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Crux Mathematicorum

Crux Mathematicorum
with Mathematical Mayhem

EDITORIAL

Have you heard of the pancake problem? You have a stack of all different size pancakes that you want to order from largest on the bottom to smallest on top. You are allowed to insert a spatula at any point in the stack and use it to flip all pancakes above it.

Let's try a small case of 3 pancakes. How many flips are required to order the following stacks?
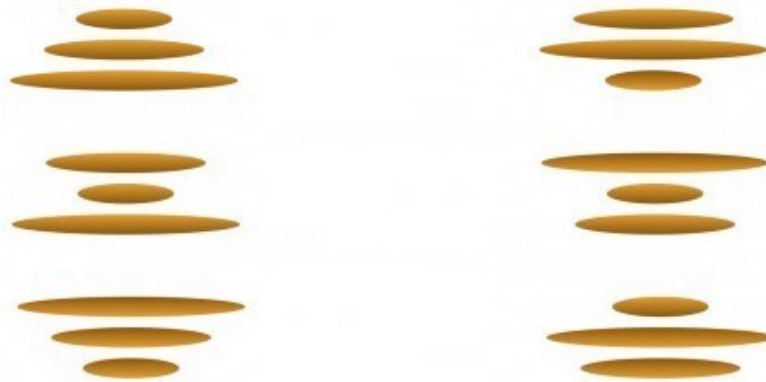
More generally, what is the maximum number of flips required for n pancakes?

This problem was first proposed by Jacob E. Goodman, under the pseudonym Harry Dweighter, in 1975 when it appeared as Elementary Problem E2569 in *American Mathematical Monthly*. Here is what we know so far. In 1979, Bill Gates and Christos Papadimitriou gave an upper bound of $\frac{5n+5}{3}$ (yes, that Bill Gates!). In 2008, the bound was improved to $\frac{18}{11}n$. In 2011, this problem was proved to be NP-hard. Not so elementary after all.

It gets even more interesting in biology context. This "flipping" operation can be applied to create reversals in the gene sequence, which allows us to study genome rearrangements in evolution. For example, cabbage is only 3 flips away from turnip! Maybe one day we will find out what came first.

Kseniya Garaschuk

MᴀᴛʜᴇᴍAᴛᴛɪᴄ
No. 7

The problems featured in this section are intended for students at the secondary school level.

> **Click here to submit solutions, comments and generalizations to any problem in this section.**

To facilitate their consideration, solutions should be received by **October 30, 2019**.

MA31. Given that the areas of an equilateral triangle with side length t and a square with side length s are equal, determine the value of $\frac{t}{s}$.

MA32. Jack and Madeline are playing a dice game. Jack rolls a 6-sided die (numbered 1 to 6) and Madeline rolls an 8-sided die (numbered 1 to 8). The person who rolls the higher number wins the game. If Jack and Madeline roll the same number, the game is replayed. If a tie occurs a second time, then Jack is declared the winner. Which person has the better chance of winning? What are the odds in favour of this person winning the game?

MA33. Observe that

$$\sqrt{2\tfrac{2}{3}} = 2\sqrt{\tfrac{2}{3}}.$$

Determine conditions for which

$$\sqrt{a\tfrac{b}{c}} = a\sqrt{\tfrac{b}{c}}.$$

where a, b, c are positive integers.

MA34. Try to replace each $*$ with a different digit from 1 to 9 so that the multiplication is correct. (Each digit from 1 to 9 must be used once.)

$$
\begin{array}{cccc}
* & * & * & * \\
\times & & & * \\
\hline
* & * & * & * \\
\end{array}
$$

Determine whether a solution is possible. If so, determine whether the solution is unique.

MA35. A polygon has angles that are all equal. If the sides of this polygon are not all equal, show that the polygon must have an even number of sides.

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Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

> ***Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA31. Étant donné que la surface d'un triangle équilatéral de côté t et celle d'un carré de côté s sont égales, déterminer la valeur de $\frac{t}{s}$.

MA32. Jacques et Madeleine jouent aux dés un peu spéciaux. Jacques utilise un dé ordinaire à 6 côtés (numérotés 1 à 6) tandis que Madeleine se sert d'un dé spécial à 8 côtés (numérotés de 1 à 8). La personne obtenant le plus gros chiffre gagne. Si Jacques et Madeleine roulent le même chiffre, on répète le jeu une seconde fois, sans déclarer de gagneur au premier tour ; si un ex aequo a lieu de nouveau, quel que soit le chiffre, Jacques est déclaré gagneur; autrement, le plus gros chiffre gagne. Laquelle personne a la meilleure chance de gagner à long terme ? Cette personne gagnante a quelle probabilité de gagner à long terme ?

MA33. Sachant que

$$\sqrt{2\tfrac{2}{3}} = 2\sqrt{\tfrac{2}{3}},$$

déterminer des conditions selon lesquelles

$$\sqrt{a\tfrac{b}{c}} = a\sqrt{\tfrac{b}{c}},$$

quelque soient a, b et c entiers positifs.

MA34. Dans le schéma qui suit, chaque $*$ représente un chiffre de 1 à 9, de façon à ce que la multiplication soit correcte, chaque chiffre de 1 à 9 étant utilisé une seule fois.

$$
\begin{array}{cccc}
* & * & * & * \\
\times & & & * \\
\hline
* & * & * & * \\
\end{array}
$$

Déterminer si une solution est possible et si elle est unique.

MA35. Les angles d'un polygone sont égaux. Si les côtés du polygone ne sont pas tous de même longueur, démontrer que le polygone doit avoir un nombre pair de côtés.

MᴀᴛʜᴇᴍAᴛᴛɪᴄ
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(2), p. 56–57.

MA6. A rectangular sheet of paper is labelled $ABCD$, with AB being one of the longer sides. The sheet is folded so that vertex A is placed exactly on top of the opposite vertex C. The fold line is XY, where X lies on AB and Y lies on CD. Prove that the triangle CXY is isosceles.

Originally problem B4 from 2018 UK Junior Mathematical Olympiad.

We received 2 submissions both of which were correct and complete. We present the solution by Richard Hess, modified by the editor.

Consider the below figure where XY is the perpendicular bisector of AC and O is the center of the rectangle $ABCD$.
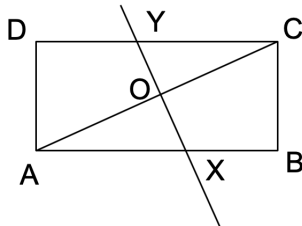
By symmetry, $OX = OY$. As XY is perpendicular to AC, it follows that CXY is an isosceles triangle where $CX = CY$.

MA7. Sixteen counters, which are black on one side and white on the other, are arranged in a 4 by 4 square. Initially all the counters are facing black side up. In one 'move', you must choose a 2 by 2 square within the square and turn all four counters over once. Describe a sequence of 'moves' of minimum length that finishes with the visible colours of the counters of the 4 by 4 square alternating (as shown in the diagram).

Originally problem B6 from 2018 UK Junior Mathematical Olympiad.

We received 2 submissions of which 1 was correct and complete. We present the solution by Richard Hess, modified by the editor.

Label the 1×1 squares left to right, bottom to top so that 1 refers to the bottom left square and 13 refers to the upper left square. First notice that the order in which the 2×2 squares are turned is inconsequential. The 1×1 squares in the bottom left and upper right begin as black, and must become white, hence the bottom left and upper right 2×2 squares must both by turned. This will be the first and second move. The squares 2,5,12, and 15 belong to exactly one 2×2 square, excluding the two squares turned in our first two moves. Since squares 2,5,12, and 15 are white after the second move, and must become black, we are forced to flip the middle bottom, middle right, middle left, and middle top 2×2 squares. After these next four moves, the checkerboard pattern is formed. It follows that 6 is the minimum number of moves.

MA8. I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm. What is the smallest square that can be made with equal numbers of each type of tile?

Originally problem B5 from 2018 UK Junior Mathematical Olympiad.

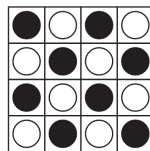*We received 2 solutions. We present a solution based on the submission by Doddy Kastanya.*

Suppose we have N squares of each type tiling a square of side length S (in cm²). Then

$$S^2 = N \cdot 1 + N \cdot 4 = 5N.$$

The smallest S that satisfies this equation is 5, which implies $N = 5$. However there is no possible arrangement of the tiles satisfying this, as can be seen from the figure below. Any 2×2 tile placed in the square covers exactly one of the four grey squares. Thus we cannot fit five 2×2 tiles into the 5×5 square.
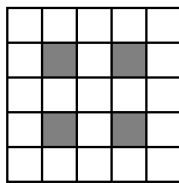
The next possible S satisfying the equation is 10, implying $N = 20$. A possible tiling is shown below.
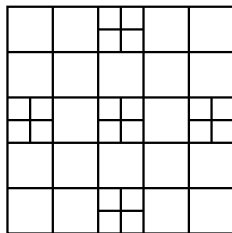
Therefore the smallest square that can be made with equal numbers of each type of tiles has a side length of 10cm.

MA9. The letters a, b, c, d, e and f represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e, \quad d + e = f.$$

Find all possible solutions for the values of a, b, c, d, e and f.

Originally problem B8 from 2018 UK Junior Mathematical Olympiad.

We received two submissions both of which were correct and complete. We present the solution by Richard Hess, modified by the editor.

Given $a + b = d$, $b + c = e$, $d + e = f$, we can combine equations to produce $f = a + c + 2b$. Given a solution (a, b, c, d, e, f) that satisfies these equations, we can interchange a and d with c and e, respectively, to find a second solution (c, b, a, e, d, f). Thus, we need only search for solutions where $a < c$. As all digits are unique in the interval 1 to 9, the smallest f can be is 7 ($a = 2, b = 1, c = 3$). Below are the 8 solutions to this problem:

$$(2, 1, 4, 3, 5, 8) \qquad (4, 1, 2, 5, 3, 8)$$
$$(2, 1, 5, 3, 6, 9) \qquad (5, 1, 2, 6, 3, 9)$$
$$(1, 2, 4, 3, 6, 9) \qquad (4, 2, 1, 6, 3, 9)$$
$$(1, 3, 2, 4, 5, 9) \qquad (2, 3, 1, 5, 4, 9)$$

MA10. An arithmetic and a geometric sequence, both consisting of only positive integral terms, share the same first two terms. Show that each term of the geometric sequence is also a term of the arithmetic sequence.

*Originally Problem **J12**, proposed by Colin Springer, from Mathematical Mayhem, 1 (1), p.19.*

We received 5 submissions, of which all were correct and complete. We present the solution by Jacob Miles, modified by the editor.

Let our arithmetic sequence be given by

$$a, a + d, a + 2d, \ldots$$

and our geometric sequence be given by

$$\alpha, \alpha r, \alpha r^2, \ldots$$

We show by proof by induction that each term αr^n of our geometric sequence is a term of our arithmetic sequence. We are given that $a = \alpha$ and $a + d = \alpha r$. Hence, the base cases of $n = 0, 1$ are satisfied. Let $n = k$. Assume αr^k is a term of our arithmetic sequence, i.e. $\alpha r^k = a + md$ for some $m \in \mathbb{N}$. We consider the case of $n = k + 1$ as follows

$$\begin{aligned} \alpha r^{k+1} &= \left(\alpha r^k\right) r \\ &= (a + md)r \\ &= ar + rmd. \end{aligned}$$

Given that $ar = a + d$, the above becomes

$$\alpha r^{k+1} = a + d + rmd$$
$$= a + (rm + 1)d.$$

The above expression of αr^{k+1} is a member of the arithmetic sequence if $rm + 1 \in \mathbb{N}$. As $m \in \mathbb{N}$ it suffices to prove that $r \in \mathbb{N}$. It is clear that $r > 0$ and that $r \in \mathbb{Q}$, for if either were not the case, we would have that $ar = a + d \notin \mathbb{N}$ which is a contradiction. We assume that $r \in \mathbb{Q}$ but $r \notin \mathbb{N}$. It follows that $r = \frac{x}{y}$ for some $x, y \in \mathbb{N}$ where $y \neq 1$. Without loss of generality, assume $\gcd(x, y) = 1$. As each term of our geometric sequence is a positive integer, we have that $ar^n \in \mathbb{N}$ for all $n \in \mathbb{N}$. Hence

$$ar^n = a\left(\frac{x}{y}\right)^n = \frac{ax^n}{y^n}.$$

We have that $y^n | ax^n$. However, since $\gcd(x, y) = 1$, it follows that $\gcd(x^n, y^n) = 1$. This implies $y^n | a$. However, as $y \neq 1$ we can choose a sufficiently large n such that $y^n > a$ causing $y^n \nmid a$. This would imply that $ar^n \notin \mathbb{N}$. By proof by contradiction we conclude that $r \in \mathbb{N}$ and that every term in our geometric sequence is in our arithmetic sequence.

PROBLEM SOLVING VIGNETTES
No.7
Shawn Godin
Careful Counting

This month we will look at problem B4 from the 2018 Canadian Open Mathematics Challenge administered by the CMS. You can check out past contests on the CMS webpage at `cms.math.ca/Competitions/COMC`.

Determine the number of 5-tuples of integers $(x_1, x_2, x_3, x_4, x_5)$ so that

(a) $x_i \geq i$ for $1 \leq i \leq 5$;

(b) $\displaystyle\sum_{i=1}^{5} x_i = 25$.

Solution 1: We will look to see if we can find any patterns by considering possible solutions in an orderly manner. Suppose that we fix $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. If we want the five numbers to add to 25, then $x_4 + x_5 = 25 - (1 + 2 + 3) = 19$. Recall that we also need $x_4 \geq 4$ and $x_5 \geq 5$. Putting this together, we get the following 11 5-tuples:

$$(1, 2, 3, 4, 15), (1, 2, 3, 5, 14), (1, 2, 3, 6, 13), \ldots, (1, 2, 3, 14, 5).$$

Next we will examine what happens when we allow x_3 to take on different values. We will keep $x_1 = 1$ and $x_2 = 2$ and let $x_3 = 4$. Using the same idea as in the first case we get 10 new 5-tuples:

$$(1, 2, 4, 4, 14), (1, 2, 4, 5, 13), (1, 2, 4, 6, 12), \ldots, (1, 2, 4, 13, 5).$$

Thus, if we fix $x_1 = 1$ and $x_2 = 2$, the total number of 5-tuples is

$$11 + 10 + 9 + \cdots + 1 = \sum_{i=1}^{11} i = 66.$$

Now, consider what happens if we fix $x_1 = 1$ and let $x_2 = 3$. If we go through the same process again, we get $10 + 9 + 8 + \cdots + 1 = 55$ more 5-tuples. So if we fix $x_1 = 1$ and let all other values vary we get

$$(11 + 10 + 9 + \cdots + 1) + (10 + 9 + 8 + \cdots + 1) + \cdots + (2 + 1) + 1 = \sum_{i=1}^{11} \sum_{j=1}^{i} j$$

5-tuples. Thus for the total problem, we will need to let x_1 vary. Letting $x_1 = 2$ and thinking through the process we get

$$(10 + 9 + 8 + \cdots + 1) + (9 + 8 + 7 + \cdots + 1) + \cdots + (2 + 1) + 1 = \sum_{i=1}^{10} \sum_{j=1}^{i} j$$

new 5-tuples. Hence looking at *all* possible solutions we must have

$$[(11 + 10 + 9 + \cdots + 1) + \cdots + (2 + 1) + 1]+$$

$$[(10 + 9 + 8 + \cdots + 1) + \cdots + (2 + 1) + 1] + \cdots + [1] = \sum_{i=1}^{11} \sum_{j=1}^{i} \sum_{k=1}^{j} k. \quad (1)$$

In Vignette #5 [2019: 45(5), p. 236-240] we introduced and proved the following formulas:

$$1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n + 1)}{2}, \quad (2)$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}. \quad (3)$$

We will add one more which will be of use to us in our solution:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{2}. \quad (4)$$

Enjoy practicing your induction by proving that the formula holds for all n.

Going back to (1), using (2), (3), and (4) we get

$$\sum_{i=1}^{11} \sum_{j=1}^{i} \sum_{k=1}^{j} k = \sum_{i=1}^{11} \sum_{j=1}^{i} \frac{j(j + 1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{11} \sum_{j=1}^{i} (j^2 + j)$$

$$= \frac{1}{2} \sum_{i=1}^{11} \left(\frac{i(i + 1)(2i + 1)}{6} + \frac{i(i + 1)}{2} \right)$$

$$= \frac{1}{6} \sum_{i=1}^{11} (i^3 + 3i^2 + 2i)$$

$$= \frac{1}{6} \left(\frac{11^2 \cdot 12^2}{4} + 3 \cdot \frac{11 \cdot 12 \cdot 23}{6} + 2 \cdot \frac{11 \cdot 12}{2} \right)$$

$$= 1001.$$

Therefore there are 1001 5-tuples that satisfy the conditions in the problem. □

Solution 2: We will look at the problem from another point of view. Suppose we wanted, for case of simplicity, to find all 3-tuples of non-negative integers (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 5$. This is a simplification of our problem by considering only 3 numbers, having a smaller sum and letting them all be any non-negative integer. For this problem we could list out all the possibilities or count them by carefully looking at cases.

