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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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A MATHEMATICAL JOURNEY

Andy Liu

In the summer of 1993, Matthew Wong of Old Scona Academic High School in Edmonton, and Daniel van Vliet of Salisbury Composite High School in Sherwood Park, were invited to attend an International Mathematics Tournament of the Towns Conference in Beloretsk, Russia, along with me. It was chaired by Professor Nikolay Konstantinov, President of the Tournament and recent winner of the Paul Erdős Award from the World Federation of National Mathematics Competitions. Professor Nikolay Vasiliev, who chairs the Problem Committee of the Tournament, was also present.

There were 60 participants in all. The 15 people from the West consisted of 1 Englishman, 2 Austrians, 3 Canadians, 4 Germans and 5 Colombians. Apart from Professor Gottfried Perz of Graz, Professor Juan Camilo Gomez and Professor Juan Carlos Vera of Bogotá, and me, the others were high school students or recent graduates. Among those from the former Eastern Block were some Bulgarians, Armenians and Estonians.

Beloretsk is in the Bashkirian Republic of Russia. It is just west of the Ural Mountains and north east of the Caspian Sea. The train ride from Moscow takes 36 hours each way. The time difference from Edmonton is 12 hours. So we had come literally to the other side of the world.

It is quite hot in Beloretsk on an August day, but comfortably cool in the morning and the evening. The population is about 75000, spread over quite a large rural area. It is not uncommon to be followed by chickens and sheep while walking on the streets. The metallurgical industry, founded in 1762, gives the local economy a big boost, but the air and water are refreshingly clean.

The Beloretsk Computer School which hosted the Conference is at the edge of the town. It consists of the original school building and a new five-floor dormitory. The three of us shared a spacious, comfortable and well-furnished room. We were next to the Austrians, with whom we shared a sink, a toilet, a shower and a refrigerator. The landscape around the school is very picturesque. A nearby river was a favourite spot for swimming, and the site of the traditional Russian tea-party by the bonfire one evening.

Our daily routine was roughly as follows. Breakfast was at 9 in the morning. From 10 to 12, there was usually a Mathematical Education Forum. From 12 to 2 was a Problem Session. Lunch was at 2. From 3 to 5 ran another Problem Session. Supper was at 7, and occasionally another Mathematics Education Forum ran from 8 to 10.

The Conference began officially on August 1, even though our appetites had already been whetted by a problem set distributed on the train. During the first few Problem Sessions, four problems were presented to the students as well as the teachers. The proposers provided some relevant background information. This was done in Russian, with adequate translation into English, which all 15 people from the West understood.

The Mathematical Education Forums being for the teachers only, the students could devote all their time to problem-solving, other than sleeping and eating. The food was good. They could work in their own rooms, in classrooms, on the meadows, or wherever they chose. They could work in teams. The English student joined the Canadians.

There were many social activities. Besides going to the river, we had a trip to the mountains, a soccer match, a table-tennis tourney and a nightly party, with cartoons on video, in Cafe 502 on the top floor of the dormitory.

The students had only until 10 o'clock in the evening of August 3 to solve the problems. Starting from August 4, solutions to those parts of the problems which had been solved were presented, along with a more challenging fifth problem.

The final deadline was at 10 o'clock in the evening of August 7. All solutions, as far as they were known to the proposers, were presented on August 8. During that afternoon, the participants were presented with diplomas, with very detailed descriptions of what they had accomplished, and whether their efforts were solo ones or in collaboration. I was most impressed with the meticulous care with which the jury had graded the students' work.

The Anglo-Canadian team did not win any prizes. We probably spent too much time socializing with the Russian students. However, I felt that this was just as important an aspect of our trip as working on the problems. The three of them did get some work done, and the jury commended them for formulating a generalization of one of the problems and making partial progress towards its solution. Matthew and Daniel continued to work on the problems even after their return.

On August 9, the last day of the Conference, solutions to the more difficult problems in this year's Tournament of the Towns were presented. Then we bade farewell to Professor V.G. Khazankin, principal of the Beloretsk Computer School, and other friends. They included Mother Khazankin, Ilia, the eight-year-old son of one of the teachers, and Alexei. He is eighty-three, a most interesting man who has collected lots of minerals and folklore from the Ural region.

We spent one night in Moscow on the way into, and two more on the way out of Russia. We stayed with Moscow mathematicians, who moved their families out temporarily so that we could have their apartments to ourselves. They are a very dedicated group. Besides running the Tournament, they organize the Independent University of Moscow, which keeps alive the famed tradition of the Moscow Mathematics Circles, without official recognition or financial support.

It was a wonderful experience, living in actual Russian homes. In the little time we had, we managed to get quite a bit of sightseeing done. We had an acute sense of the changing social fabric at a very exciting time in the history of a nation which not many have had the privilege to observe first hand.

