Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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The Solution of a Common Differential Equation

E. R. LOVE

The differential equation is frequently met, but the method of solving it described here is likely to be new to most readers.

1. Linearity

The differential equation to be solved is of the form

$$y'' + ay' + by = f(x). \tag{1}$$

Here y is an unknown function of x, each prime denotes a differentiation with respect to x, a and b are known real constants, and $f(x)$ is the sum of the given functions $q(x)\mathrm{e}^{kx}$, where the $q(x)$ are real polynomials and the k are real or complex constants. Differential equations of this kind often occur in branches of mathematical physics such as mechanics and electrical theory.

We reduce the solution of (1) to the solution of differential equations with the same left-hand side, but with particularly simple right-hand sides. This is an entirely standard approach. The contents of section 3 are also well known. On the other hand, the essence of section 2 has not previously appeared in print, while the method described in section 4 — and sketched in reference 1 — has not so far found its way into a textbook.

The key result enabling us to simplify the solution of (1) is the following one, which is usually referred to as the linearity property of the differential equation, or as the linear superposition of solutions.

Lemma 1. *Suppose that $y = y_1(x)$ and $y = y_2(x)$ are solutions of the differential equations*

$$y'' + ay' + by = f_1(x) \quad \textit{and} \quad y'' + ay' + by = f_2(x),$$

respectively, and that c_1, c_2 are any constants. Then $y = c_1y_1(x) + c_2y_2(x)$ is a solution of

$$y'' + ay' + by = c_1f_1(x) + c_2f_2(x).$$

This follows immediately from the identity

$$\frac{d}{dx}\{c_1y_1(x) + c_2y_2(x)\} = c_1y_1'(x) + c_2y_2'(x)$$

and the corresponding identity for second derivatives.

If, now, in (1), $f(x) = \sum_{r=1}^{n} q_r(x)\mathrm{e}^{k_rx}$, then to obtain a solution of (1) we need only solve each equation

$$y'' + ay' + by = q_r(x)\mathrm{e}^{k_rx}$$

for $r = 1, 2, \ldots, n$ and add these n solutions.

2. Homogenous equations

In this section we solve the simplest differential equation of the form (1), namely the so-called *homogeneous* equation

$$y'' + ay' + by = 0. \tag{2}$$

Our aim is to find the *general solution* of (2), which means a formula comprising *all* solutions. For instance the general solution of $\frac{dy}{dx} = x$ is $y = \frac{1}{2}x^2 + C$, where C is any constant. Each value of C gives a solution and every solution is of this form. The general solution is a *family* of solutions, not a single solution.

Lemma 2. *The general solution of the differential equation*

$$y' + \alpha y = 0 \tag{3}$$

is

$$y = C\mathrm{e}^{-\alpha x}, \tag{4}$$

where C is any constant.

Since $\mathrm{e}^{\alpha x} \neq 0$ for all α and x, (3) is equivalent to

$$\mathrm{e}^{\alpha x}y' + \alpha\mathrm{e}^{\alpha x}y = 0,$$

or to

$$\frac{d}{dx}(\mathrm{e}^{\alpha x}y) = 0,$$

which has the general solution $\mathrm{e}^{\alpha x}y = C$, i.e. (4).

Theorem 1. *The general solution of the differential equation* (2) *is*

$$y = A\mathrm{e}^{p_1x} + B\mathrm{e}^{p_2x}$$

if the quadratic

$$p^2 + ap + b = 0, \tag{5}$$

called the indicial equation *of* (2), *has distinct roots p_1, p_2; and it is*

$$y = (Ax + B)\mathrm{e}^{p_1x}$$

if (5) *has the repeated root p_1.*

To prove the theorem we begin by introducing the new unknown function $Y = y\mathrm{e}^{-px}$, where p is a constant to be determined. Substituting $y = Y\mathrm{e}^{px}$ in (2) we have

$$\mathrm{e}^{px}\{(Y'' + 2pY' + p^2Y) + a(Y' + pY) + bY\} = 0,$$

so that (since $\mathrm{e}^{px} \neq 0$),

$$Y'' + (2p + a)Y' + (p^2 + ap + b)Y = 0. \tag{6}$$

If p satisfies the indicial equation (5), then (6) reduces to

$$Y'' + (2p + a)Y' = 0$$

which, by Lemma 2, has the general solution

$$Y' = C\mathrm{e}^{-(2p+a)x}, \tag{7}$$

where C is any constant. Provided that $2p + a \neq 0$, the general solution of (7) is therefore

$$Y = -\frac{C}{2p + a}\mathrm{e}^{-(2p+a)x} + A,$$

where A is any constant. Hence the general solution of (2) is

$$\begin{aligned} y = Y\mathrm{e}^{px} &= (B\mathrm{e}^{-(2p+a)x} + A)\mathrm{e}^{px} \\ &= A\mathrm{e}^{px} + B\mathrm{e}^{-(p+a)x}, \end{aligned} \tag{8}$$

where we have written B instead of $-C/(2p+a)$, since this is any constant.

There are now two cases to consider.

(i) Suppose that the indicial equation (5) has two different roots, say p_1 and p_2. Since $p_1 + p_2 = -a, 2p_1 + a = p_1 - p_2 \neq 0$. So the previous paragraph applies with $p = p_1$. Thus (8) becomes

$$y = A\mathrm{e}^{p_1 x} + B\mathrm{e}^{-(p_1+a)x} = A\mathrm{e}^{p_1 x} + B\mathrm{e}^{p_2 x}$$

and this is the general solution of (2).

(ii) Suppose that (5) has the repeated root p_1. Then $a^2 - 4b = 0$ and $p_1 = -\frac{1}{2}a$. Thus (6) reduces to $Y'' = 0$ and this has the general solution

$$Y = Ax + B,$$

where A and B are any constants. So the general solution of (2) is

$$y = Y\mathrm{e}^{p_1 x} = (Ax + B)\mathrm{e}^{p_1 x},$$

and the theorem is proved.

It is, of course, easy to verify by direct substitution that $y = A\mathrm{e}^{p_1 x} + B\mathrm{e}^{p_2 x}$ is a solution of (2) when the indicial equation (5) has unequal roots p_1, p_2, and that $y = (Ax + B)\mathrm{e}^{p_1 x}$ is a solution when (5) has the repeated root p_1. This is usually all that is done in a first course on differential equations. However, the proof of Theorem 1 demonstrates that only a moderate extra effort is needed to show that these two functions are, in fact, the *general* solutions of (2) under the stated conditions, in other words that *all* solutions of (2) are of either of these two forms.

Example 1. $y'' - 2y' - 3y = 0$.

The indicial equation is $p^2 - 2p - 3 = 0$, which has roots -1 and 3. Hence the general solution is

$$y = A\mathrm{e}^{-x} + B\mathrm{e}^{3x},$$

where A and B are arbitrary constants.

Example 2. $y'' + 4y' + 4y = 0$.

The indicial equation is $p^2 + 4p + 4 = 0$, which has the repeated root -2. Hence the general solution is

$$y = (Ax + B)\mathrm{e}^{-2x}.$$

So far we have implicitly taken the roots p_1, p_2 of the indicial equation (5) to be real. However, e^{px} is handled in exactly the same way whether p is real or complex. In particular, the same differentiation formula holds. For when $p = \alpha + \mathrm{i}\beta$ (where α, β are real), e^{px} may be defined by

$$\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} = \mathrm{e}^{\alpha x}\mathrm{e}^{\mathrm{i}\beta x} = \mathrm{e}^{\alpha x}(\cos\beta x + \mathrm{i}\sin\beta x),$$

and so

$$\begin{aligned} \frac{d}{dx}\mathrm{e}^{px} &= \frac{d}{dx}\mathrm{e}^{\alpha x}(\cos\beta x + \mathrm{i}\sin\beta x) \\ &= \alpha\mathrm{e}^{\alpha x}(\cos\beta x + \mathrm{i}\sin\beta x) \\ &\quad + \mathrm{e}^{\alpha x}(-\beta\sin\beta x + \mathrm{i}\beta\cos\beta x) \\ &= (\alpha + \mathrm{i}\beta)\mathrm{e}^{\alpha x}(\cos\beta x + \mathrm{i}\sin\beta x) \\ &= p\mathrm{e}^{px}. \end{aligned}$$

It follows that the solution of the differential equation (2) is derived from the indicial equation (5) in precisely the same way whether the roots p_1, p_2 are real or complex.

When a, b are real and the roots of (5) are complex, then they are $\frac{1}{2}(-a \pm \sqrt{(a^2 - 4b)})$ with $a^2 - 4b < 0$. Thus the roots are $\alpha \pm \mathrm{i}\beta$, where $\beta \neq 0$. Hence the roots of (5) are distinct and so the solution of (2) is

$$\begin{aligned} A\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} + B\mathrm{e}^{(\alpha-\mathrm{i}\beta)x} &= \mathrm{e}^{\alpha x}\big(A(\cos\beta x + \mathrm{i}\sin\beta x) \\ &\qquad + B(\cos\beta x - \mathrm{i}\sin\beta x)\big) \\ &= \mathrm{e}^{\alpha x}(C\cos\beta x + D\sin\beta x), \end{aligned}$$

where $C = A + B, D = \mathrm{i}(A - B)$ are arbitrary constants (and C, D are real if A, B are complex conjugates).

Example 3. $y'' - 2y' + 5y = 0$.

The indicial equation is $p^2 - 2p + 5 = 0$, which has roots $1 \pm 2\mathrm{i}$. Hence the general solution is

$$y = A\mathrm{e}^{(1+2\mathrm{i})x} + B\mathrm{e}^{(1-2\mathrm{i})x} = \mathrm{e}^{x}(C\cos 2x + D\sin 2x).$$

Here A and B are arbitrary complex constants and we may call the solution involving them the general complex solution of (2). Similarly, C and D are arbitrary real constants and we may call the solution involving them the general real solution.

3. Non-homogeneous equations

When the given function $f(x)$ in (1) is not identically zero, then the differential equation (1) is called *non-homogeneous*.

The differential equation (2), which we completely solved in the last section, is not only important in its own right, but also helps with the solution of (1); for Theorem 2 below shows that the general solution of (1) is the sum of the general solution of (2) and any chosen solution of (1).

Theorem 2. *Let $y = u(x)$ be the general solution of* (2) *and let $y = v(x)$ be* some *solution of* (1). *Then $y = u(x) + v(x)$ is the general solution of* (1).

We first note that Lemma 1 immediately shows $y = u(x) + v(x)$ to be a solution of (1).

Now let $y = w(x)$ be any solution of (1). Since $y = v(x)$ is also a solution of (1), by Lemma 2, $y = w(x) - v(x)$ is a solution of (2). But $y = u(x)$ is the general solution of (2) and so $w(x) - v(x)$ is of the form $u(x)$, i.e. $w(x)$ is of the form $u(x) + v(x)$.

Thus $y = u(x) + v(x)$ is the *general* solution of (1).

Observe that the above argument is valid whatever solution $y = v(x)$ of (1) is chosen.

The function $u(x)$ of Theorem 2 is usually called the *complementary function* of (1) and $v(x)$ is called a *particular solution* (or *particular integral)* of (1).

4. Particular solutions

The remark at the end of section 1 means that, in order to obtain a particular solution of any differential equation (1) it is sufficient to be able to obtain a particular solution of every equation

$$y'' + ay' + by = q(x)e^{kx}. \tag{9}$$

There are various methods for finding these particular solutions. The one which is usually employed in a first course on differential equations requires a knowledge of the several forms that a solution of (9) takes. Such a form usually involves a few constants and their values are found by substituting the form into (9). Another, much more sophisticated, method involves the so-called D-operators, which appear rather mysterious unless a good deal of trouble is taken in putting them on a firm footing. The procedure described below relies much less on memory than the first method and is much easier to understand than the second. It is most easily described by means of examples corresponding to the three forms that the right-hand side of (9) can take.

(i) *The polynomial case*.

Here k in (9) is 0. The rule is to differentiate (9) repeatedly until the right-hand side is constant.

Example 4.

$$y'' - 2y' + y = x^2 + x - 2. \tag{10}$$

Then

$$y''' - 2y'' + y' = 2x + 1 \tag{11}$$

and

$$y'''' - 2y''' + y'' = 2. \tag{12}$$

Clearly (12) is satisfied by $y'' = 2$ since then $y''' = y'''' = 0$. Substituting these in (11) we have $0 - 4 + y' = 2x + 1$, so that $y' = 2x + 5$. The values of y' and y'' substituted in (10) finally give $2 - 2(2x + 5) + y = x^2 + x - 2$, so that

$$y = x^2 + 5x + 6;$$

and this is a particular solution of (10).

The indicial equation of $y'' - 2y' + y = 0$ is $p^2 - 2p + 1 = 0$, which has the single root $p = 1$. Hence the complementary function of (10) is $y = (Ax + B)e^x$ and so, by Theorem 2, the general solution of (10) is

$$y = (Ax + B)e^x + x^2 + 5x + 6.$$

The method that we have used for finding a particular solution of (10) is essentially formal. A justification exists but, being very much an optional extra, it is confined to the end of this article as an appendix.

The solutions of the differential equations in the following two examples present slight variants of the standard procedure illustrated in Example 4.

Example 5.

$$y'' + 4y = 4x^3. \tag{13}$$

Then

$$y''' + 4y' = 12x^2, \tag{14}$$
$$y'''' + 4y'' = 24x, \tag{15}$$
$$y''''' + 4y''' = 24. \tag{16}$$

An obvious solution of (16) is $y''' = 6$. Substituting this in (15) we get $0 + 4y'' = 24x$, i.e. $y'' = 6x$. Without appealing to (14) we now have, by (13), $6x + 4y = 4x^3$, i.e. $y = x^3 - \frac{3}{2}x$.

