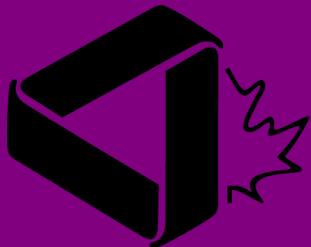


Mathematicorum

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Vol. 11, No. 3

March 1985

Sponsored by

Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa

The assistance of the publisher and the support of the Canadian Mathematical Society, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is \$25 in Canada, \$28 (or US\$21) elsewhere. Back issues: each \$2.50 in Canada, \$3.00 (or US\$2.50) elsewhere. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-10, \$19 in Canada, \$20 (or US\$15) elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

Editor: Léo Sauvé, Algonquin College, 140 Main Street, Ottawa, Ontario, Canada K1S 1C2.

Managing Editor: Kenneth S. Williams, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Assistant to the Editors: Mrs. Siu Kuen Lam.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

*

*

*

CONTENTS

The Olympiad Corner: 63	M.S. Klamkin	70
A Call (from Australia) for Articles		81
Problems - Problèmes: 1021-1030		82
Solutions: 857, 901-911, 915-917		84
<i>Crux Mathematicorum: A Call for Editors</i>		100

THE OLYMPIAD CORNER: 63

M.S. KLAMKIN

The 1985 International Mathematics Olympiad will be held in Finland from June 29 to July 11. Each invited country can send a team of up to six students. For further information, write to: The IMO-85 Organizing Committee, Department of Mathematics, University of Helsinki, Hallituskata 15, SF-00100, Helsinki, Finland.

I now extend the list, begun last month, of problems proposed by various participating countries (but unused) in past International Mathematics Olympiads. (I am grateful to L.M. Glasser for translating the problems from Mongolia and the U.S.S.R.) As usual, I solicit from all readers elegant solutions to these problems.

26. *Proposed by Australia.*

Let x_1, x_2, \dots, x_n denote n real numbers lying in the interval $[0,1]$. Show that there is a number $x \in [0,1]$ such that

$$\frac{1}{n} \sum_{i=1}^n |x - x_i| = \frac{1}{2}.$$

27. *Proposed by Australia.*

Let ABCD be a convex quadrilateral, and let A_1, B_1, C_1, D_1 be the circumcenters of triangles BCD, CDA, DAB, and ABC, respectively.

(i) Prove that either all of A_1, B_1, C_1, D_1 coincide in one point, or else they are all distinct. Assuming the latter case, show that A_1 and C_1 are on opposite sides of line B_1D_1 , and that B_1 and D_1 are on opposite sides of line A_1C_1 . (This establishes the convexity of quadrilateral $A_1B_1C_1D_1$.)

(ii) Let A_2, B_2, C_2, D_2 be the circumcenters of triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, and $A_1B_1C_1$, respectively. Show that quadrilateral $A_2B_2C_2D_2$ is similar to quadrilateral ABCD.

28. *Proposed by Belgium.*

Determine all integers x such that

$$x^4 + x^3 + x^2 + x + 1$$

is a perfect square.

29. *Proposed by Belgium.*

Determine all integer solutions (x,y) to the Diophantine equation

$$x^3 - y^3 = 2xy + 8.$$

30. *Proposed by Brazil.*

A box contains p white balls and q black balls, and beside the box lies a large pile of black balls. Two balls chosen at random (with equal likelihood) are taken out of the box. If they are of the same color, a black ball from the pile is put into the box; otherwise, the white ball is put back into the box. The procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

31. *Proposed by Bulgaria.*

Prove that, for every natural number n , the binomial coefficient $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, 3, \dots, 2n$.

32. *Proposed by Bulgaria.*

A regular n -gonal truncated pyramid with base areas S_1 and S_2 and lateral surface area S is circumscribed about a sphere. Let A be the area of the polygon whose vertices are the points of tangency of the lateral faces of the truncated pyramid with the sphere. Prove that

$$AS = 4S_1S_2 \cos^2 \frac{\pi}{n}.$$

33. *Proposed by Bulgaria.*

Given are a circle Γ and a line ℓ tangent to it at B . From a point A on Γ , a line $AP \perp \ell$ is constructed, with $P \in \ell$. If the point M is symmetric to P with respect to AB , determine the locus of M as A ranges on Γ .

34. *Proposed by Canada.*

Determine the permutation $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of $(1, 2, \dots, n)$ which maximizes

$$Q = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \dots + \alpha_n\alpha_1,$$

and also the permutation α which minimizes Q .

35. *Proposed by Canada.*

You are given an algebraic system with an addition and a multiplication for which all the laws of ordinary arithmetic are valid except commutativity of multiplication. Show that

$$(\alpha + ab^{-1}\alpha)^{-1} + (\alpha + b)^{-1} = \alpha^{-1},$$

where x^{-1} is that element for which $x^{-1}x = xx^{-1} = e$, the multiplicative identity.

36. *Proposed by Czechoslovakia.*

Let

$$S = \left\{ \frac{m+n}{\sqrt{m^2+n^2}} \mid m, n \text{ positive integers} \right\}.$$

Show that for each $(x,y) \in S \times S$ with $x < y$ there exists $z \in S$ such that $x < z < y$.

37. *Proposed by Finland.*

Four circles C, C_1, C_2, C_3 and a line ℓ are given, all in the same plane. The circles C_1, C_2, C_3 are all distinct, each touches the other two and touches also C and ℓ . If the radius of C is 1, determine the distance between its center and ℓ .

38. *Proposed by France.*

Let the numbers u_1, u_2, \dots, u_n be all positive and let

$$v_k = \sqrt[n]{u_1 u_2 \dots u_k}, \quad k = 1, 2, \dots, n.$$

Prove that

$$\sum_{k=1}^n v_k \leq e \sum_{k=1}^n u_k.$$

39. *Proposed by Mongolia.*

A country has n cities, any two of which are connected by a railroad. A railroad worker has to travel on each line exactly once. If at any stop there is a city he must reach but cannot (having already traveled on the line to or from that city), then he can fly. What is the smallest number of plane tickets he must buy?

40. *Proposed by Poland.*

Let Γ be a unit circle with center 0, and let P_1, P_2, \dots, P_n be points of Γ such that

$$\vec{OP}_1 + \vec{OP}_2 + \dots + \vec{OP}_n = \vec{0}.$$

Prove that $\overline{P_1 Q} + \overline{P_2 Q} + \dots + \overline{P_n Q} \geq n$ for all points Q .

41. *Proposed by Poland.*

A convex figure F lies inside a circle with center 0. The angle subtended by F from every point of the circle is 90° . Prove that 0 is a center of symmetry of F .

42. *Proposed by Rumania.*

If $(1 + x + x^2 + x^3 + x^4)^{496} = a_0 + a_1 x + \dots + a_{1984} x^{1984}$,

- (i) determine the g.c.d. of the coefficients $a_3, a_8, a_{13}, \dots, a_{1983}$;
- (ii) show that $10^{347} > a_{992} > 10^{340}$.

43. *Proposed by Spain.*

Solve the equation

$$\tan^2 2x + 2 \tan 2x \tan 3x - 1 = 0.$$

44. *Proposed by Sweden.*

Let n be a positive integer having at least two distinct prime factors. Show that there is a permutation $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of $(1, 2, \dots, n)$ such that

$$\sum_{k=1}^n k \cos \frac{2\pi\alpha_k}{n} = 0.$$

45. *Proposed by the U.S.A.*

Three roots of the equation

$$x^4 - px^3 + qx^2 - rx + s = 0$$

are $\tan A$, $\tan B$, $\tan C$, where A, B, C are the angles of a triangle. Determine the fourth root as a function of (only) p , q , r , and s .

46. *Proposed by the U.S.S.R.*

Let (p_{ij}) be a given $m \times n$ matrix with real entries, and let

$$A_i = \sum_{j=1}^n p_{ij} \quad \text{and} \quad B_j = \sum_{i=1}^m p_{ij}. \quad (1)$$

We say that a real number is "rounded off" if it is an integer or, if not an integer, when it is replaced by one of its two nearest neighboring integers. Show that the p_{ij} , A_i , and B_j can be rounded off so that (1) still remains valid.

47. *Proposed by the U.S.S.R.*

In the Martian language any finite ordered set of Latin letters is a word. The "Martian Word" editorial office issues a many-volume dictionary of the Martian language, in which the entries are numbered consecutively in alphabetical order. The first volume contains all the one-letter words, the second volume all the two-letter words, etc., and the numbering of the words in each successive volumes continues the numbering in the preceding one. Determine the word whose number is the sum of the numbers of the words

Prague, Olympiad, Mathematics.

48. *Proposed by the U.S.S.R.*

Let O be the center of the axis of a right circular cylinder; let A and B be diametrically opposite points in the boundary of its upper base; and let C be a boundary point of its lower base which does not lie in the plane OAB . Show that

$$\angle BOC + \angle COA + \angle AOB = 2\pi.$$

49. *Proposed by the U.S.S.R.*

Let x_1, x_2, \dots, x_n be numbers such that $1 \geq x_1 \geq x_2 \geq \dots \geq x_n > 0$. If

$0 \leq \alpha \leq 1$, prove that

$$(1 + x_1 + x_2 + \dots + x_n)^\alpha \leq 1 + x_1^\alpha + 2^{\alpha-1} x_2^\alpha + \dots + n^{\alpha-1} x_n^\alpha.$$

50. Proposed by Vietnam.

Given are the function $F(x) = ax^2 + bx + c$ and $G(x) = cx^2 + bx + a$, where

$$|F(0)| \leq 1, \quad |F(1)| \leq 1, \quad \text{and} \quad |F(-1)| \leq 1.$$

Prove that, for $|x| \leq 1$,

- (i) $|F(x)| \leq 5/4$;
- (ii) $|G(x)| \leq 2$.

