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CONTENTS

A POLYNOMIAL FUNCTIONAL EQUATION

MURRAY S. KLAMKIN

In this note, we determine all homogeneous polynomials $P(x,y)$ of degree n such that

$$\sum_{k=1}^{m} P(S-a_k,a_k) = 0, \qquad (S = a_1 + a_2 + \ldots + a_m)$$

for all real $a_1, a_2, \ldots, a_m$ and fixed $m \geq 3$, and

$$P(1,0) = 1.$$

The special case corresponding to $m=3$ was set as a problem in the 1975 International Mathematical Olympiad held at Burgas, Bulgaria. The treatment here is "simpler" than previous ones I have seen for the special case.

Letting $a_i = 1$, it follows that $x-(m-1)y$ is a factor of $P(x,y)$. Since $\Sigma(S-a_k-(m-1)a_k) = 0$, we look for a symmetric homogeneous polynomial to be the other factor. By inspection, one solution for P is

$$P_1(x,y) = (x-(m-1)y)(x+y)^{n-1}.$$

We now show that this solution is unique.

Let $P(x,y) \equiv P_1(x,y) + Q(x,y)$. Consequently, Q is also a homogeneous polynomial of degree at most n which satisfies the same functional equation as P and is subject to $Q(1,0) = 0$. To show that Q is identically zero, we use an indirect proof. We assume $Q(x,y)$ is not zero and obtain a contradiction by showing that $Q(x,y)$ will then have an infinite number of linear factors. Letting $a_1 = a$, $a_2 = b$, $a_3 = c$, and $a_k = 0$ for $k > 3$, the functional equation reduces to

$$Q(b+c,a) + Q(c+a,b) + Q(a+b,c) = 0. \tag{1}$$

As special cases of (1), we have

$$Q(b,a) + Q(a,b) + Q(a+b,0) = 0 \implies Q(1,1) = 0,$$

and

$$Q(2b,a) + 2Q(a+b,b) = 0. \tag{2}$$

From (2), we have

$$Q(r_n,1) = 0 \implies Q(r_{n+1},1) = 0,$$

where $r_{n+1} = 1 + 2/r_n$. Starting from $r_0 = 1$, the sequence (r_n) gives rise to an infinite number of linear factors. This is impossible since Q is at most of degree n.

Comment. I have recently been informed by R.C. Lyness that one of the English contestants, John Rickard (age 15), had given the same generalization previously in his Olympiad paper.

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LETTERS TO THE EDITOR

Dear Sir:

I would like to call the attention of readers to some points in my article "An Elementary Geometric Proof of the Morley Theorem", EUREKA [1977: 291-294].

First, in the preliminary analysis it would have sufficed to suppose only:

(i) BR, CR *meet* ω *at points* Y, Z *which trisect* ω,

and then to have shown (as, e.g., by the argument embodied in the proof of the lemma) that this implies Δ XYZ is equilateral.

In accordance with this, the conclusion of the lemma need merely have been:

Then the perpendicular bisectors of XP *and* XQ *trisect* ω.

Second, it should have been remarked that the proof as was given for the lemma is for the case: P lies between S and X (which corresponds to $\alpha < 15$). The proofs for the other cases (S lies between P and X, or P and S coincide, corresponding to $\alpha > 15$ and $\alpha = 15$ respectively) follow the same lines as for the first case mentioned and the conclusions are the same.

DAN SOKOLOWSKY,
Antioch College.

Sir:

The special Morley issue of EUREKA for December, 1977, is bound to arouse a great deal of interest, and perhaps you will allow some correspondence on one particular aspect of the issue, the article by Leon Bankoff.

It is inevitable that anyone attempting to survey the large number of proofs of the Morley theorem should remark that a few begin with the triangle ABC and prove, by various methods, that the intersections of adjacent trisectors of the angles produce an equilateral triangle DEF, which is Morley's theorem, but that the large majority of synthetic proofs *begin* with an equilateral triangle DEF, and then build up, again by various methods, a triangle A'B'C' similar to the given triangle ABC, the triangle A'B'C' being such that adjacent trisectors of the angles intersect in the vertices DEF of the given equilateral triangle. A similarity transformation then produces the original triangle ABC in the statement of the Morley theorem, and a proof of the Morley theorem.

Without splitting hairs over whether the first set of proofs described above are purely "direct", the second set of proofs can be called "indirect", and it is correct to say that in the case of such proofs *a converse* of the Morley theorem is proved, from which the Morley theorem is immediately deduced.

But this point is considerably obscured in Bankoff's article (bottom of p. 295) where he says, describing "indirect" proofs:

"This procedure is essentially a proof not of the theorem as stated by Morley but of its converse. An analogous situation occurs with the Steiner-Lehmus Theorem regarding two equal base angle bisectors [see EUREKA 2 (1976) 19-24]. The easily established converse does not provide a legitimate proof of the main theorem."

Bankoff seems to imply that there is a unique converse to the Morley theorem, which may be a slip of the pen, but his reference to the Steiner-Lehmus converse not being a "legitimate" proof of the main theorem is extremely disconcerting, in view of what he has already said.

Nobody but a muddled beginner in geometry would assert that the evident proof that the base angle bisectors of an isosceles triangle are equal provides a "legitimate" proof of the Steiner-Lehmus theorem, that the equality of the base angle bisectors of a triangle implies that the triangle is isosceles. Yet Bankoff asserts that this situation is *analogous* to the situation in which a proof of a converse of the Morley theorem implies a proof of the Morley theorem, thus casting doubt on all "indirect" proofs, which can hardly be his intention?

To mystify at least one reader further, Bankoff sets up criteria for an "ideal" proof, without clarifying that these are not universal, but his own, and then selects the proof by Niewenglowski as one of the two "approximating these ideals most closely", one "that follows a direct path from hypothesis to conclusion."

But the proof he cites begins with an equilateral triangle, and is therefore "indirect", and is so listed in the excellent Bibliography!

The excellent proof given by Dan Sokolowsky in the Morley issue is not "indirect", nor is the Robson proof. Whether the methods used by Robson are too advanced for an "ideal" proof is a matter of opinion. They are methods that were current in the 1920's, and they form an intrinsic part of geometry since Euclid.

It would have been a worthwhile task if Bankoff had gone through all the known Morley proofs, and correlated those that are similar in method, weeding out the inevitable rediscoveries, and so on. Interested readers could then have formed their own opinions as to their favourite proofs, and would perhaps have been ready to share this opinion with others. But *ex cathedra* statements which are self-contradictory can only create confusion.

DAN PEDOE,
University of Minnesota.

PUBLISHING NOTE

Dan Pedoe's latest book, *Geometry and the Liberal Arts*, published in 1976 in England by Penguin Books, was reviewed in EUREKA [1977 7-9]. Several readers, and this editor, experienced considerable frustration in trying to obtain a copy of the book from the British publisher. For legal reasons, the book could not be sold in North America. The editor is pleased to announce that the book is now available to North Americans in a new (hardback) edition just published by St. Martin's Press, 175 Fifth Avenue, New York, N.Y. 10010 (price: $10.95 US). Canadians should have no difficulty in obtaining a copy directly from the American publisher or through their local booksellers.

Other books by Dan Pedoe now available· *A Geometric Introduction to Linear Algebra*, Second Edition, Chelsea Publishing Co., Bronx, N.Y., 1976, *The Gentle Art of Mathematics*, Dover Publications Inc., New York, N.Y., 1973; *A Course of Geometry for Colleges and Universities*, Cambridge Universitv Press, New York, N.Y., 1970.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1978, although solutions received after that date will also be considered until the time when a solution is published.

311. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Find the unique solution of the following base ten cryptarithmic addition.

$$\begin{array}{r} \text{OTTAWA} \\ \underline{\text{CALGARY}} \\ \text{TORONTO} \end{array}.$$

312. *Proposed by R. Robinson Rowe, Sacramento, California.*

Evaluate

$$N = \{(\sqrt[16]{14163\ 17954} - 2)^2 - 3\}^2$$

to at least five significant figures.