The indicial equation $p^2 + 4 = 0$ gives $p = \pm 2i$ and so the complementary function of (13) is $Ae^{i2x} + Be^{-i2x} = C\cos 2x + D\sin 2x$. Thus the general real solution of (13) is

$$y = C\cos 2x + D\sin 2x + x^3 - \tfrac{3}{2}x.$$

Example 6.

$$y'' + 4y' = 12x^2 + 6x. \tag{17}$$

Then

$$y''' + 4y'' = 24x + 6 \tag{18}$$

and

$$y'''' + 4y''' = 24. \tag{19}$$

A solution of (19) is $y''' = 6$ and, substituting first into (18) and then into (17) we obtain $y' = 3x^2$. Hence $y = x^3 + C$, where C is an arbitrary constant. However, since any solution of (17) is sufficient for our purposes, we may take $C = 0$.

The complementary function of (17) is $A\mathrm{e}^{-4x} + B\mathrm{e}^{0x} = A\mathrm{e}^{-4x} + B$ and so the general solution is

$$y = A\mathrm{e}^{-4x} + B + x^3. \tag{20}$$

It may be noted that, if we had not taken C to be 0, the general solution of (17) would have appeared as $y = A\mathrm{e}^{-4x} + B + x^3 + C$. But $B + C$ is now one arbitrary constant, say D, and we simply obtain (20) with B replaced by D.

(ii) *The exponential case.*

Here $k \neq 0$ in (9). For the present we take k to be real, though the method works equally well when k is complex and in (iii) below k will necessarily be complex. The first step is to put $y = Y(x)\mathrm{e}^{kx}$. This yields a differential equation in Y of the type solved in (i).

When $q(x)$ is a constant, $Y(x)$ reduces to a constant unless k is a root of the indicial equation so that $y = \mathrm{e}^{kx}$ satisfies $y'' + ay' + by = 0$. These cases are illustrated in the next two examples.

Example 7.

$$y'' - 3y' + 2y = 3\mathrm{e}^{-2x}. \tag{21}$$

Putting $y = Y\mathrm{e}^{-2x}$ we have $y' = Y'\mathrm{e}^{-2x} - 2Y\mathrm{e}^{-2x}$, $y'' = Y''\mathrm{e}^{-2x} - 4Y'\mathrm{e}^{-2x} + 4Y\mathrm{e}^{-2x}$; and substitution in (21) gives

$$(Y'' - 7Y' + 12Y)\mathrm{e}^{-2x} = 3\mathrm{e}^{-2x},$$

i.e.

$$Y'' - 7Y' + 12Y = 3. \tag{22}$$

A solution of (22) is $Y = \frac{1}{4}$, so that $y = \frac{1}{4}\mathrm{e}^{-2x}$ is a particular solution of (21).

The complementary function of (21) is $A\mathrm{e}^{x} + B\mathrm{e}^{2x}$ and so the general solution of (21) is

$$y = A\mathrm{e}^{x} + B\mathrm{e}^{2x} + \tfrac{1}{4}\mathrm{e}^{-2x}.$$

Example 8.

$$y'' - 3y' + 2y = 3\mathrm{e}^{2x}. \tag{23}$$

We now put $y = Y\mathrm{e}^{2x}$, so that $y' = (Y' + 2Y)\mathrm{e}^{2x}$, $y'' = (Y'' + 4Y' + 4Y)\mathrm{e}^{2x}$. Then, from substitution in (23), $Y'' + Y' = 3$. A solution is $Y' = 3$, which implies that $Y = 3x + C$. But C may be taken to be 0 , so $y = 3x\mathrm{e}^{2x}$ is a particular solution of (23). Since the complementary function is $A\mathrm{e}^{x} + B\mathrm{e}^{2x}$ as in Example 7, the general solution is

$$y = A\mathrm{e}^{x} + (B + 3x)\mathrm{e}^{2x}.$$

It is of interest also to meet a differential equation with a less simple solution.

Example 9.

$$y'' - 2y' + 5y = 4x^3\mathrm{e}^{x}. \tag{24}$$

The substitution $y = Y\mathrm{e}^{x}$ leads to the differential equation

$$Y'' + 4Y = 4x^3$$

which was solved in Example 5. Its general solution was found to be $Y = C\cos 2x + D\sin 2x + x^3 - \frac{3}{2}x$. Therefore the general solution of (24) is

$$y = (C\cos 2x + D\sin 2x + x^3 - \tfrac{3}{2}x)\mathrm{e}^{x}.$$

(iii) *The trigonometric case.*
Since

$$\cos cx = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}cx} + \mathrm{e}^{-\mathrm{i}cx}), \qquad \sin cx = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}cx} - \mathrm{e}^{-\mathrm{i}cx}),$$

this case is essentially the one considered in (ii), though the exponentials are now complex.

Example 10.

$$y'' + y = 4x\sin x. \tag{25}$$

The differential equation can be written

$$y'' + y = -2\mathrm{i}x\mathrm{e}^{\mathrm{i}x} + 2\mathrm{i}x\mathrm{e}^{-\mathrm{i}x}$$

and taking note of Lemma 1 we need to solve

$$y'' + y = -2\mathrm{i}x\mathrm{e}^{\mathrm{i}x}, \tag{26}$$

$$y'' + y = 2\mathrm{i}x\mathrm{e}^{-\mathrm{i}x}. \tag{27}$$

In (26) we make the substitution $y = Y\mathrm{e}^{\mathrm{i}x}$. Since $y' = Y'\mathrm{e}^{\mathrm{i}x} + \mathrm{i}Y\mathrm{e}^{\mathrm{i}x}$ and $y'' = Y''\mathrm{e}^{\mathrm{i}x} + 2\mathrm{i}Y'\mathrm{e}^{\mathrm{i}x} - Y\mathrm{e}^{\mathrm{i}x}$, (26) is transformed into $Y'' + 2\mathrm{i}Y' = -2\mathrm{i}x$. Then $Y''' + 2\mathrm{i}Y'' = -2\mathrm{i}$ and this is satisfied by $Y'' = -1$. Hence $-1 + 2\mathrm{i}Y' = -2\mathrm{i}x$, i.e. $Y' = -\frac{1}{2}\mathrm{i} - x$, so that Y may be taken to be $-\frac{1}{2}\mathrm{i}x - \frac{1}{2}x^2$. Thus a solution of (26) is

$$y = (-\tfrac{1}{2}\mathrm{i}x - \tfrac{1}{2}x^2)\mathrm{e}^{\mathrm{i}x}. \tag{28}$$

A solution of (27), obtained simply by replacing i by $-\mathrm{i}$ in (28), is

$$y = (\tfrac{1}{2}\mathrm{i}x - \tfrac{1}{2}x^2)\mathrm{e}^{-\mathrm{i}x}.$$

Therefore a solution of (25) is

$$\begin{aligned} y &= (-\tfrac{1}{2}\mathrm{i}x - \tfrac{1}{2}x^2)\mathrm{e}^{\mathrm{i}x} + (\tfrac{1}{2}\mathrm{i}x - \tfrac{1}{2}x^2)\mathrm{e}^{-\mathrm{i}x} \\ &= x\sin x - x^2\cos x. \end{aligned}$$

The complementary function of (25) is $C\cos x + D\sin x$ and so the general solution is

$$y = (C - x^2)\cos x + (D + x)\sin x.$$

Finally, it may be noted that the method described for solving the second-order differential equation (1) is easily adapted to deal with higher-order equations of the same type.

Appendix

Our object is to indicate how the procedure used in section 4(i) can be validated by carrying out the justification in the case of the differential equation

$$y'' - 2y' + y = x^2 + x - 2 \qquad (10)$$

of Example 4. As before, (10) is differentiated twice to yield

$$y''' - 2y'' + y' = 2x + 1 \qquad (11)$$

and

$$y'''' - 2y''' + y'' = 2. \qquad (12)$$

We repeat the earlier remark that (12) is satisfied by $y'' = 2$ and then use integration to work backwards, showing that there is, indeed, a function y of x which satisfies additionally (11) and also (10).

First take a function y such that $y'' = 2$. Integrating (12) we get

$$y''' - 2y'' + y' = 2x + C_1,$$

where C_1 is a constant related to the chosen integral y' of y''. To obtain (11) we need to take $C_1 = 1$ and then the corresponding value of y' follows by substitution: $0 - 2.2 + y' = 2x + 1$ and so $y' = 2x + 5$.

Next, take y such that $y' = 2x + 5$ (and so also $y'' = 2$). Integrating (11) we get

$$y'' - 2y' + y = x^2 + x + C_2,$$

where C_2 is a constant related to the chosen integral y of y'. To obtain (10) we need to take $C_2 = -2$ and then the corresponding value of y follows by substitution: $2 - 2(2x + 5) + y = x^2 + x - 2$, i.e.

$$y = x^2 + 5x + 6.$$

Reference

1. E. R. Love, Particular solutions of constant coefficient linear differential equations, *Bulletin of the Institute of Mathematics and its Applications* **25** (1989), pp. 165–166. ☐

E. R. Love *was professor of pure mathematics in the University of Melbourne from 1953 until his retirement in 1977. His principal mathematical interest is (difficult) classical analysis. He enjoys conferences and is consequently an inveterate traveller. An early journey of his was by flying boat from Australia to England in 1948, taking seven days. Since his retirement the romance of his travels has diminished, but the frequency has increased; two to three European trips a year are quite common for him.*

Braintwister

4. Piles of money

I have a pile of ten current British coins consisting of at least one each of 1 p, 2 p, 10 p and 20 p. The pile is arranged with the coins of largest diameter on the bottom decreasing to the coins of smallest diameter on the top. (A 2 p coin is bigger than a 10 p coin which is bigger than a 20 p coin which is bigger than a 1 p coin.)

I wish to move the pile to a new position, moving only one coin at a time. I can form one temporary additional pile too. But in all three piles no coin is ever on top of another of smaller diameter.

The minimum number of moves it takes equals the total value, in pence, of the coins.

What is that value?

(The solution will be published next time.)

VICTOR BRYANT

The 1998 puzzle

Our annual puzzle is to express the numbers 1 to 100 with the digits of the year in order using only the operations of $+$, $-$, $\times$, $\div$, $\sqrt{}$, !, brackets and concatenation (e.g. putting the digits 1 and 9 together to make 19). Powers are not allowed. Thus, for example,

$$1 = (1 \times \sqrt{9} \times \sqrt{9}) - 8.$$

In how many ways can 1000 be expressed as the sum of at most 10 numbers constructed from a single digit? For example,

$$1000 = 888 + 88 + 8 + 8 + 8.$$

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An Estimate for π from a One-Dimensional Random Walk

DAVE HILLS and SAM WEBSTER

How to make a pi with a two pence coin!

The problem

A counter starts at zero on a straight line. A fair coin is tossed and, when the result is heads, the counter is moved one unit to the right; for tails it is moved one to the left. After m tosses, where would you expect the counter to be?

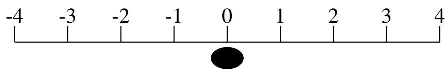

Figure 1. Random walk on the line, with heads for a jump to the right and tails for a jump to the left.

Most people would expect the counter to return to zero on average, as it has an equal chance of going either way. However, this is unlikely to be the case in any specific trial of length $m \geqslant 1$. The mean of the final position, X, of the coin would be zero but the problem concerns its distance from the centre, which is effectively the modulus of X, a random variable $X = Y - Z$, the difference of two binomial random variables Y and Z. This problem can be studied as an example of the one-dimensional random walk.

A mathematical model

Obviously, as the coin is fair, the probabilities of the counter going left or right will be $\frac{1}{2}$ each. Each throw is an independent trial with only two outcomes, each with a fixed probability of $\frac{1}{2}$, where the number of trials, m, is known. The number of moves, Y, to the right can be modelled by the binomial distribution $\mathrm{B}(m, \frac{1}{2})$; the number of moves, Z, to the left is also a binomial $\mathrm{B}(m, \frac{1}{2})$. An equation can then be written for the position, X, of the counter after m moves. Let X be this final position, with Z the number of left moves and Y the number of right moves. Then

$$X = Y - Z, \quad Z = m - Y,$$

so that

$$X = 2Y - m.$$

As Y is modelled by $\mathrm{B}(m, \frac{1}{2})$, where $p = \frac{1}{2}$ is the probability of heads, it can be approximated by the normal distribution when m is large. From the properties of the binomial we have that the mean of this normal approximation is mp and it has variance mpq (where $q = 1 - p$). Therefore $Y \sim \mathrm{N}(\frac{1}{2}m, \frac{1}{4}m)$ and $2Y \sim \mathrm{N}(m, m)$. It follows that $X = 2Y - m$ will be normally distributed as $\mathrm{N}(0, m)$. This shows that the mean of X will be zero, as most people would expect.

