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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
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Message from the Editor

We are pleased to announce a new development in the publication of *Mathematical Spectrum*. From Volume 23 onwards, each annual volume will comprise four 32-page issues published in September, November, February and May. This increase to a total of 128 pages each year will enable us to publish readers' contributions more quickly, while the new schedule will reduce the intervals between issues during the year.

To help meet the cost of publishing additional material, the subscription rate for Volume 23 will be £6.00. The dollar prices for subscribers in America and Australia will be US$13.00 and $A.16.00; the sterling equivalent of these dollar rates will be £8.00.

We are continuing to offer reduced rates for two- and three-year subscriptions because this measure has proved very popular during the past year. Readers whose subscriptions expire now will find an order form in this issue showing full details of prices up to and including Volume 25. Those who have already renewed for one or two further years will receive an order form at the appropriate time.

We look forward to the continuing support of all our readers.

Editor

Interesting Real Numbers

NICK MACKINNON, *Winchester College*

A real number is interesting if and only if it can be described by a finite English phrase (for example, 'The limit as n tends to infinity of open brackets one plus one divided by n close brackets to the power of n.')

In *Mathematical Spectrum* Volume 20 Number 3, Chris du Feu gave a proof that all natural numbers are interesting, and wondered about the real numbers. The proposition is that:

There are uncountably many dull real numbers, none of which can be exhibited.

Proof. Each finite English phrase can be assigned a natural number in the following way. The phrase is first transformed into a finite sequence using $a = 1$, $b = 2$, $\ldots$, $z = 26$ and space $= 27$. That creates the sequence $a_1, a_2, \ldots, a_n$. From the terms in the sequence create the natural number N whose prime factorisation is

$$2^{a_1} \times 3^{a_2} \times 5^{a_3} \times \ldots \times p_n^{a_n}.$$

For example, the phrase 'root two' gives the sequence 18, 15, 15, 20, 27, 20, 23, 15, from which is created the enormous natural number $2^{18}\times3^{15}\times5^{15}\times7^{20}\times11^{27}\times13^{20}\times17^{23}\times19^{15}$. The point of this construction is that it shows that the size of the set of finite English phrases is no greater than the size of the set of natural numbers because, whereas every phrase has a different natural number assigned to it, not every natural number is assigned to a phrase. (For example, 2^{28} is not assigned to a phrase.) Now, it is well known that the set of real numbers cannot be put into one-to-one correspondence with the set of natural numbers (the set of real numbers is said to be uncountable), so that there must be uncountably many real numbers that cannot be described by an English phrase. Each of these numbers is terribly dull of course.

When I first produced this proof over dinner one evening, I was immediately challenged to exhibit one of these dull numbers, but unfortunately the very act of doing so contradicts the dullness proof. They sit in eternal obscurity, Cinderellas who will never go to the ball.

Acknowledgement: Thanks to Kurt Gödel and Georg Cantor for making this dull note possible.

Powers of 5

$2+3 = 5$	$313+314+315+\ldots+937 = 5^8$
$3+4+5+6+7 = 5^2$	$938+939+940+\ldots+2187 = 5^9$
$8+9+10+\ldots+17 = 5^3$	$1563+1564+\ldots+4687 = 5^{10}$
$13+14+15+\ldots+37 = 5^4$	$4688+4689+\ldots+10\,937 = 5^{11}$
$38+39+40+\ldots+87 = 5^5$	$7813+7814+\ldots+23\,437 = 5^{12}$
$63+64+65+\ldots+187 = 5^6$	$23\,438+23\,439+\ldots+54\,687 = 5^{13}$
$188+189+190+\ldots+437 = 5^7$	$39\,063+39\,064+\ldots+117\,187 = 5^{14}$

and so on. Can you work out a general rule from this for expressing a power of 5 as the sum of consecutive positive integers?

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White to Move and Mate in Two: Order in Logic

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Keith Austin teaches pure mathematics and logic at the University of Sheffield. Whenever he gets over-enthusiastic in his logic classes then he is brought down to earth by recalling the following cartoon. Two elderly ladies are members of a jury which is being addressed by counsel. One is remarking to the other, 'His kind of logic certainly isn't my kind of logic'.

First boy: Every girl from the girls' school has been out with a boy from the boys' school.
Second boy: Lucky fellow.

The basis of this joke is the ambiguity in the first boy's statement. To see the two ways his statement can be understood we introduce some logical notation.

We write $H(x, y)$ to denote the expression,

girl x has been out with boy y.

We write

$\forall x$ to denote 'for every x';
$\exists y$ to denote 'there exists a y such that'.

The first boy meant that for every girl x there is a boy y who has been out with x. We write this as

$$\forall x\, \exists y\, H(x, y),$$

where the order, $\forall x$ first, then $\exists y$, indicates we select a girl and then find an appropriate boy.

The second boy understood the first boy to mean that there is a boy y who has been out with every girl x. We write this as

$$\exists y\, \forall x\, H(x, y),$$

where the order, $\exists y$ first, then $\forall x$, indicates we can first find a suitable boy who is appropriate for every girl.

Guest (to manager of hotel): I don't like all those mice in my room.
Manager: Tell me which ones you do like and I will get rid of the rest.

Write $L(x)$ to denote the expression,

the guest likes mouse x,

and write $\neg$ to denote 'not'.

The guest means that for every mouse x it is the case that he does not like x. We write this as

$$\forall x \neg L(x),$$

where $\neg L(x)$ means that the guest does not like mouse x.

The manager takes the guest to mean that it is not the case that for every mouse x he likes x. We write this is

$$\neg \forall x\, L(x),$$

where $\forall x\, L(x)$ means that for every mouse x the guest likes x.

H and L are called predicates, $\forall$ and $\exists$ are called quantifiers, $\neg$ is called a logical connective, and x and y are called variables. Predicates, quantifiers, connectives and variables are used to express the majority of the key ideas of logic. We shall illustrate a very common logical situation by looking first at a familiar type of chess problem (see figure 1).

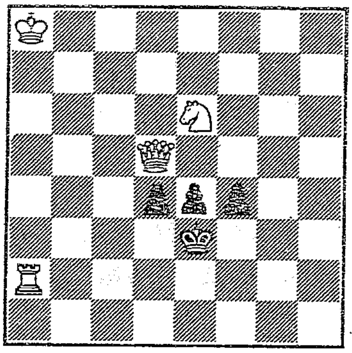

Figure 1. White to move and mate in two

What do we have to do to solve the problem? We have to find a first move for white, call it x. Having made move x, we list all the possible moves for black, call them $y_1, y_2, y_3, \ldots, y_k$. For each y_i, we make that move and then find a move for white which gives mate, call it z_i. Thus our answer looks like this

$$\begin{array}{ccccc} x & & & & \\ y_1 & y_2 & y_3 & \cdots & y_k \\ | & | & | & & | \\ z_1 & z_2 & z_3 & \cdots & z_k \end{array}$$

For this particular problem the answer is

$$\begin{array}{cccc} \text{Q} \to \text{e5} & & & \\ \text{P} \to \text{d3} & \text{P} \to \text{f3} & \text{K} \to \text{d3} & \text{K} \to \text{f3} \\ | & | & | & | \\ \text{Q takes P on f4} & \text{Q takes P on d4} & \text{Q takes P on d4} & \text{Q takes P on f4} \end{array}$$

Given any layout of pieces on the board and the instruction,

'white to move and mate in two',

we have a problem, $\mathcal{P}$. For some layouts, such as the one above, the problem $\mathcal{P}$ will have a solution, but for others there will not be a solution.

We now express the statement that problem $\mathcal{P}$ has a solution, in logical terms. Write $M(x, y, z)$ to denote the following expression.

> If we start from the given position and white makes move x, then black makes move y and then white makes move z, then the resulting position has black in checkmate.

Then the statement that $\mathcal{P}$ has a solution can be expressed as

$$\exists x\, \forall y\, \exists z\, M(x, y, z).$$

The statement that $\mathcal{P}$ does not have a solution can be expressed as

$$\neg \exists x\, \forall y\, \exists z\, M(x, y, z).$$

However, we now look at this in detail. Consider the problem in figure 2.

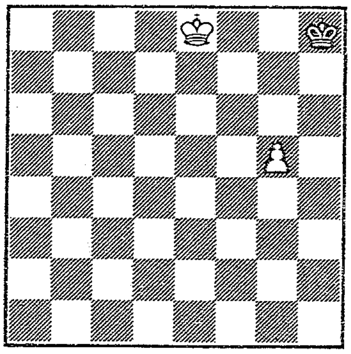

Figure 2. White to move and mate in two

What do we do to show this does not have a solution? We have to list all the possible moves for white, call them $x_1, x_2, \ldots, x_p$. For each x_i, we make that move and then find a move for black, call it y_i, so that if we make y_i and then list all the possible moves for white, $z_1^i, z_2^i, \ldots, z_{q_i}^i$, then none of them mates black. Thus our answer looks like figure 3.

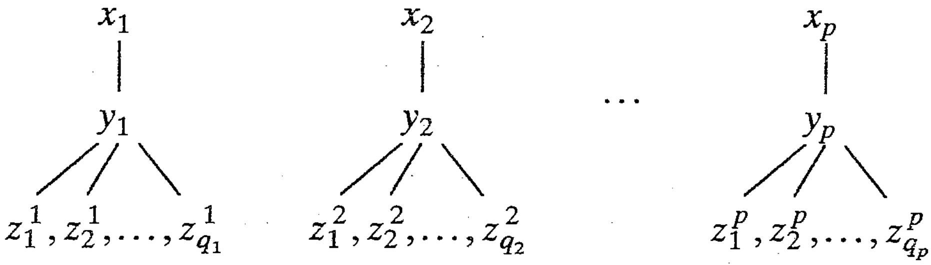

Figure 3

For our particular problem, the answer is given in figure 4.

$$\begin{array}{c}
P \to g6 \\
| \\
K \to g8 \\
K \to e7,\ K \to d8,\ P \to g7,\ K \to d7
\end{array}
\qquad\qquad
\begin{array}{c}
K \to f8 \\
| \\
K \to h7 \\
K \to e8,\ K \to e7,\ K \to f7,\ P \to g6
\end{array}$$

and similarly for K$\to$f7, K$\to$e7, K$\to$d8 and K$\to$d7.

