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CONTENTS / TABLE DES MATIÈRES

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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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THE OLYMPIAD CORNER

No. 146

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we begin with the Canadian Mathematics Olympiad for 1993 which we reproduce with the permission of the Canadian Olympiad Committee of the Canadian Mathematical Society. My thanks to Edward Wang, its chairperson, for sending me the contest, and the "official" solutions which we will discuss next issue.

1993 CANADIAN MATHEMATICS OLYMPIAD

1. Determine a triangle whose three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

2. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence $x, x+1, x+2, x+3, \ldots$.

3. In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

4. A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

5. Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and which, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even.} \end{cases}$$

Show that the sequence $y_1, y_2, y_3, \ldots$ takes on every positive integer value exactly once.

* * *

The next set of problems are from the twenty-second annual United States of America Mathematical Olympiad, written April 29, 1993. These problems are copyrighted by the committee on the American Mathematics Competition of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588–0322. As always, we welcome your original "nice" solutions and generalizations. The problems were forwarded by Cecil Rousseau.

22nd UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

April 29, 1993

Time Limit: $3\frac{1}{2}$ hours

1. For each integer $n \geq 2$ determine, with proof, which of the two positive real numbers a and b satisfying

$$a^n = a + 1, \qquad b^{2n} = b + 3a$$

is larger.

2. Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

3. Consider functions $f : [0,1] \to \mathbb{R}$ which satisfy

(i) $f(x) \geq 0$ for all x in $[0,1]$,

(ii) $f(1) = 1$,

(iii) $f(x) + f(y) \leq f(x+y)$ whenever x, y, and $x+y$ are all in $[0,1]$.

Find, with proof, the smallest constant c such that $f(x) \leq cx$ for every function f satisfying (i)–(iii) and every x in $[0,1]$.

4. Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a$, $f_2 = b$, and by letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \ldots$. (Such a sequence is said to be *log concave.*) Show that for each $n > 1$,

$$\frac{a_0 + \cdots + a_n}{n+1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \frac{a_0 + \cdots + a_{n-1}}{n} \cdot \frac{a_1 + \cdots + a_n}{n}.$$

* * *

Last issue we gave the problems of the A.I.M.E. for 1993. As promised, we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68488–0322.

1. 728	2. 580	3. 943	4. 870	5. 763
6. 495	7. 005	8. 365	9. 118	10. 250
11. 093	12. 344	13. 163	14. 448	15. 997

* * *

Now for a quick correction. In problem 8 [1993: 132], the orientation of the square labelled $ACDR$ is incorrect. It should read $ACRD$.

Next, we back up for a comment about a problem already discussed in the March number.

6. [1993: 70] *13th Austrian-Polish Mathematics Competition 1990.*

Let $P(x)$ be a polynomial with integer coefficients. Suppose that the integers $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ satisfy the conditions

$$P(x_i) = x_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1, \quad P(x_n) = x_1.$$

Show that $x_1 = x_3$.

Further comment by Murray S. Klamkin, University of Alberta.

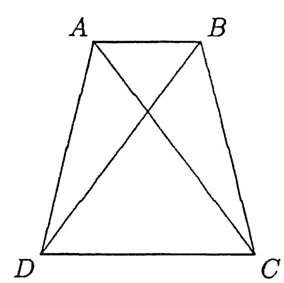
It is to be noted that there is no non-trivial solution for $n = 3$. (However, in this case the problem's supposition is false, so any conclusion is logically valid.) Also, the condition $x_1 = x_3$ should have been stated that for some i, $x_i = x_{i+2}$. We could have for $n = 4$, that $x_1 = 5$, $x_2 = 4$, $x_3 = 3$ and $x_4 = 4$.

* * *

We next turn to solutions from the readers to problems of the *1991 Australian Mathematical Olympiad* which we gave in the May 1992 number of the Corner [1992: 129–130].

1. Let $ABCD$ be a convex quadrilateral. Prove that if g is the greatest and h is the least of the distances AB, AC, AD, BC, BD, CD, then $g \geq h\sqrt{2}$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Lik Ha, Pui Ching Middle School, Hong Kong; and by Joseph Ling, University of Calgary.



If $ABCD$ is a convex quadrilateral then, since not all of angles $\angle A$, $\angle B$, $\angle C$, $\angle D$ can be less than 90° (their sum being 360°), there must be one of the angles, say $\angle A$, for which $\angle A \geq 90°$. By symmetry we may suppose $DA \geq AB$. BD may not be the greatest of the six segments, but we certainly have, since $\angle A \geq 90°$, that by the law of cosines

$$g^2 \geq BD^2 = DA^2 + AB^2 - 2(DA)(AB)\cos\angle A \geq 2AB^2 \geq 2h^2.$$

Thus $g \geq h\sqrt{2}$, which is best possible, as shown by a square.

2. Let M_n be the least common multiple of the numbers $1, 2, 3, \ldots, n$; e.g., $M_1 = 1$, $M_2 = 2$, $M_3 = 6$, $M_4 = 12$, $M_5 = 60$, $M_6 = 60$. For which positive integers n does $M_{n-1} = M_n$ hold? Prove your claim.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Wai Fun Lee, St. Mark's School, Hong Kong, and Kwong Shing Lin, Tsuen Wan Government Secondary School, Hong Kong; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We use Wang's solution with comments.

We claim that $M_{n-1} = M_n$ $(n > 1)$ if and only if n is not a prime-power. If $n = P^k$ where P is a prime and k is a positive integer, then P^{k-1} divides M_{n-1} since $P^{k-1} < M_n$, but $M_n = PM_{n-1}$ since $P^k \not| M_{n-1}$. Conversely, if n is not a prime-power, then every prime-power divisor of n, being smaller than n, must be among $1, 2, \ldots, n-1$, so $M_{n-1} = M_n$.

Remarks: (1) This problem is contained in part (a) of problem number 1252 (Least Common Multiple of $\{1, 2, \ldots, n\}$) in *Mathematics Magazine* 1986, p. 297 and 1988, p. 47.

(2) It could be proved that the only sequences of 3 consecutive positive integers which are prime-powers are (2,3,4), (3,4,5), and (7,8,9), so the longest possible sequence of consecutive prime-powers is (2,3,4,5). It follows that the longest run of strictly increasing M_i's is $M_1 < M_2 < M_3 < M_4 < M_5$. It could also be shown that there exist arbitrarily long sequences of consecutive integers with the same M value. For details, see the published solution to problem number 1252 mentioned in (1).

3. Let A, B, C be three points in the xy-plane and X, Y, Z the midpoints of the line segments AB, BC, AC respectively. Furthermore, let P be a point on the line BC so that $\angle CPZ = \angle YXZ$. Prove that AP and BC intersect in a right angle.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Joseph Ling, University of Calgary; and by Pak Kuen Lee, St. Paul's College, Hong Kong, and Wai Fun Lee, St. Mark's School, Hong Kong.

Since X, Y, and Z are the midpoints of AB, BC, and AC, it follows that ΔXYZ is similar to ΔCAB, in particular $\angle ACB = \angle YXZ$. But it is also given that $\angle YXZ = \angle CPZ$. So, $\angle ACB = \angle CPZ$. It follows that ΔZPC is isosceles and $ZP = ZC$. Now since Z is the midpoint of AC, $ZA = ZC$. Therefore P lies on the circle with centre Z and radius ZA with diameter AC. It follows that $\angle ABC$ is a right angle.

4. Prove that there is precisely one function f, that is defined for all non-zero reals, satisfying

(a) $f(x) = xf(1/x)$, for all non-zero reals x; and

(b) $f(x) + f(y) = 1 + f(x + y)$, for each pair (x, y) of non-zero reals where $x \neq -y$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Joseph Ling, University of Calgary; by Beatriz Margolis, Paris, France; and by Wai Fun Lee, St. Mark's School, Hong Kong. We use Wai Fun Lee's solution.

Setting $x = y = z/2 \neq 0$ in (b) and using (a), we get

$$1 + zf\left(\frac{1}{z}\right) = 1 + f(z) = 2f\left(\frac{z}{2}\right) = zf\left(\frac{2}{z}\right) = z\left(2f\left(\frac{1}{z}\right) - 1\right).$$

Solving for $f(1/z)$, we find $f(1/z) = (z+1)/z$. Therefore $f(z) = 1 + z$, which clearly satisfies (a) and (b).

5. Let $P_1, P_2, \ldots, P_n$ be n different points in a given plane such that each triangle $P_iP_jP_k$ $(i \neq j \neq k \neq i)$ has an area not greater than 1. Prove that there exists a triangle Δ in this plane such that

(a) Δ has an area not greater than 4; and

(b) each of the points $P_1, P_2, \ldots, P_n$ lies in the interior or on the boundary of Δ.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; and by Tak Wing Lee, St. Francis of Assisi's College, Hong Kong. We give Tak Wing Lee's solution.

Let triangle $P_iP_jP_k$ have the maximum area among all triangles with vertices in $\{P_1, P_2, \ldots, P_n\}$. No P_l can lie on the opposite side of the line through P_i parallel to P_jP_k as P_jP_k, otherwise $P_lP_jP_k$ has a greater area than $P_iP_jP_k$. Similarly, no P_l can lie on the opposite side of the line through P_j parallel to P_iP_k as P_iP_k or on the opposite side of the line through P_k parallel to P_iP_j as P_iP_j. Therefore each of the points $P_1, P_2, \ldots, P_n$ lies in the interior or on the boundary of the triangle Δ having P_i, P_j, P_k as midpoints of its sides. Since triangle $P_iP_jP_k$ has an area not greater than 1, triangle Δ has an area not greater than 4.

