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CONTENTS

SHARPENING THE NEUBERG-PEDOE INEQUALITY: I

CHIA-KUEI PENG

Let a,b,c and a',b',c' be the sides of two triangles whose areas are Δ and Δ', respectively, and let

$$H = (a^2+b^2+c^2)(a'^2+b'^2+c'^2) - 2(a^2a'^2+b^2b'^2+c^2c'^2) \tag{1}$$

$$= a^2(b'^2+c'^2-a'^2) + b^2(c'^2+a'^2-b'^2) + c^2(a'^2+b'^2-c'^2).$$

The Neuberg-Pedoe inequality states that

$$H \geq 16\Delta\Delta',$$

with equality if and only if the two triangles are similar. This inequality, one of the most beautiful inequalities connecting two triangles, has drawn the interest of many mathematicians, who have proved it by different methods and generalised or extended it in various ways (see [1-8]).

Since $\Delta^2 + \Delta'^2 \geq 2\Delta\Delta'$, it is natural to ask whether the Neuberg-Pedoe inequality can be sharpened to

$$H \geq 8(\Delta^2 + \Delta'^2) \geq 16\Delta\Delta'. \tag{2}$$

Carlitz [9] showed that the left inequality in (2) holds for some but not all pairs of triangles, and he characterised those triangle pairs for which it does not hold. We present here a double inequality similar to (2) which *does* hold for all pairs of triangles, and thus constitutes a sharpening of the Neuberg-Pedoe inequality. We show that

$$H \geq 8\left(\frac{S'}{S}\Delta^2 + \frac{S}{S'}\Delta'^2\right) \geq 16\Delta\Delta', \tag{3}$$

where $S = a^2 + b^2 + c^2$ and $S' = a'^2 + b'^2 + c'^2$.

The right inequality in (3) is a nearly immediate consequence of $(S'\Delta - S\Delta')^2 \geq 0$. To prove the left inequality, we introduce the new variables

$$x = \frac{a^2}{S},\ y = \frac{b^2}{S},\ z = \frac{c^2}{S} \quad \text{and} \quad x' = \frac{a'^2}{S'},\ y' = \frac{b'^2}{S'},\ z' = \frac{c'^2}{S'}.$$

It then follows from (1) that

$$H = SS'\{1 - 2(xx'+yy'+zz')\}.$$

On the other hand, by Heron's formula for the area of a triangle, we have

$$16\Delta^2 = S^2 - 2(a^4+b^4+c^4) = S^2\{1 - 2(x^2+y^2+z^2)\},$$

and similarly

$$16\Delta'^2 = S'^2\{1 - 2(x'^2+y'^2+z'^2)\}.$$

Using the well-known inequality

$$x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 \geq 2(xx' + yy' + zz'),$$

we finally arrive at

$$\frac{H}{SS'} \geq 8\left(\frac{\Delta^2}{S^2} + \frac{\Delta'^2}{S'^2}\right),$$

from which the desired inequality follows. Equality holds if and only if $x = x'$, $y = y'$, and $z = z'$, or, equivalently, if and only if the two triangles are similar.

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* * *

ABOUT THAT MERSENNE NUMBER...

... TIME Magazine (March 5, 1984) has sheepishly admitted that the Mersenne number $M_{251} = 2^{251} - 1$ has 76 digits and that its prime factors are $503 \times 54217 \times$ the three factors of the 69-digit integer given earlier [1984: 66].

* * *

SHARPENING THE NEUBERG-PEDOE INEQUALITY: II

GAO LING

Let ABCD and A'B'C'D' both be convex quadrilaterals inscribed in circles; let AB = a, BC = b, CD = c, and DA = d, with a similar notation for A'B'C'D'; and let F and F' denote the areas of ABCD and A'B'C'D', respectively. With

$$K = 4(ab+cd)(a'b'+c'd') - (a^2+b^2-c^2-d^2)(a'^2+b'^2-c'^2-d'^2), \tag{1}$$

we will show that

$$K \geq 16FF', \tag{2}$$

with equality if and only if corresponding angles B and B' are equal.

Since angles B and D are supplementary, we have

$$2F = (ab+cd)\sin B. \tag{3}$$

On the other hand, from the cosine law,

$$AC^2 = a^2 + b^2 - 2ab\cos B = c^2 + d^2 + 2cd\cos B,$$

and hence

$$a^2 + b^2 - c^2 - d^2 = 2(ab+cd)\cos B. \tag{4}$$

Using (3) and (4) and similar equalities for the quadrilateral A'B'C'D', we obtain

$$K - 16FF' = 4(ab+cd)(a'b'+c'd')\{1 - \cos(B-B')\},$$

and (2) follows immediately since $\cos(B-B') \leq 1$, with equality if and only if $\angle B = \angle B'$.

Inequality (2) is interesting in its own right, but we observe that it is not symmetric in its two sets of variables. An inequality analogous to (2) which is perfectly symmetric can be obtained by adding to (1) the three expressions resulting from it by successive cyclic permutations of a,b,c,d and a',b',c',d'. In this symmetric inequality (we leave it to the reader to write it out in full), equality occurs if and only if the two cyclic quadrilaterals have all corresponding angles equal (even if, as for a square and a rectangle, corresponding sides are not proportional). We do not discuss this further because inequality (2) is all we need to accomplish the main purpose of this paper.

As announced in the title, our aim is to effect a sharpening of the Neuberg-Pedoe two-triangle inequality:

$$H \geq 16\Delta\Delta',$$

where

$$F = a^2(b'^2+c'^2-a'^2) + b^2(c'^2+a'^2-b'^2) + c^2(a'^2+b'^2-c'^2),$$

Δ and Δ' being the areas of the triangles with sides a,b,c and a',b',c', respectively.

If we set $d = d' = 0$ in (1), then (2) becomes

$$4aba'b' - (a^2+b^2-c^2)(a'^2+b'^2-c'^2) \geq 16\Delta\Delta',$$

and this is easily shown to be equivalent to

$$H \geq 16\Delta\Delta' + 2(ab'-ba')^2, \tag{5}$$

with equality if and only if $\angle B = \angle B'$. If we add to (5) the results obtained from it by two successive cyclic permutations of a,b,c and a',b',c', which leave the left side unchanged, the resulting inequality is equivalent to the sought

$$H \geq 16\Delta\Delta' + \frac{2}{3}\{(bc'-cb')^2 + (ca'-ac')^2 + (ab'-ba')^2\} \geq 16\Delta\Delta',$$

with equality holding throughout if and only if the two triangles are similar.

This symmetric sharpening of the Neuberg-Pedoe two-triangle inequality was first obtained by Chen Long, a teacher at the 9th High School in Hefei.

23rd High School, Chongqing, People's Republic of China.

* * *

OF POESY AND NUMBERS

EDITH ORR

The rigid discipline of metrical structure makes poets highly sensitive to numbers. Some have even *identified* poetry with numbers (remember Longfellow's "Tell me not, in mournful numbers/ Life is but an empty dream!"?).

In an early issue of this journal, Problem 151 [1976: 195] contained a poem by the highly regarded poet Sylvia Plath (1932-1963) about a woman in the last stages of her 9-month pregnancy. No reader was perceptive enough to detect the mathematical relevance of the poem: its title was one 9-letter word, and it contained exactly 9 lines each of which had exactly 9 syllables.

Poets are still playing the numbers game. The new book *Powers of Thirteen*, by John Hollander (103 pages, Atheneum, US$13.95, paper US$6.95) contains a sequence of 169 (= 13^2) poems. Each poem is a sonnet of 13 lines (instead of the usual 14), and each line contains exactly 13 syllables. Unsurprisingly, 13 of the poems explore the number 13 and everything that it calls to mind: triskaidekaphobia (fear of the number 13), the 13-tone scale, the 13 stripes on Old Glory, 13 cards in a suit, 13 weeks in a season, etc.

Resulting in inferior poetry? Not according to reviewer David Lehman in *Newsweek* (January 23, 1984): "No one has ever doubted John Hollander's poetic virtuosity... [This book is] an audaciously original sonnet sequence that in effect reinvents the rules of the game... Here Hollander has fashioned a form remarkable for its rigor—or, rather, its supple responsiveness to his touch... These constraints liberate rather than inhibit Hollander's imagination... Above all else, *Powers of Thirteen* celebrates linguistic possibility."

