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Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Mr Pythagoras and the Tree Ring Problem

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Due to a dry climate, trees in Greece have always grown slowly, but are otherwise well behaved and nicely circular in cross-section. The sixth century B.C. Minister of Forests had just been told that the light-colored outer ring of the tree cross-section (called the sapwood by his advisers) was somehow related to the amount of precious water used by the tree. Recognizing a good thing when he heard it, the Minister promptly imposed a 'water tax' on the sapwood of all the trees in the country. It was left to the local foresters to measure all standing and recently cut trees in order to tax the sapwood area.

The young local forester soon realized that he was in deep mathematical trouble. He hurried to see an old friend of the family by the name of Pythagoras. A knock at the door with its inscribed star revealed the eccentric old man.

'Thank heavens you are here', said the forester, 'I have a tricky mathematical problem.'

'Come in, my boy', replied the old man, 'that's just what we like to hear around this place. Tell me all about it.'

After hearing the head office orders, Pythagoras asked the forester what kinds of measurements his men were prepared to make on each tree. The boy drew a sketch of a tree cross-section and explained the problems.

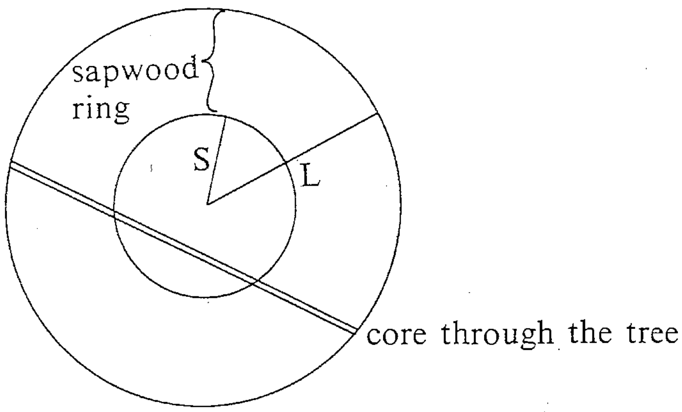


Figure 1

'Well, on the recently cut trees we can always measure the radii S and L, get the circle areas by radius^2 times 3·1, and calculate the difference. Frankly, this is a lot of trouble for those of us without much mathematical background, and I have never been certain that the men are really measuring through the true center of the circles.'

'And what about the standing trees?', questioned the old man.

'A real problem,' declared the forester, 'we can drill cores out of the tree and measure the dark and light parts, but as you can see from the sketch we often miss the true center of the tree. My men have riddled a lot of trees trying to get through the true center, and you can imagine the Minister's wrath if we don't get this job right. This is an election year.'

'Hmmmmm—perhaps a research grant of some sort would be in order here,' said Pythagoras.

'Sorry,' replied the boy, 'no funds, and I need the answer in a hurry.'

'Well son, as a favor to your family I will see what I can dream up over the weekend. Come to see me on Monday.'

The next week Pythagoras greeted the young forester at the door and led him to the library, spreading a group of neatly scribed tablets on the table. 'I think I've solved your problem,' he declared, 'you only need one measurement on the cut trees.'

'Just one?' said the boy, 'That doesn't seem right somehow.'

'No, but it is true. See here in my first drawing.'

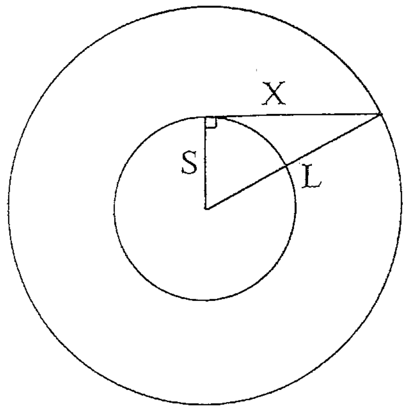


Figure 2

'Do you agree that the area of the inner circle is $S^2\pi$, while the area of the outer circle is $L^2\pi$, and that the difference between these is the ring area?'

The forester thought for a time, and replied that this was indeed what they were presently doing in the woods.

'Well, then, last weekend I worked out a proof that $S^2 + X^2 = L^2$!'

'Well, quite interesting I am sure, old chap; but hardly of any practical value', replied the boy.

'Wait now,' said Pythagoras, 'let's work it out a bit further. By multiplying each side by π we get the formula

$$S^2\pi + X^2\pi = L^2\pi;$$

shifting things around we get

$$X^2\pi = L^2\pi - S^2\pi.$$

'As you will observe, the term on the right is really (large circle area)–(small circle area), so the left term must be the ring area! Regardless of the size of the two circles involved, you need only measure the length of the line X, get the area of a circle of that radius, and you will have the answer you require. For that matter, you should calibrate a stick for your men to use, and they need never know the secret I have revealed to you.'

'Wonderful,' cried the young forester, who was by now taking a real interest in this mathematical business, 'but how about my other problem, where I have a solid tree and can't seem to hit the center of it?'

'An interesting problem, my boy', Pythagoras said with a smile, 'look at my second drawing here.'

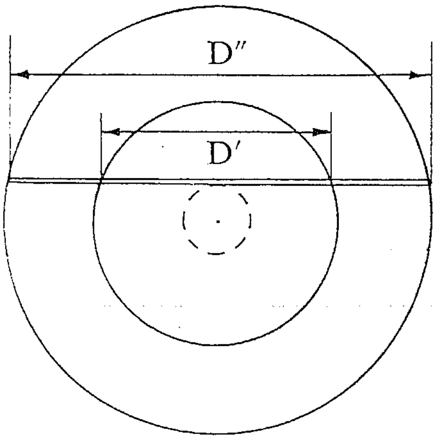


Figure 3

'Suppose you use the distance along the wood core (D') as if it were the true diameter of the solid inner circle. By how much would you miss the true area of that circle? Take your time now.'

The young man pondered a bit, then exclaimed—'Of course, I would actually be measuring a *ring* area, and would therefore miss the true area by the amount you show as a dotted circle.'

'Very good, forester', said the old man, 'and what would be the error for the larger circle if you were to use the distance D''?'

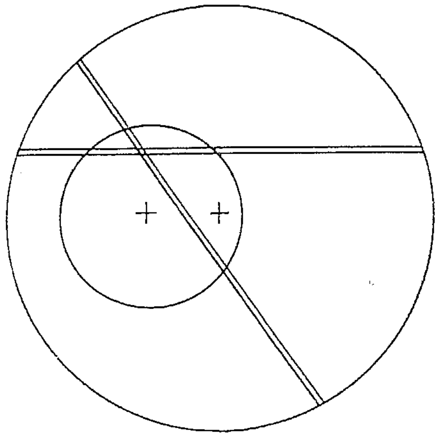
'Why, the same,' said the boy, starting to see the significance of this line of reasoning. 'It misses the center by the same amount.'

'Just so,' said Pythagoras, 'and since the errors are the same they will compensate for each other, and you will be getting the correct area for the ring that lies between them. Thus you do not care what the true circle diameters are, since you will compute the right answer no matter where you decide to measure—assuming of course that you go at least part way through both circles.

'I have continued the analysis further, and have found that even if the circles were not concentric you still get the correct answer if you can measure parallel to the direction the circles are offset, or if you measure the distances on a line passing through a point midway between the two circle centers. Can you see the principle in this diagram? This is just pure mathematics, of course, since I am quite certain that Greek trees would not grow in so ill-behaved a fashion.'

'Wonderful,' exclaimed the forester, 'I assume that you will publish at once!'

'No', replied the old man thoughtfully, 'You know I never have been sure that 3·1 is quite the correct value for π, and calculating the exact value may transcend even my powers. It would be irrational to publish before I was really sure. Perhaps a shorter version without the π term included . . . ?'



–two unbiased ways to measure ring area
Figure 4

And so it was that the young man soon became Chief Forester, and Pythagoras himself became (in secret) the head adviser to the Forest Service, keeping him in sacrificial bulls for years to come, and even allowing him to finance a small group of mathematicians to help him with the work. Although he dabbled in several fields in the intervening years, Pythagoras is still best known for his abbreviated form of the solution to the foresters' tree ring problem.

A Medley of Squares

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I feel obliged to begin with a disclaimer that may cause disappointment to some readers. The squares to be looked at below are not (I believe) an object of youthful derision. Yet they are even squarer and more conventional: in fact, just plain quadrilaterals with all their sides equal and all their angles right ones. Nevertheless, they are not without their entertainment value, as I hope to demonstrate by the presentation of five specific problems. The questions to be asked are easy enough to formulate, but none admits of a glib answer; and in several cases the solution is at present still far from complete.

1. The economic railway network

In these days, when the sole activity of British Rail that invariably conforms to schedule is the declaration of the annual deficit, my first problem is nothing if not relevant. An area 1 mile† square contains n stations, and it is planned to construct a railway network whose total length should, for reasons of economy, be as small as possible yet enable one to travel between any two stations. With how short a network

† Or 1000 miles for that matter. The reader can supply a scaling factor to taste: this does not affect the essence of the problem.

is it possible to satisfy these requirements? One's natural reaction is to say that the answer depends on the site of the stations: if they are bunched together in a cluster, quite a short network will do, while if they are scattered more widely, we should not be able to practise the same degree of economy. However, what we want to specify is a number $L(n)$, as small as possible, such that *no matter where the n stations are situated*, a network of length at most $L(n)$ can be constructed so as to link all the stations. The problem, then, is to investigate how small a value of $L(n)$ is admissible.

We might proceed as follows. (The reader is advised to supplement our description with a sketch.) Denote by A, B, C, D the vertices of the given unit square taken in order; and, to fix our ideas, suppose that AD, BC are drawn as vertical and AB, CD as horizontal lines. Take AB, AD, BC as part of the network. Denote by k an integer to be chosen later. Divide the line AB into k equal intervals, and through each point of subdivision draw a vertical line of length 1 (i.e. a line parallel to AD and extending as far as CD). The additional $k-1$ segments are also to be part of the network. So far, then, we have constructed a network of length $k+2$, but we have not yet taken account of the actual position of the stations.

Now, if a station does not lie on the network already constructed, it must lie between two adjacent vertical lines. The distance apart of any two such lines is $1/k$. Hence any station lies within a distance $1/2k$ of the existing network. We next join each of the n stations by a horizontal track to the nearest vertical line—a procedure which adds at most $n/2k$ to the total length of the network. As the result, we have a network linking all stations and of length not exceeding $k+2+n/2k$, so that

$$L(n) \leqslant k + 2 + n/2k. \tag{1}$$

The choice of k is still at our disposal, and we naturally wish to arrange for the right-hand side of (1) to be as small as possible. A little thought suggests that it is reasonable to take k as the smallest integer greater than or equal to $\sqrt{(n/2)}$. Let us see, then, what this leads to. We have

$$k = \sqrt{(n/2)} + \theta,$$

where $0 \leqslant \theta < 1$. Then $k + 2 \leqslant \sqrt{(n/2)} + 3$ and

$$\frac{n}{2k} = \frac{n}{\sqrt{(2n)} + 2\theta} \leqslant \frac{n}{\sqrt{(2n)}} = \sqrt{(n/2)}.$$

Hence, by (1),

$$L(n) \leqslant \sqrt{(2n)} + 3 \qquad (n \geqslant 2). \tag{2}$$

It should be added that this result is of interest only for large values of n. For small values of n (e.g. $n = 3$ or $n = 4$), we can certainly do better.

