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A PROBLEM ON LATTICE POINTS

by

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We are concerned in this note with the following problem.

Among any 37 lattice points in $\mathbb{Z}^3$ there are always 3 points whose centroid is a lattice point.

Starting with a simple solution using the pigeon hole principle, we decrease the bound step by step combining several ideas and basing our work on preceding solutions. We will meet some interesting principles of problem solving.

First some general considerations.

Let $P_1P_2P_3$ be a triangle of lattice points (TLP) and let (a_i, b_i, c_i) be the integer coordinates of the vertex P_i $(i = 1,2,3)$. Then the centroid of the TLP is a lattice point if and only if

$$\sum_{i=1}^{3} a_i \equiv \sum_{i=1}^{3} b_i \equiv \sum_{i=1}^{3} c_i \equiv 0 \pmod 3.$$

So we can reduce the coordinates modulo 3 and get a new TLP, whose centroid is a lattice point if and only if the centroid of the first one is.

Hence we may always assume without loss of generality that the given lattice points are all in M, where M is the set containing the $3^3 = 27$ lattice points with coordinates in $\{-1,0,1\}$.

<u>37 points</u>:

Using the pigeon hole principle we conclude that there are three possible x-coordinates, hence there are at least 13 of 37 points with the same x-coordinate. Among these 13 points we find 5 points with the same y-coordinate. For the z-coordinates of those 5 points with equal x- and y-coordinates we consider two cases.

<u>Case 1</u>: There are three points with equal z-coordinates.

<u>Case 2</u>: No three points have the same z-coordinates. Thus there are three points with pairwise different z-coordinates.

In both cases the centroid of these three points is a lattice point. □

We see that a simple reduction to the corresponding one-dimensional problem is possible. We now want to obtain bounds smaller than 37. From now on, to get a contradiction, we suppose in our proofs that there are no three points with the desired property. We start with

<u>29 points</u>:

The set $M\setminus\{(0,0,0)\}$ splits into 13 pairs of the form $\{(x,y,z),(-x,-y,-z)\}$.

Because of our assumption, no three points coincide. Therefore we find 15 different points. Without loss of generality one of them is A = (0,0,0). Furthermore, two of the remaining 14 points are in the same pair $\{(x,y,z),(-x,-y,-z)\}$. Together with A they form a TLP with integer centroid.

□

We will use the preceding idea to decrease the bound further.

<u>27 points</u>:

As before we have now a set G of 14 different points, one of them being A = (0,0,0). Each of the 13 pairs $\{(x,y,z),(-x,-y,-z)\}$ contains exactly one of the 13 remaining points. This property helps us to find points of G.

We have either $(1,0,0) \in G$ or $(-1,0,0) \in G$. Reflection in the plane $x = 0$ shows that we may assume $(1,0,0) \in G$. In the same way we get $(0,1,0) \in G$ and $(0,0,1) \in G$. Altogether:

$$\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \subset G. \tag{1}$$

Because of $(1,0,0) \in G$ and $(0,1,0) \in G$ we conclude $(-1,-1,0) \notin G$, thus $(1,1,0) \in G$. By analogy,

$$\{(1,1,0),(1,0,1),(0,1,1)\} \subset G. \tag{2}$$

$(1,0,0) \in G$ and $(0,1,1) \in G$ implies $(-1,-1,-1) \notin G$, hence

$$(1,1,1) \in G. \tag{3}$$

As G contains more than 8 points, we find one more point $B = (b_1,b_2,b_3)$ in G with at least one coordinate of value -1.

We define the points $C = (c_1,c_2,c_3)$ and $D = (d_1,d_2,d_3)$ by

$$c_i := \begin{cases} 0 & \text{if } b_i = -1 \\ b_i & \text{otherwise} \end{cases} \qquad d_i := \begin{cases} 1 & \text{if } b_i = -1 \\ b_i & \text{otherwise} \end{cases} .$$

We see that B, C, D are three pairwise different points with integer centroid. (1)-(3) implies $C,D \in G$. Contradiction. □

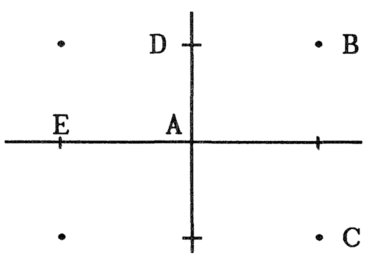
We continue this method by passing to the two-dimensional analogue.

<u>25 points</u>:

The pigeon hole principle assures us that one of the planes $z = 0$, $z = 1$ or $z = -1$ contains at least 9 of the 25 points. The contradiction then follows from the

2-dimensional analogue: Among any 9 lattice points in $\mathbb{Z}^2$ there are always 3 points whose centroid is a lattice point.

Proof. As before, let the coordinates be in $\{-1,0,1\}$. We have a set H of 5 different points, where we may let $A = (0,0)$ be one of them. The set of the remaining 8 points splits into 4 pairs of opposite vertices (with respect to the origin). Each pair contains exactly one point of H. Hence we may assume without loss of generality $B, C \in H$ and furthermore $D \in H$. Then we get $(1,0) \notin H$, so $E \in H$. Now $C, D, E \in H$ gives the desired contradiction. □



Remark: The bound 9 is optimal!

<u>23 points</u>:

Now $|G| = 12$. From the above argument we know that each of the planes $z = 0$, $z = 1$ and $z = -1$ contains exactly 4 points of G.

This is even true for all planes. To define the notion of plane, we introduce the vector space $(F_3)^3$, where F_3 is the field of 3 elements. The reduction of coordinates modulo 3 corresponds to the canonical mapping

$$\mathbb{Z}^3 \longrightarrow (F_3)^3.$$

We get the

Reformulation: Among any 23 points in $(F_3)^3$ there are 3 points with sum $(0,0,0)$.

Planes now are defined to be 2-dimensional affine subspaces of $(F_3)^3$, each of them containing consequently $3^2 = 9$ points.

The set of all planes splits into triplets of parallel planes, of which exactly one contains the origin. For each plane there is an affine mapping onto $(F_3)^2$, so it contains at most 4 different points of G. This follows from the reformulation of the 2-dimensional analogue for $(F_3)^2$.

Now we want to count the planes in $(F_3)^3$ in order to use combinatorial arguments afterwards.

Each plane passing through the origin is spanned by two linearly independent vectors v_1, v_2. There are 26 possible $v_1 \neq (0,0,0)$ and 24 possibilities for $v_2 \notin \{0, v_1, -v_1\}$. However each plane through the origin contains $8 \cdot 6$ pairs of linearly independent vectors. Therefore exactly $\frac{26 \cdot 24}{8 \cdot 6} = 13$ different planes pass through the origin. There are just as many triplets of parallel planes, hence altogether we have 39 planes.

After these geometric and combinatorial preparations we now come to the proof. Since the planes of each triplet contain the 12 different points of G, each plane, however, containing at most 4 points, we know that each of the 39 planes contains exactly 4 points of G. So we are able to count the planes in a second way. We assign to each of the $\binom{12}{3} = 220$ 3-point-subsets of G the plane spanned by them. This assignment is well-defined, because the sum of these three points is not equal to (0,0,0), so they are not on the same line. As each plane contains exactly 4 points, it is counted 4 times and we have 220/4 = 55 planes, a contradiction. □

<u>21 points</u>:

Now we may assume $|G| = 11$. We do not know any more the exact number of points in each plane. But we observe that each triplet consists of two planes containing 4 points and one plane containing 3 points. We say that the triplet is of type 4-4-3.

We count the triplets in a second way, as we did before with the planes. To each of the $\binom{11}{3} = 165$ 3-point-subsets of G we assign the triplet containing the plane determined by them (for well-definedness look above). A certain triplet is counted by some subset if and only if these three points are all in one plane of the triplet. As each triplet is of type 4-4-3, it is counted $4 + 4 + 1 = 9$ times. Hence there are $165/9 \notin \mathbb{Z}$ triplets, contradiction. □

This argument of divisibility does not even need the exact number of triplets!

We now prove the best possible bound. This time we cannot count the triplets in a second way but give an estimate.

<u>19 points</u>:

We may assume $|G| = 10$. Each triplet is of type 4-4-2 or 4-3-3.

There are $\binom{10}{3} = 120$ 3-point-subsets of G. Each of the 13 triplets is counted either 4 + 4 + 0 or 4 + 1 + 1 times, i.e., at most 8 times. But $\binom{10}{3} > 8 \cdot 13$, contradiction. □

Conclusion:

Among any 19 points in the space lattice $\mathbb{Z}^3$ there are always 3 points with integer centroid.