*

I now present solutions to some problems proposed earlier in this column.

1. [1984: 107] From the 1983 Chinese Mathematics Olympiad.

Show that

$$\operatorname{Arcsin} x + \operatorname{Arccos} x = \frac{\pi}{2}, \quad -1 \leq x \leq 1.$$

Solution by Gali Salvatore, Perkins, Québec.

Let $\alpha = \operatorname{Arcsin} x$ and $\beta = \operatorname{Arccos} x$, so that $\sin \alpha = \cos \beta = x$. Now

$$-1 \leq x \leq 0 \implies -\frac{\pi}{2} \leq \alpha \leq 0, \quad \frac{\pi}{2} \leq \beta \leq \pi$$

and

$$0 \leq x \leq 1 \implies 0 \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq \beta \leq \frac{\pi}{2};$$

and in each case $\cos \alpha = \sin \beta = \sqrt{1-x^2}$. In each case also, $0 \leq \alpha + \beta \leq \pi$ and

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta = x^2 + (1-x^2) = 1,$$

and therefore $\alpha + \beta = \pi/2$, as required.

*

2. [1984: 107] From the 1983 Chinese Mathematics Olympiad.

A function f defined on the interval $[0,1]$ satisfies

$$f(0) = f(1) \text{ and } 0 \leq x_1 \neq x_2 \leq 1 \implies |f(x_2) - f(x_1)| < |x_2 - x_1|.$$

Prove that $0 < x_1, x_2 < 1 \implies |f(x_2) - f(x_1)| < \frac{1}{2}$.

Solution by Michael W. Ecker, University of Scranton, Pennsylvania.

We give what is essentially a geometric proof. We assume without loss of generality that $f(0) = f(1) = 0$. Letting $x_1 = 0$ and 1 successively gives $|f(x_2)| < x_2$ for $x_2 \neq 0$ and $|f(x_2)| < 1-x_2$ for $x_2 \neq 1$. Thus the graph of $y = f(x)$, $0 \leq x \leq 1$,

lies inside the square shown in the figure.

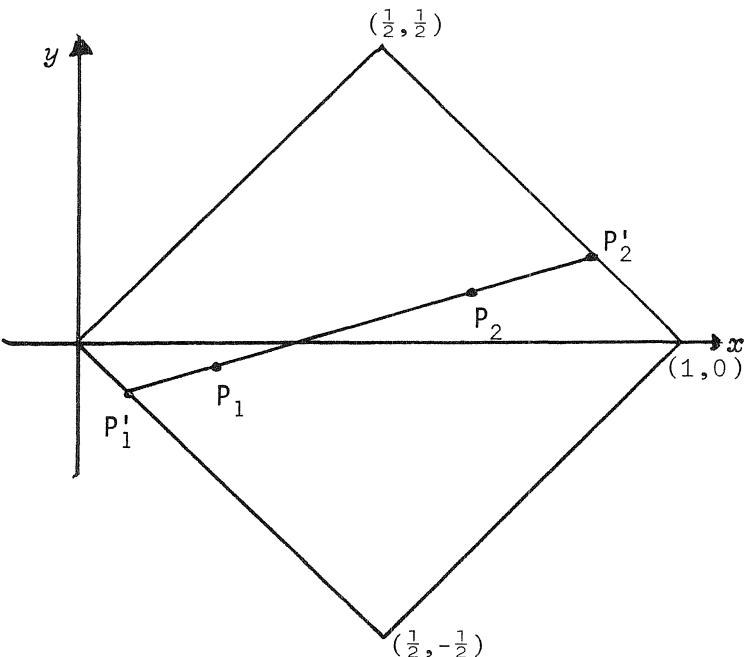
The desired result is trivially true if $x_2 = x_1$, so we assume that $x_2 \neq x_1$. Then the hypothesis

$$\left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| < 1$$

implies that the slope of the line joining any two distinct points $P_1(x_1, f(x_1))$ and $P_2(x_2, f(x_2))$ on the graph of f is numerically less than 1. If this line meets the boundary of the square in the points P'_1 and P'_2 , with abscissas x'_1 and x'_2 , respectively, then

$$|f(x_2) - f(x_1)| \leq |f(x'_2) - f(x'_1)| < \frac{1}{2}.$$

*



3. [1984: 107] From the 1983 Chinese Mathematics Olympiad.

For the quadrilateral ABCD of the figure, the following proportion holds:

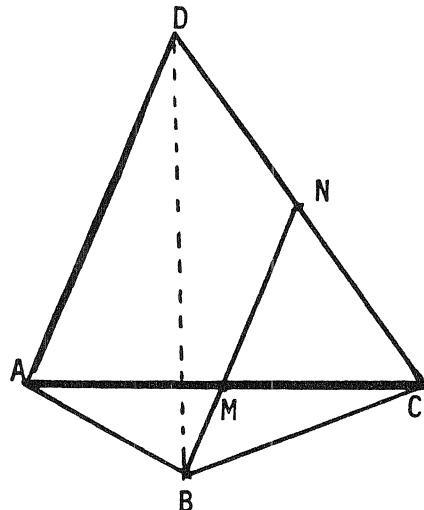
$$[ABD] : [BCD] : [ABC] = 3 : 4 : 1,$$

where the brackets denote area. If the points $M \in AC$ and $N \in CD$ are such that $AM : AC = CN : CD$ and B, M, N are collinear, prove that M and N are the midpoints of AC and CD , respectively.

Solution by Mike Molloy, student, Osgoode Township High School, Ontario.

Let $AM/AC = r$. Only one value of r is possible, since rotating BMN around B increases one of AM/AC , CN/CD and decreases the other. Thus if $r = 1/2$ is a permissible value of r , it will be the only one. Let M' and N' be the midpoints of AC and CD , respectively. Then

$$\begin{aligned} \vec{BM}' \times \vec{BN}' &= \frac{1}{4}(\vec{BA} + \vec{BC}) \times (\vec{BC} + \vec{BD}) \\ &= \frac{1}{4}(\vec{BA} \times \vec{BC} + \vec{BA} \times \vec{BD} + \vec{BC} \times \vec{BD}) \end{aligned}$$



$$\begin{aligned}
 &= \frac{1}{4}(\vec{BA} \times \vec{BC} + 3\vec{BA} \times \vec{BC} - 4\vec{BA} \times \vec{BC}) \\
 &= \vec{0},
 \end{aligned} \tag{1}$$

where in line (1) we have used the given area ratios.

Thus B, M', N' are collinear and $r = 1/2$.

*

1. [1984: 141] From the 1981 Leningrad High School Olympiad.

Find all integers x for which $|4x^2 - 12x - 27|$ is a prime number.

Solution by Paul Wagner, Chicago, Illinois.

Since $f(x) \equiv 4x^2 - 12x - 27 = (2x + 3)(2x - 9)$, a necessary condition for $|f(x)|$ to be a prime is that $|2x+3| = 1$ or $|2x-9| = 1$, that is, $x = -2, -1, 4$, or 5. Conversely, $|f(x)|$ is a prime for each of these values of x .

*

2. [1984: 141] From the 1981 Leningrad High School Olympiad.

Find $(a+b)/(a-b)$ if $a > b > 0$ and $a^2 + b^2 = 6ab$.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

From $a^2 + b^2 = 6ab$, we get $(a+b)^2 = 8ab$ and $(a-b)^2 = 4ab$. Then, since $a > b > 0$,

$$\frac{a+b}{a-b} = \sqrt{2}.$$

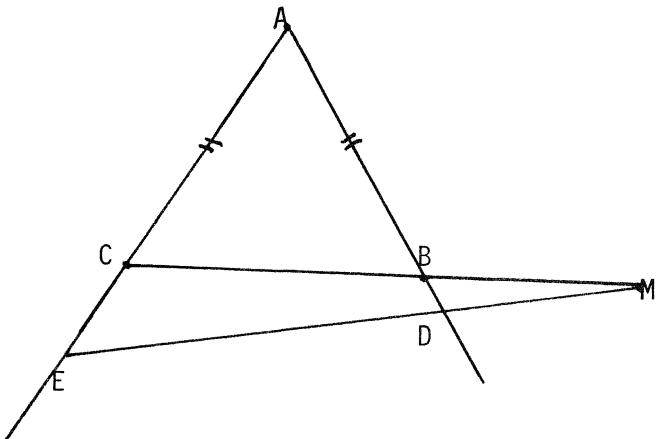
*

3. [1984: 141] From the 1981 Leningrad High School Olympiad.

From a point M outside an angle with vertex A two straight line segments are drawn, one of which cuts off on the sides of the angle two congruent segments AB and AC , and the other intersects these sides at the points D and E , respectively. Prove that $|BD|/|CE| = |MD|/|ME|$.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

Referring to the figure, we apply the theorem of Menelaus and obtain



$$\overline{AC} \cdot \overline{EM} \cdot \overline{DB} = \overline{CE} \cdot \overline{MD} \cdot \overline{BA},$$

and the desired result follows from the fact that $|AB| = |AC|$.

