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THE SKOLIAD CORNER

No. 8

R. E. WOODROW

This month we give a contest at the high school level from Saskatchewan. For many years now it has been organized and administered by Professor Gareth Griffith of the University of Saskatchewan. The contest is a pre-Olympiad contest which aids in the selection of contestants for the Canadian Mathematical Olympiad. My thanks to Professor Griffith for permission to use the contest here.

SASKATCHEWAN SENIOR MATHEMATICS CONTEST

Wednesday, February 19, 1992 — Time: 90 minutes

1. Let $f(x) = 3x^2 - \frac{1}{x+1}$.

(a) Write expressions for the following:

(i) $f(x-1)$ (ii) $f\left(\frac{1}{x}\right)$ (iii) $f(0)$ (iv) $f(2x)+1$

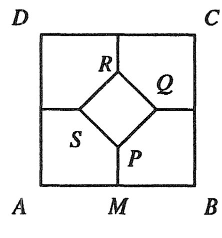
There is no need to simplify the expressions.

(b) If $g(x)$ is a function and if z is a real number such that $g(z) = 0$, we shall call z a *zero* of $g(x)$.

(i) Find a zero of the function $f(2x)+1$ in part (a) above.

(ii) How many zeros does this function have?

2. A linoleum company currently produces a product in which the pattern is a repetition of the figure, opposite.



$ABCD$ and $PQRS$ are concentric squares. The diagonals of $PQRS$ are parallel to the sides of $ABCD$. If the length of AB is one unit and if the length of PQ is $1/2$ unit, compute the length of PM where M is the midpoint of AB.

3. Let $\log_{15} 5 = a$. Write $\log_{15} 9$ as a function of a. (No calculators are necessary.)

4. Find all solutions of the equation

$$\cos 2x \sec x + \sec x + 1 = 0$$

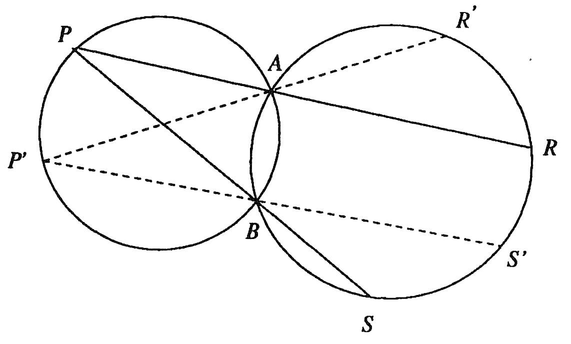
which lie in the interval $[-\pi, 2\pi]$ or $[-180°, 360°]$. (Both intervals are the same.)

5. Sketch the graphs of

$$2x^2 + y^2 = 3 \quad \text{and} \quad xy = 1$$

using the same axes for both graphs. Determine the coordinates of the points of intersection of the two curves.

6. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



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Last issue we gave the problems of the 1995 Calgary Junior Mathematics Contest. The contest was in two parts with answers only for Part A and written solutions for Part B. Accordingly we give the answers only for Part A questions, but reproduce the questions with "official" solutions for Part B.

PART A SOLUTIONS

A1.	$7/36$	A2.	$10\sqrt{2}+10\sqrt{5}+10\sqrt{10}$	A3.	675
A4.	$2^{12}=4096$	A5.	285	A6.	$6/48$
A7.	27 or 3	A8.	5	A9.	12

PART B SOLUTIONS

1. Find all the whole numbers between 1 and 100 which can be written as a sum of integers constructed by using each of the digits 0 through 9 exactly once.

(Example: $90 = 0+1+52+3+4+6+7+8+9$ is one such number.)

Solution. Of course the smallest number that can be constructed this way is

$$0+1+2+3+4+5+6+7+8+9=45.$$

Suppose we put two of these digits together to form a two-digit number, say we use the digits a and b to form the two-digit number "ab". Since "ab" is really the number $10a + b$, in putting a and b together to form "ab" we are adding $9a$ to the sum. So every time we stick two digits together we add a multiple of 9 to the sum. This says that the only possible sums that we can form are those which arise by adding a multiple of 9 to the sum 45, namely:

$$\mathbf{54, 63, 72, 81, 90, 99}$$

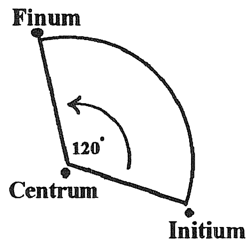
as well as **45** of course. These sums are all possible: for example,

$$54 = 0+12+3+4+5+6+7+8+9, \quad 63 = 0+1+23+4+5+6+7+8+9,$$

$$72 = 0+1+2+34+5+6+7+8+9, \quad 81 = 0+1+2+3+45+6+7+8+9,$$

$$90 = 0+1+2+3+4+56+7+8+9, \quad 99 = 0+1+2+3+4+5+67+8+9.$$

2. Two routes lead from Initium to Finum in a certain country. The Circular Route follows the arc of a circle of radius 100km centered at Centrum and passing through Initium and Finum. The Radial Route follows the radius from Initium to Centrum and then the radius from Centrum to Finum. The arc of the circular route measures 120°, and the speed limit on the Circular Route is 105 km/hr while that on the Radial Route is 100 km/hr. Which is the quicker route to take?



Solution. The length of the Radial Route is 100 + 100 = 200 km, and the top speed there is 100 km/hr, so you could drive it in 2 hours. For the Circular Route, since 120° is one-third of a complete circle, the distance of the Circular Route is

$$\frac{1}{3} \times 2 \times \pi \times 100 = \frac{200\pi}{3},$$

and the top speed there is 105 km/hr, so you could drive it in

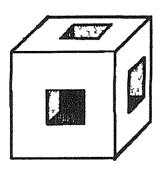
$$\frac{200\pi}{3 \times 105} = \frac{40\pi}{3 \times 21} = \frac{40\pi}{63}$$

hours. Now the question is, which is bigger,

$$2 \quad \text{or} \quad \frac{40\pi}{63}\,?$$

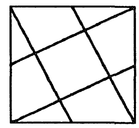
We could just use a calculator of course, but let's do it without. Since $\pi = 3.14159\ldots$ is smaller than 3.15, $40\pi/63$ will be smaller than $40 \times 3.15/63$ which is $126/63$ which is exactly 2; thus the time required to drive the Circular Route is less, so the **Circular Route** is quicker.

3. A wooden cube 9cm × 9cm × 9cm has three square holes drilled through it, each of which forms a 3cm × 3cm × 9cm tunnel through the centre of opposite faces. What is the total surface area of the exposed wood?

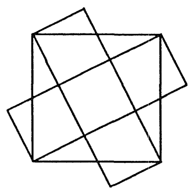


Solution. The **outside** of the cube has six faces, each of which is a 9×9 square with a 3×3 square missing, so each will have surface area $9^2 - 3^2 = 72$ square cm, for a total outside surface area of $72 \times 6 = 432$ square cm. As well, drilled into each side is a $3 \times 3 \times 3$ hole with four sides inside the cube, each side of area $3 \times 3 = 9$ square cm, so the "inside" surface area will be $9 \times 4 \times 6 = 216$ square cm. Thus the total exposed surface area is $432 + 216 =$ **648** square cm. (Note that the central $3 \times 3 \times 3$ "hole" in the cube has no "sides" to count.)

4. The large square has area 1. The inside lines join a vertex of the square to the midpoint of a side as shown. What is the area of the small central square?



Solution. The easiest way to do this problem is to draw more lines to form a tilted "cross": Now the eight little triangles in this picture are all the same size and shape, so they have the same area. Four of these triangles are inside the square and four are inside the cross, and this is the only difference between the square and the cross. Therefore the square and the cross have the same area, which is 1.



Since the small central square is obviously one-fifth of the cross, its area must be **1/5**.

There are longer ways to do this problem using similar triangles.

5. A decimal number between 0 and 1 is called an "E-number" if all of its decimal digits are even, and is called an "O-number" if all of its decimal digits are odd. For instance .028 is an E-number and .1195 is an O-number, while .037 is neither an E-number nor an O-number, because 0 is even and 3 is odd.

(a) Suppose you want to find an E-number and an O-number so that the E-number is less than the O-number, but they are as close as possible. How close can they be?

(b) How close can they be if you want the O-number less than the E-number in (a)?

Solution. (a) Look at the following table:

E-number	<	O-number	difference
.8		.9	.1
.88		.9	.02
.888		.9	.012
.8888		.9	.0112
.88888		.9	.01112
⋮		⋮	⋮

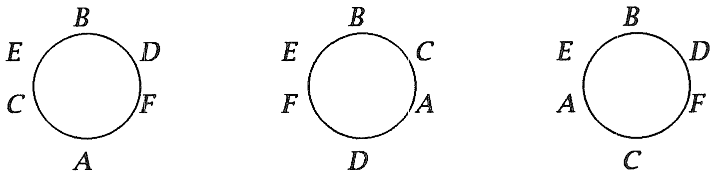
The more 8's we use in the E-number, the smaller the difference gets. But the difference never gets smaller than what it would be if we were allowed to use the infinite repeating decimal $.888\ldots$ (which equals the fraction $8/9$) for the E-number and keep $.9$ (which equals $9/10$) for the O-number; in this case the difference would be $9/10 - 8/9 = 1/90$, which is the infinite repeating decimal $.01111\ldots$. So the answer is that, by using the E-number $.88\ldots8$ with lots of 8's, and the O-number $.9$, we can get their difference to be as close to $.01111\ldots$ as we like, but no smaller (we think!).

