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CONTENTS

LÉOPOLD SAUVÉ (1921-1987)

Kenneth S. Williams

Léopold Sauvé, known to his friends as Léo, was born on December 12, 1921 in Hull, Québec. His father, Ubald Sauvé, was a detective on the morality squad of the Ottawa police force for many years. Léo came from a large family; he had five brothers and two sisters.

Léo started primary school in Hull and when later the family moved across the Ottawa River to the Sandy Hill district of Ottawa, he continued his education there. After finishing high school, Léo began work with the Government of Canada in the Topography Department located on Carling Avenue. During the war years (1939-45) Léo continued to work for the government while serving in the army reserve. In 1950 he became engaged to Carmen Chenier, a registered nurse, who lived just around the corner from Léo's home in Ottawa. They were married on May 17, 1952 and moved to Hull. During the next seven years Léo's four children were born: Claire (1953), Madeline (1955), Jacques (1957) and Luc (1959).

In 1956, with strong recommendations from Dr. Pierre Gendron, Dean of the Faculty of Science at the University of Ottawa, and Dr. Viktors Linis, Chairman of the Department of Mathematics at the University of Ottawa, Léo was offered a teaching position at St. Patrick's College. Léo accepted the appointment, and started to teach mathematics, mostly to students in the Bachelor of Science program. In 1966 the B.Sc. program was terminated at St. Patrick's College and in 1967 the college was integrated into Carleton University as a division of the Faculty of Arts. The demand for mathematics courses at St. Patrick's College dropped significantly, and Léo decided for career reasons that he should look for a position at one of the community colleges where he would have larger classes. In 1968 he accepted an appointment as a full-time teacher of mathematics at the School of Technology of Algonquin College. About this time Léo moved his family to the Alta Vista area of Ottawa so that his children could be educated both in English and French. Léo continued to teach at Algonquin College until his retirement in 1986.

During the late Fifties Léo studied mathematics at the University of Ottawa graduating with an honours degree in 1960 at the age of thirty-nine. He continued his mathematical studies at the University of Ottawa (1960-61) and at the University of Montreal (1961-62) but did not complete a graduate degree. As he did not hold a graduate degree, Léo always disclaimed being a professional mathematician. Nevertheless he was universally regarded by both his students and colleagues as being a fine mathematician and an outstanding teacher. His contributions to mathematics were recognized by awards from both the University of Waterloo and Algonquin College.

Early in 1975 an event took place which was to affect both Léo's life and the mathematical community at large. Six members of the Carleton-Ottawa Mathematics Association (R. Duff Butterill of the Ottawa Board of Education; H.G. Dworschak, F.G.B. Maskell and Léo Sauvé of Algonquin College; Viktors Linis of the University of Ottawa; and Richard J. Semple of Carleton University) met privately and decided to launch a magazine to provide a forum for the exchange of mathematical information, especially interesting problems and solutions, among the members of the mathematical community in the Ottawa region, students and teachers alike. The first issue of this magazine "*Eureka*" appeared in March 1975 with Léo Sauvé as its editor. In May 1975 the Carleton-Ottawa Mathematics Association agreed at its annual general meeting to sponsor *Eureka* in collaboration with Algonquin College. Little did Léo realize then the work he was getting into. From then until January 1986 Léo continued as editor of *Eureka* (it became *Crux Mathematicorum* in March 1978). Léo's dedication and hard work, his broad knowledge and love of mathematics, his careful eye for detail, all enabled *Crux* to grow from a four-page problem sheet to the international mathematical problem-solving journal that it is today. Léo's efforts brought a lot of joy to a great many people throughout the world who saw their problems and solutions brought to life in the pages of *Crux*. Many of these people became Léo's friends and their tributes to Léo are recorded in the issue of *Crux* dedicated to Léo (*Crux Mathematicorum* [1986:163]).

Léo was a simple man who found joy in a great many of life's pursuits. He was a musician who loved to play both classical music and blues on his piano. He was a bibliophile who loved the works of great literature. His vast collection of books, through his generosity, is now in collections at Carleton University, Université du Québec à Hull and the Université du Québec à Chicoutimi. Léo also enjoyed playing chess and card games, at which he was very good. He was a life member of the French Canadian Institute in Ottawa and played blackjack, bridge, and chess there once or twice a month. But perhaps one of his greatest joys was enjoying the fine food and wine at some of Ottawa's best restaurants.

Léo was very much a family man. He loved children and some of his happiest moments were reading and telling stories to his own children, often reading from books written in English and immediately translating them into French as he read. Léo was also very much a city person, and when his wife bought a summer cottage in the country it took Léo some time before he got around to visiting it. However, later when the opportunity arose, knowing of his wife's love of the country, he bought the mountain behind the cottage for her for the princely sum of $1500.

Léo died in hospital of a heart attack on June 19, 1987. Those of us who knew him were enriched by knowing him. We will miss him greatly.

* * *

ON A NOTE OF BOTTEMA AND GROENMAN

ROLAND H. EDDY

In [1], this journal, Bottema and Groenman have defined the Wallace point W_4 of the points A_i, $i \in I = \{1,2,3,4\}$, on the unit circle Ω as the point of concurrence of the Wallace lines I_i of A_i with respect to the triangles $A_jA_kA_l$, $\{j,k,l\} = I - \{i\}$, and have given some of its properties. These are then extended to W_n, the Wallace point of an n-tuple of points on Ω. In this follow-up note, we give some further properties of this interesting point; in particular, we list four other names by which it is known and give a brief description of each in its respective context. In what follows, we shall use the same approach as Yaglom [7] and represent the point A_i on the unit circle by the complex number $\mathbf{a}_i$, the origin of course being the center of the given circle. The complex representation of W_4 is thus $1/2 \sum_{i=1}^{4} \mathbf{a}_i$.

The following are four further interpretations given to this point:

(i) Clawson [3] calls it the *orthic center* of the cyclic quadrangle $A_1A_2A_3A_4$ and defines it as the point of concurrency of the (six) perpendiculars from the midpoint of each side to the opposite sides. If we consider, for example, the line from the midpoint of A_1A_2 perpendicular to

A_3A_4 whose equation is

$$2(z - a_3a_4\bar{z})(a_1a_2) = (a_1 + a_2)(a_1a_2 - a_3a_4),$$

it is seen that this line contains W_4 as do the other five; see Sloyan [6].

(ii) Yaglom [7] shows that the Euler (nine-point) circles of the four triangles $A_jA_kA_l$ meet in a point P and the centers of these circles lie on a circle with center P and radius 1/2, the Euler circle of the quadrangle. The point P referred to as the *Euler center* of the quadrangle is precisely the Wallace point W_4.

(iii) Droussent [4] also notes that the same four Euler circles are concurrent but calls the point of concurrence the *anticenter* of the cyclic quadrangle. He then proceeds to derive the following interesting property: "The orthocenters of the four triangles formed by the vertices of the cyclic quadrangle are the symmetries of the vertices with respect to the anticenter (Wallace point)". He also adds that the anticenter is the point of concurrence of the Wallace lines of the vertices of the quadrangle consistent with [1].

(iv) Cartuyvels [2] considers the six parallelograms $H_jA_iA_jH_i$, $\{i,j\} \subseteq I$, $i \neq j$. He notes that the point K is the midpoint of the diagonals A_iH_i, $i \in I$ and is hence the common midpoint of the parallelograms. He calls K the *orthopole* of $A_1A_2A_3A_4$ and proceeds to establish the fact that the four Wallace lines of the given quadrangle are concurrent at the orthopole. The orthopole K is thus the Wallace point W_4.

The main result in [1] is to consider an n-tuple of points on the unit circle and to show that the points

$$O, \quad W_n = \frac{1}{n-2}\sum_{i=1}^{n} a_i, \quad \text{and} \quad G_n = \frac{1}{n}\sum_{i=1}^{n} a_i$$

are collinear and further that $OG_n : OW_n = n - 2 : n$, $n \geq 3$. We extend this result by considering the orthocenter H_n of this n-gon which is defined as the point of intersection of the n circles, equal in radius to the circumcircle of the n-gon, whose centers are the orthocenters of the $(n-1)$-gons formed by $n-1$ vertices of the n-gon. Clearly, $H_n = \sum_{i=1}^{n} a_i$ and so O, W_n, G_n, and H_n are collinear on what may be termed the *Euler line* of the n-gon; see Yaglom [7]. Furthermore,

$$OG_n : OW_n : OH_n = n - 2 : n : n(n - 2).$$

The complete quadrilateral which consists of four straight lines I_i, $i \in I$, in the plane intersecting in six vertices A_{kl}, $\{k,l\} \subseteq I$, $k \neq l$, also has a Wallace point. Clawson [3] shows that the four circles C_i, $i \in I$, circumscribing the triangles T_{jkl}, $\{j,k,l\} \subseteq I$, with j, k, l all distinct, are concurrent at a point F which he calls the *focal point* of the quadrilateral. He adds further that F is also known as the Wallace or Miquel point.

