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CONTENTS

From the Editors and Publisher of *Crux Mathematicorum*

It is with great regret that we inform the readers of *Crux Mathematicorum* of the death this past summer of the founding editor of *Crux Mathematicorum*, Léo Sauvé. A complete obituary is now being prepared which we hope to include in the next issue.

THE OLYMPIAD CORNER: 87

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

First I shall relay the information that has come my way about the 28th I.M.O. (Havana). Thanks for this information go to Cecil Rousseau, who, with Gregg Patruno led the American team; to Bruce Shawyer, who, with Ron Dunkley and Ron Scoins, led the Canadian team; and to Andy Liu.

The Twenty-eighth International Mathematical Olympiad (I.M.O.) was held this year in Havana, Cuba from July 5 to 16. Teams from 42 countries took part this year for a record turnout. The maximum team size for each country was again six students, the same as for the last four years.

The 1988 and 1989 I.M.O.'s are to be held in Australia and West Germany, respectively.

The six problems of the competition were assigned equal weights of seven points each (the same as in the last six I.M.O.'s) for a maximum possible individual score of 42 and a maximum possible team score of 252. For comparison purposes see the last six I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202] and [1986: 169].

This year it required a perfect score of 42 to receive a first (gold) prize. An amazing 22 perfect scores were recorded (compared to three in 1986 and two in 1985). The second (silver) prizes went to 42 students with scores in the range 32-41 while third (bronze) prizes went to 56 students with scores in the range 18-31. Congratulations to the 22 students who placed first with a perfect score:

Name	Country
Alexanarum Barcau	Rumania
Lev Borisov	U.S.S.R.
Kevin Buzzard	Great Britain
Ralph Costa Teixeira	Brazil
Nicusor Dan	Rumania
Jordan Ellenberg	U.S.A.
Jorg Eisfeld	West Germany
Frank Goring	East Germany
Martin Harterich	West Germany
Thorsten Kleinjung	West Germany
Gerd Kunnert	East Germany
Xiong Liu	China

Vladimir Mithov	Bulgaria
Andrei Mordianu	Rumania
Igor Pujov	U.S.S.R.
Wolfgang Schwarz	West Germany
Stanislav Smirnov	U.S.S.R.
Liviu Suciu	Rumania
Jun Teng	China
Ravi Vakil	Canada
Adrian Vasiu	Rumania
Eric Wepsic	U.S.A.

That the judges awarded third prizes to students with scores of 18 when there were 22 perfect scores must raise some questions. Is Murray Klamkin's criticism, given in detail in last year's report [1986: 169] and the year before [1985: 202], valid - are there too many prizes given? Are there any flaws in the type of question and the evaluation that would favour all or nothing performances? How can one encourage new participation when "perfection" seems to be the requirement?

As the I.M.O. is officially an individual event, the compilation and comparison of team scores is unofficial, if inevitable. These team scores are compiled by adding up the individual scores of the team members. These totals are given in the following table. Congratulations to the winning team from Rumania with a nearly perfect score!

Rank	Country	Score (max 252)	Prizes 1st	2nd	3rd	Total Prizes
1.	Rumania	250	5	1	-	6
2.	West Germany	248	4	2	-	6
3.	U.S.S.R.	235	3	3	-	6
4.	East Germany	231	2	3	1	6
5.	U.S.A.	220	2	3	1	6
6.	Hungary	218	-	5	1	6
7.	Bulgaria	210	1	3	2	6
8.	China	200	2	2	2	6
9.	Czechoslovakia	192	-	4	2	6
10.	Great Britain	182	1	2	2	5
11.	Vietnam	172	-	1	5	6
12.	France	154	-	3	2	5
13.	Austria	150	-	2	3	5
14.	Netherlands	146	-	1	4	5
15.	Australia	143	-	3	-	3
16.	Canada	139	1	1	1	3
17.	Sweden	134	-	2	2	4
18.	Yugoslavia	132	-	1	3	4
19.	Brazil	116	1	-	2	3
20.	Greece	111	-	-	4	4
21.	Turkey	94	-	-	2	2
22.	Spain	91	-	-	3	3
23.	Morocco	88	-	-	3	3
24.	Cuba	83	-	-	2	2

25.	Belgium	74	–	–	1	1	
26.	Iran	70	–	–	1	1	
27.-28.	Norway	69	–	–	–	0	
27.-28.	Finland	69	–	–	2	2	
29.	Colombia	68	–	–	1	1	
30.	Mongolia	67	–	–	–	0	
31.	Poland	55	–	–	2	2	(Team of 3)
32.	Iceland	45	–	–	–	0	(Team of 4)
33.	Cyprus	42	–	–	–	0	
34.	Peru	41	–	–	–	0	
35.	Italy	35	–	–	1	1	(Team of 4)
36.	Algeria	28	–	–	–	0	
37.-38.	Luxemburg	27	–	–	1	1	(Team of 1)
37.-38.	Uruguay	27	–	–	1	1	(Team of 4)
39.	Kuwait	19	–	–	–	0	
40.	Mexico	17	–	–	–	0	(Team of 5)
41.	Nicaragua	16	–	–	–	0	
42.	Panama	7	–	–	–	0	

An encouraging development this year was the perfect paper written by Jun Teng, a girl on the Chinese team. Also, Terence Tao, the now 11 year old Australian student mentioned by Murray last year [1986: 172], moved up to a Silver prize with an almost perfect score of 40.

This year the Canadian team again placed sixteenth with a team score of 139. The team members, scores and the leaders are as follows:

Ravi D. Vakil	42	(1st prize)
Colin Springer	33	(2nd prize)
Rocky Lee	19	(3rd prize)
Stevin Fry	17	
Gavin MacBeath	17	
David Lee	11	

Leaders: Bruce Shawyer, Memorial University of Newfoundland
Ron Dunkley, University of Waterloo
Ron Scoins, University of Waterloo.

The U.S.A. team slipped a little this year to fifth place, but individual performances were excellent. The team members and leaders were

Jordan Ellenberg	42	(1st prize)
Eric Wepsic	42	(1st prize)
Robert Southworth	38	(2nd prize)
William Schneeberger	36	(2nd prize)
John Woo	32	(2nd prize)
Matthew Cook	30	(3rd prize)

Leaders: Cecil Rousseau, Memphis State University
Gregg Patruno, Columbia University.

It is interesting to note that two members of the Canadian team also wrote the 1987 U.S.A.M.O. (see [1987: 173]) and placed in the top eight. The top eight students in that competition were

Matthew M. Cook
Samuel K. Vandervelde
Ravi D. Vakil
Jeremy A. Kahn
Daniel J. Bernstein
William A. Schneeberger
Elizabeth Lee Wilmer
Rocky Lee.

We give the problems of this year's competition below. Solutions to these problems, along with those of the 1987 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1987* which may be obtained for a small charge from

Dr. W.E. Mientka, Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, U.S.A. 68588.

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THE 28TH INTERNATIONAL MATHEMATICAL OLYMPIAD

Havana, Cuba

FIRST DAY

July 10, 1987

Time allowed: 4.5 hours

1. Let $P_n(k)$ be the number of permutations of the set $\{1,\ldots,n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot P_n(k) = n!.$$

(*Remark*: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$).

2. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal area.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \le k - 1$ for all i and

$$|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1} .$$

*

SECOND DAY

July 11, 1987

Time allowed: 4.5 hours

4. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

5. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

6. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \le k \le \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \le k \le n - 2$.

*　　　　*　　　　*

We next give solutions for the problems of the 19th Canadian Mathematics Olympiad (1987) posed in the last issue of the Corner [1987: 172]. All these solutions come from R. Nowakowski, Dalhousie University, who is chairman of the Canadian Mathematics Olympiad Committee of the Canadian Mathematical Society. We also give the top eight winners in this contest.

1. Find all solutions of $a^2 + b^2 = n!$ for positive integers a, b and n with $a \le b$ and $n < 14$.

Solution.

If $n < 3$ then there is by inspection only the one solution $1^2 + 1^2 = 2!$. Suppose $n \ge 3$. Then $n!$ is a multiple of 3. If one of a and b is not a multiple of 3 then neither is $a^2 + b^2$. Therefore both a and b are divisible by 3, but then $a^2 + b^2$ is divisible by 3^2 and so n must be at least 6, i.e. there are no solutions for $n = 3,4,5$. Suppose next that $n \ge 7$. Similar arguments apply. First $n!$ is a multiple of 7, but the quadratic residues modulo 7 are 1, 2 and 4 and no two add to 0 modulo 7. Therefore, both a and b are divisible by 7 and so $a^2 + b^2$ is a multiple of 7^2. This forces $n \ge 14$. Consequently there are no solutions for $7 \le n < 14$.

The only possibility remaining is $n = 6$, with a and b multiples of 3. Considerations working modulo 4 show that both a and b must be even, i.e. $a = 6k$, $b = 6j$ and we have to solve $720 = 36k^2 + 36j^2$. Now $20 = k^2 + j^2$, $k \leq j$, has the unique solution $k = 2$, $j = 4$. This gives the solution $a = 12$, $b = 24$. Thus the two solutions for $n < 14$ are $a = b = 1$, $n = 2$ and $a = 12$, $b = 24$, $n = 6$.

Conjecture.

It is conjectured that these are the only two solutions for all n.

2. The number 1987 can be written as a three-digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z and b.

Solution.