John Hollander's talent has many facets. Just a few weeks ago, for example, appeared a new edition of *Jiggery-Pokery* (Atheneum, paper, US$6.95) edited and partly written by John Hollander and Anthony Hecht (first edition 1967). This is a compendium of Double-Dactyls, a verse form invented by Paul Pascal and the same Anthony Hecht. You don't know what is a Double-Dactyl? Here is a home-grown example (which it is our fancy to presume was written by Hollander himself while resting up from his labours on *Powers of Thirteen*, on the not unreasonable assumption that even poets can work up a powerful thirst by crunching numbers):

HOW DRY I AM

Higgledy-piggledy
Ego, John Hollander,
Sorely afflicted by
Post coitum triste,

Sought solace in numbers.
Arithmeticophile
I was, until I found
Numbers made me thirsty.

John Hollander has long been a poet of established reputation. His first volume of poetry, *A Crackling of Thorns*, won the Yale Series of Younger Poets Award for 1958. He was a recipient of a National Institute of Arts and Letters grant (1963) and has been a member of the Wesleyan University Press Poetry Board and the Bollingen Poetry Translation Prize Board. He is (or was) Professor of English at Hunter College and General Editor of the Signet Classic Poetry Series.

Mathematics is for the bards.

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* * *

THE PUZZLE CORNER

Answer to Puzzle No. 51 [1984: 52]: *Much Ado About Nothing*.

Answer to Puzzle No. 52 [1984: 52]: Circle.

* * *

THE OLYMPIAD CORNER: 53

M.S. KLAMKIN

In a previous column [1983: 205], I wrote that I would try to obtain a description of the training and selection program for the West German team in the International Mathematical Olympiad (I.M.O.). The team was coached by Arthur Engel and achieved very high standings in the last three competitions (2nd, 1st, 1st). I am grateful to Bernhard Leeb, a member of the team at the last I.M.O., for the following brief description:

WEST GERMAN NATIONAL MATHEMATICAL COMPETITION

Round 1. Four problems to be done at home from December 15 to March 15. First, second, and third prizes are awarded. Students with first and second prizes continue on to Round 2.

Round 2. Four problems to be done at home from June 15 to September 15. First, second, and third prizes are awarded. Students with first prizes continue on to Round 3. [For an example of Round 2, see [1982: 300].]

Round 3. Oral examination during first week of January. Students give one-hour talks to display mathematical talents rather than knowledge.

I.M.O. TEAM SELECTION AND TRAINING

Round 1. One test in late December and one in early January for top students from Round 2 (of the National Competition). Each test consists of two three-hour sessions with three problems each.

Round 2. The top ten to fifteen students from Round 1 (immediately above) are invited to four weekend sessions between January and May. Each session consists of three half-days of lectures and one test.

Round 3. One week of training in early June, with lectures, problem-solving sessions, and two selection tests.

Incidentally, Bernhard Leeb, who supplied the above information, wrote a perfect paper in the 1983 I.M.O. and also won a special prize for his solution to Problem 6 in that competition. This problem and his solution are given below.

6. [1983: 207] Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$b^2c(b-c)+c^2a(c-a)+a^2b(a-b)\ge 0. \qquad (1)$$

Determine when equality occurs.

Solution.

Let I denote the left member of (1). Since a cyclic permutation of (a,b,c) leaves I unchanged, there is no loss of generality in assuming that $a\ge b,c$ or $a\le b,c$. But then

$$I\equiv a(b-c)^2(b+c-a)+b(a-b)(a-c)(a+b-c)\ge 0.$$

*

In an earlier column [1982: 99], I had expressed my firm conviction that *Problem Proposing* is an important aspect of *Problem Solving*, and so I encourage students to come up with their own problems. Here are some problems proposed by students at the 1983 U.S.A. Mathematical Olympiad Practice Session held at the U.S. Military Academy at West Point. As usual, I solicit from all readers (especially secondary school students) elegant solutions and/or nontrivial extensions for possible later publication in this column.

1. *Proposed by John Steinke.*

Given six segments $S_1, S_2, \ldots, S_6$ congruent to the edges AB, AC, AD, CD, DB, BC, respectively, of a tetrahedron ABCD, show how to construct with straightedge and compass a segment whose length equals that of the *bialtitude* of the tetrahedron relative to opposite edges AB and CD (i.e., the distance between the lines AB and CD).

2. *Proposed by John Steinke.*

Given the foci F_1, F_2, and the major axis of an ellipse, show how to construct with straightedge and compass the intersection of the ellipse with a given straight line l.

3. *Proposed by Douglas Jungreis.*

Determine the maximum area of the convex hull of four circles C_i, $i = 1,2,3,4$, each of unit radius, which are placed so that C_i is tangent to C_{i+1} for $i = 1,2,3$.

4. *Proposed by Douglas Jungreis.*

If $a_0 = 1$ and

$$2a_{n+1} = \sqrt{2 + a_n - \sqrt{3-a_n^2}}, \qquad n = 0,1,2,\ldots,$$

determine a_n explicitly in terms of n.

*

I have in the past (e.g., [1984: 16]) mentioned that two excellent methematical journals at the secondary school level, the Hungarian journal *Középiskolai Matematikai Lapok* (Mathematical Journal for Secondary Schools) and the Russian journal *Kvant* (= Quantum), regularly include English translations of their proposed problems (but not of solutions). I give below some of the problems proposed in the October 1983 issue of the former journal. I solicit from all readers, for possible later publication in this column, elegant solutions with, if possible, nontrivial extensions or generalizations.

FROM *KÖZÉPISKOLAI MATEMATIKAI LAPOK* 67 (1983) 80

Gy. 2142. A 12×12 chessboard has alternating black and white squares. In one operation, every square in a single row (or column) is repainted the opposite colour (white squares repainted black and black ones white). The operation is then repeated on another row (or column). Is it possible that, after a certain number of operations, all the squares on the chessboard are black?

Gy. 2143. A *word* is any sequence of letters. Starting with the word AB, new words are formed by the repeated use of the following rules in any order of succession:

(i) If a word ends in B, add C at the end.

(ii) If a word begins with A, double the word that follows the initial A (e.g., $ABC \to ABCBC$).

(iii) If a word contains three consecutive letters B, replace them by a single C.

(iv) Omit two consecutive letters C if they occur anywhere in a word.

Consider all the words formed in this way. Does the word AC figure among them?

Gy. 2144. Determine all natural numbers n such that $2^n - 1$ equals the square or higher integral power of a natural number.

Gy. 2145. Solve the following system of equations:

$$x^3 + y^3 + z^3 = 8,$$

$$x^2 + y^2 + z^2 = 22,$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -\frac{z}{xy}.$$

Gy. 2147. Construct a triangle ABC (with sides a,b,c), given the side a, the ratio $b:c$, and the difference of the angles B and C.

Gy. 2148. Given are a circle C_1 with centre O_1 and a line l passing through the centre. Consider all circles C_2 which pass through O_1, have their centres on l, and intersect C_1 in two points. Draw the common tangents of these circles with C_1. Find the locus of the points of contact with C_2 of these tangents for all positions of C_2.

Gy. 2149. Let I be the incentre of a triangle ABC. The lines AI,BI,CI meet the circumcircle of the triangle in A_1,B_1,C_1, respectively. Prove that $AA_1 \perp B_1C_1$.

F. 2434. Prove that the equation

$$x^3 + 4x^2 + 6x + c = 0$$

cannot have three distinct real roots for any real number c.

F, 2435, Let γ be a circle with centre O. Show that, of all the triangles ABC with incircle γ, it is the equilateral triangle for which the sum $OA^2 + OB^2 + OC^2$ is minimal.

F, 2436, Prove that, for natural numbers $n > 1$,

$$\sqrt{1+\sqrt{n+\sqrt{n^2+\sqrt{n^3+\ldots+\sqrt{n^n}}}}} < n.$$

F, 2437, Every point in space is coloured either red or blue. Prove that there is a unit square with four blue vertices, or else there is one with at least three red vertices.

F, 2438, Drawing the diagonals of a convex quadrilateral, we find that, among the four triangles thus formed, three are similar to one another but not similar to the fourth. Is it true that then one of the acute angles of the fourth triangle is twice as large as an angle of the other triangles?

