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ON AN IDEA OF GROENMAN

R.H. Eddy

In *Crux* [1987: 256] and [1987: 321], the late J.T. Groenman, a prolific and much appreciated contributor, proposed the following two related problems.

1272. (solution [1988: 256]) *Let $A_1A_2A_3$ be a triangle. Let the incircle have center I and radius ρ, and meet the sides of the triangle at points P_1, P_2, P_3. Let I_1, I_2, I_3 be the excenters and ρ_1, ρ_2, ρ_3 the exradii. Prove that*

(a) *the lines I_1P_1, I_2P_2, I_3P_3 concur at a point S;*

(b) *the distances d_1, d_2, d_3 of S to the sides of the triangle satisfy*

$$d_1 : d_2 : d_3 = \rho_1 : \rho_2 : \rho_3.$$

1295. (solution [1989: 17–19]) *Let $A_1A_2A_3$ be a triangle with I_1, I_2, I_3 the excenters and B_1, B_2, B_3 the feet of the altitudes. Show that the lines I_1B_1, I_2B_2, I_3B_3 concur at a point collinear with the incenter and circumcenter of the triangle.*

A third point, with a similar construction, may be found in Nagel [9], where in Groenman's notation we may write

Let S_1, S_2, S_3 denote respectively the midpoints of the sides A_2A_3, A_3A_1, A_1A_2 of the given triangle; then the lines I_1S_1, I_2S_2, I_3S_3 concur at the MITTENPUNKT of the given triangle.

We found out about this interesting point, while on sabbatical leave in Europe in the fall of 1986, from Peter Baptist, a faculty member at Bayreuth University who has done considerable work in the geometry of the triangle, particularly as it relates to special points. (For an example of his work, see [3].) A direct translation would seem to yield the term *middlespoint* in English, and this is a good description since the point is constructed using middles, i.e., centres of circles and midpoints of line segments. This term is used in [5] and [6], but since it is rather clumsy we shall use *mittenpunkt* throughout this note.

Let us consider the point from a different angle. Consider the pair of triangles $S_1S_2S_3$ and $I_1I_2I_3$ which are in perspective with the triangle $A_1A_2A_3$ from the centroid and incentre respectively. Then the triangles $S_1S_2S_3$ and $I_1I_2I_3$ are in perspective from the mittenpunkt of $A_1A_2A_3$. While Nagel's proof is synthetic—in fact he seems to dislike other types, especially trigonometric—the problem is easily solved analytically using trilinear coordinates. Since these have appeared in *Crux* several times, further details will not be given here. It suffices to remark that the trilinear coordinates of the mittenpunkt are $(s - a_1, s - a_2, s - a_3)$ where, as usual, s is the semiperimeter of the given triangle. Since the construction seems to be something of an unnatural "buddying-up", so to speak, of the incentre and the centroid, one immediately gets the feeling that some sort of generalization is lurking around. Further justification for such a possibility is immediately obtained by replacing the centroid by the symmedian point, see *Crux* [11], which has the rather nice coordinates $(\sin\alpha, \sin\beta, \sin\gamma) = (a_1, a_2, a_3)$. It is an elementary exercise to show that the corresponding lines are again concurrent.

The following generalization is given in [6]. A point P in the interior of the given triangle determines an inscribed triangle $P_1P_2P_3$, where $P_i = A_iP \cap A_{i+1}A_{i+2}$, $i = 1, 2, 3$. (Subscripts here and below are taken modulo 3.) A second interior point Q determines a circumscribed triangle in the following manner. Consider the harmonic conjugates Q'_1, Q'_2, Q'_3 of Q_1, Q_2, Q_3 with respect to the point pairs $(A_2, A_3), (A_3, A_1)$, and (A_1, A_2) which lie on a line q, the *trilinear polar* of $Q(y_i)$ with respect to the given triangle [1]. Let $Q^i = A_{i+1}Q'_{i+1} \cap A_{i+2}Q'_{i+2}$. Then the trilinear coordinates of Q^i are $((-1)^{\delta_{ij}}y_j)$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The following theorem follows readily.

The triangles $P_1P_2P_3$ and $Q^1Q^2Q^3$ are perspective from the point $T = \cap P_iQ^i$, $i = 1, 2, 3$.

The restriction to P being in the interior of the triangle is unnecessary: see the solution of *Crux* 1541 [1991: 189]. For alternative formulations, see [1] or [2]. We are told that the theorem also appears in [8] and [10]. (We thank the referees for supplying references [1], [2], [8] and [10] which were previously unknown to us.)

For Groenman's problem **1272**, P is the Gergonne point and Q is the incentre, while for **1295**, P is the orthocentre and Q is again the incentre. Also noted in [6] is the fact that the point in **1272** is the isogonal conjugate of the mittenpunkt. This problem was discovered at the proofreading stage of [6] and thus was able to be included.

It is interesting that Groenman seemed to be approaching the same generalization. In his solution to **1295** he has P in general position while Q is still the incentre. A related class of points referred to by Nagel as *interior mittenpunkts* (defined by replacing one of the excentres by the incentre and interchanging the other two) is also given in [9]. This class is also generalized in [6]. A dual notion for T, the *mittenlinie* (middlesline), is given in [5].

For those readers who have access to a symbolic manipulation program, the following is an elementary exercise.

Under what circumstances are P, Q, and T collinear?

If the trilinear coordinates of P and Q are (z_i) and (y_i) respectively, $i = 1, 2, 3$, then it is not too difficult to show that the coordinates of $T(x_i)$ are

$$\left(y_1\left(-\frac{y_1}{z_1} + \frac{y_2}{z_2} + \frac{y_3}{z_3}\right), y_2\left(\frac{y_1}{z_1} - \frac{y_2}{z_2} + \frac{y_3}{z_3}\right), y_3\left(\frac{y_1}{z_1} + \frac{y_2}{z_2} - \frac{y_3}{z_3}\right)\right).$$

The collinearity condition then leads to the equation

$$(y_1z_2 - y_2z_1)(y_2z_3 - y_3z_2)(y_3z_1 - y_1z_3) = 0.$$

Hence $y_i/y_{i+1} = z_i/z_{i+1}$ for some i, and thus P, Q, T must lie on a line through a vertex of the given triangle.

An exhaustive search for some other references to this point has met with only one partial success. In Gallatly [7], it is mentioned only as the Lemoine point Λ of the triangle $I_1I_2I_3$. The coordinates of Λ are given as $(s - a_1, s - a_2, s - a_3)$, thus it is Nagel's mittenpunkt but under another guise. I am surprised that a point with such a pretty and ready construction seems not to have found its way into the modern geometry of the triangle. Perhaps the reader can shed some new light on the problem.

References:

[1] N. Altshiller-Court, *College Geometry*, supplementary exercise 7, page 165.

[2] Aubert et Papelier, *Exercices de Géométrie Analytique*, Vol. 1, 10th ed., Paris, 1957, problem 52, p. 35.

[3] P Baptist, Über Nagelsche Punktepaare, *Mathematische Semesterberichte* 35, 1988, 118–126.

[4] H.S.M. Coxeter, *The Real Projective Plane*, 2nd Ed., Cambridge University Press, 1961.

[5] R.H. Eddy, A Desarguesian dual for Nagel's middlespoint, *Elem. Math.* **44**/3 (1989) 79,80.

[6] R.H. Eddy, A generalization of Nagel's middlespoint, *Elem. Math.* **45**/1 (1990) 14–18.

[7] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd Ed., Francis Hodgson, London, circa 1920.

[8] S. Iwata, *Encyclopedia of Geometry*, Vol. 3, 1976, problem 676. (Japanese)

[9] C.H. von Nagel, *Untersuchugen über die wichtigsten zum Dreiecke gehörenden Kreise*, Leipzig, 1836.

[10] Y. Sawayama, *Tokyo Buturi Gakko*, 1903.

[11] Problem 1359, proposed by G.R. Veldkamp, solution (II) by R.H. Eddy, *Crux Math.* **15** (1989) 244–246.

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* * * * *

THE OLYMPIAD CORNER

No. 127

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

In recent years the September number of the Corner has been the issue in which the year's IMO is discussed. I rely on our readers and my colleagues who may attend the IMO for information, and there is frequently a panic and some delay in getting the September number out since mathematicians are notorious for forgetting to carry out routine tasks like sending off copies of questions and so on. We have also been experiencing some delays with the switch over to LaTeX and so we have decided to hold over to a later issue the 1991 IMO results. We begin this number with problems submitted to the jury but not used for the 31st IMO in China. I am indebted to Professor Andy Liu of the University of Alberta, who was involved with training the Chinese team, for taking the time to send me these problems.

1. *Proposed by Australia.*

The integer 9 can be written as a sum of two consecutive integers: $9 = 4 + 5$; moreover, it can be written as a sum of more than one consecutive integer in exactly two ways, namely $9 = 4 + 5 = 2 + 3 + 4$. Is there an integer which can be written as a sum of 1990 consecutive integers and which can be written as a sum of more than one consecutive integer in exactly 1990 ways?

2. *Proposed by Canada.*

Given n countries with 3 representatives each, a list of m committees $A(1), A(2), \ldots, A(m)$ is called a *cycle* if

(1) each committee has n members, one from each country;

(2) no two committees have the same membership;

(3) for $1 \leq i \leq m$, committee $A(i)$ and committee $A(i+1)$ have no member in common, where $A(m+1)$ denotes $A(1)$;

(4) if $1 < |i-j| < m-1$, then committees $A(i)$ and $A(j)$ have at least one member in common.

Is it possible to have a cycle of 1990 committees with 11 countries?

3. *Proposed by Czechoslovakia.*

Assume that the set of all positive integers is decomposed into r disjoint subsets $\mathbf{N} = A_1 \cup \cdots \cup A_r$. Prove that one of them, say A_i, has the following property: there exists a positive integer m such that for any k, one can find numbers $a_1, a_2, \ldots, a_k$ in A_i with $0 < a_{j+1} - a_j \leq m$, $1 \leq j \leq k-1$.

4. *Proposed by France.*

Given ΔABC with no side equal to another side, let G, K and H be its centroid, incentre and orthocentre, respectively. Prove that $\angle GKH > 90^\circ$.

5. *Proposed by Greece.*

Let $f(0) = f(1) = 0$ and

$$f(n+2) = 4^{n+2} f(n+1) - 16^{n+1} f(n) + n2^{n^2},$$

$n = 0, 1, 2, \ldots$. Show that the numbers $f(1989)$, $f(1990)$ and $f(1991)$ are divisible by 13.