Case 1: Two of the numbers are the same (there is no way they can all be the same). There are three ways that this can happen: $(0, 0, 5)$, $(1, 1, 3)$, and $(2, 2, 1)$. For each of these cases there are $\frac{3!}{2!} = 3$ ways to arrange the numbers giving $3 \cdot 3 = 9$ 3-tuples: $(0, 0, 5)$, $(0, 5, 0)$, $(5, 0, 0)$, $(1, 1, 3)$ $(1, 3, 1)$, $(3, 1, 1)$, $(2, 2, 1)$, $(2, 1, 2)$, and $(1, 2, 2)$.

Case 2: None of the numbers are the same. There are only two ways that this can happen: $(0, 1, 4)$ and $(0, 2, 3)$. For each of these cases there are $3! = 6$ ways to arrange the numbers giving $2 \cdot 6 = 12$ more 3-tuples: $(0, 1, 4)$, $(0, 4, 1)$, $(1, 0, 4)$, $(1, 4, 0)$, $(4, 0, 1)$, $(4, 1, 0)$, $(0, 2, 3)$, $(0, 3, 2)$, $(2, 0, 3)$, $(2, 3, 0)$, $(3, 0, 2)$, and $(3, 2, 0)$.

Therefore there are $9 + 12 = 21$ 3-tuples in total. We can use this method on our problem, but there will be many more cases to look at. You may (or may not!) want to see if you can identify all cases and get the correct total of 1001.

Still looking at the simplified problem, suppose we represent any particular 3-tuple with a collection of *stars and bars* (the name usually associated with this method). We will use five stars, since the total is 5 and two bars to separate them into three groups. Thus the 3-tuple $(2, 1, 2)$ would be represented by $* * | * | * *$. All stars to the left of the first bar represent x_1, the stars between the bars represent x_2 and the stars to the right of the second bar represents x_3. Similarly $* || * * * *$ would represent $(1, 0, 4)$ and $* * * * * ||$ would represent $(5, 0, 0)$.

Every 3-tuple can be represented by a unique permutation of 5 stars and 2 bars. Similarly, every permutation of 5 stars and 2 bars represents a unique 3-tuple. There is a one-to-one correspondence between the 3-tuples and the permutations of 5 stars and 2 bars. Since the total number of permutations of 5 stars and 2 bars is $\frac{7!}{5!2!} = 21$, we solved our simplified problem in a much more efficient manner.

If we return to the original problem, all permutations of 25 stars and 4 bars would give all possible 5-tuples of non-negative integers that sum to 25. This is not quite what we are after, **but** if we let $x_i = i + y_i$, for $1 \leq i \leq 5$ then $(y_1, y_2, y_3, y_4, y_5)$ is a 5-tuple of non-negative integers that add to $25 - (1 + 2 + 3 + 4 + 5) = 10$ and there is a one-to-one correspondence between the 5-tuples $(y_1, y_2, y_3, y_4, y_5)$ and the 5-tuples that we are after. That is, for example, since $(3, 1, 4, 2, 0)$ is a collection of y_is, then $(3 + 1, 1 + 2, 4 + 3, 2 + 4, 0 + 5) = (4, 3, 7, 6, 5)$ is an allowed solution to the original problem. The number of possible 5-tuples $(y_1, y_2, y_3, y_4, y_5)$ is the same as the number of permutations of 10 stars and 4 bars or

$$\frac{14!}{10!4!} = 1001.$$

$\square$

The stars and bars method yields a solution much quicker. We can summarize it as follows: the number of distinct n-tuples of non-negative integers whose sum is s is

$$\frac{(s+n-1)!}{s!(n-1)!}.$$

This can be generalized to the following statement: the number of distinct n-tuples of integers, $(x_1, x_2, \ldots, x_n)$ whose sum is s, where $x_i \leq m_i$ for $1 \leq i \leq n$ is

$$\frac{(s+(n-1) - \sum_{i=1}^{n} m_i)!}{(s - \sum_{i=1}^{n} m_i)!(n-1)!}.$$

We will explore other counting techniques in future columns.
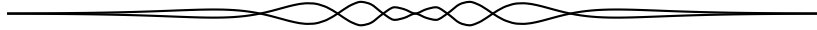
TEACHING PROBLEMS
No.4
John McLoughlin
Handshakes with a Twist

The handshake problem is perhaps the most flexible problem in my experience working with K-12 teachers. The basic example of counting handshakes lends itself to modeling the results with people as the participants. For instance, four people can each shake hands with one another. The total of 6 handshakes is reached this way at any level. The case of five people can be a new problem, or an extension can be readily grasped using the idea of a fifth person arriving in the group, thus, adding 4 handshakes to give a total of 10 handshakes. Before proceeding further, it is assumed that the reader is convinced that a total of 10 handshakes would occur if all people in a group of five shook hands with one another.

Assuming that makes sense be aware of how you would obtain the result. There are at least three viable avenues including the idea of adding additional people to simpler cases, as outlined above. You may recognize that each person in the group must shake 4 hands suggesting that there may be $5 \times 4 = 20$ handshakes required. Of course, this is not 10, and so something must be awry. Yes, each handshake involves two people and so the number of handshakes in total would equal 20 divided by 2, or 10. Do you see that in general for n people there would be $n(n-1)/2$ handshakes required? Mathematically this is equivalent to $_nC_2$ or $\binom{n}{2}$ or "n choose 2", namely, the number of ways of selecting 2 people from a group of n people.

My experience suggests that a diagrammatic approach may offer another valuable way of representing the problem. You are invited to construct a diagram with five vertices, each representing a person. Joining all possible vertices with segments will illustrate that there are 10 possible ways of connecting two of the five people. These segments represent the handshakes. Following our discussion of some of the basic handshake counting principles, we are ready to tackle an unorthodox handshake problem. The use of a diagram may be helpful in considering the core problem of our discussion as stated below. Teachers may wish to have eight people represent the characters in the problem.

> Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?

A detailed discussion of this problem appears in *Combinatorial Explorations*, a publication in the *ATOM Series* authored by Richard Hoshino and John Grant McLoughlin. "This problem is fascinating because it does not appear solvable. It is difficult to imagine that there is enough information here. However, we have all the information we need! Before reading any further, stop and attempt to solve this problem on your own."

Take some time and consider a diagram and/or a logical approach that makes it plausible to address the problem. It is not anticipated that you will necessarily solve the problem, as few of my students do in fact without some further guidance. In any case, play with the problem so that you may understand it better.

Here I will share two approaches, the first being a classroom approach and the second being the written approach.

Classroom Approach: Modeling the Problem

This approach requires 8 volunteers who are arranged into 4 couples, one of which is designated as Mr. and Mrs. Smith. Any names for the others are not a concern. It is easiest to place them in pairs square dancing, as if they are the four directions (N, S, E, W) on a compass.

Consider the important fact that Mrs. Smith received seven different answers to the number of hands shaken by the others in the group. In fact, there are only seven possible answers as the absence of a handshake with one's spouse limited the number of handshakes to a maximum of 6. That is, there must be people who accounted for each of 0, 1, 2, 3, 4, 5, and 6 handshakes. That is essential to getting started. Make sure that makes sense to you. The eighth person, Mrs. Smith, is not included on that list.

Here I ask one of the volunteers other than Mr. Smith to be the person shaking 6 hands. This person steps forward and proceeds to shake all possible hands (as in all people other than the spouse), thus, giving us a person with 6 handshakes. Do you see now that the spouse of this individual is the only person who could shake 0 hands? Hence, both members of that couple have completed their handshakes.

We proceed to identify another person other than Mr. Smith to be the person who will shake 5 hands. Observe that this person will have already counted 1 handshake and now can shake hands with Mr. and Mrs. Smith as well as with another couple. The spouse of the person shaking 5 hands will become the only person who can shake only 1 hand. Hence, these people have finished with shaking hands.

Finally, there is a volunteer who shakes more hands and becomes the person with 4 handshakes while having a spouse with only 2 handshakes. This leaves Mr. and Mrs. Smith each having shaken 3 hands. We are done with the handshakes and can definitively answer the question. How many hands did Mr. Smith shake? The answer is 3.

Written Approach: Using a Diagram

Let us model this problem graphically using a diagram. This can be thought of as a graph with 8 vertices labeled A, B, C, D, E, F, G, and H. Suppose that A is married to B, C is married to D, E is married to F, and G is married to H. Note that in the classroom model, Mr. and Mrs. Smith were positioned first. In this diagrammatic approach, their location will not be apparent until the completion of the process. That is, the coupling in pairs is required, but not the naming of any of the pairs.

Each vertex represents a person at the party. Two vertices will be joined if those two people shook hands. Again note that since no one shakes their own hand, or the hand of their spouse, a person can shake at most six hands. Thus, every person at the party shook at least 0 hands and at most 6 hands.

Consider the person who shook 6 hands. Let us assume this person is denoted as A. Hence, A shook hands with everyone at the party except for B. Represent this by drawing an edge from A to each of the other vertices in the graph, except for B. Thus, every person other than B has shaken at least one hand. Since someone at the party shook 0 hands, this implies that B must have been the person who shook 0 hands. This information is represented in the diagram below. In this diagram, the oval shape around A and B signifies that this couple has finished performing all of their handshakes.
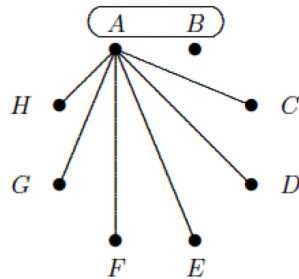
Now consider the person who shook 5 hands. Assume this person is C. Then C must shake hands with each of E, F, G, and H. This shows that everyone (other than B and D) shook at least two hands. Therefore, it follows that D must have been the person who shook exactly 1 hand, as illustrated.
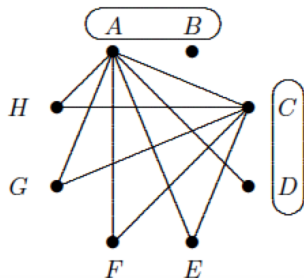
Now consider the person who shook 4 hands. Assume this person is E. Since E has already shaken two hands, E must shake hands with both G and H. Drawing edges from E to G and E to H we see that F must be the person who shook 2 hands. Further, both G and H have shaken at least three hands. The resulting diagram is shown:
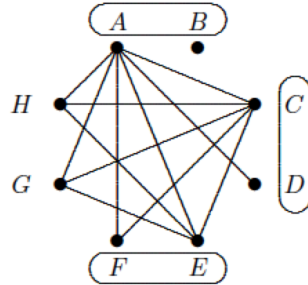
Note that G and H are married and so they do not shake hands. Thus, both G and H shook three hands. We have now indicated all the handshakes that took place at this party. However, we need to identify Mr. and Mrs. Smith. If Mrs. Smith is any of A to F, two of the individuals would have replied to her question that they shook exactly three hands. That is a contradiction because all seven replies were different. Therefore, Mrs. Smith must be either G or H. Thus, Mrs. Smith shook three hands. Likewise, Mr. Smith as her spouse shook exactly 3 hands.

Questions and thoughts for consideration with the problem

i) Considering the classroom approach, why could we not have let Mr. Smith be the volunteer who shook extra hands?

Suppose that he began by shaking 6 hands. It would have then been impossible for any of the people other than Mrs. Smith to have had 0 handshakes, thus, violating the conditions of the problem. (Do you see that we needed someone else to shake no hands?) It is more difficult to consider other cases along the way, but you may wish to ponder those also.

ii) How does one know that Mrs. Smith had to shake an odd number of hands?

iii) Observe in the concluding diagram that all couples shook a total of 6 hands. Is this a coincidence or can you explain why this must be the case?

Some problems to try

Here are a few handshake questions to consider.

1. Suppose that twenty people attended a party, and everyone shook hands with exactly three guests. How many handshakes took place?

2. Mr. and Mrs. Smith were at a party with ten other married couples. Various

handshakes took place. No one shook hands with their spouse, and of course, no one shook their own hand! After all the introductions had been made, Mrs. Smith asked the other people how many hands they shook. Surprisingly, they all gave a different answer. How many hands did Mr. Smith shake?

3. Everyone at a meeting shook hands with one another. Shortly after the meeting commenced, the chronically late character known as Tar D. arrived. Tar only managed to shake hands with some of the people present. In total, there were 59 handshakes. How many hands did Tar D. shake?

4. At a party attended by n people, various handshakes took place. Just for fun, each person shouted out the number of hands they shook. Explain why there must have been at least two people who shouted out the same number.

Reference

Hoshino, R. & Grant McLoughlin, J. (2005). *Combinatorial Explorations.* Ottawa: Canadian Mathematical Society.
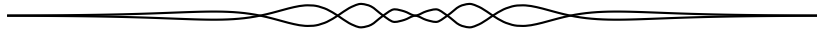
OLYMPIAD CORNER
No. 375

The problems in this section have appeared in a regional or national Olympiad.

To facilitate their consideration, solutions should be received by **October 30, 2019**.
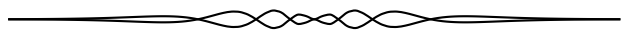
OC441. Let $f : [0, \infty) \to (0, \infty)$ be a continuous function.

(a) Prove that there exists a natural number n_0 such that for any natural number $n > n_0$ there exists a unique real number $x_n > 0$ for which

$$n \int_0^{x_n} f(t) \, dt = 1;$$

(b) Prove that the sequence $(nx_n)_{n \geq 1}$ is convergent and find its limit.

OC442. Let $H = \{1, 2, \ldots, n\}$. Are there two disjoint subsets A and B such that $A \cup B = H$ and such that the sum of the elements in A is equal to the product of the elements in B if **(a)** $n = 2016$? **(b)** $n = 2017$?

OC443. In a triangle ABC, the foot of the altitude drawn from A is T and the angle bisector of $\angle B$ intersects side AC at D. If $\angle BDA = 45°$, find $\angle DTC$.

OC444. We have n^2 empty boxes, each with a square bottom. The height and the width of each box are natural numbers in the set $\{1, 2, \ldots, n\}$. Each box differs from any other box in at least one of these two dimensions. We are allowed to insert a box into another if each dimension of the first box is smaller than the corresponding dimension of the second box and at least one of the dimensions is at least units less that the corresponding larger box dimension. In this way, we can create a sequence of boxes inserted into each other in the same orientation (i.e. the first box is inside the second, the second box is inside the third, etc.). We store each sequence of boxes on a shelf with each shelf holding one set of nested boxes. Determine the smallest number of shelves needed to store all the n^2 boxes.

OC445. There are 100 diamonds in a pile, of which 50 are genuine and 50 are fake. We invited a distinguished expert, who can recognize which diamonds are genuine. Each time we show him three diamonds, he chooses two of them and (truthfully) tells whether they are both genuine, one genuine or none genuine. Establish if we can guarantee to spot all the genuine diamonds no matter how the expert chooses the judged pair.

· ·
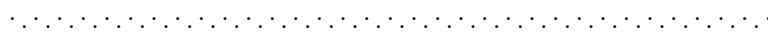
Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

> ***Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2019**.

La rédaction remercie Valérie Lapointe, Carignan, QC, d'avoir traduit les problèmes.

OC441. Soit $f : [0, \infty) \to (0, \infty)$ une fonction continue.

(a) Prouvez qu'il existe un nombre naturel n_0 tel que pour tout nombre $n > n_0$, il existe un unique nombre réel $x_n > 0$ pour lequel

$$n \int_0^{x_n} f(t) \, dt = 1;$$

(b) Prouvez que la suite $(nx_n)_{n \geq 1}$ est convergente et trouvez le résultat de sa limite.

OC442. Soit $H = \{1, 2, \ldots, n\}$. Existe-t-il deux sous-ensembles disjoints A et B tels que $A \cup B = H$ et tels que la somme des éléments dans A est égale au produit des éléments dans B si **(a)** $n = 2016$? **(b)** $n = 2017$?

OC443. Dans un triangle ABC, l'extrémité de la hauteur issue de A est T et la bissectrice de $\angle B$ intercepte le côté AC en D. Si $\angle BDA = 45°$, trouvez $\angle DTC$.

OC444. On a n^2 boîtes vides, chacune à fond carré. La hauteur et la largeur de chaque boîte est un nombre naturel de l'ensemble $\{1, 2, \ldots, n\}$. Chaque boîte est différente d'une autre sur au moins une des deux dimensions. On peut entrer une boîte dans une autre si les deux dimensions sont plus petites et qu'au moins une des deux dimensions est au moins deux unité plus petite. On peut ainsi créer une suite de boîtes à l'intérieur d'une autre (i.e. la première boîte est à l'intérieur de la deuxième, la deuxième boîte est à l'intérieur de la troisième, etc.). On range une telle suite de boîte sur une étagère. Déterminez le plus petit nombre d'étagères nécessaires pour ranger toutes les n^2 boîtes.

OC445. Il y a 100 diamants dans une pile dans laquelle 50 sont véritables et 50 sont faux. On invite un expert qui peut reconnaître quels diamants sont véritables. À chaque fois qu'on lui montre trois diamants, il en choisit deux et dit (honnêtement) s'ils sont soit tous les deux véritables, si un seul l'est ou si aucun ne l'est. Déterminez si on peut garantir de trouver tous les diamants véritables peu importe la façon dont l'expert choisit la paire jugée.

OLYMPIAD CORNER
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(1), p. 17–18; 45(2): 69–71.
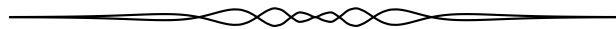
OC411. Show that for all integers $k > 1$ there is a positive integer m less than k^2 such that $2^m - m$ is divisible by k.

Originally 2017 Hungary Math Olympiad, 3rd Problem, 3rd Category, Final Round.

We received no submissions for this problem.

