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THE C.C. MACDUFFEE AWARD FOR DISTINGUISHED SERVICE

The C.C. *MacDuffee Award* is Pi Mu Epsilon's highest recognition of distinguished service. It has been presented four times during the fifty-eight year history of our fraternity. The prior recipients are:

Dr. J. Southerland Frame
Dr. Richard V. Andree
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Dr. Francis Regan

In 1972 the C.C. *MacDuffee Award for Distinguished Service* was awarded to Dr. J.C. Eaves. All of you know Dr. Eaves as the President of Pi Mu Epsilon (1966-1972). Adding Dr. Eaves to an organization is like adding yeast to a mixture. Either introduces life, promotes growth, and produces a final product with superior zest. Dr. Eaves has worked diligently and imaginatively in "promoting scholarship and mathematics" as our motto says. The number of ΠΜΕ chapters grew from 118 in 1966 to 175 in 1972 under his energetic guidance. Most of these 57 additional chapters were personally installed by Dr. Eaves -- who can relate some interesting stories concerning travel to and from various colleges and universities.

Dr. Eaves is an excellent speaker, an inspiring leader and a congenial companion in addition to being a devoted mathematician. He promotes mathematics whenever the opportunity arises, and serves the needs of mathematics at all levels -- he has served as president of Mu Alpha Theta, the national high school and junior college club, which was started in 1957 with the help of Pi Mu Epsilon. He has served as chairman of a large active university graduate program in mathematics. He has served as President of the most prestigious and active honor society in America -- Pi Mu Epsilon.

Dr. Eaves is always a welcome committee member on any project. If Dr. J.C. Eaves undertakes a job, you can be sure it will be done with a flourish, and that those who are associated with the project will enjoy and appreciate this sly humor as well as his uncanny ability to come up with feasible solutions rather than additional problems. Pi Mu Epsilon is indeed fortunate to have been included among the many projects that have been more successful because the Eaves yeast was included. It is altogether fitting and proper that Dr. Eaves' name be added to the list of illustrious recipients of Pi Mu Epsilon's highest award.

Presented at
Dartmouth College
August, 1972

J. C. Eaves

MATHEMATICAL MODELS AND THE COMPUTER[1]

by *John G. Kemeny*
President, Dartmouth College

This evening I would like to speculate on the likely impact computers will have on modeling in the coming decades. I am convinced that we are just beginning to explore this field and that there is an enormous amount we still must learn about the full power of the computer in this area.

Let me start by commenting briefly on the traditional mathematical models, those in physics. A simplified impression of the role of mathematics in science is that the physical scientist observes nature, discovers a number of facts, then formulates a model of physical reality which is usually expressed as a mathematical model. It is left to the mathematician to solve the equations of this model, draw conclusions from them, and derive answers which tell the physicist something about the future of the world or about how to apply his theory to practical situations.

As mathematicians, we are spoiled because some of the best known models of classical physics lead to closed analytic solutions. Therefore, we have a tendency to oversimplify the role of mathematics in analyzing mathematical models.

Perhaps one extreme example is Einstein's Unified Field Theory. As far as I know, the equations still have not been solved and, therefore, we do not know what Einstein's last theory says about the physical world. There is no way of evaluating whether the theory has any value or not.

At the other extreme, one says that Newton's laws lead to analytic solutions which completely determine motion in practical situations. But take as simple a problem as a rocket trip to the moon. Although the equations are known and up to a certain point you have analytic solutions, you eventually run into an elliptic integral, which can't be integrated in closed form. Therefore, even at the point where the mathematician says he has reached a solution, he has

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[1] This article is the text of an invited address by Dr. Kemeny which was given at the annual meeting of Pi Mu Epsilon at Dartmouth College, August 1972.

not really found the solution in practical terms. At this point he must turn to the computer to plot the rocket trip to the moon.

I am going to suggest that situations which are exceptional in the physical sciences may turn out to be the norm in the social sciences. Let me illustrate this with a very simple example: the population explosion.

On a simplified scale, we have no difficulty in dealing with the problem of a rapidly growing population. It is a standard exercise in calculus to assume that the rate of growth is proportional to the number of people present, express this as a differential equation, and find that the solution is the initial population times e^{kt}, k being a measure of how fast things are growing. It's very beautiful; it's even useful for long-range global predictions, but I submit that it has little to do with the problems we are facing at the present time.

Suppose you wanted to know something about the population of the world 20 years from now, and the next 20 years are critical ones. Or suppose you are interested in how many college-age students there will be 20 years from now. The nice analytic solution is totally irrelevant, because it is an equilibrium solution, and we do not have an equilibrium.

If you look at the age distribution of the population of the United States -- which in equilibrium should be nearly a straight line, which dips to zero beyond some age -- you find that the distribution is not even approximately like that. It has very significant "wiggles" in it. If we use the 1970 census and look at the people in the 30 to 40 year age group, we find a substantial dip -- the population is much smaller than it should be. If you look back in American History, this dip corresponds to the period of the Depression when people apparently decided to have fewer children.

At another point on the line there is a "bump" that corresponds to the post-World War II baby boom. Thus the nice smooth solution simply does not correspond to reality. It is off by as much as five million people in a 10 year age group and therefore the solution is not even approximately correct. The trouble with these "wiggles" is that they tend to propagate. One speaks of birth rates and death rates as though they were averages over the entire population, but this is not true. It is mostly people within a narrow age range who

have children, and it is mostly people at the high end of the age scale who die. Therefore, rates of birth and death averaged over the total population are misleading.

You have heard considerable publicity that even if Zero Population Growth (ZPG) is completely effective, as people decide to have only enough children so that in the long run the population becomes stable, it will take something of the order of magnitude of 60 years to achieve stability in the population. The reason for this is that those in the large post-War baby boom group are about to become child-bearing parents. Even if this group has only the "right" number of children, this generation is still going to have many more children than their parents' generation and we must wait until the transient effects die down.

Building all these elements into an analytic solution is difficult and not worth the effort. Yet it's an ideal problem for a computer model. We can feed into the computer the actual rather than a hypothetical distribution, use actual data about what age people are most likely to have children, and provide data on the death rates for various age groups. We can make further assumptions about the behavior of the next generation of child-bearing parents and about the likely effect on the death rate of improvements in modern medicine. And we can try out, under various assumptions, what the actual population will be 20, 30, or 50 years from now.

This is an excellent example of where a mathematical model turns into a computer model, and if one looks closely at the behavior of the model, one finds out why oversimplified discussions are very dangerous.

For example, one sometimes hears statements about the increase and decrease of the birth rate. Demographers tell us it is the child-bearing rate that counts, and if you look only at birth rates from one decade to the next, they are terribly midleading, as are death rates. At the moment, the child-bearing age group is still, more or less, from the Depression "dip" and, as such, is relatively small. If not as many children are being born, one therefore must ask how much of this is due to the smaller population in the child-bearing age group? On the other hand, a large bulge will soon be coming through the population wave. Therefore, even if this group has fewer children than their predecessors, it is quite possible that the national birth rate will shoot up.

Similarly, there has been considerable publicity given to the decline of the death rate. Here again, one must be careful to determine whether this decrease is due to better medicines and other factors enabling people to live longer, or whether it is a result of the very large number of young people in the United States who as a group are not likely to die for some time to come.

The analysis of the growth in population is the simplest example I can think of in which a computer model is crucial. Assuming you have properly constructed the computer model you will observe as you run it that the initial results suggest new questions you might not have considered if you were immersed in a beautiful, long-range analytic solution. One of the spectacular things you notice as you run the population model through a hundred years is the change in the percentage of people over age 65. A by-product of ZPG is a significant increase in the percentage of people over age 65. To some extent we would be noticing this now, but the increased longevity of people has been balanced in the last 20 years by a large birth rate, which is attributable to the post-War baby boom. Put simply, while the numerator of older people has increased, the denominator -- or total population -- has kept pace, more or less, and thus the change is not as noticeable.

If we strive for ZPG by controlling the number of young people at some reasonable level, the bulge in the system moves on to older age. You will find that while today the percentage of people over age 65 is under 10 per cent -- which is unusually high -- you will eventually have as much as 15 per cent of the total population 65 years or older. If that does indeed occur, it will make a qualitative difference in the way our society lives. If you are-planning for zero population growth -- as many people are advocating -- you must come up with entirely new plans as to what society will do when more than one out of seven people are in the retired age group.

Now let me turn to a discussion of specific computer models. I will start by mentioning briefly the work of the MIT Group under the leadership of Professor Jay Forrester, ably assisted by Professors Donella and Dennis Meadows, both of whom are now at Dartmouth College. Jay Forrester pioneered in the construction of sophisticated large scale computer models for social problems. His approach is the opposite of trying to solve all problems analytically. He by-passed the stage of the ordinary mathematical model by building his models directly inside the computer.

His models are extremely ambitious, whether he is modeling a large company, a large city, or all the problems of the world. I'm quite sure that, as with any initial model, the model is not perfect. I think he would certainly admit some imperfections, but whether the model is quite accurate is not so important. He is pioneering in an important new approach to modeling of complicated systems, an approach I think deserves an enormous amount of study. Even if his models are not quite right -- and usually the first models in science turn out to be wrong -- he has discovered some qualitative behavior that is fascinating.

I will concentrate on one aspect in particular which Forrester describes by saying that his models in the social sciences are "counter-intuitive." I can see from his examples what he means by the models being "counter-intuitive," yet the phrase bothered me at first. I couldn't see how a model could be inherently "counter-intuitive," so I tried to analyze what his statement really meant. For better or worse, here is my reconstruction of it:

Intuition is not inborn; it is a matter of training. It is my belief that these models have "counter-intuitive" behavior because our intuition, particularly our mathematical intuition, has been trained on models that behave quite differently from his. And where has our intuition been trained? Certainly in applied mathematics it has been trained primarily in the physical sciences. Forrester, then, is saying that his models have features in them that make their behavior qualitatively different from those we have become accustomed to in analyzing physical problems; therefore our intuition, built upon physical mathematics, misleads us.

Some of the elements that bring this about are the following: In physics we have been spoiled by the fact that, although there are many complicated systems, we can isolate simple sub-systems which are relatively self-contained. By way of example, let us take the composition of the universe. If the physicist had to start out by building a model of all the stars in the universe, physics would never have been born. Fortunately, there are simple sub-systems available such as the sun and the planets. But even this system is still too difficult for physics to have gotten started on.

We are even more fortunate in that the effect on the sun is so strong compared with the effect of the other planets -- not to mention the other stars -- that you can pretend, for your first study, that the sun and one planet form a closed system. I would argue that there would have been little chance of developing classical mechanics if simple, isolatable models such as the sun and one planet did not exist. Forrester also makes the important observation -- and I think he is right with regard to the social sciences -- that you cannot take complex systems apart because the individual components are too heavily inter-related.

Another thing that happens in the physical sciences -- and this probably led to the discovery of both calculus and statistics -- is that often you have a highly homogeneous system, either in time or space, so you can pretend that you have infinitely divisible time, or infinitely many similar objects, and therefore you can use calculus or statistics to find the answers. Forrester would argue that one of the major problems with social systems is that while you may have a large number of pieces in the puzzle, they are not homogenous and the differences among them are sufficiently significant so that the use of statistics can give you highly misleading answers.

The next important observation is that with good luck, many of the systems in physics, biology and economics are linear. Whenever you have a linear system, life is much easier. Any time scientists can assume something is linear, they will intentionally close their eyes to non-linearities to avoid turning an easy problem into a hopeless one. While mathematics has made important contributions to the solutions of non-linear systems, these have usually been in cases where the behavior of the system can be studied (at least locally) through linear approximations. Forrester would argue that complex social systems are highly non-linear and that linear approximations would produce qualitatively different results. He attributes this to the presence of feedback systems, though this may be an oversimplification. I feel that this aspect of the Forrester models deserves careful study by mathematicians.

The final element is chance. Chance is essentially present in all social systems and this leads to the difficult question of how you build chance into your models. Probabilists and statisticians have been doing this for a long time, but it turns out that whenever you face a problem sufficiently complicated to be of real significance in the social sciences, the probabilist runs into the same problem as the analyst: he can solve the problem in principle, but he cannot solve it in practice.

A great deal has been said about the importance of a special kind of computer model known as "simulation," where you use the computer to act out what happens in nature. Simulation has one great advantage; it quickly gives you a good, rough feeling of what goes on. Yet simulation has its shortcomings and I will mention some of them shortly.

Let's take a concrete example of simulation. Suppose you are interested in the flow of traffic in a fairly large city and you would like to do something to improve it. It is entirely possible to write a large computer program, building into it the layout of the streets, the locations of the traffic lights and one-way streets, Some information on the flow of traffic and the peculiar habits of drivers. You would also want to include the chance element so as to produce the right number of accidents at the right places and at approximately the right times.

Then you ask the question: "What can we do to improve the flow of traffic?" Let's take Manhattan as an example where the traffic situation is miserable. Since I visit Manhattan frequently, I've tried to analyze what that city does about its traffic problem and have come to the following conclusion: Every once in awhile, somebody has a brain storm and they change the timing on the lights on a few streets or make some streets one-way and then they sit back and watch the system for six months to see whether traffic flows any better. Often it does not, or if it does the improvements are insufficient to handle the increased volume of traffic.

There is absolutely no reason why the same approach to traffic control cannot be formulated into a large-scale computer model simulating in the machine what actually occurs in the streets. You would have to write a program that simulates a traffic flow just as miserable as it is in Manhattan. Then you could try out the changes in traffic lights and turn your one-way streets around, letting the computer grind away for hours to play out' six months of experience to see if traffic flow improves. If a successful pattern isn't found the first day, change the traffic lights and one-way streets again and let the computer run some more. If you are lucky, you will eventually come up with a plan that looks significantly better and then implement it. While you will have used a good deal of computer power, you will not have used a million human beings as guinea pigs. This is an excellent use of the computer because it will give accurate enough answers to spot an order-of-magnitude of change.

The computer can be bad, however, in other kinds of simulation modeling. Let me give you an example of something I did once for amusement as an illustration for a book.

I was interested in determining whether the batting order in baseball really made a significant difference. I am familiar with the Hawthorne Effect, that everytime you change something, people perform better, and in that sense there is no doubt that shuffling the batting orders may make a difference, But I wanted to see whether there was a real difference in a probabilistic sense, in the order players batted. I wrote a little model, based on an old Brooklyn Dodgers lineup. I had the Dodgers play a full season and tried to see how many runs the team scored. I built in singles, doubles triples, home runs, walks and other factors, and then tried different batting orders to see what changes occurred. There were some changes, but when I tested them they turned out not to be statistically significant.

For the fun of it, I decided to run 10 seasons for several different batting orders. Again, I tested the results carefully and found very small but statistically significant differences. I eventually ran 70 entire seasons of batting orders and again came up with small differences. I can now tell you that if you do something that's obviously bad, like having the team bat in reverse order, the team will score slightly fewer runs in a season, But the difference is ridiculously small, and only shows up if you use the stupidest possible line-ups.

The above example illustrates one of the problems encountered when using simulation models, You do not get a good feeling as to what the variance is. The best method is to run the program 10 times to determine the differences between the largest and smallest outcome, then try to estimate what the variances are most likely to be. But it is a very shaky experience when you are working with a large, complicated simulation model. If you want to play it safe, accurate results will require an enormous amount of computing time, because you are substituting mere computing power for an idea, for some evaluation, of what really goes on inside the system. This is a graphic example of brute force substituted for intelligence, and as usual, it is not a very good trade-off.

Nevertheless, I strongly believe that simulation models and other computer models are coming. Indeed, I would like to refer to a phrase we are all bored with -- that mathematics is the language of science. I'm beginning to wonder that if mathematics is the language of physical science -- which it is -- is it not possible that computer programs will become the language of social science? I say this seriously because of the interesting experiences I've had on the Dartmouth campus.

You know how difficult it is for people from two different scientific specialities to communicate at all. As a matter of fact, two mathematicians often have enough trouble communicating that the thought of a physicist and a sociologist communicating gives one a hopeless feeling. They would probably need three months just to arrive at a common language.

But I have found that people from highly diverse fields are now able to communicate in the language of high-speed computers. Any one of the general languages can be used. At Dartmouth, the language commonly used happens to be BASIC, but it could be FORTRAN or ALGOL at other institutions. The use of computer languages as languages for modeling is interesting and intriguing; their use as such would certainly break down a great deal of the language barrier amongst the disciplines.

But you need more than a common language in order to build these models. You need reasonable access to a computer, and one of the sad facts in the United States today is that most institutions do not have reasonable access to a computer. This includes some of the largest and best known institutions which may think they have accessible computers, but do *not* in any reasonable sense. Aside from those institutions that don't yet have physical access to computers, there are many others who think they are making a computer readily available if a handful of people in those communities consume enormous amounts of computer time, usually at federal expense. This is not an acceptable criterion.