6. *Proposed by Hungary.*

For a given positive integer k, denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

7. *Proposed by Iceland.*

A plane cuts a right circular cone into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the ratio of the volume of the smaller part to the volume of the whole cone.

8. *Proposed by Ireland.*

Let ABC be a triangle and ℓ the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line ℓ at E. Let the internal bisector of the angle at B meet the side AC at F and the line ℓ at G. If $GF = DE$ prove that $AC = BC$.

9. *Proposed by Japan.*

On the coordinate plane a rectangle with vertices $(0,0)$, $(m,0)$, $(0,n)$ and (m,n) is given where both m and n are odd integers. The rectangle is partitioned into triangles in such a way that

(1) each triangle in the partition has at least one side, to be called a "good" side, which lies on a line of the form $x = j$ or $y = k$ where j and k are integers, and the altitude on this side has length 1;

(2) each "bad" side is a common side of two triangles in the partition.

Prove that there exist at least two triangles in the partition each of which has two "good" sides.

10. *Proposed by Mexico.*

Determine for which positive integers k the set $X = \{1990, 1991, 1992, \ldots, 1990+k\}$ can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

11. *Proposed by the Netherlands.*

Unit cubes are made into beads by drilling a hole through them along a diagonal. The beads are put on a string in such a way that they can move freely in space under the restriction that the vertices of two neighbouring cubes are touching. Let A be the beginning vertex and B the end vertex. Let there be $p \times q \times r$ cubes on the string where $p, q, r \geq 1$.

(a) Determine for which values of p, q and r it is possible to build a block with dimensions p, q and r. Give reasons for your answer.

(b) The same as (a) with the extra condition that $A = B$.

* * *

The Canadian Mathematics Olympiad for 1991 saw two students tie for first prize. One of them went on to place in the top eight for the USAMO. The top prize winners are listed below:

First Prize	Ian Goldberg J.P. Grossman
Second Prize	Janos Csirik
Third Prize	Adam Logan
Fourth Prize	Jie Lou Kevin Kwan Peter Milley Mark Van Raamsdonk

The "official" solutions to the 1991 CMO are reproduced below with the permission of the CMO Committee of the Canadian Mathematical Society.

1991 CANADIAN MATHEMATICS OLYMPIAD

April 1991
Time: 3 hours

1. Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

Solution. Two solutions found by inspection are $(x, y, z) = (3, -1, 2)$ and $(10, 3, 7)$. Suppose a solution $(x, y, z) = (u, v, w)$ is given. Then for any positive integer k, $(x, y, z) = (k^{15}u, k^6v, k^{10}w)$ is also a solution.

2. Let n be a fixed positive integer. Find the sum of all positive integers with the following property: in base 2, it has exactly $2n$ digits consisting of n 1's and n 0's. (The first digit cannot be 0.)

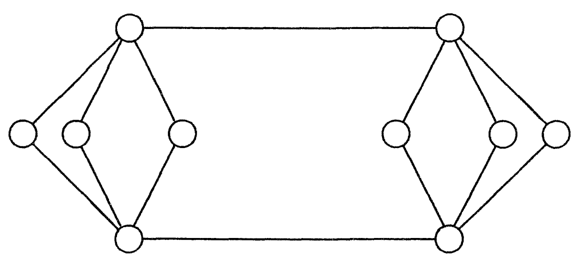
Solution. When $n = 1$, the sum is clearly 2. Let $n \geq 2$. The left digit is 1 and there are $\binom{2n-1}{n}$ possibilities for arranging $(n-1)$ 1's and n 0's in the other $2n-1$ digital positions. Consider any digital position but the first. The digit in it is 1 for $\binom{2n-2}{n}$ of the numbers and 0 for $\binom{2n-2}{n-1}$ of the numbers, so that the sum of the digits in this position is $\binom{2n-2}{n}$. Hence the sum of all the numbers is

$$\binom{2n-2}{n}(1 + 2 + \cdots + 2^{2n-2}) + \binom{2n-1}{n}2^{2n-1} = \binom{2n-2}{n}(2^{2n-1} - 1) + \binom{2n-1}{n}2^{2n-1}.$$

3. Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle.

Solution. Let O be the centre of circle C and X be the midpoint of a chord through P. Then OX is perpendicular to XP, so that X lies on that portion of the circle with diameter OP which lies within C. (When $P = O$, the locus degenerates to the single point O.)

4. Ten distinct numbers from the set $\{0, 1, 2, \ldots, 13, 14\}$ are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.



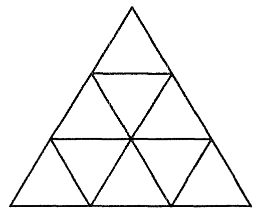
Solution. Observe that each circle has an even number of edges emanating from it. Suppose the task were possible. Then the absolute values of the differences must be

$1, 2, \ldots, 13, 14$, so that their sum is 105, an odd number. Denoting the numbers entered in the circles by x_i $(1 \le i \le 10)$, we have that

$$105 = \sum_{i<j} |x_i - x_j| \equiv \sum_{i<j} (x_i - x_j) \equiv \sum_{i<j} (x_i + x_j) \bmod 2.$$

In the final sum, each x_i is counted as often as a segment emanates from its circle, an even number of times. This yields a contradiction. Hence the task is not possible.

5. In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for $f(n)$, the number of parallelograms, for a triangle of side length n.



Solution. By symmetry, the number of parallelograms is three times the number of parallelograms whose sides are parallel to the slant sides of the triangle. Suppose we enlarge the triangle by adding an additional "subbase" line with $n + 2$ dots below its base. If we extend the sides of each parallelogram to be counted, it will meet the subbase at four distinct points. Conversely, for any choice of four points from the $n + 2$ subbase points, we can form a corresponding parallelogram by drawing lines parallel to the left slant side through the left two points and lines parallel to the right slant side through the right two points. Thus, there is a one–one correspondence between parallelograms and choices of four dots. Therefore, the total number of parallelograms of all possible orientations is $3\binom{n+2}{4}$.

* * *

The prize winners for the 1991 USA Olympiad follow. A total of 139 students from 118 schools were invited to participate.

J.P. Grossman	Toronto
Ruby Breydo	New York
Kiran Kedlaya	Washington
Joel Rosenberg	West Hartford
Robert Kleinberg	Elma
Lenhard Ng	Chapel Hill
Michail Sunitsky	New York
Dean Chung	Mountain Lakes

Notice that J.P. Grossman was also a winner of the CMO. Apparently the eligibility rules for the USAMO are changing next year to exclude Canadian entries.

* * *

In the April number of *Crux*, E.T.H. Wang asked if three was a record for the number of times one problem has reappeared in different Olympiads. Andy Liu responded by pointing me to the note "The art of borrowing problems" by René Laumen in the *World Federation Newsletter*, No. 6, August 1987. There Laumen discusses eight problems that have been used more than once, one repeated three times, but three still seems to be the record.

Andy also noticed that we didn't publish the solution to Question 2 of the *1987 Asian Pacific Mathematical Olympiad.* To complete the picture he sends in the following solution. Interestingly, the published "official solution" contains an error.

2. [1989: 131] *1989 Asian Pacific Mathematical Olympiad.*
Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solution in integers except $a = b = c = n = 0$.

Solution by Andy Liu, University of Alberta.
We must have that $6|n$, and then $3|c$. Hence $2a^2 + b^2 + 3d^2 = 10m^2$ where $n = 6m$ and $c = 3d$. If the original equation has a non-trivial solution, then this equation has one with $\gcd(a, b, d, m) = 1$. Clearly $b \equiv d \bmod 2$. Now the quadratic residues modulo 16 are 0, 1, 4 and 9. The following table gives the possibilities for $2a^2 + b^2 + 3d^2 \bmod 16$, depending on the parities of a, b and d:

	b and d odd $b^2 \equiv 1$ or 9, $3d^2 \equiv 3$ or $11 \bmod 16$	b and d even $b^2 \equiv 0$ or 4, $3d^2 \equiv 0$ or $12 \bmod 16$
a odd $2a^2 \equiv 2 \bmod 16$	$2a^2 + b^2 + 3d^2 \equiv 6$ or $14 \bmod 16$	$2a^2 + b^2 + 3d^2 \equiv 2, 6$ or $14 \bmod 16$
a even $2a^2 \equiv 0$ or $8 \bmod 16$	$2a^2 + b^2 + 3d^2 \equiv 4$ or $12 \bmod 16$	$2a^2 + b^2 + 3d^2 \equiv 0, 4, 8$ or $12 \bmod 16$

If m is odd then $10m^2 \equiv 10 \bmod 16$, and we cannot have $2a^2 + b^2 + 3d^2 = 10m^2$. Hence m is even so that $10m^2 \equiv 0$ or $8 \bmod 16$. From the above table, we must have a, b, d all even. However this contradicts that $\gcd(a, b, d, m) = 1$. It follows that $6(6a^2+3b^2+c^2) = 5n^2$ has no solution in integers except $a = b = c = n = 0$.

* * *

We turn now to solutions to problems from the "Archives".

2. [1986: 202] *1986 USAMO Training Session.*
Determine the maximum value of

$$x^3 + y^3 + z^3 - x^2y - y^2z - z^2x$$

for $0 \leq x, y, z \leq 1$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.
Let $f(x, y, z)$ denote the given expression. We show that $f(x, y, z) \leq 1$ with equality if and only if (x, y, z) equals one of the six triples: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

Suppose first that $0 \leq x \leq y \leq z \leq 1$. Then

$$1 - f(x, y, z) = (1 - z^3) + x^2(y - x) + y^2(z - y) + z^2x \geq 0$$

with equality if and only if (i) $z = 1$, (ii) $x = 0$ or $y = x$, (iii) $y = 0$ or $z = y$, and (iv) $z = 0$ or $x = 0$. From (i) and (iv) we get $x = 0$, $z = 1$ which together with (ii) and (iii) immediately yields $(x,y,z) = (0,0,1)$ or $(0,1,1)$.

Next suppose that $0 \le y \le x \le z \le 1$. Then

$$1 - f(x,y,z) = (1-z^3) + x(z^2 - x^2) + y(x^2 - y^2) + y^2 z \ge 0$$

with equality just when (i) $z = 1$, (ii) $x = 0$ or $x = z$, (iii) $y = 0$ or $x = y$, and (iv) $y = 0$ or $z = 0$. From (i) and (iv) we conclude that $z = 1$, $y = 0$. Now (ii) gives the two solutions $(x,y,z) = (0,0,1)$ and $(1,0,1)$.

Exploiting the cyclic symmetry of $f(x,y,z)$ we may reduce to one of these two cases, and this gives the six solutions listed.

