THE ACADEMY CORNER

No. 37

Bruce Shawyer

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In this issue, we have the Memorial University Undergraduate Mathematics Competition, written in September 2000.

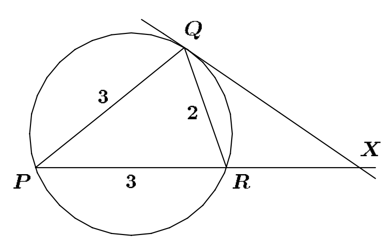
1. (a) Prove that the sum of a positive real number and its reciprocal is greater than or equal to two.

   (b) Let a be a positive real number and let x, y and z be real numbers such that $x + y + z = 0$.
   Prove that $\log_2(1 + a^x) + \log_2(1 + a^y) + \log_2(1 + a^z) \geq 3$.

2. Given any five points inside an equilateral triangle with side length 2, prove that you can always find two points whose distance apart is less than 1 unit.

3. A conical drinking glass is 12 cm deep and has a top diameter of 10 cm. The glass contains some liquid. A sphere of diameter 8 cm is placed in the glass, and it turns out that the liquid exactly fills the volume below the points where the sphere touches the cone. Find the depth of the lowest point of the sphere beneath the highest point of the liquid.

4. Triangle PQR is isosceles, with $PQ = PR = 3$ and $QR = 2$ as shown below. The tangent to the circumcircle at Q meets (the extension of) PR at X as shown. Find the length RX.



5. Sketch $y^3 - x^3 = 1$, being careful to clearly classify all extrema, inflection points and straight line asymptotes in your analysis of the curve.

6. Let a_1, a_2, ..., a_{44} be natural numbers such that $0 < a_1 < a_2 < \cdots < a_{44} \leq 125$. Prove that at least one of the differences $d_i = a_{i+1} - a_i$, $i = 1, \ldots, 43$, occurs at least ten times.

Send me your nice solutions!

Next, we present the 2000 Atlantic Provinces Council on the Sciences Mathematics Competition, written at Dalhousie University, Halifax, Nova Scotia, on 20 October 2000.

2000 APICS Math Competition

Time: 3 hours.

1. N people labelled $1, \ldots, N$ visit a mathematician's home to play the following game. They are given coins labelled $1, \ldots, N$, respectively, where coin i lands "tails", when tossed, with probability p_i, $p_i > 0$. The players throw their coin (in order $1, \ldots, N$) and the winner is simply the first player to get "tails". Now the mathematician has, under the assumption of independence, chosen the p_i's so that the game is "fair", in that every player has the same chance of winning (that is $\frac{1}{N}$, of course).

   (a) If $p_1 = 0.10$ and $N = 5$, determine values of p_2, p_3, p_4, and p_5 such that all five players have the same chance of winning the game.

   (b) Show that it is possible to have a fair game if and only if $p_1 \leq \frac{1}{N}$.

2. Let $S = \{1, 2, 3, \ldots, 3n\}$. We define a sum–3–partition of S to be a collection of n disjoint 3–subsets of S, $A_i = \{a_i, b_i, c_i\}$, $i = 1, \ldots, n$, such that the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is S, and within each triple A_i, some element is the sum of the other two. For example: $\{\{1, 5, 6\}, \{2, 9, 11\}, \{3, 7, 10\}, \{4, 8, 12\}\}$ is a sum–3–partition of $\{1, 2, 3, \ldots, 12\}$.

   (a) Find a sum–3–partition for $\{1, 2, 3, \ldots, 15\}$.

   (b) Prove that there exists no sum–3–partition for $n = 1999$.

3. Two polite but vindictive children play a game as follows. They start with a bowl containing N candies, the number known to both contestants. In turn, each child takes (if possible) one or more candies, subject to the rule that no child may take, on any one turn, more than half of what is left. The winner is not the child who gets most candy, but the last child who gets to take some.

   Thus, if there are 3 candies, the first player may take only one, as two would be more than half. The second player may take one of the remaining candies; and the first player cannot move and loses.

(a) Show that if the game begins with 2000 candies the first player wins.

(b) Show that if the game begins with $999\cdots999$ (2000 9's) candies, the first player wins.

4. Show that, if a, b, c, d, e, f are integers with absolute value less than or equal to 7, and the parabolae

$$\begin{aligned} y &= x^2 + ax + b \\ y &= x^2 + cx + d \\ y &= x^2 + ex + f \end{aligned}$$

enclose a region R of the plane, then the area of R is

(a) rational,

(b) with denominator less than 2000.

5. The three-term geometric progression $(2, 10, 50)$ is such that

$$(2 + 10 + 50) * (2 - 10 + 50) \;=\; 2^2 + 10^2 + 50^2\,.$$

(a) Generalize this (with proof) to other three-term geometric progressions.

(b) Generalize (with proof) to geometric progressions of length n.

6. Solve for all real x, $y > 0$: $2xy\ln(x/y) < x^2 - y^2$.

7. Without calculator or elaborate computation, show that

$$3^{2701} \equiv 3 \pmod{2701}\,.$$

NOTE: $2701 = 37 \times 73$.

8. An isosceles triangle has vertex A and base BC. Through a point D on AB, we draw a perpendicular to meet BC extended at E, such that $AD = CE$. If DE meets AC at F, show that the area of triangle ADF is twice that of triangle CFE.

Send me your nice solutions.

We also welcome contributions to this corner.

THE OLYMPIAD CORNER

No. 210

R.E. Woodrow

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As a first set of problems, we give the questions of the 1996-1997 Estonian Mathematical Olympiads, Final Round of the National Olympiad. Thanks go to Richard Nowakowski, Canadian Team Leader, for collecting them at the IMO in Argentina.

ESTONIAN MATHEMATICAL OLYMPIADS 1996-1997
Final Round of the National Olympiad

1. On a square table of size $3n \times 3n$ each unit square is coloured either red or blue. Each red square not lying on the edge of the table has exactly five blue squares among its eight neighbours. Each blue square not lying on the edge of the table has exactly four red squares among its eight neighbours. Find the number of red and blue squares on the table.

2. Find

(a) all quadruples of positive integers (a, k, l, m) for which the equality $a^k = a^l + a^m$ holds;

(b) all 5–tuples of positive integers (a, k, l, m, n) for which the equality $a^k = a^l + a^m + a^n$ holds.

3. Prove that, for any real numbers x and y, the following inequality holds:

$$x^2 + y^2 + 1 \; > \; x\sqrt{y^2+1} + y\sqrt{x^2+1}\,.$$

4. In a triangle ABC the values of $\tan\angle A$, $\tan\angle B$ and $\tan\angle C$ relate to each other as $1 : 2 : 3$. Find the ratio of the lengths of the sides AC and AB.

5. There are n points ($n \geq 3$) in the plane, no three of which are collinear. Is it always possible to draw a circle through three of these points so that it has no other given points

(a) in its interior? (b) in its interior nor on the circle?

6. For positive integers m, n denote $T(m,n) = \gcd\left(m, \frac{n}{\gcd(m,n)}\right)$.

(a) Prove that there exist infinitely many pairs of integers (m, n) such that $T(m, n) > 1$ and $T(n, m) > 1$.

(b) Does there exist a pair of integers (m, n) such that $T(m, n) = T(n, m) > 1$?

7. A function f satisfies the condition:

$$f(1) + f(2) + \cdots + f(n) \;=\; n^2 f(n)$$

for any positive integer n. Given that $f(1) = 999$, find $f(1997)$.

Next we give the "exercises" of the Composition de Mathématiques (Classe terminale S) from France. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina, for obtaining them for our use.

COMPOSITION DE MATHÉMATIQUES 1997
Classe Terminale S
Durée : 5 heures

1. On a placé un jeton sur chaque sommet d'un polygone régulier à 1997 côtés. Sur chacun de ces jetons est inscrit un entier relatif, la somme de ces entiers relatifs étant égale à 1. On choisit un sommet de départ et on parcourt le polygone dans le sens trigonométrique en ramassant les jetons au fur et à mesure tant que la somme des entiers inscrits sur les jetons ramassés est strictement positive.

Peut-on choisir le sommet de départ de façon à ramasser tous les jetons? Si oui, combien y a-t-il de choix possibles?

2. Une capsule spatiale a la forme du solide de révolution délimité par une sphère de centre O, de rayon R, et un cône de sommet O qui rencontre cette sphère selon un cercle de rayon r.

Quel est le volume maximal d'un cylindre droit contenu dans cette capsule, le cylindre et la capsule ayant le même axe de révolution?

3. C est un cube d'arête 1 et p est la projection orthogonale sur un plan. Quelle est la valeur maximale de l'aire de $p(C)$?

4. Etant donné un triangle ABC, on note a, b, c les longueurs de ses côtés et m, n, p les longueurs de ses médianes. Pour tout réel α strictement positif, on définit le réel $\lambda(\alpha)$ par la relation:

$$a^\alpha + b^\alpha + c^\alpha \;=\; (\lambda(\alpha))^\alpha (m^\alpha + n^\alpha + p^\alpha)\,.$$

(1°) Calculer $\lambda(2)$.

(2°) Calculer la limite de $\lambda(\alpha)$ lorsque α tend vers 0.

(3°) A quelle condition portant sur a, b, c le réel $\lambda(\alpha)$ est-il indépendant de α?

5. Dans le plan, soient A et B deux points distincts. Pour tout point C extérieur à la droite (AB), on note G l'isobarycentre du triangle ABC et I le centre de son cercle inscrit.

(1°) Soit α un réel tel que $0 < \alpha < \pi$. Quel est l'ensemble Γ des points C tels que $\left(\overrightarrow{CA}, \overrightarrow{CB}\right) = \alpha + 2k\pi$, k étant un entier? Lorsque C décrit Γ, montrer que G et I décrivent deux arcs de cercle que l'on précisera.

(2°) On suppose désormais $\pi/3 < \alpha < \pi$. Comment doit-on choisir C dans Γ pour que la distance GI soit minimale?

(3°) On note $f(\alpha)$ la distance minimale GI de la question précédente. Expliciter $f(\alpha)$ en fonction de $a = AB$ et α. Déterminer la valeur maximale de $f(\alpha)$ lorsque α décrit $\left]\frac{\pi}{3}, \pi\right[$. [Ed. A more familiar notation is $(\frac{\pi}{3}, \pi)$.]

Next we turn to readers' solutions to problems of the Turkish Team Selection Examination for the 37$^{\text{th}}$ IMO [1999 : 73].

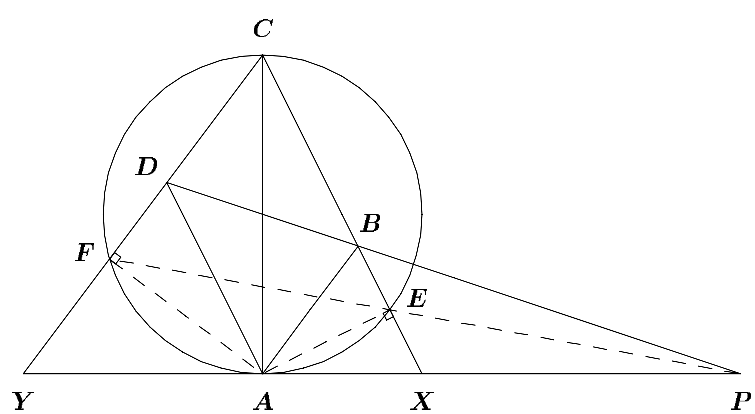
1. Let $\prod_{n=1}^{1996}(1 + nx^{3n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$ where a_1, a_2, $\ldots$, a_m are non-zero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

Solution by Mohammed Aassila, Strasbourg, France.

In general, if $k = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_p}$ then $a_k = \alpha_1 + \ldots + \alpha_p$. In particular $a_{1996} = 10 + 9 + 8 + 7 + 6 + 3 + 2 = 45$.

2. In a parallelogram $ABCD$ with $m(\hat{A}) < 90°$, the circle with diameter $[AC]$ intersects the lines CB and CD at E and F besides C, and the tangent to this circle at A intersects the line BD at P. Show that the points P, F, E are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since AC is a diameter we get $\angle AEC = \angle AFC = 90^\circ$. Since AP is tangent to the circle with diameter AC we have $AP \perp AC$.

Let X and Y be the intersections of AP with BC and CD, respectively.

Since $\angle XAC = 90^\circ$ and $AE \perp XC$, we have $AX^2 = XE \cdot XC$ and $AC^2 = EC \cdot XC$, so

$$AX^2 : AC^2 \;=\; XE : EC\,. \tag{1}$$

Similarly, we have

$$AY^2 : AC^2 \;=\; YF : FC\,. \tag{2}$$

It follows that, from (1) and (2),

$$\frac{XE}{EC}\cdot\frac{CF}{FY} \;=\; \frac{AX^2}{AC^2}\cdot\frac{AC^2}{AY^2} \;=\; \frac{AX^2}{AY^2}\,. \tag{3}$$

Since $AB \parallel CY$ and $AD \parallel XC$, we get

$$\frac{AX}{AY} \;=\; \frac{XB}{BC} \quad\text{and}\quad \frac{AX}{AY} \;=\; \frac{CD}{DY}\,.$$

Hence, we have

$$\frac{AX^2}{AY^2} \;=\; \frac{XB}{BC}\cdot\frac{CD}{DY}\,. \tag{4}$$

By Menelaus' Theorem for triangle CXY, we have

$$\frac{YP}{PX}\cdot\frac{XB}{BC}\cdot\frac{CD}{DY} \;=\; 1\,. \tag{5}$$

Thus, we obtain from (3), (4) and (5),

$$\frac{YP}{PX}\cdot\frac{XE}{EC}\cdot\frac{CF}{FY} \;=\; 1\,.$$

Therefore, P, F, E are collinear, by the converse of Menelaus' Theorem.

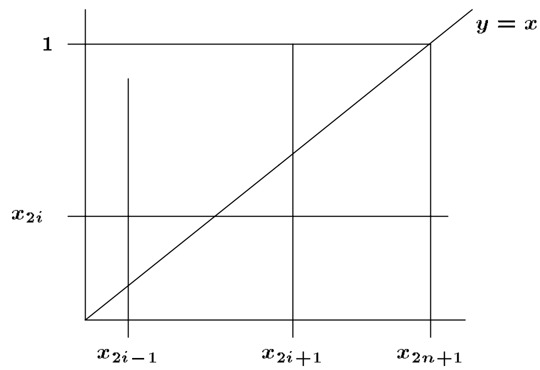
3. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ with $x_{i+1} - x_i \le h$ for $1 \le i \le 2n$, show that

$$\frac{1-h}{2} \;<\; \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \;\le\; \frac{1+h}{2}\,.$$

Solution by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.

The problem is equivalent to showing that

$$\left|\sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) - \frac{1}{2}\right| \;<\; \frac{h}{2}\,.$$



Now $\sum\limits_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) - \frac{1}{2}$ is the difference between the area of the rectangles formed by the four lines $x = x_{2i-1}$, $x = x_{2i+1}$, $y = 0$ and $y = x_{2i}$ and the triangle formed by the three lines $x = 0$, $y = 1$, $x = y$. The area contained in the rectangles but not in the triangle (respectively contained in the triangle but not in the rectangles) is a union of triangles of total base less than 1 and height $\leq h$. Hence, we have the required inequality.

4. In a convex quadrilateral $ABCD$, $\text{Area}(ABC) = \text{Area}(ADC)$ and $[AC] \cap [BD] = \{E\}$. The parallels from E to the line segments $[AD]$, $[DC]$, $[CB]$, $[BA]$ intersect $[AB]$, $[BC]$, $[CD]$, $[DA]$ at the points K, L, M, N, respectively. Compute the ratio

$$\frac{\text{Area}(KLMN)}{\text{Area}(ABCD)}.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

We denote the area of polygon $A_1A_2 \cdots A_n$ by $[A_1A_2 \cdots A_n]$. Let B', D' be the feet of perpendiculars from B, D to AC, respectively. (See figure below.)

Since $[ABC] = [ADC]$, we get $BB' = DD'$, so that

$$BE : ED \;=\; BB' : DD' \;=\; 1 : 1\,.$$

Thus, we have $BE = ED$. Since $EK \parallel DA$, we have

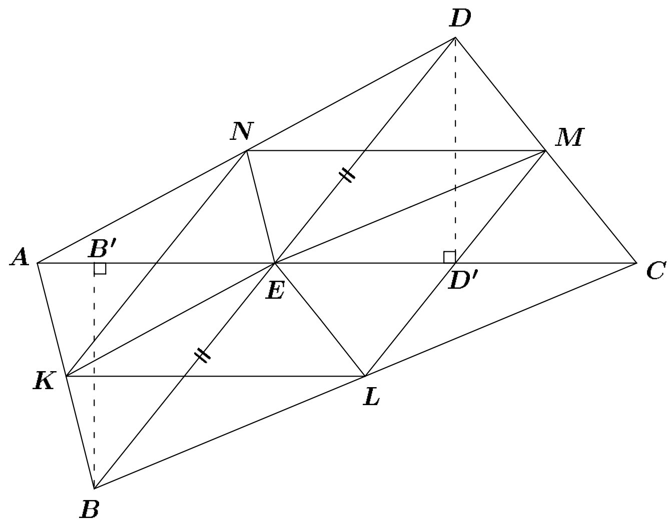
$$BK : KA = BE : ED = 1 : 1\,.$$

Hence, $BK = KA$. Similarly, we have

$$BL \;=\; LC\,, \quad CM \;=\; MD \quad \text{and} \quad DN \;=\; NA\,.$$

In triangle ABD, note that K, E and N are the mid-points of AB, BD and DA, respectively. Thus,

$$[ENK] \;=\; \tfrac{1}{4}[ABD]\,.$$



Similarly, we have

$$[EKL] = \tfrac{1}{4}[ABC], \quad [ELM] = \tfrac{1}{4}[BCD] \quad \text{and} \quad [EMN] = \tfrac{1}{4}[CDA].$$

Hence,

$$\begin{aligned} [KLMN] &= [ENK] + [EKL] + [ELM] + [EMN] \\ &= \tfrac{1}{4}[ABD] + \tfrac{1}{4}[ABC] + \tfrac{1}{4}[BCD] + \tfrac{1}{4}[CDA] \\ &= \tfrac{1}{4}([ABD] + [BCD]) + \tfrac{1}{4}([ABC] + [CDA]) \\ &= \tfrac{1}{4}[ABCD] + \tfrac{1}{4}[ABCD] \\ &= \tfrac{1}{2}[ABCD]. \end{aligned}$$

Therefore, we obtain

$$\frac{[KLMN]}{[ABCD]} = \frac{1}{2}.$$

6. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence (x_n) satisfying the condition

$$\lim_{n\to\infty} (ax_{n+1} - bx_n) = 0 \tag{1}$$

zero?

Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give the solution by Aassila.

If $b > a$, then, for $x_n = (b/a)^n$, we have $ax_{n+1} - bx_n = 0$, but $\lim_{n\to\infty}(b/a)^n = \infty$.

If $b = a$, we have the well-known example:

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$

for which we have $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, but $\{x_n\}$ does not converge to a finite limit.

Let us assume that $b < a$. We shall prove that

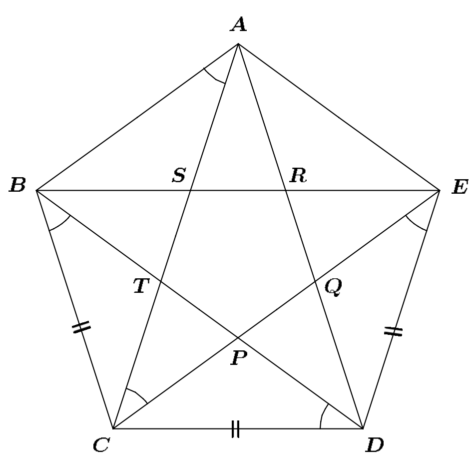
$$\underline{\lim} x_n = \overline{\lim} x_n = 0.$$

Denote by m (resp. M), the $\underline{\lim}$ (resp. $\overline{\lim}$) of $\{x_n\}$. By (1), we have $M \leq \frac{b}{a}m$, and since $m \leq M$, we deduce that $M \leq \frac{b}{a}M$, and consequently that m, $M \leq 0$. Similarly, by (1), we have that $M\frac{b}{a} \leq m$, and since $M \geq m$, we deduce that $m \geq \frac{b}{a}m$, and consequently that m, $M \geq 0$; whence, $m = M = 0$ and $\lim_{n\to\infty} x_n = 0$.

Next we turn to solutions to problems of the Australian Mathematical Olympiad 1996 [1999 : 74–75].

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.

Solution by Toshio Seimiya, Kawasaki, Japan.



As shown in the figure, we label the intersections of diagonals.

Since $BE \parallel CD$ and $AC \parallel ED$, $SCDE$ is a parallelogram, so that $CS = DE = CB$.

Hence, $\angle CBE = \angle CBS = \angle CSB = \angle DEB$.

Since $\angle CBE = \angle DEB$ and $BC = ED$, it follows that $BCDE$ is an isosceles quadrilateral, so that B, C, D, E are concyclic. Since $AB \parallel CE$, $AC \parallel DE$, we have

$$\angle BAC \;=\; \angle ACE \;=\; \angle CED \;=\; \angle CBD \;=\; \angle BDC\,.$$

Thus, A, B, C, D are concyclic, giving that A, B, C, D, E are concyclic. Since $BC \parallel AD$, we get $AB = CD$, and since $AC \parallel ED$, we have $AE = CD$. Therefore, $AB = BC = CD = DE = EA$.

Consequently, corresponding minor arcs AB, BC, CD, DE and EA are equal, and also corresponding inscribed angles are equal. We put $\angle BAC = \alpha$. Then we have

$$\angle EAB \;=\; \angle ABC \;=\; \angle BCD \;=\; \angle CDE \;=\; \angle DEA \;=\; 3\alpha\,.$$

2. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that

$$\frac{p\left(\frac{1}{2}\right) + p\left(-\frac{1}{2}\right)}{p(0)} \;=\; 1000\,.$$

Find the value of $\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}$.

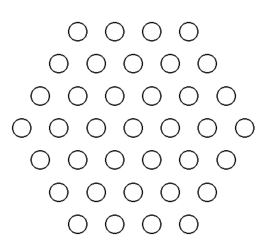
Solutions by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec; by Michel Bataille, Rouen, France; by Andrew Blinn, Western Canada High School, Calgary, Alberta; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. All the solutions were essentially the same. We give Bataille's exposition.

Note that $p(0)$ is supposed non-zero, so that r_1, r_2, r_3 are non-zero. Let $p(x) = ax^3 + bx^2 + cx + d$.

The hypothesis is: $\left(\frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d\right) + \left(-\frac{a}{8} + \frac{b}{4} - \frac{c}{2} + d\right) \;=\; 1000d$; that is, $b = 1996d$. Now,

$$\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1} \;=\; \frac{r_3 + r_1 + r_2}{r_1r_2r_3} \;=\; \frac{-b/a}{-d/a} \;=\; \frac{b}{d} \;=\; 1996\,.$$

3. A number of tubes are bundled together into a hexagonal form:



The number of tubes in the bundle can be 1, 7, 19, 37 (as shown), 61, $91, \ldots$. If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69. What is the 69^{th} number in the sequence which ends in 69?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The number is 1417969. Note first that the sequence $\{a_n\}$ is given by the formula

$$a_n \;=\; 1 + 6(1+2+3+\cdots+(n-1)) \;=\; 1 + 3n(n-1)$$

where $n \geq 1$. Clearly, a_n ends in 69 if and only if $100 \mid a_n - 69$, and $a_n \geq 69$; that is,

$$100 \mid 3n(n-1) - 68 \tag{1}$$

where $n \geq 6$.

In particular, $5 \mid 3n(n-1) - 68$ and so $3n(n-1) \equiv 68 \equiv 3 \pmod 5$. Since $(3,5) = 1$, we have $n(n-1) \equiv 1 \pmod 5$, which holds if and only if $n \equiv 3 \pmod 5$. Hence, $n = 5k+3$ for some integer $k \geq 1$. Then

$$n(n-1) \;=\; (5k+3)(5k+2) \;=\; 25k^2 + 25k + 6$$

and (1) becomes $100 \mid 75k^2 + 75k - 50$, or $4 \mid 3k^2 + 3k - 2$.

Thus, we have $3k(k+1) \equiv 2 \equiv 6 \pmod 4$. Since $(3,4) = 1$, we have $k(k+1) \equiv 2 \pmod 4$, which holds if and only if $k \equiv 1,\ 2 \pmod 4$. Thus, $k = 4t+1$ or $4t+2$, and $n = 20t+8$ or $20t+13$ for some non-negative integer t.