The reader may check that the following choice of 18 points has no three with integer centroid:

$P_1 = P_2 = (0,0,1)$

$P_3 = P_4 = (0,-1,1)$

$P_5 = P_6 = (1,1,1)$

$P_7 = P_8 = (-1,1,1)$

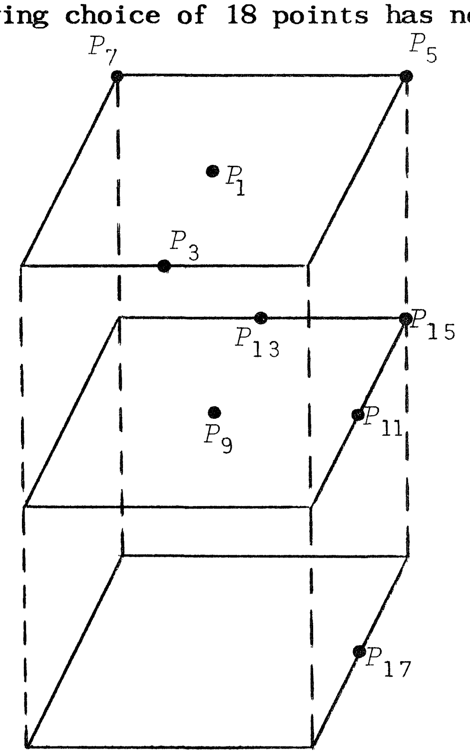
$P_9 = P_{10} = (0,0,0)$

$P_{11} = P_{12} = (1,0,0)$

$P_{13} = P_{14} = (0,1,0)$

$P_{15} = P_{16} = (1,1,0)$

$P_{17} = P_{18} = (1,0,-1)$



Finally a little problem: Using a similar combinatorial estimate as in our last proof show that in each possible example with $|G| = 9$ there occurs a triplet of type 4-4-1. So if one tries to find an example for $|G| = 9$ one must only look at the case 4-4-1!

* * *

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

I begin this issue of the Corner with a plea to those of you with access to regional/national Olympiad/pre-Olympiad problem material to send it to me as soon as possible. Fresh contest material is vital to the life of such a column as this, whose aim is a widespread dissemination of Olympiad quality problems and their elegant solutions.

The first problems we present are from the American Invitational Mathematics Examination (AIME) written March 24, 1987. The time allowed was three hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, CAMC Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588-0322.

1. An ordered pair (m,n) of non-negative integers is called "simple" if the addition $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

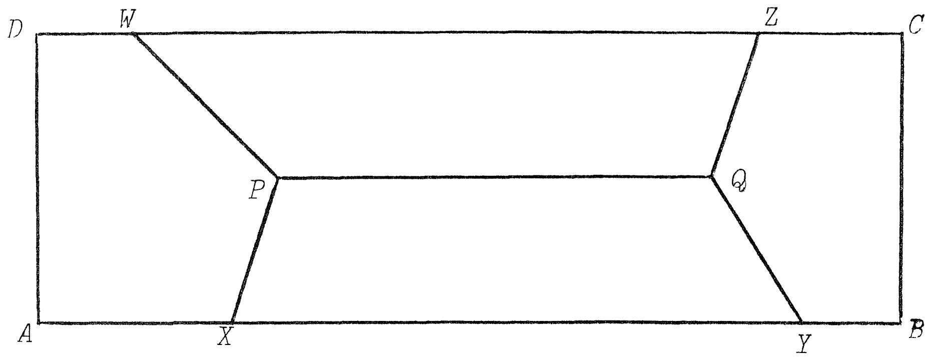
2. What is the largest possible distance between two points, one on the sphere of radius 19 with center $(-2,-10,5)$ and the other on the sphere of radius 87 with center $(12,8,-16)$?

3. By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called "nice" if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

4. Find the area of the region enclosed by the graph of
$$|x - 60| + |y| = |x/4|.$$

5. Find $3x^2y^2$ if x and y are integers such that
$$y^2 + 3x^2y^2 = 30x^2 + 517.$$

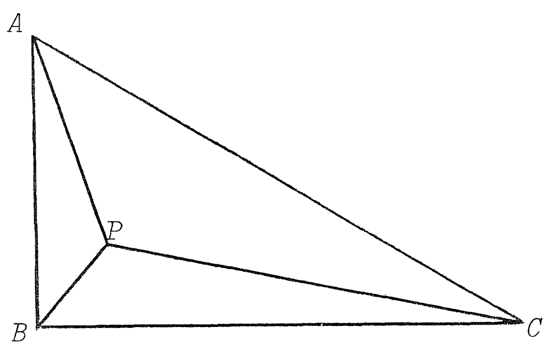
6. Rectangle $ABCD$ is divided into four parts of equal area by five segments as shown in the figure, where $XY = YB + BC + CZ = ZW = WD + DA + AX$, and PQ is parallel to AB. Find the length of AB (in cm) if $BC = 19$ cm and $PQ = 87$ cm.



7. Let $[r,s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples (a,b,c) of positive integers for which $[a,b] = 1000$, $[b,c] = 2000$, and $[c,a] = 2000$.

8. What is the largest positive integer n for which there is a unique integer k such that $\frac{8}{15} < \frac{n}{n+k} < \frac{7}{13}$?

9. Triangle ABC has right angle at B, and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.



10. Al walks down to the bottom of an escalator that is moving up and he counts 150 steps. His friend, Bob, walks up to the top of the escalator and counts 75 steps. If Al's speed of walking (in steps per unit time) is three times Bob's speed, how many steps are visible on the escalator at any given time? (Assume that this number is constant.)

11. Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive positive integers.

12. Let m be the smallest positive integer whose cube root is of the form $n + r$, where n is a positive integer and r is a positive real number less than $1/1000$. Find n.

13. A given sequence $r_1, r_2, \ldots, r_n$ of distinct real numbers can be put in ascending order by means of one or more "bubble passes". A bubble pass through a given sequence consists of comparing the second term with the first term and exchanging them if and only if the second term is smaller, then comparing the third term with the current second term and exchanging them if and only if the third term is smaller, and so on in order, through comparing the last term, r_n, with its current predecessor and exchanging them if and only if the last term is smaller.

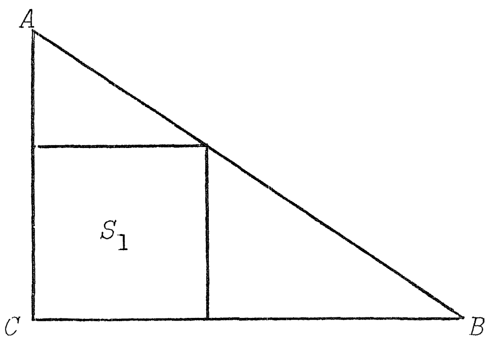
The example on the right shows how the sequence 1, 9, 8, 7 is transformed into the sequence 1, 8, 7, 9 by one bubble pass. The numbers compared at each step are underlined.

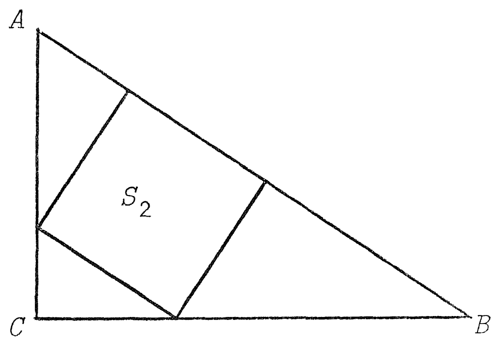
1	9	8	7
1	9	8	7
1	8	9	7
1	8	7	9

Suppose that $n = 40$, and that the terms of the initial sequence $r_1, r_2, \ldots, r_{40}$ are distinct from one another and are in random order. Let p/q, in lowest terms, be the probability that the number that begins as r_{20} will end up, after one bubble pass, in the 30th place (i.e., will have 29 terms on its left and 10 terms on its right). Find $p + q$.

14. Compute $\dfrac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}$.

15. Squares S_1 and S_2 are inscribed in right triangle ABC, as shown in the figures below. Find $AC + CB$ if area$(S_1) = 441$ and area$(S_2) = 440$.





* * *

The next set of five problems we publish are selected problems from the *Bulgarian Winter Competition*, January 16, 1986. We thank Jordan Tabov for forwarding them to us.

1. The equation $x^2 + px + q = 0$ has real roots x_1 and x_2. Determine p and q, if $\dfrac{1}{1 + x_1}$ and $\dfrac{1}{1 + x_2}$ are also roots of the same equation. (Grade 8)

2. Solve the equation $\dfrac{1}{[x]} + \dfrac{1}{[2x]} = \{x\}$ where $[x]$ denotes the greatest integer which does not exceed x, and $\{x\} = x - [x]$. (Grade 9)

3. The point C lies on the segment AB and $|AC| > |CB|$. Find the locus of the point M having the following property: there exists a circle with center C and radius $r \leq |CB|$ such that M is the point of intersection of the tangents to this circle passing through A and B. (Grade 9)

4. Given that $a \geq 1$ and b are real numbers, prove that the system

$$y = x^3 + ax + b$$
$$z = y^3 + ay + b$$
$$x = z^3 + az + b$$

has exactly one real solution. (Grade 11)

5. A regular tetrahedron $ABCD$ of edge a is given.

(a) Prove that for each point L on the segment AC there exist unique points M_L on the face ABD and N_L on the face BCD for which the perimeter $P(LM_LN_L)$ of the triangle LM_LN_L attains its minimum value.

(b) Find the minimum value of $P(LM_LN_L)$ as L describes the segment AC. (Grade 11)

* * *

We now continue with solutions that have been submitted for problems posed in the June 1985 number of the Corner.

1. [1985: 169] *The 1985 Australian Mathematical Olympiad.*

Let $f(n)$ be the sum of the first n terms of the sequence

0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, ...

(a) Give a formula for $f(n)$.

