Crux

Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 7, No. 6

June - July 1981

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Mathematics Department, the Ottawa Valley Education Liaison Council, and the University of Ottawa Mathematics Department are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $12.00. Back issues: $1.20 each. Bound volumes with index: Vols. 1&2 (combined), $12.00; Vols. 3 - 6, $12.00 each. Cheques and money orders, payable to CRUX MATHEMATICORUM (in US funds from outside Canada), should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Architecture Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Lillian Marlow.

CONTENTS

EXTENSIONS OF THE NINE-POINT CIPCLE

STANLEY COLLINGS

Abstract. Generalising from triangles in two dimensions to orthocentric simplexes in n-1 dimensions, the nine-point circle generalises to a whole family of coaxal hyperspheres. Each single hypersphere of the family is determined by the centroids of r-point subsets of the n vertices of the simplex, or equivalently by their orthocentres.

1. *Orthocentric Simplexes.*

It is well known that the altitudes of a triangle concur at a point called the orthocentre. Also that if H is the orthocentre of triangle ABC, then the midpoints of the six sides of the quadrangle ABCH lie on a circle together with the intersections $HA \cap BC$, $HB \cap CA$, $HC \cap AB$. The centre of this nine-point circle lies on the line joining H to the circumcentre O—the Euler line of triangle ABC. The centres of similitude of the circumcircle and the nine-point circle are H and the centroid G of triangle ABC. The circle on GH as diameter is the circle of similitude of triangle ABC, and it is coaxal with the circumcircle and the nine-point circle. The polar circle of triangle ABC (the circle reciprocating triangle ABC into itself) belongs to the same coaxal system.

We now turn to generalisations of triangles in higher dimensions. In 3-space this is a tetrahedron. This time the altitudes from the vertices to the opposite faces do not necessarily concur. If they do, we have an orthocentre, and the tetrahedron is said to be *orthocentric*. The generalisation in (n-1)-space is called a *simplex*. This contains n vertices and $\binom{n}{2}$ edges. Given any two edges, either they meet at a vertex, or they do not meet at all. If the simplex has an orthocentre, it is called orthocentric.

Take an orthocentric simplex $P_1P_2...P_n$ with orthocentre H, and let the altitude P_1H meet the opposite "face" at H'. Now P_1H is orthogonal to the whole face $P_2P_3...P_n$, and hence to the edge P_3P_4. Similarly P_2H is orthogonal to P_3P_4. Put the other way round, $P_3P_4 \perp HP_1$ and $P_3P_4 \perp HP_2$, therefore $P_3P_4 \perp P_1P_2$. By repetition, every two non-intersecting edges are orthogonal to each other. Conversely (as demonstrated below by vector methods), a simplex for which this property holds is necessarily orthocentric. Also, we notice that P_3P_4 is orthogonal to P_2H'. Similarly, P_2H' is orthogonal to every edge in the subspace $P_3P_4...P_n$; therefore P_2H' is an altitude of the face $P_2P_3...P_n$. By repetition of the argument, H' is the orthocentre of the face $P_2P_3...P_n$.

Take the circumcentre O as origin of vectors, and let $\vec{p}_i = \overrightarrow{OP}_i$, so that

$$|\vec{p}_1| = |\vec{p}_2| = \ldots = |\vec{p}_n| .$$

Let the point H be defined by

$$\overrightarrow{OH} = \vec{h} = (\vec{p}_1+\vec{p}_2+\ldots+\vec{p}_n)/(n-2).$$

Then

$$\begin{aligned}(n-2)\overrightarrow{P_1H}\cdot\overrightarrow{P_3P_2} &= (n-2)(\vec{h}-\vec{p}_1)\cdot(\vec{p}_2-\vec{p}_3)\\ &= \{\vec{p}_2+\vec{p}_3+\ldots+\vec{p}_n-(n-3)\vec{p}_1\}\cdot(\vec{p}_2-\vec{p}_3)\\ &= |\vec{p}_2|^2 - |\vec{p}_3|^2 + \sum_{i=4}^{n}(\vec{p}_i-\vec{p}_1)\cdot(\vec{p}_2-\vec{p}_3)\\ &= |\vec{p}_2|^2 - |\vec{p}_3|^2 \text{ if nonintersecting edges are orthogonal}\\ &= 0, \text{ as } |\vec{p}_2| = |\vec{p}_3|.\end{aligned}$$

Therefore $P_1H \perp P_2P_3$, and similarly P_1H is orthogonal to all the other edges of the face $P_2P_3\ldots P_n$; that is, P_1H is an altitude of this face. By repetition of the argument, all altitudes pass through H, so H is the orthocentre and the simplex is orthocentric.

Although we shall not use it, an interesting property of H is as follows. Because nonintersecting edges are orthogonal, any r of the vertices P_i form an orthocentric simplex lying in a subspace of $r-1$ dimensions. The remaining $n-r$ vertices determine an orthogonal subspace of $n-r-1$ dimensions. Then, for all r such that $3 \le r \le n-3$, the line joining the orthocentre of the r-simplex to the orthocentre of the $(n-r)$-simplex is the common normal to the two corresponding subspaces, and it passes through H.

Each orthocentric n-simplex has a circumcentre O and an orthocentre H. The segment joining the two may be thought of as a generalised Euler line. On it we take the points C_r defined by

$$OC_r/OH = (r-1)/r;$$

then O is the point C_1 and H is the point C_∞. Noting that $OC_{n/2}/OH = (n-2)/n$, it follows that

$$\overrightarrow{OC}_{n/2} = \sum_{i=1}^{n}\vec{p}_i/n = \overrightarrow{OG},$$

where G is the centroid of the simplex. Hence, as for a triangle, O,G,H all lie on the Euler line.

2. *Centroids and Hyperspheres.*

Let G_1 be the centroid of $P_2P_3 \ldots P_{r+1}$ and G_2 that of $P_1P_3 \ldots P_{r+1}$. We will show that G_1 and G_2 are equidistant from C_r. Since

$$r(n-2)\ \overrightarrow{OC_r} = (r-1)\ (\vec{p}_1+\vec{p}_2+\ldots+\vec{p}_n)$$

and

$$r(n-2)\overrightarrow{OG_1} = (n-2)(\vec{p}_2+\vec{p}_3+\ldots+\vec{p}_{r+1}),$$

we have

$$r(n-2)\overrightarrow{G_1C_r} = (r-1)\vec{p}_1 - (n-r-1)\vec{p}_2 - (n-r-1)\sum_{i=3}^{r+1} \vec{p}_i + (r-1)\sum_{i=r+2}^{n} \vec{p}_i\ ;$$

and similarly

$$r(n-2)\overrightarrow{G_2C_r} = -(n-r-1)\vec{p}_1 + (r-1)\vec{p}_2 - (n-r-1)\sum_{i=3}^{r+1}\vec{p}_i + (r-1)\sum_{i=r+2}^{n}\vec{p}_i.$$

If we take the scalar product of the sum and difference of these two vectors, we get

$$\begin{aligned} r^2(n-2)^2(|\overrightarrow{G_1C_r}|^2 - |\overrightarrow{G_2C_r}|^2) &= (n-2)(\vec{p}_1-\vec{p}_2)\cdot\{(2r-n)(\vec{p}_1+\vec{p}_2)-2(n-r-1)\sum_{i=3}^{r+1}\vec{p}_i + 2(r-1)\sum_{i=r+2}^{n}\vec{p}_i\} \\ &= 2(n-2)(\vec{p}_1-\vec{p}_2)\cdot \sum_{i=r+2}^{n}\ \sum_{j=3}^{r+1}(\vec{p}_i-\vec{p}_j) \\ &= 0, \end{aligned}$$

since $|\vec{p}_1| = |\vec{p}_2|$ and nonintersecting edges are orthogonal. So $|G_1C_r| = |G_2C_r|$, and by repeated application C_r is equidistant from the centroids of all r-point subsets of the vertices. Therefore all such centroids lie on a hypersphere S_r having centre C_r.

When $r = 1$, we get the circumhypersphere with centre C_1 (= O). Let its radius be R.

When $r = 2$, we get a hypersphere through the midpoints of the edges with centre C_2 at the midpoint of OH.

When $r = n-1$, we get a hypersphere through the centroids of the faces of the simplex. By inspection S_{n-1} is homothetic to S_1 having $G = C_{n/2}$ as centre of similitude, and so it has radius $R/(n-1)$. Also H is the external centre of similitude, and the midpoint of GH is C_n. Therefore S_n having GH as diameter is the hypersphere of similitude of S_1 and S_{n-1}. We notice that it has centre C_n, and that it passes through the centroid and the orthocentre of the n-point subset.

