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## Crux Mathematicorum

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## Crux Mathematicorum with Mathematical Mayhem

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# EDITORIAL

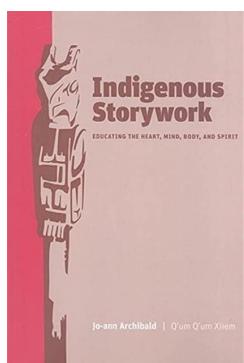
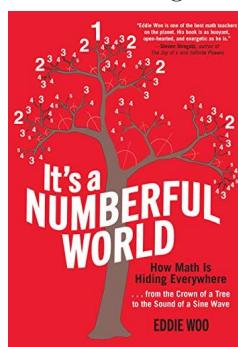
My New Year's resolution was to read more books cover-to-cover. The pandemic interfered with it first for the worse (when we were thrown into emergency remote teaching mode), then for the better (working from home is conducive to reading from home with no workplace distractions). In this editorial, I chose two to highlight.

As mathematicians, we do not need to be persuaded that math is beautiful or that it is more than just algebra. On the other hand, because we find the subject intrinsically elegant and interesting, we struggle to come up with examples of how math is, in fact, everywhere. I don't mean chicken coops, sliding ladders or even the golden ratio; rather, the opposite. What are some of the genuine mathematical appearances in our everyday lives that we can have a conversation about without getting too technical? What examples can we point out to people to describe that math is not just algebra or calculus? This is essentially what "It's a Numberful World" by Eddie Woo aims to do. In 26 short chapters, the author gives plenty of examples of mathematical patterns and how one can begin to explore them, finding intricate connections and curiosities. Woo ties in other science topics, such as chemical properties of the elements and the periodic table or development of models for the atom structure. While none of the mathematics was new to me (and yes, there's still a chapter on golden ratio), the exposition and choice of examples made for an enjoyable and educational read. As one of the reviewers at the beginning of the book points out "Eddie Woo is one helluva storyteller," which leads me to my second book...

"Indigenous Storywork" by Jo-ann Archibald. This book is a journey that will take you on a journey. You will explore the importance of oral traditions in the Coast Salish culture and the power of stories, in teaching and in life.

The author will guide you in her discovery of main principles of storytelling, a framework to study storytelling and to practice it. I didn't tell you the whole quote from a reviewer of "It's a Numberful World" above, here it is: "For a mathematician, Eddie Woo is one helluva storyteller." *For a mathematician...* I, for one, think mathematicians are wonderful storytellers: each proof, each solution is a story. These are particular types of stories told to a specific group of people, but I think we can extend our ability to tell them well in much broader contexts. We just need practice. "Indigenous Storywork" is an excellent start.

Kseniya Garaschuk



# MATHEMATTIC

## No. 15

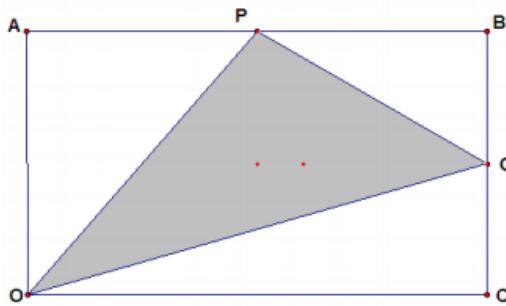
The problems featured in this section are intended for students at the secondary school level.

[Click here to submit solutions, comments and generalizations to any problem in this section.](#)

To facilitate their consideration, solutions should be received by **July 15, 2020**.



**MA71.** You are given a rectangle  $OABC$  from which you remove three right-angled triangles, leaving a fourth triangle  $OPQ$  as shaded in the diagram below.



How must you position the points  $P$  and  $Q$  so that the area of each of the three removed triangles is the same? In other words, what are the ratios  $PB : PA$  and  $QB : QC$ ?

**MA72.** Consider four numbers  $x, y, z$  and  $w$ . The first three are in arithmetic progression and the last three are in geometric progression. If  $x + w = 16$  and  $y + z = 8$ , find all possible solutions  $(x, y, z, w)$ .

**MA73.** A checkerboard is “almost tileable” if there exists some way of placing non-overlapping dominoes on the board that leaves exactly one square in each row and column uncovered. (Note that dominoes are  $2 \times 1$  tiles which may be placed in either orientation.) Prove that, for  $n \geq 3$ , an  $n \times n$  checkerboard is almost tileable if and only if  $n$  is congruent to 0 or 1 modulo 4.

**MA74.** A set of  $n$  distinct positive integers has sum 2015. If every integer in the set has the same sum of digits (in base 10), find the largest possible value of  $n$ .

**MA75.** At the Mathville Tapas restaurant, the dishes come in three types: small, medium, and large. Each dish costs an integer number of dollars, with the small dishes being the cheapest and the large dishes being the most expensive. (Tax is already included, different sizes have different prices, and the prices have stayed constant for years.) This week, Jean, Evan, and Katie order 9 small dishes, 6 medium dishes, and 8 large dishes. When the bill arrives, the following conversation occurs:

Jean: "The bill is exactly twice as much as last week."

Evan: "The bill is exactly three times as much as last month."

Katie: "If we gave the waiter a 10% tip, the total would still be less than \$100."

Find the price of the group's meal next week: 2 small dishes, 9 medium dishes, and 11 large dishes.

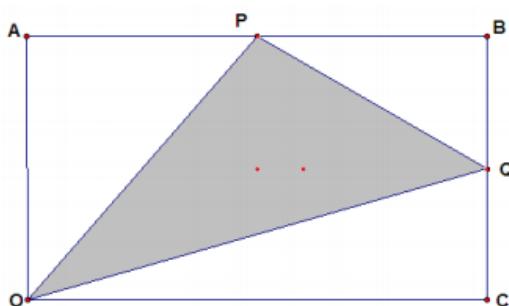
*Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.*

*Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2020**.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*

**MA71.** Trois triangles rectangles sont enlevés d'un rectangle  $OABC$ , laissant un quatrième triangle  $OPQ$ , comme indiqué ci-bas.



Déterminer comment positionner les points  $P$  et  $Q$  de façon à ce que les surfaces des trois triangles rectangles soient les mêmes. Plus précisément, déterminer les ratios  $PB : PA$  et  $QB : QC$ .

**MA72.** Soient quatre nombres  $x, y, z$  et  $w$ . Les trois premiers sont en progression arithmétique tandis que les trois derniers sont en progression géométrique. Si, de plus,  $x + w = 16$  et  $y + z = 8$ , déterminer toutes les valeurs possibles de  $(x, y, z, w)$ .

**MA73.** Un échiquier est dit “presque pavable” s'il existe une façon de presque le recouvrir de dominos non chevauchants, laissant exactement un carré vide dans chaque rangée et dans chaque colonne. (Les dominos sont des tuiles  $2 \times 1$ , placées horizontalement ou verticalement.) Démontrer que pour  $n \geq 3$  un échiquier  $n \times n$  est presque pavable si et seulement si  $n$  est congru à 0 ou 1 modulo 4.

**MA74.** Un ensemble de  $n$  entiers positifs distincts a la somme 2015. Si tout entier dans l'ensemble a la même somme de chiffres en base 10, déterminer la plus grande valeur possible de  $n$ .

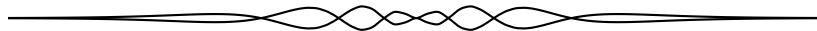
**MA75.** Un restaurant sert des plats de trois tailles: petite, médium et grande. Chaque plat coûte un nombre entier de dollars, les petits étant les moins coûteux et les grands les plus coûteux. (Les taxes sont incluses, les différentes tailles ont des prix différents, et les prix n'ont pas changé de mémoire récente.) Cette semaine Pierrette, Jeanne et Jacqueline ont commandé 9 petits plats, 6 médium et 8 de grande taille. La facture étant arrivée, voici les commentaires:

Pierrette: “La facture cette semaine est le double de celle la semaine dernière.”

Jeanne: “La facture est trois fois celle d'il y a un mois.”

Jacqueline: “Si on laissait un pourboire de 10%, le total serait toujours moins de 100 \$.”

Déterminer le coût du prochain repas: 2 petites assiettes, 9 médium et 11 de grande taille.



# MATHEMATTIC SOLUTIONS

*Statements of the problems in this section originally appear in 2019: 45(10), p. 540–541.*



**MA46.** If both  $x$  and  $y$  are integers, determine all solutions  $(x, y)$  for the equation

$$(x - 8) \cdot (x - 10) = 2^y.$$

*Adapted from question 36 on 1962 examination, MAA Problem Book II (1961–1965).*

*We received 14 submissions of which 9 were correct and complete. We present the solution by Aaratrik Basu.*

Since the right hand side of the equation is a power of 2 and, since  $a$  and  $y$  are integers, the absolute values of the multiplicands on the left hand side must be powers of 2 as well.

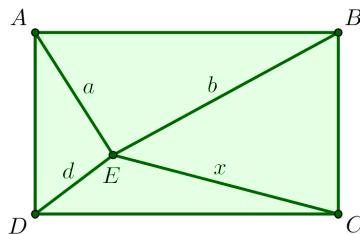
The only pair of integers that are powers of two and have a difference of two are 2 and 4.

Hence, we have,

$$x - 10 = -4 \quad \text{or} \quad 2 \quad \Rightarrow \quad x = 6 \quad \text{or} \quad 12 \quad \Rightarrow \quad y = 3.$$

Therefore, the integer solutions  $(x, y)$  to the given equation are  $(6, 3)$  and  $(12, 3)$ .

**MA47.** Let  $E$  be any point in rectangle  $ABCD$ .

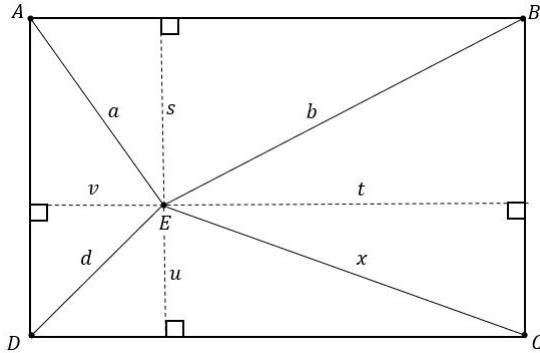


Express  $x$  in terms of  $a$ ,  $b$  and  $d$ .

*Submitted by John McLoughlin from a collection of questions with an unknown source.*

*We received 12 solutions, all of which were correct and similar in methodology. We present the solution by Ronald Martins, modified by the editor.*

Let  $s, t, u$  and  $v$  be the lengths of the perpendicular line segments from  $E$  to each side of the rectangle as shown below:



By the Pythagorean theorem, we have

$$a^2 = s^2 + v^2, \quad b^2 = t^2 + s^2, \quad x^2 = u^2 + t^2, \quad d^2 = v^2 + u^2.$$

Adding these four equations yields

$$a^2 + b^2 + d^2 + x^2 = 2(u^2 + t^2) + 2(s^2 + v^2) = 2x^2 + 2a^2 \Rightarrow x^2 = b^2 + d^2 - a^2$$

Thus  $x = \sqrt{b^2 + d^2 - a^2}$  as desired.

**MA48.** Given any triangle  $ABC$  where  $AD$  is a median of length  $m$ , prove that  $4m^2 = b^2 + c^2 + 2bc \cos A$ .

*Originally Question 1 of 1983 J.I.R. McKnight Mathematics Scholarship Paper, Scarborough Board of Education.*

*We received 11 solutions, all of which were correct. We present the solution by Tianqi Jiang, modified by the editor.*

By the Law of Cosines on  $\triangle ABC$ , we have  $a^2 = b^2 + c^2 - 2bc \cos A$ . Adding this to  $4m^2 = b^2 + c^2 + 2bc \cos A$  gives us  $a^2 + 4m^2 = 2(b^2 + c^2)$ . Thus it suffices to prove this statement instead.

By the Law of Cosines on  $\triangle ACD$  and  $\triangle ABD$ , we have

$$\begin{aligned} b^2 &= \left(\frac{a}{2}\right)^2 + m^2 - am \cos \angle ADC \\ c^2 &= \left(\frac{a}{2}\right)^2 + m^2 - am \cos \angle ADB \end{aligned} \tag{1}$$

Observe that  $\angle ADB = 180^\circ - \angle ADC$ . Thus

$$c^2 = \left(\frac{a}{2}\right)^2 + m^2 + am \cos \angle ADC \tag{2}$$

Adding our expressions for  $b^2$  and  $c^2$  in (1) and (2), respectively, yields

$$b^2 + c^2 = 2\left(\frac{a}{2}\right)^2 + 2m^2 \Rightarrow 2(b^2 + c^2) = a^2 + 4m^2.$$

This completes our proof.

**MA49.** Given that the perimeters of an equilateral triangle  $T$  and a square  $S$  are equal, determine the ratio of the area of the equilateral triangle  $T$  to the area of the square  $S$ .