There are two equations (base 10):

$$x + y + z = 1 + 9 + 8 + 7 = 25 \qquad \text{and} \qquad xb^2 + yb + z = 1987.$$

Subtracting the former from the latter gives

$$x(b^2 - 1) + y(b - 1) = 1962.$$

Since $b - 1$ divides the left-hand side, it must divide the right. Since 1987 is a three digit number base b, it follows that $10 < b < 100$. (Of course tighter bounds can be found!) The only divisor of 1962 between 10 and 99 is 18. Therefore $b = 19$ and then $x = 5$, $y = 9$, $z = 11$. (Note that in base 19, 11 would be represented by a single digit.)

Additional Problems.

How many solutions are there if $x = 0$ or $x = y = 0$ is allowed? How many if $-b < x,y,z < b$?

3. Suppose $ABCD$ is a parallelogram and E is a point between B and C on the line BC. If the triangles DEC, BED, and BAD are isosceles, what are the possible values for the angle DAB?

Solution.

Note that triangles ABD and CDB are congruent so that only the triangles CDB, BED and DEC need be considered.

First suppose that $\angle DEC = 90^\circ$ (and therefore $\angle DEB = 90^\circ$). But then since BED and DEC are isosceles, it follows that $\angle EBD = 45^\circ = \angle EDB$ and $\angle EDC = \angle ECD = 45^\circ$. This gives $\angle BCD = 45^\circ$.

Otherwise one of $\angle DEC$ and $\angle DEB$ is obtuse. The symmetry allows us to consider a triangle DXY with $\angle DEY$ obtuse, and triangles DXY, DEX and DEY isosceles (Figure 1). Note that $\angle BCD$ could correspond to either of $\angle XYD$ or $\angle YXD$. Since $\angle DEY$ is obtuse, $EY = ED$ and $\angle EYD = \angle EDY$ in either case.

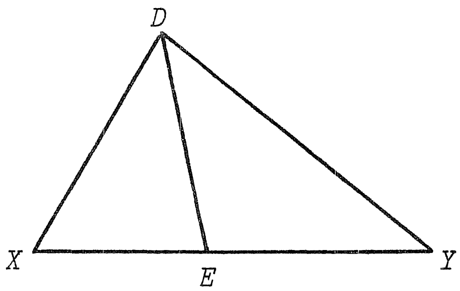


Figure 1

Consider now the large triangle DXY. First, note that if $DX = XY$ then this would force $E = X$, contrary to the hypothesis that E is between B and C. Suppose next that $DX = DY$. This leads to the situation depicted in Figure 2. The triangle DEX is isosceles and so, letting $\alpha = \angle EYD$, either $\alpha = 180^\circ - 3\alpha$ and $\alpha = 45^\circ$ or $2\alpha = 180^\circ - 3\alpha$ and $\alpha = 36^\circ$. This gives a new solution $\angle BCD = 36^\circ$.

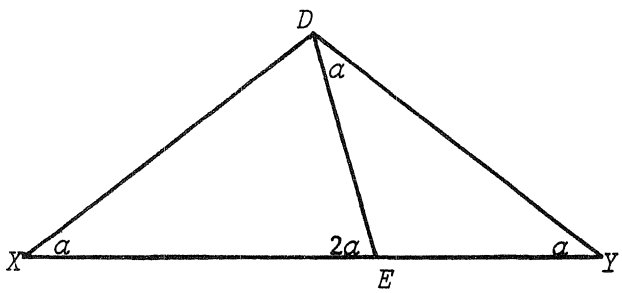


Figure 2

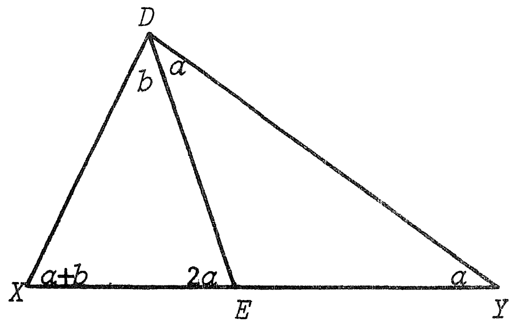


Figure 3

Suppose finally that $DY = XY$, and put $b = \angle EDX$. This leads to the situation in Figure 3 where $3\alpha + 2b = 180^\circ$. Since the triangle DEX is isosceles either $\alpha + b = 2\alpha$, i.e. $\alpha = 36^\circ$ and $\alpha + b = 72^\circ$; or $b = 2\alpha$, i.e. $\alpha = 180/7^\circ$ and $\alpha + b = 540/7^\circ$. Thus the remaining possible values of $\angle BCD$ are $180/7^\circ$, 72°, and $540/7^\circ$.

4. On a large flat field, n people are positioned so that for each person, the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even?

Solution.

If n is even it is not necessarily true that there is one dry person left. For example if $n = 2$ then they both shoot each other.

Suppose n is odd. If $n = 1$ then the result is true. Now proceed by induction. If $n > 1$ and is odd then consider a pair of people who are closest. They shoot at each other. If no other person fires at them then we are left with the case $n - 2$ and by induction at least one person remains dry. If someone fires at the pair then there are $n - 3$ squirts left and $n - 2$ people. So again one person must remain dry.

5. For every positive integer n, show that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$$

where $[x]$ is the greatest integer less than or equal to x (for example, $[2.3] = 2$, $[\pi] = 3$, and $[5] = 5$).

Solution.

Since for $n \geq 1$,

$$(\sqrt{n} + \sqrt{n+1})^2 = 2n + 1 + 2\sqrt{n(n+1)}$$

while

$$n < \sqrt{n(n+1)} < n + 1$$

we have

$$4n + 1 < (\sqrt{n} + \sqrt{n+1})^2 < 4n + 3$$

and hence

$$\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+3}. \qquad (*)$$

On the other hand there always exists an integer k satisfying

$$k^2 \leq 4n + 1 < (k+1)^2.$$

Since no square can be congruent to 2 or 3 modulo 4

$$k^2 \leq 4n + 1 < 4n + 2 < 4n + 3 < (k+1)^2$$

implying that

$$k \leq \sqrt{4n+1} < \sqrt{4n+2} < \sqrt{4n+3} < k + 1. \qquad (**)$$

Combining (*) and (**) we have

$$k \leq \sqrt{4n+1} \begin{Bmatrix} < \sqrt{4n+2} \\ < \sqrt{n} + \sqrt{n+1} \end{Bmatrix} < \sqrt{4n+3} < k + 1.$$

This shows that k is the common integer part, i.e.

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}].$$

*

Results of the Canadian Mathematics Olympiad 1987

Ravi D. Vakil	1st prize
Colin M. Springer	2nd prize
Michael Tylman	3rd prize
Stephen Fry	4th prize
Steven Gamble	4th prize
Peter A. Hallam	4th prize
Rocky Lee	4th prize
Jonathan Wong	4th prize.

* * *

Correction.

Where do the typos come from? Edward T.H. Wang has kindly pointed out the typographical error in Problem 5 of the *Sixteenth U.S.A. Mathematical Olympiad (1987)* as printed in the June number [1987: 173]. The correct statement for (a) is

(a) Prove that $A = \binom{n}{3} - \binom{d_1}{2} - \binom{d_2}{2} - \cdots - \binom{d_n}{2}$.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

1261*. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

The insphere of the tetrahedron $ABCD$ with equilateral base BCD touches the faces ACD, ADB, ABC at points E, F, G respectively. If $BE = CF = DG$, must the tetrahedron be regular? (This is an extension of *Crux* 1133 [1986: 77; 1987: 225].)

1262. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.* (Dedicated to Léo Sauvé.)

Pick a random n-digit decimal integer, leading 0's allowed, with each integer being equally likely. What is the expected number of distinct digits in the chosen integer?

1263. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokaisi, Aichi, Japan.*

Find all triangles whose sides are three consecutive integers and whose area is an integer.

1264. *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Define the *character* of a finite set of real numbers to be the sum of its maximum element and minimum element. Evaluate the average of the characters of all the nonempty subsets of $\{1,2,\ldots,n\}$.

1265. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with area F and exradii r_a, r_b, r_c, and let $A'B'C'$ be a triangle with area F' and altitudes h'_a, h'_b, h'_c. Show that

$$\frac{r_a}{h'_a} + \frac{r_b}{h'_b} + \frac{r_c}{h'_c} \geq 3\cdot\sqrt{\frac{F}{F'}} .$$

1266. *Proposed by Themistocles M. Rassias, Athens, Greece.*

Let $a_1, a_2, \ldots, a_n$ be distinct odd natural numbers, and let $\prod_{i=1}^{n} a_i$ be divisible by exactly k primes, of which p is the smallest. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < \frac{I_{p-2}}{I_{p+2k-2}}$$

where

$$I_{2m+1} = \frac{2m(2m-2)\ \ldots\ 4\cdot 2}{(2m+1)(2m-1)\ \ldots\ 3\cdot 1} .$$

1267. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with inscribed circle I of radius r. Let I_1 and J_1, of radii λ_1 and μ_1, be the two circles tangent to I and the lines A_1A_2 and A_1A_3. Analogously define circles I_2, J_2, I_3, J_3 of radii λ_2, μ_2, λ_3, μ_3, respectively.

(a) Prove that $\lambda_1\mu_1 = \lambda_2\mu_2 = \lambda_3\mu_3 = r^2$.