*

2. [1986: 230] *1986 Austrian Mathematical Olympiad.*

For $s, t \in \mathbf{N}$, let

$$M = \{(x,y) | 1 \le x \le s,\ 1 \le y \le t,\ x, y \in \mathbf{N}\}$$

be a given set of points in a plane. Determine the number of rhombuses whose vertices belong to M and whose diagonals are parallel to the x, y coordinate axes.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

The diagonals of the rhombuses must have even lengths. Consider t points spaced at unit intervals on a line. In how many ways can one select two points at even distance? There are $t-2$ pairs with distance 2, $t-4$ pairs with distance 4, and so on. This sums to give $[(t-1)^2/4]$ where $[x]$ is the integer part of x. One gets all rhombuses by combining every t-pair with every s-pair giving

$$\left[\frac{(t-1)^2}{4}\right] \cdot \left[\frac{(s-1)^2}{4}\right] .$$

3. [1986: 230; 1988: 165] *1986 Austrian Mathematical Olympiad.*

Determine the set of all values of $x_0, x_1 \in \mathbf{R}$ such that the sequence defined by

$$x_{n+1} = \frac{x_{n-1}x_n}{3x_{n-1} - 2x_n}\ , \qquad n \ge 1$$

contains infinitely many natural numbers.

Alternate solution by G.R. Veldkamp. De Bilt, The Netherlands.

From the recurrence relation, $x_n = 0$ gives $x_{n+1} = 0$ and the sequence breaks down. So we assume $x_n \ne 0$. Now put $u_n = 1/x_n$. The recurrence relation now gives $u_{n+2} - 3u_{n+1} + 2u_n = 0$, with characteristic equation $t^2 - 3t + 2 = 0$, having 1 and 2 as its roots. Hence $u_n = a + 2^n b$, where a and b do not depend on n. If $b \ne 0$ we get $\lim_{n\to\infty} |u_n| = \infty$ and so x_n tends to zero in which case the sequence is an integer only finitely often. Thus u_n and hence x_n are constants. The answer is then $x_0 = x_1 \in \mathbf{N} - \{0\}$.

*

5. [1986: 231] *1986 Austrian–Polish Mathematical Competition.*

Determine all quadruples (x, y, u, v) of real numbers satisfying the simultaneous equations

$$x^2 + y^2 + u^2 + v^2 = 4,$$
$$xu + yv = -xv - yu,$$
$$xyu + yuv + uvx + vxy = -2,$$
$$xyuv = -1.$$

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

The second equation can be rewritten

$$(x + y)(u + v) = 0$$

and the third is $xy(u + v) + (x + y)uv = -2$.

Case 1. $x + y = 0$. Then the last two equations give $x^2(u + v) = 2$ and $x^2uv = 1$. From this $u + v = 2uv$ and $x^2 = y^2 = 2/(u + v)$. Using the first equation,

$$\frac{4}{u+v} + (u + v)^2 - (u + v) = 4.$$

Setting $z = u + v$ we obtain

$$(z - 1)(z - 2)(z + 2) = z^3 - z^2 - 4z + 4 = 0.$$

This now gives three subcases.

Subcase (i): $u + v = 1 = 2uv$, no real solution.

Subcase (ii): $u + v = 2 = 2uv$, giving $u = 1$, $v = 1$, $x = \pm 1$, $y = \pm 1$ and solutions $(1, -1, 1, 1)$, $(-1, 1, 1, 1)$.

Subcase (iii): $u + v = -2 = 2uv$, and no real solution for x.

Case 2. $u + v = 0$. By symmetry we obtain $(1, 1, -1, 1)$ and $(1, 1, 1, -1)$.

8. [1986: 232] *1986 Austrian–Polish Mathematical Competition.*

An $m \times n$ matrix of distinct real numbers is given. The elements of each row are rearranged (in the same row) such that the elements are increasing from left to right. Next the elements of each column are rearranged (in the same column) such that the elements are increasing from top to bottom. Show that now the elements in each row are still increasing from left to right.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

This problem is essentially the same as question 2 on the 1980 Canadian Mathematics Olympiad where a 5×10 matrix was considered instead of an $m \times n$ matrix. The published official solution which was actually for the general case can be found on page 26 in "Report of the Twelfth CMO" (published by the CMS) [and also in *Crux* [1980: 241]].

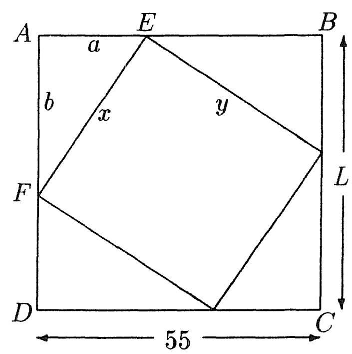
* * *

Now we turn to the November 1989 number of *Crux* and the rest of the problems proposed but not used at the 1989 IMO. The "official" solutions can be found in the booklet *30th International Mathematical Olympiad, Braunschweig, 1989,* edited by Hanns-Heinrich Langmann.

12. [1989: 260] *Proposed by Australia.*

Ali Baba the carpet merchant has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the rooms are integral numbers of feet, and that his two storerooms have the same (unknown) length, but widths of 38 and 55 feet respectively. What are the carpet's dimensions?

Correction by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.



Let the carpet have length y, width x. Let the length of the storerooms be L, and set $y/x = k$. The diagram shows the larger storeroom. By similar triangles, $BE/y = b/x$ or $BE = kb$. Similarly $DF = ka$. Now $a + kb = 55$ and $ka + b = L$, so

$$a = \frac{kL - 55}{k^2 - 1}, \quad b = \frac{55k - L}{k^2 - 1}.$$

This gives

$$x^2 = \left(\frac{kL - 55}{k^2 - 1}\right)^2 + \left(\frac{55k - L}{k^2 - 1}\right)^2$$

so

$$(k^2 - 1)^2 x^2 = (kL - 55)^2 + (55k - L)^2.$$

In the same way, the other room yields

$$(k^2 - 1)^2 x^2 = (kL - 38)^2 + (38k - L)^2.$$

Equating we have

$$(38k - L)^2 + (Lk - 38)^2 = (55k - L)^2 + (Lk - 55)^2.$$

The discriminant of this equation (after removing square factors) is $D = 4L^2 - 93^2 \equiv 3 \bmod 4$, so this equation has no rational solution.

There must be a misprint in the problem and it is likely that the widths should be 38 and 50 instead of 38 and 55. The equation then becomes

$$(38k - L)^2 + (Lk - 38)^2 = (50k - L)^2 + (Lk - 50)^2$$

which simplifies to $22k^2 - Lk + 22 = 0$. The discriminant is $D = L^2 - 44^2$ and this gives rise to rational solutions.

Editor's Note. The "official" solution in the IMO booklet starts out with the assumption that the widths are 38 and 50, even though the problem is stated as above, with widths of 38 and 55!

14. [1989: 260] *Proposed by Hungary.*

At n distinct points of a circle-shaped race course there are n cars ready to start. They cover the circle in an hour. Hearing the signal each of them selects a direction and starts immediately. If two cars meet both of them change directions and go on without loss of speed. Show that at a certain moment each car will be at its starting point.

Solution by Curtis Cooper, Central Missouri State University, Warrensburg.

We will show that after n hours, each car will be back at its starting point. Consider another car race such that if two cars meet they pass through each other instead of changing directions. Although the races are different, the "passing through" car race and the "changing direction" car race are similar in that at any given time, the points on the course occupied by cars are the same and the directions the cars at these points are moving are the same. After one hour of the "passing through" car race, each car is back at its starting point and going in its starting direction. Thus, although the cars may not be the same, we see that after one hour of the "changing direction" car race, there will be a car at each starting point and that this car will be going in the same direction that the starting car at that point went. In addition, although we may not know which car is at each starting point in one hour, we do know that the cars in the "changing direction" car race maintain the same relationship with the other cars. That is, in the "changing direction" car race, the cars in the clockwise direction and counterclockwise direction with respect to a given car are always the same. Therefore, after one hour of the "changing direction" race, the cars at the starting points on the course are some rotation (possibly $0°$) of the original cars at the starting points on the course and are going in the same directions the original cars were going. Thus after n hours, the cars in the "changing direction" car race will be back at the starting point.

* * *

In the remaining space for this month's number of the Corner, we given solutions sent in by readers to problems of the *11th Austrian-Polish Mathematics Competition* that was given in the December 1989 number of *Crux* [1989: 289–291].

1. Let $P(x)$ be a polynomial having integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.

Generalization and solution by David Singmaster, South Bank Polytechnic, London, England.

Define $M(2k) = (k!)^2$ and $M(2k+1) = k!(k+1)!$, where k is a positive integer. We show:

THEOREM. If $Q(x)$ has integer coefficients and k distinct integer roots and if $P(x) = Q(x) - y$ has an integral root with $y \neq 0$, then $|y| \geq M(k)$.

The given problem has $k = 6$, $y = 12$, so that $M(k) = 36 \not\leq |y|$, and the problem follows.

To prove the theorem, let the given roots of $Q(x)$ be $x_1, x_2, \ldots, x_k$ and let

$$B(x) = (x - x_1)(x - x_2)\ldots(x - x_k).$$

We know $Q(x)$ is a multiple of $B(x)$, say $Q(x) = A(x)B(x)$.

LEMMA. $A(x)$ has integral coefficients.

Proof. Let

$$A(x) = \sum_{i=0}^{n} a_i x^i,\ B(x) = \sum_{i=0}^{k} b_i x^i,\ Q(x) = \sum_{i=0}^{n+k} q_i x^i.$$

We know $q_i \in \mathbf{Z}$, $b_i \in \mathbf{Z}$ and $b_k = 1$. Thus $q_{n+k} = a_n b_k$ implies $a_n \in \mathbf{Z}$. Then $q_{n+k-1} = a_n b_{k-1} + a_{n-1} b_k$ gives us $a_{n-1} = q_{n+k-1} - a_n b_{k-1}$ and $a_{n-1} \in \mathbf{Z}$. Etc. □

Now suppose $P(x) = Q(x) - y$ has an integral root x_0. Since $y \neq 0$, x_0 is distinct from all the x_i and $P(x_0) = Q(x_0) - y = A(x_0)B(x_0) - y = 0$, so $y = A(x_0)B(x_0)$. From the lemma, we know $A(x_0)$ is an integer, so

$$B(x_0) = (x_0 - x_1)\ldots(x_0 - x_k)|y. \tag{1}$$

The values $x_0 - x_i$ are k distinct integers so the least absolute value of such a product is $1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdots$ (with k terms) $= M(k)$; hence $|y| \geq M(k)$. This proves the theorem.

