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The Josephus Problem

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1. Introduction

Joseph Ben Matthias (c. AD 37–100) was both an active protagonist in and chronicler of the Jewish Revolt of AD 66 to 70 against the Roman occupation of Palestine. He was appointed as military commander of the rebel forces in the Galilee region. Vespasian, commander of the Roman forces, besieged him in the fortress of Jotapata for 47 days. At the end of the siege, the leading Jews, numbering 41, including Joseph, agreed to a pact of mutual destruction. One by one they would kill one another, the order being chosen by lots. Now Joseph was not attached to the cause of the Jews as one might think and he was not a willing participant in this deadly deal. His own account is rather unclear, understandably so if one realises that he and one other survived, undoubtedly through some deceit. Upon his capture Joseph threw in his lot with the Romans, prophesying to Vespasian that he would become emperor, as indeed happened. Joseph also adopted Vespasian's family name, calling himself Flavius Josephus. Centuries later C. G. de Bachet (1591–1639) speculated about the method by which the order of extermination of the members of Joseph's group might have been decided. Bachet suggested that the participants might have been arranged in a circle and every third person eliminated. The method Bachet had in mind is illustrated in figure 1, in which six persons begin and every third is eliminated. What became known as the Josephus Problem was the problem of finding the number of the last survivor if the elimination begins with n persons.

The method suggested by Bachet is one that is commonly used in children's games to choose a child for a special role. Surely everyone recalls some rhyme of 'eeny-meeny-miny-mo' type which was used to count rhythmically in the choosing. Various examples are given in reference 1. My own recollection is that this method is excellent in that it was not obvious to those involved, until the very last, who was to be chosen. The result is more obscure yet if the rhyme is a long one but, oddly, Bachet hit exactly on the mathematically most interesting case. Counting by twos produces results that are a little too simple to be really interesting and counting by more than three produces patterns that do not seem to analyse quite as satisfactorily. Three is just right.

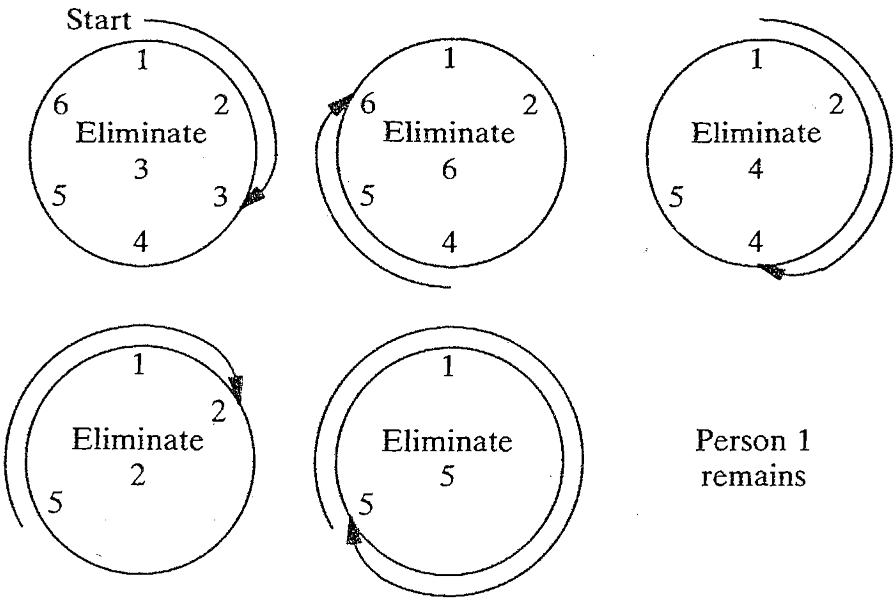

Figure 1

2. Counting by threes

Table 1

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L(n)$	2	2	1	4	1	4	7	1	4	7	10	13	2	5

Table 1 gives some data for the elimination through counting by threes. There are given various numbers of persons, n, and below $L(n)$, the number of the last survivor. The pattern that has been observed by various authors, most notably by P. G. Tait (reference 2) is this: as n increases by ones, $L(n)$ increases by threes until falling back, at certain points, to 1 or 2. The pattern is explained by the following proposition.

Proposition 1. If $L(n) = n-1$ then $L(n+1) = 1$. If $L(n) = n$ then $L(n+1) = 2$. Otherwise $L(n+1) = L(n)+3$. In all cases, $L(n+1) \equiv L(n)+3 \pmod{n+1}$.

Proof. We start with a circle of $n+1$ persons. We begin the elimination to find $L(n+1)$ by removing the third person. As we begin the elimination again we realise that we now have the Josephus Problem on n persons, the only difference being that the labelling of the persons is not the conventional one, so we introduce a new labelling (figure 2). We complete the elimination to find the last survivor. He is $L(n)'$. To find his label in the original scheme we observe that to convert from the new labels to the old labels we add three, except when the new labels are $(n-1)'$ or n'. From figure 2, n' corresponds to 2 and $(n-1)'$ to 1. Therefore the last person left, who is $L(n+1)$, is 1, if $L(n) = n-1$; 2, if $L(n) = n$ and $L(n)+3$ otherwise. This completes the proof.

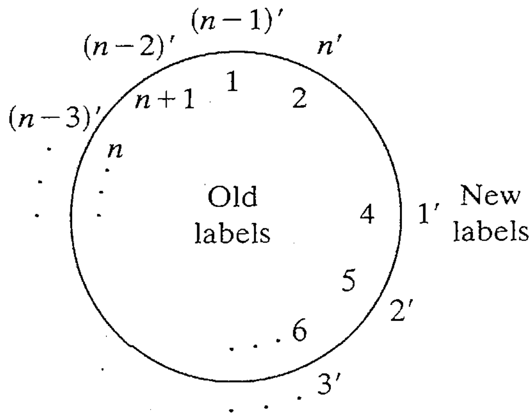

Figure 2

3. Tait's solution

Tait's solution to the problem began with the extraction of specific values of n, which I propose to call Tait numbers. The Tait numbers are those values of n for which $L(n) = 1$ or 2. These I denote by $t_1, t_2, t_3, \ldots$. Table 2 shows a partial list of Tait numbers, extracted from Table 1. Having extracted the sequence, Tait solves the problem as follows.

Table 2

Index	t_1	t_2	t_3	t_4	t_5	t_6
Tait number	2	3	4	6	9	14
$L(t_i)$	2	2	1	1	1	2

Tait's solution. To find $L(n)$. Find the largest Tait number less than or equal to n. Let this number be t_i. Record $L(t_i)$. Then, applying proposition 1, calculate this list:

$$\begin{aligned} L(t_i+1) &= L(t_i)+3, \\ L(t_i+2) &= L(t_i)+6, \\ L(t_i+3) &= L(t_i)+9, \\ &\vdots \\ L(n) &= L(t_i)+3(n-t_i). \end{aligned}$$

This last line gives $L(n)$.

We give an example.

Example. To find $L(12)$. Observe that the largest Tait number less than or equal to 12 is 9 and $L(9) = 1$. Therefore $L(10) = 4$, $L(11) = 7$ and $L(12) = 10$.

Tait's method seems to me to be only partially satisfactory. One is required to construct a sequence of numbers and the values of L at each point of the sequence. Although Tait gave a method of producing this information, the approach is cumbersome. What is really desirable is a closed formula for $L(n)$, and this is possible through the compression of all the necessary information about Tait numbers into a single constant, which I shall call η, in a remarkably compact manner. To know more we need to study the Tait numbers.

4. Tait numbers

What seems not to have been noticed is that the Tait numbers are very nearly in geometric progression. This follows from the next proposition.

Proposition 2. (a) If t_i is even then $t_{i+1} = \frac{3}{2}t_i$ and $L(t_{i+1}) = L(t_i)$.
(b) If t_i is odd and $L(t_i) = 1$ then $t_{i+1} = \frac{3}{2}t_i + \frac{1}{2}$ and $L(t_{i+1}) = 2$.
(c) If t_i is odd and $L(t_i) = 2$ then $t_{i+1} = \frac{3}{2}t_i - \frac{1}{2}$ and $L(t_{i+1}) = 1$.

Proof. All three parts are proved similarly. To illustrate the method we prove (b).

Suppose that t_i is odd and $L(t_i) = 1$. Let $t_i = 2r+1$. Then, by proposition 1, if the number of persons is increased by one, L increases by three as follows:

$$\begin{aligned} t_i &= 2r+1, & L(2r+1) &= 1, \\ t_i+1 &= 2r+2, & L(2r+2) &= 1+3, \\ t_i+2 &= 2r+3, & L(2r+3) &= 1+6, \end{aligned}$$

and so on until

$$t_i+r = 2r+(r+1) = 3r+1, \qquad L(3r+1) = 1+3r.$$

Then, by proposition 1 again, $L(3r+2) = 2$. This means that $3r+2$ is the next Tait number. But notice that $3r+2 = \frac{3}{2}(2r+1)+\frac{1}{2}$. Therefore $t_{i+1} = \frac{3}{2}t_i + \frac{1}{2}$ and $L(t_{i+1}) = 2$, as required.