*Proposed by John McLoughlin.*

*We received 10 solutions, all of which were correct. We present the solution by Viththal Ingle, modified by the editor.*

Let the side length of the equilateral triangle and square be denoted by  $a$  and  $b$ , respectively. Given the condition

$$3a = 4b \Rightarrow b = \frac{3a}{4},$$

the area of the triangle  $A_{\triangle}$  and the square  $A_{\square}$  are given by

$$A_{\triangle} = \frac{a^2\sqrt{3}}{4} \quad \text{and} \quad A_{\square} = b^2 = \frac{9a^2}{16}.$$

Thus our desired ratio is

$$\frac{A_{\triangle}}{A_{\square}} = \frac{a^2\sqrt{3}}{4} \cdot \frac{16}{9a^2} = \frac{4\sqrt{3}}{9}.$$

**MA50.** A family of straight lines is determined by the condition that the sum of the reciprocals of the  $x$  and  $y$  intercepts is a constant  $k$ . Show that all members of the family are concurrent and find the coordinates of their point of intersection.

*Submitted by John McLoughlin from a collection of questions with an unknown source.*

*We received 8 submissions, all correct. We present the solution by the Missouri State University Problem Solving Group.*

Let  $a$  and  $b$  represent the  $x$  and  $y$  intercepts respectively of a line and assume

$$\frac{1}{a} + \frac{1}{b} = k. \tag{1}$$

It follows immediately that every such line has equation

$$y = -\frac{b}{a}x + b.$$

The problem as stated is not actually correct. If  $k = 0$  we have that

$$-\frac{b}{a} = -1.$$

Since  $-\frac{b}{a}$  is the slope of every such line, we see that this family of lines consists of parallel lines with slope  $-1$ .

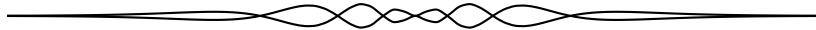
It is true when  $k \neq 0$  and all lines satisfying (1) intercept at the point

$$(1/k, 1/k).$$

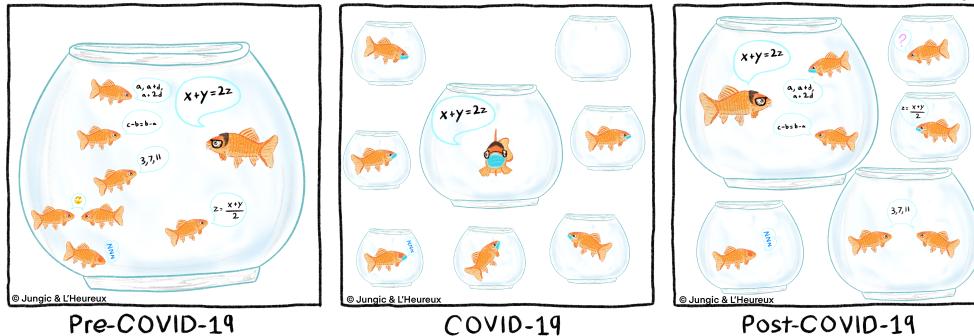
Indeed, since a line through  $(0, b)$  and  $(1/k, 1/k)$  has slope

$$m = \frac{1/k - b}{1/k} = 1 - bk = 1 - b(1/a + 1/b) = -b/a,$$

the same slope as the line satisfying (1), we conclude  $(1/k, 1/k)$  is on the line.



## COVID-19 and the Past, Present, and Future of Teaching



Some questions from the comic strip to consider:

- What was the topic of the lecture?
- How big the natural number  $n$  should be so that any 2-colouring of  $\{1, 2, \dots, n\}$  yields a monochromatic solution to  $x + y = 2z$ ?
- How big the natural number  $n$  should be so that any equinumerous 3-colouring of  $\{1, 2, \dots, 3n\}$  yields a rainbow solution to  $x + y = 2z$ ?

# TEACHING PROBLEMS

No. 10

Nat Banting

## The 100 Pets Problem

You need to purchase exactly 100 pets, and you must spend exactly \$100.00 to do so. The pets for sale (and their prices) are of three varieties: dogs (\$15.00 each), cats (\$1.00 each), and mice (\$0.25 each). How many of each pet can you purchase, if you must purchase at least one of each type?

The 100 Pets problem contains many of the characteristics that make problems of linear combinations good teaching problems. Perhaps most importantly, it is accessible, inviting mathematical activity from a variety of solvers and facilitating a variety of systematic guessing approaches as well as more formal, algebraic treatments. It is also easily extended, if needed, by changing the cost of the pets and inspecting the fallout of said adjustments – some of which involve multiple solutions or no solution at all. These benefits, which will be briefly addressed in what follows, can also be seen through the strategies outlined in the first instalment of *Teaching Problems* in Volume 45(4).

Of course, I have so far neglected to mention another key feature of the problem: its ambiguity. On first glance, it feels inaccessible to the introductory systems of equations taught at the secondary level, because there are only two equations (one for total cost and another for total pets) but three variables: dogs ( $D$ ), cats ( $C$ ), and mice ( $M$ ). However, I argue that this ambiguity is precisely what makes it a good teaching problem. Moreover, despite the fact that the 100 Pets problem initially appears inaccessible to familiar methods, the problem tends to sponsor a series of, as I call them, lurking constraints. Lurking constraints are truisms that are not directly stated in the problem, but become apparent through problem solving activity. My discussion of the 100 Pets problem will focus on these latent epiphanies, how they might sponsor solution strategies, and, in turn, how they might uncover further lurking constraints. My aim, then, is to move beyond an explanation of common student strategies by highlighting triggers that teachers might find useful when enacting the problem with learners. What follows can therefore be considered a move towards how we might *teach* a teaching problem.

The 100 Pets problem invites initial tinkering bounded only by loose rules. For instance, if students begin by purchasing one of each pet, this leaves them with \$83.75 left to spend on 97 pets. From here, students typically use tables to organize which pets they have purchased and their associated costs. Purchasing is rarely done in a random fashion, although the teacher may need to work to elicit a strategy from a learner. Some students buy in order to maximize a specific number of pets, while others increment all three pets together – purchasing one

dog, one cat, and one mouse as if they are sold as a package deal. In general, the incrementation tends to become more systematic as the activity continues. A popular emergent strategy, in my experience, is purchasing a combination that consists of exactly 100 pets and then making a series of trades – one pet traded for one other – in an attempt to equalize the dollar amount. For example, a single dog can be traded for a single cat. There is no impact on the total number of pets, but the total cost decreases by \$14.00. This strategy can also be executed in the converse: spending exactly \$100.00, then trading equal dollar amounts for different combinations of pets. For example, \$15.00 could be spent on 1 dog, or it could be swapped for 13 cats and 8 mice. That same \$15.00 could also be spent on 12 cats and 12 mice, or 11 cats and 16 mice, etc. As this trading activity picks up, a lurking constraint emerges.

*We need multiples of mice*

At some point into their work on the problem, students realize that mice must be purchased in multiples. This is a lurking constraint because nothing in the problem statement says that we must buy more than a single mouse, but no combination of non-mice can make up the \$0.25 deficit created by trading a single mouse. The dollar total (\$100.00) is a whole number and the next cheapest pet is a cat (\$1.00); therefore, mice must always be purchased and traded in multiples of four. This realization increases the efficiency of existing trading strategies, and as trading continues, the possible trades establish themselves. For the sake of brevity, let's consider the case where we first purchase 100 pets and then use trades to keep the total number of pets constant, but equalize the dollar amount to \$100.00. (Of course, a similar analysis could be performed for the converse strategy, where the cost is fixed at \$100.00 and pets are traded to achieve 100 total pets). A possible set of 100 purchased pets is given below:

	Pets purchased	Cost
5 Dogs	\$75.00	
55 Cats	\$55.00	
40 Mice	\$10.00	
Total:	100 pets	\$140.00

Once the initial 100 pets are purchased, only six possible moves can be made, each of which has a specific effect on the dollar total. The goal is then to make strategic trades to keep one total constant (quantity of pets) while the other (total cost) approaches one hundred dollars. The set of possible trades is given below.

Trade	Net Cost	Trade	Net Cost
1 Dog → 1 Cat	-\$14.00	1 Cat → 1 Dog	\$14.00
4 Cats → 4 Mice	-\$3.00	4 Mice → 4 Cats	\$3.00
4 Dogs → 4 Mice	-\$59.00	4 Mice → 4 Dogs	\$59.00

In order to set the stage for this trading strategy, an initial combination of 100 pets must be purchased. Although the price of the initial group of pets is not

important, it is difficult to ignore the impression that dogs are expensive and mice are cheap. This leads to the informal belief that any group of 100 should contain a nice balance of both cheap and expensive pets. This, naturally, leaves the cats as a nice middle-ground option. They don't tip the scale in favour of too expensive, and they don't tip the scale in favour of too many pets. In fact, they don't tip the scales at all – a realization that leads to a second lurking constraint.

*The cats don't count*

Of course, cats *do* count toward the absolute totals of both number of pets and purchase cost; however, they don't count in terms of the number-to-cost balance of any set of purchased pets. More precisely, they count equally – as one pet and one dollar. This lurking constraint holds the potential to impact the (now, quite efficient) trading strategies. Instead of trading in three denominations of pets in hopes of arriving at exactly \$100.00, the trader can now ignore cats and deal exclusively in dogs and mice. Once the total number of dogs and mice is equal to their total combined cost in dollars, the remaining amounts can be filled with cats because they increment both totals one unit at a time. In other words, if you can balance the dogs and the mice, the cats can make up the difference, as long as that balance is achieved at any number between two and ninety-nine so that at least one cat can be purchased.

This constraint also tends to sponsor a more formal, algebraic approach now that a perceived barrier has been removed. If the cats are strategically ignored, it resembles a problem that can be approached with a system of equations. We now have two variables – dogs ( $D$ ) and mice ( $M$ ) – and two equations (total cost and total pets). Of course, subtracting two equations involving cats would have resulted in the same strategy from the onset (and may have actually revealed this lurking constraint), but, in my experience, the lurking constraint makes this option all the more inviting as the problem suddenly becomes more familiar:

$$D + M = T, \quad 2 \leq T \leq 99 \text{ (number of pets)}, \quad (1)$$

$$15D + 0.25M = T, \quad 2 \leq T \leq 99 \text{ (cost of pets)}. \quad (2)$$

Equating (1) and (2), we get

$$D + M = 15D + 0.25M.$$

Multiplying by 4 and isolating for  $M$ , we get

$$\begin{aligned} 4D + 4M &= 60D + M \\ 3M &= 56D \\ M &= \frac{56}{3}D \end{aligned} \quad (3)$$

As long as the number of dogs and mice satisfy this condition, their total quantity will balance with their total cost, leaving cats to fill in the difference to one hundred. It's not the prettiest relationship, requiring a fractional number of dogs for every mouse purchased. However, it forces us to narrow the number of dogs we

decide to purchase due to a lurking constraint that has been implied throughout our activity, but never explicitly stated.

*No partial pieces of pets*

To this point, all activity has adhered to the unspoken assumption that pets must always be purchased in whole numbers; however, this requirement was only ever implied by the context of the problem. Trading fractional pieces of pets feels absurd, yet it is technically acceptable under the explicit conditions of the problem. It was always assumed that we could only deal with whole pets, but this assumption is now further refined by the denominator in (3). In order to avoid fractional pets,  $D$  must be a multiple of three. This leaves only one possibility: purchasing three dogs, because six dogs (and every multiple above six) results in too many mice. We can now see that this final lurking constraint, although implicitly upheld in the trading methods, becomes an important domain restriction in the algebraic method, moving us efficiently toward a unique solution of 3 dogs, 41 cats, and 56 mice.

Readers will deepen their understanding of the problem through solving this extension problem:

In the original problem, dogs must be bought in multiples of three – which led to a quick, unique solution. If the prices of cats and mice remain fixed, can you find a new price for dogs that allows you to buy a non-multiple of three (e.g., 2, 4, or 11, etc.) and still results in 100 pets for exactly \$100? Can you describe all such possible prices?

The 100 Pets problem is not exceedingly complex; rather, its elegance is in what it does not prescribe. The three lurking constraints detailed above are certainly not the only possible milestones of student activity; the constraints discussed here simply represent the dominant themes from my work with multiple groups of students. Ultimately, the analysis above is meant to emphasize that a problem may encourage certain solution pathways, and anticipating these common markers of student thinking better equips the teacher to act with learners. Focusing on the constraints around which action may organize will assist someone aiming to teach a teaching problem, and the 100 Pets problem provides evidence that it just might be the things that go unsaid that end up influencing the mathematical strategies of solvers.

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*Nat Banting ([nat.banting@usask.ca](mailto:nat.banting@usask.ca)) is a high school mathematics teacher currently on faculty in the Department of Curriculum Studies at the University of Saskatchewan. He shares his teaching practice across the country through various writing projects, speaking opportunities, and social media platforms – blogging at [natbanting.com/blog](http://natbanting.com/blog) and tweeting as @NatBanting.*



# OLYMPIAD CORNER

No. 383

*The problems featured in this section have appeared in a regional or national mathematical Olympiad.*

*Click here to submit solutions, comments and generalizations to any problem in this section*

*To facilitate their consideration, solutions should be received by July 15, 2020.*



**OC481.** In the plane, there are circles  $k$  and  $l$  intersecting at points  $E$  and  $F$ . The tangent to the circle  $l$  drawn from  $E$  intersects the circle  $k$  at point  $H$  ( $H \neq E$ ). On the arc  $EH$  of the circle  $k$ , which does not contain the point  $F$ , choose a point  $C$  ( $E \neq C \neq H$ ) and let  $D$  be the intersection of the line  $CE$  with the circle  $l$  ( $D \neq E$ ). Prove that triangles  $DEF$  and  $CHF$  are similar.