Figure 4

On the basis of the form of this solution we can give another logical expression for the statement that the problem does not have a solution, namely,

$$\forall x\, \exists y\, \forall z\, \neg M(x, y, z).$$

To see the connection between the two expressions for the statement that the problem does not have a solution, we return to the guest and the mice. We have seen $\forall x\, \neg L(x)$ and $\neg \forall x\, L(x)$ say different things. Now consider $\exists x\, \neg L(x)$ and $\neg \exists x\, L(x)$. First, $\exists x\, \neg L(x)$ says that there is a mouse x which the guest does not like, which is the same as saying it is not the case that the guest likes every mouse. Thus $\exists x\, \neg L(x)$ and $\neg \forall x\, L(x)$ say the same thing. Next, $\neg \exists x\, L(x)$ says it is not the case that there is a mouse x that the guest likes, which is the same as saying that, for every mouse x, the guest does not like x. Thus $\neg \exists x\, L(x)$ and $\forall x\, \neg L(x)$ say the same thing.

If we write $\equiv$ for 'says the same thing as', then we have

$$\neg \forall x\, L(x) \equiv \exists x\, \neg L(x), \qquad \neg \exists x\, L(x) \equiv \forall x\, \neg L(x).$$

These were found by a semantic approach, i.e. by considering the meaning, but they give us a syntactic rule, i.e. about symbols, namely,

if we pass a $\neg$ through a quantifier we change the quantifier.

If we apply this rule to the chess statement, we obtain

$$\begin{aligned}
\neg \exists x\, \forall y\, \exists z\, M(x, y, z) &\equiv \forall x\, \neg \forall y\, \exists z\, M(x, y, z) \\
&\equiv \forall x\, \exists y\, \neg \exists z\, M(x, y, z) \\
&\equiv \forall x\, \exists y\, \forall z\, \neg M(x, y, z).
\end{aligned}$$

The logical ideas we have just considered are clearly not confined to the chessboard, so we will look at the same ideas in another situation.

The students in a class are shown a list of ten towns. The girls are asked which towns they have visited, and the boys are asked which towns they have written letters to. The replies were as in figure 5, where a line between a student and a town indicates that student visited or wrote to that town.

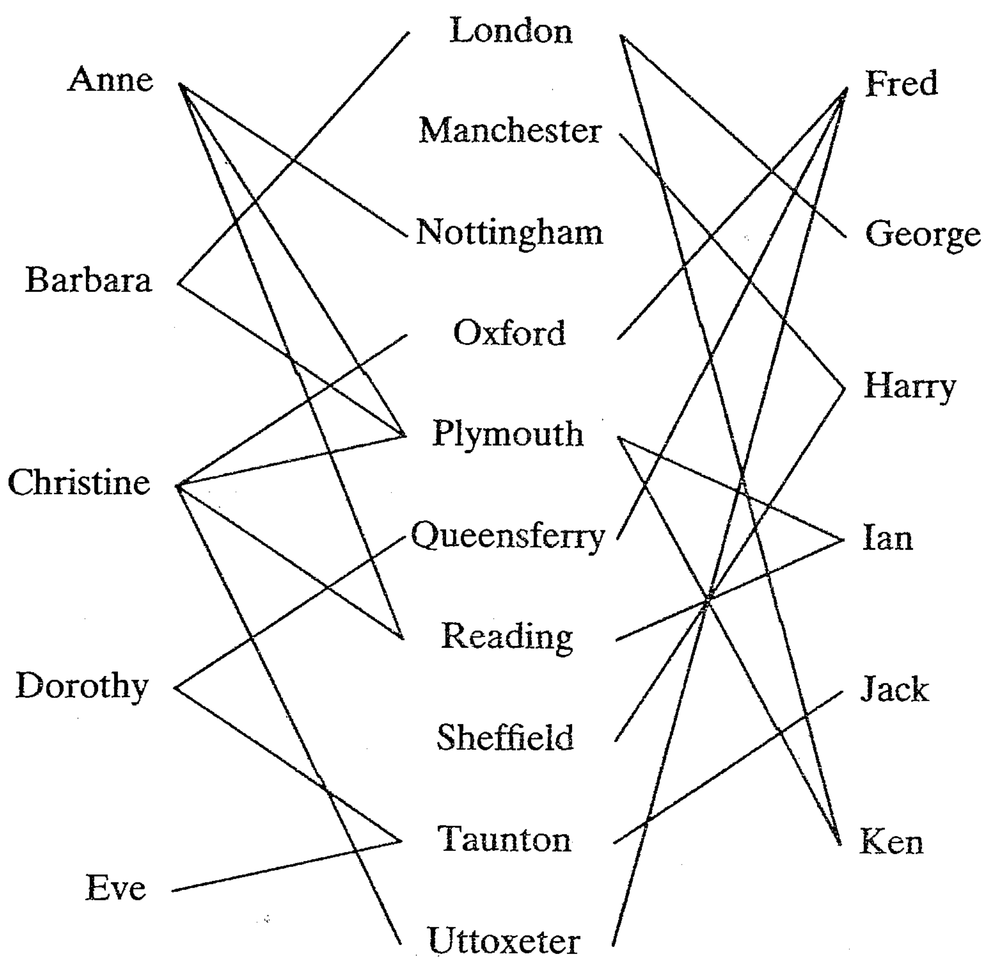

Figure 5

Write

$V(x, z)$ for 'girl x has visited town z'
$W(y, z)$ for 'boy y has written to town z'.

Is the following statement true or false:

$$\forall x\, \exists y\, \forall z\, [W(y, z) \rightarrow V(x, z)]\ ?$$

Note $[W(y, z) \rightarrow V(x, z)]$ means 'if $W(y, z)$ is true then $V(x, z)$ is true'.

To show the statement is true we have to show that for each girl x we can find a suitable boy y so that, when we consider any town z that y has written to, we find x has visited z.

Recall, from the chess situation, that we do this by producing a three-row table, as in figure 6.

On the other hand, if Ian had also written to Nottingham, then the statement in the question would have been false. We show this by finding a suitable girl x so that for each boy y there is a town z which y has written to but x has not visited. As with the chess, we express this in a three-row table, as in figure 7.

$$
\begin{tikzcd}[column sep=small]
 & x_1 = \text{Anne} \arrow[d, dash] & & & \text{Barbara} \arrow[d, dash] & & & \text{Christine} \arrow[d, dash] & & \text{Dorothy} \arrow[d, dash] & \text{Eve} \arrow[d, dash] \\
 & y_1 = \text{Ian} \arrow[dl, dash] \arrow[dr, dash] & & & \text{Ken}^{(2)} \arrow[dl, dash] \arrow[dr, dash] & & & \text{Ian} \arrow[dl, dash] \arrow[dr, dash] & & \text{Jack} \arrow[d, dash] & \text{Jack} \arrow[d, dash] \\
z_1^1 = \text{P}, & & z_2^1 = \text{R}^{(1)} & \text{L} & & \text{P} & \text{P} & & \text{R} & \text{T} & \text{T}
\end{tikzcd}
$$

Notes: (1) We only consider those z_i^1 for which $W(y_1, z_i^1)$ is true.
(2) We could have used George instead of Ken.

Figure 6

Christine

$$
\begin{tikzcd}
\text{Fred} \arrow[d, dash] & \text{George} \arrow[d, dash] & \text{Harry} \arrow[d, dash] & \text{Ian} \arrow[d, dash] & \text{Jack} \arrow[d, dash] & \text{Ken} \arrow[d, dash] \\
\text{Q} & \text{L} & \text{M} & \text{N} & \text{T} & \text{L}
\end{tikzcd}
$$

Figure 7

Finally we note that $\forall x \exists y \forall z$ is a very common combination in mathematics. For example, the statement that $1/z$ tends to 0 as z tends to infinity is taken to mean the same as

$$\forall x \exists y \forall z \left[(z > y) \rightarrow \left(\frac{1}{z} < x \right) \right],$$

where x, y and z are positive numbers.

What do we have to do to prove the expression is true? Let us look back to see what we did in the chess situation. Recall that, in chess terms, $\forall x \exists y \forall z$ is associated with success for black; our opponent chooses an x move, we choose a y move, and our opponent chooses a z move. Thus the proof that the expression is true goes as follows:

our opponent chooses a positive number x;
we choose y to be $\dfrac{1}{x}$, which is positive;
our opponent chooses a positive number z;
 if $z > y$ then $z > \dfrac{1}{x}$, so $\dfrac{1}{z} < x$ as required,
 if $z \leqslant y$ then there is nothing to prove.

Exercises. Find a key y move to show the following statements are true:

$$(a)\quad \forall x \exists y \forall z \left[(z > y) \rightarrow \left(\frac{2}{z} < x \right) \right];$$

$$(b)\quad \forall x\, \exists y\, \forall z \left[(z > y) \to \left(\frac{1}{\sqrt{z}} < x\right)\right].$$

On the other hand, suppose we wish to show an expression is not true, e.g.

$$\forall x\, \exists y\, \forall z \left[(z > y) \to \left(\frac{z}{1+z} < x\right)\right]. \tag{1}$$

Now, saying (1) is not true is the same as saying

$$\neg \forall x\, \exists y\, \forall z \left[(z > y) \to \left(\frac{z}{1+z} < x\right)\right]$$

is true, which is the same as saying

$$\exists x\, \forall y\, \exists z\, \neg \left[(z > y) \to \left(\frac{z}{1+z} < x\right)\right]$$

is true, which is the same as saying

$$\exists x\, \forall y\, \exists z \left[(z > y)\ \&\ \neg \left(\frac{z}{1+z} < x\right)\right] \tag{2}$$

is true.

Note that saying $A \to B$ is not true is the same as saying that A is true and B is not true.

Now $\exists x\, \forall y\, \exists z$ is associated with success for white; we choose an x move, our opponent chooses a y move, and we choose a z move. Thus the proof that (2), and hence (1), is true goes as follows,

> we choose x to be $\frac{1}{2}$;
> our opponent chooses a positive number y;
> we choose z to be $y+1$, which is positive;
> $z > y$ since $z = y+1$,
> $\dfrac{z}{1+z} = 1 - \dfrac{1}{1+z}$, but $1+z > 2$, so $\dfrac{1}{2} > \dfrac{1}{1+z}$, so $1 - \dfrac{1}{1+z} > \dfrac{1}{2}$,
> so $\dfrac{z}{1+z} > \dfrac{1}{2}$, so $\dfrac{z}{1+z} > x$, so $\dfrac{z}{1+z} < x$ is not true.

