Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

Differences of squares

Bob Bertuello, one of our more assiduous correspondents, recently wrote about the problem of expressing a number as a difference of two squares. He was concerned with products of numbers in arithmetic progression, such as $3 \times 7 \times 11 \times 15$. Readers may like to consider when such a number can be so written. Student readers may like to send their solution to us for possible inclusion in a future issue–see Problem 42.5.

There is what must be a fairly standard method, which perhaps deserves to be better known. Take last year for example. Suppose that we want to express 2009 as $s^2 - t^2$ for some positive integers s, t. The trick is first to factorize 2009 into its prime factors:

$$2009 = 7^2 \times 41.$$

We all know the factorization

$$s^2 - t^2 = (s+t)(s-t),$$

so the problem becomes to split the product $7^2 \times 41$ up into two factors, the larger being $s+t$ and the smaller being $s-t$. The possibilities are

$$(s+t, s-t) = (2009, 1), (287, 7), (49, 41).$$

This gives

$$(s, t) = (1005, 1004), (147, 140), (45, 4).$$

You can try this with $3 \times 7 \times 11 \times 15$. This method will give *all* ways of expressing a number as the difference of two squares.

You may wonder why we did not use the present year 2010, which is even. In this case s, t must either both be even or both be odd, so when you express 2010 as a product of two numbers, both need to be even. Maybe 2008 and 2012 are better years!

In case you should wonder why anyone would want to do this, consider Pythagorean triples such as 3, 4, 5, where $3^2 + 4^2 = 5^2$. There is a famous set of formulae which appear in Euclid's *Elements* which describe all primitive Pythagorean triples x, y, z (by 'primitive' we mean that $x, y,$ and z have no common factor). For such a triple, one of x, y must be even and the other odd, and z is odd, so we can take x even and y odd. The formulae are

$$x = 2st, \qquad y = s^2 - t^2, \qquad z = s^2 + t^2,$$

where s, t are integers with $s > t > 0$, s, t are coprime, and one of s, t is even and the other is odd.

I take it the first thing you do on New Year's Day each year is to factorize the year, much more productive than New Year's resolutions! Then, using the above technique, you can find all primitive Pythagorean triples x, y, z such that $y = 2009$. You might have some difficulty this year with $x = 2010$, but you could try $x = 2008$ or 2012 instead.

But what about putting $z = 2009$? That involves expressing 2009 as a *sum* of two squares, which is another story!

Friends in High Places

ROGER WEBSTER and GARETH WILLIAMS

A history of amicable pairs from Pythagorean times until the present day. Number mysticism, the work of Thabit ibn Qurra and Euler, the effect of electronic computation on the search for pairs, and some open conjectures.

Photograph by David Brown

Sgurr nan Ceannaichean (see Authors' note)

Pythagoras was born c. 580 BC on the Aegean island of Samos. After years spent travelling, he eventually settled in Crotona, a Greek seaport in south west Italy. Here he founded the celebrated Pythagorean School, a secret brotherhood with its own strict observances, which continued to flourish long after its leader's death c. 500 BC, spreading his teachings throughout the Greek world.

The Pythagoreans' belief that *natural numbers* held the key to the universe led them to a *number mysticism*, in which numbers were ascribed various qualities. For example, odd numbers were thought of as *male* and even ones *female*. One represented *reason*, two *opinion*, three *harmony*, four *justice*, five *marriage*, and six *creation* etc. Today raffles, the Lottery, and TV shows like *Deal or No Deal* engender a culture of *lucky numbers*.

This fanciful number mysticism led the Pythgaoreans to take the first steps in developing *number theory*, the abstract study of integers, with their early recognition of *even*, *odd*, *prime*, and *composite* numbers. They labelled numbers *abundant*, *deficient*, or *perfect* according to whether the sum of their *proper* divisors was *more than*, *less than*, or, respectively, *equal to* the number itself.

An undoubted highlight of their mathematics was the discovery of the pair of distinct numbers 220 and 284, each of which is the sum of the *proper* divisors of the other, i.e.

$$220 = 1 + 2 + 4 + 71 + 142,$$
$$284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110.$$

Such a pair is said to be *amicable* or *friendly*. The smaller number 220 is abundant and the larger 284 deficient, a result true for any amicable pair. To the number mystics, 220 and 284, each being composed of the parts of the other, symbolized perfect friendship, superstition holding that two talismans bearing them would seal friendship between the wearers. Indeed, Pythagoras described a friend as 'another I such as are 220 and 284'.

The numbers came to take on a mystical aura, playing a role in magic, astrology, sorcery, and horoscopes. Even so, commentators failed to find a single high profile appearance of this *friendliest* number pair. All the literature offers is two low profile appearances: Jacob's gift of 220 goats and 220 sheep to Esau (Genesis 32: 14), seen as a mystical way of securing friendship; and the testing of their erotic effects by eleventh-century Arab El Madschriti, who baked confections in the shapes of the two numbers, with a friend eating the smaller number, while he ate the larger, the outcome going unrecorded!

The highest profile peak in the British Isles is Ben Nevis, one of the Munros, Scottish mountains over 3000 ft, named after Sir Hugh Munro. (Munro made his ascents in the hours of darknesss, so as not to disturb the local lairds, and perhaps this is why he failed to scale all the mountains named after him!) Scottish hills between 2500 ft and 3000 ft are known as Corbetts, after J. Rooke Corbett, the first Sassenach to conquer all the Munros. (Whilst attending St John's College, Cambridge University, from 1895 to 1898, Corbett walked from Manchester to Cambridge at the beginning of each term, and back again at the end!) Although unacquainted personally, these two climbers were kindred spirits in a common quest, who may be poetically described as *friends in high places*, a friendship sealed in Pythagorean lore, for there are 220 Corbetts and 284 Munros!

Although the amicable pair $(220 : 284)$ was known in Pythagorean times, no new pair appeared until the ninth century. Legend has it that:

> Once upon a time there lived a sultan, who prided himself on his great intellect. When one of his prisoners begged to be freed, the sultan replied: 'Set me a problem and you may remain free until I have solved it.' The prisoner challenged the sultan to find an amicable pair of numbers beyond $(220 : 284)$. He then went his way and lived happily ever after, for the sultan never did solve the problem.

Progress in finding new amicable pairs only occurred when mathematicians joined the search.

The first serious contribution to the study of amicable numbers is the following remarkable result of Arab polymath Thabit ibn Qurra (826–901).

Thabit's rule. *If n is a natural number such that all the three numbers*

$$p = 3(2)^n - 1, \qquad q = 3(2)^{n+1} - 1, \qquad r = 9(2)^{2n+1} - 1$$

are prime, then the numbers $2^{n+1}pq$ and $2^{n+1}r$ form an amicable pair.

For $n = 1, 3, 6$, Thabit's Rule produces the first, second, and third *known* amicable pairs, namely

$$2^2 \cdot 5 \cdot 11 = 220 \quad \text{and} \quad 284 = 2^2 \cdot 71,$$
$$2^4 \cdot 23 \cdot 47 = 17\,296 \quad \text{and} \quad 18\,416 = 2^4 \cdot 1151,$$
$$2^7 \cdot 191 \cdot 383 = 9\,363\,584 \quad \text{and} \quad 9\,437\,056 = 2^7 \cdot 73\,727,$$

but yields no other amicable pairs for $n \le 191\,600$. The first is the classic one, the only one known at the time. Thabit deserves the credit for his breakthrough in discovering the second, for although not writing it down, the geometric example he used in proving his rule, when interpreted numerically, is *precisely* it. Al-Banna (1256–1321) and al-Farisi (1260–1320) also have legitimate claims on the latter. The honour of unearthing the third falls to Yazdi (c. 1600).

Knowledge of the amicable pair $(220 : 284)$ and its role in number mysticism reached Europe via the Arabs, and by 1550 the pair had already appeared in the works of Chuquet, Stifel, Cardan, and Tartaglia, although Thabit's rule and its consequences were unknown in the West. So it was when Fermat and Descartes began their own search for amicable pairs. Both rediscovered Thabit's rule, and in letters to Mersenne, each claimed a new pair: Fermat $(17\,296 : 18\,416)$ in 1636 and Descartes $(9\,363\,584 : 9\,437\,056)$ in 1638.

Euler's transient appearance on the scene revolutionized the search. On his entrance in 1737, just three pairs had been found in over two thousand years. On his exit in 1740 this had risen, solely by his hands, to 62, even discounting the few that later turned out to be *unfriendly*. He investigated when number pairs of a particular structure, say (apq, ars), where $p, q, r,$ and s are distinct primes and *not* divisors of a, are amicable. This particular structure led to his smallest pair

$$(2620 : 2924) = (2^2 \cdot 5 \cdot 131 : 2^2 \cdot 17 \cdot 43),$$

laying to rest expectation that any new pairs would be inordinately large. His discovery of the pair

$$(12\,285 : 14\,595) = (3^3 \cdot 5 \cdot 7 \cdot 13 : 3 \cdot 5 \cdot 7 \cdot 139)$$

laid to rest expectation that both members of an amicable pair are even.

One general result to emerge from Euler's work is the following rule.

Euler's rule. *If natural numbers k and n with $k \le n$ are such that all the three numbers*

$$p = (2^k + 1)2^{n+1-k} - 1, \qquad q = (2^k + 1)2^{n+1} - 1, \qquad r = (2^k + 1)^2 2^{2n+2-k} - 1$$

are prime, then the numbers $2^{n+1}pq$ and $2^{n+1}r$ form an amicable pair.

Case $k = 1$ is Thabit's rule. Only two pairs (k, n) with $k > 1$ are known to satisfy the above conditions, namely $(7, 7)$ and $(11, 39)$, neither of which Euler seems to have known. The first yields an amicable pair, each of whose members has 10 digits, the second yields an amicable pair, each of whose members has 40 digits.

Euler's success in developing systematic methods for finding amicable pairs encouraged a brave few to take up the search for themselves, but the thoroughness of his investigations left few easy avenues to explore. So successful had he been, that only four new pairs were discovered in the next century and a half, contributing to a grand total of 66 by the end of the nineteenth century. In 1972 Lee and Madachy (see references 1–4) published a historical survey of amicable pairs, listing all 1108 known at the time.

Electronic computation has transformed the search for amicable pairs, from a trickle into an avalanche. In their early history, only a single pair was found in a thousand years, in their recent history even one three-page article could boast *A Million New Amicable Pairs* (see reference 5). Since 1995 Jan Munch Pedersen has serviced the Internet site *Known Amicable Pairs*, http://amicable.homepage.dk/knwnc2.htm, which lists, in increasing order of lowest members, all 11 994 387 (as of September 2007) known amicable pairs, together with their discoverers, years of discovery, and prime factorizations. Its opening entries are

1. Pythagoras (500 BC): $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$,
2. Paganini (1866): $1184 = 2^5 \cdot 37$ and $1210 = 2 \cdot 5 \cdot 11^2$,
3. Euler (1747): $2620 = 2^2 \cdot 5 \cdot 131$ and $2924 = 2^2 \cdot 17 \cdot 43$.

The first and third occasion no surprise, for we have met with them before, but what of the second? Thereby hangs a tale. For two thousand years, mathematicians of the calibres of Fermat, Descartes, and Euler had scoured the heavens with their sophisticated instruments looking for amicable pairs, but failed to see what lay at their very feet, the second smallest pair (1184 : 1210). Who, then, made this startling discovery so late in the history of amicable pairs, thus ensuring a place in number theory's *hall of fame*? Yet another fairytale ending: a sixteen-year-old Italian schoolboy, Nicolo Paganini.

We have seen that the first three entries in Pedersen's 2007 list of amicable pairs have a tale to tell, but what of the last? It is the 11 994 387th and as of 2009 the largest one known. It begins

11 994 387. Jobling & Walker (2005): 1760... and 1826...,

where in *both* cases the '...' represent 24 069 further digits! It is certainly no exaggeration to say that these two members of the largest known amicable pair are indeed *friends in high places*, very high places!

Many intriguing conjectures about the structure of amicable pairs still await resolution. Amongst them include the following.

(i) The members of an amicable pair are either both even or both odd.

(ii) The members of an odd amicable pair are divisible by three.

(iii) The members of an amicable pair have a nontrivial common divisor.

Alongside attempts to solve such conjectures, the ongoing search for the largest amicable pair forges quietly ahead, unlike that for the largest known prime, which proceeds in the full blaze of media publicity. There is, however, a subtle difference in the nature of the two searches: Euclid proved that there are infinitely many primes, but no such result has been established for amicable pairs. Researchers all agree that there are infinitely many amicable pairs, but this is only a conjecture, albeit the most famous in the saga of friendly numbers.

A tongue-in-cheek comparison may be drawn between this conjecture and Fermat's Last Theorem, itself only a conjecture for three centuries. Investigators trying to prove the latter were no doubt encouraged by Fermat's own claim, that he had a truly wonderful proof, but it would not fit in the margin. Likewise, the few stalwarts attempting to prove that there are infinitely many amicable pairs may take succour from the following casual remark tucked away in Euler's 1747 paper, 'There is no doubt that infinitely many [amicable pairs] may be given'. Perhaps the conjecture that there are infinitely many amicable pairs should be dubbed Euler's Last Theorem!

Authors' note

This article was prompted by a letter of 31 October 2008 from David Gibson, Senior Officer of The Mountaineering Council of Scotland, confirming that there were 220 Corbetts and 284 Munros. Unfortunately, on the day this article was completed, *The Independent* of 11 September 2009 reported that, after re-measurement in July 2009, the height of the lowest Munro, *Sgurr nan Ceannaichean*, had been found to be 2996.82 ft. Consequently, it has been demoted to a mere Corbett, so now there are 221 Corbetts and 283 Munros. Maybe a topic for another article!

