Mathematical Spectrum

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- **Impossible magic squares**
- **Shadowing a path**
- **About hexagons**

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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A Tale of Two Series — a Dickens of an Integral

P. GLAISTER

Two interesting mathematical characters meet their reflection.

It is very often a straightforward matter to establish that an infinite series is convergent, but an entirely separate (and potentially complex) exercise to determine the sum of the series. Those series which I feel most comfortable with are alternating series of the form $a_1 - a_2 + a_3 - a_4 + \cdots$, where the terms $a_n \geq 0$. The reason for this is that if the terms decrease monotonically and tend to zero, i.e., $a_{n+1} \leq a_n$ and $a_n \to 0$ as $n \to \infty$, then the series will converge. The remaining issue, therefore, is to determine the sum of the series.

One of the most common examples in this class is the alternating harmonic series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots \tag{1}$$

whose sum is $\ln(2)$. Another popular series is

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \cdots, \tag{2}$$

a particular example of the infinite geometric series

$$x - x^2 + x^3 - x^4 + \cdots \tag{3}$$

whose sum is $x/(1+x)$, so the sum of the series (2) is $\frac{1}{3}$. The derivation of the sum of the series (3) follows a consideration of the limit of the corresponding finite geometric series and its sum, showing that (3) converges provided $1 \times 1 < 1$. Term-by-term integration of the series (3) yields, after some manipulation, the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \tag{4}$$

which can be shown to be convergent for $-1 < x \leq 1$. The series (1) is the special case of (4) for $x = 1$. Even if the sum of the series in (1) or (2) has not been determined, it is possible to estimate the sum numerically to within any desired accuracy; the number of terms required can be large or small depending on whether the series converges slowly or not. To see how this can be done, we review some important results concerning alternating series.

Set

$$S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n-1} a_n,$$

the nth partial sum of the alternating series

$$a_1 - a_2 + a_3 - \cdots, \tag{5}$$

then, as we mentioned above, if the terms a_n satisfy

(i) $a_n \geq 0,\ n \geq 1$,
(ii) $a_{n+1} \leq a_n,\ n \geq 1$,
(iii) $a_n \to 0$ as $n \to \infty$,

the series (5) converges, and $S_n \to S$ as $n \to \infty$ is the sum of (5). To prove this result it is necessary to consider the even partial sums $S_{2n}, n \geq 1$, which form an increasing sequence, and the odd partial sums $S_{2n-1},\ n \geq 1$, which form a decreasing sequence, and then show that each of these sequences converges to the same number, say S, which is then the sum of the series (5). From the proof, every even partial sum is less than or equal to S and every odd partial sum is greater than or equal to S. That is, S lies between S_{2n} and either S_{2n-1} or S_{2n+1}. Hence, for odd or even n, $|S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}$, i.e., the size of the error involved in using S_n as an approximation to S is less than or equal to the size of the first omitted term. For example, an estimate for the sum of the series (1) which is guaranteed to have an error less than 0.0005 (giving three decimal place accuracy) is obtained with the nth partial sum

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}$$

when $a_{n+1} = 1/(n+1) < 0.0005$. Thus $n \geq 2000$, showing that (1) converges slowly. The series (2) converges more rapidly, since the nth partial sum is guaranteed to have an error less than 0.0005 if $a_{n+1} = 1/2^{n+1} < 0.0005$, so that $2^{n+1} > 2000$, and hence $n \geq 10$.

The series in (1) can be thought of as the first in a sequence

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots,$$
$$\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \cdots,$$
$$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \cdots$$

and so on, or more generally

$$S(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^k}, \qquad k = 1, 2, \ldots, \tag{6}$$

so that $S(1)$ is series (1). Similarly, the series in (2) can be thought of as the first in a sequence. By rewriting (2) as

$$\frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} - \cdots$$

the next few are

$$\frac{1}{3^1} - \frac{1}{3^2} + \frac{1}{3^3} - \frac{1}{3^4} - \cdots,$$
$$\frac{1}{4^1} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} - \cdots,$$
$$\frac{1}{5^1} - \frac{1}{5^2} + \frac{1}{5^3} - \frac{1}{5^4} - \cdots$$

and so on, or more generally

$$T(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{k^n}, \qquad k = 2, 3, \ldots, \tag{7}$$

so that $T(2)$ is series (2). Both classes of series (6) and (7) are convergent, and it is possible to write down or estimate (using partial sums) the sum in all cases.

Taking the simplest case first, the sum of (7), which is a geometric series with common ratio $-1/k$, is

$$T(k) = \frac{1}{1+k}, \qquad k = 2, 3, \ldots.$$

For the companion series (6), the case $k = 1$ has already been noted with $S(1) = \ln(2)$. The case $k = 2$ is also quite well known with sum

$$S(2) = \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots.$$

Perhaps less well known is the case $k = 4$ with

$$S(4) = \frac{7\pi^4}{720} = \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \cdots.$$

More generally, for all even cases there is a formula for the sum. However, for the odd values of k, e.g.,

$$S(3) \simeq 0.901545, \qquad S(5) \simeq 0.972122,$$

we can only approximate $S(k)$.

Mathematical software such as MATHEMATICA® will compute either $S(k)$ or $T(k)$ but, in the case when k is odd, there is no formula for $S(k)$ and it can only be evaluated numerically. Readers may wish to investigate other values of k, including non-integer values. We note that the positive series corresponding to $S(k)$ is $1/1^k + 1/2^k + 1/3^k + 1/4^k + 1/5^k + \cdots$. The remarks concerning the distinction between odd and even values of k, and the numerical evaluation of the sum, apply equally to this series.

Having encountered the complementary series (6) and (7), one in which the base increases by one from term to term and the power increases by one from series to series, and the other in which the reverse happens, my thoughts turned to a series in which both the base *and* power increase. As a special example, I considered what could be said about the series

$$U = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots.$$

This series is clearly convergent by the alternating series test referred to earlier. Also, because of the rapid decrease in the size of the terms, one would expect fast convergence. Indeed, if we choose to approximate U using the nth partial sum, then with $1/(n+1)^{n+1} < 0.0005$, we obtain accuracy to three decimal places. The required value of n is 4, so that $1/1^1 - 1/2^2 + 1/3^3 - 1/4^4 = 0.783$ is an estimate for U that is accurate to three decimal places.

Although I had an accurate estimate for U, I still felt that there must be more to say about this series. MATHEMATICA gave the estimate 0.783431 by computing a numerical sum, but gave no analytical expression, and sources of mathematical formulae, of both the traditional type and the electronic, provided no assistance. Regrettably, I had to admit defeat.

Sometime later, I was preparing a first year lecture in which I had decided to introduce the problem of differentiating the function $f(x) = x^x$ by writing it as $f(x) = \exp(\ln(x^x)) = \exp(x\ln(x))$, and using the standard techniques of the chain and product rules to give $f'(x) = x^x(1 + \ln(x))$. Naturally, I contemplated the problem of determining the integral of $f(x)$, $\int x^x \,\mathrm{d}x$, deciding that I might as well discuss this when studying integration later on in the term. After much head-scratching, further recourse to sources of mathematical formulae, and even asking MATHEMATICA, I drew a blank. I suspected that this integral had no closed form, i.e. there is no elementary function $F(x)$ for which $F'(x) = x^x$. So instead, I thought that a consideration of a definite integral of x^x might be more productive, and plumped for $I = \int_0^1 x^x \,\mathrm{d}x$.

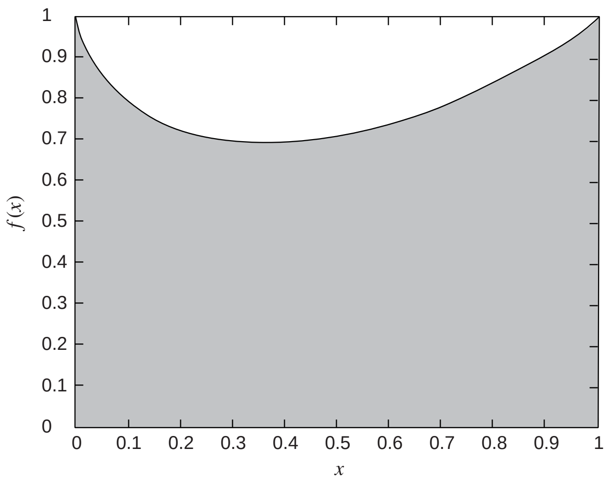


Figure 1. Graph of $y = x^x$.

Figure 1 shows the graph of $f(x) = x^x$ in [0,1], and I is the area of the shaded region. By writing x^x as $\exp(x\ln(x))$ as before, I was able to do a numerical integration to estimate I. However, because I wanted to be able to know how accurate my estimate was, I decided to determine upper bounds using rectangles above the graph, and corresponding lower bounds using rectangles below the graph. Using 10 000 rectangles (5000 to the left and 5000 to the right of the minimum at $x = 1/\mathrm{e}$) the following bounds were obtained:

$$0.783400 \le I \le 0.783461.$$

Readers might notice a possible connection between this estimate for I and the sum of the series U. At the time, I had completely forgotten about the sum U! In other circumstances, a failure to make such a connection could be disastrous. In my case I believe it was fortunate.

Therefore, having failed to make, or even notice, the connection, I endeavoured to determine the definite integral I in a non-rigorous way. Recalling the expression for the exponential function as a series, we have

$$\begin{aligned} x^x &= \exp\big(x\ln(x)\big) \\ &= 1 + x\ln(x) + \frac{\big(x\ln(x)\big)^2}{2!} + \frac{\big(x\ln(x)\big)^3}{3!} + \cdots, \end{aligned}$$

so that

$$\begin{aligned} I &= \int_0^1 x^x \, \mathrm{d}x = \int_0^1 \exp\big(x\ln(x)\big)\,\mathrm{d}x \\ &= \int_0^1 \Big(1 + x\ln(x) + \frac{\big(x\ln(x)\big)^2}{2!} + \frac{\big(x\ln(x)\big)^3}{3!} + \cdots\Big)\,\mathrm{d}x \\ &= \int_0^1 1\,\mathrm{d}x + \int_0^1 x\ln(x)\,\mathrm{d}x + \frac{1}{2!}\int_0^1 (x\ln(x))^2\,\mathrm{d}x \\ &\quad + \frac{1}{3!}\int_0^1 \big(x\ln(x)\big)^3\,\mathrm{d}x + \cdots, \end{aligned} \tag{8}$$

and the series obtained by integrating each term might give a useful formula for I.

To determine the individual terms, we could look at the first few terms separately. Alternatively, looking at the general term, if we consider

$$J_{m,n} = \int_0^1 x^m\big(\ln(x)\big)^n\,\mathrm{d}x$$

for integers $m, n \geq 0$, then integration by parts when $n > 0$ gives

$$J_{m,n} = -\frac{n}{m+1}\int_0^1 x^m\big(\ln(x)\big)^{n-1}\,\mathrm{d}x = -\frac{n}{m+1}J_{m,n-1},$$

where we have used the fact that $x^{m+1}(\ln(x))^n\big|_0^1 = 0$ since $x^{m+1}(\ln(x))^n \to 0$ as $x \to 0$. (With $x = \exp(-t)$ we have $x^{m+1}(\ln(x))^n = (-1)^n t^n/\exp((m+1)t) \to 0$ as $t \to \infty$.) Thus,

$$\begin{aligned} J_{n,n} &= \int_0^1 x^n\big(\ln(x)\big)^n\,\mathrm{d}x = -\frac{n}{n+1}J_{n,n-1} \\ &= -\frac{n}{n+1}\,\frac{-(n-1)}{n+1}J_{n,n-2} \\ &= (-1)^2\frac{n(n-1)}{(n+1)^2}J_{n,n-2} \\ &= \cdots (-1)^n n(n-1)\cdots\frac{1}{(n+1)^n}J_{n,0}, \end{aligned}$$

and since, for $n \geq 1$,

$$J_{n,0} = \int_0^1 x^n\big(\ln(x)\big)^0\,\mathrm{d}x = \int_0^1 x^n\,\mathrm{d}x = \frac{1}{n+1},,$$

we have

$$J_{n,n} = \int_0^1 x^n\big(\ln(x)\big)^n\,\mathrm{d}x = (-1)^n\frac{n!}{(n+1)^{n+1}}, \quad n \geq 1.$$

Hence, $J_{1,1} = -1!/2^2$, $J_{2,2} = 2!/3^3$, $J_{3,3} = -3!/4^4$, $J_{4,4} = 4!/5^5$, etc., and since $\int_0^1 1\,\mathrm{d}x = 1$, the formula for I in (8) becomes

$$\begin{aligned} I &= 1 - \frac{1!}{2^2} + \frac{1}{2!}\frac{2!}{3^3} - \frac{1}{3!}\frac{3!}{4^4} + \frac{1}{4!}\frac{4!}{5^5} - \cdots \\ &= \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots. \end{aligned}$$

Hence, the value of the definite integral $I = \int_0^1 x^x\,\mathrm{d}x$ is exactly the same as the sum of the series

$$U = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots,$$

a result which I found most interesting and totally unexpected. In other words, the area under the curve $y = x^x$ in [0,1] is given by the sum of the series

$$\frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \cdots;$$

or conversely, the sum of this series can be interpreted geometrically as the area shown in figure 1.