Exercises. Find key x and z moves to show that the following statements are not true:

$$(a)\quad \forall x\, \exists y\, \forall z \left[(z > y) \to \left(\frac{z}{10+z} < x\right)\right];$$

$$(b)\quad \forall x\, \exists y\, \forall z\, [(z > y) \to (z < x)].$$

Tailpiece. Good news for the hotel guest. The manager introduced a cat into the room and each mouse x became an x-mouse.

A Simple Proof of a Weaker Form of Stirling's Formula

ALEXANDER ABIAN, *Iowa State University*

Alexander Abian is a professor of mathematics at Iowa State University, Ames, Iowa, USA. He is the author of some 200 research papers in set theory, algebra, analysis and topology. Although his main interest is in axiomatic set theory and transfinite methods in mathematics, he does not draw a sharp demarcation line between pure and applied mathematics. In fact, for him, any clever reasoning is good mathematics, no matter where it is done.

In mathematics in general and in statistics and probability in particular, the *factorial of a positive integer*, denoted by $n!$, occurs quite frequently and has a special significance. Thus, it is desirable to have expressions for $n!$ which yield easily evaluated approximations to $n!$ for large n. One such expression for $n!$ is the remarkable Stirling formula:

$$n! = n^n e^{-n}\sqrt{2\pi n}\exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + \dots\right). \tag{1}$$

The higher terms in the series are determined by (17) of reference 1, and are usually quite elaborate.

What is so significant about (1) is that, even if we replace the entire power series appearing in (1) by 0, we obtain quite a satisfactory approximation to $n!$ for sufficiently large n. Readers might like to try this for $n = 10$.

Based on the facts that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} c^{1/n} = 1$$

for $c > 0$, one can readily prove that (1) implies

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e, \tag{2}$$

which we call 'a weaker form of Stirling's formula'.

We shall prove (2) independently of (1) in a simple way. Our proof is based on the fact that, for a sequence of real numbers $a_n > 0$,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a \quad \text{implies} \quad \lim_{n\to\infty} a_n^{1/n} = a. \tag{3}$$

We first prove (3).

Suppose first that $a > 0$, and let ϵ be such that $0 < \epsilon < a$. Then, according to the hypothesis of (3), there exists a natural number N such that, for every natural number p,

$$a-\epsilon \leqslant \frac{a_{N+1}}{a_N} \leqslant a+\epsilon,$$

$$a-\epsilon \leqslant \frac{a_{N+2}}{a_{N+1}} \leqslant a+\epsilon,$$

$$a-\epsilon \leqslant \frac{a_{N+3}}{a_{N+2}} \leqslant a+\epsilon,$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$a-\epsilon \leqslant \frac{a_{N+p}}{a_{N+p-1}} \leqslant a+\epsilon.$$

Multiplying the above inequalities, we have

$$(a-\epsilon)^p \leqslant \frac{a_{N+p}}{a_N} \leqslant (a+\epsilon)^p \tag{4}$$

and therefore we obtain

$$a_N^{1/(N+p)}(a-\epsilon)^{p/(N+p)} \leqslant a_{N+p}^{1/(N+p)} \leqslant a_N^{1/(N+p)}(a+\epsilon)^{p/(N+p)}. \tag{5}$$

It can be easily verified that, as $p \to \infty$, the sequence on the left in (5) converges to $a-\epsilon$; similarly, the sequence on the right converges to $a+\epsilon$. This (as (5) shows) means that, for sufficiently large p,

$$a-2\epsilon \leqslant a_{N+p}^{1/(N+p)} \leqslant a+2\epsilon. \tag{6}$$

By the definition of a limit, (6) implies that $\lim\limits_{n\to\infty} a_n^{1/n}$ exists and is equal to a. Thus, (3) is proved for $a > 0$. When $a = 0$, we can modify (4) and say that, for $\epsilon > 0$, there exists a natural number N such that, for every natural number p, $0 \leqslant a_{N+p}/a_N \leqslant \epsilon^p$, and a reasoning analogous to the case of $a > 0$ proves (3) for $a = 0$. Thus, (3) is established.

Now, to prove (2), we let $a_n = n^n/n!$ in (3) and easily derive

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = \lim_{n\to\infty}\frac{(n+1)^{n+1}n!}{(n+1)!\,n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e},$$

which proves (2), as desired.

Reference

1. Namias, V. A simple derivation of Stirling's asymptotic series. *Amer. Math. Monthly* **93**, 25–29 (1986).

Self-Altitude or Golden Triangles

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In Volume 22 Number 2 the author discussed self-median triangles. Now, in a similar spirit, he discusses self-altitude triangles. He would like to know of your investigations if they are different from his.

A *self-altitude* triangle is one in which the altitudes are proportional to the sides. The alternative name for such a triangle is a *golden triangle*: we keep you in suspense about this for a while. Suppose AL, BM and CN are the altitudes of a triangle ABC whose sides $a = BC$, $b = CA$ and $c = AB$ satisfy the ordering $c \leqslant a \leqslant b$ as in the previous article. (See figure 1.) Then

$$a \cdot AL = b \cdot BM = c \cdot CN = 2\Delta, \tag{1}$$

where Δ equals the area of triangle ABC. The altitudes acquire the ordering mentioned in the following theorem.

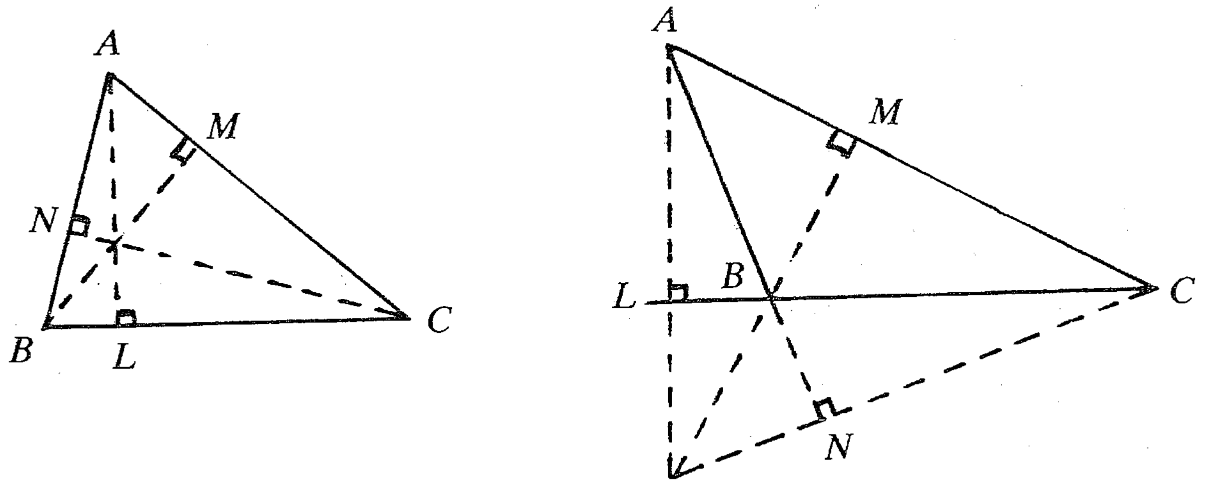

Figure 1

Theorem 1. Suppose ABC is a triangle in which $c \leqslant a \leqslant b$. Then the altitudes AL, BM and CN satisfy the relations

$$BM \leqslant AL \leqslant CN. \tag{2}$$

Proof. This is an immediate consequence of (1).

A characterisation of self-altitude triangles may be given as follows. It may be compared to the corresponding result for self-median triangles given previously.

Theorem 2. A triangle is a self-altitude triangle if and only if its sides are in geometric progression.

Proof. Consider the triangle ABC with altitudes AL, BM and CN and suppose that $c \leqslant a \leqslant b$. Assume first that the triangle is self-altitude. Therefore, by theorem 1,

$$\frac{AL}{a} = \frac{BM}{c} = \frac{CN}{b}. \tag{3}$$

Together (3) and (1) imply that

$$\frac{c}{a} = \frac{BM}{AL} = \frac{a}{b}.$$

Thus, the sides c, a and b are in geometric progression.

Conversely, if the sides c, a and b are in geometric progression, then $a^2 = bc$ and, from (1),

$$\frac{AL}{a} = \frac{2\Delta}{a^2} = \frac{2\Delta}{bc} = \frac{BM}{c} = \frac{CN}{b}$$

and so the triangle is self-altitude.

Theorem 2 at once shows that a *non-equilateral isosceles triangle cannot be a self-altitude triangle.*

Our ordering of the sides of a self-altitude triangle implies that the common ratio

$$r = \frac{c}{a} = \frac{a}{b}$$

is less than or equal to 1. Hence every self-altitude triangle (c, a, b) is similar to the triangle $(r^2, r, 1)$ for an appropriate value of r. The following theorem places bounds on r.

Theorem 3. The value r $(\leqslant 1)$ corresponds to a self-altitude triangle $(r^2, r, 1)$ if and only if $r > \frac{1}{2}(\sqrt{5}-1)$.

Proof. The lengths r^2, r and 1 are the sides of a triangle if and only if $r^2+r > 1$, from which the result follows.

For readers inquisitive about right-angled self-altitude triangles, here is the problem to solve:

> Prove that any right-angled, self-altitude triangle must be similar to the triangle whose sides are
>
> $$\tfrac{1}{2}(\sqrt{5}-1), \qquad \sqrt{\tfrac{1}{2}(\sqrt{5}-1)}, \qquad 1.$$

Many readers will no doubt be quick to spot the 'golden ratio' $\frac{1}{2}(\sqrt{5}-1)$. Ancient Greek geometers defined that a line segment AC is divided in golden section at the point B if

$$\frac{AB}{BC} = \frac{BC}{AC}, \quad \text{i.e.,} \quad \frac{c}{a} = \frac{a}{b},$$

a degenerate case of our self-altitude triangles in which $c+a = b$ or $r^2+r-1 = 0$. Hence the alternative name of *golden triangles* for self-altitude triangles.