313. *Proposed by Leon Bankoff, Los Angeles, California.*

In an R-M-S triangle ABC (that is, a triangle in which $2b^2 = c^2 + a^2$ see [1978: 14]), prove that GK, the join of the centroid and the symmedian point, is parallel to the base b.

314. *Proposed by Michael W. Ecker, City University of New York.*

Find all functions $f: R \to R$, continuous at $x = 0$, satisfying the functional relation

$$f(x) \cdot f(y) = \left[f\left(\frac{x+y}{2}\right)\right]^2$$

for all x, y in R.

315. *Proposed by Orlando Ramos, Havana, Cuba.*

Prove that, if two points are conjugate with respect to a circle, the sum of their powers is equal to the square of the distance between them.

316*. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Démontrer l'implication

$$\frac{a-x}{x-b} = \frac{a-d}{b-c}\cdot\frac{c-y}{y-d} \Longrightarrow \frac{a-y}{y-b} = \frac{a-d}{b-c}\cdot\frac{c-x}{x-d}.$$

317. *Proposed by James Gary Propp, Great Neck, N.Y.*

In triangle ABC, let D and E be the trisection points of side BC with D between B and E, let F be the midpoint of side AC, and let G be the midpoint of side AB. Let H be the intersection of segments EG and DF. Find the ratio EH : HG by means of mass points (see EUREKA [1976: 55]) or otherwise.

318. *Proposed by C.A. Davis in James Cook Mathematical Notes No.* 12 *(October* 1977*), p.* 6.

Given any triangle ABC, thinking of it as in the complex plane, two points L and N may be defined as the stationary values of a cubic that vanishes at the vertices A, B, and C. Prove that L and N are the foci of the ellipse that touches the sides of the triangle at their midpoints, which is the inscribed ellipse of maximal area.

319. *Proposed by Leigh Janes, Rocky Hill, Connecticut.*

If a solution of the following type has not yet occurred in a student paper, it soon will. "Cancelling the exponents" yields

$$\frac{37^3+13^3}{37^3+24^3} = \frac{37+13}{37+24} = \frac{50}{61},$$

which is correct.

Find necessary and sufficient conditions for the positive integer triple (A,B,C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C}.$$

320. *Proposed by Dan Sokolowsky, Antioch College.*

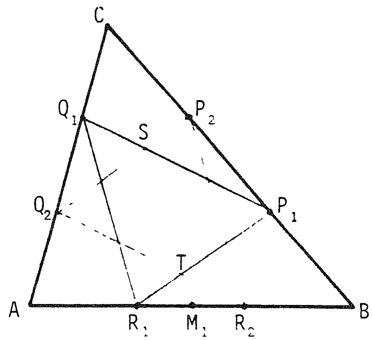
The sides of ΔABC are trisected by the points P_1, P_2, Q_1, Q_2, R_1, R_2, as shown in the figure. Show that:

(a) $\Delta P_1Q_1R_1 \cong \Delta P_2Q_2R_2$;

(b) $|P_1Q_1R_1| = \frac{1}{3}|ABC|$, where the bars denote area;

(c) the sides of Δs $P_1Q_1R_1$ and $P_2Q_2R_2$ trisect one another;

(d) If M_1 is the midpoint of AB, then C, S, T, M_1 are collinear.



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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

181. [1976: 193; 1977: 57] *Late solution: KESIRAJU SATYANARAYANA, Domalguda, Hyderabad, India.*

185. [1976: 194; 1977: 70] *Late solution: KESIRAJU SATYANARAYANA.*

189. [1976: 194; 1977: 74, 193, 252] *Late solution: KESIRAJU SATYANARAYANA.*

*　　　*　　　*

240. [1977: 105, 264, 299; 1978: 18] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Find the unique solution for this base ten cryptarithm:

$$\begin{array}{r} \text{CARL} \\ \times \quad \text{F} \\ \hline \text{GAUSS} \end{array}.$$

V. *Comment by Robert S. Johnson, Montréal, Québec.*

The following poem was inspired by this problem which, as the editor mentioned [1977: 299], has the unique distinction of having exactly three unique solutions. It is respectfully dedicated to the proposer.

PROBLEMS

A math-man who is one of the best
Likes to put our skills to the test.
　　Due to one little gem,
　　We now classify them
As unique, uniquer, and uniquest.

*　　　*　　　*

247. [1977: 131; 1978: 23] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

On page 215 of *Analytic Inequalities* by D.S. Mitrinović, the following inequality is given: if $0 < b \le a$ then

$$\frac{1}{8}\frac{(a-b)^2}{a} \le \frac{a+b}{2} - \sqrt{ab} \le \frac{1}{8}\frac{(a-b)^2}{b}.$$

Can this be generalized to the following form: if $0 < a_1 \le a_2 \le \ldots \le a_n$ then

$$k\,\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_n} \le \frac{a_1+\ldots+a_n}{n} - \sqrt[n]{a_1\ldots a_n} \le k\,\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_1},$$

where k is a constant?

II. *Comment by D.I. Cartwright and M.J. Field (reprinted with permission from James Cook Mathematical Notes No.* 13 *(November* 1977*), pp.* 11-12*).*

Let $0 < a < b$ and let m be a probability measure on $[a,b]$. Put

$$\mu = \int t\,dm, \quad \sigma^2 = \int (t-\mu)^2\,dm, \quad \text{and} \quad G = \exp\int \log t\,dm.$$

The inequality

$$\frac{\sigma^2}{2b} \le \mu - G \le \frac{\sigma^2}{2a}$$

(for the case where m consists of finitely many points of equal weight) was suggested by K.S. Williams in EUREKA [1977: 131] and proved by G. Szekeres in *JCMN* No. 12 (October 1977) [and by B.C. Rennie in EUREKA [1978: 23]]. This special case can be shown equivalent to the general case by a theorem in functional analysis. The following is a summary of a paper accepted for publication by *Proc. Amer. Math. Soc.* entitled "A Refinement of the Arithmetic Mean - Geometric Mean Inequality" by D.I. Cartwright and M.J. Field.

LEMMA. If $0 \le q \le 1$ *and* $t \ge 0$ *then*

$$1 + qt + \tfrac{1}{2}q(q-1)t^2 \le (1+t)^q \le 1 + qt + \tfrac{1}{2}q(q-1)t^2/(1+t).$$

This lemma leads to a proof of the theorem when m is a two-point distribution. Now we try to establish the theorem when m is any n-point distribution, points x_r with weights $p_r (r = 1,2,\dots,n)$. We use induction on n. Let the x_r be fixed; we assume them distinct. Consider

$$f(p) = f(p_1,\dots,p_n) = \mu - G - \Sigma\, p_k\,(x_k - \mu)^2/(2b)$$

as a function of p in the set S where all $p_k \ge 0$. Note that in this function we do not assume $\Sigma\, p_k = 1$ and also that μ and G depend on p as before, that is,

$$\mu = \Sigma\, p_k x_k \quad \text{and} \quad G = \exp \Sigma\, p_k \log x_k.$$

There is a point p^0 when f is minimized subject to $\Sigma\, p_k = 1$. If p^0 is an interior point of S then there is a Lagrange multiplier λ such that

$$\frac{\partial f}{\partial p_j} = \lambda(\partial/\partial p_j)(-1 + \Sigma\, p_k) = \lambda,$$

$$x_j - G\log x_j - (x_j - \mu)^2/(2b) = \lambda$$

for each j at the point p^0. Thus each x_j is a solution of the equation

$$\xi - G\log\xi - (\xi - \mu)^2/(2b) = \lambda.$$

Between any two roots there is by Rolle's theorem a root of

$$1 - G/\xi - (\xi - \mu)/b = 0 \quad \text{or} \quad \xi^2 - (b+\mu)\xi + bG = 0$$

which has at most one solution between a and b. It follows that there cannot be more than two distinct values of x_j and so $n \le 2$. In this case the theorem is proved. If on the other hand p^0 is not an interior point of the set S then one $p_j = 0$ and the theorem follows by induction. The other inequality is proved similarly.

Editor's comment.

