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CONTENTS

EUCLIDE PARACELSO BOMBASTO UMBUGIO

by CLAYTON W. DODGE

University of Maine at Orono

Since an article by him appeared in the October 1976 issue of EUREKA [21], it seems appropriate to write a word or two about that eminent numerologist E.P.B. Umbugio. Details of his early life and education are scanty, but we shall do our best to report accurately the life of this fabulous man. Great care will be taken to state only those facts that have been confirmed by known authorities or that have appeared in print. In any case, all statements have been carefully verified and their sources listed at the end of this paper. To the best of my knowledge that bibliography is complete. I would be grateful, however, to hear of any other sources of information about our hero.

Euclide Paracelso Bombasto Umbugio first gained international fame by planning an unusual seven-month expedition to Antarctica, whose postponement was announced in the April 1, 1913, issue of the *Erewhon Daily Howler*. He had discovered that June 13 would fall on a Friday that year, so he felt he should put off the trip until he could find a span of seven months without a Friday-the-thirteenth. His statistical analysis showed that he should have had to wait just a very few years. Since 37 years had passed when the above information was reported in *The American Mathematical Monthly*, there seemed to be some question about this genius' calculations, so readers were asked to check his work. It was found that seven consecutive months without a Friday-the-thirteenth can never occur [6].

That Professor Umbugio should have made such an error as this one in his numerological researches is unfortunate, for surely we would have learned much from his proposed expedition. But then it should be noted that he was quite young at the time, not more than 13 years old [2]. It seems reasonable that he should become 13 years old in 1913, and the fact that his trip was announced on April first strongly suggests that he attained the age of 13 on that date for, as we shall see, he is a firm believer in numerology. Hence we believe he was born April 1, 1900 although, as just stated, we have no definite confirmation of either his birth or its date.

It appears he has lived all his untumultuous[1] life in his native country of

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[1]Five *u*'s in this word: very rare. But the subject of this chronicle would undoubtedly be able to perceive more numerological implications in a word containing *nine u*'s which can be found on page 692 of the unabridged edition of *The Random House Dictionary of the English Language*. (Editor)

Guayazuela, since there is no record of his ever leaving its landlocked shores[1], but he has corresponded with the problem editors of *The American Mathematical Monthly* and other publications. Since Guayazuela is a poor country and his salary is quite meager, it is surprising that he has access to such expensive publications and that he can afford all the postage expenses. But he enters soap contests to supplement that meager income [13].

"The story is told that as a child, young Euclide loved to go camping with his father. On one occasion when he was twelve years old, he was invited on a camping trip with his father and a group of other men. The lad was delighted to be given this special treat; at last he was considered a *man* to go camping with men. So of course he was terribly anxious to justify their faith in him by doing a man's job.

"The day of the trip was a miserable, rainy day, so when they arrived at the campsite they found all the firewood completely drenched. For two long hours those men toiled to get a fire kindled. After much effort, success finally crowned their efforts and they could settle down somewhat to rest and dry off and warm up. Desiring a cup of coffee, one of the men sent young Euclide to 'put a couple of pails of water on the fire.' Unquestioningly, the lad, eager to please his elders, immediately jumped up to complete the task, although he did doubt the logic of the request. So shocked were the wet, tired men when the obedient youth started to pour the water over the burning logs that they could not even speak to stop him until he had completely emptied the second bucket.

"Several years later when he was living with his uncle Paracelso Bombasto, after whom he had been named, Umbugio was asked by the older man, a rather simple woodsman, how to reseal an envelope he had steamed open in order to insert something he had forgotten. He remarked that most of the glue on the flap of the envelope was now gone. The younger man suggested that he 'put it under something heavy.' The uncle looked about him and finally placed the letter on the floor under the bed, the heaviest object in the room." [2]

Our first glimpse into Umbugio's numerological studies appeared in April 1946, when the *Erewhon Daily Howler* said that this "famous astrologer and numerologist of Guayazuela ... predicts the end of the world for the year 2141." It seems he discovered that

$$1492^n - 1770^n - 1863^n + 2141^n$$

is divisible by 1946 for $n = 0, 1, 2, 3, \ldots$ [4]. Since 1492, 1770, and 1863 are important dates in United States history, he concluded that 2141 must be the date of the

[1]This may help dispel the long-standing mystery concerning the geographical location of the flying island of Laputa. See EUREKA, Vol. 3, No. 2, (Feb. 1977), p. 41. (Editor)

end of the world. (If it appears that this writer does not seem especially worried by this prediction of disaster, it is not that he dismisses the Professor's utterances lightly, but rather that he will be 110 years old[1] at that time.)

The good Professor has an exceptionally keen sense of humor, honed to a fine edge. For we read that one of the solvers of this doomsday prediction problem stated, "Surely the proposer of the problem must have been the joker." [4]

Numerology and number properties fascinate all of us. Surely one cannot fail to be stirred by the story of Hardy's visit to Ramanujan. When the visitor was asked what interesting numbers he had seen en route, he replied that his taxi bore the dull number 1729. Ramanujan immediately exclaimed that 1729 is far from dull; it is the smallest number that can be written as a sum of two cubes in two different ways!

In addition to his astrology [4] and analysis [19], numerology is a field of expertise for Umbugio. In the solution to problem E 961 appearing in April 1951, C.W. Trigg wondered if Umbugio had given permission to publish this problem because he knew that he, the problem's number, and its reverse are all perfect squares, and that $(961 - 169)/2 = 396$. Or did he consider the unlucky aspect of the square root of the reverse is counterbalanced by

$$1.1(1951 - 1591) = 396 \text{ and } \frac{1591 - 7(\text{for luck})}{4(\text{for April})} = 396?$$

(See [9].)

Problem E 1111, proposed by the Professor, was a most popular item. One is asked to solve this cryptarithmic division:

$$\begin{array}{r|cccccccc}
\multicolumn{1}{r}{} & & & & & & 8 & & \\ \cline{2-9}
x\,x\,x & x & x & x & x & x & x & x & x \\
\multicolumn{1}{r}{} & & x & x & x & & & & \\ \cline{3-5}
\multicolumn{1}{r}{} & & & x & x & x & x & & \\
\multicolumn{1}{r}{} & & & & x & x & x & & \\ \cline{4-7}
\multicolumn{1}{r}{} & & & & & x & x & x & x \\
\multicolumn{1}{r}{} & & & & & x & x & x & x \\ \cline{6-9}
\end{array}$$

We will not spoil the reader's pleasure by stating the solution, but we again quote Trigg's comments regarding Umbugio's perfection: "E 1111 has as many ones as the Professor has names, 4, and this times the number of letters in his name is the divisor. Thus the number of names and the number of letters are congruent modulo 9. The sum of the digits in the quotient, 25, the last two digits in the dividend, the modulus, the number of the Professor's names, and the Professor, are consecutive squares." [10]

Because of his interest in geometry, as shown by the geometric nature of many

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[1]This is what appears in the author's text. Either the figure 110 is a misprint, or the author is extremely precocious. (Editor)

of his problem proposals and other writings, he once declared the existence of a high correlation between the prominence of a mathematician in the field of geometry and double digits (such as the 11 in 1175) in his associated dates [2]. Thus Umbugio considered Fibonacci (1175? - 1250?) a great geometer. Truly Gauss (1777 - 1855), having a *triple* seven and a double five, must be the master of all geometers.

In April 1950, two problems of interest appeared: E 912 "by permission of Professor E.P.B. Umbugio, April 1, 1950" [7] and E 913 which is marked "attention Professor E.P.B. Umbugio," in addition to problem E 911 [6] which was proposed by the Professor. In the solution to E 913 we read that Umbugio wrote he was afraid to tackle this problem because 913 has 13 for its last two digits.

Also the sum of the digits is 13, and the sum of the squares of the digits is 91, which is divisible by 13 [8]. Problem E 1192 (compare with 911 and 912) is also marked "attention Professor Umbugio." [12]

In problem E 1426 we learn of his entering a contest involving the word MATHEMATICIAN. L. Bankoff pointed out that "the Professor evidently failed to notice that the number of letters in MATHEMATICIAN and the sum of the digits in E 1426 are both equal to 13!" [13] Regarding problem E 1161 Umbugio stated, "The occasion of this problem is an auspicious one, since the number of the problem has the property that

$$11^{6n+3} + 61^{6n+3}$$

is divisible by 7, a lucky number." Here Bankoff referred to Umbugio as a "square of squares," thus explaining the Professor's preference for quartic equations such as this problem involved [11].

