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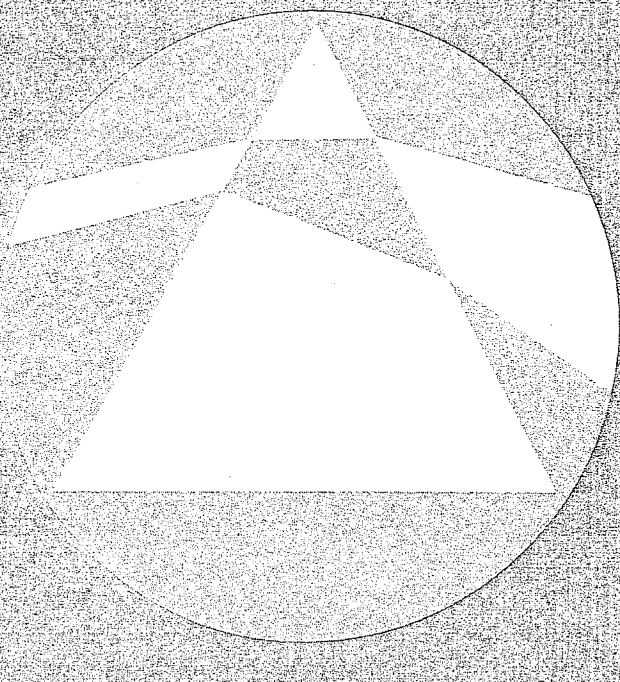

- Mathematics through problems
- The domino problem
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A magazine for students and teachers of mathematics
in schools, colleges and universities

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Mathematics Through Problems

TONY GARDINER

Technique, theory and practising routine procedures may be the staple diet of school mathematics, but problems provide the spice!

The first inkling that there is something special about mathematics often arises from the experience of tackling, and sometimes managing to solve, intriguing problems—often during adolescence. This experience may be linked to innocuous-looking, non-technical problems such as the following.

Problem 1 (BUMMPS* 1995). What is the largest amount you could have using only 1p, 2p, 5p and 10p coins, yet still be unable to make exactly 50p?

The problem is accessible to anyone who is willing to grapple with the wording. Yet it embodies a two-fold surprise. First, the answer is not 49p. Second, though it is not hard to discover what the answer seems to be, it is far from easy to give a convincing reason why the apparent maximum is a real maximum.

To be effective, there is no need for a problem to be new. The response to the following old chestnut suggested that the algebra of speed and distance is now totally neglected in most English schools!

Problem 2 (BUMMPS 1995). To get from home to school I have to cycle exactly 1 mile up a hill and exactly 1 mile down the other side. If I average 10 miles per hour on the way up, at what speed must I go down if I am to average 20 miles per hour for the whole journey? If I average u miles per hour on the way up and d miles per hour on the way down, what will be my average for the whole journey?

Most of those who worked the problem through correctly expressed genuine surprise at the result. And one can only hope that the majority (who got it wrong) kicked themselves when they received the results—together with a set of Hints and Solutions.

To make progress in mathematics one has to master basic techniques and theories—elementary arithmetic, fractions and ratio, algebra, coordinate geometry, calculus, etc. These involve serious study. But students' appetite for mathematics often stems not just from the theory (important though that is), but from the related fund of problems. This makes it especially unfortunate that areas of school mathematics that are particularly rich in good problems (such as Euclidean and coordinate geometry, probability and the tougher parts of calculus) have recently been so diluted that little remains. This makes it all the more important to provide other sources of challenging problems for youngsters to cut their teeth on.

*Birmingham University Mid-term Mathematical PuzzleS—a set of six light-hearted take-home problems for those in Years 11–13 to tackle over the ten days of February half term.

One source of such problems comes from a range of multiple-choice 'Mathematical Challenges' which are now available for all secondary age groups in the UK.

Problem 3 (UK Junior Mathematical Challenge 1994). The four digits 1, 2, 3, 4 are written in increasing order. You must insert one plus sign and one minus sign between the 1 and the 2, or between the 2 and the 3, or between the 3 and the 4, to produce expressions with different answers. How many different positive answers can be obtained in this way?

A 1; *B* 2; *C* 3; *D* 4; *E* 5.

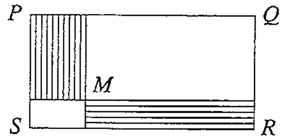
Problem 4 (UK Intermediate Mathematical Challenge 1995). A stopped clock may be useless, but it does at least tell the right time twice a day. A 'good' clock, which gains just one second each day, shows the correct time far less often. Roughly how often?

A once every 60 days; *B* once every 72 days;

C once every 360 days; *D* once every 12 years;

E once every 120 years.

Problem 5 (European Kangaroo 1995). The quadrilateral $PQRS$ is a rectangle; M is any point on the diagonal SQ. What can one say for sure about the two shaded regions?

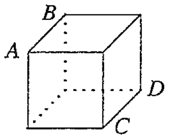


A the upper area is larger;
B the lower area is larger;
C the two regions always have equal areas;
D the two areas are equal only if M is the mid-point of SQ;
E there is insufficient information to decide.

Multiple-choice problems can whet the appetite, but they are only a beginning. Mathematics is the science of *correct reasoning*; it is not just a box of tricks for producing 'answers'. Thus at some point one has to struggle to present *reasons*, to learn to use symbols correctly, and to lay out one's solutions in the accepted way. The range of written challenges, or *Olympiads*,

In Round 1 of the 1995 British Mathematical Olympiad, the prize for the winners was a year's subscription to *Mathematical Spectrum*. We give a warm welcome to our new readers, some of whom have already sent in contributions.

provide one way for youngsters to begin to learn these skills. Consider the following problem.



Problem 6 (UK Junior Mathematical Olympiad 1993). Place the numbers 1–8 at the vertices of a cube so that the four numbers at the corners of each face always have the same sum. What do you notice about the two numbers at A and B and the two numbers at C and D? Explain clearly why this is not an accident.

This problem is aimed at 13- and 14-year-olds who have no experience of giving reasons. Yet, because this is a *mathematical* Olympiad, the marks are for the explanation—not for spotting some subjective 'pattern'. In particular, it is important to formulate what one finds in a way which makes the explanation easy to present—which probably means, for example, that one should present it in terms of $A+B$ and $C+D$, rather than $A-C$ and $D-B$. Moreover, in mathematics there is no place for vague notions of 'symmetry' or 'balance'. One wants to see students who realise that the two given edges lie in two faces with a *common edge*; and who state that, since the sum of the four numbers A, B, X and Y (say) at the corners of the top face, and the sum of the four numbers C, D, X and Y at the corners of the right-hand face are equal, and since X and Y are common to both sums, it follows that $A+B = C+D$. Nothing else will do.

Exploration can be a significant *preliminary* stage in tackling any mathematical problem; but it is only a preliminary to the real mathematics. If challenged to 'find all' numbers with some property, it is natural to explore the problem in rough, mixing logic with inspired guesswork. One may even believe one's exploration has produced a complete list. But until one's working is presented in a correct and convincing way, such a list cannot be trusted. Often, once one tries to present one's 'solution' correctly, one discovers more or less serious gaps, oversights, or hidden assumptions which need to be sorted out, or explained more clearly.

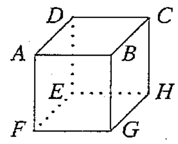
Sometimes the final solution is deliciously brief and totally clear.

Problem 7 (UK Junior Mathematical Olympiad 1992). Does there exist a two-digit number *ab* such that the difference between *ab* and its 'reverse' *ba* is a prime number?

At other times one has to work quite hard to present one's reasons clearly and succinctly—though it is only by insisting on *reasons* that one can discover the mathematics underneath the surface.

Problem 8 (UK Junior Mathematical Olympiad 1993). 31.3.93 is an interesting date, since $31\times3 = 93$. If all dates are written like this, how many years this century contain no interesting dates at all?

Perhaps the simplest way of presenting clear *reasons* is in the form of a good old-fashioned calculation in a slightly unfamiliar setting. There is nothing particularly hard about the following example; but most of the 800 candidates threw away the marks in part (a) by claiming that 'a quadrilateral with all four sides of length x has area x^2' (even though many of them drew the quadrilateral as a squidgy rhombus)! Problem-solving may lie at the heart of mathematics, but it has to build on a solid technique.



Problem 9 (British Mathematical Olympiad, Round 1, 1995). *ABCDEFGH* is a cube of side 2.

(a) Find the area of the quadrilateral *AMHN*, where M is the midpoint of *BC* and N is the midpoint of *EF*.
(b) Let P be the midpoint of *AB* and Q the midpoint of *HE*. Let *AM* meet *CP* at X and *HN* meet *FQ* at Y. Find the length of *XY*.

The *British Mathematical Olympiad* is aimed at students in their last two or three years at school who would like a tougher challenge. Round 1 (for around 800 students) seeks to present candidates with a challenge, but a manageable one; in particular, it always includes some problems, (like problem 9 above) which everyone can have a go at, as well as some more demanding problems (like problem 10 below). In Round 2 (for about 100 students) the gloves come off! The results of Round 2 are used to select 20 students for the *Final Selection Test*. In recent years it has become something of a tradition to include problems involving 'dwarfs'. I shall end this short collection with the latest example.

Problem 10 (British Mathematical Olympiad, Round 1, 1995). The seven dwarfs walk to work each morning in single file. As they go they sing their famous song: 'High-low-high-low, it's off to work we go ...'. Each day they line up so that no three successive dwarfs are either increasing or decreasing in height: thus, the line must go up-down-up-down-... or down-up-down-up-... . If they all have different heights, for how many days can they go to work like this if they insist on using a different order each day? What if Snow White always comes along too?

For those who emerge from such encounters bursting for more, there are the *International Tournament of Towns* and the *International Mathematical Olympiad*. Details of these will have to wait for another article. Anyone wanting further information about such events within the UK should send a stamped, self-addressed envelope to the author, at the School of Mathematics, University of Birmingham, Birmingham B15 2TT, UK. □

Tony Gardiner *is Reader in Mathematics and Mathematical Education at the University of Birmingham. He is founder and Director of the UK Mathematics Foundation which runs mathematical challenges on various levels for around 300 000 secondary pupils each year. Since 1990 he has been Leader of the UK International Mathematical Olympiad team, but is much more concerned about the mathematical development of the mass of able students.*

Linear and Non-Linear Oscillators

P. GLAISTER

Readers may be familiar with simple harmonic motion, where the period is independent of the amplitude. But what about other sorts of oscillations?

In A-level mechanics students usually discuss the motion of a simple linear harmonic oscillator, such as a mass on a spring, and then move on to the motion of a simple non-linear oscillator, such as a simple pendulum. In the latter case, an assumption of small amplitudes leads to a linear approximation and hence to the simple linear harmonic oscillator again.

An important feature of any non-linear oscillator is that, while the motion is periodic, the period depends on the amplitude. This is in contrast to the simple linear harmonic oscillator where the period is independent of the amplitude. Since the solution of the governing equations sometimes requires techniques beyond A-level, this fact, and the periodic nature of non-linear oscillators, is not usually mentioned to students. With the introduction into some A-level syllabuses of numerical methods for solving differential equations, however, there is now scope for combining these topics in a natural way to demonstrate these features. Indeed, it is a worthwhile exercise for students to investigate the nature of the solution and the period of oscillation of both linear and non-linear oscillators using a numerical method. A simple program on a graphics calculator can be used to display the results.

Three specific problems are considered, and the reader is left to look at suitable variations of those. A discussion on phase-plane analysis is also included as a means of determining the behaviour of the period of oscillation as a function of amplitude.

Consider the oscillator governed by the differential equation

$$\frac{d^2x}{dt^2} = f(x), \tag{1}$$

subject to the initial conditions $x(0) = A$ and $(dx/dt)(0) = 0$. Denoting Δt as a 'small' time interval, and approximations $x_1 \simeq x(\Delta t)$, $x_2 \simeq x(2\Delta t)$, ..., with $x_0 = x(0) = A$, then the derivative dx/dt can be approximated by a difference $(x_{n+1} - x_n)/\Delta t$. Similarly, the second derivative

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

can be approximated by a difference of differences

$$\frac{\dfrac{x_{n+1}-x_n}{\Delta t} - \dfrac{x_n - x_{n-1}}{\Delta t}}{\Delta t} = \frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2}.$$

Thus equation (1) can be approximated by

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2} = f(x_n), \tag{2}$$

giving a recurrence formula

$$x_{n+1} = (\Delta t)^2 f(x_n) + 2x_n - x_{n-1}, \tag{3}$$

which clearly needs two starting values, x_0 and x_1. However, since $x_0 = x(0) = A$ and $(dx/dt)(0) = 0$, which can be approximated by $(x_1 - x_0)/\Delta t = 0$, we can apply (3) with $x_0 = x_1 = A$.

Simple harmonic oscillator

Figure 1 shows the approximate solution for equation (1) obtained using equation (3) in the case $f(x) = -x$. This corresponds to a simple harmonic oscillator and the equation is linear. Three amplitudes are considered, $A = \frac{1}{2}$, 1 and 2, and the timestep $\Delta t = 0.1$. The motion is clearly periodic, with the period *independent of the amplitude*. In fact, the period for the case $f(x) = -kx$ (k constant) is given by $T = 2\pi/\sqrt{k}$, as shown below.