Finally, another example that readers may wish to consider is the positive series corresponding to U,

$$V = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots,$$

which can be shown to be convergent and its sum estimated numerically. However, since it is not an alternating series, the error bound is no longer helpful. The principal question is, of course, whether V can be expressed in terms of an integral or similar analytical expression, or in geometrical terms, as we have done for U.

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis and perturbation methods as well as mathematics and science education. Sadly, despite his many offers, neither of his children has yet been persuaded to join him on any kind of 'mathematical journey'!

Three Impossible Magic Squares

RALPH FELLOWS

Some magic squares can be constructed by straightforward methods. Other squares require considerable effort. Some descriptions define squares that cannot actually exist. This paper shows why three types of plausible-sounding magic squares can never be constructed.

Matrices: definitions and terminology

In a square matrix of even order n, each row contains n entries and each column contains n entries. The matrix as a whole contains n^2 entries. Because n is even, n^2 is divisible by four.

Because the matrix has even order, there are $\frac{1}{2}n$ odd-numbered rows, $\frac{1}{2}n$ even-numbered rows, $\frac{1}{2}n$ odd-numbered columns and $\frac{1}{2}n$ even-numbered columns.

The entries can be partitioned into four disjoint non-empty sets in the following way.

- Set A includes the entries whose row numbers are odd and whose column numbers are odd.
- Set B includes the entries whose row numbers are odd and whose column numbers are even.
- Set C includes the entries whose row numbers are even and whose column numbers are odd.
- Set D includes the entries whose row numbers are even and whose column numbers are even.

Figure 1 shows the upper left corner of a matrix with the entries labelled according to their set memberships.

A	B	A	B
C	D	C	D
A	B	A	B
C	D	C	D

Figure 1.

Because the matrix is square and of even order, all four sets are the same size, with each set containing exactly $\frac{1}{4}n^2$ entries.

Sets A and B make up the odd-numbered rows while sets C and D make up the even-numbered rows. Similarly, sets A and C make up the odd-numbered columns while sets B and D make up the even-numbered columns.

Another partitioning method combines the four letter sets just defined into two larger sets.

- The union of sets A and D is the set of *light entries*.
- The union of sets B and C is the set of *dark entries*.

Figure 2 shows the upper left corner of a matrix with the entries shaded according to this colouring scheme, which was chosen to match the colouring on a chessboard.

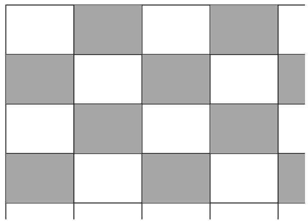


Figure 2.

Both sets are the same size, each containing exactly $\frac{1}{2}n^2$ entries.

The symbol $\sum$ is used to represent the sum of the entries in a specific set, with a subscript to identify the set. Thus $\sum_A$ stands for the sum of the entries in set A, and $\sum_{\text{dark}}$ stands for the sum of the dark entries.

Magic squares: definitions and terminology

A magic square can be defined at several levels of sophistication. The most basic property, shared by all magic squares, is that each magic square consists of a square matrix of numerical entries in which the sums of the entries in each row are all equal and the sums of the entries in each column are all equal. Because the matrix is square, there are the same number of rows and columns. And because the rows and columns are different partitions of the same entries, the sum of the entries in each row is equal to the sum of the entries in each column.

This property alone is enough to establish our first lemma.

Lemma 1. *In a magic square of even order,* $\sum_A = \sum_D$ *and* $\sum_B = \sum_C$.

Proof. As shown earlier, the union of sets A and B makes up the odd-numbered rows and the union of sets B and D makes up the even-numbered columns. Because the sum of the entries in each row is equal to the sum of the entries in each column, the sum of all the entries in all the odd-numbered rows $\sum_{A\cup B}$ equals the sum of all the entries in all the even-numbered columns $\sum_{B\cup D}$. Now we eliminate set B from both $A \cup B$ and $B \cup D$, thus subtracting $\sum_B$ from both $\sum_{A\cup B}$ and $\sum_{B\cup D}$. Thus $\sum_A = \sum_D$. A similar argument, starting with the odd-numbered rows containing sets A and B and the odd-numbered columns containing sets A and C, and then eliminating set A from both, shows that $\sum_B = \sum_C$.

Moving up a notch in sophistication, we can restrict the entries in our magic squares to the integers. This restriction allows the establishment of our second lemma.

Lemma 2. (BLADES Lemma.) *In a magic square of even order with integer entries, $\sum_{light}$ is even and $\sum_{dark}$ is even.*

Proof. As shown in Lemma 1, $\sum_A = \sum_D$. When all the entries in the square are integers, both $\sum_A$ and $\sum_D$ have integer values. Because they are equal, their sum must be even. The set of light entries is defined as the union of the entries in sets A and D, so $\sum_{\text{light}}$ is even. Similarly, because $\sum_B = \sum_C$, $\sum_{\text{dark}}$ is even. (The acronym BLADES comes from Both Light And Dark entries have Even Sums.)

Traditionally, there are several other restrictions to add to our definition, but several terms must be defined first.

Each square has two *long diagonals*, one sloping up from left to right and one sloping up from right to left, as shown in figure 3.

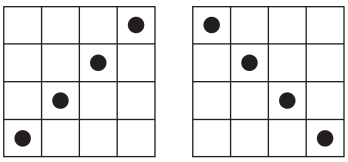


Figure 3.

In a square of order n, each long diagonal contains n entries.

Along with its two long diagonals, each square contains *broken diagonals*. Some of these slope up from left to right, as shown in figure 4.

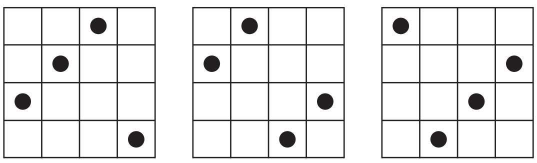


Figure 4.

The other broken diagonals slope up from right to left, as shown in Figure 5.

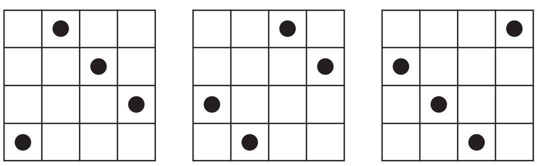


Figure 5.

Broken diagonals, like the two long diagonals, each contain n entries. Observe that each broken diagonal wraps around to the opposite border of the square when it reaches an edge. Counting long diagonals and broken diagonals together, each square of order n contains n diagonals sloping up from left to right and n diagonals sloping up from right to left.

The sum to which the entries in each row and column add up is called S, the *magic sum*.

That completes the preliminaries. Henceforth, we will be using the following definition.

Definition. A *magic square* of order n is a square matrix that meets the following requirements:

(1) Each entry is an integer.
(2) No entry is equal to any other entry.
(3) Each entry is selected from a sequence of n^2 consecutive integers.
(4) The sum of the entries in each row is the magic sum S.
(5) The sum of the entries in each column is the magic sum S.
(6) The sum of the entries in each long diagonal is the magic sum S.

Requirement (3) deserves some explanation. It is customary to number the entries in a magic square of order n with the sequence $1, 2, \ldots, n^2$. This scheme is not mandatory, however, and some authors prefer to use the sequence $0, 1, \ldots, n^2 - 1$. The examples given in this paper follow the customary numbering, but the definitions, formulas and proofs are written to show that, in squares of even order, the reasoning is valid for any sequence of consecutive integers.

Sometimes we will refer to *complementary pairs* of entries. Two entries, each the *complement* of the other, make up a complementary pair if and only if they add up to a specific sum P, which is defined as the sum of the smallest integer in the consecutive sequence added to the largest integer in the sequence. For example, in a square of order six in which the entries are numbered 1 to 36, the sum P is 37, and the entries 1 and 36 are complements, as are entries 2 and 35, 3 and 34, and so on.

The sequence of consecutive integers runs from some arbitrary integer x to $x + n^2 - 1$. Thus the sum T of all the entries in the square is given by the formula

$$T = n^2 x + \frac{n^2(n^2-1)}{2}$$

and, because the square contains n rows, the magic sum S is given by the formula

$$S = \frac{T}{n},$$

so that

$$S = nx + \tfrac{1}{2}n^3 - \tfrac{1}{2}n.$$

Because a magic square of order one contains only one integer, there is only one magic square of order one: the starting integer x in a box by itself. There are no magic squares of order two, as the reader can easily verify. There is essentially only one magic square of order three, although its entries can be rotated and reflected in several ways. (Some experimentation will show the reader that the number in the middle of the consecutive sequence must be the entry in the centre of the matrix. Once this number is positioned, the remaining numbers fall into place.) There are many magic squares of every order greater than or equal to four. Figure 6 shows some examples in which the consecutive sequence of integers starts at 1.

2	9	4
7	5	3
6	1	8

Magic square
Order three
Magic sum = 15

1	2	15	16
12	14	3	5
13	7	10	4
8	11	6	9

Magic square
Order four
Magic sum = 34

5	23	7	16	14
19	12	21	10	3
11	9	18	2	25
8	1	15	24	17
22	20	4	13	6

Magic square
Order five
Magic sum = 65

1	25	35	34	10	6
7	23	29	26	17	9
13	32	21	22	4	19
24	5	16	15	33	18
30	14	8	11	20	28
36	12	2	3	27	31

Magic square
Order six
Magic sum = 111

Figure 6.

Even-order magic square invariants

Because the three proofs that follow rely heavily on the parities (but not the exact values) of certain critical numbers, this section shows that the parities of these numbers are independent of the starting integer. Even-order squares come in two types: $4k$ squares, where $n = 4, 8, 12, \ldots$, and $4k+2$ squares, where $n = 6, 10, 14, \ldots$.

Note that these invariants apply only to magic squares of even order. They certainly do *not* hold for magic squares of odd order.

For all even-order squares

(1) Because *n is an even number*, it follows that *n^2 is divisible by four*, as stated earlier.

(2) *The total T of all n^2 entries is even.* Here $T = n^2x + \frac{1}{2}[n^2(n^2-1)]$, and because n^2 is divisible by four, both of these terms are even, and their sum T is even.

(3) *The sum P of the two entries in a complementary pair is odd.* Let x represent the smallest integer in the consecutive sequence. Then the largest integer in the sequence is $x + n^2 - 1$, and the sum of the two is $2x + n^2 - 1$. Now, $2x$ is even, as is n^2, but subtracting the 1 at the end of the expression makes the overall total odd, so P is always odd.

For $4k$ squares only

(4) *The quantity $\frac{1}{2}n$ is an even number.* Because $n = 4k$, $\frac{1}{2}n = 2k$, an even number.

(5) *The quantity $\frac{1}{4}n^2$ is an even number.* Because $n^2 = 16k^2$, $\frac{1}{4}n^2$ is even.

(6) *the magic sum S is even.* This sum is equal to $nx + \frac{1}{2}n^3 - \frac{1}{2}n$. Because n is even, nx is even, and because n^2 is divisible by four, $\frac{1}{2}n^3$ is also even, and because $\frac{1}{2}n$ is always an even number, the overall total is even, and S is always even.

For $4k+2$ squares only

(7) *The quantity $\frac{1}{2}n$ is an odd number.* Because $\frac{1}{2}n = \frac{1}{2}(4k+2) = 2k+1$, $\frac{1}{2}n$ is odd.

(8) *The quantity $\frac{1}{4}n^2$ is an odd number.* Because $n^2 = 16k + 16k + 4$, it follows that $\frac{1}{4}n^2 = 4k^2 + 4k + 1$, which is an odd number.

(9) *The magic sum S is odd.* This sum is equal to $nx + \frac{1}{2}n^3 - \frac{1}{2}n$. Because n is even, nx is even, and because n^2 is divisible by four, $\frac{1}{2}n^3$ is also even. But $\frac{1}{2}n$ is always an odd number, so the overall total is odd, and S is always odd.

Nasik squares: definitions and terminology

A *Nasik square*[1] is a magic square that meets some additional requirements. Nasik squares are also called *pandiagonal squares*, *panmagic squares* and *continuous squares*, among other things.

Definition. A *Nasik square* is a magic square in which the sums of the entries in each of the broken diagonals all equal the magic sum S.

The magic square of order one is a Nasik square. The magic square of order three, shown earlier in figure 6, is not a Nasik square. There are Nasik squares of most orders greater than or equal to four. (The orders for which Nasik squares do not exist are the subject of the upcoming proof.) Figure 7 shows two examples of Nasik squares.

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

Nasik square
Order four
Magic sum = 34

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

Nasik square
Order five
Magic sum = 65

Figure 7.