6. For each positive integer n, let

$$f(n) = \frac{1}{\sqrt[3]{n^2+2n+1} + \sqrt[3]{n^2-1} + \sqrt[3]{n^2-2n+1}}.$$

Determine the value of $f(1) + f(3) + f(5) + \cdots + f(999997) + f(999999)$.

Solutions by Christopher Bradley, Clifton College, Bristol, U.K.; by Curtis Cooper, Central Missouri State University, Warrensburg; by George Evagelopoulos, Athens, Greece; by Joseph Ling, University of Calgary; by Pak Kuen Lee, St. Paul's College, Hong Kong, and Kwong Shing Lin, Tsuen Wan Government Secondary School, Hong Kong; by P. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $A = \sqrt[3]{n+1}$ and $B = \sqrt[3]{n-1}$. Then

$$f(n) = \frac{1}{a^2+ab+b^2} = \frac{a-b}{a^3-b^3} = \frac{1}{2}(\sqrt[3]{n+1} - \sqrt[3]{n-1}).$$

Telescoping, we get, for all positive integers k,

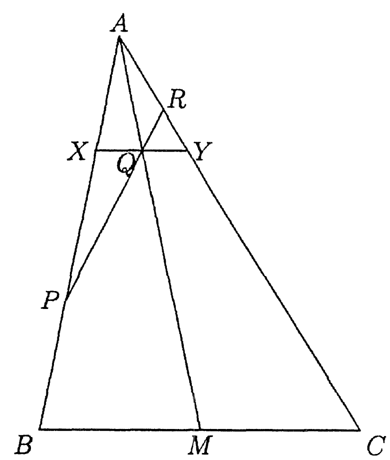
$$\sum_{n=1}^{k} f(2n-1) = \frac{1}{2}\sum_{n=1}^{k}(\sqrt[3]{2n} - \sqrt[3]{2n-2}) = \frac{1}{2}\sqrt[3]{2k}\ .$$

In particular with $k = 500000$ we get $f(1)+f(3)+f(5)+\cdots+f(999999) = \sqrt[3]{10^6}/2 = 50$.

7. In triangle ABC, let M be the midpoint of BC, and let P and R be points on AB and AC respectively. Let Q be the intersection of AM and PR. Prove that if Q is the midpoint of PR, then PR is parallel to BC.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; by Joseph Ling, University of Calgary; and by Pak Kuen Lee, St. Paul's College, Hong Kong.

Suppose that PR is not parallel to BC. Draw a line through Q and parallel to BC cutting AB and AC at X and Y respectively.



Since XY is parallel to BC, ΔAXQ is similar to ΔABM, and ΔAQY is similar to ΔAMC. Therefore

$$\frac{XQ}{BM} = \frac{AQ}{AM} = \frac{QY}{MC}\ .$$

Since $BM = MC$ it follows that $XQ = QY$. Now it is given that $PQ = QR$. Since we also have $\angle XQP = \angle YQR$, then ΔPQX must be congruent to ΔRQY. In particular $\angle PXQ = \angle RYQ$, and this is a contradiction since AB and AC are not parallel to each other.

This contradiction shows that PR must be parallel to BC.

8. Find a sequence $a_0, a_1, a_2, \ldots$ whose elements are positive and such that $a_0 = 1$ and $a_n - a_{n+1} = a_{n+2}$ for $n = 0, 1, 2, \ldots$. Prove that there is only one such sequence.

[*Editor's note.* This problem is essentially the same as *Crux* 1378 [1989: 306]. In order that we have a complete set of solutions here we nevertheless give a solution.]

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; by George Evagelopoulos, Athens, Greece; and by Pak Kuen Lee, St. Paul's College, Hong Kong, whose solution we reproduce.

The characteristic equation of the recurrence equation is $r^2 + r - 1 = 0$. Since $a_0 = 1$ and a_n is positive for every nonnegative integer n, there is a constant c such that

$$a_n = (1-c)\left(\frac{-1+\sqrt{5}}{2}\right)^n + c\left(\frac{-1-\sqrt{5}}{2}\right)^n > 0.$$

This implies that for every nonnegative integer n,

$$c\left(\left(\frac{-1-\sqrt{5}}{-1+\sqrt{5}}\right)^n - 1\right) < 1.$$

Letting $n = 2k \to \infty$, we get $c \leq 0$. Letting $n = 2k+1 \to \infty$, we get $c \geq 0$. Therefore $c = 0$ and

$$a_n = \left(\frac{-1+\sqrt{5}}{2}\right)^n.$$

* * *

To finish this month's column we give solutions from the readers to problems of *The First United Mathematics Olympiad of Senior Normal University*, Zhejiang, China [1992: 130–131].

1. Find all real solutions (x, y, z) satisfying the simultaneous equations

$$(x+2y)(x-2z) = 24, \quad (y+2x)(y-2z) = -24, \quad (z-2x)(z-2y) = -11.$$

Solution by Christopher Bradley, Clifton College, Bristol, United Kingdom.

Writing $x+y-z = a$, $y+z = b$ and $z+x = c$, the three equations reduce to

$$a^2 - b^2 = 24, \quad a^2 - c^2 = -24, \quad a^2 - (b-c)^2 = -11.$$

Eliminating a^2 gives $c^2 - b^2 = 48$ and $c^2 - (b-c)^2 = 13$. Hence $c = (b^2+13)/2b$ and $3b^4 + 166b^2 - 169 = (b^2-1)(3b^2+169) = 0$. Thus $b = 1$ or $b = -1$. When $b = 1$, $c = 7$ and $a = 5$ or $a = -5$. When $b = -1$, $c = -7$ and $a = 5$ or $a = -5$. From the first substitution one finds four solutions for x, y, z, namely $(6, 0, 1)$, $(-6, 0, -1)$, $(8/3, -10/3, 13/3)$ and $(-8/3, 10/3, -13/3)$.

2. The letters a, b, c, d, e have been written on the blackboard. Altogether 1990 letters have been written, but a given letter may have no, some or many occurrences. The following rule may be used to reduce the number of letters on the board by one.

Choose any two letters occurring on the board, erase each of them and write the letter which occurs at the intersection of the row determined by the first letter erased and the column for the second letter erased:

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	c	b	e	d
d	a	d	e	c	b
e	a	e	d	b	c

For example if c and e are erased, in order, we take the row for c and column for e and at the intersection find a d. So the c and e are replaced by d.

Does the last letter left on the board (after many repetitions of this operation) depend in any way on the order of the choices? Prove your conclusion.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by Curtis Cooper, Central Missouri State University, Warrensburg.

By relabeling a, b, c, d, e as $0, 1, 4, 2, 3$ we find that the table is just that for multiplication modulo 5. Since this is associative and commutative, the last letter remaining on the board is independent of the order of choices and is the product, modulo 5, of all the letters on the board.

3. The point P in the square $ABCD$ satisfies the following conditions:

(1) $PA < PB < PD < PC$

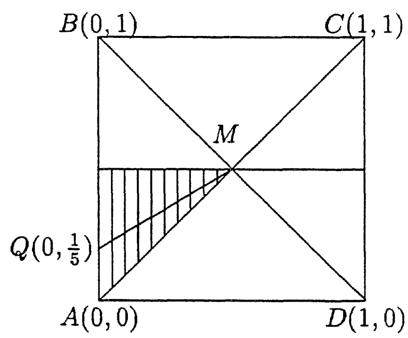
(2) $PA + PC = 2PB$

(3) $(PD)^2 = PB \times PC$.

Prove that such a point P exists, and is unique.

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K..

Set up coordinates with $A(0,0)$, $B(0,1)$, $C(1,1)$ and $D(1,0)$. Then condition (1) restricts $P(x,y)$ to the interior of the shaded region defined by $0 < x < y < 1/2$. In particular the centre, M, of the square, which satisfies conditions (2) and (3) is not allowed. Now (2) implies $PA^2 = 4PB^2 + PC^2 - 4(PB)(PC)$ which by (3) gives $PA^2 + 4PD^2 = 4PB^2 + BC^2$. In terms of the coordinates x and y we obtain



$$x^2 + y^2 + 4[(x-1)^2 + y^2] = 4[x^2 + (y-1)^2] + (x-1)^2 + (y-1)^2$$

which reduces to the line with equation $5y = 3x + 1$, satisfied by M (not allowed) and the point $Q(0, 1/5)$. [See the diagram.]

So the point P, if it exists, lies on the interior of the line segment MQ. It remains to ensure that such a point exists satisfying condition (2). Putting $y = \frac{1}{5}(1+3x)$, condition (2) requires

$$[25x^2 + (1+3x)^2]^{1/2} + [25(1-x)^2 + (4-3x)^2]^{1/2} = 2[25x^2 + (4-3x)^2]^{1/2}.$$

After squaring, rearranging, and squaring again this ugly equation reduces to

$$34x^3 - x^2 - 12x + 2 = 0.$$

Fortunately one knows that $x = 1/2$ is a solution, so factoring out $x - 1/2$ we get $17x^2 + 8x - 2 = 0$ leading to exactly one positive root, $x = (5\sqrt{2} - 4)/17$. Thus $y = (3\sqrt{2} + 1)/17$. This P satisfies $0 < x < y < 1/2$ and is the unique solution to the problem.