*

4. [1984: 141] From the 1981 Leningrad High School Olympiad.

Given that $x^x + y^y = x^y + y^x$, where x and y are natural numbers, prove that $x = y$.

Solution by K. Seymour, Toronto, Ontario.

The same conclusion holds if x and y are positive real numbers. We suppose that $x \neq y$ and obtain a contradiction. We further assume without loss of generality that $x > y$. Then

$$x^x + y^y = x^y + y^x \iff x^y(x^{x-y} - 1) = y^y(y^{x-y} - 1),$$

and the contradiction arises from $x^y > y^y$ and $x^{x-y} > y^{x-y}$.

*

5. [1984: 141] From the 1981 Leningrad High School Olympiad.

Show that in order for the diagonals of a quadrilateral to be perpendicular, it is necessary and sufficient that the midlines of the quadrilateral be congruent. (A midline is a line segment connecting the midpoints of opposite sides.)

Solution by K. Seymour, Toronto, Ontario.

If A, B, C, D are the consecutive vertices of a quadrilateral and all vectors have a common origin, then

$$\begin{aligned} \text{the midlines are congruent} &\iff \left| \frac{\vec{A} + \vec{B} - \vec{C} - \vec{D}}{2} \right| = \left| \frac{\vec{A} + \vec{D} - \vec{B} - \vec{C}}{2} \right| \\ &\iff \left(\frac{\vec{A} + \vec{B} - \vec{C} - \vec{D}}{2} \right)^2 - \left(\frac{\vec{A} + \vec{D} - \vec{B} - \vec{C}}{2} \right)^2 = 0 \\ &\iff (\vec{A} - \vec{C}) \cdot (\vec{B} - \vec{D}) = 0 \\ &\iff \text{the diagonals are perpendicular.} \end{aligned}$$

*

6. [1984: 141] From the 1981 Leningrad High School Olympiad.

Show that the two common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y = x^2/8$ are perpendicular.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

Let $y = mx + b$ be the equation of a common tangent. Since this line meets the circle in only one point, the discriminant of $(mx+b)^2 + x^2 = 2$ must vanish; and similarly for the parabola the discriminant of $mx+b = x^2/8$ must vanish. These two discriminant equations are

$$2m^2 = b^2 - 2 \quad \text{and} \quad 2m^2 = -b,$$

from which $b = -2$, then $m = \pm 1$ and the two common tangents are perpendicular.

*

7. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

The angle bisectors AD, BE, CF of a triangle ABC concur in the incenter I.

If

$$\frac{AI}{ID} = \frac{BI}{IE} = \frac{CI}{IF},$$

prove that triangle ABC is equilateral.

Solution by K. Seymour, Toronto, Ontario.

Since I is equidistant from the sides of the triangle, the equality of the given ratios implies that the three altitudes of the triangle are equal, and hence that the triangle is equilateral.

*

8. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

Which number is larger, 48^{25} or 344^{17} ?

Solution by John Morvay, Dallas, Texas.

Since

$$48^{25} = (7^2 - 1)^{25} < 7^{50} \quad \text{and} \quad 344^{17} = (7^3 + 1)^{17} > 7^{51},$$

it follows that $48^{25} < 344^{17}$.

*

9. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

Prove that the positive root of the equation

$$x(x+1)(x+2)\dots(x+1980)(x+1981) = 1$$

is less than $1/1981!$.

Solution by John Morvay, Dallas, Texas.

That the equation has exactly one positive root follows from Descartes' rule of signs. This positive root must be less than $1/1981!$, for when $x = 1/1981!$ is substituted into the left side of the equation,

$$x^{1982} + \dots + 1981!x,$$

where all coefficients are positive, the result is greater than 1.

*

10. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

Find all primes a and b such that $a^{a+1} + b^{b+1}$ is also a prime.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

Either a or b must be 2, say a . Then b is an odd prime. Now

$$b = 6n+1 \implies 8 + (6n+1)^{6n+2} \equiv 0 \pmod{3}$$

and

$$b = 6n-1 \implies 8 + (6n-1)^{6n} \equiv 0 \pmod{3}.$$

Thus the only possibility left is $b = 6n+3$ with $n = 0$, and

$$2^3 + 3^4 = 89, \text{ a prime.}$$

*

11. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

Let $0 < x < \pi/6$. Show that for all natural numbers n

$$\sin x + \tan^2 x + \sin^3 x + \dots + \tan^{2n} x < 1.4.$$

Solution by John Morvay, Dallas, Texas.

Since $\sin x < 1/2$ and $\tan x < 1/\sqrt{3}$, the sum of the finite series is less than

$$\left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \dots\right) + \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots\right) = \frac{\frac{1}{2}}{1 - \frac{1}{4}} + \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{7}{6} < 1.4.$$

*

12. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

Find the lengths of the edges of a rectangular parallelepiped with a square base if they are natural numbers and the total surface area is numerically equal to the sum of all the edge lengths.

Solution by H. Kaye, Brooklyn, N.Y.

There are eight edges of the two square bases each of length a , say, and four lateral edges each of length b , say. Then $8a+4b = 2a^2+4ab$, from which

$$2b = a(a + 2b - 4). \quad (1)$$

It follows from (1) that a is even, say $a = 2a'$, and then

$$b = a(a' + b - 2) = na, \text{ say.}$$

Now, from (1),

$$a = \frac{2n + 4}{2n + 1} = 1 + \frac{3}{2n+1}.$$

Finally, $n = 1$, and $a = b = 2$.

*

13. [1984: 142] *From the 1981 Leningrad High School Olympiad.*

In the convex quadrilateral ABCD, $\angle BAC = \angle CBD$ and $\angle ACD = \angle BDA$. Show that $|AC|^2 = |BC|^2 + |AD|^2$.

Solution by M.S.K.

With the angles θ, α, β as shown in the figure, we note that $\angle ABC = \angle ADC = \theta$ and then, by the law of sines,

$$\frac{AC}{\sin \theta} = \frac{BC}{\sin \alpha} = \frac{AD}{\sin \beta} = \lambda, \text{ say.}$$

From $[ABCD] = [DBC] + [DAB]$, where the square brackets denote area, we obtain

$$\frac{1}{2}BD \cdot AC \sin \theta = \frac{1}{2}BD \cdot BC \sin \alpha + \frac{1}{2}BD \cdot AD \sin \beta,$$

from which

$$AC \cdot \frac{AC}{\lambda} = BC \cdot \frac{BC}{\lambda} + AD \cdot \frac{AD}{\lambda},$$

and $|AC|^2 = |BC|^2 + |AD|^2$ follows.

*

14. [1984: 142] From the 1981 Leningrad High School Olympiad.

Show that $2^{b+c} + 2^{c+a} + 2^{a+b} < 2^{a+b+c+1} + 1$ for all $a, b, c > 0$.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

The given inequality is equivalent to

$$F(x, y, z) \equiv 2 + xyz - x - y - z > 0,$$

obtained by dividing throughout by 2^{a+b+c} and then setting $x = 2^{-a}$, $y = 2^{-b}$, and $z = 2^{-c}$, so that $0 < x, y, z < 1$.

Since $F(x, y, z)$ is linear in each variable, it takes on its greatest and least values at the ends of the intervals, that is, at the corners of the cube with vertices $(0, 0, 0)$, $(1, 0, 0)$, ..., $(1, 1, 0)$, ..., $(1, 1, 1)$. Since this least value is zero, $F(x, y, z) > 0$ for all points (x, y, z) in the interior of the cube.

*

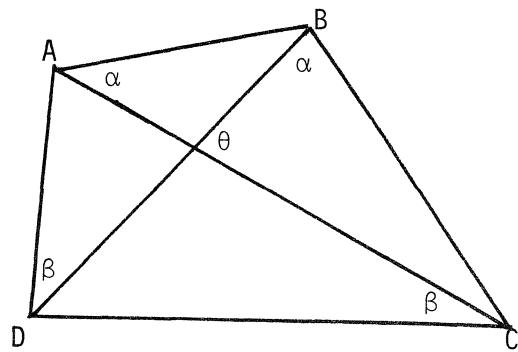
15. [1984: 142] From the 1981 Leningrad High School Olympiad.

Show that, for all positive x and y ,

$$x^3 + y^3 \leq x^3 \cdot \sqrt[3]{\frac{x}{y}} + y^3 \cdot \sqrt[3]{\frac{y}{x}}.$$

Solution by M.S.K.

