\cdots x_N]^{1/N})^t.$$

Thus, $\prod_{p=1}^{N}[f(x_p)]^{1/N} \geq f([x_1x_2\cdots x_N]^{1/N})$, which completes the proof of (6).

Clearly there are many special cases of Theorem 2 which may be of interest in particular contexts. For example,

$$f(x)f\left(\frac{1}{x}\right) \geq [f(1)]^2, \qquad x \neq 0,$$
$$f(x)f(2x)\cdots f(Nx) \geq [f(N!^{1/N}x)]^N,$$
$$f(\beta x)f(\beta^2 x)\cdots f(\beta^N x) \geq [f(\beta^{(N+1)/2}x)]^N, \quad \text{for a real positive constant } \beta.$$

4. A variational problem

We define

$$F(y_0, y_1, \ldots, y_N) = f\left(\frac{y_1}{y_0}\right)f\left(\frac{y_2}{y_1}\right)\cdots f\left(\frac{y_N}{y_{N-1}}\right),$$

where $y_0, \ldots, y_N > 0$. We suppose that the values of y_0 and y_N are specified, and for variables $y_1, y_2, \ldots, y_{N-1}$ we wish to determine the global minimum value of $F(y_0, y_1, \ldots, y_N)$.

Since F is to be minimised we begin by examining stationary values of F, characterised by $\partial F/\partial y_p = 0,\ 1 \le p \le N-1$. This yields

$$0 = \frac{\partial F}{\partial y_p}\left[f\left(\frac{y_p}{y_{p-1}}\right) f\left(\frac{y_{p+1}}{y_p}\right)\right]$$
$$= \frac{1}{y_{p-1}} f'\left(\frac{y_p}{y_{p-1}}\right) f\left(\frac{y_{p+1}}{y_p}\right) - \frac{y_{p+1}}{y_p^2} f\left(\frac{y_p}{y_{p-1}}\right) f'\left(\frac{y_{p+1}}{y_p}\right),$$

giving

$$\frac{y_p}{y_{p-1}} \frac{f'(y_p/y_{p-1})}{f(y_p/y_{p-1})} = \frac{y_{p+1}}{y_p} \frac{f'(y_{p+1}/y_p)}{f(y_{p+1}/y_p)}.$$

Apart from other possible solutions, this latter relation is clearly satisfied if $y_p/y_{p-1} = y_{p+1}/y_p$ leading to the following set of $N-1$ equalities for determining $y_1, y_2, \ldots, y_{N-1}$:

$$\frac{y_1}{y_0} = \frac{y_2}{y_1} = \cdots = \frac{y_N}{y_{N-1}}. \tag{8}$$

If we denote the value of each of the equal ratios appearing in (8) by α, it then follows that $y_p = \alpha^p y_0,\ 1 \le p \le N-1$, where α is determined by $\alpha^N = y_N/y_0$, which was initially specified. Furthermore, the corresponding stationary value of F is $[f(\alpha)]^N$. Our task now is to show that this stationary value is in fact the global minimum value of $F(y_0 \cdots y_N)$. Here the general techniques of differential calculus for distinguishing between different types of stationary points are of little value, since at best these techniques can only show the stationary value to be a local rather than a global minimum, and therefore it would be necessary to investigate whether F can possess other stationary values. However, we can now use Theorem 2 in order to prove the required result easily.

Let $x_p = y_p/y_{p-1},\ 1 \le p \le N$, and $(x_1 x_2 \cdots x_N)^{1/N} = (y_N/y_0)^{1/N} = \alpha$. It then immediately follows from (6) that, for arbitrary values of $y_1, y_2, \ldots, y_{N-1}$,

$$F(y_0, y_1, \ldots, y_N) \ge [f(\alpha)]^N,$$

showing that $[f(\alpha)]^N$ is in fact the required global minimum value of F.

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Show that, if integers x and y are such that $7x + 5y$ is divisible by 17, then so is $5x + 6y$.

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Abbas Roohol Amini

Pythagorean Rectangles

PHILIP MAYNARD

The triple (a, b, c) is a *Pythagorean triple* if $a, b, c \in \mathbb{N} = \{1, 2, \ldots\}$ and $a^2 + b^2 = c^2$. This equation can be thought of as saying that the sum of the areas of two squares is equal (numerically) to the area of another, larger, square. As suggested by the title, we generalize this result to rectangles. Specifically, we are interested in $a, b, c \in \mathbb{N}$ that satisfy

$$a(a + \lambda) + b(b + \lambda) = c(c + \lambda), \tag{1}$$

for some nonnegative integer λ. (The case $\lambda = 0$ is the Pythagorean triple case.) Note that any solution to (1) must satisfy $a + b > c$, since if $a + b \leq c$ then $a(a + \lambda) + b(b + \lambda) < (a + b)(a + b + \lambda) \leq c(c + \lambda)$, which contradicts (1). Hence, it makes sense to try and classify solutions in terms of $\xi \in \mathbb{N}$, such that $a + b = c + \xi$. In particular, for $\xi \in \mathbb{N}$ and λ a nonnegative integer, we define $N(\xi, \lambda)$ to be the number of solutions, $a, b, c \in \mathbb{N}$, to the equations

$$a(a + \lambda) + b(b + \lambda) = c(c + \lambda)$$

and

$$a + b = c + \xi.$$

(Note that the solution a, b, c is regarded as the same as the solution b, a, c.) The following lemma is simple, but crucial in our analysis.

Lemma 1 *Let $a, b, c, \xi \in \mathbb{N}$ and λ be any nonnegative integer. Then $a(a + \lambda) + b(b + \lambda) = c(c + \lambda)$ and $a + b = c + \xi$ if and only if*

$$a = \xi + \frac{\xi(\xi + \lambda)}{2(b - \xi)} \quad \textit{or, equivalently,} \quad b = \xi + \frac{\xi(\xi + \lambda)}{2(a - \xi)}, \tag{2}$$

$c = a + b - \xi$, and $\xi < a, b, c$.

Proof Assume that $a + b = c + \xi$ and $a(a + \lambda) + b(b + \lambda) = c(c + \lambda)$. Firstly, we show that $\xi < a, b, c$. It is immediate from $a(a + \lambda) + b(b + \lambda) = c(c + \lambda)$ that $c > a, b$. Then, from $a + b = c + \xi$, we have $\xi < a, b$, and also $\xi < c$. In addition, $a^2 + 2ab + b^2 = c^2 + 2c\xi + \xi^2$. Using the facts $a(a + \lambda) + b(b + \lambda) = c(c + \lambda)$ and $c = a + b - \xi$ yields

$$a = \frac{\xi(\lambda + 2b - \xi)}{2(b - \xi)} = \xi + \frac{\xi(\xi + \lambda)}{2(b - \xi)}.$$

Conversely, now assume that (2) holds along with the conditions $\xi < a, b, c$ and $a + b = c + \xi$. We must show that (1) holds. Now,

$$\begin{aligned} c(c + \lambda) &= (a + b - \xi)(a + b - \xi + \lambda) \\ &= a(a + \lambda) + b(b + \lambda) + 2ab - 2a\xi - 2b\xi - \lambda\xi + \xi^2. \end{aligned}$$

It remains to show that $2ab - 2a\xi - 2b\xi - \lambda\xi + \xi^2 = 0$. From (2), it follows that $2a(b - \xi) = 2(b - \xi)\xi + \xi(\xi + \lambda) = 2b\xi + \xi\lambda - \xi^2$, as required.

It immediately follows from Lemma 1 that if ξ is odd and λ is even then $N(\xi, \lambda) = 0$. We now count $N(\xi, \lambda)$ in all other cases. We now introduce $\tau(\cdot)$, a well-known number-theoretic function (i.e. the domain of definition is the integers). Let $n \in \mathbb{N}$, then $\tau(n)$ is the number of positive divisors of n.

