# MATHEMATICAL SPECTRUM

*A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES*

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# On Formulas for $\pi$ Involving Inverse Tangent Functions

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Here are some observations on formulas for $\pi$ involving inverse tangent functions which might interest readers. These observations throw an interesting light on Machin's formula (see below) and other related results. Formulas such as these are useful in obtaining effective series for calculating $\pi$.

*A definition.* A 'simple' formula for $\frac{1}{4}\pi$ is a linear combination of inverse tangents of unit fractions, i.e. a formula of the form

$$\tfrac{1}{4}\pi = a_1 \tan^{-1}\frac{1}{m_1} + a_2 \tan^{-1}\frac{1}{m_2} + \ldots,$$

where the $a_i$ and $m_i$ are integers.

I now state four theorems which are useful for generating 'simple' formulas. These may easily be proved using the identities

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}, \qquad \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}.$$

*Theorem* 1

$$\tan^{-1}\frac{p}{p+1} + \tan^{-1}\frac{1}{2p+1} = \frac{\pi}{4} \quad (p > 0).$$

*Theorem* 2

$$\tan^{-1}\frac{p+1}{p} - \tan^{-1}\frac{1}{2p+1} = \frac{\pi}{4} \quad (p > 0).$$

*Theorem* 3. The equation $2\tan^{-1}x = \tan^{-1}(p/q)$ has a rational solution in $x$ provided $p$ and $q$ are the legs of a Pythagorean triangle; $x$ is given by

$$\frac{-q+\sqrt{p^2+q^2}}{p}$$

(ignoring the negative solution).

*Theorem* 4

$$\tan^{-1}\frac{1}{a}-\tan^{-1}\frac{1}{b} = \tan^{-1}\frac{b-a}{ab+1},$$

where $a$ and $b$ are integers.

If $b-a = 1$ then $(b-a)/(ab+1)$ always reduces to a unit fraction. If $b-a = 2$ then $(b-a)/(ab+1)$ reduces to a unit fraction if $a$ is odd, e.g.

$$\tan^{-1}\tfrac{1}{3}-\tan^{-1}\tfrac{1}{5} = \tan^{-1}\tfrac{2}{16} = \tan^{-1}\tfrac{1}{8}.$$

Of course, there are values of $a$ and $b$ not satisfying these conditions which give unit fractions, e.g.

$$\tan^{-1}\tfrac{1}{18}-\tan^{-1}\tfrac{1}{43} = \tan^{-1}\tfrac{25}{775} = \tan^{-1}\tfrac{1}{31}.$$

**Results**

1. Putting $p = 1$ in theorem 1 yields the well-known formula

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3} = \tfrac{1}{4}\pi.$$

2. Putting $p = 3$ in theorem 1 yields the formula

$$\tan^{-1}\tfrac{3}{4}+\tan^{-1}\tfrac{1}{7} = \tfrac{1}{4}\pi.$$

Since 3 and 4 are the legs of a Pythagorean triangle we can use theorem 3 and obtain

$$\tan^{-1}\tfrac{3}{4} = 2\tan^{-1}\tfrac{1}{3}$$

and so

$$2\tan\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7} = \tfrac{1}{4}\pi.$$

3. These theorems are particularly useful when we have Pythagorean triangles with consecutive legs. Referring to pp. 328 and 329 of Beiler's book, we observe that $(119, 120, 169)$ is a Pythagorean triangle. So, by theorem 2, we obtain

$$\tan^{-1}\tfrac{120}{119}-\tan^{-1}\tfrac{1}{239} = \tfrac{1}{4}\pi.$$

By two applications of theorem 3, we obtain

$$\tan^{-1}\tfrac{120}{119} = 2\tan^{-1}\tfrac{5}{12} = 4\tan^{-1}\tfrac{1}{5}.$$

Putting these two results together we obtain Machin's formula:

$$4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239} = \tfrac{1}{4}\pi.$$

4. By theorem 4,

$$\tan^{-1}\tfrac{1}{3}-\tan^{-1}\tfrac{1}{5} = \tan^{-1}\tfrac{1}{8}.$$

Substituting for $\tan^{-1}\frac{1}{3}$ in result 2 gives

$$2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{7}+2\tan^{-1}\tfrac{1}{8} = \tfrac{1}{4}\pi.$$

Substituting for $\tan^{-1}\frac{1}{5}$ in Machin's formula produces

$$2\tan^{-1}\tfrac{1}{7}+4\tan^{-1}\tfrac{1}{8}+\tan^{-1}\tfrac{1}{239} = \tfrac{1}{4}\pi.$$

By theorem 4,

$$\tan^{-1}\tfrac{1}{7}-\tan^{-1}\tfrac{1}{8} = \tan^{-1}\tfrac{1}{57}$$

and so substituting for $\tan^{-1}\frac{1}{7}$ in the above formula yields the result

$$\tfrac{1}{4}\pi = 6\tan^{-1}\tfrac{1}{8}+2\tan^{-1}\tfrac{1}{57}+\tan^{-1}\tfrac{1}{239},$$

which is another well-known formula for $\frac{1}{4}\pi$.

If we define the 'efficiency' of a formula to be the denominator of the largest unit fraction that appears in the formula, then the efficiency of Machin's formula is 5. The significance of efficiency lies in the fact that the reason for expressing $\pi$ in terms of $\tan^{-1}$ is to obtain an approximation for $\pi$ using

$$\tan^{-1}x = x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\dots.$$

The convergence is more rapid in this series if $x$ is small. Hence the efficiency is greater the greater the denominator of the largest fraction in the formula. I would conjecture on the basis of this analysis that Machin's formula has the largest efficiency of any two-term formula. Can this be proved?

**Problems**

Readers may care to attempt the following:

1. Show that

$$\tfrac{1}{4}\pi = 12\tan^{-1}\tfrac{1}{18}+8\tan^{-1}\tfrac{1}{57}-5\tan^{-1}\tfrac{1}{239}.$$

2. Show that

$$\tfrac{1}{4}\pi = 20\tan^{-1}\tfrac{1}{57}+24\tan^{-1}\tfrac{1}{68}+12\tan^{-1}\tfrac{1}{117}-5\tan^{-1}\tfrac{1}{239}.$$

3. Show that

$$1989 = \cot\tfrac{1}{4}\pi-3\tan^{-1}\tfrac{1}{4}-\tan^{-1}\tfrac{1}{20}-\tan^{-1}\tfrac{1}{1972098}-\tan^{-1}\tfrac{1}{1976072}.$$

There is also an infinite series expansion of $\pi$ involving inverse cotangents in the problems section of this issue. I cannot remember having seen this before.

**Reference**

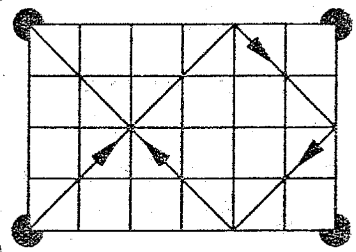
Albert H. Beiler, *Recreations in the Theory of Numbers* (Dover Books, New York, 1966).

# The Strange Billiard Table

**DAVID SHARPE,** *University of Sheffield*

The author lectures in pure mathematics, and is editor of *Mathematical Spectrum*.

The following problem was posed to GCSE students at Ryde High School, on the Isle of Wight:




The billiard table shown in the figure is a little odd. It has only four pockets and the base is divided into squares. Only one billiard ball is used, and it is always struck from the corner at 45° to the side. The ball also rebounds at 45° to the side. Investigate what happens for different numbers of squares along the sides.

Animated discussion of the problem followed over dinner, with diagrams and calculations everywhere, until the following simple solution emerged.

Suppose that the table has $m$ squares measured along $AB$ and $n$ measured along $AD$. To reach a pocket, the ball must traverse a multiple of $m$ squares along $AB$, say $rm$, and a multiple of $n$ squares along $AD$, say $sn$, and, since it is always moving at 45° to each side,

$$rm = sn.$$

The ball will go into a pocket the first time we reach $r$ and $s$ to satisfy this equation. If $m$ and $n$ have highest common factor $d$, then the smallest values of $r$ and $s$ are given by $r = n/d$ and $s = m/d$. This solution also tells us which pocket the ball enters, how many times the ball crosses the table in each direction before doing so and how many times it rebounds from the sides. If $m/d$ and $n/d$ are both odd, it traverses the board an odd number of times in the direction of side $AB$ and an odd number of times in the direction of side $AD$, and so goes into pocket $C$; if $m/d$ is even and $n/d$ is odd it goes into pocket $B$, and if $m/d$ is odd and $n/d$ is even it goes into pocket $D$. Since $m/d$ and $n/d$ have highest common factor 1, they cannot both be even, so the ball never enters pocket $A$. The number of rebounds is

$$\left(\frac{m}{d}-1\right)+\left(\frac{n}{d}-1\right) = \frac{m+n}{d}-2.$$

For example, when $m = 6$ and $n = 4$, then $d = 2$, $r = 2$ and $s = 3$, and the ball enters pocket $D$ after traversing the table twice in the direction of side $AB$ and three times in the direction of side $AD$, as shown in the diagram; it makes $\frac{1}{2}(6+4)-2 = 3$ rebounds.

# A Cinderella Property of Binomial Coefficients

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Every so often one stumbles across an obvious triviality of the kind that leaves one wondering 'Why haven't I seen this in print somewhere?'

As an example, it is well known and of widespread relevance, that if $p$ is a prime number, then the integer $\binom{p}{r}$ is divisible by $p$ for all positive integers $r$ less than $p$, but the following simple generalisation appears never to get a mention.

*Proposition.* If $n$ is any positive integer, then $\binom{n}{r}$ is divisible by $n$ for all positive integers $r$ less than $n$ and coprime with $n$ (i.e. $r$ and $n$ have no common factor apart from 1).

*Proof.*

$$n\binom{n-1}{r-1} = \frac{n(n-1)\ldots(n-r+1)}{(r-1)!} = r\binom{n}{r}.$$

Therefore $n$ divides $r\binom{n}{r}$. Hence, if $n$ and $r$ are coprime, then $n$ must divide $\binom{n}{r}$.

It would be interesting to hear of other 'Cinderella results' that readers consider to be in some way under-recognised.

# II. Infinity and Limits

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This article follows on from a previous one on infinity and enumeration, in Volume 23 Number 2.

Many people have heard of the paradox of Zeno about Achilles and the tortoise. Achilles was the Greek hero with the tender foot, and was apparently a very fast runner. The tortoise is notorious for its slowness, but not especially for its sloth. After all, Aesop's story of the tortoise and the hare contrasts the sloth of the hare to the steadiness of the tortoise, and in it the race is won by the slower animal. Similarly, Zeno arranged a paradox where the tortoise beat Achilles in a race, even though Achilles was trying very hard. The story is that Achilles challenged the tortoise to a 200 yard race. The tortoise is given 100 yards head start. Achilles runs 10 times faster than the tortoise. But, says Zeno, here is what happens to poor Achilles. Achilles runs 100 yards and reaches the point from which the tortoise started. But in the meantime, the tortoise has gone one-tenth that distance and is still 10 yards ahead of Achilles. Achilles in the next moment makes up that 10 yards while the tortoise advances another yard and is still ahead of Achilles. Achilles then runs in the next moment another yard, but the tortoise has moved ahead and is one-tenth of a yard ahead of Achilles. Achilles then runs that tenth of a yard, but the tortoise is still ahead by a little bit. And so no matter how long the race continues, Achilles trying as he might, the tortoise is always just a little bit ahead of Achilles. And so the tortoise will win the race!