On our way back, we attended the 13th International Puzzle Party in Breukelen, The Netherlands. It is an annual gathering of puzzle designers and collectors. There we met again Professor Tibor Szentiványi whom we visited in Budapest on our way to Russia, and Professor David Singmaster of England whom we later visited in London before flying home. The visit to The Netherlands was especially exciting for Daniel, who is of Dutch heritage. We stayed in Oosterhout with Professor Jan van de Craats, who has a most successful mathematics game show on television called "O! Zit dat zo".

All in all, it was a trip that has left a lasting impression on each of us, mathematically and otherwise.

BELORETSK PROBLEMS

The following are three of the five problems proposed in the Conference, and two of the five problems distributed on the train. They are reconstructed from my notes and are not the exact formulations as were presented. I invite the readers to send me nice solutions to these problems.

PROBLEM 1

Proposer: *Prof. A.A. Egorov.*
Top Prize Winner: *V. Zamjatin, high school student, Kirov, Russia.*
Other Prize Winners: *A. Barkhudarian and V. Poladian, high school students, Yerevan, Armenia; A. Bufetov, high school student from Moscow, Russia.*

Consider the following diophantine equation in x and y:

$$x^2 + (x+1)^2 + \cdots + (x+n-1)^2 = y^2, \qquad (*)$$

where n is a given positive integer.

If (*) has infinitely many solutions, we say that n is infinitely good.

(a) Prove that 2, 11, 24 and 26 are infinitely good.

(b) Prove that there are infinitely many infinitely good positive integers.

If (*) has at least one solution with $x > 0$, we say that n is very good.

(c) Prove that an infinitely good positive integer is very good.

(d) Prove that a positive integer which is very good but not infinitely good cannot be even.

(e) Prove that 49 is very good but not infinitely good.

(f) Prove that there are infinitely many positive integers which are very good but not infinitely good.

If (*) has at least one solution, we say that n is good.

(g) Prove that 25 is good but not very good.

(h) Prove that there are no other positive integers which are good but not very good.

If (*) has no solutions, we say that n is bad.

(i) Prove that 3, 4, 5, 6, 7, 8, 9 and 10 are bad.

(j) Prove that there are infinitely many bad positive integers.

(k) Devise an efficient algorithm which classifies a given positive integer as infinitely good, very good but not infinitely good, good but not very good, or bad.

PROBLEM 2

Proposer: *K.A. Knop.*

Top Prize Winners: *I. Buchkina and D. Schwarz, high school students, Moscow, Russia.*

Other Prize Winners: *Oleg Popov, high school student, Moscow, Russia; E. Tsyganov and V. Kartak, university students, Beloretsk, Russia.*

In Russia, there are 1, 2, 3, 5, 10, 15, 20 and 50 kopeck coins. To make up an integral amount, we take at every stage the largest coin not exceeding the remaining part of the amount. This method is called the Greedy Algorithm. For example, to make up 29 kopecks, the Algorithm yields $29 = 20 + 5 + 3 + 1$.

A general coinage system consists of m coins of respective integral values $1 = a_1 < a_2 < \cdots < a_m$. It is said to be suitable if for any integral amount, the number of coins used in the Greedy Algorithm is minimum.

(a) Prove that the Russian system is suitable.

(b) A new k kopeck coin is to be introduced into the Russian system. Determine all values of k for which the new system remains suitable.

(c) Prove that a general coinage system is suitable if a_{k+1} is divisible by a_k for $1 \le k \le m-1$.

(d) Prove that a general coinage system is suitable if $a_{k+1} - a_k$ is constant for $1 \le k \le m-1$.

(e) Devise an efficient algorithm for testing whether a given coinage system is suitable.

In a general coinage system S which is not necessarily suitable, denote by $f(S,k)$ the smallest number of coins required to make up the integral amount k. Denote by $g(S,n)$ the largest integral value such that $f(S,k) \le n$ whenever $k \le g(S,n)$, and by $g(m,n)$ the maximum value of $g(S,n)$ taken over all systems with m coins.

(f) Prove that $g(S,3) = 28$ if S is the Russian coinage system.

(g) Determine $g(m,n)$ for specific values of m and n, or obtain upper and lower bounds.

Suppose we are only interested in making up integral amounts up to and including 100, but we wish to do so in as efficient a way as possible.

(h) Determine the minimum values of $m \cdot \max\{f(S,k) : 1 \le k \le 100\}$ taken over all coinage systems S, where m is the number of coins in S.

(i) Determine the minimum value of $m(f(S,1) + f(S,2) + \cdots + f(S,100))$ taken over all coinage systems S, where m is the number of coins in S.

