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CONTENTS

THE OLYMPIAD CORNER: 81

R.E. WOODROW

All communications about this column should be sent directly to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

It is with some trepidation that I attempt to take over the Olympiad Corner from Professor Klamkin who has invested many hours of labour and much insight to build an active and committed readership. Murray has decided to devote more of his time to travel and writing. I am certain we all are grateful for the many hours he has put into writing the Olympiad Corner, and we wish him well in his travels. Certainly we look forward to the books he has planned - including, we hope, a work on contests and problems. For this column to continue to succeed, I must call on all those who so generously forwarded contests, problems, comments and solutions in the past to continue to do so, but to me at The University of Calgary. I am grateful that Murray has agreed to be available with advice and help because, especially in the first months, I shall be adapting myself to the role. Until the transition is completed and readers have become accustomed to forwarding problem sets and solutions to me I propose to dedicate more space to the presentation of solutions to problems posed in previous numbers.

But first we begin with the 1985 Annual High School competition of the Greek Mathematical Society, for which we thank Dimitris Vathis of Chalcis, Greece. We solicit, as usual, "nice" solutions from all readers.

1. (a) The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \left[\sqrt{9 + 2\sqrt{20}}\right]^x + \left[\sqrt{9 - 2\sqrt{20}}\right]^x - 2.$$

(i) Prove that f is an even function.

(ii) Solve the equation $f(x) = 0$.

(iii) Prove or disprove that f is onto.

(b) Consider an arbitrary function $h: \mathbb{R} \longrightarrow \mathbb{R}$ and the equation

$$h(x) - h(-x) = x. \tag{1}$$

If (1) has a finite number of solutions show that the number of solutions is an odd number.

2. Consider a straight line YY', the points $A \in YX'$, $B \in YX'$ and $O \notin YY'$. Prove that a point M lies on the line YY' if and only if there are real numbers k, ℓ with $k + \ell = 1$ such that

$$\overrightarrow{OM} = k\overrightarrow{OA} + \ell\overrightarrow{OB}.$$

Moreover find the values of k, ℓ so that

(i) M is a point of the half-line AX'.

(ii) M is a point of the half-line BX.

(iii) M is a point of the segment AB.

3. Let $S \subseteq \{(a,b)\colon a,b \in \mathbb{R}\}$ be a set such that

(i) at least one $(a,b) \in S$ has $a \neq 0$, $b \neq 0$ and $a \neq b$;

(ii) if $(x_1,y_1) \in S$, $(x_2,y_2) \in S$ and $k \in \mathbb{R}$ then

$$(x_1 + x_2, y_1 + y_2), \quad (kx_1, ky_1), \quad \text{and} \quad (x_1x_2, y_1y_2)$$

are all in S. Prove that $S = \mathbb{R}^2$.

4. For each real k let $L(k)$ be the number of solutions of the equation

$$[x] = kx - 1985$$

($[x]$ is the greatest integer that is not greater than x).

Prove that

(i) if $k > 2$ then $1 \leq L(k) \leq 2$;

(ii) if $0 < 1986k < 1$ then $L(k) = 0$;

(iii) there is at least one k such that $L(k) = 1985$.

* * *

We now present some solutions to problems published in earlier columns. A solution to the first to be presented has appeared [1986: 100] but this solution illustrates how "well known" theorems in geometry can simplify the matter at hand.

9. [1981: 42] *from 1980 Oesterreichisch-Polnischer Mathematik Weltbewerb.*

Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let D_1D_2, E_1E_2 be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined by AE_1 and AE_2.

Solution by A. Bondesen, Royal Danish School of Educational Studies, Copenhagen.

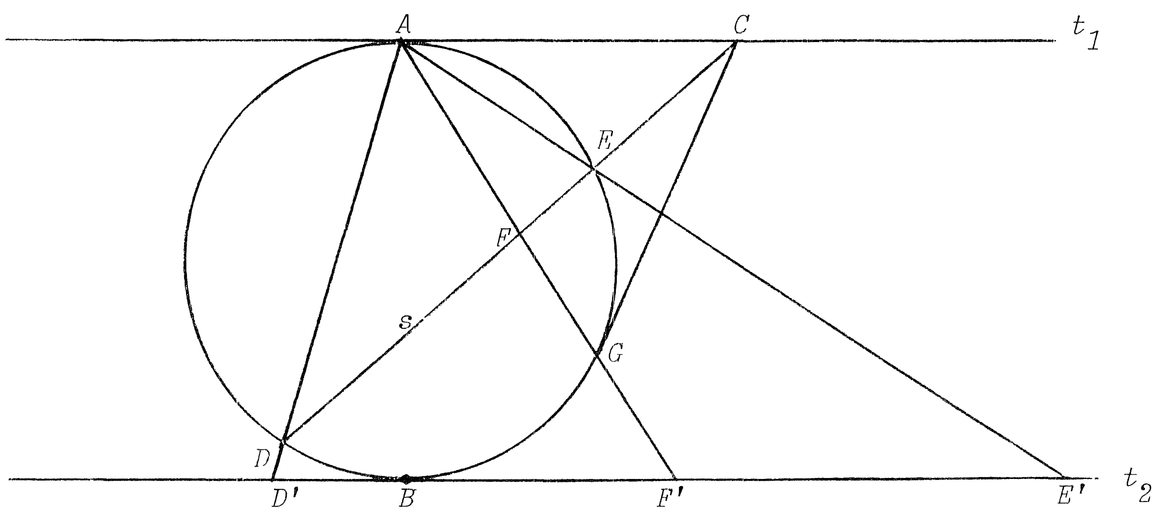
Refer to the figure below in which CG is the other tangent from C to the circle.

As is well known, [see for example Corollary 3.9.8 of 1], points F and G divide points D and E harmonically, i.e. $\frac{CE}{CD} = -\frac{FE}{FD}$. It follows from a fundamental theorem of projective geometry that the "transversal" t_2 cuts the lines AD, AF, AE and AC in the same nature, whence

$$-\frac{F'E'}{F'D'} = 1$$

(since C' is the point at ∞ "on" t_2). It follows that F' is the midpoint of segment $D'E'$.

Therefore as the secant s rotates around C from the position CG to the position CA, D' and E' move away from F', always maintaining the same distance from F'. The desired result follows.



Reference 1: Howard Eves, *A Survey of Geometry*, Allyn & Bacon, 1972.

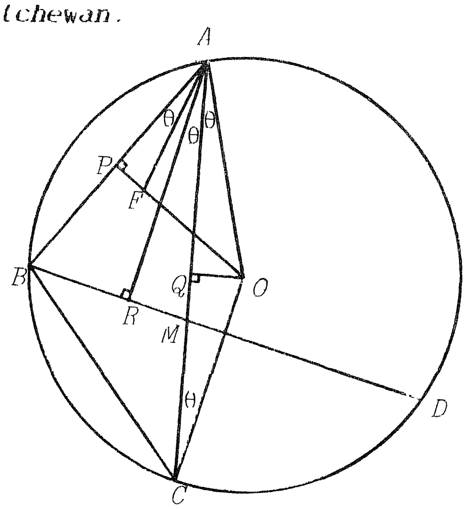
*

The second solution we present also dates back to volume 7, which, according to my records, still contains 35 problems to which we have received no solutions. Here again a "fact" from geometry intervenes at a crucial moment.

B-4. [1981: 115] *from a Bulgarian mathematical competition.*

Let the vertices A and C of a quadrilateral $ABCD$ be fixed points on a circle K, while B moves on one and D moves on the other of the two arcs of K with endpoints A and C, in such a way that always $BC = CD$. If M is the point of intersection of AC and BD and F is the circumcenter of triangle ABM, prove that the locus of F is an arc of a circle.

Solution by C. Fisher, Department of Mathematics and Statistics, The University of Regina, Saskatchewan.



First note that O (the center of K) and A must lie on the locus since these points correspond to the limiting positions of B at C and A respectively. Thus all one has to show is that $\angle AFO = \angle ABC$. This is done by showing that triangles ABC and AFO are similar by side-angle-side. Let θ be the angle between AC and the perpendicular AR to BM from A. Then $\theta = \angle OCA$ (since C is the midpoint of arc BD, making $OC \perp BD$). We also see that $\theta = \angle CAO$ since OA and OC are radii. Furthermore, θ is equal to $\angle BAF$ since for triangle ABM the angle between the line AF joining A to the circumcenter F and one side AB is equal to the angle between the altitude AR from A and the other side AM. (See, for example, Coxeter and Greitzer, *Geometry Revisited*, p.17.) Since $\angle BAC$ and $\angle FAO$ both equal θ plus or minus $\angle FAC$ (depending on whether or not AC separates B from F) they must be equal, as claimed. To see that the corresponding sides are proportional consider the two right triangles APF and AQO in the figure. Since F is equidistant from A and B, $\frac{1}{2}AB \div AF = \cos\theta$; and similarly $\frac{1}{2}AC \div AO = \cos\theta$.

