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CONTENTS

NEARLY THE LAST COMMENT ON THE STEINER-LEHMUS THEOREM

DAVID C. KAY, The University of Oklahoma

In view of recent articles in EUREKA (see [4], [5]) on proofs of the Steiner-Lehmus theorem, it appears that a very interesting observation concerning one of those proofs may have escaped attention. A slightly revised version of the inequality proof found in [1, p. 72] establishes the Steiner-Lehmus theorem in all three classical geometries simultaneously, as suggested in an exercise in [2, p. 119] and referenced in [4].

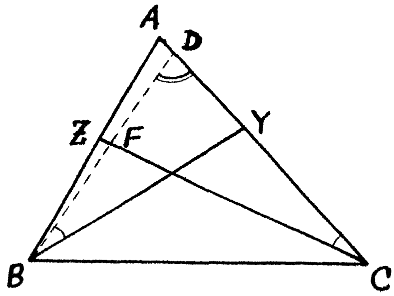
The proof which appears in [1] is due to L.M. Kelly [3] who assigned the Steiner-Lehmus proposition to this author as a problem in hyperbolic geometry when he was a graduate student at Michigan State University in 1962. Thus, the observation is not entirely the author's, and it would be interesting to learn how many EUREKA readers are also already aware of it.

One must first establish the usual inequality theorems for triangles for the two non-Euclidean geometries (in elliptic geometry we restrict the theorem to triangles whose sides are less than $\frac{1}{4}$ the spherical circumference). Then the following routine result on inequalities for triangles is established, from which an indirect statement of the Steiner-Lehmus theorem can be proved.

LEMMA. If two triangles have two angles of one congruent to two angles of the second and the included side of the first greater than that of the second, then each of the remaining two sides of the first is greater than the corresponding side of the second triangle.

THEOREM [1, p. 72]. *If one side of a triangle is greater than a second, then the angle bisector to the smaller side is the greater bisector.*

Proof (see figure). Suppose BY, CZ are angle bisectors and AC > AB. Then $\frac{1}{2}\angle C < \frac{1}{2}\angle B$ so there exists D between A and Y such that $\angle DBY = \angle ACZ$; let F be the intersection of segments BD and CZ (theorem of Pasch). Now in triangles BDY and CDF we have $\angle DBY = \angle DCF$ and $\angle BDY = \angle CDF$ but DC > DB (since, in triangle BCD, $\angle DBC = \angle DBY + \frac{1}{2}\angle B > \angle DCF + \frac{1}{2}\angle C = \angle DCB$). By the lemma, CZ > CF > BY, the desired conclusion.



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4. Léo Sauvé, "The Steiner-Lehmus Theorem", EUREKA, Vol. 2 (1976), pp. 19-24.

5. Charles W. Trigg, "A Bibliography of the Steiner-Lehmus Theorem", EUREKA, Vol. 2 (1976), pp. 191-193.

* * *

A French general?
A mysterious society?
The backbone of modern mathematics?

Oddly enough,
Bourbaki is
all of these things.

BOURBAKI

DAGMAR R. HENNEY, George Washington University

A revolution is going on in the field of mathematics and its leader is named Bourbaki. Almost every scientist has heard the name Bourbaki before and is aware of the mystery and numerous legends surrounding it. But what exactly is the truth behind these rumors?

The strangest fact about this famous mathematician is that he does not exist. A group of young French mathematicians has written numerous comprehensive volumes on mathematics using the collective pseudonym of Bourbaki. Nobody is quite certain why the name was chosen; maybe it was done in jest, or maybe to avoid using the names of all the authors who collaborated on each article published by Bourbaki.

When the French army was disastrously defeated in 1871, little did its commander, General Bourbaki, realize that some day his name might be famous and talked about by every mathematician in the world. General Bourbaki's life had been a collection of whimsical mishaps. Though at one time well thought of (he was offered a chance to become king of Greece, but declined because of his mistress's opposition) his luck soon changed. He was imprisoned in Switzerland after being defeated in battle and tried to shoot himself. Apparently he did not succeed in this, either; it is reported that at the ripe age of eighty years he challenged another retired officer to a duel.

The society that now proudly bears his name was first formed in the early 1930's in Nancy, France. It is said that at this time students at the École Supérieure were exposed to a lecture by a distinguished visitor. This visitor, called Bourbaki, was in fact an actor, and his lectures were a perfect piece of double-talk.

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The major Bourbaki publication, *The Elements of Mathematics*, appeared at the same time. The "Elements" constitutes a complete and coherent survey of modern mathematics from a sophisticated point of view. Bourbaki reorganized another fundamental field of mathematics in a giant work entitled *The Fundamental Structures of Analysis*, which has already surpassed twenty-five volumes. The subdivisions of the first part might shock the classical mathematician or general scientist who thinks of mathematics in terms of algebra, trigonometry and analytic geometry. Instead, the first part includes topology, topological vector spaces, integration, set theory, functions of a real variable, and modern algebra. Now in progress is the second part consisting of Lie algebra and commutative algebra. The third part is to be devoted to differential topology, algebraic number theory, and partial differential equations. If any layman was ever of the opinion that mathematics is a dead science, he should but take a glance at any of these volumes to be convinced that this is not so. There is even a topic in mathematics called algebraic topology which Bourbaki has not discussed because it changes so rapidly from year to year.

How is a treatise of this magnitude written? Much energy goes into the writing of each single volume, which often takes ten to twenty years to complete. The society usually meets three times a year. One person is elected to write the first draft. This draft is read during the general session and discussed at great length. Following this, a second member of the organization writes a second draft and it is read with much the same results during the following year. After ten, twelve, or fourteen years everyone involved finally comes to an agreement, and the book is turned over to the editor. The final approval of a draft is celebrated over a glass of wine. Since the Bourbaki books have been a commercial success, the members have sufficient funds to pay for travel expenses and provide food and drinks to lubricate their endeavors.

Unhappy parents of high school students have often asked who was responsible for the modern trends in mathematics. Many mathematicians share the opinion that Bourbaki represents the backbone of modern math. It is quite certain that Bourbaki is one of the leaders advocating a change of the mathematics curriculum. People all over the world are under the influence of Bourbaki since this group has collected and assimilated almost all the material available in this field. Elementary concepts which were formerly thought of as "high-powered" mathematics, such as set theory and topology (a branch of geometry which deals with properties not affected by changes in size and shape) are, thanks to Bourbaki, now often introduced on the high school level.

One of the Bourbakists, Jean Dieudonné, has warned us recently: "At the present moment the mathematics taught in our schools corresponds approximately to what was known in 1640." Are we slowly catching up with the Twentieth Century?

Several mathematicians have come to regret having cast aspersions on the existence of Bourbaki. The society does not like to have its secrets told publicly. In these matters Bourbaki will retaliate quickly. At one time an American mathematician, Boas, was asked to furnish a biography of Bourbaki, which was to be incorporated in a mathematics biography. The American returned a letter explaining that the name Bourbaki was but a pseudonym and that in fact Bourbaki did not exist. As soon as the Bourbaki group found out about this, they sent a telegram to the editor, giving in detail birthdate, birthplace and different universities attended by Bourbaki and concluding that, in fact, the American mathematician, Boas, was but a collective pseudonym for a group of young Americans, and that, in fact, Boas did not exist.

Biographical Note.