(b) Prove that $f(s + t) - f(s - t) = st$ where s and t are positive integers and $s > t$.

Solutions by Beno Arbel, School of Mathematics, Tel Aviv University, Israel; and by Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin.

(a) From the statement of the problem, it follows that $f(1) = 0$ and

(1) $f(n) = f(n-1) + [n/2]$ for each positive integer $n > 1$, where $[x]$ denotes the greatest integer in the real number x.

Next we prove that

(2) $f(2n-1) = n^2 - n$ for each positive integer n

and

(3) $f(2n) = n^2$ for each positive integer n.

Since $f(1) = 0$ and $f(2) = 1$ both (2) and (3) hold for $n = 1$. Assume that

(4) $f(2k-1) = k^2 - k$

and

(5) $f(2k) = k^2$

where k is a positive integer. Then

$$f(2k+1) = f(2k) + k = k^2 + k = (k+1)^2 - (k+1)$$

and

$$f(2k+2) = f(2k+1) + k + 1 = k^2 + k + (k+1) = (k+1)^2.$$

This establishes (2) and (3) by induction.

Now if $n = 2k-1$ then $n^2 = 4k^2 - 4k + 1$ and $[n^2/4] = k^2 - k$. For $n = 2k$, $n^2 = 4k^2$ and $[n^2/4] = k^2$. This combined with (2) and (3) gives

$$f(n) = [n^2/4] \quad \text{for} \quad n = 1,2,3,\ldots .$$

(b) There are two cases.

Case 1. $s + t = 2j$ for some positive integer j. Then

$$s - t = s + t - 2t = 2j - 2t = 2(j-t)$$

where $j > t$ since $s > t$. Also $s = 2j - t$. Thus

$$\begin{aligned} f(s+t) - f(s-t) &= f(2j) - f(2(j-t)) \\ &= j^2 - (j-t)^2 \qquad \text{(by (3))} \\ &= (2j-t)t \\ &= st. \end{aligned}$$

Case 2. $s + t = 2\ell - 1$ for some positive integer ℓ. Then

$$s - t = 2(\ell - t) - 1$$

where $\ell > t$ because $s > t$. Also $s = 2\ell - t - 1$. Now

$$\begin{aligned} f(s+t) - f(s-t) &= f(2\ell-1) - f(2(\ell-t)-1) \\ &= \ell^2 - \ell - [(\ell-t)^2 - (\ell-t)] \\ &= (2\ell - t - 1)t \\ &= st, \end{aligned}$$

completing the proof.

2. [1985: 169] *The 1985 Australian Mathematical Olympiad.*

If x, y, z are real numbers such that

$$x + y + z = 5 \qquad \text{and} \qquad yz + zx + xy = 3,$$

prove that $-1 \leq z \leq 13/3$.

Comment by George Evagelopoulos, Law student, Athens, Greece.

You can find this problem and two solutions of it in the following MAA publication: *1001 Problems in High School Mathematics - Book 4*, Problem 305, pages 12, 17.

One of the solutions is really nice. I give you this solution now!

Since

$$(x + y)^2 = (5 - z)^2$$

and

$$xy = 3 - z(x + y) = 3 - z(5 - z)$$

we have that

$$\begin{aligned} 0 \leq (x - y)^2 &= (x + y)^2 - 4xy \\ &= 25 - 10z + z^2 - 12 + 20z - 4z^2 \\ &= -3z^2 + 10z + 13 \\ &= -(z + 1)(3z - 13). \end{aligned}$$

Hence $-1 \leq z \leq 13/3$.

[*Solutions were also submitted by Beno Arbel, School of Mathematics, Tel Aviv; and by Curtis Cooper, Central Missouri State University.*]

4. [1985: 169] *The 1985 Australian Mathematical Olympiad.*

ABC is a triangle whose angles are smaller than 120°. Equilateral triangles *AFB*, *BDC*, and *CEA* are constructed on the sides of and exterior to triangle *ABC*.

(a) Prove that the lines *AD*, *BE*, and *CF* pass through one point S.

(b) Prove that $SD + SE + SF = 2(SA + SB + SC)$.

Solution by Fred A. Millar, Elkins, W.Va.

This problem is discussed in *Modern College Geometry* by David Davis, pp.63-64 (published by Addison-Wesley).

(a) Let *M*, *N*, *O* be the circumcenters of triangles *BDC*, *CEA* and *AFB* respectively.

Then $AF = AB$, $AC = AE$, $\angle FAC = \angle BAE = A + 60^\circ$, $\Delta FAC \cong \Delta BAE$, and $FC = BE$. Similarly it can be seen that $BE = AD$, so $FC = BE = AD$. The circumcircles of ΔAFB and ΔAEC have one point in common at A and will intersect in a second point S. Then $\angle BSA = 120^\circ$ since it is a supplement of $\angle F$. Also $\angle ASE = \angle ACE = 60^\circ$, so that $\angle BSE = \angle BSA + \angle ASE = 180^\circ$ and BSE is a straight line. Likewise, FSC is a straight line. But, since $\angle ASB = 120^\circ = \angle ASC$ we have $\angle BSC = 120^\circ$ and S lies on the circumcircle of $\angle BDC$. It follows that ASD is also a straight line and S is the point of intersection of the three lines AD, BE and CF and of the circumcircles of triangles AEB, BDC and CEA.

Finally consider triangle OAN where $AB/AO = \sqrt{3}$, $AE/AN = \sqrt{3}$, $\angle OAN = \angle BAE = \angle A + 60^\circ$. Thus $\Delta OAN \sim \Delta BAE$ so that $BE = ON\sqrt{3}$ and likewise $AD = NM\sqrt{3}$, $FC = OM\sqrt{3}$. Thus ΔMNO is equilateral and $\angle ASB = \angle BSC = \angle CSA = 120^\circ$.

(b) By Ptolemy's Theorem,

$$\begin{aligned} DS\cdot BC &= CS\cdot BD + BS\cdot DC, \\ ES\cdot AC &= AS\cdot CE + AE\cdot CS, \\ FS\cdot AB &= AS\cdot BF + AF\cdot BS. \end{aligned}$$

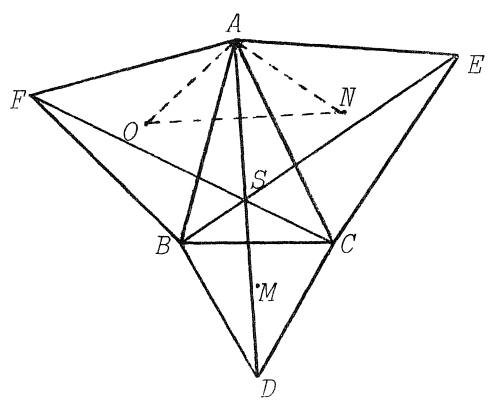
But $BC = BD = DC$, $AC = CE = AE$, and $AB = BF = AF$, so we read off

$$\begin{aligned} DS &= CS + BS, \\ ES &= AS + CS, \\ FS &= AS + BS, \end{aligned}$$

and adding we obtain

$$DS + ES + FS = 2(AS + BS + CS)$$

as desired.



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2. [1985: 170] *34th Bulgarian Mathematical Olympiad* (3rd Stage).

Determine the values of the parameter a for which the equation

$$\ell n\ 2x\ \ell n\ 3x = a$$

has two distinct solutions, and find the product of these two solutions.

Solution by Beno Arbel, School of Mathematics, Tel Aviv University, Israel.

Write the given equation as

$$(\ell n\ 2 + \ell n\ x)(\ell n\ 3 + \ell n\ x) = a.$$

This becomes

$$(\ell n\ x)^2 + (\ell n\ 2 + \ell n\ 3)(\ell n\ x) + (\ell n\ 2)(\ell n\ 3) - \alpha = 0. \qquad (*)$$

This has two distinct solutions (for $\ell n\ x$ and hence for x) just in case

$$\Lambda = (\ell n\ 2 + \ell n\ 3)^2 - 4(\ell n\ 2\ \ell n\ 3 - \alpha) > 0$$

or

$$4\alpha > 4\ell n\ 2\ \ell n\ 3 - (\ell n\ 2 + \ell n\ 3)^2.$$

This becomes

$$4\alpha > -(\ell n\ 2 - \ell n\ 3)^2$$

and

$$\alpha > -1/4(\ell n\ 2/3)^2.$$

Now if α and β are the two solutions of (*) we have

$$\ell n\ \alpha\beta = \ell n\ \alpha + \ell n\ \beta = -(\ell n\ 2 + \ell n\ 3) = -\ell n\ 6.$$

Thus

$$\alpha\beta = e^{-\ell n\ 6} = 1/6.$$

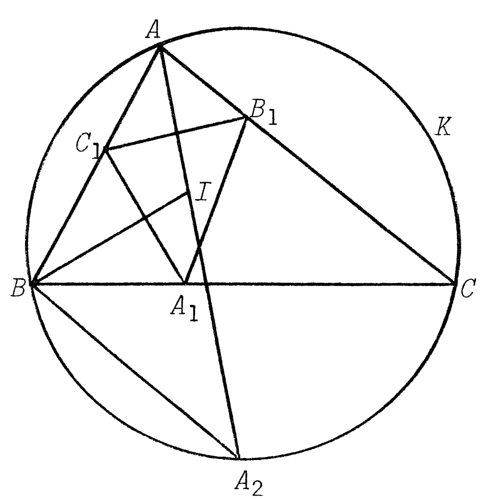
5. [1985: 170] *34th Bulgarian Mathematical Olympiad* (3rd Stage).

