Crux Mathematicorum

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Editorial Board

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

Dear ***Crux*** readers,

When was the last time you played a game of tic-tac-toe? It's been a while, hasn't it? Once you've figured out how not to lose a game, you were done with it. But have you tried expanding the board? Below, I offer you a couple of modifications of the original tic-tac-toe game with the hope of bringing back some of the game's original charm.

First, we might want to bump our two-dimensional game into three dimensions. Suppose you are playing on a $3 \times 3 \times 3$ cube consisting of 27 $1 \times 1 \times 1$ little cubes. Each turn, a player puts their symbol inside one small cube and the goal of the game is as usual: to get three of your symbols in a line. Who has the winning strategy, the first player or the second one?

Now, suppose we decide to stick to two dimensions and want to avoid simply expanding the board. So we decide to go multi scale. Let's label the cells on any 3×3 board with numbers 1 through 9 left to right from the top row down. We will play on a big 3×3 board with each cell containing a small 3×3 tic-tac-toe board. The first player gets to place their symbol in any unused cell within one of the small boards, say in small board j cell i. The next player is then forced into playing on small board i of a big board. To win, a player must win on three small boards in a line. So what's the winning strategy? (You can check out a description of this game with drawings here: `http://mathwithbaddrawings.com/ultimate-tic-tac-toe-original-post/`)

How about a game of tic-tac-toe in disguise? You have two players and numbers 1 through 9. Each turn, a player picks a number (he/she might want to write them down to keep track). The winner is the first person to have in the selection three numbers that add up to 15. Figure out how this is like tic-tac-toe and you will quickly know who has the winning strategy.

I can't take credit for these games. The first one already exists and can be purchased at game stores - just search for 3D tic-tac-toe online. The second was passed on to me by Eric Cytrynbaum, who watched graduate students playing it at the Institute for Applied Mathematics' Annual Retreat. The third, I most recently saw in a presentation by Richard Hoshino and we both originally saw it being presented by Luis Goddyn at a Combinatorial Potlatch at University of Victoria. Good games, like good gossip, travel around.

Do you have any interesting modifications of familiar games? Please share!

Kseniya Garaschuk

THE CONTEST CORNER

No. 27

Kseniya Garaschuk

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2015***, although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC131. Let D be the point on the side AC of triangle ABC. Circle of radius $2/\sqrt{3}$ is inscribed in triangle ABD and touches AB at a point M; circle of radius $\sqrt{3}$ is inscribed in triangle BCD and touches BC at a point N. Given that $|BM| = 6$ and $|BN| = 5$, find the lengths of sides of triangle ABC.

CC132. You are told the following four statements about natural numbers n and k:

a) $n + 1$ is divisible by k,

b) $n = 2k + 5$,

c) $n + k$ is divisible by 3,

d) $n + 7k$ is a prime number.

Three of these statements are true and one is not. Find all possible pairs (n, k).

CC133. Ten numbers, not necessarily unique, are written in a row. Then, under every number, we write how many numbers in this row are smaller than it. Can the second row be

a) 9 0 0 2 5 3 6 3 6 6?

b) 5 6 1 1 4 8 5 8 0 1?

CC134. Let two tangent lines from the point $M(1, 1)$ to the graph of $y = k/x$, $k < 0$ touch the graph at the points A and B. Suppose that the triangle MAB is a right-angle triangle. Find its area and the value of constant k.

CC135. Consider the following two arithmetic progressions:

$$\log a, \log b, \log c \quad \text{and} \quad \log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a.$$

Can the values a, b, c be the lengths of the sides of a triangle? If so, find the interior angles of this triangle.

. . . .

CC131. Soit D un point sur le côté AC du triangle ABC. On trace le segment BD. Le cercle inscrit dans le triangle ABD a un rayon de $2/\sqrt{3}$ et il touche le segment AB en M. Le cercle inscrit dans le triangle BCD a un rayon de $\sqrt{3}$ et il touche le segment BC en N. Sachant que $|BM| = 6$ et $|BN| = 5$, déterminer les longueurs des côtés du triangle ABC.

CC132. On nous dit que les entiers strictement positifs n et k satisfont aux quatre conditions suivantes:

a) $n + 1$ est divisible par k,

b) $n = 2k + 5$,

c) $n + k$ est divisible par 3,

d) $n + 7k$ est un nombre premier.

Sachant que les deux entiers satisfont à exactement trois des quatre conditions, déterminer tous les couples possibles (n, k).

CC133. Dix nombres, pas nécessairement distincts, sont écrits dans une rangée. Au dessous de chaque nombre, on écrit combien des nombres de la première rangée sont inférieurs à ce nombre. Est-il possible que la deuxième rangée soit

a) 9 0 0 2 5 3 6 3 6 6?

b) 5 6 1 1 4 8 5 8 0 1?

CC134. Deux droites, issues du point $M(1, 1)$, sont tangentes à la courbe d'équation $y = k/x$ $(k < 0)$ aux points A et B. Sachant que le triangle MAB est rectangle, déterminer la valeur de k et l'aire du triangle.

CC135. On considère deux suites arithmétiques,

$$\log a, \log b, \log c \quad \text{et} \quad \log a - \log 2b, \log 2b - \log 3c, \log 3c - \log a.$$

Est-il possible que a, b et c soient les longueurs des côtés d'un triangle? Si oui, déterminer les mesures des angles intérieurs de ce triangle.

CONTEST CORNER SOLUTIONS

CC81. Quadrilateral $ABCD$ has the following properties:

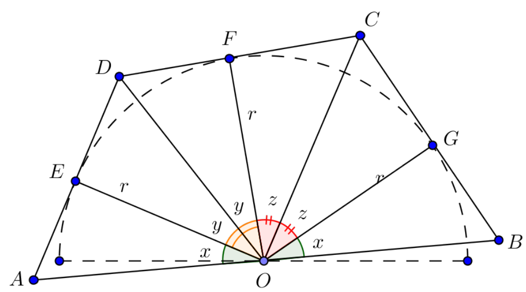
1. the mid-point O of side AB is the centre of a semicircle;
2. sides AD, DC and CB are tangent to this semicircle.

Prove that $AB^2 = 4AD \times BC$.

Originally 1998 W.J. Blundon Mathematics Contest, problem 10.

We received six solutions in total; four solutions came with the assumption that the diameter of the semicircle is the segment AB, which was not stated in the problem.

We present the solution by Šefket Arslanagić.



Assume notation as on the diagram. Since $OA = OB$, $OE = OG$, and $\angle E = \angle G = 90^o$, we have $\triangle AOE \cong \triangle BOG$. Hence $\angle AOE = \angle BOG = x$. Because tangents to the circle from an external point are equal, we have $\triangle DOE \cong \triangle DOF$ and $\triangle COG \cong \triangle COF$. Hence $\angle DOE = \angle DOF = y$ and $\angle COF = \angle COG = z$.

Also $2x + 2y + 2z = 180^o$, so $x + y + z = 90^o$, therefore $\angle OAE = \angle OAD = y + z$, $\angle ADO = x + z$, $\angle OBG = \angle OBC = y + z$, and $\angle OCB = x + y$.

From here, we have $\triangle AOD \sim \triangle BCO$, which gives us

$$\frac{AD}{OB} = \frac{OA}{BC} \iff OA \cdot OB = AD \cdot BC \iff \frac{1}{2}AB \cdot \frac{1}{2}AB = AD \cdot BC.$$

Hence, $AB^2 = 4AD \cdot BC$.

CC82. For each positive integer N, an *Eden sequence* from $\{1, 2, 3, \ldots, N\}$ is defined to be a sequence that satisfies the following conditions:

1. each of its terms is an element of the set of consecutive integers $\{1, 2, 3, \ldots, N\}$,
2. the sequence is increasing, and

3. the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from $\{1, 2, 3\}$ are

$$1 \qquad\qquad 3 \qquad\qquad 1,2 \qquad\qquad 1,2,3$$

For each positive integer N, define $e(N)$ to be the number of Eden sequences from $\{1, 2, 3, \ldots, N\}$. If $e(17) = 4180$ and $e(20) = 17710$, determine $e(18)$ and $e(19)$.

Originally 2012 Euclid Contest, problem 10b.

Solved by Titu Zvonaru and Neculai Stanciu. Below is a modified version of their solution.

We denote by $ev(N)$ the number of even length Eden sequences and by $od(N)$ the number of odd length Eden sequences for N, so $e(N) = ev(N) + od(N)$. We have $ev(1) = 0$, $od(1) = 1$ and $ev(2) = od(2) = 1$. For larger N we obtain the following.

If N is even, we have:

- $ev(N) = ev(N-1) + od(N-1) = e(N-1)$, since an even length Eden sequence for N that doesn't end on N is also an even length Eden sequence for $N-1$, while a sequence that ends on N becomes an odd length Eden sequence for $N-1$ when we remove N;
- $od(N) = od(N-1)$, since an odd length Eden sequence for N cannot end on N and is therefore also an odd length Eden sequence for $N-1$.

If N is odd, we have:

- $ev(N) = ev(N-1)$;
- $od(N) = od(N-1) + ev(N-1) + 1 = e(N-1) + 1$, with the same argument as above, except that there is also the Eden sequence formed only by the number N.

For even N, we have

$$e(N) = ev(N) + od(N) = e(N-1) + od(N-1) = e(N-1) + e(N-2) + 1,$$

which is the same as for odd N:

$$e(N) = od(N) + ev(N) = e(N-1) + 1 + ev(N-1) = e(N-1) + e(N-2) + 1.$$

From this, we can calculate

$$e(20) = e(19) + e(18) + 1 = 2e(18) + e(17) + 2,$$

from which we obtain $e(18) = 6764$ and $e(19) = 10945$.

[*We note that $e(N) + 1 = F_{n+2}$ are the Fibonacci numbers, as can be shown using the recursion for $e(N)$.*]

CC83. A map shows all Beryls Llamaburgers restaurant locations in North America. On this map, a line segment is drawn from each restaurant to the restaurant that is closest to it. Every restaurant has a unique closest neighbour. (Note that if A and B are two of the restaurants, then A may be the closest to B without B being closest to A.) Prove that no restaurant can be connected to more than five other restaurants.

Originally problem B3b from the 2004 Canadian Open Mathematics Challenge.

We present a slightly expanded solution by Titu Zvonaru and Neculai Stanciu.

Suppose A is a restaurant that is connected to six restaurants $P_1, \dots, P_6$. The points A, P_i, and P_j cannot be collinear in this order, because P_j would then not be connected to A. We assume that $P_1, \dots, P_6$ is the order in which the points appear around A in clockwise direction.

From the six angles of type $\angle P_iAP_{i+1}$ at least one must be less than or equal to $60°$; assume $\angle P_1AP_2$ is this angle. This yields that in the triangle P_1AP_2 the side P_1P_2 is not the greatest. Suppose that AP_1 is the greatest.

Then either $P_1P_2 < AP_2 < AP_1$ or $AP_2 < P_1P_2 < AP_1$. In both cases P_1 cannot be connected to A.

CC84. Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

Originally 2014 Canadian Mathematical Olympiad, problem 2.

We present the solution by Titu Zvonaru and Neculai Stanciu.

Let $m = 2p - 1, n = 2q - 1$. We will start by assuming that $p, q \geq 2$.

A red-dominated row will have at least q red squares and a blue-dominated column will have at least p blue squares. Thus, in order to have m red-dominated rows and at least $n - 1$ blue-dominated columns, there would need to be at least

$$(2p - 1)q + (2q - 2)p = 4pq - q - 2p$$

squares in the board. However, there are

$$(2p - 1)(2q - 1) = 4pq - 2p - 2q + 1$$

squares on the board. And since $p, q \geq 2$ we have

$$4pq - q - 2p > 4pq - 2p - 2q + 1.$$

Thus, we cannot have $m + n - 1$ or more as the sum.