More generally, for $x, y, m, n > 0$ consider



$$x^m + y^m \leq x^m \cdot \sqrt[n]{\frac{x}{y}} + y^m \cdot \sqrt[n]{\frac{y}{x}}.$$

Since the case $x = y$ is immediate, we can assume without loss of generality that $y = \lambda x$ with $\lambda > 1$. The inequality then reduces to

$$(\lambda^{m+1/n} - 1)(\lambda^{-1/n} - 1) \leq 0,$$

and this is clearly true. \square

A multivariate generalization is given by

$$\sum_{i=1}^p x_i^m \leq \sum_{i=1}^p x_i^m \left\{ \frac{x_i^p}{x_1 x_2 \dots x_p} \right\}^{1/n},$$

where $m, n, x_i > 0$ and p is a positive integer. We can simplify the inequality, by letting $x_i = a_i^n$, to

$$a_1 a_2 \dots a_p \sum_{i=1}^p a_i^{mn} \leq \sum_{i=1}^p a_i^{mn+p}.$$

This follows by applying the A.M.-G.M. inequality and Chebyshev's inequality:

$$\prod a_i \frac{\sum a_i^{mn}}{p} \leq \frac{\sum a_i^p}{p} \cdot \frac{\sum a_i^{mn}}{p} \leq \frac{\sum a_i^{mn+p}}{p}.$$

*

In closing, I would ask all readers submitting solutions to any of the problems in this or in past columns to clearly identify the problems by stating their numbers and the place (year and page number) where they originally appeared.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

*

*

*

A CALL (FROM AUSTRALIA) FOR ARTICLES

The Mathematical Scientist, published by the Australian Mathematical Society, is an international journal devoted to the interface between mathematics and the real world; in other words, it is concerned with mathematical modelling as applied to the physical, biological, and social sciences. Have you done some interesting work of this kind and not yet published it? If so, please send your article to any member of the following subset of the Editorial Board:

Prof. J.M. Gani, Statistics Dept., University of Kentucky, Lexington, KY 40506;
Dr. H. Ockendon, Somerville College, Oxford, OX26HD, England;
Prof. B.C. Rennie, James Cook University, N. Queensland 4811, Australia.

To save time we ask just for a clear typescript, the conversion into ready-for-printer format is left until after refereeing and acceptance.

PROBLEMS -- PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

1021. Proposed by Allan Wm. Johnson, Washington, D.C.

In the etymological decimal addition

SERGE
DE
NÎMES
DENIM

maximize \hat{N} ÎMES (where $\hat{I} = I$), the city in southern France that gave its name to denim cloth.

1022. Proposé par Armel Mercier, Université du Québec à Chicoutimi.

Soient k un entier positif et i un entier vérifiant $0 \leq i \leq k$. Montrer que

$$\sum_{j=0}^k \frac{(-1)^j [k]_j q^{j(j+1)/2} (q^i - q^j)^k}{q^{jk}} = \prod_{j=1}^k (q^j - 1),$$

où $[k]_j$ désigne le coefficient binomial de Gauss défini par $[k]_0 = 1$ et

$$[k]_j = \frac{(q^k - 1)(q^{k-1} - 1) \dots (q^{k+1-j} - 1)}{(q - 1)(q^2 - 1) \dots (q^j - 1)}, \quad 0 < j \leq k,$$

et q est une variable réelle arbitraire.

1023.* From a Trinity College, Cambridge, examination paper dated June 7, 1901.

Show that, for $n = 1, 2, 3, \dots$,

$$\sum_{k=1}^n \operatorname{Arctan} \frac{2}{k^2} = \frac{3\pi}{4} - \operatorname{Arctan} \frac{1}{n} - \operatorname{Arctan} \frac{1}{n+1}.$$

1024. Proposed by William Tunstall Pedoe, student, The High School of Dundee, Scotland.

Prove that an odd number which is a perfect square cannot be perfect.

1025. Proposed by Peter Messer, M.D., Mequon, Wisconsin.

A paper square ABCD is folded so that vertex C falls on AB and side CD is divided into two segments of lengths l and m , as shown in the figure. Find the minimum value of the ratio l/m .

1026. Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

D, E, and F are points on sides BC, CA, and AB, respectively, of triangle ABC, and AD, BE, and CF concur at point H. If H is the incenter of triangle DEF, prove that H is the orthocenter of triangle ABC. (This is the converse of a well-known property of the orthocenter.)

1027. Proposed by M.S. Klamkin, University of Alberta.

Determine all quadruples (a,b,c,d) of nonzero integers satisfying the Diophantine equation

$$abcd\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) = (a + b + c + d)^2$$

and such that $a^2 + b^2 + c^2 + d^2$ is a prime.

1028. Proposed by David Singmaster, Polytechnic of the South Bank, London, England.

Students learning modular arithmetic are pleasantly perplexed by

$$4 \cdot 5 \equiv 6 \pmod{7}.$$

Solve the following (and possibly other) generalizations:

(a) $a(a+1) \equiv a+2 \pmod{a+3}$.

(b) $a(a+1) \equiv a+2 \pmod{m}$, where m is not necessarily $a+3$.

(c) $a(a+1) \equiv (a+2)(a+3) \pmod{m}$.

1029. Proposed by Farshad Khorrami, student, The Ohio State University.

Find necessary and sufficient conditions on the complex numbers a and b so that each root of

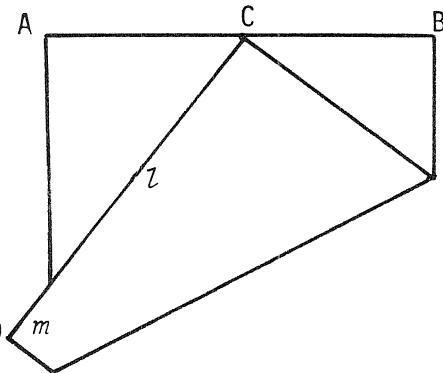
$$z^2 + az + b = 0$$

has absolute value less than 1.

1030. Proposed by J.T. Groenman, Arnhem, The Netherlands.

Given are two obtuse triangles with sides a,b,c and p,q,r , the longest sides of each being c and r , respectively. Prove that

$$ap + bq < cr.$$



*

*

*

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

857. [1983: 179; 1984: 304; 1985: 20] Proposed by Leroy F. Meyers, The Ohio State University.

(a) Given three positive integers, show how to determine algebraically (rather than by a search) the row (if any) of Pascal's triangle in which these integers occur as consecutive entries.

(b) Given two positive integers, can one similarly determine the row (if any) in which they occur as consecutive entries?

(c)* The positive integer k occurs in the row of Pascal's triangle beginning with 1, k , For which integers is this the only row in which it occurs?

III. Comment on part (c)* by David Singmaster, Polytechnic of the South Bank, London, England.

In 1971, I examined $N(\alpha)$, the number of times the integer α occurs in Pascal's triangle [1]. I showed that $N(\alpha) = O(\log \alpha)$ and conjectured that $N(\alpha)$ was bounded. The proposer's part (c)* asks to characterize those α for which $N(\alpha) = 2$. Abbott, Erdős, and Hanson [2] examined this and showed that the normal and average orders of $N(\alpha)$ are both 2. (See [4] for definitions of normal and average orders.) The normal order assertion implies that $N(\alpha) = 2$ for almost all integers (in the usual sense of density). Specifically, if $g(x)$ is the number of integers $\alpha \leq x$ such that $N(\alpha) > 2$, then they showed easily that $g(x) = O(\sqrt{x})$.

In [3], I reported on my search for repeated binomial coefficients. There are infinitely many α with $N(\alpha) \geq 6$, obtained by solving

$$\binom{n+1}{k+1} = \binom{n}{k+2},$$

leading to

$$n = F_{2i+2}F_{2i+3} - 1, \quad k = F_{2i}F_{2i+3} - 1,$$

where the F_i are the Fibonacci numbers, beginning with $F_0 = 0$. The only other non-trivial repeated binomial coefficients up to 2^{48} are the following:

$$120 = \binom{16}{2} = \binom{10}{3}$$

$$210 = \binom{21}{2} = \binom{10}{4}$$

$$1540 = \binom{56}{2} = \binom{22}{3}$$

$$7140 = \binom{120}{2} = \binom{36}{3}$$

$$11628 = \binom{153}{2} = \binom{19}{5}$$

$$24310 = \binom{221}{2} = \binom{17}{8}$$

$$3003 = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}$$

Note that the last equality is the case $i = 1$ of the above general solution. Note also that $N(3003) = 8$, and 3003 is the only known α with $N(\alpha) \geq 8$.

Perhaps some reader can investigate this question further.

REFERENCES

1. David Singmaster, "How often does an integer occur as a binomial coefficient?", *American Mathematical Monthly*, 78 (1971) 385-386.
2. H.L. Abbott, P. Erdős, and D. Hanson, "On the number of times an integer occurs as a binomial coefficient", *American Mathematical Monthly*, 81 (1974) 256-261.
3. David Singmaster, "Repeated binomial coefficients and Fibonacci numbers", *Fibonacci Quarterly*, 13 (1975) 295-298.
4. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1960, pp. 263, 356.

*

*

*

901. [1984: 18] *Proposed by Charles W. Trigg, San Diego, California.*

In

$$\text{BOXER} = \text{HITS},$$

the X doubles as a multiplication sign. Find (a) the fewest HITS, and (b) the most HITS, that the BOXER can deliver.

I. *Solution by Edwin M. Klein, University of Wisconsin-Whitewater.*

Here are the 26 (with BO > ER) computer-generated solutions to the alphametic, listed in order of increasing HITS, so solutions 1 and 26 are the answers to parts (a) and (b), respectively.