**OC482.** Let  $a_1, a_2, \dots, a_{2017}$  be real numbers such that

$$a_1 + a_2 + \cdots + a_{2017} = 2017.$$

Find the largest number of pairs  $(i, j)$  for which  $1 \leq i < j \leq 2017$  and  $a_i + a_j < 2$ .

**OC483.** Prove that for each prime number  $p > 2$ , there is exactly one positive integer  $n$  such that the number  $n^2 + np$  is a perfect square.

**OC484.** Let  $x$  be a real number with  $0 < x < 1$  and let  $0.c_1c_2c_3\dots$  be the decimal expansion of  $x$ . Denote by  $B(x)$  the set of all subsequences of  $c_1, c_2, c_3 \dots$  that consist of six consecutive digits. For instance,  $B(1/22) = \{045454, 454545, 545454\}$ .

Find the minimum number of elements of  $B(x)$  as  $x$  varies among all irrational numbers with  $0 < x < 1$ .

**OC485.** Prove that a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is increasing if and only if

$$(c-b) \int_a^b f(x) \, dx \leq (b-a) \int_b^c f(x) \, dx,$$

for all real numbers  $a < b < c$ .



*Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.*

*Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2020**.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*



**OC481.** Dans le plan, les cercles  $k$  et  $l$  intersectent en  $E$  et  $F$ . La tangente au cercle  $l$  émanant du point  $E$  intersecte le cercle  $k$  en  $H$  ( $H \neq E$ ). Sur l'arc  $EH$  du cercle  $k$ , ne contenant pas  $F$ , choisir un point  $C$  ( $E \neq C \neq H$ ); soit  $D$  le point d'intersection de la ligne  $CE$  avec le cercle  $l$  ( $D \neq E$ ). Démontrer que les triangles  $DEF$  et  $CHF$  sont similaires.

**OC482.** Soient  $a_1, a_2, \dots, a_{2017}$  des nombres réels tels que

$$a_1 + a_2 + \cdots + a_{2017} = 2017.$$

Déterminer le plus grand nombre possible de paires  $(i, j)$  telles que  $a_i + a_j < 2$  pour  $1 \leq i < j \leq 2017$ .

**OC483.** Démontrer que pour tout nombre premier  $p > 2$ , il existe exactement un entier positif  $n$  tel que  $n^2 + np$  est un carré d'entier.

**OC484.** Soit  $x$  un nombre réel tel que  $0 < x < 1$  et soit  $0.c_1c_2c_3\dots$  la représentation décimale de  $x$ . Dénoter par  $B(x)$  l'ensemble de toutes les sous suites de  $c_1, c_2, c_3\dots$  consistant de six chiffres consécutifs. Par exemple,  $B(1/22) = \{045454, 454545, 545454\}$ .

Déterminer le nombre minimum d'éléments de  $B(x)$  pour  $x$  irrationnel,  $0 < x < 1$ .

**OC485.** Démontrer qu'une fonction continue  $f : \mathbb{R} \rightarrow \mathbb{R}$  est croissante si et seulement si

$$(c-b) \int_a^b f(x) \, dx \leq (b-a) \int_b^c f(x) \, dx,$$

pour tous nombres réels  $a < b < c$ .



# OLYMPIAD CORNER SOLUTIONS

*Statements of the problems in this section originally appear in 2019: 45(10), p. 554–555.*



**OC456.** Solve the system of equations

$$\begin{aligned}(x^2 + 1)(x - 1)^2 &= 2017yz \\ (y^2 + 1)(y - 1)^2 &= 2017zx \\ (z^2 + 1)(z - 1)^2 &= 2017xy,\end{aligned}$$

where  $x \geq 1, y \geq 1, z \geq 1$ .

*Originally Problem 1, Grade 9 of the 2017 Bulgaria Math Olympiad.*

*We received 11 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.*

Multiplying the first equation by  $x$ , the second by  $y$ , and the third by  $z$ , we get

$$x(x^2 + 1)(x - 1)^2 = y(y^2 + 1)(y - 1)^2 = z(z^2 + 1)(z - 1)^2 = 2017xyz.$$

The function

$$f(x) = x(x^2 + 1)(x - 1)^2$$

is strictly increasing on  $[1, +\infty)$ , as it is a product of strictly increasing and non-negative functions.

Therefore we get  $x = y = z$  which gives

$$x(x^2 + 1)(x - 1)^2 = 2017x^3,$$

or equivalently, as  $x \neq 0$ ,

$$(x + \frac{1}{x})(x - 2 + \frac{1}{x}) = 2017.$$

Letting  $t = x + \frac{1}{x}$  we get  $t^2 - 2t - 2017 = 0$ . Since  $t \geq 2$ , then  $t = 1 + \sqrt{2018}$ . We now obtain

$$x = y = z = \frac{1}{2} \left( 1 + \sqrt{2018} + \sqrt{2015 + 2\sqrt{2028}} \right).$$

*Editor's Comment.* The Problem Solving Group of Missouri State University showed that if 2017 is replaced by an arbitrary number  $A \geq 0$ , then the system has the solution

$$x = y = z = \frac{1}{2} \left( 1 + \sqrt{1 + A} + \sqrt{(1 + \sqrt{1 + A})^2 - 4} \right).$$

**OC457.** On a blackboard are written the numbers  $1!, 2!, 3!, \dots, 2017!$ . What is the smallest among these numbers that should be deleted so that the product of all the remaining numbers is a perfect square?

*Originally Problem 3, Grade 9 of the 2017 Bulgaria Math Olympiad.*

*We received 7 submissions. We present a solution based on the submissions by Oliver Geupel and Ioannis D. Sfikas.*

First, we show that  $1! \times 2! \times \cdots \times 2017!$  and the product of any 2016 factorials selected out of the 2017 numbers cannot be perfect squares.

Let  $\nu_p(n)$  denote the exponent of prime  $p$  in the prime factorization of the natural number  $n$ . Note that the numbers 997 and 2017 are prime.

From

$$\nu_{2017}(1!) = \nu_{2017}(2!) = \cdots = \nu_{2017}(2016!) = 0 \quad \text{and} \quad \nu_{2017}(2017!) = 1,$$

it follows that

$$\nu_{2017}(1! \times 2! \times \cdots \times 2017!) = 1 \quad \text{and} \quad \nu_{2017}(n) = 1$$

for every product  $n$  of any 2016 factorials out of 2017 given numbers that includes the factor  $2017!$ . Therefore,  $1! \times 2! \times \cdots \times 2017!$  contains only one copy of the prime number 2017, and is not a perfect square. Similarly, the product of any 2016 factorials selected out of the 2017 numbers that includes the factor  $2017!$  cannot be a perfect square.

Next, let  $n = 1! \times 2! \times 3! \times \cdots \times 2016!$ . Since

$$\begin{aligned} \nu_{997}(1!) &= \nu_{997}(2!) = \cdots = \nu_{997}(996!) = 0, \\ \nu_{997}(997!) &= \nu_{997}(998!) = \cdots = \nu_{997}(1993!) = 1, \\ \nu_{997}(1994!) &= \nu_{997}(1995!) = \cdots = \nu_{997}(2016!) = 2, \end{aligned}$$

it follows that

$$\nu_{997}(n) = 997 + 2 \times 23 = 1043$$

is odd, and  $n$  is not a perfect square. In conclusion, at least two factorials have to be deleted and one of the removed factorials must be  $2017!$ .

Second, we show that exactly two factorials can be deleted.

$$\begin{aligned} 1! \times 2! \times 3! \times \cdots \times 2017! &= (1! \times 2!) \times (3! \times 4!) \times \cdots \times (2015! \times 2016!) \times 2017! \\ &= (1!^2 \times 2) \times (3!^2 \times 4) \times \cdots \times (2015!^2 \times 2016) \times 2017! \\ &= (1! \times 3! \times \cdots \times 2015!)^2 \times (2 \times 4 \times \cdots \times 2016) \times 2017! \\ &= (1! \times 3! \times \cdots \times 2015!)^2 \times 2^{1008} \times 1008! \times 2017! \end{aligned}$$

If we delete  $1008!$  and  $2017!$ , then the product of remaining numbers is a perfect square.

**OC458.** Let  $A$  be the product of eight consecutive positive integers and let  $k$  be the largest positive integer for which  $k^4 \leq A$ . Find the number  $k$  knowing that it is represented in the form  $2p^m$ , where  $p$  is a prime number and  $m$  is a positive integer.

*Originally Problem 4, Grade 9 of the 2017 Bulgaria Math Olympiad.*

We received 6 submissions of which 4 were complete. We present the solution based on the submissions of UCLan Cyprus Problem Solving Group and Problem Solving Group of Missouri State University (done independently).

First, we claim that if  $A = (n+1)(n+2) \cdots (n+8)$  for some integer number  $n \geq 0$ , then  $k = (n+2)(n+7)$ . To establish this we use the following inequalities

$$(x + \sqrt{yz})^2 \leq (x+y)(x+z) \leq \left(x + \frac{y+z}{2}\right)^2$$

that are valid for  $x, y, z$  positive real numbers. Both inequalities are equalities if and only if  $y = z$ . The inequalities follow easily from AM-GM inequality.

We write

$$A = (n^2 + 9n + 8)(n^2 + 9n + 14)(n^2 + 9n + 18)(n^2 + 9n + 20).$$

Then

$$(n^2 + 9n + \sqrt{8 \times 20})^2(n^2 + 9n + \sqrt{14 \times 18})^2 < A$$

and

$$A < \left(n^2 + 9n + \frac{8+20}{2}\right)^2 \left(n^2 + 9n + \frac{14+18}{2}\right)^2,$$

so

$$(n^2 + 9n + \sqrt[4]{8 \times 14 \times 18 \times 20})^2 < A < \left(n^2 + 9n + \frac{8+14+18+20}{2}\right)^2.$$

In conclusion,

$$(n^2 + 9n + 14)^2 < A < (n^2 + 9n + 15)^2,$$

so we must have

$$k = n^2 + 9n + 14 = (n+2)(n+7).$$

If  $(n+2)(n+7) = 2p^m$  with  $p$  prime and  $m > 0$ , then the factors  $n+2$  and  $n+7$  can be  $2p^a$  or  $p^b$ . If  $a$  and  $b$  are both positive, then  $p$  divides the factor difference:  $(n+7) - (n+2) = 5$ . So we must have  $p = 5$ ,  $n+7 = 2 \times 5$ , and  $n+2 = 5$ , leading to  $k = 50$ . If  $a = 0$ , then  $n+2 = 2$ , and  $n+7 = 7$ . So we must have  $k = 2 \times 7 = 14$ , a number of the required form. If  $b = 0$ , then  $n+2 = 1$  giving  $n = -1$ , leading to no valid solution as  $n > 0$ .

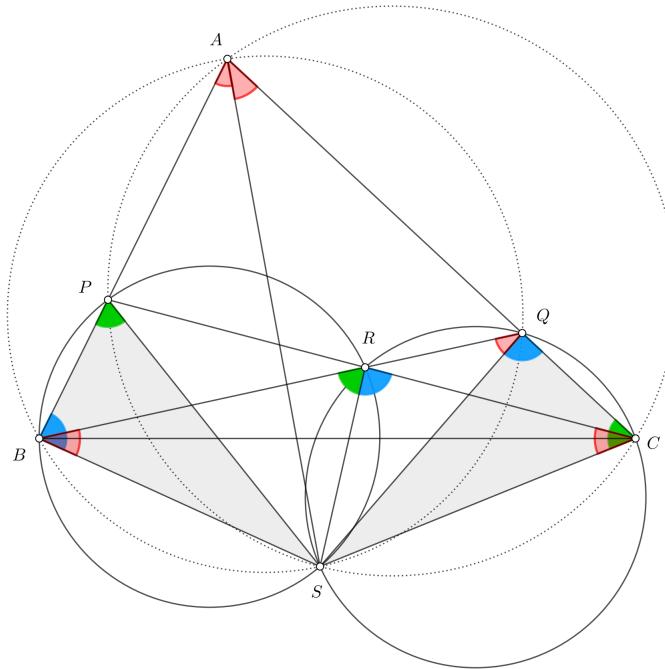
Thus the only possible values for  $k$  are  $k = 14 = 2 \times 7$  and  $k = 50 = 2 \times 5^2$ .

**OC459.** Points  $P$  and  $Q$  lie respectively on sides  $AB$  and  $AC$  of a triangle  $ABC$  such that  $BP = CQ$ . Segments  $BQ$  and  $CP$  intersect at  $R$ . The circumcircles of triangles  $BPR$  and  $CQR$  intersect again at point  $S$  different from  $R$ . Prove that point  $S$  lies on the angle bisector  $\angle BAC$ .

*Originally Problem 1, Final Round of the 2017 Poland Math Olympiad.*

*We received 13 submissions. We present two solutions.*

*Solution 1, by UCLan Cyprus Problem Solving Group.*



First, note that  $\triangle BSP$  and  $\triangle QSC$  are congruent (angle-side-angle). Indeed,  $\angle BPS = \angle BRS = \angle QCS$ ,  $PB = CQ$ , and  $\angle PBS = \angle SRC = \angle CQS$ . The angle congruency follows from concyclicity of two sets of points  $\{B, P, R, S\}$  and  $\{Q, C, S, R\}$ .

In particular,  $\triangle BSQ$  is isosceles with  $SB = SQ$  and  $\angle QBS = \angle BQS$ .