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Roger Webster *is a lecturer at Sheffield University specializing in the history of mathematics. He is a G & S devotee, revelling in this Gilbertian situation, in which a Munro lost its standing in Scottish mountaineering aristocracy, because a certain Lord High Surveyor decreed it to have dropped five feet, even though it had not moved at all!*

Gareth Williams *is a staff tutor in mathematics at the Open University specializing in topology, and is in great demand as a popularizer of mathematics. A keen mountaineer, having scaled many Munros and Corbetts, he was delighted to receive a holiday postcard, depicting a mountain scene, from his co-author beginning 'Hi 220, . . . ' and ending 'Cheers 284'.*

Equal totals with different powers

$$1 + 6 + 7 + 17 + 18 + 23 = 2 + 3 + 11 + 13 + 21 + 22,$$
$$1^2 + 6^2 + 7^2 + 18^2 + 23^2 = 2^2 + 3^2 + 11^2 + 13^2 + 21^2 + 22^2,$$
$$1^3 + 6^3 + 7^3 + 18^3 + 23^3 = 2^3 + 3^3 + 11^3 + 13^3 + 21^3 + 22^3,$$
$$1^4 + 6^4 + 7^4 + 18^4 + 23^4 = 2^4 + 3^4 + 11^4 + 13^4 + 21^4 + 22^4,$$
$$1^5 + 6^5 + 7^5 + 18^5 + 23^5 = 2^5 + 3^5 + 11^5 + 13^5 + 21^5 + 22^5.$$

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Continued Nested Radical Fractions

TEIK-CHENG LIM

1. Motivation

In spite of their long and illustrious history, the study of nested radicals and continued fractions remains relevant to date (see, e.g. [1]–[9], [11]–[16]). We define a continued nested radical fraction (CNRF) as a hybrid of a continued fraction and a nested radical:

$$\mathrm{CNRF} = \sqrt[m]{a_0 + \cfrac{b_1}{\sqrt[n]{a_1 + \cfrac{b_2}{\sqrt[n]{a_2 + \cdots}}}}}. \tag{1}$$

Conversely, we can say that nested radicals and continued fractions are special cases of a CNRF: substituting $m = -n = 2$ and $m = n = 1$ into (1) gives a nested radical (NR)

$$\mathrm{NR} = \sqrt{a_0 + b_1\sqrt{a_1 + b_2\sqrt{a_2 + b_3\sqrt{a_3 + \cdots}}}}$$

and a continued fraction (CF)

$$\mathrm{CF} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}},$$

respectively. Substituting $b_i = 1 (i = 1, 2, 3, \ldots)$ reduces the NR and CF to their simple forms:

$$\mathrm{NR} = \sqrt{a_0 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}}$$

and

$$\mathrm{CF} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \ldots].$$

2. CNRF forms of the golden ratio, silver ratio, and plastic constant

When $a_i = 1 (i - 0, 1, 2, \ldots)$, the simple forms of both the NR and CF reduce to the recursion expressions of the golden ratio:

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \tag{2}$$

and

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}, \tag{3}$$

which reduce to

$$\phi = \sqrt{1 + \phi} \tag{4}$$

and

$$\phi = 1 + \frac{1}{\phi}, \tag{5}$$

respectively, and both give $\phi^2 - \phi - 1 = 0,$ where $\phi = (1 + \sqrt{5})/2,$ the *golden ratio.* If (F_n) denotes the Fibonacci sequence, so that $F_0 = 0, F_1 = 1,$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2,$ an easy induction argument gives

$$\phi^n = F_{n-1} + F_n \phi \tag{6}$$

and

$$\phi^n = F_{n+1} + \frac{F_n}{\phi}. \tag{7}$$

Equation (6) gives an expression for ϕ as an NR:

$$\phi = \sqrt[n]{F_{n-1} + F_n \sqrt[n]{F_{n-1} + F_n \sqrt[n]{F_{n-1} + \cdots}}}. \tag{8}$$

Equation (7) gives an expression for ϕ as a CNRF:

$$\phi = \sqrt[n]{F_{n+1} + \cfrac{F_n}{\sqrt[n]{F_{n+1} + \cfrac{F_n}{\sqrt[n]{F_{n+1} + \cdots}}}}}. \tag{9}$$

Equations (2) and (3) are special cases of (8) and (9) with $n = 2$ and $n = 1,$ respectively.

A similar treatment can be applied to the *silver ratio*, $\delta,$ defined as $\delta = 1 + \sqrt{2}.$ Compared to (4) and (5), we have

$$\delta = \sqrt{1 + 2\delta} \tag{10}$$

and

$$\delta = 2 + \frac{1}{\delta}. \tag{11}$$

If (P_n) denotes the *Pell sequence*, defined by $P_0 = 0, P_1 = 1,$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2,$ an easy induction argument gives

$$\delta^n = P_{n-1} + P_n \delta \tag{12}$$

and

$$\delta^n = P_{n+1} + \frac{P_n}{\delta}. \tag{13}$$

Equations (12) and (13) enable δ to be respectively expressed as an NR and CNRF:

$$\delta = \sqrt[n]{P_{n-1} + P_n \sqrt[n]{P_{n-1} + P_n \sqrt[n]{P_{n-1} + \cdots}}}\,, \tag{14}$$

$$\delta = \sqrt[n]{P_{n+1} + \cfrac{P_n}{\sqrt[n]{P_{n+1} + \cfrac{P_n}{\sqrt[n]{P_{n+1} + \cdots}}}}}. \tag{15}$$

Substituting $n = 2$ and $n = 1$ into (14) and (15) leads to

$$\delta = \sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + 2\sqrt{1 + \cdots}}}} \quad \text{and} \quad \delta = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}},$$

respectively. The result of these expressions can also be obtained through the expansion of (10) and (11).

A third constant, the *plastic constant* p, is defined as the real root of the equation $p^3 = p+1$. Its approximate value is 1.324 719 572 4. This can be expressed as either

$$p = \sqrt[3]{1 + p} \quad \text{or} \quad p = \sqrt{1 + 1/p},$$

and these expressions give the NR

$$p = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}$$

and the CNRF

$$p = \sqrt{1 + \cfrac{1}{\sqrt{1 + \cfrac{1}{\sqrt{1 + \cdots}}}}}.$$

The golden ratio, the silver ratio, and the plastic constant k are special cases of the constant defined by

$$k = \sqrt[n+1]{1 + c \times \sqrt[n+1]{1 + c \times \sqrt[n+1]{1 + \cdots}}} = \sqrt[n]{c + \cfrac{1}{\sqrt[n]{c + \cfrac{1}{\sqrt[n]{c + \cdots}}}}}.$$

In Table 1 we give k for certain values of c and n.

Table 1 Various constants represented by k based on c and n.

c	k	
	$n = 1$	$n = 2$
1	ϕ	p
2	δ	ϕ

We illustrate, by means of examples involving the golden ratio, the faster convergence of a CNRF over CFs or NRs with certain conditions fixed. Limiting (9) to three divisions, the approximations, using $n = 1$ and $n = 2$, give values accurate to zero and three decimal places, respectively, as shown in table 2. The higher accuracy of a CNRF over a CF is attributed to the contribution of the NR component in the former. Limiting (8) and (9) to four roots for $n = 2$ and $n = 6$, we obtain the results listed in table 3. Table 3 shows two trends. Higher root order provides faster convergence to both NRs and CNRFs. In addition, the CNRF is advantageous over NRs (for fixed n) owing to the contribution of the CF component in the former.

Table 2 The approximated golden ratio of a CF and a CNRF using three divisions.

Recursion type	Recursion equation	Value	Accuracy (decimal places)
CF	$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}$	$\underline{1}.666\,6\ldots = 1.\dot{6}$	0
CNRF	$\sqrt{2+\cfrac{1}{\sqrt{2+\cfrac{1}{\sqrt{2+\cfrac{1}{\sqrt{2}}}}}}}$	$\underline{1.618}\,409\,043\,6\ldots$	3

Table 3 The approximated golden ratio of an NR and a CNRF using four roots.

Recursion type	Root order	Recursion equation	Value	Accuracy (decimal places)
NR	2	$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$	$\underline{1.5}98\,053\,182\,478\,62$	1
	6	$\sqrt[6]{5+8\times\sqrt[6]{5+8\times\sqrt[6]{5}}}$	$\underline{1.61}7\,455\,816\,235\,53$	2
CNRF	2	$\sqrt{2+\cfrac{1}{\sqrt{2+\cfrac{1}{\sqrt{2+\cfrac{1}{\sqrt{2}}}}}}}$	$\underline{1.618}\,409\,010\,436\,18$	3
	6	$\sqrt[6]{13+\cfrac{8}{\sqrt[6]{13+\cfrac{8}{\sqrt[6]{13+\cfrac{8}{\sqrt[6]{13}}}}}}}$	$\underline{1.618\,0}42\,564\,284\,72$	4

3. Conclusions

In conclusion, NRs and CFs are special cases of CNRFs. There are apparently advantages in combining roots and fractions to obtain CNRFs. Further investigation on CNRFs is recommended.

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Analysis of the Cover-Up Game

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A game that is sometimes played by school children learning to add fractions is described as follows. A six-sided die is relabeled with one face receiving the fraction $\frac{1}{2}$, one face receiving the fraction $\frac{1}{4}$, two faces receiving the fraction $\frac{1}{8}$, and two faces receiving the fraction $\frac{1}{16}$. So the probability of rolling a $\frac{1}{2}$ is $\frac{1}{6}$, as is the probability of rolling a $\frac{1}{4}$; the probability of rolling an $\frac{1}{8}$ is $\frac{1}{3}$, as is the probability of rolling a $\frac{1}{16}$.

Starting at position 0, students roll the die and move along an interval of length 1 by the amount specified by the roll's outcome. When played with a group of students, the object is to be the first to get to position 1. This game, commonly called *cover-up*, is a fun way for children to learn to add fractions.

In this article we will investigate the average play time of the cover-up game.

The rules of the game require players to land exactly at position 1. For instance, if the player is at position $\frac{7}{8}$, a roll of $\frac{1}{2}$ or $\frac{1}{4}$ causes the player to remain at $\frac{7}{8}$. A roll of $\frac{1}{8}$ will cause the player to win the game, landing him or her exactly at position 1. A roll of $\frac{1}{16}$ will move the player up to position $\frac{15}{16}$.

Since the game ends when a player lands at position 1, the cover-up game is an example of an absorbing Markov chain, $\{X_n,\ n = 0, 1, 2, \ldots\}$, where X_n may take values (positions) in the state space

$$S = \left\{0, \tfrac{1}{16}, \tfrac{1}{8}, \tfrac{3}{16}, \tfrac{1}{4}, \tfrac{5}{16}, \tfrac{3}{8}, \tfrac{7}{16}, \tfrac{1}{2}, \tfrac{9}{16}, \tfrac{5}{8}, \tfrac{11}{16}, \tfrac{3}{4}, \tfrac{13}{16}, \tfrac{7}{8}, \tfrac{15}{16}, 1\right\},$$

with position 1 being the only absorbing state. A detailed discussion of Markov chains can be found in the probability text by Ross (reference 2) or the mathematical modeling text by Mooney and Swift (reference 1). A state diagram for the possible moves from each position is shown in figure 1.

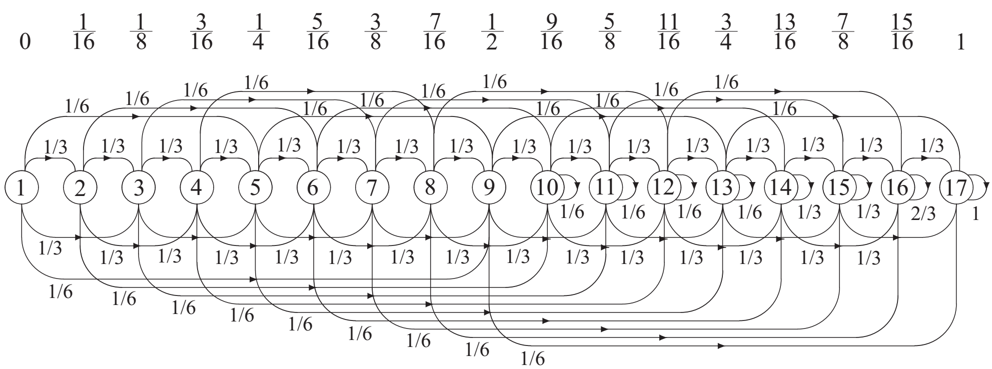

Figure 1 Cover-up state diagram. State 1 is position 0, state 2 is position $\frac{1}{16}$, state 3 is position $\frac{1}{8}, \ldots,$ state i is position $(i-1) = 16, \ldots,$ state 17 is position 1.

The state diagram can be used to produce the transition matrix $\boldsymbol{T} = \{T_{i,j}\}$ for the Markov chain, where

$$T_{i,j} = \mathrm{P}[X_{n+1} = i \mid X_n = j].$$

Note that $T_{i,j}$ is the one-step probability of moving from state j to state i.

The transition matrix $\boldsymbol{T}$ for this chain is

$$\boldsymbol{T} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{3} & 1
\end{bmatrix}.$$

(Our form for the transition matrix $T_{i,j}$ will follow that given in the text by Mooney and Swift (reference 1). Some references on Markov chains write the transition matrix as the transpose of this matrix.) With our notation, $T_{17,15} = \frac{1}{3}$ since from state 15 (position $\frac{7}{8}$), with probability $\frac{1}{3}$, we roll an $\frac{1}{8}$ and reach state 17 (position 1). Each column sums to 1, so that from each state we are accounting fully for the possible outcomes of a roll of the die.