Of course, we see that the paradox is that, as the distances decrease in Zeno's blow-by-blow account of the race, the moments that he describes are also decreasing. Actually, Achilles will reach the tortoise after he has run a distance in yards equal to $100+10+1+0.1+0.01+\ldots = 111.\bar{1} = 111+\frac{1}{9}$, while the tortoise will have run only $11.\bar{1} = 11+\frac{1}{9}$ yards. So, if Achilles is running 600 yards a minute, then it will take him very little time to catch and pass the tortoise. Indeed, if he is running 600 yards per minute, then the first moment referred to above where he runs 100 yards takes 10 seconds, the second moment takes 1 second, the next moment takes $\frac{1}{10}$ of a second, and so on. So, adding this infinite sequence of time, we get that he passes the tortoise just at the time in seconds given by $10+1+0.1+0.01+\ldots = 11.\bar{1} = 11+\frac{1}{9}$. This makes sense since to travel $111+\frac{1}{9}$ yards at 600 yards per minute (i.e. at 10 yards per second) takes $11+\frac{1}{10}+\frac{1}{90} = 11+\frac{1}{9}$ seconds. The Greeks (at least some of them) were bamboozled by Zeno's paradox because they did not have a good grasp of what it means to add together an infinite sequence of progressively smaller and smaller quantities. This was not true for all the Greeks. Archimedes and others did some good calculations of $\pi$ and areas under curves by methods of successive approximations that showed they knew how to do this type of infinitary addition. But these methods were not well understood or uniformly applied in all sciences and philosophy.

It is not nearly as troublesome now for us to understand and successfully manipulate infinite series and sequences. Sometimes, we can end up with some paradoxical results, but, if the analysis is done correctly, then we just have to learn to live with the conclusions. The simplest example of this is that $0.\bar{9} = 1$. That is, $1 = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\ldots$. So although we never reach 1 after a finite number of terms, our sum is eventually greater than any number $x$ which is less than 1. The geometric-series formula that generalizes this is the formula which says that, for a number $a$ $(0 < a < 1)$ the sum $1+a+a^2+a^3+\ldots = 1/(1-a)$. You can use this formula to check the computations above in Zeno's paradox. However, not all infinite sums of progressively smaller quantities add to a finite value. One of the most famous examples of such a series is the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$. Can you see that this series has the property that, for any fixed number $K$, a finite number of the terms of this series will sum to more than $K$?

Such calculations are related to other inductive constructions or algorithms where real numbers are obtained by infinitary processes. For example, starting with 1, add 1 and take the square root, add 1 to the result and take the square root, add 1 to this result and take the square root, etc. What do you get in the limit? That is, let $a_1 = 1$, let $a_2 = \sqrt{a_1+1}$, etc. Generally $a_{n+1} = \sqrt{a_n+1}$. If you can see why these numbers do get closer to a value $x$ as the process is continued indefinitely, then you would be justified in saying that $x = \sqrt{x+1}$. Then, by the quadratic formula, you can calculate that $x = \frac{1}{2}(1+\sqrt{5})$, an irrational number. But why do the values $a_n$ get closer and closer to some fixed quantity? A mathematician would ask this as follows: why does the sequence $(a_n)$ converge? Actually, this sequence is increasing, and it is bounded above by the number 10, for instance. So a general principle of real numbers asserts that the sequence does converge. Because of the way $x$ was obtained, we see that

$$\tfrac{1}{2}(1+\sqrt{5}) = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}}.$$

Another pleasant example of a transfinite process of real-number construction is to start with 1 and add 1, then take the reciprocal of this sum. Then add 1 and take the reciprocal of this sum, etc. That is, let $a_1 = 1$ and let $a_2 = 1/(1+1)$, and generally $a_{n+1} = 1/(1+a_n)$. Again, if we show that $(a_n)$ converges (and we can, by the way), then it is not hard to see that this process converges to a number $x$ which satisfies $x = 1/(1+x)$. So, by the quadratic formula, $x = \frac{1}{2}(\sqrt{5}-1)$. Because of the procedure we have used, we have shown that $\frac{1}{2}(\sqrt{5}-1)$ can be expressed as a *continued fraction*. That is,

$$\tfrac{1}{2}(\sqrt{5}-1) = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}.$$

You can see why these types of fractions have obtained the name of continued fractions!

A final example of carrying a procedure to infinity is a very old algorithm for square roots. It has been renamed *Newton's algorithm* because it comes from a general technique discovered by Sir Isaac Newton; but this specific instance of his technique is a much older algorithm. Suppose you want to calculate $\sqrt{2}$ by successive approximations. We start with a 'bad' guess for the answer that $a_1 = 1$. Then we choose $a_2$ equal to the average $\frac{1}{2}(a_1+2/a_1)$. You can see that if, we do this endlessly, that is, we take $a_{n+1} = \frac{1}{2}(a_n+2/a_n)$, then we generate a sequence of numbers $(a_n)$. Try it on a calculator to see what the numbers are! If you can show that the sequence $(a_n)$ has a limit $L$, then clearly $L$ would satisfy the equation $L = \frac{1}{2}(L+2/L)$. By doing a little algebra and using the quadratic formula, you can see that $L = \sqrt{2}$. This sequence of approximations does indeed converge, and it gives a way of calculating $\sqrt{2}$. This method can also be generalized to computing $\sqrt{a}$ for any positive number $a$. To do this, just let $a_1 = 1$ and generally let $a_{n+1} = \frac{1}{2}(a_n+a/a_n)$. Then the sequence $(a_n)$ converges to $\sqrt{a}$. Similar algorithms exist for computing any root of a positive number.

There is a common form to all of the preceding computational algorithms. Each is a version of the algorithm for locating a fixed point of a function $y = f(x)$ by iteration. One first chooses some $a_1$, preferably some number close to one of the points $a$ where $f(a) = a$. Then define $a_2 = f(a_1)$, $a_3 = f(a_2)$, etc. That is, $a_{n+1} = f(a_n)$ for all $n \geqslant 0$. Sometimes this algorithm fails to provide a converging sequence. But when it does (and one can figure out simple criteria for this based on some calculus), it provides a limiting value $a$ which is the desired fixed point, i.e. a point where $f(a) = a$. For example, you can use this to generate a sequence converging to the solution 4 of the equation $\frac{1}{2}x+2 = x$ by letting $f(x) = \frac{1}{2}x+2$. But try doing this with the equation $2x-2 = x$ and you will see why this method does not always work well.

In all of these constructions of numbers or functions by limits, an especially important property of real numbers is used. This property is an axiom that we inherently assume. It is a property that fails for the rational numbers, but holds for the real numbers as we like to think of them intuitively. This property (sometimes called the Least Upper Bound Axiom) says that, if we have a sequence of real numbers $a_1 \leqslant a_2 \leqslant a_3 \leqslant \ldots$ and all the numbers $a_n \leqslant b$ for some number $b$, then **there exists** a least real number $x$ such that $a_n \leqslant x$ for all $n$ (in the sense that if $b$ is any number with $a_n \leqslant b$ for all $n$, then $x \leqslant b$). You can see why this property or axiom is called the Least Upper Bound Axiom since it says that, if there is some upper bound $b$, then there an upper bound $x$ which is the least one. This fundamental principle is what 'fills in the holes' between the rational numbers. It is the axiom that tells us that numbers like $\sqrt{2}$ and e really exist. It is the axiom that says that infinitely continuing decimals actually do represent real numbers. There are many axioms or properties of real numbers that are important for telling us how addition and multiplication work, and how they interrelate with each other and with inequalities. All these axioms also hold for the rational numbers. If that were all we had, then we would have the real number system at our disposal. But the Least Upper Bound Axiom distinguishes the rational numbers from the real numbers and really makes things happen. Without this axiom, the beautiful argument of Cantor for the uncountability of the real numbers would not be meaningful.

It should be said in conclusion that, while modern mathematics was growing, specifically while calculus and differential equations were being developed in the eighteenth and nineteenth centuries, most mathematicians did not concern themselves with some of the stranger creations that would arise from their basic assumptions about the real numbers. In particular, the completeness of the real numbers as embodied in the Least Upper Bound Axiom was understood only intuitively. For example, it did not seem to them that physical reality offered any examples of functions and curves different from the piecewise smooth ones that they considered. This attitude was finally laid to rest by an example that Karl Weierstrass constructed in the late nineteenth century. He showed, by using limits, that there could be continuous functions on $[0, 1]$ which did not have tangent lines anywhere. These graphs are very crooked indeed! They were thought to be purely a mathematical curiosity until recently. Now it seems that such curves, and surfaces like them, are being considered as better models of many physical phenomena than their smoother ancestors are. For example, if you consider a coastline from a great distance, it seems to be a somewhat wiggly curve. But the closer you look, the more bends and crooks it has. In some sense, it is continuous, but too crooked to be really turning smoothly. See Mandelbrot's book on fractals (reference 1) for more about these curves.

**References**
1. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (W. H. Freeman, San Francisco, 1977).
2. J. Rosenblatt, Infinity and enumeration, *Mathematical Spectrum* **23** (1990/91), 44–54.

# Examining the Surface of a Hypercube

G. N. THWAITES, *Oakham School*

Geoff Thwaites obtained his M.A. and D.Phil. from Oxford University. He has taught at Malta University, York University and Rugby School, and is now teaching at Oakham School.

A recent GCSE project required some members of my fourth-form set to count dots spaced at 'integer points' on the edges and faces of a cube. What happens, one of my set wished to know, if you look at the corresponding problem with a four-dimensional hypercube?

They needed to find how many edges and faces such a hypercube possessed, so I drew for them a standard two-dimensional representation of a three-dimensional projection of such an object and they were surprised to discover that the three-dimensional projection possessed cubes 'built into' it. (See figure 1.)

Very well, they said, how about an $n$-dimensional hypercube? It is easy to work out the number of edges: there are $2^n$ vertices and each vertex has $n$ edges emerging from it, one parallel to each coordinate axis. Each edge emerges from two vertices and so the number of edges is $\frac{1}{2}n\times 2^n = n2^{n-1}$. But how do you find the number of squares, cubes and hypercubes on the surface? Indeed, how do you work out where the surface is?

I am always exhorting pupils to 'think geometrically' and to try to visualize the situation. However, on this occasion it turns out to be easier to think in more algebraic terms.

Define the $n$-dimensional hypercube with side-length 2 to be the set of points $(x_1, x_2, \ldots, x_n)$, where $-1 \leqslant x_i \leqslant 1$. A point must lie on the surface of the hypercube if $|x_i| = 1$ for some value of $i$. For if $|x_i| > 1$ then the point fails to satisfy the hypercube definition. Clearly we obtain an edge by fixing $n-1$ values of the $x_i$ at $+1$ or $-1$ and letting the remaining value vary between $+1$ and $-1$. This yields $n2^{n-1}$ edges, as we have seen.

We obtain an $r$-dimensional hypercube on the surface by fixing $n-r$ of the $x_i$ at $+1$ or $-1$ and letting the remaining $r$ $x_i$'s take values between $+1$ and $-1$. So the number of $r$-dimensional hypercubes on the surface is $\binom{n}{r}2^{n-r}$.

It is interesting to check these results for the four-dimensional hypercube via the projection illustrated in figure 1.

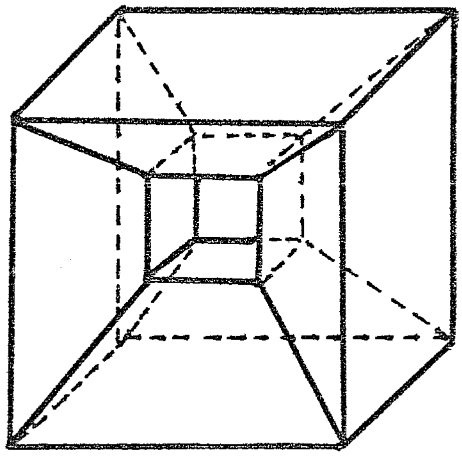


Figure 1

There are 16 vertices, $\binom{4}{1}2^3 = 32$ edges, $\binom{4}{2}2^2 = 24$ squares, $\binom{4}{3}2 = 8$ cubes, although some of the squares and cubes have changed shape a bit in the projection. It may seem odd that one of the cubes contains all the others but that this is reasonable can be seen from figure 2, which is a two-dimensional projection of an ordinary cube.

