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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER
No. 116

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with the *Canadian Mathematical Olympiad* for 1990, which we reproduce with the permission of the Canadian Mathematical Society. Many thanks to Andy Liu, The University of Alberta, for sending me a copy of the problems. We will discuss the official solutions in a later issue, but we welcome any particularly nice solutions or generalizations.

1990 CANADIAN MATHEMATICAL OLYMPIAD
April 4, 1990

1. A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1,2,3,\ldots,n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n,k) for which this is possible.

2. A set of $n(n+1)/2$ distinct numbers is arranged at random in a triangular array:

$$\begin{array}{ccccccccccc}
 & & & & & \times & & & & & \\
 & & & & \times & & \times & & & & \\
 & & & \times & & \times & & \times & & & \\
 & & & \vdots & & \vdots & & \vdots & & & \\
\times & & \times & \cdot & & \cdot & & \cdot & \times & & \times
\end{array}$$

Let M_k be the largest number in the kth row from the top. Find the probability that $M_1 < M_2 < M_3 < \cdots < M_n$.

3. Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'| \ .$$

($|A'B'|$ is the length of line segment $A'B'$, etc.)

4. A particle can travel at speeds up to 2 metres per second along the x–axis, and up to 1 metre per second elsewhere in the plane. Provide

a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

5. Suppose that a function f defined on the positive integers satisfies

$$f(1) = 1 , \qquad f(2) = 2 ,$$

$$f(n + 2) = f(n + 2 - f(n + 1)) + f(n + 1 - f(n)) \qquad (n \geq 1) .$$

(a) Show that

(i) $0 \leq f(n + 1) - f(n) \leq 1$

(ii) if $f(n)$ is odd, then $f(n + 1) = f(n) + 1$.

(b) Determine, with justification, all values of n for which

$$f(n) = 2^{10} + 1 .$$

* * *

The next set of problems we give are from the nineteenth annual *United States of America Mathematical Olympiad*, written April 24, 1990. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 08588–0322. As always we welcome your original "nice" solutions and generalizations.

19th U.S.A. MATHEMATICAL OLYMPIAD

Time limit: 3 $\frac{1}{2}$ hours

1. A certain state issues license plates consisting of six digits (from 0 through 9). The state requires that any two plates differ in at least two places. (Thus the plates $\boxed{027592}$ and $\boxed{020592}$ cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can issue.

2. A sequence of functions $\{f_n(x)\}$ is defined recursively as follows:

$$f_1(x) = \sqrt{x^2 + 48} , \qquad \text{and}$$

$$f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)} \qquad \text{for} \quad n \geq 1 .$$

(Recall that $\sqrt{\ }$ is understood to represent the positive square root.) For each positive integer n, find all real solutions of the equation $f_n(x) = 2x$.

3. Suppose that necklace A has 14 beads and necklace B has 19. Prove that, for every odd integer $n \geq 1$, there is a way to number each of

the 33 beads with an integer from the sequence

$$\{n\ ,\ n+1\ ,\ n+2\ ,\ \cdots\ ,\ n+32\}$$

so that each integer is used once, and adjacent beads correspond to relatively prime integers. (Here a "necklace" is viewed as a circle in which each bead is adjacent to two other beads.)

4. Find, with proof, the number of positive integers whose base–n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.)

5. An acute–angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extension at P and Q. Prove that the points M, N, P, Q lie on a common circle.

* * *

Next we have five Quickies from Murray S. Klamkin, The University of Alberta, Edmonton. The "quick" solutions will be discussed next issue.

1. Completely factor the polynomial

$$(y+z+w-x)(z+w+x-y)(w+x+y-z)(x+y+z-w)-16xyzw\ .$$

2. Determine all solutions (a,b,c) in natural numbers satisfying the simultaneous Diophantine equations

$$a^2 = 2(b+c) \qquad \text{and} \qquad a^6 = b^6 + c^6 + 31(b^2+c^2)\ .$$

3. If a, b, c are the sides of a triangle of area F, prove that

$$[a^2+(b-c)^2]^2 + [b^2+(c-a)^2]^2 + [c^2+(a-b)^2]^2 \geq 16F^2$$

and determine when there is equality.

4. Prove that $re^{sx} + se^{-rx} \geq 1$ where $r+s=1$ and $r,s \geq 0$.

5. A sphere is tangent to the six edges of a given tetrahedron. Prove that the three segments joining pairs of opposite points of tangency are concurrent.

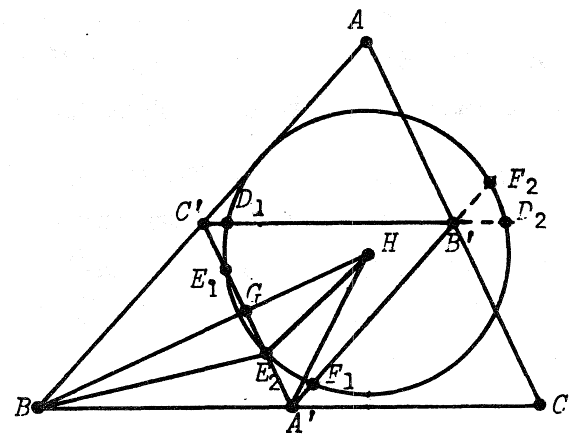
* * *

The first solutions which we give are to problems from the 1981 numbers of the Corner.

1. [1981: 268] 1981 *British Mathematical Olympiad.*

H is the orthocentre of triangle ABC. The midpoints of BC, CA, AB are A', B', C', respectively. A circle with centre H cuts the sides of triangle $A'B'C'$ (produced if necessary) in six points: D_1, D_2 on $B'C'$; E_1, E_2 on $C'A'$; and F_1, F_2 on $A'B'$. Prove that $AD_1 = AD_2 = BE_1 = BE_2 = CF_1 = CF_2$.

Solutions by Anonymous; and by George Evagelopoulos, Law student, Athens, Greece.



Since $B'C'$ is parallel to BC, AH is perpendicular to the chord D_1D_2; thus by symmetry $AD_1 = AD_2$, and similarly $BE_1 = BE_2$ and $CF_1 = CF_2$. To complete the proof it is only necessary to show that $BE_2 = CF_1$. Note that BH is perpendicular to $A'E_2$. Let them intersect at G. Then

$$\begin{aligned}(E_2B)^2 &= (BG)^2 + (E_2G)^2\\ &= ((A'B)^2 - (A'G)^2) + ((E_2H)^2 - (GH)^2)\\ &= (A'B)^2 + (E_2H)^2 - (A'H)^2.\end{aligned}$$

Similarly $(F_1C)^2 = (A'C)^2 + (F_1H)^2 - (A'H)^2$. Since $A'B = A'C$ and $E_2H = F_1H$ we have $E_2B = F_1C$ as desired.

5. [1981: 269] 1981 *British Mathematical Olympiad.*

Find the smallest possible value of $|12^m - 5^n|$, where m and n are positive integers, and prove your result.

Solutions by Anonymous; and by George Evagelopoulos, Law student, Athens, Greece.

When $m = n = 1$, $|12^m - 5^n| = 7$. We show that this is the minimum value. Since $12^m \equiv 0 \bmod 2$ and $5^n \not\equiv 0 \bmod 2$, we cannot have $|12^m - 5^n| = 0, 2, 4$ or 6. Since $12^m \equiv 0 \bmod 3$ and $5^n \not\equiv 0 \bmod 3$ we cannot have $|12^m - 5^n| = 3$. Since $12^m \not\equiv 0 \bmod 5$ and $5^n \equiv 0 \bmod 5$ we cannot have $|12^m - 5^n| = 5$. Suppose $|12^m - 5^n| = 1$. Then we have $12^m \equiv \pm 1 \bmod 5$. Since $12^1 \equiv 2 \bmod 5$, $12^2 \equiv 4 \bmod 5$, $12^3 \equiv 3 \bmod 5$ and $12^4 \equiv 1 \bmod 5$, we must have $m = 4k$ for some integer k. Similarly $5^n \equiv \pm 1 \bmod 12$, and since $5^1 \equiv 5 \bmod 12$ and $5^2 \equiv 1 \bmod 12$, we must have $n = 2l$ for some integer l. Now

$$|12^m - 5^n| = |12^{4k} - 5^{2l}| = |12^{2k} + 5^l||12^{2k} - 5^l| \neq 1.$$

6. [1981: 269] 1981 *British Mathematical Olympiad.*

Given that $a_1, a_2, \cdots, a_n$ are distinct nonzero integers and that

$$p_i = \prod_{\substack{j=1\\ j\neq i}}^{n} (a_i - a_j)\ , \qquad i = 1,2,\cdots,n\ ,$$

prove that $\displaystyle\sum_{i=1}^{n} \frac{a_i^k}{p_i}$ is an integer for every non-negative integer k.

Solution by Anonymous.