The method of establishing (2), though easy to follow, is based on quite an effective idea. It is possible to exploit this idea further and, by refining the argument, obtain the sharper estimate

$$L(n) \leqslant \sqrt{n} + \tfrac{7}{4}. \tag{3}$$

This much was achieved by 1955. And then the question lay fallow till 1977 when it was shown that, for all sufficiently large values of n,

$$L(n) \leqslant 0{\cdot}999\sqrt{n}. \tag{4}$$

The improvement of (4) over (3) may seem a very small step, but the step was really hard.

To solve the problem completely, we should need to look at lower (as well as upper) bounds of $L(n)$. Now it is not at all difficult to exhibit a distribution of n points in the unit square such that, if $\alpha < 12^{-1/4} = 0{\cdot}537\ldots$ and n is sufficiently large, then these points cannot be joined by any network whose length is smaller than $\alpha\sqrt{n}$. We conclude that

$$L(n) \geqslant \alpha\sqrt{n} \tag{5}$$

for all sufficiently large n. There is thus still a considerable gap between (4) and (5). Now it has been proved that the ratio $L(n)/\sqrt{n}$ tends to a certain limit as $n \to \infty$, and it has been conjectured that the value of this limit is $(3/4)^{1/4} = 0{\cdot}930\ldots$. A proof (or refutation) of this last conjecture will not be easy.

The emphasis in our discussion so far has been on networks joining a *large* number of points. However, the problem is also very interesting for small values of n. Thus, given three points in the plane, how does one construct the shortest network joining them? Questions such as this were treated by the formidable Swiss geometer Jakob Steiner (1796–1863), and an account of some of his investigations is included in Courant and Robbins's book *What is Mathematics?* (Oxford University Press, 1941) (see pp. 354–361). The quantitative aspect of the problem is not, however, touched on here; and the reader may himself like to attempt the determination of the value of $L(3)$.

2. Discretion and selection

Let S be a system of a finite number of similarly oriented squares of equal size in the plane. Denote by $A(S)$ the total area covered by S. This will, in general, be smaller than the sum of the areas of individual squares in S since there may well be overlaps. However, if no two squares in S have a point in common, the system S will be called *discrete*. Further, if we select certain squares in S (possibly even all of them), then the squares so selected will again constitute a system of the same kind as S; this system will be referred to as a 'subsystem' of S. In Figure 1 we have an example of a system S, with the shaded squares constituting a discrete subsystem of S.

Our question now is this: given any system S of the type described, is it always possible to select a *discrete* subsystem T of S such that the ratio $A(T)/A(S)$ exceeds a certain fixed positive constant (i.e. a number which does not depend on the particular system S)? To put it another way: does every system S possess a discrete subsystem which covers at least a fixed fraction of the area of S?

It is, in fact, not difficult to see that the answer is affirmative if the constant is chosen appropriately. Thus, given a system S, there always exists a discrete subsystem T such that

$$A(T) \geqslant \tfrac{1}{9}A(S). \tag{6}$$

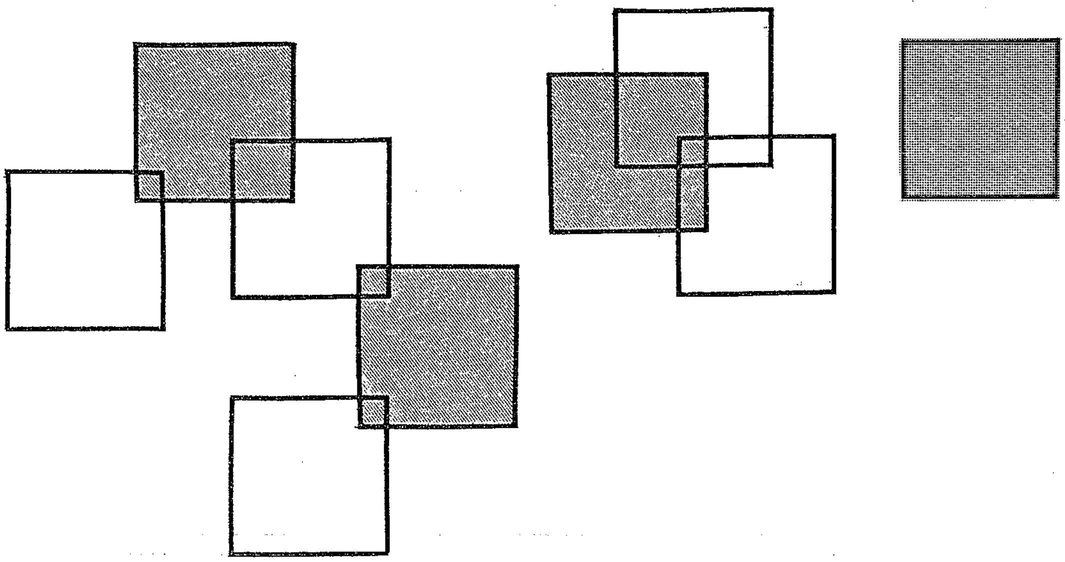

Figure 1

A proof, which not only establishes the *existence* of a suitable T but also yields an actual method of selection, runs as follows. We choose any square Q_1 in S. Any other square in S which has at least one point in common with Q_1 will be called an 'associate' of Q_1. Now all associates of Q_1 are plainly situated within a square Q_1^* (say) having the same centre and orientation as Q_1 but three times its linear size (see Figure 2). Thus $A(Q_1^*) = 9 \cdot A(Q_1)$, and the area of Q_1 is therefore at least one-ninth of the total area covered by the associates of Q_1. We now select Q_1 as a member of T and reject all its associates. Having done this, we repeat exactly the same procedure in relation to the residual system (i.e. the system left after the removal of Q_1 and its associates from S). Let this step result in the selection of a square Q_2 as a member of T. We carry on in this manner until S is exhausted and take T as the subsystem consisting of $Q_1, Q_2, \ldots$. Then T is plainly discrete and satisfies the inequality (6).

One would not expect a result obtained by so simple an argument to be best possible: nor is it. The constant 1/9 in (6) can, in fact, be replaced by 1/4, i.e. every system S contains a discrete subsystem T such that

$$A(T) \geqslant \tfrac{1}{4} A(S). \tag{7}$$

The proof of this inequality, although not 'deep', is not entirely easy; and I shall not attempt to reproduce it here. However, it *is* easy to see that (7) represents the

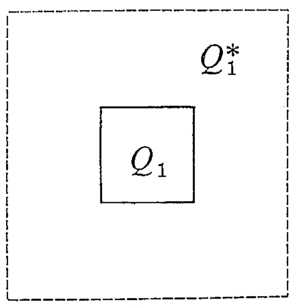

Figure 2

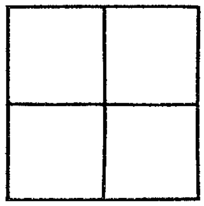

Figure 3

ultimate truth† in the sense that the number 1/4 cannot be replaced by any larger number. More precisely, given any $\lambda > 1/4$, we can exhibit a system S such that

$$A(T) < \lambda \,.\, A(S)$$

for *every* discrete subsystem T of S. The proof of this is trivial. Let S be a system comprising four squares, as shown in Figure 3. Any two of these squares have at least one point in common, and therefore a discrete subsystem can only contain a single square of S. Hence

$$A(T) = \tfrac{1}{4}A(S) < \lambda \,.\, A(S)$$

for every discrete subsystem T of this particular system S.

Let us, for a moment, revert to (6) and (7). Not surprisingly, there exist intermediate arguments which yield improvements in (6) without taking us all the way to (7). Thus, quite a simple modification of the selection procedure I described, enables us to replace 1/9 by 1/6 in (6); the reader may wish to discover for himself how this can be done.

There are many variants of the problem just discussed. For example, instead of investigating systems of similarly oriented squares of equal size, we might consider similarly oriented squares of arbitrary sizes. In that case, a minute adjustment in the earlier selection procedure leads to the conclusion that (6) remains valid. Beyond this point, however, the going gets very sticky and only a very slight improvement—the replacement of 1/9 by 5/43—has been achieved so far (and even this was done nearly twenty years ago). The 'correct' constant is probably a little smaller than 1/4; but, whatever the truth of the matter, I would not expect the problem to be settled quickly.

3. 'Small is beautiful'

We begin much as in Problem 1 but continue in a very different strain. Let Q be a unit square; let $n(\geqslant 3)$ be an integer; and let $P_1, P_2, \ldots, P_n$ be any n points on Q. We wish to show that among the $\binom{n}{3}$ triangles‡ $P_iP_jP_k$ $(1 \leqslant i < j < k \leqslant n)$, there is *always* at least one of very small area. Stated in this form, the proposition sounds excessively vague, for how small is 'very small'? Well, we shall see.

† The reader should not seek to ascribe an apocalyptic meaning to this mode of expression.

‡ If P_i, P_j, P_k happen to be collinear, the triangle $P_iP_jP_k$ is 'degenerate' and its area is 0. The possible presence of degenerate triangles in our configuration causes, of course, no difficulty.

Let us denote by $\Delta(P_i, P_j, P_k)$ the area of the triangle $P_iP_jP_k$ and by $\mu(P_1, \ldots, P_n)$ the smallest of the numbers

$$\Delta(P_i, P_j, P_k) \qquad (1 \leqslant i < j < k \leqslant n).$$

What we seek to do is to specify a function of n which, for $n \to \infty$, should tend to 0 as rapidly as possible and at the same time exceed $\mu(P_1, \ldots, P_n)$ no matter how the points $P_1, \ldots, P_n$ are distributed on Q.

Now, it is easy to show that

$$\mu(P_1, \ldots, P_n) \leqslant 1/(n-2) \qquad (n \geqslant 3). \tag{8}$$

For we may clearly suppose that no three among $P_1, \ldots, P_n$ are collinear as, in the contrary case, (8) holds trivially. Let a ray, issuing from P_1, rotate in a counterclockwise sense and let the numbering of the remaining $n-1$ points be arranged such that the ray passes through $P_2, \ldots, P_n$ in that order, with the choice of P_2 still to be decided. If the angle at P_1 (measured in a counterclockwise sense) made by every pair of 'neighbouring' rays is less than 180°, we can take any one of the $n-1$ points as P_2. Otherwise, the choice of P_2 is prescribed by the requirement that the angle $\angle P_nP_1P_2$ (taken in a counterclockwise sense) is greater than 180°. (The two cases are illustrated, for $n = 7$, in Figure 4.) For both alternatives, the $n-2$ triangles $P_1P_2P_3, P_1P_3P_4, \ldots, P_1P_{n-1}P_n$ do not overlap and, since all are in Q, they cover between them an area not exceeding 1, i.e.

$$\Delta(P_1, P_2, P_3) + \Delta(P_1, P_3, P_4) + \cdots + \Delta(P_1, P_{n-1}, P_n) \leqslant 1.$$

Hence $\Delta(P_1, P_{k-1}, P_k) \leqslant 1/(n-2)$ for some k with $3 \leqslant k \leqslant n$; and (8) is therefore established.