4. Partition the set $A = \{1, 2, 3, \ldots, n\}$ into two subsets I_1 and I_2, satisfying the following conditions:

(1) $I_1 \cap I_2 = \emptyset$,

(2) $I_1 \cup I_2 = A$,

(3) Any three numbers in I_1 (I_2 respectively) do not form an arithmetic progression.

What is the maximum value of n for which such a partition is possible?

Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.

The maximum value of n for which such a partition is possible is $n = 8$.

First note that if a partition is possible for $n = k > 1$, then it is possible for $n = k-1$ by just deleting k itself from either I_1 or I_2. This means we cannot have a situation in which the partition is not possible for one value of n and possible for a larger value of n. It also means that if we find *all* possible partitions for $n = 8$ and show that the number 9 cannot be added either to the set I_1 or to the set I_2 to give a possible partition for $n = 9$, then *no* partition is possible for $n = 9$.

So look at $n = 8$. Without loss of generality 1 belongs to I_1. Then at least one of 2, 3, 4 must belong to I_1, otherwise we have the arithmetic progression 2, 3, 4 in I_2.

Suppose first that I_1 contains both 1 *and* 2. Then I_2 contains 3. If I_1 contains 4 as well, then I_2 must contain 6 and 7. It follows that I_1 must contain both 5 and 8, giving 2, 5, 8 in I_1. So in these cases I_1 cannot contain either 3 or 4. Suppose next that I_1 contains 1, 2 and 5. Then I_2 contains 3, 4, 8. Thus I_1 contains 6 and I_2, 7. This gives the partition $I_1 = \{1,2,5,6\}$, $I_2 = \{3,4,7,8\}$. However, clearly 9 cannot be appended to I_1, because of $(1,5,9)$ or to I_2, $(7,8,9)$.

Since I_1 cannot omit all of 3, 4, 5, there are no further cases to consider in which I_1 contains both 1 and 2.

Suppose next that I_1 contains 1 *and* 3. Now first consider the situation when I_1 contains 4 as well. Then I_2 contains 2, 5 and 7. Whereupon I_1 must contain both 6 and 8, which gives the progression 4, 6, 8. Hence if I_1 contains both 1 and 3 it cannot contain 4. Nor can it contain 5. So suppose it contains 6. Then I_2 contains 2, 4, 5; I_1 has 8 and I_2 contains 7. This gives the partition $I_1 = \{1,3,6,8\}$, $I_2 = \{2,4,5,7\}$. Once again 9 cannot be added to I_1 because of 3, 6, 9 nor to I_2 because of 5, 7, 9.

Since I_1 cannot omit all of 4, 5, 6, there are no further cases to consider in which I_1 contains both 1 and 3.

Suppose I_1 contains 1 *and* 4, *and I_2,* 2 *and* 3. If I_1 now contains 5, then I_2 must contain 6. Whereupon I_1 has 8. This leads to $I_1 = \{1,4,5,8\}$, $I_2 = \{2,3,6,7\}$, but again 9 cannot be appended to either set. Suppose next that I_1 contains 1, 4 and 6 and I_2, 2, 3, 5. Clearly this case goes no further as 7 cannot be put into either I_1 nor I_2. One cannot have I_1 containing 1, 4, 7; nor can I_1 contain 1, 4, 8 without 5, 6, 7 because I_2 would then have all of these.

So $n = 8$ has precisely three partitions, and none of these extend to $n = 9$. The result follows.

5. AD', BE', CF' are the three medians of the non-obtuse triangle ABC with centroid G. They respectively intersect at D, E, F with the circumcircle of triangle ABC. Prove that $GD + GE + GF \geq 8R/3$.

Solutions by Seung-Jin Bang, Albany, California; and by Christopher J. Bradley, Clifton College, Bristol, U.K. We use Bang's solution.

Let $x = GD'$, $y = GE'$, and $z = GF'$. Then $GA = 2x$, $GB = 2y$, $GC = 2z$,

$2xGD = 2yGE = 2zGF$, and

$$3xDD' = \left(\frac{a}{2}\right)^2, \quad 3yEE' = \left(\frac{b}{2}\right)^2, \quad 3zFF' = \left(\frac{c}{2}\right)^2.$$

From $3x(GD - x) = a^2/4$, $3y(GE - y) = b^2/4$, and $3z(GF - z) = c^2/4$, we have $GD = x + a^2/(12x)$, $GE = y + b^2/(12y)$, and $GF = z + c^2/(12z)$. It is well-known that

$$3x = \sqrt{\frac{b^2 + c^2 - a^2}{2}}, \quad 3y = \sqrt{\frac{a^2 + c^2 - b^2}{2}}, \quad \text{and} \quad 3z = \sqrt{\frac{a^2 + b^2 - c^2}{2}}.$$

It follows that

$$GD + GE + GF = \frac{a^2 + b^2 + c^2}{18}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{a^2 + b^2 + c^2}{6}(f(a^2) + f(b^2) + f(c^2)),$$

where $f(t) = 2/\sqrt{2(a^2 + b^2 + c^2) - 3t}$. Since

$$\frac{f(a^2) + f(b^2) + f(c^2)}{3} \geq f\left(\frac{a^2 + b^2 + c^2}{3}\right) = \frac{2}{\sqrt{a^2 + b^2 + c^2}},$$

we have

$$GD + GE + GF \geq \sqrt{a^2 + b^2 + c^2} = 2R\sqrt{\sin^2 A + \sin^2 B + \sin^2 C}.$$

From here on we use the method of Lagrange multipliers. Let

$$g(A, B, C) = \sin^2 A + \sin^2 B + \sin^2 C - \lambda(A + B + C - \pi),$$

and $0 < A, B, C \leq \pi/2$. From $\partial g/\partial A = \partial g/\partial B = \partial g/\partial C = 0$ we have $\cos 2A = \cos 2B = \cos 2C = \lambda$ and $A = B = C = \pi/3$. Since ABC is a nonobtuse triangle, if $A \to 0^+$ then $B, C \to \pi/2$ and $g(0, \pi/2, \pi/2) = 2$. From $g(\pi/2, B, C) = 1 + \sin^2 B + \sin^2(\pi/2 - B) = 2$ and $g(\pi/3, \pi/3, \pi/3) = 9/4$ we have $g(A, B, C) \geq 2$ for all $A, B, C \leq \pi/2$ with $A + B + C = \pi/2$. It follows that $GD + GE + GF \geq 2\sqrt{2}\,R$, which is sharper than required.

* * *

That exhausts our file of solutions for the May 1992 number, and makes a logical place to stop. Send me your Olympiads, as well as your nice solutions and generalizations.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **January 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1851. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum_{i=1}^{n} x_i^2 = 1$. Prove that

$$\frac{2\sqrt{n}-1}{5\sqrt{n}-1} \leq \frac{1}{n}\sum_{i=1}^{n}\frac{x_i+2}{x_i+5} \leq \frac{2\sqrt{n}+1}{5\sqrt{n}+1}\ .$$

1852. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with $AB < AC$ and with orthocenter H. Let I^* be the incenter of ΔHBC. The line through I^* perpendicular to BC meets AB and AC at P and Q respectively. Prove that the perimeter of ΔAPQ is equal to $AC - AB$.

1853. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers which satisfies the condition

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every $n \geq 1$. Prove that either the sequence converges or $\lim_{n\to\infty} b_n = \infty$.

1854. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In any convex pentagon prove that the sum of the squares of the diagonals is less than three times the sum of the squares of the sides.

1855. *Proposed by Bernardo Recamán, United World College of Southern Africa, Mbabane, Swaziland.*

Twelve friends agree to eat out once a week. Each week they will divide themselves into 3 groups of 4 each, and each of these groups will sit together at a separate table. They have agreed to meet until any two of the friends will have sat at least once at the same table at the same time. What is the minimum number of weeks this requires?

1856. *Proposed by Jisho Kotani, Akita, Japan.*

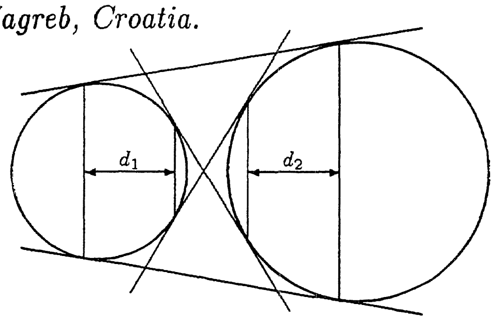
Find the rectangular brick of largest volume that can be completely wrapped in a square piece of paper of side 1 (without cutting the paper).

1857. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Prove that, for any positive integer n,

$$1 < \frac{27^n(n!)^3}{(3n+1)!} < \sqrt{2}\,.$$

1858. *Proposed by Vladimir Devidé, Zagreb, Croatia.*



The four common tangents to two circles are drawn, and their points of tangency connected by chords as shown in the diagram. (These chords are parallel, as they are all perpendicular to the line joining the centres of the circles.) Prove that $d_1 = d_2$.

1859. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Let w be any n-letter "word" ($n \geq 1$) which contains at most 10 different letters, like MATHEMATICORUM or AAAEEIIIAAOOOUUUU. Prove that you can replace the letters of w by decimal digits (different letters replaced by different digits, the first letter of w not replaced by 0) so that the resulting n-digit number is a multiple of 3.

1860*. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Prove or disprove that

$$\sum \frac{\cos[(A-B)/4]}{\cos(A/2)\cos(B/2)} \geq 4,$$

where the sum is cyclic over the angles A, B, C of a triangle.