We may rewrite $\displaystyle\sum_{i=1}^{n} \frac{a_i^k}{p_i}$ as

$$\frac{\displaystyle\sum_{i=1}^{n} \left[(-1)^{i-1} a_i^k \prod_{\substack{1\le u<v\le n\\ u,v\neq i}} (a_u - a_v) \right]}{\displaystyle\prod_{1\le u<v\le n} (a_u - a_v)}\ . \tag{1}$$

Suppose we identify a_p with a_q for some $1 \le p < q \le n$. Then all terms in the numerator of (1) vanish except for the two corresponding to $i = p$ and $i = q$. Let $q - p = t + 1$, $t \ge 0$. Then the term corresponding to $i = q$ can be written

$$(-1)^{q-1+t} a_p^k \prod_{\substack{1\le u<v\le n\\ u,v\neq p}} (a_u - a_v)\ ,$$

the extra factor $(-1)^t$ being generated because the factors $a_q - a_v$ $(= a_p - a_v)$, $q < v < p$, had to be reversed. Since $q - 1 + t = p + 2t$, the $i = p$ and $i = q$ terms are now easily seen to be equal in magnitude but opposite in sign. Hence the numerator of (1) vanishes identically so that it has a factor $a_p - a_q$. Since p and q are arbitrary, the numerator of (1) has

$$\prod_{1\le u<v\le n} (a_u - a_v)$$

as a factor so that (1) is a polynomial in $a_1, a_2, \cdots, a_n$ for any non-negative integer k. The desired result follows.

*

The next solution is to a problem from the January 1988 number of *Crux*. Curtis Cooper sent in solutions to Kürschak Competition problem 1982.2 [1988: 1], a solution for which was discussed last autumn [1989:227], and to the following problem.

<u>1984.1</u> [1988: 2] *Kürschak Competitions.*

Writing down the first four rows of Pascal's triangle in the usual way and then adding up the numbers in vertical columns, we obtain seven numbers as shown below. If we repeat this procedure with the first 1024 rows of the triangle, how many of the 2047 numbers thus obtained will be odd?

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ \hline 1 & 1 & 4 & 3 & 4 & 1 & 1 \end{array}$$

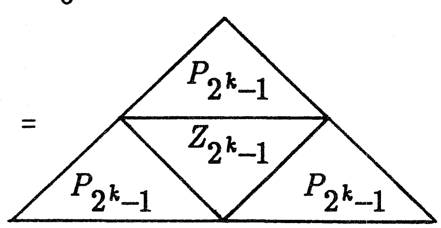
Solution by Curtis Cooper, Central Missouri State University.

Let

$$Z_k = \begin{array}{c} \overbrace{\begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \end{array}}^{k \text{ zeros}} \\ \begin{array}{cccc} 0 & 0 & & 0 \end{array} \\ \vdots \\ \begin{array}{cc} 0 & 0 \end{array} \\ 0 \end{array}$$

and let

$$P_1 = \begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}, \qquad P_{2^{k+1}-1} =$$



for any positive integer k. Then by induction on n and the facts that

$$\binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a}{b} \quad \text{and} \quad \binom{a}{0} = \binom{a}{a} = 1,$$

we have that the pattern of numbers mod 2 in rows 0 to $2^n - 1$ of Pascal's triangle is P_{2^n-1} for any positive integer n. Next, let S_{2^k-1} denote the row of $2^{k+1}-1$ binary numbers obtained by adding the columns of rows 0 to $2^k - 1$ of Pascal's triangle and reducing mod 2. Then by the above we can see that $S_1 = 111$ and

$$S_{2^{k+1}-1} = 0^{2^k} \cdot S_{2^k-1} \cdot 0^{2^k} + S_{2^k-1} \cdot 0 \cdot S_{2^k-1} \tag{1}$$

for any positive integer k, where 0^m is a row of m zeros, $\cdot$ is concatenation and $+$ is componentwise addition mod 2. Now, let n be any positive integer and

$w_n = (110)^{[(2^n-1)/3]}$ be the concatenation of $[(2^n-1)/3]$ copies of the string 110, where [ ] denotes the greatest integer function. Then by (1) and induction on n we have that

$$S_{2^n-1} = \begin{cases} w_n \cdot 1 \cdot w_n^r & \text{if } n \text{ is even}, \\ w_n \cdot 111 \cdot w_n^r & \text{if } n \text{ is odd}, \end{cases} \tag{3}$$

where w_n^r is the reverse of the row of numbers w_n. Finally, the number of odd numbers obtained in adding the columns of the first $2^{10} = 1024$ rows of Pascal's triangle is the number of 1's in $S_{2^{10}-1}$ which is

$$2 \cdot \left[\frac{2^{10}-1}{3}\right] + 1 + 2 \cdot \left[\frac{2^{10}-1}{3}\right] = 1365 .$$

*

Now we turn to the last solutions in our files for problems given in the 1988 numbers of the Corner. They are to problems from the *4th Balkan Olympiad* [1988: 289, 290].

1. Let a be a real number, and let f be a real-valued function defined on the set of all real numbers such that

$$f(x + y) = f(x)f(a - y) + f(y)f(a - x)$$

for all reals x, y. Also assume that $f(0) = 1/2$. Show that f is a constant function. (Yugoslavia)

Solutions by Botand Köszegi, student, Halifax West High School, and by M. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

For $x = y = 0$,

$$f(0) = f(0)f(a) + f(0)f(a) = 2f(0)f(a) .$$

Therefore $f(a) = 1/2$.

For $x = 0$, y arbitrary,

$$\begin{aligned} f(y) &= f(0) \cdot f(a - y) + f(y) \cdot f(a) \\ &= f(a - y)/2 + f(y)/2 \end{aligned}$$

so $f(y) = f(a - y)$ for all y.

For x given and $y = a - x$,

$$1/2 = f(a) = (f(x))^2 + (f(a - x))^2 = 2(f(x))^2 .$$

Thus $f(x) = \pm 1/2$ for each x.

Finally for $x = y = z/2$, z arbitrary,

$$f(z) = f(z/2)f(a - z/2) + f(z/2)f(a - z/2)$$
$$= 2(f(z/2))^2 = 2\cdot 1/4 = 1/2\ .$$

Thus $f(x) = 1/2$ for all x.

2. Let $x \geq 1$ and $y \geq 1$ be such that the numbers

$$a = \sqrt{x-1} + \sqrt{y-1}$$

and

$$b = \sqrt{x+1} + \sqrt{y+1}$$

are non–consecutive integers. Show that $b = a + 2$ and $x = y = 5/4$. (Romania)

Solution by M. Selby, Department of Mathematics, The University of Windsor, Ontario.

Notice that

$$f(u) = \sqrt{u+1} - \sqrt{u-1} = \frac{2}{\sqrt{u+1} + \sqrt{u-1}}$$

is a decreasing function for $u \geq 1$, so

$$b - a = \sqrt{x+1} - \sqrt{x-1} + \sqrt{y+1} - \sqrt{y-1} \leq 2\sqrt{2} < 3\ .$$

Since $b - a$ is an integer greater than 1 we have $b - a = 2$. We now have

$$\sqrt{x-1} + \sqrt{y-1} = k \tag{1}$$
$$\sqrt{x+1} + \sqrt{y+1} = k + 2 \tag{2}$$

where $x \geq 1$, $y \geq 1$ and k is an integer. We shall now show that k must be 1 and $x = y = 5/4$.

Let $u = x - 1$ and $w = y - 1$, $u \geq 0$, $w \geq 0$. Clearly $0 \leq u \leq k^2$ and $0 \leq w \leq k^2$. Solving for w in (1) and substituting into (2) we obtain

$$(k+2)\sqrt{u+2} - k\sqrt{u} = 2(k+1)\ . \tag{3}$$

We seek a solution with $0 \leq u \leq k^2$. Consider the left hand side of (3). It is a continuous function of u on $0 \leq u \leq k^2$. The maximum and minimum will occur at $u = 0$ or $u = k^2$, or where the first derivative is zero. This occurs at

$$u = \frac{k^2}{2(k+1)}\ .$$

The three values obtained are $\sqrt{2}(k+2)$, $(k+2)\sqrt{k^2+2} - k^2$, and $2\sqrt{2}\sqrt{k+1}$, respectively. Routine calculations show that $2(k+1)$ is larger than any of these values for integers $k \geq 2$. Hence the only solution is with $k = 1$.

Setting $k = 1$, (3) becomes

$$3\sqrt{u+2} - \sqrt{u} = 4$$

or

$$9(u+2) = 16 + u + 8\sqrt{u}$$

or

$$0 = 4u - 4\sqrt{u} + 1 = (2\sqrt{u} - 1)^2 \ ,$$

giving $\sqrt{u} = 1/2$ and $u = 1/4$. Thus $x - 1 = 1/4$ and $x = 5/4$. Substituting into (1) gives $\sqrt{y-1} = 1/2$ and $y = 5/4$.

3. In a triangle ABC such that

$$\sin^{23}(\alpha/2)\cos^{48}(\beta/2) = \sin^{23}(\beta/2)\cos^{48}(\alpha/2)$$

where α and β are the angles with vertices at A and B respectively, compute the ratio AC/BC. (Cyprus)

Solution by M. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Consider the function

$$f(x) = \frac{x^{23}}{(1 - x^2)^{24}}$$

on (0,1). Clearly this is a strictly increasing function and therefore one–to–one. Since we are assuming we have a triangle (non–degenerate), $0 < \alpha/2$, $\beta/2 < \pi/2$ and so $0 < \sin\alpha/2$, $\sin\beta/2 < 1$. Letting $x = \sin\alpha/2$ and $y = \sin\beta/2$, the given equation becomes $f(x) = f(y)$. Since f is one–to–one, $x = y$ and $\sin\alpha/2 = \sin\beta/2$. Now on $(0,\pi/2)$, $\sin x$ is one–to–one, therefore $\alpha/2 = \beta/2$ or $\alpha = \beta$. Thus the triangle is isosceles and $AC/BC = 1$.