If n is large, then $1/(n-2)$ is small; and consequently we can claim to have made some progress: we have shown that, among any n points on Q, three can be chosen as the vertices of a triangle of small area. But give a mathematician an ell and he is sure to demand an inch. Adopting this principle, we declare ourselves dissatisfied with (8) and ask if it is possible to replace the right-hand side by a smaller number. It has been conjectured that an inequality much tighter than (8) (for large n) is valid, namely that, for a suitable positive constant c_1,

$$\mu(P_1, \ldots, P_n) < c_1/n^2 \qquad (n \geqslant 3) \tag{9}$$

whatever the distribution of the points $P_1, \ldots, P_n$ on Q may be. (We should note that, even if c_1 is a large number, c_1n^{-2} is much smaller than $1/(n-2)$ provided that n is sufficiently large.) The inequality (9) looks quite plausible, and the realization that the task of proving or refuting it is one of exceptional difficulty may, perhaps, come as something of a shock. At present, we are nowhere near being able to decide on the truth of the conjecture (9).

Any improvement on (8) is very hard to effect. The first such improvement was found in 1950, and there the matter rested for some twenty years. More recently, progress has been resumed; and the best known result states that, for any distribution of $P_1, \ldots, P_n$, we have

$$\mu(P_1, \ldots, P_n) < c_2/n^{\alpha}, \tag{10}$$

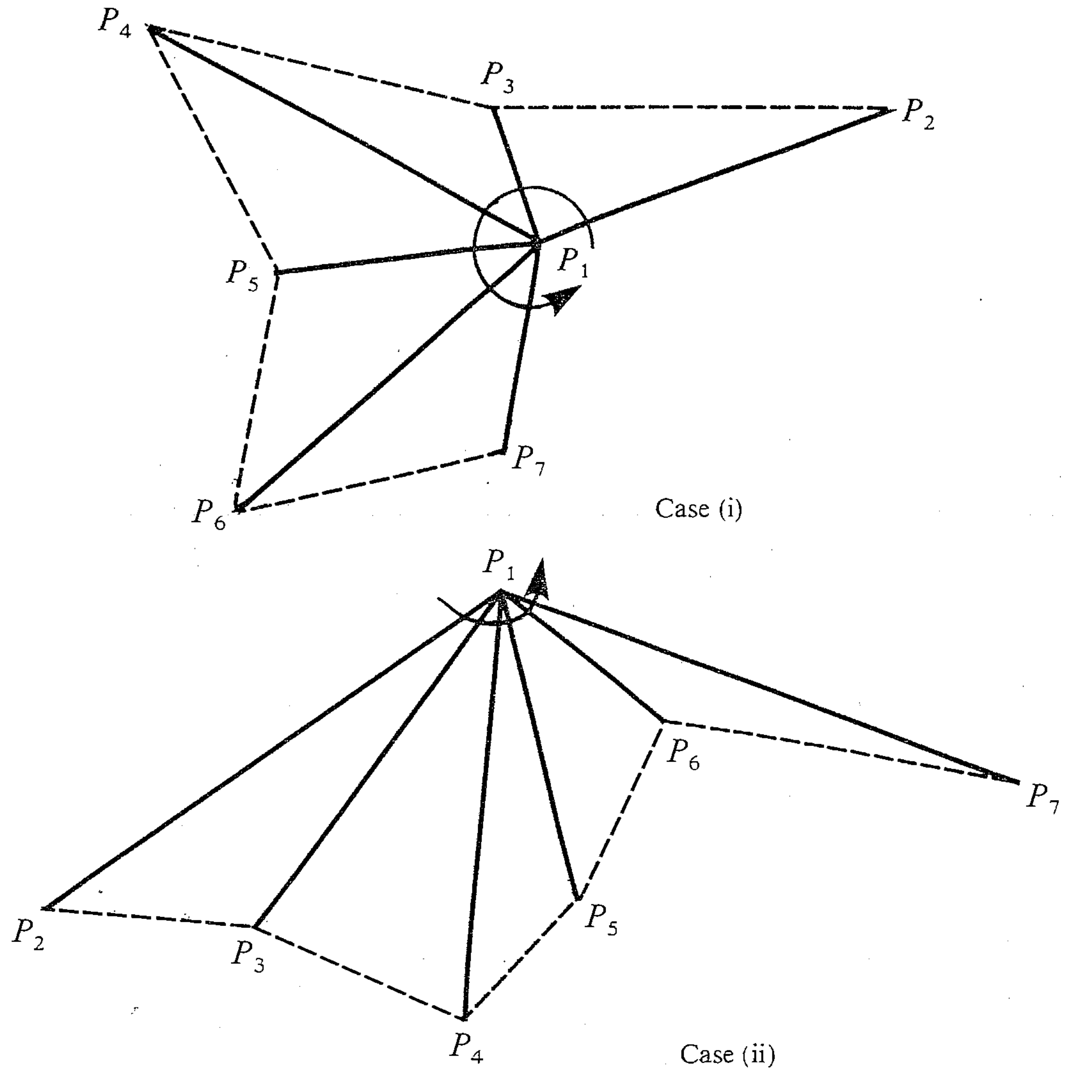

Figure 4

where α is any number smaller than $(17 - \sqrt{65})/8 = 1{\cdot}117\ldots$ and c_2 is a certain positive constant. The relation (10) is obviously superior to (8)—and the argument leading to it is one of daunting complexity—but it still falls a long way short of (9).

Finally, it is interesting to note that, if the conjecture (9) is valid, then it represents an effectively 'best possible' result and is incapable of any significant sharpening. More precisely, it can be shown that, for each $n \geqslant 3$ and a *suitable* choice of $P_1, \ldots, P_n$,

$$\mu(P_1, \ldots, P_n) > 1/8n^2. \tag{11}$$

After my harping on the difficulty of the problem, the reader may be relieved to learn that the proof of (11), though ingenious, is neither long nor hard. It is a little gem of mathematical reasoning: I omit its exhibition with reluctance and only because some knowledge of the distribution of prime numbers is needed to clinch the argument.†

† It is a far cry from areas of triangles to the distribution of prime numbers, but the hidden existence of such unlikely links is precisely one of the features that make mathematics uniquely fascinating.

4. The bouncing billiard ball

The three problems reviewed so far are all of a quantitative type. The problem to be discussed next is a mixture of quantitative and qualitative considerations, and the last problem will be seen to be wholly qualitative. It may also be of some interest to note that while the findings in Problems 1–3 derive from research carried out during the last thirty years, Problems 4 and 5 are classical.

We consider a billiard table in the shape of a square and investigate the motion of a billiard ball. The situation is, of course, conceived in highly idealized terms. The billiard ball is thought of as having no size and so is treated as a mathematical point. Further, we assume that the motion is frictionless and that no energy is lost during the impact between the ball and the edge of the table, so that, once the ball has been set in motion, it will continue to move indefinitely. Moreover, the postulate of the conservation of energy implies that when the ball strikes and then rebounds from an edge of the table, the 'angle of incidence' is equal to the 'angle of reflexion'. And finally, we shall use the convention that when the ball strikes a corner of the table, it returns along its former path.

If, now, the billiard ball is projected from any point on the table, what can we say about the nature of the path it will trace out? The problem was solved completely in 1913; and although the details of the argument are moderately sophisticated, the answer to our question can be described quite briefly. Let us denote by θ the angle between the initial line of motion of the ball and an edge of the table (it is immaterial which edge is considered). Everything then depends on the nature of θ. If $\tan \theta$ is rational (or ∞), the path is periodic, i.e. after a finite interval of time, the ball will return to its original position, moving in the same direction as initially, and everything will then be repeated *ad infinitum*. In short, the path will be a finite closed polygonal line traversed indefinitely. If, however, $\tan \theta$ is irrational, the situation is entirely different. In that case, the path is 'everywhere dense' (on the table). This means that, given any positive number ε (no matter how small), the path of the billiard ball will at some stage pass within a distance at most ε of every point of the table. Let there be no misunderstanding: I do not claim that the ball will pass *through* every point of the table but merely that it will pass *arbitrarily near* to every point.

The interested reader will find the details of the proof in Chapter 23 of Hardy and Wright, *An Introduction to the Theory of Numbers* (4th ed., Clarendon Press, 1960). Here I content myself with referring to a powerful theorem, proved by Kronecker (1823–1891) in 1884, on which the treatment of the billiard ball problem is based. Kronecker's theorem states that, if α is irrational and λ is an arbitrary real number and if $\|x\|$ denotes the distance of x from the nearest integer, then $\|n\alpha - \lambda\|$ is smaller than any preassigned positive quantity for infinitely many values of the natural number n.

Related problems present themselves almost automatically, and the most natural variants are the result of changing the shape of the billiard table. The case of a rectangular table is basically the same as that of a square, but the question is altered in an essential way, e.g. for a circular or an elliptic shape. I do not propose here to pursue the matter further and merely point to pp. 353–4 of Courant and Robbins, *What is Mathematics?* (Oxford University Press, 1941), where some extremely interesting observations on the case of an elliptic table will be found.

5. Square-filling curves: a case of morbid pathology

Among the mathematical topics studied at sixth-form level, calculus presents the greatest conceptual difficulties. Of these difficulties the pupil is normally unaware: the excitement of learning to handle a magically effective mathematical tool often makes him overlook the imprecision of his ideas, to speak of 'function', 'derivative', 'convergence' and so on in a largely intuitive manner, and to accept as conclusive a mode of reasoning that is at best plausible. This situation may well be inevitable as the pupil is repeating in his own learning process the historical experience of the community of European mathematicians. It took two hundred years for a fully articulated body of analysis to emerge from the chrysalis of rudimentary calculus, and it is therefore hardly to be expected that complete understanding will be attained instantaneously. However, at some stage the nettle has to be grasped; and that stage is normally reached in the first year of university study when an attempt is made to exhibit mathematical analysis within the framework of precise definitions and rigorous deductions. In the course of growing familiarity with 'real' mathematics, the student comes to appreciate that intuitive ideas, or rather feelings, must be kept firmly in their place (and that place is below stairs): they do not merely lack the required exactitude but often lead to demonstrably false conclusions. Thus the essence of a rigorous mathematical argument is the uncompromising removal from the formal exposition of vague or purely intuitive notions. Needless to say, intuition is indispensable when we are hunting for elusive clues or are engaged on a preliminary charting of our course of inquiry; but it has absolutely no role to play in the final demonstration. The resolve, then, to extrude intuitive notions from the formal discourse is no mere whim on the part of mathematicians: they are forced into their position by the unreliability of human intuition. This assessment can be illustrated convincingly (and often amusingly) by the construction of 'pathological' objects—objects, that is, whose existence we find at variance with the promptings of our intuition.

A concrete classical example will substantiate my general remarks. A function is said to be 'continuous' if its graph has no breaks. (This is, itself, a sloppy and inadequate definition but it will have to do for our present purpose.) Now, one's intuitive feeling suggests forcibly that a continuous function possesses a derivative (i.e. that its graph has a tangent) everywhere except possibly at some isolated points. This was, indeed, for a long time an unquestioned belief and it came as a bolt from the blue when Weierstrass (1815–1897), probably the most influential mathematician in the second half of the nineteenth century, constructed in 1861 a function everywhere continuous and nowhere differentiable. The graph of such a function is really impossible to visualize—the nearest we can get to it is by trying to think of a curve consisting of an infinity of infinitely small crinkles.