*   *   *   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1680. [1991: 238; 1992: 250] *Proposed by Zun Shan and Ji Chen, Ningbo University, China.*

If m_a, m_b, m_c are the medians and r_a, r_b, r_c the exradii of a triangle, prove that

$$\frac{r_b r_c}{m_b m_c} + \frac{r_c r_a}{m_c m_a} + \frac{r_a r_b}{m_a m_b} \geq 3.$$

III. *Solution by De-jun Zhao, Xingchang, Zhejiang, China.*

We shall give a very short proof of the problem. First note that, with s the semiperimeter,

$$r_b r_c = s(s-a) \quad \text{and} \quad 4m_b m_c \le \frac{1}{2}(4a^2+b^2+c^2), \quad \text{etc.}$$

[1992: 251], therefore

$$\sum \frac{r_b r_c}{m_b m_c} \ge \sum \frac{8s(s-a)}{4a^2+b^2+c^2},$$

where the sums are cyclic over a, b, c. Thus we need only prove that

$$H \equiv \sum 8s(s-a)(4b^2+c^2+a^2)(4c^2+a^2+b^2) - 3\prod(4a^2+b^2+c^2) \ge 0. \qquad (1)$$

Let $a = y+z$, $b = z+x$, $c = x+y$. Then

$$\begin{aligned} H &= 9xyz\sum x(y-z)^2 + \frac{25}{2}\left(\sum(x+y+z)(y-z)^2\right)^2 + 75\sum yz\sum x^2(y-z)^2 \\ &\qquad + 115(x+y+z)\sum yz\sum(y+z)(y-z)^2 \\ &\ge 0. \end{aligned}$$

IV. *Comment by Zhen Wang and Ji Chen, Ningbo University, China.*

We prove Kee-Wai Lau's inequality

$$\sum \frac{bc}{m_b^2+m_c^2} \ge 2, \qquad (2)$$

given on [1992: 253].

From $4m_a^2 = 2b^2+2c^2-a^2$, etc., (2) is equivalent to

$$\sum \frac{bc}{4a^2+b^2+c^2} \ge \frac{1}{2},$$

and this follows from

$$\begin{aligned} &2\sum bc(4b^2+c^2+a^2)(4c^2+a^2+b^2) - \prod(4a^2+b^2+c^2) \qquad (3) \\ &= -4\sum a^6 + 8\sum(a^5b+a^5c) - 21\sum(a^4b^2+a^4c^2) + 2\sum a^4bc \\ &\qquad + 34\sum b^3c^3 + 10\sum(a^3b^2c+a^3c^2b) - 78a^2b^2c^2 \\ &= \frac{1}{16}\sum(b-c)^2(c+a-b)(a+b-c)(119a^2+41b^2+41c^2-18bc) \\ &\qquad + \frac{9}{16\sum a}\left(\prod(b+c-a)\right)\sum(b-c)^2[(3a-b-c)^2+2(c+a-b)(a+b-c)] \\ &\ge 0. \end{aligned}$$

[*Editor's note.* The equalities above, and similar ones for H in solution III, have been verified by helpful colleague Len Bos using MACSYMA. But the editor hasn't the foggiest idea how they were obtained! Can anyone see an easier reason why the expressions in (1) and (3) are nonnegative?

Now, as promised on [1992: 253], here is Lau's short elegant proof of 1680 using his inequality (2).]

V. *Solution by Kee-Wai Lau, Hong Kong.*

We have

$$\sum \frac{r_b r_c}{m_b m_c} = \frac{1}{4}\sum \frac{(b+c)^2 - a^2}{m_b m_c} \geq \frac{1}{2}\sum \frac{4bc - a^2}{m_b^2 + m_c^2}$$
$$= 2\sum \frac{bc}{m_b^2 + m_c^2} - \frac{1}{2}\sum \frac{a^2}{m_b^2 + m_c^2} \geq 2 \cdot 2 - \frac{1}{2} \cdot 2 = 3,$$

by (2) and item 10.10, page 212 of D.S. Mitrinović et al, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

*   *   *   *   *

1756. [1992: 176] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

For positive integers $n \geq 3$ and $r \geq 1$, the *n-gonal number of rank r* is defined as

$$P(n,r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}\ .$$

Call a triple (a,b,c) of natural numbers, with $a \leq b < c$, an *n-gonal Pythagorean triple* if $P(n,a) + P(n,b) = P(n,c)$. When $n = 4$, we get the usual Pythagorean triple.

(i) Find an n-gonal Pythagorean triple for each n.

(ii) Consider all triangles ABC whose sides are n-gonal Pythagorean triples for some $n \geq 3$. Find the maximum and the minimum possible values of angle C.

I. *Solution to part* (i) *by P. Penning, Delft, The Netherlands.*

The condition for a triple reads:

$$(n-2)(a^2 + b^2 - c^2) = (n-4)(a+b-c). \qquad (1)$$

Choose $c = b + v$, with $v > 0$ an integer. Then since $b^2 - c^2 = -v(b+c)$ we have

$$(n-2)(a^2 - v(b+c)) = (n-4)(a-v)$$

and so

$$2b + v = b + c = \frac{a^2}{v} - \frac{n-4}{n-2}\left(\frac{a}{v} - 1\right).$$

Set $a/v - 1 = w(n-2)/2$, with $w > 0$. [Note that since $\gcd(n-4, n-2) = 1$ or 2 and $b + c$ is an integer,

$$w = \frac{2}{n-2}\left(\frac{a}{v} - 1\right)$$

must also be an integer.] Then

$$a = v\left(1 + \frac{w(n-2)}{2}\right),$$

$$b = \frac{v}{2}\left(1 + \frac{w(n-2)}{2}\right)^2 - \frac{w(n-4)}{4} - \frac{v}{2} = \frac{w}{4}\left(\frac{wv(n-2)^2}{2} + 2v(n-2) - n + 4\right),$$

and $c = v + b$, where a and b must be integers.

Now $v = 1, w = 2$ is always a solution giving

$$a = n-1, \quad b = \frac{(n-1)(n-2)}{2} + 1, \quad c = \frac{(n-1)(n-2)}{2} + 2.$$

In addition, if $n = 4k$ or $n = 4k+1$ then $v = 2, w = 1$ is also a solution giving

$$a = n, \quad b = \frac{n(n-1)}{4}, \quad c = \frac{n(n-1)}{4} + 2.$$

[Penning then gave a partial solution to (ii).—*Ed.*]

II. *Solution to part* (ii) *by H.L. Abbott, University of Alberta.*

[Abbott first solved part (i).—*Ed.*]

If a, b and c satisfy $P(n,a) + P(n,b) = P(n,c)$ then, from the cosine law [and (1)],

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a+b-c)(n-4)}{2ab(n-2)}\ . \tag{2}$$

To determine the maximum value of the angle C, note that $a = b = 2$, $c = n = 3$ satisfy (1), so that, by (2), $\max C \geq \cos^{-1}(-1/8)$ which is approximately 97.18 degrees. Now $\cos C$ is negative only when $n = 3$. It then follows from (2) and $c \geq b+1$, $b \geq a$ that

$$\cos C = \frac{c-a-b}{2ab} \geq \frac{1-a}{2ab} \geq \frac{1-b}{2b^2} \geq -\frac{1}{8}$$

with equality if and only if $a = b = 2$, $c = 3$. Thus $\max C = \cos^{-1}(-1/8)$.

To determine the minimum value of angle C, note first that $a = b = n = 5$, $c = 7$ satisfy (1), so that by (2), $\min C \leq \cos^{-1}(1/50)$ which is approximately 88.85 degrees. We may suppose that $n \geq 5$. From (2) and $a \leq b \leq c-1$ we now get

$$\cos C < \frac{a+b-c}{2ab} \leq \frac{a-1}{2ab} \leq \frac{b-1}{2b^2}\ .$$

It follows that if $b \geq 24$, then $\cos C < 1/50$. We may therefore suppose that $b \leq 23$. The conditions $a \leq b \leq 23$ and $c < a+b$ reduce the problem down to examining a finite number of cases. One finds that $\min C = \cos^{-1}(1/50)$ and the only two solutions of (1) for which the minimum is attained are $(n,a,b,c) = (5,5,5,7)$ and $(n,a,b,c) = (8,10,10,14)$.

Also solved (both parts) by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD K. GUY, University of Calgary; and the proposer ($\max C$

only in part (ii)*). Part* (i) *only solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; RICHARD I. HESS, Rancho Palos Verdes, California; SHOICHI HIROSE, Tokyo, Japan; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; and KENNETH M. WILKE, Topeka, Kansas. One reader sent in a partial solution of part* (i).

Hirose pointed out a paper of his (On some polygonal numbers which are, at the same time, the sums, differences and products of two other polygonal numbers, Fibonacci Quarterly **24** (1986) 99–106*) in which* (i) *and related results are considered, and which contains a lot of references.*

It would be nice to have a solution of (ii) *which didn't require so much calculation!*

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1758. [1992: 176] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

In Hilbert High School there are an infinite number of lockers, numbered by the natural numbers: $1, 2, 3, \ldots$. Each locker is occupied by exactly one student. The administration decides to rearrange the students so that the lockers are all still occupied by the same set of students, one to a locker, but in some other order. (Some students might not change lockers.) It turns out that the locker numbers of infinitely many students end up higher than before. Show that there are also infinitely many students whose locker numbers are lower than before.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

(1) Suppose that an infinite number of students move to higher locker numbers but that only a finite number of students move to lower locker numbers.