3. *Coaxal Properties.*

Assume that the S hyperspheres are coaxal for a simplex with $n-1$ vertices, and proceed by induction. For n vertices, we have a hypersphere S_r which meets each face in the hypersphere through the centroids of the r-point subsets of the vertices in that face. Therefore S_r meets each face in the corresponding hypersphere which we shall denote by S'_r. Taking just one face for the moment, the hyperspheres S'_1, S'_2, ..., S'_{n-1}, all of n-2 dimensions, are coaxal (induction hypothesis); therefore they have in common the points (real or imaginary) of a hypersphere Σ of n-3 dimensions. Now S_r contains S'_r for all $r < n$, therefore S_1, S_2, ..., S_{n-1} all contain Σ. Bringing in a second face, they all have other points in common as well; therefore they have in common a hypersphere of n-2 dimensions, and hence they are coaxal. But, as we have seen, S_n is the hypersphere of similitude of S_1 and S_{n-1}; therefore it is coaxal with these two and S_1, S_2, ..., S_n form a coaxal system.

To finish off the logical argument, we recall that, for $n = 3$, S_1, S_2, S_3 are known to be coaxal. This provides a base for the mathematical induction to follow.

4. *General Observations.*

The hyperspheres S have been generated by centroids of subsets. But, given any particular subset of r points and the $(r$-1)-dimensional subspace they determine, the intersection of S_r with this subspace is the corresponding hypersphere of similitude. Therefore S_r passes through not only the centroids but also the orthocentres of the r-point subsets of the n original vectors.

For all n, S_1 is the circumhypersphere and S_n the hypersphere of similitude. For $n = 3$, the intermediate S_2 is the nine-point circle. In a sense, for general n all intermediate hyperspheres S_2, S_3, ..., S_{n-1} are generalisations of the nine-point circle. A better observation is that all n hyperspheres belong on equal terms to one coaxal family (though who would have suspected the common roots to the circumcircle, the nine-point circle, and the circle of similitude of a triangle?). It may sound peculiar to refer to S_n as that hypersphere through the centroid of the n-point subset. However, uniqueness is restored if we recall that S_n passes also through the orthocentre H, and indeed has GH as diameter. Alternatively, S_n is the unique hypersphere of the coaxal system to pass through G.

Finally, it can be shown that an orthocentric simplex has a polar hypersphere which has centre H and is coaxal with the other hyperspheres. We may therefore reasonably give it the symbol S_∞.

Faculty of Mathematics, The Open University, Walton Hall, Milton Keynes, England MK7 6AA.

*   *   *

MAMA-THEMATICS

"Now, Felix", warned Frau Klein, "put down that bottle!"

"It's way past your bedtime, Maria Gaetana", complained Signora Agnesi, "and what can you be writing about at this witching hour?"

ALAN WAYNE

NINE-DIGIT PATTERNED PALINDROMIC PRIMES

CHARLES W. TRIGG

Printing all of the 5172 = 1·3·4·431 nine-digit palindromic primes in the restricted space of this publication is not practical. However, it is possible to show some sets of these primes with like characteristics as extracted from the print-out prepared by Jacques Sauvé on a PDP-11/45 computer at the University of Waterloo.

In an *undulating integer*, the alternate digits are consistently greater than or less than the digits adjacent to them, as in 526473. If only two distinct digits are present, as in 373737, the integer is *smoothly undulating* [1].

There are 41 palindromic primes of the form *abacacaba*, $a \neq b$, $a \neq c$. Seven of these are smoothly undulating, namely:

323232323	383838383	727272727	919191919	929292929
979797979	989898989			

The additional twenty-two that also undulate are:

121414121	121818121	141313141	141616141	151212151
161313161	171919171	181515181	191313191	343737343
363434363	707373707	727171727	737171737	747676747
757272757	909494909	919797919	959292959	969292969
989393989	989595989			

The twelve primes of this form that do not undulate are:

101414101	101616101	101717101	101919101	313434313
353232353	393232393	707979707	747979747	757878757
787171787	797676797			

Each of these palindromic primes with five like alternate digits is a concatenation of three three-digit palindromes. Thirteen of the external triads (101, 151, 181, 191, 313, 353, 383, 727, 757, 787, 797, 919, and 929) and five of the internal triads (191, 313, 373, 797, and 919) are themselves prime.

There are 384 palindromic primes of the form *abcbdbcba*, where $b \neq a,c,d$. Their frequencies of occurrence with various combinations of a and b are tabulated below.

Each of the four values of a begins about the same number of these primes. Because $b \neq a,c,d$, more of these primes have even b's than odd ones. Among these 384 primes are the 7 smoothly undulating primes previously mentioned, and 191 other undulating primes. In 28 of these, *acdca* smoothly undulates, for example, 14141 in 164616461. The entire set is given below the table on the next page.

a \ b	0	1	2	3	4	5	6	7	8	9	Totals
1	10	-	17	6	10	11	13	11	10	8	96
3	12	8	15	-	8	12	13	10	12	6	96
7	10	9	15	10	10	12	7	-	9	12	94
9	9	11	14	5	8	11	11	16	13	-	98
Totals	41	28	61	21	36	46	44	37	44	26	384

164616461	197919791	302030203	315131513	319131913
324232423	354535453	364636463	375737573	380838083
381838183	392939293	702070207	704070407	713171317
715171517	726272627	729272927	738373837	745474547
746474647	756575657	904090409	905090509	918191819
926292629	937393739	957595759		

Also, there are 14 nonundulating palindromic primes with four like interior alternate digits in which *acdca* smoothly undulates, namely:

126212621	127212721	138313831	178717871	346434643
356535653	357535753	732373237	750575057	954595459
962696269	975797579	983898389	987898789	

The distinct digits in two of the seven smoothly undulating primes given earlier are 2,3 and 8,9. In each of the primes

324232423 354535453 756575657 798989897 987898789

the three distinct digits are consecutive. In each of the primes

321242123 345424543 352545253 364656463 789868987

the four distinct digits are consecutive.

The distinct digits in one of the seven smoothly undulating primes are the consecutive odd digits 7 and 9. In each of the primes

315131513	357535753	357575753	375737573	375757573
735353537	753535357	957595759	975757579	975797579

the three distinct digits are consecutive odd digits. In each of the primes

173757371 357595753 957535759

the four distinct digits are consecutive odd digits.

REFERENCE

1. Charles W. Trigg, "Special Palindromic Primes", *Journal of Recreational Mathematics*, 4 (July 1971) 169-170.

2404 Loring Street, San Diego, California 92109.

* * *

NOTES ON NOTATION: III

LEROY F. MEYERS

Common among students and textbook writers is the following solution of an integration problem:

$$\int_3^5 (1+2x)dx = x+x^2\Big|_3^5 = 5+25-3-9 = 18.$$

The notation for substitution (between the first two equals signs) is, however, ambiguous. It indicates that 5 and 3 are to be substituted and the results subtracted. But must the result be 18? Couldn't it just as well be $x + 16$, or 0, or x? The trouble with the notation is that it does not clearly delimit the expression to be substituted *into*, and does not specify the letter to be substituted *for*. The first defect is often remedied by writing $(x+x^2)\Big|_3^5$, but the second defect is often left in.

I propose a return to a variant of a formerly common notation, namely to

$[F(x)]_{x:=a}^{b}$ as an abbreviation for $F(b) - F(a)$.

The square brackets surround the expression to be substituted into, and the "x:=" in the subscript specifies that the letter x is to be replaced[1]. It would be even better logically to write $[F(x)]_{x:=a}^{x:=b}$, but the convention is made that the same letter is replaced at the upper limit as at the lower limit, unless otherwise specified. (The use of the substitution sign ":=" rather than the equals sign "=" will be the subject of a further note.) The four interpretations of $x + x^2\Big|_3^5$ can now be distinguished notationally as

$$[x+x^2]_{x:=3}^{5}, \quad x+[x^2]_{x:=3}^{5}, \quad [x+x^2]_{t:=3}^{5}, \quad x+[x^2]_{t:=3}^{5}$$

(assuming that x is independent of t).

[1]To replace x by a is to substitute a for x. (En français: remplacer x par a; substituer a à x. Auf deutsch: x durch a ersetzen; a für x einsetzen.)

The notation can keep clear the substitutions needed in repeated integration, as in

$$\int_0^1\int_3^5 (1+2xy)\,dy\,dx = \int_0^1 [y+xy^2]_{y:=3}^5\,dx$$
$$= \int_0^1 (2+16x)\,dx$$
$$= [2x+8x^2]_{x:=0}^1$$
$$= 2+8 = 10,$$

as well as those needed in integration by parts, as in

$$\int_0^\pi (x\sin x)\,dx = [-x\cos x]_{x:=0}^\pi + \int_0^\pi (\cos x)\,dx$$
$$= \pi + [\sin x]_{x:=0}^\pi$$
$$= \pi.$$

(How often have you seen students write the answer as $-x\cos x$?) An alternate way of writing the last solution, which has advantages in dealing with improper integrals, is

$$\int_0^\pi (x\sin x)\,dx = [-x\cos x + \int(\cos x)\,dx]_{x:=0}^\pi$$
$$= [-x\cos x + \sin x]_{x:=0}^\pi$$
$$= \pi.$$

Dan Eustice showed me a beautiful fallacy which results from improper use of integration by parts:

$$\int_1^2 \frac{1}{x}dx = \frac{1}{x}\cdot x - \int_1^2 \left(-\frac{1}{x^2}\right)x\,dx = 1 + \int_1^2 \frac{1}{x}dx,$$

so that, by subtraction, $0 = 1$.