PROBLEM 3

Proposer: *Prof. N. Vasiliev.*

Top Prize Winners: *K. Wehrheim, T. Hauschildt, J. Wehrheim and M. Weyer, high school students, Hamburg, Germany.*

Other Prize Winners: *Yu. Belous, university student, Ekaterinburg, Russia; P. Volkov and G. Skorik, high school students, Nizhnij Tagil, Russia.*

A partition of a convex polygon into at least two triangles is called an anti-triangulation if whenever two of the triangles share a common segment, this segment is not a complete side of at least one of the two triangles.

(a) Determine all integers $k > 1$ such that there exists an anti-triangulation of a triangle into k triangles.

(b) Prove that no anti-triangulations exist for a convex polygon which is not a triangle.

(c) Generalize the result to partitions into convex n-gons not sharing common sides for $n > 3$.

(d) Generalize the result to partitions of a convex polyhedron into tetrahedra not sharing common faces or not sharing common sides.

TRAINING PROBLEM 1.

Let n be a positive integer. Each cell of a 2^n by n board is painted red or blue such that no two rows have the same colouring pattern. Since all possible colouring patterns must appear in the rows, in particular each column will have the same number of red cells as blue cells. Now suppose that some arbitrarily chosen cells are repainted white. Prove that it is then always possible to delete a number of rows, so that in the reduced board, each column again has the same number of red cells as blue cells.

TRAINING PROBLEM 2.

Let n be a positive integer. What is the largest size of a set of children such that for any subset of them, neither the total age nor the total I.Q. is divisible by n?

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THE OLYMPIAD CORNER

No. 151

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year, and another volume of *Crux Mathematicorum* begins. I hope that our facility with LaTeX is improving — I certainly want to thank Joanne Longworth whose skill at reading my writing and transforming it all into an attractive presentation is appreciated. Last year at least two errors crept by me which were pounced upon by our faithful and vigilant readers. I hope I do better in 1994.

It is also time to thank those who have contributed problem sets, solutions, comments and corrections. Without our readers' input the Corner would be a poorer effort, and much less fun to write. Among the contributors whose efforts were mentioned last year were:

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Thank you all (and anyone I left out).

*   *   *

As an Olympiad level competition we give the Japan Mathematical Olympiad 1992. Thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for collecting this contest and forwarding it to us.

1992 JAPAN MATHEMATICAL OLYMPIAD

1. Let x and y be relatively prime numbers with $xy > 1$, and let n be a positive even number. Prove that $x^n + y^n$ is not divisible by $x + y$.

2. In a triangle ABC with the area $= 1$ let D, E be points on AB, AC, and let P be the intersection of BE and CD. Determine the maximum area of ΔPDE under the condition

$$(\text{area of quadrilateral } BCED) = 2\times\ (\text{area of } \Delta PBC).$$

3. Prove the inequality

$$\sum_{k=1}^{n-1} \frac{n}{n-k} \cdot \frac{1}{2k-1} < 4. \qquad (n \geq 2)$$

4. Suppose that A is an (m, n)-matrix which satisfies the following conditions:

(1) $m \leq n$;

(2) each element of A is 0 or 1;

(3) if f is an injection from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$, then the $(i, f(i))$-element of A is zero for some i, $1 \leq i \leq m$.

Prove that there exist sets $S \subseteq \{1, \ldots, m\}$ and $T \subseteq \{1, \ldots, n\}$ which satisfy

i) the (i, j)-element is zero for any $i \in S$ and $j \in T$;

ii) $\#(S) + \#(T) > n$.

5. Let a_1, a_2, a_3, a_4 and n be positive integers such that

$$a_i \text{ is relatively prime to } n, \quad i = 1, 2, 3, 4,$$

$$(ka_1)_n + (ka_2)_n + (ka_3)_n + (ka_4)_n = 2n, \quad k = 1, 2, \ldots, n-1.$$

Prove that $(a_1)_n + (a_j)_n = n$ for some $2 \leq j \leq 4$. Here $(a)_n = a - n[a/n]$.

*　　　*　　　*

Next we conclude the solutions submitted by readers to the problems proposed to the jury but not used at the 32nd I.M.O. at Sigtuna, Sweden [1992: 226–227].

21. *Proposed by Czechoslovakia.*

Let $n \geq 2$ be a natural number and let the real numbers $p, a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_n$ satisfy $1/2 \leq p \leq 1$, $0 \leq a_i$, $0 \leq b_i \leq p$ $(i = 1, \ldots, n)$, and $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. Prove the inequality

$$\sum_{i=1}^{n} b_i \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j \leq \frac{p}{(n-1)^{n-1}} .$$

Solution by George Evagelopoulos, Athens, Greece.

Without loss of generality we can assume that $b_1 \geq b_2 \geq \cdots \geq b_n$. Denote by A_i the product $a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n$. If for $i < j$ also $A_i < A_j$, then $b_i A_i + b_j A_j \leq b_i A_j + b_j A_i$.