In place of 12, the original problem could have had any integer in the range $[1, 35]$.

Editor's Note. The problem was also solved by Seung-Jin Bang, Seoul, Republic of Korea; by Curtis Cooper, Central Missouri State University, Warrensburg; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

4. Determine all strictly monotone increasing functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the functional equation

$$f(f(x) + y) = f(x + y) + f(0)$$

for all $x, y \in \mathbf{R}$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by George Evagelopoulos, Athens, Greece; and by David C. Vaughan, Wilfrid Laurier University, Waterloo, Ontario.

Let f be a strictly increasing function on $\mathbf{R}$ satisfying the functional equation. In particular, if we set $y = -x$, we must have $f(f(x) - x) = f(0) + f(0) = 2f(0)$. Unless $f(x) - x$ is a constant we have a contradiction to the fact that f is $1-1$ since $f(f(x) - x)$ is constant. Thus $f(x) = x + c$ where $c = f(0)$. Any such f is strictly increasing, and we have

$$f(f(x) + y) = f(x + c + y) = (x + y + c) + c = f(x + y) + f(0),$$

that is, any such f is a solution.

7. In a regular octagon each side is coloured blue or yellow. From such a colouring another colouring will be obtained "in one step" as follows: if the two neighbours of a side have different colours, the "new" colour of the side will be blue, otherwise the colour will be yellow. [*Editor's note:* the colours are modified simultaneously.] Show that after a finite number, say N, of moves all sides will be coloured yellow. What is the least value of N that works for all possible colourings?

Solution by Curtis Cooper, Central Missouri State University.

Consider a colouring of the sides of a regular octagon where each side is coloured blue or yellow. Label the sides of the octagon $S_1, \ldots, S_8$ and code the colours of the sides of the regular octagon by the column vector $\mathbf{x} = (x_1, \ldots, x_8)$, where

$$x_i = \begin{cases} 1 & \text{if } S_i \text{ is coloured blue,} \\ 0 & \text{if } S_i \text{ is coloured yellow,} \end{cases}$$

for $i = 1, 2, \ldots, 8$. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $A^n\mathbf{x}$ is the coded colouring of the sides of the regular octagon after n steps, where the addition and multiplication in the matrix operations are performed mod 2. Since $A^4 = 0$ it follows that after 4 steps all sides will be coloured yellow. This is the least possible value of N that works for all possible colourings, since for $\mathbf{x} = (1, 0, 0, 0, 0, 0, 0, 0)^T$,

$$A\mathbf{x} = (0, 1, 0, 0, 0, 0, 0, 1)^T, \; A^2\mathbf{x} = (0, 0, 1, 0, 0, 0, 1, 0)^T, \; A^3\mathbf{x} = (0, 1, 0, 1, 0, 1, 0, 1)^T.$$

* * *

This completes the space for this month. Send in your nice solutions and your Olympiads.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1992**, *although solutions received after that date will also be considered until the time when a solution is published.*

1661. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ΔABC is inscribed in a circle Γ_1. Let D be a point on BC produced such that AD is tangent to Γ_1 at A. Let Γ_2 be a circle which passes through A and D, and is tangent to BD at D. Let E be the point of intersection of Γ_1 and Γ_2 other than A. Prove that $\overline{EB} : \overline{EC} = \overline{AB}^3 : \overline{AC}^3$.

1662. *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that

$$\frac{x_1^{2r+1}}{s-x_1} + \frac{x_2^{2r+1}}{s-x_2} + \cdots + \frac{x_n^{2r+1}}{s-x_n} \geq \frac{4^r}{(n-1)n^{2r-1}}(x_1x_2 + x_2x_3 + \cdots + x_nx_1)^r,$$

where $n > 3$, $r \geq 1/2$, $x_i \geq 0$ for all i, and $s = x_1 + x_2 + \cdots + x_n$. Also, find some values of n and r such that the inequality is sharp.

1663*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, C be the angles of a triangle, r its inradius and s its semiperimeter. Prove that

$$\sum \sqrt{\cot(A/2)} \leq \frac{3}{2}\sqrt{r/s}\sum \csc(A/2),$$

where the sums are cyclic over A, B, C.

1664. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.* (Dedicated to Jack Garfunkel.)

Consider two concentric circles with radii R_1 and R ($R_1 > R$) and a triangle ABC inscribed in the inner circle. Points A_1, B_1, C_1 on the outer circle are determined by extending BC, CA, AB, respectively. Prove that

$$\frac{F_1}{R_1^2} \geq \frac{F}{R^2},$$

where F_1 and F are the areas of triangles $A_1B_1C_1$ and ABC respectively, with equality when ABC is equilateral.

1665. *Proposed by P. Penning, Delft, The Netherlands.*

1665^n ends in 5 for $n \geq 1$, and in 25 for $n \geq 2$. Find the longest string of digits which ends 1665^n for all sufficiently large n.

1666. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

(a) How many ways are there to select and draw a triangle T and a quadrilateral Q around a common incircle of unit radius so that the area of $T \cap Q$ is as small as possible? (Rotations and reflections of the figure are not considered different.)

(b)* The same question, with the triangle and quadrilateral replaced by an m-gon and an n-gon, where $m, n \geq 3$.

1667. *Proposed by Stephen D. Hnidei, Windsor, Ontario.*
Evaluate

$$\sum_{n=1}^{\infty} \coth^{-1}(2^{n+1} - 2^{-n}).$$

1668. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*
What is the envelope of the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as a and b vary so that $a^2 + b^2 = 1$?

1669. *Proposed by J.P. Jones, University of Calgary.*
Suppose that q is a rational number with $|q| \leq 2$, and that

$$\frac{q + i\sqrt{4 - q^2}}{2}$$

is an nth root of unity for some n. Show that q must be an integer.

1670*. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let B_1, B_2, C_1, C_2 be points in the plane and let lines B_1B_2 and C_1C_2 intersect in A. Prove that the four points $G_{11}, G_{12}, G_{21}, G_{22}$ form the vertices of a parallelogram when G_{ij} is determined in any of the following ways: (i)G_{ij} is the centroid of ΔAB_iC_j; (ii) G_{ij} is the orthocenter of ΔAB_iC_j; (iii) G_{ij} is the circumcenter of ΔAB_iC_j.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1542*. [1990: 143] *Proposed by Murray S. Klamkin, University of Alberta.*
For fixed n, determine the minimum value of

$$C_n = |\cos\theta| + |\cos 2\theta| + \cdots + |\cos n\theta|.$$

It is conjectured that $\min C_n = [n/2]$ for $n > 2$.

Solution by Manuel Benito Muñoz and Emilio Fernández Moral, I.B. Sagasta, Logroño, and Maria Mercedes Sánchez Benito, I.B. Luis Bunuel, Alcorcón, Madrid, Spain.

We will prove that the conjecture is true for all $n > 2$ **except** for $n = 4$ and $n = 6$.

Consider the function

$$F_n(x) = \sum_{j=1}^{n} \left|\cos\left(jx\frac{\pi}{2}\right)\right|.$$

We will first prove that

$$(*) \quad \textit{for } n \geq 8 \textit{ with } n \textit{ even}, F_n(x) \geq n/2;$$

this proves the conjecture for *all* $n \geq 8$, since then

$$\min F_{n+1}(x) \geq \min F_n(x) = \frac{n}{2} = \left[\frac{n+1}{2}\right],$$

and we have

$$F_{n+1}(1) = F_n(1) + \left|\cos\frac{(n+1)\pi}{2}\right| = F_n(1) = \frac{n}{2}\,.$$

Finally, we verify that the conjecture is true for $n = 3,\ 5$ and 7:

$$F_3(x) \geq F_3(1) = 1, \quad F_5(x) \geq F_5(1) = 2, \quad F_7(x) \geq F_7(1) = 3.$$

However the minimum values for $n = 4$ and $n = 6$ are respectively

$$\min F_4(x) = F_4(1/3) = 1 + \frac{\sqrt{3}}{2} < 2,$$
$$\min F_6(x) = F_6(1/5) \approx 2.97 < 3.$$

We begin with the following lemma.

LEMMA 1. (a) $F_n(x+2) = F_n(x)$. (This allows us to study the function on $[0,2]$ only.)

(b) $F_n(1-x) = F_n(1+x)$. (This allows us to study the function on $[0,1]$ only.)

(c) $F_n(x)$ attains its relative minimum values (in the interval $[0,1]$) on points of the Farey sequence of order n (i.e., irreducible fractions in $[0,1]$ with denominator at most n) having odd numerator.

Proof. (a) Follows from

$$\left|\cos\frac{j(x+2)\pi}{2}\right| = \left|\cos\left(\frac{jx\pi}{2} + j\pi\right)\right| = \left|\cos\frac{jx\pi}{2}\right|.$$

(b) Follows from

$$\left|\cos\frac{j(1-x)\pi}{2}\right| = \left|\cos\left(\frac{j\pi}{2} - \frac{jx\pi}{2}\right)\right| = \left|\cos\left(\frac{j\pi}{2} + \frac{jx\pi}{2}\right)\right|.$$

(c) $F_n(x)$ has a derivative at all x except those x for which $\cos(jx\pi/2) = 0$ for some $j \in \{1, 2, \ldots, n\}$. On the interval $[0,1]$, these points will be of the form $x = (2k+1)/j$ with $j = 1, 2, \ldots, n$ and k an integer with $0 < 2k+1 < j$, that is, fractions of the Farey sequence of order n with odd numerator. If a/b, c/d are two adjacent fractions (with odd numerator) in the Farey sequence of order n, $F_n(x)$ is differentiable on the interval $(a/b, c/d)$ and, if we call $\operatorname{sgn}(j)$ the constant sign of $\cos(jx\pi/2)$ on this interval, then

$$F_n'(x) = -\sum_{j=1}^{n} \operatorname{sgn}(j) \cdot \frac{j\pi}{2} \sin\left(\frac{jx\pi}{2}\right),$$

$$F_n''(x) = -\sum_{j=1}^{n} \operatorname{sgn}(j) \cdot \frac{j^2\pi^2}{4} \cos\left(\frac{jx\pi}{2}\right) = \frac{-\pi^2}{4} \sum_{j=1}^{n} j^2 \left|\cos \frac{jx\pi}{2}\right| < 0.$$

Hence $F_n(x)$ can have only relative maximum points inside the interval, and the minimum value on $[a/b, c/d]$ will be attained at one of the endpoints. □

By this lemma, to prove $F_n(x) \geq n/2$ it suffices to verify that $F_n(a/b) \geq n/2$ for $0 < a < b \leq n$, a odd and a prime to b. (It is clear that $F_n(0)$ and $F_n(1)$ are both $\geq n/2$.)