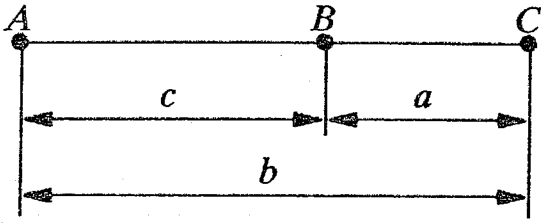

Figure 2

These self-altitude triangles are interesting in one more respect. Any three successive numbers from a geometric sequence in which the common ratio r satisfies $\frac{1}{2}(\sqrt{5}-1) < r \leqslant 1$, such as $\ldots$, 256, 192, 144, 108, 81, $\ldots$, are the lengths of the sides of a self-altitude triangle. Suppose we select two triangles that have two common sides, such as $(108, 144, 192)$ and $(81, 108, 144)$. It is easy to see that these two triangles are similar to each other and have five elements in common—three angles and two sides—yet they are incongruent.

We have shown that the self-altitude triangles are those whose sides are in geometric progression, and we have found bounds for the common ratio of this progression. We saw also in our previous article that the self-median triangles are those the squares of whose sides are in arithmetic progression. Readers may like to consider if there are corresponding bounds on the common difference of this progression. Also, it would be interesting to determine the triangles in which some other trio of concurrent lines are proportional to the sides.

More from Ramanujan's notebook

The following ingenious identity occurs in Ramanujan's notebook:

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}} = \tfrac{1}{3}\{\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}\}.$$

Can you prove it? It can be used to express various real numbers as squares of numbers involving cube roots. For example, if we put $m = n = 1$ we obtain $\sqrt[3]{5}-\sqrt[3]{4}$ as the square of $\frac{1}{3}\{5^{2/3}-4^{1/3}5^{1/3}-2^{1/3}\}$.

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Can you crack this?

A tiny folded strip of card dropped out of my Christmas cracker. On it were printed the following six tables:

1	3	5	7	9	11	13	15
17	19	21	23	25	27	29	31
33	35	37	39	41	43	45	47
49	51	53	55	57	59	61	63

2	3	6	7	10	11	14	15
18	19	22	23	26	27	30	31
34	35	38	39	42	43	46	47
50	51	54	55	58	59	62	63

4	5	6	7	12	13	14	15
20	21	22	23	28	29	30	31
36	37	38	39	44	45	46	47
52	53	54	55	60	61	62	63

8	9	10	11	12	13	14	15
24	25	26	27	28	29	30	31
40	41	42	43	44	45	46	47
56	57	58	59	60	61	62	63

16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55
56	57	58	59	60	61	62	63

A friend was to be asked to choose a number from one of the tables, and then to examine all six tables and to indicate to me in which of these his chosen number appeared. I was then to reveal what number he had chosen. To do so, I was told to add up the entries in the top left-hand corners of the indicated tables. Can you work out why this always produces the right answer?

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Which job?

Morris Kline, in *Mathematics and the Search for Knowledge* (OUP, 1986), argues that mathematics is needed for certain types of knowledge of the world, and gives examples of puzzles where he claims that use of our intuition without rigorous mathematics leads us astray. For example:

> 'Two jobs, A and B, each have starting salaries of \$1800, but A gives an annual rise of \$200 and B a semiannual rise of \$50. Which job is preferable in the long run?'

Kline suggests that our intuition would lead us to judge A to be best, whereas use of mathematics shows B to be better. Kline demonstrates this by asking us to consider how much each job pays at six-monthly intervals, and gives A: (\$) 900, 900, 1000, 1000, 1100, 1100, ..., B: 900, 950, 1000, 1050, 1100, 1150, ..., whence B pays more after the first six months!

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A Sequence Free from Powers

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The author wrote this article whilst he was a postgraduate student.

Readers may know Euclid's proof that there are infinitely many prime numbers. Assume the contrary, and denote the distinct prime numbers by $p_1, p_2, \ldots, p_t$, in increasing order. Thus $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on. Put

$$T = (p_1 p_2 \ldots p_t) + 1.$$

Clearly $p_1, p_2, \ldots, p_t$ do not divide T (in fact, the remainder when T is divided by any one of these numbers is 1), yet T must have prime factors. Hence there must be prime numbers other than $p_1, p_2, \ldots, p_t$ after all.

Denote the infinite sequence of prime numbers by $p_1, p_2, \ldots$, in increasing order and, for each positive integer n, write

$$T_n = (p_1 p_2 \ldots p_n) + 1.$$

Thus $T_1 = 3$, $T_2 = 7$, and so on. We shall prove that *none of the T_n is a power* (excluding, of course, the first power), i.e. *for each n, there are no positive integers N and $p > 1$ such that $T_n = N^p$.*

Fix n. Clearly we may suppose that $n > 2$, so that

$$T_n = (2 \times 3 \times 5 \times \ldots \times p_n) + 1 \equiv 1 \pmod{10}.$$

We suppose that N and $p > 1$ exist such that $T_n = N^p$, and derive a contradiction. We may suppose that p is prime, since if $p = rs$, where $r, s \in \mathbb{Z}$ and s is prime, then $T_r = (N^r)^s$. Since T_n is odd, N must be odd. Also, $N \not\equiv 5 \pmod{10}$ because $5 \nmid T_n$. Hence $N \equiv \pm 1, \pm 3 \pmod{10}$. If $N \equiv \pm 3 \pmod{10}$, then the powers of N congruent to $1 \pmod{10}$ are $N^4, N^8, N^{12}, \ldots$. Since p is prime, we cannot have $T_n = N^p$. If $N \equiv -1 \pmod{10}$, then the powers of N congruent to $1 \pmod{10}$ are $N^2, N^4, N^6, \ldots$. Since p is prime, this means that $p = 2$. Thus

$$T_n = N^2 = (10x - 1)^2 \quad (\text{for some } x \in \mathbb{Z}),$$

so that

$$(p_1 p_2 \ldots p_n) + 1 = 100x^2 - 20x + 1$$

$$\Rightarrow \qquad \tfrac{1}{10} p_1 p_2 \ldots p_n = 10x^2 - 2x. \qquad (1)$$

But the left-hand side of (1) is an odd integer, whereas the right-hand side is even. Hence $N \not\equiv -1 \pmod{10}$. The same argument shows that, when $N \equiv 1 \pmod{10}$, then p cannot be 2.

We are left with the case $N \equiv 1 \pmod{10}$ and p a prime greater than 2. We write $N = 10x+1$, where $x \in \mathbb{Z}$, so that

$$\begin{aligned} T_n &= (10x+1)^p \\ &= 10^p x^p \binom{p}{1} 10^{p-1} x^{p-1} + \ldots + \binom{p}{p-1} 10x + 1, \end{aligned}$$

so that

$$\tfrac{1}{10} p_1 p_2 \cdots p_n = 10^{p-1} x^p + \binom{p}{1} 10^{p-2} x^{p-1} + \ldots + \binom{p}{p-1} x. \qquad (2)$$

If $p = 5$, then 5 divides the right-hand side of (2) but not the left-hand side. (Note that a prime number p divides all the binomial coefficients $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$.) Hence $p \neq 5$. Also, $p \nmid x$, otherwise p^2 divides the right-hand side of (2) yet p^2 does not divide this left-hand side. Further, $N > p_n$. (For, if $N \leqslant p_n$, then one of $p_1, \ldots, p_n$ is a prime factor of N, yet $(p_1 p_2 \cdots p_n) + 1 = N^p$.) Hence $N^p > p_n^p$, so that

$$\begin{aligned} & (p_1 p_2 \cdots p_n) + 1 > p_n^p \\ \Rightarrow \quad & p_1 p_2 \cdots p_n \geqslant p_n p_n \cdots p_n \quad (p \text{ terms}) \\ \Rightarrow \quad & n > p \\ \Rightarrow \quad & p_n > n > p. \end{aligned}$$

Hence p divides the left-hand side of (2). But $p \nmid 10^{p-1} x^p$ and p divides every other term on the right-hand side, so that p does not divide the right-hand side. This is the final contradiction that we need to prove the result.

As a corollary of our result, we can prove that, for every $n > 1$, *T_n has at most $n-1$ factors in its prime factorization.* For suppose the contrary. Then, for some $n > 1$, T_n has at least n factors in its prime factorization. Since $p_1, \ldots, p_n$ do not divide T_n, this means that $T_n \geqslant p_{n+1}^n$, whence $T_n > p_{n+1}^n$ by our main result. Hence

$$p_{n+1}^n + 1 > p_1 p_2 \cdots p_n + 1 > p_{n+1}^n,$$

and this is clearly impossible.

My ultimate aim is to decide which T_n are prime; I believe that there are infinitely many of these.

Re-using a diary

I recently found an unused 1980 diary. In which year can I use it?

Error-Correcting Codes I

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1. Introduction

Coding theory is a new and rapidly developing area of mathematics. In this article and its sequel we shall describe some of the important applications of error-correcting codes and also show how fast decoding methods may be implemented with the help of a little algebra.

In this first article we give a brief introduction to error-correcting codes, with some simple examples, and describe three practical applications of error-correcting codes—the transmission of pictures from deep space, compact discs and computer memories. We also give an application in pure mathematics, showing how error-correcting codes played two distinct and vital roles in the recent solution of a famous long-standing problem—does there exist a projective plane of order 10?

From the mathematical point of view, much of the fascination of coding theory lies in its close links with well-established areas of pure mathematics such as algebra, number theory, combinatorics and geometry. Space does not permit us to explore all of these, and in the second article, to appear in a subsequent issue of *Mathematical Spectrum*, we consider just one such link—the algebraic decoding of a certain family of codes. First we consider two easy examples, the single-error-detecting code of International Standard Book Numbers, and then a code which could serve as a set of telephone numbers for the UK—with the facility that any single error could be automatically corrected! We then show that the decoding of more powerful codes requires the solution of a certain system of simultaneous non-linear equations. The study of these equations turns out to have a surprisingly long history. The great Indian mathematician Ramanujan showed how to solve precisely such a system in 1912 and, even more astonishingly, a Frenchman named Prony did it as long ago as 1795. We shall explain why Prony needed to solve the equations, and how he did it—with a little help from Lagrange.

2. Introduction to error-correcting codes

Error-correcting codes are used to correct errors when messages are transmitted through a noisy communication channel. For example, we may wish to send binary data (a stream of 0s and 1s) through a noisy channel as quickly and as reliably as possible. The channel might be a telephone line, a satellite communication link, or a magnetic tape or disc. The noise might be human error, lightning, thermal noise, imperfections in equipment, etc., and may result in errors, so that the data received are different from those sent. The object of an error-correcting code is to encode the data, by adding a certain amount of redundancy to the message, so that the original message can be recovered if (not too many) errors have occurred. A general digital communication system is shown in figure 1.

