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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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CARNOT AND PASCAL

Dan Pedoe

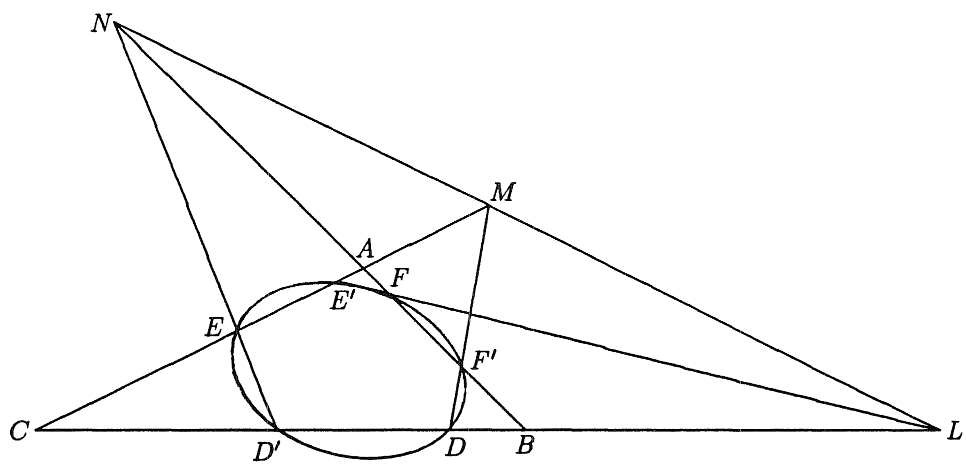
If D, D'; E, E'; and F, F' are pairs of points on the respective sides BC, CA and AB of a triangle ABC, Carnot's Theorem states that these three pairs of points lie on a conic if and only if:

$$(BD \cdot BD')(CE \cdot CE')(AF \cdot AF') = (CD \cdot CD')(AE \cdot AE')(BF \cdot BF').$$

None of the proofs I have looked at in various books mentions that this theorem is equivalent to the Pascal Theorem, which states that if

$$L = D'D \cap FE', \quad M = DF' \cap E'E, \quad N = F'F \cap ED',$$

then the three pairs of points lie on a conic if and only if L, M and N are collinear.



We use the notation of [1], defining the position-ratio of the point Z with respect to the points X and Y collinear with it as:

$$(XY)_Z = \overline{XZ}/\overline{ZY},$$

where $\overline{XZ}$ and $\overline{ZY}$ are signed lengths. This notation simplifies the applications of the Menelaus Theorem, which we use in every step of the proof. If L, M and N are collinear, then

$$(BC)_L(CA)_M(AB)_N = -1. \tag{1}$$

Since E', F and L are collinear,

$$(BC)_L(CA)_{E'}(AB)_F = -1. \tag{2}$$

Since D, F' and M are collinear,

$$(CA)_M(AB)_{F'}(BC)_D = -1. \tag{3}$$

Since D', E and N are collinear,

$$(AB)_N(BC)_{D'}(CA)_E = -1. \tag{4}$$

Using equations (2), (3) and (4), equation (1) holds if and only if

$$(AB)_F(AB)_{F'}(BC)_D(BC)_{D'}(CA)_E(CA)_{E'} = 1,$$

and this is the Carnot Theorem. So Carnot and Pascal are equivalent to each other.

Reference

[1] Dan Pedoe, The theorems of Ceva and Menelaus, *Crux Mathematicorum* (née *Eureka*) Vol. 3, No. 2 (1977) 2–4.

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In the article "Cubes of natural numbers in arithmetic progression" by K.R.S. Sastry, published in the last issue of *Crux*, there is an incorrectly drawn figure. Four "edges" in the last pentagon figure on [1992: 162] are each missing one vertex. (The *number* of vertices, 35, is given correctly.) Readers should please correct their copies. The error was ours and not Dr. Sastry's, and we apologize.

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THE OLYMPIAD CORNER

No. 137

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

My appeals from last year have had some effect: I have received information from sources about the Olympiad in Moscow, but some of it conflicts. So, I am going to postpone discussion of this year's Olympiad until a later column when I hope to have "checked sources" and received more detailed information from other faithful readers. Instead we begin this column with some of the problems proposed to the jury for the 32nd I.M.O. last year at Sigtuna, Sweden. Many thanks go to Andy Liu, University of Alberta, and also to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, who was the Canadian I.M.O. Team leader, for sending me these problems.

1. *Proposed by the Philippines.*

Let ABC be any triangle and P any point in its interior. Let P_1, P_2 be the feet of the perpendiculars from P to the two sides AC and BC. Draw AP and BP and from C drop perpendiculars to AP and BP. Let Q_1 and Q_2 be the feet of these perpendiculars. Prove that the lines Q_1P_2, Q_2P_1 and AB are concurrent.

2. *Proposed by North Korea.*

Let S be any point on the circumcircle of ΔPQR. Then the feet of the perpendiculars from S to the three sides of the triangle lie on the same straight line. Denote this line by $\ell(S, PQR)$. Suppose that the hexagon $ABCDEF$ is inscribed in a circle. Show that the four lines $\ell(A, BDF)$, $\ell(B, ACE)$, $\ell(D, ABF)$ and $\ell(E, ABC)$ intersect at one point if and only if $CDEF$ is a rectangle.

3. *Proposed by China.*

Let O be the centre of the circumsphere of a tetrahedron $ABCD$. Let L, M, N be the midpoints of BC, CA, AB respectively, and assume that $AB + BC = AD + CD$, $BC+CA = BD+AD$ and $CA+AB = CD+BD$. Prove that $\angle LOM = \angle MON = \angle NOL$.

4. *Proposed by The Netherlands.*

S is a set of n points in the plane. No three points of S are collinear. Prove that there exists a set P containing $2n-5$ points satisfying the condition: The interior of each triangle with three vertices from S contains an element of P.

5. *Proposed by France.*

In the plane we are given a set E of 1991 points; certain pairs of these points are joined by a path. We suppose that for every point of E, there exists at least 1593 other points of E to which it is joined by a path. Show that there exist six points of E every pair of which are joined by a path.

6. *Proposed by Australia.*
Prove that

$$\frac{1}{1991}\binom{1991}{0} - \frac{1}{1990}\binom{1990}{1} + \frac{1}{1989}\binom{1989}{2} - \cdots + \frac{(-1)^m}{1991-m}\binom{1991-m}{m} + \\ + \cdots - \frac{1}{996}\binom{996}{995} = \frac{1}{1991}.$$

7. *Proposed by Poland.*
Let n be any integer, $n \geq 2$. Assume that the integers $a_1, a_2, \ldots, a_n$ are not divisible by n and, moreover, that n does not divide $a_1+a_2+\cdots+a_n$. Prove that there exist at least n different sequences $(e_1, e_2, \ldots, e_n)$ consisting of zeros or ones such that $e_1a_1+e_2a_2+\cdots+e_na_n$ is divisible by n.

8. *Proposed by the U.S.S.R.*
Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence $a_1, a_2, \ldots, a_n, \ldots$ become periodic after a finite number of terms?

9. *Proposed by Ireland.*
Let a be a rational number with $0 < a < 1$ and suppose that

$$\cos(3\pi a) + 2\cos(2\pi a) = 0.$$

(Angle measurements are in radians.) Prove that $a = 2/3$.

10. *Proposed by Hong Kong.*
Let $f(x)$ be a monic polynomial of degree 1991 with integer coefficients. Define $g(x) = f^2(x) - 9$. Show that the number of distinct integer solutions of $g(x)$ cannot exceed 1995.

11. *Proposed by India.*
Let f and g be two integer-valued functions defined on the set of all integers such that

(a) $f(m + f(f(n))) = -f(f(m+1)) - n$ for all integers m and n;

(b) g is a polynomial function with integer coefficients and $g(n) = g(f(n))$ for all integers n.

Determine $f(1991)$ and the most general form of g.

12. *Proposed by the U.S.A.*
Suppose that $n \geq 2$ and $x_1, x_2, \ldots, x_n$ are real numbers between 0 and 1 (inclusive). Prove that for some index i between 1 and $n-1$ the inequality

$$x_i(1 - x_{i+1}) \geq \frac{1}{4}x_1(1 - x_n)$$

holds.

*　　　*　　　*

The Canadian Mathematics Olympiad for 1992 saw over 300 students compete. The winner, who also tied for first last year, and placed highly last year in the U.S.A.M.O. (the last year Canadians could officially compete in that Olympiad), was J.P. Grossman. The breakdown of the top prizes was

First Prize	J.P. Grossman	Fourth Prize	Felix Lo
Second Prize	Eric Lei		A.M. Nicholson
Third Prize	Kevin K.H. Cheung		R.J. Pereira
			N. Sato
			Sheila Su

The "official" solutions to the 1992 C.M.O. (the problems were given in my last column) are reproduced with the permission of the C.M.O. Committee of the Canadian Mathematical Society. My thanks to Professor Ed Barbeau, its chairman, for sending the results to me.

1992 CANADIAN MATHEMATICS OLYMPIAD

April 1992 (Time: 3 hours)

1. Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n+1$ is not an odd prime.

Solution.

Note that $1+2+\cdots+n=\frac{1}{2}n(n+1)$ divides $n!$ if and only if $n+1$ divides $2(n-1)!$. Suppose first that $n+1=2u$, an even number. If $n=1$, then $n+1$ divides $2(n-1)!$. Otherwise, $u\geq 2$ and $u=(n+1)-u\leq n-1$, so that $n+1=2u$ divides $2(n-1)!$.