LEMMA 2. If a and b are positive integers with a odd and prime to b, then $F_b(a/b) = F_b(1/b)$.

Proof. If a is odd and prime to b, then a is prime to $2b$. Thus for $j = 1, 2, \ldots, b$, we can write $ja = 2bq_j + k_j$ with q_j, k_j integers, $1 \leq |k_j| \leq b$. It is easy to see that $\{|k_j| : j = 1, 2, \ldots, b\} = \{1, 2, \ldots, b\}$ [if $|k_j| = |k_\ell|$, then $ja - 2bq_j = \pm(\ell a - 2bq_\ell)$, so $2b(q_j \pm q_\ell) = a(j \pm \ell)$; now $(a, 2b) = 1$ means $2b|(j \pm \ell)$ which is impossible if $j \neq \ell$]. Consequently

$$\begin{aligned} F_b\left(\frac{a}{b}\right) &= \sum_{j=1}^{b} \left|\cos \frac{ja\pi}{2b}\right| = \sum_{j=1}^{b} \left|\cos \frac{(2bq_j + k_j)\pi}{2b}\right| = \sum_{j=1}^{b} \left|\cos\left(q_j\pi + \frac{k_j\pi}{2b}\right)\right| \\ &= \sum_{j=1}^{b} \left|\cos \frac{k_j\pi}{2b}\right| = \sum_{j=1}^{b} \left|\cos \frac{|k_j|\pi}{2b}\right| = \sum_{j=1}^{b} \left|\cos \frac{j\pi}{2b}\right| = F_b\left(\frac{1}{b}\right). \quad \square \end{aligned}$$

As we shall see later, this lemma will reduce our proofs to searching the values of $F_n(x)$ for $x = 1/b$ with $b \leq n$.

Till now we had in mind that n was a fixed number and b was a variable number less than n. Now we must change this point of view, and we consider $b \geq 3$ fixed and n variable.

PROPOSITION 1. Let $b \geq 3$ be an integer. If $b \leq n < 2b$ with n even, then $F_n(1/b) > n/2$ except for the cases

$$F_4(1/3) \approx 1.87 < 2 \quad \text{and} \quad F_6(1/5) \approx 2.97 < 3.$$

Proof. $F_n(1/b) > n/2$ is equivalent to: the average of the terms

$$\left|\cos \frac{j\pi}{2b}\right|, \ j = 1, 2, \ldots, n,$$

is greater than $1/2$. Note that

$$\left|\cos \frac{j\pi}{2b}\right| \begin{cases} > 1/2 & \text{for } 1 \leq j < 2b/3, \\ < 1/2 & \text{for } 2b/3 < j < 4b/3, \\ > 1/2 & \text{for } 4b/3 < j < n. \end{cases}$$

Thus it suffices to show that $F_n(1/b) > n/2$ for $n =$ whichever of $[4b/3]-1$, $[4b/3]$, $[4b/3]+1$ are even, that is,

(i) for $b = 3c$, $F_{4c}(1/b) > 2c$;

(ii) for $b = 3c+1$, $F_{4c}(1/b) > 2c$ and $F_{4c+2}(1/b) > 2c+1$;

(iii) for $b = 3c+2$, $F_{4c+2}(1/b) > 2c+1$.

Since the function $|\cos(\pi x/2)|$ is concave, we can bound it on $[0,2]$ from below by the polygonal path obtained by connecting the points $(x, |\cos(\pi x/2)|)$ for $x = 0, 1/3, 2/3, 1, 4/3, 5/3, 2$ by straight lines. This gives a function $H(x)$ defined by

$$H(x) = \begin{cases} 1 + \left(\frac{3\sqrt{3}-6}{2}\right) x & = u_0 + v_0 x \qquad \text{for } x \in [0, 1/3], \\ \left(\sqrt{3} - \frac{1}{2}\right) + \left(\frac{3-3\sqrt{3}}{2}\right) x & = u_1 + v_1 x \qquad \text{for } x \in [1/3, 2/3], \\ 3/2 + (-3/2)x & = u_2 + v_2 x \qquad \text{for } x \in [2/3, 1], \\ -3/2 + (3/2)x & = u_3 + v_3 x \qquad \text{for } x \in [1, 4/3], \\ \left(\frac{5-4\sqrt{3}}{2}\right) + \left(\frac{3\sqrt{3}-3}{2}\right) x & = u_4 + v_4 x \qquad \text{for } x \in [4/3, 5/3], \\ (3\sqrt{3} - 5) + \left(\frac{6-3\sqrt{3}}{2}\right) x & = u_5 + v_5 x \qquad \text{for } x \in [5/3, 2]. \end{cases}$$

For (i),

$$\begin{aligned} F_{4c}\left(\frac{1}{3c}\right) &= \sum_{j=1}^{4c} \left|\cos\left(\frac{j\pi}{6c}\right)\right| \geq \sum_{j=1}^{4c} H\left(\frac{j}{3c}\right) = \sum_{i=0}^{3} \sum_{j=ic+1}^{(i+1)c} \left(u_i + v_i \frac{j}{3c}\right) \\ &= c \sum_{i=0}^{3} u_i + \frac{1}{3c} \sum_{i=0}^{3} v_i \frac{c[(2i+1)c+1]}{2} \\ &= c\left(\sqrt{3} + \frac{1}{2}\right) + \frac{c+1}{6} \sum_{i=0}^{3} v_i + \frac{c}{3} \sum_{i=0}^{3} i v_i \\ &= c\left(\sqrt{3} + \frac{1}{2}\right) + \frac{c+1}{6}\left(\frac{-3}{2}\right) + \frac{c}{3}\left(\frac{6 - 3\sqrt{3}}{2}\right) \\ &= \frac{c(5 + 2\sqrt{3}) - 1}{4} = 2c + \frac{(2\sqrt{3} - 3)c - 1}{4} \\ &> 2c \quad \text{for } c \geq 3, \end{aligned}$$

while for $c = 2$, by direct calculation,

$$F_8(1/6) = \sum_{j=1}^{8} \left|\cos \frac{j\pi}{12}\right| \approx 4.05 > 4.$$

For $c = 1$ appears the first special case: direct computation shows that

$$F_4(1/3) = 1 + \frac{\sqrt{3}}{2} < 2.$$

For (ii),

$$\begin{aligned} F_{4c}\left(\frac{1}{3c+1}\right) \geq \sum_{j=1}^{c} \left(u_0 + v_0 \frac{j}{3c+1}\right) + \sum_{j=c+1}^{2c} \left(u_1 + v_1 \frac{j}{3c+1}\right) + \sum_{j=2c+1}^{3c+1} \left(u_2 + v_2 \frac{j}{3c+1}\right) \\ + \sum_{j=3c+2}^{4c} \left(u_3 + v_3 \frac{j}{3c+1}\right) \end{aligned}$$

$$= c\sum_{i=0}^{3} u_i + u_2 - u_3 + \frac{1}{6c+2}[c(c+1)v_0 + (3c+1)cv_1 + (5c+2)(c+1)v_2$$
$$+(7c+2)(c-1)v_3]$$
$$= \frac{(15+6\sqrt{3})c^2 + (4\sqrt{3}-1)c}{12c+4} = 2c + \frac{(6\sqrt{3}-9)c^2 - (9-4\sqrt{3})c}{12c+4}$$
$$> 2c \quad \text{for } c \geq 2,$$

while for $c = 1$, $F_4(1/4) > 2$. Also, from the above calculation,

$$F_{4c+2}\left(\frac{1}{3c+1}\right) \geq 2c + \frac{(6\sqrt{3}-9)c^2 - (9-4\sqrt{3})c}{12c+4} + \left(\frac{3}{2b}(4c+1) - \frac{3}{2}\right)$$
$$+ \left(\frac{3\sqrt{3}-3}{2b}(4c+2) + \frac{5-4\sqrt{3}}{2}\right)$$
$$= 2c + 1 + \frac{(6\sqrt{3}-9)c^2 - (9-4\sqrt{3})c + 4\sqrt{3} - 6}{12c+4}$$
$$> 2c+1 \quad \text{for all } c \geq 1.$$

For (iii),

$$F_{4c+2}\left(\frac{1}{3c+2}\right) \geq \sum_{j=1}^{c}\left(u_0 + v_0\frac{j}{3c+2}\right) + \sum_{j=c+1}^{2c+1}\left(u_1 + v_1\frac{j}{3c+2}\right)$$
$$+ \sum_{j=2c+2}^{3c+2}\left(u_2 + v_2\frac{j}{3c+2}\right) + \sum_{j=3c+3}^{4c+2}\left(u_3 + v_3\frac{j}{3c+2}\right)$$
$$= c\sum_{i=0}^{3} u_i + u_1 + u_2 + \frac{1}{6c+4}[c(c+1)v_0 + (3c+2)(c+1)v_1$$
$$+(5c+4)(c+1)v_2 + (7c+5)cv_3]$$
$$= 2c + 1 + \frac{(6\sqrt{3}-9)c^2 + (8\sqrt{3}-15)c - (6-2\sqrt{3})}{12c+8}$$
$$> 2c+1 \quad \text{for } c \geq 2.$$

When $c = 1$ we have the other exceptional case, because

$$F_6(1/5) \approx 2.97 < 3 .$$

Thus, Proposition 1 is proved. □

PROPOSITION 2. Let a and b be positive integers such that $a < b$, a is odd, and $\gcd(a,b) = 1$. Then for all n even such that $b \leq n < 2b$, $F_n(a/b) \geq F_n(1/b)$.

