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Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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'University Mathematics'

A Conference held at the University of Newcastle, 13 March 1974

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At a recent meeting between school mathematics teachers in the North East of England and Newcastle University mathematics teachers, the idea of a sixth form conference was discussed. Its broad aim would be to indicate to lower sixth mathematics pupils what was involved in mathematics at university. The local schools encouraged the proposal and the School of Mathematics at Newcastle University arranged its first sixth form conference for the afternoon of 13 March. Invitations were sent to schools which had already shown interest in the idea, and the number of conference places, 170, was quickly filled with 150 lower sixth pupils and 20 teachers from 16 schools in the North of England.

The conference opened with an introductory lecture from Professor J. R. Ringrose, Head of the School of Mathematics at Newcastle University. He first dealt with difficulties experienced in the transition from school to university. Students not only have difficulty because of the change in ideas of their mathematical studies, but just as much because of an inevitable change of life style, occasioned by leaving home for university. They are also subjected to a rather different teaching method, consisting of lectures, which are normally the primary method of imparting knowledge, written work and tutorials (in small groups) where difficulties encountered in the lectures and the written work are discussed.

Professor Ringrose's second section dealt with the qualities we would like to see in prospective mathematics undergraduates. A reasonable level of knowledge and competence in mathematics would be vital, and an interest in mathematics itself would be needed. We would also look for a certain level of maturity. Finally, what should prospective students be looking for? Clearly they should seek courses which suit their *interests* and *ability*. Since interests often vary in a very short time, courses with great flexibility of choice in subsequent years would be advantageous. Also a consideration of non-academic facilities such as halls of residence and sports facilities would be valuable.

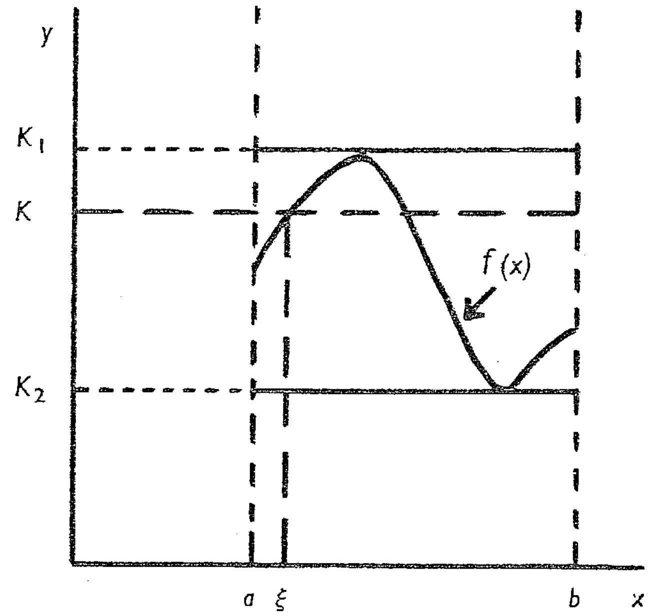
This lecture was followed by a 'typical university lecture', although no single lecture could really be classed as typical. The actual choice of a suitable university topic for lower sixth mathematics pupils presented some difficulties. The lecturer, Dr R. Johnson of the Applied Mathematics Department at Newcastle University, after some consideration chose 'Mean Value Theorems', which would only require a knowledge of simple integration. The conference members were encouraged to take notes, as in a normal university lecture, which would be of use in the tutorials which followed this lecture. Dr Johnson proved first the mean value theorem of the integral calculus (reference 1), which states that

$$\int_a^b f(x)\,dx = (b-a)f(\xi) \tag{1}$$

where ξ is some real number such that $a \leqslant \xi \leqslant b$. To prove this result, he first considered a continuous function which in the interval $[a, b]$ is bounded above and below as illustrated in the figure below; i.e., for $a \leqslant x \leqslant b$

$$K_2 \leqslant f(x) \leqslant K_1$$

where K_1 and K_2 are constants.



Now the area under the curve $y = K_1$ is clearly greater than the area under the curve $y = f(x)$ for $a \leqslant x \leqslant b$, which in turn is greater than the area under the curve $y = K_2$, $a \leqslant x \leqslant b$. Hence we can write

$$\int_a^b K_2\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b K_1\,dx$$

(the equality signs are retained so as to include the possible cases when $f(x)$ equals K_1 or K_2), and integrating the first and last integrals given, we have

$$K_2(b-a) \leqslant \int_a^b f(x)\,dx \leqslant K_1(b-a).$$

(Although the figure shows a positive function, diagrams illustrating a negative function or one that takes both signs show that the above argument is also valid in these two cases.)

Since the value of the integral $\int_a^b f(x)\,dx$ lies between $K_2(b-a)$ and $K_1(b-a)$ we can now write

$$\int_a^b f(x)\,dx = (b-a)\,K,$$

where $K_2 \leqslant K \leqslant K_1$. Referring again to the figure, we see that, since $f(x)$ is continuous, there will always be at least one value of x such that $f(x) = K$ with $a \leqslant x \leqslant b$. (In fact for the particular case illustrated there are two such values of x.) Now denote a value of x such that $f(x) = K$ by ξ, and we have the result

$$\int_a^b f(x)\,dx = (b-a)f(\xi),$$

where $a \leqslant \xi \leqslant b$, which is the statement of the first mean value theorem.

Dr Johnson then illustrated how the result could be used to estimate values of integrals which could not be integrated in the usual way. For instance, consider

$$I = \int_0^{\pi/2} [\surd(1-\tfrac{1}{4}\sin^2 x)]^{-1}\,dx$$

which cannot be integrated by the standard techniques. If we use Equation (1) with $f(x) = 1/\surd(1-\frac{1}{4}\sin^2 x)$, the mean value theorem tells us that

$$I = (\pi/2-0)/\surd(1-\tfrac{1}{4}\sin^2\xi) \tag{2}$$

where ξ is a number between 0 and $\pi/2$ whose exact value we do not know. But the maximum possible value of $1/\surd(1-\frac{1}{4}\sin^2 x)$ is given when $x = \pi/2$ and is $2/\surd 3$; and similarly the minimum value is given when $x = 0$ and this value is 1. Hence, by (2),

$$\pi/2 \leqslant I \leqslant \pi/\surd 3 \quad \text{or} \quad 1{\cdot}57 \leqslant I \leqslant 1{\cdot}81.$$

The value of I, correct to two decimal places, is in fact 1·69.

The enthusiasm of the lecturer for his subject was well received by the audience, and the lecture was followed by small group tutorials. These tutorials were led by second and third year mathematics students at Newcastle University and the time available was divided into two parts. Firstly, some tutorial problems based on the previous lecture were examined, and secondly, a broad discussion between the tutorial student and the pupils on general aspects of university life took place. Reaction to this particular session has been varied, but certainly our mathematics students welcomed the change from their normal activities.

Refreshments followed the tutorial session, and the conference concluded with a lecture by Dr N. du Plessis, senior lecturer in Pure Mathematics at Newcastle University, entitled 'The Villains of the Piece'. This was a light-hearted affair ranging from Euclid (a hero!) to Bolyai and Lobachevski (villains!) and making play with Euclid's fifth Postulate (reference 2 gives details of this postulate). He ended with the thoroughly reactionary statement that nobody knows what a set is.

Dr du Plessis' entertaining lecture was appreciated by the audience, and reaction to the whole conference from participating schools has indicated that it was both enjoyable and informative. The university staff who took part very much welcomed the opportunity to meet staff and pupils from the schools.

References

1. G. Stephenson, *Inequalities and Optimal Problems in Mathematics and the Sciences* (Longman, London, 1971), 116–119.
2. Pat Rogers, The parallel axiom, *Mathematical Spectrum* **5** (1972/73), 58–65.

What is Probability?

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Textbooks on probability and statistics often take it for granted that the idea of a random variable corresponds to something in the real world. Consequently we can apply our theoretical results to various real-life situations. If students do not understand randomness intuitively, so much the worse for them!

If we follow the history of the subject it appears that the concept of 'randomness' was a long time in developing and, moreover, it bristles with difficulties and paradoxes.

To go back to the beginnings of the subject is not only interesting as history but may also be useful in helping to attain understanding.

Some of the earliest known conceptions of probability arose among the Jews of classical and medieval times (reference 1). They considered the supreme object of all reasoning to be the establishment of justice. Now often, justice has to be done most urgently in those cases where certain knowledge is not obtainable. This opens the way for reasonable conjecture and the logic of probability.

Let us take a look at a discussion recorded in the Jerusalem Talmud dating from the fourth century. In this text two views of randomness are examined and their difficulties pointed out.

Three people, A, B, and C, put coins into a money-bag and part of it was stolen. Apparently the thief was interrupted before he could make away with the whole amount. How was the loss to be apportioned among the three owners? There was a ruling that the remaining money should be divided between the three in proportion to their original investment. This ruling was based on the following two assumptions.

(1) If a thief quickly pulls a coin out of the bag every coin has an equal chance of being drawn, regardless of whether it belongs to one or another of the partners.

(2) Small coins get mixed up evenly, so that a large enough 'random' selection of coins will reflect the composition of the original collection.

Thus it was supposed that the thief's haul, and so also what he left behind, represents a fair sample in which the distribution is the same as that of the entire original collection. 'To do justice to all, each one takes according to his investment.'

In another case, however, a different ruling had been issued. This concerned a two-storey house in which each storey belonged to a separate owner. The house collapsed and some of the building stones were stolen. Although originally the number of stones belonging to each of the two owners was unequal, it was decided to divide the loss equally between them.

These rulings were discussed by the Rabbis in the Academy who sought to discover whether the assumptions underlying the first ruling were applicable in the second case and, if not, what principle would apply.

A certain Rabbi Shammai pointed out that because building stones, unlike coins, are large, they cannot easily be distributed evenly. It may very well be that all or most of the exposed and easily accessible stones on the heap come from the upper part of the wall, if it collapsed under its own weight, or the reverse might be true in other circumstances. Other possibilities too must be taken into account. 'Because of the doubt—half is of the one and half of the other.'

In contrast to the case of coins, knowing the original proportions of ownership does not tell us the probability P_A of drawing a stone belonging to A as compared to the probability P_B of drawing a stone belonging to B. Rabbi Shammai suggested that in view of our ignorance we may assume that $P_A = P_B = \frac{1}{2}$, so that the thief may be supposed to have taken as many of A's stones as B's.

In modern times, Rabbi Shammai's proposal appears as the 'Principle of Indifference' (reference 2), which was used to evaluate the probability of a proposition when we know nothing about its correctness. This principle, it was argued, enables us to treat any proposition and its contradictory, in the absence of evidence, as two equally probable cases.

However, Rabbi Shammai's colleagues objected to his reasoning: 'On what grounds do you say that we consider the stolen ones; perhaps we should consider the remainder?'