Of course one should not jump to unfounded conclusions, but we observe that the Professor published no problems between 1960 [13] and 1967. In his 1967 entry it is stated that he has "emerged from retirement" [14], which is reasonable since he was probably 67 years old at the time. The seven-year gap is perhaps due to hurt feelings brought on by the above comments, after which he waited until a lucky heptad of years had elapsed before once again presenting his outstanding scholarship to the mathematical world. But more probably he would take such remarks as compliments, in the spirit in which they were intended. It is in any case clear he could never remain in retirement, since E 2081 mentions his "characteristic tenacity." [15]

No discussion of Euclide Umbugio would be complete without a mention of his library. From a student and old friend of his we learn of the twenty volumes that most influenced his career [19]. An examination of these titles will easily convince the reader of their owner's greatness and humanity. Other items in his vast library have been released from time to time, both publicly [18] and privately [3, 20], the latter appearing in print for the first time in the appendix to this article. "Yet there is no truth to the rumor that the original manuscript of Euclid's *Elements* resides on these shelves." [2]

How many books he himself has written is not clear, but E 766 states that Umbugio reviews books (in part) by a "spirited search for his name and quotations from his works." In all fairness we must mention that he also devotes a percentage of his review time to a "deep study of the title page and jacket" and another percentage to a "proportionally penetrating perusal of the remaining text." [5]

This very publication has been honored above all others known to the author, for in its hallowed pages appears an actual paper by the good Professor. The clear simplicity of his unubiquitous writings, and his careful avoidance of boring explanations of obvious, trivial details shine brilliantly in his unique proof of Morley's delightful theorem. The reader is not burdened with a tedious explanation of why $PZ = PY$, which is equivalent to the parallelism of YZ and BQ. He goes directly to $YM - ZL = QN$, rather than insult the reader's intelligence [21].

We suspect, although I have not had the opportunity to ask him, that Umbugio's library also contains *Exercices de Géométrie* by M.Ph. André, 14th ed.[1].

In closing it seems fitting to quote Howard Eves concerning our friend. "In 1946 ... Professor George Pólya and I thought it would be enlivening if in each April issue[1] of [*The American Mathematical Monthly*] there appeared a sort of April Fool's problem ... . It was decided that these problems would originate from a windy, verbose, but kindly numerologist, Professor Euclide Paracelso Bombasto Umbugio of Guayazuela ... . The hoax fooled many readers of the *Monthly*, and the letters received asking for his address ... would fill a little pamphlet. All inquirers were informed that Professor Umbugio moved about so much that the best way to reach him was through correspondence sent in care of the editor of the Elementary Problem Department."[17] That editor was, for 25 years, Howard Eves. Thus it appears that the Professor is related to that famous elementary problem solver S.T. Thompson [17].

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1. L. Bankoff, Letter to the editor and editorial reply, EUREKA, Vol. 2 (1976), pp. 92-93.

2. C.W. Dodge, *Numbers and Mathematics*, 2nd edition, Prindle, Weber & Schmidt (1975), pp. 155, 156, 181, 265, 302, 385, 429, 479-480.

3. C.W. Dodge, "More Umbugio Books", previously unpublished (see p. 125 in this issue).

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[1] One must be very careful about accepting as fact every statement that appears in print. It is not true that Umbugio problems appear only in April issues of the Monthly; E 1426 appeared in 1960 in the August-September issue [13].

4. E 716, *American Mathematical Monthly,* Vol. 53 (1946), p. 219, and vol. 54 (1947), p. 43.

5. E 766, *ibid.,* vol. 54 (1947), p. 223, and vol. 55 (1948), p. 30.

6. E 911, *ibid.,* vol. 57 (1950), pp. 259, 690.

7. E 912, *ibid.,* vol. 57 (1950), p. 259, and vol. 58 (1951), p. 37.

8. E 913, *ibid.,* vol. 57 (1950), p. 260, and vol. 58 (1951), p. 38.

9. E 961, *ibid.,* vol. 58 (1951), pp. 259, 700.

10. E 1111, *ibid.,* vol. 61 (1954), pp. 258, 712, and *The Otto Dunkel Memorial Problem Book,* The Mathematical Association of America (1957), p. 6.

11. E 1161, *American Mathematical Monthly,* vol. 62 (1955), pp. 254, 728.

12. E 1192, *ibid.,* vol. 62 (1955), p. 728, and vol. 63 (1956) p. 423.

13. E 1426, *ibid.,* vol. 67 (1960), p. 692, and vol. 68 (1961), p. 295.

14. E 1979, *ibid.,* vol. 74 (1967), p. 438, and vol. 75 (1968), p. 783.

15. E 2081, *ibid.,* vol. 75 (1968), p. 404, and vol. 76 (1969), p. 418.

16. E 2533, *ibid.,* vol. 82 (1975), p. 401, and vol. 83 (1976), p. 570.

17. H. Eves, *Mathematical Circles Revisited,* Prindle, Weber & Schmidt (1971), items 243° and 244°.

18. "Further Items on Umbugio's Bookshelf", *American Mathematical Monthly,* vol. 71 (1964), p. 283 (see p. 125 in this issue).

19. P. Hagis, Jr., "An Analyst's Bookshelf", *ibid.,* vol. 69 (1962), pp. 980 - 981 (see p. 123 in this issue).

20. E.S. Langford, "More Umbugio Books", previously unpublished private communication (see p. 125 in this issue).

21. E.P.B. Umbugio, "A Direct Geometrical Proof of Morley's Theorem," EUREKA, vol. 2 (1976), p. 162.

APPENDIX

1. AN ANALYST'S BOOKSHELF[1]

PETER HAGIS JR., Temple University

At the recent joint meeting of the M.A.A., A.M.A., and A.A.M. held at Bourbaki A.& M. I was most delighted to encounter my old friend and teacher, Professor E.P.B. Umbugio. On the evening of the final day, after the last paper had been presented, the professor and I, along with several other mathematicians, dined at the local inn before going our several ways. The after-dinner conversation, general at first, eventually turned to books and in particular to a discussion of great works in mathematics. Professor Umbugio, as one of the greatest of living analysts, was asked

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[1]Reprinted with permission from *The American Mathematical Monthly,* Vol. 69 (1962) pp. 980 - 981.

to name the 20 books which had most influenced his career. After several minutes of deep and considered thought he slowly ticked off the list which appears below. To the surprise of all present he mentioned none of the epoch making works of men such as Euler, Gauss, or Lagrange and none of the modern treatises which are ordinarily considered as the standard works in their fields. When questioned about these omissions the professor replied that while he was quite aware of the importance of these books in the development and expansion of mathematics in general he personally, having had neither the time nor the patience to wade through them, had been influenced by them not at all. Such a highly personalized reading list, drawn up by you or me, would, of course, be of little interest or significance to others. Such a list drawn up by an eminent analyst such as Professor Umbugio is another matter altogether. It should certainly be made available to as wide a circle of mathematicians as possible. It is with this in mind that I have decided to send the list to this journal for publication. Unfortunately, as I took no notes at the time and have not been in contact with Professor Umbugio since then, I have had to rely entirely on my memory. Therefore, some of the titles may not be exactly as stated by the professor. I, of course, take full responsibility for any such inaccuracies. Here then is Professor Umbugio's list.

1. The Jacobians and Their Struggle for Independence
2. A Ten Day Diet to Improve Indeterminate Forms
3. Cheaper by the Googol
4. 1001 Best Loved Double Integrals
5. The Torus and I
6. A Short Table of Even Primes (Abridged)
7. Will Success Spoil Runge-Kutta?
8. Dining Out in Hilbert Space
9. 100 Tasty Fillings for Empty Sets
10. Life Begins at e^{π}
11. How to Keep Condensation Points from Dripping into Open Sets
12. A Child's Garden of Tchebycheff Polynomials
13. Tom Swift and his Electric Cycloid
14. A Treasury of Matrices — Upright and Inverted
15. Improving Lipschitz Conditions in the Slums of New York
16. The Decline and Fall of e^{-x}
17. How to Prevent Rust on Riemann Surfaces
18. The Peano Postulates Transcribed for Violin and Cello

19. First Aid for Dedekind Cuts and Bruises[1]
20. A Collection of Happy Endings for Incomplete Beta Functions

2. FURTHER ITEMS ON UMBUGIO'S BOOKSHELF[2]

Insequential Analysis
Obsolescent Convergence
Deferential Equations
Natural Logs in Public Forests
History of Arcs, by Joan
Characteristics in Character Building
Eccentricities of Orbiting Mathematicians
1,000,000 Random Numbers in Ascending Order

3. MORE UMBUGIO BOOKS

E.S. LANGFORD, University of Maine at Orono

Cleaning Residues from the Complex Plane
Compactifications of Finite Spaces
Filters — for a Cleaner Topology
Tables of Closed Contour Integrals, Vol. 1: Analytic Functions
Rings and Ideals — a Modern Marriage Manual
Build Your House Yourself with Tychonoff Planks
Lifting Mappings for Fun and Profit
How to Develop a Strong Topology
Abridged Table of Square Fibonacci Numbers
Growing Exponentials in your Spare Time

4. MORE UMBUGIO BOOKS

C.W. DODGE, University of Maine at Orono

Introducing Parallel Lines — for Pleasant Meetings
Integration in Closed Neighborhoods
22/7 Computed to One Million Decimal Places

* * *

[1]An appropriate subtitle, believed to have been originated by P.D. Lax, might be "What to do Until the Mathematician Arrives." (Editor of the *Monthly*)

[2]Reprinted with permission from *The American Mathematical Monthly*, Vol. 71 (1964) p. 283.