4. Two circles κ_1, κ_2 with centres O_1, O_2 and radii 1, $\sqrt{2}$, respectively, intersect in two points A and B. Also O_1O_2 has length 2. Let AC be a chord in κ_2. Find the length of AC if the midpoint of AC lies on κ_1. (Bulgaria)

Solution found on the blackboard in the Mathematics Lounge, University of Calgary.

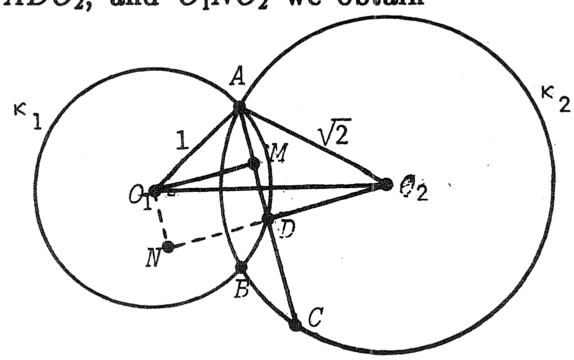
Let the midpoint of AC be D. Let M be the midpoint of AD. Now $\angle ADO_2 = \angle AMO_1 = 90^\circ$ since D and M are midpoints of chords of κ_2 and κ_1 respectively. Let the foot of the perpendicular from O_1 to O_2D (extended) be N. Let $x = \overline{AM}$, $y = \overline{O_2D}$ and $z = \overline{ND}$. Now O_1MDN is a rectangle. Applying Pythagoras' Theorem in right triangles O_1MA, ADO_2, and O_1NO_2 we obtain

$$z^2 + x^2 = 1 \ , \qquad (1)$$

$$(2x)^2 + y^2 = 2 \ , \qquad (2)$$

and

$$x^2 + (y + z)^2 = 4 \ . \qquad (3)$$



From (1) and (2),

$$4z^2 - 2 = y^2 \ . \qquad (4)$$

Using (1) and (4) in (3) we obtain

$$4z^2 - 2 + 2yz + z^2 + 1 - z^2 = 4$$

or

$$y = \frac{5 - 4z^2}{2z} . \qquad (5)$$

Now using (5) and (1) in (2),

$$4(1 - z^2) + \left(\frac{5 - 4z^2}{2z}\right)^2 = 2 ,$$

$$16z^2 - 16z^4 + 25 - 40z^2 + 16z^4 = 8z^2 ,$$

and so $z^2 = 25/32$. From (1), $x = \sqrt{7/32}$. Now $\overline{AC} = 4x = \sqrt{7/2}$.

* * *

This completes the solutions that we have on file for the 1988 numbers of the Corner. Send me your national and regional Olympiads as well as your nice solutions to problems in the Corner.

* * *

BOOK REVIEW

An Olympiad Down Under. A Report on the 29th International Mathematical Olympiad in Australia. Edited by W.P. Galvin, D.C. Hunt and P.J. O'Halloran. Australian Mathematics Foundation Ltd., Belconnen, Australia, 1988. ISBN 0–7316–5118–9. Softcover, iii + 247 pp. *Reviewed by R.E. Woodrow, University of Calgary.*

This little book should be essential reading for anyone contemplating organizing an international mathematics contest successfully. It also makes worthwhile reading for those of us who have an interest in the I.M.O. but have not been able to witness one in action. Not only are there lists of committees, rules, regulations, and statistics, but there are plenty of photographs of team leaders and contestants at work and at play. One highlight of the book is the listing of the ninety or so problems proposed by the participating nations (and a chapter where they are solved!). The six problems used in the competition [1988: 196], and the 24 other problems proposed to the jury [1988: 225, 257], all of which have been printed in *Crux*, are of course among them. Another highlight is Appendix D: the informal relay event. This was run for the fun of it after the main contest, and the questions and answers are given as well.

PROBLEMS

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **January 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1551. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find a triangle ABC with a point D on AB such that the lengths of AB, BC, CA and CD are all integers and $AD : DB = 9 : 7$, or prove that no such triangle exists.

1552*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For each integer $n \geq 2$ let

$$x_n = \left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)^n + \cdots + \left(\frac{n-1}{n}\right)^n.$$

Does the sequence $\{x_n/n\}$, $n = 2,3,\ldots$, converge?

1553. *Proposed by Murray S. Klamkin, University of Alberta.*

It has been shown by Oppenheim that if $ABCD$ is a tetrahedron of circumradius R, a, b, c are the edges of face ABC, and p, q, r are the edges AD, BD, CD, then

$$64R^4 \geq (a^2 + b^2 + c^2)(p^2 + q^2 + r^2).$$

Show more generally that, for n–dimensional simplexes,

$$(n + 1)^4R^4 \geq 4E_0E_1,$$

where E_0 is the sum of the squares of all the edges emanating from one of the vertices and E_1 is the sum of the squares of all the other edges.

1554. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Describe all finite sets $\mathscr{S}$ in the plane with the following property: if two straight lines, each of them passing through at least two points of $\mathscr{S}$, intersect in P, then P belongs to $\mathscr{S}$.

1555. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is an isosceles trapezoid, with $AD \| BC$, whose circumcircle has center O. Let $PQRS$ be a rhombus whose vertices P, Q, R, S lie on AB, BC, CD, DA respectively. Prove that Q, S and O are collinear.

1556. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Let λ and n be fixed positive integers, not both 1. Prove that the equation

$$\frac{x^2 + y^2}{\lambda xy + 1} = n^2$$

has infinitely many natural number solutions (x,y).

1557. *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

Let n be a positive integer and let $\mathcal{P}_n$ be the set of ordered pairs (a,b) of integers such that $1 \leq a \leq b \leq n$. If $f : \{1,2,\ldots,n\} \to \mathbb{R}$ is an increasing function, and $g : \mathcal{P}_n \to \mathbb{R}$ defined by

$$g(a,b) = f(a) + f(b)$$

is one–to–one, then g defines a (strict) total ordering $\prec$ on $\mathcal{P}_n$ by

$$(a,b) \prec (c,d) \quad \text{if and only if} \quad g(a,b) < g(c,d) \;.$$

Moreover $\prec$ will have the property

$$(a,b) \prec (c,d) \quad \text{whenever } a \leq c \text{ and } b \leq d \text{ (and } (a,b) \neq (c,d)) \;. \qquad (*)$$

Does every strict total ordering $\prec$ of $\mathcal{P}_n$ which satisfies (*) arise in this way?

1558. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let P be an interior point of a triangle ABC and let AP, BP, CP intersect the circumcircle of ΔABC again in A', B', C', respectively. Prove that the power p of P with respect to the circumcircle satisfies

$$|p| \geq 4rr',$$

where r, r' are the inradii of triangles ABC and $A'B'C'$.

1559. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

Find a necessary and sufficient condition on reals c and d for the roots of $x^3 + 3x^2 + cx + d = 0$ to be in arithmetic progression.

1560. *Proposed by Ilia Blaskov, Technical University, Gabrovo, Bulgaria.*

The sequence $a_2, a_3, a_4, \cdots$ of real numbers is such that, for each n, $a_n > 1$ and the equation $[a_n x] = x$ has just n different solutions. ($[x]$ denotes the greatest integer $\leq x$.) Find $\lim_{n\to\infty} a_n$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1303. [1988: 12, 1989: 26] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC and $A_1B_1C_1$ be two triangles with sides a, b, c and a_1, b_1, c_1 and inradii r and r_1, and let P be an interior point of ΔABC. Set $AP = x$, $BP = y$, $CP = z$. Prove that

$$\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq 4rr_1 .$$

III. *Further comment by Murray S. Klamkin, University of Alberta.*

A simpler proof of inequality (10) in my solution [1989: 28], i.e.

$$\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{4F}{R_1}, \tag{10}$$

where R_1 is the circumradius of $\Delta A_1B_1C_1$, is as follows. By Cauchy's inequality, we have

$$\sum_{i=1}^{n} \frac{x_i^2}{u_i} \cdot \sum_{i=1}^{n} u_i \geq \left(\sum_{i=1}^{n} x_i\right)^2, \tag{1}$$

where the x_i's and u_i's are nonnegative, with equality if and only if x_i/u_i is constant. As a special case of (1) we have

$$\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{(a + b + c)^2}{a_1 + b_1 + c_1} = \frac{2s^2}{s_1} . \tag{2}$$

Inequality (10) will now follow immediately from

$$\frac{s^2}{F} \geq \frac{2s_1}{R_1} .$$

Here the minimum value of s^2/F equals the maximum value of $2s_1/R_1$ [both equal to $3\sqrt{3}$ – see items 4.2 and 5.3 of Bottema et al, *Geometric Inequalities*], and these are taken on only for equilateral triangles.

