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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Vol. 13, No. 2
February 1987

Published by the Canadian Mathematical Society/
Publié par la Société Mathématique du Canada

The support of the University of Calgary Department of Mathematics and Statistics is gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22.50 for members of the Canadian Mathematical Society and $25 for nonmembers. Back issues: $2.75 each. Bound volumes with index: Vols. 1 & 2 (combined) and each of Vols. 3-10: $20. All prices quoted are in Canadian dollars. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content of the journal should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

ISSN 0705 - 0348.

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CONTENTS

THE OLYMPIAD CORNER: 82

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

By now many of you will have suffered the first shock/horror at the changes evident in my first issue of the Corner last month. Correspondence has not begun pouring in, but I hope to see an avalanche of contributions of new problem sets from various contests as well as elegant solutions to problems we have published.

We begin this Corner with the *First Round* of the *Dutch Mathematical Olympiad* written 7 March 1986. The answers (only) will be published next month. To permit you to score yourself the point score for category A is 2 points per problem, for category B 3 points per problem, and for category C 4 points per problem. Allow 3 hours.

A1. A cyclist has a velocity of 33.8 km/h. He uses 52 teeth on the pedal axis, and 14 teeth on the back wheel axis. Then he switches to a different gear with 54 teeth on the pedal axis and 13 teeth on the back axis. What is his velocity if his pedals have the same angular velocity as before switching?

A2. A, B, and C play a game in which after every round the loser has to double all the money of every other player. First A loses, then B, and then C. As a result, each is left with \$16.00. How much did each have before playing?

A3. How many connected puzzle pieces can be constructed by putting together five congruent equilateral triangles along common sides? (Congruent pieces are considered the same.)

A4. For banding a sparrow, a bird watcher has the choice of 5 types of bands. At most one band can be attached to each leg. How many possibilities for banding does he have?

A5. Compute the maximum area of a rhombus contained in a rectangle of sides 12 and 16.

A6. In a trapezium $ABCD$ with $AB \| CD$ we have $AB = 4$, $CD = 3$, and area of $ABCD = 14$. On AB, AD and CD points F, E, G are chosen such that $AF = FB$, $AE = ED$ and $2 \cdot DG = GC$. Compute the area of $DEFG$.

*

B1. Determine all six-digit numbers of the form 523... that are simultaneously divisible by 7, 8 and 9.

B2. One arc minute (= 1/60 degree) on the earth's surface to the north or to the south corresponds to a distance of 1 nautical mile (about 1852 metres). At what latitude on the northern hemisphere does one arc minute to the west or to the east correspond with 1/2 nautical mile?

B3. Four points are given, not in one plane. How many planes are there that have equal distance to all four points?

B4. The number $(5^2 + 9^2)(12^2 + 17^2)$ can be written as the sum of two squares of natural numbers. Give such a sum.

*

C1. Determine positive integers n, m such that

$$(n + m)^m = n^m + 1413.$$

C2. Two parallel mirrors S and T facing each other are at distance 2. A light signal departs from a point A midway between the mirrors, is reflected in S, T, and S again, and arrives at point B on mirror T. The distance AB is equal to 4. Determine the length of the path travelled by the signal.

C3. An $n \times n \times n$ cube is composed of n^3 unit cubes. Some faces of the big cube are painted. Then it is taken apart. Exactly 144 unit cubes appear with no painted faces at all. How many unit cubes have *exactly two* painted faces?

* * *

The next group of problems we pose appeared as the *Alberta High School Mathematics Scholarship Examination* given February 10, 1987. Some of these problems are on the easy side for Olympiad regulars, but we include the set to indicate the direction some "pre-Olympiad" contests are taking. Of course elegant solutions and generalizations will be considered for publication.

1. (Fibonacci, AD 1225). If a and b are two positive integers that have no common factor and such that $a + b$ is even, show that the product $ab(a - b)(a + b)$ is divisible by 24.

2. A (badly designed) table is constructed by nailing through its centre a circular disc of diameter two metres to a sphere of diameter one metre. The table tips over to bring the edge of the disc into contact with the floor. As the table rolls, the two points of contact with the floor trace out a pair of concentric circles. What are the radii of these circles?

3. Find all pairs of real numbers (x,y) that are solutions to the simultaneous equations

$$x^{x+y} = y^3 \quad \text{and} \quad y^{x+y} = x^6y^3.$$

4. Adrian thinks of an integer A between 1 and 12 inclusive and Bernice tries to guess it. With each guess B that Bernice makes, Adrian makes one of the following replies:

(a) "That's my number." if $B = A$.

(b) "You're close." if $0 < |B - A| \leq 3$.

(c) "You're not close." if $|B - A| > 3$.

Show that Bernice can always force Adrian to say "That's my number." with four or fewer guesses.

5. $ABCD$ is a rectangular sheet of paper. E is the point on the side CD such that if the paper is folded along BE, C will coincide with a point F on the side AD. G is the point of intersection of the lines AB and EF. Prove that CG is perpendicular to BD.

* * *

Next, we present the problems from the Final Round of the Bulgarian Olympiad 1985. We thank Ivan Tonov for forwarding them.

1. Let $f(x)$ be a nonconstant polynomial with integer coefficients and n and k be fixed positive integers. Prove that there are n consecutive positive integers $a, a + 1, \ldots, a + n - 1$, such that each of

$$f(a), f(a + 1), \ldots, f(a + n - 1)$$

has at least k prime divisors. (We say that the number $m = p_1^{\ell_1} p_2^{\ell_2} \ldots p_s^{\ell_s}$ has $\ell_1 + \ldots + \ell_s$ prime divisors.)

2. Find all values of the parameter a such that all roots of the equation

$$x^6 + 3x^5 + (6 - a)x^4 + (7 - 2a)x^3 + (6 - a)x^2 + 3x + 1 = 0$$

are real numbers.

3. A sphere is inscribed in a quadrilateral pyramid $MABCD$ with a base $ABCD$. The centre O of the sphere lies on the altitude MH of the pyramid. Each of the planes (ACM), (BDM) and (ABO) divides the surrounding surface into two equal parts. The ratio of the areas of the intersections of the pyramid with the planes (ACM) and (ABO) is $(\sqrt{2} + 2)$: 4. Find the dihedral angles between the planes (ACM) and (ABO) and the planes (ABM) and $(ABCD)$.

4. Let $P_1, P_2, \ldots, P_7$ be seven points in space, no four of which lie on a plane. Color each of the line segments P_iP_j, $i < j$, with one of two colors, red or black. Prove that there are two monochromatic triangles that have no common side. (We say that a triangle is monochromatic if all its sides are colored with the same color.)

Is the analogous statement true for 6 points?

5. Let ABC $(AC \neq BC)$ be a triangle with $\gamma = \angle ACB$ a given acute angle, and let M be the midpoint of AB. Point P is chosen on the line segment CM so that the bisectors of the angles PAC and PBC meet at a point Q on CM. Find the angles APB and AQB.

6. For every positive integer a let α_a be the greatest odd divisor of a. Also set

$$s_b = \sum_{a=1}^{b} \frac{\alpha_a}{a} .$$

Prove that the sequence $\{s_b/b\}_{b=1}^{\infty}$ is convergent and find its limit.

* * *

The next item is a correction forwarded by Professor Klamkin.

4. [1981: 77; 1986: 10] *From the Third Selection Test for the Rumanian team for the 1978 I.M.O.*

For a natural number $n \geq 1$, solve the equation

$\sin x \sin 2x \ldots \sin nx + \cos x \cos 2x \ldots \cos nx = 1.$

Comment by M.S.K.

J.T. Groenman has pointed out a small error in my previous solution by noting there is another solution $x = \pi$ for the case $n > 2$. I had erred in the equality condition. The last sentence should be replaced by the following:

For the equality case in Hölder's inequality, the vectors

$$(\sin^2 x, \cos^2 x),\ (\sin^2 2x, \cos^2 2x), \ldots, (\sin^2 nx, \cos^2 nx)$$

must be parallel. Thus either $x = 0$ or $x = \pi$.

* * *

Next, we present some solutions to problems posed in the March 1985 issue of the Corner.

26. [1985: 70] *Proposed by Australia.*

Let $x_1, x_2, \ldots, x_n$ denote n real numbers lying in the interval $[0,1]$. Show that there is a number $x \in [0,1]$ such that

$$\frac{1}{n}\sum_{i=1}^{n} |x - x_i| = \frac{1}{2}.$$

Solution by George Evagelopoulos, Athens, and by V.N. Murty, Pennsylvania State University.

For arbitrary a, let $f(a) = \frac{1}{n}\sum_{i=1}^{n} |a - x_i|$. Then $f(0) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $f(1) = \frac{1}{n}\sum_{i=1}^{n} (1 - x_i) = 1 - f(0)$. Thus $f(0) + f(1) = 1$. Therefore, if $f(0) > \frac{1}{2}$, then $f(1) < \frac{1}{2}$ and if $f(0) < \frac{1}{2}$ then $f(1) > \frac{1}{2}$. As f is continuous it is clear that there is x (not necessarily unique) between 0 and 1 where $f(x) = \frac{1}{2}$.