To see that we can obtain a sum of $m+n-2$, consider a board formed by tiling the upper left $(m-1)\times(n-1)$ subgrid in a red and blue checkerboard pattern and then colouring the last square in each row red and the last column in each row blue. The bottom right corner can be coloured either colour. Then the first $m-1$ rows are red-dominated and the first $n-1$ columns are blue-dominated.

When $m=1$, we can have n blue-dominated columns by colouring all squares blue. If we have n blue-dominated columns, we cannot have a red-dominated row, so this is a maximum.

Thus, the maximum is $\max(m,n)$ if m or n is 1, and $m+n-2$ otherwise.

CC85. While Lino was simplifying the fraction $\frac{A^3+B^3}{A^3+C^3}$, he cancelled the threes $\frac{A^{\cancel{3}}+B^{\cancel{3}}}{A^{\cancel{3}}+C^{\cancel{3}}}$ to obtain the fraction $\frac{A+B}{A+C}$. If $B\neq C$, determine a necessary and sufficient condition on A, B and C for Lino's method to actually yield the correct answer, ie. for $\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}$.

Originally 2005 University of Waterloo Big E contest, problem 1.

We have received several incomplete or incorrect submissions. We present the solution by Michel Bataille.

In order for the fractions to be defined, we need to have $A\neq-\omega C$, where ω is a cube root of unity.

The equation $\frac{A^3+B^3}{A^3+C^3}=\frac{A+B}{A+C}$ is equivalent to

$$\begin{aligned}&(A^3+B^3)(A+C)=(A^3+C^3)(A+B),\\&(A+B)(A+C)[(A^2-AB+B^2)-(A^2-AC+C^2)]=0,\\&(A+B)(A+C)(B-C)(B+C-A)=0.\end{aligned}$$

Since we are given $B\neq C$ and we assumed that $A\neq-\omega C$, we are left with $A=-B$ or $A=B+C$.

When $A=-B$ we get both fractions are 0, and so they are equal.

If $A=B+C$, we get

$$\frac{A^3+B^3}{A^3+C^3}=\frac{(2B+C)(B^2+BC+C^2)}{(2C+B)(B^2+BC+C^2)}.$$

Note that we cannot have $B^2+BC+C^2=0$ since this would require $B=C=0$ or $A=-\omega C$, neither of which are permitted. It follows that

$$\frac{A^3+B^3}{A^3+C^3}=\frac{2B+C}{2C+B}=\frac{A+B}{A+C}.$$

Thus, it is necessary and sufficient for $A,B,$ and C to be complex numbers satisfying $A^3\neq-C^3$, and $A=B+C$ or $A=-B$.

THE OLYMPIAD CORNER

No. 325

Nicolae Strungaru and Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2015***, although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC191. Let P be a point in the interior of triangle ABC. Extend AP, BP, and CP to meet BC, AC, and AB at D, E, and F, respectively. If triangle APF, triangle BPD and triangle CPE have equal areas, prove that P is the centroid of triangle ABC.

OC192. Find all possible values of a positive integer n for which the expression $S_n = x^n + y^n + z^n$ is constant for all real x, y, z with $xyz = 1$ and $x + y + z = 0$

OC193. Let $\{a_n\}$ be a positive integer sequence such that $a_{i+2} = a_{i+1} + a_i$ for $i \geq 1$. For positive integer n, define $\{b_n\}$ as

$$b_n = \frac{1}{a_{2n+1}} \sum_{i=1}^{4n-2} a_i.$$

Prove that b_n is a positive integer, and find the general form of b_n.

OC194. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Let $f : \mathbb{Q}^+ \to \mathbb{R}$ be a function satisfying the following three conditions:

1. for all $x, y \in \mathbb{Q}^+$, $f(x)f(y) \geq f(xy)$;
2. for all $x, y \in \mathbb{Q}^+$, $f(x + y) \geq f(x) + f(y)$;
3. there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}^+$.

OC195. Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

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OC191. Soit P un point à l'intérieur du triangle ABC. Prolongeons AP, BP et CP afin de rencontrer BC, AC et AB aux points D, E et F respectivement. Si les triangles APF, BPD et CPE ont des surfaces égales, démontrer que P est le centroïde du triangle ABC.

OC192. Déterminer toutes les valeurs possibles de l'entier positif n tel que l'expression $S_n = x^n + y^n + z^n$ est constante pour tout x, y et z réels vérifiant $xyz = 1$ et $x + y + z = 0$.

OC193. Soit $\{a_n\}$ une suite d'entiers positifs telle que $a_{i+2} = a_{i+1} + a_i$ for $i \geq 1$. Pour n entier positif, définissons $\{b_n\}$ selon

$$b_n = \frac{1}{a_{2n+1}} \sum_{i=1}^{4n-2} a_i.$$

Démontrer que b_n est entier positif et déterminer la forme générale de b_n.

OC194. Soit $\mathbb{Q}^+$ l'ensemble des nombres rationnels positifs et soit $f : \mathbb{Q}^+ \to \mathbb{R}$ une fonction satisfaisant les trois conditions :

1. pour tout $x, y \in \mathbb{Q}^+$, on a $f(x)f(y) \geq f(xy)$;
2. pour tout $x, y \in \mathbb{Q}^+$, on a $f(x + y) \geq f(x) + f(y)$;
3. il existe un nombre rationnel $a > 1$ tel que $f(a) = a$.

Démontrer que $f(x) = x$ pour tout $x \in \mathbb{Q}^+$.

OC195. Soit O le centre du cercle circonscrit du triangle aigu ABC. Soit P un point sur le côté AB tel que $\angle BOP = \angle ABC$, et soit Q un point sur le côté AC tel que $\angle COQ = \angle ACB$. Démontrer que la réflexion de BC par rapport à la ligne PQ est tangente au cercle circonscrit du triangle APQ.

OLYMPIAD SOLUTIONS

OC131. Find all $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$g(f(x) - y) = f(g(y)) + x\,,\ \forall x, y \in \mathbb{R}\,.$$

Originally from the Poland Math Olympiad 2012 Day 2 Problem 1.

We received four correct submissions. We present the solution by Michel Bataille.

For $a \in \mathbb{R}$, let $\phi_a : \mathbb{R} \to \mathbb{R}$ be defined by $\phi_a(x) = a - x$. Then, $\phi_a(\phi_a(x) - y) = x + y = \phi_a(\phi_a(y)) + x$ for all x, y, hence the pair $(f, g) = (\phi_a, \phi_a)$ is a solution. We show that there are no other solutions.

Let (f, g) be an arbitrary solution and let $a = f(0),\ b = g(0)$. From the equation with $y = 0,\ y = f(x),\ x = 0$ in succession, we readily obtain

$$g(f(x)) = x + f(b), \tag{1}$$
$$f(x + f(b)) = b - x, \tag{2}$$
$$g(a - y) = f(g(y)) \tag{3}$$

for all $x, y \in \mathbb{R}$.

Let x be any real number. Then $y = f(x)$ in (3) gives $g(a - f(x)) = b - x$ (with the help of (1) and (2)). In this relation, replacing x by $x + f(b)$ and using (2) leads to $g(a - b + x) = b - x - f(b)$, which, with $x = b - a$, yields $f(b) = a - b$. Plugging into (2) and (1), we obtain

$$f(x + a - b) = b - x, \tag{4}$$
$$g(f(x)) = x + a - b. \tag{5}$$

Substituting $x + a - b$ for x in (5), we get $g(b - x) = x + 2(a - b)$ so that $b = g(b - b) = 2a - b$. It follows that $a = b$.

Now, from (4) and (5), $f(x) = a - x = \phi_a(x)$ and $g(a - x) = x$ that is, $g(x) = a - x = \phi_a(x)$. Thus, $(f, g) = (\phi_a, \phi_a)$.

OC132. Find all primes p and q such that

$$(p + q)^p = (q - p)^{(2q-1)}.$$

Originally from the Macedonia JBMO TST 2012, Day 1, Problem 3.

We received one incorrect submission. We present the solution by Oliver Geupel.

The pair $(p, q) = (3, 5)$ is a solution and we prove that it is unique.

For, suppose that the pair (p, q) is any solution of the problem. Then, $2q - p - 1 \geq q - p > 0$. Hence, the number

$$\left(\frac{p + q}{q - p}\right)^p = (q - p)^{2q-p-1} \tag{1}$$

is an integer and we deduce that $q-p$ divides $p+q$. It follows that $q-p$ also divides $(p+q)+(q-p)=2q$. Since the positive divisors of $2q$ are the numbers 1, 2, q, and $2q$, it follows that $q-p \in \{1,\ 2,\ q,\ 2q\}$. We consider the four cases in succession.

If $q-p=1$ then $(p,q)=(2,3)$, which does not satisfy the requirements.

If $q-p=q$ then $p=0$, a contradiction.

If $q-p=2q$ then $p<0$, which is a contradiction.

Thus, $q-p=2$. By (1), we have $(p+1)^p=2^{p+3}$ and $\left(\frac{p+1}{2}\right)^p=8$, which yields $p=3$, $q=5$.

OC133. Let $f(x)=(x+a)(x+b)$ where $a,b>0$. Find the maximum of

$$F=\sum_{1\leqslant i<j\leqslant n}\min\{f(x_i),f(x_j)\},$$

where $x_1,x_2,\ldots,x_n\geqslant 0$ are real numbers satisfying $x_1+x_2+\ldots+x_n=1$.

Originally from the China Math Olympiad 2012 Day 2 Problem 2.

We received two correct submissions. We present the solution by Oliver Geupel.

Suppose $n\geq 1$. Setting

$$x_1=x_2=\cdots=x_n=\frac{1}{n},$$

we get

$$F=\frac{n-1}{2n}\cdot(na+1)(nb+1). \tag{1}$$

We prove that the value (1) is the maximum of F.

The second derivative of the function f is $f''(x)=2>0$. Hence, f is convex, so that

$$\sum_{i=1}^{n} f(x_i)\geqslant nf\left(\frac{1}{n}\right). \tag{2}$$

The second derivative of the function $g(x)=\sqrt{f(x)}$, $x>0$, is the function

$$g''(x)=\frac{-(a-b)^2}{4(x+a)^{3/2}(x+b)^{3/2}}\leq 0.$$

Thus, g is concave, so that

$$\sum_{i=1}^{n} g(x_i)\leqslant ng\left(\frac{1}{n}\right). \tag{3}$$

By (2) and (3),

$$F \leqslant \sum_{1\leqslant i<j\leqslant n} \sqrt{f(x_i)f(x_j)} = \frac{1}{2}\left[\left(\sum_{i=1}^{n} g(x_i)\right)^2 - \sum_{i=1}^{n} f(x_i)\right]$$
$$\leqslant \frac{1}{2}\left[n^2\left(g\left(\frac{1}{n}\right)\right)^2 - nf\left(\frac{1}{n}\right)\right]$$
$$= \frac{n-1}{2n}\cdot(na+1)(nb+1).$$

This completes the proof.

OC134. Let ABC be an acute-angled triangle with $AB \neq AC$. Let Γ be the circumcircle of ABC, H the orthocentre of ABC and O the centre of Γ. Let M be the midpoint of BC. The line AM meets Γ again at N and the circle with diameter AM crosses Γ again at P. Prove that the lines AP, BC and OH are concurrent if and only if $AH = HN$.