Theorem 1 *Let $\xi \in \mathbb{N}$ and λ be any nonnegative integer. If ξ is odd and λ is even then $N(\xi, \lambda) = 0$. Otherwise, if $\frac{1}{2}\xi(\xi + \lambda)$ is not a square then*

$$N(\xi, \lambda) = \frac{1}{2}\tau\left(\frac{\xi(\xi + \lambda)}{2}\right),$$

and if $\frac{1}{2}\xi(\xi + \lambda)$ is a square then

$$N(\xi, \lambda) = \frac{1}{2}\left(\tau\left(\frac{\xi(\xi + \lambda)}{2}\right) + 1\right).$$

Proof The first part is already done, so assume that $\xi(\xi+\lambda)$ is even. By Lemma 1 it follows that $N(\xi, \lambda)$ is just the number of solutions, a and b, of (2) with $\xi < a, b$ (then $c = a + b - \xi$, also $\xi < c$). To generate all the solutions of (2), we take $(a - \xi) \mid \frac{1}{2}\xi(\xi + \lambda)$ and then $b - \xi = \xi(\xi + \lambda)/2(a - \xi)$. Note also that, by (2), we have $(a - \xi)(b - \xi) = \frac{1}{2}\xi(\xi + \lambda)$. Thus, as $a - \xi$ runs through the divisors of $\frac{1}{2}\xi(\xi + \lambda)$, so does $b - \xi$ with $(a - \xi)(b - \xi) = \frac{1}{2}\xi(\xi + \lambda)$. Specifically, if $\frac{1}{2}\xi(\xi + \lambda) = \theta_1\theta_2$ then

$$a = \theta_1 + \xi, \qquad b = \theta_2 + \xi, \qquad c = \theta_1 + \theta_2 + \xi. \tag{3}$$

Note that we only have to consider positive divisors of $\frac{1}{2}\xi(\xi + \lambda)$ since, if $a - \xi < 0$ then $b - \xi < 0$ and, by (3), $a > c$ contradicts (1). Then, certainly, $\xi < a, b, c$. Since a and b are indistinguishable, we only need to let $a - \xi$ run through half of the divisors, that is $N(\xi, \lambda) = \frac{1}{2}\tau(\frac{1}{2}\xi(\xi + \lambda))$ when $\frac{1}{2}\xi(\xi + \lambda)$ is not a square, and $N(\xi, \lambda) = \frac{1}{2}(\tau(\frac{1}{2}\xi(\xi + \lambda)) + 1)$ when $\frac{1}{2}\xi(\xi + \lambda)$ is a square.

In the special case when $\lambda = 0$, we are looking at Pythagorean triples, and we obtain the following corollary.

Corollary 1 *The number of Pythagorean triples, (a, b, c), satisfying $a + b = c + \xi$ is zero if ξ is odd and $\frac{1}{2}\tau(\frac{1}{2}\xi^2)$ if ξ is even.*

The case when $\lambda = 1$ also has an interesting interpretation, here $N(\xi, 1)$ is the number of solutions to

$$(1 + 2 + \cdots + a) + (1 + 2 + \cdots + b) = (1 + 2 + \cdots + c), \tag{4}$$

for positive integers a, b, c satisfying $a + b = c + \xi$. This follows since (4) holds if and only if $\frac{1}{2}a(a + 1) + \frac{1}{2}b(b + 1) = \frac{1}{2}c(c + 1)$, i.e. if and only if $a(a + 1) + b(b + 1) = c(c + 1)$. This is just (1) with $\lambda = 1$. (Note that, assuming (4) holds, the condition $a + b = c + \xi$ is equivalent to $a^2 + b^2 + \xi = c^2$.) It follows that (4) always has at least one solution satisfying $a + b - c = \xi$, for any $\xi \in \mathbb{N}$, by Theorem 1.

Example 1 To find all solutions of (4) with $a + b = c + 6$, we must look at the factors of $\frac{1}{2}(6 \cdot 7) = 21$. There are two solutions, which we find using Theorem 1.

(i) Take $21 = 1 \cdot 21$. By (3) we have $a = 1 + 6 = 7$, $b = 21 + 6 = 27$, and $c = 1 + 21 + 6 = 28$. That is,

$$(1 + 2 + \cdots + 7) + (1 + 2 + \cdots + 27) = (1 + 2 + \cdots + 28).$$

(ii) Take $21 = 3 \cdot 7$. By (3) we have $a = 3+6 = 9$, $b = 7+6 = 13$, and $c = 3+7+6 = 16$. That is,

$$(1 + 2 + \cdots + 9) + (1 + 2 + \cdots + 13) = (1 + 2 + \cdots + 16).$$

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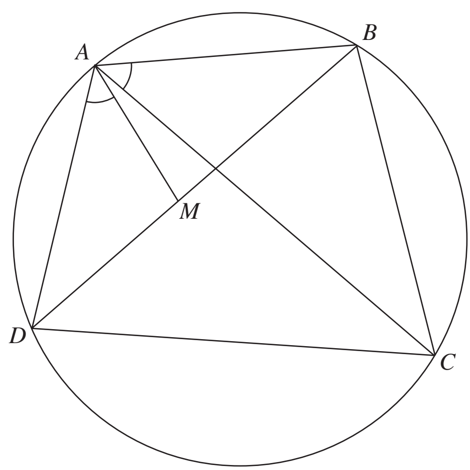
Similar triangles



Use the pairs of similar triangles ADM, ACB and ABM, ACD to prove that

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

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The Integers of James Booth

THOMAS J. OSLER and JOHN KENNEDY

In 1854, Reverend James Booth published a note (see reference 1), which was less than one page long, in which he proved that six digit integers of the form $abcabc$, for example 376 376 or 459 459, are all divisible by the numbers 7, 11, and 13. In fact,

$$abcabc = abc(10^3 + 1) = abc(7 \times 11 \times 13),$$

which proves Booth's result. We denote by $N(p, n)$ the set of all possible integers with a sequence of p digits repeated n times. Thus, all numbers in the set $N(3, 2)$ are divisible by 7, 11, and 13. The following theorem is an easy extension of the above argument.

Theorem 1 *Any divisor of* $10^p + 1$ *also divides all numbers in* $N(p, 2)$.

In table 1 we list the divisors of $10^p + 1$. From table 1 we see that 11 and 9 091 divide $10^5 + 1$; thus from Theorem 1 we know that 11 and 9 091 divide every number in $N(5, 2)$, for example 1 234 512 345 and 5 402 154 021.

It is also easy to see that you can concatenate two or more members of $N(3, 2)$ and the new number will also be divisible by 7, 11, and 13. This is because

$$abcabcdefdef = 10^6 abcabc + defdef,$$

Table 1

p	Prime divisors of $10^p + 1$	p	Prime divisors of $10^p + 1$
1	11	21	7, 11, 13, 127, 2 689, 459 691, 909 091
2	101		
3	7, 11, 13	22	89, 101, 1 052 788 969, 1 056 689 261
4	73, 137		
5	11, 9 091	23	11, 47, 139, 2 531, 549 797 184 491 917
6	101, 9 901		
7	11, 909 091	24	17, 5 882 353, 9 999 999 900 000 001
8	17, 5 882 353		
9	7, 11, 13, 19, 52 579	25	11, 251, 5 051, 9 091, 78 875 943 472 201
10	101, 3 541, 27 961		
11	11, 23, 4 093, 8 779	26	101, 521, 1 900 381 976 777 332 243 781
12	73, 137, 99 990 001		
13	11, 859, 1 058 313 049	27	7, 11, 13, 19, 52 579, 70 541 929, 14 175 966 169
14	29, 101, 281, 121 499 449		
15	7, 11, 13, 211, 241, 2 161, 9 091	28	73, 137, 7 841, 127 522 001 020 150 503 761
16	353, 449, 641, 1 409, 69 857		
17	11, 103, 4 013, 21 993 833 369	29	11, 59, 154 083 204 930 662 557 781 201 849
18	101, 9 901, 999 999 000 001		
19	11, 909 090 909 090 909 091	30	61, 101, 3 541, 9 901, 27 961, 4 188 901, 39 526 741
20	73, 137, 1 676 321, 5 964 848 081		

and the divisibility is now clear. For example, 123 123 456 456 789 789 is divisible by 7, 11, and 13. In general, we have the following theorem.

Theorem 2 *Let two or more members of $N(p, 2)$ be concatenated together to form a new number z. Then any divisor of $10^p + 1$ also divides z.*

In table 1 we see that 11 divides $10^p + 1$ whenever p is odd. Also notice that 101 divides $10^p + 1$ for $p = 2, 6, 10, 14, 18, \ldots$. These are special cases of the following theorem.

Theorem 3 *Let x be an integer (not equal to -1). Then $x + 1$ divides $x^p + 1$ if p is odd.*

Proof By direct multiplication we see that

$$\begin{aligned}
&(x+1)(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1) \\
&\quad = x(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1) + (x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1) \\
&\quad = x^p - x^{p-1} + x^{p-2} - \cdots - x^2 + x + (x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1) \\
&\quad = x^p + 1.
\end{aligned} \tag{1}$$

Thus the theorem is proved.

With $x = 10$ in (1) we see why 11 divides $10^p + 1$ for odd p, and with $x = 100$ we see why 101 divides $10^{2p} + 1$ for odd p. We can immediately obtain the following corollary.

Corollary 1 *No integer of the form $10^p + 1$, where p is an odd integer greater than 1, is prime.*

Theorem 4 *If y divides $10^q + 1$, then it divides every number in $N(pq, 2)$, where p is odd.*

Proof Let $x = 10^q$. By Theorem 3, $10^q + 1$ divides $10^{pq} + 1$, where p is an odd integer; thus, from Theorem 1, y divides every number in $N(pq, 2)$.

Just as the number 1001 was fundamental in the examination of the divisibility properties of numbers in the set $N(3, 2)$, the reader will have no trouble showing that the number 1 001 001 governs such questions for numbers in the set $N(3, 3)$. With the help of mathematical software, we can construct a table of the prime divisors of $10^{2p} + 10^p + 1$ for $p = 1, 2, 3, \ldots$, so that the divisibility of numbers in $N(p, 3)$ can be explained. Further extensions to $N(p, 4)$, $N(p, 5)$, and so on, can be studied.