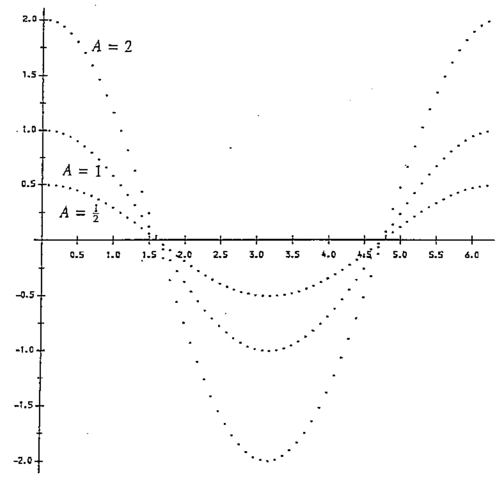


Figure 1

Simple pendulum oscillator

Figure 2 shows the approximate solution for equation (1) obtained using equation (3) in the case $f(x) = -\sin x$. This corresponds to a simple pendulum for amplitudes which are not necessarily small, and the equation is non-linear. Three amplitudes have again been considered, $A = \frac{1}{2}$, 1 and 2, and the timestep $\Delta t = 0.1$. The motion is again periodic, with a *weak* dependency of the period on the amplitude. In fact the period, T, in the case $f(x) = -k\sin x$ (k constant) is given by $T = 4K(\sin\frac{1}{2}A)/\sqrt{k}$, where K is a complete elliptic integral of the first kind which cannot be determined in terms of elementary functions. For small amplitudes,

$$T \simeq 2\pi\frac{1+\frac{1}{16}A^2}{\sqrt{k}}.$$

This function is discussed further below.

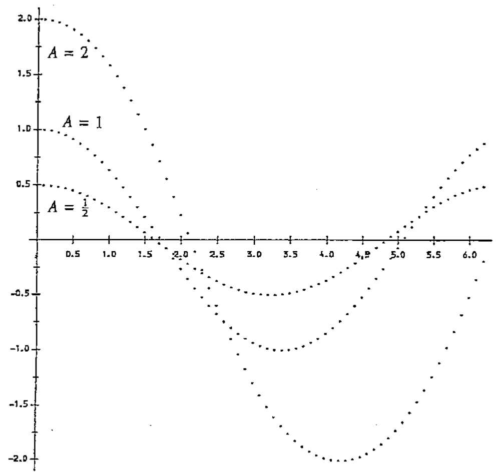


Figure 2

Simple cubic oscillator

Figure 3 shows the approximate solution for equation (1) obtained using equation (3) in the case $f(x) = -x^3$, and this is a well-known example of a non-linear oscillator. A mechanical system consisting of a mass attached to the middle of an unstretched spring with both ends fixed exhibits oscillations in the transverse direction to the length of the spring which are governed by this equation. The three amplitudes are $A = \frac{1}{2}$, 1 and 2, and the timestep is again $\Delta t = 0.1$. This time there is a *strong* dependency of the period on the amplitude with $T = 4I\sqrt{2/k}/A$, where $I \simeq 1.31$.

Similarly, any non-linear oscillator governed by equation (1) with initial conditions $x(0) = A$ and $(\mathrm{d}x/\mathrm{d}t)(0) = B$ can be investigated in a similar way using the numerical scheme (3) on a graphics calculator or equivalent. The corresponding starting values are $x_0 = x(0) = A$ and $(x_1 - x_0)/\Delta t = B$, i.e. $x_0 = A$ and $x_1 = x_0 + B\,\Delta t$. We encourage readers to try this for various functions $f(x)$. Reducing Δt gives a more accurate solution.

Phase-plane analysis

Certain second-order differential equations of the form in equation (1) can be reduced to first-order ones by the simple substitution $y = \mathrm{d}x/\mathrm{d}t$. Frequently, an analytical relationship between x and t is unobtainable, but in some cases it is possible to solve the first-order equation and obtain a relationship between x and $\mathrm{d}x/\mathrm{d}t$. By this method it may be possible to obtain valuable information about the solution of the second-order equation, even when the full solution is not known. In particular, if the motion is periodic, then the period can be determined. The relationship between x and $\mathrm{d}x/\mathrm{d}t$ can be shown as a solution curve (for given initial conditions) in the *phase-plane*, as the (x, y) plane is called.

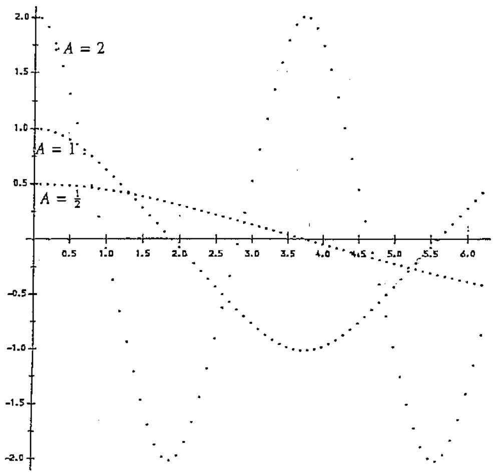


Figure 3

The direction of motion of the point $(x, \mathrm{d}x/\mathrm{d}t)$ along a solution curve (or trajectory) in the phase-plane as t increases is determined from the fact that $\mathrm{d}x/\mathrm{d}t > 0$ in the upper half plane and so x must *increase* there, and $\mathrm{d}x/\mathrm{d}t < 0$ in the lower half plane and hence x must *decrease* there. *Periodic motion* is indicated by a closed curve in the phase-plane, an example of which is shown in figure 4. With the initial conditions $x(0) = A$ and $(\mathrm{d}x/\mathrm{d}t)(0) = 0$, the trajectory begins at point a, then moves to points b, c, d and e in turn, and then returns to point a.

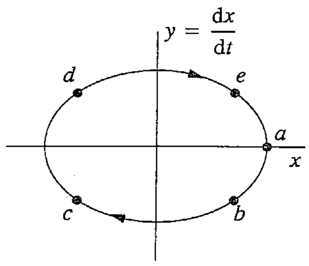


Figure 4

Consider now the three examples above.

Simple harmonic oscillator

Integrating $\mathrm{d}^2x/\mathrm{d}t^2 = -kx$ once gives the energy equation

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = -\tfrac{1}{2}kx^2 + \text{constant}, \tag{4}$$

and with initial conditions $x(0) = A$ and $(\mathrm{d}x/\mathrm{d}t)(0) = 0$, becomes

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = k(A^2 - x^2) \qquad (5)$$

or

$$\frac{x^2}{A^2} + \frac{(\mathrm{d}x/\mathrm{d}t)^2}{k^2A^2} = 1. \qquad (6)$$

The relationship in equation (6) describes an ellipse in the phase-plane, and this is shown in figure 4. The period, T, is the time taken for a point in the phase-plane to make one complete cycle of the trajectory, or $T = 4t_{1/4}$, where $t_{1/4}$ is the time taken for the point to traverse one quadrant, say the first. Thus, from equation (5), the period is given by

$$T = 4\int_0^{t_{1/4}} \mathrm{d}t = 4\int_0^A \frac{1}{\sqrt{k(A^2-x^2)}}\,\mathrm{d}x = \frac{2\pi}{\sqrt{k}}, \qquad (7)$$

where the integral has been evaluated using the substitution $x = A\sin\theta$.

Simple pendulum oscillator

Integrating $\mathrm{d}^2x/\mathrm{d}t^2 = -k\sin x$ once and applying the initial conditions $x(0) = A$ and $\mathrm{d}x/\mathrm{d}t(0) = 0$ gives the energy equation

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = k(\cos x - \cos A). \qquad (8)$$

Although the trajectory is again closed, indicating periodic motion, it is *not* an ellipse. However, the qualitative shape is that of an ellipse, i.e. an oval, similar to that in figure 4. Furthermore, equation (8) cannot be integrated exactly to give a relationship between x and t. The period T, is given from equation (8) as

$$T = 4\int_0^{t_{1/4}} \mathrm{d}t = 4\int_0^A \frac{1}{\sqrt{2k(\cos x - \cos A)}}\,\mathrm{d}x = 2\sqrt{\frac{2}{k}}\int_0^A \frac{1}{\sqrt{\cos x - \cos A}}\,\mathrm{d}x. \qquad (9)$$

Now, using $\cos x = 1 - 2\sin^2\frac{1}{2}x$, $\cos A = 1 - 2\sin^2\frac{1}{2}A$ and the substitution $\sin\frac{1}{2}x = \sin\frac{1}{2}A\sin\theta$, enables the integral in equation (9) to be simplified to

$$T = \frac{4}{\sqrt{k}}\int_0^{\frac{1}{2}\pi} \frac{1}{\sqrt{1-\sin^2\frac{1}{2}A\sin^2\theta}}\,\mathrm{d}\theta = \frac{4}{\sqrt{k}}K(\sin\tfrac{1}{2}A), \qquad (10)$$

where

$$K(m) = \int_0^{\frac{1}{2}\pi} \frac{1}{\sqrt{1-m^2\sin^2\theta}}\,\mathrm{d}\theta$$

is called a *complete elliptic integral of the first kind*. Integrals of this form cannot be determined in terms of elementary functions. For *small amplitudes* A, $\sin^2\frac{1}{2}A \simeq (\frac{1}{2}A)^2 = \frac{1}{4}A^2$ and hence

$$\frac{1}{\sqrt{1-\sin^2\frac{1}{2}A\sin^2\theta}} \simeq \frac{1}{\sqrt{1-\frac{1}{4}A^2\sin^2\theta}} \simeq 1 + \tfrac{1}{2}(\tfrac{1}{4}A^2)\sin^2\theta,$$

using the binomial expansion. Thus in equation (10)

$$T \simeq \frac{4}{\sqrt{k}}\int_0^{\frac{1}{2}\pi} (1 + \tfrac{1}{8}A^2\sin^2\theta)\,\mathrm{d}\theta = \frac{2\pi}{\sqrt{k}}(1 + \tfrac{1}{16}A^2). \qquad (11)$$

Simple cubic oscillator

The corresponding energy equation for $\mathrm{d}^2x/\mathrm{d}t^2 = -kx^3$ is

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \tfrac{1}{4}k(A^4 - x^4), \qquad (12)$$

whose trajectory is very similar to that of an ellipse. The period is given by

$$T = 4\int_0^{t_{1/4}} \mathrm{d}t = 4\sqrt{\frac{2}{k}}\int_0^A \frac{1}{\sqrt{A^4-x^4}}\,\mathrm{d}x, \qquad (13)$$

and substituting $x = Az$ in this integral gives

$$T = \frac{4}{A}\sqrt{\frac{2}{k}}\int_0^1 \frac{1}{\sqrt{1-z^4}}\,\mathrm{d}z = 4\sqrt{\frac{2}{k}}\frac{I}{A}, \qquad (14)$$

where

$$I = \int_0^1 \frac{1}{\sqrt{1-z^4}}\,\mathrm{d}z$$

can be determined numerically as $I \simeq 1.31$.

As a final exercise, readers may like to investigate the simple pendulum problem governed by $\mathrm{d}^2x/\mathrm{d}t^2 = -k\sin x$, with initial conditions $x(0) = 0$ and $(\mathrm{d}x/\mathrm{d}t)(0) = B$ (initial displacement and speed) in the case $B > 2\sqrt{k}$. Readers should sketch the trajectory in the phase-plane as well as applying the general numerical scheme mentioned above. A physical interpretation should help. Take care!

Further reading

W. Chester, *Mechanics* (Allen and Unwin, London, 1979). □

Paul Glaister *lectures in mathematics at the University of Reading. His research interests include computational fluid dynamics, numerical analysis and perturbation methods, as well as mathematics and science education. His principal and most exhausting leisure interests are his two young children.*

More About An Infinite Exponential

G. T. VICKERS

At first sight $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ may look peculiar, but surely its value increases without limit as the number of exponents increases? In fact, its behaviour is much more subtle.

This article is a continuation of 'An infinite exponential' by Bob Bertuello (reference 1) and my equally brief note 'Experiments with infinite exponents' (reference 2). In order to make this contribution self-contained we start with a review of the story so far.

Let α be any positive number and define

$$S_0(\alpha) = \alpha, \qquad S_1(\alpha) = \alpha^{\alpha}, \qquad S_2(\alpha) = \alpha^{\alpha^{\alpha}}, \qquad \dots .$$

This sequence is more conveniently written as

$$S_0(\alpha) = \alpha, \qquad S_{n+1}(\alpha) = \alpha^{S_n(\alpha)} \quad (n \geqslant 0). \qquad (1)$$

(In view of what follows, the letter x used in reference 1 is best replaced by α.) The objective is to understand the behaviour of the sequence $\{S_n\}$. In reference 1 it is stated that

(a) $S_n(\alpha)$ converges for $0 < \alpha \leqslant e^{1/e}$;
(b) $S_n(e^{1/e}) \to e$ as $n \to \infty$;
(c) $S_n(\alpha)$ diverges for $\alpha > e^{1/e}$.

In order to encourage experiments and investigation, reference 2 stated that (a) or (c) is false. Reference 2 also suggested that the more general sequence

$$a_0 = \beta, \qquad a_{n+1} = \alpha^{a_n} \quad (n \geqslant 0)$$

(with $\alpha > 0$) together with its inverse sequence

$$b_0 = \gamma, \qquad b_n = \alpha^{b_{n+1}} \quad \left(\text{or } b_{n+1} = \frac{\ln b_n}{\ln \alpha}\right) \quad (n \geqslant 0)$$

were also worthy of investigation. Finally it was stated that these sequences could be used to find approximate values to both roots of equations such as

$$2^x = 5x-1,$$

the technique needed being only simple numerical iteration (not 'integration', as unfortunately appeared in reference 2).

