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Ramanujan's Third Problem

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1. The problem

Ray Hill recently described (*Mathematical Spectrum* **20**, pages 1–8) Srinivasa Ramanujan's life and work. Ramanujan's first publication was in 1911 and consisted of three problems, which were quoted in full in Ray Hill's article. The third one intrigued me and, I am sure, many other readers. The problem is:

Show that it is possible to solve the equations:

$$\begin{aligned} x+\ \ y+\ \ z &= a \quad (1) & p^3x+q^3y+r^3z &= d \quad (4)\\ px+qy+rz &= b \quad (2) & p^4x+q^4y+r^4z &= e \quad (5)\\ p^2x+q^2y+r^2z &= c \quad (3) & p^5x+q^5y+r^5z &= f \quad (6) \end{aligned}$$

where x, y, z, p, q and r are unknowns. Solve the above when $a = 2$, $b = 3$, $c = 4$, $d = 6$, $e = 12$ and $f = 32$.

In fact, though the equations can be solved in general, there are values of a, b, c, d, e and f for which no solution is possible.

Ray Hill suggested that one should start by trying to solve the four similar equations (for x, y, p and q unknowns):

$$\begin{aligned} x+\ \ y &= a \quad (7) & p^2x+q^2y &= c \quad (9)\\ px+qy &= b \quad (8) & p^3x+q^3y &= d \quad (10) \end{aligned}$$

Try this problem before reading further.

2. Solution by substitution

Clearly there are possibilities of successive substitutions. For example, $x = a-y$ from (7), so, substituting in (8), we obtain

$$pa-py+qy = b, \qquad \text{giving} \quad y = \frac{b-pa}{q-p} \quad \text{and} \quad x = \frac{b-qa}{p-q}.$$

Then, from (9), $p^2(b-qa)-q^2(b-pa) = c(p-q)$, so

$$apq-b(p+q)+c = 0. \tag{11}$$

Similarly, from (10), $p^3(b-qa)-q^3(b-pa) = d(p-q)$, so

$$apq(p+q)-b(p^2+pq+q^2) = d,$$

which gives (substituting for apq)

$$bpq-c(p+q)+d = 0. \tag{12}$$

Now (11) and (12) give, respectively,

$$p = \frac{c-bq}{b-aq}, \qquad p = \frac{d-cq}{c-bq},$$

so that we have the quadratic

$$q^2(b^2-ac)+q(ad-bc)+(c^2-bd) = 0. \tag{13}$$

Solving (11) and (12) for q would lead to the same quadratic in p: so the roots are p and q. From these one can calculate x and y.

But successive elimination will be nastier for the original problem. On the other hand, did you notice that the coefficients of the quadratic (13) are the determinants coming from the simultaneous equations (11) and (12) for pq and $p+q$? Does that suggest anything?

3. Using the cubic with roots p, q and r

Can we find a similar cubic (with roots p, q and r) from equations (1) to (6)? Suppose we can and that it is

$$F(s) \equiv (s-p)(s-q)(s-r) \equiv s^3+As^2+Bs+C = 0. \tag{14}$$

Consider adding multiples of equations (1) to (4) together and thus obtaining cubics in p, q and r. In order to make the cubics vanish they will have to be multiples of $F(p)$, $F(q)$ and $F(r)$. So consider $C\times(1)+B\times(2)+A\times(3)+(4)$ which gives

$$\begin{aligned} Ca+Bb+Ac+d &= C(x+y+z)+B(px+qy+rz) \\ &\qquad +A(p^2x+q^2y+r^2z)+(p^3x+q^3y+r^3z) \\ &= xF(p)+yF(q)+zF(r) = 0. \end{aligned} \tag{15}$$

Similarly $C\times(2)+B\times(3)+A\times(4)+(5)$ and $C\times(3)+B\times(4)+A\times(5)+(6)$ give

$$Cb+Bc+Ad+e = xpF(p)+yqF(q)+zrF(r) = 0 \tag{16}$$

and

$$Cc+Bd+Ae+f = xp^2F(p)+yq^2F(q)+zr^2F(r) = 0. \tag{17}$$

We can now (in general) solve (15), (16) and (17) for A, B and C to find the cubic (14) with roots p, q and r. In general the cubic will have three distinct roots (possibly complex), and substituting these into equations (1) to (3) we shall find unique values of x, y and z.

Applying this method with Ramanujan's values of a to f in equations (15) to (17), we have:

$$6+4A+3B+2C = 0, \tag{15'}$$

$$12+6A+4B+3C = 0 \tag{16'}$$

and

$$32+12A+6B+4C = 0 \tag{17'}$$

which, by elimination $[(16')-(\frac{3}{2})\times(15')$ and $(17')-2\times(15')]$, give $B = 6$, $A = -5$ and finally $C = -2$. So

$$F(s) = s^3-5s^2+6s-2 = 0.$$

By inspection there is a factor $(s-1)$ leaving the quadratic factor (s^2-4s+2). So the three roots are $p = 1$, $q = 2+\sqrt{2}$ and $r = 2-\sqrt{2}$.

Substituting these in (1), (2) and (3) we have

$$x+y+z = 2, \quad x+2(y+z)+(y-z)\sqrt{2} = 3, \quad x+6(y+z)+4(y-z)\sqrt{2} = 4.$$

So

$$(y+z)+(y-z)\sqrt{2} = 1, \qquad 5(y+z)+4(y-z)\sqrt{2} = 2.$$

Then $y+z = -2$ and $y-z = \frac{3}{2}\sqrt{2}$, giving $y = -1+\frac{3}{4}\sqrt{2}$ and $z = -1-\frac{3}{4}\sqrt{2}$ and finally $x = 4$, completing the solution.

It is easy to see that equations (15), (16) and (17) have a unique solution in general, but that values of a, b, c, d, e and f can be chosen to make the equations inconsistent (for example, $a = b = c = d = e = 0$ and $f = 1$).

4. Ramanujan's own solution

Ramanujan did not give a solution with the publication of his three problems. Later he gave a method of solution for the general case of $2n$ equations. He used the fact that $1/(1-\theta p)$ can be written as an infinite series:

$$1+\theta p+\theta^2p^2+\ldots+\theta^np^n+\ldots.$$

In case some readers find infinite series difficult, I shall modify his method slightly and use the more familiar identity:

$$\frac{1-\theta^6p^6}{1-\theta p} = 1+\theta+\theta^2p^2+\theta^3p^3+\theta^4p^4+\theta^5p^5.$$

Now let

$$G(\theta) = a+\theta b+\theta^2c+\theta^3d+\theta^4e+\theta^5f$$

$$= (x+y+z)+\theta(px+qy+rz)+\theta^2(p^2x+q^2y+r^2z)+\dots$$
$$+\theta^5(p^5x+q^5y+r^5z)$$
$$= x(1+\theta p+\theta^2p^2+\dots+\theta^5p^5)$$
$$+y(1+\theta q+\theta^2q^2+\dots+\theta^5q^5)$$
$$+z(1+\theta r+\theta^2r^2+\dots+\theta^5r^5)$$
$$= \frac{x(1-\theta^6p^6)}{1-\theta p}+\frac{y(1-\theta^6q^6)}{1-\theta q}+\frac{z(1-\theta^6r^6)}{1-\theta r}.$$

So, by multiplying both sides of the equation by $(1-\theta p)(1-\theta q)(1-\theta r)$ and interchanging them, we have

$$x(1-\theta^6p^6)(1-\theta q)(1-\theta r)$$
$$+y(1-\theta^6q^6)(1-\theta r)(1-\theta p)$$
$$+z(1-\theta^6r^6)(1-\theta p)(1-\theta q)$$
$$= (1-\theta p)(1-\theta q)(1-\theta r)\,G(\theta)$$
$$= [1-\theta(p+q+r)+\theta^2(qr+rp+pq)-\theta^3pqr]$$
$$\times(a+\theta b+\theta^2c+\theta^3d+\theta^4e+\theta^5f).$$

We can now compare coefficients of powers of θ to give:

$$\theta^0:\ x+y+z = a,$$
$$\theta^1:\ -x(q+r)-y(r+p)-z(p+q) = b-a(p+q+r),$$
$$\theta^2:\ xqr+yrp+zpq = c-b(p+q+r)+a(qr+rp+pq),$$
$$\theta^3:\ 0 = d-c(p+q+r)+b(qr+rp+pq)-apqr, \tag{15''}$$
$$\theta^4:\ 0 = e-d(p+q+r)+c(qr+rp+pq)-bpqr, \tag{16''}$$
$$\theta^5:\ 0 = f-e(p+q+r)+d(qr+rp+pq)-cpqr, \tag{17''}$$
$$\theta^6:\ -xp^6-yq^6-zr^6 = -f(p+q+r)+e(qr+rp+pq)-dpqr,$$
$$\theta^7:\ xp^6(q+r)+yq^6(r+p)+zr^6(p+q) = f(qr+rp+pq)-epqr,$$
$$\theta^8:\ -xp^6qr-yq^6rp-zr^6pq = -fpqr.$$

The middle three of these equations are equivalent to (15), (16) and (17) derived earlier, since

$$A = -(p+q+r), \qquad B = qr+rp+pq, \qquad C = -pqr$$

(sums and products of the roots of a cubic). The other equations are identities easily derivable from the original equations (1) to (6).

5. A solution using vectors and determinants

Yet another way of looking at this problem is to consider the vectors

$$\boldsymbol{p} = (1\ \ p\ \ p^2\ \ p^3), \quad \boldsymbol{q} = (1\ \ q\ \ q^2\ \ q^3), \quad \boldsymbol{r} = (1\ \ r\ \ r^2\ \ r^3),$$
$$\boldsymbol{a} = (a\ \ b\ \ c\ \ d), \quad \boldsymbol{b} = (b\ \ c\ \ d\ \ e), \quad \boldsymbol{c} = (c\ \ d\ \ e\ \ f).$$

Clearly equations (1) to (4) can now be written as

$$\boldsymbol{a} = x\boldsymbol{p} + y\boldsymbol{q} + z\boldsymbol{r},$$

equations (2) to (5) as

$$\boldsymbol{b} = px\boldsymbol{p} + qy\boldsymbol{q} + rz\boldsymbol{r}$$

and equations (3) to (6) as

$$\boldsymbol{c} = p^2x\boldsymbol{p} + q^2y\boldsymbol{q} + r^2z\boldsymbol{r}.$$

Now we can eliminate $\boldsymbol{q}$ and $\boldsymbol{r}$ from these equations to give

$$\boldsymbol{c} - (r+q)\boldsymbol{b} + qr\boldsymbol{a} = x(p-r)(p-q)\boldsymbol{p}.$$