Dagmar R. Henney, Associate Professor of Mathematics, who dreamed the American dream while a German schoolgirl, is now realizing it as a young, skilled, American specialist. She has combined emigration, application, and ability to become sought after by international conferences in mathematics and highly regarded by her faculty colleagues and students at George Washington. Barred as a Jew from Nazi schools, she first learned figures and equations from her physicist father. She entered high school after the war, at age 10, and learned enough mathematics there to qualify for 63 credits when she entered the University of Miami in 1962. Within three and a half years, she had completed bachelor's and master's degrees, and married Alan Henney, now a physicist with the Naval Ordnance Laboratory. Later she was naturalized and commenced studies for the doctorate at the University of Maryland, working on additive set-valued functions in Banach spaces, a highly esoteric but important segment of pure math. She has published research papers in journals in Europe, Asia, and the United States. She is currently a member of the reviewing staff of *Mathematical Reviews, Zentralblatt für Mathematik, American Mathematical Monthly, Physics Today*. She has given invitation lectures for the University of Freiburg during sessions in Oberwolfach, Germany, of an international meeting of mathematicians. She has lectured at national Mathematics Society meetings at Miami, Cornell, New York, Chicago. At George Washington she teaches classes in calculus, finite mathematics, and measure and integration. She is adviser and a charter member of the new University chapter of Pi Mu Epsilon, national honorary mathematics fraternity, and is a member of Sigma Xi, Phi Beta Kappa, and various other scholarly groups. She has recently been named to the National Association of Science Writers. Dr. Henney, like many mathematicians, enjoys music, art, and a good game of chess. She is convinced that mathematics is *not* a "dead science," and that mathematicians in general are both useful and entertaining.

* * *

Patriotic Homonyms

There is a vast difference between America and amurca. (Charles W. Trigg)

There is a vas deferens between an American and a merkin. (Edith Orr)

THE DISTRIBUTION OF ODD NUMBERS

A Probabilistic Approach

RICHARD S. FIELD, Santa Monica, California

While the labor of countless mathematicians has been applied to the question of the distribution of prime numbers, comparable efforts have not been expended on the more basic questions regarding the incidence of *odd numbers*. Surely an examination of the parent is relevant to an analysis of the child!

It is almost universally supposed in the mathematical community (as if mathematicians were hypnotized by its monotonous regularity) that the odd-even-odd-even progression continues indefinitely. Yet it is this very human urge to extrapolate from the familiar to the unknown that has had to be overcome by such men as Einstein, in order that modern science could proceed. But there is no evidence that number theorists have advanced a whit beyond Diophantus. In fact such a typical hieroglyph as

$$(\lambda, i) = 1 \rightarrow \theta \equiv -1 \pmod{7} \neq \square$$

is illuminative of their primitive state of mind. Number theory serves in a sense as a vicarious return to the womb. Everything is orderly; proofs are plonked out as if they were lollipops; one leads to another.

Rather than subjecting the reader to any symbols, lemmas or other mind-deadening devices, we will ask him to perform an experiment. Place in a container a number of strips—say twenty—of which half are marked with even numbers and half with odd. Draw pairs randomly and multiply their numbers together. On the average, the product will be odd only one time out of four! Then draw three strips at a time and multiply the three numbers. Now an odd product occurs only one time in eight! The more strips drawn, the smaller the probability of an odd product.

All composite numbers can be represented as the product of a set of factors greater than unity: on the average, the larger the number, the more factors. If a *single one* of the factors is *even*, the number is *even*. This directly implies that the larger a number is, the more likely it is to be even. So the chance that a very large, randomly selected number is odd is extremely small. It is thus clearly implied that the density of odd numbers approaches asymptotically that of the prime numbers; if one encounters a very large odd number, he may be almost certain that it is prime!

Once again man has been tricked by his instinct to extrapolate. A chance of nature—that odds and evens occur in uniform sequence in our puny realm of experience—is misinterpreted as a natural law. If modern physics operated on this basis, we would never have gotten to the moon. To free ourselves from hidebound concepts we must apply statistical—that is, probabilistic—methods in much the same way that quantum physicists have found essential to their science.

The editor who has the courage to publish this brief paper may expect, along with the author, to be subject to the storm of indignation and anguish that follows every break from traditional thought. This price we willingly pay.

* * *

LETTER TO THE EDITOR

Dear Editor:

Rennie's article on a universal cover in the March EUREKA [1977: 62] caught my eye.

H.S.M. Coxeter says in another *Eureka* [1] that J. Pál's result, 0.8453, was slightly improved in 1954 by R.P.C. Caldwell, now at the University of Rhode Island; he cut off another piece of the regular hexagon to reduce the area to 0.8444.

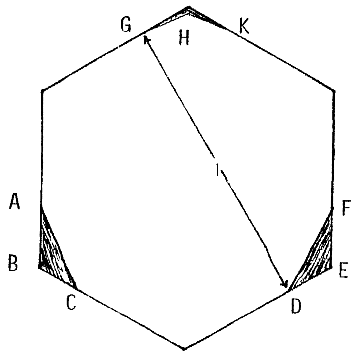
But an even better improvement of Pál's result had already been published in a 1936 paper by R. Sprague [3], a copy of which is enclosed.

C.S. Ogilvy discussed the problem in his book *Tomorrow's Math*, but the picture in his first edition is faulty. The second edition of his book [2] contains a better picture and a reference to Sprague's paper.

J.D.E. KONHAUSER,
Macalester College,
Saint Paul, Minnesota.

Editor's comment.

The figure shows a regular hexagon of side $1/\sqrt{3}$ (so that it could be circumscribed around a circle of diameter 1). Cutting off triangles ABC and DEF, whose bases AC and DF are tangent to the circle, leaves area 0.8453, which is Pál's result. Draw arcs GH and HK with radius 1 and centres D and C, respectively. Sprague showed in [3] that the area of the corner outside the two arcs GH and HK can also be cut off, reducing the area to 0.8441.



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1. H.S.M. Coxeter, "Lebesgue's Minimal Problem", *Eureka* (Journal of the Cambridge Archimedean Society), October 1958, p. 13.

2. C. Stanley Ogilvy, *Tomorrow's Math*, Second Edition, Oxford University Press, 1972, pp. 53-55, 170-171.

3. R. Sprague, "Über ein elementares Variationsproblem", *Matematisk Tidsskrift*, 1936, pp. 96-99.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1977, although solutions received after that date will also be considered until the time when a solution is published.

234. [1977: 104] Correction. The problem should read:

If $\sin\frac{2^n\pi}{13} = \pm\sin\frac{\pi}{13}$, prove that...(the rest is unchanged).

248. [1977: 131] Correction. Replace the last two sentences by the following sentence:

Prove that WQ is equal to the circumradius of Δ PFG.

251. *Proposed by Robert S. Johnson, Montréal, Québec.*

Solve the cryptarithmic addition given below. It was given to me by a friend who got it himself from a friend at least 15-20 years ago. I hope some reader can identify its unknown originator.

$$\begin{array}{r} \text{SPRING} \\ \text{RAINS} \\ \text{BRING} \\ \text{GREEN} \\ \hline \text{PLAINS} \end{array}$$

252. *Proposed by Richard S. Field, Santa Monica, California.*

Discuss the solutions, if any, of the system

$$x^y = A$$
$$y^x = A + 1,$$

where $A \geq 2$ is an integer.

253. *Proposed by David Fisher, Algonquin College, Ottawa.*

Let $x{\uparrow}y$ denote x^y. What are the last two digits of $2{\uparrow}(3{\uparrow}(4{\uparrow}5))$?

254. *Proposed by M.S. Klamkin, University of Alberta.*

(a) If $P(x)$ denotes a polynomial with integer coefficients such that

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 4000,$$

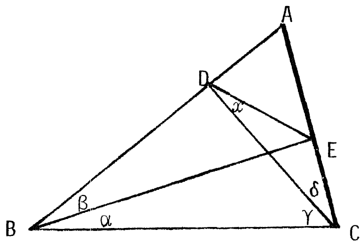
prove that the zeros of $P(x)$ cannot be integers.

(b) Prove that there is no such polynomial if

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 1000.$$

255. *Proposed by Barry Hornstein, Canarsie H.S., Brooklyn, N.Y.*

In the adjoining figure, the measures of certain angles are given. Calculate x in terms of α, β, γ, δ.



256*. *Proposed by Harry L. Nelson, Livermore, California.*

Prove that an equilateral triangle can be dissected into five isosceles triangles, n of which are equilateral, if and only if $0 \le n \le 2$. (This problem was suggested by Problem 200.)

257. *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Can one draw a line joining two distant points with a BankAmericard? (Solutions with Chargex or Master Charge also acceptable.)

258. *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

For any rational $k \neq 0$ or -1, find the value of the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/k}(n^{k-1/k} + i^{k-1/k})}{n^{k+1}}.$$

259*. *Proposed by Jacques Sauvé, University of Waterloo.*

The function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$$

is defined for all real x. Can one express $f(x)$ in closed form in terms of known (not necessarily elementary) functions?

260. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Given any triangle (other than equilateral), let P represent the projection of the incentre I on the Euler line OGNH where O, G, N, H represent respectively the circumcentre, the centroid, the centre of the nine-point circle and the orthocentre of the given triangle. Prove that P lies between G and H. In particular, prove that P coincides with N if and only if one angle of the given triangle has measure 60°.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

197. [1976:220; 1977:109] *Proposed by Charles W. Trigg, San Diego, California.*

In the octonary system, find a square number that has the form *aaabaaa*.

III. *Comment by the proposer.*

Professor Closs's comment II [1977:110] has added greatly to the appreciation of the mathematical erudition of the ancient Yuki, an Indian tribe now nearly extinct. (It should be noted that 9 is not a digit in the octonary scale, but rather that its equivalent in base eight is 11. The sum of the digits of this 3^2 is $2 = \sqrt[3]{10}$.)

The Yuki tribe has a ceremonial dance attributed to a legendary forebear named Lay Lei. The costumes worn feature elaborate helmets. Throughout the dance, the dancers form a square. The dance is accompanied by a repetitive chant,

o-mahat o-mahat o-mahat pa-wi o-mahat o-mahat o-mahat,

synchronized with three hard slaps by the left hand on the left thigh, followed by a vertical leap, and then three slaps by the right hand on the right thigh [1].

Tourists tend to think that the chant is calling attention to the dancers' fantastic headgear. Those with leanings toward the esoteric East call attention to the recurrent appearance of the mystic *Om*. Now, however, Closs's information makes it clear that the ceremony celebrates the marvelous power of squares[1] discovered by the Yuki's ancestors. Each of the hand slaps, in mystic groups of three, emphasizes the 4 spaces between the fingers. The leap following the hard left hand slaps is induced by the previous pain and glorifies the oneness of the individual. The chant itself repeats the unique square, 4441444; doubly sacred because each of its digits is a square; triply sacred since the sum of its digits is a square, $5^2 = 31$; and quadruply sacred because the sum of the digits of the sum is a square, 4.

Others contend that there is evidence that the original Yuki Indians did not come across a land bridge from Asia, but rather in seaworthy canoes that stopped over in Hawaii. There the legendary chief Yuki Lay Lei left his mark on their musical instruments and floral decorations.

The octonary system could well have gotten a start in the Middle East when the animals entered the Ark two-by-two-by-two. (There seems to be no justification for tying this in with the length *cubit*.) Noah, however, had too many solid-hoofed and web-footed creatures with no space between their toes.

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[1]Could Lay Lei be hanging from the family tree of Euclide Paracelso Bombasto Umbugio (see [2])?

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1. Isidore MacGregor Aristotle Laigpulla, "Ceremonial Dances of the Western Tribes," *Acta Apocrypha Anthropologia*, Vol. 4 (1944), pp. 414-441.

2. Charles W. Trigg, "Revealed at Last: The Face of E.P.B. Umbugio," EUREKA, Vol. 3 (1977), pp. 126-128.

* * *

208. [1977:10] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let a and b be real numbers such that $a \geq b \geq 0$. Determine a matrix X such that

$$X^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

Adapted from the solutions of Clayton W. Dodge, University of Maine at Orono; and David R. Stone, Georgia Southern College, Statesboro, Georgia.

If

$$X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \text{and} \quad X^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

then we must have

$$x^2 + yz = a = w^2 + yz,$$

$$z(x+w) = b = y(x+w),$$

which imply $x^2 = w^2$ and $(x+w)(y-z) = 0$. We consider two cases.

Case 1: solutions in which $x+w \neq 0$. Here $x \neq -w$, so we must have $x = w$ and $y = z$. Thus

$$X = \begin{bmatrix} x & y \\ y & x \end{bmatrix},$$

where x and y satisfy

$$x^2 + y^2 = a \quad \text{and} \quad 2xy = b.$$

Solving these equations simultaneously gives

$$(x,y) = (A,B),\ (-A,-B),\ (B,A),\ (-B,-A),$$

where

$$A = \frac{\sqrt{a+b} + \sqrt{a-b}}{2} = \sqrt{\frac{a + \sqrt{a^2 - b^2}}{2}},$$

$$B = \frac{\sqrt{a+b} - \sqrt{a-b}}{2} = \sqrt{\frac{a - \sqrt{a^2 - b^2}}{2}}.$$

Thus

$$X = \pm\begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \text{or} \quad \pm\begin{bmatrix} B & A \\ A & B \end{bmatrix}. \tag{1}$$

It is easy to verify, conversely, that the four matrices in (1) are in fact solutions to the problem. They are thus the only solutions for which $x + w \neq 0$.

Case 2: solutions in which $x + w = 0$. Such solutions will exist, of course, only if $b = 0$. Here we have $w = -x$, and the only other constraint is $x^2 + yz = a$. All the possibilities are included in

$$X = \begin{bmatrix} x & y \\ \dfrac{a - x^2}{y} & -x \end{bmatrix} \quad \text{with } x,y \text{ arbitrary and } y \neq 0 \tag{2}$$

and

$$X = \pm\begin{bmatrix} \sqrt{a} & 0 \\ z & -\sqrt{a} \end{bmatrix} \quad \text{with } z \text{ arbitrary.} \tag{3}$$

Conversely, it is easily verified that all the matrices in the infinite sets (2) and (3) are in fact solutions to the problem.

All solutions have thus been found.

Also solved by S. GREITZER, Rutgers University; M.S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Naubinway, Michigan; DALE STEWART, University of Ottawa; KENNETH M. WILKE, Topeka, Kansas; and the proposer (two solutions).

Editor's comment.

Most of the other solutions submitted were incomplete in that they did not give all the answers to the problem, and one escaped the stigma of being called incorrect by the clever subterfuge of being incomprehensible (at least to this dim-witted editor).

Klamkin and the proposer showed that the given matrix can be expressed in the form

$$X^2 = T\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}T^{-1},$$

where $\lambda_1 = a + b$ and $\lambda_2 = a - b$ are the eigenvalues of the given matrix. Then a square root of the matrix is given by

$$X = T\begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}T^{-1}.$$

(This is essentially the approach taken by Bellman in [1].) But they did not indicate whether and how all solutions can be found by this method.

An answer to the more general problem of finding a square root of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which had been proposed by Hüseyin Demir, was published in [3]. Harry L. Nelson observed in [2] that he found the solution given in [3] to be incomprehensible (the solver was a Canadian!). He then proceeded to give his own (quite clear but incomplete) answer to the problem, but without any indication of how he arrived at it.

Incidentally, in [2] Nelson gives reference [3] but in garbled form. Readers of the *Journal of Recreational Mathematics* will find reference [3] given correctly below.

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1. Richard Bellman, *Introduction to Matrix Analysis*, Second Edition, McGraw-Hill (1970), pp. 93-94.

2. Harry L. Nelson, "Matrix Root", *Journal of Recreational Mathematics*, Vol. 9, No. 2 (1976-77), p. 135.

3. Solution to Problem 487, *Mathematics Magazine*, Vol. 36 (1963), pp. 76-77.

* * *

209. [1977:10] *Proposed by L.F. Meyers, The Ohio State University.*

Suppose that the sequence $(a_n)_{n=1}^{\infty}$ of nonnegative real numbers converges to 0. Show that there exists a sequence $(e_n)_{n=1}^{\infty}$ each of whose terms is 1 or -1 such that $\Sigma_{n=1}^{\infty} e_n a_n$ converges.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Let $e_1 = 1$ and let $e_{k+1} = 1$ or -1 according as

$$S_k = \sum_{n=1}^{k} e_n a_n \le 0 \text{ or } > 0.$$

That is, whenever a partial sum is nonpositive, the next term is positive; whenever it is positive, the next term is negative. Now we prove that the sequence of partial sums S_k is a Cauchy sequence. Let $\varepsilon > 0$ be given. Then there is a positive integer N such that

$$k > N \implies |a_k| < \frac{\varepsilon}{2}.$$

If $e_m = 1$ (or -1) for all $m > N$, then S_m is a monotone increasing (or decreasing) sequence that is bounded above (or below) by zero, and the proof is complete since every bounded monotone sequence is a Cauchy sequence (see [2], Theorem 58.10). Otherwise choose $m > N$ such that $e_m e_{m+1} = -1$. Then we must have $|S_m| < \frac{\varepsilon}{2}$ and, if $n > m$, $|S_n| < \frac{\varepsilon}{2}$ by the definition of the e_k above. Hence

$$|S_m - S_n| \le |S_m| + |S_n| < \varepsilon,$$

so the S_k form a Cauchy sequence. Since every Cauchy sequence of real numbers has a real limit (see [2], Theorem 62.6), it follows that $\Sigma_{n=1}^{\infty} e_n a_n$ exists.