Examining the structure of the transition matrix $\boldsymbol{T}$, we see that it can be decomposed into blocks according to the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{A}_{16\times 16} & \boldsymbol{O}_{16\times 1} \\ \boldsymbol{B}_{1\times 16} & \boldsymbol{I}_{1\times 1} \end{bmatrix}.$$

This decomposition is always possible for absorbing Markov chains. In general, if a Markov chain has a absorbing states and b nonabsorbing states, we can arrange the transition matrix (by interchanging rows and/or columns as necessary) to have the form

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{A}_{b\times b} & \boldsymbol{O}_{b\times a} \\ \boldsymbol{B}_{a\times b} & \boldsymbol{I}_{a\times a} \end{bmatrix}.$$

This block decomposition is useful for computations. We can multiply $\boldsymbol{T}$ by itself algebraically by treating the blocks just as if they were numbers and the matrix as if it were a 2×2 matrix and obtain

$$\boldsymbol{T}^2 = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{B} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{B} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^2 & \boldsymbol{O} \\ \boldsymbol{BA} + \boldsymbol{B} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^2 & \boldsymbol{O} \\ \boldsymbol{B}(\boldsymbol{I} + \boldsymbol{A}) & \boldsymbol{I} \end{bmatrix}.$$

Repeating this will give

$$\boldsymbol{T}^3 = \begin{bmatrix} \boldsymbol{A}^3 & \boldsymbol{O} \\ \boldsymbol{B}(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2) & \boldsymbol{I} \end{bmatrix},$$

and, in general, we will have

$$\boldsymbol{T}^n = \begin{bmatrix} \boldsymbol{A}^n & \boldsymbol{O} \\ \boldsymbol{B}(\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{n-1}) & \boldsymbol{I} \end{bmatrix}.$$

The sum of matrices in the lower left-hand block entry,

$$\boldsymbol{B}(\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{n-1}),$$

is reminiscent of a geometric series of real numbers. It turns out that since one can show that the largest absolute value of the eigenvalues of $\boldsymbol{A}$ is less than 1, we have

$$\boldsymbol{B}(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots) = \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{A})^{-1}.$$

This gives that as, $n \to \infty$,

$$\boldsymbol{T}^n \to \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{A})^{-1} & \boldsymbol{I} \end{bmatrix}.$$

The matrix

$$\boldsymbol{F}_{b\times b} = (\boldsymbol{I}_{b\times b} - \boldsymbol{A}_{b\times b})^{-1}$$

is known as the *fundamental matrix* and it contains information about the average time to absorption. The entries $f_{i,j}$ of this matrix are the average number of times the process is in state i, given that it began in state j. A detailed discussion of this and related ideas can be found in the mathematical modeling text by Mooney and Swift (reference 1).

For the cover-up game, the fundamental matrix is

$$\boldsymbol{F} = (\boldsymbol{I}_{16,16} - \boldsymbol{A})^{-1}$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.401 & 0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.276 & 0.401 & 0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.300 & 0.276 & 0.401 & 0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.235 & 0.300 & 0.276 & 0.401 & 0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.412 & 0.235 & 0.300 & 0.276 & 0.401 & 0.259 & 0.444 & 0.333 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.381 & 0.494 & 0.282 & 0.360 & 0.331 & 0.481 & 0.311 & 0.533 & 0.4 & 1.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.466 & 0.413 & 0.513 & 0.306 & 0.382 & 0.363 & 0.502 & 0.347 & 0.56 & 0.48 & 1.2 & 0 & 0 & 0 & 0 & 0 \\
0.437 & 0.512 & 0.440 & 0.547 & 0.337 & 0.427 & 0.392 & 0.552 & 0.384 & 0.672 & 0.48 & 1.2 & 0 & 0 & 0 & 0 \\
0.524 & 0.469 & 0.530 & 0.463 & 0.568 & 0.368 & 0.447 & 0.426 & 0.578 & 0.461 & 0.672 & 0.48 & 1.2 & 0 & 0 & 0 \\
0.645 & 0.714 & 0.620 & 0.706 & 0.618 & 0.768 & 0.497 & 0.622 & 0.581 & 0.866 & 0.576 & 0.84 & 0.6 & 1.5 & 0 & 0 \\
0.776 & 0.764 & 0.804 & 0.726 & 0.800 & 0.742 & 0.847 & 0.611 & 0.719 & 0.784 & 0.924 & 0.66 & 0.9 & 0.75 & 1.5 & 0 \\
1.757 & 1.884 & 1.782 & 1.905 & 1.716 & 1.945 & 1.707 & 2.009 & 1.492 & 1.986 & 1.74 & 2.1 & 1.5 & 2.25 & 1.5 & 3
\end{bmatrix},$$

where we have rounded values to three decimal places. The sum of the entries of the jth column of the fundamental matrix $\boldsymbol{F}$ is the average number of time steps for a process initially in state j to be absorbed. For example, looking at column 16 in the fundamental matrix $\boldsymbol{F}$,

Table 1 Average number of plays until absorption in state 17 (position 1).

Current state	Position	Average number of steps to absorption
1	0	8.645
2	$\frac{1}{16}$	8.500
3	$\frac{1}{8}$	7.984
4	$\frac{3}{16}$	7.725
5	$\frac{1}{4}$	7.189
6	$\frac{5}{16}$	7.130
7	$\frac{3}{8}$	6.481
8	$\frac{7}{16}$	6.434
9	$\frac{1}{2}$	5.714
10	$\frac{9}{16}$	6.449
11	$\frac{5}{8}$	5.592
12	$\frac{11}{16}$	5.28
13	$\frac{3}{4}$	4.2
14	$\frac{13}{16}$	4.5
15	$\frac{7}{8}$	3
16	$\frac{15}{16}$	3

if we are in position $\frac{15}{16}$ (state 16), in the future we will spend zero time in earlier states. The only roll of the die that will allow us to move is a roll of $\frac{1}{16}$, and this roll will cause us to reach position 1 (state 17). A roll of $\frac{1}{16}$ occurs with probability $\frac{1}{3}$, and so the expected number of rolls until we can move is 3.

Table 1 shows the sum of each of the columns of the fundamental matrix $\boldsymbol{F}$. Starting at state 1, which corresponds to position 0, we see that it takes an average of $\sum_{i=1}^{16} f_{i,1} \approx 8.645$ plays to complete this game.

We performed a simulation of the cover-up game in which we 'played the game' one million times. The resulting distribution of the number of steps to win is shown in figure 2. As expected, the distribution for the number of moves has a long tail with a standard deviation of roughly $\sigma = 3.48$. Unusual features, such as the decrease in frequency from two steps to three steps, can be confirmed. For instance, the probability of finishing the traditional game in two steps is $\frac{1}{36}$ (a roll of $\frac{1}{2}$ twice in a row), whereas the probability of finishing the game in three steps is $\frac{1}{72}$ (a roll of $\frac{1}{2}$ and two rolls of $\frac{1}{4}$, in any order).

We close this article by proposing four problems for further investigation. The first problem details two popular variations of cover-up; to assist readers, the transition matrix for each variation has been provided.

Problem 1 We propose two variations of the cover-up game. In variation 1, we do not require a player to land exactly on position 1, but rather allow for a winning roll to overshoot position 1.

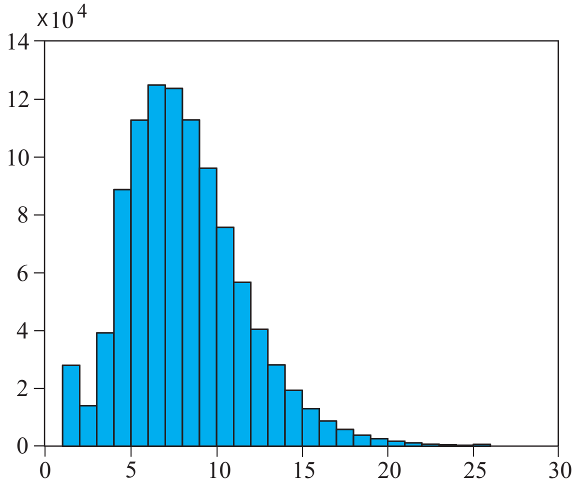

Figure 2 Distribution of the number of steps to win, based on simulations of one million games of cover-up.

In variation 2, we penalize a roll that would overshoot position 1 by requiring the player to move backward by the amount of that roll. To highlight the differences in these variations, suppose that a player is at position $\frac{3}{4}$ and rolls a $\frac{1}{2}$. The player remains at position $\frac{3}{4}$ in the traditional game, wins the game by advancing past position 1 in variation 1, and moves $\frac{1}{2}$ unit back to position $\frac{1}{4}$ in variation 2. Calculate the average number of rolls to win in each of these variations. When comparing these variations with the traditional game, one might guess that players will finish sooner with variation 1 and will take longer to finish with variation 2. Show that this is indeed the case.

Problem 2 An extraordinarily long version of this game could be played if we made another variation in which we identified position 0 with the winning position in a circular state diagram. We assume that an overshoot forces the player to continue playing from the new position. For example, a player at position $\frac{7}{8}$ that rolls a $\frac{1}{2}$ will then advance to position $\frac{3}{8}$. Calculate the average number of rolls to win in this variation.

Problem 3 Variation 1 has a fundraising interpretation: when an organization is trying to raise funds during a telethon, they would like to hit their goal, but overshooting it is fine too! Contact an organization that will give you their donation levels and number of donors at each level. Estimate how many days they will need to hold their telethon next year in order to reach a new goal. Suggest potential strategies to make the number of days less.

Problem 4 Some family board games work like the traditional cover-up game, while others are like variation 1. Consider the example of the board game *Chutes and Ladders* in which the winner must land exactly on the last square, as in the traditional cover-up game. Use the layout of the board to write down the 100×100 matrix for this game, and find the average number of steps to win the game. How much does this answer change if we use variation 1? How much does this answer change if we use variation 2?

References

1 D. D. Mooney and R. J. Swift, *A Course in Mathematical Modeling* (The Mathematical Association of America, Washington, DC, 1999).
2 S. M. Ross, *Introduction to Probability Models*, 7th edn. (Academic Press, Boston, MA, 2000).

***Randall Swift**, **Jennifer Switkes** and **Stephen Wirkus** are Professors of Mathematics and Statistics at California State Polytechnic University, Pomona, where they enjoy each others' company and working on fun problems together. The results of their semi-regular meetings have often produced solutions to many problems, not always mathematical in nature.*

Divisibility tests

There are some well-known divisibility tests such as the one which says that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Thus, 2010 is divisible by 3 because $2+0+1+0$ is. The following tests are less well known.

Divisibility by 7

A number is divisible by 7 if the number obtained by omitting the units digit less double this digit is a multiple of 7. For example, 392 is divisible by 7 because $39-4=35$ is also divisible by 7. Also, a number is divisible by 7 if the difference of the number without the last two digits and three times those two digits is a multiple of 7. For example, 38549 is divisible by 7 because $385-147=238$ is.

Divisibility by 13

A number is divisible by 13 if the sum of the number without the units digit and four times this digit is a multiple of 13. For example, 897 is divisible by 13 because $89+28=117$ and $11+28=39$ are. Similarly, a number is divisible by 13 if the sum of the number without the last two digits plus three times those two digits is a multiple of 13. For example, 4927 is divisible by 13 because $49+81=130$ is.

Divisibility by 17

A number is divisible by 17 if the difference between the number without the units digit and five times this digit is a multiple of 17. For example, 1513 is divisible by 17 because $151-15=136$ and $13-30=-17$ are. Similarly, a number is divisible by 17 if the sum of the number without the last two digits plus eight times those two digits is a multiple of 17. For example, 12 410 is divisible by 17 because $124+80=204$ and $2+32=34$ are.

Divisibility by 19

A number is divisible by 19 if the sum of the number without the units digit and two times this digit is a multiple of 19. For example, 608 is divisible by 19 because $60+16=76$ and $7+12=19$ are. Similarly, a number is divisible by 19 if the sum of the number without the last two digits plus four times those two digits is a multiple of 19. For example, 11 761 is divisible by 19 because $117+244=361$ is.

The checking process may be repeated on the reduced figures until a number is arrived at which is recognised as a multiple of the divisor being tested.
Divisibility by larger numbers is also possible.
Can the reader prove these results?

Midsomer Norton, Bath **Bob Bertuello**

Triangles and Parallelograms of Equal Area Inside the Hyperbola

ADAM M. ROMASKO and THOMAS J. OSLER

Introduction

In a previous article (see reference 1), an unusual property of the ellipse was examined. In this article we continue the study begun in reference 1 by extending that property to the hyperbola. This work was motivated by studying a paper by Euler (see reference 2), in which he found general curves that shared properties of the conic sections. In the eighteenth century, mathematicians routinely studied many properties of these curves which are no longer standard tools of a mathematician.

First we review the problem examined in reference 1. We define a few new terms, *diameters, reciprocal diameters,* and *reciprocal points*, in an ellipse.

The diameter of an ellipse is any chord that passes through the centre. In figure 1 MM' and mm' are diameters. Now start with any diameter MM'. We say that the diameter mm' is *reciprocal* to diameter MM' if it is parallel to the tangent line to the ellipse at M. If we started with diameter mm' then MM' would be the reciprocal diameter. We say that the points m and m' are reciprocal to the point M. Euler assumed that his readers were familiar with reciprocal diameters and points. He also assumed that his readers would be aware that the area of the parallelogram, $MmM'm'$, is constant, regardless of the choice of the initial diameter, and equal to $2ab$. The area of the triangles CMm and CMm' are also constant and equal $ab/2$.