Now let $n+1$ be odd and composite. Then $n+1=qr$, where $3\leq q\leq r\leq\frac{1}{3}(n+1)$. Then $3\leq q<2r\leq n-1$ (since $n\geq 9$), so that $n+1=qr$ must divide $2(n-1)!$.

Finally, if $n+1=p$ is odd and prime, then p cannot divide $2(n-1)!$. The required result follows.

2. For $x,y,z\geq 0$, establish the inequality

$$x(x-z)^2+y(y-z)^2\geq(x-z)(y-z)(x+y-z)$$

and determine when equality holds.

Solution 1.

$$\begin{aligned}
&x(x-z)^2+y(y-z)^2-(x-z)(y-z)(x+y-z)\\
&\quad=[x(x-y)(x-z)+x(y-z)(x-z)]+[y(y-x)(y-z)+y(x-z)(y-z)]\\
&\qquad\qquad-(x-z)(y-z)(x+y)+z(x-z)y-z)\\
&\quad=x(x-y)(x-z)+y(y-x)(y-z)+z(z-x)(z-y) &(1)\\
&\quad=(x-y)(x^2-xz-y^2+yz)+z(x-z)(y-z)\\
&\quad=(x-y)^2(x+y-z)+z(x-z)(y-z). &(2)
\end{aligned}$$

Expression (1) remains unchanged under any permutation, so there is no loss of generality in assuming $x\geq y\geq z\geq 0$. But then expression (2) is clearly nonnegative. Equality occurs when $x=0$, $y=z$, or $y=0$, $x=z$, or $z=0$, $x=y$, or $x=y=z$.

Solution 2.

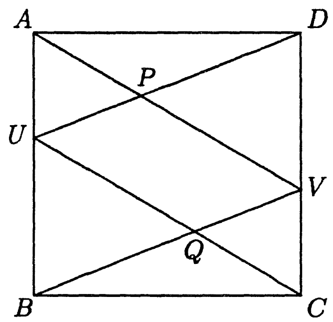
The inequality is equivalent to

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2$$

which is symmetric in x, y and z. Thus, it suffices to check the inequality for any ordering of x, y, z. When $x \geq z \geq y \geq 0$, we have

$$x(x-z)^2 + y(y-z)^2 \geq 0 \geq (x-z)(y-z)(x+y-z).$$

3. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.



Solution.

Let [.] denote area. With no loss of generality, we may assume $BU \geq CV$. Since ΔBQU is similar to ΔVQC, $UQ \geq QC$ with equality if and only if $BU = CV$. Also the distance from B to UC is greater than or equal to the distance from V to UC, with equality if and only if $BU = CV$.

Select a point E in UQ such that $QE = QC$. Then $[BEU] \geq [VEU]$, $[BEQ] = [BQC]$ and $[VEQ] = [VQC]$. Hence $[BQU] + [VQC] \geq [UQV] + [BQC]$. Since $[UQV] = [BQC]$, $[UQV] \leq \frac{1}{4}[UVCB]$. Similarly, $[UPV] \leq \frac{1}{4}[UVDA]$. Hence $[PUQV] \leq \frac{1}{4}[ABCD]$. Equality holds if and only if $BU = CV$, in which case the area of $[PUQV]$ is maximized.

4. Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

Solution 1.

The given equation is equivalent to the quartic

$$0 = x^4 + 2x^3 - x^2 - 6x - 3 = (x^2 + x + 1)^2 - 4(x+1)^2 = (x^2 + 3x + 3)(x^2 - x - 1)$$

whence $x = \frac{1}{2}(-3 \pm i\sqrt{3})$ or $x = \frac{1}{2}(1 \pm \sqrt{5})$.

Solution 2.

The transformation $w = x + 1$ transforms the equation to

$$w^2 - 2w + 1 + 1 - \frac{2}{w} + \frac{1}{w^2} = 3$$

or

$$\left(w + \frac{1}{w}\right)^2 - 2\left(w + \frac{1}{w}\right) - 3 = 0.$$

Hence

$$w + \frac{1}{w} = -1 \text{ or } w + \frac{1}{2} = 3 \text{ i.e. } w^2 + w + 1 = 0 \text{ or } w^2 - 3w + 1 = 0.$$

Therefore,

$$x = w - 1 = \frac{1}{2}(-1 \pm i\sqrt{3}) - 1 = \frac{1}{2}(-3 \pm i\sqrt{3})$$

or

$$x = w - 1 = \frac{1}{2}(3 \pm \sqrt{5}) - 1 = \frac{1}{2}(1 \pm \sqrt{5}).$$

5. A deck of $2n + 1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n + 1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k - 1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

Solution 1.

Suppose the cards labelled i are placed in positions a_i and b_i ($a_i < b_i$) counting from the left, for $i = 1, 2, \ldots, n$. Since the joker occupies position $n + 1$ and $b_i = i + a_i$,

$$\sum_{k=1}^{n} (a_k + b_k) = (1 + 2 + \cdots + n) + (n + 2) + (n + 3) + \cdots + (2n + 1)$$

whence

$$2\sum_{k=1}^{n} a_k = (n + 2) + \cdots + (2n + 1) + \frac{3n(n + 1)}{2}$$

so that $3n(n + 1) = 4[\sum_{k=1}^{n} a_k]$ whence either n or $n + 1$ is divisible by 4. (Note also that $\sum_{k=1}^{n} a_k$ is divisible by 3.)

Thus, if $n = 1, 2, 5, 6, 9, 10$, the task cannot be accomplished.

If $n = 3$, there is essentially one solution: $232J311$.

If $n = 4$, there is essentially one solution: $2423J4311$.

When $n = 7$, and $n = 8$, there are many solutions. For $n = 7$ we have

2	7	2	3	5	6	3	J	7	5	4	6	1	1	4
3	7	2	3	2	6	4	J	7	5	4	6	1	1	5
6	2	7	2	4	5	6	J	4	7	5	3	1	1	3
4	2	7	2	4	5	6	J	3	7	5	3	6	1	1
2	5	2	7	4	6	5	J	4	3	7	6	3	1	1
1	1	2	7	2	4	6	J	5	4	7	3	6	5	3
4	1	1	7	4	3	6	J	3	5	7	2	6	2	5
5	3	4	7	3	5	4	J	6	2	7	2	1	1	6
4	5	3	7	4	3	5	J	6	2	7	2	1	1	6
2	3	2	7	3	4	5	J	6	4	7	5	1	1	6

and, for $n = 8$,

7	8	4	2	6	2	4	7	J	8	6	5	3	1	1	3	5
6	8	5	7	1	1	6	5	J	8	7	4	2	3	2	4	3
2	8	2	4	6	7	5	4	J	8	6	5	7	3	1	1	3

among many others.

Solution 2.

Number the positions for the cards from left to right $1, 2, \ldots 2n+1$. There are n even positions and $n+1$ odd positions. Each pair of cards bearing the same odd integer occupies one even and one odd position, while each pair of cards bearing the same even integer occupies two positions with the same parity. Thus, the number of positions with either parity occupied by the even cards is even.

Let n be even. The joker occupies an odd position. There are $n/2$ even positions occupied by cards bearing odd integers leaving $n/2$ even positions to be occupied by cards bearing even integers. Thus $n/2$ must be even and n divisible by 4.

Let n be odd. The joker occupies an even position. There are $\frac{1}{2}(n+1)$ odd positions occupied by cards bearing an odd integer leaving $\frac{1}{2}(n+1)$ odd positions occupied by cards bearing an even integer. Thus $\frac{1}{2}(n+1)$ must be even and $n+1$ divisible by 4.

Hence, the arrangement is not possible for $n = 1, 2, 5, 6, 9, 10$. Examples illustrate that it is possible for $n = 3, 4, 7, 8$.

* * *

We now turn to solutions sent in by readers to problems of the *1990 Australian Mathematical Olympiad* [1991: 101–102]. We give below only those solutions which differ from the published "official" solutions.

2. Prove that there are infinitely many pairs of positive integers m and n such that n is a factor of $m^2 + 1$ and m is a factor of $n^2 + 1$.

Solutions by George Evagelopoulos, Athens, Greece, and by Chris Wıldhagen, Rotterdam, The Netherlands. We give Wildhagen's solution which uses the Fibonacci sequence!

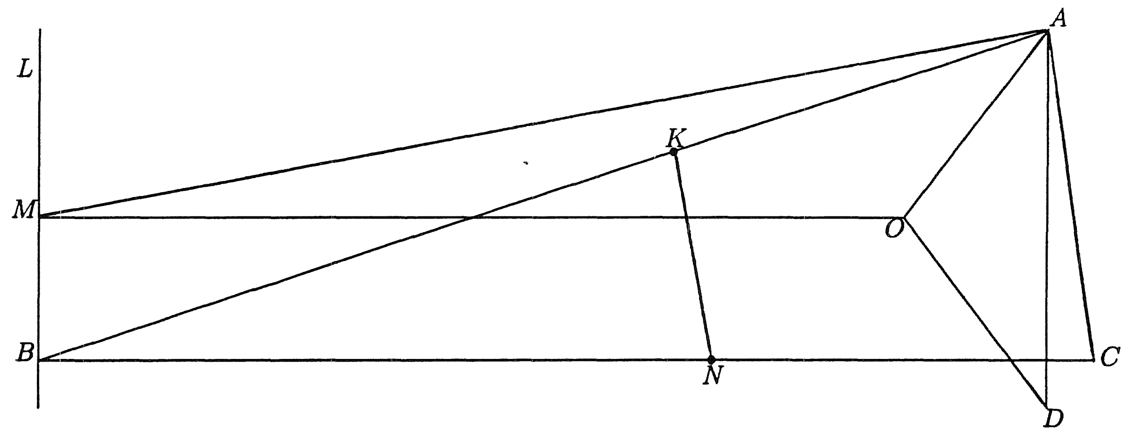
Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence, defined recursively by setting $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$, for $n \geq 1$. By using the Binet formula, it is straightforward to show that $F_{2n+1}^2 + 1 = F_{2n-1}F_{2n+3}$, for all $n \geq 1$. Then, of course, also $F_{2n-1}^2 + 1 = F_{2n-3}F_{2n+1}$, for all $n \geq 2$. From these two observations we find that

$$F_{2n-1}|(F_{2n+1})^2 + 1 \quad \text{and} \quad F_{2n+1}|F_{2n-1}^2 + 1.$$

This implies the desired result.