So the 4×4 determinant with rows $\boldsymbol{p}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is zero, since the bottom row is linearly dependent on the other rows. But this determinant can be expanded by its top row to give

$$\begin{vmatrix} 1 & p & p^2 & p^3 \\ a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{vmatrix} = 1\begin{vmatrix} b & c & d \\ c & d & e \\ d & e & f \end{vmatrix} - p\begin{vmatrix} a & c & d \\ b & d & e \\ c & e & f \end{vmatrix}$$
$$+p^2\begin{vmatrix} a & b & d \\ b & c & e \\ c & d & f \end{vmatrix} - p^3\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0,$$

which gives a cubic in p, a multiple of the cubic $F(s)$ of equation (14). With the Ramanujan values we have the determinant

$$\begin{vmatrix} 1 & p & p^2 & p^3 \\ 2 & 3 & 4 & 6 \\ 3 & 4 & 6 & 12 \\ 4 & 6 & 12 & 32 \end{vmatrix} = \begin{vmatrix} 1 & p & p^2 & p^3 \\ 2 & 3 & 4 & 6 \\ 0 & -\frac{1}{2} & 0 & 3 \\ 0 & 0 & 4 & 20 \end{vmatrix} = \begin{vmatrix} 1 & p & p^2 & p^3 \\ 2 & 0 & 0 & 4 \\ 0 & -\frac{1}{2} & 0 & 3 \\ 0 & 0 & 4 & 20 \end{vmatrix}$$
$$= 1\times4\times(-\tfrac{1}{2})\times4 + p\times2\times3\times4 + p^2\times2\times(-\tfrac{1}{2})\times20 - p^3\times2\times(-\tfrac{1}{2})\times4$$
$$= 4p^3 - 20p^2 + 24p - 8 = 4(p-1)(p-2-\sqrt{2})(p-2+\sqrt{2}) = 0.$$

(Here we have subtracted $\frac{3}{2}$ times row (2) from row (3), subtracted twice row (2) from row (4) and then added six times the new row (3) to, and subtracted the new row (4) from, row (2). Then we expanded the determinant by the top row.)

This determinantal form is the easiest form for calculating a solution provided that there is a straightforward solution. But if some of the 3×3 subdeterminants are zero, we may need to solve equations (15) to (17) in order to find the cubic (or show that there is no solution). And, if the cubic has repeated roots, we shall not find unique values of x, y and z.

6. A concluding question

Ramanujan's Third Problem was set without a solution in 1911. As Ray Hill's article points out, attempts to eliminate variables can lead one into a hopeless mess. Ramanujan himself published a solution in 1912. As Ray Hill informs us, mathematicians studying error-correcting codes rediscovered Ramanujan's method of solution. I have suggested two other methods of solution. All three methods have avoided getting into a hopeless mess. Can you find a fourth elegant solution to Ramanujan's Third Problem?

Athletic Performance Trends in Olympics

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The Olympic Games are a great sporting event, where athletes from all over the world take part to decide the best. Held every four years since 1896, barring interruptions during the World Wars, the modern Olympics provide a unique forum where world athletic standards are regularly measured. Olympic performances show an inexorable trend towards better results. The athletes are seen to run faster, jump higher and throw farther in each succeeding Olympics. This article takes a look at the Olympic performance trends in track and field events, where the efforts are recorded in absolute measures such as seconds, feet and metres. The data are taken from reference 1.

Figure 1 shows the first-place (gold medal) performances of each modern Olympics in two popular throwing events—the discus throw and shot-put. The dramatic upward trend in both events are evident from the figure. The throwing distances have more than doubled in discus and nearly doubled in

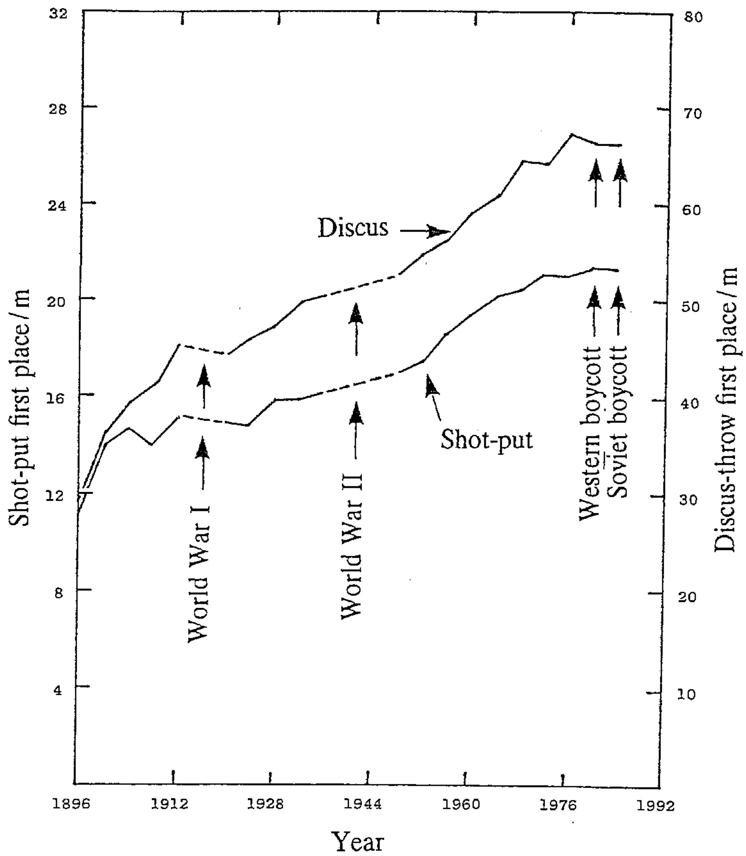

Figure 1. First-place performance in shot-put and discus

shot-put since 1896, when the first modern Olympiad was held. Also apparent in the figure is the levelling off of the first-place performances in the last two Olympiads. This was almost certainly brought about by the lower level of competition because of the Western boycott of the Olympics in 1980 and the Soviet block boycott in 1984.

Figure 1 also shows a greater slope of the discus curve, which perhaps signifies a greater importance of technique as compared with sheer strength in the discus throw. As a mathematical exercise, one can calculate the slopes of the two curves by fitting the data points with least-squares straight lines. From the general equation of a straight line with slope m and y intercept c,

$$y = mx + c, \tag{1}$$

we can obtain the normal equations

$$\sum y = m \sum x + cn \tag{2}$$

and

$$\sum xy = m \sum x^2 + c \sum x. \tag{3}$$

Here the summation runs from 1 to n, the number of data points. Solving equations (2) and (3) simultaneously, we get

$$m = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}, \tag{4}$$

and

$$c = \frac{\sum y \sum x^2 - \sum x \sum xy}{n\sum x^2 - \left(\sum x\right)^2}. \tag{5}$$

The shot-put curve of figure 1 yields a slope of 1.0625 m (3 ft $5\frac{7}{8}$ in) per decade, while the discus curve gives a slope of 3.9375 m (12 ft 11 in) per decade. It is obviously the case that future athletes have to do progressively better in order to have any chance of winning in the Olympics.

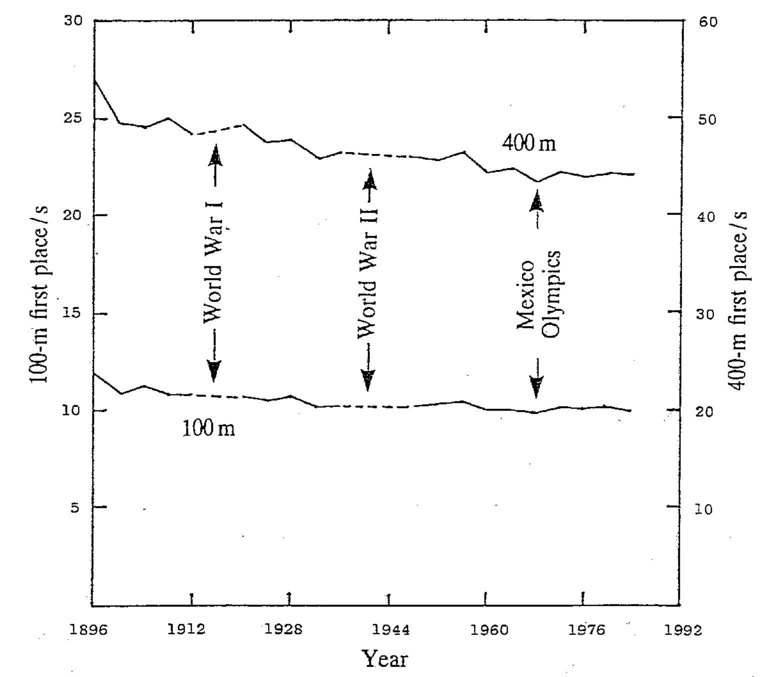

Figure 2. First-place timings in 100-m and 400-m races

Figure 2 shows the times of the first-place winners in the 100-m and 400-m sprint. The downward trends of both curves are noticeable, as is the gentler slope of the 100-m curve. The best results were returned in the 1968 Mexico Olympics where the rarefied atmosphere helped the sprint events but hurt long-distance running (reference 1). The results of the two events were, however, relatively unaffected by the 1980 and 1984 boycotts. In contrast to the relentless increase of the distances in throwing events, the running times in the sprints seem to level off. Since there is a lower limit in the running times, a rectilinear decrease is mathematically unattainable. Hence the trends in the short-distance running would be best studied by an exponential curve rather than linear regression.

The introduction of new techniques, styles or equipment can dramatically improve athletic performances. It is estimated that the use of starting blocks in the sprints gives an advantage of 0.3 m (1 ft) of running distance (reference 2). The introduction of the fibreglass pole certainly revolutionised pole vaulting. Similarly, the backward style of jumping introduced by Dick Fosbury also changed the course of high-jumping. Today, virtually no leading high-jump competitor uses the orthodox straddle style of jumping. Figure 3 shows the pole-vault results before and after the introduction of the fibreglass pole. This difference, which is easily noticeable, can be calculated as follows. We assume that the two sets of data points could be fitted with straight lines having the same slope (the slope gives a measure of the increase due to purely human factors). The difference in the y intercept of the two lines would then be attributed to the use of the fibreglass pole. We first find the least-squares straight line of the data points prior to 1964 (when conventional poles were used) from equations (4) and (5). We next find the y intercept of the data points from 1964 onwards from equation (2) using the same slope:

$$c_1 = \frac{\sum y - m\sum x}{n}. \tag{6}$$

The difference in the two intercepts, which comes out to be 0.53 m ($20\frac{7}{8}$ in) is then attributed to the use of the fibreglass pole rather than conventional pole.

A similar calculation for the high-jump event reveals that the 'Fosbury flop' style introduced in 1968 by Dick Fosbury and used exclusively from 1976 onwards produced jumps 0.06 m ($2\frac{3}{8}$ in) higher than the traditional straddle style of jumping. Even this relatively small measure of advantage has been enough to render the traditional style of jumping obsolete.

