Crux Mathematicorum

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 10, No. 4

April 1984

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa

The assistance of the publisher and the support of the Canadian Mathematical Society, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22 in Canada, $24.50 (or US$20) elsewhere. Back issues: each $2.25 in Canada, $2.50 (or US$2) elsewhere. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-9, $17 in Canada, $18.40 (or US$15) elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

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CONTENTS

PRIME ASPECTS OF 37

CHARLES W. TRIGG

37, its two digits, and its reverse are all primes in the usual sense. Thus 37 is a *prime prime*. That is, it is one of the set of primes that remain prime after successively dropping the right-hand digit [14]. It is also one of the set of primes that remain prime after successively dropping the leftmost digit. Hence 37 (and its reverse) is a *right prime prime*, a *left prime prime*, and a *double prime prime* [1].

37 is an *isolated prime* [2], since it is not one of a twin prime pair (35 and 39 being composite).

Let

$$B_1 = \frac{1}{6},\quad B_2 = \frac{1}{30},\quad \ldots,\quad B_5 = \frac{5}{66},\quad B_6 = \frac{691}{2730},\quad \ldots,\quad B_{16} = \frac{37 \cdot 208360028141}{510},\quad \ldots$$

be Bernoulli numbers, each expressed in its lowest terms. With p an odd prime, consider the set $\{B_1, B_2, \ldots, B_{u-1}\}$, where $u = (p-1)/2$. If the numerators of none of these fractions are divisible by p, then p is said to be a *regular prime*; otherwise it is said to be *irregular* [13]. Kummer in 1850 found that all the primes less than 100 are regular except 37, 59, and 67. Thus 37 is the smallest irregular prime.

37 retains its prime status, as does its reverse 73, even under Stein's whimsical extrapolation of Gulliver's account of mathematics as practiced on the flying island of Laputa [7]. This defines the *good numbers* as 1 and all natural numbers obtained from 1 by adding a multiple of 3. A good natural number is a *Lagado prime* (named after the earthbound metropolis of the Kingdom) if it has exactly two good divisors. Hence

$$37 = 1 \cdot (1 + 12 \cdot 3) \quad \text{and} \quad 73 = 1 \cdot (1 + 24 \cdot 3)$$

are Lagado primes.

An *absolute pseudoprime*, n, is a composite number such that for each integer a the number $a^n - a$ is divisible by n [6]. Such a number is $n = 37 \cdot 13 \cdot 61$.

37 is the norm of the Gaussian primes $1 + 6i$, $1 - 6i$, and their associates. 73 is the norm of the Gaussian primes $3 + 8i$, $3 - 8i$, and their associates.

Prime loops.

A *prime loop* is formed when primes are placed around a circle in such an order that each pair of adjacent primes concatenated forms a prime. Thus (37, 67) is a prime loop since 3767 and 6737 are both prime. Prime loops containing from two to ten two-digit primes exist with 37 as an element. For example:

(37, 79) (37, 19, 13, 61, 97, 43)

(37, 97, 43) (37, 19, 13, 67, 61, 97, 43)

(37, 79, 19, 31) (37, 79, 19, 13, 67, 61, 97, 43)

(37, 61, 43, 97, 67) (37, 79, 19, 13, 61, 97, 43, 73, 31)

(37, 61, 97, 43, 73, 31, 19, 13, 67, 79)

The last loop is one of eleven prime loops each of which contains all ten two-digit primes of the form $6k + 1$ once each [10].

37 expressed by primes less than 37.

$37 = 2 + 3 + 13 + 19 = 2 + 5 + 7 + 23 = 2 + 5 + 11 + 19$
$= 2 + 5 + 13 + 17 = 2 + 7 + 11 + 17 = 3 + 5 + 29$
$= 3 + 11 + 23 = 5 + 13 + 19 = 7 + 11 + 19 = 7 + 13 + 17.$

$37 = 2\cdot3 + 31 = 2\cdot5 + 3 + 7 + 17 = 2\cdot7 + 23 = 2\cdot11 + 3 + 5 + 7$
$= 2\cdot13 + 11 = 2\cdot17 + 3 = 2\cdot19 - 3 - 5 + 7$
$= 2\cdot23 + 3 + 5 - 17 = 2\cdot29 - 3 - 5 - 13 = 2\cdot31 - 5 - 7 - 13.$

$37 = 3\cdot5 + 2 + 7 + 13 = 3\cdot7 + 2(19-11) = 3\cdot11 - 19 + 23$
$= 3\cdot13 + 17 - 19 = 3\cdot17 - 2\cdot7 = 3\cdot19 - 7 - 13$
$= 3\cdot23 - 13 - 19 = 3\cdot29 - 19 - 31 = 3\cdot31 - 7(3+5).$

$37 = 2 + 5\cdot7 = 5\cdot11 + 13 - 31 = 5\cdot13 - 11 - 17$
$= 5\cdot17 - 19 - 29 = 5\cdot19 - 2\cdot29 = 5\cdot23 - 7 - 11 - 29 - 31$
$= 5\cdot29 - 7\cdot13 - 17 = 5\cdot31 - 2\cdot3\cdot11 - 23 - 29.$

For other representations of 37 by other primes, see [8].

37 in prime arithmetic progressions.

The only three arithmetic progressions of positive prime numbers containing 37 in which 37 is not the leading term are

31, 37, 43; 13, 37, 61; and 7, 37, 67, 97, 127, 157,

the last of which contains six primes. The progression 37, 73, 109 contains both 37 and its reverse.

In progressions of prime numbers greater than 3, all common differences, d, are multiples of 6. For $30 < d < 2310$, no progression of primes starting with 37 has more than five members. The progression for $d = 2310$ is

37, 2347, 4657, 6967, 9277, 11587.

Within the stated interval, the *thirty-seven* three-prime progressions have d = 36, 60, 72, 102, 156, 186, 270, 312, 336, 396, 450, 486, 606, 696, 786, 792, 816, 996, 1026, 1116, 1176, 1242, 1260, 1410, 1506, 1512, 1572, 1590, 1716, 1740, 1746, 1752, 1836, 1992, 2046, 2202, and 2256.

The sixteen four-prime progressions in the interval have d = 120, 192, 360, 420, 540, 582, 672, 882, 1050, 1092, 1200, 1542, 1632, 1662, 1710, and 1830.

The two five-prime progressions in the interval have d = 870 and 1290.

37, a prime-generated prime.

37 is the fifth term of the sequence of primes of the form $2p + 3$, where p is prime. That is,

7, 13, 17, 29, 37, 41, 61, 89, ...,

wherein 7 + 13 + 17 = 37 and (13 + 61)/2 = 37. In the expression $2 \cdot 37 + 3 = 7 \cdot 11$, each integer is prime.

37 is the fifth term of the sequence of primes of the form $3p - 2$, where p is prime. That is,

7, 13, 19, 31, 37, 67, 109, ...,

wherein -7 + 13 + 31 = 37 = (7 + 67)/2, and $3 \cdot 37 - 2 = 109$.

37 is the third term of the sequence of primes of the form $3p + 4$, where p is prime. That is,

13, 19, 37, 43, 61, 73, 97, ...,

wherein (13 + 61)/2 = 37, and a pair of reversal primes, 37 and 73, appears.

37 is the second term of the sequence of primes of the form $6p + 7$, where p is prime. That is,

19, 37, 73, 109, 181, 193, 229, ...,

wherein $229 = 7 + 3 \cdot 37 \cdot 2$, and again the reversal primes 37 and 73 appear, this time in adjacent positions.

37 and its reverse are the first two terms of the sequence of primes of the forms $9p + 10$ and $12p + 13$, where p is prime. That is,

37, 73, 109, 127, 163, 181, 271, ...

and

37, 73, 97, 241, 457, ...

We observe that $9 \cdot 37 + 10 = 7^3$ and $12 \cdot 37 + 13 = 457$.

37 in other prime sequences.

37 and its reverse 73 occupy the fifth and ninth positions in the sequence of primes of the form $4n + 1$, thus:

5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, ...

They occupy the same positions in the sequence of primes of the form $6n + 1$, thus:

7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, ...

In the sequence of primes of the form $9n + 1$, the reverses are adjacent, thus:

19, 37, 73, 109, ...

Sequences of primes of other forms in which 37, but not 73, appears are:

7, 17, 37, 47, 67, 97, 107, ..., $5n+2$;
23, 37, 79, 107, ..., $7n+2$;
37, 59, 103, ..., $11n+4$;
37, 113, 151, ..., $19n-1$.

There are 19 primes less than 10^4 of the form $n^2 + 1$, namely: 2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, 2917, 3137, 4357, 5477, 7057, 8101, and 8837. Therein, $(3+7)^2 + 1 = 101$ and $(2\cdot 37)^2 + 1 = 5477$. These two primes have digital roots of 2 and 5, which are members of the sequence. Furthermore,

$(2\cdot 5 + 101)/37 = 3$, while $(2 + 257)/37 = 7$.
$37 = 3137 - 2917 - 197 + 17 - 5 + 2$,
$37^2 = 8837 - 7057 - 401 - 17 + 5 + 2$, and
$37^3 = 5(8837 + 1297) - 17$.