3. Let ABC be a triangle and k_1 be a circle through the points A and C such that k_1 intersects AB and BC a second time in the points K and N respectively, K and N being different. Let O be the centre of k_1. Let k_2 be the circumcircle of the triangle KBN, and let the circumcircle of the triangle ABC intersect k_2 also in M, a point different from B. Prove that OM and MB are perpendicular.

Solutions by George Evagelopoulos, Athens, Greece; D.J. Smeenk, Zaltbommel, The Netherlands; and by Andy Liu, University of Alberta. We give Liu's solution.



Now $AKNC$ is cyclic with centre O. Also $KMBN$ and $AMBC$ are cyclic. Extend MN to meet k_1 in D so that $AKNCD$ is cyclic. Extend BM to L. We then find:

$$\angle LMA = 180° - \angle BMA = \angle ACB \quad \text{since } AMBC \text{ is cyclic,}$$
$$\angle ACB = \angle ADN \quad \text{since } ANDC \text{ is cyclic,}$$
$$\angle ADN = \angle NKB \quad \text{since } AKND \text{ is cyclic,}$$
$$\angle NKB = \angle NMB \quad \text{since } KMBN \text{ is cyclic.}$$

Thus $LB \| AD$ since $\angle ADN = \angle NMB$. Also

$$\angle LMA = \angle MAD \quad \text{since } LB\|AD,$$
$$MA = MD \quad \text{since } \angle ADN = \angle MAD,$$

$$
\begin{array}{ll}
MO = MO, & \\
\Delta MOA \equiv \Delta MOD & \text{since } OA = OD, \\
\angle AMO = \angle DMO & \text{since } \Delta MOA \equiv \Delta MOD, \\
\angle LMO = \angle BMO & \text{since } \angle AMO + \angle LMA = \angle DMO + \angle NMB.
\end{array}
$$

Thus $OM \perp MB$ because $\angle LMO + \angle BMO = 180°$.

4. A solitaire game is played with an even number of discs, each coloured red on one side and green on the other side. Each disc is also numbered, and there are two of each number; i.e. $\{1,1,2,2,3,3,\ldots,N,N\}$ are the labels. The discs are laid out in rows with each row having at least three discs. A move in this game consists of flipping over simultaneously two discs with the same label. Prove that for *every* initial deal or layout there is a sequence of moves that ends with a position in which no row has only red or only green sides showing.

Solutions by George Evagelopoulos, Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's solution.

Call a row monochromatic if all sides are of the same colour and dichromatic otherwise. If all rows are dichromatic, we are done. Suppose therefore that there exists some monochromatic row R_1. It suffices to show that we can decrease the number of monochromatic rows in a finite number of moves. Flip over some disc D_1 in R_1 together with its mate D_1' (the disc with same label), changing R_1 to a dichromatic row since each row contains at least three discs. If D_1' belongs to R_1, we are done. So suppose D_1' belongs to some row $R_2 \neq R_1$. This gives us the start of the following process.

Choose k maximal such that there are rows $R_1, R_2, \ldots, R_{k+1}$ and discs $D_1, D_1', \ldots, D_k, D_k'$ with the following properties:

1) D_i' is the mate of D_i, $1 \leq i \leq k$.

2) $R_1, R_2, \ldots, R_{k+1}$ are all different.

3) D_i belongs to R_i, $1 \leq i \leq k$.

4) D_i' belongs to R_{i+1}, $1 \leq i \leq k$.

5) All discs of R_i, except D_i, are of one colour.

If R_{k+1} is dichromatic, we are done. Suppose therefore that R_{k+1} is monochromatic. Pick some $D_{k+1} \neq D_k'$ in R_{k+1}. Clearly $D_{k+1}, D_{k+1}' \notin \{D_1, D_1', \ldots, D_k, D_k'\}$. Two possibilities arise:

I) $D_{k+1}' \in R_{k+1}$. By flipping over D_{k+1} and D_{k+1}' the row R_{k+1} transforms to a dichromatic one and we are done.

II) $D_{k+1}' \notin R_{k+1}$, say $D_{k+1}' \in R_{k+2}$. By maximality of k, it follows that $R_{k+2} = R_2$, for some $1 \leq i \leq k$. Now $D_{k+1}' \neq D_i'$, so by flipping over D_{k+1} and D_{k+1}', the row R_{k+1} becomes dichromatic and the row R_i remains dichromatic.

This finishes the proof.

7. For each positive integer n, let $d(n)$ be the number of distinct positive integers that divide n. Determine all positive integers for which $d(n) = n/3$ holds.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; George Evagelopoulos, Athens, Greece; Bob Prielipp, University of Wisconsin–Oshkosh; and Chris Wildhagen, Rotterdam, The Netherlands.

Let n be a positive integer. It is known that $d(n) \leq 2\sqrt{n}$. To see this notice that every divisor $k \leq \sqrt{n}$ can be paired with the divisor n/k greater than $\sqrt{n}$. Now if $n = 3d(n)$ we have that 3 divides n and $n \leq 6\sqrt{n}$ yielding $n \leq 36$. The table below gives the prime factorizations, for the multiples of 3 up to 36, and it is easy to check that the only solutions are 9, 18 and 24.

n	Factorization	$d(n)$	$n/3$
3	3	2	1
6	$2 \cdot 3$	4	2
9	3^2	3	3
12	$2^2 \cdot 3$	6	4
15	$3 \cdot 5$	4	5
18	$2 \cdot 3^2$	6	6
21	$3 \cdot 7$	4	7
24	$2^3 \cdot 3$	8	8
27	3^3	4	9
30	$2 \cdot 3 \cdot 5$	8	10
33	$3 \cdot 11$	4	11
36	$2^2 \cdot 3^2$	9	12

8. Let n be a positive integer. Prove that

$$\frac{1}{\binom{2n}{1}} - \frac{1}{\binom{2n}{2}} + \frac{1}{\binom{2n}{3}} - \cdots + \frac{(-1)^{k-1}}{\binom{2n}{k}} + \cdots + \frac{1}{\binom{2n}{2n-1}} = \frac{1}{n+1} .$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by George Evagelopoulos, Athens, Greece; by Pavlos Maragoudakis, student, University of Athens, Greece; by William Y.C. Chen, Las Alamos National Laboratory, New Mexico, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give the solution of Chen and Wang.

We show in general that for all positive integers $m \geq 2$,

$$(*) \quad \sum_{k=1}^{m-1} \frac{(-1)^{k+1}}{\binom{m}{k}} = \frac{1+(-1)^m}{m+2} = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \frac{2}{m+2} & \text{if } m \text{ is even.} \end{cases}$$

The given identity is the special case when $m = 2n$. To prove $(*)$ note that $m + 2 = (m - k + 1) + (k + 1)$. Hence

$$\begin{aligned}
(m+2)\sum_{k=1}^{m-1} & (-1)^{k+1}k!(m-k)! \\
&= \sum_{k=1}^{m-1}(-1)^{k+1}k!(m-k+1)! + \sum_{k=1}^{m-1}(-1)^{k+1}(k+1)!(m-k)! \\
&= \sum_{k=1}^{m-1}(-1)^{k+1}k!(m-k+1)! + \sum_{k=2}^{m}(-1)^{k}k!(m-k+1)! \\
&= m! + (-1)^m m! \\
&= (1+(-1)^m)m!
\end{aligned}$$

Dividing both sides by $(m+2)m!$, $(*)$ follows immediately.

Comment. The idea of expressing $m+2$ as $(m-k+1)+(k+1)$ and then telescoping the summation is used quite often (and very effectively) in many combinatorial problems. We propose to call it the "borrow-and-return" principle.

* * *

That finishes the solutions we have for the April 1991 number of the Corner, and the space we have available. Send me your contests and your nice solutions.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Codes, Puzzles and Conspiracy, by Dennis Shasha, published by W. H. Freeman and Company, New York, 1992, 241+ pages, hardbound (US$ 17.95) ISBN 0-7167-2314-X, paperback (US$ 11.95) ISBN 0-7167-2275-5. *Reviewed by Andy Liu.*

The publisher offers the reader the following

Proposition. Buy this book.

Proof by Seduction:

This inexpensive book contains 5 contest problems. If you solve them correctly by September 1, 1993, you may win a trip. There is also an optional problem which may be substituted for any of the 5 contest problems. Here, you may have to use a Proof by Abduction. Buy the book and find out what this means. □

Apart from the attraction and challenge of the contest, the reader will also enjoy the intriguing mathematical problems from cover to cover. They provide excellent practice in Proofs by Deduction, Induction, Reduction and so on. Many of them are related to computing science.