Conversely, if $n = 20t+8$, then $3n(n-1) - 68 = 1200t^2 + 900t + 100$, and if $n = 20t+13$, then $3n(n-1) - 68 = 1200t^2 + 1500t + 400$. In both cases, (1) holds. Therefore, we conclude that a_n ends in 69 if and only if $n = 20t+8$ or $20t+13$ for $t = 0,\ 1,\ 2,\ \ldots$. To find the 69^{th} such number, we put $t = 34$ into $n = 20t+8$ to obtain $n = 688$ and $a_{688} = 1 + 3 \times 688 \times 687 = 1417969$.

4. For which positive integers n can we rearrange the sequence 1, 2, $\ldots$, n to a_1, $a_2, \ldots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2$, $3, \ldots, n$?

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use Wang's solution.

The required permutations exist if and only if n is even. First of all, since $a_1 - 1 \neq 0$, we have $a_1 > 1$ and so $|a_1 - 1| = a_1 - 1$. Partition $S = \{1,\ 2,\ \ldots,\ n\}$ has $S = A \cup B$ where $A = \{i \in S \mid a_i - i \geq 0\}$ and $B = \{j \in S \mid a_j - j < 0\}$. Since $a_i - i = a_1 - 1$ for all $i \in A$ and

$a_j - j = 1 - a_1$ for all $j \in B$, we have

$$\sum_{i \in A}(a_i - i) + \sum_{j \in B}(a_j - j) \;=\; |A|(a_1 - 1) + |B|(1 - a_1) \;=\; (|A| - |B|)(a_1 - 1)\,.$$

Since

$$\sum_{i \in A}(a_i - i) + \sum_{j \in B}(a_j - j) \;=\; \sum_{k \in S} a_k - \sum_{k \in S} k \;=\; 0\,,$$

we conclude that

$$(|A| - |B|)(a_1 - 1) \;=\; 0\,,$$

and so, that $|A| = |B|$. Hence, $n = |S| = |A| + |B| = 2|A|$, showing that n must be even.

Conversely, if $n = 2k$ is even, then the permutation σ below clearly has the described property:

$$\sigma \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-1 & 2k \\ 2 & 1 & 4 & 3 & \cdots & 2k & 2k-1 \end{pmatrix}\,;$$

that is, $\sigma = (1,2)(3,4)\cdots(2k-1,2k)$ is the product of k disjoint transpositions.

Remark. Permutations with the described property are not unique in general; for example, when $n = 6$, we could take $\sigma = (14)(25)(36)$.

5. Let a_1, a_2, $\dots$, a_n be real numbers and s, a non-negative real number, such that

(i) $a_1 \le a_2 \le \cdots \le a_n$; (ii) $a_1 + a_2 + \cdots + a_n = 0$;

(iii) $|a_1| + |a_2| + \cdots + |a_n| = s$.

Prove that

$$a_n - a_1 \;\ge\; \frac{2s}{n}\,.$$

Solution by Michel Bataille, Rouen, France.

The result is clear when $s = 0$, so we will suppose $s > 0$. This implies that at least one of the a_i's is non-zero. Since $a_1 + a_2 + \cdots + a_n = 0$, the a_i's cannot all be non-negative, or all be non-positive. Thus:

$$a_1 \;=\; \min(a_i) < 0 \quad \text{and} \quad a_n \;=\; \max(a_i) > 0\,.$$

There exists $k \in \{1, 2, \dots, n-1\}$ such that $a_1 \le a_2 \le \cdots \le a_k \le 0 < a_{k+1} \le \cdots \le a_n$. Then $a_{k+1} + \cdots + a_n = -(a_1 + a_2 + \cdots + a_k) = |a_1| + |a_2| + \cdots + |a_k| = s - (|a_{k+1}| + \cdots + |a_n|) = s - (a_{k+1} + \cdots + a_n)$. Hence, $a_{k+1} + \cdots + a_n = \frac{s}{2} = -(a_1 + a_2 + \cdots + a_k)$. For $i \in \{1, 2, \dots, k\}$, $j \in \{k+1, \dots, n\}$, we have: $a_j - a_i \le a_n - a_1$ ($= \delta$, say) so that $a_n - a_1 \le \delta$, $a_n - a_2 \le \delta$, $\dots$, $a_n - a_k \le \delta$. Adding up, we get:

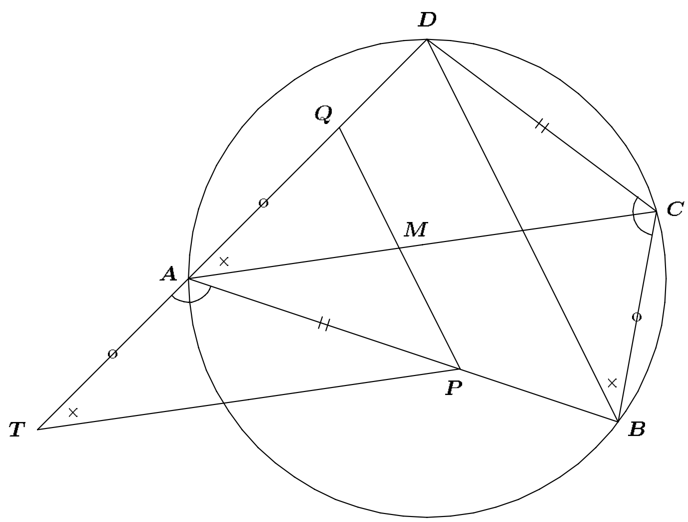
$ka_n + \frac{s}{2} \leq k\delta$. Substituting successively $a_{n-1}, \ldots, a_{k+1}$ for a_n, we get similarly:

$$ka_{n-1} + \frac{s}{2} \leq k\delta, \ldots, ka_{k+1} + \frac{s}{2} \leq k\delta.$$

Adding up again, we obtain: $k(a_n + \cdots a_{k+1}) + (n-k)\frac{s}{2} \leq (n-k)k\delta$ or $k\frac{s}{2} + (n-k)\frac{s}{2} \leq (n-k)k\delta$, which leads to $\delta \geq \frac{ns}{2k(n-k)}$. But $(n-k)+k = n$; hence, $k(n-k) \leq \frac{n^2}{4}$ and $\frac{ns}{2k(n-k)} \geq \frac{2s}{n}$. Thus, $\delta \geq \frac{2s}{n}$.

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD, respectively, such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the mid-point of PQ.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let T be a point on AD produced beyond A such that

$$AT = BC.$$

Since $AT = BC$, $AP = CD$ and $\angle TAP = \angle TAB = \angle BCD$, we get $\triangle ATP \equiv \triangle CBD$, so that

$$\angle ATP = \angle CBD.$$

Since $\angle CBD = \angle CAD$, we have

$$\angle ATP = \angle CAD.$$

Thus, $TP \parallel AC$; that is, $TP \parallel AM$.

Hence, we get $PM : MQ = TA : AQ = BC : AQ = 1 : 1$. Therefore, $PM = MQ$.

7. For each positive integer n, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let k be a positive integer and $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$. Prove that n_i is a prime for $i = 1, 2, \ldots$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bornsztein's solution.

It is well known that

$$n \text{ is a prime if and only if } \sigma(n) - n = 1\,.$$

Let k be a positive integer.

We have $\sigma(n_i) - n_i = k$, for $i = 1, 2, \ldots$. Suppose that none of the n_i are prime. Then we have $n_i = p_i q_i$ with $2 \leq q_i \leq p_i < n_i$ and $n_i > p_i \geq \sqrt{n_i}$.

We deduce that $k = \sigma(n_i) - n_i \geq 1 + \sqrt{n_i}$ for $i = 1, 2, \ldots$. But (n_i) is an infinite increasing sequence of positive integers, so $\lim_{i\to\infty} 1 + \sqrt{n_i} = \infty$, a contradiction to $k \geq 1 + \sqrt{n_i}$.

Thus, there is i_0 with n_{i_0} a prime. Then, $k = \sigma(n_{i_0}) - n_{i_0} = 1$, and, for each i, $\sigma(n_i) - n_i = 1$, making n_i a prime.

Remark by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

It is possible to have $\sigma(m) - m = \sigma(n) - n$ for $m \neq n$. For example $\sigma(26) - 26 = 16 = \sigma(12) - 12$. However, it is by no means clear whether it is true that, given any $k \in \mathbb{N}$, there are k distinct natural numbers n_1, n_2, $\ldots$, n_k such that $\sigma(n_i) - n_i = \sigma(n_j) - n_j$ for all i, $j = 1, 2, \ldots, k$.

8. Let f be a function that is defined for all integers and takes only the values 0 and 1. Suppose f has the following properties:

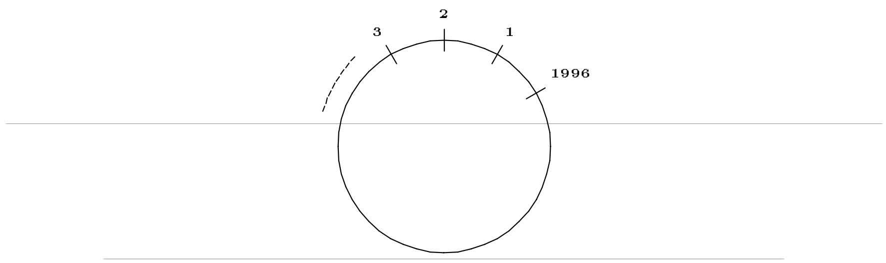
(i) $f(n + 1996) = f(n)$ for all integers n;

(ii) $f(1) + f(2) + \cdots + f(1996) = 45$.

Prove that there exists an integer t such that $f(n + t) = 0$ for all n for which $f(n) = 1$ holds.

Solutions by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's write-up.

Let C be a circle with radius $\frac{1996}{2\pi}$. The real line can be wrapped around the circle in the counterclockwise sense so that the integer i is mapped to the point A_i of the circle.



Since the circumference of the circle is 1996, if two numbers differ by 1996, they map to the same point of the circle, (that is, $A_{i+1996} = A_0$ for $i \in \mathbb{Z}$.)

From (i), the integers are represented on C by the points $A_1, \ldots, A_{1996}$, and we may colour those A_i such that $f(i) = 1$. By (ii), and since $f(n) \in \{0, 1\}$, there are exactly 45 coloured points.

If $i < j$ and $i, j \in \{1, \ldots, 1996\}$, A_i and A_j determine two arcs of lengths $j - i$ and $i - j + 1996$ (which may be equal). The possible lengths are $1, 2, 3, \ldots, 1995$. But there are 45 coloured points, and distinct coloured points determine at most $2\binom{45}{2} = 1980$ different lengths.

Then, for all i, there is $t \in \{1, \ldots, 1995\}$ such that if A_i is coloured then A_{i+t} is not coloured (with subscripts read modulo 1996).

That is, for all i, if $f(i) = 1$, then $f(i + t) = 0$.

That completes the *Olympiad Corner* for this issue. Send me your Olympiad contest materials and your nice solutions!

Quickie

In rectangle $ABCD$ with side a, arcs of circles of radii a are centred at A and B, and meet at E outside the rectangle. Find the circumradius of triangle CDE. (Thanks to Ed Barbeau for the idea!)

Solution

Draw $\triangle CDF$ congruent to $\triangle BAE$ ($BE \| CF$, $AE \| DF$).
Then $EF = BC = FC = FD = a$, giving that F is the circumcentre.
Thus, the circumradius is a.

BOOK REVIEWS

ALAN LAW

Twenty Years Before the Blackboard,
by Michael Steuben with Diane Sandford,
published by the Mathematical Association of America, 1998,
ISBN 0-88385-525-9, softcover, 174+ pages, $23.50 (U.S.)
Reviewed by **Nicholas Buck**, *College of New Caledonia, Prince George, BC.*

What makes a good mathematics teacher? The answer is not clear to me, but most students and teachers could make reasonable suggestions about what they think makes a good mathematics teacher. In my fourteen years teaching, many ideas — from a wide variety of sources — about how to improve (my) teaching have floated across my desk. The amusing thing is that (not surprisingly) these suggestions are not always consistent with each other. This just illustrates what I believe to be the case, and that is that there is no single model of a good mathematics (or any other kind of) teacher and we (or at least certain education theorists) should stop trying to construct one. One student may enjoy and respond positively to one style of teaching, while the next student may be completely turned off by the same style. One style may be appropriate for certain subjects or levels of teaching and quite inappropriate for other subjects and levels.

Take the question of how much history of mathematics to include in a mathematics course. I know mathematicians who think the history should be left out completely; and still others who think that it is important to place the material in historical context and connect it with other intellectual activities of the same era. I am in the second group, and Steuben's book will appeal to anyone of like mind, but especially those who have recently begun teaching at the high school level. That is not to say that more experienced teachers at other levels could not benefit from and enjoy reading (at least parts of) this book, but they probably already have their favourite tried-and-true anecdotes, puns, etc., (but don't oblique references to Monty Python get some strange looks these days?).

Steuben's book is a treasure trove of ideas, suggestions, problems, examples, anecdotes, humour, and other miscellanea of a mathematical nature that he has successfully incorporated into his teaching. Not many will agree with or use everything in the book, but there is something here for anyone who thinks that good teaching should be entertaining and thought provoking and not just a sequence of compartmentalized topics with intended learning outcomes, course objectives and skill sets. We all know how so many modern students — products (victims?) of the video and computer age — need to be entertained while they are learning (as Gary Trudeau has said, "they insist on a certain comfort level."). This book is full of little ideas one could try in order to catch the attention of these modern students. Of course, as one presses ahead beyond high school and lower division post-secondary mathematics courses, the lecturer necessarily focuses on the careful formulation of definitions and on proofs of theorems. This book becomes less useful at that level.

In his famous book *How To Solve It*, Georg Pólya comments amusingly on the "traditional mathematics professor of popular legend." He describes the eccentric, doddering, absent-minded professor we all have (or at least should have) experienced (enjoyed?) somewhere along the way. This professor would surely not meet the prototype of the good maths teacher promoted by many. But, as Pólya remarks, "After all, you can learn something from this traditional mathematics professor. Let us hope that the mathematics teacher from whom you cannot learn anything will not become traditional."

Archimedes — What Did He Do Besides Cry Eureka,
by Sherman Stein,
published by the Mathematical Association of America, 1999, Classroom Resource Materials Series.
ISBN 0-88385-718-9, softcover, 155+x pages, $24.95 (U.S.).
Reviewed by **C.L. Kaller**, *Kelowna, BC.*

Professor Stein states in the introductory remarks that his goal in writing the book is "*to make what I view as Archimedes' most mathematically significant discoveries accessible to the busy people of the mathematical community, whom I think of as anyone who recognizes the equation of a parabola.*"

In pursuing his goal, Stein gives an admirable presentation, in very attractive and readable form, of the accomplishments of Archimedes during the Golden Age of Greece. Without going into any extensive peripheral details which could otherwise distract the reader from the essential results of the original Archimedean investigations, the book presents the logical steps used by this great Greek mathematician when presenting his results so long ago. Starting with some brief notes on the life and times of Archimedes, the author outlines the geometric reasoning underlying the theories of the lever, centres of gravity and floating bodies, as well as finding the sums of series, areas of plane figures, volumes of solids and the very important estimation of π. The reader cannot help but be fascinated by the simple, yet powerful, ingenuity exhibited by Archimedes in his research; even as we have today much more technically advanced tools to arrive at many of these same results based on calculus, what must grip the admiration of the reader is, as Stein states, "how much Archimedes accomplished with the limited tools at his disposal some 22 centuries ago."

This small book certainly meets the MAA criterion of one of the 'Classroom Resource Materials', and should be found a useful addition to any senior high school, college and even university library. With the appendices and references which permit more knowledgeable readers to pursue in greater depth the presentations in the book, the author has achieved the goal he set for himself in making better known the great contributions of Archimedes to our mathematical culture.

The only small criticisms this reviewer has are in the usual ambiguities of geometric notations. Single capital letters, P, Q, R, $\ldots$ designate points; lower case letters designate numerical quantities. This distinction is not always followed. And when a line joining points P and Q is denoted by PQ, it would be more clear if overbars were used to indicate unambiguously the distance from P to Q; that is, $\overline{PQ}$. This would overcome the ambiguity when encountering PQ^2 or $PQ \cdot A$. But most of us have learned to tolerate and work within such inconsistencies as we do geometric work.

Letter to the Editor

Regarding: Edward T. H. Wang's note *"Some Bounds for $\phi(n)\sigma(n)$"* [2000 : 280].

It may be of interest for the audience of ***Crux Mathematicorum with Mathematical Mayhem*** to learn that some of the results of this note can be found elsewhere, already. (This indeed does no harm to their neat presentation within this note!)

- In [1], Theorem 329, the inequality $\frac{6}{\pi^2} < x_n < 1$, where $x_n = \frac{\phi(n)\sigma(n)}{n^2}$ is given ($n > 1$).

  Furthermore, $\liminf\limits_{n\to\infty} x_n = \frac{6}{\pi^2}$ and $\limsup\limits_{n\to\infty} x_n = 1$.

- By the way, the inequality $\phi(n)\sigma(n) \le n^2 - 1$ ($n > 1$) was again posed as a problem in [2].

- Finally I would like to draw the attention to the recent referential source [3] where in the first few chapters there are collected very man inequalities for almost all types of arithmetical functions. (Many of the results stem from sources not easily accessible at all. This fact indeed increases the value of this book.)

  Let me quote two examples from § I.6. (*Inequalities* by J. Sándor):

  With $\psi(n) = n \cdot \prod\limits_{p|n}\left(1 + \frac{1}{p}\right)$; that is, *Dedekind's arithmetical function*, the following hold:

  - $\psi(n)^{\phi(n)} < n^n$ for $n > 1$, and
  - $\psi(n)^{\phi(n)} > n^n$ if all prime factors of n are greater than or equal to 5.

References.

[1] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford, New York 1996, 1965.

[2] A. Makowski, Problem 3932, *Mathesis* 69 (1960), 65.

[3] D.S. Mitrinovic, J. Sándor and B. Crstici, *Handbook of Number Theory*, Dordrecht, Boston, London.

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A Spatial Problem Solved with Stereographic Projection

Shay Gueron and Oran Lang

Stereographic projection is a transformation in space, used frequently in complex analysis. It can also be interpreted as the spatial analog of inversion. In this paper, we demonstrate how the elementary properties of the stereographic projection can be used for solving a geometric problem.

The following problem was proposed to the 1999 International Mathematical Olympiad (IMO) Jury:

Problem: A finite set F of $n \geq 3$ points in space (the plane) is called **completely symmetric** if it satisfies the following condition: for every two distinct points A, B from F, the perpendicular bisector plane (the perpendicular bisector) of the segment AB is a plane (an axis) of symmetry for F. Find all completely symmetric sets.

The IMO Jury looked for a relatively simple geometric question for the IMO paper, and decided to use only the planar version of the problem. The answer in the planar variant is not surprising: any completely symmetric set consists of the vertices of a regular polygon. The straightforward generalization to space would read: a completely symmetric set consists either of the vertices of a regular polygon, or the vertices of a regular polyhedron. Surprisingly however, this is not the correct answer for the 3D–version: a completely symmetric nonplanar set consists of the vertices of a regular tetrahedron or a regular octahedron. The other regular polyhedrons, namely the cube, the regular dodecahedron and the regular icosahedron, are ruled out. This counterintuitive result is not easy to see, particularly when looking at the problem strictly as one of space geometry. We show here how stereographic projection helps to reduce the problem to a planar one, and thus, makes it easier to understand.

Stereographic Projection — Definitions and Properties

Let σ be a sphere, S a point on σ, π the tangent plane at S, and NS a diameter of σ. A stereographic projection through N is the one-to-one transformation from $\sigma \setminus \{N\}$ to π in which every point A on σ is mapped to the intersecting point A' of NA and π (see Figure 1).

Interpretation: Stereographic projection can be interpreted as the spatial analog of inversion in the plane.

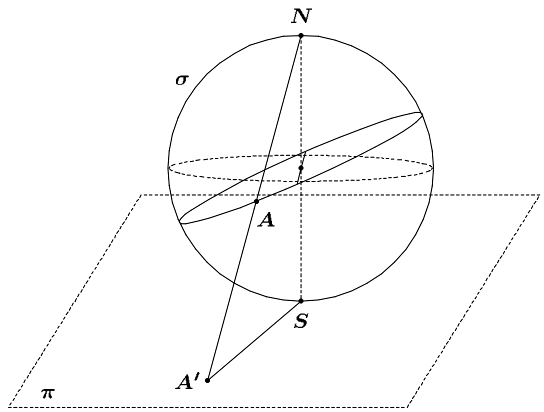

Figure 1.

Explanation: Let A be a point and A' its stereographic projection. Since $\angle NAS = \angle NSA' = 90°$ we have $\triangle NAS \sim \triangle NSA'$, and therefore $|NA| \cdot |NA'| = |NS|^2$. Since NS is a diameter of σ, the product $|NA| \cdot |NA'|$ does not depend on the choice of A. As, by definition, N, A, A' are collinear, the stereographic projection can be viewed as an inversion in space with centre N and radius $|NS|$.

Two elementary properties of the stereographic projection are used below (for proofs see [1]).

P1. For every two points A, B, and their projections A', B', we have

$$|A'B'| = |AB| \cdot \frac{|NS|^2}{|NA| \cdot |NB|}.$$

P2. A circle on σ, passing through N, is mapped under the stereographic projection to a line in π. A circle on σ, not passing through N, is mapped under the stereographic projection to a circle in π.

Inversion in the Plane — Definitions and Properties

In this paper we use the following properties of inversion in the plane (for proofs see, for example, [1]):

Definition: Let (O, r) be a circle in the plane π, with centre O and radius r. Inversion of the plane, in (O, r) is the transformation $I : \pi \setminus \{O\} \to \pi \setminus \{O\}$ that maps every point $A \neq O$ in π to the point $A' = I(A)$ lying on the ray OA, and satisfying $|OA| \cdot |OA'| = r^2$. The point O is called the centre of the inversion.

Basic Properties of Inversion

I1. $I(I(A)) = A$.

I2. A line passing through O is invariant under the inversion.

I3. A circle passing through O is mapped to a line that does not pass through O. A line not passing through O is mapped to a circle passing through O.

I4. A circle not passing through O is mapped to a circle not passing through O.

I5. Let D be a disc whose boundary is the circle C, and suppose that C passes through the inversion centre O. Denote the image of C under the inversion by $l = I(C)$ (the line l does not pass through O). Then $I(D)$ is the half-plane, with respect to l, that does not contain O. Conversely, if D is a half-plane with respect to a line l not passing through O, then $I(D)$ is the interior (disc) of the circle $C = I(l)$.

Lemma (*the stereographic projection of great circles on* σ) [2]:Let c be a great circle on σ, not passing through N. Let Γ be the cylinder touching the sphere σ along c. Draw a line through N, parallel to the axis of Γ, and let P' be its intersection with π. Then the stereographic projection of c is the circle c' lying in π, with the centre P' and the radius $|NP'|$.

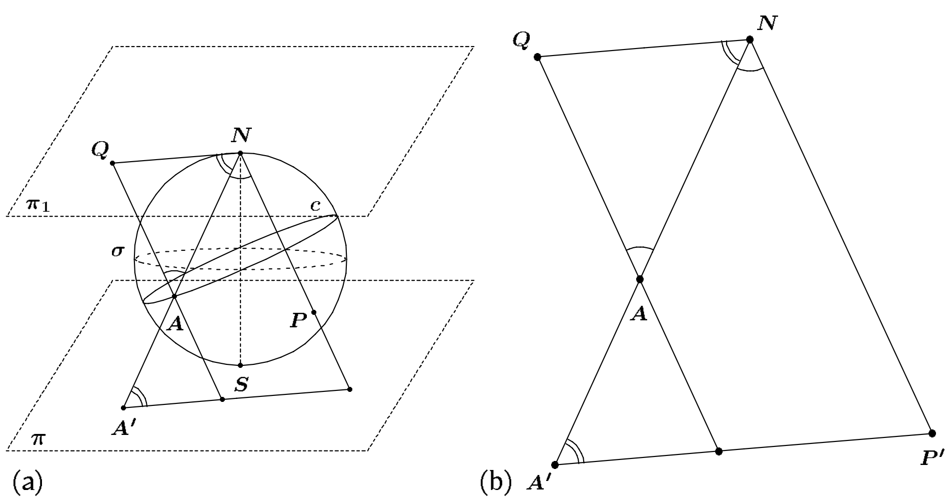

Figure 2.