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THE OLYMPIAD CORNER: 88

R.E. WOODROW

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We begin this month with some of the problems that were proposed but not used at the 28th I.M.O. in Havana. More of these will be given in upcoming numbers of the Olympiad Corner. Thanks go to Bruce Shawyer for transmitting the problems to me. As usual, readers are invited to send in solutions.

Australia 1. Let $x_1,x_2,\ldots,x_n$ be n integers and let p be a positive integer less than n. Put

$$
\begin{aligned}
S_1 &= x_1 + x_2 + \ldots + x_p , & T_1 &= x_{p+1} + x_{p+2} + \ldots + x_n , \\
S_2 &= x_2 + x_3 + \ldots + x_{p+1} , & T_2 &= x_{p+2} + \ldots + x_n + x_1 , \\
&\vdots \\
S_n &= x_n + x_1 + \ldots + x_{p-1} , & T_n &= x_p + x_{p+1} + \ldots + x_{n-1}
\end{aligned}
$$

(so the x_i "wrap around", that is, after x_n there comes x_1 again). Next let $m(a,b)$ be the number of numbers i for which S_i leaves the remainder a and T_i leaves the remainder b on division by 3, where each of a and b is 0, 1, or 2.

Show that $m(1,2)$ and $m(2,1)$ leave the same remainder when divided by 3.

Australia 2. a_1, a_2, a_3, b_1, b_2, b_3 are positive real numbers. Prove that

$$
\begin{aligned}
&(a_1b_2 + b_1a_2 + a_2b_3 + b_2a_3 + a_3b_1 + b_3a_1)^2 \\
&\qquad \geq 4(a_1a_2 + a_2a_3 + a_3a_1)(b_1b_2 + b_2b_3 + b_3b_1)
\end{aligned}
$$

and show that the two sides of the inequality are equal if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} .$$

Belgium 1. If $f:(0,\infty) \longrightarrow \mathbb{R}$ is a function having the property that $f(x) = f(1/x)$ for all $x > 0$, prove that there exists a function $u:[1,\infty) \longrightarrow \mathbb{R}$ such that

$$u\left[\frac{x + 1/x}{2}\right] = f(x) \quad \text{for all} \quad x > 0.$$

East Germany 1. In a chess tournament with $n \geq 5$ participants suppose that $[n^2/4] + 2$ games have been already played. ($[x]$ denotes the integer part of x.)

(a) Show that there are 5 players a, b, c, d, e for which the following games have been played: ab, ac, bc, ad, ae, de.

(b) Is this still true if only $[n^2/4] + 1$ games have been played?

[Use induction on n.]

Finland 1. In a Cartesian coordinate system, the circle C_1 has center $O_1 = (-2,0)$ and radius 3. Denote the point $(1,0)$ by A and the origin by O. Prove that there is a positive constant c such that for every point X which is exterior to C_1,

$$\overline{OX} - 1 \geq c \min\{\overline{AX}, \overline{AX}^2\}.$$

Find the smallest possible c.

Finland 2. Does there exist a second degree polynomial $p(x,y)$ in two variables such that every non-negative integer n equals $p(k,m)$ for one and only one ordered pair (k,m) of non-negative integers?

France 1. Let $t_1, t_2, \ldots, t_n$ be n real numbers satisfying $0 < t_1 \leq t_2 \leq \ldots \leq t_n < 1$. Prove that

$$(1 - t_n)^2\left[\frac{t_1}{(1 - t_1^2)^2} + \frac{t_2^2}{(1 - t_2^3)^2} + \ldots + \frac{t_n^n}{(1 - t_n^{n+1})^2}\right] < 1.$$

France 2. Let ABC be a triangle. For each point M of the segment BC denote by B' and C' the orthogonal projections of M on the lines AC and AB, respectively. Determine those points M for which the length of $B'C'$ is minimum.

Great Britain 1. Prove that if the equation

$$x^4 + ax^3 + bx + c = 0$$

has all its roots real then $ab \leq 0$.

Great Britain 2. Numbers $d(n,m)$, where n, m are integers and $0 \leq m \leq n$, are defined by

$$d(n,0) = d(n,n) = 1 \quad \text{for all} \quad n \geq 0$$

and

$$m \cdot d(n,m) = m \cdot d(n-1,m) + (2n-m) \cdot d(n-1,m-1)$$

for $0 < m < n$. Prove that all the $d(n,m)$ are integers.

Great Britain 3. Find, with proof, the smallest real number c with the following property: for every sequence $\{X_i\}$ of positive real numbers such that

$$X_1 + X_2 + \ldots + X_n \leq X_{n+1} \quad \text{for} \quad n = 1,2,3,\ldots,$$

we have

$$\sqrt{X_1} + \sqrt{X_2} + \ldots + \sqrt{X_n} \leq c\sqrt{X_1 + X_2 + \ldots + X_n}$$

for $n = 1,2,3,\ldots$. [c is to be independent of the X_i and independent of n.]

Greece 1. Consider the regular 1987-gon $A_1A_2\ldots A_{1987}$ with center O. Show that the sum of vectors belonging to any proper subset of $M = \{OA_j\colon\ j = 1,2,\ldots,1987\}$ is nonzero.

Greece 2. Solve the equation

$$28^x = 19^y + 87^z$$

where x, y, z are integers.

Holland 1. Given 5 real numbers u_0, u_1, u_2, u_3, u_4, prove that it is always possible to find 5 real numbers v_0, v_1, v_2, v_3, v_4 satisfying the following conditions:

(i) $u_i - v_i$ is an integer for each i;

(ii) $\sum_{0 \leq i < j \leq 4} (v_i - v_j)^2 < 4$.

Hungary 1. Does there exist a set M in the usual Euclidean space such that for any plane σ, the intersection $M \cap \sigma$ is finite and non-empty?

Iceland 1. Let S_1 and S_2 be two spheres with distinct radii which touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally as well as the two neighbouring solid spheres. What are the possible values of n?

Morocco 1. Let $\theta_1, \theta_2, \ldots, \theta_n$ be real numbers such that

$$\sin\theta_1 + \sin\theta_2 + \ldots + \sin\theta_n = 0.$$

Prove that

$$|\sin \theta_1 + 2 \sin \theta_2 + \dots + n \sin \theta_n| \leq [n^2/4],$$

where [X] is the integer part of X.

<u>**Poland 1**</u>. Find the number of partitions of the set $\{1,2,\dots,n\}$ into three subsets A_1, A_2, A_3, some of which may be empty, such that the following conditions are satisfied:

(i) after the elements of each subset have been put in ascending order, every two consecutive elements of any subset have different parity;

(ii) if A_1, A_2 and A_3 are all non-empty, then in exactly one of them the smallest number is even.

Remark. A partition is determined by a family of sets A_1, A_2, A_3 such that

$$A_1 \cup A_2 \cup A_3 = \{1,2,\dots,n\} \quad \text{and} \quad A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \phi;$$

another ordering of the sets, e.g. A_2, A_3, A_1, gives the *same* partition as A_1, A_2, A_3.

<u>**Poland 2**</u>. Let P, Q, R be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers p, q, r and a polynomial S such that $P = pS$, $Q = qS$ and $R = rS^2$.

<u>**Rumania 1**</u>. Show that the numbers 1,2,...,1987 can be coloured using 4 colours so that no arithmetical progression with 10 terms has all its members coloured the same.

<u>**Spain 1**</u>. Determine, with justification, the integer solutions of the equation

$$3z^2 = 2x^3 + 385x^2 + 256x - 58195.$$

<u>U.S.A. 1</u>. Let $r > 1$ be a real number, and let n be the largest integer less than r. Consider an arbitrary real number x with $0 \leq x \leq n/(r-1)$. By a base r expansion of x, we mean a representation of x in the form

$$x = \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \dots ,$$

where the a_i are integers with $0 \leq a_i < r$. You may assume, without proof, that every number x in the interval $0 \leq x \leq n/(r-1)$ has at least one base r expansion.

Prove that, if r is not an integer, then there exists a number p as above which has infinitely many distinct base r expansions.

<u>Vietnam 1</u>. Can a rectangular courtyard of dimension $m \times n$ be covered with tiles composed of 1×1 squares in the form of an L () if

(a) $m \times n = 1985 \times 1987$?

(b) $m \times n = 1987 \times 1989$?

<u>West Germany 1</u>. How many words with n digits can be formed from the alphabet $\{0,1,2,3,4\}$, if neighbouring digits must differ by exactly one?

<u>Yugoslavia 1</u>. Find the least number k such that for any $a \in [0,1]$ and any natural number n,

$$a^k(1 - a)^n < \frac{1}{(n + 1)^3}$$

is valid.

*

I must apologize for not knowing the countries of origin for the following problems.

<u>X 1</u>. It is given that $x = -2272$, $y = 10^3 + 10^2c + 10b + a$ and $z = 1$ satisfy the equation

$$ax + by + cz = 1$$

where a, b, c are positive integers with $a < b < c$. Find y.

<u>X 2</u>. Let PQ be a line segment of fixed length but variable position on the side BC of a triangle ABC, with the order $BPQC$, and let the lines through P, Q parallel to the lateral sides meet AC, AB at P_1, Q_1 and P_2, Q_2, respectively. Prove that the sum of the areas of the trapezoids PQQ_1P_1 and PQQ_2P_2 is independent of the position of PQ on BC.