*

Records indicate that rather fewer problems from volume 8 remain without solutions - here we find 24. For volume 9 there apparently remain only 20 problems without solutions published in later volumes of *Crux*. The next solution reduces the number of problems from volume 10 whose solutions have not been treated to 44. Let's get with it and produce elegant solutions to the remaining problems to clear the slate for '85 and beyond!

2. [1984: 74] *West Point Proposals.*

Given the foci F_1, F_2 and the major axis of an ellipse, show how to construct with straightedge and compass the intersection of the ellipse with a given straight line ℓ.

Solution by M. Molloy, Osgoode Township High School, Osgoode, Ontario

The idea is to compress in the direction of the major axis to transform the ellipse ε into a circle O, and the line ℓ into a line ℓ'. One can easily construct the intersection of O and ℓ'. These points are then transformed into the solution by the inverse stretching. To see that this can be done by straightedge and compass one may consider the following steps:

I. (i) Compression. Let the length of the major axis be a, and the distance between the foci c. It is a standard construction to construct the length b of the minor axis since $b^2 = a^2 - c^2$.

(ii) Construct C, the circle with diameter the minor axis. This is the image of ε under a compression in the direction of the major axis by a factor of $\frac{b}{a}$, with the minor axis left fixed.

(iii) Construct ℓ'. Extend the minor axis to meet ℓ at Y. Find the point of intersection X of ℓ with the major axis (extended). Construct X', the point on the major axis on the same side of the minor axis as X, but at distance $\frac{b}{a}$ times the distance of X from the minor axis. ℓ' is the line through $X'Y$. (Note Y is fixed under the compression.)

II. Construct the points of intersection of ℓ' and O.

For each of the zero, one, or two points P' of intersection of ℓ' and O construct the point P on the perpendicular from P' to the minor axis such that P lies on the same side of the axis as P' and at distance $\frac{b}{a}$ that of P' from the axis.

It is easy to verify by analytic calculations that the points P so constructed are the points of intersection of the line ℓ and the ellipse ε.

[*Editor's challenge*: A construction accompanied by a synthetic (constructive) verification would be more elegant.]

*

We now present solutions to some of the problems in Volume 11, No.2. These are all problems posed but not used for the I.M.O.

8. [1985: 37] *Proposed by Finland.*

Let F: $[0,1] \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$\begin{cases} F(2x) = b(F(x)) & 0 \le x \le \frac{1}{2} \\ F(x) = b + (1 - b)F(2x - 1) & \frac{1}{2} \le x \le 1 \end{cases}$$

where $b = (1 + c)/(2 + c)$ and $c > 0$. Prove that $0 < F(x) - x < c$ for all $x \in (0,1)$.

Correction and Solution by George Evagelopoulos, Athens, Greece.

Suppose

$$F(2x) = aF(x) \qquad 0 \le x \le \frac{1}{2} \qquad a \ne 1$$

and

$$F(x) = b + (1 - b)F(2x - 1) \qquad \frac{1}{2} \le x \le 1 \qquad b \ne 0.$$

Then

$$F(0) = aF(0) \iff F(0) = 0$$

and

$$F(1) = b + (1 - b)F(1) \iff F(1) = 1.$$

Now

$$F(1) = aF\left(\frac{1}{2}\right)$$

and

$$F\left(\frac{1}{2}\right) = b + (1 - b)F(0).$$

Thus $F\left(\frac{1}{2}\right) = b$, and $1 = ab$. Thus the problem is incorrectly stated. It should read:

Let F: $[0,1] \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$\begin{cases} F(x) = bF(2x) & 0 \le x \le \frac{1}{2} \\ F(x) = b + (1 - b)F(2x - 1) & \frac{1}{2} \le x \le 1 \end{cases}$$

where $b = (1 + c)/(2 + c)$ and $c > 0$. Prove that $0 < F(x) - x < c$ for all $x \in (0,1)$.

With this correction the solution is as follows.

As above $F(0) = 0$ and $F(1) = 1$. For $0 \le x \le \frac{1}{2}$, we get

$$F(x) - x = \frac{1 + c}{2 + c}(F(2x) - 2x) + \frac{c}{2 + c}x$$

and

$$F(x) - x - c = \frac{1 + c}{2 + c}(F(2x) - 2x - c) - \frac{c}{2 + c}(1 - x)$$

and for $\frac{1}{2} \le x \le 1$, we see that

$$F(x) - x = \frac{1}{2 + c}(F(2x - 1) - (2x - 1)) + \frac{c}{2 + c}(1 - x)$$

and

$$F(x) - x - c = \frac{1}{2 + c}(F(2x - 1) - (2x - 1) - c) - \frac{c}{2 + c}(x + c).$$

It is now easy to prove that $0 \le F(x) \le c$ for all x of the form $\frac{k}{2^n}$, which means for all x in $[0,1]$ by the continuity of F. The proof is an easy induction on n.

Finally for $x \in (0,1)$ we see that $\frac{c}{2 + x}x$, $\frac{c}{2 + c}(1 - x)$ and $\frac{c}{2 + c}(x + c)$ are all positive so the inequalities must be strict. Hence $0 < F(x) - x < c$ for all $x \in (0,1)$.

10. [1985: 370] *Proposed by Great Britain.*

If the sides a, b, c of a triangle satisfy

$$2(bc^2 + ca^2 + ab^2) = b^2c + c^2a + a^2b + 3abc$$

prove that the triangle is equilateral. Prove also that the equation can be satisfied by positive real numbers that are not the sides of a triangle.

Solution by George Evagelopoulos, Athens, Greece.

We have

$$\begin{aligned}
&2(bc^2 + ca^2 + ab^2) = b^2c + c^2a + a^2b + 3abc \\
\iff\ & a^3 + b^3 + c^3 - 3abc - a^3 - b^3 - c^3 + 2bc^2 + 2ca^2 + 2ab^2 \\
&\qquad\qquad - b^2c - c^2a - a^2b = 0 \\
\iff\ & a^3 + b^3 + c^3 - 3abc - b(a^2 - 2ab + b^2) - c(b^2 - 2bc + c^2) \\
&\qquad\qquad - a(c^2 - 2ac + a^2) = 0 \\
\iff\ & \tfrac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2] \\
&\qquad\qquad - b(a - b)^2 - c(b - c)^2 - a(c - a)^2 = 0 \\
\iff\ & (a + b + c)(a - b)^2 + (a + b + c)(b - c)^2 + (a + b + c)(c - a)^2 \\
&\qquad\qquad - 2b(a - b)^2 - 2c(b - c)^2 - 2a(c - a)^2 = 0 \\
(*)\quad \iff\ & (a - b)^2(a + c - b) + (b - c)^2(a + b - c) + (c - a)^2(b + c - a) = 0.
\end{aligned}$$

When a, b, c are the sides of a triangle we get $a + c - b > 0$, $a + b - c > 0$, and $b + c - a > 0$ so that in the case of a triangle, (*) is equivalent to $a = b = c$ and the triangle is equilateral.

In general there is no loss in assuming

$$a \geq b \geq c > 0.$$

Then (*) is equivalent to

$$(a - b)^2(a + c - b) + (b - c)^2(a + b - c) = (c - a)^2(a - b - c).$$

With the assumption $a \geq b \geq c$ the left side is positive if $a \neq c$, so a necessary condition is that $a > b + c$. To produce an example we may rewrite the original equation as a quadratic equation in a,

$$(b - 2c)a^2 + (c^2 + 3bc - 2b^2)a + (b^2c - 2bc^2) = 0.$$

Setting $b = 3$, $c = 1$ we obtain

$$a^2 - 8a + 3 = 0$$

which has the positive solution $a = 4 + \sqrt{13}$. Thus $4 + \sqrt{13}$, 3, 1 gives a nontriangular solution.

[Thanks go to J.T. Groenman for the explicit example.]

11. [1985: 37] *Proposed by Great Britain.*

Prove that there is a unique infinite sequence $\{u_0, u_1, u_2, \ldots\}$ of positive integers such that for all $n \geq 0$

$$u_n = \sum_{r=0}^{n} \binom{n + r}{r} u_{n-r} .$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

We first establish a lemma and two corollaries. With these we show that the unique solution is $u_n = 2^n$ for all n.

Lemma. $$\sum_{k=0}^{n} \binom{n + k}{n} \left(\frac{1}{2}\right)^{n+k} = 1.$$

Proof. Let n be a fixed positive integer. We toss a fair coin repeatedly until we first get $n + 1$ heads or $n + 1$ tails. This requires at least $n + 1$ and at most $2n + 1$ tosses. For $k = 0,1,2,\ldots,n$ let E_{n+k+1} be the event that exactly $n + k + 1$ tosses are required.