II. *Solution by M.S. Klamkin, University of Alberta.*

The problem here is just a special case of Dirichlet's test (see [1]):

If an oscillatory series Σe_n *has finite maximum and minimum limiting values, it will become convergent if its terms are multiplied by a decreasing sequence* (a_n) *which tends to zero as a limit.*

Also solved by DAN SOKOLOWSKY, Yellow Springs, Ohio; and the proposer.

REFERENCES

1. T.J.I'a Bromwich, *Infinite Series*, Macmillan, London, 1947, p. 59.

2. C.W. Dodge, *Numbers and Mathematics*, Second Edition, Prindle, Weber & Schmidt, Boston, 1975.

* * *

210. [1977:10] *Proposed by Murray S. Klamkin, University of Alberta.*

P, Q, R denote points on the sides BC, CA and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad (\ne 0,\ 1/2,\ 1),$$

then PQR (in some order) is similar to ABC.

I. *Adapted from the solutions submitted independently by W.J. Blundon, Memorial University of Newfoundland; and Dan Sokolowsky, Yellow Springs, Ohio.*

Consider first the case of scalene triangles and assume without loss of generality that $a > b > c$.

First note that

$$\Delta\,AQR = \Delta\,BRP = \Delta\,CPQ = k(1-k)\cdot\Delta\,ABC.$$

Now QR, RP, PQ are equal in some order to λa, λb, λc where

$$\lambda^2 = \frac{\Delta\,PQR}{\Delta\,ABC} = 1 - 3k(1-k) = 1 - 3k + 3k^2$$

and $k \ne 0,\ 1/2,\ 1$.

Apply the cosine law for C simultaneously in Δs ABC, QPC. Then

$$PQ^2 = (1-k)^2a^2 + k^2b^2 - 2k(1-k)ab\cdot\frac{a^2+b^2-c^2}{2ab}$$

$$= (1-3k+2k^2)a^2 + (2k^2-k)b^2 + (k-k^2)c^2.$$

Let PQ = λa. Then $PQ^2 - (1-3k+3k^2)a^2 = 0$, that is,

$$-k^2a^2 + (2k^2-k)b^2 + (k-k^2)c^2 = 0,$$

whence

$$k = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}.$$

Similarly, we complete the solution for k of nine such equations given by

one of QR, RP, PQ = one of λa, λb, λc

and compile the results, which may be grouped into three cases identified by the subscripts $i = 1, 2, 3$.

Case 1. $$\frac{P_1Q_1}{b} = \frac{Q_1R_1}{c} = \frac{R_1P_1}{a} \iff k_1 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2} \qquad (0 < k_1 < \tfrac{1}{2}),$$

which has a unique solution for each $a > b > c$.

Case 2. $$\frac{P_2Q_2}{c} = \frac{Q_2R_2}{a} = \frac{R_2P_2}{b} \iff k_2 = \frac{c^2 - a^2}{2c^2 - a^2 - b^2} \qquad (\tfrac{1}{2} < k_2 < 1),$$

which also has a unique solution for each $a > b > c$.

Case 3. $$\frac{P_3Q_3}{a} = \frac{Q_3R_3}{b} = \frac{R_3P_3}{c} \iff k_3 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2},$$

that is,

$$k_3 = \frac{b^2 - c^2}{(b^2-c^2) - (a^2-b^2)}.$$

This is impossible if $0 < k < 1$; but it seems artificial to restrict the points P, Q, R to lie on the *segments* BC, CA, AB, and in any case the proposal does not require it explicitly. So if we do not impose the restriction $0 < k < 1$, Case 3 also has a unique solution for each $a > b > c$.

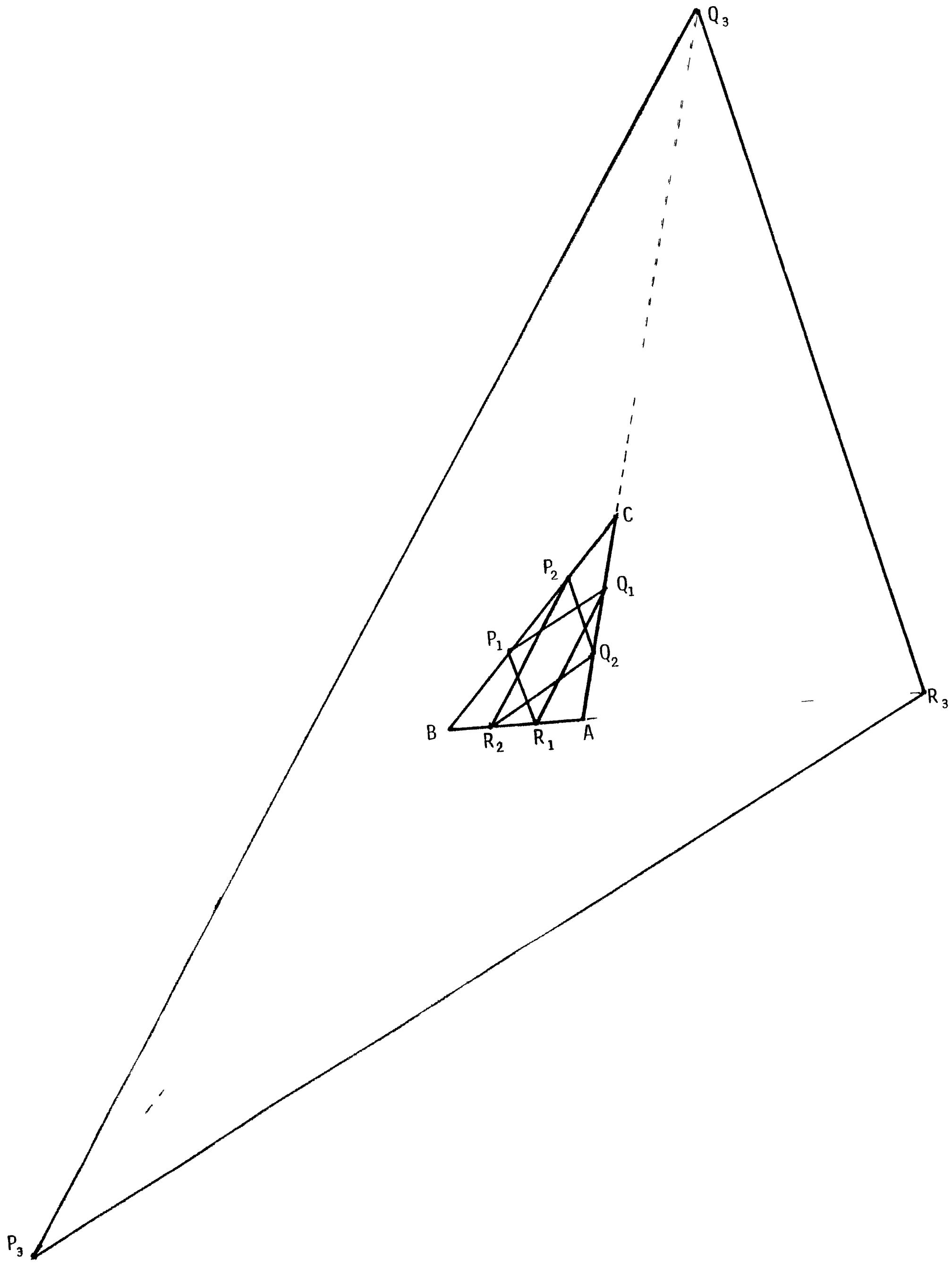
We have thus shown that for each scalene triangle ABC there are three distinct triangles PQR similar to ABC. The similarity in each case is *indirect*. This could be shown here separately for each case, but it follows all at once from solution II below.

