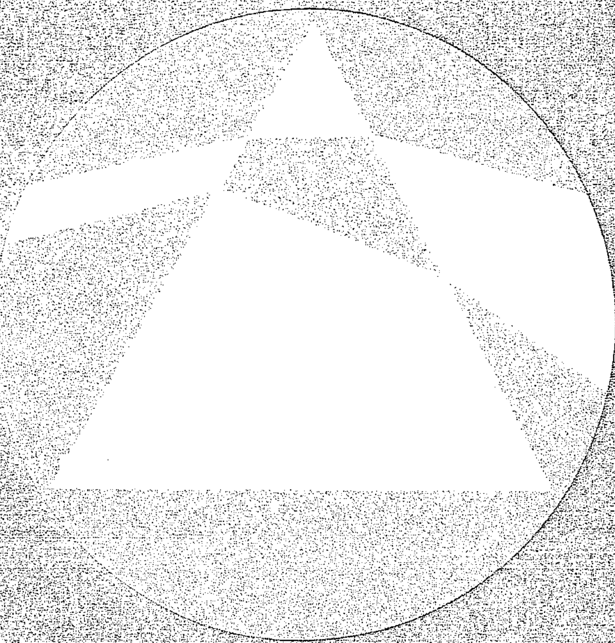
Mathematical Spectrum

1994/5 Volume 27 Number 2



- Georg Cantor, 1845–1918
- Arithmetical progressions
- Reflection problems
- An optimal dartboard

A magazine for students and teachers of mathematics in schools, colleges and universities

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year, consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column. The copyright of all published material is vested in the Applied Probability Trust.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions; prizes are available annually for students as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

All correspondence should be sent to

The Editor, Mathematical Spectrum,
Hicks Building, The University,
Sheffield S3 7RH, UK.

Subscriptions

Current volume

Volume 27, Issues 1, 2 and 3 (postage included) costs
£8.00 or US$ 12.35 or $A 17.20

These prices apply to subscribers in all parts of the world. There is now only one set of sterling rates applicable to all countries.

A discount of 10% will be allowed on all orders for five or more copies of the current volume sent to the same address.

Details of reduced prices for two- and three-year subscriptions are available on request.

Back issues

Most back issues are available; information concerning prices and a list of the articles published may be obtained from the Editor.

* * *

Enquiries about rates, subscriptions and advertisements should be directed to

The Editor, Mathematical Spectrum,
Hicks Building, The University,
Sheffield S3 7RH, UK.

Georg Cantor, 1845–1918
He Transposed Mathematics into a New Key

HAZEL PERFECT

This is a brief account of one of the great pioneers in mathematics of the nineteenth century.

Georg Cantor, the creator of the theory of infinite sets, was born just 150 years ago. Described by Bertrand Russell as one of the greatest intellects of the nineteenth century, Cantor was an imaginative genius whose work has transformed mathematical thought. It is tempting to make a little word-play with his name which, being interpreted, means precentor or leader of the singing: he spent his whole working life (albeit unwillingly) at the University of Halle, birthplace of that master of choral music, George Frederick Handel. We shall be content, however, to say that Cantor sang a new song in mathematics and added infinitely many new notes to the mathematical scale. The theory of sets has led to disquiet and disharmony in the mathematical world, but it continues to lead to new insights as some of the discords are being resolved.

Cantor's parents were both of Jewish descent but professing Christians. His father, a prosperous merchant and broker in St Petersburg, had originally come from Copenhagen, and was a Protestant; his mother, artistic and musical, was Austro-Hungarian and a Roman Catholic. When Cantor was eleven years old the family moved from Russia to Germany, first to Wiesbaden and then to Frankfurt. From an early age Cantor had the great desire to be a mathematician, but his father wanted him instead to study to become an engineer. His sensitivity, and his joy when his father gave way to his wishes, are clear from a letter written in 1862 from the University of Zürich, where he had just begun his University career: 'Now I am happy, seeing that it does not distress you any longer when I choose to follow my feelings. I hope that one day you will be proud of me, dearest father.' His father did not, however, live to see such a day; he died in 1863, and Cantor moved to Berlin University to study mathematics, physics and philosophy. Cantor's early research, and the subject of his doctoral dissertation, was in number theory, where he developed ideas suggested in the works of Gauss. However, it was in his first researches at the University of Halle, where in 1869 he had obtained a post as an unsalaried lecturer, that we can see the beginnings of his life work. Here he was encouraged by his colleague Heine to pursue a problem raised by Riemann concerning trigonometric series. The problem led him to go more deeply into the foundations of mathematical analysis and, in particular, to the construction of the first part of his theory of sets, namely the theory of point sets in Euclidean space. One writer expresses it more dramatically: 'He started with a very special problem of Riemann in very technical mathematics, but it somehow led him, step by step, to the theory of sets and to immortality.'

As we have already indicated, Cantor's whole professional career was spent at the University of Halle, his promotion to a full professorial chair coming in the year 1879. Halle itself was a city with rich cultural traditions, a picturesque town with its medieval Rathaus, the remains of its castle, and its beautiful churches. The plays of Shakespeare were already being performed there in the seventeenth century. Once considered as one of the most Catholic cities in Germany, Halle later became a stronghold of Lutheranism. The name derives from the word Hal meaning salt, in recognition of the salt wells situated in the area. The modern development of Halle began around about 1830 and growth was rapid: from a population of about 28,000 in 1840 it grew to over 200,000 in 1930, that is, in a little over Cantor's lifetime. Partly this was the result of its position as a major railway junction, but also because recent developments made it into a foremost industrial centre for the region in salt, chemicals, iron and steel. The University of Halle (later Halle-Wittenberg) was founded in 1694 and was long recognised as one of the principal seats of Protestant learning. Nowadays it is also noted for its research in applied chemistry. There is no doubt that Cantor felt lonely and somewhat isolated mathematically in Halle, although evidently the mathematicians there met regularly with those from nearby Leipzig. Salaries were comparatively low in Halle, but it was principally that it was not a mathematical centre to be compared with those in Berlin and Göttingen that Cantor repeatedly sought to move to one of these more prestigious universities. He was unsuccessful; and we shall understand something of the reason for this as we look at the nature of his revolutionary work and its reception by the more conservative mathematicians of his day. Cantor had married in 1874, and he evidently finally resigned himself to the fact that he was not to move from Halle when, twelve years later, he bought a magnificent new home for the family in the Händelstrasse there.

One of Cantor's close friends was the mathematician Richard Dedekind from Brunswick. The two men first met in 1872 in Switzerland, which Cantor visited often in his younger days; and they met again in 1874 when Cantor was on his honeymoon. Both were interested in the foundations of mathematical analysis, and both made fundamental contributions to the subject which have profoundly influenced the development of modern mathematical analysis. A recent book on fractals (*Fractals, Chaos and Power Laws*, by M. Schroeder (1990)) is dedicated to Cantor, and traces some of the basic ideas in this computer-age topic to the pathological point sets introduced by Cantor. Dedekind was fourteen years senior to Cantor and there was, to begin with, something of a master-pupil relationship between them which, as Cantor's biographer observes, is evident in their correspondence; and clearly Cantor was much influenced by Dedekind's ways of thinking. Their correspondence over the years also gives insight into their very different working methods: Dedekind, the 'classicist', wrote regularly, was systematic, his style being abstract and analytical; Cantor, the 'romantic', wrote impulsively and sporadically as his mood dictated, and Cantor's moods were liable to change from week to week. In 1881 Cantor's colleague Heine died, and Cantor proposed Dedekind as his first choice for his successor in Halle. However, Dedekind somewhat apologetically declined the offer, giving his reasons as largely financial ones, though he may also have feared a clash of personalities with the more volatile Cantor. Whatever the real reason, it is undoubtedly the case that Halle would have become a much more congenial and stimulating place for Cantor had Dedekind accepted the position. The friendship between the two maybe cooled a little after this incident, and their correspondence lapsed for a number of years. It was resumed for a while a year or two after their meeting at the Zürich mathematical congress in 1897.

Cantor's work with infinite sets and transfinite arithmetic precipitated an upheaval in mathematical thought, and his new ideas and methods did not find ready acceptance among some of his contemporaries. We have already remarked that, seemingly, Cantor's career prospects were adversely affected by this, and probably also his health. Certainly Cantor himself believed that his failure to secure a position in the University of Berlin was largely due to opposition from, among others, the eminent professor, Leopold Kronecker, who had great influence there, and who was deeply suspicious of Cantor's modes of reasoning. Cantor also lost some of his friends. Mathematicians were prepared to accept the notion of infinity in the situation of a sequence 'tending to infinity', but not at first the role of the 'actual' infinities introduced and manipulated by Cantor. For Cantor recognised a whole hierarchy of different infinities, he was able to order them in magnitude, and to perform arithmetic operations with them. As early as the seventeenth century, Galileo had noticed the pairing

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 4 & 9 & 16 & \cdots \end{array}$$

between the elements of the sets $\{1,2,3,4,\ldots\}$ and $\{1,4,9,16,\ldots\}$. We say today that sets whose elements can be so paired are similar, or that they have the same cardinal number. A child who has not yet learned to count can, nevertheless, decide whether he has the same number of sweets in his bag as there are friends at his party by such a pairing-off procedure. The difference for infinite sets is that, as in the example above, the elements of such a set can be paired off with those of a proper subset; and this fact had led Galileo to think that any comparison between infinite cardinal numbers could not be meaningful—they were just infinite. As we shall see, Cantor resolved this. An infinite set which is similar to $\{1,2,3,4,\ldots\}$ is evidently one whose elements can be written down in a list, and is said to be countable. It is not difficult to show that the rational numbers are countable. The algebraic numbers consist of all roots of all polynomial equations with integer coefficients, and so form a much wider set than the rationals; however, they too are countable. The argument usually employed to establish this remarkable fact is Cantor's original one. We associate with the polynomial $a_0+a_1x+\cdots+a_nx^n$ the positive integer $n+|a_0|+|a_1|+\cdots+|a_n|$. There will only be a finite number of polynomials corresponding to each such integer, and so we can write down all the polynomials in a list in a fairly obvious way and then, for each individual polynomial, make a list of its (finite numbers of) roots. Not every real number is algebraic (for instance e and π are not), and the question arises whether the real numbers are countable. Cantor succeeded in proving that they were not. Thus, briefly, if we look only at the real numbers between 0 and 1 and try to write them, in decimal form, in a list

$$\begin{array}{l} 0.a_{11}a_{12}a_{13}\ldots \\ 0.a_{21}a_{22}a_{23}\ldots \\ 0.a_{31}a_{32}a_{33}\ldots \\ \ldots \quad \ldots \quad \ldots \end{array}$$

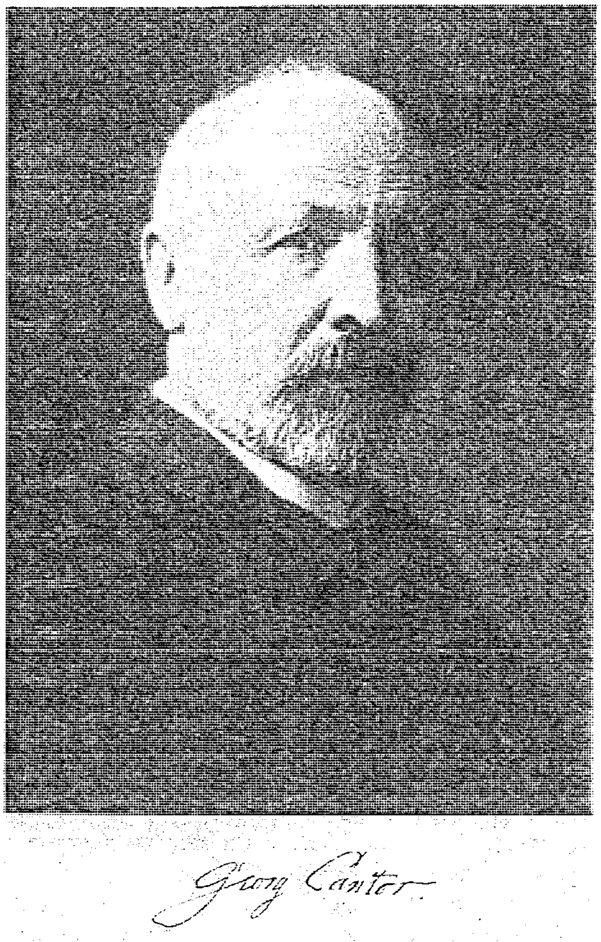
then the number

$$0.b_1b_2b_3\ldots$$

defined by the rule that, for all i,

$$b_i = \begin{cases} 1 & (\text{if } a_{ii} \neq 1), \\ 2 & (\text{if } a_{ii} = 1), \end{cases}$$

evidently does not feature in the list. So the making of a list of all real numbers is impossible. For two sets P and Q we follow Cantor and say that P has cardinal number greater than Q if a proper subset of P is similar to Q but not so P itself. Thus the set of real numbers has cardinal number greater than that of any countable set. Cantor established the existence of infinitely many different infinite cardinal numbers by an incredibly ingenious argument. (That all these cardinal numbers are different from each other is a consequence of the fact that the notion of 'greater than' has the usual properties of an order relation.) Let X be a



set (finite or infinite) and $\mathcal{P}(X)$ the collection of all subsets of X. Obviously the collection of all singleton subsets of X is similar to X. So the cardinal number of $\mathcal{P}(X)$ is greater than or equal to that of X. If it were equal to that of X then there would be a pairing $x \leftrightarrow X_x$ between the elements x and the subsets X_x of X. Suppose such a pairing exists, and let the subset A of X be defined by the rule

$$A = \{x \in X : x \notin X_x\},$$

that is, x belongs to A precisely if it does not belong to the set to which it is paired. Certainly $A = X_a$ for some a in X, since A is a subset of X. Either $a \in A$ which, by the definition of A, means that $a \notin X_a = A$; or $a \notin A$ which, again by the definition of A, means that $a \in X_a = A$. Thus we have a contradiction in either case, and this forces the conclusion that X and $\mathcal{P}(X)$ are not similar, and the cardinal number of $\mathcal{P}(X)$ is greater than that of X. Similarly, the cardinal number of $\mathcal{P}(\mathcal{P}(X))$ is greater than that of $\mathcal{P}(X)$, and so on. We mention just one more result here, this time concerned with point sets. This states that the number of points inside a square is equal to the number of points of an interval. There are some technical problems in making the proof precise, but the basic idea is so simple that it is worth remarking upon. For the unit square $\{(x,y): 0 < x,y < 1\}$ and the unit interval $\{x: 0 < x < 1\}$, essentially a pairing of elements may be obtained by associating each (x,y) of the square, where the decimal representations of x and y are $0.x_1x_2\ldots$ and $0.y_1y_2\ldots$, with the point of the interval with decimal representation $0.x_1y_1x_2y_2\ldots$. (The technical problems arise because of the non-uniqueness of a decimal representation, but are fairly easily resolved.) It is interesting to glean from Cantor's correspondence with Dedekind what were his own initial reactions to some of his results: for instance, his biographer tells us that Cantor discussed with Dedekind the problem of mapping a square on to an interval one to one and, for some considerable time, tried to prove the impossibility of this, as it appeared to be intuitively so implausible. The actual result came as a great surprise to him as it seemed to threaten the whole idea of 'dimension'; he is quoted as saying, 'I see it but I do not believe it'. Only later, when the impossibility of a *continuous* one-to-one mapping was proved, was the status of the notion of dimension restored.