37 is the sequence starter in the prime positional sequence

37 157 919 7193 72727 919913

wherein for any term n the next term is the nth prime [16].

Primitive roots of primes.
g is a primitive root of p if $g^{p-1} \equiv 1 \pmod{p}$ and $g^x \not\equiv 1 \pmod{p}$ for $0 < x < p-1$.
The twelve primitive roots of 37 are [5]

2 5 13 15 17 18 19 20 22 24 32 35.

Four of the roots are consecutive integers. Roots equidistant from the middle of the sequence sum to 37. Each pair of equidistant roots except (15, 22) contains one prime. Six of the roots are even and six are odd. There are 5 primes, four multiples of 5, and one 5th power. There are 5 roots relatively prime to 37-1, and 6 roots do not exceed (37-1)/2.

37 is a primitive root of the following fifty-eight primes less than 10^3 [5]: 59, 79, 97, 109, 113, 131, 167, 179, 193, 227, 239, 241, 251, 257, 283, 311, 313, 331, 347, 353, 383, 389, 401, 409, 431, 439, 463, 467, 479, 503, 523, 541, 547, 557, 563, 569, 587, 643, 647, 653, 661, 701, 709, 727, 757, 769, 797, 809, 827, 829, 857, 859, 907, 911, 919, 947, 967, 977. Eight of these primes are palindromes. Those of the forms $4k - 1$ and $4k + 1$ are in the ratio 33:25; those of the form $6k - 1$ and $6k + 1$ are in the ratio 34:24.

Miscellanea.

In the sequence resulting from the application of the Collatz algorithm (if it is odd, triple it and add 1; if it is even, divide it by 2) starting with 37, all odd terms save the final 1 are prime. Thus: 37, 112, 56, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20 10, 5, 16, 8, 4, 2, 1. Indeed, 5, 7, 11, 13, and 17 are consecutive primes.

In the following sequence, the first 37 positive integers have been placed so that the sum of each pair of adjacent numbers is a prime: 37 - 36 - 31 - 12 - 19 - 28 - 25 - 6 - 11 - 8 - 23 - 14 - 33 - 4 - 7 - 30 - 29 - 18 - 5 - 24 - 13 - 10 - 3 - 16 - 1 - 2 - 17 - 20 - 21 - 22 - 15 - 26 - 35 - 32 - 9 - 34 - 27. In the following addition of the thirty-six prime sums, the frequencies of occurrence (other than 1) of the prime sums precede the parentheses: (3) + (11) + (13) + 2(17) + 3(19) + 2(23) + (29) + 3(31) + 6(37) + 3(41) + 3(43) + 3(47) + (53) + (59) + 2(61) + 2(67) + (73) = 1342, and (1+2) + (3+4) = 3 + 7. The largest prime sum is the reverse of 37. As a sum, 37 occurs twice as often as the next most frequently occurring prime.

Euclid used the expression

$$Q(p) = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot\ldots\cdot p + 1,$$

where p is a prime, to demonstrate the infinitude of primes. A related expression which can be used for the same purpose is

$$R(p) = 2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot\ldots\cdot p - 1.$$

It turns out that

$$Q(37) = 181\cdot60611\cdot676421,$$

but no value of $Q(p)$ is divisible by 37. In contrast,

$$R(23) = 37\cdot131\cdot46027.$$

The smallest $R(p)$ that contains four distinct prime factors is [3]

$$R(37) = 229\cdot541\cdot1549\cdot38669.$$

Curiously enough, the sum of the digits of the four factors of $R(37)$ is $2\cdot37$.

There are exactly 37 palindromic primes less than 10^7 composed of prime digits [9]. The twelve in which 37 is imbedded are: 373, 37273, 37573, 73237, 77377, 3337333, 3722273, 3732373, 3773773, 7327237, 7352537, and 7733377.

According to Samuel Yates [15], the prime divisors of the repunit $R_{37} = (10^{37}-1)/9$ are

$$2028119\cdot247629013\cdot2212394296770203368013.$$

Every repunit consisting of $3n$ ones is divisible by 37, since $111 = 3\cdot37$.

The smallest solutions of the equation $x^2 + 3y^2 = p$, where p is a prime, are:

$(p, x, y) = (7, 2, 1), (13, 1, 2), (19, 4, 1), (31, 2, 3)$, and $(37, 5, 2)$.

The smallest solutions of the equation $x^2 + 4y^2 = p$, where p is a prime, are:

$(p, x, y) = (5, 1, 1), (13, 3, 1), (17, 1, 2), (29, 5, 1)$, and $(37, 1, 3)$.

The smallest solutions of the equation $x^3 - y^3 = p$, where p is a prime, are:

$$(p, x, y) = (7, 2, 1), (19, 3, 2), \text{ and } (37, 4, 3).$$

Curiosa.

Each of the prime digits of 37 is generally considered to be a lucky number.

37 = 7 + 3 + 7 + 3 + 7 + 3 + 7, a smoothly undulating sequence including 3 threes amongst its 7 terms.

(3 + 7)/2 = 5 involves the first four primes.

37 + 3 + 7 = 47 and 37 - 3 + 7 = 41. Both of these sums are primes with the prime digit sums 11 and 5, and have the prime digital roots 2 and 5 differing by 3 and summing to 7.

37 - 3 - 7 = 27 and 37 + 3 - 7 = 33. Both of these sums are composite, and have composite digital roots, 9 and 6, differing by 3.

The related primes 373 and 3773773 have the prime digit sums 13 and 37, respectively.

37 is the twelfth prime, its reverse 73 is the twenty-first prime, and 12 is the reverse of 21. Furthermore,

$$\frac{73 - 37}{21 - 12} = 2^2 = 7 - 3, \quad \text{and} \quad 3 \cdot 7 = 21.$$

The only set of four distinct two-digit primes composed of a pair of twin primes and their reverses is 71, 73, 17, and 37. Indeed, 17 and 37 belong to each other [11] in the sense that

$$(1 - 7)^2 + 1 = 37 \quad \text{and} \quad (3 - 7)^2 + 1 = 17.$$

Their respective digital roots are the cubes 1 and 8, which differ by 7 whereas their cube roots sum to 3.

37 also belongs to the prime 101, in the sense that $(3 + 7)^2 + 1 = 101$. [11]

The sum of the squares of the first 7 primes is

$$2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 2 \cdot 37 \cdot 3^2 = 666,$$

the number of the beast.

Each of Emma Lehmer's twin primes, $9 \cdot 2^{211} \pm 1 =$

$$2961908203\;1781708758\;78944\underline{28}60\;3230908968\;9245581404\;4\underline{37}5402749\;62432 \pm 1,$$

contains 37 even and 28 odd digits [4]. The sum of the even digits is 2·73. Both 37 and 28 are imbedded in the primes.

Among the 37^n, $n \leq 37$, there are 11 powers with prime digit sums, Σ, namely: (n, Σ) = (2, 19), (3, 19), (12, 73), (14, 109), (15, 109), (18, 127), (21, 163), (25, 127), (29, 199), (30, 181), and (37, 271). [12]

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THE OLYMPIAD CORNER: 54

M.S. KLAMKIN

I present two new problem sets this month. The first consists of the problems set at the second round of the 1983 Chinese Mathematics Olympiad. I am grateful to Gengzhe Chang (University of Science and Technology of China at Hefei) for supplying these problems in English translation. Next I give, through the courtesy of Walther Janous (Ursulinengymnasium at Innsbruck), the problems of the 1982 Austrian-Polish Mathematics Competition.

*

1983 CHINESE MATHEMATICS OLYMPIAD (Second Round)

October 16, 1983 - Time: 2 hours

1. (8 points) Show that

$$\mathrm{Arcsin}\, x + \mathrm{Arccos}\, x = \frac{\pi}{2}, \qquad -1 \le x \le 1.$$

2. (12 points) A function f defined on the interval $[0,1]$ satisfies

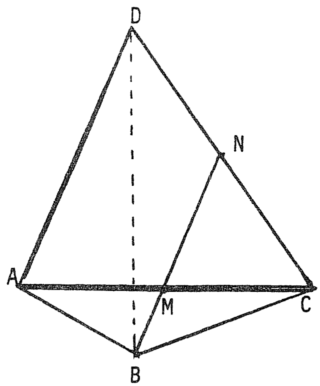
$$f(0) = f(1) \text{ and } 0 \le x_1 \ne x_2 \le 1 \implies |f(x_2)-f(x_1)| < |x_2-x_1|.$$

Prove that $0 \le x_1, x_2 \le 1 \implies |f(x_2)-f(x_1)| < \frac{1}{2}$.

3. (16 points) For the quadrilateral ABCD of the figure, the following proportion holds:

$$[ABD] : [BCD] : [ABC] = 3 : 4 : 1,$$

where the brackets denote area. If the points $M \in AC$ and $N \in CD$ are such that $AM : AC = CN : CD$ and B,M,N are collinear, prove that M and N are the midpoints of AC and CD, respectively.



4. (16 points) Determine the maximum volume of a tetrahedron whose six edges have lengths 2, 3, 3, 4, 5, and 5.