<u>X 3</u>. Let ℓ, ℓ' be two lines in 3-space, and let A, B, C be three points taken on ℓ with B as midpoint of the segment AC. If a, b, c are the distances of A, B, C from ℓ', respectively, show that

$$b \le \sqrt{\frac{a^2 + c^2}{2}}$$

with equality holding if ℓ, ℓ' are parallel.

<u>X 4</u>. Compute $\sum_{k=0}^{2n} (-1)^k a_k$ where the a_k's are the coefficients in the expansion $(1 - \sqrt{2}x + x^2)^n = \sum_{k=0}^{2n} a_k x^k$.

* * *

We finish this article by presenting solutions submitted to problems posed in early numbers of the Corner. It seems that my call for solutions has spurred some activity, as is the case with the next few solutions to problems from 1984. In the next articles I am going to try to intermingle recently received solutions with those from the backlog so that you don't have to wait too long to see your name in print.

F.2439. [1984: 76] *From Közepiskolai Matematikai Lapok 67 (1983) 80.*

A regular pyramid has a square base of edge length e, and θ is the dihedral angle between adjacent lateral faces. Find the radius of the sphere internally tangent to the four lateral faces, and also the radius of the circumsphere of the pyramid.

Apply the results to the case where the lateral faces of the pyramid are equilateral triangles.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let the pyramid be labelled $PABCD$, with $ABCD$ the square base. Let P' be the center of the square. Let M be the midpoint of the segment AD and let E be the foot of the perpendicular from A on PD, so $AE \perp DE$. By symmetry $CE \perp DE$ as well. Since $PD \perp AE$ and $PD \perp CE$, PD is perpendicular to the plane of AEC. Thus $\angle AEC = \theta$, and, since P' lies on the plane AEC (indeed, on the segment AC), $\Delta PEP'$ and $\Delta DEP'$ are right triangles. From $\Delta P'EC$

$$EC = (P'C)\csc(\theta/2) = (e/\sqrt{2})\csc(\theta/2).$$

Then

$$(DE)^2 = (DC)^2 - (EC)^2 = e^2 - ((e/\sqrt{2})\csc(\theta/2))^2 = e^2(1 - 1/2\csc^2(\theta/2)).$$

Now, since $\Delta PP'D \sim \Delta P'ED$, we have

$$PD = \frac{(P'D)^2}{ED} = \frac{e^2/2}{\sqrt{e(1 - 1/2\csc^2(\theta/2))}} = \frac{e}{\sqrt{(4 - 2\csc^2(\theta/2))}}.$$

Continuing,

$$(P'P)^2 = (PD)^2 - (P'D)^2 = \frac{e^2}{4 - 2\csc^2(\theta/2)} - \frac{e^2}{2}$$

$$= \frac{e^2(1 - \sin^2(\theta/2)}{2(2\sin^2(\theta/2) - 1)} = \frac{e^2\cos^2(\theta/2)}{-2\cos\theta}$$

or

$$P'P = (e/\sqrt{2})\cos(\theta/2)\sqrt{-\sec\theta}. \qquad (*)$$

Then

$$PM = \sqrt{(PP')^2 + (P'M)^2} = \sqrt{(e^2/2)\cos^2(\theta/2)(-\sec\theta) + e^2/4}$$
$$= (e/2)\sqrt{1 - 2\cos^2(\theta/2)\sec\theta} = (e/2)\sqrt{1 - (1 + \cos\theta)\sec\theta}$$
$$= (e/2)\sqrt{-\sec\theta}.$$

Clearly, both the circumcenter O and the incenter I (center of the inscribed circle tangent to the four faces) lie on the line PP'. Let I' be the point where the insphere touches face PAD. Then, by symmetry, I' lies on PM, and $II'\perp PM$. By the similar triangles $\Delta PI'I$ and $\Delta PP'M$ we have

$$\frac{I'I}{PI} = \frac{P'M}{PM} = \frac{e/2}{(e/2)\sqrt{-\sec\theta}} = \frac{1}{\sqrt{-\sec\theta}}.$$

It is clear that there are infinitely many such spheres unless we also assume that the sphere be tangent to the base. This means $I'I = P'I$ so that $r = I'I$ satisfies

$$\frac{r}{PP' - r} = \frac{1}{\sqrt{-\sec\theta}}.$$

Solving for r and using (*) we obtain

$$r = \frac{e}{\sqrt{2}}\cdot\cos\frac{\theta}{2}\cdot\sqrt{-\sec\theta}\cdot\frac{1}{1 + \sqrt{-\sec\theta}} = \frac{e}{\sqrt{2}}\cdot\cos\frac{\theta}{2}\cdot\frac{1}{1 + \sqrt{-\cos\theta}}.$$

Turning to the circumcircle, we see that since $OP = OM$ = circumradius,

$$OP^2 = OM^2 = (P'M)^2 + (P'O)^2 = e^2/4 + (PP' - OP)^2$$

so,

$$\text{circumradius} = OP = \frac{e^2/4 + (PP')^2}{2PP'} = \frac{e^2/4 + (e^2/2)\cos^2(\theta/2)(-\sec\theta)}{(e/\sqrt{2})\cos(\theta/2)\ \sqrt{-\sec\theta}}$$
$$= \frac{e}{4\sqrt{2}}\cdot\frac{1}{\cos(\theta/2)\sqrt{-\cos\theta}}.$$

ii) If we suppose that each lateral face of the pyramid is an equilateral triangle, then

$$EC = (CD)\sin 60^\circ = \frac{\sqrt{3}}{2}e \qquad \text{so} \qquad \sin(\theta/2) = \frac{e/\sqrt{2}}{EC} = \frac{\sqrt{6}}{3}.$$

Hence $\cos(\theta/2) = \sqrt{3}/3$ and $\cos\theta = -1/3$. From the above

$$\text{inradius} = \left[\frac{\sqrt{6} - \sqrt{2}}{4}\right]e \qquad \text{and} \qquad \text{circumradius} = \frac{3\sqrt{2}}{8}e.$$

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1. [1984: 108] *Austrian-Polish Mathematics Competition, 1982.*

Determine all pairs of natural numbers (n,k) such that

$$\text{g.c.d.}((n + 1)^k - n, (n + 1)^{k+3} - n) > 1.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

The given g.c.d. can also be written as

$$\text{g.c.d.}((n+1)^k - n, ((n+1)^{k+3} - n) - ((n+1)^k - n))$$
$$= \text{g.c.d.}((n+1)^k - n, (n+1)^k((n+1)^3 - 1))$$
$$= \text{g.c.d.}((n+1)^k - n, (n+1)^3 - 1),$$

since clearly $\text{g.c.d.}((n+1)^k - n, (n+1)^k) = 1$.

If this g.c.d. is greater than 1, then there exists a prime p such that

(i) $(n+1)^k \equiv n \bmod p$ and

(ii) $(n+1)^3 \equiv 1 \bmod p$.

Suppose $k \equiv 1 \bmod 3$; then, by (i) and (ii)

$$n \equiv [(n+1)^3]^{\frac{k-1}{3}}(n+1) \equiv n + 1 \bmod p,$$

or $0 \equiv 1 \bmod p$, which is absurd.

Suppose $k \equiv 2 \bmod 3$. Then we have

$$n \equiv [(n+1)^3]^{\frac{k-2}{3}}(n+1)^2 \equiv n^2 + 2n + 1 \bmod p, \qquad \text{or}$$

$$n^2 + n + 1 \equiv 0 \bmod p.$$

Also

$$1 \equiv (n+1)^2(n+1) \equiv n(n+1) \equiv n^2 + n \equiv -1 \bmod p.$$

This gives $p = 2$, but then $n^2 + n + 1 \equiv 0 \bmod p$ is impossible.

Finally, suppose $k \equiv 0 \bmod 3$. Then,

$$n \equiv [(n+1)^3]^{k/3} \equiv 1 \bmod p,$$
$$1 \equiv (n+1)^3 = 2^3 \bmod p.$$

Thus $p = 7$. So $n \equiv 1 \bmod 7$. Conversely $7|(n+1)^k - n$ and $7|(n+1)^3 - n$ if $n \equiv 1 \bmod 7$ and $k \equiv 0 \bmod 3$.