Proposition 2 tells us that the Tait numbers are generated as follows:

$$\begin{aligned} t_1 &= 2, \\ t_2 &= \tfrac{3}{2}t_1 + y_1, \\ t_3 &= \tfrac{3}{2}t_2 + y_2, \\ &\vdots \\ t_i &= \tfrac{3}{2}t_{i-1} + y_{i-1}. \end{aligned}$$

Here each of the $y_1, y_2, \ldots, y_{i-1}$ is a predetermined constant equal to $-\frac{1}{2}$ or 0 or $\frac{1}{2}$. If we work down the list substituting one equation into the next we obtain

$$t_i = (\tfrac{3}{2})^{i-1}t_1 + (\tfrac{3}{2})^{i-2}y_1 + (\tfrac{3}{2})^{i-3}y_2 + \ldots + y_{i-1}. \tag{1}$$

Dividing formula (1) through by $(\frac{3}{2})^i$ we obtain

$$\frac{t_i}{(\frac{3}{2})^i} = \tfrac{2}{3}t_1 + (\tfrac{2}{3})^2 y_1 + (\tfrac{2}{3})^3 y_2 + \ldots + (\tfrac{2}{3})^i y_{i-1}. \tag{2}$$

Formula (2) suggests that to study the growth of the sequence of Tait numbers it may well be worth considering the infinite sum

$$\tfrac{2}{3}t_1 + (\tfrac{2}{3})^2 y_1 + (\tfrac{2}{3})^3 y_2 + \ldots.$$

It is a simple exercise to show that this infinite sum takes a finite value, which I shall call η. Using η, formula (2) may be written as follows:

$$\frac{t_i}{(\frac{3}{2})^i} = \eta - (\tfrac{2}{3})^{i+1} y_i - (\tfrac{2}{3})^{i+2} y_{i+1} - \ldots. \tag{3}$$

This reorganises simply to

$$t_i = \eta(\tfrac{3}{2})^i - \tfrac{2}{3}y_i - (\tfrac{2}{3})^2 y_{i+1} - (\tfrac{2}{3})^3 y_{i+2} - \ldots. \tag{4}$$

If we write

$$\epsilon_i = -[\tfrac{2}{3}y_i + (\tfrac{2}{3})^2 y_{i+1} + (\tfrac{2}{3})^3 y_{i+2} + \ldots],$$

then formula (4) takes the following simple form:

$$t_i = \eta(\tfrac{3}{2})^i + \epsilon_i. \tag{5}$$

This formula (5) may be interpreted as saying that the sequence of Tait numbers comprises the geometric progression, $\eta(\frac{3}{2})^i$, plus an 'error term', ϵ_i. But how small is ϵ_i? To begin an answer, we collect some more numerical data. Table 3 is based on a value of $\eta = 1.216\,703$: what we see in it is striking. Firstly, ϵ_i is very small and so the Tait numbers are very close to the geometric progression $\eta(\frac{3}{2})^i$. Secondly, the sign of ϵ_i determines the value of $L(t_i)$, that is, if $t_i > \eta(\frac{3}{2})^i$ then $L(t_i) = 2$ and

Table 3

t_i	$\eta(\frac{3}{2})^i$	ϵ_i	$L(t_i)$
2	1.825 05	0.174 95	2
3	2.737 58	0.262 42	2
4	4.106 37	−0.106 37	1
6	6.159 56	−0.159 56	1
9	9.239 34	−0.239 34	1
14	13.859 01	0.140 99	2
21	20.788 51	0.211 49	2
31	31.182 77	−0.182 77	1

if $t_i < \eta(\frac{3}{2})^i$ then $L(t_i) = 1$. Through these properties, η remarkably encodes all the information needed to solve the Josephus Problem. We formalise these facts in proposition 3.

Proposition 3. (a) $0 < |\eta(\frac{3}{2})^i - t_i| < \frac{1}{3}$.
(b) If $t_i > \eta(\frac{3}{2})^i$ then $L(t_i) = 2$. If $t_i < \eta(\frac{3}{2})^i$ then $L(t_i) = 1$.

Proof. This proof is necessarily brief. It requires careful consideration of proposition 2, which reveals these facts about the constants y_i:

(i) the signs of the non-zero terms in the sequence $y_1, y_2, y_3, \ldots$ alternate;

(ii) there are infinitely many non-zero terms in the sequence $y_1, y_2, y_3, \ldots$.

This means that $\epsilon_i = -[\frac{2}{3}y_i + (\frac{2}{3})^2 y_{i+1} + \ldots]$, when we omit the zero terms, must be of one of two forms:

$$\text{either} \qquad \epsilon_i = -[(\tfrac{2}{3})^{a_1} \times \tfrac{1}{2} - (\tfrac{2}{3})^{a_2} \times \tfrac{1}{2} + (\tfrac{2}{3})^{a_3} \times \tfrac{1}{2} - \ldots]$$

$$\text{or} \qquad \epsilon_i = -[-(\tfrac{2}{3})^{a_1} \times \tfrac{1}{2} + (\tfrac{2}{3})^{a_2} \times \tfrac{1}{2} - (\tfrac{2}{3})^{a_3} \times \tfrac{1}{2} + \ldots],$$

where, in either case, the sequence $a_1, a_2, a_3, \ldots$ is a strictly increasing sequence of positive integers. It follows from the alternating series test (see reference 3) that the series converges and its sum lies between any two consecutive partial sums. In particular, we have

$$|\epsilon_i| < (\tfrac{2}{3})^{a_1} \times \tfrac{1}{2} \leqslant \tfrac{1}{3}.$$

If $\epsilon_i < 0$, we are concerned with the case where the first non-zero y_j, such that $j \geqslant i$, is equal to $+\frac{1}{2}$. Proposition 2 indicates that this happens only if

$$L(t_i) = L(t_{i+1}) = L(t_{i+2}) = \ldots = L(t_j) = 1 \quad \text{and} \quad L(t_{j+1}) = 2.$$

So if $\epsilon_i < 0$, that is, if $t_i < \eta(\frac{3}{2})^i$ then $L(t_i) = 1$. For precisely similar reasons, if $t_i > \eta(\frac{3}{2})^i$ then $L(t_i) = 2$. This concludes the proof of proposition 3 (b).

Corollary 4. The Tait numbers are the sequence $[\eta(\frac{3}{2})^i + \frac{1}{3}]$, where [] denotes the integer part.

Proof. By proposition 3 (a), $-\frac{1}{3} < \eta(\frac{3}{2})^i - t_i < \frac{1}{3}$. Adding $\frac{1}{3} + t_i$ to these inequalities, we obtain $t_i < \eta(\frac{3}{2})^i + \frac{1}{3} < t_i + \frac{2}{3}$. Taking integer parts, these inequalities become $[t_i] \leqslant [\eta(\frac{3}{2})^i + \frac{1}{3}] \leqslant [t_i + \frac{2}{3}]$. But since $[t_i] = [t_i + \frac{2}{3}] = t_i$, it now follows that $[t_i] = [\eta(\frac{3}{2})^i + \frac{1}{3}]$, as required.

Corollary 5. Let n be a positive integer. Then $t_i \leqslant n$ if and only if $i \leqslant [\log_{\frac{3}{2}}\{(n + \frac{1}{3})/\eta\}]$.

Proof. Let n be a positive integer and $n \geqslant t_i$. Then, by proposition 3 (a), $n \geqslant \eta(\frac{3}{2})^i - \frac{1}{3}$. This inequality reorganises to $(n + \frac{1}{3})/\eta \geqslant (\frac{3}{2})^i$, η

being positive. Taking logarithms to the base $\frac{3}{2}$ we obtain $\log_{\frac{3}{2}}\{(n+\frac{1}{3})/\eta\} \geqslant i$. Taking integer parts, we have $[\log_{\frac{3}{2}}\{(n+\frac{1}{3})/\eta\}] \geqslant [i] = i$. This constitutes half of the proof. The steps are reversible, with some minor alterations, to complete the 'if and only if' proof.

From now onwards we understand $g(n)$ to be $[\log_{\frac{3}{2}}\{(n+\frac{1}{3})/\eta\}]$.

Corollary 6. Let n be a positive integer. The largest Tait number less than or equal to n is $[\eta(\frac{3}{2})^{g(n)}+\frac{1}{3}]$.

Proof. Immediate from corollaries 4 and 5.

Corollary 7. $L(t_i) = 1+t_i-[\eta(\frac{3}{2})^i]$.

Proof. By propositions 3(a) and 3(b), if $L(t_i) = 1$, we have $t_i < \eta(\frac{3}{2})^i < t_i+\frac{1}{3}$. Therefore $t_i = [\eta(\frac{3}{2})^i]$, from which $1+t_i-[\eta(\frac{3}{2})^i] = 1 = L(t_i)$. If $L(t_i) = 2$, again from propositions 3(a) and 3(b), $t_i-\frac{1}{3} < \eta(\frac{3}{2})^i < t_i$ and so $[\eta(\frac{3}{2})^i] = t_i-1$. From this it follows that $1+t_i-[\eta(\frac{3}{2})^i] = 2 = L(t_i)$, as required.

5. The formula for $L(n)$

We are now in a position to carry out Tait's method, working with closed formulae. Let t_i be the largest Tait number less than or equal to n. Then, by Tait's method,

$$L(n) = L(t_i)+3(n-t_i). \tag{6}$$

By corollary 7, $L(t_i)$ can be rewritten and so formula (6) becomes

$$\begin{aligned} L(n) &= 1+t_i-[\eta(\tfrac{3}{2})^i]+3(n-t_i) \\ &= 1+3n-2t_i-[\eta(\tfrac{3}{2})^i]. \end{aligned} \tag{7}$$

By corollaries 5 and 6, from formula (7) we obtain the final result:

$$L(n) = 1+3n-2[\eta(\tfrac{3}{2})^{g(n)}+\tfrac{1}{3}]-[\eta(\tfrac{3}{2})^{(g(n)}], \quad \text{where } g(n) = \left[\log_{\frac{3}{2}} \frac{n+\frac{1}{3}}{\eta}\right]. \tag{8}$$

This looks daunting, but is easy to use with a calculator and the value $\eta \sim 1.216703$.

Example. To calculate $L(41)$. Set $n = 41$. Then

$$\frac{n+\frac{1}{3}}{\eta} = 33.97159 \quad \text{and} \quad \log_{\frac{3}{2}} 33.97159 = 8.69501.$$

Therefore

$$g(41) = 8, \qquad \eta(\tfrac{3}{2})^8 = 31.18277, \qquad [\eta(\tfrac{3}{2})^8+\tfrac{1}{3}] = 31, \qquad [\eta(\tfrac{3}{2})^8] = 31.$$

Therefore $L(41) = 1+3\times41-2\times31-31 = 31$. That $L(41) = 31$ was first verified by Bachet.

Formula (8) tells us exactly how to calculate $L(n)$ and is effective even if η is known only to a few decimal places. Indeed, from only six decimal places of η, $L(n)$ may be calculated for n up to 2.65×10^6. But what if we want to know who is next to last in the elimination? The beauty of formula (8) is that it will answer this question too, by slotting in a different value of η. Now η, which we might perhaps call η_1, finds the last survivor. But $\eta_2 = 1.408\,639$ (to six decimal places), when inserted into formula 8, finds the last but one to be eliminated; $\eta_3 = 2.621\,392$ (to six decimal places) finds the last but two to be eliminated for $n \geqslant 4$ (there can be a few small values of n for which formula (8) does not work, but the restrictions are minor). So it goes on: to pick out the last but k, there is a constant η_{k+1}. Formula (8) is a complete solution to the problem of elimination through counting by threes.