CLAYTON WILLARD DODGE

A Rebuttal

by E.P.B. UMBUGIO, Guayazuela

Sensing the controversial and unscrupulous nature of the preceding article, the editor of this journal very kindly allowed me the courtesy of reading it and replying to its contents at this time. He apologized for committing himself to publish such [expletive deleted] writing, but he had given his word before seeing the paper.

I am too upset even to attempt to rebut the comments Dodge has made. The reader will surely not be misled by his obvious falsehoods, and therefore I shall rectify only the one glaring false implication that will immediately destroy all others.

It is indeed not I who am the invention of someone's imagination, but rather it is he, Professor Clayton Willard Dodge, who is purely fictitious. I challenge any of the readers of this publication to state they have ever actually seen him. You cannot, for he does not exist!

C.W. Dodge was created by me several years ago to allow me to publish certain books, problems, and solutions that I did not have the time to polish up to the high standards I demand be associated with my name. For several years this arrangement worked well. Unfortunately, he became greedy and headstrong, eschewing my unusurious philosophy. Since that time he has gone his own way, slandering me at every turn.

I shall say no more, for I intend no harm to any person, real or immy.

* * *

REVEALED AT LAST: THE FACE OF E.P.B. UMBUGIO

CHARLES W. TRIGG, Professor Emeritus, Los Angeles City College

I am about to reveal for the first time to the readers of EUREKA the results of a thirty-year search for a likeness of the celebrated numerologist, Professor Euclide Paracelso Bombasto Umbugio. Since he has always stayed so close to his home, it has been most difficult to locate any photographs of Professor Umbugio, or even to find anyone who knows what he looks like.

The first break in my search, which turned out after all to be inconclusive, occurred in 1955, when I read in [1] that Leon Bankoff, D.D.S., had referred to Umbugio as "a square of squares". Upon enquiry, Dr. Bankoff revealed that he had indeed met Umbugio in his professional capacity as a dentist. As a result of persistent thumb-sucking in his formative years, Umbugio's teeth had grown crooked. Later, after he started to get famous, he became conscious of this deformity and sought to have it corrected. But there was not in all of Guyazuela a dentist with a sufficient knowledge of geometry to be able to reset the teeth parallel to each other as well as truly perpendicular to the gums. A good orthodontist was clearly needed. In desperation, the penurious professor appealed to the *Mathematical Association of America*. The *M.A.A.*, ever mindful of the welfare of its members, provided a grant that enabled Dr. Bankoff, the best geometer among American dentists, to go to Guayazuela and treat our hero. But Dr. Bankoff said that the modest professor, who wishes to avoid the limelight, had sworn him to secrecy and that he would not add anything to his cryptic utterance in [1].

I made no further progress for twenty-two years. The real breakthrough came recently when I read in EUREKA [1977: 41] that Professor Umbugio and Dr. Irving Joshua Matrix had both attended a seminar in advanced numerology at the Grand Academy of Lagado, in Laputa (see [2]). I immediately made a flying trip to the flying island of Laputa (the island met me halfway) and, after searching through all the wastepaper baskets of the Academy conference room, came up with the following portraits, thought to have been doodled by Dr. Matrix's daughter Iva (see Figures 1 and 2).

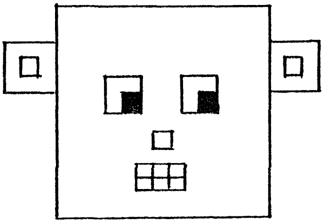

Figure 1

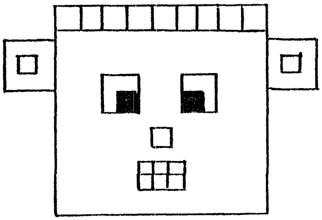

Figure 2

There is no doubt whatever that these portraits represent Professor Umbugio. The "square of squares" features, as well as the splendid orthodontal work of Dr. Bankoff which is quite apparent in the portraits, confirm it beyond the shadow of a doubt. In the first, which contains 4^2 squares, he shows his obvious leanings (or leerings) toward the left. In the second, where he exhibits 5^2 squares as a result of donning his thinking cap, he has squared off against the Euclideans and, in his talented and efficient way, is utilizing his congenital physical (matching his mental) strabismus to demonstrate the verity of Riemannian (elliptical) geometry.

Finally, a diligent search undertaken long ago through 49 museums and 64 numismatic collections recently bore fruit. I found one of 81 medallions struck off in honor of Euclide Paracelso Bombasto Umbugio by the University of Guayazuela for his sharp performance on the steering committee of their College of Animal Husbandry, of which he was the temporary[1] chairman. To prove that that's no bull, I made a rubbing of the obverse of the medallion (see Figure 3). The idealized profile of the Professor on the obverse exhibits 3^2 squares and shows him without his thinking cap but admiring his ever-present earring (not perceptible in a frontal view). The design on the reverse of the medallion deals with the operation for which he was honored, but it will not reproduce well. Since the reverse also designated the Professor as Castrato Magnificento, it seemed doubly wise not to rub it.

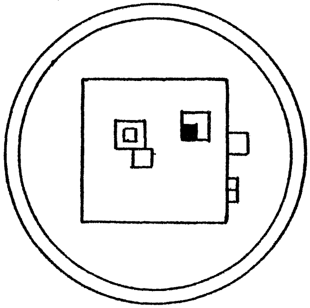

Figure 3

I hope that the publication of this article will stimulate the search for an authentic photograph of Professor Euclide Paracelso Bombasto Umbugio.

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1. Solution to Problem E 1161, *The American Mathematical Monthly*, Vol. 62 (1955), p. 728.

2. Martin Gardner, *The Incredible Dr. Matrix*, Scribner's, 1976, p. 6.

3. Charles W. Trigg, *Mathematical Quickies*, McGraw-Hill Book Co., New York (1967), pp. 10, 25, 56.[2]

* * *

Limping Limericks.

GALILEO REDIVIVUS

In sixteen hundred and thirty-three
Galileo Galilei
     Shocked the world
     When he unfurled
The secret of the starry sea.

He was renowned the world around,
And all his theories were sound;
     But the Pope, wit slow,
     Said he didn't know
His math from a scroll in the ground.

(Perpetrated by)
WILLIAM A. McWORTER, JR.,
The Ohio State University.

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[1]i.e., not for heifer and heifer.

[2]This seems to be the only reference left out of Dodge's bibliography on pages 122 - 123 of this issue. I add it here for the sake of completeness.

LETTERS TO THE EDITOR

Dear Editor:

I'm enjoying the February issue of EUREKA, especially the exchange on the millionth digit ($\neq$ decimal) of π [1977: 40 - 41] and Edith Orr's poem(?) [1977: 39]. But Miss Orr should realize that since Jimmy Who? has become President Jimmy, he may prefer not to be considered an imaginary unit!

DAVID R. STONE,
Georgia Southern College,
Statesboro, Georgia.

Dear Editor:

Thank you for sending me a copy of Professor David R. Stone's letter, and for giving me an opportunity to respond to it in the pages of your esteemed publication.

First of all, let me say that as a published poetess I resent the question mark after the word *poem* in Professor Stone's letter. This gentleman(?), who has apparently never progressed beyond the *June-moon* level of poetry, does not seem to be aware that there is such a thing as blank verse.

