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CONTENTS

ON TRIANGLES WITH GIVEN RATIOS OF ANGLE COSINES

O. BOTTEMA and LÉO SAUVÉ

Let l, m, n be given real numbers. We ask whether there exists a triangle ABC (possibly degenerate) with angles α, β, γ such that

$$\cos\alpha : \cos\beta : \cos\gamma = l : m : n.$$

The problem is trivial if $lmn = 0$, so we assume that $lmn \neq 0$. We will need the following fact:

$$1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 0 \tag{1}$$

and

$$0 < \alpha+\beta+\gamma < 2\pi, \qquad -\pi \leq \beta+\gamma-\alpha,\ \gamma+\alpha-\beta,\ \alpha+\beta-\gamma \leq \pi \tag{2}$$

are necessary and sufficient conditions for α, β, γ to be the angles of a (possibly degenerate) triangle. This is an immediate consequence of the identity

$$\begin{aligned} 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma \\ = -4\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\beta+\gamma-\alpha}{2}\cos\frac{\gamma+\alpha-\beta}{2}\cos\frac{\alpha+\beta-\gamma}{2}, \end{aligned}$$

which is valid for all angles α, β, γ [1]. We consider two cases.

Case 1: l, m, n all have the same sign. Here we may assume that

$$l : m : n = a : b : c,$$

where $a, b, c > 0$. Suppose there exists a triangle ABC such that the ratios $(\cos\alpha)/a$, $(\cos\beta)/b$, $(\cos\gamma)/c$ are all equal. Then these ratios are all nonzero and there exists an $x \neq 0$ such that

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} = \frac{1}{x}. \tag{3}$$

Now (1) holds if and only if

$$f(x) \equiv x^3 - (a^2+b^2+c^2)x - 2abc = 0. \tag{4}$$

The discriminant of (4) is

$$D = 4\{(a^2+b^2+c^2)^3 - 27a^2b^2c^2\},$$

and it follows from the A.M.-G.M. inequality that $D \geq 0$. Therefore (4) has three real roots, exactly one of which is positive (by Descartes' rule of signs). The two negative roots do not satisfy (3) since the cosines of the angles of a triangle cannot all three be negative (not even in a degenerate triangle). Let x_0 be the positive root. It is easily verified that $f(0)$, $f(a)$, $f(b)$, and $f(c)$ are all strictly negative. Thus x_0 satisfies (3) since $x_0 > \max\{a,b,c\}$ and, from (3), the angles of our assumed triangle ABC are

$$\alpha = \operatorname{Arccos}\frac{a}{x_0}, \qquad \beta = \operatorname{Arccos}\frac{b}{x_0}, \qquad \gamma = \operatorname{Arccos}\frac{c}{x_0}. \tag{5}$$

If such a triangle exists, it is therefore acute-angled and unique (to within similarity transformations).

To prove existence, we merely note from (5) that $0 < \alpha,\beta,\gamma < \pi/2$, so that

$$0 < \alpha+\beta+\gamma < \frac{3\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \beta+\gamma-\alpha,\ \gamma+\alpha-\beta,\ \alpha+\beta-\gamma < \pi,$$

and it follows from (1) and (2) that α,β,γ are the angles of a triangle.

This unique triangle ABC is easily constructed once we have found the positive root x_0 of (4). We start by drawing a circle of diameter x_0. We choose two diametrically opposite points B and C on this circle, and then, on the same side of segment BC, we cut off points P and Q such that the chords BP and CQ have lengths b and c, respectively, as shown in Figure 1. If BP and CQ meet in A, then ABC is the required triangle. For

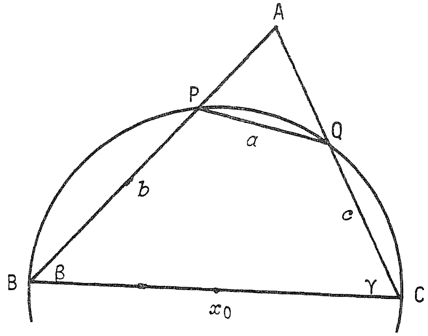

Figure 1

$$\angle PBC = \operatorname{Arccos}\frac{b}{x_0} = \beta \quad \text{and} \quad \angle QCB = \operatorname{Arccos}\frac{c}{x_0} = \gamma.$$

It is interesting to note that PQ = a. For a proof of this, see [2].

Case 2: l,m,n do not all have the same sign. Here we may assume that

$$l : m : n = -a : b : c,$$

where again $a,b,c > 0$. Suppose there exists a triangle ABC such that the ratios $(\cos\alpha)/(-a)$, $(\cos\beta)/b$, $(\cos\gamma)/c$ are all equal. Then these ratios are all nonzero and there exists a $y \neq 0$ such that

$$\frac{\cos\alpha}{-a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c} = \frac{1}{y}. \tag{6}$$

Now (1) holds if and only if

$$g(y) \equiv y^3 - (a^2+b^2+c^2)y + 2abc = 0. \tag{7}$$

The discriminant D of (7) is the same as in Case 1, and $D \geq 0$ with equality if and only if $a = b = c$. Therefore (7) has three real roots, one negative and two positive (by Descartes' rule of signs). The negative root does not satisfy (6), for no triangle can have two negative angle cosines (not even a degenerate triangle). If a,b,c are all equal, say to k, then (7) reduces to

$$y^3 - 3k^2y + 2k^3 = 0,$$

with negative root $-2k$ and double positive root k. We then have $\cos\alpha = -1$ and $\cos\beta = \cos\gamma = 1$, so ABC is the degenerate triangle with angles $(\pi,0,0)$, whose existence is not in doubt.

We now assume that a,b,c are not all equal. Then $D > 0$ and the two positive roots of (7) are distinct. We will call them y_0 and y_1, with $y_0 > y_1 > 0$. For any $y > 0$, it follows from (6) that α is an obtuse angle, so $\cos\alpha = -\cos\delta$, where $0 < \delta=\beta+\gamma < \pi/2$. Now $\beta < \delta$ and $\gamma < \delta$ imply that $\cos\beta > \cos\delta$ and $\cos\gamma > \cos\delta$. Hence an additional necessary condition for the existence of our triangle ABC is that

$$a < \min\{b,c\}. \tag{8}$$

Because $g(0) > 0$ and

$$g(a) = -a(b-c)^2 \le 0, \quad g(b) = -b(c-a)^2 < 0, \quad g(c) = -c(a-b)^2 < 0,$$

it follows that $y_1 \le \min\{a,b,c\}$, so the smaller positive root y_1 does not satisfy (6). But the larger positive root y_0 is satisfactory since $y_0 > \max\{a,b,c\}$. So the angles of our assumed triangle ABC are, from (6),

$$\alpha = \operatorname{Arccos}\frac{-a}{y_0}, \qquad \beta = \operatorname{Arccos}\frac{b}{y_0}, \qquad \gamma = \operatorname{Arccos}\frac{c}{y_0}. \tag{9}$$

If such a triangle exists, it is therefore obtuse-angled and unique (to within similarity transformations).

To prove existence, we note from (9) that

$$\frac{\pi}{2} < \alpha < \pi \quad \text{and} \quad 0 < \beta,\gamma < \frac{\pi}{2},$$

and from (8) that

$$\pi-\alpha > \beta \quad \text{and} \quad \pi-\alpha > \gamma.$$

From these inequalities we obtain

$$\frac{\pi}{2} < \alpha+\beta+\gamma < 2\pi, \qquad -\pi < \beta+\gamma-\alpha < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \gamma+\alpha-\beta,\ \alpha+\beta-\gamma < \pi,$$

and it follows from (1) ard (2) that α,β,γ are the angles of a triangle.

This unique triangle ABC is easily constructed once we have found the larger positive root y_0 of (7). We start by drawing a circle of diameter y_0. We choose two diametrically opposite points B and C, and then, on the same side of segment BC, we cut off points P and Q such that the chords BP and CQ have lengths b and c, respectively, as shown in Figure 2. If BP and CQ meet in A, then ABC is the required triangle. For

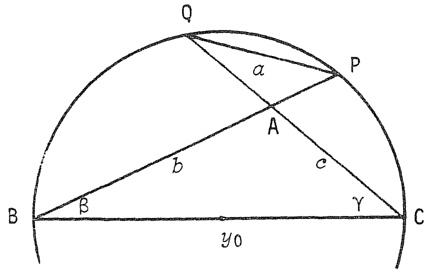

Figure 2

$$\angle PBC = \text{Arccos}\,\frac{b}{y_0} = \beta \quad \text{and} \quad \angle QCB = \text{Arccos}\,\frac{c}{y_0} = \gamma.$$

As for Case 1, it follows from [2] that PQ = a.

