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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

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Mathematical Spectrum Awards for Volume 25

Prizes have been awarded to the following student readers for contributions published in Volume 25:

Gregory Economides;

Mark French;

Dylan Gow;

Ian Tame.

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

A Network of Minimum Length

JOHN MACNEILL, *Census Computer Services*

The author is a former teacher now writing programs for Census Computer Services Ltd, Wolverhampton.

Trying to accustom pupils to investigative work, I once set the following task to a mathematics class which was the first in that school to obtain substantial escape from the likes of 'Do Exercise 7, numbers 1 to 6'.

> There are two houses on the same side of a long straight gas main. New gas pipes are to be laid to connect the houses to the gas main. The connection of the new piping to the main is to be at only one place. The length of new piping laid is to be as little as possible. Investigate.

Pressure of work meant that I had not given much time to selecting this task, and I had misgivings about it. What if some of the class had learnt in physics about how light always take the shortest possible path, for instance when travelling from a point A to a mirror to a point B? Would that reduce or enhance the suitability of the task? Or what if a previous mathematics teacher, when teaching about the geometric transformation called reflection in a line, had discussed a problem like the following?

> A girl is to run from a tree at $A(0,36)$, touch a wall (which is along the x-axis) and then run to a tree at $B(80,24)$. Where should she touch the wall so that her route is as short as possible? What is the length of this shortest route?

The elegant, calculus-free solution to the problem involves the reflection in the x-axis of one of the trees. For instance, let A_1 be $(0,-36)$ as in Figure 1. Then the route

$$A \rightarrow \text{point } P \text{ on the wall} \rightarrow B$$

obviously has the same length as the route

$$A_1 \rightarrow \text{point } P \text{ on the wall} \rightarrow B$$

since $A_1P = AP$ and this is so for every choice of the point P on the wall. We disregard the possibility of difficulty with climbing the wall. The length of the route A_1PB is minimised by simply choosing P so that A_1PB is a straight line. Naturally this choice of P also minimises the route APB. So the girl should touch the wall where A_1B cuts the x-axis, that is, at $(48,0)$ and the length of the shortest route is the distance A_1B, that is, 100 units.

I had thought at the time that the houses-and-gas-main investigation was broadly equivalent to the trees-and-wall problem, as the investigation had been devised more or less on the spur of the moment and based on that problem. At least they will gain valuable practice in the important early stages of an investigation, I was telling myself. At least they will benefit by discussing their ideas with me and with each other. There might be some useful practice in Pythagorean calculations and graph drawing. We would be able to discuss the assumptions that we made.

'Is it allowed to have the new piping going through one house to the other house?', a pupil asked.

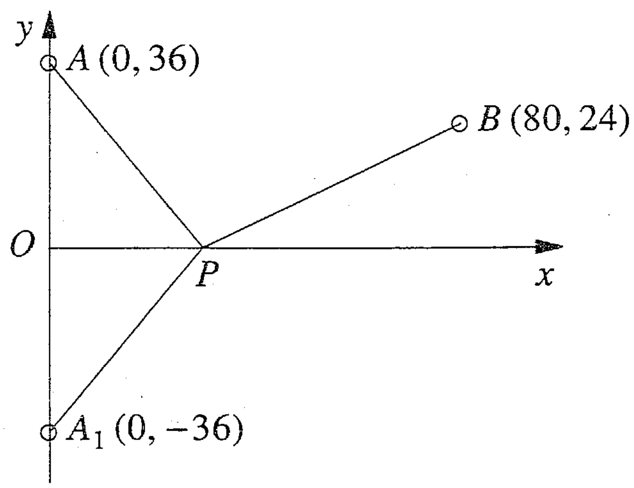

Figure 1

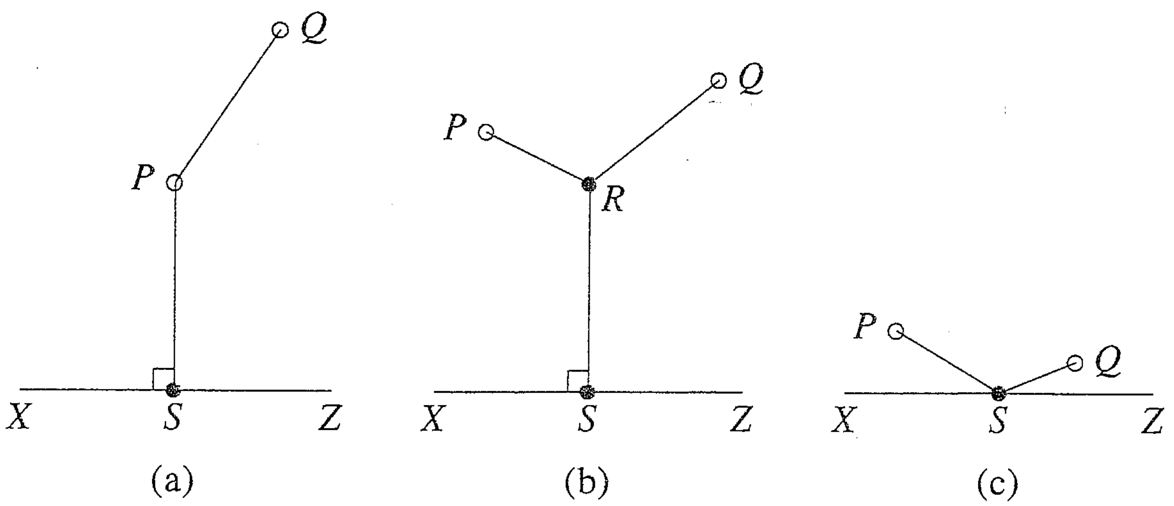

Figure 2

'Yes, if it helps.'

'Can you have the new piping in a kind of Y shape?', asked another pupil.

'Yes, if it helps.'

In investigational work even the teacher can learn something. I was being forced to realise that there are three different pipe-laying strategies, any one of which may give the minimum length of new piping: it depends on the relative positions of the houses and the gas main.

In figure 2, points P and Q represent the houses, the line XZ represents the gas main and the points R and S represent pipe junctions.

The strategy of figure 2(a)—call this *plan Γ*—may minimise the length of new piping, for instance when PQ is nearly perpendicular to XZ.

The strategy of figure 2(b)—call this *plan Y*—may minimise the length of new piping, for instance when P and Q are near each other but distant from XZ, with PQ nearly parallel to XZ.

The strategy of figure 2(c)—call this *plan V*—may minimise the length of new piping, for instance when P and Q are far apart but each is near XZ.

In a particular case, how do we know which of the plans Γ, Y or V to choose? If plan Y is the correct one, how do we locate the point R? It was unnerving to be in charge of the class but not to know all the answers; some mathematical educationists assert that the ideal investigation is one where neither teacher nor pupils know in advance what can be discovered. However, a weekend intervened and I was able to return fortified with answers and ideas which I then had to struggle to keep to myself as the class continued with the investigation.

Suppose, provisionally, that the new piping should be laid in a Y shape. Imagine making a physical model in which each house P and Q is represented by a small fixed smooth peg, the gas main XZ is represented by a long fixed smooth thin straight rod, the pipe junction R in the Y

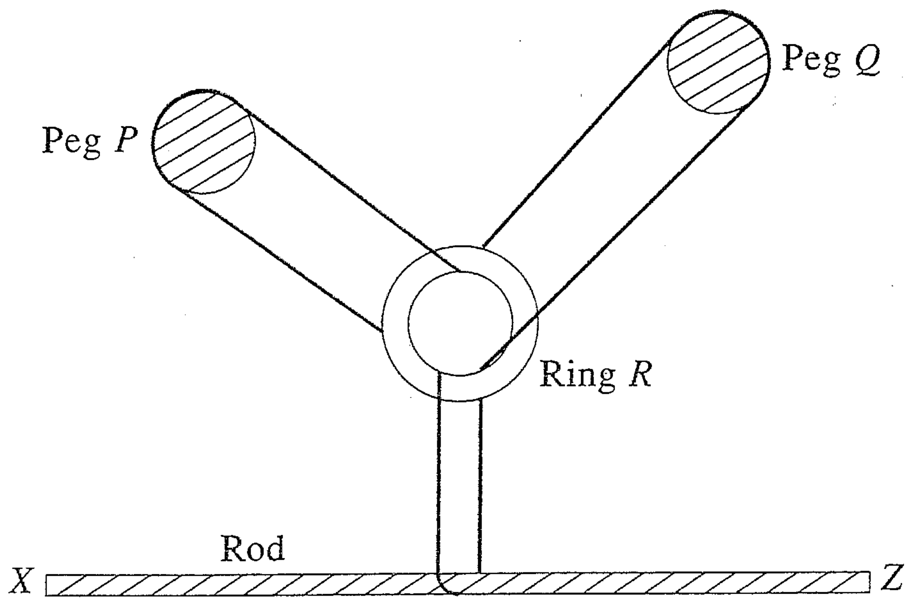

Figure 3

shape is represented by a small smooth light movable ring and the Y shape is made of a loop of light elastic string which goes round each peg, round the rod and through the ring as in figure 3. (Incidentally, notice how theoretical mechanics text has a tendency to threaten the world record for the number of consecutive adjectives.)

Assuming that the loop of elastic string stays taut for every position of the ring, the equilibrium position of the ring gives the minimum length of the loop of elastic string, which corresponds to the minimum length of new piping. This equilibrium position is the key.

When the equilibrium position is at a peg, plan Γ is needed.

When the equilibrium position is on the rod, plan V is needed.

When the equilibrium position is neither at a peg, nor on the rod, plan Y is needed.

Given that plan Γ is needed, the details are obvious.

Given that plan V is needed, the details may be found as for the trees-and-wall problem.

Given that plan Y is needed, it is easy to locate the equilibrium position. Since the tension in the elastic string is the same everywhere in the loop, for equilibrium the three angles at the junction of the Y are equal and so each is 120°. For equilibrium the elastic string obviously meets the rod at right angles. These angle sizes allow us to locate the equilibrium position.