To counter Rabbi Shammai's 'Principle of Indifference', the Talmud shows that it leads to a paradox. For if one considers the stolen stones and assumes that the probability is equal that a particular stone belongs either to A or B, the total loss is suffered equally by both partners despite the fact that their shares were unequal originally. For definiteness, let the shares of A and B be 100 and 50 respectively and suppose 50 were stolen. Each one will therefore loose 25 and their portions will now be 75 and 25 respectively.

On the other hand, one ought to be able to apply Rabbi Shammai's reasoning, if it is valid, to any stone, and therefore also to the remaining stones. Thus, for any one of the remaining stones the probability should be equal that it belongs to either A or B. In that case, A and B would each get 50, which contradicts the first calculation. Moreover, if one considers the remainder, division into half and half cannot be applied for another reason as well. For, of the 100 stones remaining, at least 50 certainly belong to A, since B never had more than 50 altogether. A's share can vary then between 50 and 100. How then can one say that the chances are equal for each stone that it belongs to A or B?

In the course of the discussion, Rabbi Yosé ben Rabbi Bon presented another puzzle.

He accepts that in the case of large stones where thorough mixing cannot easily occur one cannot indeed assume that the distribution in a sample will be the same as that in the original population. However, we do have some information to go on and one might agree that it is fair to use a rule of inference which enables us to draw a conclusion for a single instance. This rule is known in the Talmud as 'Follow the majority'. Since in the case in hand twice as many stones belong to A as belong to B, if a single stone is picked up it is more likely to belong to A than to B. We cannot say that the probability $P_A = \frac{2}{3}$ since other factors may be relevant, but surely $P_A > \frac{1}{2}$! Now since an individual stone must belong entirely to either A or B, it is fair to assign it to A. However, this procedure will lead to a paradox, since the same reasoning can be applied to every one of the 50 stolen stones individually. In every single case there is a majority probability that the stone belongs to the majority owner. However, the conjunction of these inferences is that all the stolen stones come from the one partner's share alone, and that is very unlikely and 'justice will suffer'.

In fact, other rabbis sharpen the paradox. They point out a consideration which is equivalent to dealing with a case in which, say, 101 stones were stolen. For each stone singly the majority tells us that it belongs to A. However, since A originally had only 100 stones, the conclusion that 101 were stolen from his property is patently wrong.

Various safeguards were devised to avoid the paradox in special cases. However, this is still a very live issue. What probability is high enough to justify accepting an hypothesis? Set it as high as you will, as a general rule it may lead to what is now called the *lottery-paradox* (reference 3). The probability that a particular ticket is a loser in a lottery is always high. If there are a million tickets and only one winner, the probability that any single ticket is a loser is 0·999999. Most people would say that is practically a certainty.

Is there very much that can be asserted, including empirical observation statements, that have a greater, or even as great, probability? If we take together a million statements—each of overwhelmingly high likelihood—of the form: 'Ticket n is a loser' ($1 \leqslant n \leqslant 1{,}000{,}000$), the conclusion is equivalent to 'All the tickets are losers' and that is certainly false, for there is one winner!

These and other difficulties arose very early in the history of probabilistic logic. The problems that early thinkers dealt with obviously did not involve the kind of sophisticated mathematics that even elementary statistics requires today. Nonetheless, they often enable us to get to the basic issues involved in some practical applications, as well as in the philosophical foundations of statistics.

References
1. For an extensive treatment of early sources from which are taken the examples here cited see Nachum L. Rabinovitch, *Probability and Statistical Inference in Ancient and Medieval Jewish Literature* (University of Toronto Press, 1973).
2. A number of paradoxes generated by the Principle of Indifference are presented in chapter 4 of John M. Keynes, *A Treatise on Probability* (3rd ed.) (Oxford, 1961), 41–51.
3. Henry E. Kyburg, *Probability and the Logic of Rational Belief* (Wesleyan University Press, 1961), 197.

Pitfalls of Elementary Set Theory

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You've got to be tough in this job. East end, rough area. Needs peak physical fitness, mental agility. I slammed the door quietly shut with a precisely aimed kick. 'Shut up!' I spat out. That cut the noise by half.

I went straight at it, no messing about. 'Today it's Set Theory. What's a set? A collection of objects. Just that; no more, no less.' The class went quiet—I'd caught their interest. Then I saw Bond had his hand up.

'Can I leave the room?' Bond was always leaving the room, like a dud camel. I paced slowly over and looked contemptuously down at him. One colleague had tried to call his bluff and then had to send him home in desperation. I had him taped: 'go on,' I snapped.

Then came examples of sets. I always use the set of all cows, as I love these peaceful ruminants untroubled by the curse of intelligence. But this class was thick. Take this instance. 'The collection P of all people in the room is a set,' I illustrated. That fool Wilson had his hand up. 'Is Bond in P?' Slowly and loudly I explained that as Bond wasn't in the room he wasn't in P.

The noise built up as I got onto subsets, then cut suddenly. I slid my hand into the pocket where I kept my pipe. I didn't smoke but the bowl felt good nestling in the palm of my hand. Teachers aren't allowed to carry guns. Bond had opened the door and he walked in silence to his seat. The class saw I was vigilant and did nothing. I relaxed, alert as a cat. 'For example, the set B of all boys in this room is a subset of the set P of all people in the room. Any questions?' I interrogated.

A few hands were raised but as they noticed one of the Smiths had his hand up, they were lowered again. Perhaps instinctively they realize he is so stupid that his need for education is so much greater than theirs. I nodded at him. 'Sir, is Bond in B?' Sometimes I have to fight to keep a smile off my face; Smith—and his brother —really would be happier with a wad of chewing gum in the set of all cows. 'Then why isn't he in P?' Smith howlered. Typical of inferior intelligence to concentrate on irrelevant detail and miss the point. As an old hand I knew the webs such ignorance can weave, so I changed the example to the set *K* of all books in the room.

Then I saw Harris playing with something under his desk. 'What have you got there?' I hurled. 'A book,' he sullened. The good teacher turns everything to his advantage. 'Aha! A member of the set *K*. Let's have a look at it.' It was a disgusting bedraggled collection of bad drawings called *Marvelman Meets the Mad Monster*. 'That,' I contempted, 'is not a book. Any of you think it's a book?' I wasn't surprised that the Smiths did, but most of the other boys followed their lead. 'Who agrees with me?' To receive the fawning admiration of half a dozen spotty girls does nothing for the image, and I may have lost my cool. 'O.K.' I abrupted. 'It's in *K* by 19 votes to 6.' The bell rang; I knew when not to press my luck, and I quit.

But next day I was back, standing easily before them, in command. I told them again what they'd done before (craftily I made *K* the set of 'all books which were in this room at ten-thirty yesterday') but Smith obviously didn't want to know. I moved swiftly and silently over and brought my hand down in a karate chop which would have split his desk in two; at the last moment I decided a bang would scare him enough. He looked up, startled. 'Out with it!' I ultimatumed. It was *Marvelman Meets the Mad Monster* again. 'A member of the set *K*,' I joculared. 'I don't know,' he ignoranced. I decided to conduct a staccato interrogation.

Didn't we decide it was yesterday?

That was Harris's.

Harris, where is your copy?

Sold it to Quince in the fifth form. He deals in comics.

Smith, where did you get yours?

Quince.

So it was Harris's?

Quince had three of them.

Harris, did yours have any distinguishing marks?

Two tears in the back page.

Smith's copy had three tears in the back page. My mind ran smooth and fast. I could have blustered my way out but I reckon you should never offer less than the truth even if you suspect it's too subtle for them to understand. 'Look,' I lucided, 'there's a difficult principle in set theory which says that if you can't tell the difference between two things they count as the same. For example, the set consisting of 2, 3 and 2 is just the set consisting of 2 and 3—'

Edwards had his hand up. 'Are both Smith twins in the set of people in the room?' I quietly looked at the class. One of them was there—Arthur, I think; the other wasn't. But they were both so thin they could have been used in advertisements for starvation, both at that stage in puberty when you weren't sure whether they had moustaches or not; identical. 'Well, yes,' I smiled, 'I suppose they are.'

Then Smith had his hand up. 'I can tell the difference,' he eagered.

There is a level of intelligence below which it is impossible to teach abstract ideas, and the wise man knows when he's reached it. I spoke quietly to soften the insult I was offering them. 'We'll work out the area of a triangle,' I said. 'And God help you in the examination.'

Moral: Sets aren't as obvious as you might think.

(In her recent article (reference 1), Pat Rogers pointed out that the idea that a set is simply a collection of objects is inadequate for pure mathematics. The story indicates another kind of difficulty which arises when collections of real objects are considered; if you try to describe any particular collection, I shall find an objection to your description, and if you alter your description to meet my objection, I shall find another objection, and so on. This is only worrying if you insist on viewing this kind of 'set theory' as part of pure mathematics, for most of applied mathematics has a similar peculiarity—it does not fit perfectly the world it hopes to describe, though it often fits well enough. If a pure mathematician is really pressed to say what a set is, he will talk about concepts very different from the elementary one. His approach will be very abstract, and although logicians who study the foundations of mathematics have not reached unanimity on the subject, most working mathematicians now favour the 'axiomatic' approach discussed by Pat Rogers.)

Reference

1. Pat Rogers, The continuum hypothesis. *Mathematical Spectrum* **6** (1973/74), 47–51.

The Divergence of the Simple Harmonic Series

N. A. DRAIM

I know of three different demonstrations of the fact that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots \qquad (1)$$

diverges; and I wonder whether readers of *Mathematical Spectrum* can supply any other proofs.

If we put

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \quad (n = 1, 2, \ldots),$$

we wish to show that $S_n \to \infty$ as $n \to \infty$, i.e., that S_n will be as large as we like provided only that n is sufficiently large.

The simplest proof depends on a suitable grouping of the terms in the harmonic series. For $r = 1, 2, \ldots,$

$$S_{2^r} = (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots+\left(\frac{1}{2^{r-1}+1}+\frac{1}{2^{r-1}+2}+\ldots+\frac{1}{2^r}\right).$$

There are $r+1$ groups and the numbers of terms in them are

$$1, 1, 2, 2^2, \ldots, 2^{r-1}$$

respectively. Hence

$$S_{2^r} > 1+\frac{1}{2}+2.\frac{1}{4}+4.\frac{1}{8}+\ldots+2^{r-1}.\frac{1}{2^r}$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2} = 1+\frac{1}{2}r.$$

This means that S_{2^r} can be made arbitrarily large by taking r sufficiently large, or $S_{2^r} \to \infty$ as $r \to \infty$. Since S_n increases with n, it follows that $S_n \to \infty$ as $n \to \infty$.

The second proof is particularly short, but is not as self-contained as the first one, since it involves a knowledge of integration and of the logarithmic function. For $x \geqslant r$, $1/r \geqslant 1/x$. Hence

$$\frac{1}{r} = \int_r^{r+1} \frac{1}{r}\,dx \geqslant \int_r^{r+1} \frac{1}{x}\,dx \quad (r = 1, 2, \ldots)$$

and

$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} \geqslant \int_1^2 \frac{1}{x}dx+\int_2^3 \frac{1}{x}dx+\int_3^4 \frac{1}{x}dx+\ldots+\int_n^{n+1} \frac{1}{x}dx$$

$$= \int_1^{n+1} \frac{1}{x}dx = [\log x]_1^{n+1} = \log(n+1).$$

But $\log(n+1) \to \infty$ as $n \to \infty$ and consequently $S_n \to \infty$ as $n \to \infty$.

