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CONTENTS

A SYMMETRIC QUOTIENT THEOREM

HERBERT E. SALZER

Abstract. For any nth degree polynomial $P_n(x)$, the symmetry of the polynomial part of the quotient

$$\frac{P_n(x)}{\prod_{i=1}^{m}(x-x_i)}, \quad m<n, \tag{1}$$

in the $m+1$ variables $x, x_1, \ldots, x_m$ is proved in three different ways: directly, from the remainder theorem, and from the properties of divided differences.

1. *The main theorem.* The theorem in algebra which follows is so very simple and elementary, as well as practically obvious, that I am surprised not to have found, as yet, its explicit statement in some book or article. I still believe that it must be known to many persons.

THEOREM. *The polynomial part of the quotient* (1) *for*

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \tag{2}$$

where the division is <u>*algebraic*</u> *in all the variables x and x_i (i.e., keeping x and the x_i as letters instead of numbers), is an* $(m+1)$-ary $(n-m)$-ic *that is symmetric in the $m+1$ variables $x, x_1, \ldots, x_m$.*

When that $(n-m)$-ic is arranged in powers of either x or any x_i, the coefficient of x^r or x_i^r is the same $(n-r-m)$-ic in the remaining m variables; the coefficient of either $x^r x_i^s$ or $x^s x_i^r$, and either $x_i^r x_j^s$ or $x_i^s x_j^r$, is the same $(n-r-s-m)$-ic in the remaining $m-1$ variables; with a similar result for the six permutations of r,s,t in $x^r x_i^s x_j^t$, etc. That $(n-m)$-ic does not involve any of $a_0, a_1, \ldots, a_{m-1}$ and is independent of the algebraic divisibility and remainder when $x_1, \ldots, x_m$ are assigned particular numerical values.

The proof we give now is slightly longer, but somewhat more revealing, than the alternative proofs we give later. The theorem is proved first for $m=1$. In the algebraic division of $P_n(x)$ by $x-x_1$, the coefficient of x^m, for $m=n-1, n-2, \ldots, 1, 0$, in the quotient is obtained by adding a_{m+1} to $x_1 \times$ coefficient of x^{m+1} in the quotient. This is the familiar "nesting" process for calculating $P_n(x_1)$, except that here the last step is omitted. The coefficients of $x^r x_1^s$, for $r+s=0,1,\ldots,n-1$, in the quotient may be exhibited in the following tabular form:

$$\begin{array}{c|cccccc}
 & x^0 & x^1 & x^2 & \cdots & x^{n-2} & x^{n-1} \\
\hline
x_1^0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
x_1^1 & a_2 & a_3 & a_4 & \cdots & a_n & \\
x_1^2 & a_3 & a_4 & a_5 & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
x_1^{n-2} & a_{n-1} & a_n & & & & \\
x_1^{n-1} & a_n & & & & &
\end{array}$$

From the symmetry, the coefficient of $x^r x_1^s$, which is a_{r+s+1}, is the same as the coefficient of $x^s x_1^r$.

For $m > 1$, the complete symmetry is an immediate consequence of the obvious symmetry with respect to any transposition $(x_i x_j)$ and the symmetry with respect to any transposition (xx_i) which is apparent from the theorem just proved for $m = 1$ when the algebraic division is done by $\Pi_{j=1,j\neq i}^{m}(x-x_j)$ followed by $x - x_i$. □

To illustrate this theorem, consider the polynomial parts of the quotients of

$$x^6 - 2x^5 + 7x^4 + 5x^3 + 3x^2 - 4x + 2$$

by $x - x_1$, $(x-x_1)(x-x_2)$, $(x-x_1)(x-x_2)(x-x_3)$, and $(x-x_1)(x-x_2)(x-x_3)(x-x_4)$ which, when arranged in powers of x whose coefficients show, for $m > 1$, the symmetry with respect to any $(x_i x_j)$, are respectively

$$\begin{aligned}&x^5 + (x_1-2)x^4 + (x_1^2-2x_1+7)x^3 + (x_1^3-2x_1^2+7x_1+5)x^2\\ &\quad + (x_1^4-2x_1^3+7x_1^2+5x_1+3)x + (x_1^5-2x_1^4+7x_1^3+5x_1^2+3x_1-4),\end{aligned}$$

$$\begin{aligned}&x^4 + (x_1+x_2-2)x^3 + (x_1^2+x_2^2+x_1x_2-2x_1-2x_2+7)x^2\\ &\quad + (x_1^3+x_2^3+x_1^2x_2+x_1x_2^2-2x_1^2-2x_2^2-2x_1x_2+7x_1+7x_2+5)x\\ &\quad + (x_1^4+x_2^4+x_1^2x_2^2+x_1^3x_2+x_1x_2^3-2x_1^3-2x_2^3-2x_1^2x_2-2x_1x_2^2\\ &\qquad +7x_1^2+7x_2^2+7x_1x_2+5x_1+5x_2+3),\end{aligned}$$

$$\begin{aligned}&x^3 + (x_1+x_2+x_3-2)x^2 + (x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3-2x_1-2x_2-2x_3+7)x\\ &\quad + (x_1^3+x_2^3+x_3^3+x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2+x_1x_2x_3\\ &\qquad -2x_1^2-2x_2^2-2x_3^2-2x_1x_2-2x_1x_3-2x_2x_3+7x_1+7x_2+7x_3+5),\end{aligned}$$

and

$$\begin{aligned}&x^2 + (x_1+x_2+x_3+x_4-2)x + (x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_1x_3+x_1x_4\\ &\qquad +x_2x_3+x_2x_4+x_3x_4-2x_1-2x_2-2x_3-2x_4+7).\end{aligned}$$

The symmetry with respect to any (xx_i) is readily verified.

2. *An application.* There is this corollary to the preceding theorem:

Let x_i, $i = 1,2,\ldots,m$, be any mth *degree algebraic integer together with all its conjugates. If the a_r, $r = 0,1,\ldots,n$, in (2) are integers, the polynomial part of the quotient*

$$\frac{P_n(x_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{m}(x_i - x_j)} \tag{3}$$

for some particular i, where we divide algebraically *in x_i (in the division it is immaterial whether the x_j, $j \neq i$, are kept as letters or initially given their numerical values) becomes, on putting in the numerical values of x_i and x_j, $j \neq i$, equal to an integer which is the same for every choice of x_i.*

This result is immediate from the definition of algebraic integers and the well-known relations between symmetric functions of the roots of an equation and its coefficients.

To illustrate, let x_1, x_2 be the algebraic integers $-2 \pm i\sqrt{3}$, which are the roots of $x^2 + 4x + 7 = 0$, and let

$$P_5(x) = x^5 - 2x^4 + 3x^3 - x^2 + 7x + \text{any } a_0.$$

Choosing $x_i = x_2$, the polynomial part of the quotient (i.e., ignoring the remainder) in the algebraic division by $x_2 - x_1$ is

$$x_2^4 + (x_1-2)x_2^3 + (x_1^2-2x_1+3)x_2^2 + (x_1^3-2x_1^2+3x_1-1)x_2 + (x_1^4-2x_1^3+3x_1^2-x_1+7),$$

which is symmetric in x_1 and x_2 and equal to 23.

If, after the x_j, $j \neq i$, have been given their numerical values, the algebraic division (3) for the variable x_i leaves no remainder, then, of course, the numerical quotient (3) has the same integral value as the algebraic quotient (3) after x_i is given its numerical value. But algebraic divisibility, while thus *sufficient*, is *not necessary* for the numerical quotient to be an integer. This may be seen from

$$P_2(x) = x^2 + 2x - 3, \qquad x_1 = \sqrt{3}, \qquad x_2 = -\sqrt{3},$$

where the algebraic quotient of $P_2(x_1)$ by $x_1 - x_2 = x_1 + \sqrt{3}$ consists of the polynomial part $x_1 + 2 - \sqrt{3}$ and the nonvanishing remainder $-2\sqrt{3}/(x_1 + \sqrt{3})$, which for $x_1 = \sqrt{3}$ gives for the numerical quotient the integer $2 - 1 = 1$.