Furthermore, since  $\angle BPS = \angle QCS = \angle ACS$ , then  $A, C, S, P$  are concyclic. Hence,  $\angle BAS = \angle PAS = \angle PCS = \angle RCS = \angle RQS = \angle BQS$ . Similarly,  $A, Q, S, B$  are concyclic, and  $\angle CAS = \angle QBS$ . Thus  $\angle BAS = \angle CAS$  and  $AS$  is the angle bisector of  $\angle BAC$ .

*Solution 2, by Andrea Fanchini.*

We use barycentric coordinates with reference to  $\triangle ABC$ . Let  $t = BP = CQ$ . Then  $P$  and  $Q$  have barycentric coordinates  $P(t : c - t : 0)$  and  $Q(t : 0 : b - t)$ .

The lines  $BQ$  and  $CP$  are

$$BQ : (b-t)x - tz = 0, \quad CP : (c-t)x - ty = 0$$

and intersect at

$$R = BQ \cap CP = (t : c-t : b-t).$$

The circumcircle of  $\triangle BPR$  is specified by the equation

$$a^2yz + b^2zx + c^2xy - (x+y+z) \left[ c(c-t)x + \frac{ct^2 - 2S_B t + a^2c}{b+c-t} z \right] = 0.$$

The circumcircle of  $\triangle CQR$  is specified by the equation

$$a^2yz + b^2zx + c^2xy - (x+y+z) \left[ b(b-t)x + \frac{bt^2 - 2S_C t + a^2b}{b+c-t} y \right] = 0.$$

Circumcircles of  $\triangle BPR$  and  $\triangle CQR$  intersect at point

$$S(-(b+c)t^2 + ((b+c)^2 + a^2)t - a^2(b+c) : b(b+c-t)^2 : c(b+c-t)^2).$$

$S$  lies on the angle bisector  $\angle BAC$  having equation  $cy - bz = 0$ .

*Editor's Comments.* All but two solutions adopted a geometric approach. The solvers identified several properties of the geometric configuration. For example: the circumcircles of  $\{B, P, R, S\}$  and  $\{Q, C, S, R\}$  are equal (Corneliu Manescu-Avram and Anderson Torres),  $\triangle APS$  and  $\triangle RQS$  are similar,  $\triangle AQS$  and  $\triangle PRS$  are similar (Ivko Dimitrić). From the congruency of  $\triangle BSP$  and  $\triangle QSC$ , it follows that their altitudes to sides  $PB$  and  $CQ$  are equal; hence  $S$  is equidistant from  $AB$  and  $AC$  and lies on the angle bisector of  $\angle BAC$  (Oliver Geupel and Prithwijit De).

Ioannis Sfikas mentioned the source of OC 459 and several solutions are listed at <https://artofproblemsolving.com/community/c6h1422437p7998793>

**OC460.** Prove that the set of positive integers  $\mathbb{Z}^+$  can be represented as a union of five pairwise disjoint subsets with the following property: each 5-tuple of numbers of the form  $(n, 2n, 3n, 4n, 5n)$ , where  $n \in \mathbb{Z}^+$ , contains exactly one number from each of these five subsets.

*Originally Problem 4, Final Round of the 2017 Poland Math Olympiad.*

*We received 4 correct submissions. We present the solution by C.R. Pranesachar.*

We can express every positive integer  $n$  as

$$n = 2^x 3^y 5^z p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

where  $x, y, z, r_1, r_2, \dots, r_k$  are non-negative integers and  $p_1, p_2, \dots, p_k$  are distinct primes strictly greater than 5. We define  $f(n)$

$$f(n) = x + 3y + 4z.$$

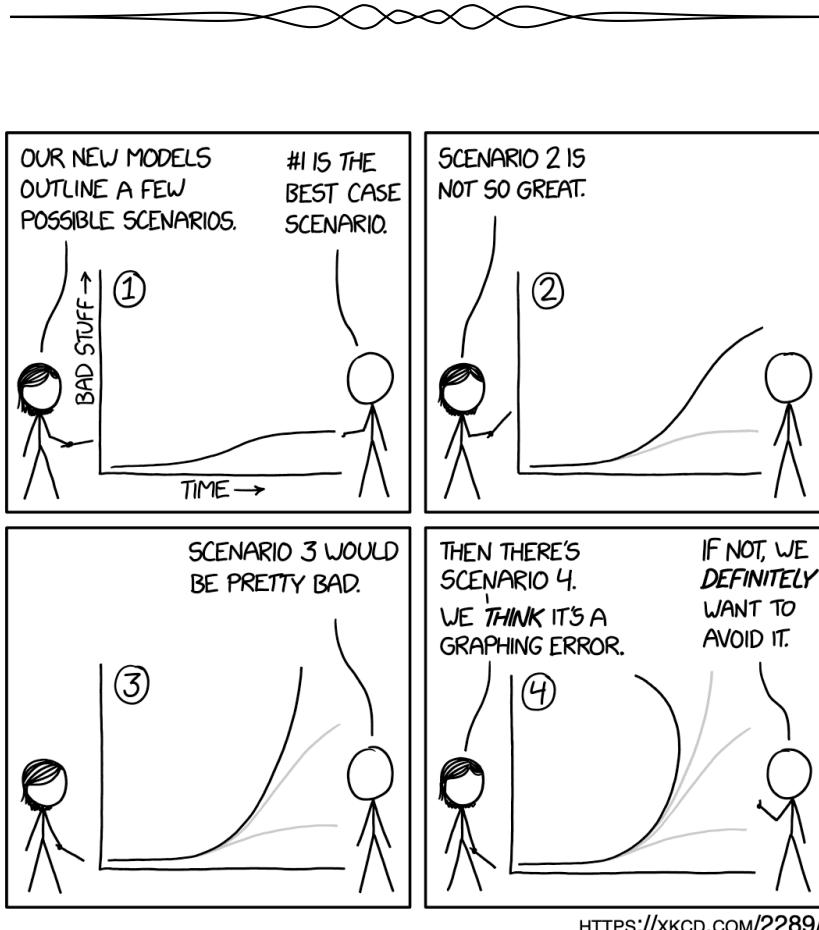
Note that

$$\begin{aligned} f(2n) &= x + 3y + 4z + 1, & f(3n) &= x + 3y + 4z + 3, \\ f(4n) &= x + 3y + 4z + 2, & f(5n) &= x + 3y + 4z + 4. \end{aligned}$$

We partition the set  $\mathbb{Z}^+$  of positive integers into 5 pairwise disjoint sets  $A_0, A_1, A_2, A_3, A_4$  according to the rule:

$$n \in A_j \quad \text{if and only if} \quad f(n) \equiv j \pmod{5}.$$

Since  $f(n), f(2n), f(3n), f(4n)$ , and  $f(5n)$  are mutually incongruent modulo 5, we have that each of  $n, 2n, 3n, 4n$ , and  $5n$  belong to one of these 5 subsets, and no two of them belong to the same subset.



# FOCUS ON...

No. 41

Michel Bataille

Inequalities *via* auxiliary functions (II)

## Introduction

In this second part, we present a selection of properties of convex functions linked to inequalities and examine some applications to problems. We recall that a function  $f : I \rightarrow \mathbb{R}$  is convex on the interval  $I$  if  $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$  whenever  $x, y$  are in  $I$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$ . The function  $f$  is concave if  $-f$  is convex. In most of the applications, the function  $f$  is twice differentiable on  $I$  and in that case,  $f$  is convex if and only if  $f''(x) \geq 0$  for all  $x \in I$ .

## Jensen's inequality

Jensen's inequality generalizes the inequality of the definition above and is easily proved by induction: Let  $f : I \rightarrow \mathbb{R}$  be convex and let  $n$  be an integer with  $n \geq 2$ . If  $\alpha_1, \dots, \alpha_n \geq 0$  with  $\alpha_1 + \dots + \alpha_n = 1$  and  $x_1, \dots, x_n$  are in the interval  $I$ , then

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \dots + \alpha_n f(x_n).$$

This inequality turns out to be very useful. Here are some examples. We start with problem **3887** [2013 : 414,416 ; 2014 : 401].

Let  $a, b$  and  $c$  be positive real numbers. Prove that

$$\frac{a^2}{bc(a^2 + ab + b^2)} + \frac{b^2}{ac(b^2 + bc + c^2)} + \frac{c^2}{ab(a^2 + ac + c^2)} \geq \frac{9}{(a+b+c)^2}.$$

Three solutions were featured, all based on algebraic manipulations and AM-GM. Here is a very simple solution involving a convex function.

Let  $f(x) = \frac{1}{1+x+x^2}$ . It is readily checked that the proposed inequality rewrites as

$$\frac{a}{a+b+c} f\left(\frac{b}{a}\right) + \frac{b}{a+b+c} f\left(\frac{c}{b}\right) + \frac{c}{a+b+c} f\left(\frac{a}{c}\right) \geq \frac{9abc}{(a+b+c)^3}. \quad (1)$$

Since  $f''(x) = 6x(x+1)(1+x+x^2)^{-3} \geq 0$  for  $x > 0$ , the function  $f$  is convex on  $(0, \infty)$  and Jensen's inequality shows that the left-hand side of (1) is greater than or equal to

$$f\left(\frac{a}{a+b+c} \cdot \frac{b}{a} + \frac{b}{a+b+c} \cdot \frac{c}{b} + \frac{c}{a+b+c} \cdot \frac{a}{c}\right) = f(1) = \frac{1}{3}.$$

As a result, it is sufficient to prove that

$$\frac{1}{3} \geq \frac{9abc}{(a+b+c)^3}.$$

We are done since this inequality follows at once from AM-GM.

Our second example is problem **3641** [2011 : 234,237 ; 2012 : 200], of which we offer a variant of solution.

Let  $0 \leq x_1, x_2, \dots, x_n < \pi/2$  be real numbers. Prove that

$$\left( \frac{1}{n} \sum_{k=1}^n \sec(x_k) \right) \left( 1 - \left( \frac{1}{n} \sum_{k=1}^n \sin(x_k) \right)^2 \right)^{1/2} \geq 1.$$

For  $k = 1, 2, \dots, n$ , let  $a_k = \sin(x_k)$ . Then,  $a_k \in [0, 1]$  and the inequality rewrites as

$$\frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1-a_k^2}} \geq \frac{1}{\sqrt{1 - \left( \frac{1}{n} \sum_{k=1}^n a_k \right)^2}},$$

that is,

$$\frac{1}{n} \sum_{k=1}^n f(a_k) \geq f \left( \frac{1}{n} \sum_{k=1}^n a_k \right)$$

where  $f : [0, 1] \rightarrow \mathbb{R}$  is defined by  $f(x) = \frac{1}{\sqrt{1-x^2}}$ . Clearly, the inequality holds if  $f$  is convex and this is indeed the case since

$$f''(x) = (1-x^2)^{-5/2}(1+2x^2) \geq 0$$

when  $x \in [0, 1]$ .

To conclude the section, we choose an example, extracted from problem **2730** [2002 : 177 ; 2003 : 186], where the function to be used is not obvious.

Let  $\text{GM}(x_1, x_2, \dots, x_n)$  denote the geometric mean of the real numbers  $x_1, x_2, \dots, x_n$ . Given positive real numbers  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ , prove that

$$\text{GM}(a_1+b_1, a_2+b_2, \dots, a_n+b_n) \geq \text{GM}(a_1, a_2, \dots, a_n) + \text{GM}(b_1, b_2, \dots, b_n).$$

We introduce the function  $f(x) = \ln(1+e^x)$ . Then  $f''(x) = \frac{e^x}{(e^x+1)^2} \geq 0$  so that  $f$  is a convex function on  $\mathbb{R}$ . It follows that, if  $x_1, x_2, \dots, x_n$  are any positive real numbers,

$$f \left( \frac{1}{n} (\ln x_1 + \dots + \ln x_n) \right) \leq \frac{1}{n} (f(\ln x_1) + \dots + f(\ln x_n)).$$

This easily writes as  $\ln(1 + \sqrt[n]{x_1 \dots x_n}) \leq \ln(\sqrt[n]{(1+x_1) \dots (1+x_n)})$  that is,

$$\text{GM}(1+x_1, \dots, 1+x_n) \geq 1 + \text{GM}(x_1, \dots, x_n).$$

Now, we have

$$\text{GM}(a_1+b_1, a_2+b_2, \dots, a_n+b_n) = \text{GM}(a_1, a_2, \dots, a_n) \cdot \text{GM}\left(1 + \frac{b_1}{a_1}, \dots, 1 + \frac{b_n}{a_n}\right)$$

and so

$$\text{GM}(a_1+b_1, a_2+b_2, \dots, a_n+b_n) \geq \text{GM}(a_1, a_2, \dots, a_n) \left(1 + \text{GM}\left(\frac{b_1}{a_1}, \dots, \frac{b_n}{a_n}\right)\right).$$

The result immediately follows.

### Popoviciu's inequality

More advanced than Jensen's is Popoviciu's inequality (see references [1] and [2]):

If  $m, n$  are integers with  $2 \leq m \leq n - 1$  and  $f$  is continuous and convex on the interval  $I$ , then

$$\begin{aligned} & m \sum_{(m)} f\left(\frac{a_{k_1} + a_{k_2} + \dots + a_{k_m}}{m}\right) \\ & \leq \binom{n-2}{m-2} \left[ \frac{n-m}{m-1} \sum_{k=1}^n f(a_k) + nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \right] \end{aligned}$$

whenever  $a_1, a_2, \dots, a_n$  are in  $I$ . [ $\sum_{(m)}$  indicates that the summation is over all  $m$ -combinations  $k_1, k_2, \dots, k_m$  of  $1, 2, \dots, n$ ].