Computer modeling is an outstanding example of a situation in which an occasional shot at the computer, or a 20-minute turn-around time, is totally useless. One of the great break-throughs in time-sharing systems is the capacity for research scientists to converse with a computer. With a 20-minute turn-around time, however, you might as well have somebody else do your computing for you. You are never going to do the work for which computer models are ideal if you don't have the opportunity to sit at a terminal, vary the parameters of your data, watch the results come out, and, if the answers raise still more questions, to begin exploring their implications fully.

In the typical batch processing system, you would never even get your program debugged because the models I am describing are sufficiently complicated that you need at least 100 runs before they work properly. If you have to wait half a day before you get another pass at it, you are going to be drawing retirement pay before the program is debugged. But beyond the obvious advantage of a time-sharing system, the fact that you literally work with the system in the same way you work with a mathematical model with paper and pencil, makes a time-sharing system absolutely crucial for the development of computer models.

Let me try to give some evaluation of the role of computers in this area. In any modeling, an obvious advantage of the computer is that it is fast, accurate and cheap. That may sound strange coming from a college president because most of us complain about how expensive computers are. But the work computers do in a given amount of time makes them incredibly cheap and they are getting cheaper every day. Secondly, it is much easier to mite a computer program than to design a good mathematical model in closed form. This has both its advantages and disadvantages. It is too tempting for many amateurs to get into the act of writing models. Nevertheless, for

those who work in this field, it is a lot easier than forming mathematical models. Still another advantage is that while in ordinary modeling you must work simultaneously in formulating the model and solving the equations, once you have devised a computer model, you can tell the computer to do the rest of the work and the machine results will give you a good feeling of how the model behaves.

These are some of the advantages, and yet computer models have a number of disadvantages. First of all, I believe all mathematicians and scientists work with a trial and error system, whether they admit it publicly or not. On the surface, the trial and error system appears ideally suited for the computer. You can mite the model and provide a large number of parameters, then ask the computer to run through all of the possible variations and tell you which variation is best. There is, however, one major catch in this novel approach: how will the computer know what 'best' means in this particular case? While the computer is capable of generating enormous amounts of data, if you are not careful, especially in a batch processing system, you are likely to get stacks of information that still need many hours of evaluation before you can arrive at a final answer.

It would be a major breakthrough in the art of computer programming if we could develop sound techniques for teaching computers how to evaluate their results. The reason we are unable to develop such techniques is because we don't exactly know how to evaluate our own work. Therefore, the trial and error methods can often be frustrating.

I can best illustrate this fact by telling a story on myself. In a course I taught last spring we focused on one of Jay Forrester's world models and discussed ways to evaluate how well off the world will be in 2100 for various values of the parameters used by Forrester. There were a great number of strategies and ways to change parameters such as population growth, pollution control, and resource depletion. What we wanted to do was try out a wide range of parameter values to see which set would produce the best possible life on earth by 2100.

Obviously, obtaining the answers would require a tremendous number of calculations, but the class insisted on trying out hundreds of variations, each requiring a computer run of the model from 1970 to 2100, This exercise was going to be the grand finale of the course, so we wrote the program and made the runs, changing the parameters each time and letting the computer keep track of the payoff function in terms of quality of life in the world. Finally the computer produced the best solution. There was a unique optimum in it, and I was happy, until I started studying the solution. The solution provided by the computer had one very interesting property: there were practically no people left.

So back we went to the computer to work through that one set of parameters to see step by step what actually happened in that particular simulation. The most horrible things you can imagine began to occur during the next 100 years -- population explosions, famines, and unparalleled starvation. As a result, almost all of the people in the world were wiped out. We wound up with a situation where a handful of survivors had all the resources on earth and no means to generate pollution. In a sense, it was a second Garden of Eden, but it's not quite the solution we had in mind when we began.

This illustration is a perfect example of what a computer model can do if the computer has no way of knowing which solution or set of conditions is best to achieve the goals we are seeking. But I did protect against this to some degree by re-running the whole program and saying to the computer, "Incidentally, if along the way the population gets wiped out, forget it."

In the end, however, this question still lingers: "What is it that we missed in the automatic recipe we gave the computer for evaluating solutions that would not have been overlooked if we had seen these thousands of printouts and had been able to follow the computations in detail?" Trial and error is dangerous if we have only crude ways of telling the computer how to evaluate success.

A second observation is that computers seem to be very poor at combinatorics. The machines are no better at combinatorics today than they were before. Computers may be faster so you can use them for combinatorial problems, but they still do miserably.

A third observation is that in spite of some spectacular success in artificial intelligence, computers are, in my opinion, still poor at pattern recognition. Here's my favorite example of this fact. Laymen are always amazed if you tell them the computer can do differentiation for you. (I don't mean numerical differentiation, but closed-form differentiation), because many people think computers can't work with formulas. Computers can and do work with formulas. To teach the machine everything you would teach in freshman calculus about differentiation would result in a large but not terribly difficult program. Obviously you will want the computer to do what the student in freshman calculus must: find the derivative and simplify it. The chain rule, which is a big stumbling block for students, is perfect for computers. Everything is fine until you ask the computer to simplify the answer, and here the machine does miserably. While it may take the average freshman several minutes to find the derivative, the computer can do it in a negligible amount of time. On the other hand, the simplification that even the average freshman can do by merely looking at the answer may take the computer many times longer than the differentiation took. What is missing is the ability to tell the computer "Now just look at the answer and you'll see what you have to do."

Finally, because computers are so fast and models so complicated, I suspect it is much harder to find errors in the model. If Newton, in his Law of Gravitation, had accidentally cubed rather than squared the distance, it would have been immediately noticeable. I have a feeling that errors of that order of magnitude can sneak by you in a computer model and it may take a long time to find them.

So what do I conclude from this discussion of mathematical models and the computer? First, computer models are a crucial new tool for the social sciences and for attacking the problems of society. Second, mathematicians and others well trained in mathematics are particularly good at formulating social science models. I want to avoid the argument of whether these models are mathematics or not, for I believe that in 50 years they will be recognized as standard mathematics. The same might be said of statistics, but, to avoid argumentation, let me simply say that like statistics, computer modeling is a field in which people with strong mathematical training do significantly better than those without such training.

But unlike statistics, we presently face one major hurdle with regard to computer models, that we don't have a general theory. We need a theory of computer models, just as we have theories for the behavior of systems of differential equations or for statistical systems. Without such theories, formulating sound computer models is too much a hit or miss method.

Therefore, let me close by recommending to those of you who are looking for a fruitful career in the coming decades to give serious study to the formulation of mathematical models, particularly computer models in the social sciences, for they have an important future role to play.

REFEREES FOR THIS ISSUE

The editorial staff sends a note of appreciation to the following persons who freely offered their time to evaluate papers submitted for publication prior to this issue: Robert C. Bueker, Western Kentucky University; Joseph Lehner, University of Pittsburgh; Kenneth Loewen, Norman, Oklahoma; Robert Strum, Neptune, New Jersey; Eugene W. Womble, Presbyterian College (South Carolina); and, members of the Mathematics Department at the University of Oklahoma, R.V. Andree, David Drennan, Stanley B. Eliason, Andy Magid, Bernard R. McDonald, and Kirby Smith.

CHARACTERIZATION OF AN ANALYTIC FUNCTION OF A QUATERNION VARIABLE

By Joseph J. Buff
New York University

In this brief paper we consider an attempt to define an analytic quaternion valued function of a quaternion variable. Fundamental to the theory of complex variables is the idea of an analytic function, a complex valued function of a complex variable which is differentiable on some region. Can we, and if so how shall we, extend this idea to quaternions?

A quaternion, of course, is a number of the form

$$X_1 1 + X_2 i + X_3 j + X_4 k, \tag{1}$$

where $X_1, \ldots, X_4$ are real "coefficients" and $1, i, j,$ and k are quantities multiplying according to the table

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

and otherwise obeying the laws of polynomial arithmetic. Now, we may consider a quaternion variable Q to be expressible in the convenient form of (1), and we may begin to study a quaternion valued, single valued function of Q which may be written in the form

$$A_1 1 + A_2 i + A_3 j + A_4 k \tag{2}$$

where again $A_1, \ldots, A_4$ are real coefficients, and in particular, each is itself a real valued, single valued function of the four real variables $X_1, \ldots, X_4$. Hence, should we have a function of this type we may write it in the informative form

$$F(Q) = A_1(X_1,\ldots,X_4)1 + A_2(X_1,\ldots,X_4)i + A_3(X_1,\ldots,X_4)j + A_4(X_1,\ldots,X_4)k,$$

exactly as is the case in the elementary theory of complex variables. Now, for clarity, we mention that in the future a *function*, written $F(Q)$, will mean a function of this type, a quaternion valued function of a quaternion variable.

Before we consider limits and derivatives, we need a topology on the space of quaternions, so we shall take an open set in quaternion space to mean simply an open set according to E^4 (euclidean space of dimension 4). Thus Q-space, being in effect E^4, is now a metric space, and we have no problem understanding the meaning of limit. In the future we shall take our function $F(Q)$ as being defined on some open, connected set, or region, of E^4.

We may now get down to business and try to define an analytic function. A reasonable definition of a derivative must involve the famous difference quotient $\Delta F/\Delta Q$. Note, however, that in Q-space since multiplication is not commutative, we must consider two distinct quantities when speaking of difference quotients, these being

$$\Delta F \cdot \frac{1}{\Delta Q} \qquad \text{and} \qquad \frac{1}{\Delta Q} \cdot \Delta F,$$

the two generally not being equal. We shall then speak of two derivatives: A right derivative shall mean

$$\lim_{\Delta Q \to 0} \Delta F \cdot \frac{1}{\Delta Q}$$

and a left derivative shall mean

$$\lim_{\Delta Q \to 0} \frac{1}{\Delta Q} \cdot \Delta F,$$

presuming the limits exist. A function shall be called right or left differentiable at a point when the appropriate derivative exists, and right or left analytic in a region when it is right or left differentiable throughout that region.

Now, we have encountered what seems to be a difficulty in defining a derivative because the two derivatives might not be the same. However, we see at once that when the two derivatives exist in a region they must be equal. For, if F is right differentiable we may let ΔQ vanish along the X_1 axis and find

$$\lim_{\Delta Q \to 0} \Delta F \cdot \frac{1}{\Delta Q} = \lim_{\Delta X_1 \to 0} \Delta F \cdot \frac{1}{\Delta X_1}$$

which limit exists. If F is left differentiable we also have

$$\lim_{\Delta Q \to 0} \frac{1}{\Delta Q} \cdot \Delta F = \lim_{\Delta X_1 \to 0} \frac{1}{\Delta X_1} \cdot \Delta F$$

which limit exists, but ΔX_1 being real, the two limits are equal.

Definition: A (single valued) quaternion valued function of a quaternion variable is said to be ***analytic*** on a region if it is both right and left differentiable on that region.

The derivative of an analytic function may also be defined.

Definition: The ***derivative*** of a function that is analytic on a region is the common value of its right and left derivative throughout that region.

We shall now introduce a restriction on the functions we are considering.

Restriction 1: The functions A_1, A_2, A_3, and A_4 are differentiable with respect to X_1, X_2, X_3, and X_4.

Now, suppose we have an analytic function $F(Q)$ defined on some region. What does $F(Q)$ look like?

Let us define, in a strictly formal sense, $\partial F/\partial X_a$ to mean

$$\frac{\partial A_1}{\partial X_a} + \frac{\partial A_2}{\partial X_a} i + \frac{\partial A_3}{\partial X_a} j + \frac{\partial A_4}{\partial X_a} k, \qquad a = 1, 2, 3, 4.$$

Now, if F is analytic it is right differentiable, and so we have:

$$\lim_{\Delta X_1 \to 0} \Delta F \cdot \frac{1}{\Delta X_1} = \lim_{\Delta X_2 \to 0} \Delta F \cdot \frac{1}{i\Delta X_2} = \lim_{\Delta X_3 \to 0} \Delta F \cdot \frac{1}{j\Delta X_3}$$

$$= \lim_{\Delta X_4 \to 0} \Delta F \cdot \frac{1}{k\Delta X_4}.$$

That is,

$$\frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial X_2} \cdot \frac{1}{i} = \frac{\partial F}{\partial X_3} \cdot \frac{1}{j} = \frac{\partial F}{\partial X_4} \cdot \frac{1}{k},$$

which gives:

$$\frac{\partial A_1}{\partial X_1} + i\frac{\partial A_2}{\partial X_1} + j\frac{\partial A_3}{\partial X_1} + k\frac{\partial A_4}{\partial X_1}$$

$$= -i\frac{\partial A_1}{\partial X_2} + \frac{\partial A_2}{\partial X_2} + k\frac{\partial A_3}{\partial X_2} - j\frac{\partial A_4}{\partial X_2}$$

$$= -j\frac{\partial A_1}{\partial X_3} - k\frac{\partial A_2}{\partial X_3} + \frac{\partial A_3}{\partial X_3} + i\frac{\partial A_4}{\partial X_3}$$

$$= -k \frac{\partial A_1}{\partial X_4} + j \frac{\partial A_2}{\partial X_4} - i \frac{\partial A_3}{\partial X_4} + \frac{\partial A_4}{\partial X_4} .$$

If we now equate the coefficients of 1, i, j, and k, respectively, we arrive at a system of equations

$$\begin{aligned}
\frac{\partial A_1}{\partial X_1} &= \frac{\partial A_2}{\partial X_2} = \frac{\partial A_3}{\partial X_3} = \frac{\partial A_4}{\partial X_4} \\
\frac{\partial A_2}{\partial X_1} &= -\frac{\partial A_1}{\partial X_2} = \frac{\partial A_4}{\partial X_3} = -\frac{\partial A_3}{\partial X_4} \\
\frac{\partial A_3}{\partial X_1} &= -\frac{\partial A_4}{\partial X_2} = -\frac{\partial A_1}{\partial X_3} = \frac{\partial A_2}{\partial X_4} \\
\frac{\partial A_4}{\partial X_1} &= \frac{\partial A_3}{\partial X_2} = -\frac{\partial A_2}{\partial X_3} = -\frac{\partial A_1}{\partial X_4} ,
\end{aligned} \tag{3}$$

which are necessary conditions if F is right differentiable.

Now let F be left differentiable. We similarly must have:

$$\lim_{\Delta X_1 \to 0} \frac{1}{\Delta X_1} \cdot \Delta F = \lim_{\Delta X_2 \to 0} \frac{1}{i\Delta X_2} \cdot \Delta F = \lim_{\Delta X_3 \to 0} \frac{1}{j\Delta X_3} \cdot \Delta F$$
$$= \lim_{\Delta X_4 \to 0} \frac{1}{k\Delta X_4} \cdot \Delta F.$$

That is,

$$\frac{\partial F}{\partial X_1} = -i \frac{\partial F}{\partial X_2} = -j \frac{\partial F}{\partial X_3} = -k \frac{\partial F}{\partial X_4},$$

and so:

$$\begin{aligned}
&\frac{\partial A_1}{\partial X_1} + i \frac{\partial A_2}{\partial X_1} + j \frac{\partial A_3}{\partial X_1} + k \frac{\partial A_4}{\partial X_1} \\
&= -i \frac{\partial A_1}{\partial X_2} + \frac{\partial A_2}{\partial X_2} - k \frac{\partial A_3}{\partial X_2} + j \frac{\partial A_4}{\partial X_2} \\
&= -j \frac{\partial A_1}{\partial X_3} + k \frac{\partial A_2}{\partial X_3} + \frac{\partial A_3}{\partial X_3} - i \frac{\partial A_4}{\partial X_3} \\
&= -k \frac{\partial A_1}{\partial X_4} - j \frac{\partial A_2}{\partial X_4} + i \frac{\partial A_3}{\partial X_4} + \frac{\partial A_4}{\partial X_4} .
\end{aligned}$$

Again equating the coefficients of 1, i, j, and k respectively we have:

$$\begin{aligned}
\frac{\partial A_1}{\partial X_1} &= \frac{\partial A_2}{\partial X_2} = \frac{\partial A_3}{\partial X_3} = \frac{\partial A_4}{\partial X_4} \\
\frac{\partial A_2}{\partial X_1} &= -\frac{\partial A_1}{\partial X_2} = -\frac{\partial A_4}{\partial X_3} = \frac{\partial A_3}{\partial X_4} \\
\frac{\partial A_3}{\partial X_1} &= \frac{\partial A_4}{\partial X_2} = -\frac{\partial A_1}{\partial X_3} = -\frac{\partial A_2}{\partial X_4} \\
\frac{\partial A_4}{\partial X_1} &= -\frac{\partial A_3}{\partial X_2} = \frac{\partial A_2}{\partial X_3} = -\frac{\partial A_1}{\partial X_4} ,
\end{aligned} \tag{4}$$

which are necessary conditions if F is left differentiable. Now, if F is analytic we combine (3) and (4) to get:

$$\begin{aligned}
\frac{\partial A_1}{\partial X_1} &= \frac{\partial A_2}{\partial X_2} = \frac{\partial A_3}{\partial X_3} = \frac{\partial A_4}{\partial X_4} \\
\frac{\partial A_2}{\partial X_1} &= \frac{\partial A_1}{\partial X_2} = \frac{\partial A_4}{\partial X_3} = \frac{\partial A_3}{\partial X_4} = 0 \\
\frac{\partial A_3}{\partial X_1} &= \frac{\partial A_4}{\partial X_2} = \frac{\partial A_1}{\partial X_3} = \frac{\partial A_2}{\partial X_4} = 0 \\
\frac{\partial A_4}{\partial X_1} &= \frac{\partial A_3}{\partial X_2} = \frac{\partial A_2}{\partial X_3} = \frac{\partial A_1}{\partial X_4} = 0 ,
\end{aligned} \tag{5}$$

which turn out to be rather strong conditions for a function to be analytic.