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.

* * *

THE OLYMPIAD CORNER: 26

M.S. KLAMKIN

Solutions to the problems posed at the Thirteenth Canadian Mathematics Olympiad (1981) are given below. They were prepared by the Olympiad Committee of the Canadian Mathematical Society.

THIRTEENTH CANADIAN MATHEMATICS OLYMPIAD

6 May 1981 - 3 hours

1. For any real number t, denote by $[t]$ the greatest integer which is less than or equal to t. For example: $[8] = 8$, $[\pi] = 3$ and $[-5/2] = -3$. Show that the following equation has no real solution:

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345.$$

Solution.

Let $f(x)$ denote the left member of the equation, and suppose there exists a real number x such that $f(x) = 12345$. Since $f(195) = 12285$, $f(196) = 12348$, and f is increasing, we must have $195 < x < 196$. Let $y = x - 195$. Then

$$f(y) = f(x) - f(195) = 12345 - 12285 = 60.$$

However, since $0 < y < 1$, so that $0 < ny < n$ and $[ny] \le n - 1$ for any positive integer n, we also have

$$f(y) \le 0 + 1 + 3 + 7 + 15 + 31 = 57,$$

a contradiction.

2. Given are a circle of radius r and a tangent line l to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to l with Q on l. Determine the maximum of the area of triangle PQR.

Solution.

Draw RS parallel to PQ with S on the circle. Since PRS is a triangle (possibly degenerate) inscribed in the circle, its maximum area is $3\sqrt{3}r^2/4$ when it is equilateral. Now the area of PQR is exactly half that of PRS and is therefore at most $3\sqrt{3}r^2/8$. This maximum is attained when angle PRQ is $\pi/6$.

3. Given a finite collection of lines in a plane P, show that it is possible to draw an arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, etc.) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

Solution.

For any finite family F of lines in the plane, we can choose a coordinate system such that no line is vertical and no two lines intersect in the half-plane $x \ge 0$. Let $y = ax + b$ be the line in F where b is maximum. The region

$$D = \{(x,y): x > 0,\ y > ax + b\}$$

can accommodate circles of arbitrary size. If any line in F intersects D, it must either intersect $x = 0$ at a point $(0,y)$ with $y > b$ or intersect $y = ax + b$ at a point (x,y) with $x \geq 0$, but neither case is possible.

On the other hand, $\{x=r : r \text{ rational}\}$ is a countable family of lines which intersects every circle in the plane.

4. $P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P(Q(x)) \equiv Q(P(x))$ for all real numbers x. If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P(P(x)) = Q(Q(x))$ also has no real solution.

Solution.

Since $P(x) = Q(x)$ has no real solution, we may assume that $P(x) > Q(x)$ for all real x. If $P(P(x)) = Q(Q(x))$ has a solution $x = a$, then we have

$$P(Q(a)) > Q(Q(a)) = P(P(a)) > Q(P(a)) = P(Q(a)),$$

a contradiction.

5. Eleven theatrical groups participated in a festival. Each day, some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival, each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last?

Solution.

Let $N = \{1,2,\ldots,n\}$ be the days of the festival and let each group be labelled with a subset $A \subset N$ where the group A performs on day x if and only if $x \in A$. In order for two groups A and B to watch each other perform at least once, we cannot have $A \subseteq B$ or $B \subseteq A$; in particular, $A \neq B$ and so different groups have different labels.

We first show that $n = 6$ is sufficient. One possible labelling for the 11 groups is $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$, and $\{6\}$.

We now show that $n = 5$ is insufficient. The label of each group would be a subset $A \subset \{1,2,3,4,5\}$ and clearly $1 \leq |A| \leq 4$. Define a *chain* to be a sequence $A_1 \subset A_2 \subset A_3 \subset A_4$ with $|A_i| = i$ for $1 \leq i \leq 4$. The number of chains is $5\cdot4\cdot3\cdot2 = 120$. A subset of size 1 or 4 appears in $4\cdot3\cdot2 = 24$ chains while a subset of size 2 or 3 appears in $2\cdot3\cdot2 = 12$ chains. Since there are 11 groups, their labels must appear at least $11\cdot12 = 132$ times in the 120 chains. By the pigeonhole principle, two of the labels, say A and B, appear in the same chain, violating the conditions $A \not\subset B$ and $B \not\subset A$.

*

I now give solutions to the problems posed at the 1981 Alberta High School Prize Examination in Mathematics. The problems and solutions were prepared by the Mathematics Department of the University of Alberta.

1. Show that the two equations

$$x^4 - x^3 + x^2 + 2x - 6 = 0$$

and

$$x^4 + x^3 + 3x^2 + 4x + 6 = 0$$

have a pair of complex roots in common.

Solution.

Let $f(x)$ and $g(x)$ denote the left members of the first and second equations, respectively. Then

$$g(x) + f(x) = 2(x^4+2x^2+3x) = 2x(x+1)(x^2-x+3)$$

and

$$g(x) - f(x) = 2(x^3+x^2+x+6) = 2(x+2)(x^2-x+3).$$

Thus

$$f(x) = (x^2-x+3)(x^2-2) \quad \text{and} \quad g(x) = (x^2-x+3)(x^2+2x+2),$$

so $f(x)$ and $g(x)$ have in common the roots of $x^2 - x + 3$, which are $\frac{1}{2}(1 \pm i\sqrt{11})$.

2. Trevor wrote down a four-digit number x, transferred the right-most digit to the extreme left to obtain a smaller four-digit number y, and then added the two numbers together to obtain a four-digit number z. The next day he was unable to find his calculations but remembered that the last three digits of z were 179. What was x?

[A four-digit number does not start with zero.]

Solution I.

Let the digits of x from left to right be a, b, c, and d. Then those of y are d, a, b, and c. Since $0 < d \le a$ and $a + d \le 9$, we must have $1 \le d \le 4$. If $d = 4$, then $c = 5$, $b = 2$, and $a = 9$, which contradicts $a + d \le 9$. If $d = 3$, then $c = 6$, $b = 1$, and $a = 0$, which contradicts $a > 0$. If $d = 2$, then $c = 7$, $b = 0$, and $a = 1$, which contradicts $a \ge d$. Hence $d = 1$ from which $c = 8$, $b = 9$, $a = 1$, and $x = 1981$.

Solution II.

With a,b,c,d as in solution I, we have

$$z = x + y = 11(100a + 10b + c + 91d)$$

and the missing digit of z is $1 - 7 + 9 = 3$. from $0 < d \le a$ and $a + d \le 3$, it follows that $d = 1$. Hence $c = 8$, $b = 9$, $a = 1$, and $x = 1981$.

3. A baseball league is made up of 20 teams. Each team plays at least once and there are no tie games. A team's average is defined to be its number of wins divided by its total number of games played.

(a) If each team played the same number of games, show that the sum of the averages of all the teams is 10.

(b) If each team did not play the same number of games, show that the sum of the averages of all the teams must be at least 1 and at most 19.

Solution.

Let w_i and l_i be, respectively, the number of wins and losses by team i, where $i = 1,2,\ldots,n$ and n is the number of teams. The ith team's average is then $a_i = w_i/(w_i + l_i)$.

(a) Suppose $w_i + l_i = M$ for all i. Then $\Sigma a_i = \Sigma w_i/M$, where all sums, here and later, are for $i = 1,2,\ldots,n$. Now $\Sigma w_i = \Sigma l_i = nM/2$, so $\Sigma a_i = n/2$. When $n = 20$, this means that the sum of the averages is 10.

(b) Suppose the number of games played by team i is $w_i + l_i = m_i$; then $\Sigma w_i = \Sigma l_i = \frac{1}{2}\Sigma m_i$. Let $m^* = \max\limits_{1 \le i \le n} m_i$. Then $m^* \le \frac{1}{2}\Sigma m_i$ and so

$$\Sigma a_i = \Sigma w_i/m_i \ge (\Sigma w_i)/m^* \ge 1.$$

Thus the sum of the averages is at least 1. Similarly, we find that $\Sigma l_i/m_i \ge 1$, and so

$$\Sigma a_i = \Sigma\{1 - (l_i/m_i)\} = n - \Sigma l_i/m_i \le n - 1.$$

For $n = 20$, this means that the sum of the averages is at most 19.

4. A farmer owns a fenced yard in the shape of a square 30 metres by 30 metres. He wishes to divide the yard into three parts of equal area, using 50 metres length of fencing. Find two different (i.e., noncongruent) ways that he can do this.

Solution.