He is equally unperceptive in his imagined reaction of President Carter to having the imaginary unit j named after him. One has only to think of the gauss, the ohm, the volt, the weber, the watt, the pascal, the newton, the joule, the coulomb, the farad, the ampere, the henry, the curie, the millihelen[1], to realize that dozens of persons would be totally forgotten today if they had not had a unit named after them. President Carter, a former nuclear engineer, is surely familiar with most of these units, and he will realize that having the imaginary unit j called a *jimmy* may be his best, possibly his only, chance of achieving immortality.

EDITH ORR,
Ottawa, Ont.

Dear Editor:

Miss Orr's use of the question mark after "gentleman" has indeed put me in my place. In apology and in view of her millihelen rating, I would like to take Miss Orr out and show her the beauty of the June moon over Georgia's pines. (In such circumstances I could ignore the capability of a mind which produces *blank* verse.)

So "jimmy" it shall be;and even though Jimmy's square shall be negative, his (fourth) power will be real.

DAVID R. STONE
Georgia Southern College,
Statesboro, Georgia.

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[1]*Editor's note.* The millihelen is the unit of female beauty. The beautiful Miss Orr, who at her best rates 998 millihelens, may be overoptimistic in assuming that all readers are familiar with this unit. The millihelen, which immortalizes Helen of Troy, is the amount of beauty required to launch one ship. See *Verbatim,* The Language Quarterly, Vol. III, No. 4, February 1977, p. 12.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than October 1, 1977.

241. *Proposed by John J. McNamee, Executive Director, Canadian Mathematical Congress.*

Solve the base ten cryptarithm:

$$(HE)(EH) = WHEW.$$

242*. *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ont.*

Give a geometrical construction for determining the focus of a parabola when two tangents and their points of contact are given.

243. *Proposé par Hippolyte Charles, Waterloo, Québec.*

a) Trouver des conditions nécessaires et suffisantes pour que le plus grand commun diviseur des entiers positifs a et b, $a > b$, soit égal à leur différence.

b) *Application.* Trouver toutes les paires d'entiers positifs dont le plus grand commun diviseur est égal à leur différence et dont le plus petit commun multiple est 180.

244*. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Solve the following problem, which can be found in *Integrated Algebra and Trigonometry*, by Fisher and Ziebur, Prentice-Hall (1957) p. 259:

A rectangular strip of carpet 3 ft. wide is laid diagonally across the floor of a room 9 ft. by 12 ft. so that each of the four corners of the strip touches a wall. How long is the strip?

245. *Proposed by Charles W. Trigg, San Diego, California.*

Find the volume of a regular tetrahedron in terms of its bimedian b. (A *bimedian* is a segment joining the midpoints of opposite edges.)

246. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Let p_i denote the ith prime and let P_n denote the product of the first n primes. Prove that the number N defined by

$$N = \frac{P_n}{p_i p_j \cdots p_r} \pm p_i p_j \cdots p_r,$$

where $p_i, p_j, \ldots, p_r$ are any of the first n primes, all different, or unity, is a prime whenever $N < p_{n+1}^2$. (This is known as Tallman's Formula.)

Example:

$$N = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{2 \cdot 3 \cdot 5} \pm 2 \cdot 3 \cdot 5 = 107 \text{ or } 47, \text{ both primes.}$$

247*. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

On page 215 of *Analytic Inequalities* by D.S. Mitrinović, the following inequality is given: if $0 < b \le a$ then

$$\frac{1}{8}\frac{(a-b)^2}{a} < \frac{a+b}{2} - \sqrt{ab} \le \frac{1}{8}\frac{(a-b)^2}{b}.$$

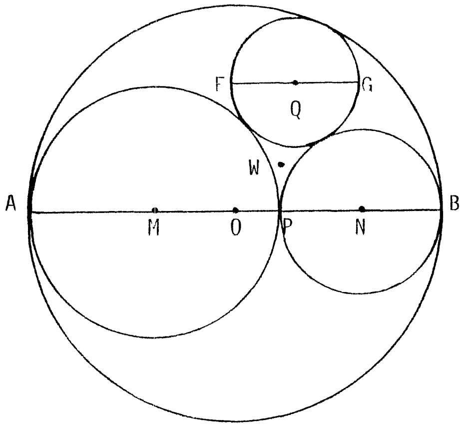
Can this be generalized to the following form: if $0 < a_1 \le a_2 \le \ldots \le a_n$ then

$$k\,\frac{\sum_{1 \le i < j \le n}(a_i - a_j)^2}{a_n} \le \frac{a_1 + \ldots + a_n}{n} - \sqrt[n]{a_1 \ldots a_n} \le K\,\frac{\sum_{1 \le i < j \le n}(a_i - a_j)^2}{a_1},$$

where k is a constant?

248. *Proposed by Dan Sokolowsky, Yellow Springs, Ohio.*

Circle (Q) is tangent to circles (O), (M), (N), as shown in the figure, and FG is the diameter of (Q) parallel to diameter AB of (O). W is the radical center of circles (M), (N), (Q). Prove that WQ is the circumradius of ΔPFG. (Note that W is not necessarily its circumcenter.)



249. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

The positive integers 1, 4, and 6 are not primes and cannot be written as sums of distinct primes. Prove or disprove that all other positive integers are either prime or can be written as sums of distinct primes.

250. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

(a) Find all pairs (m,n) of positive integers such that

$$|3^m - 2^n| = 1.$$

(b)* If $|3^m - 2^n| \neq 1$, is there always a prime between 3^m and 2^n?

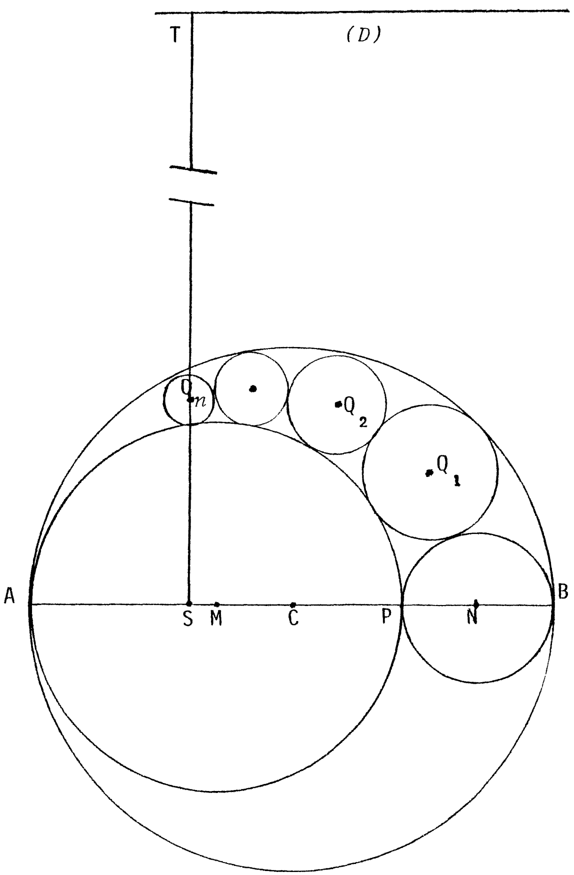
* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

177. [1976: 171; 1977: 50] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

P is a point on the diameter AB of a circle whose centre is C. On AP, BP as diameters, circles are drawn. Q is the centre of a circle which touches these three circles. What is the locus of Q as P varies?



II. *Editor's comment.*

In conversations I had with Dan Sokolowsky, Yellow Springs, Ohio, and Dan Pedoe, University of Minnesota, they pointed out to me a weakness in Salvatore's solution of this problem [1977: 50]. In that solution (and the accompanying figure on page 51), Pappus's Theorem assures us that, if circle (Q) (with radius r) is tangent to circles (M), (N), (C), then $|QS| = 2r$. In the second part of his proof, Salvatore concluded by saying that, if circle (Q) is tangent to circles (C) and (M), and if $|QS| = 2r$, then it follows from Pappus's Theorem that circle (Q) is tangent to circle (N). This is, of course, incorrect. The desired conclusion follows from a *converse* of

Pappus's Theorem, not from the theorem itself. But this converse, though true and easy to establish, has never, as far as I can tell, been mentioned in the literature.

I give below a strengthened form of Pappus's Ancient Theorem which does include the desired converse.

THEOREM. Let

(i) *Circles* (Q_1), (Q_2), ..., (Q_n) *all be tangent to circles* (C) *and* (M), *as shown in the figure;*

(ii) (Q_1) *be tangent to* (N), (Q_2) *to* (Q_1), ..., (Q_{n-1}) *to* (Q_{n-2});

(iii) *the radius of* (Q_n) *be* r_n.