The easily proved relation

$$k_{i+1} = \frac{1 - k_i}{2 - 3k_i}, \qquad i = 1, 2, 3;\ k_4 = k_1$$

shows how the three solution triangles $P_iQ_iR_i$ are related to each other.

As an example, the figure on page 162 shows a scalene triangle ABC and the


Q_3
C
P_2
Q_1
P_1
Q_2
R_3
B
R_2
R_1
A
P_3

three triangles $P_iQ_iR_i$. In that figure $a=4$, $b=3$, $c=2$ and

$$k_1 = \frac{a^2-b^2}{2a^2-b^2-c^2} = \frac{7}{19}, \qquad k_2 = \frac{1-k_1}{2-3k_1} = \frac{12}{17}, \qquad k_3 = \frac{1-k_2}{2-3k_2} = -\frac{5}{2}.$$

Extending the above results to $a \ge b \ge c$, we have for each equilateral triangle $a=b=c$ trivially a unique triangle PQR for every k. Finally, for the sets of isosceles triangles $a=b>c$ and $a>b=c$, substitution of $a=b$ and $b=c$ in the three cases above gives only the values $k = 0, 1/2, 1$, all of which are inadmissible.

II. *Solution by the proposer.*

Let a complex number representation for the vertices be z_1, z_2, z_3. Then, as well known (see [1], for example), another triangle given by w_1, w_2, w_3 is directly similar to the former one if and only if

$$\begin{vmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{vmatrix} = 0.$$

Here P, Q, R are given by rz_2+sz_3, rz_3+sz_1, rz_1+sz_2, respectively, where r and s are constants related to k. If PQR ~ ABC, then by elementary determinant operations

$$0 = \begin{vmatrix} z_1 & tz_2+z_3 & 1 \\ z_2 & tz_3+z_1 & 1 \\ z_3 & tz_1+z_2 & 1 \end{vmatrix} = \begin{vmatrix} z_1 & z+z_2(t-1) & 1 \\ z_2 & z+z_3(t-1) & 1 \\ z_3 & z+z_1(t-1) & 1 \end{vmatrix} = \begin{vmatrix} z_1 & z_2 & 1 \\ z_2 & z_3 & 1 \\ z_3 & z_1 & 1 \end{vmatrix},$$

where $t = r/s$ and $z = z_1+z_2+z_3$. The last determinant reduces to

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0.$$

We also obtain the same equation if we start out with PQR ~ CAB. The equation factors into

$$(z_1+\omega z_2+\omega^2 z_3)(z_1+\omega^2 z_2+\omega z_3) = 0,$$

where ω is a primitive cube root of unity. Either factor set equal to zero implies ABC is equilateral; for example,

$$z_1 + \omega z_2 + \omega^2 z_3 = 0 = z_1 - z_3 + \omega(z_2-z_3).$$

Also solved by R. ROBINSON ROWE, Naubinway, Michigan.

Editor's comment.

It is clear from the proposer's solution II that he intended the proposal to read: "...then PQR (in some order) is *directly* similar to ABC." His solution then shows that only equilateral triangles have the desired property. In this form, the problem can be found in Pedoe [1] (Problem 47.5), where the crucial word "directly" is also inadvertently omitted.

But solution I shows that the problem, *as stated*, with no implicit restriction to direct similarity, has a richer solution which leaves only strictly isosceles (nonequilateral) triangles ABC for which there is no distinct similar triangle PQR (due to the restriction $k \neq \frac{1}{2}$).

REFERENCE

1. D. Pedoe, *A Course of Geometry for Colleges and Universities*, The University Press, Cambridge, 1970, p. 184.

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211. [1977:42] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve the cryptarithm FGB^2 = MASKEL. Since there are nine letters involved, naturally we seek a solution where FGB is divisible by 9.

I. *Solution by Judy Lynch, 6th Grade, Bulloch Academy, Statesboro, Georgia.*

Since FGB is divisible by 9 and its digits are all different, we must have

$$F + G + B = 9 \text{ or } 18. \tag{1}$$

Also $10^5 < FGB^2 < 10^6$ gives $317 \leq FGB \leq 999$, and $F \neq M$ further reduces this to

$$317 \leq FGB \leq 948. \tag{2}$$

From the list of numbers which satisfy (1) and (2), we can eliminate

1) all numbers containing repeated digits;
2) all numbers ending in 0, 1, 5, 6 (since $B \neq L$);
3) all numbers ending in 2 or 8 which contain a 4;
4) all numbers ending in 3 which contain a 9;
5) all numbers ending in 4 which contain a 6;
6) all numbers ending in 7 which contain a 9;
7) all numbers ending in 9 which contain a 1.

Of the remaining twenty-three numbers, only

$$567^2 = 321489$$

involves nine different digits and furnishes the unique solution.

II. *Comment by R. Robinson Rowe, Naubinway, Michigan.*

This problem sure knocked the L out of MASKELL.

Also solved by DON BAKER, Presidio Junior H.S., San Francisco, California; DOUG DILLON, Brockville, Ontario; HERTA T. FREITAG, Roanoke, Virginia; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; JAMES HOLT, student, Georgia Southern College, Statesboro, Georgia; R.S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, Bronx, N.Y.; R. ROBINSON ROWE, Naubinway, Michigan (solution as well); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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212. [1977:42] *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

Find four consecutive integers which are divisible by 5, 7, 9, 11 respectively.

Solution by Doug Dillon, Brockville, Ontario; and Daniel Flegler, Waldwick H.S., Waldwick, N.J. (independently).

It is clear that

$$\frac{5 \cdot 7 \cdot 9 \cdot 11 + a_i}{2}, \quad \text{where } a_1 = 5,\ a_2 = 7,\ a_3 = 9,\ a_4 = 11,$$

gives one solution: 1735, 1736, 1737, 1738. A generalization is too obvious to need mentioning.

Also solved by DON BAKER, Presidio Junior H.S., San Francisco, California; HIPPOLYTE CHARLES, Waterloo, Québec; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; RADFORD DE PEIZA, Woburn C.I., Scarborough, Ontario; CLAYTON W. DODGE, University of Maine at Orono; HERTA T. FREITAG, Roanoke, Virginia; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; JAMES HOLT, student, Georgia Southern College, Statesboro, Georgia; R.S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, Bronx, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Naubinway, Michigan; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Many solvers, using the Chinese Remainder Theorem (sometimes known as the Formosa Theorem (see, e.g., [1])) or otherwise, showed that all solutions can be found by taking

$$1735 + 3465n, \quad n \text{ any integer}, \tag{1}$$

as the first number.

Trigg twigged that if (1) is replaced by

$$1734 + 3465n, \quad n \text{ any integer}, \tag{2}$$

then we get five consecutive integers divisible respectively by 3, 5, 7, 9, 11.

Rowe rowet that with $n = 6$ in (2) we get the string 22524,...,22530 divisible respectively by 3, 5, 7, 9, 11, 13, 15.

Dodge fudged by noting that if the problem is changed to require that the four consecutive integers be respectively divisible by 4, prime, divisible by 51, and divisible by 5, then the smallest positive solution starts with $1732 = [1000\sqrt{3}]$, the year of George Washington's birth! (By any standard, therefore, 1733 was a prime year, since that is the year in which George Washington took his first steps, fell down, and said his first words.)

REFERENCE

1. H.M. Stark, *An Introduction to Number Theory*, Markham, 1970, p. 72.

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213. [1977:42] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

(a) Prove that (in the usual notation) the sides of a triangle are in arithmetic progression if and only if $s^2 = 18Rr - 9r^2$.

(b) Find the corresponding result for geometric progression.

Solution by the proposer.