3. *Relation to the remainder theorem.* A more concise proof of the symmetric

quotient theorem for $m=1$ follows from the remainder theorem:

$$\frac{P_n(x)}{x-x_1} = Q_{n-1}(x,x_1) + \frac{P_n(x_1)}{x-x_1}\ . \tag{4}$$

It is apparent from (4) that the symmetry of $Q_{n-1}(x,x_1)$ follows from the symmetry of $\{P_n(x)-P_n(x_1)\}/(x-x_1)$. Conversely, the symmetry of $Q_{n-1}(x,x_1)$ implies the remainder theorem since, from

$$\frac{P_n(x)}{x-x_1} = Q_{n-1}(x,x_1) + \frac{Q_n(x_1)}{x-x_1}\ , \tag{5}$$

the symmetry of $Q_{n-1}(x,x_1)$ gives

$$\frac{P_n(x)-Q_n(x_1)}{x-x_1} = \frac{P_n(x_1)-Q_n(x)}{x_1-x}\ ,$$

or $P_n(x)-Q_n(x)=Q_n(x_1)-P_n(x_1)$ which, on letting $x=x_1$, yields $Q_n(x_1)=P_n(x_1)$, making (5) the same as (4). Thus the remainder theorem is equivalent to the symmetric quotient theorem for $m=1$, from which, by the argument in Part 1, it is seen to be equivalent to the theorem for $m\geq 1$.

4. *Relation to divided differences.* A third proof of the main theorem follows immediately from Newton's divided difference formula,

$$f(x) = f(x_1) + \sum_{s=1}^{m-1}\left(\prod_{j=1}^{s}(x-x_j)\right)[x_1x_2\ldots x_{s+1}] + \prod_{j=1}^{m}(x-x_j)[xx_1\ldots x_m], \tag{6}$$

where $[x_1x_2\ldots x_{s+1}]$ is the sth divided difference of $f(x)$ at the points $x_1, x_2, \ldots$ x_{s+1}. (For the definition and properties of divided differences, see [1, Ch. I, pp. 1-19] for a thorough treatment.) If, in (6), we replace $f(x)$ by $P_n(x)$ and divide by $\Pi_{j=1}^{m}(x-x_j)$, then the left side becomes (1); the last term on the right is the mth divided difference $[xx_1\ldots x_m]$, which is a polynomial in x of the $(n-m)$th degree whose symmetry in $x,x_1,\ldots,x_m$ is a property of divided differences for any function f (see [1, p. 7]); and the first m terms constitute a proper rational function. As there is only one such representation of (1), its polynomial part, $Q_{n-m}(x,x_1,\ldots,x_m)$, is identical with $[xx_1\ldots x_m]$, the mth divided difference of $P_n(x)$ at $x,x_1,\ldots,x_m$, and is thus symmetric in those $m+1$ arguments. □

For $f(x) = P_n(x)$ we can derive (6) by repeated application of the remainder theorem (4) to the successive polynomial parts of the quotients of $P_n(x)\equiv Q_n(x)$ and $x-x_1$, $Q_{n-1}(x,x_1)$ and $x-x_2$, ..., $Q_{n-m+1}(x,x_1,\ldots,x_{m-1})$ and $x-x_m$, utilizing the definition and symmetry of divided differences [1, pp. 1, 7]. The successive remainders

$$Q_{n-s}(x_{s+1},x_1,\ldots,x_s) = Q_{n-s}(x_1,\ldots,x_s,x_{s+1}), \qquad s = 0,1,\ldots,m-1$$

are the sth divided differences of $P_n(x)$ at $x_1,\ldots,x_{s+1}$.

Summarizing, we have

$$\text{polynomial part of } \frac{P_n(x)}{\prod_{i=1}^{m}(x-x_i)}$$

$$\equiv Q_{n-m}(x,x_1,\ldots,x_m) = [xx_1\ldots x_m]$$

$$= \frac{P_n(x)}{\prod_{j=1}^{m}(x-x_j)} + \sum_{i=1}^{m} \frac{P_n(x_i)}{(x_i-x)\prod_{\substack{j=1\\ j\neq i}}^{m}(x_i-x_j)} . \tag{7}$$

(For the last equality, see [1, p. 7].) From (7), the remainder term, which is $-\sum_{i=1}^{m}\ldots$, after multiplication by $\prod_{i=1}^{m}(x-x_i)$, is seen to be the m-point Lagrange interpolation polynomial form of its $(m-1)$th degree numerator that equals $P_n(x_i)$ when $x=x_i$, $i=1,2,\ldots,m$.

REFERENCE

1. L.M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan, London, 1933.

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* * *

SQUARE INTEGERS WITH DISTINCT DIGITS IN BASE EIGHT

CHARLES W. TRIGG

Inveterate digit-delvers may care to examine with me the table on the following page, which contains the seventy square integers, N^2, in the octonary system that are composed of distinct digits.

Eleven of these squares (marked with *) are permutations of sets of consecutive digits. Four contain the eight digits, but only one, $2453^2 = 6532471$, consists of the seven nonzero digits; and there a cyclic permutation of N is imbedded in N^2. In $6215^2 = 47302651$, a reverse cyclic permutation of N terminates N^2.

The squares of both 55 and 65 are composed of the four odd digits.

In the square 56731 the first three digits are consecutive in increasing order of magnitude, as are the first four digits of 345621.

N	N^2	N	N^2	N	N^2	N	N^2
1	1	65	5371	257	73641	651	540621
2	4Δ	67	5721	263	76451	667	570321
4	20Δ	77	7601	334	136420	715	637051
5	31Δ	116	13704	354	154620	733	670531
7	61Δ	124	15620	362	162304	763	746251
15	251	147	24561	416	216304	1242	1567204
16	304Δ	156	27504	436	237604	1463	2436051*
17	341	161	30741	445	247531	1751	3645021*
22	504	165	32571	453	256471	2273	5460231*
24	620	177	37401	455	260751	2334	5716420
25	671Δ	225	53271	524	341620	2316	5612704
34	1420	233	56731	527	345621*	2453	6532471*
36	1604	234	57420	536	357204	3344	13675420*
45	2531	241	62501	544	367420	4622	26713504*
47	2761	242	63204	551	376421	5277	34675201*
53	3471	254	71620	613	460571	6215	47302651*
54	3620Δ	255	72351	627	503421*		
55	3751	256	73104Δ	634	513420*		

In eight cases (those sporting a Δ), N and N^2 together contain distinct digits; and in $256^2 = 73104$, all eight digits are involved.

For $N = 1$, 7, 16, 17, 34, 77, 177, and 627, the digit sums of N and N^2 are the same.

Ten of the N's are palindromes: the trivial 1, 2, 4, 5, and 7; the repdigits 22, 55, and 77; and 161 and 242, which have the same digit sum.

Among the N's there are five pairs of consecutive integers: (1, 2), (4, 5), (24, 25), (233, 234), and (241, 242); two trios of consecutive integers: (15, 16, 17) and (53, 54, 55); and one quartet: (254, 255, 256, 257).

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THE OLYMPIAD CORNER: 12

MURRAY S. KLAMKIN

On the menu this month: one new Practice Set, No. 10, and the solutions to Practice Set 9.

PRACTICE SET 10

10-1. If a,b,c,d are positive integers, show that

$$30 \mid (a^{4b+d} - a^{4c+d}).$$

(Here $m|n$ means that m divides or is a factor of n.)

10-2. Determine the area of the region bounded by the closed curve whose points (x,y) in rectangular coordinates satisfy

$$\sqrt{(x-a)^2+y^2} \cdot \sqrt{(x+a)^2+y^2} = 2ky,$$

where the constants satisfy $k \geq a > 0$.

10-3. For $a \geq b \geq c \geq 0$, establish the inequality

$$b^m c + c^m a + a^m b \geq bc^m + ca^m + ab^m$$

(a) when m is a positive integer;

(b) find a proof valid for all real $m \geq 1$.

*

SOLUTIONS TO PRACTICE SET 9

9-1. ABCD is a plane quadrilateral. A' is symmetric to A with respect to B; B' is symmetric to B with respect to C; C' is symmetric to C with respect to D; and D' is symmetric to D with respect to A. Construct quadrilateral ABCD given the points A', B', C', D'.

Solution. We identify all points with their position vectors relative to some origin O; a point M, for example, will be represented by $\vec{m} = \overrightarrow{OM}$. (To further simplify the notation, we will leave the arrows in the quiver and write simply m for $\overrightarrow{OM}$, and use this notation for all points shown in the figure.)