F, 2439, A regular pyramid has a square base of edge length e, and θ is the dihedral angle between adjacent lateral faces. Find the radius of the sphere internally tangent to the four lateral faces, and also the radius of the circumsphere of the pyramid.

Apply the results to the case where the lateral faces of the pyramid are equilateral triangles.

P, 383, Does there exist a multiple of 5^{100} which contains no zero in its decimal representation?

P, 384, In a party of 25 members, whenever two members don't know each other, they have common acquaintances among the others. Nobody knows all the others. Prove that if we add the numbers of acquaintances of all members the sum is at least 72. Can the sum be 92? (By *acquaintance* we mean mutual acquaintance.)

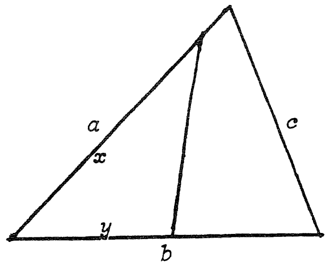
*

I now present solutions to several problems from earlier columns.

J-16, [1980: 316] *From a list of Russian "Jewish" problems.*

Construct a straight line that divides into two equal parts both the area and the perimeter of a given triangle.

Solution by Gregg Patruno, student, Princeton University.



Let a,b,c be the sides of the given triangle, and suppose a segment joining sides a and b bisects both the area and the perimeter of the triangle, as shown in the figure. Then

$$x + y = s \equiv \tfrac{1}{2}(a+b+c) \qquad \text{and} \qquad xy = \tfrac{1}{2}ab. \tag{1}$$

Now, assuming $x \geq y$, (1) holds if and only if

$$x = \frac{s + \sqrt{s^2-2ab}}{2} \qquad \text{and} \qquad y = \frac{s - \sqrt{s^2-2ab}}{2}. \tag{2}$$

It is easy to show from (2) that

$$a \geq x > 0 \iff a \geq c \qquad \text{and} \qquad b \geq y > 0 \iff c \geq b,$$

so the bisecting segment must join the longest and shortest sides of the triangle.

The straightedge and compass constructions for x and y from (2) are standard.

*

J-19. [1980:316] *From a list of Russian "Jewish" problems.*

Six points are given, one on each edge of a tetrahedron of volume 1, none of them being a vertex. Consider the four tetrahedra formed as follows: Choose one vertex of the original tetrahedron and let the remaining vertices be the three given points that lie on the three edges incident with the chosen vertex. Prove that at least one of these four tetrahedra has volume not exceeding 1/8.

Solution by M.S.K.

Let A_1,A_2,A_3,A_4 be the vertices of the tetrahedron of volume 1. For $i,j = 1,2,3,4$, $i \neq j$, let d_{ij} denote the distance from A_i to the given point on edge A_iA_j, so that $d_{ij}+d_{ji} = A_iA_j$, and let V_i be the volume of the small tetrahedron with one vertex at A_i. Since the volumes of two tetrahedra with a common trihedral angle are proportional to the products of the respective edges of the trihedral angle, we have

$$V_1 = \frac{d_{12}d_{13}d_{14}}{A_1A_2\cdot A_1A_3\cdot A_1A_4}.$$

With similar results for V_2,V_3,V_4, we obtain

$$V_1V_2V_3V_4 = \prod_{1\leq i<j\leq 4} \frac{d_{ij}d_{ji}}{(A_iA_j)^2}.$$

Since $x(a-x)/a^2 \leq \frac{1}{4}$, we therefore have

$$V_1V_2V_3V_4 \leq \left(\tfrac{1}{4}\right)^6 = \left(\tfrac{1}{8}\right)^4,$$

and so at least one of the V_i does not exceed 1/8.

*

2. [1981: 72] *From the 1979 Moscow Olympiad (for Grades 7 and 8).*

A (finite) set of weights is numbered 1, 2, 3, ... The total weight of the set is 1 kilogram. Show that, for some number k, the weight numbered k is heavier than $1/2^k$ kilogram.

Solution by John Morvay, Dallas, Texas.

Assume, on the contrary, that $w_k \le 1/2^k$ for $k = 1,2,\ldots,n$, where w_k is the weight numbered k. Then

$$w_1+w_2+\ldots+w_n \le \frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n} = 1 - \frac{1}{2^n} < 1.$$

This gives a contradiction, since the total weight is 1. Hence $w_k > 1/2^k$ for some k.

*

3. [1981: 73] *From the 1979 Moscow Olympiad (for Grades 7 and 8).*

A square is dissected into several rectangles. Show that the sum of the areas of the circles circumscribing the rectangles is no less than the area of the circle circumscribing the square.

Solution by M.S.K.

Let there be n rectangles, of dimensions $a_i \times b_i$, $i = 1,2,\ldots,n$. The circumcircle of the ith rectangle has area $\pi(a_i^2+b_i^2)/4$, and we wish to show that

$$\sum_{i=1}^{n} (a_i^2+b_i^2) \ge 2s^2,$$

where s is a side of the square. Since the n rectangles make up the square, we have

$$\sum_{i=1}^{n} a_i b_i = s^2.$$

Finally, from $a^2+b^2 \ge 2ab$ with equality if and only if $a = b$, we obtain

$$\sum_{i=1}^{n} (a_i^2+b_i^2) \ge \sum_{i=1}^{n} 2a_i b_i = 2s^2,$$

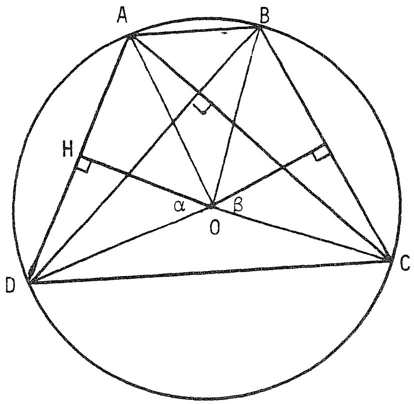
with equality if and only if all the rectangles in the decomposition are squares.

Rider. Give a 3-dimensional extension.

*

5. [1981: 73] *From the 1979 Moscow Olympiad (for Grades 8 and 9).*

Quadrilateral ABCD is inscribed in a circle with center O. The diagonals AC and BD are perpendicular. If OH is the perpendicular from O to AD, show that $OH = \frac{1}{2}BC$.



Solution by M.S.K.

Referring to the figure, we see that

$$OH = R\cos\alpha \quad \text{and} \quad BC = 2R\sin\beta,$$

where R is the circumradius. Since $AC \perp BD$, we have

$$\text{arc } AD + \text{arc } BC = 180^\circ = 2\alpha + 2\beta.$$

Hence $\sin\beta = \cos\alpha$ and $OH = \frac{1}{2}BC$.

*

8. [1981: 73] *From the 1979 Moscow Olympiad (for Grades 9 and 10).*

Can we represent (three-dimensional) space as the union of an infinite set of lines any two of which intersect?

Solution by M.S.K.

Yes. Just consider all lines passing through a single point.

*

9. [1981: 73] *From the 1979 Moscow Olympiad (for Grade 9).*

Does there exist an infinite sequence $\{a_1, a_2, a_3, \ldots\}$ of natural numbers such that no element of the sequence is the sum of any number of other elements and such that, for all n, (a) $a_n \le n^{10}$; (b) $a_n \le n\sqrt{n}$?

Solution by John Morvay, Dallas, Texas.

The answer is no for both cases, even if in (b) we have $a_n \le n^{\sqrt{n}}$. Without loss of generality, we may assume that the sequence, if it exists, is monotonically increasing. For any such sequence $\{a_n\}$ with the given forbidden sum property, we have $a_{n+1} > 2^n - 1$, since the subsequence $\{a_1, a_2, \ldots, a_n\}$ has 2^n-1 distinct nonempty subsets with distinct sums. Finally,

$$a_{n+1} > 2^n - 1 > n^{\sqrt{n}} > n^{10}$$

for n large enough.

*

J-25. [1981: 143] (Corrected) *From a list of Russian "Jewish" problems.*

In a convex quadrilateral ABCD, the sides AB and CD are congruent and the midpoints of diagonals AC and BD are distinct. Prove that the straight line through these two midpoints makes equal angles with AB and CD.

Solution by M.S.K.

(Note that in the original formulation of the problem the word "equal" was mistakenly replaced by "unequal".) In our proof, we do not assume that the quadrilateral is convex, or even simple.