Reference

1 J. Booth, On a property of numbers, *Proc. R. Soc. London,* **7** (1854/1855), pp. 42–43.

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The First Class Honours Grade: An Application of the Beta Distribution

JOHN C. B. COOPER

1. Introduction

The probability density function of the beta distribution can be expressed as

$$\beta(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}, \quad \text{where } 0 \le x \le 1 \text{ and } p, q > 0. \tag{1}$$

Alternatively, because the beta function can be expressed in terms of the gamma function, i.e. $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, we can substitute this into (1) to obtain

$$\beta(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1}.$$

The shape of this distribution is governed by the two parameters p and q; we summarise this as follows.

(i) When $p = q$, the distribution is symmetrical; as p and q become larger, the distribution becomes more peaked and the variance decreases.

(ii) When $p \neq q$, the distribution is skewed with the degree of skewness increasing as p and q become more unequal. If $p > q$, then the distribution is skewed to the left and vice versa.

(iii) When $p = q = 1$, the distribution becomes the uniform distribution and the density function has the constant value 1 over the entire interval from 0 to 1 for the random variable X.

If X has a beta distribution with parameters p and q, it can be shown that its mean μ and variance σ^2 are given by

$$\mu = \frac{p}{p+q} \quad \text{and} \quad \sigma^2 = \frac{pq}{(p+q)^2(p+q+1)}$$

(see reference 1). We can now express p and q in terms of μ and σ^2 as follows:

$$p = \mu\left(\frac{\mu(1-\mu)}{\sigma^2} - 1\right), \tag{2}$$

$$q = (1-\mu)\left(\frac{\mu(1-\mu)}{\sigma^2} - 1\right). \tag{3}$$

As an illustration of the above, suppose that $p = 3$ and $q = 2$, then the probability density function reduces to $\beta(x) = 12x^2(1-x)$ with $\mu = 0.6$ and $\sigma^2 = 0.04$. Note that, because

p is almost equal to q, the function will be almost symmetrical. We may now calculate the probability that, say, $X \leq 0.4$ from the probability distribution function as follows:

$$\int_0^{0.4} \beta(x)\,\mathrm{d}x = \int_0^{0.4} 12x^2(1-x)\,\mathrm{d}x = \left[4x^3 - 3x^4\right]_0^{0.4} = 0.1792.$$

2. Empirical application

The author has had sole responsibility for teaching and examining an honours module entitled 'Quantitative Analysis Of Financial Decisions' since its introduction 26 years ago. The module has remained substantially the same in terms of content and assessment and a total of 399 students have undertaken it to date. The precise nature of the subject is such that high examination marks are possible, and grade data indicate that the proportion (X) scoring a first class mark (i.e. an examination mark in excess of 70%) in any year is a random variable ranging from 0 to 0.53 with a mean of 0.14 and variance of 0.0224. Because a proportion has a finite range from 0 to 1, it is appropriate to try and model this data with a beta distribution.

To obtain estimates for the two parameters p and q, we substitute the sample mean $\mu = 0.14$ and the variance $\sigma^2 = 0.0224$ into (2) and (3) to obtain $p = 0.6125$ and $q = 3.7625$. This therefore suggests a steep, right-skewed distribution.

Armed with these estimated parameters, we can compute cumulative probabilities using the Simple Interactive Statistical Analysis (SISA) software package (see reference 2); these are shown in table 1. Thus, for example, the theoretical probability that, in any year, 10% or fewer students score a first class mark in this particular module is 0.54, i.e. $\mathrm{P}(X \leq 0.1) = 0.54$. In fact, in 58% of the years under consideration, 10% or fewer students scored a First. As can be seen from table 1, this particular beta distribution provides a good fit to the observed variation in X.

References

1 N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions* (John Wiley, New York, 1995).

2 http://home.clara.net/sisa/.

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Table 1

Proportion of 'Firsts' (X)	Observed percentage	Theoretical probability
≤ 0.1	0.58	0.54
≤ 0.2	0.77	0.74
≤ 0.3	0.81	0.86
≤ 0.4	0.92	0.92
≤ 0.5	0.96	0.96
≤ 0.6	1.00	0.99

Mathematics in the Classroom

Trigonometric substitutions in action

We are familiar with the powerful use of trigonometric substitutions in calculus. The role of these is not limited to calculus. In this article, we show how trigonometric substitutions can be used to prove some inequalities. These problems can also be solved in other ways.

Problem 1 If a and b are positive real numbers such that $a+b=2$, prove that $ab \le 1$

Solution Using the identity $\sin^2\theta + \cos^2\theta = 1$, we can set $a = 2\sin^2\theta$ and $b = 2\cos^2\theta$ for some θ. Then $ab = 4\sin^2\theta\cos^2\theta = \sin^2 2\theta \le 1$.

Problem 2 Let x and y be real numbers satisfying $x^2 - 4x + y^2 + 3 = 0$. Find the maximum and minimum values of $x^2 + y^2$.

Solution The given equation can be written as $(x-2)^2 + y^2 = 1$, so we can set $x - 2 = \sin\theta$ and $y = \cos\theta$. Then $x^2 + y^2 = (\sin\theta + 2)^2 + \cos^2\theta = 5 + 4\sin\theta$, so the maximum value of $x^2 + y^2$ is $5 + 4 = 9$ and its minimum value is $5 - 4 = 1$.

Problem 3 For $a, b, c \in (0, 1)$, prove that $\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$.

Solution Since $a, b, c \in (0, 1)$, we may write $a = \sin^2 A$, $b = \sin^2 B$, and $c = \sin^2 C$, for some $A, B, C \in (0, \pi/2)$. Now the inequality becomes

$$\sin A \sin B \sin C + \cos A \cos B \cos C < \sin A \sin B + \cos A \cos B = \cos(A - B) < 1.$$

Problem 4 For positive real numbers a and b, prove that $\sqrt{ab}\sqrt{cd} \le \sqrt{(a+d)(b+c)}$.

Solution The inequality can be transformed to

$$\sqrt{\frac{a}{a+d}\,\frac{b}{b+c}} + \sqrt{\frac{c}{b+c}\,\frac{d}{a+d}} \le 1.$$

Setting $a/(a+d) = \sin^2\alpha$ and $b/(b+c) = \sin^2\beta$, for $0 < \alpha, \beta < \pi/2$, the inequality holds and takes the following form:

$$\sin\alpha\sin\beta + \cos\alpha\cos\beta \le 1 \quad \text{or} \quad \cos(\alpha - \beta) \le 1.$$

Problem 5 For any real numbers x and y, prove that

$$-\frac{1}{2} \le \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \le \frac{1}{2}.$$

Solution The substitutions $x = \tan A$ and $y = \tan B$ transform the inequalities to

$$-\frac{1}{2} \le \frac{(\tan A + \tan B)(1 - \tan A \tan B)}{(1 + \tan^2 A)(1 + \tan^2 B)} \le \frac{1}{2}$$

or $-1 \le \sin 2(A + B) \le 1$, which is obviously true.

Problem 6 If $f(x) = (x^2 - 1)/(x^2 + 1)$ for every real number x, find the minimum value of $f(x)$.

Solution After substituting $x = \tan\theta$, $-\pi/2 < \theta < \pi/2$, we obtain

$$f(x) = \frac{\tan^2\theta - 1}{\tan^2\theta + 1} = -\cos 2\theta.$$

For $-\pi/2 < \theta < \pi/2$, we have $-1 < \cos 2\theta \le 1$ or $-1 \le -\cos 2\theta < 1$. Therefore, $f(x)$ has minimum value -1, which occurs at $\theta = 0$, i.e. at $x = 0$.

Problem 7 Let $0 < a, b, c < 1$ with $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \ge \frac{3\sqrt{3}}{2}.$$

Solution We may write $a = \tan(A/2)$, $b = \tan(B/2)$, and $c = \tan(C/2)$, where $0 < A, B, C < \pi/2$, so that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

From the identity, we obtain

$$\tan(X+Y+Z) = \frac{\tan X + \tan Y + \tan Z - \tan X \tan Y \tan Z}{1 - \tan X \tan Y - \tan Y \tan Z - \tan Z \tan X}$$

and

$$\tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \infty,$$

so that $A/2 + B/2 + C/2 = \pi/2$ or $A + B + C = \pi$ (i.e. A, B, and C are the angles of a triangle). Now we have to prove that

$$\frac{\tan(A/2)}{1-\tan^2(A/2)} + \frac{\tan(B/2)}{1-\tan^2(B/2)} + \frac{\tan(C/2)}{1-\tan^2(C/2)} + \ge \frac{3\sqrt{3}}{2}$$

or

$$\tan A + \tan B + \tan C \ge 3\sqrt{3}.$$

By applying the arithmetic mean-geometric mean inequality, we have

$$\frac{\tan A + \tan B + \tan C}{3} \ge (\tan A \tan B \tan C)^{1/3},$$

so that

$$\frac{\tan A + \tan B + \tan C}{3} \ge (\tan A + \tan B + \tan C)^{1/3},$$

since $\tan(A+B+C) = 0$. Hence $(\tan A + \tan B + \tan C)^{2/3} \ge 3$, whence $\tan A + \tan B + \tan C \ge 3\sqrt{3}$, as required.