Not without reason had the more conservative of Cantor's contemporaries been uneasy about some of the non-constructive arguments used by Cantor in his revolutionary work; for, during the last years of the century, paradoxes at the heart of the subject began to appear. At about this time Russell and Whitehead were writing their *Principia Mathematica*—an attempt at the logical deduction of all mathematics from set-theoretic premises. In his own words, when he had thought the work almost complete, Russell had an 'intellectual setback'. He had, in fact, become aware of Cantor's own paradox about the set of all sets. We shall describe this as it relates closely to the subset theorem demonstrated on page 26, second column. Let, then, X be the set of all sets. On the one hand, $\mathcal{P}(X)$, the set of all subsets of X (being itself a subset of X) has cardinal number not exceeding X. On the other hand, according to Cantor's theorem, the cardinal number of $\mathcal{P}(X)$ exceeds that of X. This soon led Russell to other paradoxes of a similar kind; and, to quote him again: 'There was something wrong, since such contradictions were unavoidable on ordinary premises. ... Throughout the latter half of 1901 I supposed the solution would be easy, but by the end of that time I had concluded that it was a big job' (Bertrand Russell's *Autobiography*, Volume 1). To some extent, of course, paradoxes are a function of history. Zeno's paradox of the race between Achilles and the tortoise seemed more threatening in the fifth century BC than it does to us today with our greater understanding of the infinitessimal calculus. Likewise, mathematical logicians are coming to terms with the set-theoretic paradoxes by replacing the intuitive theory of sets by a formal axiomatic theory. Cantor himself was, of course, much preoccupied with the paradoxes which his work had generated. He had propounded another problem as well, which continued to tantalise him throughout the rest of his mathematical life. In its simplest form the problem is whether or not there is a cardinal number strictly greater than that of any infinite countable set and strictly less than that of the set of real numbers (the continuum). The so-called continuum hypothesis asserts that there is not. Finding a proof or counterproof of the continuum hypothesis was regarded as a major problem at the turn of the present century and, indeed, featured as the first in the list of famous unsolved problems posed by David Hilbert at the International Congress of Mathematicians in 1900. Cantor was continually frustrated by his own unsuccessful attempts to resolve the problem. If he had been able to foresee what is the status of the problem today he would have understood better his lack of success. Not until about 1964 was its status fully clear, when it was demonstrated that both the continuum hypothesis and its negative are consistent with the usual axioms of set theory. This is a situation similar to that of Euclid's parallel postulate in geometry. Here is a hypothesis that can be neither proved nor disproved on the basis of the other axioms: either the hypothesis or its negative can be adjoined to the list of axioms to form a consistent extended set of axioms.

So what kind of person was Georg Cantor, to whom today's mathematical world owes so much? We can only continue to hint at some aspects of his character and sketch in a little more of his personal story. Physically Cantor was tall and imposing. From his mother he no doubt inherited his love of art and music; occasionally he had voiced regrets that he had not been allowed to become a violinist. Cantor's marriage in 1874 to Vally Guttmann was also the year of publication of his first revolutionary paper on the theory of sets. The marriage was evidently born of deep affection, and his wife's sunny personality was a happy counter to the serious and often melancholy temperament of her husband. They had two sons and four daughters. So Cantor's work was achieved against the background of a happy and stable family life; not, however, untouched by tragedy, for their younger son Rudolf died aged thirteen. Nevertheless, from the age of thirty-nine until his death, Cantor suffered spasmodically from depression and was often in a sanitorium. He was by nature, as we have already indicated, volatile and impulsive and subject to swings of mood; and it is likely that his depression would today be diagnosed as manic. Cantor's preoccupation with the logical problems thrown up by his work, and his fruitless attempts to prove (or disprove) the continuum hypothesis, no doubt exacerbated his condition; and perhaps also so did the bitterness towards those who questioned the validity of his work. Concerning his professional life, we read that his student lectures were sharp, clear, lively and stimulating, and he had a warm and friendly relationship with his students. He was also much sought after at mathematical conferences, where he was able to provide a stream of ideas and problems for his audience. Cantor must have had considerable organisational ability, as the first steps round about 1888 towards the founding of the German Mathematical Union (Deutsche Mathematische Vereinigung) were initiated by him, and he was its first chairman three years later. For Cantor, philosophy was an essential ingredient of mathematics, and he had a considerable knowledge of the older philosophical and theological literature on the infinite. Though he grew up in the Protestant faith, he was deeply influenced by the Catholic atmosphere in his mother's family and always wanted to be able to defend his mathematical ideas against theological objections. The year 1897, when he was fifty-two, saw Cantor's last mathematical publication, and from 1900 he was regularly allowed leave of absence from Halle. His intense intellectual activity had inevitably taken its toll, and he needed to break away from mathematics. He turned his attention to the Bacon–Shakespeare controversy in some of these periods of isolation and depression; and this work was done in no desultory fashion, for he had spent much time over many years on Elizabethan scholarship. Cantor retired from the University in 1913, and in 1918, following a heart attack, he died suddenly and painlessly in a sanitorium.

After the submission of his doctoral dissertation, Cantor was required, as was the custom, to defend the theses propounded in it. In one of these he had written: 'In re mathematica ars proponendi questionem pluris facienda est quam solvendi' (In mathematics, to propose problems is of more value than to solve them). The fundamental research which has continued after Cantor's lifetime has surely justified this philosophy.

Acknowledgement

I am indebted to Mrs Trude Taylor for the translation from German of biographical material about Cantor. □

***Hazel Perfect**, University of Sheffield, is on the advisory board of Mathematical Spectrum and has contributed articles on a number of occasions. Her main mathematical interests are in combinatorics.*

The 1995 puzzle

Readers are invited to try our annual puzzle, to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations of $+$, $-$, $\times$, $\div$, $\sqrt{}$, ! and concatenation (i.e. forming 19 from 1 and 9). For example,

$$1 = 1\times(\sqrt{9}+\sqrt{9}-5).$$

Harmonic Means and Egyptian Fractions

FRANK CHORLTON

Finding all positive integers p and q such that their harmonic mean is a given odd positive integer n involves solving the equation

$$\frac{1}{p}+\frac{1}{q}=\frac{2}{n} \tag{1}$$

for all admissible p and q. Thus $2/n$ is expressed as the sum of a pair of Egyptian fractions, i.e. those which are reciprocals of positive integers (see box). Apart from the trivial solution $p=q=n$ there is always at least one other solution whenever $n>1$, as the following analysis shows.

From equation (1) we have

$$(2p-n)(2q-n)=n^2;$$

and, without loss of generality, we may therefore write

$$2p-n=\frac{na}{b}, \qquad 2q-n=\frac{nb}{a},$$

where a and b are positive integers with $\mathrm{hcf}(a,b)=1$. And so

$$p=n\frac{a+b}{2b}, \quad q=n\frac{a+b}{2a}. \tag{2}$$

Since

$$\mathrm{hcf}(a+b,b)=\mathrm{hcf}(a+b,a)=1,$$

a and b must be divisors of n for integral p and q. This implies that $a+b$ is even. The general solution of (1) is therefore given by (2), where a and b are coprime positive integers which divide n.

For example, for $n=105$, the values $a=3$ and $b=5$ give the solution

$$\tfrac{2}{105}=\tfrac{1}{84}+\tfrac{1}{140}. \qquad \square$$

Egyptian fractions

The ancient Egyptians almost exclusively used 'unit fractions' (those with numerator 1). Thus, in Egyptian hieroglyphics,

$\frac{1}{5}$ appears as 

$\frac{1}{8}$ appears as 

$\frac{1}{20}$ appears as 

The one exception is that

$\frac{2}{3}$ appears as 

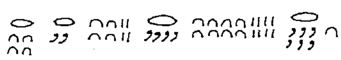
When multiplying $1/n$ (n odd) by 2, the Egyptians had to express $2/n$ as a sum of unit fractions, which were always distinct, so that they did not write

$$\frac{2}{n}=\frac{1}{n}+\frac{1}{n}.$$

A third of the Rhind Papyrus, which is 18 feet (almost 5.5 m) long, is taken up with a table expressing $2/n$ for n an odd number between 5 and 101 as a sum of unit fractions. For example, $\frac{2}{61}$ is expressed as

$$\tfrac{1}{40}+\tfrac{1}{244}+\tfrac{1}{488}+\tfrac{1}{610},$$

or, in hieroglyphics,



Frank Chorlton *retired from the Department of Computing Science and Applied Mathematics at University of Aston in Birmingham some years ago, but still does some part-time teaching there. He has contributed to Mathematical Spectrum on several occasions.*

Sums of Arithmetic Progressions

ROGER COOK and DAVID SHARPE

The article describes which natural numbers can be expressed as a sum of natural numbers in arithmetic progression with a given common difference, and explains how to find such an expression when it is possible. For example, with common difference 1,

$$3 = 1+2, \qquad 5 = 2+3, \qquad 6 = 1+2+3, \quad \text{but} \quad 4 = ?.$$

The results

From time to time, *Mathematical Spectrum* has included items in which readers have expressed powers as sums of consecutive natural numbers. (See L. B. Dutta Volume 17 Number 3 page 15, Bob Bertuello Volume 23 Number 3 page 94 and Volume 24 Number 1 pages 23–24, and Joseph McLean Volume 18 Number 2 page 57 and Volume 23 Number 2 page 60.) We have received yet more submissions on this from the last two of these assiduous correspondents. They had noted that powers of 2 cannot be so expressed. In fact, the question has little to do with powers, so we ask:

> which natural numbers can be expressed as sums of consecutive natural numbers?

Of course, it is always to be understood that there is more than one number in the sum, otherwise it is rather easy; just write $n = n$. A little experimentation should soon convince readers that

> the only natural numbers which cannot be expressed in this way are the powers of 2.

Those who wish to see a formal proof of this and the other results in this article are referred to the reference. Our aim here is to show how such expressions can be found.

Consecutive integers form an arithmetic progression with common difference 1. It is natural to consider common difference 2. Try writing out the numbers 1 to 100 and sieving out $1+3$, $2+4$, $3+6$, ..., $1+3+5$, $2+4+6$, $3+5+7$, ..., $1+3+5+7$, $2+4+6+8$, ..., and you will discover that the primes (and 1) are left. It is in fact the case that

> the natural numbers which cannot be expressed as a sum of natural numbers in arithmetic progression with common difference 2 are the primes (and 1).

By this stage readers may well have been bitten by the bug. What about arithmetic progressions with common difference 3? If we do a similar thing for the numbers 1 to 100, those which are not sums of arithmetic progressions with common difference 3 are

$$1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 10, \quad 14, \quad 16,$$
$$20, \quad 28, \quad 32, \quad 44, \quad 52, \quad 56, \quad 64, \quad 68, \quad 76, \quad 88.$$

There is no pattern here. Or is there? The powers of 2 are there. The others are

$$3,$$
$$2\times3, \quad 2\times5, \quad 2\times7,$$
$$2^2\times5, \quad 2^2\times7, \quad 2^2\times11, \quad 2^2\times13, \quad 2^2\times17, \quad 2^2\times19,$$
$$2^3\times7, \quad 2^3\times11.$$

By stopping at 100, we have somewhat obscured what is happening, but the exceptional numbers seem to be the powers of 2 and powers of 2 times an odd prime lying in a band which looks as though it may get wider as the power gets bigger: thus 2^0p (where $p = 3$), 2^1p (where $p = 3, 5, 7$) and 2^2p (where $p = 5, 7, 11, 13, 17, 19$). It turns out that the excluded numbers for common difference 3 are the powers of 2 and 2^hp, where p is an odd prime such that

$$\tfrac{1}{3}(2^{h+1}+1) < p \leqslant 3(2^{h+1}-1).$$

For example, with $h = 3$ we obtain the excluded numbers 2^3p, where p is prime and $7 \leqslant p \leqslant 43$.

We shall now give the result for general common difference d. It depends whether d is odd or even.