OC412. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y

$$f(y - xy) = f(x)y + (x - 1)^2 f(y).$$

Originally 2017 Czech-Slovakia Math Olympiad, 3rd Problem, Final Round.

We received 5 submissions of which 4 were correct. We present the solution by Sundara Narasimhan.

We evaluate the relation at $x = 0$ and $y = 1$ to find $0 = f(0)$.

We evaluate the relation at $x = 1$ and $y = 1$ to find $f(0) = f(1)$.

We evaluate the relation at $x = x$ and $y = 1$, and use $f(0) = f(1) = 0$ to find $f(1 - x) = f(x)$.

We make the substitution $1 - x = t$ in the original relation, and use $f(1-x) = f(x)$ to get for any $t \in \mathbb{R}$ and $y \in \mathbb{R}$

$$f(yt) = f(t)y + t^2 f(y).$$

We interchange y and t to get $f(ty) = f(y)t + y^2 f(t)$. Since $f(ty) = f(yt)$, we find that for any $t \in \mathbb{R}$ and $y \in \mathbb{R}$

$$(t^2 - t)f(y) = (y^2 - y)f(t).$$

We take $t = 2$ in the last relation to find that for any $y \in \mathbb{R}$

$$f(y) = \frac{f(2)}{2}(y^2 - y).$$

Therefore, the solutions of our functional equation must be of the form

$$f(x) = c(x^2 - x),$$

for some real constant c. In fact, we can check that any function of this form is a solution of the original relation. We established that the set of all functions that satisfy the original relation are $f(x) = c(x^2 - x)$, with c being a real constant.

OC413. To each sequence consisting of n zeros and n ones is assigned a number which is the number of largest segments with the same digits in it (for example, the sequence 00111001 has 4 such segments 00, 111, 00, 1). For each n, add all the numbers assigned to each sequence. Prove that the resulting sum is equal to

$$(n + 1)\binom{2n}{n}.$$

Originally 2017 Czech-Slovakia Math Olympiad, 4th Problem, Final Round.

We received one submission. We present the solution of Kathleen Lewis.

The total number of distinct sequences of n zeroes and n ones is $\binom{2n}{n}$. The number of largest same-digit segments of such sequence has a range between 2 and $2n$. The minimum number of 2 is displayed by two sequences that have all zeros together and all ones together

$$(000\ldots111\ldots, \ 111\ldots000\ldots).$$

The maximum number is displayed by two sequences that have alternating zeros and ones

$$(101010\ldots, \ 010101\ldots).$$

For a natural number j between 2 and $2n$, let N_j be the number of sequences that have exactly j largest same-digit segments.

First we show that for any j, $N_j = N_{2n+2-j}$, in other words $N_2, N_3, \ldots, N_{2n+1}$, N_{2n+2} are symmetrical about $n + 1$. In fact we can calculate N_j.

Case 1. Assume j is even, i.e. $j = 2k$ for some natural number k.

Since the sequence has j same-digit blocks, k of these are blocks of zeroes and the remaining k are blocks of ones. The sequence is uniquely determined by the points where we cut the original list of n zeros and the original list of n ones. $k - 1$ cuts need to be made to obtain k blocks, and these cuts are selected from $n - 1$ links between the original n zeros. Therefore the original sequence of n zeros can be cut in k blocks in $\binom{n-1}{k-1}$ ways. Similarly for ones. Hence

$$N_j = 2\binom{n-1}{k-1}\binom{n-1}{k-1}.$$

The number 2 was added to the above expression to account for whether the sequence starts with zero or one.

Since j is even, it follows that $2n + 2 - j = 2(n + 1 - k)$ is even, and

$$N_{2n+2-j} = 2\binom{n-1}{n-k}\binom{n-1}{n-k}$$

Using properties of binomial coefficients

$$N_{2n+2-j} = 2\binom{n-1}{(n-1)-(n-k)}\binom{n-1}{(n-1)-(n-k)} = N_j.$$

Case 2. Assume j is odd, i.e. $j = 2k + 1$ for some natural number k.

Since the sequence has j same-digit blocks, $k + 1$ of these are blocks of zeroes and the remaining k are blocks of ones, or vice versa k blocks of zeros and $k + 1$ blocks of ones. Using arguments that we invoked at case 1 we show that

$$N_j = 2\binom{n-1}{k}\binom{n-1}{k-1}.$$

Since j is odd, it follows that $2n + 2 - j = 2(n - k) + 1$ is odd, and

$$N_{2n+2-j} = 2\binom{n-1}{n-k-1}\binom{n-1}{n-k}$$

Using properties of binomial coefficients

$$N_{2n+2-j} = 2\binom{n-1}{(n-1)-(n-k-1)}\binom{n-1}{(n-1)-(n-k)} = N_j.$$

Now we can proceed to calculate the required sum

$$S = 2N_2 + 3N_3 + \cdots + (2n-1)N_{2n-1} + (2n)N_{2n}.$$

Because $N_j = N_{2n+2-j}$, we have

$$\begin{aligned}
2S &= (2N_2 + (2n)N_{2n}) + (3N_3 + (2n-1)N_{2n-1}) + \cdots + ((2n)N_{2n} + 2N_2) \\
&= (2 + 2n)N_2 + (3 + 2n - 1)N_3 + \cdots + (2n - 1 + 2)N_{2n-1} + (2n + 2)N_{2n} \\
&= 2(n+1)(N_2 + N_3 + \cdots + N_{2n-1} + N_{2n}).
\end{aligned}$$

However, $N_2 + N_3 + \cdots + N_{2n-1} + N_{2n}$ is the total number of sequences of n zeros and n ones, namely $\binom{2n}{n}$. Therefore, the sum $S = (n+1)\binom{2n}{n}$.

An interesting interpretation of this result is that the average number of largest same-digit segments in a sequence of n zeros and n ones is $n + 1$. And this is mainly due to the fact that the distribution of the number of largest same-digit segments is symmetrical about $n + 1$.

OC414. Find all prime numbers p and all positive integers a and m such that $a \leq 5p^2$ and $(p - 1)! + a = p^m$.

Originally 2017 Bulgaria Math Olympiad, 4th Problem, Grade 9-12, Final Round.

We received only one incomplete submission, which we do not present here.

OC415. Let n be a positive integer and let $f(x)$ be a polynomial of degree n with real coefficients and n distinct positive real roots. Is it possible for some integer $k \geq 2$ and for a real number a that the polynomial

$$x(x + 1)(x + 2)(x + 4)f(x) + a$$

is the k-th power of a polynomial with real coefficients?

Originally 2017 Bulgaria Math Olympiad, 5th Problem, Grade 9-12, Final Round.

We received no submissions for this problem.

OC416. Given an acute nonisosceles triangle ABC with altitudes CD, AE, BF. Points E' and F' are symmetrical to E and F with respect to points A and B, respectively. Take a point C_1 on the ray $\overrightarrow{CD}$ such that $DC_1 = 3CD$. Prove that $\angle E'C_1F' = \angle ACB$.

Originally 2017 Bulgaria Math Olympiad, 6th Problem, Grade 9-12, Final Round.

We received 3 submissions and we present 2 of them.

Solution 1, by Oliver Geupel.

We drop the constraint that triangle ABC is acute and nonisosceles, and prove the result for an arbitrary triangle ABC. Moreover, we prove the stronger result that the triangles ABC and $E'F'C_1$ are similar.

We work in the complex plane. We use lower-case letters to denote the complex-number representations of geometrical points denoted by corresponding upper-case letters. For example a is the complex number assigned to point A. We assume without loss of generality that the points A, B, and C are on the unit circle.

First we recall the result that the foot of the perpendicular from an arbitrary point P to the chord XY of the unit circle is the point specified by the complex number

$$\frac{1}{2}(p + x + y - xy\bar{p}).$$

Hence,

$$d = \frac{1}{2}\left(a + b + c - \frac{ab}{c}\right), \; e = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right), \; f = \frac{1}{2}\left(a + b + c - \frac{ca}{b}\right).$$

Moreover, since points E' and F' are symmetrical to E and F with respect to points A and B, respectively

$$e' = a + (a - e) = \frac{1}{2}\left(3a - b - c + \frac{bc}{a}\right), \; f' = b + (b - f) = \frac{1}{2}\left(3b - c - a + \frac{ca}{b}\right).$$

Also,

$$c_1 = d + 3(d - c) = 2a + 2b - c - \frac{2ab}{c}.$$

Next, we compute $a(c-b)(e'-c_1)$ and $b(c-a)(f'-c_1)$ to find that

$$a(c-b)(e'-c_1) = b(c-a)(f'-c_1)$$
$$= \frac{5}{2}(a^2b + ab^2) + \frac{1}{2}(bc^2 - cb^2) + \frac{1}{2}(ac^2 - a^2c) - 3abc - 2\frac{a^2b^2}{c}.$$

Thus,

$$\frac{e'-c_1}{f'-c_1} = \frac{b(c-a)}{a(c-b)} = \frac{(1/a) - (1/c)}{(1/b) - (1/c)} = \frac{\bar{a} - \bar{c}}{\bar{b} - \bar{c}}.$$

This equality of complex numbers implies,

$$\frac{C_1 E'}{C_1 F'} = \frac{CA}{CB} \quad \text{and} \quad \angle E'C_1 F' = \angle ACB.$$

This completes the proof.

Solution 2, by Andrea Fanchini.
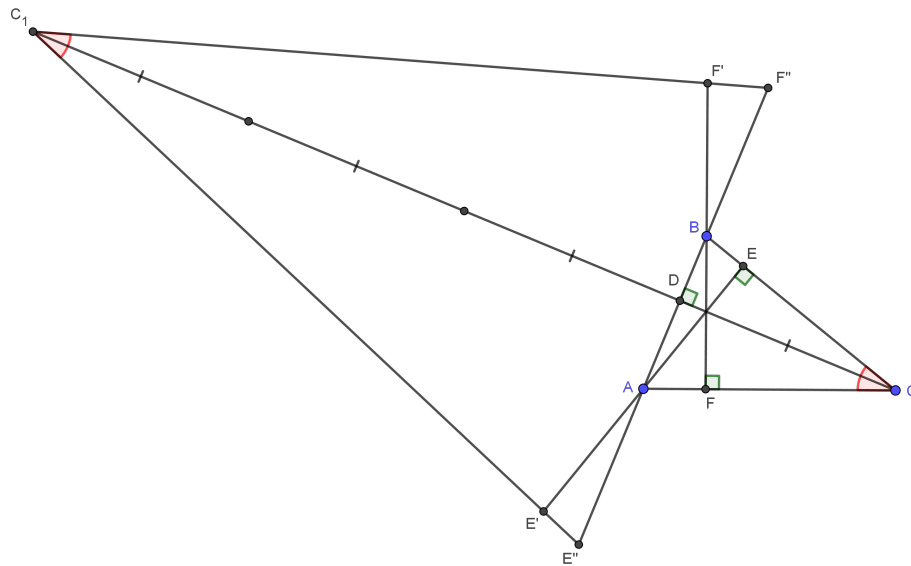
We use Conway triangle notations: S stands for twice the area of $\triangle ABC$, $S_A = S \cot \angle BAC$, $S_B = S \cot \angle ABC$, and $S_C = S \cot \angle ACB$.

We use barycentric coordinates with reference to the triangle ABC:

$$D(S_B : S_A : 0), \qquad E(0 : S_C : S_B), \qquad F(S_C : 0 : S_A)$$

$$E'(-2a^2 : S_C : S_B), \qquad F'(S_C : -2b^2 : S_A).$$

Since the point C_1 divides the segment CD in the ratio $(-4 : 3)$, it follows that

$$CC_1/C_1 D = (-4)/3 \quad \text{and} \quad C_1(4S_B : 4S_A : -3c^2).$$

Therefore, the lines C_1E', and C_1F' are

$$C_1E' : (S_AS_B + 3S^2)x + 2(S_B^2 + 3S^2)y + 4(a^2S_A + S^2)z = 0,$$

$$C_1F' : 2(S_A^2 + 3S^2)x + (S_AS_B + 3S^2)y + 4(b^2S_B + S^2)z = 0,$$

and the intersection points of these lines with the line AB are

$$E'' = C_1E' \cap AB = \left(2(S_B^2 + 3S^2) : -(S_AS_B + 3S^2) : 0\right),$$

$$F'' = C_1F' \cap AB = \left(S_AS_B + 3S^2 : -2(S_A^2 + 3S^2) : 0\right).$$

We calculate

$$\begin{aligned}
\angle E'C_1F' &= \angle F''C_1D + \angle E''C_1D \\
&= \arctan\frac{F''D}{C_1D} + \arctan\frac{E''D}{C_1D} = \arctan\frac{E''F'' \cdot C_1D}{C_1D^2 - E''D \cdot F''D},
\end{aligned}$$

where

$$E''F'' = \frac{3\left((a^2S_A + S^2)(2b^2 + S_C) + (b^2S_B + S^2)(2a^2 + S_C)\right)}{c(2a^2 + S_C)(2b^2 + S_C)},$$

$$C_1D = \frac{3S}{c}, \qquad E''D = \frac{3(a^2S_A + S^2)}{c(2a^2 + S_C)}, \qquad F''D = \frac{3(b^2S_B + S^2)}{c(2b^2 + S_C)}$$

Therefore,

$$\begin{aligned}
\angle E'C_1F' &= \arctan\frac{S\left((a^2S_A + S^2)(2b^2 + S_C) + (b^2S_B + S^2)(2a^2 + S_C)\right)}{S^2(2a^2 + S_C)(2b^2 + S_C) - (a^2S_A + S^2)(b^2S_B + S^2)} \\
&= \arctan\frac{S_C(8S_CS^2 + 3c^2S^2 + a^2b^2c^2)}{S(8S_CS^2 + 3c^2S^2 + a^2b^2c^2)} \\
&= \arctan\frac{S_C}{S} \\
&= \angle ACB.
\end{aligned}$$

OC417. Point M is the midpoint of side BC of a triangle ABC in which $AB = AC$. Point D is the orthogonal projection of M onto side AB. Circle ω is inscribed in triangle ACD and tangent to segments AD and AC at K and L, respectively. Lines tangent to ω which pass through M intersect line KL at X and Y, where points X, K, L and Y lie on KL in this order. Prove that points M, D, X and Y are concyclic.

Originally 2017 Poland Math Olympiad, 5th Problem, Final Round.

We received no submissions for this problem.

Crux Mathematicorum, Vol. 45(7), September 2019

OC418. Three sequences $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$, $(c_0, c_1, \ldots, c_{2n})$ of nonnegative real numbers are given such that for all $0 \le i, j \le n$ we have $a_i b_j \le (c_{i+j})^2$. Prove that

$$\sum_{i=0}^{n} a_i \cdot \sum_{j=0}^{n} b_j \le \left(\sum_{k=0}^{2n} c_k \right)^2 .$$

Originally 2017 Poland Math Olympiad, 6th Problem, Final Round.

We received no submissions for this problem.

OC419. Prove that there exist infinitely many positive integers m such that there exist m consecutive perfect squares with sum m^3. Determine one solution with $m > 1$.

Originally 2017 Germany Math Olympiad, 6th Problem, Final Round.

We received 6 correct submissions. We present a solution that follows the submissions of the Problem Solving Group of Missouri State University and David Manes. At the end, we include a list of examples by Dominique Mouchet.

We start by computing the difference between m^3 and the sum of m arbitrary consecutive perfect squares:

$$m^3 - \sum_{i=a}^{a+m-1} i^2 = m^3 - \left(\sum_{i=1}^{a+m-1} i^2 - \sum_{i=1}^{a-1} i^2 \right)$$

$$= m^3 - \left(\frac{(a+m-1)(a+m)(2a+2m-1)}{6} - \frac{(a-1)a(2a-1)}{6} \right)$$

$$= \frac{m \left(4m^2 - 6am + 3m - 6a^2 + 6a - 1 \right)}{6}.$$

Therefore, for any pair of positive integers (a, m) that satisfy

$$f(a, m) = 4m^2 - 6am + 3m - 6a^2 + 6a - 1 = 0,$$

we have m consecutive squares summing to m^3.

Let $a_0 = 1$ and $m_0 = 1$ and recursively define for any $n \ge 1$

$$\begin{aligned} a_n &= 11a_{n-1} + 16m_{n-1} - 5 \\ m_n &= 24a_{n-1} + 35m_{n-1} - 12. \end{aligned} \tag{1}$$

We prove by induction that $f(a_n, m_n) = 0$, $a_n \ge 1$, $m_n \ge 1$ for all integers $n \ge 1$, and $m_i \ne m_j$ for all integers $i \ne j$.

First it is easy to check $f(a_0, m_0) = f(1, 1) = 0$. Assuming that $f(a_{n-1}, m_{n-1}) = 0$, a routine, but tedious, calculation yields $f(a_n, m_n) = 0$. Second, $a_0, m_0 \ge 1$ and assuming $a_{n-1}, m_{n-1} \ge 1$, it follows that $a_n \ge 11 + 16 - 5 \ge 1$ and

$m_n \geq 24 + 35 - 12 \geq 1$. Finally, the sequence of m_n is strictly increasing since $m_n \geq 24 + 35m_{n-1} - 12 = 35m_{n-1} + 12$ and the m_n are positive. Therefore, the m_n sequence leads to infinitely many positive integers with the required property.