A trivial partition is shown in Figure 1. Figure 2 shows a modification in

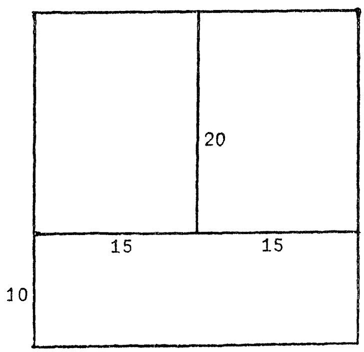


Figure 1

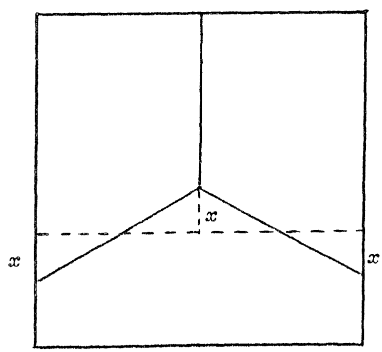


Figure 2

which the three regions are clearly of equal area. The total length of fencing in Figure 2 is

$$20 - x + 2\{(2x)^2 + 15^2\}^{\frac{1}{2}}$$

and we want this to be equal to 50. Solving this equation, we obtain $x = 0$, which corresponds to Figure 1, and $x = 4$. With $x = 4$, the total length of fencing in Figure 2 is $16 + 2\cdot 17 = 50$, as required.

5. Do *either* part (a) *or* part (b).

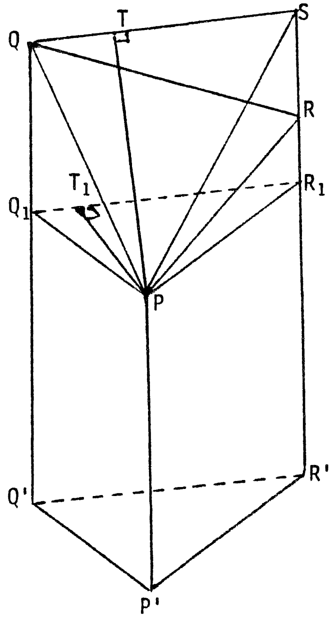
(a) If P'Q'R' is the parallel projection of a triangle PQR onto any plane, prove that the volumes of the two tetrahedra P'Q'R'P and PQRP' are the same.

(b) Prove that if $x \geq 0$, then

$$\left(\frac{x+1}{n+1}\right)^{n+1} \geq \left(\frac{x}{n}\right)^{n}, \qquad n = 1,2,3,\ldots .$$

(a) *Solution.*

Let the plane through P which is perpendicular to PP' meet QQ' and RR' in Q_1 and R_1, respectively, as shown in the figure. Choose S on RR' so that QS is parallel to Q_1R_1. Drop perpendiculars PT on QS and PT_1 on Q_1R_1, and let $\angle TPT_1 = \theta$. Then $\angle PT_1T$ is a right angle and $PT_1 = PT\cos\theta$. Hence, if vertical bars denote the area of a triangle (and, later, the volume of a tetrahedron), we have



$$|PQ_1R_1| = \tfrac{1}{2}Q_1R_1\cdot PT_1 = \tfrac{1}{2}QS\cdot PT\cos\theta = |PQS|\cos\theta.$$

Now tetrahedra PQRP' and PSQP' have the same base PQP' and equal altitudes to that base; hence

$$|PQRP'| = |PQSP'| = \frac{1}{3}|PQS|\cdot PP'\cos\theta = \frac{1}{3}|PQ_1R_1|\cdot PP'$$

$$= |PQ_1R_1P'|.$$

Similarly, $|P'Q'R'P| = |PQ_1R_1P'|$, so

$$|P'Q'R'P| = |PQRP'|,$$

as required.

(b) *Solution I.*

Let $f(x) = (x+1)^{n+1}/x^n$, $x > 0$. Then $f'(x) = (x-n)(x+1)^n/x^{n+1}$ is negative for $0 < x < n$ and positive for $x > n$. So $f(x)$ is a decreasing function of x for $0 < x < n$ and an increasing function of x for $x > n$. Since f is continuous it must then take on an absolute minimum at $x = n$. That is, for all $x > 0$,

$$\frac{(x+1)^{n+1}}{x^n} \geq \frac{(n+1)^{n+1}}{n^n} \quad \text{or} \quad \left(\frac{x+1}{n+1}\right)^{n+1} \geq \left(\frac{x}{n}\right)^n.$$

Solution II.

Let $a_1 = 1$ and $a_2 = a_3 = \ldots = a_{n+1} = x/n$. Then the arithmetic mean of the nonnegative numbers $a_1, a_2, \ldots, a_{n+1}$ is

$$\frac{1}{n+1}(a_1 + a_2 + \ldots + a_{n+1}) = \frac{x+1}{n+1}$$

while the geometric mean is

$$(a_1 a_2 \ldots a_{n+1})^{1/(n+1)} = \left(\frac{x}{n}\right)^{n/(n+1)}.$$

Since the arithmetic mean in always at least as large as the geometric mean, we have

$$\frac{x+1}{n+1} \geq \left(\frac{x}{n}\right)^{n/(n+1)}.$$

Raising both sides of this inequality to the power $n + 1$, we get

$$\left(\frac{x+1}{n+1}\right)^{n+1} \geq \left(\frac{x}{n}\right)^n .$$

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

645. *Proposed by Dmitry P. Mavlo, Moscow, U.S.S.R.*

Can the product of four successive terms of an arithmetic progression with rational terms be an exact fourth power?

646. *Proposed by J. Chris Fisher, University of Regina.*

Let M be the midpoint of a segment AB.

(a) What is the locus of a point the product of whose distances from A and B is the square of its distance from M; that is

$$\{X: |XA|\cdot|XB| = |XM|^2\}.$$

(b) In a circle γ through M and B, the three chords BP, BP', and BM satisfy

$$|BP| = |BP'| = \sqrt{2}|BM|.$$

Prove that the tangent to γ at M meets the lines BP and BP' (extended) in points X and Y, respectively, that are equidistant from M. Note that this fact suggests a construction for the locus of part (a), since X and Y satisfy

$$|XA|\cdot|XB| = |YA|\cdot|YB| = |XM|^2.$$

(c) What is the locus of a point for which the absolute value of the difference of its distances from A and B equals $\sqrt{2}$ times its distance from M; that is,

$$\{X: \|XA| - |XB\| = \sqrt{2}|XM|\}.$$

647. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A cake (a rectangular parallelepiped) with icing on the top and the four sides is to be cut (using vertical cuts only) and shared by *n* persons.

(a) If the top is square, show how to cut the cake so that each person gets the same amounts of cake and icing as everybody else.

(b) Do the same for the general case of a rectangular top.

(c)* In (b), is there a way to cut the cake so that each person's share is in one piece?

648. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given are a triangle ABC, its centroid G, and the pedal triangle PQR of its incenter I. The segments AI,BI,CI meet the incircle in U,V,W; and the segments AG,BG,CG meet the incircle in D,E,F. Let ∂ denote the perimeter of a triangle and consider the statement

$$\partial PQR \le \partial UVW \le \partial DEF.$$

(a) Prove the first inequality.

(b)* Prove the second inequality.

649. *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

The centroid and the circumcenter of a rectangle coincide. Are there other quadrangular laminae with this property?

650. *Proposed by Paul R. Beesack, Carleton University, Ottawa.*

(a) Two circular cylinders of radii r and R, where $0 < r \le R$, intersect at right angles (i.e., their central axes intersect at an angle of $\pi/2$). Find the arc length l of one of the two curves of intersection, as a definite integral.

(b) Do the same problem if the cylinders intersect at an angle γ, where $0 < \gamma < \pi/2$.

(c) Show that the arc length l in (a) satisfies

$$l \le 4r\int_0^{\pi/2} \sqrt{1+\cos^2\theta}\;d\theta < 5\pi r/2.$$

651. *Proposed by Charles W. Trigg, San Diego, California.*

It is June, the bridal month, and LOVE is busting out all over. So THEY obey the biblical injunction to go forth and multiply, resulting paradoxically in a cryptarithmic *addition* which you are asked to investigate with averted eyes.

Find out in how many ways

$$\begin{array}{r} \text{THEY} \\ \underline{\text{MADE}} \\ \text{LOVE} \end{array}$$

and in which way their LOVE was greatest.

652.* *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Let R, r, s represent respectively the circumradius, the inradius, and the semiperimeter of a triangle with angles α, β, γ. It is well known that

$$\Sigma\sin\alpha = \frac{s}{R},\quad \Sigma\cos\alpha = \frac{R+r}{R},\quad \Sigma\tan\alpha = \frac{2rs}{s^2-4R^2-4Rr-r^2}.$$

As for half angles, it is easy to prove that $\Sigma\tan(\alpha/2) = (4R+r)/s$. Find similar expressions for $\Sigma\cos(\alpha/2)$ and $\Sigma\sin(\alpha/2)$.

653. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

For every triangle ABC, show that

$$\Sigma\cos^2\frac{B-C}{2} \ge 24\Pi\sin\frac{A}{2},$$

where the sum and product are cyclic over A,B,C, with equality if and only if the triangle is equilateral.