As mentioned in my earlier solution, Tsintsifas has also shown that if $\Delta A_1B_1C_1$ is inscribed in ΔABC, then

$$\frac{a^2}{a_1} + \frac{b^2}{b_1} + \frac{c^2}{c_1} \geq \frac{4F^2}{a_1b_1c_1} . \tag{3}$$

It is easy to show that inequalities (2) and (3) are incomparable, i.e., equivalently neither $r^2 \geq 2r_1R_1$ nor $2r_1R_1 \geq r^2$ is valid. This leads to the following open problems:

For a given triangle T, determine the maximum inradius and the minimum circumradius of a triangle (a) *inscribed in T,* (b) *circumscribed about T.*

Finally, some other special cases of (1) which are similar to (2) are

$$\frac{(s-a)^2}{a_1} + \frac{(s-b)^2}{b_1} + \frac{(s-c)^2}{c_1} \ge \frac{s^2}{2s_1},$$

$$\frac{a^2}{s_1 - a_1} + \frac{b^2}{s_1 - b_1} + \frac{c^2}{s_1 - c_1} \ge \frac{4s^2}{s_1},$$

$$\frac{(s-a)^2}{s_1 - a_1} + \frac{(s-b)^2}{s_1 - b_1} + \frac{(s-c)^2}{s_1 - c_1} \ge \frac{s^2}{s_1},$$

$$\frac{a^2}{a_1^2} + \frac{b^2}{b_1^2} + \frac{c^2}{c_1^2} \ge \frac{(a+b+c)^2}{a_1^2 + b_1^2 + c_1^2}.$$

* * *

1408. [1989: 13; 1990: 87] *Proposed by Jordi Dou, Barcelona, Spain.*

Given the equilateral triangle ABC, find all positive real numbers r for which there is a point $P(r)$ such that

$$\frac{PA}{1} = \frac{PB}{r} = \frac{PC}{r^2},$$

and describe the locus of $P(r)$.

II. *Editor's comment.*

The editor should have included some mention of the proposer's beautiful results concerning this problem, which are illustrated in his awesome drawing of the locus [1990: 88]. Let Ω be the circumcircle of ΔABC, and O the circumcentre. Let L be the midpoint of OB and H the midpoint of OL (i.e. the midpoint of the altitude from B). Lines HH' and LL' are parallel to AC. Circle Γ_0 has centre B and the same radius as Ω, and so passes through O. Finally let I be the inversion with centre O which fixes Ω. Then:

(i) *the locus Λ of $P(r)$ is fixed by I;*

(ii) *Γ_0 is the osculating circle of Λ at O;*

(iii) *LL' is the asymptote of Λ and is the image of Γ_0 under I;*

(iv) *letting ω be any circle with centre on line HH' and which is orthogonal to Ω, the endpoints of any diameter of ω which (extended) passes through O will lie on Λ.*

The proposer's original problem included the instruction to "construct the asymptote of the locus of $P(r)$ and the osculating circle at O". This was simplified down to "describe the locus" by the well-meaning editor.

1431. [1989: 109] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Show that, for prime $p \equiv 1$, 3 or 5 mod 8,

$$\sum_{k=1}^{(p-1)/2} \tan \frac{k^2\pi}{p} = 3\cdot \sum_{k=1}^{(p-1)/2} \cot \frac{k^2\pi}{p} . \tag{1}$$

(b)* Prove that (1) fails for all primes $p \equiv 7$ mod 8.

I. *Solution by K.S. Williams, Carleton University.*

Let p denote an odd prime. We prove after a number of lemmas the following theorem.

THEOREM. (i) If $p \equiv 1$ mod 4 then

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = 0 .$$

(ii) If $p \equiv 3$ mod 4 and $p > 3$ then

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \left[1 - 2\left(\frac{2}{p}\right)\right]\sqrt{p}\cdot h(-p) , \qquad \sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \sqrt{p}\cdot h(-p) .$$

Here $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol of quadratic residuacity mod p. We remark that $h(-p)$ is a positive integer, and that

$$\left(\frac{2}{p}\right) = \begin{cases} +1 & \text{if } p \equiv 1 \text{ or } 7 \bmod 8 , \\ -1 & \text{if } p \equiv 3 \text{ or } 5 \bmod 8 . \end{cases}$$

Thus when $p \equiv 3$ mod 8 and $p > 3$ we have, by (ii),

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = 3\sqrt{p}\cdot h(-p) , \qquad \sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \sqrt{p}\cdot h(-p) ,$$

so that (1) holds in this case. Also, (1) is easily verified directly for $p = 3$. When $p \equiv 7$ mod 8 we have

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = -\sqrt{p}\cdot h(-p) , \qquad \sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \sqrt{p}\cdot h(-p) ,$$

so that (1) *fails*, as required in part (b). Of course, if $p \equiv 1$ or 5 mod 8 then (1) holds since, by (i), both sides are zero.

LEMMA 1. $$\sum_{k=1}^{p-1} \tan\frac{k\pi}{p} = 0 = \sum_{k=1}^{p-1} \cot\frac{k\pi}{p} .$$

Proof. The polynomial $(1 + iz)^p - (1 - iz)^p$ is of degree p, its coefficient of z^{p-1} is 0, and it has the p distinct roots $\tan(k\pi/p)$ $(k = 0,1,\cdots,p - 1)$. Hence we have

$$\sum_{k=0}^{p-1} \tan\frac{k\pi}{p} = 0 .$$

Similarly, the polynomial $(z + i)^p - (z - i)^p$ is of degree $p - 1$, its coefficient of z^{p-2} is 0, and it has the $p - 1$ distinct roots $\cot(k\pi/p)$ $(k = 1,2,\cdots,p - 1)$. Hence we have

$$\sum_{k=1}^{p-1} \cot\frac{k\pi}{p} = 0 . \qquad \square$$

LEMMA 2. (a) $$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \frac{1}{2}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right)\tan\frac{k\pi}{p} .$$

(b) $$\sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \frac{1}{2}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right)\cot\frac{k\pi}{p} .$$

Proof. (a) As

$$\left\{1^2,2^2,\cdots,\left(\frac{p-1}{2}\right)^2\right\}$$

constitute the $(p - 1)/2$ quadratic residues mod p, and $\tan(k\pi/p)$ is periodic in k with period p, we have

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \sum \tan\frac{k\pi}{p}, \quad \text{the second sum over all } 1 \le k \le p - 1, \ \left(\frac{k}{p}\right) = 1$$

$$= \frac{1}{2}\sum_{k=1}^{p-1} \tan\frac{k\pi}{p} + \frac{1}{2}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right)\tan\frac{k\pi}{p} = \frac{1}{2}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right)\tan\frac{k\pi}{p}$$

by Lemma 1.

(b) Similar to (a). $\square$

LEMMA 3. For $k = 1,2,\cdots,p - 1$ we have

(a) $$\tan\frac{k\pi}{p} = -\frac{2}{p}\sum_{n=1}^{p-1} n(-1)^n\sin\frac{2kn\pi}{p} ,$$

(b) $$\cot\frac{k\pi}{p} = -\frac{2}{p}\sum_{n=1}^{p-1} n\sin\frac{2kn\pi}{p}.$$

Proof. (a) Taking $z = e^{2k\pi i/p}$ in the identity

$$\sum_{n=1}^{p-1} n(-1)^n z^n = \frac{pz^p}{z+1} - \frac{z(z^p+1)}{(z+1)^2} \qquad (z \neq -1),$$

we obtain

$$\begin{aligned}\sum_{n=1}^{p-1} n(-1)^n e^{2kn\pi i/p} &= \frac{p}{e^{2k\pi i/p}+1} - \frac{2e^{2k\pi i/p}}{(e^{2k\pi i/p}+1)^2}\\ &= \frac{p[\cos(k\pi/p) - i\sin(k\pi/p)]}{2\cos(k\pi/p)} - \frac{1}{2\cos^2(k\pi/p)}.\end{aligned}$$

Equating imaginary parts, we deduce

$$\sum_{n=1}^{p-1} n(-1)^n \sin\frac{2kn\pi}{p} = -\frac{p}{2}\tan\frac{k\pi}{p}.$$

(b) This is done similarly by taking $z = e^{2k\pi i/p}$ in the identity

$$\sum_{n=1}^{p-1} nz^n = \frac{pz^p}{z-1} - \frac{z(z^p-1)}{(z-1)^2} \qquad (z \neq 1)$$

and equating imaginary parts (see Lemma 2 of [5]). □

LEMMA 4. For $k = 1,2,\cdots,p-1$ we have

$$\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\sin\frac{2kn\pi}{p} = \begin{cases} 0 & \text{if } p \equiv 1 \bmod 4, \\ \left(\frac{k}{p}\right)\sqrt{p} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof. This follows by equating imaginary parts in the famous result of Gauss ([3], Theorem 212 of [4]):

$$\sum_{n=1}^{p-1}\left(\frac{n}{p}\right) e^{2kn\pi i/p} = \left(\frac{k}{p}\right) i^{(p-1)^2/4}\sqrt{p}.$$

Compare also to Lemma 3 of [5]. □

LEMMA 5. (a) $$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\tan\frac{k\pi}{p} = \begin{cases} 0 & \text{if } p \equiv 1 \bmod 4, \\ -\frac{2}{\sqrt{p}}\sum_{n=1}^{p-1}(-1)^n n\left(\frac{n}{p}\right) & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(b) $$\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\cot\frac{k\pi}{p} = \begin{cases} 0 & \text{if } p \equiv 1 \bmod 4\ , \\ -\dfrac{2}{\sqrt{p}}\displaystyle\sum_{n=1}^{p-1} n\left(\frac{n}{p}\right) & \text{if } p \equiv 3 \bmod 4\ . \end{cases}$$