(b) This time, by using an O-number like $.7999\ldots9$, where there are lots of 9's, and the E-number $.8$, we can get their difference to be as small as we want.

6. Alicia, Brad, Chris, Drew, Elsie and Farid went to a Stampede breakfast together and sat around a circular table eating pancakes. Alicia ate more pancakes than the person to her right, but fewer than Brad. Brad ate fewer pancakes than the person to his right, but more than Chris. Chris ate fewer pancakes than the person to her left, but more than Drew. Drew ate more pancakes than the person to his left, but fewer than Elsie. Farid wasn't hungry, and didn't eat any pancakes at all.

Find all the possible ways they could have been seated around the table.

Solution. There are **three** possible seating arrangements:



We leave it to the reader to check that these possibilities work and are the only ones.

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That completes this number of the Skoliad Corner. Send me your opinions, suggestions, and suitable contest materials.

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THE OLYMPIAD CORNER

No. 168

R. E. WOODROW

All communications about this column should be sent to Professor R. E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we give the selected problems of the 32nd Ukrainian Mathematical Olympiad as a challenge to the readers to submit "nice" solutions. My thanks go to Georg Gunther, Sir Wilfred Grenfell College, and Canadian Team Leader at the 34th I.M.O. at Istanbul, Turkey, who sent them to me.

32nd UKRAINIAN MATHEMATICAL OLYMPIAD

March 1992 — Selected Problems

1. (8th form) Points A, B, C, D in the plane are given. It is known that distance $|AB|$ is smaller than $|CB|$ and $|DB|$ and distance $|CD|$ is smaller than $|AD|$ and $|BD|$. Prove that line segments AB and CD do not intersect.

2. (8) There are real numbers a, b, c such that $a \geq b \geq c > 0$. Prove that

$$\frac{a^2-b^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b} \geq 3a-4b+c.$$

3. (9) The circle is given. The point A is a fixed point on the circle. The point D is a fixed interior-point of the circle. Consider an arbitrary chord BC of the circle containing D. Let M be the point of intersection of medians of triangle ABC. Find the set of all possible points M.

4. (9) Given a 25×25 chess board, two players in turn put white and black figures on the squares of the board. The first player places the white figures and the second player the black ones. A figure may not be put on a square if the square is not empty or if all neighbouring squares are occupied by figures of the same colour. The player who cannot place his figure loses. Find a winning strategy for one of the players. (Remark: Squares are called neighbouring if they have a common side.)

5. (10) Prove that there are no real numbers x, y, z such that

$$x^2+4yz+2z=0$$

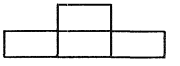
$$x+2xy+2z^2=0$$

$$2xz+y^2+y+1=0.$$

6. (10) For real numbers a, b, c, d from the closed interval $[1,2]$ prove the inequality

$$\frac{a+b}{b+c}+\frac{c+d}{d+a}\leq 4\frac{a+c}{b+d}\,.$$

7. (10) On a 100×100 chess board are situated 800 figures. Every figure occupies four squares of the board. No two figures cover the same square. Prove that it is possible to place one more such figure on the board in the same way.



8. (11) Prove that number $\operatorname{arctg}(4/3)\pi$ is irrational.

* * * * *

Next we turn to the "official" results of the 36th I.M.O. which was written in Toronto, July 19–20, 1995. My source this year is Murray Grant, Systems Administrator to the International Mathematical Olympiad 1995, and Bruce Shawyer, the Chief Operating Officer. (Last number we featured a personal account of the experience of Sam Maltby as a coordinator.) I hope that I have made no serious errors in compiling the results and transcribing names.

This year a total of 412 students from 73 countries took part. This number is up somewhat from last year. Sixty-four countries sent teams of 6 (the number invited to participate in recent years). But there were teams of each size (including I suppose 0 if you consider teams which did not come.)

The contest is officially an individual competition and the six problems were assigned equal weights of seven points each (the same as at the last 14 I.M.O.'s), for a maximum possible individual score of 42 (and a total possible of 252 for a national team of six students). (This year nine teams consisted of fewer than six students.) For comparisons see the last 14 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], [1991: 257], [1992: 263], [1993: 256] and [1994: 243].

This year there were 14 perfect scores, with the median score being 18. The jury awarded a first prize (Gold) to the thirty students who scored 37 or more on the two papers. Second (Silver) prizes went to the seventy-one students with scores from 29 to 36, and third (Bronze) prizes went to the one hundred students with scores from 19 to 28. Any student who did not receive a medal, but who scored full marks on at least one problem, was awarded honourable mention.

The jury awarded a Special Prize to Nikolay Nikolov of Bulgaria for his elegant solution to Question 6. Nikolay is in his fourth Olympiad with awards:

1995	Gold and Special Prize
1994	Silver
1993	Gold
1992	Gold

(The Special Prize was a telescope.) Because the 1996 Olympiad will be written earlier in July than is customary, he will just fail to be eligible for a fifth Olympiad.

Congratulations to the Gold Medalists.

Name	Country	Score
Mihály Bárász	Hungary	42
Péter Burcsi	Hungary	42
Cheng Chang	China	42
Egmont Koblinger	Hungary	42
Song Liu	China	42
Ciprian Manolescu	Romania	42
Maryam Mirzakhani	Iran	42
Nikolay Nikolov	Bulgaria	42
Sergei Norine	Russia	42
Dragos Oprea	Romania	42
Ovidiu Savin	Romania	42
Sug-woo Shin	South Korea	42
Ngo Dac Tuan	Vietnam	42
Chenchang Zhu	China	42
Tetsuyuki Maruoka	Japan	42
Yuly Sannikov	Ukraine	41
Lev Buhovsky	Israel	40
Dao Hai Long	Vietnam	40
Mikhail Ostrovski	Russia	39
Constantin Chiscanu	Romania	38
Haidong Wang	China	38
Dmitri Zaporojets	Russia	38
Artur Avila Cordeiro de Melo	Brazil	37
Emmanual Breuillard	France	37
Edward Thomas Crane	U.K.	37
Kyo Min Jeong	Korea	37
Joseph Samuel Myers	U.K.	37
Reza Sadeghi	Iran	37
Dmitri Tchelkak	Russia	37
Gunther Vogel	Germany	37

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Next we give the problems from this year's I.M.O. Competition. Solutions to these problems, along with those of the 1995 U.S.A. Mathematical Olympiad will appear in a booklet entitled *Mathematical Olympiads* 1995 which may be obtained for a small charge from: Dr. W.E. Mientka, Executive Director, M.A.A. Committee on H.S. Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, 68588, U.S.A.

I.M.O.

First Day — July 19, 1995

Time: 4.5 hours

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

2. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, A_2, \ldots, A_n$ in the plane, and real numbers $r_1, r_2, \ldots, r_n$ satisfying the following two conditions:

(i) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

(ii) for each triple i, j, k ($1 \leq i < j < k \leq n$) the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Second Day — July 20, 1995

Time: 4.5 hours

4. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

(i) $x_0 = x_{1995}$;

(ii) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, 2, \ldots, 1995$.

5. Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD, \qquad DE = EF = FA,$$

and

$$\angle BCD = \angle EFA = 60°.$$

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

6. Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

(i) A has exactly p elements, and

(ii) the sum of all the elements in A is divisible by p.

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As the I.M.O. is officially an individual event, the compilation of team scores is unofficial if inevitable. These totals, and prize-rewards are given in the following table.