654. *Proposed by Randall J. Covill, Maynard, Massachusetts.*

Suppose that some extraterrestrials have three hands and a total of thirteen fingers on each of the two sides of their symmetric bodies. Each hand has one or more fingers. How many different types of gloves are necessary to outfit those extraterrestrials?

655. *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.*

If $0 < a,b,c,d < 1$, prove that

$$(\Sigma a)^3 > 4bcd\Sigma a + 8a^2bcd\Sigma(1/a),$$

where the sums are cyclic over a,b,c,d.

656. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

P is an interior point of a convex region R bounded by the arcs of two intersecting circles C_1 and C_2. Construct through P a "chord" UV of R, with U on C_1 and V on C_2, such that $|PU|\cdot|PV|$ is a minimum.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

551. [1980: 183] *Proposed by Sidney Kravitz, Dover, New Jersey.*

Here is another alphametical Cook's tour for which no visas are needed:

```
 FRANCE
 GRFECE
FINLAND .
```

Solution by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.

It is at once evident that F = 1, I = 0, G = 8 or 9, and D is even. Beyond this, for lack of better insight into the relationships among the letters (or perhaps because of insufficient knowledge of the countries involved), I have had to resort to brute force, that is, to carry out in tedious slow motion what a computer could do in a flick of a bit. The work is conveniently set out in the adjoining tree, where all the branches except the last have had their growth inhibited by a duplication of digits. It shows that the unique solution is

```
 124578
 928878 .
1053456
```

F	I	E	D	C	N	A	L	R	G
1	0	2	4	3	6	8	*		
				8	6	9	*		
		3	6	2	4	7	*		
				4	8	*			
		4	8	3	6	*			
				6	2	7	*		
		6	2	3	7	*			
				4	9	5	*		
				7	5	*			
				8	7	4	*		
		7	4	2	5	*			
				6	3	*			
				8	*				
		8	6	2	5	3	*		
				3	7	5	4	*	
				7	5	4	3	2	9

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; JAMES BOWE, Erskine College, Due West, South Carolina;

CLAYTON W. DODGE, University of Maine at Orono; N. ESWARAN, Indian Institute of Technology, Kharagpur, India; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; TOMAS L. LLOYD, Technical High School of Linköping, Sweden; LAI LANE LUEY, Willowdale, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; HYMAN ROSEN, Cooper Union, New York, N.Y.; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

All solvers (including, presumably, the six who sent in only an answer) had to use a fair amount of brute force (which is most economically described in tabular form, as in our featured solution) to arrive at the unique answer. Some might say that therefore this problem is not suitable for thinking mathematicians, who can only solve it by acting as mindless computers. But the editor firmly believes (on no evidence whatsoever) that, in any alphametic that has a unique answer, there are relationships among the letters that, if the solver could only discern them, would enable him or her to eliminate most of the brute force by some mathematical finesse. If this is so, then it must be concluded that the solutions received, though all were correct, were all unsatisfactory. Mathematicians are expected to do more than just "flick their bit."

Nyon noted that the alphametic has no solution when FINLAND is replaced by IRELAND. No wonder. There is not even a solution for all the IRE in that LAND.

* * *

552. [1980: 183] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

Given positive constants a,b,c and nonnegative real variables x,y,z subject to the constraint $x + y + z = \pi$, find the maximum value of

$$f(x,y,z) \equiv a\cos x + b\cos y + c\cos z.$$

Comment by M.S. Klamkin, University of Alberta.

The result

$$a\cos x + b\cos y + c\cos z \le \frac{bc}{2a} + \frac{ca}{2b} + \frac{ab}{2c} \tag{1}$$

is already known (see [1] and the references therein) under the less restrictive condition $abc > 0$, the inequality being reversed if $abc < 0$, and equality occurring if and only if

$$a\sin x = b\sin y = c\sin z.$$

An equivalent form of (1) was given much earlier by Wolstenholme (see [2]).

More generally, the author has shown [2] that if A,B,C are the angles of a triangle, n is an integer, and x,y,z are real, then

$$x^2 + y^2 + z^2 \geq (-1)^{n+1}(2yz \cos nA + 2zx \cos nB + 2xy \cos nC), \qquad (2)$$

with equality if and only if

$$\frac{x}{\sin nA} = \frac{y}{\sin nB} = \frac{z}{\sin nC}.$$

The truth of (2) follows from its equivalence to

$$\{x + (-1)^n (y \cos nC + z \cos nB)\}^2 + (y \sin nC - z \sin nB)^2 \geq 0.$$

Finally, many known triangle inequalities can be deduced as special cases of (2). See [2] for a number of such examples.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; TERRY R. EVERSON, student, The Ohio State University; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, University of Wisconsin-Oshkosh; HYMAN ROSEN, Cooper Union, New York, N.Y.; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer. Three incorrect solutions were received.

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, pp. 23-24.

2. M.S. Klamkin, "Asymmetric Triangle Inequalities," *Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, Univ. Beograd, No. 357-380 (1971), pp. 33-44.

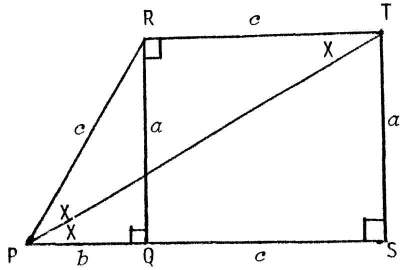
* * *

553. [1980: 183] *Proposed by Leroy F. Meyers, The Ohio State University.*

Let X be an angle strictly between 0° and 45°. Show that 2X is an acute angle of a Pythagorean triangle (a right triangle in which all sides have integer lengths) if and only if X is the smallest angle of a right triangle in which the legs (but not necessarily the hypotenuse) have integer lengths.

I. *Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

We have given $0 < X < 45°$, so 2X is an acute angle. In the adjoining figure, X is the smaller acute angle in right triangle PST and 2X is an acute angle in right triangle PQR. It is easy to see that the complete figure is uniquely determined if either triangle PQR or triangle PST is given.



Suppose triangle PQR is given, with sides of integer lengths a,b,c. Then X is the smallest angle of triangle PST (since $X < 45°$) and the legs have integer lengths a and $b + c$. (Here the length of the hypotenuse, $\sqrt{a^2+(b+c)^2}$, is not necessarily an integer.)

Conversely, suppose triangle PST is given, with legs of integer lengths a and $b + c$. In triangle PQR,

$$\begin{aligned} a^2 + b^2 = c^2 &\implies a^2 = (c+b)(c-b) \\ &\implies c-b \text{ is rational} \\ &\implies c = \{(c+b) + (c-b)\}/2 \text{ is rational} \\ &\implies b = (b+c)-c \text{ is rational.} \end{aligned}$$

Since a,b,c are all rational, 2X is an acute angle of a Pythagorean triangle (similar to triangle PQR).

II. *Solution by Terry R. Everson, student, The Ohio State University.*

Let X be the smallest angle of a nonisosceles right triangle with legs of integral lengths a and b, where $a > b$. Then $\tan X = b/a$ and

$$\tan 2X = \frac{2\tan X}{1-\tan^2 X} = \frac{2ab}{a^2-b^2},$$

so 2X is an acute angle of a Pythagorean triangle with sides

$$a^2-b^2, \quad 2ab, \quad a^2+b^2.$$

Conversely, let 2X be an acute angle of a Pythagorean triangle with legs of lengths a (opposite 2X) and b, and hypotenuse of length c. Then $\sin 2X = a/c$, $\cos 2X = b/c$, and

$$\tan X = \frac{\sin 2X}{1+\cos 2X} = \frac{a}{b+c},$$

so X is the smallest (since $a < c < b + c$) angle of a right triangle with legs of integral lengths a and $b + c$.

Also solved by LEON BANKOFF, Los Angeles, California; JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; ERNEST W. FOX, Marianopolis College, Montréal, Québec; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LAI LANE LUEY, Willowdale, Ontario; BENGT MÅNSSON, Lund, Sweden; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HYMAN ROSEN, Cooper Union, New York, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; ALAN WAYNE, Pasco-Hernando Community College, New Port Richey, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Nearly all solutions (including the proposer's) were at least partly trigonometric. The proposer had written: "This is expected to be a very elementary problem. Possibly someone will do this without trigonometry."

Professor Meyers, meet Professor Gibbs.

*　　　　*　　　　*

554. [1980: 183] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

A sequence of triangles $\{\Delta_0, \Delta_1, \Delta_2, \ldots\}$ is defined as follows: Δ_0 is a given triangle and, for each triangle Δ_n in the sequence, the vertices of Δ_{n+1} are the points of contact of the incircle of Δ_n with its sides. Prove that Δ_n "tends to" an equilateral triangle as $n \to \infty$.

Solution by Leroy F. Meyers, The Ohio State University.