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3 T. Andreescu and R. Gelca, *Mathematical Olympiad Challenges* (Birkhäuser, Boston, MA, 2000).

4 T. Andreescu, G. Dospinescu, V. Cirtoaje and M. Lascu, *Old and New Inequalities* (GIL, Romania, 2004).

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Computer Column

Permutations and combinations or 'How I learned to stop worrying and love recursion'

One of our readers, John Halsall, recently needed to find a way of printing out permutations and combinations of a set, and set about writing programs to do the job. To create programs that would work in general, though, he took a very different approach from the one you or I would use if we were trying to do the job with a pencil and paper. Looking at them, I thought that there had to be an easier way, one that was more like the pencil-and-paper method. It turns out that there is, provided that you are willing to brave the murky waters of recursive programming. The idea of recursion is very powerful and, if you have never seen it before, can look a little like magic.

Permutations and combinations

To recap briefly, John wanted a way to list the ways that b objects (for example, letters) could be selected from a set of size a (for example, a word). Say, for example, that we want to find all the ways of selecting two letters from the word CAT. If the order of the letters does not matter (so CA is the same as AC) then this is known as a combination; if the order does matter then it is known as a permutation.

There are well-known formulae for the *number* of ways of doing this, but it is less obvious how to go about listing them all systematically. This is the problem that John tackled, and solved quite ingeniously. His solution for combinations is the simplest of the two, so that is the one we will discuss here.

He realised that, since in a combination the letter order does not matter, he could keep the order the same as in the original word, and form new combinations by always moving to the right. The resulting algorithm was as follows.

(i) Form a combination from the first b letters of the word.

(ii) Move the last letter of the combination one place to the right, forming a new combination. Repeat until we reach the end of the word.

(iii) Move the second-to-last letter of the combination one place to the right. If this is not possible, try the third-to-last letter, then the one before that, until either we get to a letter that can be moved to the right or we run out of letters.

(iv) If we have run out of letters, stop. Otherwise, set the subsequent letters of the combination to the next letters of the word, in order. Go back to step 2.

To illustrate this, let us try to use it to find three-letter combinations from the word BIRD. This gives us BIR (step 1), BID (step 2), BRD (steps 3 and 4), and finally IRD (steps 3 and 4).

This works well, and it is easy to see how to turn it into a computer program, but it is quite different from the way we humans would tackle the task, which would probably be to break it down into a series of subtasks. For the BIRD example, we would first look for three-letter combinations beginning with B, then ones beginning with I, R, and D. Having decided on the first letter, we would then work through the possible second letters, and finally the possibilities for the third letter. The end result is the same list of combinations, but with quite a different perspective.

This becomes clear if we move from considering combinations to looking at permutations. In the human approach, very little changes: we simply allow ourselves to look to the left as well as the right when selecting letters. For John's algorithm, however, the situation is quite different: looking at permutations is considerably harder than handling combinations. In fact, the easiest way forward is to calculate combinations as before, then move the letters in each combination around to generate all the permutations.

How, though, do we adapt the human approach into a computer program? This, if you have not already guessed, is where recursion comes in.

Recursion

The idea behind recursion is perhaps best illustrated by an example. Imagine that you want to write a program to calculate $n!$, for some n entered by the user. The most obvious way to do this is simply to multiply together the numbers from 1 to n in sequence, as in the following example. (For simplicity, the following examples present only the essential parts of the algorithms we are considering.)

```
INPUT n%
Result% = 1
FOR i% = 1 TO n%
  Result% = Result% * i%
NEXT i%
PRINT Result%
END
```

Compare this with the following recursive algorithm for calculating $n!$, which looks very different.

```
DECLARE FUNCTION Factorial% (n%)
INPUT n%
PRINT Factorial%(n%)
END

FUNCTION Factorial% (n%)
  IF n% = 1 THEN
    Factorial% = 1
  ELSE
    Factorial% = n% * Factorial(n% - 1)
  END IF
END FUNCTION
```

The logic here is this: if $n = 1$ then the factorial is 1, otherwise it is n times the factorial of $n-1$. As with a proof by induction, we only deal with the smallest case directly, then define the others by reference to smaller cases. On the surface, it appears that we have somehow avoided doing most of the work; we have dealt with $n = 1$, but have not really said very much about the rest of the problem. In fact, we have said everything we need to say. For example, say that we run the program and input 3. `Factorial(3)` tells us that the answer is `3*Factorial(2)`, and `Factorial(2)` is defined as `2*Factorial(1)`. Since `Factorial(1)` is 1, the answer comes out as 6, as we would expect.

Applying recursion to permutations

For the factorial problem, using recursion does not really seem to be very useful: the program is longer and more complicated, with no obvious benefit. The idea, however, comes into its own when we come back to considering permutations and combinations.

Now, we are ready to write a program that mirrors the human approach.

```
DECLARE SUB Comb (Word$, b%, CurrLetter%, Result$)
INPUT Word$
INPUT b%
CALL Comb(Word$, b%, 0, "")
END

SUB Comb (Word$, b%, CurrLetter%, Result$)
  IF b% = 0 THEN
    PRINT Result$
  ELSE
    FOR n% = CurrLetter% + 1 TO LEN(Word$)
      CALL Comb(Word$, b% - 1, n%,
                 Result$ + MID$(Word$, n%, 1))
    NEXT n%
  END IF
END SUB
```

This is a little more complicated than the factorial example, but the logic essentially follows the pen-and-paper idea, i.e. to find all b-letter combinations from the input word, work through all the letters, and for each one find all the $(b-1)$-letter combinations from the remainder of the word. If b is zero then there is no more work to be done, so we simply output what we have so far and go back for more.

Again, this looks a little like we have managed to avoid doing most of the work, but in fact it contains everything that is needed. It can also easily be adapted to handle permutations instead of combinations, but we shall leave that as an exercise for the reader.

All this simplicity comes at a price, however, and it has to be paid in computer memory. Unlike John's approach, the computer needs to keep track of all the subtasks: each time we start a new one without ending one of the existing ones, we use up a little more storage space. In the BIRD example, the computer needs to remember simultaneously that it is, say, up to B in the list of possible first letters, up to I in the list of second letters, and up to R in the list of third letters, otherwise it will lose its place.

In this case, there is unlikely to be a problem – it would be a monumental combination problem indeed that used up any significant amount of a modern computer's memory – but it always needs to be borne in mind, especially in more complex programs. It is also vital to be sure that you include an end point for the recursion: there must always be at least one case that is dealt with explicitly, otherwise you might find the computer going round in a loop forever, or until it runs out of memory!

In conclusion, recursive algorithms can be a very important addition to your toolbox. Whenever you find yourself trying to solve a problem by breaking it down into a series of subproblems, and then breaking those down and down until you have something you can solve easily, reach for a recursive algorithm!

Peter Mattsson

Letters to the Editor

Dear Editor,

The rational mean

I have recently come across an item called a *rational mean*. It is based on the idea of a *mediant*, where the mediant of $(a/b, c/d)$ is $(a+c)/(b+d)$. The mediant was used in the study of Farey sequences by Moriz Stern and Achille Brocott, and in Ford's circles and Pick's theorem. The rational mean expands the meaning of a mediant to include as many terms as necessary to meet the problem in hand. For instance, to find the arithmetic mean (AM) of a number of terms, all that is required is to evaluate the rational mean (RM) of the terms having equal denominators. For example,

$$\mathrm{AM}\left\{\tfrac{1}{2}, \tfrac{3}{5}, \tfrac{7}{3}, \tfrac{2}{3}\right\} = \mathrm{RM}\left\{\tfrac{15}{30}, \tfrac{18}{30}, \tfrac{70}{30}, \tfrac{20}{30}\right\} = \tfrac{15+18+70+20}{30+30+30+30} = \tfrac{123}{120} = \tfrac{41}{40}.$$

Similarly, to find the harmonic mean (HM) of a number of terms, all that is required is to evaluate the RM of the terms having equal numerators. For example,

$$\mathrm{HM}\left\{\tfrac{1}{2}, \tfrac{3}{5}, \tfrac{7}{3}, \tfrac{2}{3}\right\} = \mathrm{RM}\left\{\tfrac{42}{84}, \tfrac{42}{70}, \tfrac{42}{18}, \tfrac{42}{63}\right\} = \tfrac{42+42+42+42}{84+70+18+63} = \tfrac{168}{235}.$$

But this is only the beginning. Say we wanted to find the 5th root of 7. We start by writing five simple terms whose product is 7. In the world of the rational mean, there are many ways of doing this but, for this case, we select

$$A = \left\{x, x, x, x, \frac{7}{x^4}\right\}.$$

Here, we are looking for an iterative formula with rapid convergence. We first try the AM

$$\mathrm{AM}\{A\} = \mathrm{RM}\left\{\frac{x^5}{x^4}, \frac{x^5}{x^4}, \frac{x^5}{x^4}, \frac{x^5}{x^4}, \frac{7}{x^4}\right\} = \frac{4x^5+7}{5x^4},$$

which is the Newton–Raphson iteration formula.