The problems are skilfully woven into what may be termed as an information-age spy-thriller. The earlier volume (same author, same publisher and equally highly recommended), "The Puzzling Adventures of Dr. Ecco", ends with the disappearance of the great omniheurist. The present volume reveals that he has been kidnapped. The reader is led on a whirlwind tour of the world (as well as the world of discrete mathematics) in picking up the trail of Dr. Ecco. Encountered on the way are various esoteric characters, including some nefarious ones in high places (shades of Ollie North!). The book also presents a continual discourse on politics, democracy and liberty.

The tale apparently does not end with this book; so we have the pleasure of waiting in anticipation for more episodes and challenging problems from this outstanding series. The review will now end with another look at the publisher's proposition.

Proof by Reproduction:

This is a problem in Chapter 34 of the book, concerning a strange Japanese cult called the Oddists. In a vote, a motion passes if and only if an odd number of members are in favour. Assuming that each member has made up his or her mind on how to vote, the Oddists nevertheless seek an oxymoronic protocol by which they can vote with open secrecy. In phase 1, each Oddist talks to all other members and exchanges some kind of information, without revealing his or her true vote. In phase 2, each Oddist may disguise his or her vote according to the information received. In phase 3, each Oddist casts openly his or her vote determined in phase 2. Can the reader design such a protocol so that the final outcome will always be the same as if every member casts his or her original vote?

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1761. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. Let D be the foot of the perpendicular from C to AB, and let M be the midpoint of CD. Let E be the foot of the perpendicular from A to BM, and let F be the foot of the perpendicular from A to CE. Prove that $AF \leq AB/3$.

1762. *Proposed by Steven Laffin, student, École J. H. Picard, and Andy Liu, University of Alberta, Edmonton.* (Dedicated to Professor David Monk, University of Edinburgh, on his sixtieth birthday.)

Starship Venture is under attack from a Zokbar fleet, and its Terrorizer is destroyed. While it can hold out, it needs a replacement to drive off the Zokbars. Starbase has spare Terrorizers, which can be taken apart into any number of components, and enough scout ships to provide transport. However, the Zokbars have n Space Octopi, each of which can capture one scout ship at a time. Starship Venture must have at least one copy of each component to reassemble a Terrorizer, but it is essential that the Zokbars should not be able to do the same. Into how many components must each Terrorizer be taken apart (assuming all are taken apart in an identical manner), and how many scout ships are needed to transport them? Give two answers:

(a) assuming that the number of components per Terrorizer is as small as possible, minimize the number of scout ships;

(b) assuming instead that the number of scout ships is as small as possible, minimize the number of components per Terrorizer.

1763. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $t \geq 0$, and for each integer $n \geq 1$ define

$$x_n = \frac{1 + t + t^2 + \cdots + t^n}{n+1}.$$

Prove that $x_1 \leq \sqrt{x_2} \leq \sqrt[3]{x_3} \leq \sqrt[4]{x_4} \leq \cdots$.

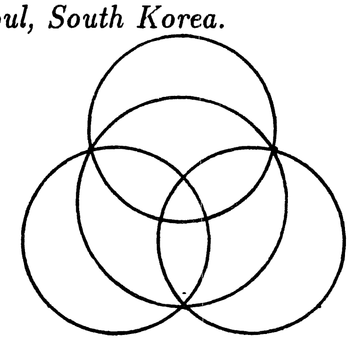
1764. *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Determine the extreme values of $a^2b + b^2c + c^2a$, where a, b, c are sides of a triangle of semiperimeter 1.

(b)* What are the extreme values of $a_1^2a_2 + a_2^2a_3 + \cdots + a_n^2a_1$, where $a_1, a_2, \ldots, a_n$ are the (consecutive) sides of an n-gon of semiperimeter 1?

1765. *Proposed by Kyu Hyon Han, student, Seoul, South Korea.*

There are four circles piled up, making a total of 10 regions. The outer circles each have 5 regions and the central circle has 7 regions. You put one of the numbers $0, 1, 2, \ldots, 9$ in each region, without reusing any number, so that the sum of the numbers in any circle is always the same value, say S. What is the smallest and the largest possible value of S?



1766*. *Proposed by Jun-hua Huang, student, 4th Middle School of Nanxian, Hunan, China.*

The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$, $x_2 = x$, and

$$x_{n+2} = xx_{n+1} + nx_n$$

for $n \geq 0$. Prove or disprove that for each $n \geq 2$, the coefficients of the polynomial $x_{n-1}x_{n+1} - x_n^2$ are all nonnegative, except for the constant term when n is odd.

1767. *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

(a) Jerry Slocum, the American puzzle collector, has the following puzzle, called Double Five, dating from about 1890. Ten coins are arranged in a circle. You are allowed to move a coin, in either direction, over two coins (piled up or not, with empty spaces between allowed) and place it on the next coin to make a pile of two. For example, from the starting position one could place coin 1 on top of coin 4, and then coin 5 on top of coin 3, etc. The object is to make five piles of two coins which are in the even locations, with blank spaces in the odd locations. Can you do it?

(b) Can you instead arrange to have the final piles of two in consecutive locations?

(c)* What about for other (even) numbers of coins?

1768. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let S be any point in the plane of triangle ABC different from a vertex. S_1, S_2, S_3 are the feet of the respective perpendiculars from S to BC, CA, AB, and l_1, l_2, l_3 are the respective perpendiculars from A, B, C to the lines S_2S_3, S_3S_1, S_1S_2.

(a) Show that l_1, l_2, l_3 are concurrent, and

(b) determine the locus of their common point as S moves along the Euler line of triangle ABC.

1769. *Proposed by Jason Colwell, student, Edmonton, Alberta.*

Suppose $Q(x)$ is a polynomial with real coefficients and zero constant term.

(a) Prove that there exists a real polynomial $P(x)$ such that

$$Q(n) = \sum_{k=1}^{n} P(k)$$

for all positive integers n.

(b) Show that $Q(-1) = -P(0)$ for all $P(x)$ and $Q(x)$ as in (a).

1770. *Proposed by F.R. Baudert, Glenstantia, South Africa.*

Find the sum to n terms of the series $6, 7, 16, 17, 26, 27, \ldots$. (Your answer should be a single expression in terms of n.)

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

993. [1984: 318; 1986: 56; 1992: 14] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let P be the product of the $n+1$ positive real numbers $x_1, x_2, \ldots, x_{n+1}$. Find a lower bound (as good as possible) for P if the x_i satisfy

(a) $\displaystyle\sum_{i=1}^{n+1} \frac{1}{1+x_i} = 1;$

(b)* $\displaystyle\sum_{i=1}^{n+1} \frac{a_i}{b_i + x_i} = 1$, where the a_i and b_i are given positive real numbers.

III. *Comment by Murray S. Klamkin, University of Alberta.*

We give a more elementary solution than the one of Maltby [1992: 14] for the case $n = 1$ of part (b). With a slightly different notation, and again using the assumption $a_i \geq 1$ for all i as on [1992: 14], we want the lower bound of xy given that

$$\frac{1+a}{1+x} + \frac{1+b}{1+y} = 1,$$

where a and b are fixed and a, b, x, y are nonnegative. Clearly we must have $x > a$ and $y > b$. Eliminating y, we have after some algebra that

$$xy = b(x-a) + \frac{a(a+1)(b+1)}{x-a} + 2ab + a + b + 1.$$

Then by the A.M.–G.M. inequality,

$$\min xy = 2ab + a + b + 1 + 2\sqrt{ab(a+1)(b+1)},$$

which agrees with Maltby. There is equality if and only if

$$b(x-a) = \frac{a(a+1)(b+1)}{x-a}$$

or

$$x = a + \sqrt{\frac{a(a+1)(b+1)}{b}} \quad \text{and symmetrically} \quad y = b + \sqrt{\frac{b(a+1)(b+1)}{a}}$$

(we are assuming $ab \neq 0$). If say $b = 0$, then (as on [1992: 14]) the best lower bound is $a + 1$ and is obtained by taking y arbitrarily small; similarly if $a = 0$ the best lower bound is $b + 1$.

* * * * *

1137*. [1986: 79, 177(revised); 1987: 228; 1988: 79] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s}$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

IV. *Solution by Ji Chen, Ningbo University, China.*

We first show that

$$4m_b m_c \geq 2a^2 - 4b^2 - 4c^2 + 9bc, \quad \text{etc.} \tag{1}$$

This holds because

$$4m_b^2 = 2c^2 + 2a^2 - b^2, \quad \text{etc.} \tag{2}$$

and

$$\begin{aligned}(2c^2 + 2a^2 - b^2)&(2a^2 + 2b^2 - c^2) - (2a^2 - 4b^2 - 4c^2 + 9bc)^2 \\ &= -18b^4 - 18c^4 + 72b^3c + 72bc^3 - 108b^2c^2 + 18c^2a^2 + 18a^2b^2 - 36a^2bc \\ &= 18(b-c)^2[a^2 - (b-c)^2] \geq 0.\end{aligned}$$

Now the given inequality is equivalent to

$$\sum \frac{1}{a} \cdot \sum m_a > \frac{15}{2}, \tag{3}$$

where the sums are cyclic over a, b, c, by median duality; see equation (1) on [1987: 228]. (3) is equivalent to

$$\left(\sum \frac{1}{a}\right)^2 \left(\sum m_a^2 + 2\sum m_b m_c\right) > \frac{225}{4},$$

which by (1) and (2) is implied by

$$\left(\sum \frac{1}{a}\right)^2 \left[\frac{3}{4}\sum a^2 + \frac{1}{2}\sum (2a^2 - 4b^2 - 4c^2 + 9bc)\right] > \frac{225}{4}.$$

This is equivalent to

$$\left(\sum \frac{1}{a}\right)^2 (2\sum bc - \sum a^2) > 25,$$

i.e.,

$$(\sum bc)^2(2\sum bc - \sum a^2) > 25a^2b^2c^2,$$

which is true because

$$\begin{aligned}(\sum bc)^2&(2\sum bc - \sum a^2) - 25a^2b^2c^2 \\ &= -\sum(a^4b^2 + a^4c^2) - 2\sum a^4bc + 2\sum b^3c^3 + 4\sum(a^3b^2c + a^3c^2b) - 16a^2b^2c^2 \\ &= \frac{1}{16}\sum(b-c)^2[a^2 - (b-c)^2][15a^2 + (b-c)^2] \\ &\qquad + \frac{1}{8}\left(\sum a^3 + 11abc\right)\prod(b+c-a) + \frac{1}{4}\prod(b+c-a)^2 \\ &> 0.\end{aligned}$$

[*Editor's note.* As in Chen's recent solution of *Crux* 1663 [1992: 188], the editor would like to see an easier proof that the above expression is positive.]