(2) From (1) it follows that there is a maximum locker number N, from which a student moved to a lower numbered locker.

(3) Consider lockers 1 to N and note from (2) that no student from a higher locker number than N moves into any of them. Thus the rearrangement of students in the first N lockers is just a permutation of the first N students in these lockers.

(4) Any rearrangement of students in lockers $N+1, N+2, \ldots$ implies a lowest numbered locker, $M > N$, from which a student moves. M cannot be filled from below or it would not be the minimum; M cannot be filled from above because then N would not be the maximum as stated in (2).

Statement (1) thus leads to a contradiction, and we're done.

Also solved by H.L. ABBOTT, University of Alberta; CHARLES ASHBACHER, Cedar Rapids, Iowa; MARGHERITA BARILE, student, Universität Essen, Germany; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; ANDY PULKSTENIS, Messiah College, Grantham, Pennsylvania; CORY PYE, student, Memorial University of Newfoundland, St. John's; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution was sent in.

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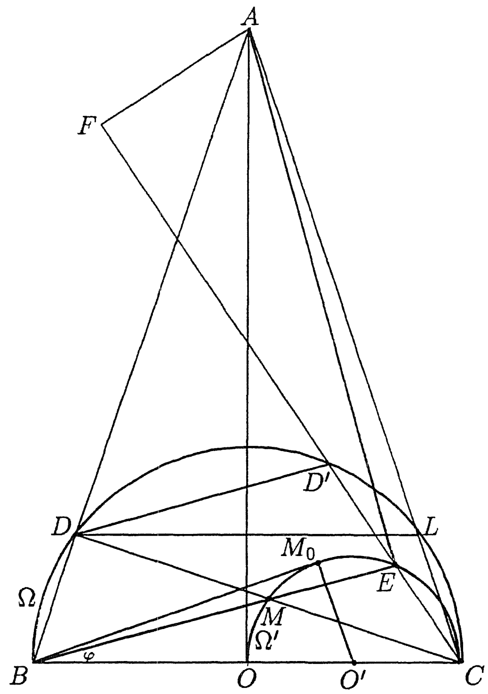
1761. [1992: 205] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$. Let D be the foot of the perpendicular from C to AB, and let M be the midpoint of CD. Let E be the foot of the perpendicular from A to BM, and let F be the foot of the perpendicular from A to CE. Prove that $AF \le AB/3$.

Solution by Jordi Dou, Barcelona, Spain.

Let Ω be the circle of diameter BC (with centre O), and Ω' the circle of diameter OC (with centre O'). Put $D' = CE \cap \Omega$, $L = AC \cap \Omega$, M_0 the point of contact of the tangent to Ω' from B, $\varphi = \angle CBM$, and $\varphi_0 = \angle CBM_0$.



The point D is on Ω, so M is on Ω'. From

$$\frac{BD}{BM} = \frac{BE}{BA} \quad (= \cos\angle ABE)$$

and

$$\frac{BO}{BA} = \frac{BD}{BC} \quad (= \cos\angle ABC),$$

we have $BM \cdot BE = BA \cdot BD = BO \cdot BC$. Thus $BM \cdot BE = BO \cdot BC$, so the point E lies on Ω'. From $DL \| BC$ and $DD' \| BE$,

$$\varphi = \angle LDD' = \angle LCD' = \angle ACF;$$

thus, since $\varphi \le \varphi_0$,

$$\frac{AF}{AB} = \frac{AF}{AC} = \sin\varphi \le \sin\varphi_0 = \frac{O'M_0}{BO'} = \frac{1}{3}\,.$$

If $M = M_0$, then $AF = AB/3$ and $OA = BC/\sqrt{2}$ [since then BM_0 and AO' are perpendicular — *Ed.*]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SEUNG-JIN BANG, Albany, California; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; L.J. HUT, Groningen, The Netherlands, WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One incorrect solution was received.

No doubt, Dou's beautiful solution will be especially appreciated by the other solvers, most of whom resorted to analytic geometry.

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1762. [1992: 206] *Proposed by Steven Laffin, student, École J. H. Picard, and Andy Liu, University of Alberta, Edmonton.* (Dedicated to Professor David Monk, University of Edinburgh, on his sixtieth birthday.)

Starship Venture is under attack from a Zokbar fleet, and its Terrorizer is destroyed. While it can hold out, it needs a replacement to drive off the Zokbars. Starbase has spare

Terrorizers, which can be taken apart into any number of components, and enough scout ships to provide transport. However, the Zokbars have n Space Octopi, each of which can capture one scout ship at a time. Starship Venture must have at least one copy of each component to reassemble a Terrorizer, but it is essential that the Zokbars should not be able to do the same. Into how many components must each Terrorizer be taken apart (assuming all are taken apart in an identical manner), and how many scout ships are needed to transport them? Give two answers:

(a) assuming that the number of components per Terrorizer is as small as possible, minimize the number of scout ships;

(b) assuming instead that the number of scout ships is as small as possible, minimize the number of components per Terrorizer.

Solution by the proposers.

Note first that the minimum number of spare Terrorizers Starbase must have is $n+1$. If there are only n of them, there will be n copies of each component. If the Zokbars capture n scout ships carrying a common component, Starship Venture will not get a complete Terrorizer. On the other hand, if there are $n+1$ spare Terrorizers, Starship Venture is guaranteed at least one copy of each component. [It is assumed throughout that no two copies of any component are ever carried by any one scout ship, else the Zokbars could still capture all copies of that component.—*Ed.*] Hence there is no advantage in having more than $n+1$ spare Terrorizers.

(a) The minimum number of components into which each Terrorizer must be taken apart is $n+1$. Otherwise, the Zokbars may get a complete Terrorizer by capturing, for each component, one scout ship that is carrying it.

If each Terrorizer is taken apart into $n+1$ components, Starbase must have $(n+1)^2$ scout ships. Each will carry one component of one Terrorizer. If there are fewer scout ships, then some scout ship must carry at least two components. The Zokbars can capture this scout ship and, for each of the other components, one scout ship carrying it.

(b) Starbase must have $2n+1$ scout ships. Otherwise, the Zokbars will end up capturing a number of scout ships equal to that which get through to Starship Venture. If one side can get a complete Terrorizer, so can the other.

If Starbase has $2n+1$ scout ships, each Terrorizer must be taken apart into $\binom{2n+1}{n}$ components, one for each subset of n of the $2n+1$ scout ships. Each component will be carried by the $n+1$ scout ships not in the subset associated with it. Whichever n scout ships the Zokbars capture, exactly one component will be missing. On the other hand, if the number of components is smaller, two distinct (not necessarily disjoint) subsets of n scout ships [i.e., at least $n+1$ scout ships altogether] will then be missing the same component. This is impossible since there are $n+1$ copies of each component.

Remark. This problem is a variant of the Couriers Problem from Dennis Shasha's *The Puzzling Adventures of Dr. Ecco*, pages 115–118 and 173–175. The book is published by W.H. Freeman and Company, New York, 1988.

To the Editor's surprise, no solutions of this problem were sent in.

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1763. [1992: 206] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $t \geq 0$, and for each integer $n \geq 1$ define

$$x_n = \frac{1+t+t^2+\cdots+t^n}{n+1}\ .$$

Prove that $x_1 \leq \sqrt{x_2} \leq \sqrt[3]{x_3} \leq \sqrt[4]{x_4} \leq \cdots$.

I. *Solution by Chris Wildhagen, Rotterdam, The Netherlands.*

Define

$$F(t) = \frac{(1+t+t^2+\cdots+t^{n-1})^n}{(1+t+t^2+\cdots+t^n)^{n-1}}\ .$$

Then for $n \geq 2$,

$$\sqrt[n-1]{x_{n-1}} \leq \sqrt[n]{x_n} \quad \Leftrightarrow \quad (x_{n-1})^n \leq (x_n)^{n-1} \quad \Leftrightarrow \quad F(t) \leq \frac{n^n}{(n+1)^{n-1}} = F(1).$$

For $t \neq 1$,

$$F(t) = \frac{(t^n-1)^n}{(t-1)(t^{n+1}-1)^{n-1}}$$

and so $F'(t) = G(t)H(t)$, where

$$G(t) = \frac{(t^n-1)^{n-1}}{(t-1)^2(t^{n+1}-1)^n}$$

and

$$\begin{aligned} H(t) &= n^2t^{n-1}(t-1)(t^{n+1}-1) - (t^n-1)[(t^{n+1}-1) + (n^2-1)(t-1)t^n] \\ &= -t^{2n} + n^2t^{n+1} + 2(1-n^2)t^n + n^2t^{n-1} - 1 \\ &= -(t^n-1)^2 + n^2t^{n-1}(t-1)^2 \\ &= -n^2(t-1)^2\left[\left(\frac{1+t+t^2+\cdots+t^{n-1}}{n}\right)^2 - \left(\sqrt[n]{1\cdot t\cdot t^2\cdots t^{n-1}}\right)^2\right] \\ &< 0 \end{aligned}$$

by the A.M.–G.M. inequality. Since $G(t) < 0$ for $t \in [0,1)$ and $G(t) > 0$ for $t \in (1,\infty)$, we deduce that $F(t)$ is strictly increasing on $[0,1)$ and strictly decreasing on $(1,\infty)$, and thus $F(t)$ has an absolute maximum at $t = 1$, proving the required result.