We may go one step further if we introduce a second restriction on the class we are **considering,** the motivation of which will become obvious at once.

Restriction 2: There exist integers a and b between 1 and 4, not equal, such that

$$\frac{\partial^2 A_a}{\partial X_a^2}, \qquad \frac{\partial^2 A_b}{\partial X_a \partial X_b}, \quad \text{and} \quad \frac{\partial^2 A_b}{\partial X_a \partial X_b}$$

exist, and the last two are continuous on the region on which F is analytic

Restriction 2 presupposes much stronger conditions than Restriction 1, and these are its consequences:

If $F(Q)$ satisfies Restriction 2, then for $a \neq b$ we have by (5)

$$\frac{\partial A_a}{\partial X_a} = \frac{\partial A_b}{\partial X_b} \quad \text{and} \quad \frac{\partial A_a}{\partial X_b} = 0,$$

so that

$$\frac{\partial^2 A_a}{\partial X_a^2} = \frac{\partial}{\partial X_a}\left(\frac{\partial A_b}{\partial X_b}\right) = \frac{\partial}{\partial X_b}\left(\frac{\partial A_b}{\partial X_a}\right) = 0,$$

by (5) and basic theorems of partial derivatives. But this very simple system of partial differential equations tells us that for each $a = 1, 2, 3, 4$,

$$A_a = CX_a + k_a,$$

where C and k_a are real constants. Thus,

$$F(Q) = C\cdot Q + K,$$

where C is a real constant and K is a quaternion constant. We have then a further characterization of an analytic function which is stated in the following proposition (just proved):

Proposition: If $F(Q)$ is a single valued, quaternion valued function of a quaternion variable, analytic on a region and satisfying Restrictions 1 and 2 on that region, then

$$F(Q) = C\cdot Q + K$$

for some real constant C and some quaternion constant K.

Finally, we observe that if F is of the form $F(Q) = C\cdot Q + K$, then F is certainly analytic, for

$$\lim_{\Delta Q \to 0} \Delta F \cdot \frac{1}{\Delta Q} = C \quad \text{and} \quad \lim_{\Delta Q \to 0} \frac{1}{\Delta Q}\cdot \Delta F = C$$

so that F is right and left differentiable on the region and thus analytic by definition, and also clearly, F satisfies Restrictions 1 and 2.

In short, an analytic function under Restriction 1 is characterized by system (5), which places great limitations on the class of such functions. If we consider functions satisfying Restriction 2 as well, our class is limited indeed. It might be worthwhile to try to characterize functions satisfying other less stringent conditions than those investigated here.

LINEAR RECURRENCE RELATIONS AND SERIES OF MATRICES

By *Kenneth Loewen*[1]
Norman, Oklahoma

1. *A Linear Recurrence*

Consider a sequence generated by a linear recurrence of the form

$$u_{n+1} = au_n + bu_{n-1}. \tag{1}$$

Assume the initial conditions $u_0 = 0$, $u_1 = 1$. This is no real restriction, for if we let $v_0 = r$ and $v_1 = s$ and for other values v_n is given by (1), then Table 1 suggests, and an induction proves, the relation

n	0	1	2	3	4
u_n	0	1	a	$a^2 + b$	$a^3 + 2ab$
v_n	r	s	$as + br$	$(a^2 + b)s + abr$	$(a^3 + 2ab)s + b(a^2 + b)r$

TABLE 1

$$v_n = su_n + rbu_{n-1} = v_1u_n + bv_0u_{n-1}.$$

Thus, the behavior of the specialized sequence $\{u_n\}$ completely determines that of the more general sequence $\{v_n\}$. (The Fibonacci Sequence $\{F_n\}$ is the case $a = b = 1$).

This recurrence can be written in terms of matrices as follows:

$$(u_{n+1}, u_n) = (u_n, u_{n-1})\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}.$$

The matrix

$$R = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$$

is a generalization of the Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

for the Fibonacci sequence. Similar to the well known relation

[1]Professor Loewen is the previous editor of this journal.

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for Fibonacci sequences, we have the relationship

$$R^n = \begin{pmatrix} u_{n+1} & u_n \\ bu_n & bu_{n-1} \end{pmatrix},$$

which can also be proved rather easily by induction.

2. *Series of Matrices*

A series of $m \times n$ matrices

$$\sum_{k=0}^{\infty} A^{(k)}$$

is said to *converge* to a matrix A if and only if for every $i = 1, \ldots, m$ and every $j = 1, \ldots, n$, the series

$$\sum_{k=0}^{\infty} a_{ij}^{(k)}$$

converges to a_{ij}.

Let fix) be any function which has a power series expansion with a positive radius of convergence, and let R by any $n \times n$ square matrix. Then we can consider the expression

$$f(tR) = \sum_{k=0}^{\infty} a_k t^k R^k \qquad (R^0 = I)$$

with t any real number. This is the function f evaluated at the square matrix tR. It is defined for every matrix tR for which the series converges. By the Cayley-Hamilton theorem a square matrix satisfies its own characteristic equation $\det(R - xI) = 0$, or

$$c_0 I + c_1 X + c_2 X^2 + \ldots + c_n X^n = 0,$$

where X denotes an $n \times n$ matrix. This means that at most n distinct powers of R are linear independent, and hence the infinite series can be written in terms of the matrices $I, R, \ldots, R^{n-1}$. A theorem of Sylvester (Reference 2, page 78) enables us to compute the coefficients of the polynomial sum of the series. If the roots of the characteristic equation are distinct, the sum can be written

$$f(tR) = \sum_{k=0}^{\infty} a_k t^k R^k = \frac{D_0 I + D_1 tR + \ldots + D_{n-1} t^{n-1} R^{n-1}}{D}$$

where D is the Vandermonde determinant of characteristic roots of tR ($r_1, \ldots, r$ are the characteristic roots of R)

$$D = \begin{vmatrix} 1 & 1 & & 1 \\ r_1 t & r_2 t & \ldots & r_n t \\ (r_1 t)^2 & (r_2 t)^2 & \ldots & (r_n t)^2 \\ \vdots & \vdots & \vdots & \vdots \\ (r_1 t)^{n-1} & (r_2 t)^{n-1} & \ldots & (r_n t)^{n-1} \end{vmatrix},$$

D_0 is the determinant obtained by replacing the first row of D by

$$f(r_1 t), \ldots, f(r_n t), \qquad (2)$$

the determinant D_1 is obtained by replacing the second row of D by (2), and so on -- a process reminiscent of Cramer's rule. (This form of the solution is given in Reference 1, page 243. Reference 2 gives a solution in another notation and develops means to cover the case with repeated roots.)

3. *An Application*

By combining the results of the first two sections using $R = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$, after noting that the characteristic roots of tR are

$$tr_1 = t\,\frac{a - \sqrt{a^2 + 4b}}{2}, \quad tr_2 = t\,\frac{a + \sqrt{a^2 + 4b}}{2},$$

then

$$D = \begin{vmatrix} 1 & 1 \\ tr_1 & tr_2 \end{vmatrix} = t\sqrt{a^2 + 4b},$$

$$D_0 = \begin{vmatrix} f(tr_1) & f(tr_2) \\ tr_1 & tr_2 \end{vmatrix},$$

$$D_1 = \begin{vmatrix} 1 & 1 \\ f(tr_1) & f(tr_2) \end{vmatrix} = f(tr_2) - f(tr_1).$$

If we consider only the *upper right entry* in all the matrices, noting that I had a zero there, we get after simplifying

$$\sum_{n=0}^{\infty} a_n u_n t^n = \frac{f(r_2 t) - f(r_1 t)}{\sqrt{a^2 + 4b}}$$

For example, since e^R converges for all square matrices R (See Reference 2, page 41), we can use

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we have

$$\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \frac{e^{tr_1} - e^{tr_2}}{\sqrt{a^2+4b}} = \frac{e^{t(a+\sqrt{a^2+4b})/2} - e^{t(a-\sqrt{a^2+4b})/2}}{\sqrt{a^2+4b}}.$$

If $a = b = 1$, we get the exponential generating function for the Fibonacci sequence

$$\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \frac{e^{t(1+\sqrt{5})/2} - e^{t(1-\sqrt{5})/2}}{\sqrt{5}}$$

Any function having a series expansion could be used for $f(x)$. For example, using the Bessel function $J_0(x)$ gives a generating function for even indices (since the series has only even powers)

$$\sum_{n=0}^{\infty} u_{2n} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = \frac{J_0(tr_2) - J_0(tr_1)}{\sqrt{a^2+4b}}$$

4. *Generalization*

The same proceedure may be used to obtain generating functions for sequences generated by higher order recurrences. Thus the sequence generated by

$$u_{n+1} = au_n + bu_{n-1} + cu_{n-2},\ u_0 = 0,\ u_1 = 0,\ u_2 = 1 \tag{3}$$

has a recursion matrix

$$R = \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \\ c & 0 & 0 \end{pmatrix}.$$

To find a simple expression for R^n consider three sets of initial conditions giving sequences u as in (3), v_n with $v_0 = 0$, $v_1 = 1$, $v_2 = 0$, and w with $w_0 = 1$, $w_1 = 0$, $w_2 = 0$. In addition let x_n have initial conditions $x_0 = r$, $x_1 = s$, $x_2 = t$. Tabulation of the first few values of each sequence is shown in Table 2.

n	0	1	2	3	4
u_n	0	0	1	a	$a^2 + b$
v_n	0	1	0	b	$ab + c$
w_n	1	0	0	c	ac
x_n	r	s	t	$at + bs + cr$	$(a^2 + b)t + (ab + c)s + acr$

TABLE 2

This suggests the relations

$$x_n = u_n t + v_n s + w_n r,$$

$$w_n = cu_{n-1}.$$

$$v_n = bu_{n-1} + w_{n-1} = bu_{n-1} + cu_{n-2}.$$

Each of these can be proved by mathematical induction. Since by definition

$$(x_n, x_{n-1}, x_{n-2})R = (x_{n+1}, x_n, x_{n-1})$$

and noting that

$$R = \begin{pmatrix} u_3 & u_2 & u_1 \\ v_3 & v_2 & v_1 \\ w_3 & w_2 & w_1 \end{pmatrix},$$

then we have

$$R^2 = \begin{pmatrix} u_3 & u_2 & u_1 \\ v_3 & v_2 & v_1 \\ w_3 & w_2 & w_1 \end{pmatrix} \cdot R = \begin{pmatrix} u_4 & u_3 & u_2 \\ v_4 & v_3 & v_2 \\ w_4 & w_3 & w_2 \end{pmatrix}.$$

In general an induction then proves

$$R^n = \begin{pmatrix} u_{n+2} & u_{n+1} & u_n \\ v_{n+2} & v_{n+1} & v_n \\ w_{n+2} & w_{n+1} & w_n \end{pmatrix} =$$

$$\begin{pmatrix} u_{n+2} & u_{n+1} & u_n \\ bu_{n+1} + cu_n & bu_n + cu_{n-1} & bu_{n-1} + cu_{n-2} \\ cu_{n+1} & cu_n & cu_{n-1} \end{pmatrix}$$

The characteristic equation of R may be found by direct computation, or by noting that by reflecting on the non-principal diagonal we get the companion matrix for the polynomial (the reflection at most changes the sign of the determinant) which set equal to zero gives

$$x^3 - ax^2 - bx - c = 0.$$

Let r_1, r_2, r_3 be its roots and set

$$D = \begin{vmatrix} 1 & 1 & 1 \\ tr_1 & tr_2 & tr_3 \\ (tr_1)^2 & (tr_2)^2 & (tr_3)^2 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ tr_1 & tr_2 & tr_3 \\ e^{tr_1} & e^{tr_2} & e^{tr_3} \end{vmatrix}$$

(we do not need explicit formulas for D_0 and D_1.) Then we can obtain an exponential type generating function by considering the ***first row, third column*** of the matrix series

$$e^{tr} = \sum_{n=1}^{\infty} R^n \frac{t^n}{n!} = (D_0 I + D_1 tR + D_2 t^2 R^2)/D .$$

Since I and R have a 0 in row 1, column 3 and R^2 has a 1 there we get

$$\sum_{n=1}^{\infty} u_n \frac{t^n}{n!} = t^2 D_2/D .$$

As an example let us look at the case $a = b = c = 1$. The sequence begins as shown in Table 3.

n	0	1	2	3	4	5	6	7	8	9
u_n	0	0	1	1	2	4	7	13	24	44

TABLE 3

The characteristic equation has one real and two complex roots. Let them be $r_1 = r$, $r_2 = p + iq$, $r_3 = p - iq$. Then we have the exponential generating function

$$\sum_{n=1}^{\infty} u_n \frac{t^n}{n} = \frac{t^2 \begin{vmatrix} 1 & 1 & 1 \\ tr & t(p+iq) & t(p-iq) \\ e^{tr} & e^{t(p+iq)} & e^{t(p-iq)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ tr & t(p+iq) & t(p-iq) \\ t^2r^2 & t^2(p+iq)^2 & t^2(p-iq)^2 \end{vmatrix}}$$

$$= \frac{(p-r)e^{pt}\sin qt - qe^{pt}\cos qt + qe^{rt}}{q((p-r)^2 + q^2)}.$$

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HAPPY NUMBERS

By Daniel P. Wensing
John Carroll University

Consider the sequence of positive integers 79, 130, 10, 1, 1, 1, $\cdots$ This sequence was constructed from 79 where each subsequent term was the sum of the squares of the digits of the preceding term. The sequence of positive integers defined in this manner shall be called the ***happiness sequence.*** The happiness sequence of the number 2 is: 2, 4, 16, 37, 58, 89, 145, 42, 20, 4, 16, $\cdots$. As one can see, unlike the happiness sequence of 79 which in a sense terminates at 1, this sequence gets caught in a repetitive cycle. Since the happiness sequence of 79 very nicely comes to 1, 79 is called a ***happy number.*** On the other hand, since the happiness sequence of 2 enters into a repetitive cycle which never returns to 1, 2 is called an ***unhappy number.***

So far, consideration of a number's happiness has been restricted to base ten. However, this concept can be expanded to any positive integer in any arbitrary base $b > 1$. For example, 47_8 would be followed by 101_8 in the happiness sequence, since $4^2 + 7^2 = 101_8$. The number 1231231_4 is happy, because it sets up the happiness sequence of: 123123_4, 130_4, 22_4, 20_4, 10_4, 1_4. Interestingly, after a number of random picks in base four, one gets the impression that ***all numbers are 'happy in base four.*** Before this conjecture can be proved it is necessary to understand some theory behind happiness sequences.

Once again restricting analysis to base ten, it can be easily seen that any five digit number is followed by, at most, a three digit number in the happiness sequence. This is demonstrated by the fact that 99999 is the five digit number producing the largest subsequent term of 405. Indeed, the more digits a number has the smaller will be the next term in the happiness sequence. In light of this, let us generalize for an n digit (positive) integer in an arbitrary base b > 1. The question is posed whether there is some consistent value for n where the next term in the happiness sequence is always less than the one preceding it? The notation $e\ldots wxyz_b$ will be used for the n digit number in base b, where all letters other than the base subscript represent digits, and $e \neq 0$.

This question can be stated mathematically as follows: For arbitrary base $b > 1$, does there exist an integer n such that for any n digit number $e\ldots wxyz_b$,

$$eb^{n-1} + \ldots + wb^3 + xb^2 + yb^1 + zb^0 > e^2 + \ldots + w^2 + x^2 + y^2 + z^2 \quad ?$$

Isolating attention on respective terms from each side of this main inequality, it can be readily seen that $yb^1 \geq y^2$, the reason being that y is a digit permissible in base b, and is consequently restricted by $0 \leq y \leq (b - 1)$. Since $yb \geq y^2$ it follows that:

$$\begin{array}{ccc} xb \geq x^2 & \text{and necessarily} & xb^2 \geq x^2, \\ wb \geq w^2 & \text{and} & wb^3 \geq w^2, \\ \vdots & \vdots & \vdots \\ eb > e^2 & \text{and} & eb^{n-1} > e^2. \end{array}$$

This follows necessarily because the left, and already greater side, is simply multiplied by some positive power of b. Consequently, with the exception of z, the respective terms on the left side of the main inequality are greater than their counterpart on the right. Since eb alone is already greater than the corresponding e^2, it suffices to show that the unused value of $(eb^{n-1} - eb)$ will, for some n, be greater than z^2. As previously mentioned, to exist as a leading digit e must be a permissible digit and non-zero. Therefore, to put the tightest restriction on the proposed inequality $eb^{n-1} - eb > z^2$, we make e as small as possible, that is, $e = 1$, and we make z as large as possible, that is $z = (b - 1)$. Thus we obtain the proposed inequality:

$$b^{n-1} - b > (b - 1)^2,$$

which is equivalent to

$$b^{n-1} - b > b^2 - 2b + 1,$$

$$b^{n-1} > b^2 - b + 1,$$

$$b^n > b^3 - b^2 + b,$$

$$b^n + b^2 > b^3 + b.$$

Obviously, $b^n + b^2 > b^3 + b$ for $n \geq 3$, so the above inequalities are each valid. This proves the following theorem:

If A is any number in a happiness sequence that is calculated using base $b > 1$ and A has three or more digits in base b, then the term following A in the sequence is smaller than A.