[1] In the middle 1800s, H. E. Dudeney, the puzzle expert, received correspondence from a Mr Frost regarding an unusual type of magic square. Mr Frost was living in India, in a town named Nasik. This type of magic square became known by the name of that town.

Observe that no Nasik square of order six is shown. The purpose of the upcoming proof is to show that a Nasik square of order six cannot be constructed.

No order $4k + 2$ Nasik squares

Theorem. *No Nasik square of order $4k + 2$ can be constructed.*

Proof. Assume that a Nasik square of order $4k + 2$ does exist, partitioned into the sets A, B, C and D and also into the light and dark entries described earlier.

For a square to be a Nasik square, the sum of the entries in each of the diagonals sloping up from right to left must be the magic sum S. We will be looking at only half these diagonals — the diagonals that make up the set of light entries. Because we are talking about only half of the diagonals, we will be dealing with $\frac{1}{2}n$ diagonals.

Because the diagonals we are considering slope up from right to left, we will start in column n with the entries $(2, n)$, $(4, n)$, ..., (n, n) and track all the diagonals at once. Because n is even, these entries all have even row and column numbers, so they are all members of set D — and they are all the members of set D in column n.

Moving along each diagonal, left one column to column $n-1$ and up one row, we reach the entries $(1, n-1)$, $(3, n-1)$, ..., $(n-1, n-1)$. These entries all have odd row and column numbers, so they are all members of set A — in fact, all the members of set A in column $n-1$.

Moving left another column to column $n - 2$ and up another row, we have a slight problem: we can't move up from cell $(1, n-1)$ to cell $(0, n-2)$. So for that broken diagonal we wrap down to cell $(n, n-2)$. Once again, all the entries have even row and column numbers, so we have gathered all the members of set D in column $n - 2$.

We continue moving up and left, wrapping down from row 1 to row n when we move from an odd-numbered column to an even-numbered column, until we reach column 1. As we go, we accumulate all the members of sets A and D and only the members of sets A and D — in other words, all the members and only the members of the set of light entries.

Upon completion, we have tracked $\frac{1}{2}n$ diagonals — an odd number of diagonals — the entries in each of which add up to the magic sum S — another odd number. So the sums of all the entries in the accumulated diagonals, or $\sum_{\text{light}}$, must be an odd number.

This is a contradiction of the BLADES lemma, which shows that $\sum_{\text{light}}$ is always even.

Therefore, no Nasik square of order $4k + 2$ can be constructed.

Associative squares: definitions and terminology

An *associative square* is a magic square that meets some additional requirements. Associative squares are also called *associated squares*. In an associative square, the entries are considered in complementary pairs.

Definition. An *associative square* is a magic square in which the two entries in each complementary pair occupy positions equidistant from the centre of the square and opposite one another across the centre.

In matrix terminology, this means that if one entry in a complementary pair is at row x and column y, its complement is at row $n + 1 - x$ and column $n + 1 - y$.

In an associative square of odd order, which contains an odd number of entries, the entry in the centre of the square must be the integer $\frac{1}{2}P$, the integer in the middle of the consecutive sequence.

The magic squares of order one and order three are associative squares. There are associative squares of most orders greater than or equal to four. (The orders for which associative squares do not exist are the subject of the upcoming proof.) Figure 8 shows two examples.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Associative square
Order four
Magic sum = 34

18	22	1	10	14
24	3	7	11	20
5	9	13	17	21
6	15	19	23	2
12	16	25	4	8

Associative square
Order five
Magic sum = 65

Figure 8.

Observe that no associative square of order six is shown. The purpose of the upcoming proof is to show that an associative square of order six cannot be constructed.

No order $4k + 2$ associative squares

Theorem. *No associative square of order $4k + 2$ can be constructed.*

Proof. Assume that an associative square of order $4k+2$ does exist, partitioned into the sets A, B, C and D and also into the light and dark entries described earlier.

By the definition of an associative square, the entries in each complementary pair must be positioned in a certain way.

One entry in a complementary pair is located in an odd row and odd column if and only if the other entry in the pair is located in an even row and even column. If one entry is in row $2x + 1$ (an odd number), then the other entry must be in row $n + 1 - (2x + 1)$, or $n - 2x$ (an even number). The same reasoning applies to columns. Thus a complementary pair with one entry in set A has its other entry in set D, and vice versa.

On the other hand, one entry in a complementary pair is located in an odd row and even column if and only if the other entry in the pair is located in an even row and odd column, by the reasoning just shown. Thus a complementary pair with one entry in set B has its other entry in set C, and vice versa.

This means that the two entries in each complementary pair are both members of the set of light entries (the union of sets A and D) or are both members of the set of dark entries (the union of sets B and C).

Now consider only the complementary pairs in the set of light entries, which contains $\frac{1}{2}n^2$ entries. Because the sum P of a complementary pair is odd, half of these entries (or $\frac{1}{4}n^2$ entries) must be even, and half of the entries (or, again, $\frac{1}{4}n^2$ entries) must be odd. And because $\frac{1}{4}n^2$ is always an odd number, there must be an odd number of odd entries in the set of light entries. This means $\sum_{\text{light}}$ must be an odd number.

This is a contradiction of the BLADES lemma, which shows that $\sum_{\text{light}}$ is always even.

Therefore, no associative square of order $4k + 2$ can be constructed.

Magic squares: Nasik, associative, neither, or both

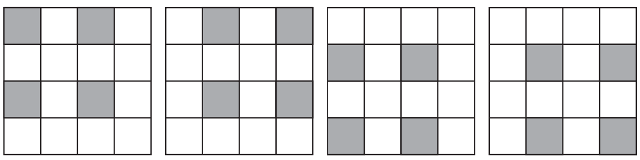
A magic square can be a Nasik square, or an associative square, or neither, or, for some orders, both. The order one square is both a Nasik square and an associative square. The order three square (figure 6) is an associative square but not a Nasik square. The three order four squares shown so far include an associative square (figure 8), a Nasik square (figure 7), and a square that is neither (figure 6) — an order four square cannot be both, as will be shown in the upcoming proof. The order five squares shown so far include a square that is both a Nasik square and an associative square (figure 7), a square that is an associative square but not a Nasik square (figure 8), and a square that is neither (figure 6). The only order six square shown (figure 6) is, of course, neither a Nasik square nor an associative square.

No order four magic square can be both Nasik and associative

As mentioned earlier, an order four magic square can be a Nasik square or an associative square, but not both. The following proof shows why.

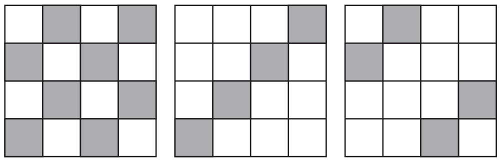
Theorem. *An order four magic square cannot be both a Nasik square and an associative square.*

Proof. Consider a Nasik square of order four, partitioned into sets A, B, C and D as described earlier. These sets are small enough that we can list the entries in each one:



Set A Set B Set C Set D

Remember that the union of sets B and C is also defined as the set of dark entries.

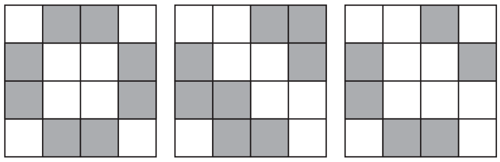


$B \cup C$ Long diagonal Broken diagonal

The set of dark entries can be defined another way: as the union of the entries in one of the long diagonals and the entries in one of the broken diagonals. Because this is a Nasik square, the sum of the entries in each of these two diagonals is the magic sum S, so it follows that $\sum_{\text{dark}} = 2S$.

Thus $\sum_B + \sum_C = 2S$. By Lemma 1, $\sum_B = \sum_C$, so $\sum_B = S$ and $\sum_C = S$.

Now consider another set E built of eight entries from two broken diagonals: the broken diagonal containing entries $(1, 2)$, $(2, 1)$, $(3, 4)$ and $(4, 3)$, and the broken diagonal containing entries $(1, 3)$, $(2, 4)$, $(3, 1)$ and $(4, 2)$. Because each broken diagonal sums to S, it follows that $\sum_E = 2S$.

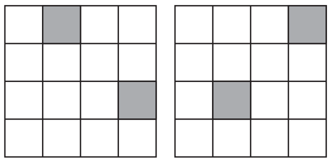


Set E Set F Set G $(E \cap F)$

Consider another set F built of eight entries from two broken diagonals: the broken diagonal containing entries $(1, 4)$, $(2, 1)$, $(3, 2)$ and $(4, 3)$, and the broken diagonal containing entries $(1, 3)$, $(2, 4)$, $(3, 1)$ and $(4, 2)$. Because each broken diagonal sums to S, it follows that $\sum_F = 2S$.

Also consider set G, the intersection of sets E and F. We do not have an exact figure for $\sum_G$ but we will not need one.

Let set H be the two entries remaining after eliminating set G from set E. Their sum $\sum_H = \sum_E - \sum_G = 2S - \sum_G$.



Set H Set I

Let set I be the two entries remaining after eliminating set G from set F. Their sum $\sum_I = \sum_F - \sum_G = 2S - \sum_G$.

Note that the union of sets H and I equals set B, so $\sum_H + \sum_I = \sum_B$. Because $\sum_H = 2S - \sum_G$ and $\sum_I = 2S - \sum_G$, it follows that $\sum_H = \sum_I$. And because $\sum_B = S$, it follows that $\sum_H = \frac{1}{2}S$ and $\sum_I = \frac{1}{2}S$.

Based on the formulas given earlier for S and P, the number $\frac{1}{2}S$ is exactly the sum P to which a complementary pair in an order four square should total. In other words, the two entries in set H are a complementary pair, and so are the two entries in set I.

In an associative square, this is not the way the entries in the complementary pair should be situated. Consider entry $(1, 4)$ in set I. In an associative square, its complement should be entry (4, 1). In a Nasik square, its complement is entry $(3, 2)$, as we have just shown. Thus an order four square cannot satisfy both definitions.

Therefore, an order four magic square cannot be both a Nasik square and an associative square.

Acknowledgement

The author extends his thanks and appreciation to Professor Thomas Moore, of the Mathematics and Computer Science Department at Bridgewater State College, for his help and encouragement in the preparation of this paper.

References

Regarding $4k+2$ Nasik and associative squares, the following article contains a short (and different) impossibility proof that covers both types of squares:

1. C. Planck, Pandiagonal magics of orders 6 and 10 with minimal numbers, *The Monist* **29** (1919), pp. 307–316.

The following book treats magic squares in general very rigorously and mentions (but does not include) a proof by an M. Raynor regarding $4k + 2$ squares:

2. Maurice Kraitchik, *Mathematical Recreations* (Norton Press, New York, 1942).

Since Frénicle de Bessy developed it in the 1600s, a standard list of all the order four magic squares has been available. Although not a proof, examination of a copy of this list shows that in fact no order four square is both Nasik and associative. Such a copy can be downloaded from the following website: `http://www.pse.che.tohoku.ac.jp/~msuzuki`.

Ralph Fellows *is a student at Bridgewater State College, Bridgewater, Massachusetts. After several decades in the field of computer technology, he has recently changed careers. He is now pursuing a profession in mathematics, where he feels his contributions can have a more immediate, practical and beneficial effect on the world around him. This paper on magic squares is his first effort along these lines.*

Shadowing a Path

GUIDO LASTERS and DAVID SHARPE

Start with a conic, an ellipse, a hyperbola, a parabola, or even a pair of straight lines. In our figures we use an ellipse. Draw any path $A_1A_2 \cdots A_n$, where $n \geq 3$, 'bouncing off' the conic as shown in figures 1 and 2. (There is no requirement that A_1A_2 and A_2A_3 make equal angles with the normal to the conic at A_2; the points $A_1, A_2, \ldots, A_n$ are arbitrary points of the conic. The usual convention applies when $A_i = A_{i+1}$, namely that the chord A_iA_{i+1} becomes the tangent to the conic at A_i.)

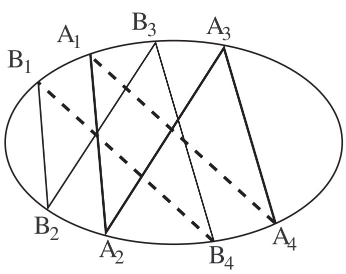


Figure 1. $n = 4$.

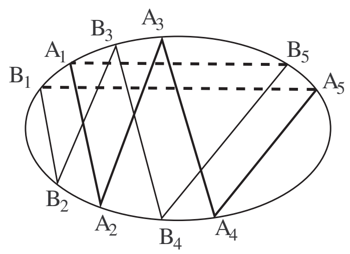


Figure 2. $n = 5$.

Now shadow this path with a path $B_1B_2 \cdots B_n$ parallel to the first. Then, if n is even, A_1A_n is parallel to B_1B_n; if n is odd, A_1B_n is parallel to B_1A_n.