Then E_{n+k+1} occurs if and only if exactly n of the previous $n + k$ tosses have the same result as the $(n + k + 1)^{st}$ toss. Thus the probability of E_{n+k+1} is

$$p(E_{n+k+1}) = 2\binom{n+k}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right) = \binom{n+k}{n}\left(\frac{1}{2}\right)^{n+k}.$$

Since exactly one of the E_{n+k+1}'s occurs, we have

$$P\left(\bigcup_{k=0}^{n} E_{n+k+1}\right) = 1 = \sum_{k=0}^{n}\binom{n+k}{n}\left(\frac{1}{2}\right)^{n+k}.$$

<u>Corollary 1</u>. $\sum_{k=0}^{n}\binom{n+k}{n}2^{n-k} = \sum_{k=0}^{n}\binom{n+k}{n}2^{n-k} = 2^{2n}.$

<u>Corollary 2</u>. $\sum_{k=1}^{n}\binom{n+k}{k}2^{n-k} = 2^{2n} - 2^n.$

We now use Corollary 2 to prove by induction that $u_n = 2^n$ for all n. Because $u_n > 0$ and $u_0^2 = u_0$, $u_0 = 1 = 2^0$. Thus $u_1^2 = u_1 + 2$ so $(u_1 - 2)(u_1 + 1) = 0$ and $u_1 = 2$.

Assume now that $u_k = 2^k$ for all non-negative integers k less than $n \geq 2$.

Now

$$u_n^2 = \sum_{k=0}^{n}\binom{n+k}{k}u_{n-k} = u_n + \sum_{k=1}^{n}\binom{n+k}{k}2^{n-k} = u_n - 2^n + 2^{2n}.$$

From this we get

$$(u_n - 2^n)(u_n + 2^n - 1) = 0.$$

It follows that $u_n = 2^n$ and the induction is complete.

<u>15</u>. [1985: 38] *Proposed by Romania.*

The set $\{1,2,\ldots,49\}$ is partitioned into three subsets. Show that at least one of the subsets contains three different numbers a, b, c such that $a + b = c$.

Solution by Curtis Cooper, Central Missouri State University.

Since $\frac{49}{3} > 16$, one of the subsets, say X, contains at least 17 elements

$$x_1 < x_2 < \ldots < x_{17}, \quad \text{say.}$$

Form the differences

$$x_2 - x_1, x_3 - x_1, \ldots, x_{17} - x_1,$$

and omit x_1, if it appears in the list. If one of the remaining differences belongs to X we are done. Otherwise, since $15/2 > 7$ one of the subsets, say

Y ($\neq X$), contains at least 8 elements from these differences

$$y_1 < y_2 < \ldots < y_8$$

where

$$y_1 = x_i - x_1 \quad \text{for some } i.$$

Consider the differences

$$y_2 - y_1 \text{ , } y_3 - y_1 \text{ , } \ldots \text{ , } y_8 - y_1$$

and omit y_1 and x_i should they appear. If one of these differences belongs to Y we are obviously done. If one of them, say $y_j - y_1 \neq y_1$, x_i belongs to X, let $y_j = x_\mu - x_1$ and note that $i \neq \mu$. Then $(x_\mu - x_1) - (x_i - x_1) = x_\mu - x_i$ belongs to X, and we are done.

Thus we may suppose that 5 distinct differences $z_1 < z_2 < \ldots < z_5$ belong to the remaining subset Z.

Now write

$$z_1 = y_j - y_1$$

and

$$y_j = x_k - x_1 \text{ .}$$

From the differences

$$z_2 - z_1 \text{ , } \quad z_3 - z_1 \text{ , } \quad z_4 - z_1 \text{ , } \quad z_5 - z_1$$

omit z_1, y_j and x_k, if they appear. The remaining difference must belong to one of X, Y or Z and, as above, we find three distinct elements a, b, c in one of X, Y or Z such that $a + b = c$.

18. [1985: 38] *Proposed by Romania.*

A polynomial $P(x)$ of degree 990 satisfies

$$P(k) = F_k \qquad k = 992, 993, \ldots, 1982$$

where $\{F_k\}$ is the Fibonacci sequence, defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n = 2,3,4,\ldots$. Prove that $P(1983) = F_{1983} - 1$.

Solution by R.E.W.

It is just as easy to prove a (slight) generalization of the desired result.

Let p be a polynomial of degree n and suppose that for $0 \leq i \leq n$ $p(K + i) = F_{L+i}$, where K and L are integers, with $L \geq n + 1$. Then $p(K + n + 1) = F_{L+n+1} - F_{L-n-1}$.

This gives the desired result with $K = L = 992$ as $F_{L-n-1} = F_1 = 1$.

We prove the result by induction on the degree n of p. If $n = 0$, then p is a constant $C = F_L$. Now $p(K + 1) = C$ as well. Since $F_{L+1} = F_L + F_{L-1}$, we see $C = F_{L+1} - F_{L-1}$ and the statement is true. For an application of induction assume the statement is true for all polynomials g of degree $n_1 < n$, and all $K_1 \geq 0$, $L_1 \geq n_1 + 1$. Fix p of degree n and $K \geq 0$, $L \geq n + 1$. Suppose $p(K + i) = F_{L+i}$, $i = 0,1,2,\ldots,n$. Set $g(x) = p(x + 1) - p(x)$, the first difference function for p. Now the degree n_1 of g is $n - 1$. Also for $i = 0,1,2,\ldots,n - 1 = n_1$

$$g(K + i) = p(K + i + 1) - p(K + i) = F_{L+i+1} - F_{L+i} = F_{L+i-1} = F_{(L-1)+i}\ .$$

Applying the induction hypothesis to g with $K_1 = K$ and $L_1 = L - 1 \geq n + 1 - 1 = (n - 1) + 1 = n_1 + 1$ we conclude that

$$g(K + n) = g(K + (n - 1) + 1) = F_{L_1+n_1+1} - F_{L_1-n_1-1} = F_{L+n-1} - F_{L-n-1}\ .$$

Now

$$\begin{aligned} p(K + n + 1) &= p(K + n) + g(K + n) \\ &= F_{L+n} + F_{L+n-1} - F_{L-n-1} \\ &= F_{L+n+1} - F_{L-n-1} \end{aligned}$$

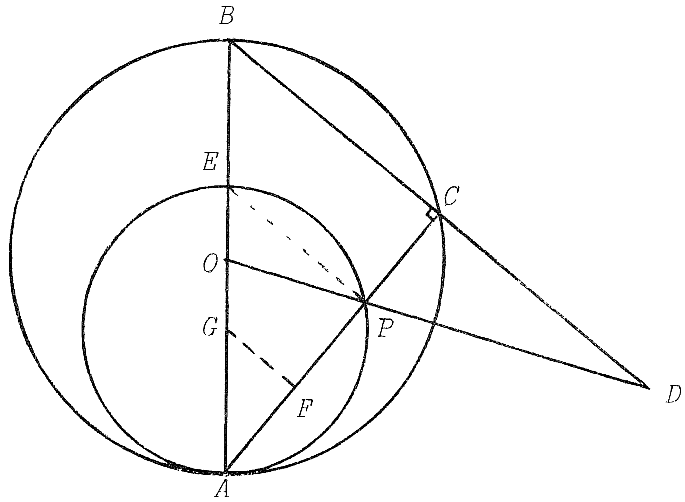
as desired.

[*Editors note*: A solution by Curtis Cooper, Central Missouri State University was also submitted which used the Newton forward difference formula for the polynomial.]

<u>20</u>. [1985: 38] *Proposed by Sweden.*

AB is the diameter of a circle γ with center O. A segment BD is bisected by the point C on γ, and AC and DO intersect at P. Prove that there is a point E on AB such that P lies on the circle with diameter AE.

Solution by Sister M. Jeanne Monk, S.N.D., Regina High School, South Euclid, Ohio.



Since C is the midpoint of DB and O the midpoint of AB, we have that P is the point of intersection of the medians of triangle ABD. Angle ACB is 90°, because it is inscribed in a semicircle. Construct the perpendicular bisector of AP, and call F the midpoint of AP. Let G be the intersection of the bisector with AB. The circle centred at G with radius AG passes through P. Label its diameter AE, i.e. $AG = GE$. We shall show that E does not depend on the segment DB - it is the point E on AB such that $AE = \frac{2}{3} AB$.

Now angle AFG is 90° and angle APE is 90° (since it is inscribed in a semicircle). Therefore triangles AFG, APE and ACB are all similar. Since P is the intersection of the medians of ADB, $AP = \frac{2}{3} AC$. Hence $AE = \frac{2}{3} AB$ as required.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before August 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

1201*. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

Prove that

$$(x+y+z)\left[\frac{xc^2}{a^2}+\frac{ya^2}{b^2}+\frac{zb^2}{c^2}\right] \ge \left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right](a^2yz+b^2zx+c^2xy),$$

where a, b, c are the sides of a triangle and x, y, z are real numbers.