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Out for a stroll

I walk from the point $(a,b) \neq (0,0)$ in a straight line to $(0,0)$, from where I walk in a straight line to $(c,d) \neq (0,0)$. I work out whether the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

is positive, negative or zero. What is the connection between my journey and the result of my calculation?

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Cyclotomic Polynomials

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Recently a reader, Dr G. de Visme, sent *Mathematical Spectrum* a problem—to show that the coefficients of all cyclotomic polynomials are 0, 1 or -1. When I discovered what a cyclotomic polynomial was, I asked what the solution was; to which the setter, very reasonably, replied that he did not know—he had set the question because he wanted to know whether the result was correct.

The cyclotomic polynomial $\phi_n(x)$ is the polynomial with leading coefficient $+1$ whose roots are the distinct roots of $x^n-1=0$ which are not roots of $x^m-1=0$ for any positive integer m less than the fixed positive integer n. Now most of the roots of $x^n-1=0$ are complex ($\exp(2\pi ri/n)$ for $r=0,1,2,\ldots,n-1$), but it turns out that the cyclotomic polynomials have real integer coefficients.

For example, $\phi_3(x)=x^2+x+1=(x-\omega)(x-\omega^2)$, because $x^3=1$ has three roots: 1, ω and ω^2 ($\omega=\exp\frac{2}{3}\pi i$) and $x=1$ satisfies $\phi_1(x)\equiv x^1-1=0$, while ω and ω^2 do not satisfy $x^1=1$ or $x^2=1$. And, of course, $x^3-1=(x-1)(x^2+x+1)$.

Similarly $x^6-1=0$ has roots $\exp(2\pi ri/6)$ ($r=0,\ldots,5$). Values of r with no common factor with 6 (i.e. $r=1$ and $r=5$) give roots of the cyclotomic polynomial $\phi_6(x)=x^2-x+1$; other values of r give roots of $x^3=1$, $x^2=1$ and $x^1=1$. Thus

$$x^6-1=\phi_6(x)\,\phi_3(x)\,\phi_2(x)\,\phi_1(x),$$

where $\phi_2(x)=x+1$. We can similarly factorise x^4-1 and $x^{12}-1$ to obtain $\phi_4(x)=x^2+1$ and $\phi_{12}(x)=x^4-x^2+1$.

As every root of x^n-1 must be a root of $\phi_n(x)$ or of some $\phi_t(x)$ (where t divides n),

$$x^n-1=\phi_n(x)\prod\phi_t(x), \tag{1}$$

in which the product is taken over all proper divisors of n.

We can use induction to show that the constant term in every $\phi_t(x)$ is $+1$ (except for $\phi_1(x)=x-1$) and that every coefficient in an integer. For suppose that the constant $\phi_n(0)$ is not $+1$, but that the constants $\phi_t(0)$ are all $+1$ for $1 < t < n$. Then put $x = 0$ in equation (1) to give $-1 = -\phi_n(0)$, contradicting our supposition.

Similarly, suppose that the coefficient of x^r in $\phi_n(x)$ is non-integral, but that all coefficients of x^q in $\phi_n(x)$ are integers for $q < r$ and that all coefficients in $\phi_t(x)$ are integers for $t < n$. Then all coefficients in the product polynomial $\prod \phi_t(x)$ are integers. Now compare the coefficients of x^r in equation (1); the left-hand side gives zero, whereas the right-hand side gives a non-integral value, contradicting our supposition. So all coefficients are integers.

Let us call the cyclotomic polynomials with coefficients 0, 1 or -1, T; it turns out that there are large cyclotomic polynomials which are not T, but many cyclotomic polynomials are T. All our examples have been T; how do we find cyclotomic polynomials which are not T?

If p is a prime then $\phi_p(x) = x^{p-1}+x^{p-2}+\ldots+x+1$, since $x^p-1 = (x-1)\phi_p(x)$. So $\phi_p(x)$ is T.

As $x^{pq}-1 = \phi_{pq}(x)\phi_p(x)\phi_q(x)\phi_1(x)$ when p and q are distinct primes,

$$\phi_{pq}(x) = \frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}.$$

The coefficients can be calculated by expanding by the binomial theorem

$$(1-x^p)^{-1} = 1+x^p+x^{2p}+\ldots.$$

The product of this expansion with the similar expansion of $(1-x^q)^{-1}$ gives the double sum

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^{mp+nq} = \sum_{r=0}^{\infty} a_r x^r.$$

Clearly $a_r \geqslant 0$ and equals the number of ways r can be expressed as $mp+nq$ (m and n non-negative integers). Now, if $a_r > 1$, then there must be non-negative numbers m, n, μ (with $m > \mu$) and ν such that $r = mp+nq = \mu p+\nu q$. So

$$(m-\mu)p = (\nu-n)q$$

and $m-\mu$ is a positive multiple of q, i.e. $m \geqslant q$ and $r = mp+nq \geqslant pq$. So $a_r = 1$ or 0 for $r < pq$.

As

$$\phi_{pq}(x) = (x^{pq+1}-x^{pq}-x+1)\sum_{r=0}^{\infty} a_r x^r,$$

the coefficient of x^r in $\phi_{pq}(x)$ is $a_r - a_{r-1} = 0$, 1, or -1 for $r < pq$. But the polynomial is of degree $(p-1)(q-1) < pq$, and so is T. (You may wonder how the higher terms all vanish; it is just like the well-known result that $(1-x)(1+x+x^2+x^3+\ldots) = 1$.)

If $r = p^n$, where p is prime and n is a positive integer, then $\phi_r(x) = \phi_p(x^{r/p})$. (Multiply this expression by $(x^{r/p}-1)$, which is satisfied by all the roots of x^r-1 which do not belong to $\phi_r(x)$.) So this polynomial is T.

So how do we find a cyclotomic polynomial which is not T?

The next simplest expression would be $\phi_{pqr}(x)$, where p, q and r are distinct primes. It turns out that $\phi_{2pq}(x) = \phi_{pq}(-x)$ when p and q are distinct odd primes, so $\phi_{2pq}(x)$ is T. See the problems section.

So we might try p, q and r distinct odd primes. Now the smallest product $pqr = 3\times5\times7 = 105$. So factorising,

$$x^{105}-1 = \phi_{105}(x)\phi_{35}(x)\phi_{21}(x)\phi_{15}(x)\phi_7(x)\phi_5(x)\phi_3(x)\phi_1(x).$$

This looks rather difficult, but the University computer in Oxford has been taught algebra and can factorise polynomials. So it was given this problem and it printed out

$$\begin{aligned}\phi_{105}(x) = {} & x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2x^{41}-x^{40}-x^{39}\\ & +x^{36}+x^{35}+x^{34}+x^{33}+x^{32}+x^{31}-x^{28}-x^{26}\\ & -x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}\\ & +x^{12}-x^9-x^8-2x^7-x^6-x^5+x^2+x+1.\end{aligned}$$

As you can see, $\phi_{105}(x)$ is of degree 48 and only the coefficients of x^{41} and x^7 are not 0 or 1 or -1. The computer then showed off by factorising $x^{3003}-1$, but I won't bother you with the answer; $\phi_{3003}(x)$ is of degree 1440 and four of the coefficients are as big as 4.

So Dr de Visme's problem was a very reasonable one—and gives rise to other problems, which can be found in the problems section.

You visit a country which has 3-cent, 7-cent and 29-cent coins. How many different amounts can you pay exactly if you have five 3-cent coins, three 7-cent coins and seven 29-cent coins?

The Girl and the Fly: A von Neumann Legend

HERBERT R. BAILEY, *Rose-Hulman Institute of Technology*

Professor Bailey received undergraduate degrees in electrical and chemical engineering from Rose-Hulman Institute in Terre Haute, Indiana, and a Ph.D. in mathematics from Purdue University. He worked in government and industry as an applied mathematician for 15 years and has been teaching mathematics for the past 25 years.

John von Neumann was born in Budapest in 1903. He made outstanding contributions in many fields ranging from abstract mathematics to applied economics, and he has been called the father of high-speed electronic computing. A very readable account of his life was written by Paul Halmos (reference 1). There are many stories attesting to his calculating skills, power of recall and quickness of mind. One of my favourite von Neumann legends concerns a girl and a fly.

Sally is 500 feet from a wall and a fly is resting on her nose. They both start towards the wall, Sally walking at 5 feet per second and the fly flying at 15 feet per second. When the fly hits the wall, it immediately reverses direction and returns to her nose where it again reverses direction. This nose-to-wall and wall-to-nose pattern continues until Sally hits the wall. How far did the fly travel?

This is a classic problem and the 'trick' solution is to note that Sally walks for 100 seconds before hitting the wall. Thus the fly flies for 100 seconds and travels (15 feet per second)$\times$(100 seconds) = 1500 feet. The legend is that, when von Neumann was given this problem by his high-school teacher, he quickly solved it. The teacher said, 'So you saw the trick.' He answered 'What trick? It was an easy series'.

In this note we derive an 'easy series' for the flight time which might have been the one von Neumann used.

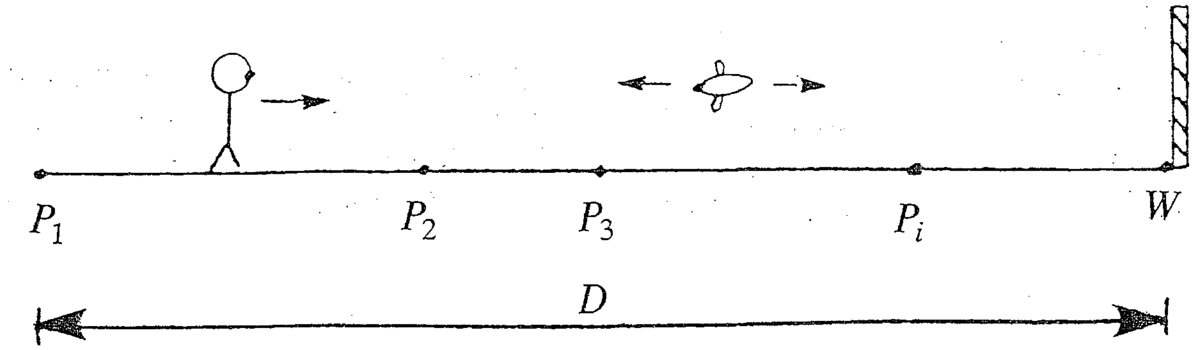


Figure 1

Let P_1 be their initial position and W be the turnaround point at the wall as shown in figure 1. Then P_1 is the point where the fly is first on Sally's nose. Let P_i $(i = 1, 2, \ldots)$ be the point corresponding to the ith contact of fly and nose and t_i be the time elapsed between contact i and contact $i+1$. Let D be the distance from P_1 to W, v_F be the fly's speed and v_S be Sally's speed $(v_F > v_S)$.