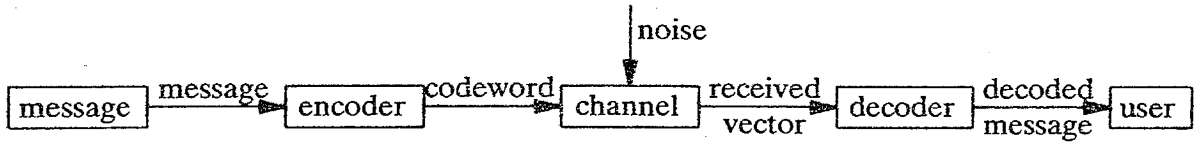


Figure 1

Let us look at a very simple example in which the only messages we wish to send are 'YES' and 'NO'.

Example 1 (See figure 2.)

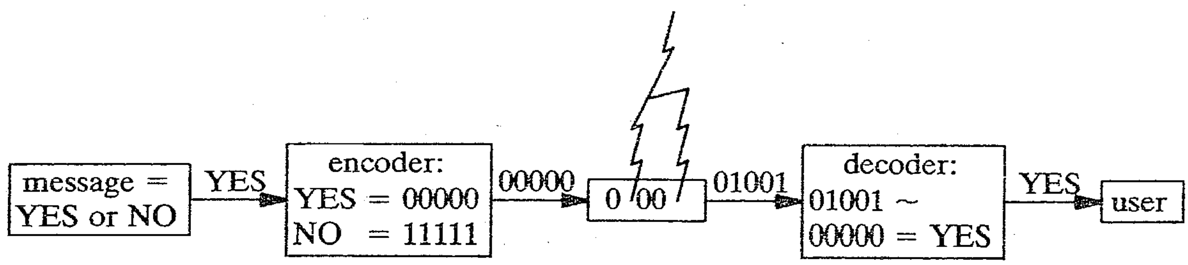


Figure 2

Here two errors have occurred and the decoder has decoded the received vector 01001 as the 'nearest' codeword, which is 00000 or YES.

A *binary code* is just a given set of *codewords*, each codeword being a sequence of 0s and 1s.

The code of example 1 consists of the codewords 00000 and 11111. If the messages YES and NO are identified with the symbols 0 and 1, respectively, then each message symbol is encoded simply by repeating the symbol five times. This is an example of how redundancy can be added to messages to protect them against noise. The extra symbols sent are themselves subject to error and so there is no way to *guarantee* accuracy; we just try to make the probability of accuracy as high as possible. The above code is good for error correction as it is guaranteed to correct up to two errors in any codeword, but it is not very efficient because the codewords contain 80% redundancy.

The set $\{0,1\}$ is called the *alphabet* of a binary code. More generally, a *q-ary code* is a given set of codewords, where each codeword is a sequence of symbols from an alphabet of q symbols.

For example, the set of all street names in the city of Salford is a 27-ary code (the space between words is the 27th symbol) and provides a good example of poor encoding, for two street names on the same estate are HILLFIELD DRIVE and MILLFIELD DRIVE.

A code in which each codeword consists of a fixed number n of symbols is called a *block code* of *length* n. From now on, our attention will be restricted to such codes.

Example 2. The set of all 10-digit telephone numbers in the United Kingdom is a 10-ary (or *decimal*) code of length 10. Little thought appears to have been given to allocating numbers in such a way that the frequency of 'wrong numbers' is minimized. Many wrong numbers are obtained by the misdialling of a single digit. The problems this can cause are illustrated by a recent news story. Because of a typing error, in which a 9 was typed as an 8, 90 000 government employees were given the wrong number to ring if they had queries about pay. This led to an unfortunate telephone subscriber in Gateshead receiving over 200 calls a day. After 7 000 calls in 28 days, British Telecom finally changed the subscriber's telephone number.

Yet it is possible to use a code of over 82 million 10-digit telephone numbers (enough for the needs of the UK) such that if just one digit of any number is misdialled the correct connection can nevertheless be made. We shall describe this code in the second article.

Example 3. Suppose we need just four codewords, e.g. to represent the four directions N, S, E and W. The fastest binary code we could use is

$$C_1 = \begin{cases} 00 = \mathrm{N} \\ 01 = \mathrm{W} \\ 10 = \mathrm{E} \\ 11 = \mathrm{S} \end{cases},$$

but this is useless for detecting or correcting errors. Now consider the length-3 code we get by adding a parity-check symbol to each codeword of C_1, i.e. we add a 0 or a 1 so as to make the total number of 1s in each codeword an even number. This gives the code

$$C_2 = \begin{cases} 000 \\ 011 \\ 101 \\ 110 \end{cases}.$$

Now, if any single error occurs, the received vector will have an odd number of 1s and so we can detect the error and ask for retransmission. C_2 is an example of a single-error *detecting* code. But if messages cannot be retransmitted, e.g. in receiving pictures from space or in playing back an old magnetic tape, we need to try to *correct* errors in received vectors. The code C_2 would be of no use here; for example, given the received vector 001 and assuming a single error, the correct codeword might be any of the first three codewords of C_2. However, by the suitable addition of two further digits to each codeword we get the length-5 code.

$$C_3 = \begin{cases} 00000 \\ 01101 \\ 10110 \\ 11011 \end{cases}.$$

Any two codewords of C_3 differ from each other in at least three places and so, if a single error occurs in any codeword, the received vector will still be closer to the transmitted codeword than to any other. The code C_3 has the facility to correct any single error. If used for error detection only, C_3 can detect any double error.

For many applications we can envisage binary block codes being used as follows. If a stream of binary data is to be sent through a noisy channel, we first break the data into blocks of k bits (binary digits) and then encode each of these blocks into a codeword of length n by adding $n-k$ check bits. In the code of example 1, each message bit is encoded as a codeword of length 5 by the addition of 4 check bits (80% redundancy) while, if we use code C_3 of example 3, each block of 2 message bits is encoded as a codeword of length 5 by the addition of 3 check bits (60% redundancy).

Example 4. Our final introductory example is a single-error correcting code, called an *array code*, which has a particularly easy decoding algorithm. Suppose the stream of binary data is grouped into blocks of 9 message bits. The 9 bits are arranged in a 3×3 array:

$$\begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix}.$$

A check bit is added to each row and column. The values of these check bits $x_{14}, x_{24}, x_{34}, x_{41}, x_{42}$ and x_{43} are chosen so as to make the number of 1s in each row and in each column an even number. Finally a further check bit x_{44} is added so that the total number of 1s in the block is made even. This results in a length-16 codeword represented by the following array, in which each row and each column has an even number of 1s:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{matrix}.$$

For example, the message block

110
101
001

is encoded as

1100
1010
0011
0101

If now a bit, say x_{23}, is incorrectly received, then the number of 1s in the second row and in the third column will no longer be even, and so a 0 at position x_{23} can be changed into a 1 or vice versa, thus correcting the error. If used for error detection, the array code is guaranteed to detect up to three errors in any codeword. The minimum undetectable error pattern consists of four errors so placed as to define a rectangle in the large array.

Although simple, the array codes are not very efficient. For example, the above array code has only 9 message bits in the 16 bits of a codeword. In 1950 Richard Hamming of Bell Telephone Laboratories described a class of error-correcting codes which can correct any single error or detect any three errors. One of these, of length 16, has 11 message bits and only five check bits.

3. Some applications of error-correcting codes

3.1. *Pictures from deep space.* In 1965, Mariner 4 was the first spaceship to photograph another planet, taking 22 photographs of Mars. Each picture was broken down into 200×200 picture elements. Each element was assigned a binary 6-tuple representing one of 64 brightness levels from white (= 000000) to black (= 111111). Interference in the transmission of such messages across vast distances of space led to rather blurred pictures being received (the power of a spaceship's transmitter is less than that of an ordinary light bulb).

From 1969 to 1976 much sharper pictures were obtained by Mariners 6 to 9, partly because each picture was now broken down into 700×832 elements, and partly because a powerful code called a Reed–Muller code was used. Each binary 6-tuple was encoded as a codeword of length 32 by the addition of 26 check bits. The Reed–Muller code used is an example of a 7-error correcting code.

In 1976, Viking 1 landed on Mars and returned high quality *colour* photographs. Surprisingly, transmission of a colour picture in the form of binary data is almost as easy as transmission of a black-and-white one. It may be achieved simply by taking the same black-and-white photograph several times, each time through a different coloured filter.

From 1979, high-quality colour pictures of Mars, Venus, Jupiter, Saturn and Uranus (and their moons) have been returned by the Voyager spacecraft. These include some exciting discoveries such as an erupting volcano on Io, one of the moons of Jupiter. In 1989 Voyager 2 passed Neptune and further spectacular pictures were received.

3.2 *Compact discs*. Just as a picture can be broken into tiny pieces and represented as data, so also can sound. For example, a compact disc is made by sampling the sound 44 100 times per second and recording each fragment by a binary vector of length 16, thus allowing $2^{16} = 65\,536$ different levels of sound for each fragment. The error-correcting scheme, devised at Philips Research Laboratories in Eindhoven in the late 1970s, involves the clever use of not one but two powerful codes called Reed–Solomon codes. The second code is able to mop up errors detected but not corrected by the first code. The information stored on the discs is broken up into blocks, or 'frames'. Each frame consists of 192 message bits to which are added 64 check bits (25% redundancy).

Errors on the disc could originate from defects in the manufacturing process, damage during use, or fingermarks or dust on the disc. A flaw such as a scratch can produce a train of errors called an error burst. To counter the effects of such error bursts, the information on the disc is 'interleaved'. This means that the bits representing one fragment of sound are scattered over a large number of frames and stored at various parts of the disc (these bits arrive together at the output by means of delay lines of differing lengths). As a result, errors coming in a large burst are distributed as small numbers of errors over a large number of codewords, and so come within the error-correcting capability of the code. The code is thus able to correct an error-burst of up to 4000 bits. To put it another way, it should be possible to drill a hole 3 mm in diameter through a compact disc with no loss of sound quality (I haven't tried this for myself!).