(a) The sides a, b, c in some order form an arithmetic progression if and only if

$$(2a-b-c)(2b-c-a)(2c-a-b) = 0,$$

that is, $(2s-3a)(2s-3b)(2s-3c) = 0$, where $2s = a+b+c$. This equation reduces to

$$8s^3 - 12s^2\Sigma a + 18s\Sigma bc - 27abc = 0.$$

By the well-known results $\Sigma bc = s^2 + 4Rr + r^2$ and $abc = 4Rrs$, the equation reduces finally to

$$s^2 = 18Rr - 9r^2.$$

(b) Here the necessary and sufficient condition is

$$(a^2-bc)(b^2-ca)(c^2-ab) = 0,$$

which is equivalent to $(t-a)(t-b)(t-c) = 0$, where $t^3 = abc$. This equation reduces to

$$t^3 - t^2\Sigma a + t\Sigma bc - abc = 0,$$

that is, $\Sigma bc = t\Sigma a$. Thus $(\Sigma bc)^3 = abc(\Sigma a)^3$, whence

$$(s^2 + 4Rr + r^2)^3 = 32Rrs^4.$$

Also solved by LEON BANKOFF, Los Angeles, California; RADFORD DE PEIZA, Woburn C.I., Scarborough, Ontario; CLAYTON W. DODGE, University of Maine at Orono; R.S. JOHNSON, Montréal, Québec (part (a) only); F.G.B. MASKELL, Algonquin College, Ottawa; R. ROBINSON ROWE, Naubinway, Michigan; DAN SOKOLOWSKY, Yellow Springs, Ohio; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia (part (a) only); and CHARLES W. TRIGG, San Diego, California (part (a) only).

* * *

214. [1977:42] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Prove that if the sequence $\{a_i\}$ is an arithmetic progression, then

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

Solution by Radford de Peiza, Woburn C.I., Scarborough, Ontario.

A trivial, but necessary, verification shows that the given equation is true if the common difference $d = 0$.

Suppose now $d \neq 0$. We have

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} = \sum_{k=1}^{n-1} \frac{\sqrt{a_{k+1}} - \sqrt{a_k}}{d} = \frac{\sqrt{a_n} - \sqrt{a_1}}{d} = \frac{a_n - a_1}{d(\sqrt{a_1} + \sqrt{a_n})} = \frac{(n-1)d}{d(\sqrt{a_1} + \sqrt{a_n})}$$

$$= \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

Also solved by DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; DANIEL FLEGLER, Waldwick H.S., Waldwick, N.J.; HERTA T. FREITAG, Roanoke, Virginia; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; F.G.B. MASKELL, Algonquin College, Ottawa; SIDNEY PENNER, Bronx Community College, Bronx, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Naubinway, Michigan; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

With the exception of one, all other solvers implicitly or explicitly made, and used, the assumption $d \neq 0$, but mentioned no separate verification for $d = 0$, which means that, technically speaking, their proofs are incomplete. A proof is a proof is a proof.

The exceptional one was Dodge. His proof was by induction. He did not make a separate verification for $d = 0$, but his proof was carefully contrived so that every step was valid even for $d = 0$.

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215. [1977:42] *Proposed by David L. Silverman, West Los Angeles, California.*

Convert the expression given below from mathematics to English, thereby obtaining the perfect scansion and rhyme scheme of a limerick:

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 5(11) = 9^2 + 0.$$

I. *Solution by R. Robinson Rowe, Naubinway, Michigan.*

The specification "perfect scansion and rhyme scheme of a limerick" is a bit ambiguous. The rhyme scheme is definite, but scansion is subordinate to the pitch and the punch. In Wood [4], we read:

> The limerick is a five-line poem, following a rigid pattern. Its 1st, 2nd and 5th lines have three feet each, while its 3rd and 4th have two only. It is usually written in amphibrach, sometimes with the addition of an extra unaccented [syllable] at the beginning, which gives it an anapestic movement.

But checking into an anthology of good limericks [5] shows that "perfect scansion" permits *either* an iamb or an amphibrach for the last foot of a line, and so I submit the following:

A dózen, / a gróss and / a scóre
And thrée times / the squáre root / of fóur,
    Divíded / by séven,
    Plus fíve times / eléven,
Is níne to / the squáre and / no móre.

Thus my scansion ends lines 1, 2 and 5 with iambs and all other feet are amphibrachs.

II. *Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.*

A dozen, a gross and a score,
With three times the square root of four,
    Divided by seven,
    Plus five times eleven,
Makes nine to the two and no more.

III. *Solution by Charles W. Trigg, San Diego, California.*

To twelve and its square and a score,
Add three times the square root of four,
    Divide that by seven,
    Add five times eleven,
Get nine squared and not a thing more.

IV. *Solution by the proposer.*

A dozen, a gross and a score
Plus three times the square root of four,
    Divided by seven,
    Plus five times eleven,
Is nine squared and not a bit more.

Also solved (with some fractured amphibrachs) by STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; R.S. JOHNSON, Montréal, Québec; LEROY F. MEYERS, The Ohio State University; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

Conrad located this problem in Hurley [2], where it is credited to a mathematical poet named Archimedes O'Toole. The proposer also gave the same reference and confessed to the editor that Archimedes O'Toole is a *nom d'emprunt* which he has used from time to time.

The limerick is the only fixed poetic form native to the English Language. *Hickory Dickory Dock* was a limerick (remember?), but that was the last time the limerick ever saw a nursery. Don Marquis is reported to have divided limericks into three kinds: "Limericks to be told when ladies are present, limericks to be told when ladies are absent and clergymen are present, and LIMERICKS."

Baring-Gould's [1] was the first publicly issued collection to combine both clean and bawdy examples. But the largest collection of limericks ever published, erotic or otherwise, is the unbuttoned one edited by Legman [3]. It contains 1739 limericks, *none* of them otherwise. It also has a 76-page historical introduction to the limerick, perhaps the fullest to be found anywhere. (The preceding paragraph was taken from it.)

In a strictly scientific spirit and as a service to readers, like an editor should, I have carefully examined each of the 1739 limericks in [3], and I am able to report that, unless my count was faulted by a zooming blood pressure, only 6 of them (Nos. 169, 910, 911, 1415, 1416, 1555) refer to mathematics (this should make us all feel very virtuous). None of them, unfortunately, can be quoted here. However, I *should* tell readers that another one (No. 901) begins and ends as follows:

There was a young girl of Topeka
...........(!)..........
So she tried it and shouted, "Eureka!"

REFERENCES

1. William S. Baring-Gould, *The Lure of the Limerick*, Clarkson N. Potter, Inc., New York, 1967.

2. James F. Hurley, Editor, *Litton's Problematical Recreations*, pp. 43, 293. (Whenever possible, the editor would appreciate receiving *complete* references, including publisher and year.)

3. G. Legman, Editor, *The Limerick*, Bell Publishing Co., New York, 1969.

4. Clement Wood, *Unabridged Rhyming Dictionary*, World Publishing Co., 1943, p. 35.

5. *The World's Best Limericks*, The Peter Pauper Press, 1951.

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216. [1977:42] *Proposed by L.F. Meyers, The Ohio State University.*

For which positive integers n is it true that

$$\sum_{k=1}^{(n-1)^2} [\sqrt[3]{kn}] = \frac{(n-1)(3n^2-7n+6)}{4} \text{ ?}$$

The brackets, as usual, denote the greatest integer function.

Solution by W. J. Blundon, Memorial University of Newfoundland.

Let S_n be the given sum. If, in accordance with usual practice, we define $\Sigma_{k=1}^{0} a_k = 0$, then the stated formula holds for $n = 1, 2$, and we can henceforth assume $n > 2$.