It remains to consider how to decide whether the equilibrium position is at a peg (plan Γ), on the rod (plan V) or neither at a peg or on the rod (plan Y).

It is useful to make a comparison with the problem of locating, for a given triangle ABC, the point S which minimises the value of

$AS+BS+CS$ (see the reference). If one of the angles of the triangle is 120° or more, then S is at that angle; otherwise S is the point in the triangle such that each of the angles ASB, BSC and CSA is 120° (plan Y for the triangle). We may think of plan Y as the basic solution for the triangle, a solution which breaks down when there is not enough 'room' in the triangle for the Y shape.

For the houses and the gas main, we may also think of plan Y as the basic solution, a solution which may break down in some cases. So when looking for the equilibrium position of the ring it is natural to start by trying to find the junction of the Y shape. We know that two arms of the Y shape are at 30° to XZ. Through P and Q draw lines at 30° to XZ as shown in figure 4, the lines intersecting at the point T, which is a candidate for being the junction of the Y shape and the equilibrium position of the ring.

If the diagram turns out to be like figure 4(a), then T fulfils our hopes; we may complete the Y shape by drawing the perpendicular from T to XZ.

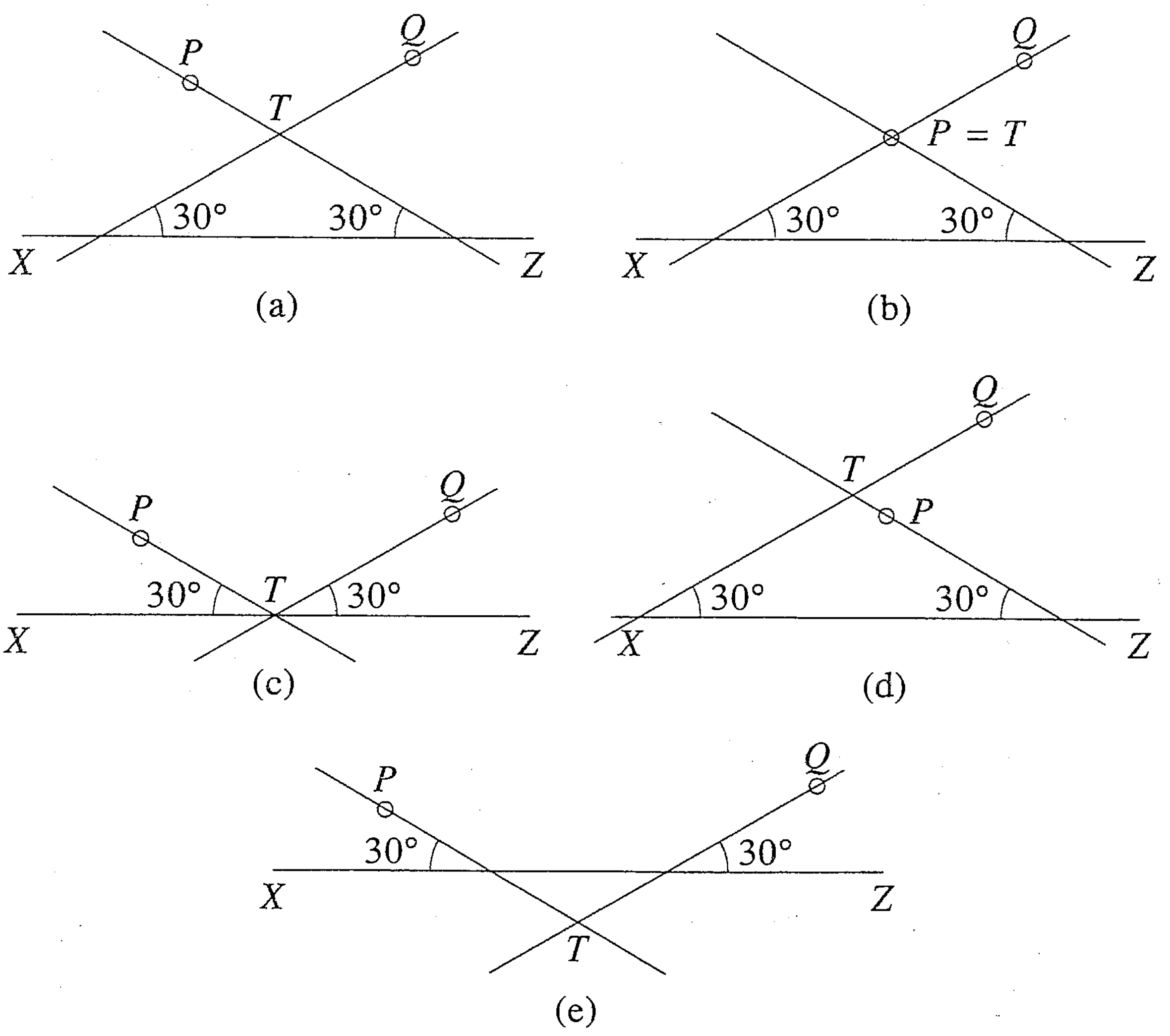

Figure 4

Figure 4 (b) may be regarded as giving a Y shape in which *PT* is of zero length. Figure 4 (b) represents the 'boundary' between plan Y and plan Γ.

Figure 4 (c) may be regarded as giving a Y shape in which the perpendicular to XZ is of zero length. Figure 4 (c) represents the boundary between plan Y and plan V.

Now let peg *Q* be movable. If *Q* is as in figure 4 (b) and then *Q* moves directly away from *XZ*, then figure 4 (d) is obtained, but the ring equilibrium position obviously remains at *P*. So plan Γ is needed for figure 4 (d).

If *Q* is as in figure 4 (c) and the *Q* moves further from *P* in a direction parallel to *XZ*, then figure 4 (e) is obtained, but the ring equilibrium position obviously remains on the rod. So plan V is needed for figure 4 (e).

So that's it. An interested reader may wish to consider the effect of discarding the condition that the connection of the new piping to the main is to be at only one point, or of discarding our implicit assumption that *P*, *Q*, *X* and *Z* are coplanar.

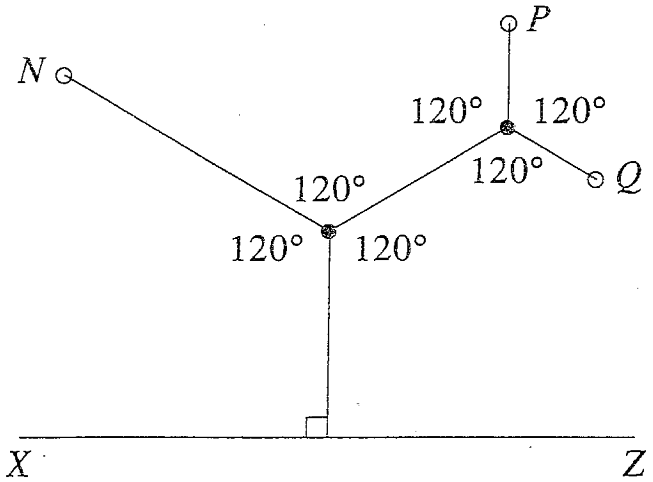

Figure 5

The ring-and-elastic-string approach is an alternative to the equal-weights-and-string approach in the reference. An advantage of the ring-and-elastic-string approach is that situations involving the possibility of more than one junction may be dealt with by having more than one ring, as is shown in figure 5, which gives an example of connecting three houses *N*, *P* and *Q* to the main *XZ*.

A physical ring-and-elastic-string model could actually be built to demonstrate the solution to a real-life problem such as finding the minimum-length network of new roads to link a given place with two existing non-straight roads, representing each of the existing roads by a rod bent into the appropriate shape.

Perhaps in future I shall be quicker to appreciate the difference between a route of minimum length and a network of minimum length. This has been trivial mathematics (Latin *trivium* = a crossroads, a place where three roads meet).

References

1. R. H. Eddy, On a problem of Thwaites, *Mathematical Spectrum*, **25** (1992/93), 3–7.

The Smarandache Function

The Smarandache function takes its name from the exiled Romanian mathematician Florentin Smarandache. It is defined as follows:

For any non-null integers n, $S(n)$ is the smallest integer such that $S(n)!$ is divisible by n.

The importance of the notion is that it characterizes a prime number, in the following way:

Let $p > 4$, then p is prime if and only if $S(p) = p$.

Open problems involving this function have inspired mathematicians around the world to study it and its applications. (See problem 26.5 on page 56 of this issue: ED.)

In 1975 Smarandache, who was then a student at the University of Craiova, was attracted by number theory and began to publish mathematical papers in this field. In 1980 his research paper 'A function in number theory', based on a special representation of the integers, was published in *Analele Universitatii Timisoara, Serie Stiinte Matematice*, Volume 18, pages 79–88. The paper was reviewed in the *Zentralblatt für Mathematik* 471.10004 (1982) by P. Kiss, and in *Mathematical Reviews* 83c.10008 (1983) by R. Meyer.

In 1988 Smarandache escaped from the Ceausescu dictatorship, stayed almost two years in a political refugee camp in Turkey, and finally emigrated to the United States.

Contributions related to the Smarandache function have been presented to several international conferences. Some published references (in chronological order) are:

1. C. Corduneanu, Abstract on the Smarandache function, *Libertas Mathematica* **9** (1989), p. 175 (Texas State University, Arlington).

2. *Smarandache Function Journal* **1** (1990) (Number Theory Publishing Company, Phoenix). Surveyed by *The International Directory of Little Magazines and Small Presses* and *Ulrich's Directory* (USA): reviewed in *Zentralblatt für Mathematik* 745.11004 (1992).

3. Mike Mudge, The Smarandache function, *Personal Computer World* (London) July 1992, p. 420.

4. Mike Mudge, A return visit to the Florentin Smarandache function, *Personal Computer World* (London) February 1993, p. 403.