1202. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let M_0, M_1 be lattice points and let M be a point such that M_0M_1M is an equilateral triangle. Let (a,b) be the coordinates of M reduced modulo 1. Prove that the set of all such pairs (a,b) is dense in the unit square $\{(x,y) \mid 0 \le x \le 1 , 0 \le y \le 1\}$, where M_0, M_1 vary over all lattice points.

1203. *Proposed by Milen N. Naydenov, Varna, Bulgaria.*

A quadrilateral inscribed in a circle of radius R and circumscribed around a circle of radius r has consecutive sides a, b, c, d, semiperimeter s and area F. Prove that

(a) $2\sqrt{F} \le s \le r + \sqrt{r^2+4R^2}$;

(b) $6F \le ab+ac+ad+bc+bd+cd \le 4r^2+4R^2+4r\sqrt{r^2+4R^2}$;

(c) $2sr^2 \le abc + abd + acd + bcd \le 2r\left[r + \sqrt{r^2 + 4R^2} \right]^2$;

(d) $4Fr^2 \le abcd \le \frac{16}{9} r^2(r^2 + 4R^2)$.

1204. *Proposed by Thomas E. Moore, Bridgewater State College, Bridgewater, Massachusetts.*

(a) Show that if n is an even perfect number, then $n - \phi(n)$ is a square (of an integer), where $\phi(n)$ is Euler's totient function.

(b) Find infinitely many n such that $n - \phi(n)$ is a square.

1205. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let triangle $A_1A_2A_3$ have sides a_1, a_2, a_3 with respective midpoints M_1, M_2, M_3. Let lines p, q, with intersections with a_i or its extension denoted P_i, Q_i respectively, have the properties that M_i is the midpoint of P_iQ_i for each i and that $p \perp q$. Find the locus of the intersection point S of p and q.

1206. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let X be a point inside triangle ABC, let Y be the isogonal conjugate of X and let I be the incenter of ΔABC. Prove that X, Y, and I colline if and only if X lies on one of the angle bisectors of ΔABC.

1207. *Proposed by G.A. Chambers and M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

If A and B are $m \times n$ and $n \times m$ matrices, respectively, with $m \ge n$, and AB is an identity matrix, prove that $m = n$. (A weaker form of this problem was proposed by the second proposer as a Quickie in *Mathematics Magazine* some years ago.)

1208. *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Let A', B', C' be points on sides BC, CA, AB, respectively, of ΔABC such that

$$\frac{A'B}{BC} = \frac{B'C}{CA} = \frac{C'A}{AB} \qquad (\ne 0, \tfrac{1}{2}, 1)$$

and so that some angle of ΔABC is equal to some angle of $\Delta A'B'C'$. Show that ΔABC and $\Delta A'B'C'$ are indirectly similar. In consequence, show that if they are directly similar then they are equilateral.

1209. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Characterize all positive integers a and b such that

$$a + b + (a,b) \leq [a,b],$$

and find when equality holds. Here (a,b) and $[a,b]$ denote respectively the g.c.d. and l.c.m. of a and b.

1210. *Proposed by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*

If A, B, C are the angles of an acute triangle, prove that

$$(\tan A + \tan B + \tan C)^2 \geq (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

976. [1984: 262; 1986: 145] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

(a) For all possible sets of n distinct points in a plane, let $T(n)$ be the maximum number of equilateral triangles having their vertices among the n points. Evaluate $T(n)$ explicitly in terms of n, or (at least) find a good upper bound for $T(n)$.

(b) If $a_n = T(n)/n$, prove or disprove that the sequence $\{a_n\}$ is monotonically increasing.

(c) Prove or disprove that $\lim_{n\to\infty} a_n = \infty$.

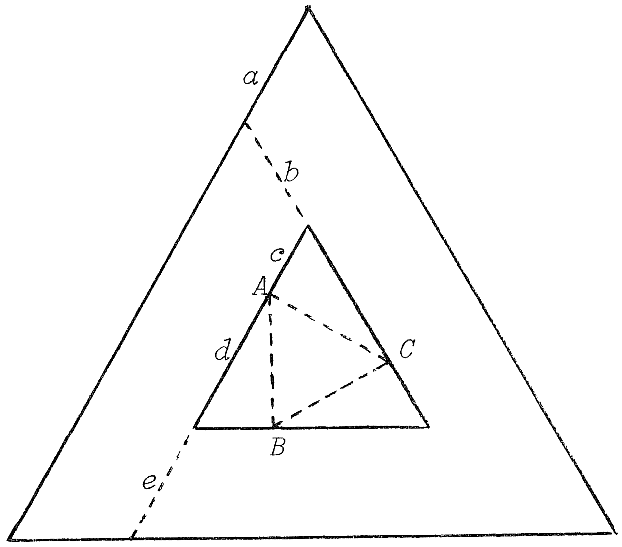
Comment by W.O. Moser, McGill University, Montréal, Québec.

The answer to question (4) [1986: 148], asking for an easy demonstration that the triangular grid with n points on each side has exactly $\binom{n+2}{4}$ equilateral triangles, is as follows.

There is a one-to-one correspondence between the set of such equilateral triangles ABC and the 5-tuples (a,b,c,d,e) of non-negative integers satisfying

$$a + b + c + d + e = n - 1, \qquad d \geq 1,$$

as illustrated by the accompanying figure.



Of course, there are

$$\left[\begin{matrix} n - 1 - 1 + 5 - 1 \\ 5 - 1 \end{matrix}\right] = \left[\begin{matrix} n + 2 \\ 4 \end{matrix}\right]$$

such 5-tuples (a,b,c,d,e).

* * *

1062. [1985: 219] *Proposed by M.S. Klamkin, University of Alberta.*

(a) Let Q be a convex quadrilateral inscribed in a circle with center O. Prove:

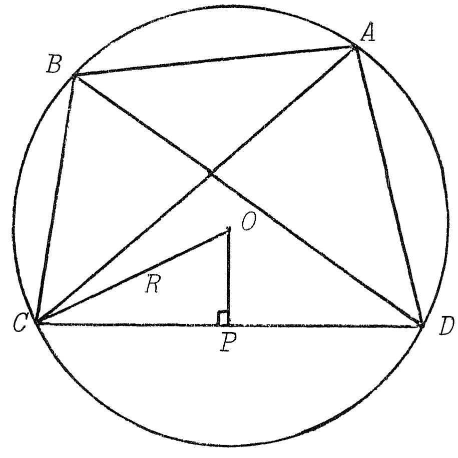
(i) If the distance of any side of Q from O is half the length of the opposite side, then the diagonals of Q are orthogonal.

(ii) Conversely, if the diagonals of Q are orthogonal, then the distance of any side of Q from O is half the length of the opposite side.

(b)* Suppose a convex quadrilateral Q inscribed in a centrosymmetric region with center O satisfies either (i) or (ii). Prove or disprove that the region must be a circle.

Solution to (a) by J.T. Groenman, Arnhem, The Netherlands.

Let $Q = ABCD$, let P be the midpoint of CD, and let R be the radius of the circle.



(ii) Given that $AC \perp BD$, we have

$$\angle CBD = 90^\circ - \angle ACB$$

and also

$$\angle COP = \angle CBD.$$

Thus

$$OP = R \cos \angle COP = R \sin \angle ACB$$

while

$$AB = 2R \sin \angle ACB,$$

and so

$$OP = \frac{1}{2}(AB).$$

Similarly for the other three sides of Q.

(i) Assuming $OP = \frac{1}{2}(AB)$, we have

$$R \cos \angle COP = \frac{1}{2}\cdot 2R \sin \angle ACB,$$

so

$$\cos \angle COP = \sin \angle ACB.$$

If O is inside Q, then

$$\angle CBD + \angle ACB = \angle COP + \angle ACB = 90^\circ,$$

and thus $AC \perp BD$. But if O is outside Q we cannot conclude this. I could expect this possibility as I only used $OP = \frac{1}{2}(AB)$ and did not use the similar relations involving the other three sides of Q.

[*Editor's note*: Groenman then went on to give a correct, but long, proof of (i), using all four of $OP = \frac{1}{2}(AB)$, etc.]

Partial solution by the proposer.

(a) Let $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$ denote vectors from O to the vertices A, B, C, D of Q.

(i) Assuming that the distance from O to AB is $\frac{1}{2}(CD)$, this says that

$$(\vec{A} + \vec{B})^2 = (\vec{C} - \vec{D})^2,$$

or that

$$\vec{A}\cdot\vec{B} + \vec{C}\cdot\vec{D} = 0,$$

since

$$\vec{A}^2 = \vec{B}^2 = \vec{C}^2 = \vec{D}^2 = R^2$$

where R is the radius of the circle. Similarly, if we assume that the distance from O to BC is $\frac{1}{2}(DA)$, we get

$$\vec{B}\cdot\vec{C} + \vec{D}\cdot\vec{A} = 0.$$

Then, since

$$(\vec{A} - \vec{C})\cdot(\vec{B} - \vec{D}) = \vec{A}\cdot\vec{B} + \vec{C}\cdot\vec{D} - \vec{B}\cdot\vec{C} - \vec{D}\cdot\vec{A} = 0,$$

we conclude $AC \perp BD$.