(b) Prove that

$$\sum_{i=1}^{3} \lambda_i + \sum_{i=1}^{3} \mu_i \geq 10r.$$

1268. *Proposed by Herta T. Freitag, Roanoke, Virginia and Dan Sokolowsky, Williamsburg, Virginia.*

Let N be an arbitrary positive integer. For each divisor d of N let $n(d)$ denote the number of divisors of d. Let S be the set of divisors of N. Prove that

$$\left[\sum_{d\in S} n(d)\right]^2 = \sum_{d\in S} (n(d))^3.$$

1269*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a non-obtuse triangle with circumcenter M and circumradius R. Let u_1, u_2, u_3 be the lengths of the parts of the cevians (through M) between M and the sides opposite to A, B, C respectively. Prove or disprove that

$$\frac{R}{2} \le \frac{u_1 + u_2 + u_3}{3} < R.$$

1270. *Proposed by Péter Ivady, Budapest, Hungary.*

Prove the inequality

$$\frac{x}{\sqrt{1+x^2}} < \tanh x < \sqrt{1 - e^{-x^2}}$$

for $x > 0$.

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1087. [1985: 289; 1987: 120] *Proposed by Robert Downes, student, Moravian College, Bethlehem, Pennsylvania.*

Let a, b, c, d be four positive numbers.

(a) There exists a regular tetrahedron $ABCD$ and a point P in space such that $PA = a$, $PB = b$, $PC = c$, and $PD = d$ if and only if a, b, c, d satisfy what condition?

(b) This condition being satisfied, calculate the edge length of the regular tetrahedron $ABCD$.

(For the corresponding problem in a plane, see Problem 39 [1975: 64; 1976: 7].)

Editor's comment.

O. Bottema has kindly pointed out that the n-dimensional version of this problem appeared as Problem 722, page 124 of *Nieuw Archief voor Wiskunde* Vol.3, no.1 (March 1985), proposed by Bottema, with his solution appearing on pages 263-264 of Vol.4, no.3 (November 1986). The method is different from our published solution, due to Klamkin.

*　　　　*　　　　*

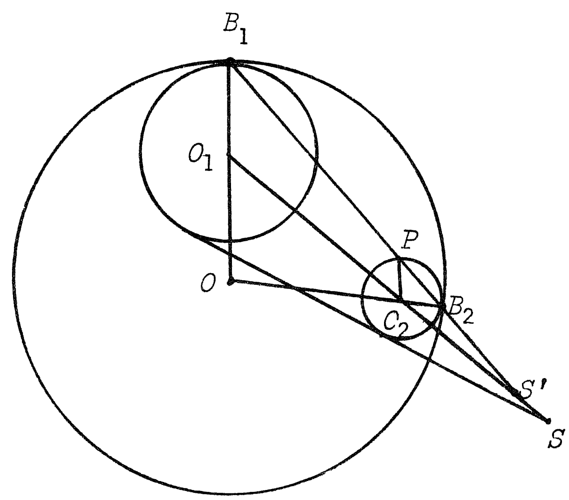
1091*. [1985: 324; 1987: 128] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let $A_1A_2A_3$ be a triangle and γ_i the excircle opposite A_i,

$i = 1,2,3$. Apollonius knew how to construct the circle Γ internally tangent to the three excircles and encompassing them. Let B_i be the point of contact of Γ and γ_i, $i = 1,2,3$. Prove that the lines A_1B_1, A_2B_2, and A_3B_3 are concurrent.

III. *Solution by Tosio Seimiya, Kawasaki, Japan.*

First we prove



Lemma. Let C_1 and C_2 be disjoint circles contained in and tangent to a circle C at points B_1 and B_2 respectively. Let S be the external homothetic center of C_1 and C_2. Then the line B_1B_2 passes through S.

Proof. Let O, O_1, O_2 be the centers of C, C_1, C_2, respectively.

The line B_1B_2 meets C_2 in a point P different from B_2, and meets O_1O_2 in a point S'. Then

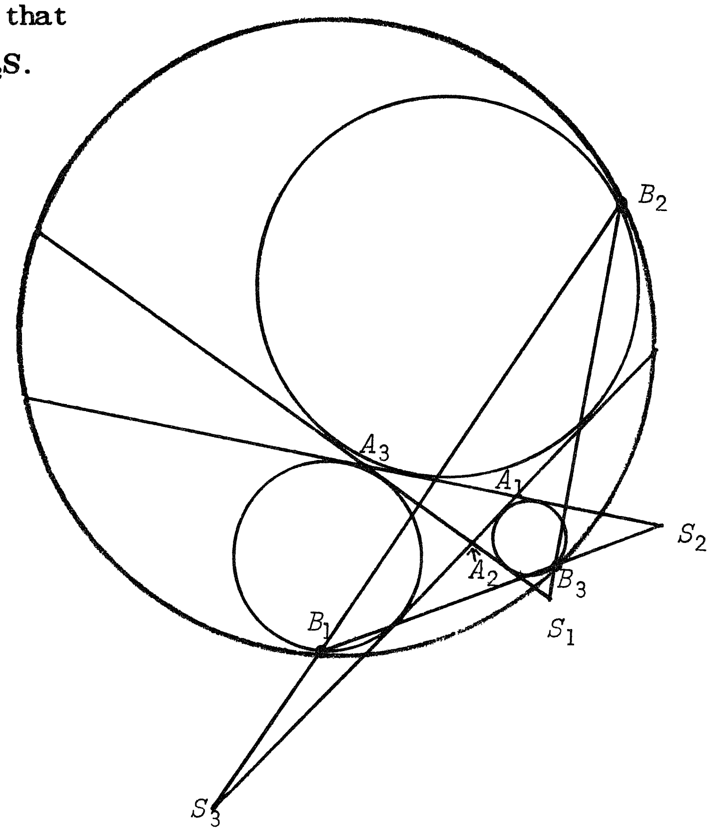
$$\angle OB_1B_2 = \angle OB_2B_1 = \angle O_2B_2P = \angle O_2PB_2,$$

showing that $B_1O_1 \parallel PO_2$. It follows that

$$O_1S' : O_2S' = O_1B_1 : O_2P = O_1S : O_2S.$$

This shows $S = S'$. □

Now let S_1, S_2, S_3 be the external homothetic centers of (C_2,C_3), (C_3,C_1), (C_1,C_2), respectively. Then it is well-known that the three points S_1, S_2, S_3 are collinear. We consider the triangles $A_1A_2A_3$ and $B_1B_2B_3$. By the lemma, the lines B_1B_2, B_2B_3, B_3B_1 meet the lines A_1A_2, A_2A_3, A_3A_1 in the points S_3, S_1, S_2, respectively. Therefore, Desargues' theorem shows that the three lines A_1B_1, A_2B_2, A_3B_3 are concurrent.



* * *

1116. [1986: 27, 77; 1987: 190] *Proposed by David Grabiner, Claremont High School, Claremont, California.*

(a) Let $f(n)$ be the smallest positive integer which is not a factor of n. Continue the series $f(n),f(f(n)),f(f(f(n)))\dots$ until you reach 2. What is the maximum length of the series?

(b) Let $g(n)$ be the second smallest positive integer which is not a factor of n. Continue the series $g(n),g(g(n)),g(g(g(n)))\dots$ until you reach 3. What is the maximum length of the series?

Comment by the proposer (now a student at Princeton University).

The conjecture which Professor Meyers proposed on [1986: 191] as a generalization of my problem is false. Using his notation, $f_6(8) = 10$, $f_6(10) = 9$, and $f_6(9) = 8$, so $f_6^m(8)$ is never 7.

What is true, however, is that $f_k^m(n)$ eventually reaches a cycle, and I can bound the number of iterations needed. Let $p_1^{a_1},\dots,p_{k+1}^{a_{k+1}}$ be $k + 1$ powers of distinct primes, with each $p_i^{a_i}$ greater than k. Since from Meyers' solution $f_k(n) = jq^b$ for some prime q and some $j \le k$, $f_k(n)$ can divide at most one of $p_1^{a_1},\dots,p_{k+1}^{a_{k+1}}$. Thus $f_k^2(n)$, and every succeeding iteration, is less than or equal to $p_{k+1}^{a_{k+1}}$, the largest of the $p_i^{a_i}$. Since we always have $f_k(n) \ge k$, there are at most $p_{k+1}^{a_{k+1}} - k$ possible values for $f_k^m(n)$ for m at least 2, and thus two of the first $p_{k+1}^{a_{k+1}} - k + 2$ iterations are the same, giving a cycle. We can find all the cycles by checking $f(n)$ for the possible values of $f^2(n)$.

Thus, for example, to show that repeated iterations of f_6 always converge to the cycle $8,10,9,8,\dots$, we need only check $f_6(7)$ through $f_6(19)$, from which we can conclude that $f_6^4(n)$ is always 8, 9, or 10 by checking the cases.

There need not, however, be just one cycle; iterations of f_9 may converge to either $10,13,10,\dots$ or $12,15,12,\dots$.

* * *

1129. [1986: 52] *Proposed by Donald Cross, Exeter, England.*

(a) Show that every positive whole number ≥ 84 can be written as the sum of three positive whole numbers in at least four ways (all twelve numbers different) such that the sum of the squares of the three numbers in any group is equal to the sum of the squares of the three numbers in each of the other groups.

(b) Same at part (a), but with "three" replaced by "four" and "twelve" by "sixteen".

(c)* Is 84 minimal in (a) and/or (b)?