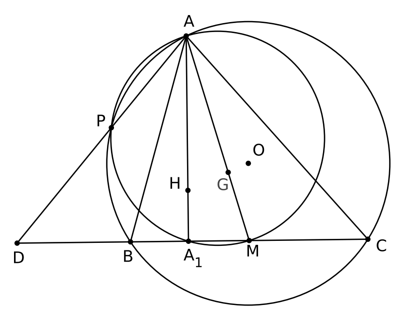
Originally from the France 2012 TST Day 1 Problem 2.

We received three correct submissions. We present the solution by Michel Bataille.

Since $AB \neq AC$, the median AM is not perpendicular to BC and O does not lie on AM. Thus, $P \neq A$ and AP intersects BC. The point of intersection D of AP and BC is on the radical axis AP of Γ and the circle γ_M with diameter AM. It follows that D has the same power with respect to these two circles and so

$$DP \cdot DA = DM \cdot DA_1 \qquad (1)$$

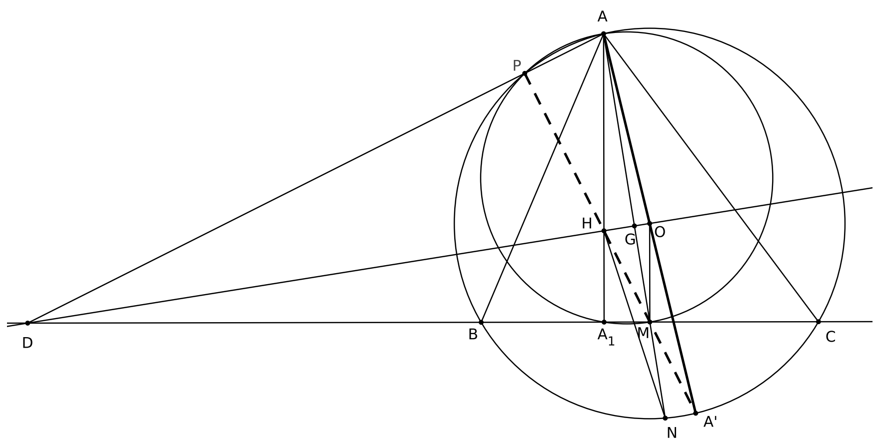
where A_1 is the foot of the altitude from A of ΔABC (A_1 is on γ_M since $\angle AA_1M = 90°$).



First, suppose that $HA = HN$. Since we also have $OA = ON$, the Euler line OH is the perpendicular bisector of AN, and, as such, is perpendicular to AM at the centroid G of ΔABC.

Now, H, G, M, A_1 are points of the circle γ_1 with diameter HM and the power of D with respect to γ_1 is $\mathcal{P}_{\gamma_1}(D) = DM.DA_1$. Similarly, H, G, A, P are points of the circle γ_2 with diameter AH, hence $\mathcal{P}_{\gamma_2}(D) = DP \cdot DA$. From (1), D has the same power with respect to γ_1 and γ_2 and therefore D is on their radical axis, which is the line $HG = OH$ since the circles intersect. We conclude that AP, BC and OH are concurrent.

Conversely, suppose that AP, BC and OH are concurrent, that is, OH passes through D. Let A' be the point of Γ diametrically opposite to A (see figure).



The homothety $h_{A'}$ with centre A' and scale factor 2 transforms O into A and M into H (since $\overrightarrow{AH} = 2\overrightarrow{OM}$, a well-known result), therefore the four points A', M, H, P lie on the perpendicular ℓ to AP at P [ℓ passes through A' because P is on Γ of which AA' is a diameter; ℓ passes through M because P is on γ_M of which AM is a diameter; and A', M, H are collinear because $h_{A'}(M) = H$]. The orthocenter H' of ΔADM is the intersection of $\ell = MP$ and AA_1, hence coincides with H. It follows that the Euler line DH is perpendicular to AM at G. In addition, O is on this Euler line and $OA = ON$, hence DH is the perpendicular bisector of AN and so $HA = HN$.

OC135. Prove that for each $n \in \mathbb{N}$ there exist natural numbers $a_1 < a_2 < \cdots < a_n$ such that $\phi(a_1) > \phi(a_2) > \cdots > \phi(a_n)$ where ϕ denotes the Euler ϕ function.

Originally from the Iran National Math Olympiad (3rd Round), 2012 Final Exam Problem 3.

We present the solution by Oliver Geupel.

The proof is by induction. The result holds for $n = 1$.

Suppose that the number $n > 1$ is such that there are natural numbers $b_1 < b_2 < \cdots < b_{n-1}$ with the property $\phi(b_1) > \phi(b_2) > \cdots > \phi(b_{n-1})$. Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$ be the infinite sequence of primes. It is well-known that

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k}\right) = 0.$$

Thus, there are distinct primes $q_1, q_2, \ldots, q_m$ such that

$$\prod_{k=1}^{m}\left(1-\frac{1}{q_k}\right) < \frac{1}{4}$$

and $\gcd(b_1 b_2 \cdots b_{n-1},\ q_1 q_2 \cdots q_m) = 1$. Therefore,

$$\phi(b_1 \cdot q_1 q_2 \cdots q_m) \le b_1 \cdot q_1 q_2 \cdots q_m \cdot \prod_{k=1}^{m}\left(1-\frac{1}{q_k}\right) < \frac{1}{4} \cdot b_1 \cdot q_1 q_2 \cdots q_m.$$

By Bertrand's postulate, there is a prime p such that

$$\begin{aligned} \phi(b_1 \cdot q_1 q_2 \cdots q_m) &< \frac{1}{4} \cdot b_1 \cdot q_1 q_2 \cdots q_m. && (1)\\ &< p - 1 = \phi(p) < p && (2)\\ &< b_1 \cdot q_1 q_2 \cdots q_m. && (3) \end{aligned}$$

Let

$$\begin{aligned} a_1 &= p,\\ a_2 &= b_1 \cdot q_1 q_2 \cdots q_m,\\ a_3 &= b_2 \cdot q_1 q_2 \cdots q_m,\\ &\ldots,\\ a_n &= b_{n-1} \cdot q_1 q_2 \cdots q_m. \end{aligned}$$

Then, $a_1 < a_2 < \cdots < a_n$. Moreover,

$$\begin{aligned} &\phi(a_1) = p - 1\\ >&\phi(a_2) = \phi(b_1)\phi(q_1 \cdots q_m)\\ &\ldots\\ >&\phi(a_n) = \phi(b_{n-1})\phi(q_1 \cdots q_m). \end{aligned}$$

This completes the induction.

Editor's Comments. It is of interest to note that if $a_n \in (0,1)$ is a sequence of real numbers, then the product $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{a_n}\right)$ is non-zero if and only if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Since the sum of the reciprocals of the primes diverges, the product mentioned in the solution above is indeed zero.

FOCUS ON...

No. 13

Michel Bataille

The Dot Product

Introduction

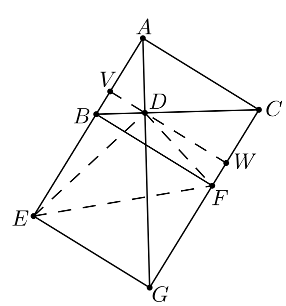
For the solver facing a geometry problem, vectors occupy a vantage point, just between synthetic and analytical methods. Operations on vectors often lead to advantageous proofs and in particular, the dot product (or scalar product) appears as an essential tool. The purpose of this number is to show this product at work through alternative solutions to several past problems.

Example 1 : a useful trick.

We begin with an exercise adapted from problem **2973** [2004 : 369, 372 ; 2005 : 465] :

> Let triangle ABC be right-angled at A and let D be the foot of the altitude from A. Let F be such that $BACF$ is a rectangle. The line CF meets the line AD at G and E is such that $BFGE$ is a rectangle. Show that ΔEDF is right-angled.

The figure presents many right angles and we are to discover one more, a favourable ground for the dot product. All the more so as we add more right angles by drawing the orthogonal projections V and W of D onto AB and CF, respectively. Note that V, D, W are collinear.



We propose a very short solution that resorts to a trick of constant use, based on the following obvious property of the dot product : $\overrightarrow{u}\cdot\overrightarrow{v} = \overrightarrow{u}\cdot(\overrightarrow{v}\pm\overrightarrow{w})$ whenever $\overrightarrow{w}\perp\overrightarrow{u}$. Using this to pass from one product to the next, we obtain

$$\overrightarrow{VE}\cdot\overrightarrow{WF} = \overrightarrow{VE}\cdot\overrightarrow{VB} = \overrightarrow{VB}\cdot\overrightarrow{DG} = \overrightarrow{VD}\cdot\overrightarrow{DG} = \overrightarrow{VD}\cdot\overrightarrow{DW}.$$

Thus, $\overrightarrow{DE}\cdot\overrightarrow{DF} = (\overrightarrow{VE}-\overrightarrow{VD})\cdot(\overrightarrow{WF}-\overrightarrow{WD}) = \overrightarrow{VE}\cdot\overrightarrow{WF}+\overrightarrow{VD}\cdot\overrightarrow{WD} = 0$ and $DE\perp DF$ follows.

Example 2 : an equivalence, directly.

Here is the statement of problem **3665** [2011 : 388, 390 ; 2012 : 293] :

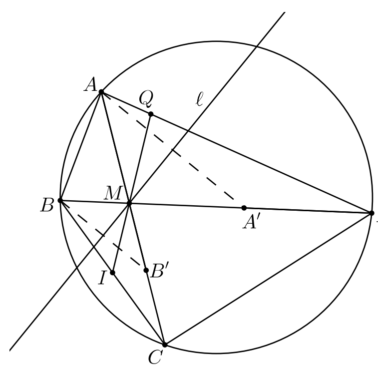
> Let the diagonals AC and BD of the cyclic quadrilateral $ABCD$ intersect at M, and the line joining M to the midpoint of BC meet AD at Q. Prove that MQ is perpendicular to AD if and only if the sides AD and BC are parallel (in which case $ABCD$ is an isosceles trapezoid), or the diagonals are perpendicular (and we have Brahmagupta's configuration).

The following proof first shows that the condition MQ perpendicular to AD is equivalent to an equality of dot products. This leads to a very direct solution. Let I be the midpoint of BC. Note that $\overrightarrow{MB} \cdot \overrightarrow{MD}$ and $\overrightarrow{MA} \cdot \overrightarrow{MC}$ are both equal to the power of M with respect to the circle through A, B, C, D. Then,

$$\begin{aligned} 2\overrightarrow{MI} \cdot \overrightarrow{AD} &= (\overrightarrow{MB} + \overrightarrow{MC}) \cdot (\overrightarrow{MD} - \overrightarrow{MA}) \\ &= \overrightarrow{MC} \cdot \overrightarrow{MD} - \overrightarrow{MA} \cdot \overrightarrow{MB}. \end{aligned}$$

It follows that MQ is perpendicular to AD if and only if

$$\overrightarrow{MC} \cdot \overrightarrow{MD} = \overrightarrow{MA} \cdot \overrightarrow{MB} \qquad (1)$$



Now, the triangles MCD and MBA are inversely similar ; more precisely, there exists an inverse similarity σ such that $\sigma(M) = M$, $\sigma(B) = C$ and $\sigma(A) = D$. If ℓ denotes the internal bisector of $\angle BMC$ and B', A' are the images of B, A in ℓ, respectively, we have $\overrightarrow{MC} = \lambda\overrightarrow{MB'}$ and $\overrightarrow{MD} = \lambda\overrightarrow{MA'}$ where the positive real number λ is the factor of σ. As a result,

$$\overrightarrow{MC} \cdot \overrightarrow{MD} = \lambda^2\overrightarrow{MB'} \cdot \overrightarrow{MA'} = \lambda^2\overrightarrow{MB} \cdot \overrightarrow{MA}$$

and (1) is equivalent to $(\lambda^2 - 1)\overrightarrow{MB} \cdot \overrightarrow{MA} = 0$; that is, to $\lambda = 1$ or $\overrightarrow{MB} \cdot \overrightarrow{MA} = 0$. The result follows since $\lambda = 1$ means that BC and AD are parallel, that is, $ABCD$ is an isosceles trapezoid and $\overrightarrow{MB} \cdot \overrightarrow{MA} = 0$ means that the diagonals of $ABCD$ are perpendicular.