For $n = 0,1,2,\ldots$, let the points of contact of the incircle of triangle $A_nB_nC_n$(otherwise known as Δ_n) with the sides be A_{n+1}, B_{n+1}, and C_{n+1}, where A_{n+1} is in the interior of the segment B_nC_n, etc.. Let the incenter of Δ_n be I_n.

Now two angles of quadrilateral $A_nB_{n+1}I_nC_{n+1}$ are right angles. Hence $\angle B_{n+1}I_nC_{n+1} = 180° - A_n$. Since I_n is the circumcenter of Δ_{n+1}, we also have

$$A_{n+1} = \tfrac{1}{2}\angle B_{n+1}I_nC_{n+1} = 90° - \tfrac{1}{2}A_n,$$

and so, by an easy induction,

$$A_{n+1} - 60° = -\tfrac{1}{2}(A_n - 60°) = \ldots = (-\tfrac{1}{2})^{n+1}(A_0-60°).$$

Hence $\lim\limits_{n\uparrow}(A_n - 60°) = 0$ or $\lim\limits_{n\uparrow}A_n = 60°$, and similarly $\lim\limits_{n\uparrow}B_n = 60°$. In other words, if n is large, then Δ_n is nearly equiangular, and so nearly equilateral.

But this does *not* show that $\lim\limits_{n\uparrow}\Delta_n$ is an equilateral triangle! In fact, as we will show, the sequence $(\Delta_0, \Delta_1, \Delta_2, \ldots)$ squeezes down to a point, hence to no triangle at all, and the conclusion of the proposal is therefore technically incorrect.

Suppose that the sides of Δ_n have lengths a_n, b_n, and c_n, and that the semiperimeter, inradius, and area are s_n, r_n, and K_n. Then $K_n = r_ns_n$ and we have $K_{n+1} = a_{n+1}b_{n+1}c_{n+1}/4r_n$ since r_n is the circumradius of Δ_{n+1}. The law of cosines applied to triangle $B_{n+1}I_nC_{n+1}$ gives

$$a_{n+1}^2 = 2r_n^2 + 2r_n^2\cos A_n = 4r_n^2\cos^2\tfrac{1}{2}A_n.$$

Hence

$$K_{n+1} = 2r_n^2(\cos\tfrac{1}{2}A_n)(\cos\tfrac{1}{2}B_n)(\cos\tfrac{1}{2}C_n)$$

and so

$$\frac{K_{n+1}}{K_n} = \frac{2r_n}{s_n}(\cos \tfrac{1}{2}A_n)(\cos \tfrac{1}{2}B_n)(\cos \tfrac{1}{2}C_n).$$

Now $2r_n < a_n$, etc., so that $r_n < \frac{1}{3}s_n$. Hence $K_{n+1}/K_n < 2/3$ if $n \geq 1$, and so $\lim_{n\uparrow} K_n = 0$. (For a near-equilateral triangle, K_{n+1}/K_n is near $\frac{1}{4}$.)

Also solved by LEON BANKOFF, Los Angeles, California; JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; TERRY R. EVERSON, student, The Ohio State University; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDRÉ LADOUCEUR, École secondaire De La Salle, Ottawa; LAI LANE LUEY, Willowdale, Ontario; BENGT MÅNSSON, Lund, Sweden; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

Editor's comment.

Bankoff revealed that an equivalent problem was proposed in 1956 and solutions by N.J. Fine and Bankoff were later published [1]. But *this* problem was carefully worded to have a correct conclusion:

Denote the sides and inradius of a triangle by a_0, b_0, c_0, *and* r_0. *The points of contact form a new triangle whose sides and inradius are* a_1, b_1, c_1, *and* r_1. *Repeating the process one obtains the sequences* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, *and* $\{r_n\}$. *Show that as* $n \to \infty$

$$\lim r_n/a_n = \lim r_n/b_n = \lim r_n/c_n = \sqrt{3}/6.$$

The problem editor made an interesting comment following the solutions of this problem: "The convergence of the angles A_n, B_n, C_n to $\pi/3$ is quite rapid. [W.B.] Carver remarked that if we start with *any triangle whatever,* all the angles of triangle $A_7B_7C_7$ will differ from $60°$ by less than $1°$."

REFERENCE

1. N.J. Fine and Leon Bankoff, solutions to Problem E 1233 (proposed by Joseph Andrushkiw), *American Mathematical Monthly,* 64 (1957) 274-275.

*　　　　*　　　　*

555. [1980: 184] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

An *n-persistent number* is an integer such that, when multiplied by $1,2,\ldots,n$, each product contains each of the ten digits at least once. (A *persistent number* is one that is n-persistent for each $n = 1,2,3,\ldots$. It is known [1979: 163] that there are no persistent numbers. The number

$$N = 526315789473684210$$

given there is n-persistent for $1 \le n \le 18$, but is not 19-persistent.)

(a) Find a 19-persistent number.

(b) Prove or disprove: for each n, there exist n-persistent numbers.

I. *Solution to part (b) by P. Erdős, Mathematical Institute of the Hungarian Academy of Sciences.*

I can prove that the density of n-persistent numbers is 1 for every n. To see this, observe that the number of integers less than or equal to y which do not contain one of the digits $0,1,\ldots,9$ is less than y^{1-c} for some $c > 0$. This is well known and easy to see. Thus the number of integers $m < x$ for which one of the integers tm, $m = 1,2,\ldots,n$, does not contain all the ten digits is less than $n(nx)^{1-c} = o(x)$.

II. *Solution by John T. Barsby, St. John's-Ravenscourt School, Winnipeg, Manitoba.*

The number N in the proposal is n-persistent for $1 \le n \le 18$. We show by induction that there are n-persistent numbers for all $n > 18$ and then illustrate our method by finding a 19-persistent number from the given 18-persistent number.

Suppose we have given a number that is n-persistent for some $n \ge 18$. Add a zero at the end unless the number already ends in zero. The resulting number, call it N, is still n-persistent. Form the product $M = (n+1)N$. If M contains all ten digits, then N is $(n+1)$-persistent and we are done. If not, form an integer Q by listing, in any order, the digits missing from M (they are all nonzero). If d is the number of digits in $n + 1$, form the number $R = Q\cdot 10^d + r$, where r is chosen so that $(n+1)|R$.

Now take N, *prefix* it with enough zeros to make the total number of digits the same as in M. To the left of those zeros place the integer $R/(n+1)$. The notation

$$P = [R/(n+1)][00\ldots00][N]$$

for the resulting concatenated integer should be easy to interpret. We claim that P is $(n+1)$-persistent. For if $1 \le k \le n$, then

$$kP = [kR/(n+1)][0\ldots0][kN],$$

and the tail $[kN]$ contains all ten digits since N is n-persistent; while

$$(n+1)P = [R][M]$$

also contains all ten digits since $[R]$ contains (at least) all the digits not in $[M]$.

To illustrate, with the 18-persistent number

$$N = 5263157894736842l0,$$

we have $M = 19N = 9999999999999999990$. We can take $Q = 12345678$ and

$$R = 1234567807 = 19\cdot 64977253.$$

The resulting 19-persistent number is $P = 6497725305263157894736842l0$.

III. *Partial solution by the proposer (revised by the editor).*

Our solution will require the use of prime reciprocals, excluding those of 2 and 5. These are, of course, periodic decimals. If the period of a prime reciprocal $1/p$ contains $p - 1$ digits, then p is said to be a *full-period prime*. There is strong evidence to the effect that there are infinitely many full-period primes, but the evidence is not yet quite conclusive. It is conjectured that approximately 3/8 of all primes have full periods [1]. It is well known that, if p is a full-period prime, then the periods of k/p, $k = 1,2,\ldots,p-1$, are the $p - 1$ cyclic permutations of the period of $1/p$. For this reason, full-period primes are also called *cyclic primes*. One other property of full-period primes will be crucial for our problem: *if $p > 10$ is a full-period prime, then each of the ten digits occurs at least once in the period of $1/p$ (and hence in each of its cyclic permutations).* To see this, note that $1/p < 1/10$ so, for any digit d, there is at least one multiple $k/p \in [d/10, (d+1)/10)$.

The *integer repetend* of a full-period prime p is $(10^p - 10)/p$. For example, 7 is a full-period prime since $1/7 = 0\cdot\overline{142857}$, and its integer repetend is $(10^7 - 10)/7 = 1428570$. It is clear that, if $(10^p - 10)/p$ contains all ten digits (as happens when $p > 10$), then so does $k(10^p - 10)/p$ for $k = 1,2,\ldots,p-1$. It follows that, if $p > 10$ is a full-period prime, then its integer repetend is a $(p-1)$-persistent number and hence n-persistent for $1 \le n \le p-1$.

The number N given in the proposal happens to be the integer repetend of the full-period prime 19, which explains why it is 18-persistent. To find a 19-persistent number, we need only take the next full-period prime, which is 23. Its integer repetend

$$4347826086956521739130$$

is 22-persistent, and hence also 19-persistent.