In view of the fact that the throwing and jumping distances continually improve with time, athletes need to improve their own performances accordingly in order to win in successive Olympics. Ralph Boston, a consistent long-jumper, participated in three consecutive Olympics and produced comparable jumps of 8.12 m, 8.03 m and 8.16 m. The first time he won the gold medal, the second time a similar jump was worth a silver medal and four years later, a slightly better jump could bring only a bronze. Al Oerter, on the other hand, managed to improve his performance enough to win four consecutive gold medals in the discus with throws of 56.36 m, 59.18 m, 61.00 m and 64.78 m, respectively.

The performance of individual athletes often shows a parabolic trend with time. For instance, Jay Sylvester competed in four successive Olympics in the discus throw. His efforts of 59.09 m, 61.78 m, 63.50 m and 61.98 m exhibit a clear parabolic trend. His maximum effort of 63.50 m fetched him

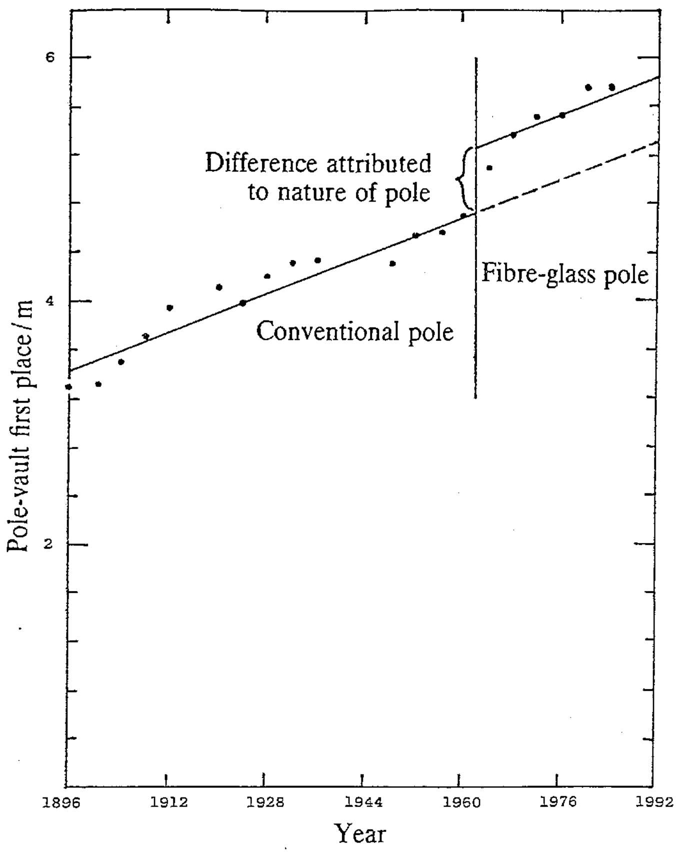

Figure 3. Least-squares lines and first-place performances in pole vault

a silver medal. Parry O'Brien also participated in four successive Olympics in the shot-put. His efforts of 17.41 m, 18.57 m, 19.11 m and 19.20 m, which brought him two golds, one silver and a fourth-place finish, show a trend along the ascending part of a parabola. In spite of the fact that he had a career best throw in the last Olympics, the increment was not large enough to earn him any medal.

Finally, is this linear trend in throwing and jumping events sustainable in the long run or will the trend eventually level off? Considering the general increase in athletic participation throughout the world and improvements in diet, physique, technique and equipment, the author's prediction is that the linear upward trend will easily be maintained at least well into the next century.

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13, 14, 15 and 15, 26, 37

JOHN MACNEILL, *The Royal School, Wolverhampton*

The author's interests include mathematical education and all mathematics which is not both difficult and non-elementary.

The area of a triangle with sides of integral length is not in general integral. The triangle with sides of length 13, 14 and 15 is of interest since it has integral sides and area; further, its sides are in arithmetic progression.

A triangle with sides a, b and c in arithmetic progression can be found by choosing values for the parameters u and v in

$$a = 3u^2+3v^2, \qquad b = 6u^2+2v^2, \qquad c = 9u^2+v^2.$$

Then the triangle has area $\Delta = 3buv$ and inradius $r = 2uv$, as can be checked using the triangle formulae

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = rs,$$

where the semi-perimeter $s = \frac{1}{2}(a+b+c)$.

Table 1

u	v	sides	r	Δ
1	1	3, 4, 5*	1	6
1	2	13, 14, 15	4	84
2	1	15, 26, 37	4	156
3	1	15, 28, 41*	3	126
1	4	25, 38, 51	8	456
3	2	39, 62, 85	12	1092
4	1	51, 98, 145	8	1176
1	5	17, 28, 39*	5	210
5	1	39, 76, 113*	5	570
2	5	61, 74, 87	20	2220
3	4	75, 86, 97	24	3096
4	3	75, 114, 153	24	4104
5	2	87, 158, 229	20	4740
6	1	111, 218, 325	12	3924
1	7	29, 52, 75*	7	546
3	5	51, 52, 53*	15	1170
7	1	75, 148, 221*	7	1554

Table 1 gives the different shapes of triangle for natural numbers u and v up to $u+v = 8$. Where the lengths calculated had a common factor it has been divided out (*). Any case such as $u = 1$, $v = 3$ which gives a triangle similar to one already found has been omitted.

The striking pattern of coefficients in the parametric expressions for a, b and c can be thought of as deriving from the 1,2,3 'triangle' which has sides in arithmetic progression and $\Delta = 0$. In a similar way we can use the 'triangle' with sides h^2, $k-h^2$ and k to suggest the more general equations

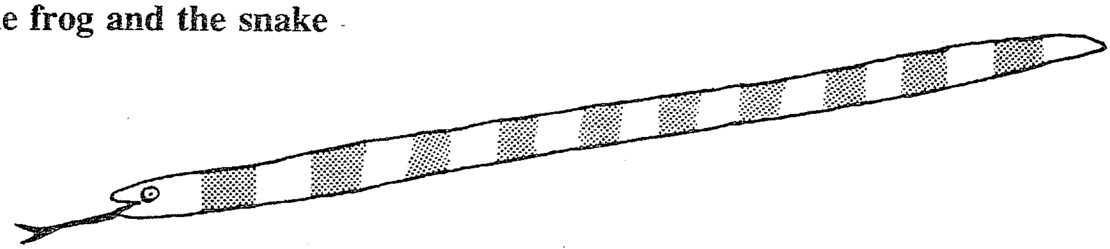
$$a = h^2ku^2+kv^2, \qquad b = (k-h^2)ku^2+(k-h^2)v^2, \qquad c = k^2u^2+h^2v^2,$$

where $k > h^2$, which give triangles with integral sides and area (sides not necessarily in arithmetic progression) and for which $\Delta = bhkuv$. In fact every triangle with integral sides and area can be found by using these equations (and scaling up or down if necessary), as can be proved by observing that a triangle similar to one with sides a', b' and c', area Δ' and semi-perimeter s' can be obtained by taking

$$h = s'-b', \qquad k = hs', \qquad u = s'-a', \qquad v = \Delta',$$

as readers may wish to check. As a final challenge, we ask whether r and s must be integral when a, b, c and Δ are integral.

The frog and the snake



One morning at sunrise, I found a frog and a snake in my garden, 18 m apart and heading for one another. Watching them over the next few days, I noticed that the frog moved ahead 3.5 m each daytime and moved back 1 m at night. The snake, being a more nocturnal creature, moved forward 2.5 m each nighttime and moved back 2 m each daytime. When and where does the snake catch the frog?

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A Fibonacci Sum

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The author was 14 years of age when he wrote this article. Besides being interested in mathematics, he is also a runner and high jumper.

1. The problem

While browsing through some problems in a popular book on Number Theory, I found the following series which had to be summed:

$$\binom{n}{1}F_1+\binom{n}{2}F_2+\ldots+\binom{n}{n}F_n, \tag{1}$$

where $\binom{n}{r}$ denotes the binomial coefficient and the F_r are the celebrated Fibonacci numbers, defined by

$$F_1 = F_2 = 1, \qquad F_{r+1} = F_r+F_{r-1} \quad (r = 2,3,\ldots).$$

Let us deal with a more general case when $F_1 = p$ and $F_2 = q$.

After trying to make the series look like a binomial expansion, I came across the following solution, which avoids induction. Define $F_0 = q-p$, and assume that

$$F_r = \lambda\alpha^r+\mu\beta^r$$

for some λ, μ, α, β and all r. Since $F_{r+1} = F_r+F_{r-1}$ we have

$$\lambda\alpha^{r+1}+\mu\beta^{r+1} = \lambda\alpha^r+\mu\beta^r+\lambda\alpha^{r-1}+\mu\beta^{r-1}.$$

This is satisfied if

$$\alpha^{r+1} = \alpha^r+\alpha^{r-1}, \qquad \beta^{r+1} = \beta^r+\beta^{r-1},$$

or

$$\alpha^2 = \alpha+1, \qquad \beta^2 = \beta+1. \tag{2}$$

Then the series (1) is

$$\begin{aligned}\sum_{r=1}^{n}\binom{n}{r}F_r &= \sum_{r=0}^{n}\binom{n}{r}F_r-F_0\\ &= \sum_{r=0}^{n}\binom{n}{r}(\lambda\alpha^r+\mu\beta^r)+p-q\\ &= \lambda\sum_{r=0}^{n}\binom{n}{r}\alpha^r+\mu\sum_{r=0}^{n}\binom{n}{r}\beta^r+p-q\\ &= \lambda(1+\alpha)^n+\mu(1+\beta)^n+p-q\end{aligned}$$

$$= \lambda\alpha^{2n}+\mu\beta^{2n}+p-q \quad \text{from (2)}$$
$$= F_{2n}+p-q.$$

Since for the 'normal' Fibonacci numbers we have $p-q = 0$, the sum of (1) is F_{2n}.

It is worthwhile to find a formula for this generalized F_r. We have $\alpha^2 = \alpha+1$ and $\beta^2 = \beta+1$, from which

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}$$

or vice versa. Now

$$p = F_1 = \lambda\alpha+\mu\beta = \lambda\left(\frac{1+\sqrt{5}}{2}\right)+\mu\left(\frac{1-\sqrt{5}}{2}\right),$$
$$q = F_2 = \lambda\alpha^2+\mu\beta^2 = \lambda\left(\frac{3+\sqrt{5}}{2}\right)+\mu\left(\frac{3-\sqrt{5}}{2}\right).$$

These give

$$\lambda = \frac{1}{2\sqrt{5}}\{(3p-q)-\sqrt{5}(p-q)\}, \qquad \mu = -\frac{1}{2\sqrt{5}}\{(3p-q)+\sqrt{5}(p-q)\},$$

so that

$$F_r = \lambda\alpha^r+\mu\beta^r$$
$$= \frac{1}{2\sqrt{5}}\left\{[(3p-q)-\sqrt{5}(p-q)]\left(\frac{1+\sqrt{5}}{2}\right)^r-[(3p-q)+\sqrt{5}(p-q)]\left(\frac{1-\sqrt{5}}{2}\right)^r\right\}.$$

Thus, when $p = q = 1$, we obtain

$$F_r = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^r-\left(\frac{1-\sqrt{5}}{2}\right)^r\right\}$$

for the rth term of the Fibonacci sequence.