I. *Solutions to (a) and (b) by the proposer.*

(a) A solution for numbers of the form $84 + 3n$ is

$$\begin{array}{c} 1+n\ ,\ 39+n\ ,\ 44+n \\ 4+n\ ,\ 29+n\ ,\ 51+n \\ 5+n\ ,\ 27+n\ ,\ 52+n \\ 12+n\ ,\ 17+n\ ,\ 55+n. \end{array}$$

For fixed $n \geq 0$, all twelve numbers are positive and distinct, each triple adds up to $84 + 3n$, and

$$(1+n)^2 + (39+n)^2 + (44+n)^2 = 3458 + 168n + 3n^2$$
$$(4+n)^2 + (29+n)^2 + (51+n)^2 = 3458 + 168n + 3n^2$$
$$(5+n)^2 + (27+n)^2 + (52+n)^2 = 3458 + 168n + 3n^2$$
$$(12+n)^2 + (17+n)^2 + (55+n)^2 = 3458 + 168n + 3n^2.$$

Similarly, we can show that a solution for numbers of the form $85 + 3n$ is

$$\begin{array}{c} 1+n\ ,\ 38+n\ ,\ 46+n \\ 2+n\ ,\ 34+n\ ,\ 49+n \\ 4+n\ ,\ 29+n\ ,\ 52+n \\ 13+n\ ,\ 16+n\ ,\ 56+n \end{array}$$

and that a solution for numbers of the form $86 + 3n$ is

$$\begin{array}{c} 1+n\ ,\ 41+n\ ,\ 44+n \\ 5+n\ ,\ 28+n\ ,\ 53+n \\ 8+n\ ,\ 23+n\ ,\ 55+n \\ 11+n\ ,\ 19+n\ ,\ 56+n. \end{array}$$

Since every whole number ≥ 84 can be written in one of these three forms, we are done.

(b) In the same way, a solution is:

$\underline{84 + 4n}$	$\underline{85 + 4n}$
$1+n\ ,\ 18+n\ ,\ 27+n\ ,\ 38+n$	$1+n\ ,\ 21+n\ ,\ 30+n\ ,\ 33+n$
$2+n\ ,\ 15+n\ ,\ 30+n\ ,\ 37+n$	$5+n\ ,\ 14+n\ ,\ 29+n\ ,\ 37+n$
$3+n\ ,\ 13+n\ ,\ 32+n\ ,\ 36+n$	$9+n\ ,\ 15+n\ ,\ 19+n\ ,\ 42+n$
$6+n\ ,\ 10+n\ ,\ 29+n\ ,\ 39+n$	$10+n\ ,\ 11+n\ ,\ 23+n\ ,\ 41+n$

$\underline{86 + 4n}$	$\underline{87 + 4n}$
$1 + n$, $19 + n$, $28 + n$, $38 + n$	$1 + n$, $24 + n$, $28 + n$, $34 + n$
$3 + n$, $14 + n$, $33 + n$, $36 + n$	$2 + n$, $20 + n$, $32 + n$, $33 + n$
$5 + n$, $15 + n$, $24 + n$, $42 + n$	$6 + n$, $14 + n$, $29 + n$, $38 + n$
$6 + n$, $12 + n$, $27 + n$, $41 + n$	$10 + n$, $13 + n$, $22 + n$, $42 + n$.

II. *Editor's comments.*

There was only one response to this problem, disappointing considering the rather nice (I think) solution of the proposer; it was a computer calculation by STEWART METCHETTE of Culver City, California, showing that for part (a), 84 is indeed the smallest whole number having the given property. Metchette in fact counted, for $N = 1$ to 157 and $m = 1$ to 8, the number $f(N,m)$ of m-tuples of triples of numbers such that for each m-tuple,

(i) all $3m$ numbers involved are distinct;

(ii) each triple adds up to N;

(iii) the sum of squares for each triple is constant.

Thus $f(N,1)$ is just the number of partitions of N into three distinct parts (exercise: find a formula for $f(N,1)$!), and part (a) asks for the smallest number N_4 such that $f(N,4) > 0$ for all $N \geq N_4$, now known to be $N_4 = 84$.

Two results from Metchette's calculations are:

(1) For each m, the smallest value of N for which $f(N,m) > 0$ appears to be a multiple of 3. For instance,

m	N	
1	6	(of course)
2	12	(here the two triples are 1, 5, 6 and 2, 3, 7, with $1 + 5 + 6 = 2 + 3 + 7 = 12$ and $1^2 + 5^2 + 6^2 = 2^2 + 3^2 + 7^2$)
3	27	
4	36	
5	87	
6	90	
7	144	
8	147	
≥ 9	?	

Also, the number of m-tuples (for $m > 1$) appears to be larger when N is a multiple of 3 than when it is not.

(2) The smallest value N_m such that $f(N,m) > 0$ for all $N \geq N_m$ is: $N_2 = 20$, $N_3 = 51$, $N_4 = 84$, $N_5 > 157$. None of $N_6, N_6, \ldots$ have even been proved to exist, much less their value determined.

I would welcome further correspondence from *Crux* readers on this problem, explaining Metchette's observations, or settling whether 84 is minimal in part (b).

* * *

<u>1130</u>. [1986: 52] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Show that

$$a^{3/2} + b^{3/2} + c^{3/2} \leq 3^{7/4}R^{3/2}$$

where a, b, c are the sides of a triangle and R is the circumradius.

I. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

We give a more general inequality as a consequence of the following inequality [1]:

$$\left[\frac{vw}{u} + \frac{wu}{v} + \frac{uv}{w}\right]^2 \geq 4(u^2\sin^2A + v^2\sin^2B + w^2\sin^2C),$$

where A, B, C are the angles of a triangle and u, v, w are arbitrary positive numbers. There is equality if and only if u, v, w form the sides of a triangle UVW satisfying $U + 2A = V + 2B = W + 2C = 180^\circ$.

By the power mean inequality,

$$\left[\frac{u^2\sin^2A + v^2\sin^2B + w^2\sin^2C}{u^2 + v^2 + w^2}\right]^{1/2} \geq \left[\frac{u^2\sin^{3/2}A + v^2\sin^{3/2}B + w^2\sin^{3/2}C}{u^2 + v^2 + w^2}\right]^{2/3}.$$

Hence

$$4^{-3/4}(u^2 + v^2 + w^2)^{1/4}\left[\frac{vw}{u} + \frac{wu}{v} + \frac{uv}{w}\right]^{3/2} \geq u^2\sin^{3/2}A + v^2\sin^{3/2}B + w^2\sin^{3/2}C.$$

Then since $\sin A = \frac{a}{2R}$, etc.,

$$R^{3/2}(u^2 + v^2 + w^2)^{1/4}\left[\frac{vw}{u} + \frac{wu}{v} + \frac{uv}{w}\right]^{3/2} \geq u^2a^{3/2} + v^2b^{3/2} + w^2c^{3/2}.$$

The proposed inequality corresponds to the special case $u = v = w$.

II. *Editor's comment.*

It was pointed out by most solvers of this problem that a more general result, namely that

$$a^k + b^k + c^k \leq 3(R\sqrt{3})^k$$

holds if and only if

$$k \leq \frac{\log 9 - \log 4}{\log 4 - \log 3} \approx 2.8188,$$

appears as 5.28 in Bottema et al, *Geometric Inequalities*. In fact this result even appeared recently on these pages [1985: 263]!

Solved by J.T. GROENMAN, *Arnhem, The Netherlands*; **WALTHER JANOUS**, *Ursulinengymnasium, Innsbruck, Austria*; M.S. KLAMKIN, *University of Alberta, Edmonton, Alberta*; EDWIN M. KLEIN, *University of Wisconsin, Whitewater, Wisconsin*; VEDULA N. MURTY, *Penn State University, Middletown, Pennsylvania*; BOB PRIELIPP, *University of Wisconsin, Oshkosh, Wisconsin*; *and the proposer*.

* * *

<u>1131</u>. [1986: 77] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let $A_1A_2A_3$ be a triangle with sides a_1, a_2, a_3 labelled as usual, and let P be a point in or out of the plane of the triangle. It is a known result that if R_1, R_2, R_3 are the distances from P to the respective vertices A_1, A_2, A_3, then a_1R_1, a_2R_2, a_3R_3 satisfy the triangle inequality, i.e.

$$a_1R_1 + a_2R_2 + a_3R_3 \geq 2a_iR_i\ , \qquad i = 1,2,3. \tag{1}$$

For the a_iR_i to form a non-obtuse triangle, we would have to satisfy

$$a_1^2R_1^2 + a_2^2R_2^2 + a_3^2R_3^2 \geq 2a_i^2R_i^2$$

which, however, need not be true. Show that nevertheless

$$a_1^2R_1^2 + a_2^2R_2^2 + a_3^2R_3^2 \geq \sqrt{2}a_i^2R_i^2$$

which is a stronger inequality than (1).

Solution by the proposer.

Since

$$2(q^2r^2 + r^2p^2 + p^2q^2) - p^4 - q^4 - r^4$$
$$= (p + q + r)(q + r - p)(r + p - q)(p + q - r),$$

we have the identity

$$z_1^4(z_2 - z_3)^4 + z_2^4(z_3 - z_1)^4 + z_3^4(z_1 - z_2)^4$$
$$= 2\{z_2^2z_3^2(z_1 - z_2)^2(z_1 - z_3)^2 + z_3^2z_1^2(z_2 - z_3)^2(z_2 - z_1)^2$$
$$+ z_1^2z_2^2(z_3 - z_1)^2(z_3 - z_2)^2\} \tag{2}$$

for arbitrary complex numbers z_1, z_2, z_3. (Put $p = z_1(z_2 - z_3)$, etc.)