Proof: Draw the plane π_1 parallel to π through N, as in Figure 2(a). Clearly, π_1 is tangent to σ at N. Let A be an arbitrary point on c. Draw the line NAA' and a line parallel to the generator of Γ through A. Denote its intersection with π_1 by Q. Since $NP' \| AQ$, the points N, P', A, Q and

A' are coplanar, and therefore $QN \| A'P'$ (see Figure 2(b)). It follows that $\angle QNA = \angle NA'P'$, $\angle NAQ = \angle A'NP'$. Since QA and QN are two tangents to σ, they have equal lengths and therefore $\angle QNA = \angle QAN$, implying that $\angle NA'P' = \angle A'NP'$ and thus, $|A'P'| = |NP'|$.

Solution of the Problem by Stereographic Projection

We are now ready to tackle our problem of finding all completely symmetric sets in space, and will do this by using stereographic projection.

Suppose that F is a completely symmetric set. We first show that all the points in F lie on a sphere. Let O be the barycentre of F. Since for every $A, B \in F$, the reflection through the perpendicular bisector plane of AB, $\tau_{A,B}$, maps F to itself, O is invariant under this reflection and therefore lies in $\tau_{A,B}$. It follows that $|OB| = |OA|$; that is, the distance $|OA|$ is the same for all points $A \in F$. Denoting $|OA| = a$, it follows that all the points of F lie on a sphere σ whose centre is O and whose radius is a. Every plane of symmetry of F passes through O, and thus, cuts a great circle from σ.

We now choose a point N in F. For each point $P \in F$, $P \neq N$, the perpendicular bisector plane $\tau_{N,P}$ is a plane of symmetry of F. Therefore, for every other point $A \in F$, there exists a point B symmetric to A with respect to $\tau_{N,P}$. If A lies in $\tau_{N,P}$, then $B = A$. Further, for every pair of points $A, B \in F$, the plane $\tau_{A,B}$ is a plane of symmetry of F, and thus, there exists a point $P \in F$ symmetric to N with respect of this plane (it is possible that $P = N$).

Now consider the stereographic projection from N that maps F to F' in π. Using the lemma, we see that the image of the great circle which is cut from σ by the plane of symmetry $\tau_{N,P}$, is a circle in π whose centre is P' and whose radius is NP'. Since $AB \| NP$, the points A, B, N, P are coplanar, and therefore by Property **P2** the projections A', B', P' are collinear. Furthermore, A' and B' lie on the same side of P', $|PB| = |NA|$ and $|PA| = |NB|$ (see Figure 3). Using Property **P1** and noting that $|NS| = 2a$, we obtain

$$\begin{aligned} |P'A'| \cdot |P'B'| &= |PA| \cdot \frac{(2a)^2}{|NA| \cdot |NP|} \cdot |PB| \cdot \frac{(2a)^2}{|NB| \cdot |NP|} \\ &= \frac{(2a)^4}{|NP|^2} = |NP'|^2 \,. \end{aligned}$$

Therefore, the points of σ in the plane of symmetry $\tau_{N,P}$ are mapped to a circle c_P in the plane, and for every point A' in F' there exists a point B' in F' such that the inversion with respect of c_P takes A' to B'.

The conclusion is that **every plane of symmetry of F defines a circle of inversion of F' with centre in F'**. In other words, F' is invariant under the inversion with respect to this circle.

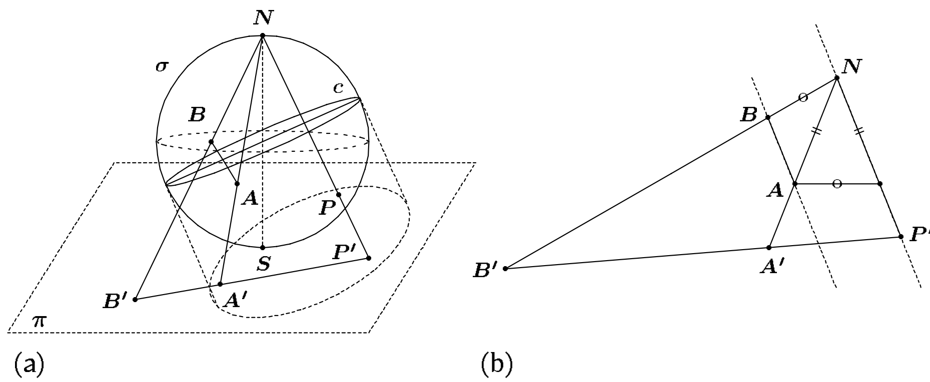

Figure 3.

Our problem is therefore reduced to finding all sets F' *in the plane* that satisfy:

Property **1**: For every two points A', $B' \in F'$, there exists a circle of inversion of F' that maps A' to B'. If this circle passes through S, it is the perpendicular bisector of $A'B'$, and considered as a circle with infinite radius. In all other cases, the centre of the circle of inversion belongs to F'.

Property **2**: For every point $A' \in F'$, there exists a circle of inversion of F' with centre A'. Using the lemma, we see that the radius of this circle is $|NA'| = \sqrt{(2a)^2 + |SA'|^2}$, where a is a constant and S is a given point in the plane (not necessarily belonging to F).

Note that for every finite set F' in the plane satisfying Properties **1** and **2**, we can construct a completely symmetric set F in space by constructing a sphere with radius a, tangent to the plane at S and performing a stereographic projection.

If the points of F' are collinear, the points of F lie on a circle, and this implies that we are dealing with the planar case of the problem. It is now easy to show that the points of F are the vertices of some regular polygon (for example, by proving that F is invariant under appropriate rotation about the centre of the circle). We now assume that the points of F' are not collinear and denote the vertices of their convex hull by $A'_1, A'_2, \ldots, A'_n$. Property **1** implies that there exists a circle of inversion that maps A'_1 to A'_2. By definition, no points of F' on the line $A'_1A'_2$ lie outside the segment $A'_1A'_2$. Therefore, the circle of inversion is the perpendicular bisector of $A'_1A'_2$ passing through S, and it is a line of symmetry of F'. Similarly, for every $1 \le i \le n$, the circle of inversion that maps A'_i to A'_{i+1} is the perpendicular bisector of the segment $A'_iA'_{i+1}$ and passes through S. Recall now that the composition of two reflections through the lines l_1 and l_2 is a rotation centred at the intersecting point of l_1 and l_2. It follows that F', as well as its convex hull, is invariant under some rotation about S. Thus, $A'_1A'_2 \cdots A'_n$ is a regular polygon with centre S.

We now denote the circle of inversion of F' centred at A'_i by c_i, the radius of this circle by r_i, and the inversion of P' with respect of this circle by $I_i(P')$. Clearly, $r_i \leq |A'_i A'_{i+1}|$ since otherwise $I_i(A'_{i+1})$ falls outside the segment $A'_i A'_{i+1}$. The solution of our problem is now obtained by considering the following three cases for n.

A. The case $n \geq 5$.

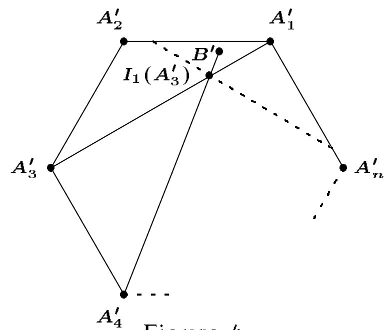

Figure 4.

Since $A'_1 A'_2 \cdots A'_n$ is a regular polygon, it is inscribed in a circle passing through A'_1. Therefore (by I3) $I_1(A'_2 A'_3 \cdots A'_n)$ are $n-1$ collinear points, and the line connecting them intersects the segments $A'_1 A'_2$ and $A'_1 A'_n$. By Property **1**, there exists a circle of inversion that maps A'_4 to $I_1(A'_3)$. The perpendicular bisector of $A'_4 I_1(A'_3)$ does not pass through S, and since $A'_1 A'_2 \cdots A'_n$ is the convex hull of F', no point on the ray $I_1(A'_3)A'_4$ lies beyond A'_4. It follows that the centre of this circle lies on the ray $A'_4 I_1(A'_3)$ beyond $I_1(A'_3)$. We denote this point by B' (by Property **1**, $B' \in F$). Consequently (by I5), $I_1(B')$ lies outside the convex hull, which is a contradiction. Therefore, we must have $n < 5$.

B. The case $n = 4$.

We first prove that $r_i = |A'_i A'_{i+1}|$. Assume that $r_i < |A'_i A'_{i+1}|$ (we showed above that $r_i \leq |A'_i A'_{i+1}|$). This implies that $I_i(A'_{i+1})$ lies on the segment $A'_i A'_{i+1}$. By Property **1**, there exists a circle of inversion of F' that maps $I_i(A'_{i+1})$ to A'_{i+2}. Since there are no points outside the segment $I_i(A'_{i+1})A'_{i+2}$, and their perpendicular bisector does not pass through S, we arrive at a contradiction, which implies that $r_i = |A'_i A'_{i+1}|$.

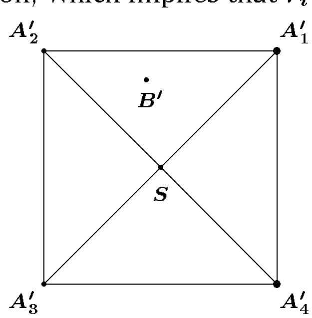

Figure 5.

Denote $|A'_i S|$ by r. Then $|A'_i S| \cdot |A'_i A'_{i+2}| = 2r^2 = |A'_i A'_{i+1}|^2$, which implies that $S = I_i(A'_{i+2})$. Hence $S \in F'$. Suppose that F' contains a point B' different from A'_1, A'_2, A'_3, A'_4 and S. With no loss of generality, we assume that B' is inside the $\triangle A'_1 A'_2 S$. It follows (by I5) that $I_1(B')$ is outside the line $A'_2 A'_3$, and this contradicts the definition of $A'_1 A'_2, \ldots, A'_n$ as the defining points of a convex hull. Consequently, the set F' consists of the four vertices of the square $A'_1 A'_2 A'_3 A'_4$ and its centre S.

Now, by Property 2, $\sqrt{2}r = \sqrt{r^2 + (2a)^2}$, which implies that $r = 2a$. Finally, simple calculations for the original set F, (for example, calculating $|NA_1|$, $|SA_1|$, $|A_1 A_2|$, $\cdots$) show that the points in F are the vertices of a regular octahedron.

C. The case $n = 3$.

As in the previous case, $r_i = |A'_i A'_{i+1}|$. Assume that there is another point B' in F'. Then (by I5) $I_1(B')$ lies outside the circle (and hence the triangle) $A'_1 A'_2 A'_3$, and we arrive at a contradiction. We conclude that F' consists of the vertices of an equilateral triangle $A'_1 A'_2 A'_3$. We denote $r = |SA_i|$, and then $\sqrt{3}r = \sqrt{r^2 + (2a)^2}$, which implies that $r = \sqrt{2}a$. By simple calculations we can now conclude that F consists of the vertices of a regular tetrahedron.

Acknowledgments

We thank an anonymous referee for a careful review and helpful suggestions.

References

[1] R.A. Johnson, Advanced Euclidean Geometry. Dover (1960).

[2] N.N. Yaglom, Geometric Transformations III, MAA series (1973).

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THE SKOLIAD CORNER

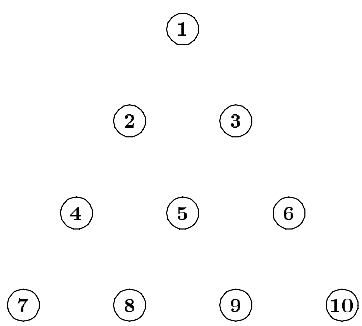
No. 50

R.E. Woodrow

Last issue we gave some of the solutions to the problems of the Preliminary Round of the British Columbia Colleges Senior High School Contest for 2000. Here, thanks to Jim Totten, The University College of the Cariboo, one of the organizers, are the rest of the "official" solutions.

6. While 10 pin bowling, Sam left 3 pins standing which formed the vertices of an equilateral triangle. How many such equilateral triangles are possible?

(a) 15 **(b)** 14 **(c)** 12 **(d)** 10 **(e)** none of these



Solution. The answer is **(a)**. Number the pins as shown in the diagram on the left. There is then one large equilateral triangle with 4 pins on a side, namely the one with vertices numbered $(1, 7, 10)$. There are also three equilateral triangles with 3 pins on a side, namely the ones whose vertices are numbered $(1, 4, 6)$, $(2, 7, 9)$, and $(3, 8, 10)$. The equilateral triangles with 2 pins on a side come in two distinct orientations, one with a single vertex above the horizontal base and one with a single vertex below the horizontal base. For the first type we have six such: $(1, 2, 3)$, $(2, 4, 5)$, $(3, 5, 6)$, $(4, 7, 8)$, $(5, 8, 9)$ and $(6, 9, 10)$. For the second type we have only three such: $(2, 3, 5)$, $(4, 5, 8)$, and $(5, 6, 9)$. This gives us a total of 13 equilateral triangles so far. However, there are two others which are skewed somewhat to the edges of the outer triangle: $(2, 6, 8)$ and $(3, 4, 9)$, which gives us a total of 15 equilateral triangles.

7. If I place a 6 cm $\times 6$ cm square on a triangle, I can cover up to 60% of the triangle. If I place the triangle on the square, I can cover up to $\frac{2}{3}$ of the square. What is the area, in cm^2, of the triangle?

(a) $22\frac{4}{5}$ **(b)** 24 **(c)** 36 **(d)** 40 **(e)** 60

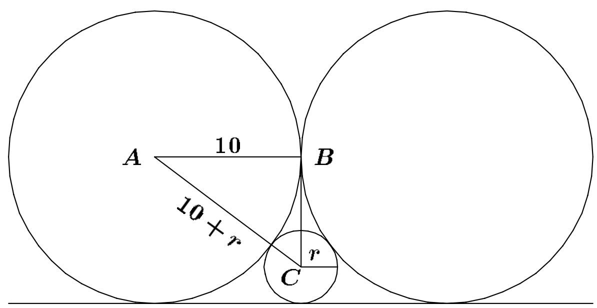
Solution. The answer is **(d)**. The critical idea here is to recognize that when the square covers as much of the triangle as possible, the triangle will also cover as much of the square as possible, and that at this point the amount of triangle covered is the same as the amount of square covered. Let A be the area of the triangle. Then $0.6A = \frac{2}{3} \cdot 36$, or $A = 40$ cm^2.

8. Two circles, each with radius 10 cm, are placed so that they are tangent to each other and a straight line. A smaller circle is nestled between them so that it is tangent to the larger circles and the line. What is the radius, in centimetres, of the smaller circle?

(a) $\sqrt{10}$ (b) 2.5 (c) $\sqrt{2}$ (d) 1 (e) none of these

Solution. The answer is (b). Let A be the centre of one of the large circles, let B be the point of contact between the two large circles and let C be the centre of the small circle. Then $AB \perp BC$, $AC = 10 + r$ and $BC = 10 - r$. From the Theorem of Pythagoras, we have

$$\begin{aligned} (10+r)^2 &= 10^2 + (10-r)^2\,, \\ 100 + 20r + r^2 &= 100 + 100 - 20r + r^2\,, \\ 40r &= 100\,, \\ r &= 2.5\,. \end{aligned}$$



9. Arrange the following in ascending order:

$$2^{5555}\,,\quad 3^{3333}\,,\quad 6^{2222}\,.$$

(a) 2^{5555}, 3^{3333}, 6^{2222}
(b) 2^{5555}, 6^{2222}, 3^{3333}
(c) 6^{2222}, 3^{3333}, 2^{5555}
(d) 3^{3333}, 6^{2222}, 2^{5555}
(e) 3^{3333}, 2^{5555}, 6^{2222}

Solution. The answer is (e). See #10 of the Junior paper – solution on [2000 : 347].

10. Given that $0 < x < y < 20$, the number of integer solutions (x, y) to the equation $2x + 3y = 50$ is:

(a) 25 (b) 16 (c) 8 (d) 5 (e) 3

Solution. The answer is (e). Clearly y must be even in order to get integer solutions. The largest possible value for y is 16 since we must have $x > 0$. When $y = 16$, we have $x = 1$. Thus, $(x, y) = (1, 16)$ is a solution. Let us consider successively smaller (even) values for y: $(x, y) = (4, 14)$, $(7, 12)$, $(10, 10)$, etc. However, the solution $(10, 10)$ and any further ones do not satisfy $y > x$. Thus, we are left with the solutions $(x, y) = (1, 16)$, $(4, 14)$, and $(7, 12)$.

11. Suppose A, B, and C are positive integers such that

$$\frac{24}{5} = A + \frac{1}{B + \frac{1}{C+1}}\,.$$

The value of $A + 2B + 3C$ equals:

(a) 9 (b) 12 (c) 15 (d) 16 (e) 20

Solution. The answer is (c). Since B and C are positive integers, we see that $B + 1/(C+1) > 1$, whence its reciprocal is smaller than 1. Therefore, A must represent the integer part of $24/5$; that is, $A = 4$. Then we have

$$\frac{4}{5} = \frac{1}{B + \frac{1}{C+1}} \quad \text{or} \quad \frac{5}{4} = B + \frac{1}{C+1}\,.$$

For exactly the same reason as above we see that B must be the integer part of $5/4$; that is, $B = 1$. Then $1/4 = 1/(C+1)$, which implies that $C = 3$. Then

$$A + 2B + 3C = 4 + 2(1) + 3(3) = 15\,.$$

12. A box contains m white balls and n black balls. Two balls are removed randomly without replacement. The probability one ball of each colour is chosen is:

(a) $\frac{mn}{(m+n)(m+n-1)}$ (b) $\frac{mn}{(m+n)^2}$ (c) $\frac{2mn}{(m+n-1)(m+n-1)}$ (d) $\frac{2mn}{(m+n)(m+n-1)}$ (e) $\frac{m(m-1)}{(m+n)(m+n-1)}$

Solution. The answer is (d). The number of ways of choosing 2 balls (without replacement) from a box with $m+n$ balls is $\binom{m+n}{2}$. The number of ways of choosing 1 white ball is m, the number of ways of choosing 1 black ball is n. Thus, the probability that one ball of each colour is chosen is:

$$\frac{mn}{\binom{m+n}{2}} = \frac{mn}{\frac{(m+n)!}{2(m+n-2)!}} = \frac{mn}{\frac{(m+n)(m+n-1)}{2}} = \frac{2mn}{(m+n)(m+n-1)}\,.$$

13. If it takes x builders y days to build z houses, how many days would it take q builders to build r houses? Assume these builders work at the same rate as the others.

(a) $\frac{qry}{xz}$ (b) $\frac{ryz}{qx}$ (c) $\frac{qz}{rxy}$ (d) $\frac{xyr}{qz}$ (e) $\frac{rz}{qxy}$

Solution. The answer is (d). Since x builders build z houses in y days, we see that x builders build z/y houses per day, or 1 builder builds $z/(xy)$ houses per day. Hence q builders build $qz/(xy)$ houses per day. Let s be the number of days needed for q builders to build r houses. Then q builders build r/s houses per day. Equating these we get $qz/(xy) = r/s$, whence $s = rxy/(qz)$ days.

14. If $x^2 + xy + x = 14$ and $y^2 + xy + y = 28$, then a possible value for the sum of $x + y$ is:

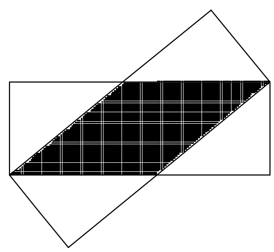
(a) -7 (b) -6 (c) 0 (d) 1 (e) 3

Solution. The answer is (a). Adding the two given equations gives:

$$\begin{aligned} x^2 + 2xy + y^2 + x + y &= 42\,; \\ \text{that is,} \qquad (x+y)^2 + (x+y) - 42 &= 0\,, \\ (x+y-6)(x+y+7) &= 0\,. \end{aligned}$$

Thus, $x + y = 6$ or $x + y = -7$.

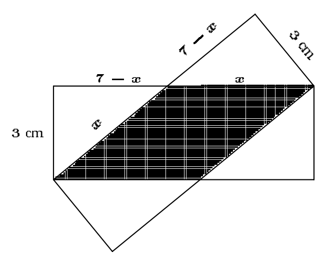
15. Two congruent rectangles each measuring 3 cm $\times 7$ cm are placed as in the figure. The area of overlap (shaded), in cm^2, is:



(a) $\frac{87}{7}$ (b) $\frac{29}{7}$ (c) $\frac{20}{7}$ (d) $\frac{21}{2}$ (e) none of these

Solution. The answer is (a). All the unshaded triangles in the diagram below are right-angled and thus are congruent. By the Theorem of Pythagoras we have

$$\begin{aligned} x^2 &= (7-x)^2 + 3^2 = 49 - 14x + x^2 + 9\,, \\ 14x &= 58 \quad \text{or} \quad x = \frac{29}{7}\,. \end{aligned}$$



The area of the shaded parallelogram is $3x = \frac{87}{7}$ cm^2.

That completes the *Skoliad Corner* for this issue. Send me suitable contest materials and suggestions for the future of the *Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M65 3G3. The electronic address is

mayhem@math.toronto.edu

Editorial

This issue brings to completion another year and another volume. It also represents my last issue as Editor of Mathematical Mayhem. I have been at this post for a long time now, and in that time, I have seen Mayhem go through several changes, most notably the merger with Crux Mathematicorum, but its spirit and purpose has remained the same.

At this point, I would like to thank all the current members of the Mayhem staff: Cyrus Hsia, Adrian Chan, Donny Cheung, Jimmy Chui, and David Savitt, as well as all the other staff members who are no longer present. Your hard work and dedication over the years have been greatly appreciated.

I would also like to thank Bruce Shawyer, who has had to put up with his share of garbled LaTeX and missed deadlines, for his patience and guidance, and in addition the CMS, without whom Mayhem would not be where it is today.

Most of all, however, I would like to thank all the people who have read and contributed to Mayhem. We greatly value your input and support, and you are the reason that Mayhem exists.

Taking over from me next year will be Shawn Godin, a high school math and physics teacher in Ottawa. His enthusiasm and vision will be a welcome addition to Mayhem, and I wish him the best of luck. Thanks again to all. I once said I believed that Mayhem was a good thing. I still do.

Naoki Sato
Mayhem Editor

Shreds and Slices

Another Proof of the Ellipse Theorem

Nikolaos Dergiades, Thessaloniki, Greece, has kindly provided a short proof of the final theorem concerning the mid-point ellipse in "Ellipses in Polygons" [2000 : 361]. We begin by recalling the theorem.

Theorem. Let ABC be a triangle in the plane, and let z_1, z_2, and z_3 be the complex numbers corresponding to the vertices A, B and C, respectively. Let $p(z) = (z - z_1)(z - z_2)(z - z_3)$.

Then the foci of the mid-point ellipse ω are the roots of the equation $p'(z) = 0$, and the centre of ω is the root of the equation $p''(z) = 0$.

Proof. Let F be one of the foci of the mid-point ellipse ω, and let z be the complex number corresponding to it. Let $\alpha = z - z_1$, $\beta = z - z_2$ and $\gamma = z - z_3$.

Let K, M and N be the mid-points of sides BC, AC and AB, respectively (See Figure 1). It is known (see [1]) that the median AK passes through the centre of ω. Similarly, so do medians BM and CN. Thus, the centre of ω is the centroid of ABC, with corresponding complex number $(z_1 + z_2 + z_3)/3$, the root of the equation $p''(z) = 0$. In order to prove that the foci are the roots of the equation $p'(z) = 0$, it is sufficient to prove that $\alpha\beta + \beta\gamma + \gamma\alpha = 0$.