2. A variation of the problem

Instead of the Fibonacci sequence, we now consider a variant of it defined as follows:

$$T_0 = 0, \qquad T_1 = T_2 = 1, \qquad T_{r+2} = T_r+T_{r-1} \quad (r = 1,2,3,\ldots),$$

with a view to summing the series

$$\binom{n}{1}T_1+\binom{n}{2}T_2+\ldots+\binom{n}{n}T_n.$$

Assume that

$$T_r = \lambda\alpha^r + \mu\beta^r + \nu\gamma^r.$$

Since $T_{r+2} = T_r + T_{r-1}$, we need

$$\lambda\alpha^{r+2} + \mu\beta^{r+2} + \nu\gamma^{r+2} = \lambda\alpha^r + \mu\beta^r + \nu\gamma^r + \lambda\alpha^{r-1} + \mu\beta^{r-1} + \nu\gamma^{r-1}.$$

This is satisfied if

$$\alpha^{r+2} = \alpha^r + \alpha^{r-1}, \qquad \beta^{r+2} = \beta^r + \beta^{r-1}, \qquad \gamma^{r+2} = \gamma^r + \gamma^{r-1},$$

that is, if α, β and γ are roots of the cubic equation

$$x^3 = x + 1.$$

(It should be noted that this equation does not have a repeated root; if it did, such a root would also be a root of the derivative $3x^2 = 1$, that is, it would be $\pm 1/\sqrt{3}$, which are easily seen not to be roots. We shall now be able to determine λ, μ and ν from the values of T_0, T_1 and T_2:

$$0 = \lambda + \mu + \nu, \qquad 1 = \lambda\alpha + \mu\beta + \nu\gamma, \qquad 1 = \lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2.)$$

Now (noting that $T_0 = 0$)

$$\begin{aligned}
\sum_{r=0}^{n} \binom{n}{r} T_r &= \sum_{r=0}^{n} \binom{n}{r} (\lambda\alpha^r + \mu\beta^r + \nu\gamma^r) \\
&= \lambda \sum_{r=0}^{n} \binom{n}{r} \alpha^r + \mu \sum_{r=0}^{n} \binom{n}{r} \beta^r + \nu \sum_{r=0}^{n} \binom{n}{r} \gamma^r \\
&= \lambda(1+\alpha)^n + \mu(1+\beta)^n + \nu(1+\gamma)^n \\
&= \lambda\alpha^{3n} + \mu\beta^{3n} + \nu\gamma^{3n} \\
&= T_{3n}.
\end{aligned}$$

We can generalize the Fibonacci sequence and the sequence (T_r) as follows. For each positive integer N, we can define the sequence $(F_{N,r})$ by

$$F_{N,0} = 0, \qquad F_{N,r} = 1 \quad (r = 1, 2, \ldots, N-1),$$
$$F_{N,r+(N-1)} = F_{N,r} + F_{N,r-1} \quad (r = 1, 2, 3, \ldots).$$

Thus $N = 2$ gives the Fibonacci sequence and $N = 3$ gives the sequence (T_r). Assume that

$$F_{N,r} = \lambda_1\alpha_1^r + \lambda_2\alpha_2^r + \ldots + \lambda_N\alpha_N^r.$$

Since

$$F_{N,r+(N-1)} = F_{N,r} + F_{N,r-1},$$

this gives

$$\sum_{i=1}^{N} \lambda_i \alpha_i^{r+N-1} = \sum_{i=1}^{N} \lambda_i \alpha_i^{r} + \sum_{i=1}^{N} \lambda_i \alpha_i^{r-1},$$

and this satisfied if the α_i are roots of the equation

$$x^N = x+1.$$

(As with the sequence (T_r), this equation has no repeated roots and this enables us to determine $\lambda_1, \lambda_2, \ldots, \lambda_N$.) Then (noting that $F_{N,0} = 0$)

$$\begin{aligned}\sum_{r=0}^{n} \binom{n}{r} F_{N,r} &= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=1}^{N} \lambda_i \alpha_i^r \\ &= \sum_{i=1}^{N} \lambda_i \sum_{r=0}^{n} \binom{n}{r} \alpha_i^r \\ &= \sum_{i=1}^{N} \lambda_i (1+\alpha_i)^n \\ &= \sum_{i=1}^{N} \lambda_i \alpha_i^{Nn} \\ &= F_{N,Nn}.\end{aligned}$$

Readers may like, for example, to verify that

$$\sum_{r=1}^{n} \binom{9}{r} F_{7,r} = F_{7,63}.$$

(The 63rd term of this sequence is 521.)

3. Ratios of successive terms

By means of a computer, the ratios of successive terms of these sequences can be calculated. We define

$$R_{N,n} = \frac{F_{N,n+1}}{F_{N,n}}.$$

It happens that

$$R_{7,498} = 1.112\,787\,525, \qquad R_{7,499} = 1.112\,797\,456.$$

This leads us to think that these ratios tend to a limit. For the Fibonacci sequence,

$$R_{2,n} = \frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2},$$

which is the positive root of the equation $x^2 = x+1$. It can, in fact, be proved generally that, for $N = 2, 3, \ldots$, the equation $x^N = x+1$ has a unique positive real root k_N and that

$$R_{N,n} \to k_N \quad \text{as } n \to \infty.$$

With $N = 7$, the positive real root of $x^7 = x+1$ can be found by means of Newton–Raphson iteration:

$$x_{n+1} = x_n - \frac{x_n^7 - x - 1}{7x_n^6 - 1}.$$

If we begin the iteration with $x_0 = 1$, the values of x_n given in table 1 are obtained. Thus $k_7 \approx 1.112\,775\,684$, which is very close to $R_{7,498}$ and $R_{7,499}$.

Table 1

n	x_n
1	1.166 666 667
2	1.120 110 132
3	1.112 929 312
4	1.112 775 753
5	1.112 775 684
6	1.112 775 684

4. A further generalization

We define the 'Tribonacci numbers' Q_r by

$$Q_0 = 1, \qquad Q_1 = Q_2 = 1, \qquad Q_{r+1} = Q_r + Q_{r-1} + Q_{r-2} \quad (r = 2,3,\ldots).$$

Assume that

$$Q_r = \lambda_1\alpha_1^r + \lambda_2\alpha_2^r + \lambda_3\alpha_3^r.$$

Then, as above, we want α_1, α_2 and α_3 to be roots of the equation

$$x^3 = x^2 + x + 1.$$

This equation has distinct roots, and this enables us to calculate λ_1, λ_2 and λ_3 from the cases $r = 0, 1, 2$. The 'Trinomial theorem' says that

$$(a+b+c)^n = \sum_{p+q+r=n} \frac{n!}{p!\,q!\,r!} a^p b^q c^r.$$

(In this sum, p, q and r are non-negative.) Now

$$\sum_{p+q+r=n} \frac{n!}{p!\,q!\,r!} Q_{2p+q} = \sum_{p+q+r=n} \frac{n!}{p!\,q!\,r!} (\lambda_1\alpha_1^{2p+q} + \lambda_2\alpha_2^{2p+q} + \lambda_3\alpha_3^{2p+q})$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} \lambda_i \sum_{p+q+r=n} \frac{n!}{p!\,q!\,r!} (\alpha_i^2)^p \alpha_i^q 1^r \\
&= \sum_{i=1}^{3} \lambda_i (\alpha_i^2 + \alpha_i + 1)^n \\
&= \sum_{i=1}^{n} \lambda_i \alpha_i^{3n} \\
&= Q_{3n}.
\end{aligned}
$$

Readers may like to verify this in the case $n = 4$, say; the twelfth 'Tribonacci number' is 504.

I leave the task of finding a corresponding sum for the 'n-bonacci numbers' to the reader.

Perfect Boxes

K. R. S. SASTRY, *Box 21862, Addis Ababa, Ethiopia*

The author obtained his bachelor's and master's degrees in mathematics from the University of Mysore, India, and is currently teaching at a secondary school in Addis Ababa. His article on Heronian triangles appeared in *Mathematical Spectrum*, Vol. 8, No. 3. One of his unsolved problems is how to interest students in mathematics in an overcrowded classroom!

1. Introduction

A polygon the lengths of whose sides are natural numbers is called a perfect polygon (reference 1) if its perimeter and area are given by the same number. For example, the right-angled triangle with side-lengths, 10, 8 and 6 is a perfect triangle because both its perimeter and area are equal to 24. Again, the square of side 4, the rhombus of side 5 and height 4, the isosceles trapezium with parallel bases 6 and 14 and each of the oblique sides 5 are some examples of perfect quadrilaterals. All perfect triangles have been determined—in fact there are just five. (Be patient! We shall find these as a by-product of our discussion.) They have been named perfect triangles only recently, but they were determined as early as 1865 (reference 2). As far as I know, the problem of determining all other perfect polygons is still open. However, instead of struggling on in the plane, let us rise high into the three-dimensional world to discover rectangular perfect boxes!

2. What is a perfect box?

First we should define what a perfect box is. By analogy with a perfect polygon, it seems natural to say that a box is perfect if the sum of the edges is equal to its volume. However, analogy is a many-sided activity. Looking again at a perfect polygon, the property of being perfect may well be due to the equality between the one-dimensional quantity, perimeter, and the higher-dimensional quantity, area. Thus a box may also be said to be perfect if the sum of its edges is equal to its total surface area, or if its total surface area equals the volume. We broaden our definition of a perfect box to include all the three types mentioned above.

3. Determination of perfect boxes

In what follows, let the natural numbers l, m and n such that $l \geqslant m \geqslant n$ denote the edges of a rectangular box. Then the statement that such a box is perfect may be given by one of the following three equations, corresponding to different definitions of what is a perfect box:

(i) $4(l+m+n) = lmn$;
(ii) $4(l+m+n) = 2(lm+mn+nl)$;
(iii) $2(lm+mn+nl) = lmn$.

Therefore, the determination of perfect boxes is equivalent to the determination of the natural number solutions (l, m, n) to these three equations. To avoid the repetition of solution strategy, we now discuss only the solution of (i).