Now let z_1, z_2, z_3 be the complex number representation of the vertices A_1, A_2, A_3, respectively, with respect to an origin P anywhere in the plane of the triangle. Since $|z_1| = R_1$, $|z_2 - z_3| = a_1$, etc., solving (2) for $z_1^4(z_2 - z_3)^4$ and applying the triangle inequality we get that

$$R_1^4a_1^4 \leq 2(R_2^2R_3^2a_2^2a_3^2 + R_3^2R_1^2a_3^2a_1^2 + R_1^2R_2^2a_1^2a_2^2) + R_2^4a_2^4 + R_3^4a_3^4\ ,$$

or

$$(a_1^2R_1^2 + a_2^2R_2^2 + a_3^2R_3^2)^2 \geq 2a_i^4R_i^4$$

which is equivalent to the proposed inequality for points P in the plane of the triangle.

To obtain the inequality for a point P out of the plane of $A_1A_2A_3$, it suffices to show that

$$a_1^2(R_1^2 + h^2) + a_2^2(R_2^2 + h^2) + a_3^2(R_3^2 + h^2) \geq \sqrt{2}a_i^2(R_i^2 + h^2) \qquad (3)$$

where h is the perpendicular distance from P to the plane of $A_1A_2A_3$, and R_i is the distance from A_i to the projection of P onto this plane. By the previous case we need only show that

$$a_1^2 + a_2^2 + a_3^2 \geq \sqrt{2}a_i^2 ,$$

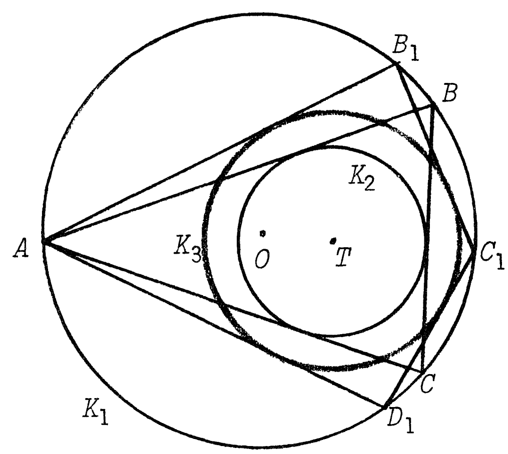
which follows from the previous case applied to the special case $R_1 = R_2 = R_3$ (when P is the circumcenter of $A_1A_2A_3$).

Finally a correction: the last line of the proposal should be changed to "which can be shown to be incomparable to (1)".

* * *

<u>1132</u>. [1986: 78] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

A triangle ABC has circumcircle K_1, with centre O and radius R, and inscribed circle K_2, with centre T and radius r. A third circle K_3 of centre T and radius r_1 has the property that there is a quadrilateral $AB_1C_1D_1$ which is both inscribed in K_1 and circumscribed about K_3. Find r_1 in terms of R and r.



Solution by Dan Sokolowsky, Williamsburg, Virginia.

Let $OT = d$. We then have

$$R^2 - d^2 = 2Rr \qquad (1)$$

from Theorem 295, page 186 of [1], and

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r_1^2} \qquad (2)$$

from Theorem 125, page 95 of [1]. From (2),

$$r_1^2 = \frac{(R^2 - d^2)^2}{2(R^2 + d^2)} . \qquad (3)$$

Using (1) in (3) and solving, we obtain

$$r_1^2 = \frac{4R^2r^2}{2(2R^2 - 2Rr)} = \frac{Rr^2}{R - r},$$

or

$$r_1 = r\sqrt{\frac{R}{R - r}}.$$

Reference:

[1] R.A. Johnson, *Advanced Euclidean Geometry*.

Also solved by the proposer. There was one incorrect solution submitted.

* * *

1133. [1986: 78] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

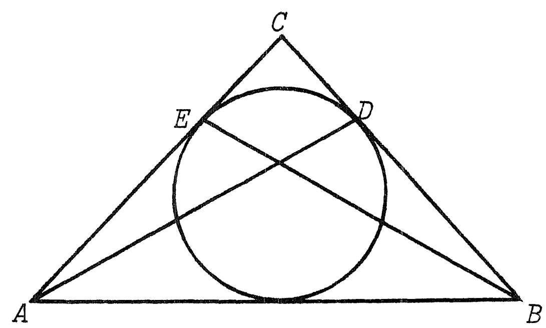
The incircle of triangle ABC touches sides BC and AC at points D and E respectively. If $AD = BE$, prove that the triangle is isosceles.

Solution by Jack Garfunkel, Flushing, N.Y.

In triangles ACD and BCE we have $\angle ACD = \angle BCE$, $CD = CE$, and $AD = BE$. Using the law of sines, we get

$$\frac{\sin CAD}{\sin C} = \frac{CD}{AD} = \frac{CE}{BE} = \frac{\sin CBE}{\sin C},$$

so $\sin CAD = \sin CBE$ and hence $\angle CAD = \angle CBE$ since these angles are acute. Hence,



$$\Delta ACD \equiv \Delta BCE$$

(by a.a.s.), and $AC = BC$.

Also solved by BENO ARBEL, *Tel Aviv University, Tel Aviv, Israel;* R.H. EDDY, *Memorial University, St. John's, Newfoundland;* J.T. GROENMAN, *Arnhem, The Netherlands;* RICHARD I. HESS, *Rancho Palos Verdes, California;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* M.S. KLAMKIN, *University of Alberta, Edmonton, Alberta;* KEE-WAI LAU, *Hong Kong;* LEROY F. MEYERS, *The Ohio State University, Columbus, Ohio (three solutions!);* DAN SOKOLOWSKY, *Williamsburg, Virginia;* KENNETH M. WILKE, *Topeka, Kansas; and the proposer. There was one incorrect solution submitted.*

For a possible extension to three dimensions, see Problem 1261, this issue.

* * *

1134. [1986: 78] *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Let n be a positive integer, and consider the set of all n-tuples $(a_1,a_2,\ldots,a_n)$ of integers for which

(i) $0 \le a_1 \le a_2 \le \ldots \le a_n < n$, and

(ii) $a_1 + a_2 + \ldots + a_n \equiv 0 \pmod{n}$.

Prove that the integers $0,1,2,\ldots,n-1$ each occur the same number of times as coordinates of elements in this set.

Solution by Michael M. Parmenter, Memorial University, St. John's, Newfoundland.

Let S denote the set of n-tuples described in the problem. Define a map f: S $\longrightarrow$ S as follows:

$$f(a_1,a_2,\ldots,a_n) = \begin{cases} (a_1+1,a_2+1,\ldots,a_n+1) & \text{if } a_n < n-1, \\ (\underbrace{0,\ldots,0}_{t \text{ times}},a_1+1,a_2+1,\ldots,a_{n-t}+1) & \\ & \text{if } a_{n-1} < a_{n-t+1} =\ldots= a_n = n-1. \end{cases}$$

(That is, just add 1 to each component of the n-tuple and reduce modulo n.) It is easy to see that f is a bijection and that the number of times an integer i occurs in S equals the number of times $i+1$ (taken mod n) occurs in f(S) = S. The result follows.

Also solved (using the same method) by LEROY F. MEYERS, *The Ohio State University, Columbus, Ohio;* DAN SOKOLOWSKY, *Williamsburg, Virginia; and the proposer.*

* * *

1135. [1986: 79] *Proposed by Jack Garfunkel, Flushing, N.Y.*

(A variation of an old problem, dedicated to Dr. Leon Bankoff.)

(a) Given equilateral triangles ABC and $A'B'C'$ in the same plane, both labeled counterclockwise, prove that triangle $M_1M_2M_3$ is equilateral, where M_1, M_2, M_3 are the midpoints of AA', BB', CC' respectively.

(b)* Given similar triangles ABC and $A'B'C'$ in the same plane, prove that triangle $M_1M_2M_3$ is similar to triangle ABC, where M_1, M_2, M_3 are as in (a).

Comment by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

More general results which include both (a) and (b) as special cases are given in the solutions and comments for *Crux* 464 [1980: 185-187].

Also solved by W.J. BLUNDON, Memorial University, St. John's, Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, University of Minnesota, Minneapolis, Minnesota; DANIEL B. SHAPIRO, Ohio State University, Columbus, Ohio; ESTHER SZEKERES, Turramurra, Australia; and (part (a)) the proposer.

* * *

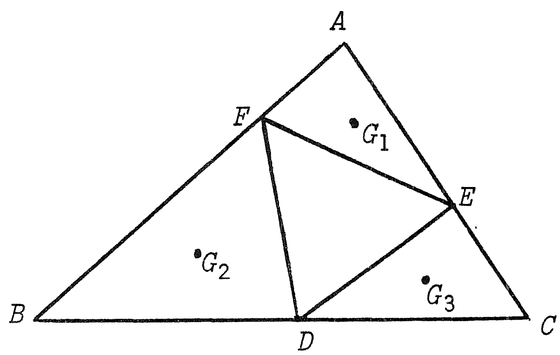
1136. [1986: 79] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and D, E, F points on BC, CA, AB respectively. Denote by G_1, G_2, G_3 the centroids of triangles AEF, BDF, CDE respectively. Prove that

$$[G_1G_2G_3] = \frac{2[ABC] + [DEF]}{9}$$

where $[M]$ stands for the area of the figure M.