Finding $E(|X|)$

To find the expectation $E(|X|)$, we require

$$E(|X|) = \int_{-m}^{m} |x| f(x) dx \Rightarrow \int_{-\infty}^{+\infty} |x| f(x) dx \quad (m \to \infty)$$

where $f(x)$ is the approximate probability density function of the variable X with $-m$ and m as the upper and lower limits of the exact distribution, tending to $-\infty$, $+\infty$, as m becomes large. To simplify the algebra, it is possible to consider $X/\sqrt{m} \approx \mathrm{N}(0, 1)$ so that we can deal with the standard normal distribution. For simplicity let $X/\sqrt{m} = W$, which means that $E(|W|)$ needs to be found. The approximate density function of W is

$$f(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \quad (-\infty < x < +\infty)$$

with

$$E(W) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} x \mathrm{e}^{-x^2/2} dx = 0.$$

Below is a graph of $x\mathrm{e}^{-x^2/2}$, which indicates why the above integral is zero. What we need is the mean of the modulus, $|W|$. This can be obtained easily as follows.

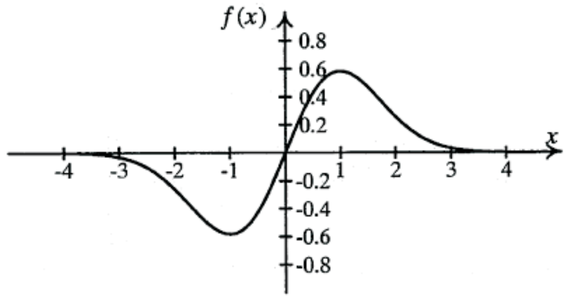

Figure 2. The graph of $x\mathrm{e}^{-x^2/2}$.

We have that

$$E(|W|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x| \mathrm{e}^{-x^2/2} dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} |x| \mathrm{e}^{-x^2/2} dx,$$

and using the substitution $u = \frac{1}{2}x^2$, $du/dx = x$, we find

$$E(|W|) = \frac{2}{\sqrt{2\pi}} \int_{0}^{+\infty} \mathrm{e}^{-u} du = \sqrt{\frac{2}{\pi}}.$$

Hence, since $X = \sqrt{m}W$, we obtain

$$E(|X|) = \sqrt{\frac{2m}{\pi}}.$$

This shows that you would expect the counter to be a distance $\sqrt{2m/\pi}$ from 0 after m trials.

The above implies that if you were only equipped with a coin, pen and paper, you could estimate π, but it would take a very long time. A computer program, such as the one below, would be quicker.

```
10 m = 1000        ; number of 'throws'
20 p = 0           ; position of counter
30 FOR t = 1 to m ; set up loop
40 IF RND(1)<0.5 THEN p = p + 1 ELSE p = p - 1
                   ; move left or right
50 NEXT t
60 PRINT ((2 * m) / (p ^ 2))
                   ; display approximation for pi
```

However, the results from this program are very poor. Even using very large values of m, it returned values from 0.1 to m (not brilliant when $m = 10000$!). Another problem with this program is that when the loop ends and $p = 0$, the program creates the error 'division by zero'.

An extension of the method

The reason that this does not generate a reasonable estimate for π is that the estimator is not consistent. Its variance does not tend to zero for large m, as can be shown mathematically. For

$$\begin{aligned}\text{Var}(|W|) &= E(|W^2|) - [E(|W|)]^2 \\ &= E(W^2) - \left(\sqrt{\frac{2}{\pi}}\right)^2 \\ &= 1 - \frac{2}{\pi},\end{aligned}$$

since

$$E(|W^2|) = \text{Var}(W) = 1$$

and

$$E(|W|) = \sqrt{2/\pi}.$$

This shows that the variance is constant, and does not tend to zero when m tends to infinity. We would not therefore expect it to result in a reasonable estimate for π. It is possible however to modify our method so that the estimator is consistent, thus giving a reasonable value of π.

By repeating the above program many times and averaging the results, one can improve the estimate for π. This is because this method provides a consistent estimator, with the variance tending to zero as m tends to infinity, as can be proved using the central limit theorem. For our case this theorem would state: for a sample of size n drawn from the distribution of $|W|$ with mean $\sqrt{2/\pi}$ and variance $1-(2/\pi)$, the distribution of the sample mean is approximately

$$N\left(\sqrt{\frac{2}{\pi}}, \frac{1-\frac{2}{\pi}}{n}\right).$$

This information would make it possible to calculate confidence intervals for the sample estimate, $\hat{\pi}$, of π. For a normal distribution, it is known that 95% of the data lie within two standard deviations of the mean. This leads to the conclusion that 95% of the values of the variable $|W|$ in a sample of size n would be expected to lie within

$$\sqrt{\frac{2}{\pi}} \pm 2\frac{\sqrt{1-\frac{2}{\pi}}}{\sqrt{n}}.$$

If $n = 10{,}000$ then 95% of the values of $|W|$ lie between 0.792 and 0.804, and since $E(|W|) = \sqrt{2/\pi}$, then for $|W| = 0.792$, the corresponding estimate of π would be $\hat{\pi} = 3.188$ while for $|W| = 0.804$, the corresponding estimate of π would be $\hat{\pi} = 3.094$. The program below leads to sample estimates $\hat{\pi}$ of π.

```
 10 u = 0              ; 'totalling' variable
 20 n = 50             ; number of 'throws'
                         each loop
 30 FOR x = 1 TO 10000 ; loop for repeating
                         the experiment
 40 p = 0              ; position of coin
 50 FOR t = 1 TO n     ; loop for 'throws'
 60 IF RND(1) < 0.5 THEN p = p + 1 ELSE p = p - 1
                       ; move left and right
 70 NEXT t
 80 u = u + ABS(p)     ; add the result to u
 90 r = (u / x) ^ 2    ; calculate value of pi
100 r = (2 * n) / r
110 PRINT r , x        ; display pi, and the
                         number of repeats
120 NEXT x
```

Of the following ten results that we obtained, nine of the values of our estimate $\hat{\pi}$ turn out to lie in the range $(3.094, 3.188)$:

$$\begin{array}{ccccc} 3.122, & 3.176, & 3.092, & 3.149, & 3.168, \\ 3.152, & 3.021, & 3.065, & 3.107, & 3.100. \end{array}$$

Thus, like the Buffon needle problem, the random walk problem can be used to generate reasonable estimates of π.

The authors would like to thank Ali Vahdati and Irena Hosford for their invaluable help and support in writing this article. □

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Pathological Double Sums

J. P. G. EWER and B. C. JONES

The authors found the following examples useful in a first course in real analysis.

Those of us who went to school before the advent of calculators can remember various methods of checking the accuracy of the arithmetical calculations we had to perform. For example, suppose a shop sells three types of item over a period of four months and the income generated is given in table 1.

Table 1

	Jan	Feb	Mar	Apr	Row sum
Item 1	21	30	42	17	110
Item 2	19	40	31	43	133
Item 3	27	22	14	15	78
Column sum	67	92	87	75	

We wish to know the total generated. Of course, we simply have to add the 12 numbers, but we need to do this without error. One way to check our answer is to add across all the rows and down all the columns.

Now if we add the right-hand column and the bottom row of the extended array, we should obtain the same answer (it is 321). If we obtain the same answer we probably have the right answer; if we do not we have made an error. This seems obvious and trivial, but is it always true?

Consider now a loan company with the capacity to offer an infinite number of loans over an infinite period. At the beginning of year 1 it loans, at no interest, a customer 1 unit of money. The customer repays half the loan (i.e. $\frac{1}{2}$ unit) after 1 year, half the remainder after another year, and so on, repaying $\frac{1}{2}$ unit, $\frac{1}{4}$ unit, $\frac{1}{8}$ unit etc. in successive years. Since $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$, the loan will be recovered.

However, at the end of the first year the company takes on another client on the same terms. Again the loan will be recovered. Every year the company takes on a new customer and evidently, in the limit, the company recovers all its money and its deficit is 0.

Now think of it another way. At the start the company is owed 1 unit. At the start of the second year it has loaned another unit and recovered $\frac{1}{2}$ unit, so for that year its deficit is $1+\frac{1}{2}$ units. Continuing in this way, its total deficit is $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$, and not 0 as argued before. We set this out as in table 2, with negative signs denoting money lent. Adding all the row sums gives a total of $0+0+0+\cdots=0$, whereas adding the column sums gives a total of $-1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}=-2$.

This is an example, inspired by an exercise on double integrals (ref. 1), where $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\neq\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$, i.e. the order in which two infinite sums are carried out may not in general be interchanged.

Table 2

	Year					
	1	2	3	4	$\cdots$	Row sums
Customer 1	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$	0
Customer 2	0	-1	$\frac{1}{2}$	$\frac{1}{4}$	$\cdots$	0
Customer 3	0	0	-1	$\frac{1}{2}$	$\cdots$	0
Customer 4	0	0	0	-1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Column sum	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\cdots$	

It is not difficult to construct further arrays with equally bizarre behaviour. We give two examples. Firstly,

$$\begin{array}{cccccc} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ -2 & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ -4 & 2 & 1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ -8 & 4 & 2 & 1 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Here $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=0$, whereas $\sum_{i=1}^{\infty}a_{ij}$ diverges for all j. If, instead, we calculate the total by summing over the a_{ij} in both the first n rows and the first n columns and then letting $n\to\infty$, we obtain $\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=-2$.

Finally, consider the array

$$\begin{array}{ccccccc} 1 & -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{32} & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & -\frac{1}{32} & \cdots \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{16} & \cdots \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{8} & \cdots \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & 1 & -\frac{1}{4} & \cdots \\ -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{4} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Then $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=0$ whereas $\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=1$.

Reference

1. J. E. Marsden and A. J. Tromba, *Vector Calculus* (Freeman, New York, 1981) p. 278. □

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Summation by Parts

FRANK CHORLTON

You have heard of integration by parts; here is summation by parts

The integration of a product of two functions may often be achieved by using integration by parts. Finite sums of products may be amenable to evaluation by an analogous process known as *summation by parts*. This is to be expected since integrals are limiting forms of summations. In this note we first obtain a suitable formula for the summation by parts of a product and then illustrate its use.

Suppose that, for $n = 1, 2, 3, \ldots$, two sequences $\{u_n\}$ and $\{v_n\}$ are prescribed and that

$$U_n = \sum_{r=1}^{n} u_r, \qquad V_n = \sum_{r=1}^{n} v_r \qquad (n \geqslant 1).$$

We also define $U_0 = V_0 = 0$, so that, for $n \geqslant 1$, $u_n = U_n - U_{n-1}$ and $v_n = V_n - V_{n-1}$. Then, for $n \geqslant 1$,

$$\begin{aligned} u_n V_n + v_n U_{n-1} &= (U_n - U_{n-1})V_n + (V_n - V_{n-1})U_{n-1} \\ &= U_n V_n - U_{n-1}V_{n-1}. \end{aligned}$$

Since $U_0 = V_0 = 0$, summing both sides from $n = 1$ to N, we have

$$\sum_{n=1}^{N} u_n V_n + \sum_{n=1}^{N} v_n U_{n-1} = U_N V_N. \tag{1}$$

Of course so simple a formula cannot be expected to be new. In fact it was used as long ago as 1826 by the great Norwegian mathematician N. H. Abel (1802–1829). We now consider a number of applications of (1).

Let us first determine the finite sum $\sum_{n=1}^{N}(n2^n)$. In (1) we take $u_n = 1$ so that $U_n = n$. Also we take $v_n = 2^n$ so that $V_n = 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$. Then (1) gives

$$\sum_{n=1}^{N}\{1(2^{n+1} - 2) + 2^n(n-1)\} = N(2^{N+1} - 2)$$

or

$$\sum_{n=1}^{N}\{(2^{n+1} - 2^n) - 2\} + \sum_{n=1}^{N}(n2^n) = N(2^{N+1} - 2).$$

The first sum on the left reduces to $2^{N+1} - 2 - 2N$. Thus

$$\sum_{n=1}^{N}(n2^n) = 2^{N+1}(N-1) + 2.$$

The technique is applicable to finding the sums of positive integral powers of the natural numbers $1, 2, \ldots, N$. We now address this problem.

In (1) we first take $u_n = v_n = 1 \quad (n = 1, 2, \ldots N)$. Then $U_n = n = V_n \quad (n = 1, 2, \ldots, N)$, and so (1) becomes $\sum_{n=1}^{N}\{1(n) + 1(n-1)\} = N^2$, i.e.

$$\sum_{n=1}^{N} n = \tfrac{1}{2}N(N+1). \tag{2}$$

Next, take $u_n = n, v_n = 1 \quad (n = 1, 2, \ldots, N)$. By (2),

$$U_n = \sum_{n=1}^{N} n = \tfrac{1}{2}N(N+1), \qquad V_N = N.$$

Hence (1) gives

$$\sum_{n=1}^{N}\{n^2 + \tfrac{1}{2}(n-1)n\} = \tfrac{1}{2}N^2(N+1).$$

Solving for $\sum_{n=1}^{N} n^2$ and using (2) gives

$$\sum_{n=1}^{N} n^2 = \tfrac{1}{6}N(N+1)(2N+1).$$

When we take $u_n = v_n = n$ we obtain $U_n = V_n = \frac{1}{2}n(n+1)$. Putting these values into (1), we find that

$$\sum_{n=1}^{N} n^3 = \{\tfrac{1}{2}N(N+1)\}^2.$$

Thus $\sum_{n=1}^{N} n^3 = \{\sum_{n=1}^{N} n\}^2$. Continuing in this way we can evaluate $\sum_{n=1}^{N} n^k$ for any positive integer k. This problem was also tackled, by different means, in references 1–3.