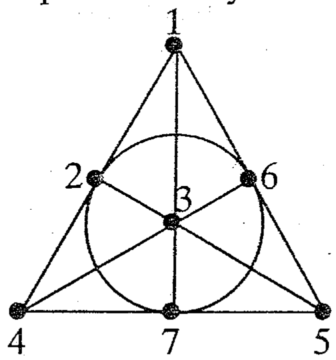
3.3. *Reliability of computer memories*. Almost all modern computers have memories that are built from silicon chips. It may surprise readers to know that a stored 0 or 1 in a memory chip can spontaneously switch to what it should not be! Such errors are usually caused by alpha particles released during radioactive decay. Experiments indicate that the mean time before a single memory cell will fail as a result of a hit by an alpha particle is about a million years. The problem is that a typical computer memory has millions of memory cells. So for a one megabyte memory, for example, the mean time before failure is only about 40 days. There is no known economically feasible way to shield a computer memory against alpha particles—we have a situation where cure is better than prevention! By using a simple single-error correcting Hamming code, and at the cost of increasing the number of bits stored by about 20% (32-bit messages get stored as 39-bit codewords), one can increase the mean time before failure of a one megabyte memory from 40 days to over 60 years. For greater reliability, one can use more powerful codes. For an interesting article on the use of error-correcting codes in computer memories, see reference 1.

3.4. *An application in pure mathematics—the non-existence of a projective plane of order 10.* The solution of one of the most famous unsolved problems in mathematics was recently announced by a team of mathematicians and computer scientists at Concordia University in Montreal. They claim to have proved the non-existence of a finite projective plane of order 10. As was the case with the solution of another famous problem—the four-colour conjecture, proved in 1976—the proof is heavily reliant on the use of computers. The proof also depends vitally on the use of error-correcting-codes—in two quite different ways!

First, we give a brief description of the problem. A *finite projective plane of order n*, with $n > 0$, is a collection of n^2+n+1 lines and n^2+n+1 points such that

1. every line contains $n+1$ points;
2. every point lies on $n+1$ lines;
3. any two distinct lines intersect in exactly one point;
4. any two distinct points lie on exactly one line.

The smallest example of a finite projective plane is a triangle, the plane of order 1, with just three lines and three points. The smallest non-trivial example is of order 2, as shown in figure 3. There are seven points labelled 1 to 7. There are also seven lines labelled $L1$ to $L7$. Six of them are straight lines but $L6$ is represented by the circle through 2, 6 and 7.



$L1 = \{1,2,4\}$, $L5 = \{5,6,1\}$,
$L2 = \{2,3,5\}$, $L6 = \{6,7,2\}$,
$L3 = \{3,4,6\}$, $L7 = \{7,1,3\}$.
$L4 = \{4,5,7\}$,

Figure 3. The finite projective plane of order 2

It is easy to show that, if there exists a certain algebraic structure called a finite field of order q, then there exists a finite projective plane of order q. (Consider the three-dimensional vector space V of ordered triples over such a field; call the two-dimensional subspaces of V 'lines' and the one-dimensional subspaces 'points'. A point p is said to lie on line L if p is a subspace of L.) It was shown by Évariste Galois (the great French mathematician, who died in a duel at the age of 20 in 1832) that there exists a finite field of order q if and only if q is a prime power. So there exists a projective plane of order n for every prime power n.

Whether or not there exists a projective plane of order n with n not a prime power has long remained an open problem. Indeed, the problem has an alternative formulation, in terms of orthogonal Latin squares, dating back to Euler in 1782. Euler's conjecture that there does not exist a projective plane of order 6 was proved by the Frenchman G. Tarry in 1901 by systematic enumeration. The next case where n is not a prime power is $n = 10$, and this case has proved far more resistant to attack.

A breakthrough came in 1970 when it was shown that a binary error-correcting code could be associated with a finite projective plane (not wishing to go into the technicalities here, I merely mention that the code is a binary linear code generated by the rows of the incidence matrix of the plane). Much was proved about the structure of such a code. This led to the feasibility of an exhaustive computer search for the projective plane of order 10, starting from a relatively small number of starting configurations and trying to complete the rest of the structure. A number of attempts were made during the 1970s and significant progress was made, but the problem withstood the challenge. In 1980, Clement Lam, Larry Thiel and Stanley Swierczy at Concordia University in Montreal began a determined attack. Nine years later a proof was announced! The computing time taken had been about three years on a VAX-11/780 and 2000 hours on a CRAY-1A.

Of course, the claimants are well aware of the difficulties associated with such a 'proof', and went to great lengths to build in checking procedures. However, they acknowledge that their result is only an experimental one, needing an independent verification or, better still, a theoretical explanation.

They are also aware of the possibility of an undetected hardware failure. A common error of this type is the random changing of bits in the computer memory (as described in application 3), which could result in the loss of a branch of the search-tree. So we see that error-correcting codes play a second vital role in the computer-generated proof—this time as a practical rather than theoretical tool—in reducing the number of undetected hardware errors to a manageable level.

Even with the protection of error-correcting codes, the CRAY-1A is reported to have undetected random errors at the rate of about one every 1000 hours of computing. This means that the part of the Concordia team's 'proof' carried out on the CRAY is almost guaranteed to contain about two random errors! The authors of the proof address this problem and argue that the possibility of hardware errors leading to a wrong conclusion is extremely small. Basically, their argument depends on the observation that, if a projective plane of order 10 exists, then it could be constructed from many different starting points, and random hardware failures would be unlikely to eliminate all of them.

For further details of the proof of the non-existence of the projective plane of order 10, reference 2 provides a good expository account while reference 3 gives the full technical details.

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Computer Column

MIKE PIFF

Polynomial division

The long division of one polynomial by another is a particularly irksome task, as readers will be well aware. However, to a computer, this is yet another trivial chore for it to take in its stride. Continuing with our excursion into *Modula2*, the following extract will divide the polynomial with coefficients a by that with coefficients b, setting q to the quotient and r to the remainder after division. For programmers in BASIC or any other reasonable language, the procedure `divide` is the one which does all the work, and its operation is very simple. All it does is to match the next coefficient of r to that in b by selecting a suitable `factor`, and then subtract b multiplied by that factor, till r is of lower degree than b.

This program is not good *Modula2*. The polynomials should be stored in what are known as *records*, together with their degrees, and also, the whole implementation of `divide` should be in a completely different module. However, first things first, and this program is only going to introduce one new feature of *Modula2*, namely, *parameter passing*. The only complication we need to dwell on is the fact that parameters q and r of `divide` are preceded by `VAR`. This indicates that those two variables are modified when the procedure is called. Thus, a and b are not changed by `divide` even though they appear to have been. The procedure takes a copy of a and b and modifies that copy. The modified copy is destroyed when the procedure terminates. On the other hand, q and r are physically changed afterwards into the quotient and remainder.

```
MODULE dividepolynomials;
FROM InOut IMPORT WriteString,WriteLn;
FROM RealInOut IMPORT ReadReal,WriteReal;
CONST  n=5;  tolerance=1.0E-20;
TYPE
   degrees=[0..n];
   polys=ARRAY degrees OF REAL;
VAR a,b,q,r:polys;   i:degrees;
PROCEDURE deg(a:polys):degrees;
   VAR   i:degrees;
   BEGIN
      i:=n;
      WHILE (i>0)AND(ABS(a[i])<tolerance) DO
         i:=i-1
      END;
      RETURN i
   END deg;
PROCEDURE zero(a:polys);
   VAR   i:degrees;
   BEGIN
      FOR i:=0 TO n DO a[i]:=0.0;END;
   END zero;
PROCEDURE divide(a,b:polys;VAR q,r:polys);
VAR   i,degr,degb :degrees;
   factor,leadb:REAL;
BEGIN
   zero(q);   r:=a;   degr:=deg(r);   degb:=deg(b);
   leadb:=b[degb];
   WHILE degr>=degb DO
      factor:=r[degr]/leadb;
      q[degr-degb]:=factor;
      FOR i:=0 TO degb DO
         r[i+degr-degb]:=r[i+degr-degb]-factor*b[i];
      END;
      degr:=degr-1;
   END;
   a[2]:=999.0;(*An unnecessary statement, to show
      that a[2] did not really change*)
END divide;
BEGIN
   WriteString('Input coeffts of a in ascending order');
   FOR i:=0 TO n DO ReadReal(a[i]) END;WriteLn;
   WriteString('Input coeffts of b in ascending order');
   FOR i:=0 TO n DO ReadReal(b[i]) END;WriteLn;
   divide(a,b,q,r);
   WriteString('Quotient has coefficients');WriteLn;
   FOR i:=0 TO n DO WriteReal(q[i],10); END;WriteLn;
   WriteString('Remainder has coefficients');WriteLn;
   FOR i:=0 TO n DO WriteReal(r[i],10); END;WriteLn;
END dividePolynomials.
```

Letters to the Editor

Dear Editor,

Modula2

I was interested in the coverage of Modula2 in the Computer Column (Volume 22 Number 2), but think I may be able to help you here. I notice that you are advising people to go out and buy a Modula2 compiler for around £50 to £60. There is no need to do this; I have a public domain compiler which comes on two $5\frac{1}{4}$ inch disks (for the PC) which I can let readers have for £5.00 including postage and disks. Readers may write to me directly; I do not mind copying the disks as required. I hope this will encourage more of your readers to take up Modula2. The two disks contain the compiler, an inbuilt text editor and full documentation. I can also supply the software on a single $3\frac{1}{2}$ inch disk if required.

Yours sincerely,

ANDY LUNNESS
(69 Bronte Avenue, Bury, Lancashire BL9 9RN)

Dear Editor,

π from Pascal's triangle

I was interested to read Alan Fearnehough's letter in *Mathematical Spectrum*, Volume 22, Number 2, pages 62–63. In that letter he showed that the quantity $2\pi/3\sqrt{3}$ may be calculated from Pascal's triangle, specifically from the numbers occurring on the central axis as indicated in bold below.

$$\begin{array}{ccccccccccccc}
&&&&&&\mathbf{1}&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&\mathbf{2}&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&\mathbf{6}&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&\mathbf{20}&&15&&6&&1
\end{array}$$

The sequence $1, 2, 6, 20, \ldots$ being labelled $A_1, A_2, A_3, \ldots$, it was shown that

$$\frac{2\pi}{3\sqrt{3}} = \sum_{r=1}^{\infty} \frac{1}{(2r-1)A_r}.$$

The summands are simple but the calculated quantity involves both $\sqrt{3}$ and π. At the cost of a slightly more complicated summand we can evaluate $\frac{1}{3}\pi$. In a letter to this magazine, Volume 21, Number 3, pages 97–98, I mentioned that $A_1, A_2, A_3, \ldots$ are the coefficients of the binomial expansion of $(1-4x)^{-\frac{1}{2}}$. That is,

$$(1-4x)^{-\frac{1}{2}} = \sum_{r=1}^{\infty} A_r x^{r-1}, \quad \text{for } |x| < \tfrac{1}{4}.$$

If we substitute $x = y^2$ we obtain the following formula:

$$(1-4y^2)^{-\frac{1}{2}} = \sum_{r=1}^{\infty} A_r y^{2r-2}, \quad \text{for } |y| < \tfrac{1}{2}.$$

Integrating with respect to y, we have

$$\tfrac{1}{2}\arcsin 2y = \sum_{r=1}^{\infty} \frac{A_r y^{2r-1}}{2r-1}, \quad \text{for } |y| < \tfrac{1}{2}.$$

Setting $y = \frac{1}{4}$ and then multiplying through by 4 we conclude that

$$\tfrac{1}{3}\pi = \sum_{r=1}^{\infty} \frac{A_r}{(2r-1)16^{r-1}}.$$

In passing, I had hoped that $\sum_{r=1}^{\infty} 1/A_r$ might prove to be a significant constant. In fact it is $\frac{2}{27}(18+\sqrt{3}\pi)$, which readers may wish to deduce from Alan Fearnehough's method.