*   *   *   *   *

1659*. [1991: 172] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

For any integer $n > 1$, prove or disprove that the largest coefficient in the expansion of

$$(1 + 2x + 3x^2 + 4x^3)^n$$

is the coefficient of x^{2n}.

Solution by Shalosh B. Ekhad, Temple University, Philadelphia, Pennsylvania.

Here is a proof of this fact. First observe that the polynomial

$$f(x) = (1 + 2x + 3x^2 + 4x^3)^n$$

is unimodal [i.e., its coefficients increase and then decrease when listed in the standard order]. In fact, it even enjoys the stronger property of log-concavity [i.e., the coefficients satisfy $c_i^2 \geq c_{i-1}c_{i+1}$ for $i = 1, 2, \ldots, 3n - 1$], because the product of two log-concave polynomials with nonnegative coefficients and without internal zeros is still a log-concave polynomial with nonnegative coefficients and without internal zeros (see [3], Proposition 2, p. 503 — my thanks to W.Y.C. Chen for informing me of this reference). Hence it suffices to show that the difference between the coefficients of x^{2n} and x^{2n-1}, and between the coefficients of x^{2n} and x^{2n+1}, are both positive. Calling these two differences $C(n)$ and $D(n)$ respectively, we have to prove that for $n > 1$, $C(n) > 0$ and $D(n) > 0$. By direct inspection, these are true for $n = 2, 3, 4$; hence, thanks to the recurrences established in the two lemmas below, these inequalities are true in general. The recurrences were found by the algorithm in [1].

[*Editor's note.* At this point the editor consulted expert colleague Len Bos for some much-needed assistance in understanding this solution. In what follows, Ekhad's solution has been freely augmented with Bos's explanation.]

The coefficient of x^k in $f(x)$ is just

$$\frac{1}{k!} f^{(k)}(0),$$

where $f^{(k)}(0)$ is the kth derivative of $f(x)$, evaluated at $x = 0$. Thus

$$C(n) = \frac{1}{(2n)!} f^{(2n)}(0) - \frac{1}{(2n-1)!} f^{(2n-1)}(0).$$

Therefore, by the Cauchy integral formula of complex variables (e.g., see Theorem 19, p. 150 of [2]),

$$C(n) = \frac{1}{2\pi i}\left(\int_C \frac{f(z)}{z^{2n+1}}\,dz - \int_C \frac{f(z)}{z^{2n}}\,dz\right) = \frac{1}{2\pi i}\int_C \frac{f(z)(1-z)}{z^{2n+1}}\,dz, \tag{1}$$

where C is some simple closed contour containing the origin. [Kids, don't try this at home.—*Ed.*] Now let

$$a(n) = 200(n+3)(n+2)(n+1), \quad b(n) = 10(n+3)(n+2)(6n+7),$$

$$c(n) = 4(n+3)(2n+3)(4n+9), \quad d(n) = 2(n+3)(2n+3)(n+4),$$

and

$$P_n(g(n)) = -a(n)g(n) - b(n)g(n+1) - c(n)g(n+2) + d(n)g(n+3)$$

where $g(n)$ is a function. Also let

$$F(n,x) = \frac{f(x)(1-x)}{x^{2n+1}} = \frac{(1+2x+3x^2+4x^3)^n(1-x)}{x^{2n+1}}\,.$$

Then the first lemma is

$$P_n(F(n,x)) = \frac{d}{dx}\,G(n,x), \tag{2}$$

where

$$\begin{aligned}G(n,x) = {}& \frac{F(n,x)(1+2x+3x^2+4x^3)}{x^5(x-1)}\cdot(64n^2x^7+288nx^7+288x^7-48n^2x^6-184nx^6\\ &-168x^6-32n^2x^5-76nx^5-12x^5-20n^2x^4-30nx^4+15x^4+20n^2x^3\\ &+150nx^3+210x^3+6n^2x^2+47nx^2+57x^2+8n^2x+44nx+48x\\ &+2n^2+11n+12).\end{aligned}$$

It follows from (1) that

$$P_n(C(n)) = \frac{1}{2\pi i}\int_C P_n(F(n,z))\,dz = \frac{1}{2\pi i}\int_C \frac{d}{dz}\,G(n,z)\,dz = 0,$$

e.g., see Corollary 2, p. 115 of [2]. In other words,

$$-a(n)C(n) - b(n)C(n+1) - c(n)C(n+2) + d(n)C(n+3) = 0$$

or

$$C(n+3) = \frac{a(n)C(n)+b(n)C(n+1)+c(n)C(n+2)}{d(n)}\,,$$

and since

$$a(n)>0,\quad b(n)>0,\quad c(n)>0,\quad d(n)>0,$$

and by direct calculation

$$C(2)>0,\quad C(3)>0,\quad C(4)>0,$$

it follows by induction that $C(n) > 0$ for all $n > 1$, as desired.

In a similar manner,

$$D(n) = \frac{1}{(2n)!}\,f^{(2n)}(0) - \frac{1}{(2n+1)!}\,f^{(2n+1)}(0) = \frac{1}{2\pi i}\int_C \frac{f(z)(z-1)}{z^{2n+2}}\,dz.$$

Putting

$$a'(n) = 100(n+2)(n+1)(4n+9),\quad b'(n) = 5(n+2)(24n^2+82n+75),$$

$$c'(n) = (2n+5)(32n^2+112n+105),\quad d'(n) = (n+3)(4n+5)(2n+7),$$

and defining

$$P_n'(g(n)) = -a'(n)g(n) - b'(n)g(n+1) - c'(n)g(n+2) + d'(n)g(n+3)$$

and

$$F'(n,x) = \frac{f(x)(x-1)}{x^{2n+2}} = \frac{(1+2x+3x^2+4x^3)^n(x-1)}{x^{2n+2}},$$

the second lemma is

$$P_n'(F'(n,x)) = \frac{d}{dx}G'(n,x), \tag{3}$$

where

$$\begin{aligned} G'(n,x) = \frac{F'(n,x)(1+2x+3x^2+4x^3)}{2x^5(x-1)} &\cdot (256n^2x^7 + 1216nx^7 + 1120x^7 - 192n^2x^6 \\ &- 592nx^6 - 840x^6 - 128n^2x^5 - 224nx^5 - 380x^5 - 80n^2x^4 - 140nx^4 \\ &- 280x^4 + 80n^2x^3 + 440nx^3 + 385x^3 + 24n^2x^2 + 134nx^2 + 100x^2 \\ &+ 32n^2x + 132nx + 115x + 8n^2 + 34n + 30). \end{aligned}$$

From this it follows as before that

$$P_n'(D(n)) = 0,$$

and we conclude that also $D(n) > 0$ for $n > 1$. [*Editor's note.* Statements (2) and (3), which Ekhad obtained via an algorithm from the Almkvist–Zeilberger paper [1], can be checked by computer, and were, and they work!]

References:

[1] G. Almkvist and D. Zeilberger, The method of differentiating under the integral sign, *J. Symbolic Computation* **10** (1990) 571–591.

[2] E.B. Saff and A.D. Snider, *Fundamentals of Complex Analysis*, Prentice-Hall, 1976.

[3] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Graph Theory and its Applications: East and West,* in *Annals N.Y. Acad. Sci.* **576** (1989) 500–535.

Two other readers, and the proposer, checked that the given result held for n less than various values. The problem was suggested by problem 5.2.1 *(proposed by Tan Seaw San) in* Menemui Matematik 5 (1983), *p.* 100, *which was to verify the case* $n = 20$.

This is a remarkable result. It would be nice to have a simpler proof, say one in which it is not necessary to write out the functions $G(n,x)$ and $G'(n,x)$!

* * * * *

1667. [1991: 208] *Proposed by Stephen D. Hnidei, Windsor, Ontario.*

Evaluate

$$\sum_{n=1}^{\infty} \coth^{-1}(2^{n+1} - 2^{-n}).$$

Solution by Carles Romero Chesa, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain.