5. (18 points) Determine

$$\min_{A,B}\ \max_{0\le x\le 3\pi/2} |\cos^2 x + 2\sin x\cos x - \sin^2 x + Ax + B|.$$

*

AUSTRIAN-POLISH MATHEMATICS COMPETITION

1st day: July 9, 1982 - Time: $4\frac{1}{2}$ hours

1. Determine all pairs of natural numbers (n,k) such that

$$\text{g.c.d.}((n+1)^k - n,\ (n+1)^{k+3} - n) > 1.$$

2. We are given a unit circle C with center M and a closed convex region R in the interior of C. From every point P of circle C, there are two tangents to the boundary of R that are inclined to each other at 60°. Prove that R is a closed circular disk with center M and radius $\frac{1}{2}$.

3. Prove that, for all natural numbers $n \geq 2$,

$$\prod_{i=1}^{n} \tan\left\{\frac{\pi}{3}\left(1 + \frac{3^i}{3^n - 1}\right)\right\} = \prod_{i=1}^{n} \cot\left\{\frac{\pi}{3}\left(1 - \frac{3^i}{3^n - 1}\right)\right\}.$$

2nd day: July 10, 1982 - Time: 4½ hours

4. N being the set of natural numbers, for every $n \in N$ let $P(n)$ denote the product of all the digits of n (in base ten). Determine whether or not the sequence $\{x_k\}$, where

$$x_1 \in N, \qquad x_{k+1} = x_k + P(x_k), \qquad k = 1,2,3,\ldots,$$

can be unbounded (i.e., for every number M, there exists an x_j such that $x_j > M$).

5. Let $\{A,B\}$ be a partition of the unit interval $I = [0,1]$ (so that $A \cup B = I$ and $A \cap B = \emptyset$), and, for any real number x, let

$$\{x\} + A = \{y \mid y = x + a,\ a \in A\}.$$

Prove that there is no real number x such that $B = \{x\} + A$.

6. Let a be a fixed natural number. Find all functions f defined on the set D of natural numbers $x \geq a$ and satisfying the functional equation

$$f(x+y) = f(x) \cdot f(y)$$

for all $x,y \in D$.

3rd day: July 11, 1982 - Time: 4 hours

(Team Competition)

(Here each team, usually consisting of six students, solves the three problems collectively and submits at most one solution per problem.)

1. Determine all triplets (x,y,z) of natural numbers, with z as small as possible, for which there exist natural numbers a,b,c,d satisfying

(i) $x^y = a^b = c^d$ with $x > a > c$,

(ii) $z = ab = cd$,

(iii) $x + y = a + b$.

2. Point X is in the interior of a given regular tetrahedron ABCD of edge

length 1. If $d(X,YZ)$ denotes the shortest distance from X to edge YZ, prove that

$$d(X,AB) + d(X,AC) + d(X,AD) + d(X,BC) + d(X,BD) + d(X,CD) \geq \frac{3}{\sqrt{2}},$$

and show that equality holds if and only if X is the center of the tetrahedron.

3. Let

$$S_n = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\sqrt{j^2+k^2}}.$$

Determine a real constant c such that, for all natural numbers $n \geq 3$,

$$n \leq S_n \leq cn.$$

Remark. The smaller the number c determined, the greater will be the number of points awarded for the solution.

*

I now present solutions to some problems published earlier in this column.

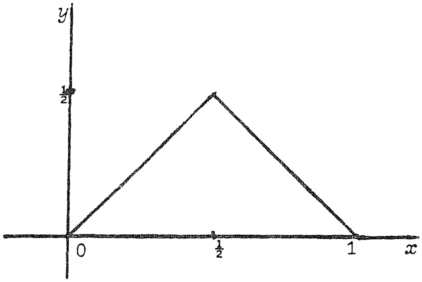
12. [1981: 74] *From a 1979 Moscow Olympiad (for Grade 10).*

The function f is defined on the interval $[0,1]$ and is twice differentiable at each point. The absolute value of the derivative is never greater than 1 on the interval, and $f(0) = f(1) = 0$. What is the greatest possible maximum value that $f(x)$ can have on the interval?

Solution by M.S.K.

The given function f is differentiable, hence continuous, on the interval $[0,1]$, so it has an absolute maximum at some point x_0, and $f'(x_0) = 0$ if x_0 is an interior point of the interval. We compare f with the function g defined by

$$g(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$



whose graph is shown in the figure. Since $f(0) = f(1) = 0$, no point of the graph of f can lie above the graph of g. For otherwise, geometrically or by the mean value theorem, $|f'(x)|$ would exceed 1 at some point. Therefore $f(x_0) \leq g(\frac{1}{2}) = \frac{1}{2}$.

We show that in fact $f(x_0) < \frac{1}{2}$. Suppose on the contrary that $f(x_0) = \frac{1}{2}$. Then we must have $x_0 = \frac{1}{2}$ and $f'(\frac{1}{2}) = 0$. Now, for $0 < h \leq \frac{1}{2}$,

$$1 = \frac{\frac{1}{2} - (\frac{1}{2}-h)}{h} = \frac{\frac{1}{2} - g(\frac{1}{2}-h)}{h} \leq \frac{f(\frac{1}{2}) - f(\frac{1}{2}-h)}{h},$$

and taking the limit as $h \to 0$ shows that $f'_-(\frac{1}{2}) \geq 1$, contradicting $f'(\frac{1}{2}) = 0$. Therefore $f(x_0) < \frac{1}{2}$.

So far we have assumed that f is only (once) differentiable on the interval $[0,1]$. Assuming that f is twice differentiable on the interval provides no new information; nothing, in particular, about the supremum of $f(x_0)$ as f ranges over all suitable functions. However, if we assume differentiability of a sufficiently high order, it is possible to find a function f which otherwise satisfies the conditions of the problem and for which $f(\frac{1}{2})$ is as close to $\frac{1}{2}$ as we please. Consider, for example, the following function f (obtained by smoothing out function g in the neighborhood of $x = \frac{1}{2}$):

$$f(x) = \begin{cases} x - \dfrac{(2x)^r}{2r}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\[2ex] (1-x) - \dfrac{(2(1-x))^r}{2r}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

This function satisfies $f(0) = f(1) = 0$, $|f'(x)| \leq 1$, and it is n times differentiable on the interval $[0,1]$ if $r > n$. Its absolute maximum is

$$f(\tfrac{1}{2}) = \frac{1}{2} - \frac{1}{2r},$$

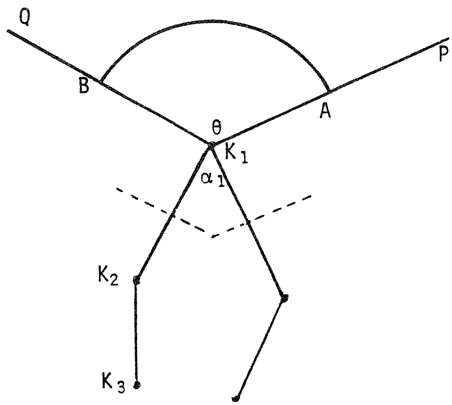
which can be made as close to $\frac{1}{2}$ as we please by taking r large enough.

2. [1982: 237] *From a 1982 Bulgarian Olympiad.*

In a plane there are n circles each of unit radius. Prove that at least one of these circles contains an arc which does not intersect any of the other circles and whose length is not less than $2\pi/n$.

Solution by George Tsintsifas, Thessaloniki, Greece.

Let $K_1, K_2, \ldots, K_m$ be the consecutive vertices and $\alpha_1, \alpha_2, \ldots, \alpha_m$ the corresponding interior angles of the convex hull of the centers of the n given circles, so that $m \leq n$. We assume that these have been labeled so that $\min \alpha_i = \alpha_1$. As a consequence, we have



$$\alpha_1 \leq \frac{(m-2)\pi}{m}. \qquad (1)$$

As shown in the figure, the perpendiculars K_1P and K_1Q to the sides of angle α_1 form an angle θ such that

$$\theta + \alpha_1 = \pi, \tag{2}$$

and it follows from (1) and (2) that

$$\theta \geq \frac{2\pi}{m} \geq \frac{2\pi}{n}.$$

Therefore K_1P and K_1Q intersect circle (K_1) in an arc AB whose length is not less than $2\pi/n$.

For $1 < i \leq n$, no circle (K_i) has a point in common with arc AB. For the points of intersection of circles K_1 and K_i, if any, lie on the perpendicular bisector of segment K_1K_i, and this perpendicular bisector cannot intersect the interior of angle AK_1B.

2. [1982: 269] *From a 1963 Peking Mathematics Contest for Grade 12.*

Nine points are randomly selected inside a square of side 1. Show that three of the points are the vertices of a triangle of area at most 1/8.

Solution by M.S.K.

Partition the square into four squares of side $\frac{1}{2}$. By the pigeonhole principle, at least one of the four squares contains three of the given points, and they are the vertices of a triangle of area at most 1/8.

Rider. Show that the maximum area of a triangle inscribed in a rectangle is half the area of the rectangle.

*

3. [1982: 269] *From a 1963 Peking Mathematics Contest for Grade 12.*

Given are $2n+3$ points in the plane, no three collinear and no four concyclic. Is it possible to construct a circle passing through three of the points so that exactly half of the remaining $2n$ points lie inside the circle? Justify your answer.

Comment by M.S.K.