From ellipse to hyperbola

We now try to change the problem of finding triangles of equal area inside an ellipse to finding such triangles inside a hyperbola. In figure 2 we see the standard hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

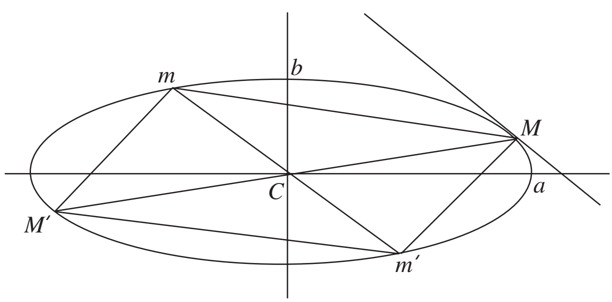

Figure 1

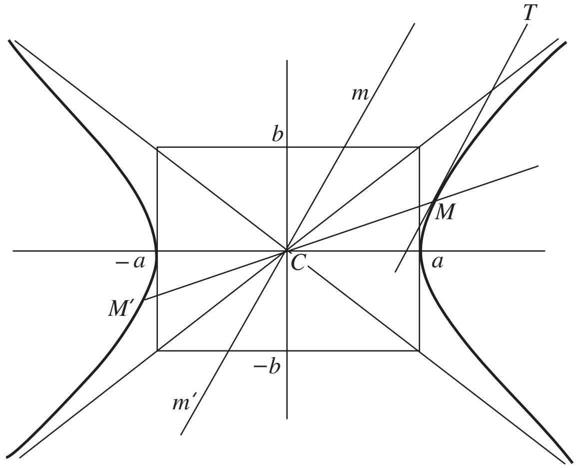

Figure 2

The line MM' is a diameter, since it is a chord of the hyperbola that passes through its centre. The tangent at point M is MT. Now we try to construct a reciprocal diameter mCm' as a chord through the centre parallel to the tangent MT. We see this line in figure 2, but there is a problem! The line cannot form a chord since it never intersects the hyperbola!

It seems natural to try adding the related curve

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \tag{2}$$

as seen in figure 3. Both (1) and (2) share the same centre and asymptotes. Now we have the diameter MM' as well as a reciprocal diameter mm'. We can ask the critical question: is the area of the triangle CMm the same number, regardless of the slope of the line CM?

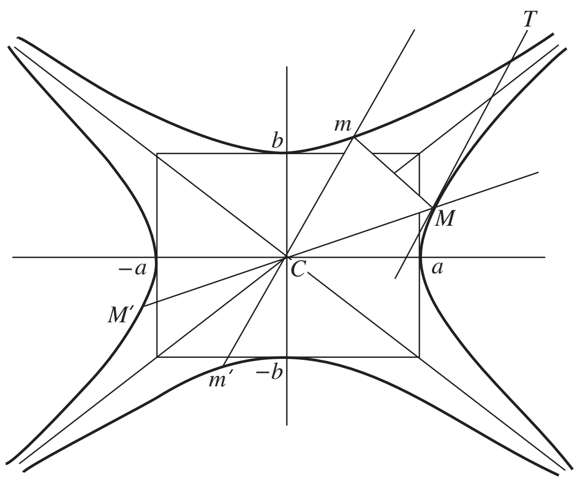

Figure 3

Parametric equations for the ellipse and the hyperbola

The usual equation for the ellipse, $x^2/a^2 + y^2/b^2 = 1,$ can also be written in parametric form using trigonometric functions

$$x = a\cos\theta \quad \text{and} \quad y = a\sin\theta.$$

This follows at once from the identity $\sin^2\theta + \cos^2\theta = 1.$

In a similar way we can write parametric equations for the hyperbola using the less familiar hyperbolic functions

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}.$$

The parametric equations for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ now become

$$x = a\cosh\phi \quad \text{and} \quad y = b\sinh\phi, \tag{3}$$

and these follow immediately from the identity $\cosh^2\phi - \sinh^2\phi = 1.$ Since the function $\cosh\phi \geq 1,$ the above equations only describe the right branch of our hyperbola. To describe the left branch, we need the second pair of parametric equations

$$x = -a\cosh\phi \quad \text{and} \quad y = b\sinh\phi,$$

Note that on both branches of our hyperbola, when the parameter $\phi = 0,$ we are at the x-axis. As the parameter increases from 0, the curve rises above the x-axis. As the parameter decreases from 0, the curve descends below the axis.

The parametric equations for the hyperbola $y^2/b^2 - x^2/a^2 = 1$ are

$$x = a\sinh\phi \quad \text{and} \quad y = b\cosh\phi \tag{4}$$

for the upper branch of the hyperbola. The lower branch is described by

$$x = a\sinh\phi \quad \text{and} \quad y = -b\cosh\phi.$$

We will also need the derivative formulae

$$\frac{\mathrm{d}(\sinh x)}{\mathrm{d}x} = \cosh x \quad \text{and} \quad \frac{\mathrm{d}(\cosh x)}{\mathrm{d}x} = \sinh x.$$

Finding reciprocal points m and M

We begin by finding the coordinates of the reciprocal point m from the coordinates of the given point M in figure 3. Call the coordinates of $M(x_1, y_1)$ and the coordinates of the reciprocal point $m(x_2, y_2)$. Thus, we have, from the parametric equations of the hyperbola (3),

$$x_1 = a\cosh\phi_1 \quad \text{and} \quad y_1 = b\sinh\phi_1, \tag{5}$$

where ϕ_1 is the value of the parameter that identifies point M.

Next we must find the slope of the tangent line at the point M. From (3) we obtain

$$\mathrm{d}x = a\sinh\phi\,\mathrm{d}\phi \quad \text{and} \quad \mathrm{d}y = b\cosh\phi\,\mathrm{d}\phi,$$

so the slope of the tangent line at M is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_M = \frac{b\cosh\phi_1}{a\sinh\phi_1}. \tag{6}$$

The equation of the line Cm has slope given by (6), and, therefore, the coordinates of point m must satisfy

$$\frac{y_2}{x_2} = \frac{b\cosh\phi_1}{a\sinh\phi_1}, \tag{7}$$

as well as the parametric equations of the hyperbola, (4), i.e.

$$x = a\sinh\phi \quad \text{and} \quad y = b\cosh\phi. \tag{8}$$

Comparing (7) and (8) we see the happy fact that the point m has the same parameter $\phi = \phi_1$ as does point M. Thus, the coordinates of point m are

$$x_2 = a\sinh\phi_1 \quad \text{and} \quad y_2 = b\cosh\phi_1. \tag{9}$$

Area of a triangle with one vertex at the origin

We now show how to find the area of the triangle CMm in figure 3 from the coordinates of its vertices $C(0,0)$, $M(x_1, y_1)$, and $m(x_2, y_2)$ shown in figure 4. We see that

$$\text{Area}\, CMm = \text{Area}\, CAm + \text{Area}\, ABMm - \text{Area}\, CBM = \frac{x_2y_2}{2} + \left(\frac{y_1+y_2}{2}\right)(x_1 - x_2) - \frac{x_1y_1}{2},$$

which simplifies to

$$\text{Area}\, CMm = \frac{x_1y_2 - x_2y_1}{2}. \tag{10}$$

Area of triangle CMm is constant

We are at last ready to prove that the area of triangle CMm is constant. Replacing x_1, y_1, x_2, and y_2 in (10) by (5) and (9) we obtain

$$\text{Area}\, CMm = \frac{1}{2}(a\cosh\phi_1 b\cosh\phi_1 - a\sinh\phi_1 b\sinh\phi_1) = \frac{ab}{2}(\cosh^2\phi_1 - \sinh^2\phi_1) = \frac{ab}{2}.$$

This shows that the area of the triangle CMm is independent of the slope of the line CM. It is interesting to note that in the previous article (see reference 1), where the ellipse $x^2/a^2 + y^2/b^2 = 1$ was used instead of the hyperbola, the same formula for the area of the triangles generated by reciprocal diameters was found.

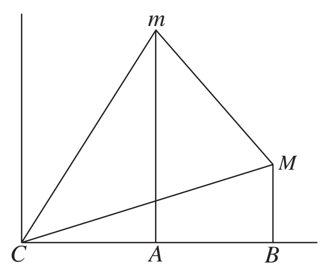

Figure 4

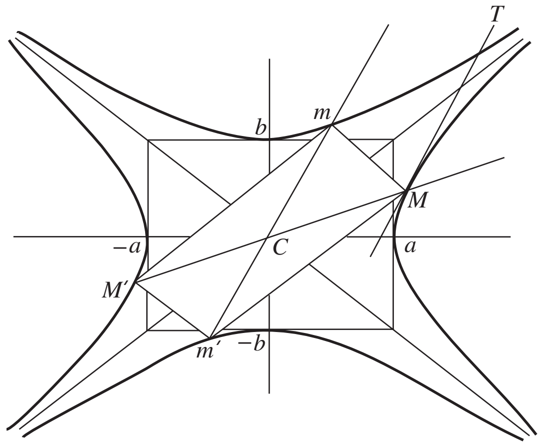

Figure 5

Parallelograms of constant area

In this section we remark that the parallelogram $MmM'm'$ shown in figure 5 also has constant area. We have shown that the triangle CMm always has area $ab/2$ when constructed using reciprocal points. Of course, the triangle $CM'm'$ has the same area.

Now suppose that we started our discussion by drawing the semi-diameter CM' rather than CM. After finding the slope and M', we would discover the reciprocal point m. The computations that follow would give us the area of triangle $CM'm$, rather than triangle CMm. These computations are nearly identical to those shown above and result in the same value for the area, $ab/2$. Thus, the area of the parallelogram $MmM'm'$ is the constant value $2ab$.

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2 E. Greve and T. J. Osler, Translation with notes of Euler's paper E83: On some properties of conic sections that are shared with infinitely many other curved lines. Available at http://www.math.dartmouth.edu/∼euler/.

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A Farmer and a Fence

PRITHWIJIT DE

1. Introduction

A farmer is facing the task of dividing his land, which is in the shape of a circle, into n regions of equal area by fencing its interior. He has with him fences of two different shapes: *straight lines* and *circles*. The choice of circle as a shape is motivated by the fact that among all closed curves in the plane enclosing a given area the circle has the least perimeter. We assume that the boundary of the land is fenced to begin with and we are only interested in minimizing the cost of fencing the new regions. The cost per unit length of fence is assumed to be 1 unit and, hence, cost minimization is equivalent to minimizing the total length of the fence. The most common approach is to dissect the field along n radial directions emanating from the centre of the land such that any pair of adjacent radii subtend equal angles at the centre. Is this the most economical option among all conceivable schemes composed of linear and circular fences for every value of n or is there a better option? In this article we will consider two basic dissecting schemes, compare the lengths of the fences under them, and construct schemes which are more efficient than these two. We will conclude the article by posing an optimization problem.

2. Dissection schemes

The first scheme, S_1, consists of dissecting the land into n sectors such that each sector subtends an equal angle, $2\pi/n$, at the centre. The second scheme, S_2, consists of dividing the land by $(n-1)$ concentric circles such that the resulting regions have the same area. For the sake of clarity, we present an illustration of the schemes in figure 1 when $n=4$.

3. Comparison of the lengths

Let the radius of the land be one kilometre. The total length of the fence under S_1 is $L_1(n)=n$, and the same under S_2 can be calculated as follows. Let C_k, $1\le k\le n-1$, be the family of concentric circles labelled from the centre outwards, and let r_k be the radius of C_k. If A_1 is the area enclosed by C_1 then $A_1=\pi r_1^2$. We also know that the area A_k enclosed by C_{k-1}

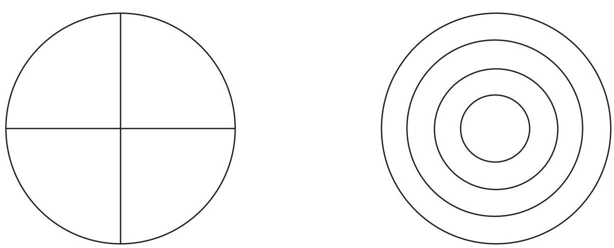

Figure 1 Dividing a land into four pieces according to S_1 and S_2.