From

$$\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

we have

$$\begin{aligned}\coth^{-1}(2^{n+1}-2^{-n}) &= \frac{1}{2}\ln\left(\frac{2^{n+1}-2^{-n}+1}{2^{n+1}-2^{-n}-1}\right)\\ &= \frac{1}{2}\ln\left(\frac{2^{2n+1}-1+2^n}{2^{2n+1}-1-2^n}\right)\\ &= \frac{1}{2}\ln\frac{(2^n+1)(2^{n+1}-1)}{(2^n-1)(2^{n+1}+1)}\ .\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n=1}^{k}\coth^{-1}(2^{n+1}-2^{-n}) &= \frac{1}{2}\sum_{n=1}^{k}\ln\frac{(2^n+1)(2^{n+1}-1)}{(2^n-1)(2^{n+1}+1)}\\ &= \frac{1}{2}\ln\prod_{n=1}^{k}\frac{(2^n+1)(2^{n+1}-1)}{(2^n-1)(2^{n+1}+1)}\\ &= \frac{1}{2}\ln\frac{(2^1+1)(2^2-1)(2^2+1)(2^3-1)\ldots(2^k+1)(2^{k+1}-1)}{(2^1-1)(2^2+1)(2^2-1)(2^3+1)\ldots(2^k-1)(2^{k+1}+1)}\\ &= \frac{1}{2}\ln\frac{3(2^{k+1}-1)}{2^{k+1}+1}\ .\end{aligned}$$

Finally we have

$$\sum_{n=1}^{\infty}\coth^{-1}(2^{n+1}-2^{-n}) = \frac{1}{2}\ln\left(\lim_{k\to\infty}\frac{3(2^{k+1}-1)}{2^{k+1}+1}\right) = \frac{1}{2}\ln 3.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; LEN BOS, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; ALBERT KURZ, student, Council Rock H.S., Holland, Pennsylvania; KEE-WAI LAU, Hong Kong; JEAN-MARIE MONIER, Lyon, France; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

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1668. [1991: 208] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

What is the envelope of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

as a and b vary so that $a^2+b^2=1$?

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Eliminating b from the two equations given in the problem statement gives

$$\frac{x^2}{a^2}+\frac{y^2}{1-a^2}=1. \tag{1}$$

The envelope of the family of ellipses is obtained by eliminating a from (1) and the derivative of (1) with respect to a. This derivative is

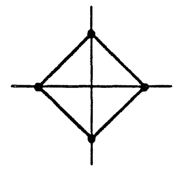
$$-\frac{2x^2}{a^3}+\frac{2ay^2}{(1-a^2)^2}=0,$$

whence $x^2(1-a^2)^2=y^2a^4$. Taking the square root of each side gives $x(1-a^2)=\pm ya^2$ or

$$a^2=\frac{x}{x\pm y}\,. \tag{2}$$

Substituting (2) into (1) gives $(x\pm y)^2=1$, which yields the four equations

$$x+y=1,\quad x-y=1,\quad x+y=-1,\quad x-y=-1.$$



These four equations define the envelope as a square connecting the points $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$, which is what one suspected all along after having drawn two degenerate ellipses and a circle ($a=1$, $a=0$, and $a=\sqrt{2}/2$).

My compliments to Dr. Rabinowitz, who continues to entertain us with interesting and elegant problems!

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

R.P. Sealy, Mount Allison University, located the problem (slightly generalized) as #15, p. 296 of Olmsted's Advanced Calculus *(Prentice–Hall, 1961). Hut points out that a much wider generalization, to find the envelope of*

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^p=1 \quad \textit{subject to} \;\; a^q+b^q=c^q \quad (c>0),$$

is a very old problem, and is worked out in the Dutch textbook F. Schuh and J.G. Rutgers, Compendium der Hoogere Wiskunde, Part IV, 1928, *pp.* 152–155. *This generalization was in fact the proposer's original problem! (The editor chose to feature only a special case.) As Hut points out, with $p = q = 2$ as in the given problem, but allowing a^2 or b^2 to be negative, one gets hyperbolas as well as ellipses, and the envelope becomes the four straight lines obtained by extending the sides of the above square.*

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1669. [1991: 208] *Proposed by J.P. Jones, University of Calgary.*

Suppose that q is a rational number with $|q| \leq 2$, and that

$$\frac{q + i\sqrt{4-q^2}}{2}$$

is an nth root of unity for some n. Show that q must be an integer.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

If $(q + i\sqrt{4-q^2})/2$ is an nth root of unity for some n, there exist $n \in \mathbf{N}$ and $k \in \mathbf{Z}$ such that

$$\frac{q + i\sqrt{4-q^2}}{2} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\ .$$

That is,

$$q = 2\cos\frac{2k\pi}{n}\ , \quad \sqrt{4-q^2} = 2\sin\frac{2k\pi}{n}\ .$$

It is known (e.g., see Corollary 3.12, p. 41 of [1]) that the only rational values of $\cos(2k\pi/n)$ are 0, $\pm 1/2$, ± 1. Then

$$\cos\frac{2k\pi}{n} = 0 \quad \Rightarrow \quad q = 0 \in \mathbf{Z};$$

$$\cos\frac{2k\pi}{n} = \pm\frac{1}{2} \quad \Rightarrow \quad q = \pm 1 \in \mathbf{Z};$$

$$\cos\frac{2k\pi}{n} = \pm 1 \quad \Rightarrow \quad q = \pm 2 \in \mathbf{Z}.$$

Reference:

[1] I. Niven, *Irrational Numbers*, Carus Mathematical Monographs 11, MAA, 1967.

II. *Solution by Robert B. Israel, University of British Columbia.*

In fact the result is true with "nth root of unity" replaced by "algebraic integer", i.e., a root of a polynomial with integer coefficients. Let $\zeta = (q + i\sqrt{4-q^2})/2$. If ζ is an algebraic integer, then so is its complex conjugate $\bar{\zeta}$. Thus $q = \zeta + \bar{\zeta}$ is an algebraic integer (the sum of two algebraic integers is an algebraic integer — e.g., see Theorem 9.12, p. 417 of [1]). But any rational number that is an algebraic integer is an ordinary integer (Theorem 9.9, p. 415 of [1]), so q is an integer.

[*Editor's note.* Israel went on to give a second proof, avoiding using the result that the sum of two algebraic integers is an algebraic integer, which, as he says, is "less than obvious".]

Reference:

[1] I. Niven, H.S. Zuckerman and H.L. Montgomery, *An Introduction to the Theory of Numbers*, fifth edition, John Wiley & Sons, 1991.

Also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; JEAN-MARIE MONIER, Lyon, France; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

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1670*. [1991: 208] *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let B_1, B_2, C_1, C_2 be points in the plane and let lines B_1B_2 and C_1C_2 intersect in A. Prove that the four points $G_{11}, G_{12}, G_{21}, G_{22}$ form the vertices of a parallelogram when G_{ij} is determined in any of the following ways: (i)G_{ij} is the centroid of ΔAB_iC_j; (ii) G_{ij} is the orthocenter of ΔAB_iC_j; (iii) G_{ij} is the circumcenter of ΔAB_iC_j.

Solution by José Yusty Pita, Madrid, Spain.

(i) Let B' be the midpoint of AB_1 and B'' the midpoint of AB_2. Join each of these points to C_1 and C_2; the barycenters (centroids) will be 1/3 of the way along these lines, and therefore $G_{11}G_{12}$ and $G_{21}G_{22}$ are parallel to C_1C_2 and equal to 1/3 of its length. Thus $G_{11}G_{12} = G_{21}G_{22}$ and they are parallel, so they form a parallelogram.

(ii) The altitudes from B_1 and B_2 will be perpendicular to AC_1 and AC_2, respectively; therefore $G_{11}G_{12}$ and $G_{21}G_{22}$ are parallel. The same argument applies to the altitudes from C_1 and C_2. They clearly form a parallelogram.

(iii) The circumcenters G_{ij} will lie on the perpendicular bisectors of AB_1 and AB_2, and also on the perpendicular bisectors of AC_1 and AC_2; thus $G_{11}G_{12}$ and $G_{21}G_{22}$ are parallel, and so are $G_{11}G_{21}$ and $G_{12}G_{22}$, so they also form a parallelogram.

Also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; CARLES ROMERO CHESA, I.B. Manuel Blancafort, La Garriga, Catalonia, Spain; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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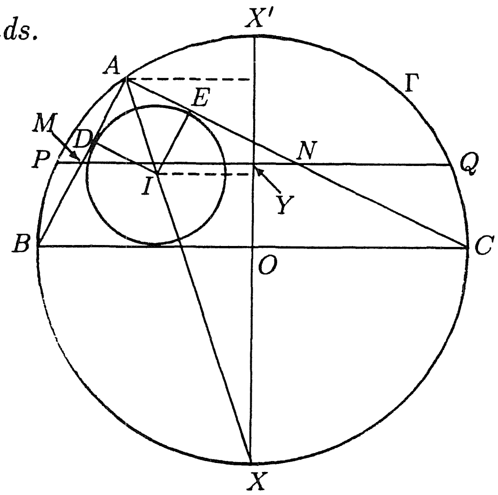
1671. [1991: 237] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A right triangle ABC with right angle at A is inscribed in a circle Γ. Let M, N be the midpoints of AB, AC, and let P, Q be the points of intersection of the line MN with Γ. Let D, E be the points where AB, AC are tangent to the incircle. Prove that D, E, P, Q are concyclic.

Solution by P. Penning, Delft, The Netherlands.



Let I be the centre and r the radius of the incircle. The line AI is the bisector of the angle at A and intersects the circle Γ in X, the midpoint of the arc BXC. Any circle passing through D and having centre on AX also passes through E. The line MN is parallel to BC. The perpendicular bisector of BC and PQ is OX, where O [the circumcentre] is the midpoint of BC. Any circle passing through P and having centre on OX also passes through Q. If $PX = EX$ then the circle with X as centre and passing through P also passes through E, D and Q.