To begin with, consider the following sequence defined by the function F:

$$a_0 = \delta, \qquad a_{n+1} = F(a_n) \quad (n \geqslant 0). \qquad (2)$$

Suppose that F has a fixed point X, i.e.

$$X = F(X) \qquad (3)$$

and set $a_n = X+x_n$, where x_n is small. Then

$$\begin{aligned} a_{n+1} &= F(a_n) = F(X+x_n) \\ &= F(X)+x_nF'(X)+\tfrac{1}{2}x_n^2F''(X)+\dots . \qquad (4) \end{aligned}$$

(Strictly it is necessary to impose certain conditions upon F, but we shall be somewhat incautious regarding technicalities.) Thus if x_n is small

$$\begin{aligned} X+x_{n+1} = a_{n+1} &\approx F(X)+x_nF'(X) \\ &= X+x_nF'(X) \end{aligned}$$

and so

$$x_{n+1} = F'(X)x_n.$$

This implies that

$$\begin{aligned} x_n = F'(X)x_{n-1} &= [F'(X)]^2x_{n-2} \\ &= \dots = [F'(X)]^n x_0. \qquad (5) \end{aligned}$$

It follows from equation (5) that if $|F'(X)| < 1$ then $\{x_n\}$ will decrease numerically to zero. In this case $a_n \to X$ as $n \to \infty$ and we say that the fixed point X is *stable*. But if $|F'(X)| > 1$ then the numbers x_n will get numerically large and so invalidate the approximation used in going from (4) to (5). The fixed point X is now said to be *unstable*. Thus the behaviour of the recurrence (2) is as follows:

(a) if δ is close to X and $|F'(X)| < 1$ then $a_n \to X$ as $n \to \infty$;
(b) if $|F'(X)| > 1$ then $\{a_n\}$ will not converge to X no matter how close to X the value of δ (provided, of course, that $\delta \neq X$).

The problem of determining how 'close' δ has to be in (a) is very difficult. It must not be imagined that $|F'(X)| < 1$ guarantees convergence from all initial values. Note also that if equation (3) does not have a root then $\{a_n\}$ defined by (2) cannot converge.

A useful geometrical technique in the investigation of recurrence equations such as (2) is *cobwebbing*.

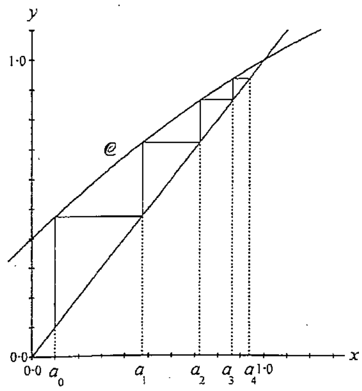


Figure 1. A simple example of cobwebbing showing how successive iterates of a recurrence relation are found graphically. The curve $\mathscr{C}$ is the graph of F and the fixed point is (1, 1).

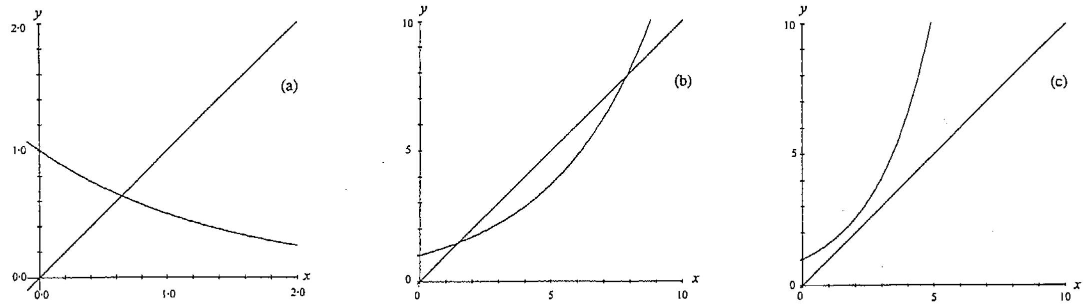


Figure 2. Graphs of $y = \alpha^x$ when α has the values (a) 0.5, (b) 1.3 and (c) 1.5.

This involves drawing the graph of the function F together with the identity function. An example is shown in figure 1 where the curve $\mathscr{C}$ is the plot of $y = F(x)$ and the first few iterates are shown starting from a_0. Notice how the line $y = x$ is used in getting the next term in the sequence. In the situation sketched, the picture is more like a staircase than a cobweb (which will appear later) and sometimes the technique is referred to as *staircasing*.

Let us now apply these results to the function

$$F(x) = \alpha^x,$$

where $\alpha > 0$. The first step is to find the fixed points of F, i.e. to solve equation (3). Now the behaviour of F depends crucially upon the value of α. Clearly if $\alpha < 1$ then equation (3) has just one root but if $\alpha > 1$ then there may be no roots, two roots or one (repeated) root. The critical case is when the two curves

$$y = x \quad \text{and} \quad y = \alpha^x$$

touch one another at the fixed point X. This occurs when their slopes are the same at X or when

$$1 = \alpha^X \ln \alpha = X \ln \alpha = \ln \alpha^X$$

$$\Rightarrow X = \alpha^X = \mathrm{e} \Rightarrow \alpha^{\mathrm{e}} = \mathrm{e} \Rightarrow \alpha = \mathrm{e}^{1/\mathrm{e}}.$$

So

$$0 < \alpha < 1 \Rightarrow X = \alpha^X \text{ has one root;}$$

$$1 < \alpha < \mathrm{e}^{1/\mathrm{e}} \Rightarrow X = \alpha^X \text{ has two roots,}$$

$$\text{say } X_1 \text{ and } X_2 \ (X_1 > X_2);$$

$$\mathrm{e}^{1/\mathrm{e}} < \alpha \Rightarrow X = \alpha^X \text{ has no roots.}$$

We have seen that whether $\{a_n\}$ can converge or not depends upon the value of $|F'(X)|$, and in particular whether it is numerically less than unity. Now for $F(x) = \alpha^x$ we have $F'(x) = \alpha^x \ln \alpha$ and so

$$F'(X) = \alpha^X \ln \alpha = X \ln \alpha = \ln \alpha^X = \ln X.$$

Thus

$$\frac{1}{\mathrm{e}} < X < \mathrm{e} \Rightarrow -1 < \ln X < 1 \Rightarrow |F'(X)| < 1$$

and

$$X < \frac{1}{\mathrm{e}} \text{ or } X > \mathrm{e} \Rightarrow |F'(X)| > 1.$$

Furthermore, the critical cases are

$$X = \frac{1}{\mathrm{e}} \Rightarrow \frac{1}{\mathrm{e}} = \alpha^{1/\mathrm{e}} \Rightarrow \alpha = \mathrm{e}^{-\mathrm{e}} \approx 0.066$$

and

$$X = \mathrm{e} \Rightarrow \mathrm{e} = \alpha^{\mathrm{e}} \Rightarrow \alpha = \mathrm{e}^{1/\mathrm{e}} \approx 1.445.$$

Hence there are four intervals of α which need to be considered.

1. $0 < \alpha < \mathrm{e}^{-\mathrm{e}}$; here there is a unique fixed point X but $F'(X) < -1$ and $\{a_n\}$ cannot converge.
2. $\mathrm{e}^{-\mathrm{e}} < \alpha < 1$; again a unique fixed point, but now $-1 < F'(X) < 0$ and $\{a_n\}$ will converge if δ is near to X.
3. $1 < \alpha < \mathrm{e}^{1/\mathrm{e}}$; now there are two fixed points, $X_1 > \mathrm{e} > X_2$ and $0 < F'(X_2) < 1 < F'(X_1)$. The inequalities for X_1 and X_2 hold because
$$\frac{\ln X_1}{X_1} = \frac{\ln X_2}{X_2} = \ln \alpha < \frac{1}{\mathrm{e}}$$
and the function $\ln x/x$ has a maximum value of $1/\mathrm{e}$ at $x = \mathrm{e}$. Thus X_1 is an unstable point and X_2 is a stable one.
4. $\alpha > \mathrm{e}^{1/\mathrm{e}}$; there is no fixed point and so $\{a_n\}$ cannot possibly converge. In fact the sequence increases without limit.

For the sequence which originally prompted this investigation, $\{S_n(\alpha)\}$, δ is α. For this sequence, it is now fairly easy to convince oneself (perhaps by using cobwebbing) that

$$0 < \alpha < \mathrm{e}^{-\mathrm{e}} \Rightarrow \{S_n(\alpha)\} \text{ does not converge;}$$

$$\mathrm{e}^{-\mathrm{e}} < \alpha < 1 \Rightarrow \{S_n(\alpha)\} \text{ converges to the unique}$$

$$\text{solution of } X = \alpha^X;$$

$$1 < \alpha < \mathrm{e}^{1/\mathrm{e}} \Rightarrow \{S_n(\alpha)\} \text{ converges to the smaller}$$

$$\text{of the two roots of } X = \alpha^X;$$

$$\alpha > \mathrm{e}^{1/\mathrm{e}} \Rightarrow \{S_n(\alpha)\} \text{ increases without limit.}$$

This leaves us with the interesting problem of describing the behaviour of $\{S_n(\alpha)\}$ when $0 < \alpha < \mathrm{e}^{-\mathrm{e}}$. The sequence is clearly bounded (in fact $0 < S_n(\alpha) < 1$ for all n), so what does it do? When I had got this far with the problem, I resorted to numerical investigations to prompt imagination and intuition. It soon became apparent that $\{S_n(\alpha)\}$ converged to a 2-cycle. This means that, for each α with $0 < \alpha < \mathrm{e}^{-\mathrm{e}}$, there is a pair of numbers (say X and Y) such that

$$\left.\begin{array}{r} S_{2n}(\alpha) \to X \\ S_{2n+1}(\alpha) \to Y \end{array}\right\} \quad \text{as } n \to \infty.$$

For example

$$\alpha = 0.05 \Rightarrow X \approx 0.1374,\ Y \approx 0.6626,$$

$$\alpha = 0.01 \Rightarrow X \approx 0.0131,\ Y \approx 0.9415.$$

Now $S_{2(n+1)}(\alpha) = \alpha^{S_{2n+1}(\alpha)} = \alpha^{(\alpha^{S_{2n}(\alpha)})}$ and so considering the even terms is equivalent to investigating the recurrence

$$a_{n+1} = F(a_n) \quad \text{with} \quad F(x) = \alpha^{(\alpha^x)}.$$

Exactly the same is true for the odd terms. Also for this new F

$$F'(x) = \alpha^{(\alpha^x)}\alpha^x(\ln\alpha)^2$$

and so if $F(X) = X$ then

$$F'(X) = X\ln X\ln\alpha = \ln X\ln\alpha^X.$$

If $S_{2n} \to X$ and $S_{2n+1} \to Y$, then $\alpha^X = Y$ and $\alpha^Y = X$, and so

$$F'(X) = \ln X\ln Y \quad \text{and} \quad X^X = \alpha^{XY} = Y^Y.$$

With the substitutions $X = e^{-x}$, $Y = e^{-y}$ the equation $X^X = Y^Y$ becomes $e^x/x = e^y/y$. Note that $0 < X < 1$ implies that $x > 0$ and similarly y is positive. It was cryptically suggested in reference 2 that it was useful to know that, for $y > x > 0$,

$$\frac{e^x}{x} = \frac{e^y}{y} \Rightarrow xy < 1.$$

(See Problem 27.8.) This result shows that $1 > F'(X) > 0$. Thus the 2-cycle is always stable. Figure 3 illustrates the 2-cycle, the thick line, when α is 0.03.

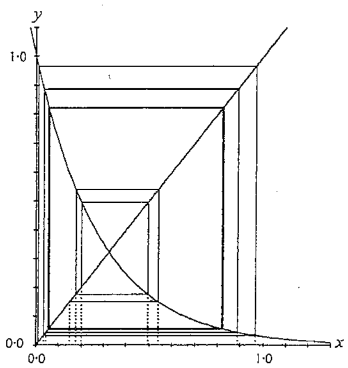


Figure 3. This shows the 2-cycle when $\alpha = 0.03$ (shown as the thick line) together with two separate sequences converging to it from the initial values 0.2 and 0.03.

The figure also shows how the cycle is approached from the inside (starting from $a_0 = 0.2$) as well as from the outside (starting from $a_0 = 0.03$). This figure also provides an explanation for the term 'cobwebbing'. For small α the cycle is close to the values 0 and 1.

More general sequences

I hope that you will be sufficiently interested in this strange sequence to try some numerical experiments and theoretical investigations. In particular try to show that if

$$f_{n+1} = ac^{f_n} + b \quad (n \geq 0) \tag{6}$$

(where $c > 0$) then, provided that $ac^b\ln c \neq 1/e$, $\{f_n\}$ has one of the following types of behaviour:

(a) a stable fixed point X, i.e. $f_n \to X$ as $n \to \infty$ provided that f_0 is close to X;
(b) a stable 2-cycle;
(c) $f_n \to \infty$ as $n \to \infty$.

Try to identify the conditions under which each will occur and verify your results with numerical examples. What happens if $ac^b\ln c = 1/e$?

Inverse sequences

The inverse of the sequence defined by equation (6) is $g_n = ac^{g_{n+1}} + b$ or

$$g_{n+1} = \frac{\ln\dfrac{g_n - b}{a}}{\ln c}. \tag{7}$$

The interesting property of inverse sequences is that an unstable point of the original sequence becomes a stable point for its inverse. Likewise a stable point is transformed into an unstable one (why?). For example, the equation

$$2^x = 5x - 1 \tag{8}$$

can be used to generate the sequence $\{x_n\}$ by

$$x_{n+1} = \frac{2^{x_n} + 1}{5}.$$

A little experimenting will show that this sequence has a stable fixed point, i.e. for suitable x_0 it converges. But equation (8) has two roots. Rewrite it to give another sequence which converges to the other root. The arithmetic involved is horrendous without any aid, trivial with a computer if you know a little programming and still quite easy if you use a calculator.

References

1. Bob Bertuello, An infinite exponential, *Mathematical Spectrum* 27 (1994/95), p. 22.
2. Glenn Vickers, Experiments with infinite exponents, *Mathematical Spectrum* 27 (1994/95), p. 34. □

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Triangle Inequalities for Rectangles

P. SHIU

A magic rectangle is one with at least one side of integer length. A well-known problem is to prove that, if a rectangle can be partitioned into a finite number of smaller magic rectangles, then the large rectangle is magic. This article gives two solutions, one more elementary than the other.