The answer is yes. For two solutions, see Ross Honsberger, *Mathematical Morsels*, Mathematical Association of America, Washington, D.C., 1978, pp. 48-51.

*

4. [1982: 269] *From a 1963 Peking Mathematics Contest for Grade 12.*

A set of 2^n objects is partitioned into a number of subsets. A move consists of transferring from one of the subsets to an equal or smaller subset a number of objects equal to the cardinality of the second subset. Prove that, irrespective of the initial partition, all the subsets can be combined into a single set by a finite number of moves.

Solution by Willie Yong, Singapore.

We use induction on n. The conclusion is immediate if $n = 1$. Suppose it holds for some $n \geq 1$. In any partition of a set of 2^{n+1} objects, the number of subsets of odd cardinality is even. Any move involving two subsets of odd cardinality results in two subsets of even cardinality (one being possibly the empty subset, which is of even cardinality). Thus, after a finite number of moves, it is possible to have all the subsets of even cardinality. We now imagine that in every subset the objects are glued together in pairs. As a result, we now have a total of 2^n objects, and the desired conclusion follows from the induction assumption.

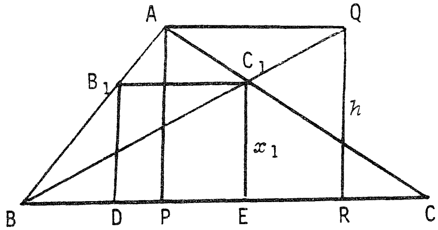
*

1. [1982: 270] *From a 1964 Peking Mathematics Contest for Grade 12.*

ABC is a triangle in which angle A is nonacute. Let B_1DEC_1 be any inscribed square of triangle ABC with B_1 on AB, C_1 on AC, and DE on BC. Let $B_2D_1E_1C_2$ be any inscribed square of triangle AB_1C_1 with B_2 on AB_1, C_2 on AC_1, and D_1E_1 on B_1C_1. The process is continued in the same way for a finite number of steps. Prove that the sum of the areas of all the inscribed squares is strictly less than half the area of triangle ABC.

Solution by M.S.K.

We first construct square APRQ where AP = h is the altitude from A (see figure). The squares B_1DEC_1 and APRQ are homothetic, with B the homothetic center. Let BP = a_1, PC = a_2, so that BC = a = $a_1 + a_2$, and let PE = y_1, $EC_1 = x_1$. By similar triangles, we have



$$\frac{x_1}{h} = \frac{a_1+y_1}{a_1+h} = \frac{a_2-y_1}{a_2},$$

from which follows

$$y_1 = \frac{ha_2}{a+h} \qquad \text{and} \qquad x_1 = \frac{ah}{a+h}.$$

Since triangle AB_1C_1 with inscribed square $B_2D_1E_1C_2$ of side x_2 is homothetic to triangle ABC with inscribed square B_1DEC_1 of side x_1, we have

$$\frac{x_1}{BC} = \frac{x_2}{B_1C_1} \qquad \text{or} \qquad x_2 = \frac{x_1^2}{a}.$$

A simple induction now shows that, if the process in performed n times, then

$$x_n = \frac{x_1^n}{a^{n-1}}, \qquad n = 1,2,3,\ldots .$$

We now show that the desired conclusion holds even if the process is performed infinitely often. The sum of the areas of all the inscribed squares is

$$\sum_{n=1}^{\infty} x_n^2 = \frac{x_1^2}{1-\frac{x_1^2}{a^2}} = \frac{h^2a^2}{a^2+2ah} < \frac{ha}{2},$$

since $2h^2a^2 < 2h^2a^2+ha^3$.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

931. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Replace the letters with digits to obtain a decimal addition:

$$\begin{array}{r} \text{PASTE} \\ \underline{\text{STAPLE}}. \\ \text{FASTEN} \end{array}$$

Then PASTE, STAPLE, or FASTEN somehow your solution to a postcard and send it along to the editor.

932. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $p \equiv 7 \pmod 8$ be a prime so there is an integer w such that

$$w^2 \equiv 2 \pmod p.$$

Prove that the value of the Legendre symbol $\left(\frac{2+w}{p}\right)$ is given by

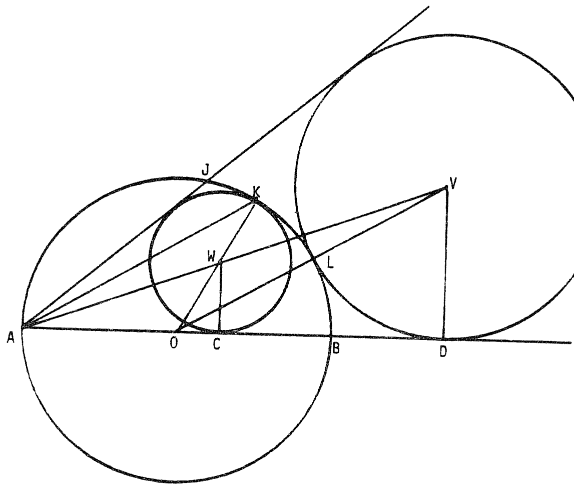
$$\left(\frac{2+w}{p}\right) = \begin{cases} +1, & \text{if } p \equiv 15 \pmod{16}, \\ -1, & \text{if } p \equiv 7 \pmod{16}. \end{cases}$$

933. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

A truncated triangular pyramid with base ABC and lateral edges AA_1, BB_1, and CC_1 is given. Let P be an arbitrary point in the plane of triangle $A_1B_1C_1$, and let Q be the common point of the planes α, β, and γ passing respectively through B_1C_1, C_1A_1, and A_1B_1, and parallel respectively to the planes PBC, PCA, and PAB. Prove that the tetrahedron ABCQ and the pyramid ABB_1A_1C have equal volumes.

934. *Proposed by Leon Bankoff, Los Angeles, California.*

As shown in the figure, the diameter AB, a variable chord AJ, and the intercepted minor arc JB of a circle (O) form a mixtilinear triangle whose inscribed circle (W) touches arc JB in K and whose mixtilinear excircle (V) touches arc JB in L. The projections of W and V upon AB are C and D, respectively. As J moves along the circumference of circle (O), the ratio of the arcs KL and LB varies.



(a) When arcs KL and LB are equal, what are their values?

(b) Show that BD is equal to the side of the inscribed square lying in the right angle of triangle ADV.

935. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

It is easy to show that (0,0) and (9,9) are the only solutions (x,y) of the equation

$$x^3 + y^3 = 18xy$$

in which $x = y$. Find a solution in integers (x,y) with $x > y$ and show that it is unique.

936. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find all eight-digit palindromes in base ten that are also palindromes in at least one of the bases two, three, ..., nine.

937. *Proposed by Jordi Dou, Barcelona, Spain.*

ABCD is a trapezoid inscribed in a circle ϕ, with AB $\|$ DC. The midpoint of AB is M, and the line DM meets the circle again in P. A line l through P meets line BC in A', line CA in B', line AB in C', and the circle again in F'.

Prove that (A'B', C'F') is a harmonic range.

938. *Proposed by Charles W. Trigg, San Diego, California.*

Is there an infinity of pronic numbers of the form $a_n b_n$ in the decimal system? (A *pronic number* is the product of two consecutive integers. The symbol x_n indicates x repeated n times. For example, $5_3 2_3 = 555222$.)

939. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Triangle ABC is acute-angled at B, and AB < AC. M being a point on the altitude AD, the lines BM and CM intersect AC and AB, respectively, in B' and C'. Prove that BB' < CC'.

940. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Show that, for any triangle ABC,

$$\sin B \sin C + \sin C \sin A + \sin A \sin B \le \frac{7}{4} + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \le \frac{9}{4}.$$

* * *

ADVENT OF THE BREAKFAST SERIAL

I was just going to say, when I was interrupted, that one of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economical and practical wisdom is an extension or variation of the following arithmetical formula: $2 + 2 = 4$. Every philosophical proposition has the more general character of the expression $a + b = c$. We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures.

The above is the opening paragraph in the first of a series of articles by Oliver Wendell Holmes (like A. Conan Doyle a medical doctor turned author, not to be confused with his equally famous son, Justice Oliver Wendell Holmes of the U.S. Supreme Court) in the early issues (1857) of a magazine that he himself christened *The Atlantic Monthly*. The articles were published under the running title "The Autocrat of the Breakfast-Table" and later collected in a book of the same name. A. Conan Doyle so admired the book that he named Sherlock Holmes after its American author.

* * *

LETTER TO EDITH ORR

Re your comment [1983: 211]: Better dat Mary shoulda spent a sheepless night.

G.S., Brooklyn, N.Y.

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

801. [1983: 21] *Proposed by Sidney Kravitz, Dover, New Jersey.*

At this time (late January 1983), Canada's peripatetic Prime Minister, the apostle of the North-South dialogue, is catching his breath at home. So let us lose no time in proposing the following alphametic before he takes off again (or bows out):

$$\begin{array}{r} \text{PIERRE} \\ \underline{\text{ELLIOTT}} \\ \text{TRUDEAU} \end{array}.$$

Solutions were received from CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

The unique answer, obtained by all solvers, is

$$\begin{array}{r} 246116 \\ \underline{6884577} \\ 7130693 \end{array}.$$

Reaching this answer required a lot of brute force on the part of the unaided solver, so the problem was more suitable as a programming exercise. Most solvers submitted only the answer: they apparently just twiddled their thumbs while they waited for a computer to burp out the answer. Apart from the proposer, who was in honour bound to prove the uniqueness of the answer (to get his problem published), only Kenneth M. Wilke, of Topeka, Kansas, was unfazed by the difficulties and submitted a full-blown solution, too long to be reproduced here (motto of Kansas: *ad astra per aspera*).