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Stacking Boxes

DAVID SHARPE, *University of Sheffield*

The author is editor of *Mathematical Spectrum*. He was presented with the problem in this article whilst spending a few days' holiday on the Isle of Wight prior to the rigours of a university term, and he spent the rest of the holiday solving it.

The pupils in Year 8 of Bishop Lovett's School, Ryde, on the Isle of Wight, were asked to investigate the number of different ways of stacking identical boxes in any number of piles next to a wall. Two ways are to be regarded as different if one cannot be obtained from the other by reflection in a vertical plane. Presumably the students were meant to draw the various possibilities for a number of boxes. The possibilities for one to five boxes are shown in figure 1.

If we denote by P_n the number of ways of stacking n boxes, we thus have

$$P_1 = 1, \qquad P_2 = 2, \qquad P_3 = 3, \qquad P_4 = 6, \qquad P_5 = 10.$$

The question is: can we find a formula for P_n?

It is clear that the problem has little to do with boxes. The aim is to determine the number of ways of expressing the natural number n as a sum of natural numbers, where two expressions are to be regarded as the same if the order of the terms in one sum is the reversal of the order of the terms in the other. Thus, for example, the expressions $3 = 2+1$ and $3 = 1+2$ are to be regarded as the same and are only counted once. However, the expressions $4 = 2+1+1$ and $4 = 1+2+1$ are to be

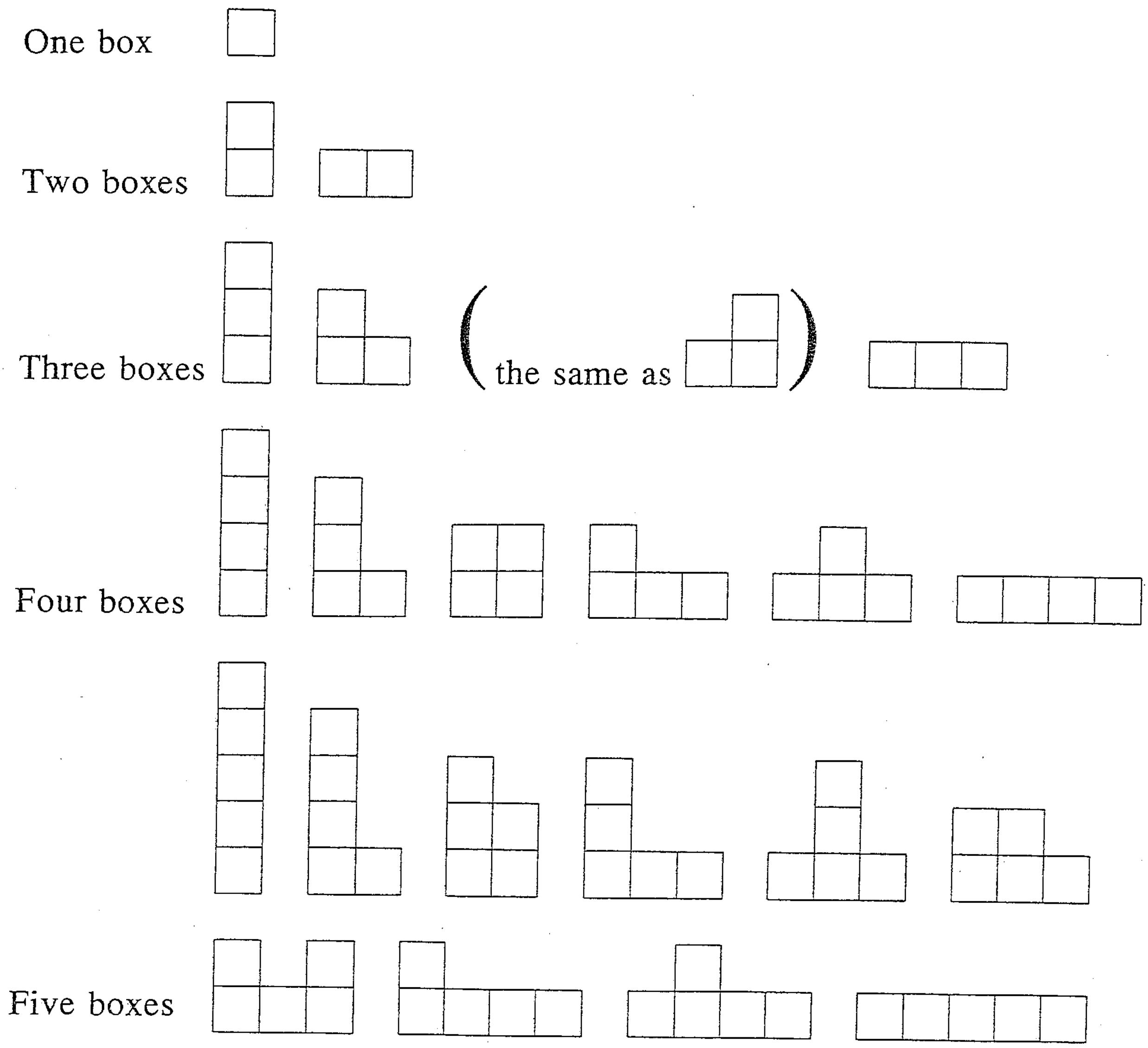

Figure 1

regarded as different. This may seem odd in a number-theoretic context, but that is where the boxes come in.

It was not clear from calculating $P_1, \ldots, P_5$, or even P_6, which was found to be 20 (I was too prone to errors to be confident of having counted the cases for $n = 7$ correctly!), what the result for P_n might be.

The situation is much simpler if we do not identify mirror images, so that, for example, $3 = 2+1$ and $3 = 1+2$ are counted as two separate expressions. If we denote by Q_n the number of expressions of n as a sum of natural numbers, where the order of the terms in the sum is taken into account, then

$$Q_1 = 1, \qquad Q_2 = 2, \qquad Q_3 = 4, \qquad Q_4 = 8,$$

and it is an obvious conjecture that $Q_n = 2^{n-1}$. To prove this, we note that each ordered partition of n can be uniquely represented by a sequence of n

dots and markers in the spaces between the dots. For example:

$$\begin{array}{lll} n=5 & 5=3+1+1 & \bullet\ \bullet\ \bullet \mid \bullet \mid \bullet \\ & 5=1+2+1+1 & \bullet \mid \bullet\ \bullet \mid \bullet \mid \bullet \end{array}$$

There are two possibilities for each of the $n-1$ spaces between the dots, either to have a marker or not. Hence there are 2^{n-1} ordered partitions of n, as conjectured. We remark in passing that the number of ordered partitions of n into k parts is the binomial coefficient

$$\binom{n-1}{k-1}.$$

The next question is: can we link P_n and Q_n? There are P_n ways of expressing n according to the rules and we can reverse the order of the terms in these expressions. If we take all these expressions together, we shall have $2P_n$ in total. These will include all the expressions listed by Q_n except that the symmetrical expressions will be included twice. Thus

$$2P_n = Q_n + R_n,$$

where R_n is the number of symmetrical expressions of n as a sum of natural numbers.

An example may help. We know that $P_4 = 6$:

$$4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+2+1 = 1+1+1+1.$$

When these are reversed in order we obtain

$$4 = 4 = 1+3 = 2+2 = 1+1+2 = 1+2+1 = 1+1+1+1.$$

If we now include all these together, we obtain the non-symmetrical expressions

$$4 = 3+1 = 2+1+1 = 1+3 = 1+1+2$$

once and the symmetrical expressions

$$4 = 4 = 2+2 = 1+2+1 = 1+1+1+1$$

twice. Here $Q_4 = 8$ and $R_4 = 4$, and $2P_4 = 12 = Q_4+R_4$.

If we can calculate R_n, we can thus determine P_n. Each symmetrical ordered partition of n may be uniquely represented by $\frac{1}{2}n+1$ dots and markers between them when n is even and by $\frac{1}{2}(n-1)+1$ dots and markers when n is odd. For example

$$\begin{array}{lllll} n=6 & \bullet \mid \bullet\ \bullet\ \bullet & \leftrightarrow & \bullet \mid \bullet\ \bullet\ \bullet\ \bullet \mid \bullet & 6=1+4+1 \\ & \bullet \mid \bullet\ \bullet \mid \bullet & \leftrightarrow & \bullet \mid \bullet\ \bullet \mid \bullet\ \bullet \mid \bullet & 6=1+2+2+1 \\ & \bullet\ \bullet\ \bullet \mid \bullet & \leftrightarrow & \bullet\ \bullet\ \bullet \mid \bullet\ \bullet\ \bullet & 6=3+3 \end{array}$$

(the pattern has been made symmetrical by reflecting in the space between the last two dots);

$$\begin{array}{llcll}
n = 7 & \bullet \mid \bullet\ \bullet\ \bullet & \leftrightarrow & \bullet \mid \bullet\ \bullet\ \bullet\ \bullet\ \bullet \mid \bullet & 7 = 1+5+1 \\
 & \bullet \mid \bullet\ \bullet \mid \bullet & \leftrightarrow & \bullet \mid \bullet\ \bullet \mid \bullet \mid \bullet\ \bullet \mid \bullet & 7 = 1+2+1+2+1 \\
 & \bullet\ \bullet\ \bullet \mid \bullet & \leftrightarrow & \bullet\ \bullet\ \bullet \mid \bullet \mid \bullet\ \bullet\ \bullet & 7 = 3+1+3
\end{array}$$

(the pattern has been made symmetrical by reflecting in the last dot). Hence the number of symmetrical partitions of n is

$$R_n = \begin{cases} 2^{\frac{1}{2}n} & \text{(if } n \text{ is even)}, \\ 2^{\frac{1}{2}(n-1)} & \text{(if } n \text{ is odd)}. \end{cases}$$

We know that

$$P_n = \tfrac{1}{2}(Q_n + R_n).$$

We have thus derived the following formula for P_n:

$$P_n = \begin{cases} 2^{n-2} + 2^{\frac{1}{2}(n-2)} & \text{(when } n \text{ is odd)}, \\ 2^{n-2} + 2^{\frac{1}{2}(n-3)} & \text{(when } n \text{ is even)}. \end{cases}$$

Thus $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, $P_4 = 6$, $P_5 = 10$, $P_6 = 20$, $P_7 = 36$, $P_8 = 72$, etc.