Hence all admissible pairs are given by

$$\{(n,k): n,k \in \mathbb{N}\ ,\ n \equiv 1 \bmod 7\ ,\ k \equiv 0 \bmod 3\}.$$

6. [1984: 108] *Austrian-Polish Mathematics Competition, 1982.*

Let a be a fixed natural number. Find all functions f defined on the set D of natural numbers $x \geq a$ and satisfying the functional equation

$$f(x + y) = f(x)\cdot f(y)$$

for all $x,y \in D$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Suppose $f(a) = 0$; then $f(a + k) = f(a)f(k) = 0$ for $k \geq a$, so $f(x) = 0$ for $x \geq 2a$. Then for any $x \geq a$, $(f(x))^2 = f(2x) = 0$, so $f(x) = 0$ identically on

D. Thus, we now assume that $f(a) \neq 0$. Setting first $(x,y) = (a,a+k+1)$ and then $(x,y) = (a+1,a+k)$ in the functional equation gives

$$f(2a+k+1) = f(a)f(a+k+1) = f(a+1)f(a+k)$$

or

$$f(a+k+1) = \frac{f(a+1)}{f(a)}f(a+k) \qquad \text{for} \qquad k \geq 0.$$

An easy induction gives

$$f(a+k) = \frac{(f(a+1))^k}{(f(a))^{k-1}}, \qquad k \geq 0.$$

In particular

$$f(2a) = \frac{(f(a+1))^a}{(f(a))^{a-1}} = f(a+a) = (f(a))^2,$$

or

$$f(a+1) = [f(a)]^{\frac{a+1}{a}}\omega$$

where ω is an ath root of unity. Thus

$$f(a+k) = (f(a))^{\frac{a+k}{a}}\omega^k,$$

or $f(n) = [f(a)]^{n/a}\omega^n$, $n \geq 0$. (Note that the previous solution $f(n) \equiv 0$ is also of this form). Conversely, such an f satisfies the given equation, by direct verification.

T1. [1984: 108] *Austrian-Polish Mathematics Team Competition, 1982.*

Determine all triplets (x,y,z) of natural numbers, with z as small as possible, for which there exist natural numbers a, b, c, d satisfying

(i) $x^y = a^b = c^d$ with $x > a > c$,

(ii) $z = ab = cd$,

(iii) $x + y = a + b$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

From (i) and (ii)

$$c^{ad} = (a^b)^a = a^{cd}, \qquad \text{so} \qquad \frac{\ln a}{a} = \frac{\ln c}{c}.$$

The function $\frac{\ln x}{x}$ is monotonic on the intervals $[1,e)$ and (e,∞). Since $c < a$ we must have $1 \leq c < e$. But $c = 1$ implies $a = 1$, which is contradictory. Thus $c = 2$. The equation $\frac{\ln a}{a} = \frac{\ln 2}{2}$ has a unique solution in (e,∞), which by inspection is $a = 4$.

By (i) $x^y = 4^b$ so x is a power of 2, say $x = 2^k$. Then

$$y = \frac{b\ \log_2 4}{\log_2 x} = \frac{2b}{k}.$$

Since $2^k = x > a = 4$, $k > 2$.

By (iii) $x + y = 2^k + 2b/k = 4 + b$ or

$$b = 2^k - 4 + \frac{2(2^k - 4)}{k - 2}. \qquad (*)$$

By (ii) $z = 4b$. Thus we wish to minimize b. For $k = 3$, $(*)$ gives $b = 12$. For $k \geq 4$, $b > 2^k - 4 \geq 12$. Hence we choose $k = 3$. Then $x = 2^3 = 8$, $y = 2(12)/3 = 8$, $z = 4(12) = 48$, and the unique solution to the problem is

$$(x,y,z) = (8,8,48).$$

T2. [1984: 108] *Austrian-Polish Mathematics Team Competition, 1982.*

Point X is in the interior of a given regular tetrahedron $ABCD$ of edge length 1. If $d(X,YZ)$ denotes the shortest distance from X to edge YZ, prove that

$$d(X,AB) + d(X,AC) + d(X,AD) + d(X,BC) + d(X,BD) + d(X,CD) \geq 3/\sqrt{2},$$

and show that equality holds if and only if X is the center of the tetrahedron.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let R and S be the midpoints of AC and BD, respectively. Let P be any point on AC, and Q be any point on BD.

Now AC is perpendicular to both DR and BR and hence to any line in plane DRB. In particular, $QR \perp AC$ and so $QP \geq QR$, with equality iff $P = R$. Similarly, $RS \perp BD$, so $QR \geq SR$ with equality iff $Q = S$. Thus, $QP \geq SR$ with equality just in case $P = R$ and $Q = S$.

Now suppose P' is the point on AC closest to X and Q' is on BD and closest to X. Then, from the triangle inequality,

$$d(X,AC) + d(X,BD) = XP' + XQ' \geq P'Q' \geq SR$$

with equality iff X lies on $P'Q'$ and $P' = R$, $Q' = S$, i.e. just in case X lies on RS. But

$$RS = \sqrt{(RB)^2 - (SB)^2} = \sqrt{(BC)^2 - (RC)^2 - (SB)^2} = 1/\sqrt{2}.$$

By symmetry, we conclude that the sum of distances in question is greater than or equal to $3/\sqrt{2}$ with equality just in case X lies on each of the three lines joining the midpoints of opposite edges. There is at most one such point X, and the center of the tetrahedron is such a point. [In vector notation the center is represented by

$$\frac{\vec{A} + \vec{B} + \vec{C} + \vec{D}}{4} = \frac{1}{2}\left[\frac{\vec{A} + \vec{C}}{2}\right] + \frac{1}{2}\left[\frac{\vec{B} + \vec{D}}{2}\right] = \frac{\vec{R} + \vec{S}}{2} = \text{midpoint of } RS, \text{ etc.}]$$

T3. [1984: 109] *Austrian-Polish Mathematics Team Competition, 1982.*

Let

$$S_n = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\sqrt{j^2 + k^2}}.$$

Determine a real constant c such that, for all natural numbers $n \geq 3$,

$$n \leq S_n \leq cn.$$

Remark. The smaller the number c determined, the greater will be the number of points awarded for the solution.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Partition the unit square $[0,1] \times [0,1]$ into n^2 subsquares by the partition points

$$\{(j/n,k/n): 1 \leq j,k \leq n\}.$$

Then a Riemann sum of the function $(x^2 + y^2)^{-1/2}$ for this partition is

$$\sum_{1 \leq j,k \leq n} (j^2/n^2 + k^2/n^2)^{-1/2}(1/n^2) = S_n/n.$$

Also $(x^2 + y^2)^{-1/2} \geq (j^2/n^2 + k^2/n^2)^{-1/2}$ on $\left[\frac{j-1}{n},\frac{j}{n}\right] \times \left[\frac{k-1}{n},\frac{k}{n}\right]$, with equality only at $(x,y) = (j/n,k/n)$. Therefore

$$\frac{S_n}{n} < \iint_{[0,1]\times[0,1]} (x^2 + y^2)^{-1/2}dx\,dy$$

and

$$\lim_{n\to\infty} \frac{S_n}{n} = \iint_{[0,1]\times[0,1]} (x^2 + y^2)^{-1/2}dx\,dy$$

giving that the smallest value of c is

$$\iint_{[0,1]\times[0,1]} (x^2 + y^2)^{-1/2}dx\,dy.$$

This integral is easily evaluated by using polar coordinates. By symmetry

$$c = 2 \iint_{0\leq y\leq x\leq 1} (x^2 + y^2)^{-1/2}dx\,dy = 2\int_0^{\pi/4}\int_0^{\sec\theta} (r^2)^{-1/2}r\,dr\,d\theta$$

$$= 2\int_0^{\pi/4} \sec\theta\,d\theta = 2\,\ell n(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = 2\,\ell n(\sqrt{2} + 1).$$

[*Editor's remark:* What about solutions not involving the calculus?]

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

<u>1271</u>. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated in memoriam to Léo Sauvé.)

Prove that

$$\sqrt{3}\ \Sigma \sin A_1/2 \geq 4\ \Sigma \sin B_1 \sin A_2/2 \sin A_3/2,$$

where $A_1A_2A_3$ and $B_1B_2B_3$ are two triangles and the sums are cyclic over their angles.

<u>1272</u>. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle. Let the incircle have center I and radius ρ, and meet the sides of the triangle at points P_1, P_2, P_3. Let I_1, I_2, I_3 be the excenters and ρ_1, ρ_2, ρ_3 the exradii. Prove that

(a) the lines I_1P_1, I_2P_2, I_3P_3 concur at a point S;

(b) the distances d_1, d_2, d_3 of S to the sides of the triangle satisfy

$$d_1:d_2:d_3 = \rho_1:\rho_2:\rho_3.$$

<u>1273</u>. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle, M an interior point, and $A'B'C'$ its pedal triangle. Denote the sides of the two triangles by a, b, c and a', b', c' respectively. Prove that

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} < 2.$$

<u>1274</u>. *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Let spheres S_1, S_2 be inscribed in a cone τ. Let a line L, not a lateral element of τ, touch S_1 and S_2, and meet τ at P and Q. Show that the length of PQ is independent of L.

1275. *Proposed by P. Penning, Delft, The Netherlands.*

On a circle C with radius R three points A_1, A_2, A_3 are chosen arbitrarily. Prove that the three circles with radius R, not coinciding with C, and passing through two of the points A_1, A_2, A_3, intersect in the orthocentre of $\Delta A_1A_2A_3$.

1276. *Proposed by Ernst v. Heydebrand, Heidenheim, Federal Republic of Germany.*

C is the right angle and X the midpoint of the hypotenuse of a nonisosceles right triangle. The incircle of the triangle touches the hypotenuse at Y, and the line CY meets the perpendicular bisector of the hypotenuse at Z. Show that $\overline{XZ} = s$, the semiperimeter of the triangle.

