Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Vol. 11, No. 10

December 1985

Publié par la Société Mathématique du Canada

The support of the Carleton University Department of Mathematics and Statistics and the University of Ottawa Department of Mathematics are gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22.50 for members of the Canadian Mathematical Society and $25 for nonmembers. Back issues: $2.75 each. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-10: $20. All prices quoted are in Canadian dollars. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content of the journal should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

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ISSN 0705 - 0348.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

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CONTENTS

THE OLYMPIAD CORNER: 70

M.S. KLAMKIN

The new problems I propose this month, for which I solicit elegant solutions from all readers, were all proposed (but unused) by various countries at the 26th International Mathematical Olympiad held this year in Finland. They were taken from the booklet *XXVI I.M.O., Results and Problems*, edited by Matti Lehtinen, to whom I am grateful for sending me a copy.

1. *Proposed by Bulgaria.*

If a,b,n are integers, with $n > 0$, prove that

$$\frac{b^{n-1}a(a+b)(a+2b)\ldots\{a+(n-1)b\}}{n!}$$

is an integer.

2. *Proposed by Brazil.*

A convex quadrilateral is inscribed in a circle of radius 1. Prove that the (positive) difference d between its perimeter and the sum of the lengths of its diagonals satisfies $0 < d < 2$.

3. *Proposed by Canada.*

Determine the maximum value of

$$\sin^2\theta_1 + \sin^2\theta_2 + \ldots + \sin^2\theta_n,$$

where $\theta_1 + \theta_2 + \ldots + \theta_n = \pi$ and all $\theta_i \geq 0$.

4. *Proposed by Canada.*

Prove that

$$\frac{x_1^2}{x_1^2+x_2x_3} + \frac{x_2^2}{x_2^2+x_3x_4} + \ldots + \frac{x_{n-1}^2}{x_{n-1}^2+x_nx_1} + \frac{x_n^2}{x_n^2+x_1x_2} \leq n-1,$$

where all $x_i > 0$.

5. *Proposed by Czechoslovakia.*

Let T be the set of all lattice points (i.e., all points with integer coordinates) in 3-dimensional space. Two such points, (x,y,z) and (u,v,w), are called *neighbors* if and only if

$$|x-u| + |y-v| + |z-w| = 1.$$

Prove that there exists a subset S of T such that, for each $P \in T$, there is exactly one point of S among P and its neighbors.

6. *Proposed by Czechoslovakia.*

Let A be a set of positive integers such that $|x-y| \geq xy/25$ for any two elements x,y of A. Prove that A contains at most nine elements. Also, give an example of such a nine-element set.

7. *Proposed by East Germany.*

Determine whether or not there exist 100 distinct lines in the plane having exactly 1985 distinct points of intersection.

8. *Proposed by France.*

Determine eight positive integers $n_1, n_2, \ldots, n_8$ with the following property: For every integer k such that $-1985 \leq k \leq 1985$, there are eight integers $\alpha_1, \alpha_2, \ldots, \alpha_8$, each belonging to the set $\{-1,0,1\}$, such that

$$\alpha_1 n_1 + \alpha_2 n_2 + \ldots + \alpha_8 n_8 = k.$$

9. *Proposed by Great Britain.*

E_1, E_2, and E_3 are three mutually intersecting ellipses, all in the same plane, with respective foci F_2 and F_3, F_3 and F_1, and F_1 and F_2, which are not all three collinear. Prove that the common chords of each pair of ellipses are concurrent.

10. *Proposed by Great Britain.*

A sequence of polynomials $\{P_m\}$, $m = 0,1,2,\ldots$, in x, y, and z is defined by $P_0(x,y,z) \equiv 1$ and, for $m > 0$, by

$$P_m(x,y,z) = (x+z)(y+z)P_{m-1}(x,y,z+1) - z^2 P_{m-1}(x,y,z).$$

Prove that each $P_m(x,y,z)$ is symmetric in x,y,z.

11. *Proposed by Iceland.*

If

$$\sum_{i=1}^{n} x_i x_{i+1} x_{i+2} x_{i+3} = 0,$$

where $x_{i+n} = x_i$ and all $x_i^2 = 1$, prove that n is divisible by 4.

12. *Proposed by Israel.*

1985 points are given inside a unit cube. Show that it is always possible to choose 32 of the points in such a way that every closed polygon (possibly degenerate) with these points as vertices has perimeter less than $8\sqrt{3}$.

13. *Proposed by Italy.*

Two persons X and Y play a game with a die. Whenever the toss of the die results in a 1 or a 2, X wins the toss; otherwise Y wins the toss. The game is won by the first player who wins two consecutive tosses. Determine the probabilities that each of the players wins the game within 5 tosses. Also, determine the probabilities of winning for each player if there is no restriction on the number of tosses.

14. *Proposed by Mongolia.*

From each of the vertices of a regular n-gon a point starts to move with constant speed along the perimeter of the n-gon in the clockwise sense. Prove that if all the points get to a vertex A at the same time, then the points will never be together again at any other vertex. Can the points be together again at A?

15. *Proposed by The Netherlands.*

The solid S is defined as the intersection of the six spheres whose diameters are the six edges of a given regular tetrahedron with unit edge length. Prove that no pair of points of S is further apart than $1/\sqrt{6}$.

16. *Proposed by Poland.*

If

$$x_n = \sqrt[2]{2 + \sqrt[3]{3 + \ldots + \sqrt[n]{n}}},$$

prove that $x_{n+1} - x_n < 1/n!$ for $n = 2,3,4,\ldots$.

17. *Proposed by Poland.*

For which k can the set $\{1,2,\ldots,k\}$ be partitioned into a prime number p of subsets such that the sum of the elements in each subset is the same?

18. *Proposed by Rumania.*

For $k \geq 2$, let $n_1, n_2, \ldots, n_k$ be positive integers such that

$$n_2 \mid (2^{n_1}-1), \quad n_3 \mid (2^{n_2}-1), \quad \ldots, \quad n_k \mid (2^{n_{k-1}}-1), \quad n_1 \mid (2^{n_k}-1).$$

Prove that $n_1 = n_2 = \ldots = n_k = 1$.

19. *Proposed by Rumania.*

Show that the sequence $\{a_n\}$ defined by $a_n = [n\sqrt{2}]$ for $n = 1,2,3,\ldots$ (where the brackets denote the greatest integer function) contains an infinite number of integral powers of 2.

20. *Proposed by Sweden.*

Two equilateral triangles are inscribed in a circle with radius r. If A is the area of the set consisting of all points interior to both triangles, prove that $2A \geq r^2\sqrt{3}$.

21. *Proposed by Sweden.*

If a,b,c are real numbers such that

$$\frac{1}{bc-a^2}+\frac{1}{ca-b^2}+\frac{1}{ab-c^2}=0,$$

prove that

$$\frac{a}{(bc-a^2)^2}+\frac{b}{(ca-b^2)^2}+\frac{c}{(ab-c^2)^2}=0.$$

22. *Proposed by Spain.*

Show how to construct a triangle ABC given the side AB, the length of segment OH and the fact that OH is parallel to AB, where O and H are the circumcenter and orthocenter, respectively, of the triangle.

23. *Proposed by Spain.*

Find all triples (x,y,z) of positive integers such that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{5}.$$

24. *Proposed by the Soviet Union.*

Decompose the number $5^{1985}-1$ into a product of three integers each of which exceeds 5^{100}.

25. *Proposed by the Soviet Union.*

34 countries participated in a Jury session of the I.M.O., each country being represented by the leader and the deputy leader of the team. Before the meeting, some of the participants greeted each other by shaking hands, but no team leader shook hands with his or her deputy. After the meeting, the leader of the Illyrian team asked every other participant (including her own deputy) the number of people they had shaken hands with, and all the answers she got were different. How many people did the Illyrian team greet?

26. *Proposed by Turkey.*

Determine the smallest positive integer n such that

(i) n has exactly 144 distinct positive divisors, and

(ii) there are ten consecutive integers among the positive divisors of n.

27. *Proposed by the U.S.A.*

Determine the range of $w(w+x)(w+y)(w+z)$, where x,y,z,w are real numbers such that

$$x+y+z+w=x^7+y^7+z^7+w^7=0.$$

28. *Proposed by the U.S.A.*

We are given n elements $a_1,a_2,\ldots,a_n$ which are organized into n pairs

$P_1, P_2, \ldots, P_n$ in such a way that two pairs P_i and P_j share exactly one element when $\{a_i, a_j\}$ is one of the pairs. Prove that every element is in exactly two of the pairs.

29. *Proposed by Vietnam.*

Prove that for every point M on the surface of a regular tetrahedron there exists a point M' such that there are at least three different curves joining M to M' on the surface with the smallest possible length among all the curves joining M to M' on the surface.