1. $46 \times 23 = 1058$	10. $58 \times 37 = 2146$
2. $57 \times 24 = 1368$	11. $56 \times 39 = 2184$
3. $49 \times 32 = 1568$	12. $65 \times 38 = 2470$
4. $69 \times 23 = 1587$	13. $65 \times 48 = 3120$
5. $42 \times 38 = 1596$	14. $82 \times 45 = 3690$
6. $52 \times 34 = 1768$	15. $64 \times 58 = 3712$
7. $58 \times 34 = 1972$	16. $85 \times 46 = 3910$
8. $52 \times 38 = 1976$	17. $68 \times 59 = 4012$
9. $54 \times 39 = 2106$	18. $76 \times 53 = 4028$

19.	$79 \times 52 = 4108$	23.	$74 \times 68 = 5032$
20.	$78 \times 59 = 4602$	24.	$87 \times 62 = 5394$
21.	$92 \times 53 = 4876$	25.	$85 \times 74 = 6290$
22.	$86 \times 57 = 4902$	26.	$95 \times 72 = 6840$

II. *Comment by Stewart Metchette, Culver City, California.*

Of the 26 solutions [as listed in solution I], only No. 4 has an odd number of HITS (one left hook was not followed by a right one); Nos. 2, 6, 10, and 15 involve the consecutive digits 1 to 8; Nos. 21 and 24 involve the consecutive digits 2 to 9; and the only digit used in all solutions is 5.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD I. HESS, Rancho Palos Verdes, California; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California; GLEN E. MILLS, Pensacola Junior College, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

This is the sort of problem that a computer can solve at the flick of a bit, and most solvers availed themselves of this fact (or at least sent in only answers, however obtained). The few who chose to slug it out by brute force for the edification of the editor managed to find few labour-saving finesse. They won the match, but they ended up exhausted on the mat, almost too weak to claim the fruits of victory.

*

*

*

902. [1984: 18] *Proposed by J. Chris Fisher, Université Libre de Bruxelles, Belgique (now at University of Regina, Saskatchewan).*

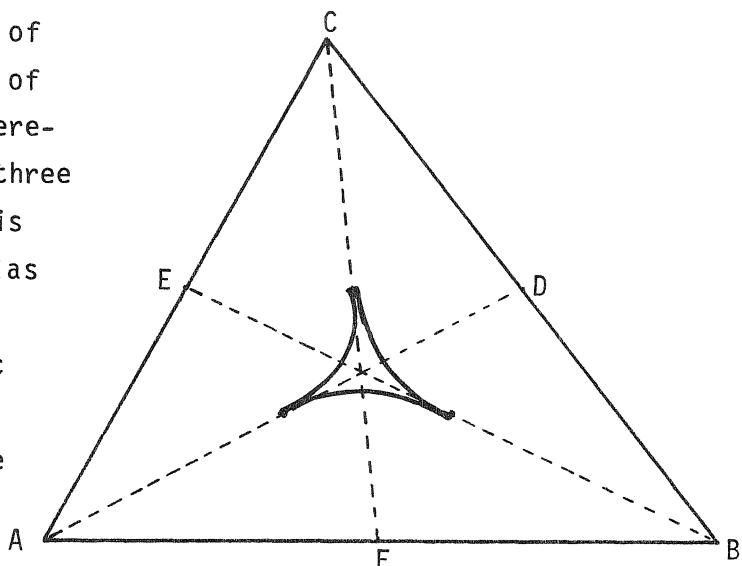
(a) For any point P on a side of a given triangle, define Q to be that point on the triangle for which PQ bisects the area. What is the locus of the midpoint of PQ?

(b) Like the curve in part (a), the locus of the midpoints of the perimeter-bisecting chords of a triangle (see Crux 674 [1982: 256]) has an orientation that is opposite to that of the given triangle. Is this a general principle? More precisely, given a triangle and a family of chords joining P(t) to Q(t), where (i) P(t) and Q(t) move counterclockwise about the triangle as t increases and (ii) $P(t) \neq Q(t)$ for any t, does the midpoint of PQ always trace a curve that is clockwise-oriented?

Solution by Jordi Dou, Barcelona, Spain.

(a) It is well known and easy to show that the variable lines which form with the sides of a fixed angle a triangle of constant area are tangent to a hyperbola whose asymptotes are the sides of the angle. Moreover, for each line the point of

contact is precisely the midpoint M of the segment PQ cut off by the sides of the angle. The desired locus is therefore a curvilinear triangle having three cusps. As shown in the figure, it is formed by arcs of the three hyperbolas having pairs of sides of the given triangle ABC as asymptotes, each arc being tangent to two of the medians AD, BE, and CF of the triangle. The three cusps are the midpoints of these medians.



As P traverses the perimeter of ABC once in the counterclockwise sense, the midpoint M of PQ traverses its locus twice in the clockwise sense, once as P traces the path AFBD (and Q traces the path DCEA), and once more as P and Q complete their circuit of the perimeter of ABC. \square

Let S be the area of the given triangle and σ the area of the region bounded by the curvilinear locus. Since affine transformations preserve area ratios, the ratio σ/S is constant for all triangles. If ABC is an isosceles right triangle with $AB = AC = 1$, it is easy to show by elementary calculus (we leave the details to the reader) that $\sigma = (\ln 8 - 2)/8$. Since $S = 1/2$ in this case, we therefore have

$$\frac{\sigma}{S} = \frac{\ln 8 - 2}{4} \approx \frac{1}{50}$$

for all triangles.

(b) As stated in the proposal, this question is trivial and the answer is no. It suffices to have $P(t)$ sufficiently close to $Q(t)$ for all t to ensure that the midpoint M of PQ moves in the same sense as P and Q. In each of the two cited examples, "PQ bisects the area" and "PQ bisects the perimeter", the correspondence between P and Q is an involution, and perhaps the proposer tacitly assumed this condition. But even with this additional condition the answer is still no. We give three examples which shed some light on this matter.

With the same notation as in the figure, we assume that when P is at A,B,D, then Q is at D,C,A, respectively. Let p and q be the distances, measured along the perimeter in the counterclockwise sense, from A to P and Q, respectively, where

$$0 < p < AB \implies AB + BD < q < AB + BC$$

and

$$AB < p < AB + BD \implies AB + BC < q < AB + BC + CA.$$

Finally, let $q = \phi(p)$, where ϕ is increasing and involutory. As P goes from A to B to D (and Q goes from D to C to A), the midpoint M of PQ traces a closed curve γ that goes from the midpoint M_0 of AD to D and back to M_0 .

Example 1. If dq/dp is strictly increasing for all p , then the curve γ is traced in the same sense as P and Q.

Example 2. If dq/dp is constant for all p , then γ consists of the segment M_0D traced in both senses.

Example 3. If dq/dp is strictly decreasing for all p , then the curve γ is traced in the sense opposite to that of P and Q.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

*

*

*

903. [1984: 19] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let ABC be an acute-angled triangle with circumcenter O and orthocenter H.

- (a) Prove that an ellipse with foci O and H can be inscribed in the triangle.
- (b) Show how to construct, with straightedge and compass, the points L, M, N where this ellipse is tangent to the sides BC, CA, AB, respectively, of the triangle.
- (c) Prove that AL, BM, CN are concurrent.

I. *Solution to part (a) by the proposer.*

More generally, it is known [1]

that a conic is determined when three tangents and a focus F are given, and then the other focus is the isogonal conjugate of F with respect to the triangle formed by the three tangents. So, for every triangle and every point F, there is a conic inscribed in the triangle, having F as one focus. The nature of this conic depends upon the position of F. According to [1], the conic is a parabola if F lies on the circumcircle of the triangle (the isogonal conjugate of F is then a point at infinity), and it is an ellipse (e) or a hyperbola (h) according to the

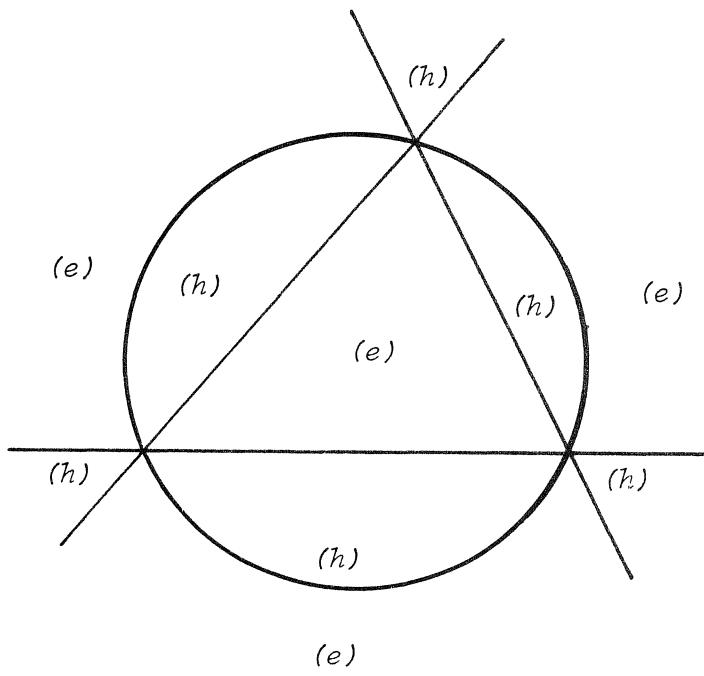


Figure 1

region in the plane in which F lies, as shown in Figure 1.

In our problem, if we take for the focus F the orthocenter H , which lies inside the acute-angled triangle, then the inscribed conic is an ellipse, and its other focus is the circumcenter O , the isogonal conjugate of H .

