PI MU EPSILON JOURNAL

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Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

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PI MU EPSILON JOURNAL

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Editorial correspondence, including books for review, chapter reports, news items and manuscripts (two copies) should be mailed to PI MU EPSILON JOURNAL, Richard L. Poss, EDITOR, St. Norbert College, 100 Grant Street, De Pere, WI 54115; E-Mail Address, possrl@sncac.snc.edu; FAX # 414-337-4098. Students submitting manuscripts are requested to identify their college or university and their class or expected graduation date. Others are requested to provide their affiliation, academic or otherwise.

Problems for solution and solutions to problems should be mailed directly to the PROBLEM EDITOR. Puzzle proposals and puzzle solutions should be mailed to the EDITOR.

The **PI MU EPSILON JOURNAL** is published at St. Norbert College twice a year–Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

1993 NATIONAL MEETING, PRELIMINARY INFORMATION

The 1993 National Meeting of Pi Mu Epsilon will be held in Vancouver, British Columbia, in Canada. The meeting will be held in conjunctions with the AMS-MAA meetings from August 15-19. Pi Mu Epsilon will again co-host this national meeting with the MAA student chapters.

The National Council of Pi Mu Epsilon has approved,. on a temporary bask, a more generous travel allowance for student speakers. The first speaker from a given chapter will be eligible for the same travel allowance as before, but if there are more than one speaker from a given chapter, the additional speakers (up to four) will be eligible for an allowance of 20% of what the first speaker receives. For example, if the distance traveled (by cat or van) is over 2400 miles (round trip distance), a single student speaker would receive $600, two student speakers would receive $720 (to share in any way they wish), three speakers would share $840, four speakers would share $960, and five or more speakers ftom this single chapter would share $1080.

The reason for trying the new scale for travel allowances is to encourage more students to speak at the national meeting. There was some concern that the fact that Vancouver is so far removed ftom most of the schools that have traditionally sent student speakers to the national meetings might cause the number of speakers to fall below what it has been. This new policy hopes to encourage chapters to send multiple speakers.

CALL FOR NOMINATIONS

Nominations are being accepted for the office of national councilor of Pi Mu Epsilon. Please send nominations to: Eileen Poiani, St. Peter's College, Jersey City, NJ 07306. Nominations must be received by November 30, 1992. Elections for Pi Mu Epsilon offices will be held in the spring of 1993. Pi Mu Epsilon sponsors will be receiving ballots for their chapters to use in the voting process.

STUDENT PAPERS

In each year that at least five student papers have been received by the Editor, prizes of $200, $100, and $50 , known as Richard V. Andree Awards, are given to student authors. All students who have not yet received a Master's Degree or higher are eligible.

There are four student papers in this issue of the *Journal*. One of the papers is "Exploring Self-Duality in Graphs," by Concetta DePaolo and Russell Martin. They prepared this paper during the National Science Foundation's Research Experience for Undergraduates Program at Worcester Polytechnic Institute in the summer of 1991. At that time, Concetta was a student at Worcester Polytechnic Institute and Russell was a student at Syracuse University.

The second paper is "Fractorials!" by Nataniel Greene. Nataniel was a junior at Carmel High School in New York when he prepared thii paper.

The thud paper is "On the Number of Invertible Matrices Over Z_{p^e}," by Mark Lancaster. Mark prepared this paper during his senior year at Hendrix College.

The final student paper k "On Transpositions Over Finite Fields," by Beth Miller. Beth prepared this paper while she was a junior at Penn State University – New Kensington Campus.

EXPLORING SELF-DUALITY IN GRAPHS

Concetta *DePaolo* and Russell Martin
Worcester Polytechnic Institute and Symcuse University

Introduction

We define a graph G on a set of vertices and a set of edges. (For those not familiar with graph theory, refer to a text such as Harary [2].) If G is drawn in the plane such that there are no edge **crossings,** then G is a plane graph, and we can also define a set of faces of G. For any such plane graph G, we can construct the geometric dual of G, denoted G^*, as follows: (1) within every face f of G, create a vertex f^* of G^*; (2) for each edge e separating faces f_i and f_j of G, let e^* be an edge joining vertices f_i^* and f_j^* in G^*. We now make a distinction between a planar graph and a plane graph. Planar graphs are graphs that *can* be drawn in the plane without crossings of the **edges,** and plane graphs **are** drawn in the plane without crossings of the edges. Notice we need a plane graph in order to construct its geometric dual.

When is a plane graph "self-dual"? Since a plane graph can be defined by its vertices, edges, and faces, it is natural to think that its dual must also be defined by exactly these if it is to be **"self-dual."** That is, not only must there be vertex and edge isomorphisms **from** G to G^* such that all adjacencies and incidences are preserved (we call these ***graphical*** isomorphisms), but there must also be an isomorphism which maps faces to faces such that if two faces both border **an** edge in G, they must also border the corresponding edge in G*. In other words, G and G*, in addition to being graphically isomorphic, must also be identically embedded in the plane. If this occurs, we say G is geometrically self-dual, and write $G \cong G^*$.

Most people, however, would not consider the embeddings of a graph when dealing with **self-**duality; a graphical isomorphism from G to G^* is usually sufficient in graph theoretical **terms.** If a graph is embedded in the plane such that it is graphically isomorphic to its dual (whether or not the dual is embedded differently), then we say G is combinatorially self-dual, and write $G \approx G^*$. Clearly, geometrical self-duality implies combinatorial self-duality.

We show that combinatorial **self-duality** is indeed weaker than geometric self-duality using the plane graph H in Figure 1 (opposite).

Notice that in H, the face inside the **parallel** edges borders two loops which are at opposite vertices. In H^*, neither face, inside or outside the parallel edges, borders two loops at **opposite** vertices. Therefore, H is not geometrically self-dual. However, by inspection, it is easy to see that H can be made geometrically **self-dual** by reembedding the bridge on the left into the outer face, and reembedding one of the bridges on the right into the inner face. Therefore, the difference between combinatorial and geometric self-duality is the graph's embedding.

There is another, still weaker, type of self-duality which we call ***abstract*** self-duality, which occurs when a plane graph G (whether or not it is geometrically or combinatorially self-dual) is embedded such that its collections of cycles remains unchanged in the dualization process. In other words, there is a one-to-one correspondence between the edges of G and the edges of G^* such that the collection of cycles is preserved. For an example of an abstractly self-dual graph, see Figure 3, graph H_1. H_1 is not combinatorially nor geometrically self-dual because it contains a vertex of degree six, while its dual does not. However, the sets of cycles of H_1 and H_1^* are identical. Clearly, combinatorial self-duality implies abstract self-duality.

When talking about cycles, it is also natural to talk about matroids, since the "cycle matroid" $M(G)$ of G corresponds to the family of all cycles of the graph G. When two graphs G and H, have the same collection of cycles, we say their cycle matroids are isomorphic. This is denoted as $M(G) \cong M(H)$. Thus, when we say G is abstractly self-dual, we mean that $M(G) \cong M(G^*)$.

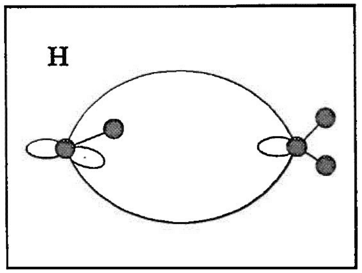


Figure 1a: A **combinatorially** self-dual **graph**

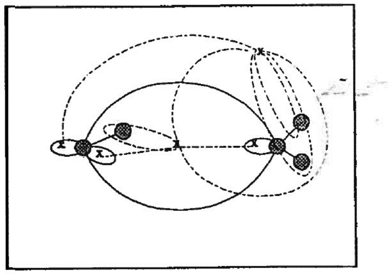


Figure 1b: The dualization process

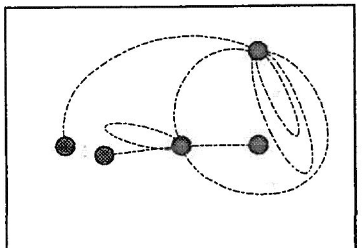


Figure 1c: The dual of H

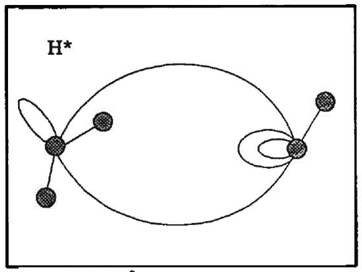


Figure 1d: H^*

The purpose of this paper is to examine and explore the relationship amongst these three concepts of self-duality. In particular, we address when the concepts are or are not equivalent. Also, we give some methods for constructing graphs with one or more of these properties. We've found that these topics are best explored according to the vertez connectivity (the minimum number of vertices whose removal disconnects the graph) **of the** graphs in question. We begin with %connected graphs.

3-connected graphs

We begin our examination of %connected graphs with the realization **of two** important theorems. The **first,** due **to** Whitney (see Harary [2]), states that if G is a 3-connected planar graph, then it has a unique plane **embedding,** that is, the faces and their bordering edges are uniquely **determined.** The second theorem, due to Welsh [3], states that if two 3-connected graphs without loops have **isomorphic** cycle matroids, then they must be graphically isomorphic.

Let us explore what these **theorems** mean with regard to the relationships between geometric, combinatorial, and abstract self-duality in 3-connected graphs. Let's begin with the assumption that G is **3-connected** and abstractly self-dual, **i.e.** $M(G) \cong M(G^*)$. Observe that G cannot contain a loop (since G^* would then have a corresponding bridge and so would be **only** 1-connected). By the Welsh theorem, we see that G and G^* must be graphically isomorphic. That is, abstract duality implies **combinatorial** self-duality. However, the Whitney theorem assures us that the embedding of a 3-connected graph is unique, **so** that $G \approx G^*$ implies that $G \cong G^*$. Therefore, we've established that for 3-connected **graphs**, the concepts geometric, combinatorid, and abstract self-duality all coincide.

2-connected graphs

When dealing with %connected graphs, the relationship amongst the types of self-duality becomes more interesting. Abstract **self-duality** no longer implies geometric **self-duality.** In fact, **2-connected graphs** with $M(G) \cong M(G^*)$ need not even be **combinatorially** self-dual! However, there does exist a **special** relationship amongst graphs with isomorphic cycle matroids. In Whitney [5, **Sec.1**] it is proved that any two graphs with isomorphic cycle matroids can be obtained from one another by a series of what are called Whitney *moves.* The significance of these moves is **that** performing them on a graph does not change its cycle matroid. We first describe a ***Whitney twist*** (or just *twist):*

Suppose vertices u and v form a **cutset** of the graph, G. Split G into two ***sections***, G_1 and G_2, by cutting the graph through u and v, forming two connected graphs, each having vertices u and v. If there is an edge uv in G, *we* arbitrarily assign it to G_1. We form the graph G' by re-identifying the vertices u in G_1 with v in G_2, and v in G_1 with u in G_2. See Figure 2 for an example of this operation. Note that G' is still 2-connected, but it need not be isomorphic to G. See **Welsh** [4] and Whitney [5] for further details on this operation.

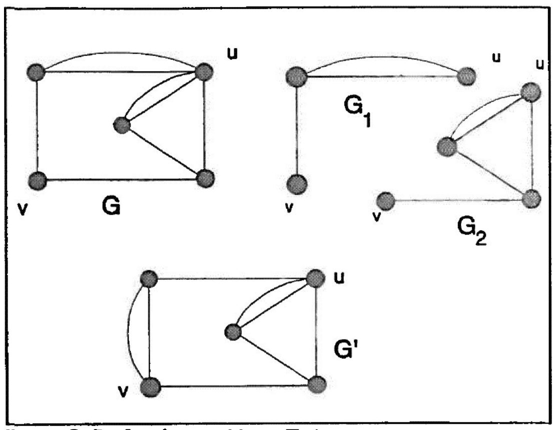


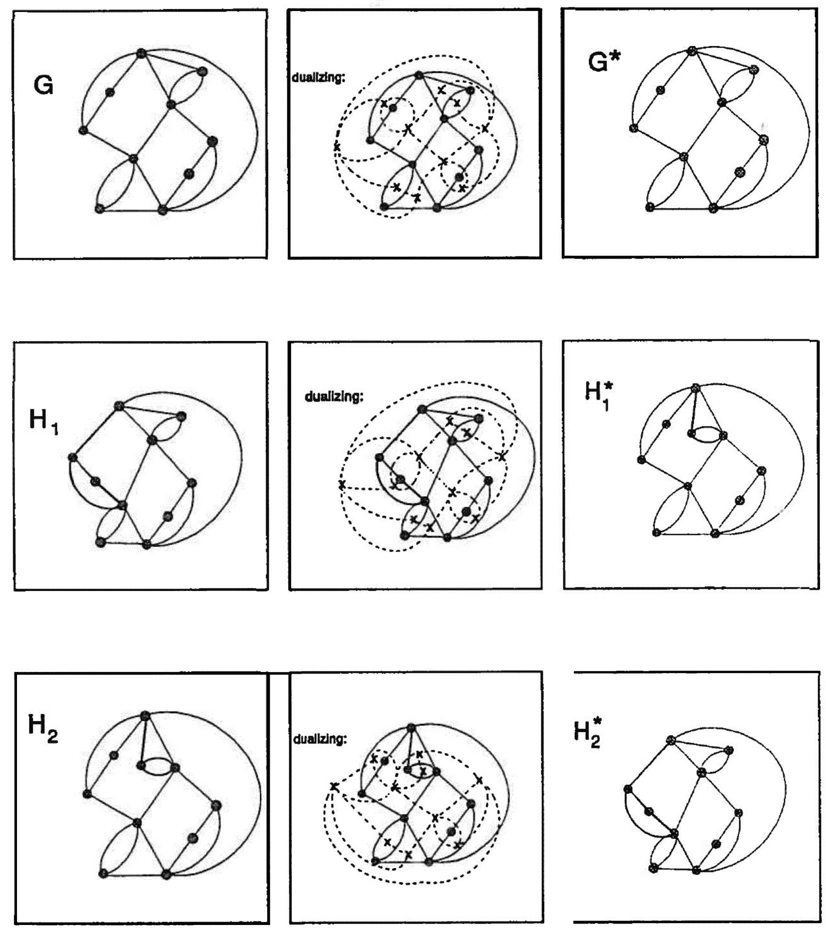
Figure 2 **Performing** a **Whitney** Twist

We **will also** consider the reembedding operation to be a Whitney move, because it does not change the cycle matroid of a graph.

We now want to explore the relationship between the two operations of twisting and **reembedding.** The following series of diagrams (Figure 3) first show a 2- connected geometrically **self-dual** graph G, the formation of its dud G*, and then a **90-degree** rotation of G^* so **as** to **show** that $G^* \cong G$. The next diagram shows H_1, a graph obtained from G by a Whitney twist on the upper left hand part of the graph. When the dual of H_1 is constructed, we find that it is graphically isomorphic to the dual of G; **however**, the **cone-** like section in the upper part of **the** graph of H_1 is embedded in a **5-sided** face in H_1^* rather than in a 4–sided face as it is in G*. Therefore, a twisting in the original **graph** has led to a reembedding of the dual.

The next graph, H_2, was obtained from G by reembedding the cone-like section in the upper part of the graph. Notice that H_2 and H_1^* are identical. We find that H_2^* is G with a twist performed in the upper left hand part. Therefore, a reembedding of the **original** graph has led to a twisting in the dual.

FIGURE 3



Due to the interchange of faces and vertices in the **dualization** process, a twisting always leads to a reembedding of the dual, and a reembedding always leads to a twisting of the dual, as is shown in the following theorem.

Theorem 1. Let G be a plane graph and let G^* be its dual. If H is a graph obtained by a series of Whitney twists from G, then H* is a reembedding of G^*. **Furthermore,** if H is a reembedding of G, then H^* can be obtained from G^* by a **series** of Whitney twists. In other words, twisting and reembedding are "dual operations."

Proof: For the **first** sentence, we observe that as a result of twisting G, none of the facial adjacencies have changed. To see this we argue **as** follows: Let f be a face of G that is bounded by edges from both G_1 and G_2, where G_1 and G_2 are the sections of G created by the **cutset** $\{u,v\}$. Let $c_1, c_2, \ldots c_n$ be the cycle of edges that bounds f. By renumbering the edges if needed, we can say that $c_1, c_2, \ldots c_k$ is the sequence of edges joining u and v in G_1, and $c_{k+1}, c_{k+2}, \ldots, c_n$ is the sequence of edges joining v to u in G_2. When we perform the twist, we identify u in G_1 to v in G_2, and v in G_1 to u in G_2. Therefore, the cycle boundiig f in H is $c_1, \ldots, c_k, c_n, c_{n-1}, \ldots, c_{k+1}$, containing the same edges as in G. We apply this same argument to each face that is bounded by edges from both G_1 and G_2. If a face is completely bounded by edges in G_1 or G_2, then the face will not be affected. Therefore, if two faces shared an edge in G, they will share the same edge in H, after the twist has been **performed.** In other words, the facial adjacencies have not been altered by the twist.

We know that adjacent faces in a graph give rise to adjacent vertices in the graph's dual. H has the same number of faces, vertices, and edges as G, so H^* has the same number of faces, vertices, and edges as G*. Furthermore, because G and H have the same adjacent faces, G* and H* have corresponding adjacent vertices. This produces a mapping between the vertex and edge sets of G* and H* that preserves the vertex adjacencies.

To show that H^* is a reembedding of G^*, consider the effect on the faces of H* by considering the effect of the twist on the vertices of H. Due to the nature of the twisting operation, **some** of the vertex adjacencies in H will be different **from** those in G. If two vertices were adjacent in G, and are not adjacent in H, then the corresponding faces of H* will not be adjacent **as** they were in G*. So, the embedding on H^* will be different from that of G^*.

When we perform a series of twists, we actually have a sequence of graphs $G, H_1, H_2, \ldots, H_{n-1}, H$. **From** the above argument, we know that the following are graphically isomorphic: $G^*, H_1^*, H_2^*, \ldots, H_{n-1}^*$, H*. Since H* is graphically isomorphic to G^*, it is just a planar **representation** of the same graph, which as stated above, need not be embedded in the same way. The second sentence of Theorem 1 can be proved in a **similar** manner. Alternatively, we can show it can be proved using duality. We have just shown that a twist in G corresponds to a reembedding in G^*. Since this is true for any plane graph G, we therefore know that a twist on the plane graph G^* wrresponds to a reembedding in $(G^*)^*$. But since $(G^*)^*$ is G, we have that a twist in G^* corresponds to a reembedding in G. ∎

Suppose G is any 2-connected, abstractly self-dual graph, If we twist G to get H, we have shown that the cycle matroid has not changed, that is, $M(H) \cong M(G)$, so that (1) $M(H) \cong M(G^*)$, since G is abstractly self-dual. Also, by Theorem 1, H^* is merely a reembedding of G^*, so that (2) $M(H^*) \cong M(G^*)$. Combining (1) and (2), we have $M(H) \cong M(H^*)$.

Therefore, if a graph is obtained from a geometrically self-dual graph by a series of Whitney twists, then the resulting graph is still abstractly self-dual (but this does not automatically mean it has to be geometrically, or even combinatorially self-dual).

In a similar manner, we can generate other graphs that are abstractly self-dual by using all of the different, diitinct planar embeddings of a geometrically self-dual graph. The duals of these graphs will be twists of the original dual. The same king of argument **as** before can be used to show that the resulting graph is still abstractly self-dual (but this does not automatically mean it has to be geometrically, or even combinatorially self-dual).

We now present a theorem which asserts that any %connected abstractly self-dual graph-can always be manipulated to form a **combinatorially** self-dual graph:

Theorem 2. If G is a **2-connected** plane graph with $M(G) \cong M(G^*)$, then there exists at least one graph H, obtained from G by a finite series of Whitney twists, such that H ≈ H*.

Proof: Since G is abstractly self-dual, by the Whitney theorem [5, **Sec. 1**], G can be twisted to get G^*. Let us perform this twist (or series of twists) to get a new graph H graphically **isomorphic** to G^*. When we twist G to get H, the dual of H, H^*, is merely a reembedding of G^*, by Theorem 1. **If** H^* **is** a reembedding of G^*, then H^* and G^* are graphically isomorphic. By construction, we also have that H and G^* are isomorphic. Therefore, H* and H are graphically isomorp c. Thus, H ≈ H*, or H is combinatoridly self-dual. ∎

Conjecture: If G is a 2-connected abstractly self-dual graph, then there exists at least one **graph** H, obtained from G by a series of twists and/or **reembeddings,** such that H is geometrically self-dual. ∎

This **conjecture** is supported by the graph G in Figure 3 which **has** sixteen twists (or **series** of twists), and sixteen embeddings of each of these twists. We drew all 256 graphs and dualized each one of them. We found **that** for each twist, there was at least one embedding of these resulting graphs that was geometrically self-dual. This example strengthens our belief that the conjecture is true. Our results on this example are summarized in Table **I**. When we perform all possible series of twists on G, we find that some geometrically isomorphic graphs result from different **series** of twists. The number and percentages in parentheses indicate the results when we exclude these duplications from our totals.

Table I

Type of Graph	Number of Graphs	Percent of Total
Geometrically **Self-Dual**	32 (20)	12.5% (12.5%)
Combinatorially (but not Geometrically) Self-Dud	24 (12)	9.375% (7.5%)
Abstractly Self-Dual **Only**	**200** (128)	78.125% (80%)
Total	256 **(160)**	100%

1-connected graphs

We have seen that a 2-connected abstractly self-dual graph does not necessarily have to be combinatorially or geometrically self-dual. The same is true for 1-connected graphs. However, like **2-**wnnected graphs, there exists a special relationship among 1-connected graphs with isomorphic cycle matroids. In Whitney [5, **Sec.1**], it is proved that all pairs of 1-connected graphs whose cycle matroids are isomorphic to one another can be obtained from one another by one of the following Whitney moves for 1-connected ***graphs:*** (1) rearrangement of the **blocks** (maximal 2-connected subgraphs; see **[2,** Chapter **3])**, by detaching at a cut vertex, and either leaving the graph diswnnected **or** reattaching the blocks at different vertices; (2) **twistings** performed within the blocks of the graph;

(3) reembeddings of the graph; and (4) some combination of the first three. Notice that none of these operations change the cycle rnatroid of the graph.

Because there are several Whitney moves in this case, the classification of 1- connected abstractly self-dual graphs is much less structured than that of its higher connected counterparts. We now state a theorem which serves as a means of classification and construction of 1-connected abstractly self-dual graphs.

Theorem 3. Let G be a plane graph which consists of k (abstractly) self-dual blocks and 2m non self-dual blocks, m of which are duals of the remaining m, joined arbitrarily at cut vertices so long as no additional cycles are created and no block is embedded inside a bounded face of another. Then G is abstractly self-dual.

Proof: Since the only cycles present in G occur within the $k+2m$ blocks, G may be thought of as a "pseudoforest" whose "branches" are the $k+2m$ blocks. Unlike a true graph-theoretical forest, this **"pseudoforest"** has cycles, but only the ones existing within the $k+2m$ blocks.