As one can guess, Popoviciu's inequality is not used very often. To give a simple example, I repeat my solution to problem 1724 posed in the June 2005 issue of *Mathematics Magazine*.

Let  $x_1, x_2, \dots, x_n$  be positive real numbers. Prove that

$$\frac{1}{n} \sum_{k=1}^n x_k - \left( \prod_{k=1}^n x_k \right)^{1/n} \leq \frac{1}{n} \sum_{1 \leq j < k \leq n} (\sqrt{x_j} - \sqrt{x_k})^2.$$

Since

$$\sum_{1 \leq j < k \leq n} (\sqrt{x_j} - \sqrt{x_k})^2 = (n-1) \sum_{k=1}^n x_k - 2 \sum_{1 \leq j < k \leq n} \sqrt{x_j} \sqrt{x_k},$$

the inequality rewrites as

$$2 \sum_{1 \leq j < k \leq n} \sqrt{x_j} \sqrt{x_k} \leq (n-2) \sum_{k=1}^n x_k + n \left( \prod_{k=1}^n x_k \right)^{1/n}.$$

This inequality holds because, setting  $a_k = \ln x_k$  for  $k = 1, 2, \dots, n$  and  $f(x) = e^x$ , it becomes

$$2 \sum_{1 \leq j < k \leq n} f\left(\frac{a_j + a_k}{2}\right) \leq (n-2) \sum_{k=1}^n f(a_k) + nf\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right),$$

which is nothing else than Popoviciu's inequality with the convex function  $f$  and  $m = 2$ .

### Hadamard's inequality

If  $a < b$  and  $f : [a, b] \rightarrow \mathbb{R}$  is convex, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt.$$

As a geometric consequence of its convexity, the graph of the function  $f$  is above the tangent at any of its points. With the point  $(\frac{a+b}{2}, f(\frac{a+b}{2}))$  of the graph, this means that

$$f(t) \geq f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right)$$

for all  $t \in [a, b]$ . The inequality follows by integrating from  $a$  to  $b$ .

Of course, Hadamard's inequality is expected to intervene when integrals are involved. It is the case in our first example, slightly adapted from problem **898** proposed in March 2009 in *The College Mathematics Journal*:

Let  $f : I \rightarrow \mathbb{R}$  be a convex function defined on an open interval  $I$  and let  $a, b \in I$  with  $a < b$ . Prove that

$$\int_0^1 f(a + (b-a)y) dy \geq \int_0^1 f\left(\frac{3a+b}{4} + \frac{b-a}{2}y\right) dy.$$

Transforming the integrals by means of the substitution  $y = \frac{x-a}{b-a}$ , we see that the inequality is equivalent to

$$\int_a^b f(x) dx \geq \int_a^b f\left(\frac{x}{2} + \frac{a+b}{4}\right) dx.$$

Now, from Jensen's inequality, we have

$$f\left(\frac{1}{2} \cdot x + \frac{1}{2} \cdot \frac{a+b}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f\left(\frac{a+b}{2}\right)$$

from which we deduce that

$$\int_a^b f\left(\frac{x}{2} + \frac{a+b}{4}\right) dx \leq \frac{1}{2} \int_a^b f(x) dx + \frac{b-a}{2} f\left(\frac{a+b}{2}\right).$$

The desired inequality follows since

$$\frac{b-a}{2} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \int_a^b f(x) dx$$

by Hadamard's inequality.

No integral is *a priori* expected in our second example, problem **11127** posed in the January 2005 issue of *The American Mathematical Monthly*:

Prove that when  $0 < x \leq y < \pi/2$

$$\left(\frac{\cos x \sin y}{\sin x \cos y}\right)^{\sin x \sin y} \leq \exp\left(\sqrt{\cos(x-y)}\left(\sqrt{\frac{\cos x}{\cos y}} - \sqrt{\frac{\cos y}{\cos x}}\right)\right).$$

Taking logarithms and using elementary trigonometry, the proposed inequality transforms into

$$\tan x \tan y (\ln(\tan y) - \ln(\tan x)) \leq \left(\sqrt{1 + \tan^2 y} - \sqrt{1 + \tan^2 x}\right) \sqrt{1 + \tan x \tan y}.$$

This obviously holds if  $x = y$  and for  $x < y$ , letting  $a = \ln(\tan x)$ ,  $b = \ln(\tan y)$ , rewrites as

$$\frac{e^{a+b}}{\sqrt{1+e^{a+b}}} \leq \frac{\sqrt{1+e^{2b}} - \sqrt{1+e^{2a}}}{b-a}. \quad (2)$$

Now, let  $f(t) = \frac{e^{2t}}{\sqrt{1+e^{2t}}}$ . Since

$$f''(t) = (1+e^{2t})^{-5/2}(e^{6t} + 2e^{4t} + 4e^{2t}) > 0,$$

$f$  is convex on  $\mathbb{R}$  and it is readily checked that (2) is just Hadamard's inequality applied to  $f$ .

### Petrovic's inequality

Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be a convex function such that  $f(0) = 0$ . Then, for any  $x_1, x_2, \dots, x_n \geq 0$ , we have

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq f(x_1 + x_2 + \cdots + x_n).$$

The proof is easy once it is remarked that if  $x \geq 0$  and  $\alpha \in [0, 1]$ , then

$$f(\alpha x) = f(\alpha x + (1-\alpha)0) \leq \alpha f(x) + (1-\alpha)f(0) = \alpha f(x).$$

It follows that if  $x_1, x_2 \geq 0$  and  $x_1 + x_2 > 0$ , then

$$\begin{aligned} f(x_1) + f(x_2) &= f\left(\frac{x_1}{x_1+x_2} \cdot (x_1+x_2)\right) + f\left(\frac{x_2}{x_1+x_2} \cdot (x_1+x_2)\right) \\ &\leq \frac{x_1}{x_1+x_2} \cdot f(x_1+x_2) + \frac{x_2}{x_1+x_2} \cdot f(x_1+x_2) \end{aligned}$$

hence  $f(x_1) + f(x_2) \leq f(x_1 + x_2)$  (and this also holds if  $x_1 + x_2 = 0$  since then  $x_1 = x_2 = 0$ ). The proof is completed by induction.

This general result easily leads to Weierstrass's inequalities:

With  $f(x) = -\ln(1+x)$ , Petrovic's inequality gives

$$\ln(1+x_1) + \ln(1+x_2) + \cdots + \ln(1+x_n) \geq \ln(1+(x_1+x_2+\cdots+x_n))$$

and so

$$\prod_{i=1}^n (1+x_i) \geq 1 + \sum_{i=1}^n x_i$$

for  $x_1, x_2, \dots, x_n \geq 0$ .

The second inequality of Weierstrass's, namely

$$\prod_{i=1}^n (1-a_i) \geq 1 - \sum_{i=1}^n a_i$$

for  $a_1, a_2, \dots, a_n \in [0, 1]$  is similarly obtained with the function  $f(x) = e^{-x} - 1$  and taking  $x_i = -\ln(1-a_i)$ .

For another application, we again consider

$$\Delta(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i - \prod_{i=1}^n x_i$$

already met in part I. If  $a_1, a_2, \dots, a_n \geq 1$ , then Petrovic's inequality applied to the function  $x \mapsto e^x - 1$  and  $x_i = \ln a_i$  leads to

$$\left( \prod_{i=1}^n a_i \right) - 1 \geq \left( \sum_{i=1}^n a_i \right) - n$$

and therefore  $\Delta(a_1, a_2, \dots, a_n) \leq n - 1$ .

In contrast, if

$$1 \leq a_1 \leq \frac{1}{a_2} \leq a_3 \leq \frac{1}{a_4} \leq \cdots \leq a_n^{(-1)^{n+1}},$$

then  $\Delta(a_1, a_2, \dots, a_n) \geq n - 1$ . This was problem **1048** in the March 2015 issue of *The College Mathematics Journal* and we propose a solution based on the following property close to Petrovic's inequality:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable convex function such that  $f(0) = 0$ .

If  $x_1, x_2, \dots, x_n$  are such that  $x_k \geq 0$  for odd  $k$ ,  $x_k \leq 0$  for even  $k$  with  $|x_1| \leq |x_2| \leq |x_3| \leq |x_4| \leq \cdots \leq |x_n|$ , then

$$f(x_1 + x_2 + \cdots + x_n) \leq f(x_1) + f(x_2) + \cdots + f(x_n).$$

*Proof.* We prove that  $f(x_1 + x_2) \leq f(x_1) + f(x_2)$  whenever  $x_1 x_2 \leq 0$ . An easy induction will then complete the proof.

Fix  $x_2$  and consider  $\phi$  defined by

$$\phi(x) = f(x + x_2) - f(x) - f(x_2).$$

Then, we have  $\phi(0) = 0$  and  $\phi'(x) = f'(x + x_2) - f'(x)$ . Recalling that  $f'$  is nondecreasing and distinguishing the cases  $x_2 \geq 0$  and  $x_2 \leq 0$ , we see that  $\phi(x_1) \leq 0$  in both cases.

Applying this inequality again with the function  $f : x \mapsto f(x) = e^x - 1$  and  $x_i = \ln a_i$  ( $i = 1, 2, \dots, n$ ) yields  $\Delta(a_1, a_2, \dots, a_n) \geq n - 1$ .

### Karamata's inequality

If  $f : I \rightarrow \mathbb{R}$  is convex and if  $x, y, z, x', y', z'$  are in the interval  $I$  and satisfy

$$x \geq y \geq z, x' \geq y' \geq z', x \geq x', x + y \geq x' + y' \quad \text{and} \quad x + y + z = x' + y' + z',$$

then

$$f(x) + f(y) + f(z) \geq f(x') + f(y') + f(z')$$

(for a proof and a generalization see [3]).

In [1999 : 17] Murray Klamkin gave a series of examples. Here is another one, a variant of solution to problem 878 proposed in the May 2008 issue of *The College Mathematics Journal*:

Let  $a, b$ , and  $c$  be the lengths of the sides and  $s$  the semi-perimeter of triangle  $ABC$ . Prove that

$$(a + b - c)^{a+b+s} (b + c - a)^{b+c+s} (c + a - b)^{c+a+s} \leq a^{\frac{a}{2}+2s} b^{\frac{b}{2}+2s} c^{\frac{c}{2}+2s}.$$

Observing that  $a + b + s = \frac{a+b-c}{2} + 2s$  (for example) and taking logarithms, the required inequality can be written as

$$f(a + b - c) + f(b + c - a) + f(c + a - b) \geq f(a) + f(b) + f(c) \quad (3)$$

where  $f(x) = -\left(\frac{x}{2} + 2s\right) \ln(x)$ .

Now, the function  $f$  is convex on the interval  $(0, 2s)$  (which contains the real numbers  $a, b, c, a + b - c, b + c - a, c + a - b$ ) and moreover, assuming  $a \geq b \geq c$ , we have

$$\begin{aligned} a + b - c &\geq c + a - b \geq b + c - a, \\ a + b - c &\geq a, \\ (a + b - c) + (c + a - b) &\geq a + b, \\ (a + b - c) + (c + a - b) + (c + a - b) &= a + b + c. \end{aligned}$$

Thus, the inequality (3) is a direct consequence of Karamata's inequality.

## Exercises

1. (problem 3062) Let  $a, b, c$  be positive real numbers such that  $a + b + c = 1$ .  
Prove that

$$(ab + bc + ca) \left( \frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{a^2 + a} \right) \geq \frac{3}{4}.$$

2. (Problem 11770 of *The American Mathematical Monthly*) Prove, for real numbers  $a, b, x, y$  with  $a > b > 1$  and  $x > y > 1$ , that

$$\frac{a^x - b^y}{x - y} > \left( \frac{a + b}{2} \right)^{\frac{x+y}{2}} \log \left( \frac{a + b}{2} \right).$$

(Hint: first apply Hadamard's inequality to the function  $t \mapsto m^t$  on  $[y, x]$ , where  $m = \frac{a+b}{2}$ ).

## References

- [1] T. Popoviciu, Sur certaines inégalités qui caractérisent les fonctions convexes, *Analele Stiintifice Univ. AL.I. Cuza Iasi Sect. I-a Mat (N.S.) 11B*, 1965, 155-164
- [2] V. Cîrtoaje, Two Generalizations of Popoviciu's Inequality, *Crux Mathematicorum with Mathematical Mayhem*, 31, N°5 Sept. 2005, 313-8.
- [3] A.W. Marshall, I. Olkin, B.C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, Springer, 2011



# PROBLEMS

*Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.*

To facilitate their consideration, solutions should be received by **July 15, 2020**.