Rank	Country	Score	Gold	Silver	Bronze	Total
1.	China	236	4	2	—	6
2.	Romania	230	4	2	—	6
3.	Russia	227	4	2	—	6
4.	Vietnam	220	2	4	—	6
5.	Hungary	210	3	1	2	6
6.	Bulgaria	207	1	4	1	6
7.	South Korea	203	1	4	1	6
8.	Iran	202	2	3	1	6
9.	Japan	183	1	3	2	6
10.	United Kingdom	180	2	1	3	6
11.	U.S.A.	178	—	3	3	6
12.	Taiwan	176	—	4	1	5
13.	Israel	71	1	2	2	5
14.	India	165	—	3	3	6
15.	Germany	162	1	3	1	5
16.	Poland	161	—	1	5	6
17.-18.	Czech Republic	154	—	1	5	6
17.-18.	Yugoslavia (Serbia-Montenegro)	154	—	2	3	5
19.	Canada	153	—	2	3	5
20.	Hong Kong	151	—	2	3	5
21.-22.	Australia	145	—	1	4	5
21.-22.	SLovakia	145	—	2	2	4
23.	Ukraine	140	1	1	1	3
24.	Morocco	138	—	1	4	5
25.	Turkey	134	—	2	3	5
26.-28.	Belarus	131	—	1	3	4
26.-28.	Italy	131	—	—	4	4
26.-28.	Singapore	131	—	2	2	4
29.	Argentina	129	—	2	2	4
30.	France	119	1	—	2	3
31.	Macedonia	117	—	1	3	4
32.-33.	Armenia	111	—	2	1	3
32.-33.	Croatia	111	—	—	3	3
34.	Thailand	107	—	1	2	3
35.	Sweden	106	—	—	2	2
36.-37.	Finland	101	—	—	3	3
36.-37.	Moldova	101	—	1	1	2
38.	Colombia	100	—	1	2	3
39.-40.	Latvia	97	—	1	1	2
39.-40.	Switzerland (Team of 5)	97	—	2	—	2
41.	South Africa	95	—	—	2	2
42.	Mongolia	91	—	—	1	1
43.	Austria	88	—	—	1	1
44.	Brazil	86	1	—	—	1
45.	The Netherlands	85	—	—	2	2
46.	New Zealand	84	—	1	1	2
47.	Belgium	83	—	—	1	1

Rank	Country	Score	Gold	Silver	Bronze	Total
48.	Georgia	79	—	1	—	—
49.	Denmark	77	—	—	1	1
50.	Lithuania	74	—	—	—	—
51.	Spain	72	—	—	1	1
52.	Norway	70	—	—	1	1
53.	Indonesia	68	—	—	1	1
54.	Greece	66	—	—	1	1
55.	Cuba (Team of 4)	59	—	—	—	—
56.	Estonia	55	—	—	—	—
57.	Kazakhstan	54	—	—	—	—
58.-59.	Cyprus	43	—	—	—	—
58.-59.	Mexico	43	—	—	1	1
60.	Slovenia (Team of 5)	42	—	—	—	—
61.	Ireland	41	—	—	—	—
62.	Macau	33	—	—	—	—
63.	Trinidad and Tobago	32	—	—	—	—
64.	Azerbaijan (Team of 3)	30	—	—	—	—
65.-66.	Kgrgyzstan	28	—	—	—	—
65.-66.	Philippines	28	—	—	1	1
67.	Portugal	26	—	—	—	—
68.	Iceland (Team of 4)	19	—	—	—	—
69.	Bosnia-Herzegovina	18	—	—	—	—
70.	Chile (Team of 2)	14	—	—	—	—
71.	Sri Lanka (Team of 1)	10	—	—	—	—
72.	Malaysia (Team of 2)	1	—	—	—	—
73.	Kuwait (Team of 2)	0	—	—	—	—

This year the Canadian Team rose to 19th place from 24th place last year. The Team members were:

Cyrus Hsia	35	Silver
Frederic Latour	31	Silver
Donny Cheung	25	Bronze
Byung Kyu Chun	25	Bronze
Alyssa Ker	19	Bronze
Lawrence Tang	18	Honourable Mention

The Team Leader this year was Professor Richard Nowakowski of Dalhousie University, with Deputy Team Leader Georg Gunther of Sir Wilfred Grenfell College.

The Chinese Team placed first this year. Its members were:

Song Liu	42	Gold
Cheng Chang	42	Gold
Chenchang Zhu	42	Gold
Haidong Wong	38	Gold
Yijun Yao	36	Silver
Yizhou Lin	36	Silver

The Chinese Team Leader was Zhang Zhusheng and the Deputy Team Leader was Wang Jie. A hearty congratulations!!

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To finish this issue of the Corner we give solutions to problems posed in the March 1994 number of the Corner. The Olympiad we gave that month is the 9th Balkan Mathematical Olympiad, [1994: 64-65] written in Athens, May, 1992.

1. (Bulgaria). Let m and n be positive integers and

$$A(m,n) = m^{3^{4n}+6} - m^{3^{4n}} - m^5 + m^3.$$

Find every n such that $A(m,n)$ is divisible by 1992 for every m.

Solutions by Panos E. Tsaoussoglou, Athens, Greece; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's version.

First we note that $1992 = 2^3 \cdot 3 \cdot 83$ (with 83 prime) and $A(m,n) = (m^2-1)(m^{3^{4n}}(m^4+m^2+1) - m^3)$. Let us seek all n such that $83 | A(2,n)$.

$$\text{Now} \qquad A(2,n) = 3 \cdot (2^{3^{4n}} \cdot 21 - 8).$$

Thus n satisfies

$$21 \cdot 2^{3^{4n}} \equiv 8 \pmod{83}, \qquad \text{so} \qquad 2^{3^{4n}} \equiv 32 \equiv 2^5 \pmod{83}.$$

Since 2 is a primitive root of 83, it follows that $3^{4n} \equiv 5 \pmod{83}$, or $(-1)^n \equiv 82$. This is clearly impossible. Therefore no n with the required properties exists.

[*Editor's note.* From the comments of Tsaoussoglou and the "official solution" that was sent to us thanks to Professor Tomescu of Romania via Professor F. Bellot Rosado, Vallodolid, Spain, the Spanish team leader at the I.M.O., it appears that the correct statement of the problem was

$$A(m,n) = m^{3^{4n}+6} - m^{3^{4n}+4} - m^5 + m^3.$$

2. (Cyprus). Prove that for each positive integer n

$$(2n^2 + 3n + 1)^n \geq 6^n (n!)^2.$$

Solutions by Seung-Jin Bang, Seoul, Korea; by Joseph Ling, The University of Calgary; by Bob Prielipp, University of Wisconsin-Oshkosh; by Panos E. Tsaoussoglou, Athens, Greece; by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. There were basically two types of solution sent in and we give a representative of each. First we give Wang's solution via the A-G Mean inequality.

By the Arithmetic Mean-Geometric Mean Inequality we have

$$\frac{n(n+1)(2n+1)}{6n} = \frac{1^2+2^2+\cdots+n^2}{n} \geq (1^2 2^2 \ldots n^2)^{1/n} = (n!)^{2/n}$$

$$\text{or} \quad (n+1)(2n+1) \geq 6(n!)^{2/n}$$

$$\text{or} \quad (2n^2 + 3n + 1)^n \geq 6^n (n!)^2$$

with equality if and only if $n = 1$.

Next we give Wildhagen's version.

For $1 \leq k \leq n$ we have $k(n+1-k) \leq ((n+1)/2)^2$, hence

$$(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \leq \left(\frac{n+1}{2}\right)^{2n}.$$

Therefore it suffices to show that

$$(2n+1)^n (n+1)^n \geq 6^n \cdot \left(\frac{n+1}{2}\right)^{2n}$$

$$\text{or} \quad (2n+1)(n+1) \geq 6\left(\frac{n+1}{2}\right)^2$$

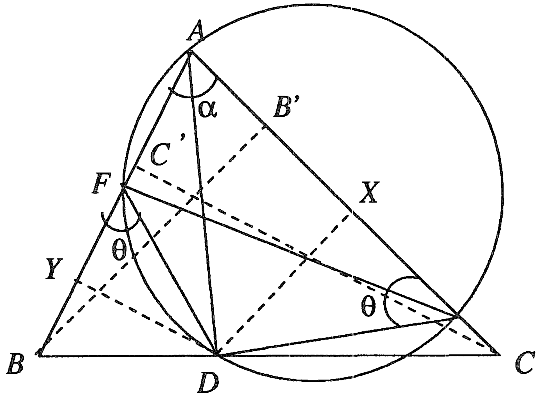
$$\text{or} \quad 2n+1 \geq \frac{3}{2}(n+1).$$

This last inequality holds trivially (and is strict unless $n = 1$).

3. (Greece). Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB respectively (different from A, B, C). If $AFDE$ is inscribable in a circle show that:

$$\frac{4(DEF)}{(ABC)} \leq \left(\frac{EF}{AD}\right)^2.$$

Solutions by Toshio Seimiya, Kawasaki, Japan; and by Panos E. Tsaoussoglou, Athens, Greece. We give Seimiya's version.