I now propose to have another bash at intuition or 'common sense' by exploring the idea of a 'curve'. What, in fact, *is* a curve? Given the usual set of coordinate axes, we might say that a curve is the graph of some function f, i.e. the set of all points (x, y), where $y = f(x)$ and x ranges over a (finite or infinite) interval. This is not yet entirely satisfactory because of complications caused by many-valued functions, and it is better to use a 'parametric representation', i.e. to think of a curve as the set of all points $(\phi(t), \psi(t))$, where ϕ, ψ are one-valued functions and t ranges over an interval. This curve is said to be continuous if ϕ, ψ are continuous functions. A familiar example is that of a parabola with the parametric representation $(at^2, 2at)$, where t ranges from $-\infty$ to $+\infty$.

At all events, however we may think of a curve, one idea is surely present in our minds. It is that, in some sense (not usually very clearly defined), a curve is a one-dimensional object and that the total area occupied by it is negligible (or, perhaps, we should say that it is equal to 0). Yet this idea—to all appearances so natural and in such close accord with our experience—is nevertheless wide of the mark. For if we take a square Q in the plane, we can produce a continuous curve Γ which passes through *every* point of Q. Thus the area occupied by Γ is far from negligible—it is the whole of Q. Intuitively, the existence of such a twisting and turning curve appears unthinkable; but the force of the mathematical argument compels our intuition to withdraw from the unequal contest. If, then, any mathematical object is to be designated as pathological, a continuous 'square-filling' curve has assuredly a clear title.

The construction of such a curve† can nowadays be accomplished very briskly; but as the reasoning is rather technical, I shall not enter into details. Let me simply record that the existence of a continuous square-filling curve was first demonstrated in 1890 by Guiseppe Peano (1858–1932), who must be regarded as one of the leading figures in the critical movement—a movement which sought to place mathematics on a secure logical foundation by freeing it from the hazards of an intuitive approach.

The existence of curves of the Peano type at once raises a number of other questions, and I shall mention just two of them, (i) Does there exist a continuous curve which passes through every point of the *entire* plane? (ii) Is there a three-dimensional analogue of Peano's discovery? As may be guessed, the answers to both questions are affirmative.

The reader has now the five problems before him, and I can rest my case. Several of the problems are still largely unsolved and all of them readily suggest fresh lines of enquiry. Possibly the reader would like to exercise his imaginative powers by framing further questions arising from or related to those I have presented for his inspection (and preferably also taking some steps towards their solution). The correspondence columns of *Mathematical Spectrum* are—so the Editor informs me—open to the communication of such material.

† For some intriguing facts, see E. P. Northrop, *Riddles in Mathematics* (Penguin Books, 1961) 139–140.

Queueing Processes

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Each one of us encounters queueing processes in everyday life. We queue up at a bus stop, join a queue in a bank, post office, ticket office, department store or supermarket. We wait in a barbershop, in the dentist's office, or at an airport. There are many other examples of queueing processes such as sharing computers, or telecommunication channels, directing the landing of airplanes at an airport, and controlling road traffic. When we lift the telephone receiver, we actually participate in a queueing process and expect that the waiting time is negligible, that is, the dialing tone sounds before we are ready to make the call.

In many queueing processes we can control the situation. We can assign more servers to a service station, we can add a new unit to a telephone exchange or we can allocate more buses to a bus route. If we want to design an efficient queueing system we should study how the waiting time and the queue size depend on the controllable parameters.

To illustrate the problems that arise, let us consider a simple queueing problem. Let us suppose that at a bus stop buses are scheduled to arrive at regular six-minute intervals. If a passenger arrives at the stop at random between 12 noon and 1 p.m., what is his expected waiting time? Let us suppose that 12 noon is an arrival time of a bus, and the arrival time of a given passenger between 12 noon and 1 p.m. has a uniform distribution over this interval of 60 minutes. By a uniform distribution we mean that if we divide the one-hour interval into subintervals of equal length, each subinterval has the same probability of containing the arrival time of the passenger. Thus, if we divide the one-hour interval into 60 one-minute intervals, then each subinterval has probability 1/60; if we divide the one-hour interval into 3600 one-second intervals, then each subinterval has the probability 1/3600 of containing the arrival time of the passenger. Let us measure time from 12 noon and denote by $\eta(t)$ the waiting time at time t, that is, the time difference between t and the arrival time of the next bus. Let $T = 60$ minutes. Figure 1 illustrates how $\eta(t)$ varies as a function of t if buses indeed arrive regularly in six-minute intervals. Immediately after the

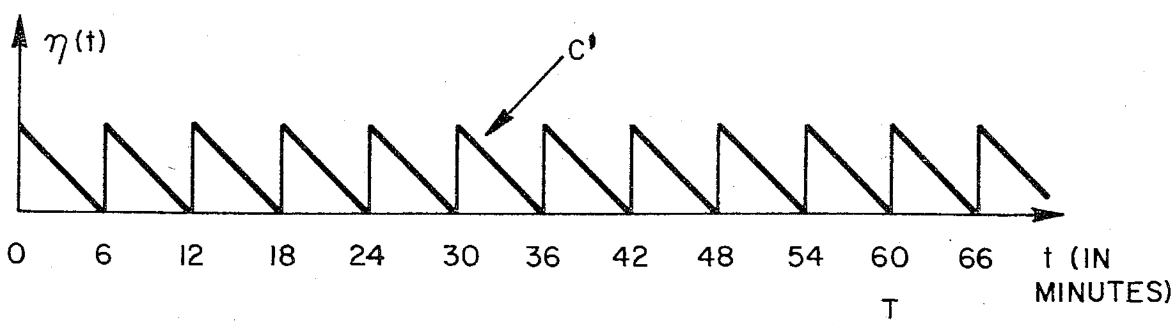

Figure 1

departure of a bus $\eta(t)$ is six minutes, and then $\eta(t)$ decreases linearly, reaching 0 at the arrival time of the next bus.

What is the expected (average, mean) waiting time of a passenger arriving at random in the interval $(0, T)$? Since the arrival time of the passenger has a uniform distribution over $(0, T)$, the expected waiting time, say W, is the area between the curve $C: (t, \eta(t))$, $0 \leqslant t \leqslant T$, and the t-axis, divided by T, that is,

$$W = \frac{1}{T}\int_0^T \eta(t)dt = \frac{10 \times 18}{60}\min = 3\min. \tag{1}$$

The integral of $\eta(t)$ from 0 to T, that is, the area below the curve C, is the total area of 10 triangles, each having an area of $(6 \times 6/2)\min^2 = 18\min^2$. The result (1) is plausible. Buses arrive regularly, and the average waiting time is one half of the inter-arrival time.

How does W change if the regular arrival pattern of the buses is altered? To study the effect of any change in the regular arrival pattern, let us consider the solution of the following general problem. Let us measure time from the departure of a bus and denote by $a_1, a_2, \ldots, a_n$ the next n consecutive inter-arrival times. Let $a_1 + a_2 + \cdots + a_n = T$. Let us suppose that a passenger arrives at random in the interval $(0, T)$ and the arrival time has a uniform distribution over $(0, T)$. Define

$$\mu = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{T}{n} \tag{2}$$

as the expected inter-arrival time of the buses and

$$\mu_2 = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \tag{3}$$

as the second moment of the inter-arrival times.

If $\eta(t)$ denotes the waiting time at time t, where t is measured from the departure time of a given bus, then $\eta(t)$ varies with t as shown in Figure 2.

As in the previous case, W, the expected waiting time of a passenger arriving at random in the interval $(0, T)$ is the area between the curve $C': (t, \eta(t))$, $0 \leqslant t \leqslant T$, and the t-axis, divided by T. The area below the curve C' is the sum of the areas of n

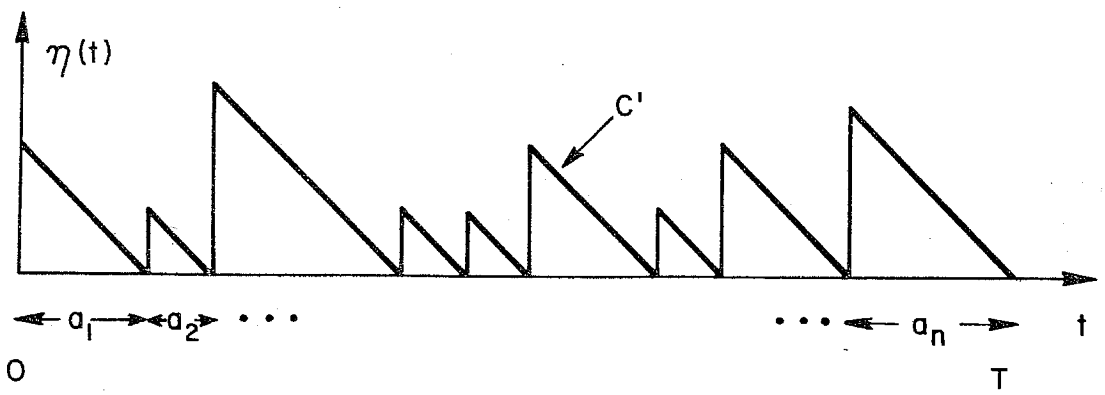

Figure 2

triangles whose areas are $a_1^2/2, a_2^2/2, \ldots, a_n^2/2$ respectively. Thus

$$W = \frac{1}{T}\int_0^T \eta(t)dt = \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{2T}. \tag{4}$$

If we use the notations (2) and (3), then (4) can be expressed in the following equivalent form:

$$W = \frac{\mu_2}{2\mu}. \tag{5}$$

If buses arrive regularly, that is, $a_1 = a_2 = \cdots = a_n$, then $\mu_2 = \mu^2$ and by (5) $W = \mu/2$. This is in agreement with the previous result (1).

Actually, we have $W = \mu/2$ if and only if buses arrive at regular intervals. Otherwise $W > \mu/2$. This follows from Schwarz's inequality which states that for all values of the $a_i (i = 1, \ldots, n)$

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geqslant \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2, \tag{6}$$

and in (6) we have equality if and only if $a_1 = a_2 = \cdots = a_n$. By (6) $\mu_2 \geqslant \mu^2$ and therefore (5) implies that always

$$W = \frac{\mu_2}{2\mu} \geqslant \frac{\mu}{2} \tag{7}$$

and $W = \mu/2$ if and only if $a_1 = a_2 = \cdots = a_n$.

By (5) we have the surprising result that the expected waiting time depends not only on the expected inter-arrival time but on the second moment of the inter-arrival times too. The expected waiting time is a minimum for regular arrivals and greater otherwise, that is, we can draw the conclusion that if we want to minimize the expected waiting time, we have to schedule the arrivals of the buses at regular intervals.