1277. *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all possible values of the expression

$$x_1x_2 + x_2x_3 + \dots + x_nx_1$$

where $n \geq 2$ and $x_i = 1$ or -1 for each i.

1278. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Find a non-constant function $f(x,y)$ such that $f(ab+a+b,c)$ is symmetric in a, b, and c.

(b)* Find a non-constant function $g(x,y)$ such that $g(ab(a+b),c)$ is symmetric in a, b, and c.

1279. *Proposed by Jordi Dou, Barcelona, Spain.*

Consider a triangle whose orthocentre lies on its incircle.

(a) Show that if one of its angles is given, the others are determined.

(b) Show that if it is isosceles, then its sides are in the proportion 4:3:3.

1280. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a triangle and let A_1, B_1, C_1 be points on BC, CA, AB, respectively, such that

$$\frac{A_1C}{BA_1} = \frac{B_1A}{CB_1} = \frac{C_1B}{AC_1} = k > 1.$$

Show that

$$\frac{k^2 - k + 1}{k(k + 1)} < \frac{\text{perimeter}(A_1B_1C_1)}{\text{perimeter}(ABC)} < \frac{k}{k + 1},$$

and that both bounds are best possible.

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1100. [1985: 326; 1987: 160] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with $C = 30^\circ$, circumcentre O and incentre I. Points D and E are chosen on BC and AC, respectively, such that $BD = AE = AB$. Prove that $DE = OI$ and $DE \perp OI$.

Editor's comment.

JORDI DOU, Barcelona, Spain, kindly points out that his proposal *Crux* 1196 [1986: 282] was intended as a generalization of *Crux* 1100, and that the published solution I of *Crux* 1100 [1987: 160], by J.T. Groenman, in fact proves this generalization. The editor was indeed informed of the connection between problems 1100 and 1196 at the time of preparing the latter for publication, but due to holiday merriment, or spots on his glasses, or both, such connection failed to register on his brain.

Dou should of course be listed among the solvers of *Crux* 1100. A separate solution of his, sent prior to his proposal of *Crux* 1196, was apparently either mislaid here or lost in the mail on the way.

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1119. [1986: 27] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The following problem, for which I have been unable to locate the source, has been circulating around DEC. A rectangle is partitioned into smaller rectangles. If each of the smaller rectangles has the property that one of its sides has integral length, prove that the original rectangle also has this property.

I. *Solution by Beno Arbel, Tel Aviv University, Tel Aviv, Israel, with small amendments by the editor.*

We must first note that the sides of the smaller rectangles are parallel to the sides of the original one. Let us choose a rectangular coordinate system and the original rectangle $ABCD$ with sides parallel to the x and y axes. Let $MNPQ$, with $M(a,b)$, $N(c,b)$, $P(c,d)$, $Q(a,d)$, be a representative "small" rectangle, a component of $ABCD$ (it may be $ABCD$ itself). Now we take the double integral of the function

$$f(x,y) = \cos 2\pi x\cdot\cos 2\pi y$$

on *MNPQ*. We get

$$I = \iint_{MNPQ} \cos 2\pi x\cdot\cos 2\pi y\ dxdy = \int_{MN} \cos 2\pi x\ dx \cdot \int_{PQ} \cos 2\pi y\ dy$$

$$= \frac{\sin 2\pi x}{2\pi}\Bigg|_a^c \cdot \frac{\sin 2\pi y}{2\pi}\Bigg|_b^d$$

$$= \frac{(\sin 2\pi c - \sin 2\pi a)(\sin 2\pi d - \sin 2\pi b)}{4\pi^2}.$$

Without loss of generality, $c - a$ is an integer n, so

$$\sin 2\pi c = \sin(2\pi a + 2\pi n) = \sin 2\pi a,$$

and hence $I = 0$. From the additive property of the integral, it follows that

$$\iint_{ABCD} \cos 2\pi x\cdot\cos 2\pi y\ dxdy = 0,$$

so that at least one of the sides of *ABCD* has integral length.

II. *Editor's comment.*

Upon seeing this problem in the February 1986 *Crux*, Stan Wagon kindly sent me a manuscript he had recently written, containing numerous, often wildly different, proofs of this same problem! None of these proofs were due to him (although he later submitted one of his own), but to N.G. de Bruijn, Sherman K. Stein, Raphael Robinson, Paul Seymour, Mihalis Yannakakis, Attila Maté, and others. The above solution of Arbel is essentially proof #2 of this paper. The paper also contains some generalizations. It has been published in the *American Math. Monthly* Vol.94 (1987) 601-617 with the title "Fourteen proofs of a result about tiling a rectangle", and is well worth a look.

Also solved by PETER GILBERT, Digital Equipment Corp., Nashua, New Hampshire; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; ESTHER SZEKERES, Turramurra, Australia; and STAN WAGON, Smith College, Northampton, Massachusetts.

Klamkin pointed out that the proposal also appeared as problem 18.9, page 92 of Mathematics Spectrum *18 (1985-86).*

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<u>1144</u>. [1986: 107] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let *ABC* be a triangle and *P* an interior point at distances x_1, x_2,

x_3 from the vertices A, B, C and distances p_1, p_2, p_3 from the sides BC, CA, AB, respectively. Show that

$$\frac{x_1x_2}{ab} + \frac{x_2x_3}{bc} + \frac{x_3x_1}{ca} \geq 4\left[\frac{p_1p_2}{ab} + \frac{p_2p_3}{bc} + \frac{p_3p_1}{ca}\right].$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We will use the different but more standard terminology

$$x_i = R_i,\ p_i = r_i, \qquad (a,b,c) = (a_1,a_2,a_3)$$

so that the inequality to be proved becomes

$$\frac{R_1R_2}{a_1a_2} + \frac{R_2R_3}{a_2a_3} + \frac{R_3R_1}{a_3a_1} \geq 4\left[\frac{r_1r_2}{a_1a_2} + \frac{r_2r_3}{a_2a_3} + \frac{r_3r_1}{a_3a_1}\right].$$

In fact we show that

$$a_1R_2R_3 + a_2R_3R_1 + a_3R_1R_2 \geq a_1a_2a_3 \geq 4(a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2). \qquad (1)$$

The left hand inequality is known and a short history and derivation of it is given in [1] (also see (5) on [1985: 279]). The right hand inequality is also known [2] and follows from the identity

$$4(a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2) = a_1a_2a_3\left[1 - \frac{(OP)^2}{R^2}\right],$$

where O is the circumcenter and R the circumradius. There is equality in the right hand inequality of (1) if $P = O$, so that there is equality in the proposed inequality if and only if the triangle is equilateral and P is its geometric center.

A known dual inequality [1] to the left hand inequality of (1) is the polar moment of inertia inequality

$$a_1R_1^2 + a_2R_2^2 + a_3R_3^2 \geq a_1a_2a_3,$$

and this has been generalized [3] to

$$a_1R_1R_1' + a_2R_2R_2' + a_3R_3R_3' \geq a_1a_2a_3$$

where P and P' are arbitrary points, not restricted to the interior or even the plane of the triangle, and $R_i' = P'A_i$.

The following dual inequalities to the right hand inequality of (1) are also given in [2]:

$$Fk^2 \geq 4Rk(a_1r_1R_1^2 + a_2r_2R_2^2 + a_3r_3R_3^2),$$

$$R(a_1r_2r_3 + a_2r_3r_1 + a_3r_1r_2)^2 \geq 8kF^2.$$

Here F is the area of the triangle, $K = R_1R_2R_3$ and $k = r_1r_2r_3$.

References:

[1] M.S. Klamkin, Triangle inequalities from the triangle inequality, *Elem. der Math.* 34 (1979) 49-55.

[2] M.S. Klamkin, An identity for simplexes and related inequalities, *Simon Stevin* 48 (1974-75) 57-64.

[3] M.S. Klamkin, Problem 77-10, *SIAM Review* 20 (1978) 400-401.

Also solved by WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

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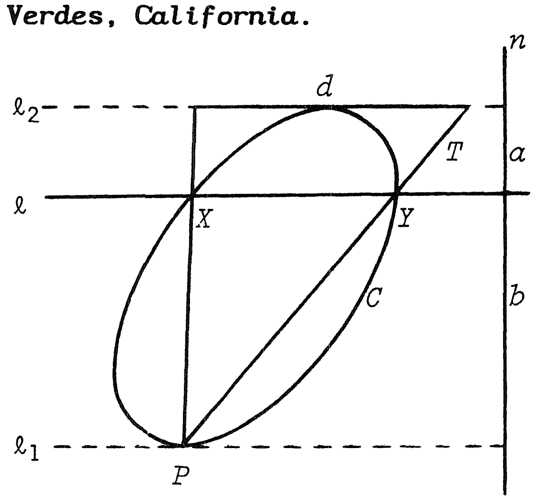
<u>1145</u>. [1986: 107] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given a plane convex figure and a straight line ℓ (in the same plane) which splits the figure into two parts whose areas are in the ratio $1:t$ ($t \geq 1$). These parts are then projected orthogonally onto a straight line n perpendicular to ℓ. Determine, in terms of t, the maximum ratio of the lengths of the two projections.