Proof. According to Lemma 2,

$$F_n\left(\frac{a}{b}\right) = F_b\left(\frac{a}{b}\right) + \sum_{j=b+1}^{n}\left|\cos\left(\frac{ja\pi}{2b}\right)\right| = F_b\left(\frac{1}{b}\right) + \sum_{j=b+1}^{n}\left|\cos\left(\frac{ja\pi}{2b}\right)\right|.$$

For each $j = b+1, \ldots, n$, find k_j such that $b+1 \leq |k_j| \leq 2b-1$ and $ja \equiv k_j \bmod 2b$. As in the proof of Lemma 2, the k_j's exist, and the numbers $|k_j|$, $j = b+1, \ldots, n$, are distinct. Also as in Lemma 2, for all j,

$$\left|\cos\left(\frac{ja\pi}{2b}\right)\right| = \left|\cos\left(\frac{|k_j|\pi}{2b}\right)\right|.$$

Thus in the summation

$$\sum_{j=b+1}^{n}\left|\cos\left(\frac{ja\pi}{2b}\right)\right|$$

we are adding $n-b$ elements of the set

$$\left\{\left|\cos\left(\frac{j\pi}{2b}\right)\right| : b+1 \leq j \leq 2b-1\right\};$$

but the smallest such sum is

$$\sum_{j=b+1}^{n}\left|\cos\left(\frac{j\pi}{2b}\right)\right|,$$

formed by the smallest $n-b$ terms. Thus

$$F_n\left(\frac{a}{b}\right) \geq F_b\left(\frac{1}{b}\right) + \sum_{j=b+1}^{n}\left|\cos\left(\frac{j\pi}{2b}\right)\right| = F_n\left(\frac{1}{b}\right). \quad \square$$

Our next step is to study the case $2b \leq n < 3b$.

PROPOSITION 3. Let a and b be positive integers where $a < b$, a is odd, and $\gcd(a,b) = 1$. Then for all even n such that $2b \leq n < 3b$, $F_n(a/b) > n/2$.

Proof We use the notation from the proof of Proposition 1. Note first that, by the symmetry of $F_n(x)$ and by Lemma 2,

$$\begin{aligned} F_{2b}\left(\frac{a}{b}\right) &= \sum_{j=1}^{b-1}\left|\cos\frac{ja\pi}{2b}\right| + 0 + \sum_{j=b+1}^{2b-1}\left|\cos\frac{ja\pi}{2b}\right| + 1 \\ &= 2F_b\left(\frac{a}{b}\right) + 1 = 2F_b\left(\frac{1}{b}\right) + 1 = F_{2b}\left(\frac{1}{b}\right). \end{aligned} \tag{1}$$

Case (i): $b = 3c$. Then

$$\begin{aligned} F_{6c}\left(\frac{1}{3c}\right) &\geq \sum_{i=0}^{5}\sum_{j=ic+1}^{(i+1)c}\left(u_i + v_i\frac{j}{3c}\right) = c\sum_{i=0}^{5}u_i + \frac{c+1}{6}\sum_{i=0}^{5}v_i + \frac{c}{3}\sum_{i=0}^{5}iv_i \\ &= c(2\sqrt{3}-2) + 0 + \frac{c}{3}(12-3\sqrt{3}) = 3c + (\sqrt{3}-1)c, \end{aligned} \tag{2}$$

and since $6c < n < 9c$,

$$F_n\left(\frac{a}{3c}\right) = F_{6c}\left(\frac{a}{3c}\right) + \sum_{j=6c+1}^{n}\left|\cos\frac{ja\pi}{6c}\right|. \tag{3}$$

But all terms $|\cos(ja\pi/6c)|$ of the above sum are greater than or equal to $1/2$, except for at most c of them. [*Editor's note.* In general, $|\cos(ja\pi/2b)| < 1/2$ for $2b \le j \le n$ if and only if j belongs to an interval of the form

$$\left(\frac{2b(t+a)}{a} + \frac{2b}{3a}\,,\ \frac{2b(t+a)}{a} + \frac{4b}{3a}\right)$$

for some nonnegative integer t, where $n < 3b$ means $t < a/2$; thus there are less than $a/2$ such intervals, each containing at most $2b/3a$ such j's, so at most $b/3$ such j's altogether.] Hence from (1), (2) and (3),

$$F_n\left(\frac{a}{3c}\right) \ge 3c + (\sqrt{3}-1)c + (n-7c)\cdot\frac{1}{2} = \frac{n}{2} + \left(\sqrt{3}-\frac{3}{2}\right)c > \frac{n}{2}\,.$$

Case (ii): $b = 3c+1$. As in the case above, we have

$$\begin{aligned}
F_{6c+2}\left(\frac{a}{3c+1}\right) &= F_{6c+2}\left(\frac{1}{3c+1}\right)\\
&\ge \sum_{j=1}^{c}\left(u_0 + v_0\frac{j}{3c+1}\right) + \sum_{j=c+1}^{2c}\left(u_1 + v_1\frac{j}{3c+1}\right)\\
&\quad + \sum_{j=2c+1}^{3c+1}\left(u_2 + v_2\frac{j}{3c+1}\right) + \sum_{j=3c+2}^{4c+1}\left(u_3 + v_3\frac{j}{3c+1}\right)\\
&\quad + \sum_{j=4c+2}^{5c+1}\left(u_4 + v_4\frac{j}{3c+1}\right) + \sum_{j=5c+2}^{6c+2}\left(u_5 + v_5\frac{j}{3c+1}\right)\\
&= c\sum_{i=0}^{5} u_i + u_2 + u_5 + \frac{1}{6c+2}[c(c+1)v_0 + c(3c+1)v_1 + (c+1)(5c+2)v_2\\
&\qquad + c(7c+3)v_3 + c(9c+3)v_4 + (c+1)(11c+4)v_5]\\
&= \frac{(6+3\sqrt{3})c^2 + (4+2\sqrt{3})c + 1}{3c+1} = 3c+1+(\sqrt{3}-1)c\left(1+\frac{1}{3c+1}\right)\\
&> 3c+1+(\sqrt{3}-1)c.
\end{aligned}$$

Now since $6c+2 < n < 9c+3$,

$$\begin{aligned}
F_n\left(\frac{a}{3c+1}\right) &= F_{6c+2}\left(\frac{a}{3c+1}\right) + \sum_{j=6c+3}^{n}\left|\cos\frac{ja\pi}{6c+2}\right|\\
&> 3c+1+(\sqrt{3}-1)c + \frac{n-7c-2}{2} > \frac{n}{2}\,,
\end{aligned}$$

because, as before [see the editor's note in case (i)], of the $n-6c-2$ terms in the summation, at most c will be less than $1/2$.

Case (iii): $b = 3c+2$. In the same way,

$$F_{6c+4}\left(\frac{a}{3c+2}\right) = F_{6c+4}\left(\frac{1}{3c+2}\right)$$

$$
\begin{aligned}
&\geq \sum_{j=1}^{c}\left(u_0+v_0\frac{j}{3c+2}\right)+\sum_{j=c+1}^{2c+1}\left(u_1+v_1\frac{j}{3c+2}\right)\\
&\quad+\sum_{j=2c+2}^{3c+2}\left(u_2+v_2\frac{j}{3c+2}\right)+\sum_{j=3c+3}^{4c+2}\left(u_3+v_3\frac{j}{3c+2}\right)\\
&\quad+\sum_{j=4c+3}^{5c+3}\left(u_4+v_4\frac{j}{3c+2}\right)+\sum_{j=5c+4}^{6c+4}\left(u_5+v_5\frac{j}{3c+2}\right)\\
&=(c+1)\sum_{i=0}^{5}u_i-u_0-u_3+\frac{1}{6c+4}[c(c+1)v_0+(c+1)(3c+2)v_1\\
&\qquad+(c+1)(5c+4)v_2+c(7c+5)v_3+(c+1)(9c+6)v_4\\
&\qquad\qquad+(c+1)(11c+8)v_5]\\
&=\frac{(6+3\sqrt{3})c^2+(8+4\sqrt{3})c+(3+\sqrt{3})}{3c+2}\\
&=3c+2+(\sqrt{3}-1)\left(c+\frac{2c+1}{3c+1}\right)>3c+2+(\sqrt{3}-1)c,
\end{aligned}
$$

and then we have, since $6c+4<n<9c+6$,

$$
\begin{aligned}
F_n\left(\frac{a}{3c+2}\right)&=F_{6c+4}\left(\frac{a}{3c+2}\right)+\sum_{j=6c+5}^{n}\left|\cos\frac{ja\pi}{6c+4}\right|\\
&>3c+2+(\sqrt{3}-1)c+\frac{n-7c-4}{2}>\frac{n}{2}\,.
\end{aligned}
$$

Thus Proposition 3 is proved. □

To establish $(*)$, it only remains to consider the case $n\geq 3b$, proceeding inductively.

PROPOSITION 4. Let a and b be positive integers with $1\leq a<b$, a odd and $\gcd(a,b)=1$. Then for all even integers n with $n\geq 3b$, $F_n(a/b)>n/2$.

Proof. Note

$$
\begin{aligned}
F_n\left(\frac{a}{b}\right)&=F_{2b}\left(\frac{a}{b}\right)+\sum_{i=1}^{n-2b}\left|\cos\frac{(2b+i)a\pi}{2b}\right|\\
&=F_{2b}\left(\frac{a}{b}\right)+\sum_{i=1}^{n-2b}\left|\cos\left(a\pi+\frac{ia\pi}{2b}\right)\right|=F_{2b}\left(\frac{a}{b}\right)+\sum_{i=1}^{n-2b}\left|\cos\frac{ia\pi}{2b}\right|\\
&=F_{2b}\left(\frac{a}{b}\right)+F_{n-2b}\left(\frac{a}{b}\right)\,. \qquad (4)
\end{aligned}
$$

If we suppose the proposition is true when $(2k-1)b\leq n<(2k+1)b$, then when $(2k+1)b\leq n<(2k+3)b$ we obtain by (4) and Proposition 3 that

$$
F_n\left(\frac{a}{b}\right)>b+\frac{n-2b}{2}=\frac{n}{2}\,,
$$

so the proposition follows provided we can establish the base case. If $b\neq 2$, 3 or 5 the proposition is true for $k=1$, i.e., in the case $b\leq n<3b$, by Propositions 1, 2 and 3.