With the exception of a few months in 1980 when Joe Clark, that hiccup of history, was Prime Minister, Pierre-Elliott Trudeau has been Prime Minister of Canada for the last 16 years. He had announced his retirement four years ago, then changed his mind. Our prescient proposer had raised the possibility that Trudeau would bow out again soon. On February 29, 1984, Trudeau announced his coming retirement. So this is the Second Leaving. If Trudeau does not change his mind again, his successor as head of the Liberal Party and Prime Minister of Canada (and possible subject of a future alphametic) will be chosen in mid-June of 1984 at a Liberal Party convention in Ottawa, Ontario (not Ottawa, Kansas, the birthplace of Senator Gary Hart, another Kansan who is reaching *ad astra per aspera*). We have enjoyed watching Trudeau pirouetting his way across the world's stage, but now *acta est fabula*. Unless...

802. [1983: 21] *Proposed by Joseph Gillis, Weizmann Institute of Science, Rehovot, Israel.*

Points P_1 and P_2 move with equal angular velocities along circles C_1 and C_2, respectively, in the plane. Prove that in general there is one point S such that $|SP_1|^2 - |SP_2|^2$ is constant (where $|SP|$ denotes the distance between S and P), and discuss the exceptional cases.

(This problem generalizes Problem 3 of the 1979 International Mathematical Olympiad [1979: 194].)

I. *Solution by Jan van de Craats, Leiden University, The Netherlands.*

We take C_1 to be the unit circle in the complex plane and t the affix of the moving point P_1 (so that $t\bar{t} = 1$). We further assume that P_2, with affix z, moves on a circle C_2 with centre m, and that $z = a$ when $t = 1$. Two cases will be considered:

Case 1: If P_1 and P_2 move in the same sense, then (see Figure 1)

$$z = m + t(a-m). \qquad (1)$$

Case 2: If P_1 and P_2 move in opposite senses, then (see Figure 2)

$$z = m + \bar{t}(a-m). \qquad (2)$$

In both cases, we look for a point S, with affix w, and a (real) constant γ such that

$$(w-t)(\bar{w}-\bar{t}) = (w-z)(\bar{w}-\bar{z}) + \gamma \qquad (3)$$

for all t such that $t\bar{t} = 1$. (Of course, if $a = 1$, then $\gamma = 0$. Case 1 is then the original Olympiad problem.) We now examine the two cases separately.

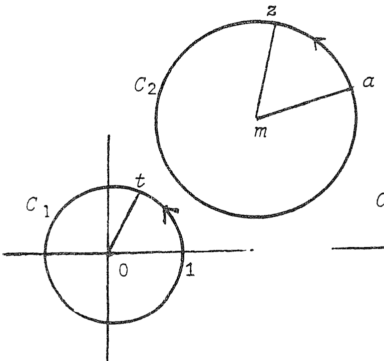


Figure 1

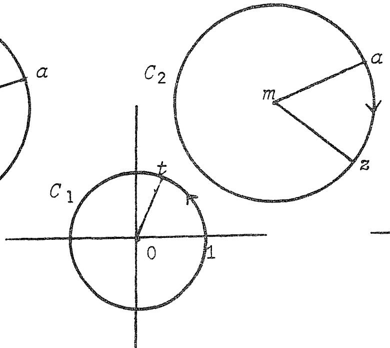


Figure 2

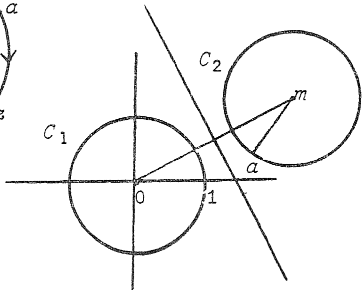


Figure 3

Case 1. Upon substituting (1) into (3), we get a quadratic polynomial in t which vanishes for all t on the unit circle, so its coefficients are all zero:

$$\bar{w} - (\bar{w}-\bar{m})(a-m) = 0, \tag{4}$$

$$w - (w-m)(\bar{a}-\bar{m}) = 0, \tag{5}$$

$$-w\bar{w} - 1 + (w-m)(\bar{w}-\bar{m}) + (a-m)(\bar{a}-\bar{m}) + \gamma = 0. \tag{6}$$

If $a-m \neq 1$, then there is only one point w given by (4) or (5), and then γ follows from (6).

If $a-m = 1$, then only $m = 0$ gives solutions, and circles C_1 and C_2 then coincide. But when the circles coincide there is one solution ($w = 0$) if $a \neq 1$ and infinitely many solutions (all complex w) if $a = 1$. In both cases $\gamma = 0$.

Case 2. Substituting (2) into (3) and proceeding as in Case 1, we obtain

$$\bar{w} - (w-m)(\bar{a}-\bar{m}) = 0, \tag{7}$$

$$w - (\bar{w}-\bar{m})(a-m) = 0, \tag{8}$$

and, again, equation (6). Substituting $\bar{w}$ from (7) into (8), we get

$$w\big((a-m)(\bar{a}-\bar{m}) - 1\big) = (a-m)\big(\bar{m} + m(\bar{a}-\bar{m})\big). \tag{9}$$

If $(a-m)(\bar{a}-\bar{m}) \neq 1$, there is only one point w given by (9), and then γ follows again from (6).

If $(a-m)(\bar{a}-\bar{m}) = 1$, then from (6) we get

$$(w-m)(\bar{w}-\bar{m}) = w\bar{w} - \gamma, \tag{10}$$

while (9) yields

$$m + \bar{m}(a-m) = 0. \tag{11}$$

If $m = 0$, then (11) is an identity and (10) gives $\gamma = 0$. Now, since $z = a\bar{t}$ from (2) and $a\bar{a} = 1$, equation (3) is equivalent to

$$w\bar{t} + \bar{w}t = w\bar{a}t + \bar{w}a\bar{t},$$

or, finally, to $w = a\bar{w}$. This represents a *line* on the origin, and a is the reflection of 1 in this line. If $m \neq 0$, then (11) is equivalent to

$$a = m - \frac{m}{\bar{m}}. \tag{12}$$

Unless (12) holds, the problem has no solution. If (12) holds, then, for *any* choice of γ, (10) represents a *line* of solutions perpendicular to the line joining 0 and m. In particular, if $\gamma = 0$, then m and a are the reflections of 0 and 1, respectively, in this line (see Figure 3).

II. *Comment by O. Bottema, Delft, The Netherlands; and J.T. Groenman, Arnhem, The Netherlands.*

Two solutions to this problem have appeared in *Nieuw Tijdschrift voor Wiskunde*: one by E.C. Buissant des Amorie [67 (March 1980) 188-190] and one by O. Bottema [68 (September 1980) 9-11].

Also solved by the COPS of Ottawa; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

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803. [1983: 21] *Proposed by Leroy F. Meyers, The Ohio State University.*

Let f be the operation which takes a positive integer n to $\frac{1}{2}n$ if n is even, and to $3n+1$ if n is odd. It is as yet unknown whether or not every positive integer n can be reduced to 1 by successively applying f to it. (This is the substance of Problem 133 [1976: 67, 144-150, 221].) In comment I (Trigg, p. 144), it is stated that it is unnecessary to carry the sequence N, fN, ffN, $fffN$, ... for N beyond any term less than N. In particular, it is not necessary to test N if N is even or is of the form $4k+1$.

Is this argument justified?

Solution by Charles W. Trigg, San Diego, California.

The argument is justified. The portion of my Comment I [1976: 144-147] mentioned in the proposal came right after I had stated that the process is known to converge to 1 for all values of $N < 10^{40}$. What followed was clearly meant to imply that, in testing numbers $N \geq 10^{40}$ *in order of magnitude*, "it is not necessary to carry the sequence for any N beyond a term $< N$." For if in the sequence of any N a term $M < N$ is encountered, then that sequence need not be carried further because from that point on it will coincide with the sequence for M *which has already been examined.* "Thus it is not necessary to test any even values [for $fN = N/2 < N$], nor [among others] odd values of the form $4k+1$, $16k+3$, or $128k+7$." For if we denote the nth iterate of f by f^n, then we have

$$f^3(4k+1) = 3k+1 < 4k+1, \quad f^6(16k+3) = 9k+2 < 16k+3, \quad f^{11}(128k+7) = 81k+5 < 128k+7.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD I. HESS, Rancho Palos Verdes, California; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; and the proposer.

Editor's comment.

Much historical and up-to-date computational information about the "$3N+1$ problem" (sometimes called the Collatz algorithm, although its actual origin is hard to pin down) can be found in Hayes [1] and Gruenberger [2]. Hayes, in particular, writes:

"...there is no need to follow the path of a number all the way to 1; once the value of N falls below the initial value the candidate can be dismissed."