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1. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 46.

2. Charles W. Trigg, solution to Crux 699, *Crux Mathematicorum*, 8 (1982) 320-322.

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* * *

CIRCLING THE SQUARE

The nine-nonzero-digit magic square

6	1	8
7	5	3
2	9	4

with its magic constant, S = 15, establishes a tenuous relationship of square to circle through its determinant, D = 360. Indeed,

$$\frac{D}{S} = \frac{360}{15} = 24 = 8\sqrt{9},$$

which, with a slight stretch of the imagination, can be construed to be an arrangement of the ten decimal digits (though you may have to turn the sheet around to see the fractured seven).

CHARLES W. TRIGG

* * *

THE OLYMPIAD CORNER: 52

M.S. KLAMKIN

I start off this month with the problems of the 1983 Brazilian Mathematical Olympiad, which I received through the courtesy of Angelo N. Barone. As usual, I welcome solutions from readers.

FIFTH BRAZILIAN MATHEMATICAL OLYMPIAD

October 1, 1983

1. Show that the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{1983}$$

has a finite number of solutions, where x,y,z are natural numbers.

2. Triangle ABC is equilateral and has side a. Squares BCPQ, CAMN, and ABRS are constructed, and they are the bases of three square pyramids with vertices V_1, V_2, V_3, all the edges being of length a. The pyramids are rotated about BC, CA, and AB until V_1,V_2,V_3 all coincide. Show that, after the rotations, MNPQRS is a regular hexagon.

3. Show that $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not an integer for any natural number $n \geq 2$.

4. Show that all the points of a circle can be coloured, each with one of two colours, in such a way that no inscribed right triangle has its three vertices all of the same colour.

5. (a) Prove that $1 \leq \sqrt[n]{n} \leq 2$ for every natural number $n \geq 1$.

   (b) Find the smallest real number k such that $1 \leq \sqrt[n]{n} \leq k$ for every natural number $n \geq 1$.

6. A sphere being given, show that the largest number of spheres congruent to and tangent to the given sphere, no two of which have any interior point in common, is at least 12 and at most 14.

   Try to refine this estimate.

*

I now give the problems set in the second parts of the 1979, 1980, and 1981 Michigan Mathematics Prize Competitions, for providing which I am grateful to Yousef Alavi. These problems are much easier than those set at the International Mathematical Olympiad (I.M.O.), so they should be accessible to a larger number of students. Note, however, that in the Michigan Competitions the average time allowed for each problem is only 20 minutes, whereas at the I.M.O. the average time allowed per problem is 90 minutes.

TWENTY-THIRD ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION - 1979

Time: 100 minutes

1. Solve for x and y, if

$$\frac{1}{x^2} + \frac{1}{xy} = \frac{1}{9} \quad \text{and} \quad \frac{1}{y^2} + \frac{1}{xy} = \frac{1}{16}.$$

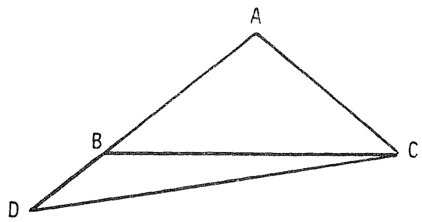
2. Find positive integers p and q, with q as small as possible, such that

$$\frac{7}{10} < \frac{p}{q} < \frac{11}{15}.$$

3. Define $a_1 = 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for all positive integers n. If $i \neq j$, prove that a_i and a_j have no common prime factor.

4. A number of points are given in the interior of a triangle. Connect these points, as well as the vertices of the triangle, by segments that do not cross one another until the interior is subdivided into smaller disjoint regions that are all triangles. It is required that each of the given points be always a vertex of any triangle containing it. Prove that the number of these smaller triangular regions is always odd.

5. ABC is an isosceles triangle with $\angle B = \angle C = 40°$. AB is extended to D so that AD = BC (see figure). Prove that $\angle BCD = 10°$.



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TWENTY-FOURTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION - 1980

Time: 100 minutes

1. On an escalator moving at constant speed a woman in a hurry walks up 9 steps as she travels from one floor to the next higher. A man in an even greater hurry runs 25 steps up the same escalator and reaches the top in half the time the woman took. How many steps does the escalator have between the two floors?

2. A circle inscribed in a right triangle divides the hypotenuse at its point of contact into segments of lengths x and y. Find, in terms of x and y,
   (a) the area of the right triangle,
   (b) the diameter of the circle.

3. Solve the system

$$\frac{1}{x^2} - \frac{1}{xy} = 30, \qquad \frac{1}{y^2} + \frac{1}{xy} = 28.$$

4. Consider the infinite sequence of positive integers

$$49,\ 4489,\ 444889,\ \ldots,$$

in which each number after the first is obtained by inserting the digits 4 and 8 (in that order) into the middle of the preceding number. Prove that all these numbers are perfect squares.

5. The sequence {100, 55, 45, 10, 35} has five terms, and each term starting with the third is the difference of the preceding two. A sequence terminates when the next term would be negative (since 10 - 35 = -25, the above example terminates with 35). Zero terms are permitted. Find a positive integer B such that the sequence $\{100, B, \ldots\}$ formed as indicated has the maximum number of terms. Generalize by showing how to find a positive integer B that will maximize the length of the sequence $\{A, B, \ldots\}$ formed as above, where A is any given positive integer.

*

TWENTY-FIFTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION - 1981

Time: 100 minutes

1. A canoeist is paddling upstream in a river when she passes a log floating downstream. She continues upstream for a while, paddling at a constant rate. She then turns around and goes downstream paddling twice as fast. She catches up to the same log two hours after she first passed it. How long did she paddle upstream?

2. Let $g(x) = 1 - \frac{1}{x}$ and define

$$g_1(x) = g(x), \quad g_{n+1}(x) = g(g_n(x)), \quad n = 1,2,3,\ldots \quad .$$

Evaluate $g_3(3)$ and $g_{1982}(1982)$.

3. Let Q denote a quadrilateral ABCD whose diagonals AC and BD intersect. If each diagonal bisects the area of Q, prove that Q must be a parallelogram.

4. Given that $(a_1, a_2, \ldots, a_7)$ and $(b_1, b_2, \ldots, b_7)$ are two permutations of the same seven integers, prove that the product

$$(a_1 - b_1)(a_2 - b_2)\ldots(a_7 - b_7)$$

is always even.

5. In analyzing the pecking order in a finite flock of chickens, we observe that for any two chickens exactly one pecks the other. We decide to call chicken K a *king* provided that, for any other chicken X, K pecks X or K pecks a third chicken Y who in turn pecks X. Prove that every such flock of chickens has at least one king. Must the king be unique?

*

Finally, I present solutions to several problems from earlier columns.

2. [1981: 15] *From the 1973 Moscow Olympiad.*

Can an integer consisting of six hundred digits 6 and any number of digits 0 be the square of another integer?

Solution by Andy Liu, University of Alberta.

Nyet! For suppose x^2 is such a square. We may then assume that x^2 ends in 6, and hence x ends in 4 or 6. Let $x = 10q+r$, with $r = 4$ or 6. Then

$$x^2 = 100q^2 + 20qr + r^2,$$

where $r^2 = 16$ or 36. In either case, the tens' digit of x^2 is odd, and so cannot be 0 or 6.

*

4. [1981: 15] *From the 1973 Moscow Olympiad.*

Three grasshoppers lie on a square ABCD, one at each of the vertices A, B, C. They start to play "symmetric leapfrog": if the grasshopper at A, for example, jumps over that at C, it lands at a point A' symmetric to A with respect to C. Can it happen that, after a number of jumps, one of the grasshoppers lands on vertex D?

Solution by Andy Liu, University of Alberta.