We denote the area of triangle PQR by (PQR). As A, F, D, E are concyclic we have

$$\angle EDF + \angle EAF = \pi, \quad \text{so } \sin\angle EDF = \sin\angle EAF = \sin\angle BAC.$$

Therefore

$$\frac{(DEF)}{(ABC)} = \frac{\frac{1}{2}DE \cdot DF \cdot \sin \angle EDF}{\frac{1}{2}AB \cdot AC \cdot \sin \angle BAC} = \frac{OE \cdot OF}{AB \cdot AC}. \tag{1}$$

Let B', X be the feet of the perpendiculars from B, D to AC, respectively, and let C', Y be the feet of the perpendiculars from C, D to AB. Because BB' is parallel to DX, we get

$$\frac{DX}{BB'} = \frac{DC}{BC} \quad \text{and} \quad \frac{DY}{CC'} = \frac{BD}{BC},$$

thus

$$\frac{DX}{BB'} \cdot \frac{DY}{CC'} = \frac{DC}{BC} \cdot \frac{BD}{BC} = \frac{BD \cdot DC}{BC^2}. \tag{2}$$

As $BC^2 = (BD + DC)^2 = (BD - DC)^2 + 4BD \cdot DC \geq 4BD \cdot DC$, we have from (2)

$$\frac{DX \cdot DY}{BB' \cdot CC'} \leq \frac{1}{4}. \tag{3}$$

We put $\angle EAF = \alpha$, and $\angle DEA = \angle DFB = \theta$, and we denote the circumradius of $AFDE$ by R, then we have $EF = 2R \sin \alpha$ and $AD = 2R \sin \theta$, so that

$$\frac{EF}{AD} = \frac{\sin \alpha}{\sin \theta}. \tag{4}$$

Because $BB' = AB \sin \alpha$, $CC' = AC \sin \alpha$, $DX = DE \sin \theta$, and $DY = DF \sin \theta$ we get

$$\frac{DX \cdot DY}{BB' \cdot CC'} = \frac{DE \cdot DF \sin^2 \theta}{AB \cdot AC \sin^2 \alpha}.$$

Hence we have from (3)

$$\frac{DE \cdot DF \sin^2 \theta}{AB \cdot AC \sin^2 \alpha} \leq \frac{1}{4},$$

i.e.

$$\frac{4DEDF}{ABAC} \leq \frac{\sin^2 \alpha}{\sin^2 \theta}. \tag{5}$$

Thus we have from (1) and (4) and (5)

$$\frac{4(DEF)}{(ABC)} \leq \left(\frac{EF}{AD}\right)^2.$$

*

No reader sent in a solution for the fourth problem and so here is a challenge to furnish a nice new solution.

* * *

That completes our file of solutions for Olympiads from the March 1994 number and the Corner for this issue. Send me your nice problem sets and solutions.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

The Book of Ingenious and Diabolical Puzzles, by Jerry Slocum and Jack Botermans. Published by Random House, New York, 1994. ISBN 0-8129-2153-4, hardcover, 128+ pages, US$20.00. *Reviewed by Daniel van Vliet, student, University of Alberta.*

This is the third in a series of books on mechanical puzzles authored by the same dynamic duo. For reviews of the earlier volumes, *Puzzles Old and New* and *New Book of Puzzles*, see [1] and [2].

The book covers a varied assortment of puzzles from around the world and from different historical periods. From the elegant simplicity of the familiar tangram to the defiant impossibility of five Coke bottles strung together by wooden arrows, readers are sure to be enchanted by the ingenuity and elegance with which these puzzles were lovingly crafted.

The puzzles are presented with both intrigue and clarity, often in majestic colours but always with an authentic feel. They are organized into the following eight categories:

(1) Put-Together Puzzles,
(2) Take-Apart Puzzles,
(3) Interlocking Puzzles,
(4) Disentanglement Puzzles,
(5) Sequential Movement Puzzles,
(6) Dexterity Puzzles,
(7) Puzzle Vessels,
(8) Impossible Objects.

Apart from the tangram mentioned above, the put-together puzzles include the famous pentominoes of Solomon Golomb. Take-apart puzzles feature many old locks, matchboxes, and other familiar household objects. The principal family of interlocking puzzles consists of the burrs, wooden blocks with notches which are to be assembled into crosses. All of the disentanglement puzzles have a strong topological flavour.

The famous Cube of Rubik and its variants head the list of sequential movement puzzles. Dexterity puzzles mostly involve rolling balls into holes by adroit manipulation. The puzzle vessels have holes on the side, and only those who know the secret can drink from them without spilling the liquid. Probably the most amazing of all types of puzzles, the impossible objects also include golf balls and tennis shoes in glass bottles with necks too thin to admit those foreign objects.

Thus the readers are treated to a comprehensive tour of the world's most fascinating puzzles, the famous alongside the obscure. The book contains information on how to make almost all of them, and how to solve a selected subset. The photography is superb. This book belongs in every public library as well as the private libraries of all puzzle lovers.

References:

[1] A. Liu, Mini-Review of various books including "Puzzles Old and New", *Crux Mathematicorum* **17** (1991) 74-77.

[2] A. Liu, Review of "New Book of Puzzles", *Crux Mathematicorum* **19** (1993) 13-14.

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LETTER TO THE EDITOR

To the Editor:

In his nice review of A. Soifer's book *Colorado Mathematical Olympiad* [1995: 198-201], Andrei Tooms discusses

> "**Problem 4.3.** Each square of a chessboard which is infinite in every direction contains a positive integer. The integer in each square equals the average of the four integers contained in the squares which lie directly above, below, left, and right of it. Show that every square of the board contains the same integer.
>
> The main condition of this problem can be written in the form
>
> $$F_{x,y} = \frac{1}{4}(F_{x-1,y} + F_{x+1,y} + F_{x,y-1} + F_{x,y+1}), \qquad (*)$$
>
> where variables $F_{x,y}$ are defined for all $x, y \in \mathbb{Z}$. In Problem 4.3 these variables are positive integers, but we may consider a more general case where they are any real numbers."

It should be pointed out that the result is not valid as stated for the general case where the numbers can be any real numbers. As a counterexample, just let $F_{x,y} = y$. Tooms inadvertently left out that the numbers should be bounded either above or below. A sophisticated proof for this bounded case is given in D. J. Newman, *A Problem Seminar*, Springer-Verlag, New York, 1982, pages 28 and 113. Incidentally, this case was submitted as a possible IMO problem by Poland some years ago but it was not used. It would be of interest to give a more elementary solution.

Regards,

Murray S. Klamkin
University of Alberta

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1996**, *although solutions received after that date will also be considered until the time when a solution is published.*

2071. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of an equilateral triangle ABC so that $PB \neq PC$, and BP and CP meet AC and AB at D and E respectively. Suppose that $PB : PC = AD : AE$. Find angle BPC.

2072. *Proposed by K. R. S. Sastry, Dodballapur, India.*

Find positive integers x, y, u, v such that

$$x^2 + y^2 = u^2 \qquad \text{and} \qquad x^2 - xy + y^2 = v^2.$$

(Equivalently, find a right-angled triangle with integral sides x, y surrounding the right angle and a triangle with sides x, y surrounding a $60°$ angle, and with the third side an integer in both cases.)

2073*. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be an interior point of an equilateral triangle $A_1A_2A_3$ with circumradius R, and let $R_1 = PA_1$, $R_2 = PA_2$, $R_3 = PA_3$. Prove or disprove that

$$R_1R_2R_3 \le \frac{9}{8}R^3.$$

Equality holds if P is the midpoint of a side. [Compare this problem with *Crux* 1895 [1995: 204].]

2074. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The number 3774 is divisible by 37, 34 and 74 but not by 77. Find another 4-digit integer $abcd$ that is divisible by the 2-digit numbers ab, ac, ad, bd and cd but is not divisible by bc.

2075. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*

ABC is a triangle with $\angle A < \angle B < \angle C$, and I is its incentre. BCL, ACM, ABN are the sides of the triangle with L on BC produced, etc., and the points L, M, N chosen so that

$$\angle CLI = \frac{1}{2}(\angle C - \angle B), \quad \angle AMI = \frac{1}{2}(\angle C - \angle A), \quad \angle BNI = \frac{1}{2}(\angle B - \angle A).$$

Prove that L, M, N are collinear.

2076. *Proposed by John Magill, Brighton, England.*

		4
	AC	
	C	24

This is a multiplicative magic square, where the product of each row, column and diagonal has the same value, ABCD. Each letter represents a digit, the same digit wherever it appears, and each cell contains an integer. Complete the square by entering the correct numbers in each of the nine cells.

2077. *Proposed by Joseph Zaks, University of Haifa, Israel.*

The determinant

$$\begin{vmatrix} z_1\overline{z_1} & z_1 & \overline{z_1} & 1 \\ z_2\overline{z_2} & z_2 & \overline{z_2} & 1 \\ z_3\overline{z_3} & z_3 & \overline{z_3} & 1 \\ z_4\overline{z_4} & z_4 & \overline{z_4} & 1 \end{vmatrix}$$

equals 0 if and only if the four complex numbers z_1, z_2, z_3, z_4 satisfy what simple geometric property?

2078*. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove or disprove that

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \le \sqrt{c(ab+1)}$$

for $a, b, c \ge 1$.

2079. *Proposed by Cristóbal Sánchez-Rubio, I. B. Penyagolosa, Castellón, Spain.*

An ellipse is inscribed in a rectangle. Prove that the contact points of the ellipse with the sides of the rectangle lie on the rectangular hyperbola which passes through the foci of the ellipse and whose asymptotes are parallel to the sides of the rectangle.

2080. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of maps f from S to S such that $f^{2080}(x) = x$ for every $x \in S$. (Here the superscript denotes iteration: $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(X))$ for all $n > 1$.)

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1980. [1994: 226] *Proposed by István Beck and Niels Bejlegaard, Stavanger, Norway.*

Find all sets of four points in the plane so that the sum of the distances from each of the points to the other three is a constant.

Solution by P. Penning, Delft, The Netherlands.