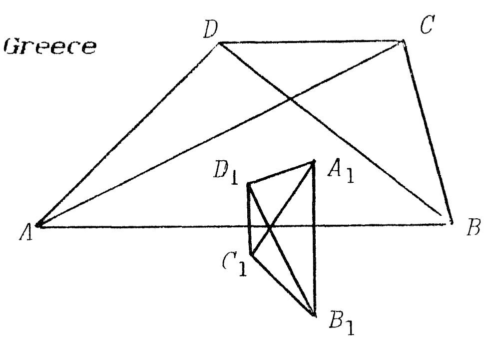
27. [1985: 70] *Proposed by Australia.*

Let $ABCD$ be a convex quadrilateral, and let A_1, B_1, C_1, D_1 be the circumcenters of triangles BCD, CDA, DAB and ABC respectively.

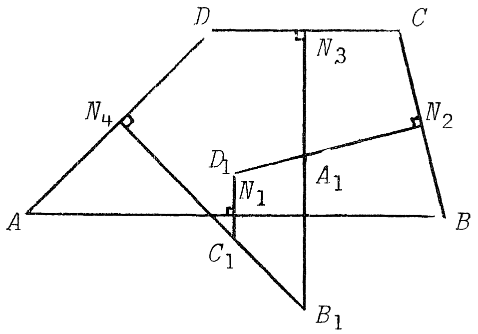
(i) Prove that either all of A_1, B_1, C_1, D_1 coincide in one point, or else they are all distinct. Assuming the latter case, show that A_1 and C_1 are on opposite sides of line B_1D_1 and that B_1 and D_1 are on opposite sides of line A_1C_1. (This establishes the convexity of quadrilateral $A_1B_1C_1D_1$.)

(ii) Let $A_2B_2C_2D_2$ be the circumcenters of triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$ and $A_1B_1C_1$, respectively. Show that quadrilateral $A_2B_2C_2D_2$ is similar to quadrilateral $ABCD$.

Solution by George Evagelopoulos, Athens, Greece



(i) If any two of the points A_1, B_1, C_1, D_1 coincide, at O say, then a circle centered at O passes through all vertices A, B, C, D of the given quadrilateral. Consequently the given quadrilateral is concyclic and all four circumcenters A_1, B_1, C_1 and D_1 coincide. Assuming this is not the case, assume, without loss of generality, that $\angle DAB + \angle BCD < 180^\circ$. Then A is outside the circumcircle of triangle BCD.



So $\overline{AA_1} > \overline{A_1C}$. Similarly $\overline{CC_1} > \overline{C_1A}$. Therefore we conclude that the perpendicular bisector of the diagonal AC separates A_1 and C_1. But since both B_1 and D_1 are on this perpendicular bisector, it is clear that A_1 and C_1 are on opposite sides of line B_1D_1. Similarly it follows that the line A_1C_1 separates B_1 and D_1, for $\overline{DD_1} < \overline{D_1B}$ and $\overline{BB_1} < \overline{B_1D}$.

(ii) Let the midpoints of the segments AB, BC, CD, DA be denoted by N_1, N_2, N_3, N_4, respectively. It follows from the cyclic quadrilateral $D_1N_1BN_2$ that $\angle A_1D_1C_1 = \angle N_1D_1N_2 = 180^\circ - \angle ABC$. The same argument holds for all the angles of $A_1B_1C_1D_1$. The angles of $A_2B_2C_2D_2$ are similarly related to those of $A_1B_1C_1D_1$. Thus

$$\angle A_2B_2C_2 = 180^\circ - \angle A_1D_1C_1 = 180^\circ - (180^\circ - \angle ABC) = \angle ABC.$$

This is true for each of the angles.

Also A_2B_2 is perpendicular to C_1D_1 and AB is perpendicular to C_1D_1. Therefore A_2B_2 is parallel to AB. Thus the sides of $A_2B_2C_2D_2$ are parallel to those of $ABCD$ in order.

Further A_1C_1 is the perpendicular bisector of BD, while B_2D_2 is the perpendicular bisector of A_1C_1. This means that the diagonal BD of $ABCD$ is parallel to the diagonal B_2D_2 of $A_2B_2C_2D_2$.

Thus the sides and diagonals of $A_2B_2C_2D_2$ are parallel to those of $ABCD$; therefore the two quadrilaterals are similar and similarly positioned.

<u>28</u>. [1985: 70] *Proposed by Belgium.*

Determine all integers x such that $x^4 + x^3 + x^2 + x + 1$ is a perfect square.

Comment: Both George Evagelopoulos of Athens and Edward T.H. Wang of Wilfrid Laurier University wrote supplying details of the history of this problem. It was proposed in 1926 by T.H. Gronwall in *Amer. Math. Monthly* (Problem 2784, 1926, p.281) and went unsolved for seven years until Professor Bennett gave a solution. Another more elegant solution was published in

Mathematical Digest (Professor J.H. Webb, Editor, July 1973). Both of these solutions are discussed in Ross Honsberger's book, *Mathematical Morsels* (Problem 74, pp.192-195, Dolciani Mathematical Expositions, No.3, M.A.A., 1978). It also was selected as one of the 400 "Best" problems (1918-1950) proposed in the *Monthly* and was included in the Otto Dunkel Memorial Problem Book (*Amer. Math. Monthly* Vol.64, No.7, Part II, 1957, p.10). The solution we present here is a variant of the second published solution that appears to be slightly simpler.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

We shall show that the only integers x such that $x^4 + x^3 + x^2 + x + 1$ is a perfect square are -1, 0, and 3.

It is easily seen that (x,y) is a solution of $x^4 + x^3 + x^2 + x + 1 = y^2$ if and only if $(x,-y)$ is as well. Hence in what follows we shall assume that $y \geq 0$.

We begin with the following collection of equivalent equations:

$$x^4 + x^3 + x^2 + x + 1 = y^2$$

$$4x^4 + 4x^3 + 4x^2 + 4x + 4 = 4y^2$$

$$(2x^2 + x)^2 + (3x^2 + 4x + 4) = (2y)^2 \tag{1}$$

$$(2x^2 + x + 1)^2 - (x - 3)(x + 1) = (2y)^2. \tag{2}$$

Note that $3x^2 + 4x + 4 > 0$ since its discriminant is $16 - 48 = -32$. Since $2x^2 + x = x(x + \frac{1}{2})$ we see that $2x^2 + x$ is nonnegative for integral x. Hence $2x^2 + x + 1 > 0$ for integral x. Now if $x > 3$, $2x^2 + x < 2y$ [by (1)] and $2y < 2x^2 + x + 1$ [by (2)]. Hence there are no solutions when $x > 3$. If $x < -1$, $2x^2 + x < 2y$ [by (1)] and $2y < 2x^2 + x + 1$ [by (2)]. Hence there are no solutions when $x < -1$. Direct calculation shows that of -1, 0, 1, 2, 3 the only solutions are -1, 0 and 3.

<u>29</u>. [1985: 70] *Proposed by Belgium.*

Determine all integer solutions (x,y) to the Diophantine equation

$$x^3 - y^3 = 2xy + 8.$$

Solution by George Evagelopoulos, Athens, Greece.

(i) If $x = 0$ then $-y^3 = 8$ so $y = -2$.

(ii) If $y = 0$ then $x^3 = 8$ so $x = 2$.

(iii) Suppose neither x nor y are zero. There are three possibilities we distinguish.

Case A. $x > 0$ and $y < 0$.

Then $x^3 = y^3 + 2xy + 8 \Longrightarrow x^3 < 2^3 \Longrightarrow x = 1$. Thus $y^3 + 2y + 7 = 0$, and this has no integral solution.

Case B. $x < 0$ and $y > 0$.

Then $y^3 - x^3 = -2xy - 8 < -2xy$ and $y^3 - x^3 = y^3 + (-x^3) > y^2 + (-x)^2 \geq -2xy$. This is impossible.

Case C. $xy > 0$.

Then $x^3 - y^3 = 2xy + 8 > 0$. Hence $x^3 - y^3 = (x - y)(x^2 + xy + y^2) > 0$ and this is equivalent to $x - y > 0$. There are now two possibilities. First supposing $x - y = 1$ then

$$(x - y)(x^2 + xy + y^2) = 2xy + 8$$

so

$$(x - y)((x - y)^2 + 3xy) = 2xy + 8. \qquad (*)$$

Then

$$(1 + 3(y + 1)y) = 2(y + 1) + 8$$

and

$$y^2 + y - 7 = 0.$$

This equation is easily seen to have no integral solutions.

Finally, suppose $x - y \geq 2$. Then

$$x^3 - y^3 = 2xy + 8$$

gives

$$2xy + 8 \geq 2(4 + 3xy)$$

from (*). This gives

$$2xy + 8 \geq 8 + 6xy$$

and

$$xy \leq 0$$

which contradicts the assumption $xy > 0$ in this case. Thus the Diophantine equation

$$x^3 - y^3 = 2xy + 8$$

has only the following two integer solutions: $(0,-2)$ and $(2,0)$.