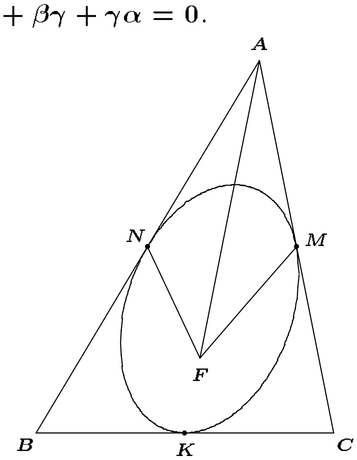


Figure 1

It is known (see [2], [3]) that AF is the bisector of $\angle MFN$, and hence,

$$\operatorname{Arg} \frac{\frac{z_1+z_2}{2} - z}{z_1 - z} = \operatorname{Arg} \frac{z_1 - z}{\frac{z_1+z_3}{2} - z},$$

or

$$\operatorname{Arg} \frac{\alpha+\beta}{2\alpha} = \operatorname{Arg} \frac{2\alpha}{\alpha+\gamma},$$

which implies that

$$\frac{\frac{\alpha+\beta}{\alpha}}{\frac{\alpha}{\alpha+\gamma}} = \frac{(\alpha+\beta)(\alpha+\gamma)}{\alpha^2}$$

is real, or the number $\dfrac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha^2}$ is real. Similarly, the numbers

$$\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\beta^2} \quad \text{and} \quad \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\gamma^2}$$

are real. Hence, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, because otherwise, two of the numbers α^2, β^2 and γ^2 would have the same argument, and hence, two of α, β and γ

would have arguments that differ by $180°$, which would mean that F lies on a side of ABC, which is impossible. ■

The previous method can be applied to the general case. As above, let an inscribed ellipse be tangent to the sides BC, AC and AB at K, M and N, respectively, and let

$$k = \frac{BK}{KC}, \quad m = \frac{CM}{MA} \quad \text{and} \quad n = \frac{AN}{NB}.$$

Then we know that AK, BM and CN are concurrent, and Ceva's Theorem gives $kmn = 1$. The complex numbers corresponding to K, M and N are

$$\frac{z_2 + kz_3}{1+k}, \quad \frac{z_3 + mz_1}{1+m} \quad \text{and} \quad \frac{z_1 + nz_2}{1+n},$$

respectively, and since AF is the bisector of $\angle MFN$, we have

$$\operatorname{Arg} \frac{\frac{z_1+nz_2}{1+n} - z}{z_1 - z} = \operatorname{Arg} \frac{z_1 - z}{\frac{z_3+mz_1}{1+m} - z}, \qquad \text{or}$$

$$\operatorname{Arg} \frac{\alpha + n\beta}{(1+n)\alpha} = \operatorname{Arg} \frac{(1+m)\alpha}{m\alpha + \gamma}.$$

As in our previous calculations, we have that the complex numbers

$$\frac{\alpha\beta + k\alpha\gamma + km\beta\gamma}{\alpha^2}, \quad \frac{\alpha\beta + k\alpha\gamma + km\beta\gamma}{\beta^2} \quad \text{and} \quad \frac{\alpha\beta + k\alpha\gamma + km\beta\gamma}{\gamma^2}$$

are real. Hence, $\alpha\beta + k\alpha\gamma + km\beta\gamma = 0$, so that the foci of the ellipse are the roots of the equation

$$(1 + k + km)z^2 - \big(z_1 + z_2 + k(z_1 + z_3) + km(z_2 + z_3)\big)\, z \\ + z_1z_2 + kz_1z_3 + kmz_2z_3 = 0,$$

and the centre of the ellipse is

$$\frac{z_1 + z_2 + k(z_1 + z_3) + km(z_2 + z_3)}{2(1 + k + km)}.$$ ■

References

[1] Cours de Mathématiques Elémentaires – Exercices de Géométrie par F.G–M, Théorème 872–2081, Cinquième Edition.

[2] Cours de Mathématiques Elémentaires – Exercices de Géométrie par F.G–M, Théorème 889 II–2113, Cinquième Edition.

[3] Poncelet, Traitè de propèrtiès projective des figures, vol. I no 461, 469.

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from the previous issue be submitted in time for issue 8 of 2001.

High School Solutions

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H261. Solve for x:

$$\left(\sqrt{7-\sqrt{48}}\right)^x+\left(\sqrt{7+\sqrt{48}}\right)^x=14\,.$$

Solution by Andrei Simion, Brooklyn Technical HS, Brooklyn, NY, USA.

First, we notice that $\sqrt{7-\sqrt{48}}=\frac{1}{\sqrt{7+\sqrt{48}}}$.

Thus, let $t=\left(\sqrt{7-\sqrt{48}}\right)^x$, and our equation becomes:

$$\begin{array}{rrcl} & t+1/t & = & 14 \\ \Longrightarrow & t^2-14t+1 & = & 0 \\ \Longrightarrow & t & = & 7\pm\sqrt{48}\,. \end{array}$$

If $t=7-\sqrt{48}$, then $x/2=1$, leading to $x=2$.

If $t=7+\sqrt{48}$, then $x/2=-1$, leading to $x=-2$.

Also solved by EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario.

Edward Wang generalized the question to solving

$\left(\sqrt{a-\sqrt{a^2-1}}\right)^x+\left(\sqrt{a+\sqrt{a^2-1}}\right)^x=2a$, *showing that* $x=\pm 2$ *are the only solutions.*

H262. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Solve the equation $x - \dfrac{x}{\sqrt{x^2-1}} = \dfrac{91}{60}$.

Solution. Let $x = \sec A$, so that $x/\sqrt{x^2-1} = \csc A$. Our equation becomes

$$\begin{array}{rrcl} & \sec A - \csc A & = & \frac{91}{60} \\ \Longrightarrow & 60(\sin A - \cos A) & = & 91 \sin A \cos A \\ \Longrightarrow & 3600(1 - 2\sin A\cos A) & = & 8281 \sin^2 A \cos^2 A\,. \end{array}$$

Letting $p = 2\sin A\cos A = \sin 2A$,

$$\begin{array}{rrcl} & 14400(1-p) & = & 8281p^2 \\ \Longrightarrow & 8281p^2 + 14400p - 14400 & = & 0 \\ \Longrightarrow & (169p - 120)(49p + 120) & = & 0 \\ \Longrightarrow & p & = & \frac{120}{169}\,. \end{array}$$

The value of $p = -120/49$ is rejected, since $p = \sin 2A \geq -1$. Continuing,

$$\begin{array}{rrcl} & \sin 2A & = & \frac{120}{169} \\ \Longrightarrow & \cos 2A & = & \pm\sqrt{1 - \left(\frac{120}{169}\right)^2} \;=\; \pm\frac{119}{169} \\ \Longrightarrow & 2\cos^2 A - 1 & = & \pm\frac{119}{169} \\ \Longrightarrow & \cos^2 A & = & \frac{25}{169} \text{ or } \frac{144}{169} \\ \Longrightarrow & \cos A & = & \pm\frac{5}{13}\,,\ \pm\frac{12}{13}\,. \end{array}$$

Since $x = \sec A$, then $x = \pm 13/5$ or $\pm 13/12$. Checking reveals that only $x = -13/12$ and $x = 13/5$ are valid solutions.

Also solved by EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta.

Note: Both of the other solutions consisted of rationalizing the equation and solving a quartic.

H263. Let ABC be an acute-angled triangle such that $a = 14$, $\sin B = 12/13$, and c, a, b form an arithmetic sequence (in that order). Find $\tan A + \tan B + \tan C$.

Solution. Let $c = 14 - t$ and $b = 14 + t$. Since ABC is acute-angled and $\sin B = 12/13$, then $\cos B = 5/13$. Using the Cosine Law,

$$\begin{array}{rrcl} & (14+t)^2 & = & (14-t)^2 + 14^2 - 2(14-t)(14)\cos B \\ \Longrightarrow & 196 + 28t + t^2 & = & 196 - 28t + t^2 + 196 - 28(14-t)(5/13) \\ \Longrightarrow & 56t - 196 & = & -28(14-t)(5/13) \\ \Longrightarrow & 13(2t-7) & = & 5(t-14) \\ \Longrightarrow & 26t - 91 & = & 5t - 70 \\ \Longrightarrow & 21t & = & 21 \\ \Longrightarrow & t & = & 1\,. \end{array}$$

Thus, ABC is a 13-14-15 triangle.

Let D be the foot of the altitude from A. Since AD is an altitude, $\sin B = AD/13 = 12/13$, so that $AD = 12$. Thus, $BD = 5$ and $DC = 9$. Also, $\tan B = 12/5$, and $\tan C = 12/9 = 4/3$.

Now,

$$\begin{aligned} \tan A &= \tan(180^\circ - B - C) \\ &= -\tan(B+C) \\ &= -\tfrac{\tan B + \tan C}{1 - \tan B \tan C} \\ &= -\tfrac{\frac{12}{5}+\frac{4}{3}}{1-\frac{16}{5}} = \tfrac{16}{11} \\ \Longrightarrow \quad \tan A + \tan B + \tan C &= \tfrac{856}{165}\,. \end{aligned}$$

Also solved by ANDREI SIMION, Brooklyn Technical HS, Brooklyn, NY, USA.

H264. Find all values of a such that $x^3 - 6x^2 + 11x + a - 6 = 0$ has exactly three integer solutions.

Solution by Lino Demasi, student, St. Ignatius HS, Thunder Bay, Ontario; and Andrei Simion, Brooklyn Technical HS, Brooklyn, NY, USA.

Let d, e, and f be the roots of the cubic $x^3 - 6x^2 + 11x + a - 6 = 0$. Hence,

$$\begin{aligned} d + e + f &= 6\,, \\ de + ef + fd &= 11\,, \\ def &= 6 - a\,. \end{aligned}$$

Notice that

$$\begin{aligned} (d+e+f)^2 &= d^2 + e^2 + f^2 + 2(de + ef + fd) \\ \Longrightarrow \quad 36 &= d^2 + e^2 + f^2 + 22 \\ \Longrightarrow \quad d^2 + e^2 + f^2 &= 14\,. \end{aligned}$$

Clearly, $-3 \le d, e, f \le 3$. But since $d + e + f = 6$, the only solution is: $\{d, e, f\} = \{1, 2, 3\}$, which implies that $6 - a = 1(2)(3) = 6$, so that $a = 0$.

Also solved by EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A237. Show that for any sequence of decimal digits that does not begin with 0, there is a Fibonacci number whose decimal representation begins with this sequence. (The Fibonacci sequence is the sequence F_n generated by the initial conditions $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

Solution. Let $\{x\}$ denote the fractional part of x; that is, $\{x\} = x - \lfloor x \rfloor$. We begin by proving a useful lemma.

Lemma: If α is irrational, then given any range $[a, b] \subseteq [0, 1)$, there exist infinitely many natural numbers n such that $\{n\alpha\} \in [a, b]$.

Proof: Let $c = b - a$. It suffices to prove that there is a natural number m such that $0 < \{m\alpha\} < c$, since then there must be an integer multiple of $\{m\alpha\}$ inside every interval of the form $[k + a, k + b]$, where k is a natural number. This gives us our infinitely many n.

We may partition the interval $[0, 1)$ into a finite number of intervals of size $< c$. Thus, by the pigeonhole principle, we must have two distinct natural numbers m_1 and m_2 such that $\{m_1\alpha\}$ and $\{m_2\alpha\}$ lie within the same interval. Then, $0 \leq \{|m_1 - m_2|\alpha\} = |\{m_1\alpha\} - \{m_2\alpha\}| < c$. If $\{|m_1 - m_2|\alpha\} = 0$, then $|m_1 - m_2|\alpha$ must be an integer, meaning that α is rational, contradiction. Thus, $0 < \{|m_1 - m_2|\alpha\} < c$, as desired.

Now, we proceed to prove the main result. We use Binet's formula,

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

which, for sufficiently large n, is the integer nearest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n .$$

Given a positive integer s, represented by a sequence of decimal digits not beginning with 0, we want to find an integer $k \geq 0$ such that

$$s \cdot 10^k \leq F_n < (s+1) \cdot 10^k .$$

Since both $s \cdot 10^k$ and $(s+1) \cdot 10^k$ are integers, this is equivalent to

$$s \cdot 10^k \leq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n < (s+1) \cdot 10^k .$$

On taking base 10 logarithms and some rearranging, we obtain the equivalent condition

$$k + \log_{10}(s\sqrt{5}) \ \le n \ \cdot \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \ < \ k + \log_{10}((s+1)\sqrt{5})\,.$$

By our lemma, we can find an infinite number of n such that

$$\{\log_{10}(s\sqrt{5})\} \ \le \ \left\{n \cdot \log_{10}\left(\frac{1+\sqrt{5}}{2}\right)\right\} \ < \ \{\log_{10}((s+1)\sqrt{5})\}\,.$$

(In the case that there is an integer between $\log_{10}\left(s\sqrt{5}\right)$ and $\log_{10}\left((s+1)\sqrt{5}\right)$, we can replace the lower bound, $\{\log_{10}\left(s\sqrt{5}\right)\}$, with 0.) This means that there will be one such n which is sufficiently large so that

$$k + \log_{10}\left(s\sqrt{5}\right) \ \le \ n \cdot \log_{10}\left(\frac{1+\sqrt{5}}{2}\right) \ < \ k + \log_{10}\left((s+1)\sqrt{5}\right)$$

for a non-negative integer k, and we are done.

A238. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at P and Q. A line through P intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at A and B, respectively, and X is the mid-point of AB. The line through Q and X intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ again at Y and Z, respectively. Prove that X is the mid-point of YZ.

(1997 Baltic Way)

Solution by Michel Bataille, Rouen, France.

Since $APYQ$ and $BPZQ$ are both concyclic, we have $\angle AYQ = \angle APQ$ and $\angle QZB = \angle QPB$. We can see that $\angle APQ + \angle QPB = \pi$, since A, P and B are collinear, and $\angle QZB + \angle BZY = \pi$, since Q, Z and Y are collinear. Thus, we have $\angle BZY = \angle APQ = \angle AYQ$.

From the fact that $APYQ$ and $BPZQ$ are both concyclic again, we also have $\angle YAP = \angle YQP = \angle ZQP = \angle ZBP$. Finally, since $|AX| = |XB|$, we conclude that AXY and BXZ are congruent triangles. Thus, $|YX| = |XZ|$, and X is the mid-point of YZ.

A239. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let a_1, a_2, $\ldots$, a_n be n distinct numbers, $n \ge 3$. Prove that

$$\sum_{i=1}^{n}\left(a_i \cdot \prod_{j\ne i}\frac{1}{a_i - a_j}\right) = 0\,.$$

Solution by Michel Bataille, Rouen, France.
Consider the rational function

$$R(x) \;=\; \sum_{i=1}^{n} \left(\frac{a_i}{x - a_i} \cdot \prod_{j \neq i} \frac{1}{a_i - a_j} \right) .$$

We may rewrite the entire sum over a common denominator to get

$$R(x) \;=\; \frac{\displaystyle\sum_{i=1}^{n} \left(a_i \cdot \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \right)}{\prod_{i=1}^{n}(x - a_i)} \;=\; \frac{P(x)}{Q(x)} \,,$$

where $P(x)$ is a polynomial of degree $\leq n - 1$ and $Q(x)$ is a polynomial of degree exactly n. However, for $1 \leq k \leq n$,

$$P(a_k) = a_k \cdot \prod_{j \neq k} \frac{a_k - a_j}{a_k - a_j} + \sum_{i \neq k} \left(a_i \cdot \frac{a_k - a_k}{a_i - a_k} \cdot \prod_{j,k \neq i} \frac{a_k - a_j}{a_i - a_j} \right) \;=\; a_k \,.$$

Since $P(x) = x$ for n distinct values of x and $\deg(P) \leq n - 1$, $P(x)$ must be identically equal to x, and, letting $u_i = a_i \cdot \prod_{j \neq i} \frac{1}{a_i - a_j}$, for convenience, we have

$$R(x) \;=\; \sum_{i=1}^{n} \frac{u_i}{x - a_i} \;=\; \frac{x}{\prod_{i=1}^{n}(x - a_i)} \,.$$

Since $xu_i = u_i(x - a_i) + u_i a_i$, we have

$$\frac{x^2}{Q(x)} \;=\; x \cdot \sum_{i=1}^{n} \frac{u_i}{x - a_i} \;=\; \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} \frac{u_i a_i}{x - a_i} \,,$$

so that

$$x^2 \;=\; \left(\sum_{i=1}^{n} u_i \right) Q(x) + \sum_{i=1}^{n} \frac{u_i a_i Q(x)}{x - a_i} \,.$$

Each $\frac{Q(x)}{x - a_i}$ is a polynomial of degree $n - 1$, but we have only one polynomial of degree n. Thus, in order for their sum to be a polynomial of degree less than 3, we must have $\sum_{i=1}^{n} u_i = 0$, as desired.

Also solved by JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; VEDULA N. MURTY, Dover, PA, USA; and CATHERINE SHEVLIN, Wallsend, England.

Murray S. Klamkin of the University of Alberta points out that this result is a special case of the identity

$$\sum_{i=1}^{n} \left(a_i^r \prod_{j \neq i} \frac{1}{a_i - a_j} \right) = 0 \,.$$

The proof given above can easily be adapted to prove this more general result.

A240. *Proposed by Mohammed Aassila, CRM, Université de Montréal, Montréal, Québec.*

Let a, b and c be integers, not all equal to 0. Show that

$$\frac{1}{4a^2+3b^2+2c^2} \le \left|\sqrt[3]{4}a+\sqrt[3]{2}b+c\right| .$$

Solution. First, note that

$$\begin{aligned}(\sqrt[3]{4}a+\sqrt[3]{2}b+c)(2\sqrt[3]{2}a^2+\sqrt[3]{4}b^2+c^2-2ab-\sqrt[3]{4}ac-\sqrt[3]{2}bc)\\ = \quad 4a^3+2b^3+c^3-6abc\,.\end{aligned}$$

By the AM–GM Inequality, $\frac{1}{3}[(\sqrt[3]{4}a)^3+(\sqrt[3]{2}b)^3+c^3] \ge \sqrt[3]{(\sqrt[3]{4}\sqrt[3]{2}abc)^3} = 2abc$, so that $4a^3+2b^3+c^3-6abc \ge 0$, with equality when $\sqrt[3]{4}a = \sqrt[3]{2}b = c$. In fact, since $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are irrational, the only time we can have integers a, b and c satisfy $\sqrt[3]{4}a = \sqrt[3]{2}b = c$ is when $a=b=c=0$. Thus, when a, b and c are integers, not all equal to 0, since $4a^3+2b^3+c^3-6abc > 0$ must also be an integer, we have $4a^3+2b^3+c^3-6abc \ge 1$.

Now, since $64 > 54 \Longrightarrow 4-3\sqrt[3]{2} > 0$ and $27 > \frac{27}{2} \Longrightarrow 3-\frac{3}{2}\sqrt[3]{4} > 0$,

$$\begin{aligned}0 &\le \left(\sqrt[6]{2}a+\frac{\sqrt[3]{2}}{\sqrt{2}}b+\frac{1}{\sqrt{2}}c\right)^2+(4-3\sqrt[3]{2})a^2+\left(3-\frac{3}{2}\sqrt[3]{4}\right)b^2+\frac{1}{2}c^2\\ \Longrightarrow 0 &\le (4-2\sqrt[3]{2})a^2+(3-\sqrt[3]{4})b^2+c^2+2ab+\sqrt[3]{4}ac+\sqrt[3]{2}bc\\ \Longrightarrow & \ 2\sqrt[3]{2}a^2+\sqrt[3]{4}b^2+c^2-2ab-\sqrt[3]{4}ac-\sqrt[3]{2}bc \le 4a^2+3b^2+2c^2\,.\end{aligned}$$

Similarly,

$$\begin{aligned}0 &\le (a-b)^2+\left(\sqrt[3]{4}a-\frac{1}{2}c\right)^2+\left(\sqrt[3]{2}b-\frac{1}{2}c\right)^2+3a^2+2b^2+\frac{5}{2}c^2\\ \Longrightarrow 0 &\le (4+2\sqrt[3]{2})a^2+(3+\sqrt[3]{4})b^2+3c^2-2ab-\sqrt[3]{4}ac-\sqrt[3]{2}bc\\ \Longrightarrow & -2\sqrt[3]{2}a^2-\sqrt[3]{4}b^2-c^2+2ab+\sqrt[3]{4}ac+\sqrt[3]{2}bc \le 4a^2+3b^2+2c^2\,.\end{aligned}$$

Thus,

$$\begin{aligned}&|2\sqrt[3]{2}a^2+\sqrt[3]{4}b^2+c^2-2ab-\sqrt[3]{4}ac-\sqrt[3]{2}bc| \le 4a^2+3b^2+2c^2\\ &\Longrightarrow |4a^3+2a^3+c^3-6abc| \le (4a^2+3b^2+2c^2)\left|\sqrt[3]{4}a+\sqrt[3]{2}b+c\right|\\ &\Longrightarrow 1 \le (4a^2+3b^2+2c^2)\left|\sqrt[3]{4}a+\sqrt[3]{2}b+c\right|\\ &\Longrightarrow \frac{1}{4a^2+3b^2+2c^2} \le \left|\sqrt[3]{4}a+\sqrt[3]{2}b+c\right|\,,\end{aligned}$$

since $4a^2+3b^2+2c^2 > 0$.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C89. *Proposed by Tal Kubo, Brown University.*

Show that the formal power series (in x and y)

$$\sum_{n=0}^{\infty} (xy)^n$$

cannot be expressed as a finite sum

$$\sum_{i=1}^{m} f_i(x) g_i(y)\,,$$

where $f_i(x)$ and $g_i(y)$ are formal power series in x and y, respectively, $1 \le i \le m$.

Solution.

To an infinite matrix $(a_{ij})_{i\in N,\, j\in N}$ one can still associate the idea of "rank": the smallest number q such that there exist q rows so that every other row is a linear combination of these q rows. (Of course, the rank might be infinite.) Note that $\mathrm{rank}(A+B) \le \mathrm{rank}(A) + \mathrm{rank}(B)$.

Any power series in two variables x and y can be represented as such a matrix: the power series

$$\sum a_{ij} x^i y^j$$

corresponds to the matrix (a_{ij}). Then any matrix corresponding to a product $f(x)g(y)$ of power series in one variable has rank 1, Thus, a finite sum of such matrices has finite rank, whereas

$$\sum_n (xy)^n$$

has infinite rank.

C90. *Proposed by Noam Elkies, Harvard University.*

Let S_1, S_2 and S_3 be three spheres in $\mathbb{R}^3$ whose centres are not collinear. Let $k \le 8$ be the number of planes which are tangent to all three spheres. Let A_i, B_i and C_i be the points of tangency between the i^{th} such tangent plane, $1 \le i \le k$, and S_1, S_2 and S_3, respectively, and let O_i be the circumcentre of triangle $A_iB_iC_i$. Prove that all the O_i are collinear. (If $k = 0$, then this statement is vacuously true.)

Solution by Michel Bataille, Rouen, France.

Let us first solve a two-dimensional analogue of this problem. Let C_1 and C_2 be two circles whose centres do not coincide. Let Q_j and r_j denote the centre and radius of C_j for $j = 1, 2$. There are up to four lines tangent to both circles. Let A_i and B_i be the points of tangency between the i^{th} line and C_1 and C_2, respectively, $i \leq 4$. We wish to prove that the mid-points O_i of the A_iB_i are collinear.

To see this, we recall the notion of the *radical axis* of two circles: the radical axis of C_1 and C_2 is the collection of points P such that

$$(Q_1P)^2 - r_1^2 = (Q_2P)^2 - r_2^2\,.$$

It is not difficult to check that the solutions to this equation form a line. To solve our two-dimensional question, we claim that each O_i lies on the radical axis of C_1 and C_2, so that they are certainly collinear. We check this claim: since A_iO_i and B_iO_i are tangent to C_1 and C_2, respectively, the triangles $A_iO_iQ_1$ and $B_iO_iQ_2$ are right triangles, and so

$$(Q_1O_i)^2 - r_1^2 = (A_iO_i)^2 = (B_iO_i)^2 = (Q_2P)^2 - r_2^2\,,$$

as desired.

We now turn to the original three-dimensional problem. For each pair of spheres S_j, S_k, with centres Q_j and Q_k and radii r_j and r_k, we define the *radical plane* to be the plane of points P such that

$$(Q_jP)^2 - r_j^2 \;=\; (Q_kP)^2 - r_k^2\,.$$

Notice that any point on the radical plane of S_1 and S_2, and which is also on the radical plane of S_2 and S_3, is automatically on the radical plane of S_1 and S_3 (by transitivity).

Moreover, the radical plane of two spheres is perpendicular to the line between their centres; since the three spheres under consideration have centres which are non-collinear, we can conclude that the three radical planes between them must jointly intersect in a single radical axis. Since the circumcentre O_i of the triangles $A_iB_iC_i$ is equidistant from each of A_i, B_i and C_i, and since each line segment O_iA_i, O_iB_i and O_iC_i is tangent, respectively, to S_1, S_2 and S_3, it follows (exactly as in the two-dimensional case) that each O_i must lie on the radical axis. Thus, the O_i are collinear.

Also solved by LAURENT LESSARD, 2nd year engineering student, University of Toronto, Toronto, Ontario; and ROMÁN FRESNEDA, Cuba.

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. The equations $x^2+5x+6 = 0$ and $x^2+5x-6 = 0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2 + 4x + 5 = 0$ and $x^2 + 4x - 5 = 0$ has integer solutions.

(a) Show that if $x^2 + px + q = 0$ and $x^2 + px - q = 0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2 = a^2 + b^2$.

(b) Determine q in terms of a and b.

(1998 Euclid, Problem 10)

Solution. First we show that p and q must be integers.