Let the points be A, B, C, D. Then

$$AB + AC + AD = L, \qquad BA + BC + BD = L,$$

$$CA + CB + CD = L, \qquad DA + DB + DC = L.$$

Subtract each of the latter three from the first:

$$AC + DA = BC + DB, \quad AB + DA = CB + DC, \quad AB + CA = DB + CD.$$

This leads [e.g., by adding two of the equations together and subtracting off the third] to:

$$AB = CD, \qquad AC = BD, \qquad AD = BC,$$

that is, the distance between any two of the points equals the distance between the other two points. This holds only for the vertices of a (possibly degenerate) rectangle. [For example, assuming that $ABDC$ is a simple (non-crossing) quadrilateral, the first two equations imply that $ABDC$ is a parallelogram, and then the third equation forces it to be a rectangle. — *Ed.*]

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; AART BLOKHUIS, Technical University Eindhoven, The Netherlands, and J. CHRIS FISHER, University of Regina; PAUL COLUCCI, student, University of Illinois; TIM CROSS, Wolverley High School, Kidderminster, U. K.; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; I. KLIMANN, student, Université de Paris 6, France; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; J. A. MCCALLUM, Medicine Hat, Alberta; VICTOR OXMAN, Haifa University, Israel; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; D. J. SMEENK, Zaltbommel, The Netherlands; DAVID C. VELLA, Skidmore College, Saratoga Springs, New York; CHRISTIAN WOLINSKI, Halifax, Nova Scotia; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposers.

Almost all of the solutions were similar to this one. Blokhuis and Fisher, Cross, McCallum, Oxman, Seimiya, Vella and the proposers all provided a more detailed discussion of why these four points form a rectangle.

Engelhaupt noted that if the four points were not coplanar then the points must form a tetrahedron with four congruent faces; that is, an isosceles *tetrahedron.*

This problem was first suggested in a more general form: find all sets of n points in the plane so that the sum of the distances from each of the points to the other $n-1$ is a constant. *We understand that Beck, Bejlegaard, Erdős and Fishburn address the more general problem in a paper "Equal distance sums in the plane", to appear soon in the Norwegian journal* Normat.

* * * * *

1985. [1994: 250] *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

Let $A_1A_2\dots A_{2n}$ be a regular $2n$-gon, $n > 1$. Translate every even-numbered vertex $A_2, A_4, \dots, A_{2n}$ by an equal (nonzero) amount to get new vertices $A_2', A_4', \dots, A_{2n}'$, and so that the new $2n$-gon $A_1A_2'A_3A_4'\dots A_{2n-1}A_{2n}'$ is still convex. Prove that the perimeter of $A_1A_2'\dots A_{2n-1}A_{2n}'$ is greater than the perimeter of $A_1A_2\dots A_{2n}$.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.

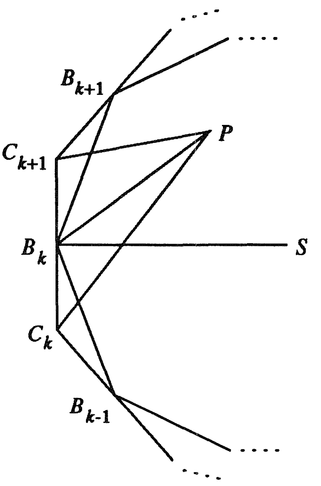
We first show the following

Lemma. Let $B_1B_2\dots B_n$ be a regular n-gon with center S and let P be any point lying in its plane. Then

$$\sum_{k=1}^{n} B_kP \ge \sum_{k=1}^{n} B_kS$$

with equality if and only if $P = S$.

Proof. Through each point B_k draw a line perpendicular to the line B_kS. These lines form another regular n-gon $C_1\dots C_n$ with the center S and the side equal to a, say. Then taking the subscripts modulo n, we obtain



$$\begin{aligned}\sum_{k=1}^{n} B_kP &= \frac{2}{a}\sum_{k=1}^{n}\frac{C_kC_{k+1}}{2}\cdot B_kP \\ &\ge \frac{2}{a}\sum_{k=1}^{n}[C_kC_{k+1}P] \ge \frac{2}{a}[C_1\dots C_n] \\ &= \frac{2}{a}\sum_{k=1}^{n}\frac{C_kC_{k+1}}{2}\cdot B_kS = \sum_{k=1}^{n} B_kS\end{aligned}$$

where $[\mathcal{F}]$ denotes the area of the figure $\mathcal{F}$. Equality holds if and only if $P = S$. This completes the proof of the lemma, and now we solve the problem.

Suppose first that n is odd. Let $B_1B_2\dots B_n$ be a regular n-gon with the center A_{2n} and such that $A_{2n-1} = B_n$ and $A_1 = B_{(n+1)/2}$. Then

$$\overrightarrow{A_{2k-1}A'_{2k}} = \overrightarrow{A_{2k+n}A'_{2k+n-1}} = \overrightarrow{B_kA'_{2n}}$$

for $k = 1, 2, \dots, n$ (the subscripts are modulo $2n$). [*Editor's note.* Here is a little explanation. Since $B_n = A_{2n-1}$, by rotation (and the symmetry of the $2n$-gon for n odd), the sets of vectors

$$\left\{\overrightarrow{A_{2n-1}A_{2n}}, \overrightarrow{A_1A_2}, \overrightarrow{A_3A_4}, \dots, \overrightarrow{A_{2n-3}A_{2n-2}}\right\},$$

$$\left\{\overrightarrow{B_nA_{2n}}, \overrightarrow{B_1A_{2n}}, \overrightarrow{B_2A_{2n}}, \dots, \overrightarrow{B_{n-1}A_{2n}}\right\}$$

and

$$\left\{\overrightarrow{A_1A_{2n}}, \overrightarrow{A_3A_2}, \overrightarrow{A_5A_4}, \dots, \overrightarrow{A_{2n-1}A_{2n-2}}\right\}$$

are equal, and thus (since the vectors $\overrightarrow{A_{2k}A'_{2k}}$ are all the same) so are the sets

$$\left\{\overrightarrow{A_{2n-1}A'_{2n}}, \overrightarrow{A_1A'_2}, \overrightarrow{A_3A'_4}, \dots, \overrightarrow{A_{2n-3}A'_{2n-2}}\right\},$$

$$\left\{\overrightarrow{B_nA'_{2n}}, \overrightarrow{B_1A'_{2n}}, \overrightarrow{B_2A'_{2n}}, \dots, \overrightarrow{B_{n-1}A'_{2n}}\right\}$$

and

$$\left\{\overrightarrow{A_1A'_{2n}}, \overrightarrow{A_3A'_2}, \overrightarrow{A_5A'_4}, \dots, \overrightarrow{A_{2n-1}A'_{2n-2}}\right\}.$$

End of explanation, part 1.] Therefore using the lemma on the n-gon $B_1\dots B_n$ with $S = A_{2n}$ and $P = A'_{2n}$, we obtain

$$\sum_{k=1}^{n}(A_{2k-1}A'_{2k} + A'_{2k}A_{2k+1}) = 2\sum_{k=1}^{n} B_kA'_{2n} > 2\sum_{k=1}^{n} B_kA_{2n} = \sum_{k=1}^{2n} A_kA_{k+1}.$$

Now suppose that n is even. Let $B_1B_2\dots B_{2n}$ be a regular $2n$-gon with the center A_{2n} and such that $A_{2n-1} = B_n$ and $A_1 = B_1$. Then

$$\overrightarrow{A'_{2k}A_{2k+1}} = \overrightarrow{A'_{2n}B_{2k+1}} \qquad \text{and} \qquad \overrightarrow{A_{2k-1}A'_{2k}} = \overrightarrow{B_{2k+n}A'_{2n}}$$

for $k = 1, 2, \dots, n$. [*Editor's note.* Similarly by rotation,

$$\left\{\overrightarrow{A_{2n}A_1}, \overrightarrow{A_2A_3}, \dots, \overrightarrow{A_{2n-2}A_{2n-1}}\right\} = \left\{\overrightarrow{A_{2n}B_1}, \overrightarrow{A_{2n}B_3}, \dots, \overrightarrow{A_{2n}B_{2n-1}}\right\}$$

and

$$\left\{\overrightarrow{A_{2n-1}A_{2n}}, \overrightarrow{A_1A_2}, \dots, \overrightarrow{A_{2n-3}A_{2n-2}}\right\} = \left\{\overrightarrow{B_nA_{2n}}, \overrightarrow{B_{n+2}A_{2n}}, \dots, \overrightarrow{B_{n-2}A_{2n}}\right\},$$

thus

$$\left\{\overrightarrow{A'_{2n}A_1}, \overrightarrow{A'_2A_3}, \dots, \overrightarrow{A'_{2n-2}A_{2n-1}}\right\} = \left\{\overrightarrow{A'_{2n}B_1}, \overrightarrow{A'_{2n}B_3}, \dots, \overrightarrow{A'_{2n}B_{2n-1}}\right\}$$

and

$$\left\{\overrightarrow{A_{2n-1}A'_{2n}}, \overrightarrow{A_1A'_2}, \ldots, \overrightarrow{A_{2n-3}A'_{2n-2}}\right\} = \left\{\overrightarrow{B_nA'_{2n}}, \overrightarrow{B_{n+2}A'_{2n}}, \ldots, \overrightarrow{B_{n-2}A'_{2n}}\right\}.$$

End of explanation, part 2!] Therefore [since n is even, and by the lemma]

$$\sum_{k=1}^{n}(A_{2k-1}A'_{2k} + A'_{2k}A_{2k+1}) = \sum_{k=1}^{2n} B_kA'_{2n} > \sum_{k=1}^{2n} B_kA_{2n} = \sum_{k=1}^{2n} A_kA_{k+1},$$

which completes the solution to the problem.