30. *Proposed by Vietnam.*

A circle with radius R and center O, and a line l, are given in the plane. The distance of O to l is d, with $d > R$. Points M and N are chosen on l such that the circle with diameter MN is externally tangent to the given circle. Show that there exists a point A in the plane such that all the segments MN are seen in a constant angle from A.

*

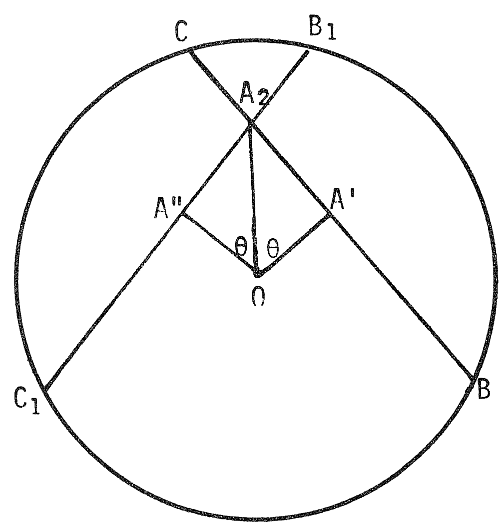
I now present solutions to some problems proposed in earlier columns.

1. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

Triangle ABC is rotated about the centre of its circumscribed circle by an angle less than 180° to form triangle $A_1B_1C_1$. If $BC \cap B_1C_1 = A_2$, $CA \cap C_1A_1 = B_2$, and $AB \cap A_1B_1 = C_2$, prove that triangles ABC and $A_2B_2C_2$ are similar.

Solution by M.S.K.

Let the angle of rotation be 2θ, and let A',B',C' and A",B",C" be the feet of the perpendiculars from the center O on BC,CA,AB and B_1C_1, C_1A_1, A_1B_1, respectively (see figure). It follows easily that $A'B'C' \sim ABC$. It now suffices to show that the configuration O-A'B'C' is similar to $O\text{-}A_2B_2C_2$. The first of the relations



$$OA_2 = \frac{OA'}{\cos\theta}, \quad OB_2 = \frac{OB'}{\cos\theta}, \quad OC_2 = \frac{OC'}{\cos\theta}$$

is clear from the figure, and the other two follow similarly. Since also

$$\angle A'OA_2 = \angle B'OB_2 = \angle C'OC_2 = \theta,$$

it follows that

$$\angle B'OC' = \angle B_2OC_2, \quad \angle C'OA' = \angle C_2OA_2, \quad \angle A'OB' = \angle A_2OB_2.$$

Thus $O\text{-}A'B'C' \sim O\text{-}A_2B_2C_2$, and then $A_2B_2C_2 \sim ABC$.

*

2. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

Two players play the following game on a triangle ABC of unit area. The first player picks a point X on side BC, then the second player picks a point Y on CA, and finally the first player picks a point Z on AB. The first player wants triangle XYZ to have the largest possible area, while the second player wants it to have the smallest possible area. What is the largest area that the first player can be sure of getting?

Solution by M.S.K.

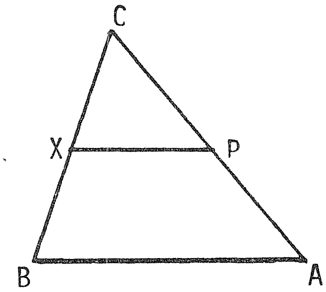


Figure 1

A point X having been chosen on side BC, let XP be parallel to AB, as shown in Figure 1. If Y is chosen on CP, then Z will be taken at A since this will maximize the altitude from Z to XY. It then follows that Y should be at P since this will minimize the altitude to XA. Similarly, if Y is chosen on PA, then Z will be taken at B and again Y should be P. We now wish to choose X so that triangle XPB, or equivalently triangle XPA, has a maximum area.

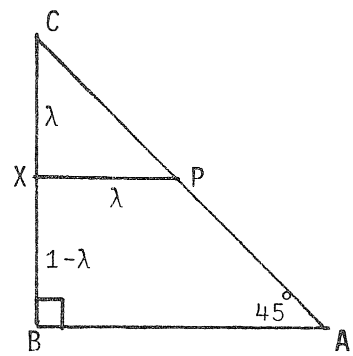


Figure 2

Although the calculation for point X is simple, it will be even simpler to take ABC as an isosceles right triangle with unit legs, as shown in Figure 2. This is permissible since we can transform the original triangle into the right triangle by an affine transformation (which preserves ratios of areas). Now the area of triangle XPB in Figure 2 is $\frac{1}{2}\lambda(1-\lambda)$, and this is maximized when $\lambda = 1/2$. Therefore the first player can always get an area of at least 1/4.

*

13. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

In a plane is given a finite set of polygons, every two of which have a common point. Show that there exists a line which intersects all the polygons.

Solution by Andy Liu, University of Alberta.

Denote the polygons by $P_1, P_2, \ldots, P_n$. Choose an arbitrary coordinate system x,y in the plane and orthogonally project each polygon onto the x-axis. Let the projection of P_i be the interval $[a_i, b_i]$, $i = 1,2,\ldots,n$. Since the polygons have pairwise nonempty intersections, so do the intervals. Let $a = \max_{1\le i\le n} a_i$. Clearly a belongs to all the intervals and the line $x = a$ is a common transversal of all the polygons.

*

14. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

Prove that, for positive a,b,c, we have

$$a^3 + b^3 + c^3 + 3abc \geq bc(b+c) + ca(c+a) + ab(a+b).$$

Solution by M.S.K.

The proposed inequality is equivalent to

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0.$$

This is the special case $t = 1$ of Schur's inequality

$$a^t(a-b)(a-c) + b^t(b-c)(b-a) + c^t(c-a)(c-b) \geq 0, \tag{1}$$

which is valid for all real numbers t when $a,b,c > 0$, with equality just when $a = b = c$. □

We give a proof of (1) for completeness. Since the left member is symmetric in a,b,c, we may assume without loss of generality that $a \geq b \geq c > 0$ when $t \geq 0$ and $0 < a \leq b \leq c$ when $t < 0$. In each case $a^t(a-b)(a-c) \geq b^t(a-b)(b-c)$, or

$$a^t(a-b)(a-c) + b^t(b-c)(b-a) \geq 0, \tag{2}$$

and

$$c^t(c-a)(c-b) \geq 0. \tag{3}$$

Now (1) follows from (2) and (3). Note that when t is an even integer, a,b,c can be negative.

*

2. [1981: 45; 1983: 309] *From the 1978 National Rumanian Mathematical Olympiad.*

Find an integer $k \geq 1$ such that the expression

$$f(k,x) \equiv \sin kx \cdot \sin^k x + \cos kx \cdot \cos^k x - \cos^k 2x$$

does not depend on x.

II. *Comment by M.S.K.*

Jordan B. Tabov notes that the above is only the first part of the Rumanian Olympiad problem. The missing second part asks for *all* integers $k \geq 1$ with the stated property. It has already been shown that $k = 3$ is the smallest such integer. We now show that $k = 3$ is the *only* such integer.

Suppose the integer $k \geq 3$ has the desired property. Since $f(k,0) = 0$, we must also have

$$f(k,\tfrac{\pi}{2}) = \sin\frac{k\pi}{2} - (-1)^k = 0.$$

Therefore $k = 4n-1$ for some positive integer n. For $k = 4n-1$ and $x = \pi/(4n-1)$, we get

$$f(k,x) = -\cos^{4n-1}\frac{\pi}{4n-1} - \cos^{4n-1}\frac{2\pi}{4n-1} = 0;$$

hence

$$\cos\frac{\pi}{4n-1} + \cos\frac{2\pi}{4n-1} = 0, \qquad \left(2\cos\frac{\pi}{4n-1} - 1\right)\left(\cos\frac{\pi}{4n-1} + 1\right) = 0,$$

and finally $\cos\{\pi/(4n-1)\} = 1/2$. Therefore $n = 1$, and $k = 3$ is the only such integer.