Nyet! Let the square be extended to an infinite square lattice. Then the grasshoppers will always land on a lattice point. Furthermore, the parities of the coordinates of the starting and landing points must be the same. Therefore no grasshopper will ever land on vertex D. What is more, no grasshopper will ever land on a lattice point that is, or was, occupied by another grasshopper.

*

5. [1981: 15] *From the 1973 Moscow Olympiad.*

A point is chosen on each side of a parallelogram in such a way that the area of the quadrilateral whose vertices are these four points is one-half the area of the parallelogram. Show that at least one of the diagonals of the quadrilateral is parallel to a side of the parallelogram.

I. *Solution by Andy Liu, University of Alberta.*

Let PQRS and ABCD be the given parallelogram and quadrilateral, respectively, as shown in Figure 1. Suppose BD $\nparallel$ QP, and let BD' $\parallel$ QP. With square brackets denoting area, we have

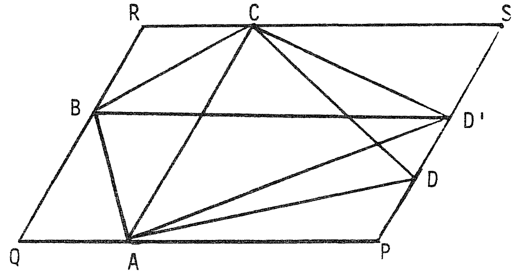


Figure 1

$$\begin{aligned}[ABC] + [ACD'] &= [ABD'] + [BCD'] \\ &= \tfrac{1}{2}[PQRS] = [ABCD] \\ &= [ABC] + [ACD].\end{aligned}$$

Hence [ACD'] = [ACD], and AC $\parallel$ PS follows from Euc. I. 39.

II. *Solution by M.S.K.*

As seen from Figure 2, where we have assumed that adjacent sides of the parallelogram, of lengths a and b, enclose an angle θ, the sum of the areas of the four corner triangles equals half the area of the parallelogram. Hence

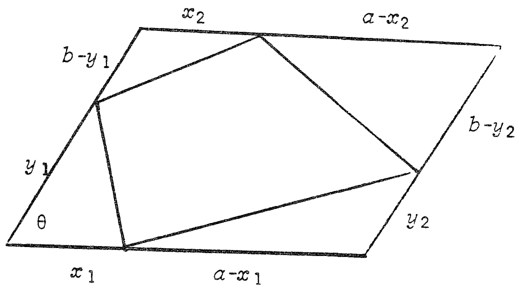


Figure 2

$$x_1y_1 + (b-y_1)x_2 + (a-x_2)(b-y_2) + y_2(a-x_1) = ab,$$

where we have cancelled the factor $\frac{1}{2}\sin\theta$ common to both sides, and this equation is equivalent to

$$(x_1 - x_2)(y_1 - y_2) = 0.$$

Therefore $x_1 = x_2$ or $y_1 = y_2$, and the desired conclusion follows. □

In this proof, it will be observed that the conclusion is independent of the values of a, b, and θ, so that we could have assumed at the outset that $a = b = 1$ and $\theta = 90^\circ$, for example. This was to be expected, for we can always transform a parallelogram into a square by two parallel projections, which preserve parallelism and ratios of areas. So there would have been no loss of generality in assuming that the parallelogram was a unit square. Parallel projections are frequently very useful in simplifying solutions to geometrical problems.

Rider. A point is chosen on each side of a rectangle of dimensions 8×15 in such a way that the perimeter of the quadrilateral whose vertices are these four points is 34. Show that at least one of the edges of the quadrilateral is parallel to a diagonal of the rectangle.

*

6. [1981: 15] *From the 1973 Moscow Olympiad.*

A square is divided into convex polygons. Show that one can further subdivide these polygons into smaller convex polygons so that, in the new division of the square, each polygon has an odd number of neighbours (*neighbours* are polygons with a common side).

Solution by Andy Liu, University of Alberta.

Triangulate all the convex polygons. All the resulting triangles have three neighbours except those, if any, which have one or two sides on the boundary of the square.

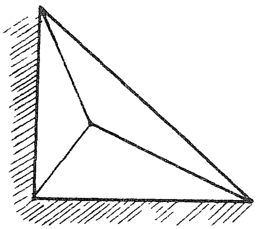


Figure 1

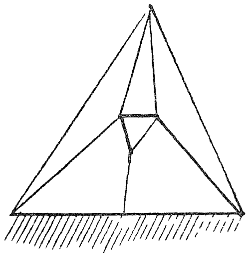


Figure 2

Those which have two sides on the boundary can be reduced by a further triangulation, as shown in Figure 1. Finally, any remaining triangle with only one side on the boundary can be reduced as in Figure 2. In the end, each polygon has an odd number of neighbours.

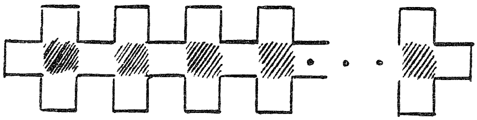
*

12. [1981: 16] *From the 1973 Moscow Olympiad.*

On an infinite chessboard, a closed simple (i.e., non-self-intersecting) path is drawn, consisting of sides of squares of the chessboard. Inside the path are k black squares. What is the largest area that can be enclosed by the path?

Solution by Andy Liu, University of Alberta.

The answer is $4k + 1$, attainable via the path illustrated in the figure. To show that this is best possible, observe that each white square in the region must be adjacent to at least one black square in the region, while with one exception each black square in the region can account for at most three white squares in the region that are not already accounted for by another black square in the region.



So the maximum area is $k + 3k + 1 = 4k + 1$.

*

1. [1983: 236] *From the 1982 Swedish Olympiad.*

Let N be a positive integer. How many solutions to the equation

$$x^2 - [x^2] = (x - [x])^2$$

are there in the interval $1 \le x \le N$?

Solution by James Semple, student, Carleton University, Ottawa.

The given equation is equivalent to $2x[x] = [x^2] + [x]^2$, or, if we set $x = a + r$, where $a = [x]$ and $0 \le r < 1$, to

$$2ar = [2ar + r^2].$$

Now $2ar$ is integral if and only if

$$r \in \left\{0, \frac{1}{2a}, \frac{2}{2a}, \ldots, \frac{2a-1}{2a}\right\},$$

and so, for each a, there are $2a$ possible values for r. For $1 \le x < N$, a runs from 1 to $N-1$ inclusively, and the corresponding number of solutions is

$$\sum_{a=1}^{N-1} 2a = N(N - 1).$$

If $x = N$, then $r = 0$, and we have one additional solution. Therefore the total number of solutions is $N^2 - N + 1$.

*

2. [1983: 236] *From the 1982 Swedish Olympiad.*

Let a,b,c be positive numbers. Prove that

$$abc \ge (b+c-a)(c+a-b)(a+b-c). \qquad (1)$$

Solution by M.S.K.

For positive a,b,c, at most one of the inequalities

$$b + c - a \ge 0, \quad c + a - b \ge 0, \quad a + b - c \ge 0 \qquad (2)$$

can fail, and then (1) holds trivially, so we may assume that (2) holds. As a consequence, a,b,c are the sides of a triangle (possibly degenerate). For pedagogical reasons, we give three proofs and a generalization of (1).

First proof. If we multiply together the obvious inequalities

$$a^2 \geq a^2-(b-c)^2 = (c+a-b)(a+b-c)$$

$$b^2 \geq b^2-(c-a)^2 = (a+b-c)(b+c-a)$$

$$c^2 \geq c^2-(a-b)^2 = (b+c-a)(c+a-b)$$

and take the square root of both sides of the resulting inequality, the desired result (1) follows, with equality if and only if $a = b = c$.

Second proof. If we multiply both sides of (1) by $a+b+c$, we obtain the equivalent inequality

$$2sabc = 2s\cdot 4Rrs \geq 16\Delta^2,$$

where R,r,s,Δ are the circumradius, inradius, semiperimeter, and area, respectively, of the triangle; and, since $\Delta = rs$, this is equivalent to the well-known

$$R \geq 2r, \qquad (3)$$

where equality holds if and only if the triangle is equilateral.