II. *Solution by Murray S. Klamkin, University of Alberta.*

By changing t to $1/t$, it follows that if the result is valid for t in $[0,1]$ then it is also valid for $t > 1$. We now first prove a stronger result, that the sequence $\{x_n\}$ is logarithmically convex for t in $[0,1]$, i.e.,

$$x_{n-1}x_{n+1} \geq x_n^2, \tag{1}$$

or equivalently that

$$(n+1)^2(1-t^n)(1-t^{n+2}) \geq n(n+2)(1-t^{n+1})^2,$$

or, expanding out and collecting terms,

$$(1-t^n)(1-t^{n+2}) \geq (n^2+2n)t^n(1-t)^2,$$

or that

$$\left(\frac{1+t+\cdots+t^{n-1}}{n}\right)\cdot\left(\frac{1+t+\cdots+t^{n+1}}{n+2}\right) \geq t^n.$$

The latter inequality will follow immediately from the A.M.–G.M. inequality:

$$\frac{1+t+\cdots+t^{m-1}}{m} \geq \left(t^{0+1+\cdots+(m-1)}\right)^{1/m} = t^{(m-1)/2} \quad \text{for } m=1,2,\ldots\ .$$

We now show that (1) implies the desired result. By repeated use of (1),

$$x_0x_1^2x_2^4x_3^6\cdots x_{r-1}^{2r-2}x_r^{r-1}x_{r+1}^r = (x_0x_2)(x_1x_3)^2(x_2x_4)^3\cdots(x_{r-1}x_{r+1})^r \geq x_1^2x_2^4x_3^6\cdots x_r^{2r}$$

or that (since $x_0=1$)

$$x_r^{1/r} \leq x_{r+1}^{1/(r+1)}.$$

Comment. The fact that $\{x_n\}$ is logarithmically convex also follows from a known theorem [1] which states that if a sequence $\{a_n\}$ is logarithmically convex, then so is the sequence $\{A_n\}$ where $A_n = \sum_{k=0}^n a_k/(n+1)$. Since it is immediate that the sequence $\{t^n\}$, $n=0,1,2,\ldots$ is logarithmically convex, so is $\{x_n\}$.

Reference:

[1] D.S. Mitrinović, I.B. Lacković, M.S. Stanković, On some convex sequences connected with N. Ozeki's results, *Publ. Electrotehn. Ser. Mat. Fiz. Univ. Beograd*, No. 634–No. 677 (1979) 3–24.

Also solved by SEUNG-JIN BANG, Albany, California; YI-MING DING, student, Ningbo University, China; F.J. FLANIGAN, San Jose State University, California; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Penn State University at Harrisburg; and the proposer.

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1764. [1992: 206] *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Determine the extreme values of $a^2b+b^2c+c^2a$, where a,b,c are sides of a triangle of semiperimeter 1.

(b)* What are the extreme values of $a_1^2a_2+a_2^2a_3+\cdots+a_n^2a_1$, where $a_1,a_2,\ldots,a_n$ are the (consecutive) sides of an n-gon of semiperimeter 1?

Solution by G.P. Henderson, Campbellcroft, Ontario.

The a_i can be the sides of a (possibly degenerate) n-gon if and only if each a_i does not exceed the sum of the other sides. That is, $a_i \leq 2-a_i$ or $a_i \leq 1$.

(a) Set

$$F(a,b,c) = a^2b + b^2c + c^2a$$

where

$$a + b + c = 2, \qquad 0 \le a, b, c \le 1.$$

We will show that

$$\frac{23}{27} = F\left(\frac{1}{3}, \frac{2}{3}, 1\right) \le F(a,b,c) \le F(1,1,0) = 1. \tag{1}$$

That is, the extreme values are 23/27 and 1.

For extrema in the interior of the feasible region, we set the derivatives of F with respect to a and b equal to zero, considering c to be a function of a and b. [For example, $c = 2 - a - b$ gives $\partial c/\partial a = -1$, so

$$\frac{\partial F}{\partial a} = 2ab + b^2\frac{\partial c}{\partial a} + 2ca\frac{\partial c}{\partial a} + c^2 = 2ab + c^2 - 2ca - b^2,$$

and similarly for $\partial F/\partial b$.—*Ed.*] We get

$$2ab + c^2 = 2bc + a^2 = 2ca + b^2.$$

The only solution in the interior is $a = b = c = 2/3$, $F = 8/9$. Since $23/27 < 8/9 < 1$, the global extrema occur on the boundary of the feasible region. Therefore at least one of a, b, c (say c) is 0 or 1. $c = 0$ implies $a = b = 1$ and $F = 1$. If $c = 1$,

$$F = a^2b + b^2 + a, \quad a + b = 1, \quad a, b \ge 0.$$

Set $b = 1 - a$. Then

$$F = -a^3 + 2a^2 - a + 1 = \frac{23}{27} + \left(a - \frac{1}{3}\right)^2\left(\frac{4}{3} - a\right) \ge \frac{23}{27}$$

and

$$F = 1 - a(1-a)^2 \le 1.$$

This completes the proof of (1).

(b) If $n \ge 4$ we will show that if $p, q \ge 1$, the extreme values of

$$F(a) = \sum_{i=1}^{n} a_i^p a_{i+1}^q$$

subject to

$$\sum_{i=1}^{n} a_i = 2, \quad 0 \le a_i \le 1, \quad a_{n+1} = a_1$$

are

$$F(1,0,1,0,0,\ldots,0) = 0 \quad \text{and} \quad F(1,1,0,0,\ldots,0) = 1.$$

The minimum value is obvious and we only need to verify the maximum for $p = q = 1$ because F is a decreasing function of p and q. We are to prove that

$$G(a) = \sum_{i=1}^{n} a_i a_{i+1} \leq 1.$$

If n is even,

$$G \leq (a_1 + a_3 + \cdots + a_{n-1})(a_2 + a_4 + \cdots + a_n).$$

This is the product of two factors whose sum is 2. Therefore $G \leq 1$. If n is odd, $n \geq 5$, choose the notation so that $a_1 \leq a_2$. Then

$$\begin{aligned} G &\leq (a_1 + a_3 + \cdots + a_n)(a_2 + a_4 + \cdots + a_{n-1}) + a_1 a_n - a_2 a_n \\ &\leq 1 + a_n(a_1 - a_2) \leq 1. \end{aligned}$$

Part (a) *also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Two other readers sent in incomplete or unconvincing solutions of part* (a). *Only Henderson was able to solve (the not too bad) part* (b)*; no doubt other readers were scared off by the "*"!*

The proposer's solution to part (a) *used his problem* 1292 *of* Mathematics Magazine *(solution on pp.* 137–139 *of the April* 1989 *issue), which contains a similar polynomial.*

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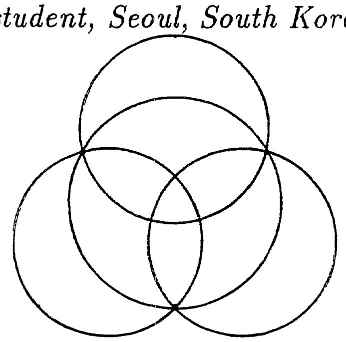
1765. [1992: 206] *Proposed by Kyu Hyon Han, student, Seoul, South Korea.*

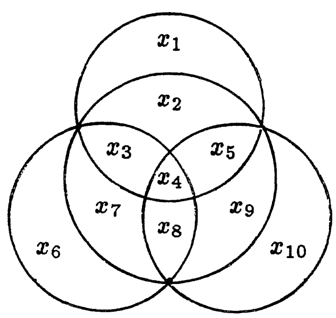
There are four circles piled up, making a total of 10 regions. The outer circles each have 5 regions and the central circle has 7 regions. You put one of the numbers $0, 1, 2, \ldots, 9$ in each region, without reusing any number, so that the sum of the numbers in any circle is always the same value, say S. What is the smallest and the largest possible value of S?



Solution by Joseph Ling, University of Calgary.

The smallest and the largest possible values of S are 21 and 25 respectively.