In time t_1 the fly goes from P_1 to W and back to P_2. Also in time t_1 Sally goes from P_1 to P_2. Thus their total distance travelled is twice the distance from P_1 to W and we have

$$t_1 v_F + t_1 v_S = 2|P_1 W| = 2D. \tag{1}$$

Likewise in time t_2 their total distance travelled is twice the distance from P_2 to W, giving

$$t_2 v_F + t_2 v_S = 2(|P_1 W| - |P_1 P_2|) = 2(D - v_S t_1) = t_1 v_F - t_1 v_S, \tag{2}$$

where (1) is used to obtain the last equality in (2). During time t_3 we have

$$t_3 v_F + t_3 v_S = 2(|P_1 W| - |P_1 P_2| - |P_2 P_3|) = 2(D - v_S t_1 - v_S t_2) = t_2 v_F - t_2 v_S, \tag{3}$$

where (2) is used to obtain the last equality in (3). In general we have

$$t_i = \frac{v_F - v_S}{v_F + v_S} t_{i-1} = K t_{i-1},$$

where $K = (v_F - v_S)/(v_F + v_S)$. The total time T is given by

$$\begin{aligned} T = \lim_{n \to \infty} \sum_{i=1}^{n} t_i = \lim_{n \to \infty} t_1 \sum_{i=0}^{n-1} K^i &= \frac{2D}{v_F + v_S} \frac{1}{1-K} \\ &= \frac{2D}{v_F + v_S} \frac{1}{1 - (v_F - v_S)/(v_F + v_S)} \\ &= \frac{D}{v_S}. \end{aligned}$$

Then the total distance travelled by the fly is $v_F T = v_F D / v_S$, as predicted by the 'trick' solution.

This is a fun problem to try. Many readers will discover the trick, and some will be able to find a series. There are a number of other ways that one might set up a series solution. If the wall is replaced by Bill walking towards Sally and the fly going from nose to nose, then the trick solution is the same but a series solution is somewhat more difficult to set up.

Reference

1. Paul Halmos, The legend of John von Neumann, *Amer. Math. Monthly* (1973), April.

A Problem in Spherical Trigonometry

PETER BRODSKY, *Catonsville, Maryland, USA*

The author writes: 'This problem arose in my work as a scientific programmer for the Hubble Space Telescope Science Institute. In this capacity I am called upon to solve problems in celestial geometry, the solutions to which are essential in predicting times when astronomical observations may be made with the telescope. (As you may know, the telescope is an orbiting observatory which was launched in April 1990.) I am also a part-time student at John Hopkins University from which I received a Master of Science degree in applied mathematics. I might add that I received little formal training in spherical trigonometry and was forced to learn "on the job".'

The space telescope is essentially a long narrow barrel with a large mirror within. To observe a celestial body, the entire barrel must be aligned toward the object. At the same time, the telescope's solar panels, which rotate about an axis perpendicular to the barrel, must be able to receive the sun's rays 'head on', i.e. at an angle of ninety degrees. Thus, to observe a star (for example), we are constrained to a particular 'roll' (i.e. rotation of the barrel about the lengthwise axis). Only at one unique roll (for a fixed barrel alignment), will the solar panel axis be in a position to receive the sun's rays at full strength. Note that, as the sun's position throughout the year changes with respect to the earth (and thus the telescope), this roll angle will be time dependent. We assume here that we are at some fixed time, with the position of the sun known.

See the diagram in figure 1: the lengthwise axis is known as V_1. The axis upon which the solar panels pivot is V_2, and the remaining axis (to form the orthogonal set) is V_3. In our parlance, the roll angle we seek is that which will place the sun in the V_1–V_3 plane.

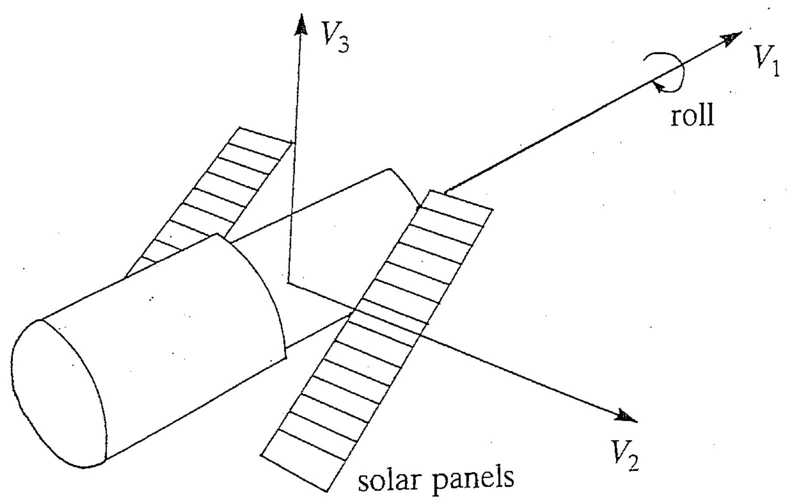


Figure 1. The Hubble space telescope

The problem can be stated as follows.

When viewed down the barrel, the celestial body will *not* in general be seen at the centre of the mirror. This offset is necessary to put the object in a particular camera's aperture. With this in mind, the astronomer needs to know the angle ϕ formed at the point T (his target) between the vector parallel to V_1V_3 and the north celestial pole. Figure 2 is an example of a view down the V_1 axis. Here N represents north, S the sun and T the target. Note the sun in the V_1-V_3 plane. Note also that, in this flat projection, V_2 and V_3 appear as straight lines which extend to infinity.

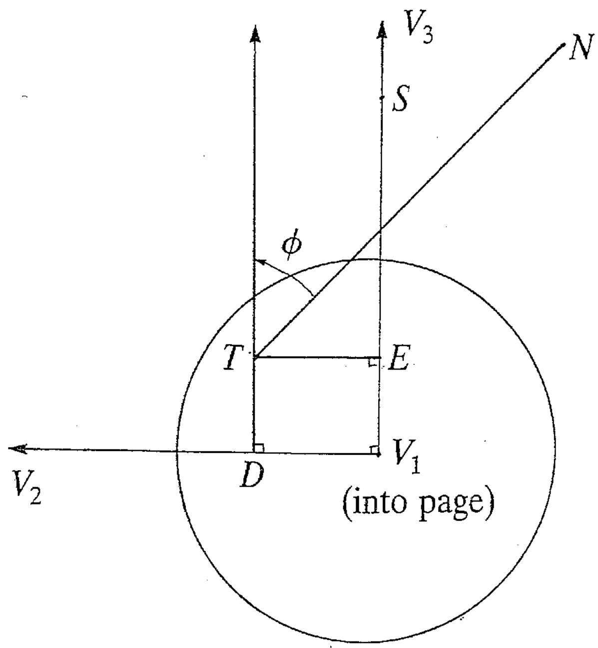


Figure 2. A view 'down the barrel'

To solve for ϕ, we project on to a unit sphere and observe that straight lines become arcs of great circles (i.e. circles whose centre is at the centre of the sphere). Remember that all triangles are now spherical triangles, whose angles do not sum to 180°. Since the sides of a spherical triangle (which are arcs of great circles) are proportional to the angles subtended at the centre of the sphere, it is usual to measure these sides by the sizes of the subtended angles (thus we may speak of an arc of 90° (say) and also of the sine/cosine of an arc). In our solution of the problem we make use of the three basic equations of spherical trigonometry, stated for a triangle with sides a, b and c, and opposite angles A, B and C.

1. Law of cosines for sides:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

2. Law of cosines for angles:

$$\cos A = \sin B \sin C \cos a - \cos B \cos C.$$

3. Law of sines:

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

In figure 3, A, B and C are the points of intersection of V_1, V_2 and V_3, respectively, with the sphere, S is the sun, T the target and N the north; D and E are the intersection points as shown.

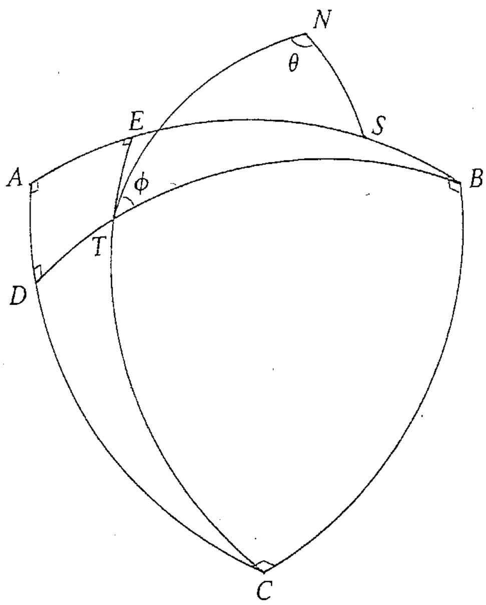


Figure 3

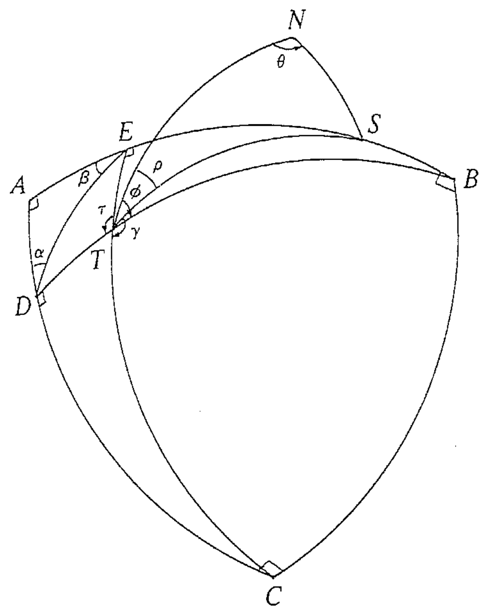


Figure 4

The problem, then, is to find ϕ when NT, NS and θ are given arbitrary values, and where

$$AB = BC = AC = 90°,$$

and

$$\angle DAE = \angle ABC = \angle BCA = \angle ADT = \angle AET = 90° \quad \text{(spherical angles)}.$$