Consequently the sum $\sum b_iA_i$ does not become smaller when we rearrange the numbers $a_1, \ldots, a_n$, so that $a_1 \le a_2 \le \cdots \le a_n$ and hence also $A_1 \ge A_2 \ge \cdots \ge A_n$, which we now assume. Then the sum $\sum b_iA_i$ with fixed a_i's is maximal given $\sum b_i = 1$, when b_1 takes the largest possible value, i.e. $b_1 = p$, and b_2 takes the next value, i.e. $b_2 = 1 - p$, whereas $b_3 = \cdots = b_n = 0$. In this case

$$\sum b_iA_i = pA_1 + (1-p)A_2 = a_3 \ldots a_n(pa_2 + (1-p)a_1) \le a_3 \ldots a_n(a_1 + a_2)p$$

since $1 - p \le p$.

Using the inequality for the geometric and arithmetic means of the $n-1$ numbers $a_3, \ldots, a_n$, $a_1 + a_2$, we get

$$a_3 \ldots a_n(a_1 + a_2) \le \frac{1}{(n-1)^{n-1}}$$

which proves the inequality.

22. *Proposed by Poland.*

Determine the maximum value of the sum

$$\sum_{i<j} x_ix_j(x_i + x_j)$$

over all n-tuples $(x_1, \ldots, x_n)$ satisfying $x_i \ge 0$ and $\sum_{i=1}^n x_i = 1$.

Solutions by Seung-Jin Bang, Albany, California; and by George Evagelopoulos, Athens, Greece. We give Bang's solution.

Let $f(x_1, \ldots, x_n)$ be the given sum. Then

$$2f(x_1, \ldots, x_n) + \sum_{k=1}^n x_k^2(x_k + x_k) = \sum_{i,j=1}^n x_ix_j(x_i + x_j)$$
$$= \sum_{i=1}^n x_i^2\left(\sum_{j=1}^n x_j\right) + \sum_{j=1}^n x_j^2\left(\sum_{i=1}^n x_i\right) = 2\sum_{k=1}^n x_k^2.$$

Thus $f(x_1, \ldots, x_n) = \sum_{k=1}^n x_k^2(1 - x_k)$. Since $x(1-x) \le 1/4$ for all x in the unit interval we have

$$f(x_1, x_2, \ldots, x_n) \le \sum_{k=1}^n x_k\left(\frac{1}{4}\right) = \frac{1}{4}.$$

Finally $f(1/2, 1/2, 0, \ldots, 0) = 1/4$ implies that $1/4$ is the required maximum value of the sum.

23. *Proposed by Finland.*

We call a set S on the real line $\mathbb{R}$ *super-invariant*, if for any stretching A of the set by the transformation taking x to $A(x) = x_0 + a(x - x_0)$ there exists a translation B, $B(x) = x + b$, such that the images of S under A and B agree; i.e., for any $x \in S$ there is a $y \in S$ such that $A(x) = B(y)$, and for any $t \in S$ there is an $u \in S$ such that $B(t) = A(u)$. Determine all super-invariant sets.

Solutions by Seung-Jin Bang, Albany, California; and by George Evagelopoulos, Athens, Greece. We give the solution of Evagelopoulos.

We assume that "stretching" entails $a \geq 0$.

One easily observes that sets consisting of one point, the complement of one point, half-lines $\{x \in \mathbb{R} : x \geq c\}$ and $\{x \in \mathbb{R} : x \leq c\}$ and their complements as well as ϕ and $\mathbb{R}$ are super-invariant. To show that these are the only possible ones, we first observe that S is super-invariant if and only if to every $a > 0$ there exists at least one b such that for all $x \in \mathbb{R}$, $x \in S$ is equivalent to $ax + b \in S$. There are two possibilities: either there is a unique $b = f(a)$ associated with each $a > 0$, or for some $a_1 \neq 0$ there are at least two b's, say b_1 and b_2, such that $x \in S \Leftrightarrow y_1 = a_1x + b_1 \in S$ and $x \in S \Leftrightarrow y_2 = a_1x + b_2 \in S$. The latter situation implies that S is periodic, i.e.

$$y \in S \Leftrightarrow \frac{y - b_1}{a_1} \in S \Leftrightarrow y + (b_2 - b_1) \in S.$$

Since S is identical to a translate of any stretching of S, for each $r \geq 0$ we have $y \in S \Leftrightarrow y + r \in S$. Such an S must necessarily equal $\mathbb{R}$.