Then $|SQ_n| = 2nr_n$ *if and only if* (Q_n) *is tangent to* (Q_{n-1}).

The *if* part of this theorem has traditionally been called Pappus's Ancient Theorem. The most illuminating proof of it is by inversion. Such a proof can be found, for example, in Eves [1] or Pedoe [2]. The truth of the converse (*only if*) part is a trivial consequence of the same proof by inversion.

The above strengthened form of Pappus's Theorem,with $n = 1$, is sufficient to validate Salvatore's original proof, but the new proof and generalization given below, inspired in part by Salvatore's proof, may perhaps be found to be more interesting

III. *Solution by Dan Pedoe, University of Minnesota; Dan Sokolowsky, Yellow Springs, Ohio; and the editor (jointly).*

Let P be a point on segment AB, and let (Q_1), (Q_2), ..., (Q_n) be a chain of circles all tangent to circles (C) and (M), with (Q_1) tangent to (N), (Q_2) to (Q_1), ..., and (Q_n) to (Q_{n-1}), as shown in the above figure. We will find the locus of the centre Q_n as P ranges over AB. (The proposed problem corresponds to $n = 1$.)

Let $|CB| = \rho$, let S be the foot of the perpendicular from Q_n upon AB, draw line *(D)* $2n\rho$ units above AB, and let SQ_n produced meet *(D)* in T. If r_n is the radius of circle (Q_n), then by Pappus's Ancient Theorem we have $|SQ_n| = 2nr_n$. Since $|CQ_n| = \rho - r_n$, we have

$$|Q_nT| = |ST| - |SQ_n| = 2n\rho - 2nr_n = 2n(\rho - r_n) = 2n|CQ_n|,$$

and so Q_n lies on an ellipse with focus C, directrix *(D)*, and eccentricity $1/2n$. More precisely, Q_n lies on the arc of this ellipse subtended by latus rectum AB. We will show that this arc (together with its reflection in AB) is the required locus.

If Q_n is any point on this elliptical arc, there is a unique circle (Q_n) tangent to circle (C); a unique circle (M) tangent internally to circle (C) at A and externally to circle (Q_n) and meeting AB again in, say, P; a unique circle (N) on PB as diameter; and uniquely determined circles (Q_1), (Q_2), ..., (Q_{n-1}) all tangent to circles (C) and (M), with (Q_1) tangent to (N), (Q_2) to (Q_1), ..., and (Q_{n-1}) to (Q_{n-2}). It now suffices to show that circle (Q_n) is tangent to circle (Q_{n-1}).

If r_n is the radius of circle (Q_n), we have

$$|SQ_n| = |ST| - |Q_nT| = 2n\rho - 2n|CQ_n| = 2n\rho - 2n(\rho - r_n) = 2nr_n,$$

and the tangency of circles (Q_n) and (Q_{n-1}) now follows from the converse part of Pappus's Ancient Theorem as given above in comment II, which completes the proof.

REFERENCES

1. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, 1972, p. 133.

2. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970, p. 89.

* * *

200. [1976: 220] *Proposed by the editor.*

(a) Prove that there exist triangles which cannot be dissected into two or three isosceles triangles.

(b) Prove or disprove that, for $n \geq 4$, every triangle can be dissected into n isosceles triangles.

Solution by Gali Salvatore, Ottawa, Ont.

(a) For $n = 2$, the dissecting line must pass through a vertex, and an investigation of the possible cases shows that the following triangles, and only those, can be dissected into 2 isosceles triangles:

(i) all right-angled triangles (draw the median to the hypotenuse);

(ii) all triangles in which one angle is twice another (see Figure 1);

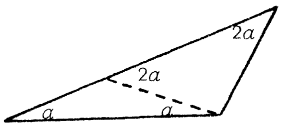


Figure 1

(iii) all triangles in which one angle is three times another (see Figure 2).

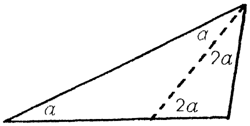


Figure 2

For $n = 3$, the dissection is always possible in at least the following cases:

(iv) all acute-angled triangles (join the circumcentre to the vertices);

(v) all right-angled triangles. Let ABC be such a triangle, with $B \leq C < A = 90°$. If $B < C$, draw $\angle BCD = B$. Then ΔBCD is isosceles, and and ΔACD can be dissected into 2 isosceles triangles as seen in (i). If $B = C$, draw

AD⊥BC and DE⊥AB.

The triangle with angles 1°, 8°, 171° is not of types (i), (ii), or (iii), and so cannot be dissected into 2 isosceles triangles. It can no more be dissected into 3 isosceles triangles; for one of the dissecting lines must pass through a vertex, but no such line exists which yields an isosceles triangle and another triangle of types (i), (ii), or (iii).

(b) For $n = 4$, the theorem holds for every triangle. For, given ΔABC in which the greatest angle is A, the altitude AH must fall within the triangle, and joining H to the midpoints of AB and AC yields 4 isosceles triangles.

We now prove by induction that the theorem holds for $n \geq 4$ for an arbitrary triangle ABC. We distinguish two cases.

(1) ΔABC is not equilateral. Suppose the theorem holds for $n = k$, where $k \geq 4$. ΔABC can be dissected into an isosceles triangle and another triangle; the latter can be dissected into k isosceles triangles, and thus ΔABC can be dissected into $k + 1$ isosceles triangles, and the induction is complete.

(2) ΔABC is equilateral. Then the theorem holds for $n = 3, 4, 5$. The cases $n = 3, 4$ have already been established. For $n = 5$, select a point D on AC such that CD < AD and draw DE||AB (see Figure 3). Then ABED is a cyclic quadrilateral and the centre O of its circumcircle lies within it. Joining O to A, B, E, D yields a dissection of ΔABC into 5 isosceles triangles. Suppose the theorem holds for $n = k$, where $k \geq 3$. If P, Q, H are the midpoints of AB, AC, BC (see Figure 4), then Δs PBH, APH, AQH are isosceles, and ΔQHC (which is equilateral) can be dissected into k isosceles triangles. Hence ΔABC can be dissected into $k + 3$ isosceles triangles, and the induction is complete.

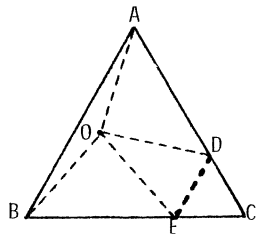


Figure 3

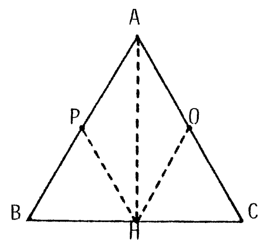


Figure 4

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence, Ottawa;* L.F. MEYERS, *The Ohio State University (partial solution);* HARRY L. NELSON, *Livermore, California;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y. (partial solution);* R. ROBINSON ROWE, *Naubinway, Michigan; and the proposer.*

Editor's comment.

This problem first appeared, with the same proposer, in the *Canadian Mathematical Bulletin,* Vol. 4 (1961) p. 189.

* * *

201. [1977: 9] *Proposed by Clayton W. Dodge, University of Maine at Orono.*
Solve the cryptarithm $LEO^2 = SAUVE$.

I. *Solution by S.L. Greitzer, Rutgers University.*

First, SAUVE is a square; hence $E = 1, 4, 9, 6$ (for obvious reasons, $E \neq 0, 5$), and the only possibilities are $(E,O) = (1,9), (4,2), (4,8), (9,3), (9,7), (6,4)$. Also $100 < LEO < 317$; hence $L = 1, 2, 3$ and we need only test $LEO = 219, 142, 148, 248, 193, 293, 197, 297, 164, 264$. Only $LEO = 248$ fills the conditions, and $(248)^2 = 61504$.

Too bad that LEO is only half perfect. If he were only 496!

II. *Solution and comment by Leroy F. Meyers, The Ohio State University.*

If the proposer was trying to show that the Editor of our journal is a square, he should have remembered that his correct name is LÉO SAUVÉ, with French accents. If so, the proposal should read: Find digits corresponding to the letters (L,E,O) and (S,A,U,V,E) such that $LÉO^2 = SAUVÉ$.