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2. P. Glaister, Determining sums recursively, *Mathematical Spectrum* **28** (1995/96) pp. 63–64.
3. F. T. Howard, Sums of powers of integers, *Mathematical Spectrum* **26** (1993/94) pp. 103–109. □

In 1982 ***Frank Chorlton*** *took early retirement from his position of senior lecturer in mathematics at Aston University. He has published many books and papers on mathematics. His other main interest is music, especially that of Bach.*

Equally Likely Sums

BARRY W. BRUNSON and RANDALL J. SWIFT

Dice are weird, but not excessively so!

Introduction

Is it possible to weight a pair of dice so as to make each of the sums $2, 3, \dots, 12$ equally likely to occur? The answer is, alas, no. This is an old question with a long history. An early source is the *American Mathematical Monthly* problem E925, 1950 by J. B. Kelly, which carried published solutions by J. Finch and P. Halmos, and by L. Moser and J. Wahab (March 1951). The books of Parzen (reference 1) and Halmos (reference 2) also consider this question. A nice discussion appears in the article by Dudewicz and Dann (reference 3), who generalize the question to more than two dice.

Motivated by the proverbial saying 'a leopard cannot change its spots…', we consider a modified question, in which we *are* allowed to change the spots on the dice, but not reweight them. That is, if we are allowed to relabel the faces of a pair of fair dice, which sets of numbers can we force to become the only possible sums, so that these sums will be equally likely? Probability generating functions and Hasse diagrams turn out to be useful tools.

Preliminaries

Labellings of die faces are denoted by brackets $[\cdot]$. A *k-set* is a collection of k numbers. A k-set will be *realizable* if it is possible to relabel a pair of fair dice so that the only possible sums are equally likely, and are those appearing in the given k-set.

Is the 12-set $\{1, 2, 3, \dots, 11, 12\}$ realizable? Yes, if we take one die labelled $[1, 2, 3, 4, 5, 6]$, and the other die labelled $[0, 0, 0, 6, 6, 6]$ (the number of times a digit is repeated is the number of faces labelled with that digit). Realizations are not necessarily unique. This 12-set may also be realized with dice labelled $[0, 2, 4, 6, 8, 10]$ and $[1, 1, 1, 2, 2, 2]$.

A simple condition sufficient to guarantee realization of a k-set is that k be a divisor of 6: that is, all 1-sets, 2-sets, 3-sets and 6-sets are realizable. To see this, just take one die with all faces labelled zero and the other die labelled with each element of the k-set repeated as needed. For instance, for the 2-set $\{13, 28\}$, label the other die $[13, 13, 13, 28, 28, 28]$.

Is our old friend the 11-set $\{2, 3, \dots, 12\}$ realizable? No. In fact, *no* 11-set is realizable. A simple condition *necessary* for the realization of a k-set is that k be a divisor of 36. Suppose that $\{s_1, s_2, \dots, s_k\}$ is realizable and consider any of the sums, say s_i. Of the 36 distinct outcomes, some m of them result in the sum s_i, with $1 \leqslant m \leqslant 36$. Thus $m/36 = 1/k$, which implies that $mk = 36$.

Is divisibility into 36 sufficient? The answer is no, and probability generating functions help show it.

Probability generating functions and 4-sets

If X is a discrete random variable, then the probability generating function (p.g.f.) of X is

$$\psi(t) = E(t^X) = \sum_x p(x)t^x \qquad \text{for} \quad 0 \leqslant t \leqslant 1,$$

where the sum is taken over all outcomes x in the sample space, and $p(x) = P(X = x)$. A detailed discussion of the properties of probability generating functions can be found in reference 4.

A key property of p.g.f.s follows from the notion of independence. If X and Y are independent random variables with respective p.g.f.s $\psi_X(t)$ and $\psi_Y(t)$, then the sum $Z = X+Y$ has p.g.f.

$$\psi_Z(t) = \psi_X(t)\psi_Y(t).$$

The key observation about dice is that the behaviour of one die is independent of the behaviour of the other die. Thus, the p.g.f. of a sum is

$$\begin{aligned} \psi(t) &= \tfrac{1}{6} \times (t^{a_1} + t^{a_2} + t^{a_3} + t^{a_4} + t^{a_5} + t^{a_6}) \\ &\quad \times \tfrac{1}{6} \times (t^{b_1} + t^{b_2} + t^{b_3} + t^{b_4} + t^{b_5} + t^{b_6}) \qquad (1) \\ &= \sum_{i=1}^{6}\sum_{j=1}^{6} \tfrac{1}{36} t^{a_i+b_j}, \end{aligned}$$

where the labels on the six faces of the first die are $a_1, a_2, \dots, a_6$, and those on the second die are $b_1, b_2, \dots, b_6$.

On the other hand, the p.g.f. for a realization of a 4-set $\{s_1, s_2, s_3, s_4\}$ is of the form

$$\phi(t) = \tfrac{1}{4} \times (t^{s_1} + t^{s_2} + t^{s_3} + t^{s_4}).$$

The two p.g.f.'s $\psi(t)$ and $\phi(t)$ will be identical if and only if

$$\tfrac{1}{4}(t^{s_1} + t^{s_2} + t^{s_3} + t^{s_4}) = \sum_{i=1}^{6}\sum_{j=1}^{6} \tfrac{1}{36} t^{a_i+b_j}.$$

The coefficients must agree, and the only way to get a factor of 1/4 on the right-hand side is with a factor of 3 from each parenthesized factor in $\psi(t)$ of equation (1). That is, $t^{a_1} + \dots + t^{a_6}$ must reduce to $3t^{a_i} + 3t^{a_j}$ for some i and j, similarly for $t^{b_1} + \dots + t^{b_6}$. Thus, each die must show only two distinct values, each appearing on three of the six faces. Using a_1, a_2, b_1, b_2 to denote those four distinct values, the p.g.f. $\psi(t)$ becomes

$$\begin{aligned}\psi(t) &= \tfrac{1}{36} \times (3t^{a_1} + 3t^{a_2}) \times (3t^{b_1} + 3t^{b_2}) \\ &= \tfrac{1}{4} \times (t^{a_1} + t^{a_2}) \times (t^{b_1} + t^{b_2}). \qquad (2)\end{aligned}$$

This yields the following characterization.

Proposition 1. *A* 4*-set is realizable if and only if it consists of two pairs having a common difference.*

For example, the 4-set $\{8, 13, 14, 19\}$ consists of two pairs having common difference $5 = 13 - 8 = 19 - 14$. This 4-set is realized, for instance, by dice with labels $[2, 2, 2, 7, 7, 7]$ and $[6, 6, 6, 12, 12, 12]$.

Proof of Proposition 1. Suppose that the 4-set $\{s_1, s_2, s_3, s_4\}$ is realizable. Then the p.g.f. for a realization of it must be of the form (2), that is

$$\begin{aligned}t^{s_1} + t^{s_2} + t^{s_3} + t^{s_4} &= (t^{a_1} + t^{a_2}) \times (t^{b_1} + t^{b_2}) \\ &= t^{a_1+b_1} + t^{a_1+b_2} + t^{a_2+b_1} + t^{a_2+b_2}.\end{aligned}$$

Assume without loss of generality that all sequences are written in increasing order. This implies that $s_1 = a_1 + b_1$ and $s_4 = a_2 + b_2$. Now $a_1 + b_2$ could be either s_2 or s_3. In the former case, $s_2 - s_1 = b_2 - b_1 = s_4 - s_3$. In the latter case, $s_2 - s_1 = a_2 - a_1 = s_4 - s_3$.

Now, suppose that $s_2 - s_1 = d = s_4 - s_3$. Then a realization of $\{s_1, s_2, s_3, s_4\}$ is given by one die labelled $[0, 0, 0, d, d, d]$ and the other die $[s_1, s_1, s_1, s_3, s_3, s_3]$. This completes the proof.

Using p.g.f.s we can show more about the realization of certain other k-sets.

Proposition 2. *All k-sets (where k divides* 36*) consisting of an arithmetic sequence are realizable.*

Proof of Proposition 2. A k-set consisting of an arithmetic sequence is of the form

$$\{a, a + d, a + 2d, a + 3d, \ldots, a + (k - 1)d\},$$

where a is the smallest number in the k-set and d is the common difference. We need only show that the statement is valid for $k = 9, 12, 18$ and 36. The p.g.f. for an arithmetic k-set is of the form

$$\psi(t) = \frac{t^a}{k}(1 + t^d + t^{2d} + t^{3d} + \ldots + t^{(k-1)d}).$$

When $k = 9$, this factorizes into

$$\frac{t^a}{9}(1 + t^d + t^{2d})(1 + t^{3d} + t^{6d}),$$

which yields for die 1 the labels $[a, a, a+d, a+d, a+2d, a+2d]$ and for die 2 $[0, 0, 3d, 3d, 6d, 6d]$.

Similar factorizations occur in the other cases, which we leave as an exercise. This completes the proof.

Hasse diagrams and 9-sets

The structure of 9-sets is richer than that of 4-sets. The concept of a Hasse diagram facilitates the analysis. The Hasse diagram of a discrete lattice depicts the order relation visually. For example, recalling the notation of Proposition 1, in which the dice had distinct faces a_1, a_2 and b_1, b_2 respectively; we obtain the Hasse diagram of the resulting 4-set $\{s_1, s_2, s_3, s_4\}$ as shown in figure 1. Edges extend from a smaller sum up to a larger sum. The absence of an edge between $a_1 + b_2$ and $a_2 + b_1$ indicates that these sums are *incomparable*: in general, we do not know which one is larger until we know the specific values of the a_i and b_i.

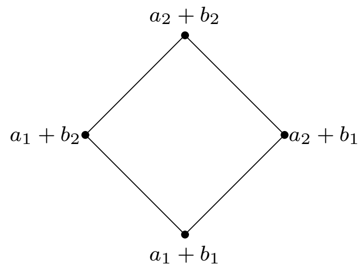

Figure 1. Hasse diagram for a 4-set.

A realization of a k-set provides those specific values, and determines which of each pair of sums is the larger. Formally, a realization of a k-set induces a *linear extension* to the lattice depicted in the Hasse diagram of the k-set. In the case of a 4-set, there is but one pair of incomparable elements; hence there are but two types of linear extensions: one in which $a_1 + b_2$ exceeds $a_2 + b_1$, and another in which $a_2 + b_1$ exceeds $a_1 + b_2$.

The p.g.f. for a realizable 9-set $\{s_1, s_2, s_3, \ldots, s_9\}$ is of the form $\psi(t) = \frac{1}{9}(t^{a_1} + t^{a_2} + t^{a_3})(t^{b_1} + t^{b_2} + t^{b_3})$, where the exponents are listed in increasing order. The elements of the 9-set occupy the vertices of the Hasse diagram in figure 2. A realization of the 9-set amounts to a linear extension of this lattice. There are many examples of non-realizable 9-sets. For instance, the 9-set $\{0, 2, 4, 8, 13, 16, 22, 28, 33\}$ has a p.g.f. of the form

$$\psi(t) = \tfrac{1}{9}(1 + t^2 + t^4 + t^8 + t^{13} + t^{16} + t^{22} + t^{28} + t^{33}),$$

which one can verify is not factorizable over the integers. Any 9-set which gives rise to a non-factorizable p.g.f. will not be realizable.

Proposition 2 guarantees that any 9-set consisting of an arithmetic sequence is realizable. There are other realizable 9-sets, for example $\{18, 20, 23, 24, 26, 29, 31, 33, 36\}$. A realization of this 9-set is given when die 1 is labelled $[0, 0, 2, 2, 5, 5]$ and die 2 $[18, 18, 24, 24, 31, 31]$.

It is possible to give a general construction procedure for realizable 9-sets. In the Hasse diagram of figure 2, mark each edge with the difference between the vertices, and number the vertices $1, 2, \ldots, 9$. The Hasse diagram of figure 3 will now be obtained.

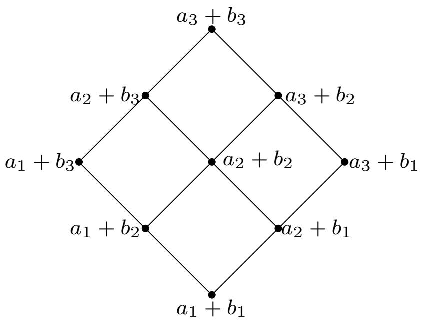

Figure 2. Hasse diagram for a 9-set.

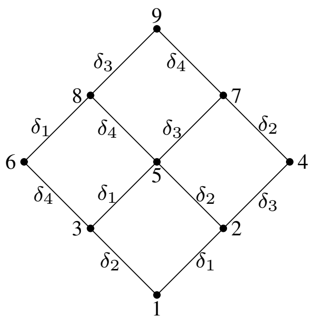

Figure 3. Hasse diagram for the construction of a 9-set.

