Crux Mathematicorum

Published by the Canadian Mathematical Society.

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 17 #8

October / octobre

1991

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Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1991 subscription rate for ten issues is \$ 17.50 for members of the Canadian Mathematical Society and \$35.00, for non-members. Back issues: \$3.50 each. Bound Volumes with index: volumes 1 & 2 (combined) and each of 3, 7, 8 & 9: \$10.00 (Volumes 4, 5, 6 & 10 are out-of-print). All prices are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

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577 King Edward
Ottawa, Ontario, Canada K1N 6N5

ACKNOWLEDGEMENTS

The support of the Department of Mathematics and Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

Published by the Canadian Mathematical Society
Printed at Ottawa Laser Copy

ISSN 0705-0348 Second Class Mail Registration Number 5432

THE OLYMPIAD CORNER

No. 128

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Since we still haven't heard from our IMO team representatives about the events in Sweden, I am putting off until the next issue a discussion of this year's contest. Anyway, we still have some of the problems from last year's IMO in China that were proposed to the jury, but not used. Once again I would like to thank Andy Liu, of the University of Alberta, who sent them to me. I also must correct a mistake I made in the last issue. Andy was an *observer* at the marking sessions in China but not a trainer of the team. I had remembered that he has helped to train the Hong Kong team in the past, and confused the rest. My apologies.

UNUSED PROBLEMS FROM THE 31ST IMO

1. *Proposed by Hungary.*

The incentre of the triangle ABC is K. The midpoint of AB is C_1 and that of AC is B_1. The lines C_1K and AC meet at B_2, the lines B_1K and AB at C_2. If the areas of the triangles AB_2C_2 and ABC are equal, what is the measure of $\angle CAB$?

2. *Proposed by Ireland.*

An eccentric mathematician has a ladder with n rungs which he always ascends and descends in the following ways: when he ascends each step, he covers a rungs, and when he descends each step, he covers b rungs, where a and b are fixed positive integers. By a sequence of ascending and descending steps, he can climb from ground level to the top rung of the ladder and come back down to ground level again. Find, with proof, the minimum value of n expressed in terms of a and b.

3. *Proposed by Norway.*

Let a and b be integers with $1 \le a \le b$, and $M = [(a+b)/2]$. Define the function $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ by

$$f(n) = \begin{cases} n+a & \text{if } n < M, \\ n-b & \text{if } n \ge M. \end{cases}$$

Let $f^1(n) = f(n)$, and $f^{i+1}(n) = f(f^i(n))$, for $i = 1, 2, \dots$. Find the smallest positive integer k such that $f^k(0) = 0$.

4. *Proposed by Poland.*

Let P be a point inside a regular tetrahedron T of unit volume. The four planes passing through P and parallel to the faces of T partition T into 14 pieces. Let $f(P)$ be the total volume of those pieces which are neither a tetrahedron nor a parallelepiped. Find the exact bounds for $f(P)$ as P varies inside T.

5. *Proposed by Poland.*

Prove that every integer $k > 1$ has a multiple which is less than k^4 that can be written in the decimal system with at most four different digits.

6. *Proposed by Romania.*

Let n be a composite positive integer and p be a proper divisor of n. Find the binary representation of the smallest positive integer N such that

$$\frac{(1 + 2^p + 2^{n-p})N - 1}{2^n}$$

is an integer.

7. *Proposed by Romania.*

Ten localities are served by two international airlines such that there exists a direct service (without stops) between any two of these localities, and all airline schedules are both ways. Prove that at least one of the airlines can offer two disjoint round trips each containing an odd number of landings.

8. *Proposed by Thailand.*

Let a, b, c and d be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}\,.$$

9. *Proposed by the U.S.A.*

Let P be a cubic polynomial with rational coefficients, and let $q_1, q_2, q_3, \ldots$ be a sequence of rational numbers such that $q_n = P(q_{n+1})$ for all $n \geq 1$. Prove that there exists $k \geq 1$ such that for all $n \geq 1$, $q_{n+k} = q_n$.

10. *Proposed by the U.S.S.R.*

Find all positive integers n for which every positive integer whose decimal representation has $n - 1$ digits 1 and one digit 7 is prime.

11. *Proposed by the U.S.S.R.*

Prove that on a coordinate plane it is impossible to draw a closed broken line such that

(1) the coordinates of each vertex are rational;
(2) the length of each edge is 1; *and*
(3) the line has an odd number of vertices.

*　　*　　*

Repeating ourselves? L.J. Upton, of Mississauga, Ontario writes pointing out that problem 2 [1991: 68–9] was previously discussed in *Eureka* (the original name of *Crux Mathematicorum*), in an article by T.M. Apostol [1977; 242–44].

*　　*　　*

Now we turn to the "archive problems".

3. [1985: 169] *1985 Australian Mathematical Olympiad.*

Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the six segments connecting them are all red.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

A 4-point set as required will be called a *red tetrahedron.* We consider two cases:

Case 1. There is a point that is on at least four blue edges. Consider the four points at the opposite ends of four blue edges emanating from this point. Since there are no blue triangles these four points constitute a red tetrahedron.

Case 2. Every point is on at most three blue edges, hence on at least five red ones. So there are at least 45 red half-edges. But there are an even number of red half-edges, hence there are at least 46. Then there is a point A that is on at least 6 red edges. Let S be the set of 6 points at the opposite ends of 6 red edges from A. As is well known, there are 3 points of S that span a monochromatic triangle, and since there are no blue triangles it must be red. These three points together with A constitute a red tetrahedron.

6. [1985: 170] *1985 Australian Mathematical Olympiad.*

Find all polynomials $f(x)$ with real coefficients such that

$$f(x) \cdot f(x+1) = f(x^2+x+1).$$

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

Substituting $x-1$ for x in the original equation

$$f(x)f(x+1) = f(x^2+x+1) \tag{1}$$

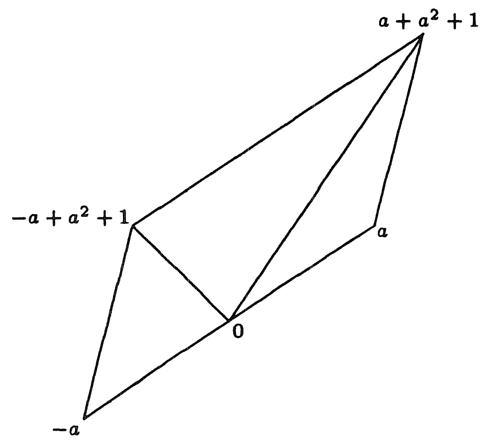
we get

$$f(x-1) \cdot f(x) = f(x^2-x+1). \tag{2}$$

There are two cases.

Case 1. If $f(x)$ is a constant polynomial we have $f(x) \equiv 0$ or $f(x) \equiv 1$.

Case 2. Suppose $f(x)$ is not a constant. Then $f(x)$ has at least one (complex) root. Let a be a root with maximum absolute value. By (1) and (2), $f(a) = 0$ implies $f(a^2+a+1) = 0$, and $f(a^2-a+1) = 0$. Thus $a \neq 0$. If $a^2+1 \neq 0$, then a, a^2+a+1, a^2-a+1, $-a$ are the vertices of a parallelogram, and $|a^2+a+1|$ or $|a^2-a+1|$ is greater than $|a|$, contradicting its choice. So $a = \pm i$, and since f has real coefficients *both* i and $-i$ are roots of $f(x)$ and $f(x) = (x^2+1)^m g(x)$ where m is a positive integer, and $g(x)$ is a polynomial which has real coefficients and is not divisible by x^2+1. By (1)



$$(x^2+1)^m g(x)\cdot(x^2+2x+2)^m g(x+1) = (x^4+2x^3+3x^2+2x+2)^m g(x^2+x+1).$$

Now

$$(x^2+1)(x^2+2x+2) = x^4+2x^3+3x^2+2x+2. \qquad (3)$$

This gives that

$$g(x)\cdot g(x+1) = g(x^2+x+1),$$

i.e. $g(x)$ satisfies the same functional equation as $f(x)$. By the argument at the beginning of this case, $g(x)$ (not being divisible by x^2+1) must be a constant polynomial, and hence, by Case 1, $g(x)\equiv 1$. Thus if $f(x)$ is non-constant and satisfies (1) we must have that

$$f(x) = (x^2+1)^m.$$

On the other hand (3) shows that (1) is then satisfied.

*

1. [1987: 71] *Second Balkan Mathematical Olympiad.*

Let O be the centre of the circle through the points A, B, C, and let D be the midpoint of AB. Let E be the centroid of the triangle ACD. Prove that the line CD is perpendicular to the line OE if and only if $AB = AC$.

Solution by G.R. Veldkamp, De Bilt, The Netherlands.

Set $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$. Then

$$\overrightarrow{OE}=\overrightarrow{OA}+\overrightarrow{OC}+\overrightarrow{OD}=\frac{3}{2}\,\overrightarrow{OA}+\frac{1}{2}\,\overrightarrow{OB}+\overrightarrow{OC}=\frac{1}{6}(3\mathbf{a}+\mathbf{b}+2\mathbf{c})$$

and

$$\overrightarrow{CD}=\frac{1}{2}\,(\overrightarrow{CA}+\overrightarrow{CB})=\frac{1}{2}\,(\overrightarrow{OA}-\overrightarrow{OC}+\overrightarrow{OB}-\overrightarrow{OC})=\frac{1}{2}\,(\mathbf{a}+\mathbf{b}-2\mathbf{c}).$$

Hence CD is perpendicular to OE if and only if $(3\mathbf{a}+\mathbf{b}+2\mathbf{c},\mathbf{a}+\mathbf{b}-2\mathbf{c})=0$. Using the fact that $(\mathbf{a},\mathbf{a})=(\mathbf{b},\mathbf{b})=(\mathbf{c},\mathbf{c})$, this is equivalent to $(\mathbf{a},\mathbf{b}-\mathbf{c})=(\mathbf{a},\mathbf{b})-(\mathbf{a},\mathbf{c})=0$. This just the condition that $OA\perp CB$, or that $AB=AC$.

[*Editor's note.* One direction of this result was discussed in [1991: 105] as a solution to problem 1 of the 1983 British Mathematical Olympiad.]

4. [1987: 72] *Second Balkan Mathematical Olympiad.*

1985 people take part in an international meeting. In any group of three there are at least two individuals who speak the same language. If each person speaks at most five languages, then prove that there are at least 200 people who speak the same language.

Solution by John Morvay, Springfield, Missouri.