> For odd d, for a natural number $n > 1$ to be expressible as a sum of natural numbers in arithmetic progression with common difference d, we require n to be not a power of 2 and also we need either $n > 2^h(2^{h+1}-1)d$ or $n > \frac{1}{2}p(p-1)d$, where 2^h is the largest power of 2 to divide n and p is the smallest odd prime factor of n. For even d, we require either n to be even and greater than d or odd and satisfying $n > \frac{1}{2}p(p-1)d$, where p is the smallest prime factor of n.

This general result seems a long way removed from the cases $d = 1, 2, 3$. Readers who wish to see how the general result reduces to the results given for the cases $d = 1, 2, 3$ may find this in the reference.

Finding expressions

We now know which integers are expressible in the way required for a common difference d, but how do we find particular expressions?

First consider the case when d is odd. We require n to be not a power of 2 and either $n > (2^{h+1}-1)d$ or $n > \frac{1}{2}p(p-1)d$. In the former case put

$$r = \frac{1}{2}\left(\frac{n}{2^h} - (2^{h+1}-1)d\right).$$

Then

$$\begin{aligned} r+(r+d)+(r+2d)+\cdots+(r+\{2^{h+1}-1\}d) &= 2^{h+1}r+\tfrac{1}{2}(2^{h+1}-1)2^{h+1}d \\ &= 2^h\left(\frac{n}{2^h}-(2^{h+1}-1)d\right)+2^h(2^{h+1}-1)d \\ &= n \end{aligned}$$

and n is a sum of 2^{h+1} terms in arithmetic progression with common difference d, the first term being

$$\frac{1}{2}\left(\frac{n}{2^h} - (2^{h+1}-1)d\right).$$

For example, put $d = 1$ and $n = 1296$. Now $1296 = 2^4\times3^4$, so $h = 4$ and $1296 > 2^4(2^5-1) = 496$. So

$$r = \frac{1}{2}\left(\frac{1296}{2^4} - (2^5-1)\right) = \tfrac{1}{2}(81-31) = 25$$

and

$$1296 = 25+26+27+\cdots+56$$

with 32 terms in the sum. With $d = 3$ and $n = 1552 = 2^4\times97$, $n > 2^4(2^5-1)\times3 = 1488$. So

$$r = \tfrac{1}{2}(\tfrac{1552}{16}-31\times3) = \tfrac{1}{2}(97-93) = 2$$

and

$$1552 = 2+5+8+11+\cdots+95,$$

a sum of 32 terms in arithmetic progression.

With d odd, n not a power of 2 and $n > \frac{1}{2}p(p-1)d$, we put

$$r = \frac{n}{p} - \tfrac{1}{2}(p-1)d.$$

Then

$$\begin{aligned} r+(r+d)+(r+2d)+\cdots+(r+\{p-1\}d) &= pr+\tfrac{1}{2}(p-1)pd \\ &= n \end{aligned}$$

and n is a sum of p terms in arithmetic progression with common difference d, the first term being $n/p-\frac{1}{2}(p-1)d$. For example, put $d = 1$ and $n = 1120 = 2^5\times5\times7$. Then $n > 2^5(2^6-1)$ is not satisfied, but $n > \frac{1}{2}(5\times4)$, so we put

$$r = \tfrac{1120}{5} - \tfrac{1}{2}\times4 = 222$$

and

$$1120 = 222+223+224+225+226,$$

a sum of five consecutive integers. With $d = 3$ and $n = 1120$, we put

$$r = \tfrac{1120}{5} - \tfrac{1}{2}\times4\times3 = 218$$

and

$$1120 = 218+221+224+227+230.$$

It is possible to have $n > 2^h(2^{h+1}-1)d$ and $n > \frac{1}{2}p(p-1)d$ holding simultaneously. For example, take $d = 1$ and $n = 1296$ as above. Then $p = 3$ and $n > \frac{1}{2}\times3(3-1)$, giving the alternative expression

$$1296 = 431+432+433,$$

a sum of a prime number of terms.

With d even, we require either n to be even and greater than d or odd and to satisfy $n > \frac{1}{2}p(p-1)d$, where p is the smallest prime factor of n. In the former case we simply write

$$n = \tfrac{1}{2}(n-d)+\tfrac{1}{2}(n+d)$$

and n is a sum of two natural numbers which differ by d. In the latter case, the same expression as in the corresponding case when d is odd can be used. For example, let $d = 2$, $n = 35$. Then $p = 5$ and $n > \frac{1}{2}p(p-1)d$. So we put

$$r = \tfrac{35}{5} - \tfrac{1}{2}\times4\times2 = 3$$

and

$$35 = 3+5+7+9+11,$$

a sum of five terms.

The result revisited

We can turn the result around. Start with a natural number $n > 1$ and ask for which values of d it can be expressed as a sum of natural numbers in arithmetic progression with common difference d. When n is a power of 2, d must be even. Apart from that, the answer is

> all odd d smaller than
>
> $$\max\left\{\frac{n}{2^h(2^{h+1}-1)}, \frac{2n}{p(p-1)}\right\}$$
>
> and all even d smaller than n when n is even and smaller than $2n/p(p-1)$ when n is odd.

Here, 2^h is the largest power of 2 to divide n and p is the smallest odd prime factor of n.

An annual puzzle

Every year readers of *Mathematical Spectrum* are challenged to express the numbers 1 to 100 in terms of the digits of the year in order. For example, in 1995

$$1 = 1\times(\sqrt{9}+\sqrt{9}-5), \qquad 2 = 1+\sqrt{9}+\sqrt{9}-5, \qquad \text{etc.}$$

As a change, we could ask:

> For which values of d can the year be expressed as a sum of natural numbers in arithmetic progression with common difference d and, when it is possible, find such an expression.

All the machinery needed to answer this question has been given in this article. To start you off,

$$\begin{aligned}1995 &= 997+998\\ &= 664+665+666\\ &= 663+665+667\\ &= 996+999\\ &= 662+665+668\\ &= 661+665+669,\end{aligned}$$

and so on.

The more interesting years are those with large powers of 2 dividing them and with their smallest odd prime factor being large.

Reference

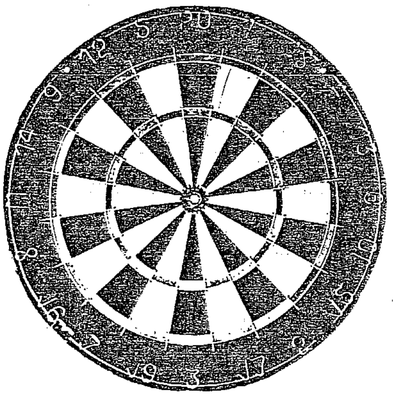
1. R. J. Cook and D. W. Sharpe, Sums of arithmetric progressions, *The Fibonacci Quarterly*, to appear. □

The authors have the good fortune (or is it misfortune!) to be editors of Mathematical Spectrum; they also teach at the University of Sheffield in their spare time.

Optimal Arrangements for a Dartboard

P. J. EVERSON and A. P. BASSOM

Have you ever wondered why the numbers on a dartboard are arranged in the way that they are? Read on.



Here we investigate one aspect of the design of a dartboard. This problem came to light during the preparation for a Royal Institution Masterclass session at the newly formed Exeter Centre in the Mathematics Department at the University of Exeter. The session itself was to cover the general area of mathematics in sport; the game of darts has many potential points of mathematical interest. One question which had to be addressed was 'Why are the numbers on the standard dartboard chosen to be where they are?'. An answer to this is very hard to find. This article examines all possible rearrangements with the aim of maximising the sum of the moduli of the difference between adjacent numbers, an idea which was first considered in reference 1. The larger this number, which we shall henceforth call the 'board number', the more a poor shot is penalised. Some properties of the board number are discussed here for a general board with m numbers, and it is necessary to consider the cases of m odd or even separately.

Case 1. The number of sections is even: $m = 2N$

Suppose we have a dartboard with $2N$ compartments labelled n_i $(i = 1,2,\dots,2N)$ with each $n_i \in \{1,\dots,2N\}$ and $n_j \neq n_k$ when $j \neq k$. For the convenience of labelling in the following, wherever n_{2N+1} is referred to the value $n_{2N+1} = n_1$ is implied.

We are interested in maximising the difference

$$D = \sum_{i=1}^{2N} |n_i - n_{i+1}| = \sum_{i=1}^{2N} \pm(n_i - n_{i+1}), \qquad (1)$$

where the $\pm$ sign is determined by whether $n_i > n_{i+1}$ or $n_{i+1} > n_i$. For any particular n_j this term appears twice in the above formula and may be associated with a plus sign twice, a minus sign twice, or maybe once with a plus and once with a minus. Therefore the coefficient of n_j may be $+2$, -2 or zero and so

$$D = 2\sum_{S_1} n_i - 2\sum_{S_2} n_i + 0\sum_{S_3} n_i,$$

where S_1, S_2, S_3 is a partition of $\{1,\dots,2N\}$. We note here that within (1) there are equal numbers of positive and negative coefficients so that the number of members of set S_1 (i.e. those n_j associated with coefficient $+2$) must be the same as that of S_2.

Our first task is to maximise D *without proving whether or not our maximum is realisable by construction of a suitable dartboard.* We assert that to achieve the maximum of D the set S_3 must be empty. For suppose not. Then since $|S_1|+|S_2|+|S_3| = 2N$ and $|S_1| = |S_2|$, $|S_3|$ must be even. If S_3 is not empty take any two members of this set, say n_α and n_β. Now these two numbers must be different, so take $n_\alpha > n_\beta$ without loss of generality. Then it is clear that the value of D may be increased by adding n_α to the set S_1 and n_β to the set S_2. Thus the maximum possible D is not realisable whilst S_3 contains members and so S_3 must be empty to achieve this maximum.

The consequence of this result is that we must have $|S_1| = |S_2| = N$ and so the required maximal value of D is obtained by considering the problem

$$D = 2\sum_{S_1} n_i - 2\sum_{S_2} n_i, \qquad (2)$$

with

$$S_1 \cup S_2 = \{1,2,\ldots,2N\}, \qquad S_1 \cap S_2 = \varnothing,$$
$$|S_1| = |S_2| = N.$$

To maximise D it is clear that we wish to maximise $\sum_{S_1} n_i$ and minimise $\sum_{S_2} n_i$. Now it is obvious that

$$\sum_{S_1} n_i \leqslant \sum_{r=N+1}^{2N} r, \qquad \sum_{S_2} n_i \geqslant \sum_{r=1}^{N} r,$$

and so by (2)

$$D \leqslant 2\sum_{r=N+1}^{2N} r - 2\sum_{r=1}^{N} r$$
$$= 2\{\tfrac{1}{2}N(3N+1)\} - 2\{\tfrac{1}{2}N(N+1)\},$$

whence $D \leqslant 2N^2$.

Although it appears that the maximum board number across all conceivable dartboards is $2N^2$, we must next prove that not only is this the theoretical maximum but also that it is attainable. This task is best accomplished by a concrete example and it is a simple calculation that the choice

$$n_{2i-1} = 2N+1-i, \qquad n_{2i} = i \qquad (i = 1,2,\ldots,N)$$

represents a possible configuration of numbers which yields a dartboard with board number $2N^2$.

Given the formulation above it is then a straightforward problem to generalise the problem at hand in order to find the number of arrangements of numbers which give the maximum board number. Equation (2) demonstrates that the maximum board number is achieved whenever S_1 consists of $N+1, N+2, \ldots, 2N$ and S_2 comprises $1,2,\ldots,N$. This is the case if and only if the elements of S_1 are interlaced with those of S_2. Without loss of generality we arbitrarily fix the position of the sector of the dartboard numbered 1. Then the required interlacing may be obtained by $N!\,(N-1)!$ distinct combinations, for the elements of S_1 may be ordered in $N!$ ways and the elements of S_2 (excepting 1 which has been fixed) ordered in $(N-1)!$ ways.

Case 2. The number of sections is odd: $m = 2N+1$

The analysis here closely mimics that of the previous case, with some slight adaptations. We outline the required modifications, but the details are omitted for brevity.

If $m = 2N+1$ then again write

$$D = 2\sum_{S_1} n_i - 2\sum_{S_2} n_i + 0\sum_{S_3} n_i,$$

and arguing as before leads to the conclusion that $S_1 \cup S_2 \cup S_3 = \{1,2,\ldots,2N+1\}$ and $|S_1| = |S_2|$. Then S_3 contains an odd number of terms and if S_3 contains three or more elements then we can always increase D by taking members from S_3 and adding to S_1 and S_2 as described in the even case above. Now, since there are an odd number of terms in S_3, S_3 must always contain one element as opposed to its being empty as before. It is now clear that to maximise D we argue that

$$\sum_{S_1} n_i \leqslant \sum_{r=N+2}^{2N+1} r, \qquad \sum_{S_2} n_i \geqslant \sum_{r=1}^{N} r,$$

and thence we conclude that $D \leqslant 2N(N+1)$. That this maximum is attainable may be verified by considering any permutation of the n_j in which the elements $\{1,2,\ldots,N\}$ separate elements $\{N+2, N+3, \ldots, 2N+1\}$ and the position of the middle element $N+1$ is immaterial. This construction also leads to the number of distinct dartboard arrangements which lead to the maximum board number $2N(N+1)$ and, after fixing the position of the sector 1 (without loss of generality) the remaining elements of $\{1,2,\ldots,N\}$ may be permuted within themselves in $(N-1)!$ ways and the elements $\{N+2, N+3, \ldots, 2N+1\}$ in a further $N!$ ways. Thus these sets may be interlaced in $N!\,(N-1)!$ ways but the element $N+1$ may be inserted anywhere within the $2N$ terms of the sequence without affecting the value of D. Thus, in total, D may be maximised by $2(N!)^2$ essentially different board arrangements.

We now briefly consider the implications of the above for a standard board with 20 numbers. In this case there are a total of 19! possible arrangements (19! = 121 645 100 408 832 000). Before realising how large this number is an attempt was made to determine the value of

$$\sum_{i=1}^{20} |n_i - n_{i+1}|,$$

for all possible boards. Just to give an idea of the size of this apparently innocuous problem, even if we had a computer capable of calculating the board numbers for 10^6 different arrangements every second, it would still take almost 4000 years to complete the search through all the possibilities.