The known vector a' is represented in terms of the unknown vectors a and b by

$$a' = b + (b-a) = 2b - a,$$

and, similarly,

$$b' = 2c - b, \quad c' = 2d - c, \quad d' = 2a - d.$$

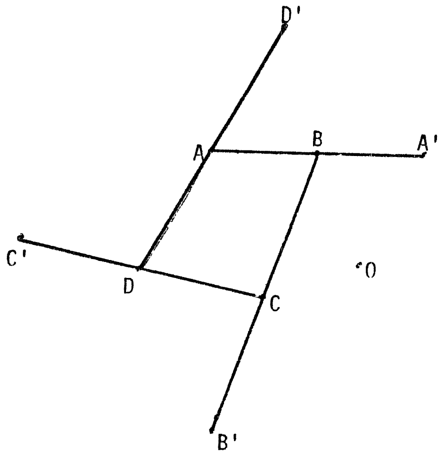
If we successively eliminate b, c, and d from these equations, we get

$$a' + 2b' = 4c - a,$$

$$a' + 2b' + 4c' = 8d - a,$$

$$a' + 2b' + 4c' + 8d' = 15a.$$

From the last equation, a can be constructed by vector addition, so point A is known. Then B is the midpoint of AA', C is the midpoint of BB', and D is the midpoint of CC'.



9-2. Tangents to a circle from an external point O meet the circle at A and B. Chord AC is constructed parallel to OB and secant OC is drawn, intersecting the circle at E. Prove that line AE bisects segment OB.

Solution. The simple solution given here is due to Dan Sokolowsky.

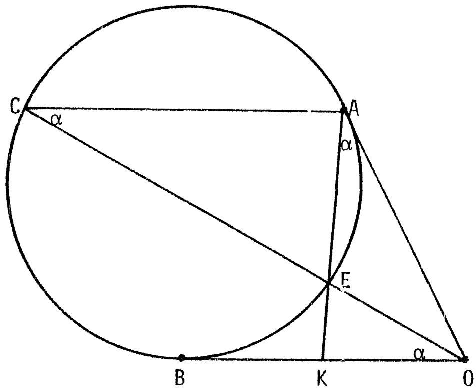
Let $AE \cap OB = K$. The three angles marked α in the figure are obviously equal; hence triangles AOK and OEK, which also have an angle at K in common, are similar. Thus,

$$\frac{OK}{AK} = \frac{EK}{OK} \qquad \text{or} \qquad OK^2 = AK \cdot EK.$$

Since KB is tangent to the circle, we now have

$$KB^2 = AK \cdot EK = OK^2,$$

and $OK = KB$, as required.



9-3. If two altitudes of a tetrahedron intersect, then the other two altitudes also intersect.

Solution. The stated result is an immediate consequence of the following two theorems which can be found in Altshiller-Court [1]:

THEOREM. If two altitudes of a tetrahedron are coplanar, the edge joining the two vertices from which these altitudes issue is orthogonal to the opposite edge of the tetrahedron.

CONVERSE THEOREM. If a pair of opposite edges of a tetrahedron are orthogonal, the two altitudes of the tetrahedron issued from the ends of each of these two edges are coplanar.

Simple synthetic proofs of these theorems can be found in the cited reference. We give an alternative vectorial proof of the first theorem:

In tetrahedron ABCD, the altitudes from A and B, say, are coplanar if and only if they intersect in a point H. Then, $\overrightarrow{HA}\cdot\overrightarrow{CD}=0$ and $\overrightarrow{HB}\cdot\overrightarrow{CD}=0$, whence

$$0 = \overrightarrow{HB}\cdot\overrightarrow{CD}-\overrightarrow{HA}\cdot\overrightarrow{CD} = (\overrightarrow{HB}-\overrightarrow{HA})\cdot\overrightarrow{CD} = \overrightarrow{AB}\cdot\overrightarrow{CD},$$

and so $AB \perp CD$. □

As a related problem, show that if one altitude intersects two other altitudes, then the four altitudes are concurrent.

REFERENCE

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Macmillan, New York, 1935, pp. 61-62; second edition, Chelsea, Bronx, N.Y., 1964, pp. 69-70.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

511. *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the following alphametic, which was inspired by the editor's comment following the solution of Crux 251 [1978: 43]:

```
   MY
 FAIR
 LADY .
ELIZA
```

You will know there is only one solution when she raises her little pinkie: in a LADY the digits are always rising.

512. *Proposed by Chris Nyberg, East York, Ontario.*

Let m and n be repdigits consisting of the *same* nonzero digit and consider the continued fraction equations

$$\sqrt{m} = n + \cfrac{2n}{n + \cfrac{n}{n + \cfrac{n}{n + \dots}}}\,, \qquad \sqrt{77} = 7 + \cfrac{7+7}{7 + \cfrac{7}{7 + \cfrac{7}{7 + \dots}}}\,.$$

The striking second equation, which is true in base ten, shows that in this base the first equation has at least the solution $m = 77$, $n = 7$. Show that, in every base $B > 3$, the first equation has a unique solution.

513. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Define the *density* $d(A)$ of a set A of natural numbers in the usual way,

$$d(A) = \lim_{n\to\infty} \frac{|\{m \in A\ :\ m \le n\}|}{n}\,,$$

provided this limit exists. (Here $|S|$ = number of elements in set S.) Also, associate to each set A of natural numbers the reciprocal series $\sum_{a\in A}(1/a)$.

(a) Can a set of density 0 have a divergent reciprocal series?

(b) Can a set of positive density have a convergent reciprocal series?

514. *Proposed by G.C.Giri, Midnapore College, West Bengal, India.*

If $\alpha + \beta + \gamma = 0$, prove that, for $n = 0,1,2,\dots$,

$$\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} = \alpha\beta\gamma(\alpha^n + \beta^n + \gamma^n) + \tfrac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}).$$

515. *Proposed by Ngo Tan, Bronx, N.Y.*

Given is a circle with center O and an inscribed triangle ABC. Diameters AA', BB', CC' are drawn. The tangent at A' meets BC in A", the tangent at B' meets CA in B", and the tangent at C' meets AB in C". Show that the points A", B", C" are collinear.

516. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Remove the last digit from a positive decimal integer, multiply the removed digit by 5, and add the product to the remaining digits of the decimal integer, thereby obtaining a new decimal integer. Repeat this process again and again, until a single digit results, as in the example

$$13258 \to 1365 \to 161 \to 21 \to 7.$$

Characterize the decimal integers for which this repetitive process terminates in the digit 7.

517.* *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given is a triangle ABC with altitudes h_a, h_b, h_c and medians m_a, m_b, m_c to sides a, b, c, respectively. Prove that

$$\frac{h_b}{m_c} + \frac{h_c}{m_a} + \frac{h_a}{m_b} \le 3,$$

with equality if and only if the triangle is equilateral.

518. *Proposed by Charles W. Trigg, San Diego, California.*

The sequence of positive integers is partitioned into the groups

$$1,\ (2,3),\ (4,5,6),\ (7,8,9,10),\ (11,12,13,14,15),\ \ldots\ .$$

Find the sum of the integers in the nth group.

519. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Let

$$\phi(x) = \frac{\sin \pi x}{x(1-x)}, \qquad 0 < x < 1.$$

Prove that ϕ is increasing for $0 < x < \frac{1}{2}$ and decreasing for $\frac{1}{2} < x < 1$.

520. *Proposed by M.S. Klamkin, University of Alberta.*

If two chords of a conic are mutually bisecting, prove that the conic cannot be a parabola.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problem.

269. [1977: 190; 1978: 79] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Let $<\sqrt{10}>$ denote the fractional part of $\sqrt{10}$. Prove that for any positive integer n there exists an integer I_n such that

$$(<\sqrt{10}>)^n = \sqrt{I_n+1} - \sqrt{I_n} .$$

IV. *Comment by the MaScoT Problems Group, University of Toronto.*

In his comment III, M.S. Klamkin found an explicit expression for the nonnegative integers I_n which satisfy

$$(\sqrt{a+1} - \sqrt{a})^n = \sqrt{I_n+1} - \sqrt{I_n}, \qquad n = 0,1,2,\ldots, \tag{1}$$

where a is a positive integer (the proposed problem had $a = 9$). But in his proof he assumed that the existence of the sequence of nonnegative integers $\{I_n\}$ had first been demonstrated, by induction (as in solution I) or otherwise. We show here that the two tasks can easily be accomplished simultaneously: proving the existence of a unique nonnegative integer I_n satisfying (1) and finding an explicit expression for it, for $n = 0,1,2,\ldots$.