We first prove this for the case $n = 3$. In figure 3, we have a conic and six points A_1, B_2, A_3 and B_1, A_2, B_3 of the conic. Pappus' theorem in projective geometry says that the three points where A_1A_2 and B_1B_2 cross, where B_2B_3 and A_2A_3 cross, and where A_1B_3 and B_1A_3 cross, are collinear. Or, to put it another way, if the paths $A_1A_2A_3$ and $B_1B_2B_3$ cross on a straight line ℓ, then A_1B_3 and B_1A_3 also cross on ℓ. If we now let ℓ be the line at infinity, then two lines meeting at a point of ℓ means that they are parallel. Hence, if $A_1A_2A_3$ and $B_1B_2B_3$ are parallel paths, A_1B_3 and B_1A_3

are parallel. This is our result when $n = 3$.

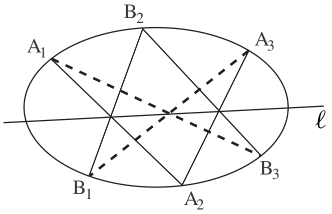


Figure 3. $n = 3$.

We can now prove the result for a general value of n (≥ 3) by induction. We have proved it for the initial case $n = 3$, so we consider a value of $n > 3$ and assume that the result is true for $n - 1$. The proof of the inductive step depends on whether n is even or odd.

First consider the case when n is even. We have parallel paths $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$. Remove the final points A_n and B_n. Since paths $A_1A_2 \cdots A_{n-1}$ and $B_1B_2 \cdots B_{n-1}$ now each have $n - 1$ points, and $n - 1$ is odd, the inductive hypothesis tells us that A_1B_{n-1} and B_1A_{n-1} are parallel. But now $A_1B_{n-1}B_n$ and $B_1A_{n-1}A_n$ are parallel paths, so the case $n = 3$, which we have proved, tells us that A_1A_n and B_1B_n are parallel.

Now consider the case when n is odd. We again consider parallel paths $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ and remove A_n and B_n. This time, since $n - 1$ is even, the inductive hypothesis tells us that A_1A_{n-1} and B_1B_{n-1} are parallel. But now $A_1A_{n-1}A_n$ and $B_1B_{n-1}B_n$ are parallel paths and the case $n = 3$ tells us that A_1B_n and B_1A_n are parallel.

The proof of the inductive step is now complete. Quite an intriguing induction!

Guido Lasters *lives in Tienen, Belgium, and* ***David Sharpe*** *teaches at the University of Sheffield, UK, and is editor of* Mathematical Spectrum. *This article is one of a growing number from these authors, usually with a geometrical theme.*

Is $\pi(6521) = 6! + 5! + 2! + 1!$ Unique?

CHRIS K. CALDWELL and G. L. HONAKER, JR.

The prime counting function, $\pi(x)$, counts exactly how many primes there are less than or equal to x. The second author discovered the following 'curio' (see reference 4):

$$\pi(6521) = 6! + 5! + 2! + 1!.$$

If we write the positive integer x in base 10, that is,

$$x = a_k \cdots a_2a_1a_0 \quad (\text{with } a_k > 0),$$

then we can ask whether there are any other prime solutions to

$$f(x) := \sum_{i=0}^{k} a_i! = \pi(x). \tag{1}$$

How many solutions can be generated if we allow x to be composite? Is there an upper bound on how far we need to look? What if we work in a base other than 10 or use other functions? Below we provide answers to these questions, and then suggest new areas for further investigation.

Searching for another

By the prime number theorem (reference 7, pp. 225–227), the prime counting function $\pi(x)$ is asymptotic to $x/\ln x$. In fact, Dusart (reference 5) has shown that, when $x \geq 599$,

$$\frac{x}{\ln x}\Big(1 + \frac{0.992}{\ln x}\Big) < \pi(x) < \frac{x}{\ln x}\Big(1 + \frac{1.2762}{\ln x}\Big).$$

The factorial $a_i!$ is at most 9! for each of the $[1 + \log x]$ digits of x, so any solution x to (1) must satisfy

$$\frac{x}{\ln x}\Big(1 + \frac{0.992}{\ln x}\Big) < \pi(x) = f(x) \leq 9!\Big[1 + \frac{\ln x}{\ln 10}\Big]. \tag{2}$$

This statement is false for $x > 48\,657\,759$, so this is an upper bound for solutions. If x is an eight-digit solution beginning with 4, then the second digit is at most 8 and we can use the tighter bound

$$f(x) \leq 4! + 8! + 9!\,6 < \pi(40\,000\,000) = 2\,433\,654$$

to see that there are no such solutions. Now, we know $x < 40\,000\,000$. After checking to see that $39\,999\,999$ does not work, we note that for $N_1 = (3.8)10^7 \leq x < 39\,999\,999$ we have

$$f(x) \leq 3! + 8! + 9!\,6 < \pi(N_1) = 2\,318\,966.$$

Similarly, for $N_2 = (3.6)10^7 \leq x < N_1$ we have

$$f(x) \leq 3! + 7! + 9!\,6 < \pi(N_2) = 2\,204\,262.$$

Therefore, there are no solutions with $x \geq N_2$.

For $N_3 = (3.0)10^7 \le x < N_2$, first we check the cases where x ends in six 9s individually; then for the remaining integers x we have

$$f(x) \le 3! + 5! + 8! + 9!\,5 < \pi(N_3) = 1\,857\,859.$$

A check of the integers $x \le N_3$ using the public domain program UBASIC (see reference 2) gives the following 23 solutions:

6500, 6501, 6510, 6511, **6521**, 12 066, 50 372, 175 677, 553 783, **5 224 903**, 5 224 923, 5 246 963, 5 302 479, 5 854 093, 5 854 409, 5 854 419, 5 854 429, 5 854 493, 5 855 904, 5 864 049, 5 865 393, 10 990 544, 11 071 599

(see reference 8, sequence A049529). Of these, only 6521 and 5 224 903 are prime (see reference 6, p. 11). Here, and below, we highlight prime numbers in bold.

Bases other than 10

We can write x in a base B other than 10

$$x = b_k \cdots b_2 b_1 b_0 \quad (\text{with } b_k > 0)$$

and ask whether the equation

$$g(x) := \sum_{i=0}^{k} b_i! = \pi(x) \tag{3}$$

has any solutions. Now, $b_i! \le (B-1)!$, so we can replace the inequality (2) with

$$\frac{x}{\ln x} < \pi(x) = g(x) \le (B-1)!\Big[1 + \frac{\ln x}{\ln B}\Big]. \tag{4}$$

Omitting the factor $1 + (0.992/\ln x)$ from (2) ensures that the left-most inequality holds for $x \ge 11$ rather than $x = 599$.

For each value of B the right side of (4) grows like a multiple of $\ln x$, whereas the left-hand side grows like $x/\ln x$; therefore the inequality is false for all large x. So there is a value $x_0(B)$ such that any solution satisfies $x \le x_0(B)$. We will show that we can take $x_0(B) = 2BB!\ln B$ for all bases $B > 2$. Since (4) is already false at $x = 13$ for $B = 2$, we may take $x_0(2) = 13$.

First note that for any solution x we have $x \ge B$ (otherwise $x! = \pi(x)$), so (4) yields

$$\frac{x}{\ln x} < (B-1)!\Big(1 + \frac{\ln x}{\ln B}\Big) \le \frac{2(B-1)!\ln x}{\ln B}. \tag{5}$$

We next show that $x < B^B$ (for $B \ge 3$). Otherwise, since $x/(\ln x)^2$ is an increasing function for $x > e^2$, the inequality above divided by $\ln x$ gives:

$$\frac{B^2}{B^2(\ln B)^2} \le \frac{x}{(\ln x)^2} < \frac{2(B-1)!}{\ln B} < \frac{2B}{\ln B}\Big(\frac{B}{e}\Big)^{B-1}.$$

The last inequality comes from $\ln(n-1)! \le n\ln n - n + 1$ (posed as a problem in the problems section). But this gives $e^{B-1} < 2B^2\ln B$, which is false for $B \ge 6$. For the remaining bases 3, 4 and 5, we can verify $x < B^B$ individually using (4). Finally, upon multiplying (5) by $\ln x$ and using our result $\ln x < B\ln B$, we have

$$x < 2(B-1)!B^2\ln B,$$

which is the desired bound.

We used UBASIC and a slightly sharpened form of the bound above to list all of the solutions for various small bases; the result of this search is in Table 1.

Table 1. Solutions in other bases.

Base B	Solutions written in base 10
2	**3**, **5**, 6, 8, 9, 10
3	**3**, 4, **5**, 6, 8
4	4, 6, 10, **19**, 27, 63
5	**101**, **229**, 374
6	18, 20, 134, 731, 737, 789, 1547
7	**5501**, 5690, 6530, **6719**, 6726, **6733**, 13 180, 14 395
8	**19**, 844, 5530, 13 174, 49 336, 49 337, 58 341, 58 348, **64 921**, 106108, **114 599**
9	21, **103**, 364, 851, 105 712, 105 721, 105 730, 493 832,494 055, 494 056, **495 491**, 495 524, 550 620, 550 622,550 654, **560 437**, 1 029 375, 1 029 376, 1 029 459,1 031 285, 1 041 084, 1 041 085, 1 041 128, 1 041 411
11	5704, 5715, 6705, 106 022, 107 114, 5 456 695, 5 927 793, 5 927 804, 5 927 815, 5 927 825, 16 981 728,61 924 436, 61 934 787, **62 009 933**, 63 370 216,67 733 027, 67 733 038, 129 294 118, 134 549 464,134 549 475, 134 549 486, 134 551 268, 136 058 582,136 058 583, 197 958 265

Alternatively, we could choose an integer x and ask if there is any base B for which the equation (3) has a solution. Clearly $x \ge B$. If we find the least integer n such that $n! \ge \pi(x)$, then we know that $b_0 = (x \mod B) \le n$, so B is a divisor of $x - i$ for some $i \le n$. For each x we then have a relative short list of possible bases. In this way, we find all of the prime integers $x \le 160\,000\,000$ such that (3) holds (x and B are written in base 10):

$$\begin{aligned}(x, B) \in \{&(3, 2), (3, 3), (5, 2), (5, 3), (17, 14), (19, 4),\\ &(19, 8), (97, 24), (97, 93), (101, 5), (103, 9),\\ &(229, 5), (661, 132), (661, 656), (673, 334),\\ &(701, 232), (5449, 908), (5449, 5443),\\ &(5501, 7), (6473, 1078), (6521, 10), (6719, 7),\\ &(6733, 7), (49\,037, 49\,030), (49\,043, 24\,518),\\ &(49\,277, 7039), (56\,809, 9467), (64\,921, 8),\\ &(114\,599, 8), (484\,061, 484\,053),\\ &(485\,909, 60\,738), (495\,491, 9), (560\,437, 9),\\ &(5\,222\,447, 5\,222\,438), (5\,222\,501, 2\,611\,246),\\ &(5\,222\,837, 1\,305\,707), (5\,224\,451, 580\,494),\end{aligned}$$

$$(5\,224\,903, 10), (5\,378\,437, 15), (6\,480\,811, 15), (61\,194\,733, 61\,194\,723), (61\,285\,057, 6\,128\,505), (62\,009\,933, 11), (67\,717\,891, 7\,524\,209)\big\}.$$

There are infinitely many such solutions! To see this, let p_n be the nth prime. Then $(x, B) = (p_{n!+1}, p_{n!+1} - n)$ is a solution to (3).

The multifactorials

Instead of the factorial function, we could use the double factorial function $n!!$ (see reference 1, p. 258) or its generalization — the multifactorial function. These are defined inductively for integers n as follows:

$$n! = \begin{cases} 1 & n \le 1, \\ n \cdot (n-1)! & n > 1 \end{cases} \quad (n \text{ factorial}),$$

$$n!! = \begin{cases} 1 & n \le 1 \\ n \cdot (n-2)!! & n > 1 \end{cases} \quad (n \text{ double factorial}),$$

$$n!!! = \begin{cases} 1 & n \le 1 \\ n \cdot (n-3)!!! & n > 1 \end{cases} \quad (n \text{ triple factorial}),$$

and, in general,

$$n!_k = \begin{cases} 1 & n \le 1 \\ n!k = n \cdot (n-k)!_k & n > 1 \end{cases} \quad (n\ k\text{-factorial}).$$

For example, $13!!! = 13!_3 = 13 \times 10 \times 7 \times 4 \times 1$ and $23!_4 = 23 \times 19 \times 15 \times 11 \times 7 \times 3$.