If we start with a non-negative integer α as our smallest possible sum, then vertex 1 in the Hasse diagram of figure 3 is $a_1 + b_1 = \alpha$. From vertex 1, there are two possible paths, to either vertex 2 or 3. For the 9-set, we require that vertex 2 be incomparable to vertex 3 and that, in particular, the corresponding sums must be different. This results in the condition that $\delta_1 = (a_2 - a_1) \neq \delta_2 = (b_2 - b_1)$. Similarly, from vertex 2, there are two possible paths, one to vertex 4 and one to vertex 5. The condition that $a_2 + b_2$ (at vertex 5) be different from $a_3 + b_1$ (at vertex 4) requires that $\delta_2 \neq \delta_3 = (a_3 - a_2)$. If we continue this analysis, we will find that the increments $\delta_1, \delta_2, \delta_3$ and δ_4, must satisfy the condition that $\{\delta_1, \delta_3, \delta_1 + \delta_3\}$ and $\{\delta_2, \delta_4, \delta_2 + \delta_4\}$ are disjoint. Thus, we obtain the realizable 9-set as

$$\begin{aligned}&\{\alpha, \alpha + \delta_1, \alpha + \delta_2, \alpha + \delta_1 + \delta_2,\\ &\alpha + \delta_1 + \delta_3, \alpha + \delta_2 + \delta_4, \alpha + \delta_1 + \delta_2 + \delta_3,\\ &\alpha + \delta_1 + \delta_2 + \delta_4, \alpha + \delta_1 + \delta_2 + \delta_3 + \delta_4\}. \qquad (3)\end{aligned}$$

Note that these sums might not appear in increasing order. Using this incremental structure and comparing this 9-set to the Hasse diagram in figure 3, we obtain a realization of this 9-set when die 1 is labelled

$$[\alpha, \alpha, \alpha + \delta_1, \alpha + \delta_1, \alpha + \delta_1 + \delta_3, \alpha + \delta_1 + \delta_3]$$

and die 2 is labelled

$$[0, 0, \delta_2, \delta_2, \delta_2 + \delta_4, \delta_2 + \delta_4].$$

(Other realizations are possible.)

For instance, using (3) with $\alpha = 3$, $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3$, and $\delta_4 = 5$, so that the above conditions are satisfied, we obtain a realization of the 9-set $\{3, 4, 5, 6, 7, 9, 10, 11, 14\}$ when die 1 is labelled $[3, 3, 4, 4, 7, 7]$ and die 2 $[0, 0, 2, 2, 7, 7]$.

Using this construction with appropriate choices of α and $\delta_1, \ldots, \delta_4$ it is possible to obtain 9-sets with no arithmetic subsequences. Letting $\alpha = 18$, $\delta_1 = 2, \delta_2 = 6, \delta_3 = 3$, and $\delta_4 = 7$ we obtain the 9-set $\{18, 20, 23, 24, 26, 29, 31, 33, 36\}$, which is the 9-set we obtained earlier.

Tetrahedra and further generalizations

We conclude with a brief consideration of equally likely sums for a pair of tetrahedra, and invite the interested reader to explore, for these and other pairs of regular polyhedra, the types of questions we posed above. There are sixteen possible outcomes for a pair of tetrahedral dice, and a k-set cannot be realizable unless k is a divisor of 16. The k-sets for $k = 1, 2, 4$ are all readily seen to be realizable, so $k = 8$ is the first case of interest. The 8-set $\{2, 5, 7, 9, 10, 12, 13, 16\}$ has at least two realizations, with tetrahedra labelled $[0, 0, 3, 3]$ and $[2, 7, 9, 13]$ or, alternatively, labelled $[2, 2, 5, 5]$ and $[0, 5, 7, 11]$.

The reader may wish to consider the following exercises.

1. Restrict attention to the digits $S = \{1, 2, 3, \ldots, 36\}$. Suppose we select a 4-set at random from S. What is the probability that it will be realizable?
2. Complete the proof of Proposition 2 by exhibiting factorizations of the p.g.f.'s when $k = 12, 18$, and 36.
3. As in Exercise 1, restrict attention to the digits $S = \{1, 2, 3, \ldots, 36\}$. Suppose a k-set is selected at random from S. What is the probability it will be an arithmetic sequence?
4. Using Hasse diagrams for 12- and 18-sets, obtain conditions, similar to those obtained for 9-sets, which provide general construction procedures for 12- and 18-sets.
5. There are five regular polyhedra. Are all k-sets consisting of an arithmetic sequence realizable for any pair of polyhedra?

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Tame Vampires

F. W. ROUSH and D. G. ROGERS

Dedicated to the memory of Sir Alexander Oppenheim (4 February 1903–13 December 1997)

How some elementary number theory helps tame Clifford Pickover's notion of *vampire numbers*.

All the integers we consider here are expressed in base 10, although we could equally well work in some other base. Suppose that x and y are d-digit integers, not both divisible by 10, whose product $v = xy$ is a $2d$-digit integer. If the digits of v are exactly the digits of x and y taken together in some order, then we say that v is a *vampire* with *fangs* x and y. We take our lead here from Clifford Pickover, the proponent of this theme and terminology (see references 3–5). Pickover thinks of such a number v as having a secret, hidden property, unsuspected amongst the other numbers, rather like the vampires in our midst, as envisaged by the popular novelist, Anne Rice. For example, for $d = 2$, there are just the following seven pairs of fangs:

$$\begin{array}{ll} \text{(i)}\ \{21, 60\},\ v = 1{,}260; & \text{(ii)}\ \{15, 93\},\ v = 1{,}395; \\ \text{(iii)}\ \{35, 41\},\ v = 1{,}435; & \text{(iv)}\ \{30, 51\},\ v = 1{,}530; \\ \text{(v)}\ \{21, 87\},\ v = 1{,}827; & \text{(vi)}\ \{27, 81\},\ v = 2{,}187; \\ \text{(vii)}\ \{80, 86\},\ v = 6{,}880. & \end{array} \tag{1}$$

(Martin Gardner has recently written extensively about properties of the number 2,187 in reference 2; he notices that $21 \times 87 = 1{,}827$, but misses $2{,}187 = 27 \times 81$.)

Many challenging questions can be raised concerning vampires; that they hold fascination for the general reader is confirmed by the review (reference 8) of Clifford Pickover's book (reference 5), which singles out this topic for special comment (reference 5 is also the focus of a sustained pedagogical discussion; see reference 9). As a case in point, we adduce Arlin Anderson's question in reference 1 of finding vampires with identical fangs (so, in the notation above, $x = y$ and $v = x^2$; we look at this in the binary case in a sequel — see reference 6). However, if we are simply interested in showing that there are vampires for arbitrarily large d, it is natural to look for instances in which the digits of the fangs appear in some obvious way.

We describe here one way to concoct fangs, based on observing some patterns in early examples, and justified by some elementary number theory. Besides its intrinsic interest, this concoction illustrates rather neatly an operational difference between mathematicians and computer-based researchers, showing how knowledge of some mathematics can still be an advantage. For, when we showed our results to Clifford Pickover, after first convincing him we had something worth looking at, he exclaimed: 'Wow! If your equations work, that's amazing. I've not seen something like that', as though the equations would still have to be run through on the computer! Indeed, on further consideration, he was less impressed, being tempted to modify the definitions to exclude our concoction. But then perhaps vampire numbers would only be left with a ghostly fascination.

It so happens that, with four of the seven pairs of fangs for $d = 2$ listed in (1), we can pick the fangs x and y so that

$$v = x^* \times 10^d + y \tag{2}$$

where, for x not divisible by 10, x^* is the d-digit integer having the same digits as x but in reverse order. The appearance of the digits of the fangs in the vampire in (2) is so transparent as to encourage investigation of this special case for general d.

Perhaps the simplest family of examples is given by taking, for positive integral k,

$$x = 25 \times 10^k + 1, \qquad y = 100(10^{k+1} + 52)/25, \tag{3}$$

so

$$\begin{aligned} v = xy &= (10^{k+1} + 52)10^{k+2} + 100(10^{k+1} + 52)/25 \\ &= x^* \times 10^{k+2} + y. \end{aligned}$$

As x and y are both $(k + 2)$-digit integers, (2) holds with $d = k + 2$. More explicitly, for $k = 1, 2, 3$ in (3), we find

$$\begin{aligned} 251 \times 608 &= 152{,}608, \\ 2{,}501 \times 4{,}208 &= 10{,}524{,}208, \\ 25{,}001 \times 40{,}208 &= 1{,}005{,}240{,}208. \end{aligned}$$

Thus, this family of examples already illustrates a phenomenon allowed under our definition, namely that of adding zeros to a given pair of fangs to produce new fangs. To eliminate this kind of replication, we might consider suitable equivalence classes of fangs. In contrast, the pair of fangs $\{21, 87\}$ cannot be extended by the insertion of zeros.

But, with some fangs, it is far from clear whether they can be extended to give new fangs in this way, raising the further problem of identifying a class of what might be termed indecomposable fangs. Even restricting attention to vampires v related to their fangs x and y by (2), rather subtle, number theoretic patterns arise, as is shown by the following further examples in which k is again a positive integer:

$$x = 52 \times 10^{6k-1} + 1, \quad y = 100(10^{6k} + 25)/52; \tag{4}$$

and

$$x = 92 \times 10^{22k-6} + 1, \quad y = 100(10^{22k-5} + 29)/92. \tag{5}$$

Of course, (4) is suggested by (3); that y is an integer in this case turns on the fact that

$$10^6 \equiv 1 \pmod{13},$$

so that $10^{6k} + 25$ is divisible by 13. Similarly, (5) represents a further modification, relying on the congruences

$$10^{17} \equiv -6, \quad 10^{22} \equiv 1 \pmod{23}.$$

The pairs of fangs in (3), (4) and (5) can all be subsumed under one general rubric. Let n be an m-digit integer not divisible by 10, and recall that n^* is then the m-digit integer having the same digits as n but in reverse order. With multiplication in mind, consider now

$$x = n \times 10^{t-m+1} + 1, \quad y = 10^m(10^t + n^*)/n,$$

where $t \geqslant m$ is to be chosen later, depending on n; note that, at least, x is an integer not divisible by 10 having $t+1$ digits and, if y is an integer, it has no fewer than this number of digits. Computing the product $v = xy$, we find that

$$\begin{aligned} v = xy &= y + ny \times 10^{t-m+1} \\ &= y + (10^t + n^*) \times 10^{t+1} \\ &= y + x^* \times 10^{t+1}, \end{aligned}$$

since reversing the digits of x gives $x^* = 10^t + n^*$ in view of the definition of x.

We now have to ensure that y is an integer with $t+1$ digits, where we have t at our disposal. For, once that is established, our computation of v shows it to be a $(2t+2)$-digit integer in which, reading from right to left, the digits of y appear in order followed by those of x in reverse order, and thus that v is a vampire with fangs x and y of the type sought.

Writing $q = \gcd(n, 10^m)$ and $p = n/q$, suppose that there is a solution t of the congruence

$$10^t + n^* \equiv 0 \pmod{p}. \tag{6}$$

(Of course, if there is one solution, there are infinitely many, so there is no problem in requiring t to be sufficiently large to make our arguments work.) Then, y will be an integer with at least $t+1$ digits. Now, if $t \geqslant m$, then $10^m(10^t + n^*)$ and $n \times 10^{t+1}$ are both integers with $m+t+1$ digits. If, more strongly, $t \geqslant 2m-1$, then comparison of the leading m digits of these two integers shows that

$$10^m(10^t + n^*) < n \times 10^{t+1}, \tag{7}$$

since n is not divisible by 10. Hence, at least when $t \geqslant 2m-1$, $y < 10^{t+1}$, and so y has *exactly* $t+1$ digits. As worst case examples for our argument we cite $n = 10^{m-1} + 1$.

In the case where $n \geqslant 2 \times 10^{m-1}$, the restriction on t for (7) to hold, and thus for y to be a $(t+1)$-digit integer, can be relaxed to $t \geqslant m$. Indeed, now it is enough to look at the leading digit on each side of the inequality (7) to see that it holds. As it happens, the pairs of fangs in (3), (4) and (5) all come under this case.

In the previous two paragraphs, for simplicity of argument, we have required more of t or of n than may be strictly necessary if the digits of n are known in greater detail. That some additional requirements are necessary to ensure that y has the right number of digits is shown by the example $n = 10{,}891$. Here, $m = 5, q = \gcd(n, 10^m) = 1$ and $p = n$. Moreover, (6) holds with $t = 5$, that is $t = m$, since

$$10^5 + n^* = 119{,}801 = 11 \times 10{,}891 = 11n.$$

Thus $y = 11 \times 10^5$. In this example, then, y has 7 digits rather than $t+1 = 6$ digits. But, for this n, it can also be shown that any solution t of (6) with $t > 5$ leads to a y with $t+1$ digits. (This example is the start of a whole further line of enquiry; see reference 7.)

A more serious problem here is that (6) does not always have solutions. However, solutions of (6) are guaranteed at least when p is a prime not dividing n^* and 10 is a primitive root modulo p (that is, all positive integers can be expressed as integral powers of 10 modulo p).

As an example where our method fails, because (6) has no solutions, consider the case $n = 13$. Here $m = 2$, so $q = \gcd(13, 100) = 1$ and $p = 13/1 = 13$. Also, $n^* = 31$, so (6) becomes

$$10^t + 31 \equiv 0 \pmod{13},$$

that is

$$10^t \equiv 8 \pmod{13}.$$

But it is easy to check that powers of 10 modulo 13 take only the values 1, 3, 4, 9, 10 and 12, showing that (6) is impossible.

On the other hand, taking $n = 26$, again $m = 2$, but now $q = 2$ and $p = 13$, while $n^* = 62$. This time (6) becomes

$$10^t + 62 \equiv 0 \pmod{13},$$

that is,

$$10^t \equiv 3 \pmod{13},$$

which we have just observed does have solutions. In fact, this time we can take any $t \equiv 4 \pmod{6}$. (Note that we have already seen that 10 is *not* a primitive root modulo 13, so primitivity is a convenient sufficient condition to ensure things work, but by no means a necessary one.)

The examples in (3), (4) and (5) are now exercises after this latter form.

As a matter of record, in 19 of the 149 pairs of fangs with $d = 3$, we can pick the fangs x and y so that (compare (2))

$$v = xy = z \times 10^d + y,$$

where z is a d-digit integer whose digits are a permutation of those of x. Indeed, although reversal of the digits of x is by far the most common case (in 10 out of the 19 pairs), all permutations other than the identity are represented.