Since the midpoint, M, of AC and the midpoint, N, of BD are distinct, ABCD is not a parallelogram. We assume that the vertices have been labeled so that AB and DC intersect, say in O. Our proof uses vectors, all with origin O, and for any point T we will use the notation $\vec{OT} = \vec{t}$ and $|\vec{t}| = t$.

If the common length of vectors $\vec{AB}$ and $\vec{DC}$ is k, then we have

$$\vec{b} = \vec{a}(1+\frac{k}{a}) \qquad \text{and} \qquad \vec{c} = \vec{d}(1+\frac{k}{d}).$$

Now line MN is not parallel to AB or CD, so let it intersect those lines in R and S, respectively. We then have, for some real numbers x, y, λ, μ,

$$\vec{r} = x\vec{a} = \vec{m} + \lambda(\vec{n}-\vec{m}) \tag{1}$$

and

$$\vec{s} = y\vec{d} = \vec{m} + \mu(\vec{n}-\vec{m}). \tag{2}$$

Now we use $2\vec{m} = \vec{a}+\vec{c}$ and $2\vec{n} = \vec{b}+\vec{d}$ to express (1) in terms of the linearly independent vectors $\vec{a}$ and $\vec{d}$:

$$2x\vec{a} = \vec{a} + (1+\frac{k}{d})\vec{d} + \lambda(\frac{k\vec{a}}{a}-\frac{k\vec{d}}{d}).$$

Therefore

$$1 + \frac{k}{d} = \frac{\lambda k}{d} \qquad \text{and} \qquad 2x = 1 + \frac{\lambda k}{a} = \frac{a+d+k}{a},$$

and so

$$|\vec{OR}| = |\vec{r}| = |x\vec{a}| = \frac{a+d+k}{2}.$$

We find similarly from (2) that $|\vec{OS}| = (a+d+k)/2$. Finally, since triangle ORS is isosceles, line MN makes equal angles with AB and CD. □

Although this vector solution is direct and easy, it would be interesting to have a simple synthetic proof, at least when the quadrilateral is convex.

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J-31. [1981: 143] *From a list of Russian "Jewish" problems.*

Solve $x(3y - 5) = y^2 + 1$ in integers.

Solution by M.S.K.

With $3y - 5 = m$, the given equation is equivalent to

$$9x = m + 10 + \frac{34}{m},$$

and the only possibilities for m are ± 1, ± 2, ± 17, ± 34. But $3|(m+5)$, so m must be of the form $3n+1$, reducing the possibilities for m to 1, 34, -2, -17, and the possible solutions to

$$(x, y) = (5, 2), (5, 13), (-1, 1), (-1, -4),$$

each of which is satisfactory.

*

J-35. [1981: 144] *From a list of Russian "Jewish" problems.*

Given are three disjoint noncongruent spheres.

(a) Show that, for any two of the spheres, their common external tangents all intersect in a point.

(b) The three spheres pairwise determine three points as in (a). Prove that these points are collinear.

Solution by M.S.K.

These results hold more generally for three pairwise homothetic disjoint noncongruent bodies.

(a) First note that there are common externally tangent lines to two homothetic bodies (even two spheres) which do not even intersect. However, if we choose only those externally tangent lines which touch the two bodies in pairs of corresponding points, then these lines are all concurrent in the homothetic center (the external one, in case there is also an internal one, as with two spheres). This follows immediately from the definition of homothecy. It is an easy corollary that any two similar and similarly placed bodies are homothetic.

A proof for two spheres can be given *ab initio* by considering any plane section of the two spheres containing their line of centers. The intersection will be two disjoint great circles, one on each sphere, whose common external tangents, by symmetry, intersect on the line of centers at a fixed distance from any one of the centers.

(b) Here the known more general result is that if three figures, not necessarily in a plane, are homothetic in pairs, then their external homothetic centers are collinear.

Proof. Let A be a point of the first figure and A',A" its corresponding points in the other two figures. Then the three external homothetic centers lie in the plane AA'A". Now let B be any other point of the first figure not in the plane AA'A", and let B',B" be its corresponding points in the other two figures. (Note that, if the three figures all lie in the same plane, we could make them 3-dimensional by adding a perpendicular spike at three corresponding points with lengths proportional to the maximum diameter of each figure.) The three external homothetic centers also lie in the plane BB'B" and, consequently, they lie on the line of intersection of the planes AA'A" and BB'B". □

The collinearity is not restricted to external homothetic centers. It is also known that a line joining two of the centers (external or internal) of three circles taken in pairs passes through a third homothetic center.

*

J-36. [1981: 144] *From a list of Russian "Jewish" problems.*

A point O lies in the base ABC of a tetrahedron P-ABC. Prove that the sum of the angles formed by the line OP and the edges PA, PB, and PC is less than the sum of the face angles at vertex P and greater than half this sum.

Solution by M.S.K.

It suffices to assume that PO is an interior ray of a trihedral angle whose edges are the rays PA,PB,PC. With α,β,γ denoting the angles BPC,CPA,APB, respectively, and α',β',γ' the angles OPA,OPB,OPC, respectively, we must show that

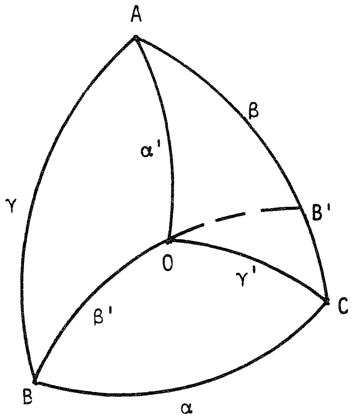
$$\frac{\alpha+\beta+\gamma}{2} < \alpha'+\beta'+\gamma' < \alpha+\beta+\gamma. \qquad (1)$$

Since the sum of two face angles of a trihedral angle is greater than the third, we have

$$\beta' + \gamma' > \alpha, \quad \gamma' + \alpha' > \beta, \quad \alpha' + \beta' > \gamma,$$

and adding these three inequalities yields the first inequality in (1).

To obtain the other inequality, we use the duality between trihedral angles and spherical triangles. (This is often useful since certain results can be proved more easily in one of the two representations.) We draw a unit sphere centered at P intersecting the rays emanating from P in O,A,B,C. The intersection of the sphere with the trihedral angle is a spherical triangle ABC which can be triangulated by great circle arcs meeting in O. If all face angles of the trihedral angles are measured in radians, then (the radius of the sphere being unity) the lengths of all connecting arcs are as shown in the figure.



Produce arc BO to meet arc AC in B'. Since all the spherical triangles we are concerned with here are convex, we have

$$\text{arc } OB' + \text{arc } B'C > \gamma' \quad \text{and} \quad \gamma + \text{arc } AB' > \beta' + \text{arc } OB',$$

and adding these gives an inequality equivalent to $\beta+\gamma > \beta'+\gamma'$. Similarly, $\gamma+\alpha > \gamma'+\alpha'$, $\alpha+\beta > \alpha'+\beta'$, and the desired inequality follows by addition of these three inequalities.

*

3. [1981: 236] *Problem proposed (by Canada, but unused) at the 1981 I.M.O.*

Let $\{f_n\}$ be the Fibonacci sequence $\{1,1,2,3,5,...\}$.

(a) Find all pairs (a,b) of real numbers such that, for each n, $af_n + bf_{n+1}$ is a number of the sequence.

(b) Find all pairs (u,v) of positive real numbers such that, for each n, $uf_n^2 + vf_{n+1}^2$ is a number of the sequence.

Solution by Gregg Patruno, student, Princeton University.

For typographical convenience, we write $f(n)$ instead of f_n for $n = 1,2,3,...$.