Let $p^2 - 4q = m^2$ and $p^2 + 4q = n^2$, for positive reals m and n. Now, the solutions to the first quadratic equation $x^2+px+q = 0$ are $(-p\pm m)/2$. The sum of both solutions must be an integer. Thus, $-p$ must be an integer, meaning that p must be an integer. The difference of the solutions must also be an integer. Thus, m must be an integer. (Similarly, from the second quadratic equation, n must be an integer as well.) Now, since $x^2+px+q = 0$ has an integer solution, we must have q being an integer (this last point can be seen by substituting one of the solutions into the equation.) Thus, p and q must be integers.

Examining the equations $p^2 - 4q = m^2$ and $p^2 + 4q = n^2$ modulo 4, we note that m and n have the same parity as p.

Adding, we get $2p^2 = m^2 + n^2$. Thus,

$$p^2 = (m^2 + n^2)/2 = [(n + m)/2]^2 + [(n - m)/2]^2 ,$$

which is a sum of two squares of integers, since m and n have the same parity. We can let $a = (n + m)/2$ and $b = (n - m)/2$.

To find q in terms of a and b, we also know that $8q = n^2 - m^2$, or, equivalently, $q = [(n + m)/2][(n - m)/2]/2 = ab/2$.

Any reader who wishes to contribute to the **Problem of the Month** section can feel free to do so by emailing a problem and any nice solution(s) to **jimmy.chui@utoronto.ca**, or sending hard copy, care of the Mayhem Editor.

Constructive Geometry — Part II

Cyrus Hsia

In part I of this series [2000 : 231] we laid the foundations of building basic geometric objects. We started with two simple tools: a straightedge and a collapsible compass. Though deceptively simple, their usefulness is demonstrated over and over again. Using only these objects we were able to construct all the tasks given in part I. We continue here with a selection of problems from various sources that require only these two tools. We also noted in part I that many geometric construction problems can be readily done by using results from other constructions that we have done. Here, we will discuss one result named Apollonius' Theorem that is useful in many geometric constructions.

We begin with a nice and easy problem that appeared on the Internet.

Problem 1

Given a circle with centre O and a point P outside the circle, construct a point Q on the circle so that angle PQO is 60°.

Solution

Here is the construction:

1. Construct an equilateral triangle on edge OP. Label the third vertex R and the centroid S. Remember, we can use the results that we have obtained in previous exercises. Constructing an equilateral triangle was problem 6 in part I of this series. Now how can you construct the centroid of an equilateral triangle?

2. Construct the circumcircle of triangle OSP. Again refer to results in the previous article that you can use to help do this. First, find the centre of the circle.

3. This circle intersects the circle with centre O in two points. Take Q to be the point on the opposite side of the line OP from point S. See Figure 1.

Proof that angle PQO is 60°: Since O, P, Q and S are concyclic we have $\angle OQP = 180^\circ - \angle OSP = 60^\circ$. Why is $\angle OSP = 120^\circ$?

The following is a problem that appeared in a Hungarian Contest.

Problem 2

Given a right triangle ABC, construct a point N inside the triangle such that the angles NBC, NCA and NAB are equal.

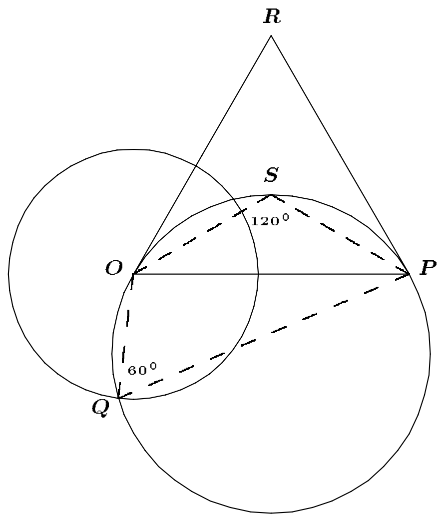


Figure 1

(*Hungarian Problem Book I, Problem 1895/2*)

Solution

Again, the idea is to work the problem backwards to get a sense of the construction that is needed. Consider Figure 2a.

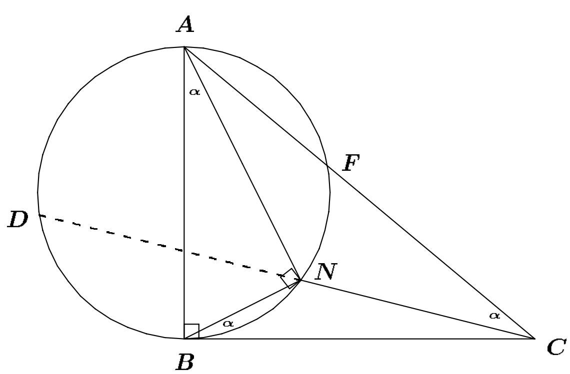


Figure 2a

Suppose N in Figure 2b satisfies the conditions. Angle ABC is the right angle. Let $\angle NBC = \angle NCA = \angle NAB = \alpha$. Now, $\angle ABN = \angle ABC - \angle NBC = 90° - \alpha$. Thus, $\angle ANB = 180° - \angle NAB - \angle ABN = 90°$. We conclude from this that N must be on the circumference of the circle with diameter AB. Thus, when we come to constructing point N, we will likely construct a circle with diameter AB.

Now extending CN to the other side of the circle to point D, we see that $\angle NDA = \angle NBA = 90° - \alpha$.

This implies that $\angle DAC = 180° - \angle ADC - \angle ACD = 90°$. Let AC intersect the circle at F. Angles DAF and DAC are both right angles. Thus, DF is a diameter of the circle as well. From this we see that by extending F through the centre O to the other side of the circle and connecting DC we would have N on the intersection of DC and the circle.

Here is the construction:

1. Using AB as diameter, construct a circle.
2. Let F be the intersection of the circle with AC. Extend F through the centre to D. Connect DC, and let N be the intersection of the circle with DC extended. See Figure 2b.

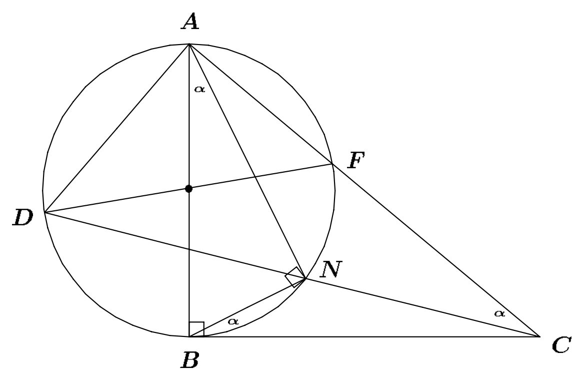


Figure 2b

Now, here is the proof that this point N satisfies the conditions in the problem. Let $\angle NBC = \alpha$. Angle NBA is then $90° - \alpha$. Since $\angle ANB$ is then a right angle, this implies that $\angle NAB$ is also α. Then $\angle NDA = \angle NBA = 90° - \alpha$. Since $\angle DAF$ is right, $\angle DAC$ is a right angle in triangle DAC. Thus, $\angle ACD = 180° - \angle DAC - \angle NDA = \alpha$. Thus, $\angle NBC$, $\angle NCA$ and $\angle NAB$ are equal as required.

APOLLONIUS

Here is a problem that comes in many disguises and is used in many problems.

Problem 3A

Construct the locus of all points P such that $AP : PB = 1 : 2$ for given points A and B. Describe this locus.

If we had to guess what the locus would look like, the two simplest ones would be a circle or a line. And that is indeed what the answers are in the general case. This result is known as Apollonius' Theorem. See [1] and [2] for more on this.

Apollonius' Theorem

Given a line segment AB, the locus of all points C such that the ratio AC to CB is a constant is either a straight line or a circle.

See if you can spot the use of this theorem in the following problems.

Problem 3B

A segment AB is given and a line m crossing it. Determine the point C on m such that m bisects angle ACB.

Problem 3C

Given three disjoint circles in the plane, construct a point on the plane such that all three circles subtend the same angle at the point.

Correspondence Program, Geometry Problem Set, Problem 19

Problem 3D

Four points A, B, C and D are given on a straight line. Construct a pair of parallel lines through A and B, and another pair through C and D, so that these pairs of parallel lines intersect in the vertices of a square.

Hint

Draw in the four circles with diameters AB, BC, CD and AD. Notice that two opposite vertices of the square lie on the circles with diameters BC and AD. Now look to see if you can apply Apollonius' Theorem.

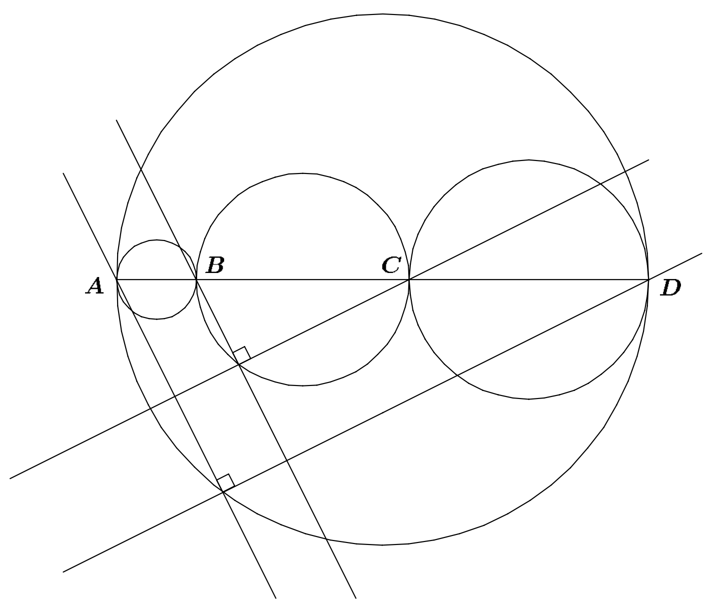


Figure 3

Now for some exercises. All of the following constructions should be performed using straight edge and compass only.

Exercises

1. Given 3 three parallel lines, construct an equilateral triangle with a vertex on each of the three lines.

2. Given 3 concentric circles, construct an equilateral triangle with a vertex on each of the three circles.
   Proposed for the 26th IMO

3. Let P be one of the two points of intersection of two intersecting circles. Construct the line l through P, not containing the common chord, such that the two circles cut off equal segments on l.

4. An army captain wishes to station an observer equally distant from two specified points and a straight road. Can this always be done? Locate any possible stations. In other words, how many points are there in the Euclidean plane which are equidistant from two given points and a given line? Find them with straightedge and compass, if possible.

5. Show how to divide a circle into nine regions of equal area, using a straight-edge and compass.

6. Construct the following special centres of a triangle: the centroid, incentre, orthocentre, circumcentre, Gergonne Point, and the Fermat Point. We saw in problem 1 the construction of all these points in an equilateral triangle because they are all the same point! Now do it for any given triangle.

References

1. Barbeau, Edward J., ATOM (A Taste of Mathematics), Volume 1, pp. 47–48. Canadian Mathematics Society, 1997.

2. Grossman, J.P., "Ye Olde Geometry Shoppe – Part II", Mathematical Mayhem, Volume 6 Issue 3, p. 10.

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PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 May 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2589. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 2, 3, \ldots$, evaluate $\displaystyle\sum_{k=1}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{n}{k-1}$.

2590. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

For $n = 1, 2, \ldots$, prove that $\displaystyle\prod_{k=1}^{n} \binom{n}{k}^2 \le \left(\frac{1}{n+1}\binom{2n}{n}\right)^n$.

2591. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Two players, A and B, each toss n fair coins, and two other players, C and D, toss $n-1$ and $n+1$ fair coins, respectively.

For each $n = 2, 3, \ldots$, prove that the two events:

A gets exactly one head more than B

and

C and D get exactly the same number of heads

are equally likely.

Find the probability of these events.

2592. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Describe all numbers, which can be represented in the form of $\dfrac{a^3+b^3}{c^3+d^3}$, where a, b, c, d are natural numbers.

2593. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Let $S(a)$ denote the sum of the digits of the natural number a. Let k and n be natural numbers with $(n,3)=1$. Prove that there exists a natural number m which is divisible by n and $S(m)=k$ if either

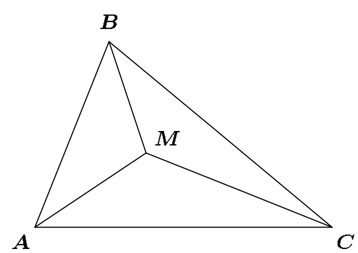
(a) $k > n-2$; or

(b) $k > S^2(n)+7S(n)-9$.

2594. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given a point M inside the triangle ABC (see diagram), prove that

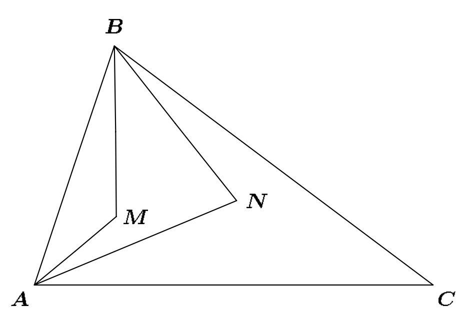
$$\min(MA,\ MB,\ MC)+MA+MB+MC \ <\ AB+BC+AC\,.$$



2595. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*

Given that M and N are points inside the triangle ABC such that $\angle MAB=\angle NAC$ and $\angle MBA=\angle NBC$, prove that

$$\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BA\cdot BC}+\frac{CM\cdot CN}{CA\cdot CB} \;=\; 1\,.$$



2596. *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*

Write $r \ll s$ if there is an integer k satisfying $r<k<s$. Find, as a function of n ($n\geq 2$) the least positive integer k satisfying

$$\frac{k}{n} \ll \frac{k}{n-1} \ll \frac{k}{n-2} \ll \ldots \ll \frac{k}{2} \ll k\,.$$

2597. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*

Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that $|\angle PBC - \angle PCB| \;\le\; \arcsin\left(2\sin\left(\frac{|\angle PAB - \angle PAC|}{2}\right)\right) - \left(\frac{|\angle PAB - \angle PAC|}{2}\right) \;\le\; |\angle PAB - \angle PAC|$.

Show that the left inequality cannot be improved in the sense that there is a position Q of P on the ray AP giving an equality.

(Thus, the inequality in **2255** is improved.)

2598. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose that AD, BE and CF are the internal angle bisectors of $\triangle ABC$, with D on BC, E on CA and F on AB. Write $a = BC$, $b = CA$, $c = AB$, $x = AE$ and $y = AF$. We are given that $x + y = a$. Prove that:

(a) $a^2 = bc$;

(b) $\frac{1}{x} - \frac{1}{y} \;=\; \frac{1}{b} - \frac{1}{c}$;

(c) $\frac{1}{x} + \frac{1}{y} \;=\; \left(\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2$;

(d) $AD < c$.

2599. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Let P be a point inside the triangle ABC and let AP, BP, CP meet the sides BC, CA, AB at L, M, N, respectively. Show that the following two conditions are equivalent:

$$\frac{1}{AP} + \frac{1}{PL} \;=\; \frac{1}{BP} + \frac{1}{PM} \;=\; \frac{1}{CP} + \frac{1}{CN}\,;$$

$$\angle APN = \angle NPB = \angle BPL = \angle LPC = \angle CPM = \angle MPA = 60^\circ\,.$$

2600. *Proposed by Svetlozar Doichev, Stara Zagora, Bulgaria.*

Find all real numbers x such that, if a and b are the lengths of sides of a triangle with medians from the mid-points of these sides of lengths m_a and m_b, respectively, then the equalities $a + xm_a \;=\; b + xm_b$ and $a \;=\; b$ are equivalent.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2468. [1999 : 367] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For $c > 0$, let x, y, $z > 0$ satisfy

$$xy + yz + zx + xyz \;=\; c\,. \tag{1}$$

Determine the set of all $c > 0$ such that whenever (1) holds, then we have

$$x + y + z \;\geq\; xy + yz + zx\,.$$

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

The positive reals c asked for are those satisfying $c \leq 4$.

From (1), we have $xy < c$ and

$$z \;=\; \frac{c - xy}{x + y + xy}\,.$$

Substituting this value of z into the given inequality, we see that we must determine those positive reals c such that

$$x + y + \frac{c - xy}{x + y + xy} \;\geq\; xy + \frac{(x+y)(c-xy)}{x + y + xy}\,,$$

or

$$(x+y)(x+y+xy) + (c - xy) \;\geq\; xy(x+y+xy) + (x+y)(c-xy)\,,$$

which simplifies to

$$A(x,y) := \;(x+y)^2 - (xy)^2 + (c - xy)(1 - x - y) \;\geq\; 0\,, \tag{2}$$

whenever x, $y > 0$ such that $xy < c$.

Taking $x = c/2$ and letting y approach zero, we obtain

$$\frac{c^2}{4} + c\left(1 - \frac{c}{2}\right) \;\geq\; 0\,,$$

which means that $c \leq 4$.

Now suppose that $c \leq 4$. Let x, $y > 0$ such that $xy < c$.

If $(x-1)(y-1) \geq 0$, then (2) follows from

$$\begin{aligned} A(x,y) &= (x-y)^2 + 4xy - (xy)^2 + (c-xy)\Big((x-1)(y-1) - xy\Big) \\ &= (x-y)^2 + (4-c)xy + (c-xy)(x-1)(y-1) \;\geq\; 0\,. \end{aligned}$$

If $(x-1)(y-1) < 0$, then it follows from

$$\begin{aligned} A(x,y) &= (x+y-2)^2 + 4(x+y-1) - (xy)^2 + (c-xy)(1-x-y) \\ &= (x+y-2)^2 + (c-4-xy)(1-x-y) - (xy)^2 \\ &= (x+y-2)^2 + (c-4-xy)\Big((x-1)(y-1) - xy\Big) - (xy)^2 \\ &= (x+y-2)^2 + (4-c)xy + (c-4-xy)(x-1)(y-1) \;\geq\; 0\,. \end{aligned}$$

This proves the above statement. [Ed. see also [2000 : 337]]

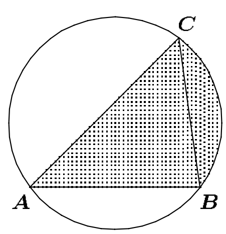
Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong, China; and the proposer. There was one incorrect solution received.

The problem, by the same proposer, had also been published as Aufgabe 61 in the Austrian journal Wissenschaftliche Nachrichten, *Vol.* ***107*** (1998) *p. 36, with solution in Vol.* ***110*** (1999) *pp. 25–26.*

The case $c = 4$ was a problem in the 1996 Vietnamese Mathematical Olympiad*; see problem 6 on* [1999 : 8].

2477. [1999 : 429] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given a non-degenerate $\triangle ABC$ with circumcircle Γ, let r_A be the inradius of the region bounded by BA, AC and $\mathbf{arc}(CB)$ (so that the region includes the triangle).



Similarly, define r_B and r_C. As usual, r and R are the inradius and circumradius of $\triangle ABC$.

Prove that

(a) $\dfrac{64}{27}r^3 \;\leq\; r_A r_B r_C \;\leq\; \dfrac{32}{27}Rr^2\,;$

(b) $\dfrac{16}{3}r^2 \;\leq\; r_B r_C + r_C r_A + r_A r_B \;\leq\; \dfrac{8}{3}Rr\,;$

(c) $4r \;\le\; r_A + r_B + r_C \;\le\; \frac{4}{3}(R+r)\,,$

with equality occurring in all cases if and only if $\triangle ABC$ is equilateral.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

It is known ([1], [2] and [3]) that

$$r_A \;=\; r\sec^2\left(\frac{A}{2}\right), \qquad r_B \;=\; r\sec^2\left(\frac{B}{2}\right), \qquad r_C \;=\; r\sec^2\left(\frac{C}{2}\right).$$

If s denotes the semiperimeter of $\triangle ABC$, then [1996 : 130]

$$\sum \cos A \;=\; \frac{R+r}{R}\,, \qquad \sum \cos A \cos B \;=\; \frac{s^2-4R^2+r^2}{4R^2}\,,$$

and

$$\prod \cos A \;=\; \frac{s^2-4R^2-4Rr-r^2}{4R^2}\,,$$

where sums and products are cyclic over the angles A, B, C. Hence,

$$\begin{aligned}\prod \cos^2\left(\frac{A}{2}\right) &= \frac{1}{8}\prod(1+\cos A) \\ &= \frac{1}{8}\left(1+\sum\cos A+\sum\cos A\cos B+\prod\cos A\right) \\ &= \frac{s^2}{16R^2}\,,\end{aligned}$$

giving

$$r_A r_B r_C \;=\; \frac{16R^2r^3}{s^2}\,. \tag{1}$$

On [1996 : 78], Seimiya showed that

$$\sum \sec^2\left(\frac{A}{2}\right) \;=\; 1+\left(\frac{4R+r}{s}\right)^2\,,$$

which implies that

$$r_A+r_B+r_C \;=\; r\left(1+\left(\frac{4R+r}{s}\right)^2\right)\,. \tag{2}$$

From the above identities, we also have

$$\sum \cos^2\left(\frac{A}{2}\right) \;=\; \frac{1}{2}\sum(1+\cos A) \;=\; \frac{4R+r}{2R}\,,$$

so that

$$\begin{aligned}\sum \sec^2\left(\frac{A}{2}\right)\sec^2\left(\frac{B}{2}\right) &= \prod\sec^2\left(\frac{A}{2}\right)\cdot\sum\cos^2\left(\frac{A}{2}\right) \\ &= 8\left(\frac{4R^2+Rr}{s^2}\right)\,.\end{aligned}$$

Hence,

$$r_B r_C + r_C r_A + r_A r_B \;=\; \frac{8(4R^2 + Rr)r^2}{s^2}\,. \tag{3}$$

Now recall Gerretsen's inequalities (see [1996 : 130]);

$$16Rr - 5r^2 \;\le\; s^2 \;\le\; 4R^2 + 4Rr + 3r^2\,,$$

where, on both sides, equality holds only when the triangle ABC is equilateral. The desired inequalities (a), including the conditions for equality, follow from (1) and the estimates

$$\left(\frac{27}{2}\right) Rr \;\le\; s^2 \le \left(\frac{27}{4}\right) R^2\,,$$

which, by Euler's inequality $2r \le R$, are weaker than Gerretsen's inequalities. Similarly, (b) follows from (3) and

$$12Rr + 3r^2 \;\le\; s^2 \;\le\; 6R^2 + \left(\frac{3}{2}\right) Rr\,,$$

and (c) follows from (2) and

$$12Rr + 3r^2 \;\le\; s^2 \;\le\; \frac{16R^2 + 8Rr + r^2}{3}\,,$$

both including the conditions for equality.

References:

[1] L. Bankoff, A Mixtilinear Adventure, *Crux Mathematicorum* **9** (1983) 2–7.

[2] C. V. Durell and A. Robson, *Advanced Trigonometry*, 23.

[3] P. Yiu, Mixtilinear Incircles, *Amer. Math. Monthly* **106** (1999) 952–955.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; MICHAEL LAMBROU, University of Crete, Crete, Greece; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Using Gerretsen's inequalities themselves in (1), (2) *and* (3) *would give stronger (though more complicated) bounds than are asked for in this problem.*

The expressions for r_A, r_B, r_C *given at the beginning of the above solution also appeared in Solution* II *of* Crux 1224 [1988: 147], *as pointed out by Tsintsifas* (*who had proposed* 1224). *In fact they are most recently in Solution* II *of* Crux with Mayhem *problem* 2464 [2000 : 432]; in particular, see pages 435–436. *Moreover, the lower bound of part* (c) *of the current problem occurs in the statement of* 1224(b), *and also in Solution* II *of* 2464.

2480. [1999 : 430] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

Writing $\phi(n)$ for Euler's totient function, evaluate

$$\sum_{d|n} d \sum_{k|d} \frac{\phi(k)\phi(d/k)}{k} .$$

Solution by Kee-Wai Lau, Hong Kong.

Making use of the well-known relation $\sum_{d|n} \phi(d) = n$, we see that

$$\begin{aligned} \sum_{d|n} d \sum_{k|d} \frac{\phi(k)\phi(d/k)}{k} &= \sum_{\substack{k,t \\ kt|n}} t\phi(t)\phi(k), \qquad (\text{set } d = kt) \\ &= \sum_{t|n} t\phi(t) \sum_{k|(n/t)} \phi(k) \;=\; \sum_{t|n} t\phi(t)(n/t) \\ &= n \sum_{t|n} \phi(t) \;=\; n^2 . \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; SOUTHWEST MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP; KENNETH M. WILKE, Topeka, KS, USA; and the proposer.