In general, to find an n-persistent number for a given n, we can take any full-period prime $p > \max\{n, 10\}$, assuming one exists. Its integer repetend $(10^p - 10)/p$ will be $(p-1)$-persistent, and hence n-persistent since $n \le p-1$.

Also solved by SIDNEY KRAVITZ, Dover, New Jersey (part (a) only); and ROBERT A. STUMP, Hopewell, Virginia. Two incorrect solutions were received.

Editor's comment.

Two unanswered questions seem worthy of consideration at this point. The first is, for a given n what is the smallest n-persistent number? It is clear that 1023456789 is the smallest 1-persistent number, and it happens to be also the smallest 2-persistent number. But what are the smallest 3-persistent, 4-persistent, ... numbers?

For a given n, one can always painstakingly build up an n-persistent number by some inductive process (as in our solution II, for example). But the resulting number is wastefully large, and wastefully long is the time required to generate it. So our second question is, for a given n how can one construct *directly* an n-persistent number? The proposer's method, using full-period primes, will be a satisfactory answer to this question as soon as it is known definitely that there are infinitely many such primes.

One of our incorrect solvers, call him Mr. X, thought he had "definitely settled" our second question. He was wrong, but his method has some promising aspects and can perhaps be refined to give a satisfactory answer.

Mr. X observed, correctly, that, for *any* prime $p > 10$, the length λ of the period of $1/p$ divides $p-1$ and that the periods of the multiples k/p, $k = 1,2,\ldots,p-1$, are all cyclic permutations of one of $(p-1)/\lambda$ distinct periods which collectively contain all ten digits. He illustrated this with $p = 13$. Here we have $1/13 = 0.\overline{076923}$ and $2/13 = 0.\overline{153846}$, so $\lambda = 6$ and the $12/6 = 2$ periods collectively contain all ten digits. Furthermore, the periods of all multiples $k/13$, $k = 1,2,\ldots,12$, are all cyclic permutations of one or the other of these two periods, as is easily verified. According to Mr. X, it is now obvious, is it not? that if we concatenate the "integer repetends" of these two periods, the resulting number

$$N = 076923153846$$

will be 12-persistent. (Alas and alack! It is not even 7-persistent! Not to speak of the forbidden initial zero.) And what is true of 13 is true of any prime $p > 10$, is it not? Thus, given any $n \ge 10$ one can from *any* prime $p > n$ generate in this way a number that is $(p-1)$-persistent and hence n-persistent. Thus ran his seat-of-the-pants argument. (Euripides, eh Mr. X?)

As we said earlier, the method has some promising aspects, and readers are invited to see if it can be refined and made correct. Perhaps even Mr. X can repair the damage. (Eumenides, Mr. X?)

In the meantime, let us console ourselves with a look into Yates [2], who has tabulated the period lengths of all primes (except 2 and 5, whose reciprocals are terminating decimals) up to and including 1370471, which is a total of 105000 primes. The largest full-period prime listed by Yates is 1370459. Thanks to the proposer's solution, we *know* that the integer

$$\frac{10^{1370459}-10}{1370459}$$

is a 1370458-persistent number! That should hold us for a while. At least until someone proves that there are infinitely many full-period primes (or else until the rip in Mr. X's pants is mended).

REFERENCES

1. Samuel Yates, "Full-Period Primes," *Journal of Recreational Mathematics*, 3 (1970) 221-225.

2. ————, *Prime Period Lengths*, published in 1975 by the author.

* * *

556*. [1980: 184] *Proposed by Paul Erdös, Mathematical Institute, Hungarian Academy of Sciences.*

Every baby knows that

$$\frac{(n+1)(n+2)\ldots(2n)}{n(n-1)\ldots2.1}$$

is an integer. Prove that for every k there is an integer n for which

$$\frac{(n+1)(n+2)\ldots(2n-k)}{n(n-1)\ldots(n-k+1)} \qquad (1)$$

is an integer. Furthermore, show that if (1) is an integer, then $k = o(n)$, that is, $k/n \to 0$.

Editor's comment.

No solution has been received for this problem, which therefore remains open. The proposer had subsequently written: "I will be glad to send you a solution of Problem 556 if no solution is sent in." If he reads this, will he please send us his solution? Other readers are invited to try and beat him to it.

* * *

557. [1980: 184] *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

For the geometric progression 2, 14, 98, 686, 4802, we have

$$(2+14+98+686+4802)(2-14+98-686+4802) = 2^2+14^2+98^2+686^2+4802^2.$$

Prove that infinitely many geometric progressions have this property.

I. *Solution de André Ladouceur, École secondaire De La Salle, Ottawa, Ontario.*

Notons $S(a,r,n)$ la somme d'une progression géométrique de n termes, de premier terme a et de raison r. Nous allons chercher dans quels cas la propriété

$$S(a,r,n,)\cdot S(a,-r,n) \;=\; S(a^2,r^2,n) \qquad (1)$$

est vérifiée. (La progression géométrique donnée correspond au cas $a = 2$, $r = 7$, $n = 5$.)

Si $a = 0$, cette propriété est évidemment vérifiée pour tout r et tout n. Elle est aussi vérifiée pour tout a et tout n si $r = 0$. Supposons donc que $a \neq 0$ et $r \neq 0$.

Si $r = \pm1$, la propriété (1) équivaut à

$$na\cdot\tfrac{1}{2}\{1-(-1)^n\}a = na^2,$$

qui est vérifiée si et seulement si n est impair.

Si $r \neq \pm 1$, la propriété (1) équivaut à

$$\frac{a(1-r^n)}{1-r}\cdot\frac{a\{1-(-r)^n\}}{1+r} = \frac{a^2(1-r^{2n})}{1-r^2}$$

ou à

$$1-(-r)^n = 1+r^n,$$

qui est aussi vérifiée si et seulement si n est impair.

Donc, outre les cas triviaux $a = 0$ et $r = 0$, la propriété désirée est vérifiée si et seulement si la progression donnée contient un nombre impair de termes.

II. *Comment by Hyman Rosen, Cooper Union, New York, N.Y.*

If $|r| < 1$, then [property (1)] holds even if $n = \infty$. For then it is equivalent to

$$\frac{a}{1-r}\cdot\frac{a}{1+r} = \frac{a^2}{1-r^2}\ .$$

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; CLAYTON W. DODGE, University of Maine at Orono; TERRY R. EVERSON, student, The Ohio State University; ERNEST W. FOX, Marianopolis College, Montréal, Québec; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; BOB PRIELIPP, University of Wisconsin-Oshkosh; HYMAN ROSEN, Cooper Union, New York, N.Y.; MATS RÖYTER, student, Chalmers University of Technology, Gothenburg, Sweden; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DONALD P. SKOW, Griffin & Brand, Inc., McAllen, Texas; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

*　　　　*　　　　*

558. [1980: 184] *Proposed by Andy Liu, University of Regina.*

(a) Find all n such that an $n\times n$ square can be tiled with L-tetrominoes.

(b) What if X-pentominoes are also available, in addition to L-tetrominoes?

(The term PENTOMINOES has been, since 15 April 1975, a registered trademark of Solomon W. Golomb.)

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

(a) I will prove somewhat more than the proposal requires, namely, that an $m\times n$ rectangle can be tiled with L-tetrominoes if and only if $m>1$, $n>1$, and $8|mn$. (Restricted to $n\times n$ squares, the condition becomes $4|n$.)

First, it is clear that if $m = 1$ or $n = 1$ not even one L-tetromino will fit, and that the rectangle cannot be tiled if $4\nmid mn$.

Next, we note that an L-tetromino is composed of two dominoes, one horizontal and one vertical, no matter how it is oriented. Thus, any rectangle which can be tiled with L-tetrominoes must be tilable with an equal number of horizontal and vertical dominoes. Now suppose that $mn = 8t + 4$. Then the number of dominoes will be $4t + 2$ and the number of horizontal as well as the number of vertical dominoes will be $2t + 1$, that is, odd. Let the rectangle be oriented so that the vertical dimension is even. Now no matter how the horizontal dominoes are placed, there will be at least two columns which have an odd number of uncovered squares, and these cannot be covered with the vertical dominoes.

Now suppose that $8|mn$. If both m and n are even, one of them is divisible by 4, and the rectangle can be tiled with 2×4 rectangles, each of which can be subdivided into two L-tetrominoes as shown in Figure 1. If one of m and n is odd, the other is divisible by 8, and the rectangle can be tiled with a combination of 2×4 and 3×8 rectangles. The 3×8 rectangles can be subdivided into six L-tetrominoes as shown in Figure 2.

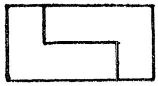


Figure 1

(b) Color the $n\times n$ square black and white, checkerboard-fashion. Since the X-pentomino covers one black square and four white, or one white square and four black, while the L-tetromino covers two black and two white, it is clear that the pentominoes must be used in pairs or not at all.