An alternative way to find the recursive solution (1), is to write the equation $f(a, m) = 0$ in an equivalent form

$$3(m + 2a - 1)^2 - 11m^2 = 1.$$

The substitution $x = m + 2a - 1$ yields a Pell equation $3x^2 - 11m^2 = 1$. For an arbitrary Pell equation $cx^2 - dy^2 = 1$, the Pell resolvent is defined to be $u^2 - cdv^2 = 1$. Therefore, the Pell resolvent for $3x^2 - 11m^2 = 1$ is $u^2 - 33v^2 = 1$ with fundamental solution $(u_1, v_1) = (23, 4)$. Let $u_0 = 1$ and $v_0 = 0$. The general solution (u_n, v_n) for the Pell resolvent is recursively given for $n \geq 1$ by

$$
\begin{aligned}
u_{n+1} &= u_1 u_n + cdv_1 v_n = 23u_n + 132v_n \qquad (2)\\
v_{n+1} &= v_1 u_n + u_1 v_n = 4u_n + 23v_n.
\end{aligned}
$$

Note that v_n is always an even integer and u_n is odd for each integer $n \geq 0$.

The general solution (x_n, m_n) for $3x^2 - 11m^2 = 1$ in terms of the solution of the resolvent is given by

$$
\begin{aligned}
x_n &= x_0 u_n + dm_0 v_n = 2u_n + 11v_n \qquad (3)\\
m_n &= m_0 u_n + cx_0 v_n = u_n + 6v_n.
\end{aligned}
$$

Observe that x_n is an even integer and m_n is an odd integer for any $n \geq 0$. As a result, for x_n and m_n defined by (2), the equation $x_n = m_n + 2a - 1$ admits an integer solution a. Moreover, the recursive formulas (2) that define the u_n and v_n sequences can be used to derive the recursive formulas (1) for the a_n and m_n sequences.

We end by listing several examples.

n	m_n	a_n	Sum
0	1	1	$1^2 = 1^3$
1	47	22	$\underbrace{22^2 + 23^2 + \cdots + 68^2}_{47 \text{ terms}} = 47^3$
2	2161	989	$\underbrace{989^2 + 990^2 + \cdots + 3149^2}_{2161 \text{ terms}} = 2161^3$
3	99359	45450	$\underbrace{45450^2 + 45451^2 + \cdots + 144808^2}_{99359 \text{ terms}} = 99359^3$
4	4568353	2089689	$\underbrace{2089689^2 + 2089690^2 + \cdots + 6658041^2}_{4568353 \text{ terms}} = 4568353^3$
5	210044879	96080222	$\underbrace{96080222^2 + 96080223^2 + \cdots + 306125100^2}_{210044879 \text{ terms}} = 210044879^3$

OC420. General Tilly and the Duke of Wallenstein play "Divide and rule!" (Divide et impera!). To this end, they arrange N tin soldiers in M companies and command them by turns. Both of them must give a command and execute it in their turn.

Only two commands are possible: The command "Divide!" chooses one company and divides it into two companies, where the commander is free to choose their size, the only condition being that both companies must contain at least one tin soldier. On the other hand, the command "Rule!" removes exactly one tin soldier from each company.

The game is lost if in your turn you can't give a command without losing a company. Wallenstein starts to command.

(a) Can he force Tilly to lose if they start with 7 companies of 7 tin soldiers each?

(b) Who loses if they start with $M \geq 1$ companies consisting of $n_1 \geq 1$, $n_2 \geq 1$, $\ldots$, $n_M \geq 1$ $(n_1 + n_2 + \cdots + n_M = N)$ tin soldiers?

Originally 2017 Germany Math Olympiad, 3rd Problem, Final Round.

We received 1 submission. We present the solution by Jeremy Mirmina.

We discuss the winning strategy of the game based on the parities (odd/even) of the number of tin soldiers N, the number of companies M, and the difference $I = N - M$.

First, notice the following. When Move 1 ("Divide!") is played N remains the same, M decreases by one, and I decreases by one and switches parity. When Move 2 ("Rule!") is played N decreases by M, M remains the same, and I decreases by M.

In the next table we summarise the changes in the parities of I, N, and M after Move 1 or Move 2 are played. We assume that the game did not end, and that Move 1 and Move 2 can be played.

I	N	M		Move 1			Move 2	
			new I	new N	new M	new I	new N	new M
odd	odd	even	even	odd	odd	odd	odd	even
odd	even	odd	even	even	even	even	odd	odd
even	even	even	odd	even	odd	even	even	even
even	odd	odd	odd	odd	even	odd	even	odd

Second, notice that if there is at least one company with exactly one soldier the two players can only use Move 1 and for exactly I times. This is because Move 1 will be applied $n_i - 1$ times to split a company with n_i soldiers into n_i companies, each with only one soldier.

Third, notice that the game ends when all companies have exactly one soldier.

Case 1. Assume I is odd, i.e. either N is even and M is odd or M is even and N is odd. Then the first player (Wallenstein) has a winning strategy.

Because $N \neq M$, he can choose a company with more than one soldier. He plays Move 1, and splits this company into one soldier and the rest. At this point, none of the two players can use Move 2, as it will make them lose. Move 1 is played exactly $I = N - M$ times until $I = 0$ and the game ends. The player who starts with I odd will continue to play with I odd and his opponent will play with I even. Eventually his opponent will receive the configuration with $I = 0$ and will lose. For this reason a player who moves with an even I, never wants to change it into odd on his opponent's turn.

Case 2. Assume I is even, and both N and M are odd. Then the second player (Tilly) has a winning strategy.

Based on the table above, the first player starts with an even I and regardless of his move he changes the parity of I to odd on his opponent's turn. Hence, the second player has always a winning strategy (see Case 1).

This answers part (a) of the problem, since $N = 7 \times 7 = 49$ is an odd number of soldiers and $M = 7$ is an odd number of companies. Wallenstein cannot force Tilly to loose, and Tilly, the second player, has a winning strategy.

Case 3. Assume I is even, and both N and M are even.

In this case, neither of the two players is interested in playing Move 1, which results in a winning configuration for the opponent. They play Move 2 for as long as they can. In fact, Move 2 can be played $m - 1$ times, where $m = \min\{n_i : i = 1, 2, \ldots, M\}$ is the size of the company with the lowest number of soldiers. If m is even then the first player (Wallenstein) has a winning strategy, and if m is odd then the second player (Tilly) has a winning strategy.

The conclusions of cases 1, 2, and 3 answer part (b).
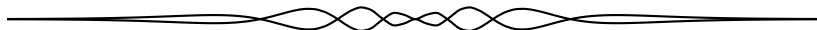
FOCUS ON...

No. 37
Michel Bataille
Geometry with Complex Numbers (II)

Introduction

In this second part, we continue to present various interventions of the complex numbers in geometry problems. We begin with regular polygons, an obvious domain of application. Then, we will consider similarities, either direct or opposite, as they can be simply represented using complex numbers, and we conclude with a look at areas.

Complex numbers and regular polygons

It is well-known that the n-th roots of a nonzero complex number are the affixes of the vertices of a regular n-gon. In particular, the n-th roots of unity $\exp(2k\pi i/n)$, $k = 0, 1, \ldots, n-1$, correspond to an n-gon inscribed in the unit circle Γ, with centre O and radius 1. Here is a first illustration, a problem proposed in the December 2017 issue of *Mathematics Magazine*:

> Let n be an integer, $n \geq 2$. Let $A_1 A_2 A_3 \cdots A_{2n+1}$ be a regular polygon with $2n+1$ sides. Let P be the intersection of the segments $\overline{A_2 A_{n+2}}$ and $\overline{A_3 A_{n+3}}$. Prove that
>
> $$(A_1 P)^2 = (A_2 A_3)^2 + (A_3 P)^2.$$

We may suppose that for $k = 1, 2, \ldots, 2n+1$, A_k is the point with complex affix w^{k-1} where $w = \exp\left(\frac{2\pi i}{2n+1}\right)$. Let p be the affix of P. We readily obtain

$$(A_2 A_3)^2 = |w^2 - w|^2 = |w - 1|^2 = (w-1)(\overline{w} - 1) = 2 - w - \overline{w} = 2 - w - w^{2n}.$$

and

$$\begin{aligned}
(A_1 P)^2 - (A_3 P)^2 &= |p - 1|^2 - |p - w^2|^2 \\
&= (p-1)(\overline{p} - 1) - (p - w^2)(\overline{p} - w^{2n-1}) \\
&= pw^{2n-1} - p + \overline{p}w^2 - \overline{p}.
\end{aligned}$$

Now, from the equations of the lines $A_2 A_{n+2}$ and $A_3 A_{n+3}$, we deduce that $pw^n + \overline{p}w = 1 + w^{n+1}$ and $pw^{n-1} + \overline{p}w^2 = 1 + w^{n+1}$. It follows that

$$pw^{2n-1} - p = w^n(pw^{n-1} + \overline{p}w^2) - w^{n+1}(pw^n + \overline{p}w) = w^n + 1 - (w^{n+1} + w)$$

$$\overline{p}w^2 - \overline{p} = (pw^{n-1} + \overline{p}w^2) - \frac{1}{w}(pw^n + \overline{p}w) = 1 + w^{n+1} - (w^{2n} + w^n)$$

and by addition,

$$(A_1P)^2 - (A_3P)^2 = 2 - w - w^{2n} = (A_2A_3)^2.$$

As a second example, we prove a vectorial result about the projections of a point on the sidelines of a regular polygon:

Let the consecutive vertices of a regular n-gon be denoted $A_0, \ldots, A_{n-1}$, in order, and let $A_n = A_0$. Let B_k be the projection of a point M onto the line A_kA_{k+1}. Show that

$$\sum_{k=0}^{n-1} \overrightarrow{MB_k} = \frac{n}{2}\overrightarrow{MO}.$$

Again, we suppose that the affix of A_k is w^k with $w = \exp(2\pi i/n)$. Let m be the affix of M and let C_k be the midpoint of A_kA_{k+1}.

We remark that $\sum_{k=0}^{n-1} \overrightarrow{MC_k} = n\overrightarrow{MO}$; for example because

$$\sum_{k=0}^{n-1} \left(\frac{w^k + w^{k+1}}{2} - m \right) = \left(\sum_{k=0}^{n-1} w^k \right) - nm = \frac{1 - w^n}{1 - w} - nm = -nm$$

and

$$\overrightarrow{C_kB_k} = \frac{(\overrightarrow{OM} \cdot \overrightarrow{A_kA_{k+1}})}{\|\overrightarrow{A_kA_{k+1}}\|^2} \overrightarrow{A_kA_{k+1}}.$$

With the notations $m_k, \delta_k = w^{k+1} - w^k$ for the complex affixes of $\overrightarrow{C_kB_k}, \overrightarrow{A_kA_{k+1}}$, respectively, the latter gives

$$m_k = (\text{Re}(m\overline{\delta_k}))\frac{\delta_k}{\delta_k\overline{\delta_k}} = \frac{1}{2} \cdot \frac{(m\overline{\delta_k} + \overline{m}\delta_k)}{\overline{\delta_k}} = \frac{1}{2}(m - \overline{m}w^{2k+1}).$$

Since

$$\sum_{k=0}^{n-1} w^{2k+1} = w \cdot \frac{1 - (w^n)^2}{1 - w^2} = 0,$$

we obtain $\sum_{k=0}^{n-1} m_k = \frac{n}{2} \cdot m$, that is,

$$\sum_{k=0}^{n-1} \overrightarrow{C_kB_k} = \frac{n}{2}\overrightarrow{OM}.$$

The desired result then follows from

$$\sum_{k=0}^{n-1} \overrightarrow{MB_k} = \sum_{k=0}^{n-1} \overrightarrow{MC_k} + \sum_{k=0}^{n-1} \overrightarrow{C_kB_k} = n\overrightarrow{MO} + \frac{n}{2}\overrightarrow{OM}.$$

(The established result will be used later, in the paragraph devoted to areas.)

Complex numbers and similarities

A similarity with factor $k > 0$ is a transformation of the plane such that the distance between the images of M, N is kMN for any points M, N of the plane. The similarities with factor 1 are the isometries while those of factor $k \neq 1$ are of the form $\mathcal{H} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{H}$ where $\mathcal{H}$ is a homothety with factor k and centre Ω and $\mathcal{I}$ is an isometry such that $\mathcal{I}(\Omega) = \Omega$. The similarity is direct or opposite according as $\mathcal{I}$ is a displacement or a reflection in a line (that is, according as it preserves orientation or not). Embedding all this in the complex plane, it can be showed that a direct (resp. opposite) similarity transforms a point M with affix m into the point M' with affix $m' = am + b$ (resp. $m' = a\overline{m} + b$) for some complex numbers $a \neq 0, b$ independent of m. To illustrate the power of this complex representation, we consider two examples, the first one being adapted from an exercise set at the French final high-school exam long ago:

> Let OAB be a triangle and suppose that points P and Q are such that the triangles $OA'B', AB'P, BQA'$ are directly similar to the triangles OAB, ABO, BOA, respectively. Show that O is the midpoint of PQ.

Taking O as the origin and using the corresponding lower-case letter for the affix of any other point, the hypotheses imply that for some complex numbers u, u_1, u_2, we have $a' = ua$, $b' = ub, b' = u_1 b + p$, $a = u_1 a + p$, $a' = u_2 a + q$, $b = u_2 b + q$ (for example, the latter because some direct similarity transforms A into A', B into B and O into Q). The elimination of u_1 and u_2 leads to $p(a - b) = ab(u - 1)$ and $q(b - a) = ab(u - 1)$. The desired relation $p + q = 0$ follows.

As a second example, we offer a variant of solution to problem **3401** [2009 : 42, 44 ; 2010 : 49]:

> Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle EAD$ and $\angle BCA = \angle EDA$, and let the lines CB and DE intersect in the point F. Prove that the midpoints of CD, BE, and AF are collinear.
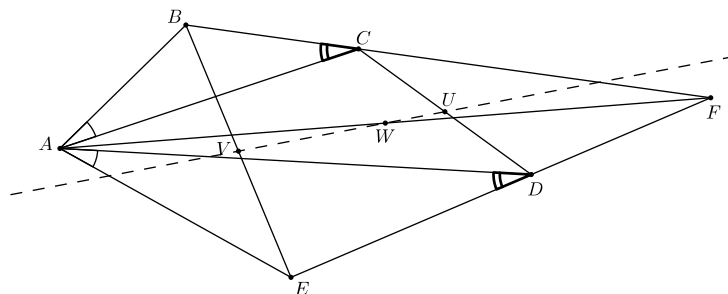
As remarked in the featured geometric solution, what matters is the fact that the triangles ABC and AED are oppositely similar. Therefore, taking the point A as the origin, there exists a complex number ω such that $e = \omega\overline{b}$ and $d = \omega\overline{c}$. Let U, V, W be the midpoints of CD, BE, AF, respectively. Their affixes are $u = \frac{1}{2}(c + \omega\overline{c})$, $v = \frac{1}{2}(b + \omega\overline{b})$, and $w = \frac{1}{2} \cdot f$.

The equations of the lines BC and DE are readily obtained:

$$\bar{z}(c - b) - z(\bar{c} - \bar{b}) = (\bar{b}c - b\bar{c})$$

and

$$\omega\bar{z}(\bar{c} - \bar{b}) - \bar{\omega}z(c - b) = |\omega|^2(b\bar{c} - \bar{b}c).$$

This said, U, V, W are collinear if and only if

$$(\bar{u} - \bar{w})(v - w) = (u - w)(\bar{v} - \bar{w})$$

or

$$(\bar{c} + \bar{\omega}c - \bar{f})(b + \omega\bar{b} - f) = (c + \omega\bar{c} - f)(\bar{b} + \bar{\omega}b - \bar{f}).$$

Expanding and arranging, this condition can be written as

$$f(\bar{b} - \bar{c}) - \bar{f}(b - c) + b\bar{c} - \bar{b}c = \omega\bar{f}(\bar{b} - \bar{c}) - \bar{\omega}f(b - c) + |\omega|^2(b\bar{c} - \bar{b}c).$$

This certainly holds since both sides are 0 (the left-hand side because F is on BC and the right one because F is on DE). The conclusion follows.

Complex numbers and area

We start with the following known expression of the area $[ABCD]$ of a quadrilateral $ABCD$:

$$[ABCD] = \pm\frac{1}{2}AC \cdot BD \cdot \sin(\angle(\overrightarrow{AC}, \overrightarrow{BD}))$$

where the sign is $+$ if and only if the quadrilateral is positively oriented, and $\angle(\overrightarrow{AC}, \overrightarrow{BD})$ is the directed angle from $\overrightarrow{AC}$ to $\overrightarrow{BD}$. If a, b, c, d are the affixes of the vertices, we obtain the following formula:

$$[ABCD] = \pm\frac{1}{2}\text{Im}((d - b)(\bar{c} - \bar{a})),$$

of which, for convenience, we repeat the proof.