A triangle ABC of area S is inscribed in a circle K of radius 1. The orthogonal projections of the incenter I of ABC on the sides BC, CA, and AB are A_1, B_1, and C_1, respectively, and S_1 is the area of triangle $A_1B_1C_1$. If the line AI meets K in A_2, prove that

$$4S_1 = AI \cdot A_2B \cdot S.$$

Solution by Beno Arbel, School of Mathematics, Tel Aviv University, Tel Aviv, Israel.

Let us denote, as usual, the radius of the inscribed circle by r and of the circumscribed circle by R (in this case $R = 1$).



It is easily seen that

$$S_1 = \frac{r^2}{2}(\sin A + \sin B + \sin C). \qquad (1)$$

Let a, b, c denote the lengths of the sides of ΔABC opposite A, B, C, respectively. From $S = \frac{a+b+c}{2}r$ and the law of sines we obtain

$$\begin{aligned} S &= rR(\sin A + \sin B + \sin C) \\ &= r(\sin A + \sin B + \sin C) \qquad (2) \end{aligned}$$

since $R = 1$.

Now, from chord A_2C, $\angle A_2BC = \angle A_2AC = A/2$. Chord AB gives $\angle BA_2A = C$. Thus $\angle IBA_2 = A/2 + B/2$ and

$$\angle BIA_2 = 180^\circ - (B/2 + A/2 + C) = A + B + C - B/2 - A/2 - C = A/2 + B/2.$$

Thus $A_2B = IA_2$.

Moreover $IA_2 = \frac{r}{2 \sin B/2 \sin C/2}$, which is obtained from ΔIBA_2 after remarking that $BI = \frac{r}{\sin B/2}$. So

$$AI \cdot A_2B = AI \cdot IA_2 = \frac{r^2}{2 \sin A/2 \sin B/2 \sin C/2}. \qquad (3)$$

From (1), (2), and (3) the equality to be demonstrated becomes

$$2r^2(\sin A + \sin B + \sin C) = \frac{r^3(\sin A + \sin B + \sin C)}{2 \sin A/2 \sin B/2 \sin C/2}.$$

After cancellation this becomes

$$r = 4 \sin A/2 \sin B/2 \sin C/2$$

which follows from the general relation

$$r = 4R \sin A/2 \sin B/2 \sin C/2$$

which may be found on p.186 of *Advanced Euclidean Geometry*, R. Johnson, Dover, 1960.

* * *

Several problems from the June 1985 number remain without elegant solutions covered in this column. Solutions are welcome! Also, no solutions are on file for the problems from the 16th Austrian Mathematical Olympiad (September 1985 issue of *Crux*). Next issue we will continue with solutions to problems from the October 1985 Olympiad Corner, but most remain without elegant solutions submitted. Here is the challenge (especially for high school students) to come up with a good solution and have it presented in this column.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

<u>1231</u>. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

On the planet of Lyre the inhabitants carefully recognize special years when their age is of the form p^2q where p and q are different prime numbers. On Lyre one is a student until he reaches a special year immediately following a special year; he then becomes a master until he reaches a year that is the third in a row of consecutive special years; he then becomes a sage until he dies in a special year that is the fourth in a row of consecutive special years.

(a) When does one become a master?

(b) When does one become a sage?

(c) How long do the inhabitants of Lyre live?

(d)* Do five special years ever occur consecutively?

<u>1232</u>. *Proposed by Esther and George Szekeres, University of New South Wales, Kensington, Australia.*

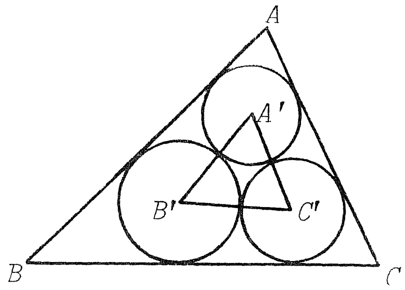
Let n be a positive integer not equal to 1, 2, 3, 6, or 15. Show that there is a positive integer $x \le [n/2] - 1$ such that both x and $2x + 1$ are relatively prime to n.

<u>1233</u>. *Proposed by Jordan Stoyanov, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In the plane we have given the line ℓ: $y = 43/25\ x + 25/43$. For $\epsilon > 0$ denote by S_ϵ the ϵ-neighbourhood of ℓ, i.e. S_ϵ is the strip containing all points in the plane whose distance to ℓ is not greater than ϵ. Find a value for ϵ such that S_ϵ contains no points with integer coordinates.

1234*. *Proposed by Jack Garfunkel, Flushing, New York.*

Given the Malfatti configuration of three circles inscribed in triangle ABC as shown, let A', B', C' be the centers of the three circles, and let r and r' be the inradii of triangles ABC and $A'B'C'$ respectively. Prove that



$$r \le (1 + \sqrt{3})r'.$$

Equality is attained when ABC is equilateral.

1235. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

In triangle ABC, D and E are the feet of the altitudes of BC and AC respectively, K and L are the midpoints of BC and AC respectively, H is the orthocentre, O is the circumcentre. Prove that if $LD \parallel EK$ then $EK \parallel HO$. Does the converse hold?

1236. *Proposed by Gordon Fick, University of Calgary, Calgary, Alberta.*

Prove *without calculus* that if $0 \le \theta \le 1$, and $0 \le y \le n$ where y and n are integers, then

$$\theta^y(1 - \theta)^{n-y} \le (y/n)^y(1 - y/n)^{n-y}.$$

In statistics, this says that the sample proportion is the maximum likelihood estimator of the population proportion. To the best of my knowledge, all mathematical statistics texts prove this result with calculus.

1237*. *Proposed by Niels Bejlegaard, Stavanger, Norway.*

If m_a, m_b, m_c denote the medians to the sides a, b, c of a triangle ABC, and s is the semiperimeter of ABC, show that

$$\Sigma\, a \cos A \le \frac{2}{3} \Sigma\, m_a \sin A \le s,$$

where the sums are cyclic.

1238. *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

Let $A = a^4$ where a is a positive integer. Find all positive integers x such that

$$A^{15x+1} \equiv A \bmod 6814407600,$$

or prove that there are none.

1239. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find all points whose pedal triangles with respect to a given triangle are isosceles and right-angled.

1240. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find distinct positive integers a, b, c such that

$$a + b + c, \qquad ab + bc + ca, \qquad abc$$

forms an arithmetic progression.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1019. [1985: 51; 1986: 152] *Proposed by Weixuan Li and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine the largest constant k such that the inequality

$$x \le \alpha \sin x + (1 - \alpha)\tan x$$

holds for all $\alpha \le k$ and for all $x \in [0,\pi/2)$.

(The inequality obtained when α is replaced by 2/3 is the Snell-Huygens inequality, which is fully discussed in Problem 115 [1976: 98-99, 111-113, 137-138].)

Editor's comment. To the published solution I should probably have added the remark that, as $\sin x \le \tan x$ for $0 \le x < \pi/2$,

$$\alpha \sin x + (1 - \alpha)\tan x = \tan x - \alpha(\tan x - \sin x)$$

increases for fixed $x \in [0,\pi/2)$ as α decreases. Thus since the given inequality holds for $\alpha = 2/3$ it will hold for all $\alpha < 2/3$.

* * *

1087. [1985: 289] *Proposed by Robert Downes, student, Moravian College, Bethlehem, Pennsylvania.*

Let a, b, c, d be four positive numbers.

(a) There exists a regular tetrahedron $ABCD$ and a point P in space such that $PA = a$, $PB = b$, $PC = c$, and $PD = d$ if and only if a, b, c, d satisfy what condition?

(b) This condition being satisfied, calculate the edge length of the regular tetrahedron $ABCD$.

(For the corresponding problem in a plane, see Problem 39 [1975: 64; 1976: 7].)

Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

We generalize the problem to show that, for positive numbers $a_0, a_1, \ldots, a_n$, there exists a regular simplex S: $A_0A_1\ldots A_n$ and a point P in its space such that $PA_i = a_i$, $i = 0,1,\ldots,n$, if and only if

$$I \equiv \left\{ \sum_{i=0}^{n} a_i^2 \right\}^2 - n \sum_{i=0}^{n} a_i^4 \geq 0. \tag{1}$$

The side length a of S is then given by

$$na^2 = \sum_{i=0}^{n} a_i^2 \pm \sqrt{(n+1)I}. \tag{2}$$

We choose the origin of a vector coordinate system at the circumcenter O of S, and for a point Z we denote $\overrightarrow{OZ}$ by $\mathbf{Z}$. It now follows that the distance between points $U = (u_0, u_1, \ldots, u_n)$ and $V = (v_0, v_1, \ldots, v_n)$ is given by

$$\begin{aligned}(UV)^2 &= (\mathbf{U} - \mathbf{V})^2 = \left[\sum_{i=0}^{n} (u_i - v_i)\mathbf{A_i} \right]^2 \\ &= \sum_{i=0}^{n} (u_i - v_i)^2 \mathbf{A}_\mathbf{i}^2 + 2 \sum_{i \neq j} (u_i - v_i)(u_j - v_j)\mathbf{A_i} \cdot \mathbf{A_j}.\end{aligned}$$

Since $\mathbf{A}_\mathbf{i}^2 = R^2$, where R is the circumradius of S, and $2\mathbf{A_i} \cdot \mathbf{A_j} = 2R^2 - a^2$,

$$\begin{aligned}(UV)^2 &= R^2 \sum_{i=0}^{n} (u_i - v_i)^2 + \sum_{i \neq j} (u_i - v_i)(u_j - v_j)(2R^2 - a^2) \\ &= R^2 \left\{ \sum_{i=0}^{n} (u_i - v_i) \right\}^2 - a^2 \sum_{i \neq j} (u_i - v_i)(u_j - v_j) \\ &= -a^2 \sum_{i \neq j} (u_i - v_i)(u_j - v_j).\end{aligned} \tag{3}$$

Letting the barycentric coordinates of P be $(x_0, x_1, \ldots, x_n)$, and since the coordinates of A_0 are $(1,0,\ldots,0)$, it follows from (3) that

$$\begin{aligned}-\left[\frac{a_0}{a}\right]^2 &= \sum_{i \neq 0} (x_0 - 1)x_i + \sum_{i \neq j \neq 0} x_i x_j \\ &= \sum_{i \neq j} x_i x_j - \sum_{i \neq 0} x_i \\ &= x_0 - 1 + T\end{aligned}$$

where $T = \sum_{i \neq j} x_i x_j$. Similarly,

$$-\left[\frac{a_i}{a}\right]^2 = x_i - 1 + T, \qquad i = 0,1,\ldots,n. \tag{4}$$

Then summing over i in (4), we get

$$-\frac{1}{a^2}\sum_{i=0}^{n} a_i^2 = 1 + (n+1)(T-1)$$

or

$$(n+1)T = n - \frac{Q}{a^2} \tag{5}$$

where

$$Q = \sum_{i=0}^{n} a_i^2 .$$

Also from (4), we get

$$x_i = x_0 + \frac{a_0^2 - a_i^2}{a^2}, \tag{6}$$

and then summing over i again,

$$1 = (n+1)x_0 + \frac{(n+1)a_0^2}{a^2} - \frac{Q}{a^2},$$

so

$$x_0 = \frac{1}{n+1} + \frac{Q}{a^2(n+1)} - \frac{a_0^2}{a^2}.$$

Thus from (6),

$$x_i = \frac{1}{n+1} + \frac{Q}{a^2(n+1)} - \frac{a_i^2}{a^2}. \tag{7}$$

On substituting (7) into (5) and simplifying, we obtain

$$(n+1)\sum_{i\neq j}\left\{\left[\frac{1}{n+1} + \frac{Q}{a^2(n+1)} - \frac{a_i^2}{a^2}\right]\left[\frac{1}{n+1} + \frac{Q}{a^2(n+1)} - \frac{a_j^2}{a^2}\right]\right\} = n - \frac{Q}{a^2}$$

$$(n+1)\left\{\left[\frac{1}{n+1} + \frac{Q}{a^2(n+1)}\right]^2\binom{n+1}{2} - \frac{n}{a^2}\left[\frac{1}{n+1} + \frac{Q}{a^2(n+1)}\right]\sum_{i=0}^{n} a_i^2 + \frac{1}{a^4}\sum_{i\neq j} a_i^2 a_j^2\right\} = n - \frac{Q}{a^2}$$

$$n(a^2+Q)^2 - 2nQ(a^2+Q) + 2(n+1)\sum_{i\neq j} a_i^2 a_j^2 = 2na^4 - 2Qa^2$$

and finally

$$na^4 - 2Qa^2 + nQ^2 - 2(n+1)\sum_{i\neq j} a_i^2 a_j^2 = 0.$$

On solving for a^2, we obtain (1) and (2).

Setting $n = 2$ in (1), we recover the plane case:

$$(a_0^2 + a_1^2 + a_2^2)^2 \geq 2(a_0^4 + a_1^4 + a_2^4)$$

$$2(a_1^2a_2^2 + a_2^2a_0^2 + a_0^2a_1^2) \geq a_0^4 + a_1^4 + a_2^4$$

(see [1975: 66]) and finally

$$(a_0 + a_1 + a_2)(a_0 + a_1 - a_2)(a_1 + a_2 - a_0)(a_2 + a_0 - a_1) \geq 0,$$

showing that for $n = 2$ the condition for existence is equivalent to a_0, a_1, a_2 satisfying the triangle inequality. [Editor's note: this gives a solution to *Crux* 1214 [1987: 52].]

The given problem corresponds to the case $n = 3$. From (1) and (2), and using the original notation, we get the solutions

(a) $(a^2 + b^2 + c^2 + d^2)^2 \geq 3(a^4 + b^4 + c^4 + d^4)$,

(b) $$3(AB)^2 = a^2 + b^2 + c^2 + d^2 \pm 2\sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 3(a^4 + b^4 + c^4 + d^4)}\ .$$

Formula (b) can be interpreted as giving the edge length of a regular simplex whose vertices lie on $n + 1$ given concentric spheres. Using a limit procedure, we can also determine the edge length of a regular simplex which has its $n + 1$ vertices on $n + 1$ given parallel hyperplanes. Assume that the successive distances between the hyperplanes are $d_1, d_2, \ldots, d_n$. Now set

$$S_r = d_1 + d_2 + \ldots + d_r\ ,$$

and let $a_0 = R$ and $a_r = R + S_r$, $r = 1,2,\ldots,n$. By letting $R \longrightarrow \infty$, the $n + 1$ spheres of radii $a_0, a_1, \ldots, a_n$ will go into $n + 1$ hyperplanes with the desired spacings. Since I in (1) will also go to infinity, there will be solutions for every spacing. In order for a in (2) to approach a limit, we have to choose the minus sign in the $\pm$. Multiplying the top and bottom of (2) by its conjugate, we get $a^2 = N/D$ where

$$N = \frac{Q^2 - (n+1)I}{n} = \frac{\left[\sum_{i=0}^{n} a_i^2\right]^2 - (n+1)\left[\left[\sum_{i=0}^{n} a_i^2\right]^2 - n\sum_{i=0}^{n} a_i^4\right]}{n}$$

$$= (n+1)\sum_{i=0}^{n} a_i^4 - \left[\sum_{i=0}^{n} a_i^2\right]^2 = \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{(a_i^2 - a_j^2)^2}{2}$$

$$= \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}(S_i - S_j)^2(S_i + S_j + 2R)^2$$

and

$$D = Q + \sqrt{(n+1)I} = \sum_{i=0}^{n} a_i^2 + \sqrt{(n+1)\left[\left[\sum_{i=0}^{n} a_i^2\right]^2 - n\sum_{i=0}^{n} a_i^4\right]}$$

$$= \sum_{i=0}^{n}(R + S_i)^2 + \sqrt{(n+1)\left[\left[\sum_{i=0}^{n}(R + S_i)^2\right]^2 - n\sum_{i=0}^{n}(R + S_i)^4\right]} .$$

We now divide N and D by R^2 and let $R \longrightarrow \infty$ to give that

$$a^2 = \frac{2\sum_{i=0}^{n}\sum_{j=0}^{n}(S_i - S_j)^2}{(n+1) + \sqrt{(n+1)[(n+1)^2 - n(n+1)]}}$$

$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{n}(S_i - S_j)^2}{n+1}$$

$$= 2\sum_{i=0}^{n} S_i^2 - \frac{2}{n+1}\left[\sum_{i=0}^{n} S_i\right]^2 .$$

There is also another solution obtained by taking the spacings d_i in reverse order. These results generalize the known special case for $n = 2$, and also the case $n = 3$ which is to appear as a problem in the *Amer. Math. Monthly*.

Also solved by the proposer.

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<u>1088</u>*. [1985: 289] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If R, r, s are the circumradius, inradius, and semiperimeter, respectively, of a triangle with largest angle A, prove or disprove that

$$s \gtreqless 2R + r \qquad \text{according as} \qquad A \lesseqgtr 90^\circ .$$

Editor's comment. As was noted by several respondents, this problem appears as 11.27(1°) in Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1968. A proof of the result follows immediately from the known equality

$$4R^2\cos A \cos B \cos C \;=\; s^2 - (2R + r)^2.$$

Other proofs and references were also given, the oldest supplied by Klamkin: Problem 13207, R.F. Davis, *Educational Times* 66 (1897) 98.

Solved by W.J. BLUNDON, Memorial University of Newfoundland, St. John's, Newfoundland; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri; C. FESTRAETS-HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, New York; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; and ESTHER SZEKERES, Turramurra, Australia.

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<u>1089</u>. [1985: 290] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find the range of the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(\theta) \;=\; \sum_{k=1}^{\infty} 3^{-k} \cos k\theta, \qquad \theta \in \mathbb{R}.$$

Solution by M.A. Selby, University of Windsor, Windsor, Ontario.