Thus we need only prove $PX = EX$. Let $x = \angle AXO$, let h be the altitude from A of ΔABC, and put the radius of the circumcircle equal to 1. In right triangle $XX'A$ (XX' a diameter of Γ),

$$2\cos x = AX = \frac{1+h}{\cos x}\,, \tag{1}$$

because the projection of AX on XX' has length $1+h$. By the cosine rule in ΔEAX,

$$\begin{aligned}(EX)^2 &= (AX)^2 + (AE)^2 - \sqrt{2}\,AE\cdot AX\\ &= 4\cos^2 x + r^2 - 2\sqrt{2}\,r\cos x\\ &= (\sqrt{2}\,\cos x - r)^2 + 2\cos^2 x.\end{aligned} \tag{2}$$

Here

$$r + 1 = \sqrt{2}\,\cos x. \tag{3}$$

[Since $\angle IBC + \angle ICB = 45°$, $\angle BIC + \frac{1}{2}\angle BXC = 180°$ and thus X is the centre of circle BIC; hence

$$\cos x = \frac{r+1}{XI} = \frac{r+1}{XB} = \frac{r+1}{\sqrt{2}}\,;$$

apologies if there is an easier proof of (3).—*Ed.*] By (3) and (1), (2) becomes

$$(EX)^2 = 1 + 2\cos^2 x = 2 + h,$$

while, from right triangles PYX and PYO,

$$(PX)^2 = \left(1 + \frac{h}{2}\right)^2 + \left(1 - \frac{h^2}{4}\right) = 2 + h = (EX)^2,$$

so $PX = EX$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA

ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; DAVID HANKIN, Brooklyn, New York; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; SHAILESH A. SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. There was one incorrect solution sent in.

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1673. [1991: 237] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

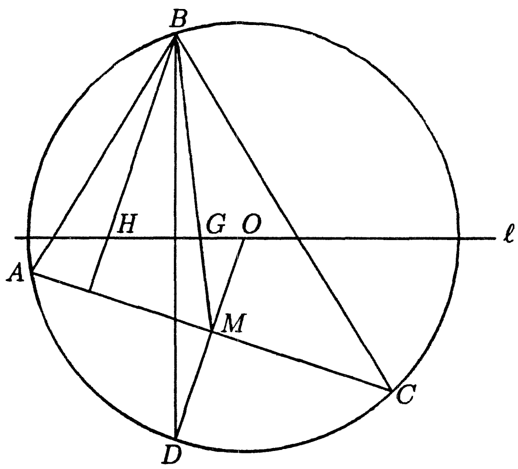
Triangle ABC is nonequilateral and has angle $\beta = 60°$. A' is an arbitrary point of line BA, not coinciding with B or A. C' is an arbitrary point of BC, not coinciding with B or C.

(a) Show that the Euler lines of ΔABC and $\Delta A'BC'$ are parallel or coinciding.

(b) In the case of coincidence, show that the circumcircles of all such triangles $A'BC'$ meet the circumcircle of ABC at a fixed point.

I. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

Soient M le milieu de AC, D le milieu de l'arc AC, et O, G, H le centre du cercle circonscrit, le centre de gravité et l'orthocentre du triangle ABC.



(a) $OM = R\cos 60° = R/2$, d'où $BH = 2OM = R$ [since $BH \| OM$, triangles BHG and MOG are similar, and $HG = 2GO$.—*Ed.*], et le quadrilatère $HBOD$ où $BH = BO = OD = R$ est un losange. La droite d'Euler ℓ est une diagonale de ce losange et donc est perpendiculaire à l'autre diagonale, BD, qui est la bissectrice de l'angle B. Le même raisonnement s'appliquant au triangle $A'BC'$, on aura aussi ℓ', droite d'Euler de $A'BC'$, perpendiculaire à la bissectrice de l'angle B. D'où $\ell \,\|\, \ell'$.

(b) Si $\ell = \ell'$, alors la droite OO' (O' le centre du cercle circonscrit du triangle $A'BC'$) est médiatrice du segment BD et les deux cercles circonscrits se coupent en B et en D.

II. *Comment by Jordi Dou, Barcelona, Spain.*

[Dou first solved the problem, in a way similar to the above.—*Ed.*]

An easy characterization of triangles $A'BC'$, considered in (b), whose Euler lines coincide with ℓ is that *sides $A'C'$ are tangent to the parabola π with focus D and directrix ℓ.* This is because the point symmetric to D with respect to side $C'A'$ is the circumcentre of $A'BC'$ and lies on ℓ. D and ℓ are fixed, thus sides $C'A'$ are tangent to π.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

Many solvers noted the result, given in solution I, *that if $\angle B = 60°$ then the Euler line of $\triangle ABC$ is perpendicular to the bisector of $\angle B$. Bellot Rosado and López Chamorro, in fact, point out that this (with its converse) is part* (a) *of problem* 1232 *of* Mathematics Magazine *(solution on pages* 43–45 *of the February* 1987 *issue), proposed by none other than current proposer Smeenk and departed* Crux *stalwart J.T. Groenman!*

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1674. [1991: 237] *Proposed by Murray S. Klamkin, University of Alberta.*

Given positive real numbers r, s and an integer $n > r/s$, find positive $x_1, x_2, \ldots, x_n$ so as to minimize

$$\left(\frac{1}{x_1^r} + \frac{1}{x_2^r} + \cdots + \frac{1}{x_n^r}\right)(1+x_1)^s(1+x_2)^s \ldots (1+x_n)^s.$$

Solution by Kee-Wai Lau, Hong Kong.

By the arithmetic mean – geometric mean inequality,

$$\sum_{k=1}^{n} x_k^{-r} \geq n \prod_{k=1}^{n} x_k^{-r/n}.$$

Hence the function of the problem is greater than or equal to

$$n \prod_{k=1}^{n} (1+x_k)^s x_k^{-r/n}.$$

For $x > 0$ let $f(x) = (1+x)^s x^{-r/n}$. Then $\lim_{x\to 0^+} f(x) = \infty$, and $\lim_{x\to\infty} f(x) = \infty$ because $n > r/s$. Thus $f(x)$ attains its minimum when $f'(x) = 0$, i.e.

$$0 = \frac{d}{dx}\frac{(1+x)^s}{x^{r/s}} = \frac{(1+x)^{s-1}}{nx^{1+r/n}}(nsx - rx - r)$$

or $x = r/(ns - r)$. It follows that the function of the problem is minimized if and only if

$$x_1 = x_2 = \cdots = x_n = \frac{r}{ns - r}.$$

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1675. [1991: 237] *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

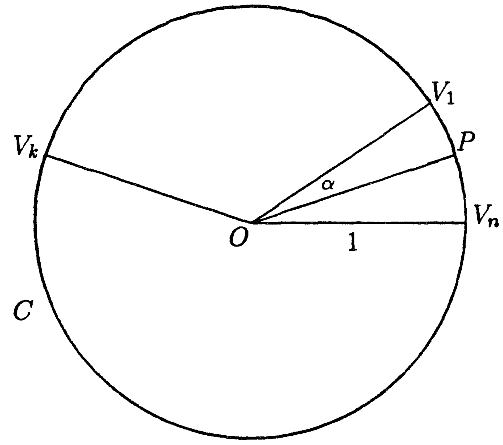
Let $V_1, V_2, \ldots, V_n$ denote the vertices of a regular n-gon inscribed in a unit circle C where $n \geq 3$, and let P be an arbitrary point on C. It is known that $\sum_{k=1}^{n} \overline{PV_k}^2$ is a constant.

(a) Show that $\sum_{k=1}^{n} \overline{PV_k}^4$ is also a constant.

(b) Does there exist a value of $m \neq 1, 2$ and a value of $n \geq 3$ such that $\sum_{k=1}^{n} \overline{PV_k}^{2m}$ is independent of P?

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let $P \in \mathrm{arc}(V_n V_1)$ and put $\alpha = \angle POV_1$.



Then

$$\angle POV_k = \alpha + \frac{2(k-1)\pi}{n}$$

and hence

$$\overline{PV_k} = 2\sin\left(\frac{\alpha}{2} + \frac{(k-1)\pi}{n}\right). \qquad (1)$$

In order to deal with $\sum_{k=1}^{n} \overline{PV_k}^{2m}$ we therefore have to deal with

$$\sum_{j=0}^{n-1} 4^m \sin^{2m}\left(\frac{\alpha}{2} + \frac{j\pi}{n}\right). \qquad (2)$$

Now for any t

$$4^m \sin^{2m} t = \binom{2m}{m} + 2\sum_{p=0}^{m-1}(-1)^{m+p}\binom{2m}{p}\cos 2(m-p)t$$

(see [1], item 1.320.1). Hence, via (2),

$$\sum_{k=1}^{n} \overline{PV_k}^{2m} = n\binom{2m}{m} + 2\sum_{p=0}^{m-1}(-1)^{m+p}\binom{2m}{p}\sum_{j=0}^{n-1}\cos\left((m-p)\alpha + 2(m-p)\frac{j\pi}{n}\right). \qquad (3)$$

Next we have the following general formula for cosine sums:

$$\sum_{j=0}^{n-1}\cos(a + jh) = \frac{\sin(nh/2)\cos[a + (n-1)h/2]}{\sin(h/2)}$$

(see [1], item 1.341.3). Thus

$$\sum_{j=0}^{n-1}\cos\left((m-p)\alpha + 2(m-p)\frac{j\pi}{n}\right)$$
$$= \frac{\sin[(m-p)\pi]\cos[(m-p)\alpha + (n-1)(m-p)\pi/n]}{\sin[(m-p)\pi/n]}. \qquad (4)$$

Whenever $m \leq n-1$, we get

$$\frac{\pi}{n} \leq \frac{(m-p)\pi}{n} \leq \frac{(n-1)\pi}{n} < \pi$$

for all $0 \leq p \leq m-1$, thus $\sin[(m-p)\pi/n] \neq 0$ while $\sin[(m-p)\pi] = 0$, and hence by (4)

$$\sum_{j=0}^{n-1} \cos\left((m-p)\alpha + 2(m-p)\frac{j\pi}{n}\right) = 0.$$

By (3) this leads finally to

$$\sum_{k=1}^{n} \overline{PV_k}^{2m} = n\binom{2m}{m},$$

independent of P whenever $n \geq m+1$. In particular for $m = 1, 2$, and for $n \geq 3$, we get the results

$$\sum_{k=1}^{n} \overline{PV_k}^2 = 2n, \quad \sum_{k=1}^{n} \overline{PV_k}^4 = 6n.$$

Reference:

[1] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products,* Academic Press, 1980.