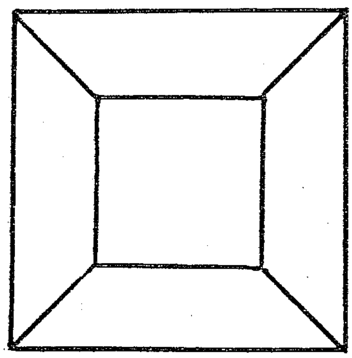


Figure 2

# Sir Ronald Aylmer Fisher, 1890–1962

J. C. GOWER, *University of Leiden*

Until retirement in 1990, John Gower headed the Biomathematics Division and Statistics Department at Rothamsted Experimental Station. Currently in the Department of Data Theory in the University of Leiden, he is Secretary of the Sir Ronald Fisher Memorial Committee of Great Britain.

**Introduction**

Ronald Aylmer Fisher was born on 17 February 1890. This year his centenary has been celebrated at special scientific meetings, by the publication of his statistical correspondence and by commemorative articles. His college, Gonville and Caius, Cambridge, has unveiled a stained glass window of a Latin Square–a reference to his work on experimental design—and the Royal Society featured his life and work in their June soirée.

The titles of his six books indicate his varied and fundamental interests: *Statistical Methods for Research Workers* (first edition 1925, fourteenth edition 1971); *The Genetical Theory of Natural Selection* (1930); *The Design of Experiments* (first edition 1935, fourth edition 1964); *Statistical Tables for Biological, Agricultural and Medical Research* (with F. Yates, first edition 1938, seventh edition 1966); *The Theory of Inbreeding* (1949); *Statistical Methods and Scientific Inference* (1956). In addition, Fisher published some 285 scientific papers, mostly in statistics (129) and genetics (140) which were reprinted in five volumes by the University of Adelaide in 1970. His collected correspondence on genetical matters was published in 1986 and his collected statistical correspondence in 1990. Fisher was showered with honours, including a knighthood in 1953, election to the Royal Society, the award of its Royal Medal and the Copley Medal (its highest award), as well as over a dozen more medals and honorary degrees from institutions throughout the world.

Fisher was primarily a scientist and only secondarily a mathematician. Appropriate tools needed to express his scientific ideas had to be developed. Here Fisher, like illustrious predecessors such as Newton and Fourier, developed the mathematics needed to underpin his scientific research; the new methods have since found applications far beyond their origins. His research developed largely in response to biological problems in agriculture and genetics. Working within these fields, Fisher laid the foundations of much of modern statistics, provided a mathematical framework for Darwin's evolutionary theory and made important contributions to the theory of scientific inference. Although he set out several fundamental principles, Fisher was not much concerned with mathematical rigour or generality; he was especially scathing about post-war developments in mathematical statistics, which he saw as being of little relevance to statisticians.

**Early years**

One of Fisher's daughters, Joan Fisher Box, has written an excellent biography of her father (reference 1); much of what follows is drawn from that source. R. A. Fisher was born into a prosperous family; they lived in Heath House, on Hampstead Hill. He was the last of five brothers, one of whom died young, and three sisters. Fisher showed his abilities at an early age: there are stories of his being read Sir Robert Ball's astronomy at the age of five and grasping the complex three-dimensional geometry of planetary movements. After one term at his preparatory school in Hampstead he was moved into a form of boys two or three years older. Nevertheless he came first in all subjects except French, in which he came last!

When Fisher moved to Harrow school, he continued to show his brilliance, taking the school prize in mathematics in 1906. Joan Fisher Box tells of a Harrow master who on being asked to name the 10 or 12 cleverest boys who had passed through his hands said, 'It would be difficult to do so but he could divide all those he taught into two groups: one contained a single outstanding boy R. A. Fisher, the other all the rest'. Throughout his life Fisher suffered from poor eyesight and to avoid eyestrain received tuition without the use of pencil and paper. As a result, he developed a remarkable ability to think geometrically, which later led to the solution of several difficult problems in the evaluation of certain probability distributions. The eyesight defect may also have been at the root of Fisher's disconcerting ability to produce results intuitively, without all the detailed logical steps of conventional mathematics.

In 1909 Fisher won a scholarship to Gonville and Caius College. He was undecided whether to read biology or mathematics but eventually chose the latter because 'I suppose, without being sure, that a mathematical technique with biological interests is rather firmer than a biological technique with mathematical interests'. Fisher passed out a Wrangler (first class honours in mathematics) with a distinction in the paper on optics. He spent much of his time as an undergraduate in developing his interest in genetics. At about that time Mendel's work on heritability had been rediscovered and the existence of genes and chromosomes hypothesised. Many considered that this discrete mechanism of inheritance was inconsistent with Darwin's theory of natural selection. A further confusing factor was the discovery of mutation which was thought to be the consequence of environmental factors. By 1911 Fisher had seen how these apparently conflicting issues could be welded into a unified theory, and had begun to develop the underpinning mathematics that 20 years later was published in his *Genetical Theory of Natural Selection*. In 1911 Fisher, still an undergraduate, expressed his views at a meeting of the Cambridge University Eugenics Society of which he was chairman, founder-member and principal instigator. Eugenics was a new science and did not then carry the racial overtones it later acquired as a result of activities in Nazi Germany. It was concerned with all heritable aspects of the improvement of the human condition; because environmental effects were then thought to be heritable, genetic, social and environmental aspects were all included. Fisher was particularly concerned that the legislation in the UK at the time disfavoured large families in the more successful part of the nation which, eventually, could prove disastrous. He waged a long campaign in favour of family allowances. Whatever one's views, it is undeniable that the Eugenic Movement (initiated towards the end of the nineteenth century by Francis Galton, a half-cousin of Charles Darwin) had a tremendous influence on the growth of statistics (see reference 2). Measurement problems often stimulate advances in statistics. The highly variable biological measurements required in eugenic studies stimulated Fisher's unprecedented advances.

Fisher's involvement in eugenics put him in touch with Major Leonard Darwin, President of the Eugenic Education Society of London and Charles Darwin's youngest son. Darwin soon appreciated Fisher's worth and tactfully helped both financially and by offering good advice. On leaving Cambridge in 1913 Fisher spent the summer farming in Canada and then took a job with a London insurance company. At the outbreak of war in 1914 he volunteered for the army but was rejected because of his poor sight—such are the mysteries of natural selection! As war work he decided to teach at school, which he did badly and did not enjoy. At the same time his interest in farming increased and he settled down, in what seems remarkably like a farming commune, with his 17-year-old wife, a female friend known as Gudrunna and her five-year-old daughter Kestrel. In Joan Fisher Box's biography this reads as being an idyllic time but it was not a financial success; Fisher was never interested in money.

At the end of the war Fisher was 28 years old and had never held a scientific job but he had continued his scientific work and had published some good scientific papers. Already in 1912, as an undergraduate, he had published his first paper, showing that estimates of the parameters of a frequency distribution could be obtained by maximising what he later termed likelihood. Fisher was later to extend this method greatly, and this was but one of several important statistical advances that were to bring him into conflict with Karl Pearson who strongly advocated estimation by the method of moments. Karl Pearson was Galton Professor of Eugenics at University College, London, the founder of *Biometrika* and the leading (virtually the only) academic statistician of his time. For half a century Pearson was a man of enormous influence in statistical circles but Fisher's ideas of heritability conflicted with Pearson's, and woe betide anybody who conflicted with Pearson. An early triumph was Fisher's 1915 paper, his first and last publication in *Biometrika*, on the distribution of the correlation coefficient from independent samples. This had been derived a few years earlier by Student (W. S. Gosset) using heavily empirical methods. Gosset was to become one of Fisher's staunchest supporters. He was involved with field experiments on barley which required yields to be averaged over a small number of plots, while Pearson's methods were targeted at large samples. Fisher used for the first time his highly geometrical arguments and obtained a solution to a problem that had defeated the best analytic efforts of others. He was later to develop small-sample theory almost single-handed, using ever more intricate extensions to the geometrical methods first used in his 1915 paper. His most important genetical work of this period, 'The correlation to be expected between relatives on the supposition of Mendelian inheritance' was acceptable neither to *Biometrika* (its natural home) nor to the Royal Society but, thanks to Leonard Darwin, was published by the Royal Society of Edinburgh in 1918. By reconciling the views of warring camps Fisher could please neither side. Although Fisher had made several notable contributions, they attracted little notice at the time and prospects looked bleak.

**Rothamsted**

Rothamsted Experimental Station is about 25 miles north west of London, in Hertfordshire, and on the edge of what is now the commuter town of Harpenden. In 1919 the then director of Rothamsted, Sir John Russell, a distinguished agricultural chemist, was looking for someone to analyse the accumulated results of 75 years of field experiments. He hoped that a mathematician with some experience of handling large quantities of data might be able to extract further information than had already been obtained. After a fruitless search his enquiries eventually brought mention of Fisher, whose tutor at Gonville and Caius provided a reference to the effect that Fisher could have become a first-class mathematician had he stuck to the ropes, but he would not. Russell later wrote, 'That looked like the type of man we wanted, so I invited him to join us.' The appointment was for six months and, in accepting, Fisher turned down the possibility of employment at overseas universities and an invitation to join Karl Pearson. Not only was Fisher the type of man wanted by Russell, but the job at Rothamsted suited Fisher very well. It offered him the opportunity to break new ground and the freedom to develop his ideas in a biological setting. The work was not genetical but, as we have seen, Fisher was already interested in agriculture and hoped to pursue his other interests.

The initial six months grew to 14 years, during which Fisher revolutionised the whole science of statistics, building the foundations of the subject as it is known today.

At Rothamsted Fisher set about analysing the accumulated experimental results; from this came his development of a class of orthogonal polynomials, and the formalisation of the analysis of variance. The latter had already been presented in a simple form in the 1918 paper. Analysis of variance is essentially an extension of Huygens' principle, with moments of inertia replaced by the concept of variance (a term introduced by Fisher) and the centroid replaced by the sample mean. Associated with it are certain significance tests, especially those concerned with variance-ratios. At that time there was certainly a need for significance tests to correct the tendency of some agronomists to believe that small chance differences of yield indicated a definite superiority of one treatment or variety over another. Fisher, encouraged by Gosset, developed the mathematical forms of the necessary small-sample probability distributions, continuing the work started in his 1915 paper. Thus Fisher obtained in 1921 the distribution of the correlation coefficient in small samples, in 1922 the true form of the chi-squared distribution for contingency tables, again in 1922 the distribution of the regression coefficient and in 1928 the multiple correlation coefficient. Not only were these obtained in the null case (i.e. zero population values) but also when this is not so—the so-called non-central distributions. Fisher's geometrical arguments linked all these distributional problems. The chi-squared distribution was not new, but Fisher's work showed that Pearson had failed to allow for the linear constraints implicit in achieving given row and column sums of a contingency table. For small tables this could lead to serious underestimation of significance. What it certainly did lead to was a serious row with Karl Pearson. To make matters worse, the exact distributions found by Fisher made redundant the heavy numerical work done by his colleagues under Pearson's supervision—the so-called Cooperative Study. From this time relations with Karl Pearson were beyond rescue.