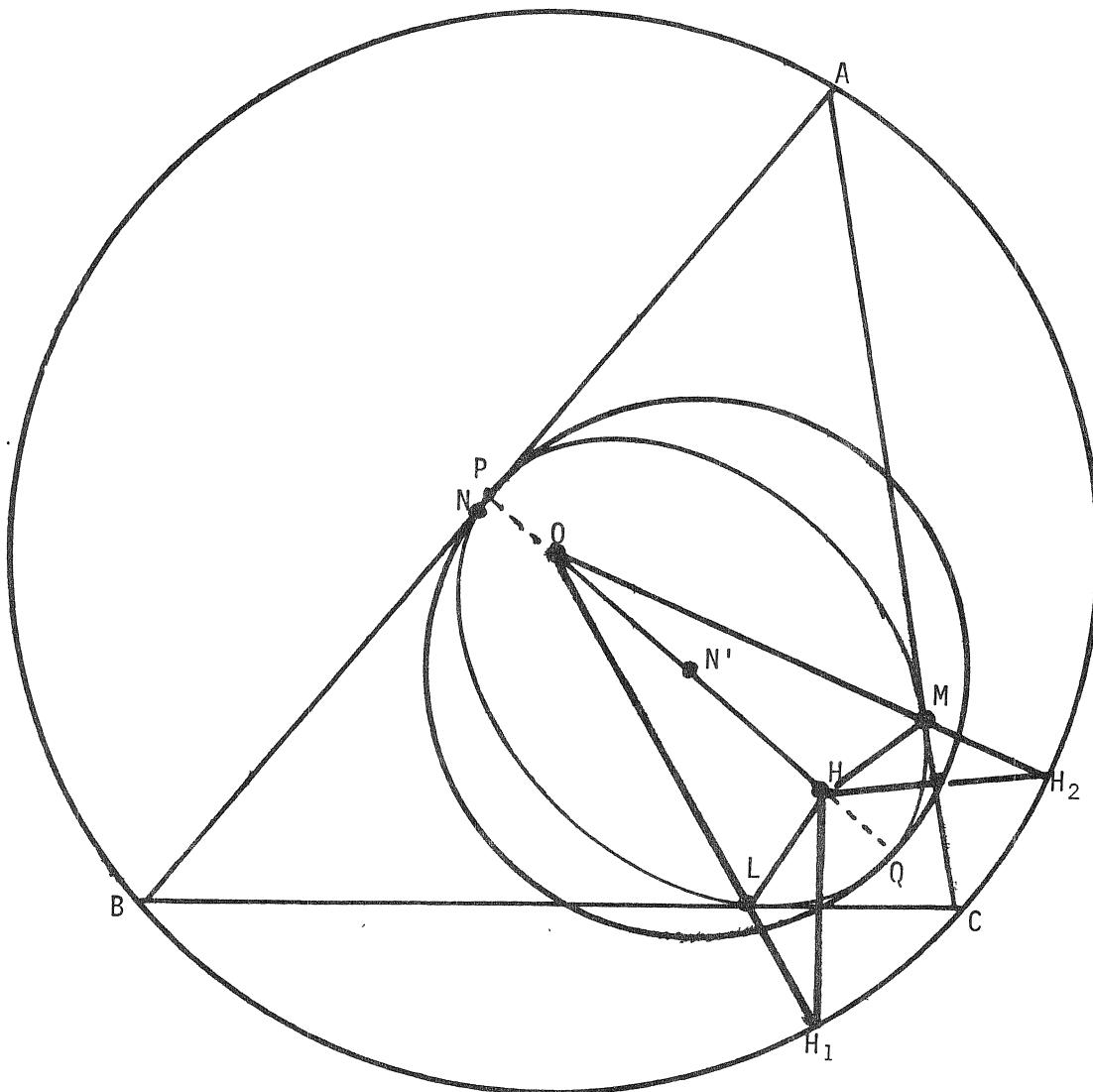


Figure 2

II. Solution to parts (b) and (c) by Leon Bankoff, Los Angeles, California.

(b) One of the standard properties of an ellipse is that any tangent makes equal angles with the focal radii at the point of contact. Accordingly if, as shown in Figure 2, H_1 is the reflection of H in side BC , then $OH_1 \cap BC = L$, the point of contact of the ellipse with side BC . The straightedge and compass construction of L is therefore obvious. Similarly, $OH_2 \cap CA = M$ and $OH_3 \cap AB = N$, where H_2 and H_3 are the reflections of H in CA and AB , respectively.

(c) As noted earlier in this journal [1977: 114], the concurrency of AL,BM,CN is an immediate consequence of Brianchon's Theorem, when ABC is considered as a degenerate hexagon. \square

Figure 2 illustrates the relationship between the circumcircle, the nine-point circle, and the ellipse with foci H and O inscribed in acute triangle ABC.

The nine-point circle turns out to be the auxiliary circle of the inscribed ellipse, whose major axis PQ is therefore equal to the circumradius of the triangle.

The reflections of H in the sides of the triangle lie on the circumcircle, thus suggesting an alternate method for locating the points L,M,N. The nine-point circle, centered at N', the midpoint of OH, with radius half that of the circumcircle, can be drawn, and it passes through the feet of the three altitudes of ABC; these altitudes can then be extended to H_1, H_2, H_3 on the circumcircle and, as before, we have $OH_1 \cap BC = L$, etc.

Also solved by LEON BANKOFF, Los Angeles, California (also part (a)); JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; DAN PEDOE, University of Minnesota; BASIL C. RENNIE, James Cook University of North Queensland, Australia; and the proposer (also parts (b) and (c)).

REFERENCE

1. Luigi Cremona, *Elements of Projective Geometry*, Dover, New York, 1960, pp. 260-261.

*

*

*

904. [1984: 19] Proposed by George Tsintsifas, Thessaloniki, Greece.

Let M be any point in the plane of a given triangle ABC. The cevians AM,BM,CM intersect the lines BC,CA,AB in A',B',C' , respectively. Find the locus of the points M such that

$$[MCB'] + [MAC'] + [MBA'] = [MC'B] + [MA'C] + [MB'A], \quad (1)$$

where the square brackets denote the signed area of a triangle.

Solution by W.J. Blundon, Memorial University of Newfoundland.

In areal coordinates, the coordinates of the vertices of ABC as triangle of reference, and of an arbitrary point M in the plane, are

$$A(1,0,0), \quad B(0,1,0), \quad C(0,0,1), \quad \text{and} \quad M(x,y,z) \quad \text{where} \quad x+y+z = 1.$$

Thus the coordinates of A' are $(0,k,1-k)$ for some k , and the collinearity of A,M, A' gives

$$\begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ 0 & k & 1-k \end{vmatrix} = 0, \quad \text{whence} \quad k = \frac{y}{y+z}.$$

With similar results for B' and C' , we have

$$A' = \frac{1}{y+z}(0,y,z), \quad B' = \frac{1}{z+x}(x,0,z), \quad C' = \frac{1}{x+y}(x,y,0).$$

We now assume without loss of generality that $[ABC] = 1$. Then (see the article by O. Bottema in this journal [1982: 228-231])

$$[MCB'] = \frac{1}{z+x} \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ x & 0 & z \end{vmatrix} = \frac{xy}{z+x}, \quad [MC'B] = \frac{1}{x+y} \begin{vmatrix} x & y & z \\ x & y & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{zx}{x+y}.$$

With these and similar results, equation (1) in the proposal becomes

$$\frac{xy}{z+x} + \frac{yz}{x+y} + \frac{zx}{y+z} = \frac{zx}{x+y} + \frac{xy}{y+z} + \frac{yz}{z+x},$$

an equation equivalent to $(y-z)(z-x)(x-y)(x+y+z) = 0$, or, since $x+y+z = 1$, to

$$(y - z)(z - x)(x - y) = 0.$$

Thus $M(x,y,z)$ lies on the locus if and only if at least two of x,y,z are equal. The required locus is therefore

$$l \cap m \cap n,$$

where l,m,n are the lines containing the medians through A,B,C , respectively.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, University of Minnesota; and the proposer.

*

*

*

905. [1984: 19] Proposed by J.T. Groenman, Arnhem, The Netherlands.

Let ABC be a triangle that is not right-angled at B or C . Let D be the foot of the perpendicular from A upon BC , and let M and N be the feet of the perpendiculars from D upon AB and AC , respectively.

(a) Prove that, if $\angle A = 90^\circ$, then $\angle BMC = \angle BNC$. (This problem is given without proof in M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, New York, 1968, p. 95, Ex. 8.)

(b) Prove or disprove the converse of part (a).

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

We show that $\angle BMC = \angle BNC$ whether or not $\angle A = 90^\circ$, and this implies that part (a) is true and its converse is false.

The proof is simple. From

$$AD^2 = AM \cdot AB = AN \cdot AC,$$

it follows that quadrilateral $BMNC$ is cyclic, and hence that $\angle BMC = \angle BNC$.

Also solved by JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

*

*

*

906. ^{*} [1984: 19] Proposed by E.J. Eckert and P.D. Vestergaard, Institute of Electronic Systems, Aalborg University Centre, Denmark.

Let P denote the set of rational points on the unit circle C , that is, the set of all points $(r/t, s/t)$ where r, s, t are integers, $t > 0$, and $r^2+s^2 = t^2$. It is known that P is dense in C . Let T be the subset of P for which t is prime. Is T dense in C ?

Editor's comment.

No solution was received for this problem, which therefore remains open. For a proof that P is dense in C , see Problem 109 [1976: 81].

*

*

*

907. [1984: 19] Proposed by Kenneth S. Williams, Carleton University, Ottawa.

The four consecutive positive integers 76, 77, 78, 79 are such that no one of them is expressible as the sum of two squares. Prove that there are infinitely many such quadruples of consecutive integers.