Both parts also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; RICHARD I. HESS, Rancho Palos Verdes, California; P. PENNING, Delft, The Netherlands; and the proposers. Part (a) *only solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; MURRAY S. KLAMKIN, University of Alberta; and SHOBHIT SONAKIYA, Kanpur, India.*

Penning obtained the same general result as Janous. Hess did it for several specific values of $m > 2$, while the proposers worked out the case $m = 3$, $n = 4$. Bluskov points out that the (most) general result:

$$\sum_{k=1}^{n} \overline{PV_k}^p \textit{ is independent of } P \textit{ if and only if } p = 2, 4, \ldots, 2(n-1)$$

(p a positive integer), was solved by Oleg Mushkarov in his article "Trigonometric polynomials and regular n-gons", in the Bulgarian journal Mathematics 8 (1982) 11–16. *Bellot Rosado and López Chamorro spotted the case $n = 3$ of part* (a) *as a problem on the* 1979 *Vietnam Mathematical Olympiad.*

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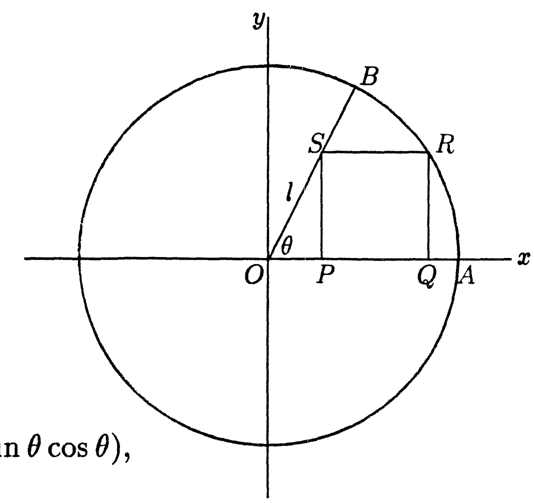
1676. [1991: 238] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

OA is a fixed radius and OB a variable radius of a unit circle, such that $\angle AOB \leq 90°$. $PQRS$ is a square inscribed in the sector OAB so that PQ lies along OA. Determine the minimum length of OS.

Solution by Phil Reiss, student, University of Manitoba, Winnipeg.

We wish to express l, the length of OS, as a function of $\theta = \angle AOB$. As the diagram makes clear, $P = (l\cos\theta, 0)$ and $S = (l\cos\theta, l\sin\theta)$. Also, R has y-coordinate $l\sin\theta$ and lies on the circle $x^2+y^2=1$, so



$$R = (\sqrt{1-l^2\sin^2\theta}, l\sin\theta).$$

Since $PQRS$ is a square, we have $PS = SR$, so

$$l\sin\theta = \sqrt{1-l^2\sin^2\theta} - l\cos\theta,$$
$$1 - l^2\sin^2\theta = l^2(\sin\theta+\cos\theta)^2 = l^2(1+2\sin\theta\cos\theta),$$

$$\begin{aligned} 1 &= l^2(1+2\sin\theta\cos\theta+\sin^2\theta) \\ &= l^2\left(1+\sin 2\theta + \frac{1-\cos 2\theta}{2}\right) = l^2\left(\frac{3}{2}+\sin 2\theta - \frac{1}{2}\cos 2\theta\right), \end{aligned}$$

and so

$$l^2 = \left(\frac{3}{2}+\sin 2\theta - \frac{1}{2}\cos 2\theta\right)^{-1}. \tag{1}$$

From (1) we see that we can minimize l^2, and thus l, by maximizing

$$f(\theta) = \frac{3}{2}+\sin 2\theta - \frac{1}{2}\cos 2\theta.$$

$f'(\theta_0) = 0$ gives $2\cos 2\theta_0 + \sin 2\theta_0 = 0$ and thus

$$\tan 2\theta_0 = -2. \tag{2}$$

Given that $0° \le \theta_0 \le 90°$, (2) implies

$$\sin 2\theta_0 = 2/\sqrt{5}, \quad \cos 2\theta_0 = -1/\sqrt{5}. \tag{3}$$

Clearly, $90° < 2\theta_0 < 180°$ for this θ_0, so that $f''(\theta_0) = 2\cos 2\theta_0 - 4\sin 2\theta_0 < 0$. Thus this θ_0 indeed produces a maximum value for $f(\theta)$, so we may minimize OS by substituting the values from (3) into (1) to obtain

$$\begin{aligned} OS = l &= \left(\frac{3}{2}+\frac{2}{\sqrt{5}}+\frac{1}{2\sqrt{5}}\right)^{-1/2} = \left(\frac{3\sqrt{5}+5}{2\sqrt{5}}\right)^{-1/2} \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^{-1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \frac{\sqrt{5}-1}{2}. \end{aligned}$$

Also solved by HAYO AHLBURG, Benidorm, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, *Franz-Ludwig-Gymnasium, Bamberg, Germany; DAVID HANKIN, Brooklyn, New York; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There were four incorrect solutions sent in.*

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1677. [1991: 238] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Evaluate (without rearranging)

$$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\cdots .$$

I. *Solution by Jeremy Bem, student, Ithaca High School, Ithaca, N.Y.*

Let

$$P(x)=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{2x^6}{6}+\cdots.$$

Then

$$P'(x)=1+x-2x^2+x^3+x^4-2x^5+\cdots=\frac{1+x-2x^2}{1-x^3}=\frac{1+2x}{1+x+x^2} .$$

Thus

$$P(x)=\int P'(x)\,dx=\int\frac{1+2x}{1+x+x^2}\,dx=\ln(1+x+x^2)+C.$$

$P(0)=0$, so $C=0$. Thus $P(1)=\ln 3$, and this is the desired series.

II. *Solution by Cory Pye, student, Memorial University of Newfoundland, St. John's.*

Let S_n represent the nth partial sum; then

$$\begin{aligned}S_{3n} &= 1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\cdots+\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{2}{3n}\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{3n}\right)-\left(\frac{3}{3}+\frac{3}{6}+\frac{3}{9}+\cdots+\frac{3}{3n}\right)\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{3n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\\ &= \ln 3n+\gamma+o(1)-(\ln n+\gamma+o(1))\\ &= \ln 3n-\ln n+o(1)=\ln 3+o(1),\end{aligned}$$

where γ is Euler's constant. Hence the required sum is

$$\lim_{n\to\infty}S_{3n}=\ln 3.$$

Also solved by H.L. ABBOTT, University of Alberta; HAYO AHLBURG, Benidorm, Spain; NIELS BEJLEGAARD, Stavanger, Norway; ILIYA BLUSKOV, Technical

University, Gabrovo, Bulgaria; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen; C. FESTRAETS-HAMOIR, Brussels, Belgium; EUGENE A. HERMAN, Grinnell College, Grinnell, Iowa; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; PHIL REISS, student, University of Manitoba, Winnipeg; ROBERT E. SHAFER, Palo Alto, California; D.J. SMEENK, Zaltbommel, The Netherlands; SHOBHIT SONAKIYA, Kanpur, India; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

A generalization (to a series with sum $\ln k$*) was found by Janous and by Klamkin. Kuczma noted the interesting series*

$$-1+\frac{1}{2}+\frac{2}{3}+\frac{1}{4}-\frac{1}{5}-\frac{2}{6}-\frac{1}{7}+\frac{1}{8}+\frac{2}{9}+\cdots=0,$$

which is just $P(-1)$ *in Bem's solution* I.

*　　*　　*　　*　　*

AWARDS FOR M.S. KLAMKIN AND M.E. KUCZMA

Readers will be pleased to hear that prolific *Crux* regulars Murray S. Klamkin, University of Alberta, and Marcin E. Kuczma, University of Warsaw, have recently been awarded the 1992 David Hilbert International Award, presented for their role in "the development of mathematical challenges at the international level". They are in heady company: the third recipient of this years' awards is Martin Gardner! The awards were presented at the International Congress on Mathematical Education at Laval University in Québec this past August. (Unfortunately, Gardner could not attend.) The editor can report, having been in the audience, that the awards consist of a rather large, heavy and attractive framed certificate, and a medal, both containing a picture of Hilbert. By the way, one of the three recipients of the inaugural David Hilbert International Awards in 1991 was Ed Barbeau of the University of Toronto. Congratulations to Murray and Marcin, and belated congratulations to Ed, for these well-deserved honours.

At the same ceremony at Laval was also presented the 1992 Paul Erdős National Awards, to Luis Davidson of Cuba, Nikolay Konstantinov of Russia, and John Webb of South Africa. Readers may recognize Professor Konstantinov as one of the founders of the Tournament of the Towns.

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