REFERENCES

1. Brian Hayes, "Computer Recreations", *Scientific American*, January 1984.

2. Fred Gruenberger, "Computer Recreations", *Scientific American*, April 1984.

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804*. [1983: 22] *Proposed by V.N. Murty, Pennsylvania State University.*

Let $P_n(x)$ be a polynomial of degree $n \geq 2$ with real coefficients, leading coefficient $a \neq 0$, and n real zeros x_i with

$$x_1 \leq x_2 \leq \ldots \leq x_n.$$

It is easily verified that

$$\int_{x_1}^{x_2} |P_2(x)|\ dx = \frac{|a|}{6}(x_2-x_1)^3$$

and (more tediously) that

$$\int_{x_1}^{x_3} |P_3(x)|\ dx = \frac{|a|}{12}\{(x_2-x_1)^3(3x_3-\Sigma x_i) + (x_3-x_2)^3(\Sigma x_i-3x_1)\},$$

where the indicated sum is for $i = 1,2,3$.

Find a "nice" compact formula for

$$\int_{x_1}^{x_n} |P_n(x)|\ dx.$$

Editor's comment.

No solution was received for this problem, which therefore remains open. Writing the result for $n = 2$ in the form

$$\int_{x_1}^{x_2} |P_2(x)|\ dx = \frac{|a|}{6}(x_2-x_1)^2(2x_2-\Sigma x_i)$$

and comparing with the result for $n = 3$ shows an emerging pattern, but it is not easy to see how the pattern will develop for $n > 3$. It might help to know the result for $n = 4$.

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805. [1983: 22] *Proposed by M.S. Klamkin, University of Alberta.*

If $x,y,z > 0$, prove that

$$\frac{x+y+z}{3\sqrt{3}} \geq \frac{yz+zx+xy}{\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}+\sqrt{x^2+xy+y^2}},$$

with equality if and only if $x = y = z$.

Solution by Vedula N. Murty, Pennsylvania State University, Capitol Campus; and the proposer (independently).

The inequalities

$$l \equiv \sqrt{y^2+yz+z^2} \ge \frac{\sqrt{3}}{2}(y+z),$$

$$m \equiv \sqrt{z^2+zx+x^2} \ge \frac{\sqrt{3}}{2}(z+x),$$

and

$$n \equiv \sqrt{x^2+xy+y^2} \ge \frac{\sqrt{3}}{2}(x+y)$$

are equivalent to $(y-z)^2 \ge 0$, $(z-x)^2 \ge 0$, and $(x-y)^2 \ge 0$, respectively. Hence

$$\frac{1}{\sqrt{3}} \ge \frac{x+y+z}{l+m+n}. \qquad (1)$$

Also

$$\frac{(x+y+z)^2}{3} \ge yz+zx+xy \qquad (2)$$

is equivalent to $(y-z)^2 + (z-x)^2 + (x-y)^2 \ge 0$. Now, from (1) and (2),

$$\frac{x+y+z}{3\sqrt{3}} \ge \frac{(x+y+z)^2}{3(l+m+n)} \ge \frac{yz+zx+xy}{l+m+n},$$

which establishes and sharpens the proposed inequality. Equality clearly holds throughout if and only if $x = y = z$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer (second solution).

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806. [1983: 22] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, India.*

Let LMN be the cevian triangle of the point S for the triangle ABC (i.e., the lines AS,BS,CS meet BC,CA,AB in L,M,N, respectively). It is trivially true that

S is the centroid of ΔABC $\Longrightarrow$ S is the centroid of ΔLMN.

Prove the converse.

Solution by Howard Eves, University of Maine.

Denote by P the (real or ideal) point of intersection of MN and BC, and by Q the point of intersection of AL and MN. In the complete quadrilateral BCMN-AP, (NM,QP) and (BC,LP) are harmonic ranges. If S is the centroid of ΔLMN, then Q is the midpoint of NM, and P is an ideal point at infinity. It then follows that L must be the midpoint of BC, and AL is a median of ΔABC. Similarly, BM and CN are medians of ΔABC, and S is the centroid of ΔABC.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

Let $\sigma = A_0A_1\ldots A_n$ be a simplex in n-dimensional Euclidean space, and let $\sigma' = A_0'A_1'\ldots A_n'$ be the cevian simplex of the point S for the simplex σ (i.e., for $i = 0,1,\ldots,n$ the line A_iS intersects the $(n-1)$-face opposite A_i in A_i'). Tsintsifas used barycentric coordinates to prove that

$$\text{S is the centroid of } \sigma \implies \text{S is the centroid of } \sigma'.$$

He then simply added that "the converse is proved in the same way".

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807. [1983: 22] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Three balls are taken at random from an urn that contains w white balls and r red balls. The probability that the three balls are white is p. If the urn had contained one more white ball, the probability of three white balls would have been $4p/3$. Find all possible values of the pair (w,r).

I. *Solution by Richard Rhoad, New Trier High School, Winnetka, Illinois.*

We assume that the three balls are taken without replacement and, to eliminate trivial solutions, that $w \geq 3$ (there are infinitely many solutions if $w = 0$ or 1, and none if $w = 2$). From the given data, we have

$$\frac{\binom{w+1}{3}}{\binom{w+r+1}{3}} = \frac{4}{3}\cdot\frac{\binom{w}{3}}{\binom{w+r}{3}}, \tag{1}$$

an equation that is soon found to be equivalent to

$$(11-w)r = (w-2)(w+1),$$

so solutions exist only if $w < 11$. With this restriction, the equation is equivalent to

$$r = -w - 10 + \frac{108}{11-w},$$

and we need test only $w = 10, 9, 8, 7, 5$. Each of these is satisfactory, and we have the solutions

$$(w,r) = (10,88),\ (9,35),\ (8,18),\ (7,10),\ (5,3).$$

II. *Comment by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

The proposer wisely did not specify that the balls are taken without replacement. He didn't have to, because there is no solution with replacement. For then the equation corresponding to (1) is

$$\left(\frac{w+1}{w+r+1}\right)^3 = \frac{4}{3}\left(\frac{w}{w+r}\right)^3.$$

This is equivalent to

$$\frac{w+1}{w+r+1}\cdot\frac{w+r}{w} = \left(\frac{4}{3}\right)^{1/3},$$

with left side rational and right side irrational.

Also solved by SAM BAETHGE, San Antonio, Texas; BARNARD H. BISSINGER, Pennsylvania State University, Capitol Campus (three solutions); W.J. BLUNDON, Memorial University of Newfoundland; RAIZAH BUJANG, Pennsylvania State University, Capitol Campus; S.C. CHAN, Singapore; the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; THOMAS F. HALLEY, student, Pennsylvania State University, Capitol Campus; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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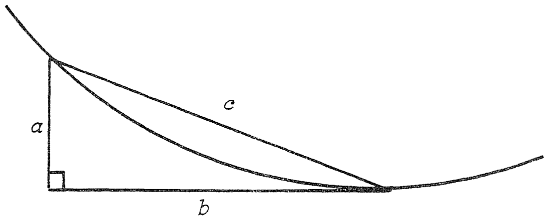
808.* [1983: 22] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Find the length of the largest circular arc contained within the right triangle with sides $a \le b < c$.

Comment by the proposer.

I do not have a definitive solution to this. Surprisingly, the incircle is not necessarily the largest circular arc. As the ratio $b:a$ gets large, a circular arc like the one shown in the figure (tangent to one side of the right angle) gets to be larger than the incircle. If it can be shown that these are the only two cases to consider, then the problem shouldn't be too hard.



One incorrect solution was received.

Editor's comment.

The proposer's comment should be helpful in arriving at a solution to this open problem.

* * *

809. [1983: 23] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let A and B be two angles of a triangle, with opposite sides a and b, respectively. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}(\cos 2nA - \cos 2nB) = \ln\frac{b}{a}.$$

Solution by S.C. Chan, Singapore; Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio; and Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire (independently).

The series summation

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln 2\sin\frac{\theta}{2}, \qquad 0 < \theta < 2\pi$$

is given by Jolley [1], who states that a proof can be found in [2]. This result yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n}(\cos 2nA - \cos 2nB) &= -\ln 2\sin A + \ln 2\sin B\\ &= \ln\left(\frac{\sin B}{\sin A}\right)\\ &= \ln\frac{b}{a},\end{aligned}$$

the last equality being a consequence of the law of sines.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; the COPS of Ottawa; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

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810. [1983: 23] *Proposed by Charles W. Trigg, San Diego, California.*

Place different integers on the vertices of a triangle, chosen so that the sums of integers on the extremities of each side of the triangle will be squares.

I. *Historical note by Howard Eves, University of Maine.*

Problem 6, Book III (using T.L. Heath's numbering) of Diophantus' *Arithmetica* (ca. A.D. 250?) asks: Find three numbers such that their sum is a square and the sum of any pair is a square. Diophantus' answer: 80, 320, 41.