Partitioning of magic rectangles

By a magic rectangle we mean a rectangle at least one of whose sides has integer length. Suppose that a rectangle R can be partitioned into a finite number of (smaller) magic rectangles, all with sides parallel to those of R. The problem is to show that R itself must be a magic rectangle. Actually it is not necessary to stipulate that the smaller rectangles must have their sides parallel to those of R, for this can be shown to follow from the fact that R is partitioned into a *finite* number of rectangles.

This 'magic' problem has been around mathematics departments for some time, and there is now a variety of solutions. We shall give the short and elegant solution favoured by number theorists and then go on to our own more elementary solution.

An elegant solution

This depends on the notion of a double integral over a rectangle R with sides parallel to the coordinate axes.

If R has bottom left vertex (a, b) and λ and μ are the lengths of the sides parallel to the x and y axes, respectively, then R is the set of points

$$\{(x, y) : a \leqslant x \leqslant a+\lambda,\ b \leqslant y \leqslant b+\mu\}.$$

Now let $f(x, y)$ be a continuous function defined on R. Then

$$\int_a^{a+\lambda} f(x, y)\,\mathrm{d}x \quad \text{and} \quad \int_b^{b+\mu} f(x, y)\,\mathrm{d}y$$

are functions only of y and only of x, respectively; and it is a known standard result that

$$\int_b^{b+\mu}\left(\int_a^{a+\lambda} f(x, y)\,\mathrm{d}x\right)\mathrm{d}y = \int_a^{a+\lambda}\left(\int_b^{b+\mu} f(x, y)\,\mathrm{d}y\right)\mathrm{d}x. \quad (1)$$

In other words, it is immaterial in which order the two integrations are performed. The common value of the two *repeated integrals* in (1) is now defined to be the *double integral* of f over R and is written

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y. \quad (2)$$

Double integrals are *additive* in the sense that, if R_1 and R_2 are adjacent rectangles such that $R = R_1 \cup R_2$ is again a rectangle as in figure 1, then

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{R_2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

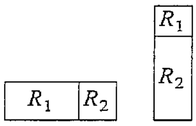


Figure 1

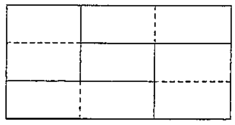


Figure 2

It follows more generally that, if R is partitioned into non-overlapping rectangles $R_1, \ldots, R_n$ (all with sides parallel to the axes), then

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \cdots + \iint_{R_n} f(x, y)\,\mathrm{d}x\,\mathrm{d}y; \quad (3)$$

for, if necessary, suitable further partitioning may be carried out, as in figure 2, until the union of any two adjacent rectangles is again a rectangle.

A particularly important special case of the double integral (2) occurs when $f(x, y)$ takes the simple form

$$f(x, y) = g(x)h(y), \quad (4)$$

where g and h are functions of single variables only; for it then follows easily from the expression of the double integral as either of the repeated integrals in (1) that

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \left(\int_a^{a+\lambda} g(x)\,\mathrm{d}x\right)\left(\int_b^{b+\mu} h(y)\,\mathrm{d}y\right). \quad (5)$$

There is one final remark. The function f in the double integral (2) may take complex values: if $f = p+\mathrm{i}q$, where p and q are real-valued functions, (2) may simply be defined as

$$\iint_R p(x, y)\,\mathrm{d}x\,\mathrm{d}y + \mathrm{i}\iint_R q(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, when f is of the form (4), then (5) will hold even if f, g and h are complex valued.

We are now ready to tackle the magic problem.

For every rectangle R with sides parallel to the axes we put

$$A(R) = \iint_R e^{2\pi i(x+y)}\,dx\,dy. \tag{6}$$

If $R = \{(x,y) : a \leqslant x \leqslant a+\lambda,\ b \leqslant y \leqslant b+\mu\}$, then

$$\begin{aligned} A(R) &= \left(\int_a^{a+\lambda} e^{2\pi ix}\,dx\right)\left(\int_b^{b+\mu} e^{2\pi iy}\,dy\right) \\ &= -\frac{e^{2\pi i(a+b)}(e^{2\pi i\lambda}-1)(e^{2\pi i\mu}-1)}{4\pi^2}. \end{aligned} \tag{7}$$

Hence $A(R) = 0$ if and only if at least one of the numbers λ and μ is an integer, i.e. if and only if R is a magic rectangle.

Now suppose that the rectangle R is partitioned into magic rectangles $R_1, \ldots, R_n$. Then, by (3),

$$A(R) = A(R_1)+\cdots+A(R_n) = 0+\cdots+0 = 0,$$

so that R itself is a magic rectangle.

Some comments on the solution

Many readers are probably wondering what makes the extraordinarily simple proof of the last section work. For instance, is the use of complex numbers and of the complex exponential function really necessary?

First of all we note that the real function $|A(R)|$, where $A(R)$ is defined by (6), is a 'magic area' in the sense that $|A(R)| = 0$ if and only if R is a magic rectangle. Moreover, by (7),

$$|A(R)| = \frac{|\sin\pi\lambda\sin\pi\mu|}{\pi^2}$$

since

$$\begin{aligned} |e^{2\pi i\lambda}-1|^2 &= |(\cos 2\pi\lambda-1)+i\sin 2\pi\lambda|^2 \\ &= (\cos 2\pi\lambda-1)^2+\sin^2 2\pi\lambda \\ &= 2(1-\cos 2\pi\lambda) \\ &= 4\sin^2\pi\lambda \end{aligned}$$

and $|e^{2\pi i\mu}-1|^2 = 4\sin^2\pi\mu$. However, $|A(R)|$ is not additive, i.e. the identity

$$|A(R_1\cup R_2)| = |A(R_1)|+|A(R_2)|$$

does not hold whenever R_1 and R_2 are adjacent rectangles such that $R_1\cup R_2$ is a rectangle. Actually from the point of view of the magic problem this does not matter since the *inequality*

$$|A(R_1\cup R_2)| \leqslant |A(R_1)|+|A(R_2)| \tag{8}$$

(proved below), which generalises to

$$|A(R)| \leqslant |A(R_1)|+\cdots+|A(R_n)| \tag{9}$$

when R is partitioned into $R_1, \ldots, R_n$, is sufficient. For if $|A(R_1)|, \ldots, |A(R_n)|$ are all zero, then

$$0 \leqslant |A(R)| \leqslant 0+\cdots+0 = 0$$

and so $|A(R)| = 0$.

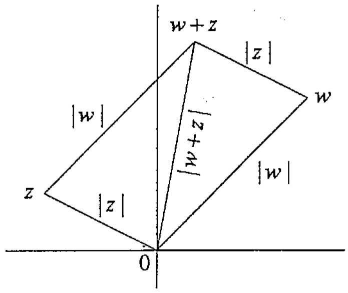


Figure 3

The basis of (8) is the *triangle inequality*

$$|w+z| \leqslant |w|+|z|, \tag{10}$$

which holds for all complex numbers w and z. The intuitive proof, illustrated in figure 3, simply amounts to the fact that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Then, by (10),

$$|A(R_1\cup R_2)| = |A(R_1)+A(R_2)| \leqslant |A(R_1)|+|A(R_2)|,$$

i.e. (8) holds.

It is easy to see that (10) can be generalised to

$$|z_1+\cdots+z_n| \leqslant |z_1|+\cdots+|z_n|$$

and that this inequality leads to (9).

The function $|A(R)|$ which satisfies (8) is called *subadditive*.

We now have the real magic area $|A(R)|$ which is also subadditive, and so solves the magic problem. However, subadditivity is proved by the use of complex numbers and functions; and in the rest of this article we wish to show that there is a real magic area $B(R)$ which can be proved to be subadditive by the use of real methods alone.

The one-dimensional magic problem

This is not really a problem at all, for it merely says that, if an interval I can be partitioned into a finite number of intervals each of which has integral length, then I has integral length. Nevertheless we now introduce a real subadditive magic length which can be applied to the one-dimensional magic problem because it easily leads to a real subadditive magic area. It is the 'nearest integer' function $\|x\|$ defined for $0 \leqslant x < 1$ by

$$\|x\| = \begin{cases} x & (\text{if } 0 \leqslant x < \tfrac{1}{2}), \\ 1-x & (\text{if } \tfrac{1}{2} \leqslant x < 1), \end{cases}$$

and for all other values of x by stipulating that $\|x\|$ has period 1. The graph is shown in figure 4.

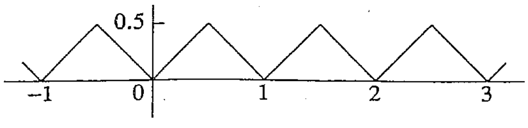


Figure 4. The graph of $y = \|x\|$.

Clearly $\|l\|$ is a magic length, but what is not so obvious is that it is also subadditive, i.e. that

$$\|l_1+\cdots+l_n\| \leqslant \|l_1\|+\cdots+\|l_n\|.$$

The result is easily proved by induction once the 'triangle inequality' corresponding to the case $n = 2$ has been established. In view of the periodicity of the function we may assume that $l_1, l_2 < 1$, and then consideration of the separate cases corresponding to various sizes of l_1, l_2 and l_1+l_2 yields the desired inequality. However, this approach is rather tedious and we give a more elegant proof which can be adapted for use in the two-dimensional magic problem.

The proof in the second section suggests that $\|x\|$ should be represented as an integral. This is easy since the graph of $\|x\|$ has gradient 1 or -1 at each point x such that $2x$ is not an integer. Thus we set

$$f(t) = \begin{cases} 1 & (\text{if } 0 \leqslant t < \tfrac{1}{2}), \\ -1 & (\text{if } \tfrac{1}{2} \leqslant t < 1), \end{cases}$$

and extend the definition of $f(t)$ to all other t by stipulating that $f(t)$ has period 1. The graph of $f(t)$ is shown in figure 5; and clearly

$$\|x\| = \int_0^x f(t)\ \mathrm{d}t.$$

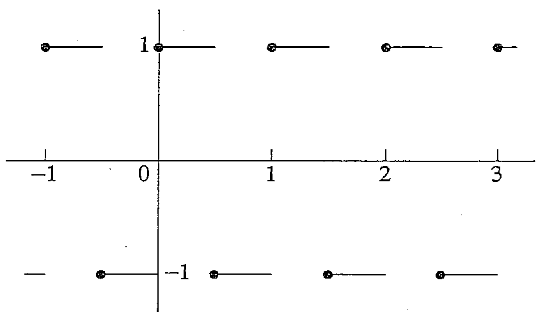


Figure 5. The graph of $y = f(x)$.

Note that when α is not an integer

$$\int_\alpha^{\alpha+x} f(t)\ \mathrm{d}t$$

can be zero without x being an integer (e.g. when $\alpha = -\frac{1}{4}$ and $x = \frac{1}{2}$), so this integral is not a magic area.

To establish the subadditivity of $\|x\|$ we first prove a lemma.

Lemma. For all real α and β,

$$\left|\int_\alpha^{\alpha+\beta} f(t)\ \mathrm{d}t\right| \leqslant \int_0^\beta f(t)\ \mathrm{d}t. \tag{11}$$

Remark. When $\beta \geqslant 0$, (11) is intuitively true; for in the right integral $f(t)$ starts with the value 1 and retains it for the longest possible interval, so the integral should be maximised.

Proof. Since

$$\int_\gamma^{\gamma+1} f(t)\ \mathrm{d}t = 0$$

for all γ, we may assume that $0 \leqslant \beta < 1$. If $0 \leqslant \beta \leqslant \frac{1}{2}$,

$$\left|\int_\alpha^{\alpha+\beta} f(t)\ \mathrm{d}t\right| \leqslant \int_\alpha^{\alpha+\beta} |f(t)|\ \mathrm{d}t$$
$$= \int_\alpha^{\alpha+\beta} \mathrm{d}t = \beta = \int_0^\beta f(t)\ \mathrm{d}t.$$

If $\frac{1}{2} < \beta < 1$, we consider the parts J and K of the interval $\alpha < t < \alpha+\beta$ in which $f(t) = 1$ and $f(t) = -1$, respectively. Clearly J and K each have total length at most $\frac{1}{2}$ and consequently at least $\beta-\frac{1}{2}$. Therefore

$$\left|\int_\alpha^{\alpha+\beta} f(t)\ \mathrm{d}t\right| \leqslant \tfrac{1}{2}-(\beta-\tfrac{1}{2})$$
$$= 1-\beta = \|\beta\| = \int_0^\beta f(t)\ \mathrm{d}t.$$

Thus the lemma is proved.

The subadditivity of $\|x\|$ now follows at once:

$$\begin{aligned}\|l_1+l_2+\cdots+l_n\| &= \int_0^{l_1+l_2+\cdots+l_n} f(t)\ \mathrm{d}t \\ &= \int_0^{l_1} f(t)\ \mathrm{d}t + \int_{l_1}^{l_1+l_2} f(t)\ \mathrm{d}t + \cdots \\ &\qquad + \int_{l_1+\cdots+l_{n-1}}^{l_1+\cdots+l_n} f(t)\ \mathrm{d}t \\ &\leqslant \int_0^{l_1} f(t)\ \mathrm{d}t + \int_0^{l_2} f(t)\ \mathrm{d}t + \cdots \\ &\qquad + \int_0^{l_n} f(t)\ \mathrm{d}t \\ &= \|l_1\|+\|l_2\|+\cdots+\|l_n\|.\end{aligned}$$

A real subadditive magic area

For any rectangle R with sides parallel to the axes and of lengths λ and μ, we define

$$B(R) = \|\lambda\|\,\|\mu\|.$$

The surface $z = \|x\|\,\|y\|$ is illustrated in figure 6.