Solution by Leroy F. Meyers, The Ohio State University.

More generally, for each positive integer k it is possible to find infinitely many sets of k consecutive positive integers none of which is a sum of two squares of integers. It is known [1] that a positive integer n is not the sum of two squares of integers just when there is a prime $p \equiv 3 \pmod{4}$ which appears with an odd exponent in the canonical factorization of n into a product of powers of distinct primes.

Let q_1, q_2, \dots, q_k be k distinct primes, with each $q_j \equiv 3 \pmod{4}$. (It is always possible to find k such primes since there are infinitely many of them, by Dirichlet's Theorem.) By the Chinese Remainder Theorem, find a positive integer x such that, for all $j = 1, 2, \dots, k$,

$$x \equiv q_j - j \pmod{q_j^2}.$$

Then $x+j$ is divisible by q_j but not by q_j^2 (since $q_j > j$ if the primes are listed in increasing order), and so $x+j$ is not the sum of two squares of integers. Hence each of the k positive integers

$$x + 1, x + 2, \dots, x + k$$

is not expressible as the sum of two squares of integers, and all positive integers congruent to these modulo $(q_1 q_2 \dots q_k)^2$ are likewise not so expressible.

For $k = 1, 2, \dots, 7$, the smallest k -tuples of consecutive positive integers that are not sums of two squares of integers are

		3					
	6	7					
21	22	23					
21	22	23	24				
75	76	77	78	79			
91	92	93	94	95	96		
186	187	188	189	190	191	192	

Also solved by CLAYTON W. DODGE, University of Maine at Orono; F. DAVID HAMMER, Palo Alto, California; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; EDWIN M. KLEIN, University of Wisconsin-Whitewater; BOB PRIELIPP, University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer. A comment was received from J.T. GROENMAN, Arnhem, The Netherlands.

REFERENCE

1. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers* (Third Edition), John Wiley & Sons, New York, 1972, p. 117.

*

*

*

908. [1984: 19] Proposed by M.S. Klamkin, University of Alberta.

Determine the maximum value of

$$P \equiv \sin^\alpha A \cdot \sin^\beta B \cdot \sin^\gamma C,$$

where A, B, C are the angles of a triangle and α, β, γ are given positive numbers.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The function $P(A, B, C)$ is continuous and nonnegative over the compact set

$$A + B + C = \pi, \quad 0 \leq A, B, C \leq \pi, \tag{1}$$

and vanishes just on its boundary. Consequently, P attains a maximum value at some interior point of the region (1), where $0 < A, B, C < \pi$. We use the method of Lagrange multipliers to find this maximum value.

Let

$$F(A, B, C, \lambda) \equiv P(A, B, C) - \lambda(A+B+C-\pi).$$

Then, since $\partial F / \partial \lambda = 0$, the maximum value of F will occur when

$$\frac{\partial F}{\partial A} = \frac{P\alpha \cos A}{\sin A} - \lambda = 0, \quad \frac{\partial F}{\partial B} = \frac{P\beta \cos B}{\sin B} - \lambda = 0, \quad \frac{\partial F}{\partial C} = \frac{P\gamma \cos C}{\sin C} - \lambda = 0. \tag{2}$$

We show that $\cos A \cos B \cos C \neq 0$. If $\cos A = 0$, for example, then $\lambda = 0$, so also $\cos B = \cos C = 0$ and $A+B+C = 3\pi/2$, a contradiction. Now, from (2), $\lambda \neq 0$ and

$$\left(\frac{P}{\lambda}\right) = \frac{\tan A}{\alpha} = \frac{\tan B}{\beta} = \frac{\tan C}{\gamma} = \frac{\tan A + \tan B + \tan C}{\alpha + \beta + \gamma} = k, \text{ say.}$$

Since $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, we therefore have

$$k(\alpha + \beta + \gamma) = k^3 \alpha \beta \gamma,$$

so $k^2 = (\alpha + \beta + \gamma)/\alpha \beta \gamma$ and

$$\tan^2 A = \frac{\alpha(\alpha+\beta+\gamma)}{\beta\gamma}, \quad \tan^2 B = \frac{\beta(\alpha+\beta+\gamma)}{\gamma\alpha}, \quad \tan^2 C = \frac{\gamma(\alpha+\beta+\gamma)}{\alpha\beta}. \quad (3)$$

Finally, from $\sin A = \sqrt{\tan^2 A / (1 + \tan^2 A)}$, etc., we obtain

$$\sin A = \sqrt{\frac{\alpha(\alpha+\beta+\gamma)}{(\alpha+\beta)(\alpha+\gamma)}}, \quad \sin B = \sqrt{\frac{\beta(\alpha+\beta+\gamma)}{(\beta+\gamma)(\beta+\alpha)}}, \quad \sin C = \sqrt{\frac{\gamma(\alpha+\beta+\gamma)}{(\gamma+\alpha)(\gamma+\beta)}},$$

and so

$$P_{\max} = \left\{ \frac{\alpha(\alpha+\beta+\gamma)}{(\alpha+\beta)(\alpha+\gamma)} \right\}^{\alpha/2} \cdot \left\{ \frac{\beta(\alpha+\beta+\gamma)}{(\beta+\gamma)(\beta+\alpha)} \right\}^{\beta/2} \cdot \left\{ \frac{\gamma(\alpha+\beta+\gamma)}{(\gamma+\alpha)(\gamma+\beta)} \right\}^{\gamma/2}.$$

In particular, for $\alpha = \beta = \gamma = 1$ we get $P_{\max} = 3\sqrt{3}/8$. This is item 2.7 in the "Bottema Bible" [1].

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; N.Y. WONG, Hong Kong; and the proposer.

Editor's comment.

The proposer noted that a closely related problem appears without solution in Hardy [2], where it is credited to Math. Tripos. 1935. This problem asked only to show that (3) holds when P is a maximum.

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, p. 19.
2. G.H. Hardy, *A Course of Pure Mathematics*, Cambridge University Press, 1958, p. 311.

*

*

*

909.* [1984: 20] Proposed by Stan Wagon, Smith College, Northampton, Massachusetts.

For which positive integers n is it true that, whenever an integer's decimal expansion contains only zeros and ones, with exactly n ones, then the integer is not a perfect square?

Comment by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.

All positive integers are congruent to the sum of their digits modulo 9, and all perfect squares are congruent to 0, 1, 4, or 7 modulo 9. So an integer con-

taining only zeros and n ones cannot be a perfect square if $n \not\equiv 0, 1, 4, \text{ or } 7 \pmod{9}$. This eliminates many from consideration, and strong computer evidence suggests that nearly all the remaining ones can be eliminated as well. In fact, we conjecture that the only perfect squares containing only zeros and ones are those of the form 100^k , where k is a nonnegative integer, for which $n = 1$.

It is easy to find all r -digit numbers whose squares end with r zeros or ones. For example, for $r = 1$ only 0, 1, and 9 have this property. To check all 2-digit candidates, we need only check those 2-digit numbers that end in 0, 1, or 9. In each case, there are only 10 numbers to check. Proceeding in a similar manner, once we have found all k -digit numbers whose squares end with k zeros or ones, we can quickly check for $(k+1)$ -digit numbers by tacking on each of the digits from 0 to 9 to the left of the k -digit numbers, and testing the squares of the results.

Using this algorithm, we have tested all numbers up to 10^{36} . Other than the trivial ones of the form 100^k , we did not find a single number of up to 36 digits that was a perfect square and consisted entirely of zeros and ones. As we keep adding digits on the left, it becomes even more unlikely that such a perfect square will be found.

Some near misses:

Comments were received from FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and the proposer.

910. [1984: 20] *Proposed by O. Bottema, Delft, The Netherlands.*

Determine the locus of the centers of the conics through the incenter and the three excenters of a given triangle.

I. *Solution by Dan Pedoe, University of Minnesota.*

Solutions are provided by Problem 14 in Durell [1] and Problems 80.8 and 80.9 in Pedoe [2].

The incenter is the orthocenter of the triangle formed by the three excenters, so that all conics through the four points are rectangular hyperbolas, and the locus of centers for these conics is the nine-point circle of the triangle formed by any three of the four points, and this is the circumcircle of the original triangle.

II. *Comment by Roland H. Eddy, Memorial University of Newfoundland.*

More generally, Smith [3] uses trilinear coordinates to find the locus of the centers of the conics through four arbitrary points. When the four points are the incenter and the three excenters, the locus turns out to be the circumcircle of the given triangle.

The actual proposal is given in Sommerville [4].

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; R.C. LYNESS, Southwold, Suffolk, England; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Editor's comment.

A related problem stated and proved in Milne [5]: *The locus of the foci of all parabolas touching the three sides of a triangle is the circumcircle of the triangle.*

REFERENCES

1. C.V. Durell, *Projective Geometry*, Macmillan, London, 1945, p. 203.
2. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, New York, 1970, p. 355.
3. Charles Smith, *An Elementary Treatise on Conic Sections*, Macmillan, London, 1927, p. 366.
4. D.M.Y. Sommerville, *Analytical Conics*, G. Bell and Sons, London, 1961, p. 180.
5. W.P. Milne, *Homogeneous Coordinates*, Edward Arnold & Co., London, 1924, p. 118.