Third proof. This proof utilizes a powerful method for handling a large class of triangle inequalities. If a,b,c are the sides of a triangle (possibly degenerate), then

$$x = s-a \geq 0, \quad y = s-b \geq 0, \quad z = s-c \geq 0.$$

Dually, if $x,y,z \geq 0$, then

$$a = y+z, \quad b = z+x, \quad c = x+y$$

are the sides of a triangle. Consequently,

$$I(a,b,c) \geq 0 \iff I(y+z,z+x,x+y) \geq 0 \qquad (4)$$

and

$$J(x,y,z) \geq 0 \iff J(s-a,s-b,s-c) \geq 0.$$

Using (4), we obtain from (1) the equivalent and well-known inequality

$$(y+z)(z+x)(x+y) \geq 8xyz, \qquad (5)$$

(which follows from $y+z \geq 2\sqrt{yz}$, etc.).

(Readers will find it useful to apply this method to Problem 6 of the 1983 I.M.O. [1983: 207].)

Generalization. We were led to a generalization of (1) by the following geometric interpretation (see [1]). If one considers a related triangle A'B'C' with sides

$$a' = \frac{b+c}{2}, \quad b' = \frac{c+a}{2}, \quad c' = \frac{a+b}{2},$$

then, since $s' = s$ and A'B'C' is "closer" to being an equilateral triangle than the original triangle, we intuitively expect to have the area inequality $\Delta' \geq \Delta$, and it can be shown that this is actually equivalent to (1).

More generally, if

$$a = \sum_{i=1}^{n} w_i a_i, \qquad b = \sum_{i=1}^{n} w_i b_i, \qquad c = \sum_{i=1}^{n} w_i c_i$$

are the sides of a triangle, where a_i, b_i, c_i are the respective sides of n given triangles $A_iB_iC_i$, $w_i \geq 0$, and $\Sigma w_i = 1$, then

$$\sqrt{\Delta} \geq \sum_{i=1}^{n} w_i \sqrt{\Delta_i}\,, \tag{6}$$

with equality if and only if the n given triangles are directly similar. □

For still further extensions, see [2]. Problem B-6 of the 1982 William Lowell Putnam Mathematical Competition [3] is the special case of (6) for two triangles with $w_1 = w_2 = \frac{1}{2}$. Our first proof of (1) is essentially that of A. Padoa (1925), which can be found in [4]. A more immediate (and earlier) proof of (1) follows from

$$abc - 8(s-a)(s-b)(s-c) = \Sigma(s-a)(b-c)^2,$$

which Coxeter [5] credits to Peano (1902), and, using (4), a proof of (5) follows likewise from

$$(y+z)(z+x)(x+y) - 8xyz = \Sigma x(y-z)^2.$$

The original proof of (5) appears to be that of Cesàro (1880), according to [4]. In the same paper [5] , Coxeter outlines a still earlier proof of (1) due to Lehmus (1820). Finally, we mention that, although (3) is usually credited to Euler (1765), MacKay [6] credits it to Chapple (1746).

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3. *Mathematics Magazine*, 56 (1983) 190.

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5. H.S.M. Coxeter, "The Lehmus Inequality", submitted to *Aequationes Mathematicae*.

6. J.S. MacKay, *Proceedings of the Edinburgh Mathematical Society*, 5 (1887) 62-63.

*

3. [1983: 236] *From the 1982 Swedish Olympiad.*

Suppose one can find a point P in the interior of the quadrilateral ABCD such that the four triangles PAB, PBC, PCD, and PDA have the same area. Show that P is on one of the diagonals AC or BD.

I. *Adapted from the solution of Ravi Ramakrishna, student, Essex Junction High School, Vermont.*

LEMMA. Let RST be a triangle and M the midpoint of side ST. Then the locus of all points P in the plane of the triangle such that [PRS] = [PTR], where the square brackets denote signed area, is the *line* RM.

Proof. The desired result follows from

$$\begin{aligned}[PRS] = [PTR] \quad &\iff \quad \tfrac{1}{2}\vec{PR}\times\vec{PS} = \tfrac{1}{2}\vec{PT}\times\vec{PR} \\ &\iff \quad \vec{PR}\times\tfrac{1}{2}(\vec{PS}+\vec{PT}) = \vec{0} \\ &\iff \quad \vec{PR}\times\vec{PM} = \vec{0} \\ &\iff \quad R,P,M \text{ are collinear.} \quad \square\end{aligned}$$

For the problem under consideration, we do not assume that the given quadrilateral is convex, or even simple. We prove the stronger result that the desired conclusion follows if we assume only that the signed areas satisfy

$$[PAB] = [PBC] \quad \text{and} \quad [PCD] = [PDA], \tag{1}$$

where P is not necessarily an interior point.

Let M be the midpoint of AC. It follows from (1) and the lemma that the lines BP and DP both go through M. Hence either P = M (and P is on AC), or else B,P,D are collinear (and P is on BD).

II. *Comment by M.S.K.*

Of related interest are Problem 3 on page 42 of this issue and the following theorem of Léon Anne [1977: 194-195]:

Let ABCD *be a quadrilateral (convex or not, simple or not, degenerate or not), and let* M *and* N *be the midpoints of the diagonals* AC *and* BD, *respectively. Then* *the locus of the points* P *such that*

$$[PAB] + [PCD] = [PBC] + [PDA],$$

where the square brackets denote signed area, is (i) the entire plane if M = N *(i.e., if* ABCD *is a parallelogram), or (ii) the line* MN *if* M ≠ N.

A simple vectorial proof, similar to that of the above lemma, suffices to establish this theorem in full generality.

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4. [1983: 236] *From the 1982 Swedish Olympiad.*

In the triangle ABC the sides are AB = 33 cm, AC = 21 cm, and BC = m cm, where m is an integer. It is possible to find a point D on AB and a point E on AC such that

$$AD = DE = EC = n \text{ cm},$$

where n is an integer. What values can m take?

Solution by M.S.K.

By the law of cosines applied to triangles ADE and ABC, we get

$$n^2 = n^2 + (21-n)^2 - 2n(21-n)\cos A \qquad (1)$$

and

$$m^2 = 33^2 + 21^2 - 2\cdot 33\cdot 21\cos A. \qquad (2)$$

It follows from (1) that $n = 21$ or $\cos A = (21-n)/2n$.

If $n = 21$, then E coincides with A, and the integer m is satisfactory if and only if $21+m > 33$ and $33+21 > m$, or $12 < m < 54$ (or $12 \le m \le 54$ if degenerate triangles are allowed).

If $n \neq 21$, then, from (2) with $\cos A = (21-n)/2n$,

$$m^2 = 33^2 + 21^2 - \frac{21\cdot 33(21-n)}{n}. \qquad (3)$$

Thus $21\cdot 33(21-n)/n$ must be an integer, and so n is one of 1, 3, 7, 9, 11. But only $n = 7$ and $n = 11$ yield integer values of m in (3): $m = 12$ and $m = 30$, respectively.

The answer to our problem is therefore $12 \le m \le 54$ if degenerate triangles are allowed, or $12 \le m < 54$ if they are not.

*

5. [1983: 236] *From the 1982 Swedish Olympiad.*

In an orthonormal coordinate system one considers the points (x,y), where x and y are integers with $1 \le x \le 12$, $1 \le y \le 12$. Each of these 144 points is coloured red, white, or blue. Show that there is a rectangle with sides parallel to the axes and having all its vertices the same colour.

Solution by Fred Galvin, University of Kansas.

We prove a stronger result: that the desired conclusion still holds if $1 \le x \le 12$ and $1 \le y \le 10$.