My favourite proof is a version of the oldest one, due to Jacob Bernoulli. It is quite sophisticated, depending on the summation of *double* series (though Bernoulli did not recognize this).

Let the entries in the doubly infinite array

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array} \qquad (2)$$

be non-negative. Moreover suppose that each row forms a convergent series, say

$$\begin{aligned} a_{11}+a_{12}+a_{13}+\ldots &= R_1, \\ a_{21}+a_{22}+a_{23}+\ldots &= R_2, \\ a_{31}+a_{32}+a_{33}+\ldots &= R_3, \\ &\text{etc;} \end{aligned}$$

and that the series

$$R_1+R_2+R_3+\ldots$$

converges to sum S. Then each column of (2) forms a convergent series, and if the sum of the nth column is C_n,

$$C_1+C_2+C_3+\ldots$$

also converges to S. Conversely, if the sums of the columns form a convergent series, then so do the sums of the rows, and the series of rows and the series of columns have the same sum. In other words, the members of the array (2) may be summed by rows or by columns. Proofs of this statement may be found in many books on infinite series.

There is one other result we require, but it is easily proved. We need to know that, for every positive integer m, the series

$$\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+2)(m+3)}+\ldots$$

converges and has sum $1/m$. This follows from the relation

$$\frac{1}{m(m+1)}+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+2)(m+3)}+\ldots+\frac{1}{n(n+1)}$$

$$=\left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\left(\frac{1}{m+2}-\frac{1}{m+3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\frac{1}{m}-\frac{1}{n}\to\frac{1}{m} \quad \text{as } n\to\infty.$$

We are now ready for our last proof of the divergence of the series (1). Suppose that, on the contrary, the series converges and has sum S, say. In the array

$$\begin{array}{ccccc} \frac{1}{1.2} & \frac{1}{2.3} & \frac{1}{3.4} & \frac{1}{4.5} & \ldots \\[2ex] 0 & \frac{1}{2.3} & \frac{1}{3.4} & \frac{1}{4.5} & \ldots \\[2ex] 0 & 0 & \frac{1}{3.4} & \frac{1}{4.5} & \ldots \\[2ex] 0 & 0 & 0 & \frac{1}{4.5} & \ldots \\[2ex] \cdot & \cdot & \cdot & \cdot & \ldots \end{array} \qquad (3)$$

the rows form convergent series of sums 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,...; and so the elements of (3), when added by rows, give the sum

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots = S.$$

On the other hand, the columns of (3) have sums $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$,...; and so the elements of (3), when added by columns, give the sum

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots = S-1.$$

We therefore have

$$S = S-1,$$

which is impossible. Therefore the series (1) cannot converge.

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Dynamic Programming

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1. Introduction

Dynamic programming is a general method for solving problems that involve a sequence of decisions. The essential features of dynamic programming can be discussed through a simple network, containing *nodes* and *arcs*, shown in Figure 1. Nodes are represented by circles, and an arc joins two nodes; note that there is a sense of direction. For example, we can move from node 1 to node 4 but not from node 4 to node 1. The number against the arcs represent distances, and the problem is to find the shortest path from node 1 to node 8.

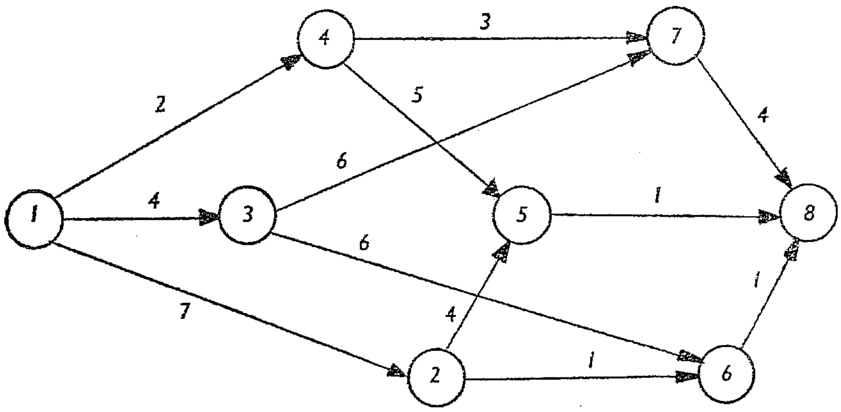

Figure 1. Shortest path problem.

It is not too difficult to see that the shortest path from node 1 to node 8 is that linking nodes 1, 4, 5 and 8, and that the total distance is 8 units. But suppose that the problem had, say, 80 nodes, and we wanted a formal method of solution which was easy and efficient when carried out by hand and also suitable for programming on a computer. Such a method is dynamic programming, and it works as follows.

Let $r(i,j)$ be the distance from a node i to an adjacent node j. For example, in Figure 1, $r(3,7) = 6$. Let $f(i)$ be the shortest distance from node i to node 8, or, in general, to the terminal node. Initially we do not know these shortest distances, and a main part of dynamic programming is concerned with calculating quantities like $f(i)$. We can conveniently put $f(i)$ values, as we get them, into boxes above the corresponding nodes.

We start by putting $f(8) = 0$. For nodes 5, 6, and 7, which connect directly to node 8, it follows easily that $f(5) = 1$, $f(6) = 1$, $f(7) = 4$. Now consider node 4. From node 4 we might go to node 7, and then take the shortest path from there, or we might go to node 5 and then take the shortest path from there. If we go to 7 the distance covered is

$$r(4,7)+f(7) = 3+4 = 7. \tag{1}$$

If we go to 5 it is

$$r(4,5)+f(5) = 5+1 = 6. \tag{2}$$

It is best to go to 5, and the shortest distance is 6 units, so $f(4) = 6$. As well as noting this value on the diagram we put a bar on the arc which gives the shortest distance, so that we can identify it later. At this point in the computation the diagram is marked up as shown in Figure 2.

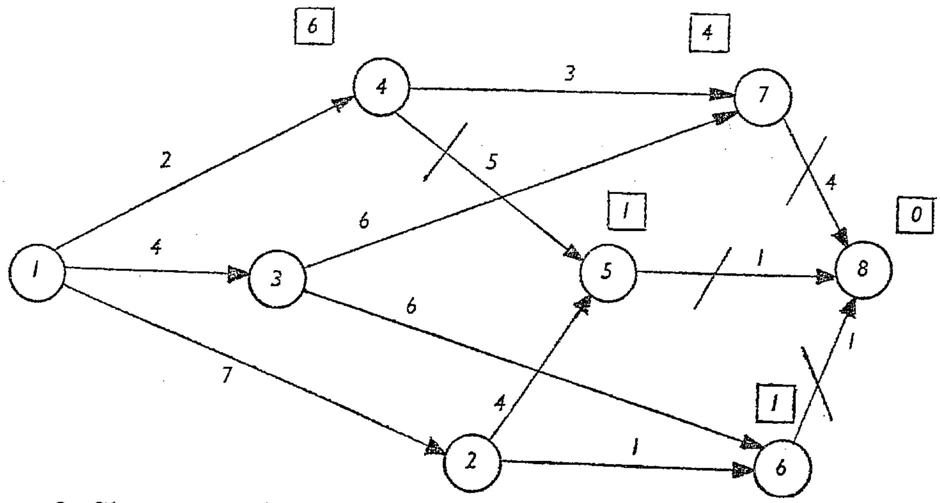

Figure 2. Shortest path problem with calculations of shortest distances to node 8 from nodes 4, 5, 6, 7, 8.

Continuing with the calculation, at node 3, $f(3)$ is the minimum of

$$r(3,7)+f(7) = 6+4 = 10 \tag{3}$$

or

$$r(3,6)+f(6) = 6+1 = 7, \tag{4}$$

giving $f(3) = 7$.

At node 2, $f(2)$ is the minimum of

$$r(2,5)+f(5) = 4+1 = 5$$

and (5)

$$r(2,6)+f(6) = 1+1 = 2$$

and hence $f(2) = 2$. Note that the last sentence can be written more compactly as

$$f(2) = \min[r(2,j)+f(j)], \quad j \in \{5, 6\}. \tag{6}$$

We finish the calculation by computing $f(1)$, where

$$f(1) = \min \begin{bmatrix} r(1,2)+f(2) \\ r(1,3)+f(3) \\ r(1,4)+f(4) \end{bmatrix} = \min \begin{bmatrix} 7+2 \\ 4+7 \\ 2+6 \end{bmatrix} = 8. \tag{7}$$

Hence the shortest distance through the network is 8 units. To trace out the optimal path we start from node 1 and take the arc with the bar, then from the node thus reached we again take the arc with the bar (drinking enthusiasts will find this easy to remember). As we go we mark a second bar on the arcs thus chosen, so that the final diagram, Figure 3, clearly indicates the optimal path, nodes 1, 4, 5, 8.

Also note that, by writing Equation (6) for a general state i, we have the following equation which is called the dynamic programming recurrence relation

$$f(i) = \min[r(i,j)+f(j)], \quad j > i. \tag{8}$$

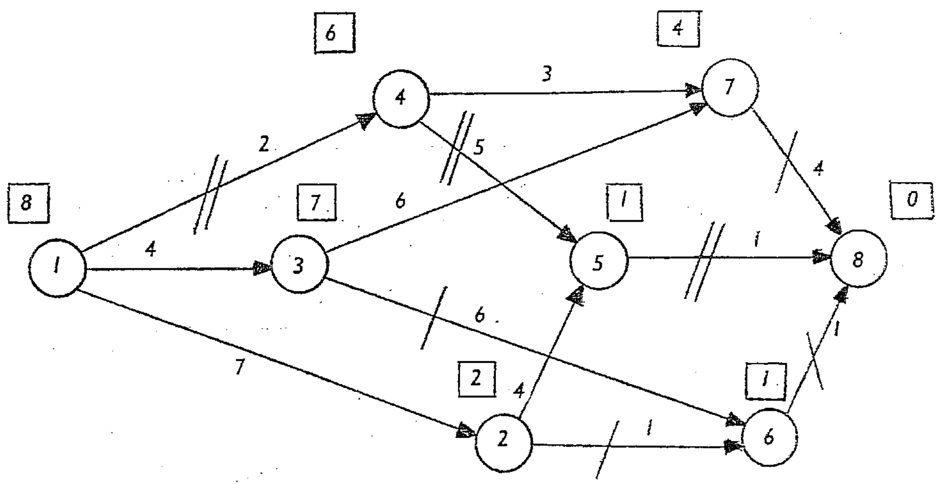

Figure 3. Shortest path problem.
(In this figure, the boxed numbers are shortest distances to the terminal node. A bar indicates an arc which is optimal from its preceding node. Double bars indicate the optimal path.)