If É is interpreted as E', the successor of E, an analysis [similar to that in solution I] shows that there are exactly two solutions:

$$(L,E,O) = (2,3,8),\ (S,A,U,V,E) = (6,1,5,0,3),\quad LÉO^2 = 248^2 = 61504 = SAUVÉ$$

and

$$(L,E,O) = (3,0,1),\ (S,A,U,V,E) = (9,6,7,2,0),\quad LÉO^2 = 311^2 = 96721 = SAUVÉ.$$

Normally, I don't solve cryptarithms, and only the name of our sympathetic editor induced me to try my hand at this one. In fact, the last cryptarithm I solved was in the *Journal of Recreational Mathematics* several years ago, and the only reason I took the trouble to solve it then was that it contained the name of another very sympathetic character. It was

$$BUY + YOUR + BOOTS + FROM + ME + ME + ME = MEYERS,$$

a reference to a Toronto bootmaker in the last chapter or two of A. Conan Doyle's *The Hound of the Baskervilles*.

III. *Solution by Andrejs Dunkels, University of Luleå, Sweden.*

To begin with, I thought, let me try a reasonable three-digit number on my pocket calculator. And so I tried 248: $248^2 = 61504$, which, to my surprise, turned out to be a solution. An analysis [similar to that in solution I] soon convinced me that I had accidentally stumbled upon the only solution.

No wonder one obtains a unique solution to a problem devoted to a unique editor of a unique journal [*editorial blushes*].

IV. *Solution and comment by the proposer.*

[After an analysis similar to that in solution I], we find that

$$LEO^2 = 248^2 = 61504 = SAUVE$$

is the unique solution.

We note the nice geometric progression in LEO. Furthermore, if we add to SAUVE the annual subscription price of 600¢, we obtain 62104 which tells us the relationship of LEO SAUVE to EUREKA: SLAVE. In any case we 122 01254 248, because when you divide out his evenness (powers of 2) he remains prime, in fact $2^5 - 1$.

Also solved by JAMES HOLT, *student, Georgia Southern College, Statesboro, Georgia;* R.S. JOHNSON, *Montréal, Québec;* F.G.B. MASKELL, *Algonquin College, Ottawa;* LEROY F. MEYERS, *The Ohio State University (second solution);* R. ROBINSON ROWE, *Naubinway, Michigan;* CHARLES W. TRIGG, *San Diego, California;* KENNETH M. WILKE, *Topeka, Kansas; and* KENNETH S. WILLIAMS, *Carleton University, Ottawa.*

Editor's comment.

Meyers also mentioned that EDITH = ORR^2 has one solution: $20736 = 144^2$, and that the following have no solution: FRED^2 = MASKELL, ROBINSON = ROWE^2, ROY^2 = MEYERS, and LEON^2 = BANKOFF (which is virtually isomorphic to FRED^2 = MASKELL). Possibly Meyers was trying to help the proposer exhaust the list of EUREKA personalities, to hasten the time when the proposer can turn his mind to more useful pursuits.

I agree with Professor Greitzer that the editor is only half perfect. But who are we two against so many!

* * *

202. [1977: 9] *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Prove that any real number can be approximated within any $\varepsilon > 0$ as the difference of the square roots of two natural numbers.

I. *Solution by L.F. Meyers, The Ohio State University.*

Let r be a real number, let $\varepsilon > 0$ be given, and let k be any integer greater than $1/4\varepsilon^2$. There exists a positive integer n such that

$$\sqrt{n} \le |r| + \sqrt{k} < \sqrt{n+1}, \tag{1}$$

and since $n + 1 > k$, we have $n \ge k > 1/4\varepsilon^2$. It now follows from (1) that

$$0 \le |r| - (\sqrt{n} - \sqrt{k}) < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} < \varepsilon,$$

since $n > 1/4\varepsilon^2$. Thus r lies within ε of $\pm(\sqrt{n} - \sqrt{k})$, the sign depending on that of r. Note that there are infinitely many solutions.

II. *Solution by M.S. Klamkin, University of Alberta.*

Let x denote the real number (with say $x > 0$). Then, if the positive integer n is large enough

$$\sqrt{[(x+\sqrt{n})^2]}-\sqrt{n}$$

is a suitable approximant to x ([ ] denotes the greatest integer function). For

$$x+\sqrt{n}-\sqrt{[(x+\sqrt{n})^2]}=\frac{(x+\sqrt{n})^2-[(x+\sqrt{n})^2]}{x+\sqrt{n}+\sqrt{[(x+\sqrt{n})^2]}},$$

and since the numerator on the right is less than 1, the error in the approximation can be made arbitrarily small by choosing n large enough.

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* R. ROBINSON ROWE, *Naubinway, Michigan;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia; and the proposer.*

Editor's comment.

This problem is equivalent to the following, proposed by Solomon W. Golomb in [1]:

Let N denote the set of natural numbers. What is the set of limit points of the set $\{\sqrt{a}-\sqrt{b} : a,b \in N\}$?

Several proofs that the answer to the question is the set of all real numbers were published in [2]. One of them, by Arthur Freund, was similar to our solution II, and another astonishingly simple one by Julian H. Blau ran as follows:

The set contains all integral multiples of each of its members. Since it has arbitrarily small members $\sqrt{n+1}-\sqrt{n}$, every real number is a limit point of it.

REFERENCES

1. Problem E 2506, *The American Mathematical Monthly*, Vol. 81(1974) p. 1111.
2. Solution to Problem E 2506, *ibid.*, Vol. 83 (1976) p. 60.

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203. [1977: 9] *Proposed by Charles W. Trigg, San Diego, California.*

On page 157 of his *Amusements in Mathematics*, Dover (1958), H.E. Dudeney makes the statement: "If you add together the digits of any number, and then as often as necessary, add the digits of the result, you must ultimately get a number composed of one figure. This last number I call the *digital root*."

Prove or disprove: The digital root of every even perfect number greater than 6 is 1.

I. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

Every even perfect number is of the form

$$A = 2^{n-1}(2^n - 1) \tag{1}$$

where both n and $2^n - 1$ are primes. I will show that a number of the form (1) has digital root 1 (that is, is congruent to 1 modulo 9), not only when A is an even perfect number greater than 6, but also whenever n is odd. In this generalized form, the problem is not new. It was proposed in 1973 by Everett Casteel, and a solution by Lois J. Reid was published in [1]. The solution I give is different from that in [1].

If $n = 2k + 1$, $k \geq 0$, then

$$A - 1 = 2^{2k}(2^{2k+1} - 1) - 1 = 2 \cdot 4^{2k} - 4^k - 1 = (2 \cdot 4^k + 1)(4^k - 1).$$

Since $4^k = (3 + 1)^k \equiv 1 \pmod 3$, we have

$$2 \cdot 4^k + 1 \equiv 4^k - 1 \equiv 0 \pmod 3;$$

hence $A - 1 \equiv 0 \pmod 9$ and the desired result follows.

II. *Adapted from a comment by H.L. Ridge, University of Toronto.*

A fascinating property of the even perfect numbers greater than 6 is their relationship with sums of consecutive odd cubes:

$$28 = 1^3 + 3^3 \qquad (2 \text{ cubes})$$

$$496 = 1^3 + 3^3 + 5^3 + 7^3 \qquad (4 \text{ cubes})$$

$$8128 = 1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3 \qquad (8 \text{ cubes})$$

The pattern that seems to be developing is surely not accidental, and it is tempting to conjecture that the next even perfect number would be the sum of 16 consecutive odd cubes. Looking more closely into the matter, we find

$$\sum_{i=1}^{n} (2i - 1)^3 = n^2(2n^2 - 1),$$

which, for $n = 2^k$, becomes

$$\sum_{i=1}^{2^k} (2i - 1)^3 = 2^{2k}(2^{2k+1} - 1).$$

This is just formula (1) for odd n, and we know from solution I that these numbers are not all even perfect numbers, although they all have digital root 1. So we are not too surprised to find that

$$\sum_{i=1}^{16} (2i - 1)^3 = 130816$$

is not a perfect number.

All we can be sure of is that for $k = 0, 1, 2, \ldots,$

$$2^{2k}(2^{2k+1} - 1) \tag{2}$$

has digital root 1, is the sum of the first 2^k consecutive odd cubes, and is an even perfect number if $2^{2k+1}-1$ is a prime (which implies that $2k+1$ is also a prime).

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* R.S. JOHNSON, *Montreal, Quebec;* M.S. KLAMKIN, *University of Alberta;* M.M. PARMENTER, *Memorial University of Newfoundland;* MAURICE POIRIER, *École secondaire Garneau, Vanier, Ont.;* BOB PRIELIPP, *The University of Wisconsin-Oshkosh;* H.L. RIDGE, *University of Toronto, (solution as well);* R. ROBINSON ROWE, *Naubinway, Michigan;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia;* KENNETH M. WILKE, *Topeka, Kansas (second solution); and the proposer.*

Editor's comment.