(ii) Letting $\angle AOB = \alpha$ and $\angle COD = \beta$, we have by hypothesis

$$\text{arc } AB + \text{arc } CD = \pi = \alpha + \beta.$$

Now

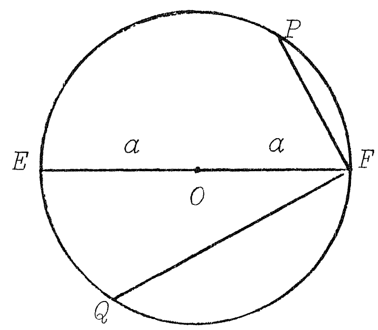
$$\begin{aligned}\vec{A}\cdot\vec{B} + \vec{C}\cdot\vec{D} &= R^2(\cos\alpha + \cos\beta)\\ &= 2R^2\cos\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]\\ &= 0,\end{aligned}$$

and so

$$(\vec{A} + \vec{B})^2 = (\vec{C} - \vec{D})^2,$$

which says that the distance from O to AB is $\frac{1}{2}(CD)$. Similarly, we obtain that the distance of O from any side is half the opposite side.

(b) Let C be the boundary of an oval (a smooth closed convex region) which is centrosymmetric with center O. Suppose that any convex quadrilateral Q inscribed in C and having orthogonal diagonals also has the property that the distance of any side of Q from O is half the length of the opposite side. We show that the oval must be a circle.



Let EF denote a maximum length chord of C. Then it is known that EF contains O (see 1.1 of [1986: 266]) and incidentally is also a binormal to C, i.e., EF is normal to the oval at each end. Now let P be any point of C other than E or F and let FQ be the chord of C perpendicular to FP. Since PF and QF are two orthogonal diagonals of the (degenerate) quadrilateral PFQ, $EF = 2(OF)$ must equal PQ. Thus PQ is also a maximum length chord of C and thus must contain O. Since P is an arbitrary point on the oval and is a constant distance from O, the oval is a circle.

It is an open question whether C must be a circle, supposing that any convex quadrilateral Q inscribed in C, such that the distance of any side of Q from O is half the length of the opposite side, also has the property that its diagonals are orthogonal.

Also solved (part (a)) by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and D.J. SMEENK, Zaltbommel, The Netherlands. In both cases, however, their proofs for (a)(i) *were valid only when O was inside Q.*

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<u>1063</u>. [1985: 220] *Proposed by Andy Liu, University of Alberta.*

The following is an excerpt from U.I. Lydna's *Medieval Medicine*:

The Chief of a village on Pagan Island was seriously ill. The Oracle revealed that he could only be cured by a potion containing exactly *five* herbs, at least four of which must be of *quint*essential nature. Unfortunately, the Oracle did not reveal what a quintessential herb was, and nobody on Pagan Island knew.

The Grand Alpharmist gathered a number of herbs and concocted sixty-eight potions, each containing exactly five herbs. In an effort to include as many combinations as possible, each trio of herbs was used in exactly one potion. The Oracle was consulted again, but it revealed only that each of the potions contained at least one quintessential herb.

The Chief's condition had deteriorated so much that further delay would prove fatal. The Grand Alpharmist therefore administered one dose of each potion, hoping that one of them would contain the necessary four quintessential herbs.

What was the fate of the Chief?

Solution by Duane Broline, University of Evansville, Evansville, Indiana.

Let H denote the set of herbs, T the set of triples of herbs, P the set of potions and Q the set of quintessential herbs. A typical element of any of these sets will be represented by the appropriate lower case letter. Let $P(i)$ denote the number of potions containing exactly i quintessential herbs. We show that $P(4) + 4P(5)$ is not congruent to zero modulo five, and thus at least one potion contains four or five quintessential herbs. In human language, the Chief was cured and lived happily ever after!

First, since each potion contains $\begin{bmatrix}5\\3\end{bmatrix}$ triples of herbs and each triple is in exactly one potion,

$$\begin{bmatrix}|H|\\3\end{bmatrix} = |T| = \begin{bmatrix}5\\3\end{bmatrix}\cdot 68.$$

It follows that $|H| = 17$, i.e. there are 17 herbs.

Next, let h be a fixed herb, and count the number of ordered pairs of the form (t,p) where $h \in t$ and $t \subset p$, to obtain

$$\begin{bmatrix}16\\2\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}\cdot k$$

where k is the number of potions containing h. Thus $k = 20$.

Let $u = |Q|$ be the number of quintessential herbs. A count of the number of ordered pairs (q,p) with $q \in Q$, $p \in P$ and $q \in p$, yields

$$u \cdot 20 = P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5). \tag{1}$$

Similarly, if h and h' are two fixed herbs then by considering ordered pairs of the form (t,p) with $\{h,h'\} \subset t \subset p$, it follows that each pair of herbs is in five potions. By examining ordered pairs of the form $(\{q,q'\},p)$ where $q,q' \in Q$ and $\{q,q'\} \subset p$, we obtain

$$\binom{u}{2} \cdot 5 = P(2) + 3P(3) + 6P(4) + 10P(5). \tag{2}$$

Since every triple of herbs is in a unique potion, we obtain

$$\binom{u}{3} = P(3) + 4P(4) + 10P(5). \tag{3}$$

Finally, there are sixty-eight potions altogether, so

$$68 = P(1) + P(2) + P(3) + P(4) + P(5). \tag{4}$$

Combining equations (1) to (4) gives

$$\binom{u}{3} - 5\binom{u}{2} + 20u - 68 = P(4) + 4P(5). \tag{5}$$

As $\binom{u}{3}$ is congruent to either 0, 1, or 4 modulo 5, the left side of (5) is congruent to 2, 3, or 1 modulo 5, and the result follows.

Also solved by CHARLES L. CHRISTMAS, Georgia Southern College, Statesboro, Georgia; WILLIAM A. MCWORTER JR. and LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; REMBERT N. PARKER, student, University of Evansville, Evansville, Indiana; and the proposer. There was one incorrect solution received.

McWorter and Meyers point out that sets of 68 five-element subsets of a 17-element set, with each triple of elements occurring in exactly one subset, do in fact exist; one such set is

(1,2,3,4,17),	(5,6,7,8,17),	(9,10,11,12,17),	(13,14,15,16,17),
(1,5,9,13,17),	(2,6,10,14,17),	(3,7,11,15,17),	(4,8,12,16,17),
(1,6,11,16,17),	(1,8,10,15,17),	(1,7,12,14,17),	(2,5,12,15,17),
(2,7,9,16,17),	(2,8,11,13,17),	(3,8,9,14,17),	(3,6,12,13,17),
(3,5,10,16,17),	(4,7,10,13,17),	(4,5,11,14,17),	(4,6,9,15,17),
(1,2,5,7,10),	(1,2,6,8,9),	(1,2,11,14,15),	(1,2,12,13,16),
(1,3,5,6,14),	(1,3,7,8,16),	(1,3,9,12,15),	(1,3,10,11,13),
(1,4,5,15,16),	(1,4,6,10,12),	(1,4,7,9,11),	(1,4,8,13,14),
(2,3,5,9,11),	(2,3,6,15,16),	(2,3,7,13,14),	(2,3,8,10,12),
(2,4,5,6,13),	(2,4,9,12,14),	(2,4,7,8,15),	(2,4,10,11,16),

(3,4,9,13,16),	*(3,4,10,14,15)*,	*(3,4,6,8,11)*,	*(3,4,5,7,12)*,
(1,5,8,11,12),	*(1,6,7,13,15)*,	*(1,9,10,14,16)*,	*(2,5,8,14,16)*,
(2,6,7,11,12),	*(2,9,10,13,15)*,	*(3,5,8,13,15)*,	*(3,6,7,9,10)*,
(3,11,12,14,16),	*(4,5,8,9,10)*,	*(4,6,7,14,16)*,	*(4,11,12,13,15)*,
(5,6,9,12,16),	*(5,6,10,11,15)*,	*(5,7,9,14,15)*,	*(5,7,11,13,16)*,
(5,10,12,13,14),	*(6,8,10,13,16)*,	*(6,8,12,14,15)*,	*(7,8,9,12,13)*,
(7,8,10,11,14),	*(6,9,11,13,14)*,	*(7,10,12,15,16)*,	*(8,9,11,15,16)*,

which is reproduced on page 326 of Eugen Netto, Lehrbuch der Combinatorik, *reprint of 2nd (1927) edition, Chelsea Publishing Company, New York. Selecting enough of the herbs 1-17 to be quintessential so that every potion contains at least one of them (for example, make all herbs quintessential) will result in a collection of potions satisfying the conditions of the problem, and so must result in at least one potion containing at least four quintessential herbs. McWorter and Meyers also observe that by choosing the quintessential herbs to be 1, 2, 4, 5, 7, 8, 11, and 13, each of the 68 potions contains at least one but not more than four quintessential herbs.*

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<u>1064</u>. [1985: 220] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Triangles ABC and DEF are similar, with angles $A = D$, $B = E$, $C = F$ and ratio of similitude $\lambda = EF/BC$. Triangle DEF is inscribed in triangle ABC, with D, E, F on the *lines* BC, CA, AB, not necessarily respectively. Three cases can be considered:

Case 1: $D \in BC$, $E \in CA$, $F \in AB$;

Case 2: $D \in CA$, $E \in AB$, $F \in BC$;

Case 3: $D \in AB$, $E \in BC$, $F \in CA$.