For $x \geq 0$, the function $\sqrt{x+1} - \sqrt{x}$ decreases strictly in value from 1 (when $x = 0$) to 0 (when $x \to \infty$). Since, for any fixed nonnegative integer n, we have $0 < (\sqrt{a+1} - \sqrt{a})^n \leq 1$, it follows that there is a unique nonnegative real number I_n satisfying (1). To find it, we first write (1) in the equivalent form

$$\sqrt{I_n} + (\sqrt{a+1} - \sqrt{a})^n = \sqrt{I_n+1},$$

then we square twice and rearrange as required to obtain first

$$\sqrt{I_n} = \tfrac{1}{2}\{(\sqrt{a+1} + \sqrt{a})^n - (\sqrt{a+1} - \sqrt{a})^n\} \tag{2}$$

and then

$$I_n = \tfrac{1}{4}\{(\sqrt{a+1} + \sqrt{a})^{2n} + (\sqrt{a+1} - \sqrt{a})^{2n} - 2\}.$$

Now the binomial theorem gives Klamkin's expression for I_n,

$$I_n = \tfrac{1}{2}\left\{\sum_{r=0}^{n}\binom{2n}{2r}(a+1)^{n-r}a^r - 1\right\}, \tag{3}$$

and we have only left to show that (3) is an integer. But this follows immediately from the fact that the sum of the first and last terms of the summation in (3) is necessarily odd, while the remaining terms of the summation are all even.

For a given n (and given a), it is usually very awkward to calculate I_n from (3). For fairly small n, it is somewhat easier to find it by using a recurrence relation such as the one W.J. Blundon found (for $a = 9$) in his comment II:

$$I_0 = 0, \qquad I_1 = 9; \qquad I_{n+1} = 38I_n - I_{n-1} + 18, \qquad n = 1,2,3,\ldots . \tag{4}$$

But different values of a require different recurrence relations; and even when one is available the calculation is time-consuming for large n. We offer a formula with which, for any given n (and any given a), the exact value of I_n can be found directly and almost instantly on a pocket calculator, the only limitation being the capacity of the calculator itself.

If we set $u = \ln(\sqrt{a+1}+\sqrt{a})$, then $-u = \ln(\sqrt{a+1}-\sqrt{a})$, and we get, from (2), $\sqrt{I_n} = \sinh nu$, from which

$$I_n = \sinh^2\{n \ln(\sqrt{a+1}+\sqrt{a})\}. \tag{5}$$

For example, when $a = 9$, we get from (5)

$$I_6 = \sinh^2\{6 \ln(\sqrt{10}+3)\} = 749609640,$$

which agrees with the value Blundon found from (4).

In a comment following the earlier solution and comments, the editor mentions the following interesting generalization due to Lois Thompson:

If R is a positive integer, $[\sqrt{R}]$ the integral part of its square root, and $R - [\sqrt{R}]^2 = k$, then there exists a sequence $\{I_n\}$ of nonnegative integers such that

$$(\langle\sqrt{R}\rangle)^n = \sqrt{I_n + k^n} - \sqrt{I_n}, \qquad n = 0,1,2,\ldots .$$

The editor then gives an explicit expression for I_n similar to (3) and a recurrence relation similar to (4). Proceeding as we did to find (5), we get

$$I_n = \left(\frac{e^{nu} - k^n e^{-nu}}{2}\right)^2, \qquad n = 0,1,2,\ldots, \tag{6}$$

where $u = \ln(\sqrt{R}+[\sqrt{R}])$. When $k = 1$, this reduces to something very like (5). For example, when $R = 7$, then $k = 3$, $u = \ln(\sqrt{7}+2)$, and (6) yields $I_6 = 25270000$.

(The MaScoT group is an assortment of adherents of the Problem-Solving course taught by H.L. Ridge and E.J. Barbeau, in the University of Toronto Master of Science in Teaching programme. The cast for this comment included Brian Lapcevic (to whom we owe in particular the hyperbolic representation (5)), Brian Dorrepaal, Marv Hill, Sandy Richardson, Al Waters, and Paul Zolis.)

A comment was also received from the proposer.

* * *

387*. [1978: 251; 1979: 201] *Proposed by Harry D. Ruderman, Hunter College Campus School, New York.*

N persons lock arms to dance in a circle the traditional Israeli Hora. After a break they lock arms to dance a second round. Let $P(N)$ be the probability that for the second round no dancer locks arms with a dancer previously locked to in the first round. Find $\lim_{N\to\infty} P(N)$.

Abstracted from a partial solution by G.P. Henderson, Campbellcroft, Ontario.

We will use n instead of N for the number of dancers and reserve N for another use. Let the n dancers be given the labels 1, 2, ..., n. In any round, the clockwise order of the dancers corresponds to a permutation

$$x = (x_1, x_2, \ldots, x_n)$$

of the labels in which $x_1 = 1$. We may choose the labels so that the permutation for the first round is the identity permutation $(1,2,\ldots, n)$.

For any permutation x, we define the set of neighbours

$$N(x) \equiv \{(x_1,x_2), (x_2,x_3), \ldots, (x_{n-1},x_n), (x_n,x_1)\}.$$

The set of forbidden pairs for the second round is

$$F(n) \equiv \{(1,2), (2,3), \ldots, (n-1,n), (n,1), (2,1), (3,2), \ldots, (1,n)\}.$$

Let $A(n)$ be the set of permutations x which satisfy the conditions for the second round; then

$$A(n) = \{x : x_1 = 1 \text{ and } N(x) \cap F(n) = \emptyset\}.$$

If $A(n)$ has $a(n)$ members, then the probability that the second round satisfies the conditions of the problem is

$$P(n) = \frac{a(n)}{(n-1)!}.$$

We can show that $a(n)$ and two other functions, to be defined below, satisfy a system of three difference equations. These can be solved to obtain an explicit formula for $a(n)$. Using this, the required probability, $P = \lim_{n\to\infty} P(n)$, can be expressed as the sum of an infinite series. This sum agrees with e^{-2} to 9 decimal places, but we have not been able to prove that the exact value of the sum is in fact e^{-2}.

The two other functions we need are $b(n)$, the number of permutations in the set

$$B(n) \equiv \{x : x_1 = 1 \text{ and } N(x) \cap F(n) = \{(n,1)\}\};$$

and $c(n)$, the number of permutations in the set $C(n)$ of all permutations x such that $x_1 = 1$, $x_2 \neq 2$, $x_{n-1} \neq n-1$, $x_n = n$ and such that $N(x) \cap F(n)$ has exactly two members: $(n,1)$ and one other member of $F(n)$. These two functions and $a(n)$ will be found to satisfy the difference equations

$$a(n+1) = (n-4)a(n) + 4(n-3)b(n) + 2c(n),$$

$$b(n+1) = a(n) + 2b(n) + b(n-1),$$

$$c(n+1) = 2(n-2)b(n) + 2b(n-1) + 2b(n-2) + 2c(n) + c(n-1).$$

A lengthy calculation enables us to eliminate the functions b and c from these equations and then to obtain

$$P = \lim_{n\to\infty} \frac{a(n)}{(n-1)!} = 2\sum_{i=0}^{\infty} \frac{h(i)}{(i+4)!}, \tag{1}$$

where

$$h(i) = \sum_{r=0}^{i} \binom{i+1}{r+1}(-2)^r E(-1,r) \quad \text{and} \quad E(-1,r) = \sum_{s=0}^{r} \frac{(-1)^s}{s!}.$$

The series (1) converges quite rapidly. The first 12 terms give $P \approx 0.135335283$. This agrees with e^{-2} to 9 decimal places, but we don't know if the exact value of P is e^{-2}.

A comment was received from BASIL C. RENNIE, James Cook University of North Queensland, Australia.

Editor's comment.

I quote from Feller [1]: "The following problem with many variants and a surprising solution goes back to Montmort (1708). It has been generalized by Laplace and many other authors. Two equivalent decks of n cards each are put into random order and matched against each other. If a card occupies the same place in both decks, we speak of a *match* (*coincidence* or *rencontre*)." Variants of this problem involve n letters and their n addressed envelopes, or n persons and their n hats in a checkroom. Feller then goes on to show that if $P(n)$ is the probability of *no* match occurring, then $\lim_{n\to\infty} P(n) = e^{-1}$.