But this can be improved upon. This time, we try the HM

$$\begin{aligned}\mathrm{HM}\{A\} &= \mathrm{HM}\left\{x, x, x, x, \frac{7}{x^4}\right\}\\ &= \mathrm{RM}\left\{\frac{7x}{7}, \frac{7x}{7}, \frac{7x}{7}, \frac{7x}{7}, \frac{7x}{x^5}\right\}\\ &= \frac{35x}{28+x^5}.\end{aligned}$$

This also forms an iteration, but it can be improved by adding a linear combination of the above AM. Thus, let the required iteration formula be

$$\begin{aligned}\mathrm{RM}\left\{\mathrm{HM}, \frac{x}{x(\mathrm{AM})}\right\} &= \mathrm{RM}\left\{\frac{35x}{28+x^5}, \frac{x}{x}\left(\frac{4x^5+7}{5x^4}\right)\right\}\\ &= \frac{42x+4x^6}{28+6x^5}\end{aligned} \tag{1}$$

Table 1

Newton–Raphson iteration	Equation (1)
$x_0 = 1$	$x_0 = 1$
$x_1 = 2.2$	$x_1 = 1.352\,941\,176$
$x_2 = 1.819\,763\,677$	$x_2 = 1.473\,871\,186$
$x_3 = 1.583\,474\,83$	$x_3 = 1.475\,773\,155$
$x_4 = 1.489\,460\,974$	$x_4 = 1.475\,773\,162$
$x_5 = 1.476\,022\,436$	$x_5 = 1.475\,773\,162$
$x_6 = 1.475\,773\,246$	$\cdots$
$x_7 = 1.475\,773\,162$	$\cdots$
$x_8 = 1.475\,773\,162$	$\cdots$

We compare this formula with Newton–Raphson iteration with $x_0 = 1$ in table 1. As we can see, (1) converges almost twice as fast as Newton–Raphson iteration.

If we follow the above carefully, we can arrive at a general iterative formula for finding the nth root of a number P (or the root of a polynomial expression P). Let $A = \{x, x, \ldots, x, P/x^{n-1}\}$, with $(n-1)$ xs, so that the product of the n terms is P. Then we obtain

$$\begin{aligned}\mathrm{HM}\{A\} &= \mathrm{HM}\left\{x, x, \ldots, x, \frac{P}{x^{n-1}}\right\}\\ &= \mathrm{RM}\left\{\frac{Px}{P}, \frac{Px}{P}, \ldots, \frac{Px}{P}, \frac{Px}{x^n}\right\}\\ &= \frac{nPx}{(n-1)P + x^n}\end{aligned}$$

and

$$\begin{aligned}\mathrm{AM}\{A\} &= \mathrm{AM}\left\{x, x, \ldots, x, \frac{P}{x^{n-1}}\right\}\\ &= \mathrm{RM}\left\{\frac{x^n}{x^{n-1}}, \frac{x^n}{x^{n-1}}, \ldots, \frac{x^n}{x^{n-1}}, \frac{P}{x^{n-1}}\right\}\\ &= \frac{P + (n-1)x^n}{nx^{n-1}}.\end{aligned}$$

We now do a linear combination of the HM and AM using what is called (for want of a better name) the *form factor* x/x (which alters the form but not the value of the AM). Thus, the required formula is

$$\begin{aligned}\mathrm{RM}\left\{\mathrm{HM}, \frac{x}{x(\mathrm{AM})}\right\} &= \left\{\frac{nPx}{(n-1)P + x^n}, \frac{x}{x}\frac{P + (n-1)x^n}{nx^{n-1}}\right\}\\ &= \frac{(n+1)Px + (n-1)x^{n+1}}{(n-1)P + (n+1)x^n}.\end{aligned}$$

It is now easy to find the nth root of a number by successive iteration.

Example 1 Find the 3.57th root of 21.53. Here, $n = 3.57$ and $P = 21.53$. Let $x_0 = 2$; we obtain the following results:

$$x_0 = 2,$$
$$x_1 = 2.352\,308\,692,$$
$$x_2 = 2.362\,662\,308,$$
$$x_3 = 2.362\,662\,504,$$
$$x_4 = 2.362\,662\,504.$$

Thus, we have a solution in three iterations.

Example 2 To find a root of a polynomial equation such as $x^5 + 3x^2 - 1000 = 0$, we proceed as follows. Let $x^5 = 1000 - 3x^2 = P$, say, and $n = 5$ so, by substituting for P, the formula becomes

$$x_{r+1} = \frac{3000x_r - 9x_r^3 + 2x_r^6}{2000 - 6x_r^2 + 3x_r^5}.$$

Using this formula with $x_0 = 1$, we find $x_9 = 3.943\,217\,72$, correct to eight decimal places. In contrast, Newton–Raphson iteration

$$x_{r+1} = \frac{4x_r^5 + 3x_r^2 + 1000}{5x_r^4 + 6x_r},$$

produces the same answer at x_{19}.

Much of the rational mean process remains to be strengthened by laying down any restrictions that may be applied to parts of the process. However, considering that no calculus was involved and only simple arithmetic applied, the method deserves attention. The above is just a taste of the full power and methods that are available by using the rational mean.

Reference

1 D. Gómez Morín, The Fifth Arithmetical Operation, In *La Quinta Operación Aritmética* (In Spanish; ISBN 980-12-1671-9).

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Dear Editor,

Brahmagupta's problems, Pythagorean solutions and Heron triangles

I have received responses from two readers to my article *Brahmagupta's Problems, Pythagorean Solutions and Heron Triangles* (*Math. Spectrum*, Volume 38, Number 2). The first reader's response was a nice observation that the two Pythagorean triangles of figure 3 in my article of the realistic interpretation of Problem 1 have the same excircles opposite the vertices E and C.

Let us consider figure 1. We have $\mathrm{BE} + \mathrm{EC} = \mathrm{BD} + \mathrm{DC}$, which becomes

$$\mathrm{BE} + (\mathrm{AC} - \mathrm{AE}) = (\mathrm{AD} - \mathrm{AB}) + \mathrm{DC}, \qquad \mathrm{AB} + \mathrm{BE} - \mathrm{AE} = \mathrm{AD} + \mathrm{DC} - \mathrm{AC}.$$

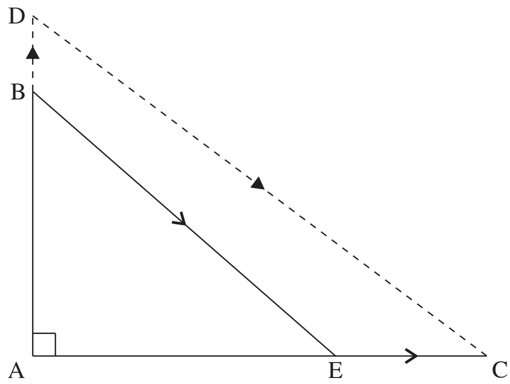

Figure 1 $BD + DC = BE + EC$.

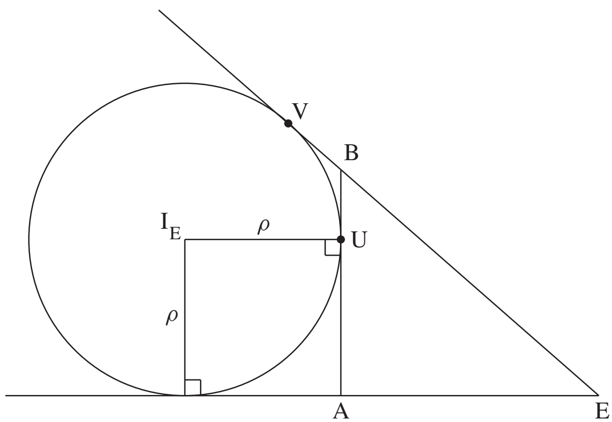

Figure 2 Excircle opposite vertex E of $\triangle ABE$.

Now consider the excircle opposite vertex E of $\triangle ABE$, with centre I_E, points of contact U and V with the sides AB and EB respectively, and exradius ρ (see figure 2). Then we obtain

$$\begin{aligned} AB + BE - AE &= (AU + BU = BV) + BE - AE \\ &= AU + (EV = AE + \rho) - AE \\ &= 2\rho \\ &= AD + DC - AC. \end{aligned}$$

Hence, $\triangle ADC$ has the same excircle opposite vertex C. Incidentally, this provides a different solution to the realistic interpretation of Problem 1.

> Find two Pythagorean triangles that have a common excircle opposite specific vertices (for example, E and C).

The reader may recall that this interpretation was suggested by the Editor.