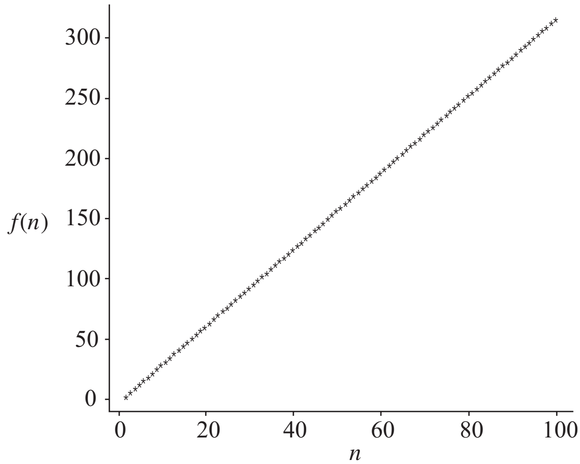

Figure 2 Plot of $f(n)$ against n.

and C_k, where $2 \le k \le n-1$, is A_1. Thus, we can write $\pi(r_k^2 - r_{k-1}^2) = \pi r_1^2$. Cancelling the common factor π from both sides gives

$$r_k^2 - r_{k-1}^2 = r_1^2$$

for $2 \le k \le n-1$. Let $r_n = 1$. Then

$$r_n^2 - r_{n-1}^2 = r_1^2.$$

Adding the $(n-1)$ equations described above we obtain $nr_1^2 = 1$, which gives $r_1 = \sqrt{1/n}$, and $r_k = \sqrt{k/n}$ for $2 \le k \le n-1$. The total length of the fence is $L_2(n) = 2\pi \sum_{k=1}^{n-1} \sqrt{k/n}$. How does $L_2(n)$ compare with $L_1(n)$? We write

$$f(n) = L_2(n) - L_1(n) = 2\pi \sum_{k=1}^{n-1} \sqrt{\frac{k}{n}} - n. \tag{1}$$

Exact computation of the expression on the right-hand side of (1) is cumbersome and a numerical estimate is a natural alternative. The first step in a numerical estimation is to fix a range of the parameter n. To convince ourselves of the sign of $f(n)$, we need to study the behaviour of $f(n)$ as n varies over a considerable range. A reasonable range is between 2 and 100. Figure 2 shows the relationship between $f(n)$ and n for $2 \le n \le 100$. We see that $f(n)$ increases with n in an approximately linear fashion. The slope of the line is a little more than 3, showing that S_2 is worse than S_1 by a factor slightly greater than 3.

4. Marriage of scheme 1 and scheme 2

The analysis so far clearly highlights the fact that S_1 is better than S_2. But the question is whether a scheme better than S_1 can be found. One way of developing a new scheme is to combine the two existing schemes. But how should we set about solving the problem. Let us revisit figure 1 and try to marry the two schemes. Assume, for the moment, that we are supposed to make four slices.

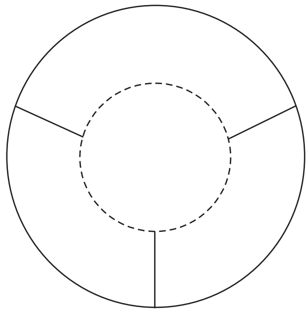

Figure 3 Dissecting a piece of land into four pieces under a scheme obtained by marrying S_1 and S_2.

Figure 3 illustrates a possible way of combining S_1 and S_2.

Step 1. Imagine dividing the land into three equal parts by drawing three radii. The angle between any two radii is 120º.

Step 2. Draw a circle with the centre of the land as its centre and a radius which has to be determined subject to the constraint that the pieces are of equal area.

The length of the fence can be ascertained in the following manner. As the regions are of equal area, the area of the circle is one quarter of the area of the land, implying that its radius is $\frac{1}{2}$ kilometre. Thus, the length of the fence is $\pi + \frac{3}{2} = 4.6415$ kilometres, slightly greater than 4 kilometres, the length of the fence under S_1. If instead of four regions we were to dissect it into n regions using the new scheme then the total length of the fence would be

$$L_3(n) = 2\pi\sqrt{\frac{1}{n}} + (n-1)\left(1 - \sqrt{\frac{1}{n}}\right).$$

Is $L_3(n)$ less than $L_1(n)$? Let us consider the difference $L_3(n) - L_1(n)$. Call it $g(n)$. How does $g(n)$ change with n? Figure 4 depicts the relationship.

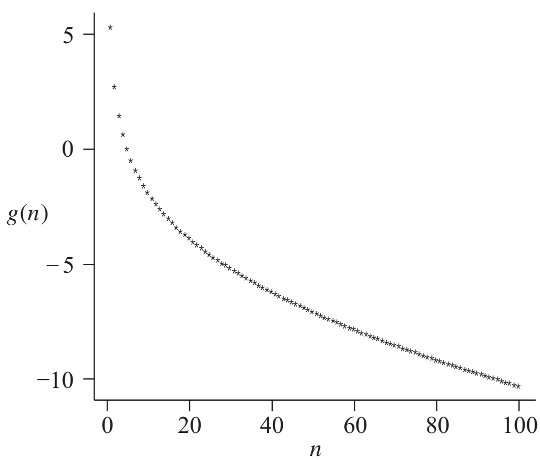

Figure 4 Plot of $g(n)$ against n.

Observe that $g(n) = (2\pi + 1)/\sqrt{n} - \sqrt{n} - 1$ and if $h(n) = g(n) + 1$ then it is easy to see that

$$|h(n)| \leq \sqrt{n} + \frac{2\pi + 1}{\sqrt{n}} \leq 2(\pi + 1)\sqrt{n}. \tag{2}$$

We see from figure 4 that the new scheme is better than S_1 for $n > 5$. The reduction in total length of the fence brought about by the new scheme increases with n. This behaviour can be explained heuristically. As n increases, the number of radial fences under S_1 increases too. The circular fence under the new scheme removes a fraction of the length of each radius and contributes an arc, shorter than the total length removed, leading to a net reduction in the total length. Also, (2) shows that the decrease in length is $O(\sqrt{n})$.

The next question that naturally comes to our minds is: could this new scheme be improved by introducing more circular fences if we want to dissect the land into more than five pieces? In view of S_2 we observe that the required number of circular fences has to be less than $(n-1)$ and, by the preceding comparison, it has to be at least 1. Therefore, for a given n, we may try to find a number $k(n)$ of circular fences, which would minimize the total length. Let $S_{3,k}$ be the scheme in which k circular fences are erected, where $1 \leq k \leq n-2$. The total length of the fence is given by

$$L_{3,k}(n) = 2\pi \sum_{j=1}^{k} \sqrt{\frac{j}{n}} + (n-k)\left(1 - \sqrt{\frac{k}{n}}\right). \tag{3}$$

Let

$$d(k, n) = L_{3,k}(n) - L_1(n).$$

We will first determine values of $k(n)$ for a series of values of n. For any particular n, $k(n)$ is that value of k in (3) for which the expression attains a minimum value. We allow n to vary between 5 and 100. Figure 5(a) displays the graph of $L_{3,k}(n)$ for a particular n and $1 \leq k \leq n-2$, and figure 5(b) depicts the behaviour of $k(n)$ over the range of n. The percentage reduction obtained by erecting $k(n)$ circular fences as a function of n is shown in figure 6.

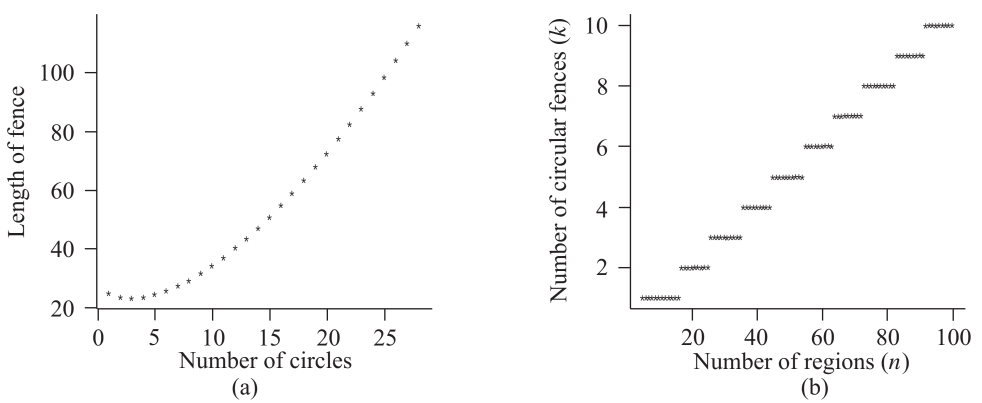

Figure 5 (a) Length of the fence under scheme 3 corresponding to $n = 30$. The minimum number of circular fences needed is 3. (b) $k(n)$ as a function of the number of regions.

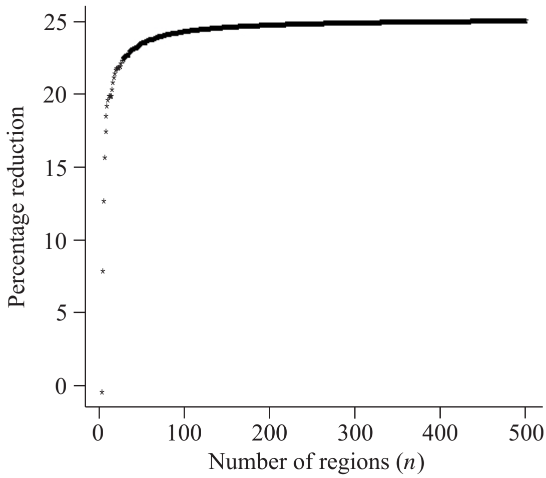

Figure 6 Percentage reduction obtained by erecting $k(n)$ circular fences as a function of the number of regions n.

Figure 5(b) shows that the number of circular fences increases with the number of regions in a stepwise fashion and, with the increase in the number of regions (n), the removal of the linear portions is under-compensated by the addition of circular fences.

The rate of increase of the percentage reduction is very sharp for small values of n but declines rapidly as n increases. The trend conveys an interesting message. This says that, for large values of n, replacing linear fences by circular fences does not reduce the total cost beyond the 25% margin, as indicated in figure 6.

5. A new scheme and an interesting problem

Now we introduce scheme S_4. In $\{S_{3,k} : 1 \le k \le n-2\}$ we put linear fences between the last circular fence and the boundary of the field. A new scheme may be conjured by modifying them in the following manner. Assume for the moment that $k = 2$.

1. Leave the innermost circular fence in $S_{3,2}$ as it is.

2. Erect m linear fences between the first and second circular fences.

3. Erect $n - m - 1$ linear fences between the second circular fence and the boundary.

See figure 7 for an illustration.

Since the regions are of the same area, the radius of the innermost circular fence is $r_1 = 1\sqrt{n}$. If r_2 is the radius of the second fence then we should have $\pi(r_2^2 - r_1^2) = m\pi r_1^2$, which gives $r_2 = \sqrt{(m+1)/n}$. The total length of the fence turns out to be

$$L_{4,2}(m,n) = 2\pi\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{m+1}{n}}\right) + m\left(\sqrt{\frac{m+1}{n}} - \sqrt{\frac{1}{n}}\right) + (n-m-1)\left(1 - \sqrt{\frac{m+1}{n}}\right).$$

Since m and n are positive integers, it is easier to find out the minimum using numerical methods rather than by algebraic methods, which may turn out to be cumbersome. But the more important question is the following.

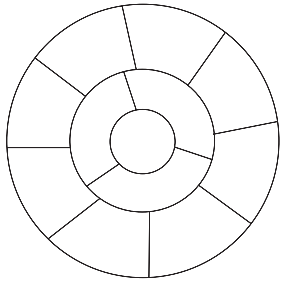

Figure 7 An illustration of the scheme $S_{3,2}$ with $m = 3$ and $n = 12$.

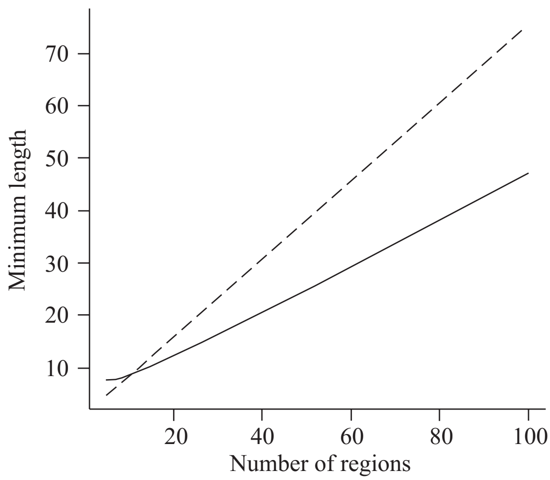

Figure 8 Minimum lengths under the new scheme (*solid line*) and under $\{S_{3,k} : 1 \le k \le n-2\}$ (*dashed line*).

Question Fix a value of n. Is the minimum length obtained under S_4 less than that obtained under $\{S_{3,k} : 1 \le k \le n-2\}$?

To answer this question, we need to determine the minimum value of $L_{4,2}(m, n)$ and compare it with the minimum value of the same under S_4. Figure 8 shows that the minimum length under the new scheme (solid line) is less than that obtained under $\{S_{3,k} : 1 \le k \le n-2\}$, except for small values of n. Also, the difference increases with n, which is indicative of the fact that S_4 is more efficient for large values of n. Note that in S_4 we have only two circular fences. Can the length be made smaller by adding more circular fences to this scheme?

Let k be arbitrary and greater than 2. Suppose that m_j linear fences are erected between the jth and $(j+1)$th circular fences, where $1 \le j \le k-1$, and let m_k be the number of fences between the kth circular fence and the boundary of the field. The radius of the first circle is $r_1 = 1\sqrt{n}$. The radii of the remaining circles may be calculated by the following system of equations:

$$r_{j+1}^2 - r_j^2 = m_j r_1^2, \qquad 1 \le j \le k-1. \tag{4}$$

Solving (4) we obtain for

$$r_{j+1} = \sqrt{\frac{m_1 + m_2 + \cdots + m_j + 1}{n}} \quad \text{for } 1 \le j \le k-1.$$

The length of the fence turns out to be

$$L = 2\pi \left(\sqrt{\frac{1}{n}} + \sum_{j=1}^{k-1} \sqrt{\frac{1 + \sum_{l=1}^{j} m_l}{n}} \right) + m_1 \left(\sqrt{\frac{m_1 + 1}{n}} - \sqrt{\frac{1}{n}} \right)$$
$$+ \sum_{j=2}^{k-1} m_j \left(\sqrt{\frac{1 + \sum_{l=1}^{j} m_j}{n}} - \sqrt{\frac{1 + \sum_{l=1}^{j-1} m_l}{n}} \right) + m_k \left(1 - \sqrt{\frac{1 + \sum_{j=1}^{k-1} m_j}{n}} \right). \quad (5)$$

In (5) L is actually a function of $k, n, m_1, m_2, \ldots, m_k$, and the problem of minimizing L for given values of k and n is equivalent to a multivariable optimization problem subject to the following constraints:

(i) $m_1, \ldots, m_k$ are integers between 1 and n,

(ii) $m_1 + m_2 + \cdots + m_k = n - 1$.