Remark. The assumption that $A_1A'_2\ldots A_{2n-1}A'_{2n}$ is convex was not necessary.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; and the proposers.

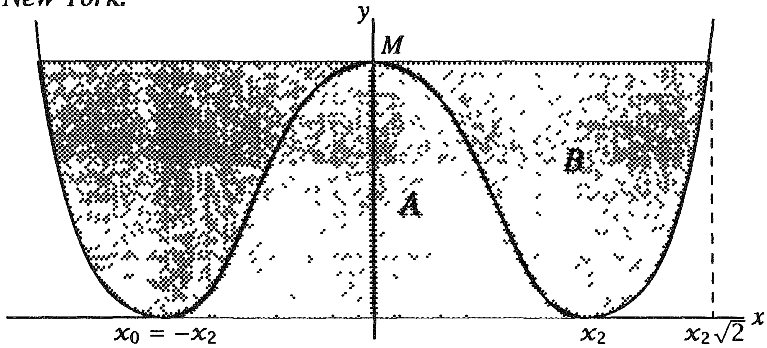
These other solvers first prove that the $2n$-gons $A_1A_2\ldots A_{2n}$ and $A_1A'_2\ldots A_{2n-1}A'_{2n}$ have the same area (the proposers require convexity of the second $2n$-gon for this), and then use the fact that the regular m-gon has the smallest perimeter among all m-gons of fixed area to obtain the result. The proposers conjecture that the inequality holds even if $A_1A'_2\ldots A_{2n-1}A'_{2n}$ is not convex, and as the above solution shows, this is indeed true. The problem was suggested by a problem in the Fall 1993 Tournament of the Towns.

* * * * *

1986. [1994: 250] *Proposed by Jisho Kotani, Akita, Japan.*

Suppose that the 4th-degree polynomial $p(x)$ has three local extrema, at $x = x_0$, x_1 and x_2, so that $p(x_0) = p(x_2) = m$ and $p(x_1) = M$, where $m < M$. Let A be the area of the region bounded by $y = m$ and $y = p(x)$, and let B be the area of the region bounded by $y = p(x)$ and $y = M$. Find B/A.

Solution by Charles R. Diminnie, St. Bonaventure University, St. Bonaventure, New York.



Since A and B are unaffected by translation in the x or y direction, we may assume without loss of generality that $x_1 = 0$ and $m = 0$. Using Taylor's Formula, the conditions $p(x_0) = p'(x_0) = 0$ and $p(x_2) = p'(x_2) = 0$ imply that $p(x)$ is divisible by $(x - x_0)^2$ and $(x - x_2)^2$. Since the degree of $p(x)$ is 4, we must have $p(x) = a(x - x_0)^2(x - x_2)^2$, for some $a \neq 0$. Then, the condition $p'(0) = 0$ implies that $-2ax_0x_2(x_0 + x_2) = 0$, i.e., $x_0 = -x_2$. Using the condition $p(0) = M$, we get

$$p(x) = a(x^4 - 2x_2^2x^2 + x_2^4) = ax^4 - 2ax_2^2x^2 + M$$

[so $M = ax_2^4$]. This last expression implies that $p(x) = M$ when $x = 0, \pm x_2\sqrt{2}$ (see the Figure). Finally, since $p(x)$ is symmetric with respect to the y-axis,

$$A = 2\int_0^{x_2} p(x)\,dx = 2\left(\frac{ax_2^5}{5} - \frac{2ax_2^2x_2^3}{3} + ax_2^4x_2\right) = \frac{16ax_2^5}{15},$$

$$\begin{aligned} B &= 2\int_0^{x_2\sqrt{2}} (M - p(x))\,dx = 2a\int_0^{x_2\sqrt{2}} (2x_2^2x^2 - x^4)\,dx \\ &= 2a\left(\frac{2x_2^2x_2^3 2\sqrt{2}}{3} - \frac{x_2^5 4\sqrt{2}}{5}\right) = \frac{16ax_2^5\sqrt{2}}{15}, \end{aligned}$$

and

$$\frac{B}{A} = \sqrt{2}.$$

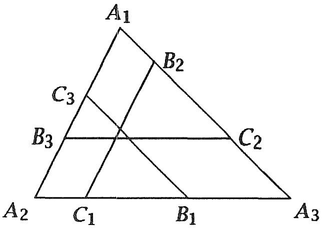
Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; KEITH EKBLAW, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; PAUL PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Four incorrect solutions were received.

* * * * *

1987. [1994: 250] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the figure, $B_2C_1 \| A_1A_2$, $B_3C_2 \| A_2A_3$ and $B_1C_3 \| A_3A_1$. Prove that B_2C_1, B_3C_2 and B_1C_3 are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1.$$



Solution by Christopher J. Bradley, Clifton College, Bristol, U. K.

Choose A_1 as origin and let $\overrightarrow{A_1A_2} = \mathbf{x}$ and $\overrightarrow{A_1A_3} = \mathbf{y}$. Suppose

$$\overrightarrow{A_1C_3} = p\mathbf{x}, \quad \overrightarrow{A_1B_2} = q\mathbf{y}, \qquad \text{where } p, q \neq 0.$$

Then, since C_3B_1 is parallel to A_1A_3 and B_1 lies on A_2A_3, the position vector for B_1 must be $p\mathbf{x} + (1-p)\mathbf{y}$. Similarly, the position vector for C_1 must be $q\mathbf{y} + (1-q)\mathbf{x}$. Let the intersection of C_3B_1 and B_2C_1 be P; since $A_1C_3PB_2$ is a parallelogram the position vector for P is $p\mathbf{x} + q\mathbf{y}$. Since B_3C_2 is parallel to A_2A_3 it follows that if B_3 has position vector $u\mathbf{x}$ then C_2 has position vector $u\mathbf{y}$. Also $u \neq 1$ since B_3C_2 is distinct from A_2A_3.

The equation of B_3C_2 is $\mathbf{r} = tu\mathbf{y} + (u - tu)\mathbf{x}$, $t \in \mathbb{R}$, and this line passes through P if and only if $u = p + q$ (with $t = q/(p+q)$). In other words,

$$B_1C_3,\ B_2C_1,\ B_3C_2 \quad \text{are concurrent if and only if} \quad u = p + q.$$

In terms of p, q, u we have

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = \frac{p}{u-p} \cdot \frac{q}{1-p-q} \cdot \frac{u-1}{q-u}.$$

This product equals 1 if and only if $pq(u-1) = (u-p)(1-p-q)(q-u)$, which holds (after some arithmetic) if and only if

$$u = p + q \quad \text{or} \quad u = \frac{pq}{p+q-1}.$$

Thus the condition that the product be 1 is apparently necessary but not sufficient. (For example, let $p = q = -u = 1/3$.) If the given lines were required to intersect the interior of the triangle, then u could not equal $pq/(p+q-1)$ since the three intersection points are all inside the triangle if and only if $0 < p, q, p+q < 1$. We conclude therefore: *if B_1, B_2, B_3 are points on the interior of the sides of $\Delta A_1A_2A_3$ then B_1C_3, B_2C_1, B_3C_2 are concurrent if and only if*

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1.$$

This is not entirely satisfactory because we would like a condition involving directed line segments that would be necessary and sufficient whether or not the given lines meet the interior of the triangle. With the usual sign conventions for directed line segments, one such condition is

$$\frac{A_1B_3}{A_1A_2} + \frac{A_2B_1}{A_2A_3} + \frac{A_3B_2}{A_3A_1} = 2,$$

since this gives the equation $u + (1-p) + (1-q) = 2$, which holds if and only if $u = p + q$, which (as we saw) holds if and only if the given lines are concurrent.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I. B. Leopoldo Cano, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, Kanpur, India (a partial solution); D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Note that the problem is unclear about whether the sides of the given triangle should be considered line segments or infinite lines. Among the solvers, only Bradley (in the featured solution) dealt with both interpretations. Most other solvers went along with the proposer's tacit assumption that the sides are line segments.

Bellot and López provide two solutions; in one they find the area of the triangle bounded by B_1C_3, B_2C_1, and B_3C_2 to be

$$\text{Area}\,(\Delta A_1A_2A_3)\cdot\left(\frac{A_1B_3}{A_1A_2}+\frac{A_2B_1}{A_2A_3}+\frac{A_3B_2}{A_3A_1}-2\right)^2 .$$

Note how this expression gives geometric meaning to Bradley's formula (given above). They provide references [1] *and* [3]*, which are concerned with areas related to our problem.*

Konečný observes that our problem is a special case of a theorem of Klamkin and Liu [2]*, although he admits that it is easier to obtain our result directly than to derive it as a corollary. Nevertheless, he found that the formula they obtained on page* 50 *of this article (to solve an earlier problem posed by Gülicher) leads to a rather nice condition for concurrency that is equivalent to Bradley's:*

$$\frac{B_3A_2}{A_1A_2}+\frac{B_1A_3}{A_2A_3}+\frac{B_2A_1}{A_3A_1}=1 .$$

References:

[1] M. N. Aref and W. Wernick, *Problems and Solutions in Euclidean Geometry.* Dover, 1968, problem 4.18, pages 120–121.