Assume then that b is a function of a, $b = f(a)$. Then for any a_1 and a_2

$$s \in S \Leftrightarrow a_1x + f(a_1) \in S \Leftrightarrow a_1a_2x + a_2f(a_1) + f(a_2) \in S,$$

and

$$x \in S \Leftrightarrow a_2x + f(a_2) \in S \Leftrightarrow a_1a_2x + a_1f(a_2) + f(a_1) \in S.$$

Assuming there is $x > -c$ with $x \in S$, one can solve $y = ax + c(a-1)$ for $y > -c$, obtaining $a > 0$, and thus show that $S \supset (-c, \infty)$. From this it is easy to deduce that the only super-invariant sets are those mentioned.

24. *Proposed by Bulgaria.*

Two students A and B are playing the following game. Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that the referee asks student A: "Can you tell the integer written by the other student?" If A answers "no", the referee puts the same question to the student B. If B answers "no", the referee puts the question back to A, etc. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer "yes".

Solution by George Evagelopoulos, Athens, Greece.

Let a and b be the integers written by A and B respectively and let $x < y$ be the two integers written by the referee. First we prove the following two propositions.

(i) If both A and B know that $\lambda < b < \mu$ and if A answers "no" then both A and B will know that $y - \mu < a < x - \lambda$.

(ii) If both A and B know that $\lambda < b < \mu$ and if B answers "no" then both A and B will know that $y - \mu < b < x - \lambda$.

Indeed, if $a \leq y - \mu$ then $a + b < a + \mu \leq y$. Therefore $a + b = x$ and A can answer "yes". Also if $a \geq x - \lambda$, then $a + b > a + \lambda \geq x$. Hence $a + b = y$ and A can answer "yes". This proves (i), and the proof of (ii) is completely analogous.

Assume that neither A nor B ever answers "yes". We shall prove that this leads to a contradiction. Initially both A and B know that $0 < b < y$. Put $a_0 = 0$ and $b_0 = y$ and assume that at some stage both A and B know that $a_k < b < b_k$ and A is about to be queried. Since we have assumed that A will answer "no" it follows from (i) that both A and B will know that $y - b_k < a < x - a_k$. By assumption B will now answer "no" and by (ii) both A and B will know that $y - (x - a_k) < b < x - (y - b_k)$, i.e. $a_k + (y - x) < b < b_k - (y - x)$. Setting $a_{k+1} = a_k + (y - x)$ and $b_{k+1} = b_k - (y - x)$ we see that this can be iterated. Since $y - x > 0$ we will have $a_{k_0} \geq b_{k_0}$ for some k_0. This contradicts $a_{k_0} < b < b_{k_0}$. This proves that after a finite number of questions A or B will answer "yes".

* * *

I have received a number of solutions to I.M.O. problems from Sigtuna from readers who feel their solutions may be more elegant than the ones given. Before deciding whether to use these submissions, I want to have a more careful look. Also, I want to move on just now to readers' solutions to problems of the 1990 *Dutch Mathematical Olympiad, Second Round* [1992: 267].

1. Prove for every integer $n > 1$:

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1) < n^n.$$

[*Editor's note.* The original problem as we received it had $n - 1$ in place of $2n - 1$.]

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Joel Brenner, Palo Alto, California and Horst Alzer, Waldröl, Germany; by Curtis Cooper, Central Missouri State University, Warrensburg; by Bob Prielipp, University of Wisconsin–Oshkosh; by Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's extension of the result.

The inequality is an immediate consequence of the identity $1+3+5+\cdots+(2n-1) = n^2$ and the A.M.–G.M. Inequality since

$$n^n = \left(\frac{1+3+\cdots+2n-1}{n}\right)^n > 1 \cdot 3 \cdot \ldots \cdot (2n-1).$$

We now prove the stronger result

$$1 \cdot 3 \cdot \ldots \cdot (2n-1) < 2n^{n-1} \tag{$*$}$$

for all natural numbers n. When $n = 1, 2, 3$ we have $1 < 2$, $3 < 4$, and $15 < 18$, respectively. Suppose $(*)$ holds for some $n \geq 3$. Then

$$1 \cdot 3 \cdot \ldots \cdot (2n+1) < 2(2n+1)n^{n-1}. \tag{1}$$

Now $n \geq 3$ gives $10n > 27$ and $64n > 54n + 27$ whence $(64/27)n > 2n + 1$. Thus $2n + 1 < n(1 + (1/3))^3 \leq n(1 + (1/n))^n$ since the sequence $\{(1 + (1/n))^n\}$, $n = 1, 2, 3, \ldots$ is increasing. Thus

$$(2n+1)n^{n-1} < n^n\left(1 + \frac{1}{n}\right)^n = (n+1)^n. \tag{2}$$

From (1) and (2), we obtain $1 \cdot 3 \cdot \ldots \cdot (2n+1) < 2(n+1)^n$, completing the induction.

Comment by Brenner and Alzer.