*

*

*

911. [1984: 53] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the following synonymous alphametic in the smallest possible base:

HARD
SHARP.
HARSH

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

From the column sums we have

$$\begin{aligned} D + P &= H + bi, \\ 2R &= S + bj - i, \\ 2A &= R + bk - j, \\ 2H &= A + bm - k, \\ S &= H - m, \end{aligned}$$

where $b \geq 6$ is the base and i, j, k, m are the carries. It is immediately obvious that $m = 1$, so $H = S+1$. Then it is easily shown that

$$7S = (b-2)(j+2k+4) - i.$$

If $i = 0$, then either $b \equiv 9 \pmod{7}$ or $j = k = 1$. If $j = k = 1$, then $R = A = H = b-1$, and there is therefore no solution. If $b = 9$, then

$$S = j + 2k + 4, \quad R = 5j + k + 2, \quad A = 2j + 5k + 1.$$

The only solution is with $j = 1, k = 0$ which, with $H = S+1$, gives

$$S = 5, \quad H = 6, \quad R = 7, \quad A = 3, \quad \{D, P\} = \{2, 4\}.$$

If $i = 1$, then $b \neq 9$, nor can $j = k = 1$. There are no solutions for $b < 11$.

Therefore the smallest possible base is 9, and the only solutions in that base are

$$\begin{array}{ccc} 6372 & & 6374 \\ \underline{56374} & \text{and} & \underline{56372} \\ 63756 & & 63756 \end{array}$$

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay; CLAYTON W. DODGE, University of Maine at Orono; RICHARD I. HESS, Rancho Palos Verdes, California; ANDY LIU, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

*

*

*

915. [1984: 54] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If $x+y+z+w = 180^\circ$, prove or disprove that

$$\sin(x+y) + \sin(y+z) + \sin(z+w) + \sin(w+x) \geq \sin 2x + \sin 2y + \sin 2z + \sin 2w,$$

with equality just when $x = y = z = w$.

Solution by M.S. Klamkin, University of Alberta.

The proposal as stated is easily disproved. For $x = 270^\circ$, $y = -90^\circ$, and $z = w = 0^\circ$, the left side of the inequality is negative and the right side zero. However, the inequality may hold if some of x, y, z, w are negative, even if $x+y+z+w \neq 180^\circ$,

and, as we will see from the generalization proved below, it always holds whenever each of the four terms on the left is nonnegative, that is, whenever

$$2k_1\pi \leq x+y \leq (2k_1+1)\pi, \quad 2k_2\pi \leq y+z \leq (2k_2+1)\pi,$$

$$2k_3\pi \leq z+w \leq (2k_3+1)\pi, \quad 2k_4\pi \leq w+x \leq (2k_4+1)\pi,$$

where k_1, k_2, k_3, k_4 are integers. In particular, it always holds if $x, y, z, w \geq 0$ and $x+y+z+w = 180^\circ$, with equality when $x = y = z = w$.

Let the angles x_i , $i = 1, 2, \dots, n$ (with $x_{n+1} = x_1$) be such that

$$2k_i\pi \leq x_i + x_{i+1} \leq (2k_i+1)\pi,$$

where the k_i are integers. Then

$$\sin(x_1+x_2) + \sin(x_2+x_3) + \dots + \sin(x_n+x_1) \geq \sin 2x_1 + \sin 2x_2 + \dots + \sin 2x_n. \quad (1)$$

The proof is simple:

$$\begin{aligned} 2 \sum_{i=1}^n \sin 2x_i &= \sum_{\text{cyclic}} (\sin 2x_1 + \sin 2x_2) = 2 \sum_{\text{cyclic}} \sin(x_1+x_2) \cos(x_1-x_2) \\ &\leq 2 \sum_{\text{cyclic}} \sin(x_1+x_2), \end{aligned}$$

and (1) follows, with equality when each $x_i - x_{i+1}$ is an integral multiple of 2π . \square

If all $x_i > 0$ and $x_1+x_2+\dots+x_n = 180^\circ$, there is the following geometric interpretation of (1). Let $P \equiv A_1 A_2 \dots A_n$ be a positively oriented convex polygon inscribed in a circle with center 0, and let $\angle A_i O A_{i+1} = 2x_i$. Also, let P' be the associated convex polygon whose vertices A'_i are the midpoints of the arcs $A_i A_{i+1}$. Then (1) is equivalent to the statement

$$\text{area of } P' \geq \text{area of } P.$$

This inequality appears in the author's paper [1], and no doubt has also appeared much earlier. It also follows immediately that

$$\text{perimeter of } P' \geq \text{perimeter of } P.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and ESTHER SZEKERES, Turramurra, New South Wales, Australia. Comments (pointing out that the problem as stated was incorrect) were received from RICHARD I. HESS, Rancho Palos Verdes, California; FLORENTIN SMARANDACHE, Lycée Sidi El Hassan Lyoussi, Sefrou, Maroc; and JORDAN B. TABOV, Sofia, Bulgaria.

REFERENCE

1. M.S. Klamkin, "On Some Geometric Inequalities", *Mathematics Teacher*, 60 (1967) 323-328.

*

*

*

916. [1984: 54] Proposed by J.T. Groenman, Arnhem, The Netherlands.

Find $\lim_{n \rightarrow \infty} S_n$ if

$$S_n = \sum_{k=1}^n \frac{1}{2^k} \tan \frac{\pi}{2^{k+1}}.$$

Solution by Frank P. Battles, Massachusetts Maritime Academy, Buzzard's Bay; and Vedula N. Murty, Pennsylvania State University, Capitol Campus (independently).

It is stated and proved in Hobson [1] that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \tan \frac{\theta}{2^k} = \frac{1}{\theta} - \cot \theta.$$

Setting $\theta = \pi/2$, we then obtain

$$\lim_{n \rightarrow \infty} S_n = \frac{2}{\pi}.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CURTIS COOPER, Central Missouri State University at Warrensburg; RICHARD I. HESS, Rancho Palos Verdes, California (partial solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; RICHARD PARRIS, Phillips Exeter Academy, New Hampshire; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; and the proposer.

REFERENCE

1. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 129.

*

*

*

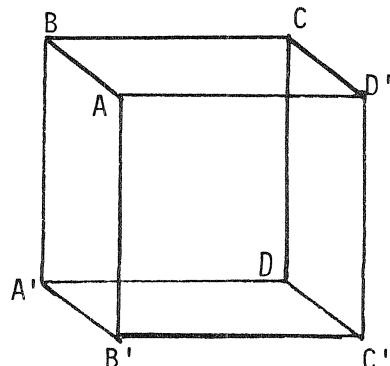
917. [1984: 54] Proposed by Rick Moorhouse, University of Toronto.

How can the eight vertices of a cube be divided into two sets of four forming two directly congruent tetrahedra such that the four vertices of each tetrahedron lie in the planes of the four faces of the other?

Solution by the proposer.

Let the eight vertices be named so that three of the faces are ABCD', CD'C'D, C'DA'B' (and consequently ABCDA'B'C'D' is a Hamiltonian path), as shown in the figure. Then the two congruent "orthoschemes" ABCD and A'B'C'D' are the required "Möbius tetrahedra".

Also solved by CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; ANDY LIU, University of Alberta; RICHARD PARRIS, Phillips Exeter Academy, New Hampshire; and STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire.



Editor's comment.

An *orthoscheme* is the name given by Schläfli to a tetrahedron whose four faces are all right triangles [1, p. 156]. "Möbius tetrahedra" refers to the following theorem of Möbius [1, p. 258]: *If the four vertices of one tetrahedron lie respectively in the four face planes of another, while three vertices of the second lie in three face planes of the first, then the remaining vertex of the second lies in the remaining face plane of the first.*

REFERENCE

1. H.S.M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1969.

*

*

*

CRUX MATHEMATICORUM
A Call for Editors

The mathematical community has been saddened to learn of the recent death of Fred Maskell, who served as Managing Editor of *Crux Mathematicorum* for many years. The Canadian Mathematical Society extends sympathy to Mr. Maskell's family and many friends while acknowledging with sincere thanks the tremendous contribution both he and Editor Léo Sauvé have made to mathematics through their service with *Crux*.

Crux Mathematicorum seems now to have reached a crossroads in its existence and the C.M.S. has been asked to assume responsibility for its publication. This is a unique problem-solving journal with an international reputation devoted to mathematics at all levels and it would well complement the existing research publications of the C.M.S. The level of financing required is considerable, however. In addition, the Society is reluctant to begin a new volume year without the assurance that it will have editors for years to come. Editors of the *Canadian Journal of Mathematics* and the *Canadian Mathematical Bulletin* now serve five-year terms.

Editing a publication such as *Crux Mathematicorum* requires special and diverse talents. An editor must have the inclination and ability to compile articles, problems and letters into a monthly issue of interest to a wide audience of mathematics educators. Besides the amalgamation of submitted solutions to the same problem, editorial comments and historical information are often necessary. In the future operation of the journal, it is envisaged that word processing facilities at the editor's institution would be virtually essential, thereby enabling the monthly dispatching of camera-ready copy to a managing editor's office in Ottawa.

Suggestions concerning the future of *Crux Mathematicorum* and/or applications for the editorship (commencing January 1986) should be sent as soon as possible to:

Professor Marcel Deruaz, Chairman,
CMS Ad Hoc Committee on *Crux Mathematicorum*,
Department of Mathematics,
University of Ottawa,
Ottawa, Ontario,
K1N 9B4

*

*

*