II. *Joint solution by Malcolm A. Smith and David R. Stone, Georgia Southern College, Statesboro, Georgia (revised by the editor).*

Suppose the triple (A,B,C) of distinct integers is a solution. Then there exists a triple (a,b,c) of integers (which we assume without loss of generality to be nonnegative) such that

$$B + C = a^2, \quad C + A = b^2, \quad A + B = c^2, \tag{1}$$

and from this easily follows

$$A = \frac{b^2 + c^2 - a^2}{2}, \quad B = \frac{c^2 + a^2 - b^2}{2}, \quad C = \frac{a^2 + b^2 - c^2}{2}. \tag{2}$$

We conclude from (2) that a,b,c are distinct and their sum is even. Conversely, for any triple (a,b,c) of distinct nonnegative integers whose sum is even, the triple (A,B,C) defined by (2) consists of distinct integers that satisfy (1).

Let D be the set of all triples (a,b,c) of distinct nonnegative integers whose sum is even, and let E be the set of all triples (A,B,C) of distinct integers that are solutions to our problem. It follows from the preceding paragraph that there is a bijection from D onto E, and that for every $(a,b,c) \in D$ the corresponding $(A,B,C) \in E$ is given by (2). We now show that E is the union of infinite sequences of solutions

$$(A_n, B_n, C_n), \quad n = 1,2,3,\ldots, \tag{3}$$

some of which may have interesting properties.

Let F be the set of all triples (x,y,z) of distinct integers such that

$$y+z+2x \neq -4n, \quad z+x+2y \neq -4n, \quad x+y+2z \neq -4n, \quad n = 1,2,3,\ldots, \tag{4}$$

and, for each $(x,y,z) \in F$, let the sequence (3) be defined by

$$\begin{cases} A_n = 2(n+x)^2 + (z-x)(x-y), \\ B_n = 2(n+y)^2 + (x-y)(y-z), \\ C_n = 2(n+z)^2 + (y-z)(z-x). \end{cases} \tag{5}$$

Now

$$B_n - C_n = (y-z)(4n+y+z+2x) = 0 \iff y = z,$$

and it follows from this and two similar results that A_n, B_n, C_n are distinct integers. And since also

$$\begin{cases} B_n + C_n = (2n + y + z)^2, \\ C_n + A_n = (2n + z + x)^2, \\ A_n + B_n = (2n + x + y)^2, \end{cases} \tag{6}$$

it follows that $(A_n, B_n, C_n) \in E$ for $n = 1,2,3,\ldots$. Every solution $(A,B,C) \in E$ is the nth term of some sequence (3). For let $(A,B.C)$ be the solution corresponding to $(a,b,c) \in D$. If we set

$$x = \frac{b+c-a}{2} - n, \qquad y = \frac{c+a-b}{2} - n, \qquad z = \frac{a+b-c}{2} - n, \tag{7}$$

then x,y,z are distinct integers, they satisfy (4), and

$$2n+y+z = a, \quad 2n+z+x = b, \quad 2n+x+y = c.$$

It now follows from (6) that (A,B,C) is the nth term of the sequence (3) defined by (5) for the triple (x,y,z) given in (7).

One interesting sequence is that resulting from the choice $(x,y,z) = (1,0,-1)$,

$$A_n = 2n^2 + 4n, \quad B_n = 2n^2 + 1, \quad C_n = 2n^2 - 4n,$$

in which the sums in pairs are consecutive squares:

$$B_n + C_n = (2n - 1)^2, \quad C_n + A_n = (2n)^2, \quad A_n + B_n = (2n + 1)^2.$$

The choice $(x,y,z) = (3,1,-1)$ yields

$$A_n = 2n^2 + 12n + 10, \quad B_n = 2n^2 + 4n + 6, \quad C_n = 2n^2 - 4n - 6,$$

and the sums in pairs are the squares of consecutive even integers:

$$B_n + C_n = (2n)^2, \quad C_n + A_n = (2n + 2)^2, \quad A_n + B_n = (2n + 4)^2.$$

Other interesting sequences can be singled out at will.

III. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

There are many ways to place three integers in the desired manner. For example, there are exactly 16 solutions in positive integers less than 100:

(2, 23, 98)	(5, 20, 44)	(12, 52, 69)	(26, 74, 95)
(2, 34, 47)	(6, 19, 30)	(14, 35, 86)	(29, 52, 92)
(3, 22, 78)	(6, 75, 94)	(15, 34, 66)	(30, 51, 70)
(4, 21, 60)	(10, 54, 90)	(16, 33, 48)	(48, 73, 96)

To make the problem more challenging, additional restrictions should be imposed. For example, one can require that the three numbers be in arithmetic progression (see [1]), that the sum of all three be a perfect square (see [2]), or that the differences between each two also be squares (see [3]).

The problem can be extended in several ways:

(a) There are many ways to put integers on the vertices of a quadrilateral so that the sums of the integers on each side will be squares. For example,

$$(1, 3, 33, 48) \quad \text{and} \quad (4, 5, 76, 45).$$

It should be even easier for polygons with more sides.

(b) Suppose, instead, that we want to place integers at the vertices of a tetrahedron so that the numbers on each face sum to a square. Again, there are many ways to do this. If the numbers are all positive and less than 100, there are exactly three solutions:

$$(1, 22, 41, 58), \quad (9, 34, 57, 78), \quad \text{and} \quad (14, 41, 66, 89).$$

(c) Suppose we wish to place integers at the vertices of a tetrahedron so that the numbers on each edge sum to a square. Call the numbers a, b, c, and d. We pick a number that is the sum of two squares in multiple ways. For example,

$$5525 = 55^2 + 50^2 = 62^2 + 41^2 = 70^2 + 25^2.$$

Then we solve the simultaneous equations

$$a+b = 55^2, \quad c+d = 50^2, \quad a+c = 62^2, \quad a+d = 70^2, \quad b+d = 41^2, \quad b+c = 25^2.$$

The system is consistent because the last two equations can be derived from the first four, and solving the latter yields the solution

$$(a, b, c, d) = (3122, -97, 722, 1778).$$

Also solved by ELWYN ADAMS, Gainesville, Florida; SAM BAETHGE, San Antonio, Texas; HIPPOLYTE CHARLES, Waterloo, Québec; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; BOB PRIELIPP, University of Wisconsin-Oshkosh; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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811. [1983: 45] *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Some say that's how all the trouble started. But it must have been very tempting, for that Adam's APPLE was truly prime!

$$\begin{array}{r} \text{ADAM} \\ \text{ATE} \\ \text{THAT} \\ \text{RED} \\ \hline \text{APPLE} \end{array}$$

Solution by Edwin M. Klein, University of Wisconsin-Whitewater.

It is clear that A = 1, T ≥ 7, E = 3, 7, or 9, and M+T+D = 10 or 20.

If T = 7, then P = 0 and M+D = 13. Since E = 9 implies L = 0 = P, we must have E = 3, but then L = 4 and D+H+R = 18, and no three of the remaining digits sum to 18. Therefore T ≠ 7.

If T = 8, then P = 0 and D+H+R = 8, an impossible sum with the remaining digits. So T ≠ 8.

Therefore T = 9, P = 2, and M+D = 11. If E = 7, then L = 0 and APPLE = 12207 is not prime. Hence E = 3, L = 6, and D+H+R = 20. If D = 4, then H+R = 16, an impossible sum with the remaining digits. Therefore D = 7 and M = 4. Finally, H+R = 13 and the digits {H,R} = {5,8} are interchangeable.

Thus there are two solutions with a unique prime APPLE:

$$\begin{array}{r} 1714 \\ 193 \\ 9519 \\ \underline{837} \\ 12263 \end{array} \qquad \text{and} \qquad \begin{array}{r} 1714 \\ 193 \\ 9819 \\ \underline{537} \\ 12263 \end{array}.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; ALLAN WM. JOHNSON JR., Washington, D.C.; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer. One incorrect solution was received.

Editor's comment.

Dodge noted that the solution is unique except for "the interchange of the H & R Block", a useful reminder that this issue of *Crux* is that of the Income Tax month, April 1984.

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812. [1983: 45] *Proposed by Dan Sokolowsky, California State University at Los Angeles.*

Let C be a given circle, and let C_i, i = 1,2,3,4, be circles such that

(i) C_i is tangent to C at A_i for i = 1,2,3,4;

(ii) C_i is tangent to C_{i+1} for i = 1,2,3.

Furthermore, let l be a line tangent to C at the other extremity of the diameter of C through A_1, and, for i = 2,3,4, let A_1A_i intersect l in P_i.

Prove that, if C, C_1, and C_4 are fixed, then the ratio of unsigned lengths P_2P_3/P_3P_4 is constant for all circles C_2 and C_3 that satisfy (i) and (ii).

Solution by Dan Pedoe, University of Minnesota.

Let A be the other extremity of the diameter of C through A_1. Take the circle with centre A_1 and radius A_1A as the circle of inversion, and invert the given configuration. Then the circle C inverts into the line l; the circle C_1 inverts into a line parallel to l, which we call C_1'; the circle C_2 inverts into a circle C_2' touching line C_1' and touching l at $A_1A_2 \cap l = P_2$; the circle C_3 inverts into a circle C_3' touching circle C_2' and touching l at $A_1A_3 \cap l = P_3$; and finally the circle C_4 inverts into a circle C_4' touching circle C_3' and touching l at $A_1A_4 \cap l = P_4$.