We recast (i) as

$$l = \frac{4(m+n)}{mn-4} \quad (mn > 4). \tag{1}$$

Then the constraint $l \geqslant m$ is equivalent to

$$\frac{4(m+n)}{mn-4} \geqslant m.$$

After simplification, this inequality reduces to

$$n \leqslant \frac{8m}{m^2-4} \quad (m > 2). \tag{2}$$

Since n is a natural number, its least value is 1. Therefore, we have

$$1 \leqslant n \leqslant \frac{8m}{m^2-4} \quad (m > 2).$$

This yields $m^2-4 \leqslant 8m$, which implies that $m \leqslant 4+2\sqrt{5}$. Now m is a natural number too. Hence we find that

$$2 < m < 9.$$

When $m = 3$, (2) yields $n \leqslant \frac{24}{5}$. However, $m \geqslant n$ and $mn > 4$. Therefore $n = 3$ or 2 only. We now go back to (1) to find the values of l.

$$m = 3 = n \Rightarrow l = \tfrac{24}{5}, \quad \text{an inadmissible value.}$$

$$m = 3, n = 2 \Rightarrow l = 10, \quad \text{an admissible value.}$$

By continuing this case-by-case examination for $m = 4$, 5, 6, 7 and 8, we obtain the following solution triplets:

$$(l,m,n) = (10,3,2),\ (6,4,2),\ (24,5,1),\ (14,6,1),\ (9,8,1). \qquad \text{(S i)}$$

Likewise, we also find from (ii) and (iii), respectively, that

$$(l,m,n) = (2,2,2),\ (4,2,1) \qquad \text{(S ii)}$$

and

$$(l,m,n) = (10,5,5),\ (20,5,4),\ (6,6,6),\ (12,6,4),\ (42,7,3),$$
$$(8,8,4),\ (24,8,3),\ (18,9,3),\ (15,10,3),\ (12,12,3). \qquad \text{(S iii)}$$

Thus we have, respectively, 5, 2 and 10 perfect boxes corresponding to the three possible definitions. Three of these, namely, $(20,5,4)$, $(12,6,4)$ and $(8,8,4)$ are *super perfect boxes* because they have integral space diagonals!

There are *five* perfect triangles and there are *five* solutions in (S i). Is this a coincidence? Suppose that a, b and c are the sides of a perfect triangle with $a \leqslant b \leqslant c$, and put

$$2s = a+b+c, \qquad s-a = l, \qquad s-b = m, \qquad s-c = n.$$

Then it is easily verified that

$$s = l+m+n, \qquad a = m+n, \qquad b = n+l, \qquad c = l+m, \qquad l \geqslant m \geqslant n.$$

We can use Heron's formula for the area of a triangle to write down the condition that the triangle is perfect, namely,

$$2s = \sqrt{s(s-a)(s-b)(s-c)},$$

or

$$4(l+m+n) = lmn.$$

Lo and behold, this is our equation (i)! Thus, the five perfect triangles hidden in (S i) are easily unearthed by

$$(a,b,c) = (m+n, n+l, l+m)$$
$$= (5,12,13),\ (6,8,10),\ (6,25,29),\ (7,15,20),\ (9,10,17).$$

At this point it is natural to investigate questions like: what integral boxes (boxes with integral dimensions) have the sums of the edges an integral multiple of the volume? Can we find a characterization of such boxes? This amounts to finding natural-number solutions (l, m, n, λ) to the equation

$$\text{(i}'\text{)}\quad 4(l+m+n) = \lambda(lmn)$$

(with $l \geqslant m \geqslant n$). The surprising fact is that λ can assume very few natural numbers and that, for each such λ, the number of boxes (l, m, n) is finite!

Theorem 1. The values of λ for which there is an integral box which has the sum of its edges equal to λ times its volume are $\lambda = 1, 2, 3, 4, 5, 6, 8$ and 12.

Proof. We proceed with (i′) and obtain, as was done in the solution of (i),

$$l = \frac{4(m+n)}{\lambda mn - 4} \geqslant m, \qquad \lambda mn > 4,$$

$$1 \leqslant n \leqslant \frac{8m}{\lambda m^2 - 4}, \qquad 1 \leqslant m \leqslant \frac{4+2\sqrt{\lambda+4}}{\lambda}, \qquad 0 < \lambda \leqslant 12.$$

This shows only that λ is not greater than 12. We have already examined the case $\lambda = 1$ in (Si). We should now go on to examine in a similar fashion the possible solutions for $2 \leqslant \lambda \leqslant 12$. We leave this analysis to interested readers. It then shows that the values $\lambda = 7, 9, 10$ and 11 do not yield natural number solutions (l, m, n). Table 1 shows all possible solutions. In like manner, or perhaps in some ingenious way, we invite readers to prove the following theorems.

Table 1

λ	1	2	3	4	5	6	8	12
(l, m, n)	(Si)	$(8,3,1)$	$(6,2,1)$	$(3,2,1)$	$(8,1,1)$	$(4,1,1)$	$(2,1,1)$	$(1,1,1)$
		$(4,2,2)$	$(2,2,2)$		$(2,2,1)$			
		$(5,4,1)$						

Theorem 2. The values of λ for which there is an integral box which has the sum of its edges equal to λ times its total surface area are $\lambda = 1, 2$.

Table 2 shows all possible solutions.

Table 2

λ	1	2
(l, m, n)	(Sii)	$(1,1,1)$

Theorem 3. The values of λ for which there is an integral box which has its total surface area equal to λ times its volume are $\lambda = 1, 2, 3, 4, 5$ and 6.

Table 3 shows all possible solutions.

Table 3

λ	1	2	3	4	5	6
(l, m, n)	(S iii)	$(6,3,2)$	$(6,3,1)$	$(2,2,1)$	$(2,1,1)$	$(1,1,1)$
		$(4,4,2)$	$(4,4,1)$			
		$(3,3,3)$	$(2,2,2)$			

4. More perfect investigations

In the plane we can generalise our notion of perfect polygons to circles and say that a circle is perfect if its circumference and area are equal. This gives the radius of a perfect circle as 2. In three dimensions we might say that a sphere is perfect if its total surface area is equal to its volume. This gives the radius of a perfect sphere as 3. We can also determine all perfect right-circular cylinders having the same total surface area as volume. This requires solving the equation $2\pi r^2 + 2\pi rh = \pi r^2 h$ in natural numbers r and h. It is also possible to determine perfect right circular cones. Now a rectangular box is a rectangular prism. Our solutions (S i), (S ii) and (S iii) determine all perfect rectangular prisms. Naturally, other questions arise for investigation, as for example the following.

1. Can we determine all perfect triangular prisms? This seems to involve those celebrated triangles, the Heronian triangles, whose sides and areas are all natural numbers. The examples with base triangle $(17, 25, 26)$ are lateral edges of length 3 in which the total surface area equals the volume shows that such prisms do exist.
2. What other perfect solids (in our sense) such as perfect tetrahedrons, perfect rectangular pyramids can be determined?

References

1. Richard L. Francis, Polygons, both perfect and regular, *The Two Year College Mathematics Journal,* May 1975, pp. 20–21.

2. Solution of E 2420, *The American Mathematical Monthly,* June–July 1974, p. 662.

Who knows Nanette?

KEITH AUSTIN, *University of Sheffield*

There are eight boys, Alan, Bob, Charles, David, Eric, Frank, George and Harry, and eight girls, Janet, Kate, Laura, Mary, Nanette, Pat, Ruth and Susan, in the village. Bob and Frank both know Janet, and possibly some other girls. Alan and Eric both know Mary, and possibly some other girls. Harry knows Kate and Ruth, and possibly some other girls. David knows all the girls except Laura and Pat. George knows all the girls except Pat and Susan. Charles knows all the girls except Laura and Susan.

There are four cycles in the village. Whenever a group of boys, which may be as small as one boy, go out for a cycle ride then all the girls they know come to wave them off. Interestingly, there are always at least as many girls waving as boys riding.

On St. Valentine's day each boy wanted to send a card to a girl he knew, he did not mind which. However, the boys were unable to do this in such a way that every girl received a card.

Who knows Nanette?

A comment on the problem

During the 1930s a mathematician at King's College, Cambridge, Philip Hall, considered the general question of pairing off a given collection of girls and boys. In his generalisation he allowed there to be different numbers of boys and girls, and his aim was to pair each boy with a girl that he knew.

He soon realised that for this to be possible the following condition would have to be satisfied:

$$\left.\begin{array}{l}\text{for any group of boys, they must know between them}\\ \text{at least as many girls as there are boys in the group.}\end{array}\right\} (C)$$

It was in the next step that Philip Hall made his breakthrough. He wondered whether the satisfying of condition (C) would in fact guarantee that a pairing-off could be made.

He decided that his speculative idea was possibly true, and so he set about the task of proving it. Eventually he constructed a proof and it was published in 1935 in the *Journal of the London Mathematical Society*.

Computer Column

MIKE PIFF

A dynamical problem

The following program was suggested to me by an article in *Scientific American* last year, at which the guests all had their preferred distance from one another, but all wished to be near the table with the drinks on. I leave it to you to decide what is going on. Perhaps you could rewrite `PROCInitialise` so that the guests are placed at random.

You could also experiment with other functions to replace `FNg(m,m1,d)`. Here, `FNg` represents the 'force' on `m` caused by `m1` being at distance `d` from him. Does this suggest anything to you?

Happy programming!

```
  1 REM Dynamical Program
  2 REM (c) Mike Piff and
  3 REM Mathematical Spectrum
 10 *SHADOW
 20 MODE1
 30PROCInitialise
 40 PROCDisplay
 50REPEAT
 60 PROCMoveMen
 70 t=INKEY(1)
 80UNTIL t<>-1
 90PROCFinish
100END
110DEF PROCDisplay
120LOCAL m
130FOR m=1 TO Pop
140MOVE X(m)*scale-16,Y(m)*scale+16
150PRINT CHR$(m+ASC("0"))
160NEXT
170ENDPROC
180DEF PROCFinish
190VDU4:CLG
200ENDPROC
210DEF PROCInitialise
220 const=0.336
230Pop=4:scale=40
240DIM X(Pop),Y(Pop),NewX(Pop),
   NewY(Pop)
250RoomWidth=32:RoomHeight=25
260RESTORE
270FOR m=1 TO Pop
280 READ X(m),Y(m)
290NEXT
300DATA 30,20
310DATA 20,6
320DATA 6,20
330 DATA 16,12
340GCOL3,1
350VDU5
360ENDPROC
370DEF PROCMoveMen
380FOR m=1 TO Pop
390 PROCEvaluate(m)
400 NewX(m)=BestX:NewY(m)=BestY
410NEXT
420 PROCDisplay
430FOR m=1 TO Pop
440 X(m)=NewX(m):Y(m)=NewY(m)
450NEXT
460 PROCDisplay
470ENDPROC
480DEF PROCEvaluate(m)
490x=X(m):y=Y(m):BestValue=
   FNValue(m,x,y):BestX=x:BestY=y
500FOR i=x-1 TO x+1
510 FOR j=y-1 TO y+1
520  IF (i>=1)AND(i<=RoomWidth)
   AND(j>=1)AND(j<=RoomHeight)
       THEN v=FNValue(m,i,j)ELSE v=0
530  IF v>BestValue THEN
   BestValue=v:BestX=i:BestY=j
540NEXT:NEXT
550ENDPROC
560DEF FNValue(m,i,j)
570LOCAL v,d,x,y
580v=0
590FOR m1=1 TO Pop
600  x=ABS(i-X(m1)):y=ABS(j-Y(m1))
610 IF x>y THEN d=x+const*y
    ELSE d=y+const*x
620  IF m1<>m THEN v=v+FNg(m,m1,d)
630 NEXT
640 =v
650 DEF FNg(m,m1,d)
660 LOCAL g
670 g=1
680 IF (m=1) THEN g=g-((m1=2)*ABS(d-5)
   +(m1=3)*ABS(d-1)+(m1=4)*FNbig(d))
690 IF (m=2) THEN g=g-((m1=1)*ABS(d-1)
   +(m1=3)*ABS(d-5)+(m1=4)*FNbig(d))
700 IF (m=3) THEN g=g-((m1=1)*ABS(d-5)
   +(m1=2)*ABS(d-1)+(m1=4)*FNbig(d))
710 =1/g
720 DEF FNbig(d)=(d-1)*(d-1)
```