Yours sincerely,
IAN RICHARDS
(Penwith Sixth Form College, Penzance)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

22.9 (Submitted by Victor Nicola, South Bank Polytechnic)
An anthill is in the shape of a right circular cone. The diameter of the base of the cone is 20 mm and its side is 30 mm. An ant wishes to go (above ground) from a point A on the base of the hill to the point B diametrically opposite. If the ant can move at the rate of 1 mm/s, how long would the journey take if it took the fastest route?

22.10 (Submitted by J. Prasad, National University of Lesotho)
Prove that

$$\sum_{k=0}^{n} (-1)^{n-k} 2^{2k} \binom{n+k}{2k} = 2n+1.$$

22.11 (Submitted by P. Glaister, University of Reading))
A ball is thrown vertically upwards in a uniform gravitational field and experiences a force of resistance per unit mass of magnitude proportional to its speed. How does the time taken by the ball to reach the highest point compare with the time taken for it to return from the highest point back to the point of projection?

22.12 (Submitted by Ahmet Ozban, Karadeniz Technical University, Turkey) Let α be an $(n+1)$-digit number ($n \geqslant 1$) with digits $x_0, x_1, \ldots, x_n$, so that

$$\alpha = x_n \times 10^n + x_{n-1} \times 10^{n-1} + \ldots + x_1 \times 10 + x_0.$$

Prove that

$$\alpha \geqslant \sum_{i=0}^{n} x_i + \prod_{i=0}^{n} x_i$$

and determine for which numbers there is equality.

Solutions to Problems in Volume 22 Number 1

22.1. Show that

$$s_1 = 1, \qquad s_1 = 1, \qquad s_n = s_{s_{n-1}} + s_{n-s_{n-1}} \quad (n > 2),$$

defines a sequence and that, for $n > 1$, $s_n = s_{n-1}$ or $s_{n-1}+1$.

Solution by Amites Sarkar (Winchester College)

Clearly s_1 and s_2 are defined and $s_2 = s_1$. Consider an integer $k \geqslant 2$ and assume inductively that $s_1, \ldots, s_k$ are defined and that the given property holds for all values of n from 2 to k. Then $1 \leqslant s_n \leqslant n$ for all n from 1 to k. Hence $s_k \leqslant k$ and $k+1-s_k \leqslant k$, so that s_{k+1} is defined by the formula $s_{k+1} = s_{s_k} + s_{k+1-s_k}$. We now consider four cases.

(1) $s_k = s_{k-1}$, $s_{k-s_{k-1}+1} = s_{k-s_{k-1}}$. Then

$$s_{k+1} = s_{s_k} + s_{k+1-s_k} = s_{s_{k-1}} + s_{k-s_{k-1}} = s_k.$$

(2) $s_k = s_{k-1}$, $s_{k-s_{k-1}+1} = s_{k-s_{k-1}}+1$. Then

$$s_{k+1} = s_{s_k} + s_{k+1-s_k} = s_{s_{k-1}} + s_{k+1-s_{k-1}} = s_{s_{k-1}} + s_{k-s_{k-1}} + 1 = s_k + 1.$$

(3) $s_k = s_{k-1}+1$, $s_{s_{k-1}+1} = s_{s_{k-1}}$. Then

$$s_{k+1} = s_{s_k} + s_{k+1-s_k} = s_{s_{k-1}+1} + s_{k-s_{k-1}} = s_{s_{k-1}} + s_{k-s_{k-1}} = s_k.$$

(4) $s_k = s_{k-1}+1$, $s_{s_{k-1}+1} = s_{s_{k-1}}+1$. Then

$$s_{k+1} = s_{s_k} + s_{k+1-s_k} = s_{s_{k-1}+1} + s_{k-s_{k-1}} = s_{s_{k-1}} + 1 + s_{k-s_{k-1}} = s_k + 1.$$

This completes the inductive step.

Also solved by Gregory Economides (University of Newcastle upon Tyne) and W. J. Church (Gresham's School, Holt).

22.2. A triangle with angles A, B and C and opposite sides of lengths a, b and c is *rational* if all the ratios a/b, b/c and c/a are rational. If $\cos A$, $\cos B$ and $\cos C$ are all rational, show that the triangle is rational.

Solution by Gregory Economides

We have

$$a\cos B + b\cos A = c, \qquad b\cos C + c\cos B = a.$$

Hence

$$\begin{aligned} b\cos C+(a\cos B+b\cos A)\cos B &= a \\ \Rightarrow \quad \frac{a}{b} &= \frac{\cos A\cos B+\cos C}{1-\cos^2 B}. \end{aligned}$$

There are similar expressions for b/c and c/a. Hence, if $\cos A$, $\cos B$ and $\cos C$ are all rational, a/b, b/c and c/a are rational and the triangle is rational.

Also solved by Amites Sarkar.

22.3. The power P transferred from a cell of e.m.f. E and internal resistance r to a purely resistive load of resistance R is given by

$$P = \frac{E^2R}{(R+r)^2}.$$

Find, *without calculus*, the value of R for which P attains its maximum value.

Solution by Amites Sarkar

The power P is maximized when

$$\frac{E^2}{P}-2r = R+\frac{r^2}{R}$$

is minimized. But

$$0 \leqslant \left(\sqrt{R}-\frac{r}{\sqrt{R}}\right)^2 = R+\frac{r^2}{R}-2r,$$

so that $R+(r^2/R) \geqslant 2r$ and $R+(r^2/R)$ is minimized when $R+(r^2/R) = 2r$, i.e. when $R = r$.

Also solved by Aimee Davidson (Form 4C, Hutchesons' Grammar School, Glasgow) and W. J. Church.

22.4. Evaluate the integral

$$\int_0^1 \frac{x^2-1}{\ln x}\,\mathrm{d}x.$$

Solution by Amites Sarkar

For any $\beta > 0$

$$\int_{\alpha=0}^{\beta}\int_{x=0}^{1} x^\alpha\,\mathrm{d}x\,\mathrm{d}\alpha = \int_{\alpha=0}^{\beta}\frac{1}{\alpha+1}\,\mathrm{d}\alpha = \ln(\beta+1).$$

But also

$$\begin{aligned} \int_{\alpha=0}^{\beta}\int_{x=0}^{1} x^\alpha\,\mathrm{d}x\,\mathrm{d}\alpha &= \int_{x=0}^{1}\int_{\alpha=0}^{\beta} x^\alpha\,\mathrm{d}\alpha\,\mathrm{d}x \\ &= \int_{x=0}^{1}\left[\frac{x^\alpha}{\ln x}\right]_{\alpha=0}^{\beta}\mathrm{d}x \\ &= \int_{x=0}^{1}\frac{x^\beta-1}{\ln x}\,\mathrm{d}x. \end{aligned}$$

Hence

$$\int_0^1 \frac{x^\beta - 1}{\ln x}\,dx = \ln(\beta+1).$$

Now put $\beta = 2$ to give the answer, $\ln 3$.

Also solved by Gregory Economides and W. J. Church.

Reviews

Dictionary of Mathematics. By E. J. BOROWSKI and J. M. BORWEIN. Collins, London and Glasgow, 1989. Pp. xi+659. Paperback £5·95. (ISBN 0-00-434347-6).

In the introduction it is explained that this dictionary is mainly intended for undergraduates who, it is hoped, will find in it not only the terms that figure in their lecture courses, but also those they might meet when exploring neighbouring territories. Since syllabuses vary enormously in the English-speaking world, and statistics, numerical analysis and applied mathematics as well as pure mathematics are covered, the resulting dictionary is necessarily quite a bulky volume. Some of its scope is indicated by entries such as 'abacus', 'bar-chart', 'catastrophe theory', 'dihedral group', 'Hooke's law', 'Laurent series', 'Napierian logarithm', 'round-off error', 'step function', 'Waring's problem'.

A welcome feature of the dictionary is the inclusion of short biographies of mathematicians. However, it is not quite clear on what basis the selection has been made. The major figures of the past are there, but the twentieth century is represented less systematically. Ramanujan has a longish entry; Ramsey's theorem is stated, but Ramsey himself gets no mention; Hardy's paragraph is mainly devoted to the mathematically trivial Hardy–Weinberg ratio from genetics; and the great Littlewood is altogether absent.

There are three appendices: one on symbols and conventions, then a table of derivatives and integrals, and finally the famous list of 23 problems posed by Hilbert in 1901, together with an indication of the progress that has since been made in solving them.

Every author of a book is faced by the question of what to include and what to leave out, but for the compilers of dictionaries the problem is particularly acute. In spite of my grouse about the biographies, I think that very much the right balance has been struck in this work. Undergraduates will find it invaluable and, at £5·95, it can hardly be out of their reach. It can even be afforded by professional mathematicians for whom it will provide many hours' informative browsing.

University of Sheffield H. BURKILL

The Historical Roots of Elementary Mathematics. By LUCAS N. H. BUNT, PHILLIP S. JONES and JACK B. BEDIENT. Dover Publications, New York, 1988. Pp. xiii + 299. Paperback £5·95. (ISBN 0-486-25563-8).