Since line C_1' is fixed, circle C_2' has a fixed radius, say R. Circle C_4' also has a fixed radius, say r. Let ρ be the radius of circle C_3'. Since C_2' and C_3' touch externally, we have

$$(P_2P_3)^2 = (R+\rho)^2 - (R-\rho)^2 = 4R\rho;$$

and since C_3' and C_4' touch externally, we also have

$$(P_3P_4)^2 = (\rho+r)^2 - (\rho-r)^2 = 4r\rho.$$

Hence $P_2P_3/P_3P_4 = \sqrt{R/r}$, which is a constant.

Also solved by JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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813. [1983: 45] *Proposed by Charles W. Trigg, San Diego, California.* 31
The array on the right is a "staircase" of primes of the form 331
3_k1. When 3 is replaced by some other digit, the furthest any staircase 3331
of primes goes is 6661, since 66661 = 7·9523. 33331
How much further does the 3_k1 staircase go before a composite number 333331
appears? Subsequent to that, what is the next prime in the staircase? 3333331

Solutions were received from LEON BANKOFF, Los Angeles, California; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

The following information was abstracted from the solutions received.

All factors appearing in the table (see next page) are primes.

The first part of the table, where complete factorizations are given, was taken from Kierstead's solution. All his results are confirmed by other solvers except the last, $3_{17}1$ = prime, which, says Kierstead, required 40 minutes on a fast computer. No other solver gave complete results beyond $3_{16}1$.

The second part of the table, where only the smallest prime factor is given, was taken from Feder's solution. The question mark after $3_{30}1$ presumably indicates that this number, if not prime, has no "small" prime factor. Our proposer, a Count of

$3_1 1$ = prime
$3_2 1$ = prime
$3_3 1$ = prime
$3_4 1$ = prime
$3_5 1$ = prime
$3_6 1$ = prime
$3_7 1$ = prime
$3_8 1 = 17 \cdot 19607843$
$3_9 1 = 673 \cdot 4952947$
$3_{10} 1 = 307 \cdot 108577633$
$3_{11} 1 = 19 \cdot 83 \cdot 211371803$
$3_{12} 1 = 523 \cdot 3049 \cdot 2090353$
$3_{13} 1 = 607 \cdot 1511 \cdot 1997 \cdot 18199$
$3_{14} 1 = 181 \cdot 1841620626151$
$3_{15} 1 = 199 \cdot 16750418760469$
$3_{16} 1 = 31 \cdot 1499 \cdot 717324094199$
$3_{17} 1$ = prime
$3_{18} 1 = 1009 \cdot \ldots$
$3_{19} 1 = 29 \cdot \ldots$
$3_{20} 1 = 23 \cdot \ldots$
$3_{21} 1 = 177943 \cdot \ldots$
$3_{22} 1 = 61 \cdot \ldots$
$3_{23} 1 = 312929 \cdot \ldots$
$3_{24} 1 = 17 \cdot \ldots$
$3_{25} 1 = 821 \cdot \ldots$
$3_{26} 1 = 353 \cdot \ldots$
$3_{27} 1 = 363941 \cdot \ldots$
$3_{28} 1 = 829 \cdot \ldots$
$3_{29} 1 = 19 \cdot \ldots$
$3_{30} 1 = ?$
$3_{31} 1 = 31 \cdot \ldots$

Digit Delvers, will be happy to learn that 31 divides $3_{31}1$, and that 19 and 29 take in each other's washing.

Wilke showed that infinitely many of the numbers $3_k 1$ are composite, and in doing so he found much interesting information. He reasoned as follows:

Since

$$3_k 1 = \frac{10^{k+1} - 7}{3}, \qquad k = 1,2,3,\ldots,$$

a prime $p > 3$ divides $3_k 1$ if and only if $10^{k+1} \equiv 7 \pmod{p}$. He then used the theory of indices (and the table of indices in the *CRC Standard Mathematical Tables*) to show that

$17 | 3_k 1$ for $k = 8 + 16i$, $i = 0,1,2,\ldots,$
$19 | 3_k 1$ for $k = 11 + 18i$,
$23 | 3_k 1$ for $k = 20 + 22i$,
$29 | 3_k 1$ for $k = 19 + 28i$,
$31 | 3_k 1$ for $k = 1 + 30i$ and $16 + 30i$,
$47 | 3_k 1$ for $k = 37 + 46i$,
$59 | 3_k 1$ for $k = 43 + 58i$,
$61 | 3_k 1$ for $k = 22 + 60i$,
$83 | 3_k 1$ for $k = 11 + 82i$ and $52 + 82i$,
$97 | 3_k 1$ for $k = 52 + 96i$,

and that no other primes $p < 100$ divide any of the numbers 3_k1. Our proposer may be disappointed to learn that his pet prime 37 (see his article in this issue) does not divide *any* of the numbers 3_k1. Not a very useful pet for a numerologist to have around the house.

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814. [1983: 45] *Proposed by Leon Bankoff, Los Angeles, California.*

Let D denote the point on BC cut by the internal bisector of angle BAC in the Heronian triangle whose sides are AB = 14, BC = 13, CA = 15. With D as center, describe the circle touching AC in L and cutting the extension of AD in J. Show that AJ/AL = $(\sqrt{5}+1)/2$, the Golden Ratio.

I. *Solution by the proposer.*

The location of the point D is a red herring. Instead, let D be *any* point on the line l which bisects angle BAC. With L and J as defined in the proposal, the ratio AJ/AL is constant for, as D varies on l, the circles (D) are all homothetic with homothetic center A. In particular, when circle (D) is the incircle of triangle ABC, whose radius is

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{8\cdot 7\cdot 6}{21}} = 4,$$

then AL = $s-b$ = 8, AD = $4\sqrt{5}$, AJ = $4(\sqrt{5}+1)$, and so

$$\frac{\text{AJ}}{\text{AL}} = \frac{\sqrt{5}+1}{2}.$$

II. *April 1 "solution" by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Major premise: The Golden Ratio pops up in the most unexpected places.

Minor premise: The ratio AJ/AL in the Heronian 13-14-15 triangle is a most unexpected place.

Conclusion: AJ/AL = the Golden Ratio.

What I really want to express is my wonder at the Golden Ratio showing up here between 13, 14, and 15, and my admiration for Dr. Bankoff who discovered it there. Congratulations, Dr. Bankoff!

Also solved by ELWYN ADAMS, Gainesville, Florida; SAM BAETHGE, San Antonio, Texas; W.J. BLUNDON, Memorial University of Newfoundland; the COPS of Ottawa; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; MALCOLM A. SMITH, Georgia Southern College, Statesboro, Georgia; et ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec.

Editor's comment.

Only Dou gave evidence of being aware of the red herring. For the others, *April fool!*

815. [1983: 46] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a,b,c, internal angle bisectors t_a, t_b, t_c, and semiperimeter s. Prove that the following inequalities hold, with equality if and only if the triangle is equilateral:

(a) $$\sqrt{3}\left(\frac{1}{at_a} + \frac{1}{bt_b} + \frac{1}{ct_c}\right) \ge \frac{4s}{abc};$$

(b) $$3\sqrt{3}\cdot\frac{\frac{1}{at_a} + \frac{1}{bt_b} + \frac{1}{ct_c}}{at_a + bt_b + ct_c} \ge 4\sqrt{\frac{2s}{(abc)^3}}.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

(a) We will use the following known results, in which T_a, T_b, T_c are the internal angle bisectors extended until they are chords of the circumcircle, R is the circumradius, and the sums (here and later) are cyclic over A,B,C and a,b,c:

$$bc = T_a t_a, \text{ etc.}; \quad T_a = 2R\cos\frac{B-C}{2}, \text{ etc.}; \ \Sigma\cos\frac{B-C}{2} \ge \frac{2}{\sqrt{3}}\Sigma\sin A.$$

For the first result, see [1976: 233]; for the second, see [1982: 115]; the third is Garfunkel's inequality [1982: 67, 138]. From these results, we get

$$\sqrt{3}\Sigma\frac{1}{at_a} = \frac{\sqrt{3}}{abc}\Sigma T_a = \frac{2\sqrt{3}R}{abc}\Sigma\cos\frac{B-C}{2} \ge \frac{2}{abc}\Sigma 2R\sin A = \frac{2}{abc}\Sigma a = \frac{4s}{abc}.$$

As in Garfunkel's inequality, equality holds if and only if ABC is equilateral.

(b) Here all we need is

$$\frac{1}{3}(at_a + bt_b + ct_c) \le \sqrt{\frac{abcs}{2}}, \tag{1}$$

obtained by setting $\lambda = 1$ in Inequality 8.12 on page 76 of the Bottema Bible, *Geometric Inequalities.* The inequality we seek here is obtained by dividing corresponding members of (1) and the inequality of part (a). As in (1), equality holds if and only if ABC is equilateral.

Also solved by JACK GARFUNKEL, Flushing, N.Y.; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

It is clear from the proposer's solution that he is having his little joke in part (b). It is easy to manufacture, by multiplying together two or more simpler inequalities, ever more monstrous inequalities that will stump everyone but their creator. To be fair, proposed inequalities should be, or at least be thought to be, "prime". In this case, fortunately, nearly all solvers were able to find, in their well-thumbed Bottema Bible, the second "factor" that led to (b).

April fool to *you*, Mr. Proposer.

END OF THE APRIL ISSUE