Solution We begin by solving for arc TS using law 1 (see figure 4):

$$TS = \arccos(\cos NT \cos NS + \sin NT \sin NS \cos\theta).$$

Then ρ, the angle measured at T from north to the sun, is found by law 3 to be

$$\rho = \arcsin\frac{\sin NS \sin\theta}{\sin TS}.$$

We now turn our attention to the quadrilateral formed by A, E, T and D. We wish to find τ, the angle at the target between E and D. The right angles at A, E and D will help to simplify this. First find DE; by law 1 and the right angle at A we get

$$DE = \arccos(\cos AE \cos AD).$$

Then, drawing α and β as shown, by law 3 and using the right angle at A we obtain

$$\alpha = \arcsin\frac{\sin AE}{\sin DE}, \qquad \beta = \arcsin\frac{\sin AD}{\sin DE}.$$

Finally, we use the right angles at D and E along with law 2 and the fact that $\sin(90-X) = \cos X$ for any X to obtain the expression for τ:

$$\tau = \arccos(\cos\alpha\cos\beta\cos DE - \sin\alpha\sin\beta).$$

We also need ET, which is easily found by law 3:

$$ET = \arcsin\frac{\sin ED \cos\alpha}{\sin\tau}.$$

Now τ can be reflected across T to become the angle at T between B and C. With this and ρ it is clear that, if we can find γ, the angle at T between S and C, then we have all the necessary information to compute ϕ. To solve for γ, first note that, since $AC = CB = 90°$, any arc of a great circle drawn from C to AB must also be $90°$. Thus, $CE = CS = 90°$. So, using law 1 we have

$$\begin{aligned} 0 = \cos 90° &= \cos SC \\ &= \cos ST \cos(90-ET) + \sin ST \sin(90-ET)\cos\gamma \\ &= \cos ST \sin ET + \sin ST \cos ET \cos\gamma. \end{aligned}$$

Solving for γ, we have

$$\gamma = \arccos\left(-\frac{\cos ST \sin ET}{\sin ST \cos ET}\right) = \arccos\left(-\frac{\tan ET}{\tan ST}\right).$$

Finally, we solve for the desired angle, ϕ:

$$\phi = \rho + \gamma - \tau.$$

How many integers between 101 and 500 are divisible by 3 or by 7 (or both)?

A Divisibility Test for Large Numbers

DIMITRI PAPADOPOULOS, *Thessaloniki, Greece*

Until his retirement in 1984, the author taught in schools in Cairo and Jerusalem as well as in his homeland, Greece. He also worked for the Schools Mathematics Project in Britain.

In previous articles by L. B. Dutta and Y. L. Cheung, Volumes 20 Number 1 and 22 Number 2, simple criteria were given for testing whether a large integer N is divisible by 7. In this article we show how similar ideas may be used to test N for divisibility by other primes p.

For a prime p the non-zero residues form a cyclic group and the number 10 will generate a subgroup. For example, when $p = 37$ the subgroup contains 10, 26 ($\equiv 100 \bmod 37$) and 1 ($\equiv 1000 \bmod 37$). The least positive integer h with $10^h \equiv 1 \pmod{p}$ is called the order of 10 modulo p. Then the subgroup generated by 10 contains h elements. Since the order of any subgroup must divide the order of the group, we see that $h \mid p-1$.

As an example, suppose that we wish to test

$$N = 6041\,278\,328\,213\,713\,762\,510\,925\,741\,357\,670\,113\,576$$

for divisibility by the primes 37, 29 and 1297.

(a) For $p = 37$ we have $10^3 \equiv 1 \pmod{p}$ and so we separate N into 3-digit parts and find their sum:

$$\begin{aligned} N &\equiv 576+113+670+357+741+925+510+762+713+213 \\ &\qquad\qquad +328+278+041+6 \\ &\equiv 6233 \pmod{37}. \end{aligned}$$

Repeating this step we get

$$N \equiv 6+233 \equiv 239 \equiv 17 \pmod{37}.$$

Thus N is not divisible by 37.

(b) For $p = 29$, 10 is a primitive root, i.e. the order of 10 modulo 29 is 28. We have $10^{28} \equiv 1$, $10^{14} \equiv -1$ and $10^7 \equiv 17 \pmod{29}$. Since $10^{28} \equiv 1 \pmod{29}$, we separate off the last 28 digits and add the remaining part:

$$\begin{array}{r} 3713\,762\,510\,925\,741\,357\,670\,113\,576 \\ +\quad 604\,127\,832\,821 \\ \hline 3713\,762\,510\,925\,741\,961\,797\,946\,397\;. \end{array}$$

The result is separated into two 14-digit parts and then we find their difference (since $10^{14} \equiv -1 \pmod{29}$):

$$\begin{array}{r} 41\,961\,797\,946\,397 \\ -\underline{37\,137\,625\,109\,257} \\ 4\,824\,172\,837\,140 \end{array}.$$

The new result is now separated into two 7-digit parts. Since $10^7 \equiv 17 \pmod{29}$, the second one is multiplied by 17 and the product is added to the first 7-digit part:

$$\begin{aligned} N \equiv 4824\,172\,837\,140 &\equiv 482\,417 \times 10^7 + 2837\,140 \\ &\equiv 482\,417 \times 17 + 2837\,140 \equiv 11\,038\,229 \pmod{29}. \end{aligned}$$

We repeat the last step:

$$\begin{aligned} N \equiv 11\,038\,229 &\equiv 1 \times 10^7 + 1038\,229 \\ &\equiv 1 \times 17 + 1038\,229 \equiv 1038\,246 \pmod{29}. \end{aligned}$$

Using a pocket calculator we easily find that

$$1038\,246 \equiv 17 \pmod{29}.$$

Hence N is not divisible by 29.

(c) For $p = 1297$ again we find that 10 is a primitive root, but we easily see that

$$10^8 \equiv 3 \pmod{1297}, \qquad 10^{16} \equiv 9 \pmod{1297},$$
$$10^{24} \equiv 27 \pmod{1297}, \qquad 10^{32} \equiv 81 \pmod{1297}.$$

We separate N into 8-digit parts and proceed as shown:

$$\begin{aligned} N &= 60\,412\,783 \times 10^{32} + 28\,213\,713 \times 10^{24} + 76\,251\,092 \times 10^{16} \\ &\qquad + 57\,413\,576 \times 10^8 + 70\,113\,576 \\ &\equiv 60\,412\,783 \times 81 + 28\,213\,713 \times 27 + 76\,251\,092 \times 9 \\ &\qquad + 57\,413\,576 \times 3 + 70\,113\,576 \\ &\equiv 6583\,819\,806 \pmod{1297}. \end{aligned}$$

Repeating this step we get

$$\begin{aligned} N = 65 \times 10^8 + 83\,819\,806 &\equiv 65 \times 3 + 83\,819\,806 \\ &\equiv 83\,820\,001 \pmod{1297}. \end{aligned}$$

Using a pocket calculator we easily find that

$$N \equiv 79 \pmod{1297}.$$

Thus N is not divisible by 1297.

Self-Cannibalistic Snake

PAUL BELCHER, *United World College of the Atlantic*

Dr Belcher is Head of Mathematics at Atlantic College.

A student of mine, Peter Peli, wondered if a snake could completely devour itself. There was a mythical serpent of ancient Egypt, called Ouroboros, which put its tail in its mouth and continually devoured itself. Whilst biologically it was obvious that the snake could not do this, Peter decided to look at the problem mathematically.

Let us suppose that the original length of the snake is 1 (in suitable units) and is as shown in figure 1. Let us also suppose that the snake always takes the form of a circle, centre the origin, and that the head only moves in the negative x direction. So after some time the snake will look as shown in figure 2. Let the end of the snake's tail, in a general position, have polar coordinates (r, θ). Initially $\theta = 2\pi$ and, for the snake fully to devour itself, finally $r = 0$. Since $r\theta$ always gives the original length of the snake, which was 1, we have $r\theta = 1$. The path described by the end of the tail of the snake is thus given by $r = 1/\theta$.

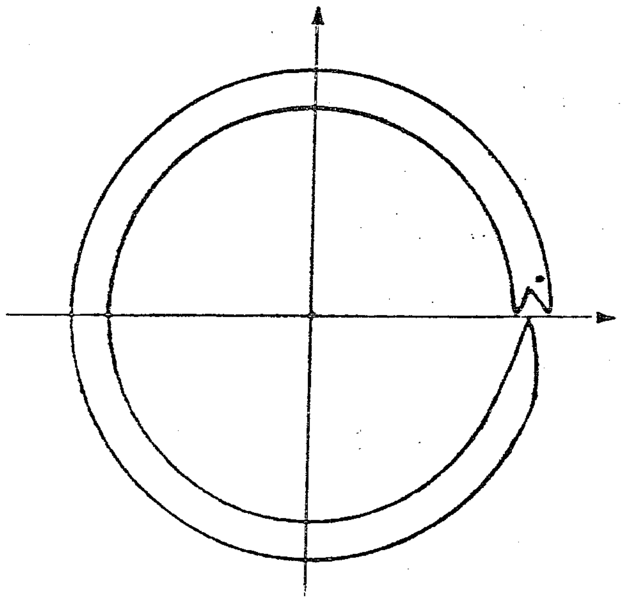


Figure 1

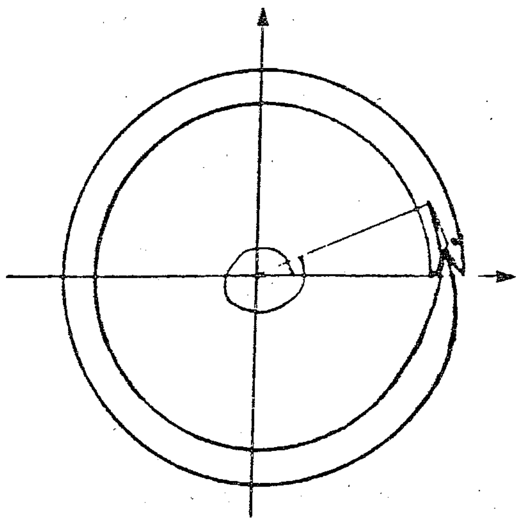


Figure 2

For the snake fully to devour itself, the length of the curve traced out by the end of its tail, using the formula

$$\int \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta$$

is

$$C = \int_{2\pi}^{\infty} \sqrt{\frac{1}{\theta^2} + \left(\frac{-1}{\theta^2}\right)^2}\, \mathrm{d}\theta = \int_{2\pi}^{\infty} \frac{\sqrt{\theta^2+1}}{\theta^2}\, \mathrm{d}\theta.$$

Using the substitution $\theta = \sinh z$, we have

$$\begin{aligned} C &= \left[\operatorname{arsinh}\theta - \frac{\sqrt{\theta^2+1}}{\theta}\right]_{2\pi}^{\infty} \\ &= \lim_{L\to\infty}\left(\operatorname{arsinh} L - \frac{\sqrt{L^2+1}}{L}\right) - \left(\operatorname{arsinh} 2\pi - \frac{\sqrt{1+4\pi^2}}{2\pi}\right). \end{aligned}$$

So we see that the curve has infinite length. This means that it is impossible for the snake fully to eat itself in a finite amount of time, since, according to Einstein, the end of the tail of the snake cannot move faster than the speed of light (without the snake getting severe stomach ache!).