One can ask whether the righthand side (n^n) can be replaced by $n^{\varepsilon n}$ for some $\varepsilon < 1$. The answer is No! We establish this negative proposition by showing that

$$\lim_{n\to\infty} \frac{\log(1 \cdot 3 \cdot \ldots \cdot (2n-1))}{n \log n} = 1.$$

This uses the inequality

$$0 < \log \Gamma(x) - \left(x - \frac{1}{2}\right) \log x - \frac{1}{2} \log 2\pi < \frac{1}{x}, \quad x > 1 \tag{$**$}$$

which may be found for example in "Some inequalities involving $(r!)^{1/r}$" by H. Minc and L. Sathre, *Proc. Edinburgh Math. Soc.* **14** (1964/65), 41–46.

Set

$$x_n = \frac{1}{n \log n} \sum_{i=1}^{n} \log(2i) \quad \text{and} \quad y_n = \frac{1}{n \log n} \sum_{i=1}^{2n} \log i.$$

From ($**$), we get for $n > 1$

$$0 < \frac{\log \Gamma(2n+1)}{n \log n} - \frac{(2n+1/2)\log(2n+1)}{n \log n} + \frac{2n+1-\frac{1}{2}\log(2\pi)}{n \log n} < \frac{1}{n \log n \cdot (2n+1)}.$$

We know that

$$\frac{2n+\frac{1}{2}}{n} \frac{\log(2n+1)}{\log n} \to 2$$

as $n \to \infty$ and conclude that

$$y_n = \frac{\log \Gamma(2n+1)}{n \log n} \to 2, \quad (n \to \infty).$$

Further

$$x_n = \frac{n \log 2 + \log n!}{n \log n} = \frac{\log 2}{\log n} + \frac{\log \Gamma(n+1)}{n \log n} \tag{3}$$

We now exploit ($**$) again to obtain

$$0 < \frac{\log \Gamma(n+1)}{n \log n} - \frac{(n+\frac{1}{2})\log(n+1)}{n \log n} + \frac{n+1-\frac{1}{2}\log 2\pi}{n \log n} < \frac{1}{n \log n \cdot (n+1)}.$$

From this we conclude that as $n \to \infty$

$$\frac{\log \Gamma(n+1)}{n \log n} \to 1, \quad \text{and} \quad x_n \to 1.$$

Therefore

$$\frac{1}{n \log n} \sum_{i=1}^{n} \log(2i-1) = y_n - x_n \to 2 - 1 = 1,$$

as $n \to \infty$.

2. The numbers $a_1, a_2, a_3, a_4, \ldots$ are defined as follows:

$$a_1 = \frac{3}{2} \quad \text{and} \quad a_{n+1} = \frac{3a_n^2 + 4a_n - 3}{4a_n^2}.$$

(a) Prove that for all n holds: $a_n > 1$ and $a_{n+1} < a_n$.
(b) From (a) it follows that $\lim_{n\to\infty} a_n$ exists. Determine this limit.
(c) Determine $\lim_{n\to\infty} a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.: Beatriz Margolis, Paris, France; and by Michael Selby, University of Windsor. We use Margolis' solution.

(a) Now

$$3 - a_{n+1} = \frac{9a_n^2 - 4a_n + 3}{4a_n^2} = \frac{(3a_n - 2/3)^2 + 23/9a_n}{4a_n^2} > 0. \tag{1}$$

Clearly $a_1 > 1$. Assume $a_k > 1$. Then

$$a_{k+1} - 1 = \frac{-a_k^2 + 4a_k - 3}{4a_k^2} = \frac{(3 - a_k)(a_k - 1)}{4a_k^2} > 0$$

using (1) and the induction hypothesis. Therefore $a_n > 1$, for all n. (2)

Now

$$a_n - a_{n+1} = \frac{4a_n^3 - 3a_n^2 - 4a_n + 3}{4a_n^2} = \frac{(4a_n - 3)(a_n^2 - 1)}{4a_n^2} > 0 \tag{3}$$

using (2). This establishes (a).

(b) From (a), $\{a_n\}$ is decreasing and bounded below by 1. Therefore it converges, to $a \geq 1$, say. Taking the limit as $n \to \infty$ in (3)

$$0 = \frac{(4a - 3)(a^2 - 1)}{4a^2}$$

so that $a = 1$ (since $a = 3/4, -1$ are ruled out by $a \geq 1$).

(c) Let $P_n = \prod_{k=1}^n a_k$. Now $a_n^2(4a_{n+1} - 3) = 4a_n^2 a_{n+1} - 3a_n^2 = 4a_n - 3$, so $a_n^2 = (4a_n - 3)/(4a_{n+1} - 3)$ and

$$P_n^2 = \prod_{k=1}^n a_k^2 = \prod_{k=1}^n \frac{4a_k - 3}{4a_{k+1} - 3} = \frac{4a_1 - 3}{4a_{n+1} - 3} = \frac{3}{4a_{n+1} - 3}.$$

Thus

$$P_n = \sqrt{\frac{3}{4a_{n+1} - 3}} \to \sqrt{3}$$

as $n \to \infty$ by the result in (b).