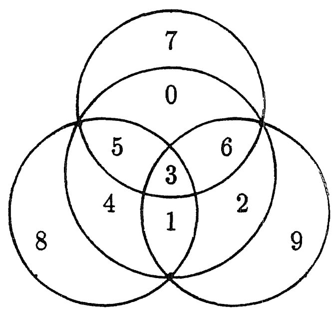
For reference, we fix the notation as follows:



Considering the center circle, we have

$$S = x_2 + x_3 + x_4 + x_5 + x_7 + x_8 + x_9 \geq 0 + 1 + 2 + 3 + 4 + 5 + 6 = 21.$$

This minimum value of S is attained by the following:



Next, equating the sum of the entries in each of the outside circles with that of the center circle and cancelling the common portions, we get

$$x_1 = x_7 + x_8 + x_9, \quad x_6 = x_2 + x_5 + x_9, \quad x_{10} = x_2 + x_3 + x_7.$$

Hence

$$x_1 + x_6 + x_{10} = 2(x_2 + x_7 + x_9) + (x_3 + x_5 + x_8). \tag{1}$$

On the other hand, since S = sum of entries in the center circle,

$$S + (x_1 + x_6 + x_{10}) = 0 + 1 + 2 + \cdots + 9 = 45. \tag{2}$$

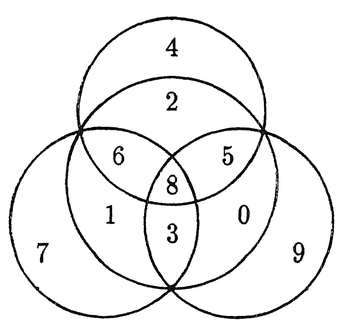
So, by (1) and (2),

$$\begin{aligned} 45 + (x_2 + x_7 + x_9) &= x_4 + (x_1 + x_6 + x_{10}) + 2(x_2 + x_7 + x_9) + (x_3 + x_5 + x_8) \\ &= x_4 + 2(x_1 + x_6 + x_{10}) = x_4 + 2(45 - S). \end{aligned}$$

Hence $2S + (x_2 + x_7 + x_9) = 45 + x_4$. Therefore

$$2S + 3 \leq 2S + (x_2 + x_7 + x_9) = 45 + x_4 \leq 45 + 9 = 54.$$

So $S \leq (54 - 3)/2 = 25.5$. Thus, the largest possible value of S is 25, which is attained by the following:



Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; FRANCIS CHAN, Waterloo, LIAN-XIANG WANG, University of Waterloo, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; DAG JONSSON, Uppsala, Sweden; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; P.E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

As several solvers figured out, there are altogether 14 *ways to put the numbers* 0 *to* 9 *on the diagram according to the problem, and each integer* S *from* 21 *to* 25 *occurs in at least one of these solutions.*

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1766*. [1992: 206] *Proposed by Jun-hua Huang, student, 4th Middle School of Nanxian, Hunan, China.*

The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$, $x_2 = x$, and

$$x_{n+2} = xx_{n+1} + nx_n$$

for $n \geq 0$. Prove or disprove that for each $n \geq 2$, the coefficients of the polynomial $x_{n-1}x_{n+1} - x_n^2$ are all nonnegative, except for the constant term when n is odd.

Solution by G.P. Henderson, Campbellcroft, Ontario.

We will prove that the conjecture is true. Set $x^2 = y$, and

$$x_{2n+1} = P_n(y), \quad x_{2n+2} = xQ_n(y), \quad n = 0, 1, \ldots$$

where P_n and Q_n are polynomials. The given difference equation becomes

$$P_n = yQ_{n-1} + (2n-1)P_{n-1}, \quad Q_n = P_n + 2nQ_{n-1}, \quad n = 1, 2, \ldots,$$

$$P_0 = Q_0 = 1.$$

It is easily verified that the solution of these is

$$P_n = \sum_{r=0}^{n} p(n,r)y^r, \quad Q_n = \sum_{r=0}^{n} q(n,r)y^r$$

where

$$p(n,r) = \binom{n}{r}\frac{c_n}{c_r}, \quad q(n,r) = \binom{n}{r}\frac{c_{n+1}}{c_{r+1}}, \quad n = 0, 1, \ldots$$

and

$$c_n = 1 \cdot 3 \cdot 5 \cdots (2n-1), \quad c_0 = 1.$$

[*Editor's note.* First observe that these formulas are correct for $n = 0$, and then use induction on n, assuming that the formulas are correct for both P_n and Q_n for $n < k$. Then

$$\begin{aligned}
yQ_{k-1} + (2k-1)P_{k-1} &= \sum_{r=0}^{k-1} \binom{k-1}{r} \frac{c_k}{c_{r+1}} y^{r+1} + (2k-1)\sum_{r=0}^{k-1} \binom{k-1}{r} \frac{c_{k-1}}{c_r} y^r \\
&= \sum_{r=1}^{k} \binom{k-1}{r-1} \frac{c_k}{c_r} y^r + (2k-1)\sum_{r=0}^{k-1} \binom{k-1}{r} \frac{c_{k-1}}{c_r} y^r \\
&= \sum_{r=1}^{k-1} \frac{(k-1)!}{r!(k-r)!} \frac{c_{k-1}}{c_r} [r(2k-1) + (2k-1)(k-r)] y^r \\
&\qquad\qquad + y^k + (2k-1)c_{k-1} \\
&= \sum_{r=1}^{k-1} \frac{(k-1)!}{r!(k-r)!} \frac{c_{k-1}}{c_r} k(2k-1) y^r + y^k + c_k \\
&= \sum_{r=0}^{k} \binom{k}{r} \frac{c_k}{c_r} y^r = P_k,
\end{aligned}$$

followed by a similar argument for Q_k.]

Set

$$u_n = x_{n-1}x_{n+1} - x_n^2.$$

We check that the constant term in this is positive if n is even and negative if n is odd. Since

$$u_{2n} = x_{2n-1}x_{2n+1} - x_{2n}^2 = P_{n-1}P_n - x^2Q_{n-1}^2$$

and

$$u_{2n+1} = x_{2n}x_{2n+2} - x_{2n+1}^2 = x^2Q_{n-1}Q_n - P_n^2,$$

the constant term in u_{2n} is $p(n-1,0)p(n,0) = c_{n-1}c_n > 0$ and in u_{2n+1} it is $-p(n,0)^2 < 0$.

For $n \geq 4$, we have

$$\begin{aligned}
u_n &= x_{n-1}[xx_n + (n-1)x_{n-1}] - x_n[xx_{n-1} + (n-2)x_{n-2}] \\
&= (n-1)x_{n-1}^2 - (n-2)x_nx_{n-2} \\
&= (n-1)x_{n-1}[xx_{n-2} + (n-3)x_{n-3}] - (n-2)x_{n-2}[xx_{n-1} + (n-2)x_{n-2}] \\
&= xx_{n-1}x_{n-2} + (n^2-4n+3)x_{n-1}x_{n-3} - (n^2-4n+4)x_{n-2}^2 \\
&= (n-2)^2u_{n-2} + x_{n-1}(xx_{n-2} - x_{n-3}).
\end{aligned}$$

The conjecture is true for $u_2 = 1$ and $u_3 = x^2 - 1$. We will have a proof by induction if we show that

$$v_n = x_{n-1}(xx_{n-2} - x_{n-3}), \quad n = 4, 5, \ldots$$

has nonnegative coefficients except perhaps for the constant term.

For $n = 2, 3, \ldots,$

$$\begin{aligned} v_{2n+1} &= x_{2n}(xx_{2n-1} - x_{2n-2}) = yQ_{n-1}(P_{n-1} - Q_{n-2}) \\ &= yQ_{n-1}\sum_{r=0}^{n-1}[p(n-1,r) - q(n-2,r)]y^r. \end{aligned}$$

In the sum, the constant term is zero and the coefficient of y^r, $r > 0$, is

$$\binom{n-1}{r}\frac{c_{n-1}}{c_r} - \binom{n-2}{r}\frac{c_{n-1}}{c_{r+1}} = \binom{n-2}{r-1}\frac{c_n}{c_{r+1}} \geq 0.$$

Since Q_{n-1} has nonnegative coefficients, so also does v_{2n+1}.

Again for $n = 2, 3, \ldots,$

$$\begin{aligned} v_{2n} &= x_{2n-1}(xx_{2n-2} - x_{2n-3}) = P_{n-1}(yQ_{n-2} - P_{n-2}) \\ &= P_{n-1}\left[\sum_{r=0}^{n-2} q(n-2,r)y^{r+1} - \sum_{r=0}^{n-2} p(n-2,r)y^r\right]. \end{aligned}$$

For $r = 0, 1, \ldots, n-2$, the coefficient of y^{r+1} in the square bracket is

$$q(n-2,r) - p(n-2,r+1) = \binom{n-2}{r}\frac{c_{n-1}}{c_{r+1}} - \binom{n-2}{r+1}\frac{c_{n-2}}{c_{r+1}} = \binom{n-1}{r+1}\frac{c_{n-2}}{c_r}$$

and the constant term in the square bracket is $-p(n-2,0)$. Hence

$$v_{2n} = \sum_{s=0}^{n-1} p(n-1,s)y^s\left[-p(n-2,0) + \sum_{r=0}^{n-2}\binom{n-1}{r+1}\frac{c_{n-2}}{c_r}y^{r+1}\right].$$

For $t = 1, 2, \ldots, n-1$, the coefficient of y^t in this is

$$-p(n-2,0)p(n-1,t) + \sum_{r=0}^{t-1}\binom{n-1}{r+1}\frac{c_{n-2}}{c_r}p(n-1,t-r-1).$$

Using only the last term of the sum, the coefficient is at least

$$-c_{n-2}\binom{n-1}{t}\frac{c_{n-1}}{c_t} + \binom{n-1}{t}\frac{c_{n-2}}{c_{t-1}}c_{n-1} = 2(t-1)\binom{n-1}{t}\frac{c_{n-1}c_{n-2}}{c_t} \geq 0.$$

For $t \geq n$, the term $-p(n-2,0)$ is not involved and the coefficient of y^t is nonnegative.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; YI-MING DING and WEI-PING QIU, students, Ningbo University, Ningbo, China; and A. HOLLEMAN, Monnickendam, The Netherlands.

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1767. [1992: 207] *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

(a) Jerry Slocum, the American puzzle collector, has the following puzzle, called Double Five, dating from about 1890. Ten coins are arranged in a circle. You are allowed to move a coin, in either direction, over two coins (piled up or not, with empty spaces between allowed) and place it on the next coin to make a pile of two. For example, from the starting position one could place coin 1 on top of coin 4, and then coin 5 on top of coin 3, etc. The object is to make five piles of two coins which are in the even locations, with blank spaces in the odd locations. Can you do it?

(b) Can you instead arrange to have the final piles of two in consecutive locations?

(c)* What about for other (even) numbers of coins?

Solution by H. L. Abbott, University of Alberta.

(a) $5 \to 2,\ 7 \to 10,\ 3 \to 8,\ 1 \to 4,\ 9 \to 6.$

(b) $1 \to 4,\ 8 \to 2,\ 10 \to 6,\ 9 \to 3,\ 7 \to 5.$

(c) (I) *Coins ending in consecutive locations.*

In general, let $n = 2k \geq 8$ be such that the coins can end up in some k consecutive locations. The coins are labelled 1 through n in a clockwise manner. There is no loss in supposing that on the first move coin 1 is placed on coin 4. There are then three possibilities for the coins that do not move: (i) $2, 3, \ldots, k+1$, (ii) $3, 4, \ldots, k+2$ and (iii) $4, 5, \ldots, k+3$.

Let coin w be placed on coin 5. If (i) holds then w must jump $6, 7, \ldots, k+1$ and this forces $k = 6$. If (ii) holds then w must jump $6, 7, \ldots, k+2$ so that $k = 5$. If (iii) holds then either w must jump $6, 7, \ldots, k+3$ yielding $k = 4$, or $w = 3$, in which case if coin u is placed on coin 6, u must jump $7, 8, \ldots, k+3$ and this gives $k = 5$. It follows that there are no solutions for $n \geq 14$ and by experimentation, it is found that there is no solution for $n = 6$. For $n = 4$, 8, 10 and 12 the following sequences of moves work.

$n = 4:\ 1 \to 4,\ 2 \to 3.$

$n = 8:\ 1 \to 4,\ 3 \to 7,\ 2 \to 5,\ 8 \to 6.$

$n = 10:\ 1 \to 4,\ 8 \to 2,\ 10 \to 6,\ 9 \to 3,\ 7 \to 5.$

$n = 12:\ 1 \to 4,\ 8 \to 5,\ 10 \to 2,\ 12 \to 7,\ 11 \to 3,\ 9 \to 6.$

(II) *Coins ending in even numbered locations.*

Consider first the variant of the problem where the coins are arranged in a row from left to right. Coin 1 cannot move past coin 6. If 1 were to be placed on 6 then 3 and 5 must first be moved to the right of 6, but no coin can be placed on 2. It follows that 1 must be placed on 4. If 1 jumps 2 and 3, then 3 must be moved before any coin can be placed on 2. In order for 3 to move, 5 must first move to the right and 3 must be placed on 6. But now no coin can reach 2. It follows then that 3 must be moved before 1 is moved. Suppose 3 ends up on 6. 3 must then jump 4 and 5 and 2 must be covered before 1 is moved, but there is no way of doing this. Therefore 3 must move to 8. This forces 5 and 7 to move first and they must move to someplace other than 4, 6 and 8. If both 5 and 7 move to the right then no coin can reach 2. Since 7 cannot move to 2, then 7 must move to the right and 5 must move to 2, jumping 3 and 4 in the process. After 5 and 7 are moved 3 must jump 4 and 6 to land on 8 and 1 is then moved to 4. If 7 moves to 12 then both 9 and 11 must have moved to the right and now nothing can reach 6. Therefore 7 must have gone to 10 and and 9 to 6. That is, the following sequence of moves (apart from an interchange of the first two or the last two moves) is forced: $5 \to 2$, $7 \to 10$, $3 \to 8$, $1 \to 4$, $9 \to 6$. It follows then that $n = 10m$ for some integer m and the blocks of length 10 are arranged according to the scheme just described.

Now suppose that the coins are arranged in a circle ordered in a clockwise sense. If $n = 4m$, $m \geq 1$, the following scheme works: $1 \to 4$, $5 \to 8$, $9 \to 12$, $\ldots$, $4m-3 \to 4m$, $3 \to 6$, $7 \to 10$, $11 \to 14$, $\ldots$, $4m-5 \to 4m-2$, $4m-1 \to 2$. That there is a solution for $n = 10m$, $m \geq 1$, follows from the discussion of the linear problem. By trial, there is no solution for $n = 6$. Suppose that for some $m \geq 3$, there is a solution for $n = 4m+2$. There is no loss in assuming that $1 \to 4$ is the first move. Then there are three possibilities: (i) 3 goes to $4m$, (ii) 3 goes to $4m+2$, (iii) 3 goes to 6.

If (i) holds then before 3 can move, $4m+1$ must move to $4m-2$ (if it moved to $4m-4$ then both $4m-1$ and $4m-3$ must be moved anticlockwise and now nothing can reach $4m+2$). 3 then moves to $4m$ and the moves $4m-1 \to 4m+2$ and $5 \to 2$ are forced. This leaves coins 6 through $4m-3$ to be dealt with as a linear arrangement. Therefore $n = 10k$ for some integer k. If (ii) holds then the sequence of moves $4m+1 \to 4m-2$, $4m-1 \to 2$, $3 \to 4m+2$, $4m-3 \to 4m$ is forced. This leaves coins 5 through $4m-4$ to be dealt with as a linear arrangement. Therefore, again, $n = 10k$ for some integer k. If (iii) holds, then before 3 can move 5 must be moved clockwise. 5 cannot move past 10. If it is moved to 10 then 7 and 9 must first be moved past 10. 7 cannot go past 12 and so must be placed on 12 forcing 9 to go to 14. But 9 and 11 must move before 7 does and 11 and 13 must move before 9 does — all of these moves must be clockwise. However, now no coin can reach 8. Therefore 5 must move to 8. Also it must move before 3 does. Thus the sequence $5 \to 8$, $3 \to 6$ is forced. Now 7 must move to 10. The same analysis shows that $9 \to 12$, $7 \to 10$ is forced and this can be continued to show that the sequence $13 \to 16$, $11 \to 14$, $\ldots$, $4m-3 \to 4m$, $4m-5 \to 4m-2$ is forced. But now $4m+1$ cannot move. Thus (iii) does not arise.

Therefore a sequence of moves which leaves coins on the even numbered positions exists if and only if $n = 4k$ or $n = 10k$ for some integer k.

Move sequences for $n = 4k$ and $n = 10k$ were also given by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany. Solutions for 10 coins (plus some other cases) were found by RICHARD I. HESS, Rancho Palos Verdes, California; ANDY LIU, University of Alberta; and the proposer. Solutions for 10 or less coins but allowing more than 2 coins in a pile were found by CHARLES ASHBACHER, Cedar Rapids, Iowa; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

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1769. [1992: 207] *Proposed by Jason Colwell, student, Edmonton, Alberta.*

Suppose $Q(x)$ is a polynomial with real coefficients and zero constant term.

(a) Prove that there exists a real polynomial $P(x)$ such that

$$Q(n) = \sum_{k=1}^{n} P(k)$$

for all positive integers n.

(b) Show that $Q(-1) = -P(0)$ for all $P(x)$ and $Q(x)$ as in (a).

Solution by Kee-Wai Lau, Hong Kong.

(a) It is easy to see that one can take $P(x)$ to be $Q(x) - Q(x-1)$.

(b) Let $P(x)$ be any polynomial such that $Q(n) = \sum_{k=1}^{n} P(k)$ for all positive integers n. Then

$$P(n) = \sum_{k=1}^{n} P(k) - \sum_{k=1}^{n-1} P(k) = Q(n) - Q(n-1)$$

for $n > 1$. So the polynomial $P(x) - Q(x) + Q(x-1)$ has infinitely many zeros and is therefore identically zero. It follows that $P(x) \equiv Q(x) - Q(x-1)$. Since $Q(0) = 0$, so $Q(-1) = -P(0)$.

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; F.J. FLANIGAN, San Jose State University, California; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HANS LAUSCH, Monash University, Clayton, Australia; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario, and WAN-DI WEI, University of Nebraska, Lincoln; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Part (a) *only solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria.*

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1770. [1992: 207] *Proposed by F.R. Baudert, Glenstantia, South Africa.*

Find the sum to n terms of the series $6, 7, 16, 17, 26, 27, \ldots$. (Your answer should be a single expression in terms of n.)

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

Let a_i be the ith member of the given series. Then

$$a_1 = 4 + 2, \quad a_2 = 9 - 2, \quad a_3 = 14 + 2, \quad a_4 = 19 - 2, \quad \ldots,$$

or in general,

$$a_i = b_i - 2(-1)^i,$$

where b_i is the ith member of the arithmetical series $4, 9, 14, 19, \ldots$; thus $b_i = 5i - 1$. Let s_n be the required sum. Then for even n we get

$$s_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = \frac{n}{2}(b_1 + b_n) = \frac{n(5n+3)}{2} \ ,$$

whereas for odd n

$$s_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i + 2 = \frac{n}{2}(b_1 + b_n) + 2 = \frac{n(5n+3)}{2} + 2.$$

Hence it follows that

$$s_n = \frac{n(5n+3)}{2} + 1 + (-1)^{n-1}.$$

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; R.H. EDDY, Memorial University of Newfoundland, St. John's; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; HERTA T. FREITAG, Roanoke, Virginia; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; STEPHEN D. HNIDEI, student, University of Windsor; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; HANS LAUSCH, Monash University, Clayton, Australia; JOSEPH LING, University of Calgary; DAVID E. MANES, State University of New York, Oneonta; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Dover, Pennsylvania; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One partial solution was sent in.

Battles, Janous, and Klamkin sent in generalizations.

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Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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