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CONTENTS

PANDIAGONAL MAGIC SQUARE EQUATIONS

ALLAN WM. JOHNSON JR.

A *magic square of order n* is an $n \times n$ block of numbers whose rows, columns, and main diagonals all add up to the same constant, called the *magic sum*. If we represent the numbers in a magic square by the algebraic symbols $A, B, C, \ldots$ and use the symbol Σ for the magic sum, then $A+B+C+D = \Sigma$, $E+F+G+H = \Sigma$, etc., are the algebraic equations for the magic square

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

. (1)

A magic square equation can be represented as a square array consisting of the coefficients of the symbols in the equation. For the magic square (1), the equation $A+B+C+D = \Sigma$ becomes the coefficient array

1	1	1	1	
				1

.

Here an extra block is attached to the lower right-hand corner to hold the coefficient of the magic sum.

This coefficient array notation is introduced not only because it is very compact but also because it allows magic square equations to be easily manipulated and provides a convenient mechanism to derive some remarkable relationships among the numbers in a magic square. For example, the square equation at the top of the next page constitutes a quick proof that the four numbers in the center of a fourth-order magic square add up to the magic sum.

Pandiagonal magic squares add up to the magic sum along the broken diagonals as well as the main diagonals. When broken diagonal equations are included in the coefficient array, it is not always clear which diagonal is meant, so broken dia-

1			1	
	1	1		
	1	1		
1			1	2

+

	1	1		
	1	1		
	1	1		
	1	1		2

+

-1	-1	-1	-1	
-1	-1	-1	-1	-2

=

	2	2		
	2	2		
				2

gonals are indicated with circled numbers and arrows, as in the following proof that if the magic square (1) is pandiagonal, then $G+J+2K+L+O = (3/2)\Sigma$:

3 3

	6		6	
6		6		
	6		6	
6		6		12

+

4

8				
	4		4	
		8		
	4		4	8

+

		4		
		4		
4	4	8	4	
		4		8

(2)

+

	-2		-2	
-2	-4	-2	-4	
	-2		-2	
-2	-4	-2	-4	-8

+

-8	-4	-4	-4	
-4				
-4				
-4				-8

=

		8		
	8	16	8	
		8		12

.

The last square in (2) contains four numbers in a diamond-like configuration. It turns out that every even-order pandiagonal magic square has an equation whose coefficients are arranged in the shape of a diamond. For a pandiagonal magic square of order $2n$, this diamond equation is formed as follows:

(a) Number the rows in the $(n+1)$-st column from the bottom up as follows:

$$1, 2, 3, \ldots, n-1, n, n-1, \ldots, 3, 2, 1.$$

(b) Number the columns in the $(n+1)$-st row from right to left as follows:

$$1, 2, 3, \ldots, n-1, n, n-1, \ldots, 3, 2, 1.$$

(c) Between numbers that are equal, copy the number along the broken diagonal that connects the equal numbers.

The coefficient of the magic sum for this diamond is $(2n^2+1)/6$. For example, the diamond equations for pandiagonal magic squares of orders 6 and 8 are

			1			
		1	2	1		
	1	2	3	2	1	
		1	2	1		
			1			19/6

and

				1				
			1	2	1			
		1	2	3	2	1		
	1	2	3	4	3	2	1	
		1	2	3	2	1		
			1	2	1			
				1				11/2

.

Before proving the existence of this diamond for all even-order pandiagonal magic squares, we develop some interesting consequences of the diamond.

THEOREM 1. 2 divides the magic sum of every pandiagonal magic square of order 2n that is composed of integers.

Proof. The diamond represents a linear combination of magic square cells adding up to $(2n^2+1)/6$ magic sums. The linear combination is an integer because the cells are integers. Hence $(2n^2+1)\Sigma/6$ must be an integer, which is possible only if $2|\Sigma$.

THEOREM 2. 6 divides the magic sum of every pandiagonal magic square of order 6k that is composed of integers.

Proof. As in the proof of Theorem 1, $(2n^2+1)\Sigma/6$ must be an integer. If $n = 3k-1$ or $n = 3k+1$, then $3|(2n^2+1)$, but $3\nmid(2n^2+1)$ when $n = 3k$.

THEOREM 3. Removing a main diagonal from a pandiagonal magic square of order 2n leaves behind two triangles of numbers, each of which sums up to $(2n-1)/2$ magic sums.

Proof. We prove this theorem for order 6 by a method that generalizes to any even order. A pandiagonal magic square keeps its pandiagonal property when a column of the magic square is relocated from the right side to the left side, an operation equivalent to moving a column of the coefficient array from the left side to the right side. Move a column of the diamond's coefficient array and subtract the result from the diamond. The result is shown at the top of the next page.

Rotating a pandiagonal magic square 90° counterclockwise is equivalent to rotating the coefficient array 90° clockwise. Move half the columns of (3) from

			1			
		1	2	1		
	1	2	3	2	1	
		1	2	1		
			1			19/6

−

		1				
	1	2	1			
1	2	3	2	1		
	1	2	1			
		1				19/6

=

		-1	1			
	-1	-1	1	1		
-1	-1	-1	1	1	1	
	-1	-1	1	1		
		-1	1			0

. (3)

the right side to the left side, then rotate 90° and add:

1					-1	
1	1			-1	-1	
1	1	1	-1	-1	-1	
1	1			-1	-1	
1					-1	

+

1	1	1	1	1		
	1	1	1			
		1				
		-1				
	-1	-1	-1			
-1	-1	-1	-1	-1		

=

1	1	1	1	1		
1	1	1	1		-1	
1	1	1		-1	-1	
1	1		-1	-1	-1	
1		-1	-1	-1	-1	
	-1	-1	-1	-1	-1	

.

This shows that the two triangles of numbers have equal sums. Because the two triangles together add up to $2n-1$ magic sums, it follows that each triangle is composed of numbers summing up to $(2n-1)/2$ magic sums. □

A construction method is employed to prove that a diamond equation exists for all pandiagonal magic squares of even order. For one of order $2n$, we selectively multiply the $2n$ row equations and the $2n$ column equations by the algebraic unknowns $R_0, R_1, R_2, \ldots, R_n$ and sum the resulting equations to form a coefficient array, to which are added the $2n$ northeast/southwest (NE/SW) diagonal equations and the $2n$ northwest/southeast (NW/SE) diagonal equations after they have been selectively multiplied by the algebraic unknowns $V_0, V_1, V_2, \ldots, V_n$. The selective multiplication produces a coefficient array with the following appearance:

(a) V_0 occurs in the NE/SW and NW/SE broken diagonals starting at row 1/column $(n+1)$.

(b) V_n occurs in the NW/SE main diagonal and in the NE/SW broken diagonal starting at row 1/column 1.

(c) For $i = 2,3,\ldots,n$, V_{n+1-i} occurs in the NE/SW and NW/SE broken diagonals starting at row 1/column i and at row 1/column $(2n+2-i)$.

(d) R_n occurs in the top row (row 1) and in the leftmost column (column 1).

(e) R_0 occurs in row $(n+1)$ and in column $(n+1)$.

(f) For $i = 2,3,\ldots,n$, R_{n+1-i} occurs in row i, in row $(2n+2-i)$, in column i, and in column $(2n+2-i)$.

The resulting order-8 coefficient array is shown at the top of the next page.

The coefficient array thus constructed has symmetrical properties facilitating the writing of algebraic equations in the $2n+2$ unknowns $V_0, V_1, \ldots, V_n, R_0, R_1, \ldots, R_n$, whose values can then be determined in a way that turns the coefficient array into the diamond equation. The $(2n-1)\times(2n-1)$ square in the lower right-hand corner of the $2n\times 2n$ coefficient array has cells that are symmetrical with respect to its center column, to its center row, and to each of its main diagonals. The $(n+1)\times(n+1)$ square in the upper left-hand corner of the $2n\times 2n$ coefficient array has cells that are symmetrical with respect to its NW/SE main diagonal. When the cell in the upper left-hand corner is eliminated, the cells in the reduced top row (of the $2n\times 2n$ coefficient array) are symmetrical with respect to its center cell, and the cells in the reduced leftmost column are also symmetrical with respect to its center cell. These symmetries mean that we know what is in every cell of the $2n\times 2n$ coefficient array when we know what is in the cells located on or above the NW/SE main diagonal of the $(n+1)\times(n+1)$ square in the upper left-hand corner of the $2n\times 2n$ coefficient array.