<u>30</u>. [1985: 71] *Proposed by Brazil.*

A box contains p white balls and q black balls, and beside the box lies a large pile of black balls. Two balls chosen at random (with equal likelihood) are taken out of the box. If they are of the same color, a black ball from the pile is put into the box; otherwise, the white ball is put back into the box. The procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this last ball is white?

Comment: Solutions were submitted by C. Cooper, Central Missouri State University; George Evagelopoulos, Athens, Greece; and John Morvay, Dallas, Texas. A solution was published in February 1986 [1986: 22] where it is solved as Problem 3 of the 1983 Australian Olympiad.

C. Cooper points out that the problem with white and black beans appears as the "Coffee Can Problem" in *Science of Programming* by David Gries, who attributes it to Carel Scholten. Is this the true origin?

32. [1985: 71] *Proposed by Bulgaria.*

A regular n-gonal truncated pyramid with base areas S_1 and S_2 and lateral surface area S is circumscribed about a sphere. Let A be the area of the polygon whose vertices are the points of tangency of the lateral faces of the truncated pyramid with the sphere. Prove that

$$AS = 4S_1S_2 \cos^2 \frac{\pi}{n} .$$

Solution by George Evagelopoulos, Athens, Greece.

Denote by n the radius of the sphere and by V and h the volume and the height of the pyramid, respectively. Then we have

$$V = \frac{h}{3}(S_1 + S_2 + \sqrt{S_1S_2}) = \frac{2r}{3}(S_1 + S_2 + \sqrt{S_1S_2}).$$

But

$$V = \frac{r}{3}(S_1 + S_2 + S),$$

too. Therefore

$$\frac{2r}{3}(S_1 + S_2 + \sqrt{S_1S_2}) = \frac{r}{3}(S_1 + S_2 + S).$$

Equivalently

$$S = S_1 + S_2 + 2\sqrt{S_1S_2} . \tag{1}$$

Denote by O_1 and O_2 the centres of the bases S_1 and S_2, respectively, and by QQ', RR' two edges of the respective bases, such that $QQ'RR'$ is a side face. Let P be the point of tangency of this face with the sphere. Let C denote the centre of the sphere. In triangles CPQ and CO_1Q, $\angle CPQ$ and $\angle CO_1Q$ are 90° as Q lies in the intersection of the planes tangent to the sphere at P and O_1. These triangles have common hypotenuse CA, and $\overline{CO_1} = \overline{CP}$. Thus $\overline{PQ} = \overline{O_1Q}$. Similarly $\overline{PQ'} = \overline{O_1Q'}$ $(= \overline{O_1Q})$ so that triangles $QQ'O_1$ and $QQ'P$ are congruent.

Similarly, triangles $RR'O_2$ and $RR'P$ are congruent. Since triangles $RR'O_2$ and $QQ'O_1$ are similar and isosceles we conclude that $\angle Q'QP$ and $\angle RR'P$ are equal and so P lies on the diagonal QR'. In fact, symmetry tells us that P is the point of intersection of the diagonals of the face $QQ'RR'$.

Furthermore, since the pyramid is regular, the points of the lateral faces lie in a plane parallel to the bases. The plane intersects the pyramid in a regular n-gon. Denoting by A' and c the area and the side, respectively, of this n-gon, we have

$$c = \frac{2ab}{a + b} \quad \text{where} \quad a = \overline{QQ'} \quad \text{and} \quad b = \overline{RR'}. \tag{2}$$

From equalities (1) and (2) and the equations

$$\frac{S_1}{A'} = \frac{a^2}{c^2}, \quad \frac{S_2}{A'} = \frac{b^2}{c^2}, \quad \frac{S_1}{S_2} = \frac{a^2}{b^2}$$

we obtain

$$A' = \frac{c^2}{ab}\sqrt{S_1S_2} = \frac{4ab}{(a + b)^2}\sqrt{S_1S_2} = \frac{4S_1S_2}{(\sqrt{S_1} + \sqrt{S_2})^2} .$$

Applying (1) we obtain

$$A' = \frac{4S_1S_2}{S} .$$

Finally, since

$$A = A' \cos^2 \frac{\pi}{n}$$

we obtain

$$AS = 4S_1S_2 \cos^2 \frac{\pi}{n} ,$$

completing the proof.

35. [1985: 71] *Proposed by Canada.*

You are given an algebraic system with an addition and a multiplication for which all the laws of ordinary arithmetic are valid except commutativity of multiplication. Show that

$$(a + ab^{-1}a)^{-1} + (a + b)^{-1} = a^{-1}$$

where x^{-1} is that element for which $x^{-1}x = xx^{-1} = e$, the multiplicative identity.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $x = (a + ab^{-1}a)^{-1}$ and $y = (a + b)^{-1}$. Then $x(a + ab^{-1}a) = (a + ab^{-1}a)x = e$ and $(a + b)y = y(a + b) = e$. Then

$$xab^{-1}(b + a) = e \tag{1}$$

$$(b + a)b^{-1}ax = e \tag{2}$$

$$ay + by = e \tag{3}$$

$$ya + yb = e. \tag{4}$$

From (1) $b + a = ba^{-1}x^{-1}$. This together with $a + b = y^{-1}$ yields $ba^{-1}x^{-1} = y^{-1}$ or $yb = xa$. Substituting this into (4) we obtain $ya + xa = e$ or

$$(y + x)a = e. \tag{5}$$

Similarly (2) leads to $b + a = x^{-1}a^{-1}b$ and hence $x^{-1}a^{-1}b = y^{-1}$. Thus $by = ax$. Substitution into (3) gives $ay + ax = e$ or

$$a(y + x) = e. \tag{6}$$

Taking (5) and (6) together we see that

$$x + y = a^{-1}$$

as desired.

36. [1985: 71] *Proposed by Czechoslovakia.*

Let

$$S = \left\{ \frac{m + n}{\sqrt{m^2 + n^2}} \;\middle|\; m, n \text{ positive integers} \right\}.$$

Show that for each $(x,y) \in S \times S$ with $x < y$ there exists $z \in S$ such that $x < z < y$.

Solution by Peter Andrews and Edward T.H. Wang of Wilfrid Laurier University, Waterloo, Ontario, and also by George Evagelopoulos, Athens, Greece.

Since all the elements of S are positive, instead of showing that for every $(x,y) \in S \times S$ with $x < y$ there is $z \in S$ such that $x < z < y$, it suffices to show that the squares of elements of S possess the property. Now

$$\left[\frac{m + n}{\sqrt{m^2 + n^2}} \right]^2 = \frac{m^2 + 2mn + n^2}{m^2 + n^2} = 1 + 2\,\frac{mn}{m^2 + n^2}.$$

It therefore suffices to solve the problem for the set

$$T = \left\{ \frac{mn}{m^2 + n^2} \;\middle|\; m, n \text{ positive integers} \right\}.$$

Let $(s,t) \in T \times T$ where $s = \dfrac{mn}{m^2 + n^2} < \dfrac{ab}{a^2 + b^2} = t$. Without any loss of generality we may assume that $m \le n$ and $a \le b$. Consider the function $f(x) = \dfrac{x}{1 + x^2}$. It is easily verified that f is strictly increasing on $[0,1]$. Indeed for $0 \le x < y \le 1$, $xy < 1$ and $xy(y - x) < (y - x)$, so $xy^2 + x < x^2y + y$ and $\dfrac{x}{1 + x^2} < \dfrac{y}{1 + y^2}$. Hence for all $c,d \in [0,1]$,

$$f(c) < f(d) \iff c < d.$$

Since

$$f\left(\frac{m}{n}\right) = \frac{mn}{m^2 + n^2} < \frac{ab}{a^2 + b^2} = f\left(\frac{a}{b}\right)$$

we get $\frac{m}{n} < \frac{a}{b}$. Therefore, if we choose *any* rational number $\frac{p}{q}$ where p and q are positive integers satisfying $\frac{m}{n} < \frac{p}{q} < \frac{a}{b}$ (for example $\frac{1}{2}\left(\frac{m}{n} + \frac{a}{b}\right)$) then the conclusion follows immediately from $f\left(\frac{m}{n}\right) < f\left(\frac{p}{q}\right) < f\left(\frac{a}{b}\right)$.

<u>43</u>. [1985: 72] *Proposed by Spain.*

Solve the equation

$$\tan^2 2x + 2 \tan 2x \tan 3x - 1 = 0.$$

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, and also by Curtis Cooper, Central Missouri State University.

Since $\cos 2x \ne 0$,

$$\begin{aligned}
&\tan^2 2x + 2 \tan 2x \tan 3x - 1 = 0\\
&\iff 2 \tan 2x \tan 3x = 1 - \tan^2 2x = 1 - \frac{\sin^2 2x}{\cos^2 2x} = \frac{\cos 4x}{\cos^2 2x}\\
&\iff 2 \sin 2x \cos 2x \tan 3x = \cos 4x\\
&\iff \sin 4x \tan 3x = \cos 4x.
\end{aligned}$$

If $\cos 4x = 0$ then $\sin 4x \ne 0$ so $\tan 3x = 0$. Thus $\sin 3x = 0$, and

$$\cos 7x = \cos 4x \cos 3x - \sin 4x \sin 3x = 0.$$

Otherwise, $\tan 4x \tan 3x = 1$ which can only hold if $\tan 7x$ is undefined. This also gives $\cos 7x = 0$. In either case, the solutions are given by $x = \frac{k\pi}{7} + \frac{\pi}{14}$ where k is any integer.