Since there are 120 points and three colours, at least 40 points are of the same colour, say red. Now let R be the set of red points,

$$n_x = |\{y:\ (x,y) \in R\}|, \qquad x = 1,2,\ldots,12,$$

and

$$S_x = \{(y_1,y_2):\ (x,\ y_1) \in R,\ (x,\ y_2) \in R,\ y_1 < y_2\}.$$

Since $n_1+n_2+\ldots+n_{12} \ge 40$, the sum given below is minimized when the red points are distributed as equally as possible among the 12 columns:

$$\sum_{x=1}^{12} \binom{n_x}{2} \ge 8\binom{3}{2} + 4\binom{4}{2} = 48.$$

Therefore

$$\sum_{x=1}^{12} |S_x| = \sum_{x=1}^{12} \binom{n_x}{2} \ge 48 > \binom{10}{2},$$

and this implies that the sets $S_1,\ S_2,\ \ldots,\ S_{12}$ are not all pairwise disjoint. If $S_{x_1} \cap S_{x_2} \ne \emptyset$, we choose $(y_1,y_2) \in S_{x_1} \cap S_{x_2}$. The points

$$(x_1,y_1),\quad (x_1,y_2),\quad (x_2,y_1),\quad (x_2,y_2)$$

are then all coloured red, and they are the vertices of a rectangle with sides parallel to the axes. □

The same conclusion follows from similar arguments for arrays of dimensions 11×11, 13×7, 16×5, and 19×4. However, for $12{\times}n$ arrays, the conclusion holds if and only if $n \ge 10$. The following 12×9 array, where we represent the colours by 1,2,3, provides the needed counterexample:

1	2	3	2	3	1	2	3	1	1	2	3
2	3	1	3	1	2	3	1	2	1	2	3
3	1	2	1	2	3	1	2	3	1	2	3
2	3	1	1	2	3	2	3	1	2	3	1
3	1	2	2	3	1	3	1	2	2	3	1
1	2	3	3	1	2	1	2	3	2	3	1
2	3	1	2	3	1	1	2	3	3	1	2
3	1	2	3	1	2	2	3	1	3	1	2
1	2	3	1	2	3	3	1	2	3	1	2

6. [1983: 236] *From the 1982 Swedish Olympiad.*

If $0 \le a \le 1$ and $0 \le x \le \pi$, prove that

$$(2a-1)\sin x + (1-a)\sin(1-a)x \ge 0.$$

Solution by Danny Ullman, University of California, Berkeley.

It is a simple calculus exercise to show that $(\sin x)/x$ is a decreasing function for $x \in (0,\pi)$. Thus

$$\frac{\sin(1-a)x}{(1-a)x} \ge \frac{\sin x}{x}, \qquad 0 \le a < 1,\ 0 < x < \pi,$$

from which we get

$$\frac{\sin(1-a)x}{\sin x} \ge 1-a \ge \frac{1-2a}{1-a}$$

and then the desired

$$(2a-1)\sin x + (1-a)\sin(1-a)x \ge 0, \tag{1}$$

which is valid for (at least) $0 \le a < 1$ and $0 < x < \pi$. Moreover, (1) holds for all $x \in [0,\pi]$ if $a = 1$, and it holds for all $a \in [0,1]$ if $x = 0$ or $x = \pi$. Hence (1) holds if $0 \le a \le 1$ and $0 \le x \le \pi$. Equality occurs just when $a = 0$ or $x = 0$ or $(a,x) = (1,\pi)$.

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

THE PUZZLE CORNER

Puzzle No. 51: Enigmatic rebus (*4 *3 *5 *7)

Oh, look at what above is hid!
Let's make a fuss! Do tell it, shout it
The way that William Shakespeare did!
If not without it, what about it?

Puzzle No. 52: Rebus (6)

O

From William Blake, we quote a point of view:
"If you have form'd a REBUS to go into,
Go into it yourself and see how you would do."

ALAN WAYNE, Holiday, Florida

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

911. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the following synonymical alphametic in the smallest possible base:

$$\begin{array}{r} \text{HARD} \\ \underline{\text{SHARP}}. \\ \text{HARSH} \end{array}$$

912.* *Proposed by Shmuel Avital, Technion - Israel Institute of Technology, Haifa, Israel.*

It is not difficult to verify that the nonlinear recursive relation

$$a_{n-1}a_{n+1} = a_n + 1, \qquad n = 1,2,3,\ldots,$$

generates a *periodic* sequence of period length 5 for any given a_0 and a_1. Show how to construct nonlinear recursive relations which will generate periodic sequences of any given period length k.

913.* *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let

$$f_n(x) = x^n + 2x^{n-1} + 3x^{n-2} + 4x^{n-3} + \ldots + nx + (n+1).$$

Prove or disprove that the discriminant of $f_n(x)$ is

$$(-1)^{n(n-1)/2}\cdot 2(n+2)^{n-1}(n+1)^{n-2}.$$

914. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

If $a,b,c > 0$, then the equation $x^3 - (a^2+b^2+c^2)x - 2abc = 0$ has a unique positive root x_0. Prove that

$$\frac{2}{3}(a+b+c) \le x_0 < a+b+c.$$

915*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If $x+y+z+w = 180°$, prove or disprove that

$$\sin(x+y) + \sin(y+z) + \sin(z+w) + \sin(w+x) \geq \sin 2x + \sin 2y + \sin 2z + \sin 2w,$$

with equality just when $x = y = z = w$.

916. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find $\lim_{n\to\infty} S_n$ if

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} \tan \frac{\pi}{2^{k+1}}.$$

917. *Proposed by Rick Moorhouse, University of Toronto.*

How can the eight vertices of a cube be divided into two sets of four forming two directly congruent tetrahedra such that the four vertices of each tetrahedron lie in the planes of the four faces of the other?

918. *Proposed by John P. Hoyt, Lancaster, Pennsylvania; and Leroy F. Meyers, The Ohio State University.*

Let k be a positive integer and g a polynomial of degree at most $k-2$. Show that

$$\sum_{n=0}^{\infty} \frac{g(n)}{(n+1)(n+2)\ldots(n+k)}$$

converges to a rational number.

919. *Proposed by Jordi Dou, Barcelona, Spain.*

Show how to construct a point P which is the centroid of triangle A'B'C', where A',B',C' are the orthogonal projections of P upon three given lines a,b,c, respectively.

920. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If a triangle of unit base and unit altitude is in the unit square, show that the base of the triangle must be one side of the square.

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A NEW ADDRESS FOR THE EDITOR

From now on, communications to the editor should be addressed as follows: Léo Sauvé, Algonquin College, 140 Main Street, Ottawa, Ontario, Canada K1S 1C2. The letters will still go to the same place as before, because the editor's office has not been moved. It is just that the powers that be at the College have decided that the back door of the building (on Main Street) should, for its long meritorious service, be promoted to the tenured rank of Front Door, and have informed the Post Office accordingly.

The office of the managing editor, F.G.B. Maskell, is on another campus and his address remains unchanged (see front page).

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

786. [1982: 277] *Proposed by O. Bottema, Delft, The Netherlands.*

Let r_1,r_2,r_3 be arbitrarily chosen positive numbers. Prove that there exists a (real) triangle whose exradii are r_1,r_2,r_3, and calculate the sides of this triangle.

Solution by Howard Eves, University of Maine at Orono.

Given $r_1,r_2,r_3 > 0$, we show that there is one and only one (real) triangle whose exradii are r_1,r_2,r_3, and that its sides are

$$a = \frac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}, \qquad b = \frac{r_2(r_3+r_1)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}, \qquad c = \frac{r_3(r_1+r_2)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}}. \qquad (1)$$

Proof of uniqueness.

Suppose there exists a triangle with sides a,b,c and exradii $r_a = r_1$, $r_b = r_2$, $r_c = r_3$, and let r,s,K denote the inradius, semiperimeter, and area of this triangle. We then have the known relations (see, e.g., Johnson [1])

$$K^2 = rr_1r_2r_3; \quad K = rs; \quad K = r_1(s-a), \text{ etc.}; \quad \frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}.$$

From the first two relations we find

$$s^2 = \frac{r_1r_2r_3}{r},$$

whence, using the fourth relation,

$$s^2 = r_2r_3 + r_3r_1 + r_1r_2.$$

But, from the first three relations,

$$rr_1r_2r_3 = rr_1s(s-a), \qquad \text{or} \qquad s(s-a) = r_2r_3,$$

and it follows that

$$a = s - \frac{r_2r_3}{s} = \frac{s^2 - r_2r_3}{s} = \frac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}},$$

with similar expressions for b and c.

Proof of existence.

If the three positive numbers a,b,c are defined by (1), then

$$b+c-a = \frac{r_1(r_2+r_3)}{\sqrt{r_2r_3+r_3r_1+r_1r_2}} > 0,$$

and similarly $c+a-b > 0$ and $a+b-c > 0$. So there *is* a triangle with side lengths a,b,c, and its semiperimeter is

$$s = \tfrac{1}{2}(a+b+c) = \sqrt{r_2r_3+r_3r_1+r_1r_2}.$$

Let K be the area of this triangle and r_a, r_b, r_c its exradii. From

$$s - a = \frac{r_2r_3}{s}, \qquad s - b = \frac{r_3r_1}{s}, \qquad s - c = \frac{r_1r_2}{s},$$

Heron's formula gives the area

$$K = \frac{r_1r_2r_3}{s}.$$

Finally,

$$r_a = \frac{K}{s-a} = r_1, \qquad r_b = \frac{K}{s-b} = r_2, \qquad r_c = \frac{K}{s-c} = r_3,$$

and the solution is complete.