V_4	V_3	V_2	V_1	V_0	V_1	V_2	V_3	
V_4V_4 R_4R_4	V_3V_3 R_3R_4	V_2V_2 R_2R_4	V_1V_1 R_1R_4	V_0V_0 R_0R_4	V_1V_1 R_1R_4	V_2V_2 R_2R_4	V_3V_3 R_3R_4	$\leftarrow R_4$
V_3V_3 R_3R_4	V_2V_4 R_3R_3	V_1V_3 R_2R_3	V_0V_2 R_1R_3	V_1V_1 R_0R_3	V_0V_2 R_1R_3	V_1V_3 R_2R_3	V_2V_4 R_3R_3	$\leftarrow R_3$
V_2V_2 R_2R_4	V_1V_3 R_2R_3	V_0V_4 R_2R_2	V_1V_3 R_1R_2	V_2V_2 R_0R_2	V_1V_3 R_1R_2	V_0V_4 R_2R_2	V_1V_3 R_2R_3	$\leftarrow R_2$
V_1V_1 R_1R_4	V_0V_2 R_1R_3	V_1V_3 R_1R_2	V_2V_4 R_1R_1	V_3V_3 R_0R_1	V_2V_4 R_1R_1	V_1V_3 R_1R_2	V_0V_2 R_1R_3	$\leftarrow R_1$
V_0V_0 R_0R_4	V_1V_1 R_0R_3	V_2V_2 R_0R_2	V_3V_3 R_0R_1	V_4V_4 R_0R_0	V_3V_3 R_0R_1	V_2V_2 R_0R_2	V_1V_1 R_0R_3	$\leftarrow R_0$
V_1V_1 R_1R_4	V_0V_2 R_1R_3	V_1V_3 R_1R_2	V_2V_4 R_1R_1	V_3V_3 R_0R_1	V_2V_4 R_1R_1	V_1V_3 R_1R_2	V_0V_2 R_1R_3	$\leftarrow R_1$
V_2V_2 R_2R_4	V_1V_3 R_2R_3	V_0V_4 R_2R_2	V_1V_3 R_1R_2	V_2V_2 R_0R_2	V_1V_3 R_1R_2	V_0V_4 R_2R_2	V_1V_3 R_2R_3	$\leftarrow R_2$
V_3V_3 R_3R_4	V_2V_4 R_3R_3	V_1V_3 R_2R_3	V_0V_2 R_1R_3	V_1V_1 R_0R_3	V_0V_2 R_1R_3	V_1V_3 R_2R_3	V_2V_4 R_3R_3	$\leftarrow R_3$
$\uparrow R_4$	$\uparrow R_3$	$\uparrow R_2$	$\uparrow R_1$	$\uparrow R_0$	$\uparrow R_1$	$\uparrow R_2$	$\uparrow R_3$	

(4)

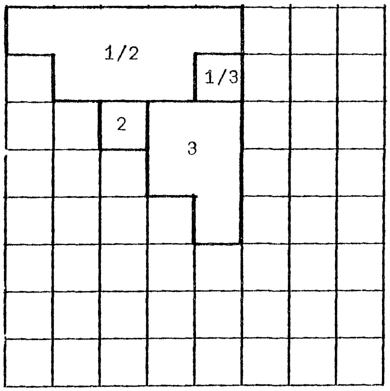
To treat the cells located on or above the NW/SE main diagonal of the $(n+1)\times(n+1)$ square, we first separate these cells into three groups:

(a) Group 1 contains the cells in the top two rows.

(b) Group 2 contains the cells that are on or to the left of the NE/SW main diagonal.

(c) Group 3 contains the cells located to the right of the NE/SW main diagonal.

The diagram on the right shows the three cell groups for the order-8 coefficient array.



For a coefficient array like (4) to equal the diamond equation, the following equations for the cells in the top row of group 1 must be true:

$$V_m + V_m + R_m + R_n = 0, \qquad m = 0,1,\ldots,n. \tag{5}$$

For the second row of cells in group 1, the equations are

$$V_{m-1} + V_{m+1} + R_m + R_{n-1} = 0, \quad m = 1,2,\dots,n-1 \tag{6}$$

and

$$V_1 + V_1 + R_0 + R_{n-1} = x \neq 0. \tag{7}$$

To derive values of $V_1, V_2, \dots, V_n, R_0, R_1, \dots, R_n$ in terms of x and V_0, we subtract corresponding equations of (6) from (5), thereby obtaining

$$(V_m - V_{m-1}) - (V_{m+1} - V_m) + (R_n - R_{n-1}) = 0, \quad m = 1,2,\dots,n-1. \tag{8}$$

Adding up these equations gives

$$(V_1 - V_0) - (V_n - V_{n-1}) + (n-1)(R_n - R_{n-1}) = 0. \tag{9}$$

Subtracting equation (5) at $m = n-1$ from equation (5) at $m = n$ yields

$$V_n - V_{n-1} = -\tfrac{1}{2}(R_n - R_{n-1}). \tag{10}$$

Subtracting equation (7) from equation (5) at $m = 0$ gives

$$V_1 - V_0 = \tfrac{1}{2}x + \tfrac{1}{2}(R_n - R_{n-1}). \tag{11}$$

On substituting equations (10) and (11) into (9), we obtain

$$R_n - R_{n-1} = -\frac{x}{2n}. \tag{12}$$

Substituting (12) into (11) yields

$$V_1 = V_0 + \frac{(2n-1)x}{4n}.$$

Substituting equation (12) into (8) and then solving (8) for $V_2, V_3, \dots, V_n$ give

$$V_m = V_0 + \frac{m(2n-m)x}{4n}, \quad m = 0,1,\dots,n. \tag{13}$$

Equation (5) at $m = n$ shows that $R_n = -V_n$ and so, by (13),

$$R_n = -V_0 - \frac{nx}{4}. \tag{14}$$

Finally, we obtain R_m by substituting equations (13) and (14) into (5):

$$R_m = -V_0 + \frac{\{n^2 - 2m(2n-m)\}x}{4n}, \quad m = 0,1,\dots,n. \tag{15}$$

Having determined values for V_m and R_m, we turn to cells of group 2 and group 3 to see what their values are.

The NE/SW broken diagonals starting at row 1/column $(n+1-i)$ separate the cells of group 2 into subsets

$$V_i + V_{i+2k} + R_{i+k} + R_{n-k}, \quad i = 0,1,\dots,n; \quad k = 0,1,2,\dots\ .$$

Substituting equations (13) and (15) shows that this expression is identically zero.

The NE/SW broken diagonals starting at row $(i+1)$/column $(n+1)$ separate the cells of group 3 into subsets

$$D_i = V_i + V_{i+2k} + R_k + R_{n-i-k}, \quad i = 1,2,\ldots,n; \quad k = 0,1,2,\ldots .$$

Substituting equations (13) and (15) shows that $D_i = ix$.

This completes the proof that a diamond equation exists for pandiagonal magic squares of all even orders. To determine how many magic sums the diamond equation represents, we add up the multipliers that were selectively applied to the magic square's diagonal equations before they were summed to obtain the diamond equation,

$$2V_0 + 4(V_1+V_2+\ldots+V_{n-1}) + 2V_n = 4nV_0 + \frac{(2n-1)(2n+1)x}{6} ;$$

and we also add up the multipliers used on the magic square's row and column equations,

$$2R_0 + 4(R_1+R_2+\ldots+R_{n-1}) + 2R_n = -4nV_0 - \frac{(n-1)(n+1)x}{3} .$$

Hence the diamond equation equals

$$\frac{(2n^2+1)x}{6} \text{ magic sums.} \tag{16}$$

The values of V_m and R_m computed by equations (13) and (15) contain fractions. To obtain integral values for V_m and R_m, we set $x = 4n$ and, to further simplify the resulting equations, we put $V_0 = 0$. This reduces (13), (15), and (16) to

$$\left.\begin{aligned} V_m &= m(2n-m), \\ R_m &= n^2 - 2m(2n-m), \end{aligned}\right\} \quad m = 0,1,\ldots,n$$

and

$$\frac{2n(2n^2+1)}{3} \text{ magic sums,}$$

which, for $n = 2$, yields 12 magic sums as in (2).

Does a diamond-like equation exist for all pandiagonal magic squares of odd order? The answer is yes, and the construction is similar to that for squares of even order. We illustrate the construction for pandiagonal magic squares of order 7 and then give the equations for V_m and R_m, the verification of which we leave to the reader. (The reader should now turn to the diagram at the top of the next page.)

For pandiagonal magic squares of order $2n+1$, the diamond equals $n(n+1)x/6$ magic sums and, for $m = 0,1,\ldots,n$, V_m and R_m are given by the equations