**4541.** *Proposed by Michel Bataille.*

Let  $a, b, c$  be positive real numbers such that  $abc = 1$  and let  $S_k = a^k + b^k + c^k$ . Prove that

$$\frac{S_2 + S_4}{2} \geq 1 + \sqrt{1 + S_3}.$$

**4542.** *Proposed by Leonard Giugiuc and Alexander Bogomolny.*

Let  $ABC$  be a triangle with centroid  $G$ . Denote by  $D, E$  and  $F$  the midpoints of the sides  $BC, CA$  and  $AB$  respectively. Find the points  $M$  on the plane of  $ABC$  such that

$$MA + MB + MC + 3MG = 2(MD + ME + MF).$$

**4543.** *Proposed by Cherng-tiao Perng.*

Let  $n = 4k + 2$  ( $k \geq 1$ ) be an integer and  $A_1 A_2 \cdots A_n$  be a polygon with parallel opposite sides, i.e.

$$A_i A_{i+1} \parallel A_{n/2+i} A_{n/2+i+1}, \quad i = 1, 2, \dots, n/2,$$

where one sets  $A_{n+1} = A_1$ . Starting with a point  $B_1$  and a circle  $C$  through  $B_1$ , define  $B_2, B_3, \dots, B_{n+1}$  inductively by requiring that the circle  $(A_i A_{i+1} B_i)$  intersects  $C$  again at  $B_{i+1}$ , for  $i = 1, 2, \dots, n$ . Prove that  $B_{n+1} = B_1$ .

**4544.** *Proposed by Burgherea Zaharia.*

Calculate

$$\int_1^2 \ln\left(\frac{x^4 + 4}{x^2 + 4}\right) \frac{dx}{x}.$$

**4545.** *Proposed by Mihaela Berindeanu.*

Solve the following equation over  $\mathbb{N}$ :

$$6^n - 19 = \left[ 5\sqrt{n^2 + 4n} \right].$$

**4546.** *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

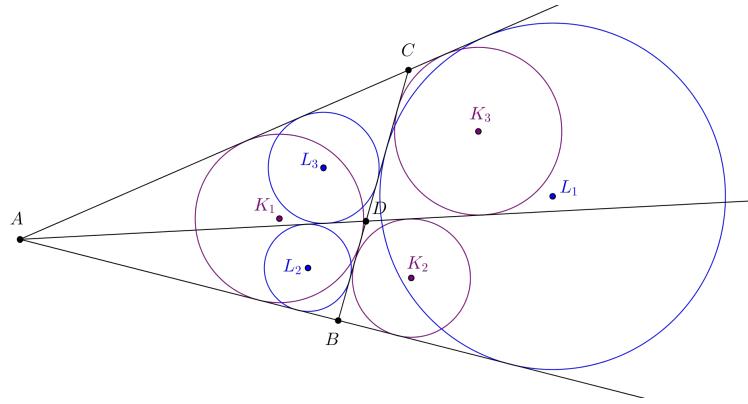
Let  $D$  be a point on the side  $BC$  of triangle  $ABC$  and consider the following tri-tangent circles:

$(K_1, k_1)$  is the incircle and  $(L_1, l_1)$  is the  $A$ -excircle of  $ABC$

$(L_2, l_2)$  is the incircle and  $(K_2, k_2)$  is the  $A$ -excircle of  $ABD$

$(L_3, l_3)$  is the incircle and  $(K_3, k_3)$  is the  $A$ -excircle of  $ACD$

Prove that  $\text{Area}(\triangle K_1 K_2 K_3) = \text{Area}(\triangle L_1 L_2 L_3)$ .



**4547.** *Proposed by George Stoica, modified by the Editorial Board.*

Consider the complex numbers  $a, b, c$  such that  $|a| = |b| = |c| = 1$ . Prove that if

$$|a + b - c|^2 + |b + c - a|^2 + |c + a - b|^2 = 12,$$

then  $a, b, c$  represent the vertices of an equilateral triangle inscribed in the unit circle.

**4548.** *Proposed by Lazea Darius.*

Find the maximum  $k$  for which

$$ab + bc + ca + k(a - b)^2(b - c)^2(c - a)^2 \leq 3$$

for all non-negative real numbers  $a, b, c$  such that  $a + b + c = 3$ .

**4549.** *Proposed by Lorian Saceanu, Leonard Giugiuc and Kadir Altintas.*

Let  $a_k$  and  $b_k$  be real numbers for  $k = 1, 2, \dots, n$ . Prove that

$$\sqrt{\sum_{k=1}^n (2a_k - b_k)^2} + \sqrt{\sum_{k=1}^n (2b_k - a_k)^2} \geq \sqrt{\sum_{k=1}^n a_k^2} + \sqrt{\sum_{k=1}^n b_k^2}.$$

**4550.** *Proposed by Leonard Giugiuc and Kunihiko Chikaya.*

Let  $\alpha$  be a real number greater than 2 and let  $x, y$  and  $z$  be positive real numbers such that  $x \geq y \geq z$ . Prove that

$$\frac{(x^\alpha - y^\alpha)(y^\alpha - z^\alpha)(z^\alpha - x^\alpha)}{(x^{\alpha-1} + y^{\alpha-1})(y^{\alpha-1} + z^{\alpha-1})(z^{\alpha-1} + x^{\alpha-1})} \geq \frac{\alpha^3}{24}((x-y)^3 + (y-z)^3 + (z-x)^3).$$

When does equality hold?

[View Details](#) | [Edit](#) | [Delete](#)

*Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.*

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2020**.*

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*



**4541.** *Proposé par Michel Bataille.*

Soient  $a, b, c$  des nombres réels positifs tels que  $abc = 1$  et  $S_k = a^k + b^k + c^k$ . Démontrer que

$$\frac{S_2 + S_4}{2} \geq 1 + \sqrt{1 + S_3}.$$

**4542.** *Proposé par Leonard Giugiuc et Alexander Bogomolny.*

Soit  $ABC$  un triangle de centre de gravité  $G$ . Dénoter  $D, E$  et  $F$  les mi-points des côtés  $BC, CA$  et  $AB$  respectivement. Déterminer les points  $M$  dans le plan tels que

$$MA + MB + MC + 3MG \equiv 2(MD + ME + MF).$$

4543 Proposé par Cherna-tiao Perna

Soit  $n = 4k + 2$  ( $k \geq 1$ ) un entier et soit  $A_1A_2\cdots A_n$  un polygone avec côtés opposés parallèles, i.e.

$$A_i A_{i+1} \parallel A_{n/2+i} A_{n/2+i+1}, \quad i \equiv 1, 2, \dots, n/2,$$

où on pose  $A_{n+1} = A_1$ . À partir d'un point  $B_1$  et un cercle  $C$ , définissons  $B_2, B_3, \dots, B_{n+1}$  inductivement en exigeant que le cercle  $(A_i A_{i+1} B_i)$  intersecte  $C$  une seconde fois en  $B_{i+1}$ , pour  $i = 1, 2, \dots, n$ . Démontrer que  $B_{n+1} = B_1$ .

**4544.** Proposé par Burgherea Zaharia.

Calculer

$$\int_1^2 \ln\left(\frac{x^4 + 4}{x^2 + 4}\right) \frac{dx}{x}.$$

**4545.** Proposé par Mihaela Berindeanu.

Résoudre dans  $\mathbb{N}$ :

$$6^n - 19 = \left[ 5\sqrt{n^2 + 4n} \right].$$

**4546.** Proposé par Thanos Kalogerakis, Leonard Giugiuc et Kadir Altintas.

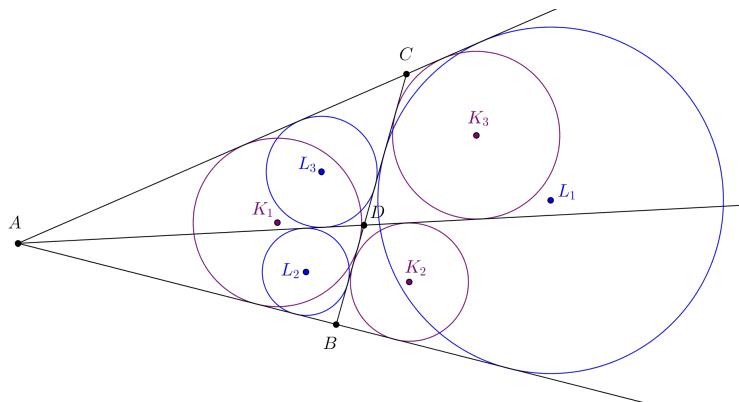
Soit  $D$  un point sur le côté  $BC$  du triangle  $ABC$  et considérer les cercles suivants:

$(K_1, k_1)$  est le cercle inscrit de  $ABC$  et  $(L_1, l_1)$  est le cercle exinscrit de  $ABC$ , opposé à  $A$

$(L_2, l_2)$  est le cercle inscrit de  $ABD$  et  $(K_2, k_2)$  est le cercle exinscrit de  $ABD$ , opposé à  $A$

$(L_3, l_3)$  est le cercle inscrit de  $ACD$  et  $(K_3, k_3)$  est le cercle exinscrit de  $ACD$ , opposé à  $A$

Démontrer que  $\text{Surface}(\triangle K_1 K_2 K_3) = \text{Surface}(\triangle L_1 L_2 L_3)$ .



**4547.** Proposé par George Stoica, modifié par le conseil.

Soient  $a, b, c$  des nombres complexes tels que  $|a| = |b| = |c| = 1$ . Démontrer que si

$$|a + b - c|^2 + |b + c - a|^2 + |c + a - b|^2 = 12,$$

alors  $a, b, c$  sont les sommets d'un triangle équilatéral inscrit dans le cercle unité.

**4548.** *Proposé par Lazea Darius.*

Soient  $a, b, c$  des nombres réels non-négatifs tels que  $a + b + c = 3$ . Déterminer le  $k$  maximal pour lequel

$$ab + bc + ca + k(a - b)^2(b - c)^2(c - a)^2 \leq 3.$$

**4549.** *Proposé par Lorian Saceanu, Leonard Giugiuc et Kadir Altintas.*

Soient  $a_k$  et  $b_k$  des nombres réels,  $k = 1, 2, \dots, n$ . Démontrer que

$$\sqrt{\sum_{k=1}^n (2a_k - b_k)^2} + \sqrt{\sum_{k=1}^n (2b_k - a_k)^2} \geq \sqrt{\sum_{k=1}^n a_k^2} + \sqrt{\sum_{k=1}^n b_k^2}.$$

**4550.** *Proposé par Leonard Giugiuc et Kunihiko Chikaya.*

Soit  $\alpha$  un nombre réel supérieur à 2 et soient  $x, y$  et  $z$  des nombres réels positifs tels que  $x \geq y \geq z$ . Démontrer que

$$\frac{(x^\alpha - y^\alpha)(y^\alpha - z^\alpha)(z^\alpha - x^\alpha)}{(x^{\alpha-1} + y^{\alpha-1})(y^{\alpha-1} + z^{\alpha-1})(z^{\alpha-1} + x^{\alpha-1})} \geq \frac{\alpha^3}{24}((x-y)^3 + (y-z)^3 + (z-x)^3).$$

Quand l'égalité tient-elle ?



# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2019: 45(10), p. 564–567.*



**4491.** *Proposed by Lorian Saceanu.*

Let  $a, b, c$  be the side lengths of acute-angled triangle  $ABC$  lying opposite of angles  $\angle A, \angle B, \angle C$ , respectively. Let  $r$  be the inradius of  $ABC$  and let  $R$  be its circumradius. Prove that

$$\frac{a\angle A + b\angle B + c\angle C}{a+b+c} \leq \arccos \frac{r}{R}.$$

*We received 13 solutions, one of which was incorrect. We present the solution by Leonard Giugiuc.*

The function  $\arccos$  is strictly concave on the interval  $(0, 1)$ ; hence,

$$\frac{aA + bB + cC}{a+b+c} = \sum_{\text{cyc}} \frac{a}{a+b+c} \cdot \arccos \left( \frac{b^2 + c^2 - a^2}{2bc} \right).$$

By Jensen's inequality,

$$\begin{aligned} \sum_{\text{cyc}} \frac{a}{a+b+c} \cdot \arccos \left( \frac{b^2 + c^2 - a^2}{2bc} \right) &\leq \arccos \left( \sum_{\text{cyc}} \frac{a}{a+b+c} \cdot \frac{b^2 + c^2 - a^2}{2bc} \right) \\ &= \arccos \left( \frac{16[ABC]^2}{2abc(a+b+c)} \right) \\ &= \arccos \left( \frac{4[ABC]}{abc} \cdot \frac{2[ABC]}{a+b+c} \right) \\ &= \arccos \frac{r}{R}. \end{aligned}$$

**4492.** *Proposed by George Stoica.*

Find the number of classes  $\hat{u}$  in  $\mathbb{Z}_n$  ( $n \geq 2$ ) with the property that both  $\hat{u}$  and  $\hat{u} - \hat{1}$  have multiplicative inverses in  $\mathbb{Z}_n$ .

*We received 13 correct solutions. We present the solution from Marie-Nicole Gras that handles a generalization of the problem.*

Let  $k$  be a positive integer. We will determine the number  $f(n)$  of elements  $\hat{u}$  in  $\mathbb{Z}_n$  for which

$$\hat{u}, \hat{u} - \hat{1}, \hat{u} - \hat{2}, \dots, \hat{u} - \widehat{(k-1)}$$

all have inverses in  $\mathbb{Z}_n$ .

Observe that  $f$  is multiplicative, i.e.,  $f(mn) = f(m)f(n)$  whenever  $\gcd(m, n) = 1$ . This follows from the Chinese Remainder Theorem whereby there is a one-one onto correspondence between elements  $\hat{u} \in \mathbb{Z}_{mn}$  and  $(\hat{v}, \hat{w}) \in \mathbb{Z}_m \times \mathbb{Z}_n$  implemented by  $u \equiv v \pmod{m}$  and  $u \equiv w \pmod{n}$ . Any number  $u$  is coprime with  $mn$  iff it is coprime with both  $m$  and  $n$  separately iff  $v$  is coprime with  $m$  and  $w$  is coprime with  $n$ . Thus, if  $n = \prod p^a$  is the prime factor decomposition of  $n$ , then  $f(n) = \prod f(p^a)$ .