Finally, for the remaining cases $b = 2$, 3 and 5 we could begin the induction on the interval $3b \le n < 5b$, since by direct computation

$$F_6(1/2) \approx 3.1 > 3\ ,\ F_{10}(1/3) \approx 5.5 > 5\ ,\ F_{16}(1/5) \approx 9.2 > 8.$$

[*Editor's note.* Here are a few more details. For $b = 2$ and a odd, one finds as in the proof of Proposition 3 that

$$F_6\left(\frac{a}{2}\right) = 3F_2\left(\frac{a}{2}\right) + 1 = 3F_2\left(\frac{1}{2}\right) + 1 = F_6\left(\frac{1}{2}\right)\ ,$$

and thus for $6 \le n < 10$,

$$\begin{aligned} F_n\left(\frac{a}{2}\right) &= F_6\left(\frac{a}{2}\right) + \sum_{j=7}^{n}\left|\cos\frac{ja\pi}{4}\right| = F_6\left(\frac{1}{2}\right) + \sum_{j=7}^{n}\left|\cos\frac{ja\pi}{4}\right| \\ &> 3 + (n-6)\frac{1}{2} \quad \text{(as is easily checked)} \\ &= \frac{n}{2}\ . \end{aligned}$$

Similarly for $b = 3$, a odd and not divisible by 3, we use

$$F_{10}\left(\frac{a}{3}\right) = 3F_3\left(\frac{a}{3}\right) + \frac{3}{2} = 3F_3\left(\frac{1}{3}\right) + \frac{3}{2} = F_{10}\left(\frac{1}{3}\right)$$

to get, for $9 \le n < 15$,

$$F_n\left(\frac{a}{3}\right) = F_{10}\left(\frac{a}{3}\right) + \sum_{j=11}^{n}\left|\cos\frac{ja\pi}{6}\right| > 5 + \frac{n-10}{2} = \frac{n}{2}\ .$$

For $b = 5$, a odd and not divisible by 5, we must use

$$F_{16}\left(\frac{a}{5}\right) \ge 3F_5\left(\frac{a}{5}\right) + 1 + \cos\frac{2\pi}{5} = 3F_5\left(\frac{1}{5}\right) + 1 + \cos\frac{2\pi}{5} = F_{16}\left(\frac{1}{5}\right)$$

to get, for $15 \le n < 25$,

$$F_n\left(\frac{a}{5}\right) = F_{16}\left(\frac{a}{5}\right) + \sum_{j=17}^{n}\left|\cos\frac{ja\pi}{10}\right| > 8 + \frac{n-16}{2} = \frac{n}{2}\ .] \qquad \square$$

To finish, we need to verify that $F_3(x) \ge 1$, $F_5(x) \ge 2$ and $F_7(x) \ge 3$ for all x. It is possible to do this quite easily with the aid of a computer. By Lemma 1 it suffices to compute the values on the terms of the Farey sequence with odd numerator. The following tables give the results.

x	1/3	1/2	1
$F_3(x)$	1.36+	1.41+	1

x	1/5	1/4	1/3	1/2	3/4	3/5	1
$F_5(x)$	2.65+	2.39+	2.73+	3.12+	2.93+	2.65+	2

x	1/7	1/6	1/5	1/4	1/3	3/7	1/2	3/5	5/7	3/4	5/6	1
$F_7(x)$	3.93+	3.55+	3.55+	4.02+	4.59+	3.93+	3.82+	4.41+	3.93+	4.02+	4.26+	3

No other solutions for this problem were received. The examples refuting the proposer's conjecture for $n = 4$ and 6 were also found by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria ($n = 4$ only); and CHARLTON WANG, student, Waterloo Collegiate Institute, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario. Hess gave the correct answer to the problem without proof.

The proposer also asks for the maximum value of

$$|\sin\theta| + |\sin 2\theta| + \cdots + |\sin n\theta|$$

for fixed n. The editor is grateful he didn't make this Crux 1542, *part (b)! However, readers may like to work on it.*

Janous recalls a somewhat similar problem in a recent U.S.S.R. Olympiad, but has no more details. Maybe a reader can add more information?

* * * * *

1546. [1990: 144] *Proposed by Graham Denham, student, University of Alberta.*

Prove that for every positive integer n and every positive real x,

$$\sum_{k=1}^{n} \frac{x^{k^2}}{k} \geq x^{n(n+1)/2}.$$

I. *Solution by Kee-Wai Lau, Hong Kong.*

We first show that for $y > 0$,

$$f_n(y) := 1 + \frac{y^{n+2}}{n+1} - y^2 > 0.$$

By differentiation it is easy to check that $f_n(y)$ attains at

$$y = \left(\frac{2(n+1)}{n+2}\right)^{1/n}$$

its minimum value of

$$\begin{aligned}1 + \frac{1}{n+1}\left(\frac{2(n+1)}{n+2}\right)^{1+\frac{2}{n}} - \left(\frac{2(n+1)}{n+2}\right)^{\frac{2}{n}} &= 1 + \frac{1}{n+1}\left(\frac{2(n+1)}{n+2}\right)^{\frac{2}{n}}\left(\frac{2(n+1)}{n+2} - (n+1)\right) \\ &= 1 - \frac{n}{n+2}\left(\frac{2(n+1)}{n+2}\right)^{\frac{2}{n}}.\end{aligned}$$

This minimum value is positive because for $n \geq 2$,

$$\left(\frac{n+2}{n}\right)^{n/2} = \left(1 + \frac{1}{n/2}\right)^{n/2} \geq 2 > \frac{2(n+1)}{n+2}.$$

We now prove the inequality of the problem by induction. For $n = 1$ the inequality reduces to an identity. Suppose that the inequality holds for $n = m \geq 1$. Then

$$\begin{aligned}\sum_{k=1}^{m+1} \frac{x^{k^2}}{k} - x^{(m+1)(m+2)/2} &\geq x^{m(m+1)/2} + \frac{x^{(m+1)^2}}{m+1} - x^{(m+1)(m+2)/2} \\ &= x^{m(m+1)/2}\left(1 + \frac{x^{(m+1)(m+2)/2}}{m+1} - x^{m+1}\right) \\ &= x^{m(m+1)/2} f_m\left(x^{(m+1)/2}\right) > 0.\end{aligned}$$

This completes the solution of the problem.

II. *Solution by the proposer.*

By the A.M.-G.M. inequality,

$$\frac{\sum_{k=1}^{n} kx^{k^2}}{\sum_{k=1}^{n} k} \geq \left(x^{\sum_{k=1}^{n} k^3}\right)^{1/\sum_{k=1}^{n} k}.$$

Since

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2,$$

we get

$$\sum_{k=1}^{n} kx^{k^2} \geq \frac{n(n+1)}{2} x^{n(n+1)/2}.$$

Finally,

$$\sum_{k=1}^{n} \frac{x^{k^2}}{k} = \sum_{k=1}^{n} \int_0^x kx^{k^2-1}\,dx \geq \frac{n(n+1)}{2} \int_0^x x^{n(n+1)/2\,-\,1}\,dx = x^{n(n+1)/2}.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; NICOS DIAMANTIS, student, University of Patras, Greece; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado: RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium. Innsbruck, Austria; WEIXUAN LI, University of Ottawa, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and JEAN-MARIE MONIER, Lyon, France.

Janous's solution was the same as the proposer's. Janous and Falkowitz gave the generalization (proved as in II): if $\lambda_1, \cdots, \lambda_n, \mu_1, \cdots, \mu_n > 0$ *satisfy*

$$\sum_{k=1}^{n} \lambda_k \mu_k^2 = \left(\sum_{k=1}^{n} \lambda_k \mu_k\right)^2,$$

then

$$\sum_{k=1}^{n} \lambda_k x^{\mu_k} \geq x^{\sum_{k=1}^{n} \lambda_k \mu_k}$$

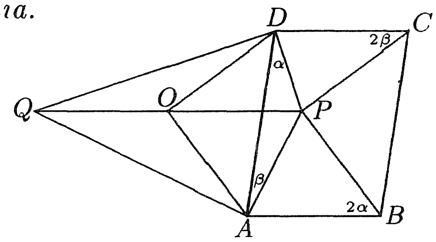
for all $x \geq 0$.

* * * * *

1547. [1990: 144] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P be an interior point of a parallelogram $ABCD$, such that $\angle ABP = 2\angle ADP$ and $\angle DCP = 2\angle DAP$. Prove that $AB = PB = PC$.

Solution by Dan Sokolowsky, Williamsburg, Virginia.



Let $\angle ADP = \alpha$ and $\angle DAP = \beta$, so that $\angle ABP = 2\alpha$ and $\angle DCP = 2\beta$. Draw $PO||AB$, PO cutting AD, with $PO = AB = CD$. Then $ABPO$, $CDOP$ are parallelograms, so $\angle AOP = 2\alpha$ and $\angle DOP = 2\beta$. Extend PO to Q with $OQ = OA$. Then

$$\angle OQA = \angle OAQ = \alpha = \angle ADP,$$

so A, P, D, Q lie on a circle κ. Hence $\angle DQP = \angle DAP = \beta$, so

$$\angle ODQ = \angle DOP - \angle DQO = 2\beta - \beta = \beta,$$

and hence $OD = OQ = OA$, which implies O is the center of κ. Then

$$AB = OP = OA = OD = PB = PC.$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; K.R.S. SASTRY, Addis Ababa, Ethiopia; BRUCE SHAWYER, Memorial University of Newfoundland; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer (whose solution was somewhat similar to Sokolowsky's).

Hut and Sastry gave converses. Sastry also showed that the result still holds if P is outside the parallelogram but on the same side of the line AD as B and C are; this can be proved as above. Finally, Sastry noted that such a point P exists if and only if BC $\leq$ 2AB.

* * * * *

1548*. [1990: 144] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a_1, a_2 be given positive constants and define a sequence $a_3, a_4, a_5, \ldots$ by

$$a_n = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}, \quad n > 2.$$

Show that $\lim_{n\to\infty} a_n$ exists and find this limit.

I. *Comment by the editor.*

Three readers, Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Marcin E. Kuczma, Warszawa, Poland; and John Lindsey, Northern Illinois University, Dekalb, noticed that this problem is equivalent to problem E3388 of the *American Math. Monthly* (p. 428 of the May 1990 issue), via the transformation $a_i = \sqrt{2}/x_i$. The editor will therefore wait to see what the *Monthly* publishes before deciding which solution, if any, to feature in *Crux*. The list of solvers of *Crux* 1548 will also be given at that time. It shouldn't be too long a wait, although the *Monthly* seems to be even further behind in their solutions than we are!

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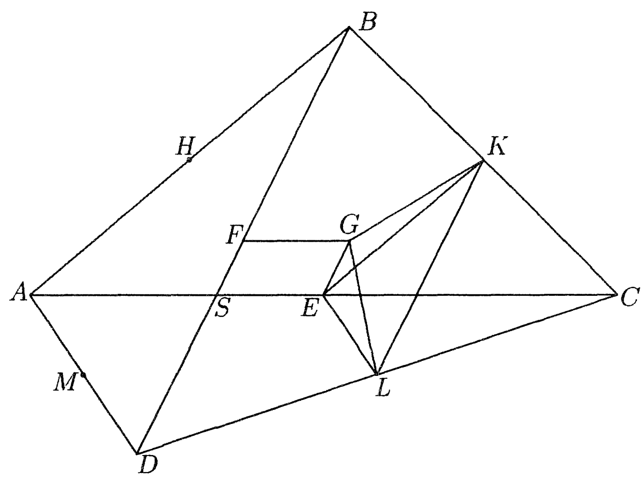
1549. [1990: 144] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In quadrilateral $ABCD$, E and F are the midpoints of AC and BD respectively. S is the intersection point of AC and BD. H, K, L, M are the midpoints of AB, BC, CD, DA respectively. Point G is such that $FSEG$ is a parallelogram. Show that lines GH, GK, GL, GM divide $ABCD$ into four regions of equal area.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

Il s'agit d'un théorème connu sous le nom de THEOREME DE BRUNE (*2000 théorèmes et problèmes de géométrie*, A. Dalle, éd. La Procure-Namur). En voici une démonstration.