*

3. [1984: 215: 1985: 117] *From the 1984 British Mathematical Olympiad.*

(i) Prove that, for all positive integers m,

$$\left(2 - \frac{1}{m}\right)\left(2 - \frac{3}{m}\right)\left(2 - \frac{5}{m}\right)\cdots\left(2 - \frac{2m-1}{m}\right) \le m!.$$

(ii) Prove that if a,b,c,d,e are positive real numbers, then

$$\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4 \ge \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e}.$$

II. *(More elegant) solution of part (i) by Beno Arbel, Tel Aviv University, Israel.*

We prove a much stronger result, according to which the $m!$ in the proposed inequality can be replaced by 1. By the A.M.-G.M. inequality,

$$\prod_{k=1}^{m}\left(2 - \frac{2k-1}{m}\right) \le \left\{\frac{1}{m}\sum_{k=1}^{m}\left(2 - \frac{2k-1}{m}\right)\right\}^m = 1. \quad \square$$

On the other hand, if $m \ge 4$, then

$$\prod_{k=1}^{m}\left(3 - \frac{2k-1}{m}\right) \le \left\{\frac{1}{m}\sum_{k=1}^{m}\left(3 - \frac{2k-1}{m}\right)\right\}^m = 2^m < m!.$$

*

20. [1985: 5] *From the 1984 All-Union Olympiad (proposed by I.K. Zhuk and I.V. Voronovich).*

The numbers +1 or -1 are written in each of the nine unit squares of a 3×3 sheet of squared paper. Then the number in each unit square is replaced by the product of the numbers of its neighbours (two unit squares are *neighbours* if they have a common side). Prove that after a finite number of such operations each unit square will contain a +1.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

After at most four operations, each unit square will contain a +1. To prove this requires only simple calculations. Suppose the nine squares initially contain $a,b,c,\ldots,i$, as in the figure. For a corner square such as the one containing a, four operations give

a	b	c
d	e	f
g	h	i

$$a \to bd \to cg \to bdfh \to 1.$$

For a boundary square such as the one containing b, they give

$$b \to ace \to bh \to acgi \to 1.$$

And for the central square the results are

$$e \to bdfh \to 1 \to 1 \to 1.$$

*

21. [1985: 5] *From the 1984 All-Union Olympiad (proposed by L.P. Kuptsov).*

Three circles C_1,C_2,C_3 of radii r_1,r_2,r_3 in the plane are exterior to one another and $r_1 > r_2$, $r_1 > r_3$. The two tangents to C_3 are drawn from the intersection point of the exterior tangents to C_1 and C_2, and the two tangents to C_2 from the intersection point of the exterior tangents to C_1 and C_3. Prove that these two pairs of tangents (to C_3 and C_2) form a quadrilateral into which a circle can be inscribed. Find its radius.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

[Readers are urged to make their own generous-size figure to follow this solution.]

Let O_i be the center of C_i, $i = 1,2,3$; A the intersection of the exterior tangents to C_1 and C_2; B the intersection of the exterior tangents to C_1 and C_3; O the intersection of AO_3 and BO_2; D the intersection of OO_1 and AB; and x = AD, y = DB, $s = O_2O$, t = OB, $u = O_3O$, v = OA. Note that, from similar triangles,

$$\frac{BO_3}{O_3O_1} = \frac{r_3}{r_1-r_3}, \quad \frac{BO_1}{O_1O_3} = \frac{r_1}{r_1-r_3}, \quad \frac{O_1O_2}{O_2A} = \frac{r_1-r_2}{r_2}, \quad \frac{O_2O_1}{O_1A} = \frac{r_1-r_2}{r_1}.$$

Applying Ceva's theorem to triangle ABO_1 and cevian point O, we get

$$\frac{x}{y}\cdot\frac{r_3}{r_1-r_3}\cdot\frac{r_1-r_2}{r_2} = 1. \tag{1}$$

Then applying Menelaus's theorem to triangle ABO_2 and transveral DOO_1 gives

$$\frac{x}{y}\cdot\frac{t}{s}\cdot\frac{r_1-r_2}{r_1} = 1, \tag{2}$$

and doing the same to triangle ABO_3 and transveral DOO_1 gives

$$\frac{x}{y}\cdot\frac{r_1}{r_1-r_3}\cdot\frac{u}{v} = 1, \tag{3}$$

From (1) and (2) we get

$$\frac{t}{s+t}\cdot r_2 = \frac{r_1r_2r_3}{r_1r_2 + r_1r_3 - r_2r_3}$$

and from (1) and (3)

$$\frac{v}{u+v}\cdot r_3 = \frac{r_1r_2r_3}{r_1r_2 + r_1r_3 - r_2r_3}.$$

Hence a circle with center O and radius

$$\frac{r_1r_2r_3}{r_1r_2 + r_1r_3 - r_2r_3}$$

will touch the tangents from A to C_3 as well as the tangents from B to C_2.

*

22. [1985: 5] *From the 1984 All-Union Olympiad (proposed by A.V. Andjans).*

A math teacher wrote the quadratic trinomial $x^2 + 10x + 20$ on the blackboard. Then each student in turn either increased by 1 or decreased by 1 either the constant term or the coefficient of x. Finally the trinomial $x^2 + 20x + 10$ appeared. Will a quadratic trinomial with integer roots necessarily appear on the blackboard in the process?

Solution by Lones Smith, Nepean, Ontario.

We begin with $x^2 + bx + c$ where $b-c = -10$ and end up with $b-c = +10$, and at each stage $b-c$ changes by ±1. Clearly, at one stage $b-c = 1$. Since our quadratic trinomial is then

$$x^2 + (c+1)x + c \equiv (x+1)(x+c),$$

it will have integer roots in the process.

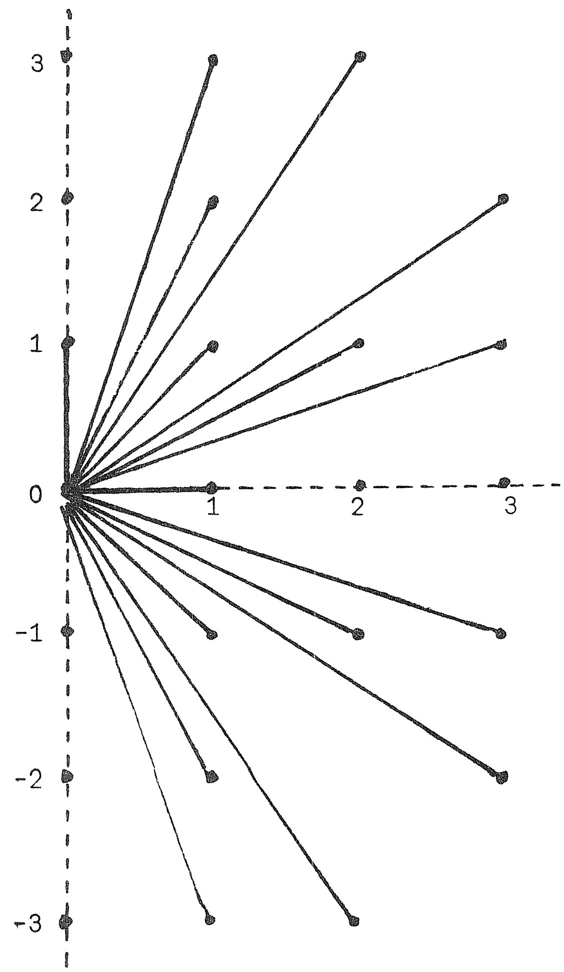
*

3. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

The vertices of a convex 32-gon lie on the points of a square lattice whose squares have sides of unit length. Find the smallest perimeter such a figure can have.

Solution by Andy Liu, University of Alberta.

If we allow polygons that are not strictly convex, the minimum perimeter is trivially 32 and is realized in rectangles of dimensions



1×15, 2×14, ..., 7×9, 8×8. For strictly convex polygons, there cannot be three parallel edges. The figure on the preceding page yields 16 edges with different slopes. Their combined length is

$$2(1 + \sqrt{2} + 2\sqrt{5} + 2\sqrt{10} + 2\sqrt{13}) \approx 40.844. \tag{1}$$

These are the shortest edges except for those of lengths 2, 3, $2\sqrt{2}$. But using the latter edges will prevent the use of edges of lengths 1 and $\sqrt{2}$, which are shorter. Finally, our polygon is a symmetric one using each of the edges in the figure twice. Hence the smallest perimeter is twice (1), or approximately 81.688.

*

4. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

On a 13 by 13 square piece of graph paper the centres of 53 of the 169 squares are chosen. Show that there will always be 4 of these 53 points which are the vertices of a rectangle whose sides are parallel to those of the paper.

Solution by Andy Liu, University of Alberta.

For $i = 1,2,\ldots,13$, let a_i be the number of points chosen in the ith row. Then $\Sigma a_i = 53$. (Sums, here and later, are for $i = 1$ to 13.) To each pair of chosen points in the same row we attach the label (j,k) if they are in the jth and kth columns. The total number of such pairs of numbers is

$$\Sigma\binom{a_i}{2} = \tfrac{1}{2}\Sigma a_i^2 - \tfrac{1}{2}\Sigma a_i \geq \frac{1}{26}(\Sigma a_i)^2 - \tfrac{1}{2}\Sigma a_i > 78 = \binom{13}{2},$$

where we have used the power mean inequality. It follows from the pigeonhole principle that two of these pairs of numbers are identical, and the two pairs of points corresponding to these two pairs of numbers are the vertices of a rectangle with sides parallel to the grid. □

Such a rectangle can be avoided if only 52 points are chosen. This will occur, for example, if the 52 points are chosen according to the (13,13,4,4,1) design: in a 13×13 configuration, pick the points so that there are 4 in each row and 4 in each column, with only one column coincidence for each pair of rows and only one row coincidence for each pair of columns. See Problem 9 below where the (7,7,3,3,1) design is illustrated. (This problem is solved in Ross Honsberger, *Mathematical Gems III*, M.A.A., 1985, pp. 2-4. See also the block designs in Marshall Hall, *Combinatorial Theory*, Blaisdell, 1967, Appendix 1, p. 291.)