Fisher's 1922 paper 'On the mathematical foundations of theoretical statistics' appeared in the *Transactions of the Royal Society*. This gave the fully worked out development of his 1912 ideas on likelihood and began the serious study of estimation theory. It introduced several totally new ideas such as the concepts of efficiency, sufficiency, consistency and information. Fisher showed that maximum-likelihood estimators are *asymptotically* most efficient and are *asymptotically* consistent, when such statistics exist. Maximum likelihood remains a cornerstone of estimation theory, though now it has many accretions—conditional likelihood, marginal likelihood, residual likelihood. A major, and unresolved, issue in estimation theory is whether or not to allow for vague prior information about plausible parameter values. If this prior information can be quantified, a mechanism for incorporating it with sample information exists in Bayes' theorem (1763). Likelihood is the probability $p(y|t)$ of getting a sample $y$ given parameters $t$. Bayesian estimation combines likelihood with so-called inverse probability $p(t|y)$. Many, including Fisher, point out that although $t$ may be unknown, it is certainly not a random variable so statements about its probability are meaningless. Even when $p(t)$ is regarded as being well defined, a further difficulty is whether or not it is legitimate to express ignorance of plausible parameter values by equi-probability, as is often done to obtain a working prior. Fisher was an implacable enemy of inverse probability estimation. His very important philosophical writings on statistical inference are summed up in his 1956 book, *Statistical Methods and Scientific Inference*.

Quite soon after starting at Rothamsted, and again encouraged by Gosset, Fisher began to develop ideas on the design of experiments. Fisher enunciated three basic principles, known facetiously as the three R's of experimentation—randomisation, replication and blocking. Randomisation was to eliminate systematic effects and justify the assumption of independence of errors required in his small-sample tests. Replication was to provide a solid basis for determining the basic experimental variation, or error as it is often misleadingly termed. Blocking is to confine sets of treatments to a limited geographical area, thereby reducing the effects of differences, or trends, in fertility in different parts of a field. Scientists had long believed that nature should be asked only one question at a time; Fisher said not so. If in agriculture you ask only one question, you will get only one answer in a year, the time of a growing season. Further if you want to know how different treatments interact, you must apply them simultaneously. The information could be obtained through what Fisher termed *factorial experiments* which in their simplest form require all combinations of several levels of several treatments. These treatment-combinations had to be assigned to blocks and randomised over blocks. Further, blocks should be kept small or they would not fulfil their purpose of local control. A full factorial experiment can be very large and uneconomical; agricultural practice often requires that treatments be superimposed on smaller parts of a larger area having a previously applied treatment. Fisher had solutions to these and other difficulties; he discovered the possibilities of split-plots, of confounding, of fractional replication and of partial confounding—incomplete block designs were developed by his younger colleague and successor at Rothamsted, Frank Yates. These developments required combinatorial mathematics, also needed for the underpinning of the randomisation theory necessary to justify the small-sample tests in the new designs.

This work was done in the context of cereal experiments. New problems occurred in other fields—variety trials, perennial crops, animal experiments, clinical trials, industrial experiments and so on, but these were developed by others. Fisher summed up his contribution in his book *Experimental Design*, first published in 1935 shortly after he left Rothamsted. In showing that the way in which data can be collected can increase efficiency and ease of interpretation, he had made one of his most important practical contributions.

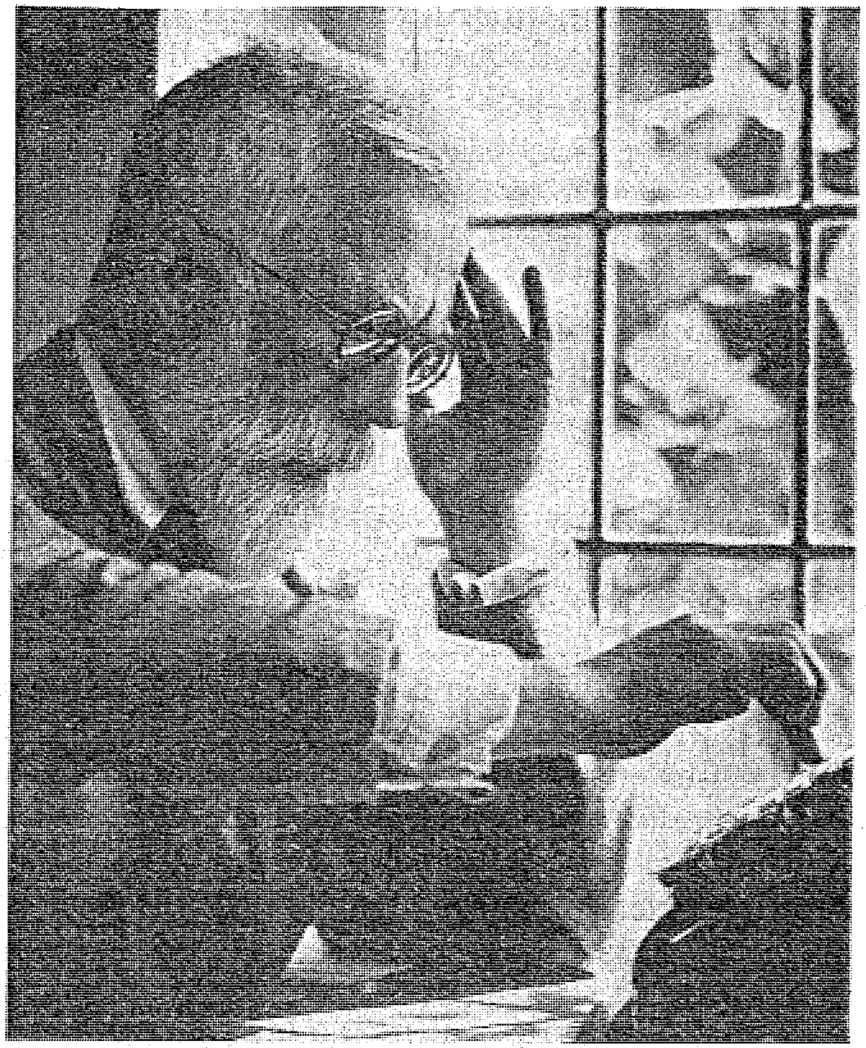
Fisher was very closely involved with the agricultural research at Rothamsted from which many of his ideas derived directly. His papers are often illustrated by practical examples. His book *Statistical Methods for Research Workers*, first published in 1925, was intended as a practical handbook for research workers in the experimental sciences. It made available the new methods, but in view of the intended readership did not go into mathematical details. It is not easy reading, partly because there are so many novel ideas and partly because of Fisher's style. It got almost universally poor reviews but nevertheless went into 14 editions; clearly it filled a need. The non-mathematical nature of *Statistical Methods* was consistent with Fisher's propensity, even in his theoretical work, to give only the sketchiest of proofs. George Barnard tells how, as a young mathematical student, he told Fisher that he was considering statistics as a career and asked what he should read. Fisher's reply was that *Statistical Methods* contained many unproved results and that Barnard as a mathematician should be able to prove them and learn a lot in the process. Professor Barnard was able to tell Fisher 30 years later that he had just completed the task. Fisher's work soon became widely known, and agricultural researchers from Britain and abroad came to Rothamsted to learn of the new methods; before long a new brand of statistical professional also came. As with all revolutions, there was resistance from the more conservative elements, but mostly they were gradually won over.

The Fishers lived in a large house in Harpenden, where the family eventually grew to two sons and six daughters. It was said that Fisher was one of the few eugenicists who had the courage of their convictions; no wonder he was interested in family allowances. Throughout his period at Rothamsted, Fisher had continued his genetical work at home. There was a large collection of mice, eventually running to 60 cages, tended by the whole family, who also helped with maintaining the records of breeding experiments spanning many years. At times other animals were kept—goats, snails, chickens, dogs. For the significance of his work in genetics I quote from a note prepared by Sir Walter Bodmer for the Royal Society Soirée: 'In the early 1920s he published classical papers demonstrating for the first time a model of heterozygote advantage for balanced polymorphism, and providing the first mathematical analysis of the effects of random fluctuation of gene frequency changes on the probability of survival of a new mutation. His book *The Genetical Theory of Natural Selection* remains perhaps to this day the most important book on evolution written since Darwin. Many a terse paragraph or two in this book have laid the basis of whole areas of population genetics including, for example, ideas on the interaction between genetic linkage and selection, the evolution of the sex ratio, models for sexual selection and mimicry, and the whole basis for socio-biology.'

**University College, Cambridge and final years**

On Karl Pearson's retirement in 1934, Fisher became the new Galton Professor at University College. That statistical teaching had been given to Pearson's son Egon caused inevitable friction, but did not stop Fisher doing statistical research any more than Rothamsted had put a halt to his genetics. Fisher was now established, with a growing international reputation. Already, in 1929, he had been elected as a mathematician to fellowship of the Royal Society. A steady stream of visitors came to work with him and Fisher himself made many overseas visits, notably to the USA. Through his friendship with P. C. Mahalanobis, founder of the Indian Statistical Institute, Fisher was able to make important contributions to the establishment of statistical teaching in India. So Fisherian ideas spread throughout the English-speaking world. Nevertheless Fisher continued to be involved in controversy, sometimes bitter.

New statistical ideas continued to be developed. Of particular importance is discriminant analysis, which is concerned with the use of multiple measurements to enable the optimal assignment of samples to one of several known classes; discriminant analysis is at the root of present-day methods for automatic medical diagnosis. Also at this time Fisher developed methods for associating numerical scores with qualitative observations which have been much developed in the social sciences. Work continued on statistical distributions and in a remarkable example of independent discovery, Fisher was one of four who within a single year, 1939–40, found the distribution of the maximum eigenvalue of a Wishart matrix. Wishart had been one of Fisher's later staff at Rothamsted, which he left in 1931 for a readership in statistics at the Cambridge School of Agriculture, giving one example of how Fisher's ideas spread. In 1932 he had found the joint distribution of the entries in a sums-of-squares-and-products matrix, thus extending Fisher's earlier work on the correlation coefficient. The Wishart distribution and the distribution of the eigenvalues of a Wishart matrix are the foundation stones of multivariate distribution theory. A much later contribution of distribution theory was the 1955 paper on distribution on a sphere, which Fisher derived to study data



Sir Ronald Aylmer Fisher 1890–1962
(Photograph by A. Barrington-Brown ARPS)

on residual magnetism, the principal evidence in support of the modern geological theory of plate tectonics. This illustrates how most of Fisher's highly original work was done in response to practical scientific problems.

On the genetical front, one of the consequences of his Galton Professorship was that for the first time Fisher had the possibility of establishing his own research team—provided he could find the funds. These he obtained through the Rockefeller Foundation via the Medical Research Council, to set up a Galton Serological Unit to do research on the genetics of blood, especially human blood. The ABO blood groups, discovered about 1900, soon became vital for blood transfusion services. Other blood groups, controlled by different genes, were discovered later. The correct blood groups of any patient or blood donor could only be established by a series of serological tests. Reactions to sera are recorded as absent, a trace, weak, positive and strong. It was within this context that Fisher developed his methods for quantifying qualitative data. The Galton Unit did important work in unravelling the complex genetical interplay of the growing number of different blood group series. At the outbreak of war the unit was moved to Cambridge as part of the national blood transfusion service. Despite this essential war work, the research activities continued; indeed it was now possible to get access to vast amounts of data, through RAF recruits and from maternity hospitals. This was useful in establishing the relationship of blood groups in mother and child, and for compiling an atlas of variations in blood-group frequencies throughout the UK—later extended globally. That there were major regional differences was of special interest to Fisher from the evolutionary point of view. During the war years information was growing on the very complicated rhesus blood groups: it makes a fascinating detective story how Fisher deduced that the complex was controlled by three closely linked genes, each with two alleles. This model not only fitted the available evidence but also predicted certain reactions which were subsequently observed.