Example 3 : a helpful interpretation.

Our third example is from the 25th Albanian Mathematical Olympiad [2007 : 279 ; 2008 : 222].

> In an acute-angled triangle ABC, let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB, respectively. Prove that $d_a + d_b + d_c \leq 3r$, where r is the radius of the incircle of triangle ABC.

A vectorial version of the required inequality (see (2) below) will provide a solution completely different from the featured one. We use the familiar notations for the sides and angles of ΔABC and denote by D, E, F the projection of the incentre I onto BC, CA, AB, respectively. If A_1 is the projection of H onto BC, we have

$$\overrightarrow{ID} \cdot \overrightarrow{IH} = ID^2 + \overrightarrow{ID} \cdot \overrightarrow{DH} = \overrightarrow{ID}^2 + \overrightarrow{ID} \cdot \overrightarrow{A_1H} = r^2 - rd_a = r(r - d_a)$$

(since ΔABC is acute-angled, H is interior to ΔABC and so $\overrightarrow{ID} \cdot \overrightarrow{A_1H} < 0$). Similar results hold for $\overrightarrow{IE} \cdot \overrightarrow{IH}$ and $\overrightarrow{IF} \cdot \overrightarrow{IH}$ and the desired inequality is equivalent to

$$\overrightarrow{IH} \cdot (\overrightarrow{ID} + \overrightarrow{IE} + \overrightarrow{IF}) \geq 0. \tag{2}$$

Let $\overrightarrow{u} = \frac{1}{ID}\overrightarrow{ID},\ \overrightarrow{v} = \frac{1}{IE}\overrightarrow{IE},\ \overrightarrow{w} = \frac{1}{IF}\overrightarrow{IF}$. Then, $\overrightarrow{ID} + \overrightarrow{IE} + \overrightarrow{IF} = r(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w})$ and since $(a + b + c)I = aA + bB + cC$, we have

$$(a + b + c)\overrightarrow{IH} = a\overrightarrow{AH} + b\overrightarrow{BH} + c\overrightarrow{CH} = 2R(a(\cos A)\overrightarrow{u} + b(\cos B)\overrightarrow{v} + c(\cos C)\overrightarrow{w})$$

or, using the law of sines,

$$(a + b + c)\overrightarrow{IH} = 2R^2((\sin 2A)\overrightarrow{u} + (\sin 2B)\overrightarrow{v} + (\sin 2C)\overrightarrow{w}),$$

where R is the circumradius.

Thus (2) is equivalent to

$$((\sin 2A)\overrightarrow{u} + (\sin 2B)\overrightarrow{v} + (\sin 2C)\overrightarrow{w}) \cdot (\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}) \geq 0. \tag{3}$$

Now, observing that for example, $\overrightarrow{u} \cdot \overrightarrow{v} = ||\overrightarrow{u}||\,||\overrightarrow{v}|| \cos(\pi - C) = -\cos C$, the left side of (3) rewrites as

$$\sin 2A + \sin 2B + \sin 2C \\ - (\sin 2A + \sin 2B)\cos C - (\sin 2B + \sin 2C)\cos A - (\sin 2C + \sin 2A)\cos B.$$

That is,

$$(\sin 2A)(1 - \cos(B - C)) + (\sin 2B)(1 - \cos(C - A)) + (\sin 2C)(1 - \cos(A - B)).$$

For example, $(\sin 2A + \sin 2B)\cos C = 2\sin C\cos(A - B)\cos C = \sin 2C\cos(A - B)$.

Inequality (3) follows since A, B, C being acute, we have $\sin 2A, \sin 2B, \sin 2C > 0$. Clearly, equality holds if and only if $A = B = C$, that is, ΔABC is equilateral.

Example 4 : another direct equivalence.

As a final example, we show how vectors and the dot product lead to a straightforward solution to problem **3347** [2008 : 241, 243 ; 2009 : 250] :

> Let $A_1A_2A_3A_4$ be a convex quadrilateral. Let B_i be a point on A_iA_{i+1} for $i \in \{1, 2, 3, 4\}$, where the subscripts are taken modulo 4, such that
>
> $$\frac{B_1A_1}{B_1A_2} = \frac{B_3A_4}{B_3A_3} = \frac{A_1A_4}{A_2A_3} \quad \text{and} \quad \frac{B_2A_2}{B_2A_3} = \frac{B_4A_1}{B_4A_4} = \frac{A_1A_2}{A_3A_4}.$$
>
> Prove that $B_1B_3 \perp B_2B_4$ if and only if $A_1A_2A_3A_4$ is a cyclic quadrilateral.

Setting $a_i = A_iA_{i+1}$, the vectorial relations $a_{i+1}\overrightarrow{B_iA_i} + a_{i+3}\overrightarrow{B_iA_{i+1}} = \overrightarrow{0}$ $(i = 1, 2, 3, 4)$ summarize the hypotheses on the points B_i. We readily deduce

$$(a_2 + a_4)\overrightarrow{B_1B_3} = a_2\overrightarrow{A_1A_4} + a_4\overrightarrow{A_2A_3} \quad \text{and} \quad (a_1 + a_3)\overrightarrow{B_2B_4} = a_1\overrightarrow{A_3A_4} + a_3\overrightarrow{A_2A_1}.$$

If α_i is the angle of the quadrilateral at vertex A_i, the dot product $\overrightarrow{B_1B_3} \cdot \overrightarrow{B_2B_4}$ is given by

$$\frac{a_1a_2a_3a_4}{(a_2 + a_4)(a_1 + a_3)} \left(\cos\alpha_2 + \cos\alpha_4 - \cos\alpha_1 - \cos\alpha_3\right).$$

Now, using $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$, we have that $\cos\alpha_2 + \cos\alpha_4 = \cos\alpha_1 + \cos\alpha_3$ is equivalent to

$$\cos\left(\frac{\alpha_1 + \alpha_3}{2}\right) \sin\left(\frac{\alpha_3 + \alpha_4}{2}\right) \sin\left(\frac{\alpha_2 + \alpha_3}{2}\right) = 0.$$

Thus, $B_1B_3 \perp B_2B_4$ is equivalent to $\alpha_1 + \alpha_3 = \pi$; that is, to $A_1A_2A_3A_4$ cyclic.

Exercises

As usual, we conclude this number with a couple of exercises. The reader is invited to solve them with the help of the dot product.

(a) Let the angle bisectors of triangle ABC meet its sides at D, E, F. Show that ΔDEF is right-angled if and only if one of the angles of ΔABC equals $120°$. [This problem was set by M. Genin in the bulletin of the APMEP (French math teachers' association) in 2006, but no solution ever appeared.]

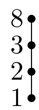
(b) (11th Iberoamerican Mathematical Olympiad 1996, problem 6)
We have n distinct points $A_1, \ldots, A_n$ in the plane. To each point A_i a real number $\lambda_i \neq 0$ is assigned, in such a way that $A_iA_j^2 = \lambda_i + \lambda_j$ for all i, j with $i \neq j$. Show that $n \leq 4$ and that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0$ if $n = 4$. (Hint : the λ_i are not so mysterious ; first show that $\lambda_i = \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k}$ whenever i, j, k are distinct.)

Partially ordered sets

Bill Sands

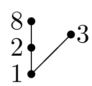
A *partially ordered set*, or *poset* for short, is a collection of elements, drawn as small circles, linked together by straight lines, where if an element a is connected to an element b by a straight line, *and a is* **lower** *on the page than b is*, then a is thought of as *less* than b in some way. There are many examples in mathematics (or real life) when objects are ordered in some way, and they are examples of posets.

Example 1 Take some numbers, say $1, 2, 3$ and 8. We can order them by the usual $<$ and get the poset



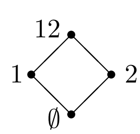
Here, the line from 3 to 8 indicates that 3 is less than 8 (under the ordering $<$), because 3 is lower on the page than 8 is. It is not necessary that the line be vertical, only that it end at a higher point on the page than where it started. Also, the length of the line is unimportant. Note that 2 is less than 8, but we have not drawn a line from 2 to 8 ; this is because there is an element 3 in between, so from the facts that $2 < 3$ and $3 < 8$, illustrated by the lines between 2 and 3 and between 3 and 8, we can deduce that 2 is less than 8, so we don't need a line to tell us that. (Or, in other words, the *upward path* from 2 through 3 to 8 tells us that $2 < 8$.)

Example 2 Next, use the same numbers, but order them by "divides into" rather than $<$. That is, 1 divides into 2, but 2 does not divide into 3, and so on. This time we get the poset



Notice that, since 2 does not divide into 3, there is no line from 2 to 3 which goes up the page. It is unimportant whether the element 3 is higher or lower on the page than 2 is.

Example 3 Suppose we take a set, say $\{1, 2\}$, and list all its subsets. We get $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, where $\emptyset$ is the symbol for the *empty set*, by which we mean the set with no elements in it. Usually we will save ourselves some notation by dropping the braces and commas in each set, so we will write the above sets as $\{\emptyset, 1, 2, 12\}$. Now suppose we order these sets by $\subset$ (containment), so that $1 \subset 12$ for example (and of course the empty set is contained in every set). Then we get the following poset :



Notice that once again we have not drawn a line from $\emptyset$ to 12, even though $\emptyset \subset 12$ is true; this is because either of the two paths from $\emptyset$ to 12 already in the picture will be enough to show this. We always omit all such unnecessary lines from posets. Suppose we use three or four elements in our set instead of two, and still list all subsets and order them by $\subset$: what do the posets look like then?

Maybe the simplest question we could ask about posets is : how many are there of a certain size? Here we don't care what the elements are called, only what the pictures look like. And things like length and slope of lines don't matter either. For example, there is only one poset with one element (a dot), and just two with two elements, namely

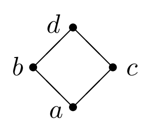
 and

There are 5 posets with 3 elements and 16 posets with 4 elements : find them.

Problem 1 How many posets with five elements are there?

Chains and Antichains

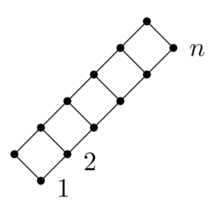
A poset in which every two elements are ordered one way or the other is called a *chain*. A chain with n elements in it is denoted by $\mathbf{n}$. In fact, a subset of the elements of a poset, which form a chain by themselves, is also called a chain. For example, the poset



is not a chain, because b is not less than c or bigger than c. It has lots of chains in it, though; each single element is a chain, so is $\{a, b\}$ and most other two-element subsets, even $\{a, d\}$ is a chain because $a < d$ is true, but $\{b, c\}$ is not a chain. Also, $\{a, b, d\}$ and $\{a, c, d\}$ are chains. Finally, the empty set $\emptyset$ is also considered a chain in any poset.

How many chains does the poset $\mathbf{n}$ have?

Problem 2 How many chains does the poset

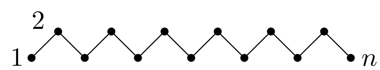


have?

An *antichain* is a poset, or a subset of a poset, in which **no** two elements are ordered. A poset with n elements which is an antichain is denoted n.

How many antichains does $\mathbf{n}$ have ? How many does n have ?