Now the significance of this can be easily realized. It can be reasoned directly from the theorem that any positive integer composed of more than two digits must eventually produce a one or two digit number in its happiness sequence. This leads to the fact that there can be no cycle which does not contain a one or two digit number. Therefore, all and only cycles in base b are determined by the happiness sequence of the one and two digit numbers. So, by examining only the one and two digit numbers *all* cycles will be uncovered. Upon inspection of base four, all fifteen one and two digit positive integers are found to be happy numbers. Consequently, there are no cycles in base four and all positive integers must be happy in base four.

Through further use of this analysis one can easily determine all cycles in a given base and discover much about the happiness of all positive integers in any base. For example, here is a list of all cycles in the number bases 2 through 10:

Base 2:	None (a "happy base")
Base 3:	(12), (22), and (2,11)
Base 4:	None (a "happy base")
Base 5:	(23), (33), (4,31,20)
Base 6:	(5,41,25,45,105,42,32,21)
Base 7:	(13), (34), (44), (63), (2,4,22,11), (16,52,41,23)
Base 8:	(24), (64), (4,20), (5,31,12), (15,32)
Base 9:	(45), (55), (58,108,72), (82,75)
Base 10:	(4,16,37,58,89,145,42,20)

The author has generated by computer all the happy numbers from 1 to 100 in bases 2 through 10. For base 10, the happy numbers are:

1,7,10,13,19,23,28,31,32,44,49,68,70,79,82,86,91,94,97,100.

GOLDBACH'S CONJECTURE

By *Christopher Scussel*[1]
Michigan State University

Goldbach's conjecture, that every even number is the sum of two prime numbers, has remained merely a conjecture for about two hundred years. This is in keeping with the usual difficulty in relating primes or sums of primes to any algebraic quantities, such as squares. In order to prove the conjecture true, of course, one must prove for each even number the existence of a pair of primes which sum to that number, and it is toward this goal that most "proofs" are oriented, although so far unsuccessfully. At this point, a pair of primes has been found for every even number so tested, so empirically, at least, the conjecture is true. In light of this, another attack presents itself: Since all the even numbers tested have a corresponding prime number pair, why not investigate just how many such pairs exist for any given even number? This is the topic of the remainder of this paper.

Some definitions are in order. First, the set of primes, denoted P, shall be defined to consist of all odd prime numbers, including 1. Thus,

$$P = \{1,3,5,7,11,13, \ldots\} .$$

The set of natural numbers N will be $\{1,2,3, \ldots\}$, while N_e shall denote the set of even natural numbers, $\{2,4,6,8, \ldots\}$. The number of ways in which a given even number n can be represented as the sum of two primes will be called the Goldbach multiplicity of n, denoted $Gm(n)$. Thus Gm is a function from the even natural numbers to the set $N \cup \{0\}$ (note that Goldbach's conjecture is true if and only if the function Gm has no zeros on the even natural numbers). Precisely de-

[1]The author wishes to thank the Department of Mathematics, Michigan State University, for its assistance in obtaining computer time for this study. The computer time was obtained from the Computer Laboratory, Michigan State University, which is supported by a grant from the National Science Foundation.

fined,

$$Gm(n) = \left|\{(a,b) \mid a + b = n;\ a,b \in P;\ a \le b\}\right|, \quad n \in N_e .$$

For example, $14 = 1 + 13 = 3 + 11 = 7 + 7$, so $Gm(14) = 3$. The Euler totient function, or ϕ-function, is the number of natural numbers less than and relatively prime to a given natural number n. Note that for any natural number n, if $n = \prod_{i=1}^{k} p_i^{a_i}$ is the standard form for n, where the a_i are natural numbers and the p_i are distinct numbers from the set $(P \cup \{2\}) - \{1\}$, then the ϕ-function can be defined as follows:

$$\phi(n) = \prod_{i=1}^{k} (p_i - 1)(p_i)^{a_i - 1}$$

Having established all necessary definitions, a quick way to get an idea of the characteristics of the Gm function would be to construct a graph of it. Such a graph, generated by computer, is shown in Fig. 1. Although the Gm function varies wildly, there is a gradual, but definite, upward trend. This can be explained using analytic number theory as follows. There are, approximately, $\frac{m}{\log m}$ primes less than a given natural number m. Further, the probability that a natural number m is prime is approximately $\frac{1}{\log m}$. Now, in order to roughly determine $Gm(n)$, where $n \in N_e$, consider that there are about

$$\frac{\frac{n}{2}}{\log(\frac{n}{2})} = \frac{n}{2\log(\frac{n}{2})}$$

primes less than $n/2$. For any of these primes, say p, if $n - p$ is also prime, then two primes have been found that sum to n, and thus are counted in calculating $Gm(n)$. Each of these numbers $n - p$ lies in the interval $[n/2,n]$, thus the probability that any one of these numbers is prime is approximated by $\frac{1}{\log(3n/4)}$, since $(3/4)n$ is the midpoint of the interval $[n/2,n]$. So, the $\frac{n}{2\log(n/2)}$ primes less than $n/2$ generate $\frac{n}{2\log(n/2)}$ numbers $n - p$, each of which has probability $\frac{1}{\log(3n/4)}$ of being prime. Thus, there should be about $\frac{n}{2\log(n/2)} \frac{1}{\log 3n/4}$ coincidences of p and $n - p$ both being prime. However, since all of the

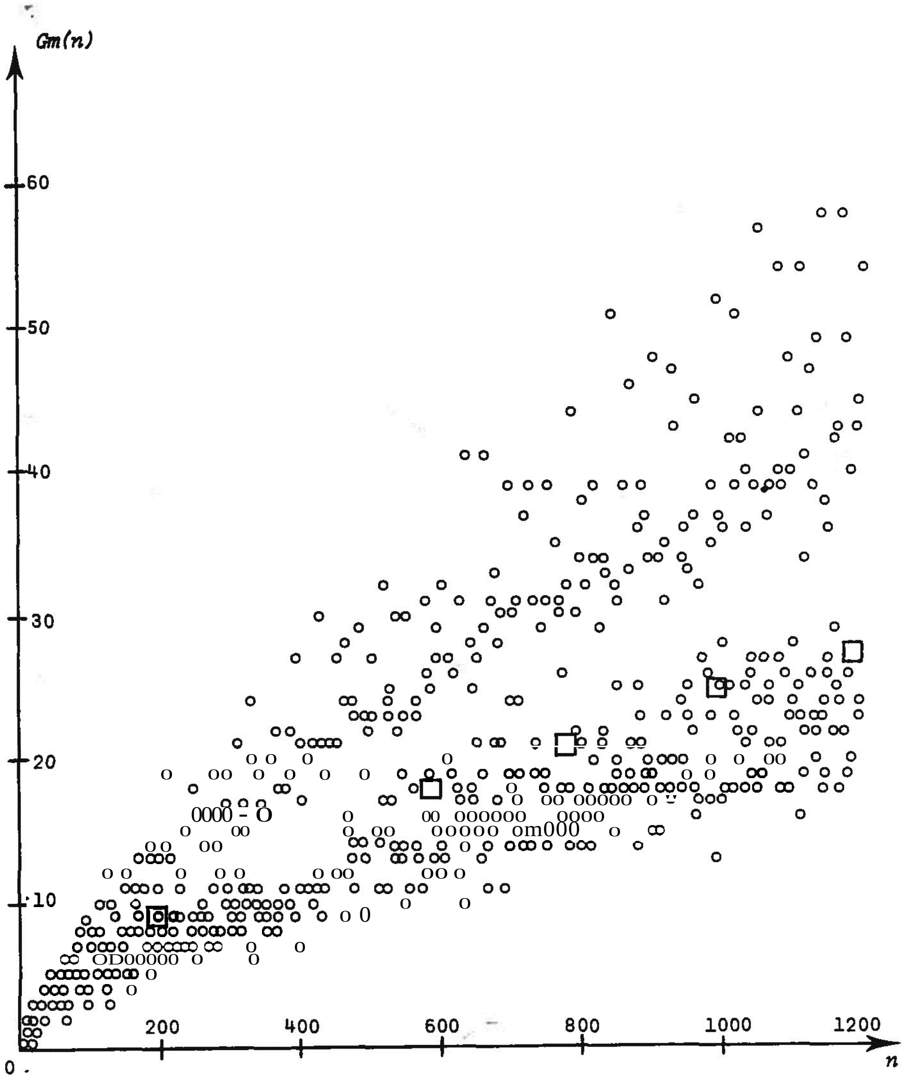

FIGURE 1

numbers p are odd, and n is even, it follows that $n - p$ must also be odd, and knowledge of this fact doubles the probability that each $n - p$ is prime, since it eliminates 2 as a possible divisor. Taking this into consideration, the estimate of the number of coincidences of p and $n - p$ being prime changes to

$$\frac{n}{2\log(\frac{n}{2})}\cdot\frac{2}{\log(\frac{3n}{4})} = \frac{n}{\log(\frac{n}{2})\log(\frac{3n}{4})}$$

But the number of such coincidences is just $Gm(n)$, which leads to the estimating function,

$$Gm(n) \approx \frac{n}{\log(\frac{n}{2})}\cdot\frac{1}{\log(\frac{3n}{4})} .$$

Several points of this approximating function have been plotted (as squares) in Fig. 1, and it seems to agree quite well with the "trend" of the actual Gm function, although of course it cannot follow the wild gyrations of the Gm function.

This behavior having been explained, attention turns to other features of the graph. Note that the graph seems to be divided into upper and lower arms. The lower arm seems to be only about half as high as the higher arm, but it is about twice as dense, in terms of number of points plotted. Why should this be? The explanation this time comes from congruences. Consider the addition of primes modulo 3. All primes (with the exception of 3) fall into one of the two congruence classes 1 or 2. So, a table of the addition of primes modulo 3 would look like this:

+	1	2
1	2	0
2	0	1

The 0 congruence class occurs twice as often as either of the other classes. Thus, assuming the primes are evenly distributed between congruence classes 1 and 2 (roughly the case), choosing two primes at random and adding them will yield a sum congruent to 0 modulo 3 with probability 1/2. Since all sums of two primes (below 1200) were of necessity used in constructing Fig. 1, it would be expected that this bias toward the 0 congruence class would show up there. Indeed, it does. Since a number in the 0 congruence class is twice as likely to be the sum of two primes than a number which is not in that class, it follows that if m and n are even naturals which differ only slightly in magnitude but m is congruent to 0 modulo 3 and n is not, then $Gm(m) \approx 2Gm(n)$, very approximately. This nicely explains the two arms of Fig. 1. The conclusion is further substantiated by investigation of the actual values of the Gm function. The author has constructed by computer a table for $Gm(n)$ where n ranges over the even integers from 0 to 20,000.

However, noting that a disparity in the addition table of the primes modulo 3 was responsible for a large portion of the variation found in the Gm function, it seems natural to consider the consequences of using moduli other than 3. In general, consider a modulus $m \in N$, $m \neq 1$. Taking P modulo m yields exactly $\phi(m)$ infinite congruence classes of primes: Choose any $m' \in N$ such that $0 < m' < m$ and m' and m are coprime. Then the set $A = \{mn + m' \mid n \in N\}$ contains infinitely many primes (Dirichlet). On the other hand, if m' and m are not coprime, then all numbers in the set A have the greatest common divisor of m and m' as a divisor. Thus the congruence class m' of the primes modulo m can contain at most one number, namely m' itself, and then only if m' is prime, and thus a divisor of m. Since the number of primes which divide the modulus is of necessity quite small with respect to the modulus, such "singular" classes will not be considered for the remainder of this discussion.

Having established the existence of $\phi(m)$ classes of primes modulo m, constructing the corresponding addition table of primes modulo m reveals a curious thing (we let $n_1, n_2, n_3, \ldots, n_{\phi(m)}$ denote the $\phi(m)$ natural numbers coprime to m). Observe that n_i is coprime to m if and

+	n_1	n_2	n_3	...	$n_{\phi(m)}$
n_1	2				0
n_2					
n_3					
$n_{\phi(m)}$	0				$m - 2$

only if $m - n_i$ is coprime to m. This symmetry results in the diagonal entries of the table being 0, and that zeros can be only on the diagonal follows easily. Thus there are $\phi(m)$ entries of 0 and $\phi^2(m) - \phi(m)$ entries which are natural numbers less than m. So the "advantage" of the 0 congruence class becomes larger as m increases, if $\phi(m)$ remains approximately constant, and the "advantage" decreases as $\phi(m)$ increases,

if m stays approximately constant. In general, then, the "advantage" given to the 0 congruence class amounts to a factor of $\frac{m}{\phi(m)}$.

Now we apply this to the estimation of $Gm(n)$, where $n \in N_e$. Using n in the argument above yields a factor of $\frac{n}{\phi(n)}$. This factor cannot be simply concatenated with the approximation $Gm(n) \approx \frac{n}{\log(n/2)} \frac{1}{\log(3n/4)}$, for the following reason: As illustrated in Fig. 1, the present approximation of the Gm function represents an average value of Gm at any particular point, as it should, in order to be a true approximation. This means that although Gm and its approximation differ considerably, on the whole they both account for about the same number of sums of primes. Thus, to multiply the approximation by a factor which is always greater than 1 would shift it up, away from the average value of the Gm function. Now, if $m \in N$ and $m \neq 1$, then $\phi(m)$ is at most $m - 1$, thus $\frac{m}{\phi(m)} > 1$ for all natural numbers m except 1. Thus, this factor cannot simply be concatenated with the present approximation. This can be taken care of by dividing the factor by its own average value. It is known that for any natural n, $\phi(n) \approx \frac{6}{\pi^2} n$. In this case, however, n is always even (since $Gm(n)$ is not defined otherwise). Now, from the formula given for the +-function in the definitions, or from just a moment of thought, $\phi(n)$ for n odd should be about twice as large as $\phi(m)$ for m an even number of about the same magnitude. If m and n are even and odd natural numbers, respectively, then let

$$am \approx \phi(m) \quad \text{and} \quad bn \approx \phi(n) ,$$

where a and b are constant for all m and n. Now, the average of a and b should be $\frac{6}{\pi^2}$, and b should be twice a. So,

$$2a = b$$

$$\frac{a + b}{2} = \frac{6}{\pi^2} ,$$

and thus $a = \frac{4}{\pi^2}$. Hence, if n is an even natural, $\phi(n) \approx \frac{4}{\pi^2} n$. The average, or "expected" value of $\frac{n}{\phi(n)}$ for $n \in N_e$ is thus $\frac{n}{(4/\pi^2)n} = \frac{\pi^2}{4}$.

Now, again for $n \in N_e$, $\frac{n}{\phi(n)} \cdot \frac{1}{(\pi^2/4)}$ would have an "expected" value of 1, but would still vary in accordance with the previously mentioned bias toward the 0 congruence class. This leads to the final approximation presented here:

$$Gm(n) \approx \frac{n}{\log(\frac{n}{2})} \cdot \frac{1}{\log(\frac{3n}{4})} \cdot \frac{n}{\phi(n)} \cdot \frac{4}{\pi^2} \cdot$$

Table 1 compares actual values of the Gm function with this final approximating function, and shows the approximation to be a quite good one, at least in the range shown.

n	$Gm(n)$	$\frac{n}{\log(\frac{n}{2})} \cdot \frac{1}{\log(\frac{3n}{4})} \cdot \frac{n}{\phi(n)} \cdot \frac{4}{\pi^2}$
10	2	2.43
100	6	5.29
1000	28	22.6
10000	128	125.
100000	810	793.
1000000	5402	5472.

TABLE 1

Although all of this really cannot prove the conjecture, it does provide an interesting insight into the problem, and into additive number theory, in general.