This book tries to relate the arithmetic, algebra and geometry that is taught in school today to its original roots. 'Do long division as the ancient Egyptians did! Solve quadratic equations like the Babylonians! Study geometry just as students did in Euclid's day!' is what the blurb promises. There is much in this book of use to the teacher of mathematics, especially an extended discussion of a Babylonian tablet describing the solution of a pair of simultaneous equations, one quadratic. There is too much geometry for the modern British audience, but that isn't the authors' fault. More serious is the prevailing dullness and worthiness. It is like eating yoghurt, very nutritious and digestible, but I wouldn't want to live on it. The authors try to liven it up with anecdotes about the mathematicians but often these are no more than legends. There are exercises which vary very greatly in difficulty. All the original materials and much more can be found in *The History of Mathematics—a Reader* edited by J. Fauvel and J. Gray and published by Macmillan and the Open University Press at £14·25. This latter book is really exciting and I read great chunks of it to my classes who especially like the match report of Cardano *versus* Tartaglia, and Berkeley tearing into Newton.

Winchester College NICK MACKINNON

Alice in Numberland: A Student's Guide to the Enjoyment of Higher Mathematics. By JOHN BAYLIS AND ROD HAGGARTY. Macmillan Education, Basingstoke, 1988. £8·95. (ISBN 0-333-44242-3).

This book seeks to provide an entertaining first glimpse of higher mathematics for anyone embarking on a degree course in the subject, and to show with the aid of humorous dialogue and reader involvement that rigour and abstraction can be both satisfying and enjoyable. Rather than covering vast syllabus areas, the book digs deeply into a few key topics centred around the beginnings of number theory, abstract algebra and analysis. The authors are to be congratulated on achieving a large measure of success in their difficult but worthy task, and in demonstrating that a reader-friendly style need not be incompatible with doing serious mathematics.

Whereas topics such as logic, permutations and induction are well suited for the able sixth-former, others such as Cauchy sequences, the Schroeder–Bernstein theorem and the construction of the real numbers from the rationals are more appropriate for the undergraduate. This causes no problem, however, since the book does not have to be read in the order in which it is written; depending on the knowledge and interests of the reader, there are convenient places to change the route, and these are clearly signposted. Just occasionally, the exposition dips below its generally high level. Sometimes central ideas are introduced at inappropriate moments, and at other times terms are introduced that are never defined. For example, in the proof that a real sequence is convergent if and only if it is Cauchy, much use is made of the triangle inequality and the notion of a subsequence; the former is discussed only after the proof, whereas the latter is never defined! There are instances where much space is devoted to establishing relatively simple results, immediately followed by more difficult results presented with no comment at all. There are a few errors, misprints and ambiguities in the text, and often mathematical formulae have been displayed with the printer in mind, rather than the reader. Despite the few reservations just mentioned, this refreshingly novel book can be recommended to any student or teacher who is interested in bridging the gap between the technique-based mathematics of the school and the proof-orientated mathematics of the university.

University of Sheffield R. J. WEBSTER

Introduction to Proofs in Mathematics. By JAMES FRANKLIN and ALBERT DAOUD. Prentice Hall of Australia, Sydney, 1988. $A.16·95. (ISBN 0-7428-1009-9).

The essence of mathematics is proof. Yet it seems to many teachers that students dislike being asked to prove, preferring problems with numerical content. If the teachers are right, why should this be so? I can think of two reasons.

(1) Many students do not understand the logical rules governing proof.
(2) Many students do not have the strategic sense to develop the sequence of statements in a proof to its conclusion. This is a failing of skill in the combination of the ideas.

This book attacks, for the most part, the first of these questions—it is a text in informal logic in which the logic of proof is discussed, without much symbolism. We find explanation of such as what induction is, how to prove a statement using 'if and only if', and the use of quantifiers. It goes on to the more sophisticated logic of analysis and abstract algebra. The logic is illustrated by examples, of which there are many, chosen largely from first-year undergraduate mathematics. Within its own aims it is successful, for it is clear and well produced, but I do not think that the student will want to refer to it more than occasionally. The real problem lies within (2) above. The insight needed to construct a proof successfully will stem from different sources in different people. Yet it is possible to reconstruct one's own thoughts and motivation, the cross-fertilisation of ideas, and spell them out in detail for the student, in answer to the perennial question, 'How did you know to do that?'. It is tiresome and exacting, but if we teachers are failing, it is this that needs to be done. (It would help too if we did not write out proofs, as is thought proper, so seamlessly as to conceal how we ever arrived at them.) Now this book misses the main chance. A variety of different proof techniques are shown and there is a section entitled 'Insight', but it lasts just two pages. In my view it should take up more than half the book and then the book would be a true mathematicians' bible, something that one could hand to undergraduates or bright school students and say 'Read this, and your work will be better for it'. As it is, it is a book to which one might sometimes direct a student in case of difficulty with the grammar of mathematics. It is a good source of small logical problems. If the quality of writing in this could be brought to bear on a second volume, a volume which would be harder to write, called *'How to think of Proofs in Mathematics'*, then together they would represent a considerable achievement.

Penwith Sixth Form College I. M. RICHARDS

For All Practical Purposes: Introduction to Contemporary Mathematics. By COMAP (Project Director SOLOMON GARFUNKEL). W. H. Freeman and Company, Oxford, 1988. Pp. xii + 450. Hardback £23·95. (ISBN 0-7167-1830-8).

Here is a book which answers the questions 'What is mathematics used for?' and 'What do mathematicians do, practice or believe in?' Written by thirteen contributing authors for the Consortium for Mathematics and its Applications, it stresses the connections between contemporary mathematics and modern society at a level which portrays contemporary mathematical thinking for the non-specialist; it aims to develop conceptual understanding and appreciation rather than computational expertise.

The self-contained main topic sections are management science, statistics, social choice, the geometry of size and shape, and mathematics for computer science. The lavishly illustrated material will appeal to readers with a wide range of mathematical experience and objectives. Each chapter includes optional sections, problems providing practice in calculation, and review vocabulary; there are also notes about famous mathematicians and comments from people who put contemporary mathematics to practical use. This is a fascinating book for the general reader and would be a valuable addition for the school library.

University of Sheffield MAVIS HITCHCOCK

Advanced Level Statistics: An Integrated Course. 2nd Edition. By A. FRANCIS. Stanley Thornes Ltd, Cheltenham, 1988. Pp. 711. Paperback £9·95. (ISBN 0-85950-813-7).

This is a new edition, with additions, of a well-used and much-liked textbook. The topics which have been included since the first edition are logarithmic transformations to linear form for regression, use of binomial and Poisson tables and Poisson and normal graph paper, acceptance sampling techniques, control charts, elements of experimental design, and finally an appendix of statistical tables. Otherwise the text is just as in the first edition.

This textbook has a proven track record. It has plenty of worked examples, a clear exposition, emphasis on important results, relevant pure maths topics integrated at the appropriate points, and many exercises of A-level standard. It successfully aims 'to present an up-to-date text which covers the broad needs of all the examining boards'; the only topic that I need and could not find is the significance testing of the regression coefficients. However, if you are looking for a source of ideas for statistics practicals and projects, then this book will disappoint. But as a text that students can read, understand and learn from, it ranks with the best of its kind.

Solihull Sixth Form College CAROL NIXON

Riddles of the Sphinx and Other Mathematical Puzzle Tales, by MARTIN GARDNER. Mathematical Association of America/Wiley, 1987. Pp. 164. £13·00. (ISBN 0-88385-632-8).

It is a great pleasure to have yet another collection of puzzles from the grandmaster himself. The material in this book was drawn from the author's column in Isaac Asimov's Science Fiction Magazine—one of the best loved features of the magazine for over ten years.

For me the highlights of the present collection include:

3 On to Charmian, which leads to the Erdös–Szekeres theorem;

23 Dirac's Scissors, which introduces us to a delightful topological puzzle invented by Dirac to explain one of the strangest properties of the electron;

26 Blues in the night, where a famous murder trial is in progress on planet Chromo;

34 Time-reversed worlds, where Professor Cracker is writing a treatise on communication between the inhabitants of two worlds, each with a different time direction.

But all the riddles are good and many lead into areas of mathematics such as geometry, the theory of numbers, probability, combinatorics and logic, and to areas of science such as relativity, quantum mechanics and geology.

This book charms, informs, inspires, puzzles and delights, and the reader can dip in, almost anywhere, and get hooked by the natural lucidity of style and the friendly tone which are so characteristic of Martin Gardner. The book is well presented and beautifully printed, and both the publishers and I guarantee that you will be amused by the stories and settings that Martin Gardner has devised to raise these questions. This book will enrich any bookshelf.

Sixth Form, Royal Grammar School, Newcastle upon Tyne — GREGORY D. ECONOMIDES

Thomas Gray Philosopher Cat, by PHILIP J. DAVIS. Souvenir Press, London, 1989. Pp. 143. £12.95. (ISBN 0-285-62947-6).

If you find *belles lettres* a bit twee and the half-mythical life of a Cambridge Senior Common Room irrelevant, then you may miss a treat. This elegant little book, based on the true story of a real cat, manages in its gentle way to confront deep problems of modern life and thought. Only in a more open musing on the function of a university does Professor Davis's touch briefly desert him. His sketches of this city and the fens are vivid and crisp, the conversations worth overhearing. Trying not to give away the plot, I must say that I found his account of the high point of the life of his hero's human, Lucas Fysst, quite moving. It came as no surprise, as I reached the end, to find that Professor Davis was co-author of that marvellous book *The Mathematical Experience*. But look before you buy; your bookseller *ought* to have a copy.

The Perse School, Cambridge — JONATHAN L. G. PINHEY

L O N D O N   M A T H E M A T I C A L   S O C I E T Y

1990 POPULAR LECTURES

University of Sheffield - Friday 22 June
Imperial College London - Friday 29 June

Professor R.A. Bailey
Designing Experiments with Allowance for Interfering Neighbours

Dr P.J. Giblin
Geometry and Computers

Popular lectures are once again being organised by the London Mathematical Society. The lectures are given by mathematicians in a way that is accessible to a wide audience of teachers, sixth formers and people with a general interest in mathematics.

The Sheffield lectures will be given in Lecture Theatre 1, Medical School, Royal Hallamshire Hospital, Sheffield S10 2JF, on Friday 22 June commencing at 7.00 p.m. Enquiries to Dr A.K. Austin, Department of Pure Mathematics, University of Sheffield, Sheffield S3 7RH. Admission is free.

The London lectures will be at Imperial College London on Friday 29 June commencing at 7.30 p.m. Admission is free, by ticket obtainable in advance. Write to Miss S.M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V ONL. A stamped addressed envelope would be much appreciated.

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