Proof. (a) By Lemmas 3 and 4 we have

$$\begin{aligned}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\tan\frac{k\pi}{p} &= -\frac{2}{p}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\sum_{n=1}^{p-1} n(-1)^n\sin\frac{2kn\pi}{p} \\ &= -\frac{2}{p}\sum_{n=1}^{p-1} n(-1)^n\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\sin\frac{2kn\pi}{p} \\ &= \begin{cases} 0 & \text{if } p \equiv 1 \bmod 4\ , \\ -\dfrac{2}{\sqrt{p}}\displaystyle\sum_{n=1}^{p-1}(-1)^n n\left(\frac{n}{p}\right) & \text{if } p \equiv 3 \bmod 4\ . \end{cases}\end{aligned}$$

(b) Similar to (a). □

LEMMA 6. If $p \equiv 3 \bmod 4$ we have

$$\sum_{n=1}^{p-1} n\left(\frac{n}{p}\right) = -\frac{p}{3}\left[2+\left(\frac{2}{p}\right)\right]\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right),$$

$$\sum_{n=1}^{(p-1)/2} n\left(\frac{n}{p}\right) = \frac{p}{6}\left[1-\left(\frac{2}{p}\right)\right]\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right).$$

Proof. See proof of Lemma 1 in [5]. □

LEMMA 7. If $p \equiv 3 \bmod 4$ we have

$$\sum_{n=1}^{p-1}(-1)^n n\left(\frac{n}{p}\right) = p\left(\frac{2}{p}\right)\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right).$$

Proof. By Lemma 6 and the properties of quadratic residues,

$$\begin{aligned}\sum_{n=1}^{p-1}(-1)^n n\left(\frac{n}{p}\right) &= \sum_{\substack{n=1 \\ n \text{ even}}}^{p-1} n\left(\frac{n}{p}\right) - \sum_{\substack{n=1 \\ n \text{ odd}}}^{p-1} n\left(\frac{n}{p}\right) = 2\sum_{\substack{n=1 \\ n \text{ even}}}^{p-1} n\left(\frac{n}{p}\right) - \sum_{n=1}^{p-1} n\left(\frac{n}{p}\right) \\ &= 4\left(\frac{2}{p}\right)\sum_{n=1}^{(p-1)/2} n\left(\frac{n}{p}\right) - \sum_{n=1}^{p-1} n\left(\frac{n}{p}\right)\end{aligned}$$

$$= \frac{2p}{3}\left(\frac{2}{p}\right)\left[1 - \left(\frac{2}{p}\right)\right] \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) + \frac{p}{3}\left[2 + \left(\frac{2}{p}\right)\right] \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right)$$

$$= p\left(\frac{2}{p}\right) \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) . \qquad \square$$

LEMMA 8. If $p \equiv 3 \bmod 4$ and $p > 3$ then

$$\left[2 - \left(\frac{2}{p}\right)\right]^{-1} \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) = h(-p) ,$$

where $h(-p)$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Proof. This is a famous result proved by Dirichlet [2] in 1839. See for example Theorem 3, page 346 of [1], or p. 168 of [4]. $\square$

Proof of Theorem. If $p \equiv 1 \bmod 4$ we have, by Lemmas 2 and 5,

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \frac{1}{2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \tan\frac{k\pi}{p} = 0 ,$$

$$\sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \frac{1}{2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot\frac{k\pi}{p} = 0 .$$

If $p \equiv 3 \bmod 4$ and $p > 3$ we have, by Lemmas 2, 5, 6, 7 and 8,

$$\sum_{k=1}^{(p-1)/2} \tan\frac{k^2\pi}{p} = \frac{1}{2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \tan\frac{k\pi}{p} = -\frac{1}{\sqrt{p}} \sum_{n=1}^{p-1} (-1)^n n \left(\frac{n}{p}\right)$$

$$= -\frac{1}{\sqrt{p}}\left(\frac{2}{p}\right) p \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) = \left[1 - 2\left(\frac{2}{p}\right)\right] \sqrt{p} \cdot h(-p)$$

and

$$\sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p} = \frac{1}{2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot\frac{k\pi}{p} = -\frac{1}{\sqrt{p}} \sum_{n=1}^{p-1} n \left(\frac{n}{p}\right)$$

$$= -\frac{1}{\sqrt{p}} \cdot \frac{-p}{3}\left[2 + \left(\frac{2}{p}\right)\right] \sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) = \sqrt{p} \cdot h(-p) .$$

This completes the proof of the theorem. $\square$

References:

[1] Z.I. Borevich and I.R. Shafarevich, *Number Theory*, Academic Press, 1966.

[2] P.G.L. Dirichlet, Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres, *Journal für die reine und angewandte Mathematik* 19 (1839) 324–369.
[3] C.F. Gauss, *Werke* (Vol. 2), Gottingen (1876) 9–45.
[4] Edmund Landau, *Elementary Number Theory*, Chelsea, 1966.
[5] A.L. Whiteman, Theorems on quadratic residues, *Math. Magazine* 23 (1949/50) 71–74.

II. *Solution to (a) by the proposer.*

Via

$$\cot 2x = \frac{\cot x - \tan x}{2},$$

(1) becomes

$$\sum_{k=1}^{(p-1)/2} \cot\frac{2k^2\pi}{p} = -\sum_{k=1}^{(p-1)/2} \cot\frac{k^2\pi}{p}. \qquad (2)$$

We note $\cot(\pi - x) = -\cot x$.

If -2 is a quadratic residue mod p, then for $1 \le j \le (p-1)/2$ there exists $1 \le k \le (p-1)/2$ such that $k^2 \equiv -2j^2 \bmod p$. (It's always possible to get $1 \le k \le (p-1)/2$ as the congruence is satisfied by k and $p - k$ as well.) Obviously $j_1 \ne j_2 \iff k_1 \ne k_2$. Thus from

$$\cot\frac{2j^2\pi}{p} = -\cot\frac{k^2\pi}{p}$$

(2) follows. Now

$$\left(-\frac{2}{p}\right) = \left(-\frac{1}{p}\right)\left(\frac{2}{p}\right) = (-1)^{(p-1)/2}\cdot(-1)^{(p^2-1)/8} = 1$$

if $p \equiv 1$ or $3 \bmod 8$, so (2) holds in these cases.

Let $p \equiv 5 \bmod 8$. Then $\left(-\frac{1}{p}\right) = 1$. Thus for each $1 \le j \le (p-1)/2$ the congruence $k^2 \equiv -j^2 \bmod p$ is solvable with $1 \le k \le (p-1)/2$, $k \ne j$. Consequently both sides of (2) vanish.

* * *

<u>1432</u>. [1989: 110] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

If the Nagel point of a triangle lies on the incircle, prove that the sum of two of the sides of the triangle equals three times the third side.

I. *Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany.*

Let I be the incenter, G the centroid and N the Nagel point of the triangle ABC. Also let a, b, c, s, r, R be the sides, semiperimeter, inradius, and circumradius. In [1978: 59] we read

$$\overline{IG}^2 = \frac{1}{9}(s^2 - 16Rr + 5r^2)$$

and in [1] $\overline{IN} = 3\overline{IG}$, so that

$$\overline{IN}^2 = s^2 - 16Rr + 5r^2. \tag{1}$$

Furthermore it is known that

$$abc = 4Rrs\ , \qquad ab + ac + bc = s^2 + 4Rr + r^2\ .$$

Thus

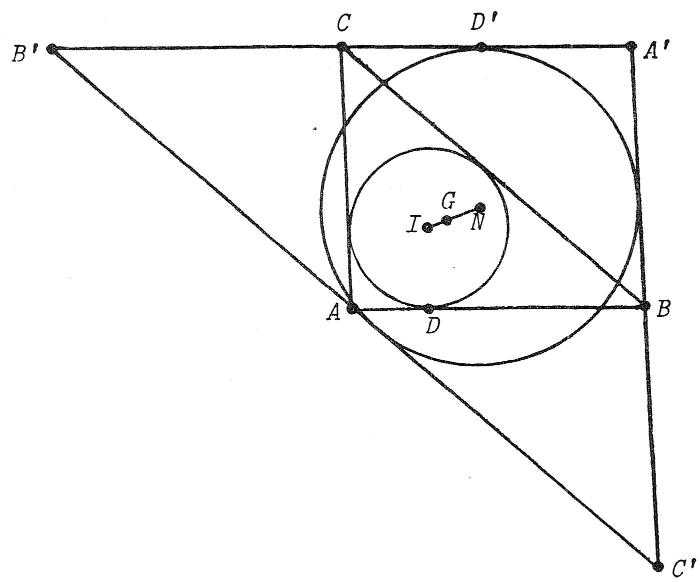
$$\begin{aligned}
\overline{IN} = r \quad (N \text{ lies on the incircle}) &\iff s^2 - 16Rr + 4r^2 = 0 \\
&\iff s^3 + (4s^3 + 16Rrs + 4r^2s) - 4s^3 - 32Rrs = 0 \\
&\iff s^3 + 4s(ab + ac + bc) - (2a + 2b + 2c)s^2 - 8abc = 0 \\
&\iff (s - 2a)(s - 2b)(s - 2c) = 0 \\
&\iff s = 2a \quad \text{or} \quad 2b \quad \text{or} \quad 2c \\
&\iff b + c = 3a \quad \text{or} \quad a + c = 3b \quad \text{or} \quad a + b = 3c.
\end{aligned}$$

Reference:

[1] P. Baptist, Nagelpunkte und Eulersche Geraden, *Didaktik der Mathematik* 1982, 114–122.