Problem 3 How many antichains does the poset

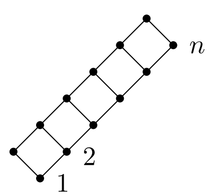


have ?

Suppose that the largest chain of a poset has n elements. Prove that the poset can be partitioned into n antichains, but no fewer.

Problem 4 Suppose that the largest antichain of a poset has n elements. Prove that the poset can be partitioned into n chains, but no fewer. (This is called *Dilworth's Theorem*, and is harder than the previous question.)

The maximal chains of a poset do not have to be all the same size, but if they are, we say that the poset is *graded*, and each element of the poset has a height. The elements at the same height form a *level* of the poset. For instance, the poset



is graded, because all maximal chains have $n + 1$ elements. The bottom element has height 0 and the top element height n. There are $n + 1$ levels. Note that every level is an antichain, but not every antichain is in only one level.

Find a graded poset which has only one antichain of largest size, and this antichain is not a level.

Now we'll look again at the poset of all subsets of a set, ordered by $\subset$. The poset of all subsets of the set $\{1, 2, \ldots, n\}$ is denoted B_n.

What does B_n look like ? Show it is graded. How many levels does it have ? What are the sizes of the levels ? Show that the biggest level is the middle one(s).

It is true that in B_n, no antichain is larger than the largest level. This result is called *Sperner's Theorem.*

Problem 5 Prove Sperner's Theorem three ways.

Proof 1. Suppose $\mathcal{A}$ is an antichain in B_n. Let p_k be the number of members of $\mathcal{A}$ of size k, for each integer k. For each member A of $\mathcal{A}$ which has size k, count the number of permutations of the set $\{1, 2, \ldots, n\}$ which begin with the elements of

A in some order. Then show that

$$\sum_{k=0}^{n} k!(n-k)!p_k \leq n! \qquad \text{and thus} \qquad \sum_{k=0}^{n} \frac{p_k}{\binom{n}{k}} \leq 1 \ .$$

Now note that $|\mathcal{A}| = \sum_{k=0}^{n} p_k$ and hence show that $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Proof 2. The idea of this proof is that if $\mathcal{A}$ is an antichain in B_n which contains sets of size different from $(n-1)/2$ (if n is odd) or $n/2$ (if n is even), then such sets can be replaced by just as many other sets closer in size to $\lfloor n/2 \rfloor$ while still getting an antichain. In this way all sets can be replaced by sets all of the same size. To make this proof work we need to prove the following. Suppose that $\mathcal{B}$ is any collection of k-element subsets of $\{1, 2, \ldots, n\}$, where $k < n$. Let $\nabla\mathcal{B}$ denote the collection of all $(k+1)$-element subsets of $\{1, 2, \ldots, n\}$ which contain some member of $\mathcal{B}$. We now count the number of ordered pairs (B, D) where $B \in \mathcal{B}$, $D \in \nabla\mathcal{B}$ and $B \subset D$. Show that this number is exactly $(n-k)|\mathcal{B}|$ and is at most $(k+1)|\nabla\mathcal{B}|$, and thus prove that

$$|\nabla\mathcal{B}| \geq \frac{n-k}{k+1}|\mathcal{B}|.$$

Conclude that if $k \leq (n-1)/2$ then $|\nabla\mathcal{B}| \geq |\mathcal{B}|$. Using a symmetrical result for $k \geq (n+1)/2$, get Sperner's Theorem.

Proof 3. First we define a *symmetric chain* to be a chain in B_n that contains a set from every level from s to $n-s$, for some $s \leq n/2$. The idea of this proof is to show that B_n can be partitioned into disjoint symmetric chains, from which the proof will follow quickly. The proof that B_n is a disjoint union of symmetric chains can be done by induction on n, using that B_n is the disjoint union of B_{n-1} and the set of all subsets of B_n containing the element n. The trick is to figure out how to turn the symmetric chains in B_{n-1} (which exist by induction hypothesis) into symmetric chains in B_n.

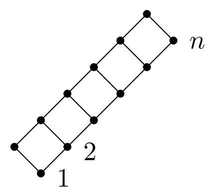
For more information about Sperner's Theorem and its proofs, see [1].

Linear Extensions

Here's another example of a poset. Suppose there are a number of tennis players, and they play a number of matches. Each two players play each other at most once, and we assume that there are no draws. We also assume that the players are naturally ordered by ability, and that the better player always beats the worse player. (Of course, this ordering is not known in advance.) Then the result of a number of matches can be illustrated by a poset, where the elements are the players, and for any players a and b, a line from a up to b in the poset means that b has beaten a (so we know that b is better than a). Before any matches have been played, the poset will be an antichain. When everyone has played everyone (or probably earlier than that), the poset will be a chain.

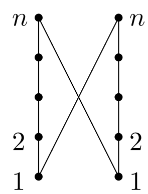
Now suppose only some matches have been played. Suppose we'd like to guess at what the final ordering of the players, from worst to best, might be. We would need to form a chain using all the players which preserves all the ordering found so far. Such a chain is called a *linear extension* of the poset.

How many linear extensions does $\mathbf{n}$ have ? How many does n have ? How many does the poset



have ? (You obtain a sequence of numbers known as the Catalan numbers.)

Problem 6 How many linear extensions does the poset



have ?

The next problem shows that the posets B_n contain every poset inside them.

Problem 7 Show that every (finite) poset is a subset of some B_n.

The *smallest* positive integer n so that poset P is a subset of B_n is called the *dimension* of P. This is also the smallest number of linear extensions of P whose "intersection" is P. Find the dimensions of the various posets considered above.

Two Open Questions

To finish, here are two questions which are not hard to understand but which are still unsolved. So don't spend too much time on them !

Question 1 Look again at the tennis players poset. Suppose some games have been played, so we have a poset P, but that we still don't know the complete ordering of the players (so P is not a chain). It seems reasonable to define the *probability* that player a is worse than player b to be

$$p(a < b) = \frac{\text{number of linear extensions of } P \text{ in which } a < b}{\text{total number of linear extensions of } P}.$$

Suppose we wanted to find two players a and b for which $p(a < b)$ is $1/2$. Of course exactly $1/2$ isn't always possible, but how close can we always get ? There is a small poset in which the best we can do is find a and b so that $p(a < b) = 1/3$ (or $2/3$). So here is the unsolved problem [2] :

For every finite poset P which is not a chain, are there always two elements a and b so that $p(a < b)$ is between $1/3$ and $2/3$?

It has been proved [3] that this statement is correct for the numbers $(5 - \sqrt{5})/10$ and $(5 + \sqrt{5})/10$!

Question 2 For each positive integer n,

- find a poset containing exactly n antichains. (Easy.)
- show that there is a poset whose largest chain has at most two elements and which contains exactly n antichains. (Solved by Václav Linek [4] when he was an undergrad student at the University of Calgary.)
- find a poset containing exactly n chains. (Easy.)

And finally, the unsolved problem :

- show that there is a poset whose largest antichain has at most two elements and which contains exactly n chains. (Unknown if true, but it probably is.)

References

[1] I. Anderson, *Combinatorics of Finite Sets*, Clarendon Press, Oxford, 1987.

[2] S.S. Kislicyn, Finite partially ordered sets and their corresponding permutation sets, *Mat. Zametki* **4** (1968), 511-518.

[3] G.R. Brightwell, S. Felsner, and W.T. Trotter, Balancing pairs and the cross product conjecture, *Order* 12 (1995), 327-349.

[4] V. Linek, Bipartite graphs can have any number of independent sets, *Discrete Math.* **76** (1989), no. 2, 131-136.

*This article (slightly adapted) was originally a handout to accompany a lecture given by the author at the Canadian Mathematical Society Winter Training Camp at York University in January 1999. Readers may remember that Václav Linek, mentioned near the end of the article, served a term as Editor-In-Chief of **Crux** some years later.*

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2015**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

3960. *Proposed by George Apostolopoulos. Correction.*

Let a, b, c be nonnegative real numbers such that $a + b + c = 4$. Prove that

$$\frac{a^2b}{3a^2 + b^2 + 4ac} + \frac{b^2c}{3b^2 + c^2 + 4ab} + \frac{c^2a}{3c^2 + a^2 + 4bc} \leq \frac{1}{2}.$$

3961. *Proposed by Michel Bataille.*

In a triangle ABC, let $\angle A \geq \angle B \geq \angle C$ and suppose that

$$\sin 4A + \sin 4B + \sin 4C = 2(\sin 2A + \sin 2B + \sin 2C).$$

Find all possible values of $\cos A$.

3962. *Proposed by Michel Bataille.*

Let ABC be a nonequilateral triangle, Γ its circumcircle and ℓ its Euler line. Let its medians from A, B, C meet Γ again at A_1, B_1, C_1, respectively, and let $M = t_B \cap t_C$, $N = t_C \cap t_A$, $P = t_A \cap t_B$ where t_A, t_B, t_C are the tangents to Γ at A, B, C, respectively.

Prove that the lines MA_1, NB_1, PC_1 and ℓ are concurrent or parallel and that the latter occurs if and only if $\cos A \cos B \cos C = -\frac{1}{4}$.

3963. *Proposed by D. M. Bătineţu and Neculai Stanciu.*

Let $A \in M_n(\mathbb{R})$ such that $A^2 = 0_n \in M_n(\mathbb{R})$ and let $x, y \in \mathbb{R}$ such that $4y \geq x^2$. Prove that $det(xA + yI_n) \geq 0$.

3964. *Proposed by George Apostolopoulos.*

Let P be an arbitrary point inside a triangle ABC. Let a, b and c be the distances from P to the sides BC, AC and AB, respectively. Prove that

$$\frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^4}{\sin^4 A + \sin^4 B + \sin^4 C} \leq 12R^2,$$

where R denotes the circumradius of ABC. When does the equality occur?

3965. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(\ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{(-1)^n}{n}\right)x^n$$

and its value at x for each x in this interval.

3966. *Proposed by Dao Hoang Viet.*

Let x and y be the legs and h the hypotenuse of a right triangle. Prove that

$$\frac{1}{2h+x+y}+\frac{1}{h+2x+y}+\frac{1}{h+x+2y}<\frac{h}{2xy}.$$

3967. *Proposed by Marcel Chiriţă.*

Determine all positive integers a,b and c that satisfy the following equation:

$$(a+b)!=4(b+c)!+18(a+c)!$$

3968. *Proposed by Michal Kremzer.*

Let $\{a\}=a-[a]$, where $[a]$ is the greatest integer function. Show that if a is real and $a(a-2\{a\})$ is an integer, then a is an integer.

3969. *Proposed by Marcel Chiriţă.*

Determine the functions $f:(\frac{8}{9},\infty)\mapsto\mathbb{R}$ continuous at $x=1$ such that

$$f(9x-8)-2f(3x-2)+f(x)=4x-4+\ln\frac{9x^2-8x}{(3x-2)^2}\qquad\text{for all } x\in\left(\frac{8}{9},\infty\right).$$

3970. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{10a^3+9}+\frac{1}{10b^3+9}+\frac{1}{10c^3+9}\geq\frac{3}{19}.$$

. . . .

3960. *Proposé par George Apostolopoulos. Correction.*

Soient a, b, c des nombres réels non négatifs tels que $a+b+c=4$. Démontrer que

$$\frac{a^2b}{3a^2+b^2+4ac}+\frac{b^2c}{3b^2+c^2+4ab}+\frac{c^2a}{3c^2+a^2+4bc}\le\frac{1}{2}.$$

3961. *Proposé par Michel Bataille.*

On considère un triangle ABC pour lequel $\angle A \ge \angle B \ge \angle C$ et

$$\sin 4A+\sin 4B+\sin 4C=2(\sin 2A+\sin 2B+\sin 2C).$$

Déterminer toutes les valeurs possibles de $\cos A$.