(a) Suppose (a,b) is such a pair of real numbers. Then for each n there is an integer i_n such that

$$af(n) + bf(n+1) = f(i_n). \tag{1}$$

Since

$$\begin{aligned} f(i_n) + f(i_{n+1}) &= \big(af(n)+bf(n+1)\big) + \big(af(n+1)+bf(n+2)\big) \\ &= af(n+2) + bf(n+3) \\ &= f(i_{n+2}), \end{aligned}$$

the sequence $\{i_1,i_2,i_3,...\}$ must consist of consecutive integers, provided $i_1 > 1$ (because $f(1)+f(3) = f(4)$ but $1,3,4$ are not consecutive integers). Setting $n = 1$ and $n = 2$ successively in (1), we obtain

$$a + b = f(i_1) \qquad \text{and} \qquad a + 2b = f(i_1+1),$$

from which follows

$$a = 2f(i_1) - f(i_1+1) = f(i_1) - f(i_1-1) \tag{2}$$

and

$$b = f(i_1+1) - f(i_1) = f(i_1-1). \tag{3}$$

Conversely, let the real numbers a and b be defined by the expressions on the right in (2) and (3) for some $i_1 > 1$. Using the well-known relation

$$f(m+n) = f(m-1)f(n) + f(m)f(n+1), \tag{4}$$

it is easy to show by induction on n that

$$af(n) + bf(n+1) = f(n+i_1-1). \tag{5}$$

Finally, we note that, if a and b are defined by the first equalities in (2) and (3), then (5) holds for all n even if $i_1 = 1$.

(b) Suppose (u,v) is such a pair of positive real numbers. Then for each n there is an integer j_n such that

$$uf^2(n) + vf^2(n+1) = f(j_n). \tag{6}$$

It follows from

$$\begin{aligned} f^2(n+1) - f^2(n) &= \big(f(n+1) + f(n)\big)\big(f(n+1) - f(n)\big) \\ &= f(n+2)\big(2f(n+1) - f(n+2)\big) \\ &= \big(f(n+1) + f(n+2)\big)^2 - f^2(n+1) - 2f^2(n+2) \\ &= f^2(n+3) - f^2(n+1) - 2f^2(n+2), \end{aligned}$$

or, equivalently, from

$$f^2(n+3) = 2f^2(n+2) + 2f^2(n+1) - f^2(n),$$

that

$$f(j_{n+3}) = uf^2(n+3) + vf^2(n+4) = 2f(j_{n+2}) + 2f(j_{n+1}) - f(j_n).$$

From this, and the observation that $u,v > 0$ implies that $j_1 < j_2 < \ldots$, we obtain the inequalities

$$\begin{aligned} f(j_{n+3}) &> 2f(j_{n+2}) - f(j_n) > f(j_{n+2}) + f(j_{n+2}) - f(j_{n+2}-2) \\ &= f(j_{n+2}) + f(j_{n+2}-1) = f(j_{n+2}+1) \end{aligned}$$

and

$$\begin{aligned} f(j_{n+3}) &< 2f(j_{n+2}) + 2f(j_{n+1}) < \big(f(j_{n+2}+1)+f(j_{n+2})\big) + \big(f(j_{n+2})+f(j_{n+2}-1)\big) \\ &= f(j_{n+2}+2) + f(j_{n+2}+1) = f(j_{n+2}+3). \end{aligned}$$

Consequently, $j_{n+3} = j_{n+2} + 2$. Setting $n = 3,4,5$ successively in (6), we get

$$4u + 9v = f(j_3), \qquad 9u + 25v = f(j_3+2), \qquad 25u + 64v = f(j_3+4). \tag{7}$$

Now the easily verified identity $f(n) = 3f(n+2) - f(n+4)$ implies that

$$4u + 9v = 3(9u+25v) - (25u+64v) = 2u + 11v,$$

and hence that $u = v$. Equations (7) now become

$$13u = f(j_3), \qquad 34u = f(j_3+2), \qquad 89u = f(j_3+4),$$

from which we conclude that $j_3 = 7$ and $u = v = 1$.

Conversely, if $u = v = 1$, then $uf^2(n) + vf^2(n+1) = f(2n+1)$ for all n. This is easily verified by setting $m = n+1$ in (4).

*

4. [1981: 236] *Problem proposed (by Colombia, but unused) at the 1981 I.M.O.*

A cube is assembled with 27 white cubes. The larger cube is then painted black on the outside and disassembled. A blind man reassembles it. What is the probability that the cube is now completely black on the outside? Give an approximation of the size of your answer.

Solution by Gregg Patruno, student, Princeton University.

The total number of ways of positioning the cubes (disregarding the orientation of each) is 27!. Of these, only 6!12!8! ways could possibly result in an all-black surface. (This outcome would require the 6 singly-painted cubes to be on the faces of the large cube but not on the edges, the 12 doubly-painted cubes to be on the edges but not at the corners, and the 8 triply-painted cubes to be in the eight corner positions.) Hence the probability of successfully positioning the 27 cubes is 6!12!8!/27!.

Once this has been accomplished, the cubes must then be oriented so that all the black surfaces are showing. For the center cube, this presents no problem. For each of the singly-painted cubes, however, there is only a 1/6 probability of a black face showing, for a combined probability of $(1/6)^6$. Likewise, there is a $(1/12)^{12}$ probability for the doubly-painted cubes to have their black edges where they should be, and a $(1/8)^8$ probability that the triply-painted cubes have their black corners in the right places. Thus the probability of successfully orienting the 27 cubes, given their proper positioning, is $1/6^6 12^{12} 8^8$. Finally, the probability of a blind man correctly assembling a black cube from scratch is

$$\frac{\frac{6!}{6^6}\cdot\frac{12!}{12^{12}}\cdot\frac{8!}{8^8}}{27!} \approx 1.8298 \times 10^{-37}.$$

To understand just how small this number is, imagine the entire blind population of the world (say 0.1% of 4 billion) busily assembling $3\times3\times3$ cubes at breakneck speed (say 4 a minute). Even if they could work nonstop, the average time between successful assemblies would be approximately 6.5×10^{23} years.

*

1. [1982: 269] *From a 1962 Peking Mathematics Contest for Grade 12.*

Evaluate

$$m! + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \ldots + \frac{(m+n)!}{n!}.$$

Solution by John Morvay, Dallas, Texas.

The required sum is

$$S = m!\sum_{k=0}^{n}\binom{m+k}{k} = m!\sum_{k=0}^{n}\left\{\binom{m+k+1}{k} - \binom{m+k}{k-1}\right\} = m!\binom{m+n+1}{n}.$$

2. [1982: 269] *From a 1962 Peking Mathematics Contest for Grade 12.*

Six circles in a plane are such that the center of each circle is outside the other circles. Show that these six circles have empty intersection.

Solution by M.S.K.

Our proof is indirect. Assume that there is a point P of common intersection and consider the six rays PC_i, $i = 1,2,\ldots,6$, where the C_i are the centers of the given circles. By the pigeonhole principle, at least one of the six angles formed by the rays does not exceed $60°$, say $\angle C_1PC_2$. Then $\max\{PC_1,PC_2\} \geq C_1C_2$, and a center of one circle lies on or inside another circle, contrary to the hypothesis. □

For a related 3-dimensional problem, see Crux 826* [1983: 79].

*

1. [1982: 269] *From a 1963 Peking Mathematics Contest for Grade 12.*

A polynomial $P(x)$ with integral coefficients takes on the value 2 for four distinct integral values of x. Show that $P(x)$ is never equal to 1, 3, 5, 7, or 9 for any integral value of x.

Solution by Willie Yong, Singapore.

Let $Q(x) \equiv P(x) - 2$. Then

$$Q(x) \equiv (x-x_1)(x-x_2)(x-x_3)(x-x_4)R(x),$$

where x_1,x_2,x_3,x_4 are distinct integers and $P(x)$ is a polynomial with integral coefficients. If

$$P(x) = 1,\ 3,\ 5,\ 7, \text{ or } 9,$$

then

$$Q(x) = -1,\ 1,\ 3,\ 5, \text{ or } 7, \tag{1}$$

respectively. Since none of the numbers on the right in (1) can be the product of five integers at least four of which are distinct, we conclude that $P(x) \neq 1, 3, 5, 7$, or 9 for any integer x. □

Problem 7 [1983: 305] is similar and similarly solved.

*

4. [1982: 270] *From a 1964 Peking Mathematics Contest for Grade 12.*

Show that a circle of diameter D can cover a parallelogram with perimeter $2D$. Show that the same circle can cover any planar region with perimeter $2D$.

Comment by M.S.K.

More generally, any continuous closed space curve C of length L is contained in a ball of radius $R \leq L/4$, and equality is forced only if C is a *needle*, i.e., a line segment of length $L/2$ traversed twice.

For a very simple proof of this result, and for related results and references, see G.D. Chakerian and M.S. Klamkin, "Minimal Covers for Closed Curves", *Mathematics Magazine*, 46 (1973) 55-61.