Calculator Button Pushing

PAUL BELCHER, *United World College of the Atlantic*

Dr Belcher is Head of Mathematics at Atlantic College.

Whilst idly (or creatively) pressing the buttons on his calculator, a student of mine, Peter Peli, discovered that, if he started with a natural number n and then pressed the arsinh button, followed by the cosh button, and repeated these two operations a total of $2n+1$ times, he ended up with the number $n+1$. If we let Ch stand for the function $x \mapsto \cosh x$ and ASh stand for the function $x \mapsto \operatorname{arsinh} x$, the result is

$$[\mathrm{Ch}\circ\mathrm{ASh}]^{2n+1}(n) = n+1 \quad (n \in \mathbb{N}\cup\{0\}).$$

The proof is as follows. Let $x \in \mathbb{R}$, $x \geqslant 0$. Then

$$\begin{aligned} \mathrm{Ch}(\mathrm{ASh}(\sqrt{x})) &= \mathrm{Ch}(\ln(\sqrt{x}+\sqrt{x+1})) \\ &= \frac{\sqrt{x}+\sqrt{x+1}}{2} + \frac{1}{2(\sqrt{x}+\sqrt{x+1})} \\ &= \sqrt{x+1}. \end{aligned}$$

So

$$[\mathrm{Ch}\circ\mathrm{ASh}]^{y}(\sqrt{x}) = \sqrt{x+y} \quad (y \in \mathbb{N})$$

and hence

$$[\mathrm{Ch}\circ\mathrm{ASh}]^{2n+1}(\sqrt{n^2}) = \sqrt{n^2+2n+1} = n+1.$$

A similar result is

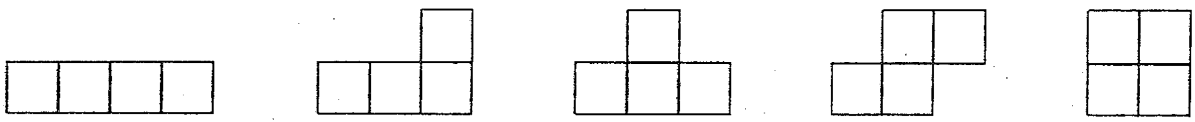
$$[\mathrm{Sh}\circ\mathrm{ACh}]^{2n-1}(n) = n-1 \quad (n \in \mathbb{N}),$$

where Sh stands for the function $x \mapsto \sinh x$ and ACh stands for the function $x \mapsto \operatorname{arcosh} x$. This can be proved in the same fashion as the first result, or as follows. For $n \in \mathbb{N}$,

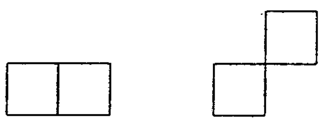
$$\begin{aligned} n-1 &= [[\mathrm{Sh}\circ\mathrm{ACh}]^{2n-1}\circ[\mathrm{Ch}\circ\mathrm{ASh}]^{2n-1}](n-1) \\ &= [\mathrm{Sh}\circ\mathrm{ACh}]^{2n-1}([\mathrm{Ch}\circ\mathrm{ASh}]^{2n-1}(n-1)) \\ &= [\mathrm{Sh}\circ\mathrm{ACh}]^{2n-1}(n). \end{aligned}$$

Corner polyominoes

The polyominoes, developed by Solomon W. Golomb, are connected arrangements of squares which are joined edge to edge. The five tetrominoes, i.e. polyominoes consisting of four squares, are shown below.



Suppose we also allow joining corner to corner. There would then be two dominoes, rather than one.



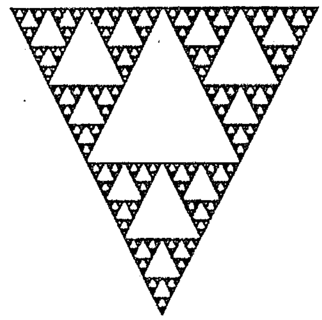
How many 'corner' trominoes and tetrominoes are there?

DAVID SINGMASTER
South Bank Polytechnic,
London

Computer Column

MIKE PIFF

Sierpinski's gasket



Sketch a triangle on a piece of paper, and mark one corner of it. Now randomly select any corner, move half way towards that corner, and mark your current position. Again choose a random corner, and move half way towards it, and carry on repeating the process. What do you think the resulting pattern will look like? Here is a picture of it, and a Modula-2 program to draw it. It uses the utilities developed in the last issue, and assumes a VGA monitor. Simply adjust the size of the screen for any other resolution.

```
MODULE Sierpinski;
FROM Graphics IMPORT BeginGraph,
EndGraph, PutPixel;
FROM InOut IMPORT Read;
FROM Utils IMPORT KeyPressed, rand;
CONST
  minpoint=1; maxpoint=3;
  x=1; y=2;
  screenrows=479; screencols=639;
  hsize=screencols; vsize=screenrows;
  white=15;
TYPE
  indices=[minpoint .. maxpoint];
  coords=[x .. y];
  points=ARRAY coords OF INTEGER;
  fixedpoints=ARRAY indices OF points;
VAR
  fpoint:fixedpoints;
  point:points;
  i:indices;
  dummy:CHAR;
PROCEDURE Init;
BEGIN
  fpoint[1][x]:=0; fpoint[1][y]:=0;
  fpoint[2][x]:=hsize; fpoint[2][y]:=0;
  fpoint[3][x]:=hsize DIV 2;
  fpoint[3][y]:=vsize;
  point:=fpoint[1];
END Init;
PROCEDURE MidPoint(p1,p2:points;
VAR p3:points);
BEGIN
  p3[x]:=(p1[x]+p2[x]) DIV 2;
  p3[y]:=(p1[y]+p2[y]) DIV 2;
END MidPoint;
BEGIN
  BeginGraph; Init;
  REPEAT
    PutPixel(point[x], point[y], white);
    i:=1+TRUNC(rand()×
      FLOAT(maxpoint));
    MidPoint(point, fpoint[i], point);
  UNTIL KeyPressed();
  Read(dummy); EndGraph;
END Sierpinski.
```

$$91 = 1+\{(\sqrt{9}!)!\div(\sqrt{9}!+2)\}$$

MIKE WENBLE
Sonning Common,
Oxon

(See Volume 24 Number 3 page 70.)

Letters to the Editor

Dear Editor,

On fast convergence to π

In *Mathematical Spectrum*, Volume 23 Number 2, P. Glaister described an iteration sequence with third-order convergence to π. Readers may be interested in a convergence result for iteration sequences which can be applied to obtain sequences with a high order of convergence to any required number.

If $x_{n+1} = f(x_n)$ is convergent to a solution x^* of the equation $x = f(x)$, then, writing $\epsilon_n = x_n - x^*$ gives

$$\begin{aligned}\epsilon_{n+1} = x_{n+1} - x^* &= f(x_n) - x^* \\ &= \left(f(x^*) + \epsilon_n f'(x^*) + \frac{\epsilon_n^2}{2!} f''(x^*) + \frac{\epsilon_n^3}{3!} f'''(x^*) + \dots\right) - x^*\end{aligned}$$

i.e.

$$\epsilon_{n+1} = \epsilon_n f'(x^*) + \frac{\epsilon_n^2}{2!} f''(x^*) + \frac{\epsilon_n^3}{3!} f'''(x^*) + \dots . \tag{1}$$

If the first m derivatives of f in (1) are zero at $x = x^*$, then the sequence $x_{n+1} = f(x_n)$ has convergence of order $m+1$; also the magnitude of the coefficient $f^{(m+1)}(x^*)/(m+1)!$ of ϵ_n^{m+1} determines the relative speed of convergence among those sequences which possess order $m+1$ convergence.

In Glaister's article

$$f(x) = x - \tan x \quad \text{and} \quad x_{n+1} = f(x_n), \tag{2}$$

with $f'(\pi) = f''(\pi) = 0$ and $f'''(\pi) = -2$, and therefore $\epsilon_{n+1} = -\frac{1}{3}\epsilon_n^3$. To obtain higher-order convergence, we can add a suitable term to $x - \tan x$ to ensure that further derivatives in (1) are zero in value at $x = \pi$. Consider $g(x) = x - \tan x + \frac{1}{3}\tan^3 x$. It is easily shown that the first *four* derivatives are zero with $g^{(5)} = 24$ at $x = \pi$. The sequence

$$x_{n+1} = g(x_n) \tag{3}$$

has fifth-order convergence to π, with $\epsilon_{n+1} = \frac{1}{5}\epsilon_n^5$.

Another sequence, easily constructed from (1), and with third-order convergence to π, is

$$x_{n+1} = h(x_n), \tag{4}$$

where $h(x) = x + \sin x$. This sequence has $h'(\pi) = h''(\pi) = 0$ and $h'''(\pi) = 1$, and so $\epsilon_{n+1} = \frac{1}{6}\epsilon_n^3$, giving a better relative speed of convergence than Glaister's sequence (2). It also converges over a larger interval, $0 < x_0 < 2\pi$, as compared to $1.98 < x_0 < 4.30$. (Glaister's article should have given 4.3 instead of 4.73.) The following table compares the performance of the sequences (2), (3) and (4), and the range of initial values x_0 for which each sequence converges.