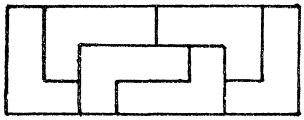


Figure 2

A necessary condition for the tiling is

$$n^2 = 4t + 5p, \qquad (1)$$

where t and p are the numbers of tetrominoes and pentominoes, respectively. If n is odd, it is clear from (1) that p must be odd. But we have just shown that p must be even. Thus there is no possible tiling if n is odd. If $n \equiv 0 \pmod 4$, the square can be tiled without any pentominoes, as shown in part (a). If $n \equiv 2 \pmod 4$, then n^2 is divisible by 4 and we see from (1) that the pentominoes must be used in groups of four. As a matter of fact, one group of four is sufficient. Figure 3 shows that a 6×6 square can be tiled with four pentominoes and four tetrominoes. If $n > 6$, the square can be subdivided into a 6×6 square, an $(n-6)\times(n-6)$ square, and two $(n-6)\times 6$ rectangles. All but the 6×6 square can be tiled as shown in part (a), and the 6×6 square is tiled as in Figure 3.

Thus all $n\times n$ squares for which n is even, except for the 2×2 square, can be tiled with L-tetrominoes and X-pentominoes.

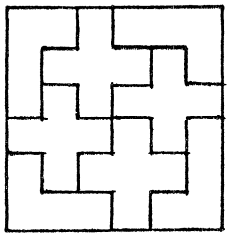


Figure 3

Also solved by the proposer.

* * *

559. [1980: 184] *Proposed by Charles W. Trigg, San Diego, California.*

Are there any positive integers k such that the expansion of 2^k in the decimal system terminates with k?

I. *Solution by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

For positive integers b and n, we will say that the positive integer k_n (allowing initial zeros) is *expomorphic relative to base* b if k_n has exactly n decimal digits and if

$$b^{k_n} \equiv k_n \pmod{10^n},$$

that is, if b^{k_n} terminates in k_n. The results given in Table 1 below appear to confirm the truth of the following theorem, for which we offer no proof:

THEOREM. If k_n *is expomorphic relative to base* b *and if*

$$b^{k_n} \equiv k_{n+1} \pmod{10^{n+1}},$$

then k_{n+1} *is also expomorphic relative to base* b.

If true, this theorem provides an easy way of determining successively an infinite sequence of expomorphic integers relative to a given base as soon as one of them has been found: if k_n is expomorphic relative to base b, then the next one in the sequence, k_{n+1}, consists of the last $n+1$ digits of b^{k_n}.

It is easy to see that there are no expomorphic integers relative to base 10, and there appear to be none relative to base 8. Table 1 exhibits expomorphic integers k_n for $1 \le n \le 5$ relative to the remaining bases $b \le 11$.

Table 1

$n = 1$		$3^{7} = \ldots(8)7$	$4^{6} = \ldots(9)6$	$5^{5} = \ldots(2)5$
$n = 2$	$2^{36} = \ldots(7)36$	$3^{87} = \ldots(3)87$	$4^{96} = \ldots(8)96$	$5^{25} = \ldots(1)25$
$n = 3$	$2^{736} = \ldots(8)736$	$3^{387} = \ldots(5)387$	$4^{896} = \ldots(8)896$	$5^{125} = \ldots(3)125$
$n = 4$	$2^{8736} = \ldots(4)8736$	$3^{5387} = \ldots(9)5387$	$4^{8896} = \ldots(2)8896$	$5^{3125} = \ldots(0)3125$
$n = 5$	$2^{48736} = \ldots48736$	$3^{95387} = \ldots95387$	$4^{28896} = \ldots28896$	$5^{03125} = \ldots03125$

$n = 1$	$6^{6} = \ldots(5)6$	$7^{3} = \ldots(4)3$	$9^{9} = \ldots(8)9$	$11^{1} = (1)1$
$n = 2$	$6^{56} = \ldots(6)56$	$7^{43} = \ldots(3)43$	$9^{89} = \ldots(2)89$	$11^{11} = \ldots(6)11$
$n = 3$	$6^{656} = \ldots(8)656$	$7^{343} = \ldots(2)343$	$9^{289} = \ldots(5)289$	$11^{611} = \ldots(6)611$
$n = 4$	$6^{8656} = \ldots(3)8656$	$7^{2343} = \ldots(7)2343$	$9^{5289} = \ldots(4)5289$	$11^{6611} = \ldots(6)6611$
$n = 5$	$6^{38656} = \ldots38656$	$7^{72343} = \ldots72343$	$9^{45289} = \ldots45289$	$11^{66611} = \ldots66611$

The method used in generating k_{n+1} from k_n is the same as that used in generating *odd* automorphic numbers. It will be recalled that a positive integer is *automorphic* if its square terminates in the integer. More precisely, in the decimal system, the n-digit positive integer a_n (initial zeros allowed) is automorphic if $a_n^2 \equiv a_n \pmod{10^n}$. (See [1] and [2] for an extensive discussion of automorphic numbers.) For each $n = 1,2,3,\ldots$, there is exactly one even automorphic number E_n and one odd automorphic number O_n (excluding trivial solutions such as 1, 01, 001, ...), and $E_n + O_n = 10^n + 1$. [In this connection, see Crux 596 [1981:18]. (Editor)]. The first six pairs of automorphic numbers are listed in Table 2.

Table 2

n	E_n	O_n	O_n^2
1	6	5	(2)5
2	76	25	(6)25
3	376	625	39(0)625
4	9376	0625	3(9)0625
5	09376	90625	8212(8)90625
6	109376	890625	79321(2)890625

II. *Adapted from part of the solution by Kenneth M. Wilke, Topeka, Kansas.*

The discussion here will relate only to expomorphic integers relative to the base $b = 2$, as in the proposal.

If ϕ is the Euler function and the integers x and n satisfy $x \geq n \geq 1$, then

$$2^{x+\phi(5^n)} \equiv 2^x \pmod{10^n}. \tag{1}$$

This follows from

$$2^x(2^{\phi(5^n)} - 1) \equiv 0 \pmod{10^n}, \tag{2}$$

which is true because $2^n | 2^x$ and $2^{\phi(5^n)} - 1 \equiv 0 \pmod{5^n}$ by Euler's generalization of Fermat's Theorem.

If we put $x = \phi(5^n)$ in (1), it follows that $2^{\phi(5^n)}$ is a solution of the congruence $a^2 \equiv a \pmod{10^n}$, and so

$$2^{\phi(5^n)} \equiv E_n \pmod{10^n}, \tag{3}$$

where E_n is the n-digit even automorphic number. If k_n is the n-digit expomorphic integer relative to base $b = 2$, so that $2^{k_n} \equiv k_n \pmod{10^n}$, it follows from (2) and (3) that

$$k_n(E_n - 1) \equiv 0 \pmod{10^n}.$$

This relation reduces the work involved in finding k_n when k_{n-1} is known, provided E_n is also known.

At my request, Harold Hladky (who now works for the Ebonite Corporation in Louisville, Kentucky) prepared a computer program based on the more direct approach of using for k_{n+1} the last $n + 1$ digits of 2^{k_n}. The [enclosed] print-out gives all values of k_n for $2 \le n \le 25$. It shows, in particular, that

$$2^{3338098615075353432948736} \quad \text{ends in} \quad 3338098615075353432948736.$$

You'd better believe it.

Also solved by LEON BANKOFF, Los Angeles, California; JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; W.J. BLUNDON, Memorial University of Newfoundland; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; SIDNEY KRAVITZ, Dover, New Jersey; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; JACQUES SAUVÉ, University of Waterloo; ROBERT A. STUMP, Hopewell, Virginia; STEPHEN WISMATH, University of Lethbridge, Alberta; and the proposer. One incorrect solution was received.

Editor's comment.

"There appear to be [no expomorphic integers] relative to base 8?" Izzatso? Well, unless the editor's abacus has gone haywire, it "appears" that

$$8^{56} = \ldots(8)56, \quad 8^{856} = \ldots(5)856, \quad 8^{5856} = \ldots(2)5856, \quad 8^{25856} = \ldots25856$$

and, to rub salt into the wound, that

$$88^{16} = \ldots(6)16, \quad 88^{616} = \ldots(1)616, \quad 88^{1616} = \ldots(5)1616, \quad 88^{51616} = \ldots51616.$$

The special case for $b=2$ of the highly plausible theorem in solution I was proved by Meyers and Wilke. They (and other readers) are invited now to prove the theorem in its full generality. They might at the same time characterize the integers b relative to which there *are* expomorphic integers (probably $b \not\equiv 0 \pmod{10}$). The editor will be glad to reopen the problem when a satisfactory proof has been received.

Incidentally, the number of digits in the integer mentioned at the end of solution II is exactly

$$1\,004\,867\,871\,682\,025\,871\,722\,839.$$

You'd better believe it.

REFERENCES

1. Vernon De Guerre and R.A. Fairbairn, "Automorphic Numbers," *Journal of Mathematical Recreations,* July 1968, pp. 173-179.

2. R.A. Fairbairn, "More on Automorphic Numbers," *Journal of Mathematical Recreations,* July 1969, pp. 170-174.

* * *