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8. [1982: 301] *Proposed by John Steinke.*

Determine all six-digit integers n such that n is a perfect square and the number formed by the last three digits of n exceeds the number formed by the first three digits of n by 1. (n might look like 123124.)

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Let $N^2 = n$. Since n is a six-digit integer, we have $317 \le N \le 999$. If $\overline{abc}$ is the number formed by the first three digits of n, then $n - 1$ is the six-digit number $\overline{abcabc}$. Hence $N^2 - 1 = \overline{abc}\cdot 1001$. With this information, it took only 7 seconds for a computer to come up with the only four solutions:

N	428	573	727	846
N^2	183184	328329	528529	715716

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PUBLISHING NOTE

We wish to draw the attention of our readers to the recently published *MATHDISK TWO* and *WORKBOOK FOR MATHDISK TWO*, written by one of our contributors, Clark Kimberling.

MATHDISK TWO is a collection of 36 programs for Apple II+ and Apple IIe computers, written by a mathematics teacher for others of his kind and their students. The disk and its 145-page workbook break new ground in the land of microcomputer software. For example, the program "Analyze Conic" enables user-input coefficients A, B, C, D, E, F for the general conic equation

$$AX^2 + BXY + CY^2 + DX + EY + F = 0,$$

and then prints, in both rotated-translated and original coordinates, all the usual things: equations, vertices, foci, directrices, eccentricity, asymptotes, and more. Another program graphs conics, as many as you wish to see on one screen. Another finds points of intersection of conics.

The workbook includes program listings, sample runs, documentations, 200 exercises, "bytes of history", and 28 figures produced from *MATHDISK TWO*. These include marvelous envelopes, spirograph curves, and graphs in polar coordinates and parametric equations.

MATHDISK TWO sells for \$18.95 (\$7.00 for additional copies), and *WORKBOOK FOR MATHDISK TWO* for \$9.95 (\$6.00 for additional copies). Add 10% for postage and handling. University of Evansville Press, P.O. Box 329, Evansville, Indiana 47702.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

921. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In U.S. liquid measure, a gill is 4 fluid ounces (fl oz) and a pint is 4 gills, that is:

$$\begin{array}{r}\mathrm{FLOZ}\\ 4\\ \hline \mathrm{GILL}\end{array} \quad \text{and} \quad \begin{array}{r}\mathrm{GILL}\\ 4\\ \hline \mathrm{PINT}\end{array}.$$

Solve these alphametical multiplications independently without reusing the digit 4.

922.* *Proposed by A.W. Goodman, University of South Florida.*

Let

$$S_n(z) = \frac{n(n-1)}{2} + \sum_{k=1}^{n-1}(n-k)^2 z^k,$$

where $z = e^{i\theta}$. Prove that, for all real θ,

$$\mathrm{Re}\,(S_n(z)) = \frac{\sin\theta}{2(1-\cos\theta)^2}(n\sin\theta - \sin n\theta) \geq 0.$$

923. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let the ordered triple (a,b,c) denote the triangle whose side lengths are a,b,c. Similarity being an equivalence relation on the set of all triangles, let the ordered ratios $a{:}b{:}c$ (which we will call a *triclass*) denote the equivalence class of all triangles (a',b',c') such that

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'};$$

and let T be the set of all triclasses. A multiplication $\circ$ on T is defined by

$$a{:}b{:}c \circ \alpha{:}\beta{:}\gamma = a\alpha+(c-b)(\gamma-\beta) : b\beta+(c-a)(\gamma-\alpha) : c\gamma+(b-a)(\beta-\alpha).$$

(a) Prove that $(T,\circ)$ is a group.

(b) If $\hat{T}$ is the set of all $a:b:c$ in T such that a,b,c are integers, prove that every triclass in $\hat{T}$ is a unique product (to within order of factors) of "prime" triclasses (relative to the multiplication induced on $\hat{T}$).

924. *Proposed by Charles W. Trigg, San Diego, California.*

In a 3-by-3 array, when the sums, S, of the elements in the four 2-by-2 subarrays are the same, the large square is said to be gnomon-magic.

Find all third-order gnomon-magic squares in which the elements are consecutive digits, and the digits in one of the 2-by-2 arrays form an arithmetic progression.

925. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The points A_i, $i = 1,2,3$, are the vertices of a triangle with sides a_i and median lines m_i. Through a point P in the plane, the lines parallel to m_i intersect a_i in S_i. Find the locus of P if the three points S_i are collinear.

926. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let P be a fixed point inside an ellipse, L a variable chord through P, and L' the chord through P that is perpendicular to L. If P divides L into two segments of lengths m and n, and if P divides L' into two segments of lengths r and s, prove that $1/mn + 1/rs$ is a constant.

927. *Proposed by J.L. Brenner, Palo Alto, California, and Lorraine L. Foster, California State University, Northridge, California.*

Find all sets of four integers x,y,z,w such that

$$3^x + 3^y = 2^z + 2^w.$$

928*. *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, People's Republic of China.*

Through a given point in space, construct a plane that bisects the total surface area and the volume of a given tetrahedron.

929. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Given a triangle ABC, find all interior points P such that, if AP,BP,CP meet the circumcircle again in A_1,B_1,C_1, respectively, then triangles ABC and $A_1B_1C_1$ are congruent.

930. *Proposed by the Cops of Ottawa.*

Does there exist a tetrahedron such that all its edge lengths, all its face areas, and its volume are integers? If so, give a numerical example.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

783. [1982: 277; 1984: 30] *Proposed by R.C. Lyness, Southwold, Suffolk, England.*

Let n be a fixed natural number. We are interested in finding an infinite sequence $(v_0,v_1,v_2,\dots)$ of strictly increasing positive integers, and a finite sequence $(u_0,u_1,\dots,u_n)$ of nonzero integers such that, for all integers $m \ge n$,

$$u_0^2v_m + u_1^2v_{m-1} + \dots + u_n^2v_{m-n} = u_0v_m^2 + u_1v_{m-1}^2 + \dots + u_nv_{m-n}^2. \qquad (1)$$

(a) Prove that (1) holds if

$$u_r = \text{coefficient of } x^r \text{ in } (1-x)^n$$

and

$$v_r = \text{coefficient of } x^r \text{ in } (1-x)^{-n-1}.$$

(b)* Find other sequences (u_r) and (v_r) for which (1) holds.

II. *Comment by M.S. Klamkin, University of Alberta.*

In a comment following the solution of this problem, the editor asks for a proof of the following identity used in the solution:

$$\sum_{r=0}^{n}\binom{n}{r}^2\binom{m+r}{2n} = \binom{m}{n}^2, \qquad m \ge n.$$

Letting $r = n-j$ and $m = k+n$, we have equivalently

$$\sum_{j=0}^{n}\binom{n}{j}^2\binom{k+2n-j}{2n} = \binom{k+n}{n}^2,$$

or, interchanging the roles of n and k,

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k} = \binom{n+k}{k}^2. \qquad (2)$$

In this form, the identity occurs without proof in an 1867 book by the Chinese mathematician Le-Jen Shoo. Proofs of (2) were given about thirty years ago by L. Carlitz, L.K. Hua, G. Huszár, G. Szekeres, L. Takács, and P. Turán [1]. Shortly thereafter, J. Surányi [2] gave a combinatorial proof of the generalization

$$\sum_{j=0}^{k}\binom{k}{j}\binom{h}{j}\binom{n+k+h-j}{k+h} = \binom{n+k}{k}\binom{n+h}{h}, \qquad (3)$$

and he noted that (3) is a special case of Saalschütz's theorem.

Editor's comment.

The proofs in [1] and [2] are not particularly easy, and some are not even elementary, so it is all the more surprising that our "three worthies" [1984: 31] coyly withheld their own "easy" proofs.

REFERENCES

1. *Matematikai Lapok*, 5 (1954) 1-6; 6 (1955) 27-29, 36-38, 219-220.

2. János Surányi, "On a problem of old Chinese mathematics", *Publ. Math.*, 4 (1956) 195-197.

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794. [1982: 303] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Determine the positive integers n for which

(a) $$\lim_{x\to 0}\frac{\exp(\tan^n x) - \exp(\sin^n x)}{x^{n+2}} = 1\ ;$$

(b) $$\lim_{x\to 0}\frac{\exp(x^n) - \exp(\sin^n x)}{x^{n+2}} = 1\ ;$$

(c) $$\lim_{x\to 0}\frac{\exp(\tan^n x) - \exp(x^n)}{x^{n+2}} = 1.$$

Solution by the Cops of Ottawa.