We might expect that even if $W > \mu/2$ for irregular intervals, we always have $W \leqslant \mu$, that is, W cannot exceed the expected inter-arrival time. To support this view we might reason that the worst case occurs if a passenger just misses a bus. Then the waiting time is precisely the next inter-arrival time which has the expectation μ. In any other case the waiting time is less than an inter-arrival time. Let us consider an example: suppose that $n = 10$, $T = 60$ min and the inter-arrival times $a_1, a_2, \ldots, a_{10}$ are 1, 4, 5, 1, 5, 33, 5, 1, 4, 1 minutes. Then

$$\mu = \frac{1 + 4 + \cdots + 1}{10}\,\text{min} = \frac{60\,\text{min}}{10} = 6\,\text{min},$$

$$\mu_2 = \frac{1^2 + 4^2 + \cdots + 1^2}{10}\,\text{min}^2 = \frac{1200\,\text{min}^2}{10} = 120\,\text{min}^2$$

and

$$W = \frac{\mu_2}{2\mu} = \frac{120}{12}\,\text{min} = 10\,\text{min}.$$

This is really a surprising result: the expected waiting time ($W = 10$ min) exceeds the expected inter-arrival time ($\mu = 6$ min). The result is surprising only at first sight. If we take into consideration that it is more probable that the arrival time of a passenger falls in a longer inter-arrival than in a shorter one, then we can see why W can be greater than μ.

In queueing theory there are many situations where we encounter similar surprising phenomena. In order to understand these completely and not to draw false conclusions, we must study the relation between the inter-arrival times and the waiting times more thoroughly.

Let us return to the process discussed above when the inter-arrival times are $a_1, a_2, \ldots, a_n$, $a_1 + a_2 + \cdots + a_n = T$ and the arrival time of a passenger has a uniform distribution over the interval $(0, T)$. Denote by $N(x)$ the number of inter-arrival times no larger than x among the n inter-arrival times $a_1, a_2, \ldots, a_n$. Then

$$F(x) = \frac{N(x)}{n} \tag{8}$$

is the probability that an inter-arrival time chosen at random is smaller than or equal to x. We say that $F(x)$ is the distribution function of the inter-arrival times.

Denote by $W(x)$ the probability that the waiting time of a passenger arriving at random in the interval $(0, T)$ is no larger than x, that is, $W(x)$ is the distribution function of the waiting time. If $x \geqslant 0$, we have

$$W(x) = \frac{1}{T} \sum_{i=1}^{n} \min(a_i, x) \tag{9}$$

where $\min(a, x) = a$ if $a \leqslant x$ and $\min(a, x) = x$ if $x \leqslant a$. For the waiting time is no larger than x if the arrival time falls either in an inter-arrival interval of length smaller than or equal to x or in a subinterval of length x of an inter-arrival interval of length greater than x.

We can easily express $W(x)$ with the aid of $F(x)$. First, let us illustrate $N(u)$ for $u \geqslant 0$ as a function of u. In Figure 3: $n = 10$, $a_1 = 4$, $a_2 = 2$, $a_3 = 7$, $a_4 = 9$, $a_5 = 5$, $a_6 = 6$, $a_7 = 3$, $a_8 = 10$, $a_9 = 6$, $a_{10} = 8$ min, and $T = 60$ min. Then $\mu = 6$ min, $\mu_2 = 42\text{ min}^2$ and $W = 3{\cdot}5$ min.

We can express the sum

$$\sum_{i=1}^{n} \min(a_i, x)$$

as the area of the shaded set in Figure 4. In Figure 4, $x = 7$ min. The shaded set consists of n rectangles whose areas are $a_i \times 1$ if $a_i \leqslant x$ and $x \times 1$ if $a_i \geqslant x$ for $i = 1, 2, \ldots, n$. On the other hand the area of the shaded set is

$$nx - \int_0^x N(u)du = \int_0^x [n - N(u)]du \tag{10}$$

for $x \geqslant 0$. Thus

$$\frac{1}{n} \sum_{i=1}^{n} \min(a_i, x) = \int_0^x \left[1 - \frac{N(u)}{n}\right] du = \int_0^x [1 - F(u)]du \tag{11}$$

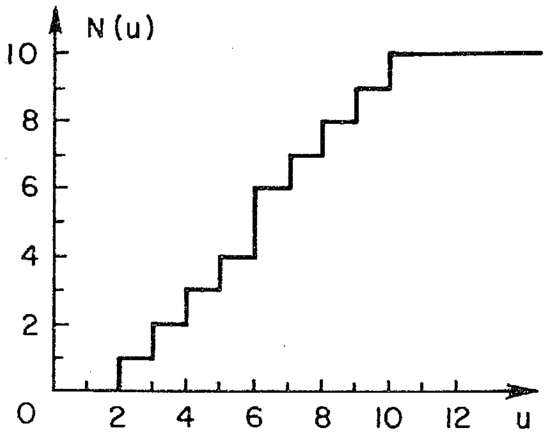

Figure 3

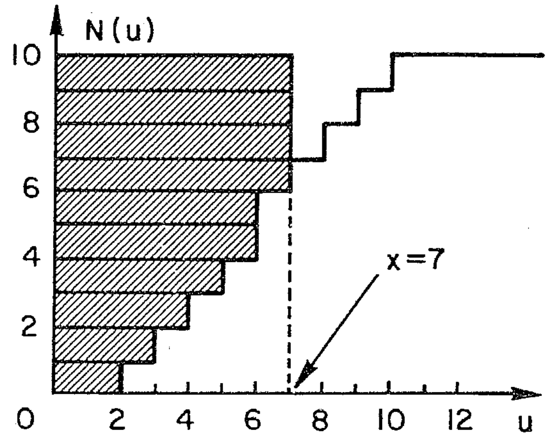

Figure 4

for $x \geqslant 0$ and by (2) and (9) we get

$$W(x) = \frac{1}{\mu}\int_0^x [1 - F(u)]du \tag{12}$$

for $x \geqslant 0$. This is an important result. Formula (12) expresses a simple relation between the distribution function of the inter-arrival times and the distribution function of the waiting time. By (12) we can study the effect of irregular arrivals of the buses on the waiting time of a passenger.

There is a simple relation between the moments of the inter-arrival times and the moments of the waiting time. We define

$$\mu_k = \frac{a_1^k + a_2^k + \cdots + a_n^k}{n} \tag{13}$$

for $k = 1, 2, \ldots$ as the kth moment of the inter-arrival times. In particular, $\mu_1 = \mu$. Then the kth moment of the waiting time is given by

$$W_k = \frac{1}{T}\int_0^T [\eta(t)]^k dt = \frac{a_1^{k+1} + a_2^{k+1} + \cdots + a_n^{k+1}}{(k+1)T} = \frac{\mu_{k+1}}{(k+1)\mu} \tag{14}$$

for $k = 1, 2, \ldots$.

In the above discussion we assumed that the inter-arrival times of the buses are given constants, not random variables. If the buses arrive at random and the inter-arrival times are random variables, then the above results can be applied for all realizations of these random variables, and we can conclude that formulas (12) and (14) remain valid for random inter-arrival times too.

In conclusion, let us mention one more example which illustrates how probability theory can help in designing efficient queueing systems. This is the case of telephone exchanges. The result which we mention was found in 1917 by A. K. Erlang (1878–1929), a Danish mathematician and telephone engineer working at the Copenhagen Telephone Company. He was a pioneer in the use of probability theory for designing telephone exchanges (see E. Brockmeyer, H. L. Halstrøm and A. Jensen (reference 1)). Here we mention one of his remarkable results which has been used ever since in the design of telephone exchanges.

Let us assume that in a telephone exchange there are m available lines. Calls arrive at random. Denote by λ the expected number of calls arriving in an interval of unit length; λ is the density of calls. If an arriving call finds a free line, a connection is made immediately. If every line is busy, an arriving call is lost. The holding times are assumed to be independent and identically distributed random variables. Denote by α the expected length of a holding time, that is the average length of a conversation. What is the probability that an arriving call finds every line busy? In other words, what is the probability that a call is lost? Denote this probability by P_m. If we want to design an adequate telephone exchange, then for given values of λ and α, we should choose m so large that P_m, the probability of loss, will be very small, say $<0{\cdot}001$. To solve this problem we should determine P_m as a function of the parameters λ, α and m. Erlang solved this problem in the case where the arrivals of calls satisfy the following conditions: the number of arrivals in disjoint time intervals are independent random variables; the distribution of the number of arrivals in any interval depends only on the length of the interval; and arrivals occur singly. In this case we say that calls arrive in accordance with a Poisson process and the probability that exactly k calls arrive in an interval of length t is given by

$$e^{-\lambda t}\frac{(\lambda t)^k}{k\,!} \tag{15}$$

for $k = 0, 1, 2, \ldots$, where λ is the density of arrivals.

It is intuitively clear and easy to prove that

$$P_1 = \frac{\lambda\alpha}{1 + \lambda\alpha}. \tag{16}$$

For $m = 2, 3, \ldots$ the probabilities P_m can be determined by the recurrence formula

$$P_m = \frac{\lambda\alpha P_{m-1}}{m + \lambda\alpha P_{m-1}}. \tag{17}$$

This implies that

$$P_m = \frac{(\lambda\alpha)^m/m\,!}{\sum\limits_{j=0}^{m} (\lambda\alpha)^j/j\,!} \tag{18}$$

for $m = 1, 2, 3, \ldots$; this is the celebrated Erlang formula. There are extensive tables for (18) which make it possible to determine the probability that a call is lost provided that we know the arrival density λ, the average holding time α, and the number m of available lines. For example, if $\lambda\alpha = 10$ (say $\lambda = 5$/min and $\alpha = 2$ min), then $P_{18} = 0{\cdot}0071424382$, $P_{21} = 0{\cdot}0008892323$, $P_{24} = 0{\cdot}0000731762$, $P_{27} = 0{\cdot}0000041694, \ldots$. If we have $m = 24$ lines, then the probability of a lost call is less than $0{\cdot}0001$.

Erlang deduced formula (18) in some particular cases and conjectured that it was true regardless of the form of the distribution function of the holding times. The truth of this conjecture was later demonstrated by É. Vaulot, F. Pollaczek, C. Palm, L. Kosten, B. A. Sevastyanov and others. For a simple proof of Erlang's formula the reader is referred to an article by the author (reference 2).

References

1. E. Brockmeyer, H. L. Halstrøm and A. Jensen, The life and works of A. K. Erlang, *Trans. Danish Acad. Tech. Sci.* **2** (1948), 1–277.
2. L. Takács, On Erlang's formula, *Ann. Math. Statist.* **40** (1969), 71–78.

Towards the Abstract

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1. Our plan

We propose to consider a very basic kind of human experience, namely, the experience of dealing with *sets* (classes) of objects. We shall show that out of the study of such simple surroundings mathematics grows by the process of observation, experimentation, discovery and invention.

We shall construct a kind of algebra whose elements are sets (of objects) and we shall study in detail the properties of such an algebra. We shall then introduce other examples of algebraic systems, and we shall propose a way of comparing such systems. We shall discover that mathematical systems can resemble each other in some fundamental way, and we shall be led to a discussion of the process of abstraction (the algebra of sets, the algebra of divisors and the algebra of logic, all leading to the abstract system called Boolean algebra).

2. Some basic questions concerning sets

In each discussion we select a set I of objects which we agree to consider and which we call our *universe of discourse*. As the elements of our mathematical system we shall take the *subsets* of our universe of discourse.