One of the earliest and most intricate attempts to use higher geometry to solve the problem of duplicating the cube (that is, finding a construction for the edge of a cube having twice the volume of a cube with a given edge) was given by Archytas (*ca.* 400 B.C.). The method involves finding a point of intersection of a ***right circular cylinder***, a ***torus of zero diameter***, and a ***right circular cone!***

NARCISSISTIC NUMBERS

By Victor G. Feser
St. Louis University

An anonymous mathematician once discovered that 153 is the sum of the cubes of its digits: $153 = 1^3 + 5^3 + 3^3$. Probably soon afterwards, three similar integers were found: 370, 371, and 407. G. H. Hardy, in his famous book *A Mathematician's Apology*, cites these examples with the comment: "There is nothing in these odd facts which appeals to the mathematician." In terms of general theory, he is correct; still, in at least two respects there is something of interest here: first of all, any empirical observation serves as a starting point for various generalizations, and these in turn lead to specializations, so that many related results and ideas may be developed (this is the main thesis of George Polya's work *Mathematics and Plausible Reasoning*); and secondly, in considering problems of this type we have the chance to apply and practice many basic techniques of number theory-- and for that matter, of logic.

This article is concerned somewhat with the second aspect-- though most of the details are left to the reader. Primarily we shall consider various levels of increasing generality.

0. The curious example of 153 having once been discovered, it becomes an almost trivial generalization to look for other three-digit integers equal to the sum of the cubes of their digits. Symbolically: let N be an integer with n digits, $a_1a_2\ldots a_n$. Then we are looking for solutions for:

$$N = \sum_{i=1}^{3} \left(a_i^3\right) \qquad (0.1)$$

1. A more significant step is to vary the number of digits or the power. Thus we have* first of all, the problem

$$N = \sum_{i=1}^{n} \left(a_i^3\right), \qquad (1.1)$$

and more generally:

$$N = \sum_{i=1}^{n} \left(a_i^k\right), \qquad (1.2)$$

k a positive integer. It can be shown that if k is a non-positive integer, then N = 1 is the unique and trivial solution.

Before we list some solutions for these cases, let us introduce some of the terminology found in the literature: An integer of n digits equal to the sum of the k^{th} power of its digits is called a *perfect digital invariant (PDI)*, of *order k*. If further $n = k$ (as in the examples cited above), the integer is called a *pluperfect digital invariant (PPDI)*. Some general facts are known: The number of *PPDI*'s is finite: in fact, no such integer can have more than 59 digits [5]. On the other hand, the number of *PDI*'s may possibly be infinite [6]. All *PDI*'s so far discovered are composite. The integers 0 and 1 are trivial *PDI*'s for all orders. There are no *PDI*'s (and thus no *PPDI*'s) of order 2—this will be shown shortly. At least one PPDI exists for every order from 3 through 10, e.g.: order 6: 548,834; order 10: 4,679,307,774. The search for solutions for orders 6 and up has been done by computer. A report is to be published soon for orders 11 through 15 [7]; apparently further solutions have been found.

Now to show that there exist no (non-trivial) solutions for order 2: if N is a one-digit number, we have merely $a_1 = a_1^2$, whence the trivial solutions. If N has four digits, then $N \geq 1000$; but each digit is ≤ 9, so the sum of the squares of the digits is $4 \cdot 9^2 \leq 324$. By induction it obviously follows that there is no solution if N has more than 4 digits. This leaves only two- and three-digit numbers to consider.

If N is a two-digit number, let its digit representation be denoted by AB, for convenience. We want $10A + B = A^2 + B^2$, or $A(10 - A) = B(B - 1)$. Here $A \neq 0$, and therefore $B \neq 0$, $B \neq 1$. Now one of the factors on the right side is even, the other odd; therefore at least one factor on the left side is also even, but then obviously both of them are. Thus 4 divides the left side, so 4 is a divisor of either B or $B - 1$ since 2 cannot divide both B and $B - 1$. Thus either:

$B = 4$ or 8		or:	$B - 1 = 4$ or 8
$B - 1 = 3$ or 7			$B = 5$ or 9
$B(B - 1) = 12$ or 56			$B(B - 1) = 20$ or 72

These four cases are easily handled empirically: no solution exists.

The three-digit case will be presented only partially. The sum of the three squares is $3 \cdot 9^2 \leq 243$, so $100 \leq N \leq 243$. Since the maximum for any digit is 9, with a square of 81, the sum of the other two squares must total at least 19. This means that no solution can contain any of the following pairs of digits: 00, 01, 02, 03, 04; 11, 12, 13, 14; 22, 23; 33. Immediately we have $155 \leq N \leq 199$. One may now finish up empirically, or look for further devices: in fact there is an elegant approach that quickly disposes of the entire three-digit case.

One comment: for higher powers it would seem very difficult to work out solutions, but in fact it has been done by hand up through order 5, so some reasonable methods do exist.

2. Continuing to generalize, we may vary the exponent within the expression. We may for instance take exponents in arithmetic progression:

$$N = \sum_{i=1}^{n} \left(a_i^{k \pm \ell i}\right), \tag{2.1}$$

where $k \pm \ell i$ is a non-negative integer for all i. If $\ell = 0$, this is again (1.2). When $\ell = 1$, many solutions are known, e.g.: $2427 = 2^1 + 4^2 + 2^3 + 7^4$ (ascending), and $332 = 3^5 + 3^4 + 2^3$ (descending). No work seems to have been done for $\ell > 1$.

A more striking possibility is the "self-power":

$$N = \sum_{i=1}^{n} \left(a_i^{a_i}\right) \tag{2.2}$$

A non-trivial solution exists: $3435 = 3^3 + 4^4 + 3^3 + 5^5$. If we define $0^0 = 0$, then 438,579,088 is also a solution; but if we define $0^0 = 1$, then there is no solution (as verified by computer).

A further possibility: let the exponents be the digits of the number in some permutation. To reduce the wearisomely large number of cases, we might consider only the cyclic permutations:

$$N = \sum_{i=1}^{n} \left(a_i^{a_{(i+\ell)}}\right), \tag{2.3}$$

where $(i + \ell)$ is reduced to modulo n.

3. For the next level of generalization, let us move away from powers of the digits to more general functions. One obvious function to try is the factorial:

$$N = \sum_{i=1}^{n} \left(a_i!\right). \tag{3.1}$$

Besides the trivial 1 and 2, only two solutions exist: 40585, and a certain three-digit number that we leave to the enjoyment of the reader. An upper limit for N is readily established, so that a computer search can be, and has been, made.

- Another function to try is summation itself:

$$N = \sum_{i=1}^{n} \left(\sum_{j=0}^{a_i} j \right) \tag{3.2}$$

That is to say, if $N = 367$, for instance, we consider $\sum_{j=0}^{3} j + \sum_{j=0}^{6} j + \sum_{j=0}^{7} j = 6 + 21 + 28$ (which is not 367). In fact, no solution exists. But continuing in the same vein, consider this idea for higher powers:

$$N = \sum_{i=1}^{n} \left(\sum_{j=0}^{a_i} \left(j^k \right) \right) \tag{3.3}$$

For $k = 2$, two solutions exist: 290 and 291. The analysis of this case has not been carried any further (for $k > 2$).

Another function: instead of the sum of the k^{th} powers, consider the k^{th} power of the sum of the digits:

$$N = \left(\sum_{i=1}^{n} a_i \right)^k . \tag{3.4}$$

This generalization is formally obvious from (1.2), and as in that case, k must be a positive integer. There exists an impressive list of solutions [3]: from the simple $81 = (8 + 1)^2$, $512 = (5 + 1 + 2)^3$, and $2401 = (2 + 4 + 0 + 1)^4$, through $205{,}962{,}976 = 46^5$ and $52{,}523{,}350{,}144 = 34^7$, all the way up to 207^{20}. The number of solutions may well be infinite, since k may be set arbitrarily large.

Another case occurs in the literature, but seems less interesting because too many solutions exist:

$$N = b \left(\sum_{i=1}^{n} a_i \right). \tag{3.5}$$

(See [1], [2].) A well-known test of divisibility states that N is divisible by 3 (or 9) if and only if the sum of its digits is a multiple of 3 (or 9); thus any such integer satisfies (3.5). Another infinite family of solutions consists of integers ending in 0, with the sum of the digits being 10 (or 30). Other families can be developed.

4. Next, we may consider not a summation at all, but rather a product defined on the digits of the number:

$$N = \prod_{i=1}^{n} \left(a_i^{\,k} \right), \tag{4.1}$$

or, analogous to earlier expressions:

$$N = \left(\prod_{i=1}^{n} a_i \right)^k \tag{4.2}$$

But in fact these two cases are identical! It would seem that there is no solution for $k = 1$. No digit of N can be 0 (except in the trivial case that N itself is 0). If $k = 2$, and some digit of N is 5, then every digit of N is odd. The analysis for this case has not been carried any further.

Another case to consider is:

$$N = \prod_{i=1}^{n} \left(a_i + b \right), \tag{4.3}$$

b an integer. Solutions do exist: for $b = 2$, $35 = (3 + 2)(5 + 2)$ and $56 = (5 + 2)(6 + 2)$; for $b = 6$, $840 = (8 + 6)(4 + 6)(0 + 6)$. A question for the reader: can b be negative?

5. The final generalization we consider needs no detailed explanation: rather it serves to place the whole discussion in a broader perspective. If we write N as $a_1 a_2 \ldots a_n$, then we mean:

$$N = \sum_{i=1}^{n} a_i 10^{n-i} .$$

In other words, *every* integer in positional notation is *ipso facto* a "function of its digits"-- though this does not satisfy the definition of "narcissistic number". But we do have the sweeping generalization: simply repeat all the previous cases in different bases.

The various functions we have mentioned here certainly do not exhaust the list of possibilities. Where does one draw the line in admitting functions to consideration? We suggest that here, as so often in more serious mathematical work, the criterion is the indefinable one of elegance; in other words, it is merely a matter of esthetics. We have also mentioned here a number of problems which have not been completely solved, which we shall leave for the enjoyment of our readers.

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ANNOUNCEMENT OF NEW AWARDS BY LOCAL CHAPTERS

PENNSYLVANIA BETA at Bucknell University announces the *Professor John S. Gold Mathematics Competition,* in honor of a most enthusiastic and dedicated teacher at Bucknell University for nearly 50 years, will be administered by the Department of Mathematics at Bucknell. The competition will consist of a two and one-half hour examination on pre-calculus material, and will be open to all high schools in the counties of Columbia, Lycoming, Montour, Northumberland, Snyder, and Union. Each high school may enter a team of three students or individual students numbering two or less, The mathematics library of the school entering the highest scoring team will receive a prize of $100, and the five highest scoring individuals will each receive a copy of the four-volume set of The World of Mathematics, and the next five individuals will be awarded honorable mention. The highest scoring individuals from each school will receive a certificate of recognition.

WEST VIRGINIA ALPHA at West Virginia University plans to present a *Pi Mu Epsilon Award Certificate* to each student at West Virginia University who completes the basic three- or four-semester calculus sequence with a straight "A" average.

EDITORIAL NOTE

Local chapters are urged to send us the names of their awards winners in order for us to publish them for further honor and recognition to those students who are achieving excellence in mathematics at some level.

THE PARTITION FUNCTION AND CONGRUENCES

By Nicholas U. Migliozzi[1]
University of Connecticut

A function which has many interesting properties is the partition function $p(n)$. It is defined as the number of unrestricted partitions of n, where two partitions are equal if they differ only in the order of their summands. For example,

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Thus, $p(4) = 5$. (For convenience, we define $p(0) = 1$.)

It can be shown that $p(n)$ is the coefficient of x^n in the infinite product $f(x)^{-1} = (1 - x)^{-1}(1 - x^2)^{-1}\cdots$. This can be seen using the fact, obtained from the binomial expansion, that $(1 - x^n)^{-1}$ may be written in the form $\sum_{j=0}^{\infty} x^{jn}$. Thus

$$\begin{aligned} f(x)^{-1} &= \prod_{n=1}^{\infty} (1 - x^n)^{-1} \\ &= \prod_{n=1}^{\infty} (1 + x^n + x^{2n} + x^{3n} + \cdots) \\ &= (1 + x^1 + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots. \end{aligned}$$

Now if we expand this product formally, to a term x^n each $1/(1 - x^i)$ must contribute one and only one factor; and if x^{ji} contributes as a factor to some x^n it is because j of the i's are summands in a partition of n, and conversely, each partition of n corresponds to a unique set of factors, one from each $1/(1 - x^i)$ whose product is x^n. That is, we have the following scheme:

$$(1 + x^1 + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots.$$

number of 1's contributed   number of 2's contributed   number of 3's contributed

Example: Suppose we desire to find $p(5)$. If we analyze the contributing terms and the corresponding partitions we find (with 1's counting as zero terms in each case):

[1] Written while the author was an undergraduate at Rutgers University.

(1) $1\cdot1\cdot1\cdot1\cdot x^5$ or 5
(2) $x\cdot1\cdot1\cdot x^4\cdot1$ or $1 + 4$
(3) $1\cdot x^2\cdot x^3\cdot1\cdot1$ or $2 + 3$
(4) $x^2\cdot1\cdot x^3\cdot1\cdot1$ or $1 + 1 + 3$
(5) $x\cdot x^4\cdot1\cdot1\cdot1$ or $1 + 2 + 2$
(6) $x^3\cdot x^2\cdot1\cdot1\cdot1$ or $1 + 1 + 1 + 2$
(7) $x^5\cdot1\cdot1\cdot1\cdot1$ or $1 + 1 + 1 + 1 + 1$

Thus $p(5) = 7$. Thus we have shown in a formal sense that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}.$$

Although the above seems to be a perfect means of obtaining $p(n)$, one finds that for large values of n the process is an extremely lengthy one. However there is a theorem which is obtained using one of the following famous indentities.

(1) Euler's Identity:

$$f(x) = \prod_{k=1}^{\infty} (1 - x^k) = \sum_{k=0}^{\infty} (-1)^k x^{k(3k\pm1)/2}, \qquad |x| < 1$$

(2) Jacobi's Identity:

$$f(x)^3 = \prod_{k=1}^{\infty} (1 - x^k)^3 = \sum_{k=0}^{\infty} (-1)^k (2k + 1)x^{k(k+1)/2}, \qquad |x| < 1$$

The proofs of the above may be found in [3]. We have:

Theorem 1: If $n \geq 1$ then

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - \cdots$$

$$= \sum_{j=0}^{\infty} (-1)^{j+1} p(n - \tfrac{1}{2}(3j^2 \pm j)), \quad 3$$

where the sum extends over all positive integers for which the arguments of the partition function are non-negative.

Proof: Using Euler's identify we have for $|x| < 1$

$$\sum_{j=0}^{\infty} (-1)^j x^{j(3j\mp1)/2} \sum_{k=0}^{\infty} p(k)x^k = f(x)f(x)^{-1} = 1.$$

That is,

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots) \sum_{k=0}^{\infty} p(k)x^k = 1,$$

or

$$\sum_{n=0}^{\infty} (p(n) - p(n - 1) - p(n - 2) + p(n - 5) + \cdots)x^n = 1.$$

Thus

$$p(n) - p(n - 1) - p(n - 2) + p(n - 5) + p(n - 7) - \cdots = 0,$$

or

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + \cdots.$$

The above theorem serves as an algorithm, and using it we present a computer program which generates $p(n)$ for $1 \leq n \leq 100$:

```
*NAME P(N)
      DIMENSION IP(100)
      IO = 5
      DO 22 N = 1,100
      IP(N) = 0
      DO 3 J = 1,50
      IJN = 0.5*(3*(J**2) - J)
      IF (N - IJN) 2,15,10
   15 IP(N) = IP(N) + ((-1)**(J + 1))
      GO TO 2
   10 L = N - IJN
      IP(N) = IP(N) + ((-1)**(J+1))*IP(L)
      IJP = 0.5*(3*(J**2) + J)
      IF (N - IJP) 2,16,11
   16 IP(N) = IP(N) + ((-1)**(J + 1))
      GO TO 18
   11 M = N - IJP
      IP(N) = IP(N) + ((-1)**(J+1))*IP(M)
   18 IF (J-50) 3,79,79
    3 CONTINUE
    2 WRITE (IO,5) N,IP(N)
    5 FORMAT (5X,I5,10X,I22)
   22 CONTINUE
   79 CALL EXIT
      END
```

Next we discuss congruences of the partition function. The most up-to-date result concerning congruences of the partition function is the following theorem (we remind the reader that $[x]$ denotes the greatest integer n such that $n \leq x$):

Theorem 2:

(a) If $24m \equiv 1 \pmod{5^n}$, then $p(m) \equiv 0 \pmod{5^n}$

(b) If $24m \equiv 1 \pmod{7^n}$, then $p(m) \equiv 0 \pmod{7^{[(n+2)/2]}}$

(c) If $24m \equiv 1 \pmod{11^n}$, then $p(m) \equiv 0 \pmod{11^n}$

(The proofs of (a) and (b) are found in [8], and a proof of (c) is found in [1].)