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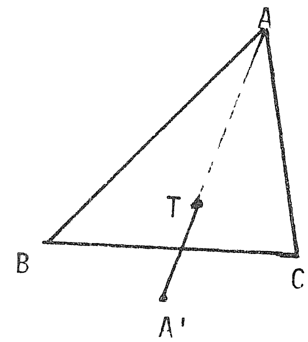
5. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

Three ants crawl along the sides of a triangle ABC in such a way that the centroid of the triangle they form at any given moment remains fixed. Show that this centroid coincides with the centroid of triangle ABC if one of the ants travels along the entire perimeter of triangle ABC.

Solution by Andy Liu, University of Alberta.

Let T be the centroid of the three ants, and suppose T does not coincide with the centroid of the triangle. Now consider the three cevians through T. For at least one vertex, say A, AT is longer than 2/3 of the cevian through A. Then the point A' on AT produced such that AT = 2TA' lies outside the triangle (see figure). Since T is the centroid of the three ants, when one of them (the one that moves all around ABC) is at A, the midpoint of the other two ants must coincide with A'. However, this is impossible if the ants are on the perimeter of the triangle. Therefore T coincides with the centroid of triangle ABC.



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6. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

A certain number of 0's, 1's, and 2's are written on a blackboard. Two unequal digits are erased and the third digit is written in their place (e.g., 2 is written if 0 and 1 are erased). This operation is repeated until no two distinct digits remain on the blackboard. Show that if only one digit remains at the end of the game, then this digit is independent of the order in which the digits were erased.

Solution by Andy Liu, University of Alberta.

Let x, y, and z be three variables denoting the numbers of 0's, 1's, and 2's, respectively. With each operation, the parities of x, y, and z all change. If $x \equiv y \equiv z \pmod 2$, then it is impossible to have only one digit remaining. Hence exactly two of x,y,z have the same parity, say $x \not\equiv y \equiv z \pmod 2$. In this case, the sole remaining digit must be 0, independently of the order in which the digits were erased.

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7. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

In a convex hexagon $A_1A_2A_3A_4A_5A_6$, let B_1,B_2,B_3,B_4,B_5,B_6 be the midpoints of diagonals $A_6A_2, A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1$, respectively. Show that if hexagon $B_1B_2B_3B_4B_5B_6$ is convex, then its area is $\frac{1}{4}$ the area of $A_1A_2A_3A_4A_5A_6$.

Solution by M.S.K.

Suppose the B-polygon is convex and let O be one of its interior points. Then O is also an interior point of the A-polygon. For $i = 1,2,\ldots,6$, we will let $\vec{A}_i$ and $\vec{B}_i$ denote the vectors $O\vec{A}_i$ and $O\vec{B}_i$, respectively. We then have

$$2\vec{B}_i = \vec{A}_{i-1} + \vec{A}_{i+1}, \tag{1}$$

where $\vec{A}_0 = \vec{A}_6$. With the brackets denoting area, we have

$$[A_1A_2\ldots A_6] = \frac{1}{2}|\vec{A}_1\times\vec{A}_2 + \vec{A}_2\times\vec{A}_3 + \ldots + \vec{A}_6\times\vec{A}_1|.$$

Finally, using (1) and cancelling terms like $\vec{A}_1\times\vec{A}_6 + \vec{A}_6\times\vec{A}_1 = \vec{0}$, we get

$$\begin{aligned}[B_1B_2\ldots B_6] &= \frac{1}{2}|\vec{B}_1\times\vec{B}_2 + \vec{B}_2\times\vec{B}_3 + \ldots + \vec{B}_6\times\vec{B}_1|\\ &= \frac{1}{8}|\vec{A}_1\times\vec{A}_2 + \vec{A}_2\times\vec{A}_3 + \ldots + \vec{A}_6\times\vec{A}_1|\\ &= \frac{1}{4}[A_1A_2\ldots A_6].\end{aligned}$$

*

8. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

Show that with the digits 1 and 2 one can form 2^{n+1} numbers, each having 2^n digits, and every two of which differ in at least 2^{n-1} places.

Solution by Andy Liu, University of Alberta.

For any positive integer k, let A_k be a set of positive integers such that

(a) every digit of each number in A_k is either 1 or 2;

(b) $|A_k| = 2^{k+1}$;

(c) each number in A_k has 2^k digits;

(d) each pair of numbers in A_k differs in exactly 2^{k-1} places or exactly 2^k places.

Trivially,

$$A_1 = \{11, 12, 21, 22\}$$

is such a set. Let

$$A_2' = \{1111, 1212, 2121, 2222\}, \qquad A_2'' = \{1122, 1221, 2112, 2211\},$$

and

$$A_2 = A_2' \cup A_2''.$$

Thus A_2' is obtained from A_1 by appending to each number of A_1 a copy of itself; and A_2'' is obtained from A_1 by appending to each number of A_1 a copy of itself with the 1's and 2's interchanged. In general, let

$$A_n = A_n' \cup A_n''$$

be obtained from A_{n-1} in an analogous manner.

Our proof is by induction. Suppose A_{n-1} has all four of the properties (a)-(d). Then, trivially, A_n has the three properties (a)-(c). Now consider two numbers in A_n.

If they are both from A_n' or both from A_n'', then they differ in the ith place, where $1 \le i \le 2^{n-1}$, if and only if they differ in the $(2^{n-1}+i)$th place. If one number is from A_n' while the other is from A_n'', then they differ in the ith place, $1 \le i \le 2^{n-1}$, if and only if they are identical in the $(2^{n-1}+i)$th place. It follows that A_n has property (d) as well, and the induction is complete.

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9. [1980: 274] *From the 9th U.S.S.R. National Olympiad (1974).*

On a 7×7 square piece of graph paper, the centres of k of the 49 squares are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest k for which this is possible?

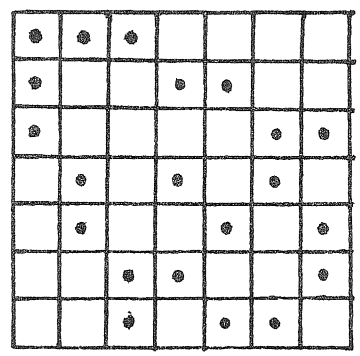
Solution by Andy Liu, University of Alberta.

We first show that if 22 points are chosen, then four of them will be the vertices of a rectangle with sides parallel to the grid. The argument is the same as that used in Problem 4 (see page 314 in this issue). The key inequality is replaced by

$$\Sigma\binom{a_i}{2} = \tfrac{1}{2}\Sigma a_i^2 - \tfrac{1}{2}\Sigma a_i \ge \tfrac{1}{14}(\Sigma a_i)^2 - \tfrac{1}{2}\Sigma a_i > 21 = \binom{7}{2},$$

where all sums are for $i = 1$ to 7.

A rectangle can be avoided if only 21 points are chosen. This will occur, for example, if the 21 points are chosen according to the (7,7,3,3,1) design: in a 7×7 configuration, pick the points so that there are 3 in each row and 3 in each column, with only one column coincidence for each pair of rows and only one row coincidence for each pair of columns. The design is illustrated in the figure. (This problem is solved in Ross Honsberger, *Mathematical Gems III*, M.A.A., 1965, pp. 4-6. See also the Marshall Hall reference at the end of Problem 4.)



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10. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

A large cube measuring k units on each edge is to be formed of smaller unit cubes, each coloured either black or white. Can this be done so that for any unit cube exactly two of its neighbours have the same colour as the unit cube itself? (Two cubes are called *neighbours* if they share a common face.)

Solution by Andy Liu, University of Alberta.

The arrangement is possible if and only if k is even. For k even, partition the top layer into 1×2×2 blocks. All cubes in the same block receive the same colour and adjacent blocks are given different colours. For succeeding layers, each cube receives the colour different from that of the cube directly above. It now follows easily that each cube has exactly two neighbours, both on the same layer, with the same colour as itself.

We now use a parity argument to show that no such colouring is possible if k is odd. Our proof is indirect. Assume that such a colouring exists. Starting from any unit cube, it is possible to trace a closed path of cubes of the same colour. Since each cube has exactly two neighbours of the same colour, each of these paths consists of an even number of cubes, and furthermore the sets of cubes of different paths are disjoint. Therefore the total number of unit cubes is even, and we have a contradiction.

*

11. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

A horizontal strip is given in the plane, bounded by straight lines, and n lines are drawn intersecting this strip. Every two of these lines intersect inside the strip and no three of them are concurrent. Consider all paths starting on the lower edge of the strip, passing along segments of the given lines, and ending on the upper edge of the strip, which have the following property: travelling along such a path, we are always going upward, and when we come to the point of intersection of two of the lines we must change over to the other line to continue following the path. Show that, among these paths,

(a) at least $\frac{1}{2}n$ of them have no point in common;

(b) there is some path consisting of at least n segments;

(c) there is some path passing along at most $\frac{1}{2}n + 1$ of the lines;

(d) there is some path which passes along each of the n lines.