47. [1985: 73] *Proposed by the U.S.S.R.*

In the Martian language any finite ordered set of Latin letters is a word. The "Martian Word" editorial office issues a many-volume dictionary of the Martian language, in which the entries are numbered consecutively in alphabetical order. The first volume contains all the one-letter words, the second volume all the two-letter words, etc., and the numbering of the words in each successive volume continues the numbering in the preceding one. Determine the word whose number is the sum of the numbers of the words

Prague, Olympiad, Mathematics.

Solution by R.E.W.

For definiteness and simplicity, we assume "Latin alphabet" refers to the alphabet currently used for English and modern Romance languages, i.e. that there are 26 letters. (Actually ancient Latin only used an alphabet of 23 or 24 letters.) We begin with the following observation. If $b \geq 2$ is an integer then every *positive* integer n can be written in the form

$$n = \alpha_1 b^{\ell} + \alpha_2 b^{\ell-1} + \ldots + \alpha_{\ell+1} b^0, \quad 0 < \alpha_i \leq b \quad \text{for} \quad i = 1,2,\ldots,\ell.$$

This can be encoded by the word $\alpha_1\alpha_2\ldots\alpha_{\ell+1}$ in letters $\underline{1},\underline{2},\ldots,\underline{b}$. One should note the difference between this expression and the standard base b expression

$$n = \gamma_1 b^{\ell} + \gamma_2 b^{\ell-1} + \ldots + \gamma_{\ell+1} b^0$$

where $\gamma_1 \neq 0$ and $0 \leq \gamma_i < b$, which can be expressed as the string $\gamma_1\gamma_2\ldots\gamma_{\ell+1}$ in "digits" $0,1,2,\ldots,b-1$.

For example, with $b = 3$, the standard base 3 expression for $n = 11$ is 102. Written in the new way (using 3 but no zeros) $n = 11$ corresponds to 32 since $11 = 3\cdot3^1 + 2\cdot3^0$.

A little reflection will show how to change between standard base b expressions and the "zero-less" expression and show that this "zero-less" expression is unique. To emphasize that we are using this "zero-less" expression for n in base b we write

$$n \sim (\alpha_1,\ldots,\alpha_{\ell+1})'_b \quad \text{for} \quad n = \alpha_1 b^{\ell} + \alpha_2 b^{\ell-1} + \ldots + \alpha_{\ell+1} b^0$$

(where $0 < \alpha_i \leq b$ for $i = 1,2,\ldots,\ell+1$). Via this expression $(\alpha_1,\ldots,\alpha_{\ell+1})'_b$ corresponds to the n^{th} number where $n = \alpha_1 b^{\ell} + \ldots + \alpha_{\ell+1} b^0$. Notice that the ordering of the numbers is just that of the dictionary listing of the words

$(\alpha_1,\alpha_2,\ldots,\alpha_{\ell+1})$.

What about addition rules for these expressions? If $n \sim (\alpha_1,\ldots,\alpha_{\ell+1})_b'$ and $m \sim (\gamma_1,\ldots,\gamma_{j+1})_b'$ what sequence corresponds to $m + n$? A little reflection tells us that we write the addition as if it were the standard way, but only carry to the column to the left when the sum exceeds b (instead of exceeds $b - 1$ as is usual). For example, with $b = 3$, $n = 11 \sim (32)_3'$ and $m = 34 \sim (321)_3'$ we see that $m + n = 45 \sim (1123)_3'$ calculated by

$$\begin{array}{r} (32)_3' \\ (321)_3' \\ \hline (1123)_3' \end{array}.$$

The problem at hand corresponds to working with $b = 26$ where $A,B,C,\ldots,Z$ correspond to $1,2,3,\ldots,26$. With this the answer is given by

$$\begin{array}{r} (\mathrm{PRAGUE})_{26}' \\ (\mathrm{OLYMPIAD})_{26}' \\ (\mathrm{MATHEMATICS})_{26}' \\ \hline (\mathrm{MATWSCGKYZB})_{26}' \end{array}.$$

Thus the word is MATWSCGKYZB.

[*Note*: Curtis Cooper, Central Missouri State University, also submitted a solution using standard expressions, base 27, but with addition "jumping zeros".]

<u>49</u>. [1985: 73] *Proposed by the U.S.S.R.*

Let $x_1,x_2,\ldots,x_n$ be numbers such that

$$1 \ge x_1 \ge x_2 \ge \cdots \ge x_n > 0.$$

If $0 \le a \le 1$ prove that

$$(1 + x_1 + x_2 + \ldots + x_n)^a \le 1 + x_1^a + 2^{a-1}x_2^a + \ldots + n^{a-1}x_n^a .$$

Solution by George Evagelopoulos, Athens, Greece.

The inequality is obviously true when $n = 0$. Suppose that, for $n = k$,

$$(1 + x_1 + x_2 + \ldots + x_k)^a \le 1 + x_1^a + 2^{a-1}x_2^a + \ldots + k^{a-1}x_k^a .$$

We shall prove the result for $n = k + 1$. Since $0 \le a \le 1$ we have that

$$(1 + x_1 + x_2 + \ldots + x_{k+1})^a - (1 + x_1 + \ldots + x_k)^a$$

$$= (1 + x_1 + \ldots + x_k)^a \left[\left[1 + \frac{x_{k+1}}{1 + x_1 + \ldots + x_k}\right]^a - 1\right]$$

$$\le (1 + x_1 + \ldots + x_k)^a \left[\left[1 + \frac{x_{k+1}}{1 + x_1 + \ldots + x_k}\right] - 1\right]$$

$$= (1 + x_1 + \ldots + x_k)^{a-1} x_{k+1} .$$

Now, this inequality is equivalent to

$$(1 + x_1 + \ldots + x_{k+1})^a \le (1 + x_1 + \ldots + x_k)^a + (1 + x_1 + \ldots + x_k)^{a-1} x_{k+1} .$$

Since also $1 + x_1 + \ldots + x_k \ge (k + 1)x_{k+1}$ and $a - 1 \le 0$, we get

$$(1 + x_1 + \ldots + x_k)^{a-1} x_{k+1} \le [(k + 1)x_{k+1}]^{a-1} x_{k+1} .$$

Then

$$(1 + x_1 + \ldots + x_{k+1})^a \le (1 + x_1 + \ldots + x_k)^a + [(k + 1)x_{k+1}]^{a-1} x_{k+1} .$$

Equivalently

$$(1 + x_1 + \ldots + x_{k+1})^a \le (1 + x_1 + \ldots + x_k)^a + (k + 1)^{a-1} x_{k+1}^a .$$

Applying the induction hypothesis,

$$(1 + x_1 + \ldots + x_{k+1})^a \le 1 + x_1 + 2^{a-1} x_2^a + \ldots + k^{a-1} x_k^a + (k + 1)^{a-1} x_{k+1}^a$$

completing the proof.

<u>50</u>. [1985: 74] *Proposed by Vietnam.*

Given are the functions $F(x) = ax^2 + bx + c$ and $G(x) = cx^2 + bx + a$ where

$$|F(0)| \le 1, \quad |F(1)| \le 1, \quad \text{and} \quad |F(-1)| \le 1.$$

Prove that, for $|x| \le 1$,

(i) $|F(x)| \le 5/4$;

(ii) $|G(x)| \le 2$.

Solutions submitted separately by George Evagelopoulos, Athens, Greece, V N Murty, Pennsylvania State University, and A Tamanas, Thornton Heath, Surrey, England.

(i) Choosing the coefficients A, B, C suitably, the function $F(x)$ can be written in the form

$$F(x) = Ax(x+1) + Bx(x-1) + C(x^2-1).$$

Setting x successively equal to 0, 1 and -1 we obtain

$$C = -F(0), \quad A = \frac{F(1)}{2} \quad \text{and} \quad B = \frac{F(-1)}{2}.$$

Then

$$\begin{aligned} |F(x)| &\le \left|\frac{F(1)}{2}\right||x(x+1)| + \left|\frac{F(-1)}{2}\right||x(x-1)| + |-F(0)||x^2-1| \\ &\le \frac{1}{2}|x(x+1)| + \frac{1}{2}|x(x-1)| + |1-x^2| \\ &= \begin{cases} \frac{5}{4} - (x+\frac{1}{2})^2 & -1 \le x \le 0 \\ \frac{5}{4} - (x-\frac{1}{2})^2 & 0 < x \le 1 \end{cases} \end{aligned}$$

which gives $|F(x)| \le \frac{5}{4}$ on $[-1,1]$.

(ii) For $x \ne 0$, we have

$$G(x) = x^2F\left[\frac{1}{x}\right].$$

Consequently

$$G(x) = \frac{F(1)}{2}(1+x) + \frac{F(-1)}{2}(1-x) + F(0)(x^2-1).$$

Then for $|x| \le 1$, we get

$$|G(x)| \le \frac{1}{2}(1+x) + \frac{1}{2}(1-x) + (1-x^2)$$

and thus

$$|G(x)| \le 2 - x^2 \le 2,$$

completing the proof.