It is important to note that in the above theorem, m and n are positive integers. Also note that withn $= 1$ in each part of the above theorem one obtains the three famous congruences of Ramanujan, namely:

(1) $p(5z + 4) \equiv 0 \pmod 5$

(2) $p(7z + 5) \equiv 0 \pmod 7$

(3) $p(11z + 6) \equiv 0 \pmod{11}$

For example, if $n = 1$ in Theorem 2(a) we have $24m \equiv 1 \pmod 5$ implies $m \equiv 4 \pmod 5$, which implies $m = 5z + 4$, which implies $p(5z + 4) \equiv 0 \pmod 5$. Of course these may be proved directly. Using Euler's Identity and Jacobi's Identity (1) and (2) may be proved (for the proofs see [4]). A similar proof of (3) is found in [9].

There are very few congruences known for $p(n)$ for the prime 13. In [6] one finds the congruence:

For $n \equiv 6 \pmod{13}$, $p(13^2 n - 7) \equiv 6p(n) \pmod{13}$.

The same author, Newman, also gives in [7] the congruence:

$$\text{For } (n,6) = 1,\ p\left(84n^2 - \frac{(n^2-1)}{24}\right) \equiv 0 \pmod{13}.$$

In the recent paper [2], one finds the congruences:

For n=1

(a) $p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13}$

(b) $p(1685441105467 99n - 6950975499605) \equiv 0 \pmod{13^2}$

No congruences involving only the partition function are known for primes 17, 19, or higher.

Likewise for smaller primes, namely 2 and 3, no congruences involving only the partition function are known. In fact as of this date, other than checking a table of $p(n)$, there is no way of knowing whether $p(n)$ is even or odd for a given n. However in [5] there is the following:

Theorem 3: $p(n)$ is even and odd infinitely often.

Proof: We have from Theorem 1,

(1) $p(n) - p(n-1) - p(n-2) + p(n-5) + \cdots = 0$,

where the general term is given by $(-1)^k p[n - \frac{1}{2}k(3k \pm 1)]$. Now suppose that $p(n) \equiv 0 \pmod 2$ for all $n \geq a$. With $n = \frac{1}{2}a(3a - 1)$, (1) becomes

$$p[\tfrac{1}{2}a(3a-1)] - p[\tfrac{1}{2}a(3a-1) - 1] - \cdots \pm p(2a-1) \mp p(0) = 0,$$

and since $p(0) = 1$ we have a contradiction (modulo 2). Likewise, suppose $p(n) \equiv 1 \pmod 2$ for all $n \geq b$. One obtains a contradiction by

taking $n = b(3b + 1)/2$ in (1), since the left-hand side contains an odd number of odd terms.

One can devote a lifetime studying the properties of $p(n)$, and the years will have been spent in a fascinating area, where an infinite number of questions remain to be answered.

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WILL YOUR CHAPTER BE REPRESENTED IN MISSOULA?

It is time to be making plans to send an undergraduate delegate or speaker from your chapter to attend the annual meeting of Pi Mu Epsilon at the University of Montana, Missoula, Montana during August 20-22, 1973. Each speaker who presents a paper will receive travel funds of up to \$300, and each delegate, up to 150. Chapters desiring to participate must apply for these funds at the National Office.

A GENERAL TEST FOR DIVISIBILITY

Robb T. *Koether*
University of Richmond

A topic of much interest in number theory is that of divisibility. Under this topic there has been much research done to find methods to test the divisibility of one number by another. Let us recall some of the better-known methods of testing for divisibility by 3, 9, 7, 13, 11, and 17.

To test a number for divisibility by 3, find the sum of the digits of the number. If this sum is divisible by 3, then so is the original number, and conversely.

A test for divisibility by 9 is similar except that the sum of the digits is tested for divisibility by 9. The original number is divisible by 9 if and only if this sum is divisible by 9.

To find whether a number is divisible by 7, multiply the units digit by 2 and subtract this product from the number that results from deleting the units digit from the given number. The original number is divisible by 7 if and only if this difference is divisible by 7.

To test for divisibility by 13, multiply the units digit by 4 and add this product to the number obtained by deleting the units digit from the given number. (This process may be repeated until a 2-digit number results.) If the result is divisible by 13, then so is the original number, and conversely.

To test for divisibility by 11, alternately add and subtract the digits of the number. The number is divisible by 11 if and only if the result is divisible by 11.

To test for divisibility by 17, multiply the units digit of the number by 51 and subtract this product from the number. The number is divisible by 17 if and only if this difference is divisible by 17.

The general test for divisibility developed below includes each of the above tests as special cases. First, a lemma from number theory will be stated.

<u>*Lemma*</u>: Let a and b be positive integers such that $(a,b) = 1$. Then there exist integers r and s such that $ar + bs = 1$.

The general test is contained in the following result:

<u>*Theorem*</u>: Let $n = 10a + b$ be a positive integer, with $0 \le b \le 9$; let d be a positive integer such that $(d,10) = 1$; let integers r and s satisfy the equation $dr + 10s = 1$. Then d divides n if and only if d divides $(a + bs)$.

Proof: Suppose $d|n$. Then $d|(10a + b)$; hence there exists an integer t such that $dt = 10a + b = 10a + b(dr + 10s) = 10a + bdr + 10bs$. Then $d(t - br) = 10a + 10bs = 10(a + bs)$; hence $d|10(a + bs)$. But it was assumed that $(d,10) = 1$, and it follows that $d|(a + bs)$.

Now suppose $d|(a + bs)$. Then there exists an integer k such that $dk = a + bs$. It follows that $d(10k) = 10a + 10bs$. Then $d(10k + br) = 10a + 10bs + dbr = 10a + b(10s + dr) = 10a + b = n$. Hence $d|n$.

The second part of the proof tells us that to test n for divisibility by d, we find values for r and s such that $dr + 10s = 1$, multiply b (the last digit of n) by s and add this to a (the number that results from deleting the units digit from n). If $a + bs$ is divisible by d, then we know that n is divisible by d. The first part of the proof tells us that if d does not divide $a + bs$, then d does not divide n.

For example, let us test 1219 for divisibility by 23. We find that $(23)(-3) + (10)(7) = 1$ therefore $r = -3$ and $s = 7$. (Also $d = 23$, $a = 121$, and $b = 9$.) Now $a + bs = 121 + 9(7) = 184$. To test 184 for divisibility by 23, we test $18 + 4(7) = 46$ for divisibility by 23. But $23|46$; therefore $23|184$, and hence $23|1219$.

In the theorem it was stated that $(d,10) = 1$. This eliminates all multiples of 2 and 5. But this really makes the method no less general because divisibility by 2 and 5 is easily checked and then any such factors can be deleted from d, after which the method is applicable.

Because d is relatively prime to 10, the last digit of d must be 1, 3, 7, or 9. Table I shows the four possible forms of d, the corresponding forms of the equation $dr + 10s = 1$, where r and s have the smallest absolute value of all possible r and s, and the corresponding forms of s.

Table 2 contains values of d and corresponding values of s. One can see that the values of s form an arithmetic progression.

d	$dr + 10s = 1$	s
$10t + 1$	$(10t + 1)(1) + (10)(-t) = 1$	$-t$
$10t + 3$	$(10t + 3)(-3) + (10)(3t + 1) = 1$	$3t + 1$
$10t + 7$	$(10t + 7)(3) + (10)(-3t - 2) = 1$	$-3t - 2$
$10t + 9$	$(10t + 9)(-1) + (10)(t + 1) = 1$	$t + 1$

TABLE 1

d	s	d	s	d	s	d	s
1	0	3	1	7	-2	9	1
11	-1	13	4	17	-5	19	2
21	-2	23	7	27	-8	29	3
31	-3	33	10	37	-11	39	4

TABLE 2

It is easily shown that this test proves each of the specific tests cited at the beginning of this paper.

First, if $d = 3$, the units digit, b, of $n = 10a + b$ is multiplied by 1 and added to a. After this is repeated several times, the result is the sum of the digits, which is then tested for divisibility by 3. If a "1" is carried in any of these additions, a "10" is being subtracted and a "1" is being added, which is equivalent to subtracting 9. Since 3 divides 9, the result is not affected.

When testing for divisibility by 9, the value of s is also 1, so again the sum of the digits is found. Also as before, if 1 is carried in the addition, it results in 9 being subtracted from the sum, which will not affect the result.

Using this test for 11, the final digit, b, is multiplied by -1 and added to a. After this process is repeated several times the result is the same as that of alternately adding and subtracting the digits of the original number. In the case where one must "borrow" during the subtraction, 10 is added to the final digit of the difference and 1 is subtracted from the next-to-last digit. But on the next subtraction, the "10" which was added is now subtracted, resulting in

11 being subtracted, which will not affect the result.

We can see from the values of s when d equals 7 or 13 that these two methods are identical to the tests for 7 and 13 shown earlier.

In the test shown previously for 17, after subtracting, the last digit of the difference will always be zero, which can then be ignored. The only part to be concerned with then was the product of 5 and the units digit, which was subtracted from all but the units digit. With this new method, when $d = 17$, then $s = -5$. So after multiplying the last digit by -5 and adding that to the number represented by the remaining digits, the same result is obtained as when multiplying the units digit by 51 and subtracting from the original number.

To illustrate the compact form in which a complete test can be displayed, let us test 22306426 for divisibility by 89. Note that $89 = 10(8) + 9$, hence $t = 8$ and $s = 8 + 1 = 9$. The computation can be arranged as follows:

```
22306426
      54
--------
 2230696
      54
--------
  223123
      27
--------
   22339
      81
--------
    2314
      36
--------
     267
      63
--------
      89
```

Now $89 | 89$, therefore $89 | 22306426$.

This general test for divisibility can be extended to numbers in any base. In base b, the divisor d will take the form $bt + n$ where $0 \le n \le b - 1$. The equation $dr + 10s = 1$ becomes $dr + bs = 1$. A general solution for r and s in this equation is:

$$r = \frac{1 - bk}{n}, \qquad s = \left(\frac{bk - 1}{n}\right) t + k$$

where k is chosen so that r and s are integers. It is now the value of $\left(\frac{bk - 1}{n}\right) t + k$ that is used to perform the divisibility test. The proof of the theorem for base b is identical to that for base 10, with the restriction that $(d, b) = 1$.

As an illustration, consider the test of $(4611)_7$ for divisibility by $(24)_7$. In this case, $b = 7$, $t = 2$, $n = 4$; thus

$$s = \left(\frac{7k - 1}{4}\right) 2 + k.$$

A suitable value for k is 3, giving s a value of 13, which is represented as $(16)_7$. We now perform the test in base 7, with all arithmetic being done, of course, in base 7.

$$\begin{array}{r} 4611 \\ \underline{16} \\ 510 \\ \underline{0} \\ 51 \\ \underline{16} \\ 24 \end{array}$$

And $(24)_7 | (24)_7$, therefore $(24)_7 | (4611)_7$.

NEW KEY-PINS AVAILABLE

Because of increased costs, the Balfour Company has recently produced a new key-pin for Pi Mu Epsilon which is identical in appearance to the old one, but contains less gold. The National Office is now distributing these pins at the special price of $5.00 per pin, post paid to anywhere in the United States. Be sure to indicate the Chapter into which you *were* initiated and the approximate date of *your* initiation. Gold pins are still available from our authorized jeweler, L. G. Balfour Company, but the new gold finish pins are available only from the national office:

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601 Elm Avenue, Room 423
University of Oklahoma
Norman, Oklahoma 73069

THE PERMUTATION GAME

By *Thomas Fournelle*
University of Illinois[1]

The permutation game (for lack of a better name) generally consists of 15 numbered squares confined in a larger square just large enough to hold 16 squares. In the diagram (see Fig. 1), "X" denotes the blank square. The 15 squares are free to move up and down, right

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	X

FIGURE 1

or left, but they can only be moved when the blank square is adjacent and cannot be moved outside of the confining square. When playing this children's game the question naturally arises as to the types of permutations it is possible to obtain with the simple moves which are allowed.[2]

We use the standard notation for permutations[3], e.g., $P = (a_1, a_2, \ldots, a_n)$ is the cyclic permutation such that $(a_1)P = a_2$, $(a_2)P = a_3$, ..., $(a_n)P = a_1$. A cyclic permutation of the form $P = (a_1, a_2)$ is called a transposition. Any permutation of a finite number of elements can be decomposed into a product of transpositions, and the number of transpositions in any such decomposition is congruent modulo 2 to the number of

---

[1]This article was written while the author was an undergraduate at St. Louis University.

[2]For another explanation of the permutation game see: Introduction to Contemporary Algebra, Marvin L. Tomber (Prentice-Hall, pp 350 ff).

[3]For more on the theory of permutations see: Herstein's Topics in Algebra, or any other introductory algebra text.

transpositions in any other decomposition. Thus, we define a permutation to be even if the number of transpositions in any decomposition is even. Otherwise, the permutation is said to be odd. We shall study only those permutations which leave the blank square invariant.

Proposition 1: Any permutation obtainable in the permutation game (which leaves the blank square invariant) is even.

Proof: We first note that to obtain a permutation we move some square into the place of the blank square, thereby moving the blank square. Then we move another square into the new position of the blank square, moving the blank square again, and so on. Thus the blank square is tracing a path and we have only to study these paths. Also, the blank square eventually comes to rest at its initial position, since we are considering permutations which leave it invariant.

We study first "simple closed paths" of the blank square, that is, those paths which have no self-intersections until the blank square reaches its original position. For every move to the left, there must be one to the right. For every move up, there must be a move down, and so on. Thus, the blank square moves an even number of times. Let us call the first square in the path of the blank square a, the second a_2, and so on, up to the last square in the path, which we call a_n (see Fig. 2). As the blank square moves around its path, a moves to the initial

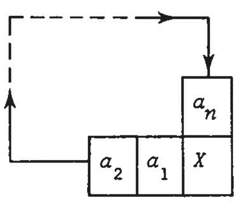

FIGURE 2

position of the blank square, a_2 moves to the initial position of a_1, and so on, until a_n moves to the initial position of a_{n-1}. Then the blank square is in the initial position of a_n and has one more move to make. It returns to its initial position and moves a_1 to the initial position of a. Thus, the permutation obtained is

$$(a_1, a_n, a_{n-1}, \ldots, a_3, a_2) = (a_1, a_n)(a_1, a_{n-1}) \ldots (a_1, a_3)(a_1, a_2).$$

The blank square moves an even number of times, but it moves a twice. Since each move of the blank square affects one square, there must be an odd number of such squares, so n is odd. Therefore, the permutation obtained must be even, as can be seen from the above decomposition into transpositions.

We now use induction on the number of self-intersections made by the path of the blank square itself. We have dealt with the case $n = 0$. Let P be a permutation obtained from a path of the blank square with n self-intersections. Let a be the square at which the path makes its first self-intersection (see Fig. 3). We decompose P into the product of permutations RST, where R is the permutation obtained by moving the

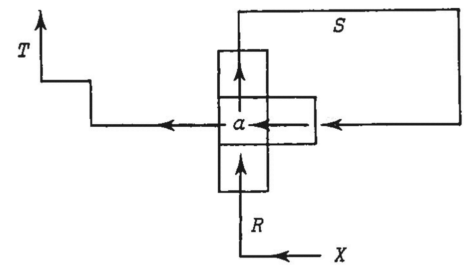

FIGURE 3

blank square along the path of P until it reaches the first self-intersection a, S is the permutation obtained by continuing the movement of the blank square along the path of P until it again reaches a, and T is the permutation which completes the path of the blank square, as does P. R and S must permute distinct squares on the board. For if they do not, the paths of R and S must intersect. But they intersect only at a, since this is the first self-intersection of the entire path, and the path of S begins at a and ends at a with no self-intersections. Therefore, the only square which both R and S can move is a, but as can be seen from Fig. 3, S does not move a at all. Therefore, $RS = SR$ and hence

$$P = RST = SRT = S(RT).$$

S is an even permutation by the case for $n = 0$. RT is even by the induction hypothesis since it has fewer than n self-intersections. Since the product of even permutations is even, P is even, ending the proof.

We see at once that we cannot obtain the permutation of just two squares on the board, since this is an odd permutation. Also, starting with Fig. 1, we cannot obtain the permutation of Fig. 4, since this permutation is

$$P = (1,15)(2,14)(3,13)(4,12)(5,11)(6,10)(7,9),$$

which is odd.

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	X

FIGURE 4

Proposition 2: Any even permutation (leaving the blank square invariant) is obtainable.

Proof: Any distinct squares a, b, and c arranged in the configuration of Fig. 5 can be cyclically permuted as indicated. By returning to

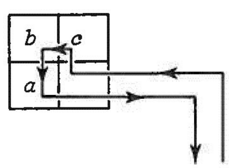

FIGURE 5

its original position along the same path by which it reached a, b, and c, the blank square affects only a, b, and c and it permutes these cyclically. If a, b, and c are not in the configuration of Fig. 5, we can always put them in such a configuration by a permutation P (which also may affect many other squares on the board). We may then apply a cyclic permutation C to a, b, and c in their new positions. It follows that PCP^{-1} cyclically permutes a, b, and c and leaves the rest of the board invariant. Thus, for any distinct a, b, and c we may obtain the cyclic permutation $(a,b,c) = (a,b)(a,c)$. But $(a,b)(c,d) = (a,b)(a,c)(a,c)(c,d) = (a,b)(a,c)(c,a)(c,d) = (a,b,c)(c,a,d)$. Thus, for distinct a, b, c, and d we may obtain the permutations $(a,b)(a,c)$ and $(a,b)(c,d)$. If P is any even permutation we may decompose it into an even number of transpositions. The first two of these are of the form $(a,b)(a,c)$ or $(a,b)(c,d)$ and thus are obtainable, and similarly for the rest of the pairs in the decomposition. Therefore, P is the product of obtainable permutations and is obtainable, proving the theorem.

Propositions 1 and 2 give us the following

Corollary: A permutation leaving the blank square invariant is obtainable iff it is even.

COUNTEREXAMPLE TO "THEOREM" PUBLISHED IN LAST ISSUE

Robert W. Quackenbush of the University of Monitoba has pointed out that the main result of the note "Basis for an Algebraic System" by J. E. Cain, Jr. (this Journal, Vol. 5, No. 7, 1972, pp. 319-320) is in error. The following is a counter-example:

Let A be the positive integers and let $F = \{f\}$, where f is of degree 2 and is defined by

$$f(m,n) = m + 1, \quad \text{if } m \geq n,$$
$$= 1, \quad \text{otherwise.}$$

It is easily seen that the algebra $(A;F)$ is not finitely generated. However, $(A;F)$ is generated by any ***infinite subset,*** and as such, cannot have a minimal generating set (basis).

The flaw in the argument as originally presented occurs when it is a assumed that a maximal independent subset generates the entire algebra.

Magic Squares Within Magic Squares

by *Joseph M. Moser*
California State University at San Diego

In [1] Strum exhibits two five by five magic squares which contain three by three magic squares. In Fig. 1 below we have exhibited a seven by seven magic square that contains within it a five by five magic square and a three by three square as exhibited by Strum in [1].

Since all entries are of the form $n + (qc + p)b$ and n and b are distinct integers it is sufficient to have $c \geq 6$ to insure that all entries be distinct. If one uses large enough integers for q and p, this author is of the opinion that any $(2m + 1)$ by $(2m + 1)$ magic square can be obtained that will have the property of containing within it successive magic squares.

$n-(4c-2)b$	$n-(4c+1)b$	$n-(3c-1)b$	$n-(3c+2)b$	$n+(5c-3)b$	$n+(5c+1)b$	$n+(4c+2)b$
$n-5cb$	$n-(2c-1)b$	$n-2cb$	$n-(2c+2)b$	$n+4cb$	$n+(2c+1)b$	$n+5cb$
$n+(5c+2)b$	$n-3cb$	$n-b$	$n-(c-1)b$	$n+cb$	$n+3cb$	$n-(5c+2)b$
$n+(4c-3)b$	$n+(3c+1)b$	$n+(c+1)b$	n	$n-(c+1)b$	$n-(3c+1)b$	$n-(4c-3)b$
$n-(c-2)b$	$n+(4c-1)b$	$n-cb$	$n+(c-1)b$	$n+b$	$n-(4c-1)b$	$n+(c-2)b$
$n+(5c-1)b$	$n-(2c+1)b$	$n+2cb$	$n+(2c+2)b$	$n-4cb$	$n+(2c-1)b$	$n-(5c-1)b$
$n-(4c+2)b$	$n+(4c+1)b$	$n+(3c-1)b$	$n+(3c+2)b$	$n-(5c-3)b$	$n-(5c+1)b$	$n+(4c-2)b$

FIGURE 1

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BRIEF REVIEW OF TWO NEW JOURNALS OF GEOMETRY

<u>**Geometriae Dedicata**</u>. D. Reidel Publishing Company, Holland. Vol. 1, No. 1, November 1972 (published yearly). $40.63 per volume to institutions, $24.38 per volume to individuals. (Editorial office: H. Freudenthal, Mathematisch Instituut der Rijksumiversiteit Utrecht, Utrecht, Budapestlaan - The Netherlands.) As its name implies, this journal is dedicated to geometry, a perhaps obsolete subdivision of mathematical research in the classical sense of the term. However, the following topics appear to be among those which the editors are presently identifying as belonging to this area, judging from the articles to be published in the first two issues: Classical Geometry (such as Projective Geometry, and Euclidean and Noneuclidean Geometry), Convexity, Algebraic Geometry, Finite Geometry, Transformation Groups, Lie Theory, Tesselation Theory, and Axiomatic Geometry (or Foundations). Apparently, topics in topology will be avoided -- even Geometric Topology (general curve theory, manifold theory, and problems pertaining to the topology of E^n, $n \geq 2$), as are topics in Graph Theory, although this point is not made clear in the editorial policy statement. The only statement which attempts to identify those areas acceptable to Dedicata reads "... most people and, in particular, geometers will agree that there still exists something that rightly may be termed geometry, if no longer as a well defined domain then certainly as a specific attitude of the creative mind, which distinguishes itself from other attitudes in mathematical research."

Obviously, as far as Dedicata is concerned, the final decision as to what constitutes a topic in geometry and what does not rests with the very distinguished members of the editorial board, which includes such renowned geometers as Hans Freudenthal (chairman), A. Barlotti, H. S. M. Coxeter, Branko Grünbaum, G. Hajos, D. G. Higman, D. R. Hughes, A. V. Pogorelov, G. C. Shephard, J. Tits, and K. Yano.

<u>**Journal of Geometry**</u>. Birkhauser Verlag, Basel and Stuttgart. Vol. 1, No. 1. DM 35 pervolume or DM 25 per single copy. (Editorial office: Universität Bochum, Mathematisches Institut, 463 Bochum, Germany.)

Similar to *Geometriae Dedicata* in purpose and coverage, this journal is devoted to the publication of current developments in geometry, "particularly of recent results in Foundations of Geometry, Geometric Algebra, Finite Geometries, Combinatorial Geometry, and special geometries. Although Geometry is a discipline dominating the interest and efforts of a great many mathematicians throughout the world, to date there has been no journal devoted specifically to these topics. It is hoped that the 'Journal of Geometry' will help to fill the gap" (from the editorial policy statement). The editorial board consists of the following prominent geometers: R. Artzy, M. Barner, A. Barlotti, W. Benz, R. C. Bose, H. Crapo, H. Karzel, R. Lingenberg, R. Rado, and G. Tallini.

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PROBLEM DEPARTMENT

Edited by Leon Bankoff
Los Angeles, California

This department welcomes problems believed to be new and, as a rule, demanding no greater ability in problem solving than that of the average member of the Fraternity. Occasionally we shall publish problems that should challenge the ability of the advanced undergraduate or candidate for the Master's Degree. Old problems characterized by novel and elegant methods of solution are also acceptable. Proposals should be accompanied by solutions, if available, and by any information that will assist the editor. Contributors of proposals and solutions are requested to enclose a self-addressed postcard to expedite acknowledgement.

Solutions should be submitted on separate sheets containing the name and address of the solver and should be mailed before the end of November 1973.

Address all communications concerning problems to Dr. Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California 90048.

Problems for Solution

292. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York.*

If perpendiculars are constructed at the points of tangency of the incircle of a triangle and extended outward to equal lengths, then the join of their endpoints form a triangle perspective with the given triangle.

293. *Proposed by Iw Kowarski, Morgan State College, Baltimore, Maryland.*

Prove that $N = 53^{103} + 103^{53}$ is divisible by 78.

294. *Proposed by Charles W. Trigg, San Diego, California.*

Show that *ABCD* is a square (Fig. 1).

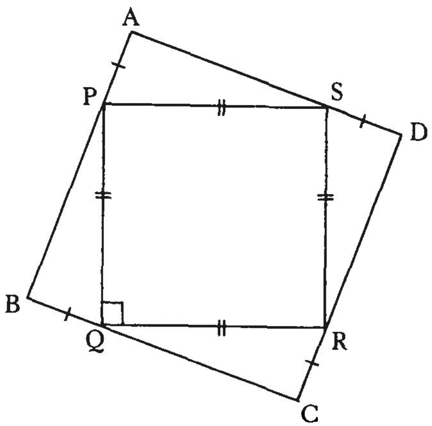

FIGURE 1

295. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Determine an equation of a regular dodecagon (the extended sides are not to be included).

296. *Proposed by Solomon W. Golomb, University of Southern California, Department of, Electrical Engineering.*

1) Combine 2, 5, and 6 to make four 2's.
2) Combine 2, 5, and 6 to make four 4's.
3) Combine 2, 5, and 6 to make four 5's.
4) Combine 2, 5, and 6 to make four 7's.
5) Combine 1, 5, and 6 to make four 7's

297. *Proposed by Roger E. Kuehl, Kansas City, Missouri.*

A traffic engineer is confronted with the problem of connecting two non-parallel straight roads by an S-shaped curve formed by arcs of two equal tangent circles, one tangent to the first road at a selected

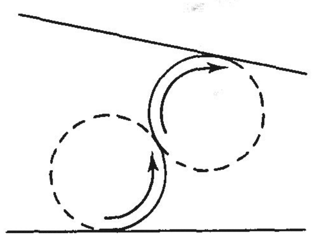

FIGURE 2

point and the other touching the second road at a given point (Fig. 2).

1) Determine the radius of the equal circles synthetically, trigonometrically or analytically.

2) If the figure lends itself to an Euclidean construction, how would one go about it?

298. *Proposed by Paul Erdös, Budapest, Hungary and Jan Mycielski, University of, Colorado, Boulder, Colorado*

Prove that

1) $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{n} + \sqrt[3]{n} + \ldots + \sqrt[n]{n}\right) = 1$

2) $\lim_{n\to\infty} \frac{1}{n}\left(n^{1/\log 3} + n^{1/\log 4} + \ldots + n^{1/\log n}\right) = e$

299. *Proposed by David L. Silverman, West lob Angeles, California.*

On the back of an envelope you see the results of an interrupted game by two players whom you know to be tic-tac-toe experts. It is generally recognized that the expert never puts himself into a potentially losing position and always wins if his opponent gives him the opportunity. There are 2 X's and 2 O's on the diagram. It is impossible to deduce whose move it is. Neglecting symmetry, what is the position?

300. *Proposed by the. Problem Editor.*

It can be shown with difficulty that if the opposite angles of a skew quadrilateral are equal in pairs, the opposite sides are also equal in pairs. (The reward of instant immortality is offered the solver who can prove this without difficulty). If two opposite sides of a *skew* quadrilateral are equal and the other two unequal, is it possible to have one pair of opposite angles equal?

301. *Proposed by Neal Jacobs, Bronx, New York.*

One-fifteenth can be expressed in "decimals" in many *ways*, for example, as $\overline{.0421}$ in base eight, or as $\overline{.013}$ in base five. Show that in any base n, the "decimal" for one-fifteenth will have no more than four recurring digits.

302. *Proposed by David L. Silverman, West Los Angeles, California and Alfred E. Neuman, Mu Alpha Delta Fraternity, New York.*

A tapestry is hung on a wall so that its upper edge is a units and its lower edge b units above the observer's eye-level. Show that in order to obtain the most favorable view the observer should stand at a distance $\sqrt{ab}$ from the wall.

Solutions

266. [Fall 1971] *Proposed by Frank P. Miller, Pennsylvania State University.*

Prove or disprove that the only integral solution of the equation $r^2 + 3s^2 + 4t^2$ is the trivial one, $r = s = t$.

Solution by Charles W. Trigg, San Diego, California.

Only *positive* integral solutions need be sought. The given equation may be written in the form

$$(2t + r)(2t - r) = 3s^2 = mn,$$

where $m > n$, Then matching factors we have

$$2t + r = m \quad \text{and} \quad 2t - r = n,$$

whereupon $t = (m + n)/4$ and $r = (m - n)/2$. Thus, for a solution in integers to exist, m and n (the factors of $3s^2$) must have the same parity.

If $m = 3$ and $n = s^2$, then $s = 1 = t = r$.

If $m = 3s$ and $n = s$, then $s = t = r$.

Non-trivial solutions may be obtained from the other pairings, $(m, n) = (s^2, 3)$ and $(3s^2, 1)$. Consider the four forms that s may have: $4k + 1$, $4k + 3$, $4k + 2$, or $4k$, where $k > 0$.

Case Ia. $s = 4k + 1$, $m = (4k + 1)^2$, $n = 3$.
Then $t = 4k^2 + 2k + 1$, $r = 8k^2 + 4k - 1$.

Case Ib. $s = 4k + 1$, $m = 3(4k + 1)^2$, $n = 1$.
Then $t = 12k^2 + 6k + 1$, $r = 24k^2 + 12k + 1$.

Case IIa. $s = 4k + 3$, $m = (4k + 3)^2$, $n = 3$.
Then $t = 4k^2 + 6k + 3$, $r = 8k^2 + 12k + 3$.

Case IIb. $s = 4k + 3$, $m = 3(4k + 3)^2$, $n = 1$.
Then $t = 12k^2 + 18k + 7$, $r = 24k^2 + 36k + 13$.
Indeed, if $k = 0$, $s = 3$, $t = 7$, $r = 13$.

Case IIIa. $s = 4k + 2$, $m = 2(2k + 1)^2$, $n = 6$.
Then $t = 2(k^2 + k + 1)$, $r = 2(2k^2 + 2k - 1)$.

Case IIIb. $s = 4k + 2$, $m = 6(2k + 1)^2$, $n = 2$.
Then $t = 2(3k^2 + 3k + 1)$, $r = 2(6k^2 + 6k + 1)$.

Case IVa. $s = 4k$, $m = 4k^2$, $n = 12$.
Then $t = k^2 + 3$, $r = 2(k^2 - 3)$.

Case IVb. $s = 4k$, $m = 12k^2$, $n = 4$.
Then $t = 3k^2 + 1$, $r = 2(3k^2 - 1)$.
Indeed, if $k = 1$, $s = 4 = t = r$, a trivial solution.

Thus there is at least one non-trivial integral solution of $r^2 + 3s^2 = 4t^2$ for every integer value of s except ± 1, ± 2, and ± 4. [Any combination of the positive and negative signs in the solutions $(\pm a, \pm a, \pm a)$ is considered to be a trivial solution]. If $s = 0$, $r = \pm 2t$.

These do not comprise all of the integral solutions of the given equation. If s is composite, factorization of $3s^2$ into m and n in other ways is possible. The more factors s has, the more solutions are possible. Examples from each of the four cases follow:

Case I. $s = 4k + 1 = 21 = 3(7)$, $m = 49$, $n = 27$, $t = 19$, $r = 11$.

Case II. $s = 4k + 3 = 15 = 3(5)$, $m = 27$, $n = 25$, $t = 13$, $r = 1$.

Case III. $s = 4k + 2 = 30 = 2(3)(5)$, $m = 54$, $n = 50$, $t = 26$, $r = 2$.

Case IV. $s = 4k = 60 = 4(3)(5)$, $m = 108$, $n = 100$, $t = 52$, $r = 4$;
$m = 300$, $n = 36$, $t = 84$ $r = 132$.

Also solved by JEANETTE BICKLEY, *St. Louis Missouri;* K. BURKE, *Seton Hall University, South Orange, N. J.;* CHARLIE CARTER, *University of Richmond, Virginia;* THOMAS CATO, JR., *Adelphi University;* ROBERT C. GEBHARDT, *Hopatcong, N. J.;* CHARLES H. LINCOLN, *Fayetteville, N. C.;* C. B. A. PECK, *State College, Pennsylvania;* BOB PRIELIPP, *University of, Wisconsin, Oshkosh; and the Proposer.*

Editor's Note.:

Most of the solvers submitted valid solutions consisting of a single counter-example that disproved the statement. Carter considered the Fermat-Pell Equation $r^2 + (s\sqrt{3})^2 = (2t)^2$ and obtained solutions from the penultimate convergents in the continued fraction expansion of $\sqrt{3}$.

270. [Spring 1972] *Proposed by Leonard Carlitz, Duke University.*

Let α, β, γ denote the angles of a triangle. Show that $\cot\frac{1}{2}\alpha + \cot\frac{1}{2}\beta + \cot\frac{1}{2}\gamma \geq 3(\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta + \tan\frac{1}{2}\gamma) \geq 2(\sin\alpha + \sin\beta + \sin\gamma)$

Solution by Alfred E. Neuman, Mu Alpha Delta Fraternity, New York.

Using the relations $(\Sigma \cot \alpha/2)^2 \geq 3\Sigma \cot \alpha/2 \cot \beta/2$ and $\Sigma \cot \alpha/2 = \Pi \cot \alpha/2$, we have

$$\sum\cot\frac{1}{2}\alpha \geq \frac{3\sum\cot\frac{1}{2}\alpha\cot\frac{1}{2}\beta}{\sum\cot\frac{1}{2}\alpha} \equiv \frac{3\sum\cot\frac{1}{2}\alpha\cot\frac{1}{2}\beta}{\Pi\cot\frac{1}{2}\alpha} \equiv 3\sum\tan\frac{1}{2}\alpha$$

Since $\Sigma\tan\alpha/2 \geq \sqrt{3}$ and $\Sigma\sin\alpha \leq 3\sqrt{3}/2$, with equality only when $a = \beta = \gamma$, it follows that $3\Sigma\tan\alpha/2 \geq 2\Sigma\sin a$.

Also solved bq FRANK WEST, *University of Nevada, Reno, and the Proposer.*

271. [Spring 1972] *Proposed bq Solomon W. Golomb, California Institute of Technologq and the University of Southern California.*

Assume that birthdays are uniformly distributed throughout the year. In a group of n people selected at random, what is the probability that all have their birthdays within a half-year interval? (This half-year interval is allowed to start on any day of the year, in attempting to fit all n birthdays into such an interval.)

Solution bq the Proposer

The probability is $n/2^{n-1}$ because any of the n birthdays can be used, mutually exclusively, to start the six-month interval, in which case the probability of all the other birthdays falling into the interval thus begun is $1/2^{n-1}$.

More generally, if all the birthdays are to fall in some fraction a of a year, where $0 < a \leq 1/2$, the probability is $n\alpha^{n-1}$. For the case $a > 1/2$, the solution becomes much more complicated.

Also solved bq MASAO JOHNSON, *Occidental College, Los Angeles;* N. J. KUENZI, *Oshkosh, Wisconsin;* SID SPITAL, *Hayward, California;* *and* FRANK WEST, *University of Nevada, Reno.* Some of the submitted solutions differed from the Proposer's solution.

272. [Spring 1972] *Proposed bq Charles* W. *Trigg, San Diego, California.*

A timely crytarithm is the calendar verity

$$7(\overline{DAY}) = \overline{WEEK}.$$

The-letters in some order represent consecutive positive digits. What are they?

Solution bq Catherine A. *Yee, Ohio State University, Columbus.*

Since the letters D, A, Y, W, E, K represent consecutive positive digits, we know that their range must equal 6. Because of the range restriction; we have ($Y \neq 1$, $K \neq 7$) and ($Y \neq 9$, $K \neq 3$). Also since Y and K are distinct, $Y \neq 5$ and $K \neq 5$.

The largest number that DAY can represent is 987, and since the product of 7 and 987 is equal to 6909, W cannot exceed 6.

These facts can be used to shorten the execution time of any computer program for solving the cyptarithm.

Below is a FORTRAN program and output from a WATFIV compiler on an IBM370/165. The letters Y, E, K, W, A, D correspond to the digits 2, 3 4, 5, 6, 7.

```
      IMPLICIT INTEGER(A-Z)
C D, A, Y, W, E, K ARE ALL POSITIVE INTEGERS
C W IS LESS THAN OR EOUAL TO 6
      DO 3 W = 1, 6
      DO 2 E = 1, 9
      IF (E .EQ. W) GO TO 2
      DO 1 K = 1, 9
C K IS NOT EQUAL TO 3 OR 5 OR 7
      IF (K.EQ.3 .OR. K.EQ.5 .OR. K.EQ.7) GO TO 1
      IF (K.EQ.W .OR. K.EQ.E) GO TO 1
      WEEK = W*1000 + E*100 + E*10 + K
      IF (WEEK/7*7 .NE. WEEK) GO TO 1
      DAY = WEEK/7
      Y = MOD(DAY,10)
C Y IS NOT EQUAL TO 1 OR 5 OR 9
      IF(Y.EQ.0 .OR. Y.EQ.1 .OR. Y.EQ.5 .OR. Y.EQ.9) GO TO 1
      IF(Y.EQ.W .OR. Y.EQ.E .OR. Y.EQ.K) GO TO 1
      DA = DAY/10
      A = MOD(DA,10)
      IF(A.EQ.0) GO TO 1
      IF(A.EQ.W .OR. A.EQ.E .OR. A.EQ.K .OR. A.EQ.Y) GO TO 1
      D = DA/10
      IF(D.EQ.0) GO TO 1
      IF(D.EQ.W .OR. D.EQ.E .OR. D.EQ.K .OR. D.EQ.Y .OR. D.EQ.A)GO TO 1
      MAX = MAXO(D,A,Y,W,E,K)
      MIN = MINO(D,A,Y,W,E,K)
      IF(MAX-MIN .GT. 5) GO TO 1
      WRITE(6,101)DAY, WEEK
  101 FORMAT(10X,'DAY = ',I3/10X,'WEEK = ',I4)
    1 CONTINUE
    2 CONTINUE
    3 CONTINUE
      STOP
      END
```

```
DAY = 762
WEEK = 5334
```

Also solved bq RICHARD L. ENISON, *New York Cay;* MIKE FIDDES, *South Dakota School of Mines and Technologq;* R. C. GEBHARDT, *Hopatcong N.J.;* MASAO JOHNSON, *Occidental College, Lob Angeles;* JAMES R. METZ, *Springfield, Illinois:* JAMES REBHORN, *Lebanon Valley College, Annville, Pennsylvania;* FRANK WEST, *University of Nevada, Reno;* GREGORY WULCZYN, *Lewisburg, Pennsylvania;* and the *Proposer.*

273. [Spring 1972] *Phopobed by Charles W. Trigg, Sun Diego, California.*

Twelve toothpicks can be arranged to form four congruent equilateral triangles. Rearrange the toothpicks to form ten triangles of the same size.

Solution by R. C. Gebhardt, James R. Metz, Richard D. Stratton and the Proposer.

Form a regular tetrahedron with six toothpicks and use the other six toothpicks (three to a face) to build tetrahedra on two of the faces of the first tetrahedron.

Editor's Note

A two-dimensional solution of a modified version of the problem was offered by FRANK WEST, of the University of Nevada, Reno. Each triangle in the plane configuration is one-fourth the area of the tetrahedral face in the thee-dimensional solution (Fig. 3).

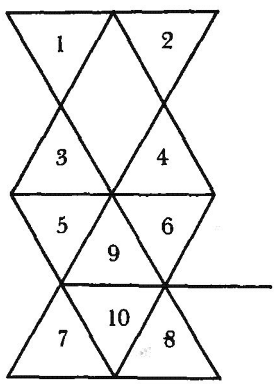

FIGURE 3

274. [Spring 1972] *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

Find the value of

$$\sum_{i=1}^{\infty} \frac{\sum_{j=1}^{k}\binom{k}{j} i^{k-j}}{(i^k)(i+1)^k}$$

for an arbitrary integer $k \geq 1$.

Solution by N. J. Kuenzi, Oshkosh, Wisconsin.

First note that for any integer $k \geq 1$,

$$\sum_{j=1}^{k}\binom{k}{j} i^{k-j} = (1+i)^k - i^k$$

Hence,

$$\sum_{i=1}^{\infty}\left(\frac{\sum_{j=1}^{k}\binom{k}{j} i^{k-j}}{i^k(i+1)^k}\right) = \sum_{i=1}^{\infty}\left(\frac{(1+i)^k - i^k}{i^k(1+i)^k}\right)$$

$$= \sum_{i=1}^{\infty}\left(\frac{1}{i^k} - \frac{1}{(1+i)^k}\right)$$

$$= 1.$$

Practically identical solutions were submitted by RICHARD L. ENISON, *Nw Yohk City;* MASAO JOHNSON, *Occidental College, Lob Angeles;* DONALD KNIGHT, *Cleveland, Ohio;* M. J. KNIGHT, *California Institute of Technology;* BOB PRIELIPP, *University of Wisconsin, Oshkosh;* KENNETH ROSEN, *Farmington, Michigan;* SID SPITAL, *Hayward, California;* ANN STEFFEN, *University of Oklahoma, Norman, Oklahoma;* T. PAUL TURIEL, *State University of Nw York at Potsdam;* FRANK WEST, *University of Nevada, Reno;* GREGORY WULCZYN, *Lewisburg, Pennsylvania;* ROBERT MILLER, *University of California at Lob Angeles; and the Proposer.*

275. [Spring 1972] *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.*

If $t(n) = n(n + 1)/2$, show that there are an infinite number of solutions in positive integers of

$$\sum_{i=0}^{r-1} t(a+i) = \sum_{i=0}^{s-1} t(a+r+i).$$

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh.

Some particular solutions of the given equation are:

$$t(1) + t(2) + t(3) = t(4),$$

$$t(5) + t(6) + t(7) + t(8) = t(9) + t(10),$$

$$t(11) + t(12) + t(13) + t(14) + t(15) = t(16) + t(17) + t(18),$$

and

$$t(19) + t(20) + t(21) + t(22) + t(23) + t(24) = t(25) + t(26) + t(27) + t(28).$$

This leads us to conjecture that

$$\sum_{i=0}^{s+1} t(s^2 + s - 1 + i) = \sum_{i=0}^{s-1} t(s^2 + 2s + 1 + i),$$

where s is an arbitrary positive integer. We now proceed to prove our conjecture. Using mathematical induction, it can easily be established that:

$$T_n = \sum_{k=1}^{n} t(k) = \frac{n(n+1)(n+2)}{6}$$

[It may be of interest to note that T_n is the n^{th} tetrahedral (or pyramidal) number.] Thus,

$$\begin{aligned}\sum_{i=0}^{s+1} t(s^2 + s - 1 + i) &= T_{s^2+2s} - T_{s^2+s-2}\\ &= (s^6 + 6s^5 + 15s^4 + 20s^3 + 14s^2 + 4s)\\ &\quad - (s^6 + 3s^5 - 5s^3 - s^2 + 2s)\\ &= 3s^5 + 15s^4 + 25s^3 + 15s^2 + 2s.\end{aligned}$$

Also

$$\begin{aligned}\sum_{i=0}^{s-1} t(s^2 + 2s + 1 + i) &= T_{s^2+3s} - T_{s^2+2s}\\ &= (s^6 + 9s^5 + 30s^4 + 45s^3 + 29s^2 + 6s)\\ &\quad - (s^6 + 6s^5 + 15s^4 + 20s^3 + 14s^2 + 4s)\\ &= 3s^5 + 15s^4 + 25s^3 + 15s^2 + 2s.\end{aligned}$$

Therefore the given equation has an infinite number of solutions in positive integers.

Also solved by FRANK WEST, *University of Nevada, Reno, and the Proposer.*

276 [Spring 1972] *Proposed by R. S. Luthar, University of Wisconsin, Waukesha.*

Find a such that the roots of $z^3 + (2 + a)z^2 - az - 2a + 4 = 0$ lie along the line $y = x$.

Solution by Sid Spital, Hayward, California.

If the roots be along $y = x$, they are of the form $z = re^{i\pi/4}$, r real. Substitution in the given equation then yields a cubic in r,

$$(*) \qquad r^3 + (2 + a)e^{-i\pi/4}r^2 - ae^{-i\pi/2}r - (2a - 4)e^{-i3\pi/4} = 0,$$

all of whose roots must be real, and therefore all of whose coefficients must be real. This can only be satisfied if $a = 2i$, making (*) become

$$r^3 + 2\sqrt{2}r^2 - 2r - 4\sqrt{2} = (r^2 - 2)(r + 2\sqrt{2}) = 0$$

which, as required, has only real roots.

Also solved by BOB PRIELIPP, *University of Wisconsin, Oshkosh;* FRANK WEST, *University of Nevada, Reno; and the Proposer.*

277. [Spring 1972] *Proposed by Alfred E. Neuman (without solution) Mu Alpha Delta Fraternity, New York.*

According to Morley's Theorem, the intersections of the adjacent internal angle trisectors of a triangle are the vertices of an equilateral triangle. If the configuration is modified so that the trisectors of one of the angles are omitted, as shown in Figure 4, show that the connector DE of the two intersections bisects the angle BDC.

I. Amalgam of Solutions by Sid Spital, Hayward, California; David C. Kay, University of Oklahoma, Norman, Oklahoma; Leonard Carlitz, Duke University; and the Proposer.

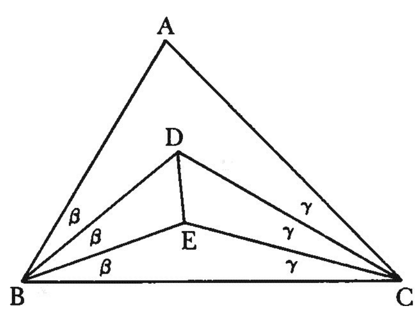

FIGURE 4

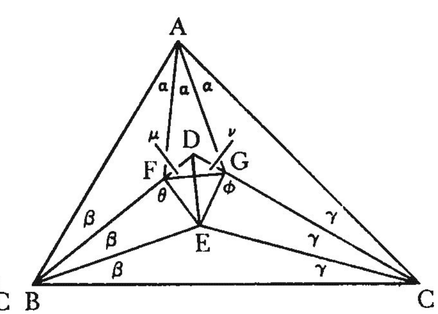

FIGURE 5

Three applications of the Law of Sines give:

$$DE \sin \angle BDE = BE \sin \beta = CE \sin \gamma = DE \sin \angle CDE.$$

Hence $\angle BDE = \angle CDE$. Carlitz remarked that the stated result (DE bisects the angle BDC) is proved by K. Venkatachaliengar, American *Mathematical*

Monthly, 65(1958), 612-613. (Fig. 4)

II. *Solution by David C. Kay, University of Oklahoma, Norman, Oklahoma.*

Let triangle FGE be the Morley triangle of trangle ABC (Fig. 5). Common proofs of Morley's theorem ultimately show that

$$\theta = \frac{\pi}{3} + \frac{A}{3} = \phi$$

(see Coxeter, *Introduction to Geometry*, Wiley 1961, p. 25, for example). Hence, since BFD and CGD are straight angles,

$$\mu + \frac{\pi}{3} + \theta = \pi = \nu + \frac{\pi}{3} + \phi$$

or $\mu = \nu$, and $FD = DG$. Hence, the perpendicular bisector of FG passes through D and E and bisects the angle FDG. That is, DE bisects the angle BDC

III. *Amalgam of Solutions by Masao Johnson, Occidental College, Lob Angeles, California; M. J. Knight, California Institute of Technology, Pasadena, California; Charles W. Trigg, San Diego, California; Frank West, University of Nevada, Reno; and the Proposer.*

Since the bisectors of the angles of a triangle are concurrent, DE bisects angle BDC of triangle BDC.

278. [Spring 1972] *Proposed by Paul Erdös, University of Waterloo, Ontario, Canada.*

Prove that every integer $\le n!$ is the sum of $< n$ distinct divisors of $n!$ Try to improve the result for large n; for example, let $f(n)$ be the smallest integer so that every integer $\le n!$ is the sum of $f(n)$ or fewer distinct divisors of n. We know $f(n) < n$. Prove $n - f(n) \to \infty$.

No solution has been received. One would be welcome.

279. [Spring 1972] *Proposed by Stanley Rabinowitz, Polytechnic Institute of Brooklyn.*

Let F_0 $F_1, F_2, \ldots$ be a sequence such that for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} .$$

An Amalgam of Solutions by Sid Spital, Hayward, California and Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

For suitable A and B, we may write:

$$F_n = A\alpha^n + B\beta^n, \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

and

$$1 + \alpha = \alpha^2, \quad 1 + \beta = \beta^2$$

Hence

$$\sum_{k=0}^{n} \binom{n}{k} F_k = A(1+\alpha)^n + B(1+\beta)^n$$

$$= A\alpha^{2n} + B\beta^{2n} = F_{2n} .$$

Also solved by HYMAN CHANSKY, *University of Maryland;* RICHARD L. ENISON, *New York City;* MASAO JOHNSON, *Occidental College, Lob Angeles;* N. J. KUENZI, *Oshkosh, Wisconsin;* PETER A. LINDSTROM, *Genesee Community College, Batavia, New York;* BOB PRIELIPP, *University of Wisconsin-Oshkosh;* KENNETH ROSEN, *Farmington, Michigan;* FRANK WEST, *University of Nevada, Reno; and the Proposer.*

280. [Spring 1972] *Proposed by Kenneth Rosen, University of Michigan.*

Find all solutions in integers of the Diophantine equation

$$x^3 + 17x^2y + 73xy^2 + 15y^3 + x^3y^3 = 10{,}000.$$

Solution by the Proposer, with a similar solution by Frank West, University of Nevada, Reno.

If the above equation is satisfied we have:

$$x^3 + 3x^2y + 3xy^2 + y^3 + x^3y^3 \equiv 4 \pmod 7$$

$$(x+y)^3 + x^3y^3 \equiv 4 \pmod 7.$$

This congruence is of the form $k^3 + j^3 \equiv 4 \pmod 7$. However, the cubic residues of 7 are 0, 1 and 6; hence the only possibilities for the residue of the sum of two cubes are 0, 1, 2, 3, 5 and 6. It is impossible for 4 to be the residue of the sum of two cubes modulo 7. Hence the equation has no solutions.

Editor's Note

Bob Prielipp called attention to several misprints in his published solution to Problem 248, on page 298 of the Spring 1972 issue. In the published solution all of the 2 symbols which appear *after* the reference to Beckenbach and Bellman's book should be > symbols.