There is not a simple formula for the number of *unordered* partitions of n (where, for example, $4 = 2+1+1 = 1+2+1 = 1+1+2$ are all regarded as the same). There is, however, one result that we can relate to stacking boxes. As an example, suppose that we stack 15 boxes in four piles as in figure 2. If we consider these as horizontal rather than vertical piles, we have stacked 15 boxes in piles, the height of the largest pile being 4. Numerically, we have passed from the partition

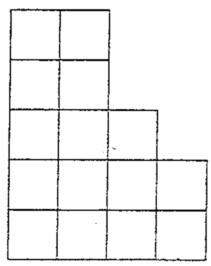

Figure 2

$$15 = 5+5+3+2$$

of 15 into four parts to

$$15 = 4+4+3+2+2$$

in which the largest number is 4. And this process can clearly be reversed.

It should be clear from this example that the number of ways of stacking n boxes in k piles, where the order of the piles is not taken into account (so that we can arrange them in decreasing order as in the example), is the same as the number of ways of stacking the boxes where the highest stack contains precisely k boxes.

Law, Food and Geometry

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Dr Katan trained as a physical chemist and then qualified as a chemical engineer. He worked for Shell for many years before leaving to set up his own consulting company. He is a leading international expert on food packaging and migration. The authors began collaborating at Brunel University in 1986, and this survived Professor Chatwin's move (or migration?) to Sheffield in 1991.

1. Migration

Plastics used in food packaging contain substances that can migrate from the package into the contents. Examples include residual monomers, plasticisers and colouring materials. Unfortunately some of these substances may have undesirable effects if too much migration occurs. Such effects range from unpleasant (but harmless) taste or smell to toxicity.

A famous example—fortunately no longer applicable—arose when airlines served spirits in miniature bottles made of PVC (polyvinyl chloride). In the early 1970s passengers started to complain that their drinks had strange tastes. The reason was found to be that residual VCM (vinyl chloride monomer) had migrated from the bottle into the drink. At about the same time, VCM was identified as a human carcinogen. The resulting uproar caused regulatory authorities to insist on substantial reductions in VCM levels in PVC used for making packages for food and drink.

But the use of plastic packaging has increased vastly since 1970, and is still increasing. The food industry now uses many different plastics, and there are many different substances (*migrants*) that migrate from them into many different foods. (Note that throughout this article the word *food* also refers to *drink.*) The amount and rate of migration depend on the properties of all three components—plastic, migrant, food—and on external factors such as temperature.

Naturally, governments are concerned to ensure that packaged food is as safe as possible. Therefore most of them, and also the EC (European Community), have regulations that specify permitted migrants and are intended to keep their quantities in packaged food below prescribed safety levels. For each permitted migrant, the safety level is determined from toxicological testing and from an assessment of the annual consumption per person of food that could contain it—*specific migration.* In addition, a level is set which is intended to limit the total quantity of all migrants present in any single item of food—*global*, or *overall, migration*.

This article is about the way in which these safety levels are given numerically. To mathematicians (and many other scientists) it is perhaps natural to express them as *concentrations* in the food, described, for example, in units of kg/m^3 (kilograms of migrant per cubic metre of food), or as a mass ratio such as mg/kg (milligrams of migrant per kilogram of food). If used exclusively, this practice would be entirely satisfactory.

But some sets of regulations, including those considered later, also express migration limits in another way, namely as the mass of migrant crossing each unit of area of the interface between the food and plastic (*migrant flux*). The units used are often mg/dm^2 (milligrams of food per square decimetre of the interface). The reason for using this measure appears to be historical; the results of laboratory tests have often been expressed as fluxes rather than concentrations.

Regulations which quote fluxes appear to assume (sometimes implicitly) that a given value of migrant flux can be equated to a unique concentration. Unfortunately this is a myth with no scientific basis; the concentration for a prescribed flux depends on the *geometry* (size and shape) of the package, as simple calculations show.

2. Extracts from EC regulations

The EC has adopted many regulations on migration. One of these is Directive 90/128/EEC (see *Official Journal of the EEC*, No L 75 of 21 March 1990). This applies to 'plastics materials and articles ... which, in the finished product state, are intended to come into contact ... with foodstuffs ...' (Article 1).

Article 2 of the Directive deals with global migration and an extract from it is:

> Plastics materials ... shall not transfer their constituents to foodstuffs in quantities exceeding 10 milligrams per square decimetre of surface area ... (overall migration limit). However this limit shall be 60 milligrams of the constituents released per kilogram ... in the following cases:
> (a) articles which are containers ... with a capacity of not less than 500 millilitres (ml) and not more than 10 litres (l); ...

These restrictions on global migration are supplemented by limits on specific migration, and Annex I states that 'the sum of all specific migration ... shall not exceed the overall migration limit' (i.e. the limit given in the extract from Article 2 quoted above). Annex II, which is regularly updated, lists those 'monomers and other starting substances' that can be used to make food packages, and also gives specific migration limits.

Specific migration is also dealt with in Article 4, which includes:

> The specific migration limits in the list set out in Annex II are expressed in mg/kg. However, such limits are expressed in mg/dm^2 in the following cases:
> (a) articles which are containers ... with a capacity of less than 500 ml or more than 10 l;
> (b) ...
> In these cases, the limits set out in Annex II, expressed in mg/kg shall be divided by the conventional conversion factor of 6 in order to express them in mg/dm^2.

Many readers may suppose that we have miscopied the last sentence, but that is not so; others may be laughing! As we show below, it is ludicrous.

In the next two sections we give some simple calculations to expose some of the serious faults in these regulations. It will be supposed that the food has density $10^3\,kg/m^3$, the same as water, and all the packages considered here will have the form of the shell of a circular cylinder of radius r and height h—see figure 1. This is not too different from many real packages, for example bottles. The main conclusions do not depend on these assumptions, as readers can easily see for themselves.

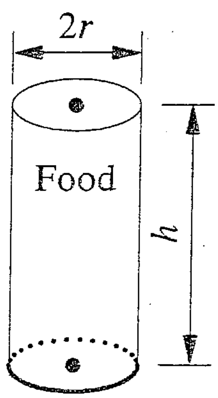


Figure 1

3. The effect of size

Suppose that the plastic contains a migrant for which the specific migration limit, given in Annex II of the Directive, is $30\,mg/kg = 3\times10^{-5}$.

In the first case, take $r = 10^{-2}\,m$ and $h = 6\times10^{-2}\,m$ so that the package capacity is $\pi r^2 h \approx 1.88\times10^{-5}\,m^3 = 18.8\,ml$. This is typical of the size in which food flavouring like vanilla essence is sold (and of the miniature PVC bottles in which airlines served spirits in the 1970s). The mass of food (of density $10^3\,kg/m^3$) is therefore about $1.88\times10^{-2}\,kg$. Since the capacity is less than 500 ml, the manufacturer is entitled by Article 4 of the Directive to 'convert' 30 mg/kg into a flux of $5\,mg/dm^2 = 5\times10^{-4}\,kg/m^2$. The internal surface area of the package is $2\pi r^2 + 2\pi rh \approx 4.40\times10^{-3}\,m^2$. Assuming that the whole area is in contact with food, the total mass of migrant in the food when migration to the permitted maximum flux has occurred is about $5\times10^{-4}\times4.40\times10^{-3}\,kg = 2.20\times10^{-6}\,kg$. The consequent concentration of migrant in the food is therefore about $(2.20\times10^{-6})/(1.88\times10^{-2}) \approx 1.17\times10^{-4} = 117\,mg/kg$, which is not the same as the nominal specific limit of 30 mg/kg which provided the starting point: in fact it is almost four times as much!

Conversely, if the package has precisely the same shape but linear dimensions 10 times as great, its capacity is about 18.8 l, which is the sort of size in which commercial cooking oil is sold. The manufacturer is again empowered by Article 4 to divide by 6. By a calculation like that above, it will be found that the maximum concentration in the food is now about 11.7 mg/kg, significantly less than the starting point.

From the point of view of food safety, it is important to record that, for all packages of this precise shape ($h = 6r$), of capacity less than 500 ml, the permitted 'rule' of division by 6 leads to concentrations in excess of 30 mg/kg. Indeed, the concentration increases indefinitely as r decreases (i.e. the concentration tends to infinity as r tends to zero).

The main conclusions are clear (and do not depend on the choice of $h = 6r$). It is not possible to 'convert' mg/kg into mg/dm^2 by division by 6 (or any other numerical factor). Article 4 of the Directive is therefore potentially dangerous since it does not prevent concentrations of migrant that are higher than the specified (and safe) maxima.

4. The effect of shape

An equally important geometrical feature is shape. In this section all the packages will have capacity 500 ml, so that the mass of the food is 5×10^{-1} kg.

The precise shape of the cylindrical package depends on the ratio of its radius r to its height h. When $h = r = r_0 \approx 5.42\times10^{-2}$ m, the capacity is 500 ml. The capacity is still 500 ml if $h = x^2 r_0$ and $r = x^{-1} r_0$ for any positive value of x— see figure 2. As x varies, so does the precise shape of the package. Table 1 gives some package dimensions for different values of x. The case of $h = 6r$ considered in section 3 occurs when $x = 6^{\frac{1}{3}} \approx 1.82$, so that $r \approx 2.98\times10^{-2}$ m.