Solution by the proposer.



We first prove that

$$\overrightarrow{G_1G_2} = \frac{\overrightarrow{AB} + \overrightarrow{ED}}{3}. \qquad (1)$$

Indeed,

$$\overrightarrow{G_1G_2} = \overrightarrow{G_1A} + \overrightarrow{AB} + \overrightarrow{BG_2}$$
$$\overrightarrow{G_1G_2} = \overrightarrow{G_1F} + \overrightarrow{FG_2}$$
$$\overrightarrow{G_1G_2} = \overrightarrow{G_1E} + \overrightarrow{ED} + \overrightarrow{DG_2}$$

from which follows

$$3\overrightarrow{G_1G_2} = (\overrightarrow{G_1A} + \overrightarrow{G_1F} + \overrightarrow{G_1E}) + \overrightarrow{AB} + \overrightarrow{ED} + (\overrightarrow{BG_2} + \overrightarrow{FG_2} + \overrightarrow{DG_2}).$$

But

$$\overrightarrow{G_1A} + \overrightarrow{G_1F} + \overrightarrow{G_1E} = \vec{0} = \overrightarrow{BG_2} + \overrightarrow{FG_2} + \overrightarrow{DG_2},$$

and hence (1) follows.

Similarly we can take

$$\overrightarrow{G_1G_3} = \frac{\overrightarrow{AC} + \overrightarrow{FD}}{3}.$$

Thus we have

$$\begin{aligned}|\overrightarrow{G_1G_2} \times \overrightarrow{G_1G_3}| &= \tfrac{1}{9}|(\overrightarrow{AB} + \overrightarrow{ED}) \times (\overrightarrow{AC} + \overrightarrow{FD})| \\ &= \tfrac{1}{9}|(\overrightarrow{AB} \times \overrightarrow{AC}) + (\overrightarrow{AB} \times \overrightarrow{FD}) + (\overrightarrow{ED} \times \overrightarrow{AC}) + (\overrightarrow{ED} \times \overrightarrow{FD})| \\ &= \tfrac{1}{9}[\,|\overrightarrow{AB} \times \overrightarrow{AC}| + |\overrightarrow{AB} \times \overrightarrow{FD}| + |\overrightarrow{ED} \times \overrightarrow{AC}| + |\overrightarrow{ED} \times \overrightarrow{FD}|\,],\end{aligned}$$

since all four cross products are perpendicular to the plane of the triangle and in the same direction. Hence, dividing by 2,

$$[G_1G_2G_3] = \frac{1}{9}([ABC] + [ADB] + [ADC] + [DEF])$$
$$= \frac{2[ABC] + [DEF]}{9}.$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; and DAN SOKOLOWSKY, Williamsburg, Virginia.

This was one case where none of the submitted solutions were as elegant as the proposer's!

* * *

1137*. [1986: 79, 177 (revised)] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s}$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

Partial solution by Wolfgang Gmeiner, Millstatt, Austria and the proposer.

We first apply Klamkin's median duality, i.e.,

$$I(a,b,c,m_a,m_b,m_c) \geq 0$$

is a valid triangle-inequality if and only if

$$I(m_a,m_b,m_c,3a/4,3b/4,3c/4) \geq 0$$

is so. (cf. Klamkin's solution to Aufgabe 677, *Elem. der Math.* 28 (1973) 130.)

Then the proposed inequality becomes

$$\frac{4}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\cdot\frac{1}{2}(m_a + m_b + m_c) > 5,$$

i.e.,

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{2m_a}{3} + \frac{2m_b}{3} + \frac{2m_c}{3}\right) > 5. \qquad (1)$$

Inequality (1) can be viewed as a special case of the following more general triangle inequality:

Let P be an interior point of a triangle ABC and R_1, R_2, R_3 the distances from P to the vertices A, B, C, respectively. Then

$$\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right](R_1 + R_2 + R_3) > 5. \tag{2}$$

In order to get (1), we assign $P = G$, the center of gravity of triangle ABC. Then $R_1 = 2m_a/3$, etc.

To prove (2) it is sufficient to choose P such that $R_1 + R_2 + R_3$ is minimal. We distinguish now two cases.

(i) $\max(A,B,C) \geq 120^\circ$. Then, by [1], item 12.55, if e.g. $A \geq 120^\circ$, then

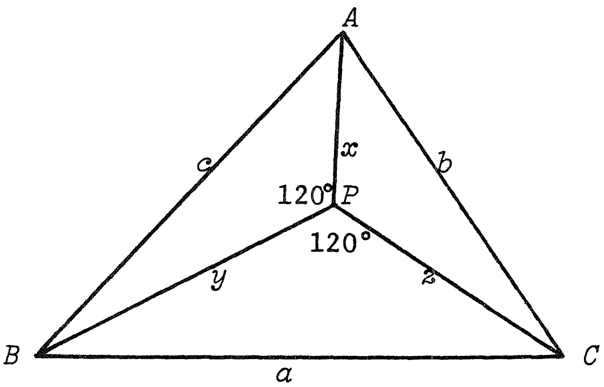
$$R_1 + R_2 + R_3 \geq b + c.$$

Since

$$\left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right](b + c) = 2 + \frac{b}{c} + \frac{c}{b} + \frac{b + c}{a} > 2 + 2 + 1 = 5,$$

inequality (2) is shown for this case. Equality holds in (2) for the degenerate triangle $a = 2b = 2c$, $P = A$.

(ii) $\max(A,B,C) < 120^\circ$. Then the point P with $R_1 + R_2 + R_3$ minimal is Torricelli's point (sometimes also named Fermat's point), i.e., the point subtending 120° with all three sides of the triangle (cf. e.g. [2], page 24). Introducing x, y, z for R_1, R_2, R_3, respectively, we get by the law of cosines



$$c = \sqrt{x^2 + y^2 + xy}$$
$$b = \sqrt{x^2 + z^2 + xz}$$
$$a = \sqrt{y^2 + z^2 + yz},$$

and (2) now becomes

$$f(x,y,z) := (x + y + z)\left[\frac{1}{\sqrt{x^2+xy+y^2}} + \frac{1}{\sqrt{x^2+xz+z^2}} + \frac{1}{\sqrt{y^2+yz+z^2}}\right] > 5 \tag{3}$$

for all $x,y,z > 0$.

We *did not* succeed in proving (3). Extensive computer-searches even suggest the validity of the better estimation:

$$f(x,y,z) > 4 + \frac{2}{\sqrt{3}} \tag{4}$$

for $x,y,z > 0$. If true this would be sharp, since

$$f\left[\frac{t}{2},\frac{t}{2},0\right] = 4 + \frac{2}{\sqrt{3}}, \qquad t > 0.$$

Unfortunately enough, item 12.8 of [1], i.e.

"If $a = \max(a,b,c)$ then $R_1 + R_2 + R_3 > b + c$." (5)

is *not* true in general. (If it were, then the general proof of (2) would go along the same lines as (i) above.) For a counterexample to (5), take

$a = b = c$, and choose P close to the midpoint of side a.

References:

[1] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.

[2] R. Honsberger, *Mathematical Gems*, Washington, 1973.

Editor's comment.

It remains to show (4), or at least (3). Readers?

The original inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{3\sqrt{3}}{s}$$

fails, as can be seen by considering the degenerate triangle $a = 0$, $b = c > 0$. Then $m_a = b$, $m_b = m_c = \frac{b}{2}$, so

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} = \frac{5}{b} = \frac{5}{s} .$$

This also shows that the revised inequality, if true, is sharp.

Counter-examples (usually the above) to the original inequality were found by NIELS BEJLEGAARD, Stavanger, Norway; SVETOSLAV BILCHEV and EMILIA VELIKOVA, Russe, Bulgaria; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; PETER WATSON-HURTHIG, Columbia College, Burnaby, B.C.; and the proposer.

* * *

<u>1138</u>. [1986: 79] *Proposed by L.J. Upton, Mississauga, Ontario.*

You are given four discs A, B, C, D of identical appearance, but weighing 1, 2, 3, and 4 units not necessarily respectively. Determine the weights of the discs in four weighings on a 2-tray balance (no extra weights supplied).

Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

Observe the results of the three weighings $A + B$ vs. $C + D$, $A + C$ vs. $B + D$, and $A + D$ vs. $B + C$. Since

$$4 + 3 > 4 + 2 > 4 + 1 = 2 + 3 > 1 + 3 > 1 + 2,$$

there will be exactly one "even" weighing, and the common weight on the two "heavy" sides must be 4, and the common weight on the two "light" sides must be 1. Then compare the remaining two weights to see which one is 2 and which one is 3.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; BENO ARBEL, Tel Aviv University, Tel Aviv, Israel; J.T. GROENMAN, Arnhem, The Netherlands;

RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; DAN SOKOLOWSKY, Williamsburg, Virginia; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

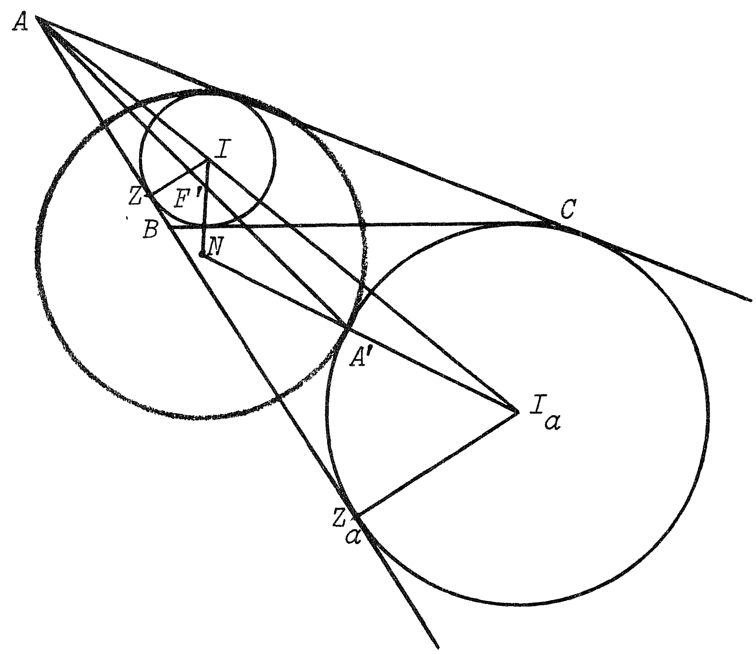
* * *

1139. [1986: 79] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle and let A', B', C' be the touch points of the nine-point circle with the A-excircle, B-excircle, and C-excircle, respectively. Prove that AA', BB', CC' concur in a point F', and that F' is collinear with the centers of the incircle and nine-point circle.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

The figure shows ΔABC, its incircle $I(r)$ touching AB at Z, its A-excircle $I_a(r_a)$ touching AB at Z_a, and its ninepoint circle $N(q)$ touching $I_a(r_a)$ at A'.



Let s be the semiperimeter and R the circumradius of ΔABC. Then the following is well known (see, e.g., pp.11-13, 20 of H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*):

$$AZ = s - a\ , \quad AZ_a = s \tag{1}$$

$$r_a(s - a) = rs \tag{2}$$

$$q = R/2. \tag{3}$$

Note that A, I, I_a are collinear, and in the order $A - I - I_a$. Thus with reference to ΔNII_a, the line AA' meets side II_a externally at A, side NI_a internally at A', hence it meets side NI internally at some point F'.

Applying the Menelaus theorem to ΔNII_a with transversal AA', we obtain

$$\frac{IF'}{F'N}\cdot\frac{NA'}{A'I_a}\cdot\frac{I_aA}{IA} = 1,$$

so

$$\frac{IF'}{F'N} = \frac{A'I_a}{NA'}\cdot\frac{IA}{I_aA} . \tag{4}$$

We also have $\Delta AZI \sim \Delta AZ_aI_a$, so using (1),

$$\frac{IA}{I_aA} = \frac{AZ}{AZ_a} = \frac{s-a}{s} . \tag{5}$$

Since $A'I_a = r_a$ and $NA' = q = R/2$, using (2) and (5) in (4) we obtain

$$\frac{IF'}{F'N} = \frac{2r_a(s-a)}{Rs} = \frac{2rs}{Rs} = \frac{2r}{R} . \tag{6}$$

The ratio in (6) defines a unique point F' of the segment NI. The same argument applied to BB' and CC' would lead to the same equation (6) for the points on NI through which they pass. It follows that all three lines AA', BB', CC' are concurrent at the point F' on NI defined by (6).

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

The problem is also contained in Theorem 5 of Andrew P. Guinand's paper "Graves triads in the geometry of the triangle", Journal of Geometry 6/2 *(1975) 131-142. I thank the proposer for since pointing this out.*

* * *

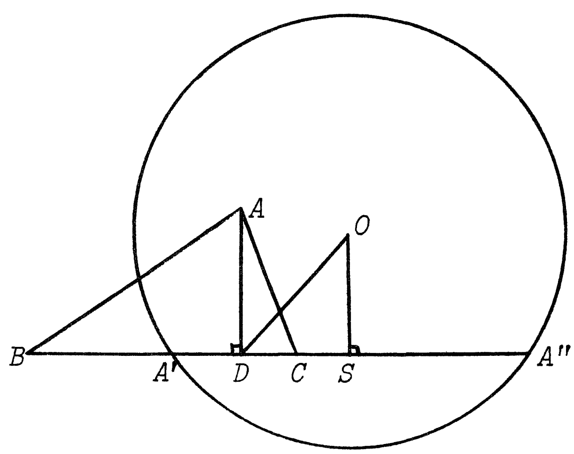
<u>1140</u>. [1986: 79] *Proposed by Jordi Dou, Barcelona, Spain.*

Given triangle ABC, construct a circle which cuts (extended) lines BC, CA, AB in pairs of points A' and A'', B' and B'', C' and C'' respectively such that angles A'AA'', $B'BB''$, C'CC'' are all right angles.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

We first prove

(I) Let a circle K with center O and radius r meet side BC of ΔABC at A' and A''. Let D be the foot of the perpendicular from A to BC, and let $h_a = AD$, $x = OD$. Then $\angle A'AA'' = 90^\circ$ if and only if $r^2 = h_a^2 + x^2$.



To prove this, let S be the midpoint of $A'A''$, so $OS \perp A'A''$. Then

$$\angle A'AA'' = 90^\circ \Leftrightarrow (A'S)^2 = (AS)^2$$
$$\Leftrightarrow (A'O)^2 - (OS)^2 = (AD)^2 + (DS)^2$$
$$\Leftrightarrow r^2 = (A'O)^2 = (AD)^2 + (DS)^2 + (OS)^2 = (AD)^2 + (OD)^2$$
$$= h_a^2 + x^2.$$

To apply this, given a ΔABC let $h_a = AD$, $h_b = BE$, $h_c = CF$, and for any point O let $OD = x$, $OE = y$, $OF = z$. By (I), if a circle $K = O(r)$ solves the problem, we must have

$$r^2 = h_a^2 + x^2 \tag{1}$$

$$r^2 = h_b^2 + y^2 \tag{2}$$

$$r^2 = h_c^2 + z^2, \tag{3}$$

which imply

$$x^2 - y^2 = h_b^2 - h_a^2 \tag{4}$$

$$y^2 - z^2 = h_c^2 - h_b^2 \tag{5}$$

$$z^2 - x^2 = h_a^2 - h_c^2. \tag{6}$$

If $D \neq E$, the locus of points O satisfying (4) is well known to be a line, call it L_1, perpendicular to DE, and easily constructed. (For details see [1], [2].) Likewise if $E \neq F$ the points O satisfying (5) form a line L_2 perpendicular to EF, while if $D \neq F$ those satisfying (6) form a line L_3 perpendicular to DF.

We have two cases:

(i) ΔABC is not a right triangle.

Then D, E, F are distinct non-collinear points, and no two of L_1, L_2, L_3 are parallel. In fact they are concurrent as can be seen by noting that any two of the equations (4) - (6) imply the third, so that if, say, L_1 and L_2 meet at O, then O satisfies (4) and (5), hence (6), implying L_3 also passes through O. Thus O can be constructed as the intersection of any two of these three lines.

Since (4) - (6) hold at such a point O we have there

$$h_a^2 + x^2 = h_b^2 + y^2 = h_c^2 + z^2 \tag{7}$$

so if we construct r so as to satisfy (1) (which must necessarily be satisfied), then (2) and (3) are satisfied as well, by virtue of (7). By (I), the corresponding circle $K = O(r)$ solves our problem. Moreover, the argument shows that in this case O and r are unique, hence so is the solution.

(ii) ΔABC is a right triangle.

We can suppose $\angle C = 90^\circ$, in which case $h_a = b$, $h_b = a$, while D and E coincide (with C) so $x = y$ for any point O. Here the lines DF and EF coincide (with CF), so the corresponding lines L_2 and L_3 are parallel. We consider the following two subcases:

(a) $a \neq b$.

Hence $h_a \neq h_b$. Since $x = y$ for any point O, there exist no points satisfying (4), which implies that in this case there is no solution for our problem. Also implied is the fact that L_2 and L_3 are not coincident since any point they had in common would satisfy (4).

Note however that (5) holds for any point O on L_2, so that at any such point

$$h_b^2 + y^2 = h_c^2 + z^2.$$

Thus if at this point O we construct the corresponding r satisfying (2) it satisfies (3) as well, hence by (I), for the corresponding circle $K = O(r)$ we have

$$\angle B'BB'' = \angle C'CC'' = 90^\circ. \tag{8}$$

We can likewise argue that for any point O on L_3 there is a circle $K = O(r)$ such that

$$\angle A'AA'' = \angle C'CC'' = 90^\circ. \tag{9}$$

Since O is arbitrary in either case, there are infinitely many circles K satisfying either (8) or (9), but as the argument shows, there are none satisfying both in common.

(b) $a = b$.

Then $h_a = h_b$, and since $x = y$, any point O satisfying either (5) or (6) satisfies the other, implying that here L_2 and L_3 coincide, call them L. Then by the same argument as in the preceding case, for any point O on L we can show there is a circle $K = O(r)$ for which both (8) and (9) hold, hence is a solution. O being arbitrary, it follows that in this case there are infinitely many solutions. It is easy to show that the point C lies on L, hence that L is the line through C perpendicular to CF (equivalently, that L is parallel to AB).

References.

[1] R.A. Johnson, *Advanced Euclidean Geometry*, pp.31-33.

[2] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, pp.30-34.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; and the proposer.