Démontrons que l'aire de $GKCL$ vaut le quart de l'aire de $ABCD$.

Joignons EK, EL, KL. L'homothétie de centre C et de rapport 2 applique $CKEL$ sur $CBAD$ (puisque K, E, L sont les milieux respectifs de BC, AC, DC), d'où

$$\text{aire}(CKEL) = \frac{1}{4}\,\text{aire}(CBAD).$$

$GE \parallel KL$ ($\parallel BD$), d'où aire(GKL) = aire(EKL),

$$\begin{aligned}\text{aire}(GKCL) &= \text{aire}(GKL) + \text{aire}(KLC) = \text{aire}(EKL) + \text{aire}(KLC)\\ &= \text{aire}(EKCL) = \frac{1}{4}\text{aire}(ABCD).\end{aligned}$$

On démontre de même que

$$\text{aire}(GHBK) = \text{aire}(GMAH) = \text{aire}(GLDM) = \frac{1}{4}\text{aire}(ABCD).$$

Also solved by JORDI DOU, Barcelona, Spain; Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; E. SZEKERES, Turramurra, Australia; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP, Tucson; and the proposer.

The proposer found the problem in Journal de Mathématiques Élémentaires, *October 1917, no. 8494.*

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1550. [1990: 144] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A = [-1,1]$. Find all functions $f : A \to A$ such that

$$|x\,f(y) - y\,f(x)| \geq |x - y|$$

for all $x, y \in A$.

Solution by R.P. Sealy, Mount Allison University.

There are four such functions:

$$f_1(x) = 1 \quad \text{for all } x \in A, \qquad f_2(x) = -1 \quad \text{for all } x \in A,$$

$$f_3(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ -1 & \text{for } x = 0, \end{cases} \qquad f_4(x) = \begin{cases} -1 & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

That these four functions satisfy the condition is easily checked. We show that there are no others.

Letting $x = 0$ and $y = 1$, we see that $|f(0)| = 1$. Letting $x = -y \neq 0$, we get $|f(-x) + f(x)| \geq 2$. which implies (since $|f(x)| \leq 1$) that $f(-x) = f(x) = \pm 1$. Hence $|f(x)| = 1$ for all $x \in A$. Suppose there exists $x, y \in A$ satisfying $xy \neq 0$ and $f(x)f(y) = -1$. Then (since $f(-x) = f(x)$) there exists $x, y \in A$ satisfying $xy < 0$ and $f(x)f(y) = -1$. Then $|x + y| \geq |x - y|$, which is impossible for $xy < 0$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; NICOS D. DIAMANTIS, student, University of Patras, Greece; MATHEW ENGLANDER, student, University of Waterloo; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; C. FESTRAETS-HAMOIR, Brussels. Belgium: WALTHER JANOUS. Ursulinengymnasium, Innsbruck, Austria; JEAN-MARIE MONIER, Lyon, France; M. PARMENTER, Memorial University of Newfoundland; and the proposer.

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1551. [1990: 171] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find a triangle ABC with a point D on AB such that the lengths of AB, BC, CA and CD are all integers and $AD : DB = 9 : 7$, or prove that no such triangle exists.

I. *Solution by Hayo Ahlburg, Benidorm, Spain.*

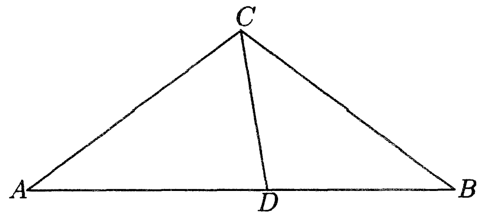
Let $AD = kx$, $DB = \ell x$, $AD : DB = k : \ell$ (given), $AB = (k + \ell)x$, $BC = y$, $CA = z$, $CD = t$. These lengths are related by Stewart's Theorem [1]

$$(k + \ell)(t^2 + k\ell x^2) = ky^2 + \ell z^2,$$

in our case ($k = 9$, $\ell = 7$)

$$16t^2 + 1008x^2 = 9y^2 + 7z^2. \tag{1}$$

While the general case of such a quaternary quadratic equation is rather involved [2], I note that for the special case of isosceles triangles ($y = z$) (1) can be written simply as



$$63x^2 = y^2 - t^2.$$

With $x = 1$ (i.e., $AD = 9$ and $DB = 7$) we find the two solutions

$y+t$	$y-t$	AB	$y = BC = CA$	$t = CD$
21	3	16	12	9
63	1	16	32	31 .

With $x > 1$, there can be many ways to split the product $63x^2$ into two factors, leading to more solutions.

References:
[1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Math. Library, Vol. 19, M.A.A., 1967, p. 6.
[2] L.J. Mordell, *Diophantine Equations*, Academic Press, 1969, pp. 49–51.

II. *Solution by P. Penning, Delft, The Netherlands.*

Consider the situation that ΔABC is similar to ΔACD. Then

$$\frac{AB}{AC} = \frac{BC}{CD} = \frac{AC}{AD}.$$

Write $AD = 9f$, $DB = 7f$, $CD = x$, $BC = a$, $AC = b$, where f is an integer determining the scale. Then

$$\frac{16f}{b} = \frac{a}{x} = \frac{b}{9f},$$

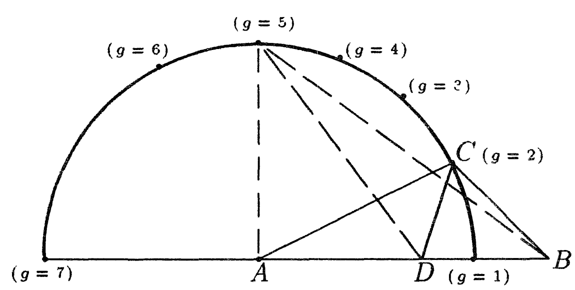
or

$$b = 12f \text{ and } \frac{x}{a} = \frac{3}{4}.$$

Points C lie on a circle with A as centre and radius $b = 12f$. With g an integer in the range $\{1, 2, \ldots, 7\}$, we have

$$a = 4gf, \quad b = 12f, \quad c = 16f, \quad x = 3gf.$$

On the boundaries ($g = 1$ and $g = 7$) ΔABC is flat.



Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; JEAN-

MARIE MONIER, Lyon, France; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

Many different examples were given. Festraets-Hamoir and Smeenk came up with Ahlburg's second isosceles triangle (in Solution I); incidentally the case $g = 3$ of Penning's solution yields the first. The right-angled triangle corresponding to $g = 5$ of Penning's solution was also found by Hut and Yusty.

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1552*. [1990: 171] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For each integer $n \geq 2$ let

$$x_n = \left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n + \cdots + \left(\frac{n-1}{n}\right)^n .$$

Does the sequence $\{x_n/n\}$, $n = 2, 3, \ldots$, converge?

Solution by Richard Katz, California State University, Los Angeles.

Yes. In fact,

$$\lim_{n\to\infty} \frac{x_n}{n} = \int_0^1 e^{-1/x}\,dx.$$

To prove this, let

$$\underline{L} = \liminf_{n\to\infty} \frac{x_n}{n}\ , \qquad \overline{L} = \limsup_{n\to\infty} \frac{x_n}{n}\ .$$

We will show that

(i) $\overline{L} \leq \int_0^1 e^{-1/x}\,dx$, and

(ii) for $0 < \delta < 1$ and $0 < \epsilon < e^{-1}$, $\int_\delta^1 (e^{-1} - \epsilon)^{1/x}\,dx \leq \underline{L}$.

Clearly (i) and (ii) imply that

$$\underline{L} = \overline{L} = \lim_{n\to\infty} \frac{x_n}{n} = \int_0^1 e^{-1/x}\,dx.$$

For (i), note that (via the Riemann sum)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{-n/j} = \int_0^1 e^{-1/x}\,dx.$$

Now, $[(j-1)/j]^j$ increases to e^{-1} as $j \to \infty$, so

$$\left(\frac{j-1}{j}\right)^j < e^{-1} \quad \text{for all } j.$$

Hence

$$x_n = \sum_{j=1}^{n} \left(\frac{j-1}{j}\right)^n < \sum_{j=1}^{n} e^{-n/j} \quad \text{for all } n,$$

and thus

$$\overline{L} = \limsup_{n\to\infty} \frac{x_n}{n} \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{-n/j} = \int_0^1 e^{-1/x}\, dx.$$

Thus (i) is proved.

To prove (ii), given $0 < \delta < 1$ and $0 < \epsilon < e^{-1}$, choose N such that

$$\left(\frac{j-1}{j}\right)^j > e^{-1} - \epsilon \quad \text{for all } j \ge N.$$

Then for n large enough so that $N/n < \delta$, we have

$$\frac{1}{n} \sum_{j=[\delta n]}^{n} (e^{-1} - \epsilon)^{n/j} \le \frac{1}{n} \sum_{j=1}^{n} \left(\frac{j-1}{j}\right)^n = \frac{x_n}{n}.$$

Taking the lim inf gives (ii). [Since $\delta - 1/n < [\delta n]/n \le \delta$, the lim inf of the left side is, via Riemann sums, the integral in (ii).]

The method given above can be used to prove the following slightly more general result. *Let* $0 \le a_j \le 1$ *for* $j = 1, 2, \ldots,$ *and suppose* $(a_j)^j \to a$. *Put*

$$x_n = \sum_{j=1}^{n} (a_j)^n$$

for each n. *Then*

$$\lim_{n\to\infty} \frac{x_n}{n} = \int_0^1 a^{1/x}\, dx.$$

Also solved by H.L. ABBOTT, University of Alberta; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and JOHN H. LINDSEY, University of Calgary. A partial solution from Wolfgang Gmeiner was sent in by the proposer. There was also one incorrect solution submitted.

All solvers in fact found the limit. Falkowitz also gave a generalization, although a weaker one than that of Katz. Abbott wonders whether x_n/n *increases (if so this would instantly prove that* $\lim_{n\to\infty}(x_n/n)$ *exists), and whether* $x_{n+m} \le x_n + x_m$ *holds for all* $n, m \ge 2$.

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