3. Given is a function $f : x \longrightarrow ax^4 + bx^3 + cx^2 + dx$ with the following properties:

- $a, b, c, d > 0$;
- $f(x)$ is an integer for all $x \in \{-2, -1, 0, 1, 2\}$;
- $f(1) = 1$ and $f(5) = 70$.

(a) Prove: $a = 1/24$, $b = 1/4$, $c = 11/24$, $d = 1/4$.

(b) Prove: $f(x)$ is an integer for every integer x.

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; Michael Selby, University of Windsor; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

From the information that $f(x)$ is an integer for $x = \pm 1$ we have $2a+2c$ and $2b+2d$ integers. That $f(x)$ is an integer for $x = \pm 2$ implies that $32a+8c$ and $16b+4d$ are integers. Thus $24a$, $12b$, $24c$, $12d$ are integers. So put $a = \alpha/24$, $b = \beta/12$, $c = \gamma/24$ and $d = \delta/12$, where α, β, γ, δ are integers. Now $f(x)$ integral at $x = \pm 1$ ensures that $\alpha+2\beta+\gamma+2\delta \mid 24$ and $\alpha - 2\beta + \gamma - 2\delta \mid 24$. It follows that $\alpha + \gamma \mid 24$ and $\beta + \delta \mid 12$.

Now $f(1) = 1$ so $\alpha+2\beta+\gamma+2\delta = 24$ and $f(5) = 70$ so $125\alpha+50\beta+5\gamma+2\delta = 336$. From these $31\alpha + 12\beta + \gamma = 78$. The conditions $a, b, c, d > 0$ and $\alpha + \gamma \mid 24$, $\beta + \delta \mid 12$ eliminate all possibilities except $\alpha = 1$, $\beta = 3$, $\gamma = 11$, $\delta = 3$, whereupon

$$f(x) = \frac{1}{24}(x^4 + 6x^3 + 11x^2 + 6x) = \frac{1}{24}x(x+1)(x+2)(x+3).$$

As this is equal to $\binom{x+3}{4}$ it is integral for all integers $x \geq 1$.

For $x = 0$ we have $f(x) = 0$. Likewise for $x = -1, -2, -3$. For $x \leq -4$ we may put $x = -4 - y$, where $y \geq 0$ and

$$f(x) = \frac{1}{24}(y+1)(y+2)(y+3)(y+4) = \binom{y+4}{4}$$

which is integral for all $y \geq 0$.

4. Given is a regular 7-gon $ABCDEFG$. The sides have length 1. Prove for the diagonals AC and AD:

$$\frac{1}{AC} + \frac{1}{AD} = 1.$$

Solutions and comments by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Christopher J. Bradley, Clifton College, Bristol, U.K.; by Leon Bankoff, Los Angeles, California; and by Weixuan Li and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. First Li and Wang's solution.

Inscribe the regular 7-gon in a circle of radius r centered at the origin O. Put A on the positive x-axis and orient the vertices in the counterclockwise direction. Since $r \sin 2\pi/14 = 1/2$ we get $r = 1/(2\sin(\pi/7))$. Since

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = r(\cos 4\pi/7 + i \sin 4\pi/7) - r$$

we have

$$AC = |\overrightarrow{AC}| = \sqrt{r^2\left(\left(\cos\frac{4\pi}{7}-1\right)^2+\sin^2\frac{4\pi}{7}\right)}$$
$$= r\sqrt{2\left(\frac{1-\cos 4\pi}{7}\right)} = r\sqrt{4\sin^2\frac{4\pi}{7}} = 2r\sin\frac{2\pi}{7}.$$

Similarly from

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = r\left(\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}\right) - r,$$

we get

$$AD = |\overrightarrow{AD}| = 2r\sin\frac{3\pi}{7}.$$

Hence

$$\frac{1}{AC}+\frac{1}{AD} = \frac{1}{2r}\left(\frac{1}{\sin\frac{2\pi}{7}}+\frac{1}{\sin\frac{3\pi}{7}}\right) = \sin\frac{\pi}{7}\cdot\frac{\sin\frac{3\pi}{7}+\sin\frac{2\pi}{7}}{2\sin\frac{\pi}{7}\cos\frac{\pi}{7}\sin\frac{3\pi}{7}}$$
$$= \frac{\sin\frac{3\pi}{7}+\sin\frac{2\pi}{7}}{\sin\frac{4\pi}{7}+\sin\frac{2\pi}{7}} = 1,$$

since $\sin(3\pi/7) = \sin(4\pi/7)$.