The approach above can also be used to bound the integers to check for the multifactorials. Using the double factorial function, we have four solutions: 34, 6288, 10 982 and 11 978. For the triple factorial function, we have these four solutions: 45, 117, 127 and 2199. If we restrict ourselves to prime solutions, then there are only two additional solutions provided by all of the multifactorial functions:

$$\pi(127) = 1!!! + 2!!! + 7!!!$$

and

$$\pi(97) = 9!_7 + 7!_7.$$

Other functions

If we just count the digits, there is one solution: 2 ($\pi(2) = 1$, and 2 has 1 digit). If we add the digits, then there are four solutions: 0, 15, 27 and 39 (none of which is prime). Using higher powers, we find the following prime solutions:

$$\pi(93\,701) = 9^4 + 3^4 + 7^4 + 0^4 + 1^4,$$
$$\pi(1\,776\,839) = 1^5 + 7^5 + 7^5 + 6^5 + 8^5 + 3^5 + 9^5,$$
$$\pi(1\,264\,061) = 1^6 + 2^6 + 6^6 + 4^6 + 0^6 + 6^6 + 1^6,$$
$$\pi(\mathbf{34\,543}) = 3^3 + 4^4 + 5^5 + 4^4 + 3^3.$$

Note also that the prime 34 543, found by the first author, is palindromic (see reference 3).

Questions for the reader

Why add the terms corresponding to each digit? We could multiply:

$$\pi(1321) = 1^3 \cdot 3^3 \cdot 2^3 \cdot 1^3,$$

or alternate signs:

$$\pi(19) = -1 + 9, \qquad \pi(53) = 5^2 - 3^2,$$
$$\pi(227) = 2^2 - 2^2 + 7^2, \qquad \pi(929) = 9^2 - 2^2 + 9^2,$$
$$\pi(47\,501) = -4! + 7! - 5! + 0! - 1!.$$

How about backwards exponentiation: $\pi(17) = 7^1$ and $\pi(23) = 3^2$?

Exploring other functions, such as the sum of divisors function, may also prove interesting. In all such cases, the authors would be pleased to hear of your results.

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The first author is a professor of mathematics at University of Tennessee at Martin. He lives on a small 'farm' in rural northwest Tennessee with his wife, five children, two cats and numerous chickens. The second author is a schoolteacher and amateur number theorist. He is an avid chess player.

About Hexagons

JENS CARTENSEN

For some reason it is not often that we encounter theorems about hexagons. I have for some time collected some of these rather beautiful and unusual 'hexagonal' theorems and I present them here for the enjoyment of the readers. A *cyclic hexagon* is a hexagon whose vertices all lie on a circle, as in figure 1.

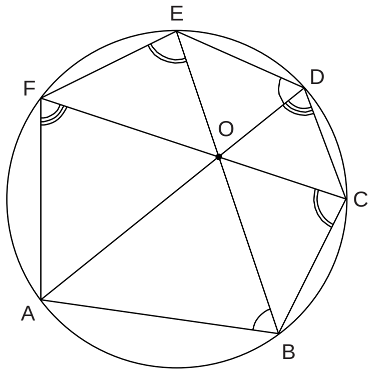


Figure 1.

Theorem 1. *In a cyclic hexagon* ABCDEF *the diagonals* AD*,* BE *and* CF *intersect in a point if and only if*

$$\mathrm{AB}\cdot\mathrm{CD}\cdot\mathrm{EF}=\mathrm{BC}\cdot\mathrm{DE}\cdot\mathrm{FA}.$$

Proof. Suppose first that the diagonals intersect in a point O as in figure 1. Then $\triangle$ABO and $\triangle$EDO are similar so that

$$\frac{\mathrm{AB}}{\mathrm{DE}}=\frac{\mathrm{AO}}{\mathrm{EO}}.$$

Also $\triangle$BCO and $\triangle$FEO are similar and $\triangle$CDO and $\triangle$AFO are similar so that

$$\frac{\mathrm{CD}}{\mathrm{FA}}=\frac{\mathrm{CO}}{\mathrm{AO}}\quad\text{and}\quad\frac{\mathrm{EF}}{\mathrm{BC}}=\frac{\mathrm{EO}}{\mathrm{CO}}.$$

Multiplication of these equations yields

$$\frac{\mathrm{AB}}{\mathrm{DE}}\cdot\frac{\mathrm{CD}}{\mathrm{FA}}\cdot\frac{\mathrm{EF}}{\mathrm{BC}}=\frac{\mathrm{AO}}{\mathrm{EO}}\cdot\frac{\mathrm{CO}}{\mathrm{AO}}\cdot\frac{\mathrm{EO}}{\mathrm{CO}}=1,$$

so that

$$\mathrm{AB}\cdot\mathrm{CD}\cdot\mathrm{EF}=\mathrm{BC}\cdot\mathrm{DE}\cdot\mathrm{FA}.$$

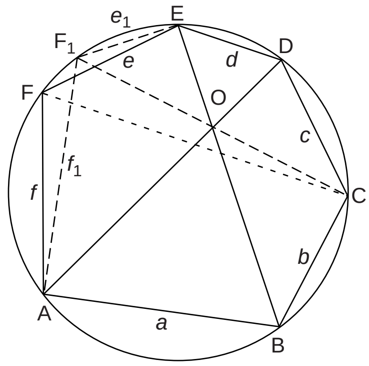


Figure 2.

Suppose on the other hand that the sides in a cyclic hexagon satisfy

$$a\cdot c\cdot e=b\cdot d\cdot f,\tag{9}$$

as in figure 2. Let O be the intersection of the diagonals AD and BE and suppose that the line CO intersects the circle at F_1. According to the first part of the theorem we have

$$a\cdot c\cdot e_1=b\cdot d\cdot f_1\,.\tag{10}$$

Using (1) and (2) we deduce that

$$a\cdot c\cdot e\cdot f_1=b\cdot d\cdot f\cdot f_1=b\cdot d\cdot f_1\cdot f=a\cdot c\cdot e_1\cdot f,$$

and thus
$$e\cdot f_1=e_1\cdot f.\tag{3}$$

We now use the theorem of Ptolemy: a quadrilateral is cyclic if and only if the sum of the products of the opposite sides equals the product of the diagonals. The quadrilateral $\mathrm{AEF_1F}$ is cyclic, so

$$\mathrm{AF_1}\cdot\mathrm{EF}=\mathrm{AE}\cdot\mathrm{FF_1}+\mathrm{EF_1}\cdot\mathrm{AF}\iff f_1\cdot e=\mathrm{AE}\cdot\mathrm{FF_1}+e_1\cdot f.$$

Comparing this with (3) we get $\mathrm{AE}\cdot\mathrm{FF_1}=0$, so F coincides with F_1. Therefore, CF passes through O.

Theorem 2. *Suppose that the vertices of a cyclic hexagon* $\mathrm{A_1A_2A_3A_4A_5A_6}$ *are the endpoints of three diameters. Then the feet of the perpendiculars from a point* P *of the circumcircle on the sides of the hexagon form a (concave) hexagon* $\mathrm{Q_1Q_2Q_3Q_4Q_5Q_6}$ *with* right *angles.*

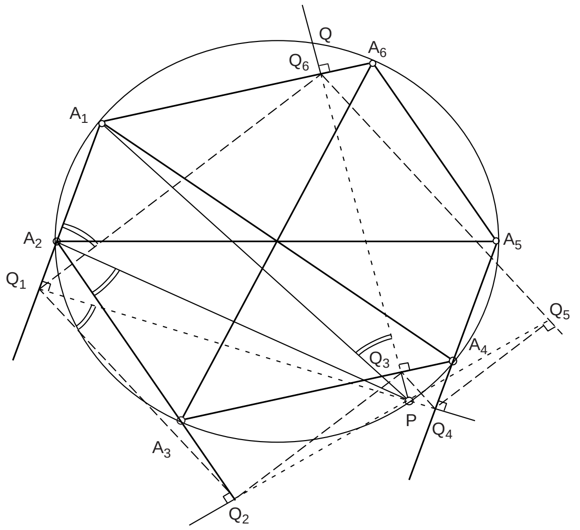


Figure 3.

Proof. Consider figure 3. The quadrilateral $A_1Q_1PQ_6$ is cyclic since it consists of two right triangles. Thus, we have

$$\angle A_1Q_1Q_6 = \angle A_1PQ_6\,.$$

The opposite sides of the hexagon are parallel, so the feet of the perpendiculars from P on the sides fall on three lines: PQ_3Q_6, Q_1PQ_4 and Q_2PQ_5. The line PQ_3Q_6 cuts equal parts off the chords A_3A_4 and A_1A_6, so the arcs $\overset{\frown}{A_3P}$ and $\overset{\frown}{A_1Q}$ are equal. Therefore,

$$\angle A_1PQ_6 = \angle PA_2A_3 = \angle PA_2Q_2\,.$$

Next we see that the quadrilateral $A_2Q_1Q_2P$ is cyclic because of the right angles at Q_1 and Q_2. In the circumcircle of this quadrilateral we have

$$\angle PA_2Q_2 = \angle PQ_1Q_2\,.$$

We have now established that

$$\angle A_1Q_1Q_6 = \angle PQ_1Q_2\,. \tag{4}$$

Now consider the angles surrounding Q_1. We see that

$$\angle A_1Q_1Q_2 = \angle A_1Q_1P + \angle PQ_1Q_2 = 90^\circ + \angle PQ_1Q_2$$

and (4) gives us:

$$\begin{aligned}\angle A_1Q_1Q_2 &= \angle Q_6Q_1Q_2 + \angle A_1Q_1Q_6\\ &= \angle Q_6Q_1Q_2 + \angle PQ_1Q_2\,.\end{aligned}$$

The two last equations yield

$$\angle Q_6Q_1Q_2 = 90^\circ.$$

The other angles in the hexagon $Q_1Q_2Q_3Q_4Q_5Q_6$ are treated in the same way.

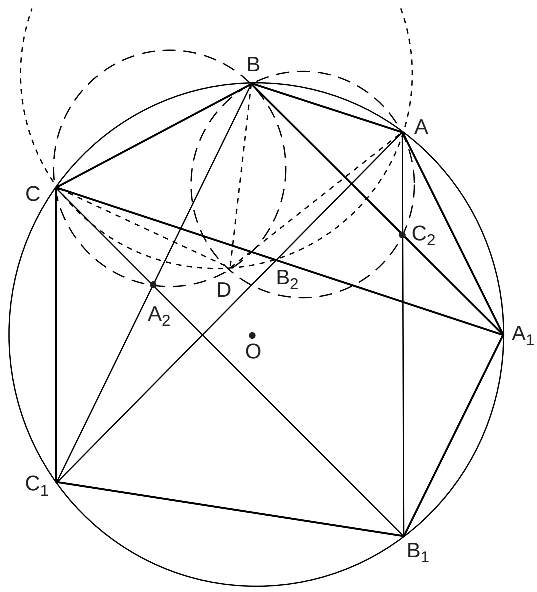


Figure 4.

Theorem 3. *In the cyclic hexagon* $ABCC_1B_1A_1$ *of figure 4 we denote by* A_2 *the intersection of* BC_1 *and* CB_1*, by* B_2 *the intersection of* CA_1 *and* AC_1 *and finally by* C_2 *the intersection of* AB_1 *and* BA_1.

a) *The circle through* B, C, A_2*, the circle through* A, B, C_2 *and the circle through* A, C, B_2 *intersect in one point* D.
b) *The points* A_2, B_2 *and* C_2 *are collinear.*

Proof. a) Let D be the intersection of the circles through B, C, A_2 and A, B, C_2. Then

$$\angle ADB = \angle AC_2B \quad \text{and} \quad \angle BDC = \angle BA_2C.$$

In the circumcircle of the hexagon $\angle AC_2B$ is an interior angle. The measure in radians of an interior angle is half of the sum of the measures of the arcs which it and its opposite angle subtend, i.e.

$$\angle AC_2B = \frac{1}{2r}(\overset{\frown}{AB} + \overset{\frown}{A_1B_1}),$$

where r is the radius of the circumcircle (see the problem in the problems section). Similarly,

$$\angle BA_2C = \frac{1}{2r}(\overset{\frown}{BC} + \overset{\frown}{B_1C_1}).$$

Thus, we get

$$\begin{aligned}\angle ADC = \angle ADB + \angle BDC &= \angle AC_2B + \angle BA_2C\\ &= \frac{1}{2r}(\overset{\frown}{AC} + \overset{\frown}{A_1C_1}).\end{aligned}$$

In the circumcircle of the hexagon $\angle AB_2C$ is an interior angle, so that

$$\angle AB_2C = \frac{1}{2r}(\overset{\frown}{AC} + \overset{\frown}{A_1C_1}).$$

But then $\angle ADC = \angle AB_2C$, which implies that D is on the circle through A, B_2 and C.

b) For the sake of simplicity we put $\angle BAB_1 = u$ and $\angle BCB_1 = v$. Since the quadrilateral BCB_1A is cyclic, $u + v = 180°$. Since BDC_2A is cyclic

$$\angle BDC_2 = 180° - u = v.$$

Similarly, BDA_2C is cyclic, so that

$$\angle BDA_2 = 180° - v = u.$$

But this gives us

$$\angle BDC_2 + \angle BDA_2 = u + v = 180°.$$

Therefore, A_2, D and B_2 are collinear. Similarly, we can prove that D, B_2 and C_2 are collinear.