Let $z = e^{i\theta}/3$; then

$$\mathfrak{Re}(z^k) \;=\; \frac{\cos k\theta}{3^k}\;.$$

Now using the geometric series

$$\sum_{k=1}^{\infty} z^k = \frac{z}{1 - z}\,, \qquad |z| < 1,$$

we get

$$f(\theta) \;=\; \sum_{k=1}^{\infty} \mathfrak{Re}(z^k) \;=\; \mathfrak{Re}\left[\sum_{k=1}^{\infty} z^k\right] \;=\; \mathfrak{Re}\left[\frac{z}{1 - z}\right]\;.$$

A simple calculation shows

$$f(\theta) \;=\; \mathfrak{Re}\left[\frac{z}{1 - z}\right] \;=\; \frac{3\cos\theta - 1}{10 - 6\cos\theta}\;.$$

Also,

$$2f(\theta) + 1 \;=\; \frac{4}{5 - 3\cos\theta} \;=\; g(\theta).$$

It is clear that $1/2 \le g(\theta) \le 2$, hence $-1/4 \le f(\theta) \le 1/2$. Thus the range of f is $[-1/4, 1/2]$.

Editor's note. Several solvers showed more generally that the range of the function

$$f(\theta) = \sum_{k=1}^{\infty} a^{-k} \cos k\theta, \qquad \theta \in \mathbb{R}, \qquad |a| > 1,$$

is

$$\left[-\frac{1}{|a| + 1}, \frac{1}{|a| - 1} \right] .$$

The same proof works.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; KARL DILCHER, Dalhousie University, Halifax, Nova Scotia; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; and the proposer.

* * *

1090. [1985: 290] *Proposed by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.*

Let Γ be a circle with center O, and A a fixed point distinct from O in the plane of Γ. If P is a variable point on Γ and AP meets Γ again in Q, find the locus of the circumcenter of triangle POQ as P ranges over Γ.

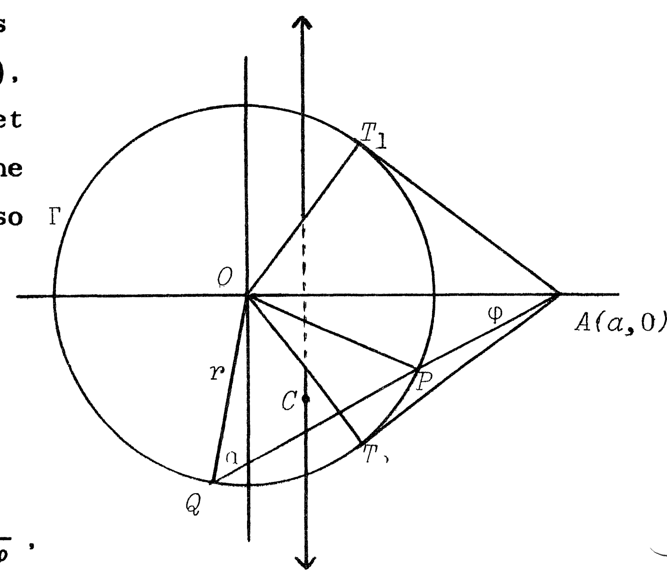
I. *Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

We introduce a Cartesian frame as shown. Let O be the origin and A = $(a,0)$, where $a > 0$ and $a \neq r$, the radius of Γ. Set $\angle OAP = \varphi$ and $\angle OQP = \alpha$, and let R be the circumradius of ΔOPQ. Then $r = 2R \sin \alpha$, so

$$R = \frac{r}{2 \sin \alpha} .$$

From ΔOAQ, $r : a = \sin \varphi : \sin \alpha$, so

$$\sin \alpha = \frac{a \sin \varphi}{r} .$$



Thus

$$R = \frac{r^2}{2a \sin \varphi} ,$$

which implies that the circumcentre C of ΔOPQ has x-coordinate

$$x_C = R \sin \varphi = \frac{r^2}{2a} ,$$

a constant. Thus the locus of C is (part of) the vertical line $x = \frac{r^2}{2a}$. We distinguish two cases:

(i) $a > r$ (A outside Γ).

The y-coordinate y_C of C satisfies

$$|y_C| = R|\cos \varphi| = \frac{r^2}{2a|\tan \varphi|} ,$$

and as

$$|\tan \varphi| \le \frac{r}{\sqrt{a^2 - r^2}}$$

we find

$$|y_C| \ge \frac{r\sqrt{a^2 - r^2}}{2a} ,$$

with equality if and only if $P = Q = T_1$ or T_2, the points of tangency from A to Γ. In this case ΔOPQ is degenerate with circumcentre the midpoint of OT_1 or OT_2. Thus these midpoints are the extreme points of the locus, which consists of all points on the line $x = \frac{r^2}{2a}$ not between them.

(ii) $a < r$ (A inside Γ).

Now the locus of C is the entire line $x = \frac{r^2}{2a}$.

II. *Solution by Dan Pedoe, University of Minnesota, Minneapolis, Minnesota.*

The locus of the circumcenter of triangle POQ is a line perpendicular to the line OA. For if the circumcircle of ΔPOQ intersects the line OA again in A', we have

$$AP \cdot AQ = AA' \cdot AO.$$

Since $AP \cdot AQ$ is constant, A' is a fixed point on AO. The circumcenter of the circle OA'PQ is equidistant from O and from A', and therefore moves on the perpendicular bisector of the segment OA'.

This method of solution shows that O can be any point distinct from A, and not necessarily the center of the given circle for a similar theorem to hold.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; KEE-WAI LAU, Hong Kong; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Smeenk and Tsintsifas noted that if A *lies outside* Γ *then the locus consists of two infinite rays and not a complete line; all other solvers have gaps in their proofs, there being none in their answers.*

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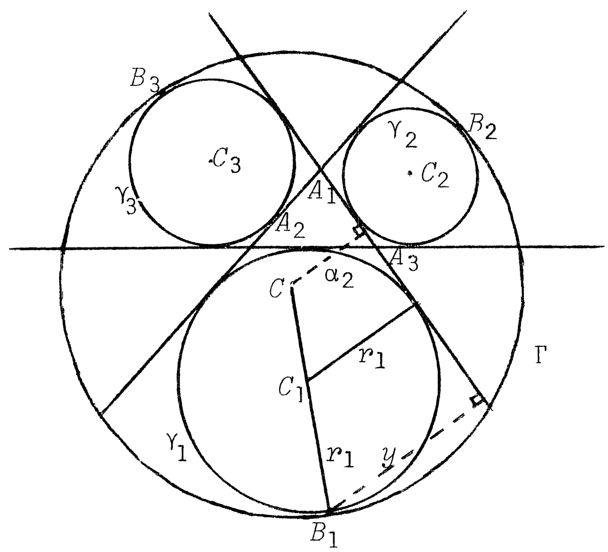
1091*. [1985: 324] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let $A_1A_2A_3$ be a triangle and γ_i the excircle opposite A_i, $i = 1,2,3$. Apollonius knew how to construct the circle Γ internally tangent to the three excircles and encompassing them. Let B_i be the point of contact of Γ and γ_i, $i = 1,2,3$. Prove that the lines A_1B_1, A_2B_2, and A_3B_3 are concurrent.

Solutions by Shiko Iwata and Hidetosi Fukagawa, Gifu and Aichi, Japan.



I. We use absolute trilinear coordinates $P(x_1,x_2,x_3)$ relative to $\Delta A_1A_2A_3$; that is, x_1 is the (signed) distance from the point P to the line A_2A_3, etc. Then $(-r_1,r_1,r_1)$ is the center C_1 of the excircle γ_1 of radius r_1. Let $(\alpha_1,\alpha_2,\alpha_3)$ be the center C of the circle Γ of radius R. The point B_1 divides the segment CC_1 externally into the ratio R: r_1. By similar triangles, the distance y from B_1 to A_1A_3 satisfies

$$\frac{y - r_1}{r_1} = \frac{r_1 - \alpha_2}{R - r_1},$$

so

$$y = \frac{r_1(R - \alpha_2)}{R - r_1}.$$

With similar calculations, we obtain that the trilinear coordinates of B_1 are

$$\left[\frac{r_1(R + \alpha_1)}{R - r_1}, \frac{r_1(R - \alpha_2)}{R - r_1}, \frac{r_1(R - \alpha_3)}{R - r_1}\right].$$

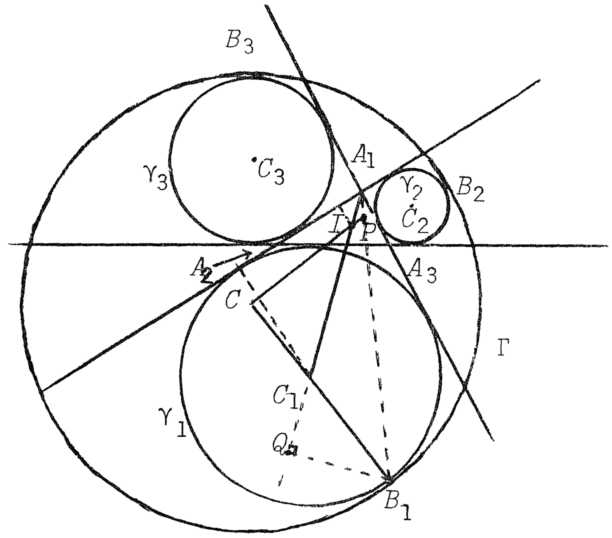
Switching to nonabsolute coordinates, we have

$$B_1 = (R + \alpha_1, R - \alpha_2, R - \alpha_3),$$

and since $A_1 = (1,0,0)$, the trilinear coordinates of any point on A_1B_1 are

$$(*, R - \alpha_2, R - \alpha_3).$$

Likewise, $(R - \alpha_1, *, R - \alpha_3)$ and $(R - \alpha_1, R - \alpha_2, *)$ are the coordinates of points on A_2B_2 and A_3B_3 respectively. Accordingly, the lines A_1B_1, A_2B_2, A_3B_3 meet at the point with coordinates $(R - \alpha_1, R - \alpha_2, R - \alpha_3)$.