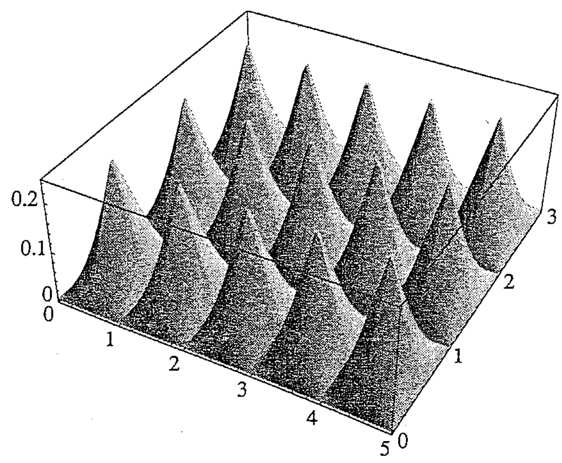


Figure 6 The surface $z = \|x\|\,\|y\|$.

Since $\|l\|$ is a magic length, $B(R)$ is clearly a magic area, i.e. $B(R) = 0$ if and only if R is a magic rectangle. However, we still need to prove that $B(R)$ is subadditive. To do so we introduce the auxiliary function $C(R)$ defined for

$$R = \{(x,y) : a \leqslant x \leqslant a+\lambda,\ b \leqslant y \leqslant b+\mu\} \qquad (12)$$

by

$$\begin{aligned} C(R) &= \left(\int_a^{a+\lambda} f(x)\,\mathrm{d}x\right)\left(\int_b^{b+\mu} f(y)\,\mathrm{d}y\right) \\ &= \iint_R f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Then $C(R)$, as a double integral, is additive.

When $(a,b) = (0,0)$, evidently $C(R) = B(R)$. In general, by the lemma of the previous section,

$$\begin{aligned} |C(R)| &= \left|\int_a^{a+\lambda} f(x)\,\mathrm{d}x\right|\left|\int_b^{b+\mu} f(y)\,\mathrm{d}y\right| \\ &\leqslant \left(\int_0^{\lambda} f(x)\,\mathrm{d}x\right)\left(\int_0^{\mu} f(y)\,\mathrm{d}y\right) = B(R). \quad (13) \end{aligned}$$

Now suppose that R, given by (12), is partitioned into rectangles $R_1,\ldots,R_n$. Let

$$R^* = \{(x,y) : 0 \leqslant x \leqslant \lambda,\ 0 \leqslant y \leqslant \mu\},$$

so that R^* is the translate of R whose bottom left vertex coincides with the origin. The given partition of R induces a partition of R^* into rectangles $R_1',\ldots,R_n'$, say, and we have, by (13),

$$\begin{aligned} B(R) = C(R^*) &= C(R_1')+\cdots+C(R_n') \\ &\leqslant B(R_1')+\cdots+B(R_n') \\ &= B(R_1)+\cdots+B(R_n). \end{aligned}$$

Thus $B(R)$ is subadditive, and the two-dimensional magic rectangle problem has been solved by the use of real functions only.

Acknowledgements

I should like to thank Professor Ian Stewart for our conversation concerning the problem, and also the editors for their help in making the article more accessible to readers. □

Peter Shiu *is a senior lecturer in pure mathematics at Loughborough University. He is a member of the British Mathematical Olympiad Committee which organises the training and selection of talented mathematics students from schools to represent the United Kingdom in the International Mathematical Olympiad. He was the Leader of the 1990 UK Team for the IMO in Beijing, and he is currently the Chair of the Problem Group in the BMOC. His research interest is in number theory, and he has written several previous articles in Mathematical Spectrum.*

Computer Column

Convergence of series

When we add together a finite number of finite numbers, we always get a finite result. But what happens when we add together an infinite number of finite quantities? Well, it depends ... .

The problem is, it depends on the sizes of those quantities, or more precisely, the sizes of *most* of them. Take for instance the sum.

$$1+1+1+1+\cdots$$

Most people would agree that the result is infinity. However, if we take

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\cdots,$$

it is a non-trivial result that the result is $\frac{1}{6}\pi^2$. The following QBASIC program tends to confirm this.

```
s = 0
nterms = 1000
DEF fnterm (i) = 1/i^2
FOR i = 1 TO nterms
s = s + fnterm(i)
NEXT
PRINT s
END
```

This sets the sum s to zero, sets the number of terms to 1000, defines the formula for the general term, $1/i^2$, and proceeds to add together 1000 such terms by brute force. The result it prints tends to confirm the convergence of this series.

But what confidence can we put in such an empirical calculation? Change the above so that it reads $1/i$ instead, and what do we get? Anything much different from $1/i^{1.0001}$? Yet one series diverges and the other converges! Try again with $(-1)^i/i$, and what happens? We can have very little faith in such computed sums, but a great deal of faith in precise mathematical arguments.

Mike Piff □

Mathematics in the Classroom

The aim of this regular feature is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

More on graphic calculators

Dr David Poole, of the Department of Mathematics and Computer Science at the University of Salford, has sent the following comment on the use of graphic calculators mentioned in last September's *Mathematics in the Classroom*.

> May I recommend a valuable use of graphic calculators, especially for all potential physicists and engineers not to mention mathematicians. In my experience, first-year university students have difficulty understanding the significance of differential equation solutions such as
>
> $$e^{-kt}(A\cos\omega t+B\sin\omega t),$$
>
> although these are very common. Sixth formers could helpfully use graphic calculators to gain an understanding of, for example, $\sin t$, $\sin 2t$, $\sin(2t+\alpha)$, $\sin(2t-\alpha)$ followed by e^{-t} multiplying each. Good scope for practicals here! In physics and engineering 'disaster' is associated with t rather than e^{-t} in front, but many engineering students find it hard to relate this to the easily accepted physical phenomena with which they are familiar (resonance). Compared to 10 years ago, familiarity with such things has disappeared. Incidentally, learning that
>
> $$A\cos\omega t+B\sin\omega t = C\sin(\omega t+\alpha)$$
>
> and hence can be understood (pictured!) would be useful indeed—it would certainly benefit a lot of entrants to science and engineering courses in higher education.

Are mathematics A-levels harder or easier than they used to be?

Dr Poole in his letter comments that there are skills that students no longer have on arrival at university that they did exhibit 10 years ago. A recent article in the *Guardian* (17 January 1995) reports on the results of a survey carried out by David Burghes of Exeter University which concludes that young people face tougher A-level mathematics hurdles than they did 40 years ago; that questions are now harder and the range of ideas covered is broader.

Professor Burghes has set out to test claims of a group of mathematics professors that students no longer have the knowledge and skills expected of them a few years ago. To test these claims he gave five papers going back to 1910 to a panel of examiners and mathematicians. Their collective assessment was that the 1930 matriculation examination was the hardest whilst the 1951 A level mathematics paper was the easiest. The 1990 examination was ranked second in terms of level of difficulty. Professor Burghes is quoted as saying, 'It cannot be justifiably claimed that standards have been slipping. In fact the survey data indicate that the standard of A-level papers has been progressively increasing since 1951.' He argues that the perceived decline in mathematical standards may be a result of looking back over past achievements with rose-tinted spectacles.

The basis on which these conclusions are drawn is in my view questionable. As an A-level teacher for the past 10 years I have seen many changes. The technological advances of the calculator have caused immeasurable change, much of which is reflected in the reduced emphasis on curve sketching questions; teaching and learning styles have undergone radical adaptations in response to the differing skills and knowledge of students who experienced GCSE success; textbooks have come and gone in the search for accessibility. But whether standards have risen or fallen, all depends on the standards against which you are measuring, and the aims and objectives of the syllabus on offer. The following three examples reproduced in the *Guardian* come from the papers used in the Exeter survey.

1910 Matriculation examination

Simplify

$$\frac{(x+a)(x-a^{2/3}x^{1/3}+a^{1/3}x^{2/3}-a)}{x^{5/3}-a^{5/3}+xa^{2/3}-ax^{2/3}+x^{1/3}a^{1/3}-a^{1/3}x^{4/3}}.$$

1951 A-level paper

(a) Find values of A, B and C for which

$$\frac{x+4}{x^2(x+2)} \equiv \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}.$$

(b) If A and G are, respectively, the arithmetic and geometric means of $p/(p+1)$ and $1/(p-1)$, prove that $4(A^2-G^4)=1$.

1990 A-level paper

The function f is defined by

$$f\colon x \to \ln(2x-3) \quad (x\in\mathbb{R},\ x>\tfrac{3}{2}).$$

Write in a similar form

(a) the inverse function f^{-1};
(b) the derived function $g = \mathrm{d}f/\mathrm{d}x$
(in each case stating the domain of the function).
(c) In separate diagrams, sketch the curves with equations

$$y = f(x), \qquad y = f^{-1}(x), \qquad y = g(x).$$

On each diagram write the equations of any asymptotes and the coordinates of any point at which the curve meets the coordinate axes.

(d) Evaluate $fgf(8)$, giving your answer to 2 decimal places.

So what do you think about the changing standards?

Carol Nixon □

The Domino Problem

CHRIS HOLT

In this problem we have an unlimited supply of 2×1 dominoes and have to find in how many different ways they can be put together to make a $3\times 2n$ rectangle. We denote this number by T_n. From figure 1, $T_1 = 3$ and we may also put $T_0 = 1$.

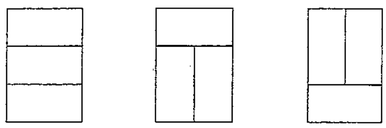


Figure 1

For $n > 2$, there are five ways of starting a $3\times 2n$ rectangle, as in figure 2.

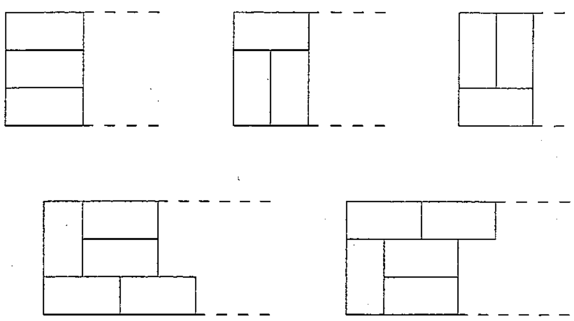


Figure 2

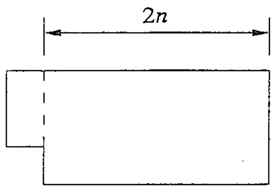


Figure 3

If we denote by G_n the number of ways of covering figure 3, then figure 2 gives that

$$T_n = 3T_{n-1} + 2G_{n-2}.$$

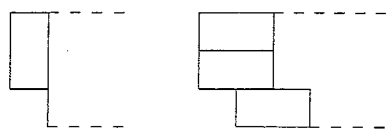


Figure 4

But figure 4 tells us that

$$G_n = T_n + G_{n-1}.$$

Hence

$$G_{n-2} = \tfrac{1}{2}(T_n - 3T_{n-1}),$$

and we can substitute for G_n and G_{n-1} and relabel to give

$$T_n = 4T_{n-1} - T_{n-2}$$

for $n \geqslant 2$, with $T_0 = 1$ and $T_1 = 3$. Thus, for example, $T_2 = 11$. Readers may like to sketch the 11 3×4 arrangements.

There is a standard way to solve the recurrence relation

$$T_n - 4T_{n-1} + T_{n-2} = 0.$$

The solutions of the quadratic equation $\lambda^2 - 4\lambda + 1 = 0$ are $\lambda = 2\pm\sqrt{3}$, so the general solution is

$$T_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n.$$

We can find the constants A and B from the initial conditions $T_0 = 1$ and $T_1 = 3$ to give

$$T_n = \frac{\sqrt{3}+1}{2\sqrt{3}}(2+\sqrt{3})^n + \frac{\sqrt{3}-1}{2\sqrt{3}}(2-\sqrt{3})^n.$$

But why should

$$T_n + T_{n-2} = 4T_{n-1}?$$

Can any reader supply a geometrical argument to show this? □

Chris Holt is in year 9 at Torquay Boys Grammar School.

Letters to the Editor

Dear Editor,

Crossing deserts

With reference to Harold Boas's letter in Volume 26 Number 4, page 122, the problem is far older than he has given. There are quite ancient versions using camels or porters which eat part of their load, but the object is to get as much as possible across the desert. A version appears in Alcuin, 9C, as problem 52 (see *Mathematical Gazette* Volume 76, No. 475, March 1992, pages 102–126). It is a bit unclear—apparently the camels don't eat when unloaded, or they are dispatched when their role is done. An unpublished manuscript of Luca Pacioli, from around 1500, has some similar problems, but the only published description discusses just one of the problems and says the others are similar. The described problem says one has 90 apples to transport 30 miles, but one eats one apple per mile and one can only carry a maximum of 30 apples at a time. Cardan's *Practica Arithmeticae Generalis*, 1539, Chapter 66, Section 57, is a complex problem of carrying food and material up the Tower of Babel. I don't really understand it, but he seems to require 15 625 porters!

Coming to problems of the type discussed in the articles of Gow and Hinderer (*Mathematical Spectrum* Volume 25 Number 3, pages 84–86, and Volume 26 Number 4, pages 100–102), the earliest example I know is from 1907 with two explorers who can each carry provisions for 12 days, but they cannot make any depots (A. Cyril Pearson, *The Twentieth Century Standard Puzzle Book*, Routledge, London, 1907, Part II, pages 139 and 216). I have a reference to a 1910 problem by Sam Loyd which I haven't yet seen. W. W. Rouse Ball (*Mathematical Recreations and Essays*, 5th edition, Macmillan, London, 1911) introduces Exploration Problems and distinguishes two versions with n explorers who can carry food for d days:

(a) if they cannot make depots, they can get one man $nd/(n+1)$ days into the desert and back;

(b) if depots are permitted, they can get a man

$$\tfrac{1}{2}d\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

into the desert and back.