Surprisingly, even though there are 10! 9! = 1316 818 944 000 arrangements that would give the maximum board number for a 20-sector dartboard, the conventional dartboard in everyday use does not attain the theoretical maximum of 200 but has a board number of 198. Inspection of the dartboard readily reveals the 'defect' which leads to this. We recall that, to achieve the maximum, the numbers 11–20 should interlace those of 1–10, but on the standard board 11 and 14 are adjacent and so too are 6 and 10. If these 'imperfections' were removed (for example by inserting 14 between 6 and 10) then the maximum would be achieved.

The method of solution used here is of course only one way of defining the difficulty of a board. Examples of other measures that could be used include examining the squares of the differences of adjacent sectors or by not only considering the immediate neighbours to any particular sector but accounting also for near neighbours.

Considering the standard board once more, top professional players always concentrate on the treble 20 when aiming for a high score. However, if you are playing at a lower standard, with a much lower degree of accuracy, this may well not be the best tactics in order to maximise your expected score. With the sector 1 immediately alongside 20, small errors are frequently harshly penalised. For the inexpert player, choosing to aim at section 14, which has 9 and 11 as adjacent sectors, could well give a better long-term result. This is, of course, the area in which the standard dartboard fails to conform to the requirements of achieving the maximum board number. Ideas of probability can obviously be used to answer questions such as these, and the Masterclass session on this topic extended the work in this way.

In the derivation of the results here only the single numbers were considered, but of course in practice the board is more complex, containing doubles, trebles and the inner and outer bull. In a fuller investigation these complications should be accounted for, especially considering the fact that players concentrate their aim on the treble ring when attempting to maximise their score.

References

1. K. Selkirk, Redesigning the dartboard, *Mathematical Gazette* **60** (1976), 171–178. □

*Both **Phil Everson** and **Andrew Bassom** are lecturers in applied mathematics at the University of Exeter: their main research interests are in areas of fluid mechanics. Outside work Andrew Bassom spends much of his time dinghy sailing whilst Phil Everson has completed several half-marathons.*

Experiments with Infinite Exponents

This note seeks to bring together three of the contributions to the last issue of *Mathematical Spectrum* (Volume 27 Number 1). On page 22 a few tantalising pieces of information are given regarding the sequence

$$S_0(x) = x, \quad S_1(x) = x^x, \quad S_2(x) = x^{x^x}, \quad \ldots .$$

This sequence is more neatly expressed as

$$f_0 = \alpha, \quad f_{n+1} = \alpha^{f_n} \quad (n \geqslant 0),$$

which is a special case of

$$a_0 = \beta, \quad a_{n+1} = \alpha^{a_n} \quad (n \geqslant 0).$$

Here x and α must be positive: only real sequences are being considered. These sequences provide excellent material for using graphic calculators and (I hope) for classroom ideas (see the Mathematics in the Classroom item on page 17 of Volume 27 Number 1). The article on page 11 about programming your computer to play music provides an alternative way of using BASIC, on a BBC Micro or IBM PC. Line 60 of Pete Dixon's program is the key one to modify. Keeping the musical effects is not a bad idea.

Try a few values of α and β and see what happens. Only three assertions were made about $S_n(x)$, namely:

1. $S_n(x)$ converges for $0 < x \leqslant e^{1/e}$;
2. $S_n(e^{1/e}) \to e$ as $n \to \infty$;
3. $S_n(x)$ diverges for $x > e^{1/e}$.

You should discover that one of these statements is false (in fact the second one is correct).

The inverse sequence

$$b_0 = \gamma, \quad b_n = \alpha^{b_{n+1}} \quad \left(\text{or } b_{n+1} = \frac{\ln b_n}{\ln \alpha}\right) \quad (n \geqslant 0)$$

is also worthy of investigation. None of these sequences will give you the cacophony of Pete Dixon's program, but neither should you get a monotone for the ultimate behaviour.

If your investigations run along similar lines to mine, then you should be able to find both roots of

$$2^x = 5x - 1$$

by rather simple numerical integration. Also, the following result is needed at one stage:

$$\text{if } \frac{e^x}{x} = \frac{e^y}{y}, \quad \text{where } y > x > 0, \quad \text{then } xy < 1.$$

I hope that the above will encourage you to try your own experiments. In the next issue of *Mathematical Spectrum* there will be a full account, with explanations.

Glenn Vickers
(University of Sheffield)

Reflection Problems

I. M. RICHARDS

It is suprising where you can get, starting with a billiard table.

Introduction

This article was prompted by David Sharpe's article in *Mathematical Spectrum* (reference 2), entitled 'The strange billiard table'. That article relates the behaviour of a ball struck at 45° to the sides, from the corner of an m by n billiard table, as shown in figure 1. Let $d = \text{hcf}(m, n)$. The ball shown projected in figure 1 will eventually arrive in

pocket B, if m/d is even and n/d is odd,
pocket C, if m/d is odd and n/d is odd,
pocket D, if m/d is odd and n/d is even.

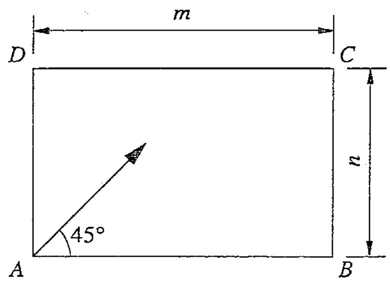


Figure 1

I too have considered this problem and seeing it again has put me in mind to assemble some of the ideas I have had about problems of reflection, whether those of a ball on a billiard table or a light ray bouncing between mirrors, where the angle of incidence is equal to the angle of reflection.

A principle of reflection problems

It is often useful in reflection problems, when the ray meets the mirror, to reflect the mirror system and not the ray, which is permitted to continue in a straight line. Here below is a simple problem to illustrate this.

Problem. Figure 2 shows a ray bouncing between mirrors M_1 and M_2. What is the shortest distance between the path of the ray and the intersection of the mirrors at A?

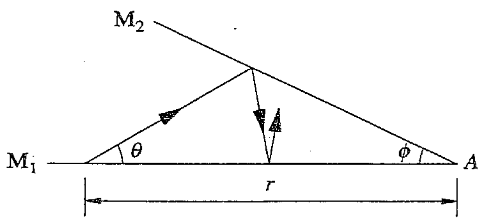


Figure 2

Solution. When the ray first strikes M_2, we reflect not the ray but M_1 in M_2, as shown in figure 3. By symmetry, the angle that the continuing line makes with M_2 is the same as that of the reflected ray, shown by the dotted line. So the behaviour of the continuing ray is exactly that of the reflected ray, but in mirror image. Other angles and distances are preserved by the reflection. We can repeat this process indefinitely, as shown in figure 4.

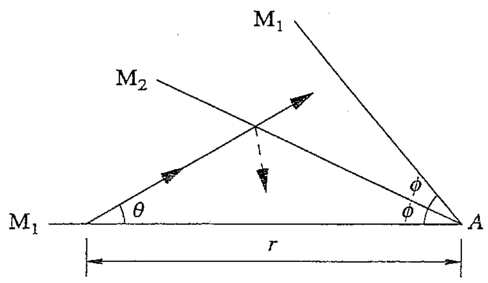


Figure 3

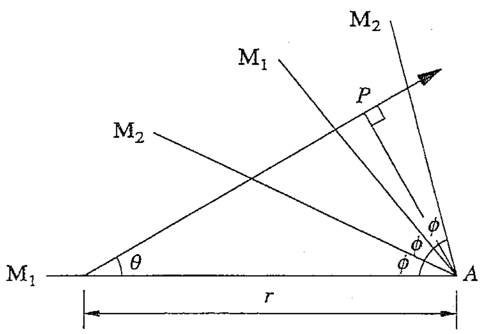


Figure 4

In figure 4, the point P is the point of closest approach of the ray's path to A. It is now obvious that $PA = r \sin \theta$.

If this problem were done by following the reflected ray it would be rather more difficult. A similar method may be applied to provide a neat proof of the 'strange billiard table' result quoted above. Readers may like to construct a solution along these lines.

To prove that $\sqrt{2}$ is irrational through a billiard shot

I felt sure that the strange billiard table could be used to prove other results. This below is the only one that I have thought of, but it amused me to use a billiard shot to prove a mathematical result.

Suppose that $\sqrt{2}$ is rational. Then there exist integers p and q, with a highest common factor of 1, such that $p/q = \sqrt{2}$. Therefore $p^2/q^2 = 2/1$. This means that the proportions of a p^2 by q^2 table are the same as those of a 2 by 1 table (coincidentally the proportions of a standard billiard table). We now perform the shot in figure 5. An unremarkable shot, but the result would have been the same on a p^2 by q^2 table. Therefore, by the strange billiard table result, p^2 is even and q^2 is odd. Since p is even, $p^2 \equiv 0 \pmod 4$. But $p^2 = 2q^2$ and $2q^2 \equiv 2 \pmod 4$, q^2 being odd. Therefore $p^2 \equiv 2 \pmod 4$. This contradiction implies that $\sqrt{2}$ cannot be rational.

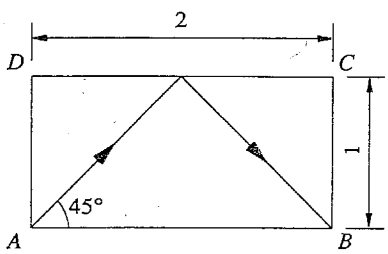


Figure 5

Beatty's theorem

There is a strikingly elegant theorem of S. Beatty to be found in R. Honsberger's book *Ingenuity in Mathematics* (reference 1), an outstanding collection of mathematical essays. Beatty's theorem tells us more about the strange billiard table but in that context it is an instance of a more general result. Hereafter $[x]$ denotes the integer part of x.

Beatty's theorem. Let x be a positive irrational number. Set $a = 1+x$ and $b = 1+1/x$. The two sequences

$$[a],\ [2a],\ [3a],\ [4a],\ \ldots,$$

$$[b],\ [2b],\ [3b],\ [4b],\ \ldots$$

are complementary, that is, these sequences, if combined, list the positive integers exactly, each integer occurring once.

Consider now a $1+x$ by $1+1/x$ table upon which a ball is projected at 45° to the sides, from a corner vertex. As remarked earlier, we may allow the ball's path to pass unreflected but reflect the table instead. The result is a picture of the ball's path passing across a grid as in figure 6. Note that on this table, in which the sides are in the ratio of 1 to x, the ball makes infinitely many incidences, unlike the table of rational proportions.

We mark off a node on the ball's path every $\sqrt{2}$ units, so that in moving from one node to the next, we move one unit vertically and one horizontally. Now take an incidence of the ball with a vertical line, such as that at P. The horizontal distance travelled is clearly a multiple of a, say qa. Since the nodes are separated by a horizontal distance of 1, then $[qa]$ is the number of nodes through which the ball has passed so far. We can now interpret the Beatty sequence $[a]$,

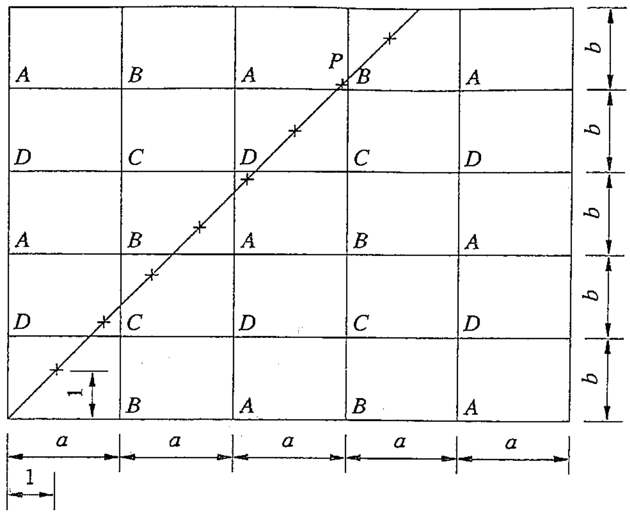


Figure 6

$[2a]$, $[3a]$, $[4a]$, ... It gives the numbers of nodes through which the ball has passed prior to the incidences with the verticals. Likewise, $[b]$, $[2b]$, $[3b]$, $[4b]$, ... gives the numbers of nodes through which the ball has passed upon its various incidences with the horizontals. But, by Beatty's theorem, these sequences include each integer once and once only. So, as we pass from one incidence to the next, we must pass through exactly one node in order that we pass from one integer to the next in the interlocking sequences. What this means is that the nodes interlace with the incidences. Between every two incidences there is exactly one node.

This provides an answer to the question, 'Exactly how far should one permit the ball to travel in order to ensure exactly n incidences?' This question would be untidy to answer with an exact formula for the distance up to the nth incidence, but we can now state a neat result as follows.

> On a $1+x$ by $1+1/x$ table, if the ball travels a distance $(n+1)\sqrt{2}$, then it will make n incidences. (Here the angle of projection is 45° and x is irrational.)

Note that the factor $n+1$ occurs rather than n because there is no incidence before the first node. This result is remarkable because the incidences occur irregularly, sometimes close together, sometimes further apart. Indeed it is quite easy to prove that incidences occur arbitrarily close to one another, and yet this result manages to separate all the incidences!

One might think that this result is not very general in referring to a $1+x$ by $1+1/x$ table, but we can apply to figure 6 a scaling factor of $x/(x+1)$. The side lengths then become x and 1 and the distance between nodes $[x/(x+1)]\sqrt{2}$. It is now clear that we may restate the above result.

> On an x by 1 table, if the ball travels a distance $(n+1)[x/(x+1)]\sqrt{2}$, then it will make n incidences. (Here the angle of projection is 45° and x is irrational.)