We say that a set A is a *subset* of a set B or that A is contained in B if and only if every element of A is an element of B. This does not exclude the case when $A = B$. If A is contained in B we write $A \subset B$. If p is an element of A we write $p \in A$. We indicate implication (if, then) by the double arrow $\Rightarrow$ and equivalence (if and only if) or twofold implication by $\Leftrightarrow$. Thus, in shorthand,

$$A \subset B \Leftrightarrow (p \in A \Rightarrow p \in B).$$

A set may be described by exhibiting its elements or the names of these elements. Thus we may describe the set of all students in the freshman mathematics class by referring to the full list I, where I consists of all students in our college. If the universe of discourse I is taken to be the set of all integers, we may speak of the set

$$A = \{1, 2, 5, 10\},$$

or the set

$$B = \{1, 3, 7, 9\}.$$

A set may also be described by indicating the characteristic properties of its elements, i.e. the properties which are possessed by all the elements of the given set and by no other elements. Thus the last two sets may be described as follows:

$A = \{$all positive integral divisors of $10\}$,
$B = \{$all x such that x is integral, $1 \leqslant x \leqslant 10$, x is relatively prime to $10\}$,

or, in shorthand,

$A = \{x{:}\, x$ positive integer, $x \mid 10\}$,
$B = \{x{:}\, x$ integral, $1 \leqslant x \leqslant 10$, $(x, 10) = 1\}$.

Here $\{x{:}\, x$ has property $P\}$ is read as *all* x such that x has the property P.

Problem 1. The following sets are described in terms of the characteristic properties of their elements. Describe these sets by exhibiting their elements. In each case a suitable universe of discourse is presupposed.

$A = \{x{:}\, x$ a prime number, $1 < x < 50\}$,
$B = \{z{:}\, z$ a complex number, $z^4 = 1\}$,
$C = \{$all right triangles with integral sides m, n, k and with the area $A < 200\}$,
$D = \{c{:}\, c$ a coefficient in the expansion of $(x + y)^{11}\}$.

Problem 2. Describe the following sets by giving the characteristic properties of their elements:
$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$,
$B = \{1, 7, 11, 13, 17, 19, 23, 29\}$,
$C = \{1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i\}$,
$D = \{6, 28, 496, 8128\}$.

Given two sets A and B (in our chosen universe of discourse I) it is natural to inquire about the set of elements which A and B have in common (the *intersection* of A and B) and also about the totality of all elements contained in at least one of the two sets A, B (the *union* of A and B). We denote the intersection of A and B by $A \cap B$ and their union by $A \cup B$. The complement of a set A is the set A' of all the elements in I but not in A. In shorthand,

$D = A \cap B \Leftrightarrow (p \in D \Leftrightarrow p \in A$ and $p \in B)$,
$U = A \cup B \Leftrightarrow (p \in U \Leftrightarrow p \in A$ or $p \in B)$,
$C = A' \Leftrightarrow (p \in C \Leftrightarrow p \notin A) \Rightarrow A' \cup A = I.$

We should note that the disjunction 'or' is used by mathematicians in the sense of 'one or the other *or both*.' In common parlance the last alternative is most often excluded. The inquiry 'Are you a man or a mouse?' excludes the possibility of both alternatives holding true simultaneously.

3. The algebra of sets

We are well on the way to constructing our mathematical system. We have on hand the elements of our system. These are the subsets of our universe of discourse I. The operations of taking the union and the intersection of two sets remind one of the familiar binary operations of arithmetic. However, we still have some subtle adjustments to make before we can achieve our end.

We recall that a *binary operation* in a set S is given by a rule which assigns a *unique element* of S to *every pair* of elements of S. We see that the taking of the union determines a binary operation in S, where S is the *set of all the subsets of* I having at least one element. Taking the intersection of two sets A and B in S yields another set in S only when A and B have elements in common. To provide two disjoint sets with an intersection we introduce a fictitious subset $\varnothing$ called the *null set* (popularly referred to as the set without elements) and write $A \cap B = \varnothing$ if A and B are disjoint. With this convention the taking of the intersection is a binary operation in the set $S = \mathscr{P}(I)$ of all subsets of I *including the null subset* $\varnothing$. The real mathematical meaning of the null set will become clearer as our discussion develops. We shall find it convenient to take $A \cup \varnothing = A = \varnothing \cup A$ for every $A \in \mathscr{P}(I)$. Then the union is also a binary operation in $\mathscr{P}(I)$.

If we agree that $\varnothing' = I$ and $I' = \varnothing$, then taking the complement of a set is a *unary operation* in $\mathscr{P}(I)$, i.e. to *every element* A of $\mathscr{P}(I)$ it assigns a *unique element* of $\mathscr{P}(I)$.

4. Conjecture, counterexample, proof

Problem 3. Let $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{1, 3, 5, 7\}$, $D = \{7, 8, 9, 10\}$, $E = \{1, 2, 3, 4, 5, 6\}$. What sets are determined by the following algebraic expressions: $(A \cup B) \cap C$, $(A \cap C) \cup (B \cap C)$, $A \cup B$, $B \cup A$, $A \cap C$, $(A \cap D) \cup C$, $C \cap A$, $(B \cap C) \cup A$, $(B \cup A) \cap (C \cup A)$, $A' \cap B'$, $(A \cup B)'$, $C' \cup D'$, $(A \cap B) \cap C$, $(C \cap D)'$, $A \cap (B \cap C)$, $B \cup (C \cup D)$, $B \cup E$, $(B \cup C) \cup D$, $E \cap E$? Here the parentheses indicate, as usual, the order in which the operations are to be carried out.

Would the reader venture any guesses regarding the properties of the operations of our new algebra on the basis of the above calculations?

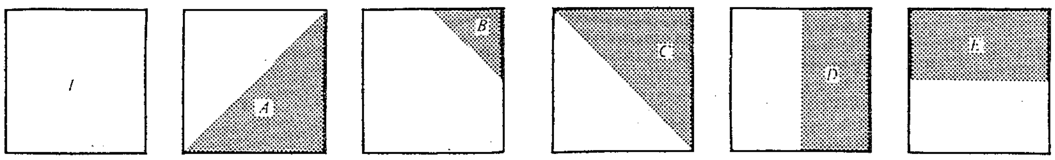


Figure 1

Problem 4. Let I consist of all points in the left-hand square in Figure 1 and let A, B, C, D, E be the subsets of I indicated by the shading. In a separate figure for each expression indicate by shading the sets determined by each algebraic expression given in Problem 3. Do the results of the new 'calculations' give additional support to the guesses suggested by the results in Problem 3?

We observe that in Problem 3, $E \cap B = \{3, 4, 5\} = B$. Also in Problem 4 $E \cap B = B$. We conjecture that $X \cap Y = Y$ for every pair of sets X and Y. That is, we conjecture that the last equality is *an identity*. However, substituting the sets A and B in Problem 3 for X and Y respectively we find that $A \cap B = \{3\} \neq B$. Thus we have found an example which is contrary to our conjecture (*a counterexample*) and have shown through this that $X \cap Y = Y$ is not an identity.

We see that in order to disprove the truth of a universal assertion we need to find but one counterexample.

It does not suffice, however, to find a number of instances for which our assertion holds true in order to be sure that it holds universally. Consider the equality (12)† $X' \cup Y' = (X \cap Y)'$ for example. This equality is verified by every pair of sets in Problem 3. This fact does not serve as a guarantee that greater perseverance will not produce a counterexample. To show that (12) is indeed an identity we must resort to a detailed analysis of its meaning.

To arrange our discussion in a neat manner we observe that the very definition of the equality of two sets as sets having exactly the same elements may be written in the form

$$U = V \Leftrightarrow [(p \in U \Rightarrow p \in V) \text{ and } (p \in V \Rightarrow p \in U)]$$

which means that

$$U = V \Leftrightarrow U \subset V \text{ and } U \supset V. \qquad (*)$$

We next observe that our basic definitions yield the following chain of implications for every X and Y:

$$p \in X' \cup Y' \Leftrightarrow p \in X' \text{ or } p \in Y' \Leftrightarrow p \notin X \text{ or } p \notin Y \Leftrightarrow p \notin X \cap Y \Leftrightarrow p \in (X \cap Y)'.$$

Reading this argument from left to right we see that $X' \cup Y' \subset (X \cap Y)'$. Reading from right to left shows that $X' \cup Y' \supset (X \cap Y)'$. That (12) is an identity follows then from $(*)$.

Problem 5. Which of the following equalities are identical equalities (identities)? In each case either find a counterexample or give an 'elementwise' proof such as the one given for (12).†‡

(1) $X \cap X = X$, (2) $X \cup Y = Y \cup X$, (3) $(X \cup Y)' = X' \cup Y'$,
(4) $X \cap (Y \cap Z) = (X \cap Y) \cap Z$, (5) $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$,
(6) $\varnothing \cap X = \varnothing$, (7) $I \cup X = I$, (8) $X \cup X' = I$, (9) $X \cap X' = \varnothing$,
(10) $X \cup Y = Y$, (11) $X \cup (Y \cap Z) = (X \cup Y) \cap Z$, (12) $(X \cap Y)' = X' \cup Y'$,
(13) $(X \cup Y)' = X' \cap Y'$, (14) $X \cap (X \cup Y) = X$,
(15) $(X' \cup Y')' \cup (X' \cup Y)' = X$, (16) $X \cup (X \cap Y) = X$, (17) $X \cup X = X$,
(18) $X \cap Y = Y \cap X$, (19) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$,
(20) $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, (21) $\varnothing \cup X = X$, (22) $I \cap X = X$,
(23) $(X')' = X$.

† The numbers in round brackets here and below refer to the lists in Problems 5 and 6.
‡ We suggest that at a first reading, it is sufficient to look in detail at (say) the odd-numbered examples only.

Problem 6. Which of the following statements hold true universally?

(24) $X \cap Y = X \Leftrightarrow X \subset Y \Leftrightarrow X \cup Y = Y$, (25) $X \subset X$,
(26) $X \subset Y$ and $Y \subset Z \Rightarrow X \subset Z$, (27) $\varnothing \subset X$, (28) $X \subset I$,
(29) $A \cap X = A \cap Y$ and $A \cup X = A \cup Y \Rightarrow X = Y$, (30) $X \subset X \cup Y$,
(31) $X \supset X \cap Y$, (32) $T = (X \cap T') \cup (X' \cap T) \Leftrightarrow X = \varnothing$,
(33) $X \subset A' \Leftrightarrow X \cap A = \varnothing$, (34) $Y \supset A' \Leftrightarrow A \cup Y = I$,
(35) $X \cap Y' = Z \cap Z' \Leftrightarrow X \cap Y = X$, (36) $X \subset Y \Leftrightarrow X' \supset Y'$,
(37) $X \subset Y$ and $X \subset Z \Rightarrow X \subset Y \cap Z$, (38) $X \subset Z$ and $Y \subset Z \Rightarrow X \cup Y \subset Z$,
(39) $X = Y \Leftrightarrow X \subset Y$ and $Y \subset X$, (40) $X \subset Y \Leftrightarrow X' \cup Y = I$,
(41) $X \subset Y \Rightarrow X \cup Z \subset Y \cup Z$, (42) $X \subset Y \Rightarrow X \cap Z \subset Y \cap Z$.