The destination of the paper of Cartwright and Field was incorrectly reported in [1978: 26] as the *Duke Mathematical Journal.*

* * *

250. [1977: 132] *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

(a) Find all pairs (m,n) of positive integers such that

$$|3^m - 2^n| = 1.$$

(b)* If $|3^m - 2^n| \ne 1$, is there always a prime between 3^m and 2^n?

I. *Solution of part (a) by Robert Bétancourt, New York, N.Y.*

For $m = 1$ or 2, all the solutions are quickly found; they are

$$(m,n) = (1,1), \quad (1,2), \quad \text{or} \quad (2,3).$$

We prove there are no others, making use of the following fact, which is easily established:

$$3^m \equiv 1 \text{ or } 3 \pmod 8 \text{ according as } m \text{ is even or odd.}$$

Suppose

$$3^m + 1 = 2^n \quad \text{or} \quad 3^m - 1 = 2^n,$$

where $m > 2$, and hence $n > 3$. In the first case, we get

$$3^m + 1 \equiv 0 \quad \text{and} \quad 3^m \equiv -1 \pmod 8, \tag{1}$$

which is impossible. The second case yields

$$3^m - 1 \equiv 0 \quad \text{and} \quad 3^m \equiv 1 \pmod 8.$$

But this can only occur when m is even, say $m = 2k$, where $k > 1$, and then we have

$$2^n = 3^{2k} - 1 = (3^k - 1)(3^k + 1).$$

This implies that $3^k + 1$ is a power of 2, say $3^k + 1 = 2^r$ with $r > 3$ (since $k > 1$), and we are led again to the impossible relation (1) with k instead of m.

II. *Solution de la partie (a) par F.G.B. Maskell, Collège Algonquin, Ottawa.*

Posons $m = 4q + r$, $0 \le r \le 3$, la valeur $r = 0$ étant exclue si $q = 0$. On a alors

$$3^m = (5 \cdot 16 + 1)^q \cdot 3^r \equiv 1,\ 3,\ 9,\ 11 \pmod{16} \qquad (2)$$

selon que $r = 0, 1, 2, 3$.

Pour $q = 0$, l'équation $3^m - 2^n = \pm 1$ a les solutions $(m,n) = (1,1)$ et $(1,2)$ quand $r = 1$, la solution $(2,3)$ quand $r = 2$, et aucune solution quand $r = 3$.

Ces solutions sont les seules. Car il découle de (2) que

$$q > 0 \implies 3^m \ne 2^n - 1$$

et

$$q > 0 \implies 3^m \ne 2^n + 1.$$

Cette dernière implication est évidente si $r \ne 0$; mais elle vaut aussi si $r = 0$, car on a alors $3^m \equiv 1 \pmod{80}$, et donc $3^m \ne 2^n + 1$.

III. *Comment on part (b)* by Leroy F. Meyers, The Ohio State University.*

It is unlikely that this problem can be solved at present. For every positive integer m there is a positive integer n such that $2^n < 3^m < 2^{n+1}$, and it follows from part (a) that when $m > 2$ we have $2^n + 1 < 3^m < 2^{n+1} - 1$. It is known [6] that, by Bertrand's "postulate" and related theorems, there are at least two primes between 2^n and 2^{n+1} if $n \ge 2$. However it will not be easy to prove that two of those primes, say p_i and p_j, must satisfy

$$2^n + 1 \le p_i < 3^m < p_j \le 2^{n+1} - 1.$$

Solutions to part (a) were also submitted by LOUIS H. CAIROLI, graduate student, Kansas State University, Manhattan, Kansas; STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; DOUG DILLON, Brockville, Ontario; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Washburn University, Topeka, Kansas; and the proposer.

Comments on part (b)* were also submitted by F.G.B. MASKELL, Algonquin College, Ottawa; BOB PRIELIPP, The University of Wisconsin-Oshkosh; and R. ROBINSON ROWE, Sacramento, California.

Editor's comment.

Several readers pointed out that part (a) appears in Sierpiński [7], as two corollaries in Prielipp [5], and that a generalization, with arbitrary primes p and q instead of 3 and 2, was a problem in the 1976 Putnam Examination (see [3] for a solution).

The information in this paragraph came from Dodge. In [2] can be found the following result: if $n > 1$, then $2^n - 1$ is never a square, cube, or higher power; and it is pointed out in the comments in [2] that in 1887 C. de Polignac [4] proved that $a^n - 2^k = \pm 1$ is impossible unless $a = 3$, $n = 1$ or $n = 2$. In [1] is established the following stronger result: $p^n - c^m = 1$ has no solution in integers c, m, n, p with p prime, $m > 1$, $n > 1$ other than $3^2 - 2^3 = 1$.

For part (b)* Maskell and Rowe both adduced some evidence (but no proof) to support the conjecture

$$\lim_{m,n\to\infty} |3^m - 2^n| = \infty.$$

This would seem to indicate that (b)* is unlikely to be disproved by searching for an abnormally small value of the difference $|3^m - 2^n|$.

Note that there are pairs of positive integers (x,y) other than $(3,2)$ such that, for some m and n, $|x^m - y^n| \neq 1$ and there are no primes between x^m and y^n. The first such occurrence is between $11^2 = 121$ and $5^3 = 125$, and the next is between $3^7 = 2187$ and $13^3 = 2197$.

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4. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, 1952, Vol. II, p. 753.

5. Robert Prielipp, Consecutive Prime Power Numbers, *The Pentagon*, 33 (Spring 1974) 104-106.

6. W. Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964, pp. 131-137.

7. ________, 250 *Problems in Elementary Number Theory*, American Elsevier, 1970, pp. 16, 96, Problems 184 and 185.

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251. [1977: 154] *Proposed by Robert S. Johnson, Montréal, Québec.*

Solve the cryptarithmic addition given below. It was given to me by a friend who got it himself from a friend at least 15-20 years ago. I hope some reader can identify its unknown originator.

```
SPRING
 RAINS
 BRING
 GREEN
------
PLAINS
```

I. *Solution by Kenneth M. Wilke, Washburn University, Topeka, Kansas.*

We will first solve the easier auxiliary cryptarithm

$$\begin{array}{r} \text{SPRING} \\ \text{R0000} \\ \text{BRING} \\ \underline{\text{GREEN}} \\ \text{PL0000} \end{array}$$

with zeros replacing AINS, and a solution of the original, more poetic one will follow instantly.

Let c_i denote the carry from the ith column to the next column on the left. Our auxiliary cryptarithm yields the relations

$$2G + N = 10c_1, \tag{1}$$

$$2N + E + c_1 = 10c_2, \tag{2}$$

$$2I + E + c_2 = 10c_3, \tag{3}$$

$$3R + c_3 = 10c_4, \tag{4}$$

$$P + R + B + G + c_4 = L + 10c_5, \tag{5}$$

$$S + c_5 = P. \tag{6}$$

From (2) and (3), we get

$$2(I - N) = 10(c_3 - c_2) + (c_1 - c_2). \tag{7}$$

It is clear that $1 \le c_i \le 2$ for i = 1, 2, 3, 4. Hence (4) implies that $c_4 = c_3$, and (7) implies that $c_1 = c_2$ and $c_3 - c_2 = \pm 1$. However, $c_3 - c_2 = -1$ leads to $c_1 = c_2 = 2$ and $c_3 = c_4 = 1$, $R = 3$, $N = I + 5$ and even, and

$$N = 6, \quad E = 6 = N \qquad \text{or} \qquad N = 8, \quad I = 3 = R.$$

Thus we must have $c_3 - c_2 = 1$, whence $c_1 = c_2 = 1$ and $c_3 = c_4 = 2$. Now (4) and (7) give $R = 6$ and $I = N + 5$, so that $N = 0$, 2, or 4 (for N must be even by (1)). But $N \ne 0$, since

$$N = 0 \implies G = 5, \ I = 5 = G.$$

On the other hand, $N = 2$ implies $G = 4$, $E = 5$, $I = 7$, and (5) becomes

$$P + B + 12 = L + 10c_5. \tag{8}$$

Now, in view of (6) and the unused digits, we must have either $c_5 = 1$, $S = 8$, $P = 9$, contradicting (8); or $c_5 = 2$, $S = 1$, $P = 3$, and none of the left-over digits then satisfy (8).

Thus we must have $N = 4$, $G = 3$, $E = 1$, $I = 9$, and (5) becomes

$$P + B + 11 = L + 10c_5.$$

This last relation, considered along with (6) and the unused digits, implies $c_5 = 2$, $S = 5$, $P = 7$, $L = 0$, $B = 2$, leaving $A = 8$ for the original cryptarithm.

The unique solutions of the two cryptarithms are

$$\begin{array}{r} 576943 \\ 60000 \\ 26943 \\ \underline{36114} \\ 700000 \end{array} \quad \text{and} \quad \begin{array}{r} 576943 \\ 68945 \\ 26943 \\ \underline{36114} \\ 708945 \end{array} .$$

II. *Comment by Clayton W. Dodge, University of Maine at Orono.*

Clearly, the last word of the cryptarithm suggests that it originated in Georgia, but the magnitude of the numbers indicates we are not dealing with peanuts!

Also solved by LOUIS H. CAIROLI, graduate student, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono (solution as well); the following students in the class of JACK LeSAGE, Eastview Secondary School, Barrie, Ontario: HARRY BINNENDYK, GLENN GAUDER, TIM MURPHY, BRUCE MURRELL, and GARY STUNDEN (independently); RAMA KRISHNA MANDAN, Hindustan Mineral Products, Bombay, India; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; R. ROBINSON ROWE, Sacramento, California; CHARLES W. TRIGG, San Diego, California; and the proposer.

Editor's comment.

Our ingenious featured solution suggests that, in *My Fair Lady*, Professor Higgins would have been well advised to start by teaching Eliza the easier song: *The rooo in Spooo falls mooolу in the plooo.*

It is now impossible to avoid mentioning that if, as is not improbable, the present Spanish king was generally good in school but weak in elementary geometry, then we could say: *The reign in Spain fails mainly in the plane.*

The proposer says that this beautiful cryptarithm goes back a couple of decades. But, as often happens with mathematical problems, it has been "in the air" in recent years. Nyon discovered it in the British journal *Games & Puzzles*, No. 40 (September 1975), pp. 39, 50. Readers interested in learning more about this journal can write to Davis Pritchard, Editor, *Games & Puzzles*, 11 Tottenham Court Rd., London, W1A 4XF, England.

* * *

252. [1977: 154] *Proposed by Richard S. Field, Santa Monica, California.*

Discuss the solutions, if any, of the system

$$x^y = A$$
$$y^x = A + 1,$$

where $A \geq 2$ is an integer.

Solution by R. Robinson Rowe, Sacramento, California (revised by the editor).

We will ignore the restriction to integral $A \geq 2$ and discuss the existence and number of solutions for real $A \geq 0$. For $A = 0$, the pair $(0, y)$ is a solution for every $y > 0$. For $A > 0$, the given system is equivalent to

$$y = \frac{\ln A}{\ln x}, \tag{1a}$$

$$y = (A+1)^{\frac{1}{x}}; \tag{1b}$$

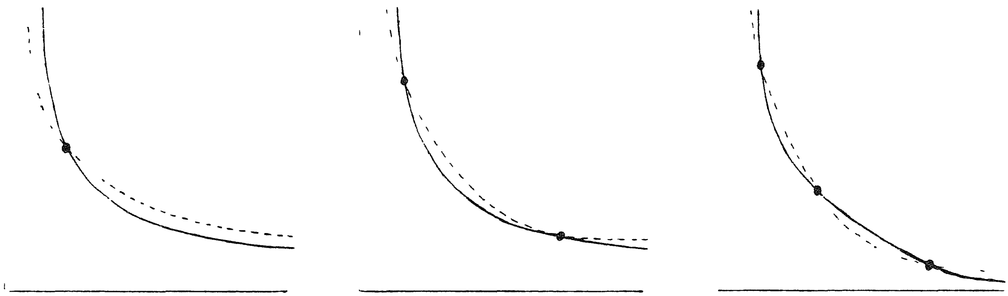
so to solve the system simultaneously we must find values of x which satisfy

$$\frac{\ln A}{\ln x} = (A+1)^{\frac{1}{x}}, \tag{2}$$

and the corresponding values of y are then given by (1).

Since it appeared hopeless to express the solutions of (2) in closed form, I used numerical methods to approximate them. My calculations indicate that (2) has two solutions if $A \approx 18.12$, one solution if $A < 18.12$, and three solutions if $A > 18.12$. It is very revealing to plot separate curves for the values of x and y obtained from (2) and (1), using A as the abscissa for each (see Figure 1). For any given A, the corresponding solution(s) (x_i, y_i) is (are) found by taking the ordinates corresponding to A on the curves x_i and y_i. Note that the curves x_2, x_3 (and also y_2, y_3) join at a point of abscissa $A \approx 18.12$. Note also that as $A \to \infty$ we have

$$x_1 \to 1, \quad y_1 \to \infty; \qquad x_2 \to y_2 \to \infty; \qquad x_3 \to \infty, \quad y_3 \to 1. \tag{3}$$



$A < 18.12$ $\qquad$ $A \approx 18.12$ $\qquad$ $A > 18.12$

Figure 2

Figure 2 shows three graphs of equations (1a) and (1b) (solid line for (1a), dotted line for (1b)), one for some $A < 18.12$, one for $A \approx 18.12$, and one for some $A > 18.12$ (not drawn to scale). A glance at this figure shows why the number of

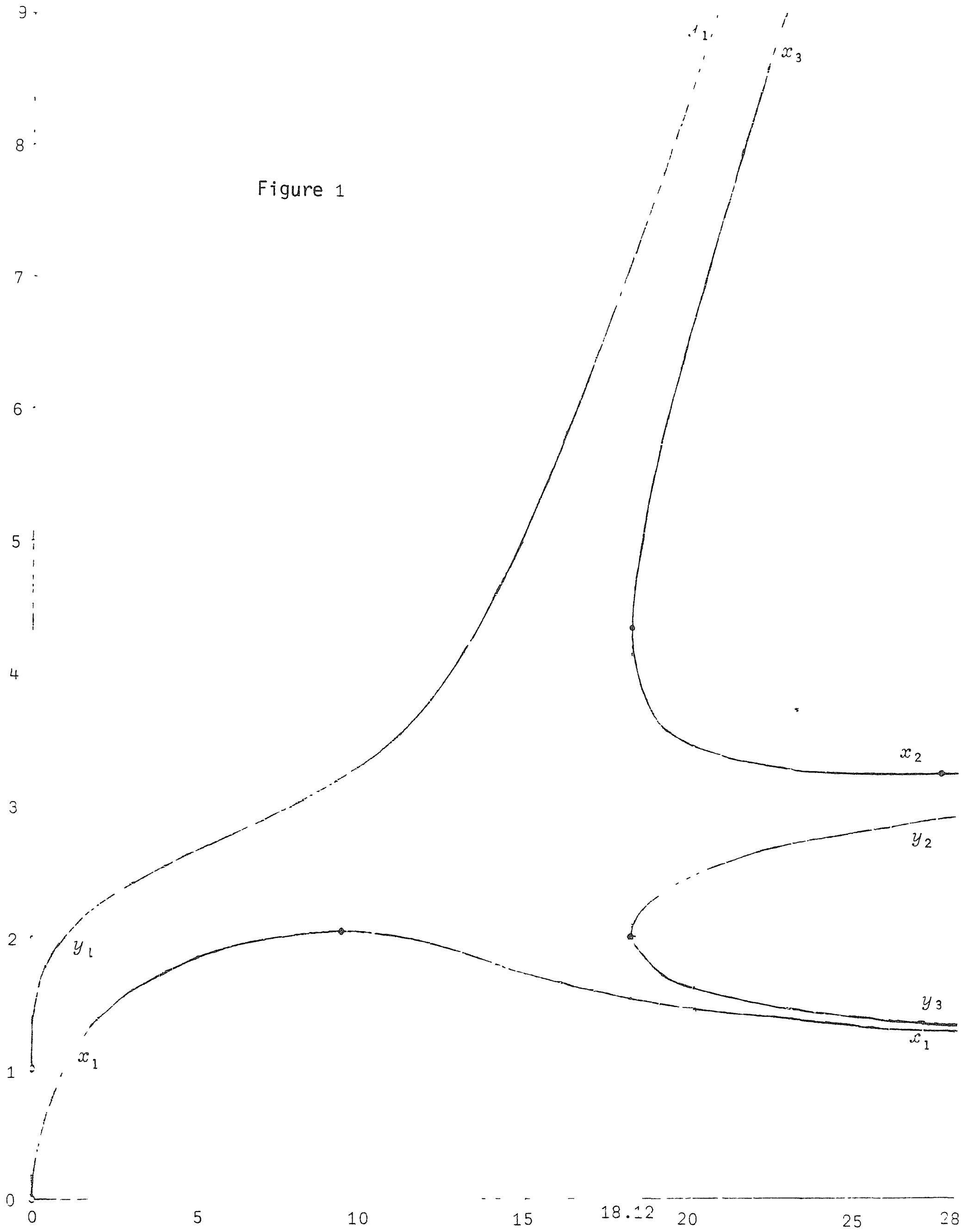


Figure 1
x_3
x_2
y_2
y_1
y_3
x_1
x_1
9
8
7
6
5
4
3
2
1
0
0
5
10
15
18.12
20
25
28

solutions increases from one to two to three.

Coming back to Figure 1, I give below a table containing approximations to some of the solutions (x_i, y_i), for selected values of $A \leq 28$, that were used to draw the figure. Note the integral solutions $(1,2)$ for $A = 1$ and $(2,3)$ for $A = 8$. These are very likely the only integral solutions, but I leave the proof of this conjecture to others.

A	$\frac{1}{2}$	1	5	8	10
(x_1, y_1)	(0.68,1.80)	(1,2)	(1.83,2.66)	(2,3)	(2.01,3.30)

A	15	18.12	20	25	28
(x_1, y_1)	(1.69,5.16)	(1.50,7.14)	(1.42,8.51)	(1.30,12.27)	(1.26,14.57)
(x_2, y_2)		(4.31,1.98)	(3.41,2.44)	(3.21,2.76)	(3.20,2.87)
(x_3, y_3)			(6.68,1.58)	(10.84,1.35)	(13.27,1.29)

Some readers may be amused to verify on their pocket calculator that, for example, the solutions for $A = 20$, which are, more accurately,

$$x_1 \approx 1.4222\ 35407, \qquad y_1 \approx 8.5050\ 49057,$$
$$x_2 \approx 3.4087\ 17748, \qquad y_2 \approx 2.4428\ 31166,$$
$$x_3 \approx 6.6830\ 47413, \qquad y_3 \approx 1.5770\ 54720,$$

do indeed satisfy the equations $x^y = 20$ and $y^x = 21$. As a means of confirming (3), I also give the solutions for $A = 1000$; these are

$$x_1 \approx 1.0072\ 80581, \qquad y_1 \approx 952.24\ 14642,$$
$$x_2 \approx 4.5567\ 02211, \qquad y_2 \approx 4.5547\ 66645,$$
$$x_3 \approx 951.24\ 03422, \qquad y_3 \approx 1.0072\ 89330.$$

These will be found to satisfy the equations $x^y = 1000$ and $y^x = 1001$.

Other items of interest in Figure 1: the curve x_1 has an absolute maximum at $A \approx 9.33866$, with corresponding solution

$$x_1 \approx 2.0180\ 66189, \qquad y_1 \approx 3.1819\ 34745;$$

and the curve x_2 has an absolute minimum at $A \approx 27.4514$, with corresponding solution

$$x_2 \approx 3.1959\ 93831, \qquad y_2 \approx 2.8508\ 67193.$$

Finally, it may be helpful to have a more accurate description of the points where x_2 joins x_3 and y_2 joins y_3. Their common abscissa is $A \approx 18.123\ 25202$ and their ordinates are

$$x_2 \approx 4.3134\ 49709, \qquad y_2 \approx 1.9820\ 21262.$$

Also solved by the proposer. Partial solutions were submitted by ROBERT S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; HERMAN NYON, Paramaribo, Surinam; and KENNETH M. WILKE, Washburn University, Topeka.

Editor's comment.

This problem, interesting though it is for real $x > 0$ and $y > 0$, becomes even more so when x and y are restricted to be positive integers. Then it is equivalent to solving the Diophantine equation

$$x^y - y^x = 1. \tag{4}$$

(Here I have interchanged the rôles of x and y, in conformity with the notation to be used later in this comment.) Maskell proved that the only solutions of (4) are $(x,y) = (2,1)$ and $(3,2)$, thus confirming Rowe's conjecture; but Wilke showed that this result was already known by outlining a beautiful proof of Andrzej Schinzel from Sierpiński [14]. However, Dickson [7] reports that Moret-Blanc had already solved (4) in 1876. Equation (4) also appears as an exercise in Nagell [11].

But this is not the end of the story, far from it. Equation (4) is a special case of the vastly more difficult Diophantine equation known as the *Catalan equation*,

$$x^m - y^n = 1. \tag{5}$$

Catalan conjectured in [5] that the only solution of (5) is $3^2 - 2^3 = 1$. (He tacitly assumed $m > 1$ and $n > 1$ to exclude trivial solutions.) Even today, 134 years later, the Catalan conjecture has been neither proved nor disproved, although in a sense, as we shall see, it is *nearly* settled.

13.

NOTE

extraite d'une lettre adressée à l'éditeur par Mr. *E. Catalan*, Répétiteur à l'école polytechnique de Paris.

„Je vous prie, Monsieur, de vouloir bien énoncer, dans votre recueil, le „théorème suivant, que je crois vrai, bien que je n'aie pas encore réussi à „le démontrer complètement: d'autres seront peut-être plus heureux:

„*Deux nombres entiers consécutifs, autres que* 8 *et* 9, *ne peuvent être* „*des puissances exactes; autrement dit:* l'équation $x^m - y^n = 1$, dans „laquelle les inconnues sont entières et positives, n'admet qu'une seule „solution."

As a matter of historical interest, I reproduce above Catalan's conjecture, exactly as it appeared in [5].

The conjecture was verified over the years for various special cases of (5) in addition to (4). Léo Hebraeus, also known as Lévi ben Gerson (1228-1344), in answer to a question of Philipp von Vitry, proved that the equation $3^m - 2^n = 1$ has only the solution $(m,n) = (2,3)$; and Euler showed that the equation $x^2 - y^3 = 1$ has only the solution $(x,y) = (3,2)$. LeVeque [9] proved that, for given x and y, equation (5) has at most one solution (m,n); and Siegel proved in 1929 that, for given m and n, equation (5) has only a finite number of solutions (x,y). Baker [2] showed how to find all solutions (x,y) for given m and n. It is shown in [1] that (5) has only the expected unique solution when x is a prime. It is known that if (5) has a solution other than the known one, then in such a solution $x > 10^{11}$, $y > 10^{11}$ [8] and $m > 5$, $n > 5$. A decisive step was recently taken by the young Dutch mathematician Robert Tijdeman, who showed the existence of a calculable universal constant C such that

$$y^n < x^m < C \tag{6}$$

for every solution (x,y,m,n) of (5) and so these solutions must be finite in number. Finally, quite recently Michel Langevin has found an upper bound for C; it is

$$C \le \exp(\exp(\exp(\exp(730)))). \tag{7}$$

All that now remains to be done to settle the Catalan conjecture is, for each quadruple (x,y,m,n) which satisfies (6) and (7), to verify if it is a solution of (5). It is in this sense that we can say that the Catalan conjecture is *nearly* settled. But a computer doing a million such verifications every millionth of a second would only do an infinitesimal portion of the job in a million years!

All the information in the preceding paragraph that is not otherwise referenced was obtained from [16], a reference sent to me by Christian Françoise, Algonquin College.

In the light of the recent history of the Catalan conjecture, perhaps some readers can explain the following statement from Dickson [7]:

> *Housel*[89] *proved Catalan's*[38] *empirical theorem that two consecutive integers, other than* 8 *or* 9, *can not be exact powers* [*with exponents* > 1].

(The superscripts 89 and 38 refer to [6] and [5] respectively.)

The Catalan conjecture is also mentioned in [12] (sent by Johnson), as well as in [3], [4], [10], [13], and [15]. The editor is particularly grateful to Wilke, who sent in photocopies of several very helpful references, and to H.G. Dworschak, Algonquin College, who helped with some of the calculations and figures.

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253. [1977: 154] *Proposed by David Fisher, Algonquin College, Ottawa.*

Let $x{\uparrow}y$ denote x^y. What are the last two digits of $2{\uparrow}(3{\uparrow}(4{\uparrow}5))$?

I. *Solution de Hippolyte Charles, Waterloo, Québec.*

D'après le théorème d'Euler-Fermat,

$$2^{20} = 2^{\phi(25)} \equiv 1 \pmod{25}$$

et donc $2^{22} \equiv 2^2 \pmod{100}$. Par suite

$$a \equiv b \pmod{20},\ a > 1,\ b > 1 \implies 2^a \equiv 2^b \pmod{100}.$$

Or $3^8 = 3^{\phi(20)} \equiv 1 \pmod{20}$; et puisque $8 \mid 4^5$ on a $3\uparrow(4\uparrow5) \equiv 1 \pmod{20}$. Donc

$$2\uparrow(3\uparrow(4\uparrow5)) \equiv 2^{21} = 2097152 \equiv 52 \pmod{100},$$

et les deux derniers chiffres sont 5 et 2.

II. *Comment by R. Robinson Rowe, Sacramento, California.*

$2\uparrow(3\uparrow(4\uparrow5))$ has more than 1.124×10^{488} digits, and is thus greater than the fourth power of a googol.

Also solved by LEON BANKOFF, Los Angeles, California; PAUL J. CAMPBELL for the BELOIT COLLEGE SOLVERS, Beloit, Wisconsin; W.J. BLUNDON, Memorial University of Newfoundland; LOUIS H. CAIROLI, graduate student, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; HERTA T. FREITAG, Roanoke, Virginia; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ROBERT S. JOHNSON, Montréal, Québec; F.G.B. MASKELL, Algonquin College, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; R. ROBINSON ROWE, Sacramento, California (solution as well); DAVID R. STONE, University of Kentucky, Lexington; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Washburn University, Topeka; and the proposer.

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254. [1977: 155] *Proposed by M.S. Klamkin, University of Alberta.*

(a) If $P(x)$ denotes a polynomial with integer coefficients such that

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 4000,$$

prove that the zeros of $P(x)$ cannot be integers.

(b) Prove that there is no such polynomial if

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 1000.$$

Solution by L.F. Meyers, The Ohio State University.

A generalization yielding both parts is proved.

Let a, b, c, d, k be integers, with $k \neq 0$. Suppose P is a polynomial with integral coefficients such that

$$P(d-k) = ak, \quad P(d) = bk, \quad \text{and} \quad P(d+k) = ck.$$

If we define Q by $Q(x) = P(x+d)$, then Q is a polynomial with integral coefficients and

$$Q(-k) = ak, \quad Q(0) = bk, \quad \text{and} \quad Q(k) = ck.$$

Now the polynomial Q_1 of lowest degree such that

$$Q_1(-k) = ak, \quad Q_1(0) = bk, \quad \text{and} \quad Q_1(k) = ck$$

is given, by the Lagrange interpolation formula or otherwise, by

$$Q_1(x) = \frac{(a-2b+c)x^2 + (c-a)kx + 2bk^2}{2k}.$$

Hence the polynomial $Q - Q_1$ has zeros at $-k$, 0, and k, and so

$$Q(x) - Q_1(x) = (x+k)x(x-k)R(x)$$

for some polynomial R, or

$$2kQ(x) - (a-2b+c)x^2 - (c-a)kx - 2bk^2 = (x^3 - k^2x)S(x), \tag{1}$$

where $S = 2kR$. Now S, being the quotient of a polynomial with integral coefficients by a polynomial with integral coefficients and leading coefficient 1, must itself have integral coefficients. Equating the coefficients of x^2 on the two sides of (1) yields

$$2kq - (a-2b+c) = -k^2s,$$

where q and s are the coefficients of x^2 in $Q(x)$ and x in $S(x)$, respectively, and it follows that

$$a - 2b + c \text{ is divisible by } k. \tag{2}$$

If $c - a$, and hence $c + a$, is even, then slightly more can be proved. Instead of (1) we now have

$$kQ(x) - \frac{a-2b+c}{2}x^2 - \frac{c-a}{2}kx - bk^2 = (x^3 - k^2x)S(x), \tag{1'}$$

where now $S = kR$. Then, as before, S has integral coefficients and, after comparing the coefficients of x^2, we obtain

$$kq - \frac{a-2b+c}{2} = -k^2s,$$

and so

$$\frac{a-2b+c}{2} \text{ is divisible by } k. \tag{2'}$$

Note that (2) and (2') are independent of d, which can therefore be any integer.

In part (a) we have $a=1$, $b=2$, $c=4$, so that $a-2b+c = 1$. Hence, by (2), k is a divisor of 1, that is, $k=\pm1$.

In part (b) we have $a=1$, $b=2$, $c=1$, so that $\frac{a-2b+c}{2} = -1$. Since $c-a$ is even, we can now use (2') to conclude that k is a divisor of -1, that is, $k=\pm1$.

Since $k=1000$ in both parts of the proposal, each part yields a contradiction. Hence there is no polynomial with integral coefficients which satisfies either of the given conditions, and so none of the zeros of P can be integers; also, all of them must be integers. (For an analogous situation, see Problem 138 [1976: 157-158]).

Also solved by PAUL J. CAMPBELL for the BELOIT COLLEGE SOLVERS, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; and the proposer.

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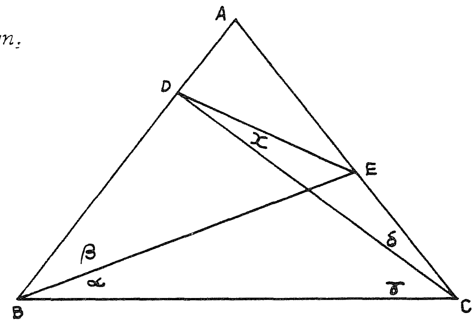

255. [1977: 155] *Proposed by Barry Hornstein, Canarsie H.S., Brooklyn, N.Y.*

In the adjoining figure, the measures of certain angles are given. Calculate x in terms of α, β, γ, δ.

I. *Solution by Kesiraju Satyanarayana, Domalguda, Hyderabad, India.*

From triangles CDE, BCD, and BCE, we have, by the law of sines,

$$\frac{\sin(x+\delta)}{\sin x} = \frac{CD}{CE} = \frac{CD}{CB}\cdot\frac{CB}{CE} = \frac{\sin(\alpha+\beta)}{\sin(\alpha+\beta+\gamma)}\cdot\frac{\sin(\alpha+\gamma+\delta)}{\sin\alpha};$$

hence

$$\cos\delta + \cot x\sin\delta = \frac{\sin(\alpha+\beta)\sin(\alpha+\gamma+\delta)}{\sin\alpha\sin(\alpha+\beta+\gamma)}$$

and

$$\cot x = \frac{\sin(\alpha+\beta)\sin(\alpha+\gamma+\delta)}{\sin\alpha\sin\delta\sin(\alpha+\beta+\gamma)} - \cot\delta, \qquad (1)$$

from which x can be calculated uniquely since $0 < x < \pi$.

II. *Comment by Leon Bankoff, Los Angeles, California.*

Equation (1) provides a convenient check on the result of Problem 134 [1976: 68, 151, 173, 222; 1977: 12, 44] as well as the result of an earlier version to be found in the references given in [1976: 153] and [1977: 12]. Additional references for this earlier version are given below ([1]-[4]). A related problem is discussed in [5].

Also solved by LEON BANKOFF, Los Angeles, California (two solutions as well); L.F. MEYERS, The Ohio State University; and the proposer. One incorrect solution was received.

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256. [1977: 155] *Proposed by Harry L. Nelson, Livermore, California.*

Prove that an equilateral triangle can be dissected into five isosceles triangles, n of which are equilateral, if and only if $0 \le n \le 2$. (This problem was suggested by Problem 200.)

Partial solution by Robert S. Johnson, Montréal, Québec.

The dissection is possible if $n = 0$ (Figure 1), $n = 1$ (Figures 2 or 3), and $n = 2$ (Figure 4).

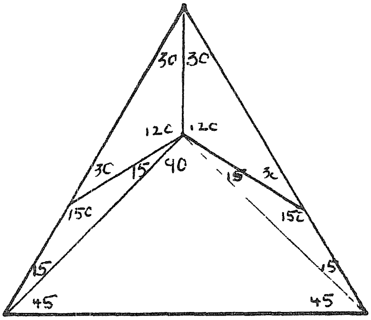

Figure 1

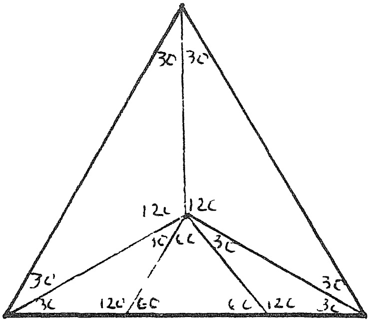

Figure 2

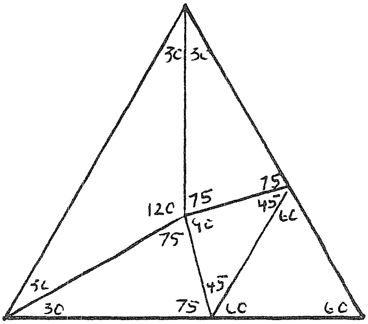

Figure 3

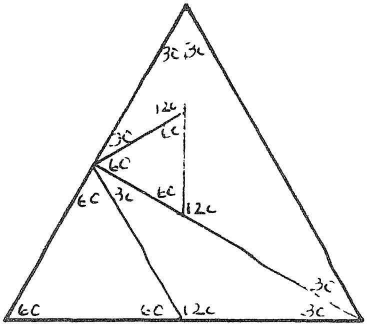

Figure 4

Suppose a dissection is possible for $n = 5$. Then the angles of the original triangle belong to 3 of the 5 equilateral triangles in the dissection. When these 3 equilateral triangles are removed, what is left is either an equilateral triangle or a quadrilateral, neither of which can be dissected into 2 equilateral triangles; or else a polygon with at least 5 sides, which cannot be dissected into 2 triangles of any kind. Hence the dissection is not possible for $n = 5$.

I tend to agree with the proposer's conjecture that the dissection is not possible for $n = 3$ or 4, but have no satisfactory proof to offer.

Also partially solved by the proposer.

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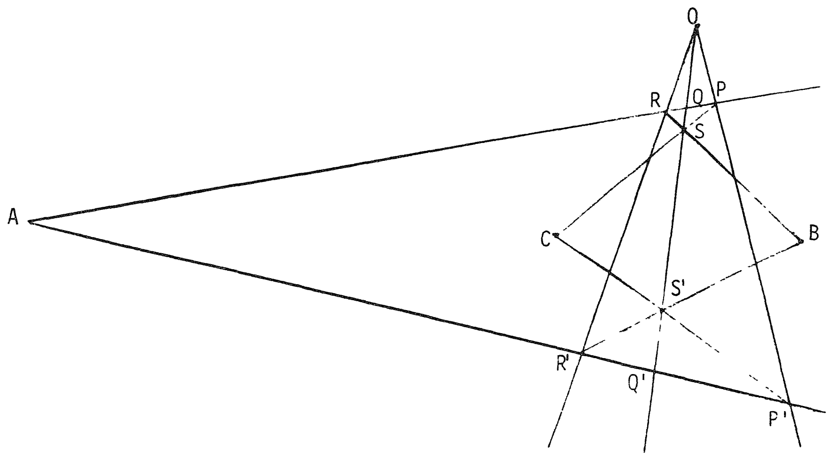
257. [1977: 155] *Proposed by W.A. McWorter, Jr., The Ohio State University.*

Can one draw a line joining two distant points with a BankAmericard? (Solutions with Chargex or Master Charge also acceptable.)

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

When presented by Howard Eves in a geometry class about twenty years ago, this problem was entitled "the epsilon ruler problem," which title I borrowed for use in my own book [1]. The problem is of interest to surveyors who must construct the line on two points when an obstacle intervenes (a mountain, building, etc.) or who must construct the line through a point and the inaccessible point of intersection of two given lines. The construction presented here is readily adapted to these surveying needs.

Let the distant points be A and B as shown in the figure. By trial and success, draw two rays from A toward B so the distance from B to each ray is considerably less than the length of the available straightedge and so that B lies between the



rays. Near B, but outside the angle formed by the rays, draw three rays from a point O to cut the two rays from A in points P and P', Q and Q', and R and R' in such a way that all three rays lie on the same side of B with OPP' closest to B, OQQ' next, and ORR' farthest from B, but still within reach of the straightedge. Let BR and BR' cut OQQ' at points S and S'. Then draw PS and P'S' to meet at C, the desired point collinear with A and B.

The proof is easy. By construction, we have made triangles PRS and P'R'S' copolar at O. Hence, by Desargues' two-triangle theorem, they are also coaxial; that is, points A, B, and C are collinear.

The procedure can be repeated as often as necessary until A is within reach.

II. *"Solution" by the problem-solving group at Larry's Bar in Columbus, Ohio.*

The simplest solution with a BankAmericard: Overdraw!

III. *"Solution" by R. Robinson Rowe, Sacramento, California.*

The proposal, with psychic foresight, allowed two alternatives to BankAmericards, for the latter are now passé. The Bank of America has replaced them with VISA cards.

One way to draw a line joining two distant points with such a card is to present it at an airline ticket office, engage passage from, say, New York to London, get aboard with a ball of string 4000 miles long, and pay it out as the plane carries you from one point to the other. In case the plane does not fly a geodesic, hold the string tight so that it corrects for any deviation.

IV. *"Solution" by Robert S. Johnson, Montréal, Québec.*

Go to a bank which honours the card and draw a line of credit from A (the amount of money you have) to B (the amount of money you need).

Also solved by LOUIS H. CAIROLI, graduate student, Kansas State University, Manhattan, Kansas; ROBERT S. JOHNSON, Montréal, Québec (several additional "solutions"); L.F. MEYERS, The Ohio State University; R. ROBINSON ROWE, Sacramento, California (second "solution"); and the proposer.

Editor's comment.

The only serious solution that was nonprojective was that of Meyers. He used opposite edges of the card to draw parallel lines and adjacent edges to draw perpendicular lines. Nonprojective, to be sure; but also non-Euclidean. The problem also appeared recently in [2].

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2. *James Cook Mathematical Notes*, No. 2 (January 1976), p. 2, proposal by C.F. Moppert; solution by B.B. Newman in *JCMN* No. 3 (March 1976), p. 2.

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258. [1977: 155] *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

For any rational $k \neq 0$ or -1, find the value of the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/k}(n^{k-1/k}+i^{k-1/k})}{n^{k+1}}.$$

Solution by L.F. Meyers, The Ohio State University.

If $k=0$, the series is undefined, and it is meaningless to speak of its limit. But the problem is meaningful for all other real values of k, not merely for rational $k \neq -1$.

The required limit can be rewritten in the form

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^{1/k} + \left(\frac{i}{n}\right)^{k}\right]. \tag{1}$$

If $k>0$, then (1) is the limit of a Riemann sum of the continuous function $x^{1/k}+x^k$ on the interval $[0,1]$, and is thus equal to the Riemann integral

$$\int_0^1 (x^{1/k}+x^k)dx = \frac{k}{k+1} + \frac{1}{k+1} = 1.$$

If $k<0$, then set $j=-k>0$. We now have

$$\left(\frac{i}{n}\right)^{1/k} + \left(\frac{i}{n}\right)^{k} = \left(\frac{n}{i}\right)^{1/j} + \left(\frac{n}{i}\right)^{j} \geq \left(\frac{n}{i}\right)^{\max\{1/j,j\}} \geq \frac{n}{i},$$

and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^{1/k} + \left(\frac{i}{n}\right)^{k}\right] \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{n}{i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = +\infty,$$

and the given series diverges to $+\infty$ by comparison with the harmonic series.

Also solved by DANIEL ROKHSAR, Susan Wagner H.S., Staten Island, N.Y.; DAVID R. STONE, University of Kentucky, Lexington; and the proposer.

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259. [1977: 155] *Proposed by Jacques Sauvé, graduate student, University of Waterloo.*

The function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$$

is defined for all real x. Can one express $f(x)$ in closed form in terms of known (not necessarily elementary) functions?

I. *Solution by L.F. Meyers, The Ohio State University.*

The Bessel function J_0 of order zero is defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2};$$

hence

$$J_0(2jx) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = f(x).$$

(I use the jimmy for $\sqrt{-1}$, since Jacques is an engineer.)

II. *Solution by Viktors Linis, University of Ottawa.*

The given function f is defined for all real (and complex!) x, and satisfies the differential equation

$$xf'' + f' - 4xf = 0, \tag{1}$$

with initial conditions $f(0) = 1$, $f'(0) = 0$. Equation (1) is a special case of Bessel's equation with general solution

$$f(x) = c_1 J_0(2ix) + c_2 Y_0(2ix),$$

where J_0 and Y_0 are Bessel functions of the first and second kind, respectively. The initial conditions give $c_1 = 1$, $c_2 = 0$, and so $f(x) = J_0(2ix)$ for all real x.

Editor's comment.

The function f in the proposal is a real-valued function of a real variable x, and it may come as a surprise to see it expressed in terms of the function J_0 with complex arguments. But indeed f can be expressed in terms of a function with real arguments, for example by using the function $I_n(z)$, called a *modified* (or *hyperbolic*) Bessel function of the first kind of index n, which satisfies (see [1], p. 116)

$$I_n(z) = i^{-n} J_n(iz)$$

whenever n is not a negative integer. For $n = 0$ and $z = 2x$, we get

$$I_0(2x) = J_0(2ix) = f(x).$$

Explicitly, we have ([1], p. 121, Ex. 16), whenever $\mathrm{Re}(n) > -\frac{1}{2}$,

$$I_n(z) = \frac{2(\frac{1}{2}z)^n}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin^{2n}\phi \cosh(z\cos\phi)\,d\phi;$$

hence, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ([1], p. 21), we have

$$f(x) = I_0(2x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cosh(2x\cos\phi)\,d\phi.$$

The proposer wrote that he first came across this function in an examination at the University of Waterloo, in which one of the questions asked for the value of $f(2)$. To ten significant figures, we have

$$f(2) = I_0(4) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cosh(4\cos\phi)\,d\phi \approx 11.301\ 92195.$$

The decimal value on the right was obtained, to the indicated accuracy, by adding only 11 terms of the defining series for $f(2)$. Values of the function $I_0(x)$ have been tabulated, for example in [2] where values are given, to 5 significant figures, for $0 \le x \le 8$.

REFERENCES

1. Earl D. Rainville, *Special Functions*, Chelsea Pub. Co., Bronx, N.Y., 1971.

2. *Standard Mathematical Tables*, 23rd Edition, CRC Press, Cleveland, 1975, p. 540.

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260. [1977: 155] *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Given any triangle (other than equilateral), let P represent the projection of the incentre I on the Euler line OGNH where O, G, N, H represent respectively the circumcentre, the centroid, the centre of the nine-point circle and the orthocentre of the given triangle. Prove that P lies between G and H. In particular, prove that P coincides with N if and only if one angle of the given triangle has measure 60°.

Solution by the proposer.

To establish the first result, we shall need the following relations, expressed in the usual notation (R, r, s = circumradius, inradius, semiperimeter), which are valid for every triangle:

$$\begin{aligned} GH &= \tfrac{2}{3}OH, \quad R \ge 2r, \\ IH^2 &= 4R^2 + 4Rr + 3r^2 - s^2, \\ OH^2 &= 9R^2 + 8Rr + 2r^2 - 2s^2, \\ IG^2 &= \tfrac{1}{9}(s^2 - 16Rr + 5r^2). \end{aligned}$$

The first two are well-known (in the second, equality holds only for equilateral triangles), and the remaining ones are easily derivable from known relations. To include isosceles triangles, in which case I lies on the Euler line, we admit the degenerate triangle GIH.

From the above relations, we have, for nonequilateral triangles,

$$9(GH^2 - IG^2 - IH^2) = 12r(R - 2r) > 0.$$

This implies that the cosine of angle GIH is negative, so GIH is an obtuse (or straight) angle, and P must therefore lie between G and H.

To prove the second result, we shall use the following additional relations, where α, β, γ are the angles of the given triangle:

$$OI^2 = R^2 - 2Rr,$$

$$\Sigma \cos\alpha = \frac{R+r}{R},$$

$$\Sigma \cos\alpha\cos\beta = \frac{s^2 - 4R^2 + r^2}{4R^2},$$

$$\Pi \cos\alpha = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}.$$

The first of these is well-known, and the others are easily derivable from known relations. A straightforward calculation yields

$$(2\cos\alpha - 1)(2\cos\beta - 1)(2\cos\gamma - 1) = \frac{s^2 - 3(R+r)^2}{R^2},$$

which vanishes if and only if $s = (R+r)\sqrt{3}$. Now, from the well-known relation $ON = \frac{1}{2}OH$, it follows that P coincides with N if and only if OI = IH, that is,

$$R^2 - 2Rr = 4R^2 + 4Rr + 3r^2 - s^2,$$

which is equivalent to $s = (R+r)\sqrt{3}$. Hence P coincides with N if and only if one angle of the given triangle has measure 60°.

Also solved by LEON BANKOFF, Los Angeles, California; CLAYTON W. DODGE, University of Maine at Orono; and KESIRAJU SATYANARAYANA, Domalguda, Hyderabad, India.

Editor's comment.

Dodge pointed out that this problem is a natural extension of an earlier problem by the same proposer [1], in which reference can be found several of the relations used here. As the proposer mentions in his solution, the relations used here that are not "well-known" are easily derivable from better-known ones. Any good trigonometry book ([2] or [3], for example) can serve as a starting point.

Since angle GIH is obtuse, it follows *a fortiori* that ΔOIH is obtuse-angled. Bankoff and Satyanarayana both proved this fact by other methods, and Bankoff pointed out that still another proof can be found in [4]. So what is one to make of the following "theorem" given (but not proved) by no less an authority than Hobson [3, p. 201]? (Only the notation has been changed to conform to our own.)

If OIH *is an equilateral triangle, shew that* $\cos\alpha + \cos\beta + \cos\gamma = \frac{3}{2}$.

The implication is trivially true, since the hypothesis is false, but that can hardly be the proof that Hobson had in mind. This anomaly was brought to the attention of the editor by Satyanarayana.

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1. Anders Bager, Solution to Problem E 2282 (proposed by W.J. Blundon), *American Mathematical Monthly*, 79 (1972) 397.

2. H.S. Hall and S.R. Knight, *Elementary Trigonometry*, Macmillan Co. of Canada, Toronto, 1928.

3. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957.

4. Rev. E.M. Radford, *Mathematical Problem Papers*, Cambridge Univ. Press, 1925, Section LXV, Problem 6, p. 339 of the Solution Book.

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ANOTHER TEN-DIGIT MAN

I have been collecting mathematical puzzles and problems for about fifty years. Charles W. Trigg's "The Ten-Digit Man" [1977: 64] sent me scurrying to my notebooks where I found another Ten-Digit Man (pictured on the right). I discovered it, I no longer recall where, more than twenty-five years ago.

HERMAN NYON,
Paramaribo, Surinam.