Also solved by SAM BAETHGE, San Antonio, Texas; LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; the COPS of Ottawa (two solutions); JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

Several solvers proved existence (all that the problem really asked for), but they successfully resisted the mathematician's normal urge to prove also uniqueness. Murty noted that formulas (1) are given without proof in Hobson [2].

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1. Roger A. Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, Dover, New York, 1960, pp. 189, 190.

2. E.W. Hobson, *A Treatise on Plane & Advanced Trigonometry*, Dover, New York, 1957, p. 194, Ex. 2(α).

* * *

787. [1982: 278] *Proposed by J. Walter Lynch, Georgia Southern College.*

(a) Given two sides, a and b, of a triangle, what should be the length of the third side, x, in order that the area enclosed be a maximum?

(b) Given three sides, a, b, and c, of a quadrilateral, what should be the length of the fourth side, x, in order that the area enclosed be a maximum?

Solution by Howard Eves, University of Maine at Orono.

Let us assume that, given n-1 consecutive sides $a_1, a_2, \ldots, a_{n-1}$ of a

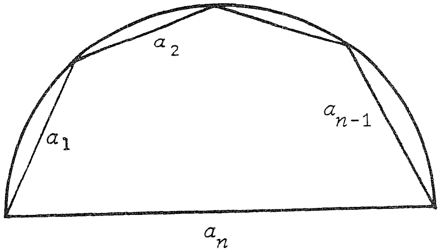

Figure 1

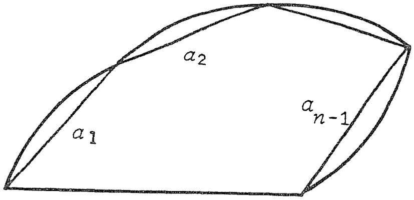

Figure 2

convex n-gon, there exists an nth side a_n such that the vertices of the n-gon lie on a semicircle having side a_n as diameter (see Figure 1). We show that this polygon, P, if it exists, has an area greater than that of any other convex n-gon having $a_1, a_2, \ldots, a_{n-1}$ for $n-1$ consecutive sides. To this end, deform the polygon P, along with the circular segments attached to its sides $a_1, a_2, \ldots,$ a_{n-1}, into any other convex polygon P' (see Figure 2). By the Dido theorem (see, e.g., Kazarinoff [1]), which says that the semicircle encloses a greater area against a straight line than any other curve of the same length, the area of P plus the attached circular segments is greater than the area of P' plus the same attached circular segments. It follows that the area of P is greater than the area of P'. We now consider the two cases $n = 3$ and $n = 4$, two cases for which the existence of P can easily be shown.

(a) For $n = 3$, set $a_1 = a$, $a_2 = b$, $a_3 = x$. Form the right triangle having legs a and b and call the hypotenuse x. This triangle, being inscribable in a semicircle, is the sought maximum polygon P. By the Pythagorean theorem,

$$x = \sqrt{a^2 + b^2}.$$

(b) For $n = 4$, set $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = x$, and denote the two diagonals of the quadrilateral by m and n. There exists a quadrilateral, with these parts, inscribable in a semicircle of diameter x if and only if, by the theorems of Ptolemy and Pythagoras, x is a positive real number such that (see Figure 3)

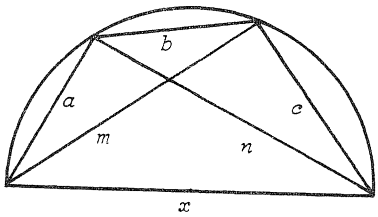

Figure 3

$$bx + ac \ (= mn) = \sqrt{x^2 - c^2} \cdot \sqrt{x^2 - a^2},$$

an equation that is easily shown to be equivalent to

$$x^3 - (a^2+b^2+c^2)x - 2abc = 0. \tag{1}$$

(The symmetry of the coefficients in (1) shows that the same equation would be obtained if the sides a,b,c were consecutive in any order.) Now the rule of signs of Descartes assures us that (1) has one and only one positive root. This positive root can be expressed exactly in closed form by using the Cardano-Tartaglia formula. However, because some values of a,b,c (e.g., 1,2,3) yield the irreducible case of the cubic, a good approximation will be easier to obtain if the exact value of the positive root is first given in trigonometric form:

$$x = 2\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\cos\left\{\frac{1}{3}\arccos\frac{3\sqrt{3}abc}{(a^2+b^2+c^2)^{3/2}}\right\}. \tag{2}$$

With a pocket calculator, a good approximation to the positive root can be obtained from (2) far more quickly than, say, from Horner's method. If a,b,c = 1,2,3, for example, we find that x = 4.11309058... .

In conclusion, it is interesting to note that for given a,b,c the side x, and hence the desired quadrilateral, is not always constructible with Euclidean tools. Thus, if a,b,c = 1,2,3, then (1) has rational coefficients but no rational root, and hence, for this case, x is not constructible with Euclidean tools.

Also solved by LEON BANKOFF, Los Angeles, California; the COPS of Ottawa; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; VIKTORS LINIS, University of Ottawa; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and the proposer. In addition, three incorrect solutions were received.

Editor's comment.

It turns out that parts (a) and (b) of this problem are special cases of a more general problem that is well known to very few people. Bankoff found it in an 1891 book by M'Clelland [2], where only a vague indication for a solution is given:

When all the sides but one of a polygon of any order are given in magnitude, the area is a maximum when the circle on the closing side as diameter passes through the remaining vertices.

M'Clelland then treats in detail the special case of the quadrilateral, with a solution similar to ours.

Essentially the same general problem resurfaced 37 years later, in 1928, proposed by Elmer Schuyler as follows [3]:

In the Plane and Solid Geometry *by Wentworth-Smith (revised edition) occurs the following proposition: Of all polygons with sides all given but one (and in a definite order), the maximum can be inscribed in a semicircle which has the undetermined side for its diameter. Prove that there is one and only one maximum. Given the lengths of the sides and their order, compute the radius of the semicircle.*

This problem remained unsolved for 33 years. Finally, in 1961, a solution by C.F. Pinzka was published [4]. He gave a very simple solution based on a powerful result that he claimed was "well known", in spite of a 33-year universal amnesia, but for which he gave no reference. He also treated in detail the special case of the quadrilateral, with a solution similar to ours.

For related problems, see Crux 699 [1982: 320] and the article by Bottema and Sauvé in this issue.

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1. Nicholas D. Kazarinoff, *Geometric Inequalities*, New Mathematical Library, No. 4, The Mathematical Association of America, p. 67.

2. William J. M'Clelland, *A Treatise on the Geometry of the Circle*, Macmillan and Co., London, 1891, p. 20.

3. Problem 3355 (proposed by Elmer Schuyler), *American Mathematical Monthly*, 35 (1928) 564.

4. C.F. Pinzka, Solution to Problem 3355, *American Mathematical Monthly*, 68 (1961) 814.

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788. [1982: 278] *Proposed by Meir Feder, Haifa, Israel.*

A *pandigital* integer is a (decimal) integer containing each of the ten digits exactly once.

(a) If m and n are distinct pandigital perfect squares, what is the smallest possible value of $|\sqrt{m} - \sqrt{n}|$?

(b) Find two pandigital perfect squares m and n for which this minimum value of $|\sqrt{m} - \sqrt{n}|$ is attained.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

(a) A pandigital integer must be divisible by 9, since its digit sum is 45, which is divisible by 9. Therefore the square root of a pandigital perfect square must be divisible by 3, and the difference between two of them must be at least 3.

(b) For

$$m = 65634^2 = 4307821956 \quad \text{and} \quad n = 65637^2 = 4308215769,$$

we have $|\sqrt{m} - \sqrt{n}| = 3$.

Also solved by the proposer. One incorrect solution was received.

Editor's comment.