Seiffert observes that the problem can be generalized to the Jordan totient function defined as

$$J_s(n) \;=\; n^s \prod_{\substack{p|n \\ p \text{ prime}}} (1 - p^{-s})$$

(note that $J_1 = \phi$.) In this case the new problem would be to evaluate:

$$\sum_{d|n} d^s \sum_{k|d} \frac{J_s(k)J_s(d/k)}{k^s} .$$

Seiffert shows that the value must be n^{2s} using the fact that

$$\sum_{d|n} J_s(d) \;=\; n^s .$$

2481. [1999 : 430] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that A, B, C are 2×2 commutative matrices. Prove that

$$\det\left((A+B+C)(A^3+B^3+C^3-3ABC)\right) \;\geq\; 0 .$$

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece; and by Kee-Wai Lau, Hong Kong (independently).*

We show that the inequality is in fact true for $n \times n$ real commutative matrices A, B, C.

Let P and Q be any two real $n \times n$ commutative matrices.

We have

$$\begin{aligned}
\det\left(P^2 + Q^2\right) &= \det\left((P+iQ)(P-iQ)\right) \\
&= \det(P+iQ)\det(P-iQ) = \det(P+iQ)\overline{\det(P+iQ)} \\
&= |\det(P+iQ)|^2 \ \geq 0\,.
\end{aligned}$$

Putting $P = \frac{\sqrt{3}}{2}(A-C)$ and $Q = \frac{1}{2}(A-2B+C)$, we have

$$\det(A^2+B^2+C^2-AB-BC-CA) \;=\; \det(P^2+Q^2) \;\geq\; 0\,.$$

Hence

$$\begin{aligned}
&\det\left((A+B+C)(A^3+B^3+C^3-3ABC)\right) \\
&\quad= \det\left((A+B+C)^2\right)\det(A^2+B^2+C^2-AB-BC-CA) \\
&\quad= \left(\det(A+B+C)\right)^2\det(A^2+B^2+C^2-AB-BC-CA) \;\geq\; 0
\end{aligned}$$

as claimed.

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

We prove that the inequality holds for real commutative $n \times n$ matrices A, B, C.

$$\begin{aligned}
\text{Let}\quad D &= (\sqrt{3}+i)(A-B)+(\sqrt{3}-i)(B-C) \\
\text{and}\quad E &= (\sqrt{3}-i)(A-B)+(\sqrt{3}+i)(B-C)\,.
\end{aligned}$$

Then it is easily verified that

$$DE \;=\; 4(A^2+B^2+C^2-AB-BC-CA)\,.$$

Note that $E = \overline{D}$, where $\overline{D}$ denotes the $n \times n$ matrix obtained from D when every entry is replaced by its conjugate.

Hence $\det(DE) = \det(D)\det(\overline{D}) = \det(D)\ \overline{\det(D)} = \ |\det(D)|^2\ \geq 0$.

Therefore,

$$\begin{aligned}
&\det\left((A+B+C)(A^3+B^3+C^3-3ABC)\right) \\
&\quad= \det\left((A+B+C)^2\right)\det(A^2+B^2+C^2-AB-BC-CA) \\
&\quad= \left(\det(A+B+C)\right)^2\det\left(\tfrac{1}{4}\,DE\right) \\
&\quad= \left(\tfrac{1}{4}\right)^n\left(\det(A+B+C)\right)^2\det(DE) \;\geq\; 0\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Bataille, Lambrou and Seiffert all pointed out that the given inequality need not hold if A, B, C are not necessarily real matrices. Both Lambrou and Seiffert gave the simple counterexample: $A = B = 0$ and $C = wI_n$ where I_n denotes the $n \times n$ identity matrix and w is a complex number such that $w^8 = -1$.

2482. [1999 : 430] *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that p, q, r are complex numbers. Prove that

$$|p+q| + |q+r| + |r+p| \;\le\; |p| + |q| + |r| + |p+q+r| \,.$$

Editor's comment.

Most solvers, and others who submitted only comments (Mohammed Aassila, Joe Howard, Walther Janous, Murray S. Klamkin, Heinz-Jürgen Seiffert) noted that this is a version of Hlawka's Inequality, and that further generalizations of it may be found in D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993, pp. 521-534 and 544-551; and D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, pp. 171-173. Aassila commented that this problem is equivalent to one proposed by France, but not used by the jury, at IMO 1987, and that Murray Klamkin's solution appeared in [1989 : 102]. Bataille remarked that a generalization to n complex numbers is given as Problem 1550 in *Mathematics Magazine*, Vol. 72, No. 3, June 1991, p. 239. Woo gave a geometric argument. Several solvers raised the question of when equality occurs, and in that regard, Romero refers us to p. 23 of L. Missotte, *1850 exercises de mathématiques pour l'oral du CAPES de mathématiques et des concours des Grandes Ecoles*, Dunod University, Paris, 1978. Finally, Janous remarked that Prof. Hlawka is still alive and, though now almost blind, is one of the still quite productive Nestors [Ed. an elderly and distinguished wise person, a wise counsellor] of Austrian mathematics.

Solved by MICHEL BATAILLE, Rouen, France; G.P. HENDERSON, Garden Hill, Campbellcroft, Ontario; MICHAEL LAMBROU, University of Crete, Crete, Greece; VEDULA N. MURTY, Visakhapatnam, India; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ANDREI SIMION, student, Brooklyn Technical HS, Brooklyn, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect solution.

2483. [1999 : 430] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*
Suppose that $0 \le A$, B, C and $A + B + C \le \pi$. Show that

$$0 \;\le\; A - \sin A - \sin B - \sin C + \sin(A+B) + \sin(A+C) \;\le \pi \,.$$

There are, of course, similar inequalities with the angles permuted cyclically.

[The proposer notes that this came up during an attempt to generalize problem 2383.]

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

Let $S(A,B,C)$ denote the middle term of the desired inequalities. From the known trigonometric identities

$$\sin x - \sin y \;=\; 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

and

$$\cos x + \cos y \;=\; 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\,,$$

it follows that

$$\begin{aligned}
&\sin(A+B) - \sin B + \sin(A+C) - \sin C\\
&\quad=\; 2\cos\left(\frac{A+2B}{2}\right)\sin\left(\frac{A}{2}\right) + 2\cos\left(\frac{A+2C}{2}\right)\sin\left(\frac{A}{2}\right)\\
&\quad=\; 2\sin\left(\frac{A}{2}\right)\left[\cos\left(\frac{A+2B}{2}\right) + \cos\left(\frac{A+2C}{2}\right)\right]\\
&\quad=\; 4\cos\left(\frac{A+B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{A}{2}\right).
\end{aligned}$$

Hence

$$S(A,B,C) \;=\; A - \sin A + 4\cos\left(\frac{A+B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{A}{2}\right)\,.$$

Under the given conditions, we thus have

$$S(A,B,C) \;\geq\; A - \sin A \;\geq\; 0$$

and

$$\begin{aligned}
S(A,B,C) \;&\leq\; A - \sin A + 4\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)\\
&=\; A + \sin A = A + \sin(\pi - A)\\
&\leq\; A + (\pi - A) \;=\; \pi\,.
\end{aligned}$$

This proves the desired inequalities. From the proof, we see that there is equality on the left hand side if and only if $A = 0$, and on the right hand side only when $A = \pi$ (and $B = C = 0$).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; THOMAS JANG, Southwest Missouri State University, Springfield, Missouri, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; and the proposer. Another solver sent in a solution similar to Seiffert's, but containing an error, small and correctible, but fatal nevertheless!

2484. [1999: 430] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given a square $ABCD$, suppose that E is a point on AB produced beyond B, that F is a point on AD produced beyond D, and that $EF = 2AB$. Let P and Q be the intersections of EF with BC and CD, respectively. Prove that

(a) $\triangle APQ$ is acute-angled;

(b) $\angle PAQ \geq 45^\circ$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let M be the mid-point of EF. Then $AM = ME = MF = AB$. Hence, M lies on the quadrant BMD of the circle with AB as radius and A as centre and, moreover, that quadrant lies inside the square. EF must either touch the quadrant at M or intersect it at two points M and N, both being inside the square [since P and Q are outside the circle]. Therefore the mid-point H of MN also lies inside the square. Then $\angle APE > \angle AHE = 90^\circ$ and $\angle AQF > \angle AHF = 90^\circ$. Hence, $\triangle APQ$ is acute-angled.

Let $P'Q'$ be the tangent to the quadrant BMD; that is, parallel to PQ, cutting BC at P' and CD at Q', and touching the quadrant at H'. Then H lies on AH' and therefore $AH \leq AB$. Hence, P' has to be between C and P while Q' has to be between C and Q. Hence, $\angle PAQ \geq \angle P'AQ'$, which proves the claim since $\angle P'AQ' = 90^\circ/2 = 45^\circ$: AP' bisects $\angle BAH'$ [since P' is the intersection point of the tangents to the quadrant at B and at H'] while, similarly, AQ' bisects $\angle DAH'$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

2485. [1999: 431] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral with $AB = BC = CD$. Let P be the intersection of the diagonals AC and BD. Suppose that $AP : BD = DP : AC$.

Prove that either $BC \parallel AD$ or $AB \perp CD$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Let $AB = BC = CD = x$, $\angle BAC = \alpha$ and $\angle CDB = \beta$. Then $\angle BCA = \alpha$, $\angle CBD = \beta$, $\angle APB = \angle DPC = \alpha + \beta$, $\angle ABP = 180^\circ - 2\alpha - \beta$ and $\angle DCP = 180^\circ - 2\beta - \alpha$. Also, from triangles CAB and DBC, we obtain $AC = 2x\cos\alpha$ and $BD = 2x\cos\beta$, respectively. Applying the Sine Law to triangles ABP and DCP,

$$AP = \frac{x\sin(2\alpha+\beta)}{\sin(\alpha+\beta)} \quad \text{and} \quad DP = \frac{x\sin(2\beta+\alpha)}{\sin(\alpha+\beta)}.$$

The condition $\frac{AP}{BD} = \frac{DP}{AC}$ now implies

$$\frac{\sin(2\alpha+\beta)}{\sin(\alpha+\beta)\cos\beta} \;=\; \frac{\sin(2\beta+\alpha)}{\sin(\alpha+\beta)\cos\alpha}\,.$$

This gives

$$\sin(2\alpha+\beta)\cos\alpha \;=\; \sin(2\beta+\alpha)\cos\beta\,,$$

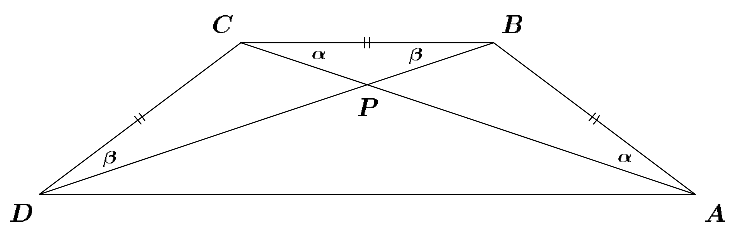
or,

$$\sin(3\alpha+\beta) \;=\; \sin(3\beta+\alpha)\,.$$

There are two possibilities:

either $3\alpha+\beta = 3\beta+\alpha$, or $(3\alpha+\beta)+(3\beta+\alpha) = 180^\circ$.

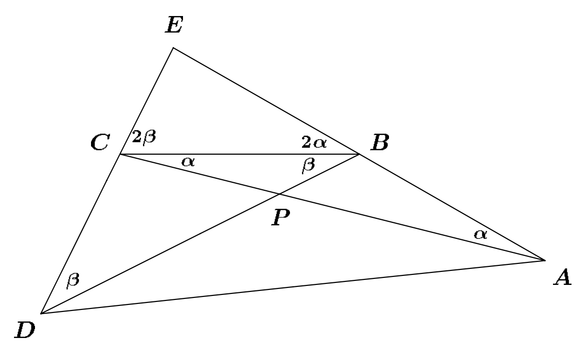
(1) If $3\alpha+\beta = 3\beta+\alpha$, then $\alpha = \beta$, so that $ABCD$ can be inscribed in a circle. Since $AB = CD$, it follows that $BC \parallel AD$.



(2) If $(3\alpha+\beta)+(3\beta+\alpha) = 180^\circ$, then $\alpha+\beta = 45^\circ$. Let E be the intersection point of AB and CD.

Then $\angle CBE = \angle BCA + \angle BAC = 2\alpha$. Similarly, $\angle BCE = 2\beta$.

Therefore, $\angle BEC = 180^\circ - 2\alpha - 2\beta = 90^\circ$, which shows that $AB \perp CD$.



Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was also one incorrect solution submitted.

2486. [1999: 431] *Proposed by Joe Howard, New Mexico Highlands University, Las Vegas, NM, USA.*

It is well known that $\cos(20°)\cos(40°)\cos(80°) = \frac{1}{8}$.

Show that $\sin(20°)\sin(40°)\sin(80°) = \frac{\sqrt{3}}{8}$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

We have

$$\begin{aligned} 4\sin 20°\sin 40°\sin 80° &= 2[\cos 20° - \cos 60°]\sin 80° \\ &= 2\cos 20°\sin 80° - \sin 80° \\ &= \sin 100° + \sin 60° - \sin 100° \\ &= \frac{\sqrt{3}}{2}, \end{aligned}$$

and hence, $\sin 20°\sin 40°\sin 80° = \frac{\sqrt{3}}{8}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (5 solutions); MICHEL BATAILLE, Rouen, France; FRANK BATTLES, Massachusetts Maritime Academy, MA, USA; SOUMYA KANTI DAS BHAUMIK, student, Angelo State University, TX, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JONATHAN CAMPBELL, student, Chapel Hill High School, NC, USA; JENN CARRUTHERS, Burlington, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; PETER HURTHIG, Columbia College, Vancouver, BC (2 solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; R. LAUMEN, Antwerp, Belgium (2 solutions); GERRY LEVERSHA, St. Paul's School, London, England; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Brooklyn Technical High School, NY, USA; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Several solvers noted that generalizations of this problem or the problem itself can be found in various sources, for example, Sinefakopoulos refers to D.O. Shklarsky, N.N. Chentzov and I.M. Yaglom, The USSR Olympiad Problem Book, Dover, *NY, 1993, p. 35, problem 232, (a). Wilke mentioned the equality*

$$\prod_{k=1}^{(n-1)/2} 2\sin\frac{k\pi}{n} = \sqrt{n},$$

found in T. Nagell, Number Theory, *Chelsea, NY, 1964, p.173.*

2487. [1999 : 431] *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

If a, b, c, d are distinct real numbers, prove that

$$\frac{a^4+1}{(a-b)(a-c)(a-d)}+\frac{b^4+1}{(b-a)(b-c)(b-d)}$$
$$+\frac{c^4+1}{(c-a)(c-b)(c-d)}+\frac{d^4+1}{(d-a)(d-b)(d-c)} \;=\; a+b+c+d\,.$$

I. *Composite solution by Vedula N. Murty, Visakhapatnam, India and Peter Y. Woo, Biola University, La Mirada, CA, USA.*

The left hand side of the given identity can be written as N/D where

$$\begin{aligned} D &= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \text{ and} \\ N &= (a^4+1)(b-c)(b-d)(c-d)-(b^4+1)(a-c)(a-d)(c-d) \\ &\quad +(c^4+1)(a-b)(a-d)(b-d)-(d^4+1)(a-b)(a-c)(b-c)\,. \end{aligned}$$

By the Factor Theorem, it is readily verified that each factor of D is also a factor of N.

Since both D and N are antisymmetric polynomials of degrees six and seven, respectively, the quotient N/D must be a symmetric polynomial in a, b, c and d of degree one.

Hence $N=k(a+b+c+d)(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ for some constant k.

Comparing the coefficient of a^4 we see that $k=1$ and the conclusion follows.

II. *Solution by Mangho Ahuja, Southeast Missouri State University, Cape Girardeau, MO, USA.*

Let A, B, C and D denote the four terms of the left hand side of the identity to be proved. We need to show that

$$A+B+C+D \;=\; a+b+c+d\,.$$

Consider $F(x)=\dfrac{x^4+1-(x-a)(x-b)(x-c)(x-d)}{(x-a)(x-b)(x-c)(x-d)}$.

Using the method of partial fractions we easily find that

$$F(x) \;=\; \sum_{\text{cyclic}} \frac{a^4+1}{(a-b)(a-c)(a-d)}\cdot\frac{1}{x-a} \;=\; \sum_{\text{cyclic}} \frac{A}{x-a}\,.$$

Multiplying the equation above by $(x-a)(x-b)(x-c)(x-d)$ we get

$$(x^4+1)-(x-a)(x-b)(x-c)(x-d) \;=\; \sum_{\text{cyclic}} A(x-b)(x-c)(x-d)\,.$$

Comparing the coefficient of x^3 we get $a+b+c+d = A+B+C+D$, as claimed.

III. *Solution and generalization by Michael Lambrou, University of Crete, Crete, Greece (modified slightly by the editor).*

We show more generally that if $a_1, a_2, \ldots, a_n$ are distinct complex numbers, $n \geq 2$ and if $S_n(k) = \sum_{i=1}^{n} \dfrac{a_i^k}{\prod_{j \neq i}(a_i - a_j)}$, then

$$S_n(k) = \begin{cases} 0 & \text{if} \quad 0 \leq k \leq n-2\,, \\ 1 & \text{if} \quad k = n-1\,, \\ \sum_{i=1}^{n} a_i & \text{if} \quad k = n\,. \end{cases}$$

Since the left side of the given identity is $S_4(4) + S_4(0)$, it equals $\sum_{i=1}^{4} a_i$ as claimed. For $0 \leq k \leq n-2$, using partial fractions, it is easy to see that

$$\frac{x^{k+1}}{\prod_{j=1}^{n}(x - a_j)} = \sum_{i=1}^{n} \frac{a_i^{k+1}}{\prod_{j \neq i}(a_i - a_j)} \cdot \frac{1}{x - a_i}\,. \tag{1}$$

Letting $x = 0$, we get $S_n(k) = 0$ as claimed.

In particular, for $k = n-2$, (1) becomes

$$x^{n-1} = \sum_{i=1}^{n} a_i^{n-1} \prod_{j \neq i} \left(\frac{x - a_j}{a_i - a_j} \right)\,. \tag{2}$$

Comparing the coefficients of x^{n-1} on both sides of (2) then yields

$$1 = \sum_{i=1}^{n} \frac{a_i^{n-1}}{\prod_{j \neq i}(a_i - a_j)} = S_n(n-1)\,.$$

It remains to show that $S_n(n) = \sum_{i=1}^{n} a_i$.

By long division, it is easily seen that

$$\frac{x^n}{\prod_{j=1}^{n}(x - a_j)} = 1 + \frac{\left(\sum_{j=1}^{n} a_j \right) x^{n-1} + f(x)}{\prod_{j=1}^{n}(x - a_j)}\,, \tag{3}$$

where $f(x) = \sum_{k=0}^{n-2} A_k x^k$ is a polynomial of degree at most $n-2$.

For each fixed $i = 1, 2, \ldots, n$, multiplying both sides of (3) by $x - a_i$ and letting $x = a_i$ we get:

$$\frac{a_i^n}{\prod_{j \neq i}(a_i - a_j)} = \frac{\left(\sum_{j=1}^{n} a_j\right) a_i^{n-1} + f(a_i)}{\prod_{j \neq i}(a_i - a_j)}. \tag{4}$$

Adding up (4) for $i = 1, 2, \ldots, n$, and using the facts that

$$S_n(n-1) = 1, S_n(n-2) = \ldots = S_n(1) = S_n(0) = 0,$$

we then obtain

$$\begin{aligned} S_n(n) &= \left(\sum_{j=1}^{n} a_j\right) S_n(n-1) + A_{n-2}S_n(n-2) \\ &\qquad + \ldots + A_1 S_n(1) + A_0 S_n(0) \\ &= \sum_{j=1}^{n} a_j, \end{aligned}$$

as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; ÓSCAR CIAURRI, Universidad La Rioja, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; H. N. GUPTA and J. CHRIS FISHER, University of Regina, Regina, Saskatchewan; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; R. LAUMEN, Antwerp, Belgium; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Brooklyn Technical HS, Brooklyn, NY, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; JEREMY YOUNG, student, Nottingham High School, Nottingham, UK; and the proposer.

Gupta and Fisher showed that the result holds if the 1*'s in the denominators are replaced by a constant* k*. Using Lagrange's Interpolation Theorem, Seiffert also obtained the more general result that* $S_n(n) + cS_n(0) = \left(\sum_{i=1}^{n} a_i\right)$ *where* c *is any complex constant. See notations in* III *above. The special case of this when* $c = 1$ *was also obtained by Ciaurri by using the theory of residue in complex analysis. Of course, all of these generalizations are contained in Solution* III *above. The most general result was given by Janous who, also using the theory of residue, showed that if* m *denotes any non-negative integer, then* $S_n(m) = \sum a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}$ *where the summation is over all non-negative integers* $\alpha_1, \alpha_2, \ldots \alpha_n$ *such that* $\alpha_1 + \alpha_2 + \cdots + \alpha_n = m - n + 1$*. It is easily seen that this result implies the one given in* III *above. Laumen pointed out that in the article* Shreds and Slices: Cyclic Sums the Easy Way *which appeared in* Mathematical Mayhem (*Vol. 8, issue 4, p. 3*)

[*Ed: Written by Naoki Sato*], *it was proved that* $\sum\limits_{\text{cyclic}} \dfrac{a^4}{(a-b)(a-c)(a-d)} = a+b+c+d$. *Hence to prove the given identity, it suffices to show that* $\sum\limits_{\text{cyclic}} \dfrac{1}{(a-b)(a-c)(a-d)} = 0$ *which can be easily verified by straightforward computations.*

Murray S. Klamkin, University of Alberta, Edmonton, Alberta commented that the identity is a special case of some more general and known results about cyclic sums. Using the notation of Solution III *above, this result, which can be found in* A Treatise on the Theory of Determinants *by T. Muir (Dover, NY, 1960, pp. 329–331), essentially states that* $S_n(k) = 0$ *if* $0 \le k < n-1$ *and equals the complete symmetric function of the* a_i*'s of degree* $k-n+1$ *if* $k \ge n-1$. (*For example, when* $k = n+1$, *the sum is* $\sum\limits_{i=1}^{n} a_i^2 + \sum\limits_{i \ne j} a_i a_j$.) *For* $k > n$, *this extends the results obtained by Lambrou in* Solution III *above.*

2488. [1999 : 431] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S_n = \mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_{n+1}$ be a simplex in $\mathbb{E}^n$, and M a point in S_n. It is known that there are real positive numbers λ_1, λ_2, $\ldots$, λ_{n+1} such that $\sum\limits_{j=1}^{n+1} \lambda_j = 1$ and $M = \sum\limits_{j=1}^{n+1} \lambda_j \mathbf{A}_j$ (here, by a point $\mathbf{P}$, we mean the position vector $\overrightarrow{\mathbf{OP}}$). Suppose also that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n+1}$, and let $\mathbf{B}_k = \dfrac{1}{k} \sum\limits_{j=1}^{k} \mathbf{A}_j$.

Prove that

$$M \in \text{ convex cover of } \{\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{n+1}\}\,;$$

that is, there are real positive numbers μ_1, μ_2, $\ldots$, μ_{n+1} such that

$$M = \sum_{k=1}^{n+1} \mu_k \mathbf{B}_k\,.$$

Note the necessary condition for a convex cover: $\sum\limits_{k=1}^{n+1} \mu_k = 1$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Comment. The condition given in the original problem was "$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$". This has been corrected to "$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n+1}$". Furthermore, $\mu_1, \mu_2, \ldots, \mu_{n+1}$ being "real positive numbers" has to be weakened to "real non-negative numbers" as the following example shows. Let $n = 2$ and $\lambda_1 = \lambda_2 = 1/2$. Then $M = (A_1 + A_2)/2$, $B_1 = A_1$ and $B_2 = (A_1 + A_2)/2 = M$, whence $M = 0 \cdot B_1 + 1 \cdot B_2$!

[Ed. The editor, not the proposer, was responsible for these oversights.]