[*Editor's note.* Some solvers gave p. 280 of D.M. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989 as a reference for equation (1).]

II. (*Partial?*) *Solution by L.J. Hut, Groningen, The Netherlands.*

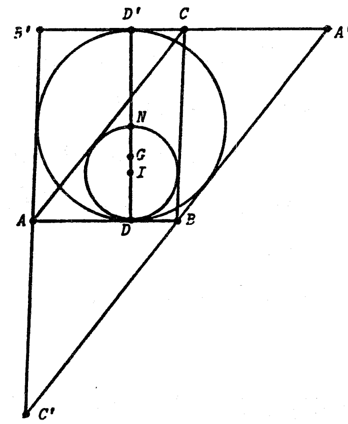


I, G and N are collinear, with $IG{:}GN = 1{:}2$. Let $A'B'C'$ be the triangle formed by drawing lines through A, B, C parallel to the opposite sides of ΔABC. Then A, G and A' are collinear with $2AG = GA'$, and similarly for B and C; that is, ABC and $A'B'C'$ are homothetic with center G and ratio -2. This transformation thus sends I to N, so N is the center of the incircle of $\Delta A'B'C'$.

If N lies on the incircle of ΔABC, then (since the inradius of $\Delta A'B'C'$ is twice the inradius of ΔABC) the two incircles will touch each other. Letting D, D'

be the points where the two incircles touch AB, $A'B'$, respectively, we have $DG = 2GD'$, $ID \perp AB$, $ND' \perp A'B'$.

[*Editor's note.* Here Hut makes the claim that the incircles *must touch each other at D*, so that the points $DIGND'$ form a straight line perpendicular to AB and $A'B'$. *The editor cannot verify this claim*, although it seems to be upheld in diagrams he has constructed. (We ignore the further problem of how edge AB is selected over the other two edges of ΔABC.) Can Hut or any other reader fill in the details?]



Now, letting r be the inradius, s the semiperimeter, and Δ the area of ΔABC,

$$DD' = 4r = \frac{4\Delta}{s} = \frac{2DD' \cdot c}{s},$$

so that

$$a + b + c = 2s = 4c$$

or

$$a + b = 3c .$$

III. *Comment by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

Reading old issues of *Mathematics Magazine* I have seen in Vol. 14, February 1940, p. 281 the problem 259, proposed by Walter B. Clarke and solved by C.W. Trigg: "Construct a triangle whose verbicenter lies on its incircle." This verbicenter (curious name!) is precisely the point of Nagel of the triangle, and in the published solution (which includes another proof of *Crux* 1432) two other related problems are cited: problem 121, October 1936, p. 56 and problem 172, February 1938, p. 249 (in the same journal, of course). The construction is the following: "Let $\underline{a}$ be the base of the triangle, and with a radius b such that $a < b < 2a$ and one extremity of $\underline{a}$ as center describe a circle. With the other extremity as center and radius equal to $3a - b$ describe another circle. The intersections of the circles determine the alternate third vertices of the required triangle. It will be noted that the locus of A for a fixed $\underline{a}$ is an ellipse with semi-axes $3a/2$ and $a\sqrt{2}$."

Also solved by HARRY ALEXIEV, Zlatograd, Bulgaria; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

As Solution I shows, the converse of the problem holds as well. This was also pointed out by Shirali and Sokolowsky.

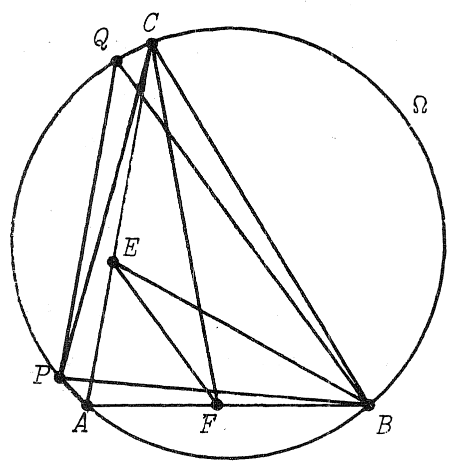
The problem was due to the proposer and appeared in the Dutch book Hoofdstukken uit de Elementaire Meetkunde (*second edition, 1987*) *of O. Bottema.*

* * *

1433. [1989: 110] *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let ABC be a triangle with sides $a > b > c$ and circumcircle Ω. Its internal angle–bisectors meet BC, CA and AB at D, E and F respectively. The line through B parallel to EF meets Ω again at Q, and P is on Ω such that $QP \| AC$. Prove that $\overline{PC} = \overline{PA} + \overline{PB}$.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany.



Since BE and CF are angle–bisectors,

$$\overline{AE} = \frac{bc}{a+c}, \quad \overline{AF} = \frac{bc}{a+b}.$$

Also, $\angle FAE = \angle BPC$ and, since $\overline{AP} = \overline{QC}$, $\angle ABP = \angle QBC$ and hence

$$\angle AFE = \angle ABQ = \angle PBC.$$

Thus ΔAFE is similar to ΔPBC, so

$$\overline{PB} : \overline{PC} = \overline{AF} : \overline{AE} = (a + c):(a + b),$$

or

$$b\overline{PB} = a\overline{PC} + c\overline{PC} - a\overline{PB}. \qquad (1)$$

From the theorem of Ptolemäus follows, regarding the quadrilateral $ABCP$,

$$b\overline{PB} = a\overline{PA} + c\overline{PC}. \qquad (2)$$

Subtracting (1) and (2) one gets $\overline{PA} + \overline{PB} = \overline{PC}$.

Also solved by P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. Sokolowsky's solution was the same as Engelhaupt's above. The others were only slightly different.

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1434. [1989: 110] *Proposed by Harvey Abbott and Murray S. Klamkin, University of Alberta.*

It is known that

$$\frac{(3m)!(3n)!}{m!n!(m+n)!(n+m)!}, \quad \frac{(4m)!(4n)!}{m!n!(2m+n)!(2n+m)!}, \quad \frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

are all integers for positive integers m, n.

(i) Find positive integers m, n such that

$$I(m,n) = \frac{(6m)!(6n)!}{m!n!(4m+n)!(4n+m)!}$$

is *not* an integer.

(ii) Let A be the set of pairs (m,n), with $n \le m$, for which $I(m,n)$ is not an integer, and let $A(x)$ be the number of pairs in A satisfying $1 \le n \le m \le x$. Show that A has positive lower density in the sense that

$$\liminf_{x \to \infty} \frac{A(x)}{x^2} > 0 .$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

The clue to (ii) is the observation that the set

$$Q = \{(x,y) : 0 \le y \le x \le 1 \text{ , } [6x] + [6y] < [4x+y] + [4y+x]\} \tag{1}$$

is nonempty (here $[x]$ is the greatest integer $\le x$); namely,

$$Q = \{(x,y) : 4x + y \ge 2 \text{ , } 4y + x \ge 1 \text{ , } x < 1/2 \text{ , } y < 1/6\} \tag{2}$$

is the quadrangle with vertices

$$\left(\frac{1}{2},\frac{1}{6}\right) , \left(\frac{11}{24},\frac{1}{6}\right) , \left(\frac{7}{15},\frac{2}{15}\right) , \left(\frac{1}{2},\frac{1}{8}\right) .$$

(There is no need to verify that the sets (1) and (2) are equal; it suffices for the sequel to take (2) as the definition of Q.) What follows is routine. Choose a prime $p > 2$ and suppose (m,n) is a lattice point in

$$pQ = \{(px,py) : (x,y) \in Q\}.$$

By (2),

$$n < m \text{ , } \frac{6m}{p} < 3 \text{ , } \frac{6n}{p} < 1 \text{ , } \frac{4m+n}{p} \ge 2 \text{ , } \frac{4n+m}{p} \ge 1 . \tag{3}$$

Thus p enters the denominator of $I(m,n)$ with the exponent

$$\left[\frac{m}{p}\right] + \left[\frac{n}{p}\right] + \left[\frac{4m+n}{p}\right] + \left[\frac{4n+m}{p}\right] \ge 3$$

and numerator with the exponent

$$\left[\frac{6m}{p}\right] + \left[\frac{6n}{p}\right] \le 2$$

(there are no higher order terms, as p^2 exceeds $6m$). Consequently

$$\bigcup_{p \text{ prime}} (\mathbb{Z}^2 \cap pQ) \subseteq A .$$

The claim hence follows immediately: denoting by $p(x)$ the greatest prime $\le x$ we have asymptotically $p(x) \approx x$ and so

$$\frac{A(x)}{x^2} \ge \frac{A(p(x))}{x^2} \ge \frac{|\mathbb{Z}^2 \cap p(x)Q|}{x^2} \approx \text{Area } Q \left(= \frac{1}{720}\right)$$

as $x \to \infty$.