The largest summand in S_n is $n-1$. Let N_t, $1 \le t \le n-1$, denote the number of times summand t occurs in S_n, so that

$$S_n = \sum_{t=1}^{n-1} tN_t. \tag{1}$$

Since

$$n-1 \le \sqrt[3]{kn} < n \iff \frac{(n-1)^3}{n} \le k < n^2,$$

we have, in view of the range of k in S_n,

$$N_{n-1} = (n-1)^2 - \left[\frac{(n-1)^3}{n}\right].$$

For $1 \le t \le n-2$, since

$$t \le \sqrt[3]{kn} < t+1 \iff \frac{t^3}{n} \le k < \frac{(t+1)^3}{n},$$

we conclude that

$$N_t = \left[\frac{(t+1)^3}{n}\right] - \left[\frac{t^3}{n}\right] \quad\text{or}\quad \left[\frac{(t+1)^3}{n}\right] - \left[\frac{t^3}{n}\right] + 1$$

according as $n \nmid t^3$ or $n \mid t^3$; and similarly

$$N_{t-1} = \left[\frac{t^3}{n}\right] - \left[\frac{(t-1)^3}{n}\right] \quad\text{or}\quad \left[\frac{t^3}{n}\right] - \left[\frac{(t-1)^3}{n}\right] - 1$$

according as $n \nmid t^3$ or $n \mid t^3$. Thus S_n is increased by 1 for each t such that $n \mid t^3$.

Let λ be the number of terms in the sequence $1^3, 2^3, \ldots, (n-2)^3$ which are divisible by n. Substituting the above results in (1), we have

$$\begin{aligned} S_n &= \lambda + \sum_{t=1}^{n-2} t\left\{\left[\frac{(t+1)^3}{n}\right] - \left[\frac{t^3}{n}\right]\right\} + (n-1)\left\{(n-1)^2 - \left[\frac{(n-1)^3}{n}\right]\right\} \\ &= \lambda + (n-1)^3 - \sum_{t=1}^{n-1} \left[\frac{t^3}{n}\right]. \end{aligned} \tag{2}$$

If T_n denotes the sum in (2), we can sum T_n in reverse order and get

$$2T_n = \sum_{t=1}^{n-1}\left\{\left[\frac{(n-t)^3}{n}\right] + \left[\frac{t^3}{n}\right]\right\} = \sum_{t=1}^{n-1}(n^2 - 3nt + 3t^2) + \sum_{t=1}^{n-1}\left\{\left[\frac{t^3}{n}\right] + \left[-\frac{t^3}{n}\right]\right\}.$$

The first of these sums is

$$n^2(n-1) - 3n \cdot \frac{n(n-1)}{2} + 3 \cdot \frac{n(n-1)(2n-1)}{6} = \frac{1}{2}n(n-1)^2;$$

and since

$$\left[\frac{t^3}{n}\right] + \left[-\frac{t^3}{n}\right] = \begin{cases} 0 & \text{if } n \mid t^3 \\ -1 & \text{if } n \nmid t^3 \end{cases},$$

the second sum is $-(n-1-\lambda)$. Thus

$$4T_n = n(n-1)^2 - 2(n-1-\lambda) = (n-2)(n-1)(n+1) + 2\lambda.$$

Now, from (2),

$$4S_n = 4\lambda + 4(n-1)^3 - 4T_n = (n-1)(3n^2 - 7n + 6) + 2\lambda,$$

so that

$$S_n = \frac{(n-1)(3n^2 - 7n + 6) + 2\lambda}{4},$$

where λ is the number of terms of the sequence $1^3, 2^3, \ldots, (n-2)^3$ which are divisible by n.

Thus the relation in the proposal holds if and only if $\lambda = 0$. We will show that

$$\lambda = 0 \iff n \text{ is squarefree.}$$

If n is not squarefree, there is a prime p such that $p^2 \mid n$; then $n = p^2a$ and, for $t = pa$, we have

$$n \mid t^3, \quad 1 \le t \le p^2a - 2 = n - 2,$$

so $\lambda \ne 0$.

If $\lambda \ne 0$, there is an integer t, $1 \le t \le n-2$, such that $n \mid t^3$; then n is not squarefree for otherwise $n \mid t$ and $n \le t$, a contradiction.

Also solved by DOUG DILLON, Brockville, Ontario; R. ROBINSON ROWE, Naubinway, Michigan (partial solution); DAN SOKOLOWSKY, Yellow Springs, Ohio; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer.

Editor's comment.

In a letter accompanying his proposal, the proposer wrote: "The problem appears in Uspensky and Heaslet [1], where the reader is asked to prove that equality holds if n is an odd prime greater than 3. (*Are* there any even primes greater than 3?)

However, equality holds for many other numbers, as my solution shows."

REFERENCE

1. Uspensky and Heaslet, *Elementary Number Theory*, McGraw-Hill, New York and London, 1939, p. 243, Problem 8.

* * *

217. [1977:43] *Proposed by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

Find all integer solutions of $n^2(n-1)^2 = 4(m^2-1)$.

Solution by Herta T. Freitag, Roanoke, Virginia.

Since m^2-1 must be a perfect square, $m = \pm 1$. Therefore

$$\{(n,m)\} = \{(0,1), (0,-1), (1,1), (1,-1)\}.$$

Also solved by DON BAKER, Presidio Junior H.S., San Francisco, California; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; R.S. JOHNSON, Montréal, Québec; LEROY F. MEYERS, The Ohio State University; SIDNEY PENNER, Bronx Community College, Bronx, N.Y.; BOB PRIELIPP, The University of Wisconsin-Oshkosh; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Naubinway, Michigan; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas;and the proposer.

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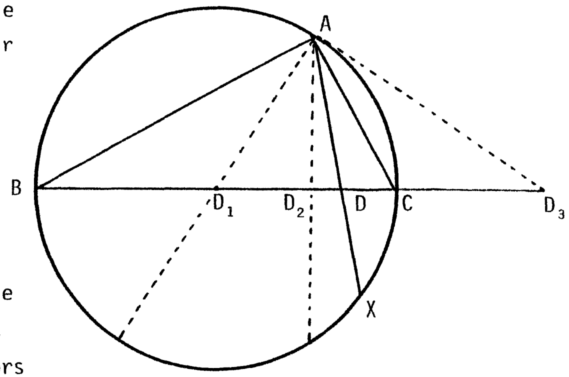
218. [1977:43] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Everyone knows that the altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. The median to the hypotenuse also has this property. Does any other segment from vertex to hypotenuse have the property?

Solution de F.G.B. Maskell, Collège Algonquin, Ottawa.



Soit ABC un triangle rectangle en A avec son cercle circonscrit (voir la figure). Nous allons, par une interprétation libérale du problème, trouver un point D de la *droite* BC tel que le segment AD soit moyenne proportionnelle entre les segments BD et DC.

Supposons d'abord que D divise le segment BC intérieurement, et soit ADX la corde correspondante. On a alors

$$AD^2 = BD \cdot DC = AD \cdot DX,$$

d'où AD = DX et AD est médiane ou hauteur du ΔABC.

Si D divise BC extérieurement, la relation $AD^2 = BD \cdot CD$ exige que AD soit tangente au cercle.

Il y a donc trois segments ayant la propriété désirée:

AD_1, la médiane;
AD_2, la hauteur;
AD_3, la tangente en A au cercle circonscrit.

Also solved by STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; RADFORD DE PEIZA, Woburn C.I., Scarborough, Ontario; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; R.S. JOHNSON, Montréal, Québec (partial solution); F.G.B. MASKELL, Algonquin College, Ottawa (second solution); LEROY F. MEYERS, The Ohio State University; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Naubinway, Michigan; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

None of the other solvers found that the tangent AD_3 was also a solution to the problem, so we have a home run for Algonquin College!

Of course, as mentioned in the above solution, this requires a liberal interpretation of the problem, but the continued progress of mathematics is due in large part to such liberal interpretations of mathematical situations.

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219. [1977:43] *Proposed by R. Robinson Rowe, Sacramento, California.*

Find the least integer N which satisfies

$$N = a^{a+2b} = b^{b+2a}, \quad a \neq b.$$

Composite solution pieced out from the contributions of several solvers identified below.

The proposer did not completely specify the domains of N, a, b; so we will consider several cases. We assume without loss of generality that $a < b$ in each case.

Case 1. N positive integer and a, b integers.

The least value of N is 1, which occurs for $(a,b) = (-2,1)$. (Griffiths, Meyers, Rennie, Stone)

Case 2. N, a, b positive integers.