Stone located this problem in [2], and the solution in [1] was followed by a list of about 60 "other solvers", including three who solved it once more for us.

Twenty-four even perfect numbers are known. The first is 6, and the others correspond to the following values of k in (2): 1, 2, 3, 6, 8, 9, 15, 30, 44, 53, 63, 260, 303, 639, 1101, 1140, 1608, 2126, 2211, 4844, 4970, 5606, 9968. The last and largest of these,

$$2^{19936}(2^{19937}-1),$$

is a number of 12003 digits ending in 56. It was discovered in 1971. It is not known if odd perfect numbers exist or if there are infinitely many even perfect numbers.

Much fascinating information about perfect numbers can be found in [3]. For more up-to-date information, see [4].

REFERENCES

1. Solution to Problem 870, *Mathematics Magazine,* Vol. 47 (1974) p. 112.

2. Underwood Dudley, *Elementary Number Theory,* W.H. Freeman and Co., San Francisco, 1969, p. 61, No. 15.

3. Albert H. Beiler, *Recreations in the Theory of Numbers,* Dover, 1964, pp. 11 - 25.

4. David M. Burton, *Elementary Number Theory,* Allyn and Bacon, 1976, pp. 218 - 235.

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204. [1977: 10] *Proposed by R. Robinson Rowe, Sacramento, California.*

A common 8×10-inch plate of coordinate paper is $W=80$ spaces wide by $L=100$ spaces long with 8000 small squares.

(a) Including larger ones, how many squares are there?

(b) How many oblongs (nonsquare rectangles) are there?

Solution by Clayton W. Dodge, University of Maine at Orono.

A segment of length k units can be placed in any one of $(n+1)-k$ positions along a grid line of length $n\geq k$.

(a) Thus, by considering two adjacent edges of the square, we see that there are $(81-k)(101-k)$ positions for such a square in the 80 by 100 grid. Letting k run from 1 to 80 we find that the total number of squares is

$$\sum_{k=1}^{80}(81-k)(101-k) = \sum_{k=1}^{80} 8181 - 182\sum_{k=1}^{80} k + \sum_{k=1}^{80} k^2$$

$$= 80 \cdot 8181 - 182 \cdot \frac{80 \cdot 81}{2} + \frac{80 \cdot 81 \cdot 161}{6}$$

$$= 238680.$$

More generally, for a $W \times L$ grid with $W \le L$, the number of squares is

$$\sum_{k=1}^{W}(W+1-k)(L+1-k) = \sum_{k=1}^{W} k(L-W+k) = (L-W)\sum_{k=1}^{W} k + \sum_{k=1}^{W} k^2$$

$$= \frac{(L-W)\,W\,(W+1)}{2} + \frac{W(W+1)(2W+1)}{6}$$

$$= \frac{W(W+1)(3L+1-W)}{6}.$$

(b) The total number of rectangles (including squares) in the W by L grid is

$$\sum_{m=1}^{W}\sum_{k=1}^{L}(W+1-m)(L+1-k) = \sum_{m=1}^{W}\sum_{k=1}^{L} mk = \sum_{m=1}^{W} m \cdot \sum_{k=1}^{L} k = \frac{W(W+1)}{2} \cdot \frac{L(L+1)}{2}$$

$$= \frac{WL(W+1)(L+1)}{4}. \qquad (1)$$

If we subtract the expression of part (a) from that above, we get as the number of nonsquare oblongs:

$$\frac{WL(W+1)(L+1)}{4} - \frac{W(W+1)(3L+1-W)}{6} = \frac{W(W+1)(3L^2-3L-2+2W)}{12}.$$

For $W = 80$ and $L = 100$ we get 16,123,320 oblongs.

Also solved by DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.; and the proposer.*

Editor's comment.

Dodge's was the only completely correct solution received. Rokhsar's answer to part (a) (238,600) was wrong, and so was the proposer's answer to part (b) (16,117,335). If they don't believe me they can start counting!

The proposer noted that this problem generalizes a problem in Hunter and Madachy [1], which says for an $N \times N$ grid the total number of rectangles (including squares) is $(N^2+N)^2/4$. Our own formula (1) reduces to this when $W = L = N$.

REFERENCE

1. J.A.H. Hunter and J.S. Madachy, *Mathematical Diversions*, Dover, 1975, pp. 89, 129.

* * *

205. [1977: 10] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Find the least common multiple of the numbers

$$(29!)(37!) \quad \text{and} \quad (23!)(41!).$$

I. *Solution by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

With the notations (a,b) and $[a,b]$ for the g.c.d. and l.c.m., respectively, of the pair $\{a,b\}$, it is known that

$$(ca,\ cb) = c(a,b) \quad \text{and} \quad [a,b] = \frac{ab}{(a,b)}.$$

Thus, if $n < q < r < m$, we have

$$(n!m!,\ q!r!) = n!r!\left(\frac{m!}{r!}, \frac{q!}{n!}\right),$$

and so

$$[n!m!,\ q!r!] = \frac{m!q!}{\left(\frac{m!}{r!}, \frac{q!}{n!}\right)}. \qquad (1)$$

In the present problem, the denominator in (1) becomes

$$\left(\frac{41!}{37!}, \frac{29!}{23!}\right) = (41 \cdot 40 \cdot 39 \cdot 38,\ 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24)$$

$$= 2^4 \cdot 3 \cdot 5 \cdot 13,$$

and so

$$[23!41!,\ 29!37!] = \frac{29!41!}{2^4 \cdot 3 \cdot 5 \cdot 13} = 19 \cdot 41 \cdot 29!37!\ .$$

II. *Solution by Kenneth S. Williams, Carleton University, Ottawa.*

Since

$$\frac{29!}{23!} \cdot 19 \cdot 41 = \frac{41!}{37!} \cdot 2^2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 29$$

and (in the notation of solution I)

$$(19 \cdot 41,\ 2^7 \cdot 3^3 \cdot 5 \cdot 7 \cdot 29) = 1,$$

it follows that

$$\left[\frac{29!}{23!}, \frac{41!}{37!}\right] = \frac{29!}{23!} \cdot 19 \cdot 41,$$

and so

$$[29!37!,\ 23!41!] = 29!37!\,19 \cdot 41.$$

Also solved by HERTA T. FREITAG, *Roanoke, Virginia;* BOB PRIELIPP, *The University* *of Wisconsin-Oshkosh;* R. ROBINSON ROWE, *Naubinway, Michigan;* CHARLES W. TRIGG, *San Diego, California; and* KENNETH M. WILKE, *Topeka, Kansas. Two incorrect solutions were received.*

* * *

206. [1977: 10] *Proposed by Dan Pedoe, University of Minnesota.*

A circle intersects the sides BC, CA and AB of a triangle ABC in the pairs of points X, X', Y, Y' and Z, Z' respectively. If the perpendiculars at X, Y and Z to the respective sides BC, CA and AB are concurrent at a point P, prove that the respective perpendiculars at X', Y' and Z' to the sides BC, CA and AB are concurrent at a point P'.

Solution by Dan Sokolowsky, Yellow Springs, Ohio.

We prove the following theorem, which shows that the property described in this problem holds for more general configurations, not just for triangles.

THEOREM. Let γ *be a circle with centre* O, *let* P *and* P' *be two points symmetric with respect to* O, *and suppose* γ *meets each of a set of lines* L_i *at points* X_i, X_i'. *Then the perpendiculars to each line* L_i *at* X_i *all pass through* P *if and only if the perpendiculars to* L_i *at* X_i' *all pass through* P'.

Proof. It is sufficient to show that the conclusion holds for an arbitrary line L of the set, which meets γ at X, X' (see Figure 1).

Suppose the perpendicular to L at X goes through P, and let the perpendicular to L at X' meet line PO in Q. If $OX'' \perp L$, then $XX'' = X''X'$, and hence $PO = OQ$ since $PX \parallel OX'' \parallel QX'$. Thus $Q = P'$ and the perpendicular to L at X' goes through P'.

The converse can be proved by repeating the above argument with X, X' and P, P' interchanged.