For Case 1, it is known that $\lambda \geq \frac{1}{2}$ (see *Crux* 606 [1982: 24, 108]). Prove that, for each of Cases 2 and 3,

$$\lambda \geq \sin \omega,$$

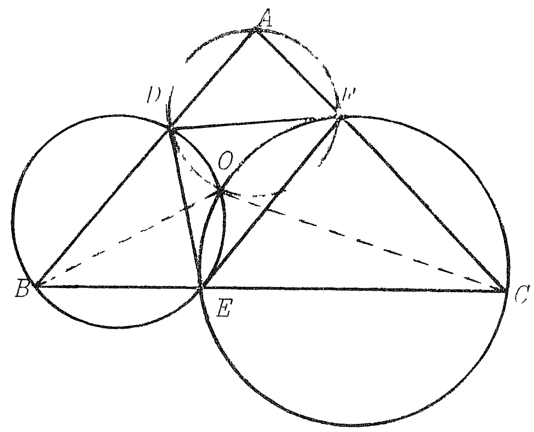
where ω is the Brocard angle of triangle ABC. (This inequality also holds *a fortiori* for Case 1, since $\omega \leq 30^\circ$.)

Solution by the proposer.

We do Case 3; Case 2 is similar.

The center of similitude of the triangles ABC and DEF is the common point O of the circles ADF, BED and CFE ([1], page 23). But then

$$\begin{aligned}\angle BOC &= \angle BOE + \angle EOC\\ &= \angle BDE + \angle EFC\\ &= \angle E + \angle A\\ &= \angle B + \angle A,\end{aligned}$$



thus

$$\angle OBC + \angle OCB = \angle C$$

and so

$$\angle OBC = \angle OCA.$$

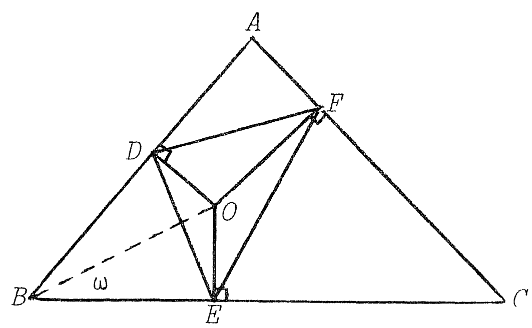
Similarly

$$\angle OCA = \angle OAB,$$

so O is the Brocard point and $\omega = \angle OBC$ the Brocard angle of ΔABC ([1], page 264).

The smallest ratio of similitude will occur when DEF is the pedal triangle of the point O, and therefore

$$\lambda \geq \frac{OE}{OB} = \sin \omega.$$



Reference:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.

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<u>1065</u>. [1985: 220] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

The orthocenter H of an orthocentric tetrahedron $ABCD$ lies inside the tetrahedron. If X ranges over all the points of space, find the minimum value of

$$f(X) = \{BCD\}\cdot AX + \{CDA\}\cdot BX + \{DAB\}\cdot CX + \{ABC\}\cdot DX,$$

where the braces denote the (unsigned) area of a triangle.

(This is an extension to 3 dimensions of *Crux* 866 [1984: 327].)

Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

An equivalent problem "The Steensholt inequality for a tetrahedron" was proposed as Problem E1264 in *Amer. Math. Monthly* 64 (1957) 744-745 with published solutions by N.D. Kazarinoff and myself. Subsequently, this was generalized in a paper of J. Schopp, "The inequality of Steensholt for an n-dimensional simplex", *Amer. Math. Monthly* 66 (1959) 896-897. For the convenience of the reader, we give the generalization.

Letting $A_0A_1A_2\ldots A_n$ be an n-dimensional simplex of volume V and $\{A_i\}$ denote the volume of the $(n-1)$-dimensional face opposite A_i, we prove that

$$f(X) = \Sigma\ \{A_i\}\cdot A_iX \geq n^2V$$

for all points X of n-space, with equality if and only if the simplex is orthocentric with an interior orthocenter H and X coincides with H.

Let r_i denote the signed distance from a point X to the face opposite A_i (if A_i and X are on the same side of this face, then r_i is positive, otherwise it is negative). Now

$$A_iX + r_i \geq h_i$$

where h_i is the altitude from A_i. Thus

$$f(x) \geq \Sigma\ \{A_i\}(h_i - r_i).$$

But since for each i

$$h_i\{A_i\} = nV = \Sigma\ r_j\{A_j\},$$

we have

$$f(x) \geq n(n+1)V - nV = n^2V.$$

Equality holds if and only if

$$A_iX + r_i = h_i$$

for all i, which means that X must be an interior orthocenter of the simplex.

Also solved by the proposer.

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1066*. [1985: 221] *Proposed by D.S. Mitrinovic, University of Belgrade, Yugoslavia.*

Consider the inequality

$$(y^p + z^p - x^p)(z^p + x^p - y^p)(x^p + y^p - z^p)$$
$$\leq (y^q + z^q - x^q)^r(z^q + x^q - y^q)^r(x^q + y^q - z^q)^r.$$

(a) Prove that the inequality holds for all real x, y, z if $(p,q,r) = (2,1,2)$.

(b) Determine all triples (p,q,r) of natural numbers for each of which the inequality holds for all real x, y, z.

Partial solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We claim that if r is even and $p = qr$, then the above inequality holds. In particular, we get part (a).

Let

$$f(a,b,c) = (b + c - a)(c + a - b)(a + b - c);$$

then the inequality in question reads

$$f(x^p, y^p, z^p) \le [f(x^q, y^q, z^q)]^r. \tag{1}$$

We put $x = y = z = u$ and get

$$u^{3p} \le u^{3qr}.$$

For $u > 1$ this implies $p \le qr$, but for $0 < u < 1$ we get $p \ge qr$. Hence if (1) holds for all x, y, z we must have $p = qr$.

Of course for $r = 1$ and $p = q$, (1) holds. We now assume $r \ge 2$, and show that p must be even for (1) to hold. Indeed, let p be odd. Then q and r are odd also. If (1) holds for all x, y, z then also

$$f((-x)^p, (-y)^p, (-z)^p) \le [f((-x)^q, (-y)^q, (-z)^q)]^r,$$

that is,

$$f(x^p, y^p, z^p) \ge [f(x^q, y^q, z^q)]^r,$$

holds for all x, y, z. Thus

$$f(x^p, y^p, z^p) = [f(x^q, y^q, z^q)]^r$$

for all x, y, z, which (by putting $z^q = x^q + y^q$ for example) implies $r = 1$ and $p = q$.

We need one more result. In "Inequalities involving elements of triangles, quadrilaterals or tetrahedra", *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* 461-497 (1974) 257-263, Oppenheim proved that if a, b, c are the sides of a triangle and, for $s \ge 1$, F_s denotes the area of the triangle with sides $a^{1/s}$, $b^{1/s}$, $c^{1/s}$, then

$$\left[\frac{4F_s}{\sqrt{3}}\right]^s \ge \left[\frac{4F_t}{\sqrt{3}}\right]^t \tag{2}$$

for $s > t \ge 1$. By the Heron formula, (2) reads equivalently

$$\frac{[f(a^{1/s}, b^{1/s}, c^{1/s})]^s}{[f(a^{1/t}, b^{1/t}, c^{1/t})]^t} \ge \frac{[(a^{1/t} + b^{1/t} + c^{1/t})/3]^t}{[(a^{1/s} + b^{1/s} + c^{1/s})/3]^s}. \tag{3}$$

By the general mean-inequality, the right-hand side of (3) is ≥ 1. Therefore (3) yields, for $s > t \ge 1$,

$$[f(a^{1/s},b^{1/s},c^{1/s})]^s \geq [f(a^{1/t},b^{1/t},c^{1/t})]^t. \tag{4}$$

Now suppose that r is even and $p = qr$. Since the right-hand side of (1) is always nonnegative, (1) is only interesting in the case that $f(x^p,y^p,z^p) > 0$, i.e., x^p, y^p, z^p form a triangle. As

$$|x^q| = (x^p)^{1/r}, \quad \text{etc.},$$

$|x^q|$, $|y^q|$, $|z^q|$ also form a triangle.