For those seeking the world record for large vampires, the ones produced in this vein will not at heart be satisfying. Hence our title.

We are grateful to Arlin Anderson and Clifford Pickover for helpful discussions, especially on matters of definition, and for sharing with us unpublished material; to Tony Davie for further information; and to Colin Ramsay and Arnold Adelberg.

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Fred Roush *is based at Alabama State University, in Montgomery. He and his family are great subscribers to magazines and periodicals of all descriptions. It was when* ***Douglas Rogers*** *was visiting, in May 1995, and they were clearing a place to sit down amongst all the back issues, that they came upon Clifford Pickover's problem page in the newly arrived June issue of* Discover *– clearly an opportune environment for the research reported here.*

Powerless Sequences

K. PRAKASH

The sequences in this article arose from the author's interest in Fermat's Last Theorem, which has now been proved.

1. Introduction

In Volume 22, Number 3, pages 92–93 of *Mathematical Spectrum*, I showed that a certain sequence of natural numbers contains no powers, i.e. natural numbers which are powers of natural numbers where the power is greater than one. We shall show in this article that the sequence (2^n-1), i.e. $1, 3, 7, 15, 31, 63, \ldots$ contains only one power, namely 1, and that the sequence (2^n+1), i.e. $3, 5, 9, 17, 33, 65, \ldots$ contains only one power, namely 9. We shall also show that, if the sequence (p^n+1) contains a power, where p is an odd prime, then p must be a Mersenne prime, i.e. a prime of the form 2^q-1, where q is necessarily prime.

2. The sequence (2^n-1)

Suppose that $2^n-1 = N^a$ for some $N, a \in \mathbb{N}$ with $a > 1$, where $n > 1$. If p is a prime factor of a, then $2^n - 1 = (N^{a/p})^p$, so we may suppose that a is a prime, which we write as p.

Consider first the case $p = 2$. If $2^n-1 = N^2$, then N is odd so

$$N^2 \equiv 1 \pmod 4.$$

But

$$2^n - 1 \equiv -1 \pmod 4,$$

so the sequence contains no square after its first term. Suppose now that

$$2^n - 1 = N^p,$$

where p is an odd prime. Then

$$2^n = N^p + 1 = (N+1)S, \tag{1}$$

where

$$\begin{aligned} S &= \frac{N^p+1}{N+1} \\ &= \frac{[(N+1)-1]^p+1}{N+1} \\ &= (N+1)^{p-1} - \binom{p}{1}(N+1)^{p-2} + \binom{p}{2}(N+1)^{p-3} \\ &\quad - \cdots + \binom{p}{p-1} \end{aligned}$$

so

$$S \equiv p \pmod{N+1}. \tag{2}$$

Now $N > 1$ so $S > 1$ so $2|N+1$ and $2|S$ from (1). Hence, from (2), $2|p$. But p is an odd prime. This proves our assertion about the sequence (2^n-1).

3. The sequence $(2^n + 1)$

We write $2^n + 1 = N^q$. As before, we may suppose that q is prime. Now

$$2^n = N^q - 1 = (N-1)T, \tag{3}$$

where

$$\begin{aligned} T &= \frac{N^q - 1}{N-1} \\ &= \frac{[(N-1)+1]^q - 1}{N-1} \\ &= (N-1)^{q-1} + \binom{q}{1}(N-1)^{q-2} + \binom{q}{2}(N-1)^{q-3} \\ &\quad + \cdots + \binom{q}{q-1} \end{aligned} \tag{4}$$

so $T \equiv q \pmod{N-1}$. Thus any common divisor of T and $N-1$ must divide q so the greatest common divisor of T and $N-1$ is 1 or q. If T and $N-1$ have greatest common divisor 1, then, since $q > 1$, (4) gives that $T > 1$ and so $N - 1 = 1$ from (3). Hence $N = 2$ and $2^n = 2^q - 1$, which is clearly impossible. Thus T and $N-1$ have greatest common divisor q. But, from (3), q divides 2^n, so $q = 2$. Hence, from (4), $T = N + 1$ and, from (3),

$$N - 1 = 2^r, \quad T = 2^s$$

for some $r, s \geqslant 1$. Thus $2^s = 2^r + 2$ so $2^{s-1} = 2^{r-1} + 1$. The only way that two powers of 2 can differ by 1 is for $r - 1 = 0, s - 1 = 1$, i.e. $r = 1$ and $s = 2$, so $N - 1 = 2$, i.e. $N = 3$ and $2^n + 1 = 3^2 = 9$, i.e. the only term in the sequence which is a power is its third term, namely 9.

4. The sequence $(p^n + 1)$

Now let p be an odd prime number and suppose that $p^n + 1 = N^q$. As before, we may suppose that q is prime. Then

$$p^n = N^q - 1 = (N-1)T, \tag{5}$$

where

$$\begin{aligned} T &= \frac{N^q - 1}{N-1} \\ &= \frac{(N-1+1)^q - 1}{N-1} \\ &= (N-1)^{q-1} + \binom{q}{1}(N-1)^{q-2} \\ &\quad + \cdots + \binom{q}{q-1}, \end{aligned} \tag{6}$$

so that

$$T \equiv q \pmod{N-1} \tag{7}$$

and

$$T \equiv (N-1)^{q-1} + q \pmod{(N-1)q}. \tag{8}$$

It follows from (7) that T, $N-1$ have greatest common divisor 1 or q because q is prime. Suppose that their greatest common divisor is q. From (5), $q | p^n$ so that $q = p$ and $N - 1 = p^r$, $T = p^s$ for some $r, s \geqslant 1$. Also (8) gives that

$$p^s \equiv p^{r(p-1)} + p \pmod{p^{r+1}}.$$

If $s > 1$ it follows that $p^2 | p$, which is not so. Hence $s = 1$. But, from (6),

$$T \geqslant \frac{N^2 - 1}{N-1} = N + 1 > N - 1,$$

so $s > r$. Hence $r = 0$, which is not so.

Hence $N - 1, T$ are coprime. Since $T > N - 1$, the only possibility from (5) is that $N - 1 = 1$ so $N = 2$ and $p^n + 1 = 2^q$ so $p^n = 2^q - 1$. In section 2 we showed that $2^q - 1$ cannot be a power because $q > 1$, so $n = 1$ and $p = 2^q - 1$, i.e. p is a Mersenne prime. □

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Solution to Braintwister 3

(We have the powers)

Answer: 3, 7 and 18.

Solution: Let the number in the calculator be x. Then the first two answers are

$$a = x + \frac{1}{x} \qquad \text{and} \qquad b = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = a^2 - 2.$$

For a and $a^2 - 2$ to be single-digit integers with the second one bigger we must have $a = 3$ and $b = 7$.

Notice that a little bit of algebra shows you that the third term satisfies

$$c = x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 1\right) = a(b-1) = 18.$$

You might like to note that the x in the calculator's memory had to satisfy $x + \frac{1}{x} = 3$, which has roots $\frac{1}{2}(3 + \sqrt{5})$ and $\frac{1}{2}(3 - \sqrt{5})$.

VICTOR BRYANT

Mathematics in the Classroom

Conditional probability and Bayes' theorem

Back in February 1997, *The Numbers Game* on BBC2 television turned its attention to Bayes' theorem and Professor Adrian Smith explained the concept through an intriguingly simple problem. He invited viewers to consider three cards: the first has two red sides, the second has two green sides, the third has one red and one green side. One of these three cards is selected at random and the upper side is seen to be red. The question to be answered is: what is the probability that the underside is red too?

You may be tempted at this stage to think that the answer to this question is 1/2, but a little more thought should convince you that the answer is really 2/3.

This provides us with a good illustration of Bayes' theorem at work. The theorem describes the way in which probabilities change in response to some prior information being available. In the example of the three cards described above, we may apply the theorem in the following way. We need to define two events:

A: underside of card is red,

B: topside of card is red.

Then Bayes' theorem states that

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)},$$

where $P(A \mid B)$ denotes the conditional probability of A occurring if it is known that B has already occurred.

For our card problem, $P(A \text{ and } B) = 1/3$ and

$$\begin{aligned} P(B) &= P(\text{red on topside} \mid RR)P(RR) \\ &\quad + P(\text{red on topside} \mid RG)P(RG) \\ &= (1/3) + [(1/2) \times (1/3)] \\ &= 1/2. \end{aligned}$$

Hence we see that $P(A \mid B) = (1/3)/(1/2) = 2/3$.

A classroom practical: the lady or the tiger

Watching this on TV (and also reading David Sharpe's article on similar procedures; see reference 1) reminded me of the following famous short story that I first gleaned from David Maycock (reference 2), and on which a useful probability practical exercise was based.

A king, having discovered his daughter's romance with a certain courtier, has placed the unfortunate young man on trial. Now this king enjoyed administering a curious kind of justice, which usually took the form of placing the prisoner in front of two identical doors, behind one of which was a hungry tiger and behind the other a desirable young lady. The prisoner chose one door and opened it. If the tiger sprang through the door, the man's fate was considered a just punishment for his crime. If the lady stepped forth, the man's innocence was rewarded by a marriage ceremony performed on the spot.

For the case of his daughter's suitor, the king decides to toughen up the test by placing the courtier in front of *three* pairs of doors. Behind one pair of doors are two hungry tigers, behind another are one tiger and one lady, and behind the third are two ladies who are identical twins dressed exactly alike.

The trial is to proceed as follows. The courtier must first choose a pair of doors. Then he selects one of the two and opens it. If the tiger emerges, that will end the proceedings. If the lady emerges, the door would immediately be slammed shut. The lady and her unknown partner (either her twin sister or a tiger) would then be secretly rearranged in the same two rooms, one to a room at random. The courtier would then be given a second choice between the same two doors without knowing whether the arrangement was different to or the same as before. If he chose a tiger, that was that again; if he chose a lady the door would be slammed shut, the rearranging repeated and the courtier given a third and final choice of one of the same two doors. If successful in his last choice, he would marry the lady and his ordeal would be over.

The day of the trial arrived and all went according to plan. Twice the courtier selected a lady, and the question now posed is: exactly what probability does the courtier have of finding a lady on his third choice of door?

At this stage the students usually hazard a guess as to the answer to this question.

Finding an answer

A tree diagram shows that the four main outcomes facing the courtier are:

T LT LLT LLL.

We then use cards and dice to simulate these outcomes: from a pack of cards, three 10s are used to represent the tigers and three queens to represent the ladies. These six cards are placed in pairs (TT, LT, LL) face down on the desk and a die used to determine which pair has been chosen. The die is rolled again to see if it is a left-hand or a right-hand card that is to be selected. If a 10 occurs, the result T is recorded and another simulation started. If a queen occurs, the two cards are shuffled and the die is thrown again to determine a second choice from this pair of cards. This is repeated until either a tiger occurs (LT, LLT) or three ladies consecutively (LLL). The simulation is repeated many times, class results are collected together and the conditional probability is estimated from the data.

Alternatively, a tree diagram can be drawn up to find the probability distribution of the four outcomes, from which it becomes clear that $P(\mathrm{L}_3 \mid \mathrm{L}_1\mathrm{L}_2)$ is 9/10, considerably larger than most students guess it to be.

References

1. D. Sharpe, The daughter's dilemma, *Mathematical Spectrum* **29**, (1996/7) p. 29.
2. *Looking at Statistics*. 'A' level AHEAD maths and physics INSET video. (Aston University, 1990).

Carol Nixon

Letters to the Editor

Dear Editor,

Problems 28.4 and 28.5

These problems asked for all solutions in integers of the equations $2^n + n^2 = m^2$ and $3^n + n^3 = m^3$; the former has the only solution $n = 6$, $m = 10$ and the latter has no solution. In his Letter to the Editor from *Mathematical Spectrum* **29**, pp. 66-67, Toby Gee shows that the only solution of the equation $p^k = m^p - n^p$, where $m,\ n,\ k$ are natural numbers and p is a prime, is given by $2^6 = 10^2 - 6^2$.

Concerning the equation $2^n + n^2 = m^2$ we prove here the following result.

Theorem. *The equation*

$$2^y + y^2 = z^n \tag{1}$$

has no solutions (y, z, n) such that y is odd and $n > 1$.

Proof. The given equation has no solution (y, z, n) with $n > 1$, and y odd, $y < 5$. Assume now that $y \geqslant 5$ and that it has a solution (y, z, n). We may assume that n is prime. We first show that n is odd. Indeed, assume that $(y,\ z)$ is a positive solution of $y^2 + 2^y = z^2$ with both y and z odd. Then $(z+y)(z-y) = 2^y$. Since $\gcd(z+y,\ z-y) = 2$ it follows that $z - y = 2$ and $z + y = 2^{y-1}$. Hence, $y = 2^{y-2} - 1$. However, one can easily check that $2^{y-2} - 1 > y$ for $y \geqslant 5$.