The restriction that x is irrational may be omitted. If x is rational, then the ball eventually ends up in a pocket at a vertex, but up to that point the same rule applies. We can generalise further by varying the angle of projection. If the angle of projection is α to the side of length x, we can apply a scaling factor of $\tan\alpha$ parallel to that side, to restore the angle of projection to 45°. We then read off the distance between the nodes and rescale to restore the angle of projection. The details are left to the reader to fill in, but it should be found that the most general form of the result is the following.

If the ball is projected at an angle α to the side of length x, on an x by y table, then if the ball travels

$$(n+1)\frac{xy}{x\sin\alpha+y\cos\alpha},$$

it will make n incidences, provided that, in the case where $x\tan\alpha/y$ is rational, it has not already entered a pocket.

In passing, in the case where $x\tan\alpha/y$ is rational, one may speculate about the number of incidences prior to the ball entering a pocket. This problem is equivalent to a *Mathematical Spectrum* problem, number 16.4, in Volume 16 Number 2 (1983/84).

An unsolved problem

Having observed what occurs with a ball bouncing around within a rectangle, we might speculate about a similar problem where the ball is reflected within another polygon. We might observe the behaviour as in the first case in figure 7, where the ball encounters a vertex. We might encounter the second case in figure 7, where the ball follows a closed path.

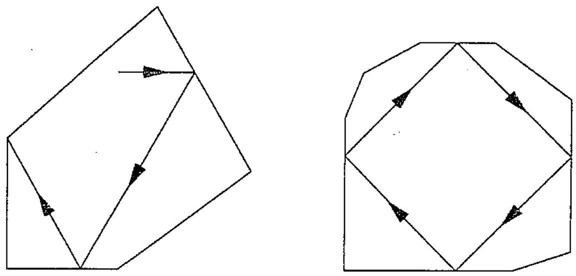


Figure 7

What else may occur? I believe that the only alternative is that the ball bounces around in a rather random fashion. There are many ways of formulating this conjecture, but I think that if the ball does not follow the examples of figure 7 then it makes incidences arbitrarily close to any point of the polygon, no matter what the polygon. Readers may like to verify this for simple classes of shapes, such as triangles, rectangles or some other shapes which exhibit symmetries. Or perhaps I am wrong and a reader can supply a counter-example.

References

1. R. Honsberger, *Ingenuity in Mathematics*, pp. 93–100. (Mathematical Association of America, Washington, DC).
2. David Sharpe, 'The strange billiard table' *Mathematical Spectrum* **23** (1990/91), p. 68. □

***Ian M. Richards** teaches at Penwith College, Penzance.*

Computer Column

Josephus' problem

In Volume 24 of *Mathematical Spectrum*, I. M. Richards described the Josephus problem, in which n people stand in a circle and the sth person round the circle is eliminated repeatedly. J. M. Edwards, a retired Head of Mathematics from Lincoln, has contributed two BASIC programs to illustrate this elimination process. I have slightly modified one of them to work in QBasic, and you may need to change it yourself if your own BASIC uses `REPEAT...UNTIL` rather than `DO...LOOP` (or even `LABEL...GOTO`)!

```
Q=500
DIM A(Q)
DO
      INPUT "Number and Step ";N,S
      FOR Z=0 TO Q: A(Z)=0: NEXT Z
      X=0: M=0
      DO
            C=0
            DO
                  DO
                        X=(X+1) MOD N
                  LOOP UNTIL A(X)=0
                  C=C+1
            LOOP UNTIL C=S
            A(X)=1
            IF X=0 THEN X=N
            PRINT X;
            M=M+1
      LOOP UNTIL M=N
LOOP
```

Mike Piff □

Difference and Differential Equations

G. N. THWAITES

The 'post-box problem' leads to a discussion of difference and differential equations, and the interrelationship between them.

A well-known post-box problem is the following.

The disarrangement problem. Suppose that n letters are to be posted into n pigeonholes, that each letter goes into a different slot, and that there is only one correct way for this to happen. In how many ways is it possible to post the letters so that every one is in the wrong place?

Calling this number u_n, we can easily check by direct calculation that $u_1 = 0$, $u_2 = 1$, $u_3 = 2$, and so on. One way to approach the problem further is to try to find a *difference equation* (sometimes called a *recurrence relation*) which expresses u_n in terms of u_{n-1}, u_{n-2}, and so on. Thus, for $n \geqslant 3$, suppose that the letter for the first pigeonhole goes into the kth (where $2 \leqslant k \leqslant n$). If the kth letter then goes into the first pigeonhole there are u_{n-2} ways of disarranging the remaining $n-2$ letters. If, however, the kth letter is not permitted to go into the first pigeonhole then there are u_{n-1} ways of disarranging the 2nd to the nth letters. Since k may take $n-1$ values, we have

$$u_n = (n-1)(u_{n-1}+u_{n-2}).$$

What is now required is to find a formula for u_n which will fit the difference equation and also the initial conditions $u_1 = 0$ and $u_2 = 1$. This problem was solved by a colleague of mine, Philip Colville, by using *ad hoc* methods. He then raised the question of whether general methods exist for solving recurrence relations of this type.

In general the answer would appear to be no, but one approach to reformulating such a problem is to use generating functions. If we have a sequence $u_0, u_1, u_2, \ldots$ then its generating function is

$$y = u_0+u_1x+u_2x^2+\cdots+u_nx^n+\cdots,$$

provided that the power series converges for all sufficiently small x, i.e. has radius of convergence $R > 0$. (Such functions may be familiar to some readers from probability theory, where, for example, $u_n = \Pr(X = n)$ for an appropriate random variable X.)

To see how such a function may help in the study of difference equations let us start by considering the special case of second-order difference equations with constant coefficients:

$$au_n+bu_{n-1}+cu_{n-2} = 0 \quad (n \geqslant 2,\ a \neq 0).$$

Let

$$y = u_0+u_1x+u_2x^2+\cdots+u_nx^n+\cdots.$$

We now try to create an algebraic expresssion involving y for which the coefficient of x^n, for sufficiently large n, is $au_n+bu_{n-1}+cu_{n-2}$ and so equals zero. We have

$$\begin{aligned} ay+bxy+cx^2y &= au_0+au_1x+au_2x^2+\cdots+au_nx^n+\cdots \\ &\quad +bu_0x+bu_1x^2+\cdots+bu_{n-1}x^n+\cdots \\ &\quad +cu_0x^2+\cdots+cu_{n-2}x^n+\cdots \\ &= au_0+(au_1+bu_0)x. \end{aligned}$$

Thus

$$y = \frac{au_0+(au_1+bu_0)x}{a+bx+cx^2}.$$

If the equation $at^2+bt+c = 0$ has unequal roots α and β, then for suitable A and B,

$$\begin{aligned} y &= \frac{au_0+(au_1+bu_0)x}{a(1-\alpha x)(1-\beta x)} \\ &= \frac{A}{1-\alpha x}+\frac{B}{1-\beta x} \\ &= A(1+\alpha x+\alpha^2x^2+\cdots)+B(1+\beta x+\beta^2x^2+\cdots). \end{aligned}$$

Thus $u_n = A\alpha^n+B\beta^n$. If the roots are equal, then we have

$$\begin{aligned} y &= \frac{au_0+(au_1+bu_0)}{a(1-\alpha x)^2} \\ &= \frac{A}{1-\alpha x}+\frac{B}{(1-\alpha x)^2} \\ &= A(1+\alpha x+\cdots) \\ &\quad +B(1+2\alpha x+3\alpha^2x^2+\cdots+(n+1)\alpha^nx^n+\cdots). \end{aligned}$$

Thus $u_n = A\alpha^n+B(n+1)\alpha^n$, more usually written as $(C+Dn)\alpha^n$. This result obviously generalises to higher-order relations and it also becomes clear where the terms of the form $n\alpha^n$ come from.

We can now consider how difference equations and differential equations may be related. Note first that, if $y = u_0+u_1x+\cdots+u_nx^n+\cdots$, then

$$\frac{dy}{dx} = u_1+2u_2x+\cdots+nu_nx^{n-1}+\cdots$$

and

$$\frac{d^2y}{dx^2} = 2u_2+\cdots+n(n-1)u_nx^{n-2}+\cdots$$

and so on.

To illustrate the situation let us again consider a special case. Suppose that y is a solution of the second-order linear differential equation with constant coefficients

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+b\frac{\mathrm{d}y}{\mathrm{d}x}+cy=0.$$

Substituting for y and its differential coefficients in terms of x and looking at the coefficient of x^{n-2} we see that

$$an(n-1)u_n+b(n-1)u_{n-1}+cu_{n-2}=0 \quad (n\geqslant 2),$$

so that the numbers u_i are related by a difference equation. As it happens, this difference equation can be solved quite easily. Multiply it through by $(n-2)!$ and set $v_k=k!\,u_k$ for $k\geqslant 0$. Then $av_n+bv_{n-1}+cv_{n-2}=0$ for $n\geqslant 2$ and so we know what v_n must be from our previous work. Since $u_n=v_n/n!$ we have

$$y=\sum_{n=0}^{\infty}\frac{A\alpha^n+B\beta^n}{n!}x^n=A\mathrm{e}^{\alpha x}+B\mathrm{e}^{\beta x},$$

for unequal roots, and

$$\begin{aligned}y&=\sum_{n=0}^{\infty}\frac{A\alpha^n+Bn\alpha^n}{n!}x^n\\&=A\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!}+B\alpha x\sum_{n=1}^{\infty}\frac{(\alpha x)^{n-1}}{(n-1)!}\\&=A\mathrm{e}^{\alpha x}+Cx\mathrm{e}^{\alpha x}\end{aligned}$$

for equal roots, where $C=B\alpha$. This shows, incidentally, why the results for the two types of equation are so closely related. More importantly, it is clear from the above equations that the process can be reversed, and knowledge of the solution of the differential equation can be used to solve the corresponding difference equation.

Let us now consider a difference equation of the form

$$f(n)u_n+g(n)u_{n-1}+h(n)u_{n-2}=0, \tag{1}$$

where f, g and h are polynomials in n. We shall see that the generating function of the sequence $\{u_i\}$ is a solution of a differential equation of the form

$$p_m(x)\frac{\mathrm{d}^my}{\mathrm{d}x^m}+\cdots+p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+p_0(x)y=q(x), \tag{2}$$

where the $p_i(x)$ and $q(x)$ are polynomials in x. The idea is to select the $p_i(x)$ in such a way that, when the generating function is substituted for y, the coefficient of x^n (or x^{n-1} or x^{n-2}, etc.) in the left-hand side of (2) is equal to the expression (1), for sufficiently large n, and so equals zero. The terms for the smaller values of n then form the polynomial $q(x)$. The method is most easily described by means of an example.

Consider the expression

$$F(n)=(n+1)u_n+(n^2+n+1)u_{n-1}-nu_{n-2}.$$

Now $(n+1)u_n$ is the coefficient of x^n in

$$\begin{aligned}&x(u_1+2u_2x+\cdots+nu_nx^{n-1}+\cdots)\\&\qquad+(u_0+u_1x+\cdots+u_nx^n+\cdots)=x\frac{\mathrm{d}y}{\mathrm{d}x}+y,\end{aligned}$$

if $n\geqslant 1$. Next,

$$(n^2+n+1)u_{n-1}=[(n-1)(n-2)+4(n-1)+3]u_{n-1}$$

is the coefficient of x^{n-1} in

$$\begin{aligned}&x^2(2u_2+3\times 2u_3x+\cdots+(n-1)(n-2)u_{n-1}x^{n-3}+\cdots)\\&\quad+4x(u_1+2u_2x+\cdots+(n-1)u_{n-1}x^{n-2}+\cdots)\\&\quad+3(u_0+u_1x+\cdots+u_{n-1}x^{n-1}+\cdots)\\&=x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}+3y,\end{aligned}$$

if $n\geqslant 3$, that is, it is the coefficient of x^n in

$$x\left(x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}+3y\right).$$

Finally, $nu_{n-2}=[(n-2)+2]u_{n-2}$ is the coefficient of x^{n-2} in

$$\begin{aligned}&x(u_1+2u_2x+\cdots+(n-2)u_{n-2}x^{n-3}+\cdots)\\&\quad+2(u_0+u_1x+\cdots+u_{n-2}x^{n-2}+\cdots)\\&=x\frac{\mathrm{d}y}{\mathrm{d}x}+2y,\end{aligned}$$

if $n\geqslant 3$, that is, it is the coefficient of x^n in

$$x^2\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\right).$$

Thus, if $n\geqslant 3$, $F(n)$ is the coefficient of x^n in

$$\begin{aligned}q(x)&=\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)+x\left(x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}y}{\mathrm{d}x}+3y\right)-x^2\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\right)\\&=x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-(x^3-4x^2-x)\frac{\mathrm{d}y}{\mathrm{d}x}-(2x^2-3x-1)y.\end{aligned}$$

Suppose now that $F(n)=0$ for $n\geqslant 2$. Then, when y is written as the generating function

$$u_0+u_1x+\cdots+u_nx^n+\cdots,$$

the coefficients of x^n in $q(x)$ are all zero for $n\geqslant 3$, so that $q(x)$ reduces to a quadratic. Thus y satisfies a relatively simple differential equation.