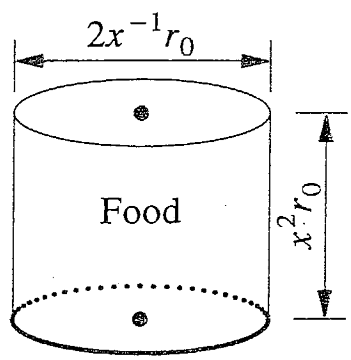


Figure 2

Table 1. Dimensions of the cylindrical package for different values of x.

x	$h\,/\,10^{-2}$ m	$r\,/\,10^{-2}$ m
0.5	1.35	10.84
1	5.42	5.42
2	21.68	2.71

Suppose that the specific migration limit is once more 30 mg/kg or, after 'conversion', a flux of 5×10^{-4} kg/m^2—see section 3. Since the total internal surface area is

$$2\pi r^2 + 2\pi rh = 2\pi r_0^2(x^{-2}+x),$$

the mass of migrant in the food after migration to the limit has occurred is

$$10^{-3}\pi r_0^2(x^{-2}+x)\,\mathrm{kg/m^2} \approx 9.23\times10^{-6}(x^{-2}+x)\,\mathrm{kg}.$$

The concentration C of migrant in the food is then given by

$$C \approx \frac{9.23\times10^{-6}(x^{-2}+x)}{5\times10^{-1}} \approx 18.46(x^{-2}+x)\,\mathrm{mg/kg}. \qquad (1)$$

A graph of C against x is shown in figure 3.

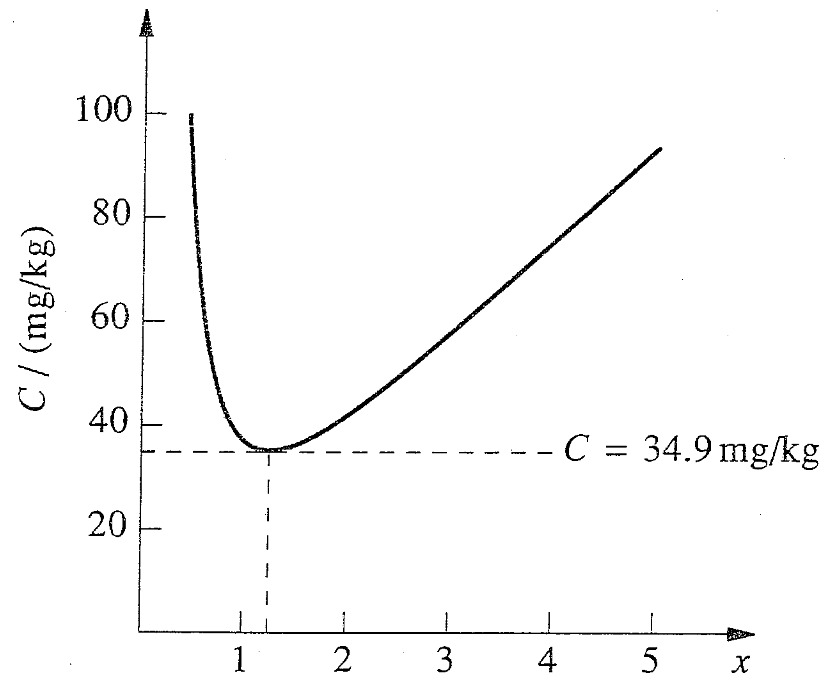


Figure 3

It is clear from figure 3 (see also section 5 below) that C varies a great deal with x and that, in particular:

- C has a least value of about 34.9 mg/kg when x is about 1.25 (in this case $r = \frac{1}{2}h \approx 4.30\times10^{-2}$ m);
- for all other values of x, C is greater than 34.9 mg/kg and therefore greater than the specific migration limit of 30 mg/kg—yet another example of the impossibility of converting from mg/kg to mg/dm^2;
- for large values of x, $C \approx 18.5x$ mg/kg (since x^{-2} in (1) is then much less than x) and so increases indefinitely with x, i.e. as the package becomes thinner, more and more like a piece of spaghetti;
- similarly, for small values of x, $C \approx 18.5x^{-2}$ mg/kg and also increases indefinitely as x decreases, i.e. as the package becomes fatter, more and more like a pizza base.

This detailed calculation illustrates the important general conclusion that specifying the flux of migrant does not specify the concentration of migrant in food of a given capacity unless the package shape is also known.

5. Technical interlude on the minimum value of $x^{-2}+x$

Readers who have studied calculus and stationary values will be able to show that $x^{-2}+x$ has a minimum value of $3\times2^{-\frac{2}{3}} \approx 1.89$ when $x = 2^{\frac{1}{3}} \approx 1.26$.

But it is not always necessary to use calculus to find maxima and minima, and $x^{-2}+x$ provides an example. By multiplying out the brackets on the right-hand side, it will be found that

$$x^{-2}+x = 3\times2^{-\frac{2}{3}}+2^{-\frac{2}{3}}x^{-2}(x-2^{\frac{1}{3}})^2(2^{\frac{2}{3}}x+1) \tag{2}$$

for all values of x ($\neq 0$).

Since x is positive (by definition—see figure 2), $(2^{\frac{2}{3}}x+1)$ is also positive. Also $(x-2^{\frac{1}{3}})^2$ is never negative because it is a square. Therefore

$$2^{-\frac{2}{3}}x^{-2}(x-2^{\frac{1}{3}})^2(2^{\frac{2}{3}}x+1)$$

on the right-hand side of (2) is never negative; its least value is obviously 0 when $x = 2^{\frac{1}{3}}$. So it can be seen from (2) that the minimum value of (2) is $3\times2^{-\frac{2}{3}}$ when $x = 2^{\frac{1}{3}}$. (It is always a good question to ask mathematicians where apparently mysterious expressions like the right-hand side of (2) come from! In this case the answer is that it was found using the general result that the arithmetic mean of a set of positive quantities is never less than their geometric mean—here three quantities were used: x^{-2}, $\frac{1}{2}x$ and $\frac{1}{2}x$.)

Since the minimum value of $(x^{-2}+x)$ is $3\times2^{-\frac{2}{3}}$, the minimum value of C in (1) is about $18.46\times3\times2^{-\frac{2}{3}}$mg/kg ≈ 34.9 mg/kg, consistent with figure 3.

6. Concluding note

Despite the simplicity of the work above, the basic, and serious, faults that it illustrates are obviously not apparent to the lawyers or to the scientists who advise them—mainly analytical chemists and toxicologists. Moreover, despite strenuous attempts, we have not yet been able to convince them that the regulations require significant revision. This has been a depressing and salutary experience. Not only must the mathematical modelling be good, but also its meaning must be communicated effectively; the latter is often far harder than the former!

Summing the Series $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ Using Pascal's Identity

SEYAMAK JAFARI, *Chemical Engineering, AIT, Ahwaz, Iran*

The author teaches engineering and physics.

We recall Pascal's identity

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

for $1 \leqslant r \leqslant n$, where

$$\binom{n}{r}=\frac{n!}{r!\,(n-r)!},$$

the binomial coefficient. Hence

$$\begin{aligned}
1+2+3+\cdots+n &= \binom{1}{1}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{n}{1}\\
&= \binom{2}{2}+\binom{2}{1}+\binom{3}{1}+\cdots+\binom{n}{1}\\
&= \binom{3}{2}+\binom{3}{1}+\binom{4}{1}+\cdots+\binom{n}{1}\\
&= \binom{4}{2}+\binom{4}{1}+\cdots+\binom{n}{1}\\
&= \ldots \qquad \ldots\\
&= \binom{n}{2}+\binom{n}{1}\\
&= \binom{n+1}{2}\\
&= \tfrac{1}{2}n(n+1),
\end{aligned}$$

the familiar sum.

Also,

$$\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}=\binom{3}{3}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n}{2}$$

$$
\begin{aligned}
&= \binom{4}{3} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2} \\
&= \binom{5}{3} + \binom{5}{2} + \binom{6}{2} + \cdots + \binom{n}{2} \\
&= \ldots \qquad \ldots \qquad \ldots \\
&= \binom{n}{3} + \binom{n}{2} \\
&= \binom{n+1}{3}.
\end{aligned}
$$

Hence

$$\binom{2}{2} + \binom{2}{2} + \binom{3}{2} + \binom{3}{2} + \binom{4}{2} + \binom{4}{2} + \cdots + \binom{n}{2} + \binom{n}{2} = 2\binom{n+1}{3}. \qquad (*)$$

Now

$$\binom{r}{2} + \binom{r+1}{2} = \tfrac{1}{2}r(r-1) + \tfrac{1}{2}(r+1)r = r^2,$$

so that (*) gives

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 - \binom{n+1}{2} = 2\binom{n+1}{3},$$

whence

$$
\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + n^2 &= 2\binom{n+1}{3} + \binom{n+1}{2} \\
&= 2 \times \tfrac{1}{6}(n+1)n(n-1) + \tfrac{1}{2}(n+1)n \\
&= \tfrac{1}{6}n(n+1)[2n-2+3] \\
&= \tfrac{1}{6}n(n+1)(2n+1),
\end{aligned}
$$

the well-known formula.

Powerless Arithmetic Progressions

Find a non-constant arithmetic progression of positive integers which contains no powers of integers except the first.

DES MACHALE
(University College,
Cork, Ireland)

A Proof of the Arithmetic–Geometric Mean Inequality

ALEXANDER ABIAN, *Iowa State University*
ESFANDIAR ESLAMI, *Kerman University*

Alexander Abian is a professor of mathematics at Iowa State University in the USA. Esfandiar Eslami is a professor of mathematics and head of the mathematics department at Kerman University in Iran. They both believe that the heart of the matter in a complicated mathematical item is very often an idea of stark beauty which, however, is often hidden under thick layers of sophistication.

We give here a rather simple proof of the arithmetic–geometric mean inequality, which is one of the most fundamental inequalities in mathematics. In its simple form it says that the arithmetic mean of a given finite set of non-negative real numbers is always greater than the geometric mean of these numbers unless these numbers are all equal, in which case the two means are also equal.