Let $\alpha = \arg(d - b)$ and $\beta = \arg(c - a)$. Then, $\sin(\angle(\overrightarrow{AC}, \overrightarrow{BD})) = \sin(\alpha - \beta)$, hence

$$[ABCD] = \pm\frac{1}{2}|c - a| \cdot |d - b|\sin(\alpha - \beta)$$

$$= \pm\frac{1}{2}|c - a| \cdot |d - b|\text{Im}(e^{i(\alpha - \beta)})$$

$$= \pm\frac{1}{2}\text{Im}(|d - b|e^{i\alpha} \cdot |c - a|e^{-i\beta}),$$

that is,

$$[ABCD] = \pm\frac{1}{2}\text{Im}\left((d - b)(\overline{c - a})\right).$$

Note that taking $d = c$, we obtain a formula for the area of $\triangle ABC$:

$$[ABC] = \pm\frac{1}{2}\text{Im}\left((c - b)(\bar{c} - \bar{a})\right).$$

We will see these results at work in two examples. First, we again consider the projections B_k of a point M on the sides of a regular n-gon $A_0 A_1 \ldots A_{n-1}$ (with $A_0 = A_n$) and we suppose that the point B_k lies on the segment $A_k A_{k+1}$ for $k = 0, 1, \ldots, n-1$ (see the second problem of the second paragraph). We will show that $S_1 = S_2$ where

$$S_1 = \sum_{k=0}^{n-1} [MA_k B_k] \quad \text{and} \quad S_2 = \sum_{k=0}^{n-1} [MA_{k+1} B_k].$$

We use the notations used earlier and denote by b_k and c_k the affixes of B_k and C_k. Observing that $\Delta M A_k B_k$ and $\Delta M A_{k+1} B_k$ have opposite orientations, proving that $S_1 - S_2 = 0$ amounts to proving that

$$\sum_{k=0}^{n-1} \mathrm{Im}[(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k})] = 0.$$

Now, we have

$$(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k}) = 2(m - b_k)(\overline{c_k} - \overline{b_k})$$
$$= 2(m - b_k)(\overline{c_k} - \overline{m}) + 2|m - b_k|^2$$

and we remark that $(m - b_k)\overline{c_k}$ is a real number (since OC_k is parallel to MB_k). It follows that

$$\sum_{k=0}^{n-1} \mathrm{Im}[(m - b_k)(\overline{w_k} - \overline{b_k}) + (m - b_k)(\overline{w_{k+1}} - \overline{b_k})] = -2 \cdot \mathrm{Im} \left(\overline{m} \sum_{k=0}^{n-1} (m - b_k) \right).$$

But we have $\sum_{k=0}^{n-1} (m - b_k) = \frac{n}{2} \cdot m$ (from $\sum_{k=0}^{n-1} \overrightarrow{MB_k} = \frac{n}{2} \overrightarrow{MO}$ proved earlier) so that

$$-2 \cdot \mathrm{Im}(\overline{m} \sum_{k=0}^{n-1} (m - b_k)) = -n \mathrm{Im}(m\overline{m}) = 0$$

and consequently $S_1 - S_2 = 0$.

Our second example is adapted from a problem set in the *Mathematical Gazette* in 2017:

> Squares are described externally on the sides of a convex quadrilateral *ABCD*. Prove that the line segments joining the centres of opposite squares are perpendicular and that the length of each line segment is $\sqrt{2S + \frac{1}{2}(x^2 + y^2)}$, where S is the area of *ABCD* and $x = AC$, $y = BD$.
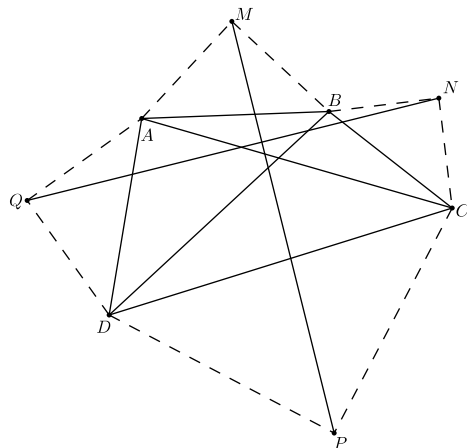
Assuming that $ABCD$ is clockwise oriented and with obvious notations, we have $b-m = i(a-m)$, hence $(1-i)m = b-ia$. Similarly, $(1-i)n = c-ib$, $(1-i)p = d-ic$ and $(1-i)q = a - id$. It follows that $q - n = -i(p - m)$ so that $NQ = PM$ and $NQ \perp PM$. Furthermore,

$$2MP^2 = |(1 - i)(p - m)|^2 = [d - b - i(c - a)] \cdot [\overline{d} - \overline{b} + i(\overline{c} - \overline{a})] = \alpha + i\beta$$

where $\alpha = |d - b|^2 + |c - a|^2 = x^2 + y^2$ and

$$\beta = (d - b)(\overline{c} - \overline{a}) - (\overline{d} - \overline{b})(c - a) = 2i\text{Im}[((d - b)(\overline{c} - \overline{a})] = -4i[ABCD].$$

Finally, $2MP^2 = x^2 + y^2 + 4S$, as desired.

As usual, we end the number with a couple of exercises.

Exercises

1. Let C be a point distinct from the vertices of a triangle OAB. Suppose that ΔOCD and ΔCAE are directly similar to ΔOAB. Prove that $CDBE$ is a parallelogram.

2. Use complex numbers to solve problem **3898**: On the extension of the side AB of the regular pentagon $ABCDE$, let the points F and G be placed in the order F, A, B, G so that $AG = BF = AC$. Compare the area of triangle FGD to the area of pentagon $ABCDE$.
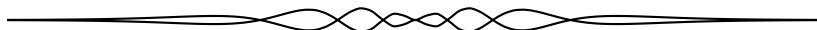
PROBLEMS

To facilitate their consideration, solutions should be received by **October 30, 2019**.

4461. *Proposed by Marian Dinca, Leonard Giugiuc and Daniel Sitaru.*

Let u, v and w be distinct complex numbers such that $\dfrac{w - u}{v - u}$ is not a real number. Consider a complex number $z = \alpha u + \beta v + \gamma w$, where $\alpha, \beta, \gamma > 0$ are real numbers such that $\alpha + \beta + \gamma = 1$. Prove that

$$(|z - v| + |w - u|)^2 + (|z - w| + |u - v|)^2 > (|z - u| + |v - w|)^2.$$

4462. *Proposed by George Apostolopoulos.*

Let a, b, c be the lengths of the sides of triangle ABC with inradius r and circumradius R. Show that

$$\frac{a^2}{b + c} + \frac{b^2}{a + c} + \frac{c^2}{a + b} \leq \frac{3\sqrt{6}R}{4r}\sqrt{R(R - r)}.$$

4463. *Proposed by Max A. Alekseyev*

For all integers $n > m \geq 0$, prove that

$$\sum_{k=0}^{n}(-1)^k \binom{2n + 1}{n - k}(2k + 1)^{2m+1} = 0.$$

4464. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

Let ABC be a triangle with external angle bisectors k, l and m to angles A, B and C, respectively. Projections of A on l and m are L and P, respectively. Similarly, projections of B on m and k are N and K and projections of C on k and l are Q and M. Show that the points M, N, P, Q, K and L are concyclic.

4465. *Proposed by Nguyen Viet Hung.*

Let ABC be a triangle with centroid G and medians m_a, m_b, m_c. Rays AG, BG, CG intersect the circumcircle at A_1, B_1, C_1 respectively. Prove that

$$\frac{\text{Area}[A_1 B_1 C_1]}{\text{Area}[ABC]} = \frac{(a^2 + b^2 + c^2)^3}{(8m_a m_b m_c)^2}.$$

4466. *Proposed by Arsalan Wares.*

Let A be a regular hexagon with vertices $A_k, k = 1, 2, \ldots, 6$. There are two congruent overlapping squares inside A. Each of the squares shares one vertex with A and two vertices of each square lie on opposite sides of hexagon A as in the figure:
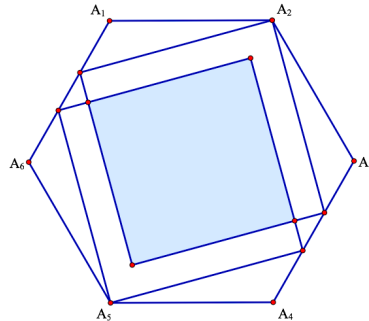
Find the exact area of the shaded region, if the length of each side of hexagon A is 2.

4467. *Proposed by Paul Bracken.*

Show that for $x > 0$,

$$\arctan x \cdot \arctan \frac{1}{x} > \frac{x}{2(x^2 + 1)}.$$

(*Ed.*: Take a look at the problem 4327.)

4468. *Proposed by Florin Stanescu.*

Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) + f'(0) = f(1)$. Show that there exists $c \in (0, 1)$ such that

$$\frac{c}{2} f(c) = \int_0^c f(x) dx.$$

4469. *Proposed by Leonard Giugiuc and Dan-Stefan Marinescu.*

Let ABC be a triangle and let P be an interior point of ABC. Denote by R_a, R_b, R_c the circumradii of the triangles PBC, PCA and PAB, respectively. Prove that $R_a R_b R_c \geq PA \cdot PB \cdot PC$.

4470. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be three distinct complex numbers such that $|a| = |b| = |c| = 1$ and $|a + b + c| \leq 1$. Prove that $|a^2 + bc| \geq |b + c|$.

. . . .

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 octobre 2019**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4461. *Proposé par Marian Dinca, Leonard Giugiuc et Daniel Sitaru.*

Soient u, v et w des nombres complexes distincts tels que $\dfrac{w - u}{v - u}$ n'est pas un nombre réel. Considérons alors un nombre complexe $z = \alpha u + \beta v + \gamma w$, où $\alpha, \beta, \gamma > 0$ sont des nombres réels tels que $\alpha + \beta + \gamma = 1$. Démontrer que

$$(|z - v| + |w - u|)^2 + (|z - w| + |u - v|)^2 > (|z - u| + |v - w|)^2.$$

4462. *Proposé par George Apostolopoulos.*

Soient a, b, c les longueurs des côtés du triangle ABC, où r est le rayon du cercle inscrit et R est le rayon du cercle circonscrit. Démontrer que

$$\frac{a^2}{b + c} + \frac{b^2}{a + c} + \frac{c^2}{a + b} \le \frac{3\sqrt{6}R}{4r}\sqrt{R(R - r)}.$$

4463. *Proposé par Max A. Alekseyev*

Pour entiers $n > m \ge 0$, démontrer que

$$\sum_{k=0}^{n}(-1)^k \binom{2n + 1}{n - k}(2k + 1)^{2m+1} = 0.$$

4464. *Proposé par Borislav Mirchev et Leonard Giugiuc.*

Soit ABC un triangle dont les bissectrices externes des angles A, B et C sont k, l et m respectivement. Les projections de A vers l et m sont L et P, respectivement. De façon similaire, les projections de B vers m et k sont N et K respectivement, et les projections de C vers k et l sont Q et M. Démontrer que les points M, N, P, Q, K et L sont cocycliques.

4465. *Proposé par Nguyen Viet Hung.*

Soit ABC un triangle avec centroïde G et médianes m_a, m_b, m_c. Les rayons AG, BG, CG intersectent le cercle circonscrit en A_1, B_1, C_1 respectivement. Démontrer que

$$\frac{\text{Area}[A_1 B_1 C_1]}{\text{Area}[ABC]} = \frac{(a^2 + b^2 + c^2)^3}{(8m_a m_b m_c)^2}.$$

4466. *Proposé par Arsalan Wares.*

Soit A un hexagone régulier de sommets $A_k, k = 1, 2, \ldots, 6$. Deux carrés congrus se chevauchent à l'intérieur de A. Chacun des carrés partage un sommet avec A et deux sommets de chaque carré se trouvent sur des côtés opposés de A, tel qu'indiqué ci-bas:
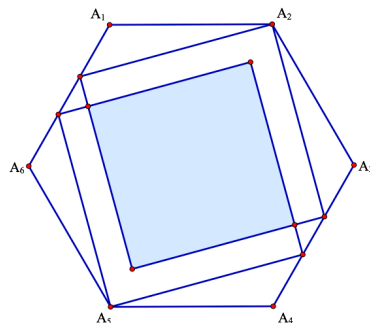
Déterminer la valeur exacte de la surface colorée, si les côtés de A sont de longueur 2.

4467. *Proposé par Paul Bracken.*

Démontrer que pour $x > 0$

$$\arctan x \cdot \arctan \frac{1}{x} > \frac{x}{2(x^2 + 1)}.$$

(*Note*: Voir le probléme 4327.)

4468. *Proposé par Florin Stanescu.*

Soit $f : [0, 1] \to \mathbb{R}$ une fonction différentiable telle que f' est continue et que $f(0) + f'(0) = f(1)$. Démontrer qu'il existe $c \in (0, 1)$ tel que

$$\frac{c}{2}f(c) = \int_0^c f(x)dx.$$

4469. *Proposé par Leonard Giugiuc et Dan-Stefan Marinescu.*

Soit ABC un triangle et soit P un point dans son intérieur. Dénotons par R_a, R_b, R_c les rayons des cercles circonscrits des triangles PBC, PCA, PAB, respectivement. Démontrer que $R_a R_b R_c \geq PA \cdot PB \cdot PC$.

4470. *Proposé par Leonard Giugiuc et Diana Trailescu.*

Soient a, b et c trois nombres complexes distincts tels que $|a| = |b| = |c| = 1$ et $|a + b + c| \leq 1$. Démontrer que $|a^2 + bc| \geq |b + c|$.

Solutions /417

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(1), p. 33–37; 45(2), p. 85–89.
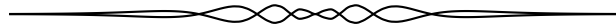
4401. *Proposed by Ruben Dario and Leonard Giugiuc.*

Let D and E be the centres of squares erected externally on the sides AB and AC, respectively, of an arbitrary triangle ABC, and define F and G to be the intersections of the line BC with lines perpendicular to ED at D and at E. Prove that the resulting segments BF and CG are congruent.

We received 11 solutions. We present the solution by Madhav Modak.
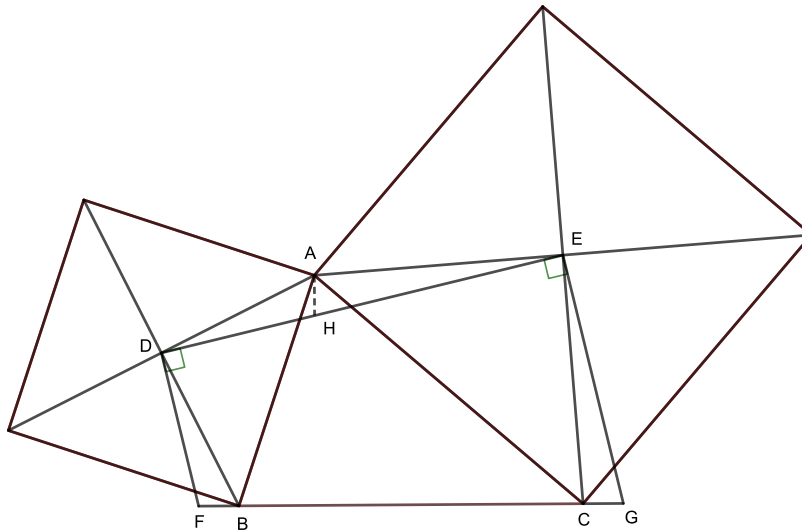
Let H be the point on DE such that $DH = DF$. Since D is the centre of the square on side AB we have $DB \perp DA$ and $DB = DA$. Further, we are given that $DF \perp DH$, and so $\angle ADH = 90° - \angle HDB = \angle BDF$. It follows that $\triangle ADH$ and $\triangle BDF$ are congruent. Hence $FB = HA$. Moreover,

$$\angle AHE = 180° - \angle AHD = 180° - \angle BFD,$$

and using the fact that $DF \| EG$ we conclude that $\angle AHE = \angle ECG$. As before, since $AE \perp EC$ and $HE \perp EG$, we get that $\angle AEH = \angle CEG$. Using the fact that $AE = EC$, we get that $\triangle AEH$ and $\triangle CEG$ are congruent, so $AH = CG$. Therefore, $FB = CG$, as desired.

4402. *Proposed by Peter Y. Woo.*

Consider a rectangular carpet $ABCD$ lying on top of floor tiled with 8 square tiles with side length of 1 foot each (as shown in the diagram).
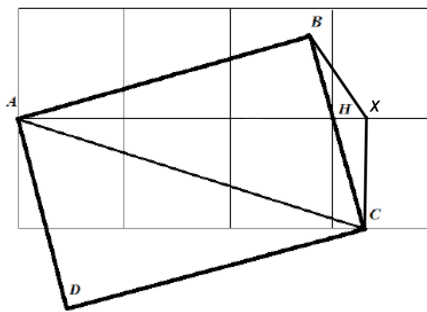
Suppose AH bisects $\angle BAC$. Express $\tan \angle BAH$ as the sum of a rational number and the square root of a rational number.

We received 15 submissions, all correct. We present the solution by Jirapat Kaewkam, enhanced by the editor.

Let CX be perpendicular to the horizontal line l extended from AH with X being on l. Connect CX (see figure). Since $\angle ABC = \angle AXC = 90°$, we see that A, B, X, C are concyclic so $\angle XCH = \angle BAX = \angle XAC$. Hence $\Delta XCH \sim \Delta XAC$ from which it follows that

$$\frac{AX}{CX} = \frac{CX}{HX} \quad \text{or} \quad (AX)(HX) = (CX)^2 = 1,$$

so $(HX + 3)(HX) = 1$. Solving

$$(HX)^2 + 3(HX) - 1 = 0,$$

we then obtain

$$\tan(\angle BAH) = \tan(\angle XCH) = \frac{HX}{CX} = HX = \frac{-3 + \sqrt{13}}{2} = -\frac{3}{2} + \sqrt{\frac{13}{4}}.$$

4403. *Proposed by Michel Bataille.*

Let m be an integer with $m > 1$. Evaluate in closed form

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n+1}{k+1} \frac{k}{m+k}.$$

We received 8 submissions, all of which were correct and complete. We present two solutions.

Solution 1, by the proposer.