2. Computational efficiency

As an indication of the computational efficiency of the procedure just described, consider a shortest path problem with n nodes. Suppose that there is an arc directed from node i to node j for all $j > i$. To solve this problem by dynamic programming we process each arc once, making one addition and one comparison per car. There are $\frac{1}{2}n(n-1)$ arcs and so the total number of computational steps (counting each addition or comparison as one step) is $n(n-1)$. On the other hand, if we calculated the length of every path separately this would require $n-1$ additions and one comparison per path, and there are $(n-1)!$ paths. The total number of computational steps would be $n!$ Thus for $n = 80$, dynamic programming requires a mere 6320 steps, whereas complete enumeration requires 80!, an astronomical number.

3. Application

Although the shortest path example contains the essential features of dynamic programming, it gives little indication of the theoretical generality of the technique or of its practical importance. It can be shown, for example, that linear programming, the calculus of variations, and many problems in Markov processes and queueing theory are special cases of dynamic programming. This does not necessarily imply that dynamic programming is always the preferred computational technique, since this will depend on the details of the problem structure.

Perhaps the most important applications are in the field of management (see reference 2). Here one is often concerned with problems which involve a sequence of decisions in time or space in a way comparable to the shortest path problem. Frequently, these are discrete decisions (e.g., whether or not to place an order for stock, whether to repair or replace a machine, whether to make metal sheet in one standard size or another) which means that we cannot use the calculus or other continuous-variable optimization techniques. The production planning problem in the next section is an example.

4. A production planning problem

A boatbuilding company has orders for boats of a certain type to be delivered at the end of the months shown.

Month	No. of boats
1. February	1
2. March	2
3. April	5
4. May	3

The construction of any given boat takes place over a period of one calendar month. Stock is zero at the begining of February and is to be zero after the May delivery. If boats are built in a particular month there is an overhead cost of 4 units. Stock holding costs 1 unit per boat per complete month, not counting the month during which the boat is being built. Orders must be met. In what months should boats be built and in what quantities in order to minimize costs?

It will probably not be obvious at first that this problem is similar in principle to the shortest path problem of Section 1. As a preliminary step in the formulation we need to realise that, for the problem as stated,

(i) whenever we build boats we should build sufficient to provide stock for an exact number of months;

(ii) boats should not be built unless the present stock is exhausted.

We then identify node i of a progressive network with state i of the current system, which corresponds to having zero stock at the start of month i. Here February is month 1, March is month 2, etc. Let $r(i,j)$ be the cost of overheads and storage associated with producing in month i sufficient boats to last until the start of month j. For example, if in February we build 8 boats this will last until the start of May; the costs in the period February to April inclusive will then be as follows.

Month	Boats built	Boats stored	Cost
1. February	8	0	4
2. March	0	7	7
3. April	0	5	5
		Total cost	16

Hence $r(1,4) = 16$. Similar calculations for other combinations of i and j, $i < j$, give the costs indicated in Figure 4, where the number against the arc connecting node i with node j is the cost $r(i,j)$.

Thus we have reduced the problem to a 'shortest path' structure. Applying dynamic programming to Figure 4 we find that the solution is to build 3 boats in February and 8 in April and that the total cost of overheads and storage is then 13 units.

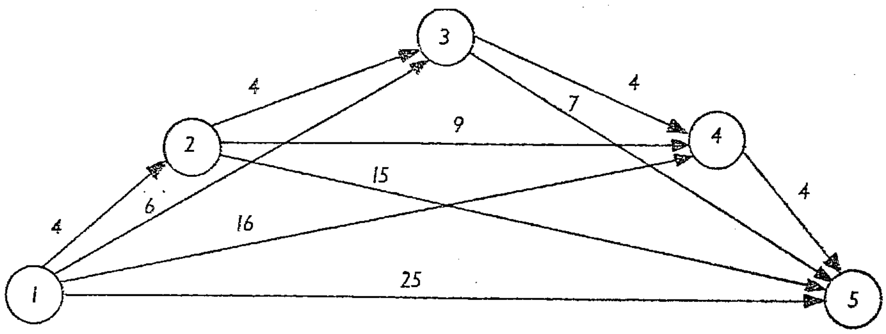

Figure 4. Production planning problem: network diagram.

5. Computer programs

With larger problems the calculation involved in dynamic programming becomes tedious. It is reasonably easy to write a program in a general purpose computing language such as Fortran for the simple type of shortest path problem described here. However, it is even simpler to use Dynacode (reference 3) which is a computing system developed specifically for dynamic programming. Dynacode contains diagnostic tests which guard against a wrong formulation, it has built in array and file handling facilities which facilitate the solution of larger problems, and it allows for a great many variations in structure, for example stagewise problems, discounted returns, probabilistic transitions and infinite planning horizon. The Dynacode input for the production planning problem is shown in Table 1.

TABLE 1

Production planning problem: Dynacode input

```
DYNACODE PLANNING PROBLEM
MINIMIZE
DETERMINISTIC
NONSTATIONARY
FINITE STAGE
PROGRESSIVE
TERMINAL STATES 1
INITIAL STATE 1
*PRODUCTION PLANNING PROBLEM
*STATE IS MONTH, 1 = FEB, 2 = MAR, ETC.
*ACTION IS BUILD-QUANTITY IN TERMS OF MONTHS OF SUPPLY
*VALUE IS TOTAL REMAINING COST
DATA
0, 5, 0, 0, 0
1, 4, 1, 4, 5
1, 3, 1, 4, 4
1, 3, 2, 7, 5,
1, 2, 1, 4, 3,
1, 2, 2, 9, 4
1, 2, 3, 15, 5
1, 1, 1, 4, 2
1, 1, 2, 6, 3
1, 1, 3, 16, 4
1, 1, 4, 25, 5
END
```

In Table 1 the first statement gives the problem title. The next seven statements indicate that we have a minimization problem with deterministic transitions, based on a finite, progressive network, with one terminal state and with an initial state which is labelled state 1. The four statements which start with asterisks are called title statements. They are used to give an interpretation of the formulation of the current problem and they appear also on the output table. The data consist in general of an entry of the form n, i, k, r, j where

n is 0 if the state is terminal and is 1 otherwise,
i is the state label and corresponds to the node number in Figure 4,
k is the action label and is the number of months' supply of boats built,
r corresponds to $r(i,j)$ in Figure 4,
is the adjacent state to state i under action k.

The corresponding Dynacode output is shown in Table 2. This contains the title statements from the input, followed by a table giving the optimal process. This table has four columns which in general contain values of the variables n, i^*, k^*, f where

n has the same interpretation as on the input,
i^* is a state on the optimal path,
k^* is the optimal action in state i^*,
f is the optimal value of state i^*.

TABLE 2

Production planning problem: Dynacode output

```
*PRODUCTION PLANNING PROBLEM
*STATE IS MONTH, FEB = 1, MAR = 2, ETC.
*ACTION IS BUILD-QUANTITY IN TERMS OF MONTHS OF SUPPLY
*VALUE IS TOTAL REMAINING COST
*
OPTIMAL PROCESS
*
*         STAGE      STATE      ACTION      VALUE
            1          1           2          13
            1          3           2           7
            0          5           0           0
```

Thus Table 2 tells us that the optimal path goes through states 1, 3, 5 and that in state 1 (February) we should build 2 months' boats and in state 3 (April) we should build 2 months' boats. The total cost is $f(1) = 13$. This is the same result as was obtained by hand.

6. Bibliographic notes

Dynamic programming was pioneered by the American mathematician Richard Bellman who published a book with that title in 1957. A major extension of the technique into the area of stochastic problems was made by Ronald Howard in 1960. Reference 2 is a recent textbook of dynamic programming, and reference 3 contains details of the Dynacode computing system.

References

1. R. E. Bellman, *Dynamic Programming* (Princeton University Press, 1957).
2. N. A. J. Hastings, *Dynamic Programming with Management Applications* (Butterworths, London, and Crane-Russak, New York, 1973).
3. N. A. J. Hastings, *Dynacode Dynamic Programming System Handbook* (Management Centre, University of Bradford, 1974).
4. R. A. Howard, *Dynamic Programming and Markov Processes* (Wiley, New York, 1960).

The Sweep of a Logging Truck

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1. Introduction

I have recently been on study leave in British Columbia, Canada where the most important industry is logging. Pines grow to a height of 100 feet or more over an area the size of Europe but with a much more rugged terrain and climate. One of the major problems of the logging industry is how to transport the trees, after they have been cut down, to the coast where the sawmills are located. Use is made of rivers, lakes and sea inlets but most of the trees are transported at some stage of their journey on a logging truck. These trucks are very powerful and are usually operated only on private logging roads built by the logging companies. Such roads do not have a tarmacadam surface but are literally carved out of the forest. Because of the mountainous topography these roads will almost certainly contain many sharp bends and steep grades and their construction and maintenance are very expensive. The cost is roughly proportional to the surface area of the road and hence they are made as narrow as possible, usually single track with passing places for returning empty trucks. There are interesting timetabling problems which can be formulated about the most efficient frequencies at which loaded and unloaded trucks should be dispatched along a given section but I was asked for advice about the actual road design.

The problem concerns not so much new logging roads as necessary modifications to existing roads so that they can handle more powerful trucks carrying longer loads. Thus power is available to pull a trailer loaded with 100 ft trees but the roads are designed for trailers with 50 ft trees. This gives rise to no difficulties on a straight road but on a bend the road has to be wider in order that the load may go smoothly round. The question is how much wider and on what path should the truck be steered through the bend. In general the bends are not too acute and are designed so that the truck can negotiate them at a reasonable speed. However, from the mathematical model we develop we shall be able to discuss the manoeuvring possibilities of the truck on a very tight corner such as might be experienced at low speeds on loading or turning round. It is this need for manoeuvrability which determines the basic design of the truck and is assumed to be prescribed.

2. Mathematical model

A typical logging truck consists of a tractor and trailer freely pivoted together. The tractor has a front steering axle and a pair of rear axles above which is mounted a horizontal circular platform free to rotate about a symmetrically placed vertical axis. This platform is called a bunk and the trees or logs are chained to it. The trailer has only a pair of rear axles with an associated bunk to which the logs are also chained. Thus the distance between the bunks is a constant of the motion and the pull on the trailer is communicated by the logs and bunks rather than through the pivot. The trailer chassis consists solely of two metal cylinders which can slide one inside the other and which join the bunk to the pivot. Thus the chassis length can vary during the motion and enables the tractor and trailer to move independently. The configuration is shown schematically in Figure 1. A and B are the axes of the bunks, distance h apart; XY is the tree with AX equal to λh; C is the pivot with AC equal to ah. A typical value for a is 0·3 and this value can be adjusted before a truck is loaded; in a simple towing situation a would be zero. FF is the tractor front axle, PP and QQ are the tractor rear axles, RR and SS are the trailer rear axles, all of length $2L$. We assume the truck has width $2L$ and the load has width $2W$ at its rear end X. The sweep of the truck is the maximum deviation of the load from the path steered by the tractor and the maximum value obviously occurs at the rear end X.