We recall that the arithmetic mean of n real numbers $a_1, \ldots, a_n$ is their average, i.e. $(a_1 + \cdots + a_n)/n$, and the geometric mean of n non-negative real numbers $a_1, \ldots, a_n$ is the unique non-negative nth root of their product, i.e. $(a_1 \cdots a_n)^{1/n}$.

Thus, in its simple form, the arithmetic–geometric mean inequality says that, for any n non-negative real numbers $a_1, \ldots, a_n$,

$$\frac{a_1 + \cdots + a_n}{n} > (a_1 \cdots a_n)^{1/n},$$

unless $a_1 = \cdots = a_n$ (in which case the inequality becomes an equality).

For $n = 1$, the proof of the above statement is trivial (the equality holds). For $n = 2$ its proof follows from the fact that

$$\tfrac{1}{2}(a_1 + a_2) - \sqrt{a_1 a_2} = \tfrac{1}{2}(\sqrt{a_1} - \sqrt{a_2})^2 > 0 \quad (\text{unless } a_1 = a_2).$$

However, it is worth noting that there is no corresponding simple proof for arbitrary n.

Theorem 1. For any n non-negative real numbers $a_1, \ldots, a_n$,

$$\frac{a_1 + \cdots + a_n}{n} > (a_1 \cdots a_n)^{1/n} \tag{1}$$

unless

$$a_1 = \cdots = a_n, \tag{2}$$

in which case (1) becomes an equality.

Proof. If one of the a_i (but not all) is 0 then (1) holds trivially. If all the a_i are 0 then (2) holds trivially. So, in what follows we let, in the statement of theorem 1, all the a_i be positive real numbers.

We use induction. As mentioned above, the statement of theorem 1 is true for $n = 1$. Thus we assume the statement of the theorem for a particular n and show that with the added a_{n+1} we have

$$\frac{a_1+\cdots+a_n+a_{n+1}}{n+1} > (a_1\cdots a_n a_{n+1})^{1/(n+1)}, \tag{3}$$

unless

$$a_1 = \cdots = a_n = a_{n+1}, \tag{4}$$

in which case (3) becomes an equality. Obviously (3) is equivalent to

$$a_1+\cdots+a_n+a_{n+1}-(n+1)(a_1\cdots a_n a_{n+1})^{1/(n+1)} > 0. \tag{5}$$

In the left-hand side of (5), we keep $a_1,\dots,a_n$ fixed and replace a_{n+1} by the variable x and consider the resulting function f given by

$$f(x) = a_1+\cdots+a_n+x-(n+1)(a_1\cdots a_n x)^{1/(n+1)}. \tag{6}$$

Since all the a_i are positive, we let $x > 0$ in (6). Now

$$f'(x) = 1-(a_1\cdots a_n)^{1/(n+1)}x^{-n/(n+1)}. \tag{7}$$

Thus,

$$f'(x) = 0 \quad \text{for} \quad (a_1\cdots a_n)^{1/n} = x_0, \text{ say.} \tag{8}$$

From (7) and (8) we see that $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$. Hence, $f(x)$ for $x > 0$ has a unique minimum m_0 which is attained at x_0. But then, from (6) and (8), we see that the value m_0 of the minimum of $f(x)$ is given by

$$m_0 = f(x_0) = a_1+\cdots+a_n-n(a_1\cdots a_n)^{1/n}, \tag{9}$$

which is greater than 0 by virtue of our induction hypothesis (1), provided all the a_i are not equal. Since for $x > 0$ the minimum of $f(x)$ is greater than 0, we see that $f(x) > 0$ for $x > 0$ from which, in view of (6), we see that (5) and (3) follow readily. On the other hand, if all the a_i are equal then from (8) and (9) it follows, respectively, that $a_1 = \cdots = a_n = x_0$ and that $m_0 = f(x_0) = 0$. Thus $f(a_{n+1}) > 0$, provided $a_{n+1} \neq x_0 = a_1 = \cdots = a_n$ from which, in view of (6), we see that (5) and (3) again follow readily. On the other hand, if $a_{n+1} = x_0 = a_1 = \cdots = a_n$ then, clearly, (4) holds.

Thus theorem 1 is proved.

Next we consider a stronger version of theorem 1.

Theorem 2. For any n non-negative real numbers $a_1,\ldots,a_n$ and any n non-negative real numbers $r_1,\ldots,r_n$ with $r_1+\cdots+r_n > 0$,

$$\frac{r_1 a_1+\cdots r_n a_n}{r_1+\cdots+r_n} > (a_1^{r_1}\cdots a_n^{r_n})^{1/(r_1+\cdots+r_n)}, \tag{12}$$

unless

$$a_1 = \cdots = a_n, \tag{13}$$

in which case (12) becomes an equality.

Proof. The proof of theorem 2 can be given on the lines of the proof of theorem 1, where however, instead of the function f given in (6), the function F given below must be considered:

$$\begin{aligned} F(x) = {} & r_1 a_1+\cdots+r_n a_n+r_{n+1}x \\ & -(r_1+\cdots+r_{n+1})(a_1^{r_1}\cdots a_n^{r_n}x^{r_{n+1}})^{1/(r_1+\cdots+r_{n+1})}. \end{aligned}$$

The minimum of $F(x)$ is achieved at

$$(a_1^{r_1}\cdots a_n^{r_n})^{1/(r_1+\cdots+r_n)} = x_0,$$

say, and the value of this minimum is

$$r_1 a_1+\cdots+r_n a_n-(r_1+\cdots r_n)(a_1^{r_1}\cdots a_n^{r_n})^{1/(r_1+\cdots+r_n)},$$

which is always non-negative and serves as the essential step in completing the inductive proof of theorem 2 (as in the case of theorem 1).

Remark. We would like to note that in the book by Hardy et al. (see the reference, page 108) an outline proof of the arithmetic–geometric mean inequality is given based on the theory of maxima and minima of functions of several real variables. A complete proof along those lines would be quite involved and cumbersome. As shown above, our proof achieves the same goal by invoking the theory of minima of a function of a single real variable.

Reference

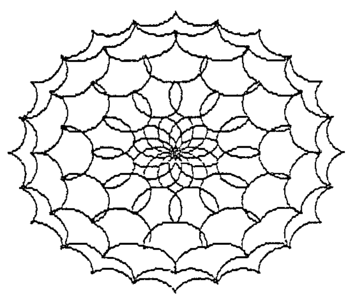
1. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, 1973).

Computer Column

MIKE PIFF

Rotation of a curve about a point

This article is a departure from the recent topics. John Littlewood from Nottingham has sent in a GW-BASIC program which draws a closed curve and then rotates it through various angles to draw an attractive pattern. His example uses the hypocycloid as the base curve. Here is the output produced by his program. A short article by John Littlewood on the epi/hypocycloid will appear in a forthcoming issue of *Mathematical Spectrum*. As usual, if anyone else wants to contribute to this column then please send in your program on a 3.5-inch disk to *Mathematical Spectrum*. If you would like to submit electronically, I can be reached via `M.Piff@sheffield.ac.uk`.

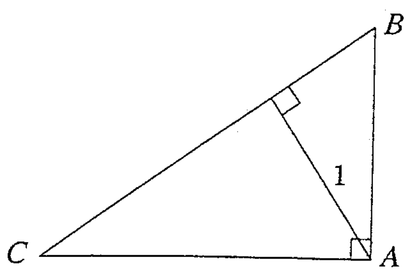


```
10 CLS:REM"JOHN"
20 SCREEN 2:PI=22/7:R=40
30 CIRCLE(300,100),0:T=12
40 FOR D = 0 TO 330 STEP 30
50 E=PI*D/180:H=-1
60 FOR A = 0 TO 360:B=PI*A/180:C=B*(H+T)
70 X=300+R*(1+H/T)*COS(B)-H*R*COS(C)/T
80 Y=100+R*(1+H/T)*SIN(B)-R*SIN(C)/T
90 REM"MOVE POINT TO ORIGIN"
100 X1=X-300-R:Y1=Y-100
110 REM"ROTATE"
120 X2=X1*COS(E)-Y1*SIN(E)
130 Y2=X1*SIN(E)+Y1*COS(E)
140 REM"MOVE POINT TO CENTRE OF SCREEN"
150 X3=X2+300 :REM"correct for aspect ratio"
155 X3=(X3-300)*2.4+300
160 Y3=Y2+100:LINE-(X3,Y3)
170 NEXT A:NEXT D
```

Pseudo-Pythagoras

Show that

$$BC^2 = AB^2 \times AC^2.$$



SEYAMAK JAFARI
(Ahwaz, Iran)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues, and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

26.4 (Submitted by Sinefakopoulos Achilleas, University of Athens)
Let ABC be a triangle and let D and E be the points on BC such that $\angle ADB$ is a right angle and $\angle DAB = \angle EAC$. Prove that

$$\text{area } \triangle EAC > \text{area } \triangle DAB \Leftrightarrow AC > AB.$$

26.5 The *Smarandache function* $\eta : \mathbb{N} \to \mathbb{N}$ is defined by $\eta(n) =$ the smallest positive integer m such that n divides $m!$.
(a) Calculate $\eta(p^{p+1})$, where p is prime.
(b) Find all possible integers n such that $\eta(n) = 10$.
(c) Prove that, for every real number k, there is a positive integer n such that

$$\frac{n}{\eta(n)} > k.$$

Does $n/\eta(n) \to \infty$ as $n \to \infty$? (Question (a) was submitted by Pedro Melendez of Belo Horizonte, Brazil, (b) and (c) by Thomas Martin, Phoenix, Arizona, USA.)

26.6 (Submitted by Gregory Economides, University of Newcastle upon Tyne Medical School)
Evaluate

$$\int_{x=1}^{\infty} \int_{y=1/x}^{2x} 2y^2 \exp\left\{-\left(x+\frac{1}{x}\right)y\right\} \, dy \, dx.$$

Solutions to Problems in Volume 25 Number 4

25.10 Does there exist a plane through the origin which intersects the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

with $a \geqslant b \geqslant c > 0$, in a circle?