Let

$$S_n = \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k}.$$

We show that

$$S_n = \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).$$

Using the well-known identity

$$\sum_{j=k}^{n}\binom{j}{k} = \binom{n+1}{k+1}$$

and changing the order of summation, we obtain

$$S_n = \sum_{j=1}^{n}\sum_{k=1}^{j}(-1)^{k-1}\frac{k}{m+k}\binom{j}{k}.$$

Taking the relation $k\binom{j}{k} = j\binom{j-1}{k-1}$ into account yields

$$S_n = \sum_{j=1}^{n} j \sum_{k=1}^{j}(-1)^{k-1}\frac{1}{m+k}\binom{j-1}{k-1}. \tag{1}$$

Now we use the closed form

$$\sum_{k=1}^{j}(-1)^{k-1}\frac{1}{m+k}\binom{j-1}{k-1} = \frac{(j-1)!}{(m+1)(m+2)\cdots(m+j)}$$

which readily follows from the decomposition of $\frac{1}{(m+1)(m+2)\cdots(m+j)}$ into partial fractions. Back to (1), this leads to

$$S_n = \sum_{j=1}^{n}\frac{j!}{(m+1)(m+2)\cdots(m+j)} = m!\sum_{j=1}^{n}\frac{1}{(j+1)(j+2)\cdots(j+m)}. \tag{2}$$

But we have

$$\frac{1}{(j+1)(j+2)\cdots(j+m)}$$
$$= \frac{1}{m-1}\left(\frac{1}{(j+1)(j+2)\cdots(j+m-1)} - \frac{1}{(j+2)(j+3)\cdots(j+m)}\right).$$

So that the last sum in (2) is telescopic and therefore

$$S_n = \frac{m!}{m-1}\left(\frac{1}{m!} - \frac{1}{(n+2)\cdots(n+m)}\right) = \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).$$

Solution 2, by Paul Bracken, Brian Bradie, Madhav Modak, CR Pranesachar, and Daniel Văcaru, all done independently.

We have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k}$$

$$=\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1} - m\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{1}{m+k}. \qquad (3)$$

Using binomial expansion,

$$0 = (1-1)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k = \sum_{k=-1}^{n}(-1)^{k+1}\binom{n+1}{k+1}$$

$$= -n + \sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}. \qquad (4)$$

Again by binomial expansion

$$(1-x)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k x^k = 1 - (n+1)x + \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{k+1}.$$

It follows that

$$x^{m-2}((1-x)^{n+1} - 1 + (n+1)x) = \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{m+k-1}.$$

Since $m > 1$, we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{1}{m+k} = \int_0^1 \sum_{k=1}^{n}\binom{n+1}{k+1}(-1)^{k-1}x^{m+k-1}\ dx$$

$$= \int_0^1 x^{m-2}((1-x)^{n+1} - 1 + (n+1)x)\ dx.$$

The integral $\int_0^1 x^{m-2}(1-x)^{n+1}\ dx$ is a beta function with the value $\dfrac{(m-2)!(n+1)!}{(m+n)!}$.
Therefore the above integral has the value

$$\frac{(m-2)!(n+1)!}{(m+n)!} + \frac{n+1}{m} - \frac{1}{m-1}.$$

Combining this with the result in (4) into (3) we have

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k+1}\frac{k}{m+k} = n - m\left(\frac{(m-2)!(n+1)!}{(m+n)!} + \frac{n+1}{m} - \frac{1}{m-1}\right)$$

$$= \frac{1}{m-1}\left(1 - \frac{n+1}{\binom{m+n}{m}}\right).$$

4404. *Proposed by Nguyen Viet Hung.*

Let x, y and z be integers such that $x > 0, z > 0$ and $x + y > 0$. Find all the solutions to the equation

$$x^4 + y^4 + (x + y)^4 = 2(z^2 + 40).$$

We received 11 submissions, 9 of which were correct and complete. We present the solution by Brian D. Beasley.

The given equation is equivalent to

$$(x^2 + xy + y^2)^2 = z^2 + 40.$$

Letting

$$n = x^2 + xy + y^2,$$

we note that $n > 0$ and

$$(n + z)(n - z) = 40.$$

Since $z > 0$, this implies

$$(n + z, n - z) = (40, 1), \ (20, 2), \ (10, 4), \ \text{or} \ (8, 5).$$

But

$$(n + z) + (n - z) = 2n$$

must be even, so $2n = 22$ or $2n = 14$, and hence $(n, z) = (11, 9)$ or $(n, z) = (7, 3)$.

If $n = x(x + y) + y^2 = 11$ with $x > 0$ and $x + y > 0$, then $y \in \{0, \pm 1, \pm 2, \pm 3\}$. But none of these values for y will yield an integer value for x.

If $n = x(x + y) + y^2 = 7$ with $x > 0$ and $x + y > 0$, then $y \in \{0, \pm 1, \pm 2\}$. Four of these five values for y yield a positive integer value for x. Thus there are four solutions for (x, y, z) to the original equation:

$$(1, 2, 3), \ (2, 1, 3), \ (3, -1, 3), \ (3, -2, 3).$$

4405. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let ABC be a triangle and let K be a point inside ABC. Suppose that BK intersects AC in F and CK intersects AB in E. Let M be the midpoint of BE, N be the midpoint of CF and suppose that MN intersects BK at P. Show that the midpoints of AF, EK and MP are collinear.

All 7 of the submissions we received were correct, but two of them relied on a computer. We present the solution by Andrea Fanchini.

We use barycentric coordinates with respect to triangle ABC. Working backwards, for the points E and F to be

$$E = CK \cap AB = (m, 1 - m, 0), \qquad F = BK \cap AC = (1 - n, 0, n)$$

(where m, n are parameters with $0 < m, n < 1$), we must have

$$BK : nx - (1-n)z = 0, \quad CK : (1-m)x - my = 0,$$

and, finally,

$$K\left(m(1-n) : (1-m)(1-n) : mn\right).$$

The midpoints of BE and of CF are then

$$M = (m : 2 - m : 0), \qquad N = (1 - n : 0 : 1 + n).$$

The line $MN : (2-m)(1+n)x - m(1+n)y - (1-n)(2-m)z = 0$ intersects BK at P, so that

$$P = \left(m(1 - n^2) : (2-m)(1-n) : mn(1+n)\right).$$

Finally, the midpoints of AF, EK and MP are

$$M_{AF} = (2 - n : 0 : n),$$

$$M_{EK} = (m(mn - 2n + 2) : (1-m)(mn - 2n + 2) : mn),$$

and

$$M_{MP} = (m(mn - n^2 - n + 2) : (2-m)(mn - 2n + 2) : mn(1+n)).$$

Because

$$\begin{vmatrix} 2 - n & 0 & n \\ m(mn - 2n + 2) & (1-m)(mn - 2n + 2) & mn \\ m(mn - n^2 - n + 2) & (2-m)(mn - 2n + 2) & mn(1+n) \end{vmatrix} = 0,$$

it follows that these midpoints are collinear.

Editor's comments. Note that there is no need to restrict K to the interior of the triangle: it could any point in the plane except B or C. In other words, we can allow m and n to be any real numbers except $m \neq 0$ and $n \neq 1$, and the result continues to hold.

It is interesting to compare our problem with an extended version of Hjelmslev's theorem:

> When all the points P on one line are related by a similarity to all the points P' on a second line, then the points X dividing the segments PP' in a fixed ratio $PX : XP'$ are distinct and collinear or else they all coincide.

See, for example, F. G.-M., *Exercices de géométrie*, 4th ed., Theorem 1146d, page 473. Does any reader see an easy direct proof that there exists a similarity that takes the points A, E, M to the points F, K, P ?

4406. *Proposed by Bill Sands.*

Four trees are situated at the corners of a rectangle. You are standing outside the rectangle, the nearest point of the rectangle being the midpoint of one of its sides, 1 metre away from you. To you in this position, the four trees appear to be equally spaced apart.

a) Find the side lengths of the rectangle, assuming that they are positive integers.

b) Suppose that the rectangle is a square. Find the length of its side.
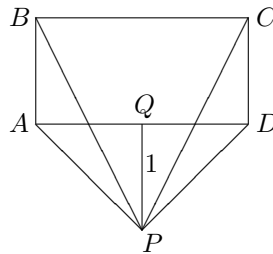
We received 8 correct solutions. We present two different approaches.

Solution 1, by Roy Barbara and C.R. Pranesachar (independently).

Let $\theta = \angle QPC$, so that

$$2\theta = \angle APB = \angle BPC = \angle CPD$$

and $3\theta = \angle QPD$. Let the length of BC be u and the length of CD be $v-1$. Then $\tan\theta = u/(2v)$ and

$$\frac{u}{2} = \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \frac{12uv^2 - u^3}{2(4v^3 - 3u^2v)}.$$

Hence $4v^3 - 3u^2v = 12v^2 - u^2$, whereupon $u^2(3v-1) = 4v^2(v-3)$.

(a) Let u and v be positive integers. Since $3v-1$ and v^2 are coprime, $3v-1$ must divide $4(v-3)$. Because

$$2(3v-1) - 4(v-3) = 2v + 10 > 0,$$

we must have $3v-1 = 4(v-3)$ and $v = 11$. Hence $u = v = 11$ and the respective lengths of CD and BC are 10 and 11.

An alternative argument begins by rewriting the foregoing equation as

$$27u^2 = 36v^2 - 96v - 32 - \frac{32}{3v-1}.$$

The right side is positive and $0 \neq 4v^2(v-3)$, so that v exceeds 3. Since $3v-1$ divides 32, the only possibility is $v = 11$.

(b) If $ABCD$ is square, then $v = u + 1$ and so

$$0 = 4(u+1)^2(u-2) - u^2(3u+2)$$
$$= u^3 - 2u^2 - 12u - 8$$
$$= (u+2)(u^2 - 4u - 4).$$

Hence the sidelength of the square is $u = 2(1 + \sqrt{2})$.

Solution 2, by Daniel Vacaru.

Let the respective lengths of AP, AB and AD be s, t, u. Let $\theta = \angle BPQ$. Since $PQ \| AB$, $\angle ABP = \theta$. Also $\angle APB = 2\theta$ and $\angle APQ = 3\theta$. By the Law of Sines, $\sin 2\theta / t = \sin \theta / s$, whereupon $\cos \theta = t/(2s)$.

Since

$$\frac{1}{s} = \cos 3\theta = 4\cos^3 \theta - 3\cos \theta = \frac{t^3 - 3ts^2}{2s^3},$$

then $(3t+2)s^2 = t^3$. Setting $s^2 = 1 + (u^2/4)$ yields that

$$(3t+2)(4+u^2) = 4t^3 \qquad \Longrightarrow \qquad (3t+2)u^2 = 4t^3 - 12t - 8.$$

(a) Let t and u be positive integers. Rewrite the equation as

$$27u^2 = 36t^2 - 24t - 92 - \frac{32}{3t+2}.$$

Since $3t+2$ divides 32, either $t = 2$ or $t = 10$. The first option leads to $(t,u) = (2,0)$ which is inadmissible, and the second to $(t,u) = (10,11)$.

(b) When $u = t$, the equation becomes

$$0 = t^3 - 2t^2 - 12t - 8 = (t+2)(t^2 - 4t - 4),$$

and the value $2(1+\sqrt{2})$ for the sidelength of the square.

4407. *Proposed by Mihaela Berindeanu.*

Circle C_1 lies outside circle C_2 and is tangent to it at E. Take arbitrary points B and D different from E on the common tangent line. Let the second tangent from B to C_1 touch it at M and to C_2 touch it at N, while the second tangents from D to those circles touch them at Q and P, respectively. If the orthocenters of the triangles MNQ and PNQ are H_1 and H_2, prove that $\overrightarrow{H_1 H_2} = \overrightarrow{MP}$.

We received 6 submissions of which 5 were correct. We present the solution by Oliver Geupel.
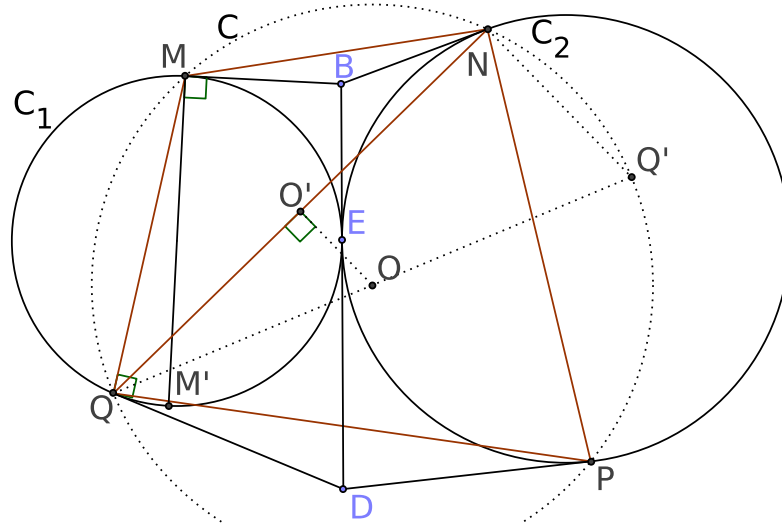
Let point M' be the antipode of M on C_1. We use directed angles ($\measuredangle$) modulo $180°$. In $\Delta QM'M$ and its circumcircle we have

$$\measuredangle QMM' = \measuredangle MQM' + \measuredangle QM'M = 90° + \measuredangle QEM.$$

Since the points E, M, and N all lie on a common circle centered at B, we have that $\measuredangle MEN = \frac{1}{2}\measuredangle MBN$, so

$$\measuredangle BMN = 90° + \measuredangle MEN.$$

Hence,

$$\begin{aligned}
\measuredangle QMN &= \measuredangle QMM' + \measuredangle M'MB + \measuredangle BMN \\
&= 90° + \measuredangle QEM + 90° + 90° + \measuredangle MEN \\
&= 90° + \measuredangle QEN.
\end{aligned}$$

Similarly (replacing Q, M, N, B by N, P, Q, D, respectively),

$$\measuredangle QPN = 90° + \measuredangle QEN.$$

Thus, $\measuredangle QMN = \measuredangle QPN$, which implies that the points M, N P, and Q all lie on a common circle, say, C with center O.

Let O' be the midpoint of the segment NQ. Let point Q' be the antipode of Q on C. The segment $O'O$ joins the midpoints of two sides of the right triangle NQQ'. Hence $2\overrightarrow{O'O} = \overrightarrow{NQ'}$. On the other hand, the lines MQ' and H_1N are both perpendicular to MQ, and the lines $Q'N$ and MH_1 are both perpendicular to NQ. Hence, the quadrilateral $H_1MQ'N$ is a parallelogram, from which we deduce $\overrightarrow{NQ'} = \overrightarrow{H_1M}$. It follows that $\overrightarrow{H_1M} = 2\overrightarrow{O'O}$. Analogously, $\overrightarrow{H_2P} = 2\overrightarrow{O'O}$. We conclude that

$$\overrightarrow{H_1M} = \overrightarrow{H_2P}.$$

Consequently, the quadrilateral H_1MPH_2 is a parallelogram, which proves the desired result.

Editor's comments. The person who submitted the faulty solution misread the problem, labeling the figure so that N and Q are on C_2. Interestingly, the result $\overrightarrow{H_1H_2} = \overrightarrow{MP}$ continues to hold in the modified problem, but because the segment MP is now a chord of C_1, the proof becomes somewhat easier.

4408. *Proposed by Leonard Giugiuc, Dan Stefan Marinescu and Daniel Sitaru.*

Let $\alpha \in (0,1] \cup [2, \infty)$ be a real number and let a, b and c be non-negative real numbers with $a + b + c = 1$. Prove that

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a+b)^\alpha + (b+c)^\alpha + (c+a)^\alpha.$$

We received 4 submissions. One of the submitted solutions was incomplete. We present the proof by Ioannis D. Sfikas.

We give a proof based on the following proposition by Leonard Giugiuc. (See *Hlawka's Inequalities for a class of functions*, Romanian Mathematical Magazine, 2016, by Daniel Sitaru and Leonard Giugiuc.)

Proposition: Let $f(x) : [0, \infty) \to \infty$ be a differentiable function such that $f(0) = 0$ and $f'(x)$ is convex. Then for all nonnegative $x, y, z \in \mathbb{R}$,

$$f(x) + f(y) + f(z) + f(x+y+z) \geq f(x+y) + f(y+z) + f(z+x).$$

Proof of the current problem: Define $f(x) : [0, \infty) \to \mathbb{R}$ by $f(x) = x^\alpha$. Then $f(0) = 0$ and $f'''(x) = \alpha(\alpha - 1)(\alpha - 2)x^{\alpha - 3} \geq 0$ since $(\alpha - 1)(\alpha - 2) \geq 0$ for $\alpha \in (0, 1] \cup [2, \infty)$.

Hence $f'(x)$ is convex. We then have, by the Proposition, that

$$f(a) + f(b) + f(c) + f(a+b+c) \geq f(a+b) + f(b+c) + f(c+a)$$

so

$$a^\alpha + b^\alpha + c^\alpha + 1 \geq (a+b)^\alpha + (b+c)^\alpha + (c+a)^\alpha$$

follows.