Solution by Richard I. Hess, Rancho Palos Verdes, California.



Given a convex region C, let the line ℓ divide the region into two parts of areas A and B, having projections onto n of lengths a and b, respectively. Then we want to maximize b/a subject to $B/A = t \geq 1$. Equivalently, we wish to minimize B/A subject to $b/a = s \geq 1$.

Let C intersect ℓ in two points X and Y. Let ℓ_1 and ℓ_2 be lines parallel to ℓ and touching C at either end as shown, and let P be any point of C on ℓ_1. Form a triangle T with one vertex P and opposite side on ℓ_2 by extending PX and PY to ℓ_2. Then the values of a and b for T are the same as for C, while A has increased to A' and B has decreased to B'. (Possibly A = A' and/or $B = B'$.) Thus for fixed $s = b/a$, $t = B/A$ is minimized when the convex region is a triangle and the line ℓ is parallel to one of the sides. For this case,

$$A' + B' = \frac{d}{2}(a + b),$$

$$B' = \frac{d}{2}\cdot\frac{b}{a + b}\cdot b,$$

and so

$$A' = \frac{d}{2}\cdot\frac{a^2 + 2ab}{a + b}$$

where d is the length of the side along ℓ_2. Thus

$$t = \frac{B'}{A'} = \frac{b^2}{a^2 + 2ab} = \frac{s^2}{1 + 2s}\ .$$

Solving, we get that

$$s = t + \sqrt{t(t+1)}$$

is the maximum value of b/a given $t = B/A$.

Also solved by DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

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<u>1146</u>. [1986: 107] *Proposed by Jordi Dou, Barcelona, Spain.*

Let AD, BE, CF be cevians of ΔABC and V the foot of the bisector of $\angle A$. Prove that the conic through $DEFAV$ is perpendicular to AV at A.

Solution by Dan Pedoe, University of Minnesota, Minneapolis, Minnesota.

First a little notation, and the statements of three well-known theorems.

If the points P, Q, R are collinear, we write the position-ratio $\overline{PQ}/\overline{QR}$ of Q with respect to P and R as $(PR)_Q$. Now suppose that D, D', E, E', F, F' are points on the respective sides BC, CA, AB of triangle ABC, not coinciding with vertices. Then we recall:

Ceva's Theorem: AD, BE and CF are concurrent if and only if

$$(BC)_D(CA)_E(AB)_F = 1.$$

Menelaus' Theorem: D', E' and F' are collinear if and only if

$$(BC)_{D'}(CA)_{E'}(AB)_{F'} = -1.$$

Carnot's Theorem: The six points lie on a conic if and only if

$$(BC)_D(BC)_{D'}(CA)_E(CA)_{E'}(AB)_F(AB)_{F'} = 1.$$

Assume that D, D', E, E', F, F' lie on a conic, and that AD, BE and CF are *concurrent* cevians (this corrects the statement of the problem). Then from Ceva and Carnot we have

$$(BC)_{D'}(CA)_{E'}(AB)_{F'} = 1.$$

Let the line $E'F'$ intersect BC in G. Then from Menelaus,

$$(BC)_G(CA)_{E'}(AB)_{F'} = -1,$$

and therefore

$$(BC)_{D'} = -(BC)_G.$$

The points D' and G are therefore harmonic conjugates with respect to B and C. If AD' is the internal bisector of angle BAC, then AG is the external bisector, that is, $AG \perp AD'$.

To obtain the conic of the problem we consider the limiting case, when E' and F' approach A. Then the line $E'F'$ becomes the tangent to the conic at A, so AD' is perpendicular to the conic at A.

Also solved by J.T. GROENMAN, *Arnhem, The Netherlands;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria; and the proposer.*

*   *   *

<u>1147</u>. [1986: 108] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is shown in Z.A. Melzak's *Companion to Concrete Mathematics II* (Wiley Interscience, N.Y., 1976, p.81) that

$$\int_0^\infty \frac{2 - 2\cos u - u\sin u}{u^4}\,du = \frac{\pi}{12},$$

and it is noted that "this is quite simply obtained by residues and complex integration, but it is not quite so simple to obtain by real-variable methods alone". Obtain this result by real-variable methods alone.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let I denote the improper integral. Using integration by parts we obtain

$$I = -\frac{1}{3}\cdot\frac{2 - 2\cos u - u\sin u}{u^3}\Bigg|_0^\infty + \frac{1}{3}\int_0^\infty \frac{\sin u - u\cos u}{u^3}\,du.$$

By L'Hospital's rule we find

$$\lim_{u\to 0^+} \frac{2 - 2\cos u - u\sin u}{u^3} = \lim_{u\to 0^+} \frac{\sin u - u\cos u}{3u^2} = \lim_{u\to 0^+} \frac{u\sin u}{6u} = 0.$$

Thus

$$I = \frac{1}{3}\int_0^\infty \frac{\sin u - u\cos u}{u^3}\,du,$$

and another application of integration by parts yields

$$I = -\frac{1}{6}\lim_{u\to 0^+} \frac{\sin u - u\cos u}{u^2} + \frac{1}{6}\int_0^\infty \frac{u\sin u}{u^2}\,du$$

$$= \frac{1}{6}\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{12}.$$

Whether the above proof can be considered as "by real-variable methods alone" depends on whether one accepts the identity

$$\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}$$

which usually is obtained by complex contour integral and residue calculus.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Penn State University, Middletown, Pennsylvania; ROBERT E. SHAFER, Berkeley, California; and the proposer.

Most solvers mentioned that there are real-variable methods for doing the last integral above; the proposer referred to pp.189-195 of J. Edwards, A Treatise on the Integral Calculus II, *Chelsea, N.Y., 1954.*

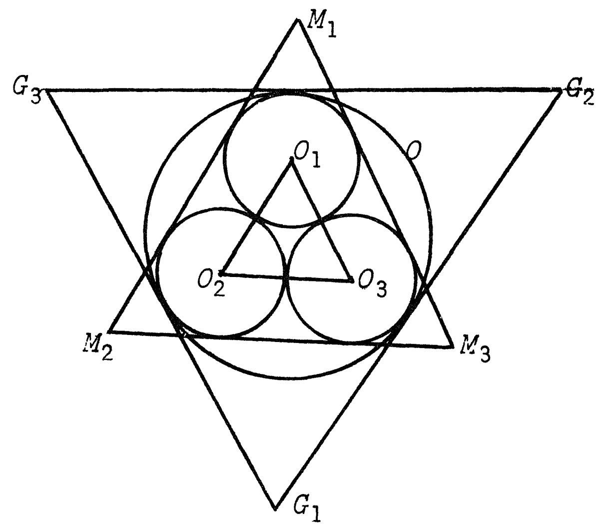
* * *

1150*. [1986: 108] *Proposed by Jack Garfunkel, Flushing, N.Y.*

In the figure, $\Delta M_1M_2M_3$ and the three circles with centers O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles and the sides of triangle $G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that

$$P(\Delta G_1G_2G_3) \geq P(\Delta M_1M_2M_3) + P(\Delta O_1O_2O_3),$$

where P stands for perimeter. Equality is attained when $\Delta O_1O_2O_3$ is equilateral.



Editor's comment.

No solutions for this problem have been received. A similar fate befell *Crux* 1077 [1987: 93]. It seems "Malfatti configuration" problems are tough! Maybe somebody out there would like to give this one another try.

* * *

1151*. [1986: 138] *Proposed by Jack Garfunkel, Flushing, N.Y.*

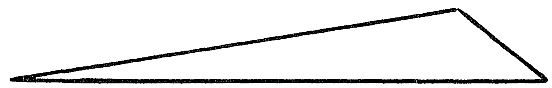
Prove (or disprove) that for an obtuse triangle ABC,

$$m_a + m_b + m_c \leq s\sqrt{3}$$

where m_a, m_b, m_c denote the medians to sides a, b, c and s denotes the semiperimeter of ΔABC. Equality is attained in the equilateral triangle.

I. *Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

The inequality is *not* universally valid. Consider an obtuse triangle in which two side-lengths are very close to 2, and the third side is very short.



Then $m_a + m_b + m_c$ is close to $2 + 1 + 1 = 4$, and s is close to 2. But $4 > 2\sqrt{3}$, so if "very close" and "very short" are close enough and short enough, respectively, the above inequality cannot hold.

II. *Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

[Klamkin first gave more or less the same counter-example. - Ed.]

It is known that

$$\frac{3s}{2} < m_a + m_b + m_c < 2s$$

(see 8.1 of O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Gröningen, 1969). The above counterexample shows that the upper bound proportionality constant 2 cannot be decreased even for obtuse triangles. As the lower bound is attained for degenerate obtuse triangles of sides $2a$, $2a$, $4a$, it too is sharp for obtuse triangles.

Also solved by SVETOSLAV BILCHEV and EMILIA VELIKOVA, Russe, Bulgaria; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; VEDULA N. MURTY, Penn State University, Middletown, Pennsylvania; and G.R. VELDKAMP, De Bilt, The Netherlands. There was one incorrect solution submitted.