Now, from $B_k = (A_1 + \cdots + A_k)/k$, $M = \lambda_1 A_1 + \cdots + \lambda_{n+1} A_{n+1}$, and the required $M = \mu_1 B_1 + \cdots + \mu_{n+1} B_{n+1}$, we infer that

$$\begin{aligned} M &= \left(\mu_1 + \frac{\mu_2}{2} + \cdots + \frac{\mu_{n+1}}{n+1}\right) A_1 + \left(\frac{\mu_2}{2} + \cdots + \frac{\mu_{n+1}}{n+1}\right) A_2 \\ & \quad + \cdots + \left(\frac{\mu_n}{n} + \frac{\mu_{n+1}}{n+1}\right) A_n + \left(\frac{\mu_{n+1}}{n+1}\right) A_{n+1}\,. \end{aligned}$$

Whence, "looking from the back", we get:

$$\begin{aligned} \mu_{n+1} &= (n+1)\lambda_{n+1}\,, \\ \mu_n &= n(\lambda_n - \lambda_{n+1})\,, \\ \mu_{n-1} &= (n-1)(\lambda_{n-1} - \lambda_n)\,, \\ & \vdots \\ \mu_2 &= 2(\lambda_2 - \lambda_3)\,, \quad \text{and finally} \\ \mu_1 &= \lambda_1 - \lambda_2\,. \end{aligned}$$

Of course,

$$\begin{aligned} \mu_1 + \cdots + \mu_{n+1} &= (\lambda_1 - \lambda_2) + (2\lambda_2 - 2\lambda_3) + \cdots \\ & \qquad \cdots + (n\lambda_n - n\lambda_{n+1}) + (n+1)\lambda_{n+1} \\ &= \lambda_1 + \cdots + \lambda_{n+1} \;=\; 1\,. \end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOSÉ LUIS DIAZ, Universitat Politècnica de Catalunya, Terrassa, Spain; and the proposer.

All solvers found Janous's two corrections to the statement of the problem, as well as the correction mentioned on [2000 : 179].

2489. [1999: 505] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*

The set of twelve vertices of a regular icosahedron can be partitioned into three sets of four vertices, each being such that none of the sets have their four vertices forming a golden rectangle. In how many different ways can this be done?

Solution by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain.

Let τ be the positive solution of the equation $\tau^2 = 1 + \tau$; that is, $\tau = \frac{\sqrt{5}+1}{2}$. Golden rectangles are those similar to one with side lengths 1 and τ.

One regular icosahedron inscribed into the edges of the regular octahedron of vertices $(0,0,\pm\tau^2)$, $(0,\pm\tau^2,0)$, $(\pm\tau^2,0,0)$ (and that divides those sections into golden sections), has the following twelve vertices (following H.S.M. Coxeter, *Fundamentos de Geometría*, Spanish version of the *Introduction to Geometry*, Limusa, 1988, pp. 193–195; see also, by the same author, *Regular Polytopes*, Dover, 1973, pp. 50–53):

$$\begin{array}{ll} 1\ (0,\tau,1) & \overline{1}\ (0,-\tau,-1) \\ 2\ (1,0,\tau) & \overline{2}\ (-1,0,-\tau) \\ 3\ (\tau,1,0) & \overline{3}\ (-\tau,-1,0) \\ 4\ (-\tau,1,0) & \overline{4}\ (\tau,-1,0) \\ 5\ (-1,0,\tau) & \overline{5}\ (1,0,-\tau) \\ 6\ (0,-\tau,1) & \overline{6}\ (0,\tau,-1) \end{array}$$

(Point $\overline{n}$ is the point opposite to n.)

Let us space out these twelve vertices on 4 layers, according to Euclid *xiii*, 16:

- Layer 1. – The point 1.
- Layer 2. – Points 2, 3, 4, 5, 6. (These are the vertices of a regular plane pentagon.)
- Layer 3. – Points $\overline{2}$, $\overline{3}$, $\overline{4}$, $\overline{5}$, $\overline{6}$. (These form, with the vertices of layer 2, a pentagonal antiprism.)
- Layer 4. – The point $\overline{1}$.

The distances from point 1 to any of the vertices of layer 2 are all equal to 2 (edges of the regular icosahedron), since, for example, $12^2 = 1 + \tau^2 + (1-\tau)^2 = 4$.

The distances from point 1 to any of the vertices of layer 3 are all equal to 2τ, since, for example, $1\overline{2}^2 = 1 + \tau^2 + (1+\tau)^2 = 4\tau^2$.

Further $1\overline{1}^2 = 4\tau^2 + 4 = 4\tau + 8$, so that the distance $1\overline{1}$ is equal to $2\sqrt{2+\tau}$.

Therefore, the unique golden rectangles formed by four vertices of the above regular icosahedron are built with two sides of length 2 (two edges) and two sides of length 2τ. But any given edge, say the 12 edge, is only parallel to its opposite edge (the edge $\overline{12}$), so that the only golden rectangles formed by four vertices of any regular icosahedron are the fifteen rectangles formed from two opposite edges of the polyhedron. For our icosahedron, these are the following (the sequence of vertices listed is not necessarily the geometric one):

$$\begin{array}{lllll} 12\overline{12} & 23\overline{23} & 34\overline{34} & 45\overline{45} & 56\overline{56} \\ 13\overline{13} & 24\overline{24} & 35\overline{35} & 46\overline{46} & \\ 14\overline{14} & 25\overline{25} & 36\overline{36} & & \\ 15\overline{15} & 26\overline{26} & & & \\ 16\overline{16} & & & & \end{array}$$

And finally we count the number of partitions of the twelve vertices of the polyhedron into three sets of vertices:

- **Total number of such partitions**:

$$\frac{1}{3!} \cdot \binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4} \;=\; 5775\,.$$

- **Partitions with three golden rectangles** (Number of partitions for which all three sets of vertices form golden rectangles):

$$\frac{1}{3!} \cdot 15 \cdot 6 \cdot 1 \;=\; 15\,.$$

- **Partitions with exactly two golden rectangles**: 0 (there are none.)

- **Partitions with exactly one golden rectangle**:

$$\frac{1}{3!} \cdot 15 \cdot \left[\binom{8}{4} - 6\right] \cdot 1 \;=\; 160\,,$$

where the factor $1/3!$ in each expression is present because the order of choosing the three sets is irrelevant. Thus, our answer to the proposed question is : $5775 - 15 - 160 = 5600$ different partitions.

There was one incorrect solution.

2490. [1999 : 505] *Proposed by Mihály Bencze, Brasov, Romania.*
Let $\alpha > 1$. Denote by x_n the only positive root of the equation:

$$(x+n^2)(2x+n^2)(3x+n^2)\cdots(nx+n^2) \;=\; \alpha\, n^{2n}\,.$$

Find $\lim\limits_{n\to\infty} x_n$.

Solution by Kee-Wai Lau, Hong Kong, China.

For $x \geq 0$ let $f(x) = \left(1+\frac{x}{n^2}\right)\left(1+\frac{2x}{n^2}\right)\cdots\left(1+\frac{nx}{n^2}\right) - \alpha$. Then the given equation is equivalent to $f(x) = 0$.

Since f is strictly increasing for $x > 0$, $f(0) < 0$ and $f(x) \to \infty$ as $x \to \infty$, the equation $f(x) = 0$ has a unique positive root x_n.

We note that $\alpha = f(x_n) + \alpha > \sum\limits_{k=1}^{n} \dfrac{kx_n}{n^2} = \dfrac{(n+1)x_n}{2n} > \dfrac{x_n}{2}$, and so, we have that $x_n < 2\alpha$.

It is well known that

$$x - \frac{x^2}{2} \;<\; \ln(1+x) < x \;\text{ for }\; 0 < x < 1\,. \tag{1}$$

If $n > 2\alpha$, then $x_n < n$, and so, we have that $0 < \dfrac{kx_n}{n^2} < 1$ for all $k = 1, 2, \ldots, n$.

Letting $x = \dfrac{kx_n}{n^2}$ in (1), and summing the inequalities from $k = 1$ to $k = n$, we obtain

$$-\frac{1}{2}\sum_{k=1}^{n}\left(\frac{kx_n}{n^2}\right)^2 \;<\; \sum_{k=1}^{n}\left(\ln\left(1+\frac{kx_n}{n^2}\right) - \frac{kx_n}{n^2}\right) \;<\; 0\,,$$

or

$$-\frac{(n+1)(2n+1)x_n^2}{12n^3} \;<\; \ln\alpha - \frac{(n+1)x_n}{2n} \;<\; 0\,,$$

whenever $n > 2\alpha$.

Letting $n \to \infty$, we then have

$$0 \;\le\; \lim_{n\to\infty}\left(\ln\alpha - \frac{x_n}{2}\right) \;\le\; 0\,,$$

from which it follows immediately that $\lim\limits_{n\to\infty} x_n = 2\ln\alpha$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Most of the other submitted solutions are far more complicated than the one given above. Among the methods used were: differentiation, integration, Intermediate Value Theorem, Mean Value Theorem, AM–GM Inequality, Jensen's Inequality, Weierstrass' Inequality, (generalized) Bernoulli's Inequality, Bolzano's Theorem, uniform boundedness, and majorization!

Konečný gave a one-line "proof" based on the "fact" that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{kx}{n^2}\right) \;=\; e^{x/2}$$

which he believed "must be well known", but could not find a reference. Neither could this editor. Can any reader supply a reference or a proof of this?

2491. [1999: 505] *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are two geometric sequences for which

$$\sum_{k=1}^{n} f(a_k) \;<\; 0 \;<\; \sum_{k=1}^{n} f(b_k)\,.$$

Prove that there exists a geometric sequence $\{c_k\}_{k=1}^n$ for which

$$\sum_{k=1}^{n} f(c_k) \;=\; 0\,.$$

Compilation of essentially identical solutions by Manuel Benito and Emilio Fernandez, I.B. Praxedes Mateo Sagasta, Logroño, Spain; Jonathan Campbell, student, Chapel Hill High School, Chapel Hill, NC, USA; Michael Lambrou, University of Crete, Crete, Greece; and Kee-Wai Lau, Hong Kong, China.

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = \sum_{k=1}^{n} f(xy^k)$. This is a continuous function. Thus, there exist (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2$ satisfying $g(x_1,y_1) < 0$ and $g(x_2,y_2) > 0$. By the Intermediate Value Theorem, there exists $(p_1,q_1) \in \mathbb{R}^2$ satisfying $g(p_1,q_1) = 0$. Take $c_k = p_1 q_1^{k-1}$ for $k = 1, 2, \ldots, n$, and we are done.

Also solved by MICHEL BATAILLE, Rouen, France; NIKOLAOS DERGIADES, Thessaloniki, Greece; KEITH EKBLAW, Walla Walla, WA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one incomplete solution.

2492. [1999: 506] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, suppose that $\angle BAC$ is a right angle. Let I be the incentre of $\triangle ABC$, and suppose that D and E are the intersections of BI and CI with AC and AB, respectively. Let points P and Q be on BC such that $IP \| AB$ and $IQ \| AC$.

Prove that $BE + CD = 2PQ$.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The converse is also true. More precisely, we see that

for any $\triangle ABC$, $BE + CD = 2PQ$ if and only if A is a right angle.

Since $\triangle IPQ \sim \triangle ABC$ and the altitude to side PQ of $\triangle IPQ$ is the inradius of $\triangle ABC$, $PQ : a = r : h_a$. Using $[ABC] = ah_a = r(a+b+c)$, we deduce that $PQ/a = a/(a+b+c)$, so that

$$PQ = \frac{a^2}{a+b+c}.$$

Since $BE : EA = a : b$, we get

$$BE = \frac{ac}{a+b}.$$

Similarly,

$$CD = \frac{ab}{a+c}.$$

Therefore, the following six statements are equivalent:

$$BE + CD = 2PQ,$$

$$\begin{aligned}
\frac{ac}{a+b}+\frac{ab}{a+c} &= \frac{2a^2}{a+b+c}\,,\\
\frac{c}{a+b}+\frac{b}{a+c}-\frac{2a}{a+b+c} &= 0\,,\\
\frac{(2a+b+c)(b^2+c^2-a^2)}{(a+b)(a+c)(a+b+c)} &= 0\,,\\
b^2+c^2-a^2 &= 0\,,
\end{aligned}$$

$\triangle ABC$ has a right angle at A.

II. *Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Reflect D in CI giving D', and reflect E in BI giving E'. Since

$$\angle BIC \;=\; 180^\circ - \frac{\angle B+\angle C}{2} \;=\; 180^\circ - 45^\circ \;=\; 135^\circ\,,$$

we have

$$\angle BIE \;=\; \angle CID \;=\; 45^\circ\,,\quad \angle EIE' \;=\; \angle DID' \;=\; 90^\circ\,.$$

Furthermore, since $IP||AB$, we have $\angle BIP = \angle IBE = \angle PBI$, which means that triangle IBP is isosceles, and consequently, that P, as the intersection of BC with the perpendicular bisector of BI, is the circumcentre of the right triangle IBD'. From this, it follows that

$$PB \;=\; PI \;=\; PD'\,.$$

Analogously, Q is the circumcentre of the right triangle CIE', so that

$$QC \;=\; QI \;=\; QE'\,.$$

Finally, this means that

$$\begin{aligned}
BE+CD &= BE'+CD' \;=\; (BQ-QE')+(CP-PD')\\
&= (BQ-PD')+(CP-QE')\\
&= (BQ-PB)+(CP-CQ)\\
&= 2PQ\,.
\end{aligned}$$

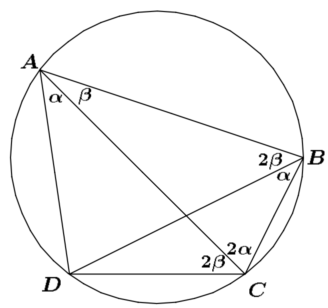
Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; HO-JOO LEE, student, Kwangwoon University, Kangwon-Do, South Korea; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; ANDRÉ LOUIS SOUZA DE ARAÚJO, Rio de Janeiro; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; CHOONGYUP SUNG, Pusan, Korea; ALBERT WHITE, Bonaventure, NY; PETER Y. WOO, Biola University, La Mirada, CA, USA; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer.

Only Seiffert and Janous mentioned that the converse also holds.

2493. [1999: 506] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that $ABCD$ is a convex cyclic quadrilateral, that $\angle ACB = 2\angle CAD$, and that $\angle ACD = 2\angle BAC$.

Prove that $BC + CD = AC$.

Solution by Jeremy Young, student, University of Cambridge, Cambridge, UK.



Let $\angle CAD = \alpha$ and $\angle BAC = \beta$. Then $\angle ACB = 2\alpha$ and $\angle ACD = 2\beta$. Since the quadrilateral $ABCD$ is cyclic, we have that $\angle DAB + \angle DCB = 180^\circ$, which gives $(\alpha + \beta) + (2\alpha + 2\beta) = 180^\circ$; that is, $\alpha + \beta = 60^\circ$. Let R be the circumradius of the quadrilateral $ABCD$. By the Sine Rule applied to $\triangle ABC$, $BC = 2R\sin\beta$. Similarly, from $\triangle ADC$, $CD = 2R\sin\alpha$. Then

$$\begin{aligned} BC + CD &= 2R\sin\beta + 2R\sin\alpha \;=\; 2R\left(\sin\beta + \sin(60^\circ - \beta)\right) \\ &= 2R\left(\sin\beta + \tfrac{\sqrt{3}}{2}\cos\beta - \tfrac{1}{2}\sin\beta\right) \\ &= 2R\left(\tfrac{1}{2}\sin\beta + \tfrac{\sqrt{3}}{2}\cos\beta\right) \\ &= 2R\sin(60^\circ + \beta) \;=\; 2R\sin(\alpha + 2\beta) \\ &= 2R\sin\angle ABC \;=\; AC\,, \end{aligned}$$

as desired.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; HENRY J. PAN, student, East York Collegiate Institute, Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria(two solutions); HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, *Greece; D.J. SMEENK, Zaltbommel, the Netherlands; ANDRÉ LOUIS SOUZA de ARAÚJO, Instituto Militar de Engenharia, Brazil; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; CHOONGYUP SUNG, Pusan Science High School, Pusan, Korea; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

Most of the submitted solutions are similar to the one given above.

2494. [1999 : 506] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given $\triangle ABC$ with $AB < AC$, let I be the incentre and M be the mid-point of BC. The line MI meets AB and AC at P and Q, respectively. A tangent to the incircle meets sides AB and AC at D and E, respectively.

Prove that $\dfrac{AP}{BD} + \dfrac{AQ}{CE} = \dfrac{PQ}{2MI}$.

Solution by the proposer.

Let L and N be the mid-points of BE and CD, respectively. Since M is the mid-point of BC, we have $ML||CE$, $ML = \frac{1}{2}CE$, and $MN||BD$, $MN = \frac{1}{2}BD$. We put $\angle APQ = \theta$ and $\angle AQP = \phi$. Since $MN||PA$, we get $\angle NMI = \angle APQ = \theta$. Similarly, we have $\angle LMI = \angle AQP = \phi$. Since quadrilateral $BCED$ has an incircle with centre I, we have that L, N and I are collinear by Newton's Theorem. By the Law of Sines for $\triangle APQ$, we have

$$\frac{AP}{\sin\phi} = \frac{AQ}{\sin\theta} = \frac{PQ}{\sin(\theta+\phi)}\,. \qquad (1)$$

[*Editor's comment*. For the next step we require that I lies between L and N: because D and E are on the sides AB and AC, L and N are on the sides of the mid-point triangle of $\triangle ABC$, while I is always inside that triangle.]

Since I is between L and M, we have $[LMN] = [LMI] + [NMI]$, (where $[XYZ]$ denotes the area of triangle XYZ). Therefore,

$$\frac{1}{2}MN\cdot ML\sin(\theta+\phi) = \frac{1}{2}ML\cdot MI\sin\phi + \frac{1}{2}MN\cdot MI\sin\theta\,.$$

Dividing both sides by $ML\cdot MI\cdot MN$, we get

$$\frac{\sin(\theta+\phi)}{2MI} = \frac{\sin\phi}{2MN} + \frac{\sin\theta}{2ML}\,. \qquad (2)$$

From (1) and (2), we obtain

$$\frac{PQ}{2MI} = \frac{AP}{2MN} + \frac{AQ}{2ML}\,.$$

Since $2MN = BD$ and $2ML = CE$, we have

$$\frac{PQ}{2MI} = \frac{AP}{BD} + \frac{AQ}{CE}\,.$$

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; and CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK.

2495. [1999 : 506, 2000 : 238] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let P be the interior isodynamic point of $\triangle ABC$; that is, $\dfrac{AP}{bc} = \dfrac{BP}{ca} = \dfrac{CP}{ab}$ (a, b, c are the side lengths, BC, CA, AB, of $\triangle ABC$).

Prove that the pedal triangle of P has area $\dfrac{\sqrt{3}}{d^2}F^2$, where F is the area of $\triangle ABC$ and $d^2 = \dfrac{a^2+b^2+c^2}{2} + 2\sqrt{3}F$.

I. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

Denote the projection of P onto BC, CA and AB by K, L and M, respectively. Let $\lambda > 0$ be the common ratio that defines P; that is,

$$PA \;=\; \lambda bc\,,\; PB \;=\; \lambda ca\,,\; PC \;=\; \lambda ab\,.$$

Because BP is the diameter of the circumcircle of $\triangle BKM$, the Sine Law applied to that triangle implies

$$KM \;=\; PB\sin\beta \;=\; \lambda ac\sin\beta \;=\; 2\lambda F \tag{1}$$

(where F is the area of $\triangle ABC$). In the same way $LM = LK = 2\lambda F$. Thus, the pedal triangle KLM is equilateral and its area $[KLM] = (\sqrt{3}/4)KM^2$. Moreover, $60^\circ = \angle PKM + \angle PKL = \angle PBM + \angle PCL$, so that

$$\angle BPC \;=\; 180^\circ - \angle PBC - \angle PCB \;=\; 180^\circ - (\beta+\gamma-60^\circ) \;=\; \alpha + 60^\circ\,.$$

Define B' to be the point outside $\triangle ABC$ for which $\triangle ACB'$ is equilateral. The Cosine Law for $\triangle ABB'$ yields

$$\begin{aligned} BB'^2 &= b^2+c^2-2bc\cos(\alpha+60^\circ) \\ &= b^2+c^2-bc\cos\alpha+2bc(\sqrt{3}/2)\sin\alpha \\ &= b^2+c^2+\frac{a^2-b^2-c^2}{2}+2\sqrt{3}F \\ &= \frac{a^2+b^2+c^2}{2}+2\sqrt{3}F \\ &= d^2\,. \end{aligned}$$

Since $PB : PC = AB : AB' = c : b$ and $\angle BPC = \angle BAB'$, $\triangle PBC \sim \triangle ABB'$ and therefore, $PB : a = c : d$, so that $PB = \dfrac{ac}{d}$. This together with (1) implies that

$$KM \;=\; PB\sin\beta \;=\; \frac{ac\sin\beta}{d} \;=\; \frac{2F}{d}\,.$$

Thus, $[KLM] = \left(\dfrac{\sqrt{3}}{4}\right)KM^2 = \dfrac{F^2\sqrt{3}}{d^2}$, as desired.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

The formula for the area $[KLM]$ of the pedal triangle of an isodynamic point is derived in [2, p. 24] in the form

$$[KLM] = \frac{F}{2(\cot\omega\cot 60^\circ \pm 1)},$$

where ω is the Brocard angle, "$+$" is used for our interior isodynamic point, and "$-$" for the exterior isodynamic point. The desired form comes from replacing $\cot\omega$ by its equivalent [4, p. 266]

$$\cot\omega = \frac{a^2+b^2+c^2}{4F}.$$

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

This problem is connected with a variety of familiar results. An arbitrary point P inside a triangle, whose distances from the vertices are x, y and z, determines a pedal triangle whose sides are $\frac{ax}{2R}$, $\frac{by}{2R}$ and $\frac{cz}{2R}$ [1, p. 23]. *Thus, we deduce immediately that the pedal triangle in this problem is equilateral. See also* **2377** [1999 : 438], *which deals specifically with equilateral pedal triangles. More formulas for the area of a pedal triangle can be found in the solutions to* **1076** [1987 : 62-64]; *or see* [3, p. 235].

Here is one further connection. The three segments that join a vertex of $\triangle ABC$ to the remote vertex of an equilateral triangle erected externally on the opposite side all have length d and pass through one isogonic centre — the point from which the angles subtended by the sides are all 120° [4, pp. 218-221 *and* 295-296]. *This point is the isogonal conjugate of our interior isodynamic point; it is perhaps more familiar as the Fermat-Torricelli point: when no angle of $\triangle ABC$ is as great as 120°, this point minimizes the sum $x+y+z$, thus resolving a problem Fermat proposed to Torricelli.*

References

[1] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*. Math. Assoc. of America New Mathematical Library 19, 1967.

[2] W. Gallatly, *The Modern Geometry of the Triangle*. Hodgson, London, 1910.

[3] Ross Honsberger, *From Erdős to Kiev: Problems of Olympiad Caliber*. (Dolciani Mathematical Expositions, No. 17). Math. Assoc. of America, 1995.

[4] R.A. Johnson, *Advanced Euclidean Geometry*. Dover, N.Y., 1960.

2496. [1999 : 506] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a triangle ABC, let C_A be the circle tangent to the sides AB, AC and to the circumcircle internally. Define C_B and C_C analogously. Find the triangle, unique up to similarity, for which the inradius and the radii of the three circles C_A, C_B and C_C are in arithmetic progression.

Solution by the proposer.

We denote by r, r_A, r_B and r_C the radii of the incircle and the circles C_A, C_B and C_C, respectively. It is known [Leon Bankhoff, *A Mixtilinear Adventure*, Crux Math. 9 (1983), 2–7] that

$$\begin{aligned} r_A &= r\sec^2\tfrac{A}{2} = r\left(1+\tan^2\tfrac{A}{2}\right), \\ r_B &= r\sec^2\tfrac{B}{2} = r\left(1+\tan^2\tfrac{B}{2}\right), \\ r_C &= r\sec^2\tfrac{C}{2} = r\left(1+\tan^2\tfrac{C}{2}\right). \end{aligned} \qquad (1)$$

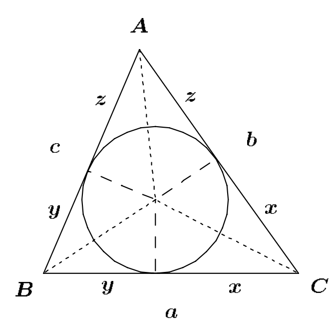
If we assume, without loss of generality, that $r \le r_A \le r_B \le r_C$, then the condition that these radii are in arithmetic progression implies that $r_A = r + d$, $r_B = r + 2d$ and $r_C = r + 3d$ for some common difference d.

Using (1), it follows that $d = r\tan^2\frac{A}{2}$, $2d = r\tan^2\frac{B}{2}$ and $3d = r\tan^2\frac{C}{2}$.