Number of iterates required to approximate π to within 10^{-9} starting at $x = x_0$

x_0	$x_{n+1} = f(x_n)$	$x_{n+1} = g(x_n)$	$x_{n+1} = h(x_n)$
3.2	two	two	two
$\frac{22}{7}$	two	one	one
3.142	one	one	one
3.14	two	one	two
3.0	three	two	two
Convergence	$x_0 \in [1.98, 4.3]$	$x_0 \in [2.10, 4.18]$	$x_0 \in [0, 6.28]$

Yours sincerely,

JOHN MOONEY
(Mathematics Department,
Glasgow Polytechnic)

Dear Editor,

Formula 1 motor racing

The bath model discussed by David Grigg (*Mathematical Spectrum* Volume 24 Number 3, pages 88–89) reminds me of the 'Nigel Mansell' problem for two particular reasons. Firstly the problem requires similar techniques to solve, and secondly, the answer is also not obvious.

In the practice session of the British Grand Prix at Silverstone, the British driver Nigel Mansell completes two laps, each at an average speed of 100 miles/hour, whereas Brazil's Ayrton Senna completes the first lap at an average speed of 90 miles/hour and the second lap at an average speed of 110 miles/hour. Who has the greater average speed over two laps, and who should be in front on the starting grid?

If the length of one lap is d, and t_1, v_1 and t_2, v_2 denote the time and speed for the first and second laps, respectively (in consistent units), then $d = v_1 t_1 = v_2 t_2$. The average speed over the two laps is then

$$v = \frac{2d}{t_1 + t_2} = \frac{2d}{\dfrac{d}{v_1} + \dfrac{d}{v_2}} = \frac{2}{\dfrac{1}{v_1} + \dfrac{1}{v_2}},$$

i.e. $2/v = 1/v_1 + 1/v_2$, and v is the harmonic mean of v_1 and v_2. For Mansell, $v_1 = v_2 = 100$ miles/hour, and so $v = 100$ miles/hour, which is obvious. For Senna, however, $v_1 = 90$ miles/hour and $v_2 = 110$ miles/hour, and so $v = 2/(\frac{1}{90} + \frac{1}{110}) = 99$ miles/hour, and so Mansell has the greater average speed, which is *not* obvious. Moreover, Mansell's time for the two laps is $2d/100 < 2d/99$ which is Senna's time for the two laps, and hence Mansell starts ahead of Senna on the starting grid, as one would expect!

Yours sincerely,

P. GLAISTER
(Department of Mathematics
University of Reading)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

24.10 (Submitted by Feng Yuefeng, The Middle School affiliated to the Hunan Teachers' University, Changsha, China)
Obtain a formula for the sum of the cubes of n integers in arithmetic progression.

24.11 (Submitted by Seung-Jin Bang, Seoul)
Evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^4 x}{\sqrt{2}-\sin 2x}\,\mathrm{d}x.$$

24.12 (Submitted by Dermot Roaf, Exeter College, Oxford—see his article on cyclotomic polynomials in this issue)
Show that

(i) $\phi_{2p}(x) = \phi_p(-x)$, where p is an odd prime;
(ii) $\phi_{2pq}(x) = \phi_{pq}(-x)$, where p and q are distinct odd primes;
(iii) if $r = p^n$, where p is prime and n is a positive integer, then $\phi_r(1) = p$, but that, for all other values of r greater than 1, $\phi_r(1) = 1$.

Solutions to Problems in Volume 24 Number 2

24.4 Prove that

$$\prod_{k=0}^{n} 2\cos 2^k\theta = 2\sum_{k=0}^{2^n-1} \cos(2k+1)\theta.$$

Solutions by Oliver Johnson (King Edward's School, Birmingham)

$$\begin{aligned}2\sum_{k=0}^{2^n-1}\cos(2k+1)\theta &= \frac{1}{\sin\theta}\sum_{k=0}^{2^n-1} 2\cos(2k+1)\theta\sin\theta\\ &= \frac{1}{\sin\theta}[\sin 2\theta+(\sin 4\theta-\sin 2\theta)+(\sin 6\theta-\sin 4\theta)\\ &\qquad +\ldots+\{\sin 2^{n+1}\theta-\sin(2^{n+1}-2)\theta\}]\\ &= \frac{\sin 2^{n+1}\theta}{\sin\theta}\end{aligned}$$

$$= 2\cos 2^n\theta \frac{\sin 2^n\theta}{\sin\theta}$$

$$= 2\cos 2^n\theta \times 2\cos 2^{n-1}\theta \frac{\sin 2^{n-1}\theta}{\sin\theta}$$

$$= \dots$$

$$= \prod_{k=0}^{n} 2\cos 2^k\theta.$$

Alternatively, the formula can be proved by induction on n. The result is true when $n = 0$. Assume that it is true when $n = r$. Then

$$\prod_{k=0}^{r+1} 2\cos 2^k\theta = 2[\cos\theta + \cos 3\theta + \dots + \cos(2^{r+1}-1)\theta]2\cos 2^{r+1}\theta$$

$$= 2[\{\cos(2^{r+1}+1)\theta + \cos(2^{r+1}-1)\theta\}$$
$$+ \{\cos(2^{r+1}+3)\theta + \cos(2^{r+1}-3)\theta\}$$
$$+ \dots + \{\cos(2^{r+2}-1)\theta + \cos\theta\}]$$

$$= 2\sum_{k=0}^{2^{r+1}-1} \cos(2k+1)\theta,$$

which gives the result when $n = r+1$. This proves the inductive step.

24.5 There are n students present at a mathematics lecture. Every two students are either friends of each other or strangers to each other. No two friends have a friend in common. Every two strangers have two and only two friends in common. Show that each student has the same number of friends at the lecture. (The penultimate sentence was omitted in error in Volume 24 Number 2; the correct version appeared in Volume 24 Number 3.)

Solution by Gregory Economides (a student at the University of Newcastle upon Tyne Medical School), who submitted the problem.

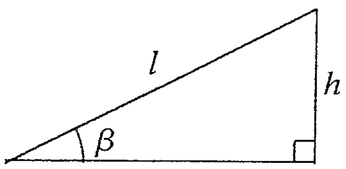
Suppose that student X has the maximal number k of friends, $X_1, \dots, X_k$. Denote by $Y_1, \dots, Y_{n-k-1}$ the strangers to X, and write $S_X = \{X_1, \dots, X_k\}$, $S_Y = \{Y_1, \dots, Y_{n-k-1}\}$. The members of S_X are mutual strangers, since they have X as a common friend. Hence each pair in S_X have a mutual friend other than X who will belong to S_Y. Moreover, the additional friends associated with different pairs in S_X must be different. For if some Y in S_Y was friend to the distinct pairs X_g, X_h and X_i, X_j, then X and Y would have at least three friends in common. Hence every friend of X has at least $k-1$ friends in S_Y and so at least k friends in all and therefore exactly k friends. Now consider Y in S_Y. Since X and Y have a common friend, Y must be a friend of some X_i. But X_i can take the role of X, so that every friend of X_i, in particular Y, has exactly k friends. Hence every student has exactly k friends.

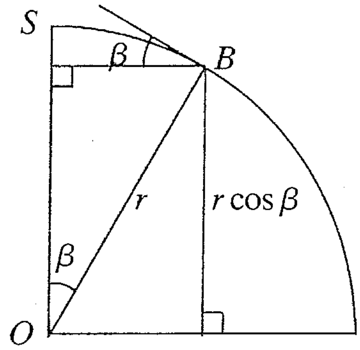
24.6 Mike the mountaineering mite is at a point on the rim of the horizontal base of a right circular cone of radius r and height h. He wants to climb up to the vertex of the cone but unfortunately the direct route is too steep for him since he can only follow a path which never makes an angle greater than β with the horizontal. Find the length of Mike's shortest route to the vertex.

Later we see Mike at a point on the rim of the horizontal base of a hemisphere of radius r. He wants to climb up to the highest point of the hemisphere. The restriction on the steepness of his path still applies. Find the length of Mike's shortest route to the highest point of the hemisphere.

Solution by Oliver Johnson

For the cone, Mike has to gain height h without making an angle greater than β with the horizontal, so the length l of the shortest route is $h \operatorname{cosec} \beta$.





For the hemisphere, Mike has to reach a point B on the hemisphere at which the tangent to the hemisphere in a vertical plane through the centre makes an angle β with the horizontal. This he does by rising through a height $r \cos \beta$ at an angle β with the horizontal, so the length is $r \cos \beta \operatorname{cosec} \beta = r \cot \beta$. He then proceeds directly from B to the summit S in distance $r\beta$, so the shortest route is $r(\beta + \cot \beta)$.

Reviews

Huygens and Barrow, Newton and Hooke. By V. I. ARNOL'D. (English translation by E. J. F. PRIMROSE.) Birkhäuser Verlag, Basel, 1990. Pp. 118 + 6 colour plates. DM28/SFr. 24 (ISBN 3-7643-2383-3).

This book by one of the greatest living mathematicians grew out of a lecture he gave to commemorate the 300th anniversary of Newton's *Principia*. There is some carefully researched historical material on the great seventeenth century mathematicians. Then the author draws links to three very different areas of recent research in which some of their ideas, way ahead of their time, bear fruit.

The first two chapters are mainly historical. There is a typical example of the kind of geometrical argument used by Newton to circumvent the calculus, some correspondence with Hooke in which Newton makes an awful error regarding the hypothetical motion of a particle inside the earth, correctly solved by Hooke, and in which Hooke suggests to Newton an inverse-square law of attraction, and also a discussion of the dispute over priority between Newton and Leibniz, and of the contribution made by Newton's teacher, Barrow.

Chapter 3 relates Huygen's study of wavefronts and evolvents/involutes, via the symmetry group of the icosahedron, to the modern study of quasi-crystals, structures found in X-ray studies showing apparent pentagonal symmetry. It is here that the striking colour plates have their place.