* * *

<u>1141</u>. [1986: 106, 138 (corrected)] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

Disjoint, non-touching spheres O_1 and O_2 are inside and tangent to a sphere O. Four spheres S_1, S_2, S_3, S_4, each tangent to two of the others as well as to O_1, O_2, and O, are packed in a ring in that order inside O and around O_1 and O_2. Show that

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$$

where r_i is the radius of S_i.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Let C be the point of tangency of O_1 and O. Under inversion in the unit sphere at C, let O, O_1, O_2, S_1, S_2, S_3, S_4 be mapped onto ω, ω_1, ω_2, σ_1, σ_2, σ_3, σ_4, respectively. Then ω and ω_1 are parallel planes; ω_2 is a sphere touching ω but not ω_1; and each σ_i is a sphere touching ω, ω_1 and ω_2, and two other σ_i's. But then the σ_i's have the same radius r (= half the distance between ω and ω_1), and their centers are at the corners of a square, say with center D and diagonal d. Put $CD = b$, and let the distance from C to the center of σ_i be a_i $(i = 1,2,3,4)$.

Clearly, for each i,

$$r_i = \frac{1}{2}\left[\frac{1}{a_i - r} - \frac{1}{a_i + r}\right] = \frac{r}{a_i^2 - r^2} \tag{1}$$

(Figure 1). Besides,

$$\frac{2a_1^2 + 2a_3^2 - d^2}{4} = b^2 = \frac{2a_2^2 + 2a_4^2 - d^2}{4}$$

(Figure 2), whence

$$a_1^2 + a_3^2 = a_2^2 + a_4^2,$$

and so from (1),

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}.$$

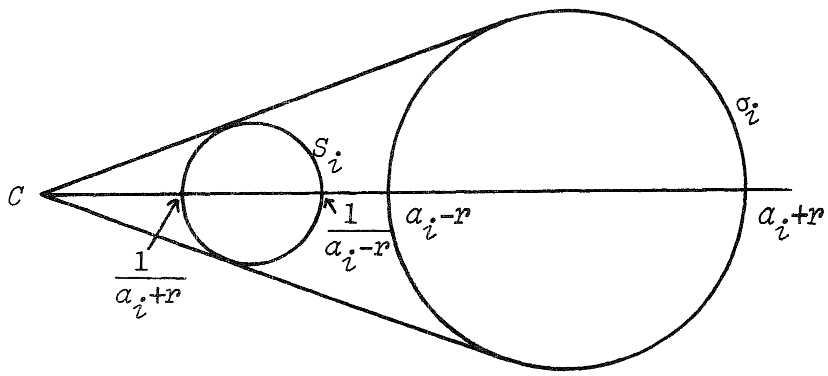


Figure 1

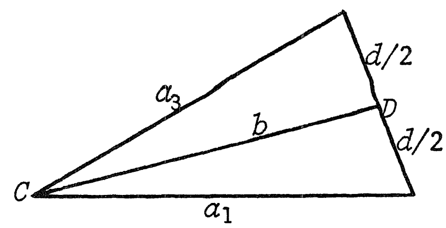


Figure 2

II. *Remark by Jordi Dou, Barcelona, Spain.*

[Dou first solved the problem, using the above method. - Ed.]

The proposed equality is valid if O_1, O_2 are disjoint spheres interior to O, whether or not they are tangent to O. The demonstration, which is similar, considers the inversion I which transforms O_1, O_2 into concentric spheres. Observe that the radii of the resulting spheres $I(S_i)$ are equal and their centers form the vertices of a square, etc.

Also solved by the proposer.

The problem was taken from a lost 1823 Sangaku *and was quoted in the 1832 book* Kokon Sankan.

* * *

<u>1142</u>. [1986: 107] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Suppose ABC is a triangle whose median point lies on its inscribed circle.

(a) Find an equation relating the sides a, b, c of ΔABC.

(b) Assume $a \geq b \geq c$. Find an upper bound for a/c.

(c) Give an example of a triangle with *integral* sides having the above property.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

(a) From the proof of 5.8 in the Bottema bible *Geometric Inequalities*, we read that

$$(GI)^2 = r^2 - \frac{1}{12}(\Sigma a)^2 + \frac{2}{9}\Sigma a^2,$$

where G is the median point, I the incentre, and r the inradius, and the sums are cyclic over a, b, c. We want $GI = r$, or

$$\frac{1}{12}(\Sigma a)^2 = \frac{2}{9}\Sigma a^2,$$

i.e.,

$$8(a^2 + b^2 + c^2) = 3(a + b + c)^2. \qquad (1)$$

(b) Denoting $x = a/c$, $y = b/c$, (1) becomes

$$8(x^2 + y^2 + 1) = 3(x + y + 1)^2$$

or

$$5x^2 - 6xy + 5y^2 - 6x - 6y + 5 = 0. \qquad (2)$$

Investigation shows that (2) represents an ellipse. As $a \geq b \geq c$ and $a \leq b + c$ we have

$$x \geq y \qquad \text{and} \qquad x \leq y + 1. \qquad (3)$$

In order to find the extreme values of x we write (2) as

$$5y^2 - 6y(x + 1) + 5x^2 - 6x + 5 = 0.$$

Then x is extreme if the discriminant

$$D(x) = 36(x + 1)^2 - 4\cdot 5(5x^2 - 6x + 5)$$

vanishes, i.e.

$$9(x + 1)^2 - 5(5x^2 - 6x + 5) = 0,$$

$$x^2 - 3x + 1 = 0.$$

The solutions

$$x = 3/2 + \sqrt{5}/2\ , \qquad y = 3/2 + 3\sqrt{5}/10$$

satisfy (3), so the upper bound of a/c is $3/2 + \sqrt{5}/2$.

(c) Just trying, we find that $(x,y) = (2.6,2)$ satisfies (2), so we can take $a = 13$, $b = 10$, $c = 5$. Another solution is $a = b = 5$, $c = 2$.

Also solved by R.H. EDDY, Memorial University, St. John's, Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

* * *

<u>1143</u>. [1986: 107] *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

(a) Given integers $k \geq 0$ and $\ell \geq 1$, characterize all natural numbers n such that $\begin{bmatrix} n \\ k+i-1 \end{bmatrix}$ divides $\begin{bmatrix} n \\ k+i \end{bmatrix}$ for $i = 1,2,\ldots\ell$.

(b) Given a natural number n, determine the largest integer ℓ such that for some integer $k \geq 0$, $\begin{bmatrix} n \\ k+i-1 \end{bmatrix}$ divides $\begin{bmatrix} n \\ k+i \end{bmatrix}$ for $i = 1,2,\ldots\ell$.

Solution by the proposers.

(a) Since

$$\frac{\begin{bmatrix} n \\ k+i \end{bmatrix}}{\begin{bmatrix} n \\ k+i-1 \end{bmatrix}} = \frac{n-k-i+1}{k+i} = \frac{n+1}{k+i} - 1,$$

we obtain

$$n + 1 \equiv 0 \bmod (k + i)$$

or

$$n \equiv -1 \bmod (k + i).$$

Thus the required n are characterized by

$$n \equiv -1 \bmod \ell\mathrm{cm}(k + 1, k + 2, \ldots, k + \ell).$$

(b) Note first that if an integer m is divisible by each of

$$k + 1, k + 2, \ldots, k + \ell \tag{1}$$

where $k \geq 0$ and $\ell \geq 1$, then m is divisible by each of

$$1, 2, \ldots, \ell.$$

For if t is an integer with $1 \leq t \leq \ell$, then one of the numbers in (1) must be a multiple of t. Consequently, we may assume without loss of generality that $k = 0$. By the same arguments used in (a), we get $n + 1 \equiv 0 \pmod i$ for each of $i = 1,2,\dots\ell$. Therefore, the largest ℓ is the least positive integer ℓ with the property that $1,2,\dots,\ell$ are all divisors of $n + 1$, but $\ell + 1$ does not divide $n + 1$.

Remarks.

(i) Part (a) shows the rather interesting fact that there exist arbitrarily long sequences of "consecutive" binomial coefficients, each dividing into the next.

(ii) Coincidentally, our answer to (b) is related to the function $f(n)$ defined by D. Grabiner in part (a) of *Crux* 1116 [1987: 190]. In terms of that function, the largest ℓ we are looking for is $f(n + 1) - 1$. Therefore an alternative description of our ℓ would be: $\ell = p^\alpha - 1$ where p^α is the smallest prime power not dividing $n + 1$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

* * *

1149. [1986: 108] *Proposed by Lanny Semenko, Erewhon, Alberta.*

Solve the base ten alphametic

$$\text{HOT}^\circ\text{C} = \text{COOL}^\circ\text{F}.$$

(Do not, of course, replace C (Celsius) and F (Fahrenheit) by digits.)

Solution by Charles W. Trigg, San Diego, California.

We have to solve the alphametic equation

$$\frac{9}{5}(\text{HOT}) + 32 = \text{COOL}.$$

Clearly, HOT is a multiple of 5, and each side of the equation is $\equiv 5 \pmod 9$. The maximum temperature possible is 985°C = 1805°F, so C = 1. Thus we need to examine only the following corresponding temperatures in the two systems: (F,C) = (1004,540), (1220,660), (1229,665), (1337,725), (1445,785), (1553,845), and (1778,970). The only temperature readings with like middle digits are the unique:

$$970^\circ\text{C} = 1778^\circ\text{F}.$$

Also solved by FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; GLEN E. MILLS, Valencia Junior College, Orlando, Florida; J. SUCK, Essen, Federal Republic of Germany; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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