The student should test his powers of observation and his capacity for initiative on the examples in Problems 3–6. He should go through the various stages involved in the process of mathematical discovery, viz. 1) experimentation and observation, 2) making a conjecture, 3) testing for possible counterexamples, 4) justification.

5. Interdependence of properties of a mathematical system

Problem 7. Show that for every two sets X and Y in I, the four sets $X \cap Y$, $X \cap Y'$, $X' \cap Y'$, $X' \cap Y$ form a partition of I, i.e. these sets are pairwise disjoint (which means that no two of them have elements in common) and their union is I.

It can be shown directly without resorting to an elementwise argument that some universal statements are consequences of one or more other universal statements. Thus (1) is a consequence of (31), (37) and (39). On the other hand (29) can be deduced from (31), (18), (24), (2) and (5).

Problem 8. Justify the steps in the above derivation of (1) and (29).

We shall speak of this type of derivation as a relative argument, to distinguish it from the elementwise method of proof. The relative argument brings out (indeed is based on) the pattern of dependence which exists among various true statements in a given mathematical system. It is the study of these patterns that a mathematician has in mind when he speaks of the study of the structure of such systems.

6. The game of a 'reduced inventory': the first step towards abstraction

Let us list all the statements in Problems 5 and 6 (or, say, the odd-numbered ones) which can be proved to hold true by an elementwise argument. Then let us play the following game. We shall allow ourselves to drop a statement from this list if its truth follows by a relative argument from a number of other true statements in our listing. From what is left we can again drop a true statement provided it is derivable from others which have not yet been dropped from the list, and so on. This process will terminate either when the remaining statements can no longer be derived from each other or through the limitations of the player's ingenuity.

We may, as an example, start our game as follows: We drop (29) from our initial list since it follows from (31), (24), (2), (5) and (18). Since (1) is a consequence of (31),

(37) and (39), none of which has been dropped, we may drop (1) in addition to (29). Next, we note that (8) follows from (25), (2) and (40), all of which still remain in our listing. Hence (8) can also be dropped.

Proceeding in this manner we can go as far as our ingenuity will take us.

We should note that if every statement in the reduced list which is left at the end of the game is proved by an elementwise argument, then all other statements on the original list can be deduced from those on the reduced list by the usually less cumbersome relative argument.

Different players are likely to arrive at different though equally useful reduced lists. Birkhoff and MacLane (reference 1) use (1), (17), (2), (18), (4), (19), (5), (20), (24), (7), (28), (21), (22), (6), (27), (8), (9), (12), (13), (23). Rosenbloom (reference 3) essentially uses (18), (4), (35) and Courant and Robbins (reference 2) suggest (2), (19), (15).

A relative argument involving only the identities in a reduced list of statements is called *algebraic manipulation.*

Problem 9. Show using algebraic manipulation that

$$(A \cap B)' = (A \cap B') \cup (A' \cap B) \cup (A' \cap B') \text{ is an identity.}$$

Proof. $(A \cap B)' = A' \cup B' = (A' \cap I) \cup (B' \cap I) = [(A' \cap B) \cup (A \cap B')] \cup [(B' \cap A) \cup (B' \cap A')] = (A \cap B') \cup (A' \cap B) \cup (A' \cap B')$.

Justify each step of the above proof by references to appropriate identities in Problem 5. Comment on the identity just proved in the light of the partition described in Problem 7.

7. Systems with identical reduced inventories of properties: the second step toward abstraction

Let us now make what will seem, at first glance, to be a digression but which will prove to be the promised thought-provoking surprise of our discussion.

Problem 10. Show that the taking of the greatest common divisor (a, b) of two integers a and b is a binary operation in the set S if 1) S is the set of positive integers; 2) $S = S_n$ is the *set of all positive divisors* of $n = 6$, or of $n = 12$, or of $n = 30$ or of an arbitrary integer n. Show that for each of the above sets S, the taking of the lowest common multiple $[a, b]$ of two integers a and b is also a binary operation.

Problem 11. To conform with the usual custom of placing the symbol for binary operations between the two elements involved, we shall write $a \wedge b = (a, b) =$ g.c.d. of a and b, and $a \vee b = [a, b] =$ l.c.m. of a and b. If a divides c, we write $a < c$. Show that the following statements hold true in S_6, S_{12}, S_n (n an integer) and S:

(1) $x \wedge x = x$, (2) $x \vee y = y \vee x$, (4) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, (5) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$, (20) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, (24) $x \wedge y = x \Leftrightarrow x < y \Leftrightarrow x \vee y = y$. Observe the similarity with the like numbered statements in Problems 5 and 6.

What identities in our present systems correspond to (21) and (22) in Problem 5? In other words, what elements in our system behave like $\varnothing$ and I in the algebra of sets? In what way is this behaviour the same as that of zero relative to addition and unity relative to multiplication in ordinary arithmetic in the set of all integers $\mathbf{Z}$?

What statements in Problems 5 and 6 have a bearing on the conditions

$$a \wedge x = 1, a \vee x = n (\text{in } S_n)? \qquad (\ast\ast)$$

Does $(\ast\ast)$ have a solution x for every $a \in S_{12}$? Show that $(\ast\ast)$ does have a solution x for every $a \in S_6$. Show that the same is true for S_{30}. Can one generalize this? In view of this discussion, how will you define a' in S_6, in S_{30}?

Show (the second step toward abstraction) that all the statements involving $\wedge$, $\vee$, $<$, and $'$ in S_6 (also in S_{30}) which correspond to true statements in Problems 5 and 6 are also true. For what integers n can this be said of S_n? (The answer is given at the end of the article.) Note that in order to prove our assertion *it suffices to compare corresponding reduced systems.*

If we do not inquire into the nature of the elements (sets of objects in Problems 5 and 6, positive divisors of an integer n in the example above) but are concerned only with the properties of binary or unary operations and of some binary relations such as $\subset$ or $<$, and if these have the same properties as the operations and relations in our two examples, viz., the algebra of sets and the 'algebra of divisors', which we introduced, then we say that we study Boolean† algebras and that we deal with an abstract theory of which our examples serve as *realizations.*

$\wedge$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

$\vee$	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

a	a'
1	6
2	3
3	2
6	1

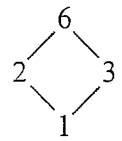


Figure 2

It is said that Boolean algebras are obtained by the process of generalization. The question arises as to how general is 'general'. It is interesting to note that for each Boolean algebra of a finite number of elements one can obtain a realization as an algebra of sets which gives a *true (faithful) representation* of our Boolean algebra. We shall try to convey this subtle idea by means of an example.

To bring out clearly the meaning of what we have in mind, we make use of the familiar representation of operations through the use of tables, in the same way in which multiplication in arithmetic is represented by the multiplication tables.

Thus, the tables for $\wedge$, $\vee$, the table for $'$ and the charts for $<$ in the case of S_6 are given in Figure 2.

† After the English logician George Boole (1815–1864).

On the other hand, the tables for $\cap$, $\cup$, the table for complementation, and the chart for $\subset$ for $I = \{1, 2\}$ are given in Figure 3.

$\cap$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{1\}$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{2\}$	$\varnothing$	$\varnothing$	$\{2\}$	$\{2\}$
$\{1, 2\}$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$

$\cup$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\varnothing$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1, 2\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$	$\{1, 2\}$
$\{2\}$	$\{2\}$	$\{1, 2\}$	$\{2\}$	$\{1, 2\}$
$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$

A	A'
$\varnothing$	$\{1, 2\}$
$\{1\}$	$\{2\}$
$\{2\}$	$\{1\}$
$\{1, 2\}$	$\varnothing$

Figure 3

We notice that the two systems are 'identical except for notation'. More precisely we can establish a 'one-to-one' correspondence $[1 \leftrightarrow \varnothing, 2 \leftrightarrow \{1\}, 3 \leftrightarrow \{2\}, 6 \leftrightarrow I]$ so that if $A \leftrightarrow a$, $B \leftrightarrow b$, then $A \cap B \leftrightarrow a \wedge b$, $A \cup B \leftrightarrow a \vee b$, $A' \leftrightarrow a'$, $A \subset B \Leftrightarrow a < b$. Hence the corresponding statements in the two systems either both hold true or both are false.

Problem 12. Show that the Boolean algebra of the positive divisors of 30 (the case S_{30}) is faithfully represented by the algebra of subsets of $I = \{1, 2, 3\}$.

We say that two systems which are faithful replicas of each other represent one and the same *abstract* mathematical system.

Note that the algebra of sets with $I = \{1, 2, 3\}$ is *essentially different* from the algebra of positive divisors of 6, *in the sense of not being faithful replicas of each other.*

Other realizations of Boolean algebras are used in logic and, through that, in the design of the high-speed digital computers (reference 4). It is primarily this last application that shifted the position of Boolean algebras away from 'pure mathematics' to make it one of the more popular subjects in 'applied mathematics'.

(The answer to the question on p. 94 is that n must contain no repeated factor.)

References

1. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (Macmillan, London, 1963).
2. R. Courant and H. E. Robbins, *What is Mathematics?* (Oxford University Press, London, 1961).
3. P. Rosenbloom, *The Elements of Mathematical Logic* (Dover, New York, 1950).
4. I. Adler, *Thinking Machines* (Dobson, London, 1961).

Letter to the Editor

Dear Editor,

Heronian triangles

Mr John Strange has made an interesting contribution (Volume 10, No. 1, pp. 15–24) to the study of Heronian triangles. Part of his article concerned the generation of Heronian triangles by means of formulae for the sides such as the one that I gave in a previous letter (Volume 9, No. 2, pp. 58–59), namely

$$(x + y)|xy - z^2|, \qquad x(y^2 + z^2), \qquad y(z^2 + x^2), \tag{1}$$

where x, y, z are positive integers. In this connection the following result is of interest.

(*) Let k, a, b, c, be positive integers. Then the triangle with sides a, b, c, is Heronian if and only if the triangle with sides ka, kb, kc is Heronian.

It is clear that, if the triangle a, b, c, is Heronian, then so is the triangle ka, kb, kc. The opposite implication is, however, not so obvious, and the proof is of some interest.

Put

$$\phi(a,b,c) = (a + b + c)(-a + b + c)(a - b + c)(a + b - c),$$

so that the criterion for the triangle with integral sides a, b, c to be Heronian is that $\phi(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ is a perfect square.

Now suppose that the triangle ka, kb, kc is Heronian. Then

$$\tfrac{1}{16}k^4\phi(a,b,c) = \phi(\tfrac{1}{2}ka, \tfrac{1}{2}kb, \tfrac{1}{2}kc)$$

is a perfect square, say m^2, and so

$$k^4\phi(a,b,c) = 16m^2.$$

Since the right side is a perfect square, so is the left side and therefore $\phi(a,b,c)$ is a perfect square. We wish to show that $\phi(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ is also a perfect square. When a, b, c are all even, then $\phi(a,b,c)$ is divisible by 16 and so $\phi(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c) = \frac{1}{16}\phi(a,b,c)$ is a perfect square. We shall show that, in fact, a, b, c *must* all be even.