Here $b = ka$, with rational $k > 1$; then the given relation holds if and only if

$$a = k^{\frac{k+2}{k-1}} = k^{1+\frac{3}{k-1}} \quad \text{and} \quad b = k^{\frac{2k+1}{k-1}} = k^{2+\frac{3}{k-1}}. \qquad (1)$$

It is clear that a and b are integers if and only if both k and $3/(k-1)$ are integers, that is, if and only if $k = 2$ or 4. Now

$$k = 2 \implies (a,b) = (16,32) \implies N = 16^{80} = 32^{64} = 2^{320}$$

and

$$k = 4 \implies (a,b) = (16,64) \implies N = 16^{144} = 64^{96} = 2^{576}.$$

Thus the least integral solution in this case is $N = 2^{320}$, but we might as well have been asked for *all* solutions since there are only two. (Meyers, Stone, Wilke, proposer)

Case 3. N positive integer and a, b positive reals.

Relations (1) still hold with real $k > 1$, and we have

$$N = (a^a)^{2k+1} = k^{\frac{(k+2)(2k+1)}{k-1} \cdot k^{\frac{k+2}{k-1}}}. \tag{2}$$

But for ease of handling we consider instead

$$\log\log N = \log(k+2)(2k+1) + (k+2)\left(\frac{\log k}{k-1}\right) + \log\left(\frac{\log k}{k-1}\right) = \phi(k).$$

It is fairly obvious (and it can be verified by calculus) that $\phi(k)$ is increasing for all $k > 1$ and that $\lim_{k\to\infty} \phi(k) = \infty$. Furthermore, for $k = 1 + u$, $0 < u < 1$, a Maclaurin expansion gives

$$\phi(k) = \phi(1+u) = 3 + 2\log 3 + \frac{31}{72}u^2 + \ldots,$$

so that

$$\lim_{k\to 1^+} \phi(k) = 3 + 2\log 3 = \log 9e^3.$$

Since (2) is continuous for all $k > 1$, N will assume all integral values greater than $\exp(9e^3)$, the least of which is

$$N = \left[e^{9e^3}\right] + 1 = 3.216182\ldots \times 10^{78},$$

a 79-digit integer. (Linis, Rennie, Stone)

Case 4. N negative rational and a, b integers.

Here a and b must both be negative, and also both odd since $a + 2b$ and $b + 2a$ must be odd. If we set $A = -b$ and $B = -a$, then $A < B$ (since $a < b$) and we need

$$N = (-B)^{-(B+2A)} = (-A)^{-(A+2B)},$$

that is,

$$-\frac{1}{N} = A^{A+2B} = B^{B+2A}.$$

We can now use the argument of Case 2, since nowhere in that argument was use made of the fact that N was an integer. Applied to the present case, it enables us to conclude that the existence of a negative rational solution N would require $A = 16$, which is impossible since A must be odd. Thus there are no solutions in this case. (Meyers)

Solutions were submitted by W.J. BLUNDON, Memorial University of Newfoundland; RADFORD DE PEIZA, Woburn C.I., Scarborough, Ontario; T.J. GRIFFITHS, A.B. Lucas Secondary School, London, Ontario; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; BASIL C. RENNIE, James Cook University of North Queensland, Australia; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

It is clear from the proposer's solution that Case 2 is the one he had in mind, but his failure to be more specific resulted in a much richer problem. Sometimes less is more in mathematics.

Rowe noted that the least solution in Case 2 is a 97-digit number which he partially computed as

$$2^{320} = 2\ 135\ 987\ 035..........936\ 576.$$

He could have saved himself the trouble, for Trigg reported that the full value of 2^{320} (as well as that of the 174-digit 2^{576}) can be found in one of the 75 volumes containing the first 33,219 powers of 2 which Rudolf Ondrejka has deposited in the library of Long Island University. They include all the powers of 2 with fewer than 10,001 decimal digits.

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220. [1977:43] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

C is a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. A chord through B meets CD at T and arc AD at U. Prove that there is a circle tangent to CD at T and to arc AD at U.

Solution and comments by Leon Bankoff, Los Angeles, California.

Let S denote the intersection of the radius UO with the line through T parallel to AB (see Figure 1). In the similar triangles UST and UOB, it is

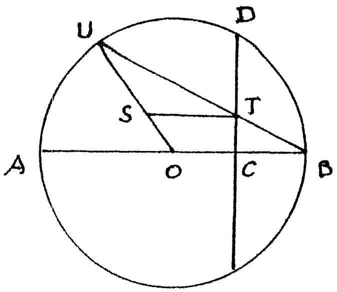

Figure 1.

evident that UO = OB and that US = ST. It is now easily seen that the circle (S)ST, or (S)SU, is tangent to DC at T and to arc AD at U.

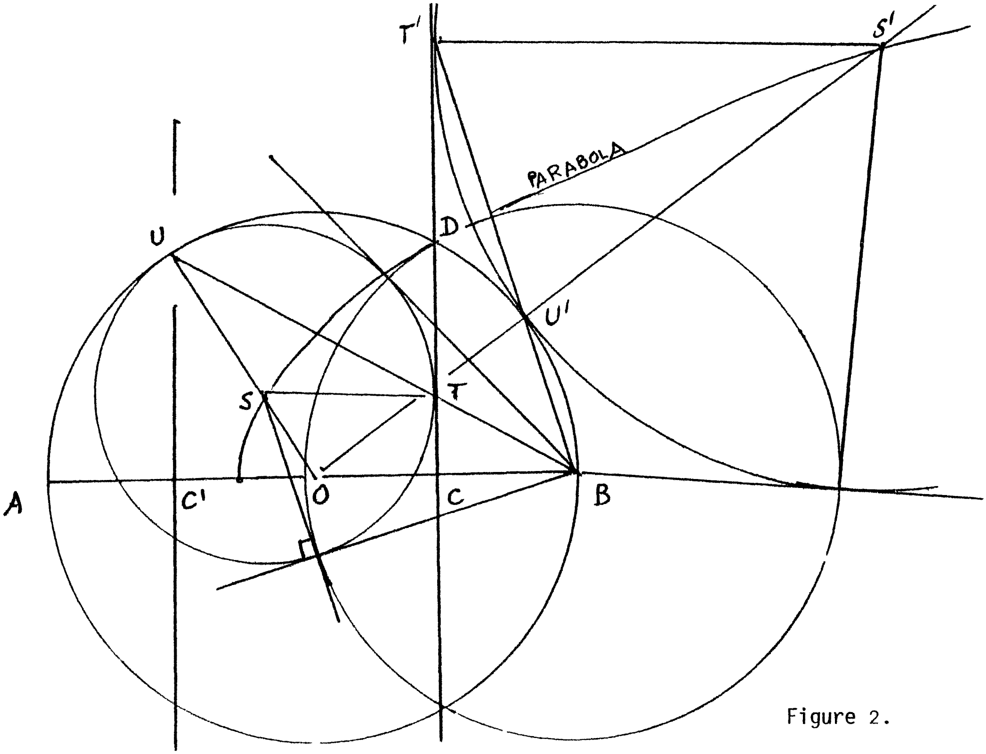

Figure 2.

Items of Interest: a) The same proof applies literally to the generalization involving extensions of chord CD and arc AD (see Figure 2).

b) Irrespective of the location of T on CD or its extension, the length of the tangent from B to the circle (S)ST is always equal to BD. The reason for this is that U and T are inverse points with respect to the circle of inversion (B)BD, which cuts the family of variable circles (S) orthogonally.

c) While the points of tangency from B to the circles (S) lie on the circle of inversion, it is interesting to observe that the centers of the variable circles (S)ST as T varies along CD follow the path of a parabola having O as its focus, with its directrix parallel to CD and cutting AB at C', where CC' = AO.

Also solved by DOUG DILLON, CLAYTON W. DODGE, ROLAND H. EDDY, HERTA T. FREITAG, T.J. GRIFFITHS, GILBERT W. KESSLER, LEROY F. MEYERS, SIDNEY PENNER, SAHIB RAM MANDAN, R. ROBINSON ROWE, CHARLES W. TRIGG, and the proposer.

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