An example of (i) would be $m = 11$, $n = 3$, which satisfy (3) for $p = 23$.

Also solved by the proposers. Part (i) only solved by RICHARD I. HESS, Rancho Palos Verdes, California; and SAM MALTBY, student, Calgary.

The proposers ask whether $\lim_{x\to\infty}(A(x)/x^2)$ *exists.*

The problem of showing that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral occurred in the 1975 *U.S.A. Mathematical Olympiad (and was suggested by Klamkin).*

* * *

1435. [1989: 110] *Proposed by J.B. Romero Marquez, Universidad de Valladolid, Valladolid, Spain.*

Find all pairs of integers x, y such that

$$(xy - 1)^2 = (x + 1)^2 + (y + 1)^2.$$

Solution by C. Festraets–Hamoir, Brussels, Belgium.

Posons

$$X = x + 1\ , \qquad Y = y + 1\ .$$

L'équation s'écrit alors

$$(XY - X - Y)^2 = X^2 + Y^2,$$

$$XY(XY - 2X - 2Y + 2) = 0\ .$$

(1) $\quad X = 0 \Longrightarrow \boxed{x = -1,\ y \in \mathbb{Z}}\ .$

(2) $\quad Y = 0 \Longrightarrow \boxed{y = -1,\ x \in \mathbb{Z}}\ .$

(3) $\quad XY - 2X - 2Y + 2 = 0 \Longleftrightarrow Y = \dfrac{2X-2}{X-2} = 2 + \dfrac{2}{X-2}\ .$

$$Y \in \mathbb{Z} \Longleftrightarrow X - 2 = \pm 1 \quad \text{ou} \quad X - 2 = \pm 2$$

$$\Longleftrightarrow X = 3 \quad \text{ou} \quad X = 1 \quad \text{ou} \quad X = 4 \quad \text{ou} \quad X = 0\ .$$

$X = 3$ donne $Y = 4$; $\boxed{x = 2 \ \text{ et } \ y = 3}$.

$X = 1$ donne $Y = 0$; voir (2) .

$X = 4$ donne $Y = 3$; $\boxed{x = 3 \ \text{ et } \ y = 2}$.

$X = 0$; voir (1) .

Also solved by HAYO AHLBURG, Benidorm, Spain; SEUNG–JIN BANG, Seoul, Republic of Korea; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, *Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; GUO-GANG GAO, student, Universite de Montreal; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; LARRY HOEHN, Austin Peay State University, Clarksville, Tennessee; JILL HOUGHTON, Sydney, Australia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; S.J. MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin-Oshkosh; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; W.R. UTZ, University of Missouri, Columbia; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University; and the proposer. There was also one anonymous solver. Two other readers gave partial solutions.*

* * *

1436. [1989: 110] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

A point P lies on the circumcircle Γ of a triangle ABC, P not coinciding with one of the vertices. Circles Γ_1 and Γ_2 pass through P and are tangent to AB at B, and to AC at C, respectively. Γ_1 and Γ_2 intersect at P and at Q.

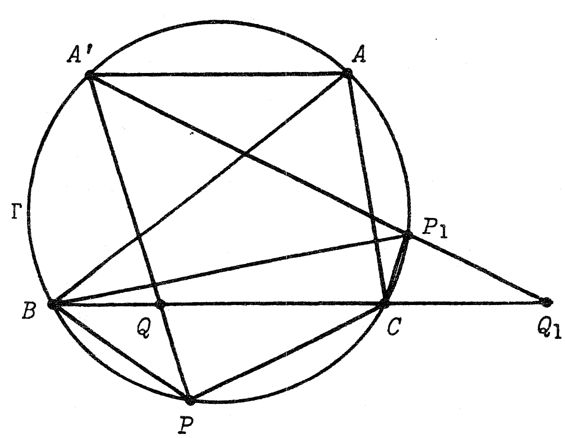
(a) Show that Q lies on the line BC.

(b) Show that as P varies over Γ the line PQ passes through a fixed point on Γ.

Solution by Hidetosi Fukagawa, Aichi, Japan.

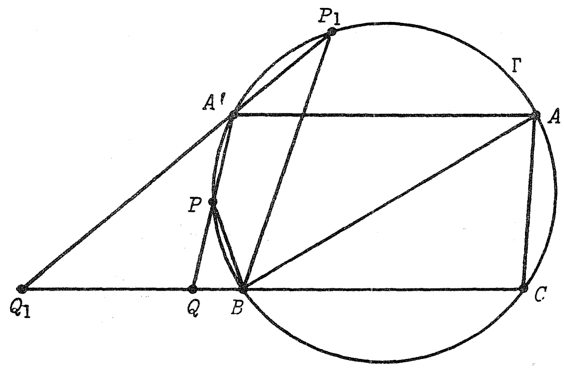
Let A' be the point on Γ such that $AA' \| BC$. This point A' is the desired point in (b). For any point P on Γ [two examples are given in the figure], let $A'P$ meet BC in Q. Then, since

$$\angle BPA' = \angle A'AB = \angle ABQ \ , \qquad (1)$$



the circle BPQ is tangent to AB at B and so must be Γ_1. Likewise, the circle CPQ is Γ_2. In conclusion, Γ_1 and Γ_2 intersect in the point Q on BC. This shows (a).

[*Editor's comment.* This solution shares with perhaps all others received the inconvenience of not applying to every case. However nearly all solvers realized this, some making comments to the effect that other cases are similar. Fukagawa's solution was the shortest and also adapts itself well to handling the remaining locations of P. There are two, represented in the figure by P and P_1. Here, equation (1) above should be replaced with



$$\angle BPA' = 180^\circ - \angle A'AB = \angle ABQ$$

and

$$\angle BP_1A' = \angle A'AB = 180^\circ - \angle ABQ_1$$

respectively, and in both cases the proof is otherwise the same.]

Also solved by JORDI DOU, Barcelona, Spain; JILL HOUGHTON, Sydney, Australia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH A. SHIRALI, Rishi Valley School, India; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer, who also credits J.T. Groenman with a solution.

The proposer found the problem in Journal de Mathématiques Elémentaires, *Paris* 1920, *no.* 9086.

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<u>**1437**</u>. [1989: 111] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be an equilateral triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. If

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B},$$

prove that ABC is equilateral.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Given an equilateral triangle $A'B'C'$ and a positive real number r, there exists a triangle XYZ, and only one, such that its sides YZ, ZX, XY pass through A', B', C' respectively and such that

$$\frac{YA'}{A'Z} = \frac{ZB'}{B'X} = \frac{XC'}{C'Y} = r. \tag{1}$$

To demonstrate this, it suffices to note that Y must be the double point (or centre) Y_0 of the homothety $H = H_a{*}H_b{*}H_c$, where H_a, H_b, H_c are the homotheties of ratio $-1/r$ and centres A', B', C' respectively. It is clear that $Z_0 = H_a(Y_0)$ is equivalent to

$$\frac{A'Z_0}{A'Y_0} = -\frac{1}{r}, \qquad \text{i.e.} \quad \frac{Y_0A'}{A'Z_0} = r ,$$

and similarly $X_0 = H_b(Z_0)$ and $Y_0 = H_c(X_0)$ are equivalent to

$$\frac{Z_0B'}{B'X_0} = r \qquad \text{and} \qquad \frac{X_0C'}{C'Y_0} = r ;$$

thus

$$Y_0 = H_c(H_b(H_a(Y_0))) ,$$

and it follows that $X_0Y_0Z_0$ is the unique triangle with the required condition (1). Moreover it is obvious that we can construct an equilateral triangle T circumscribed about $A'B'C'$ having the condition (1). Therefore triangle $X_0Y_0Z_0$ coincides with T.

II. *Solution by Dan Pedoe, Minneapolis, Minnesota.*

Without any calculations, we can see that the theorem implied by this problem is true. We consider the (by now) time–honoured method of projecting a triangle, by orthogonal projection, into an equilateral triangle (see [1]).

There is an ellipse which touches the sides of ΔABC at its midpoints. This ellipse is a circle if and only if the given triangle is equilateral. An orthogonal projection of the inscribed ellipse into a circle maps ΔABC onto an equilateral triangle.

But the defining property of $\Delta A'B'C'$ with respect to ΔABC is unaltered by orthogonal projection (the ratio in which a point divides a segment is unchanged). Since the projection of ABC is equilateral, it is therefore clear that the projection of $A'B'C'$ must also be equilateral.

The circle which touches the sides of $A'B'C'$ at the midpoints therefore projects into a circle. The orthogonal projection must therefore be the identity, and since the projection of ABC is equilateral, ΔABC must itself be equilateral.

Reference:

[1] Dan Pedoe, Homogeneous coordinates and the Lhuilier theorem, *Crux Mathematicorum* 9 (1983) 160–165.

III. *Solution by Murray S. Klamkin, University of Alberta.*

The given conditions are equivalent to the points A', B', C' dividing the sides BC, CA, AB in the same ratio and in the same sense. We generalize the problem to a regular n-gon $B_1B_2\cdots B_n$ inscribed in an n-gon $A_1A_2\cdots A_n$ with B_i on

A_iA_{i+1} (here $A_{n+1} = A_1$) and dividing each side in the same ratio and the same sense. We show that $A_1A_2\cdots A_n$ is a regular n-gon except possibly for the case when n is even and the points B_i are midpoints of their respective sides A_iA_{i+1}.