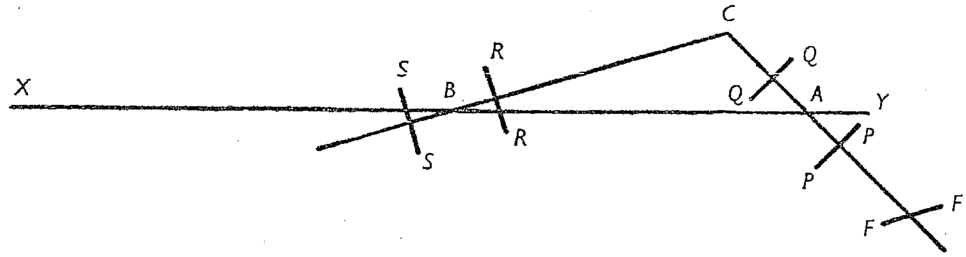

Figure 1. Truck configuration.

We consider a single track road of width $2\beta h$ and a typical bend in the road is assumed to be a circular arc of radius h/α centre O as shown in Figure 2. For simplicity we assume that the truck enters the bend with the tractor and trailer in

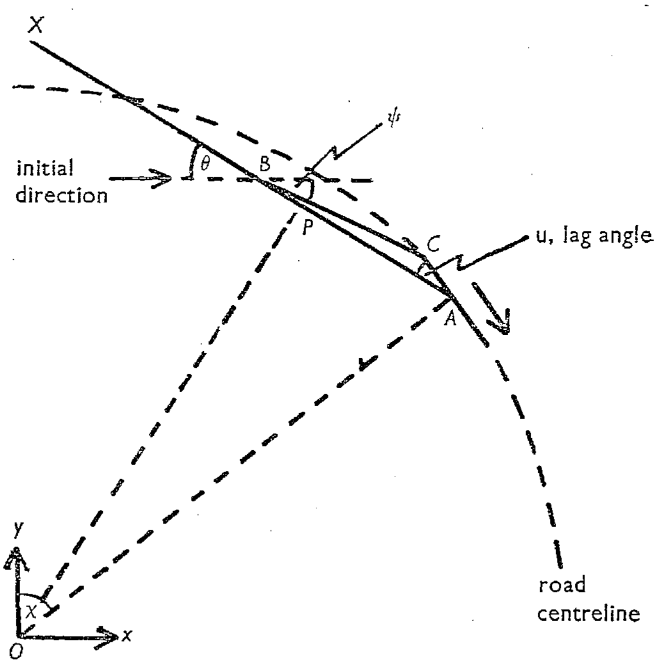

Figure 2. Road configuration.

line and that the driver steers so that the axis of the front bunk A follows the centre line of the road. The position of A round the bend can be defined by the angle χ which the tractor AC makes with the initial direction, and it is convenient to define x, y axes, origin O, parallel and perpendicular to this initial direction. At a general position the log will make an angle θ with the initial direction and the log angle BAC between the log and tractor we define to be $u = \chi - \theta$. The required road half-width h to allow for the sweep of the log end is then $OX - OA + W$. On the inside of the bend the required road width will be $OA + L - OP$ where OP is the perpendicular from O onto AB.

To complete the mathematical model we need one further assumption that there is little or no sideways slip on the wheels†. In fact for a pair of axles rigidly joined there must be some slip on each wheel but we require the mean slip to be zero so that the centres A and B of the axle pairs have no velocity components normal to the tractor and trailer centreline respectively. For the tractor this condition implies that the tractor centreline AC is tangential to the circular path of A so that OA is perpendicular to AC and χ is the angle moved by A round the bend. An alternative definition of the curvature of a bend used in reports about highway design is to define it by the angle $N°$ round the bend which corresponds to an arc length of 100 feet. In our notation

$$N = \frac{180}{\pi} \frac{100\alpha}{h}, \tag{1}$$

where h is measured in feet. Thus for $h = 30$ ft and $\alpha = 0{\cdot}1$, there is a 19° bend, and for $h = 40$ ft and $\alpha = 1{\cdot}0$, $N = 142°$. We need only consider values of α in the range $0 < \alpha < 1$ for all practical driving situations. In loading or turning round manoeuvers, it will be desirable to have α as large as possible.

The log length λh will be greater than h but is unlikely to exceed $3h$ in a practical situation so that we only consider values of λ in the range $1 < \lambda < 3$; a will be assumed given in the range $0 \leqslant a < 0{\cdot}5$; the actual value of h can be adjusted from load to load but is usually in the range of 30–40 feet.

3. Mathematical analysis

Since the tractor wheels do not slip the position of A is given by

$$x = \frac{h}{\alpha} \sin \chi, \quad y = \frac{h}{\alpha} \cos \chi.$$

The position of B is then given by

$$X = \frac{h}{\alpha} \sin \chi - h \cos \theta, \quad Y = \frac{h}{\alpha} \cos \chi + h \sin \theta. \tag{2}$$

† The first time I looked at the problem I was not told that the trailer chassis length was variable so the wheels had to slip. The analysis is then more complicated and requires the use of simple dynamical principles, that is resolving forces and taking moments for the system. The results I obtained might be relevant for a logging truck on ice with the trailer chassis mechanism frozen solid!

Since the trailer wheels do not slip, the point B moves in the direction BC and

$$\frac{dY}{dX} = -\tan\psi, \tag{3}$$

where ψ is the angle BC makes with the initial direction. From the geometry of triangle ABC

$$\frac{\sin(\chi-\psi)}{h} = \frac{\sin(\theta-\psi)}{ah} = \frac{\sin u}{bh}. \tag{4}$$

ψ, θ and u are all functions of χ and from (3), using (2),

$$\left(-\frac{h}{\alpha}\sin\chi + h\frac{d\theta}{d\chi}\cos\theta\right)\cos\psi + \left(\frac{h}{\alpha}\cos\chi + h\frac{d\theta}{d\chi}\sin\theta\right)\sin\psi = 0.$$

This reduces to

$$\sin(\chi-\psi) = \alpha\frac{d\theta}{d\chi}\cos(\theta-\psi). \tag{5}$$

If we now eliminate ψ between Equations (4) and (5) with a given, we obtain a first-order non-linear differential equation for θ or u in terms of χ, recalling that $u+\theta = \chi$. Initial conditions will be that $\theta = 0$ when $\chi = 0$.

After some simple manipulation we obtain from (4) and (5),

$$\frac{du}{d\chi} = 1 - \frac{\sin u}{\alpha(1-a\cos u)}; \quad u(0) = 0. \tag{6}$$

This can be very easily integrated numerically and typical solution curves are sketched in Figure 3 for appropriate values of α. They have the property that $du/d\chi > 0$ for all values of χ, and u has a limit c for large values of χ. The value

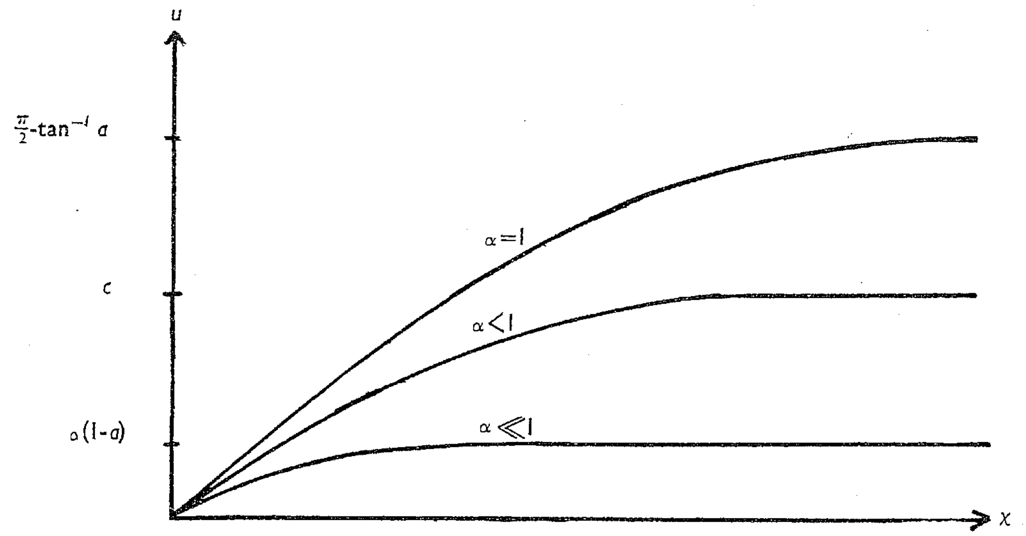

Figure 3. Lag angle.

of c will be given by $\alpha(1-a\cos c) = \sin c$, which reduces to

$$c = \arcsin\left\{\frac{\alpha(1-a(1-\alpha^2+a^2\alpha^2)^{\frac{1}{2}})}{1+a^2\alpha^2}\right\}. \tag{7}$$

When $\alpha = 1$, $c = \frac{1}{2}\pi - 2\arctan a$; and when $\alpha \ll 1$, $c \sim \alpha(1-a)$. The limiting value c is approached exponentially so that $c-u \sim e^{-\gamma\chi}$ where

$$\gamma = \frac{\alpha(\cos c - a)}{\sin^2 c}.$$

For small α,

$$\gamma \sim \frac{1}{\alpha(1-a)} \text{ and is large;}$$

for $\alpha = 1$,

$$\gamma = \frac{a(1+a^2)}{1-a^2}.$$

We can now calculate the sweep of the log end, given by $OX - OA + W$ in Figure 2. By simple geometry

$$\begin{aligned} OX^2 &= \left(\frac{h}{\alpha}\sin\chi - \lambda h\cos\theta\right)^2 + \left(\frac{h}{\alpha}\cos\chi + \lambda h\sin\theta\right)^2 \\ &= h^2\left(\frac{1}{\alpha^2} + \lambda^2 - 2\frac{\lambda}{\alpha}\sin u\right). \end{aligned} \tag{8}$$

The sweep decreases as χ increases since the lag angle u increases. Thus the maximum road width βh required is given by

$$\beta = \left(\lambda^2 + \frac{1}{\alpha^2}\right)^{\frac{1}{2}} - \frac{1}{\alpha} + \frac{W}{h}.$$

Curves of $\beta - (W/h)$ against α for various values of λ are shown in Figure 4.

Also shown in Figure 4 as a broken line is the required value of $\beta - (L/h)$ for clearance on the inside of a bend for any truck on any bend. At any point, for a given truck, the required

$$\beta = \frac{1}{\alpha}(1-\cos u) + \frac{L}{h} < \frac{1}{\alpha}(1-\cos c) + \frac{L}{h}$$

for any point on the bend. Since c decreases with a so that the simple towing situation $a = 0$ produces the greatest lag angle,

$$\beta - \frac{L}{h} < \frac{1}{\alpha}(1-\cos c)_{a=0} = \frac{1-(1-\alpha^2)^{\frac{1}{2}}}{\alpha}. \tag{10}$$

It is this expression which is shown in Figure 4 as a broken line, together with the value for $a = 0{\cdot}3$.