This is the more common form and is one of the cases solved by Hinderer.

Henry Dudeney has a version (*Modern Puzzles*, C. Arthur Pearson, 1926, Problem 49, pages 21 and 111; also published in *536 Puzzles and Curious Problems*, Scribners, New York, Problem 76, pages 22 and 240) without depots, but petrol can only be transferred in unopened gallon cans. R. M. Abraham (*Diversions and Pastimes*, Constable, London, 1933, Problem 34, pages 13 and 25; Dover reprint with the same title, later changed to *Tricks and Amusements with Coins, Cards, String, Paper and Matches*, Dover, New York, 1954, Problem 34, pages 9–10 and 112) gives Ball's form (a) with $n = 4$, $d = 5$ and mentions the general case. As Boas says, the problem attracted a great deal of attention during the 1940s. The first paper of this phase was by O. Helmer (Problem in logistics: The Jeep problem, Project Rand Report RA-15015, 1 December 1946, 7 pp.).

Perhaps the most interesting set of these problems occurs in Pierre Berloquin's *Le Jardin du Sphinx*, Dunod, Paris, 1981, translated by Charles Scribner Jr. as *The Garden of the Sphinx*, Scribners, New York, 1985. I will cite the latter version. He has five problems: 1, 40, 80, 141 and 150, all with $d = 5$; we want to get a man across a desert of width 4, and sometimes back. Problem 1 is Ball's form (a), with $n = 4$ men, using 20 days' water. Problem 40 is Ball's form (b), but using only whole day trips, using 14 days' water. Problem 80 is Ball's form (b), optimized for width 4, using $11\frac{1}{2}$ days' water. Problem 141 uses depots and bearers who don't return, as in Alcuin?? You can get one man, who is the only one to return, a distance

$$d\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

into the desert this way. He gives the optimum form for width 4, using $9\frac{1}{2}$ days' water. Problem 150 is like problem 141, except that no one returns! You can get him

$$d\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

into the desert this way. The optimum here uses 4 days' water.

Yours sincerely,

DAVID SINGMASTER
(South Bank University,
London SE1 0AA, UK)

Dear Editor,

Kepler's polygonal well

Tamara Curnow has studied a 'polygonal well' pattern in *Mathematical Spectrum* Volume 26 Number 4. I published a version in the *Newsletter of the London Mathematical Society* Volume 122 (October 1985) page 12, noting that I had been told that Kepler considered it. There were several responses pointing out other appearances of the problem, one of which must have been where I originally saw it as a student but had forgotten it.

Kepler does indeed consider the pattern, but he does not draw it, in the original Preface to the Reader in his *Mysterium Cosmographicum* of 1596. He was looking for patterns to model the orbits of planets and this was the last planar version before he went to his famous model with spheres and polyhedra. The problem appears in E. Kasner and J. R. Newman, *Mathematics and the Imagination*, Simon and Schuster, New York, 1940, pages 310–312, but they conjecture that the limiting radius is about $\frac{1}{12}$. This was noted in Problem 4793, *American Mathematical Monthly* Volume 65 (1958), page 451 and Volume 66 (1959), pages 242–243, where the limiting radius was determined as 0.11494 to five figures. As Curnow notes, the convergence is very slow and one needs to use rather high-powered techniques. Several distinguished computers have posed or worked on the problem. C. J. Bouwkamp found the reciprocal limiting radius as 8.700 036 625 208 194 5... in 1965. This is the limiting radius if we go outward rather than inward as Curnow did. Richard Hamming gives the problem in his *Computers and Society*, McGraw-Hill, New York, 1972. It also appeared in *Popular Computing* Number 16 (July 1974) page 11, with answer by John Wrench in Number 89 (August 1980) page 20, corrected and improved by Herman P. Robinson in Number 91 (October 1980) page 19 giving the reciprocal value as 8.700 036 625 208 194 503 222 409 859 113 004 971 193 297 949 742 892 092 1... .

Clive J. Grimstone (Note 64.6, *Mathematical Gazette* Volume 64 Number 428, June 1980, pages 120–121) obtained a radius between 0.114 9419 and 0.114 9421. The problem was used in a TV program 'Shrinking Polygons' for the Open University course M203, made in 1986. Ralph P. Boas used it in his *Invitation to Complex Analysis*, Random House, New York, 1987, pages 233 and 237. It will also appear (or has appeared?) in Robert M. Young's *Excursions in Calculus*, a publication of the Mathematical Association of America.

Drawing the increasing pattern to 30-gons gives a rather handsome picture (Figure 1). I put a vertex of each polygon on the x-axis. Then the adjacent vertices generate an oval shape. Alternatively, one can make each polygon touch the preceding one, which gives a rather swirled effect (Figure 2). The following BASIC program will provide the output.

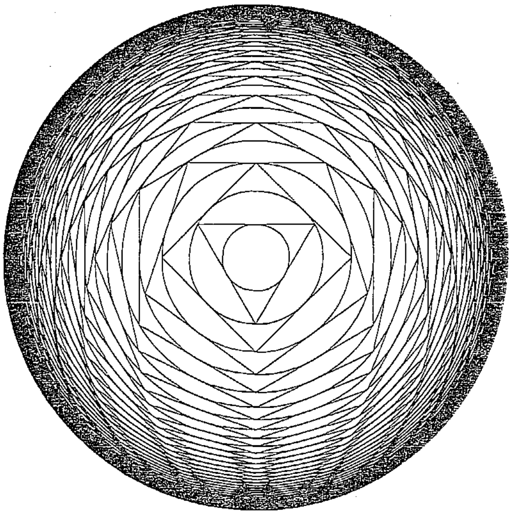


Figure 1

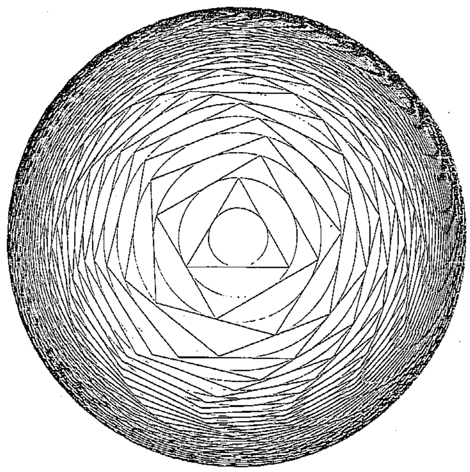


Figure 2

```
10 REM KEPLER. BAS
20 REM This draws the Keplerian polygonal pattern.
100 PRINT "Input R0 and NMAX."
105 REM R0 = initial radius; NMAX = number of polygons drawn.
110 INPUT R, NMAX
120 PI = 3.1415926535
200 SCREEN 9
210 CLS
220 REM These statements initialize and clear the graphics and screen.
230 REM The centre of my graphics screen is at (320,175).
250 CIRCLE (320,175),R,15
260 REM This draws a circle of radius R and centre (320,175).  This
270 REM automatically adapts to the aspect ratio of pixels, so it
280 REM appears circular on the screen.
300 FOR N = 3 TO NMAX
310 TH = PI/N
320 TH2 = TH + TH
330 R = R/COS(TH)
335 REM This computes the radius of the next circle and polygon.
340 CIRCLE (320,175),R,15
347 This positions the 'pen' at the initial point for the polygon.
350 FOR K = 1 TO N
360 AL = K*TH2
370 LINE - (320 + R*COS(AL),175 + .73*R*SIN(AL)),15
373 REM This draws from the last pen position to the given coordinates,
374 REM which are the coordinates of the next point on the polygon.
375 REM The factor .73 is required to compensate for the pixel aspect
376 REM ratio.
380 NEXT K
390 NEXT N
```

Yours sincerely,

DAVID SINGMASTER
(South Bank University,
London SE1 0AA, UK)

Dear Editor,

The bicycle wheel and Langley's adventitious angles

Following W. M. Pickering's piece on bicycle wheels, (Volume 27 Number 1, pages 1–4) he and others may like to know of the book by Jobst Brandt, *The Bicycle Wheel* (Avocet Inc., California, 1983, ISBN 0 96072361 7), which includes a formula for spoke length equivalent to the one given by Pickering. It also includes several other interesting mechanical results, an analysis of the dynamic changes to wheel shape caused by pedalling, braking, etc., as well as being a guide to the craft of wheel-building—in itself a fascinating combination of art and science with a practical outcome.

The problem 'Langley's adventitious angles' (Volume 27, Number 1, page 7) is an old chestnut which I remember from student days. I have not seen the Penguin book, but wonder if it refers to the elegant solution obtained by construction? Construct X on CD so that $\angle XBC = 20°$. Show that $EB = BC = BX = EX = XD$ and so find the angles in triangle EDX.

Yours sincerely,

ANDREW JOBBINGS
(Head of Mathematics,
Bradford Grammar School,
Bradford BD9 4JP, UK)

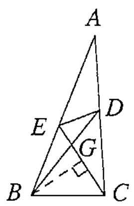
Dear Editor,

Langley's (?) adventitious angles

This puzzle is an exercise (given without solution) in the book *Geometry for Sixth Forms* by C. O. Tuckey and F. J. Swan. I have the second edition, published in 1955. The authors call it 'Mahatma's Puzzle' and state that it was published in December 1938 but unfortunately they don't say where.

They also give an interesting extension of the problem by making one of the angles arbitrary.

$$\angle BAC = 2x,$$
$$\angle ECA = 30°,$$
$$\angle ECB = 30° + 2x,$$
$$\angle EBD = 30° - x,$$
$$\angle DBC = 90° - 3x.$$



The required angle $\angle EDB$ still turns out to be 30°.

All of the useful isosceles triangles which arise in the adventitious case are no longer available, but a solution is readily found using a construction line and two applications of the sine rule.

Let $\angle EDB = \alpha$, put $BC = 1$ and draw BG perpendicular to EC. The following angles are required:

$$\angle ABD = 30 - x, \qquad \angle EDC = \alpha + 30 + x,$$
$$\angle GBC = 60 - 2x, \qquad \angle BEC = 30 + 2x.$$

Since $\angle BEC = \angle BCE$, $BE = 1$. Further, $EC = 2GC = 2\sin(60 - 2x)$. Applying the sine rule to triangles BED and CED gives

$$\frac{DE}{\sin(30-x)} = \frac{1}{\sin\alpha} \quad \text{and} \quad \frac{DE}{\sin 30} = \frac{2\sin(60-2x)}{\sin(\alpha+30+x)}.$$

Elimination of DE and a little trigonometric manipulation leads to

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

and so

$$\alpha = 30°.$$

Yours sincerely,

BARRY MARTIN
(Computer Science and Applied Mathematics,
Aston University, Birmingham B4 7ET, UK)

Dear Editor,

Langley's adventitious angles

I was interested to see the panel about 'Langley's adventitious angles' on page 7 of *Mathematical Spectrum* Volume 27 Number 1. I can fill in some of the background.

E. M. Langley taught mathematics at Bedford Modern School, where he had also been a pupil. He was a prominent member of the Mathematical Association for many years, serving as Hon. Secretary from 1885 to 1893 and as founder editor of the *Mathematical Gazette* for its first six issues in 1894–5. He posed the problem of the adventitious angles in the *Gazette* for October 1922 (Volume 11 Number 160). Various solutions sent in by twelve readers were published in May 1923 (Volume 11 Number 164); most of these were trigonometrical, but three, as well as Langley's own, used methods of pure geometry.

The problem surfaced again in the *Gazette* in 1975, while I was editor. An article by Colin Tripp in June of that year (Volume 59 Number 408) generalised the problem by asking what other angles BAC, DBC, ECB other than 20°, 60°, 50° would enable $\angle EDB$ to be found. With the restriction to integral numbers of degrees for all the angles, he conjectured that there were 53 solutions, some of these described by their geometrical forms as 'kites' or 'fans'. This again generated a large reader response, which was summarised in two further articles in March 1977 (Volume 61 Number 415) and October 1978 (Volume 61 Number 421); this last issue also included an article by John Rigby generalising the problem still further, by removing the requirement for the triangle ABC to be isosceles and concentrating on the quadrangle $BCDE$ with four angles given.

Langley died in 1933, and in the issue for October that year the *Gazette* published two obituary notices. One of these was by E. T. Bell, of the California Institute of Technology, who had been a pupil of Langley at Bedford Modern. It is a nice coincidence that the issue of *Spectrum* reviving Langley's problem also contains a review of Constance Reid's book about Bell. I quote one sentence from his appreciation:

> Since then I have listened to many lectures by other mathematicians, some of them famous, but I have yet to hear a mathematical subject presented with the force, the conviction, the clarity and the all-sufficient brevity which characterized Mr Langley's expositions.

The other notice, which is more about Langley the man rather than Langley the mathematician, was by J. P. Kirkman. (Is there another coincidence here? I wonder if he was of the same family as T. P. Kirkman, famous for the 'fifteen schoolgirls' problem, who is mentioned in *Spectrum* in the panel next to Langley's problem.) This reveals Langley as a man of wide interests, and especially as a botanist: he became an authority on hybridisation, and a cultivated blackberry was named 'Edward Langley' in acknowledgement of his researches in that field.

Yours sincerely,

DOUGLAS QUADLING
(12 Archway Court, Barton Road,
Cambridge CB3 9LW, UK)

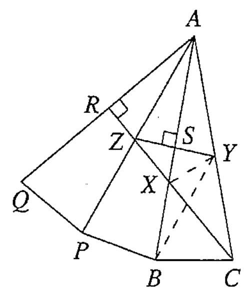
[Norman Routledge sent essentially three solutions, one of which is given below.