Meanwhile Fisher himself, with the remnants of his Eugenics Department, was evacuated to Rothamsted where he renewed contact with his old colleagues. Fisher still resided at his Harpenden home. The death in the RAF of George, the older of his two sons, was a great personal blow, from which Fisher perhaps never fully recovered, and which contributed to his separation from his wife. In 1943 Fisher was appointed to the Arthur Balfour Chair of Genetics at Cambridge. Fisher could not expect to develop his new department until after the war. However, he believed that the university would then give him good support. In the event he was to be disappointed; not even the Serological Unit survived in Cambridge. To compensate for the loss of serology Fisher pioneered research on bacterial genetics but, although many important discoveries were made, the project was starved of resources. Work continued on breeding lines of mice, the plant *Lythrum*, and poultry. These inbreeding experiments were done to produce pure lines with particular genetic characteristics—eventually over 20 separate lines of mice were maintained. This work led to a mathematical book, *The Theory of Inbreeding*, published in 1949. While at Cambridge Fisher published *Statistical Methods and Scientific Inference*, the culmination of 40 years' thinking. During this time his ideas had evolved mainly in emphasis rather than substance. For example, in *The Design of Experiments* (1935) Fisher had written 'Every experiment may be said to exist only in order to give the facts a chance of disproving the null hypothesis'. More often experimenters know that the null hypothesis is false and are seeking to measure the size of effects. In his later years Fisher recognised that in some quarters, too much importance was attached to significance tests.

After his retirement in 1959 Fisher moved to Adelaide where he had been invited by Alf Cornish, Chief of the Division of Mathematical Statistics of the Commonwealth Scientific and Industrial Research Organisation. Cornish had been a student of Fisher's in University College days, where they had worked on $k$-statistics. Fisher had a happy time in Adelaide where he continued to develop his ideas on scientific inference. He died there on 2 August 1962.

Fisher was a remarkable scientist, possibly to be ranked with Newton and Darwin. It is sometimes said that he had two major careers, one in statistics and one in genetics, but within each he pioneered several areas, any one of which could in itself have formed the basis of a distinguished career—small-sample theory, theory of estimation, design of experiments, theory of scientific inference, discriminant analysis, evolutionary genetics, population genetics, serology, and bacterial genetics.

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**A man and his sons**

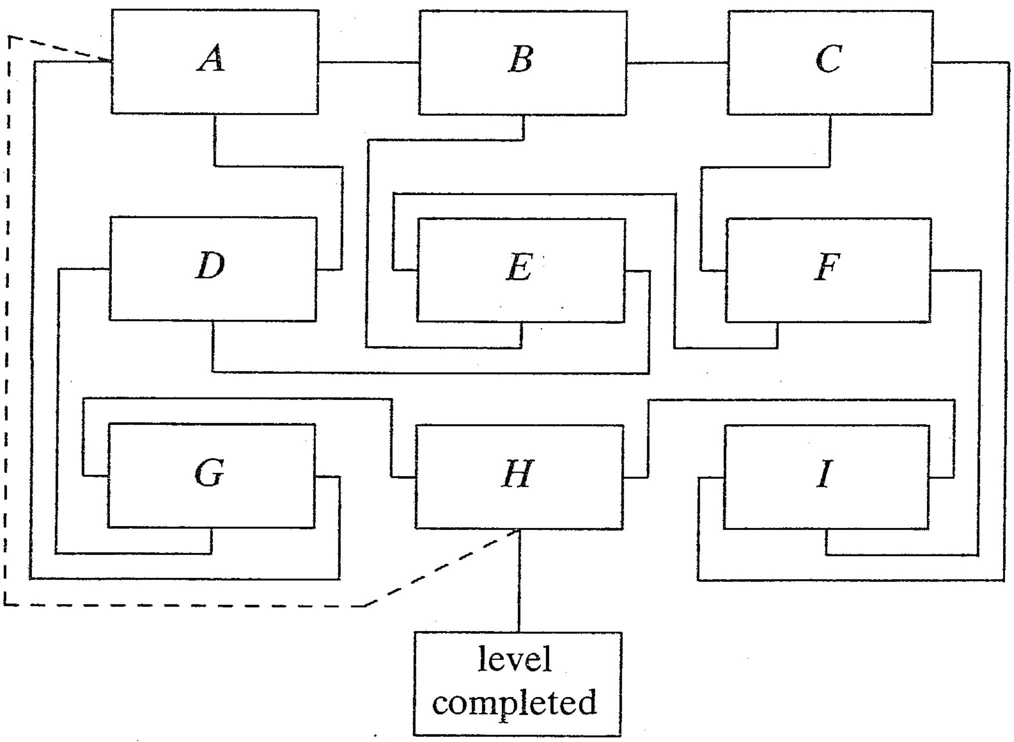
A man has three sons. The age of the youngest times the sum of the ages of the other two is 1311; the age of the second son times the sum of the ages of the other two is 1767. How old are the sons?

# Get Found!

**KEITH E. HELLMAN**, *University of Chicago*

The author wrote this article when he was a third-year undergraduate in the Mathematics Department at the University of Chicago.

The computer game 'Get Lost' described by J. N. MacNeill in Volume 22 Number 1 of *Mathematical Spectrum* consists of a maze (figure 1), and four levels. The game begins by randomly placing an explorer in one of nine rooms ($A$ to $I$). Each room looks identical and has three doors, which I shall denote as west ($W$), south ($S$), and east ($E$). (They were originally left, bottom, and right). The explorer never sees the voyage between the two rooms. However, on levels 1 and 3, he does know by which door a room is entered. On levels 3 and 4 the explorer must travel through all nine rooms before exiting; if he uses the $S$ door of room $H$ before this, he enters room $A$ through the $W$ door. The problem posed by Mr MacNeill is to devise a method of exiting all four levels.




The quickest levels to escape from are 1 and 3; in these levels the explorer knows by which door a room is entered. Only three moves, at most, are needed to pinpoint an explorer's location in the maze. A useful exercise before beginning this algorithm is to make three charts, one for each possible exit ($W$, $S$ or $E$), listing the entrance door used and the room entered. An example for $S$ exits is shown in figure 2.

Room exited

| | | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|---|
| | W | | | F | | | E | D | A | |
| Entry door | S | | E | | | B | | | | |
| | E | D | | | E | | | | | F |

Table contents: Room entered

Figure 2

The algorithm, although long and involved, has a straightforward and systematic construction. Movement is denoted by $W$ for west exit, $S$ for south exit, etc; comments are indicated by an asterisk. Escape routes are listed together for levels 1 and 3.

*Algorithm 1.0 and 3.0* (for levels 1 and 3)

```
S                                            * Exit via south door.
If entry = S                                 * Entered by a south door?
Then E                                       * Then exit east door ....
      If entry = W
      Then   Position is room C, and rooms
             E and B have been visited
             L1: EES                         * L1 = level 1 escape route.
             L3: SEESSWWS                    * L3 = level 3 escape route.
       Else  Position is room D, and rooms   * Written as (D; BE)
             B and E have been visited         from now on.
             L1: WWS
             L3: EWWESWEES
       End
Else   If entry = E
       Then S
             If entry = S
             Then  (B; DE)
                   L1: WWWS
                   L3: EEWESWEES
             Else  (E; AD) or (E; IF)
                   If entry = W
                   Then  (E; AD)
                         L1: EWWS
                         L3: SESEEWWS
                   Else  (E; IF)
                         L1: EWWS
                         L3: SEWWSWWS
```

```
                End
            End
        Else  S
            If entry = S
            Then  (B; FE)
                L1: WWWS
                L3: EEESSWWS
            Else  If entry = W
                Then  (E; CF)
                    L1: EWWS
                    L3: SWSWWEES
                Else  S
                    If entry = S
                    Then  (B; GDE)
                        L1: WWWS
                        L3: WEESEES
                    Else  (E; HAD)
                        L1: EWWS
                        L3: WEWWWWWS
                    End
                End
            End
        End
End
```

Now consider the scenario of levels 2 and 4. The explorer does not know entry doors, and does not know which room he is in. His key to finding himself is the symmetry of the maze itself. Consider the *S* exit of room *H* locked and reflect the maze on an axis through the *S* exits of rooms *B*, *E* and *H*; indeed the maze is now symmetric. This symmetry leads to three special 'trapping' passages: *W* from rooms *G* and *H*, *S* from *B* and *E*, and *E* from *H* and *I*. I call these trapping passages because, if the proper exit is used, the explorer will only cycle between the respective rooms. Notice that, from any room, three consecutive exits through *S* doors guarantees that the explorer must be in either room *B* or *E*. Something similar happens when using consecutive *W* or *E* doors, but then it requires at least five consecutive exits (this is easily verified by the reader: due to symmetry, only rooms *A*, *B*, *D*, *E*, *G*, and *H* need to be checked for either *W* or *E*). The next hurdle is finding the sequence of exits that will guarantee the explorer to be in only one room, regardless of which room (*B* or *E*), (*G* or *H*), or (*H* or *I*) he begins in.

Once again, symmetry is the key to the problem. The question would not be difficult if an explorer began in rooms *A* and *I*, or *B* and *H*, or *C* and $G$, or $D$ and $F$; symmetry would allow us to circle the maze until rooms $D$ and $F$ were occupied and then one $S$ exit would guarantee the explorer to be in room $E$. This fact motivates the sequence of $WW$ (or $EE$) which places the explorer in rooms $C$ and $G$ (or $A$ and $I$, respectively). Two $S$ exits now guarantees room $E$. A similar sequence of three exits also exist if beginning in rooms ($G$ or $H$), or ($H$ or $I$). The asymetrical $S$ exit from room $H$ creates the two-move $SW$ sequence that guarantees a position in room $G$. These position-finding sequences are the heart of algorithms 1.1, 2.0, 3.1 and 4.0, listed below with comments.

*Algorithm* 1.1

| | |
|---|---|
| *SSS* | * Guarantees room *B* or *E*. |
| *W* | |
| If entry door = *E* | |
| Then *WW* | * Explorer is in room *A* and goes to *H*. |
| Else *EE* | * Explorer is in room *F* and goes to *H*. |
| End | |
| *S* | * Exit to level 2. |

*Algorithm* 2.0

| | |
|---|---|
| *WWWWW* | * Guarantees room *G* or *H*. |
| *S* | |
| If level not complete | * If level not complete then position |
| Then (*D*) *WWS* | is room *D*; go to *H* and exit. |
| End | |

*Algorithm* 3.1

| | |
|---|---|
| *SSS* | * Guarantees room *B* or *E*. |
| *W* | |
| If entry door = *E* | |
| Then *WSSWWEE* | * Explorer is in room *A* and has been in room *B*. Visit all rooms while going to *H*. |
| Else *EWWWSWW* | * Explorer is in room *F* and has been in room *E*. Visit all rooms while going to *H*. |
| End | |
| *S* | * Exit to level 4 |

*Algorithm* 4.0

| | |
|---|---|
| *SSSWWSS* | * Guarantees explorer in room *E*. |
| *EWEEESEES* | * Visit all rooms before exiting *H*. |

An algorithm similar to 4.0 (also requiring 16 moves) can be constructed using the trapping sequence of algorithm 2.0. Notice that algorithm 1.0 takes from 5 to 7 moves depending on the room where you begin, whereas the number of moves in algorithm 1.1 is fixed at 7; likewise algorithm 3.0 requires 10 to 11 exits, and algorithm 3.1 has a fixed number of 12 moves. Depending upon how the scoring of the computer game is set up (fastest in time, smallest number of exits, or a combination of the two), and the program (is a buffer file of seven characters available?). Algorithms 1.0 and 2.0 may not be the lowest scoring choices because of the number of decisions that would have to be made and the number of alternative moves that might have to be memorized.

# Powerless Arithmetic Progressions

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In a previous article (Volume 22, Number 3, pages 92–93), I proved that the famous sequence whose $n$th term is $(p_1p_2...p_n)+1$, where $p_1,p_2,...,p_n$ denote the first $n$ primes in increasing order, possesses no powers (excluding, of course, first powers). Here we shall use this sequence to prove that, *given an integer* $K > 1$, *there is an arithmetic progression of positive integers which has no powers up to and including Kth powers.*

We start with the infinite sequence $p_1,p_2,...$ of prime numbers, in increasing order. A famous theorem of Dirichlet states that, for coprime positive integers $a$ and $b$, there are infinitely many prime numbers of the form $ar+b$, where $r$ is a positive integer. We do not need the full force of this theorem. Choose $j$ so that $p_{j+1} > K$; the reason for this choice will become clear later. Then Dirichlet's result tells us that there is an $r$ such that

$$P = (p_1p_2...p_j)r+1$$

is prime.