Solution by Andy Liu, University of Alberta.

Let the lines be $l_1, l_2, \ldots, l_n$, intersecting the lower edge of the strip at the respective points $A_1, A_2, \ldots, A_n$ and the upper edge at $B_1, B_2, \ldots, B_n$. We may assume that $A_1, A_2, \ldots, A_n$ lie in that order from left to right. Since every pair of lines intersects within the strip, the points $B_n, B_{n-1}, \ldots, B_1$ lie in that order from left to right. Let $C_j^i = C_i^j$ be the point of intersection of l_i and l_j. Then l_i is divided into n segments by

$$C_1^i, \quad C_2^i, \quad \ldots, \quad C_{i-1}^i, \quad C_{i+1}^i, \quad \ldots, \quad C_n^i,$$

though not necessarily in that order. If i is odd, colour the segments along l_i alternately red and blue, with the bottom segment red. If i is even, colour the segments along l_i alternately blue and red, with the bottom segment blue. For each A_k between A_i and A_j, l_k intersects one of $A_iC_j^i$ and $A_jC_j^i$. For each A_k not between A_i and A_j, l_k intersects either both or neither of $A_iC_j^i$ and $A_jC_j^i$. It follows that

(*) *at any intersection, the two segments to the left are of one colour while the two segments to the right are of the other colour.*

Denote by p_i the path which starts from A_i. By (*), the entire path is of the same colour. Moreover, p_i is red if i is odd and blue if i is even. By (*) again, paths of the same colour cannot intersect. Hence at least $\frac{1}{2}n$ of them have no point in common, proving (a).

Note that there are altogether n^2 segments, and each lies on some path. Since the number of paths is n, the average number of segments per path is n. Hence there is some path consisting of at least n segments, proving (b).

We call C_j^i *exposed* if it lies on $p_1 \cup p_n$ and *protected* otherwise. We claim that each l_i contains at most two exposed intersection points. If l_i contains at least three exposed points, then two of them, say C_j^i below C_k^i, must both lie on p_1, say. There cannot be any C_t^i between C_j^i and C_k^i, as otherwise either $A_tC_t^i$ protects C_j^i or $B_tC_t^i$ protects C_k^i. All points along $A_iC_j^i$ are protected by $A_jC_k^j$ and $A_kC_k^j$. Similarly, all points along $B_iC_k^i$ are protected by $B_kC_k^j$ and $B_jC_k^j$. Thus it is impossible for l_i to contain a third exposed intersection point. It now follows that at most n intersection points can lie on $p_1 \cup p_n$, and so the total number of segments along p_1 and p_n is at most $n+2$. Thus either p_1 or p_n has at most $\frac{1}{2}n+1$ segments along at most $\frac{1}{2}n+1$ of the lines, proving (c).

Finally, note that p_k will end at B_{n+1-k} by (*). Let $k = [(n+1)/2]$. Then A_k and B_{n+1-k} lie on opposite sides of l_i for every $i \neq k$ or $n+1-k$. Hence p_k crosses l_i, but it is impossible for p_k not to pass along l_i for at least one segment. It follows that p_k passes along each of the n lines, proving (d).

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12. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

Given is a polynomial $P(x)$ whose coefficients are (i) natural numbers, (ii) integers. Denote by a_n the sum of the digits in the decimal representation of $P(n)$. Show that there is some number which occurs infinitely often in the sequence $a_1, a_2, a_3, \ldots$.

Solution by M.S.K. and Andy Liu, University of Alberta.

(i) Let k be a natural number such that 10^k exceeds all the coefficients of $P(x)$. Then, for $n = 10^{k+i}$, $i = 0,1,2,\ldots$, the sum a_n of the digits of $P(n)$ is equal to the sum of the digits of the fixed coefficients of $P(x)$.

(ii) For a sufficiently large natural number α, the coefficients of $P(x+\alpha)$ are all natural numbers. (Just let α be larger than the real part of each root of $P(x)$.) Now apply (i) to $P(x+\alpha)$.

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15. [1980: 275] *From the 9th U.S.S.R. National Olympiad (1974).*

Quadrilateral ABCD is inscribed in a circle. It is rotated about the centre of the circle through an angle less than 180° to form quadrilateral $A_1B_1C_1D_1$. Show that the points

$$AB \cap A_1B_1, \quad BC \cap B_1C_1, \quad CD \cap C_1D_1, \quad DA \cap D_1A_1$$

are the vertices of a parallelogram.

Solution by M.S.K.

Let

$$AB \cap A_1B_1 = A_2, \quad BC \cap B_1C_1 = B_2, \quad CD \cap C_1D_1 = C_2, \quad DA \cap D_1A_1 = D_2.$$

Also let the feet of the perpendiculars from center O to AB,BC,CD,DA and A_1B_1,B_1C_1, C_1D_1,D_1A_1 be A',B',C',D' and A",B",C",D", respectively. It follows from Problem 1 (page 308 in this issue) that

$$O\text{-}A'B'C'D' \sim O\text{-}A_2B_2C_2D_2.$$

Finally, since it follows easily that A'B'C'D' is a parallelogram, so is $A_2B_2C_2D_2$.

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16. [1980: 276] *From the 9th U.S.S.R. National Olympiad (1974).*

Twenty teams are participating in the competition for the championships both of Europe and the world in a certain sport. Among them, there are k European teams (the results of their competitions for world champion count also towards the European championship). The tournament is conducted in round robin fashion. What is the largest value of k for which it is possible that the team getting the (strictly) largest number of points towards the European championship also gets the (strictly) smallest number of points towards the world championship, if the sport involved is

(a) hockey (0 for a loss, 1 for a tie, 2 for a win);
(b) volleyball (0 for a loss, 1 for a win, no ties).

Solution by Andy Liu, University of Alberta.

(a) The number of points in the average score for the world championship is

$$\frac{\binom{20}{2}\cdot 2}{20} = 19.$$

Hence, to have the (strictly) smallest number of points, a team in the world championship can have at most 18 points. If there are at least 19 European teams, then the average score in the European championship is at least 18 points, so a team with at most 18 points cannot have the (strictly) largest number of points. Therefore $k < 19$.

	European															World					Scores	
Teams	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	V	W	X	Y	Z	E	W
A		1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	8	8
B	0		0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	6	11
C	0	1		1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	7	9
D	0	1	0		1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	7	9
E	0	1	0	0		1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	7	9
F	0	1	0	0	0		1	1	1	1	1	1	0	0	0	0	0	0	1	1	7	9
G	0	1	0	0	0	0		1	1	1	1	1	1	0	0	0	0	0	1	1	7	9
H	0	1	0	0	0	0	0		1	1	1	1	1	1	0	0	0	0	1	1	7	9
I	0	1	0	0	0	0	0	0		1	1	1	1	1	1	0	0	0	1	1	7	9
J	1	0	1	0	0	0	0	0	0		1	1	1	1	1	1	1	1	0	0	7	10
K	1	0	1	1	0	0	0	0	0	0		1	1	1	1	1	1	1	0	0	7	10
L	1	0	1	1	1	0	0	0	0	0	0		1	1	1	1	1	1	0	0	7	10
M	1	0	1	1	1	1	0	0	0	0	0	0		1	1	1	1	1	0	0	7	10
N	1	0	1	1	1	1	1	0	0	0	0	0	0		1	1	1	1	0	0	7	10
O	1	0	1	1	1	1	1	1	0	0	0	0	0	0		1	1	1	0	0	7	10
V	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0		1	1	0	0		10
W	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0		1	1	0		10
X	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0		1	1		10
Y	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0		1		9
Z	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0			9

It is possible to have $k = 18$. Let A and B be two of the European teams. All games are ties except that A beats B, B beats both non-European teams, and both of these beat A. In the world championship, A has 18 points, B has 20 points, while every other team has 19 points. In the European championship, A has 18 points, B has 16 points, while every other team has 17 points.

(b) In the world championship, the average score is $9\frac{1}{2}$ points. Hence, to have the (strictly) smallest number of points, a team can have at most 8 points. If $k \geq 16$, then the average score in the European championship is at least $7\frac{1}{2}$ points, so a team with at most 8 points cannot have the (strictly) largest number of points. Hence $k < 16$.

It is possible to have $k = 15$. A possible table of standings is given above (in which a 1 means that the row team has beaten the column team, etc.). Team A has the (strictly) smallest number of points in the world championship but the (strictly) largest number of points in the European championship.