Next let us return to the post-box problem. We have

$$u_n-(n-1)u_{n-1}-(n-1)u_{n-2}=0$$

for $n\geqslant 3$. It is convenient to define $u_0=1$ to fit in with the recurrence relation. Unfortunately the values of u_n grow so quickly that the radius of convergence of the generating function is 0. However, if we let $v_n=u_n/n!$ then the equation becomes

$$n!\,v_n-(n-1)(n-1)!\,v_{n-1}-(n-1)(n-2)!\,v_{n-2}=0,$$

i.e.

$$nv_n-(n-1)v_{n-1}-v_{n-2}=0,$$

and this equation does not pose such a problem. Let

$x = v_0 + v_1 x + \cdots$. Now

$$\begin{aligned}\frac{dy}{dx} - x\frac{dy}{dx} - xy &= v_1 + 2v_2 x + \cdots + \qquad nv_n x^{n-1} + \cdots \\ &\quad - v_1 x - \cdots - (n-1)v_{n-1}x^{n-1} - \cdots \\ &\quad - v_0 x - \cdots - \qquad v_{n-2}x^{n-1} - \cdots \\ &= v_1.\end{aligned}$$

Therefore

$$\frac{dy}{dx} - \frac{x}{1-x}y = \frac{v_1}{1-x}.$$

The integrating factor is

$$\exp\left\{\int -\frac{x}{1-x}\,dx\right\} = (1-x)e^x,$$

so that

$$\frac{d}{dx}[(1-x)e^x y] = v_1 e^x.$$

Integrating, we have

$$(1-x)e^x y = v_1 e^x + c.$$

Remembering that $y = v_0$ when $x = 0$, we obtain

$$(1-x)e^x y = v_1 e^x + (v_0 - v_1).$$

Thus

$$\begin{aligned}y &= v_1(1-x)^{-1} + (v_0 - v_1)e^{-x}(1-x)^{-1} \\ &= v_1(1 + x + \cdots + x^n + \cdots) \\ &\quad + (v_0 - v_1)\left(1 - \frac{x}{1!} + \cdots + \frac{(-x)^n}{n!} + \cdots\right) \\ &\qquad \times (1 + x + \cdots + x^n + \cdots)\end{aligned}$$

and, from the term in x^n,

$$v_n = v_1 + (v_0 - v_1)\left(1 - 1 + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right).$$

Remembering that $u_n = v_n n!$, so that $u_0 = v_0$ and $u_1 = v_1$, we obtain our solution

$$u_n = n!\left\{u_1 + (u_0 - u_1)\left(1 - 1 + \cdots + \frac{(-1)^n}{n!}\right)\right\}.$$

For the post-box problem itself, $u_0 = 1$ and $u_1 = 0$, so that

$$\begin{aligned}u_n &= n!\left(1 - 1 + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right) \\ &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n.\end{aligned}$$

Although the form of the solution is not a 'closed' one, the series $1 - 1 + 1/2! - \cdots + (-1)^n/n!$ is nearly e^{-1} and so $u_n \approx n!/e$. (In fact, for $n \geqslant 2$, u_n is the nearest integer to $n!/e$.)

Even in cases where it is easier to solve the difference equation rather than the corresponding differential equation the process can yield interesting results. For example, consider the relation $na_n + 3a_{n-3} = 0$ with its equation

$$\frac{dy}{dx} = 3x^2 y = a_1 + 2a_2 x.$$

The integrating factor is e^{x^3} and the solution is

$$y = a_0 e^{-x^3} + a_1 e^{-x^3}\int e^{x^3}\,dx + 2a_2 e^{-x^3}\int xe^{x^3}\,dx. \quad (3)$$

The three terms on the right supply the powers x^{3n}, x^{3n+1} and x^{3n+2}, respectively. From the difference equation the coefficient of x^{3n+1} is

$$a_1\frac{(-3)^n}{4\times 7\times\cdots\times(3n+1)};$$

this coefficient arises from the second term in the right-hand side of (3) by multiplying together the Maclaurin expansion of e^{-x^3} and the term-by-term integration of the Maclaurin expansion of e^{x^3}, i.e.

$$a_1\left(\sum_{k=0}^{\infty}(-1)^k\frac{(x^3)^k}{k!}\right)\left(\sum_{r=0}^{\infty}\frac{x^{3r+1}}{r!\,(3r-1)}\right).$$

This gives us the formula

$$\frac{(-3)^n}{4\times 7\times\cdots\times(3n+1)} = \sum_{r=0}^{n}\frac{(-1)^{n-r}}{(n-r)!\,r!\,(3r+1)}.$$

The right-hand side is

$$\frac{1}{n!}\int_0^1 (x^3-1)^n\,dx,$$

as readers may like to verify. Thus we have

$$\int_0^1 (1-x^3)^n\,dx = \frac{3^n(n!)}{4\times 7\times\cdots\times(3n+1)}.$$

A similar formula may be obtained from the coefficient of x^{3n+2}. Indeed it is easy to generalise the whole thing by considering $na_n + ma_{n-m} = 0$, and showing that

$$\int_0^1 x^{k-1}(1-x^m)^n\,dx = \frac{m^n(n!)}{k(m+k)(2m+k)\cdots(nm+k)} \qquad (1 \leqslant k \leqslant m).$$

This result can be extended to all positive integer values of k but is more usually proved by writing $x^{k-1}(1-x^m)^n$ as $x^{k-1}(1-x^m)^{n-1} - x^k x^{m-1}(1-x^m)^{n-1}$ and using integration by parts to obtain—yes, you've guessed it!—a difference equation. □

Geoff Thwaites *obtained his first degree and his doctorate from Balliol College, Oxford. He has taught at York University, Malta University and Rugby School, and is now Head of Mathematics at Oakham School. His major research interest at the moment lies in trying to understand the marking guidelines for Key Stage 4 coursework.*

Mathematics in the Classroom

The aim of this regular feature is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

Investigations in the teaching of A-level mathematics

Nationally, the number of A-level mathematicians is in decline. Although this trend has not yet been reflected in the numbers at my college, I am very conscious that we see very few students progressing to a degree course in mathematics. So, is there anything that can be done at sixth-form level to nurture and develop the fascination for and enjoyment of the subject that has led many sixteen-year-olds to take mathematics as an A-level choice? Equally, how can we instil a real interest and understanding in those whose choice of this A-level stems from the perception that mathematics is a useful subject even though they may not particularly enjoy it?

One way to achieve these objectives is to ensure that students actively participate in their own learning. The problem solving which is routine in the teaching of mathematics is one obvious example of this active participation. Investigations or practical work can also provide many opportunities for developing a student's understanding and for increasing his or her satisfaction.

For the teacher, the prospect of launching into investigational work can be quite daunting, and time pressures of completing the syllabus have frequently ensured that class teaching must take priority. However, the reduction in content of the new A-level syllabuses should alleviate this problem and leave more time in which to extend the range of teaching styles. Indeed, some of the coursework tasks in the new syllabus that we have just adopted could be described as investigational activities.

Here are some areas in which we have found investigations to be very worthwhile as part of the learning process.

Differentiating from first principles

Consider the curve $y = x^2$. Suppose we want to find the gradient of this curve at the point $P(2, 4)$. First find the gradient of the straight line that joins P to the point Q whose coordinates are $(3, 9)$. Then reduce the x coordinate of Q in steps of 0.1 so that Q moves closer to P. Find the gradient of PQ each time and then again when the x coordinate of Q is 2.01. Complete the table:

Q	Gradient PQ
$(3, 9)$	5
$(2.9, 8.41)$	.
$(2.8, 7.84)$	.
⋮	⋮
$(2.1, 4.41)$	.
$(2.01, 4.0401)$	.

- What value do you think the gradient is tending towards?
- So what do you think the gradient of the curve is at the point $(2, 4)$?

Now change the coordinates of P to $(3, 9)$ and Q to $(4, 16)$ and repeat the procedure.

- What do you think the gradient is at the point $(3, 9)$?
- Can you now find a formula for generating the value of the gradient at any point on this curve?

Test your theory on another point. Now generalise: see if you can derive your formula for the gradient by taking P as the general point (x, x^2) and Q as the point $(x+h, (x+h)^2)$.

- What is the gradient of PQ?
- What happens to the value of this gradient if $h \to 0$? Does this agree with your earlier findings?

Now investigate the gradient of the curve $y = x^3$.

- Are these two results sufficient for you to write down an expression for the gradient of the curve $y = x^n$ at the general point (x, x^n)?

Remember that you also know the result for the case $n = 1$, so you have three results from which to generalise.

The double angle formulae

Plot the graphs of $y = \sin x$ and $y = \sin 2x$ on the same axes, using values of x from 0° increasing in steps of 30° to 360°. Can you spot a connection between $\sin 2x$ and $\sin x$? Plot a graph of $y = \sin 2x/\sin x$ for values of x from 0° to 360°. Try some values of x close to 0—say 0.1, 0.05, 0.005—so that you can find the value of $\sin 2x/\sin x$ when $x = 0$.

- Does the graph of $\sin 2x/\sin x$ resemble another graph you know?

Use this result to find a formula for $\sin 2x$.

- Can you also find a formula for $\cos 2x$?

The natural logarithm as an integral

Draw the graph of $y = 1/x$ for $x = 1$ to $x = 10$ as accurately as you can on a piece of 2-mm graph paper. The function $F(a)$ is defined to be the area under this graph between the lines $x = 1$, $x = a$ and the x-axis. Write down the value of $F(1)$. By counting squares and working out the appropriate areas, complete the following table:

a	1	2	3	4	5	6	7	8	9	10
$F(a)$	.	.	.	.	.	.	.	.	.	.

Use the table to check that the following are true:

$$F(6) = F(2) + F(3),$$

$$F(8) = F(2) + F(4),$$

$$F(10) = F(2)+F(5).$$

It would seem in general that it may be true that

$$F(ab) = F(a)+F(b).$$

Similarly check that

$$F(4) = 2F(2),$$

$$F(8) = 3F(2),$$

$$F(9) = 2F(3).$$

- What generalisation can you suggest for $F(a^n)$?
- What function have you met with these properties?

Now plot the values $(a, F(a))$ from the table and join them with a smooth curve. You should recognise this graph.

- Can you write down its equation?
- So what have you discovered about the value of

$$\int_1^a \frac{1}{x}\,dx?$$

Loci in the Argand diagram

Sketch the following graphs:

(a) $x^2+y^2 = 4$;
(b) $(x-1)^2+y^2 = 4$;
(c) $x^2+(y-2)^2 = 4$;
(d) $(x-1)^2+(y-2)^2 = 4$.

If $z = x+iy$ then $|z| = \sqrt{x^2+y^2}$. Express the following in terms of x and y:

(a) $|z| = 2$;
(b) $|z-1| = 2$;
(c) $|z-2i| = 2$;
(d) $|z-(1+2i)| = 2$.

Use these results to identify the curve represented by $|z-(a+ib)| = r$. Now draw quick sketches of the locus of $P(x,y)$, where P represents z in an Argand diagram if:

(a) $|z-3i| = 4$;
(b) $|z+2| = 3$;
(c) $|z-1+i| = 1$;
(d) $|z+3-5i| = 2$.

Finally ...

More ideas for investigations can be found in *These Have Worked For Us At A-level* and *These Have Also Worked For Us At A-level*, both published by the Association of Teachers of Mathematics, and also in Susan Pirie's *Mathematical Investigations in Your Classroom*, published by Macmillan. But if there are any others that have worked well for you, we should be pleased to hear from you.

Carol Nixon □

An Example of Uncorrelated Dependent Random Variables

MICHAEL CAIN

Several examples exist of two dependent random variables which are uncorrelated. Many of these are rather contrived, but here we outline one in which the variables concerned have simple descriptions in the familiar die-throwing context.

Let X_1 and X_2 be the scores obtained on two throws of a fair six-sided die, define Z to be the number of sixes obtained and let Y be an indicator variable with value determined by whether the total X_1+X_2 is even or odd, so that $Y = 0$ if X_1+X_2 is even and $Y = 1$ if X_1+X_2 is odd.

The joint probability function $p(y,z)$ of Y and Z is

		z = 0	1	2
y	0	$\frac{13}{36}$	$\frac{4}{36}$	$\frac{1}{36}$
	1	$\frac{12}{36}$	$\frac{6}{36}$	0

The derivation of these probabilities, from the two-dimensional sample space of the 36 possible outcomes of (X_1, X_2), is left as an exercise for the reader.

Particularly in view of the zero entry in the last cell, i.e. $p(1,2) = 0$, Y and Z are clearly not independent. However, $\text{cov}(Y,Z) = 0$, as can readily be seen, since

$$E(Y) = \tfrac{18}{36} = \tfrac{1}{2},$$

$$E(Z) = \tfrac{10}{36}\times 1+\tfrac{1}{36}\times 2 = \tfrac{12}{36} = \tfrac{1}{3},$$

$$E(YZ) = \tfrac{6}{36} = \tfrac{1}{6},$$

so that

$$E(YZ)-E(Y)E(Z) = \tfrac{1}{6}-\tfrac{1}{2}\times\tfrac{1}{3} = 0.$$

We thus have that $\rho(Y,Z) = 0$, but although Y and Z are uncorrelated, they are not independent. □

Michael Cain *is in the Department of Economics, University of Wales, Aberystwyth.*

Letters to the Editor

Dear Editor,

Palindromic numbers

Answers to both questions raised by Alex J.-C. Chen on page 85 of Volume 26 Number 3 are given in a forthcoming article by Eric A. Schmidt, 'The nth positive integer palindrome generator', to appear in the *Undergraduate Journal of Mathematics*.

Yours sincerely,
F. J. PAPP
(Department of Mathematics,
University of Michigan,
Dearborn, MI 48128, USA)

Dear Editor

Palindromic numbers

In Volume 26 Number 3 page 85, Alex Chen asked how many palindromic numbers there are up to 14041. For a given palindromic number x, if x is the Nth palindromic number, I would like to propose the following formula for x. Denote the number of digits in x by n and denote by k the 'left half' of x. Thus, when $x = 1661$, $k = 16$; when $x = 14041$, we put $k = 140$. Then

$$N = \begin{cases} k+10^{\frac{1}{2}n}-10 & (n \text{ even}), \\ k+10^{\frac{1}{2}(n-1)}-10 & (n \text{ odd}). \end{cases}$$

For example, when $x = 1661$, $N = 16+100-10 = 106$; when $x = 14041$, $N = 140+100-10 = 230$.