[*Editor's plea: Please submit comments, solutions and corrections, and remember to print your name clearly on the submission so that we can give credit where it is due.*]

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

* * *

1211. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Let $f(n)$ be the smallest integer greater than n such that $6^{f(n)}$ ends in the digits of 6^n. For example, $f(2) = 7$ since $6^7 = 279936$ is the next smallest power of 6 ending in $6^2 = 36$. Find a formula for $f(n)$.

1212. *Proposed by Svetoslav Bilchev, Technical University, and Emilia Velikova, Mathematikalgymnasium, Russe, Bulgaria.*

Prove that

$$\frac{u}{v+w}\cdot\frac{bc}{s-a} + \frac{v}{w+u}\cdot\frac{ca}{s-b} + \frac{w}{u+v}\cdot\frac{ab}{s-c} \geq a + b + c$$

where a, b, c are the sides of a triangle and s is its semiperimeter, and u, v, w are arbitrary positive real numbers.

1213*. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

In *Math. Gazette* 68 (1984) 222, P. Stanbury noted the two close approximations $e^6 \approx \pi^5 + \pi^4$ and $\pi^9/e^8 \approx 10$. Can one show without a calculator that (i) $e^6 > \pi^5 + \pi^4$ and (ii) $\pi^9/e^8 < 10$?

1214. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be an equilateral triangle and let P be an interior point. Show that there is a triangle with side lengths PA_1, PA_2, PA_3.

1215. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a, b, c be nonnegative real numbers with $a + b + c = 1$. Show that

$$ab + bc + ca \le a^3 + b^3 + c^3 + 6abc \le a^2 + b^2 + c^2$$
$$\le 2(a^3 + b^3 + c^3) + 3abc,$$

and for each inequality determine all cases when equality holds.

1216*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

1217. *Proposed by Niels Bejlegaard, Stavanger, Norway.*

Given are two lines ℓ_1 and ℓ_2 intersecting at A, and a point P in the same plane, where P does not lie on either angle bisector at A. Also given is a positive real number r.

(a) Construct a line through P, intersecting ℓ_1 and ℓ_2 at B and C respectively, such that $AB + AC = r$.

(b) Construct a line through P, intersecting ℓ_1 and ℓ_2 at B and C respectively, such that $|AB - AC| = r$.

1218*. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*

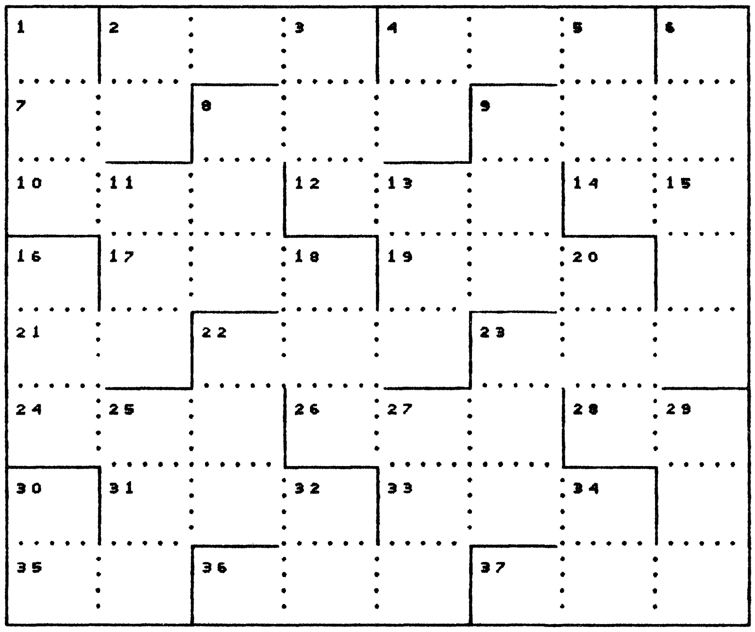
Let F_1 be the area of the orthic triangle of an acute triangle of area F and circumradius R. Prove that

$$F_1 \le \frac{4F^3}{27R^4}.$$

1219. *Proposed by Herta Freitag, Roanoke, Virginia, and Dan Sokolowsky, Williamsburg, Virginia.* (Dedicated to Léo Sauvé.)

Let the incircle of ΔABC touch AB at D, and let E be a point of side AC. Prove that the incircles of triangles ADE, BCE, and BDE have a common tangent.

<u>1220</u>. *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*



31B, 1D, 30D	14A, 28A, 6U	27D, 36B, 23B	24A, 12A, 32D
27D, 2B, 6D	18U, 12B, 7A	31A, 33A, 9B	8U, 17A, 20D
8U, 29D, 3D	15D, 8A, 28A	19B, 22D, 13U	22B, 18D, 16U
4U, 35A, 2D	31B, 33B, 9A	14A, 34U, 21A	19B, 37B, 25D
5D, 4A, 11D	31A, 10A, 26A	15U, 3D, 34U	9U, 23U, 20D

The twenty clues are triples (a,b,c) of two- and three-digit numbers, which form the sides of primitive integer triangles ABC with angle B twice the size of angle A.

A = across, B = back, D = down, U = up. For example, 31B has its hundreds and units digits in the squares labelled 32 and 31 respectively; 4U has its units digit in the square labelled 4.

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1069. [1985: 221] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

The point P lies in the plane of but outside a scalene triangle ABC. Show that there exist exactly two pairs (s,t), with $0 \le s,t < \pi$, such that the distances from P to sides BC, CA, AB, respectively, are proportional to

$$\sin(A - s)\sin(A - t) : \sin(B - s)\sin(B - t) : \sin(C - s)\sin(C - t), \qquad (1)$$

that is, such that (1) is a set of trilinear coordinates for P.

Solution by the proposer.

We begin with a result on parametric representation of lines in trilinear coordinates:

For any point $\alpha_1 : \beta_1 : \gamma_1$ except A, B, and C, the points given by

$$\alpha_t : \beta_t : \gamma_t = \alpha_1 \sin(A - t) : \beta_1 \sin(B - t) : \gamma_1 \sin(C - t), \qquad (2)$$

with $0 \le t < \pi$, form a line.

(More often, lines are represented by equations of the form $\ell\alpha + m\beta + n\gamma = 0$.)

To prove this, note that

$$\begin{vmatrix} \alpha_t & \beta_t & \gamma_t \\ \alpha_1 \sin A & \beta_1 \sin B & \gamma_1 \sin C \\ \alpha_1 \cos A & \beta_1 \cos B & \gamma_1 \cos C \end{vmatrix} = 0 \qquad (3)$$

(since $\cos t \times$ row 2 $- \sin t \times$ row 3 = row 1). This implies (item 4615 of [1] or page 8 of [2]) that $\alpha_t : \beta_t : \gamma_t$ lies on the line joining the points $\alpha_1 \sin A : \beta_1 \sin B : \gamma_1 \sin C$ and $\alpha_1 \cos A : \beta_1 \cos B : \gamma_1 \cos C$.

Conversely, any $\alpha : \beta : \gamma$ on this line causes (3) to hold, whence numbers p and q must exist satisfying

$$\alpha = p\alpha_1 \sin A + q\alpha_1 \cos A,$$
$$\beta = p\beta_1 \sin B + q\beta_1 \cos B,$$
$$\gamma = p\gamma_1 \sin C + q\gamma_1 \cos C.$$

Let t be the number satisfying

$$0 \le t < 2\pi, \quad \cos t = \frac{p}{\sqrt{p^2 + q^2}}, \quad \sin t = \frac{-q}{\sqrt{p^2 + q^2}}.$$

Then

$$\alpha\colon \beta\colon \gamma = \alpha_1\sin(A - t)\colon \beta_1\sin(B - t)\colon \gamma_1\sin(C - t).$$

If $t \ge \pi$ then $\alpha\colon \beta\colon \gamma$ remains fixed if $t - \pi$ is substituted for t.

Next we show that for any point $\alpha_1\colon \beta_1\colon \gamma_1$ except A, B, C, the points given by

$$\alpha_s\colon \beta_s\colon \gamma_s = [\alpha_1\sin(A - s)]^2\colon [\beta_1\sin(B - s)]^2\colon [\gamma_1\sin(C - s)]^2, \qquad (4)$$

$0 \le s < \pi$, form an ellipse inscribed in ABC. The line L in (2) can be written in the form

$$\alpha\sqrt{\ell} \pm \beta\sqrt{m} \pm \gamma\sqrt{n} = 0.$$

Thus, for each $\alpha_s\colon \beta_s\colon \gamma_s$ on L, the point $\alpha_s^2\colon \beta_s^2\colon \gamma_s^2$ satisfies

$$\sqrt{\ell\alpha} \pm \sqrt{m\beta} \pm \sqrt{n\gamma} = 0.$$

This is the equation of an ellipse inscribed in ABC (item 4740 of [1] or page 40 of [2]).