Reflect $\triangle ABC$ in AB, giving $\triangle ABP$.

Reflect $\triangle ABP$ in AP, giving $\triangle AQP$.

Draw the perpendicular bisector of AQ, cutting AQ in R, and AB in X.

Draw the perpendicular bisector of AB, cutting AB in S, and AC in Y.



The perpendicular bisectors of the sides of $\triangle AQB$ meet, at Z, say, on AP.

Since $\triangle AQC$ is equilateral, RX must go through C, and $\angle XCY = 30°$. Thus $\angle XCB = 50°$ and X is E.

Since $\angle SAY$ reflects in SY to $\angle SBY$, then $\angle SBY = 20°$ and $\angle YBC = 60°$, so Y is D.

In $\triangle ARX$, two of the angles are 90° and 40°, so $\angle RXA = 50°$. This angle reflects in AX to $\angle SXY$, which is thus 50°. (It is easily seen that Z reflects to Y.) Hence in $\triangle BXY$, $\angle XYB = 30°$, or $\angle EDB = 30°$, as required. ED.]

Dear Editor,

Sums of arithmetic progressions

In Volume 27 Number 2, pages 30–32, Roger Cook and David Sharpe gave a result that described which natural numbers $N > 1$ are expressible as a sum of natural numbers in arithmetic progression with a given common difference d. The following are necessary and sufficient conditions to guarantee a solution of the arithmetic progression

$$N = r+(r+d)+(r+2d)+\cdots+(r+sd):$$

(a) $s+1$ divides $2N$;
(b) $2N/s(s+1) > d$;
(c) $2N/s(s+1) \equiv sd \pmod 2$.

The proof of this is quite easy. The following are easy consequences.

Corollary 1. If $s+1$ is an odd divisor of N, then N will be the sum of an arithmetic progression of $s+1$ terms and common difference d for all values of d less than $2N/s(s+1)$, and for no others.

Corollary 2. If $s+1$ is an even divisor of $2N$, then N will be the sum of an arithmetic progression with $s+1$ terms and common difference d for even d smaller than $2N/s(s+1)$ when $2N/(s+1)$ is even and for odd d smaller than $2N/s(s+1)$ when $2N/(s+1)$ is odd, and for no others.

Corollary 3. If N is a power of 2, then it can never be the sum of an arithmetic progression with an odd common difference.

Corollary 4. A natural number N is the sum of (more than one) consecutive natural numbers if and only if it is not a power of 2.

Corollary 5. A natural number $N > 1$ is a sum of natural numbers which form an arithmetic progression with common difference 2 if and only if it is not prime.

Yours sincerely,

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(52 Violet Street,
Waterside,
Londonderry BT47 2AP, UK)

[The difference between Ken Adams' result and the result given by Roger Cook and David Sharpe is that their result does not involve the parameter s. ED.]

Dear Editor,

Further extensions to the 1994 problem

In *Mathematical Spectrum* Volume 27 Number 1, pages 15–16, Mike Wenble defines $S_n(i,j)$ as the set of integers between 1 and n that can be formed by the digits i and j and the arithmetical operators including 'square root', 'factorial' and concatenation. His article continued by giving a table showing the orders of $S_{100}(i,j)$ for all i and j from 1 to 9 and invited readers to verify this. I have now done so, and I hope that he will be pleased to learn that my results are the same as his.

I then went on to define the obvious extension $S_n(i,j,k,l)$ and wrote a program to evaluate $S_{151}(1,9,8,9)$ to $S_{151}(1,9,9,7)$. I enclose a summary of the results. Again, I would be interested to hear of any verification of these.

It is interesting to compare my results with those obtained by your readers in the annual *Mathematical Spectrum* puzzle. My results for 1989 and 1992 are the same as those of your readers. For 1993 I have found two answers that your readers failed to find:

$$65 = -1+\sqrt{[(\sqrt{9})!\times\{(\sqrt{9})!+(3!)!\}]},$$

$$67 = 1+\sqrt{[(\sqrt{9})!\times\{(\sqrt{9})!+(3!)!\}]}.$$

For 1994 all numbers up to 150 are possible. I am unable to find any reader results for 1990 or 1991.

Summary of results

```
0056 FINDALL(1,9,8,9)
Missing numbers are: 132 133 142 148 151

0061 FINDALL(1,9,9,0)
Missing numbers are: 51 65 67 68 69 74 75 77
102 104 105 106 107 108 131 132 134 135 136
137 138 140 141 142 146 147 148 149 150 151

0004 FINDALL(1,9,9,1)
Missing numbers are: 51 65 66 68 69 74 75 77 92
102 104 105 106 107 108 109 131 132 134 135 136
137 138 140 141 142 146 147 148 149 150 151

0013 FINDALL(1,9,9,2)
Missing numbers are: 69 75 93 104 105 134 135
136 137 140 141 147 148 149 150 151

0018 FINDALL(1,9,9,3)
Missing numbers are: 68 70 131 133 137 142 143
145 146 148 149 151

0027 FINDALL(1,9,9,4)
No missing numbers

0029 FINDALL(1,9,9,5)
Missing numbers are: 81 82

0038 FINDALL(1,9,9,6)
Missing numbers are: 68 70 95 97 131 133 137
141 142 143 145 146 148 149 151

0048 FINDALL(1,9,9,7)
Missing numbers are: 108 109 111 115 123 129
131 132 134 135 138 139 142 146 148 149 150
```

Yours sincerely,

MIKE SWAIN
(157 Old Woosehill Lane,
Wokingham, Berks RG11 2UN, UK)

Dear Editor,

Arithmetical functions

Let $\pi(x)$ denote the number of primes not greater than x, so that

$$\pi(x) = \sum_{\substack{p\leqslant x\\ p\text{ prime}}} 1 = \sum_{n\leqslant x} a(n),$$

where

$$a(n) = \begin{cases} 1 & (n \text{ prime}),\\ 0 & (\text{otherwise}).\end{cases}$$

Various formulae for $\pi(x)$ can be obtained using different expressions for $a(n)$. Firstly we consider functions α with $0 \leqslant \alpha(n) \leqslant 1$ and $\alpha(n) = 1$ if and only if n is prime, for example we can take

$$\alpha(n) = \frac{\phi(n)}{n-1},$$

$$\alpha(n) = \frac{2}{d(n)},$$

or

$$\alpha(n) = \frac{n+1}{\sigma(n)},$$

where $\phi(n)$ is Euler's function, $d(n)$ is the number of divisors of n and $\sigma(n)$ is their sum. We then take $a(n) = [\alpha(n)]$.

More generally, we can use any function α such that $0 \leqslant \alpha(n) < 2$ and $\alpha(n) \geqslant 1$ if and only if n is prime. Examples of this type are

$$\alpha(n) = \frac{\phi(n)+d(n)}{n+1} \quad\text{or}\quad \alpha(n) = \frac{2n}{\phi(n)+\sigma(n)}.$$

The interested reader will no doubt be able to construct many more examples, using basic arithmetic functions of number theory. It is also possible to use Wilson's theorem: for $n > 1$,

$$(n-1)!+1 \equiv 0 \pmod n$$

if and only if n is prime. This gives that

$$\left|\cos\frac{\{(n-1)!+1\}\pi}{n}\right| = 1$$

if and only if n is prime, and hence

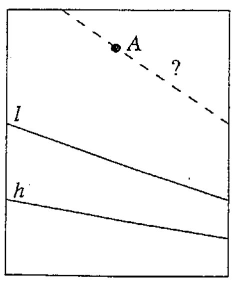
$$\pi(x) = \sum_{2\leqslant n\leqslant x}\left[\left|\cos\frac{\{(n-1)!+1\}\pi}{n}\right|\right].$$

Yours sincerely,

F. SAJDAK
(Student at the
University of Auckland,
New Zealand)

Oh Yes It Is!

We recently published an article on Fermat's last theorem (*Mathematical Spectrum* Volume 26 Number 3, pages 65–73), stimulated by Andrew Wiles' claim to have proved it. Unfortunately, by the time our article appeared in print experts had detected a flaw in Wiles' original proof and he had retracted it. However, by the summer of 1994 he had repaired the proof and submitted it to the *Annals of Mathematics*. The proof has now been refereed by eminent mathematicians who are agreed that it is valid. So the answer to whether or nor Fermat's last theorem is a theorem is 'Oh yes it is!'.



A straight line h drawn on a piece of paper is called 'the horizon'. Straight lines are said to be 'parallel' if they intersect on the horizon (produced). Given a straight line l and a point A on the paper, how would you construct the straight line through A parallel (in the above sense) to l? The construction should be confined to the piece of paper.

G. Lasters
(Tienen, Belgium)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

27.9 In the diagram, BD and CE bisect angles ABC and ACB respectively, and $BD = CE$. Prove that triangle ABC is isosceles.
(Submitted by B. J. Hulbert, Reading)

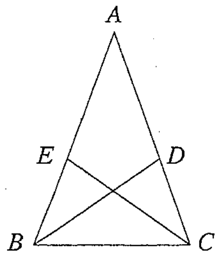


Figure 27.9

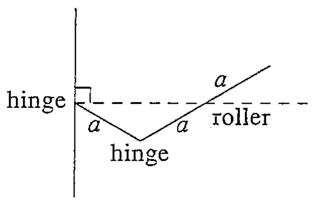


Figure 27.10

27.10 A door has two leaves, one being twice the size of the other, joined by a hinge. The smaller leaf is hinged to the wall and the larger has a roller half way along which moves along the dotted line shown. How much floor space is needed, and what is the locus of a point on the door when it is opened?
(Submitted by Jamaludin Md Ali, Penang, Malysia.)

27.11 Find a formula for the volume of a tetrahedron in terms of the lengths of the three edges which meet at a vertex and the three face angles at that vertex.
(Submitted by J. A. Scott, Chippenham.)

27.12 A circular table with radius r is pushed into the corner of a square room. A point P on the edge of the table is m units from one wall and n units from the other, where $m, n < r$. Prove that, if m, n and r are all integers and m and n are coprime, then either m or n is a perfect square and the other is even.
(Submitted by Sammy Yu, age 14, and Jimmy Yu, age 12, special students at the University of South Dakota.)

Solutions to Problems in Volume 27 Number 1

27.1 Prove that there are infinitely many odd integers n such that no term of the infinite sequence

$$1994^{1994}+1, \quad 1994^{1994^{1994}}+1, \quad 1994^{1994^{1994^{1994}}}+1, \quad \ldots$$

is divisible by n.

Solution

$1994 \equiv -1 \pmod 3$, so all the numbers in the sequence are congruent to 2 (mod 3) and so are not divisible by 3. Hence none of them is divisible by $3m$ for any odd integer m.

Also solved by Vytautas Paskunas, Christ's Hospital School, Horsham, Edward Waldron, St Edward's School, Oxford, and Richard Edlin, Westminster School.

27.2 Imagine 1994 trees standing in a circle, with a starling in each tree. Every few minutes, two of the birds fly in opposite directions round the circle to the neighbouring tree. Show that at no time are all the birds in the same tree.

Solution and extension by Rosemary Sexton, Kendrick School, Reading

Number the trees 1 to 1994 in order round the circle. After n moves, let the total number of birds in odd-numbered trees be k_n. Then $k_0 = 997$. In any move, either two birds leave odd-numbered trees or two birds leave even-numbered trees or one bird leaves an odd-numbered tree and one leaves an even-numbered tree. Hence

$$k_{n+1} = k_n - 2 \text{ or } k_n + 2 \text{ or } k_n.$$

It follows that k_n is always odd, so it can never be 0 or 1994. Hence at no time can all the birds be in the same tree. Note that this proof works equally well if two birds fly in the same direction round the circle to a neighbouring tree. It also works whenever the number of trees is twice an odd number.

Suppose now that we have n trees. For which values of n is it possible for all the birds to be in the same tree at the same time? Number the trees 0 to $n-1$ in order round the circle. Denote by b_{ij} the number of birds in tree i after j moves, and put

$$s_j = \sum_{i=0}^{n-1} i b_{ij}.$$

Suppose that, on the $(j+1)$th move, the bird in tree x moves to tree $x+1 \pmod n$ and the bird in tree y moves to tree $y-1 \pmod n$. Then

$$b_{x,j+1} = b_{xj} - 1, \qquad b_{y,j+1} = b_{yj} - 1$$

and, modulo n in the first suffix,

$$b_{x+1,j+1} = b_{x+1,j} + 1, \qquad b_{y-1,j+1} = b_{y-1,j} + 1$$

and $b_{i,j+1} = b_{ij}$ for all other i. Hence

$$s_{j+1} \equiv s_j - x - y + (x+1) + (y-1) \pmod n$$

i.e.

$$s_{j+1} \equiv s_j \pmod n.$$

Hence, after k moves,

$$s_k \equiv s_{k-1} \equiv \ldots \equiv s_0 \pmod n.$$

But

$$s_0 = 1 + 2 + \ldots + (n-1) = \tfrac{1}{2}(n-1)n$$

and so

$$s_k \equiv \tfrac{1}{2}(n-1)n \pmod n.$$

Suppose that, after k moves, all the birds are in tree m. Then $s_k = mn$ and

$$\tfrac{1}{2}(n-1)n \equiv mn \equiv 0 \pmod{n}.$$

Hence $\frac{1}{2}(n-1)$ is an integer, so n must be odd.

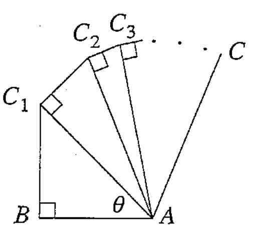
If n is odd, then it is possible for all the birds to be in tree $\frac{1}{2}(n-1)$ at the same time as follows. The birds in trees 0 and $n-1$ both move in $\frac{1}{2}(n-1)$ moves to tree $\frac{1}{2}(n-1)$. Then the birds in trees 1 and $n-2$ move in $\frac{1}{2}(n-3)$ moves to tree $\frac{1}{2}(n-1)$. And so on.