3962. *Proposé par Michel Bataille.*

Soit ABC un triangle non équilatéral, ℓ sa droite d'Euler et Γ le cercle circonscrit au triangle. Les médianes issues des sommets A, B et C coupent Γ de nouveau aux points respectifs A_1, B_1 et C_1. Soit $M=t_B\cap t_C$, $N=t_C\cap t_A$ et $P=t_A\cap t_B$, t_A, t_B et t_C étant les tangentes à Γ aux points respectifs A, B et C.

Démontrer que les droites MA_1, NB_1, PC_1 et ℓ sont concourantes ou parallèles et qu'elles sont parallèles si et seulement si $\cos A\cos B\cos C=-\frac{1}{4}$.

3963. *Proposé par D. M. Bătineţu et Neculai Stanciu.*

Soit $A\in M_n(\mathbb{R})$ de manière que $A^2=0_n\in M_n(\mathbb{R})$ et soit $x,y\in\mathbb{R}$ de manière que $4y\ge x^2$. Démontrer que $det(xA+yI_n)\ge 0$.

3964. *Proposé par George Apostolopoulos.*

On considère un point quelconque P à l'intérieur d'un triangle ABC. Soit a, b et c les distances depuis P jusqu'aux côtés respectifs BC, AC et AB. Démontrer que

$$\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^4}{\sin^4 A+\sin^4 B+\sin^4 C}\le 12R^2,$$

R étant le rayon du cercle circonscrit au triangle ABC. À quelles conditions y a-t-il égalité ?

3965. *Proposé par Ovidiu Furdui et Alina Sîntămărian.*

Déterminer le domaine de convergence de la série entière

$$\sum_{n=1}^{\infty}\left(\ln\frac{1}{2}+1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{(-1)^n}{n}\right)x^n,$$

ainsi que sa valeur en fonction de x dans ce domaine.

3966. *Proposé par Dao Hoang Viet.*

Soit h la longueur de l'hypoténuse d'un triangle rectangle et x et y les longueurs des cathètes. Démontrer que

$$\frac{1}{2h+x+y} + \frac{1}{h+2x+y} + \frac{1}{h+x+2y} < \frac{h}{2xy}.$$

3967. *Proposé par Marcel Chiriţă.*

Déterminer tous les entiers strictement positifs a, b et c qui vérifient l'équation :

$$(a+b)! = 4(b+c)! + 18(a+c)!$$

3968. *Proposé par Michal Kremzer.*

Soit $\{a\} = a - [a]$, $[a]$ étant la partie entière de a. Démontrer que si a est un réel et $a(a - 2\{a\})$ est un entier, alors a est un entier.

3969. *Proposé par Marcel Chiriţă.*

Déterminer les fonctions $f : (\frac{8}{9}, \infty) \mapsto \mathbb{R}$ qui sont continues en $x = 1$ et qui vérifient

$$f(9x-8) - 2f(3x-2) + f(x) = 4x - 4 + \ln\frac{9x^2-8x}{(3x-2)^2} \quad \text{pour tout } x \in (\frac{8}{9}, \infty).$$

3970. *Proposé par Nermin Hodžić et Salem Malikić.*

Soit a, b, c des réels non négatifs tels que $a + b + c = 3$. Démontrer que

$$\frac{1}{10a^3+9} + \frac{1}{10b^3+9} + \frac{1}{10c^3+9} \geq \frac{3}{19}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3861. *Proposed by Bill Sands.*

Prove that, for any positive real number x, there is a $\sqrt{2}\times x$ rectangle on the plane with each corner having at least one integer coordinate.

We present a solution based on all six submitted solutions.

Set $t = x/\sqrt{2}$ and consider the points

$$A = (0, t),\ B = (t, 0),\ C = (t + 1, 1) \text{ and } D = (1, t + 1).$$

Then AB and CD are parallel of length x and BC and DA are parallel of length $\sqrt{2}$. It is easy to check that $\overrightarrow{AB} \cdot \overrightarrow{BC}$ is equal to 0, making $ABCD$ a rectangle.

Editor's comment. Does the result remain true if $\sqrt{2}$ is replaced by $\sqrt{3}$?

3862. *Proposed by Mehmet Şahin.*

Let triangle ABC be right angled, with $\angle BAC = 90^\circ$ and altitude AD. Let K and L be on AB and CA so that DK and DL are bisectors of angles $\angle BDA$ and $\angle CDA$ respectively. Let M and N be the feet of the perpendiculars from K and L, respectively, to BC. Prove that $KM + NL = AD$.

We received 22 correct submissions and one faulty submission.

Solution 1, a combination of the similar solutions submitted by half the solvers.

Since the bisector DK of angle BDA divides the segment AB into segments BK and AK proportional in length to the segments BD and AD, we have

$$\frac{BK}{AB} = \frac{BD}{AD + BD}.$$

The altitude AD of the right triangle ABC satisfies $AD^2 = BD \cdot CD$, whence

$$\frac{BD}{AD + BD} = \frac{BD}{\sqrt{BD \cdot CD} + BD} = \frac{\sqrt{BD}}{\sqrt{CD} + \sqrt{BD}}.$$

The triangles BKM and BAD are similar, so that, finally,

$$\frac{KM}{AD} = \frac{BK}{BA} = \frac{\sqrt{BD}}{\sqrt{CD} + \sqrt{BD}}.$$

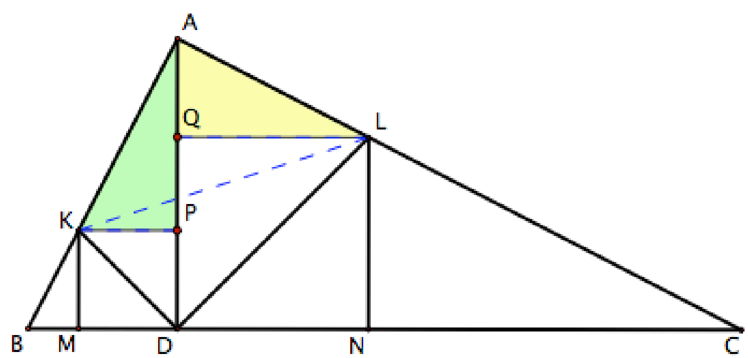
Similarly,

$$\frac{NL}{AD} = \frac{\sqrt{CD}}{\sqrt{CD} + \sqrt{BD}}.$$

Thus,

$$\frac{KM}{AD} + \frac{NL}{AD} = 1,$$

and $KM + NL = AD$, as desired.



Solution 2, a composite of similar solutions by Sefket Arslanagić and Joel Schlosberg.

Let P and Q be the feet of the perpendiculars to AD from K and L, respectively. Then $KMDP$ and $LNDQ$ are rectangles (since each has three angles known to be $90°$) ; in fact, they are squares because the diagonals from D bisect the right angles there. Thus,

$$KM = KP \quad \text{and} \quad LN = QD. \tag{1}$$

Note that also $\angle LDK = 90°$ (because DK and DL both bisect the right angles formed by AD at D). It follows that $AKDL$ is cyclic (because of the right angles at A and D), whence

$$\angle LKA = \angle LDA = \frac{1}{2}\angle CDA = 45°,$$

and we deduce that triangle AKL, with its right angle at A, is isosceles with congruent legs AK and AL. Since $\angle LQA = 90° = \angle APK$, and $\angle QAL = 90° - \angle KAP = \angle PKA$, it follows that triangles AQL and KPA are congruent. Thus $KP = AQ$; but we know from (1) that $KM = KP$ and $NL = QD$. We have, therefore,

$$KM + NL = KP + QD = AQ + QD = AD$$

as desired.

3863. *Proposed by Michel Bataille.*

Let a, b, c be real numbers such that $a^2 + b^2 + c^2 \leq 1$. Prove that

$$a^2b(b-c) + b^2c(c-a) + c^2a(a-b) \geq \frac{(b-c)^2(c-a)^2(a-b)^2}{2} .$$

We received ten correct submissions. We present the identical solution by Arkady Alt and Adnan Ali, done independently.

Since $a^2 + b^2 + c^2 \le 1$, we have by Cauchy-Schwarz inequality that

$$\begin{aligned}
&2(a^2b(b-c) + b^2c(c-a) + c^2a(a-b)) \\
&\quad = (ab-bc)^2 + (bc-ca)^2 + (ca-ab)^2 \\
&\quad \ge (b^2+c^2+a^2)((ab-bc)^2 + (bc-ca)^2 + (ca-ab)^2) \\
&\quad \ge (b(ab-bc) + c(bc-ca) + a(ca-ab))^2 \\
&\quad = (ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a)^2 \\
&\quad = ((a-b)(b-c)(c-a))^2,
\end{aligned}$$

which completes the proof.

3864. *Proposed by Cristinel Mortici.*

For every positive integer m, denote by $m!!$ the product of all positive integers with same parity as m, which are less than or equal to m. Let $n \ge 1$ be an integer. Prove that

$$(-1)^n(2n)!! - (2n-1)!! + (2n+1)!!\sum_{k=1}^{n}\frac{1}{2k-1}$$

is divisible by $(2n+1)^2$.

One incorrect solution was received. We present the solution of the proposer.

Let

$$p(x) = (x-1)(x-3)\dots(x-(2n-1)).$$

Let n be even. Then

$$p(x) = x^2q(x) - (2n-1)!!\sum_{k=1}^{n}\frac{1}{2k-1}x + (2n-1)!! \tag{1}$$

for some polynomial $q(x)$. For $x = 2n+1$, we get

$$(2n)!! = (2n+1)^2q(2n+1) - (2n+1)!!\sum_{k=1}^{n}\frac{1}{2k-1} + (2n-1)!!,$$

$$(2n)!! - (2n-1)!! + (2n+1)!!\sum_{k=1}^{n}\frac{1}{2k-1} = (2n+1)^2q(2n+1).$$

If n is odd, then (1) becomes

$$p(x) = x^2q(x) + (2n-1)!!\sum_{k=1}^{n}\frac{1}{2k-1}x - (2n-1)!!$$

and the conclusion follows by again taking $x = 2n+1$.

3865. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC

$$\sum_{\text{cyclic}} \frac{1}{1+\cot^3\left(\frac{A}{2}\right)} \leq \frac{3R}{2(r+s)}$$

where s, r, and R are the semiperimeter, the inradius and the circumradius of ABC, respectively.

The inequality is not correct and there was an error in the proof submitted by the proposer. Five solvers provided counterexamples. One submitter provided a proof for acute triangles, but the reference provided appeared to have flawed inequalities and it is possible that the proposed inequality is valid in the other direction. We provide a list of the more accessible counterexamples.

We can use the identity $\cot(\theta/2) = (1+\cos\theta)/(\sin\theta)$ to compute the cotangents of the half angles.

Counterexample 1, by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie.

Let ABC be a right triangle with $\angle C = 90^\circ$. We have that $a^2+b^2=c^2$, $R=c/2$, $a+b=r+s$, $\cot(A/2)=(b+c)/a$ and $\cot(B/2)=(a+c)/b$. The right side of the inequality is $3c/(4(a+b))$.

When $(a,b,c)=(3,4,5)$, then $\cot(A/2)=3$, $\cot(B/2)=2$ and $\cot(C/2)=1$. The left side of the inequality is $1/2+1/9+1/28=15/28+1/9$ while the right side is $15/28$, so the inequality fails.

Counterexample 2, by Šefket Arslanagić.