Let $f(x)$, $g(x)$, and $h(x)$ denote the functions whose limits are given in (a), (b), and (c), respectively. Using the familiar Maclaurin expansions of the functions tan, sin, and exp, we soon find

$$f(x) = \frac{n}{2} + o(x^2),$$

$$g(x) = \frac{x^{n-2}}{2} + \frac{n}{6} + o(x^2),$$

$$h(x) = f(x) - g(x) = -\frac{x^{n-2}}{2} + \frac{n}{3} + o(x^2).$$

Observing first that $\lim_{x\to 0} g(x) \neq 1$ and $\lim_{x\to 0} h(x) \neq 1$ if $n = 1$ or 2, we find that

$$\lim_{x\to 0} f(x) = 1 \iff n = 2,$$

$$\lim_{x\to 0} g(x) = 1 \iff n = 6,$$

$$\lim_{x\to 0} h(x) = 1 \iff n = 3.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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795. [1982: 303] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC, let t_a, t_b, t_c be the lengths of its internal angle bisectors, and let T_a, T_b, T_c be the lengths of these bisectors extended to the circumcircle of the triangle. Prove that

$$T_a + T_b + T_c \ge \frac{4}{3}(t_a + t_b + t_c).$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

The following chain of inequalities, where a,b,c are the sides and s the semiperimeter of the triangle, considerably sharpens the proposed inequality:

$$T_a + T_b + T_c \ge \frac{4s}{\sqrt{3}} \ge \frac{4}{3}\sqrt{s}(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}) \ge \frac{4}{3}(t_a + t_b + t_c).$$

The first inequality was established in this journal by Groenman [1982: 116], and the last two follow from Item 8.9 on page 75 of *Geometric Inequalities*, the Bottema Bible, where they are credited to Santaló (1943).

Also solved by LEON BANKOFF, Los Angeles, California (two solutions); W.J. BLUNDON, Memorial University of Newfoundland; J.T GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and the proposer.

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796. [1982: 303] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

For an integer $n > 1$, let $f(n)$ be the product of all the positive divisors of n other than n itself. Characterize the fixed points of f, that is, characterize the integers $n > 1$ such that $f(n) = n$. (For example, $f(8) = 1\cdot 2\cdot 4 = 8$.)

Comment by Bob Prielipp, University of Wisconsin-Oshkosh.

This problem appears, with a simple solution, as Exercise 2 on page 174 of Sierpiński's *Elementary Theory of Numbers* (Hafner Pub. Co., New York, 1964). The answer is that n is a fixed point of f if and only if n is the cube of a prime or the product of two distinct primes.

Also solved by SAM BAETHGE, San Antonio, Texas; W.J. BLUNDON, Memorial University of Newfoundland; KENT BOKLAN, student, Massachusetts Institute of Technology; the COPS of Ottawa; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; OLIVIER LAFITTE, élève au Lycée Michel Montaigne, Bordeaux, France; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; LAWRENCE SOMER, Washington, D.C.; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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797. [1982: 304] *Proposed by H. Kestelman, University College, London, England.*

Show that the trace of a real $n\times n$ matrix A is equal to $x^{\mathsf T}Ax$ for some real column vector x with $x^{\mathsf T}x = n$.

Solution by the proposer.

Let $M = A + A^T$; M is real and symmetric and so there is a real diagonal matrix D and a real orthogonal matrix Ω such that $M = \Omega D\Omega^T$. If u is the column vector with all its components 1, and $x = \Omega u$, then

$$x^T Mx = u^T\Omega^T\Omega D\Omega^T\Omega u = u^T Du = \operatorname{tr} D = \operatorname{tr} M.$$

Since $\operatorname{tr} M = 2\operatorname{tr} A$ and $x^T Mx = 2x^T Ax$, this implies $x^T Ax = \operatorname{tr} A$; at the same time

$$x^T x = u^T\Omega^T\Omega u = u^T u = n.$$

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798. [1982: 304] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

For a nonnegative integer n, evaluate

$$I_n \equiv \int_0^1 \binom{x}{n} dx.$$

Solutions were received from CURTIS COOPER, Central Missouri State University, Warrensburg; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India. Comments were received from LEROY F. MEYERS, The Ohio State University; and the proposer.

Editor's comment.

The proposer painstakingly worked out the exact values of I_n for $n = 0,1,2,\ldots,10$. It is clear that $I_0 = 1$. For $n \geq 1$, Satyanarayana easily derived the formula

$$I_n = \frac{1}{n!}\left(\frac{1}{n+1} - \frac{s_1}{n} + \frac{s_2}{n-1} - \ldots + (-1)^{n-1}\frac{s_{n-1}}{2}\right),$$

where s_k is the sum of the products of the numbers $1,2,\ldots,n-1$ taken k at a time. This can obviously be rewritten in the form

$$I_n = \frac{1}{n!}\left(\frac{s(n,1)}{2} + \frac{s(n,2)}{3} + \ldots + \frac{s(n,n)}{n+1}\right), \tag{1}$$

where the $s(n,k)$ are Stirling numbers of the first kind.

Cooper derived (not so easily) the following recurrence relation, which is much more useful than (1) for computational purposes,

$$I_n = \frac{1}{2}I_{n-1} - \frac{1}{3}I_{n-2} + \ldots + \frac{(-1)^{n+1}}{n+1}I_0, \tag{2}$$

from which $I_1, I_2, I_3, \ldots$ can be found in succession.

Meyers alerted the editor to Comtet [1], where useful information could be found. It turned out, however, that Comtet [2], a revised and enlarged English translation of [1], was even more useful. It contained, in addition to formulas equivalent to (1) and (2), the following information, which is as definitive as we are likely to get for this interesting problem:

(a) All the I_n can be found from the generating function $t/\log(1+t)$:

$$\sum_{n=0}^{\infty} I_n t^n = \frac{t}{\log(1+t)}$$

$$= 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \frac{1}{24}t^3 - \frac{19}{720}t^4 + \frac{3}{160}t^5 - \frac{863}{60480}t^6 + \frac{275}{24192}t^7$$

$$- \frac{33953}{3628800}t^8 + \frac{8183}{1036800}t^9 - \frac{3250433}{479001600}t^{10} + \cdots .$$

(All the coefficients given here agree with the values of I_n calculated by the proposer.)

(b) When $n \to \infty$, we have

$$I_n \sim \frac{(-1)^{n+1}}{n(\log n)^2}.$$

Comtet does not derive these results, but he gives the references [Liénard, 1946], [Nyström, 1930], and [Wachs, 1947]. Unfortunately, although [2] contains a 25-page bibliography (more than 1000 titles), the names Liénard, Nyström, and Wachs nowhere appear in it. The text of [1] was updated in [2] but not, apparently, the bibliography.

REFERENCES

1. Louis Comtet, *Analyse Combinatoire*, Presses Universitaires de France, 1970, Tome Second, p. 141. Ex. 14.

2. ________, *Advanced Combinatorics*, D. Reidel Pub. Co., Dordrecht, The Netherlands, 1974, pp. 293-294.

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799. [1982: 304] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Can the 25 consecutive primes 1327, ..., 1523 be rearranged into a fifth-order magic square?

Solution by the proposer.