The last part of the theorem is really Pascal's theorem in projective geometry. It is also true if the hexagon is inscribed in one of the other conic sections.

Theorem 4. *Let* ABCDEF *be a cyclic hexagon in which* AB = BC = CD = DE*, and where* BF *is a diameter in the circumcircle (see figure 5). If* AE *and* CF *intersect in* X*, then* AF $\perp$ DX.

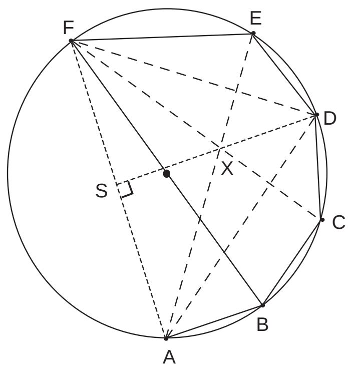


Figure 5.

Proof. The angle $\angle BCF$ subtends a diameter, so BC $\perp$ CF. As AB = BC = CD, we have BC$\|$AD. Thus, CF $\perp$ AD. But this implies that DF $\perp$ AE, because the lines CF and DF intersect under the same angle as do the lines AD and AE. This means that the point X is the orthocentre of $\triangle ADF$, so that AF $\perp$ DX.

Theorem 5. *Let* $P_1P_2P_3P_4P_5P_6$ *be a* regular *hexagon and* X *a point on the circumcircle. If* A, B *and* C *are the feet of the perpendiculars from* X *to the diagonals* P_1P_4, P_2P_5 *and* P_3P_6*, then the area of* $\triangle ABC$ *does not depend on the position of* X.

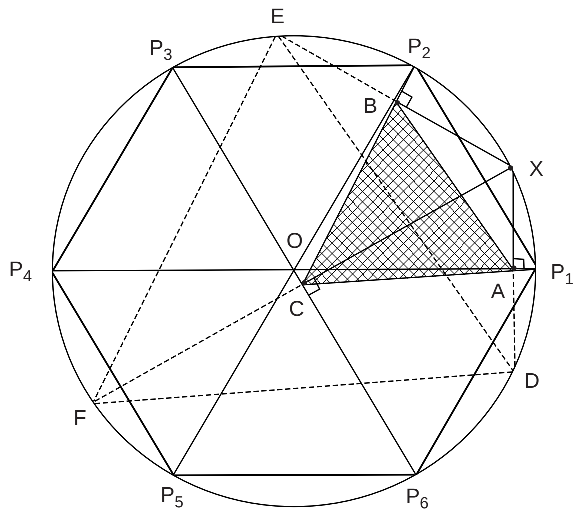


Figure 6.

Proof. The lines XA, XB and XC intersect the circumcircle in D, E and F (see figure 6). Since D, E and F are the reflections of X in the chords P_1P_4, P_2P_5 and P_3P_6, we have XA = AD, XB = BE and XC = CF, so that $\triangle ABC$ can be obtained from $\triangle DEF$ by a dilatation with a scaling factor of $\frac{1}{2}$ from the point X.

Since XB is perpendicular to OP_2 and XC is perpendicular to OP_3,

$$60° = \angle P_2OP_3 = \angle BXC = \angle EXF = \angle EDF.$$

Similarly,

$$60° = \angle P_1OP_6 = \angle CXA = \angle FXD = \angle FED.$$

Thus, $\triangle EFD$ is equilateral and inscribed in the circle. Therefore, its area is independent of the position of X and the same is true for $\triangle ABC$.

Theorem 6. *In the hexagon* ABCDEF *the lines connecting the intersections of the medians of the 'corner triangles'* ABC, BCD, CDE, DEF, EFA, FAB *form a hexagon whose opposite sides are equal and parallel.*

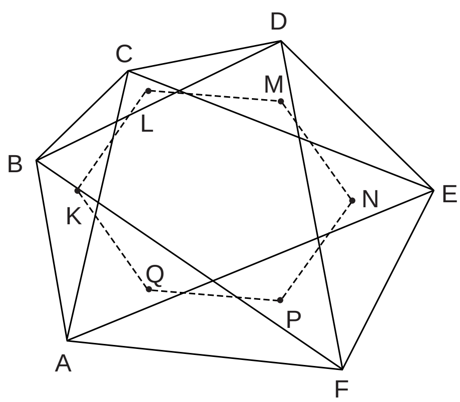


Figure 7.

Proof. It turns out to be practical to use vectors. Let O be the origin of position vectors in the plane. The intersections of the medians in the triangles are K, L, M, N, P, Q respectively, and if we denote the position vectors to the vertices of the hexagon by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, etc., we have

$$\overrightarrow{\mathrm{OK}} = \tfrac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) \quad \text{and} \quad \overrightarrow{\mathrm{OQ}} = \tfrac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{f}),$$

so that

$$\overrightarrow{\mathrm{KQ}} = \tfrac{1}{3}(\boldsymbol{f}-\boldsymbol{c}).$$

Similarly,

$$\overrightarrow{\mathrm{OM}} = \tfrac{1}{3}(\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}) \quad \text{and} \quad \overrightarrow{\mathrm{ON}} = \tfrac{1}{3}(\boldsymbol{d}+\boldsymbol{e}+\boldsymbol{f}),$$

so that

$$\overrightarrow{\mathrm{MN}} = \tfrac{1}{3}(\boldsymbol{f}-\boldsymbol{c}).$$

The other pairs of opposite sides are treated similarly.

The author is a teacher in a 'gymnasium' (or high school) in Copenhagen. He has published textbooks in mathematics and a great number of expository articles (or mathematical 'snapshots') in the magazine of the Danish Mathematics Teachers' Association.

Solution to Braintwister 12

(Magic magic square)

Numerically

	15	

$\therefore$ row sum $= 45$

Number of Letters

	7	

$\therefore$ row sum $= 21$

The lines including 15 must have two other numbers which add up to 30 and which use 14 letters altogether. The only pairs are

2, 28   5, 25   8, 22   (11, 19 or 12, 18).

(We cannot use both 11, 19 and 12, 18 because that would give two entries of six letters each.) So the other eight entries in the square include a *highest value of* 28.

5	28	12
22	15	8
18	2	25

VICTOR BRYANT

The Arithmetic mean–Geometric mean Inequality — a geometric proof at a glance

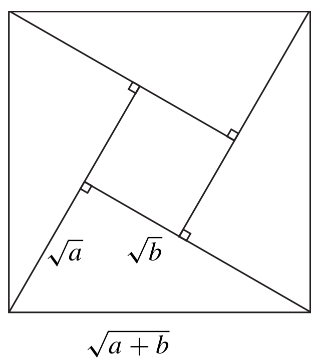
The area of the large square is greater than or equal to 4 times the area of the right-angled triangle, so

$$a+b \geq 4 \times \frac{1}{2}\sqrt{ab},$$

and so

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

SIMON MCVITTIE
Hills Road Sixth Form College, Cambridge



Mathematics in the Classroom

Curriculum 2000

September 2000 saw the introduction of the new curriculum for post-16 education. Instead of the traditional three or four A-level programme, students are now being offered a four or five AS-level programme in Year 12. They also have opportunities to have the three key skills of Application of Number (AON), Information Technology and Communication accredited in a way which will enhance their UCAS point collection. It is anticipated that these students will go on to carry three or four of their AS choices through to full A-levels in the second year of their sixth form experience. So a typical sixth form student now has a much fuller and broader programme of study than that enjoyed by their predecessors.

Application of Number key skills

Key skills are the skills that all A-level students practice all the time in both their studies and their leisure-time pursuits. They are deemed to be crucial for life-long learning and said to be valued by employers as well as higher education institutions. Each A-level subject, whatever the discipline, is thought to present opportunities for students to develop the appropriate skills. For Application of Number these can be summarised as:

- plan and interpret information;
- carry out calculations;
- interpret results and present findings.

Lots of publications are now beginning to appear which assist teachers in using classroom activities to generate the evidence that is required to demonstrate the acquisition of these skills. Reference 1 provides several interesting investigational ideas that are appropriate for A-level Maths students, delivering the curriculum at the same time as achieving this key skills evidence. One of these ideas works well as an individual activity aimed at introducing the need for numerical as well as analytical methods for the solution of some equations. It is reproduced here with a recommendation to give it a try.

The ladders problem[2]

Two ladders of length 2m and 3m are being used to decorate a corridor (see figure 1). They cross at 1m vertically above the floor as shown in the figure. How wide is the corridor? Work through the following steps to find a solution.

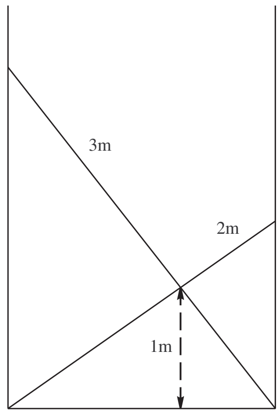


Figure 1. The ladders problem.

- Use geometry and algebra to find an equation with one variable, the width. This equation cannot be solved algebraically.
- Solve the equation graphically and numerically, giving your answers to an appropriate degree of accuracy (this is a requirement of AON at Level 3).
- Which method do you prefer and why? (Justifying methods is also a requirement of AON.)
- If you find more than one solution, justify your choice of the 'correct' solution and explain where any others have come from.
- Check your solution from an accurate scale drawing.
- Write up your findings in a way that you consider to be most appropriate for presenting your results, showing all working, diagrams and checking procedures. (Further requirements of AON.)
- You may wish to prepare and give a presentation to the class on your work.

Carol Nixon

Reference

1. Key Skills Support Programme, *Key Skills in A-level Mathematics* (National Extension College, Cambridge, 2000).

[2] Readers may already have considered this problem, which appeared in Volume 33, Number 1 as Problem 33.2 — Ed.

Computer Column

Are You Being Served?

In the last issue we looked at how to write a JavaScript calculator for a web page. As we saw, JavaScript is most often used as a client-side technology: it is the browser rather than the server that does the processing. This is fine for simple applications like a calculator, but more interesting tasks generally require server-side solutions.

There are several ways of using a server to generate dynamic web pages; if you have access to a Microsoft Windows NT server, then you may wish to consider using ASP — Active Server Pages. (If you do not have access to an NT server, and you wish to experiment with ASP scripts, then note that ASP runs on Personal Web Server for Windows 95 and later. Furthermore, although ASP is a Microsoft technology, software exists that enables ASP extensions to run on a variety of other web servers and operating systems.)

You create an ASP file in exactly the same way as you create an HTML file. However, as well as containing HTML tags, the ASP file contains script within `<%` and `%>` delimiters. Furthermore, the file has a `.asp` instead of a `.htm` extension. When a browser requests an ASP file, the server recognises the `.asp` extension and sends it to the ASP component on the server. The ASP component then reads the file and interprets any scripts contained therein; the result is then returned to the browser as plain HTML. To the user, the resulting page looks like any other HTML page. Using ASP it is thus easy for you to create web pages 'on the fly', and to customise pages for users depending upon their actions: so different users see different pages. ASP also gives you the capability of easily connecting a web page to a database (which is one of the reasons that on-line shops are often powered by ASP technology).

ASP supports a variety of scripting languages, but perhaps the most widely used language for this purpose is VBscript. If you are familiar with Visual Basic, then you should be able to create an ASP with little difficulty. A very simple example of an Active Server Page is shown in appendix 1.

What happens in this example is that a user requests the file `hello.asp` from the server. Before the file is sent back to the user, however, the server processes all the ASP code between the `<%` and `%>` delimiters. (The `<%=` and `%>` delimiters — note the equals sign — simply instruct the server to evaluate an expression and return the result to the browser.) The server returns an HTML page for display in the user's browser; but the HTML it returns depends upon the time of day. If it is before noon at the server's location, it might return the code given in appendix 2; if it is after noon, it might return the code given in appendix 3.

The ASP example of appendix 1 is extremely simple, and you might well argue that client-side JavaScript would do this particular job much better (for instance, it is preferable to have a greeting based on the time at the client rather than the time at the server!) Similarly, although the JavaScript calculator that we developed in the last issue could quite easily be rewritten using VBscript in an ASP, there would be little advantage in doing so. But remember that ASP allows you to put extremely sophisticated VBscript between the `<%` and `%>` delimiters. It opens up a host of possibilities for making your web pages dynamic and interactive, possibilities that are not available to client-side technologies.

You could use ASP technology to develop quite ambitious web-based mathematical applications. For example, several recent articles in *Mathematical Spectrum* have described how to use Excel spreadsheets in the mathematics classroom. If you want to display your spreadsheet in a web page — well, ASP lets you do it quickly and easily. The sorts of web page you can develop with ASP are limited mainly by your imagination and programming skill.