When $(A,B,C)=(60^\circ,30^\circ,90^\circ)$, then $\cot(A/2)=\sqrt{3}$, $\cot(B/2)=2+\sqrt{3}$ and $\cot(C/2)=1$. If we take $(a,b,c)=(\sqrt{3},1,2)$, then $R=1$, $r+s=1+\sqrt{3}$ and $3R/(2(r+s))=(3/4)(\sqrt{3}-1)=0.54904$. The left side is equal to $0.1614+0.0189+0.5>0.6$, so again the result fails.

Counterexample 3, by Šefket Arslanagić and Tuti Zvonaru (independently).

Let $(A,B,C)=(120^\circ,30^\circ,30^\circ)$. We can take $(a,b,c)=(2\sqrt{3},2,2)$. In this case, $\cot(A/2)=1/\sqrt{3}=0.57735$ and $\cot(B/2)=\cot(C/2)=3.73205$. The left side is equal to 0.87636. We have that $s=2+\sqrt{3}$, $\Delta=\sqrt{3}$, $r=\sqrt{3}(2-\sqrt{3})=2\sqrt{3}-3$ and $R=abc/4\Delta=2$. Since $r+s=3\sqrt{3}-1$, the right side is equal to $3/(3\sqrt{3}-1)=0.71494$ and the result fails.

Counterexample 4, by Šefket Arslanagić.

Let $(a,b,c)=(14,13,15)$. This can be regarded as the juxtaposition of $(9,12,15)$ and $(5,12,13)$ right triangles along the side 12. Here we have that $\Delta=84$, $s=21$, $r=4$, $R=65/8$, so the right side of the inequality is $39/80=0.4875$. We have $\sin B=4/5$, $\cos B=3/5$, $\cot(B/2)=2$, $\sin C=12/13$, $\cos C=5/13$ and $\cot(C/2)=3/2$.

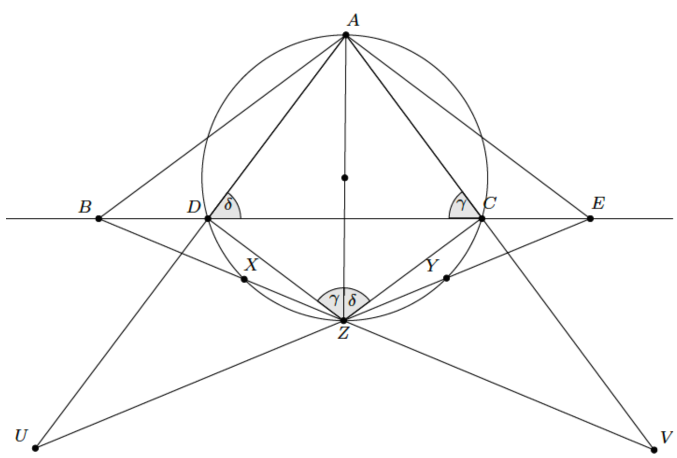
Finally, angle A can be split into two angles, one whose sine and cosine are $3/5$ and $4/5$, and one whose sine and cosine are $5/13$ and $12/13$. Using angle sum formulae, we find that $\sin A = 56/65$, $\cos A = 33/65$, $\cot A/2 = 33/56$. The left side turns out to be $0.83013 + 0.11111 + 0.22857 = 1.1698$, so the inequality again fails.

3866. *Proposed by Michel Bataille.*

Distinct points B, C, D, E on a line ℓ are such that $\angle BAC = \angle DAE = 90^\circ$ for some point A. Let X, Y on the circumcircle of ΔCAD be such that $\angle AXB = \angle AYE = 90^\circ$. If BX intersects line AC at V and EY intersects line AD at U, prove that UV is parallel to ℓ.

We received six correct submissions and one incomplete solution. We present two solutions.

Solution 1, by Oliver Geupel.



Let Z be the point on the circle $\Gamma = (CAD)$ with the property that the line segment AZ is a diameter of Γ. Since the point X lies on the circle Γ, we have $\angle ZXA = 90^\circ = \angle AXB = \angle AXV$, so that the points B, Z, and V are collinear. Similarly, the points E, Z, and U are collinear. Hence, the lines BV and EU intersect at point Z. Since point C lies on Γ, we have $\angle ACZ = 90^\circ = \angle BAC$. Hence the lines AB and CZ are parallel. Thus,

$$\frac{BV}{ZV} = \frac{AB}{CZ}. \tag{1}$$

Analogously,

$$\frac{EU}{ZU} = \frac{AE}{DZ}. \tag{2}$$

Since the points A, C, Z, and D are concyclic, $\angle ACD = \angle AZD$ and $\angle CDA = \angle CZA$. Denote these angle measures by γ and δ, respectively. Then,

$$\frac{AB}{AE} = \frac{AC\tan\gamma}{AD\tan\delta} = \frac{AC\cot\delta}{AD\cot\gamma} = \frac{CZ}{DZ}. \tag{3}$$

Putting the equations (1), (2), and (3) together, we conclude

$$\frac{BV}{ZV} = \frac{AB}{CZ} = \frac{AE}{DZ} = \frac{EU}{ZU}.$$

Consequently, the lines UV and ℓ are parallel.

Solution 2, by Prithwijit De, modified and expanded by the editor.

Place the figure on a Cartesian plane such that A is at the origin and l is the line with equation $y = -1$. Let lines AB and AD have equations $y = m_1x$ and $y = m_2x$ respectively (note that $m_1 \neq m_2$). Then the equations of the lines through AC and AE are $y = -x/m_1$ and $y = -x/m_2$ respectively (since $AC \perp AB$, $AE \perp AD$, and both lines go through the origin). The point C is on both the line l and the line through AC, and hence its coordinates are $(m_1, -1)$. Similarly, the coordinates of D are $(-1/m_2, -1)$. It follows that the equation of the circumcircle of CAD is

$$x^2 + y^2 + (1/m_2 - m_1)x + (m_1/m_2 + 1)y = 0. \tag{4}$$

We now find the equation of the line through BX in order to be able to figure out the coordinates of V. Suppose the coordinates of X are (α_1, β_1). Let λ_1 be the slope of the line through BX. The point B is on both the line l and the line $y = m_1x$, so its coordinates are $(-\frac{1}{m_1}, -1)$. Since $\angle AXB = 90^\circ$, the slope of the line through AX is the negative reciprocal of the slope of the line through BX; that is, $\frac{\beta_1}{\alpha_1} \cdot \frac{\beta_1+1}{\alpha_1+\frac{1}{m_1}} = -1$. We rearrange this equation to $\beta_1^2 = -\alpha_1^2 - \frac{\alpha_1}{m_1} - \beta_1$. On the other hand, from the fact that X is on the circle (4), we have $\alpha_1^2 + \beta_1^2 + \frac{\alpha_1}{m_2} - \alpha_1 m_1 + \frac{\beta_1 m_1}{m_2} + \beta_1 = 0$. Substituting $\beta_1^2 = -\alpha_1^2 - \frac{\alpha_1}{m_1} - \beta_1$ in this expression, and dividing the result by β_1, gives us $\frac{\alpha_1}{\beta_1} \cdot \left(\frac{1}{m_2} - \frac{1}{m_1}\right) - \frac{\alpha_1}{\beta_1} \cdot m_1 + \frac{m_1}{m_2} = 0$. Solve for $\frac{\alpha_1}{\beta_1}$, and conclude that

$$\lambda_1 = -\frac{\alpha_1}{\beta_1} = \frac{m_1^2}{m_1 - m_2(m_1^2+1)}. \tag{5}$$

Finally, since the point V is the intersection of the lines through BX and AC, its coordinates can be found by solving the pair of equations

$$y + 1 = \lambda_1 \left(x + \frac{1}{m_1}\right); \quad y = -\frac{x}{m_1}. \tag{6}$$

In particular, the y-coordinate of V is $\frac{m_2}{m_1 - m_2}$.

Repeat the same process for the point U. Denote by (α_2, β_2) the coordinates of Y. First, using $AY \perp YE$ and the fact that Y is on the circle (4), we find that the slope of the line through EY is

$$\lambda_2 = -\frac{\alpha_2}{\beta_2} = \frac{m_1}{m_2^2 - m_1 m_2 + 1}. \tag{7}$$

Then, since U is the intersection of the lines through EY and AD, we need to solve the system of equations

$$y + 1 = \lambda_2(x - m_2); \quad y = m_2 x. \tag{8}$$

We find that the y-coordinate of U is $\frac{m_2}{m_1-m_2}$. We can thus calculate that the slope of UV is zero, which means that UV is parallel to l.

3867. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $(a_n)_{n\geq 1}$ be a positive real sequence and $a > 0$ such that

$$\lim_{n\to\infty}(a_n - a\cdot n!) = b > 0.$$

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right).$$

We received four correct submissions and one incorrect solution. We present the solution of Paolo Perfetti, modified by the editor.

Note that $a_n = b + an! + o(1)$, and so

$$(a_n)^{\frac{1}{n}} = (b + an! + o(1))^{\frac{1}{n}} = (an!)^{\frac{1}{n}}\left(1 + \frac{b+o(1)}{an!}\right)^{\frac{1}{n}} \sim (an!)^{\frac{1}{n}}.$$

Using Stirling's formula, $n! = (n/e)^n\cdot\sqrt{2\pi n}\cdot(1+o(1))$, we can write

$$\begin{aligned}(an!)^{\frac{1}{n}} &= a^{\frac{1}{n}}\cdot\frac{n}{e}(\sqrt{2\pi n})^{\frac{1}{n}}(1+o(1))^{\frac{1}{n}}\\ &= \frac{n}{e}\cdot\exp\left(\frac{\ln(a\cdot\sqrt{2\pi n}\cdot(1+o(1)))}{n}\right).\end{aligned}$$

Continuing from the last equality, the Taylor expansion for the exponential function then gives us

$$\begin{aligned}(an!)^{\frac{1}{n}} &= \frac{n}{e}\left(1 + \frac{\ln(a\cdot\sqrt{2\pi n}\cdot(1+o(1)))}{n} + O\left(\frac{\ln^2 n}{n^2}\right)\right)\\ &= \frac{n}{e} + \frac{1}{e}\cdot\ln(a\cdot\sqrt{2\pi n}\cdot(1+o(1))) + O\left(\frac{\ln^2 n}{n}\right).\end{aligned}$$

Hence

$$\begin{aligned}(a(n+1)!)^{\frac{1}{n+1}} - (an!)^{\frac{1}{n}} &= \tfrac{1}{e} + \tfrac{1}{e}\cdot\ln\left(\tfrac{a\cdot\sqrt{2\pi(n+1)}\cdot(1+o(1))}{a\cdot\sqrt{2\pi n}\cdot(1+o(1))}\right) + O\left(\tfrac{\ln^2 n}{n}\right)\\ &\to \tfrac{1}{e} \text{ as } n\to\infty.\end{aligned}$$

Since $\sqrt[n+1]{a_{n+1}} \sim (a(n+1)!)^{\frac{1}{n+1}}$ and $\sqrt[n]{a_n} \sim (an!)^{\frac{1}{n}}$ (see the beginning of the proof), it follows that $\lim\limits_{n\to\infty}\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \frac{1}{e}$ as well.

3868. *Proposed by Iliya Bluskov.*

Determine the maximum value of $f(x,y,z) = xy + yz + zx - xyz$ subject to the constraint $x^2+y^2+z^2+xyz = 4$, where x, y and z are real numbers in the interval $(0,2)$.

We received 12 correct submissions and one incorrect submission. We present one of the three solutions submitted by Šefket Arslanagić modified by the editor.