* * *

<u>1152</u>. [1986: 138] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\sum \cos\frac{\alpha}{2} \le \frac{\sqrt{3}}{2}\sum \cos\frac{1}{4}(\beta - \gamma)$$

where α, β, γ are the angles of a triangle and the sums are cyclic over these angles.

Solution by Kee-Wai Lau, Hong Kong.

Let

$$A = \frac{\pi - \alpha}{2}, \quad B = \frac{\pi - \beta}{2}, \quad C = \frac{\pi - \gamma}{2},$$

so that A, B, C are again the angles of a triangle. We then have $\cos \alpha/2 = \sin A$, etc., so that

$$\sum \cos \alpha/2 = \sum \sin A.$$

Using the result of *Crux* 613 [1982: 55, 67, 138], that

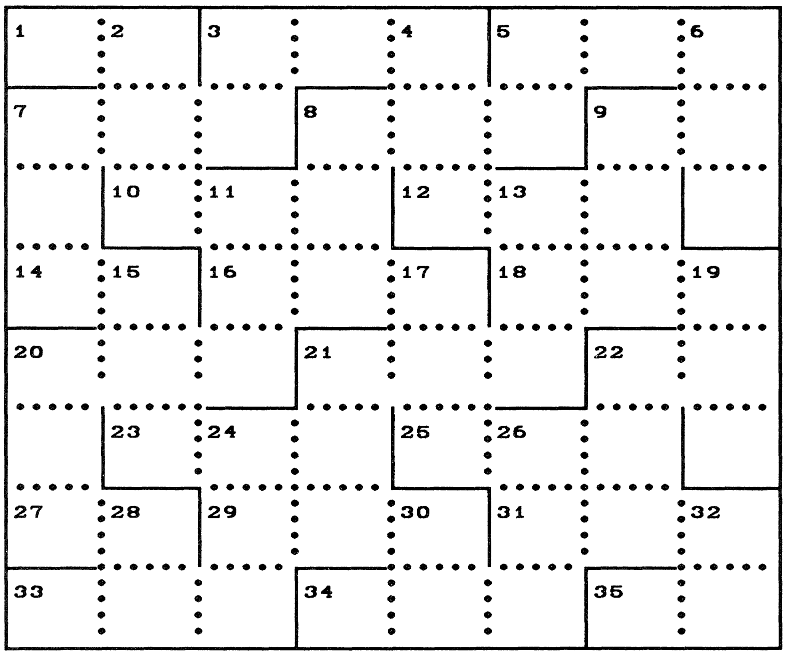
$$\frac{2}{\sqrt{3}} \Sigma \sin A \le \Sigma \cos \left[\frac{B-C}{2}\right] ,$$

we easily obtain the desired inequality.

Also solved (usually by the same method) by SVETOSLAV BILCHEV *and* EMILIA VELIKOVA, *Russe, Bulgaria;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* M.S. KLAMKIN, *University of Alberta, Edmonton, Alberta;* VEDULA N. MURTY, *Penn State University, Middletown, Pennsylvania;* GEORGE TSINTSIFAS, *Thessaloniki, Greece; and the proposer.*

* * *

<u>1153</u>. *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*



1B	3D	9B	29B	7A	21D	12B	11U	20U
2D	6D	5B	19U	15D	7D	22D	18B	15U
27A	2D	26D	20A	8D	8A	16A	31A	33A
5D	3A	25B	30D	14A	9A	16B	24D	23B
28D	35A	3U	30U	9U	13D	22A	32U	32D
4U	21A	21D	19U	17D	10A	32U	34A	33A

The answers are distinct 2- and 3-digit decimal numbers, none beginning with zero. Each of the above sets of answers is a primitive Pythagorean triple, in increasing size, so that the third member is the hypotenuse. A = across, B = back, D = down, U = up. For example, 1B has its tens & units digits in the squares labelled 2 & 1 respectively; 11U is a 3-digit number with its tens & units digits in squares 16 & 11 respectively.

Solution.

(1) 2	(2) 1	(3) 3	0	(4) 8	(5) 7	5	(6) 5
(7) 6	6	5	(8) 9	6	5	(9) 7	3
2	(10) 5	(11) 0	5	(12) 1	(13) 4	3	2
(14) 5	(15) 5	(16) 2	7	(17) 3	(18) 4	4	(19) 6
(20) 1	2	4	(21) 7	7	5	(22) 3	3
4	(23) 7	(24) 9	9	(25) 7	(26) 1	3	3
(27) 5	(28) 2	(29) 2	3	(30) 4	(31) 7	3	(32) 6
(33) 7	8	5	(34) 7	8	3	(35) 4	5

Found by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; GLEN E. MILLS, Colonial High School, Orlando, Florida; JÜRGEN WOLFF, Steinheim, Federal Republic of Germany; ANNELIESE ZIMMERMANN, Bonn, Federal Republic of Germany; and the proposer.

* * *

<u>1154</u>. [1986: 139] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, and C be the angles of an arbitrary triangle. Determine the best lower and upper bounds of the function

$$f(A,B,C) = \Sigma \sin A/2 - \Sigma \sin A/2 \sin B/2$$

(where the summations are cyclic over A, B, C) and decide whether they are attained.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

We have

$$(\Sigma \sin A/2 - 1)^2 = \Sigma \sin^2 A/2 + 2\,\Sigma \sin B/2 \sin C/2 - 2\,\Sigma \sin A/2 + 1.$$

Thus

$$f(A,B,C) = 1/2[\Sigma \sin^2 A/2 + 1 - (\Sigma \sin A/2 - 1)^2]. \qquad (1)$$

The relations

$$\Sigma \sin^2 A/2 = 1 - 2\Pi \sin A/2 \qquad (2)$$

and

$$\Sigma \sin A/2 - 1 = 4\Pi \sin(45^\circ - A/4) \qquad (3)$$

are easily obtained from the identities

$$\sin^2 A/2 = \frac{1 - \cos A}{2}$$

$$\sin A + \sin B + \sin C - \sin(A + B + C) = 4\Pi \sin \frac{B + C}{2}$$

$$\cos A + \cos B + \cos C + \cos(A + B + C) = 4\Pi \cos \frac{B + C}{2} .$$

From (1), (2), and (3),

$$f(A,B,C) = 1/2[(1 - 2\Pi \sin A/2) + 1 - (4\Pi \sin(45^\circ - A/4))^2]$$
$$= 1 - \Pi \sin A/2 - 8\Pi \sin^2(45^\circ - A/4).$$

Now, it is well known that

$$0 < \Pi \sin A/2 \le 1/8, \tag{4}$$

from which we can deduce

$$0 < \Pi \sin(45^\circ - A/4) \le 1/8 \tag{5}$$

by considering the triangle whose angles are $90^\circ - A/2$, $90^\circ - B/2$, $90^\circ - C/2$. Thus

$$3/4 \le f(A,B,C) < 1. \tag{6}$$

The upper bounds of (4) and (5), and hence the lower bound of (6), are attained for equilateral triangles. The products in (4) and (5) tend towards the lower bound 0 if one of the angles of the triangle tends to 180°, so for these triangles the upper bound of (6) is approached.

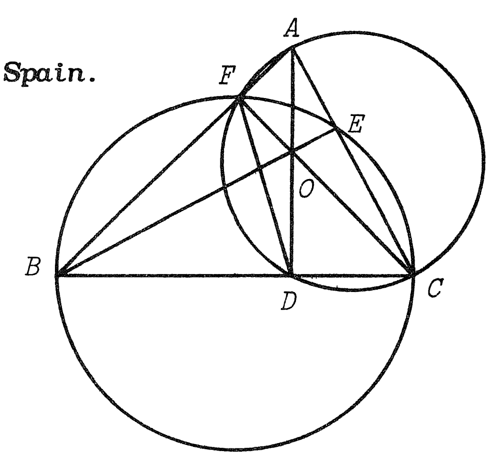
Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; and the proposer. Less-than-best bounds were found by two other readers.

* * *

<u>1155</u>. [1986: 140] *Proposed by Roger Izard, Dallas, Texas.*

In triangle ABC cevians AD, BE, and CF meet at point O. Points F, B, C, and E are concyclic. Points A, F, D, and C are also concyclic. Show that AD, BE, and CF are altitudes.

Solution by Jordi Dou, Barcelona, Spain.



Since $\angle ADF = \angle ACF$ from $ACDF$, and $\angle ACF = \angle EBA$ from $FECB$, we have $\angle ODF = \angle FBO$. Thus $ODBF$ is concyclic and

$$\angle BFO + \angle BDO = 180^\circ.$$

Since $\angle AFC = \angle ADC$ from $ACDF$, we have $\angle BFO = \angle BDO$, and thus

$$\angle BFO = \angle BDO = 90^\circ.$$

Hence O is the orthocentre.