$V_3 \quad V_2 \quad V_1 \quad V_0 \quad V_0 \quad V_1 \quad V_2 \quad V_3$

V_2V_3 R_3R_3	V_1V_2 R_2R_3	V_0V_1 R_1R_3	V_0V_0 R_0R_3	V_0V_1 R_1R_3	V_1V_2 R_2R_3	V_2V_3 R_3R_3	$\leftarrow R_3$
V_1V_2 R_2R_3	V_0V_3 R_2R_2	V_0V_2 R_1R_2	V_1V_1 R_0R_2	V_0V_2 R_1R_2	V_0V_3 R_2R_2	V_1V_2 R_2R_3	$\leftarrow R_2$
V_0V_1 R_1R_3	V_0V_2 R_1R_2	V_1V_3 R_1R_1	V_2V_2 R_0R_1	V_1V_3 R_1R_1	V_0V_2 R_1R_2	V_0V_1 R_1R_3	$\leftarrow R_1$
V_0V_0 R_0R_3	V_1V_1 R_0R_2	V_2V_2 R_0R_1	V_3V_3 R_0R_0	V_2V_2 R_0R_1	V_1V_1 R_0R_2	V_0V_0 R_0R_3	$\leftarrow R_0$
V_0V_1 R_1R_3	V_0V_2 R_1R_2	V_1V_3 R_1R_1	V_2V_2 R_0R_1	V_1V_3 R_1R_1	V_0V_2 R_1R_2	V_0V_1 R_1R_3	$\leftarrow R_1$
V_1V_2 R_2R_3	V_0V_3 R_2R_2	V_0V_2 R_1R_2	V_1V_1 R_0R_2	V_0V_2 R_1R_2	V_0V_3 R_2R_2	V_1V_2 R_2R_3	$\leftarrow R_2$
V_2V_3 R_3R_3	V_1V_2 R_2R_3	V_0V_1 R_1R_3	V_0V_0 R_0R_3	V_0V_1 R_1R_3	V_1V_2 R_2R_3	V_2V_3 R_3R_3	$\leftarrow R_3$
$\uparrow$ R_3	$\uparrow$ R_2	$\uparrow$ R_1	$\uparrow$ R_0	$\uparrow$ R_1	$\uparrow$ R_2	$\uparrow$ R_3	

$$V_{2k} = V_0 + \frac{k(n-k)x}{2n+1}, \qquad k = 0,1,2,\ldots,$$

$$V_{2k+1} = V_0 + \frac{(k+1)(n-k)x}{2n+1},$$

$$R_m = -V_0 + \frac{\{n(n+1)-2m(2n+1-m)\}x}{4(2n+1)}.$$

The reader is also invited to prove these interesting consequences of the diamond:

THEOREM 4. 3 *divides the magic sum of every pandiagonal magic square of order* $3(2k+1)$ *that is composed of integers.*

THEOREM 5. Removing a main diagonal from a pandiagonal magic square of order $2n+1$ *leaves two triangles of numbers, each of which sums up to n magic sums.*

THEOREM 6. In every pandiagonal magic square of order $2n+1$, $8n$ *of the* $(2n+1)^2$ *cells can each be expressed as a linear combination of the remaining* $(2n-1)^2$ *cells. For* $n > 1$, $2n+1$ *of the* $(2n-1)^2$ *cells can be located in the fourth row, with the rest of the* $(2n-1)^2$ *cells located below the fourth row in a cell rectangle of dimensions* $2n-3$ *rows by* $2n$ *columns.*

We end with the analogue of Theorem 6 for magic squares of even order.

THEOREM 7. In every pandiagonal magic square of order $2n$, $8n-5$ *of the* $(2n)^2$ *cells can each be expressed as a linear combination of the remaining*

$(2n-2)^2+1$ cells. Two of the $(2n-2)^2+1$ cells can be located in the first column at rows 3 and 4, with the rest of the $(2n-2)^2+1$ cells located to the right of the first column and below row 3 in a cell rectangle of dimensions $2n-3$ rows by $2n-1$ columns.

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THE OLYMPIAD CORNER: 29

M.S. KLAMKIN

First a correction. In the 1981 International Mathematical Olympiad, U.S. team member James R. Roche won a Second Prize Award, not a Third Prize Award as reported earlier in this column [1981: 222].

I give two 1981 Olympiad problem sets this month, one from Hungary and one from England. I invite readers to send me, for possible publication in this column, copies of other recent competitions, as well as elegant solutions to the problems proposed in the present and in past columns.

I am grateful to L. Csirmaz for the English version of the Hungarian problems given below. (I am not sure what First, Second, and Third Versions imply, and I do not know the time allotted for these problems.)

1981 HUNGARIAN MATHEMATICAL OLYMPIAD (SECOND ROUND)

First Version

1. For which natural numbers n is $2^8 + 2^{11} + 2^n$ a perfect square?

2. The real numbers x and y satisfy $0 < x < 1$ and $x+y = 1$. Determine the maximum and minimum values of the expression

$$x\cdot\frac{1+x^2}{1+x} + y\cdot\frac{1+y^2}{1+y} .$$

3. A frustum of a certain triangular pyramid has lower base of area A, upper base of area B (where $B < A$), and the sum of the areas of its lateral faces is P. The frustum is such that it can be divided, by a plane parallel to the bases, into two smaller frustums in each of which a sphere can be inscribed. Prove that

$$P = (\sqrt{A} + \sqrt{B})(\sqrt[4]{A} + \sqrt[4]{B})^2 .$$

Second Version

1. Make pairs from the medians of the faces of a tetrahedron in such a way

that medians starting from the same midpoint of an edge form a pair. Suppose that in each pair the two medians have equal lengths. How many different lengths of these medians can there be?

2. Let

$$f(x) = \begin{cases} \sin \pi x, & \text{if } x < 0 \\ f(x-1)+1, & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \cos \pi x, & \text{if } x < \frac{1}{2} \\ g(x-1)+1, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

Solve the equation $f(x) = g(x)$.

3. Denote by $f(k)$ the number of zeros in the decimal representation of the natural number k. Compute

$$S_n = \sum_{k=1}^{n} 2^{f(k)},$$

where $n = 10^{10}-1$.

Third Version

1. Six points are given on a circle. Choosing any three of them (this can be done in 20 ways), the orthocentre of the triangle determined by these three points is connected by a straight line to the centroid of the triangle determined by the remaining three points. Prove that these 20 lines all go through a fixed point.

2. Let n be a positive integer, and let $f(n)$ denote the number of triplets consisting of three different positive integers the sum of which is exactly n. (Two triplets are considered to be identical if they differ only by the order of their elements.) For which n is $f(n)$ an even number?

3. Construct (and prove your result) a polynomial $P(x)$ with integral coefficients such that

$$|P(x) - 0.5| < \frac{1}{1981}$$

for every real number x in the interval $[0.19, 0.81]$.

I am grateful to Willie S.M. Yong for sending me the problems of the 1981 British Mathematical Olympiad which appear below.

BRITISH MATHEMATICAL OLYMPIAD

19 March 1981
Time allowed: $3\frac{1}{2}$ hours

1. H is the orthocentre of triangle ABC. The midpoints of BC,CA,AB are A',B',C', respectively. A circle with centre H cuts the sides of triangle A'B'C' (produced if necessary) in six points: D_1,D_2 on B'C'; E_1,E_2 on C'A'; and

F_1, F_2 on $A'B'$.

Prove that $AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2$.

2. Given are the positive integers m and n. S_m is the sum of m terms of the series

$$(n+1) - (n+1)(n+3) + (n+1)(n+2)(n+4) - (n+1)(n+2)(n+3)(n+5) + \ldots,$$

where the terms alternate in sign and each, after the first, is the product of consecutive integers with the last but one omitted.

Prove that S_m is divisible by $m!$ but not necessarily by $m!(n+1)$.

3. Given that a,b,c are positive numbers, prove that

$$\text{(i) } a^3 + b^3 + c^3 \geq b^2c + c^2a + a^2b;$$

$$\text{(ii) } abc \geq (b+c-a)(c+a-b)(a+b-c).$$

4. n points are given in general position in space (i.e., no four are coplanar), and S is the set of all tetrahedra whose vertices are four of the n points.

Prove that, if a plane does not pass through any of the n points, then that plane cannot cut more than $n^2(n-2)^2/64$ of the tetrahedra of S in *quadrilateral* cross sections.

5. Find the smallest possible value of $|12^m - 5^n|$, where m and n are positive integers, and prove your result.

6. Given that $a_1, a_2, \ldots, a_n$ are distinct nonzero integers and that

$$p_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} (a_i - a_j), \quad i = 1,2,\ldots,n,$$

prove that $\sum_{i=1}^{n} a_i^k/p_i$ is an integer for every nonnegative integer k.

*

I now present solutions to some problems published earlier in this column. I would appreciate receiving from readers solutions to the problems which still remain unsolved, or more elegant solutions to those already solved. Two of the solutions are by Noam D. Elkies who, as a member of the winning U.S.A. team at the 1981 International Mathematical Olympiad, achieved a perfect score; so these problems were duck soup to him. He is 14 years old.

J-22. [1981: 143] Can a spatial figure have exactly six axes of symmetry?

Solution by Noam D. Elkies, student, Stuyvesant H.S., New York, N.Y.

Yes: a regular pentagonal prism.

J-23. [1981: 143] Three given circles, O_1, O_2, O_3, intersect pairwise: O_1 and O_2 at points A and B, O_2 and O_3 at points C and D, and O_3 and O_1 at points E and F. Prove that the straight lines AB, CD, and EF intersect at a point.

Solution.

This is a well-known theorem: "*The radical axes of three circles, taken in pairs, are concurrent*. For, the point in which any two radical axes intersect has equal power with regard to all three circles, and therefore lies on the third radical axis. The theorem is evidently still valid in the various special cases, namely if one or more of the circles be null, or if two of them be concentric, or if their centers be collinear." [R.A. Johnson, *Advanced Euclidean Geometry*, Dover, New York, 1960, p. 32.]

J-24. [1981: 143] A point P is selected in the base BCD of a given tetrahedron A-BCD (not necessarily regular) and lines are drawn through it parallel to the edges AB, AC, AD, intersecting the faces of the tetrahedron in other points U, V, W. Find the point P of base BCD for which the volume of tetrahedron P-UVW is a maximum.

Solution.

We give a very direct vectorial solution. A being the origin of vectors, let the position vectors of the points B,C,D,P be $\vec{b},\vec{c},\vec{d},\vec{p}$, respectively. Then there are nonnegative scalars x,y,z such that

$$\vec{p} = x\vec{b} + y\vec{c} + z\vec{d}, \qquad x + y + z = 1,$$

and it follows that the points U,V,W have the position vectors

$$\vec{u} = y\vec{c} + z\vec{d}, \quad \vec{v} = z\vec{d} + x\vec{b}, \quad \vec{w} = x\vec{b} + y\vec{c}.$$

Using square brackets to denote the volume of a tetrahedron, we now have

$$[\text{P-UVW}] = \frac{1}{6}|\vec{PU}\cdot\vec{PV}\times\vec{PW}| = \frac{1}{6}|x\vec{b}\cdot y\vec{c}\times z\vec{d}| = xyz[\text{A-BCD}].$$

By the A.M.-G.M. inequality, xyz is a maximum when $x = y = z = 1/3$. Thus the desired point P is the centroid of triangle BCD.

J-27. [1981: 143] Solve the inequality $2xy \ln(x/y) < x^2-y^2$.

Solution.

It is clear that x and y must have the same sign. If we set $t = x/y$, then the given inequality is equivalent to

$$F(t) \equiv t - \frac{1}{t} - 2\ln t > 0.$$

Since $F'(t) = (1-1/t)^2 \geq 0$ and $F(1) = 0$, the inequality holds if and only if $t = x/y > 1$.

J-28. [1981: 143] The lengths of the sides of a convex quadrilateral are, in order, a, b, c, d and its area is S. Prove that $2S \leq ac + bd$.

Solution.

We have $2S = ef \sin\theta$, where e and f are the lengths of the diagonals and θ is the angle between them. Then, by Ptolemy's inequality, we have

$$2S = ef\sin\theta \leq ef \leq ac + bd,$$

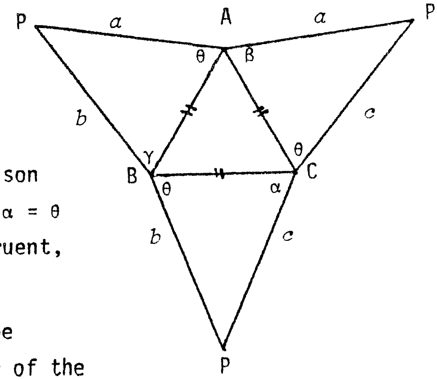
and the desired inequality is established.

There is equality if and only if $\theta = 90^\circ$ and the quadrilateral is cyclic.

J-29. [1981: 143] The base ABC of a pyramid P-ABC is an equilateral triangle. If the angles PAB, PBC, and PCA are all congruent, prove that P-ABC is regular.

Solution by Noam D. Elkies, student, Stuyvesant H.S., New York, N.Y.

First we clarify a slight ambiguity in the proposal. We are asked to show that P-ABC is *a regular pyramid*, not necessarily a regular tetrahedron. A regular pyramid is one whose base is a regular polygon and whose lateral faces make equal angles with the base (or having the foot of its altitude at the center of the base).



Suppose the pyramid is cut open and laid flat, with elements labeled as shown in the figure. It suffices to show that $a = b = c$. Without loss of generality, we may assume that $a \leq b \leq c$. Then triangle PBC shows that $\alpha \leq \theta$. On the other hand, a comparison of triangles PCB and PCA shows that $\theta \leq \alpha$. So $\alpha = \theta$ and $b = c$. Now triangles PCA and PAB are congruent, so $\beta = \theta = \gamma$, and $a = b = c$ follows.

J-32. [1981: 143] What conditions must be satisfied by the coefficients u,v,w of the polynomial

$$x^3 - ux^2 + vx - w \qquad (1)$$

in order that line segments whose lengths are roots of the polynomial can form a triangle.

Solution.

Let a,b,c be the roots of (1). Then a,b,c are the lengths of the sides of a triangle if and only if (i) they are all real, (ii) they are all positive, and (iii) they satisfy the triangle inequalities.

(i) The roots are all real if and only if the discriminant Δ of (1) is non-positive, that is, if and only if [1, pp. 179-180]

$$27\Delta = 27w^2 - 18uvw + 4v^3 + 4wu^3 - u^2v^2 \le 0. \tag{2}$$

(ii) When (2) is satisfied, the roots are all positive if and only if

$$u > 0, \quad v > 0, \quad w > 0. \tag{3}$$

(iii) When (2) and (3) are satisfied, the triangle inequalities hold if and only if

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c) = 2\Sigma b^2c^2 - \Sigma a^4 > 0. \tag{4}$$

Since $u = \Sigma a$, $v = \Sigma bc$, $w = abc$, and [1, p. 95]

$$\Sigma b^2c^2 = v^2 - 2wu, \quad \Sigma a^4 = u^4 - 4u^2v + 2v^2 + 4wu,$$

condition (4) is equivalent to

$$4u^2v - u^4 - 8wu > 0. \tag{5}$$

Thus (2), (3), and (5) are the required necessary and sufficient conditions.

REFERENCE

1. S. Barnard and J.M. Child, *Higher Algebra*, Macmillan, London, 1964.

J-34. [1981: 144] ABC is a triangle of perimeter p. The tangent to its incircle which is parallel to BC meets AB in E and AC in F. Among all triangles of perimeter p, is there one for which EF is of maximum length?

Solution.

We use customary notation for triangle ABC: a,b,c,s for the sides and semi-perimeter, h_a for the altitude from A, r for the inradius, and K for the area.

From similar triangles AEF and ABC, we have

$$\frac{\text{EF}}{h_a - 2r} = \frac{a}{h_a};$$

hence

$$\text{EF} = \frac{a}{h_a}(h_a - 2r) = \frac{a^2}{2K}\left(\frac{2K}{a} - \frac{2K}{s}\right) = \frac{a(s-a)}{s}.$$

Since $a+(s-a) = s = p/2$ is constant, $a(s-a)$ takes on its maximum value when $a = s-a$, that is, when $a = s/2 = p/4$, and then EF $= p/8$. This maximum is attained for the infinite class of triangles for which $a = p/4$ and $b+c = 3p/4$.

J-37. [1981: 144] If a,b,c,d are, in order, the sides of a convex quadrilateral and S is its area, prove that

$$S \le \left\{\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+d}{2}\right)\left(\frac{d+a}{2}\right)\right\}^{\frac{1}{2}}.$$

Solution.

It follows from Bretschneider's formula that, for sides of given lengths, the area S is a maximum when the quadrilateral is cyclic, in which case we have

$$S_{\max} = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where s is the semiperimeter. Hence it suffices to show that

$$(s-a)(s-b)(s-c)(s-d) \le \left(s-\frac{c+d}{2}\right)\left(s-\frac{d+a}{2}\right)\left(s-\frac{a+b}{2}\right)\left(s-\frac{b+c}{2}\right).$$

This follows from the multiplication of the four obvious inequalities

$$(s-a)(s-b) \le \left(s-\frac{a+b}{2}\right)^2,$$
$$(s-b)(s-c) \le \left(s-\frac{b+c}{2}\right)^2,$$
$$(s-c)(s-d) \le \left(s-\frac{c+d}{2}\right)^2,$$
$$(s-d)(s-a) \le \left(s-\frac{d+a}{2}\right)^2.$$

There is equality if and only if $a = b = c = d$, that is, if and only if the given quadrilateral is a square. □

More generally, one can show that

$$(s-a_1)(s-a_2)(s-a_3)(s-a_4) \le (s-b_1)(s-b_2)(s-b_3)(s-b_4),$$

where $s = \Sigma a_i/2$ and

$$b_i = w_i a_1 + w_{i+1}a_2 + w_{i+2}a_3 + w_{i+3}a_4, \quad i = 1,2,3,4,$$

in which $w_{i+4} = w_i$, $\Sigma w_i = 1$, and $w_i \ge 0$. (This and other extensions are to appear in a joint paper by M.S. Klamkin and Clarke Carroll, Australian National University.)

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

POSTSCRIPT TO "THREE MORE PROOFS OF ROUTH'S THEOREM"

I have just come across a direct vectorial proof of Routh's Theorem which is essentially the same as one of the proofs that A. Liu and I gave recently in this journal [1981: 199-203]. It can be found in A.S.B. Holland, "Concurrencies and Areas in a Triangle," *Elemente der Mathematik* 22 (1967) 49-55.

M.S. KLAMKIN

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

660. [1981: 204] Correction: The displayed equation should read

$$DL + EM + FN = s\tan\frac{A}{2}.$$

681. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Of all the girls that are so smart,
  There's none like pretty Sally.
She is the darling of my heart,
  And she lives in our alley.
And when our alley the sunlight dapples,
  My darling SALLY

$$\begin{array}{r} \text{SALLY} \\ \text{SELLS} \\ \text{RIPE} \\ \hline \text{APPLES} \end{array} .$$

(With apologies to Henry Carey (c. 1687 - 1743).)

682. *Proposed by Robert C. Lyness, Southwold, Suffolk, England.*

Triangle ABC is acute-angled and Δ_1 is its orthic triangle (its vertices are the feet of the altitudes of triangle ABC). Δ_2 is the triangular hull of the three excircles of triangle ABC (that is, its sides are the external common tangents of the three pairs of excircles that are not sides of triangle ABC).

Prove that the area of triangle Δ_2 is at least 100 times the area of triangle Δ_1.

683. *Proposed by Kaidy Tan, Fukien Teachers University, Foochow, China.*

Triangle ABC has AB > AC, and the internal bisector of angle A meets BC at T. Let P be any point other than T on *line* AT, and suppose lines BP,CP intersect lines AC,AB in D,E, respectively. Prove that BD > CE or BD < CE according as P lies on the same side or on the opposite side of BC as A.

684. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let O be the origin of the lattice plane, and let $M(p,q)$ be a lattice point with relatively prime positive coordinates (with $q > 1$). For $i = 1,2,\ldots,q-1$, let P_i and Q_i be the lattice points, both with ordinate i, that are respectively the left and right endpoints of the horizontal unit segment intersecting OM. Finally, let $P_iQ_i \cap OM = M_i$.

(a) Calculate

$$S_1 = \sum_{i=1}^{q-1} \overline{P_iM_i}.$$

(b) Find the minimum value of $\overline{P_iM_i}$ for $1 \le i \le q-1$.

(c) Show that $\overline{P_sM_s} + \overline{P_{q-s}M_{q-s}} = 1, \quad 1 \le s \le q-1.$

(d) Calculate

$$S_2 = \sum_{i=1}^{q-1} \frac{\overline{P_iM_i}}{\overline{M_iQ_i}}.$$

(e) Show that the area of a simple triangle is $\frac{1}{2}$. (A *simple triangle* is one whose vertices are lattice points and which has no other lattice point in its interior or on its perimeter.)

685. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a triangle ABC with internal angle bisectors t_a, t_b, t_c meeting a,b,c in U,V,W, respectively; and medians m_a, m_b, m_c meeting a,b,c in L,M,N, respectively. Let

$$m_a \cap t_b = \text{P}, \quad m_b \cap t_c = \text{Q}, \quad m_c \cap t_a = \text{R}.$$

Crux 588 [1980: 317] asks for a proof of the equality

$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = 8.$$

Establish here the inequality

$$\frac{AR}{RU}\cdot\frac{BP}{PV}\cdot\frac{CQ}{QW} \ge 8,$$

with equality if and only if the triangle is equilateral.

686. *Proposed by Charles W. Trigg, San Diego, California.*

Without using calculus, analytic geometry, or trigonometry, find the area of the region which is common to the four quadrants that have the vertices of a square as centers and a side of the square as a common radius.

[A solution using analytic geometry appears in *School Science and Mathematics*, 78 (April 1978) 355.]

687. *Proposed jointly by J. Chris Fisher, University of Regina; and Roger Servranckx, University of Saskatchewan at Saskatoon.*

(a) Show that there exists a number γ such that the equation

$$c^{c^{x}} = x$$

has three solutions whenever $0 < c < \gamma$.

(b) How many solutions does the equation

$$c^{c^{\cdot^{\cdot^{c^{x}}}}} = x$$

have when there are $2n$ c's in the ladder and $0 < c < \gamma$?

688. *Proposed by Robert A. Stump, Hopewell, Virginia.*

Let $\circ$ denote a binary operation on the set of all real numbers such that, for all real numbers a,b,c,

(i) $0\circ a = -a$; (ii) $a\circ(b\circ c) = c\circ(b\circ a)$.

Show that $a\circ(b\circ c) = (a\circ b)\circ(-c)$.

689. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let m_a, m_b, m_c denote the lengths of the medians to sides a,b,c, respectively, of triangle ABC, and let M_a, M_b, M_c denote the lengths of these medians extended to the circumcircle of the triangle. Prove that

$$M_a/m_a + M_b/m_b + M_c/m_c \geq 4.$$

690. *Proposé par Hippolyte Charles, Waterloo, Québec.*

n étant un entier positif donné, soit

$$S_k = n^{n+k} - \binom{n}{1}(n-1)^{n+k} + \binom{n}{2}(n-2)^{n+k} - \ldots + (-1)^{n-1}\binom{n}{n-1}.$$

Calculer S_k pour $k = 0,1,2,3$.

674. [1981: 239] (Correction) The second sentence should read: The incircle of the medial triangle touches its sides in R, S, T (R being on B'C', etc.).

* * *

MAMA-THEMATICS

Frau Hilbert, to her son: "David! Stop staring into space!"

Napier's mother: "Make no bones about it, John, computation is as easy as rolling off a log."

ALAN WAYNE, Holiday, Florida

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

492. [1979: 291; 1980: 291; 1981: 50, 117] *Proposed by Dan Pedoe, University of Minnesota.*

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}AB$ are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{2}AB$?

III. *Comment on part (b) by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

We consider here the possibility of finding the third vertex C of an equilateral triangle ABC, given only the two points A and B (not the entire segment AB) and a rusty compass of span $r < \frac{1}{2}AB$. The circles with centers A and B do not intersect; hence no new "distinguished" points can be constructed and, in attempting to solve the problem, we will have to pick "arbitrary" points in the plane and use them as centers to draw additional circles. We will show that the problem can always be solved in practice. Our technique will be based on the following lemma, which was suggested by a problem of Bankoff [1]. (The notation $A \triangledown B = C$ used in the lemma means that ABC is an equilateral triangle with the vertices labeled A,B,C in counterclockwise order.)

LEMMA. Let A, B, P *be any three distinct points in the plane. If*

$$P \triangledown A = Q, \quad B \triangledown P = R, \quad R \triangledown A = S, \quad Q \triangledown S = C,$$

then $B \triangledown A = C$.

Proof. We assume that the points are imbedded in the complex plane and, to simplify the notation, we identify each point with its affix (see Figure 1). It is known (see, e.g., [2]) that if $X \triangledown Y = Z$, then $Z = -X\omega - Y\omega^2$, where $\omega = e^{2\pi i/3}$. Easy calculations (using $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$) now give

$$Q = -P\omega - A\omega^2,$$

$$R = -B\omega - P\omega^2,$$

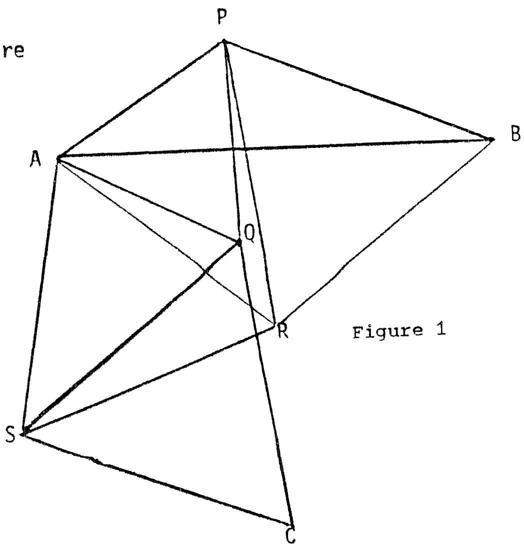


Figure 1

$$S = -R\omega - A\omega^2 = P + (B-A)\omega^2,$$
$$C = -Q\omega - S\omega^2 = -B\omega - A\omega^2,$$

and $B \triangledown A = C$ follows. □

How does this help with our problem? Let us say that we can *reproduce* a distance d if, given two points X and Y that are d units apart, we can construct the point $Z = Y \triangledown X$ with our rusty compass. Prior work on this problem [1980: 291] shows that we can reproduce any distance $d \le 2r$.

Let A and B be two points such that $AB > 2r$. If we can pick a point P in the plane such that none of the lengths PA, PB, AR, and QS (with P,Q,R,S as in the lemma) exceeds $2r$, then we can construct Q, R, S, and C in succession and thereby reproduce AB. For example, Figure 1 shows how to reproduce AB with a rusty compass of span $r = (1/3)AB$, since none of the distances PA, PB, AR, and QS exceeds $(2/3)AB$.

Suppose we could prove the following theorem for some fixed $\varepsilon > 0$:

THEOREM T_ε. *If we can reproduce any distance that does not exceed* δ, *then we can reproduce the distance* $(1+\varepsilon)\delta$.

Then it would follow that we could reproduce *any* distance. For we already know we can reproduce any distance $d \le 2r$, and any distance $D > 2r$ can be built up from some distance $d \le 2r$ by a finite number, n, of successive applications of Theorem T_ε. (Pick some $n > \log_{1+\varepsilon}(D/2r)$.)

But how do we pick the point P given A and B? If we were fortunate enough to pick for P the center P* of the equilateral triangle with vertices A, B, and $C' = A \triangledown B$ (see Figure 2), then the distances PA, PB, AR, and QS would all equal $d/\sqrt{3} \approx 0.577d$, where $d = AB$. But we cannot pick this point out exactly by eye; however, it is easy to pick a random point P fairly close to P*. By a continuity argument, if P is close enough to P*, then $\max\{PA,PB,AR,QS\}$ can be made arbitrarily close to $d/\sqrt{3}$.

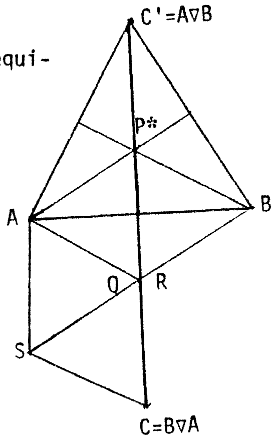


Figure 2

In practice, it is very easy to pick a random point P by eye so that $\max\{PA,PB,AR,QS\} \le 0.9d$. Even if we were to pick for P the midpoint of AB, then we would have

$$\max\{PA,PB,AR,QS\} = d\sqrt{3}/2 \approx 0.866d < 0.9d.$$

Thus, in practice, Theorem T_ε is true for $1+\varepsilon = 1/0.9$, or $\varepsilon = 1/9$. We have therefore given a practical method of reproducing any distance. Unfortunately, we have not specified an effective method of picking a point P other than the vague instruction: *Pick a random point* P *by eye somewhere near the center of triangle* ABC', *where* C' = A ∇ B.

An *effective method* (i.e., one involving no randomness) would follow from the following conjecture, which I have been unable to prove:

CONJECTURE. If P *is a point such that* PA = r *and* $\angle$PAB $\leq 30°$, *then*

$$\max\{PA, PB, AR, QS\} \leq d/(1+\varepsilon)$$

for some fixed $\varepsilon > 0$ *independent of* d = AB *and* r.

If the conjecture is true, the point P can be used to prove Theorem T_ε. We could effectively construct such a point P by drawing a circle around A, picking any point P_1 on this circle, and then swinging six successive arcs with the rusty compass to obtain the vertices of a regular hexagon $P_1P_2P_3P_4P_5P_6$ (see Figure 3). At least one of these six points P_i must be such that angle P_iAB does not exceed $30°$. Then we just apply the algorithm of the lemma using $P = P_i$, $1 \leq i \leq 6$, and abort the operation any time we encounter a distance greater than $2r$. By our conjecture, one of these six tries would be guaranteed to succeed.

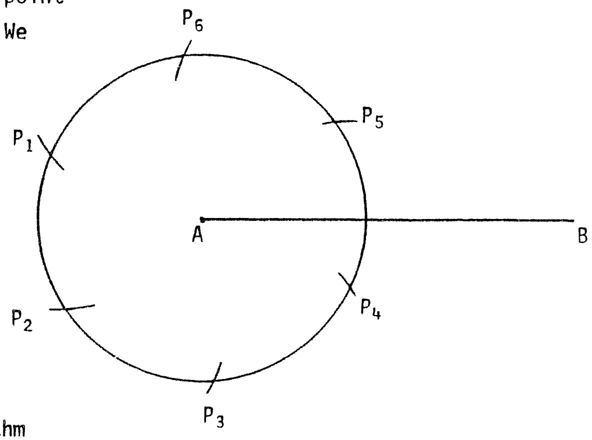


Figure 3

A comment on part (b) was also received from BIKASH K. GHOSH, Bombay, India.

REFERENCES

1. Leon Bankoff (proposer), Problem 438, *Mathematics Magazine*, 34 (1961) 174.

2. A.M. Gleason, R.E. Greenwood, and L.M. Kelly, *The William Lowell Putnam Mathematical Competition Problems and Solutions 1938-1964*, Mathematical Association of America, 1980, p. 498.

* * *

526. [1980: 78; 1981: 87] *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

The following are examples of *chains* of lengths 4 and 5, respectively:

25, 225, 1225, 81225

25, 625, 5625, 75625, 275625.

In each chain, each link is a perfect square, and each link (after the first) is obtained by prefixing a single digit to its predecessor.

Are there chains of length n for $n = 6,7,8,\ldots$?

II. *Solution by L. Csirmaz, Mathematical Institute, Hungarian Academy of Sciences.*

The answer to the question is NO. In fact, we show that the only chains of length at least 3 are those listed in the earlier partial solution [1981: 88]. This list was exhaustive for squares less than 10^{14}, so we now restrict our attention to squares greater than 10^{14}. For such squares, if there is a chain of length at least 3, then it has a subchain of length 3, and we show that no such chain of length 3 exists (with links containing no initial or final zeros).

Suppose, on the contrary, that $(\alpha^2, \beta^2, \gamma^2)$ is such a chain, where $10\nmid\alpha$ and

$$10^{k-1} < \alpha^2 < 10^k, \quad k > 13; \tag{1}$$

then there are nonzero digits a and b such that

$$10^k a + \alpha^2 = \beta^2 \quad \text{and} \quad 10^{k+1}b + \beta^2 = \gamma^2.$$

From $10^k a = (\beta+\alpha)(\beta-\alpha)$, we get

$$\beta + \alpha = a_1 2^i 5^j \quad \text{and} \quad \beta - \alpha = a_2 2^{k-i} 5^{k-j}, \tag{2}$$

where $0 \le i,j \le k$ and a_1,a_2 are nonzero digits such that

$$1 \le a_1 \le a_2 \le a \quad \text{and} \quad a_1 a_2 = a. \tag{3}$$

Using $0.4\cdot10^k < 4\alpha^2 < 4\cdot10^k$ and $2a\cdot10^k = 2\beta^2 - 2\alpha^2$, the first of which follows from (1), we obtain

$$(0.4+2a)10^k < (\beta+\alpha)^2 + (\beta-\alpha)^2 < (4+2a)10^k. \tag{4}$$

If we set $(\beta+\alpha)^2/10^k = a_1^2 2^{2i}5^{2j}/10^k = A$, then $(\beta-\alpha)^2/10^k = a^2/A$, and (4) is equivalent to

$$0.4+2a < A + \frac{a^2}{A} < 4+2a. \tag{5}$$

Observing that $(\beta+\alpha)^2 > \beta^2 > 10^k a$, so that $A > a$, we can solve (5) to get

$$0.2+a+\sqrt{(0.2+a)^2-a^2} < A < 2+a+\sqrt{(2+a)^2-a^2},$$

from which follows

$$\frac{0.2+a+\sqrt{(0.2+a)^2-a^2}}{a_1^2} < \frac{2^{2i}5^{2j}}{10^k} < \frac{2+a+\sqrt{(2+a)^2-a^2}}{a_1^2}. \tag{6}$$

Of the pairs (a_1,a) which satisfy (3), the pair (3,9) minimizes the left member of (6) and the pair (1,9) maximizes the right member; hence

$$1.234 < \frac{2^{2i}5^{2j}}{10^k} < 17.325,$$

which implies that

$$1.111\cdot 10^{k/2} < 2^i 5^j < 4.163\cdot 10^{k/2}. \qquad (7)$$

Since $10\nmid\alpha$, we cannot simultaneously have $1 \le j < k$ and $2 \le i < k-1$. However, $j \ne 0$ in (7) because $1.111\cdot 10^{k/2} > 3^k > 2^i$; and $j \ne k$ because $(5/\sqrt{10})^k > 4.163$ if $k \ge 4$ (and we have $k > 13$). Therefore $1 \le j < k$ and, consequently, either $i < 2$ or $i \ge k-1$.

If $i < 2$, then $1 \le 2^i \le 2$ and, from (7),

$$0.555\cdot 10^{k/2} < 5^j < 4.163\cdot 10^{k/2};$$

hence (replacing j by $k-j$ in (2)) we have

$$\beta + \alpha = s_1 5^{k-j} \quad \text{and} \quad \beta - \alpha = s_2 2^{k-1}5^j,$$

where $0 < s_1,s_2 < 20$ and

$$-0.3658+0.7153k < k-j < 0.8862+0.7154k.$$

Similarly, if $i \ge k-1$, then $\frac{1}{2}\cdot 2^k \le 2^i \le 2^k$ and

$$1.111(5/2)^{k/2} < 5^j < 8.326(5/2)^{k/2},$$

so that

$$\beta - \alpha = s_1 5^{k-j} \quad \text{and} \quad \beta + \alpha = s_2 2^{k-1}5^j,$$

with $0 < s_1,s_2 < 20$ and

$$0.0654+0.2846k < j < 1.3169+0.2847k.$$

Summing up each of these two cases, we get

$$2\beta = s_1 5^{k-j} + s_2 2^{k-1}5^j$$

and

$$2\beta = |s_1 5^{k-j} - s_2 2^{k-1}5^j|$$

with

$$-0.8862+0.2846k < j < 1.3169+0.2847k. \qquad (8)$$

The same result holds with $k+1,\beta,\gamma$ instead of k,α,β; that is, there are integers s_1',s_2', with $0 < s_1',s_2' < 20$, and j' such that

$$2\beta = |s_1' 5^{k+1-j'} - s_2' 2^k 5^{j'}|$$

with

$$-0.8862+0.2846(k+1) < j' < 1.3169+0.2847(k+1). \tag{9}$$

Now, changing the signs of s_1' and s_2' if necessary, we arrive at

$$(2B =)\ s_1 5^{k-j} + s_2 2^{k-1} 5^j = s_1' 5^{k+1-j'} - s_2' 2^k 5^{j'},$$

that is,

$$2^{k-1}(s_2 5^j + 2s_2' 5^{j'}) = s_1' 5^{k+1-j'} - s_1 5^{k-j}. \tag{10}$$

From (8) and (9) we know that $k+1-j'$ and $k-j$ both exceed

$$l = -1.3169 + 0.7153k;$$

therefore the right member of (10) is divisible by $5^{[l]}$. The left member is not divisible by 5^{j+1} if $j < j'$, nor by $5^{j'+1}$ if $j' < j$, nor by 5^{j+3} if $j = j'$. (In the last case, the left member equals $2^{k-1}5^j(s_2+2s_2')$, and the claim follows from $|s_2+ 2s_2'| < 20+2\cdot 20 = 60$.) In any case, the left member is not divisible by $5^{[l']}$, where

$$l' = 3 + 1.3169 + 0.2847k,$$

and so $l < l'$. But we also have $l \geq l'$ if $k \geq 14$, and this is the desired contradiction. □

A modification of the above proof would show that there are infinitely many chains of length 2.

* * *

556. [1980: 184; 1981: 189, 241] *Proposed by Paul Erdös, Mathematical Institute, Hungarian Academy of Sciences.*

Every baby knows that

$$\frac{(n+1)(n+2)\ldots(2n)}{n(n-1)\ldots2.1}$$

is an integer. Prove that for every k there is an integer n for which

$$\frac{(n+1)(n+2)\ldots(2n-k)}{n(n-1)\ldots(n-k+1)} \tag{1}$$

is an integer. Furthermore, show that if (1) is an integer, then $k = o(n)$, that is, $k/n \to 0$.

II. *Solution by L. Csirmaz, Mathematical Institute, Hungarian Academy of Sciences.*

Denote by $n(k)$ the smallest integer $n > k$ for which

$$\frac{(n+1)(n+2)\dots(2n-k)}{n(n-1)\dots(n-k+1)} = \frac{(2n-k)!(n-k)!}{(n!)^2} \tag{1'}$$

is an integer.

First we prove that $n(k)$ exists for every k. We easily find that $n(1) = 6$, $n(2) = 6$, and $n(3) = 9$. For $k \geq 4$, we claim that (1') is an integer for $n = k! - 2$. From this it will follow that $n(k)$ exists for every k, and that $n(k) \leq k! - 2$ when $k \geq 4$.

Observe first that for the stated values of k and n we have $\sqrt{n} < n/2-k$, $k < n$, and there is no prime p such that $n-k/2 < p \leq n$. Let

$$f(x) = \left[\frac{2n-k}{x}\right] + \left[\frac{n-k}{x}\right] - 2\left[\frac{n}{x}\right] .$$

It is easy to verify that

$$f(x) \begin{cases} \geq \ 1, & \text{if } 0 < x \leq \frac{n}{2} - k, \\ \geq \ 0, & \text{if } \frac{n}{2} - k < x \leq n - \frac{k}{2}, \\ = -1, & \text{if } n - \frac{k}{2} < x \leq n, \\ \geq \ 0, & \text{if } x > n. \end{cases}$$

Now let p be any prime. The exponent of p in $s!$ is $\Sigma_{i=1}^{\infty}[s/p^i]$; therefore its exponent in (1') is

$$e_p = \sum_{i=1}^{\infty} f(p^i) . \tag{2}$$

We prove that $e_p \geq 0$, and from this our claim follows. We have $f(p^i) \geq 0$ except possibly if $n-k/2 < p^i \leq n$. In that case we must have $i \geq 2$ since there is no prime in that interval. At the same time we have

$$p^{i-1} \leq \frac{p^i}{2} \leq \frac{n}{2} < n - \frac{k}{2} \quad \text{and} \quad p^{i+1} \geq 2p^i > 2n-k > n;$$

so $f(p^i) = -1$ in (2) for at most one i. In this case, however,

$$2 \leq p \leq p^{i/2} = \sqrt{p^i} \leq \sqrt{n} < \frac{n}{2} - k,$$

that is, $f(p) \geq 1$. Thus $f(p) + f(p^i) \geq 0$, and $e^p \geq 0$ since the remaining terms are nonnegative.

Now let $\varepsilon > 0$ and suppose that $k/n(k) > \varepsilon$ for infinitely many k. This will lead to a contradiction. It is well known that there exists a threshold number $n_0(\varepsilon)$ such that for every $n > n_0(\varepsilon)$ there is a prime p between $n - (\varepsilon/2)n$ and n. Now choose k so large that $k/n(k) > \varepsilon$ and $n(k) > n_0(\varepsilon)$. Then there is a prime p such that $n-k/2 < p < n$. In the right member of (1'), the numerator is divisible by p only, while the denominator is divisible by p^2. Therefore (1') cannot be an integer, and we have our contradiction. We conclude that $\lim_{k\to\infty} k/n(k) = 0$.

A partial solution was received from BIKASH K. GHOSH, Bombay, India.

* * *

572. [1980: 251] *Proposed by Paul Erdős, Technion - I.I.T., Haifa, Israel.*

It was proved in Crux 458 [1980: 157] that, if ϕ is the Euler function and the integer $t > 1$, then each solution n of the equation

$$\phi(n) = n - t \qquad (1)$$

satisfies $t + 2 \le n \le t^2$.

Let $F(t)$ be the *number* of solutions of (1). Estimate $F(t)$ as well as you can from above and below.

Editor's comment.

The following solution was received, in slightly different form, through the intermediary of the proposer. The editor has added, in square brackets and small type, a few remarks drawing attention to points where some confirmation or clarification may still be needed.

Solution by Imre Z. Rursa, Mathematical Institute, Hungarian Academy of Sciences.

Our solution will require the following two lemmas.

LEMMA 1. If $n = p_1^{a_1}\ldots p_r^{a_r}$, where the p_i are distinct primes and the a_i are positive integers, then

$$\phi(n) = \frac{n}{d}\phi(d),$$

where $d = p_1\ldots p_r$.

Proof. We have

$$\begin{aligned}\phi(n) &= (p_1^{a_1}-p_1^{a_1-1})\ldots(p_r^{a_r}-p_r^{a_r-1})\\ &= p_1^{a_1-1}\ldots p_r^{a_r-1}(p_1-1)\ldots(p_r-1)\\ &= \frac{p_1^{a_1}\ldots p_r^{a_r}}{p_1\ldots p_r}(p_1-1)\ldots(p_r-1)\\ &= \frac{n}{d}\phi(d).\end{aligned}$$

LEMMA 2. The number $\omega(n)$ of distinct prime factors of n satisfies

$$\omega(n) = O(\log n).$$

Proof. With the notation of Lemma 1, we have

$$\log n = a_1 \log p_1 + \dots + a_r \log p_r$$
$$\geq \log 2 + \dots + \log 2$$
$$= r \log 2,$$

so $r \leq \log n/\log 2$, that is, $\omega(n) = O(\log n)$. □

The answer to our problem is contained in the following

THEOREM. Let $F(t)$ denote the number of solutions of the equation

$$n - \phi(n) = t, \tag{1}$$

where ϕ is Euler's function and t is a positive integer. Then we have

$$F(t) = O(t \log\log t/\log^2 t) \tag{2}$$

but

$$F(t) \neq o(t \log\log t/\log^2 t). \tag{3}$$

Proof. Let $F_0(t)$ denote the number of solutions of the form $n = pq$, where p and q are distinct primes. Then (1) becomes

$$p + q = t + 1.$$

From Theorem 3.11, page 117, of Halberstam and Richert's *Sieve Methods* (Academic Press, 1974), we have

$$F_0(t) = O(t \log\log t/\log^2 t), \tag{4}$$

and it is known [no reference was given for this and I have been unable to find one] that

$$F_0(t) \neq o(t \log\log t/\log^2 t). \tag{5}$$

We will prove that

$$F(t) - F_0(t) = O(t/(\log t)^c) \tag{6}$$

for every fixed $c > 0$. Then (4) and (5) together with (6) imply (2) and (3).

We consider in turn the possible forms of the solutions of (1) other than the form $n = pq$, with p and q distinct primes. These are

(a) $n = p^\alpha$;
(b) $n = p^\alpha q$, $\alpha > 1$;
(c) $n = p^\alpha q^\beta$, $\alpha > 1$, $\beta > 1$;
(d) numbers n having at least three distinct prime factors.

For solutions of the form (a), (1) becomes

$$p^{\alpha-1} = t,$$

which has at most one solution.

For solutions of the form (b), (1) becomes

$$q = \frac{t}{p^{\alpha-1}} - (p-1).$$

Thus the only primes p which can give rise to solutions of (1) are those which divide t. Hence the number of solutions of the form (b) is, by Lemma 2, less than

$$\omega(t) = O(\log t).$$

For solutions of the form (c), (1) becomes

$$p + q - 1 = \frac{t}{p^{\alpha-1}q^{\beta-1}}.$$

Here the only primes p and q which can give rise to solutions of (1) are those which divide t. Hence the number of solutions of the form (c) is less than

$$\{\omega(t)\}^2 = O(\log^2 t).$$

We now turn our attention to solutions of (1) of the form (d). Let $G(t)$ be the number of solutions of (1) when n is restricted to *square-free* numbers having at least three prime factors. We will show later that

$$G(t) = O\left(t/(\log t)^c\right). \tag{7}$$

Assuming (7) for the moment, we obtain the number of not necessarily square-free solutions of (1) of the form (d). Let m be the product of the distinct prime divisors of n, and set $n = md$. Then

$$\begin{aligned} n - \phi(n) &= md - \phi(md) \\ &= md - \phi(m)d \qquad \text{(by Lemma 1)} \\ &= (m - \phi(m))d, \end{aligned}$$

so that (1) becomes

$$m - \phi(m) = \frac{t}{d}.$$

Hence the total number of solutions of type (d) is

$$\begin{aligned} &\leq \sum_{d \mid t} G(t/d) \\ &= \sum_{e \mid t} G(e) \quad \text{[here some care is needed to ensure that } e > 1] \\ &= O\Big(\sum_{e \mid t} \frac{e}{(\log e)^{c+1}}\Big) \quad \text{(for a fixed } c) \\ &= O\Big(\omega(t) \cdot \frac{t}{(\log t)^{c+1}}\Big) \end{aligned}$$

$$= O(\log t \cdot \frac{t}{(\log t)^{c+1}})$$

$$= O(\frac{t}{(\log t)^{c}}),$$

completing the proof of (6).

We now turn to the proof of (7). Let p be a prime divisor of n and set $n = pk$. (Recall that n is assumed square-free, so $p \nmid k$.) We have

$$\begin{aligned} t = n - \phi(n) &= pk - \phi(pk) \\ &= pk - (p-1)\phi(k) \\ &= k + (p-1)(k - \phi(k)), \end{aligned}$$

so that

$$p - 1 = \frac{t-k}{k-\phi(k)} . \tag{8}$$

Since $p-1 > 0$, it follows from (8) that

$$k < t \quad \text{and} \quad k-\phi(k) \mid t-k. \tag{9}$$

We will estimate the number of solutions k of (9); and n is then uniquely determined by means of (8) and $n = pk$. Let q be the largest prime divisor of k, and set $k = ql$; then $l > 1$ since n has at least three prime divisors. We have

$$\begin{aligned} k - \phi(k) &= ql - \phi(ql) \\ &= ql - (q-1)\phi(l) \\ &= q(l-\phi(l)) + \phi(l), \end{aligned}$$

and this divides $t - ql$ by (9). Set

$$u = q(l-\phi(l)) + \phi(l)$$

and

$$v = t(l-\phi(l)) + l\phi(l),$$

so that $u \mid t-ql$. Then

$$t \equiv ql \pmod u$$

and

$$q(l-\phi(l)) \equiv -\phi(l) \pmod u,$$

so

$$\begin{aligned} qv &= q\{t(l-\phi(l)) + l\phi(l)\} \\ &\equiv -ql\phi(l) + ql\phi(l) \\ &\equiv 0 \pmod u. \end{aligned}$$

Suppose $(q,u) > 1$. Then, as q is prime, we have $q \mid u$, so $q \mid t$ and $q \mid \phi(l)$. [We would like this to be impossible or yield only a small number of k, but it is not clear why this should be so.] Hence we must have $(q,u) = 1$ and so $u|v$. Now

$$v = tl - (t-l)\phi(l) \le tl,$$

as $l < k < t$, that is, $v \le t^2$. Hence for u the number of possibilities is $d(v)$, the number of divisors of v. From Theorem 317 of Hardy and Wright's *An Introduction to the Theory of Numbers* (Fourth Edition, Oxford, 1960), we have

$$d(v) < 2^{c\log v/\log\log v} < 2^{d\log t/\log\log t} = t^{e/\log\log t}.$$

As $l - \phi(l) \ne 0$, thus u determines q; therefore the total number of solutions for which

$$l \le t^{1-f/\log\log t}$$

is

$$\le t^{e/\log\log t}\cdot t^{1-f/\log\log t} = t^{1-(f-e)/\log\log t} = O(t/(\log t)^c).$$

If $l > t^{1-g/\log\log t}$, then

$$q = \frac{k}{l} \le \frac{t}{l} < t^{g/\log\log t}.$$

As q was maximal, k is a product of primes, all less than $t^{g/\log\log t}$. The number of such numbers up to t is known to be $O(t/(\log t)^c)$. [A reference for this result would have been helpful.] This completes the proof.

Editor's comment.

The proposer had written: "It is an old unsolved problem of Sierpiński and myself that infinitely many integers are not of the form $n - \phi(n)$."

* * *

582. [1980: 283; 1981: 254] *Proposed by Allan Wm. Johnson Jr., Washington D.C.*

In how many ways can five distinct digits A, B, C, D, E be formed into four decimal integers AB, CDE, EDC, BA for which the mirror-image multiplication

$$AB\cdot CDE = EDC\cdot BA$$

is true? (For example, the mirror-image multiplication $AB\cdot CD = DC\cdot BA$ is true for $13\cdot 62 = 26\cdot 31$.)

III. *Solution by Clayton W. Dodge, University of Maine at Orono.*

The given equation can be written in the form

$$(10A+B)(100C+10D+E) = (100E+10D+C)(10B+A),$$

which is equivalent to

$$111(A\cdot C-B\cdot E) = 10(A\cdot E+B\cdot D-A\cdot D-B\cdot C).$$

Since $10|A\cdot C-B\cdot E$, $A\cdot C\neq B\cdot E$ implies

$$A\cdot E+B\cdot D-A\cdot D-B\cdot C = A(E-D) + B(D-C) > 111,$$

which is impossible. Hence $A\cdot C = B\cdot E$ and $C = B\cdot E/A$. Now

$$A\cdot E+B\cdot D = A\cdot D+B\cdot C = A\cdot D+B^2\cdot E/A$$

which, since $A-B \neq 0$, reduces to

$$D = \frac{E(A+B)}{A} = E + C.$$

Clearly, B and E cannot be too large because $D = E+C < 10$, so we try various values of A, B, and E, keeping $B\cdot E/A$ an integer. The possibilities are soon exhausted, and we find that there are exactly 11 solutions (if solutions $AB\cdot CDE = EDC\cdot BA$ and $BA\cdot EDC = CDE\cdot AB$ are considered to be identical).

[The 11 solutions are those listed on page 255, excepting the last two in the first column, which have repeated digits. (Editor)]

IV. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

If Clayton W. Dodge, Charles W. Trigg, Kenneth M. Wilke, and the proposer all say that there are only 11 solutions, then it is a fairly safe bet that there are only 11 solutions. In fact, the two "solutions" given at the bottom of the first column on page 255 are invalid since they do not contain five distinct digits, as required.

I employed a Rube Goldberg contraption of my own (also known as a VAX-11/780) to confirm that, in fact, there are no other solutions.

While I was at it, I also decided to look at the related problem of finding six distinct digits such that

$$ABC\cdot DEF = FED\cdot CBA.$$

I found that there are essentially only 3 solutions and that they can be written down in the following interesting pattern:

$$143\cdot 682 = 286\cdot 341,$$
$$431\cdot 268 = 862\cdot 134,$$
$$314\cdot 826 = 628\cdot 413.$$

Additional comments have (so far) been received from RICHARD V. ANDREE, University of Oklahoma; CHARLES W. TRIGG, San Diego, California; and the proposer.

Editor's comment.

The editor had momentarily forgotten that the problem required distinct digits. Then he licked his chops and made snide remarks about Rube Goldberg contraptions [1981: 255].

Dodge, Trigg, Wilke, and the proposer had given the 11 correct solutions, no more and no less. In some cases, their processes for squeezing out solutions may have been elaborate but, like all Rube Goldberg contraptions, they *worked.*

* * *

583. [1980: 283; 1981: 256] Inadvertently omitted from the list of solvers: J.T. GROENMAN, Arnhem, The Netherlands.

* * *

584. [1980: 283] *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

If a triangle is isosceles, then its centroid, circumcentre, and the centre of an escribed circle are collinear. Prove the converse.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let ABC be a triangle with circumcentre O, centroid G, orthocentre H, and excentre I_2 opposite angle B. We assume that O, G, and I_2 are collinear.

If O = G, then O = G = H (since GH = 2OG) and the triangle is equilateral, hence isosceles. If O ≠ G, then O,G,H are collinear on the Euler line of the triangle, and hence O,H,I_2 are collinear. Let M_1,M_3 be the midpoints of BC,BA; F_1,F_3 the feet of the altitudes on BC,BA; and let the escribed circle with centre I_2 meet BC,BA produced in T_1,T_3. We know that $BT_1 = BT_3 = s$, the semiperimeter of the triangle. Thus

$$\begin{aligned} OH:OI_2 &= M_1F_1:M_1T_1 = (M_1C-F_1C):(BT_1-BM_1) \\ &= (\tfrac{a}{2}-b\cos C):(s-\tfrac{a}{2}) = (a^2-2ab\cos C):a(b+c) \\ &= (c^2-b^2):a(b+c) = (c-b):a, \end{aligned}$$

and similarly

$$OH:OI_2 = M_3F_3:M_3T_3 = (a-b):c.$$

Thus $(c-b):a = (a-b):c$, from which $(c-a)(c+a-b) = 0$. Since $c+a-b > 0$, we must have $c = a$, and the triangle is isosceles.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

* * *

THE PUZZLE CORNER

Answer to Puzzle No. 3 [1981: 237]:

$$\begin{array}{r} 19463 \\ 4 \\ \hline 077852 \end{array}.$$

Answer to Puzzle No. 4 [1981: 237]: Pieces of eight.

* * *