Editor's comments: It is easy to find a counterexample to show that the proposed inequality needs not be true if $\alpha \in (1, 2)$; e.g., if $\alpha = \frac{3}{2}, a = b = \frac{2}{5}, c = \frac{1}{5}$, then

$$c^\alpha + b^\alpha + c^\alpha + 1 \approx 2(0.4)^{1.5} + (0.2)^{1.5} + 1 = 1.5954 \dots$$

while

$$(a+b)^\alpha + (b+c)^\alpha + (c+a)^\alpha \approx (0.8)^{1.5} + 2(0.6)^{1.5} = 1.6451 \dots,$$

so LHS < RHS.

4409. *Proposed by Christian Chiser.*

Let A and B be two matrices in $M_2(\mathbb{R})$ such that $A^2 = O_2$ and B is invertible. Prove that the polynomial $P = \det(xB^2 - AB + BA)$ has all integer roots.

We received 6 submissions of which 4 were correct and complete. As stated this problem is false and 4 counterexamples were provided. The solution by Ivko Dimitrić featured here provides a general solution for the roots and when such roots are integer.

The statement is shown to be false by the following counter-example with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

For these matrices, $A^2 = O_2$,

$$xB^2 - AB + BA = \begin{bmatrix} x & 1 \\ 1 & 4x \end{bmatrix}$$

and the roots of $P(x) = 4x^2 - 1$ are non-integers, $x = \pm\frac{1}{2}$.

Nevertheless, it is possible to determine the roots of P in general and examine when they will be integers.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From

$$A^2 = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we see that if $a + d \neq 0$ then $b = c = 0$, which immediately yields also $a = d = 0$ and $a + d = 0$, a contradiction! Thus, tr $A = a + d = 0$, so $d = -a$ and $a^2 + bc = 0$.

Let $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 = -bc$. Then we compute

$$-AB + BA = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} - \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$= \begin{bmatrix} cq - br & b(p - s) - 2aq \\ 2ar - c(p - s) & br - cq \end{bmatrix}.$$

Further,

$$xB^2 = x\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} (p^2 + qr)x & q(p + s)x \\ r(p + s)x & (s^2 + qr)x \end{bmatrix}$$

Hence,

$$xB^2 - AB + BA = \begin{bmatrix} (p^2 + qr)x - (br - cq) & q(p + s)x + b(p - s) - 2aq \\ r(p + s)x - c(p - s) + 2ar & (s^2 + qr)x + (br - cq) \end{bmatrix}$$

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and

$$P(x) = [(p^2 + qr)x - (br - cq)]\,[(s^2 + qr)x + (br - cq)]$$
$$- [q(p + s)x + b(p - s) - 2aq]\,[r(p + s)x - c(p - s) + 2ar].$$

The coefficient of x^2 in this quadratic trinomial is

$$(p^2 + qr)(s^2 + qr) - qr(p + s)^2 = (ps - qr)^2 = (\det B)^2$$

and the coefficient of x is reduced to

$$(p^2 + qr)(br - cq) - (s^2 + qr)(br - cq)$$
$$+ q(p + s)[c(p - s) - 2ar] + r(p + s)[2aq - b(p - s)]$$
$$= (br - cq)(p^2 - s^2) + (p + s)[cq(p - s) - br(p - s)]$$
$$= (br - cq)(p^2 - s^2) - (p + s)(p - s)(br - cq)$$
$$= 0.$$

Finally, the constant term of $P(x)$ equals

$$-(br - cq)^2 + [b(p - s) - 2aq]\,[c(p - s) - 2ar]$$
$$= -(br - cq)^2 + bc(p - s)^2 - 2a(br + cq)(p - s) + 4a^2qr$$
$$= -(br - cq)^2 + (br + cq)^2 - [a(p - s) + (br + cq)]^2 + 4a^2qr$$
$$= -[a(p - s) + (br + cq)]^2 + 4bcqr + 4a^2qr$$
$$= -[a(p - s) + (br + cq)]^2,$$

since $bc = -a^2$. Hence,

$$P(x) = (ps - qr)^2 x^2 - [a(p - s) + (br + cq)]^2$$

has roots

$$x = \pm\,\frac{a(p - s) + (br + cq)}{ps - qr}.$$

The roots are integer if and only if the quotient on the right hand side is an integer, in particular, when $A, B \in M_2(\mathbb{Z})$ and $A^2 = O_2$, $\det B = \pm 1$, but in general the roots are non-integers.

4410. *Proposed by Daniel Sitaru.*

Prove that

$$\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}\,dx < \sqrt{2} - \frac{\pi}{4}.$$

We received 6 correct solutions. There were 10 additional solutions that can be considered weakly correct in that either they obtained a different upper bound for the integral and then gave a numerical argument that this did not exceed the desired

bound, or based estimates on a series expansion. There was one incorrect solution. We present two solutions following different approaches.

Solution 1, by Michel Bataille and Àngel Plaza (independently).

The substitution $u = (\pi/4) - x$ leads to

$$\int_0^{\pi/4} \sqrt{\sin 2x} \, dx = \int_0^{\pi/4} \sqrt{\cos 2u} \, du.$$

From the Cauchy-Schwarz Inequality,

$$1 + \sqrt{\cos 2x} < \sqrt{2}(1 + \cos 2x)^{1/2} = 2 \cos x.$$

Therefore

$$\frac{\pi}{4} + \int_0^{\pi/4} \sqrt{\sin 2x} \, dx = \int_0^{\pi/4} (1 + \sqrt{\cos 2x}) \, dx < 2 \int_0^{\pi/4} \cos x dx = \sqrt{2}.$$

The result follows.

Solution 2, Brian Bradie and Daniel Vicaru (independently).

By the Root-Mean-Square (or the Jensen) Inequality,

$$\frac{1 + \sqrt{\sin 2x}}{2} < \sqrt{\frac{1 + \sin 2x}{2}} = \frac{\cos x + \sin x}{\sqrt{2}} = \sin\left(x + \frac{\pi}{4}\right).$$

Hence

$$\frac{\pi}{4} + \int_0^{\pi/4} \sqrt{\sin 2x} \, dx = \int_0^{\pi/4} (1 + \sqrt{\sin 2x}) \, dx = 2\left[-\cos\left(x + \frac{\pi}{4}\right)\right]_0^{\pi/4} = \sqrt{2},$$

from which the result follows.

4411. *Proposed by Michel Bataille.*

Let n be a positive integer. Find the largest constant C_n such that

$$\frac{(xy)^n}{z^{n+1}} + \frac{(yz)^n}{x^{n+1}} + \frac{(zx)^n}{y^{n+1}} \geq C_n \left(\max(x, y, z)\right)^{n-1}$$

holds for all real numbers x, y, z satisfying $xyz > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$.

We received 4 solutions, 3 of which were correct. We present the solution by Walther Janous.

The two conditions $xyz > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ imply that two of the three variables x, y, and z have to be negative. As the inequality is symmetric, we let $x > 0$.

Then, say, $y = -s$, and $z = -t$, with s and t positive. But then $\max\{x, y, z\} = x$, and $\frac{1}{x} - \frac{1}{s} - \frac{1}{t} = 0$ leads to

$$x = \frac{st}{s+t}.$$

The inequality under consideration is successively equivalent to

$$\frac{(-sx)^n}{(-t)^{n+1}} + \frac{(st)^n}{x^{n+1}} + \frac{(-tx)^n}{(-s)^{n+1}} \geq C_n \cdot x^{n-1}$$

$$\frac{(st)^n}{x^{n+1}} - \left(\frac{s^n}{t^{n+1}} + \frac{t^n}{s^{n+1}} \right) \cdot x^n \geq C_n \cdot x^{n-1}$$

$$\frac{(s+t)^{n+1}}{st} - \frac{s^{2n+1} + t^{2n+1}}{(st) \cdot (s+t)^n} \geq C_n \cdot \left(\frac{st}{s+t} \right)^{n-1}$$

$$\frac{(s+t)^{2n}}{(st)^n} - \frac{s^{2n+1} + t^{2n+1}}{(st)^n \cdot (s+t)} \geq C_n.$$

As the left-hand expression is homogeneous of degree 0, we may and do let $t = 1$, resulting in the inequality

$$\frac{(t+1)^{2n}}{t^n} - \frac{t^{2n+1} + 1}{t^n \cdot (t+1)} \geq C_n$$

for $t > 0$; that is, upon expanding, we have successively

$$\frac{\displaystyle\sum_{j=0}^{2n} \binom{2n}{j} \cdot t^j - \sum_{j=0}^{2n} t^j}{t^n} \geq C_n,$$

$$\frac{\displaystyle\sum_{j=1}^{2n-1} \left(\binom{2n}{j} - 1 \right) \cdot t^j}{t^n} \geq C_n,$$

$$\sum_{j=1}^{n-1} \left(\binom{2n}{j} - 1 \right) \cdot \left(t^{n-j} + \frac{1}{t^{n-j}} \right) + \left(\binom{2n}{n} - 1 \right) \geq C_n.$$

Since $w + 1/w \geq 2$ for all $w > 0$, the left-hand sum attains its least value for $t = 1$. Therefore, the best constant C_n has the value

$$C_n = \frac{(1+1)^{2n}}{1^n} - \frac{1^{2n+1} + 1}{1^n \cdot (1+1)} = 2^{2n} - 1,$$

and the proof is complete.

4412. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with incenter I. If I_a, I_b, I_c are the excenters of ABC, show that $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ if and only if ABC is equilateral.

We received 10 submissions, all of which were correct, and we present the solution by Cristóbal Sánchez-Rubio with some details added by the editor.

The desired result follows quickly from three familiar theorems; each is elementary and easy to prove for an arbitrary triangle ABC (acute or not).

1. For a point P in the plane of triangle $I_aI_bI_c$, $\overrightarrow{PI_a} + \overrightarrow{PI_b} + \overrightarrow{PI_c} = \overrightarrow{0}$ if and only if P is the centroid of $\Delta I_aI_bI_c$.

2. The incenter of the given triangle ABC is the orthocenter of $\Delta I_aI_bI_c$.

3. A triangle is equilateral if and only if its centroid and orthocenter coincide.

Consequently, $\overrightarrow{II_a} + \overrightarrow{II_b} + \overrightarrow{II_c} = \overrightarrow{0}$ if and only if $\Delta I_aI_bI_c$ is equilateral. But if either triangle $I_aI_bI_c$ or ABC is equilateral, its sides would be parallel to the sides of the other, whence the other would also be equilateral.

4413. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let ABC be a triangle with incenter I and circumcircle ω. The lines AI, BI, CI intersect ω a second time at M, N, P, respectively. Also suppose that NP intersects AB and AC at E and F, respectively. We define points G, H, J and D analogously (see the picture). Show that if $EF = GH = JD$, then triangle ABC is equilateral.
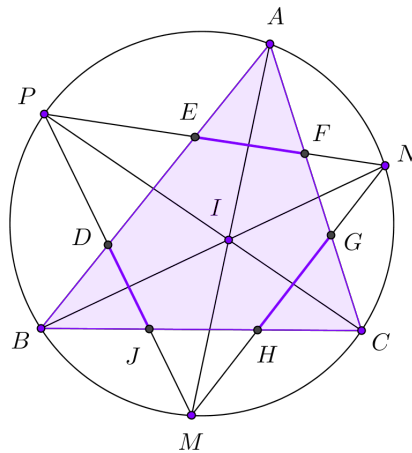
We received 9 submissions, all correct, and present a composite of similar solutions from Prithwijit De and Jirapat Kaewkam, done independently.

Let $T = AI \cap NP$. Observe that in triangle AIP,

$$\angle APT = B/2 = \angle IPT \quad \text{and} \quad \angle IAP = A/2 + C/2 = \angle AIP.$$

Thus, ΔAIP is isosceles and PT bisects its vertex angle, so that $PT \perp AI$ and $AT = TI = AI/2$. Moreover, in ΔAEF the bisector AT of the angle at A is perpendicular to the base, whence $ET = TF = EF/2$. Thus

$$EF = 2ET = 2AT\tan(A/2) = \frac{AI\sin(A/2)}{\cos(A/2)} = \frac{r}{\cos(A/2)},$$

where r is the inradius of triangle ABC. Similarly,

$$GH = \frac{r}{\cos(C/2)} \quad \text{and} \quad JD = \frac{r}{\cos(B/2)}.$$

It follows that $EF = GH = JD$ if and only if $\cos(A/2) = \cos(B/2) = \cos(C/2)$, which (because $0 < \frac{A}{2}, \frac{B}{2}, \frac{C}{2} < 90°$) is equivalent to $A = B = C = 60°$. In other words, ΔABC is equilateral if and only if $EF = GH = JD$.

4414. *Proposed by Konstantin Knop.*

Let α and β be a pair of circles that intersect in points P and Q, and let the diameter AA' of α lie on the same line as the diameter BB' of β such that the end points lie in the order $AB'A'B$. Suppose that PB' intersects α again at the point C, that PA' intersects β again at D, and that the lines AD and BC intersect at R. Prove that the line QR intersects the segment AB at its midpoint.
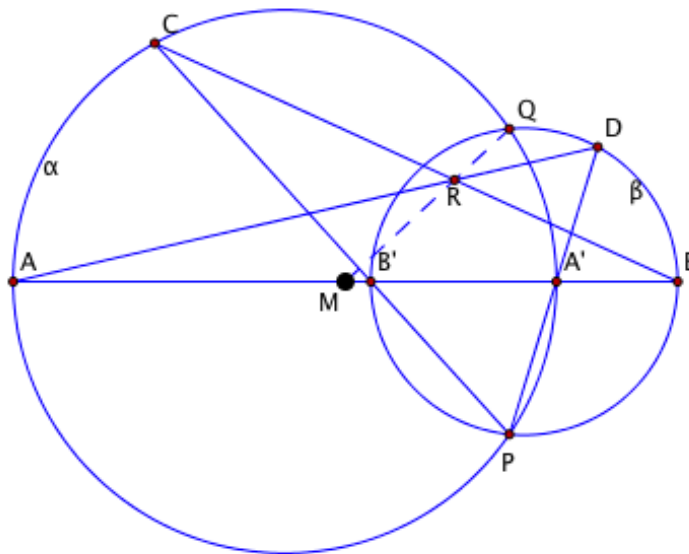
We received 2 solutions to this problem. However, both submissions utilized brute force calculations to achieve the result. We leave the problem open in hopes to receive a more insightful solution. Please email your submissions directly to crux-editors@ cms. math. ca .

4415. *Proposed by Titu Zvonaru.*

Let ABC be an acute-angled triangle with $AB < AC$, where AD is the altitude from A, O is the circumcenter and M and N are the midpoints of the sides BC and AB, respectively. The line AO intersects the line MN at X. Prove that DX is parallel to OC.

We received 13 correct solutions. Of those, 7 gave a synthetic argument; 4 used analytic geometry and 1 used barycentric coordinates.
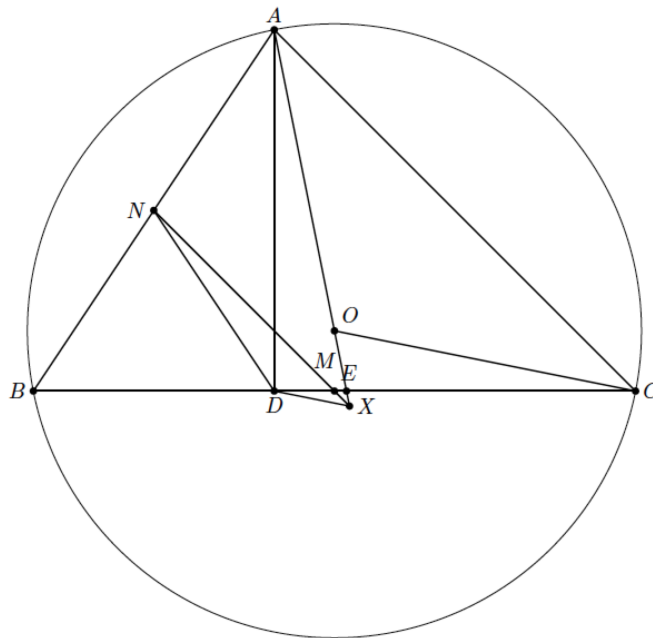
Solution 1, by Dimitrić Ivko.

Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$. Since $\angle AOC = 2\beta$ and $OA = OC$, then $\angle XAC = 90° - \beta$. Since $NM \| AC$, then $\angle NXA = \angle XAC = 90° - \beta$. Since $BN = ND$, then $\angle NDA = 90° - \angle NDB = 90° - \beta$.

Thus, $\angle NXA = \angle NDA$, and so $ANDX$ is cyclic. Hence

$$\angle AXD = 180° - \angle AND = \angle BND = 180° - 2\beta = \angle XOC.$$

It follows that $DX \| OC$.

(*Note from Sushanth Sathish Kumar:* Once $ANDX$ is proved to be cyclic, $DX \| OC$ follows from $\angle NXD = \angle NAD = 90° - \beta = \angle OCA$ and $NX \| AC$.)

Solution 2, by Vijaya Prasad Nalluri.

Let BC and AX intersect at E. Since $NX \| AC$, triangles MEX and CEA are similar, so that $XE : AE = ME : EC$. Since $AD \| OM$, triangles MOE and

DAE are similar, so that $AE : OE = DE : ME$. Hence $XE : OE = DE : EC$. Therefore, triangles XED and OEC are similar, so that $\angle DXE = \angle COE$. This equality of alternate angles implies that $DX \| OC$.

Solution 3, by Prithwijit De.