II. Let I be the incenter and r the inradius of $\Delta A_1A_2A_3$. Then A_1, I, C_1 are collinear. We produce the segment IC to meet A_1B_1 at P. Let Q be a point on line A_1IC_1 such that $B_1Q \parallel IC$. Then by similar triangles,

$$\frac{PI}{B_1Q} = \frac{A_1I}{A_1Q} = \frac{A_1I}{A_1I + IQ},$$

$$\frac{A_1C_1}{A_1I} = \frac{r_1}{r},$$

and

$$\frac{IQ - IC_1}{IC_1} = \frac{QC_1}{IC_1} = \frac{B_1Q}{IC} = \frac{B_1C_1}{CC_1} = \frac{r_1}{R - r_1},$$

so that

$$B_1Q = \frac{r_1}{R - r_1} \cdot IC$$

and

$$IQ = \left[\frac{r_1}{R - r_1} + 1\right] IC_1 = \frac{R}{R - r_1} \cdot IC_1 .$$

Thus

$$PI = \frac{A_1I \cdot B_1Q}{A_1I + IQ} = \frac{A_1I \cdot IC \cdot \dfrac{r_1}{R - r_1}}{A_1I + IC_1 \cdot \dfrac{R}{R - r_1}} = \frac{r_1 \cdot A_1I \cdot IC}{R(A_1I + IC_1) - r_1 \cdot A_1I}$$

$$= \frac{r_1 \cdot A_1I \cdot IC}{R \cdot A_1C_1 - r_1 \cdot A_1I} = \frac{IC}{\dfrac{R}{r_1} \cdot \dfrac{A_1C}{A_1I} - 1} = \frac{IC}{\dfrac{R}{r_1} \cdot \dfrac{r_1}{r} - 1} = \frac{IC}{\dfrac{R}{r} - 1},$$

a formula which does not depend on the circle γ_1. This means that A_1B_1, A_2B_2, and A_3B_3 will all pass through the point P.

Remark: A Japanese wooden tablet dating from 1797 contained the following formula for R:

$$R = \frac{s^4}{4r_1r_2r_3} + \frac{r_1r_2r_3}{4s^2},$$

where s is the semiperimeter of $\Delta A_1A_2A_3$. The solution of the time is rather long and complicated. We hope that your readers will find a simpler solution. [*I will print the most elegant solution I receive. - Ed.*]

* * *

<u>1092</u>. [1985: 325] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let n be a natural number. Show that, for all natural numbers m, the sum

$$S_m = \sum_{k=1}^{n} \left[\sec \frac{2k\pi}{2n+1}\right]^m$$

is an integer.

Solution by Len Bos, University of Calgary, Calgary, Alberta.

For $n \geq 0$ we let

$$P_n(x) = 2^n \prod_{k=1}^{n} \left[x - \cos \frac{2k\pi}{2n+1}\right]$$

so that $P_0(x) = 1$ and $P_1(x) = 2x + 1$. We first claim that

$$P_{n+1} = 2xP_n - P_{n-1}, \qquad n \geq 1, \tag{1}$$

the proof of which we defer to the end. Note that this recurrence implies that the coefficients of each $P_n(x)$ are all integers and, since $P_0(0) = 1$, $P_1(0) = -1$, and

$$P_{n+1}(0) = -P_{n-1}(0)$$

for all $n \geq 1$, that $P_n(0) = \pm 1$ for all n.

Now

$$\begin{aligned} S_m &= \sum_{k=1}^{n} \frac{1}{\cos^m \frac{2k\pi}{2n+1}} \\ &= -\sum_{k=1}^{n} \left.\frac{1}{\left[x - \cos \frac{2k\pi}{2n+1}\right]^m}\right|_{x=0} \\ &= -\frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} \left.\sum_{k=1}^{n} \frac{1}{x - \cos \frac{2k\pi}{2n+1}}\right|_{x=0} \\ &= \frac{(-1)^m}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} \left.\left[\frac{P_n'(x)}{P_n(x)}\right]\right|_{x=0} . \end{aligned} \tag{2}$$

Next expand $\frac{1}{P_n(x)}$ in a Taylor series about $x = 0$, say we get $\sum_{j=0}^{\infty} a_j x^j$.

Then

$$P_n(x) \sum_{j=0}^{\infty} a_j x^j \equiv 1.$$

But

$$P_n(x) = \pm 1 + b_1 x + b_2 x^2 + \dots + b_n x^n$$

where the b_i are all integers. It follows by an easy induction that the a_j are all integers as well. Hence

$$\frac{P_n'(x)}{P_n(x)} = P_n(x) \cdot \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} d_j x^j$$

for some integers d_j, which implies by (2) that

$$\begin{aligned} S_m &= \frac{(-1)^m}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} \left[\sum_{j=0}^{\infty} d_j x^j \right] \Bigg|_{x=0} \\ &= \frac{(-1)^m}{(m-1)!} (m-1)! d_{m-1} \\ &= (-1)^m d_{m-1} , \end{aligned}$$

an integer for all m.

It remains to prove (1). Define $q_n(x)$ by $q_0 = 1$, $q_1 = 2x + 1$, and

$$q_{n+1} = 2xq_n - q_{n-1} . \tag{3}$$

Clearly q_n is a polynomial of degree n with leading coefficient 2^n. Hence, to show that $q_n = P_n$, we need only show that

$$q_n \left[\cos \frac{2k\pi}{2n+1}\right] = 0, \qquad 1 \le k \le n.$$

By solving the recurrence relation (3) we see that

$$\begin{aligned} q_n(x) = \frac{1}{2(1-x)} \Big[&(1 - x - i\sqrt{1-x^2})(x + i\sqrt{1-x^2})^n \\ &+ (1 - x + i\sqrt{1-x^2})(x - i\sqrt{1-x^2})^n \Big] \end{aligned}$$

so for $0 < \theta < \pi$,

$$\begin{aligned} q_n(\cos\theta) &= \frac{1}{2(1-\cos\theta)} \left[(1 - e^{i\theta})e^{ni\theta} + (1 - e^{-i\theta})e^{-ni\theta} \right] \\ &= \frac{1}{2(1-\cos\theta)} [2\cos n\theta - 2\cos(n+1)\theta] \\ &= \frac{1}{1-\cos\theta} \left[2 \sin \frac{2n+1}{2} \theta \sin \frac{\theta}{2} \right] . \end{aligned}$$

Hence

$$q_n\left[\cos\frac{2k\pi}{2n+1}\right] = \frac{1}{1-\cos\frac{2k\pi}{2n+1}}\left[2\sin k\pi \quad \sin\frac{k\pi}{2n+1}\right]$$

$$= 0$$

for each k.

Also solved by the proposer.

* * *

<u>1093</u>*. [1985: 325] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Prove that

$$\left\{\frac{\Sigma \sin A}{\Sigma \cos(A/2)}\right\}^3 \geq 8\,\Pi \sin\frac{A}{2},$$

where the sums and product are cyclic over the angles A, B, C of a triangle. When does equality occur?

Comments by Vedula N. Murty, Pennsylvania State University, Middletown, Pennsylvania.

The proposed inequality need not hold for all triangles. For example, consider the triangle with $A = \pi/12$, $B = \pi/8$, $C = 19\pi/24$. Then

$$\left\{\frac{\Sigma \sin A}{\Sigma \cos(A/2)}\right\}^3 \cong 0.161961766$$

while

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \cong 0.192904037.$$

However, the proposed inequality does hold for type I triangles, i.e. those satisfying

$$A \leq B \leq C, \qquad B \geq \pi/3.$$

The inequality is equivalent to

$$\frac{y^3}{2x} \geq (\Sigma \cos(A/2))^3, \tag{1}$$

where

$$y = s/R, \qquad x = r/R,$$

s, r, R being the semiperimeter, inradius, and circumradius, respectively, of the triangle. For any triangle, we have the classical Kooistra inequality

$$2 < \Sigma \cos A/2 \leq 3\sqrt{3}/2\ ;$$

hence (1) is established if we demonstrate that

$$\frac{y^3}{2x} \geq \left[\frac{3\sqrt{3}}{2}\right]^3. \tag{2}$$

For triangles of type I we have

$$y \geq \sqrt{3}(1 + x),$$

and thus

$$y^3 \geq (\sqrt{3})^3(1 + x)^3.$$

Hence (2) is established if

$$(\sqrt{3})^3(1 + x)^3 \geq 2x\left[\frac{3\sqrt{3}}{2}\right]^3,$$

which simplified becomes

$$4(1 + x)^3 \geq 27x,$$
$$4x^3 + 12x^2 - 15x + 4 \geq 0,$$
$$(2x - 1)^2(x + 4) \geq 0,$$

which is true.