Now  $f(p^a)$  is the number of nonnegative integers  $r$  less than  $p^a$  for which none of  $r, r - 1, \dots, r - (k - 1)$  is a multiple of  $p$ . If  $k \geq p$ , there are no such integers. If  $k < p$ , such integers have the form  $bp + c$  where  $0 \leq b \leq p^{a-1} - 1$  and  $k \leq c \leq p - 1$ . Therefore  $f(p^a) = p^{a-1}(p - k)$ .

Hence

$$f(n) = \begin{cases} 0, & \text{if } k \text{ exceeds some prime divisor of } n; \\ \prod p^{a-1}(p - k) = n \prod \left(1 - \frac{k}{p}\right), & \text{otherwise.} \end{cases}$$

In the particular case of the problem,  $k = 2$  and the answer is

$$n \prod \left(1 - \frac{2}{p}\right).$$

*Editor's Comment.* As Walther Janous pointed out, the original result dates back 150 years and can be found in V. Schemmel, Über relative Primzahlen, *Journal für die reine und angewandte Mathematik* **70** (1869), p. 191-192. The problem turns up in an interesting context some 60 years later, as D. N. Lehmer comes across the same situation in the construction of magic squares with special properties. See in particular pages 538 and 539 of D.N. Lehmer, On the congruences connected with certain magic squares, *Transactions of the A.M.S.* **31** (1929), p. 529-551.

Oliver Geupel tracked down this paper which ventures into the same territory: Henry L. Alder, A generalization of the Euler  $\phi$ -function, *American Mathematical Monthly* **65** (1958), 690-692.

### 4493. Proposed by Nguyen Viet Hung.

Find all real numbers  $x, y$  such that

$$\left\{ \frac{x+2y+1}{x^2+y^2+7} \right\} = \frac{1}{2}$$

where  $\{a\}$  denotes the fractional part of  $a$ .

We received 33 solutions. We present the solution by Vincent Blevins.

The only pair of real numbers satisfying this condition is  $(1, 2)$ .

Let  $x$  and  $y$  be two real numbers such that the ratio  $\frac{x+2y+1}{x^2+y^2+7}$  has fractional part  $\frac{1}{2}$ , i.e.,

$$\frac{x+2y+1}{x^2+y^2+7} = n + \frac{1}{2}$$

for some integer  $n$ . Multiplying both sides of this equation by the denominator and subtracting the numerator yields the equation

$$(2n+1)x^2 + (2n+1)y^2 - 2x - 4y + 14n + 5 = 0.$$

After completing the square and simplifying, we have

$$\left(x - \frac{1}{2n+1}\right)^2 + \left(y - \frac{2}{2n+1}\right)^2 = -\frac{28n^2 + 24n}{(2n+1)^2}.$$

If  $n$  is a nonzero integer, then the fraction on the right side of the previous equality is negative. As the sum of two squares can never be negative, there can be no pair of real solutions in this case. On the other hand, if  $n = 0$ , then

$$(x-1)^2 + (y-2)^2 = 0.$$

A sum of two squares can only be zero if both terms in the sum are zero, i.e., the pair  $(1, 2)$  is the only solution.

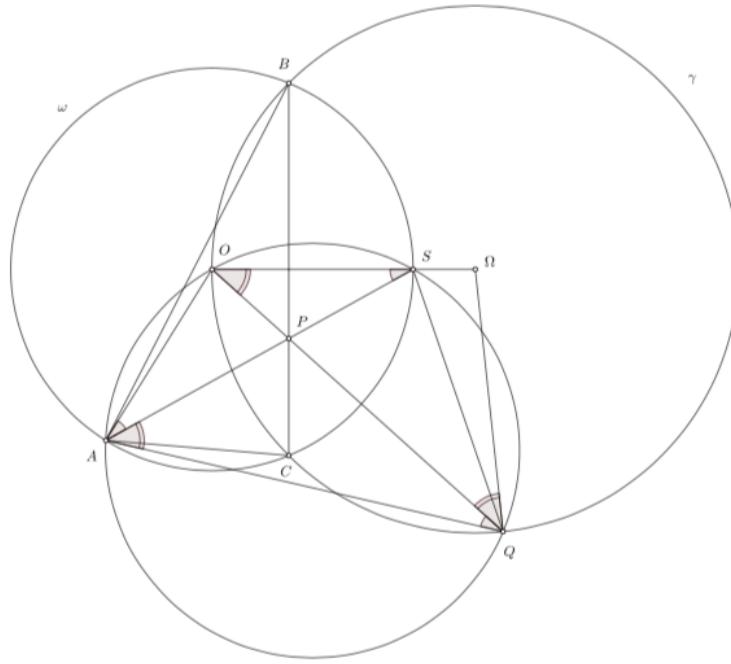
#### 4494. Proposed by Michel Bataille.

Let  $O$  be the circumcentre of a triangle  $ABC$  such that  $\angle BAC \neq 90^\circ$  and let  $\gamma$  be the circumcircle of  $\triangle BOC$  and  $\Omega$  its centre. If  $P$  is a point of the side  $BC$ , let  $Q$  denote the point of intersection other than  $O$  of the line  $OP$  and  $\gamma$ . For which  $P$  do the lines  $OA$  and  $\Omega Q$  intersect at  $M$  such that  $MA = MQ$ ?

*We received 8 submissions, all of which were correct, and feature a composite of similar solutions submitted by the UCLan Cyprus Problem Solving Group and by Xunhan Zheng.*

We will show that  $OA$  and  $\Omega Q$  intersect at a point  $M$  such that  $MA = MQ$  if and only if  $AP$  is the bisector of  $\angle BAC$ . This will force us to assume that  $AB \neq AC$  (because if  $AB$  were to equal  $AC$  then  $A, O, P, Q, \Omega$  would lie on a line and, consequently,  $M$  would not be defined). All angles are assumed to be directed angles between lines.

Let  $S$  be the other point of intersection of  $AP$  with the circumcircle  $\omega$  of  $ABC$ . Note that because we assume that  $P$  is between  $B$  and  $C$ , it follows that  $A$  and  $S$  are on opposite sides of the line  $BC$ .



The power of  $P$  with respect to  $\omega$  and to  $\gamma$  gives us  $AP \cdot PS = BP \cdot PC = OP \cdot PQ$ , whence  $A, S, O, Q$  lie on a circle. It follows that

$$\angle ASO = \angle AQO \quad \text{and} \quad \angle SAQ = \angle SOQ.$$

We have

$$\begin{aligned} \angle OAQ &= \angle OAS + \angle SAQ \\ &= \angle ASO + \angle SAQ \quad (OS = OA) \\ &= \angle AQO + \angle SOQ \quad (A, O, S, Q \text{ concyclic}), \end{aligned}$$

and

$$\begin{aligned} \angle AQ\Omega &= \angle AQO + \angle OQ\Omega \\ &= \angle AQO + \angle \Omega OQ \quad (\Omega O = \Omega Q). \end{aligned}$$

Note that  $OA$  and  $\Omega Q$  intersect at a point  $M$  such that  $MA = MQ$  if and only if  $\angle OAQ = \angle MAQ = \angle AQM = \angle AQ\Omega$ . Thus,

$$\begin{aligned} \angle OAQ = \angle AQ\Omega &\iff \angle SOQ = \angle \Omega OQ \\ &\iff O, S, Q \text{ collinear} \\ &\iff S \text{ lies on the perpendicular bisector of } BC \\ &\quad \text{with } A \text{ and } S \text{ separated by } BC \\ &\iff AP \text{ is the bisector of } \angle BAC. \end{aligned}$$

**4495.** *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Prove Mihaileanu's theorem: Given a point  $P$  inside triangle  $ABC$  set  $x = [PBC]$ ,  $y = [PCA]$ , and  $z = [PAB]$ , where square brackets denote area. If  $M$  is a point on side  $AB$  and  $N$  is a point on side  $AC$ , then line  $MN$  contains  $P$  if and only if

$$y \cdot \frac{BM}{MA} + z \cdot \frac{CN}{NA} = x.$$

[Editor: See Solution 2 of 4445.]

*We received 15 submissions, all of which were correct, and feature one example of each of the two most popular approaches.*

*Solution 1, by Marie-Nicole Gras.*

We put  $\angle A = \angle BAC$ ,  $\theta = \angle PAC$  and  $\omega = \angle BAP$ ; since  $P$  is inside  $\triangle ABC$ , we have  $\theta + \omega = \angle A$ .

The line  $MN$  contains  $P$  if and only if

$$[APN] + [AMP] = [AMN]$$

$$\begin{aligned} &\iff \frac{1}{2} AP \cdot AN \sin(\theta) + \frac{1}{2} AM \cdot AP \sin(\omega) = \frac{1}{2} AM \cdot AN \sin(\angle A) \\ &\iff \frac{1}{2} \frac{AP}{AM} \sin(\theta) + \frac{1}{2} \frac{AP}{AN} \sin(\omega) = \frac{1}{2} \sin(\angle A) \\ &\iff \frac{1}{2} AB \cdot AC \frac{AP}{AM} \sin(\theta) + \frac{1}{2} AB \cdot AC \frac{AP}{AN} \sin(\omega) = \frac{1}{2} AB \cdot AC \sin(\angle A) \\ &\iff [APC] \frac{AB}{AM} + [ABP] \frac{AC}{AN} = [ABC] \\ &\iff y \frac{AB}{AM} + z \frac{AC}{AN} = x + y + z \\ &\iff y \frac{AM + MB}{AM} + z \frac{AN + NC}{AN} = x + y + z \\ &\iff y \left(1 + \frac{MB}{AM}\right) + z \left(1 + \frac{NC}{AN}\right) = x + y + z \\ &\iff y \frac{MB}{AM} + z \frac{NC}{AN} = x, \end{aligned}$$

and the theorem is proved.

*Solution 2, by the UCLan Problem Solving Group.*

Using barycentric coordinates (relative to  $\Delta ABC$ ) we have

$$P = (x : y : z), \quad M = (BM : MA : 0), \quad \text{and} \quad N = (CN : 0 : NA).$$

These three points are collinear if and only if

$$\begin{vmatrix} x & y & z \\ BM & MA & 0 \\ CN & 0 & NA \end{vmatrix} = 0.$$

Expanding, we get

$$x(MA)(NA) - y(BM)(NA) - z(MA)(CN) = 0,$$

which is equivalent to

$$x = y \frac{BM}{MA} + z \frac{CN}{NA}.$$

*Editor's comments.* Michel Bataille observed that Mihaileanu's theorem is a result known (without any attribution) to those familiar with barycentric coordinates. It can be found in "Focus On ... No. 4: The Barycentric Equation of a Line" [Vol. 38:9, 2012: 367-368]. There you can also find further details concerning the method behind our second featured solution. Janous adds that if one uses signed areas and directed distances, our result holds for an arbitrary point  $P$  in the plane of  $\Delta ABC$  while  $M$  and  $N$  are points different from a vertex on the lines  $AB$  and  $AC$ , respectively.

#### 4496. Proposed by Leonard Giugiuc.

Let  $a$  and  $b$  be two fixed numbers such that  $0 < a < b$ . We consider the function

$$f : [a, b] \times [a, b] \times [a, b] \rightarrow \mathbb{R}, \quad f(x, y, z) = (x + y + z) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right).$$

Find the maximum value of the function.

We received 15 correct solutions, one incorrect solution and one that appealed to a complicated obscure general inequality. Most of the solutions relied in some way on the convexity of  $f(x, y, z)$  with respect to each variable. Several solvers appealed to Gireaux' Theorem, which asserts that a function defined on a box convex in each variable assumes its maximum on the edges of the box; the first solution based on this approach is self-contained. The second solution is due to Oliver Geupel; Roy Barbara had the same approach differing only in detail.

*Solution 1.*

Fix  $(x, y) \in [a, b] \times [a, b]$  and let  $u = x + y$ , so that  $\frac{1}{x} + \frac{1}{y} = \frac{u}{xy}$ . Then

$$f(x, y, z) = (u + z) \left( \frac{u}{xy} + \frac{1}{z} \right) = \frac{u^2}{xy} + 1 + u \left( \frac{z}{xy} + \frac{1}{z} \right).$$

The function  $(z/xy) + (1/z)$  is convex for  $z \in [a, b]$  with its minimum at  $z = \sqrt{xy} \in [a, b]$  and its maximum at either  $z = a$  or  $z = b$ . A similar statement holds for  $x$  for fixed  $(y, z)$  and  $y$  for fixed  $(x, z)$ .