Letters to the Editor

Dear Editor,

Pascal's roots

Here is something for those with an interest in the Golden ratio:

$$0.618034\ldots = \tfrac{1}{2}(\sqrt{5}-1) = \binom{1}{1}\left(\frac{1}{5}\right)+\binom{3}{2}\left(\frac{1}{5}\right)^2+\binom{5}{3}\left(\frac{1}{5}\right)^3+\ldots.$$

The reason for this derives from Pascal's triangle, and I was reminded of it by the letter in *Mathematical Spectrum,* Volume 21, Number 1, pages 25–26 from R. F. Talbot. The letter described the fact that the coefficients in the expansion of $(1+x)^{-n}$, where n is a natural number, can be found by extending Pascal's triangle. I would point out that there is one other binomial expansion that can be read from Pascal's triangle. It is the expansion of $(1-4x)^{-\frac{1}{2}}$. Figure 1 shows the triangle with the coefficients indicated in bold. These are precisely the numbers of the form $\binom{2r}{r}$.

$$\begin{array}{ccccccccccccc}
&&&&&&\mathbf{1}&&&&&&\\
&&&&&1&&1&&&&&\\
&&&&1&&\mathbf{2}&&1&&&&\\
&&&1&&3&&3&&1&&&\\
&&1&&4&&\mathbf{6}&&4&&1&&\\
&1&&5&&10&&10&&5&&1&\\
1&&6&&15&&\mathbf{20}&&15&&6&&1
\end{array}$$

Figure 1. Coefficients of $(1-4x)^{-\frac{1}{2}}$ on Pascal's triangle

I leave readers to work out why these entries are indeed the coefficients of that binomial expansion.

Let us give the sequence $1, 2, 6, 20, \ldots$, the labels $A_1, A_2, A_3, \ldots$. Then Pascal's triangle gives a method of evaluating square roots. If we substitute $x = (y-1)/4y$, we get:

$$\begin{aligned}(1-4x)^{-\frac{1}{2}} &= \left(1-4\left[\frac{y-1}{4y}\right]\right)^{-\frac{1}{2}}\\ &= \left(1-\frac{y-1}{y}\right)^{-\frac{1}{2}}\\ &= \left(\frac{1}{y}\right)^{-\frac{1}{2}}\\ &= \sqrt{y}.\end{aligned}$$

So, if we set $y = 2$, $(y-1)/4y = \frac{1}{8}$ and so

$$\sqrt{2} = A_1+A_2(\tfrac{1}{8})+A_3(\tfrac{1}{8})^2+\ldots.$$

If $y = 3$, $(y-1)/4y = \frac{1}{6}$ and

$$\sqrt{3} = A_1 + A_2(\tfrac{1}{6}) + A_3(\tfrac{1}{6})^2 + \dots .$$

If $y = 5$, $(y-1)/4y = \frac{1}{5}$ and

$$\sqrt{5} = A_1 + A_2(\tfrac{1}{5}) + A_3(\tfrac{1}{5})^2 + \dots .$$

Indeed, the square root of any number may be expressed in this way from Pascal's triangle. The formula above for the Golden ratio may be obtained by omitting A_1, to get an expression for $\sqrt{5}-1$, and then halving each coefficient. Now the entries on the central axis of Pascal's triangle can be simply halved by moving to the nearest entries on the row above, as shown.

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&\mathbf{1}&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&\mathbf{3}&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&\mathbf{10}&&10&&5&&1& \\
1&&6&&15&\vdots&20&&15&&6&&1
\end{array}$$

Figure 2. Halving the numbers $\binom{2r}{r}$ gives this column

So the vertical column indicated in bold, which consists of the numbers $\binom{2r-1}{r}$, gives the coefficients in our expression for the Golden ratio. Finally, if the above means of evaluating $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. are not sufficiently rapidly convergent, many other substitutions in the binomial expansion will give rapid convergence. For example, with $x = \frac{1}{20}$,

$$\sqrt{5} = 2[A_1 + A_2(\tfrac{1}{20}) + A_3(\tfrac{1}{20})^2 + \dots].$$

This expansion gives $\sqrt{5}$ to three decimal places from only five easily calculated terms.

Yours sincerely,

I. M. Richards
Penwith Sixth Form College,
Penzance,
Cornwall.

Dear Editor,

$$x^n + y^n = z^m$$

In *Mathematical Spectrum* Volume 20 Number 2 pp. 56–57, Mike Swain defined a primitive solution of the equation $x^n+y^n = z^m$ to be a triple of positive integers (x,y,z) satisfying the equation and such that $\mathrm{hcf}(x,y,z) = 1$. On this basis, $(10,5,5)$, although a solution of the equation $x^2+y^2 = z^3$, is not a primitive solution. Because the equation is not homogeneous in x, y and z, perhaps a better definition of a primitive solution would be a solution (x,y,z) such that there is no integer $a > 1$ such that $a^m|x$, $a^m|y$ and $a^n|z$. (If there is such an integer a, then $(x/a^m, y/a^m, z/a^n)$ is also a solution of the equation.) With this definition, $(10,5,5)$ is a primitive solution of the equation $x^2+y^2 = z^3$. Mike Swain gives the formulae $x = |s^3-3st^2|$, $y = |t^3-3s^2t|$, $z = s^2+t^2$, where s and t are positive integers such that hcf$(s,t) = 1$ and one of s and t is even and the other is odd. These do give primitive solutions, but do not include the solution $(10,5,5)$. Indeed, $(p[p^2+q^2], q[p^2+q^2], p^2+q^2)$ is a solution, where p and q are positive integers; $p = 2$, $q = 1$ gives the solution $(10,5,5)$.

Mike Swain also listed some solutions of the equation $x^3+y^3 = z^2$, but was unable to recognize any pattern. If you add x and y for his solutions, you obtain either a square or 3 times a square, as seen in the tables.

Table 1

x	y	$x+y$	z
2	1	$3 = 3\times1^2$	3
37	11	$48 = 3\times4^2$	228
433	242	$675 = 3\times15^2$	9765
877	851	$1728 = 3\times24^2$	35928
1177	23	$1200 = 3\times20^2$	40380
1201	122	$1323 = 3\times21^2$	41643

Table 2

x	y	$x+y$	z
65	56	$121 = 11^2$	671
112	57	$169 = 13^2$	1261
312	217	$529 = 23^2$	6371
1064	305	$1369 = 37^2$	35113

It may be noted that, in table 2, $x+y$ is always the square of a prime and that either x or y is divisible by 7. I have proved that, when (x,y,z) is a solution of $x^3+y^3 = z^3$ such that $x+y$ is a perfect square, then either x or y is divisible by 7. My proof is quite involved.

Yours sincerely,

K. R. S. Sastry
Box 21862,
Addis Ababa,
Ethiopia

Dear Editor,

Divisibility by n

May we add a comment to your recent article and correspondence on divisibility by 7.

Let r be a natural number not divisible by 2 or 5 and let m be the integer of smallest magnitude for which $r|10m-1$. To test a natural number n in decimal form for divisibility by r, remove the last digit d and add md, giving $\frac{1}{10}(n-d)+md = n'$, say. Thus, $n = 10n'-(10m-1)d$ and so $r|n \Leftrightarrow r|n'$. If either $m > 0$ and $n > 10m-1$ or $m < 0$ and $n > -9(10m-1)$, then $n > n' > 0$, and we can repeat the procedure. Table 1 lists m for the first few primes.

Table 1

r	3	7	11	13	17	19	23	29	31	37	41
m	1	−2	−1	4	−5	2	7	3	−3	−11	−4

For instance, checking 964 838 for divisibility by 7 yields the sequence 96 467 (i.e. $96483-2\times8$), 9632, 959, 77. Since 77 is divisible by 7 but not by 3, the same is true of 964 838 because we are actually checking for divisibility by $|10m-1|$, which is 21 in this case.

We draw attention to the tests for 3 and 11.

The process clearly generalizes to any base.

When $m = 4$, it gives a partition of $\{1, 2, \ldots, 39\}$ into the following cycles:

1	4	16	25	22	10
2	8	32	11	5	20
3	12	9	36	27	30
6	24	18	33	15	21
7	28	34	19	37	31
13					
14	17	29	38	35	23
26					
39					

Table 2 shows the cycles for positive m up to 15. Their multiplicities are given in brackets.

Table 2

m	1	2	3	4	5
$10m-1$	9	19	29	39	49
No of fixed points	9	1	1	3	1
Cycle lengths	–	18(1)	28(1)	6(6)	42(1) 6(1)
m	6	7	8	9	10
$10m-1$	59	69	79	89	99
No of fixed points	1	3	1	1	9
Cycle lengths	58(1)	22(3)	13(6)	44(2)	2(45)
m	11	12	13	14	15
$10m-1$	109	119	129	139	149
No of fixed points	1	1	3	1	1
Cycle lengths	108(1)	6(1) 16(1) 48(2)	21(6)	46(3)	148(1)

Yours sincerely

A. J. DOUGLAS AND G. T. VICKERS
(University of Sheffield)

Correction to the article 'Summing the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$' by Nicholas Shea in Volume 21 Number 2

We are sorry that a section was printed out of order. The last two lines of page 50 and the top two-thirds of page 51 should follow on after page 49 in Section 1, followed by Section 2. Also, the series expansion of $\arcsin x$ in Section 3 is incorrect; it should be

$$\arcsin x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1\times 3}{2\times 4}.\frac{x^5}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}.\frac{x^7}{7} + \ldots .$$

That makes the rest of the argument in this section invalid.

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

21.9. (Submitted by Gregory Economides, Sixth Form, Royal Grammar School, Newcastle upon Tyne)
The speed u of projection of a particle projected to clear a wall of height H when the point of projection is at a distance D from the base of the wall, is given by

$$u^2 = \frac{gD^2}{D\sin 2\theta - H - H\cos 2\theta},$$

where θ is the angle of projection. (Since $\tan\theta > H/D$, $u^2 > 0$.) Find, *without the use of calculus*, the minimum speed of projection.

21.10. (Submitted by M. A. Hamieh, The Lebanese University, Beirut)
Let E be a finite non-empty set of natural numbers and, for each non-empty subset X of E, denote by $\|X\|$ the sum of the elements of X. What is $\sum_X \|X\|$, where the sum is taken over all non-empty subsets of E?