Thus it is possible for all the birds to be in the same tree if and only if the number of trees is odd.

Also solved by Junji Inaba, William Hulme's Grammar School, Manchester, Vytautas Paskunas, Toby Gee, Frome Community College and Richard Edlin.

27.3 In the figure, the sequence of angles is $\theta, \theta/2, \theta/2^2, \theta/2^3, \ldots, \theta/2^n$, and C is the limiting point of the sequence $C_1, C_2, C_3, \ldots$. Express AC in terms of AB and θ.

$$\begin{aligned} \angle BAC_1 &= \theta \\ \angle C_1AC_2 &= \theta/2 \\ \angle C_2AC_3 &= \theta/2^2 \\ \cdots &= \cdots \\ \angle C_iAC_{i+1} &= \theta/2^i \end{aligned}$$



Solution by Junji Inaba

$$\begin{aligned} AC_n &= AB \sec\theta \sec\frac{\theta}{2} \sec\frac{\theta}{2^2} \cdots \sec\frac{\theta}{2^{n-1}} \\ &= \frac{AB \sin\frac{\theta}{2^{n-1}}}{\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^{n-1}}\sin\frac{\theta}{2^{n-1}}} \\ &= \frac{2AB \sin\frac{\theta}{2^{n-1}}}{\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^{n-2}}\sin\frac{\theta}{2^{n-2}}} \\ &= \frac{2^2AB \sin\frac{\theta}{2^{n-1}}}{\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^{n-3}}\sin\frac{\theta}{2^{n-3}}} \\ &= \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &= \frac{2^nAB\sin\frac{\theta}{2^{n-1}}}{\sin 2\theta} \\ &= \frac{2\theta AB}{\sin 2\theta} \times \frac{\sin\frac{\theta}{2^{n-1}}}{\frac{\theta}{2^{n-1}}} \\ &\to \frac{2\theta AB}{\sin 2\theta} \quad \text{as } n \to \infty, \end{aligned}$$

so

$$AC = AB\frac{2\theta}{\sin 2\theta}.$$

Also solved by Vytautas Paskunas.

27.4 Determine, for all positive integral values of k, the behaviour as $x \to 0+$ (i.e. as x tends to 0 through positive values) of

$$\frac{1}{\sin^k x} - \frac{1}{x^k}.$$

Solution

$$\begin{aligned} &\frac{1}{\sin^k x} - \frac{1}{x^k} \\ &= \frac{x^k - \sin^k x}{x^k \sin^k x} \\ &= \frac{x-\sin x}{x^k \sin^k x}(x^{k-1} + x^{k-2}\sin x + x^{k-3}\sin^2 x + \cdots + \sin^{k-1} x) \\ &= \frac{x-\sin x}{x^3}\frac{x}{\sin x}\left[\left(\frac{x}{\sin x}\right)^{k-1} + \left(\frac{x}{\sin x}\right)^{k-2} + \cdots + 1\right]x^{2-k}. \end{aligned}$$

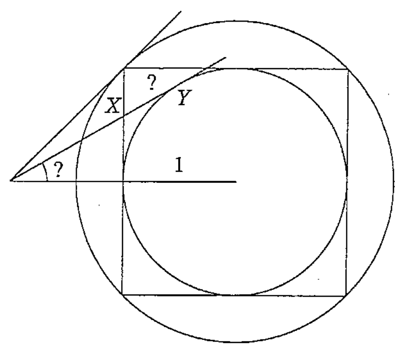
Now $x/\sin x \to 1$ as $x \to 0$ and

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} = \tfrac{1}{6}.$$

Also $x^{2-k} \to 0$ as $x \to 0$ when $k = 1$, $x^{2-k} = 1$ when $k = 2$ and $x^{2-k} \to \infty$ as $x \to 0+$ when $k \geqslant 3$. Hence, as $x \to 0+$,

$$\frac{1}{\sin^k x} - \frac{1}{x^k} \to \begin{cases} 0 & (\text{if } k = 1), \\ \frac{1}{3} & (\text{if } k = 2), \\ \infty & (\text{if } k \geqslant 3). \end{cases}$$

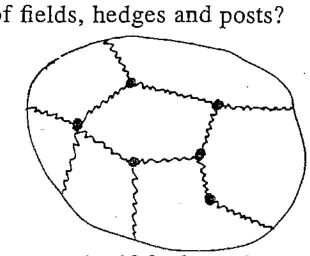
Also solved by Vytautas Paskunas. □



What are the angle and the distance XY?

Seyamark Jafari
(Ahwaz, Iran)

An island is divided into a field separated by hedges, and a post is put at each place where hedges meet. Is there a relationship between the number of fields, hedges and posts?



7 fields, 12 hedges, 6 posts

Reviews

The Knot Book. By C. C. ADAMS. W. H. Freeman & Co. Ltd., New York, 1994. Pp. xiii+306. Hardback \$32.95. (ISBN 0-7167-2393-X).

Take a length of string, tangle it up in some way and splice together the two ends: that is what mathematicians mean by a knot. The first problem when presented with such a knot is whether it can be disentangled and layed out to form a plain circle (the 'unknot'); if after some fiddling you succeed, the answer is clear, but if not, you are in an uncertain position. Either you have a genuinely knotted knot, or you simply lack the patience or dexterity to unknot it. Then if you are given two knots you can ask if they can be manipulated until they are identical; again all is well if you succeed but uncertain if you fail.

The basic idea of knot theory is to find an effective way to distinguish one knot from another when they are different, and to do this one defines an algebraic 'invariant' (such as a number or polynomial) associated with a knot. A useful invariant will come with a reasonably practical method of calculation. For example, given a particular knot K you might lay it down on the table, and from that particular view you might calculate a polynomial $f[K]$ by some routine method. If you pick up the knot and fiddle around some more and put it down again, it will look different, but you can use the same method and, although the steps in the calculation will be different, the answer will be the same polynomial $f[K]$: the answer is *invariant* under manipulation of the knot.

It is easy to tell when two polynomials are different, so that if $f[K_1] \neq f[K_2]$ it follows that K_1 can never be manipulated so that it is K_2: the two knots are genuinely different. (Of course it might happen that $f[K_1] = f[K_2]$, and then you are none the wiser; that is quite rare, and you can either seek a different invariant to distinguish the two knots, or else carry on trying to manipulate one to be like the other.)

The first surprising thing is that there are really lots of invariants one can define; many have been known and studied since the time of Poincaré (1854–1912). However until the mid 1980's all methods of defining invariants involved rather sophisticated mathematics and needed a course in algebraic topology to explain. Then two things happened: V. F. R. Jones found a new invariant which could be *calculated* rather easily, and not long afterwards L. Kauffman found that it could also be proved to be an invariant by completely elementary arguments (explained in detail on pages 149–153 in the book).

This has given rise to many exciting new ideas in knot theory and topology, and to connections with physics and other areas of mathematics. Furthermore it has also opened the way to allow knot theory to be presented to a wider audience: there are many new books, courses and magazine articles explaining these ideas. The present book is probably the most extended attempt to explain knot theory to a general audience.

The book mentions an enormous number of different ideas, which makes for fascinating and very stimulating reading. Each of the ten chapters covers a rather different aspect of knots, and the variety may be suggested by the titles: 1. Introduction, 2. Tabulating knots, 3. Invariants of knots, 4. Surfaces and knots, 5. Types of knots, 6. Polynomials, 7. Biology, Chemistry and Physics, 8. Knots, links and graphs, 9.Topology, 10. Higher-dimensional knotting. The book finishes with a tabulation of knots and a useful annotated bibliography.

With this coverage in 300 pages it is no criticism that the treatment is sometimes swift or superficial. However, the exposition is somewhat uneven, with some explanations directed at the novice and others assuming partial familiarity. In particular, the author often describes elegant geometric arguments, and readers who are not familiar with the ideas may find themselves a little short of explanation. I would have preferred a little more indication about how much harder some things are than others, and when things are being glossed over.

Quibbles apart, this book gives a fascinating glimpse into a very active and accessible corner of topology. Anyone with a vivid spatial imagination and a little background in mathematical ways of thought will find plenty to interest them, and the effort will be well rewarded. Nonetheless, I suspect the book would be best used to provide background or accompaniment to a lecture course such as the one that gave rise to it.

University of Sheffield J. P. C. GREENLEES

Mathematics: A Human Endeavor, 3rd edn. By HAROLD R. JACOBS. W. H. Freeman & Co. Ltd., New York, 1994. Pp. xii+678. Hardback £23.95. (ISBN 0-7167-2425-X).

This book will be familiar to many. Now in its third edition, "Mathematics: A Human Endeavor" is a book dedicated to all those who firmly believe that an interesting book on mathematics is a contradiction in terms. Indeed, it is subtitled 'A book for those who think they don't like the subject' and that is precisely what it is, a book that aims to both inform and entertain, and, in so doing, get across the message that mathematics can be great fun.

Here is a basic grounding in fundamental mathematical ideas. Inductive and deductive reasoning, the idea of functions, logarithms, the regular polyhedra, a discussion on statistics are just a flavour for what this book has to offer.

In general, chapters begin with a bit of preliminary chat, but the bulk of the book consists of set exercises for the student. Jacobs aims, for the main, to draw on real-life situations. For example, a hyperbola is introduced as a curve formed when a supersonic plane's conical shock wave intersects with the ground. While getting the student to discover how a function works, Jacobs encourages them to work with a formula which approximates the length of an ocean wave as a function of its speed. At the end of each chapter, the material is consolidated with a section for further exploration and a summary and review.

The book is rounded off with a chapter on topology and networks, including a look at the problem of the seven bridges at Königsberg and the fascinating properties of the Möbius strip. This use of recreational mathematics to stimulate the student's interest is typical throughout, for this is one of those books which, at the right level, you can flop open at any page and find something that will occupy you for half an hour. The emphasis is definitely on the student working things out for him or herself, and the text is peppered with many interesting little puzzles to tangle with.

What impressed me in particular about this book is the author's eagerness to acquaint the student with important milestones in mathematics and to familiarise the reader with the names of great mathematicians, and the multitude of different applications in which he demonstrates mathematics at work, or rather, in which he gets the student to uncover mathematics at work.

Every page is lively and colourful. 982 illustrations in 678 pages, many of which are cartoons which are actually funny, make this a bright and attractive book that does not scare off potential readers at the very sight of it. The style is not didactic but encouraging and enthusiastic, the description of every concept benefiting wonderfully from the imagination of a man who could explain complex analysis to a village idiot. Anyone considering a career in teaching mathematics could do a lot worse than have a read through this book.

Above all, it is, as mathematics should be, terrific fun. There is even a foreword by Martin Gardner—now what more could you ask for?

Student at the University of Bristol — MARK BLYTH

Other books received

Technical Calculus with Analytic Geometry. By JUDITH L. GERSTING. Dover, New York, 1992. Pp. viii+501. Paperback £13.95 (ISBN 0-486-67343-X).

This is a republication of a book first published in 1984. Designed with the users of mathematics in mind, this student-friendly text contains a wealth of examples. Special features are the ubiquitous owls offering words of advice, some more helpful than others, and the evocative pictures at the head of each chapter.

Representation Theory of Finite Groups. By MARTIN BURROW. Dover, New York, 1993. Pp. viii+184. Paperback £6.95 (ISBN 0-486-67487-8).

This is a republication of a textbook which first appeared in 1965. □

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LONDON MATHEMATICAL SOCIETY

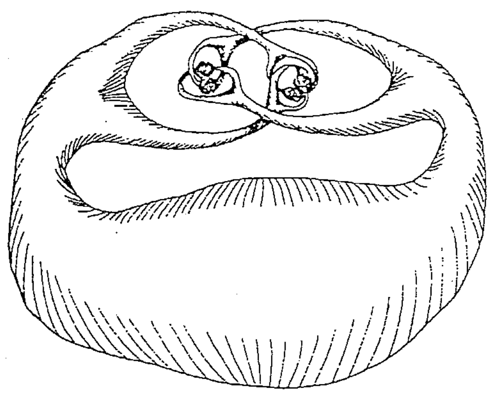
1995 POPULAR LECTURES

Manchester University - Friday 2 June
Edinburgh University- Thursday 15 June
Imperial College - Friday 30 June

Professor Peter Saunders

MEASURING THE MARIGOLD

"What mathematics can tell us about flowers ... and fruit flies ... and Friesians ... "



Professor Nigel Ray

WILD GEOMETRY

"Tame geometry is epitomised by familiar shapes of everyday life. Wild geometry is much more mysterious, and in this talk, we introduce some of its objects, and discuss their growing importance."

EDINBURGH UNIVERSITY Commences at 2.00 pm, 3.00 pm refreshments, ends at 4.30 pm. Lecture Theatre A, David Hume Tower, George Square, Edinburgh. Admission is free. Enquires to Dr P. Heywood, Department of Mathematics & Statistics, Edinburgh University, James Clerk Maxwell Building, The King's Buildings, Edinburgh EH9 3JZ.

MANCHESTER Commences at 7.00 pm, 8.00 pm refreshments, ends at 9.30 pm. Lecture Theatre B, Roscoe Building, University of Manchester. Admission free, with ticket in advance. Apply by Friday 26 May to Dr M. Prest, Department of Mathematics, University of Manchester M13 9PL. A stamped addressed envelope would be appreciated.

LONDON Commences at 7.30 pm, 8.30 pm refreshments, ends at 10.00 pm. The Great Hall, Sherfield Building, Imperial College, South Kensington, London SW7. Admission free, with ticket in advance. Apply by Monday 26 June to Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL. A stamped addressed envelope would be appreciated.

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