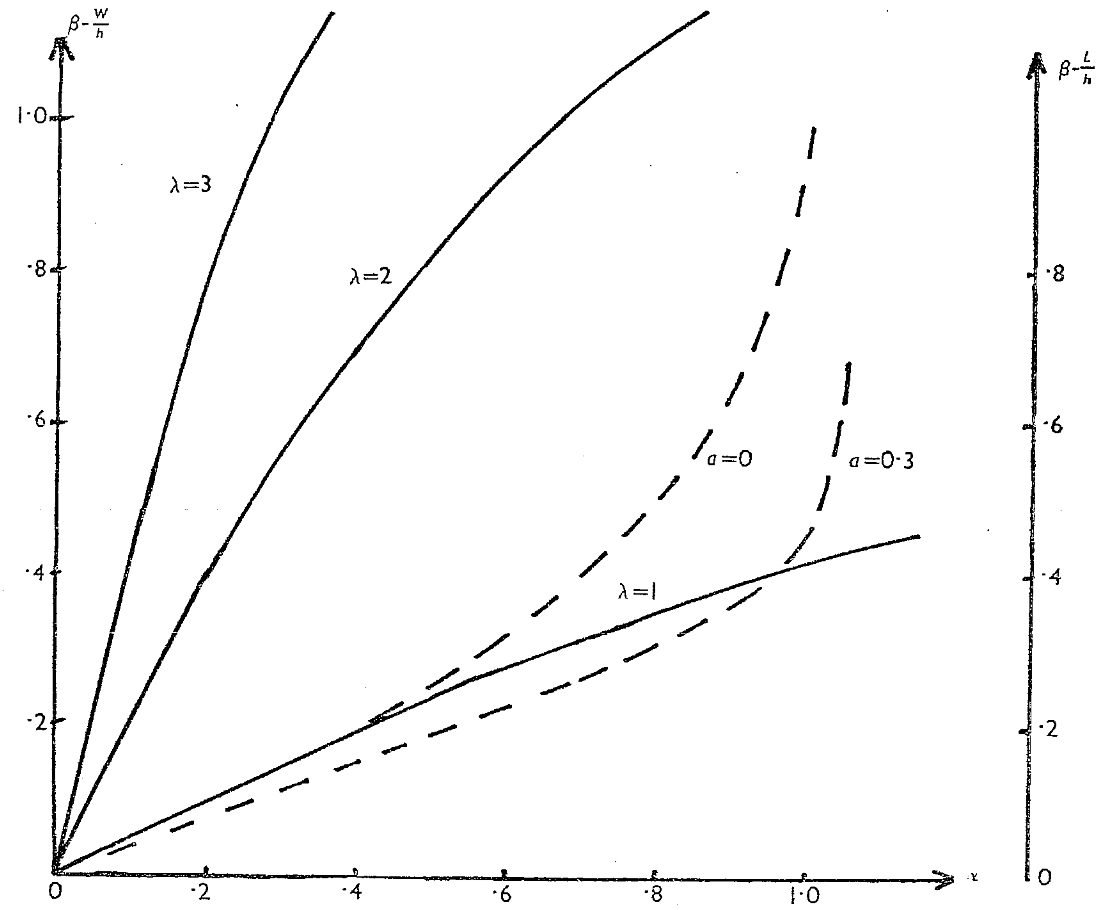

Figure 4. Road half-width.

4. Results and conclusions

A typical example to illustrate the results is as follows. For a truck with a bunk separation of 40 ft, on a curve of radius 200 ft, then $\alpha = 0{\cdot}2$ and it is a 28° bend. If the truck width is 8 ft and the log end width is 4 ft, then for a log of length 80 ft from the front bunk $\lambda = 2$, $W/h = 0{\cdot}05$ and $L/h = 0{\cdot}10$. Thus, from Figure 4, $\beta = 0{\cdot}45$ for the outside of the bend and $\beta = 0{\cdot}2$ for the inside of the bend, to allow for any value of a. The necessary theoretical road half-widths are 18 ft on the outside and 8 ft on the inside of the bend. For a log of length 48 ft from the front bunk and a log end width of 6 ft, $\lambda = 1{\cdot}2$ and $\beta = 0{\cdot}22$ for the outside and is unchanged for the inside. Necessary load half-widths are therefore 8·8 ft and 8 ft respectively. Thus the road needs to be widened by at least 9·2 ft to accommodate the longer load and the driver should steer a path which is no longer roughly down the middle of the road. This may not be so easy to achieve in practice and a road width of 36 ft, that is widened by 20 ft, would be necessary for a bad driver. Thus the major savings would come from employing only skilful drivers who can judge how close to the inside of the bend they should steer. This is not a trivial skill since the typical speeds of these trucks are 30–40 m.p.h. and these speeds are necessary because of the long distances to be travelled.

Our theory has shown that the sweep is largest as the truck enters the bend since thereafter the lag angle increases and assists the truck in accommodating to the bend. This conclusion is still valid when a truck enters a bend with its tractor and trailer not in line, such as when it enters from a reverse curve. The results of Figure 4 are only valid for initial conditions of a truck and trailer in line. If there is an initial lag angle c_0 due to a reverse curve, Equation (6) must be modified to have $u(0) = -c_0$. In this case the necessary road width is given by

$$\beta = \left(\lambda^2 + \frac{1}{\alpha^2} + 2\frac{\lambda}{\alpha}\sin c_0\right)^{\frac{1}{2}} - \frac{1}{\alpha} + \frac{W}{h}. \tag{11}$$

In our first example if the truck entered the bend from a reverse curve of radius 200 ft, there would be an initial negative lag angle c_0, possibly as large as arc sin 0·2. The value of β from (11) would now be 0·85 so that the sweep becomes 34 ft. The truck alignment on entering the bend is clearly the most important feature in the determination of road width and some design allowance must be made for the possibility of negative non-alignment even on a straight road. A compensating feature is that by the time the log end enters the bend there will have been a change in the lag angle due to the curved motion of the tractor. Thus the value of u used in (11) will not be as large as c_0. In addition on the reverse curve the negative lag will not have attained the limiting value c_0. Both these effects are small, except for bends of large curvature, and give more clearance. Rather than calculate the correction due to them they can be used as a safety margin. There will, in fact, have to be a large safety margin of extra road width at points where the road has an inflection, that is a bend is entered from a reverse curve. The best path for the tractor with a long load in switching from the inside of the reverse curve to the inside of the approaching bend is a complicated calculation from our theory and is unlikely to be achieved in practice except by an automatic driver with a programmed steering path.

The value of the variable trailer length mechanism occurs on bends of large curvature. Thus in simple towing when $a = 0$ it is not possible to negotiate a bend with $\alpha > 1{\cdot}0$, that is of radius of curvature equal to the bunk separation length. With a general value for a, however, values of $\alpha > 1{\cdot}0$ are possible and satisfy $\alpha(1 - a\cos c) = \sin c$. Thus α has a maximum value of $(1-a^2)^{-\frac{1}{2}}$ and for $a = 0{\cdot}5$, an extreme value in practice, α has a maximum of 1·25 and a curve whose radius of curvature is 4/5 the bunk separation length can be negotiated. For values of $\alpha > 0{\cdot}5$ it is clear from Figure 4 that considerable saving in road width is achieved by increasing the value of a and it seems likely that in the original truck design when loads were such that λ was not much greater than 1·0, a was chosen so that the necessary road half-width on the inside of any bend was always smaller than that on the outside. Thus a driver was expected to drive down the middle of the road with the knowledge that he was most likely to touch the verge on the outside of the bend with his rear trailer axles. With a long load he will have to drive so that the log end just misses the road edge on the outside of the bend. Even if this skill can be acquired the longer loads will require substantial road widening and it is not clear that it is economical to use the extra power available.

The simple mathematical model proposed in this article depends on the condition of no mean slip on the axles. In a different context such as a car pulling a caravan, the dynamics of the tyres are more important since the weight is so much less. There can be oscillations in the mean slip of the axle which contribute to the observed phenomenon of caravans swaying alarmingly on winding country roads. To construct an appropriate mathematical model for the dynamics of a tyre is difficult and is one of the many problems which challenge the applied mathematician.

Letters to the Editor

Dear Editor,

In response to the question of the title of Underwood Dudley's article in Volume 6 No. 2 of *Mathematical Spectrum* 'Who Was the First Non-Euclidean?', it may be that Gauss, Bolyai, Lobachevsky, *et al.* are all late-comers by a considerable margin.

The readers of this article in *Mathematical Spectrum* will find the following article of considerable interest and relevance: Imre Tóth, Non-Euclidean Geometry before Euclid, *Scientific American* CCXXI No. 5 (1969), 87–98.

The summary preceding the article reads as follows: 'Certain works of Aristotle, written 2,000 years before the advent of non-Euclidean geometry, contain references to the possibility of a non-Euclidean approach to the famous problem of parallels.'

Yours sincerely,

F. J. RAPP
(University of Lethbridge,
Alberta, Canada)

Dear Editor,

New syllabuses for statistics in schools

I was asked by COSE (the Committee on Statistical Education) to write some impressions of the meeting they organized, jointly with the Royal Statistical Society, on 3 November 1973. Two things emerged clearly from a lively and well-attended conference: first, that there is widespread concern about school statistics, and second that something needs to be done for those teachers of mathematics who are not happy in that field. Just what should be done is not clear yet, but I found some features of the new syllabuses encouraging.

Professor Downton (University of Birmingham) introduced the first session, and we were plunged almost at once into a clear and sometimes moving talk from Mr Owen (H.M. Inspector), on the teaching of statistics in first and middle schools. It is remarkable how much is done at this stage to lay foundations for more formal work in probability and statistics. To achieve understanding the teacher needs to be aware of the statistical content of the work, and the various stages of development—otherwise the feeling of progress, and much of the point of the exercise, may be lost. But all the most important ideas can be, and often are, introduced at this level.

Mr Durran (Winchester College) followed with some stimulating quotations, and some examples from his rogues' gallery of examination questions. His criticism was not just cheap sniping, but concerned the setting of questions which test things irrelevant to the main stream of statistics, or which are unlike genuine examples. What is needed is more experience of variability, more critical looking at data, more thinking about models, and more testing of models by simulation. All attempts to determine how statistics can be properly taught in schools have, so far, failed, and a major survey of this area is needed. Many important points were raised in the discussion that followed: relations with industry, inclusion of projects, teaching by non-mathematicians; but these are too long to report here.

Dr Evans (University College of Wales, Aberystwyth) showed how the Welsh Joint Education Committee had met the increasing demand for Pure Mathematics with Statistics by devising a syllabus comparable in its mathematical demands with Pure Mathematics with Mechanics. It is intended to be a sound introduction to further study and later use, assumes no prior knowledge, and is formulated in considerable detail. Statistical inference forms the core of the syllabus; unusual features are emphasis on conditional probability, and on expectation, the use of joint probability distributions to establish properties of expectation and variance, and comparison of various unbiased estimators. A full copy of his talk can be obtained directly from him.

Mr Warwick (London University School Examinations Department) and Mr Pennycuick (Chislehurst and Sidcup Grammar School) reported on a pilot statistics paper in the London mathematics A level examination, examined for the first time this year. The new feature is that candidates are encouraged to bring project work into the examination room, and that most of the questions are to be answered, at least in part, by reference to it. There are real difficulties in the scheme: time planning and carrying out suitable projects, types of material needed by teachers, and the difficulty of devising half-questions which make project experience necessary. But preliminary indications seem satisfactory. Though perhaps simpler in content, the course is more demanding for teachers, and needs greater maturity from students, than more usual ones.

Professor Gani (University of Sheffield) directed a final discussion, where some of the unanswered questions were vigorously raised:

How closely is statistics related to mathematics?

How can teachers be suitably trained? Should some be statisticians?

Who is going to produce suitable textbooks?

How can teachers' fears of the subject be laid to rest?

How can examinations test real familiarity with statistical material?

It was a full but useful day. It drew out contrasting views on the subject, aired genuine difficulties, gave us two new schemes, and aroused the widespread interest in the problems COSE was set up to study. I remain grateful to all those responsible for a highly constructive conference. Let us hope more action follows!

Yours sincerely,

L. E. ELLIS

(Marlborough College)

Dear Editor,

Complex variable and fluid flow—a new intuitive link

As a student I was always vexed by the *minus* sign which 'intrudes' into the pair of results:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{(CR1)} \quad \text{and} \quad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{(div).}$$

These are respectively the first Cauchy–Riemann equation and the condition for zero divergence in two-dimensional fluid flow with velocity components (u, v). A similar *minus* sign occurs as between the second Cauchy–Riemann equation and curl, in an obvious notation. Here is an interpretation.

If $w = f(z)$ is differentiable we have $dw = f'(z)\,dz$ in the neighbourhood of a fixed z. In fluid flow, w is the complex velocity $u+iv$ at this point z, and $w+dw$ that at a neighbouring $z+dz$ so that dw is the relative velocity.

This relative velocity field dw may be indicated by an arrow drawn at any point $z+dz$, and the circle $|dz| = h$ is a convenient set of these points. If $f'(z)$ has modulus k and angle α then $dw = f'(z)\,dz$ to sufficient accuracy, and so dw is dz enlarged by a factor k and turned through α. The relative velocity field dw is then as shown on Figure 1 which is drawn for the particular values: k about 0·4 and α about 30°.

Clearly this field has positive divergence since the flow is outwards at all points and equally it has a curl in the anticlockwise sense. But the rotational symmetry of the field makes it intuitively obvious that, overall, the *horizontal* components du and *vertical* components dv make equal contributions to the divergence. A similar remark applies to the curl. To assist your intuition, just consider (Figure 2) the du at a point on the

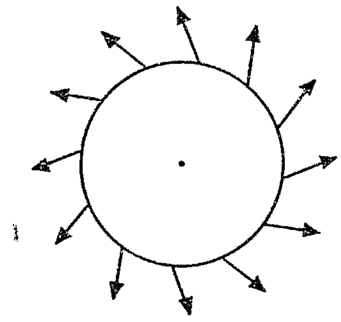

Figure 1

circle where $\arg(dz) = \theta$ and the dv where $\arg(dz) = \theta+\frac{1}{2}\pi$; these are equal, with value $hk\cos(\theta+\alpha)$, and make equal angles $(\pi-\theta)$ with the corresponding radii dz, so that they contribute equally to the divergence or to the curl.

Since du, dv make equal contributions, du and $-dv$ would make no contribution. In other words, the velocity field $u-iv$ has *zero* divergence and *zero* curl.

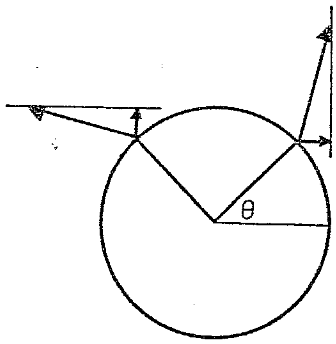

Figure 2

Hence, to create a two-dimensional irrotational incompressible flow leaving aside the potential for the moment, write down any differentiable function, $u+iv$, of z and consider $(u, -v)$; this is your velocity field.

The above argument is correct to the first order of small quantities but may be strengthened by considering $\delta w = f'(z)\,dz+\frac{1}{2}f''(z)\,dz^2$. The additional velocity field represented by the term $(dz)^2$ prescribes equal vector velocities at points dz and $-dz$ and hence equal components Δu, say, and equal Δv at these points. Since these points are diametrically opposite on our circle it is obvious (Figure 3) that this pair of velocities

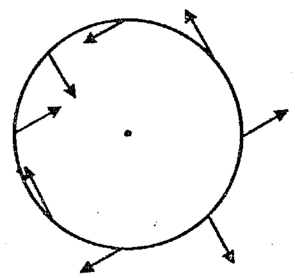

Figure 3

Δu contribute zero to the divergence and zero to the curl; the same is true of Δv and thus of $-\Delta v$. So finally the terms in dz^2 contribute zero divergence and zero curl. As a matter of fact the same sort of argument applies to the term in $(dz)^3$, and so on, but a full treatment along these lines would strain the present informal geometrical atmosphere.

Yours sincerely,
H. M. FINUCAN
(University of Queensland, Australia)

Problems Submitted by Readers

Dear Editor,

A tactical problem in aerial warfare

Country A has invaded Country B without warning. A has a line of S.A.M. bases at intervals close to their common frontier. B has some supersonic airplanes fitted with an anti-S.A.M. device which leads the missiles astray. B wishes to destroy an S.A.M. base and for this purpose has a powerful laser beam of destructive magnitude which can be fixed at any angle to the flight axis of the plane. If this were a right-angle the plane must fly in a circle and this flight path could easily be found by the ground-based computer range finders of A.

B has therefore arranged that their plane shall fly in a logarithmic spiral ($r = e^{a\theta}$) with the enemy target at the origin and the laser beam fixed at an angle ϕ to the flight axis of the plane, where $a = \cot\phi$, so that the laser beam always passes through the target at the origin.

The following questions are set for solution.

(a) How many fixations of position of the plane must the ground range finder of A make to determine the curve on which the B plane is travelling?

(b) Is it possible to fix the constant a and the speed of the plane so that the sonic boom also would pass through the target for at least a few seconds?

Yours sincerely,
W. H. CARTER (Major)
(69 Viceroy Court, Lord Street,
Southport, Lancs PR8 1PW)

Dear Editor,

Ptolemy's Theorem

Your readers may be interested in providing two proofs of Ptolemy's Theorem. Ptolemy of Alexandria (A.D. 87–165) wrote the *Almagest*, a compendium of astronomy, and produced some interesting geometrical proofs in it. Would your readers be interested in providing a geometric and a trigonometric proof to the following theorem?

Theorem: In a cyclic quadrilateral the product of the two diagonals is equal to the sum of the products of the opposite sides.

Yours sincerely,
A. B. PATEL
(V. S. Patel College of Arts and Science,
Billimora, India)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in a subsequent issue. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

In future, solutions to problems will be given in the second issue following that in which the problems are set. Thus the solutions to problems in Volume 6 No. 2 will appear in Volume 7 No. 2 (January 1975).

This change is necessary because *Mathematical Spectrum* is now published in annual volumes of three issues instead of two, so that the time between the publication of one issue and the press date for the next is too short to allow readers time to send in their solutions.

Problems

7.1. Two projectiles are fired from a point O at the same time. Describe how the direction and length of the straight line joining the projectiles vary with time during the subsequent flight. (Air resistance can be neglected.)

7.2. Let $A_1 A_2 \ldots A_n$ be a regular plane polygon with centre O, and let P be a point in the plane outside the circumcircle of the polygon. Compare the geometric mean of the lengths $A_r P$ $(1 \leqslant r \leqslant n)$ with the length OP in the following two cases: (i) when OP passes through a vertex of the polygon; (ii) when OP bisects a side of the polygon.

7.3. (Submitted by B. G. Eke, University of Sheffield.) The real numbers $a_1, a_2, a_3, \ldots$ are positive, less than 1 and such that

$$a_n < \tfrac{1}{2}(a_{n-1} + a_{n+1})$$

for $n = 2, 3, \ldots$. Show that a_n tends to a limit as n tends to infinity.

Book Reviews

Studies in Structure. By JOAN M. HOLLAND. Macmillan Publishers, London, 1972. Pp. v+235. £1·95.

The study of structures plays an important role in mathematics, for it enables one to answer such questions as: 'What are the essential features of the real number system which makes it possible to do calculus?' (or of the complex number system which make it possible to find roots of polynomial equations). When one knows the answers, it is possible to apply, for example, the methods of the calculus to problems having a certain 'family resemblance', which is what structure is all about. It is rather like taking a clock to pieces to see what makes it tick. Unfortunately some books on mathematical structure resemble the activities of a clock repairer who has forgotten that the function of a clock is to tell the time (and to have aesthetic appeal!). So it was a pleasant surprise to find that the author has approached her task by investigating genuinely interesting topics and developing the structural ideas in the context of those examples. The horological simile is left to the reader.

The structures with which this book is concerned are the basic algebraic ones of group, ring and field and quite properly the author introduces them through examples drawn from the theory of numbers and geometry. That indeed was where they originated, but the examples chosen are unusual enough to attract the attention of readers who are already familiar with the structures, as well as those meeting them for the first time, and the book leaves one with the impression of an enthusiastic author who actually enjoyed writing the book.

The first three chapters contain introductory material on algebra, which could offer refreshing revision to pupils working for A-level.

In Chapter 4, we meet the author's favourite topic: the Fibonacci numbers. The sequence 1, 1, 2, 3, 5, 8, 13,... in which each term after the second is obtained by adding the previous two terms, has excited the interest of professional and amateur mathematicians for centuries and one readily shares that fascination in this book. As well as the familiar curiosities of phyllotaxis (leaf arrangement in plants) and the well known connection with the golden section and approximation to the number $f = (-1+\sqrt{5})/2$, which are interestingly dealt with here, there are less well known investigations on the properties of the sequence $(f+1)^n$, where n runs through all integers. Just as one can use a prime number to produce a field from the integers (modular arithmetic); so one can reduce the terms of that sequence modulo a prime to obtain—what? The answer is: sometimes a field and sometimes only a ring. The criterion for the two cases involves the possibility of finding a 'square root of 5 to the given prime modulus'; so introducing the important concept of quadratic reciprocity.

There are chapters on finite fields, on permutation groups, on the symmetric group and bell-ringing and on groups of symmetries of patterns. And all the time interesting new ideas to think about. In short, the book ought to be in school and college libraries: and it is even worth buying for oneself.

University of Nottingham. J. V. ARMITAGE

Introducing Real Analysis. By DAVID FOWLER. Transworld Publishers Ltd, London, 1973. Pp. iv+101. £0·80.

This is a lovely book, written by a mathematician who knows the flavour and direction of his subject at university level. His clear understanding and interest are easily seen, and he has brought these within the reach of readers wishing to appreciate not only analysis but also the attitude of mind of a professional mathematician. The book could profitably be read by students at school interested in mathematics, possibly before they reach the sixth form; I would recommend very strongly that anyone intending to study mathematics at university should read it beforehand; so many of our prospective mathematicians find the transition from school to university mathematics something of a shock.

The book begins by introducing the student to questions relating local and global properties of real valued functions of a real variable. In particular, the problem of whether a locally constant real valued function, defined on an interval, is a constant function leads naturally to the basic difference between the real numbers and the rational numbers, namely that of completeness. The need for the concept of 'neighbourhood' arises naturally, and this is used in a well-motivated discussion of continuity. The usual theorems on continuous real valued functions defined on a closed bounded interval are well presented and proved with a minimum of unnecessary concepts.

The chapter on differentiability leads up to the mean value theorem and to Taylor's theorem. The definition of differentiability used is one which generalises naturally to functions defined on subsets of Euclidean space of more than one dimension, a situation which the author has kept well in mind, although it falls outside the scope of this book.

I believe Dr Fowler has achieved his aim of bringing mathematical analysis within the scope of anyone interested enough to read his book.

University of Durham R. S. ROBERTS

Introduction to Mathematical Analysis. By C. R. J. CLAPHAM. Routledge and Kegan Paul, London, 1973. Pp. 83. £0·75.

The book is an introduction to some of the most important topics in real analysis. The topics covered are the axioms for the real numbers, sequences, series, continuous functions, differentiable functions, and the Riemann integral.

Little previous knowledge is assumed, and the treatment is readable and thorough, leaving no gaps, so that one is able to follow the thread in each chapter.

In the chapter on differentiable functions Rolle's Theorem and the mean value theorem are proved. It is very helpful that, for each theorem, the geometrical meaning is explained before the statement and proof are given. At the beginning of the proof of the mean value theorem, there is a good explanation of why the particular function defined is chosen.

The final and longest chapter is concerned with the Riemann integral. In the introduction to this section, the author explains why he has chosen this approach to integration in preference to the one used at an elementary level where it is common to define integration as the reverse of differentiation and to connect it to the idea of areas under curves.

There is a good stock of examples in the main text of the book to illustrate the definitions and theorems introduced. Sometimes the completion of a proof is left as an exercise to the reader. The first time a term is introduced, heavy type is used.

At the end of each chapter there are between eight and fourteen exercises, with answers at the back to those questions requiring a definite answer rather than, say, a proof. These exercises both test what has been presented in the chapter and introduce new results that follow on.

The book would certainly be very useful to undergraduates in mathematics for courses in calculus and real analysis.

Grey College, Durham D. J. OVERTON

Trigonometry: An Analytic Approach. By I. DROOYAN, W. HADEL and C. C. CARICO. Collier–Macmillan Ltd, London, 1973. (Second Edition.) Pp. ix+397. £4·95.

Say 'Trigonometry' to pupils in this country, and their first reaction probably involves right-angled triangles. They would certainly be surprised to find that, in this book on trigonometry, right-angled triangles first appear on p. 178! In fact, this book illustrates neatly the difference between the traditional teaching method (which proceeds by easy stages from an intuitional approach based on the known and familiar to the formal, axiomatic treatment of an abstract system), and the reverse order which is sometimes urged today.

Beginning with set notation and the definition of a function as a set of ordered pairs, the book first introduces the circular functions (including inverse functions) in considerable detail, together with the associated formulae (usually known as trig formulae), including $\cos(x_1+x_2)$, etc. It is only after this careful and thorough development, half-way through the book, that angles are introduced (the *measure* of an angle is carefully explained); this is followed by trigonometric ratios and the trigonometric functions in their full generality, culminating in work on the solution of triangles. Two final chapters deal with vectors and complex numbers including De Moivre's Theorem. There are four Appendices; the first is a set of tables, the second a list of axioms and properties of the real numbers, the third is devoted to logarithmic functions and the use of base 10 logarithms for calculations, and the last to special triangle formulae such as $\sin A/2$ in terms of s.

The text is beautifully printed in two colours and the logical development is carefully carried out; this together with the 'Selected Answers' make for a book from which a pupil could work on his own with a little guidance. It will be clear from the order and method of treatment that the book is unlikely to be used for class teaching in schools. A further difficulty is that all examples drawn from mechanics use imperial units. But a copy in the library would be valuable for all teachers and sixth form pupils to browse through, *and* to study.

Clifton College JOHN HERSEE

Complex Numbers. By J. WILLIAMS. George Allen & Unwin Ltd, London, 1972. Pp. 88. £1·90, hardback; £0·80, paperback.

In reviewing a book for students, a reviewer usually attempts to answer such questions as: what was the purpose of this book, to whom is it addressed, and does it seem to fulfil its purpose? It is the advertised intention of this book to 'provide an inexpensive source of fully solved problems' in complex numbers, catering mainly for 'first year . . . students in mathematics, engineering and physical science'. It is suggested that the book be used in conjunction with standard lecture courses; the volume claims to cover 'all the essential manipulation of complex numbers up to . . . simple mapping', and 'corresponds to a course read by all [*sic*] first year undergraduate students of mathematics, physical science and engineering and those at equivalent levels in colleges of technology . . .'.

Whether these students need such a book I am not in a position to say. Certainly it is unsuitable at school level, or for those meeting complex numbers for the first time; complete absence of motivation, from the very outset, disqualifies it on that score. At £1·90 for 88 pages it can scarcely be described as inexpensive, and it is difficult to see how the substantial hard cover can be justified. The paperback version is adequate for the purpose the book will serve, but at £0·80 it is not cheap.

In general the material is sound and carefully presented. Printing and layout tends to be too compressed (e.g., p. 43); the book would have benefited from a more generous allowance of space, for example, by the use of two-line fractions (see p. 50, problem 3.29).

The examples tend to be a bit disjointed and appear in somewhat haphazard sequence as if chosen at random. Some of the solutions are quite neat and brief, some elegant and some ingenious. But alas, there are many that are unnecessarily long-winded and laborious, largely owing to the author's reluctance to use a geometric approach.

This is a serious defect and adversely affects the value of the book in so far as a large number of problems could have been solved much more easily by using a geometric approach. This illustrates the difficulties of the student who has not been taught a certain body of geometry at school. Perhaps the best way of redeeming the book would be by inserting a short preface to read: 'the book is specially addressed to students whose background of school geometry is weak, and who would thus find difficulty if confronted with a geometric treatment . . .'!

One cannot resist the conclusion, however, that they might be better employed learning some geometry.

Royal Grammar School, Newcastle upon Tyne F. J. BUDDEN

Numerical Mathematics. By A. J. MOAKES. Macmillan, London, 1973. (Third Edition.) Pp. xi+96. £0·90.

This revision comes conveniently at a time when it is not easy to find suitable books for sixth-form numerical work. Most either omit important topics or are aimed at undergraduates.

Numerical Mathematics covers all four of the main topics found in most modern syllabuses (notably M.E.I.); these are the solution of $f(x) = 0$ by iterative methods, the solution of a set of linear equations, numerical integration and numerical differentiation. The text would serve a sixth form or a college of education well.

The significant addition is a comprehensive set of revision exercises, many of them from past examination papers. Textual revisions are also welcome: the use of a table of differences to detect errors is elaborated, and von Mises' iteration is introduced. This saves some of the labour in Newton's iteration. An appendix has been added on books of function tables, and the most useful bibliography has been updated.

The text may appear to be primarily about the hand-calculating machine. It does teach the techniques and some subtleties of its use yet it aims higher than a mere functional level. It forms an introduction to numerical mathematics, its sophistications and its limitations; and unashamedly the hand-calculating machine is used as the teaching aid.

Help is given to the student in the important area of work layout, and stress is laid on the need for checking procedures. Some will find the format unusual at first. From Chapter 4 onwards the left-hand pages are reserved for notes and the right-hand for the exercises. One finds oneself wanting to read both pages simultaneously.

The notes are useful and contain some gems. Some points could have been elaborated, but this is where the theory underlying the techniques can be developed in more detail in the classroom situation. In any case the text would then have become long and cumbersome, also dearer than its present respectable price of £0·90.

St Paul's School, London S.W.13 R. W. BAYLIS

A Practical Course in SL/I, Subset of PL/I. By C. TOMASSO. Foulsham (W.) & Co, Ltd, Slough, 1971. Pp. 144. £1·00.

This is a useful book for introducing sixth formers to computing through SL/I, and would enable students to write worthwhile programs in both the business and scientific fields. No detailed knowledge of computers is assumed, the exercises are well graded and the book is suitable for individual working, preferably with computer access readily available.

Durham Technical College G. WILLMOTT

Notes on Contributors

David Burghes was first an undergraduate, later a postgraduate student, and finally a Lecturer in Applied Mathematics at the University of Sheffield; but three years ago he moved to the University of Newcastle. His principal field of research is the subject with the formidable appellation of magnetohydrodynamics. He has a growing interest in mathematical education.

Nachum L. Rabinovitch is Principal of Jews' College, London, and Professor of Rabbinic Literature there. He has been a Lecturer in Mathematics at the University of Toronto and has published numerous papers on the history of mathematics as well as a book on early developments in probability in addition to books on rabbinics and religion.

John Pym, a graduate of Birmingham and Cambridge, is a Reader in Pure Mathematics in the University of Sheffield and the author of numerous papers, predominantly in modern analysis. He also maintains a lively interest in mathematical logic and the foundations of mathematics, although in these fields he is anxious not to relinquish his amateur status.

Nicholas A. Draim graduated from the U.S. Naval Academy in 1922 and later obtained the degree of M.Sc. in naval architecture at Massachusetts Institute of Technology and LL.B. from Georgetown Law School. He is a Captain, U.S. Navy, Retired, with a varied naval and legal career. Captain Draim is the inventor and holder of early patents in aviation design and construction. He is an amateur mathematician with published contributions in *The Higher Arithmetic* by H. Davenport, and *The Mathematics Teacher*.

N. A. J. Hastings, a graduate of Cambridge and Birmingham, is a Senior Lecturer in Management Science at the University of Bradford. His interests lie in the application of mathematical and computational methods to management problems and, in particular, he has worked on the development of the Dynacode dynamic programming system and the Netcode project scheduling system. His hobbies include surfing and fell walking.

A. B. Tayler is Fellow and Tutor in Mathematics at St Catherine's College, Oxford, and recently spent a year as Visiting Professor at the University of Victoria, British Columbia, Canada. His research interests are in mathematical modelling, differential equations and fluid mechanics, and he actively solicits novel mathematical problems through the Oxford Study Groups with Industry. He has been an organiser of the Oxford Scholarship and entrance procedure, and initiated the negotiations which have resulted in his college becoming co-educational from 1974.

Contents

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