In view of (ii) the minimization of L for known k and n involves only $m_1, \ldots, m_{k-1}$. The reader is invited to propose a solution to this seemingly difficult optimization problem.

6. Conclusion

In this article we began with two elementary fencing schemes and along the way introduced new schemes, which are essentially ramifications of combinations of the two basic schemes we started with. For small values of n, S_1 is more efficient than the rest of the schemes introduced. For large values of n, the most efficient scheme according to us is S_4, but this is probably not the best because the solution to the unsolved optimization problem posed in the previous section may yield a better answer.

The problem discussed in this article is indeed a practical one. The choice of the shapes of the fences had to be restricted to basic geometric shapes to avoid mathematical complexities and it was common sense rather than mathematical concepts that led to the choice of the two initial schemes, S_1 and S_2. But the beauty lay in experimenting with the schemes in order to derive a better one. This also gave rise to an unsolved optimization problem. Thus, we see that, often, simple real-life problems may lead to the exploration of interesting mathematical ideas through experimentation and can serve as origins of research problems.

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A Generalized Pell Triangle

THOMAS KOSHY

Pell numbers, P_n, and Pell–Lucas numbers, Q_n, are often defined recursively:

$$P_1 = 1, \qquad P_2 = 2, \qquad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3.$$
$$Q_1 = 1, \qquad Q_2 = 3, \qquad Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 3.$$

They can also be defined explicitly using the Binet-like formulae:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \qquad \text{and} \qquad Q_n = \frac{\gamma^n + \delta^n}{2},$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the solutions of the equation $x^2 = 2x + 1$ (see reference 1). Note that $\gamma\delta = -1$. Using these formulae, it can be shown that

$$P_m + Q_m = P_{m+1}, \qquad Q_{m-1} + Q_m = 2P_m,$$
$$2\sum_{i=1}^{k} P_i = Q_{k+1} - 1, \qquad \sum_{i=1}^{k} Q_i = P_{k+1} - 1.$$

The Pell families can also be extended to zero and negative subscripts. For example, $P_0 = 0$, $Q_0 = 1$, and $P_{-1} = 1$.

In this article we will establish some interesting relationships involving the Pell families and the triangular numbers

$$t_k = \frac{k(k+1)}{2},$$

where $k \geq 1$. To this end, first we define the *generalized Pell numbers*, G_n, recursively:

$$G_1 = a, \qquad G_2 = b, \qquad G_n = 2G_{n-1} + G_{n-2}, \quad n \geq 3,$$

where a and b are arbitrary integers. It can be shown by strong induction that

$$G_n = P_{n-1}b + P_{n-2}a,$$

where $n \geq 1$.

Next, we construct a triangular array using the numbers G_n, as in figure 1. Let s_n denote

$$\begin{array}{ccccc} & & a & & \\ & b & & 2b+a & \\ 5b+2a & & 12b+5a & & 29b+12a \end{array}$$

Figure 1 A generalized Pell triangle.

the sum of the elements in the top n rows of the array. Then

$$\begin{aligned} s_n &= \sum_{i=1}^{t_n} G_i \\ &= \sum_{i=1}^{t_n} (P_{i-1}b + P_{i-2}a) \\ &= b\sum_{i=1}^{t_n-1} P_i + a + a\sum_{i=1}^{t_n-2} P_i \\ &= b\frac{Q_{t_n}-1}{2} + a + a\frac{Q_{(t_n-1)}-1}{2} \\ &= \frac{b(Q_{t_n}-1)+a(Q_{(t_n-1)}+1)}{2}. \end{aligned}$$

For example,

$$2s_3 = b(Q_6-1)+a(Q_5+1) = 98b+42a, \qquad s_3 = 49b+21a.$$

The sum r_n of the elements in row n is given by

$$\begin{aligned} 2r_n &= 2s_n - 2s_{n-1} \\ &= [b(Q_{t_n}-1)+a(Q_{t_n-1}+1)] - [b(Q_{t_{n-1}}-1)+a(Q_{(t_{n-1})-1}+1)] \\ &= b(Q_{t_n}-Q_{t_{n-1}}) + a(Q_{t_n-1}-Q_{(t_{n-1})-1}). \end{aligned}$$

In particular, let $a=1$ and $b=2$, so $G_n = P_n$. The sum S_n of the elements in the top n rows of the Pell numbers in figure 2 is given by

$$2S_n = 2(Q_{t_n}-1)+Q_{t_n-1}+1 = 2Q_{t_n}+Q_{t_n-1}-1,$$

and the nth row sum R_n is given by

$$2R_n = 2(Q_{t_n}-Q_{t_{n-1}}) + Q_{t_n-1} - Q_{(t_{n-1})-1}.$$

For example,

$$2S_4 = 2Q_{t_4}+Q_{t_4-1}-1 = 2Q_{10}+Q_9-1 = 2\cdot 3363+1393-1 = 8118,$$

				1				
			2		5			
		12		29		70		
	169		408		985		2378	
5741		13 860		33 461		80 782		195 025

Figure 2 A Pell triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 3 & & 7 & & & \\
 & & 17 & & 41 & & 99 & & \\
 & 239 & & 577 & & 1393 & & 3363 & \\
8119 & & 19\,601 & & 47\,321 & & 114\,243 & & 275\,807
\end{array}
$$

Figure 3 A Pell–Lucas triangle.

so $S_4 = 4059$; and

$$
\begin{aligned}
2R_5 &= 2(Q_{t_5} - Q_{t_4}) + Q_{t_5-1} - Q_{t_4-1} \\
&= 2(Q_{15} - Q_{10}) + Q_{14} - Q_9 \\
&= 2(275\,807 - 3363) + 114\,243 - 1393 \\
&= 657\,738,
\end{aligned}
$$

so $R_5 = 328\,869$.

Suppose that $a = 1$ and $b = 3$, so $G_n = Q_n$. Then the sum S'_n of the top n rows and the nth row sum of the Pell–Lucas array in figure 3 can be computed as follows:

$$
\begin{gathered}
2S'_n = 3(Q_{t_n} - 1) + Q_{t_n-1} + 1 = 2Q_{t_n} + (Q_{t_n} + Q_{t_n-1}) - 2 = 2Q_{t_n} + 2P_{t_n} - 2, \\
S'_n = P_{t_n+1} - 1,
\end{gathered}
$$

and

$$
\begin{aligned}
2R'_n &= 3(Q_{t_n} - Q_{t_{n-1}}) + Q_{t_n-1} - Q_{(t_{n-1})-1} \\
&= 2(Q_{t_n} - Q_{t_{n-1}}) + (Q_{t_n} + Q_{t_n-1}) - (Q_{t_{n-1}} + Q_{(t_{n-1})-1}) \\
&= 2(Q_{t_n} - Q_{t_{n-1}}) + 2P_{t_n} - 2P_{t_{n-1}}, \\
R'_n &= Q_{t_n} - Q_{t_{n-1}} + P_{t_n} - P_{t_{n-1}} \\
&= (P_{t_n} + Q_{t_n}) - (P_{t_{n-1}} + Q_{t_{n-1}}) \\
&= P_{t_n+1} - P_{t_{n-1}+1}.
\end{aligned}
$$

For example,

$$
S'_4 = P_{t_4+1} - 1 = P_{11} - 1 = 5740
$$

and

$$
R'_4 = P_{t_4+1} - P_{t_3+1} = P_{11} - P_7 = 5741 - 169 = 5572.
$$

Reference

1 M. Bicknell, A primer of the Pell sequence and related sequences, *Fibonacci Quart.* **13** (1975), pp. 345–349.

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A Mathematical Haystack Without Fifth Powers

M. A. NYBLOM

1. Introduction

A mathematical haystack was defined in reference 2 as an infinite sequence of positive integers that 'conceals' all but a finite set of prime numbers, where, for example, the concealed primes may be factors of the numbers listed. The nature of this concealment means that any sequence which includes or implies the primes either explicitly or as an obvious factor fails to satisfy the haystack condition. One, but not an immediately apparent, example of a mathematical haystack is the sequence of *repunit numbers*, which is registered in 'The On-Line Encyclopedia of Integer Sequences' under A002275. A repunit number, denoted by R_n, is represented by n 1s, $111\ldots11$. That this sequence of repunit numbers is a haystack can be established via the well-known result of elementary number theory known as Fermat's little theorem, which states that if p is prime and a, p are relatively prime, then $a^{p-1} \equiv 1(\operatorname{mod} p)$. In particular, we can apply this theorem to show that $p|R_{p-1}$ for every prime $p > 5$. Specifically, $10^{p-1} \equiv 1(\operatorname{mod} p)$ implies that $p|(10^{p-1} - 1)$, or $p|999\ldots99$, so that $p|111\ldots11$, or $p|R_{p-1}$, because p and 9 are coprime. In addition, as $3|R_3$, we conclude that every prime other than 2 or 5 is a divisor of some repunit number R_n, where $n > 2$. Hence, the sequence

$$1, 11, 111, 1111, 11111, 111111, 1111111, \ldots$$

conceals most of the primes as factors, and so is a mathematical haystack. Despite its seemingly innocuous appearance, many open problems abound in connection with the sequence of repunit numbers. A notoriously difficult problem related to the haystack condition is the existence of infinitely many prime repunits. At present, the only known prime repunit numbers are R_2, R_{19}, R_{23}, R_{317}, and R_{1031} (see reference 4). However, of interest to us here is perhaps a far more tractable problem concerning the nonexistence of perfect powers in the sequence of repunit numbers. Recall that a perfect power is a positive integer of the form m^n, $n \geq 2$. It is easy to show that every repunit R_n, $n > 1$, cannot be a square since an odd square must be congruent to $1(\operatorname{mod} 4)$, but clearly $R_n \not\equiv 1(\operatorname{mod} 4)$ for $n > 1$. Hence, no such repunit can be of the form m^n, where n is a positive even integer. To the author's knowledge, the only result connected with the nonexistence of odd exponent perfect power repunits was proved for the case of those exponents divisible by 3 (see references 1 and 3), and, moreover, required the use of the theory of p-adic numbers. It is thus rather surprising that we can now present the following purely elementary proof for the nonexistence of fifth power repunits greater than unity.

2. An elementary proof

Suppose that the equation $R_n = (10^n - 1)/9 = y^5$ has an integer solution (n, y), $n, y > 1$. First observe that y cannot be an even integer, so $y \equiv 9, 7, 5, 3,$ or $1(\operatorname{mod} 10)$, whence

$y^5 \equiv 9, 7, 5, 3,$ or $1 \pmod{10}$, respectively. Hence, as $R_n \equiv 1 \pmod{10}$, the equation $R_n = y^5$ can only be solvable if $y \equiv 1 \pmod{10}$. Writing $y = 10N + 1$, and expanding and rearranging the equation $R_n = (10^n - 1)/9 = (10N + 1)^5$, we find that

$$R_{n-1} = \frac{10^{n-1} - 1}{9} = 10^4 N^5 + \binom{5}{1} 10^3 N^4 + \binom{5}{2} 10^2 N^3 + \binom{5}{3} 10 N^2 + \binom{5}{4} N. \qquad (1)$$

The right-hand side of (1) must be divisible by 5, but the left-hand side of (1) is not. Thus, the equation $R_n = y^5$ has no integer solutions when $n, y > 1$ other than the trivial $(n, y) = (1, 1)$.

It is unfortunate that the above argument cannot be extended to other prime powers $p > 5$, thus leaving open the problem of the nonexistence of all perfect powers in the mathematical haystack of repunit numbers.

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Using all the digits

$$\begin{aligned} 9 &= \frac{75\,249}{8361} = \frac{58\,239}{6471} = \frac{57\,429}{6381}, \\ 100 &= 75 + 24 + \frac{3}{6} + \frac{9}{18} \\ &= 91 + \frac{7524}{836} = 91 + \frac{5742}{638} = 91 + \frac{5823}{647} \\ &= 94 + 5 + \frac{38}{76} + \frac{1}{2} = 94 + \frac{1578}{263} \\ &= 96 + \frac{2148}{537} = 96 + \frac{1428}{357} = 96 + \frac{1752}{438}. \end{aligned}$$

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A Power Slide

P. GLAISTER

Think of a real number $x > 0$, find its xth root, i.e. $p = x^{1/x}$, and then generate the sequence

$$p^p,\ p^{p^p},\ p^{p^{p^p}},\ \ldots. \tag{1}$$

Before you do this you need to know the order of performing the operations to make the terms in (1) unambiguous. Therefore, we take p^{p^p} to mean $p^{(p^p)}$, etc., rather than $(p^p)^p = p^{(p^2)}$, which is something entirely different, unless $p = 2$ of course! Do you notice any interesting behaviour? For example, with $x = 2$, $p = 2^{1/2} = \sqrt{2}$ and the sequence is

$$\sqrt{2},\ \sqrt{2}^{\sqrt{2}},\ \sqrt{2}^{\sqrt{2}^{\sqrt{2}}},\ \ldots;$$

do the terms approach a limit and, if so, what is it? Similarly, what about when $x = \frac{1}{2}$, in which case $p = \frac{1}{2}^2 = \frac{1}{4}$ and the sequence is

$$\tfrac{1}{4},\ \tfrac{1}{4}^{1/4},\ \tfrac{1}{4}^{(1/4)^{(1/4)}},\ \ldots.$$

The sequence in (1) can be thought of as a sequence of functions

$$f_n(x) = (x^{1/x})^{\underbrace{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot^{(x^{1/x})}}}}}}_{n}}, \qquad n = 0, 1, 2, \ldots,$$

i.e. $f_0(x) = (x^{1/x})$, $f_1(x) = (x^{1/x})^{(x^{1/x})}$, etc., which can be defined recursively as

$$f_n(x) = (x^{1/x})^{f_{n-1}(x)}, \quad n = 1, 2, \ldots, \qquad f_0(x) = (x^{1/x}). \tag{2}$$

It is therefore the nature of

$$f(x) = (x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}} = \lim_{n\to\infty} f_n(x)$$

that we are interested in.