From this it follows that

$$\tan\tfrac{B}{2} = \sqrt{2}\tan\tfrac{A}{2} \quad\text{and}\quad \tan\tfrac{C}{2} = \sqrt{3}\tan\tfrac{A}{2}. \qquad (2)$$

If we let a, b and c be the sides opposite angles A, B and C, respectively, we know that we can write $a = x + y$, $b = x + z$ and $c = y + z$, as shown in the diagram to the right. We notice that $\tan\frac{A}{2} = \frac{r}{z}$, $\tan\frac{B}{2} = \frac{r}{y}$ and $\tan\frac{C}{2} = \frac{r}{x}$. Using (2) above, we easily deduce that $x : y : z = \sqrt{2} : \sqrt{3} : \sqrt{6}$, and hence, it follows that the radii are in the desired arithmetic progression if the three sides of the triangle are in the ratio



$$a : b : c = \sqrt{2}+\sqrt{3} : \sqrt{2}+\sqrt{6} : \sqrt{3}+\sqrt{6}. \qquad (3)$$

Furthermore, we know that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$, which, in combination with (2), gives us $\tan^2\frac{A}{2} = \left(\sqrt{2}+\sqrt{3}+\sqrt{6}\right)^{-1}$. This tells us that the angles in the triangle are

$$\begin{aligned} A &= 2\tan^{-1}\frac{1}{\sqrt{\sqrt{2}+\sqrt{3}+\sqrt{6}}} \approx 45.83^\circ, \\ B &= 2\tan^{-1}\frac{\sqrt{2}}{\sqrt{\sqrt{2}+\sqrt{3}+\sqrt{6}}} \approx 61.75^\circ \quad\text{and} \\ C &= 2\tan^{-1}\frac{\sqrt{3}}{\sqrt{\sqrt{2}+\sqrt{3}+\sqrt{6}}} \approx 72.42^\circ. \end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, *Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA. RICHARD I. HESS, Rancho Palos Verdes, CA, USA replaced "circumcircle" in the statement of the problem by "incircle", and gave a solution in this case.*

All of the correct solutions that were submitted made use of the formula $r_A = r\sec^2\frac{A}{2}$, *or of a variant of this. This formula was either assumed to be known (with suitable references to the literature), or was derived as part of the solution. Seiffert used (1) to deduce the answer in the form* $b = \left(\sqrt{3}+1\right)\left(\sqrt{6}-2\right)a$ *and* $c = \left(\sqrt{2}+1\right)\left(3-\sqrt{6}\right)a$ *[which can be re-scaled to give (3)]. Woo used an inversion argument, by inverting in the circle centred at* A *with radius* $\sqrt{bc}$. *If this inversion takes* B *to* B' *and* C *to* C', *then the circumcircle is taken to the line* $B'C'$ *and the circle* C_A *is inverted into the excircle of* $\triangle ABC$, *which lies opposite angle* A. *He then showed that* $r_A = r_a\frac{bc}{s^2}$, *where* r_a *is the radius of this excircle, and* s *is the semi-perimeter. Finally, letting* K *be the area of* $\triangle ABC$, *he showed that* $r_A = \frac{Kbc}{s^2(s-a)} = \frac{rbc}{s(s-a)}$ *and* $\tan\frac{A}{2} = \frac{r}{s-a}$ *to deduce that* $\frac{bc}{s(s-a)} = \sec^2\frac{A}{2}$, *and hence, that* $r_A = r\sec^2\frac{A}{2}$. *[Woo notes that he used the same argument to solve problem* **2464**.*] Bradley used a power-of-point argument to deduce that* $(r_A - r)\csc^2\frac{A}{2} = r_A$, *from which* $r_A = r\sec^2\frac{A}{2}$ *follows at once. Benito and E. Fernández use elementary means to show that* $r_A = \frac{bc}{s}\tan\frac{A}{2}$. *They complete their proof by showing that* $\frac{b}{c} = \frac{\sqrt{6}}{3}\left(\sqrt{2}-1\right)\left(\sqrt{3}+1\right)$ *and* $\frac{a}{c} = \frac{\sqrt{3}}{3}\left(\sqrt{2}-1\right)\left(\sqrt{3}+\sqrt{2}\right)$ *[which can be re-scaled to give (3)].*

2497. [1999: 506] *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

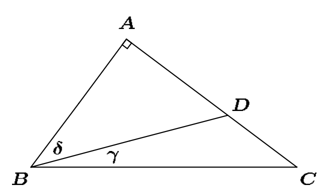
Given $\triangle ABC$ and a point D on AC, let $\angle ABD = \delta$ and $\angle DBC = \gamma$. Find all values of $\angle BAC$ for which $\frac{\delta}{\gamma} > \frac{AD}{DC}$.

Solution by the proposer.

First, suppose that $\angle BAC = \frac{\pi}{2}$.

Since

$$\frac{AD}{DC} = \frac{\frac{AD}{AB}}{\frac{AC}{AB} - \frac{AD}{AB}} = \frac{\tan\delta}{\tan(\gamma+\delta) - \tan\delta},$$



the given inequality can be written as

$$\frac{\delta}{\gamma} > \frac{\tan\delta}{\tan(\gamma+\delta) - \tan\delta} \qquad \left(\text{where } 0 < \gamma+\delta < \tfrac{\pi}{2}\right).$$

Using Lagranges's Mean Value Theorem, this is equivalent to

$$\frac{\tan(\gamma+\delta) - \tan\delta}{(\gamma+\delta) - \delta} > \frac{\tan\delta - \tan 0}{\delta - 0},$$

or

$$\frac{d}{dt}\tan t\bigg|_{t=\xi_1} > \frac{d}{dt}\tan t\bigg|_{t=\xi_2} \qquad \left(0 < \xi_2 < \delta < \xi_1 < \gamma+\delta < \tfrac{\pi}{2}\right);$$

that is,

$$\frac{1}{\cos^2\xi_1} > \frac{1}{\cos^2\xi_2}\,, \quad \text{or} \quad \cos\xi_1 < \cos\xi_2\,, \quad \text{which is true.}$$

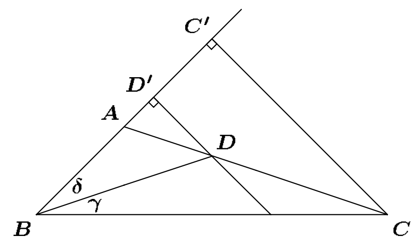
Now suppose that $\angle BAC > \frac{\pi}{2}$. Let D' and C' be points on AB such that $DD' \perp BA$ and $CC' \perp BA$.

Then

$$\tan(\gamma+\delta) = \frac{CC'}{BC'} \qquad (1)$$

and

$$\frac{1}{\tan\delta} = \frac{BD'}{DD'} < \frac{BC'}{DD'}\,. \qquad (2)$$



By multiplying (1) and (2) we get

$$\frac{\tan(\gamma+\delta)}{\tan\delta} < \frac{CC'}{DD'} = \frac{AC}{AD} \qquad \text{since } \triangle ADD' \sim \triangle ACC'.$$

This gives, equivalently,

$$\frac{\tan(\gamma+\delta)-\tan\delta}{\tan\delta} < \frac{AC-AD}{AD} = \frac{DC}{AD}\,,$$

or

$$\frac{\tan\delta}{\tan(\gamma+\delta)-\tan\delta} > \frac{AD}{DC}\,.$$

Thus, to prove that $\dfrac{\delta}{\gamma} > \dfrac{AD}{DC}$, it is sufficient to prove that, for $0 < \gamma+\delta < \frac{\pi}{2}$,

$$\frac{\delta}{\gamma} > \frac{\tan\delta}{\tan(\gamma+\delta)-\tan\delta}\,.$$

But this inequality has been proved previously, and so, the given inequality is also valid for every $\angle BAC > \frac{\pi}{2}$. [Ed. remember $\angle BAC < \pi$.]

Lastly, suppose that $\angle BAC < \frac{\pi}{2}$. Then the inequality is not valid. To show this, it is sufficient to create a counterexample for such an angle.

We construct $\triangle ABC$ with $\angle BCA > \angle CAB$, so that $BA > BC$. Let BD be the bisector of $\angle ABC$, so that $\frac{\delta}{\gamma} = 1$. But $\frac{AD}{DC} = \frac{BA}{BC} > 1$, giving $\frac{\delta}{\gamma} < \frac{AD}{DC}$.

In conclusion, the given inequality is valid for every $\triangle ABC$ with $\angle BAC \geq \frac{\pi}{2}$.

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; and CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK. There was one partial solution.

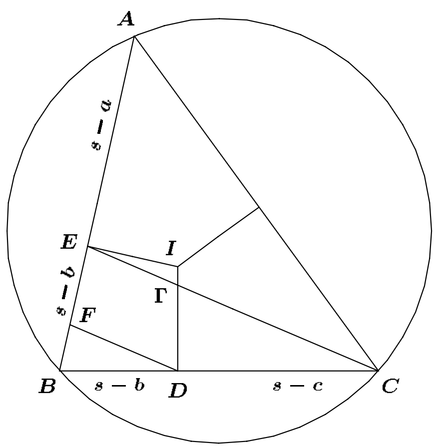
2498. [1999 : 507] *Proposed by K.R.S. Sastry, Dodballapur, India.*

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

In an integer triangle ABC, prove that the Gergonne point Γ bisects the Gergonne cevian AD if and only if b, c, $\frac{1}{2}|3a - b - c|$ form a triangle where the measure of the angle between b and c is $\frac{\pi}{3}$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

See the figure below. The Gergonne cevian through C intersects AB at E. Point F lies on AB so that $DF||CE$. Thus, $A\Gamma : \Gamma D = AE : EF$. Also, $AE = s - a$, $EB = s - b = BD$ and $DC = s - c$, where s is the semiperimeter.



Now Γ is the mid-point of AD if and only if $EF = AE = s - a$. Since $EF : EB = DC : BC$, this is equivalent to $a(s - a) = (s - b)(s - c)$, or to

$$3a^2 - b^2 - c^2 + 2bc - 2ac - 2ab \;=\; 0\,. \qquad (1)$$

In a triangle with sides b and c, and an angle $\pi/3$ between b and c, for the third side a', we have ${a'}^2 = b^2 + c^2 - bc$. Thus, we are to show $4b^2 + 4c^2 - 4bc = (3a - b - c)^2$, or

$$3a^2 - b^2 - c^2 + 2bc - 2ac - 2ab \;=\; 0\,. \qquad (2)$$

(1) and (2) are identical; that is it!

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Most solvers noted that the triangle need not be integer-sided, as can be seen from the above proof. Also, many noted that the absolute value signs can be dropped, since (for example) from (1), it follows that $3a^2 - 2ac - 2ab \geq 0$, so that $3a - b - c$ must be positive.

Janous asserts that the condition of the problem is also equivalent to $r_a = r_b + r_c$; that is, one exradius is the sum of the other two, which readers may enjoy establishing for themselves.

2499. [1999 : 507] *Proposed by K.R.S. Sastry, Dodballapur, India.*

A Gergonne cevian is the line segment from a vertex of a triangle to the point of contact, on the opposite side, of the incircle. The Gergonne point is the point of concurrency of the Gergonne cevians.

Prove or disprove:

two Gergonne cevians may be perpendicular to each other.

Solution by Toshio Seimiya, Kawasaki, Japan.

More precisely, we shall prove that *each side of a triangle subtends an obtuse angle at the Gergonne point*; that is, we prove that if P is the Gergonne point of $\triangle ABC$, then $\angle BPC > 90^\circ$, $\angle CPA > 90^\circ$, $\angle APB > 90^\circ$.

The incircle touches BC, CA and AB at D, E and F, respectively. Then $AE = AF$, $BF = BD$, $CD = CE$, and AD, BE and CF are concurrent at P. The proof is by contradiction. We assume that $\angle BPC \leq 90^\circ$. Then

$$BP^2 + CP^2 \;\geq\; BC^2 \,. \tag{1}$$

Moreover, we have $\angle CPE \geq 90^\circ$. Therefore, $CE > CP$. Similarly, we get $BF > BP$. Thus,

$$BP^2 + CP^2 \;<\; BF^2 + CE^2 \;=\; BD^2 + CD^2 \;<\; (BD + CD)^2 \;=\; BC^2 \,.$$

That is, $BP^2 + CP^2 < BC^2$, which contradicts (1). Therefore $\angle BPC \leq 90^\circ$ is not true. Thus, we have $\angle BPC > 90^\circ$, as desired. Similarly, we have $\angle CPA > 90^\circ$ and $\angle APB > 90^\circ$.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer. There was one incorrect solution.

Sastry's solution exploited a related problem of his (#227, *College Math. J.* **15**:2 (March, 1984) 165-166). *For a convex quadrangle such as $BCEF$, $BC^2 + EF^2 = BF^2 + CE^2$ if and only if $BE \perp CF$.*

2500. [1999 : 507] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
In the lattice plane, the unit circle is the incircle of $\triangle ABC$.

Determine all possible triangles ABC.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.
Let a, b and c be the side lengths, s the semiperimeter, Δ the area and r the inradius of $\triangle ABC$. Without loss of generality, let us assume

$$a \;\leq\; b \;\leq\; c\,. \tag{1}$$

It is well known that

$$\Delta \;=\; sr \;=\; \sqrt{s(s-a)(s-b)(s-c)}\,.$$

Since $r = 1$, we obtain

$$\Delta \;=\; s\,, \tag{2}$$

or,

$$(b+c-a)(c+a-b)(a+b-c) \;=\; 4(a+b+c)\,. \tag{3}$$

In the lattice plane, the area of a triangle is a rational number, by Pick's Theorem; hence, (2) implies that s must also be a rational number. Since A, B and C are lattice points, then a, b and c are square roots of integers. Consequently, s can be rational only when a, b and c are positive integers. Let

$$x \;=\; b+c-a\,, \quad y \;=\; c+a-b \quad \text{and} \quad z \;=\; a+b-c\,. \tag{4}$$

Then

$$a \;=\; \frac{y+z}{2}\,, \quad b \;=\; \frac{z+x}{2} \quad \text{and} \quad c \;=\; \frac{x+y}{2}\,. \tag{5}$$

Equation (3) becomes

$$xyz \;=\; 4(x+y+z)\,. \tag{6}$$

From (1), (4), (5) and (6), it follows that x, y and z are positive even integers with $z \leq y \leq x$. Then

$$xyz = 4(x+y+z) \;\leq\; 12x \;\Longrightarrow\; yz \;\leq\; 12 \;\Longrightarrow\; z \;=\; 2\,.$$

From (6), $x = 2 + \frac{16}{2y-4}$, which implies $y = 4$ and $x = 6$. This gives the only triangle with the required property: *the right triangle with sides* $a = 3$, $b = 4$ *and* $c = 5$. The distance between the vertex of the right angle and the origin is $\sqrt{2}$, so that the vertex of the right angle must be at one of the points $(\pm 1, \pm 1)$. This gives 4 possible locations of the triangle in the lattice plane.

Four other locations can be obtained by reflections about the lines through the origin and the vertex of a right angle.

Also solved by MANUEL BENITO and EMILIO FERNANDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer. There was also one incorrect solution submitted. Yiu pointed out that the following fact is known: The only triangle with integer-valued sides and with inradius $r = 1$ is the $(3, 4, 5)$ triangle. *Yiu refers to the editor's note to Problem 1168, American Mathematical Monthly, 63 (1956), p. 43-44.*

Problems from One Hundred Years Ago

Preliminary Examination for the Army, 1888

Time allowed — 2 hours

1. Add $2\frac{6}{7}$ to $4\frac{11}{35}$.
2. Subtract $3\frac{5}{8}$ from $6\frac{7}{68}$.
3. Multiply $\frac{11}{42}$ by $\frac{28}{55}$.
4. Divide $\frac{13}{96}$ by $\frac{65}{84}$.
5. Add together 4.30726, $.076428$, 371.864 and 20.0472.
6. Subtract 47.063782 from 701.04681.
7. Multiply 40.637 by $.028403$.
8. Divide 8.31183 by 23.05.
9. Reduce 1.047 of 2 weeks 5 hours to minutes and the decimal of a minute.
10. In 347693 inches, how many miles, furlongs, pols, yards, etc.?
11. What would a tax of 3s. 11d. in the pound amount to in £480?
12. Find the simple interest of £11175 in $2\frac{1}{2}$ years at 2 per cent per annum.

How life has changed!!

YEAR END FINALE

Again, a year has flown by! It is difficult to realize that I have now done this job for five years. My term has two years to go until the end of 2002, when you will have a new Editor-in-Chief.

There are many people that I wish to thank most sincerely for particular contributions. Again, first and foremost is BILL SANDS. Bill is of such value to me and to the continuance of ***CRUX with MAYHEM***. As well, I thank most sincerely, ILIYA BLUSKOV, ROLAND EDDY, CHRIS FISHER, CLAYTON HALFYARD, GEORG GUNTHER, BILL SANDS, JIM TOTTEN, and EDWARD WANG, for their regular yeoman service in assessing the solutions; BRUCE GILLIGAN, ED BARBEAU, CHRIS FISHER, RICHARD McINTOSH, DONALD RIDEOUT, DIETER RUOFF, GARY SNEDDON, for ensuring that we have quality articles; ALAN LAW, for ensuring that we have quality book reviews, ROBERT WOODROW, who carries the heavy load of two corners, and RICHARD GUY for sage advice whenever necessary.

The editors of the ***MAYHEM*** section, NAOKI SATO, CYRUS HSIA, ADRIAN CHAN, DONNY CHEUNG, JIMMY CHUI and DAVID SAVITT, all do a sterling job. We look forward to welcoming SHAWN GODIN, the incoming Mayhem Editor.

I also thank all those who assist with proofreading. The quality of all these people is a vital part of what makes ***CRUX with MAYHEM*** what it is. Thank you one and all.

As well, I would like to give special thanks to our Associate Editor, CLAYTON HALFYARD, for continuous sage advice, and for keeping me from printing too many typographical and mathematical errors; and to my colleagues, YURI BAHTURIN, DAVID PIKE, ROLAND EDDY, EDGAR GOODAIRE, MAURICE OLESON, MIKE PARMENTER, DONALD RIDEOUT, NABIL SHALABY, in the Department of Mathematics and Statistics at Memorial University, and to JOHN GRANT McLOUGHLIN, Faculty of Education, Memorial University, for their occasional sage advice. I have also been helped by some Memorial University students, KARELYN DAVIS, ALASDAIR GRAHAM, as well as WISE Summer students, ALLISON BLUNDON and MEGAN CHAYTOR.

The staff of the Department of Mathematics and Statistics at Memorial University deserve special mention for their excellent work and support: ROS ENGLISH, MENIE KAVANAGH, WANDA HEATH, and LEONCE MORRISSEY; as well as the computer and networking expertise of DWAYNE HART and CRAIG SQUIRES.

Also the LaTeX expertise of JOANNE LONGWORTH at the University of Calgary, the ***MAYHEM*** staff, and all others who produce material, is much appreciated.

Thanks to BOB QUACKENBUSH as Managing Editor. The CMS's TEX Editor, MICHAEL DOOB has been very helpful in ensuring that the printed master copies are up to the standard required for the U of T Press, who continue to print a fine product.

The online version of ***CRUX with MAYHEM*** continues to attract attention. We recommend it highly to you. Thanks are due to LOKI JORGENSON, JUDI BORWEIN, and the rest of the team at SFU who are responsible for this.

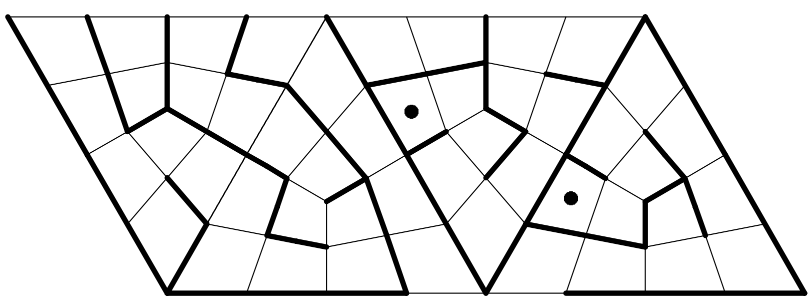
Finally, I would like to express real and heartfelt thanks to the Head of my Department, HERBERT GASKILL, and to the Dean of Science, BOB LUCAS. Without their support and understanding, I would not be able to do the job of Editor-in-Chief.

Last but not least, I send my thanks to you, the readers of ***CRUX with MAYHEM***. Without you, ***CRUX with MAYHEM*** would not be what it is. Keep those contributions and letters coming in. We need your ARTICLES, PROPOSALS and SOLUTIONS to keep ***CRUX with MAYHEM*** alive and well. I do enjoy knowing you all.

A Maze in Three Dimensions

Izador Hafner

Here is a maze on a tetrahedron, given as an unfolded plane plan. Can you solve it? What is the path from one dot to the other?
[Ed. We will print a solution in a subsequent issue. Let us know if you would like more of these amazing mazes.]



Izidor Hafner
Faculty of Electrical Engineering
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1000 Ljubljana, Slovenia

F.G.-M. Mystery solved?

Professor Paul Yiu has provided us with the following quotation:

Some of us often wondered who F.G.-M. was. [See, for example, Hyacinthos Messages 46 and 50]. I have recently acquired a copy of Gabay's 1997 reprint of

F.G.-M.: Trigonométrie (4th edition), 1890.

The inside front cover of the book contains the following interesting note.

Signification des initiales

Lorsqu'un Frère des Écoles Chrétiennes écrivait un livre, son nom n'était pas mentionné, mais on indiquait les initiales du Supérieur général en fonction.

C'est ainsi que les Exercices de géométrie descriptive écrits par le Frère GABRIEL-MARIE, furent publiés pour la première fois en 1877 sous les initiales F.I.-C.

F.I.-C.: Frère IRLIDE – Jean-Pierre CAZENEUVE (nom civil) – Supérieur de 1875 à 1884.

La troisième édition a été publiée en 1893 sous les initiales F.J.

F.J.: Frère JOSEPH – Jean-Marie JOSSERAND (nom civil) – Supérieur de 1884 à 1897.

Par une heureuse coïncidence, la quatrième et la cinquième édition ont été publiées en 1909 et 1920 sous les initiales du véritable auteur, F.G.-M.

F.G.-M.: Frère GABRIEL-MARIE – Edmond BRUNHES (nom civil), 1836-1916 – Supérieur de 1897 à 1913.

There is an endnote explaining that

Ces précisions ont été aimablement communiquées par l'Association La Salle à Paris et par le Centre Scolaire Jean-Baptiste de la Salle à Lyon.

ENGLISH TRANSLATION

What the initials mean

When a Christian Brother wrote a book, his name would not appear anywhere. Instead, the initials of the superior general in office were used.

Hence the Exercices de géométrie descriptive written by brother GABRIEL-MARIE were first published in 1877 under the initials F.I.-C.

F.I.-C.: Frère IRLIDE – Jean-Pierre CAZENEUVE (civil name) – Superior from 1875 to 1884.

The third edition was published in 1893 under the initials F.J.

F.J.: Frère JOSEPH – Jean-Marie JOSSERAND (civil name) – Superior from 1884 to 1897.

It is a happy coincidence that the fourth and fifth editions were published in 1909 and 1920 under the initials of the real author, F.G.-M.

F.G.-M.: Frère GABRIEL-MARIE – Edmond BRUNHES (civil name), 1836-1916 – Superior from 1897 to 1913.

There is an endnote explaining that

These explanations were kindly provided by the Association La Salle in Paris and the Centre Scolaire Jean-Baptiste de la Salle in Lyon.

Announcement

Florida Atlantic University announces the publication of

Forum Geometricorum

an electronic journal devoted to classical Euclidean geometry and related areas, freely accessible to the internet community. The first papers will appear in early 2001. Please visit

http://www.math.fau.edu/ForumGeom/

and enter your free subscription, and support Forum Geometricorum by submitting your papers in Euclidean geometry.

Thanks to Professor Paul Yiu for providing this information.

Dr. Herta Freitag

We have just learned of the passing of a Dr. Herta Freitag, a contributor to ***Crux Mathematicorum*** from 1977 to 1993. Dr. Freitag died in January 2000 at the age of 91.

She was born and raised in Vienna, Austria. Despite having had bad experiences with Mathematics in her early schooling, she became enamoured of Mathematics at age 12 because "this is the one subject where I don't need to memorize, but can just think things out myself".

She graduated in Mathematics and Physics from the University of Vienna in 1934. Circumstances in Europe at that time resulted in a move to England to await an immigration visa to the United States. She moved there in 1944.

After a brief stint as a school teacher in upstate New York, she attended Columbia University, graduating with a master's degree in 1948, and she stayed there to complete her doctorate in 1953. She then accepted a position at Hollins College, (now Hollins University), Roanoke, Virginia, where she spent her entire academic career, "retiring" in 1971.

She always described herself as a "student of Mathematics, albeit, quite an addicted one". She was very interested in problem posing and solving, as our readers will be well aware. She never gave up, leading a very active life, and publishing into the 1990's, including two papers in 1999.

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Faites-nous parvenir votre renouvellement avant le 1er mars 2001 pour éviter tout retard et ne pas manquer de numéro.