Suppose that, in our problem, we require merely that in the second round no dancer have the same person *on his or her right* as in the first round. The similarity with the Montmort problem now makes it easy to conjecture that the answer to our problem would then be e^{-1}, and that the answer to our problem as originally stated would be e^{-2}. Proving these statements is another matter. Rennie was thinking of the hats in the checkroom problem when, after some heuristic musings, he wrote: "If anybody were running a book on the answer to this problem, I might be tempted to put a crafty dollar on e^{-2}."

Only the bare bones of Henderson's partial solution were given above. In complete and intricate detail, it would have occupied 6 or 7 pages. Nevertheless, it represents a great step forward, for it shows that the exact answer to our problem is very likely to be e^{-2}. All that remains to be done now is for some other reader to take the final step, to bridge the gap in Henderson's solution or else, inspired by the knowledge of the probable answer, to blaze an entirely new (and, the editor fervently hopes, shorter) path to the answer.

Readers wishing to communicate with G.P. Henderson about his partial solution may do so by writing to him at Garden Hill, P.O. Box 18, R.R. 1, Campbellcroft, Ontario, Canada L0A 1B0.

REFERENCE

1. William Feller, *An Introduction to Probability Theory and Its Applications*, Volume 1, Third Edition, John Wiley & Sons, New York, 1968, p. 100.

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427. [1979: 77; 1980: 31] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A corridor of width a intersects a corridor of width b to form an "L". A rectangular plate is to be taken along one corridor, around the corner and along the other corridor with the plate being kept in a horizontal plane. Among all the plates for which this is possible, find those of maximum area.

II. *Comment by M.S. Klamkin, University of Alberta.*

A more difficult and unsolved problem is to determine the greatest area that can get around the corner if there is no restriction on the shape of the area. Another unsolved problem is to determine the longest convex arc which can be taken around the corner. These two problems are discussed, and conjectures and partial results are given, in *SIAM Review* Problem 66-11*[1]. Although the problem was submitted by the well-known problemist Leo Moser, it is likely that the problem originated elsewhere (see Hammersley [2], where is also given the best known result for the area problem, $\pi/2 + 2/\pi$, when the two corridors are both of unit width.) The difficulty of these problems is compounded when the action takes place in a three-dimensional hallway of height h, and it is desired to find the maximum volume, the maximum planar area, and the longest planar convex arc which can negotiate the corner.

Editor's comment.

The unrestricted shape area problem mentioned by Klamkin was recently resurrected in *James Cook Mathematical Notes*, where it was proposed by M.J.C. and B.J.W. Baker. The proposers' best result, which appeared in *J.C.M.N.* [3], was also $\pi/2 + 2/\pi$.

REFERENCES

1. Problem 66-11*, "Moving Furniture Through a Hallway," proposed by Leo Moser, *SIAM Review*, 11 (1969) 75-78, 12 (1970) 582-586.

2. J.M. Hammersley, "On the enfeeblement of mathematical skills by 'Modern Mathematics' and by similar soft intellectual trash in schools and universities," *Bull. Inst. Math. & Appl.*, 1968, pp. 66-85.

3. *James Cook Mathematical Notes*, No. 21, Vol. 2, January 1980, pp. 79-80.

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428. [1979: 77] *Proposed by J.A. Spencer, Magrath, Alberta.*

Let AOB be a right-angled triangle with legs $OA = 2OB$ (see figure). Use it to find an economical Euclidean construction of a regular pentagon whose side is not equal to any side of ΔAOB. "Economical" means here using the smallest possible number of Euclidean operations: setting a compass, striking an arc, drawing a line.

Solution by Hayo Ahlburg, Benidorm, Alicante, Spain.

Between the side a and the diagonal d of a regular pentagon, there is the relationship $a = d(\sqrt{5}-1)/2$.

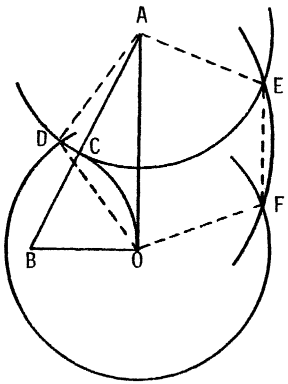
This suggests the following construction:

1. Strike an arc B(BO), that is, with centre B and radius BO, meeting AB in C.

2. Strike an arc A(AC).

3. Strike an arc O(AC), meeting arc A(AC) in D as shown.

4. Strike an arc D(OA), meeting arcs A(AC) and O(AC) in E and F as shown.



Then ADOFE is the required regular pentagon. The construction required 3 compass settings, striking 4 arcs, and, of course, drawing 5 lines for the sides of the pentagon: 12 Euclidean operations in all.

Proof. If we take $OB = 1$, then $d = OA = 2$, $AB = \sqrt{5}$, and $a = AD = \sqrt{5}-1$; hence $a = d(\sqrt{5}-1)/2$.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain (12 operations); and the proposer (14 operations).

* * *

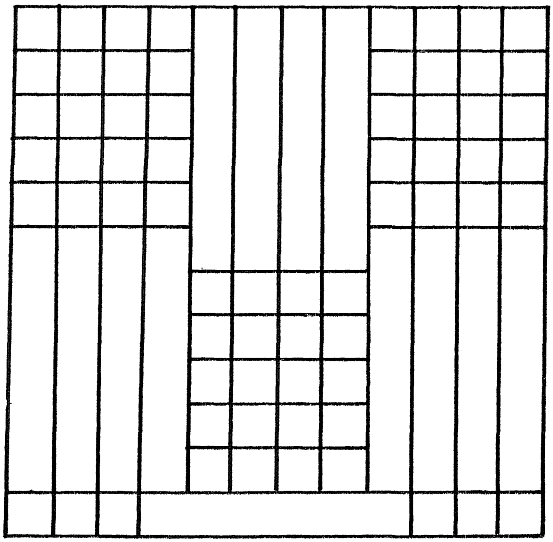
429. [1979: 77] *Proposed by M.S. Klamkin and A. Liu, both from the University of Alberta.*

On a $2n\times 2n$ board we place $n\times 1$ polyominoes (each covering exactly n unit squares of the board) until no more $n\times 1$ polyominoes can be accommodated. What is the maximum number of squares that can be left vacant?

This problem generalizes Crux 282 [1978: 114].

Solution by the proposers.

The figure (which illustrates the case $n=6$ and can easily be generalized) shows that the $2n\times 2n$ board can be *blocked* by $2n+1$ $n\times 1$ polyominoes (so that no more can be accommodated).



To show that $2n+1$ is minimal, assume that the board is blocked by $2n$ polyominoes. Clearly, not all of them can be of the same orientation (horizontal or vertical).

We now state the evident fact that if, for any $k\le n$, the kth line (row or column) from an edge of the board contains a polyomino, then so must the ith line from the same edge for all $i<k$. Hence the lines not containing a polyomino must be consecutive if parallel.

Let there be p rows and q columns that do not contain a polyomino. Note that $p\ge 1$ and $q\ge 1$. Then there is a $p\times q$ rectangle on the board which is uncovered. Since there are $2n$ polyominoes and $4n$ lines altogether, we must have $p+q\ge 2n$ and either $p\ge n$ or $q\ge n$. This shows that an additional polyomino can be accommodated and provides the needed contradiction.

It is clear that the number of squares left vacant is maximum when the number of polyominoes used is minimum. Hence the maximum number of squares left vacant is

$$(2n)^2-n(2n+1) = 2n^2-n.$$

* * *

430. [1979: 78] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) For $n=1$, 8^{-n} equals a decimal fraction whose digits sum to 8. Prove that 8^{-n} for $n=2,3,4,\ldots$ never again equals a decimal fraction whose digits sum to 8.

(b) The cube of 8 has decimal digits that sum to 8. For $n=4,5,6,\ldots$, is there another 8^n whose decimal digits sum to 8?

I. *Solution by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

(a) Since $8^{-n}=(125/1000)^n$, the significant digits of 8^{-n} are the same as those of 125^n. If the digits of 125^n sum to 8, then

$$125^n \equiv (-1)^n = -1 \pmod 9,$$

so we need consider only odd n. From $125^2 \cdot 125 \equiv 125 \pmod{1000}$, it follows by induction that 125^n ends in 125 for all odd n. Since $1+2+5=8$, we conclude that the digits of 125^n, and hence those of 8^{-n}, sum to 8 if and only if n is odd and 125^n contains exactly three digits, that is, if and only if $n=1$.