Let the k abstractly self-dual blocks be denoted by B_i $(i = 1, 2, \ldots k)$. Similarly, let the 2m non self-dual pairs be denoted by H_j and H_j^* $(j = 1, 2, \ldots m)$. The faces of the plane **embeddings** of G are the bounded faces of each B_i, H_j, and H_j^*, together with one unbounded face (call it F) surrounding G. The vertices of G are precisely those of the blocks (counting cut vertices only once, even though they may be in many blocks). The edges of G are precisely the edges of the blocks. When we form G^*, each B_i^*, each H_j^*, and each $(H_j^*)^* \cong H_j$ will have the vertex corresponding to F as one of its vertices (because each has faces adjacent to F in G).

We note that $G \not\approx G^*$ unless all k abstractly self-dual blocks are combinatorially self-dual, and G was assembled in such a way that all of the $k+2m$ blocks had one common vertex. (See graph G in Figure 4.) Otherwise, at least one of the pairs of blocks are vertex **disioint** in G, whereas they share **a common** vertex **in** G^* (graphs H and H^* in Figure 4).

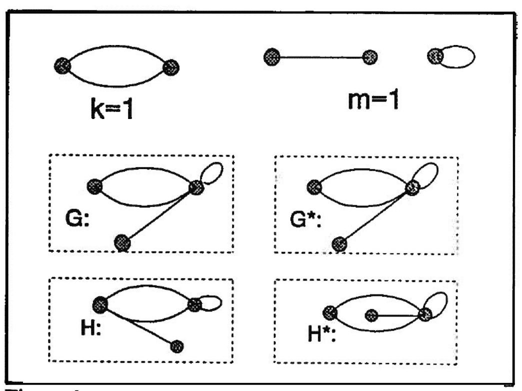


Figure 4

Each of the original 2m blocks is still a block of G*. Therefore, their cycles are still intact and so each of their individual cycle matroids is intact. Also, since the k blocks are abstractly **self-**dual, their cycles, and therefore their cycle matroids, also remain intact in G^*. The cycle matroid of G consists of the cycles of B_i, H_j, and H_j^*, and no others. Furthermore, the cycle matroid of G^* is precisely the same (because every two blocks have exactly the one vertex corresponding to F in common, so no cycles besides those inherent in the original blocks are present). Therefore, $M(G) \cong M(G^*)$. ■

We see that this theorem also gives a method for constructing geometrically self-dual 1-connected graphs. We do so by first asserting that the k abstractly self-dual components must be ***geometrically*** self-dual, and second, by attaching the $k+2m$ blocks such that (1) they all share one common vertex, (2) that no two share any other vertex, and (3) that no block is embedded inside the face of another. The theorem also tells us how to construct 1-connected graphs which may not be geometrically self-dual, but that will always be abstractly self-dual, by attaching the blocks (with no additional restrictions on the k self-dual blocks) in a relatively arbitrary fashion, without creating any new cycles. See Servatius and Christopher **[3]**, and Archdeacon and Richter **[1]** for other methods of creating geometrically self-dual graphs.

Since any 1-connected, but not 2-connected, abstractly self-dual graph is made up of some collection of $k+2m$ blocks, the next task is to determine which, and how many, arrangements of these blocks will result in a graph that is either combinatorially or geometrically self-dual. In other words, how do we perform the Whitney operations on a 1-connected abstractly self-dual graph to manipulate it so that it is self-dual in a stricter sense? For certain sets of blocks, there may be several such resulting graphs. For others, there may by only the one mentioned above. Because of the possibly large number of arrangements of the blocks of a 1-connected graph, this question is difficult to answer in general, but also promises to be an interesting endeavor.

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This paper is the result of joint research between the two authors during the National Science Foundation's Research Experience for Undergraduates Program which was held at Worcester Polytechnic Institute in the summer of 1991. At the time that they prepared the paper, Concetta DePaolo was a student at Worcester Polytechnic Institute and Russell Martin was a student at Syracuse University. They are currently enrolled in graduate programs at Rutgers and Clemson Universities, respectively.

MATCHING PRIZE FUND

If your chapter presents award for Outstanding Mathematical Papers or for Student Achievement on Mathematics, you may apply to the National Office for an amount equal to that spent by your Chapter, up to a maximum of fifty dollars. Contact Professor Robert Woodside, Secretary-Treasurer. ■

FRACTOBJAL!

Nataniel S. Greene
Yeshiva University

INTRODUCTION

The fractorial has emerged out of two separate investigations. The first was to see whether the operation of factorial could be defined for non-integer values. If $3! = 6$ and $4! = 24$, then what does $3\frac{1}{2}!$ equal? If $3\frac{1}{2}!$ has a value, then mustn't there be an inverse factorial operation that could answer questions like $x! = 29$?

The second investigation involved factorial expansions. When $4!$ is expanded, we have $4! = (4)(3)(2)(1)$; the common difference between each factor is 1. The question was: how are expansions with common differences other that 1, such as $(8)(6)(4)(2)$ or $(10)(7)(4)(1)$, related to the factorial?

A study of these two questions led to a unification of ideas. Fractional factorial and products of arithmetic sequences like the ones mentioned above can be expressed using a single notation: $a!_b$ (read "a fractorial step b"), where a and b are positive real numbers and b is the common difference between the factors. The name "fractorial" was suggested by Mr. Paul Eckhardt, the chairman of the mathematics department of Carmel High School, who amalgamated the words "fractional factorial." I am grateful to my former teachers: Mr. Eckhardt for his suggestions, and Mr. Anthony Iannotta for his encouragement.

Using fractorial notation, the normal factorial now becomes a specific case of $a!_b$ when a is a natural number and $b = 1$. From this new definition springs forth a wealth of new relationships to explore.

Definition 1. $a! = a(a-1)(a-2)\ldots(3)(2)(1)$. This is the normal definition of factorial, where a is a natural number.

Example. $5! = (5)(4)(3)(2)(1)$

Definition 2. $0! = 1$

Definition 3. $a!_b = a(a-b)(a-2b)\cdots(a-kb)$, where a and b are positive real numbers, $a \geq b$, and k is the largest whole number such that $0 \leq k < (a/b)$. Once again, $a!_b$ is read as "a fractorial step b," b being referred to as the step. When $b = 1$, $a!_b$ may simply be called "a factorial."

Examples.

$$2!_{(1/2)} = (2)(1\tfrac{1}{2})(1)(\tfrac{1}{2}) = 1\tfrac{1}{2}$$

$$(2\tfrac{1}{2})! = (2\tfrac{1}{2})(1\tfrac{1}{2})(\tfrac{1}{2}) = 1\tfrac{7}{8}$$

$$3!_{(2/3)} = (3)(2\tfrac{1}{3})(1\tfrac{2}{3})(1)(\tfrac{1}{3}) = 3\tfrac{8}{9}$$

$$1!_{(1/4)} = (1)(3/4)(1/2)(1/4) = 3/32$$

$$(3/2)!_{(1/4)} = (3/2)(5/4)(1)(3/4)(1/2)(1/4) = 45/256$$

Definition 4. When a and b are natural numbers, $a!_b$ is called a "perfect fractorial."

Examples. These numbers are perfect fractorials:

$$8!_2 = (8)(6)(4)(2) = 384$$
$$11!_3 = (11)(8)(5)(2) = 880$$
$$9!_3 = (9)(6)(3) = 192$$
$$5!_4 = (5)(1) = 5$$
$$4!_1 = 4! = (4)(3)(2)(1) = 24$$

A multitude of notations for factorials and generalized factorials have been introduced during the last couple of centuries. For a more detailed discussion of these notations, one can refer to Charles Jordan's Calculus of Finite Differences (Chelsea Publishing Co., New York, 1947). The notation used by Jordan in his book is of particular interest. He defines the "generalised factorial of degree n":

$$(a)_{n,b} = a(a-b)(a-2b)\cdots(a-nb+b)$$

This is essentially the definition of the fractorial. The way in which the two definitions differ, however, is on their solutions to the question: How many factors should there be in a given fractorial's product? Jordan allows the number of factors, n, to be an arbitrary independent variable, while in fractorial, n is a function completely determined by the values of a and b. A detailed development of this follows below. In this development, n will refer to the number of factors in the given fractorial, and k will refer to the number of steps.

FORMULAS FOR COMPUTATION AND SIMPLIFICATION

As we saw from the examples that followed Definition 4, the factors of a fractorial's descending product must naturally stop before reaching zero. This is the reason for defining $k < (a/b)$. The last factor $(a - kb)$ is always > 0. This piece of information can be used to derive an important inequality (Theorem 1) and a formula for the number of factors in a given fractorial's expansion (Theorem 2).

If $(a - kb) > 0$ and if a and b are restricted to the natural numbers, then the following is also naturally true: $(a - kb) \geq 1$. Since the last factor must be the least possible integer ≥ 1, $(a - kb)$ must be $< (b + 1)$. These two statements can be combined to form the compound inequality:

$$1 \leq (a - kb) < (b + 1) \tag{1}$$

This statement can be generalized for the rational expression $(a/c)!_{(b/d)}$, whose last factor is $[a/c - (kb)/d]$. Replacing a by ad and b with bc, where a, b, c, and d are natural numbers, inequality (1) becomes

$$1 \leq (ad - kbc) < (bc + 1) \tag{2}$$

Dividing by cd gives the following:

$$1/(cd) \leq [a/c - (kb)/d] < [b/d + 1/(cd)] \tag{3}$$

Theorem 1. For all natural numbers a, b, c, and d, the number (n) of factors in the fractorial expression $(a/c)!_{(b/d)}$ satisfies the inequality $n \leq (ad + bc - 1)/(bc) < (n + 1)$.

Proof: Given $1/(cd) \leq [a/c - (kb)/d] < [b/d + 1/(cd)]$, from (3),

$$(kb)/d \leq [a/c - 1/(cd)] < [(kb)/d + b/d],$$

and thus

$$k \leq [(ad)/(bc) - 1/(bc)] < (k + 1)$$
$$(k + 1) \leq [(ad)/(bc) - 1/(bc) + 1] < (k + 2)$$

Since $n = (k + 1)$, we have $n \leq (ad + bc - 1)/(bc) < (n + 1)$.

Example 1. $x!_3$ has 13 factors. What are the possible values for x?

$$\begin{aligned} n &\le (a+b-1)/b < (n+1), (c=d=1) \\ 13 &\le (x+3-1)/3 < 14 \\ 13 &\le (x+2)/3 < 14 \\ 39 &\le x+2 < 42 \\ 37 &\le x < 40 \end{aligned}$$

Example 2. $99!_{(x/2)}$ has 19 factors. Find all possible values for x.

$$\begin{aligned} n &\le (ad+bc-1)/(bc) < (n+1) \\ 19 &\le [(99)(2)+x-1]/x < 20 \\ 19x &\le (197+x) < 20x \\ 18x &\le 197 < 19x \\ 10\tfrac{7}{19} &< x \le 10\tfrac{17}{18} \end{aligned}$$

Theorem 2. The number of factors (n) in the expansion of $(a/c)!_{(b/d)}$ equals $\mathrm{int}[(ad+bc-1)/(bc)]$, where a, b, c, and d are natural numbers and $\mathrm{int}(x)$ takes any positive real number x and truncates its decimal mantissa. [**E.g.,** int(3.895) = **3.**]

Proof: We know that $n \le (ad+bc-1)/(bc) < (n+1)$. The middle expression is a rational number trapped between two whole numbers. We can make it equal to n by chopping off its decimal mantissa:

$$\begin{aligned} \mathrm{int}(n) &= \mathrm{int}[(ad+bc-1)/(bc)] < \mathrm{int}(n+1) \\ n &= \mathrm{int}[(ad+bc-1)/(bc)] \end{aligned}$$

Example. How many factors does $97!_{(3/11)}$ have?

$$n = \mathrm{int}[((97)(11)+3-1)/3] = \mathrm{int}[1069/3] = 356$$

Corollary 1. When a and b are natural numbers, the number of factors in $a!_b$ is $\mathrm{int}[(a+b-1)/b]$.

Example. How many factors does $89!_4$ have?

$$n = \mathrm{int}[(89+4-1)/4] = 23$$

Theorem 3. If ad is divisible by bc, then the last factor in the expansion of $(a/c)!_{(b/d)}$ equals the step and the number of factors equals $(ad)/(bc)$. Proof: Assume $(ad)/(bc)$ is an integer. Then, by Theorem 2, if n is the number of factors,

$$n = \mathrm{int}[(ad+bc-1)/(bc)] = \mathrm{int}[(ad)/(bc)+1-1/(bc)] = (ad)/(bc).$$

Thus, the last factor is

$$a/c-(n-1)(b/d) = a/c-[(ad)/(bc)-1](b/d) = a/c-(adb)/(bcd)+b/d = b/d,$$

which is the step size. It is also apparent, by moving backwards through the proof, that if the last factor equals the step, then the number of factors equals $(ad)/(bc)$, and, therefore, ad is divisible by bc.

Example. How many factors does $81!_3$ have?

This problem can be answered quickly upon inspection using Theorem 3. We see that 81 is divisible by 3, and consequently the answer is 27. Therefore, if you need to know the number of factors in a fractorial expansion, first see if ad is divisible by bc. If it is not, only then use Theorem 2. Theorem 2 will give the correct answer in all cases, but using it to compute n can be a bit more tedious.

Theorem 4. If a and b are real numbers such that $a \ge 0$ and $b > 0$, then $(a+b)!_b = a!_b(a+b)$. (This theorem gives us a recursive definition for the fractorial.)

Proof: $(a+b)!_b = (a+b)[a(a-b)\cdots(a-kb)] = a!_b(a+b)$.

Theorem 5. If a is a nonnegative integer and b is any positive **real** number, then $(ab)!_b = a!b^a$.

Proof:

$$\begin{aligned} (ab)!_b &= ab(ab-b)(ab-2b)\cdots(3b)(2b)(b) \\ &= [a(a-1)(a-2)\cdots(3)(2)(1)][(b)(b)\cdots(b)] \\ &= a!b^a \end{aligned}$$

Examples. $10!_2 = [(5)(2)]!_2 = (5!)2^5$ and $18!_3 = [(6)(3)]!_3 = (6!)3^6$.

Corollary 1. $b!_b = b$.

Proof: Let a = 1.

Corollary 2. $0!_b = 1$.

Proof. Let a = 0, so that $(0!_b = (0!)b^0$. This is a very interesting situation. When we derive $0!_b$ from Theorem 5, we see that whether this expression has any meaning or not depends entirely on whether 0! has meaning. Since we define 0! to be **1**, $0!_b$ must also = 1.

It is also interesting to see what happens when we ask how many factors there are in $0!_b$ using the formula $n = \mathrm{int}[(a+b-1)/b]$. Letting a = 0, we have $n = \mathrm{int}[(b-1)/b]$ which is zero for $b \ge 1$. Therefore, this formula will give a reasonable answer to our question when $b \ge 1$. There are no factors in $0!_b$'s expansion, simply because $0!_b$ has no expansion.

Corollary 3. If a > 0 and $(ax)!_x = p$, then $x = [p/(a!)]^{1/a}$.

Proof; If $(ax)!_x = p$, then $(a!)x^a = p$ and so $x = [p/(a!)]^{1/a}$.

Example. If $(2x)!_x = 10$, then $x = [(10)/(2!)]^{1/2} = \sqrt{5}$.

Corollary 4. If $a \ge 0$, $p > 0$, and $(ax)!_x = (px)!_x$, then $x = [(a!)/(p!)]^{1/(p-a)}$.

Proof. If $(ax)!_x = (px)!_x$, then $(a!)x^a = (p!)x^p$. Thus $(x^p)/(x^a) = (a!)/(p!)$, which means that $x^{(p-a)} = (a!)/(p!)$ and so $x = [(a!)/(p!)]^{1/(p-a)}$.

Example. If $(5x)!_x = (9x)!_x$, then $x = [(5!)/(9!)]^{1/(9-5)} = 2(189^{1/4})$

Theorem 6. If a and b are integers such that $a \ge 0$ and $b > 0$ and x is any positive real number, then $(ax)!_{bx} = (a!_b)x^n$.

Proof:

$$\begin{aligned} (ax)!_{bx} &= ax(ax-bx)(ax-2bx)\cdots(ax-kbx) \\ &= [a(a-b)(a-2b)\cdots(a-kb)][(x)(x)\cdots(x)] \\ &= (a!_b)x^n \end{aligned}$$

Examples. $27!_6 = [(9)(3)]!_{(2)(3)} = (9!_2)3^5$ and $38!_4 = [(19)(2)]!_{(2)(2)} = (19!_2)2^{10}$

Corollary 1. *If* $a > 0$ and $(ax)!_{bx} = p$, then $x = [p/(a!_b)]^{1/n}$.

Proof: If $(ax)!_{bx} = p$, then $(a!_b)x^n = p$, and so $x = [p/(a!_b)]^{1/n}$.

Example. If $(3x)!_{2x} = 8$, then $x = [8/(3!_2)]^{1/2} = \sqrt{8/3}$.

Corollary 2. If $a \geq 0$, $p > 0$, and $(ax)!_{bx} = (px)!_{qx}$, then $x = [(p!_q)/(a!_b)]^{1/(n_1 - n_2)}$, where n_1 and n_2 are the number of factors in $(ax)!_{bx}$ and $(px)!_{qx}$, respectively.

Proof: Suppose $(ax)!_{bx} = (px)!_{qx}$. Then $(a!_b)x^{n_1} = (p!_q)x^{n_2}$. Dividing both sides by x^{n_2}, we have $(x^{n_1})/(x^{n_2}) = (p!_q)/(a!_b)$, or $x^{n_1 - n_2} = (p!_q)/(a!_b)$. And so $x = [(p!_q)/(a!_b)]^{1/(n_1 - n_2)}$.

Example. Solve for x: $(17x)!_{2x} = (12x)!_{5x}$.

$$x = [(12!_5)/(17!_2)]^{1/(n_1 - n_2)}$$
$$n_1 = \text{int}[(17 + 2 - 1)/2] = 9 \qquad n_2 = \text{int}[(12 + 5 - 1)/5] = 3$$
$$(n_1 - n_2) = 6, \text{ so that}$$
$$x = [(12!_5)/(17!_2)]^{1/6}$$

It is interesting to note that if the number of factors in $(ax)!_{bx}$ equals the number of factors in $(px)!_{qx}$, then $(n_1 - n_2) = 0$ and the value of x becomes undefined. In that case our original question becomes meaningless and we realize that there is no value for x that will make this statement true.

Theorem 7. If a, b, c, and d are whole numbers where $a \geq 0$ and b, c, and $d > 0$, then $(a/c)!_{b/d} = [(ad)!_{bc}]/[(cd)^n]$, where n is the number of factors in the fractorial.

Proof:

$$\begin{aligned}(a/c)!_{b/d} &= (a/c)(a/c - b/d)(a/c - 2b/d) \cdots (a/c - kb/d) \\ &= [(cd)^n/(cd)^n][(a/c)(a/c - b/d)(a/c - 2b/d) \cdots (a/c - kb/d)] \\ &= [(ad)(ad - bc)(ad - 2bc) \cdots (ad - kbc)]/(cd)^n \\ &= (ad)!_{bc}/(cd)^n\end{aligned}$$

Thus by merely establishing a common denominator, the theorem is proved. This is a very important theorem because it allows one to compute with greater ease the fractorial whose step is also a quotient.

Corollary 1. $(a/c)! = (a!_c)/c^n$.

Examples.

$$(8/2)! = (8!_2)/(2^{(8/2)}) = 384/16 = 24$$
$$6!_{(3/2)} = (12!_3)/(12^{(12/3)}) = 1944/16 = 243/2$$
$$(9/4)!_{(2/3)} = (27!_3)/12^4 = 209/256$$
$$(7/2)! = (7!_2)/(2^4) = 105/16$$
$$(2/3)!_{(1/9)} = (18!)_3/(27^6) = ([(6!)3^6]/27^6 = (6!)/9^6 = 80/59049$$

Corollary 2. $1!_{(b/d)} = (d!_b)/(d^n)$.

Proof: Let $a/c = 1$.

Examples. $1!_{(1/5)} = (5!)/(5^5) = 24/625$. $\quad 1!_{(2/5)} = (5!_2)/(5^3) = 3/25$.

Theorem 8: Suppose a and b are rational numbers and h is any natural number such that $1 \leq h < a/b$, then $a!_b = a!_b(a - b)!_{bh}(a - 2b)!_{bh} \cdots [a - (h - 1)b]!_{bh}$.

Proof: $a!_b = a(a - b)(a - 2b) \cdots (a - kb)$. Let h be a natural number such that $1 \leq h < a/b$. $a!_b$ can now be rewritten as:

$$a!_b = a(a - b)(a - 2b) \ldots [a - (h - 1)b] \cdots (a - kb).$$

these factors can be regrouped in the following manner:

1. Let every hth factor, after and including a, be grouped together: $a(a - bh)(a - 2bh) \cdots (a - kbh)$.
2. Let every hth factor, after and including $(a - b)$, be grouped together: $(a - b)(a - b - bh)(a - b - 2bh) \cdots (a - b - kbh)$.
3. Let the same be done to each successive factor, up to and including the hth factor $[a - (h - 1)b]$. For the hth factor this gives:

$$[a - (h - 1)b][a - (h - 1)b - bh][a - (h - 1)b - 2bh] \cdots [a - (h - 1)b - kbh].$$

We now have the following equality:

$$\begin{aligned} a!_b = {} & [a(a - bh)(a - 2bh) \cdots (a - kbh)] \\ & [(a - b)(a - b - bh)(a - b - 2bh) \ldots (a - b - kbh)] \\ & .[(a - 2b)(a - 2b - bh)(a - 2b - 2bh) \cdot \quad (a - 2b - kbh)] \\ & \\ & .[(a - (h - 1)b)(a - (h - 1)b - bh)(a - (h - 1)b - 2bh) \cdots (a - (h - 1)b - kbh)]\end{aligned}$$

Which when simplified becomes:

$$a!_b = a!_{bh}(a - b)!_{bh}(a - 2b)!_{bh} \cdots [a - (h - 1)b]!_{bh}.$$

What this theorem says is that you can break up a large fractorial into products of smaller fractorials, and you can multiply smaller fractorials to create a larger fractorial.

Examples.

$$10! = (10!_3)(9!_3)(8!_3) \quad \text{i.e., } [(10)(9)(8) \cdots (1)] = [(10)(7)(4)(1)][(9)(6)(3)][(8)(5)(2)]$$
$$13!_2 = (13!_4)(11!_4)$$
$$13!_{(1/2)} = (13!)(12\tfrac{1}{2}!)$$
$$8!_{(1/4)} = (8!)(7\tfrac{3}{4}!)(7\tfrac{1}{4}!)$$
$$3!_{(1/3)} = (3!)(2\tfrac{2}{3}!)(2\tfrac{1}{3}!).$$

THE FRACTORIAL ROOT

Definition 5. In order to solve the equation $y!_b = x$ for y, where x and y are positive real numbers, a new algebraic operation is needed to reverse or undo the fractorial process. We call the inverse operation of the fractorial the "**fractorial** root" and it is designated in the following manner: $y = x{\scriptstyle\mathrm{i}}_b$, *read* " $y = x$ fractorial root step *b*" or "$y =$ the fractorial root of x step *b*." Although the fractorial is itself a function, its inverse is only a relation. For instance, although $3! = 6$, there are infinite number of solutions to the equation $x! = 6$. The solution set is $S = \{3, 3.4738, 4.1766, \ldots\}$. In this case, 3 is referred to as the "principal fractorial root."

Definition 6. In general, the principal fractorial root of a positive real number R is the smallest value of x, where $(x \geq b)$ such that $x!_b = R$. In the examples that follow, we will mainly be concerned with solving for the principal fractorial root.

SOLVING FOR THE FRACTORIAL ROOT

Example 1. Solve for x: $x! = 40$. In other words, we are trying to find the factorial root of *40* (the fractorial root, step *1*, of *40)*.

STEP *1.* Trap $x!$ between two perfect fractorials,

Since $4! = 24$ and $5! = 120$, $4! < x! < 5!$. By our definition of a principal fractorial root, it must also be true that $4 < x < 5$. What *we* are actually saying is that *4* plus a decimal mantissa, m, equals x.

STEP *2.* Set up an equation.

We have two alternatives. We can either solve for m, using the equation:

$$(4+m)(3+m)(2+m)(1+m)(m) = 40$$

or we can solve directly for x using an alternate equation:

$$(x)(x-1)(x-2)(x-3)(x-4) = 40.$$

I prefer this second method.

It is important to note that when $x!_b$ is trapped between two perfect fractorials $a!_b$ and $(a+1)!_b$, the number of factors in its expansion is equal to the number of factors in the expansion of the higher order fractorial.

Solving for x using Newton's method, we obtain the principal fractorial root, *40)~4.5897*.

In general, after finding the principal fractorial root, all other fractorial roots can be obtained by increasing the number of factors in the expansion of $x!_b$. In this case, $(x)(x-1)(x-2)(x-3)(x-4)(x-5) = 40$ and $(x)(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = 40$ will yield two other possible solutions to the equation $x! = 40$.

Example 2. Solve for x: $x!_2 = 200$.

STEP *1.* Trap $x!_2$ Since $7!_2 = 105$ and $8!_2 = 384$, $7!_2 < x!_2 < 8!_2$ and $7 < x < 8$.

STEP *2.* Set up an equation: Either $(7+m)(5+m)(3+m)(1+m) = 200$ and $x = (7+m)$, or $x(x-2)(x-4)(x-6) = 200$. Solving for x in either case gives 200¡2 *as 7.4381.*

Example 3. Solve for x: $x!_{(3/2)} = 20$. This problem is slightly different from the first two in that in this case the step is a non-integer. Problems like this will require a little extra work.

STEP *1.* Re-express the problem using Theorem *7:* $x!_{(3/2)} = 20$ becomes $[(2x)!_3]/(2^n) = 20$.

STEP *2.* Trap $[(2x)!_3]/(2^n)$ between two perfect fractorials by substituting in for x two appropriate consecutive integers. If $x = 4$, then $n = 3$ and the expression becomes $(8!_3)/(2^3) = 10$. If $x = 5$, then $n = 4$ and the expression becomes $(10!_3)/(2^4) = 35$. Thus, $(8!_3)/8 < [(2x)!_3]/(2^n) < (10!_3)/16$, and $4 < x < 5$.

STEP *3.* Find n. It is still unknown whether $n = 3$ or $n = 4$. If we let $x = 4\frac{1}{2}$, then the expression becomes $(9!_3)/(2^3) = 20\frac{1}{4}$. Since $[(2x)!_3]/(2^n) < 20\frac{1}{4}$, we conclude that $n = 3$.

STEP *4.* Set up an equation. $[(2x)!_3]/8 = 20$, or $(2x)!_3 = 160$, and so $2x(2x-3)(2x-6) = 160$, and finally, x *as 4.4899.*

The FACTORIAL ROOTS OF ONE. It is interesting to solve the equation $x! = 1$. Since $0! = 1$, one might assume that *0* is the principal fractorial root. However, by definition of the principal fractorial root, x must be $\geq$ the step. While *0* is one value for x¡, the principal fractorial root is *1*. Solving for another possible value we have: $x(x-1) = 1$ and thus $x = (\sqrt{5}+1)/2$, the golden ratio! Thus the golden ration, which has the tendency of appearing in the most unexpected places, is a factorial root of *1*.

SOLVING FOR THE UNKNOWN STEP.

The technique for doing this is quite similar to that of solving for the fractorial root. Although, while there are an infinite number of solutions to $x!_b = R$, there is only one value of x that will make the equation $a!_x = R$ true.

Example 1. Solve for x: $10!_x = 105$.

STEP *1.* Trap $10!_x$ between two perfect fractorials. Since $10!_5 = 5 =$ and $10!_4 = 120$, $10!_5 < 10!_x < 10!_4$. We conclude from this that $4 < x < 5$.

STEP *2.* Set up an equation. Since $10(10-x)(10-2x) = 105$, we have $4z^2 - 60x + 179 = 0$, and so $x \approx 4.1088$.

Example 2. Solve for x: $9!_x = 66$.

STEP *1.* $9!_3 = 162$ and $9!_4 = 45$. Therefore, $9!_4 < 9!_x < 9!_3$ and $3 < x < 4$.

STEP 2 $9(9-x)(9-2x) = 66$, thus $6x^2 - 81x + 221 = 0$, and so $x \approx 3.7955$.

DISCONTINUITY, STIRLING'S FORMULA, AND THE GAMMA FUNCTION.

The **following** properties hold true, **making** the fractorial discontinuous when x is any multiple *(n)* of the step:

1. $\lim_{x^-}(x!_b) = (nb)!_b$ *2.* $\lim_{x^+}(x!_b) = 0$

It is important to note that Stirling's formula, $a! \approx \sqrt{2\pi a}(a/e)^a$, is only a valid approximation for a! when a is a whole number.

The reason for this is that , while the function $f(a) = a!$ is discontinuous when a is a whole number, the approximation formula is a smooth function for all positive values of a. It would make an interesting problem to derive a formula to Stirling's that would approximate $a!_b$ for whole numbers a and b.

An interesting question to consider is how the fractorial function relates to the gamma function. It is true that $\Gamma(x+1) = x!$ for every whole number x. Nevertheless, it is important to realize that these are two markedly different functions. $\Gamma(x)$ is continuous for $x \geq 0$, while $(z!)$is not. $\Gamma(x)$ is defined for negative values of x, and for $0 < x < 1$, while $x!$ is not. The fact that the gamma function has the value of the factorial for all whole numbers x, should not imply that the factorial must naturally behave like the gamma function for fractional and negative values of x.

CONCLUSION.

We have seen that the concept of the generalized factorial operation called the fractorial implies a number of new theorems and algebraic techniques. As a useful notation, the fractorial also offers the ability to simplify, manipulate, and compute with greater ease the long, space-consuming products that sometimes appear in formulas. It would be interesting to see whether, as a mathematical idea in and of itself, the fractorial has new insights to offer to combinatorics or the sciences.

The author prepared this paper while he was a junior at Carmel High School in Carmel, NY. He is currently enrolled at Yeshiva University under an early admission program.

ON TRANSPOSITIONS OVER FINITE FIELDS

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Let K denote the finite field of order $q = p^n$. A polynomial $f(z)$ in $K[x]$ represents a function $F : K \to K$ if $F(b) = f(b)$ for all b in K. Two polynomials f and g represent the same function F if and only if

$$f(x) \equiv g(x) \pmod{x^q - x} \tag{1}$$

Further, by the **Lagrange** interpolation formula [2, p. 369], every function over K can be represented by a polynomial $f(x)$ in $K[x]$.

Now a polynomial $f(x)$ in $K[x]$ is called a **permutation polynomial** if the function represented by f is one-to-one. Hence, if we identify polynomials related by (1), the permutation polynomials over K form a group, isomorphic to the symmetric group S_q. Thus every permutation polynomial is the product (composition) of finitely many transpositions.

The purpose of this note is to point out that transpositions over K are represented by "nice" polynomials. More precisely, if $T_{a,b}$ denotes the transposition over K defined by

$$T_{a,b}(x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ x & \text{otherwise.} \end{cases}$$

then we will show that

$$T_{a,b}(x) \equiv (a - b)\sum_{k=1}^{q-1}(a^k - b^k)x^{q-1-k} + x \pmod{x^q - x}. \tag{2}$$

To prove equation (2), we will use an alternative method to the **Lagrange** formula. This second method uses the fact that $x^{q-1} = 1$ for all nonzero elements of the field K. We will also need the fact that $px = 0$ for all x in K, where p denotes the characteristic of the field.

We are ready for our result.

THEOREM: With notation as above,

$$T_{a,b}(x) \equiv (a - b)\sum_{k=1}^{q-1}(a^k - b^k)x^{q-1-k} + x \pmod{x^q - x}.$$

Proof:

$K^* = K - \{0\}$ is a cyclic group of order $q - 1$. Hence,

$$(x - a)^{q-1} = \begin{cases} 0 & \text{if } x = a \\ 1 & \text{if } x \neq a. \end{cases}$$

So,

$$T_{a,b}(x) = (b - a)[1 - (x - a)^{q-1}] + (a - b)[1 - (x - b)^{q-1}] + x$$

for all x in K. Now, combining the binomial formula $(r - s)^n = \sum_{i=0}^{n}\binom{n}{i}r^i s^{n-i}$ and the fact that $px = 0$ for all x in K, we obtain:

$$(x - a)^q = x^q - a^q.$$

Therefore,

$$\begin{aligned} T_{a,b}(x) &= (b - a)[1 - (z - a)^{q-1}] + (a - b)[1 - (x - b)^{q-1}] + x \\ &= (a - b)[(x - a)^{q-1} - (x - b)^{q-1}] + x \\ &= \begin{cases} (a - b)\left[\frac{(x-a)^q}{x-a} - \frac{(x-b)^q}{x-b}\right] + x & \text{if } x \neq a, b \\ b & \text{if } x = a \\ a & \text{if } x = b \end{cases} \\ &= \begin{cases} (a - b)\left[\frac{x^q-a^q}{z-a} - \frac{x^q-b^q}{z-b}\right] + x & \text{if } x \neq a, b \\ b & \text{if } x = a \\ a & \text{if } x = b \end{cases} \\ &= \begin{cases} (a - b)\left[\sum_{k=1}^{q-1}(a^k - b^k)x^{q-1-k}\right] + x & \text{if } x \neq a, b \\ b & \text{if } x = a \\ a & \text{if } x = b \end{cases} \\ &= (a - b)\left[\sum_{k=1}^{q-1}(a^k - b^k)x^{q-1-k}\right] + x \quad \text{for all } x \text{ in } K. \end{aligned}$$

COROLLARY: $\deg(T_{a,b}(x)) = q - 2$.

Note: The reader can find further information concerning finite fields in references [1] and [2] and concerning permutation polynomials in [2] and [3]. Reference [3] is an exellent survey where current open problems on the topic are discussed.

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The author prepared this paper, under the supervision of Professor Javier *Gomez-Calderon,* while she was a junior at Penn Stale University - New Kensington Campus.

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ON THE NUMBER OF INVERTIBLE MATRICES OVER $\mathbf{Z}_{p^e}$

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Invertible matrices play an important role in cryptography and coding theory. To encode a message **from** plaintext S into **codetext** T, one first divides the message into blocks of length **n,** using buffer characters when necessary to make all the **blocks** the same length. Let S' be this buffered message. Next, using S', one can create a n x $\frac{|S'|}{n}$ matrix (where $|S'|$ is the number of characters is S') with each column representing a block of length n. Finally, one uses an invertible matrix M to encode this message in **codetext** T. The decoding process is where the existence of M^{-1} is essential. To decode a message, one takes the **codetext** T in matrix form and multiplies it by M^{-1} to find the original message S.

For example, suppose we wish to encode the message "MATHISFUN" in blocks of length two. Since this message is of length nine, we must add a buffer character that will not be confused with the original message, such as "Q" or "Z." Thus we send the message "MATHISFUNQ" in blocks of 2 using the invertible encoding matrix, for example,

$$\begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}.$$

We write the 26-letter alphabet as A = 0, B = 1, C = 2, ..., Z = 25. Using arithmetic modulo 26, the encoding process for our message is

$$\begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix} \cdot \begin{bmatrix} M & T & I & F & N \\ A & H & S & U & Q \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix} \cdot \begin{bmatrix} 12 & 19 & 8 & 5 & 13 \\ 0 & 7 & 18 & 20 & 16 \end{bmatrix} = \begin{bmatrix} 24 & 21 & 2 & 6 & 2 \\ 8 & 4 & 14 & 25 & 13 \end{bmatrix}.$$

Thus, "MATHISFUNQ" is now encoded as "YIVECOGZCN." To translate "YIVECOGZCN" back into plaintext, we first compute the inverse of

$$\begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix},$$

which is

$$\begin{bmatrix} 13 & -5 \\ 13 & 2 \\ -5 & \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ 21 & 2 \end{bmatrix} \text{ (modulo 26).}$$

Hence, the decoding process is

$$\begin{bmatrix} 13 & 21 \\ 21 & 2 \end{bmatrix} \cdot \begin{bmatrix} 24 & 21 & 2 & 6 & 2 \\ 8 & 4 & 14 & 25 & 13 \end{bmatrix} = \begin{bmatrix} 12 & 19 & 8 & 5 & 13 \\ 0 & 7 & 18 & 20 & 16 \end{bmatrix},$$

which is the original message "MATHISFUNQ."

If the n x n encoding matrix M is over a finite field (i.e., the alphabet's length is a prime number), then we have an invertible matrix if, and only if, $\det(M) \# 0$. However, most alphabets probably do not have a prime number of characters, so M is usually over a finite ring. This poses a problem, for M may be non-invertible even though $\det(M) \# 0$. Such **a** result is possible due to a ring's zero divisors.

Since we have seen that invertible matrices are necessary for the decoding process, it is nice to know just how many we can create. An interesting result from determining the number of 1 x 1 and 2 x 2 invertible matrices is that the probability of choosing an invertible matrix from the total number of matrices over $\mathbf{Z}_p$ is the same as the probability of choosing an invertible matrix from the total number of matrices possible over $\mathbf{Z}_{p^e}$, where p is prime and e > 1.

First, let us count the n x n matrices that are invertible over the field $\mathbf{Z}_p$, where p is prime. The total number of ways to fill the entries of the first n x 1 column is p^n. However, for a matrix to be invertible, there cannot exist a column filled with zeros. Thus, there exist $p^n - 1$ acceptable ways to create the first column of an invertible matrix. Again, the total number of ways to fill the entries of the second n x 1 column is p^n. However, to create an invertible matrix, we cannot choose any multiple of the first column. Hence, there exist a total of $p^n - p$ acceptable ways to create the second column. Continuing in this manner, we see that the number of invertible n x n matrices is given by

$$(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1}). \tag{1}$$

Hence, the probability of choosing an invertible n x n matrix from the set of all n x n matrices possible is

$$\frac{(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{p^{n^2}}. \tag{2}$$

We now turn to the task of trying to determine the number of n x n invertible matrices over the ring $\mathbf{Z}_{p^e}$.

Proposition 1. Let $M_{1\times1}$ be the set of 1 x 1 matrices over $\mathbf{Z}_{p^e}$, where p is prime and $e > 1$. The number of invertible matrices in $M_{1\times1}$ is p' $\left(\frac{p-1}{p}\right)$.

Proof: Obviously, a 1 x 1 matrix is invertible if, and only if, its single entry is invertible over $\mathbf{Z}_{p^e}$. Using **Euler's φ-function,** we see that the number of invertible elements in $\mathbf{Z}_{p^e}$ (i.e. the number of elements that are relatively prime to p^e is given by

$$\varphi(p^e) = p^e\left(1 - \frac{1}{p}\right) = p^e\left(\frac{p-1}{p}\right)$$

Hence, the probability that an invertible matrix is picked is

$$\frac{p^e\left(\frac{p-1}{p}\right)}{p^e} = \frac{p-1}{p}.$$

Interestingly enough, $\frac{p-1}{p}$ is also the probability of choosing an invertible matrix from $M_{1\times1}$ over $\mathbf{Z}_p$.

As motivation for counting the 2 x 2 invertible matrices over $\mathbf{Z}_{p^e}$, where p is prime and e > 1, we will first consider the problem of counting the number of 2 x 2 invertible matrices over $\mathbf{Z}_{2^e}$, where $e > 1$.

Let $M_{2\times2}$ be the set of 2 x 2 matrices over $\mathbf{Z}_{2^e}$.

Suppose that A 6 $M_{2\times2}$ such that

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix},$$

so $\det(A) = a_1b_2 - b_1a_2$. Hence, to be assured that A^{-1} exists, we must have $[\det(A)]^{-1}$ 6 $\mathbf{Z}_{2^e}$. Notice that "$a \in \mathbf{Z}_{2^e}$ is invertible," "$a \in \mathbf{Z}_{2^e}$ is an odd number," and "$a \in \mathbf{Z}_{2^e}$ is relatively prime to 2^e" are equivalent statements. Thus, to count all the invertible matrices in $M_{2\times2}$, one only needs to count all the **differences** of products $a_1b_2 - b_1a_2$ that yield an odd number. To this end, we **see** that $\det(A)$ is an odd number when a_1b_2 is odd and b_1a_2 is even, or vice versa. Hence, we can create a sequence of independent steps that will count the number of ways in which $\det(A) = a_1b_2 - b_1a_2$ can be an odd number, where $N = 2^e$:

Step 1. Choose one of a_1b_2 or b_1a_2 to be odd. (*The other* product *is forced to be even*). There are 2 ways to do this step.

Step 2. For the product that is odd, choose two numbers that are relatively prime to 2^e (i.e. odd times odd is odd). There are $[\varphi(N)]^2$ ways to do this step.

Step 3. For the product that is even, choose two numbers whose product is even. There are $N^2 - [\varphi(N)]^2$ ways to do this step.

Thus the number of invertible matrices in $M_{2\times 2}$ over $\mathbf{Z}_{2^e}$ is

$$2[\varphi(N)]^2\left[N^2 - [\varphi(N)]^2\right] \tag{3}$$

Hence, the probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_{2^e}$ is

$$\begin{aligned}
\frac{2[\varphi(N)]^2\left[N^2 - [\varphi(N)]^2\right]}{N^4} &= \frac{2[\varphi(2^e)]^2\left[(2^e)^2 - [\varphi(2^e)]^2\right]}{2^{4e}} \\
&= \frac{2\left[2^{e-1}\right]\left[2^{2e} - \left[2^{e-1}\right]^2\right]}{2^{4e}} \\
&= \frac{2\cdot 2^{4e-2}\left[1 - 2^{-2}\right]}{2^{4e}} \\
&= 2\cdot 2^{-2}\left[1 - 2^{-2}\right] = \frac{3}{8}.
\end{aligned}$$

Notice that the probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_2$ is

$$\frac{(p^2-1)(p^2-p)}{p^4} = \frac{(2^2-1)(2^2-2)}{2^4} = \frac{3}{8}.$$

This result provides a basis for the following proposition.

Proposition 2. The probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_p$ is equal to the probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_{p^e}$, where p is prime and $e > 1$.

Proof; To prove this conjecture, we must determine the number of invertible matrices in $M_{2\times 2}$ over $\mathbf{Z}_{p^e}$, where p is prime, $p > 2$, and $e > 1$. For convenience of notation, let $N = p^e$. Note that invertible elements of $\mathbf{Z}_N$ can now be odd or even. For instance, in $\mathbf{Z}_{3^2}$, both 7 and 2 are invertible. Also note that under the constraints of N, $2[\varphi(N)]^2\left[N^2 - [\varphi(N)]^2\right]$ counts the ways to get an invertible number when one of a_1b_2 or b_1a_2 is invertible and the other is not invertible (i.e. invertible minus non-invertible equals invertible over positive powers of a single prime).

Hence, we now need to determine the number of differences of products that are invertible where each of a_1b_2 and b_1a_2 is invertible as well. Thus, we can create a sequence of independent steps that will count the number of ways in which $\det(A) = a_1b_2 - b_1a_2$ can be invertible when each of a_1b_2 and b_1a_2 is invertible.

Step 1. Choose i to be an invertible element of $\mathbf{Z}_N$. There are $\varphi(N)$ ways to do this step.

Step 2. Suppose that a and b are invertible elements of $\mathbf{Z}_N$ such that $a - b = i$. We know that for each i, there exist N ways to write i as a difference. Of those N differences, we know that a non- invertible number appears in a difference a total of $2[N - \varphi(N)]$ times. Thus, there are $N - 2[N - \varphi(N)]$ choices of differences in which both numbers of the differences are invertible. Hence, there are $[2\varphi(N) - N]$ ways to do this step.

Step 3. For each difference $a - b = i$, where each of a and b are invertible, we need to count the number of ways in which a could be a product of two invertible elements of $\mathbf{Z}_N$, call them α_1 and α_2. There are $\varphi(N)$ choices for α_1. Notice that for any choice of α_1, α_2 is uniquely determined. Thus there are only $\varphi(N)$ products that create a. Likewise, b can be determined from $\varphi(N)$ products. Hence, there are $[\varphi(N)]^2$ ways to do this step.

Thus, the number of ways in which $\det(A) = a_1b_2 - b_1a_2$ can be invertible when each of a_1b_2 and b_1a_2 is invertible is $\varphi(N)[2\varphi(N) - N][\varphi(N)]^2$. Hence, the number of invertible matrices in $M_{2\times 2}$ over $\mathbf{Z}_N$ ($N = p^e$, where p is prime, $p > 2$, and $e > 1$) is

$$2[\varphi(N)]^2\left[N^2 - [\varphi(N)]^2\right] + \varphi(N)[2\varphi(N) - N][\varphi(N)]^2. \tag{4}$$

When $N = 2^e$ for $e > 1$, we have $\varphi(2^e) = 2^{e-1}$. Thus $[2\varphi(N) - N] = \left[2\cdot 2^{e-1} - 2^e\right] = 0$, which reduces equation (4) to equation (3). Hence, we can state that equation (4) holds for $N = p^e$, where p is prime and $e > 1$. After some algebra, equation (4) simplifies to

$$N[\varphi(N)]^2[2N - \varphi(N)]. \tag{5}$$

Thus, for p prime and $e > 1$, the probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_{p^e}$ is

$$\begin{aligned}
\frac{N[\varphi(N)]^2[2N - \varphi(N)]}{N^4} &= \frac{p^e[\varphi(p^e)]^2[2p^e - \varphi(p^e)]}{p^{4e}} \\
&= \frac{p^e\left[p^e - p^{e-1}\right]^2\left[2p^e - (p - p^{e-1})\right]}{p^{4e}} \\
&= \frac{p^e\left[p^e - p^{e-1}\right]^2\left[p^e + p^{e-1}\right]}{p^{4e}} \\
&= \frac{[p-1]^2[p+1]}{p^3} \\
&= \frac{(p^2-1)(p-1)}{p^3} \\
&= \frac{(p^2-1)(p^2-p)}{p^4}
\end{aligned}$$

This is exactly the same probability of choosing an invertible matrix from $M_{2\times 2}$ over $\mathbf{Z}_p$ using equation (2). This proves Proposition 2.

Using the counting techniques from the proof of Proposition 2, we can count the number of invertible matrices that have special forms. For example, suppose we want to count the number of invertible matrices over $\mathbf{Z}_N$ ($N = p^e$, where p is prime and $e > 1$) of the form

$$A = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

First, we note that $\det(A) = (a_1b_2 - b_1a_2)c_3$. To be sure that A is invertible, both $a_1b_2 - b_1a_2$ and c_3 must be invertible over $\mathbf{Z}_N$. From equation (5), we know that the number of ways in which $a_1b_2 - b_1a_2$ is invertible is $N[\varphi(N)]^2[2N - \varphi(N)]$. Also, c_3 has $\varphi(N)$ ways to be invertible. Hence, the number of invertible matrices of the form

$$\begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

is $N[\varphi(N)]^2[2N-\varphi(N)]\varphi(N)$.

In general, the following conjecture is true:

Conjecture 3. The probability of choosing an invertible matrix from $M_{n\times n}$ over $\mathbb{Z}_p$ is equal to the probability of choosing an invertible matrix from $M_{n\times n}$ over $\mathbb{Z}_{p^e}$, where $n \geq 3$, p is prime, and $e > 1$.

This conjecture can he proven using the techniques of Kohlitz [1, Exercises 16-20, pp. 77-78].

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Mark Lancaster prepared this paper during his senior year at Hendrix College; his work on related topics continued into the following summer at the University of Tennessee (Knoxville), with Dr. David E. Dobbs as his advisor.

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2409 new members joined Pi Mu Epsilon during the 1991-92 academic year,

THE 5-STEP PROBABILITY SOLVER

Hang-Ling Chang and Paul J. Fairbanks
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1. Introduction

The Monte Hall problem, which originated on the game show "Let's Make a Deal," has stirred a renewed interest in the technique of solving prohahility problems. The authors believe their 5-step probability solver fills a gap in this area, in which intuition can often he deceiving. In the next section, we will illustrate the procedure for using the five steps to solve two such problems. In Section 3, the authors will use the 5-step method to show why the contestant on "Let's Make a Deal" doubles her prohahility of winning by using what is generally considered the non-intuitive strategy. This procedure should provide the rigorous solution of most, if not all, probability problems. It should also eliminate any possible controversy which could arise if an intuitive procedure is applied.

2. The 5 Steps

A surprising result concerning Acquired Immunity Deficiency Syndrome (AIDS) testing will be used to introduce the five steps.

Step 1: Identify all the partition rules.

Step 2: Define all basic events created by the partition rules.

Step 3: Formulate all known information using probability statements and the events defined.

Step 4: Formulate the questions using probability statements and the events defined.

Step 5: Apply prohahility formulas to find the solution(s) to the problem.

Let us assume that one percent of the US population has Human Immuno Virus (HIV). Let us further assume that a test has been developed which gives a positive result 98% of the time when the patient has HIV. This same test gives a negative result 97% of the time when the patient does not have the virus. We assume that the test is always conclusive; hence for patients that do not have the virus, the test gives a positive result 3% of the time. The question we wish to answer is: " What is the probability that a given person has HIV if he or she tests positive?"

Step 1: Partition the sample space, the population of the US, into those people that have HIV and those that do not. We also partition the sample space into those that test positive and those that test negative.

Step 2: Basic events are events defined in terms of exactly one partition rule. Suppose that a person is selected at random. We define V = "people with HIV", and T = "people that test positive". We previously assumed that this test always gives a positive or a negative result. We further assume that a person either has or doesn't have HIV. Therefore, V' = "people that don't have HIV," and T' = "people that test negative." Figure 1 summarizes the first two steps in our procedure.

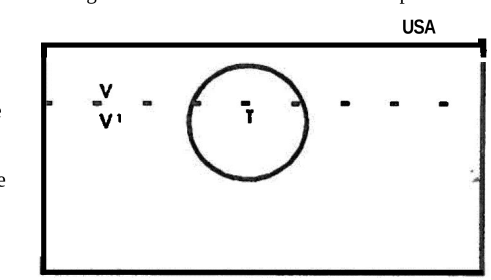

Figure 1: Partitioning of the Sample Space
V = People with HIV
T = People who tested positive

Step 3: $P(V) = 0.01$ (by assumption), and the given conditional probabilities are $P(T|V) = 0.98$ and $P(T'|V') = 0.97$.

Step 4: We wish to know the probability that a person who tests positive actually has HIV: $P(V|T)$.

Step 5: Using Bayes' Rule,

$$P(V|T) = \frac{P(V)P(T|V)}{P(V)P(T|V) + P(V')P(T|V')}$$
$$= \frac{(0.01)(0.98)}{(0.01)(0.98) + (0.99)(0.03)}$$
$$= 0.25$$

Hence, despite the fact that the test seems to be quite accurate based on the given information, only one-fourth of those people who test positive actually had contracted HIV.

The next example illustrates the use of the five steps for a typical elementary probability problem: "As the buzzer sounds at the end of a basketball game, the Celtics' center, Tom Moore, is fouled and is awarded two free throws. Moore makes 80% of his foul shots, and since he is as cool as ice, we can safely assume that his probability of making the second one is not affected by how he did on the first shot. The score is Pistons 100 and Celtics 99. What is the probability that the game will be tied and go into overtime?"

Step 1: For the first shot, partition the sample space into the two basic events – make it or miss it. Do the same for the second shot.

Step 2: Let M_1 = "make the 1st shot," and M_2 = "make the 2nd shot." Naturally, the complements will be the act of missing the respective shots.

Step 3: $P(M_1) = P(M_2) = 0.80$

Step 4: We wish to know the probability of a tie after regulation, which equates to the probability of making exactly one of the two free throws.

Step 5:

$$P(M_1 \text{ and } M_2') + P(M_1' \text{ and } M_2) = P(M_1)P(M_2') + P(M_1')P(M_2)$$
$$= (0.80)(0.20) + (0.20)(0.80)$$
$$= 0.32$$

3. Resolving The Monte Hall Controversy

In one version of the Monte Hall example, the host asks the contestant to choose one of three boxes in an attempt to find the one box which contains the key to a new car. Before the contestant opens the box she chose, the host always knowingly opens an empty box from the two that remain. Now the contestant is offered the opportunity to trade her box for the one remaining. Should she? Generally, the intuitive answer is that "It doesn't matter"; but we will use the 5-step probability solver to show why she should switch.

There are three boxes and they are equally likely to contain the key to the new car. Thus, there should be no argument that the contestant's probability of winning is, and remains, one-third if she has made up her mind to not switch. We will now apply our five steps to the experiment which consists of choosing a box and then switching to the remaining box. Since we are considering the case in which the contestant always chooses a box and then switches, this is a two-step experiment.

Step 1: Two partition rules are evident. The first one partitions the sample space of all possible outcomes of the experiment into two events: the event of obtaining the box with the key and the event of obtaining an empty box. The second one partitions the sample space created by the act of switching. In either case, two events are created: obtaining the box with the key or obtaining an empty box.

Step 2: Let K_f = "the first box chosen contains the key," and K_s = "the box switched to contains the key." The other events can now be defined as complements of these two.

Step 3: Assuming that the three boxes have the same chance of being chosen by the contestant, the known information is: $P(K_f) = 1/3$, $P(K_s|K_f) = 0$, and $P(K_s|K_f') = 1$.

Step 4: The answer to the question "Should the contestant switch to the third box?" depends solely on the values of $P(K_f)$ and $P(K_s)$. If $P(K_s)$ exceeds $P(K_f)$, then it is to the contestant's advantage to switch.

Step 5: Since $P(K_f) = 1/3$ is already given, we will derive $P(K_s)$ as follows:

$$P(K_s) = P(K_s \text{ and } K_f) + P(K_s \text{ and } K_f')$$
$$= P(K_f)P(K_s|K_f) + P(K_f')P(K_s|K_f')$$
$$= (1/3)(0) + (2/3)(1)$$
$$= 2/3$$

Therefore, switching doubles the contestant's chance of winning the car from 1/3 to 2/3.

4. Conclusion

Ever since its initial publication in the "Ask Marilyn" column (Parade, Sept. 9, 1990), the Monte Hall problem has instigated heated discussions concerning probability problems. However simple these problems may appear, they can be very deceptive if not handled with care. We have found throughout our years of teaching probability that the 5-step solution procedure presented here has never failed to provide clear, non-controversial, and, most importantly, correct answers.

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ON THE MATCHING PROBLEM IN PROBABILITY

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This paper presents standard results in the classic matching problem in probability. These results should be a part of any good introductory course in the theory of probability, and can be found in a number of excellent textbooks. The method of derivation employed in this article may be original and definitely simplifies the proofs, but the main results are well known to mathematicians and statisticians.

In the classic matching problem in probability, we imagine that n men (each wearing a hat), arrive at a social function, and each checks his hat at the door. At the end of the evening, the men are given their hats back **completely at random.** We assume that the n men are distinct, and that their hats are distinct. For mathematical convenience, we assume that the men are numbered from 1 to n, and the hats are also numbered from 1 to n. When they arrived, man number i was wearing hat number i, $i = 1,2,3,\dots,n$. When they leave, each man will be wearing a randomly selected hat, thereby generating a random permutation of the integers $1,2,3,\dots,n$.

In the spirit of probability, let us define events $A_1, A_2, \dots, A_n$ by the following description: A_i is the event that man numbered i leaves with his own hat. If A_i occurs, we will say that a match occurs at i. Evidently,

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

for $i = 1,2,\dots,n$.

In the material below, AB means the intersection of the sets A and B, and A^c is the complement of the set A.

Now $A_1 \cup A_2 \cup \dots \cup A_n$ is the event that at least one match occurs, and $A_1^c A_2^c \cdots A_n^c$ is the event that no matches occur; i.e., no one leaves with his own hat. These two events are complementary and are related by the equation

$$P(A_1^c A_2^c \cdots A_n^c) = 1 - P(A_1 \cup A_2 \cup \dots \cup A_n).$$

Using the usual counting rules, we see that

$$P(A_1) = \frac{(n-1)!}{n!},$$

$$P(A_1 A_2) = \frac{(n-2)!}{n!},$$

$$P(A_1 A_2 A_3) = \frac{(n-3)!}{n!},$$

and so forth. [We assume that the reader knows that the number of permutations $i_1, i_2, i_3, \dots, i_n$ of $1,2,3,\dots,n$ is $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$].

Now let's return to the probability of at least one match. (The following equations are a generalization of the result $P(A \cup B) = P(A) + P(B) - P(AB)$ to the union of n sets, and can be verified by induction.)

$$P(\text{at least one match}) = P(A_1 \cup A_2 \cup \dots \cup A_n)$$

$$= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n}\sum P(A_i A_j) + \sum_{1 \le i < j < k \le n}\sum\sum P(A_i A_j A_k) - \dots$$

$$\dots + (-1)^{n+1} P(A_1 A_2 \cdots A_n).$$

In the above equations, the first summation is over all single events, the double summation is over all **pairwise** intersections, the third summation is over all triple intersections, and so forth. Thus the single sum contains $\binom{n}{1}$ terms, the double sum contains $\binom{n}{2}$ **terms,** the triple sum contains $\binom{n}{3}$ terms, etc.

Now, by symmetry, $P(A_i) = P(A_1)$ for all i, $P(A_i A_j) = P(A_1 A_2)$ for all $i < j$, $P(A_i A_j A_k) = P(A_1 A_2 A_3)$ for all $i < j < k$, and so forth. Consequently,

$$P(\text{at least one match}) = \binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} - \dots + (-1)^{n+1}\binom{n}{n}\frac{1}{n!}$$

$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!},$$

where the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0,1,2,\dots,n.$$

Recalling that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots,$$

we see that, for large n,

$$P(\text{at least one match}) \approx 1 - e^{-1}$$

Actually, since we have an alternating series, n doesn't even need to be large in order for the approximation to be good. Also, since the event "no matches occur" is the complement of the event "at least one match occurs," we see that

$$P(\text{no matches occur}) = 1 - P(\text{at least one match occurs}) \approx 1 - (1 - e^{-1}) = e^{-1}.$$

By a **derangement** of $1,2,3,\dots,n$ we mean a permutation of $1,2,3,\dots,n$ with no integer in its natural position; i.e., no matches occur. [For example, 3,1,2 is a derangement of 1,2,3 ; but 1,3,2 is not.] Let D_n be the number of derangements of $1,2,3,\dots,n$. Then

$$P(\text{no matches occur}) = \frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}.$$

Consequently,

$$D_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right\} \quad \text{and} \quad \lim_{n\to\infty}\frac{D_n}{n!} = e^{-1}$$

Using the concept of derangements will expedite the calculation of additional probabilities of interest. For example:

$$P(\text{exactly one match occurs}) = P(A_1 A_2^c \cdots A_n^c) + P(A_1^c A_2 A_3^c \cdots A_n^c) + \dots + P(A_1^c A_2^c \cdots A_{n-1}^c A_n).$$

Now, by symmetry, all of the terms in the above equation are equal, so that

$$P(\text{exactly one match occurs}) = n\,P(A_1 A_2^c \cdots A_n^c) = n\left\{\frac{D_{n-1}}{n!}\right\} = \frac{D_{n-1}}{(n-1)!}.$$

This last result followed by counting the number of derangements of $2,3,\dots,n$, and dividing by $n!$. Thus

$$P(\text{exactly one match}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{n-1}}{(n-1)!} \approx e^{-1}.$$

In a similar vein,

$$P(\text{exactly two matches}) = P(A_1A_2A_3^c\cdots A_n^c)+\ldots+P(A_1^c\cdots A_{n-2}^cA_{n-1}A_n).$$

The right-hand side involves $\binom{n}{2}$ terms, all of which are equal. Thus, we have

$$\begin{aligned}
P(\text{exactly two matches}) &= \binom{n}{2}P(A_1A_2A_3^c\cdots A_n^c) = \frac{n!}{2!(n-2)!}\,\frac{D_{n-2}}{n!}\\
&= \frac{1}{2!}\,\frac{D_{n-2}}{(n-2)!}\\
&= \frac{1}{2!}\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^{n-2}}{(n-2)!}\right\}\\
&\approx \frac{e^{-1}}{2!}.
\end{aligned}$$

In a similar fashion, and with the help of derangements, we can see that

$$\begin{aligned}
P(\text{exactly } k \text{ matches occur}) &= \frac{1}{k!}\,\frac{D_{n-k}}{(n-k)!}\\
&= \frac{1}{k!}\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^{n-k}}{(n-k)!}\right\}\\
&\approx \frac{e^{-1}}{k!}.
\end{aligned}$$

for $0\le k\le n$. If we define the random variable X to be the number of matches that occur, then

$$\lim_{n\to\infty} P(X=k)=\frac{e^{-1}}{k!},\ k=0,1,2,3,\ldots$$

and we have found both the exact and the asymptotic distributions of X. As the reader may have noticed, this last equation could be written

$$P(X=k)\approx\frac{e^{-1}}{k!}=\frac{1^k}{k!}e^{-1},\ k=0,1,2,3,\ldots.$$

Thus, X is approximately Poisson distributed with a mean of one. This gives the result $E(X)$ as 1 for all n; in other words, the average number of matches is always approximately one, regardless of how many men are present.

Interestingly enough, the average number of matches is always one, regardless of how many men are present; i.e., $E(X)=1$ for all n. This can be deduced simply as follows. Write $X = X_1+X_2+\ldots+X_n$, where X_i is 1 if the i-th man gets his own hat back, and 0 if not. Since the mean of X is $E(X)=E(X_1)+E(X_2)+\ldots+E(X_n)$, and since $E(X_i)=0\cdot P(X_i=0)+1\cdot P(X_i=1)=P(X_i)=1/n$ for $i=1,2,3,\ldots,n$, it follows that $E(X)=1/n+1/n+\ldots1/n=1$ for every n. Thus, no matter now many men are involved, on the average, one man receives his own hat.

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THE SERIES FOR LN 2

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If $x>0$, then

$$x-1\ge \ln x\ge 1-\frac{1}{x} \tag{1}$$

This simple proposition can be used to derive the formula

$$\ln 2 = 1-\frac12+\frac13-\frac14+\cdots+\frac{1}{2n-1}-\frac{1}{2n}+\cdots \tag{2}$$

The left side of (1) follows immediately from the fact that $f(x)=x-1-\ln x$ has an absolute minimum at $x=1$, because $f'(x)=1-1/x=0$ iff $x=1$, and $f''(x)=1/x^2>0$ for $x>0$. If we write $1/x$ for x, then the expression $x-1\ge \ln x$ becomes $x\ln x\ge x-1$, and we have (1).

The standard derivation of (2) uses the series for $\ln(1+x)$, which is normally not covered until the second course in calculus. However, a proof based on (1) can be presented early in the first course. Using (1), we have

$$\begin{aligned}
1-\frac12+\frac13-\cdots+\frac{1}{2n-1} &= 1+\frac12+\frac13+\cdots+\frac{1}{2n-1}-2\left(\frac12+\frac14+\cdots+\frac{1}{2n-2}\right)\\
&= 1+\frac12+\frac13+\cdots+\frac{1}{2n-1}-\left(1+\frac12+\frac13+\cdots+\frac{1}{n-1}\right)\\
&= \frac1n+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\\
&= \left(\frac{n+1}{n}-1\right)+\left(\frac{n+2}{n+1}-1\right)+\cdots+\left(\frac{2n}{2n-1}-1\right)\\
&> \ln\frac{n+1}{n}+\ln\frac{n+2}{n+1}+\cdots+\ln\frac{2n}{2n-1}\\
&= \ln\left(\frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdots\frac{2n}{2n-1}\right)\\
&= \ln 2\\
&> \left(1-\frac{n}{n+1}\right)+\left(1-\frac{n+1}{n+2}\right)+\cdots+\left(1-\frac{2n-1}{2n}\right)\\
&= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\\
&= 1+\frac12+\frac13+\cdots+\frac{1}{2n}-2\left(\frac12+\frac14+\cdots+\frac{1}{2n}\right)\\
&= 1-\frac12+\frac13-\frac14+\cdots+\frac{1}{2n-1}-\frac{1}{2n}.
\end{aligned}$$

Hence,

$$\ln 2 > 1-\frac12+\frac13-\frac14+\cdots+\frac{1}{2n-1}-\frac{1}{2n}>\ln 2-\frac{1}{2n}.$$

Letting $n\to\infty$ gives (2).

PACKING PROBLEMS WITH SPHERES

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A dentist once wrote me to ask how it was possible to pack more that 1000 1-inch (spherical) marbles into a right rectangular box 10" x 10" x 10". Actually, there is room for at least 1188 marbles.

The two-dimensional packing problems discussed in this note are of two types. In the first type, the space to be covered is a rectangle with integral width; the "spheres" are disks of unit diameter. In the second type, the width is not an integer.

Case 1, integral width If the width is 1, the number of disks that fits is clearly [s], where the "box" is a rectangle of area s. If the width is 2, the arrangement of Figure 1 is different.

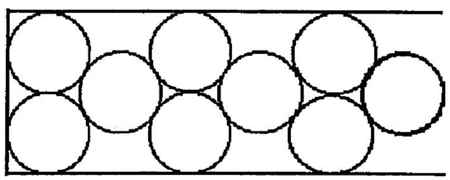

Figure 1

Each even-numbered row is moved successively $1 - \frac{1}{2}\sqrt{3} \approx 0.134$ units closer to the left-hand edge. If there are k even-numbered rows (and thus 2k rows in all) the packing can gain room for an extra two disks if k is large enough, but loses k disks when compared to the (square) lattice packing. The loss outweighs any possible gain.

The same argument and conclusion hold for widths 3, 4, 5, 6, and 7. For width 8, there will be a gain after 16 pairs of rows, no net loss or gain after 17 pairs of rows, a loss after 18-32 pairs of rows, and a gain after 32 pairs of rows. Since $1 - \frac{1}{2}\sqrt{3}$ is not rational, the pattern of net gains and losses is not a regular sequence of integers, but has some hiccups.

If the width of the rectangle exceeds 15 units, there is always a gain from the 17th pair of rows onwards. If the width lies between 10 and 15 units, the number of pairs of rows that entail losses becomes sparser and sparser.

Case 2, non-integral width. If the width of the rectangle is not an integer, a simple analysis can be used to calculate a "packing constant" that will measure the density of the best packing. Here are two examples.

Example 1. Suppose the width lies between 1 and 2. (See Figure 2.)

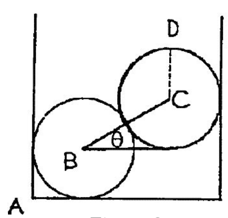

Figure 2

If the width w is precisely 2, the packing constant is 2; this means that two disks can fit into each integral unit of height. In this case, the angle θ from BC (the segment with end-points at the centers of two kissing circles) to the horizontal is $0°$. We shall take θ as a parameter. the y-coordinate of D (the top of the second disk) is equal to the sum of the y-components of the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$, that is, $\frac{1}{2} + \sin\theta + \frac{1}{2}$, or $1 + \sin\theta$. A reasonable definition of " packing constant" is $2/(1 + \sin\theta)$. This is, to repeat, the number of disks per unit height of the rectangle.

Example 2. Suppose the width lies between 2 and 3. (See Figure 3.)

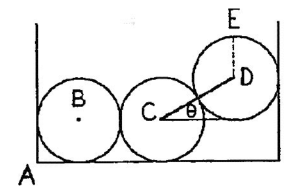

Figure 3

If the width is precisely 3, the packing constant is 3. In any other case, an appropriate parameter is the angle from the horizontal to the line segment connecting the centers of the second and third circles. The y-coordinate of F is $1 + \sin\theta$. The packing constant is, in this case, $3/(1 + \sin\theta)$. (If the width of the rectangle is between k and k+1, the packing constant is $(k + 1)/(1 + \sin\theta)$.]

Return to Figure 2. If the height of the rectangle is less that $1 + 1 + \sin\theta$, it will not be possible to fit more than one disk into the (truncated) space. What the packing constant measures is the limit, as the height of the rectangle increases indefinitely, of the number of disks per unit height.

Covering the entire plane. It has been proved that, for a two- dimensional plane infinite in both directions, close-packing is the densest possible disposition of disks. (See Figure 4.)

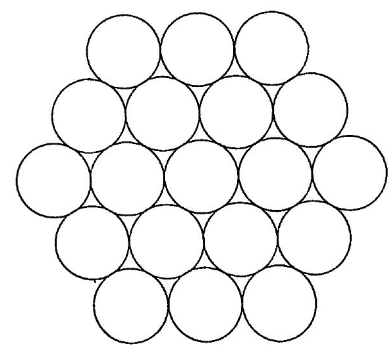

Figure 4

Paradoxically, a mathematical proof that close-packing in three dimensional space is densest was not published until 1963. *(American Mathematical Society Symposia Proceedings 7*, pp. 58-71.) In nine dimensions, the question is difficult.

The author wishes to thank J. W. Downs and the late J. D. E. Konhauser for their help with, this paper.

ON AN ELEMENTARY METHOD OF FINDING THE MINIMUM VALUE OF $\sum_{j=1}^{n} x_j^\alpha$, SUBJECT TO THE CONDITION $\sum_{j=1}^{n} x_j = a$, WHERE a IS A POSITIVE INTEGER

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We shall consider the following problem:

Problem. Find the minimum value of $f(x_1,\ldots,x_n) = \sum_{j=1}^n x_j^\alpha$, subject to the constraint conditions $\sum_{j=1}^n x_j = a$ and $x_j > 0$ for $j = 1,\ldots,n$, where a is a real number greater than or equal to one.

Although some solutions for the problem are already known (see references), they require involved calculations. However, if we restrict a to be an integer, we can give a very elementary method of finding the minimum value sought in the problem.

First, we present two lemmas.

Lemma 1. If x and a are positive real numbers, then $x^{\alpha+1} - x^\alpha \geq x - 1$, and the equality holds if and only if $x = 1$.

Proof: Let us set $T = x^{\alpha+1} - x^\alpha - (x-1)$. Then T can be factored in the form $(x^\alpha - 1)(x-1)$.

(i) If $x > 1$, then $x^\alpha - 1 > 0$ and $x - 1 > 0$. Hence, we have $T > 0$.

(ii) If $0 < x < 1$, then $x^\alpha - 1 < 0$ and $x - 1 < 0$. Hence, we also have $T > 0$.

By the way, we can easily check that T is equal to zero if and only if $x = 1$. Therefore, we have the desired result. ■

Lemma 2. If $x_j > 0$ $(j = 1,\ldots,n)$ and $\sum_{j=1}^n x_j = n$, then

$$\sum_{j=1}^n x_j^{\alpha+1} \geq \sum_{j=1}^n x_j^\alpha,$$

and the equality holds if and only if $x_1,\ldots,x_n$ are all equal to 1.

Proof:

$$\begin{aligned}\sum_{j=1}^n x_j^{\alpha+1} - \sum_{j=1}^n x_j^\alpha &= \sum_{j=1}^n (x_j^{\alpha+1} - x_j^\alpha)\\ &\geq \sum_{j=1}^n (x_j - 1) \quad \text{(by Lemma 1)}\\ &= \left(\sum_{j=1}^n x_j\right) - n\\ &= 0.\end{aligned}$$

This proves that $\sum_{j=1}^n x_j^{\alpha+1} \geq \sum_{j=1}^n x_j^\alpha$. It is clear from Lemma 1 that equality holds if and only if $x_1 = x_2 = \cdots = x_n = 1$. ■

Theorem. If $x_1,\ldots,x_n$ are positive real numbers and $\sum_{j=1}^n x_j = a$, then the minimum value of $f(x_1,\ldots,x_n) = \sum_{j=1}^n x_j^\alpha$, where α is a positive integer, is $\dfrac{a^\alpha}{n^{\alpha-1}}$.

Proof: If $a = 1$, the result is trivial. Hence, without loss of generality, we can assume that $a \geq 2$. Since $\sum_{j=1}^n x_j = a$ can be written in the form $\sum_{j=1}^n \frac{nx_j}{a} = n$, we obtain

$$\sum_{j=1}^n \left(\frac{nx_j}{a}\right)^\alpha \geq \sum_{j=1}^n \left(\frac{nx_j}{a}\right)^{\alpha-1} \geq \cdots \geq \sum_{j=1}^n \left(\frac{nx_j}{a}\right)$$

by Lemma 2. Therefore, we have

$$\begin{aligned}f(x_1,\ldots,x_n) &= \sum_{j=1}^n x_j^\alpha\\ &= \left(\frac{a^\alpha}{n^\alpha}\right)\sum_{j=1}^n \left(\frac{nx_j}{a}\right)^\alpha\\ &\geq \left(\frac{a^\alpha}{n^\alpha}\right)\sum_{j=1}^n \left(\frac{nx_j}{a}\right)\\ &= \frac{a^\alpha n}{n^\alpha}\\ &= \frac{a^\alpha}{n^{\alpha-1}}.\end{aligned}$$

As before, we can easily see, by Lemma 2, that the equality holds if and only if $x_1 = \ldots = x_n = \frac{a}{n}$. This completes the proof of the theorem. ■

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INQUIRIES

Inquiries about certificates, pins, posters, matching prize funds, support for regional meetings, and travel support for national meetings should be directed to the Secretary-Treasurer, Robert M. Woodside, Department of Mathematics, East Carolina University, Greenville, NC 27858, 919-757-6414.

INSTANTANEOUS CENTERS AND THEIR GEOMETRY

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Consider the following problem: "Define a tangent line to a curve at a point *A*." Many students, even in college level, will respond: "A line perpendicular to the radius." Even in the case of a circle, that statement is not a definition, but a theorem. So this is where we would like to start.

1. THEOREM. Let (C) be a circle with center C and let *A* be a point on (C). Then the tangent line to (C) at *A* is perpendicular to the line segment CA.

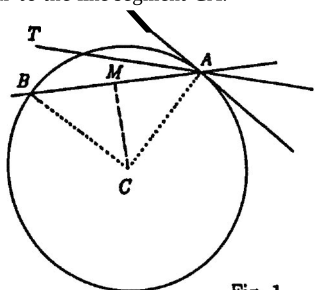

Fig. 1

The proof may look convincing, but it is not easy. If fact, we rely somewhat on intuition. Choose a line that intersects (C) at two points, *A* and *B*. (See Figure *1*.) Let M be the midpoint of the line segment *AB*. Then CM is perpendicular to *AB*. When *B* approaches *A*, the point *M* also approaches *A* and CM, which is always the perpendicular bisector of *AB*, will remain perpendicular to the limiting position of *AB*, that is, *AT*, the tangent line to (C).

One may ask: "Was this a proof?" Probably not a very good one, but it introduces the idea of approaching a limit, which is as old as the geometry itself.

2. TANGENT LINES TO AN ELLIPSE. Let F_1 and F_2 be foci of an ellipse. (See Figure 2.)

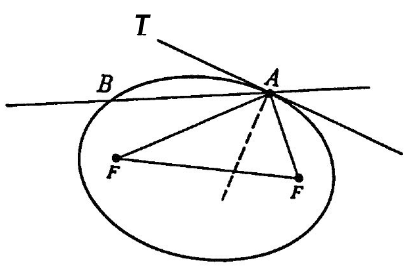

Fig. 2

Then the tangent line at a point *A* of the ellipse is the limiting position of a line *AB* when *B* approaches A. One can prove that the tangent line *AT* is perpendicular to the bisector of the angle F_1AF_2. We shall leave the proof to the reader. What may be interesting here is that this angle bisector does not pass through a fixed point. For this reason, we can not call this angle bisector a radius of the ellipse.

3. INSTANTANEOUS CENTERS. In machinery, all sorts of wheels roll over one another; many of them are not circular. We shall study these wheels through the concept of instantaneous centers. Let a plane (lamina) move over a fixed plane. Then there is an instantaneous center of rotation C. The locus of *C* in the fixed plane is called the base curve, and the locus of *C* in the moving plane is called the rolling curve. Actually one curve rolls over the other. We shall study these curvet geometrically.

4. THE POSITION OF A LAMINA. The position of a plane is completely fixed if three non-collinear points of it are fixed. Since we are interested in a moving plane, we only need the position of two points of the variable plane on the fixed plane. Here we rely on intuition. The reader may supply a proof.

5. OBTAINING THE CENTERS. Let *A* and *B* be two points attached to a lamina moving over a fixed plane. (See Figure 3.)

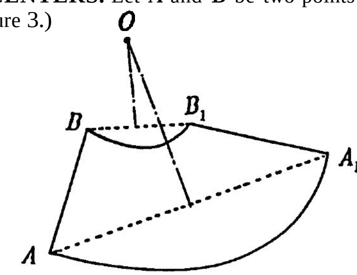

Fig. 3

Let A_1 and B_1 be new positions of A and *B*. Then $A_1B_1 = AB$. Let *O* be the point of intersection of the perpendicular bisectors of AA_1 and BB_1. (Special cases will be discussed later.) Then

$$OA = OA_1, \quad \text{and} \quad OB = OB_1.$$

So we may say that *AB* has rotated about *O* through an angle $BOB_1 = AOA_1$. We now define *C*, the instantaneous center by

$$C = \lim_{\substack{A_1 \to A \\ B_1 \to B}} O.$$

Special Cases. If the moving plane has a fixed point C in the fixed plane, then C is the instantaneous center. So the variable plane rotates about *C*.

If every point A of the variable plane moves on a straight line parallel to a fixed line d of the base plane, then the moving lamina will be shifting parallel to d. (See Figure 4.) In this case we may say that the instantaneous center is at infinity.

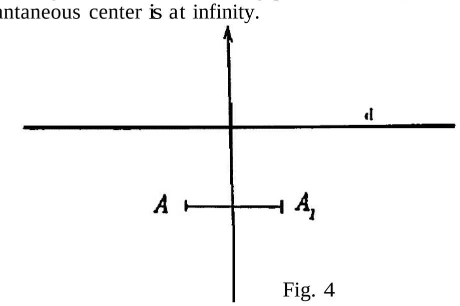

Fig. 4

7. THEOREM. Let a plane move so that a point *A* of it remains on a curve Γ. Then C, the instantaneous center, is on the normal line of Γ at *A*.

Proof: Let A_1 be a displacement of A. (See Figure 5.) Then the center of rotation is on the perpendicular bisector of AA_1. As A_1 approaches A, this perpendicular becomes the normal at A.

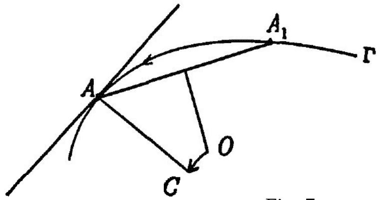

Fig. 5

BASE AND ROLLING CURVES. Let a plane move over a **fixed** plane. Then the locus of C, the instantaneous center of rotation in the fixed plane, is called the base *curve* and locus of C in the moving plane is called the ***rolling curve.*** We shall study the subject through examples.

9. GEOMETRIC TREATMENTS. Some examples will be studied,

Example 1. Let A and B be two fixed points of a moving plane such that A moves on the ***z-axis,*** B on the y-axis, and the length of $AB = l$ remains constant. Obtain the base and rolling curves.

Solution. By Theorem 7, the instantaneous center of rotation C is the point of intersection of the perpendicular to the ***z-axis*** at A and the perpendicular to the y-axis at B. (See Figure **6.**) Since $OC = AB = l$, the circle of diameter l which passes through the origin **rolls** over the circle of center O and radius l.

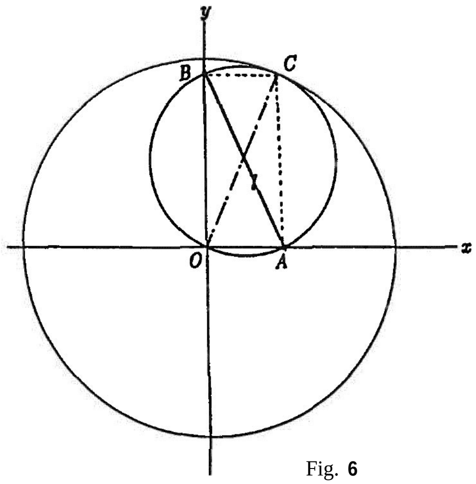

Fig. **6**

Example 2. In Example 1, one may substitute the y-axis by an axis t and get similar results. We **shall** give a few hints. Let the angle between Ox and Ot be a. (See Figure 7.)

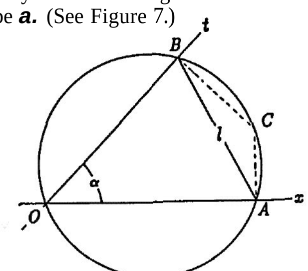

Fig. 7

We draw the perpendicular **line** to Oz at A and the perpendicular line to Ot at B. Then the point of intersection of these perpendicular lines is C, the instantaneous center of rotation. Note that the four points O, A, C, and B are on a circle, the angle ACB is $\pi - a$ and the length of $AB = l$ is constant. We leave it to the reader to show that OC is constant and thus obtain the base and rolling curves.

Example 3. Example 1 can be looked at in a different way. Let A move on the x-axis and D move on the circle of radius $l/2$ with center at O (See Figure 8.) Then we obtain the same result as in Example 1. We omit the proof.

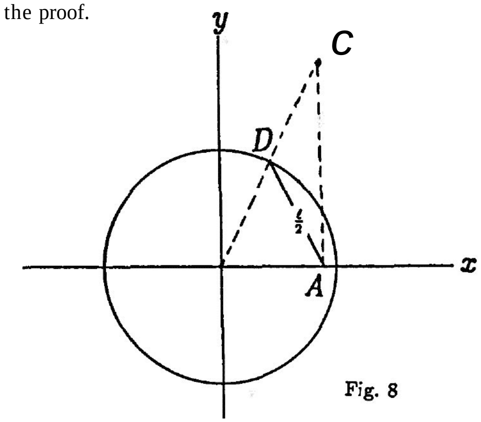

Fig. 8

10. ALGEBRAIC APPROACH. We shall give an example.

A plane connected to a line segment $AB = l$, where the length of l is greater than r, moves such that A is ou the circle $x^2 + y^2 = r^2$ and B is on the z-axis. (See Figure 9).

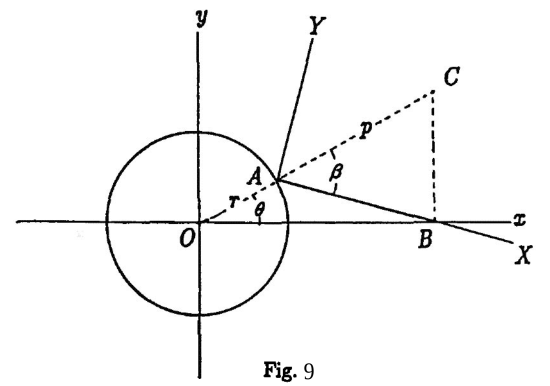

Fig. 9

Obtain the base and rolling curves.

Solution: Clearly C, the instantaneous center of rotation, is the point of intersection of OA and the perpendicular to the z-axis at B. Let C = (x, y) in the base plane. Let the angle $BOC = 0$.

Then

$$l^2 = r^2 + x^2 - 2rx\cos\theta. \tag{1}$$

Since

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \tag{2}$$

we can simplify *(1)* to

$$(x^2+r^2-l^2)\sqrt{x^2+y^2} = 2rx^2 \tag{3}$$

Note that in the special case $l = r$ the equation *(3)* changes to

$$x^2+y^2 = 4r^2 \quad \text{and} \quad x = 0. \tag{4}$$

Now for the rolling curve let AB as AX be the X-*axis* and AY perpendicular **to** it be the Y-*axis* in the variable plane. (See Figure 9). Let $AC =$ p and the angle $CAB = \beta$. Then

$$X = p\cos\beta \qquad Y = p\sin\beta \tag{5}$$

Note that p is variable. In the right triangle OBC we have

$$CB^2 = (r+p)^2 - OB^2. \tag{6}$$

Also in the triangle ABC we have

$$CB^2 = l^2 + p^2 - 2lp\cos\beta \tag{7}$$

In the triangle OAB the angle $OAB = \pi - \beta$. Thus

$$OB^2 = r^2 + l^2 + 2rl\cos\beta. \tag{8}$$

From (6), (7), and *(8)* we obtain

$$l^2 + p^2 - 2lp\cos\beta = (r+p)^2 - (r^2 + l^2 + 2rl\cos\beta). \tag{9}$$

Simplifying and letting $p\cos\beta = X$ and $p\sin\beta = Y$, we get

$$(l^2 - lX)\sqrt{X^2+Y^2} = r(X^2+Y^2) - rlX \tag{10}$$

which is the equation of the rolling curve. Rationalizing and simplifying, we obtain

$$r^2(X^2+Y^2)^2 - \left[(l^2-lX)^2 - 2r^2lX\right](X^2+Y^2) + r^2l^2X^2 = 0. \tag{11}$$

The case $r = l$ **amy** be interesting. We leave it to the reader.

One may approach the problem by the use of polar coordinates. We leave it to the reader.

SOME INTERESTING PROBLEMS: Let (A) and (B) be two fixed circles of equal radii in the same plane. **(See Figure 10.)**

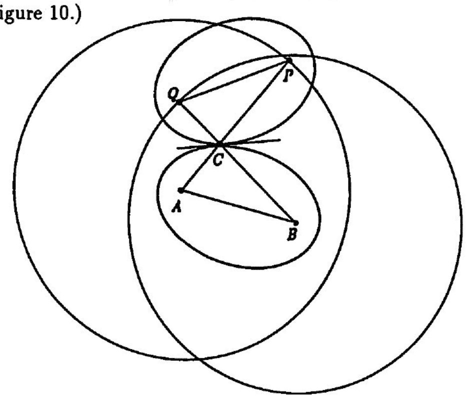

Fig. *10*

A lamina moves over the plane of the circles so that two fixed points P and Q of it move respectively on *(A)* and (B), and $PQ = AB$. Obtain the base and rolling curves.

Solutions: We shall study the case that *(A)* and *(B)* intersect. It is clear that C, the instantaneous center of rotation, is the point of **intersection** of AP and BQ. Let the radius of *(A)* and *(B)* be a. Then one observes that

$$CA + CB = a \quad \text{and} \quad CQ + CP = a. \tag{12}$$

This implies that the base curve is on an ellipse with foci A and B, and the rolling curve is an **ellipse** of the same size with foci P and Q.

In our solution we have chosen two intersecting circles. The reader may look into the case where the circles do not intersect.

MESSAGE FROM THE SECRETARY-TREASURER

Copies of the new, revised Constitution and Bylaws are now available. The prices are: *$1.50* for each of the first four copies and $1 for each copy thereafter. **I.e., $(1.50** ***n)*** for n C 4 and $\$(n+2)$ for $n \geq 4$.

The videotape of Professor Joseph A. **Gallian's** AMS-MAA-PME Invited Address, "The **Mathematics** of Identification Numbers," given as part of **PME's** 75th Anniversary Celebration at Boulder, **CO,** in August, *1989,* is also still available. The tape may be borrowed free of charge by PME chapters, and by others upon an advance payment of *$10.* Please contact my office if you desire to borrow the tape, telling me the date on which you would like to use it. I prefer to mail the tape directly to faculty advisors, and expect them to take tesponsibility for returning it to my office. Please submit your request in writing and include a phone number **and** a time that I might reach you if there are problems. **Robert** M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC *27858.*

CHANGES OF ADDRESS

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MAXIMAL ELEMENTS AND UPPER BOUNDS IN POSETS

Jayanthi Ganapathy
University of Wisconsin – *Oshkosh*

This article is a collection of **questions** about **maximal** elements and upper bounds in a partially ordered **set (POSET),** and answers to **those** questions, accompanied by proofs, and, in some cases, examples.

The inspiration for this article is a **discussion** that ensued, after definitions and establiihed results, such **aa** Zorn's Lemma, were introduced in a course taught by the author. An investigation of answers to questions raised by the atudenta resulted in the contents of this article.

For the benefit of **those** readers who are unfamiliar or out of touch with this topic, some definitions and a statement of Zorn's Lemma follow:

Definitions: (See Abbot, **1969**; Yosida, 1978; Kirilov and Guishiani, 1982; or **Morash,** 1987.)

I. A set P is said to be partially ordered by a binary relation $\prec$ if, for each **a,** b, and *c* in ***P***, we have

$$\text{(i) } a \prec a \quad \text{(reflexivity)}$$
$$\text{(ii) } a \prec b \text{ and } b \prec c \implies a \prec c \quad \text{(transitivity)}$$
$$\text{(iii) } a \prec b \text{ and } b \prec a \implies a = b \quad \text{(anti-symmetry)}$$

II. A set P with a partial ordering $\prec$ is a **partially** ordered set **(POSET)**.

III. A partial ordering $\prec$ on a set P is a hear ordering if, for each pair of elements **a** and b in ***P***, either $a \prec b$ or $b \prec a$.

IV. A set P with a linear ordering $\prec$ is a **linearly** ordered set **(LOSET)**.

V. Let S be a subset of a **POSET** $(P, \prec)$. An element u in *P* is an upper bound for S if $s \prec u$ for all $s \in S$.

VI. An element *m* in a poset $(P, \prec)$ is a **maximal** element of P if, for any $a \in P$, $m \prec a \implies a = m$.

VII. If $(P, \prec)$ is a **POSET** and **a,** b are in ***P***, then the least upper bound of **a** and ***b***, denoted by $a \vee b$, is defined as follows: $a \prec a \vee b$, $b \prec a \vee b$ and, for any element $s \in P$ such that $a \prec s$ and $b \prec s$, $a \vee b \prec s$.

VIII. A POSET $(P, \prec)$ is a lattice if, for each pair of elements **a** and b in ***P***, the least upper bound $a \vee b$ exists in P.

Zorn's **Lemma.** (See **Hrbacek** and **Jech,** 1984; Morash, 1987; or Pinter, 1971.) If $(P, \prec)$ is a **POSET** in which every linearly ordered subset has an upper bound, then P has a maximal element.

Two questions that arise naturally from **Zorn's** Lemma are the following:

Question **I.** Can a **POSET** $(P, \prec)$ have a maximal element even if the hypothesis of Zorn's Lemma is not satisfied?

The following example provides an answer:

Example **I.** Let $P = \{a + bi : a^2 + b^2 < 1, a > 0, b > 0\} \cup \{1\} \cup \{i\}$, where $i^2 = -1$. A partial ordering $\prec$ is defined on P **as** follows: $a + bi \prec c + di$ if and only if $a \le c$ **and** $b \le d$. (See Figure 1.)

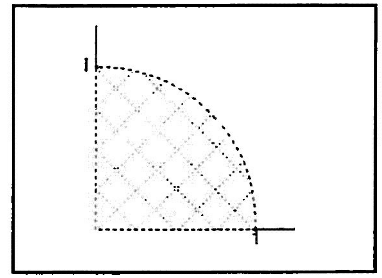

Figure 1

A subset S of P is defined as **follows:** $S = \{a + bi \in P : a = \frac{1}{2}\}$. (See Figure 2.)

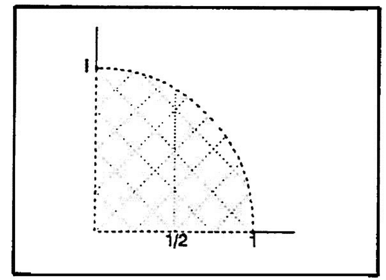

Figure 2

This set $(P, \prec)$ fails to satisfy the hypothesis of Zorn's Lemma since the subset S **has** no upper bound in *P*, although S is linearly ordered. However, P **has** maximal elements 1 and *i*.

Question **II.** Can a **POSET** $(P, \prec)$ have more than one maximal element whether or not the hypothesis of Zorn's Lemma is satisfied?

Example I provides part of the answer. Example **II,** below, completes the answer to Question **II.**

Example II. Let $P = \{1, 2, 3, 4\}$, where a partial ordering $\prec$ is defined as follows: for **a,** b in ***P***, $a \prec b$ if and only if **a** divides **b**. Note that the hypothesis of Zorn's **Lemma** is satisfied and that there are two maximal elements, namely 3 and 4, in P.

The next question concerns the uniqueness of a maximal element. The examples above illustrate the fact that a maximal element, when it exists, is not always unique.

Question **III.** What are some conditions on a **POSET** P, containing a maximal element m, that will guarantee the uniqueness of ***m?***

If $\prec$ is a linear ordering on a set P that contains a maximal element ***m***, then ***m*** must be unique. For, if m' is also a maximal element, then either $m \prec m'$ or $m' \prec m$. Since ***m*** is maximal, $m \prec m' \implies m = m'$ and since m' is maximal, $m' \prec m \longrightarrow m' = m$. Thus, in either case, $m' = m$, and the **maximal** element is unique.

If $(P, \prec)$ is a partially ordered lattice containing a maximal element ***m***, then again ***m*** must be unique. For, if m' is also a maximal element, then m 4 $m \vee m'$ and $m' \prec m \vee m'$. Since ***m*** and

m' are maximal elements, the preceding observations imply that $m = m \vee m' = m'$, thus proving that m is unique.

The proof of **Zorn's Lemma,** which uses the Axiom of Choice, involves constructing a particular linearly ordered subset of the given **POSET** and then establishing that an upper bound of that subset is a maximal element of the given **POSET.** (See Hrbacek and Jech, 1984, or Pinter, 1971.)

In **Example** I of this article, neither of the two maximal elements of the set is an upper bound for any subset that contains elements other that 1 and i.

These observations arouse one's curiosity about the possibility of a maximal element, when it exists, being an upper bound for some or each subset of the original set. This leads to the final question of this article:

Question IV. If a **POSET** $(P, \prec)$ has a maximal element m, what are some situations in which m is an upper bound for some or all subsets of P?

It follows easily from the proof which immediately follows Question III that the unique maximal element of a **LOSET,** when it exist, is an upper bound for each subset of the **LOSET.** In fact, m is the greatest element of the **LOSET** in the sense that $a \prec m$ for all a in the **LOSET.**

If $(P, \prec)$ is a partially ordered lattice containing a unique maximal element m, then m will be an upper bound for each subset s of P. For, if S is any subset of P and $a \in S$, then $m \prec a \vee m$, which, since m is maximal, implies that $m = a \vee m$. But we also have $a \prec a \vee m$. Thus $a \prec m$.

If $(P, \prec)$ is a **POSET** containing a maximal element m, then m is an upper bound for any linearly ordered subset S of P that contains m. Thii can be proved as follows: If S is such a subset of P, then, for any $a \in S$, $a \prec m$ or $m \prec a$. But $m \prec a$ would imply that $m = a$, since m is maximal. Thus, in either case, $a \prec m$.

Summarizing our results, we conclude that in a partially ordered lattice or **LOSET,** a maximal element, if it exists, is unique and is the upper bound for each subset of the original set. In an arbitrary **POSET** containing a maximal element, uniqueness of the maximal element is not guaranteed. If a **POSET** contains a maximal element m, then for any linearly ordered subset that contains m, m is an upper bound; in fact, m will be the only maximal element that subset will contain. As illustrated by the examples in this article, a **POSET,** in general, can have more than one maximal element and, in general, a maximal element does not have to be an upper bound for any subset that contains elements other than the maximal element itself.

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3. A. A. Kirilov and A. D. Guishiani, *Theorems and Problems in Functional Analysis,* Springer-Verlag, 1982.
4. R. P. Morash, *Bridge to Abstract Mathematics: Mathematical Pmof and Structures,* Random House, 1987.
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6. K. Yosida, *Functional Analysis, 5th Edition,* Springer- Verlag, 1978.

LETTER TO THE EDITOR

Dear Editor:

I'd like to make a comment on the Spring '92 issue of the *Pi Mu Epsilon Journal.* First, (concerning) the article by Russell Euler, "A Closed form for a Family of Summations." It is obvious that

$$\frac{1}{n(n-1)\cdots(n-p)} = \frac{1}{p}\left[\frac{1}{n-p} - \frac{1}{n}\right]\left[\frac{1}{(n-1)\cdots(n-p+1)}\right],$$

from which it follows that

$$\binom{n}{p}^{-1} = \frac{p}{p-1}\left[\binom{n-1}{p-1}^{-1} - \binom{n}{p-1}^{-1}\right]$$

Thus the paper only gives a special case of

$$\sum_{n=r}^{\infty}\binom{n}{p}^{-1} = \frac{p}{p-1}\binom{r-1}{p-1}^{-1}$$

when $r = p$.

Another comment is that Andrew Cusumano's article is really a solution to a Putnam Problem (1966) also quoted by Bender and Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* problem 5.59.

I would appreciate your passing these comments to your readers.

David Ivy, Baltimore, MD

ATTENTION FACULTY ADVISORS

To have your chapter's report published, send copies to Robert M. Woodside, Secretary-Treasurer, Department of Mathematics, East Carolina University, Greenville, NC 27858 and to Richard L. Poss, Editor, St. Norbert College, De Pere, WI 54115.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department **welcomes** problems believed to be new and at a level appropriate for the readers of this **journal.** Old problems displaying novel and elegant methods of solution are **also** invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. **An** asterisk (*) preceding a problem number indicates that the proposer did not submit a **solution.**

All communicationsshould be addressed to C. W. Dodge, Math. Dept., University of Maine, Orono, ME **04469.** Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side **only)** properly identified with name and address. Solutions to problems in this issue should be mailed by July **1, 1993.**

Problems for Solution

777. [Spring **1992]** ***Corrected.*** *Proposed by* ***Seung-Jin*** *Bong* ***Seoul, Korea.***
It is **well known** that, for $n \geq 2$, $\ln(n + 1)$ c $S_n < 1 + \ln n$, where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

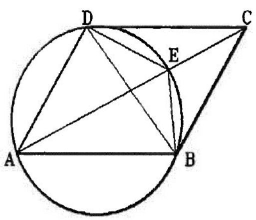
It is also **known** (***Crux*** *Mothemoticomm* **11 (1985) p. 109)** that, for $n \geq 2$,

$$n(n + 1)^{1/n} - n < S_n \leq n - (n - 1)n^{-1/(n-1)}.$$

Prove that

$$\ln(n + 1) < n(n + 1)^{1/n} - n \quad\text{and}\quad n - (n - 1)n^{-1/(n-1)} < 1 + \ln n$$

for all $n \geq 2$.



Problem *780*

780. [Spring 1992] *Corrected Proposed by* ***R S. Luthar, University*** *of* ***Wisconsin Center, Janesville, Wisconsin.***
Let $ABCD$ be a parallelogram with $\angle A = 60°$. **Let** the **circle** through A, B, and D intersect AC at E. See the figure. Prove that $BD^2 + AB\cdot AD = AE\cdot AC$.

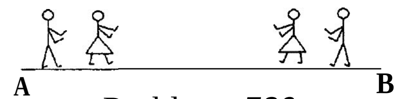
784. *Proposed by* ***Alan Wayne, Holiday, Florida.***

Restore the enciphered digits in the decimal computation:

$$(TWO)(TWO + TWO) = EIGHT.$$

785. *Proposed by* ***Charles Ashbacher, Cedar Rapids, Iowa, and*** *dedicated to the* ***memory*** *of Joseph* ***Konhauser.*** *Student* ***solutions are especially*** *solicited.*
A tiling of the plane by non-overlappiig, non-congruent rectangles $P_1, P_2, \ldots$ is **defined** in,&=-following way: P_1 **is** an arbitraryx by y rectangle; $P_2, P_3, \ldots$ are all squares such that the side of each square P_{k+2} **is** equal to the sum of **thc** sides of the two prenous squares P_k and P_{k+1} for all $k > 1$. Show this tiling.

786. ***Proposed*** *by* ***Dmitry*** *P. Mavlo, Moscow, Russia*
From **two towns** A and B, **48** km apart, **two** groups of hikers march toward each other starting at the same time. The group leaving A marches at **4 km/hr** by marches of not more than 6 hr at one time. The group from B hikes at *6* **km/hr** for not more than *2* **hr** *at* a time. After **marching *t* hr,** the **first** group must rest for at least t **hr.** The second group has to rest not less than $2t$ hr **after** t **hr** of **hiking.** Find the least time until the two groups meet and describe the hiking patterns necessary for that solution.



Problem *786*

787. *Proposed by R S.* ***Luthar,*** *University of Wsconsin Center,* ***Janesville, Wisconsin.***
If a, b, c, d are the **roots** of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

then evaluate the expression

$$(a + b + c - 2d)(b + c + d - 2a)(c + d + a - 2b)(d + a + b - 2c)$$

in terms of p, q, r, and s.

788. *Proposed by the late Jock* ***Garfunkel,*** *Flushing New* ***York.***
Given positive numbers x, y, z such that $x + y + z = 1$, prove that

$$xy + yz + zx \geq x^2y^2 + y^2z^2 + z^2x^2 + 8xyz.$$

789. *Proposed by* ***David Iny, Baltimore, Maryland.***
Evaluate the integral

$$\int_0^1 \left(\frac{y - 1}{\sqrt{y}\,\ln y}\right) dy.$$

790. ***Proposed*** *by* ***Florentin Smarandache, Phoenix, Arizona.***
In base *6* how many **digits** does the nth prime **contain?**

791. ***Proposed*** *by* ***Seung-Jin*** *Bong* ***Seoul,*** *Republic of Korea*
Prove that $2^n + 1$, where n is a nonnegative integer, is never a multiple of **143.**

792. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea*

Given any thirteen distinct real numbers, prove that there exists at least one subset $\{x, y, z\}$ of three of them such that

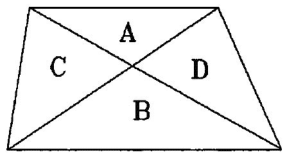
$$0 < \frac{(x-y)(y-z)(x-z)}{(1+xy)(1+yz)(1+xz)} < \frac{1}{3\sqrt{3}}.$$

793. *Proposed by Dieter Bennewitz, Koblenz, Germany.*

Given any trapezoid, its diagonals divide its interior area into four triangular areas: A and B adjacent to the parallel bases, and C and D adjacent to the nonparallel sides, as shown in the figure.

a) Prove that the areas C and D are equal and that $A \cdot B = C \cdot D$.

b) Find area C in terms of the lengths of the altitude and the bases of the trapezoid.



Problem 793

794. *Proposed by Peter A. Lindstrom, North Lake College, Irving, Texas.*

For $-3 \le x \le 6$, show that 2π is equal to the sum of the zeros of

$$f(x) = \sin(x + \cos x).$$

795. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all solutions on the interval $[0, 2\pi]$ to

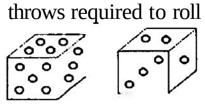
$$2\cos^3 x - 2\cos x + 1 = 0.$$

796. *Proposed by Michael W. Ecker, Clarks Summit, Pennsylvania.*

a) A die is thrown until a prescribed face (e.g. say 3) shows. What is the mathematically expected number of throws required for this to occur?

b) Same question, but suppose a throw now consists of rolling 2 dice. In particular, should we expect this expectation to be half that of part (a)?

c) What is the smallest whole number of dice needed to constitute one throw, if we wish to have the mathematically expected number of throws required to roll our prescribed number not exceed 2?



Solutions

754. [Spring 1991, Spring 1992] *Proposed by Seung-Jin Bang, Seoul. Korea.*

Let $a_1 = a_2 = 1$, $a_3 = 2$, and $a_{n+1} = a_n \cdot a_{n-1} + a_{n-2}$ for $n > 3$. Show that

$$a_{n+2}a_na_{n-2} - a_{n+2}a_{n-1}^2 - a_{n+1}^2a_{n-2} + 2a_{n+1}a_na_{n-1} - a_n^3 + 3 = 0.$$

Additional Editor's comment. In the Editor's comment I stated that only Klamkin spotted my error, that the problem should read $n > 3$. The word "only" should have been omitted. This error was also spotted by ROBERT C. GEBHARDT, *Hopatcong, NJ,* HENRY S. LIEBERMAN, *Waban, MA,* WILLIAM H. PEIRCE, *Rangeley, ME, and* MOHAMMAD P. SHAIKH, *Western Michigan University, Kalamazoo. I shall do my penance yet another $n > 3$ times.*

758. [Fall, 1991] *Proposed by Charles Ashbacher, Hiawatha, Iowa.*

Solve this base ten alphametric which celebrates Leonhard Euler's contributions to graph theory:

$$E + V*GRAPH = EULER.$$

Solution by Alma College Problem Solving Group, Alma College, Alma, Michigan.

It *will* be helpful to rewrite the problem in the form

$$\begin{array}{r} GRAPH \\ \times \quad V \\ \hline EULxx \\ + \quad E \\ \hline EULER. \end{array}$$

The use of L in line 3 of the display is justified by noting that it could be $L - 1$ only if the E in the tens place in line 5 were 0, but E cannot be zero since it begins a word. It is readily seen that V cannot be 0, 1, or 9, that $G \ne 0$, that E cannot be 0, 1, or 2, and that $H \ne 0$. From the first and last columns we see that

$$G \times V\ (+\ carry) = E \quad \text{and} \quad H \times V + E \equiv R \pmod{10}.$$

Now G, R, H, E, and V are distinct only when $V = 2$, 3, 4, or 7 (found after testing all possible combinations of G, R, and E). The one case where V = 7, the three cases where V = 4, and the six cases where $V = 3$ are not solvable, leaving only $V = 2$. Then G can be only 1, 3, or 4.

If $G = 1$, then $E = 3$, and $R = 5$, 7, or 9. Then $H = 6$, 7, or 8. No allowable combination yields a solution.

If G = 4, then $E = 8$ or 9. If $E = 8$, then $R = 0$ and $H = 6$, and no solution results. If $E = 9$, then $P = 4$ or 9, both of which are taken. So $G \ne 4$.

We have $G = 3$, and $E = 6$ or 7. If $E = 7$, then $P = 8$ and $R = 5$ or 9. If $R = 9$, then $U = 8$ or 9, so R $\ne$ 9. If $R = 5$, then $H = 4$ and $U = 0$ or 1. No combination of the remaining numbers will satisfy A and L.

Thus $G = 3$ and $E = 6$. Now the unique solution $6 + 2 \times 34079 = 68164$ *can* be found.

Also solved by SCOTT H. BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* REX H. WU, *Brooklyn, NY, and the* PROPOSER.

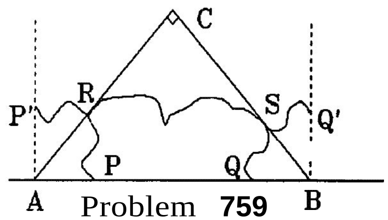
759. [Fall, 1991] *Proposed by John E. Wetzel, University of Illinois, Urbana, Illinois.*

Call a plane arc *special* if it has length 1 and lies on one side of a line through its end points. Show that any special arc can be contained in an isosceles right triangle of hypotenuse 1.

I. *Solution by the Proposer.*

Given the plane arc PQ, lying all on one side in line PQ, circumscribe an isosceles right triangle ABC, with right angle at C and hypotenuse AB lying on line PQ, about the arc, as shown in the figure for this problem. Reflect the sub-arcs PR and SQ in the legs AC and BC respectively, obtaining the sub-arcs $P'R$ and $S'Q$. It is now dear that the length of the given arc $PRSQ$, which is 1 unit, is equal to the length of the arc $P'RSQ'$, which in turn is greater than or equal to the length of the hypotenuse AB.

Moral: Pause and reflect.



Problem **759**

II. *Comment by **Murray** S. **Klamkin,** University of Alberta, Edmonton, **Alberta,** Canada.*

This is a special case of the "worm" problem. For related results and references, see H. T. Croft, K. J. Falconer, and R. **K.** Guy, *Unsolved Problems in **Geometry,*** New York: Springer-Verlag, 1991, pp. 129-130.

For a dosed curve of length 1 which is the **boundary** of a convex set Γ, we have that there is a circumscribing triangle with angles **a,** 0, and γ whose perimeter P satisfies the inequality

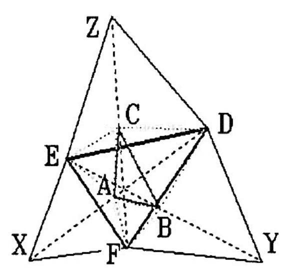
$$P \le \frac{1}{2\pi}\,\frac{(\sin\alpha + \sin\beta + \sin\gamma)^2}{\sin a \,\sin\beta\, \sin y}$$

Equality occurs if Γ is a circle. See H. G. **Eggleston,** *Problems in Euclidean Space,* New York: Pergamon, 1957, p. 157.

Also solved by **MARK** EVANS, *Louisville, **KY,** for the case where the arc is an arc of a circle, and by* **REX** H. **WU,** *Brooklyn, **NY.***

760. **[Fall, 1991]** *Proposed by John E. Wetzel, University of Illinois, **Urbana,** Illinois.*

Napoleon's theorem is concerned with erecting equilateral triangles outwardly on the sides of a given **triangle *ABC.*** Then *DEF* is the triangle formed by the third vertices of these equilateral triangles BCD, CAE, and ABF. Lemoine asked in 1868 if one **can** reconstruct triangle ABC when only triangle *DEF* is given. Shortly afterward, Keipert showed that the construction is to erect outward equilateral triangles ***EFX,*** *FDY*, and *DEZ* on triangle *DEF*, and then *A*, *B*, and *C* are the midpoints of the segments *DX*, ***EY,*** and ***FZ.*** His proof **was** quite tedious. Find a simple proof of **Keipert's** construction.



Problem **760**

I. *Solution by the Proposer.*

The accompanying figure shows a given triangle *ABC*, the third vertices *D*, *E*, and *F* of equilateral triangles erected on its sides, and the third vertices *X*, ***Y,*** and *Z* of equilateral triangles erected on the sides of triangle *DEF.* Consider the mapping α that is the product of the three 60° **counterclockwise** rotations about points *D*, *F*, and *E* in that order. Then α is a **halfturn. Since** the three rotations applied in turn map point C to *B*, then to *A*, and back to C again, we have $\alpha(C) = C$ *and* it follows that α is a **halfturn** about C. We apply the product of the three halfturns to point Z, **noting that** *Z* maps to *E*, then to *X*, and finally to *F*. Hence $\alpha(Z) = F$, so *C* is the midpoint of ***FZ.*** Similarly *A* and *B* are the midpoints of *DX* and N.

II. *Solution by Paul S. **Bruckman, Edmonds,** Washington.*

Let *a*, ***b,*** *c*, etc. denote the complex representations for the vertices *A*, *B*, ***C,*** etc., and let μ = cis $(\pi/3) = 1/2 + i\sqrt{3}/2$. Then $\mu^2 = -1/2 + i\sqrt{3}/2$, so $\mu - \mu^2 = 1$. Now we have

$$d - c = \mu(b - c), \text{ so that } d = \mu b + (1 - \mu)c.$$

Similarly,

$$e = \mu c + (1 - \mu)a,\ f = \mu a + (1 - \mu)b, \text{ and } x = \mu e + (1 - \mu)f.$$

Now the affix of the midpoint of *DX* is

$$(½)(d + x) = (½)\{\mu b + (1 - \mu)c + \mu[\mu c + (1 - \mu)a] + (1 - \mu)[\mu a + (1 - \mu)b]\}$$

$$= (½)\{a(2\mu - 2\mu^2) + b(1 - \mu + \mu^2) + c(1 - \mu + \mu^2)\} = a.$$

Hence A is the midpoint of *DX*. **Similarly,** *B* and *C* are the midpoints of *EY* and ***FZ.***

Also solved by RICHARD I. **HESS,** *Rancho **Palos** Verdes, **CA,*** MURRAY S. KLAMKIN, *University **of Alberta,** Canada, and* **REX** *H.* WU, *Brooklyn, NY.*

*Editorial comment. Wetzel noted that **Kiepert's argument** is not very pretty. It used Ptolemy's theorem on the cyclic quadrilaterals BCDP, etc., where P is the point of intersection of the three lines AD, BE, and CF. **Wu** pointed cut the delightful article by Wetzel "Converses of Napoleon's Theorem" in The American Mathematical Monthly, April 1992, pp. 339-351. His proof appears on p. 342.*

***762.** [Fall, **1991]** *Proposed* by ***Hao-Nhien** Qui **Vu,** Purdue University, Lafayette, Indiana.*

Following Cantor, we assume a list of the rationals in **[0,1) can** be made. Each rational is listed as a terminating decimal if possible, or as a repeating decimal. Thus numerals ending in **nonterminating** repeating 9's are not permitted. Define a new number x such that the k**th** place of x is 5 if the kth place in the kth number in the list is not 5, and is 4 otherwise. So, for example, if the list starts with **0.5, 0.32,** 0.666666, then x = **0.455....** Show that the number x must be irrational and therefore **this** process does not prove the rationals are not denumerable. Saying that x is irrational because the rationals are countable, however amusing, is not sufficient.

*Solution by **Charles Ashbacher,** Cedar Rapids, Iowa.*

The proof assumes that the phrase "a list of the rationals in **[0,1) can** be made" means that a complete list **can** be made. This assumption is not unreasonable, given that we are following Cantor. Thus assume that x is rational. Since $0 < x < 1$, there must be a number y in the list such that $x = y$. The method of construction of x, however, guarantees that x and y must differ in at least one decimal place. **Since** x is an infinite decimal whose digits are **all** 4s and **5s,** $x \neq y$. This contradiction shows that x cannot be rational.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* PAUL S. BRUCKMAN, *Edmonds, WA, and* VICTOR G. FESER, *University of Mary, Bismarck, ND.*

Editorial comment. This pmblem was proposed by Vu when he was a sophomore at Purdue. After holding it for several years, I decided to run it in hopes that it might elicit some interesting comments. I received "The problem seems to be incorrect" and "It seems to be complete nonsense." Oh well, it was a nice try.

763. [Fall, 1991] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Find all real solutions to the equation

$$(x^2 - 7x + 11)^{x^2-11x+30} = 1.$$

I. Solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

If $a^b = 1$, then $|a|^b = 1$, so let us solve

$$|x^2 - 7x + 11|^{x^2-11x+30} = 1,$$

and consequently,

$$(x^2 - 11x + 30)\ln|x^2 - 7x + 11| = 0.$$

This implies that $x^2 - 11x + 30 = 0$ or $x^2 - 7x + 11 = \pm 1$. Hence $x = 5, 6, 2, 3,$ or 4. Each of these values checks in the given equation.

II. Solution by Barbara Lehman, Brigantine, New Jersey.

Any values that make the exponent zero without making the base zero will satisfy the equation. Thus solve $x^2 - 11x + 30 = 0$ and get $x = 5$ or 6. Any values that make the base 1 will work, so solve $x^2 - 7x + 11 = 1$ and get $x = 2$ or 5. Finally, any values that make the base -1 and the exponent an even integer will also suffice, so solve $x^2 - 7x + 11 = -1$ and obtain $x = 3$ or 4, both of which produce even exponents. So the solutions are $x = 2, 3, 4, 5, 6$.

This is an excellent problem to solve on a graphing calculator. I plan to present it to my pre-calculus class.

III. *Comment by the Proposer.*

If one starts with

$$\log(x^2 - 7x + 11)^{x^2-11x+30} = \log 1 = 0,$$

one is easily led to cases where $x^2 - 11x + 30 = 0$ or $x^2 - 7x + 11 = 1$, but not readily to $x^2 - 7x + 11 = -1$.

IV. Comment by Elizabeth Andy, Limerick, Maine.

Solve *a* to the *b* equals 1,
A problem that's easy and fun!
But base -1
Must also be done,
Or you cannot say you have won!

So when problems seem easy to you,
Just be careful to think them all through.
When writing that letter,
Think, "Can I do better?"
Then please write everything you need to write in that last line
to make sure you have done all that you need to do!

Gold stars to each of the following persons for solving the problem correctly the first time: ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* CHARLES ASHBACHER, *Cedar Rapids, IA,* SEUNG-JIN BANG, *Seoul, Korea,* BARRY BRUNSON, *Western Kentucky University, Bowling Given;* JAMES E. CAMPBELL, *University of Missouri-Columbia,* ROBERT C. GEBHARDT, *Hopatcong, NJ,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* HENRY S. LIEBERMAN, *Waban,* MA, DAVID E. MANES, *SUNY at Oneonta,* BOB PRIELIPP, *University of Wisconsin-Oshkosh,* MICHAEL A. VITALE, *St. Bonaventure University, NY, and the* PROPOSER.

Silver stars to these contributors for also obtaining all the solutions: FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* PAUL S. BRUCKMAN, *Edmonds, WA,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* JAYANTHI GANAPATHY, *University of Wisconsin-Oshkosh,* PETER A. LINDSTROM, *North Lake College, Irving, TX,* THOMAS MITCHELL, *Southern Illinois University at Carbondale, and* YOSHINOBU MURAYOSHI, *Eugene, OR, and* KENNETH M. WILKE, *Topeka, KS.*

Bronze stars to fourteen partial solvers who shall remain nameless.

Editorial comment. Most partial solvers missed the third case mentioned by the proposer, many by using logarithms, but some overlooked that case without the aid of logarithms. A couple of people arrived at irrational roots by setting the base equal to 1 and setting the exponent equal to 1, and obtaining four roots for the two polynomial equations. The fallacy here is that those two simultaneous equations do not have a common root as is required by an "and" statement, so no solution results. The pmblem was easy - so easy that oversights were inevitable.

764. [Fall, 1991] *Proposed by William K. Delaney, S.J., Loyola Marymount University, Los Angeles, California.*

Evaluate the indefinite integral

$$\int (x + 1)e^x \ln x \, dx.$$

I. Solution by the Proposer.

This is an unusual instance of double integration by parts. First, use parts with $u = x + 1$ and $dv = e^x\,dx$ to find that $\int (x + 1)\,e^x\,dx = xe^x + C$. Again integrate by parts, using $u = \ln x$ and $dv = (x + 1)\,e^x\,dx$ this time, to get

$$\int (x + 1)e^x \ln x\,dx = xe^x \ln x - \int xe^x\left(\frac{1}{x}\right)dx$$
$$= xe^x \ln x - e^x + C.$$

This approach works equally well for any integral of the form $\int P(x)\,e^{ax} \ln x\,dx$, provided x is a factor of $\int P(x)\,e^{ax}\,dx$.

II. Solution by Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

By letting $u = e^x \ln x$ and $dv = dx$ and integrating by parts, we get

$$\int e^x \ln x\,dx = xe^x \ln x - \int x\left(\frac{e^x}{x} + e^x \ln x\right)dx$$
$$= xe^x \tan x - e^x - \int xe^x \ln x\,dx.$$

Thus

$$\int (x+1)e^x \ln x\, dx = e^x(x \ln x - 1) + C.$$

III. *Generalizations by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

The following two integration formulas follow by differentiation or by successive integration by parts:

(1) $$\int (\ln x)\{DxDy\}\, dx = (\ln x)\{xDy\} - y + C,$$

(2) $$\int (\ln^2 x)\{DxDxDy\}\, dx = (\ln^2 x)\{(xD)^2 y\} - 2(\ln x)\{xDy\} + 2y + C.$$

We now let $y = e^{ax}F(x)$ to give

(1') $$\int (e^{ax} \ln x)\{F' + aF + x(F'' + 2aF' + a^2F)\}\, dx$$

$$= xe^{ax}(\ln x)(F' + aF) - e^{ax}F + C,$$

(2') $$\int (e^{ax} \ln^2 x)\{F' + aF + 3x(F'' + 2aF' + a^2F)$$

$$+ x^2(F''' + 3aF'' + 3a^2F' + a^3F)\}\, dx$$

$$= e^{ax}(\ln^2 x)\{x(F' + aF) + x^2(F'' + 2aF' + a^2F)\}$$

$$- 2xe^{ax}(\ln x)(F' + aF) + 2e^{ax}F + C.$$

As special cases, we let $F = 1$ and $a = 1$ in (1') and (2') to obtain

$$\int (x+1)e^x \ln x\, dx = (x \ln x - 1)e^x + C,$$

$$\int (x^2 + 3x + 1)e^x \ln^2 x\, dx = \{(x + x^2)\ln^2 x - 2x \ln x + 2\}e^x + C.$$

As another special case, we let $a = 0$ and $F = exp(x^2)$ in (1') to give

$$\int 4(x^3 + x)\, exp(x^2) \ln x\, dx = (2x^2 \ln x - 1)\, exp(x^2) + C.$$

Now

(3) $$\int z^n(D^{n+1}y)\, dz$$

$$= z^nD^ny - nz^{n-1}D^{n-1}y + n(n-1)z^{n-2}D^{n-2}y + \ldots + (-1)^n n!y + C,$$

which can be verified by differentiation or obtained by repeated integration by parts. Take $z = \ln x$ and $n = 1$ or 2 in (3) to get (1) and (2). Remember that, since $z = \ln x$, then $dz = (1/x)\, dx$, so each D should be replaced by xD.

If we let $z = \ln x$ and $y = e^{ax}F$, then (3) becomes

(4) $$\int (\ln^n x)\{D(xD)^n e^{ax}F\}\, dx$$

$$= (\ln^n x)(xD)^n e^{ax}F - n(\ln^{n-1} x)(xD)^{n-1}e^{ax}F + \ldots.$$

Note that $(xD)^n$ can be expanded in terms of x^kD^k [1], i.e.,

$$(xD)^n = \sum_{k=1}^{n} S_n^k x^k D^k,$$

where the S_n^k are Stirling numbers of the second kind. Also, one can evaluate $D^k(e^{ax}F)$ by means of the exponential shift theorem, i.e.,

$$D^k(e^{ax}F) = e^{ax}(D + a)^kF.$$

Reference

1. C. Jordan, *Calculus of Finite Differences*, Chelsea, N.Y., 1947, pp. 170, 196.

IV. *Comment by David E. Penney, University of Georgia, Athens, Georgia*

The integral $\int (x + a)e^x \ln x\, dx$ appears to be nonelementary if $a \neq 1$.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI*, CHARLES ASHBACHER, *Cedar Rapids, IA*, MOHAMMAD K. AZARIAN, *University of Evansville, IN*, SEUNG-JIN BANG, *Seoul, Korea*, PRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay*, SCOTT H. BROWN, *Stuart Middle School, FL*, PAUL S. BRUCKMAN, *Edmonds, WA*, BARRY BRUNSON, *Western Kentucky University, Bowling Green*, JAMES E. CAMPBELL, *University of Missouri-Columbia*, RUSSELL EULER, *Northwest Missouri State University, Maryville*, GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ*, MARK EVANS, *Louisville, KY*, JAYANTHI GANAPATHY, *University of Wisconsin-Oshkosh*, ROBERT C. GEBHARDT, *Hopatcong, NJ*, STEPHEN I. GENDLER, *Clarion University of Pennsylvania*, RICHARD I. HESS, *Rancho Palos Verdes, CA*, HENRY S. LIEBERMAN, *Waban*, MA, PETER A. LINDSTROM *(2 solutions) North Lake College, Irving, TX*, CARRIE LONGSHAW, *Southeast Missouri State University, Cape Girardeau*, DAVID E. MANES, *SUNY at Oneonta*, LAURA ANN MCSWEENEY, *Brockton*, MA, YOSHINOBU MURAYOSHI, *Eugene, OR*, WILLIAM MYERS, *Belmont Abbey College, NC*, DAVID E. PENNEY, The University of Georgia, Athens, MIKE PINTER, *Belmont College, Nashville, TN*, BOB PRIELIPP, *University of Wisconsin-Oshkosh*, JAY SLOTNICK, *Alma College, MI*, MICHAEL A. VITALE, *St. Bonaventure University, NY*, STAN WAGON, *Macalester College, St. Paul, MN, and* REX H. WU, *Brooklyn, NY. One incorrect solution was received.*

765. [Fall, 1991] *Proposed by the late Charles W. Trigg, San Diego, California.*

Find a square integer in base 4 that is a concatenation of two like integers.

Solution by Kenneth M. Wilke, Topeka, Kansas.

The desired integer n must satisfy the equation

(1) $$n^2 = 4^kA + A = (4^k + 1)A,$$

where k is the number of digits in A. Clearly then, $4^{k-1} \le A < 4^k$. The smallest positive integer of the form $4^k + 1$ that has a square factor is $4^5 + 1 = 1025 = 5^2 \cdot 41$. Thus A has the form $41t^2$ and $256 \le 41t^2 < 1024$, which is satisfied for $t = 3$ or 4. Therefore, we have the two solutions

$$615^2 = (21213_4)^2 = 1130111301_4 \text{ and } 820^2 = (30310_4)^2 = 2210022100_4.$$

If we allow leading zeros, then we may take $t = 1$ or 2, obtaining

$$205^2 = (3031_4)^2 = 0022100221_4 \text{ and } 410^2 = (22122_4)^2 = 0221002210_4.$$

Also solved by CHARLES ASHBACHER, *Cedar Rapids, IA*, SEUNG-JIN BANG, *Seoul, Korea*, SCOTT H. BROWN, *Stuart Middle School, FL*, PAUL S. BRUCKMAN, *Edmonds, WA*, JAMES

E. CAMPBELL, *University of Missouri-Columbia,* MARK EVANS, *Louisville, KY,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania,* RICHARD I. HESS, *Rancho Polos Verdes, CA,* REX H. WU, *Brooklyn, NY, and the* PROPOSER.

766. [Fall, **1991]** *Proposed* by *Murray S. Klamkin, University of Alberta. Edmonton, Alberta, Canada.*

Determine

$$\frac{d^n}{dx^n}(x^n \ln^2 x)$$

at $x = e$.

Solution by Seung-Jin Bang, Seoul, Republic of Korea

We introduce the differential operators $D = d/dx$ and $\mathcal{D} = d/dt$. Letting $x = e^t$, then we have $Dy = e^{-t}\mathcal{D}y$ and $x^n \ln^2 x = t^2e^{nt}$. Now

$$D(x^n \ln^2 x) = e^{-t}\mathcal{D}t^2e^{nt} = e^{(n-1)t}(\mathcal{D} + n)t^2,$$

$$D^2(x^n \ln^2 x) = e^{(n-2)t}(\mathcal{D} + n - 1)(\mathcal{D} + n)t^2, \ldots,$$

$$D^n(x^n \ln^2 x) = (\mathcal{D} + 1)(\mathcal{D} + 2)\cdots(\mathcal{D} + n - 1)(\mathcal{D} + n)t^2$$

It follows that

$$D^n(x^n \ln^2 x)|_{x=e} = \left[n!t^2 + 2n!\left(\sum_{i=1}^{n}\frac{1}{i}\right)t + 2n!\left(\sum_{i>j}\frac{1}{ij}\right)\right]\Big|_{t=1}$$

$$= n!\left[1 + 2\sum_{i=1}^{n}\frac{1}{i} + \left(\sum_{i=1}^{n}\frac{1}{i}\right)^2 - \sum_{i=1}^{n}\frac{1}{i^2}\right] = n!\left[\left(1 + \sum_{i=1}^{n}\frac{1}{i}\right)^2 - \sum_{i=1}^{n}\frac{1}{i^2}\right].$$

Also solved by CHARLES ASHBACHER, *Cedar Rapids, IA, FRANK* P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* SCOTT H. BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* RICHARD I. HESS, *Rancho Polos Verdes, CA,* HENRY S. LIEBERMAN, *Waban, MA,* WILLIAM *H.* PEIRCE, *Rangeley, ME,* REX H. WU, *Brooklyn, NY,* PAUL YIU, *Florida Atlantic University, Boca Raton, and the* PROPOSER.

767. [Fall, **1991]** *Proposed by J. L. Brenner, Palo Alto, California.*

Let a_0 and a_1 be positive integers, and for $n \geq 2$, define

$$a_n = \frac{a_{n-1}^2}{a_{n-2}}.$$

For what choices of a_0 and a_1 will all the a_n be integers?

Solution by Rube R. Czech, Attica, New York.

The solution is that a_0 divides a_1. *We* have that

$$a_2 = \frac{a_1^2}{a_0} = a_0\left(\frac{a_1}{a_0}\right)^2, \quad a_3 = \frac{a_2^2}{a_1} = a_0\left(\frac{a_1}{a_0}\right)^3,$$

$$a_4 = \frac{a_3^2}{a_2} = a_0\left(\frac{a_1}{a_0}\right)^4, \ldots, a_n = a_0\left(\frac{a_1}{a_0}\right)^n, \ldots,$$

a formula readily verified by mathematical induction. To that end, the formula is clearly true for $n = 0$ and for $n = 1$. It is then easy to show that, if it is true for $n = k$ and for $n = k + 1$, then it is also true for $n = k + 2$.

If we have that a_0 divides a_1, then a_1/a_0 is an integer and it follows from our formula for a_n that a_n is an integer or a product of integers, and hence it is always an integer.

Suppose now that a_0 does not divide a_1. Then there is a prime p and natural numbers n and k such that $k < n$, p^n divides a_0, p^k divides a_1 but p^{k+1} does not divide a_1. Then we count the number of factors of p in a_{n+1}, which must be nonnegative for a_{n+1} to be an integer. By our formula for a_n we see there are $n + 1$ factors of a_1 in the numerator and n factors of a_0 in the denominator. Thus there are $(n + 1)k - n^2 \leq (n + 1)(n - 1) - n^2 = -1$ factors of p in a_{n+1}, so a_{n+1} is a fraction that, in reduced terms, has a factor of p in its denominator.

Hence it is both necessary and sufficient that a_0 divide a_1.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI,* CHARLES ASHBACHER, *Cedar Rapids, IA,* MOHAMMAD *K.* AZARIAN, *University of Evansville, IN,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* PAUL S. BRUCKMAN, *Edmonds, WA,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS, *Louisville, KY,* VICTOR G. FESER, *University of Mary, Bismarck, ND,* STEPHEN I. GENDLER, *Clarion University of Pennsylvania.* RICHARD I. HESS, *Rancho Polos Verdes, CA,* MURRAY S. KLAMKIN, *University of Alberta, Canada,* HENRY S. LIEBERMAN, *Waban,* MA, PETER A. LINDSTROM, *North Lake College. Irving, TX,* DAVID E. MANES, *SUNY at Oneonta,* YOSHINOBU MURAYOSHI *(two solutions), Eugene, OR,* WILLIAM H. PEIRCE, *Rangeley, ME,* KENNETH M. WILKE, *Topeka. KS,* REX H. WU, *Brooklyn, NY,* PAUL YIU, *Florida Atlantic University, Boca Raton, and the* PROPOSER.

768. [Fall, 1991] *Proposed by the late Jack Garfunkel, Flushing, New York.*

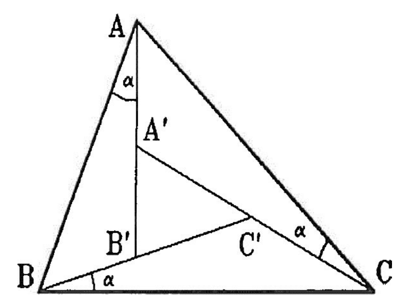
Given a triangle ABC, draw rays inwardly from each vertex to form a triangle $A'B'C'$ such that B', C', A' lie on rays AA', BB', CC', respectively, and

$$\sphericalangle BAB' = \sphericalangle ACA' = \sphericalangle CBC' = \alpha,$$

as shown in the figure. Prove that:

a) Triangle $A'B'C'$ is similar to triangle ABC.

b) The ratio of similitude is $\cos\alpha - \sin\alpha \cot\omega$, where ω is the **Brocard** angle of triangle ABC.



Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida.

a) Note that

$$\sphericalangle B'A'C' = \sphericalangle A'AC + \sphericalangle A'CA = \sphericalangle A'AC + \alpha = \sphericalangle BAC,$$

and similarly $\sphericalangle A'B'C' = \sphericalangle ABC$, so triangles ABC and $A'B'C$ are similar.

b) Apply the law of sines to triangles $AA'C$, BCC', and ABC to get

$$A'C = \frac{b \sin(A - \alpha)}{\sin A} \quad \text{and} \quad CC' = \frac{a \sin\alpha}{\sin C} = \frac{b \sin A \sin\alpha}{\sin B \sin C}$$

It follows that

$$\frac{A'C'}{AC} = \frac{\sin(A - a)}{\sin A} - \frac{\sin(C + B) \sin\alpha}{\sin B \sin C}$$
$$= \cos a - \sin\alpha[\cot A + \cot B + \cot C] = \cos a - \sin\alpha \cot\omega,$$

where ω is the Brocard angle of the triangle ABC and it is well known that

$$\cot\omega = \cot A + \cot B + \cot C.$$

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI.* CHARLES ASHBACHER, *Cedar Rapids,* LA, PAUL S. BRUCKMAN, *Edmonds, WA,* RICHARD *I.* HESS, *Rancho Palos Verdes, CA,* MURRAY S. KLAMKIN, *University of Alberta, Canada,* KANDASAMY MUTHUVEL, *University of Wisconsin-Oshkosh,* BOB PRIELIPP, *University of Wisconsin-Oshkosh, and the* PROPOSER.

***Editorial** comment.* I *apologize for not defining the **Brocard angle** in the statement of the problem. I assumed it was more well known than it apparently is. Its full definition and many properties can be found in either reference below. The **Brocard** angle* ID *is* $\angle MAB = \angle MBC = \angle MCA$ *formed by the unique point **M** from which such equal angles are subtended inside triangle ABC. Any triangle ABC has two **Brocard** points **M** and **N**, the second one subtending equal angles* $\angle NBA = \angle NCB = \angle NAC$. *Furthermore,* $\angle MAB = \angle NBA =$ ID. *The **Brocard point M** for **triangle ABC is** defined as the unique point of intersection of the three circles **through** B and C and tangent to **CA**, through C and A and tangent to **AB**, and through A and B and tangent to BC. That* cot ID = *cot A* + *cot B* + *cot C is found **in** [1].*

References

1. John **Casey,** *A Sequel to **Euclid**,* **Hodges, Figgis, &** Co., *1888,* pp. *172, 177.*
2. *N.* A. Court, *College Geometry,* Johnson Publishing Co., 1925, pp. *243-247.*

769. **[Fall, 1991]** *Proposed by R. S. **Luthar**, University of Wisconsin Center, **Janesville**, Wisconsin.*

If ABC is a triangle in which $c^2 = 4ab \cos A \cos B$, Prove that the triangle is isosceles.

*I. Solution by Bob **Prielipp**, University of **Wisconsin-Oshkosh, Oshkosh**, Wisconsin.*

By the law of cosines we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{and} \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

Now the following equalities are all equivalent:

$$c^2 = 4ab \cos A \cos B,$$

$$c^2 = 4ab\frac{b^2 + c^2 - a^2}{2bc}\frac{c^2 + a^2 - b^2}{2ca},$$

$$c^4 = [I? \cdot (a^2 - b^2)][c^2 + (a^2 \cdot b^2)],$$

$$(a^2 \cdot b^2)^2 = 0,$$

$$a = b.$$

Hence I? = $4ab \cos A \cos B$ if and only of $a = b$, that is, if and only if the triangle is isosceles with apex C.

II. *Solution by George P. **Evanovich**, Saint Peter's College, Jersey City, New Jersey.*

Because c is the sum of the projections of sides AC and BC on AB, then we have $c = b \cos A + a \cos B$. Then

$$4ab \cos A \cos B = c^2 = b^2 \cos^2 A + 2ab \cos A \cos B + a^2 \cos^2 B,$$

$$0 = (b \cos A - a \cos B)^2,$$

whence $AC = BC$ and the triangle is isosceles.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, *MI.* CHARLES ASHBACHER, *Cedar Rapids, IA,* MOHAMMAD *K.* AZARIAN, *University of Evansville, IN,* SEUNG-JIN BANG, *Seoul, Korea,* FRANK P. BATTLES, *Massachusetts Maritime Academy, Buzzards Bay,* DIETER BENNEWITZ, *Koblenz, Germany,* SCOTT H. BROWN, *Stuart Middle School, FL,* PAUL S. BRUCKMAN, *Edmonds, WA,* DAN DIMINNIE, *Allegheny College, Meadville, PA,* RUSSELL EULER, *Northwest Missouri State University, Maryville,* JAYANTHI GANAPATHY, *University of Wisconsin-Oshkosh,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* MURRAY S. KLAMKIN, *University of Alberta,* Canada, HENRY S. LIEBERMAN, *Waban,* AM, DAVID E. MANES, *SUNY at Oneonta,* YOSHINOBU MURAYOSHI *(two solutions), Eugene, OR,* KANDASAMY MUTHUVEL, *University of Wisconsin-Oshkosh,* MICHAEL A. VITALE, *St. Bonaventure University, NY,* KENNETH M. WILKE, *Topeka, KS,* REX *H.* WU, *Brooklyn, NY, and the* PROPOSER.

***770. [Fall,** 1991] *Proposed by Robert C. **Gebhardt, Hopatcong**, New Jersey.*

A deck of cards, numbered from *1* to n, is dealt at random to n persons. Then a second similar deck is dealt to the same n persons. What is the probability that at least one of then persons received two cards with the same number?

Solution by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts.

That the first deck is dealt out is irrelevant; one simply needs a set of n persons or other objects numbered **1** to n. The problem is then isomorphic to the **"hatcheck"** problem, a well known problem of **matchings** and derangements. See, for example, *Applied Combinatorics* by Fred Roberts, pp. 203-205. The desired probability is

$$\sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j!},$$

which approaches $1 - 1/e$ as $n \to \infty$.