Also solved by AAGE BONDESEN, *Royal Danish School of Educational Studies, Copenhagen, Denmark;* C. FESTRAETS-HAMOIR, *Brussels, Belgium;* JACK GARFUNKEL, *Flushing, N.Y.;* J.T. GROENMAN, *Arnhem, The Netherlands;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* KEE-WAI LAU, *Hong Kong;* MALCOLM SMITH, *Georgia Southern College, Statesboro, Georgia;* DAN SOKOLOWSKY, *Williamsburg, Virginia;* GEORGE TSINTSIFAS, *Thessaloniki, Greece;* G.R. VELDKAMP, *De Bilt, The Netherlands; and the proposer.*

*　　　　*　　　　*

1156. [1986: 140] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

At any point P of an ellipse with semiaxes a and b $(a > b)$, draw a normal line and let Q be the other meeting point. Find the least value of length PQ, in terms of a and b.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

We generalize the problem by finding the shortest normal of an n-dimensional ellipsoid. Intuitively, one expects that for a sufficiently "fat" ellipsoid the shortest normal will be the minor axis, while for a sufficiently "thin" ellipsoid the shortest normal will be smaller than the minor axis and originate from points near the ends of the major axis.

Let the equation of the ellipsoid be

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} = 1, \tag{1}$$

and without loss of generality let $0 < a_1 \le a_2 \le \ldots \le a_n$. The direction numbers of the normal at point $P:(h_1,h_2,\ldots,h_n)$ on the ellipsoid are

$$\left[\frac{h_1}{a_1^2}, \frac{h_2}{a_2^2}, \ldots, \frac{h_n}{a_n^2}\right],$$

and its equation in parametric form is

$$\frac{(x_1 - h_1)a_1^2}{h_1} = \frac{(x_2 - h_2)a_2^2}{h_2} = \ldots = \frac{(x_n - h_n)a_n^2}{h_n} = \lambda. \tag{2}$$

Point Q, which is the other end point of the normal, is gotten by solving (1) and (2) simultaneously, i.e.

$$\sum_{i=1}^{n} \frac{h_i^2(1 + \lambda/a_i^2)^2}{a_i^2} = 1,$$

or

$$\lambda^2 \sum_{i=1}^{n} \frac{h_i^2}{a_i^6} + 2\lambda \sum_{i=1}^{n} \frac{h_i^2}{a_i^4} = 0. \tag{3}$$

From (2), the length N of the normal is given by

$$N^2 = \sum_{i=1}^{n} (x_i - h_i)^2 = \lambda^2 \sum_{i=1}^{n} \frac{h_i^2}{a_i^4}$$

where

$$\lambda = \frac{-2 \sum_{i=1}^{n} h_i^2/a_i^4}{\sum_{i=1}^{n} h_i^2/a_i^6}$$

is the nonzero root of (3). Hence

$$N^2 = \frac{4\left[\sum_{i=1}^{n} h_i^2/a_i^4\right]^3}{\left[\sum_{i=1}^{n} h_i^2/a_i^6\right]^2}.$$

Letting $t_i = h_i/a_i$, our problem analytically is now to minimize

$$\frac{4\left[\sum_{i=1}^{n} t_i^2/a_i^2\right]^3}{\left[\sum_{i=1}^{n} t_i^2/a_i^4\right]^2}$$

subject to the constraint $\sum_{i=1}^{n} t_i^2 = 1$.

To solve this problem, we first solve the two-dimensional case. Putting $x = t_1^2$, $y = t_2^2$, we wish to minimize

$$N^2 = \frac{4\left(\frac{x}{a^2} + \frac{y}{b^2}\right)^3}{\left(\frac{x}{a^4} + \frac{y}{b^4}\right)^2}$$

where $x + y = 1$, $x \geq 0$, $y \geq 0$, and $0 < b < a$. Then letting

$$r = \frac{a^2}{b^2} - 1, \qquad s = \frac{a^4}{b^4} - 1,$$

and replacing x by $1 - y$, we get

$$\left[\frac{N}{2a}\right]^2 = \frac{\left[\frac{1-y}{a^2} + \frac{y}{b^2}\right]^3}{a^2\left[\frac{1-y}{a^4} + \frac{y}{b^4}\right]^2} = \frac{\left(1 - y + \frac{a^2}{b^2}y\right)^3}{\left(1 - y + \frac{a^4}{b^4}y\right)^2} = \frac{(1 + ry)^3}{(1 + sy)^2} . \tag{4}$$

The numerator of the derivative of the right hand side with respect to y is

$$\begin{aligned}(1 + sy)^2 \cdot 3(1 + ry)^2 \cdot r &- (1 + ry)^3 \cdot 2(1 + sy) \cdot s \\ &= (1 + sy)(1 + ry)^2[3r(1 + sy) - 2s(1 + ry)] \\ &= (1 + sy)(1 + ry)^2[rsy - (2s - 3r)].\end{aligned}$$

Now

$$2s - 3r = \frac{2a^4}{b^4} - 2 - \frac{3a^2}{b^2} + 3 = \left(\frac{a^2}{b^2} - 1\right)\left(\frac{2a^2}{b^2} - 1\right) > 0, \tag{5}$$

so that there is a minimum at

$$y = \frac{2s - 3r}{rs} \tag{6}$$

provided this value is ≤ 1, i.e., from (5),

$$\left(\frac{a^2}{b^2} - 1\right)\left(\frac{2a^2}{b^2} - 1\right) \leq \left(\frac{a^2}{b^2} - 1\right)\left(\frac{a^4}{b^4} - 1\right)$$

or

$$\frac{2a^2}{b^2} \leq \frac{a^4}{b^4}$$

or

$$2b^2 \leq a^2.$$

If (6) holds, then by (4),

$$\left[\frac{N}{2a}\right]^2 = \frac{\left[1 + \frac{2s - 3r}{s}\right]^3}{\left[1 + \frac{2s - 3r}{r}\right]^2} = \frac{27(s - r)^3/s^3}{4(s - r)^2/r^2} = \frac{27r^2(s - r)}{4s^3}$$

$$= \frac{27\left[\frac{a^2}{b^2} - 1\right]^2\left[\frac{a^4}{b^4} - \frac{a^2}{b^2}\right]}{4\left[\frac{a^4}{b^4} + 1\right]^3} = \frac{27a^2b^4(a^2 - b^2)^3}{4(a^4 - b^4)^3} = \frac{27a^2b^4}{4(a^2 + b^2)^3} ,$$

so that

$$N^2 = \frac{27a^4b^4}{(a^2 + b^2)^3} .$$

There is also an endpoint minimum corresponding to $y = 1$, giving

$$N^2 = \frac{4a^2(1 + r)^3}{(1 + s)^2} = \frac{4a^2(a^2/b^2)^3}{(a^4/b^4)^2} = 4b^2$$

by (4). By the A.M.-G.M. inequality,

$$\left[\frac{\frac{a^2}{2b^2} + \frac{a^2}{2b^2} + 1}{3}\right]^3 \geq \frac{a^4}{4b^4} ,$$

so

$$4b^2 \geq \frac{27a^4}{b^2\left[\frac{a^2}{b^2} + 1\right]^3} = \frac{27a^4b^4}{(a^2 + b^2)^3},$$

with equality if and only if $a^2 = 2b^2$. Consequently,

$$\min N^2 = \begin{cases} \dfrac{27a^4b^4}{(a^2 + b^2)^3} & \text{if } a^2 \geq 2b^2 \\ 4b^2 & \text{if } a^2 \leq 2b^2. \end{cases}$$

This answers the proposed problem.

We now solve the n-dimensional case by reducing it to the previous two-dimensional case. Note that the plane determined by the normal line and the center of the ellipsoid intersects the ellipsoid in an ellipse with the same center and normal line. Since the lengths of the radius vectors from the center to the ellipsoid can vary only from a_1 to a_n, this ellipse will have semiaxes between a_1 and a_n, and so the extremal normals must lie in the plane of the semiaxes a_1 and a_n. Thus in this case we get the same minimal value for N^2 as before, with a replaced by a_n and b replaced by a_1. As a homogeneous inequality, we have equivalently that

$$4\left[\sum_{i=1}^{n} \frac{t_i^2}{a_i^2}\right]^3 \geq \frac{27a_1^4a_n^4}{(a_1^2 + a_n^2)^3}\left[\sum_{i=1}^{n} t_i^2\right]^2\left[\sum_{i=1}^{n} \frac{t_i^2}{a_i^4}\right]^2$$

provided $a_n^2 \geq 2a_1^2$, otherwise the first fraction on the right hand side is replaced by $4a_1^2$ (we still assume $a_1 = \min\{a_i\}$ and $a_n = \max\{a_i\}$).

For a related problem (solved in a similar fashion) of finding the maximum distance a normal line to a given ellipsoid can be from the center of the ellipsoid, see M.S. Klamkin and R.G. McLenaghan, An ellipse inequality, *Mathematics Magazine* 50 (1977) 261-263.

Also solved by AAGE BONDESEN, Royal Danish School of Educational Studies, Copenhagen, Denmark; KEE-WAI LAU, Hong Kong; ROBERT LYNESS, Southwold, Suffolk, England; CHRISTOPHER ODGERS, Camosun College, Victoria, B.C.; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. There were two incorrect solutions submitted.

The problem was taken from a still existing Japanese mathematical wooden tablet hung in 1912.

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