Let the vertices A_i have the complex representation z_i with origin at the centroid G of $A_1A_2\cdots A_n$. Then the B_i are given by $az_i + bz_{i+1}$ where a and b are given nonzero real constants with sum 1. It follows immediately that the centroid of $B_1B_2\cdots B_n$ is also G. Since $B_1B_2\cdots B_n$ is regular,

$$az_{i+1} + bz_{i+2} = \lambda(az_i + bz_{i+1}) , \qquad i = 1,2,\cdots,n , \tag{1}$$

where λ is a primitive nth root of unity and $z_{n+i} = z_i$. We can rewrite (1) as

$$a(z_{i+1} - \lambda z_i) = -b(z_{i+2} - \lambda z_{i+1}) , \qquad i = 1,2,\cdots,n .$$

It now follows by multiplying the latter set of equations that

$$(a^n - (-1)^n b^n) \prod_{i=1}^{n} (z_{i+1} - \lambda z_i) = 0 .$$

Then, unless n is even and $a = b = 1/2$, $z_{i+1} - \lambda z_i = 0$ for all i which implies that $A_1A_2\cdots A_n$ is regular. On the other hand, for the case $n = 4$ and $a = b = 1/2$, $A_1A_2A_3A_4$ need not be regular, as it is easy to show that the midpoints of the sides of any quadrilateral whose diagonals are equal and orthogonal are vertices of a square.

More generally, let $A_1A_2\cdots A_n$ be a polygon P_a whose vertices have the complex number representation A_i with respect to an origin which is the centroid G of P_a, and let $B_1B_2\cdots B_n$ be an associated polygon P_b defined by

$$B_i = \lambda_1 A_i + \lambda_2 A_{i+1} + \cdots + \lambda_n A_{i+n-1} , \qquad i = 1,2,\cdots,n , \tag{2}$$

where the λ_i's are fixed numbers and may even be complex. It follows immediately that G is also the centroid of P_b, and also that if P_a is regular, then so is P_b. We now determine conditions on the λ_i's so that we have the converse result P_b regular $\Rightarrow$ P_a regular. One simple condition is that the A_i's be determined uniquely from the B_i's via (2). In matrix form, (2) can be rewritten as $\Omega A = B$ where

$$A = [A_1, A_2, \cdots, A_n] \qquad \text{and} \qquad B = [B_1, B_2, \cdots, B_n]$$

are column vectors and Ω is the circulant matrix

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_2 & \lambda_3 & \cdots & \lambda_1 \\ \vdots & & & \vdots \\ \lambda_n & \lambda_1 & \cdots & \lambda_{n-1} \end{bmatrix} .$$

Equivalently the desired condition is that Ω be nonsingular, i.e. $\det \Omega \neq 0$, in which case $A = \Omega^{-1}B$. Since it is known [1] that the inverse of a (nonsingular) circulant matrix is also circulant, it then follows that P_b regular $\Rightarrow$ P_a regular.

As is known [1] det Ω has the n factors

$$\lambda_1 + \lambda_2\omega^r + \lambda_3\omega^{2r} + \cdots + \lambda_n\omega^{(n-1)r}, \qquad r = 1,2,\cdots,n ,$$

where ω is a primitive nth root of unity. As a special case, let $\lambda_1 = \lambda_2 = \cdots = \lambda_m \neq 0$ and $\lambda_j = 0$ for $j > m$. Then as is known (e.g. see page 3 of [2]) $\Omega \neq 0$ if and only if m is relatively prime to n.

References:

[1] P.J. Davis, *Circulant Matrices*, Wiley–Interscience, N.Y., 1979, p.74.

[2] M.S. Klamkin, On equilateral and equiangular polygons, *Crux Mathematicorum* 7 (1981) 2–5.

Also solved by JACK GARFUNKEL, Flushing, N.Y.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; DAN PEDOE, Minneapolis, Minnesota (a second solution); TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There were also two incorrect solutions submitted.

Klamkin's generalization to n–gons (unless n is even and midpoints are taken) was also obtained by Seimiya.

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1438. [1989: 111] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

AB is a fixed chord in a given circle, and C is a variable point on the circle, other than A and B. Prove that

$$\frac{\sin A + \cos B}{\sin\left(\frac{A-B}{2}\right) + \cos\left(\frac{A-B}{2}\right)}$$

is independent of C, where the angles A and B are the angles of ΔABC.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

We have

$$\frac{\sin A + \cos B}{\sin\left(\frac{A-B}{2}\right) + \cos\left(\frac{A-B}{2}\right)} = \frac{\sin A + \sin(90^\circ + B)}{\sin\left(\frac{A-B}{2}\right) + \sin\left(90^\circ - \frac{A-B}{2}\right)}$$

$$= \frac{2\sin\left(\frac{A+B+90^\circ}{2}\right)\cos\left(\frac{90^\circ + B - A}{2}\right)}{2\sin 45^\circ \cos\left(\frac{90^\circ + B - A}{2}\right)}$$

$$= \sqrt{2}\sin\left(\frac{A+B+90^\circ}{2}\right) = \sqrt{2}\sin\left(\frac{270^\circ - C}{2}\right) ,$$

because $A + B + C = 180^\circ$. But angle C is constant for all the positions of point C on the circumference (on one of the arcs AB), and we are done.

Note. If the point C is, for example, below the chord AB, the angle C to consider is $180^\circ - C$, and then the final expression involves $\sin(45^\circ + C/2)$ which equals $\sin(135^\circ - C/2)$.

Also solved by BENO ARBEL, Tel Aviv University, Israel; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; SAM MALTBY, student, Calgary; TAT Y. NGAI, University College of Cape Breton, Sydney, Nova Scotia; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

Not all the above solvers considered (or at least mentioned) the two essentially different locations of point C!

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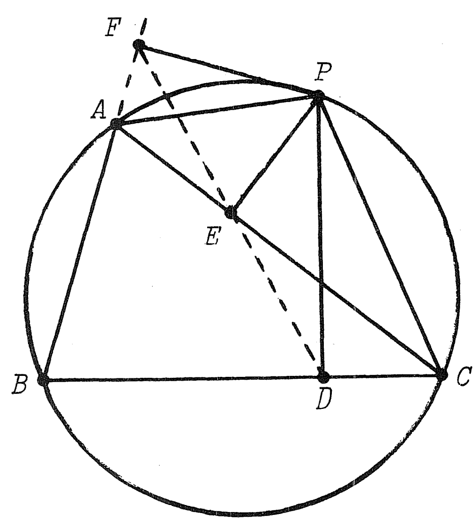
1442. [1989: 148] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle. If P is a point on the circumcircle, and D, E, F are the feet of the perpendiculars from P to BC, AC, AB respectively, then it is well known that D, E, F are collinear (Wallace line or Simson line). Find P such that E is the midpoint of the segment DF.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.



Suppose the problem solved, and let DEF be the Simson line of P. The quadrilateral $PCDE$ is cyclic (because $\angle PDC = \angle PEC = 90^\circ$), and so

$$PE = PC \sin \angle PCE = PC \sin \angle PDE\ ,$$

whence

$$PC = \frac{PE}{\sin \angle PDE} = \frac{ED}{\sin \angle EPD} = \frac{ED}{\sin C}\ ,$$

by using the sinus law in ΔPDE. Therefore $ED = PC \sin C$, and analogously $EF = PA \sin A$. But as $DE = EF$, we obtain

$$\frac{AP}{PC} = \frac{\sin C}{\sin A} = \frac{c}{a}$$

by the sinus law in ΔABC. So P belongs to the Apollonius circle which divides AC internally and externally in the ratio c/a. But P also belongs to the circumcircle of ABC; the intersection of both circles gives the points B and P.

More generally, if it were required that $DE = k \cdot EF$, where k is a fixed positive real, then it suffices to draw the Apollonius circle which divides AC in the ratio $c:ka$.

Remark. This problem is not new. It appears as ex. 7.4, p. 196 of Aref–Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, New York, 1968; and also as ex. 9d), p. 121 of Altshiller–Court, *College Geometry*, Johnson Pub. Co., 1925.

Also solved by HARRY ALEXIEV, Zlatograd, Bulgaria; C. FESTRAETS–HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; K.R.S. SASTRY, Addis Ababa, Ethiopia; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

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LETTER TO THE EDITOR

Dear Sir,

I have been subscribing to *Crux Mathematicorum* for some years, and I really enjoy the journal. However I am a little bit disappointed in Mr. Andy Liu's Mini–Reviews. In my view, *Crux Mathematicorum* should continue to be a journal only for problem solving. There are so many other journals that have book reviews that I do not think it is necessary to have them in *Crux Mathematicorum*. But if you should continue with Andy Liu's Mini–Reviews it is necessary that the books still are in stock. For example, many books from MIR, which Mr. Liu wrote about some time ago [1989: 142], have been out of stock for many years. I also think that it is important to have the ISBN number in the review. And for all readers in foreign countries it would be a great help to have the address of the publisher. Perhaps it would be a good idea, and more in style with the journal, only to review problem books.

Yours sincerely,

Bengt Ahlin
Bandhagen, Sweden

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