The second response I received was about generating the complete set of Heron triangles from the twice-broken bamboo. This was an extension of Problem 2 in my article. The second reader felt that my equation (7), i.e.

$$\mathrm{BD} = (ps - qr)(pr + qs), \qquad \mathrm{DF} = pq(r^2 + s^2), \qquad \mathrm{BF} = rs(p^2 + q^2), \tag{7}$$
$$\text{area } \triangle\mathrm{DBF} = pqrs(ps - qr)(pr + qs),$$

should be subjected to the constraints $p > q$, $r > s$, $ps - qr > 0$, and therefore yield only obtuse Heron triangles. There is an important difference to be noted here.

Equation (7) was derived using trigonometry. The trigonometric approach is always more powerful than the geometric approach in the following sense. If the reader wishes to describe the complete set of Heron triangles then the following three cases will have to be considered:

(i) the Heron triangles containing a right angle, i.e. Pythagorean triangles,

(ii) acute Heron triangles,

(iii) obtuse Heron triangles.

However, in the trigonometric approach we obtain just one description and the preceding three cases become just special instances of that one description. In other words, if (7) had been obtained using figure 4 of my article, then the second reader is correct in imposing the restrictions $p > q$, $r > s$, $ps - qr > 0$, and they yield only obtuse Heron triangles. In the trigonometric approach the bamboo in figure 4 disappears; what matters are the angles of $\triangle\mathrm{BDF}$. It is up to us whether to retain the earlier domain – in which case we obtain only obtuse Heron triangles – or to enlarge the domain to generate the complete set of Heron triangles. To enlarge the domain, we do *not* impose $p > q$ and $r > s$. To make side BD positive, we either use $ps > qr$ or use the absolute value $|ps - qr|$. The following examples show that (7) yields *all* types of Heron triangles in the enlarged domain (the standard convention is to remove the gcd if $\gcd(a, b, c) > 1$ except in the solution of special problems).

Example 1 The case $(p, q) = (3, 1)$, $(r, s) = (1, 2)$ and $(p, q) = (3, 2)$, $(r, s) = (1, 1)$ yields the Pythagorean triangles $(5, 3, 4)$ and $(5, 12, 13)$ respectively.

Example 2 The case $(p, q) = (3, 2)$ and $(r, s) = (2, 3)$ yields the isosceles Heron triangle $(10, 13, 13)$. More generally, $(r, s) = (q, p)$ yields the isosceles Heron triangle $(2(p^2 - q^2), p^2 + q^2, p^2 + q^2)$.

Example 3 The case $(p, q) = (4, 3)$, $(r, s) = (1, 3)$ and $(p, q) = (5, 4)$, $(r, s) = (1, 2)$ yields the acute Heron triangles $(39, 40, 25)$ and $(39, 50, 41)$ respectively.

Example 4 The case $(p, q) = (3, 1)$, $(r, s) = (2, 1)$ and $(p, q) = (3, 1)$, $(r, s) = (4, 1)$ yields the obtuse Heron triangles $(7, 15, 20)$ and $(13, 51, 40)$ respectively.

I wish to express gratitude to both correspondents. The reader may correspond with the author for clarification of any points that may arise on reading this letter.

Yours sincerely,

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Dear Editor,

A special case?

I am writing in response to Muneer Jebreel's question in *Mathematical Spectrum*, Volume 38, Number 3, p. 135. He asked if the identities

$$1^5 + 2^5 + \cdots + 13^5 = 11^2(1^3 + 2^3 + \cdots + 13^3) = 1001^2 \tag{1}$$

are a special case of something more general. The answer is yes, but it may be limited to sums of cubes and fifth powers or cubes and first powers.

Let $s_i(n) = k^p s_j(n)$, where i, j, k, p, and n are all positive integers and

$$s_i(n) = \sum_{r=1}^{n} r^i. \tag{2}$$

Clearly, $n = k = 1$ and $i = j$, $k = 1$ always satisfy this equation and will therefore be ignored in the rest of this letter. We may now assume that $i > j$. It can be shown that if, regarded as polynomials with rational coefficients, $s_j(n)$ is not a factor of $s_i(n)$ then there can only be a finite number of solutions of (2) for any given pair (i, j). This is because if both sides of the equation are divided by $s_j(n)$ we end up with a remainder term which is numerically less than 1 for sufficiently large n and so cannot be an integer. For example,

$$\frac{8s_7(n)}{s_5(n)} = 6n^2 + 6n - 5 + \frac{3}{2n^2 + 2n - 1}.$$

Now, $0 < 3/(2n^2 + 2n - 1) < 1$ if $n > 1$ so $s_7(n)/s_5(n)$ cannot be an integer except when $n = 1$. For all $i, j < 7$, I found the only case where we get an integer ratio, for $n > 1$, is $s_7(n)/s_2(n) = 4\,030\,325$ when $n = 25$. So

$$1^7 + 2^7 + \cdots + 25^7 = 4\,030\,325(1^2 + 2^2 + \cdots + 25^2).$$

Now, again working within the domain of polynomials with rational coefficients, we have $s_1(n)|s_j(n)$ for all j, $s_2(n)|s_j(n)$ for even j, and $s_3(n)|s_j(n)$ for odd $j > 1$. In fact, I believe we need only consider three cases, since when $i > j > 3$ I conjecture that $s_j(n)$ is never a factor of $s_i(n)$. (Could a reader help here?) The three cases to consider are

(i) $j = 1$,

(ii) $j = 2$, i even,

(iii) $j = 3$, i odd.

Each of the cases reduces to a Diophantine equation of the form $f(n) = ak^p$. The cases $(i, j) = (2, 1), (3, 1), (4, 2)$, and $(5, 3)$ reduce to quadratic Diophantine equations, for which there are elementary solving techniques; see, for example, reference 1. The method depends on finding successive convergents of appropriate continued fractions; I summarise the method as follows.

- $(i, j) = (2, 1)$. We have $n = (3q^2 - 1)/2$ and $k = q$. For example, if $q = 3$ and $n = 13$ we get
$$1^2 + 2^2 + \cdots + 13^2 = 3^2(1 + 2 + \cdots + 13).$$

- $(i, j) = (3, 1)$. This reduces to finding the square triangular numbers, which involves solving $x^2 - 2y^2 = 1$. The first nontrivial example is

$$1^3 + 2^3 + \cdots + 8^3 = 6^2(1 + 2 + \cdots + 8) = 36^2.$$

- $(i, j) = (4, 2)$. This reduces to solving $3x^2 - 5y^2 = 7$. The smallest pair satisfying this is $x = 2$ and $y = 1$; the rest can be built from this using the recurrence relations $x_{m+1} = 4x_m + 5y_m$, $y_{m+1} = 3x_m + 4y_m$ and $n = (x_m - 1)/2$, $k = y_m/2$. These require x_m to be odd and y_m to be even so we take alternate pairs of x and y. The first few pairs for (x, y) are $(2, 1)$, $(13, 10)$, $(102, 79)$, and $(803, 622)$. Of these, only $(13, 10)$ and $(803, 622)$ work. The former yields $n = 6$ and $k = 5$ giving

$$1^4 + 2^4 + \cdots + 6^4 = 5^2(1^2 + 2^2 + \cdots + 6^2).$$

- $(i, j) = (5, 3)$. This reduces to finding solutions of $3x^2 - 2y^2 = 1$. The smallest pair satisfying this is $x = 1$ and $y = 1$; the rest can be built from this using the recurrence relations $x_{m+1} = 5x_m + 4y_m$, $y_{m+1} = 6x_m + 5y_m$, and $n = (3x_m - 1)/2$, $k = y_m$. The first few pairs for (x, y) are $(1, 1)$, $(9, 11)$, and $(89, 109)$. The pair $(1, 1)$ is trivial; the pair $(9, 11)$ yields (1). Note that in this case, because $s_3(n) = n^2(n+1)^2/4$ is a perfect square (it is, in fact, the square of $s_1(n)$), we get the 1001^2 in (1) from

$$11s_1(13) = 11(1 + 2 + \cdots + 13) = 1001.$$

The pair $(89, 109)$ gives $n = 133$ and $k = 109$, so we obtain

$$\begin{aligned} 1^5 + 2^5 + \cdots + 133^5 &= 109^2(1^3 + 2^3 + \cdots + 133^3) \\ &= (109(1 + 2 + \cdots + 13))^2 \\ &= 913\,896\,491\,101^2. \end{aligned}$$

I have examined the other cases for $i, j < 7$ with $n \leq 2000$, using EXCEL®, and have found no other solutions of (2) with $p = 2$.