Chapter 4 discusses triumphs of Newtonian gravity theory, including the prediction of Gor'kavyi and Fridman, from an analysis of the resonance structure of the ring system of Uranus, of a whole series of satellites—predictions subsequently confirmed by Voyager 2. It discusses the role of chaotic behaviour in the formation of (Kirkwood) gaps in planetary rings and in the asteroid belt. It also discusses the three-body problem and questions of stability in the solar system.

Parts of this book will be inaccessible to readers without at least two or three years of undergraduate mathematics behind them, weaving together as they do, apart from Newton's laws, topics including potential theory, Hamiltonian dynamics, multidimensional Fourier transforms, representation theory of groups, and differential and algebraic geometry. At times the discussion is quite condensed—with the right background, one will find it clear and concise, but otherwise it may be incomprehensible.

Thus, whilst I enjoyed the historical section and the author's sense of humour, and found much of interest in the later chapters, I would hesitate to recommend the book as a whole to all readers of *Mathematical Spectrum*. However, a reader with a suitable mathematical background and with an interest in history, or in the solar system, or in crystal symmetry, or just in beautiful mathematics, will find the book interesting and enjoyable.

Third-year mathematics, Queens' College, Cambridge JEREMY BYGOTT

Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise. By MANFRED SCHROEDER. W. H. Freeman, New York, 1991. Pp. xviii + 429. £24.95 (ISBN 0-7167-2136-8).

As a genetics student I once felt the need for a non-integral number of degrees of freedom in a chi-squared test, so I asked the professor if this was permitted. 'By all means', he said, 'if you don't mind a space of non-integral dimensions'. (His name was R. A. Fisher.) We no longer blanch at such notions: fractals have seen to that.

Here is a book which is rather more than just another popular account of fractals and chaos. It is very wide-ranging in its coverage, quite deep in some of its treatments, and full of interesting asides, mathematical, historical, and sometimes merely anecdotal. The index reads like a *Who's Who* of mathematics—from Anderson localization, Bethe lattices and Caley trees down to Zipf's law; everything seems to possess fractal connections.

Excellent value for money, complete with eight colour plates and numerous black-and-white figures.

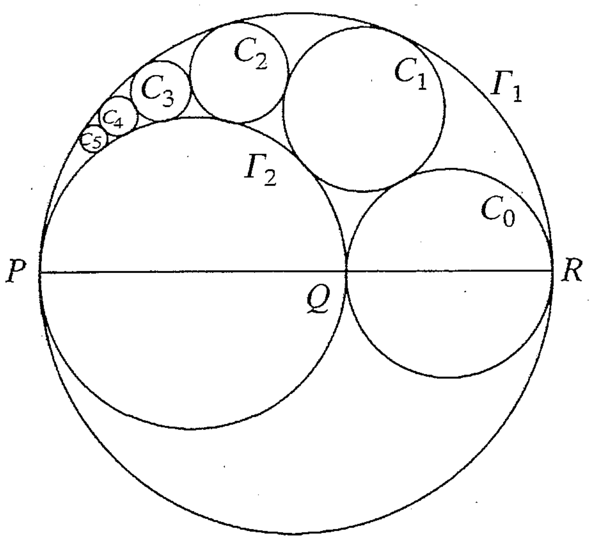
Gonville and Caius College, Cambridge A. W. F. EDWARDS

Work Out Pure Mathematics A-level. By B. HAINES AND R. HAINES. Macmillan Education Ltd, Basingstoke, 1991. Pp. x+246. £7.50 (ISBN 0-333-54385-8).

This new edition which claims to be modified to help the post-GCSE student is written by A-level examiners. There are two pages on 'How to use the book', 'Revision' and 'The Examination', followed by 19 chapters. Each chapter has the following format: fact sheet, worked examples, exercises, outline solutions to the exercises. The questions are A-level type, a few of them being actual past questions. The book is laid out well and easy to follow. I could recommend it to conscientious A-level students who wish to complement their A-level textbook and the notes taken from their teacher. I find it is often the very hard-working student, who lacks mathematical insight and is worried by the examination itself, who gains most from this type of revision guide. A competitor to this book is the *Longman Revise Guide, Mathematics*, by Michael Kenwood and Cyril Mass, also A-level examiners. This guide also includes a very useful table of which items are in the syllabus of which examination board; with 309 pages and 27 chapters it is probably better value at £7.95.

United World College of the Atlantic PAUL BELCHER

Excursions in Geometry. By C. STANLEY OGILVY. Dover, New York, 1991. Pp. vi+178. Paperback £4.20 (ISBN 0-486-26530-7).



I opened this book for the first time at page 55, on which the following theorem is proved. Circle Γ_2 touches circle Γ_1 internally at P, circle C_0 touches Γ_2 at Q and Γ_1 at R, while PQR is a diameter of Γ_1; C_1 touches Γ_1, Γ_2 and C_0, C_2 touches Γ_1, Γ_2 and C_1 (and isn't C_0) etc. Then the perpendicular distance from the centre of C_n to PQR is n times the diameter of C_n. You may not agree, but I think this is a pretty good theorem. It certainly started me thinking. Could I prove the case $n = 1$? How big is C_{314}? Where does C_{271828} sit?

Originally published in 1969, this book is full of such pretty results. I couldn't find a single boring one. The explanation is tucked away in the introduction: 'It is regrettable that so few non-trivial theorems can be proved within the framework of the traditional geometry course when so many startlingly good ones lie just around the corner .... It is my purpose to present some of these to you ....' To my delight I found the presentation unrigorous and incomplete and the later chapters in a fairly haphazard order. Anything that appears self-evident or easy to prove isn't proved, leaving all the space to the exciting bits. Included are chapters on

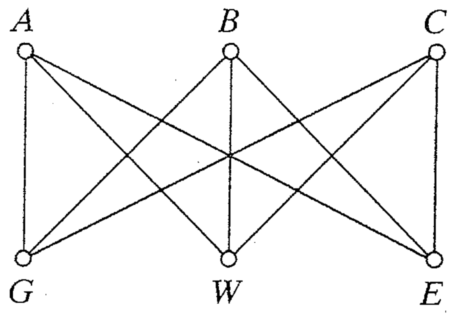
Soddy's hexlet, projective geometry inversion and angle trisection. There is also a discussion of some unsolved problems in geometry at the end.

The style is conversational and entertaining and the author poses many questions along the way. Some of the answers are in the notes at the back of the book, which also contain references and alternative proofs.

The only mathematical background necessary for this book is some basic algebra and very basic geometry (enough to understand the first paragraph of this review will do). There is nothing very difficult inside (not counting the unsolved problems), but you would have to be really clever to find it all trivial. I found it very enjoyable.

Trinity College, Cambridge AMITES SARKAR

Graphs and their uses. By OYSTEIN ORE (revised and updated by ROBIN J. WILSON). New Mathematical Library 34, The Mathematical Association of America, 1990. Pp. viii + 153. Paperback £14.50 (ISBN 0-88385-635-2).



This is one of a series of volumes written by distinguished mathematicians for sixth formers/high school students on topics not usually covered in a school syllabus. A 'graph' here is not a boring old parabola, but a finite set of points called its 'vertices' and lines joining vertices called 'edges'. For example, the diagram shows the graph arising from the famous problem of joining three houses A, B and C to three utilities, gas, water and electricity (G, W, E). The question is: can this be done without the edges crossing? Or again, how many colours are needed to colour a map so that every two countries with a common boundary are coloured differently? Answer: 4. But what if the map is on the surface of a torus (or rubber ring)? Answer: 7. Problems like these and many others are described in a 'non-textbooky' way. This volume should yield valuable rewards to readers prepared to try things for themselves with a pen and paper at the ready.

University of Sheffield DAVID SHARPE

Can You Win? By MIKE ORKIN. W. H. Freeman, New York, 1991. Pp. 181. Paperback £8.95 (ISBN 0-7167-2155-4).

Mum told me that you never win at gambling but, to her consternation, Dad taught me the games. Mike Orkin's book *Can You Win?* improves on these lessons by being a clear gambler's manual. Using simple probability and a colloquial treatment of the weak law of large numbers, he analyses seven types of gambling. He writes as an insider explaining the rules of such games as roulette, blackjack and lotteries, explaining common practice with some affection, but going on to calculate consistent expected losses.

US terms are used throughout the book, but to the mathematician this is irrelevant, as the deductions come from clearly laid-out rules. Any chapter will help motivate lessons in probability at about GCSE level, problems and answers being provided.

The final chapter on the prisoner's dilemma is excellent. This problem is of great importance in the real world; for example, in economics and genetics. In a non-zero-sum interaction involving cooperation and competition, so-called 'nasty' strategies are not evolutionarily stable. He explains this, and other elementary results, with an accessibility which stems from his informal style. For a class it would be an interesting example of computer-driven mathematics.

My father's gambling side appreciated Orkin's enthusiasm for an entertaining flutter. I intend to test his 'no lose' blackjack system. And on the question 'Can you win?', he systematically proves my mother was right.

University College, Oxford NICK SHEA

Archimedes' Revenge: The Joys and Perils of Mathematics. By PAUL HOFFMAN. Penguin, London, 1991. Pp. viii + 285. Paperback £6.99 (ISBN 0-14-012506-X).

In his introduction Paul Hoffman says that his aim is 'to sketch the range and scope of mathematics'. He divides the book into four sections: numbers, shape, machines and 'one man, one vote'.

The first section covers number theory, dealing with the number of the beast, and various conjectures about primes, as well as cryptography, and other aspects of number theory. The section on shape includes tessellation, three-dimensional packing, the Möbius strip, and infinite minimal surfaces. The third section, on machines, explains the system of Boolean algebra, and shows what can and cannot be done by computers. I found the last section most interesting: it shows how different electoral systems are fundamentally unsound—my favourite paradox was that an increase in the support of a candidate can lose him the election—and then does the same for systems of apportionment.

This book requires minimal mathematical knowledge and is written clearly and simply. Paul Hoffman has succeeded in doing what he set out to do: his book is well worth reading.

The College, Winchester ANDREW THOMSON

Introduction to Number Theory with Computing. By R. B. J. T. ALLENBY AND E. J. REDFERN. Edward Arnold, London, 1989. Pp. x + 310. £14.95 (ISBN 0-7131-3661-8).

A nicely written textbook which neatly combines the mathematics of numbers with computer programs which can be used on a home computer. Recommended for undergraduates.

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