Another result is that 'the non-zero residue-classes modulo a prime number $p$ form a cylic group under multiplication'. (For example, if $p = 7$ the non-zero residue-classes are $\bar{1},\bar{2},...,\bar{6}$, with $\bar{3}^1 = \bar{3}$, $\bar{3}^2 = \bar{2}$, $\bar{3}^3 = \bar{6}$, $\bar{3}^4 = \bar{4}$, $\bar{3}^5 = \bar{5}$ and $\bar{3}^6 = \bar{1}$, so that $\bar{3}$ generates the group.) For the prime number $p$ given above, let the integer $a$ be such that the residue-class $\bar{a}$ generates this group. We assert that the arithmetic progression $s_n = pn+a$ contains no powers up to and including $K$th powers.

Suppose that $s_n = m^q$ for some positive integers $m$ and $q \geqslant 2$. We shall show that $q > k$, and this will prove our result. Now $a$ and $p$ are coprime, so that $s_n$ and $p$ are coprime, and in turn $m$ and $p$ are coprime. Thus the residue-class $\bar{m}$ in $\mathbb{Z}_p$ is not zero and so, by choice of $a$, $\bar{m} = \bar{a}^i$

for some positive integer $i$. Thus $\bar{a} = \bar{s}_n = \bar{m}^q = \bar{a}^{iq}$, so that $\bar{a}^{iq-1} = \bar{1}$. Since $\bar{a}$ has order $p-1$, this means that $p-1$ divides $iq-1$, so that

$$iq-1 = (p-1)b$$

for some integer $b$. Thus

$$iq = (p_1p_2\dots p_j)rb+1.$$

From this we see that $p_1, p_2, \dots, p_j$ cannot divide $q$, so that $q \geqslant p_{j+1}$. But we chose $j$ so that $p_{j+1} > K$, so that $q > K$, as required.

Another way of expressing our result is that, given an integer $K > 1$, we have found a linear polynomial $a+px$ in $x$ with non-negative integer coefficients which, when evaluated at positive integers, does not take values which are powers up to and including $K$th powers. For a polynomial of arbitrary degree $t$ with the same property, we may take $a+px^t$. In a subsequent article, I shall show how to find polynomials with non-negative integer coefficients which, when evaluated at positive integers, take no values which are powers.

It would be nice to know whether there is an arithmetic progression which contains no powers.

# Computer Column

MIKE PIFF

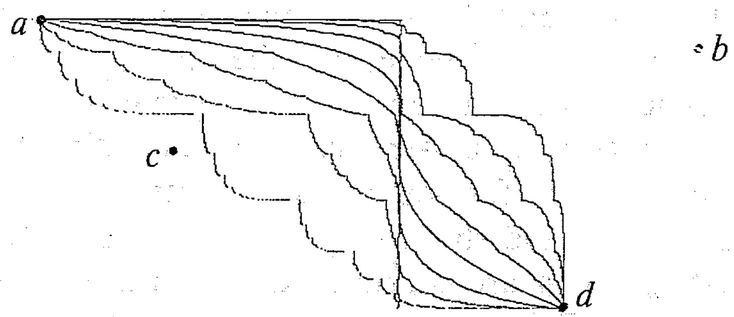
### Bézier curves and Bernshteĭn polynomials

Four points $a$, $b$, $c$ and $d$ are given in the plane. We wish to find a curve through $a$ and $d$ which starts in the direction $ab$ and ends in the direction $cd$. Clearly a cubic will do, and in vector form its equation is

$$z = (1-t)^3\boldsymbol{a}+3(1-t)^2t\boldsymbol{b}+3(1-t)t^2\boldsymbol{c}+t^3\boldsymbol{d},$$

plotted for $0 \leqslant t \leqslant 1$, the *Bernshteĭn polynomial.*

Surprisingly, the curve can be drawn iteratively by the following process. Let the midpoints of $ab$, $bc$ and $cd$ be $e$, $f$ and $g$, the midpoints of $ef$ and $fg$ be $h$ and $i$, and finally the midpoint of $hi$ be $j$. Plot $j$, and repeat the plotting on the sequences $a, e, h, j$ and $j, i, g, d$!



The following program plots the curve by this method. The four pairs of coordinates have to be input first, separated by spaces. The program presupposes the existence of a definition module for graphical output, containing routines to start and end graphical output and plot a pixel.

The value $t = 0.5$ gives the smooth curve we want. Other values of $t$ ranging from 0.1 to 0.9 are also plotted, just to see what effect this has. The 'midpoint' for these values of $t$ is a fraction $t$ of the way along each segment.

A sample screen dump is included.

```
MODULE Bernshtein;
FROM Graphics IMPORT BeginGraph,
EndGraph, PutPixel;
FROM InOut IMPORT Read, ReadCard,
WriteString, WriteLn;
CONST
  White=1;
TYPE
  coords=RECORD x,y:REAL END;
VAR
  a,b,c,d:coords;
  t:REAL; i:INTEGER;
PROCEDURE mid(a,b:coords;VAR c:coords);
BEGIN
  c.x:=(t×a.x+(1.0−t)×b.x);
  c.y:=(a.y+b.y)/2.0;
END mid;
PROCEDURE Plot(a:coords);
BEGIN
PutPixel(TRUNC(a.x),TRUNC(a.y),White);
END Plot;
PROCEDURE ReadCoords(VAR a:coords);
VAR ax,ay:CARDINAL;
BEGIN
  WriteString
  ('Give␣a␣pair␣of␣coordinates␣x␣y␣␣'); WriteLn;
  WriteString('x='); ReadCard(ax); WriteLn;
  WriteString('y='); ReadCard(ay); WriteLn;
  a.x:=FLOAT(ax); a.y:=FLOAT(ay);
END ReadCoords;
PROCEDURE dist(a,b:coords):REAL;
BEGIN
  RETURN ABS(a.x−b.x)
  +ABS(a.y−b.y);
END dist;
PROCEDURE DrawCurve
  (a,b,c,d:coords);
VAR
  e,f,g,h,i,j:coords;
BEGIN
  mid(a,b,e); mid(b,c,f);
  mid(c,d,g); mid(e,f,h);
  mid(f,g,i); mid(h,i,j);
  Plot(j);
  IF dist(a,j)>0.5 THEN
    DrawCurve(a,e,h,j);
    DrawCurve(j,i,g,d);
  END;
END DrawCurve;
PROCEDURE Pause;
VAR ch:CHAR;
BEGIN
  Read(ch);
END Pause;
BEGIN
  ReadCoords(a); ReadCoords(b);
  ReadCoords(c); ReadCoords(d);
  BeginGraph;
  Plot(a); Plot(b); Plot(c); Plot(d);
  FOR i:=1 TO 9 DO
    t:=FLOAT(i)/10.0; DrawCurve(a,b,c,d);
  END;
  Pause; EndGraph;
END Bernshtein.
```

### The 1991 Puzzle

The aim is to construct as many of the numbers 1 to 100 as possible from the digits of the year in order, using only $+$, $-$, $\times$, $\div$, $\surd$, ! and concatenation (i.e. constructing the number 19 from 1 and 9, for example). To start you off, $1 = 1\times(9-9+1)$.

# Letters to the Editor

Dear Editor,

*Sums of consecutive integers*

Reading Joseph McLean's letter on page 60 of Volume 23 Number 2 of *Mathematical Spectrum* caused me to have another look at L. B. Dutta's list of powers of 5 on page 78 of Volume 22 Number 3.

I have discovered two very simple recurrence relations that give the first and last of the consecutive integers in such cases. These apply if a table of powers of an odd number is being compiled and will complement the explicit formulae given by J. McLean on page 57 of Volume 18 Number 2.

Let $m \geqslant 3$ be an odd integer. The sequences of integers $(a_n)$, $(b_n)$ are defined by

$$a_0 = 1, \qquad a_1 = \tfrac{1}{2}(m-1), \qquad a_n = ma_{n-2}-a_1 \quad (n \geqslant 2);$$

$$b_0 = 1, \qquad b_1 = \tfrac{1}{2}(m+1), \qquad b_n = mb_{n-2}+a_1 \quad (n \geqslant 2).$$

Then, for $n \geqslant 1$,

$$m^n = a_n + \ldots + b_n$$

as a sum of consecutive integers. For example,

$$13^1 = 6+7 \qquad (a_1 = 6,\ b_1 = 7),$$

$$13^2 = 7+\ldots+19 \qquad (a_2 = 13\times1-6,\ b_2 = 13\times1+6),$$

$$13^3 = 72+\ldots+97 \qquad (a_3 = 13\times6-6,\ b_3 = 13\times7+6),$$

$$13^4 = 85+\ldots+253 \qquad (a_4 = 13\times7-6,\ b_4 = 13\times19+6).$$

Further, the number, $s_n$, of terms in this sum is given by $s_0 = 1$, $s_1 = 2$, $s_n = ms_{n-2}$ for $n \geqslant 2$.

Yours sincerely,

BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
Bath, Avon BA3 2RG)

Dear Editor,

*The Gudermannian function*

Readers may be interested in the definition of this function since, although it seems virtually unknown, it is perhaps not without interest, as its significance is that it can be used to express hyperbolic functions in terms of circular functions without recourse to imaginary numbers. The results $\cosh x = \cos \mathrm{i}x$ and $\sinh x = -\mathrm{i}\sin \mathrm{i}x$, where $\mathrm{i}^2 = -1$, are of course well known and are fundamental results in the theory of functions of a complex variable. The Gudermannian function is most simply defined by $\mathrm{gd}(x) = \arctan \sinh x$, so that $-\frac{1}{2}\pi < \mathrm{gd}(x) < \frac{1}{2}\pi$ for all $x$. Then, obviously, $\sinh x = \tan \mathrm{gd}(x)$ and it follows easily that $\cosh x = \sec \mathrm{gd}(x)$. Formulae for other hyperbolic functions are immediate consequences of these relations.

It is also of interest that $\mathrm{gd}'(x) = \operatorname{sech} x$, so that

$$\mathrm{gd}(x) = \int_0^x \frac{1}{\cosh u}\, du.$$

Other formulae are $\mathrm{gd}(x) = 2 \arctan \tanh \frac{1}{2}x = 2 \arctan \exp(x) - \frac{1}{2}\pi$. The geometrical background of the Gudermannian function is explored in an article by J. M. H. Peters which appeared in the *Mathematical Gazette* Volume 68, Number 445, pages 192–196, 1984.

Yours sincerely,
TERRY S. GRIGGS
(Lancashire Polytechnic)

# Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

## Problems

23.7 (Submitted by Alan Fearnehough, Portsmouth Sixth Form College—see the article in this issue)
Show that

$$\pi = 4 \sum_{r=1}^{\infty} \cot^{-1} 2r^2.$$

23.8 (Submitted by Jeremy Bygott, Queens' College, Cambridge)
Factorize the $n \times n$ determinant

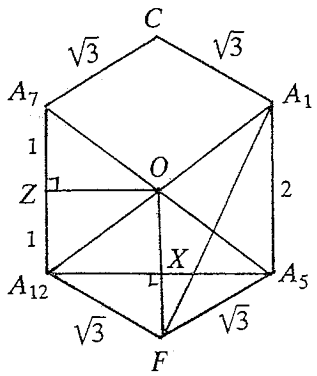
$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{(n-1)2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{vmatrix}.$$

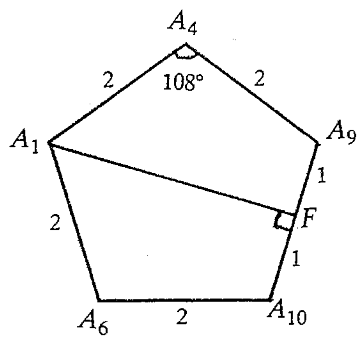
23.9 (Submitted by P. Glaister, University of Reading)
A ball is hit in the air with speed $U$ and caught at the same height. Determine the maximum length of the path travelled by the ball over all possible angles of projection.