Case (i): x^q, y^q, z^q have the same sign. Noting that

$$[f(a,b,c)]^r = [f(-a,-b,-c)]^r, \tag{5}$$

we may and do assume that x^q, y^q, $z^q > 0$. Then putting $t = 1$, $s = r$, $a = x^p$ etc., (4) yields

$$[f(x^q,y^q,z^q)]^r \geq f(x^p,y^p,z^p),$$

i.e., (1).

Case (ii): only two of x^q, y^q, z^q have the same sign. By (5) we may and do assume $x^q < 0$, $y^q > 0$, $z^q > 0$. Then $x^q = -t^q$ where $t > 0$, and

$$\begin{aligned}[f(x^q,y^q,z^q)]^r &= [(t^q + y^q + z^q)(t^q + y^q - z^q)(t^q - y^q + z^q)]^r \\ &\geq [f(t^q,y^q,z^q)]^r \\ &\geq f(t^p,y^p,z^p) \\ &= f(x^p,y^p,z^p),\end{aligned}$$

the last inequality holding because of case (i). This finishes the proof that the given inequality holds when r is even and $p = qr$.

The only case left unsettled is $p = qr$, r odd > 1, q even. As in Case (i) above, we arrive at the validity of (1) if x^p, y^p, z^p form a triangle. What is more, as x^q, y^q, z^q are always nonnegative (1) also holds if only x^q, y^q, z^q form a triangle. Therefore, it remains to deal with the inequality (putting $u = x^q$, $v = y^q$, $w = z^q$)

$$f(u^r,v^r,w^r) \leq [f(u,v,w)]^r,$$

that is,

$$\begin{aligned}(u^r + v^r - w^r)(u^r - v^r + w^r)(-u^r + v^r + w^r) \\ \leq [(u + v - w)(u - v + w)(-u + v + w)]^r\end{aligned}$$

where $r > 1$ is odd, $u,v,w \geq 0$ and (say) $u + v \leq w$.

I could *not* settle this case!

Comment on part (a) by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

If x^2, y^2, z^2 do not satisfy the triangle inequality, the left-hand side of the inequality is negative and the inequality holds trivially. So we can assume that x, y, and z are the lengths of the sides of some non-obtuse triangle (possibly degenerate). For this case, the inequality is known (see page 10 of M.S. Klamkin, Notes on inequalities involving triangles or tetrahedrons, *Publications de la Faculté d'Electrotechnique de l'Université à Belgrade*, No.330-No.337 (1970) 1-15), and was shown to be equivalent to

$$\left(\frac{r}{R}\right)^2 \geq 2\cos X \cos Y \cos Z = \left(\frac{r}{R}\right)^2 - 2(IH)^2.$$

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<u>1067</u>. [1985: 221] *Proposed by Jack Garfunkel, Flushing, N.Y.*

(a)* If $x, y, z > 0$, prove that

$$\frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(yz + zx + xy)} \leq \frac{3 + \sqrt{3}}{9}.$$

(b) Let r be the inradius of a triangle and r_1, r_2, r_3 the radii of its three Malfatti circles (see *Crux* 618 [1982: 82]). Deduce from (a) that

$$r \leq (r_1 + r_2 + r_3)\cdot\frac{(3 + \sqrt{3})}{9}.$$

Solution by Chung-lie Wang, University of Regina, Regina, Saskatchewan.

We present two proofs of the inequality in (a), the second actually being a generalization.

For the first, let

$$f(x,y,z) = \frac{xyz(\ x + y + z + \sqrt{x^2 + y^2 + z^2}\ )}{(x^2 + y^2 + z^2)(yz + zx + xy)}.$$

By setting $x = sz$, $y = tz$ for positive s, t and noting that

$$(s + t + 1)^2 = s^2 + t^2 + 1 + 2(s + t + st),$$

we have

$$f = f(s,t) = \frac{st(s + t + 1 + \sqrt{s^2 + t^2 + 1}\ )}{(s^2 + t^2 + 1)(s + t + st)}$$

$$= \frac{2st(s + t + 1 + \sqrt{s^2 + t^2 + 1}\ )}{(s^2 + t^2 + 1)[(s + t + 1)^2 - (s^2 + t^2 + 1)]}$$

$$= \frac{2st}{(s^2 + t^2 + 1)(s + t + 1 - \sqrt{s^2 + t^2 + 1}\,)} \,. \tag{1}$$

Now, logarithmic partial differentiation of (1) yields

$$\frac{f_s}{f} = \frac{1}{s} - \frac{2s}{s^2 + t^2 + 1} - \frac{1 - s(s^2 + t^2 + 1)^{-1/2}}{s + t + 1 - \sqrt{s^2 + t^2 + 1}}$$

and

$$\frac{f_t}{f} = \frac{1}{t} - \frac{2t}{s^2 + t^2 + 1} - \frac{1 - t(s^2 + t^2 + 1)^{-1/2}}{s + t + 1 - \sqrt{s^2 + t^2 + 1}} \,.$$

Setting $f_s = f_t = 0$, we obtain that either $s = t$ or

$$\frac{1}{st} - \frac{2}{s^2 + t^2 + 1} + \frac{1}{(s + t + 1 - \sqrt{s^2 + t^2 + 1}\,)\sqrt{s^2 + t^2 + 1}} = 0. \tag{2}$$

But

$$s + t + 1 > \sqrt{s^2 + t^2 + 1}$$

and

$$\frac{1}{st} > \frac{2}{s^2 + t^2 + 1}$$

both hold for $s,t > 0$, so (2) has no solution. Thus $s = t$, and

$$\frac{1}{s} - \frac{2s}{2s^2 + 1} = \frac{1 - s(2s^2 + 1)^{-1/2}}{2s + 1 - \sqrt{2s^2 + 1}}$$

$$\frac{1}{s(2s^2 + 1)} = \frac{1 - s(2s^2 + 1)^{-1/2}}{2s + 1 - \sqrt{2s^2 + 1}}$$

$$2s + 1 - \sqrt{2s^2 + 1} = s(2s^2 + 1) - s^2\sqrt{2s^2 + 1}$$

$$(s^2 - 1)\sqrt{2s^2 + 1} = 2s^3 - s - 1 = (s - 1)(2s^2 + 2s + 1)$$

so either $s = 1$ or

$$(s + 1)\sqrt{2s^2 + 1} = 2s^2 + 2s + 1$$

$$(s^2 + 2s + 1)(2s^2 + 1) = (2s^2 + 2s + 1)^2$$

which has no positive solution. Thus the only critical point of $f(s,t)$ is $s = t = 1$. Moreover, a straightforward manipulation (with somewhat tedious details omitted) yields

$$f_{ss} = f_{tt} = -2f_{st} = -\frac{2(4+\sqrt{3})}{27} \qquad \text{at } s = t = 1.$$

Consequently, since $f_{ss} < 0$ and

$$f_{st}^2 - f_{ss}f_{tt} = -\frac{3}{4} f_{ss}^2 < 0,$$

f has relative maximum $\frac{3+\sqrt{3}}{9}$ at $s = t = 1$. Hence*,

$$f(s,t) \le (3+\sqrt{3})/9$$

for all $s,t > 0$, and part (a) follows.

[*Editor's note*: At this point the solver proved by "brute force" that f has its absolute maximum at $s = t = 1$. A question for the readers: is there a theorem of the sort "If $f\colon \mathbb{R}^2 \to \mathbb{R}$ has exactly one critical point P, and f has a relative maximum at P, *and f also has the property* ($\cdots$), then f attains its absolute maximum at P." which will complete the above proof?]

We now present a generalization of (a), with a different proof.

Let

$$s(t) = \left[\sum_{j=1}^{n} a_j x_j^t\right]^{1/t}$$

and

$$a = \sum_{j=1}^{n} a_j$$

where $a_j, x_j > 0$. Then we claim

$$\frac{s(q) + s(p)}{s^2(p)s^{-1}(r)} \le \frac{a^{1/q} + a^{1/p}}{a^{2/p - 1/r}} \tag{3}$$

for $r \le q \le p$, with equality holding if and only if $x_1 = \ldots = x_n$. Part (a) follows with $n = 3$, $a_1 = a_2 = a_3 = 1$, $a = 3$, $p = 2$, $q = 1$, $r = -1$.