To conclude, if we require further identities in the exact format of (1), they almost certainly only arise from sums of cubes and fifth powers or cubes and first powers and there are an infinite number of each type. If desired, explicit formulae can be given covering all cases. For $(i, j) = (5, 3)$ they are

$$k = \frac{(\sqrt{6} - 2)(5 + 2\sqrt{6})^m - (\sqrt{6} + 2)(5 - 2\sqrt{6})^m}{4}$$

and

$$n = \frac{\sqrt{6}((\sqrt{6} - 2)(5 + 2\sqrt{6})^m + (\sqrt{6} + 2)(5 - 2\sqrt{6})^m)}{8} - \frac{1}{2}.$$

These formulae give all possible solutions for $(i, j) = (5, 3)$. Or, equivalently,

$$k = \left\lfloor \frac{(\sqrt{6} - 2)(5 + 2\sqrt{6})^m}{4} \right\rfloor$$

and

$$n = \left\lfloor \frac{\sqrt{6}(\sqrt{6}-2)(5+2\sqrt{6})^m}{8} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Note that $n/k \to \sqrt{6}/2$ as $m \to \infty$.

Reference

1 D. M. Burton, *Elementary Number Theory* (Allyn and Bacon, Boston, MA, 1980).

Yours sincerely,

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Dear Editor,

Magic squares – a general method

Vinod Tyagi's item *Making magic squares* (*Math. Spectrum,* Volume 38, Number 3, p. 123) reminds me of a method for making magic squares I discovered many years ago when recovering from flu. I had previously learned general methods for finding $n \times n$ magic squares, using the integers from 1 to n^2. My schoolteacher had shown me the method if n is odd; one of my students had shown me (and the class) the method for any multiple of 4, when I myself had been a teacher for about ten years. But I had never learned a method for n when $n \equiv 2 \bmod 4$, so I decided to try and find a 6×6 magic square. I came up with the following general method, illustrated for $n = 4$.

(1) A *diagonalised Latin square* is defined as an $n \times n$ matrix made up of n distinct letters arranged so that each row, column, and main diagonal contains each letter once and once only.

(2) Construct any such Latin square, using the letters A, B, C, and D.

(3) Construct a *conjugate diagonalised Latin square* by replacing the four As by one each of the letters A, B, C, and D; perform similar replacements for each of the other three letters.

(4) For the first square make the substitution $A = 0$, $B = 4$, $C = 8$, and $D = 12$ (these may be permuted in any way).

(5) For the second square make the substitution $A = 1$, $B = 2$, $C = 3$, and $D = 4$ (these may be permuted in any way).

(6) 'Add' the two squares, which are regarded as matrices.

(7) We now have a magic square with each row, column, and diagonal totalling 34. It is, I think, quite easy to see why the method works.

Here is the detailed working for one example.

(2)	A	B	C	D	(3)	A	B	C	D
	D	C	B	A		C	D	A	B
	B	A	D	C		D	C	B	A
	C	D	A	B		B	A	D	C

(4)	0	4	8	12	(5)	1	2	3	4	(6)	**1**	**6**	**11**	**16**
	12	8	4	0		3	4	1	2		**15**	**12**	**5**	**2**
	4	0	12	8		4	3	2	1		**8**	**3**	**14**	**9**
	8	12	0	4		2	1	4	3		**10**	**13**	**4**	**7**

I leave it as an exercise for the reader to use this method to construct a 6×6 magic square. It is not too hard, although it may take a fair bit of trial and error to succeed.

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

39.5 The squares $ABCD$ and $A_1B_1C_1D_1$ in the plane have equal sides. Prove that

$$AA_1^2 + CC_1^2 = BB_1^2 + DD_1^2.$$

(Submitted by Abbas Roohol Aminy, Sirjan, Iran)

39.6 A set S with n elements, where n is a positive integer, has m distinct subsets $A_1, \dots, A_m$ such that $A_i \bigcup A_j = S$ whenever $i \neq j$. Show that $m \leq n+1$ and that if $m = n+1$ then $A_1, \dots, A_m$ are uniquely determined up to order.

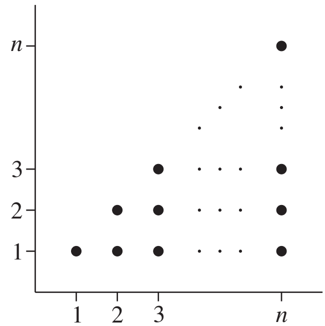
(Submitted by H. A. Shah Ali, Tehran, Iran)

39.7 Denote the lengths of the sides of a triangle ABC by a, b, and c and the lengths of its medians by d, e, and f. Express d, e, and f in terms of a, b, and c.

Show how to construct a triangle $A'B'C'$ from ABC with sides d, e, and f, and deduce that a similar construction applied to triangle $A'B'C'$ will give a triangle similar to ABC.

(Submitted by J. A. Scott, Chippenham, UK)

39.8 Two points P and Q are chosen at random from the array of $\frac{1}{2}n(n+1)$ points as shown below.



Determine the probabilities that the line PQ

(a) is parallel to, (b) is perpendicular to,
(c) makes an acute angle with, (d) makes an obtuse angle with

the positive direction of the x-axis.

(Submitted by M. A. Khan, Lucknow, India)

Solutions to Problems in Volume 38 Number 3

38.9 Let $P(x)$ be a polynomial with integer coefficients and constant term 5928 such that $P(5) = 2006$. Prove that 12, 19, and 26 are not roots of $P(x)$.

Solution by Mihály Bencze, who proposed the problem

For any x and y, we have

$$P(x) - P(y) = (x - y)Q(x, y),$$

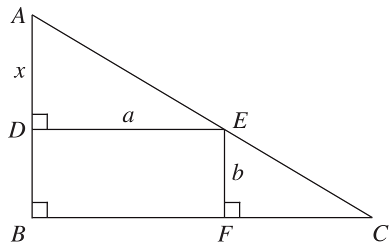
where $Q(x, y)$ is a polynomial in x and y with integer coefficients. Thus,

$$\begin{aligned} P(12) - P(5) &= 7Q(12, 5), \\ P(19) - P(5) &= 14Q(19, 5), \\ P(26) - P(5) &= 21Q(26, 5). \end{aligned}$$

Since $P(5) = 2006$ so is not divisible by 7, none of $P(12)$, $P(19)$, or $P(26)$ can be zero. (Any integer root of P would have to divide its constant term, and $5928 = 12 \times 19 \times 26$, so that 12, 19, and 26 are possible candidates as roots of P. Apart from this, 5928 is a red herring!)

38.10 A right-angled triangle has a point on its hypotenuse at fixed distances a and b from its other two sides. What is its minimum area?

Solution by Bor-Yann Chen, University of California, Irvine



Put $AD = x$. By similar triangles, we have

$$\frac{AD}{DE} = \frac{EF}{FC},$$

so that

$$FC = \frac{ab}{x},$$

and the area of $\triangle ABC$ is

$$A = \frac{1}{2}\left(a + \frac{ab}{x}\right)(x+b) = \frac{a(x+b)^2}{2x}.$$

Then we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{a(x+b)}{x} - \frac{a(x+b)^2}{2x^2} = \frac{a(x+b)(x-b)}{2x^2},$$

and this is zero when $x = b$. Also,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{2ax}{2x^2} + \frac{a}{2}(x^2-b^2)\left(-\frac{2}{x^3}\right),$$

which is positive when $x = b$. Hence, the area is at a minimum when $x = b$, with minimum value $2ab$.

38.11 For a positive real number x, determine the number of triangular numbers less than or equal to x. (A 'triangular number' is a positive integer of the form $1 + 2 + \cdots + n$, where n is a positive integer.)

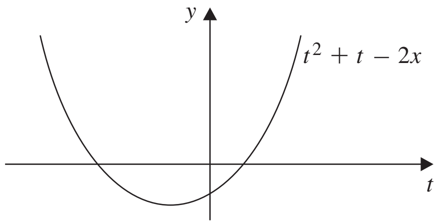
Solution by Michael Nyblom, Melbourne, Australia

We want the largest positive integer n such that

$$\tfrac{1}{2}n(n+1) \leq x,$$

i.e.

$$n^2 + n - 2x \leq 0.$$



The graph of $y(t) = t^2 + t - 2x$ is a parabola as shown above which crosses the t-axis at points

$$t = \frac{-1 \pm \sqrt{1+8x}}{2},$$

so the largest positive integer n such that $n^2 + n - 2x \leq 0$ is

$$\left\lfloor \frac{-1 \pm \sqrt{1+8x}}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer-part function.

38.12 Determine all natural numbers n for which there exists a permutation $a_1, \ldots, a_n$ of $0, 1, \ldots, n-1$ such that the remainders when $a_1, a_1+a_2, a_1+a_2+a_3, \ldots, a_1+a_2+\cdots+a_n$ are divided by n are all distinct.

Solution by H. A. Shah Ali, who proposed the problem

If $a_k = 0$ for some $k > 1$, then $a_1 + \cdots + a_{k-1} = a_1 + \cdots + a_k$, and this is not possible. Hence, $a_1 = 0$. Now,

$$a_1 + \cdots + a_n = 0 + 1 + \cdots + (n-1) = \tfrac{1}{2}(n-1)n;$$

so, if n is odd, $a_1 + \cdots + a_n$ is divisible by n, so that a_1 and $a_1 + \cdots + a_n$ have the same remainder on division by n, namely zero. Hence, n must be even. For any even n, there is such a permutation of $0, 1, \ldots, n-1$, namely

$$0, n-1, 2, n-3, \ldots, n-2, 1.$$

Modulo n, these numbers are

$$0, -1, 2, -3, 4, -5, \ldots, n-2, -(n-1),$$

and the remainder sequence modulo n is

$$0, -1, 1, -2, 2, -3, \ldots,$$

giving the n possible distinct remainders modulo n.