# Solutions to Problems in Volume 23 Number 1

23.1 Denote by $O$ the centroid of a regular icosahedron with vertices $A_1, A_2, \ldots, A_{12}$, labelled so that $|A_1A_i| \leqslant |A_1A_j|$ for $i < j$. Determine the angle $A_1OA_2$.

*Solution* 1 by Nicholas Saltmarsh (Gresham's School, Holt)




Consider an icosahedron of side 2, divided down a plane of symmetry to give the hexagon $A_1A_5FA_{12}A_7C$ shown. Consider also the regular pentagon $A_1A_6A_{10}A_9A_4$ formed by five edges. In order to find the required $\angle A_1OA_2$, or $\angle A_7OA_{12}$, we must first find $A_1F$ by reference to this pentagon:

$$\begin{aligned} A_1A_{10}^2 &= A_1A_6^2 + A_6A_{10}^2 - 2A_1A_6 \times A_6A_{10} \cos 108° \\ &= 8 - 8\cos 108°. \end{aligned}$$

Hence

$$A_1F^2 = A_1A_{10}^2 - A_{10}F^2 = 7 - 8\cos 108°.$$

Thus

$$\cos \angle A_1A_5F = \frac{A_1A_5^2 + A_5F^2 - A_1F^2}{2A_1A_5 \times A_5F} = \frac{2\cos 108°}{\sqrt{3}}.$$

Next

$$\begin{aligned} A_5X = A_5F\cos(\angle A_1A_5F - 90°) &= A_5F \sin \angle A_1A_5F \\ &= \sqrt{3}\sqrt{1 - \tfrac{4}{3}\cos^2 108°} = \sqrt{3 - 4\cos^2 108°}. \end{aligned}$$

Hence

$$\tan \angle A_{12}OZ = \frac{A_{12}Z}{OZ} = \frac{A_{12}Z}{A_5X} = \frac{1}{\sqrt{3 - 4\cos^2 108°}}.$$

To simplify this, we must find an expression for $\cos^2 108°$. Now

$$\cos 5\times 18° + \mathrm{i}\sin 5\times 18° = (\cos 18° + \mathrm{i}\sin 18°)^5$$

by de Moivre's theorem, so that, if we equate real parts, we obtain

$$0 = \cos^5 18° - 10\cos^3 18° \sin^2 18° + 5\cos 18° \sin^4 18°,$$

which simplifies to

$$16\cos^4 18^\circ - 20\cos^2 18^\circ + 5 = 0,$$

whence

$$\cos^2 18^\circ = \tfrac{1}{8}(5+\sqrt{5}).$$

Hence

$$\cos 36^\circ = 2\cos^2 18^\circ - 1 = \tfrac{1}{4}(1+\sqrt{5}), \qquad \cos 72^\circ = 2\cos^2 36^\circ - 1 = \tfrac{1}{4}(\sqrt{5}-1),$$

so that

$$\cos 108^\circ = -\tfrac{1}{4}(\sqrt{5}-1).$$

Hence

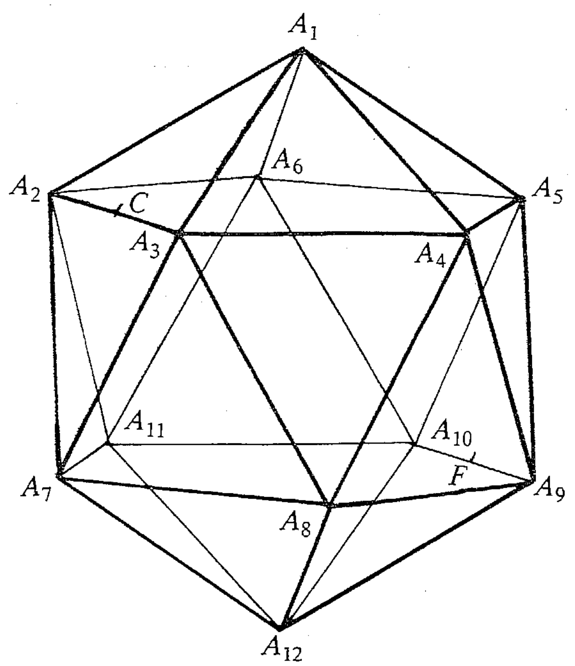
$$\tan \angle A_{12}OZ = \left\{3 - 4\left(\frac{\sqrt{5}-1}{4}\right)^2\right\}^{-\frac{1}{2}} = \left(\frac{3+\sqrt{5}}{2}\right)^{-\frac{1}{2}} = \frac{2}{1+\sqrt{5}},$$

so that

$$\angle A_7OA_{12} = 2\tan^{-1}\frac{2}{1+\sqrt{5}} = \tan^{-1}\frac{2\left(\dfrac{2}{1+\sqrt{5}}\right)}{1-\left(\dfrac{2}{1+\sqrt{5}}\right)^2} = \tan^{-1}2,$$

which is approximately $63^\circ 26'$.

*Solution* 2 by Amites Sarkar (Trinity College, Cambridge), who proposed the problem.




Put $\angle A_1OA_2 = d$. In the diagram, $A_9$ is the opposite vertex to $A_2$, so $\angle A_1OA_9 = \pi - \alpha$, and so also $\angle A_1OA_7 = \pi - \alpha$. If we put $OA_1 = OA_2 = \ldots = OA_{12} = 1$, it follows from the triangles $A_1OA_2$ and $A_1OA_7$ that

$$A_1A_2 = 2\sin\tfrac{1}{2}\alpha \quad \text{and} \quad A_1A_7 = 2\sin\tfrac{1}{2}(\pi-\alpha) = 2\cos\tfrac{1}{2}\alpha.$$

But $A_1A_2A_7A_8A_4$ forms a regular pentagon. It follows that

$$A_1A_7 = 2A_1A_2\cos 36^\circ = \tfrac{1}{2}(1+\sqrt{5})A_1A_2$$

(see Solution 1). Hence

$$\tan\tfrac{1}{2}\alpha = \frac{A_1A_2}{A_1A_7} = \frac{2}{1+\sqrt{5}},$$

whence

$$\alpha = 2\tan^{-1}\frac{2}{1+\sqrt{5}} = \tan^{-1}2,$$

as in Solution 1.

23.2 Find the sum of the series

$$\sum_{r=1}^{n} \frac{1}{\sin 2^r x},$$

where $x$ is a real number and $2^r x$ is not an integer multiple of $\pi$ for $r = 0, 1, \ldots, n$.

*Solution* by Paul de Sa (Christ's College, Cambridge)

We have

$$\operatorname{cosec} x + \cot x = \frac{1+\cos x}{\sin x} = \frac{2\cos^2\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \cot\tfrac{1}{2}x$$

when $x$ is not an integer multiple of $\pi$, so that

$$\operatorname{cosec} x = \cot\tfrac{1}{2}x = \cot x.$$

Hence, with the restriction given on $x$,

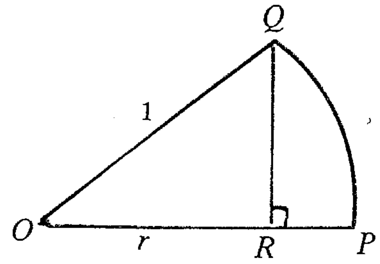
$$\begin{aligned}&\operatorname{cosec} 2x + \operatorname{cosec} 4x + \operatorname{cosec} 8x + \ldots + \operatorname{cosec} 2^n x\\ &\quad = (\cot x - \cot 2x) + (\cot 2x - \cot 4x) + \ldots + (\cot 2^{n-1}x - \cot 2^n x)\\ &\quad = \cot x - \cot 2^n x.\end{aligned}$$

Also solved by Jeremy Bygott (Queen's College, Cambridge), M. Movahhedian (Isfahan University of Technology, Iran), Oliver Johnson (King Edward's School, Birmingham), Selemon Zicke (International Community School, Addis Ababa, Ethiopia), and by Nicholas Saltmarsh, who points out that, by use of the identities $\sin ix = i\sinh x$ and $\cos ix = \cosh x$, we obtain the formula

$$\sum_{r=1}^{n} (\sinh 2^r x)^{-1} = \coth x - \coth 2^n x.$$

23.3 Let $PQ$ be an arc of a circle, centre $O$, with angle $POQ < 90^\circ$, and denote by $R$ the foot of the perpendicular from $Q$ to $OP$. Show that the volume of the cone obtained by rotating $OQR$ through $360^\circ$ about $OR$ is equal to the volume of the solid obtained by rotating $RPQ$ through $360^\circ$ about $RP$ if and only if $R$ divides $OP$ in the golden ratio.

*Solution* 1 by Jeremy Bygott




We may suppose that the circle has radius 1. Let $OR = r$, so that $QR = \sqrt{1-r^2}$, and denote by $V_1$ and $V_2$ the volumes of the solids of revolution of $OQR$ and $RQP$ about $OP$ respectively. Then

$$V_1 = \int_0^r \pi\left(\sqrt{1-r^2}\frac{x}{r}\right)^2 \mathrm{d}x = \tfrac{1}{3}\pi r(1-r^2)$$

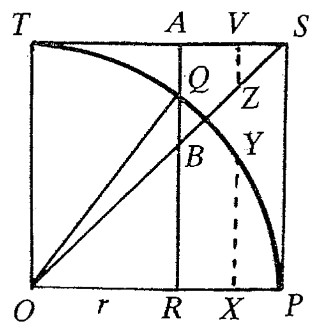
and

$$V_2 = \int_r^1 \pi(1-x^2)\ \mathrm{d}x = \pi\left[x-\tfrac{1}{3}x^3\right]_r^1 = \pi(\tfrac{2}{3}-r+\tfrac{1}{3}r^3).$$

Hence

$$\begin{aligned} V_1 = V_2 &\Leftrightarrow r-r^3 = 2-3r+r^3 \\ &\Leftrightarrow r^3-2r+1 = 0 \\ &\Leftrightarrow (r-1)(r^2+r-1) = 0 \\ &\Leftrightarrow r^2+r-1 = 0 \\ &\Leftrightarrow \frac{r}{1} = \frac{1-r}{r} \\ &\Leftrightarrow \frac{OR}{OP} = \frac{RP}{OR} \\ &\Leftrightarrow R \text{ divides } OP \text{ in the golden ratio.} \end{aligned}$$

*Solution* 2 by Amites Sarkar, employing methods that predate the calculus. As before, we take the circle to have radius 1, and consider the line parallel to $OT$ distance $x$ from $OT$, shown dotted on the diagram. The segment $XY$ of this line when rotated about $OP$ gives a disc of area $\pi(1-x^2)$, and this is the same as the area of the annulus formed by rotating the segment $ZV$ about $OP$. It follows by



Cavalieri's principle that the volume of the solid formed by rotating $RQP$ about $RP$ is equal to the volume of the solid formed by rotating the triangle $BSA$ about $OP$. By a theorem of Pappus, the volume of a solid of revolution is the area of the figure that is rotated multiplied by the distance travelled by its centre of mass. Hence the condition for the two volumes to be equal is

$$\tfrac{1}{2}OR \times RQ \times \tfrac{1}{3}RQ = \tfrac{1}{2}AB \times AS(AR - \tfrac{1}{3}AB)$$

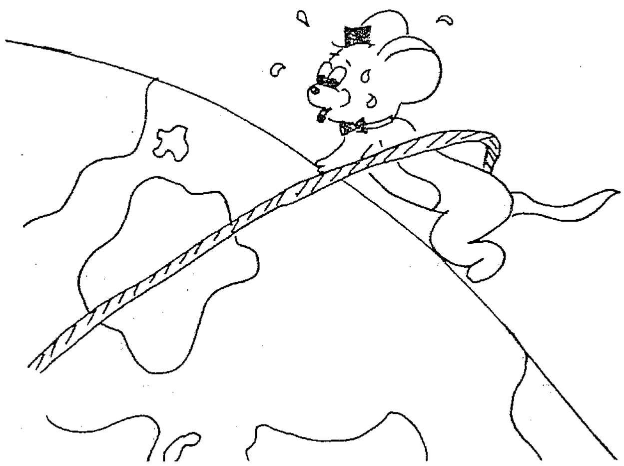
(cancelling $2\pi$ from both sides)

$$\begin{aligned}
&\Leftrightarrow OR \times RQ^2 = AB \times AS(3AR - AB)\\
&\Leftrightarrow OR(OQ^2 - OR^2) = RP^2[3OQ - (OQ - OR)]\\
&\Leftrightarrow OR \times RP(OQ + OR) = RP^2(2OQ + OR)\\
&\Leftrightarrow OR(1 + OR) = (1 - OR)(2 + OR)\\
&\Leftrightarrow OR^2 + OR - 1 = 0,
\end{aligned}$$

which is the condition that $R$ divides $OP$ in the golden ratio.