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17. [1980: 276] *From the 9th U.S.S.R. National Olympiad (1974).*

Given real numbers

$$a_1, a_2, \ldots, a_m \quad \text{and} \quad b_1, b_2, \ldots, b_n,$$

and positive numbers

$$p_1, p_2, \ldots, p_m \quad \text{and} \quad q_1, q_2, \ldots, q_n,$$

we form an $m\times n$ array in which the entry in the ith row ($i = 1,2,\ldots,m$) and jth column ($j = 1,2,\ldots,n$) is

$$\frac{a_i + b_j}{p_i + q_j}.$$

Show that in such an array there is some entry which is no less than any other in the same row and no greater thanany other in the same column

(a) when $m=2$ and $n=2$,

(b) for arbitrary m and n.

Solution by Andy Liu, University of Alberta.

For typographical convenience we let

$$e(i,j) = \frac{a_i + b_j}{p_i + q_j}.$$

(a) We may assume without loss of generality that $e(1,1)$ is no greater than the other three entries, so that, in particular,

$$e(1,1) \le e(1,2) \tag{1}$$

and

$$e(1,1) \le e(2,1). \tag{2}$$

We now assume that there is no entry $e(i,j)$ with the desired property and find a contradiction. If $e(1,2) \le e(2,2)$, then $e(1,2)$ has the desired property, so we must have

$$e(2,2) < e(1,2). \tag{3}$$

If $e(2,2) \ge e(2,1)$, then $e(2,2)$ has the desired property, so we must have

$$e(2,2) < e(2,1). \tag{4}$$

Since all the p_i's and q_j's are positive, we can clear the denominators in each of inequalities (1) to (4) and obtain four equivalent inequalities (the one corresponding to (1), for example, being

$$a_1p_1 + a_1q_2 + b_1p_1 + b_1q_2 \le a_1p_1 + a_1q_1 + b_2p_1 + b_2q_1).$$

Finally, adding these four new inequalities yields an inequality equivalent to $0 < 0$, and we have our contradiction.

(b) We pick the smallest entry in each column and then pick the greatest among these. We may assume without loss of generality that the resulting entry is $e(1,1)$. We now assume that there is no entry $e(i,j)$ with the desired property and find a contradiction. Since $e(1,1)$ does not have the desired property, we must have

$$e(1,1) < e(1,j) \tag{5}$$

for some $j > 1$. Now let $e(i,j)$ be the smallest entry in the jth column. Then $e(i,j) \le e(1,1)$, so $i > 1$ and

$$e(i,j) < e(1,j). \tag{6}$$

Note that

$$e(1,1) \le e(i,1), \tag{7}$$

from which it follows that

$$e(i,j) \le e(i,1). \tag{8}$$

It is now easily verified that inequalities (5) to (8) yield the same contradiction as inequalities (1) to (4) in part (a).

*

The following problems published in this column in Vol. 7 (1981) are still open, and readers are urged to send in elegant solutions.

Pages 15-16, 36th Moscow Olympiad (1973): Nos. 1, 8, 9, 10, 11, 13, 15.

Page 42, Austrian-Polish Competition (1980): Nos. 6, 7, 9.

Page 43, Mersch Competition (1980): Nos. 2, 5.

Page 44, Mariehamn Competition (1980): Nos. 1, 4, 5.

Pages 46-47, Rumanian Olympiad (1978): 11th class, Nos. 1, 2, 3; 12th class, No. 2.

Pages 74-78, Rumanian Selection Tests (1978): First Selection Test, Nos. 2, 5, 6, 7, 8, 9; Second Selection Test, Nos. 1, 2, 3, 4; Third Selection Test, Nos. 1, 2, 3, 4, 5, 6, 7, 8; Fourth Selection Test, Nos. 1, 2, 3, 4, 6.

Pages 114-115, Bulgarian Competitions: Nos. B-1, B-4, B-6.

Page 143, Russian "Jewish" problems: Nos. J-26, J-30.

Pages 236-237, 1981 unused I.M.O. problems: Nos. 1, 13.

Pages 268-269, British Olympiad (1981): Nos. 1, 5, 6.

Editor's note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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NEW GEOMETRY PROGRAM FOR THE IBM PC

Named EUCLID, this new program for the IBM Personal Computer will do all possible Euclidean constructions and conics, except for extreme cases.

Keys F1 to F5 on the keyboard allow the user to plot points on the screen, join them with segments, extend segments or choose random points on them, plot circles, and compute points of intersection of lines and circles. These capabilities are sufficient to carry out virtually all straightedge-and-compass constructions.

Keys F6 and F7 bisect angles and construct perpendiculars, and Key F8 provides access to other time-savers: triangle analysis, including plotting of circumcircle, incircle, and excircles, if desired; instant polygons, regular or random, up to 15 vertices; the unique conic through any five distinct points; nine conic-plotting options for user-chosen foci, vertices, etc.; points of intersection of any two conics, or of a conic and a line.

All points are automatically labeled when input or computed, and Key F9 allows labels to be erased or changed. Key F10 lists in double-precision all labeled points.

The program EUCLID serves two purposes: students can practise (and teachers demonstrate) constructions; and students and teachers can enjoy making discoveries, e.g., Problems 1042 [1985: 146] and 1091* [on this page]. (For the latter problem, the EUCLID program has supported the conjecture on every choice of random triangles tried, always with 14 decimal-place agreement on the point of concurrence, so the theorem is very likely true.)

EUCLID runs on IBM Personal Computers with 256K and DOS 2.1 or later. To order the EUCLID disk and manual, send check for $37.95 or purchase order to University of Evansville Press, 1800 Lincoln Avenue, Evansville, Indiana 47714.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1091*. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let $A_1A_2A_3$ be a triangle and γ_i the excircle opposite A_i, $i = 1,2,3$. Apollonius knew how to construct the circle Γ internally tangent to the three excircles and encompassing them. Let B_i be the point of contact of Γ and γ_i, $i = 1,2,3$. Prove that the lines A_1B_1, A_2B_2, and A_3B_3 are concurrent.

1092. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n be a natural number. Show that, for all natural numbers m, the sum

$$S_m = \sum_{k=1}^{n} \left(\sec \frac{2k\pi}{2n+1}\right)^m$$

is an integer.

1093*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Prove that

$$\left\{\frac{\Sigma \sin A}{\Sigma \cos (A/2)}\right\}^3 \geq 8\Pi \sin \frac{A}{2},$$

where the sums and product are cyclic over the angles A,B,C of a triangle. When does equality occur?

1094. *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

A skew quadrilateral consists of two triangular surfaces ABC and ABD with common edge AB. Prove that

$$\text{dihedral angle with edge AB} = \text{Arccos}\left(\frac{\tan S}{\tan C}\right) + \text{Arccos}\left(\frac{\tan S}{\tan D}\right),$$

where S is half the central angle of chord AB in the circumsphere ACBD.

1095. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $N_n = \{1,2,\ldots,n\}$, where $n \geq 4$. A subset A of N_n with $|A| \geq 2$ is called an *RC-set* (relatively composite) if $(a,b) > 1$ for all $a,b \in A$. Let $f(n)$ be the maximum cardinality of all RC-sets A in N_n. Determine $f(n)$ and find all RC-sets in N_n of cardinality $f(n)$.

1096. *Proposed by M.S. Klamkin, University of Alberta.*

Determine the maximum and minimum values of

$$S \equiv \cos\frac{A}{4}\cos\frac{B}{4}\cos\frac{C}{4} + \sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4},$$

where A,B,C are the angles of a triangle. (No calculus, please!)

1097. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let AD,BE,CF be the angle bisectors and AM,BN,CP the medians of a triangle ABC. Prove that

$$\vec{AD}\cdot\vec{AM} + \vec{BE}\cdot\vec{BN} + \vec{CF}\cdot\vec{CP} = s^2,$$

where s is the semiperimeter.

1098. *Proposed by Jordi Dou, Barcelona, Spain.*

Characterize all trapezoids for which the circumscribed ellipse of minimal area is a circle.

1099. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find a solution to each of the Diophantine equations

$$x^2 - 726y^2 = 3 \quad \text{and} \quad x^2 - 363y^2 = -2$$

in which the integers x and y are both greater than 500.

1100. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with $C = 30°$, circumcentre O and incentre I. Points D and E are chosen on BC and AC, respectively, such that BD = AE = AB. Prove that DE = OI and DE ⊥ OI.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

971. [1984: 261] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the piscine alphametic

$$\text{FISH + FISH + FISH + ... + FISH = SHOAL.}$$

This SHOAL has 73 FISH.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Clearly, F = 1 and S = 7, 8, or 9. If S = 9, then I = 2, H = 4, and L = 2 (duplication). If S = 8, then I = 1 (duplication). Therefore S = 7, I = 0, H = 8, and we have the unique solution

$$1078 \cdot 73 = 78694.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; FRED A. MILLER, Elkins, West Virginia; GLEN E. MILLS, Maitland, Florida; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; JAMES WATSON, student, Erskine College, Due West, South Carolina; PETER WATSON-HURTHIG, Columbia College, Vancouver; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

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972. [1984: 261] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

(a) Prove that two equilateral triangles of unit side cannot be placed inside a unit square without overlapping.