(b) The answer if NO. For suppose that, for some $n \geq 4$, the digits of 8^n sum to 8; then 8^n contains at least four digits, the last of which is not an 8, and

$$8^n \equiv (-1)^n = -1 \pmod 9,$$

so n must be odd. From

$$8^4 \cdot 8 \equiv 8 \quad \text{and} \quad 8^4 \cdot 8^3 = 8^4 \cdot 512 \equiv 2 \pmod{10},$$

it follows by induction that 8^{4k+1} ends in 8 and 8^{4k+3} ends in 2 for all nonnegative integers k. So we must have $n=4k+3$ for some $k \geq 1$, and 8^n ends in 2. Now the number formed by the last three digits of 8^n must be divisible by 8, end in 2, and have a digit sum less than 8. The only numbers with these properties are easily found to be

$$032, \quad 112, \quad 232, \quad 312.$$

We shall investigate each of these possibilities and see that each of them leads to a contradiction. Since

$$8^7 = 2097152 \quad \text{and} \quad 8^{11} = 8589934592,$$

both of which have digit sums greater than 8, we may assume that $n=4k+3 \geq 15$. Thus $8^n \geq 8^{15} = 3.518\ldots \times 10^{13}$, so 8^n has at least 14 digits. It follows that, at least for $j=1,2,\ldots,14$, the number formed by the last j digits of 8^n is divisible by 2^j since 8^n is so divisible.

Case 1. Suppose 8^n ends in 232. The possibilities are

$$8^n \text{ ends in } 0232 \quad \text{or} \quad 8^n = 1232.$$

Both must be rejected since $2^4 \nmid 0232$ and 1232 has fewer than 14 digits.

Case 2. Suppose 8^n ends in 312. The possibilities here are

$$8^n \text{ ends in one of } 0312,\ 1312 \quad \text{or} \quad 8^n = 2312.$$

Since $2^4 \nmid 0312$, $2^4 \mid 1312$, and 2312 has fewer than 14 digits, the only possibility retained is an ending of 1312. Further possibilities are now

$$8^n \text{ ends in } 01312 \quad \text{or} \quad 8^n = 11312.$$

The first possibility is retained since $2^5 \mid 01312$, but the second is rejected since 11312 has fewer than 14 digits. Finally, we have the possibilities

$$8^n \text{ ends in } 001312 \quad \text{or} \quad 8^n = 101312,$$

but both must be rejected since $2^6 \nmid 001312$ and 101312 has fewer than 14 digits.

Case 3. Suppose 8^n ends in 032. Proceeding as in Cases 1 and 2 (the method should now be clear), all the possibilities are soon eliminated, the last to go being

$$8^n \text{ ends in } 001100032 \quad \text{or} \quad 8^n = 101100032,$$

because $2^9 \nmid 001100032$ and 101100032 has fewer than 14 digits.

Case 4. Suppose 8^n ends in 112. The possibilities here are slightly more numerous, but they can all be eliminated by hand in far less time than it would take to program a computer to do the job. The last to go are

$$8^n \text{ ends in } 001010010112 \quad \text{or} \quad 8^n = 101010010112,$$

because $2^{12} \nmid 001010010112$ and 101010010112 has fewer than 14 digits. (It is here that we *need* to know that 8^n has at least 14 digits. This knowledge was not necessary to eliminate the possibilities

$$8^n = 1232, \quad 8^n = 2312, \quad \ldots, \quad 8^n = 101010010112$$

in the four cases, since it is easily verified that these numbers are not powers of 8.)

II. *Comment by the proposer.*

It has surely been known for a long time that 8^3 has a digit sum of 8. Dickson [1] credits Moret-Blanc with proving in 1879 that 1, 8, 17, 18, 26, 27 are the only numbers equal to the digit sums of their cubes (the problem had been proposed by

Laisant in 1878). But that 8^3 is the *only* power of 8 (other than 8 itself) with a digit sum of 8 is, to the best of my knowledge, a new result.

It is easy to show that 1, 7, 22, 25, 28, 36 are the only numbers equal to the digit sums of their fourth powers, and that, for a given power $p = 2,3,4,\ldots$, at most a finite set of numbers greater than 1 can possibly equal the digit sums of their pth powers. But I have been unable to prove that this set is nonempty for every power $p = 2,3,4,\ldots$, as is suggested by the following numerical evidence:

$$28^5 = 17210368,$$

$$45^6 = 8303765625,$$

$$31^7 = 27512614111,$$

$$54^8 = 72301961339136,$$

$$54^9 = 3904305912313344,$$

$$82^{10} = 137448031335960586 24.$$

It is easy to prove that there are infinitely many numbers *not* equal to the digit sum of any pth power (if $m \not\equiv 0 \pmod 3$, then no power p of $3m$ has a digit sum of $3m$ because the digit sum is a multiple of 9). I have been unable to resolve whether or not infinitely many numbers *do* have a power whose digits sum to the number.

It does not appear to be unusual for numbers to have more than one power whose digits sum to the number; for example:

$18^3 = 5832$	$46^5 = 205962976$
$18^6 = 34012224$	$46^8 = 20047612231936$
$18^7 = 612220032$	
$28^4 = 614656$	$54^8 = 72301961339136$
$28^5 = 17210368$	$54^9 = 3904305912313344$

But for 8, as we have seen, only 8^3 has a digit sum of 8.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh (part (a) only); CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The proposer's solution to part (b) used much heavier number-theoretical machinery, and he had to appeal to a computer at the heart of his proof. Most of the remaining solutions to part (b) were not quite complete, a fact acknowledged by some but not all of the solvers themselves.

Stone observed that "since 8^n has so many digits for large n, it is inconceivable that they would not sum to more than 8. But the improbable can happen: 8^{11663} ends in 0010112." And the proposer's computer came up with 8^{149163}, a 134708-digit number ending in 02010112. At the other end, we know from Crux 50 [1975: 78] that there are infinitely many powers of 8 that start with 1 followed by as many millions of zeros as we please. Here the improbable is *bound* to happen.

REFERENCE

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Volume I, Chelsea, New York, 1966, p. 457.

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431. [1979: 107] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

The following decimal alphametic is dedicated to Erwin Just, Problem Editor of the *Two-Year College Mathematics Journal*, who modestly refused to publish it in his own journal:

$$\begin{array}{r} \text{YES} \\ \text{YES} \\ \underline{\text{JUST}} \\ \text{ERWIN} \end{array}.$$

ERWIN is, of course, unique.

Solution by the proposer.

Clearly, $E=1$, $R=0$, and $J=8$ or 9. The relations to be satisfied are

$$\begin{aligned} 2S+T &= N+10x, \\ 2+S+x &= I+10y, \\ 2Y+U+y &= W+10z, \\ J+z &= 10, \end{aligned} \tag{1}$$

with $0 \le x \le 2$, $0 \le y \le 1$, and $1 \le z \le 2$.

Suppose $S=2$. Then $y=0$ and $(x,I)=(0,4)$ or $(1,5)$. If $(x,I)=(0,4)$, then

$(T,N,J,z) = (3,7,9,1)$, $(3,7,8,2)$, or $(5,9,8,2)$. If $(x,I) = (1,5)$, then $(T,N,J,z) = (9,3,8,2)$. But in all four cases no values remain to satisfy (1).

Suppose $S = 3$. Then $(T,N,J,y,I) = (2,8,9,0,5)$, $(8,4,9,0,6)$, or $(9,5,8,0,6)$. Again, no values remain to satisfy (1) in all three cases.

Suppose $S = 4$. There $x = 1$, $y = 0$, and $I = 7$. Consequently, $(T,N,J) = (5,3,8)$, $(5,3,9)$, or $(8,6,9)$, and in each case no values are left to satisfy (1).

If $S = 5$, then $T = N$.

Suppose $S = 6$. Then $x = 1$ and $I = 9$, whence $J = 8$ and $z = 2$. Again, equation (1) cannot hold.

If $S = 7$, then $I = R = 0$ or $I = E = 1$.

Finally, suppose $S = 9$; then $J = 8$, $z = 2$, $y = 1$, and once again equation (1) cannot hold.

It follows that $S = 8$ and $J = 9$; then $I = 2$, $T = 7$, $N = 3$, and $(Y,U,W) = (4,6,5)$. The unique solution is

$$\begin{array}{r} 418 \\ 418 \\ 9687 \\ \hline 10523 \end{array}.$$

Also solved by LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; J.A. McCALLUM, Medicine Hat, Alberta; CHARLES W. TRIGG, San Diego, California; and KENNETH M. WILKE, Topeka, Kansas. One incorrect solution was received, and EDITH ORR submitted a comment.