Now each point P lying outside ABC lies outside the ellipse (4). Hence P lies on two lines tangent to the ellipse. Consequently, there exist s_1 and s_2 such that $\alpha_{s_1}\colon \beta_{s_1}\colon \gamma_{s_1}$ and $\alpha_{s_2}\colon \beta_{s_2}\colon \gamma_{s_2}$, given by (4), are the points of tangency. The two tangent lines are then

$$\sin(A - s_i)\sin(A - t)\colon \sin(B - s_i)\sin(B - t)\colon \sin(C - s_i)\sin(C - t),$$

$0 \le t < \pi$, $i = 1,2$. Accordingly, there must exist t_1 and t_2 such that (s_1,t_1) and (s_2,t_2) are pairs (s,t) such that (1) is a set of trilinear coordinates for P. Clearly, no other pair (s,t) determines the point P.

References:

[1] George S. Carr, *Formulas and Theorems in Pure Mathematics*, 2nd ed., Chelsea, New York, 1970.

[2] W.P. Milne, *Homogeneous Coordinates*, 3rd ed., Edward Arnold, London, 1931.

One partial solution was received.

* * *

<u>1073</u>. [1985: 249] *Proposed by Jordi Dou, Barcelona, Spain.*

Let K be an interior point of triangle ABC. Through a point P in the plane of the triangle, parallels to the cevians AK, BK, CK are drawn to meet BC, CA, AB in L, M, N, respectively. If the points L, M, N are collinear,

(a) prove that the locus of P is an ellipse;

(b) construct the centre of this ellipse.

(This problem generalizes *Crux* 925 [1985: 154], which is the special case when $K = G$, the centroid of triangle ABC.)

I. *Solution by O. Bottema, Delft, The Netherlands.*

(a) The problem is one of affine geometry; therefore we introduce homogeneous barycentric coordinates x, y, z with respect to ABC. Thus $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$. Let the given point K be (p,q,r) and the variable point P be (x_0,y_0,z_0). Then the equation of the line AK is

$$ry - qz = 0,$$

and its intersection with the line ℓ: $x + y + z = 0$ at infinity is

$$(q + r,-q,-r).$$

Thus the line through P parallel to AK is

$$(qz_0 - ry_0)x + [rx_0 + (q + r)z_0]y - [(q + r)y_0 + qx_0]z = 0,$$

and its intersection L with $x = 0$ is

$$(0,(q + r)y_0 + qx_0,rx_0 + (q + r)z_0).$$

The coordinates of M and N follow by cyclic permutation. Hence L, M, N are collinear if

$$\begin{vmatrix} 0 & (q + r)y_0 + qx_0 & rx_0 + (q + r)z_0 \\ py_0 + (r + p)x_0 & 0 & (r + p)z_0 + ry_0 \\ (p + q)x_0 + pz_0 & qz_0 + (p + q)y_0 & 0 \end{vmatrix} = 0. \qquad (1)$$

If we add the second the third columns to the first, the elements of the latter have the factor $x_0 + y_0 + z_0 \neq 0$, and so (1) reduces to

$$\begin{vmatrix} q + r & qx_0 + (q + r)y_0 & rx_0 + (q + r)z_0 \\ r + p & 0 & ry_0 + (r + p)z_0 \\ p + q & qz_0 + (p + q)y_0 & 0 \end{vmatrix} = 0,$$

or

$$\begin{aligned} -(q + r)[ry_0 + (r + p)z_0][qz_0 + (p + q)y_0] \\ + (p + r)[rx_0 + (q + r)z_0][qz_0 + (p + q)y_0] \\ + (p + q)[qx_0 + (q + r)y_0][ry_0 + (r + p)z_0] = 0 \end{aligned}$$

which simplifies to

$$[(p+q)(p+r)(q+r) - qr(q+r)]y_0z_0 + q(p+r)[r+(p+q)]x_0z_0 + r(p+q)[(p+r)+q]x_0y_0 = 0,$$

$$(q+r)(p^2+pq+pr)y_0z_0 + q(p+r)(p+q+r)x_0z_0 + r(p+q)(p+q+r)x_0y_0 = 0,$$

and finally

$$p(q+r)y_0z_0 + q(p+r)x_0z_0 + r(p+q)x_0y_0 = 0.$$

So we obtain for the locus of P the equation

$$p(q+r)yz + q(p+r)xz + r(p+q)xy = 0, \qquad (2)$$

a conic through the vertices A, B, C. Its intersection with the line ℓ at infinity is found by putting $z = -(x+y)$ in (1), obtaining

$$-p(q+r)y(x+y) - q(r+p)x(x+y) + r(p+q)xy = 0$$

which simplifies to

$$q(r+p)x^2 + 2pqxy + p(q+r)y^2 = 0. \qquad (3)$$

The discriminant of (3) is

$$4p^2q^2 - 4pq(p+r)(q+r) = -4pqr(p+q+r)$$

which is negative, because K is an interior point of the triangle and so p, q, r are all positive. Hence the conic is an ellipse.

(b) If $Q = (m_1, m_2, m_3)$ is the centre of the above ellipse, the polar line with respect to the ellipse reads

$$[m_2r(p+q) + m_3q(r+p)]x + [m_3p(q+r) + m_1r(p+q)]y + [m_1q(r+p) + m_2p(q+r)]z = 0. \qquad (4)$$

Since (4) must be the line ℓ at infinity, we have

$$m_2r(p+q) + m_3q(r+p) = m_3p(q+r) + m_1r(p+q) = m_1q(r+p) + m_2p(q+r),$$

two homogeneous linear equations for m_1, m_2, m_3 with the solutions

$$m_1 = q+r\ , \quad m_2 = r+p\ , \quad m_3 = p+q.$$

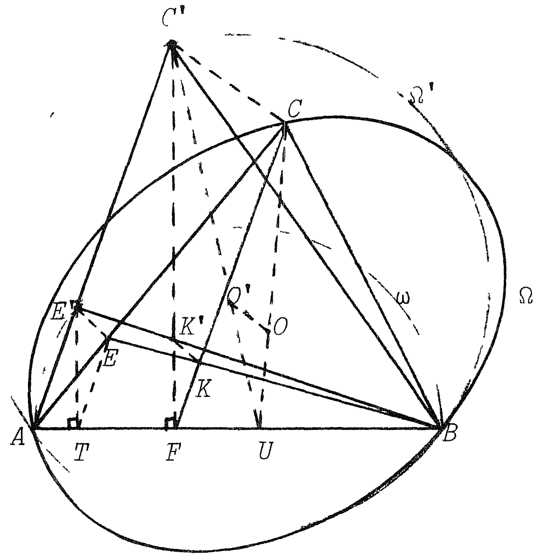
Thus

$$m_1 : m_2 : m_3 = q+r \ : \ r+p \ : \ p+q,$$

which determines the centre Q.

II. *Solution by the proposer*

Extend CK to meet AB at F, and BK to meet AC at E, and let T be on AB so that $ET \| CF$. Draw the semicircle ω with diameter AB and let the perpendicular to AB at T meet ω at E'.



Now let φ be the homological affinity with axis AB such that $\varphi(E) = E'$. Then $C' = \varphi(C)$ is found by intersecting AE' with the line through C parallel to EE'. Naturally $BE' \perp AC'$, and (since the triangles FCC' and TEE' are homothetic) $C'F \perp AB$. Since K is the intersection of CF and BE, $K' = \varphi(K)$ will be the intersection of $C'F$ and BE', and so K' is the orthocentre of ABC'. Thus the points $P' = \varphi(P)$ will have the property that the feet of the altitudes from P' to the sides of ABC' are collinear, which by Simson's theorem implies that the locus of P' is the circumcircle Ω' of ABC'. Therefore the required locus of P is $\varphi^{-1}(\Omega') = \Omega$, which is an ellipse through A, B, C. The centre of Ω is $O = \varphi^{-1}(O')$, where O' is the circumcentre of ABC'. Letting $C'O'$ meet AB at U, O will be the intersection of CU with the line through O' parallel to EE'.

* * *

1074. [1985: 249] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with circumcenter O. Prove that

(a) there are two points P in the plane of the triangle such that

$$PA^2 : PB^2 : PC^2 = \sec^2 A : \sec^2 B : \sec^2 C \ ;$$

(b) these two points and O are collinear;

(c) these two points are inverses with respect to the circumcircle of the triangle.

Solution by Robert Lyness, Southwold, England.

Let

$$k_1 = \left| \frac{\sec B}{\sec C} \right| , \quad k_2 = \left| \frac{\sec C}{\sec A} \right| , \quad k_3 = \left| \frac{\sec A}{\sec B} \right| ;$$

we wish to find points P such that

$$\frac{PB}{PC} = k_1 , \quad \frac{PC}{PA} = k_2 , \quad \frac{PA}{PB} = k_3 .$$

The set of points $\{Z: \frac{ZB}{ZC} = k_1\}$ is the Apollonius circle S_1 whose diameter is X_1Y_1, where

$$\overrightarrow{BX}_1 = k_1\overrightarrow{X_1C} , \quad \overrightarrow{BY}_1 = -k_1\overrightarrow{Y_1C}. \tag{1}$$

From (1) it follows that B, C divide X_1, Y_1 harmonically, that B, C are inverse points with regard to S_1, and that any circle through B and C cuts S_1 orthogonally.

Similarly, any circle through C and A cuts the circle

$$S_2 = \{Z: \frac{ZC}{ZA} = k_2\}$$

orthogonally.

If S_1 and S_2 intersect, say in (real) points P, Q, then

$$\frac{PA}{PB} = \frac{QA}{QB} = \frac{1}{k_1k_2} = k_3 ,$$

so P and Q lie on the circle

$$S_3 = \{Z: \frac{ZA}{ZB} = k_3\},$$

which cuts any circle through A and B orthogonally. Thus P and Q satisfy the condition of part (a).

PQ is the common chord of S_1, S_2, S_3 which is part of their radical axis. The circumcircle, radius R, of ΔABC cuts all three circles orthogonally so its centre O is their radical centre. Hence O, P, Q are collinear. Finally, the orthogonality of the circles makes $OP\cdot OQ = R^2$, and P, Q are thus inverse points with regard to the circumcircle.

More generally, PA: PB: PC = QA: QB: QC if and only if P and Q are inverse points with regard to the circumcircle of ΔABC.

Also solved by JORDI DOU, Barcelona, Spain; and the proposer.

Lyness and Dou both observed that the points P asked for in the problem need not exist. The proposer also pointed out the generalization given above by Lyness.

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<u>1075</u>. [1985: 249] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with circumcenter O and incenter I, and let DEF be the pedal triangle of an interior point M of triangle ABC (with D on BC, etc.). Prove that

$$OM \geq OI \Leftrightarrow r' \leq \frac{r}{2} ,$$

where r and r' are the inradii of triangles ABC and DEF, respectively.

Comments by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

We show that

$$OM \geq OI \Longrightarrow r' \leq \frac{r}{2}\ ;$$

however, the reverse implication will not hold.

We use the standard indicial notation of $A_1A_2A_3$ for the triangle; a_1, a_2, a_3 the respective sides; r_1, r_2, r_3 the distances from M to the respective sides; and R_1, R_2, R_3 the distances from M to the respective vertices.

We first determine an expression for $(OM)^2$. Using barycentric coordinates with respect to ABC,

$$\overrightarrow{OM} = \sum \frac{a_1r_1\overrightarrow{OA_1}}{2F}$$

where F is the area of the given triangle and the sums here and subsequently are cyclic over the indices 1, 2, 3. Then, using $\overrightarrow{OA_1}^2 = R^2$, $2\overrightarrow{A_2}\cdot\overrightarrow{A_3} = 2R^2 - a_1^2$, etc. and $2F = \Sigma\ a_1r_1$, $a_1a_2a_3 = 4FR$,

$$\begin{aligned}(OM)^2 = \overrightarrow{OM}^2 &= \frac{R^2\ \Sigma\ a_1^2r_1^2 + \Sigma\ a_2a_3r_2r_3(2R^2 - a_1^2)}{4F^2}\\ &= \frac{R^2(\Sigma\ a_1r_1)^2}{4F^2} - \frac{4FR}{4F^2}\ \Sigma\ a_1r_2r_3\\ &= R^2 - \frac{R}{F}\ \Sigma\ a_1r_2r_3.\end{aligned}$$

Also, it is known that $(OI)^2 = R^2 - 2Rr$. Thus

$$(OM)^2 \geq (OI)^2 \Longleftrightarrow r \geq \frac{\Sigma\ a_1r_2r_3}{2F}\ . \tag{1}$$

We now find an expression for r'. Since

$$2[DEF] = \Sigma\ r_2r_3\sin A_1 = \frac{\Sigma\ a_1r_2r_3}{2R}$$

and

$$EF = R_1\sin A_1 = \frac{a_1R_1}{2R}\ , \quad \text{etc.,}$$

we get

$$r' = \frac{[DEF]}{s} = \frac{\Sigma\ a_1r_2r_3}{\Sigma\ a_1R_1}\ ,$$

where s is the semiperimeter of DEF. Thus

$$r' \leq \frac{r}{2} \Longleftrightarrow r \geq \frac{2\ \Sigma\ a_1r_2r_3}{\Sigma\ a_1R_1}\ . \tag{2}$$

Since it is known that

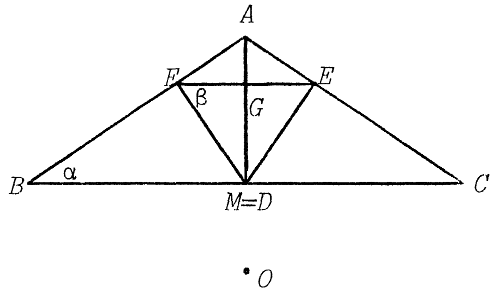
$$\Sigma\, a_i R_i \geq 4F = 2\, \Sigma\, a_i r_i \tag{3}$$

(12.19 of O. Bottema et al, *Geometric Inequalities*), from (1) and (2) we can conclude that

$$OM \geq OI \implies r' \leq \frac{r}{2}\,.$$

However, the converse would require the reverse inequality in (3).

[*Editor's comment*: In fact a specific counterexample is as follows. Let ABC be isosceles with A obtuse, and choose M to be the midpoint of BC. Then $D = M$, and since O is outside ABC, $OM < OI$. From



$$r = BD \tan\frac{\alpha}{2}\,,$$

$$r' = FG \tan\frac{\beta}{2} = FG \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{FG\left(1 - \tan\frac{\alpha}{2}\right)}{1 + \tan\frac{\alpha}{2}}\,,$$

and

$$FG = AF \cos\alpha = AD \sin\alpha\cos\alpha = BD \tan\alpha\sin\alpha\cos\alpha = BD\sin^2\alpha,$$

we get

$$\frac{r}{r'} = \frac{\tan\frac{\alpha}{2}\left(1 + \tan\frac{\alpha}{2}\right)}{\sin^2\alpha\left(1 - \tan\frac{\alpha}{2}\right)}$$

and since

$$\lim_{\alpha\to 0}\frac{r}{r'} = \lim_{\alpha\to 0}\frac{1}{2\alpha} = \infty\,,$$

$\frac{r}{r'}$ will be greater than 2 for sufficiently small α ($\alpha = 23^\circ$ is small enough).]

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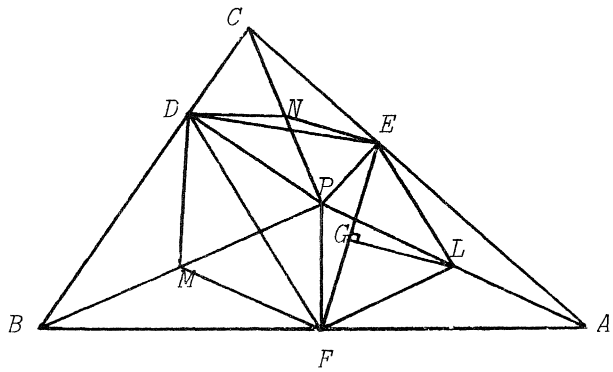
<u>1076</u>. [1985: 249] *Proposed by M.S. Klamkin, University of Alberta.*

Let x, y, z denote the distances from an interior point P of a given triangle ABC to the respective vertices A, B, C; and let K be the area of the pedal triangle of P with respect to ABC. Show that

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C + 8K$$

is a constant (independent of P).

I. *Solution by George Tsintsifas, Thessaloniki, Greece.*



We denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB. The center L of the circle $AEPF$ is the midpoint of the segment AP.

We can write

$$x^2\sin A = (2x\sin A)(x\cos A)$$
$$= 2\overline{EF}\cdot 2\overline{LG},$$

where G is the orthogonal projection of L on EF, and we take $\overline{LG}$ to be negative if A is obtuse. From this we easily obtain

$$x^2\sin A = 8[FKE],$$

where as usual $[Q]$ denotes the area of the figure Q, and we take $[FKE]$ to be negative if A is obtuse. Letting M, N be the midpoints of the segments BP, CP respectively, we similarly obtain

$$y^2\sin 2B = 8[FMD]$$

and

$$z^2\sin 2C = 8[DNE]$$

(we may assume B and C are acute). Hence

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C + 8K = 8[DNELFM].$$

But it is elementary to see that

$$[ABC] = 2[DNELFM],$$

that is,

$$x^2\sin 2A + y^2\sin 2B + z^2\sin 2C + 8K = 4[ABC]. \qquad (1)$$

II. *Remarks by the proposer.*

(i) For an equilateral triangle of side 2, the above identity (1) becomes

$$x^2 + y^2 + z^2 + 4(qr + rp + pq) = 8$$

where p, q, r are the respective distances from P to the sides a, b, c.

(ii) Since

$$K = \frac{R^2 - \overline{OP}^2}{4R^2} F$$

where $F = [ABC]$ and O is the circumcenter, (1) is also equivalent to

$$\Sigma\ x^2 \sin 2A = 2F + \frac{2(\overline{OP})^2 F}{R^2},$$

(where the sums here and subsequently are cyclic), and so

$$\Sigma\ x^2 \sin 2A \geq 2F$$

or

$$\Sigma\ x^2 a^2 (b^2 + c^2 - a^2) \geq 16R^2F^2, \tag{2}$$

with equality if and only if $P = O$. Since xa, yb, zc determine a triangle (possibly degenerate), it follows by the Neuberg-Pedoe inequality [1] that

$$\Sigma\ x^2 a^2 (b^2 + c^2 - a^2) \geq 16FF_0, \tag{3}$$

where F_0 is the area of a triangle of sides xa, yb, zc. By the result in *Crux* 872 [1984: 334] that $R^2F \geq F_0$, (2) is stronger than (3).

Reference:

[1] O. Bottema, M.S. Klamkin, Joint triangle inequalities, *Simon Stevin* 48 (1974-75) 3-8.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; and the proposer.

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<u>1078</u>. [1985: 250] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Prove that

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot \frac{1}{k} = \sum_{k=1}^{n} \frac{2^k - 1}{k}.$$

I. *Solution by Beno Arbel, Tel-Aviv University, Israel.*

Let

$$S_n = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \cdot \frac{1}{k}.$$

By substitution of the known identity

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

we obtain

$$S_n = \sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot \frac{1}{k} + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot \frac{1}{k+1}$$

$$= S_{n-1} + \frac{1}{n} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix},$$

since

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} \cdot \frac{1}{k+1} = \begin{bmatrix} n \\ k+1 \end{bmatrix} \cdot \frac{1}{n}.$$

Then

$$S_n = S_{n-1} + \frac{1}{n}(2^n - 1),$$

a recursion formula which gives the desired result.

By the same technique, one can prove

$$\sum_{k=1}^{n} (-1)^{k+1} \begin{bmatrix} n \\ k \end{bmatrix} \cdot \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k}$$

and

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{m}{m+k} = \begin{bmatrix} m+n \\ n \end{bmatrix}^{-1}.$$

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The posed problem is a special case ($a = 0, b = 1$) of the following more general identity. Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$, $a < b$. Then

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{b^k - a^k}{k} = \sum_{k=1}^{n} \frac{(b+1)^k - (a+1)^k}{k}.$$

Indeed,

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{b^k - a^k}{k} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \int_a^b x^{k-1} dx = \int_a^b \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \cdot \frac{1}{x}\, dx.$$

$$= \int_a^b \frac{(x+1)^n - 1}{x}\, dx = \int_{a+1}^{b+1} \frac{t^n - 1}{t - 1}\, dt$$

$$= \sum_{k=1}^{n} \int_{a+1}^{b+1} t^{k-1} dt = \sum_{k=1}^{n} \frac{(b+1)^k - (a+1)^k}{k}.$$

III. *Comment by Karl Dilcher, Dalhousie University, Halifax, Nova Scotia.*

A considerably more general form of this identity is listed as 1.11 in H.W. Gould, *Combinatorial Identities*, Morgantown, W.Va., 1972; namely

$$\sum_{k=0}^{n-1} \begin{bmatrix} z \\ k \end{bmatrix} \frac{x^{n-k}}{n-k} = \sum_{k=1}^{n} \begin{bmatrix} z-k \\ n-k \end{bmatrix} \frac{(x+1)^k - 1}{k}.$$

With $z = n$ and $x = 1$ we get the identity in question.

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; M. BROZINSKY and J. GARFUNKEL, Queensborough Community College, Bayside, New York; LES DAVISON, Laurentian University, Sudbury, Ontario; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; PETER ROSS, University of Santa Clara, Santa Clara, California; M. SELBY, University of Windsor, Windsor, Ontario; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

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1079. [1985: 250] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let

$$g(a,b,c) = \sum \frac{a}{a+2b}\cdot\frac{b-4c}{b+2c},$$

where the sum is cyclic over the sides a, b, c of a triangle.

(a) Prove that $\frac{-5}{3} < g(a,b,c) \le -1$.

(b)* Find the greatest lower bound of $g(a,b,c)$.

I. *Solution to part (a) by the proposer.*

The upper bound $g(a,b,c) \le -1$ is equivalent to

$$\frac{a}{a+2b}\cdot\frac{b-4c}{b+2c} + \frac{b}{b+2c}\cdot\frac{c-4a}{c+2a} + \frac{c}{c+2a}\cdot\frac{a-4b}{a+2b} \le -1 ,$$

or

$$a(b-4c)(c+2a) + b(c-4a)(a+2b) + c(a-4b)(b+2c) \le -(a+2b)(b+2c)(c+2a),$$

or

$$12abc \le 4ab^2 + 4bc^2 + 4ca^2,$$

or

$$abc \le \frac{ab^2 + bc^2 + ca^2}{3},$$

which holds by the A.M.-G.M. inequality.

Similarly, the lower bound is equivalent to

$$3a(b-4c)(c+2a) + 3b(c-4a)(a+2b) + 3c(a-4b)(b+2c) > -5(a+2b)(b+2c)(c+2a)$$

or

$$54abc > 4ab^2 - 4a^2b + 4bc^2 - 4b^2c + 4ca^2 - 4c^2a$$

or

$$\frac{27}{2} > \frac{b-a}{c} + \frac{c-b}{a} + \frac{a-c}{b} .$$

This inequality holds since, by the triangle inequality, $b - a < c$ etc.

II. *Solution to (b) by Richard I. Hess, Rancho Palos Verdes, California.*

Putting $a = b$ and c much smaller than a, g will be approximately $\frac{1}{3} - 2 + 0 = \frac{-5}{3}$. Thus the lower limit $\frac{-5}{3}$ for g can be approached arbitrarily closely.

Also note that putting $a = b = c$ yields the upper bound $g = -1$.

Also solved (upper bound of (a)) by J.T. GROENMAN, Arnhem, The Netherlands; and (part (b)) STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire.

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1080. [1986: 11] (Corrected) *Proposed by D.S. Mitrinovic, University of Belgrade, Yugoslavia.*

Determine the maximum value of

$$f(a,b,c) = \left| \frac{b-c}{b+c} + \frac{c-a}{c+a} + \frac{a-b}{a+b} \right| ,$$

where a, b, c are the side lengths of a nondegenerate triangle.

Solution by B.C. Rennie, Burnside, Australia

If $a \geq b \geq c$, then

$$\frac{b-c}{b+c} + \frac{a-b}{a+b} + \frac{c-a}{c+a} = \frac{b-c}{b+c}\cdot\frac{a-b}{a+b}\cdot\frac{a-c}{a+c} \geq 0$$

(see the comment following *Crux* 883 [1985: 30]). Thus putting

$$x = b - c \geq 0, \quad y = a - b \geq 0, \quad z = b + c - a \geq 0,$$

we have that

$$f(a,b,c) = \frac{xy(x+y)}{(x+2y+2z)(x+3y+2z)(2x+3y+2z)} .$$

The supremum for nonnegative z is at $z = 0$. What can we say about the supremum for nonnegative x and y? The function is homogeneous and continuous in x and y and attains its maximum for some positive x and y. Therefore we may set $y = 1$ and maximize

$$F(x) = \frac{x(x + 1)}{(x + 2)(x + 3)(2x + 3)}$$

for positive x. By logarithmic differentiation, we wish to solve

$$\frac{1}{x} + \frac{1}{x + 1} = \frac{1}{x + 2} + \frac{1}{x + 3} + \frac{1}{2x + 3},$$

which simplifies to

$$\frac{2x + 1}{x(x + 1)} = \frac{6x^2 + 26x + 27}{(x + 2)(x + 3)(2x + 3)}$$

and finally to

$$x^4 + 2x^3 - 7x^2 - 18x - 9 = 0.$$

To solve this, put $t = x + 1$, giving

$$t^4 - 2t^3 - 7t^2 - 2t + 1 = 0,$$

then put $v = t + \frac{1}{t}$, giving

$$v^2 - 2v - 9 = 0.$$

The required root is $v = 1 + \sqrt{10}$, so $t = \frac{s}{2}$ where

$$s = (1 + \sqrt{2})(1 + \sqrt{5}).$$

The required bound is then

$$F\left[\frac{s - 2}{2}\right] = \frac{s(s - 2)}{(s + 1)(s + 2)(s + 4)}.$$

Simplification is made easier by observing that

$$\frac{(s + 1)(s + 4)}{s} = 2\left[\frac{s}{2} + \frac{2}{s} + \frac{5}{2}\right] = 2(1 + \sqrt{10} + \frac{5}{2})$$
$$= 7 + 2\sqrt{10} = (\sqrt{2} + \sqrt{5})^2,$$

and we find the value

$$\frac{\sqrt{10} - 3}{\sqrt{2} + \sqrt{5}} \approx 0.044456.$$

Note that this is the least upper bound for $f(a,b,c)$ over all triangles but is not attained by any nondegenerate triangle.

Also solved by ERNEST J. ECKERT, Aalborg University, Aalborg, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; and KEE-WAI LAU, Hong Kong.

The original uncorrected problem [1985: 250] was seen to have no solution (i.e. f is unbounded) by O. Bottema, J.T. Groenman, Friend H. Kierstead Jr., Leroy F. Meyers, and Edward T.H. Wang. Meyers also pointed out that if a, b, c are arbitrary positive real numbers the least upper bound of f is 1, approached when one of the variables is 0 and another approaches 0.

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