Assign coordinates: $O \sim (0,0)$, $A \sim (p,q)$, $B \sim (-b,k)$, $C \sim (b,k)$. Then $M \sim (0,k)$ and $D \sim (p,k)$. The equation of the line AO is $y = (q/p)x$. Since MN passes through $(0,k)$ and has the same slope as AC, its equation is

$$y = k + \left(\frac{k-q}{b-p}\right)x.$$

The point X where MN and AO intersect has coordinates

$$\left(\frac{kp(b-p)}{qb-pk}, \frac{kq(b-p)}{qb-pk}\right).$$

The slope of the line DX is

$$\frac{\frac{kq(b-p)}{qb-pk} - k}{\frac{kp(b-p)}{qb-pk} - p} = \frac{kp(k-q)}{pb(k-q)} = \frac{k}{b},$$

which is the slope of AC. The result follows.

4416. *Proposed by Nguyen Viet Hung.*

Let ABC be an acute triangle with orthocentre H. Denote by r_a, r_b, r_c the exradii opposite the vertices A, B, C, and by r_1, r_2, r_3 the inradii of triangles BHC, CHA, AHB, respectively. Prove that

$$r_1 + r_2 + r_3 + r_a + r_b + r_c = a + b + c.$$

We received 9 submissions, all correct, and present the solution by Kee-Wai Lau.

We start with standard formulas for the inradius and an exradius of a triangle ABC in terms of its circumradius R:

$$r = R(\cos A + \cos B + \cos C - 1) \tag{1}$$

and

$$r_a = R(1 + \cos B + \cos C - \cos A). \tag{2}$$

By (2) and the corresponding expressions for r_b and r_c we obtain

$$r_a + r_b + r_c = R(3 + \cos A + \cos B + \cos C). \tag{3}$$

We now show that

$$r_1 = R(\sin B + \sin C - \cos A - 1). \tag{4}$$

Note that the angles of ΔBCH are $\frac{\pi}{2} - C$, $\frac{\pi}{2} - B$, and $B + C$, while its circumradius is

$$\frac{BC}{2\sin(B+C)} = \frac{a}{2\sin A} = R.$$

We get formula (4) by replacing A by $B + C$, B by $\frac{\pi}{2} - C$, and C by $\frac{\pi}{2} - B$ in (1). This, together with the corresponding expressions for r_2 and r_3, gives us

$$r_1 + r_2 + r_3 = R(2\sin A + 2\sin B + 2\sin C - \cos A - \cos B - \cos C - 3). \quad (5)$$

Adding together (3) and (5), we obtain

$$r_1 + r_2 + r_3 + r_a + r_b + r_c = 2R(\sin A + \sin B + \sin C) = a + b + c,$$

as desired.

4417. *Proposed by Dan Stefan Marinescu, Daniel Sitaru and Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $abc \geq 1$. Further, let x, y and z be real numbers such that $xy + yz + zz \geq 3$. Prove that

$$(y^2 + z^2)a + (z^2 + x^2)b + (x^2 + y^2)c \geq 6.$$

We received 4 solutions, 3 of which were correct. We present the solution by Walther Janous.

We shall prove a more general result, namely:

$$\left(y^2 + z^2\right)a + \left(z^2 + x^2\right)b + \left(x^2 + y^2\right)c \geq 2(xy + yz + zx).$$

This will follow if we show that Q, defined by

$$Q = (b+c)x^2 + (c+a)y^2 + (a+b)z^2 - 2xy - 2yz - 2zx$$

is a positive semi-definite quadratic form. But its corresponding symmetric matrix equals

$$M = \begin{bmatrix} b+c & -1 & -1 \\ -1 & c+a & -1 \\ -1 & -1 & a+b \end{bmatrix}.$$

We thus have to check only its three principal minors.

- The inequality $b + c > 0$ is clear.

- Next we have to show that

$$\begin{vmatrix} b+c & -1 \\ -1 & c+a \end{vmatrix} > 0,$$

that is

$$ab + ac + bc + c^2 > 1.$$

By the AM-GM inequality, we get even more:

$$ab + ac + bc + c^2 > ab + ac + bc \geq 3(abacbc)^{1/3} = 3(abc)^{2/3} \geq 3.$$

- Finally, the inequality $\det M \geq 0$ has to be verified; that is,

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc - 2a - 2b - 2c - 2 \geq 0,$$

or equivalently,

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 2abc \geq 2a + 2b + 2c + 2.$$

Because of $abc \geq 1$, it certainly will follow from

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 2a + 2b + 2c. \tag{1}$$

Since

$$\frac{5}{6}\left(a^2b + a^2c\right) + \frac{1}{6}\left(b^2c + bc^2\right) = (b+c)\cdot\left(\frac{5}{6}a^2 + \frac{1}{6}bc\right),$$

the left-hand expression in (1) can be written as

$$(b+c)\cdot\left(\frac{5}{6}a^2 + \frac{1}{6}bc\right) + (c+a)\cdot\left(\frac{5}{6}b^2 + \frac{1}{6}ca\right) + (a+b)\cdot\left(\frac{5}{6}c^2 + \frac{1}{6}ab\right).$$

But the AM-GM inequality yields

$$(b+c)\cdot\left(\frac{5}{6}a^2 + \frac{1}{6}bc\right) \geq 2\sqrt{bc}\cdot\left(a^2\right)^{5/6}\cdot(bc)^{1/6}$$
$$= 2a^{5/3}(bc)^{2/3}$$
$$= 2a(abc)^{2/3} \geq 2a,$$

and two similar inequalities for the other two summands. This completes the proof.

Remark: It is a bit disturbing to have a, b, c limited by the constraint $abc \geq 1$. We shall remove it as follows. Let a, b, and c be arbitrary positive real numbers. Then the three numbers

$$a_1 = \frac{a}{(abc)^{1/3}}, b_1 = \frac{b}{(abc)^{1/3}}, c_1 = \frac{c}{(abc)^{1/3}}$$

satisfy the condition

$$a_1b_1c_1 = 1 \geq 1,$$

whence by what we have already shown,

$$(b_1 + c_1)\,x^2 + (c_1 + a_1)\,y^2 + (a_1 + b_1)\,z^2 \geq 2(xy + yz + zx);$$

that is, there holds even more generally

$$(b+c)x^2 + (c+a)y^2 + (a+b)z^2 \geq 2(abc)^{1/3}(xy + yz + zx)$$

for all positive real numbers a, b, and c and all real numbers x, y, and z. Furthermore, the various applications of the AM-GM inequality show that equality occurs if and only if $a = b = c$ and $x = y = z$.

4418. *Proposed by Daniel Sitaru.*

Consider a convex cyclic quadrilateral with sides a, b, c, d and area S. Prove that

$$\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c} \geq 64S^2.$$

We received 7 correct solutions. We present 5 of them.

We make some preliminary remarks. The formula for the area S of a quadrilateral with sides a, b, c, d and perimeter $2s = a + b + c + d$ is

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta},$$

where θ is half the sum of two opposite angles. This is dominated by the area of a cyclic quadrilateral with the same sides, namely

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$
$$= \frac{1}{4}\sqrt{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}$$
$$= \frac{1}{4}\sqrt{[(a+b)^2 - (c-d)^2][(c+d)^2 - (a-b)^2]}$$
$$= \frac{1}{4}\sqrt{[(a+c)^2 - (b-d)^2][(b+d)^2 - (a-c)^2]}.$$

The statement of the problem remains true for noncyclic quadrilaterals.

Solution 1, by Oliver Geupel.

Let

$$(w, x, y, z) = (s-a, s-b, s-c, s-d).$$

Then

$$(a+b, b+c, c+d, d+a) = (y+z, z+w, w+x, x+y).$$

Applying the arithmetic-geometric means inequality twice, we find that

$$\frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}$$
$$= \frac{(y+z)^5}{w+x} + \frac{(z+w)^5}{x+y} + \frac{(w+x)^5}{y+z} + \frac{(x+y)^5}{z+w}$$
$$\geq 4(y+z)(z+w)(w+x)(x+y)$$
$$\geq 4(2\sqrt{yz})(2\sqrt{zw})(2\sqrt{wx})(2\sqrt{xy})$$
$$= 64xyzw \geq 64S^2.$$

Equality holds if and only if the quadrilateral is a square.

Solution 2, by Šefket Arslanagić.

By the arithmetic-geometric means inequality,

$$S \leq \sqrt{(s-a)(s-b)}\sqrt{(s-c)(s-d)}$$

$$\leq \frac{1}{4}(2s-a-b)(2s-c-d) = \frac{1}{4}(c+d)(a+b).$$

Similarly, $S \leq \frac{1}{4}(b+c)(a+d)$. Therefore

$$64S^2 = 4(16S^2)$$

$$\leq 4(a+b)(b+c)(c+a)(d+a)$$

$$= 4\left[\frac{(a+b)^5}{c+d} \cdot \frac{(b+c)^5}{d+a} \cdot \frac{(c+d)^5}{a+b} \cdot \frac{(d+a)^5}{b+c}\right]^{1/4}$$

$$\leq \frac{(a+b)^5}{c+d} + \frac{(b+c)^5}{d+a} + \frac{(c+d)^5}{a+b} + \frac{(d+a)^5}{b+c}.$$

Solution 3, by C.R. Pranesachar.

By the arithmetic-geometric means inequality,

$$\frac{(a+b)^5}{c+d} + \frac{(c+d)^2}{a+b} \geq 2[(a+b)^2(c+d)^2]$$

$$\geq 2[(a+b)^2 - (c-d)^2][(c+d)^2 - (a-b)^2]$$

$$\geq 32S^2.$$

A similar inequality holds for the other two terms of the left side and the result follows.

Solution 4, by Leonard Giugiuc and Digby Smith, independently. We have:

$$64S^2 = 64(s-a)(s-b)(s-c)(s-d)$$

$$\leq 64\left[\frac{(s-a)+(s-b)+(s-c)+(s-d)}{4}\right]^4$$

$$= 64\left(\frac{2s}{4}\right)^4$$

$$= 4s^4.$$

From an instance of the Hölder inequality, for positive x, y, z, t, m, n, p, q,

$$\left(\frac{x^5}{m} + \frac{y^5}{n} + \frac{z^5}{p} + \frac{t^5}{q}\right)(m+n+p+q)(1+1+1+1)^3 \geq (x+y+z+t)^5,$$

applied to

$$(x, y, z, t; m, n, p, q) = (a + b, b + c, c + d, d + a; c + d, d + a, a + b, b + c),$$

we find that the left side is not less than

$$\frac{2^5(a + b + c + d)^5}{4^3 \cdot 2(a + b + c + d)} = \frac{2^{10} s^5}{2^8 s} = 4s^4 \geq 64S^2.$$

Solution 5, by Walther Janous.

We prove a more general result: *Let $p > q > 0$ and $p + q \geq 1$. Then*

$$\frac{(a + b)^p}{(c + d)^q} + \frac{(b + c)^p}{(d + a)^q} + \frac{(c + d)^p}{(a + b)^q} + \frac{(d + a)^p}{(b + c)^q} \geq 2^{p-q+2} S^{(p-q)/2}.$$

Applying the arithmetic-geometric means inequality to the denominator yields

$$\frac{(a + b)^p}{(c + d)^q} + \frac{(c + d)^p}{(a + b)^q} = \frac{(a + b)^{p+q} + (c + d)^{p+q}}{[(a + b)(c + d)]^q}$$

$$\geq 2^{2q} \cdot \frac{[(a + b)^{p+q} + (c + d)^{p+q}]}{(a + b + c + d)^{2q}},$$

with an analogous inequality for the other two terms on the left side. Using the convexity of x^{p+q}, we see that the left side is not less than

$$2^{2q} \left[\frac{(a + b)^{p+q} + (b + c)^{p+q} + (c + d)^{p+q} + (d + a)^{p+q}}{(a + b + c + d)^{2q}} \right]$$

$$\geq \frac{2^{2q} \cdot 4}{(a + b + c + d)^{2q}} \left[\frac{(a + b) + (b + c) + (c + d) + (d + a)}{4} \right]^{p+q}$$

$$= \frac{2^{2q+2}}{(a + b + c + d)^{2q}} \left[\frac{a + b + c + d}{2} \right]^{p+q}$$

$$= 2^{q-p+2}(a + b + c + d)^{p-q}.$$

On the other hand, from the AM-GM inequality [as in Solution 4],

$$S \leq \frac{(a + b + c + d)^2}{4}$$

whereupon

$$2^{p-q+2} S^{(p-q)/2} \leq 2^{p-q+2} \left[\frac{(a + b + c + d)^{p-q}}{2^{2(p-q)}} \right]$$

$$= 2^{q-p+2}(a + b + c + d)^{p-q}.$$

The result follows.

4419. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90°$. Let D on the hypotenuse BC produced beyond C be such that $CD = CB + BA$. The internal bisector of $\angle ABC$ intersects the line through the midpoints of AB and AC at T. Prove that $\angle TCA = \angle CDA$.

We received 13 solutions. We present the solution by Mihai Miculiţa and Titu Zvonaru.
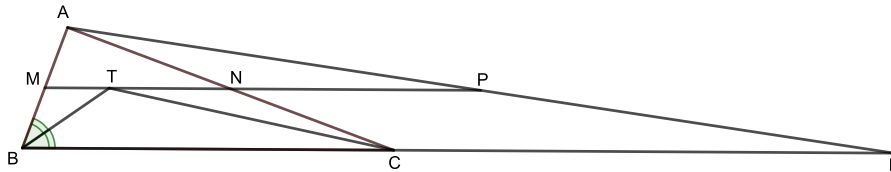
Use a, b and c to denote the lengths of the sides of the triangle. Let M and N be the midpoints of AB and AC, respectively, and P be the intersection of line MN with line AD. Note that $MN \| BC$ and $MN = \frac{a}{2}$; also, P must be the midpoint of AD.

Since BT is the bisector of $\angle CBA$, so $\angle CBT = \angle TBA$, and $MN \| BC$ gives us $\angle MTB = \angle TBC$. So $\triangle MTB$ is isosceles, giving us $MT = MB = c/2$, and hence $NT = MN - MT = \frac{a-c}{2}$.

Since NP joins the midpoints of AC and AD, using $CD = a + c$ we get $NP = \frac{a+c}{2}$.

Thus
$$NT \cdot NP = NA \cdot NC \quad \Longleftrightarrow \quad (a-c)(a+c) = b^2,$$

which holds by the Pythagorean Theorem. We deduce that the quadrilateral $PCTA$ is cyclic, thus $\angle TCA = \angle TPA$.

Finally, since $NP \| CD$ we have $\angle TPA = \angle CDA$, allowing us to conclude $\angle TCA = \angle CDA$.

4420. *Proposed by Leonard Giugiuc and Marian Dinca.*

Let $A_0 A_1 \ldots A_{n-1}$, $n \geq 10$ be a regular polygon inscribed in a circle of radius r centered at O. Consider the closed disks $\omega(A_k), k = 0, \ldots, n-1$ centered at A_k of radius r. Prove that
$$\bigcap_{k=0}^{n-1} \omega(A_k) = \{O\}.$$

*We received 4 solutions. Presented is the one by Walther Janous, lightly edited.*

We show that the statement holds for all $n \geq 3$. First of all, it's clear that O is an element of the intersection under consideration. We now distinguish two cases.

If n is even, then the disks $\omega(A_0)$ and $\omega(A_{n/2})$ are tangent to each other.

If n is odd, let $n = 2k + 1$. We consider $\omega(A_0)$ and its opposite disks $\omega(A_k)$ and $\omega(A_{k+1})$. Suppose w.l.o.g. that A_0 is north of O and the vertices of the polygon are ordered counterclockwise. Then the boundary circles of $\omega(A_0)$ and $\omega(A_k)$ intersect in O (the southernmost point of $\omega(A_0)$ and a point P northwest of O.
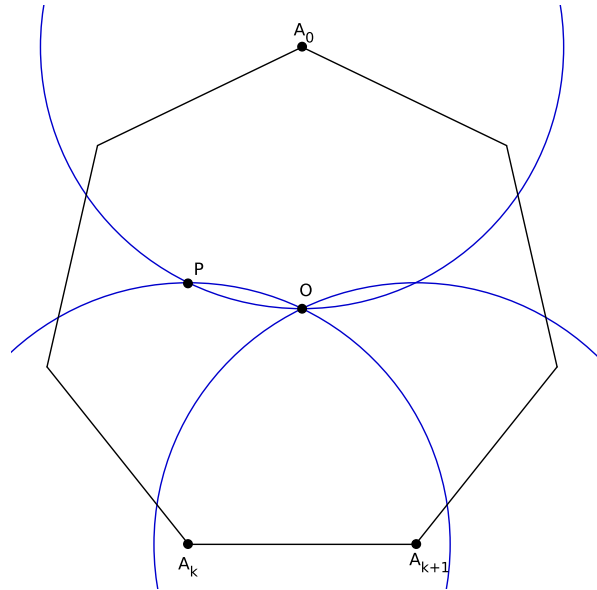
Thus the intersection $\omega(A_0) \cap \omega(A_k)$ is contained in the western hemisphere. By a similar argument the intersection $\omega(A_0) \cap \omega(A_{k+1})$ is contained in the eastern hemisphere. Therefore the intersection $\omega(A_0) \cap \omega(A_k) \cap \omega(A_{k+1})$ consists of only O and the claim follows.
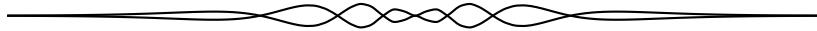
SNAPSHOT

Mathematician Marco Buratti in Val di Rabbi, a subvalley of Val di Sole, proving that certain graphs can survive at the altitude of 3000 meters (see the T-shirt).