Let $u = \frac{x}{2}$, $v = \frac{y}{2}$ and $w = \frac{z}{2}$. Then $u, v, w \in (0, 1)$,

$$f(x, y, z) = xy + yz + zx - xyz = 4(uv + vw + wu - 2uvw) \tag{1}$$

and

$$u^2 + v^2 + w^2 + 2uvw = 1. \tag{2}$$

We prove that

$$uv + vw + wu - 2uvw \leq \frac{1}{2}. \tag{3}$$

Note first that among u, v, w, at least two of them, say u and v, are such that $u \geq \frac{1}{2}$, $v \geq \frac{1}{2}$ or $u \leq \frac{1}{2}$, $v \leq \frac{1}{2}$. In either case, we have $(2u-1)(2v-1) \geq 0$, so

$$u + v - 2uv \leq \frac{1}{2}. \tag{4}$$

Since $u^2 + v^2 \geq 2uv$, we have from (2) that

$$1 \geq 2uv + w^2 + 2uvw \quad \text{or} \quad 2uv(1+w) \leq 1 - w^2,$$

which implies that $2uv \leq 1 - w$, so

$$w \leq 1 - 2uv. \tag{5}$$

Multiplying (4) and (5), we then have

$$vw + wu - 2uvw \leq \frac{1}{2} - uv$$

establishing (3). Then from (1), we obtain $f(x, y, z) \leq 2$ and the upper bound is clearly attained when $x = y = z = 1$.

3869. *Proposed by Dao Thanh Oai.*

It is known that any nondegenerate conic that passes through the vertices of a triangle ABC and its orthocentre H must be a rectangular hyperbola whose centre lies on the triangle's nine-point circle. Prove that the centre of the hyperbola is the midpoint of the segment that joins H to the fourth point (different from A, B, and C) where the hyperbola intersects the circumcircle.

We received four correct submissions. We present the solution by Michel Bataille.

Let $\mathcal{H}$ be a rectangular hyperbola through A, B, C and H. We choose the asymptotes of $\mathcal{H}$ as axes of an orthonormal system with origin at the centre O of $\mathcal{H}$ so that the equation of $\mathcal{H}$ is $xy = k$ for some nonzero real number k. Let $A(a, \frac{k}{a}), B(b, \frac{k}{b}), C(c, \frac{k}{c})$ where a, b, c are distinct, nonzero real numbers. Since the altitude from A is perpendicular to $\overrightarrow{BC}$ and passes through $(a, k/a)$, its equation is $bcx - ky = abc - \frac{k^2}{a}$. Similarly, the equation of the altitude from B is $cax - ky = abc - \frac{k^2}{b}$ and it readily follows that

$$H\left(x_H = \frac{-k^2}{abc}, y_H = \frac{-abc}{k}\right) \qquad (1).$$

Now, let $x^2 + y^2 - 2\alpha x - 2\beta y + \gamma = 0$ be the equation of the circumcircle $\mathcal{C}$ of ΔABC. Substituting $\frac{k}{x}$ for y, we readily see that the abscissas of the four points of $\mathcal{C} \cap \mathcal{H}$ are the solutions to

$$x^4 - 2\alpha x^3 + \gamma x^2 - 2\beta kx + k^2 = 0$$

Since the product of these solution is k^2 and three of them are a, b, c, the fourth solution is $\frac{k^2}{abc}$. As a result, $\mathcal{C} \cap \mathcal{H} = \{A, B, C, K\}$ where

$$K\left(\frac{k^2}{abc}, \frac{abc}{k}\right).$$

Comparing with (1), we conclude that O is the midpoint of the segment HK.

3870. *Proposed by Ovidiu Furdui.*

Calculate

$$\int_0^1 \ln^2(\sqrt{x} + \sqrt{1-x})\, dx\,.$$

We received five correct submissions. We present the solution by Michel Bataille, somewhat expanded by the editor.

Let I be the given integral. The substitution $x = \sin^2\left(\frac{\pi}{4} + \frac{y}{2}\right)$ $(y \in [-\frac{\pi}{2}, \frac{\pi}{2}])$ gives $dx = \frac{1}{2}(\cos y)dy$, $\sqrt{x} + \sqrt{1-x} = \sqrt{2}\cos\frac{y}{2} = \sqrt{1+\cos y}$, using the double angle formulas for sine and cosine, the phase angle relation between sine and cosine, and the formula for the cosine of a sum of angles. Then we have :

$$I = \frac{1}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\ln(1+\cos y))^2\ (\cos y)\, dy = \frac{1}{4}\, J,$$

where $J = \int\limits_0^{\frac{\pi}{2}} (\ln(1+\cos y))^2\ (\cos y)\, dy$. An integration by parts yields

$$\begin{aligned} J &= \left[(\sin y)(\ln(1+\cos y))^2\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} (\sin y)\cdot(2\ln(1+\cos y))\cdot\frac{-\sin y}{1+\cos y}\, dy \\ &= 2(K - L), \end{aligned}$$

where $K = \int_0^{\frac{\pi}{2}} \ln(1+\cos y)\, dy$ and $L = \int_0^{\frac{\pi}{2}} (\cos y)\ln(1+\cos y)\, dy$.
A similar integration by parts gives

$$L = \int_0^{\pi/2} (1 - \cos y)\, dy = \frac{\pi}{2} - 1.$$

As for K, we obtain

$$\begin{aligned} K &= \int_0^{\frac{\pi}{2}} \ln\left(2\cos^2\frac{y}{2}\right)\, dy = \frac{\pi\ln(2)}{2} + 2\int_0^{\frac{\pi}{2}} \ln\left(\cos\frac{y}{2}\right)\, dy \\ &= \frac{\pi\ln(2)}{2} + 4\int_0^{\frac{\pi}{4}} \ln(\cos x)\, dx. \end{aligned}$$

Now, we compute the last integral. Recall that the Catalan constant G is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Using a power series representation for $\arctan(x)$ over $[0,1]$ (derived by using the power series for $\frac{1}{1+t^2}$) and integrating term-by-term, we have

$$G = \int_0^1 \frac{\tan^{-1}(x)}{x}\, dx.$$

Now, let

$$U = \int_0^{\frac{\pi}{4}} \ln(\cos x)\, dx \quad \text{and} \quad V = \int_0^{\frac{\pi}{4}} \ln(\sin x)\, dx.$$

Then

$$\begin{aligned} V + U &= \int_0^{\pi/4} \ln\left(\frac{1}{2}\sin(2x)\right) dx = \frac{\pi}{4}\ln(1/2) + \int_0^{\pi/4} \ln(\sin(2x))dx \\ &= -\frac{\pi}{4}\ln(2) + \frac{1}{2}\int_0^{\pi/2} \ln(\sin u)du. \end{aligned}$$

To compute this quantity, we observe that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin u)du = \int_{\frac{-\pi}{4}}^{0} \ln(\sin(x+\frac{\pi}{2}))dx = \int_0^{\frac{\pi}{4}} \ln(\cos x)dx = U$$

using symmetry and the phase relation. Then we obtain :

$$\int_0^{\pi/2} \ln(\sin u)du = \int_0^{\frac{\pi}{4}} \ln(\sin u)du + \int_0^{\frac{\pi}{4}} \ln(\cos u)du = U + V,$$

and therefore $U + V = \dfrac{-\pi}{2}\ln(2)$, upon rearranging. On the other hand, we have :

$$\begin{aligned} V - U &= \int_0^{\pi/4} \ln(\tan x)dx = [x\ln(\tan x)]_0^{\pi/4} - \int_0^{\pi/4} x\,\frac{1+\tan^2 x}{\tan x}dx \\ &= -2\int_0^{\pi/4} \frac{x}{\sin(2x)}dx = -\frac{1}{2}\int_0^{\pi/2} \frac{u}{\sin(u)}du = -G, \end{aligned}$$

where the last equality follows from

$$\int_0^{\pi/2} \frac{u}{\sin(u)} du = 2\int_0^1 \frac{\tan^{-1}(x)}{x}\, dx$$

using the substitution $u = 2\tan^{-1}(x)$. Thus, $2U = -\frac{\pi}{2}\ln(2) + G$ (and as a bonus, $2V = -\frac{\pi}{2}\ln(2) - G$).

Collecting our previous results, we obtain

$$J = 2\left(\frac{\pi\ln(2)}{2} + 2G - \pi\ln(2) + 1 - \frac{\pi}{2}\right)$$

and then, to conclude $I = G + \dfrac{1}{2} - \dfrac{\pi}{4}(1 + \ln(2))$, as desired.

Editor's Comments. This problem was solved by 4 solvers : M. Bataille, D. Koukakis, K.-W. Lau and P. Perfetti. Bataille had previously computed $\int_0^{\frac{\pi}{4}} \ln(\cos x)\, dx$ in his solution to ***Crux*** problem **2793** [volume 2003, page 522], and in *Math. Gazette*, Vol. 86, March 2004, p. 156, and so cited that ; Koukakis cited the value of $\int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx$ (as did Bataille in his solution to **2793**). The above proof relates the two quantities, in what is hopefully an insightful way. All three of Bataille, Koukakis, and Lau started their solutions with different trigonometric substitutions, which end up in the same place, but have different journeys. Perfetti's solution, however, uses no trigonometry, but instead utilizes some contour integration, residues, and some nifty analysis.

Math Quotes

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

Galilei, Galileo (1564 - 1642) in "Opere Il Saggiatore", p. 171.

SOLUTIONS

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

George Apostolopoulos, Messolonghi, Greece : 3964
Michel Bataille, Rouen, France : 3961, 3962
D. M. Bătineţu, Bucharest and Neculai Stanciu, Buzău, Romania : 3963
Marcel Chiriţă, Bucharest, Romania : 3967, 3969
Dao Hoang Viet, Pleiku city, Viet Nam : 3966
Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania : 3965
Nermin Hodžić, Dobošnica, Bosnia and Herzegovina and Salem Malikić, Simon Fraser University, Burnaby, BC : 3970
Michal Kremzer, Gliwice, Silesia, Poland : 3968

Solvers - individuals

Adnan Ali, Student in A.E.C.S-4, Mumbai, India : 3862, **3863**, 3866
Arkady Alt, San Jose, CA, USA : **3863**, 3867, 3868
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina : **CC81**, **3862**, 3863, **3865**, 3866, **3868** (3 solutions)
Roy Barbara, Lebanese University, Fanar, Lebanon : **3861**
Ricardo Barroso Campos, University of Seville, Seville, Spain : 3862
Michel Bataille, Rouen, France : CC81, **CC85**, **OC131**, OC133, **OC134**, 3862, 3863, 3866, 3867, 3868, **3869**, **3870**
Iliya Bluskov, University of Northern British Columbia, Prince George, BC : 3868
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : 3862 (2 solutions)
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India : 3862, **3866**
Ovidiu Furdui, Campia Turzii, Cluj, Romania : 3870
Oliver Geupel, Brühl, NRW, Germany : OC131, **OC132**, **OC133**, OC134, **OC135**, 3862, **3866**, 3868, 3869
John G. Heuver, Grande Prairie, AB : 3862
Dag Jonsson, Uppsala, Sweden : 3862
Václav Konečný, Big Rapids, MI, USA : **3861**
Dimitrios Koukakis, Kilkis, Greece : 3862, 3870
Kee-Wai Lau, Hong Kong, China : 3870
Kathleen E. Lewis, University of the Gambia, Brikama, Gambia : **3861**
Joseph M. Ling, University of Calgary, Calgary, AB : OC131
Salem Malikić, student, Simon Fraser University, Burnaby, BC : 3862, 3863, 3865, 3868
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India : 3862, 3868
Cristinel Mortici, Valahia University of Târgovişte, Romania : **3864**
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Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy : 3863, **3867**, 3868, 3870
Angel Plaza, University of Las Palmas de Gran Canaria, Spain : 3867
Mehmet Şahin, Ankara, Turkey : 3862

Solvers - collaborations