The proposer stated (presumably after a computer search) that the answer given in part (b) is unique.

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789. [1982: 278] *Proposed by H. Kestelman, University College, London, England.*

If A and B are square matrices (any orders), then they have a common eigenvalue if and only if $AX = XB$ for some $X \neq 0$.

Solution by the proposer.

Suppose $Av = \lambda v$ and $w^{\mathsf T}B = \lambda w^{\mathsf T}$, where v and w are nonzero column vectors, and set $X = vw^{\mathsf T}$. Then $X \neq 0$ and

$$AX = \lambda vw^{\mathsf T} = vw^{\mathsf T}B = XB.$$

Conversely, suppose $AX = XB$ and $X \neq 0$. By induction, $f(A)X = Xf(B)$ for every polynomial f. Take, say,

$$f(t) = \det(tI-B) = (t-c_1)(t-c_2)\ldots(t-c_m).$$

By the Cayley-Hamilton theorem, $f(B) = 0$, and so

$$(A-c_1I)(A-c_2I)\ldots(A-c_mI)x = 0$$

for some $x \neq 0$. This implies that $A-c_rI$ is singular for some r, and then c_r is an eigenvalue of A and of B.

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790. [1982: 278] *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*

Let ABC be a triangle with sides a,b,c in the usual order, and let l_a,l_b,l_c and l'_a,l'_b,l'_c be two sets of concurrent cevians, with l_a,l_b,l_c intersecting a,b,c in L,M,N, respectively. If

$$l_a \cap l'_b = \mathrm{P}, \quad l_b \cap l'_c = \mathrm{Q}, \quad l_c \cap l'_a = \mathrm{R},$$

prove that, independently of the choice of concurrent cevians l'_a,l'_b,l'_c, we have

$$\frac{\mathrm{AP}}{\mathrm{PL}}\cdot\frac{\mathrm{BQ}}{\mathrm{QM}}\cdot\frac{\mathrm{CR}}{\mathrm{RN}} = \frac{abc}{\mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN}} \geq 8, \tag{1}$$

with equality occurring just when l_a,l_b,l_c are the medians of the triangle.

(This problem extends Crux 588 [1981: 306].)

Solution by Jordi Dou, Barcelona, Spain.

Let $\mathrm{X} = l_a \cap l_b \cap l_c$ and $\mathrm{X}' = l'_a \cap l'_b \cap l'_c$. We will show that the desired conclusion is true exactly as stated provided the wording of the problem is interpreted strictly, with "l_a,l_b,l_c intersecting *sides* a,b,c" implying that X (but not necessarily X') is an interior point of triangle ABC. But for now we interpret "sides" loosely to mean "sides or sides produced", and allow both X and X' to range over the entire plane.

Let l_a', l_b', l_c' intersect sides a, b, c in L', M', N', respectively. If we apply the theorem of Menelaus to triangles BCN, CAL, ABM and transversals AL', BM', CN', respectively, we obtain

$$\frac{BL'}{L'C}\cdot\frac{CR}{RN}\cdot\frac{NA}{AB} = -1, \qquad \frac{CM'}{M'A}\cdot\frac{AP}{PL}\cdot\frac{LB}{BC} = -1, \qquad \frac{AN'}{N'B}\cdot\frac{BQ}{QM}\cdot\frac{MC}{CA} = -1.$$

If we multiply these three relations and note that, by Ceva's theorem,

$$\frac{BL'}{L'C}\cdot\frac{CM'}{M'A}\cdot\frac{AN'}{N'B} = +1,$$

we obtain

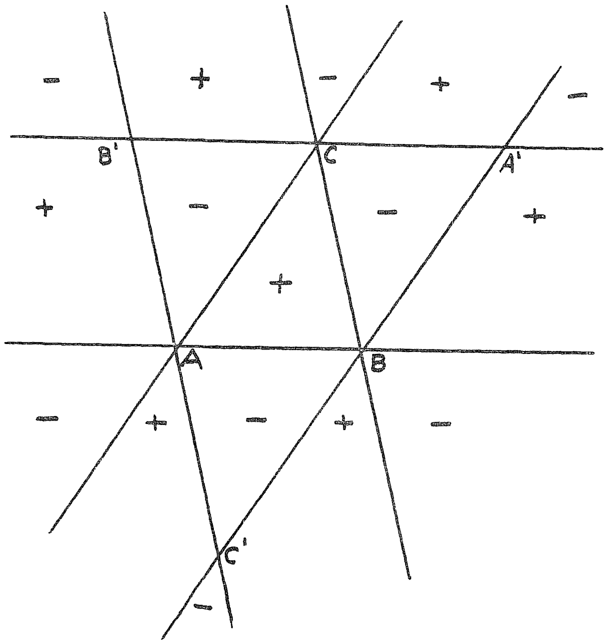
$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN}\cdot\frac{LB}{BC}\cdot\frac{MC}{CA}\cdot\frac{NA}{AB} = -1,$$

which is equivalent to

$$\frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = \frac{BC\cdot CA\cdot AB}{BL\cdot CM\cdot AN} = \frac{abc}{BL\cdot CM\cdot AN} \equiv \phi(X), \qquad (2)$$

which establishes the first part of (1).

Observe that the value of (2) depends upon X alone and is thus independent of X'. The figure (where A'B'C' is the antimedial triangle of ABC) shows the sign of $\phi(X)$ according to the position of X in the plane. $\phi(X)$ vanishes along the lines B'C', C'A', A'B', and there are infinite discontinuities along the lines BC, CA, AB. The function has only one relative extremum, and it occurs in the interior of triangle ABC. In fact we prove that, if X is an interior point of triangle ABC, then



$$\phi(X) \geq 8, \qquad (3)$$

with equality if and only if X is the centroid of the triangle, and this will complete the proof of (1). The result (3) is not new. It appears (in equivalent form) without proof in Bottema [1], where it is credited to S.I. Zetel' in a Russian journal (1962).

Let

$$\frac{BL}{LC} = \alpha, \qquad \frac{CM}{MA} = \beta, \qquad \frac{AN}{NB} = \gamma,$$

where $\alpha,\beta,\gamma > 0$ and $\alpha\beta\gamma = 1$ (by Ceva's theorem). Then

$$\begin{aligned}\phi(X) &= \frac{BC}{BL}\cdot\frac{CA}{CM}\cdot\frac{AB}{AN}\\ &= (1+\frac{1}{\alpha})(1+\frac{1}{\beta})(1+\alpha\beta)\\ &= (1+1) + (\alpha+\frac{1}{\alpha}) + (\beta+\frac{1}{\beta}) + (\alpha\beta+\frac{1}{\alpha\beta})\\ &\ge 8,\end{aligned}$$

with equality if and only if $\alpha = \beta = \gamma = 1$, that is, if and only if X is the centroid of the triangle.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

REFERENCE

1. O. Bottema, *et al.*, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, pp. 127-128.

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791. [1982: 302] *Proposed by Alan Wayne, Holiday, Florida.*

At whom might this be shouted?

$$\begin{array}{r} SO\\ SCAT\\ \underline{SCAT}\\ 5032 \end{array}$$

The sum of this base ten cryptarithm will answer the question.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Clearly $\underline{S = 2}$. Since the carry from the tens' column is even, we have carry+2+2A = 3, so $\underline{A = 0}$, and then $\underline{C = 5}$ follows. Finally, O+2T = 12, and only $\underline{O = 6}$ and $\underline{T = 3}$ will do. Hence the only possible solution is 5032 = CATS, which is not, after all, totally unexpected. That this *is* a solution is verified from the addition

$$\begin{array}{r} 26\\ 2503\\ \underline{2503}\\ 5032 \end{array}. \qquad (1)$$

II. *Solution by Charles W. Trigg, San Diego, California.*

Remembering the vehement (if not SCATological) ejaculations of some ailurophobic contemporaries, we equate 5032 to CATS. Then, noting that $12 - 2\cdot3 = 6$, we establish the solution [(1)] faster than the vocalized letters can chase out the 5032. We note that O+T gives the fabled number of lives of a 503 as well as the number of tails in a cat-o'-n-tails.