21.11. (Submitted by Russell Euler, Northwest Missouri State University)
Prove that

$$\frac{1}{1}\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

where $\binom{n}{r}$ denotes the binomial coefficient.

21.12. (Submitted by A. J. Douglas and G. T. Vickers, University of Sheffield)
Show that, for all positive integers i and j with $i < j$, the sum

$$\frac{1}{i} + \frac{1}{i+1} + \ldots + \frac{1}{j}$$

is not an integer. (We remark that, since the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges and $n^{-1} \to 0$ as $n \to \infty$, there are partial sums

$$1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

which are arbitrarily close to integers.)

Hint: Bertrand's postulate says that, if p_r denotes the rth prime number, then $p_r < p_{r+1} < 2p_r$ for every r.

Solutions to Problems in Volume 21 Number 1

21.1 Prove that, for every positive real number r and negative integer s,

$$\frac{r+s+1}{r+1} \leqslant \left(\frac{r+1}{r}\right)^s.$$

Solution by Gregory Economides (Royal Grammar School, Newcastle upon Tyne)

First we prove Bernoulli's inequality, namely: if x is a real number, $x \geqslant -1$, and n is a natural number, then $(1+x)^n \geqslant 1+nx$. This is true when $n = 1$, so we assume, inductively, that $(1+x)^k \geqslant 1+kx$. Since $1+x \geqslant 0$, we can multiply by $1+x$ to give

$$\begin{aligned}(1+x)^{k+1} &\geqslant (1+kx)(1+x)\\ &= 1+(k+1)x+kx^2\\ &\geqslant 1+(k+1)x,\end{aligned}$$

which completes the inductive step. Thus Bernoulli's inequality holds for all natural numbers n. Now

$$\begin{aligned}\left(\frac{r+1}{r}\right)^s - \frac{r+s+1}{r+1} &= \left(1-\frac{1}{r+1}\right)^{-s} - \left(1+\frac{s}{r+1}\right)\\ &= \left(1-\frac{1}{r+1}\right)^n - \left(1-\frac{n}{r+1}\right), \quad \text{where } n = -s,\\ &\geqslant 0, \quad \text{by Bernoulli's inequality.}\end{aligned}$$

Also solved by Amites Sarkar (Winchester College), Nicholas Shea (University College, Oxford) and Dylan Gow (Oakham School).

21.2. Show that there is a positive integer N such that $x!$ is not a highly composite number when x is an integer greater than N.

Solution by Gregory Economides

Let $x \geqslant 4$. We can express $x!$ in the form

$$x! = 2^{r_1}3^{r_2}\ldots p_n^{r_n},$$

where p_n denotes the nth prime number and the exponents r_i are given by

$$r_i = \left[\frac{x}{p_i}\right] + \left[\frac{x}{p_i^2}\right] + \ldots + \left[\frac{x}{p_i^k}\right]$$

for $1 \leqslant i \leqslant n$, where $[\alpha]$ denotes the greatest integer less than or equal to α and k is the positive integer such that $p_i^k \leqslant x < p_i^{k+1}$. Now

$$\tfrac{7}{8}x! = 2^{r_1-3}3^{r_2}5^{r_3}7^{r_4+1}\ldots$$

and $\frac{7}{8}x!$ is a positive integer less than $x!$. If

$$(r_1-3+1)(r_2+1)(r_3+1)(r_4+1+1)\ldots(r_n+1) \geqslant (r_1+1)(r_2+1)\ldots(r_n+1),$$

i.e. if

$$3r_4+5 \leqslant r_1, \tag{1}$$

then $d(x!) \leqslant d(\frac{7}{8}x!)$, where $d(\beta)$ denotes the number of positive divisors of the positive integer β. Now

$$r_1 = \left[\frac{x}{2}\right]+\left[\frac{x}{2^2}\right]+\ldots+\left[\frac{x}{2^k}\right],$$

where $2^k \leqslant x < 2^{k+1}$, so that

$$\begin{aligned} r_1 &\geqslant \left(\frac{x}{2}-\frac{1}{2}\right)+\left(\frac{x}{2^2}-\frac{3}{2^2}\right)+\ldots+\left(\frac{x}{2^k}-\frac{2^k-1}{2^k}\right) \\ &= (x+1)\left(1-\frac{1}{2^k}\right)-k \\ &\geqslant x-k-1 \quad (\text{because } x \leqslant 2^{k+1}-1). \end{aligned}$$

Thus

$$r_1 \geqslant x-\frac{\ln x}{\ln 2}-1 \quad (\text{because } 2^k \leqslant x). \tag{2}$$

Also,

$$\begin{aligned} r_4 &= \left[\frac{x}{7}\right]+\left[\frac{x}{7^2}\right]+\ldots+\left[\frac{x}{7^m}\right] \quad (\text{where } 7^m \leqslant x < 7^{m+1}) \\ &\leqslant \frac{x}{7}+\frac{x}{7^2}+\ldots+\frac{x}{7^m} \\ &= \frac{x}{6}\left(1-\frac{1}{7^m}\right). \end{aligned}$$

Thus

$$r_4 < \tfrac{1}{6}x. \tag{3}$$

It follows from (2) and (3) that, if

$$3\times\tfrac{1}{6}x+5 \leqslant x-\frac{\ln x}{\ln 2}-1,$$

i.e. if

$$x-\frac{2\ln x}{\ln 2}-12 \geqslant 0,$$

then inequality (1) is satisfied. Put

$$f(x) = x-\frac{2\ln x}{\ln 2}-12.$$

Then

$$f'(x) = 1-\frac{2}{x\ln 2} > 0 \quad \text{when } x > 2,$$

so that $f(x)$ is strictly increasing when $x > 2$. Since $f(21) > 0$ [$f(20) < 0$], we have that $f(x) > 0$ when $x \geqslant 21$. Thus, when $x \geqslant 21$, $d(x!) < d(\frac{7}{8}x!)$ and $x!$ is not a highly composite number.

It can easily be shown that

$$d(\tfrac{3}{4}\times 8!) = d(8!), \qquad d(\tfrac{5}{6}\times 9!) > d(9!),$$

that $d(\frac{7}{8}\times x!) \geqslant d(x!)$ when $10 \leqslant x \leqslant 20$, and that $2!, 3!, \ldots, 7!$ *are* highly composite numbers.

21.3. Denote by $P_n(x)$ the polynomial $x^{n+1}+(n-x)(x+1)^n$, where n is a positive integer. Prove that

(i) when n is odd, $P_n(x) > 0$ for all real numbers x;

(ii) when n is even, $P_n(x)$ has exactly one real root.

Solution by Gregory Economides

We can write $P_n(x)$ as

$$P_n(x) = x^{n+1}+(n+1)(x+1)^n-(x+1)^{n+1},$$

whence

$$P_n'(x) = (n+1)\{x^n+n(x+1)^{n-1}-(x+1)^n\},$$

so that

$$P_n'(x) = (n+1)\,P_{n-1}(x). \tag{*}$$

The identity

$$a^{n+1}-b^{n+1} = (a-b)(a^n+a^{n-1}b+a^{n-2}b^2+\ldots+b^n)$$

gives that

$$(x+1)^{n+1}-x^{n+1} = (x+1)^n+(x+1)^{n-1}x+(x+1)^{n-2}x^2+\ldots+x^n.$$

We consider the difference

$$(x+1)^{n-r}x^r-(x+1)^{n-r-1}x^{r+1} = (x+1)^{n-1}\left(\frac{x}{x+1}\right)^r$$

for $r = 0, 1, 2, \ldots, n-1$. If n is odd and either $x > 0$ or $x < -1$, then this expression is positive, so that

$$(x+1)^n > (x+1)^{n-1}x > (x+1)^{n-2}x^2 > \ldots > x^n.$$

Hence

$$(x+1)^{n+1}-x^{n+1} < (n+1)(x+1)^n,$$

which gives that $P_n(x) > 0$. If n is odd and $-1 \leqslant x < 0$, then $x^{n+1} > 0$ and $(n-x)(x+1)^n \geqslant 0$. So again $P_n(x) > 0$. Finally, $P_n(0) = n > 0$. This proves (i).

If n is even, we know from (i) that $P_{n-1}(x) > 0$ for all x, and (*) now gives that $P_n'(x) > 0$ for all x. Thus $P_n(x)$, a polynomial of odd degree $n-1$, is an increasing function of x and so it has exactly one real root.

21.4. Determine the number of sequences of n terms using 0, 1 and 2 such that there are no two consecutive 1's and no two consecutive 2's.

Solution by Gregory Economides

Let $u_{n,0}$, $u_{n,1}$ and $u_{n,2}$ denote the number of sequences of n terms beginning with 0, 1 and 2, respectively, such that there are no two consecutive 1's and no two consecutive 2's and let $u_n = u_{n,0}+u_{n,1}+u_{n,2}$. Then, for $n \geqslant 1$,

$$u_{n,0} = u_{n-1}, \qquad u_{n,1} = u_{n-1,0}+u_{n-1,2}, \qquad u_{n,2} = u_{n-1,0}+u_{n-1,1},$$

so that

$$\begin{aligned} u_n &= u_{n,0}+u_{n,1}+u_{n,2} \\ &= u_{n-1}+2u_{n-1,0}+u_{n-1,1}+u_{n-1,2} \\ &= 2u_{n-1}+u_{n-1,0} \\ &= 2u_{n-1}+u_{n-2}, \quad \text{when } n \geqslant 2. \end{aligned}$$

The quadratic equation $x^2 = 2x+1$ has roots $1 \pm \sqrt{2}$, so that the general solution of this difference equation is

$$u_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n,$$

where A and B are independent of n. With $u_1 = 3$ and $u_2 = 7$, this gives

$$u_n = \tfrac{1}{2}\{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}\}.$$

Also solved by Amites Sarkar, Nicholas Shea and Dylan Gow.

Reviews

Tales of Physicists and Mathematicians. By S. G. GINDIKIN (translated from the Russian by Alan Shuchat). Birkhäuser, Basel. Pp. 157. SFR.48·00. ISBN 3-7643-3317-0.

This book makes no attempt to be a comprehensive history of science or mathematics. Instead it concentrates on a small number of people and fundamental ideas in mathematics and physics from the mid-sixteenth century to the mid-nineteenth century. The chapters are centred on Cardano (and Tartaglia), Galileo, Huygens, Pascal and Gauss. The Russian author's enjoyment of his subject matter is evident, and he communicates this in a fascinating way. He writes in his foreward that 'there is no greater satisfaction than following the flight of fancy of a genius, no matter how long ago he lived'. The book is unusual in the way that biographical details and historical context are blended with a serious attempt to describe the mathematics involved. So there is mathematical 'meat' to stretch the reader, for example in Huygens' work on the isochronous pendulum or in Gauss' proof of the constructibility of the regular 17-gon. Some but not all of the mathematics goes well beyond school level, but it is presented so that an able and interested sixth-former will understand sufficient to whet his or her appetite for more.