Editor's comment.

ERWIN is, of course, unique, and so is Erwin, as readers of the *Two-Year College Mathematics Journal* can testify.

Edith Orr dipped her pen in acid and wrote: "Erwin Just's modesty in refusing to publish this problem in his own journal is highly commendable, especially in the light of *our* editor's use of his own name in Crux 201 [1977: 136]." This is an unsavory episode upon which we will not dwell.

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432. [1979: 108] *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x + x \sin x}{x^2 + \cos^2 x}\, dx.$$

Solution by Gali Salvatore, Perkins, Québec.

The integrand, call it f, is an even function which is defined and continuous over the entire real axis; hence the required integral is

$$I = 2\int_0^{\infty} f(x)dx = 2\lim_{b\uparrow\infty}\int_0^b f(x)dx, \tag{1}$$

provided this limit exists.

The function g given by $g(x) = \arctan(-\cos x/x)$ is defined and differentiable for all $x \neq 0$, and it is easy to verify that $f(x) = g'(x)$ for all $x \neq 0$. It follows that, if $0 < a < b$, then

$$\int_a^b f(x)dx = g(b) - g(a).$$

Now, from (1),

$$I = 2\lim_{b\uparrow\infty}\Big\{\lim_{a\downarrow 0}\{g(b) - g(a)\}\Big\} = 2\lim_{b\uparrow\infty}\{g(b) - (-\pi/2)\} = 2\{0 - (-\pi/2)\} = \pi.$$

Also solved by E.J. BARBEAU, University of Toronto; CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; LEROY F. MEYERS, The Ohio State University; V.N. MURTY, Pennsylvania State University, Capitol Campus; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

All solvers arrived at the correct answer. Those who are satisfied merely with correct answers should look at the back of textbooks, where millions of answers can be found, most of them correct. But a correct *solution* consists of more than just the mindless manipulation of symbols with no thought given to the legitimacy of the operations they are made to perform, as long as they lead to the correct answer. This journal aims to supplement, not to replace, classroom and textbook. So let us leave in classroom and textbook the simulacrum of mathematics that goes under the name of first-year calculus.

Observe that only one limit is involved in the improper integral (1), because the integrand f is defined and continuous over the entire interval of integration. It may be necessary to introduce more limits later on to evaluate the integral: this depends upon the choice of antiderivative (or indefinite integral). Now there is no getting around the fact that finding antiderivatives is essentially a guessing game. The game is further complicated by the fact that, for a given integrand, there are infinitely many antiderivatives, and some may be more convenient than others for the job at hand.

The antiderivative g used in our featured solution was easy enough to guess. It is undefined at $x=0$ and therefore required the use of another limit later on to evaluate the integral. Four solvers found it more convenient to replace the given integrand $f(x)$ by

$$\frac{\sec x + x\sec x\tan x}{x^2\sec^2 x+1},$$

for which the antiderivative arctan $(x\sec x)$ was easier to guess. But in doing so they introduced infinitely many discontinuities in the integrand, each of which would necessitate the taking of two limits (one on each side) to evaluate the integral. Three of the four solvers blithely ignored those discontinuities and romped home to the correct answer far ahead of the plodding and meticulous fourth solver.

But in the long run the tortoise always wins.

Moral. There is more to mathematics than just getting the right answer. So don't expect any medals from this editor if you cancel the 6 in $\frac{16}{64}$ to obtain the correct answer $\frac{1}{4}$. Because the next time you may be faced with something like $\frac{26}{65}$ or $\frac{19}{95}$. Hmmm. Well, you know what I mean.

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433. [1979: 108] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

An exam question asked: *How many distinct* 5*-letter words can be formed using the letters A, A, A, B, B, B?*

A student misread the question and determined instead the number of distinct 6-letter words using these same letters, yet obtained the correct answer. Was this accidental or is it a special case of a more general pattern? Explain.

Solution by Clayton W. Dodge, University of Maine at Orono.

There is a more general pattern. Whether or not there are duplicated letters is irrelevant. Given any n letters, distinct or not, let S_{n-1} be the set of all distinct $(n-1)$-letter words, and S_n the set of all distinct n-letter words, that can be formed from these letters. We will show that S_{n-1} and S_n each contain the same number of words by finding a bijection from S_{n-1} onto S_n.

The relation $\rho \subseteq S_{n-1}\times S_n$ is defined as follows: for any $x\in S_{n-1}$ and $y\in S_n$, the pair $(x,y)\in\rho$ if and only if y is the n-letter word obtained by adjoining *at the end* of the $(n-1)$-letter word x any one letter that was left unused in forming the word x. Since exactly one letter was left unused in forming any $x\in S_{n-1}$, we have

$$(x,y) \in \rho \quad \text{and} \quad (x,z) \in \rho \implies y = z,$$

so ρ is a mapping from S_{n-1} into S_n, and we can write $(x,y) \in \rho$ in the form $y = \rho(x)$. Adjoining a letter at the end of two distinct $(n-1)$-letter words always produces two distinct n-letter words; hence

$$x \neq y \implies \rho(x) \neq \rho(y),$$

and ρ is an injective mapping. Finally, removing the last letter from any n-letter word obviously produces an $(n-1)$-letter word; in other words, for any $y \in S_n$, there exists an $x \in S_{n-1}$ such that $y = \rho(x)$. Thus ρ is a surjection onto S_n and hence a bijection from S_{n-1} onto S_n.

Also solved by LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, University of New Mexico, Albuquerque, N.M.; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DONALD P. SKOW, Griffin & Brand Inc., McAllen, Texas; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer (two solutions).

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434. [1979: 108] *Proposed by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

(a) It is not hard to show by Bertrand's Postulate that all the solutions in positive integers x, y, m, n of the equation

$$(m!)^x = (n!)^y$$

are given by $m = n = 1$; and $m = n$, $x = y$. Find such a proof.

(b)* Prove the same result without using Bertrand's Postulate or equivalent results from number theory.

Solution to part (a) *by Malcolm A. Smith, Georgia Southern College, Statesboro, Georgia.*

If $m = 1$ or $n = 1$, then $m = n = 1$ follows from

$$(m!)^x = (n!)^y; \tag{1}$$

so we assume $m,n \geq 2$. Let p be the greatest prime not exceeding m. From (1) and the uniqueness of factorization, it follows that p is also the greatest prime not exceeding n. Now Bertrand's Postulate implies that $p \leq m,n < 2p$, so p occurs to exactly the first power in both $m!$ and $n!$. Hence, from (1), $p^x = p^y$, so $x = y$ and $m = n$.

Part (a) was also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

Part (b) remains open.

One solver had submitted a solution to part (b) based on the theorem that, for $m > 1$, $m!$ is never a square or higher power. But he retracted that solution when he discovered [1,2] that this theorem is a corollary of Bertrand's Postulate (which, incidentally, is no longer a mere "postulate" since it was proved by Chebyshev in 1852).

REFERENCES

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Volume II, Chelsea, New York, 1952, p. 679.

2. W. Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1964, p. 138.

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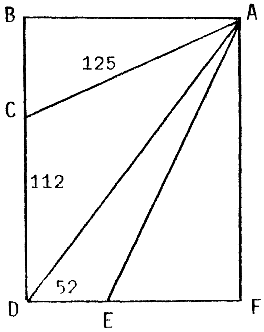
435. [1979: 108] *Proposed by J.A.H. Hunter, Toronto, Ontario.*



This little problem was inspired by the late R. Robinson Rowe's tale of the ardent jogger (Crux 356 [1978: 160; 1979: 80]). In rectangle ABDF, we have AC = 125, CD = 112, DE = 52, as shown in the figure, and AB, AD, and AF are also integral. Evaluate EF.

Solution by Friend H. Kierstead, Cuyahoga Falls, Ohio.

AB is integral, and so is BC = AF - CD; hence ABC is a Pythagorean triangle with hypotenuse 125. From the well-known fact that the sides of all Pythagorean triangles are given by

$$k(m^2 - n^2), \qquad 2kmn, \qquad k(m^2 + n^2),$$

where m and n are relatively prime, of different parity, and $m > n$, and the additional fact that here k can only be 1, 5, or 25, we find that ABC must be one of the following three triangles:

$$(35,\ 120,\ 125), \qquad (44,\ 117,\ 125), \qquad (75,\ 100,\ 125).$$

Since AB > 52, its only possible values are AB = 75, 100, 117, 120. But only AB = 117 (with $BC = 44$) leads to an integral value $AD = 195$ in triangle ABD. We conclude that $EF = 117 - 52 = 65$.