Solution by Gregory Economides

The plane $z = kx$ intersects the ellipsoid where

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{k^2x^2}{c^2} = 1,$$

i.e.

$$y^2 = b^2\left[1-\left(\frac{1}{a^2}+\frac{k^2}{c^2}\right)x^2\right].$$

Now

$$\begin{aligned} x^2+y^2+z^2 &= x^2+b^2\left[1-\left(\frac{1}{a^2}+\frac{k^2}{c^2}\right)x^2\right]+k^2x^2 \\ &= b^2+x^2\left[1-\frac{b^2}{a^2}-k^2\left(\frac{b^2}{c^2}-1\right)\right], \end{aligned}$$

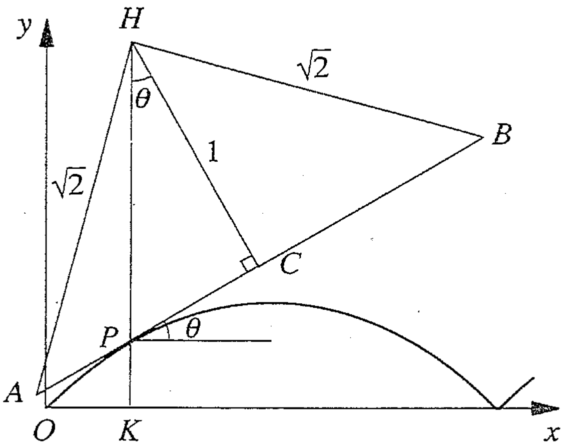
and this is equal to b^2 if

$$k^2 = \frac{c^2(a^2-b^2)}{a^2(b^2-c^2)}.$$

Hence the two planes $a^2(b^2-c^2)z^2 = c^2(a^2-b^2)x^2$ intersect the ellipsoid in a circular cross-section of radius b. (If $b = c$, then the plane $x = 0$ intersects the ellipsoid in the circle $x = 0$, $y^2+z^2 = b^2$ of radius b.)

25.11 Archimedes, a mathematics student, has a cart with square wheels, each with a side of length 2 units. The cart can remain stationary on a rough road whatever the position of the wheels. Find an equation of the curve describing the road, and show that the cart can move along the road without bumping.

Solution by Gregory Economides, who proposed the problem



Let AB be the side of the square wheel in contact with the road at the point P, H the centre of the square and K the point where HP produced cuts the x-axis—see the figure. We choose the origin O at the point where A was on the x-axis. Since the wheels do not slip,

$$AP = \text{arc}\, OP = s \quad \text{(say)}.$$

Then

$$\frac{dy}{dx} = \tan\theta = \frac{PC}{HC} = 1-s = 1-\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

Hence

$$\frac{d^2y}{dx^2} = -\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

If we put $z = dy/dx$, we can integrate this equation to give

$$\int \frac{1}{\sqrt{1+z^2}}\, dz = -x+c,$$

where c is a constant, i.e.

$$z = \sinh(-x+c)$$

or

$$\frac{dy}{dx} = \sinh(-x+c).$$

When $x = 0$, $dy/dx = 1$, so $\sinh c = 1$. Now

$$y = -\cosh(-x+c)+d$$

and $y = 0$ when $x = 0$, so that $d = \cosh c = \sqrt{2}$. Hence

$$y = \sqrt{2}-\sqrt{2}\cosh x+\sinh x.$$

Further,

$$\begin{aligned} HK &= HP+PK \\ &= \sec\theta+y \\ &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}+y \\ &= -\frac{d^2y}{dx^2}+y \\ &= \sqrt{2}\cosh x-\sinh x+\sqrt{2}-\sqrt{2}\cosh x+\sinh x \\ &= \sqrt{2}, \end{aligned}$$

so the centre of the wheel is a constant height above the x-axis and the cart does not bump.

25.12 Evaluate

$$\lim_{\lambda\to\infty}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma}.$$

Solution 1 by Gregory Economides
For $\lambda > 0$,

$$\ln\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma} = \gamma\left[\ln\left(\frac{1-\frac{\alpha}{\lambda}}{\frac{1}{\lambda}}\right)-\ln\left(\frac{1-\frac{\beta}{\lambda}}{\frac{1}{\lambda}}\right)\right].$$

Put $\mu = 1/\lambda$. By L'Hôspital's rule,

$$\begin{aligned}\lim_{\lambda\to\infty}\ln\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma} &= \lim_{\mu\to 0}\gamma\left(\frac{\ln(1-\alpha\mu)}{\mu}-\frac{\ln(1-\beta\mu)}{\mu}\right)\\ &= \lim_{\mu\to 0}\gamma\left(\frac{-\alpha}{1-\alpha\mu}-\frac{-\beta}{1-\beta\mu}\right)\\ &= \lim_{\mu\to 0}\frac{\gamma(\beta-\alpha)}{(1-\alpha\mu)(1-\beta\mu)}\\ &= \gamma(\beta-\alpha).\end{aligned}$$

Hence

$$\lim_{\lambda\to\infty}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma} = \exp\{\gamma(\beta-\alpha)\}.$$

Solution 2 by Sammy and Jimmy Yu (7th and 6th graders, Vermillion Middle School, South Dakota, USA)

Suppose first that $\gamma \neq 0$. Then

$$\begin{aligned}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma} &= \left[\left(1+\frac{(\beta-\alpha)|\gamma|}{(\lambda-\beta)|\gamma|}\right)^{(\lambda-\beta)|\gamma|}\right]^{\gamma/|\gamma|}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\beta\gamma}\\ &= \left[\left(1+\frac{(\beta-\alpha)|\gamma|}{x}\right)^{x}\right]^{\gamma/|\gamma|}\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\beta\gamma},\end{aligned}$$

where $x = (\lambda-\beta)|\gamma|$. As $\lambda \to \infty$, so also $x \to \infty$. We now use the well-known limit

$$\lim_{x\to\infty}\left(1+\frac{y}{x}\right)^{x} = \mathrm{e}^{y}$$

to give that, as $\lambda \to \infty$,

$$\left(\frac{\lambda-\alpha}{\lambda-\beta}\right)^{\lambda\gamma} \to [\exp\{(\beta-\alpha)|\gamma|\}]^{\gamma/|\gamma|}\times 1 = \exp\{(\beta-\alpha)\gamma\}.$$

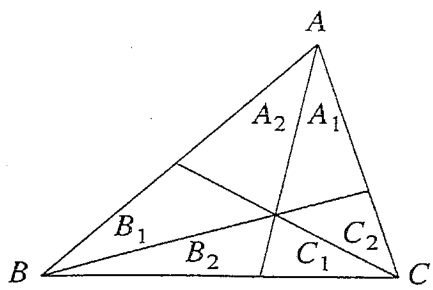
The result is trivially true when $\gamma = 0$.

Also solved by T. O. Womack (Winchester College) and Sinefakopoulos Achilleas.

More pseudo-Pythagoras

Show that

$$\frac{\sin A_1}{\sin A_2}\times\frac{\sin B_1}{\sin B_2}\times\frac{\sin C_1}{\sin C_2} = 1.$$



SEYAMAK JAFARI
(Ahwaz, Iran)

Reviews

Decision and Discrete Mathematics. By CHRIS COMPTON AND GEOFF RIGBY. Hodder and Stoughton, London, 1992. Pp. 139. Paperback £7.99 (ISBN 0-340-57171-3).

This is the companion text to the 'Decision and Discrete Mathematics' component of MEI Structured Mathematics, examined by the Oxford and Cambridge Board. Six components are required for an A-level in mathematics. We are told that the book 'introduces the reader to the relevance of algorithms in a variety of contexts, including manufacturing and service industries, medicine and everyday life'. Ten chapters cover a wide range of material, beginning with an introduction to algorithms, and the associated ideas of efficiency, recursion and heuristics. The high initial standard is maintained throughout the book with 'Sorting and searching', including quicksort and binary search; four chapters on network problems, Dijkstra, Kruskal, Prim; 'Simulation'; 'Decision analysis'; and finally two chapters on critical paths. The style throughout is practical and experimental, with emphasis on heuristics. Numerous exercises are set using 'real' data. These are supplemented with 'investigations', which are designed to provide ideas for the coursework element. At times I felt the style was too loose: for example on page 85, '... if we are able to start and finish at the same node (traversable) it (the network) must have no odd nodes' and then on page 86 'If there are no odd nodes in the network, the network is traversable.' There should at least be some indication that a proof is missing here. (There is a short proof using posets.) That said, there is plenty to interest the pure-minded. My only other gripe is the omission of an index.

Certainly I would have welcomed this course in preference to statistics when I did A-levels. The subject matter is refreshingly different from the conventional. Those taking the course who are not conversant with a language such as Pascal would gain from an A-level computer language course. Business Studies, I am sure, would likewise be a strong companion. The book itself is suitable for self-study: perhaps for adult education? But the authors envisage a teacher supervising a group of students with their own books. More generally, the book would be a useful source of ideas for the committed teacher.

Undergraduate at Trinity College, Cambridge DYLAN MENZIES-GOW

What's Happening in the Mathematical Sciences, Volume 1. American Mathematical Society, Providence, RI (ISBN 0-8218-8999-0).

How can you do research in mathematics? Is there anything left to discover (or invent)? What do mathematicians do? And why do they do it? And what use is it, anyway? If these are the sorts of questions that you ask or have been asked of you, then this attractively produced issue from the American Mathematical Society is just what you have been looking for. It is the first of hopefully many such to be produced annually to describe some work currently being done by mathematicians and its relevance to the world in which we live. It covers such topics as mathematical models of the heart, computational complexity, shapes and sounds, mathematical models of the environment, higher-dimensional geometry, curves of shortest length on surfaces, mathematical models for the growth of a crystal, computer programs (written by a student) for modelling the growth of plants, testing numbers for primality and colouring of maps.