Assume now that $n = p \geqslant 3$ is an odd prime. Write

$$\left(y + 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2}\right) \cdot \left(y - 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2}\right) = z^n.$$

Since $\mathbb{Z}[\mathrm{i}\sqrt{2}]$ is Euclidean, and

$$\gcd\left(y + 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2},\ y - 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2}\right) = 1,$$

it follows that there exists $a, b \in \mathbb{Z}$ such that

$$y + 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2} = \left(a + b\mathrm{i}\sqrt{2}\right)^n \tag{2}$$

and

$$y - 2^{(y-1)/2} \cdot \mathrm{i}\sqrt{2} = \left(a - b\mathrm{i}\sqrt{2}\right)^n. \tag{3}$$

From (2) and (3) it follows that

$$y = \frac{(a + b\mathrm{i}\sqrt{2})^n + (a - b\mathrm{i}\sqrt{2})^n}{2} \equiv a^n \pmod 2 \tag{4}$$

and

$$2^{(y-1)/2} = \frac{(a + b\mathrm{i}\sqrt{2})^n - (a - b\mathrm{i}\sqrt{2})^n}{2\sqrt{2}\mathrm{i}}. \tag{5}$$

From (4) we conclude that a is odd. From (5) it follows that

$$2^{(y-1)/2} = b(na^{n-1} + s),$$

where s is even. Since both n and a are odd, it follows that $na^{n-1} + s$ is odd as well. Hence, $b = 2^{(y-1)/2}$. Equation (1) can now be rewritten as

$$y^2 + 2^y = z^n = \left((a + b\mathrm{i}\sqrt{2})(a - b\mathrm{i}\sqrt{2})\right)^n = (a^2 + 2b^2)^n$$

or

$$y^2 + 2^y = (a^2 + 2^y)^n > 2^{ny} \geqslant 2^{3y}. \tag{6}$$

Inequality (6) implies that

$$y^2 > 2^{3y} - 2^y = 2^y(2^{2y} - 1) > 2^y,$$

which is false for $y \geqslant 5$.

Yours sincerely,

FLORIAN LUCA
(Department of Mathematics,
Syracuse University, NY)

Dear Editor,

$183184 = 428^2$

(Volume 30 Number 1 page 9)

K. R. S. Sastry did not specify the number of digits for the 'root' number in his '$183184 = 428 \times 428$' poser. In obvious notation, the first few solutions are $(183, 328, 528, 715)$, (6099), $(13224, 40495)$, $(106755, 453288)$. This was a simple and interesting exercise in BASIC programming — I used UBASIC, which easily handles the large numbers involved.

Yours sincerely,

ALAN D. COX
(Pen-y-Maes, Ostrey Hill,
St Clears, Dyfed, SA33 4AJ)

Dear Editor,

An identity for Fibonacci numbers

Most proofs of identities for Fibonacci numbers are number-theoretic, so readers may be interested in a combinatorial approach.

Recall that the Fibonacci sequence is defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n > 0.$$

We shall prove the identity

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n.$$

We first write $K_n = F_{n+1}$, so that

$$K_1 = 1, K_2 = 2 \text{ and } K_{n+2} = K_{n+1} + K_n \quad \text{for } n > 0.$$

The identity now becomes

$$K_{m+n} = K_m K_n + K_{m-1} K_{n-1} \qquad (m, n > 1).$$

We prove by induction that K_n is the number of ways of tiling an $n \times 1$ rectangle with 1×1 and 2×1 tiles. The result is true for $n = 1$ and $n = 2$. Suppose that it is true for $n = r$ and $n = r + 1$. Any $(r + 2) \times 1$ rectangle can be tiles by adding a 2×1 tile to a $r \times 1$ tiling or by adding a 1×1 tile to an $(r+1) \times 1$ tiling, so the number of tilings of an $(r+2) \times 1$ rectangle is $K_r + K_{r+1} = K_{r+2}$. This proves the inductive step.

Now consider an $(m + n) \times 1$ rectangle, divided into an $m \times 1$ and an $n \times 1$ rectangle. There are $K_m K_n$ tilings such that no 2×1 tile crosses the division and $K_{m-1} K_{n-1}$ where a 2×1 tile crosses the division. Hence the total number of tilings is

$$K_m K_n + K_{m-1} K_{n-1} = K_{m+n},$$

which proves the identity.

Yours sincerely,
Mansur Boase
(Student, St Paul's School, London)

Dear Editor,

The primality of the Lucas numbers

The Fibonacci sequence is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_1 = F_2 = 1$. It is known that, if a Fibonacci number F_m is prime then m must also be prime (see reference 1, Theorem 179).

The Lucas sequence, defined by the recurrence relation $L_{n+2} = L_{n+1} + L_n$ with initial conditions $L_1 = 1$, $L_2 = 3$, is perhaps less well known to readers. We note that the Lucas numbers can be extended backwards to negative values using the recurrence

n	-4	-3	-2	-1	0	1	2	3	4
L_n	7	-4	3	-1	2	1	3	4	7

and see that $L_{-n} = (-1)^n L_n$.

The analogue of the result on Fibonacci numbers is that if L_m is prime then m is either prime or a power of 2. To see this, suppose that $m > 0$ is neither prime nor a power of 2. Then $m = kq$ for some odd number q with $1 < q < m$. We consider the Lucas sequence $L_n \mod L_k$; it must be of the form

n	$-k-1$	$-k$	$-k+1$	$-k+2$	$\dots$	$k-1$	k	$k+1$	$k+2$
L_n (mod L_k)	L_{-k-1}	0	L_{-k-1}	L_{-k-1}	$\dots$	L_{k+1}	0	L_{k+1}	L_{k+1}

Since $L_{k+1} = \pm L_{-k-1}$, the sequence

$$0, L_{k+1}, L_{k+1}, \dots$$

is either identical with

$$0, L_{-k-1}, L_{-k-1}, \dots$$

or the negative of it. Therefore 0 is repeated every $2k$ places in this sequence, at the values $n = qk$ with q odd. Thus $L_m \equiv 0 \mod L_k$, so L_m cannot be prime.

Reference

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Clarendon, Oxford, 1938).

Yours sincerely,
Mansur Boase

Dear Editor,

Regular polygon rings

I enjoyed reading Professor R. A. Dunlap's article on regular polygon rings in *Mathematical Spectrum* **30** (1997/8) pp. 13–15. His analysis of rings of regular polygons in which the central figure is also a regular polygon identifies the four cases in which each central vertex is the common point of 3 edges, but omits three other possibilities. Each central vertex may be a common point of 4, 5 or 6 edges. (No more are possible because no more than 6 coplanar regular polygons can meet at a point.) Using the methods described in his article it can be shown that this leads to the following additional cases.

(a) A ring of 8 squares surrounding a central square.

(b) A ring of 18 equilateral triangles surrounding a central hexagon.

(c) A ring of 12 equilateral triangles surrounding a central triangle.

Yours sincerely,
K. Robin McLean
(Department of Education, University of Liverpool, Liverpool L69 3BX)

Dear Editor,

Difficult questions

In *Mathematical Spectrum* Volume 29 Number 2 page 41, Kamlesh Gaya asks the readers to help him solve the following *difficult questions*.

1. How many positive integers less than 10^{10} have exactly one digit equal to a square and have the sum of the digits equal to a square?

2. How many positive integers less than 10^{10} have all their digits equal to a prime and have the sum of their digits equal to a prime?

The number of numbers with n digits satisfying the conditions in the first problem is given by the sum of the coefficients of the square powers of x in the expansion of

$$[n(x+x^4+x^9)+n-1] \\ \times (x^2+x^3+x^5+x^6+x^7+x^8)^{n-1}.$$

To find the number of numbers satisfying the conditions up a prescribed number of digits we merely sum the numbers for each of the separate number of digits up to that limit.

Mathematica generates the following output, where the first column is the number of digits, the second column is the number of numbers satisfying the property with that number of digits and the third column is the cumulative sum.

Digits	Number	Total
1	3	3
2	8	11
3	54	65
4	396	461
5	2526	2987
6	16773	19760
7	108304	128064
8	694010	822074
9	4431028	5253102
10	28077793	**33330895**
11	174756728	208087623
12	1111820096	1319907719
13	6916761170	8236668889
14	42439566183	50676235072
15	268009758304	318685993376
16	1670350105002	1989036098378
17	10115116199929	12104152298307
18	62591214874360	74695367172667
19	394298668115751	468994035288418
20	2412193823307801	2881187858596219

The answer therefore to Kamlesh's first question is 33330895.

The second problem can be done in a similar fashion. The generating function this time is

$$(x^2+x^3+x^5+x^7)^n,$$

and the sum of the coefficients of prime powers of x gives the number of numbers with n digits which satisfy the conditions of problem two. The following *Mathematica* code:

$$\begin{aligned}
&\text{pridig}[n_] := \text{pridig}[n] = \\
&\quad \text{Module}[\{x,y\}, y = \text{Expand}[(x^2+x^3+x^5+x^7)^n]; \\
&\quad \text{Apply}[\text{Plus, Map}[\text{Coefficient}[y,x,\#]\&, \\
&\qquad \text{Prime}[\text{Range}[\text{PrimePi}[7n]+1]]]]]; \\
&\quad \text{cumsum} = 0;\ \text{TableForm}[\text{Table}[\{i, \text{pridig}[i], \\
&\qquad \text{cumsum} = \text{cumsum} + \text{pridig}[i]\}, \{i, 20\}]]
\end{aligned}$$

produces:

Digits	Number	Total
1	4	4
2	4	8
3	24	32
4	80	112
5	290	402
6	984	1386
7	3927	5313
8	15880	21193
9	66996	88189
10	267450	**355639**
11	990451	1346090
12	3645864	4991954
13	14393925	19385879
14	59925922	79311801
15	250872105	330183906
16	1025162768	1355346674
17	4036944423	5392291097
18	15309045720	20701336817
19	56799957778	77501294595
20	213660353140	291161647735

which is in the same format as the first table. Hence the answer to problem two is 355639.

Yours sincerely,
ROBERT PAGE
(Bradford Grammar School,
Bradford BD9 4JP)

Dear Editor,

The harmonic series

Many readers will be aware that the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, but they may not realize how slow the divergence is. I executed the following C program and after 10^9 terms it only reached 21.30.

```
#include <math.h>
#include <stdio.h>
double k, S;

main()
{
    S=0;
    for(k=1; k<=1.0e9; k++)
    {
        S=S + 1/k;
    }
    printf("%f\n", S);
}
```

Yours sincerely,
ALLEN BROWN
(98 Histon Road,
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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

30.5 Let a, b, c be positive real numbers. Prove that

$$(a+b+c)abc \geqslant 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

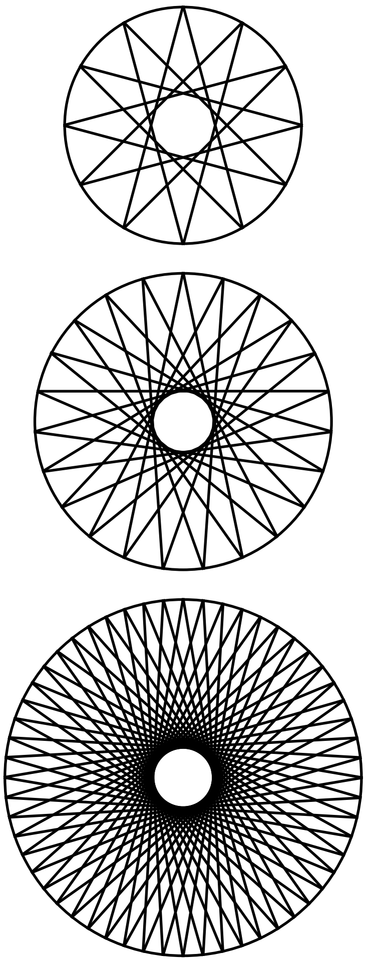
(Submitted by Mansur Boase, St Paul's School, London)

30.6 Given a triangle ABC, three equilateral triangles ABD, BCE and CAF are drawn external to the triangle ABC. Show that triangles ABC and DEF have the same centroid.

(Submitted by J. A. Scott, Chippenham)

30.7 In the three diagrams shown, determine the ratios of the radii of the two circles.

(Submitted by Seyamack Jafari, Ahwaz, Iran)

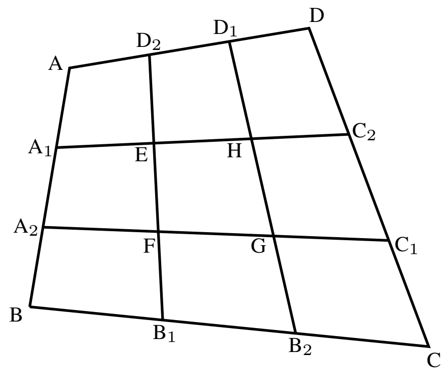


30.8 Find the maximum area of a quadrilateral with sides of given lengths a, b, c, d in cyclic order.

(Submitted by J. Smart, Sheffield)

Solutions to Problems in Volume 29 Number 3

29.9 In the figure, $\mathrm{DD_1 = D_1D_2 = D_2A}$, $\mathrm{AA_1 = A_1A_2 = A_2B}$, $\mathrm{BB_1 = B_1B_2 = B_2C}$, $\mathrm{CC_1 = C_1C_2 = C_2D}$. Determine the area of the quadrilateral EFGH in terms of the area of ABCD.