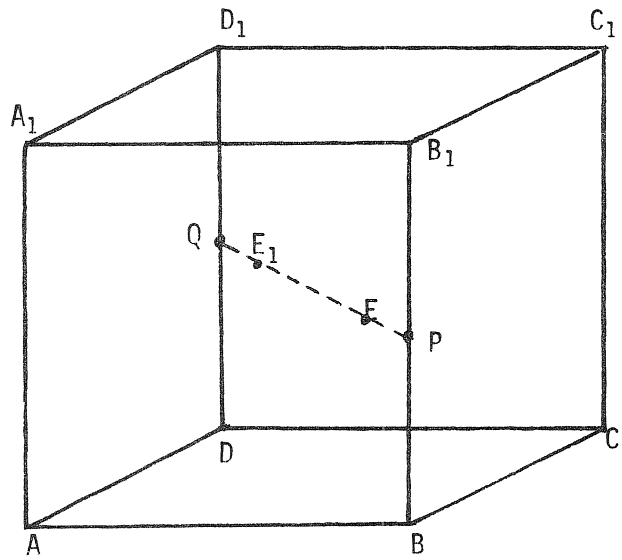
(b) What is the maximum number of regular tetrahedra of unit edge that can be packed without overlapping inside a unit cube?

(c) Generalize to higher dimensions.

Partial solution by Jordan B. Tabov, Sofia, Bulgaria.

(a) The *diameter* (least upper bound of the set of distances between pairs of points) of a unit square is $\sqrt{2}$, and it is easy to see that the diameter of the configuration consisting of two nonoverlapping equilateral triangles of unit side is at least $\sqrt{3}$. Since $\sqrt{3} > \sqrt{2}$, it follows that the statement in part (a) is true.

(b) At least two regular tetrahedra of unit edge can be packed without overlapping inside a unit cube. To see this, let $ABCDA_1B_1C_1D_1$ be a unit cube (see figure). P and Q being the midpoints of edges BB_1 and DD_1, respectively, choose points E and E_1 on PQ such that $PE = QF_1 = (\sqrt{2}-1)/2$. It is easy to verify that tetrahedra AA_1EE_1 and CC_1FE_1 are regular of unit edge, and they are placed without overlapping inside the unit cube.



Also partially solved by GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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973. [1984: 262] *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Evaluate $\lim_{n\to\infty} P_n$ if $P_1 = 4$ and

$$P_{n+1} = 2^{n+1}\sqrt{2}\sqrt{1-\sqrt{1-\left(\frac{P_n}{2^{n+1}}\right)^2}}, \qquad n = 1,2,3,\ldots .$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Via induction, it quickly follows that $P_n = 2^{n+1}\sin(\pi/2^n)$, $n = 1,2,3,\ldots$. Thus

$$\lim_{n\to\infty} P_n = 2\pi\cdot\lim_{n\to\infty}\frac{\sin(\pi/2^n)}{\pi/2^n} = 2\pi.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; KARL DILCHER, Dalhousie University, Halifax; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; PETER WATSON-HURTHIG, Columbia College, Vancouver; and the proposer.

Editor's comment.

Several solvers showed that P_n is the perimeter of a regular 2^n-gon inscribed in a circle of unit radius, and the required limit follows at once.

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974. [1984: 262] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the following double inequality, where A,B,C are the angles of any triangle:

$$\cos A \cos B \cos C \le 8 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} \le \frac{1}{8}.$$

The inequality involving the first and third members and that involving the second and third members are both well known. Prove the inequality involving the first and second members.

Comment by M.S. Klamkin, University of Alberta.

The proposed inequality is equivalent to

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) - \cos A \cos B \cos C \ge 0,$$

an inequality already established in Problem 836 [1984: 228].

Also solved by LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; ROLAND H. EDDY, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; and the proposer.

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975. [1984: 262] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Parabolas $y^2 = 2px$ and $x^2 = 2qy$ are given. A triangle with vertices $P_i(x_i,y_i)$, $i = 1,2,3$, is inscribed in $y^2 = 2px$ and the lines containing two of its sides are tangent to $x^2 = 2qy$.

(a) Prove that $2p^2q = y_1y_2y_3$.

(b) Deduce from (a), or otherwise, that the line containing the third side of the triangle is also tangent to $x^2 = 2qy$.

I. *Solution by Gali Salvatore, Perkins, Québec.*

(a) Since two of the lines P_2P_3, P_3P_1, and P_1P_2 are tangent to $x^2 = 2qy$, we may assume that $P_i \ne (0,0)$ for $i = 1,2,3$ and that the two tangents are both nonvertical (and so each has a slope). We represent the parabola $y^2 = 2px$ parametrically by $x = 2pt^2$, $y = 2pt$, so

$$P_i = (2pt_i^2,\ 2pt_i), \qquad i = 1,2,3,$$

where t_1,t_2,t_3 are distinct nonzero real numbers. We note that the nonvertical line

$y = mx + b$ is tangent to $x^2 = 2qy$ if and only if

$$qm^2 + 2b = 0. \tag{1}$$

The equation of P_1P_2 is

$$x - (t_1+t_2)y + 2pt_1t_2 = 0.$$

Hence, using (1), P_1P_2 is tangent to $x^2 = 2qy$ if and only if $t_1+t_2 \neq 0$ and

$$q + 4pt_1t_2(t_1 + t_2) = 0. \tag{2}$$

Similarly, P_1P_3 is tangent to $x^2 = 2qy$ if and only if $t_1+t_3 \neq 0$ and

$$q + 4pt_1t_3(t_1 + t_3) = 0. \tag{3}$$

Subtracting (2) and (3) yields

$$4pt_1(t_2 - t_3)(t_1 + t_2 + t_3) = 0,$$

or, since $4pt_1(t_2 - t_3) \neq 0$,

$$t_1 + t_2 + t_3 = 0. \tag{4}$$

From (4) and (2) or (3), we get

$$q = 4pt_1t_2t_3, \tag{5}$$

so

$$2p^2q = 2pt_1 \cdot 2pt_2 \cdot 2pt_3 = y_1y_2y_3.$$

Note that (4) and (5) must both hold if P_1P_2 and P_1P_3 are both tangent to $x^2 = 2qy$.

(b) We assume that P_1P_2 and P_1P_3 are both tangent to $x^2 = 2qy$, so (4) and (5) both hold. The line P_2P_3 is tangent to $x^2 = 2qy$ if and only if $t_2+t_3 \neq 0$, which is guaranteed by (4) since $t_1 \neq 0$, and

$$q + 4pt_2t_3(t_2 + t_3) = 0,$$

which follows from (5) and (4).

II. *Joint comment on part (b) by J. Chris Fisher, University of Regina; and Michal Szurek, (Visiting Professor from) University of Warsaw.*

This is a special case of a result known as *Poncelet's Porism* [7, Vol. 1, p. 316]:

Two conics C, C' and a point $P_0 \in C$ exterior to C' are given. Define P_1 to be the point where one of the tangents from P_0 to C' meets C, and P_j ($j \geq 2$) to be the point (different from P_{j-2}) where the tangent to C' from P_{j-1} meets C. If $P_n = P_0$ for some n, then the polygon closes with n vertices from any starting point P_0 of C.

There are many proofs of this porism [1, Vol. 2, p. 55; Vol. 5, p. 65], [2], [3], [5, p. 55], [6], [7]. The pair of parabolas given in the problem satisfy the hypothesis with $n = 3$. One way to see this is to consider the degenerate triangle with P_0 at the origin and $P_1 = P_2$ where the x-axis meets the line at infinity.

An analytic approach (due to Cayley [2], [3]) is outlined in [8, Articles 370 and 376]. The condition that $n = 3$ for the two conics

$$C:\quad ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and

$$C':\ a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0$$

is that $\theta = 4\Delta\theta'$, where

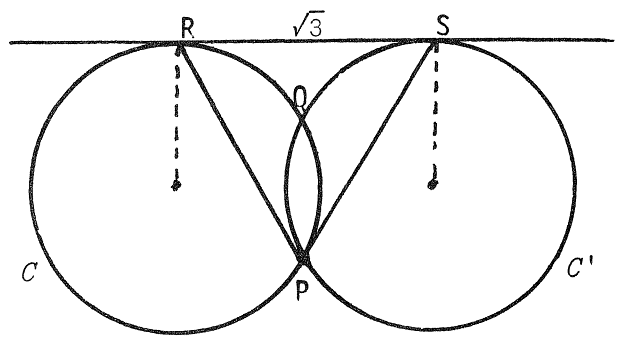
$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2,$$

$$\theta = (bc-f^2)a' + (ca-g^2)b' + (ab-h^2)c' + 2(gh-af)f' + 2(hf-bg)g' + 2(fg-ch)h',$$

and θ' is obtained from θ by interchanging the primed and unprimed letters. In the given problem $\theta = \theta' = 0$, so the condition is satisfied.

Readers might be interested in other special cases that are more or less elementary.

(i) If C is the circumcircle of a given triangle and C' is either the incircle or an excircle, then any other triangle inscribed in C that has two sides tangent to C' will also have its third side tangent. (This is an immediate consequence of Euler's formula $d^2 = R^2 \pm 2Rr$.)



Note that the original problem with parabolas is projectively equivalent to the case of unit circles whose centres are separated by a distance of $d = \sqrt{3}$. This may be seen as follows: Call the points of intersection of the two circles P and Q, and let R and S be the points where the tangents to each circle at P meet the other circle. One shows that RS is a common tangent to the circles (as in the figure). Let ϕ be the projectivity that takes P to the origin, R and S to the points at infinity of the x- and y-axes, and Q to the second point of intersection of the two parabolas. The proof that ϕ takes the circles to the parabolas is completed by invoking the theorem that there is exactly one conic tangent to a pair of lines at specified points that passes through a point not on either line.

(ii) The case when C and C' are nonintersecting circles and $n \geq 3$. (An inversion that takes C and C' to concentric circles takes the polygon to a chain of n evenly spaced congruent circles passing through the common centre.)

(iii) If two triangles have six distinct vertices, all lying on a conic, there is a polarity for which both triangles are self-polar. Conversely, if two triangles, with no vertex of either on a side of the other, are self-polar for a given polarity, then their six vertices lie on a conic and their six sides touch another conic [4, Theorems 9.41 and 9.42].

Also solved by S.C. CHAN, Singapore; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

REFERENCES

1. Henry F. Baker, *Principles of Geometry*, Frederick Ungar, New York, 1968.

2. Arthur Cayley, *Philosophical Mag.*, 6 (1853) 99-102.

3. ________, *Philosophical Trans. Royal Soc. London*, 151 (1861) 225-239.

4. H.S.M. Coxeter, *Projective Geometry*, Blaisdell, 1964 (or University of Toronto Press, 1974).

5. H.G. Forder, *Geometry, An Introduction*, Harper Torchbooks, New York, 1962.

6. Phillip Griffiths and Joseph Harris, "On Cayley's explicit solution to Poncelet's porism", *L'Enseignement Math.*, 24 (1978) 31-40.

7. J.V. Poncelet, *Traité des propriétés projectives des figures*, Mett-Paris, 1865.

8. George Salmon, *A Treatise on Conic Sections*, 6th ed., Chelsea, New York, 1954.

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977. [1984: 262] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $f(n) = 2n^2 + 14n + 25$. It is easy to verify that $f(17) = 29^2$. Find two more positive integers n such that $f(n)$ is a perfect square.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

We will show that there are, in fact, infinitely many such integers. We seek nonnegative integers n such that

$$f(n) \equiv (n+3)^2 + (n+4)^2 = p^2$$

for some positive integer p. Thus the problem is equivalent to the well-known problem of finding Pythagorean triangles whose legs differ by unity. This problem is fully discussed in Beiler [1]. The first six solutions are given in the adjoining table. Additional solutions can be obtained from

n	$f(n) = p^2$
0	5^2
17	29^2
116	169^2
693	985^2
4056	5741^2
23657	33461^2

$$n_i = 6n_{i-1} - n_{i-2} + 14 \quad \text{and} \quad p_i = 6p_{i-1} - p_{i-2}.$$

But why bother? Beiler has provided us with a table of the first 100 solutions, the last of which corresponds to an n with 77 digits.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antoanio, Texas; S.C. CHAN, Singapore; CURTIS COOPER, Central Missouri State University, Warrensburg; KARL DILCHER, Dalhousie University, Halifax; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California (two solutions); FRED A. MILLER, Elkins, West Virginia; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; DAVID R. STONE, Georgia Southern College, Statesboro; CHARLES W. TRIGG, San Diego, California; PETER WATSON-HURTHIG, Columbia College, Vancouver; and the proposer (two solutions). Two incorrect solutions were received.

REFERENCE

1. Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover, New York, 1964, pp. 122-124 and Table 103, pp. 328-329.

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979. [1984: 263] *Proposed by Andy Liu, University of Alberta.*

Determine the smallest positive integer m such that

$$529^n + m\cdot 132^n$$

is divisible by 262417 for all odd positive integers n.

I. *Solution by Curtis Cooper, Central Missouri State University, Warrensburg.*

It is clear from

$$529 + 1984\cdot 132 = 262417 \tag{1}$$

that no $m < 1984$ will do. Moreover, it follows from (1) that

$$529^n + 1984\cdot 132^n \equiv 0 \pmod{262417} \tag{2}$$

holds for $n = 1$, and from $529^2 \equiv 132^2 \pmod{262417}$ and an easy induction that (2) holds for all odd positive integers n. Thus the required number is $m = 1984$.

II. *Solution by Peter Watson-Hurthig, Columbia College, Vancouver.*

It follows from

$$a^{2k+1} + mb^{2k+1} = a(a^{2k} - b^{2k}) + b^{2k}(a + mb)$$

that a^2-b^2 divides $a^{2k+1} + b^{2k+1}$ for any nonnegative integer k if and only if it divides $b^{2k}(a + mb)$. With $a = 529$ and $b = 132$, we have

$$a^2 - b^2 = (a + b)(a - b) = 661\cdot 397 = 262417.$$

Thus 262417 divides $529^{2k+1} + m\cdot 132^{2k+1}$ for any nonnegative integer k if and only if it divides $132^{2k}(529 + 132m)$. Since 262417 is relatively prime to 132, the necessary and sufficient condition is that 262417 divide $529 + 132m$. The smallest m for which this occurs is

$$m = \frac{262417 - 529}{132} = 1984.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; KARL DILCHER, Dalhousie University, Halifax; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austra; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; ALEKSANDAR ZOROVIĆ, Elizabethtown High School, Kentucky; and the proposer.

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980. [1984: 263] *Proposed by Leon Bankoff, Los Angeles, California.*

Show that

$$\frac{\Pi \sin A}{\Sigma \sin A} + \Sigma \sin^2\frac{A}{2} = 1,$$

where the sums and product are cyclic over the angles A,B,C of a triangle.

Composite of the solutions of Blundon, Eddy, Garfunkel, Janous, Klamkin, and Murty (who are further identified below).

The proposed equality follows immediately from the following known results, where R,r,s have their usual meanings:

$$\Pi \sin A = \frac{rs}{2R^2}, \quad \Sigma \sin A = \frac{s}{R}, \quad \Sigma \sin^2\frac{A}{2} = 1 - \frac{r}{2R}.$$

Solutions were received from W.J. BLUNDON, Memorial University of Newfoundland; DUANE BROLINE, University of Evansville, Indiana; ROLAND H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; FRED A. MILLER, Elkins, West Virginia; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; PETER WATSON-HURTHIG, Columbia College, Vancouver; ALEKSANDAR ZOROVIĆ, Elizabethtown High School, Kentucky; and the proposer.

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981. [1984: 291] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the doubly true multiplication

$$5 \cdot \text{ELEVEN} = \text{FIFTY5}$$

where FIVE is, of course, prime.

Solution by Edwin M. Klein, University of Wisconsin-Whitewater.

Clearly F = 1 and {T,I} = {0,5}. Since L and V produce the same carry when multiplied by 5, we must have

$$\{L,V\} = \{2,3\},\quad \{6,7\},\quad \text{or}\quad \{8,9\};$$

but the last possibility must be excluded because it leads to the duplication F = 9. Now

$$L = 2 \implies I = 0,\ T = 5,\ F = 6,\ V = 3 \implies \text{FIVE} = 6031 = 37\cdot 163;$$
$$L = 6 \implies I = 0,\ T = 5,\ F = 8,\ V = 7 \implies \text{FIVE} = 8071 = 7\cdot 1153;$$
$$L = 7 \implies I = 5,\ T = 0,\ F = 8,\ V = 6 \implies \text{FIVE} = 8561 = 7\cdot 1223.$$

Therefore L = 3, I = 5, T = 0, F = 6, V = 2, so FIVE = 6521, a prime, and then N = 7, Y = 8. The unique reconstruction is

$$5\cdot 131217 = 656085.$$

Also solved by JAMES BOWE, Erskine College, Due West, South Carolina; KENNETH ALLAN BRADLEY, student, The Ohio State University; MARGARET A. BROWNING, student, The Ohio State University; J. ALAN BURKHOLDER, student, The Ohio State University; DAVID J. COLEMAN, student, The Ohio State University; RICHARD I. HESS, Rancho Palos Verdes, California; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts; LORI KRAUS, student, The Ohio State University; KATHI MAUK, student, The Ohio State University; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Maitland, Florida; MARJORIE ANNE NELSON, student, The Ohio State University; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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ARTICLES AND NOTES

PROPOSALS AND SOLUTIONS

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PROPOSERS AND SOLVERS

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