Suppose there exists such a fifth-order magic square. The given 25 consecutive primes

1327	1399	1433	1459	1489
1361	1409	1439	1471	1493
1367	1423	1447	1481	1499
1373	1427	1451	1483	1511
1381	1429	1453	1487	1523

sum to 36015, so the magic sum is 36015/5 = 7203. Since all the primes involved are congruent to 1 or 5 modulo 6, and $7203 \equiv 3 \pmod 6$, each row and column must be congruent modulo 6 to some permutation of (1,1,1,1,5) or (5,5,5,5,1). Suppose there are x rows congruent to a permutation of (1,1,1,1,5); then there are $5-x$ rows congruent to a permutation of (5,5,5,5,1), and the square contains $5+3x$ primes of the form $6k+1$. Exactly 11 of the given primes are of the form $6k+1$: 1327, 1381, 1399, 1423, 1429, 1447, 1453, 1459, 1471, 1483, and 1489. Therefore $x=2$, and the square contains two rows of the form (1,1,1,1,5) and three rows of the form (5,5,5,5,1). So, to within permutations of entire rows and internal permutations of the elements of each row, the square (reduced modulo 6) has the following appearance:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & 1 & 1 & 5 \\ 5 & 5 & 5 & 5 & 1 \\ 5 & 5 & 5 & 5 & 1 \\ 5 & 5 & 5 & 5 & 1 \end{array} \tag{1}$$

By the same argument used for rows, the square must contain three columns congruent to permutations of (5,5,5,5,1). Since it is clear that only two such columns can be obtained by internal permutations of each row in (1), we have a contradiction and the desired square does not exist. □

And who cares, you may ask, whether or not the 25 consecutive primes 1327, ..., 1523 rearrange into a fifth-order magic square? These 25 primes are preceded by the 16 primes 1229, ..., 1321 which rearrange into a fourth-order magic square, and followed by the 36 primes 1531, ..., 1787 which rearrange into a sixth-order magic square:

1231	1259	1321	1307
1301	1291	1229	1297
1283	1279	1319	1237
1303	1289	1249	1277

1543	1699	1601	1787	1637	1657
1609	1597	1667	1723	1607	1721
1709	1567	1753	1553	1559	1783
1583	1621	1697	1741	1669	1613
1733	1663	1579	1571	1759	1619
1747	1777	1627	1549	1693	1531

In an endeavor to obtain 77 consecutive primes that separate into consecutive segments of 16, 25, and 36 consecutive primes which rearrange into consecutive-prime magic squares of consecutive order, I spent a frustrating weekend on my TRS-80 microcomputer trying to fit the 25 primes 1327, ..., 1523 into a fifth-order magic square. This problem was born when I abandoned attempting to fit the primes and sought instead to prove that the fit is impossible. Later I succeeded in discovering 77 consecutive primes of the desired type, which were reported in [1].

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio.

REFERENCE

1. Allan Wm. Johnson Jr., "Consecutive-Prime Magic Squares", *Journal of Recreational Mathematics*, 15 (1982-83) 17-18.

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800. [1982: 304] *Proposed by Charles W. Trigg, San Diego, California.*

Find triads of consecutive triangular numbers whose products are square numbers.

Solution by Lawrence Somer, Washington, D.C. (revised by the editor).

More generally, we find k-tuples of consecutive triangular numbers whose products are squares, for fixed $k \geq 1$. For $n = 1,2,3,\ldots$, let $T_n = n(n+1)/2$ denote the nth triangular number. Then

$$P(n,k) \equiv T_n T_{n+1} \ldots T_{n+k-1} = \frac{n(n+k)}{2^k}\cdot(n+1)^2(n+2)^2\ldots(n+k-1)^2, \qquad (1)$$

and we wish to find all natural numbers n such that $P(n,k)$ is a square. We will show that there are only finitely many solutions if k is even and infinitely many if k is odd.

Case 1: k even. Let $k = 2l$. It is clear from (1) that $P(n,k)$ is a square if and only if $n(n+2l) = y^2$ for some natural number y, or

$$n = -l + \sqrt{l^2+y^2}; \qquad (2)$$

and this, in turn, is possible if and only if $l^2+y^2 = z^2$ for some natural number z. Now from

$$(z - y)(z + y) = l^2$$

and the fact that l^2 has a finite number of divisors, we conclude that there are only finitely many possibilities for y and z, and hence for n. To find the satisfactory values of n, we set $d_1 = z-y$ and $d_2 = z+y$. Then

$$d_1 \equiv d_2 \pmod 2, \qquad d_1 < d_2, \qquad \text{and} \qquad d_1 d_2 = l^2. \qquad (3)$$

Since $y = (d_2-d_1)/2$, we obtain from (2)

$$n = -l + \sqrt{l^2 + \left(\frac{d_2-d_1}{2}\right)^2}. \qquad (4)$$

We conclude that, if $k = 2l$, then $P(n,k)$ is a square if and only if n is given by (4) for some pair (d_1,d_2) satisfying (3).

It is easy to verify that there is no solution if k = 2 or 4; one solution if k = 6, 8, 10, 12, or 14; two solutions if k = 16 or 18; one solution if k = 20; etc. Some of the results are set out in the following table.

k	l^2	(d_1,d_2)	n	$P(n,k)$
6	9	(1,9)	2	1260^2
8	16	(2,8)	1	7560^2
10	25	(1,25)	8	3308104800^2
12	36	(2,18)	4	6810804000^2
14	49	(1,49)	18	240814160266680000^2
16	64	(4,16)	2	4168212048000^2
		(2,32)	9	1001829765736800000^2
18	81	(3,27)	6	841537003218912000^2
		(1,81)	32	$1806022478454406721308800000^2$
20	100	(2,50)	16	$115750890181751688864000000^2$

Case 2: k odd. It is clear from (1) that $P(n,k)$ is a square if and only if $n(n+k) = 2t^2$ for some natural number t, or

$$n = \frac{-k + \sqrt{k^2+8t^2}}{2}; \tag{5}$$

and this, in turn, is possible if and only if

$$w^2 - 8t^2 = k^2 \tag{6}$$

for some *odd* natural number w.

One infinite set of solutions is obtained by assuming that $w \equiv t \equiv 0 \pmod{k}$. We then set $w = ku$, $t = kv$, and (6) is equivalent to the Pell equation

$$u^2 - 8v^2 = 1,$$

whose solutions are

$$(u_1,v_1) = (3,1) \quad \text{and} \quad (u_{i+1}, v_{i+1}) = (3u_i+8v_i,\ u_i+3v_i), \quad i = 1,2,3,\ldots .$$

The solutions of (6) are then

$$(w_i,\ t_i) = (ku_i,\ kv_i),$$

and then, from (5),

$$n_i = \frac{-k + ku_i}{2}, \qquad i = 1,2,3,\ldots . \tag{7}$$

We now claim that (7) gives all the solutions for n if 2 is a quadratic nonresidue modulo k. This follows because (6) implies that

$$w^2 \equiv 8t^2 = 2\cdot 4t^2 \pmod{k},$$

which is impossible if 2 is a quadratic nonresidue modulo k and if it is not the case that $w \equiv t \equiv 0 \pmod{k}$. In particular, (7) gives all the solutions for n if $k = 3$.

If (w_1,t_1) is any solution of (6) for which $w_1 \equiv t_1 \equiv 0 \pmod{k}$ does not hold, then another infinite set of solutions of (6) is given by

$$(w_{i+1}, t_{i+1}) = (3w_i+8t_i, w_i+3t_i), \qquad i = 1,2,3,\ldots .$$

According to Nagell [1], it suffices to seek for such a solution (w_1,t_1) within the bounds

$$0 < w_1 < k\sqrt{2}, \qquad 0 \le t_1 < \frac{k}{2\sqrt{2}}.$$

We now show how to find explicitly all the solutions n_i for the case $k = 3$. It is easy to verify that the u_i satisfy the recurrence relation

$$u_1 = 3, \quad u_2 = 17, \quad u_{i+2} = 6u_{i+1} - u_i, \qquad i = 1,2,3,\ldots ;$$

and then, from (7),

$$n_1 = 3, \quad n_2 = 24, \quad n_{i+2} = 6n_{i+1} - n_i + 6, \qquad i = 1,2,3,\ldots .$$

The first few values of n_i and the resulting $P(n_i,3)$ are set out in the following table.

i	n_i	$P(n_i,3)$
1	3	30^2
2	24	5850^2
3	147	1157730^2
4	864	229221540^2
5	5043	45384688830^2
6	29400	8985939059790^2

Also solved by SAM BAETHGE, San Antonio, Texas; W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; DAVID R. STONE, Georgia Southern College, Statesboro; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

Stone noted that the case $k = 1$ (finding triangular numbers that are squares) is a very old problem and that many references to it can be found in Dickson's *History of the Theory of Numbers* (Volume II, Chapter 1). In particular, Euler solved it in 1732-33 and returned to it (and related problems) several times over a 50-year period. Stone also found in the same reference (page 34) an outline of a solution of the case $k = 3$ (our problem) by L. Aubry (1911). *Nihil sub sole novum.*

REFERENCE

1. Trygve Nagell, *Introduction to Number Theory*, Chelsea, New York, 1964, p. 206.

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