Appendix 1. A very simple Active Server Page

```
<html><head>
<TITLE>hello.asp</TITLE>
</head>
<body bgcolor="#CCFFFF">
Today is <%=now%> and we are working
hard in the APT office<br>

<%if hour(now())>11 THEN%>
Good Day from us all here in Sheffield!
<%ELSE%>
Good Morning from us all here in
Sheffield!
<%END IF%>
</body></html>
```

Appendix 2. HTML returned from the ASP (before noon)

```
<html><head>
<TITLE>hello.asp</TITLE>
</head>
<body bgcolor="#CCFFFF">
Today is Wed 10:45am and we are working
hard in the APT office<br>
Good Morning from us all here in
Sheffield!
</body></html>
```

Appendix 3. HTML returned from the ASP (after noon)

```
<html><head>
<TITLE>hello.asp</TITLE>
</head>
<body bgcolor="#CCFFFF">
Today is Wed 04:30pm and we are working
hard in the APT office<br>
Good Day from us all here in Sheffield!
</body></html>
```

Stephen Webb

Letters to the Editor

Dear Editor,

A curious prime sequence (Volume 32, Number 3, p. 67)

The given sequence can be extended (in two ways) by three terms at the beginning:

5				
$5+2=7$				
$7+4=11$				given:
$11+2=13$				5
$13+4=17$				$5+8=13$
$17+2=19$	or	$17+6=23$		$13+10=23$
$19+4=23$		$23+8=31$	same as	$23+8=31$
$23+6=29$		$31+6=37$		
$29+8=37$		$37+4=41$		
$37+6=43$		$41+2=43$		
$43+4=47$	same as	$43+4=47$		

and it can be extended by four terms near the end:

given:		
$1163+68=1231$		
$1231+70=1301$	or	$1231+66=1297$
$1301+72=1373$		$1297+64=1361$
$1373+74=1447$		$1361+62=1423$
$1447+76=1523$		$1423+64=1487$
$1523+78=1601$		$1487+62=1549$
		$1549+60=1609$
		$1609+58=1667$
		$1667+56=1723$
		$1723+54=1777.$

This is the best I have managed so far.

Yours sincerely,

C. R. Jeggo
(27 York Road,
Woking,
Surrey GU22 7XH.)

Dear Editor,

A curious prime sequence

Bablu Chandra Dey's curious prime sequence is, from 41 onwards, the well-known 'prime generator' n^2+n+41. It can be extended by starting with 5, $5+2=7$, $7+4=11$, $11+2=13$, $13+4=17$, $17+6=23$ and continuing as given by Mr Dey.

Yours sincerely,

Alastair Summers
(Stamford School,
Lincolnshire.)

Dear Editor,

A curious prime sequence

The seventh term of the sequence is $43 = 41+1+1$, and from then on the sequence just lists all the primes of the famous Euler polynomial x^2+x+41, for $x < 40$. This idea is 228 years old.

As far as the editor's question is concerned, on the one hand Baker and Stark have proved that x^2+x+A cannot be a prime for $x = 0, 1, \ldots, A-1$ for any $A > 41$, but on the other hand there is the famous conjecture of Erdös which states that there are arbitrarily long sequences of primes in arithmetic progression. It is reasonable to believe that a modification of this to the sequence of gaps $\{g, g+2, g, g+2, \ldots\}$, instead of the sequence of gaps $\{g, g, g, g, \ldots\}$, could be made arbitrarily long as well.

Yours sincerely,

Filip Saidak
(Dept of Mathematics and Statistics,
Queen's University,
Kingston,
Ontario,
Canada.)

Dear Editor,

The Syracuse Algorithm

The Syracuse Algorithm is a well-known pastime with numbers. It is also known as 'Hailstones', or the '$3N+1$' problem.

You start with any positive integer greater than 1, and then cycle through the following steps until you reach the number 1:

- if the number is even, divide it by 2;
- otherwise, multiply the number by 3 and add 1.

So if we start with 3, for example, we reach 1 after 7 steps:

$$3, 10, 5, 16, 8, 4, 2, 1,$$

and if we start with 17, we reach 1 after 12 steps:

$$17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

We call the number of steps taken to reach the number 1 the *step count*, so that 7 is the step count of 3, and 12 is the step count of 17.

Several numbers can share the same step count. So, for example, the numbers that have a step count of 12 are:

$$17, 96, 104, 106, 113, 640, 672, 680, 682, 4096.$$

Note that the largest of these numbers is 4096, which equals 2^{12}. It is always true that, for any step count c, the largest number with that step count will be 2^c.

The set of numbers that share the same step count fall into clusters. Taking our example of the numbers with a step count of 12, we have:

$$\{17\}, \qquad \{96, 104, 106, 113\},$$
$$\{640, 672, 680, 682\}, \qquad \{4096\}.$$

It will be noted that the values in one cluster are roughly 6 times the values in the previous cluster.

For any number m, with a step count of c, and with $m < 2^c$, we can calculate a larger number which shares the same step count c, by using the following procedure:

(i) express m as (an odd number) $\times 2^N$;
(ii) calculate $(6m) + 2^{N+1}$.

It is an easy exercise to show that m and the larger number do indeed have the same step count.

If this larger number is less than 2^c, we can use this procedure again to calculate a still larger number. Thus, starting with the number 17 we can calculate successively:

$$17, 104, 640, 4096.$$

Since the number we calculate can never be larger than 2^c, our number sequence must eventually end exactly at 2^c.

Now, it seems to be true that the Syracuse Algorithm can be applied to any number, and the number will eventually reduce to 1 in a finite number of steps. If this is the case, then the above procedure can be applied to any number which is greater than 1, and which is not itself a power of 2, and eventually a value will be reached which is equal to some power of 2. That power will be the step count of the original number.

Thus, we can be sure, for example, that the above procedure applied to the number 27 will eventually end up as the very large number 2^{111}, because the step count of 27 is 111.

Further reading

D. G. Wells, *The Penguin Dictionary of Curious and Interesting Numbers* (Penguin, London, 1997). (See the entry '27' on p. 26.)

M. E. Lines, *Think of a Number* (Adam Hilger, Bristol, 1990). (See Chapter 3.)

Yours sincerely,
B. J. HULBERT
(Fourways,
Micklands Road,
Caversham,
Reading RG4 6LT.)

Mary Cannell

Mary Cannell, who died in April 2000 at the age of 86, has been awarded a posthumous honorary degree by Nottingham Trent University for her outstanding contribution to lifelong learning.

It wasn't until her retirement that she became a remarkable historian of mathematics and secured a lasting reputation for herself. She focused her energies into researching, articulating and promoting the life and career of George Green, the nineteenth century scientist and mathematician.

While many mathematicians have learned of Green's theorem and Green's functions, it wasn't until the 1970s, when Cannell and colleagues worked towards restoring his memory, that knowledge of George Green himself became widespread.

Miss Cannell played a major rôle in the restoration of Green's Mill — a converted windmill in Sneinton, Nottingham, which now hosts science exhibitions and public lectures. She also instigated the installation of a memorial window at Gonville and Caius College, Cambridge, and a memorial plaque in Westminster in honour of Green

Her acclaimed biography *George Green: Mathematician and Physicist 1793–1841*[3] was published in 1993.

Brought up and educated in Liverpool, Cannell studied French with history at Liverpool University. Graduating with a postgraduate diploma in education, she began her career teaching French in English schools. She also taught English at schools in France and later taught at the British Army Foundation College during the war. Her experience of lecturing to British troops led to her spending the rest of her career in higher education, specifically in teacher training.

[3] Readers will find a review of this book in Volume 26, Number 2, 1994/94 and articles on Green in Volume 20, Number 2, 1987/88 and Volume 25, Number 3, 1992/93 — Ed.

Problems and Solutions

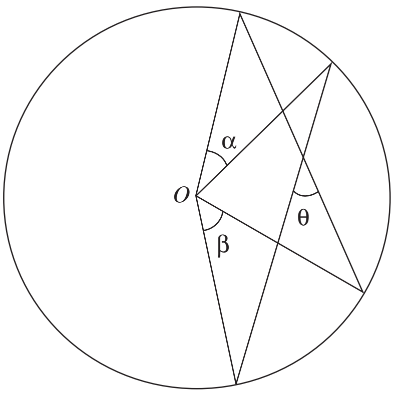
Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

33.5 Prove that, for all integers $n > 1$, $\ln(n-1)! < n\ln n - n + 1$.

(See the article by Chris Caldwell and G. Honaker in this issue)

33.6



Express θ in terms of α and β. (O is the centre of the given circle.)

(See the article by Jens Cartensen in this issue)

33.7 The 'complement' of a positive integer a is defined to be $10^n - a$, where n is the number of digits in the decimal representation of a. For example, the complement of 975 is 25. Determine the number of n-digit positive integers which are divisible by their complements.

(Submitted by Ahmet Özban, Kirikkale University, Turkey)

33.8 Let m, n be positive integers with gcd $(m, (n-1)!) = 1$, and let $a_1, \dots, a_n$ be integers whose sum is not divisible by m. Prove that there is a permutation $b_1, \dots, b_n$ of $a_1, \dots, a_n$ such that none of $b_1, b_1 + b_2, b_1 + b_2 + b_3, \dots, b_1 + b_2 + \cdots + b_n$ is divisible by m.

(Submitted by H. Shah Ali, Tehran)

Solutions to Problems in Volume 32 Number 3

32.9 Deduce formula (2) from formula (1) on page 50 (Volume 32, No. 3), in the article 'François Viète and the quest for π' by G. C. Bush.

Solution independently by Bor-Yann Chen (University of California, Irvine) and Zhan Su (Nottingham High School)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \dots \qquad (1)$$

so

$$\frac{\pi}{2} = \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \dots,$$

$$\frac{\pi^2}{4} = \frac{4}{2} \cdot \frac{2^2}{2+\sqrt{2}} \cdot \frac{2^2}{2+\sqrt{2+\sqrt{2}}} \cdot \dots,$$

and so

$$\frac{\pi^2}{8} = \frac{2^2}{2+\sqrt{2}} \cdot \frac{2^2}{2+\sqrt{2+\sqrt{2}}} \cdot \dots. \qquad (3)$$

We now multiply (1) and (3) term-by-term to give

$$\frac{\pi}{4} = \frac{2\sqrt{2}}{2+\sqrt{2}} \cdot \frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}} \cdot \frac{2\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \dots,$$

which is formula (2).

32.10 The three points U, V, W have distinct x-coordinates u, v, w respectively and $k = v - u$. Construct the straight line through U parallel to VW and denote by P the point on this line with x-coordinate $w + k$. Construct the straight line through W parallel to PV and denote by Q the point on this line with x-coordinate $v + k$. Show that the straight line through V parallel to UQ is the tangent at V to the unique curve with equation of the form $y = ax^2 + bx + c$ which passes through U, V, W.

Solution by Chun Chung Tang (Impington Village College, Cambridge)

The line through U parallel to $\overrightarrow{\mathrm{VW}}$ is

$$\begin{pmatrix} u \\ au^2 + bu + c \end{pmatrix} + \lambda \begin{pmatrix} w - v \\ (aw^2 + bw + c) - (av^2 + bv + c) \end{pmatrix},$$

and P is the point on this line with x-coordinate $w + k$. Thus, for P,

$$w + k = u + \lambda(w - v) \qquad (k = v - u)$$

and

$$\lambda = \frac{w + k - u}{w - v}.$$

Hence,

$$\overrightarrow{\mathrm{OP}} = \begin{pmatrix} w + k \\ au^2 + bu + c + (w + k - u)(aw + av + b) \end{pmatrix},$$

and so

$$\overrightarrow{\mathrm{PV}} = \overrightarrow{\mathrm{OV}} - \overrightarrow{\mathrm{OP}}$$
$$= \begin{pmatrix} v-w-k \\ (av^2+bv+c)-(au^2+bu+c)-(w+k-u)(aw+av+b) \end{pmatrix}$$
$$= \begin{pmatrix} v-w-k \\ a(v^2-u^2-(w+v)(w+k-u))+b(v-u-w-k+u) \end{pmatrix}$$
$$= \begin{pmatrix} u-w \\ a\big(k(v+u)-k(w+v)-(w+v)(w-u)\big)+b(v-w-k) \end{pmatrix}$$
$$= \begin{pmatrix} u-w \\ a(u-w)(k+w+v)+b(u-w) \end{pmatrix}.$$

The line through W parallel to $\overrightarrow{\mathrm{PV}}$ is

$$\begin{pmatrix} w \\ aw^2+bw+c \end{pmatrix} + \mu \begin{pmatrix} u-w \\ a(u-w)(k+w+v)+b(u-w) \end{pmatrix}$$

and Q is the point on this line with x-coordinate $v+k$ so, for Q,

$$v+k = w+\mu(u-w)$$

and

$$\mu = \frac{v+k-w}{u-w}.$$

Hence

$$\overrightarrow{\mathrm{OQ}} = \begin{pmatrix} v+k \\ (aw^2+bw+c)+a(k+w+v)(v+k-w)+b(v+k-w) \end{pmatrix},$$

and so

$$\overrightarrow{\mathrm{UQ}} = \begin{pmatrix} v+k-u \\ a(w^2-u^2)+b(w-u)+a(v+k)^2-aw^2+b(v+k-w) \end{pmatrix}$$
$$= \begin{pmatrix} v+k-u \\ a(v+k+u)(v+k-u)+b(v+k-u) \end{pmatrix}$$
$$= \begin{pmatrix} 2(v-u) \\ 2(v-u)(2av+b) \end{pmatrix},$$

which has gradient $2av+b$. This is also the gradient of the tangent to the curve $y = ax^2+bx+c$ at V, and this proves what is asked.