The author succeeds in presenting mathematics as a human endeavour with all the disappointments, frustrations and excitement that that involves. I found, for example, the description of Olaf Römer's calculation in 1676 of the speed of light and his dramatic demonstration most interesting. The chapters are also spiced with intriguing biographical details. I had not known that the new-born baby Cardano needed to be revived in a bath of warm wine! Nor that it was Pascal who first had the idea for a wheelbarrow.

I recommend this book as a stimulating and enjoyable introduction to further reading in the history of science and mathematics.

Fettes College, Edinburgh C. B. G. ASH

From One to Zero. By GEORGES IFRAH. Penguin Books, London, 1987. Pp. xvi + 497. £8·95 (ISBN 0-14-009919-0).

This is a highly entertaining book in which the author takes the reader on a fascinating journey of exploration into the evolution of numbers and number systems. This comprehensive history of number is written in an engaging, lucid style, is lavishly illustrated and presupposes only the most elementary of knowledge.

Can animals count? Which god was dismembered to reappear as a notation for fractions? What is the oldest zero in history? Who invented an elaborate system of coloured knots for recording numbers? Where was a tadpole used to denote 100 000? The answers to these, and many more questions, are to be found in this book which can be strongly recommended to anyone wishing to know the origins of the Hindu–Arabic numerals we use today and to any teacher of mathematics searching for interesting material to enliven his lessons. Do you understand the title of the book? If not, then perhaps you should go out and buy a copy too!

University of Sheffield R. J. WEBSTER

Rubik's Cubic Compendium. By ERNÖ RUBIK, TAMÁS VARGA, GERZSON KÉRI, GYÖRGY MARX and TAMÁS VEKERDY. Oxford University Press, 1989. Pp. xii + 226. Hardback £14·95. ISBN 0-19-853202-4.

The apparent simplicity of the Rubik's cube conceals great complexity, as Ernö Rubik himself discovered in 1974. A few years later, when the cube had swept across the world, winning many awards in the process, millions of other people were wrestling with the same complexity. The cube had become the theme of articles in journals, newspapers and magazines, the subject of television programmes, an analogy for quarks, and a very elegant yet complicated example of a permutation group.

This book first appeared in Hungarian during the early phase of cubemania in 1981. It is written in seven chapters, with an introduction by David Singmaster, in which he explains why the cube became so popular so rapidly. The seventh chapter, an afterword also written by Singmaster, gives a brief history of the cube since 1981, and describes some subtle variations of the cube, such as Rubik's octahedron.

The first chapter is an autobiographical account of the invention of the cube by Ernö Rubik. It is written in a frank and modest style and every step in the process of invention is described in detail. Chapters 2 and 3 both contain different restoration methods, although I found the method described in chapter 3 more useful than that in the second chapter. The latter is supplemented with many exercises designed to familiarise the reader with various elementary properties of the cube. Some of these exercises lead the reader into useful sequences of moves for restoring the cube which are not covered in the text. Solutions to these are provided. Chapter 3 describes some algorithms that restore the cube in a small number of moves, such as the Cambridge method (requiring at most 96 moves in its original form) and Thistlethwaite's algorithm which can be modified to provide a solution in at most 50 moves. The problem of restoring the related $3 \times 3 \times 2$ magic domino is also considered. At the end of this chapter there are voluminous tables giving important sequences of moves which perform various simple transformations.

The authors of the previous two chapters collaborate in chapter 4 to produce an essay on the mathematics of the cube. The first section gives impossibility proofs of some theorems (such as 'It is not possible to flip a single edge without moving any other edges'). The second section, supplemented with fascinating colour diagrams, explains some basic group theory connected with the cube. Chapter 5 is called 'The Universe of the Cube', the title being fairly self-explanatory, and chapter 6 is about the psychology of the cube.

A brief exercise in error-hunting revealed a few trivial copying errors (e.g. '5.2×10^{26}' on page 153 should read '5.2×10^{20}' and 'Black', part of the caption to figure 2.11 on page 26 should probably be 'Back'). However, there were a few serious mistakes: the captions to the diagrams in figure 3.5, page 102, should read, from left to right, '$(FL^1D^2LF^1T^2)^2$, $F(FL^1D^2LF^1T^2)^2F^1$, $B(FL^1D^2LF^1T^2)^2B^1$.' Also, I found six mistakes in the tables in chapter 3. The correct versions for the sequences numbered below are:

3.1.9	$FRTR^1F^2LFL^2TLT^1$
3.2.2	$R^1T^2R^2TR^1T^1R^1T^2LFRF^1L^1$
3.2.9	$FRTR^1T^1F^1F^1L^1T^1LTF$
3.2.11*c*	$F^2LFTFT^2F^1L^1T^1B^1T^2BLF^1L^1F^2$
3.4.7*b*	$LT^1FRTR^1F^2LFL^2T$
3.4.10	$R^2B^1RBTF^1TFT^2RTR^1T^1R^2$

Parts of the book require a familiarity with some basic terminology in group theory, although most of the chapters can be enjoyed by the devoted non-mathematical reader, as there is little technical mathematics in the book. Perhaps this is the reason why it is so entertaining to read, for the authors view the cube not only from a mathematical point of view, but from all other angles as well.

Winchester College AMITES SARKAR

A Concise History of Mathematics. By DIRK J. STRUIK. Fourth revised edition. Dover Publications, New York, 1987. Pp. xii+228. £6·75 paperback. ISBN 0-486-60255-9.

With such an enormous task it is inevitable that this most compact reference history should be only a sketch of the unfolding of a few trends in the development of mathematics from the first available records to the middle of the twentieth century. Undoubtedly, too, many mathematicians have been omitted, notably those from Russia and the Far East. Dr Struik sets the major contributors to mathematical knowledge, including Euclid, Archimedes, Newton, Leibnitz, Euler, Gauss, Cauchy and Riemann, along with their findings, in the social and cultural context of the day. The early supremacy of the French, Germans and Italians is without question. It is stimulating to read of each's mathematical fertility, breadth of interest and how contemporaries related to each other. Often there was mutual encouragement, with admiration, but sometimes there was bitter rivalry.

Instances are given when even a 'great' like Euler has a lapse, showing, for example, a lack of concern with rigour when concluding that

$$n+n^2+\ldots = \frac{n}{1-n}$$

and

$$1+\frac{1}{n}+\frac{1}{n^2}+\ldots = \frac{n}{n-1}$$

together imply

$$\ldots+\frac{1}{n^2}+\frac{1}{n}+1+n+n^2+\ldots = 0.$$

The chapters giving the development up to the nineteenth century are very readable by interested sixth-formers, but the modern topics might make more sense if a mathematical dictionary were to hand.

This would make a good book for the library, containing as it does a multitude of biographical references. Moreover, there are facsimiles of pages from original publications which might provide a challenge to interpret.

Totton College, Southampton — A. J. DAVIES

Probability: An Introduction. By SAMUEL GOLDBERG. Dover Publications Inc, New York. Pp. xiv+315. £6·35.

This book is a reissue of one first published in 1960. The author, then Emeritus Professor of Mathematics at Oberlin College and one-time student of William Feller, set out to provide an introduction to probability that required no knowledge of calculus by the reader but only a good background in high-school algebra. Consequently, the book restricts itself to covering finite probability spaces.

The author adopts the approach of defining probability as a measure on a sample space and the language of set theory is used throughout. However, the author makes a considerable effort to maintain the connection between the abstract approach and applications. For instance, there is an illustration of the application of probability theory to population genetics and the factors influencing evolution. Using the method of difference equations the author demonstrates the Hardy–Weinberg law, i.e. the distribution of the three genotypes (with respect to a single gene) is fixed after one generation.

The book covers random variables, probability functions, mean, variance, joint probability functions, covariance, correlation and sample means. The final chapter restricts its discussion of probability distributions to the binomial distribution including the testing of statistical hypothesis and an example of decision-making.

This is a very readable account of basic probability theory with a large number of clearly explained worked examples. It deals with matters which are usually glossed over in A-level text books such as the fact that if the probability of an event is zero it does not necessarily mean that it is impossible. This lays the ground work for the study of continuous probability spaces. The book would make suitable reading for good sixth formers who intend to study probability and statistics in higher education. With its collection of 360 problems with answers supplied to half of them it also makes an excellent resource book for teachers.

Portsmouth Sixth Form College L. A. FEARNEHOUGH

Bivariate Data. Edited by A. C. BAJPAI. Micros in Mathematics Education Statistics Unit 3. John Wiley Software. £21·75 (ISBN 0-471-91747-8). 2 program discs for BBC micro B or master, teacher's notes, user's manual.

This unit is concerned with the description and analysis of bivariate data. Correlation, linear and non-linear regression models are examined in detail. The properties of the bivariate normal distribution are dealt with and there is an excellent section in which simple additive and multiplicative seasonal models are applied to time series using both simulated and published data sets.

The package begins with a useful guide for inexperienced users and it progresses unambiguously through the material, making full use of graphics and colour. The package is very easy to use. Throughout, emphasis is put on learning by experiment; the laws for linear combinations of normal distributions are determined by the user, and series of data sets allow users to get a 'feel' for the properties and limitations of various correlation coefficients, regression models and time series models.

Part 1 consists of a computer simulation of stopping distances, and it seems somewhat artificial; a bug also occurs here after changing the variables which are plotted. Many of the computer calculations take a considerable time and could be accelerated by using an assembler. The method of data entry (by moving a point of the screen) could be usefully augmented by a numerical direct-entry facility to allow real data to be used.

The program is best suited to use by an individual. The long calculations needed to experiment with data in the classroom are avoided, although the whole package takes a long while to complete. Consultation with a teacher is required to confirm, formalise and explain the results obtained, and to detail the methods of calculation which are used. However, a thorough student would finish with a detailed knowledge and true understanding of the statistics covered, to a standard beyond that required for A-level mathematics.

6th former at Gresham's School, Holt, Norfolk NICHOLAS SHEA

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L O N D O N   M A T H E M A T I C A L   S O C I E T Y

1989 POPULAR LECTURES

Professor F.G. Leppington
Asymptotic Approximation Simplified

Dr R.J. Wilson
Stamping through Mathematics

Liverpool University - Friday 23 June
Imperial College London - Friday 30 June

Popular lectures are once again being organised by the London Mathematical Society. The lectures are given by mathematicians in a way that is accessible to a wide audience of teachers, sixth formers and people with a general interest in mathematics.

The lectures will be given at the University of Liverpool on Friday 23 June at 5.00 p.m. and at Imperial College London on Friday 30 June at 7.30 p.m.

Admission is free, by ticket obtainable in advance. For the Liverpool venue write to Dr I.R. Porteous, Department of Pure Mathematics, University of Liverpool, PO Box 147, Liverpool L69 3BX and for the London venue write to Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V ONL. A stamped addressed envelope would be much appreciated.

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