From the monotonicity of weighted means, we have

$$\frac{s(r)}{a^{1/r}} \le \frac{s(q)}{a^{1/q}} \le \frac{s(p)}{a^{1/p}}$$

with equality if and only if $x_1 = \ldots = x_n$. Thus

$$\frac{s(q)}{s(p)} \le a^{1/q - 1/p}$$

and

$$\frac{1}{s(p)s^{-1}(r)} = \frac{s(r)}{s(p)} \le a^{1/r - 1/p} ,$$

and so

$$\frac{s(q) + s(p)}{s^2(p)s^{-1}(r)} \le a^{1/q - 1/p} + a^{1/r - 1/p} = \frac{a^{1/q} + a^{1/p}}{a^{2/p - 1/r}} .$$

A continuous model of inequality (3) can be readily stated as follows (with an almost evident proof omitted). Let ϕ and μ be two continuous positive functions on an interval (b,c) of the real line. Then

$$\frac{S(q) + S(p)}{S^2(p)S^{-1}(r)} \le \frac{A^{1/q} + A^{1/p}}{A^{2/p - 1/r}}$$

for $r \le q \le p$, where

$$S(t) = \left[\int_b^c \phi^t(x)\mu(x)dx \right]^{1/t} ,$$

$$A = \int_b^c \mu(x)dx ,$$

with equality if and only if ϕ is a constant function on (b,c).

Finally, to prove (b), we put $x^2 = r_1$, $y^2 = r_2$, $z^2 = r_3$ in (a) and get

$$\frac{\sqrt{r_1r_2r_3}(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3})}{(r_1 + r_2 + r_3)(\sqrt{r_2r_3} + \sqrt{r_3r_1} + \sqrt{r_1r_2})} \le \frac{3 + \sqrt{3}}{9} .$$

From *Crux* 618 [1982: 84] we read

$$r = \frac{(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3})\sqrt{r_1r_2r_3}}{\sqrt{r_2r_3} + \sqrt{r_3r_1} + \sqrt{r_1r_2}} ,$$

and (b) follows.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; and (part (b)) by the proposer.

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1068. [1985: 221] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

A triangle ABC has sides a, b, c in the usual order. Prove that

$$\ell n\ c = \ell n\ a - \left\{ \frac{b}{a} \cos C + \frac{b^2}{2a^2} \cos 2C + \frac{b^3}{3a^3} \cos 3C + \ldots \right\}.$$

(This problem is not new. A reference will be given with the solution.)

Solution by Kenneth S. Williams, Carleton University, Ottawa, Ontario.

For $b \le a$ the series

$$\frac{b}{a} \cos C + \frac{b^2}{2a^2} \cos 2C + \ldots$$

converges, and

$$\begin{aligned}\frac{b}{a} \cos C &+ \frac{b^2}{2a^2} \cos 2C + \ldots \\ &= \mathrm{Re} \left[\frac{b}{a} e^{iC} + \frac{b^2}{2a^2} e^{2iC} + \ldots \right] \\ &= -\ \mathrm{Re}\ \ell n\left[1 - \frac{b}{a} e^{iC}\right] \\ &= -\ \ell n\left|1 - \frac{b}{a} e^{iC}\right| \\ &= -\ \ell n\sqrt{(1 - \frac{b}{a} \cos C)^2 + \frac{b^2}{a^2} \sin^2 C} \\ &= -\ \frac{1}{2}\ \ell n\left[1 - \frac{2b}{a} \cos C + \frac{b^2}{a^2}\right] \\ &= -\ \frac{1}{2}\ \ell n\left[\frac{c^2}{a^2}\right] \\ &= \ell n\ a - \ell n\ c,\end{aligned}$$

which gives the required result.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; and the proposer.

The problem came from page 33 of Plane Trigonometry, part II, *Rev. J.W. Colenso (Bishop of Natal), New Edition London (Longmans, Green and Co.).*

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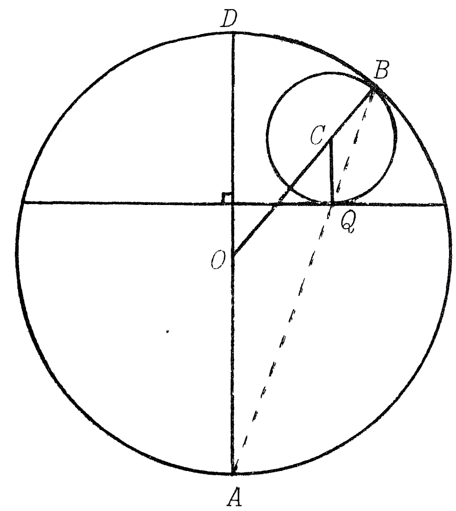
<u>1070</u>. [1985: 221] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let O be the center of an n-dimensional sphere. An $(n - 1)$-dimensional hyperplane, H, intersects the sphere (O) forming two segments. Another n-dimensional sphere, with center C, is inscribed in one of these segments, touching sphere (O) at point B and touching hyperplane H at point Q. Let AD be the diameter of sphere (O) that is perpendicular to hyperplane H, the points A and B being on opposite sides of H. Prove that A, Q, and B colline.

Solution by the proposer.



Since the two spheres are tangent, O, C, and B colline. Lines OB and OA determine a plane P. Since CQ is perpendicular to H and OA is also perpendicular to H, CQ must lie in plane P. BCQ and BOA are isosceles triangles, and $\angle BCQ = \angle BOA$. Therefore these two triangles are similar. Hence $\angle CBQ = \angle OBA$. Thus B, Q, A colline.

Also solved by JOHN FLATMAN, Timmins, Ontario.

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<u>1071</u>. [1985: 248] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

The cubic meter, or *stere*, is a measure of volume in the metric system:

```
    1
CUBIC
METER
-----
STERE .
```

Solve this decimal addition without reusing the digit 1.

Solution.

```
    1
29082
36564
-----
75646 .
```

Found by JOHN FLATMAN, Timmins, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; J.A. MCCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, *Valencia Community College, Orlando, Florida*; *J. SUCK, Essen, Federal Republic of Germany*; *KENNETH M. WILKE, Topeka, Kansas*; *and the proposer.*

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<u>1072</u>. [1985: 248] *Proposed by Herta T. Freitag, Roanoke, Virginia.*

For $n = 1,2,3,\ldots$, a sequence of triangles $A_nB_nC_n$ has sides (in the usual order)

$$a_n = n^2 + n + 1, \quad b_n = 2n + 1, \quad c_n = n(n + 2).$$

A point D_n is chosen on line A_nB_n such that $\angle A_nC_nD_n = 60^\circ$. Let

$$r_n = \frac{[D_nB_nC_n]}{[A_nD_nC_n]},$$

where the square brackets denote signed area. Find all pairs of positive integers m, n, if any, such that $r_mr_n = 1$.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

From the law of cosines we obtain

$$\begin{aligned}\cos A_n &= \frac{b_n^2 + c_n^2 - a_n^2}{2b_nc_n} \\ &= \frac{(2n + 1)^2 + n^2(n + 2)^2 - (n^2 + n + 1)^2}{2n(n + 2)(2n + 1)} \\ &= \frac{2n^3 + 5n^2 + 2n}{2n(n + 2)(2n + 1)} \\ &= \frac{1}{2},\end{aligned}$$

so that $A_n = 60^\circ$. Therefore $A_nB_nD_n$ is an equilateral triangle and

$$\begin{aligned}A_nD_n &= 2n + 1, \\ D_nB_n &= n(n + 2) - (2n + 1) = n^2 - 1.\end{aligned}$$

Thus

$$\begin{aligned}[D_nB_nC_n] &= (n^2 - 1)h, \\ [A_nD_nC_n] &= (2n + 1)h,\end{aligned}$$

where h is the altitude from C_n to A_nB_n, and

$$r_n = \frac{n^2 - 1}{2n + 1} .$$

Calculation of r_n for the first few values of n gives

n	r_n
1	0
2	3/5
3	8/7
4	5/3 .

Since it is clear that r_n monotonically increases with increasing n, the only (m,n) pairs are $(2,4)$, $(4,2)$, and the somewhat dubious $(1,\infty)$.

Also solved by SAM BAETHGE, San Antonio, Texas; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J.T. GROENMAN, Arnhem, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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A MESSAGE FROM THE EDITOR

As you will already have seen, another transition occurs in *Crux Mathematicorum* this issue, with Robert Woodrow taking over from Murray Klamkin as writer of the Olympiad Corner.

Murray will be greatly missed. I would like to wish him a happy retirement, and thank him for his huge contribution to *Crux* over the last eight years, a contribution by no means limited to the Olympiad Corner. I edited only nine of his eighty Corners, but in my struggles to understand his concise, clever, and nearly always correct arguments, have already become thoroughly intimidated by his talent for algebraic manipulation and triangle inequalities, to mention just two areas. Despite all this, I can't be too unhappy at the prospect of Murray's retirement, as he tells me that, without the Corner to produce, he will have more time to propose and solve problems. This bodes well for *Crux* readers!

And so may I introduce Robert Woodrow. Rob, a friend and colleague here at the University of Calgary, has long been interested and involved in mathematics education in schools and university, and in mathematics contests in particular. He will, I'm sure, keep the Olympiad Corner the very interesting feature of *Crux* that you have come to expect. You, in turn, can make his job easier by bombarding his mailbox with solutions. He won't mind a bit. Welcome, Rob.

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