Also solved by CHARLES ASHBACHER *(computer solution), Cedar Rapids, IA,* PAUL S. BRUCKMAN, *Edmonds, WA,* GEORGE P. EVANOVICH, *Saint Peter's College, Jersey City, NJ,* MARK EVANS *(computer solution), Louisville, KY,* RICHARD I. HESS, *Rancho Palos Verdes, CA,* MURRAY S. KLAMKIN, *University of Alberta, Canada, and* REX H. WU, Brooklyn, *NY. One incorrect solution was received.*

Evanovich gave the reference Niven, Mathematics of Choice, pp. 78-80; Hess listed W. W. Rouse Ball, Mathematical Essays; Klamkin found David and Barton, Combinatorial Chance, P. 105; Wu located Constantine, Combinatorial Theory and Statistical Design, and the editor used Munroe, Theory of Probability, pp. 70-72.

The ***Pi Mu Epsilon Journal*** was founded in **1949** and is dedicated to undergraduate and beginning graduate students interested in mathematics. Submitted articles, announcements, and contributions to the Puzzle Section and Problem Department of the ***Journal*** should be directed toward this group.

Undergraduates and beginning graduate students are urged to submit papers to the ***Journal*** for consideration and possible publication. Student papers are given top priority. Expository articles by professionals in all areas of mathematics are especially welcome. Some guidelines are:

1. Papers must be correct and honest.
2. Most readers of the ***Pi Mu Epsilon Journal*** are undergraduates; papers should be directed to them.
3. With rare exceptions, papers should be of general interest.
4. Assumed definitions, concepts, theorems, and notations should be part of the average undergraduate curriculum.
5. Papers should not exceed **10** pages in length.
6. Figures provided by the author should be camera-ready.
7. Papers should be submitted in duplicate to the Editor.

RALPH P. BOAS – IN MEMORIAM

Ralph P. Boas, Jr., died in Seattle on July **25, 1992.** He was a former President of the MAA and a former editor of ***The American Mathematical Monthly.*** He was a distinguished research mathematician, publishing many articles and several books. He taught at Northwestern University for many years and was quite active in MAA matters. This Editor is grateful to him for his help in reviewing papers and his valuable insights into editing a mathematics journal. For a more complete biography, see FOCUS, the newsletter of the MAA, September, **1992.**

PUZZLE SECTION

The Editor thanks all those who sent in solutions to Joe Konhauser's puzzles that appeared in the Spring, **1992,** issue of the ***Journal.*** Since the solutions had already appeared in earlier issues of the ***Journal,*** the newly submitted solutions and the names of the solvers will not be printed.

SOLUTION TO MATHACROSTIC NO. 34 (SPRING, 1992)

WORDS:

A.	cooeebird	M.	the monster
B.	abyselpha	N.	eviscerate
C.	semeiosis	O.	Rosamond's Bower
D.	Theaetetus of Athens	P.	episode
E.	intention	Q.	airtight
F.	ace in the hole	R.	lightship
G.	leitmotiv	S.	in the limelight
H.	tensegrity	T.	tooth and nail
I.	even vertex	U.	inedited
J.	reverie	V.	ergonomics
K.	neve	W.	string theory
L.	ana		

AUTHOR AND TITLE: CASTI – ALTERNATE REALITIES

QUOTATION: In science, there is a method to get at the scheme of things – observation, hypothesis, and experiment. In religion, there is a method too – divine enlightenment. However, the religious method is not repeatable nor is it necessarily available to every interested investigator.

SOLVERS: THOMAS F. BANCHOFF, Brown University; JEANETTE BICKLEY, St. Louis Community College at Meramec, MO; CHARLES R. DIMINNIE, St. Bonaventure University, NY; META HARRSEN, New Hope, PA; TED KAUFMAN, Brooklyn, NY; CHARLOTTE MAINES, Rochester, NY; STEPHANIE SLOYAN, Georgian Court College, NJ; JOSEPH C. TESTEN, Mobile, AL;

MATHACROSTIC NO. 35

Proposed by the late Joseph D. E. Konhauser. This is the last known acrostic that he constructed,

The **227** letters to be entered in the numbered spaces in the grid will be identical to those in the **28** keyed words at the matching numbers. The key numbers have been entered in the diagram to assist in constructing the solution. When completed, the initial letters on the Words will give the name of an author and the title of a book; the completed grid will be a quotation from that book.

Solutions to Mathacrostic No. 35 should be sent to: Richard Poss, ***Pi Mu Epsilon Journal,*** St. Norbert College, **100** Grant Street, De Pere, WI **54115.** Solutions must be received by March **1.**

DEFINITIONS	WORDS
A Alice B. Toklas said it has "the pungency of a high-born radish bred to a low-brow cucumber"	127 36 138 159 92 5 116 170
8. rising	173 181 65 101 14 201 153 191 21
C. flip	8 38 190 223 83
D. crisscross	53 9 66 133 155 115 40 194 91
E. to redden	64 46 161 88 221 19 209 27 149 75 103
F. first two words of English title of Marcel Duchamp's controversial 1912 painting (2 ids.)	34 189 227 214 218 136 148 140 99 89 81 119 41 114
6. medley	202 164 31 100 130 185 105 157 42 71
H. laughing jackass	60 145 13 45 51 205 2 106 79 23
I. raucous noise; squawk	160 59 48 124
J. a whitefish of Irish lakes	30 67 179 50 82 200
K. composite plant having heads with both disk and ray flowers	47 10 20 203 76
L. billingsgate	197 167 225 117 151 212 74 68 216
M. good for nothing	37 43 132 126 94 206 107 32
N. a moral or an emotional pang	168 95 199 44 135 61
0. layer of earth called "zone of mobility"	217 93 4 169 28 131 55 104 17 141 78 152 180
P. comet with period 17.97 years, first observed in 1913	186 18 109 1 204 97 49
Q. tenor	90 16 139 86 121
R. times past (2 ids.)	162 7 52 158 220 80 120 25
S. The _______ Scar, Caroline Bird's fascinating look at the Great Depression	224 110 198 176 3 33 142 207 163
T. it was discovered by the French. glamorized by Times Square, and immortalized by Las Vegas (2 wds.)	193 72 122 113 171 184 118 150
U. sensation	143 73 102 96 15 182 156 62
V. to fidget	24 39 166 63 215 196 226
W. to cover here and there	108 77 87 22 187 35 165 57 172
X. a pool or puddle	11 175 128 26 213
Y. platelike	84 154 58 29 12 98 219 177 195 70 146
Z. not connected by conjunctions	134 54 174 210 111 125 188 85 144
a friendly goblin or brownie of Scandinavian folklore	56 112 208 183 222
b. last song jointly written by Richard Rogers and Oscar Hammerstein II	178 211 69 129 137 147 192 123 6

1 P	2 H	3 S	4 O		5 A	6 b		7 R	8 C		9 D	10 K	11 X	12 Y
13 H	14 B	15 U	16 Q		17 O	18 P	19 E	20 K	21 B	22 W	23 H	24 V	25 R	26 X
	27 E	28 O	29 Y	30 J	31 G	32 M		33 S	34 F	35 W	36 A		37 M	38 C
39 V		40 D	41 F	42 G		43 M	44 N	45 H	46 E	47 K	48 I	49 P		50 J
51 H	52 R	53 D	54 Z		55 O	56 a	57 W		58 Y	59 I	60 H	61 N	62 U	
63 V	64 E	65 B	66 D	67 J	68 L	69 b	70 Y	71 G	72 T	73 U		74 L	75 E	
76 K	77 W	78 O	79 H	80 R	81 F	82 J	83 C		84 Y	85 Z	86 Q	87 W		88 E
89 F	90 Q		91 D	92 A	93 O	94 M	95 N	96 U	97 P	98 Y	99 F		100 G	101 B
102 U	103 E		104 O	105 G	106 H	107 M	108 W	109 P	110 S	111 Z	112 a	113 T	114 F	115 D
	116 A	117 L	118 T	119 F	120 R		121 Q	122 T		123 b	124 I	125 Z	126 M	127 A
	128 X		129 b	130 G	131 O	132 M	133 D	134 Z	135 N	136 F		137 b	138 A	139 Q
140 F	141 O		142 S	143 U	144 Z	145 H	146 Y	147 b	148 F		149 E	150 T	151 L	152 O
153 B	154 Y	155 D	156 U	157 G	158 R	159 A	160 I		161 E	162 R	163 S	164 G	165 W	
166 V	167 L		168 N	169 O	170 A	171 T		172 W	173 B	174 Z		175 X	176 S	177 Y
178 b	179 J	180 O	181 B	182 U		183 a	184 T	185 G	186 P	187 W		188 Z	189 F	190 C
191 B		192 b	193 T	194 D	195 Y		196 V	197 L	198 S	199 N	200 J	201 B		202 G
203 K	204 P	205 H	206 M	207 S	208 a		209 E	210 Z	211 b		212 L	213 X	214 F	
215 V	216 L	217 O	218 F		219 Y	220 R		221 E	222 a	223 C	224 S	225 L	226 V	227 F

THE 1992 NATIONAL PI MU EPSILON MEETING

This year's national meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections. The reason for this departure from tradition was that, because of ICME having its meeting in Quebec, Canada, the MAA and AMS did not hold their usual combined summer meeting. Pi Mu Epsilon hosted the meeting on the campus of Miami University, in Oxford, Ohio. The meeting ran from August 5 through 8.

The J. Sutherland Frame Lecturer was Underwood Dudley, from DePauw University. The title of his talk was "Angle Trisectors." The MAA Invited Address was by Peter Hilton, from the State University of New York at Binghamton. He provided "Another Look at Fibonacci and Lucas Numbers." In addition to these invited addresses, there were three minicourses available to the students and faculty who attended the meeting. These were: "Tilings by Hand and Computer," by Doris Schattschneider of Moravian University; "Variations on a Spiral," by David Kullman of Miami University (this minicourse was presented twice); and "Environmental Mathematics," by Ben A. Fusaro of Salisbury State University.

At the annual Pi Mu Epsilon banquet, David Ballew, President of Pi Mu Epsilon, gave tribute to Joseph D. E. Konhauser, who passed away last February. Joe, a former editor of this *Journal,* passed away in February. He was a National Councilor for Pi Mu Epsilon at the time of his death. (See the Spring, 1992, issue of this *Journal,* page 349.) Joe's unexpired term on the national council is being filled by Robert S. Smith, from Miami University (Ohio). In addition to his other contributions to Pi Mu Epsilon, Robert Smith was in charge of all the local arrangements for this year's meeting at Miami University. President Ballew also introduced J. Kevin Colligan, a representative of the National Security Agency. The NSA has again given Pi Mu Epsilon a generous grant to distribute to student speakers to help defray the cost of their travel to the national meeting.

The Pi Mu Epsilon Council held its annual meeting on Friday, August 7. The Council noted that this first joint meeting with the MAA student chapters was quite successful and agreed to work with the MAA student chapters to again co-host the meeting in 1993. Because the 1993 summer meeting will be in Vancouver, British Columbia, Canada, and this site is so far from most of the schools which have usually sent student speakers, the Council approved, on a temporary basis, a more generous travel allowance for student speakers. (See note on page 421.) The Council hopes that this experimental plan will help to continue the large number of student speakers at the national meetings. There were a total of 31 Pi Mu Epsilon student speakers at this year's meeting.

PROGRAM – STUDENT PAPER SESSIONS

Interesting Properties of Some Graph Products	Ray Adams Massachusetts Alpha Worcester Polytechnic Institute
On Maximizing the Product of Partitions	Jeffery John Boats New York Omega St. Bonaventure University
On Directed A-Cycles in n-Tournaments	John Davenport Ohio Delta Miami University

Grassman Algebras, Functional Integrals, and the Hubbard Model	Anthony F. DeLia Florida Theta University of Central Florida
Two Dimensional Analysis of Heat Flux in a Copper Plate Using the Finite Difference Method	James A. DiLellio Ohio Nu University of Akron
Elaboration on Usefulness of Constructively Limited and Irresolvable Demonstrations	Vladimir Dimitrijevic Ohio Xi Youngstown State University
JMP-ing into Data Analysis and Exploration	William Duckworth Ohio Delta Miami University
A Number-Theoretic Identity Arising from Burnside's Orbit Formula	Francis Fling Kansas Beta Kansas State University
A Generating Function for Nilpotent Pairs in a Finite Group	Mike Galloy Indiana Gamma Rose-Hulman Institute of Technology
On the Packing Graph	Raitis Grinsbergs Minnesota Gamma Macalester College
Elements of Hyperbolic Geometry	Tony Hinrichs Indiana Gamma Rose-Hulman Institute of Technology
Mathematics in Advanced Macro-Economics	Barry E. Jones Ohio Delta Miami University
Primitive Pythagorean Triples	Dennis Keeler Ohio Delta Miami University
The Connectivity of Interior of 3-Regular 3-Connected Bipartite Planar Graphs	Dylan T. G. KhooLim Ohio Delta Miami University
Buffon's Needle Problem	Susan Koppenol Louisiana Delta Southeastern Louisiana University
An Introduction to Elliptic Integrals	Amy Krebsbach Wisconsin Delta St. Norbert College

Cooperative Learning in Mathematics Education
Elizabeth **Kuehner**
Pennsylvania Omicron
Moravian College

A SLATEC Compatible Subroutine for Spline Approximation Using General Basis Functions and Constraints
Mark P. Kust
Michigan Epsilon
Western Michigan University

Patchwork Mathematics
Shelly L. Martin
Ohio Xi
Youngstown State University

Some Combinatorial Results Arising from Complete Digestion of Proteins
Jennifer Miners
Arkansas Beta
Hendrix College

Development of a Power Outage Emergency Response System
Mike **Ochrtman**
Oklahoma Beta
Oklahoma State University

A Numerical Model Including PID Control of a Multizone Crystal Growth Furnace
Charles H. **Panzarella**
Ohio Nu
University of Akron

Completely Positive Matrices
Keith Rhodes
Ohio Delta
Miami University

Bounding the Sneech Population
Melissa A. Smith
Ohio Lambda
John Carroll University

Patterns of Periodicity in the Mandelbrot Set
Michael J. South
Georgia Epsilon
Valdosta State College

Probabilities Associated with Plinko
Traca E. Tithof
Ohio Xi
Youngstown State University

A Tool for Solving 2-Dimensional Systems of Equations
Christina T. **Tsiaparas**
Ohio Xi
Youngstown State University

Fluid Diffusion in the Brain
David D. Turner
Washington Zeta
Eastern Washington University

Crystallographic Groups in the Plane
Linda M. **Vargo**
Illinois Iota
Illinois Benedictine College

John Napier and His Definition of Logarithm
Daniel L. Viar
Arkansas Alpha
University of Arkansas

Dynamics of a Quadratic Family * In Pictures *
Connie **Yarema**
Texas Mu
East Texas State University

AMERICAN MATHEMATICAL SOCIETY • FOUNDED 1888 •

For the fourth consecutive year, the American Mathematical Society has given Pi Mu Epsilon a grant to be used as prize money for excellent student presentations. As always, there were many excellent presentations, and four of them were selected to receive prizes of $100 each. The winning speakers were:

Jeffery John Boats, St. Bonaventure University
On Maximizing the Product of Partitions

Francis Fung, Kansas State University
A Number-Theoretic Identity Arising from Burnside's Orbit Formula

Susan Koppenol, Southeastern Louisiana University
Buffon's Needle Problem

Daniel L. **Viar,** University of Arkansas
John Napier and His Definition of Logarithm

Pi Mu Epsilon is again grateful to the American Mathematical Society for the generous gift that has made these awards possible.

GLEANINGS FROM THE CHAPTER REPORTS

MICHIGAN ZETA (University of Michigan, Dearborn) We continued our Focus on Faculty series for a third year. Three faculty members of the Mathematics Department presented lectures: on the future of computers in college mathematics, Rivest's coin tossing problem, and geometric modeling. To assist students of all levels, The Chapter sponsored two math advising sessions. Professors representing the areas of sewndary education, statistics, computer science, and applied mathematics aided students in future course selection and possible career and graduate school paths. In addition, a representative from the University's Career Planning and Placement Office was on hand to advise prospective graduates on possible careers which utilize a mathematics degree. We sponsored two student/faculty mixers this past year. The April mixer was a faculty thank-you luncheon. To aid those students that might pursue more advanced degrees, the chapter organized a graduate school announcement library. On a social level, the chapter had two game nights with pizza and other refreshments. The chapter also sponsored a dinner at the end of each semester.

NEW MEXICO ALPHA (New Mexico State University) The chapter conducted its fourth annual NMSU Math Challenge on Saturday, April 4, 1992. This contest, taken by high school students from around the region, had three strands: an individual contest, a team bowl contest, and a team modelling contest. In the Individual Contest, 181 students took the qualifying round at their own high schools. Forty-four students were invited to take the second round on campus. The first place winner, Russell Kehl, received an HP 48S calculator. The two second place winners, Jeffrey Miller and Xin Wang, were given TI-81 calculators. The third place winner, Charles Hardin, and the two fourth place winners, Michael Martinez and Tim Fox, received books. Nine four-member teams took part in the Team Competition. First place winners were the Cruces Conics. The Ream Modelling Competition was new this year. Two faculty members wrote an original problem *Monttoring Meteor Impacts on the Moon.* Eleven 2-4 members from four area high schools spent a weekend solving it. The top five team were invited to present their solutions orally to the judges and interested spectators on Team Day. The top team received a plaque for their school and the members (Jeff Miller, Charles Hardin, and Steven Bennett) were each given asubscription to *Quantum* magazine.

OHIO ZETA (The University of Dayton) The chapter was very active this year. Some of the highlights are as follows: At the Annual Pi Mu Epsilon Regional Conference held in September, 1991, at Miami University, Oxford, OH, three students gave talks. They were David Jessup. David Kass, and Kristine Fromm. A Number of students attended and gave talks at the spring meeting of the Ohio Section of the MAA held at the University of Dayton. The students are David Jessup, David Kass, Kristine Fromm, Kristen Toft, Marni Ryder, and Thomas Szendrey. David Jessup gave a talk at the Summer AMS-MAA meeting held in Orono, ME. David Jessup and David Kass jointly received this year's Faculty Award for Excellence in Mathematics. Jeff Oliver was the recipient of this year's Sophomore Class Award.

OHIO NU (The University of Akron) The chapter held its annual induction banquet on April 24, 1992. A number of awards were presented at this banquet. David Johnson, Brian Van Pelt, and Asim Yarkan received one-year memberships in the American Society. Jayashree Dorairaj, Polychronis Papageras, and Lucy Pramudji received one-year memberships to the Society of Industrial and Applied Mathematics. Jan Spears received a one-year membership to the American Statistical Association. Melissa Jolly and Kelly Kerata received one-year memberships to the Mathematical Association of America. The Samuel Selby Scholarships ($500 each) were awarded to James Dilellio and Zhaolin Mao. The William Beyer Statistics Scholarship ($400) was awarded to William Blue. The Mary Maxwell Memorial Scholarships in Mathematics ($400 each) were awarded to Dawn Holgate, Joseph Ramey, Carl Stitz, and Gary Traicoff. Jonobie Baker, the Western Reserve Science Day Winner - Mathematical Sciences Category, received a $50 U.S. savings bond.

OHIO XI (Youngstown State University) Eleven students attended the National Pi Mu Epsilon Meeting at Orono, ME. Eight of these students presented papers: Jim Baglama, James Banoczi, Hester Brosag, Sharyn Campbell, Dimitros Chalop, Heather DeSimone, Linda Hughes, and Marguerite Nedreberg. Three of the speakers received prizes: Heather DeSimone, Linda Hughes, and Marguerite Nedreberg. The chapter had its initiation of new members on November 6. On November 20 we had our annual book sale. The books were donated by professors. The chapter made a profit of $260. We had a Christmas party on December 15. There was another initiation of new members at the meeting of January 29. Other activities during the year were: a second book sale and a sweatshirt sale. The final initiation of new members was held on April 15.

TENNESSEE GAMMA (Middle Tennessee State University) The chapter began the 1991-92 year with its semi-annual pizza party. New members were initiated and officers elected. In October, Tom Ingram, University of Missouri at Raleigh, gave a presentation on "Dynamical Systems." The semester ended with a combined Christmas party with the two other mathematics clubs. At the March 12 meeting, there was a panel discussion titled "Is There Life After a Master's Degree in Mathematics?" The five panelists were all former MTSU Master of Science degree students. They were Joy Whitenack (who is working on an Ed. D. at Vanderbilt), Lori Henslee, Amy Wildsmith, Michael Mogensen Vermillion (who are currently working on Ph. D.'s at Vanderbilt), and Susan Calvert (instructor at Motlow State Community College).

For the 1992 Tom Vickrey Mathematics Project Competition, Dawn Woodard (first place) gave her paper titled "Automorphism Groups of the Hasse Subgroup Diagrams for Cyclic Groups with Order Divisible by Exactly Two Primes." Gary Estep (sewnd Place) presented his project "Application of Infinite Series to Fractals." Dawn Luna and Robert Ralston presented their paper "Two Finite Self-Dual Geometries," and Kevin Gipson gave his paper "Minimal Surfaces." We also had four of our members present their papers at the Hendrix-Rhodes-Sewanee Undergraduate Mathematics Symposium at Memphis. Finally, in April, we proctored the Junior High Contest held at MTSU, which is our annual fundraiser. The year ended with another combined picnic with members of the other two mathematics clubs.

Seventh Annual

MORAVIAN COLLEGE
STUDENT MATHEMATICS CONFERENCE

Bethlehem, Pennsylvania

Saturday, February 13, 1993

We invite you to join us, whether to present a talk or just to listen and socialize. The conference will begin at 9:00 a.m. and continue into late afternoon. After an invited address, the remainder of the day will be devoted to undergraduate student talks. Talks may be fifteen or thirty minutes long. They may be on any topic related to mathematics, operations research! statistics or computing. We encourage students doing research or honors work to present their work here. We also welcome expository talks, talks about interesting problems or applications and talks about internships, field studies and summer employment. We need your title, time of presentation (15 or 30 minutes) and a 50 word (approximate) abstract by February 5, 1993.

Sponsored by the **Moravian College Chapter of Pi Mu Epsilon** and the **Lehigh Valley Association of Independent Colleges.**

Please contact: Fred Schultheis
Department of Mathematics, Moravian College
1200 Main St. Bethlehem! PA 18018
(Telephone: (215) 861-3925)

Announcing the 10th Annual
Rose-Hulman Institute of Technology
Conference on Undergraduate Mathematics

Friday and Saturday, March 19th and 20th, 1993
on the Rose-Hulman campus in Terre Haute, Indiana

Featuring Keynote Speaker

Carla Savage

of North Carolina State University

and

A *CAYLEY* SHORTCOURSE

offered by

Gary Sherman

of the Rose-Hulman Institute of Technology

Carla Savage has been studying Gray codes, and variants, and their relationship to graph theory, group theory, and discrete mathematics. Gray codes are schemes for listing combinatorial objects so that successive objects differ in a small, specified way. Professor Savage has also worked in the area of parallel algorithms and architectures. She is a popular speaker, whose work has been partially supported by NSF and NSA.

Gary Sherman directs a Summer NSF–REU program, which features computational group theory. *CAYLEY* is a software package which allows the user to do computations in groups, rings, and fields. Anyone interested in computational algebra or discrete mathematics will find this shortcourse of interest.

Undergraduate students are encouraged to submit abstracts of papers, in any area of the mathematical sciences, for presentation.

For more information, contact: Bart Goddard
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