The writer, Barry Riprà, succeeds admirably in giving readers without the technical know-how a flavour of the work being done and helping them to share his obvious enthusiasm and that of the workers.

And, after all that, single issues are free, apart from postage of \$7.00 (\$13.50 if you want it sent airmail). Write to

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A magazine to put in the rack in your sixth-form common room alongside *Mathematical Spectrum*.

University of Sheffield — DAVID SHARPE

Harrap's Maths Mini Dictionary. Edited by NIGEL ANDREWS. Chambers, Edinburgh, 1993. Pp. xii+276. Paperback £3.99 (ISBN 0-245-60472-3).

A handy pocket dictionary designed to be used by students taking GCSE mathematics in the UK (aged 16), with references to the National Curriculum Attainment Target. But it goes well beyond GCSE in its entries. A useful stocking filler at Christmas. However, care should be taken in its use. For example, see (or perhaps it would be better not to see!) the incorrect or misleading figures for 'asymptotes', 'inflection' and 'node'; or the dreadful entry on 'convergence'; or the entry on the 'Königsberg bridge problem', where it is suggested that a graph can have one vertex of odd degree; or the use of the symbol $\delta y/\delta x$ for 'partial derivative'. These howlers need to be corrected in a revised edition.

University of Sheffield — DAVID SHARPE

George Green Mathematician and Physicist 1793–1841: The Background to his Life and Work. By D. M. CANNELL. Athlone Press, London, 1993. Pp. xxvi+265. Hardback £35.00 (ISBN 0-485-11433-X).

Why has the same picture of a windmill appeared in two issues of *Mathematical Spectrum* (Volume 20, Number 2, 1987/88 and Volume 25, Number 3, 1992/93)? This is the windmill in Sneinton, Nottingham, where George Green (1793–1841) was miller. Most of his life was spent in obscurity in Sneinton; no picture of Green could be unearthed for this biography. What education he had until he went as an undergraduate to Gonville and Caius College, Cambridge, at the mature age of 40, can only be surmised, but he was largely self-taught. Yet, at the age of 35, while still in Sneinton, he wrote 'An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism' containing startlingly new ideas with applications not only to electricity and magnetism but also in our own century to semiconductors, superconductors, atomic and nuclear forces and quantum mechanics. As Einstein remarked, Green anticipated the work of other mathematicians, especially Gauss. His essay went almost completely unnoticed, as did his later mathematical works. His struggle against the prejudices of his day, the discovery after his death of his work by Lord Kelvin, and the efforts in our own day to obtain due recognition for one of the finest and most original mathematicians that Britain has ever produced are movingly told in the present volume. Unfortunately, its high price will prevent most readers from obtaining their own copy, a poignant reminder perhaps of the battle over the years to make the name of George Green known.

Happily, however, the battle has finally been won. Generations of students of mathematics and physics know his name as they grapple with Green's theorem and Green's functions. But do they know who Green was? What would George Green have thought if he could go now to Westminster Abbey and see a plaque inscribed with his name alongside such greats as Isaac Newton and Michael Faraday; and then go to Sneinton to see his windmill in action and coffee mugs decorated with his mathematics being sold in the visitors' centre? It's a funny world! Read this book if you can get hold of a copy. If not, read the article in *Mathematical Spectrum* by Professor Lawrie Challis to whom the book is dedicated and who has done more than anyone in our own day to obtain public recognition for George Green. Then go to Westminster Abbey and honour a great mathematician. Finally, go to Sneinton. You can even buy a mug!

University of Sheffield DAVID SHARPE

MEI: Statistics Book I. By ANTHONY ECCLES, ALAN GRAHAM AND ROGER PORKESS. Hodder and Stoughton, London, 1993. Pp. iv + 140. Paperback £6.99 (ISBN 0-340-57170-5).

The number of students who successfully make the transition from GCSE to an 'A'-level mathematics course is sadly in decline but several innovations are under way which are aiming to reduce this trend. The MEI Structured Mathematics is one such modular scheme which sets out to make mathematics at this level 'accessible, interesting and relevant to a wide range of students', and if this, the first in a series of books written to support the statistics component of this new syllabus, is representative of the scheme then these aims are certainly destined to be accomplished.

The book contains five sections: Exploring Data; Probability; Selections; Binomial Distribution; and Hypothesis Testing using the Binomial Distribution, this last chapter providing an unusual but welcome contribution to a text that does not start the exploration of continuous variable theory. The authors are intent on interpretation of data rather than emphasising routine calculations, and in this way they get quickly on to some more advanced ideas of analysis which is usually where I find that students' interests can be more easily sustained.

Attention is immediately seized by the opening example presented in the form of an article from a fictional local paper reporting another road accident involving a young cyclist. This device is used throughout the book and is very successful in offering reality as well as human interest to the analysis of the story, and no doubt will leave the student with the means to digest and evaluate similar articles that he or she might subsequently encounter. This has to be a better approach than the standard contrived examples that usually abound in first texts; it is certainly a much more interesting one, and indeed the whole book has lots of appeal in terms of its layout as well as its content.

As a teacher I should certainly like to use this book and I feel sure that students will have a lot of fun acquiring some valuable skills through it. My only anxiety would be a doubt that we could afford to equip our students with the six books that will form the full series of support texts for just the statistics component of this new syllabus.

Solihull Sixth Form College CAROL NIXON

The Magic of Numbers. By ERIC TEMPLE BELL. Dover Publications, New York, 1991. Pp. vii+418. Paperback £8.95 (ISBN 0-486-26788-1).

This is a Dover paperbook edition of a work that was first published in 1946. In this highly entertaining book, Bell investigates the history of two dilemmas concerning the nature of mathematical truth that have beset philosophers from ancient times until the present day. Firstly, is mathematics a free human creation or does it exist independently of our minds, lying external to us waiting to be discovered? Secondly, can the laws of nature only be obtained by experiment and observation or can they be deduced from purely theoretical considerations? The hero of Bell's story is Pythagoras, the 6th century BC Greek Philosopher, who founded a culture based on the doctrine that *everything is number*, and the various ways in which such number mysticism has influenced the development of religion, philosophy, science and mathematics down through the ages are fully explored. Along the way, the reader is introduced to the views of such luminaries as Plato, Roger Bacon, Giordano Bruno, Galileo, Newton, Berkeley, Saccheri, Lobachewsky and Einstein. This lucidly written, non-technical account of what may be thought by some to be rather dry topics succeeds in being lively, stimulating, thought-provoking, amusing, mischievous, and often infuriating, but not once dull. Bell never allows the facts, or a lack of them, to get in the way of a good story; he was, after all, the author of some fifteen science-fiction novels! Those familiar with his *Men of Mathematics* will know what to expect; others will be in for a pleasant surprise.

University of Sheffield ROGER WEBSTER

Other books received

Theory of Approximation. By N. I. ACHIESER. Dover, New York, 1992. Pp. x+307. Paperback £8.95 (ISBN 0-486-67129-1).

This Dover edition is a republication of the work first published by Frederick Ungar Publishing Company, New York, in 1956.

Singular Integral Equations. By N. I. MUSKHELISHVILI. Dover, New York, 1992. Pp. 447. Paperback £12.95 (ISBN 0-486-66893-2).

This Dover edition is a republication of the second edition published in 1953 by P. Noordhoff, NV, Groningen, Holland, based on the second Russian edition of 1946.

Linear Algebra: A Geometric Approach. By E. SERNESI. Chapman and Hall, London, 1993. Translated by J. MONTALDI from the original Italian language edition published in 1989 by Bollati Boringhieri editore s.p.a. Pp. 384. Hardback £35.95 (ISBN 0-412-40670-5), paperback £16.95 (ISBN 0-412-40680-2).

This textbook is suitable for second-year mathematics undergraduates.

The Development of Mathematics. By E. T. BELL. Dover, New York, 1992. Pp. xiv + 637. Paperback £12.95 (ISBN 0-486-27239-7).

This is a reprint of a volume first published in 1940. Generations of readers will be familiar with the pithy comments of the author in the development of mathematics from his classic *Men of Mathematics*.

Flatland: A Romance of Many Dimensions. By EDWIN A. ABBOTT. Dover, New York, thrift edition, 1992. Pp. xii + 83. Paperback £0.95 (ISBN 0-486-27263-X).

This modestly priced volume is a reprint of a book first published way back in 1884. To quote from the introduction 'Though the crowded years go by, this ... tale shows no signs of age. It remains as spry as ever, a timeless classic of perennial fascination that seems to have been written for today. Like all great art, it defies the tyrant time!'

Mathematical Morphology and Image Processing. By EDWARD R. DOUGHERTY. Marcel Dekker, New York, 1993. Pp. 552. Hardback $160.00 (ISBN 0-8247-8724-2).

This text is for researchers and practitioners in the field of image processing.

Mathematics Dictionary. Van Nostrand Reinhold, New York, 1992. Pp. vii + 548. Hardback £33.50 (ISBN 0-442-00741-8), paperback £19.95 (ISBN 0-442-01241-1).

Among the new topics listed in this fifth edition are: Catalan numbers, catastrophe theory, chaos, exotic spheres, fractals, fuzzy logic, Mandelbrot set, polyominos. A useful source of reference which might whet a browser's appetite to find out more about a topic.

Number Words and Number Symbols: A Cultural History of Numbers. By KARL MENNINGER. Dover, New York, 1992. Pp. xiii+480. Paperback £14.95 (ISBN 0-486-27096-3).

This is a republication of the English translation (MIT Press, Cambridge, MA, 1969) of the 1957–58 revised German edition issued by the Vandenhoek and Ruprecht Publishing Company, Göttingen. Although the book is written for the lover of intellectual and cultural history, it contains a wealth of fascinating detail likely to interest mathematics students at all levels.

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