The assertion is surely true if some participant has a common language with the other 1984, since $1984/5 > 200$. Thus we assume that some pair $\{P_1, P_2\}$ have no common language. This pair forms 1983 triads with the remaining participants, each of which must have a common language with P_1 or P_2 (or both). It follows that one of the pair, say P_1, has a common language with each of at least 992 participants. Since P_1 only speaks at most five languages, some one of them is spoken by at least 199 of the 992 people. Then that language is spoken by at least $199 + 1 = 200$ people, including P_1.

* * *

We now give solutions for some of the problems proposed but not used on the 1987 IMO in Havana, Cuba. These were given in the October and November 1987 numbers of the Corner.

Finland 1. [1987: 246]

In a Cartesian coordinate system, the circle C_1 has center $O_1 = (-2, 0)$ and radius 3. Denote the point $(1, 0)$ by A and the origin by O. Prove that there is a positive constant c such that for any point X which is exterior to C_1,

$$\overline{OX} - 1 \geq c \ \min\{\overline{AX}, \overline{AX}^2\}.$$

Find the smallest possible c.

Solution by George Evagelopoulos, Athens, Greece.

Denote by D_1 and D_2 the disks bounded by C_1 and the circle C_2 with center A and radius 1. Clearly, $\min\{\overline{AX}, \overline{AX}^2\} = \overline{AX}$ if $X \notin D_2$ and $\min\{\overline{AX}, \overline{AX}^2\} = \overline{AX}^2$ if $X \in D_2$.

If $X \notin D_1 \cup D_2$, set $t = \overline{OX}/\overline{AX}$. Then X lies on the Apollonius circle S_t. On S_t, $(\overline{OX} - 1)/\overline{AX} = t - 1/\overline{AX}$ is minimized when $\overline{AX}$ is minimal; this is clearly the case when X is on the boundary of $D_1 \cup D_2$. If X is on C_2, $\overline{AX} = 1$ and $t - 1/\overline{AX} = t - 1$ is minimized when X is as close to O as possible; this means that X is the intersection X_0 of C_1 and C_2. By some elementary trigonometry, $t = \overline{OX_0} = \sqrt{5/3}$. If $X \in C_1$, one calculates that

$$t - 1/\overline{AX} = \frac{\sqrt{1 + 24\sin^2(\omega/2)} - 1}{6\sin(\omega/2)},$$

where ω is the angle XO_1A. This is an increasing function of ω. So even here $t - 1/\overline{AX}$ is minimized at X_0.

If $X \in D_2 \backslash D_1$, we again consider S_t such that $X \in S_t$. On S_t, $(\overline{OX} - 1)/\overline{AX}^2 = t/\overline{AX} - 1/\overline{AX}^2$. This function of $\overline{AX}$ assumes its minimum either when AX takes its largest value or when it takes its smallest value, i.e., either on the boundary C_2 or the x-axis, where it reduces in either case to $t - 1$ and is minimized at X_0, or on the boundary C_1, where its expression is

$$\frac{\sqrt{1 + 24\sin^2(\omega/2)} - 1}{[6\sin(\omega/2)]^2}.$$

This decreases with ω and is minimized at X_0. So one can choose $c = \sqrt{5/3} - 1$. It is also the smallest possible value of c.

Poland 2. [1987: 248]

Let P, Q, R be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers p, q, r and a polynomial S such that $P = pS$, $Q = qS$ and $R = rS^2$.

Solution by George Evagelopoulos, Athens, Greece.

We prove by induction on $h(P,Q,R) = \deg P^4 + \deg Q^4 + \deg R^2$ that the conclusion holds for

$$\xi P^4 + Q^4 = R^2, \quad \text{with } \xi \in \{-1,+1\}. \tag{1}$$

The case $h(P,Q,R) = 0$ is obvious. Let us assume $h(P,Q,R) > 0$.

Suppose that an irreducible polynomial F divides two of the polynomials P^4, Q^4, and R^2. Then F divides the third one, and the uniqueness of factorization implies $F|P$, $F|Q$ and $F^2|R$, which readily completes the induction step.

Now suppose that P, Q, R are pairwise co-prime. Then $-\xi P^4 = (Q^2 - R)(Q^2 + R)$ where $Q^2 - R$ and $Q^2 + R$ are co-prime. Hence, by the uniqueness of factorization, there are polynomials A, B such that

$$Q^2 - R = \xi_1 A^4, \quad Q^2 + R = \xi_2 B^4 \tag{2}$$

where $\xi_1, \xi_2 \in \{-1,+1\}$. Adding up these two equations we get

$$\xi_1 A^4 + \xi_2 B^4 = C^2, \quad \text{where } C = \sqrt{2}Q.$$

Note that $\xi_1 = 1$ or $\xi_2 = 1$; otherwise $A = B = C = 0$, and then, by (1) and (2) $P = Q = R = 0$. Furthermore A^4, B^4 and C^2 are pairwise co-prime, because A^4 is co-prime with B^4. Finally $h(A,B,C) < h(P,Q,R)$ because

$$\begin{aligned} h(A,B,C) &= \deg A^4 + \deg B^4 + \deg C^2 = \deg P^4 + \deg C^2 \\ &= \deg P^4 + \frac{1}{2}\deg Q^4 = h(P,Q,R) - \frac{1}{2}\deg Q^4 - \deg R^2, \end{aligned}$$

and if Q and R were constant, so would be P. Now, the induction hypothesis implies that A, B, C are constant. Hence, by (1) and (2), also P, Q and R are constant.

Poland 1. [1987: 278]

Let F be a one-to-one mapping of the plane into itself which maps closed rectangles into closed rectangles. Show that F maps squares into squares. Continuity of F is not assumed.

Solution by George Evagelopoulos, Athens, Greece.

We consider an arbitrary rectangle $ABCD$. Let O be the center of the rectangle, and X, Y, Z, T the midpoints of the sides AB, BC, CD, DA respectively. Let P, P_{AB}, P_{BC}, P_{CD}, P_{DA} denote the rectangles $ABCD$, $ABYT$, $BCZX$, $CDTY$, $DAXZ$, and a, b the segments YT, XZ, respectively. Thus we have

$$P = P_{AB} \cup P_{CD} = P_{BC} \cup P_{DA}, \quad a = P_{AB} \cap P_{CD}, \quad b = P_{BC} \cap P_{DA}.$$

We denote by $Q, Q_{AB}, Q_{BC}, Q_{CD}, Q_{DA}, a', b'$ the respective images, i.e. $Q = F(P)$, $Q_{AB} = F(P_{AB})$, etc. We have

(1) $Q = Q_{AB} \cup Q_{CD}, \quad a' = Q_{AB} \cap Q_{CD}$.

This implies that

(2) Q_{AB} and Q_{CD} are rectangles such that for some two parallel sides of Q one of them is a side of Q_{AB} and another is a side of Q_{CD}.

This in turn implies that

(3a) a' is a line segment whose endpoints lie on sides of the rectangle Q and which is parallel to its two sides, or

(3b) a' is a rectangle whose vertices lie on sides of Q and whose sides are parallel to the sides of Q.

From (3a) and (3b), and from similar conditions for the set b', it follows that

(4) the set $a' \cap b'$ consists of a single point iff the sets a' and b' are perpendicular line segments which are parallel to sides of the rectangle Q and whose endpoints lie on sides of Q.

Since $F(O) = a' \cap b'$, the set $a' \cap b'$ is a single point. From this, from the definition of a, b, a', b', and from (4) we obtain the following

Lemma: *F maps line segments into line segments, sides of rectangles into sides of rectangles, vertices of rectangles into vertices of rectangles, and perpendicular line segments into perpendicular line segments.*

From the lemma it follows that if $ABCD$ is a square, then $F(A)F(B)F(C)F(D)$ is its image and it is a rectangle. F maps diagonals of $ABCD$ onto diagonals of its image. Since AC and BD are perpendicular, their images $F(A)F(C)$ and $F(B)F(D)$ are perpendicular, too. But this is possible if and only if $F(A)F(B)F(C)F(D)$ is a square.

* * *

We now return to solutions for problems from the December 1989 number of the Corner. We discuss the first four problems of the *24th Spanish Mathematics Olympiad* [1989: 291].

1. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence of integers such that $x_1 = 1$, $x_n < x_{n+1}$ for all $n \geq 1$ and $x_{n+1} \leq 2n$ for all $n \geq 1$. Show that for each positive integer k there exist two terms x_r, x_s of the sequence such that $x_r - x_s = k$.

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

Let k be a positive integer. Partition $\{1, 2, 3, \ldots, 2k\}$ into k pairs $\{1, k+1\}$, $\{2, k+2\}, \{3, k+3\}, \ldots, \{k, 2k\}$. Since $1 = x_1 < x_2 < \ldots < x_{k+1} \leq 2k$, by the pigeon hole principle some 2 members of $\{x_1, x_2, \ldots, x_{k+1}\}$ must comprise one of the pairs. The result follows.

2. We choose n points ($n > 3$) on a circle, numbered from 1 to n in any order. We say that two non-adjacent points A and B are related if, in one of the arcs with A and B as endpoints, all the points are marked with numbers smaller than those of A and B. Show that the number of pairs of related points is exactly $n - 3$.

Editor's comment. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, points out that this problem is identical to problem 3 of the 1987 Arany Daniel Competition [1989: 5]. A solution was submitted by Curtis Cooper, Central Missouri State University, Warrensburg, but we have already given a solution for the earlier occurrence. See [1990: 264–265].

3. Show that $25x + 3y$ and $3x + 7y$ are multiples of 41 for the same integer values of x and y.

Editor's comment. Several readers pointed out that the problem as stated can not be correct. These included Richard A. Gibbs, Fort Lewis College; Richard K. Guy, University of Calgary; Stewart Metchette, Culver City, CA; Bob Prielipp, University of Wisconsin–Oshkosh; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. Guy points out that $25x + 3y$ is a multiple of 41 just if $x \equiv 13y \bmod 41$ (or $y \equiv 19x \bmod 41$), while $3x + 7y$ is a multiple of 41 just if $x \equiv 25y$ (or $y \equiv 23x$) mod 41. The only solutions are $x \equiv y \equiv 0 \bmod 41$. Gibbs suggested that perhaps the 7 should be replaced by 2, since $2(25x + 3y) = 41x + 3(3x + 2y)$. This gives $25x + 3y$ is a multiple of 41 iff $3x + 2y$ is, since 41 is prime. Bob Prielipp suggested that the correct statement may have been that $25x + 3y$ and $31x + 7y$ are multiples of 41 for the same values of x and y. This is since $25x+3y \equiv 0 \bmod 41$ iff $16(25x+3y) \equiv 0 \bmod 41$ iff $31x+7y \equiv 0 \bmod 41$, since $16\cdot 25 \equiv 31$ and $16 \cdot 3 \equiv 7 \bmod 41$.

4. The celebrated Fibonacci sequence is defined by

$$a_1 = 1,\ a_2 = 2,\ a_i = a_{i-2} + a_{i-1}\ (i > 2).$$

Express a_{2n} in terms of only a_{n-1}, a_n, and a_{n+1}.

Solutions by Richard K. Guy, University of Calgary; O. Johnson, King Edward's School, Birmingham, England; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's presentation.

We first show that for all $n \geq 2$ and all k with $2 \leq k < 2n - 1$

$$a_{2n} = a_k a_{2n-k} + a_{k-1} a_{2n-k-1}. \qquad (*)$$

When $k = 2$ or $2n - 2$, the right side of (*) equals $a_2 a_{2n-2} + a_1 a_{2n-3} = 2a_{2n-2} + a_{2n-3} = a_{2n-2} + a_{2n-1} = a_{2n}$. Suppose that (*) holds for some k with $2 \leq k < 2n - 2$. Then we have

$$\begin{aligned} a_{2n} &= a_k(a_{2n-k-1} + a_{2n-k-2}) + a_{k-1}a_{2n-k-1} \\ &= (a_k + a_{k-1})a_{2n-k-1} + a_k a_{2n-k-2} \\ &= a_{k+1}a_{2n-(k+1)} + a_k a_{2n-k-2}, \end{aligned}$$

completing the induction. Setting $k = n$ in (*) we obtain

$$\begin{aligned} a_{2n} = a_n^2 + a_{n-1}^2 &= (a_n + a_{n-1})^2 - 2a_n a_{n-1} \\ &= a_{n+1}^2 - 2a_n a_{n-1}. \end{aligned}$$

Remarks: (1) The expressions remain valid for $n = 1$ provided we set $a_0 = 1$.

(2) Clearly there are *many* other such expressions, e.g., using $a_{n-1} = a_{n+1} - a_n$ we have $a_{2n} = a_{n+1}^2 - 2a_{n+1}a_n + 2a_n^2$ and $a_{2n} = a_n^2 + a_{n-1}^2$ can be written as $a_{2n} = (a_{n+1} - a_{n-1})^2 + a_{n-1}^2 = a_{n+1}^2 - 2a_{n+1}a_{n-1} + 2a_{n-1}^2$, etc.

* * *

We now turn to problems from the January 1990 number of *Crux* with solutions for the problems of the *Singapore Mathematical Society Interschool Mathematical Competition, 1988 (Part B)* [1990: 4–5].

1. Let $f(x)$ be a polynomial of degree n such that $f(k) = \frac{k}{k+1}$ for each $k = 0, 1, 2, \ldots, n$. Find $f(n+1)$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea, and Murray S. Klamkin, University of Alberta.

Let $g(x) = (x+1)f(x) - x$. Then the given condition becomes $g(0) = g(1) = \cdots = g(n) = 0$. It follows that $g(x) = kx(x-1)\cdots(x-n)$ and

$$(x+1)f(x) = x + kx(x-1)\cdots(x-n).$$

Putting $x = -1$, we have $k = \frac{(-1)^{n+1}}{(n+1)!}$. We conclude that

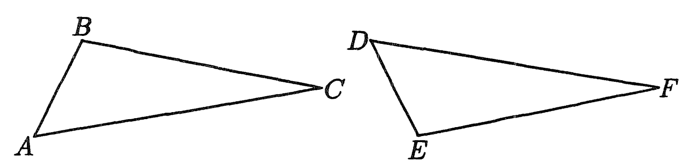
$$f(x) = \frac{x + \frac{(-1)^{n+1}}{(n+1)!}x(x-1)\cdots(x-n)}{x+1}.$$

Thus

$$f(n+1) = \frac{n+1+(-1)^{n+1}}{n+2} = \begin{cases} 1 & n \text{ odd} \\ \frac{n}{n+2} & n \text{ even.} \end{cases}$$

Editor's note. Murray Klamkin points out that this problem has appeared previously, for example in M.S. Klamkin, *USA Mathematical Olympiads 1972–1986*, MAA, 1988, pp. 20–21. Slightly less elementary solutions were sent in by Duane M. Broline, Eastern Illinois University, Charleston; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

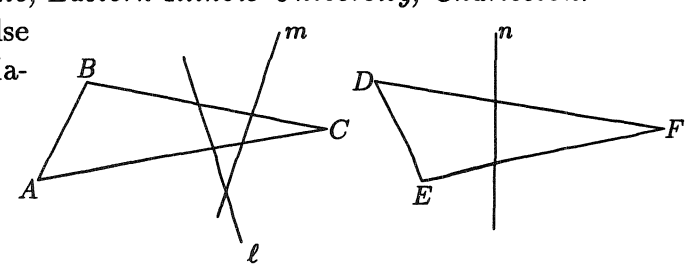
2. Suppose ΔABC and ΔDEF in the figure are congruent. Prove that the perpendicular bisectors of AD, BE, and CF intersect at the same point.



Comment by Duane M. Broline, Eastern Illinois University, Charleston.

The problem is obviously false as stated, as the accompanying diagram illustrates.

ℓ is the $\perp$ bisector of AD.
m is the $\perp$ bisector of BE.
n is the $\perp$ bisector of CF.



However, there is a result about perpendicular bisectors of congruent triangles:

Let ΔABC be a triangle in the plane and R any other point. If ΔDEF is the image of ΔABC under any isometry which fixes R, then the perpendicular bisectors of AD, BE and CF intersect at R.

(This result follows since the perpendicular bisector of a line segment is the locus of all points which are equidistant from the two endpoints.)

3. Find all positive integers n such that P_n is divisible by 5, where $P_n = 1 + 2^n + 3^n + 4^n$. Justify your answer.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Stewart Metchette, Culver City, California; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Michael Selby, University of Windsor.

Working mod 5, $P_n \equiv 1+2^n+(-2)^n+(-1)^n$. If n is odd, $P_n \equiv 1+2^n-2^n-1 \equiv 0$ and so $5|P_n$. If $n = 2(2k+1)$, $2^n \equiv 4^{2k+1} \equiv -1$. So

$$P_n \equiv 1 + 2^n + 2^n + 1 \equiv 1 - 1 - 1 + 1 \equiv 0,$$

and $5|P_n$. Finally suppose $n = 4m$. Then $2^n \equiv (2^4)^m \equiv 1^m \equiv 1$ and $P_n \equiv 1+1+1+1 \equiv 4 (\not\equiv 0)$. Thus P_n is divisible by 5 just in case n is odd or twice an odd number.

Alternate Solution by Duane M. Broline, Eastern Illinois University, Charleston; by Murray S. Klamkin, University of Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

$P_n = 1 + 2^n + (5-2)^n + (5-1)^n$. If n is odd, then $5|P_n$ by expansion of the last two terms using the binomial theorem. If n is even, $n = 2m$, we get $P_n = 2(1+2^{2m}) + 5k$, by the same method. Now $1 + 2^{2m} = 1 + (5-1)^m$. For the latter to be divisible by 5, m must be odd. Summarizing, n must be odd or twice an odd number.

4. Prove that for any positive integer n, any set of $n+1$ distinct integers chosen from the integers $1, 2, \ldots, 2n$ always contains 2 distinct integers such that one of them is a multiple of the other.

Editor's note. Comments and solutions were received from Seung-Jin Bang, Seoul, Republic of Korea; Duane M. Broline, Eastern Illinois University, Charleston, Illinois; Murray S. Klamkin, University of Alberta; Bob Prielipp, University of Wisconsin–Oshkosh; and Michael Selby, University of Windsor.

This problem should be fairly well known. Murray Klamkin points out that with $n = 100$ it is given in D.O. Shklyarsky, N.N. Chentsov, I.M. Yaglom, *Selected Problems and Theorems in Elementary Mathematics*, Mir Publishers, Moscow, 1979. There the authors give two proofs, one by induction and the combinatorial proof below.

Solution. Let $x_1, x_2, \ldots, x_{n+1}$ be the $n+1$ chosen integers. Then $x_i = 2^{r_i} m_i$ with $r_i \geq 0$ and m_i odd for $1 \leq i \leq n+1$. Since there are only n odd numbers up to $2n$ we must have $m_i = m_j$ for some $i \neq j$. Then x_i divides x_j or x_j divides x_i according to whether $r_i < r_j$ or $r_i > r_j$.

5. Find all positive integers x, y, z satisfying the equation $5(xy + yz + zx) = 4xyz$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Duane M. Broline, Eastern Illinois University, Charleston, Illinois; by Stewart Metchette, Culver City, California; by Bob Prielipp, University of Wisconsin–Oshkosh; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. The write-up given is Wang's.

The given equation can be rewritten as

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{5} \qquad (*)$$

Without loss of generality, we may assume that $1 \le x \le y \le z$. Since x, y and z are positive $x = 1$ is clearly impossible. On the other hand, if $x \ge 4$, then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le \frac{3}{4} < \frac{4}{5}.$$

Thus $(*)$ can have integer solutions only if $x = 2$ or $x = 3$.

When $x = 3$, $(*)$ becomes $\frac{1}{y}+\frac{1}{z}=\frac{7}{15}$. If $y \ge 5$, then $\frac{1}{y}+\frac{1}{z} \le \frac{2}{5} < \frac{7}{15}$. Thus $y = 3$ or 4. In either case, we can easily find that the corresponding value for z is not an integer.

When $x = 2$, $(*)$ becomes $\frac{1}{y}+\frac{1}{z}=\frac{3}{10}$. If $y \le 3$, then $\frac{1}{y}+\frac{1}{z} > \frac{1}{3} > \frac{3}{10}$. If $y \ge 7$ then $\frac{1}{y}+\frac{1}{z} \le \frac{2}{7} < \frac{3}{10}$. Thus $y = 4, 5$, or 6.

For $y = 4$, we solve and get $z = 20$, and a solution $(2, 4, 20)$. For $y = 5$, we have $z = 10$ and the solution $(2, 5, 10)$. For $y = 6$ we find $z = 15/2$, which is not an integer.

To summarize, the given equation has exactly 12 solutions obtained by permuting the entries of each of the two ordered triples $(2, 4, 20)$ and $(2, 5, 10)$.

* * *

This completes the Corner for this month. Send me your nice solutions!

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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

More Mathematical Morsels, by R. Honsberger, published by the Mathematical Association of America, Washington, 1991, ISBN 0-88385-313-2, softcover, 322+ pages. *Reviewed by Andy Liu, University of Alberta.*

This may be considered a sequel to the author's earlier problem anthology *Mathematical Morsels.* However, in the present volume, almost all of the 57 morsels appeared previously in *Crux Mathematicorum.* The author has foraged in its fertile soil before. See for instance his *Mathematical Gems III*, especially Chapter 7.

Let us examine first the material from outside *Crux.* Morsel 51 presents a new proof of a theorem of Moessner due to K. Post. Morsel 55 illustrates with an example the

"Probabilistic Methods in Combinatorics" discussed by P. Erdős and J. Spencer in their monograph of that title (the reference is inadvertently omitted). Morsel 56 is based on a student project by W. K. Chan on point sets not determining right triangles. The original papers were by A. Seidenberg and H. L. Abbott. Morsel 57 reexamines a morsel from the author's earlier *Mathematical Morsels.*

This is excellent material. Its presence, however, also serves to keep "*Crux Mathematicorum*" off the cover of the book. Nevertheless, it is gratifying to see the labour of love by Léo Sauvé and Fred Maskell duly acknowledged in the Preface.

Practically all of the remaining 53 morsels are taken from the regular Problem Sections of *Crux.* (A reference to *Crux* 1119 [1987: 258] should have been made in Morsel 49, which is based on an outstanding expository article by S. Wagon.) The only other exceptions are Morsels 1, 26 and 46, which are taken from the Olympiad Corners in 1979. Why they are not included in the first of eight sections titled "Gleanings from Murray Klamkin's Olympiad Corners, 1979–1986" is puzzling.

The author certainly has good taste in the choice of material. For instance, it would have been a grievous omission if Gregg Patruno's brilliant proof of Archimedes' "Broken-chord Theorem" had not been included. Happily, the readers are reacquainted with this gem as Morsel 8.

In the Preface, the author freely admits that great liberty has been taken with the work of the original contributors to *Crux.* The author's tendency is to take apart an argument and analyse it step by step. Depending on the intended audience, this is a valuable service. However, there are times when the "down marketing" may have gone a bit too far. It would also seem desirable to have some sort of classification of the problems by subject matter, along the line of S. Rabinowitz's ambitious project (see his "Letter to the Editor" on [1991: 96]).

The proposers and solvers of the problems in the regular Problem Sections are listed at the end of the book. There are only some minor glitches; for instance, the references for Morsels 5, 6 and 10 have been permuted, and in the reference to Morsel 29 the Hungarian journal *KOMAL* is called a Russian journal. On the other hand, there is no complete or consistent acknowledgement of the original contributors to the Olympiad Corners. This is particularly confusing since the author has included solutions to a small number of problems to which no solutions have yet appeared in *Crux.*

The book is certainly up to the author's high standards, and is a good addition to the bookshelves, especially since the Canadian Mathematical Society has regrettably stopped putting out bound volumes of *Crux.*

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1992,** *although solutions received after that date will also be considered until the time when a solution is published.*

1671. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A right triangle ABC with right angle at A is inscribed in a circle Γ. Let M, N be the midpoints of AB, AC, and let P, Q be the points of intersection of the line MN with Γ. Let D, E be the points where AB, AC are tangent to the incircle. Prove that D, E, P, Q are concyclic.

1672. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that for positive real numbers a, b, c, x, y, z,

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq 3\left(\frac{xy+yz+zx}{x+y+z}\right),$$

and determine when equality holds.

1673. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC is nonequilateral and has angle $\beta = 60°$. A' is an arbitrary point of line BA, not coinciding with B or A. C' is an arbitrary point of BC, not coinciding with B or C.

(a) Show that the Euler lines of ΔABC and $\Delta A'BC'$ are parallel or coinciding.

(b) In the case of coincidence, show that the circumcircles of all such triangles $A'BC'$ meet the circumcircle of ABC at a fixed point.

1674. *Proposed by Murray S. Klamkin, University of Alberta.*

Given positive real numbers r, s and an integer $n > r/s$, find positive $x_1, x_2, \ldots, x_n$ so as to minimize

$$\left(\frac{1}{x_1^r} + \frac{1}{x_2^r} + \cdots + \frac{1}{x_n^r}\right)(1+x_1)^s(1+x_2)^s \ldots (1+x_n)^s.$$

1675. *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $V_1, V_2, \ldots, V_n$ denote the vertices of a regular n-gon inscribed in a unit circle C where $n \geq 3$, and let P be an arbitrary point on C. It is known that $\sum_{k=1}^{n} \overline{PV_k}^2$ is a constant.

(a) Show that $\sum_{k=1}^{n} \overline{PV_k}^4$ is also a constant.

(b) Does there exist a value of $m \neq 1, 2$ and a value of $n \geq 3$ such that $\sum_{k=1}^{n} \overline{PV_k}^{2m}$ is independent of P?

1676. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

OA is a fixed radius and OB a variable radius of a unit circle, such that $\angle AOB \le 90°$. $PQRS$ is a square inscribed in the sector OAB so that PQ lies along OA. Determine the minimum length of OS.

1677. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Evaluate (without rearranging)

$$1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\cdots .$$

1678. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

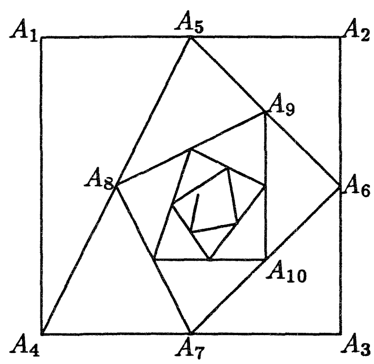
Show that

$$\sqrt{s}(\sqrt{a}+\sqrt{b}+\sqrt{c}) \le \sqrt{2}(r_a+r_b+r_c),$$

where a, b, c are the sides of a triangle, s the semiperimeter, and r_a, r_b, r_c the exradii.

1679. *Proposed by Len Bos and Bill Sands, University of Calgary.*

$A_1A_2A_3A_4$ is a unit square in the plane, with $A_1(0,1)$, $A_2(1,1)$, $A_3(1,0)$, $A_4(0,0)$. A_5 is the midpoint of A_1A_2, A_6 the midpoint of A_2A_3, A_7 the midpoint of A_3A_4, A_8 the midpoint of A_4A_5, and so on. This forms a spiral polygonal path $A_1A_2A_3A_4A_5A_6A_7A_8\ldots$ converging to a unique point inside the square. Find the coordinates of this point.



1680. *Proposed by Zun Shan and Ji Chen, Ningbo University, China.*

If m_a, m_b, m_c are the medians and r_a, r_b, r_c the exradii of a triangle, prove that

$$\frac{r_br_c}{m_bm_c}+\frac{r_cr_a}{m_cm_a}+\frac{r_ar_b}{m_am_b} \ge 3.$$

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1448. [1989: 149; 1990: 222] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of a triangle, prove that

$$\frac{2}{3}\left(\sum \sin\frac{A}{2}\right)^2 \ge \sum \cos A,$$

with equality when $A = B = C$.

[*Editor's note.* The statement of this problem is false, but Murray Klamkin conjectured on [1990: 222] that it should be true for *acute* triangles. Three readers have since sent in proofs of this conjecture. Two of them contain different "best possible" results, so here they are.]

II. *Solution by G.P. Henderson, Campbellcroft, Ontario.*

We choose the notation so that $A \geq B \geq C$. Then we prove:

(a) if $B < 2\arcsin\left(\frac{\sqrt{3}-1}{4}\right) \cong 21.1°$, the inequality is false;

(b) if $2\arcsin\left(\frac{\sqrt{3}-1}{4}\right) \leq B < 30°$, it is false for small values of C and becomes true as C increases toward B;

(c) if $B \geq 30°$, the inequality is true.

For acute triangles, $B \geq 45°$. Therefore M.S. Klamkin's conjecture that the inequality is true for such triangles is correct.

Set

$$X = \sin(C/2), \quad v = \sin(B/2), \quad Y = \sin(A/2).$$

Then

$$0 < v < \sqrt{2}/2 \tag{1}$$

and

$$0 < X \leq v \leq Y < 1. \tag{2}$$

Since $\sum A/2 = 90°$,

$$X^2 + 2vXY + Y^2 = 1 - v^2. \tag{3}$$

In terms of X, Y and v the proposed inequality is

$$8X^2 + 4XY + 8Y^2 + 4vX + 4vY + 8v^2 - 9 \geq 0. \tag{4}$$

Both (3) and (4) are simpler if we rotate the XY-axes through 45° and change the scales. Set

$$X = x - y, \quad Y = x + y.$$

(2) and (3) become

$$|x - v| \leq y < x \tag{5}$$

and

$$2(1+v)x^2 + 2(1-v)y^2 = 1 - v^2. \tag{6}$$

The inequality is now

$$20x^2 + 12y^2 + 8vx + 8v^2 - 9 \geq 0. \tag{7}$$

Geometrically, for a given v satisfying (1), a certain arc of the ellipse (6) is to be outside the ellipse (7).

We use (6) to eliminate y. From (5), $y \geq 0$. Therefore

$$y = \sqrt{\frac{1 - v^2 - 2(1+v)x^2}{2(1-v)}}. \tag{8}$$

The first part of (5) is equivalent to

$$(2x+1-v)(2x+2v^2-v-1) \le 0.$$

The first factor can be omitted because $x > 0$ (from (5)) and $v < 1$. From the second part of (5),

$$4x^2 > 1 - v^2.$$

Thus (5) is equivalent to

$$x_1 < x \le x_2 \tag{9}$$

where

$$x_1 = \frac{\sqrt{1-v^2}}{2}$$

and

$$x_2 = \frac{1+v-2v^2}{2} = \frac{(1-v)(1+2v)}{2}.$$

For these values of x, the expression under the radical sign in (8) is positive. Using (8), (7) becomes

$$f(x) = 8(1-4v)x^2 + 8v(1-v)x - 8v^3 + 2v^2 + 9v - 3 \ge 0. \tag{10}$$

We are to determine the values of x and v that satisfy (1), (9) and (10). We find

$$\begin{aligned} f(x_1) &= (1-v)(4v\sqrt{1-v^2}-1) = \left(1 - \sin\frac{B}{2}\right)(2\sin B - 1), \\ f(x_2) &= (1-v)(32v^4 - 16v^3 - 12v^2 + 8v - 1) \\ &= \frac{1}{2}(1-v)(2v-1)^2(4v+1+\sqrt{3})(4v+1-\sqrt{3}). \end{aligned}$$

Case (a): $B < 2\arcsin(\sqrt{3}-1)/4$.

We have

$$0 < v < (\sqrt{3}-1)/4 < 1/4, \quad f(x_1) < 0, \quad f(x_2) < 0,$$

and f is convex. Therefore (10) is false for $x_1 < x \le x_2$.

Case (b): $2\arcsin(\sqrt{3}-1)/4 \le B < 30^\circ$.

The sign of f changes from negative to positive as x increases from x_1 to x_2, that is, as C increases from 0 to B. Therefore (10) is false for small values of C and becomes true as C approaches B.

Case (c): $B \ge 30^\circ$.

We have

$$\frac{1}{4} < \frac{\sqrt{6}-\sqrt{2}}{4} \le v < \frac{\sqrt{2}}{2}, \quad f(x_1) \ge 0, \quad f(x_2) \ge 0,$$

and f is concave. Therefore (10) is true for $x_1 < x \le x_2$.

III. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

While failing in its full generality, the inequality is indeed valid for acute triangles, as conjectured by M.S. Klamkin. Actually, we prove that *the inequality holds in every triangle of angles not exceeding* $\arccos((1-4\sqrt{3})/8) \approx 137.8°$; *the bound cannot be improved upon.*

Assume $A \ge B \ge C$ and let

$$\varphi = \frac{B+C}{2}, \quad \psi = \frac{B-C}{2}, \quad t = 2\cos\varphi - 1, \quad x = \cos(\psi/2).$$

Then $A = \pi - 2\varphi$, $B = \varphi + \psi$, $C = \varphi - \psi$, $\pi/3 \ge \varphi \ge \psi \ge 0$, $0 \le t \le 1$, $0 \le x \le 1$, $\cos\psi = 2x^2 - 1$, $\cos\varphi = (1+t)/2$, and we have

$$\begin{aligned}
\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} &= \sin\frac{\pi - 2\varphi}{2} + 2\sin\frac{\varphi}{2}\cos\frac{\psi}{2} \\
&= \cos\varphi + 2\sqrt{\frac{1-\cos\varphi}{2}} \cdot \cos\frac{\psi}{2} \\
&= \frac{1+t}{2} + \sqrt{1-t}\, x, \\
\cos A + \cos B + \cos C &= (1 - 2\cos^2\varphi) + 2\cos\varphi\cos\psi \\
&= 1 - \frac{(1+t)^2}{2} + (1+t)(2x^2 - 1).
\end{aligned}$$

After small manipulation, the inequality under investigation takes on the form

$$f(t,x) = 4(1+2t)x^2 - 2(1+t)\sqrt{1-t}\, x - (2 + 7t + 2t^2) \le 0.$$

For $t, x \in [0,1]$,

$$f(t,1) - f(t,x) = 2(1-x)[2(1+2t)(1+x) - (1+t)\sqrt{1-t}] \ge 0,$$

with equality only for $x = 1$. So it suffices to examine f for $x = 1$. With some routine calculation we arrive at

$$f(t,1)\begin{cases} < 0 & \text{for } t \in (0, \sqrt{3}/2), \\ = 0 & \text{for } t = 0 \text{ and } t = \sqrt{3}/2, \\ > 0 & \text{for } t \in (\sqrt{3}/2, 1]. \end{cases}$$

The "border value" $t = \sqrt{3}/2$ corresponds to

$$\cos A = -\cos 2\varphi = 1 - 2\cos^2\varphi = 1 - (1+t)^2/2 = (1 - 4\sqrt{3})/8;$$

clearly, $t = 0$ corresponds to $A = \pi/3$, and $x = 1$ to $B = C$.

Conclusion: writing $\alpha = \arccos((1-4\sqrt{3})/8) \approx 137.8°$,

if $A \le \alpha$, the inequality is true for all B, C;

if $A > \alpha$, the inequality can fail to hold, and certainly does so when $B = C$.

For $A < \alpha$, equality requires $t = 0$ and $x = 1$, which occur when the triangle is equilateral.

Klamkin's conjecture was also proved by JOHN LINDSEY, Northern Illinois University, Dekalb.

The editor did not succeed in combining the above proofs. Maybe some reader can easily derive one result from the other. Note, by the way, that Kuczma's upper bound on A and Henderson's lower bound on B are related:

$$\arccos\left(\frac{1-4\sqrt{3}}{8}\right) = 180^\circ - 4\arcsin\left(\frac{\sqrt{3}-1}{4}\right),$$

corresponding to the isosceles triangle with

$$A = \arccos\left(\frac{1-4\sqrt{3}}{8}\right) \approx 137.8^\circ, \quad B = C = 2\arcsin\left(\frac{\sqrt{3}-1}{4}\right) \approx 21.1^\circ,$$

for which equality holds in the problem.

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1553. [1990: 171] *Proposed by Murray S. Klamkin, University of Alberta.*

It has been shown by Oppenheim that if $ABCD$ is a tetrahedron of circumradius R, a, b, c are the edges of face ABC, and p, q, r are the edges AD, BD, CD, then

$$64R^4 \geq (a^2+b^2+c^2)(p^2+q^2+r^2).$$

Show more generally that, for n-dimensional simplexes,

$$(n+1)^4 R^4 \geq 4E_0E_1,$$

where E_0 is the sum of the squares of all the edges emanating from one of the vertices and E_1 is the sum of the squares of all the other edges.

I. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

A slightly stronger estimate can be obtained, namely,

$$E_0E_1 \leq \left(\frac{4}{3}n\right)^3 R^4. \tag{1}$$

The right-hand expression in (1) is less than the claimed one $(n+1)^4R^4/4$, except for $n = 3$, when the two values are equal.

Let $\mathbf{v}_0, \ldots, \mathbf{v}_n$ be the vectors from the circumcenter to the vertices. Denote by $\mathbf{w}$ the centroid of the system $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. So

$$|\mathbf{v}_i| = R \quad (i = 0, 1, \ldots, n);$$

$$\mathbf{w} = \frac{1}{n}(\mathbf{v}_1 + \cdots + \mathbf{v}_n), \quad |\mathbf{w}| \leq R.$$

By definition,

$$E_1 = \frac{1}{2}\sum_{i,j=1}^{n} |\mathbf{v}_i - \mathbf{v}_j|^2 = \sum_{i,j=1}^{n}(R^2 - \mathbf{v}_i\cdot\mathbf{v}_j) = n^2R^2 - \left(\sum_{i=1}^{n}\mathbf{v}_i\right)^2 = n^2(R^2 - |\mathbf{w}|^2);$$

$$E_0 = \sum_{i=1}^{n}|\mathbf{v}_0 - \mathbf{v}_i|^2 = \sum_{i=1}^{n}(2R^2 - 2\mathbf{v}_0\cdot\mathbf{v}_i) = 2n(R^2 - \mathbf{v}_0\cdot\mathbf{w});$$

dots denote inner (scalar) products of vectors. Since $\mathbf{v}_0\cdot\mathbf{w} \geq -R|\mathbf{w}|$, and using the A.M.-G.M. inequality, we obtain

$$\begin{aligned} E_0E_1 &= 2n^3(R^2 - \mathbf{v}_0\cdot\mathbf{w})(R^2 - |\mathbf{w}|^2) \leq 2n^3(R^2 + R|\mathbf{w}|)(R^2 - |\mathbf{w}|^2) \\ &= 8n^3R\left[(R - |\mathbf{w}|)^{1/3}\left(\frac{R+|\mathbf{w}|}{2}\right)^{2/3}\right]^3 \\ &\leq 8n^3R\left[\frac{1}{3}(R - |\mathbf{w}|) + \frac{2}{3}\left(\frac{R+|\mathbf{w}|}{2}\right)\right]^3 = \left(\frac{4}{3}n\right)^3 R^4. \end{aligned}$$

Remark. Assume $n \geq 2$. Then the estimate in (1) is sharp; it turns into equality if and only if $3\mathbf{w} = -\mathbf{v}_0$. Note that, except for $n = 2$ and 3, there are many nonisometric optimal configurations and that, except for $n = 3$, the regular simplex is *not* among them.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let O be the circumcenter and G the centroid of the simplex $A_0\cdots A_n$. Then the following identities are more or less familiar, at least for triangles and tetrahedra:

$$\sum_{i=0}^{n}\overline{GA_i}^2 = (n+1)(R^2 - \overline{OG}^2), \qquad E_0 + E_1 = (n+1)\sum_{i=0}^{n}\overline{GA_i}^2$$

(see, e.g., various simplex-problems by Murray Klamkin and/or George Tsintsifas in *Crux*, or Mitrinović et al, *Recent Advances in Geometric Inequalities*, p. 493 and p. 502). Therefore

$$E_0 + E_1 = (n+1)^2(R^2 - \overline{OG}^2)$$

and we get the better inequality

$$4E_0E_1 \leq (E_0 + E_1)^2 = (n+1)^4(R^2 - \overline{OG}^2)^2.$$

Also solved by G.P. HENDERSON, Campbellcroft, Ontario; and the proposer.

With this "formidable four" (the three solvers plus the proposer) in action, it's not surprising that all of them found stronger results! Henderson in fact gave the same improvement as Kuczma, with the same remarks at the end. In another direction, the proposer showed that $W^4R^4 \geq 4E_0'E_1'$ *where*

$$E_0' = \sum_{j=1}^{n} w_0w_ja_{0j}^2 \quad \textit{and} \quad E_1' = \sum_{1\leq i<j}^{n} w_iw_ja_{ij}^2$$

are weighted *sums,* $W = \sum_{i=0}^{n} w_i$ *the sum of the weights, and where* a_{ij} *is the length of the edge between vertices* i *and* j.

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1554. [1990: 171] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Describe all finite sets $\mathcal{S}$ in the plane with the following property: if two straight lines, each of them passing through at least two points of $\mathcal{S}$, intersect in P, then P belongs to $\mathcal{S}$.

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Suppose that a finite set $\mathcal{F}$ of points in the plane has exactly one of the following properties:

I: the points of $\mathcal{F}$ are collinear;

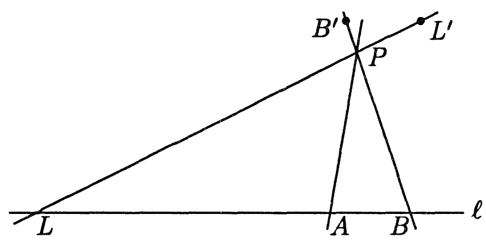
II: $|\mathcal{F}| = 5$, and the points of $\mathcal{F}$ are the vertices of a parallelogram together with the point of intersection of the diagonals;

III: all points of $\mathcal{F}$, except one, are on a straight line.

Clearly such a set has the "closure property" as required by the problem. Conversely, each set $\mathcal{F}$ obeying the conditions of the problem satisfies I, II or III as we shall show.

So take a finite set $\mathcal{F}$ with the closure property. We may assume that $|\mathcal{F}| \geq 4$, else $\mathcal{F}$ satisfies I or III and we are done. If A, B, C, D are 4 points of $\mathcal{F}$, no 3 of which are collinear, then we can group them in two pairs, say $\{A, B\}$ and $\{C, D\}$, such that the two lines AB and CD intersect. This observation shows that $\mathcal{F}$ contains 3 collinear points lying on a line ℓ, say.

For each point $L \in \mathcal{F} \cap \ell$, and each point P of $\mathcal{F}$ not on ℓ, let $\theta(P, L)$ be the non-obtuse angle between PL and ℓ (we assume that I doesn't hold, else we are done). Choose P and L such that $\theta(P, L)$ is minimal and if there are several choices for P choose the one with the distance $d(P, L)$ between P and L minimal. Let A and B be two other points on ℓ belonging to $\mathcal{F}$. Choose points L' on line PL and B' on line PB, not necessarily in $\mathcal{F}$, such that P lies between L and L' and between B and B'.



We claim that $\mathcal{F}$ contains no point different from P and on the same side of ℓ as P. This follows from the following six facts:

(i) $\mathcal{F}$ contains no point in (PLB) = the interior of angle PLB (by minimality of $\theta(P, L)$);

(ii) $\mathcal{F}$ contains no point on the open half-line $\overrightarrow{PL'}$ (the line through A and any point of $\overrightarrow{PL'}$ intersects (PB) = the interior of segment $PB \subseteq (PLB)$; now use (i));

(iii) $\mathcal{F}$ contains no point in $(B'PL')$ (the line through B and any point of $(B'PL')$ intersects $\overrightarrow{PL'}$; now use (ii));

(iv) $\mathcal{F}$ contains no point on (PL) (by minimality of $d(P, L)$);

(v) $\mathcal{F}$ contains no point on $\overrightarrow{PB'}$ (the line through A and any point of $\overrightarrow{PB'}$ intersects (PL); now see (iv));

(vi) $\mathcal{F}$ contains no point in (PBL) (the line through B and any point of (PBL) intersects (PL); now see (iv)).

If $\mathcal{F}$ contains no point on that side of ℓ which doesn't contain P, then by the claim we see that $\mathcal{F}$ satisfies III. So suppose that $\mathcal{F}$ *does* contain such a point; call it Q. By the claim, Q is the only point of $\mathcal{F}$ on that side. Now it's easy to see that ℓ contains exactly 3 points, else one can create a new point of $\mathcal{F}$ on some side of ℓ. Moreover $PQ \cap \ell = A$, $QL||PB$, $QB||PL$. Thus $\mathcal{F}$ satisfies II.

Also solved by JORDI DOU, Barcelona, Spain; and the proposer. Another reader sent in the correct solution without proof. There was also one incorrect solution received.

Walther Janous recalls the problem in an article in Kvant *"a long time ago", and also a similar* Monthly *problem, but could not supply details.*

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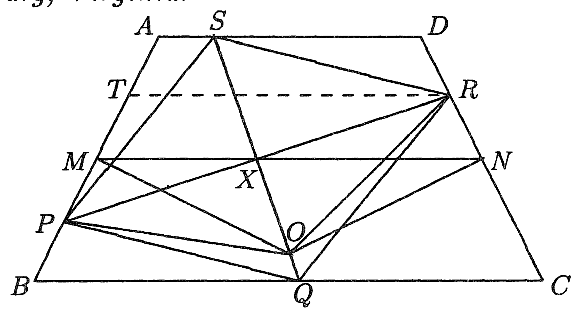
1555. [1990: 172] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is an isosceles trapezoid, with $AD||BC$, whose circumcircle has center O. Let $PQRS$ be a rhombus whose vertices P, Q, R, S lie on AB, BC, CD, DA respectively. Prove that Q, S and O are collinear.

Solution by Dan Sokolowsky, Williamsburg, Virginia.



Let M be the midpoint of AB and N the midpoint of CD. It is then easily seen that $MN||BC$, and that MN bisects QS, hence that MN passes through the common midpoint X of QS and PR. Let $RT||MN$ (as shown). Then, since $PX = XR$, $PM = MT$. Clearly $MT = RN$, so $PM = RN$. Also, since $AB = CD$, $OM = ON$, while $\angle OMP = \angle ONR = 90°$. Hence $\Delta OMP \cong \Delta ONR$, so $OP = OR$. Then, since $XP = XR$, OX is the perpendicular bisector of PR, as is QS. Hence the lines OX and QS coincide, which implies that Q, S and O are collinear.

Also solved by JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1556. [1990: 172] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Let λ and n be fixed positive integers, not both 1. Prove that the equation

$$\frac{x^2 + y^2}{\lambda xy + 1} = n^2$$

has infinitely many natural number solutions (x, y).

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

Multiplying both sides of the equation of the problem statement by $4(\lambda xy+1)$ and rearranging terms gives

$$4x^2 - 4n^2\lambda xy = 4n^2 - 4y^2.$$

Completing the square on the left side gives

$$(2x - n^2\lambda y)^2 = 4n^2 - 4y^2 + n^4\lambda^2 y^2$$

or

$$(2x - n^2\lambda y)^2 - (n^4\lambda^2 - 4)y^2 = 4n^2. \tag{1}$$

If $n = 1$ and $\lambda = 2$, the second term on the left vanishes and we have the infinite set of solutions $x = y \pm 1$. With any other values for n and λ, the multiplier of y^2 in (1) is not a perfect square, so we have a Pell equation. One solution is $x = n$, $y = n^3\lambda$, as can be shown by substitution into (1). And it is well known that a Pell equation with at least one solution has an infinite number of solutions.

Also solved by H.L. ABBOTT, University of Alberta; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

The problem is related to (and inspired by) problem 6 *of the* 1988 *IMO* [1988 : 197] *(see* Crux 1420 [1990:122] *for another such problem).*

The proposer also observes that, for $\lambda n^2 > 2$, any solution of the given equation yields a right triangle of sides

$$4\lambda n^2,\ (2x - \lambda n^2 y)n - 4,\ (2x - \lambda n^2 y)n + 4.$$

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1557. [1990: 172] *Proposed by David Singmaster, South Bank Polytechnic, London, England.*

Let n be a positive integer and let $\mathcal{P}_n$ be the set of ordered pairs (a,b) of integers such that $1 \le a \le b \le n$. If $f : \{1,2,\dots,n\} \to \mathbf{R}$ is an increasing function, and $g : \mathcal{P}_n \to \mathbf{R}$ defined by

$$g(a,b) = f(a) + f(b)$$

is one-to-one, then g defines a (strict) total ordering $\prec$ on $\mathcal{P}_n$ by

$$(a,b) \prec (c,d) \text{ if and only if } g(a,b) < g(c,d)\ .$$

Moreover $\prec$ will have the property

$$(a,b) \prec (c,d) \text{ whenever } a \le c \text{ and } b \le d \text{ (and } (a,b) \ne (c,d))\ . \qquad (*)$$

Does every strict total ordering $\prec$ of $\mathcal{P}_n$ which satisfies $(*)$ arise in this way?

Solution by Jean-Marie Monier, Lyon, France.

The answer is no, for $n = 4$ for example. Consider the strict total ordering $\prec$ defined on $\mathcal{P}_4$ by:

$$(1,1) \prec (1,2) \prec (2,2) \prec (1,3) \prec (1,4) \prec (2,3) \prec (3,3) \prec (2,4) \prec (3,4) \prec (4,4).$$

$\prec$ satisfies $(*)$. Suppose there exists $f : \{1,2,3,4\} \to \mathbf{R}$ and $g : \mathcal{P}_4 \to \mathbf{R}$ as above such that

$$(a,b) \prec (c,d) \iff g(a,b) < g(c,d).$$

Then we have

$$f(2) + f(2) = g(2,2) < g(1,3) = f(1) + f(3),$$
$$f(1) + f(4) < f(2) + f(3),$$
$$f(3) + f(3) < f(2) + f(4).$$

By summing we get a contradiction.

A similar counterexample was found by MARCIN E. KUCZMA, Warszawa, Poland; and by the proposer and the editor (jointly).

The problem was inspired by W.R. Ransom's problem 3471 *of the* Amer. Math. Monthly, *solution in Vol.* 38 (1931) 474–475, *which contains an example of an ordering (for $n = 4$) which* does *arise in the above way.*

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1558. [1990: 172] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let P be an interior point of a triangle ABC and let AP, BP, CP intersect the circumcircle of ΔABC again in A', B', C', respectively. Prove that the power p of P with respect to the circumcircle satisfies

$$|p| \geq 4rr',$$

where r, r' are the inradii of triangles ABC and $A'B'C'$.

Solution by Murray S. Klamkin, University of Alberta.

We change the notation by letting $(A_1, A_2, A_3) = (A, B, C)$ and $(A_1', A_2', A_3') = (A', B', C')$. For an interior point P, $p = R^2 - (OP)^2 \geq 0$, where R is the circumradius and O the circumcenter. Since $R_iR_i' = p$ for $i = 1,2,3$, where as usual $A_iP = R_i$ and $A_i'P = R_i'$, we have [from the similar triangles $A_3'A_2'P$ and A_2A_3P] that

$$\frac{a_1'}{a_1} = \frac{R_2'}{R_3} = \frac{R_1R_2R_2'}{R_1R_2R_3} = \frac{R_1p}{R_1R_2R_3}, \text{ etc.},$$

where a_i, a_i' are the sides of triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ respectively, and hence

$$a_i' = \frac{a_iR_ip}{K}$$

where $K = R_1R_2R_3$. So aside from the proportionality factor p/K, $\Delta A_1'A_2'A_3'$ is gotten by inversion from $\Delta A_1A_2A_3$ (see p. 294 of Mitrinović et al, *Recent Advances in Geometric*

Inequalities, or other references given re *Crux* 1514 on [1991: 117]). From the same references we see that, with F, F' the areas of triangles $A_1A_2A_3$ and $A'_1A'_2A'_3$ and r_i, r'_i the distances from P to their sides, and disregarding the factor p/K,

$$F' = \frac{1}{2}\sum a'_i r'_i = \frac{1}{2}\sum a_i r_i R_i^2 \quad \text{and} \quad r' = \frac{2F'}{\sum a_i R_i},$$

so that, accounting for the factor p/K,

$$r' = \frac{p}{K}\left(\frac{\sum a_i r_i R_i^2}{\sum a_i R_i}\right).$$

It is also known that

$$\sum a_i r_i R_i^2 = 2pF$$

[equation (6) in M.S. Klamkin, An identity for simplexes and related inequalities, *Simon Stevin* 48 (1974–75) 57–64]. Hence the given inequality can be rewritten as

$$P \geq \frac{4rp}{K}\left(\frac{\sum a_i r_i R_i^2}{\sum a_i R_i}\right)$$

or

$$R_1R_2R_3\sum a_i R_i \geq 4r\sum a_i r_i R_i^2 = 8rpF.$$

The latter now follows immediately from the product of the two known inequalities

$$R_1R_2R_3 \geq 2rp$$

(*Crux* 1327 [1989: 123]) and the Steensholt inequality

$$\sum a_i R_i \geq 4F$$

(item 12.19 of Bottema et al, *Geometric Inequalities*).

A related inequality is

$$sR_1R_2R_2 \geq \sum a_i r_i R_i^2$$

occurring on [1990: 127]. Problems 1514 [1991: 116] and 1543 [1991: 190] are also closely related.

Also solved by the proposer.

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1559. [1990: 172] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

Find a necessary and sufficient condition on reals c and d for the roots of $x^3+3x^2+cx+d=0$ to be in arithmetic progression.

Solution by Beatriz Margolis, Paris, France.

We claim that the necessary and sufficient condition is

$$c - d = 2. \tag{1}$$

Assume the roots of the given equation to be $a - r$, a, $a + r$. Relations between roots and coefficients yield

$$(a - r)a(a + r) = -d, \tag{2}$$

$$(a - r)a + a(a + r) + (a + r)(a - r) = c, \tag{3}$$

$$(a - r) + a + (a + r) = -3. \tag{4}$$

Hence by (4) $a = -1$, so that by (2) and (3)

$$1 - r^2 = d \quad \text{and} \quad 3 - r^2 = c.$$

Therefore condition (1) is necessary.

Assume (1) holds. Then the given equation reads

$$x^3 + 3x^2 + (2 + d)x + d = 0.$$

We see by inspection that $x = -1$ is a solution, so that we may factorize to obtain

$$(x + 1)(x^2 + 2x + d) = 0.$$

Now the solutions to $x^2 + 2x + d = 0$ are $x = -1 \pm \sqrt{1 - d}$. In other words, if (2) holds, the roots of the given equation are

$$-1 - \sqrt{1 - d}, \ -1, \ -1 + \sqrt{1 - d},$$

i.e., they form an arithmetic progression, and condition (2) is sufficient.

Observe that the necessary and sufficient conditions to have a *real* arithmetic progression are $c - d = 2$ and $d \leq 1$ (or $c \leq 3$).

Also solved by HAYO AHLBURG, Benidorm, Spain; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward's School, Birmingham, England; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; JEAN-MARIE MONIER, Lyon, France; M.M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; CORY PYE, student, Memorial University of Newfoundland; KENNETH M. WILKE, Topeka, Kansas; TARA YASENCHAK, Messiah College, Grantham, Pennsylvania; and the proposer. Three incorrect solutions were also sent in.

Lau points out that a more general result (replace the 3 *by an arbitrary coefficient) appears as problem* 7*, section* 1.5 *of E.J. Barbeau,* Polynomials, *Springer-Verlag,* 1989, *with solution on p.* 251.

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1560. [1990: 172] *Proposed by Ilia Blaskov, Technical University, Gabrovo, Bulgaria.*

The sequence $a_2, a_3, a_4, \ldots$ of real numbers is such that, for each n, $a_n > 1$ and the equation $[a_n x] = x$ has just n different solutions. ($[x]$ denotes the greatest integer $\leq x$.) Find $\lim_{n\to\infty} a_n$.

Solution by Margherita Barile, student, Università degli Studi di Genova, Italy.

The only solution of $[a_1 x] = x$ must be $x = 0$. Let now $n > 1$. If $[a_n x] = x$ then $x \geq 0$, because $x < 0$ implies $a_n x < x$, since $a_n > 1$. As $[a_n 0] = 0$, $[a_n x] = x$ has exactly n different solutions if and only if it has $n-1$ different positive solutions.

Let $(a_n) = a_n - [a_n]$ (so $(a_n) \geq 0$). Then $a_n x = [a_n]x + (a_n)x$, thus for $x > 0$ an integer,

$$\begin{aligned}[a_n x] = x &\iff x \leq [a_n]x + (a_n)x < x + 1 \\ &\iff [a_n] = 1 \text{ and } (a_n)x < 1,\end{aligned}$$

which is true for exactly $n-1$ different integers $x > 0$ if and only if

$$[a_n] = 1 \quad \text{and} \quad \frac{1}{n} \leq (a_n) < \frac{1}{n-1}.$$

This implies

$$1 + \frac{1}{n} \leq a_n < 1 + \frac{1}{n-1}\ ,$$

so that we immediately conclude

$$\lim_{n\to\infty} a_n = 1.$$

Note. The value of a_1 does not influence the result, but we can observe that the hypothesis is true for a_1 if $a_1 \geq 2$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incomplete solution was sent in.

Janous remarks that the same conclusion holds whenever the number of solutions of $[a_n x] = x$ *goes to infinity with* n.

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1561. [1990: 204] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine an infinite class of integer triples (x, y, z) satisfying the Diophantine equation

$$x^2 + y^2 + z^2 = 2yz + 2zx + 2xy - 3.$$

Solution by Hayo Ahlburg, Benidorm, Spain.

The well-publicized identity (see Léo Sauvé's footnote on [1976: 176])

$$1^2 + (n^2 - n + 1)^2 + (n^2 + n + 1)^2 \\ = 2(n^2 - n + 1) + 2(n^2 - n + 1)(n^2 + n + 1) + 2(n^2 + n + 1) - 3 \quad (1)$$

is one answer to this problem if we choose n to be any integer.

To find all solutions, we rearrange the original equation and get

$$z^2 - 2(x + y)z + (x - y)^2 + 3 = 0$$

and

$$z = x + y \pm \sqrt{4xy - 3}.$$

To make z an integer, $4xy-3$ must be the square of an odd number $2n+1$, i.e., $4n^2+4n+1 = 4xy - 3$ or

$$n^2 + n + 1 = xy,$$

where n can be any integer. We can choose x and y as factors of $n^2 + n + 1$ (this can sometimes be done in several ways), and with

$$z = x + y \pm (2n + 1)$$

this solves the original equation. Due to the symmetry of the problem, any permutation of the values for x, y and z is also a solution. n, x, y and z can of course also have negative values. Using both signs for x, y, z, and for $2n + 1$ in the expression for z leads to duplications. But using + signs throughout already gives an infinite number of solutions. Factoring $n^2 + n + 1$ into $x = 1$ and $y = n^2 + n + 1$, and with $z = x + y - (2n + 1)$, we get equation (1).

A nice special group is the series

$$1, 1, 3, 7, 19, 49, 129, 337, 883, 2311, 6051, \ldots,$$

where each term has the form $F_{2n+1} - F_nF_{n+1}$ made up of Fibonacci numbers. Any three consecutive numbers of this series form a solution.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; GUO-GANG GAO, Université de Montréal; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; O. JOHNSON, student, King Edward's School, Birmingham, England; DAG JONSSON, Uppsala, Sweden; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JOHN H. LINDSEY, Northern Illinois University, Dekalb; STEWART METCHETTE, Culver City, California; JEAN-MARIE MONIER, Lyon, France; VEDULA N. MURTY, Penn State at

Harrisburg; P. PENNING, Delft, The Netherlands; SHAILESH A. SHIRALI, Rishi Valley School, India; W.R. UTZ, Columbia, Missouri; EDWARD T.H. WANG, Wilfrid Laurier University, and WAN-DI WEI, University of Waterloo; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

According to the proposer, the problem is equivalent to finding all triangles of sides $\sqrt{x}, \sqrt{y}, \sqrt{z}$ (x, y, z positive integers) with the same area as an equilateral triangle of unit sides.

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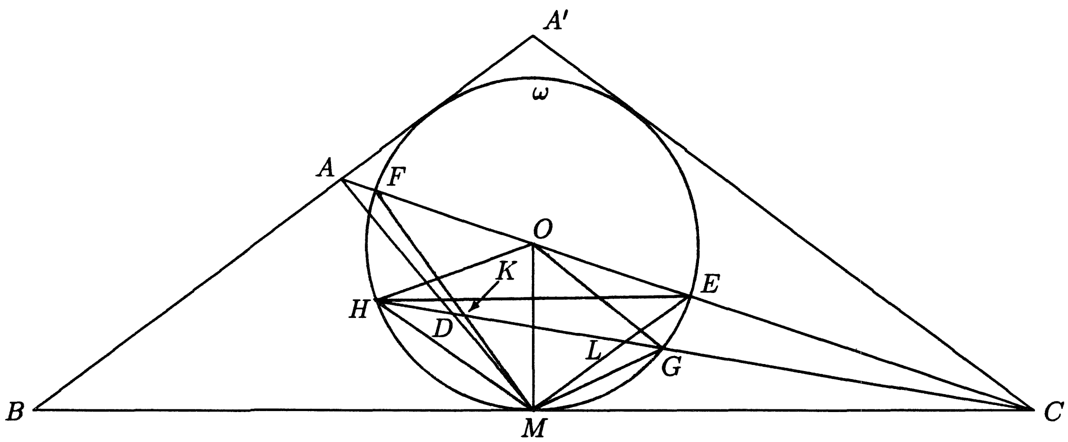
1562. [1990: 204] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let M be the midpoint of BC of a triangle ABC such that $\angle B = 2\angle C$, and let D be the intersection of the internal bisector of angle C with AM. Prove that $\angle MDC \le 45°$.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Consider the isosceles triangle $A'BC$, where A' is on BA extended such that $\angle BCA' = \angle CBA$. The centre O of the incircle ω of $A'BC$ is on CA. Let $\{E, F\} = \omega \cap CA$ (in the order $CEOFA$), $\{G, H\} = \omega \cap CD$ (in the order $CGDH$), $K = MF \cap CD$, $L = ME \cap CD$, as shown.



We have

$$\angle FMH - \angle EMG = \angle FEH - \angle EHG = \angle ECH = \angle MCH = \angle MGH - \angle MHG,$$

therefore

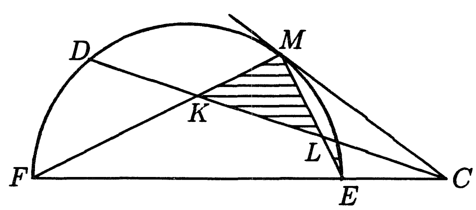
$$\angle MKG = \angle FMH + \angle MHG = \angle MGH + \angle EMG = \angle MLH.$$

Since $\angle FME = 90°$, we conclude $\angle MKL = \angle MLK = 45°$. Thus

$$\angle MDC = \angle MKL - \angle DMK \le 45°.$$

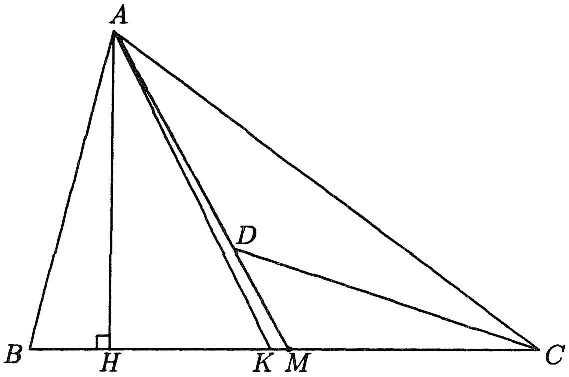
Equality holds only when $A = F$, i.e. $A'BC$ is equilateral, i.e. $\angle B = 60°$, $\angle C = 30°$, $\angle A = 90°$.

Note: the solution is based on the nice property illustrated at the right, namely, *if CD is the bisector of $\angle C$, then KLM is an isosceles right triangle.*



II. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

$\angle MDC = \angle MAC + C/2$, ainsi



$$\angle MDC \le 45° \iff \angle MAC \le \frac{90° - C}{2}\,.$$

Désignons par H le pied de la hauteur issue de A, $\angle HAC = 90° - C$. Il faut donc démontrer que $\angle MAC \le \angle HAC/2$, autrement dit que la bissectrice intérieure de $\angle HAC$ coupe BC en un point K tel que $KC \ge MC$. On a

$$\frac{HK}{KC} = \frac{AH}{AC} = \sin C,$$

donc

$$\frac{HC}{KC} = \frac{HK + KC}{KC} = \sin C + 1$$

et

$$KC = \frac{HC}{\sin C + 1} = \frac{b\cos C}{\sin C + 1}\,.$$

Ainsi

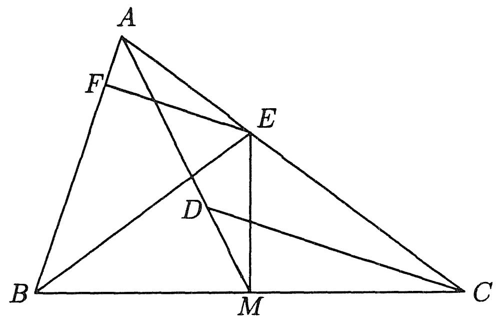
$$\begin{aligned} KC \ge MC &\iff \frac{b\cos C}{\sin C + 1} \ge \frac{a}{2} = \frac{b\sin A}{2\sin B} = \frac{b\sin(B+C)}{2\sin 2C} = \frac{b\sin 3C}{2\sin 2C} \\ &\iff 2\cos C\sin 2C \ge (\sin C + 1)\sin 3C \\ &\iff \sin 3C + \sin C \ge \sin C\sin 3C + \sin 3C \\ &\iff 1 \ge \sin 3C, \end{aligned}$$

ce qui est vrai.

III. *Solution by the proposer.*

Note first that

$$\begin{aligned} 2\angle MDC &= 2(\angle MAC + \angle ACD) \\ &= 2\angle MAC + \angle ACM \\ &= \angle MAC + \angle AMB. \qquad (1) \end{aligned}$$



Let E be the intersection of the bisector of $\angle B$ with AC; then we get

$$\angle EBC = \frac{1}{2}\angle ABC = \angle ECB.$$

As M is the midpoint of BC, we get $EM \perp BC$. Let F be the foot of the perpendicular from E to AB. Since $\angle ABE = \angle EBM$, we have $EF = EM$. As $EF \perp AB$ we have $AE \ge EF$, therefore we get $AE \ge EM$. Thus we obtain $\angle AME \ge \angle MAE$, i.e., $90° - \angle AMB \ge \angle MAC$. Therefore

$$90° \ge \angle AMB + \angle MAC. \qquad (2)$$

From (1) and (2) we get $90° \ge 2\angle MDC$, consequently we have $\angle MDC \le 45°$.

Also solved by HAYO AHLBURG, Benidorm, Spain; ILIA BLASKOV, Technical University, Gabrovo, Bulgaria; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; LJUBOMIR LJUBENOV, Stara Zagora, Bulgaria; VEDULA N. MURTY, Penn State Harrisburg; P. PENNING, Delft, The Netherlands; SHAILESH A. SHIRALI, Rishi Valley School, India; and D.J. SMEENK, Zaltbommel, The Netherlands.

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1563*. [1990: 204] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $N \geq 2$. For each positive integer n the number $A_n > 1$ is implicitly defined by

$$1 = \sum_{k=1}^{n} \frac{1}{A_n k^N - 1} .$$

Show that the sequence $A_1, A_2, A_3, \ldots$ converges.

Solution by John H. Lindsey, Northern Illinois University, Dekalb.

$A_n > 1$ is defined since the given sum is decreasing in A_n and has limits ∞ and 0 as $A_n \to 1^+$ and $A_n \to \infty$ respectively.

Suppose for some n, $A_{n+1} \leq A_n$. Then

$$1 = \sum_{k=1}^{n} \frac{1}{a_n k^N - 1} \leq \sum_{k=1}^{n} \frac{1}{A_{n+1} k^N - 1} < \sum_{k=1}^{n+1} \frac{1}{A_{n+1} k^N - 1} = 1,$$

a contradiction. Thus $\{A_n\}$ increases, and it suffices to show $A_n \leq 3$ for all n. So suppose some $A_n > 3$. Then

$$\begin{aligned} 1 &= \sum_{k=1}^{n} \frac{1}{A_n k^N - 1} < \sum_{k=1}^{n} \frac{1}{3k^N - 1} \leq \sum_{k=1}^{n} \frac{1}{2k^N} \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2} \\ &< \frac{1}{2}\left(1 + \sum_{k=2}^{n} \frac{1}{(k-1)k}\right) = \frac{1}{2}\left(1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)\right) = \frac{1}{2}\left(1 + 1 - \frac{1}{n}\right) < 1, \end{aligned}$$

a contradiction.

Also solved by MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; MORDECHAI FALKOWITZ, Tel-Aviv, Israel; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; ROBERT B. ISRAEL, University of British Columbia; RICHARD KATZ, California State University, Los Angeles; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and CHRIS WILDHAGEN, Rotterdam, The Netherlands.

Several solvers observed that the result is true for any $N > 1$ and/or found better bounds for A_n. Falkowitz points out that problem E3348 *of the* American Mathematical Monthly *(1989, p. 735) can be shown to be similar to this problem.*

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1565. [1990: 205] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

From the set of vertices of the n-dimensional cube choose three at random. Let p_n be the probability that they span a right-angled triangle. Find the asymptotic behavior of p_n as $n \to \infty$.

Solution by Shailesh A. Shirali, Rishi Valley School, India.

Coordinatise n-space and assume without loss that the vertices of the n-dimensional cube are the 2^n possible points all of whose coordinates are 0 or 1. Let us now count the total number of 3-sets of vertices that can serve as the vertices of a right-angled triangle. This equals $2^n q(n)$ where $q(n)$ is the number of such 3-sets in which the "elbow" of the right angle is the origin $O(0,\cdots,0)$. Now if A, B are vertices of the unit cube and $\angle AOB = 90°$, then the scalar product $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$. Considering the placement of 1's in the two vectors (the 1's must occur in disjoint positions if the scalar product is to be 0), it is clear that $q(n)$ can be equivalently defined as the total number of (unordered) pairs of nonempty disjoint subsets of a given n-set. It follows that

$$q(n) = \frac{1}{2}\sum_{r=1}^{n-1} C(n,r)(2^{n-r}-1)\,, \quad \text{where } C(n,r) = \frac{n!}{r!(n-r)!}\,.$$

This summation is easily evaluated, for $\sum_{r=1}^{n-1} C(n,r) = 2^n - 2$ while by the binomial theorem

$$\sum_{r=1}^{n-1} C(n,r)2^{n-r} = \sum_{r=1}^{n-1} C(n,r)2^r = 3^n - 2^n - 1.$$

Therefore we find that

$$q(n) = \frac{1}{2}(3^n - 2\cdot 2^n + 1).$$

Finally the required probability must equal

$$p(n) = \frac{2^n q(n)}{C(2^n,3)} = \frac{3(3^n - 2\cdot 2^n + 1)}{(2^n-1)(2^n-2)}\,.$$

Thus for large n, $p(n)$ is approximately equal to $3(3/4)^n$. In particular, $p(n)$ tends to 0 as n tends to ∞.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOHN H. LINDSEY, Northern Illinois University, Dekalb; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

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1566. [1990: 205] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Find all Heronian triangles ABC (i.e. with integer sides and area) such that the lengths OA, AH, OH are in arithmetic progression, where O and H are respectively the circumcenter and the orthocenter of ΔABC.

Solution by Vedula N. Murty, Penn State Harrisburg.

I claim that (with integer sides and area) a triangle satisfying the condition on OA, AH, OH cannot exist. Here is my proof.

We know

$$OA = R \text{ (circumradius)}, \; AH = 2R\cos A, \; OH = R\sqrt{1 - 8\cos A\cos B\cos C}$$

(e.g., see Hobson, *A Treatise on Trigonometry*, Cambridge Univ. Press). If OA, AH, OH are to be in arithmetic progression, then we must have $2AH = OA + OH$, or

$$4\cos A = 1 + \sqrt{1 - 8\cos A\cos B\cos C},$$

$$(4\cos A - 1)^2 = 1 - 8\cos A\cos B\cos C,$$

and so

$$8\cos A(2\cos A - 1 + \cos B\cos C) = 0.$$

Therefore either $\cos A = 0$ or $1 - 2\cos A = \cos B\cos C$.

If $\cos A = 0$ then $A = \pi/2$, so A and H coincide. In this case the lengths OA, AH, OH would be $R, 0, R$ which are not in arithmetic progression.

Hence we must have $1 - 2\cos A = \cos B\cos C$. Replacing $\cos A$, $\cos B$, $\cos C$ by

$$\frac{b^2 + c^2 - a^2}{2bc}, \quad \frac{c^2 + a^2 - b^2}{2ca}, \quad \frac{a^2 + b^2 - c^2}{2ab}$$

respectively (where a, b, c are the sides of the triangle), and multiplying both sides by $4a^2bc$, we obtain

$$4a^2bc - 4a^2(b^2 + c^2 - a^2) = (a^2 - (b^2 - c^2))(a^2 + (b^2 - c^2)) = a^4 - (b^2 - c^2)^2$$

or

$$3a^4 - 4a^2(b^2 + c^2 - bc) + (b^2 - c^2)^2 = 0.$$

This quadratic equation in a^2 has two roots $a^2 = (b - c)^2$, $a^2 = (b + c)^2/3$, i.e. either $a = b - c$ or $c - b$, which is impossible as the triangle will be degenerate, or $b + c = \sqrt{3}a$, which is also impossible if a, b, c are integers. Thus the problem as given has no solution!

Also solved by P. PENNING, Delft, The Netherlands; and the proposer.

The above proof does not use that the area of the triangle is an integer. Penning (whose solution was the same) noticed this.

Hayo Ahlburg, Benidorm, Spain, looked at all *permutations of the lengths OA, AH, OH, but was not able to determine if any of them yield an arithmetic progression (in a Heronian triangle). Can the readers help?*

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