Counterexamples were also found by RICHARD I. HESS, Rancho Palos Verdes, California; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

Hess calculates that the minimum of

$$\left\{\frac{\Sigma \sin A}{\Sigma \cos(A/2)}\right\}^3 - 8 \Pi \sin A/2$$

is approximately -0.039249058, *occurring at* $A = B = 17.238^\circ$, $C = 145.524^\circ$.

* * *

<u>1094</u>. [1985: 325] *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

A skew quadrilateral consists of two triangular surfaces ABC and ABD with common edge AB. Prove that

$$\text{dihedral angle with edge } AB = \operatorname{Arccos}\left[\frac{\tan S}{\tan C}\right] + \operatorname{Arccos}\left[\frac{\tan S}{\tan D}\right],$$

where S is half the central angle of chord AB in the circumsphere $ACBD$.

Solution by Jordan B. Tabov, Sofia, Bulgaria.

In this solution we assume that the angles C, D, $S \neq \pi/2$, although it is possible to find suitable interpretations of the problem and the result in the cases when one or more of these angles equals $\pi/2$.

Denote by M the midpoint of the segment AB, by O_1 and O_2 the circumcenters of triangles ABC and ABD, by C_1 and D_1 the midpoints of the arcs ACB and ADB of the circumcircles of triangles ABC and ABD respectively. Note that C_1O_1 and D_1O_2 are perpendicular to AB at M.

First we find (Figure 1) that

$$O_1M = \frac{AB}{2|\tan C|}$$

and in fact that

$$\overrightarrow{O_1M} = -\frac{AB}{2O_1C_1\tan C}\,\overrightarrow{O_1C_1}.$$

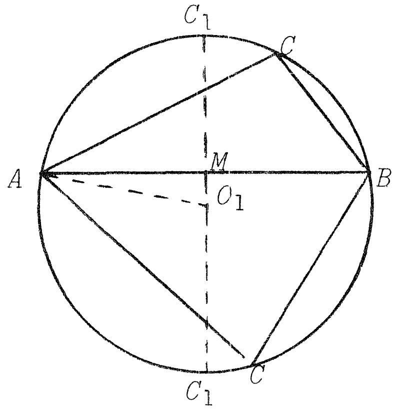


Figure 1

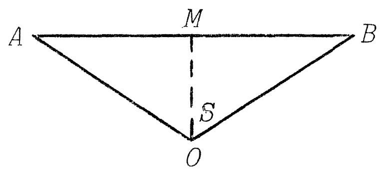


Figure 2

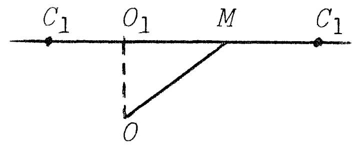


Figure 3

Further, denoting by O the center of the circumsphere, we have (see Figure 2)

$$OM = \frac{AB}{2\ \tan S}\ .$$

Also $OO_1 \perp O_1M$. Hence (see Figure 3)

$$\cos \angle OMC_1 \ = \ \frac{AB}{2\ \tan C} \div \frac{AB}{2\ \tan S} \ = \ \frac{\tan S}{\tan C}\ ;$$

consequently

$$\angle OMC_1 = \operatorname{Arccos}\left[\frac{\tan S}{\tan C}\right]\ ,$$

and similarly

$$\angle OMD_1 = \operatorname{Arccos}\left[\frac{\tan S}{\tan D}\right]\ .$$

Since the plane C_1MD_1 (which passes through O as well) is perpendicular to AB, the dihedral angle with edge AB is equal to $\angle C_1MD_1$. Depending on the position of M, C_1 and D_1 we have one of the following cases:

$$(1)\quad \angle C_1MD_1 \ = \ |\angle OMC_1 - \angle OMD_1|$$
$$= \left|\operatorname{Arccos}\left[\frac{\tan S}{\tan C}\right] - \operatorname{Arccos}\left[\frac{\tan S}{\tan D}\right]\right|$$

(see Figure 4);

$$(2)\quad \angle C_1MD_1 \ = \ \operatorname{Arccos}\left[\frac{\tan S}{\tan C}\right] + \operatorname{Arccos}\left[\frac{\tan S}{\tan D}\right]$$

(see Figure 5);

$$(3)\quad \angle C_1MD_1 \ = \ 2\pi - \operatorname{Arccos}\left[\frac{\tan S}{\tan C}\right] - \operatorname{Arccos}\left[\frac{\tan S}{\tan D}\right]$$

(see Figure 6).

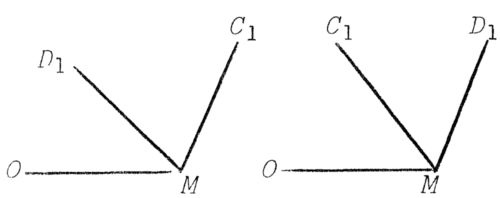


Figure 4

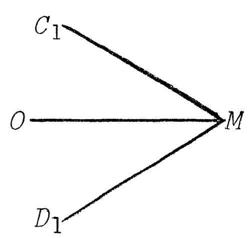


Figure 5

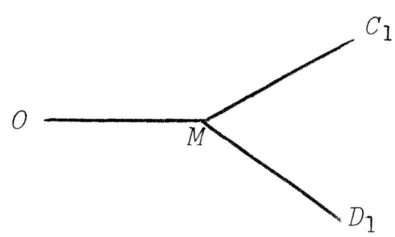


Figure 6

Thus the proposed formula corresponds to our result in case 2; in the other two cases we found formulae which differ in details.

Also solved by the proposer.

* * *

1095. [1985: 325] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $N_n = \{1,2,\ldots,n\}$, where $n \geq 4$. A subset A of N_n with $|A| \geq 2$ is called an *RC-set* (relatively composite) if $(a,b) > 1$ for all $a,b \in A$. Let $f(n)$ be the maximum cardinality of all RC-sets A in N_n. Determine $f(n)$ and find all RC-sets in N_n of cardinality $f(n)$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We show that $f(n) = [n/2]$ and that A is uniquely determined such that $|A| = f(n)$.

As $1 \notin A$ and no two successive natural numbers can belong to A, it follows that $f(n) \leq [n/2]$. The set

$$A = \{2,4,\ldots,2[n/2]\}$$

shows $f(n) = [n/2]$. If there would exist another set A with $|A| = f(n)$, then $A = \{3,5,\ldots\}$, contradiction!

Also solved by RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer.

* * *

1097. [1985: 325] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let *AD*, *BE*, *CF* be the angle bisectors and *AM*, *BN*, *CP* the medians of a triangle *ABC*. Prove that

$$\overrightarrow{AD}\cdot\overrightarrow{AM} + \overrightarrow{BE}\cdot\overrightarrow{BN} + \overrightarrow{CF}\cdot\overrightarrow{CP} = s^2,$$

where s is the semiperimeter.

Solution by Kee-Wai Lau, Hong Kong.

Let $AB = c$, $BC = a$, $AC = b$ so that $s = (a + b + c)/2$. The desired result follows immediately from the following three identities:

$$\overrightarrow{AD}\cdot\overrightarrow{AM} = s(s - a) \qquad (1)$$
$$\overrightarrow{BE}\cdot\overrightarrow{BN} = s(s - b) \qquad (2)$$
$$\overrightarrow{CF}\cdot\overrightarrow{CP} = s(s - c). \qquad (3)$$

We prove (1) only as the proofs of (2) and (3) are similar. Since AD is an angle bisector, we have

$$\frac{AB}{AC} = \frac{BD}{DC}$$

so that

$$c(a - BD) = b(BD)$$

or

$$BD = \frac{ac}{c + b}.$$

Hence

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AB} + \frac{c}{b + c}\overrightarrow{BC}.$$

Also

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}.$$

Therefore, using $|\overrightarrow{AB}| = c$, $|\overrightarrow{BC}| = a$, and

$$\overrightarrow{AB}\cdot\overrightarrow{BC} = -ca\cos B = \frac{b^2 - a^2 - c^2}{2},$$

we have

$$\begin{aligned}
\overrightarrow{AD}\cdot\overrightarrow{AM} &= |\overrightarrow{AB}|^2 + \left(\frac{1}{2} + \frac{c}{b + c}\right)\overrightarrow{AB}\cdot\overrightarrow{BC} + \frac{c}{2(b + c)}|\overrightarrow{BC}|^2 \\
&= c^2 + \left(\frac{1}{2} + \frac{c}{b + c}\right)\frac{b^2 - a^2 - c^2}{2} + \frac{ca^2}{2(b + c)} \\
&= \frac{4c^2(b + c) + (b + 3c)(b^2 - a^2 - c^2) + 2ca^2}{4(b + c)} \\
&= \frac{(b + c)^3 - a^2(b + c)}{4(b + c)} = \frac{(b + c)^2 - a^2}{4} \\
&= \frac{b + c + a}{2}\cdot\frac{b + c - a}{2} = s(s - a).
\end{aligned}$$

Also solved by BENO ARBEL, Tel Aviv University, Tel Aviv, Israel; R.H. EDDY, Memorial University, St. John's, Newfoundland; H. FUKAGAWA, Yokosuka High School, Tokai-city, Aichi, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; and the proposer. Two respondents misread the vectors in the proposal as being scalar distances.

* * *

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