To compute $x^{1/x}$ for any real number $x > 0$, we use its most general definition

$$p = x^{1/x} = \mathrm{e}^{(1/x)\ln x},$$

and, similarly, for the sequence in (2), we have

$$\begin{gathered} f_n(x) = (x^{1/x})^{f_{n-1}(x)} = \mathrm{e}^{f_{n-1}(x)\ln(x^{1/x})} = \mathrm{e}^{(1/x)f_{n-1}(x)\ln x}, \\ f_0(x) = (x^{1/x}) = \mathrm{e}^{(1/x)\ln x}. \end{gathered} \tag{3}$$

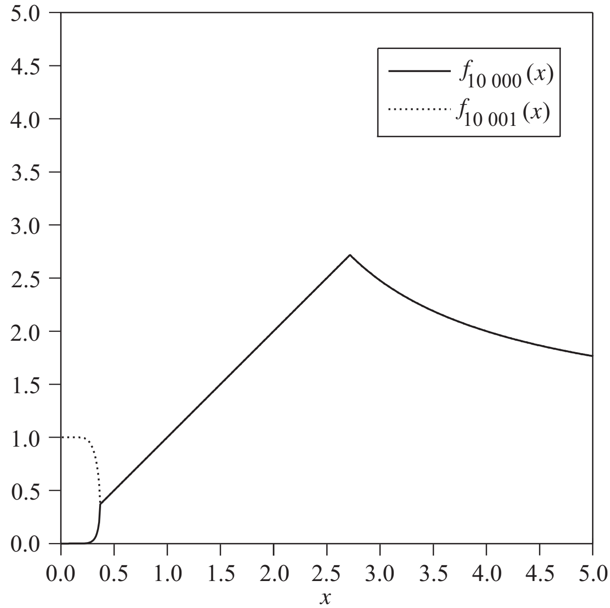

Figure 1 Graphs of $f_{10\,000}$ and $f_{10\,001}$.

To identify the nature of the sequence and its limit, we consider two successive terms, $f_{10\,000}(x)$ and $f_{10\,001}(x)$, as shown in figure 1, from which we can identify the nature of the sequence and its limit. In particular, we see that the sequence appears not to converge for all x.

First we see that, for small values of x, the sequence oscillates between two values, and in fact this point of bifurcation is where $x = 1/\mathrm{e}$. For values beyond this, the limit is $f(x) \equiv x$, until it reaches a maximum value, which is where $x = \mathrm{e}$, and decays thereafter. Indeed, from (3),

$$f_0'(x) = \frac{1}{x^2}(1 - \ln x) f_0(x),$$

while

$$f_n'(x) = \left[\frac{\ln x}{x} f_{n-1}'(x) + \frac{1}{x^2}(1 - \ln x) f_{n-1}(x)\right] f_n(x).$$

Thus, f_0 has a stationary point at $x = \mathrm{e}$, and, by induction, so has f_n for all n. We also see from the graph that this is a maximum point. Therefore, we have

$$(x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}} \equiv x, \qquad \mathrm{e}^{-1} \le x \le \mathrm{e},$$

with extreme values

$$(\mathrm{e}^{1/\mathrm{e}})^{(\mathrm{e}^{1/\mathrm{e}})^{(\mathrm{e}^{1/\mathrm{e}})^{\cdot^{\cdot^{\cdot}}}}} \equiv \mathrm{e}, \qquad \left(\frac{1}{\mathrm{e}^{\mathrm{e}}}\right)^{(1/\mathrm{e}^{\mathrm{e}})^{(1/\mathrm{e}^{\mathrm{e}})^{\cdot^{\cdot^{\cdot}}}}} \equiv \frac{1}{\mathrm{e}}.$$

Also, we have

$$f(2)=\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=2 \quad \text{and} \quad f\left(\tfrac{1}{2}\right)=\tfrac{1}{4}^{(1/4)^{(1/4)^{\cdot^{\cdot^{\cdot}}}}}=\tfrac{1}{2},$$

which were the values we started with. Furthermore, we have

$$f(4)=\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}=2=f(2),$$

as can be seen from figure 1.

The limit function is useful because it can be used to represent the inversion of

$$y^x=x^y. \tag{4}$$

From (4) we have

$$y=(x^y)^{1/x}=x^{y/x}=(x^{1/x})^y=(x^{1/x})^{(x^{1/x})^y}=\cdots=(x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}}, \tag{5}$$

and, conversely, if $y=(x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}}$ then

$$\ln y=(x^{1/x})^{(x^{1/x})^{(x^{1/x})^{\cdot^{\cdot^{\cdot}}}}}\ln(x^{1/x})=y\ln(x^{1/x})=\left(\frac{y}{x}\right)\ln x,$$

and, hence, $\ln(y^x)=\ln(x^y)$, i.e. $y^x=x^y$. For example, the well-known result $2^4=4^2$ is represented by the fact that $f(4)=2$, and, for $x>1$, there is one nontrivial solution $(y\neq x)$ for y (except for $x=\mathrm{e}$). For $0<x<1/\mathrm{e}$, the sequence does not converge, oscillating between two values, but these still mean something concrete. For example $f_n(\frac{1}{3})$ oscillates between 0.0785 and 0.7719 (approximately) for large n, meaning that

$$0.3333^{0.0785}=0.7719^{0.3333} \quad \text{and} \quad 0.3333^{0.7719}=0.0785^{0.3333}.$$

We note that in (5) an iteration sequence of the form

$$y_n=\mathrm{e}^{(\ln x/x)y_{n-1}} \tag{6}$$

is being generated. If we take $g(y)=\mathrm{e}^{(\ln x/x)y}$ then $|g'(x)|<1$ for $\mathrm{e}^{-1}<x<\mathrm{e}$, and, thus, x is an attracting fixed point in this range, and the sequence in (6) converges to the fixed point with $y_n\to x$.

Finally, for the interested reader, we note that the classical way of writing the inversion of (5) is in terms of the Lambert W function (see reference 1), which is defined implicitly by

$$W(x)\mathrm{e}^{W(x)}=x,$$

and which we leave the reader to show leads to

$$y=-\frac{x}{\ln x}W\left(\frac{-\ln x}{x}\right).$$

The graph of $f(x)$ reminds me of a children's slide, so I like to call it the *power slide*.

Acknowledgement

The author is grateful to a referee for making suggestions which have resulted in improvements to this article.

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Mathematical Spectrum Awards for Volume 41

Prizes have been awarded to the following student readers for contributions in Volume 41:

Aviv Adler
for the article 'Fundamental Transformations of Sudoku Grids'
(with Ilan Adler);

Robert Buonpastore
for the article 'Triangles and Parallelograms of Equal Area in an Ellipse'
(with Thomas J. Osler);

Robert P. Gove
for the article 'On the Natural Exponential Function'
(with Jan Rychtář);

Robert Combs and **Jon Kuhl**
for the article 'Exploring Ideas for Improving the Convergence Rate in Gauss– Seidel Iteration' (with Jennifer Switkes).

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Mathematics in the Classroom

Filling cuboidal holes with bricks

1. Introduction

I was, as ever, looking for extension material for my further mathematics classes when I came across one of the supplementary combinatorial problems in reference 1 asking for the number of ways of filling a $2 \times 2 \times 12$ hole with twenty four $1 \times 1 \times 2$ bricks. It occurred to me, because of the scope for both variation and generalisation, that this idea of counting the number of ways in which holes can be filled with bricks might lead to a challenging classroom activity.

We consider here two variations of the above problem, the solutions of which would be accessible to most further mathematics students. Enumerating possible arrangements of bricks gives students the opportunity to develop their problem-solving skills, encounter methods for solving linear recurrence relations, apply proof by induction, explore the floor function, utilise identities involving the binomial coefficients, and even meet Gauss' hypergeometric function. I have carried out this activity with one of my classes, and found that it also promoted both teamwork and independence of thought. Since some students did have problems visualising the arrangements, I introduced a practical element to the lesson by providing them with plastic bricks.

2. Scenario 1

We shall start by calculating a_n, the number of ways of filling a $2 \times 2 \times n$ hole with $1 \times 2 \times 2$ bricks. It is possible to do this by considering the separate layers of bricks in the hole. By an 'm-layer' we mean a collection of bricks that completely fills exactly m levels of the hole, but does not contain a subcollection completely filling exactly k levels for some $k < m$. In this case there are only two possibilities. Any layer will either be one unit thick (a 1-layer), consisting of just one flat brick, or two units thick (a 2-layer), consisting of two upright bricks. The 1-layers have just one possible orientation while the 2-layers have two.

We may obtain an expression for a_n via a simple recurrence relation. Each arrangement of bricks in a $2 \times 2 \times (n-1)$ hole gives rise, by adding one flat brick, to exactly a_{n-1} of the a_n ways of filling a $2 \times 2 \times n$ hole. On the other hand, for each arrangement in a $2 \times 2 \times (n-2)$ hole we obtain, by adding two upright bricks, exactly $2a_{n-2}$ of the arrangements enumerated by a_n. Then, on noting that all possible ways of filling a $2 \times 2 \times n$ hole have been counted exactly once, it is evident that

$$a_n = a_{n-1} + 2a_{n-2} \quad \text{for } n \geq 2$$

(where $a_1 = 1$ and $a_0 = 1$ by definition).

In order to solve this recurrence relation, we try a solution of the form $a_n = \alpha^n$. We then require that

$$\alpha^n = \alpha^{n-1} + 2\alpha^{n-2},$$

the nonzero roots of which are given by $\alpha_1 = 2$ and $\alpha_2 = -1$. The general solution is thus

$$a_n = 2^n c + (-1)^n d.$$

The coefficients c and d may be found by using the initial conditions $a_0 = a_1 = 1$, and, finally, we obtain the result

$$a_n = \tfrac{1}{3}(2^{n+1} + (-1)^n). \tag{1}$$

3. Scenario 2

Next we work out b_n, the number of ways of filling a $2 \times 3 \times n$ hole with $1 \times 2 \times 2$ bricks. First, by considerations of volume, it is clearly not possible to fill a $2 \times 3 \times n$ hole with $1 \times 2 \times 2$ bricks when n is odd. Thus, $b_{2k-1} = 0$ for $k = 1, 2, 3, \ldots$. The situation with regard to the layers is a little more complicated than it was for the first scenario. At this point it might be useful for the students to play around with the bricks until they can see what is going on. They will soon find that there are five possible 2-layers. What about the number of $2k$-layers when $k > 1$? There are in fact only two possible $2k$-layers for any $k > 1$. Not all will be convinced of the truth of this initially, and indeed this point did bring about plenty of lively discussion amongst my own students.

By considering the ways in which a $2 \times 3 \times n$ hole can be filled with layers of $1 \times 2 \times 2$ bricks, the following linear recurrence relation for b_{2k} emerges:

$$b_{2k} = 5b_{2k-2} + 2b_{2k-4} + 2b_{2k-6} + \cdots + 2b_2 + 2 \quad \text{for } k > 1.$$

On subtracting the recurrence relation for b_{2k} from the one for b_{2k+2}, we have $b_{2k+2} - b_{2k} = 5b_{2k} - 3b_{2k-2}$ and, thus,

$$b_{2k+2} - 6b_{2k} + 3b_{2k-2} = 0.$$

This can be solved using methods similar to those used in the previous section, and we end up with

$$b_{2k} = \tfrac{1}{6}((3 + \sqrt{6})^{k+1} + (3 - \sqrt{6})^{k+1}). \tag{2}$$

4. Further thoughts

(a) Although the method used in section 2 to obtain a_n is probably the most straightforward one, it is, from my own experience at least, not necessarily the one that students tend to gravitate towards. Some will notice that $2^k \binom{n-k}{k}$ gives the number of ways of arranging the bricks such that there are exactly k 2-layers along the length of the hole. From this, it follows that

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^k \binom{n-k}{k}, \tag{3}$$

where $\lfloor x \rfloor$ is the floor function, denoting the greatest integer not exceeding x. It now makes a rather stiff challenge for the students to show by induction that (3) leads to (1). For examples of inductive proofs involving sums of binomial coefficients, students might like to refer to reference 2 or 3.

(b) It might be interesting next to let the students consider the corresponding problem for holes and bricks of other dimensions. Are the techniques we have used above generally applicable? It would appear, for example, that the problem of finding the number of ways of filling a $3 \times 3 \times n$ hole with $1 \times 1 \times 3$ bricks is considerably more complex than the scenarios we have considered here. In addition, students might be encouraged to develop algorithms that would enable these enumerations to be carried out on a computer.