The proof of the formula is not difficult. First suppose that n is even. There are k palindromic numbers with an even number of digits up to x (reflect $1, 2, 3, \ldots, k$ to give $11, 22, 33, \ldots, x$). There are 90 three-digit palindromic numbers (reflect $10, 11, \ldots, 99$ in their last digits to give $101, 111, \ldots, 999$), 900 five-digit palindromic numbers (reflect $100, \ldots, 999$ in their last digits), and so on up to $9\times 10^{\frac{1}{2}n-1}$ $(n-1)$-digit palindromic numbers. If we add these numbers we obtain

$$\begin{aligned} N &= k+90+900+\cdots+9\times 10^{\frac{1}{2}n-1} \\ &= k+90(1+10+10^2+\cdots+10^{\frac{1}{2}n-2}) \\ &= k+90\frac{10^{\frac{1}{2}n-1}-1}{9} \\ &= k+10^{\frac{1}{2}n}-10. \end{aligned}$$

Now suppose that n is odd. There are $k-9$ palindromic numbers with an odd number of digits up to x (reflect $10, 11, \ldots, k$ in the last digit to give $101, 111, \ldots, x$). There are 9 two-digit palindromic numbers (reflect $1, 2, \ldots, 9$ to give $11, 22, \ldots, 99$), 90 four-digit palindromic numbers (reflect $10, 11, \ldots, 99$ to give $1001, 1111, \ldots, 9999$), and so on to give $9\times 10^{\frac{1}{2}(n-3)}$ $(n-1)$-digit palindromic numbers. Now

$$\begin{aligned} N &= k-9+9+90+900+\cdots+9\times 10^{\frac{1}{2}(n-3)} \\ &= k+90(1+10+10^2+\cdots 10^{\frac{1}{2}(n-5)}) \\ &= k+90\frac{10^{\frac{1}{2}(n-3)}-1)}{9} \\ &= k+10^{\frac{1}{2}(n-1)}-10. \end{aligned}$$

Yours sincerely,
GUSTAAF LAHOUSSE
(St-Donatuslaan 4,
B-1850 Grimbergen, Belgium)

Dear Editor

The Smarandache function

In recent issues of *Mathematical Spectrum*, problems on the Smarandache function S have appeared. For a positive integer n, $S(n)$ is defined to be the smallest positive integer such that $S(n)!$ is divisible by n. Using Henry Ibstedt's table of Smarandache functions (see the reference), I have noticed that, for all values of n up to 4800, there is always a prime between $S(n)$ and $S(n+1)$ (including possibly $S(n)$ and $S(n+1)$), but I have not been able to give a proof of this result for all n. It might be useful to apply Breusch's theorem (for $n \geqslant 48$, there exists a prime number between n and $\frac{9}{8}n$) rather than Bertrand's postulate/Tchebychev's theorem (there exists a prime between n and $2n$).

It would also be interesting to have an asymptotic formula for the number of these primes.

Yours sincerely,
I. M. RADU
(Bucharest)

Reference

1. Henry Ibstedt, The F. Smarandache function $S(n)$: programs, tables, graphs, comments. *Smarandache Function Journal* 2-3 (1993), 38–71.

Dear Editor

The Smarandache function, inter alia

I have just seen the short contribution on page 84 of Volume 26 Number 3 of *Mathematical Spectrum* by J. Rodriguez. A few months ago I had a problem published in the *American Mathematical Monthly* asking for a proof that, for almost all n, $S(n) = P(n)$, the greatest prime factor of n—the proof is quite easy. It is immediate that, for squarefree n, $S(n) = P(n)$ and from this and the fact that $p_{n+1}/p_n \to 1$ $(2, 3, 5, \ldots$ is the sequence of consecutive primes) it is easy to see that, for every k, there is an increasing sequence of integers $n_1 < n_2 < \cdots < n_k$ for which $S(n_1) > S(n_2) > \cdots > S(n_k)$.* To see this let $p_1, p_2, \ldots, p_l$ be a sequence of large primes such that $p_1 < 2p_2 < \cdots < kp_k$ but $p_1 > p_2 > \cdots > p_k$. It immediately follows from $p_{n+1}/p_n \to 1$ that such a sequence exists and it is easy to see that $S(rp_r) = p_r$ if all the p's are greater than k.

*This has also been shown independently in letters from Khalid Khan and Pål Gronås in Volume 27 Number 1, pages 20 and 21. Ed.

It is also of interest to see how many terms $n_1 < n_2 < \cdots < n_k$ with $S(n_1) > \cdots > S(n_k)$ we can take with $n_k < n$; i.e. can we find a reasonably dense sequence of descending values of the Smarandache function. If one uses $p_{n+1} < p_n + p_n^{1-\varepsilon}$ one can prove that there is a sequence of length n^c of integers $n_1 < n_2 < \cdots < n_k < n$ $(k > n^c)$ and $S(n_1) > S(n_2) > \cdots > S(n_k)$. I do not see how to get the largest c for which such a sequence exists. More generally, put $f(n) = \max k$ for which $n_1 < n_2 < \cdots < n_k \leqslant n$ and $S(n_1) > \cdots > S(n_k)$.

Also it is easy to find for every k a sequence $n_1, n_2, \ldots, n_k$ for which $S(n_1), S(n_2), \ldots, S(n_k)$ form two arbitrary permutations.

I certainly do not see how to solve the problem posed by Yau on page 85 of the same issue.

Nearly 60 years ago Obláth and I proved that, for $k > 2$,

$$x^k \pm y^k = n! \quad ((x,y) = 1)$$

has only a finite number of solutions (in fact no solutions). We could never drop the condition $(x,y) = 1$.

An old (and probably hopeless) conjecture states that $n!+1 = x^2$ is only possible for $n = 4$, 5 and 7. I offer a reward of £500 for a proof and £100 for a counterexample. I would be glad to see a proof that $n!+1 = p^2$ has no solutions (except 4, 5 and 7).

Can any of your readers find an asymptotic formula for

$$\sum_{\substack{n < x \\ S(n) > P(n)}} S(n)^2 ?$$

(Clearly $S(n) \geqslant P(n)$.)

An old and hopeless conjecture of mine states that the density of the integers for which $P(n) > P(n+1)$ is $\frac{1}{2}$. I offer a reward of £500 for a proof or disproof. (See Elliott's *Probabilistic Number Theory*, published by Springer-Verlag.)

Yours sincerely

PAUL ERDŐS
(Mathematical Institute,
Hungarian Academy of Sciences,
Budapest)

Paul Erdős, who recently celebrated his eightieth birthday, is one of the world's most prolific mathematicians. He is the author or co-author of over 1300 mathematical papers, and is famous for his contributions to number theory, combinatorics and probability. Mathematicians assign themselves Erdős numbers as follows: Paul Erdős has Erdős number 0, anyone who has written a joint paper with Paul Erdős has Erdős number 1. You have Erdős number E when you have written a joint paper with someone who has Erdős number $E-1$ but not written a joint paper with anyone having lower Erdős number.

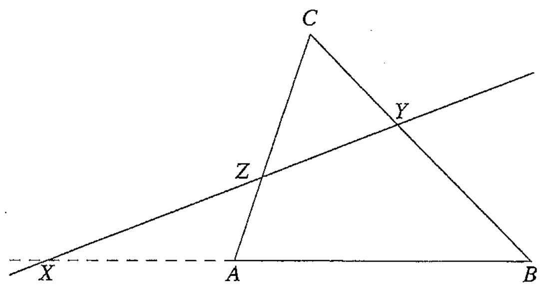
Fields Medals

The Fields Medals, the most prestigious awards in mathematics, are announced every four years at the International Congress of Mathematicians. Comparable to the Nobel Prize, the Fields Medal is reserved for mathematicians under the age of 40 and is awarded for work of a seminal nature, pointing the way to current and future progress in mathematics research.

Fields Medals awarded at the 1994 International Congress of Mathematicians in Zürich, Switzerland, went to:

- **Jean Bourgain** of the Institut des Hautes Études Scientifiques (Paris), the Institute for Advanced Study (Princeton, NJ) and the University of Illinois at Urbana–Champaign;
- **Pierre-Louis Lions** of the University of Paris–Dauphine;
- **Jean-Christophe Yoccoz** of the University of Paris–Sud, Orsay;
- **Efim Isaakovich Zelmanov** of the University of Wisconsin–Madison.

Extending Menelaus' theorem



Menelaus' theorem for a triangle says that

$$\frac{AX}{XB} \times \frac{BY}{YC} \times \frac{CZ}{ZA} = -1$$

(using directed measurements). Can you extend it to a quadrilateral, or a pentagon or, in general, an n-sided polygon?

Seyamark Jafari
(Ahwaz, Iran)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

27.5 Let X, Y and Z be points on sides BC, CA and AB, respectively, of an equilateral triangle ABC with $BX = CY = AZ = m$ units and $XC = YA = ZB = n$ units, where m and n are integers that are prime. Prove that, if $\triangle XYZ$ has integral sides, then X, Y and Z are the midpoints of the sides of $\triangle ABC$.
(Submitted by Sammy Yu, age 14, and Jimmy Yu, age 12, middle school students and special students at the University of South Dakota, USA.)

27.6 Find all real-valued functions satisfying the functional equation

$$f(x)+f\left(\frac{1}{1-x}\right) = \frac{x}{x-1}$$

for all $x \in \mathbb{R}$ ($x \neq 0, 1$).
(Submitted by Gregory Economides, University of Newcastle upon Tyne Medical School.)

27.7 What is the maximum possible degree of a polynomial in t given by the $n\times n$ determinant

$$\begin{array}{cc}\left|\begin{array}{ccc|ccc} t & & & & & \\ & \ddots & & & & \\ & & t & & & \\ \hline & & & 0 & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \end{array}\right| & \begin{array}{c} \\ \\ \\ \uparrow \\ m \\ \downarrow \end{array} \\ \qquad\qquad\qquad \leftarrow \; m \; \rightarrow & \end{array}$$

where the unspecified entries are arbitrary constants?
(Submitted by J. A. Scott, Chippenham.)

27.8 Prove that, if $e^x/x = e^y/y$, where $y > x > 0$, then $xy < 1$.
(Submitted by Glenn Vickers, University of Sheffield.)

Solutions to Problems in Volume 26 Number 4

26.10 Let $m \geqslant 13$ be an integer and $f\colon \mathbb{N} \to \mathbb{N}$ a mapping such that

$$f(n) = \begin{cases} n-m+13 & (\text{if } n > m^2), \\ f(f(n+m-12)) & (\text{if } n \leqslant m^2). \end{cases}$$

Find all m such that $f(1994) = 1994$.

Solution by David Pirnes (University of Cambridge).

I will show that $f(n) = f(m^2+1) = m^2-m+14$ for $n \leqslant m^2$.

(a) When $m^2-m+13 \leqslant n \leqslant m^2$, so that $n \leqslant m^2$ and $n+m-12 > m^2$, we have

$$\begin{aligned} f(n) &= f(f(n+m-12)) \\ &= f(n+m-12-m+13) = f(n+1), \end{aligned}$$

so

$$\begin{aligned} f(m^2-m+13) &= f(m^2-m+14) \\ &= \dots \qquad \dots \\ &= f(m^2+1) = m^2-m+14. \end{aligned}$$

(b) Assume that $f(n) = m^2-m+14$ for $l \leqslant n \leqslant m^2+1$, where $l \leqslant m^2-m+13$. Then

$$\begin{aligned} f(l-1) &= f(f(l+m-13)) \\ &= f(m^2-m+14) = m^2-m+14 \end{aligned}$$

(because $l \leqslant l+m-13 \leqslant m^2$).

Thus, by (a) and (b), we have shown by induction that $f(n) = m^2-m+14$ for all $n \leqslant m^2$.

Case 1: $1994 \leqslant m^2$. $f(1994) = m^2-m+14 = 1994$ gives that $(m-45)(m+44) = 0$, so $m = 45$ (and so $m^2 \geqslant 1994$).

Case 2: $1994 > m^2$. $f(1994) = 1994-m+13 = 1994$ gives that $m = 13$ (and so m^2 is indeed smaller than 1994).

Thus $f(1994) = 1994$ if and only if $m = 13$ or 45.

Also solved by Gregory Economides and Khalid Khan (London School of Economics).

26.11 Find the sum of the infinite series

$$1-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{15}+\tfrac{1}{17}-\tfrac{1}{23}+\tfrac{1}{25}-\cdots.$$

Solution by Sammy and Jimmy Yu.