Reviews

Common Errors in Statistics (and How to Avoid Them). By Phillip I. Good and James W. Hardin. John Wiley, Chichester, 2nd edn., 2006. Paperback, 254 pages, £26.50 (ISBN 0-471-79431-8).

The first edition of this book appeared in 2003 and received much acclaim for being easily understood, highly informative, enjoyable to read, and full of good advice. It was recognised as being written for people who define good practice rather than those who seek to emulate it. Indeed, the stated purpose of this text is to provide a mathematically rigorous but readily understandable foundation for statistical procedures, for, as the authors intelligently observe, access to statistical software will no more make one a statistician than access to a chainsaw will make one a lumberjack. Wise words indeed!

This second edition builds on the work of the first edition in analysing common mistakes, exposing popular myths, and assisting readers to select the most appropriate statistical technique for the task in hand. Two new chapters entitled 'Interpreting Reports' and 'Which Regression Method?' have been added along with further elaboration of key topics such as hypothesis testing and multivariate regression. These revisions improve an already useful text.

This book progresses the authors' mission of emphasising the great value of statistical tools when appropriately applied, and is therefore essential reading for anyone who is required to analyse data and who is not entirely confident about the best way to do this.

Carol Nixon

Math Made Visual. Creating Images for Understanding Mathematics. By Claudi Alsina and Roger Nelsen. MAA, Washington, DC, 2006. Hardback, 190 pages, \$49.95 (ISBN 0-88385-746-4).

'How can you cut a hole in a piece of paper large enough to walk through?'. 'What is the easiest way of measuring the diagonal of a box?'. 'How can the formula for the sums of the squares of the first n natural numbers be visually demonstrated?'.

These are just three of the questions posed and answered in this amazing teaching resource and mathematically educative book! J. E. Littlewood said

> A heavy warning used to be given that pictures are not rigorous, this has never had its bluff called and has permanently frightened its victims.

This book calls this bluff. It demonstrates how to present geometrical and algebraic results visually. Each result is clear from a diagram without any calculation or advanced logic being required. The book is very well structured and written in three parts. In Part I, each of the 20 chapters deals with a different method which is well illustrated and then followed up with an adequate number of exercises, to help and encourage the reader to create their own visual demonstrations. Some of the chapter headings are 'Representing numbers by lengths of segments', 'Employing Isometry', 'Overlaying Tiles', and 'Ingenuity in 3D'. The mathematics is expressed concisely, but also rigorously and clearly. Some language learned at degree level is assumed but hardly dominates, as the meat of the book is in the pictures!

The material relates to syllabuses from lower secondary to early University level, but a good deal could be presented in some form to students at any level. Included are visual demonstrations of a host of identities relating to Fibonacci and other sequences, many standard inequalities, elementary algebraic and trigonometric identities, unfamiliar proofs of Pythagoras and other geometrical theorems, space filling problems, properties of 3D models, tangrams, and much more.

Part II contains some history, observations, and some useful practical applications amongst other interesting material, for example, quadric surfaces. Part III contains many hints and solutions to the exercises. There is also an extensive bibliography.

Alastair Summers

Visual Statistics: Seeing Data with Dynamic Interactive Graphics. By Forrest W. Young, Pedro M. Valero-Mora and Michael Friendly. John Wiley, Hoboken, NJ, 2006. Hardback, 364 pages, £52.95 (ISBN 0-471-68160-1).

In the authors' own words, this book sets out to outline 'a system which presents the user with a visual environment for statistical analysis that supports instantaneous interaction and uses dynamic graphics to communicate information about the data'. The book consists of three parts as follows.

Part I: Introduction

1. Introduction
2. Examples

Part II: See Data – The Process

3. Interfaces and Environments
4. Tools and Techniques

Part III: Seeing Data – Objects

5. Seeing Frequency Data
6. Seeing Univariate Data
7. Seeing Bivariate Data
8. Seeing Multivariate Data
9. Seeing Missing Values

These are followed by around ten pages of references, an Author Index, and a Subject Index.

The authors have successfully achieved their aim, and produced a book which will be of great value to all statisticians interested in interactive graphics.

Australian National University, Canberra **Joe Gani**

99 Points of Intersection. By Hans Walser (translated from the original German by Peter Hilton and Jean Pedersen). MAA, Washington, DC, 2006. Hardback, 168 pages, $48.50 (ISBN 0-88385-553-4).

This book is in three parts. Part I gives a few examples to illustrate the book's theme, which is a study of concurrence. Part II contains 99 pictures with occasional comment but each clearly illustrating some theorem of concurrence, all involving straight lines, points, and circles. The diagrams vary from simple to quite complicated. The most amazing are in the introduction, which contains some fascinating optical effects reminiscent of fractal diagrams. Part III has a very useful section for students on methods of proof, which will aid attempts on proofs of the

99 results; it also gives selective proofs. The whole book can be surveyed in a few minutes, but this is deceptive, as an attempt to prove all the results might take years! The author points out that, while some of the theorems can be solved by hand, some will require computer assisted software. He puts out a challenge to readers to discover an elementary proof of number 79, which had eluded him in spite of years of effort. I have one criticism, highlighted by number 79. It took me some time to realise that shading was used to indicate similarity. The glossary of symbols on page 17 omits this. Also it does indicate that equal and parallel lengths will be marked in the diagrams, but then they never are. There is also an extensive bibliography.

Most of the mathematics required for the proofs is generally of university or very advanced school standard, but some is accessible to school students and the results could in any case be a useful resource for school teachers as well as for university teachers. It is a fascinating book.

Alastair Summers

Other books received

Thinking Maths. Cognitive Acceleration in Mathematics Education. By Mundher Adhami and Michael Shayer. Heinemann, Oxford, 2006. Paperback, 240 pages, £95.00 (ISBN 0-435307-80-0).

Category Theory. By Steve Awodey. Oxford University Press, 2006. Hardback, 268 pages, £65.00 (ISBN 0-19-856861-4).

Threading Homology Through Algebra: Selected Patterns. By Giandomenico Boffi and David A. Buchsbaum. Oxford University Press, 2006. Hardback, 268 pages, £60.00 (ISBN 0-19-852499-4).

Latent Curve Models: A Structural Equation Perspective. By Kenneth A. Bollen and Patrick J. Curran. John Wiley, Chichester, 2006. Hardback, 312 pages, £55.95 (ISBN 0-471-45592-X).

AQA GCSE Mathematics: Foundation Module 1. By Sue Chandler and Ewart Smith. Heinemann, Oxford, 2006. Paperback, 152 pages, £9.50 (ISBN 0-435-80719-6).

Mathematical Geophysics: An Introduction to Rotating Fluids and the Navier-Stokes Equations. By Jean-Yves Chemin, Benoit Desjardins, Isabelle Gallagher and Emmanuel Grenier. Oxford University Press, 2006. Hardback, 262 pages, £45.00 (ISBN 0-19-857-133-X).

Differential and Integral Equations. By Peter Collins. Oxford University Press, 2006. Paperback, 386 pages, £27.50 (ISBN 0-19-929789-4).

USA And International Mathematical Olympiads 2005. By Zuming Feng. MAA, Washington, DC, 2006. Paperback, 100 pages, $31.95 (ISBN 0-88385-823-1).

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The Structure of Models of Peano Arithmetic. By Roman Kossak. Oxford University Press, 2006. Hardback, 326 pages, £50.00 (ISBN 0-19-856827-4).

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Topology Now! By Robert Messer and Philip D. Straffin. The Mathematical Association of America, Washington, DC, 2006. Hardback, 250 pages, $49.95 (ISBN 0-88385-744-8).

An Invitation to Modern Number Theory. By Steven J. Miller and Ramin Takloo-Bighash. Princeton University Press, 2006. Hardback, 519 pages, £32.50 (ISBN 0-691-12060-9).

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AQA GCSE Mathematics: Foundation Student Book. By Glyn Payne, Ron Holt, Mavis Rayment and Ian Robinson. Heinemann, Oxford, 2006. Paperback, 560 pages, £16.99 (ISBN 0-435-21044-0).

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Scottish Secondary Mathematics: Red 1 Student Book. By Tom Sanaghan, Jim Pennel, Carol Munro, Carole Ford, John Dalton and James Cairns. Heinemann, Oxford, 2006. Paperback, 248 pages, £12.75 (ISBN 04-350-4011-1).

Bayesian Networks and Probabilistic Inference in Forensic Science. By Franco Taroni, Colin Aitken, Paolo Garbolino and Alex Biedermann. John Wiley, Chichester, 2006. Hardback, 372 pages, £55.00 (ISBN 0-470-09173-8).

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