Also solved by Paul de Sa and Nicholas Saltmarsh.

# Mouse tales



Imagine that a length of string is stretched tightly round the equator of the earth. It is then cut, extended by a metre, and positioned to form a circle above the equator. Can a mouse squeeze between the string and the earth? What happens if the earth is replaced by a ping-pong ball?

R. J. WEBSTER
University of Sheffield

# Reviews

**Game, Set and Math.** By IAN STEWART. Basil Blackwell, Oxford, 1989. Pp. viii + 191. Hardback £12.95 (ISBN 0-631-17114-2).

This book is a selection of twelve articles originally published in the 'Visions mathématiques' column of *Pour la Science*, the French translation of *Scientific American*. The author wrote them in English, and they were subsequently translated into French by the editor. Since then Dr Stewart has edited and updated them, just as Martin Gardner used to, and restored the English puns.

The result is this jolly book. Beneath the informal presentations, the range of material is vast, from geometric measure theory through algebraic topology to Markov chains. A typical chapter starts in an often fictitious dialogue between colourful, sometimes fictitious, characters. Gradually, a problem, a theme, a concept, or a technique is introduced. Soon, the dialogue thins and the mathematics begins to take over. At this stage, the characters serve either to reassure, in that they are just as confused as the reader, or to point out alternative ways of viewing an idea. Whenever familiarity with an idea helps further understanding, questions are provided in italics. If the reader feels the need for a table, formula, or diagram, it is supplied: thus the book contains a graph of the probability of your winning a match of tennis against your fixed probability of winning a point, a diagram of a Grunbaum polyhedron cleverly split into four, and intricate patterns in Pascal's triangle to ten different moduli. The author also suggests several avenues of further study. At one place, for example, a net for a model of the Boy surface is pictured. The two hours I spent making the thing out of cornflakes box cardboard and glue were certainly worth it and gave a new meaning to the author's description of it in terms of three strangely interrelated doughnuts. At times I felt that the pace was either a little slow or a little fast, but the author's skilful explanation helped iron out these irregularities. Chapters 5 and 12 are different from the rest and consist of a series of puzzles, some of which are quite sophisticated.

This book is suitable for absolutely everyone. I thoroughly enjoyed it.

Trinity College, Cambridge AMITES SARKAR

**To Mock a Mockingbird and Other Logic Puzzles.** By RAYMOND SMULLYAN. Oxford University Press, 1990. Pp. x + 246. £5.95 (ISBN 0-19-286095-X).

This book represents a courageous effort to explain some tough logic and metamathematics by way of a staircase of carefully graded puzzles. In my case, the job was successful. Hardened by disastrous failures at the earlier problems, I tackled those in the chapters on Curry's paradox and Godel's theorem only to find the logical complexities reduced to practical, and hence more familiar, problems of friendly creatures in the book. My previous hangups over self-reference and undecidability vanished on reading the solutions to these. The puzzles were quite difficult, you see, and I couldn't do very many on my own.

The first two parts of the book are written in a similar vein to Smullyan's earlier puzzle books and the puzzles are from lands as far afield as the Island of Knights and Knaves or Subterranea, both populated by several variants of liars and truth-tellers. Once I'd broken into these, I was totally caught up in the excitement. After a fantastically hard puzzle about the Fountain of Youth, the scene changed to an enchanted forest inhabited by some not too heavily disguised set theory. The elements of the various sets are birds who respond in clever ways if you name another bird to them. Soon, I got to know a few of these, but my memory was jogged by the helpful who's who of birds at the back of the book. The stage was then set for a succession of short chapters introducing the reader to some quite recent ideas in combinatorial logic. At this point I got rather confused but, carried by the plot, I felt my way towards the climax, a chapter on 'Ideal birds'.

I liked the book. It probably helps if you read Chapters 14 to 25 very slowly.

Trinity College, Cambridge — AMITES SARKAR

**Practical Statistics.** By MARY ROUNCEFIELD and PETER HOLMES. Macmillan Education, Basingstoke, 1989. Pp. ii+346. £5.95 (ISBN 0-333-47344-2).

**Practical Statistics: A Teachers' Guide to the Course.** By MARY ROUNCEFIELD and PETER HOLMES. Macmillan Education, Basingstoke. 1989. Pp. vi+43. £4.00 (ISBN 0-333-51561-7).

If you believe that statistics is a very practical subject, although often taught in a theoretical way, and you are in need of some simple ideas for demonstrating the concepts of the subject through practicals that can be easily carried out in the classroom, then *Practical Statistics* is the book for you. In view of the dearth of statistics teachers with any formal training in their subject area, there has long been a need for a book such as this one. Anyone who has attended any of the courses at the Centre for Statistical Education in Sheffield may have already had a taste of some of the ideas expounded in this text and will undoubtedly welcome more of the same. The sixteen chapters are linked to specific standard topics found in most AS- and A-level Statistics syllabuses. The practicals have been piloted by a network of schools and colleges and modified in response to their findings. Some of the suggested experiments can yield data that illustrate several ideas and you are encouraged by the authors to keep data collected for one purpose so that it can be used at a later stage to demonstrate a different concept. Easy to read and with methods that are simple to follow, this book is highly recommended as a must for anyone involved in the teaching of statistics.

In sharp contrast to its companion volume, *Practical Statistics: A Teacher's Guide to the Course* is very poor value. Half of its pages consist of a reproduction of the photocopiable pages of *Practical Statistics*, and the remainder, containing a summary of its sixteen chapters, although useful to have, adds little to the exposition of the original, and is certainly not worth the price.

Solihull Sixth Form College — CAROL NIXON

**Excursions in Number Theory.** By C. STANLEY OGILVY and JOHN T. ANDERSON. Dover, New York, 1988. Pp. 168. £4.20 (ISBN 0-486-25778-9).

This Dover edition, first published in 1988, is an unabridged and corrected republication of the work first published by Oxford University Press in 1966. This superb volume invites readers to join a stimulating journey into the beauty of number theory. Beginning with familiar notions, the authors develop the concepts necessary for an understanding of complex subjects. Included are discussions of number patterns, prime numbers, irrationals and iterations, continued fractions and calculating prodigies.

This book is obviously aimed at those for whom mathematics is excitement and delight. No special mathematical training is needed—just high school mathematics. The latter is tested on page 78 where the reader is told that most elementary mathematics texts contain serious blunders, one of which is the unqualified statement that, for all $x$, $a$ and $b$,

$$(x^a)^b = (x^b)^a$$

and on page 82 the reader is warned to guard against falling into certain formal traps. The identity

$$(a+b)^2 \equiv a^2+2ab+b^2$$

might lead one to think that, because all numbers of the form $a^2+2ab+b^2$ are squares, any number of the form $a^2+3ab+b^2$ could never be a perfect square.

The answer to the question 'What perfect squares are also triangular numbers?', obtained on page 126 by means of the continued-fraction expansion for $\sqrt{2}$, might be of interest to teachers involved with GCSE coursework.

It would be very difficult to find another book which covers this range of material in such detail, with no mathematical background beyond high school mathematics being required. I recommend this book unreservedly to all with a fondness for numbers and an inquisitive mind.

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GREGORY D. ECONOMIDES

**Numbers Through the Ages.** Edited by GRAHAM FLEGG. Macmillan Education, Basingstoke, in association with the Open University, 1989. Pp. viii + 227. Hardback £30.00 (ISBN 0-333-49130-0); paperback £10.95 (ISBN 0-333-49131-9).

Fundamental to any attempt to form a consistent method of dealing with such practical physical quantities as size and amount is the ability to give recognisable spoken and written representations of different values, and be able to tally, measure, or in some other way render quantities into these values. Thus do we have the concepts of 'number' and 'counting'. This book aims to describe the development and evolution of these twin ideas, from antiquity to the present day.

Having said this, however, the first main topic discussed, that of counting systems, takes its examples from across most of history, by describing primitive methods of counting that exist today amongst tribes in remote areas of Africa, South America and the Pacific Islands as well as those used, for instance, by the ancient civilisations of Egypt, Babylon and Sumer. Here, one is struck by the fact that a system in use today by certain African tribesmen, and which fails to proceed further than 5 or 6, can be so inferior to methods used thousands of years ago. It is also interesting that the Babylonians, at a time when they were numerically advanced enough to develop a full counting system, chose the base 60 rather than the now popular base 10. In a later chapter it is shown that, when the Babylonians improved their number symbolism, calculations were no more difficult for them than our own arithmetic is to us. Perhaps the world would be more mathematically advanced now if it had adopted the Babylonian system universally in those days.

The next chapter concentrates on number words as spoken in everyday language. It is fascinating to follow the paths of diverging pronunciation across time and distance from the ancient Indo-European original, via such differing languages as Celtic, Gothic, Latin, Greek, Sanskrit and Vedic, to modern English, German, Russian, Iranian and Hindu. In the following chapter on written numbers, that is, numerical symbolism, alongside such exotic examples as Chinese and Mayan can be seen the gradual development of the modern decimal digits. Thus the modern 3 arises originally from a corruption of the obvious symbol $\equiv$. One can imagine a hurried Hindu scribe, in order to save time, scribbling the number 3 quickly without lifting stylus from paper.

The importance of having a place-value system, in which, as with modern numbers, the relative position of a digit within a number denotes a certain multiple of the base, is stressed. Combined with this, the introduction of a separate digit representing a 'no' value, i.e. zero, makes calculation a much simpler task. Many older forms of representing numbers which did not have these two attributes made calculation complicated and unwieldy: imagine trying to multiply two Roman numerals without recourse to decimals!

The book is based on a series of lectures given at the Open University by various lecturers, and there is therefore a degree of repetition between chapters. However, each chapter is well written and contains a wealth of fascinating details. It should be realised that this book is not for light reading, nor is it a mathematics textbook in the normal sense, but an in-depth historical survey full of maps and diagrams, extracts from learned sources and photographs. The look of the text is good and misprints thin on the ground (I only found two). I have two slight complaints; in Chapter 5, the number of examples of Babylonian numerical calculations is a trifle excessive, and in Chapter 6, the last section on calculating machines, which ends with a short discussion on the modern computer, seems oddly out of place after what has gone before. All in all, however, the book is a welcome read for those interested not just in mathematical results but in mathematics.

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JOSEPH MCLEAN

**Other books received**

**Innumeracy**. By JOHN ALLEN PAULOS. Penguin, London, 1990. £4.99 paperback. This is the paperback version. The original hardback was reviewed in *Mathematical Spectrum* Volume 23, pages 63–64.

**Basic Statistical Computing**. By D. COOKE, A. H. CRAVEN AND G. M. CLARKE. Edward Arnold, London, 1990. Pp. xiii + 178. £12.95 (ISBN 0-34053919-4).

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