Not required by the problem, but also of interest, is that $AE = 169$, also an integer.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; J.T. GROENMAN, Groningen, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; EDGAR LACHANCE, Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; V.N. MURTY, Pennsylvania State University, Capitol Campus; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; MATS RÖYTER, Chalmers Tekniska Högskola, Gothenburg, Sweden; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer. One incorrect solution was received.

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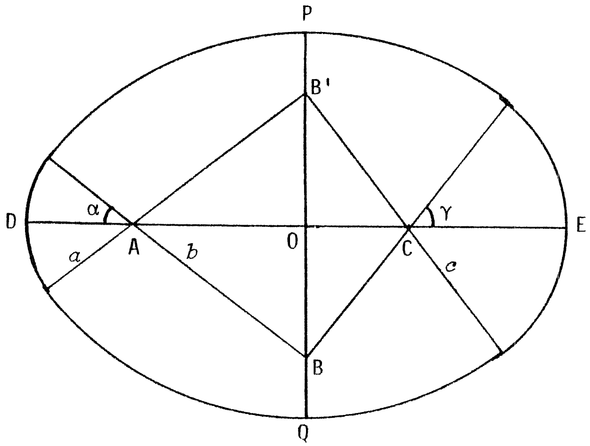
436. [1979: 109] *Proposed by the late R. Robinson Rowe, Naubinway, Michigan.*

A hen's egg is an ovoid of revolution and its oval longitudinal section can be approximated by four circular arcs as shown in the figure.



(a) Given the half-central angles α and γ at A and C, the length $DE = L$ and the width $PQ = W$, find the three radii a, b, c.

(b) Calculate the three radii if

$\alpha = \sin^{-1} 0.6$, $\gamma = \sin^{-1} 0.8$,

$L = 50$ mm, $W = 36$ mm.

I. *Solution by Clayton W. Dodge, University of Maine at Orono.*

(a) Let $OB = h$, so that $b = h + W/2$. From right triangles ABO and CBO, we get

$$b - a = h \csc\alpha, \qquad b - c = h\csc\gamma, \qquad OA = h\cot\alpha, \qquad OC = h\cot\gamma.$$

Now we have

$$\begin{aligned} L &= DA + OA + OC + CE \\ &= a + h\cot\alpha + h\cot\gamma + c \end{aligned}$$

$$= b - h\csc\alpha + h\cot\alpha + h\cot\gamma + b - h\csc\gamma$$

$$= h + W/2 - h\csc\alpha + h\cot\alpha + h\cot\gamma + h + W/2 - h\csc\gamma,$$

from which we get

$$h = \frac{L - W}{2 + \cot\alpha - \csc\alpha + \cot\gamma - \csc\gamma}.$$

Now the values of b, a, and c are readily calculated from the equations

$$b = h + W/2, \qquad a = b - h\csc\alpha, \qquad c = b - h\csc\gamma.$$

(b) With the given values of α and γ, we have

$$\cot\alpha = \frac{4}{3}, \quad \csc\alpha = \frac{5}{3}, \quad \cot\gamma = \frac{3}{4}, \quad \csc\gamma = \frac{5}{4},$$

from which we get $h = 12$ mm, and then

$$b = 30 \text{ mm}, \qquad a = 10 \text{ mm}, \qquad c = 15 \text{ mm}.$$

II. *Comment by the proposer.*

The dimensions given in part (b), with which it turns out that this egg is eggzact, were not chosen quite arbitrarily. They are a very close approximation to an egg graded as LARGE in Michigan. The former grades of SMALL, STANDARD, and LARGE have been replaced by MEDIUM, LARGE, and EXTRA LARGE. There are no SMALL eggs any more!

Also solved by FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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437. [1979: 109] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Since Professor E.P.B. Umbugio retired, the University of Guayazuela has been able to set aside all of his salary of 50 million Guayazbucks a year. Finally this account accumulated enough money to purchase a basic pocket electronic calculator, which the good Professor Emeritus has been working with since he is the only one at the U. of G. who understands how to use it. The Great Numerologist has been trying to find all Pythagorean triangles having the hypotenuse divisible by 7. He feels it will be good luck to construct his retirement cottage out of these triangles only. To date he has found only the two triangles 21, 28, 35 and 35, 84, 91. Help the professor by finding all such triangles, and especially all primitive ones (triangles having no common factor greater than 1 in their three sides).

I. *Solution by Leroy F. Meyers, The Ohio State University.*

If the professor's cottage is to be constructed entirely out of primitive Pythagorean triangles, then the cottage will be primitive indeed, in fact entirely deflated (to use a word frequently employed in disparagement of the good professor). For there are no primitive triangles with hypotenuse divisible by 7.

To see this, suppose $a^2+b^2=c^2$ and c is divisible by 7. Since a and b must each be congruent to one of $0, \pm 1, \pm 2, \pm 3$ modulo 7, their squares must each be congruent to one of 0, 1, 4, 2 modulo 7, and the only sum of two of these which is congruent to 0 modulo 7 is $0+0$. This shows that a and b are both divisible by 7.

But the professor can still build his cottage, for there are plenty of non-primitive Pythagorean triangles with hypotenuse divisible by 7. He can use any primitive triangle with sides

$$m^2-n^2, \qquad 2mn, \qquad m^2+n^2,$$

where m and n are relatively prime, of different parity, $m>n$, and multiply each side by the same multiple of 7.

Guayazuela is a hot country. So the professor can still adhere to his original plan by using primitive triangles for the windows.

II. *Quotation by John A. Winterink, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico.*

Summa duorum quadratorum $aa+bb$ per nullum numerum primum huius formae $4n-1$ unquam dividi potest, nisi utriusque radix seorsim a et b sit divisibilis per $4n-1$. (Euler)

III. *Comment by Charles W. Trigg, San Diego, California.*

Unfortunately, Professor Umbugio's library [1977: 123-125, 186-187; 1978: 188] does not include Dickson's three-volume *History of the Theory of Numbers*. For, on page 228 of Volume II of that authoritative work, the good professor could have found the following: "Fermat stated that he had proved that ... x^2+y^2 is divisible by no prime $4n-1$ if x and y are relatively prime." Had he known, the Great Numerologist would have abandoned his search for primitive triangles with hypotenuses divisible by any of the "lucky" numbers 3, 7, or 11, or a host of others. Then again, he might not.

A late bulletin from Guayazuela reports that, frustrated by not finding any primitive triangles to cover his roof, the professor has come down with a bad case of the shingles. Fellow sufferers will sympathize.

IV. *Comment by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

Hopefully[1], the professor had not planned to shingle his roof with primitive triangles, for there are none such (and my namesake hurricane[2] would dampen the professor and short out his new calculator).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, University of New Mexico, Albuquerque, N.M.; ALLAN WM. JOHNSON, JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico; and the proposer.

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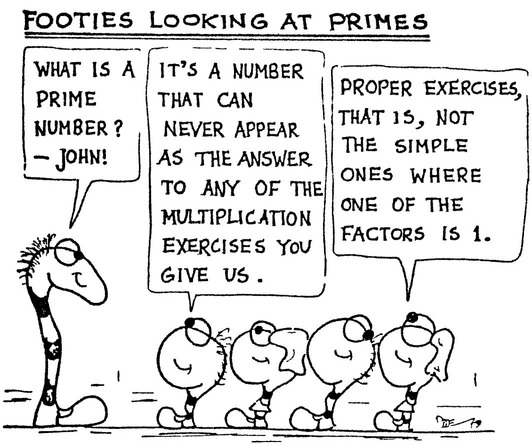

ANDREJS DUNKELS, University of Luleå, Sweden
KLAUS HÄRTIG, Humboldt Universität, Berlin, DDR

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[1]Hopelessly, this French-speaking editor decides to break a lance against the bastardization of the English language, as exemplified by the above use of the adverb "hopefully", which is currently fashionable in circles that should know better. What is the precise grammatical role of "hopefully" in the above sentence? What verb, adjective, or other adverb does it modify?

[2]Hurricane David was in the news at about the time this problem was published.