Solution by Jeremy Young, Nottingham High School

$$\overrightarrow{\mathrm{CA}} = \overrightarrow{\mathrm{CB}} + \overrightarrow{\mathrm{BA}} = 3\overrightarrow{\mathrm{B_1B}} + 3\overrightarrow{\mathrm{BA_2}} = 3\overrightarrow{\mathrm{B_1A_2}}$$

and

$$\overrightarrow{\mathrm{CA}} = \overrightarrow{\mathrm{CD}} + \overrightarrow{\mathrm{DA}} = \tfrac{3}{2}\overrightarrow{\mathrm{C_1D}} + \tfrac{3}{2}\overrightarrow{\mathrm{DD_2}} = \tfrac{3}{2}\overrightarrow{\mathrm{C_1D_2}},$$

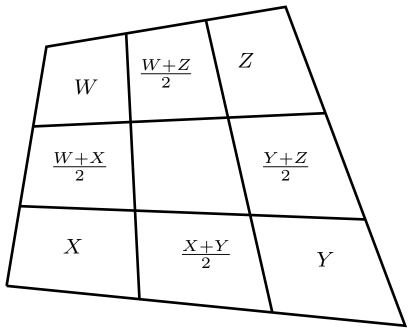
so $\mathrm{B_1A_2}$ and $\mathrm{C_1D_2}$ are parallel and $\mathrm{B_1A_2} = \frac{1}{2}\mathrm{C_1D_2}$. Thus triangles $\mathrm{FA_2B_1}$ and $\mathrm{FC_1D_2}$ are similar and $\mathrm{D_2F = 2FB_1}$, $\mathrm{C_1F = 2FA_2}$. If this is repeated for $\mathrm{A_1D_2}$ and $\mathrm{B_1C_2}$, $\mathrm{D_1C_2}$ and $\mathrm{A_1B_2}$, $\mathrm{C_1B_2}$ and $\mathrm{D_1A_2}$, we see that

$$\mathrm{A_1E = EH = HC_2}, \qquad \mathrm{A_2F = FG = GC_1},$$
$$\mathrm{B_1F = FE = ED_2}, \qquad \mathrm{B_2G = GH = HD_1}.$$

It now follows that, in terms of area,

$$\mathrm{AA_1ED_2 + A_2BB_1F = 2A_1A_2FE},$$

and similarly for three other directions. We now have areas as shown in the figure below.



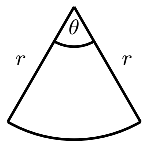
By a similar process, the area of the central quadrilateral is

$$\frac{1}{2}\left(\frac{W+X}{2}+\frac{Y+Z}{2}\right)=\tfrac{1}{4}(W+X+Y+Z).$$

Now

$$\text{area ABCD} = \tfrac{9}{4}(W+X+Y+Z) = 9\,\text{area EFGH}.$$

29.10 A piece of wire of length ℓ is bent into the shape of a sector of a circle. Find the maximum area of the sector.



Solution by Shen Lei, Millfield School

$$\left(\sqrt{\theta}-\frac{2}{\sqrt{\theta}}\right)^2 \geqslant 0$$

so

$$\theta+\frac{4}{\theta}-4\geqslant 0$$

so

$$\theta+\frac{4}{\theta}+4\geqslant 8$$

so

$$\theta^2+4\theta+4\geqslant 8\theta$$

so

$$\frac{\theta}{(\theta+2)^2}\leqslant\frac{1}{8},$$

with equality when $\theta=2$. Now $\ell=2r+r\theta$, so

$$r=\frac{\ell}{\theta+2}.$$

The area is

$$\tfrac{1}{2}r^2\theta=\frac{\theta\ell^2}{2(\theta+2)^2}\leqslant\frac{\ell^2}{16},$$

with equality when $\theta=2$, so the maximum area is $\ell^2/16$.

Also solved by Zhao Yueyang (Millfield School), Gourab Datta (Highgate School, London), Minh Can (University of Southern California, Los Angeles), Konstantin Ardakov (Christchurch College, Oxford), Jeremy Young.

29.11 Solve the simultaneous equations

$$ax+by=2 \tag{1}$$
$$ax^2+by^2=20 \tag{2}$$
$$ax^3+by^3=56 \tag{3}$$
$$ax^4+by^4=272. \tag{4}$$

Solution independently by Minh Can and Konstantin Ardakov

$x(1)-(2)$ gives

$$by(x-y)=2(x-10). \tag{5}$$

$x(2)-(3)$ gives

$$by^2(x-y)=20x-56. \tag{6}$$

From (5), (6), $by(x-y)\neq 0$, $x\neq 10$ and

$$y=\frac{20x-56}{2x-20}.$$

$x(3)-(4)$ gives

$$by^3(x-y)=56x-272, \tag{7}$$

and (6), (7) give that $20x\neq 56$ and

$$y=\frac{56x-272}{20x-56}.$$

Hence

$$\frac{56x-272}{20x-56}=\frac{20x-56}{2x-20},$$

which reduces to

$$x^2-2x-8=0$$

or

$$(x-4)(x+2)=0,$$

so $x=4$ or -2. Thus $y=-2$ or 4 respectively. Both solutions give $a=b=1$. Hence $(a,b,x,y)=(1,1,4,-2)$ or $(1,1,-2,4)$. It can be verified directly that these are both solutions.

Also solved by Zhao Yueyang, Shen Lei, Gourab Datta and Jeremy Young.

29.12 Simplify

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C},$$

where A, B, C are the angles of a triangle, and find all non-zero real numbers α for which

$$\frac{\sin \alpha A + \sin \alpha B + \sin \alpha C}{\sin A \sin B \sin C}$$

is independent of A, B and C.

Solution by Konstantin Ardakov

With standard notation,

$$\begin{aligned}
&\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C}\\
&\quad = \frac{2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C}{\sin A \sin B \sin C}\\
&\quad = 2\left(\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B}\right)\\
&\quad = 2\left[\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{b}{2R}\frac{c}{2R}} + \frac{\frac{c^2+a^2-b^2}{2ca}}{\frac{c}{2R}\frac{a}{2R}} + \frac{\frac{a^2+b^2-c^2}{2ab}}{\frac{a}{2R}\frac{b}{2R}}\right]\\
&\quad = \frac{4R^2}{a^2b^2c^2}\big(a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2)\\
&\qquad + c^2(a^2+b^2-c^2)\big)\\
&\quad = \frac{-4R^2}{a^2b^2c^2}\big(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2\big)\\
&\quad = \frac{-4R^2}{a^2b^2c^2}\big((c^2-a^2-b^2)^2-4a^2b^2\big)\\
&\quad = \frac{-4R^2}{a^2b^2c^2}\big(4a^2b^2\cos^2 C-4a^2b^2\big)\\
&\quad = \frac{16R^2\sin^2 C}{c^2}\\
&\quad = 4.
\end{aligned}$$

Suppose now that

$$\frac{\sin \alpha A + \sin \alpha B + \sin \alpha C}{\sin A \sin B \sin C} = k,$$

where k is independent of A, B, C. If we differentiate partially with respect to A keeping C fixed and using the fact that $A+B+C=\pi$, we obtain

$$\begin{aligned}
\alpha\cos\alpha A - \alpha\cos\alpha B &= k(\cos A\sin B\sin C\\
&\qquad - \sin A\cos B\sin C)\\
&= k\sin(B-A)\sin C\\
&= k\sin(B-A)\sin(B+A)\\
&= \tfrac{1}{2}k(\cos 2A - \cos 2B).
\end{aligned}$$

If we repeat this procedure we obtain

$$-\alpha^2\sin\alpha A - \alpha^2\sin\alpha B = k(-\sin 2A - \sin 2B).$$

If we do this a third time we obtain

$$\alpha^3\cos\alpha A - \alpha^3\cos\alpha B = 2k(\cos 2A - \cos 2B).$$

Hence

$$\alpha^2\tfrac{1}{2}k(\cos 2A - \cos 2B) = 2k(\cos 2A - \cos 2B).$$

If we choose A, B so that $\cos 2A \neq \cos 2B$ we obtain $\alpha = \pm 2$ ($k \neq 0$ because $\alpha \neq 0$). We have already shown that the expression is independent of A, B, C when $\alpha = 2$, and so also when $\alpha = -2$. Hence these are precisely the values of α for which the expression is independent of A, B, C.

***Mathematical Spectrum* Awards**
for Volume 29

Prizes have been awarded to the following student readers for various contributions published in Volume 29:

Mansur Boase
Toby Gee
Junji Inaba

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of \$50 for articles, and up to \$25 for letters, solutions to problems, and other items.

If two numbers add up to or differ by a multiple of 50, then their squares have the same final two digits; e.g.

$$68^2 = 4624, \quad 32^2 = 1024, \quad 118^2 = 13924.$$

Can you prove this?

BOB BERTUELLO
Bath

$c(x)$ and $s(x)$ are two functions such that $c'(x) = -s(x)$ and $s'(x) = c(x)$. Prove that $c(x)^2 + s(x)^2$ is constant.

JUNJI INABA
(Student, William Hulme's
Grammar School, Manchester)

Reviews

The Ingenious Mind of Nature By George M. Hall. Plenum, New York, 1997. Pp. 450. Paperback $29.95 (ISBN 0-306-45571-4).

This book bears the subtitle 'Deciphering the Patterns of Man, Society and the Universe' and its author claims to have made original contributions in finding general principles which apply to *all* systems. He quickly introduces, on page 7, the main 'thesis' of the book, and I can do no fairer than quote it: *patterns of related elements in space — systems — combined with the attributes of the elements, constitute programs by which those patterns efficaciously devolve into order or disorder as the case may be*. He names this *physiogenesis*, but by this stage the game is up. The correct name is *mumbo-jumbo*, and we have here another case of an emperor sporting his new clothes.

Lest you think this harsh, let me offer a few telling extracts to illustrate the extent of the author's scholarship. On page 106 we find that topography is a branch of mathematics. Page 139 talks of 'lines of logic which wrap back on themselves to prove their own axioms', and a footnote refers to Gödel's proof 'which may or may not cover the inability of theorems to prove their own axioms'. Dimensional analysis, on page 186 teaches that, of kinematic measures, *velocity* is the easiest to understand: it means 'the *units of length* travelled per unit of time'. Over the page another footnote explains that energy can be defined tautologically as a time cross-section of power.

Page 96 marks the start of a section rich in priceless gems. We read of Newton rejecting the idea of absolute space, Einstein's reference frame being the speed of light itself, and *Fourier's* formalization of the doctrine of conservation of energy (no mention of Helmholtz).

A glossary and an appendix attempt clarification of technical terms. Two examples will suffice: 'Space is a dimensioned void that extends to infinity', and 'In effect, energy may be an effect'. The author's staple reference throughout the book is *The Harper Dictionary of Modern Thought*, along with *Encyclopaedia Britannica* and *Great Books of the Western World*. Clearly not a book to be taken seriously, but possibly suitable for the dentist's waiting room.

George M. Hall, we are told on the cover, is an adjunct professor at Pima Community College, Tucson AZ, where he teaches physics, history, psychology, sociology and economics. He has written numerous books and articles.

Queen Mary's Grammar School — Stephen Rout

Other books received

Interdisciplinary Lively Application Projects. Edited by D. C. Arney. Mathematical Association of America, Washington, DC, 1997. Pp. xii+222. Paperback $27.50 (ISBN 0-88385-706-5).

To quote from the preface: 'The ILAPs (Interdisciplinary Lively Application Projects) in this volume have taken 6–10 hours of student effort on average. However, as with all large or open-ended problems, there will be a large variation in time spent by different student teams'. The eight projects are:

Getting Fit with Mathematics
Decked Out (designing a verandah for a building)
Parachute Panic
Flying with Differential Equations
Planning a Backpacking Trip
Smog in the Los Angeles Basin
Beams and Bridges
Contaminant Transport

Some of the pages are suitable for duplicating as handouts for students, sample solutions are given and there is advice on writing reports. Some of the mathematics is quite advanced, e.g. Fourier series in the projects on contaminants.

Introductory Discrete Mathematics. By V. K. Balakrishnan. Dover, New York, 1997. Pp. xiv+256. Paperback $9.95 (ISBN 0-486-69115-2).

This concise text offers an introduction to discrete mathematics for undergraduate students in computer science and mathematics. Mathematics educators consider it vital that their students be exposed to a course in discrete methods that introduces them to combinatorial mathematics and to algebraic and logical structures focusing on the interplay between computer science and mathematics. The present volume emphasizes combinatorics, graph theory with applications to some standard network optimization problems, and algorithms to solve these problems.

Precalculus: Functions and Graphs. By M. A. Munem and J. P. Yizze. Worth, New York, 1997. Pp. xvii+686. Hardback $29.95 (ISBN 1-57259-157-9).

The sixth edition of a textbook covering such topics as basic algebra, coordinate geometry, trigonometry, linear equations and linear inequalities, conics, finite arithmetic and geometric series — but not calculus!

Principia Mathematica to *56. By Alfred North Whitehead and Bertrand Russel. Cambridge University Press, Cambridge, 1997. Pp. 410. Paperback. $32.50 (ISBN 0-521-62606-4).

Principia Mathematica is one of the most famous mathematics books of all time, in which Whitehead and Russell sought to deduce the fundamental propositions of logic and mathematics from a small number of logical premises and primitive ideas. This book contains an abridged text of Volume 1. For most of our readers, it will be a volume to browse through with reverent awe rather than to read.

Resources for Teaching Linear Algebra. Edited by David Carlson et al. Mathematical Association of America, Washington, DC, 1997. Pp. x+287. Paperback $ 34.95 (ISBN 0-88385-150-4).

As the title implies, this is a resource book for teachers of linear algebra. There are articles from many teachers and users of the subject which teachers will find stimulating.

Foundations of Advanced Mathematics. By Diana Cowey et al. Hodder and Stoughton, London, 1997. Pp. 177. Paperback $9.99 (ISBN 0-340-65855-X).

This attractively produced volume is part of the MEI Structured Mathematics programme, and is intended as an access course for advanced level mathematicians or as a GNVQ mathematics unit. There are chapters on Calculations, Algebra, Graphs, Trigonometry, and Statistics and Probability.

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