[2] M. S. Klamkin and Andy Liu, Simultaneous generalizations of the theorems of Ceva and Menelaus. *Math. Mag.* 65:1 (1992), 48–52.

[3] V. V. Prasolov, *Zadachi po Planimetrii, Vol. 1.* Nauka, Moscow, 1991, problem 1.35.

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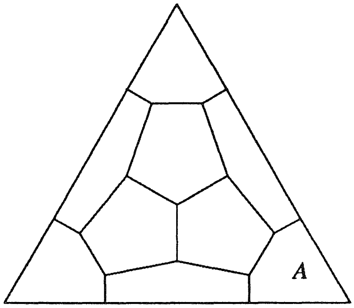
1988. [1994: 251] *Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.*

Show that any triangle can be dissected into 19 or fewer convex pentagons of equal area.

Solution by Cyrus Hsia, student, Woburn Collegiate, Scarborough, Ontario.

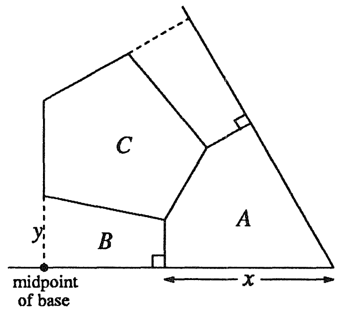
We claim that any triangle can be dissected into 9 convex pentagons of equal area.

Take an equilateral triangle and dissect it as follows:



This can be done if we consider the corner A. We can find some x such that the area of A is $1/9$ the area of the triangle. Once we have this we can find y so that the area of B is $1/18$ the area of the triangle. Then the area of C will be

$$\frac{1}{3} - 2\cdot\frac{1}{18} - \frac{1}{9} = \frac{1}{9}$$



the area of the triangle. By symmetry all nine pentagons will have equal areas. Further all other triangles are shears of the equilateral triangle which preserves the ratio of areas, and we are done.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.

Hess's solution contained the same dissection into nine pentagons. Is a dissection into fewer convex pentagons possible?

* * * * *

1989*. [1994: 251] *Proposed by Ignotus, Engelberg, Switzerland.*

The sequence of non-negative integers $0, 1, 3, 0, 4, 9, 3, 10, \ldots$ is defined as follows: $a_0 = 0$ and

$$a_n = \begin{cases} a_{n-1} - n & \text{if } a_{n-1} \geq n \\ a_{n-1} + n & \text{otherwise} \end{cases}$$

for $n \geq 1$. Does every non-negative integer occur in the sequence? [*Editor's note.* This problem is closely related to problem 12, proposed but not used at the 1993 IMO; see [1994: 241].]

Solution by Andy Liu, University of Alberta.

Suppose $a_n = 0$ for some $n \geq 0$. We claim that for $0 \leq k \leq n$,

$$\text{(i) } a_{n+2k+1} = n - k + 1 \qquad \text{and} \quad \text{(ii) } a_{n+2k+2} = 2n + k + 3.$$

This is the case for $k = 0$ since $a_{n+1} = n + 1$ and $a_{n+2} = 2n + 3$. Suppose the results hold for some $k \geq 0$. Then since $2n + k + 3 \geq n + 2k + 3$ we get

$$a_{n+2(k+1)+1} = a_{n+2k+3} = a_{n+2k+2} - (n + 2k + 3) = n - k = n - (k + 1) + 1,$$

$$a_{n+2(k+1)+2} = a_{n+2k+4} = a_{n+2k+3} + (n + 2k + 4) = 2n + (k + 1) + 3.$$

This completes the inductive argument.

It follows [from (ii) with $k = n$] that $a_{3n+3} = a_{3n+2} - (3n + 3) = 0$ if $a_n = 0$. Since $a_0 = 0$, the sequence contains 0 infinitely many times. Also, if $a_n = 0$ then [from the above, and setting $k = 0, 1, 2, \ldots, n$ in (i)] $a_{n+1}, a_{n+3}, \ldots, a_{3n+3}$ run through $n + 1, n, n - 1, \ldots, 0$. It follows that the sequence contains every non-negative integer infinitely many times.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge; CHARLES ASHBACHER, Cedar Rapids, Iowa; MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I. B. Sagasta, Logroño, Spain (2 solutions); DONNY CHEUNG, student, St. John's-Ravenscourt School, Winnipeg, Manitoba; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; P. PENNING, Delft, The Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's; R. P. SEALY, Mount Allison University, Sackville, New Brunswick; and ASHISH KR. SINGH, Kanpur, India.

As in the solution above, most solvers showed that every non-negative integer occurs infinitely often in the sequence. This was, in fact, the question raised by the proposer originally.

Since, as indicated in the solution above, $a_{3n+3} = 0$ wherever $a_n = 0$, it follows immediately that $a_n = 0$ for $n = \frac{1}{2}(3^k - 3)$ for all $k = 1, 2, 3, \ldots$. Hess, Jackson, Pye and Sealy all showed that $a_n = n$ whenever $n = \frac{1}{2}(3^k - 1)$ for $k = 1, 2, 3, \ldots$.

* * * * *

1990. [1994: 251] *Proposed by Leng Gangsong, Hunan Educational Institute, Changsha, China.*

Let r be the inradius of a tetrahedron $A_1A_2A_3A_4$, and let r_1, r_2, r_3, r_4 be the inradii of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. Prove that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \leq \frac{2}{r^2},$$

with equality if the tetrahedron is regular.

Solution by the proposer.

Let S_1, S_2, S_3, S_4 be the areas of the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$; let α, β, γ be the dihedral angles at the edges A_2A_3, A_2A_4, A_3A_4; and let h_1 be the altitude from the vertex A_1 of the tetrahedron and $h_1' = A_1E$ be the altitude of the face $A_1A_2A_3$. Then

$$\begin{aligned} h_1 &= h_1' \sin\alpha = \frac{2S_4}{A_2A_3}\sqrt{(1+\cos\alpha)(1-\cos\alpha)} \\ &= \frac{2}{A_2A_3}\sqrt{(S_4+S_4\cos\alpha)(S_4-S_4\cos\alpha)}\,, \end{aligned} \tag{1}$$

and analogously

$$h_1 = \frac{2}{A_2A_4}\sqrt{(S_3+S_3\cos\beta)(S_3-S_3\cos\beta)}\,, \tag{2}$$

$$h_1 = \frac{2}{A_3A_4}\sqrt{(S_2+S_2\cos\gamma)(S_2-S_2\cos\gamma)}\,. \tag{3}$$

From (1), (2) and (3) we get

$$h_1 = \frac{2}{A_2A_3+A_2A_4+A_3A_4}\cdot Q \tag{4}$$

where

$$\begin{aligned} Q = \sqrt{(S_4+S_4\cos\alpha)(S_4-S_4\cos\alpha)} &+ \sqrt{(S_3+S_3\cos\beta)(S_3-S_3\cos\beta)} \\ &+ \sqrt{(S_2+S_2\cos\gamma)(S_2-S_2\cos\gamma)}\,. \end{aligned}$$

According to Cauchy's inequality we have

$$\begin{aligned} Q &\le \left\{(S_4+S_4\cos\alpha)+(S_3+S_3\cos\beta)+(S_2+S_2\cos\gamma)\right\}^{1/2} \\ &\qquad \cdot\left\{(S_4-S_4\cos\alpha)+(S_3-S_3\cos\beta)+(S_2-S_2\cos\gamma)\right\}^{1/2} \\ &= \left\{S_4+S_3+S_2+(S_4\cos\alpha+S_3\cos\beta+S_2\cos\gamma)\right\}^{1/2} \\ &\qquad \cdot\left\{S_4+S_3+S_2-(S_4\cos\alpha+S_3\cos\beta+S_2\cos\gamma)\right\}^{1/2} \\ &= (S_4+S_3+S_2+S_1)^{1/2}(S_4+S_3+S_2-S_1)^{1/2} = S^{1/2}(S-2S_1)^{1/2} \end{aligned} \tag{5}$$

where $S = S_1+S_2+S_3+S_4$. From (4) and (5), we get

$$h_1 \le \frac{2\sqrt{S(S-2S_1)}}{A_2A_3+A_2A_4+A_3A_4}\,.$$

Therefore [since $rS = h_1S_1$ = three times the volume of the tetrahedron] we obtain

$$r = \frac{h_1 S_1}{S} \le \frac{2S_1}{A_2A_3 + A_2A_4 + A_3A_4} \cdot \frac{\sqrt{S(S-2S_1)}}{S} = r_1 \cdot \sqrt{\frac{S-2S_1}{S}}\,.$$

In the same manner, we have

$$r \le r_i \sqrt{\frac{S-2S_i}{S}}\,, \quad i = 1,2,3,4,$$

hence

$$\sum_{i=1}^{4} \frac{1}{r_i^2} \le \frac{1}{r^2} \sum_{i=1}^{4} \frac{S-2S_i}{S} = \frac{2}{r^2}\,.$$

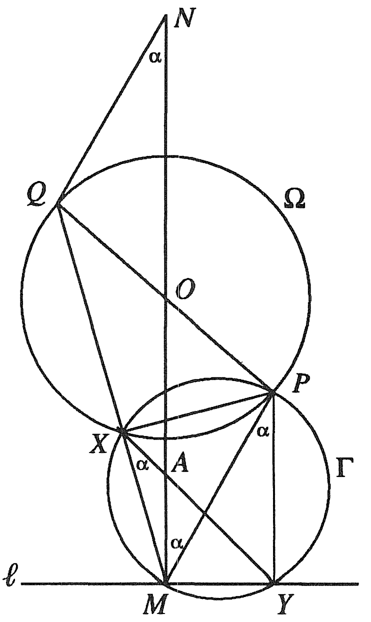
A partial solution and a comment were received.