Hence, the maximum must occur when the values of  $x, y, z$  belong to  $\{a, b\}$ . By symmetry, we need examine only  $f(a, a, a) = 9$  and

$$f(a, a, b) = \frac{(2a+b)(a+2b)}{ab} = 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right) \geq 9.$$

Thus the maximum value is

$$5 + 2\left(\frac{a}{b} + \frac{b}{a}\right) = 9 - 2\left(\sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}}\right)^2.$$

*Solution 2, by Oliver Geupel and Roy Barbara (done independently).*

Without loss of generality, suppose that  $0 < a \leq x \leq y \leq z \leq b$ . Then

$$\begin{aligned} f(a, b, b) - f(x, b, b) &= \frac{2}{abx}(x-a)(b^2-ax); \\ f(x, b, b) - f(x, z, z) &= \frac{2}{bxz}(b-z)(bz-x^2); \\ f(x, z, z) - f(x, y, z) &= \frac{1}{xyz}(x+z)(y-x)(z-y). \end{aligned}$$

Adding these equations yields that

$$\frac{(a+2b)(2a+b)}{ab} = f(a, b, b) \geq f(x, y, z) \quad \forall (x, y, z) \in [a, b]^3.$$

*Editor's Comments.* Various solvers pointed out that the result is far from new. G. Stoica proved in a 1986 paper in *Gazeta Matematica Seria B* **91**, 151-154 that the maximum of  $(\sum_{i=1}^n x_i)(\sum_i 1/x_i)$  on the box  $[a, b]^n$  is  $n^2(a+b)^2/(4ab)$  when  $n$  is even and  $[n^2(a+b)^2 - (a-b)^2]/(4ab)$  when  $n$  is odd. Walther Janous pointed out that this in turn is a special case of the 1948 Kantorovich Inequality, which has a Wikipedia entry. In 1914, P. Schweitzer provided an upper bound. Daniel Vacaru draws attention to a 1972 paper by Alexandru Lupaş that follows up on this work and provides a complicated form of the maximum. With five variables, this was the fifth problem on the 1977 USAMO, where the maximum was given as  $25 + 6(\sqrt{b/a} - \sqrt{a/b})^2$ .

#### 4497. Proposed by Hoang Le Nhat Tung.

Let  $a, b, c$  be positive real numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{4(a^2 + b^2 + c^2)}{ab + bc + ca} + \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}.$$

We received 20 submissions, 19 of which were correct, and the other one used Maple-based outputs. We present the proof by Madhav Modak, modified slightly by the editor.

Let  $E$  denote the left side of the given inequality. Then

$$\begin{aligned} E + 6 &= \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} + 6 = \frac{a^2 + b^2}{ab} + 2 + \frac{b^2 + c^2}{bc} + 2 + \frac{c^2 + a^2}{ca} + 2 \\ &= \frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} \\ &\geq \frac{(a+b+b+c+c+a)^2}{ab+bc+ca} = \frac{4(a+b+c)^2}{ab+bc+ca}, \end{aligned}$$

by the Cauchy-Schwarz inequality. (*Ed: What the solver used here is actually a special case of the C-S Inequality, commonly known as Titu's Lemma.*) Hence

$$\begin{aligned} E - 2 &\geq \frac{4(a+b+c)^2}{ab+bc+ca} - 8 \\ \text{or } E - 2 &= \frac{4(a^2 + b^2 + c^2)}{ab+bc+ca}. \end{aligned} \tag{1}$$

Next we have  $a^2 + b^2 + c^2 \geq ab + bc + ca$ , so

$$2 \geq \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}. \tag{2}$$

The result follows from (1) + (2).

The equality holds in (1) if and only if  $(a+b)/ab = (b+c)/bc = (c+a)/ca$  or  $a = b = c$  and equality holds in (2) if and only if  $a/b = b/c = c/a$  or  $a = b = c$ . Hence, equality holds in the given inequality if and only if  $a = b = c$ .

#### 4498. Proposed by Sergey Sadov.

Consider the function  $f(x) = 1/(x^2 + 1)$  for  $x > 0$ . Prove that there exists  $n$  such that the  $n$ th derivative  $f^{(n)}(x)$  does not have constant sign for  $x > 2019$ .

We received 14 submissions, of which 13 were correct and one was incomplete. We present the solution provided by Brian Bradie.

Let

$$f(x) = \frac{1}{x^2 + 1} = \frac{1}{2i} \left( \frac{1}{x-i} - \frac{1}{x+i} \right).$$

Then

$$f^{(n)}(x) = \frac{(-1)^n n!}{2i} \left( \frac{1}{(x-i)^{n+1}} - \frac{1}{(x+i)^{n+1}} \right) = \frac{(-1)^n n!}{2i} \cdot \frac{(x+i)^{n+1} - (x-i)^{n+1}}{(x^2 + 1)^{n+1}}.$$

It follows that  $f^{(n)}(x) = 0$  when  $(x+i)^{n+1} = (x-i)^{n+1}$ ; that is, when

$$x = -i + (x-i)e^{2\pi ki/(n+1)}$$

for  $k = 1, 2, \dots, n$ . Solving for  $x$  yields

$$x = \cot \frac{\pi k}{n+1}$$

for  $k = 1, 2, \dots, n$ . The numerator of  $f^{(n)}$  is a polynomial of degree  $n$  with  $n$  zeros, so each zero must be simple and therefore  $f^{(n)}(x)$  must change sign at each zero. For  $0 < x < \pi$ ,  $\cot x$  is a decreasing function so the largest zero of  $f^{(n)}$  is

$$x = \cot \frac{\pi}{n+1}.$$

Now,

$$\cot \frac{\pi}{n+1} > 2019 \quad \text{whenever} \quad n > \frac{\pi}{\cot^{-1} 2019} - 1.$$

Thus, for any integer

$$n \geq 6342 > \frac{\pi}{\cot^{-1} 2019} - 1,$$

$f^{(n)}(x)$  changes sign at least once for  $x > 2019$ .

#### 4499. Proposed by H. A. ShahAli.

Prove that the following system of Diophantine equations has infinitely many unproportional solutions in positive integers:

$$\begin{cases} a + b + c + d = e + f + g, \\ a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2. \end{cases}$$

We received 13 correct solutions, and one incomplete solution. We present 7 different variants below.

*Solution 1, by Corneliu Manescu-Avram.*

Let  $a, b, c$  be integers with greatest common divisor 1. Then

$$(a, b, c, d; e, f, g) = (a, b, c, a+b+c; a+b, b+c, c+a)$$

satisfies the system. There is an infinite set of these solutions that are pairwise nonproportional.

*Solution 2, by the proposer.*

The system is satisfied by

$$(a, b, c, d; e, f, g) = (u+v+w, -u+v+w, u-v+w, u+v-w; 2u, 2v, 2w).$$

*Solution 3, by Marie Nicole Gras.*

The system is satisfied by

$$(a, b, c, d; e, f, g) = (1, 2, c, c+3; 3, c+1, c+2)$$

for arbitrary integer  $c$ , giving nonproportional sets.

*Solution 4, by David Manes.*

The system is satisfied by

$$(a, b, c, d; e, f, g) = (1, 1, c, c - 2; 2, c - 1, c - 1)$$

for arbitrary integer  $c$ , giving nonproportional sets.

*Solution 5, by Roy Barbara, Brian Beasley, UCLan Cyprus Problem Solving Group, Oliver Geupel, and Walther Janous (independently).*

Begin with a solution of the system  $a + b + c = e + f$ ,  $a^2 + b^2 + c^2 = e^2 + f^2$  and then make  $d = g$ . This method produces solutions such as the following:  $(1, 2, 6, d; 4, 5, d)$ ,  $(1, 1, 4, d; 3, 3, d)$  and  $(2f - 2, f - 2, 1, d; 2f - 3, f, d)$ .

*Solution 6, by C.R. Pranesachar.*

Let  $n$  be an integer. Then the system is satisfied by

$$(a, b, c, d; e, f, g) = (2n^2 + 10n + 2, 2n - 1, n + 2, n + 3; 2n^2 + 10n + 1, 3n + 4, n + 1).$$

The strategy for finding this is to first eliminate  $g$  from the two equations, and then get a rational expression for  $a$  in terms of the remaining variables. With  $(b, c, d; e, f) = (n + k, n + 2, n + 3; 3n + 4, n + 1)$ , we find that

$$a = 2(2n^2 - kn + 4n + 1)/(n - k).$$

*Solution 7, by the Missouri State University Problem Solving Group.*

More generally, when  $m \geq n \geq 2$  we can solve the simultaneous system

$$\sum_{i=1}^m a_i^r = \sum_{i=1}^n b_i^r$$

with  $r = 1, 2$ . With  $k = m - n + 2$  this is satisfied by

$$a_1 = 2t^2 + 2(k - 1)t + (k - 1)(k - 2), \quad a_i = 2t \quad (2 \leq i \leq m),$$

$$b_1 = 2[t^2 + (k - 1)t], \quad b_2 = (k - 1)(2t + k - 2), \quad b_i = 2t \quad (3 \leq i \leq n).$$

Since

$$\frac{b_2}{a_2} = \frac{(k - 1)(2t + k - 2)}{2t} = k - 1 + \frac{(k - 1)(k - 2)}{2t},$$

it is clear than we can get an infinite nonproportional set of solutions.

*Editor's Comments.* The system is satisfied by  $(a, b, c, d; e, f, g) = (1, 2, 4, 7; 3, 5, 6)$ . As Ioannis Sfikas points out, this solution has an interesting generalization. The two quadruples  $(1, 2, 4, 7)$ ,  $(0, 3, 5, 6)$  comprising all the numbers from 0 to 7 have the same cardinality and the same sums and square sums. If we add 8 to one

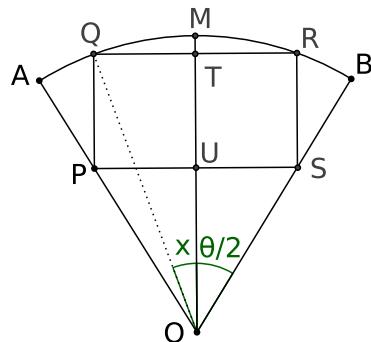
set and append the result to the other, we get the pair  $(1, 2, 4, 7, 8, 11, 13, 14), (0, 3, 5, 6, 9, 10, 12, 15)$  comprising all the numbers from 0 to 15 with the same cardinality that have the same sums, square sums and cube sums. We can continue this process to obtain, for each positive integer  $n$  a partition of the numbers from 0 to  $2^n - 1$  inclusive into two sets with  $2^{n-1}$  elements whose  $k$ th power sums are equal for  $0 \leq k \leq n - 1$ . Note that if we add any constant  $c$  to each element of the sets, they retain the same property.

This reminds us of the Tarry-Escott problem which is to determine the values of the positive integer  $m$  for which two sets of integers of size  $m$  can be found for which the sums of the  $k$ th powers are equal for  $0 \leq k \leq m - 1$ . Some solutions for small values of  $m$  are  $\{(1, 5, 6), (2, 3, 7)\}, \{(1, 5, 8, 12), (2, 3, 10, 11)\}, \{(1, 5, 9, 17, 18), (2, 3, 11, 15, 19)\}$ . This problem, with a rich history revealed by an online search, has hooked people for over a century and a half.

#### 4500. Proposed by Chudamani R. Pranesachar.

Let  $AB$  be an arc of a circle with radius  $r$  and centre  $O$ , its angle subtended at the center, denoted  $\theta$ , being less than  $\pi$ . Let  $M$  be the mid-point of the shorter arc  $AB$ . Points  $P$  on radius  $OA$ ,  $S$  on radius  $OB$ ,  $Q$  and  $R$  on arc  $AB$  are taken such that  $PQRS$  is a rectangle. Prove that when the area of  $PQRS$  is maximum, the line segments  $OQ$ ,  $OM$ ,  $OR$  divide angle  $AOB$  into four equal parts of common value  $\frac{\theta}{4}$ ). Determine this maximum area in terms of  $r$  and  $\theta$ .

We received 13 submissions, all of which were correct, and we feature the solution by Oliver Geupel.



If  $T$  is the midpoint of the chord  $QR$ , the line  $OT$  is perpendicular to the chord  $QR$  and to the opposite side of the rectangle, namely  $PS$ . Hence,  $OP = OS$ . Let  $x = \angle MOR = \angle MOQ$  where  $0 < x < \theta/2$ , and let  $U$  be the midpoint of the side  $PS$ . We have

$$OT = QT \cot x, \quad OU = PU \cot \frac{\theta}{2}, \quad TU = OT - OU,$$

and

$$PU = QT = r \sin x, \quad PS = 2r \sin x.$$

Therefore,

$$TU = r \sin x \left( \cot x - \cot \frac{\theta}{2} \right) = r \sin x \cdot \frac{\sin(\theta/2 - x)}{\sin(\theta/2) \sin x} = r \frac{\sin(\theta/2 - x)}{\sin(\theta/2)}.$$

Consequently,

$$\begin{aligned} [PQRS] &= PS \cdot TU = \frac{2r^2}{\sin(\theta/2)} \sin x \sin \left( \frac{\theta}{2} - x \right) \\ &= \frac{r^2}{\sin(\theta/2)} \left( \cos \left( 2x - \frac{\theta}{2} \right) - \cos \frac{\theta}{2} \right) \\ &\leq \frac{r^2}{\sin(\theta/2)} \left( 1 - \cos \frac{\theta}{2} \right) = r^2 \left( \csc \frac{\theta}{2} - \cot \frac{\theta}{2} \right) = r^2 \tan \frac{\theta}{4}. \end{aligned}$$

Consequently, the maximum area is  $r^2 \tan(\theta/4)$  which is attained if and only if  $\cos(2x - \frac{\theta}{2}) = 1$ , which implies that  $2x - \frac{\theta}{2} = 0$  and, thus,  $x = \frac{\theta}{4}$ , as desired.