32.11 Two people arrange to meet at a certain place in town between 3 pm and 4 pm for tea. They decide that each will wait for up to 10 minutes between these times and then leave if the other has not turned up. If all times between 3 pm and 4 pm are equally likely for the arrival of each person, what is the probability that they will meet?

Solution by Chun Chung Tang

If A arrives between 3.10 pm and 3.50 pm, then the probability that A meets B is $\frac{2}{6}$, so the probability that they meet and that A arrives between these times is $\frac{4}{6} \times \frac{2}{6} = \frac{2}{9}$. Now consider t where $0 < t < 10$. The probability that A arrives during a small interval of time δt just after t minutes past 3 is $\frac{1}{60}\delta t$. The probability that A meets B, given A's arrival time in this interval, is $\frac{1}{6}+\frac{1}{60}t$, so the overall probability is $\left(\frac{1}{6}+\frac{1}{60}t\right)\frac{1}{60}\delta t$. Hence, the probability that they will meet and that A arrives in the first 10 minutes is

$$\int_0^{10} \frac{1}{60}\left(\frac{1}{6}+\frac{t}{60}\right)\mathrm{d}t = \frac{1}{360}\left[t+\frac{t^2}{20}\right]_0^{10} = \frac{15}{360} = \frac{1}{24}.$$

By symmetry, the probability that A arrives after 3.50 pm and meets B is also $\frac{1}{24}$, so the probability that A and B meet is $\frac{1}{24}+\frac{2}{9}+\frac{1}{24} = \frac{11}{36}$.

32.12 What are the last two digits of the following numbers, where m, n are positive integers?

$$(10n-5)^{2m}, \quad (20n-5)^{2m+1}, \quad (20n-15)^m \quad \text{with } m \geq 2.$$

Solution by Chun Chung Tang.

$$(10n-5)^{2m} = \big(100(n^2-n)+25\big)^m.$$

By use of the Binomial Theorem, we see that the last two digits are 25,

$$\begin{aligned}(20n-5)^{2m+1} &= \big((20n-5)^2\big)^m(20n-5) \\ &= \big(100(4n^2-2n)+25\big)^m(20n-5) \\ &= (100x+25)(20n-5) \text{ for some } x \in \mathbb{N} \\ &= 100(20nx+5n-5x-2)+75,\end{aligned}$$

so the last two digits are 75. For $k \in \mathbb{N}$,

$$(20n-15)^{2k} = \big(100(4n^2-6n+2)+25\big)^k,$$

so the last two digits are 25.

$$\begin{aligned}(20n-15)^{2k+1} &= (100y+25)(20n-15) \text{ for some } y \in \mathbb{N} \\ &= 2000ny+100(5n-15y-4)+25\end{aligned}$$

and the last two digits are again 25. Hence, the last two digits of $(20n-15)^m$ when $m \geq 2$ are 25.

Also solved by Zhan Su.

***Mathematical Spectrum* Awards for Volume 32**

Prizes have been awarded to the following student readers for contributions in Volume 32:

Zhang Yun
for his article 'An introduction to geometric inequalities' (pages 35–36);

Jeremy Young
for his contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems and other items.

Reviews

Cryptorunes. By CLIFFORD A. PICKOVER. Pomegranate Communications, Rohnert Park, CA, 2000. Pp. 99. Paperback $13.95 (ISBN 0-7649-1251-8).

Wvxlwv nv

This very simple code summarises the whole book, which consists largely of quotations and aphorisms written in ciphers, and the reader is invited to solve them. No previous knowledge of cryptography is required; one only needs some curiosity to be attracted by the book.

Code deciphering is already an attraction; the enjoyment is further enhanced by such a diverse collection of quotations and aphorisms — Jesus, Einstein, Pascal, Goethe, St Augustine and Camus. Answers to all but the last seventeen puzzles are given at the back of the book, Clifford Pickover offers a prize for the first five readers who figure out the meanings of all the seventeen puzzles!

The book is certainly entertaining. Pickover provides some background information on cryptography and the basic techniques for deciphering codes with references to history and well-known novels; for example he mentions the use of Navajo language for secure communications in the Second World War and 'The Adventure of the Dancing Men' in Sherlock Holmes. Those who love code-cracking will definitely enjoy the challenges presented in the book. For others who are less interested, me for instance, it is still stimulating to read about the subject. The language is friendly and everything is well explained. I found the section on exchanging messages with extraterrestrials particularly appealing. According to the author, Soviet researchers in the 1970s suggested sending the message

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

because it is 'mind-catching', with sums on both sides equal to 365, while he himself prefers

$$1 + \mathrm{e}^{\mathrm{i}\pi} = 0.$$

Euler would certainly agree with him.

Student, Impington Village College — CHUN CHUNG TANG

Geometry from Africa: Mathematical and Educational Explorations. By PAULUS GERDES. MAA, Washington, 1999. Pp. xv+210. Paperback $39.95 (ISBN 0-88385-715-4).

Starting with a generous introduction to African ornaments and artefacts, the author explores the geometrical values inherent in these African designs throughout the book. Proofs for Pythagoras' theorem spring from Mozambican headrest decoration; construction of magic squares from mat weaving patterns; determination of volumes of regular shapes from woven baskets and many more. He devotes a whole chapter (one fourth of the book) to a declining sand drawing tradition known as *sona* in the Chokwe language. 'During nearly half a century among the Ngangela the American missionary Pearson found only four men who had real knowledge of *sona* sand drawing.' There are plenty of suggestions for its educational uses. In fact, one of the aims of the book seems to be to encourage the reader to explore mathematical ideas through the study of African designs. It is recommended to mathematics educators who may find the problems helpful in their teaching.

This book is particularly appealing to readers who have an interest in both art and mathematics, although most parts of the book do not require specialist knowledge in either field. The extensive amount of research is reflected by the wide variety of African motifs shown and the author's detailed discussions on their geometrical values. Progress may be slow for the general reader considering the amount of mathematics involved. Constant concentration is often a necessity. Fortunately, the chapters and the sections can be read independently.

It is not surprising that such a theme is supported by a liberal collection of diagrams and illustrations. I feel that the layout would have been improved if the diagrams had been on the same page as or a page facing the text.

Student, Impington Village College — CHUN CHUNG TANG

The Context Problem Book VI: American High School Mathematics Examinations 1989–1994. By LEO J. SCHNEIDER. MAA, Washington, 2000. Pp. 232. Paperback $21.95 (ISBN 0-88385-642-5).

This book contains five American High School Mathematics Examination papers (with answers), as well as a section on how the problems are constructed and developed from first thoughts to a final question, and a section on how to solve mathematical problems for competitions.

Each of the examination papers contains thirty questions with multiple answers. The questions range from the easy to the very hard. The first few questions are easy and they get harder as you progress through the paper, with the very hardest questions right at the end. The questions test a range of topics covering the whole mathematics syllabus and beyond. Each paper is accessible to candidates with a wide range of ability, from the good to the very best. The answers provide a difficult choice if only a vague idea of the answer is known, and they cannot be guessed.

The solutions to the problems provide a clear explanation of how the question could have been tackled. Some of the solutions to harder problems provide a variety of ways that the problem can be approached. The solutions are easy to follow and, even if you had no thought on how to solve the problem, the solution can be understood.

The section on how problems are constructed has no real relevance to the preceding examination papers, but is an interesting section on how the construction of problems is developed. It starts with a simple idea and shows how it is modified and adapted until it becomes the final problem.

The final section on how to solve competition mathematics is more interesting. It contains methods and approaches for solving the most difficult of problems. It has a range of methods covering a vast number of mathematical topics. Some of the methods are complex and highly advanced, but the explanation of how to use them is very clear and easy to follow. Having read this section, some of the harder problems become easier to do, and some that seemed impossible became possible.

Altogether, this book is useful for anyone who is thinking of entering mathematics competitions, and an interesting and stimulating challenge for everyone else.

Stamford School SIMON MUMFORD

Random Walks of George Pólya. By GERALD L. ALEXANDERSON. MAA, Washington, 2000. Pp. 320. Paperback \$29.95 (ISBN 0-88385-528-3).

Generations of students will be familiar with the name of George Pólya from one slim volume entitled *How to Solve It*. Not for Pólya the vast abstract structures of modern mathematics. For him, mathematics was all about solving problems. And he did, hundreds of them. His other main fascination was to try to pass this art on to others; hence the book upon which so many of us have been brought up.

George Pólya was born in 1887 and his life spanned almost a century. He was one of the giants of 20th century mathematics. This biography fascinatingly tracks his life and work. It is full of photographs of mathematicians, appropriately so, for Pólya was an enthusiastic photographer whose picture album was published posthumously in 1987.

As a sample of Pólya's problems, here is his orchard problem: 'How thick must the trunks of the trees in a regularly spaced circular forest grow if they are to block completely the view from the centre?' (p. 59).

As an illustration of the elegance of his mathematics, consider his proof of the arithmetic mean–geometric mean inequality (p. 75). For positive real numbers $a_1, \ldots, a_n$, write

$$A = \frac{a_1 + \cdots + a_n}{n}, \qquad G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Now, $\mathrm{e}^x \geq 1 + x$ for $x \geq 0$, so $\mathrm{e}^{(a_i/A)-1} \geq a_i/A$. Hence,

$$\mathrm{e}^{(a_1/A)-1}\mathrm{e}^{(a_2/A)-1} \cdots \mathrm{e}^{(a_n/A)-1} \geq \frac{a_1 a_2 \cdots a_n}{A^n},$$

so

$$\mathrm{e}^{((a_1+\cdots+a_n)/A)-n} \geq \frac{a_1 a_2 \cdots a_n}{A^n},$$

and so

$$\mathrm{e}^{n-n} \geq \frac{a_1 a_2 \cdots a_n}{A^n},$$

thus

$$A^n \geq G^n,$$

whence

$$A \geq G.$$

Isn't that beautiful!

Anyone who aspires to teach mathematics could learn with profit from Pólya. One of his maxims was to be as concrete as possible. To quote: 'Pólya was not only a mathematician, he was also a teacher. For him it was not sufficient to solve a problem — he had to study it until he saw the solution clearly so that he could put it in a form easily accessible' (p. 94). The final quote of this biography sums up its subject: 'Great teachers do not simply teach us to do; they teach us to be .... A master teacher does not follow the syllabus as much as invent it; he does not cover the ground, he makes it bloom' (p. 163). If only ... !

The University of Sheffield DAVID SHARPE

Mathematical Mysteries: The Beauty and Magic of Numbers. By CALVIN C. CLAWSON. Perseus, Oxford, 2000. Pp. 313. Paperback £11.95 (ISBN 0-7382-0259-2).

Since the age of 7 (as long as I can remember) I have suffered from a rare condition known as 'mathematical enthusiasm'. Throughout my life I have tried to suppress the symptoms (as it seemed the thing to do) with no success.

Calvin Clawson also suffers from this condition, but rather than lock himself away in a mathematical sanatorium (otherwise known as a university), he seems to want to spread the condition to other people. The biggest problem with this is that his affliction has taken a highly contagious (and rare) form! If you do not want to catch the condition (or like me are trying to suppress the symptoms), I suggest you stay well away from *Mathematical Mysteries*.

All the factors known to aggravate the condition are there: sequences, prime numbers, the 'golden mean' and Srinivasa Ramanujan (who achieved notoriety in the film 'Good Will Hunting'). Other less well known (but no less interesting) factors are also there: Kurt Gödel (and his world of strange logic), Georg Riemann (answers to his hypothesis on a postcard), and many others.

If you haven't read an interesting maths book before, then here is your chance. But don't read the last page first!

Rickmansworth DANIEL BROWN

Other books received

Fundamentals of University Mathematics. By COLIN MCGREGOR, JOHN NIMMO AND WILSON STROTHERS. Horwood, Chichester, 2000. Pp. 558. Paperback £25.00 (ISBN 1-898563-10-1).

The second edition of a well-used textbook whose title well describes its content.

The Beginnings and Evolution of Algebra. By ISABELLA BASHMAKOVA AND GALINA SMIRNOVA. MAA, Washington, 2000. Pp. 179. Paperback \$24.95 (ISBN 0-88385-329-9).

Five More Golden Rules. By JOHN L. CASTI. John Wiley, Chichester, 2000. Pp. 268. Hardback £18.50 (ISBN 0-471-32233-4).

The Star and Cross Polyhedra. By PATRICK TAYLOR. Nattygrafix, Ipswich, 2000. Pp. 80. Paperback £6.00 (ISBN 0-9516701-5-8).

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