*   *   *   *   *

1991. [1994: 284] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Ω is a fixed circle with center O. Let M be the foot of the perpendicular from O to a fixed line ℓ, and let P be a variable point on Ω. Let Γ be the circle with diameter PM, intersecting Ω and ℓ again at X and Y respectively. Prove that the line XY always passes through a fixed point.

Solution by Waldemar Pompe, student, University of Warsaw, Poland.



Let N be a symmetric point to M with respect to O, let PQ be a diameter of Ω and let $A = XY \cap MN$. We show that the length of MA is constant, so A is the required fixed point. Since PM is a diameter of Γ, PY and MN are parallel and also M, X, Q are collinear. Moreover, since $PO = QO$ and $MO = NO$, $QN \parallel PM$. If M lies outside Ω, we obtain

$$\angle MXA = \angle MPY = \angle PMN = \angle MNQ$$

(if M lies inside the circle Ω, we similarly prove that $\angle MXA = \angle MNQ$), which means that X, A, N, Q are concyclic. Therefore $MX \cdot MQ = MA \cdot MN$, which gives

$$MA = \frac{MX \cdot MQ}{MN}\,.$$

The product $MX \cdot MQ$ is of course constant, $MN = 2MO$ is constant too, so MA is constant, as required.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; GERD BARON, Technische Universität Wien, Austria (two solutions); FRANCISCO BELLOT ROSADO, I. B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U. K.; CYRUS HSIA, student, Woburn Collegiate, Scarborough, Ontario; PAUL PENNING, Delft, The Netherlands; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D. J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Most solvers used analytic geometry, though the proposer's solution was somewhat similar to Pompe's.

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1992. [1994: 285] *Proposed by K. R. S. Sastry, Dodballapur, India.*

Find all pairs of cubic polynomials

$$x^3 + ax^2 + bx + c \quad \text{and} \quad x^3 + bx^2 + ax + c,$$

where a and b are positive integers and c is a nonzero integer, so that they have three integer roots each, exactly one of which is common.

Solution by Robert B. Israel, University of British Columbia.

The only solution pairs are

$$\{x^3 + 11x^2 + 8x - 20 = (x-1)(x+10)(x+2),$$
$$x^3 + 8x^2 + 11x - 20 = (x-1)(x+4)(x+5)\}$$

and

$$\{x^3 + 13x^2 + 10x - 24 = (x-1)(x+12)(x+2),$$
$$x^3 + 10x^2 + 13x - 24 = (x-1)(x+3)(x+8)\}.$$

Moreover, if we remove the requirement that a and b be positive, the only additional pairs are

$$\{x^3 - n^2x^2 - x + n^2 = (x-1)(x+1)(x-n^2),$$
$$x^3 - x^2 - n^2x + n^2 = (x-1)(x+n)(x-n)\}$$

for integers $n \geq 2$.

We can write the polynomials as

$$(x-A)(x-B)(x-C) \quad \text{and} \quad (x-A)(x-D)(x-E)$$

where A, B, C, D and E are integers, B and C different from D and E. For c to be the same in both polynomials, we need $ABC = ADE$. Since $c \neq 0$ we have $BC = DE$, and none of A, B, C, D and E can be 0. From the coefficients of x^2 and x, we obtain

$$AB + AC + BC = -A - D - E \quad \text{and} \quad AD + AE + DE = -A - B - C.$$

Subtracting these,

$$0 = AB + AC - AD - AE - B - C + D + E = (A-1)(B + C - D - E),$$

so $A = 1$ or $B + C = D + E$.

Now if $B + C = D + E$ and $BC = DE$, we get

$$(B-C)^2 = (B+C)^2 - 4BC = (D-E)^2,$$

i.e. $B - C = \pm(D - E)$, and then $\{B, C\} = \{D, E\}$. So we conclude that $A = 1$.

Now substituting $A = 1$ in our equations we have $B+C+BC = -1-D-E$ and $BC = DE$. Solving for B and D we get

$$B = -\frac{E(C+E+1)}{C+E+CE}, \qquad D = -\frac{C(C+E+1)}{C+E+CE}.$$

We can assume without loss of generality that B has the largest absolute value among B, C, D and E. Since $BC = DE$, C has the least absolute value. Moreover, we must actually have $|B| > |D|$ and $|B| > |E|$; for instance, B and E must be different, so if $|B| = |E|$, we must have $E = -B$, $D = -C$ and $B + C + BC = -1 + B + C$, i.e. $BC = -1$ (which implies $B = D = \pm 1$, and is not allowed). By symmetry, $|B| > |D|$ too, and thus $|E| > |C|$.

From $|B| > |E|$ and the equation for B in terms of C and E, we have $|C+E+1| > |C+E+CE|$. Now $|C+E+1| \leq |C| + |E| + 1$ and $|C+E+CE| \geq |CE| - |C| - |E|$, so

$$(|C|-2)(|E|-2) = |CE| - 2|C| - 2|E| + 4 < 5.$$

Therefore $|C| \leq 3$ (if $|C| \geq 4$ we would have from $|E| > |C|$ that $|E| \geq 5$, and $(|C|-2)(|E|-2) \geq 6$). If $C > 0$ the inequality $|C+E+1| > |C+(1+C)E|$ is equivalent to

$$-\frac{2C+1}{C+2} < E < \frac{1}{C},$$

which implies $-2 < E < 1$, and that is impossible since $|E| > |C|$. Thus we are left with the cases $C = -3$, -2 and -1.

If $C = -3$, we have

$$D = \frac{6-3E}{3+2E} = -\frac{3}{2} + \frac{21}{6+4E}.$$

Note that this is a decreasing function of E on $(-\infty, -3/2)$ and $(-3/2, \infty)$. If $E > -3/2$ we would need $E \geq 4$, and then D is between $-6/11$ and $-3/2$, which is not allowed [because $|D| > |C|$]. If $E < -3/2$ we would need $E \leq -4$, and D is between -3.6 and $-3/2$, again not allowed.

If $C = -2$, we have

$$D = \frac{2-2E}{2+E} = -2 + \frac{6}{2+E}.$$

If this is an integer, $2+E$ must be one of -6, -3, -2, -1, 1, 2, 3 or 6. The cases where $|D|>2$ and $|E|>2$ are $E=-8$ and -5, which correspond to our two solutions, and $E=-4$ and -3, which are the same solutions with D and E interchanged.

If $C=-1$, we have $D=-E$ and $B=E^2$. This yields the family of solutions

$$\Big\{x^3-n^2x^2-x+n^2=(x-1)(x+1)(x-n^2),$$
$$x^3-x^2-n^2x+n^2=(x-1)(x+n)(x-n)\Big\}$$

where $n=|E|\ge 2$.

Also solved by GERD BARON, Technische Universät Wien, Austria; C. J. BRADLEY, Clifton College, Bristol, U. K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; F. J. FLANIGAN, San Jose State University, San Jose, California; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; and the proposer. There were three incorrect solutions submitted.

Janous also solved the more general problem with no restriction on the sign of a and b. The proposer also identified a single example with $c=0$, namely

$$\Big\{x^3+5x^2+6x=x(x+2)(x+3),\quad x^3+6x^2+5x=x(x+1)(x+5)\Big\},$$

but was unable to show that no other pairs exist. Flanigan points out that $c=0$ corresponds to Problem 251 in Mathematics Magazine. *A solution by E. P. Starke appears on page 284 of the May/June issue in 1956 (which proves that the pair listed above is the only example with $c=0$).*

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LETTER TO THE EDITOR

To the Editor:

You probably have seen the twenty-page writeup on Leon Bankoff in *The College Mathematics Journal*, Vol. 23, No. 2 (March 1992), and realize what a remarkable person Leon is. Dentist (he practised till he was 85), guitar player, jazz pianist, sculptor and excellent mathematician, to name a few of his talents.

Leon will be 87 on December 13. Could you put a brief but heartfelt **HAPPY BIRTHDAY** greeting in an upcoming issue of *Crux*?

Regards,

Name Withheld On Request

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