(c) When carrying out mathematical explorations such as these, it is often possible to veer off at tangents in order to pursue other fruitful lines of inquiry. Particularly able and interested students might like to explore the connection between (2) and Gauss' hypergeometric function, given by

$$F(a, b; c\colon z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots .$$

This function arises in many physical problems. It appears, for example, in connection with certain aspects of solar structure. For appropriate choices of a, b, and c, $F(a, b; c\colon z)$ reduces to elementary functions. To take an example, $F(1, 1; 2\colon z) = -\ln(1-z)/z$. Of relevance to our problem is the result

$$b_{2k} = 3^k F\left(-\frac{k+1}{2}, -\frac{k}{2}; \frac{1}{2}\colon \frac{2}{3}\right),$$

a proof of which might provide an interesting diversion for a bright student.

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Letters to the Editor

Dear Editor,

Programming in BASIC on a Nintendo DS Console

Programming in BASIC is an activity integrated into many mathematics courses. For a long time, all personal computers came with a BASIC compiler pre-installed. GW-BASIC and later QBASIC came with MS-DOS. Other computers, like the Sinclair ZX Spectrum, came with a ROM-based BASIC compiler. Those were the golden days of BASIC programming. A large amount of literature was written to support the educational uses of computer programming in BASIC, benefiting both students and teachers. Unfortunately, with the advent of the Windows operating systems, the very simple and easy to learn BASIC programming language has started to lose ground.

I have found an inexpensive way to implement programming in BASIC on the very popular Nintendo DS gaming console. The first step is to buy a Nintendo DS Slot 1 memory cartridge, like, for example, the U2 DS SDHC MicroSD/TF Card Multimedia Adapter for NDS/DS Lite from www.dealextreme.com that costs about $7. The second step is to buy a MicroSD memory card. The final step is to download the ZXDS program from http://zxds.raxoft.cz/ and install it on the memory card. ZXDS is an emulator of the Sinclair ZX Spectrum computer for the Nintendo DS, written by Patrik Rak. The touch screen of the Nintendo DS is used as a keyboard, while the upper LCD screen is used as a computer monitor.

The World of Spectrum web site (www.worldofspectrum.org) maintains a large collection of books about the Sinclair ZX computers, programming in BASIC and in machine code, and some applications to maths, science, and game design. Some of these books, out of print for a long time, and with the permission of the copyright holder, are now posted online as pdf files.

The Nintendo DS is already being used in schools in Japan to help teach English (see reference 1). It is time to start making use of this powerful technology in mathematics classrooms too!

Yours sincerely,

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Reference

1 Nintendo DS teaches English at Japanese schools, *eSchool News* (2008), p. 19.

Dear Editor,

Summing a series of Fibonacci numbers

Here we present a method of summing the first n terms of the series

$$F_1 - F_4 + F_7 - F_{10} + \cdots,$$

where F_n denotes the Fibonacci sequence. First take the case of odd n. We express this sum in two ways:

$$S_n = F_1 - (F_5 - F_3) + F_7 - (F_{11} - F_9) + \cdots - (F_{3n-4} - F_{3n-6}) + F_{3n-2}$$

and

$$S_n = (F_2 - F_0) - F_4 + (F_8 - F_6) - F_{10} + \cdots - F_{3n-5} + (F_{3n-1} - F_{3n-3}).$$

We add these two expressions to give

$$\begin{aligned} 2S_n &= F_1 - F_0 + (F_2 + F_3 - F_4) - (F_5 + F_6 - F_7) + \cdots - (F_{3n-4} + F_{3n-3} - F_{3n-2}) \\ &\quad + F_{3n-1} \\ &= 1 + F_{3n-1}, \end{aligned}$$

so that $S_n = \frac{1}{2}(1 + F_{3n-1})$. When n is even, a similar argument gives $S_n = \frac{1}{2}(1 - F_{3n-1})$.

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

42.5 Which products of numbers in arithmetic progression are the difference of two squares?

(Submitted by Bob Bertuello, Midsomer Norton, Bath)

42.6 What are the last six digits of 6249^{6249}?

(Submitted by Abbas Roohol Amini, Sirjan, Iran)

42.7 Solve the equation

$$\sqrt{x+6\sqrt{x}+8}+\sqrt{x+3\sqrt{x}+2}=\sqrt{4x+18\sqrt{x}+18}.$$

(Submitted by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain)

42.8 Let $\sigma = k+\sqrt{k^2\pm 1}$, where k is a positive integer. What is $\tan\pi\sigma \pm \tan\pi\sigma^{-1}$?

(Submitted by J. A. Scott, Chippenham)

Solutions to Problems in Volume 41 Number 3

41.9 Determine the area enclosed by the central region of the curve with equation

$$r=\theta+\frac{1}{\theta} \qquad (\theta>0)$$

in polar coordinates.

Solution by George Bignall, Nottingham High School and Daniel Fretwell, University of Sheffield (independently)

A plot of the curve is given in figure 1.

Denote by α and $\alpha+2\pi$ the values of θ at which the curve intersects itself, so that

$$\alpha+\frac{1}{\alpha}=(\alpha+2\pi)+\frac{1}{\alpha+2\pi},$$

which simplifies to

$$\alpha^2+2\pi\alpha-1=0,$$

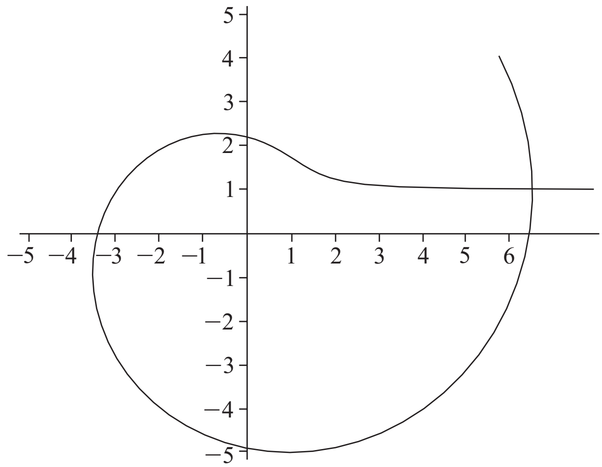

Figure 1

so that

$$\alpha = -\pi + \sqrt{\pi^2 + 1}.$$

Hence, the area enclosed by the curve is

$$\int_{\alpha}^{\alpha+2\pi} \frac{1}{2} r^2 \, \mathrm{d}\theta = \frac{1}{2} \int_{\sqrt{\pi^2+1}-\pi}^{\sqrt{\pi^2+1}+\pi} \left(\theta + \frac{1}{\theta}\right)^2 \mathrm{d}\theta,$$

which works out to $\frac{4}{3}\pi^3 + 4\pi$.

41.10 What is the sum of the infinite series

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{f_{2n+1}},$$

where (f_n) denotes the Fibonacci sequence? (Note: $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$ for all $n \geq 2.$)

Solution by Abbas Roohol Amini, who proposed the problem

In the formula

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y},$$

we put $x = \tan^{-1} A$ and $y = \tan^{-1} B$ to give

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1} \left(\frac{A - B}{1 + AB} \right).$$

Thus,

$$\tan^{-1} f_{2n+2} - \tan^{-1} f_{2n} = \tan^{-1} \frac{f_{2n+2} - f_{2n}}{1 + f_{2n+2} f_{2n}} = \tan^{-1} \frac{f_{2n+1}}{f_{2n+1}^2} = \tan^{-1} \frac{1}{f_{2n+1}}$$

(replacing n by $2n+1$ in the note). Hence,

$$\begin{aligned}\sum_{n=1}^{N}\tan^{-1}\frac{1}{f_{2n+1}} &= (\tan^{-1} f_4 - \tan^{-1} f_2) + (\tan^{-1} f_6 - \tan^{-1} f_4) + \cdots \\ &\quad + (\tan^{-1} f_{2N+2} - \tan^{-1} f_{2N}) \\ &= \tan^{-1} f_{2N+2} - \tan^{-1} f_2 \\ &\to \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4} \quad \text{as } N \to \infty.\end{aligned}$$

41.11 Determine all positive integers n for which

$$\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}$$

is an integer.

Solution by Daniel Fretwell, University of Sheffield

Denote the given number by x. Suppose that x is an integer. We have

$$x^2 = n + x$$

so that

$$x = \tfrac{1}{2}(1+\sqrt{1+4n}),$$

whence

$$1 + 4n = (2x-1)^2 = 4x^2 - 4x + 1$$

and $n = x(x-1)$, a product of two consecutive positive integers. Conversely, if $n = x(x-1)$, where $x > 1$ is an integer, then $x^2 = n + x$ and

$$x = \sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}\,,$$

an integer.

Also solved by Gian Paolo Almirante (Milan, Italy), Bor-Yann Chen (University of California, Irvine), Abbas Roohol Amini (Sirjan, Iran), and George Bignall (Nottingham High School).

41.12 For a triangle ABC, prove that

$$\cos^2\frac{A}{2} = \frac{a(s-a)}{bc}$$

(where $s = \frac{1}{2}(a+b+c)$) and that $\cos^2(A/2)$, $\cos^2(B/2)$, and $\cos^2(C/2)$ are the sides of a triangle.

Solution by George Bignall, Nottingham High School

By the cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

so that

$$\begin{aligned}\cos^2 \frac{A}{2} &= \frac{\cos A + 1}{2}\\ &= \frac{b^2 + c^2 - a^2 + 2bc}{4bc}\\ &= \frac{(b+c)^2 - a^2}{4bc}\\ &= \frac{(b+c+a)(b+c-a)}{4bc}\\ &= \frac{s(s-a)}{bc}.\end{aligned}$$

Hence,

$$\begin{aligned}\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} &= \frac{s(s-a)}{bc} + \frac{s(s-b)}{ca} - \frac{s(s-c)}{ab}\\ &= \frac{s}{abc}(a(s-a) + b(s-b) - c(s-c))\\ &= \frac{s}{2abc}(a(b+c-a) + b(c+a-b) - c(a+b-c))\\ &= \frac{s}{2abc}(c^2 - a^2 - b^2 + 2ab)\\ &= \frac{s}{c}(1 - \cos C)\\ &> 0,\end{aligned}$$

so that $\cos^2(A/2), \cos^2(B/2),$ and $\cos^2(C/2)$ are the lengths of the sides of a triangle.

The first part was also solved by Abbas Roohol Amini.

Reviews

Aha! Solutions. By Martin Erickson. The Mathematical Association of America, 2009. Hardback, 230 pages, \$61.50 (ISBN 9780-88385-829-5).

Martin Erickson's 'Aha! Solutions' offers a selection of 100 problems, designed to delight and instruct the reader. The problems cover a great variety of topics, such as number theory, geometry, probability and calculus. Each of these problems is followed by a concise solution, and a discussion of the mathematics involved, which provides a greater understanding of the wider implications of the solution. Individual taste will play a significant part in how the book is read, with some topics appealing greatly, and others containing problems appearing initially inaccessible. I found trying a few problems from different sections the most enjoyable way of approaching the book, as tackling a whole topic at a time could prove slightly daunting.

Personally, I particularly enjoyed the problems involving number theory and calculus, as these were the simplest to comprehend, given the brief explanation of the problem. The understanding of a problem, though, will depend largely upon the knowledge of the reader. As a sixth form student myself, some problems required a leap of faith, where Erickson's solutions involved mathematics with which I was unfamiliar. As the sections advanced, those devoted to geometry became, in my opinion, a much greater challenge. This was mostly due to the inherent difficulties in trying to convey the manipulation of sometimes multi-dimensional objects concisely.

It was the Toolkit, however, which I considered the most useful section. As a glossary of a diverse selection of mathematical terms, it not only proved very useful in solving numerous problems (especially if unfamiliar with the topic), but also in developing a broader knowledge of mathematical results.

The varying difficulty of the problems allows the book to be appreciated by anyone with even just a rudimentary grasp of mathematics; however, it would be a grave mistake to expect a series of gentle problems, easing into a few advanced problems by the end of the book. I found myself stretched by a few topics within the intermediate problems, whereas by the advanced section some of the solutions themselves required substantial deciphering. Potentially this could be viewed as limiting, however, it is this sense of challenge which encourages perseverance and can instil a desire to gain a deeper understanding of the mathematics involved.

Sixth Form, Nottingham High School **George Bignall**

When Less Is More. By Claudi Alsina and Roger B. Nelsen. The Mathematical Association of America, 2009. Hardback, 164 pages, $58.95 (ISBN 978-0-88385-342-9).

Does the feeling lurk inside some mathematicians that to prove an equality is somehow more worthwhile than to confirm an inequality? Here is a book that begs to differ. This extraordinary quotation comes early on—'Inequality is the only bearable thing; the monotony of equality can only lead us to boredom' (Francis Picabia).

The authors tell their story beautifully, beginning with the foundations provided by the arithmetic mean-geometric mean inequality before playfully going on to enlighten many areas of algebra, geometry, and number theory. There is something inexorable about their exposition; the authors have clearly lived with these inequalities for years, and have decided, with great care, upon the most aesthetic juxtaposition that they can create. The sense is of an unfolding narrative, enlivened by historical snippets and real-world applications; one wonders as one reads when one's favourite inequality will appear in the tale, and in this book it is sure to find remarkable bedfellows.

There are so many gems that it is hard to pick favourites, but the Erdos–Mordell inequality presented here is a fine example of a theorem where an elementary proof is hard to find, but which eventually yields to the correct insight. Using Euler's formula linking V, F, and E alongside linear programming leads to surprising results. Throughout the diagrams are, of course, essential and apt, and given Roger Nelsen's celebrated previous work on proofs without words, this is as it should be. A truly joyous book; if you buy a copy, people will ask you why you are smiling.

Jonny Griffiths

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