To avoid the possible cat-astrophe of an undiscovered second solution, we examine... [here follows a solution similar to solution I (Editor)]. Thus, the intuitive reconstruction previously given is unique, and, in a sense, lets the cat out of the bag. Or was it from a cat-a-comb, a hairy experience? Not likely, since a cat-aclysm would have been needed to overcome its cat-aleptic state with no cat-aplasm available.

Is .5032 the cat-a-log of 3.186?

Also solved by SAM BAETHGE, San Antonio, Texas; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

What is there left for the editor to say after Trigg's cat-achrestic solution and comment? One thing: we regret that our one regular contributor from Cat-alonia (Jordi Dou) did not submit a solution.

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792. [1982: 302] *Proposed by E.J. Barbeau, University of Toronto.*

The number 144 has the remarkable property that, with only the three exceptions

$$2448 = 17\cdot144, \qquad 2736 = 19\cdot144, \qquad 3312 = 23\cdot144,$$

each of its first 50 multiples differs from the first perfect square not less than it by a perfect square.

Show that, for any positive integer n, there is a number m such that n consecutive positive multiples

$$km,\ (k+1)m,\ \ldots,\ (k+n-1)m$$

have the property that each differs from the smallest perfect square not less than it by a perfect square.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

We wish to investigate the range of values of the integer x for which mx is the difference of two squares,

$$mx = c^2 - b^2,$$

and

$$(c-1)^2 - mx \le 0. \qquad (1)$$

If an integer is the difference of two squares, it must be odd or a multiple of 4. Therefore, if the range of x is to include both odd and even values, then m must be a multiple of 4. So we set $m = 4p$ and obtain

$$4px = (c+b)(c-b). \qquad (2)$$

Although, for particular values of p and x, $4px$ may be broken into two factors in many ways, we can always set

$$2p = c + b, \qquad 2x = c - b,$$

whence

$$c = p + x, \qquad b = p - x.$$

Now (1) becomes

$$f(x) \equiv (c-1)^2 - 4px = (p+x-1)^2 - 4px = x^2 - 2(p+1)x + (p-1)^2 \le 0. \; (3)$$

The integer x satisfies (3) if and only if $x_1 \le x \le x_2$, where

$$x_1 = p+1-2\sqrt{p} = \tfrac{1}{4}m+1-\sqrt{m} \qquad \text{and} \qquad x_2 = p+1+2\sqrt{p} = \tfrac{1}{4}m+1+\sqrt{m}$$

are the zeros of f. The number of integer values of x for which (3) holds is therefore

$$r = 1 + 2[\sqrt{m}],$$

and m is a solution to our problem for all values of $n \le r$. But since m can be chosen as large as we wish, there is an m with the required properties for any given n. □

When $m = 144$, then $r = 25$, and all multiples of 144 from

$$mx_1 = 25\times144 \qquad \text{to} \qquad mx_2 = 49\times144$$

have the desired property. The other 22 solutions mentioned in the proposal arise from factoring (2) in other ways. So the remarkable property of 144 is due in part to its many factors.

Also solved by the COPS of Ottawa; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; LAWRENCE SOMER, Washington, D.C; and the proposer.

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793, [1982: 303] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Consider the following double inequality for the Riemann Zeta function: for $n = 1,2,3,\ldots$,

$$\frac{1}{(s-1)(n+1)(n+2)\ldots(n+s-1)} + \zeta_n(s) < \zeta(s) < \zeta_n(s) + \frac{1}{(s-1)n(n+1)\ldots(n+s-2)}, \quad (1)$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad \text{and} \qquad \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}.$$

Go as far as you can in determining for which of the integers $s = 2,3,4,\ldots$ the inequalities (1) hold.

(N.D. Kazarinoff asks for a proof that (1) holds for $s = 2$ in his *Analytic Inequalities*, Holt, Rinehart & Winston, 1964, page 79; and Norman Schaumberger asks for a proof or disproof that (1) holds for $s = 3$ in *The Two-Year College Mathematics Journal*, 12 (1981) 336.)

Solution by Gali Salvatore, Perkins, Québec.

Let $f_n(s)$ and $g_n(s)$ denote the left and right members, respectively, of (1). The sequences $\{f_n(s)\}$ and $\{g_n(s)\}$ both converge to $\zeta(s)$, since each is the sum of a null sequence and the sequence $\{\zeta_n(s)\}$ which converges (by definition) to $\zeta(s)$. We will show that

$$f_n(2) < \zeta(2) < g_n(2), \qquad n = 1,2,3,\ldots; \quad (2)$$

$$f_n(3) < \zeta(3) < g_n(3), \qquad n = 1,2,3,\ldots; \quad (3)$$

$$f_n(s) < g_n(s) < \zeta(s), \qquad s > 3,\ n = 1,2,3,\ldots. \quad (4)$$

Thus the question raised in our problem will have been settled for *all* $s > 1$. Our proof of (2) will provide a solution to Kazarinoff's problem, and our proof of (3) will confirm the truth of Schaumberger's conjecture. (We call it a conjecture only because a published solution is long overdue in the *Two-Year College Mathematics Journal* (now called the *College Mathematics Journal*).) We will need the following results, the first of which is quite obvious:

(a) $f_n(s) < g_n(s)$ for $s > 1$ and $n = 1,2,3,\ldots$.

(b) $$f_{n+1}(s) - f_n(s) = \frac{1}{(n+1)^s} + \frac{1}{(s-1)(n+2)(n+3)\ldots(n+s)} - \frac{1}{(s-1)(n+1)(n+2)\ldots(n+s-1)}$$

$$= \frac{1}{(n+1)^s} - \frac{1}{(n+1)(n+2)\ldots(n+s)}$$

$$> 0,$$

for $s > 1$ and $n = 1,2,3,\ldots$.

(c) $g_{n+1}(2) - g_n(2) = \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)^2} < 0$ for $n = 1,2,3,\ldots$.

(d) $g_{n+1}(3) - g_n(3) = \frac{1}{(n+1)^3} + \frac{1}{2(n+1)(n+2)} - \frac{1}{2n(n+1)} = -\frac{1}{n(n+1)^3(n+2)} < 0$

for $n = 1,2,3,\ldots$.

(e) $$g_{n+1}(s) - g_n(s) = \frac{1}{(n+1)^s} + \frac{1}{(s-1)(n+1)(n+2)\ldots(n+s-1)} - \frac{1}{(s-1)n(n+1)\ldots(n+s-2)}$$

$$= \frac{1}{(n+1)^s} - \frac{1}{n(n+1)(n+2)\ldots(n+s-1)}$$

$$> 0,$$

for $s > 3$ and $n = 1,2,3,\ldots$, since $s > 3$ implies that $(n+1)^2 < n(n+s-1)$.

Now the sequence $\{f_n(2)\}$ is strictly increasing by (b), the sequence $\{g_n(2)\}$ is strictly decreasing by (c), and each sequence converges to $\zeta(2)$. This establishes (2).

Similarly, the sequence $\{f_n(3)\}$ is strictly increasing by (b), the sequence $\{g_n(3)\}$ is strictly decreasing by (d), and each sequence converges to $\zeta(3)$. This establishes (3).

For $s > 3$, the sequence $\{g_n(s)\}$ is strictly increasing by (e) and it converges to $\zeta(s)$. This, together with (a), establishes (4).

Also solved (partially) by the proposer.

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ANOTHER MERSENNE NUMBER BITES THE DUST?

According to TIME Magazine (February 13, 1984), Gustavus Simmons, James Davis, and Diane Holdridge, all from Sandia National Laboratories in Albuquerque, assisted by a Cray computer (for 32 hours and 12 minutes), have found that the prime factorization of the 69-digit Mersenne number

$$M_{251} = 2^{251} - 1$$
$$= 132\ 686\ 104\ 398\ 972\ 053\ 177\ 608\ 575\ 506\ 090\ 561$$
$$429\ 353\ 935\ 989\ 033\ 525\ 802\ 891\ 469\ 459\ 697$$

is

$$178\ 230\ 287\ 214\ 063\ 289\ 511$$
$$\times\ 61\ 676\ 882\ 198\ 695\ 257\ 501\ 367$$
$$\times\ 12\ 070\ 396\ 178\ 249\ 893\ 039\ 969\ 681.$$

Unfortunately, $[251 \log 2] = 75$, so M_{251} should contain 76 digits, not 69. And it should end in 247, not 697.

Another TIME reporter bites the dust?

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