First, let

$$a + b + c = t, \qquad -a + b + c = u, \qquad a - b + c = v, \qquad a + b - c = w. \tag{2}$$

Then $t - u = 2a$, $t - v = 2b$, $t - w = 2c$ and so t, u, v, w have the same parity, i.e. they are either all even or all odd.

Next, we note that the set of all odd integers can be divided into two disjoint sets P, Q, where

P is the set of integers of the form $4n + 1$,

Q is the set of integers of the form $4n + 3$.

Moreover, if $p, p' \in P$ and $q, q' \in Q$, then

$$pp' \in P, \qquad qq' \in P, \qquad pq \in Q.$$

Listing all the possibilities for the integers u, v, w given by (2), with a, b, c all odd, we can draw up a membership table as follows:

u	v	w	$t = u + v + w$	$uvwt$
P	P	P	Q	Q
P	P	Q	P	Q
P	Q	Q	Q	Q
Q	Q	Q	P	Q

On the other hand, if $\phi(a, b, c) = uvwt$ is a perfect square, then it must belong to P. The contradiction shows that, when $\phi(a, b, c)$ is a perfect square, then a, b, c are all even, so that $\phi(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ is also a perfect square. This proves the statement (*).

In his article Mr Strange made the point that the Heronian triangle with sides 10, 39, 35 is not produced by my formulae (1). However, the similar triangle 50, 195, 175 is given by $x = 3$, $y = 7$, $z = 4$, and so it follows immediately from (*) that the 10, 39, 35 triangle is Heronian.

Finally, I might mention that, contrary to a remark made by Mr Strange, my method for obtaining the expressions (1) is not based on placing two right-angled triangles together.

Yours sincerely,

A. R. PARGETER
(Blundell's School, Tiverton, Devon)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

For background and terminology to problems 10.7 and 10.8, see Sections 2 and 5 of the article 'A Medley of Squares', on pp. 72–81.

10.7 Let S be any finite system of similarly oriented squares of equal size in the plane, and denote by $A(S)$ the total area covered by S. Show that it is always possible to find a discrete subsystem T of S such that $A(T) \geqslant \frac{1}{6}A(S)$.

10.8 Accepting as known the existence of a continuous square-filling curve, demonstrate the existence of a continuous curve which passes through every point of the entire plane.

10.9 (Submitted by J. R. Alexander, University of Southampton.) The following algorithm describes a geometrical procedure:

(0) take any triangle ABC;
(1) circumscribe a circle around ABC;
(2) draw tangents l, m, n at A, B, C;
(3) let $A = m \cap n$, $B = n \cap l$, $C = l \cap m$;
(4) go to (1).

Describe the angles of $\triangle ABC$ after reaching (3) for the nth time, and determine under what circumstances the angle at A takes its initial value again.

Now begin with a cyclic quadrilateral $ABCD$ instead of a triangle, and carry out the analogous construction. Show that, if it is possible to pass beyond (1) for the second time, then

$$AB^2 + CD^2 = d^2,$$

where d is the diameter of the circle circumscribing $ABCD$.

Solutions of Problems in Volume 10, Number 1

10.1. Show that the sum of the lengths of the diagonals of a plane quadrilateral exceeds the sums of the lengths of two opposite sides.

Solution by John Ramsden (University of Aston)

In Figure A

$$AC + BD = AE + EC + BE + ED$$

and

$$AE + ED > AD,$$
$$BE + EC > BC.$$

Hence

$$AC + BD > AD + BC.$$

Also solved by Hema Murty (Carlton University, Ottawa); John Savva (Winchester College).

10.2. A pyramid on a triangular base has the length of each sloping side one unit and the length of each base side $\sqrt{2}$ units. The point P is a point of the base, distant d_1, d_2, d_3 units from the base vertices. Determine the distance of P from the apex of the pyramid.

Solution

Choose rectangular axes $Oxyz$ as shown, and denote the distance of P from the apex O by d. Let the coordinates of P be (α, β, γ). Then

$$d_1^2 = (\alpha - 1)^2 + \beta^2 + \gamma^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 1,$$
$$d_2^2 = \alpha^2 + (\beta - 1)^2 + \gamma^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\beta + 1,$$
$$d_3^2 = \alpha^2 + \beta^2 + (\gamma - 1)^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\gamma + 1.$$

Since $d^2 = \alpha^2 + \beta^2 + \gamma^2$, we have

$$d_1^2 + d_2^2 + d_3^2 = 3d^2 - 2(\alpha + \beta + \gamma) + 3.$$

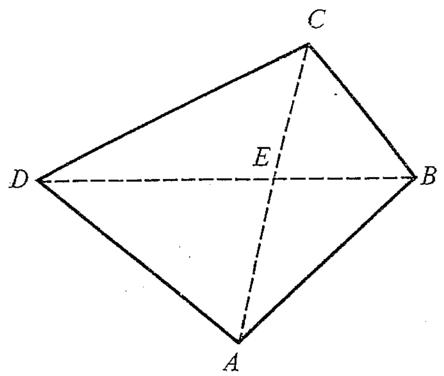
The equation of the plane of the base is $x + y + z = 1$, so $\alpha + \beta + \gamma = 1$. Hence

$$d_1^2 + d_2^2 + d_3^2 = 3d^2 + 1$$

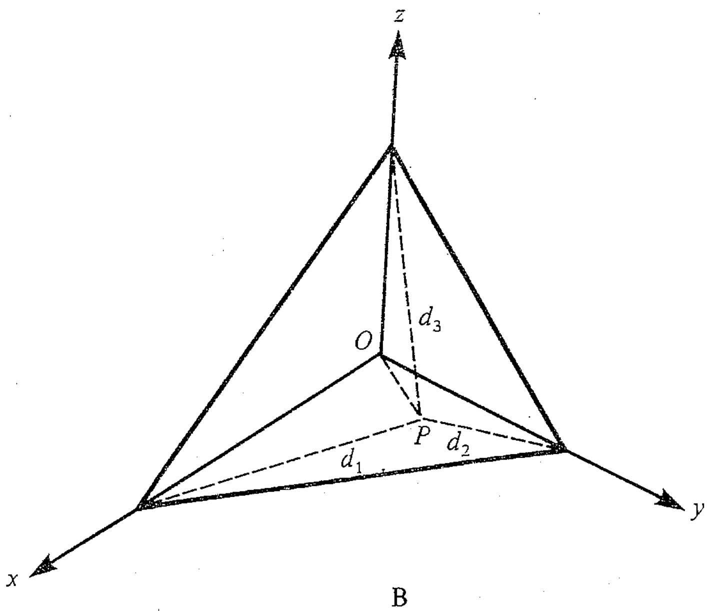
and

$$d = \sqrt{\tfrac{1}{3}(d_1^2 + d_2^2 + d_3^2 - 1)}.$$

Also solved by John Ramsden.



A



B

10.3. The positive† real numbers p, q, r are such that $q \neq r$ and $2p = q + r$. Show that

$$\frac{p^{q+r}}{q^q r^r} < 1.$$

† The word 'positive' was omitted in error from the original question.

Solution

Consider the q numbers $1/q, \ldots, 1/q$ and the r numbers $1/r, \ldots, 1/r$. Since $q \neq r$, their geometric mean is smaller than their arithmetic mean, so that

$$\left[\left(\frac{1}{q}\right)^q\left(\frac{1}{r}\right)^r\right]^{1/(q+r)} < \frac{2}{q+r}, \quad \frac{1}{q^q r^r} < \left(\frac{2}{q+r}\right)^{q+r}, \quad \frac{p^{q+r}}{q^q r^r} < 1.$$

Book Reviews

Vector and Tensor Methods. By FRANK CHORLTON. John Wiley & Sons Ltd., Chichester, 1976. Pp. 332. £8.00 cloth, £3.50 paper.

Since vector algebra and analysis were first introduced into undergraduate courses, by Cunningham in Cambridge and Milne and Chapman in London, there has been a steady increase in the volume of vectorial work presented to science and engineering students at university. More recently, revision of A-level syllabuses has meant that such work is now expected to be covered in some detail before arrival at university.

Chorlton's book covers most of the work needed in a university course. In 332 pages it should do so! Much of the material is standard and of course must look very similar to that in many earlier standard works. It is clear, well written and well printed. Little is left to the imagination and initiative of the reader, a point which should prove of value to the self-taught.

However, some of the sections seem unusual; for example the section on the solution of vector equations is based on the motion of linear dependence rather than on physical concepts. Further, to treat mechanics, electromagnetism and hydrodynamics in a work of this size always leads to unevenness; Maxwell's equations get five pages only and the Galilean transformation is completely missed.

The short introduction to tensors is satisfactory as far as it goes, but again space inhibits a promising start.

As an introduction to vector methods in mathematically based sciences this work is comparable to many others published ten or twenty years ago. It would have been more appropriate then than now when emphasis in undergraduate studies is moving into very different fields.

Queen Mary College, London — C. PLUMPTON

Adventures with Your Hand Calculator. By LENNART RADE and BURT A. KAUFMAN. Cemrel, Inc., St. Louis, 1977. Pp. iv + 131.

This is a superb book which is aimed at a readership of a slightly lower age than the average reader of *Mathematical Spectrum*, but it would have interest for those readers who own a programmable calculator.

The book consists of twenty 'adventures' which are activities designed to interest and stimulate at the eleven to sixteen age range. It does this very well, having just the right amount of open-ended work to be stimulating rather than daunting, and would be excellent to give to the pupil who finishes first and who has an odd ten minutes or quarter of an hour to fill at the end of a lesson. Different adventures will appeal to different readers; I enjoyed those based on a simple random number generator and my son and I now play backgammon using a programmable calculator instead of dice.

The interest for older readers will probably come in the challenge of fitting some of the exercises onto a programmable calculator. For instance, which integers between 0 and 100 have a perfect square all of whose digits are odd? And having got the answer, prove it!

University of Durham — H. NEILL

College Algebra. By J. S. RATTI. Collier Macmillan, London, 1977. Pp. vi + 282.

This book covers the work done in algebra in the first and second year sixth form in the UK, but not in sufficient detail nor with sufficiently challenging exercises to make it a useful text book.

Calculus with Analytic Geometry (3rd Edition). By MURRAY H. PROTTEN and CHARLES B. MORREY, Jr. Addison-Wesley, London, 1977. Pp. ix + 920. £12.80.

College Calculus with Analytic Geometry (3rd Edition). By MURRAY H. PROTTEN and CHARLES B. MORREY Jr. Addison-Wesley, London, 1977. Pp. ix + 559. £12.80.

The first of these two books which now appear in their third edition is a subset of the second, which includes, at its far end, chapters on Infinite Series, Partial Derivatives and Multiple Integration. They are attractive, well-produced books in the best American tradition covering calculus and its applications from the earliest stages. This new edition has included extra, more challenging exercises.

Essentials of Trigonometry (2nd Edition). By IRVING DROOYAL, WALTER HADEL and CHARLES C. CARICO. Collier Macmillan, London, 1977. Pp. viii + 328. £9.40.

In this second edition which covers Trigonometry through the usual addition identities to complex numbers, the authors have rewritten fairly substantial pieces of text and reviewed and revised exercises.

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