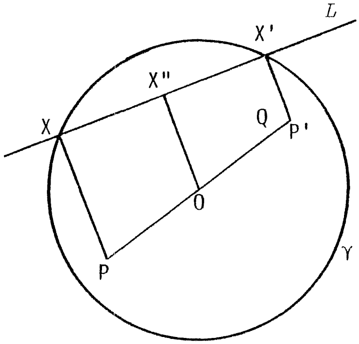


Figure 1

Also solved by J.D. DIXON, *Haliburton Highlands Secondary School, Haliburton, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* ROLAND H. EDDY, *Memorial University of Newfoundland;* SAHIB RAM MANDAN, *Indian Institute of Technology, Kharagpur, India; and the proposer.*

Editor's comment.

The proposer showed that when the perpendiculars at X, Y, Z meet at P (see Figure 2), then, as M ranges over the circle, the chords $MM' \perp PM$ form an envelope of an ellipse inscribed in ΔABC, which has P as one focus; and that the perpendiculars at M' all pass through P', which

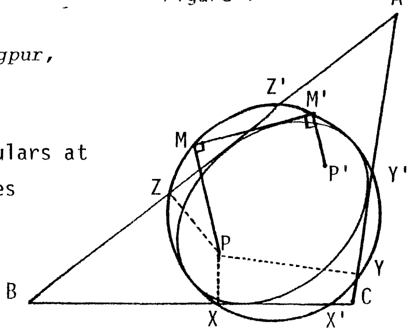


Figure 2

is the other focus of the ellipse. For details see [1].[1]

Dodge showed that our problem is a nearly immediate consequence of a theorem about isogonal conjugate points, which can be found in [2].

REFERENCES

1. Dan Pedoe, *Geometry and the Liberal Arts*, Penguin Books, 1976, p. 207.

2. Nathan Altshiller Court, *College Geometry*, Barnes and Noble, 1952, p. 271, Theorem 641.

* * *

207. [1977: 10] *Proposed by Ross Honsberger, University of Waterloo.*

Prove that $\frac{2r+5}{r+2}$ is always a better approximation to $\sqrt{5}$ than r.

I. *Solution by J. Walter Lynch, Georgia Southern College, Statesboro, Georgia.*

The theorem is true if r is positive and rational, which may be what the proposer had in mind but omitted to state. But it also holds for some negative and some irrational values of r.

We will find all values of r for which

$$\left|\frac{2r+5}{r+2}-\sqrt{5}\right| < |r-\sqrt{5}|. \qquad (1)$$

The left member of (1) is easily transformed to

$$\left|\frac{2-\sqrt{5}}{r+2}\right| |r-\sqrt{5}|;$$

hence (1) holds whenever

$$|r-\sqrt{5}| \neq 0 \quad \text{and} \quad \left|\frac{2-\sqrt{5}}{r+2}\right| < 1,$$

that is, whenever $r < -\sqrt{5}$ and $r > \sqrt{5}-4$, $r \neq \sqrt{5}$.

II. *Solution and comments by M.S. Klamkin, University of Alberta.*

This problem is not new. I had made up a class of these problems for a high school text in algebra by R.E. Johnson *et al.* (Addison-Wesley) many years ago. More generally, we can determine integers a, b, c, d (nonuniquely) such that $(ar+b)/(cr+d)$ is an always better *rational* approximation to the square root of a

[1] If you can find a copy. This book was recently reviewed in EUREKA [1977: 7], and it was then thought that it would be distributed in Canada by Penguin Books Canada Ltd. I have since learned that copyright restrictions will make the expected Canadian distribution impossible, and that an American edition by St. Martin's Press is now in preparation. Canadians should have no difficulty getting a copy of the American edition when it is available.

nonsquare positive integer N than the nonnegative rational approximation r. (Note that the statement of problem 207 is incomplete. It is false if r is negative or if r is irrational.)

Proof. We wish to satisfy

$$\left|\frac{ar+b}{cr+d}-\sqrt{N}\right| < |r-\sqrt{N}|.$$

Since r can be arbitrarily close to $\sqrt{N}$, we must satisfy

$$\frac{a\sqrt{N}+b}{c\sqrt{N}+d}=\sqrt{N},$$

whence $a=d$, $b=cN$. Then we must satisfy

$$\left|\frac{ar+cN}{cr+a}-\sqrt{N}\right| = |r-\sqrt{N}|\left|\frac{a-c\sqrt{N}}{a+cr}\right| < |r-\sqrt{N}|.$$

Since $|a-c\sqrt{N}| < |a+cr|$ for all prescribed r, a and c must have the same sign. Thus it is necessary and sufficient that

$$0<\frac{c}{a}\sqrt{N}<2.$$

For $N=5$, clearly $a=2$, $c=1$ is satisfactory. Better approximations can be obtained by choosing a/c closer to $\sqrt{5}$, by iterating $r_{m+1}=(2r_m+5)/(r_m+2)$, e.g., $(a,c)=(9,4)$, $(38,17)$.

To obtain always better rational approximations to $N^{1/m}$ (m an integer >2), we would have to start with the form

$$\frac{a_0r^{m-1}+a_1r^{m-2}+\ldots+a_m}{b_0r^{m-1}+b_1r^{m-2}+\ldots+b_m},$$

which takes on the value $N^{1/m}$ for $r=N^{1/m}$. In particular I had set the following problem on the 1st U.S.A. Mathematical Olympiad, 1972:

Let R denote a nonnegative rational number. Determine a fixed set of integers a, b, c, d, e, f such that for every choice of R,

$$\left|\frac{aR^2+bR+c}{dR^2+eR+f}-\sqrt[3]{2}\right| < |R-\sqrt[3]{2}|.$$

Proceeding as before, two such always better approximants are

$$\frac{2R^2+2R+2}{R^2+2R+2} \quad\text{and}\quad \frac{4R^2+5R+6}{3R^2+4R+5}.$$

Also solved by W.J. BLUNDON, *Memorial University of Newfoundland;* CLAYTON W. DODGE, *University of Maine at Orono;* S.L. GREITZER, *Rutgers University;* R.S. JOHNSON, *Montreal, Que.;* L.F. MEYERS, *The Ohio State University;* R. ROBINSON ROWE, *Naubinway, Michigan;* KENNETH S. WILLIAMS, *Carleton University, Ottawa; and the proposer.*

* * *

ANNOUNCEMENTS

1. R.S. Johnson, Montreal, Quebec, identifies reference [5] in [1977: 74] as being from the *Journal of Recreational Mathematics*, Vol. 3, No. 3, July 1970, pp. 176-178.

2. The bound copies of the combined EUREKA Volumes 1 and 2 (1975-1976) are now ready. The page size has been reduced to approximately $6\frac{1}{2} \times 8\frac{1}{2}$ in., dozens of minor errors and misprints have been corrected, and a full index is provided. Readers who had ordered a copy along with their 1977 subscription should receive it within a few days, if they have not received it already. Other readers can get a copy by sending a $10 cheque or money order in Canadian or U.S. funds (payable to the Carleton-Ottawa Mathematics Association) to F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ont., Canada K1S 0C5.

You can help EUREKA by purchasing a few copies of this book for yourself and your friends, and by seeing to it that a copy gets into your high school, college, or university library.

* * *

WHY DID METHUSELAH MISS THE BOAT?

According to Genesis 5:25 - 29 and 7:6, "... Methuselah lived 187 years, and begat Lamech. ... And all the days of Methuselah were 969 years; and he died. And Lamech lived 182 years, and begat a son: And he called his name Noah, ... And Noah was 600 years old when the flood of waters was upon the earth."

Consequently, as H.S.M. Coxeter has observed, in the year of the flood, Methuselah's age was

$$187 + 182 + 600 \quad \text{or} \quad 969.$$

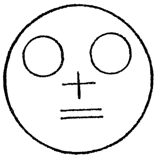
The question arises: Did Methuselah die by drowning, from shock over being left behind by his grandson Noah, from fright over the imminence of the Flood, or just from old age?

CHARLES W. TRIGG[1]

* * *

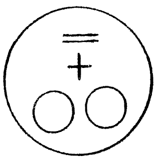
VARIATIONS ON A THEME BY BANKOFF V

The theme by Leon Bankoff



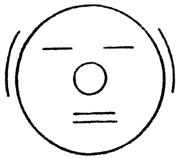
$0 + 0 = 0$

Variation No. 10
by H.G. Dworschak



"Take me to your leader"

Variation No. 11
by B.C. Rennie



$-(-0) = 0$ or

"algebra makes me sleepy"

[1]Reprinted by permission from the (January 1974) *Wesley Palms Breezes*, San Diego, California.