Put

$$\begin{aligned} S_A &= 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\cdots, \\ S_B &= 1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\cdots, \end{aligned}$$

so that the given series S is given by

$$S = \tfrac{1}{2}(S_A+S_B).$$

Now

$$\frac{1}{1+t^2} = 1-t^2+t^4-t^6+\cdots+(-1)^n t^{2n}+\frac{(-1)^{n+1}t^{2n+2}}{1+t^2}$$

and

$$\frac{1}{1+t^4} = 1-t^4+t^8-t^{12}+\cdots+(-1)^n t^{4n}+\frac{(-1)^{n+1}t^{4n+4}}{1+t^4},$$

so that

$$\int_0^1 \frac{1}{1+t^2}\,dt = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)^n}{2n+1}+R_1$$

and

$$\int_0^1 \frac{1+t^2}{1+t^4}\,dt$$

$$= \int_0^1 \biggl(1+t^2-t^4-t^6+t^8+t^{10}-\cdots+(-1)^n t^{4n}$$

$$+(-1)^n t^{4n+2}+(-1)^{n+1}\frac{t^{4n+4}(1+t^2)}{1+t^4}\biggr)\,dt$$

$$= 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\cdots+\frac{(-1)^n}{4n+1}+\frac{(-1)^n}{4n+3}+R_2,$$

where

$$R_1 = \int_0^1 \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\,dt,$$

$$R_2 = \int_0^1 \frac{(-1)^{n+1}(t^{4n+4}+t^{4n+6})}{1+t^4}\,dt.$$

Hence

$$|R_1| \leqslant \int_0^1 t^{2n+2}\,dt = \frac{1}{2n+3} \to 0 \quad (\text{as } n\to\infty)$$

and

$$|R_2| \leqslant \int_0^1 (t^{4n+4}+t^{4n+6})\,dt$$

$$= \frac{1}{4n+5}+\frac{1}{4n+7} \to 0 \quad (\text{as } n\to\infty).$$

Therefore

$$S_A = \int_0^1 \frac{1}{1+t^2}\,dt = [\tan^{-1}t]_0^1 = \tfrac{1}{4}\pi$$

and

$$S_B = \int_0^1 \frac{1+t^2}{1+t^4}\,dt = \frac{1}{\sqrt{2}}\int_0^\infty \frac{1}{1+u^2}\,du$$

$$= \frac{1}{\sqrt{2}}[\tan^{-1}u]_0^\infty = \frac{\pi}{2\sqrt{2}}$$

(using the substitution $u = \sqrt{2}t/(1-t^2)$). Hence

$$S = \tfrac{1}{8}\pi(1+\sqrt{2}).$$

Also solved by David Pirnes and Khalid Khan.

26.12 Prove that, for $n > 2$,

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} > \tfrac{1}{2}\log_2 n.$$

Solution by David Pirnes.

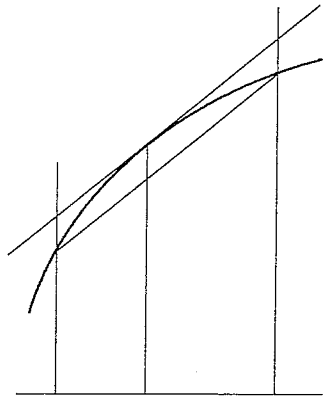
We use induction on n. The result holds when $n = 3$ (LHS $= \frac{5}{6} \approx 0.83$, RHS ≈ 0.79). Now let $r \geqslant 3$ and assume that the result holds when $n = r$. Now, by the mean value theorem, for some s between r and $r+1$,

$$\log_2(r+1)-\log_2 r = \left[\frac{d}{dx}(\log_2 x)\right]_{x=s}$$

$$= \left[\frac{1}{x\log_e 2}\right]_{x=s}$$

$$\leqslant \frac{1}{r\log_e 2}$$

$$< \frac{2}{r+1}.$$



(The last inequality holds when $r+1 < 2r\log_e 2$, i.e. when $r > 1/(2\log_e 2-1) \approx 2.58$.) By the inductive hypothesis

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r} > \tfrac{1}{2}\log_e r,$$

so

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}+\frac{1}{r+1} > \tfrac{1}{2}\log_e r+\frac{1}{r+1}$$

$$> \tfrac{1}{2}\log_e r+\tfrac{1}{2}[\log_2(r+1)-\log_2 r]$$

$$= \tfrac{1}{2}\log_2(r+1),$$

which proves the inductive step. Hence the result holds for all $n \geqslant 3$.

Also solved by Gregory Economides and Khalid Khan. □

Mathematical Spectrum Awards for Volume 26

Prizes have been awarded to the following student readers for contributions published in Volume 26:

- **Tamara Curnow** for her article 'Falling down a polygonal well' (pages 110–118);
- **Gregory Economides** and **Sammy and Jimmy Yu** for other contributions.

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Reviews

Cryptology. By ALBRECHT BEUTELSPACHER. Mathematical Association of America, Washington, DC, 1994. Pp. xvi+156. Paperback \$25.50. (ISBN 0-88385-504-6).

If, like me, you never got past the stage of sending highly secret and sensitive information to your older sister using lemon juice and a matchstick, then this book is definitely for you.

Cryptology, or cryptography, is the business of designing methods of enciphering messages, as distinguished from cryptanalysis, which deals with the breaking of those messages. This intriguing book, then, introduces the reader to this world of covert subterfuge and wily ingenuity by demonstrating several of the systems that have been developed and the efforts made to crack them. It ranges from one of the simplest, the Vigenère cypher, and how it can be attacked with statistical analysis, to the rather more complicated RSA algorithm, a cypher system that has its roots set in some results from number theory.

It is an absorbing subject. Danger lurks at every turn in the person of the devilishly cunning Mr X, who will use all his wit and guile to foil your efforts. For example, can you be sure that the message you have received came from whom you think it did? Can you stop Mr X from intercepting and revising your messages? With the irresistible approach of the world of the information superhighway, applications in the commercial world are becoming increasingly important. Can you be sure that satellite television is only being watched by those who have paid for it, and not by some masochistic cryptanalyst who has tapped into the signal?

This book is not particularly heavy going mathematically. The style is informal and the plentiful diagrams illustrate the key ideas most effectively. Exercises at the ends of the chapters explore further the points raised in the text, and there is a long list of relevant articles and books at the back.

The age of self-destructing cassette tapes divulging impossibly difficult missions is over. More advanced than *The Secret Agent's Handbook*, this is a lucid and entertaining book which is difficult to leave alone. It is worth some serious thought.

Student at the University of Bristol MARK BLYTH

MEI: Mechanics Book 2. By J. S. BERRY, E. GRAHAM AND R. PORKESS. Hodder and Stoughton, London, 1994. Pp. 137. Paperback £6.99. (ISBN 0-370-57301-5).

This book forms part of the MEI Structured Mathematics series, which is 'designed to increase substantially the number of people taking the subject post-GCSE, by making it accessible, interesting and relevant to a wide range of students'. There are chapters on friction, moments, centre of mass, energy and work, and momentum and impulse. A final short chapter deals with frameworks, and there are answers to selected exercises.

The revitalisation of mechanics teaching at A-level has been one of the most refreshing curriculum innovations of the last decade, and the influence of the important work done by the Leeds/Sheffield/Manchester *Mechanics in Action* project can be seen in this text. In particular, major concepts are developed through the practical activities of 'investigations' (rather open-ended) and 'experiments' (more prescribed). In addition, some of the material is presented in 'real-life' situations (e.g. abundant references to squash rackets, karate experts, etc.), and this emphasis also permeates the copious exercises. Each chapter concludes with a useful summary of 'key points'. All of this, combined with the attractive typographical style of the series, contributes to a very user-friendly feel, which, however, never patronises the student.

A welcome feature is the inclusion of brief historical notes, which enable readers to appreciate that Watt and Joule, for instance, are more than just names of SI units. (Newton's ideas, of course, pervade the whole text, but this volume includes no specific historic reference to him.) Another welcome aspect is the attention given to the concept of *model*, particularly the emphasis on the assumptions made in constructing one (on pages 4–5, for instance). In this way the student can acquire a critical awareness of the modelling process by considering the justifications for the assumptions made.

MEI has a strong tradition in mechanics, and this text, although inviting and having many merits, unfortunately does not seem to me to fully uphold that tradition. I believe that the student ultimately has to face up to the conceptual demands of the subject, and thus deserves texts which offer the minimum potential for conceptual confusion. Despite the appealingly innovative aspects of this book, it does not appear that enough care has been taken to protect the student from such pitfalls.

Head of Mathematics NEIL BIBBY
Sir James Henderson British School of Milan

Statistics for Engineering Problem Solving. By STEPHEN B. VARDEMAN. PWS Publishing Company, Boston, MA, 1993. Pp. 712. Hardback £22.95. (ISBN 0-534-92871-4).

While there are now a number of excellent books for engineers explaining the statistical ideas of experimental design or quality control, these books are usually too expensive and too specialised to recommend as the main course text for a service course in statistics. On the other hand, undergraduate texts in engineering statistics are often very technique orientated and fail to give sufficient coverage of topics such as experimentation and quality improvement. Stephen Vardeman has made a serious attempt to relate the content of his book to the needs of professional engineers and to present the material in a way that shows a student how statistical ideas can be used to help solve engineering problems.

On the whole the content of this book seems well suited to courses for engineers. In addition to the usual coverage of descriptive statistics, sampling and regression, the author has included several topics of particular importance to engineers—control charts (Shewhart, but not Cusum); experimental design, including factorial, fractional factorial, response surface and mixture designs;

predictions and tolerance intervals and analysis of means. Rather surprisingly, the book does not contain any coverage of robust design. Material on the probability rules and on the ideas of conditional probability and independence are placed in an appendix, along with a very brief coverage of the Weibull model. In view of the importance of this material in reliability analysis, I would personally prefer to see it included in the main text and explained in rather more detail.

I was particularly impressed by the way in which the author presents statistical ideas in the context of the solution of engineering problems. For example, in introducing graphical procedures and summary statistics, these techniques are used as a basis for comparison between machines, laboratories and so on. This contrasts sharply with many other texts in which graphs are drawn and statistics calculated with no attempt to interpret them.

Many students experience difficulties in grasping statistical concepts. The author of this text explains statistical ideas very carefully and avoids glossing over difficult points. For example, in discussing confidence intervals, he is careful to distinguish between a classical confidence interval and a probability distribution for the mean. At times, I felt that this approach led to explanations that were too lengthy and technical for the intended audience and that could perhaps have been simplified by the use of a diagram.

The author illustrates the use of computer software through several pieces of Minitab output, though the computer analyses are usually presented after a problem has been analysed manually.

Two aspects of the book that I did not particularly like were the rather impersonal writing style and the page layout which places section headings along with figure and table captions in the left-hand margin; in reading the book I often found myself drifting from one section to another without noticing the start of a new section.

Despite the reservations mentioned above, this book is original enough in its content and approach to be worth inspection by anyone looking for a statistics text for use with engineers.

Loughborough University — RICHARD BUXTON

Memorabilia Mathematica: The Philomath's Quotation Book (reprint of 1914 edition). By ROBERT EDOUARD MORITZ. Mathematical Association of America, Washington, DC, 1993. Pp. vii+410. Paperback $24.00 (ISBN 0-88385-321-3).

Out of the Mouths of Mathematicians: A Quotation Book for Philomaths. Edited by ROSEMARY SCHMALZ. Mathematical Association of America, Washington, DC, 1993. Pp. x+294. Paperback $29.00 (ISBN 0-88385-509-7).

The first of these books was published in 1914, and the second is its sequel, concentrating on quotes from this century. They both contain hundreds of short quotations, passages and anecdotes by and about mathematicians arranged into sections such as 'The nature of mathematics', 'Mathematics education', 'Persons and anecdotes', 'Exhortations to aspiring mathematicians' and 'Mathematics and the computer'.

Rosemary Schmalz states that she wanted '... to compile a volume that can be used by researchers to facilitate a literature search, by writers to emphasise or substantiate a point, by teachers to amuse their students, and by all readers, in particular young readers, to get the flavour of mathematics and to whet their appetites and make them eager to learn more about it.' Both Schmalz and Moritz have clearly satisfied this aim.

If these books only serve to encourage people to read some of the literature from which the quotations have been culled, they could still be considered to be an unqualified success. But in addition these books will provoke thought and discussion, amuse and give pleasure to their readers. They should be on every mathematician's bookshelf.

Student at St. John's College, Oxford — MARK FRENCH

A Mathematical Pandora's Box. By BRIAN BOLT. Cambridge University Press, 1993. Pp. 126. Paperback £7.95 (ISBN 0-521-44619-8).

This book marks a slight departure from the standard Brian Bolt formula. Intermingled with the usual collection of problems (some new, some old, some easy, some hard) there are some discussions on mathematical phenomena. We have, for example, details of a drill bit to drill square holes, artificial gravity in space stations, rational approximations to $\sqrt{N}$ and angle trisectors.

Nevertheless, the problems form the bulk of the book. The mathematical level of these is rarely above GCSE, the emphasis being on 'creative thinking', although many of the problems are somewhat stereotyped.

Overall a reasonable book for those below the sixth form—some will enjoy the problems and benefit from the discussions.

Student at St. John's College, Oxford — MARK FRENCH

Other books received

Essentials of Statistical Methods, in 41 pages. By T. P. HUTCHINSON. Rumsby Scientific Publishing, Sydney, 1993. Pp. ii+41, A4 format. Paperback £4.00, A$10.00, US$8.00, Can.$9.00, NZ$13.00.

An introductory statistics course gets as far as some techniques of inference—the testing of hypotheses and the construction of confidence intervals. That is the subject of Part III of this book. In preparation for this, the student needs to know about data description and about probability. These are covered in Parts I and II. Available from Rumsby Scientific Publishing, PO Box Q355, Queen Victoria Building, Sydney, NSW 2000, Australia. A price reduction of 35% is offered on orders for eight or more copies. (Please add 15% if delivery by airmail is required.)

Elementary Classical Analysis. By JERROLD R. MARSDEN AND MICHAEL J. HOFFMAN. W. H. Freeman and Company, Oxford, 1993, second edition. Pp. xiv+738. Hardback £21.95 (ISBN 0-7167-2105-8).

This is the second edition of a textbook suitable for second- and third-year undergraduates in a UK university.

Einstein Metrics and Yang–Mills Connections. By TOSHIKI MABUCHI AND SHIGERU MUKAI. Marcel Dekker, New York, 1993. Pp. viii+224. Paperback $99.75 (ISBN 0-8247-9069-3).

Problem Solving and Investigations. By DAVID WELLS. Third edition, 1993, £2.00 post paid, available from Rain Press, 6 Camarthen Road, Westbury on Trym, Bristol, BS9 4DU.

This personal critique is a third, enlarged edition of one which first appeared in 1986 concerning the current state of mathematics education. □

Mathematical Spectrum

1994/5 Volume 27 Number 2

ISSN 0025-5653

Published by the Applied Probability Trust
Typeset by The Pi-Squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK