Crux Mathematicorum

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Crux Mathematicorum
with Mathematical Mayhem

EDITORIAL

Einstein said, "You do not really understand something unless you can explain it to your grandmother" and I tend to agree. Sure, some explanations might take a while, but if you truly understand something, you should know it to the core and be able to explain it all the way from the basics up.

In mathematics though, it is most beneficial not to just understand a process or a concept, but to be comfortable moving between different representations of the same object in order to call upon the most applicable technique for any given problem. It is this flexibility that allows for creativity within our subject and often yields the most surprising results. In this sense, mathematics has long since become an interdisciplinary subject all in itself: many innovations now occur when two or more mathematical areas, often seemingly disconnected, are involved in the process. This ability to choose one representation and technique over the other is most useful in problem solving on any level.

So here are some graphical representations of mathematical results that would ordinarily be written algebraically. See if you can figure out what statement each picture embodies.
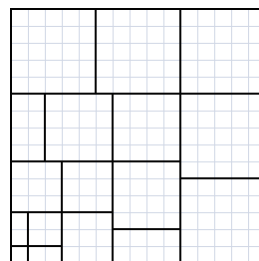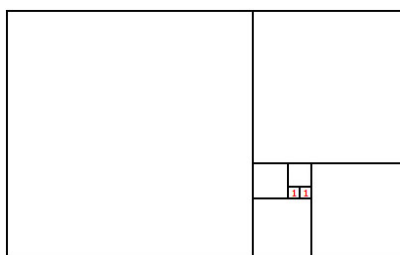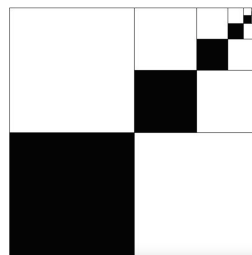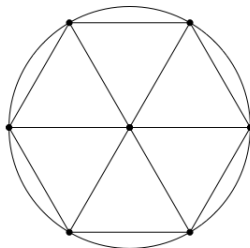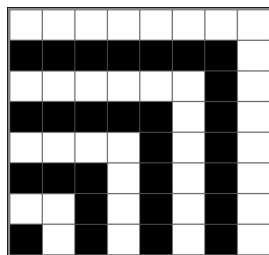
Thanks to `http://mathoverflow.net/` for the inspiration and some graphics.

Kseniya Garaschuk

THE CONTEST CORNER
No. 35
Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **August 1, 2016**, *although late solutions will also be considered until a solution is published.*
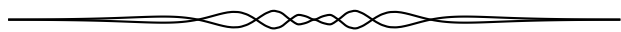
CC171. The zeroes of the polynomial $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all the possible values of the number a?

CC172. What is the area of regular hexagon $ABCDEF$ with $A(0,0)$ and $C(7,1)$?

CC173. In the following figure, an isosceles triangle with $AB = 12$ is divided into 4 polygons of equal area using segments perpendicular to AB. Find x.
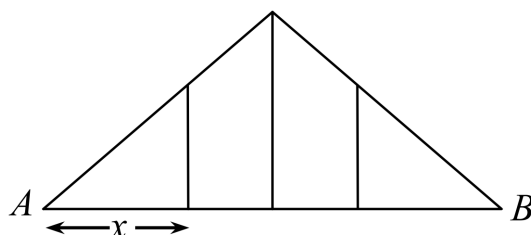
CC174. Evaluate $\sqrt{1111 - 22}$ and $\sqrt{111111 - 222}$. Conjecture the result for $\sqrt{111111111111111111111111 - 222222222222}$ and prove it.

CC175. Twenty two mathematics contests were held with five prizes given out for each one. The organizers notice that for each pair of contests, there is exactly one participant who has won a prize in both contests. Show that one of the participants has won a prize in each of the contests.

. . . .

CC171. Les zéros du polynôme $f(x) = x^2 - ax + 2a$ sont des nombres entiers. Quelle est la somme des valeurs possibles de a?

CC172. Quelle est l'aire de l'hexagone régulier $ABCDEF$ avec $A(0;0)$ et $C(7;1)$?

CC173. Dans la figure suivante, le triangle isocèle avec $AB = 12$ est divisé en 4 polygones de même aire en n'utilisant que des segments perpendiculaires à AB. Quelle est la longueur x?
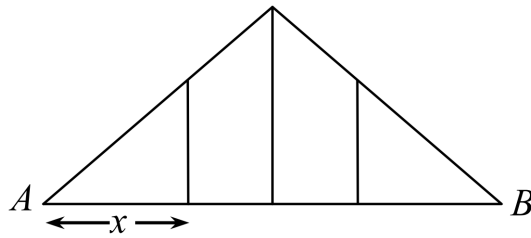
CC174. Calculer $\sqrt{1111 - 22}$ et $\sqrt{111111 - 222}$. Conjecturer le résultat de $\sqrt{111111111111111111111111 - 222222222222}$ et le démonter.

CC175. Vingt-deux concours de mathématique ont eu lieu, et pour chacun d'entre eux on a remis cinq prix. Les organisateurs s'aperçoivent alors que pour chaque paire de concours, il y a exactement un participant qui a gagné un prix dans ces deux concours. Montrer qu'un des participants a gagné un prix dans chacun des concours.

Math Quotes

If you do not or cannot humanize the learning of mathematics through joy, shared discoveries and mutual bewilderment/wonder FIRST, then it's not just students you will lose to disengagement. You will also lose teachers. Mathematics has become so procedural and sterile, having all the fun of the smell of antiseptics in an operating room.

At least be passionate. Leave your emotions about math tattooed all over the chalkboard. If you don't care, then why should your students...

By Sunil Singh.

CONTEST CORNER
SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(5), p. 188–189.

⎯⎯⎯◇◇◇◇◇⎯⎯⎯

CC121. Towns A and B are situated on two straight roads intersecting at the angle of $\angle ACB = 60°$. One way to get from A to B is by taking the bus which goes from A to C to B; this takes 11 minutes. Alternatively, you can walk from A directly to B, which takes an hour and 10 minutes. Finally, you can first walk from A to the road on which B is situated and then take the bus to B, but this takes longer still even if the bus comes immediately.

Find the distance from A to the intersection C if you walk at the speed of 3 km/h and the bus drives at the speed of 30 km/h.

Originally question 1 on the 1969 entrance exam to the mathematical department of the Moscow State University.

We received two correct solutions and one incorrect submission. We present the solution by Digby Smith, slightly modified by the editor.

Denote the distance AC by x, BC by y and AC by z. Using the "distance = speed × time" formula, it follows that $z = 3 \cdot (70/60) = 7/2$ km and $x + y = 30 \cdot (11/60) = 11/2$ km. Hence $y = 11/2 - x$. Applying the cosine law, it follows that

$$z^2 \quad = x^2 + y^2 - 2xy\cos(60°) \Leftrightarrow$$

$$\tfrac{49}{4} \quad = x^2 + \left(\tfrac{11}{2} - x\right)^2 - 2x\left(\tfrac{11}{2} - x\right)\cdot\tfrac{1}{2} \Leftrightarrow$$

$$\tfrac{49}{4} \quad = x^2 + \tfrac{121}{4} - 11x + x^2 - \tfrac{11}{2}x + x^2.$$

Rearranging,

$$3x^2 - \tfrac{33}{2}x + 18 \quad = 0 \Leftrightarrow$$
$$3(2x^2 - 11x + 12) \quad = 0 \Leftrightarrow$$
$$(2x - 3)(x - 4) \quad = 0.$$

Hence $x = 4$ km (which gives $y = \tfrac{3}{2}$ km), or $x = \tfrac{3}{2}$ km (which gives $y = 4$ km). We now need to verify the condition regarding the walking time when we combine the two modes of transportation. Let D be the closest point on CB to A. Then $AD = x\sin(60°)$ and $CD = x\cos(60°)$. The time needed to walk the distance AD and take the bus from D to B is (in minutes)

$$20AD + 2DB = 10\sqrt{3} \cdot x + 2DB.$$

Whether $DB = CD - CB = \tfrac{x}{2} - y$ or $DB = CB - CD = y - \tfrac{x}{2}$ depends on whether D is closer to C than B or not. We consider the two possible values of x calculated above.

If $x = 4$ then CD is longer than CB ($\triangle ABC$ is obtuse). That is,

$$DB = CD - CB = 2 - \frac{3}{2} = \frac{1}{2},$$

and the total time to walk from A to D and then take the bus to B is $40\sqrt{3} + 1$ minutes. It is easy to see that this is greater than the 70 minutes needed to walk from A to C and then from C to B.

If $x = \frac{3}{2}$ then CD is shorter than CB ($\triangle ABC$ is acute). That is,

$$DB = CB - CD = 4 - \frac{3}{4} = \frac{13}{4},$$

and the total time to walk from A to D and then take the bus to B is $15\sqrt{3} + \frac{13}{2}$ minutes. This is easily seen to be less than $15 \cdot 2 + \frac{14}{2} = 37$, which is much less than the 70 minutes needed to walk from A to C and then from C to B.

Hence only $x = 4$ satisfies the additional mixed mode of transportation time constraint. We conclude that the distance from A to C must be 4km.

CC122. The sequence $\{x_n\}$ is given by the following recursion formula:

$$x_1 = \frac{a}{2}, \quad x_n = \frac{a}{2} + \frac{x_{n-1}^2}{2}, \quad n \geq 2, \quad 0 < a < 1.$$

Find the limit of the sequence.

Originally question 12 on the 1977 entrance exam to the Moscow Physics Institute.

We received seven correct solutions and one incomplete submission. We present the solution by Henry Ricardo.

Using mathematical induction, we show that $x_n < x_{n+1} < 1$ for every positive integer n, thereby proving that the sequence is monotonic and bounded and hence has a limit.

When $n = 1$, we see that

$$x_1 = \frac{a}{2} < \frac{a}{2} + \frac{(a/2)^2}{2} = x_2 < \frac{1}{2} + \frac{1}{8} < 1.$$

Now suppose that $x_N < x_{N+1} < 1$ for some $N \geq 2$. Then, since $0 < a < 1$,

$$x_{N+1} = \frac{a}{2} + \frac{x_N^2}{2} < \overbrace{\frac{a}{2} + \frac{x_{N+1}^2}{2}}^{x_{N+2}} < \frac{a}{2} + \frac{1}{2} < 1.$$

and the induction is complete.

If $\lim_{n\to\infty} x_n = L$, then the recursion formula gives us $L = \frac{a}{2} + \frac{L^2}{2}$, or $L^2 - 2L + a = 0$, yielding $L = 1 \pm \sqrt{1-a}$. Since $x_n < 1$ for all n, we conclude that $L = 1 - \sqrt{1-a}$.

CC123. Find how many pairs of integers (x, y) satisfy the inequality

$$2^{x^2} + 2^{y^2} < 2^{1976}.$$

Originally question 5 on the 1976 entrance exam to the All-republican Distance Education Moscow School.

There were no correct submissions for this problem and one incorrect submission.

CC124. In a chess tournament, every participant played every other participant exactly once. In each game each participant scored 1 point for the win, 0.5 points for the tie and 0 points for the loss. At the end of the tournament, you discovered that in any group of any three participants there is one who, in the games against the other two, got 1.5 points. What is the maximum possible number of participants the tournament could have had?

Originally question 3 on the 1999 entrance exam to the mathematical-mechanical department of the Belorussian State University.

We received no solutions to this problem.

CC125. Orthogonal projections of a triangle ABC onto two perpendicular planes are equilateral triangles with side length 1. If the median AD of triangle ABC has length $\sqrt{\frac{9}{8}}$, find BC.

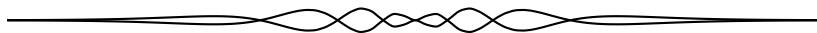*Originally question 4 on the 1969 entrance exam to the mathematical-mechanical department of Moscow State University.*

We received no solutions to this problem.

THE OLYMPIAD CORNER
No. 333
Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **August 1, 2016***, although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

OC231. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x + y) + xy$$

for all $x, y \in \mathbb{R}$.

OC232. Given a positive integer m, prove that there exists a positive integer n_0 such that all first digits after the decimal point of $\sqrt{n^2 + 817n + m}$ in decimal representation are equal, for all integers $n > n_0$.

OC233. Let ω be a circle with centre A and radius R. On the circumference of ω four distinct points B, C, G, H are taken in that order in such a way that G lies on the extended B-median of the triangle ABC, and H lies on the extension of the altitude of ABC from B. Let X be the intersection of the straight lines AC and GH. Show that the segment AX has length $2R$.

OC234. Let N be an integer, $N > 2$. Arnold and Bernold play the following game. There are initially N tokens on a pile. Arnold plays first and removes k tokens from the pile, $1 \leq k < N$. Then Bernold removes m tokens from the pile, $1 \leq m \leq 2k$ and so on, that is, each player, on their turn, removes a number of tokens from the pile that is between 1 and twice the number of tokens his opponent took last. The player that removes the last token wins.

For each value of N, find which player has a winning strategy and describe it.

OC235. Prove that there is a constant $c > 0$ with the following property: if a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \ldots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵∴∵

OC231. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$f(x)f(y) = f(x + y) + xy$$

pour tout $x, y \in \mathbb{R}$.

OC232. Soit m un entier positif. Démontrer qu'il existe un entier positif n_0 tel que le premier chiffre après la décimale dans $\sqrt{n^2 + 817n + m}$ est le même pour tout $n > n_0$.

OC233. Soit ω un cercle avec centre A et rayon R. Sur la circonférence de ω sont placés, dans cet ordre, quatre points distincts B, C, G et H, de façon à ce que G se trouve sur la B-médiane prolongée du triangle ABC et H se trouve sur l'altitude prolongée du triangle ABC à partir de B. Soit X l'intersection des lignes AC et GH. Démontrer que le segment AX est de longueur $2R$.

OC234. Soit N un entier, $N > 2$. Arnold et Bernold s'amusent au jeu suivant; au départ, N jetons se trouvent dans un tas. Arnold joue premier et enlève k jetons du tas, où $1 \le k < N$. Ensuite Bernold enlève m jetons du tas, où $1 \le m \le 2k$. Par la suite, en alternance, chaque joueur enlève un nombre de jetons entre 1 et deux fois le nombre de jetons enlevés par l'autre joueur à l'étape précédente. Le joueur qui enlève le dernier jeton gagne.

Pour chaque valeur de N, déterminer le joueur gagnant et décrire sa stratégie.

OC235. Démontrer qu'il existe une constante $c > 0$ avec la propriété suivante: si a, b, n sont des entiers positifs tels que $\gcd(a + i, b + j) > 1$ pour tout $i, j \in \{0, 1, \ldots, n\}$, alors

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$
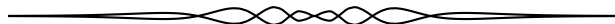
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(5), p. 194–195.

OC171. Find the 3-digit number for which the ratio between the number and the sum of its digits is minimal.

Originally problem 1 from the 2013 Albania Team Selection Test.

We received eight correct solutions. We present the solution by Andrea Fanchini.

Our number has the form $100a + 10b + c$. We have to minimize

$$\frac{100a + 10b + c}{a + b + c} = 1 + \frac{99a + 9b}{a + b + c}.$$

This ratio is minimized if $c = 9$, so we have

$$1 + \frac{99a + 9b}{a + b + c} = 1 + \frac{99a + 9b}{a + b + 9} = 10 + \frac{90a - 81}{a + b + 9}.$$

Now to minimize the ratio we have to maximize b, so $b = 9$

$$10 + \frac{90a - 81}{a + b + 9} = 10 + \frac{90a - 81}{a + 18} = 100 - \frac{1701}{a + 18}.$$

Finally to minimize the ratio we have to minimize a, so $a = 1$. Therefore the requested number is 199.

OC172. Determine all polynomials $P(x)$ with real coefficients such that

$$(x + 1)P(x - 1) - (x - 1)P(x)$$

is a constant polynomial.

Originally problem 1 from the 2013 Canadian Mathematical Olympiad.

We received five correct solutions. We present the solution by Šefket Arslangić.

Let $c \in \mathbb{R}$ be the constant such that

$$(x + 1)P(x - 1) - (x - 1)P(x) = c = \tfrac{c}{2}((x + 1) - (x - 1)).$$

Rearranging gives

$$(x + 1)(P(x - 1) - \tfrac{c}{2}) = (x - 1)(P(x) - \tfrac{c}{2}).$$

Let $Q(x) = P(x) - \tfrac{c}{2}$. Then,

$$(x + 1)Q(x - 1) = (x - 1)Q(x)$$

and from here $x = -1$ implies that $Q(-1) = 0$ and $x = 1$ implies that $Q(0) = 0$. Thus $Q(x) = x(x+1)R(x)$ for some real polynomial $R(x)$. By the above equation, we see that

$$(x+1)x(x-1)R(x-1) = (x+1)Q(x-1) = (x-1)Q(x) = (x-1)x(x+1)R(x).$$

Isolating gives

$$(x+1)x(x-1)(R(x-1) - R(x)) = 0$$

which holds for all values of x. Thus, $R(x) = a$ for some real constant a. Hence, since $P(x) = Q(x) + \frac{c}{2}$, we have that

$$P(x) = ax(x+1) + \frac{c}{2}$$

as desired.

OC173. Each integer is colored with one of two colors, red or blue. It is known that, for every finite set A of consecutive integers, the absolute value of the difference between the number of red and blue integers in the set A is at most 1000. Prove that there exists a set of 2000 consecutive integers in which there are exactly 1000 red numbers and 1000 numbers blue.

Originally problem 3 from the 2013 Italian Mathematical Olympiad.

We received no solutions to this problem.

Hint. Denote by a_k the difference of red numbers to blue numbers in the set of consecutive integers beginning with k and ending with $k + 1999$ (a total of 2000 integers). Justify why $a_k > 0$ for all k and then consider the numbers from 1 to 2002000.

OC174. Suppose that a and b are two distinct positive real numbers withe the property that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for all positive integers n. Prove that a and b are integers.

Originally problem 1 from day 2 of the 2013 Romania Team Selection Test 2013.

We received no solutions to this problem.

Hint. First show that $\lim_{n \to \infty} \frac{\lfloor nb \rfloor}{\lfloor na \rfloor} = \frac{b}{a}$.

OC175. The circumcircle of triangle ABC has centre O. The point P is the midpoint of the arc $\frown BAC$ and QP is a diameter. Let I be the incentre of the triangle ABC and let D be the intersection of PI and BC. The circumcircle of $\triangle AID$ and the extension of PA meet at F. E is a point on the line segment PD such that $DE = DQ$. Let r, R be the radius of the inscribed circle and circumcircle of $\triangle ABC$, respectively.

If $\angle AEF = \angle APE$, prove that

$$\sin^2 (\angle BAC) = \frac{2r}{R}.$$

Originally problem 2 from day 5 of the 2013 China Team Selection Test.

We received no solutions to this problem.

Hint. Show that $\angle DQP = \angle DQI$. Show next that $OI = OM$ where M is the midpoint of BC by arguing using similarity.
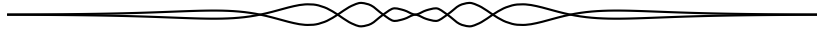
Constructing perfect squares

Problem 1.

Take the number 16 and insert the number 15 between its two digits: we get the number 1156. Into the middle of this number, insert another 15 to get 111556, then 11115556 and so on. Prove that all such numbers of the form $11\ldots155\ldots56$ are perfect squares. Consider the same process, but with numbers that result from inserting 48 into 49; that is, prove that numbers 4489, 444889, 44448889 and so on are perfect squares.

Problem 2.

Take the number 961 and insert n zeros between its digits. Show that the resulting number is a perfect square for any value of n. Find other 3-digit numbers with the same property.

Problems by I. Suev, from Kvant, 1981 (3), p.15.

FOCUS ON...

No. 17
Michel Bataille
Congruences (I)

Introduction and first examples

In many problems involving integers (divisibility, Diophantine equations, etc.), an ingenious appeal to congruences can be most helpful. Choosing the appropriate modulus is often the key, and sometimes, a very short solution follows! Here are some simple examples.

> If integers a, b, c satisfy $a^2 + b^2 = c^2$, show that at least one of a, b, c is a multiple of 5.

The choice of the modulus, 5, is obvious! Now, suppose that $a \not\equiv 0 \pmod 5$ and $b \not\equiv 0 \pmod 5$. Then a^2 and b^2 are congruent to 1 or 4, hence $a^2 + b^2$ is congruent to $0, 2$ or 3. Since squares are not congruent to 2 or 3, we must have $c^2 \equiv 0$ and so c is a multiple of 5.

> Show that $2^k + 9^m = 7^n$ is impossible with positive integers k, m, n.

Consider the equality modulo 7: $2^k + 2^m \equiv 0$. Now, the powers of 2 modulo 7 are 1 or 2 or 4. Since the sum of two of them is never a multiple of 7, the equality cannot hold.

> If k is an integer and $k > 1$, show that $2^k - 1$ is not the sum of two perfect squares.

Assume that for some integers m, n, we have $2^k - 1 = m^2 + n^2$. Since $2^k - 1$ is odd, m and n are of opposite parity, say m even and n odd and so, modulo 4, $m^2 \equiv 0$, $n^2 \equiv 1$ and $m^2 + n^2 \equiv 1$. This contradicts $2^k - 1 \equiv 3$ (since $k > 1$).

We now proceed to a selection of more elaborate examples.

Around divisibility

The problem of finding the rightmost digits of a natural number appears now and then. It is naturally linked to the divisibility by $10, 100, 1000, \ldots$ A good example is problem **2578** [2000 : 429 ; 2001 : 538] (slightly reworded):

> For $n \in \mathbb{N}$, determine the three rightmost digits of the number $\dfrac{1 + 5^{2n+1}}{6}$.

Let A be the given number. Then $A = \frac{1 - (-5)^{2n+1}}{1 - (-5)} = 1 - 5 + 5^2 - \cdots + 5^{2n}$. Modulo 10, $5^k \equiv 5$ for any integer $k \geq 1$, hence $A \equiv 1 - 5 + 5 - \cdots + 5 = 1$. The units digit of A is 1.

The tens digit of A is just the units digit of $B = \frac{A-1}{10}$. Now,

$$
\begin{aligned}
B = \frac{5^{2n-1} - 5^{2n-2} + \cdots + 5 - 1}{2} &= \frac{5^{2n-2}(5-1) + 5^{2n-4}(5-1) + \cdots + (5-1)}{2} \\
&= 2(5^{2n-2} + 5^{2n-4} + \cdots + 1).
\end{aligned}
$$

Thus, $B \equiv 0 + 0 + \cdots + 2 \pmod{10}$ and the tens digit of A is 2.

Lastly, we consider $C = \frac{B-2}{10} = \frac{A-21}{100}$. We readily see that $C = 5 + 5^3 + \cdots + 5^{2n-3}$, a sum of $n-1$ terms. It follows that, modulo 10, $C \equiv 0$ if n is odd and $C \equiv 5$ if n is even. The hundreds digit of A is 0 or 5 according as n is odd or even.

Our second example is inspired by problem **3294** [2007 : 486, 488 ; 2008 : 496]:

> Let T_k denote the kth triangular number $\frac{k(k+1)}{2}$. Show that the number $T_m T_n (T_m - T_n)(4T_m + 4T_n - 1)$ is divisible by 90 for all positive integers m and n.

Let $A(m,n) = T_m T_n (T_m - T_n)(4T_m + 4T_n - 1)$. We have to prove that $A(m,n)$ is divisible by $2, 5$ and 9.

(a) Clearly $A(n,m)$ is even since at least one of the numbers $T_n, T_m, T_m - T_n$ is even.

(b) From $2T_k = k(k+1)$, it is readily deduced that

$$
\begin{aligned}
T_k &\equiv 0 \pmod{5} \quad \text{if} \quad k \equiv 0 \text{ or } 4 \pmod{5}, \\
T_k &\equiv 1 \pmod{5} \quad \text{if} \quad k \equiv 1 \text{ or } 3 \pmod{5}, \\
T_k &\equiv 3 \pmod{5} \quad \text{if} \quad k \equiv 2 \pmod{5}.
\end{aligned}
$$

Thus, $T_n T_m \equiv 0 \pmod{5}$ if T_n or T_m is divisible by 5, $T_m - T_n \equiv 0 \pmod{5}$ if $T_n \equiv T_m \pmod{5}$ and otherwise $4(T_n + T_m) - 1 \equiv 4(3+1) - 1 \equiv 0 \pmod{5}$. As a result, $A(n,m)$ is divisible by 5 for all m, n.

(c) Again, it is easily seen that

$$
\begin{aligned}
T_k &\equiv 0 \pmod{3} \quad \text{if} \quad k \equiv 0 \text{ or } 2 \pmod{3}, \\
T_k &\equiv 1 \pmod{3} \quad \text{if} \quad k \equiv 1 \pmod{3}.
\end{aligned}
$$

Now, if $T_n \equiv T_m \equiv 0 \pmod{3}$, $A(n,m)$ is clearly divisible by 9; if one of T_n and T_m is congruent to 0 while the other is congruent to 1, then $T_n T_m (4T_n + 4T_m - 1)$ is divisible by 9; lastly, if $T_n \equiv T_m \equiv 1 \pmod{3}$, then $n \equiv m \equiv 1 \pmod{3}$ and from $2(T_m - T_n) = (m-n)(n+m+1)$, we see that $2(T_m - T_n)$ is divisible by 9, hence $T_m - T_n$ is divisible by 9 and so is $A(n,m)$.

A Diophantine equation

The following equation was set at the Olympiad of Turkey in 2005 [2009 : 145 ; 2010 : 228]:

Find all triples (m, n, k) of nonnegative integers such that $5^m + 7^n = k^3$.

We propose a solution a bit simpler than Oliver Geupel's featured solution.

The triple $(0, 1, 2)$ is obviously a solution for (m, n, k). We show that there are no other solutions.

First, assume that (m, n, k) is a solution with $m \geq 1$.

We cannot have $n = 0$ because $5^m + 1 \equiv 2 \pmod 4$ whereas a cube is congruent to $0, 1$, or 3 modulo 4. Thus $n \geq 1$ and so $k \geq 3$. Note that k is not a multiple of 7 (since 7 does not divide 5^m), hence $k^3 \equiv 1$ or $-1 \pmod 7$ and so $5^m \equiv 1$ or $-1 \pmod 7$. A quick study of the powers of 5 modulo 7 shows that m must be a multiple of 3, say $m = 3r$ for some integer $r \geq 1$.

The equation now rewrites as $7^n = (k - 5^r)(k^2 + k \cdot 5^r + 5^{2r})$, implying that $k - 5^r = 7^s$ for some integer s with $0 \leq s < n$. It follows that

$$7^{n-s} = k^2 + k \cdot 5^r + 5^{2r} = (5^r + 7^s)^2 + 5^r(5^r + 7^s) + 5^{2r} = 7^{2s} + 3 \cdot 7^s \cdot 5^r + 3 \cdot 5^{2r}.$$

Since 7 does not divide $3 \cdot 5^{2r}$, we must have $s = 0$ and so $7^n = 1 + 3 \cdot 5^r + 3 \cdot 5^{2r}$. Clearly $r > 1$, hence $7^n \equiv 1 \pmod{25}$, which calls for $n = 4u$ for some integer $u \geq 1$. But this leads to a contradiction since then $5^m + 7^{4u} \equiv 2 \pmod 4$. Thus, there is no solution with $m \geq 1$.

In the case when $m = 0$, the equation becomes $7^n = (k - 1)(k^2 + k + 1)$ and this easily leads to $k = 2$ and $n = 1$. (Details are left to the reader.)

Modulo 8

We devote our last paragraph to the modulus 8 because of the following property that all problem-solvers know (or should know – my advice to beginners!): the square of an odd integer is always congruent to 1 modulo 8. (It is also wise to remember that the square of an even integer is congruent to 0 or 4 modulo 8.)

In our first example (set by Prithwijit De in *The Mathematical Gazette* in 2007), we shall also use the following corollary: $x^4 \equiv 0$ or 1 modulo 16 according as x is even or odd.

Let N be an odd positive integer. Show that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4 = 8N + 1$$

has no integer solution.

Assume that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is a solution. Since the sum

$$S = x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4$$

is odd, the number m of odd integers among $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ must be an odd integer with $1 \leq m \leq 7$. It follows that $S \equiv m \pmod 8$ and so $m = 1$ (since

$8N + 1 \equiv 1 \pmod{8}$). Say x_1 is odd and $x_2, \ldots, x_8$ are even. But this implies $S \equiv 1 \pmod{16}$ while $8N + 1 \equiv 9 \pmod{16}$ (since N is odd). We have attained the desired contradiction.

A timely resort to the property is shown in the following variant of Edward Wang's solution to a problem of the 2000 Hungarian Olympiad [2004 : 204 ; 2006 : 35]:

> Find the integer solutions of $5x^2 - 14y^2 = 11z^2$.

Obviously $x = y = z = 0$ is a solution. We show that there are no other solutions. Assume that (x, y, z) is a solution with $(x, y, z) \neq (0, 0, 0)$ and let $d = \gcd(x, y, z)$. Then, $d \neq 0$ and $x = du$, $y = dv$, $z = dw$ where u, v, w are integers such that $\gcd(u, v, w) = 1$.

Clearly, u, v, w satisfy $5u^2 = 14v^2 + 11w^2$. If w were even, say $w = 2k$, then $5u^2$ would be even and so u would be even, say $u = 2\ell$. But this yields $10\ell^2 = 7v^2 + 22k^2$ which implies that v would be even as well. We have reached a contradiction with $\gcd(u, v, w) = 1$. Thus w is odd. The integer u is also odd (u even implies w even). It follows that u^2 and w^2 are congruent to 1 modulo 8 and so, $14v^2 = 5u^2 - 11w^2 \equiv 2 \pmod{8}$ in contradiction with the obvious results: $14v^2 \equiv 0 \pmod{8}$ if v is even, $14v^2 \equiv 6 \pmod{8}$ if v is odd. This completes the proof. In the forthcoming second part, we will consider properties and problems related to congruences modulo a prime. As usual, we end our number with a few exercises for practice.

Exercises

1. Show that if k is a positive odd integer and $2^k + 3^k = a^n$ for some integers a, n with $n \geq 2$, then k is a multiple of 5.

2. Let m, n be integers such that $m > n \geq 1$ and suppose that $m(m+n) = k^2 + \ell^2$ and $n(m - n) = 2(k^2 - \ell^2)$ for some integers k, ℓ. Prove that m, n have the same parity.

3. Find all odd positive integers a, b, c, d, n such that $a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n$.
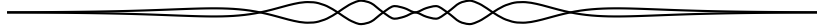
Introduction to Inequalities

Jacob Tsimerman

We give an introduction to some common inequalities that come up in problem solving. For a more thorough introduction, see the book *Inequalities – A Mathematical Olympiad Approach* by R. Bulajich Manfrino, J.-A. Gómez Ortega and R. Valdez Delgado.

1 Theorems that are good to know

A lot of mileage can be gotten from the basic fact that the square of a real number a is non-negative. For instance,

$$(a - b)^2 \geq 0 \rightarrow a^2 + b^2 \geq 2ab$$

and we immediately derive the basic **AM-GM** inequality. Likewise, consider the famous

Cauchy-Schwarz inequality: $(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \geq (\sum_{i=1}^{n} a_i b_i)^2$.

To prove the inequality, just note that we can rewrite the LHS-RHS as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2,$$

which, being a sum of squares, is clearly positive.

Jensen's inequality: *Suppose f(x) is a continuous function satisfying the inequality $\frac{f(x)+f(y)}{2} \geq f(\frac{x+y}{2})$ for all x,y. Then for all x_i we have*

$$\sum_{i=1}^{n} f(x_i) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

The condition $f(x) + f(y) \geq 2f(\frac{x+y}{2})$ is known as convexity*, and we say that f is a* convex *function.*

To become proficient at Jensen's inequality, it's good to know a bunch of convex functions. Here are three very useful examples. Note that if $f(x)$ is convex, then $cf(x)$ is convex as long as $c \geq 0$. Negative c aren't allowed!

1. x^n for $n \geq 1$, or $n \leq 0$ $x \geq 0$;

2. $-\sin(x)$ for $0 \leq x \leq \pi$;

3. c^x for $c > 1$ and $x \geq 1$.

1.1 Generalizations of AM-GM

Weighted AM-GM: For $w_i \geq 0$ and $x_i \geq 0$, we have $\sum_{i=1}^{n} w_i x_i \geq \prod_{i=1}^{n} x_i^{w_i}$.

Remark. The weighted AM-GM (WAM-GM) inequality is in some sense the most natural and convenient phrasing of AM-GM. Try proving it (or at least getting the idea of the proof, which is at least as important) by first doing the case of integer weights w_i, then rational weights, and then deducing it for real weights by continuity.

Hölder's inequality: For $a_i, b_i, \alpha, \beta \geq 0$, we have

$$\left(\sum_{i=1}^{n} a_i \right)^{\alpha} \left(\sum_{i=1}^{n} b_i \right)^{\beta} \geq \left(\sum_{i=1}^{n} a_i^{\frac{\alpha}{\alpha+\beta}} b_i^{\frac{\beta}{\alpha+\beta}} \right)^{\alpha+\beta}.$$

It is easy to get discouraged by the generality of Hölder's inequality, but in practise it is very clean and intuitive, so it's good to play around with examples so as to get comfortable with it. For example, use Hölder's to prove the following:

Exercise: Prove that $(a_1 + a_2)(b_1 + b_2)(c_1 + c_2) \geq \left((a_1 b_1 c_1)^{\frac{1}{3}} + (a_2 b_2 c_2)^{\frac{1}{3}} \right)^3$, if all variables are positive.

Power-Mean inequality: If $x \geq y > 0$, and $a_1, a_2, \ldots, a_n \geq 0$, then

$$\left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{\frac{1}{x}} \geq \left(\frac{a_1^y + a_2^y + \cdots + a_n^y}{n} \right)^{\frac{1}{y}}.$$

This is actually a generalization of AM-GM, in the limiting case of $x = 1, y = 0$.

No inequality toolbox is complete without knowing Schur's inequality, which is a nice bridge from the Jensen-type inequalities we've been studying to rearrangement inequalities.

1.2 A powerful 3-variable inequality

Schur's Inequality: If $a, b, c \geq 0$, then

$$a^3 + b^3 + c^3 + 3abc \geq a^2 b + b^2 a + a^2 c + c^2 a + b^2 c + c^2 b.$$

An indication that Schur's inequality is powerful is that it cannot be deduced from applications of AM-GM to monomials, due to the 'weak' term abc on the LHS. It is very instructive to go through the proof of Schur's. To start, break symmetry by assuming without loss of generality, $a \geq b \geq c$. The equation then rearranges as

$$a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0,$$

which we can then rearrange to

$$(a-b)(a(a-c) - b(b-c)) + c(c-a)(c-b) \geq 0,$$

but this is clearly true, since every term on the LHS is positive (convince yourself of this!). This proof is very short, but proofs like these are tricky to come up with, since by ordering the variables the symmetry is lost. There are a few general inequalities based on the idea of ordering the variables.

Rearrangement inequality: If $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$ and π is any permutation of $1, 2, ..., n$, then

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\pi(i)}.$$

Chebyshev's inequality: With the same assumptions as above, we get

$$\sum_{i=1}^{n} a_i b_i \geq \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n}$$

Chebyshev's inequality can of course be deduced with n applications of the rearrangement inequality. One of the reasons that the above are so powerful is they make **no assumptions on positivity!**

2   A quirky use of geometric progressions

As you probably know, the sum of geometric series $1 + x + x^2 + \cdots + x^n$ is $\frac{1-x^{n+1}}{1-x}$. Consequently, if you encounter an infinite sum $1 + x + x^2 + \cdots$, then you have the formula

$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x} \tag{1}$$

as long as $|x| < 1$, ensuring convergence. Summing an infinite geometric series might set off some alarm bells for some of you, and words like 'convergence' may or may not come to mind. We won't worry about it in these notes; safe to say the above can be made precise in a rigorous mathematical setting.

For our purposes, what matters is that one can use (1) to prove interesting inequalities. For example, solve the following inequality by expanding into a geometric series: for $0 < x, y, z < 1$, prove

$$\frac{1}{1-x^2} + \frac{1}{1-y^2} + \frac{1}{1-z^2} \geq \frac{1}{1-xy} + \frac{1}{1-xz} + \frac{1}{1-yz}.$$

3 Problems

1. Nesbitt's inquality: Suppose $x, y, z > 0$. Prove that

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2}.$$

   Many inequalities can be reduced to this one as the final step of the solution, so it is good to keep it in the back of your mind.

2. Prove for $x \geq 0$, we have $x^2 - 3 \geq -\frac{2}{x}$.

3. Prove that for $p, q, r, a, b, c \geq 0, p + q + r = 1$, we have

$$a + b + c \geq \sum_{\text{cyc}} a^p b^q c^r,$$

   where $\sum_{\text{cyc}} a^p b^q c^r = a^p b^q c^r + a^q b^r c^p + a^r b^p c^q$.

4. Prove that for $a, b, c, d \geq 0, abcd = 1$, we get $a^2 + b^2 + c^2 + d^2 \geq 4$.

5. Prove that for $a_i \geq 0$, we have $\prod_{i=1}^{n}(1 + a_i) \geq \left(1 + (a_1 a_2 \cdots a_n)^{\frac{1}{n}}\right)^n$.

6. (APMO 2004/5) Prove that if $x, y, z \geq 0$ then

$$(2 + x^2)(2 + y^2)(2 + z^2) \geq 9(xy + xz + xz).$$

7. Let $a_1, a_2, \ldots, a_n$ be positive reals with sum 1. Prove that

$$a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n \leq \frac{1}{4}$$

   and determine the cases of equality.

8. Prove that if $a, b, c \geq 0$ with $abc = 1$, then

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ac}{a^5 + c^5 + ac} \leq 1.$$

9. Let $a, b, c \geq 0, a + b + c = 1$. Prove that

$$\sqrt{ab + c} + \sqrt{ac + b} + \sqrt{bc + a} \geq 1 + \sqrt{ab} + \sqrt{bc} + \sqrt{ac}.$$

10. Let $a, b, c \geq 0$. Prove that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{a^3 + c^3 + abc} + \frac{1}{b^3 + c^3 + abc} \leq \frac{1}{abc}.$$

11. (IMO 1975) If a, b, c, d are positive reals, prove that

$$1 < \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d} \leq 2,$$

and all intermediate values are achieved.

12. (RUSSIA 2004) $x_1, x_2, \ldots, x_n, n > 3$ are positive real numbers with product of 1. Prove that

$$\frac{1}{1 + x_1 + x_1 x_2} + \frac{1}{1 + x_2 + x_2 x_3} + \cdots + \frac{1}{1 + x_n + x_n x_1} \geq 1.$$

Hint: the product 1 condition is annoying. Can you reparametrize and get rid of it?

13. (Japan 2005/3) Let $a, b, c \geq 0, a + b + c = 1$. Prove that

$$a\sqrt[3]{1 + b - c} + b\sqrt[3]{1 + c - a} + c\sqrt[3]{1 + a - b} \leq 1.$$

14. Suppose the real quadratic form $Q(\mathbf{x}) = \sum_{i,j=1}^{n} a_{i,j} x_i x_j$ is non-negative for all real numbers x_i, x_j. Prove you can write $Q(\mathbf{x})$ as a sum of squares of linear forms.

15. (Baltic Way 2004) Prove that if p, q, r are positive real numbers with product 1, then for all natural numbers n we have

$$\frac{1}{p^n + q^n + 1} + \frac{1}{p^n + r^n + 1} + \frac{1}{q^n + r^n + 1} \leq 1.$$

16. (Iran 1996) Prove the following inequality for positive real numbers x, y, z:

$$(xy + xz + yz)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(x+z)^2}\right) \geq \frac{9}{4}.$$

17. For $a, b, c, d \geq 0$, prove that $(ab)^{\frac{1}{3}} + (cd)^{\frac{1}{3}} \leq (a + c + d)^{\frac{1}{3}} (a + c + b)^{\frac{1}{3}}$.

18. (Mathlinks) If $x, y, z \geq 0$, prove

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \leq \frac{3}{2} \frac{x^2 + y^2 + z^2}{xy + xz + yz}.$$

19. (Baltic Way 2008) Let a, b, c be positive real numbers with $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a^2}{2 + b + c^2} + \frac{b^2}{2 + c + a^2} + \frac{c^2}{2 + a + b^2} \geq \frac{(a + b + c)^2}{12}.$$

20. (USAMO 00/6) If $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$, then show that

$$\sum_{i,j} \min(a_i a_j, b_i b_j) \leq \sum_{i,j} \min(a_i b_j, b_i a_j).$$

4   Hints

1. Try expanding, or using Cauchy.

2. Try WAM-GM, can you see the weights?

3. Try WAM-GM applied a few times.

4. Try homogenizing, then see if something strikes you.

5. You can do this using Hölder's if you pick the right weights!

6. Remarkably, this can be done entirely with WAM-GM and Schur's inequality to handle the nasty $x^2y^2z^2$ term.

7. Try simple transformations, like flipping a_i and a_{i+1}, and argue by the extremal principle.

8. Set $a = \frac{1}{x}$, etc..

9. First homogenize, then use Hölder's within each square-root.

10. Expand! It's not as scary as it seems.

11. Modify one variable at a time.

12. Set $x_1 = \frac{y_1}{y_2}$, etc . . .

13. Use Hölder's after a substitution.

14. Try induction.

15. Does this look like a different problem from this set?

16. Use extremal methods.

17. Use Hölder's, but not in the obvious way.

18. Either expand, or use a tricky Cauchy-Schwarz.

19. Another Cauchy-Schwarz.

20. This problem is really hard! Induction works, but it is quite hard to set up properly. Good luck!

. . . .
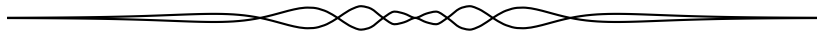
PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **August 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.
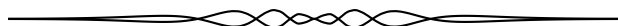
4041. *Proposed by Arkady Alt.*

Let a, b and c be the side lengths of a triangle ABC. Let AA', BB' and CC' be the heights of the triangle and let $a_p = B'C', b_p = C'A'$ and $c_p = A'B'$ be the sides of the orthic triangle. Prove that:

a) $a^2 (b_p + c_p) + b^2 (c_p + a_p) + c^2 (a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, where s is the semiperimeter of ABC.

4042. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be real numbers in $[0, \pi/2]$ such that $a + b + c = \pi$. Prove the inequality

$$2\sqrt{2}\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \geq \sqrt{\cos a \cos b \cos c}.$$

4043. *Proposed by Michel Bataille.*

Suppose that the lines m and n intersect at A and are not perpendicular. Let B be a point on n, with $B \neq A$. If F is a point of m, distinct from A, show that there exists a unique conic $\mathcal{C}_F$ with focus F and focal axis BF, intersecting n orthogonally at A. Given $\epsilon > 0$, how many of the conics $\mathcal{C}_F$ have eccentricity ϵ?

4044. *Proposed by Dragoljub Milošević.*

Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \leq \frac{27}{7}.$$

4045. *Proposed by Galav Kapoor.*

Suppose that we have a natural number n such that $n \geq 10$. Show that by changing at most one digit of n, we can compose a number of the form $x^2 + y^2 + 10z^2$, where x, y, z are integers.
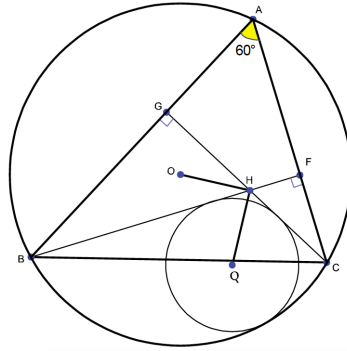
4046. *Proposed by Michel Bataille.*

Let a, b, c be nonnegative real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 1$. Prove that

$$a^2 + b^2 + c^2 + 7(ab + bc + ca) \geq \sqrt{8(a+b)(b+c)(c+a)}.$$

4047. *Proposed by Abdilkadir Altinaş.*

Let ABC be a triangle with circumcircle O, orthocenter H and $\angle BAC = 60°$. Suppose the circle with centre Q is tangent to BH, CH and the circumcircle of ABC. Show that $OH \perp HQ$.

4048. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 2$ be an integer and let $a_k \geq 1$ be real numbers, $1 \leq k \leq n$. Prove the inequality

$$a_1 a_2 \cdots a_n - \frac{1}{a_1 a_2 \cdots a_n} \geq \left(a_1 - \frac{1}{a_1}\right) + \left(a_2 - \frac{1}{a_2}\right) + \cdots + \left(a_n - \frac{1}{a_n}\right)$$

and study equality cases.

4049. *Proposed by Mihaela Berindeanu.*

Evaluate

$$\int \frac{\sin x - x \cos x}{(x + \sin x)(x + 2 \sin x)} dx$$

for all $x \in (0, \pi/2)$.

4050. *Proposed by Mehtaab Sawhney.*

Prove that

$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} = \binom{4n}{2n}^2$$

for all nonnegative integers n.

$\cdot\,\cdot$

4041. *Proposé par Arkady Alt.*

Soit a, b et c les longueurs des côtés du triangle ABC et soit AA', BB' et CC' les hauteurs du triangle. De plus, soit $a_p = B'C'$, $b_p = C'A'$ et $c_p = A'B'$ les longueurs des côtés de son triangle orthique. Démontrer que:

a) $a^2(b_p + c_p) + b^2(c_p + a_p) + c^2(a_p + b_p) = 3abc$;

b) $a_p + b_p + c_p \leq s$, s étant le demi-périmètre du triangle ABC. (On a utilisé s au lieu de p pour éviter la confusion avec l'indice p.)

4042. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Soit a, b et c des réels dans l'intervalle $[0, \pi/2]$ tels que $a + b + c = \pi$. Démontrer que

$$2\sqrt{2}\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \geq \sqrt{\cos a \cos b \cos c}.$$

4043. *Proposé par Michel Bataille.*

Soit m et n deux droites non perpendiculaires qui se coupent en A. Soit B un point sur n, $B \neq A$. Étant donné un point F sur m, distinct de A, démontrer qu'il existe exactement une conique $\mathcal{C}_F$, ayant pour foyer F et pour axe focal BF, qui coupe n perpendiculairement en A. Étant donné ϵ ($\epsilon > 0$), combien des coniques $\mathcal{C}_F$ ont pour excentricité ϵ?

4044. *Proposé par Dragoljub Milošević.*

Soit x, y et z des réels strictement positifs tels que $x + y + z = 1$. Démontrer que

$$\frac{x+1}{x^3+1} + \frac{y+1}{y^3+1} + \frac{z+1}{z^3+1} \leq \frac{27}{7}.$$

4045. *Proposé par Galav Kapoor.*

Soit un entier n ($n \geq 10$). Démontrer qu'en changeant au plus un des chiffres de n, il est possible d'obtenir un nombre de la forme $x^2 + y^2 + 10z^2$, x, y et z étant des entiers.

4046. *Proposé par Michel Bataille.*

Soit a, b et c des réels non négatifs tels que $\sqrt{a} + \sqrt{b} + \sqrt{c} \geq 1$. Démontrer que

$$a^2 + b^2 + c^2 + 7(ab + bc + ca) \geq \sqrt{8(a+b)(b+c)(c+a)}.$$

4047. *Proposé par Abdilkadir Altinaş.*

Soit un triangle ABC avec $\angle BAC = 60°$, H son orthocentre et O le centre du cercle circonscrit au triangle. On considère le cercle de centre Q qui est tangent à BH, à CH et au cercle circonscrit au triangle ABC. Démontrer que $OH \perp HQ$.

4048. *Proposé par Leonard Giugiuc and Daniel Sitaru.*

Soit n un entier $(n \geq 2)$ et soit a_k des réels $(a_k \geq 1, 1 \leq k \leq n)$. Démontrer que

$$a_1 a_2 \cdots a_n - \frac{1}{a_1 a_2 \cdots a_n} \geq \left(a_1 - \frac{1}{a_1}\right) + \left(a_2 - \frac{1}{a_2}\right) + \cdots + \left(a_n - \frac{1}{a_n}\right)$$

et considérer les cas où il y a égalité.

4049. *Proposé par Mihaela Berindeanu.*

Évaluer

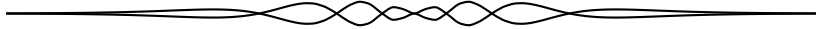$$\int \frac{\sin x - x \cos x}{(x + \sin x)(x + 2\sin x)} dx$$

pour tout réel x dans l'intervalle $(0, \pi/2)$.

4050. *Proposé par Mehtaab Sawhney.*

Démontrer que

$$\sum_{k=0}^{2n} \binom{4n}{k, k, 2n-k, 2n-k} = \binom{4n}{2n}^2$$

pour tout entier non négatif n.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014: 40(5), p. 212–216.

3941. *Proposed by Gabriel Dinu Ovidiu. Correction.*

Prove that

$$\frac{x_1}{x_2 + x_3 + x_4 + \cdots + x_n - x_1} + \frac{x_2}{x_1 + x_3 + x_4 + \cdots + x_n - x_2} + \cdots$$

$$+ \frac{x_n}{x_1 + x_2 + x_3 + \cdots + x_{n-1} - x_n} \geq \frac{n}{n-2},$$

where $x_i \in \mathbb{R}^+$, $x_i \neq 0$, $x_i < x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, $n > 2$.

Editor's Comments. There was a slight error in the posing of the problem; the condition applies to $i = 1$ as well. However, everyone picked this up and two submitters gave counterexamples to the problem as originally stated.

There were 16 correct solutions and one flawed solution. Seven solvers applied the Cauchy-Schwarz Inequality along the lines of Solution 1. Five solvers used Jensen's Inequality for a convex function along the lines of Solution 2.

Solution 1, by Titu Zvonaru.

Let

$$s = \sum_{i=1}^{n} x_i \quad \text{and} \quad t = \sum_{i=1}^{n} x_i^2.$$

We make two applications of the Cauchy-Schwarz Inequality to

$$(a_i, b_i) = (x_i/\sqrt{sx_i - 2x_i^2}, \sqrt{sx_i - 2x_i^2})$$

and to $(a_i, b_i) = (1, x_i)$ to obtain the desired lower bound for the left side:

$$\sum_{i=1}^{n} \frac{x_i}{s - 2x_i} = \sum_{i=1}^{n} \frac{x_i^2}{sx_i - 2x_i^2} \geq \frac{s^2}{s^2 - 2t} \geq \frac{s^2}{s^2 - (2s^2/n)} = \frac{n}{n-2}.$$

Equality holds if and only if all the x_i are equal.

Editor's Comments. Two other solvers followed this approach but used Bergström's inequality $\sum(x_i^2/a_i) \geq (\sum x_i)^2 / \sum a_i$ with $a_i > 0$ instead of that of Cauchy-Schwarz for the initial inequality.

Solution 2, by various solvers.

With $s = \sum_{i=1}^{n} x_i$, the given condition is that $x_i < s/2$ for each i. The function $f(x) = x/(s - 2x)$ is convex on the interval $(0, s/2)$, so that, by Jensen's inequality,

$$\sum_{i=1}^{n} f(x_i) \geq nf\left(\frac{s}{n}\right) = \frac{n}{n-2},$$

as desired.

3942. *Proposed by Marcel Chiriţă.*

Consider a sequence $(x_n)_{n \geq 1}$ with $x_1 = 1$ and $x_{n+1} = \frac{1}{n+1}\left(x_n + \frac{1}{x_n}\right)$ for $n \geq 1$. Find $x_n\sqrt{n}$.

We received two correct solutions. We present the solution by Michel Bataille.

Editor's Comments. The proposer intended that the solver should determine $\lfloor x_n\sqrt{n} \rfloor$. However, Bataille interpreted the problem as determining the the limit of $x_n\sqrt{n}$ as n tends to infinity. His solution to this problem follows; it is straightforward to show that the greatest integer of $x_n\sqrt{n}$ is thus always 1.

We first prove by induction that, for $n \geq 2$, $\frac{1}{n} \leq x_n^2 \leq \frac{1}{n-1}$.

Since $x_2 = 1$, the base case holds. Suppose the inequality holds for $n \geq 2$. Since the function $f(t) = t + \frac{1}{t}$ decreases on $(0, 1)$,

$$f\left(\frac{1}{n-1}\right) \leq f(x_n^2) \leq f\left(\frac{1}{n}\right).$$

It follows that

$$\frac{f\left(\frac{1}{n-1}\right) + 2}{(n+1)^2} \leq x_{n+1}^2 = \frac{f(x_n^2) + 2}{(n+1)^2} \leq \frac{f\left(\frac{1}{n}\right) + 2}{(n+1)^2},$$

from which

$$\frac{\frac{1}{n-1} + n + 1}{(n+1)^2} \leq x_{n+1}^2 \leq \frac{\frac{1}{n} + n + 2}{(n+1)^2}.$$

Thus

$$\frac{1}{n+1} < \frac{n^2}{n^2 - 1} \cdot \frac{1}{n+1} \leq x_n^2 \leq \frac{(n+1)^2}{n(n+1)^2} = \frac{1}{n}.$$

This completes the induction step.

Since $x_n \geq 0$ for all n, we obtain that

$$1 \leq x_n\sqrt{n} \leq \sqrt{\frac{n}{n-1}}$$

from which we deduce that $\lim_{n \to \infty} x_n\sqrt{n} = 1$.

3943. *Proposed by Michel Bataille.*

Let a be a positive real number. Evaluate

$$\lim_{n\to\infty} \left(n \cdot \int_0^a \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt \right).$$

We received four submissions of which three were correct and complete. We present the solution by Paolo Perfetti.

The limit is equal to $\frac{1}{2\tanh a}$.

Let $\delta_n \to 0$ a sequence to be chosen later. We have

$$n \left(\int_0^a \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt \right) = n \int_0^{a-\delta_n} \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt + n \int_{a-\delta_n}^a \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt.$$

Furthermore,

$$0 < n \int_0^{a-\delta_n} \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt \le n(a - \delta_n) \left(\frac{\cosh(a - \delta_n)}{\cosh a} \right)^{2n+1},$$

since the function $\cosh x$ increases for $x > 0$. Moreover,

$$\cosh(a - \delta_n) = \cosh a - \delta_n \sinh a + o(\delta_n)$$

so that

$$
\begin{aligned}
n(a - \delta_n) \left(\frac{\cosh(a - \delta_n)}{\cosh a} \right)^{2n+1} &= n(a - \delta_n) \left(1 - \tanh a \delta_n + o(\delta_n) \right)^{2n+1} \\
&= (a - \delta_n) e^{\ln n + (2n+1)\ln(1 - \delta_n \tanh a + o(\delta_n))} \\
&= (a - \delta_n) e^{\ln n + (2n+1)(-\delta_n \tanh a + o(\delta_n))} \\
&= (a - \delta_n) e^{\ln n + (2n+1)(-\delta_n \tanh a)(1 + o(1))} \\
&\to 0
\end{aligned}
$$

if we choose δ_n in such a way that

$$\ln n + (2n + 1)(-\delta_n \tanh a) \to -\infty$$

(for instance, $\delta_n = an^{-\frac{2}{3}}$). Thus, we have obtained

$$\lim_{n\to\infty} n \int_0^{a-\delta_n} \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt = 0.$$

Now the integral between $a - \delta_n$ and a:

$$n \int_{a-\delta_n}^{a} \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt = n \int_{a-\delta_n}^{a} e^{(2n+1)(\ln \cosh t - \ln \cosh a)} dt$$

$$= n \int_{a-\delta_n}^{a} e^{(2n+1)(\ln(\cosh a + (t-a) \sinh a + O((t-a)^2)) - \ln \cosh a)} dt$$

$$= n \int_{a-\delta_n}^{a} e^{(2n+1)(\ln(1 + (t-a) \tanh a + O((t-a)^2)))} dt$$

$$= n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + O((t-a)^2))} dt$$

The last integral can be bounded as

$$n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + C_1 (t-a)^2)} dt \leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + O((t-a)^2))} dt$$

$$\leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + C_2 (t-a)^2)} dt,$$

where $C_1, C_2 > 0$. Therefore,

$$n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a)} dt \leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + O((t-a)^2)))} dt$$

$$\leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + C_2 \delta_n^2)} dt.$$

Now, we assume $n \delta_n^2 \to 0$ ($\delta_n = a n^{-2/3}$ is good) and obtain

$$n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a)} dt \leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a + O((t-a)^2)))} dt$$

$$\leq n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a)} dt \cdot e^{(2n+1) C_2 \delta_n^2} dt.$$

It follows that

$$\lim_{n \to \infty} n \left(\int_{0}^{a} \left(\frac{\cosh t}{\cosh a} \right)^{2n+1} dt \right) = \lim_{n \to \infty} n \int_{a-\delta_n}^{a} e^{(2n+1)((t-a) \tanh a)} dt$$

$$= \lim_{n \to \infty} \frac{n(1 - e^{-(2n+1)\delta_n \tanh a})}{(2n+1) \tanh a}$$

$$= \lim_{n \to \infty} \frac{n}{(2n+1) \tanh a}$$

$$= \frac{1}{2 \tanh a},$$

since

$$\lim_{n \to \infty} e^{-(2n+1)\delta_n \tanh a} = 0.$$

3944. *Proposed by Bill Sands.*

With the elimination of the penny in Canada, purchase totals in stores are rounded off to the nearest multiple of 5 cents. For example, if you bought several items in a store with total price \$9.97, you would only pay \$9.95, but if your items totalled to \$9.98 then you would pay \$10. Suppose you go into a dollar store and want to buy 123 items worth 1 cent, 2 cents, 3 cents, ..., \$1.23. You are allowed to group the 123 items into any number of groups of any sizes, and each group would be a separate purchase. How could you group the 123 items so as to pay the smallest possible total amount?

We received four correct and three incomplete solutions.

We present a modified and shortened version of the solutions of Joseph DiMuro, Salem Malikić, and Michael Parmenter and Bruce Shawyer.

First we note that subtracting multiples of five cents from the prices does not affect the rounding, so to simplify the problem, we assume that we have 25 items each costing one, two, or three cents and 24 items each costing zero or four cents. We claim that maximum savings can be achieved by paying for each of the zero, one, and two cent items as well as one three cent item separately and grouping the remaining items into 24 groups of one item of three cents and one item of four cents.

We begin by assuming that the items have been grouped together such that the saving is maximum. We will make changes to the groupings without increasing the price paid until we reach our solution above. If a group can be split into two groups of total prices $5s + i$ and $5t + j$, where $i = 0, 1, 2$, then we split it. It can be checked easily that this split does not increase the total price. Therefore we may assume that all the items that cost at most two cents are bought separately. Furthermore the remaining groups consist either of exactly one three cent item and one four cent item, of one or two three cent items, or of up to three four cent items. Now if we have a group of only three cent items and one of only four cent items, we remove one item from each group and put the two items in a new group. Again we won't increase the total price (check a few cases). By this process we eventually arrive at our solution.

To calculate our savings, we note that we save one cent for each one cent item, two cents for each two cent item and for each group of two items, and we pay two cents extra for the single three cent item. In total we save \$1.21 compared to the original overall price of \$76.26 (or \$1.20 compared to the \$76.25 we would have to pay).

*As an addendum we present the proposer's proof that the above scheme is optimal.*

Notice that the 24 items costing a multiple of five cents can be added to any purchase without changing the amount saved from that purchase. Thus we can ignore these items. Next, in each purchase, you cannot save more than two cents. The 25 items costing $5r + 2$ cents are the only ones which when bought separately

will save you two cents. Thus each of the remaining 74 items can contribute at most one cent to the total saving, whether bought separately or in combination with one or more other items. Therefore the total saving can be at most $25 \cdot 2 + 74 \cdot 1 = 124$ cents, so you must pay at least \$76.26 - \$1.24 = \$75.02. But you have to pay a multiple of 5 cents, thus at least \$75.05.

3945. *Proposed by J. Chris Fisher.*

Given circles (A) and (B) with centres A and B, and a circle (C) with centre C that meets (A) in points A_1 and A_2 that are not on (B), and meets (B) in points B_1 and B_2 that are not on (A), prove that the unique conic with foci A and B that is tangent to the perpendicular bisector of $A_2 B_2$ is tangent also to the perpendicular bisector ℓ of $A_1 B_1$.

We received two correct submissions. We present the solution by Michel Bataille.
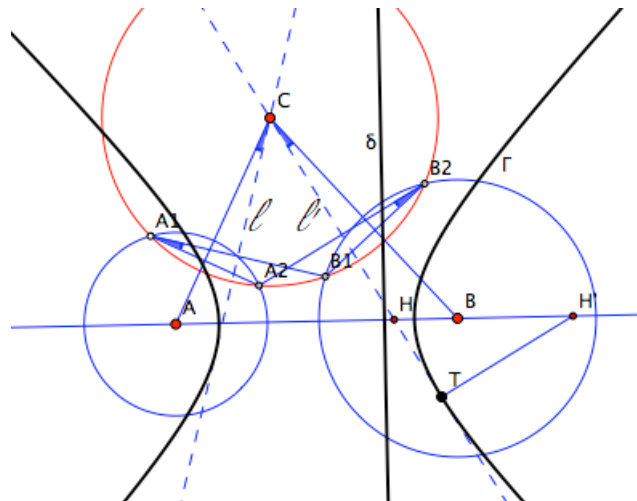
The line CA through the centres of the circles (C) and (A) is perpendicular to $A_1 A_2$. Similarly, the line CB is perpendicular to $B_1 B_2$. Also, $\ell \perp A_1 B_1$ and $\ell' \perp A_2 B_2$ where ℓ' is the perpendicular bisector of $A_2 B_2$. If $\angle(\cdot, \cdot)$ denotes the directed angle of lines, we thus have

$$\angle(CA, \ell) = \angle(A_1 A_2, A_1 B_1) \quad \text{and} \quad \angle(CB, \ell') = \angle(B_1 B_2, A_2 B_2).$$

It follows that

$$\begin{aligned}
\angle(CA, \ell) + \angle(CB, \ell') &= \angle(A_1 A_2, A_1 B_1) + \angle(B_1 B_2, A_2 B_2) \\
&= \angle(B_2 A_2, B_2 B_1) + \angle(B_1 B_2, A_2 B_2) \\
&= 0,
\end{aligned}$$

where the second equality holds since A_1, B_2, A_2, B_1 are concyclic.

As a result, the lines ℓ and ℓ' are isogonal conjugates with respect to the lines CA, CB.

Now, if Γ is the conic with foci A, B tangent to ℓ', let m be the second tangent to Γ passing through C. From a known theorem (often attributed to Poncelet), ℓ' and m are isogonal conjugates with respect to CA, CB (see [1], theorem 17, p.42). Thus $m = \ell$ and ℓ is also tangent to Γ.

Note. To obtain Γ, let ℓ' intersect the focal axis AB at H and let H' be the harmonic conjugate of H with respect to A, B. Then H' is the point of intersection with AB of the normal to Γ which is perpendicular to ℓ'. Thus the point of contact T of ℓ' with Γ is the second point of intersection of ℓ' and the circle with diameter HH'. The eccentricity of Γ is $e = \frac{BH'}{BT}$ (see Theorem 4 p.18 in C.V. Durell, *A Concise Geometrical Conics*, Macmillan and Co, 1952). The directrix δ associated with B is easily obtained from T and e since $e = \frac{TB}{d(T,\delta)}$.

3946. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC

a) $\dfrac{a^2}{w_b w_c} + \dfrac{b^2}{w_a w_c} + \dfrac{c^2}{w_a w_b} \geq 4,$

b) $\left(\dfrac{a}{w_b w_c}\right)^2 + \left(\dfrac{b}{w_a w_c}\right)^2 + \left(\dfrac{c}{w_a w_b}\right)^2 \geq \left(\dfrac{4}{3R}\right)^2,$

where R is the circumradius of ABC and w_a, w_b, w_c are the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c, respectively.

We received 14 correct solutions and present a composite based on three nearly identical solutions. We present a solution by Miguel Amengual Covas, Cao Minh Quang, and Titu Zvonaru, done independently.

a) It is well known that $w_a \leq \sqrt{s(s-a)}$ and two similar results (Ed.: see item 8.8 on p. 75 of [1]). Hence we have, by the AM-GM inequality, that

$$w_b w_c \leq \sqrt{s^2(s-b)(s-c)} = s\sqrt{(s-b)(s-c)} \leq s\left(\frac{s-b+s-c}{2}\right) = \frac{sa}{2}.$$

Therefore, $\sum \dfrac{a^2}{w_b w_c} \geq \dfrac{2}{s}\sum a = 4.$

b) Since $w_b w_c \leq \dfrac{sa}{2}$, we have $\left(\dfrac{a}{w_b w_c}\right)^2 \geq \dfrac{4}{s^2}$. By the well-known inequality $a+b+c \leq 3\sqrt{3}R$ (Ed.: see item 5.3 on p. 49 of [1]), we then have

$$\sum \left(\frac{a}{w_b w_c}\right)^2 \geq \frac{12}{s^2} = \frac{48}{(a+b+c)^2} \geq \frac{48}{27R^2} = \left(\frac{4}{3R}\right)^2.$$

References: [1] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

3947. *Proposed by Michel Bataille.*

Let $A_1 A_2 A_3$ be a non-isosceles triangle and I its incenter. For $i = 1, 2, 3$, let D_i be the projection of I onto $A_{i+1} A_{i+2}$ and U_i, V_i be the respective projections of A_{i+1}, A_{i+2} onto the line IA_i (indices are taken modulo 3). Prove that

(a) $\dfrac{U_1 D_1}{V_1 D_1} \cdot \dfrac{U_2 D_2}{V_2 D_2} \cdot \dfrac{U_3 D_3}{V_3 D_3} = \dfrac{U_1 D_2}{V_1 D_3} \cdot \dfrac{U_2 D_3}{V_2 D_1} \cdot \dfrac{U_3 D_1}{V_3 D_2} = 1,$

(b) $\dfrac{[D_1 U_1 V_1]}{\sin^2 \frac{\alpha_2 - \alpha_3}{2}} + \dfrac{[D_2 U_2 V_2]}{\sin^2 \frac{\alpha_3 - \alpha_1}{2}} + \dfrac{[D_3 U_3 V_3]}{\sin^2 \frac{\alpha_1 - \alpha_2}{2}} = [A_1 A_2 A_3]$, where α_i is the angle of $\triangle A_1 A_2 A_3$ at vertex A_i $(i = 1, 2, 3)$ and $[XYZ]$ denotes the area of $\triangle XYZ$.

For this question, we received three correct solutions and one incomplete submission. We present the solution by Titu Zvonaru, slightly expanded by the editor.
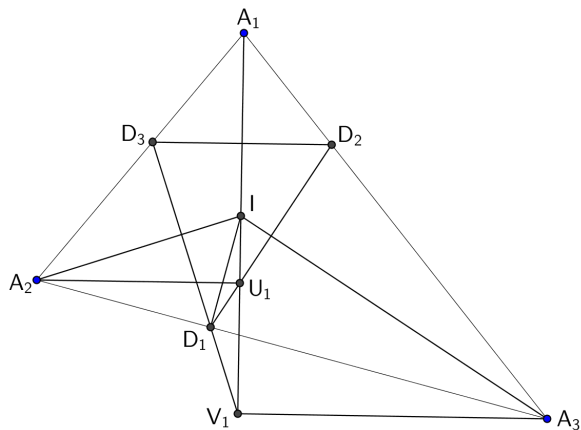
Denote by a_i the length of the side of the triangle opposite $\angle A_i$, and by s the semiperimeter (i.e. $s = \frac{a_1 + a_2 + a_3}{2}$).

The quadrilateral $A_2 D_1 U_1 I$ is cyclic (since $\angle A_2 D_1 I = \angle A_2 U_1 I = 90°$), and so $\angle D_1 U_1 V_1 = \angle D_1 A_2 I = \frac{\alpha_2}{2}$. The quadrilateral $A_3 V_1 D_1 I$ is also cyclic (since $\angle A_3 V_1 I = \angle A_3 D_1 I = 90°$), and so $\angle D_1 V_1 U_1 = \angle D_1 A_3 I = \frac{\alpha_3}{2}$. Finally, in $\triangle D_1 U_1 V_1$ we have $\angle U_1 D_1 V_1 = 180° - \frac{\alpha_2}{2} - \frac{\alpha_3}{2} = 90° + \frac{\alpha_1}{2}$, so we can conclude that $\sin(\angle U_1 D_1 V_1) = \cos \frac{\alpha_1}{2}$. Apply the Sine Law to $\triangle D_1 U_1 V_1$ to get

$$\frac{D_1 U_1}{\sin \frac{\alpha_3}{2}} = \frac{D_1 V_1}{\sin \frac{\alpha_2}{2}} = \frac{U_1 V_1}{\cos \frac{\alpha_1}{2}}. \tag{1}$$

(a) We can rearrange (1) to obtain $\frac{U_1 D_1}{V_1 D_1} = \frac{\sin \frac{\alpha_3}{2}}{\sin \frac{\alpha_2}{2}}$. Similarly, we find formulas for the remaining ratios, and we get

$$\frac{U_1 D_1}{V_1 D_1} \cdot \frac{U_2 V_2}{V_2 D_2} \cdot \frac{U_3 D_3}{V_3 D_3} = \frac{\sin \frac{\alpha_3}{2}}{\sin \frac{\alpha_2}{2}} \cdot \frac{\sin \frac{\alpha_1}{2}}{\sin \frac{\alpha_3}{2}} \cdot \frac{\sin \frac{\alpha_2}{2}}{\sin \frac{\alpha_1}{2}} = 1.$$

To prove the second equality, we will start by applying the Cosine Law to $\triangle A_1 D_2 U_1$. Tangent segments from a point to a circle are equal, and so $A_1 D_2 = A_1 D_3$, $A_3 D_2 = A_3 D_1$ and $A_2 D_1 = A_2 D_3$. Since $A_2 D_1 + D_1 A_3 = a_1$, we can conclude that $A_1 D_2 = s - a_1$. From the right-angled $\triangle A_1 U_1 A_2$ we can calculate that $A_1 U_1 = a_3 \cos \frac{\alpha_1}{2}$. Finally, we know that $\angle U_1 A_1 D_1 = \frac{\alpha_1}{2}$ and so from the Cosine Law we get

$$(U_1 D_2)^2 = a_3^2 \cos^2 \frac{\alpha_1}{2} + (s - a_1)^2 - 2a_3(s - a_1) \cos^2 \frac{\alpha_1}{2}.$$

Use the formula $\cos^2 \frac{\alpha_1}{2} = \frac{s(s - a_1)}{a_2 a_3}$ and simplify to get

$$(U_1 D_2)^2 = \frac{a_1(s - a_1)(s - a_2)}{a_2} = a_1^2 \sin^2 \frac{\alpha_3}{2},$$

where in the last step we used $\sin^2 \frac{\alpha_3}{2} = \frac{(s - a_1)(s - a_2)}{a_1 a_2}$. Thus we can conclude that $U_1 D_2 = a_1 \sin \frac{\alpha_3}{2}$, and similarly we can apply the Cosine Law to $\triangle A_1 V_1 A_3$ to conclude that $V_1 D_3 = a_1 \sin \frac{\alpha_2}{2}$. Repeat the process with the remaining factors to get

$$\frac{U_1 D_2}{V_1 D_3} \cdot \frac{U_2 D_3}{V_2 D_1} \cdot \frac{U_3 D_1}{V_3 D_2} = \frac{\sin \frac{\alpha_3}{2}}{\sin \frac{\alpha_2}{2}} \cdot \frac{\sin \frac{\alpha_1}{2}}{\sin \frac{\alpha_3}{2}} \cdot \frac{\sin \frac{\alpha_2}{2}}{\sin \frac{\alpha_1}{2}} = 1.$$

(b) From the right angled triangles $A_1 V_1 A_3$ and $A_1 U_1 A_2$ we can calculate $A_1 V_1 = a_2 \cos \frac{\alpha_1}{2}$ and $A_1 U_1 = a_3 \cos \frac{\alpha_1}{2}$. Regardless of whether U_1 or V_1 is closer to A_1, we can combine these two formulas to get

$$(U_1 V_1)^2 = (a_2 - a_3)^2 \cos^2 \frac{\alpha_1}{2}. \tag{2}$$

Denote the circumradius of $\triangle A_1 A_2 A_3$ by R. From basic circle geometry, one can easily deduce that $a_i = 2R \sin \alpha_i$. Using trigonometric identities for sums of angles we can write

$$\begin{aligned}(a_2 - a_3)^2 = 4R^2(\sin \alpha_2 - \sin \alpha_3)^2 \quad &= 16R^2 \sin^2 \frac{\alpha_2 - \alpha_3}{2} \cos^2 \frac{\alpha_2 + \alpha_3}{2} \\ &= 16R^2 \sin^2 \frac{\alpha_2 - \alpha_3}{2} \sin^2 \frac{\alpha_1}{2},\end{aligned}$$

where for the last equality we used the fact that $\frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} = 90°$. Combining this with the formula for $U_1 V_1$ in (2), we get

$$(U_1 V_1)^2 = 16R^2 \sin^2 \frac{\alpha_2 - \alpha_3}{2} \sin^2 \frac{\alpha_1}{2} \cos^2 \frac{\alpha_1}{2}.$$

Apply the sine area formula to $\triangle D_1 U_1 V_1$, and use (1) to rewrite the side lengths in terms of $U_1 V_1$:

$$[D_1 U_1 V_1] = \frac{1}{2} \cdot D_1 U_1 \cdot D_1 V_1 \cdot \sin \angle U_1 D_1 V_1 = \frac{(U_1 V_1)^2 \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} \sin \frac{\alpha_2 + \alpha_3}{2}}{2 \cos^2 \frac{\alpha_1}{2}}.$$

Substitute our earlier formula for $U_1 V_1$ in terms of the circumradius and use the fact that $\sin \frac{\alpha_2 + \alpha_3}{2} = \cos \frac{\alpha_1}{2}$ to get

$$\frac{[D_1 U_1 V_1]}{\sin^2 \frac{\alpha_2 - \alpha_3}{2}} = 8R^2 \sin^2 \frac{\alpha_1}{2} \cos \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} = 4R^2 \sin \alpha_1 \sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2}.$$

Repeat with the remaining area terms. Finally, $\sin \frac{\alpha_1}{2} \sin \frac{\alpha_2}{2} \sin \frac{\alpha_3}{2} = \frac{[A_1 A_2 A_3]}{4Rs}$ is a known area formula, and so we have

$$
\begin{aligned}
\frac{[D_1 U_1 V_1]}{\sin^2 \frac{\alpha_2 - \alpha_3}{2}} + \frac{[D_2 U_2 V_2]}{\sin^2 \frac{\alpha_3 - \alpha_1}{2}} + \frac{[D_3 U_3 V_3]}{\sin^2 \frac{\alpha_1 - \alpha_2}{2}} &= \frac{[A_1 A_2 A_3]}{4Rs} \cdot 4R^2 (\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3) \\
&= \frac{[A_1 A_2 A_3]}{4Rs} \cdot 4R^2 (\frac{a_1}{2R} + \frac{a_2}{2R} + \frac{a_3}{2R}) \\
&= \frac{[A_1 A_2 A_3]}{4Rs} \cdot 4R^2 \cdot \frac{s}{R} \\
&= [A_1 A_2 A_3],
\end{aligned}
$$

concluding the proof.

Editor's Comments. It is interesting to observe that D_i, U_i, D_{i+1} and D_i, D_{i+1}, V_{i+1} are collinear (as can be shown either by analytic geometry, or by angle chasing). This was a key part of the proposer's solution, but not used in any of the other received solutions.

3948. *Proposed by George Apostolopoulos.*

Let $a_1, a_2, \ldots a_n$ be real numbers such that $a_1 > a_2 > \ldots > a_n$. Prove that

$$
\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n} + a_1 - a_n \geq 2(n-1).
$$

When does the equality hold?

We received 25 submissions. We present the solution by Prithwijit De.

Let $b_k = a_k - a_{k+1}$ for $k = 1, 2, \ldots, n-1$. Then the left-hand side of the inequality reduces to

$$
\sum_{k=1}^{n-1} \left(\frac{1}{b_k} + b_k \right).
$$

Notice that for all k, we have $b_k > 0$ and $\frac{1}{b_k} + b_k \geq 2$. Hence

$$
\sum_{k=1}^{n-1} \left(\frac{1}{b_k} + b_k \right) \geq 2(n-1).
$$

Equality occurs if and only if $b_k = 1$ for all k. That is if and only if $a_k = a_{k+1} + 1$ for $k = 1, 2, \ldots, n-1$.

3949. *Proposed by Arkady Alt.*

For any positive real a and b, find

$$
\lim_{n \to \infty} \left((n+1) \left(\frac{a^{\frac{1}{n+1}} + b^{\frac{1}{n+1}}}{2} \right)^{n+1} - n \left(\frac{a^{\frac{1}{n}} + b^{\frac{1}{n}}}{2} \right)^n \right).
$$

We received nine submissions, all of which were correct. We present the solution by Michel Bataille.

We prove that the limit is $\sqrt{ab}$.

Let $U_n = \left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^n$ and $V_n = (n+1)U_{n+1} - nU_n$.

First, we consider the case $a = b$. Then, $\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2} = a^{\frac{1}{n}}$ for every positive integer n, so that

$$V_n = (n+1)a - na = a$$

and $\lim\limits_{n \to \infty} V_n = \sqrt{a^2}$ is obvious.

From now on, we suppose $a \neq b$. We adopt the notation α for $\ln\left(\frac{a}{b}\right)$.

As $n \to \infty$, we may write

$$U_n = \left(\frac{b^{\frac{1}{n}}\left(1+\left(\frac{a}{b}\right)^{\frac{1}{n}}\right)}{2}\right)^n$$

$$= \frac{b}{2^n}\left(1+e^{\frac{\alpha}{n}}\right)^n$$

$$= \frac{b}{2^n}\left(2 + \frac{\alpha}{n} + \frac{\alpha^2}{2n^2} + o(1/n^2)\right)^n$$

$$= b\left(1 + \frac{\alpha}{2n} + \frac{\alpha^2}{4n^2} + o(1/n^2)\right)^n.$$

In consequence,

$$\ln(U_n) = \ln(b) + n\ln\left(1 + \frac{\alpha}{2n} + \frac{\alpha^2}{4n^2} + o(1/n^2)\right)$$

$$= \ln(b) + n\left(\frac{\alpha}{2n} + \frac{\alpha^2}{4n^2} - \frac{1}{2}\cdot\frac{\alpha^2}{4n^2} + o(1/n^2)\right)$$

$$= \ln(b) + \frac{\alpha}{2} + \frac{\alpha^2}{8n} + o(1/n),$$

and so

$$U_n = be^{\alpha/2}e^{\alpha^2/8n+o(1/n)} = be^{\alpha/2}\left(1 + \frac{\alpha^2}{8n} + o(1/n)\right).$$

As a result, we obtain

$$nU_n = nbe^{\alpha/2} + \frac{\alpha^2 be^{\alpha/2}}{8} + o(1)$$

and

$$V_n = (n+1)be^{\alpha/2} - nbe^{\alpha/2} + o(1) = be^{\alpha/2} + o(1).$$

We conclude that

$$\lim\limits_{n \to \infty} V_n = be^{\alpha/2} = b\cdot\sqrt{\frac{a}{b}} = \sqrt{ab}.$$

3950. *Proposed by Cristinel Mortici.*

Let $A \subset \{1, 2, 3, ..., n\}$ be a $(\left[\frac{n}{3}\right] + 2)$-element set, not containing two consecutive numbers, where $[\cdot]$ denotes the greatest integer function. Prove that there exist elements $x < y < z$ of A such that either (x, y, z) is an arithmetic progression, or $(x, y, z - 1)$ is an arithmetic progression.

There were three submitted solutions for this problem, all of which were correct. We present two solutions, covering the two submitted solution methods.

Solution 1, by Kathleen Lewis.

Let $k = [n/3] + 2$ and let $A = \{a_1, a_2, \ldots, a_k\}$. Define $b_i = a_{i+1} - a_i$ for $i = 1, 2 \ldots, k - 1$. Then

$$\sum_{i=1}^{k-1} b_i = a_k - a_1 \leq n - 1.$$

Since the set A contains no consecutive integers, $b_i \geq 2$ for all i. Since $[n/3] \geq (n - 2)/3$, then $k - 1 \geq (n + 1)/3$. If $b_i \geq 3$ for all i, then

$$\sum_{i=1}^{k-1} b_i \geq n + 1.$$

This is 2 more than the sum can be, so there must be at least two 2's among the b_i's. If any of the b_i's are larger than 3, there would need to be an additional 2 to compensate. So the number of twos on the list of b_i's is at least two more than the number of b_i's that are greater than 3. It is possible that $b_{k-1} = 2$, but all of the other twos must have another element following them. Not all of the successors of twos can be larger than 3, since there are too many twos. Therefore, at least one of the twos is followed by either a 2 or a 3. If $b_i = b_{i+1} = 2$, then (a_i, a_{i+1}, a_{i+2}) is an arithmetic progression. If $b_i = 2$ and $b_{i+1} = 3$, then $(a_i, a_{i+1}, a_{i+2} - 1)$ is an arithmetic progression.

Solution 2, by Trey Smith.

Let $x < y < z$ be integers. We say the triple (x, y, z) is *good* if and only if they are not consecutive integers and either (x, y, z) or $(x, y, z - 1)$ is an arithmetic progression.

The result will be proved using strong induction on n.

Observe that for $n = 1, 2, 3, 4, 6$, there is no set A containing at least three elements that will satisfy the hypothesis. It is reasonably easy to show that for $n = 5, 7, 8, 9, 10, 11, 12$ any set A that does satisfy the hypothesis will contain a good triplet.

Let $n > 12$, and $k = (\left[\frac{n}{3}\right] + 2)$. Suppose that the set $A = \{a_1, a_2, a_3, \ldots, a_k\}$ satisfies the hypothesis. If (a_1, a_2, a_3) is a good triplet, the result holds. Suppose, then, that (a_1, a_2, a_3) is not a good triplet. There are two cases to consider.

Case 1: $a_2 - a_1 \geq 3$. Let $B = \{a_2 - 3, a_3 - 3, a_4 - 3, \ldots, a_k - 3\}$. Set B satisfies the hypothesis for $n - 3$, and by the inductive assumption, contains a good triplet (x, y, z). Then $(x + 3, y + 3, z + 3)$ is a good triplet in A and the result holds.

Case 2: $a_2 - a_1 = 2$. Then we have that $a_3 - a_2 \geq 4$ or else (a_1, a_2, a_3) is good. Let $B = \{a_3 - 6, a_4 - 6, a_5 - 6, \ldots, a_k - 6\}$. B satisfies the hypothesis for $n - 6$, and so contains a good triplet (x, y, z). Then $(x + 6, y + 6, z + 6)$ is a good triplet in A and, again, the result holds.

Editor's Comments. In all of these solutions, one might draw a connection to the pigeonhole principle. Namely, the idea of forcing a particular collection of elements of A to exist by finding either too many distinct numbers (as in the proposer's solution), too many elements of the set far away from the previous element (in the solution by Lewis), or too many elements larger than the smallest two (in the solutions by Smith and Barbara). Of course, the wonderful thing about the pigeonhole principle is that it can be used in many different ways.
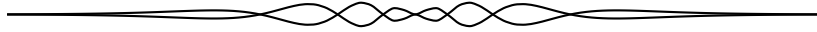
AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Arkady Alt, San Jose, CA, USA : 4041

Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey: 4047

Michel Bataille, Rouen, France: 4043, 4046

Mihaela Berindeanu, Bucharest, Romania : 4049

Leonard Giugiuc and Daniel Sitaru, Romania : 4048

Leonard Giugiuc and Diana Trailescu, Romania : 4042

Galav Kapoor, India : 4045

Dragoljub Milošević, Gornji Milanovac, Serbia : 4044

Mehtaab Sawhney, Commack High School, NY, USA : 4050

Solvers - individuals

Arkady Alt, San Jose, CA, USA : CC122, 3941, 3948, 3949

Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : **3946**

G. G. N. Angilella, Dipartimento di Fisica e Astronomia and Scuola Superiore di
    Catania, Universit'a di Catania, Italy : CC122

George Apostolopoulos, Messolonghi, Greece : 3946, 3948

Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : OC171,
    OC172, 3941, 3946, 3948

Fernando Ballesta Yague, I.E.S. Infante don Juan Manuel, Murcia, Spain : CC122

Roy Barbara, Lebanese University, Fanar, Lebanon : 3950

Michel Bataille, Rouen, France : 3941, **3942**, 3943, **3945**, 3946, 3947, 3948, **3949**, 3953

Marcel Chiriţă, Bucharest, Romania : 3942

Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : **3948**

Joseph DiMuro, Biola University, La Mirada, CA, USA : **3944**, 3948

Andrea Fanchini, Cantu, Italy : 3946, 3947, **OC171**

J. Chris Fisher, Cleveland Heights, OH, USA : 3945

Oliver Geupel, Brühl, NRW, Germany : OC172

John G. Heuver, Grande Prairie, AB : 3941, 3948

Dag Jonsson, Uppsala, Sweden : 3946

Kee-Wai Lau, Hong Kong, China : 3948, 3949

William LeVine, Summer@Brown Program, Brown University, Providence, RI, USA :
    OC171

Kathleen Lewis, University of the Gambia, Brikama, Republic of the Gambia : **3950**

Joseph M. Ling, Department of Mathematics and Statistics, University of Calgary,
    Calgary, AB : OC172

Salem Malikić, student, Simon Fraser University, Burnaby, BC : 3941, **3944**, 3946, 3948

Dragoljub Milošević, Gornji Milanovac, Serbia : 3941, 3946, 3948

Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India : 3948

Cristinel Mortici, Valahia University of Târgovişte, Romania: 3950

Somasundaram Muralidharan, Chennai, India : OC171

Gabriel Dinu Ovidiu, Bălceşti, Romania : 3941

Solvers - collaborations