Editor's Note. Bankoff sent an article by himself and Garfunkel as reference, and Amengual gives the following bibliographic information:

A solution of this problem may be found in Leon Bankoff and Jack Garfunkel, "The Heptagonal Triangle", *Mathematics Magazine* **46** (Jan–Feb 1973), 13.

The problem also appears in:

Problem E1222, "The Regular Heptagon", Item number 7 in the February 1957 issue of the *American Mathematical Monthly*, **112**.

V. Gusev, V. Litvinenko, A. Mordkovich, *Solving Problems in Geometry*, Mir Publishers, 1988, Problem Number 56.

G. Doroteev, M. Potapov, N. Rozov, *Elementary Mathematics*, Mir Publishers, Problem 19, 368.

And it was a problem proposed in the Escuela de Ayudantes de Telecomunicación of Spain (*Gaceta Matemática* Suplement, Madrid, 1949, 64).

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That's all the space for this month. Send me your pre-Olympiads, Olympiads, and nice solutions.

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BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

The Canadian Mathematical Olympiad 1969–1993, edited by Michael Doob, translated into French by Claude Laflamme. Published by the Canadian Mathematical Society, Ottawa, 1993. xix+262 pages, paperback, ISBN 0-919558-05-4. *Reviewed by Robert Geretschläger, Bundesrealgymnasium, and Gottfried Perz, Pestalozzigymnasium, both of Graz, Austria.*

The 25th anniversary of the Canadian Mathematical Olympiad is being used as a (highly welcome) excuse by the Canadian Mathematical Society to publish this updated collection of all problems posed in the CMO thus far. For non-Canadians, it is interesting to note that the CMS has decided to publish English and French versions of all problems and solutions in one volume.

Whereas the problems will not be new to regular *Crux* readers, and many of them have already been published by the CMS in two slim volumes, this book offers a complete overview of the CMO so far, including prize winners, chairs and members of the Olympiad Committee. There is also a short history of the competition by Ed Barbeau, which includes in particular some interesting footnotes on the evolution of several questions.

This book offers many excellent problems. Obviously, a great deal of effort has been put into the solutions, and a number of different ones are offered for several problems. Specifically, it contains many fine non-standard problems that cannot easily be categorized. These are very well suited to teaching logical problem-solving rather than algorithm crunching.

With so many questions, a problem that was bound to creep in is the necessary brevity of some solutions. While this may not be a cause of undue stress for an experienced mathematician, students using the book on their own may not find these proofs terribly easy to follow.

Another superficial quibble is the fact that the solutions always follow immediately after the problems, making it difficult to tell the difference between questions and answers at first glance. (Perhaps using different typefaces could have helped?) On the other hand, it is very easy to find the problems posed in a specific year because the years are printed in inverse type on the edges of the pages.

Browsing through the list of prize winners, it is interesting to note that a huge majority comes from Ontario schools. This poses the question: Is there something in the Ontario school-system better suited to encouraging mathematically talented students to excel (or just participate) in Olympiad-type competitions than elsewhere? What is it? Could it be transferred to other schools and school-systems? This phenomenon may be worth a more detailed study.[1]

[1][For readers not familiar with Canadian geography, it should be pointed out that Ontario is easily the largest Canadian province, with between 35% and 40% of Canada's population, and also that most of these people are concentrated in a small area. Actually, according to a just released survey of school students in Canada (*Calgary Herald*, December 17) the province where students performed best in mathematics was Québec, with Ontario and most other provinces bunched together in the middle of the pack.—*Ed.*]

All told, this volume is an excellent addition to any collection of books on mathematical problems, and should not be absent from the shelves of any school library.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1901. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous even function such that $f(0) = 0$ and $f(x+y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Must f be monotonic on $\mathbb{R}^+$?

1902. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcircle Γ. Let P be a variable point on the arc ACB of Γ, other than A, B, C. X and Y are points on the rays AP and BP respectively such that $AX = AC$ and $BY = BC$. Prove that the line XY always passes through a fixed point.

1903. *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let $n > 1$ be an integer. How many permutations $(a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$ are there such that

$$1 \mid a_1 - a_2, \quad 2 \mid a_2 - a_3, \quad \ldots \quad , \quad n-1 \mid a_{n-1} - a_n?$$

1904. *Proposed by Kee-Wai Lau, Hong Kong.*

If m_a, m_b, m_c are the medians of a triangle with sides a, b, c, prove that

$$m_a(bc - a^2) + m_b(ca - b^2) + m_c(ab - c^2) \ge 0.$$

1905. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*
Find all real solutions of the equation

$$\sqrt{x_1^2+x_2^2+\cdots+x_n^2}=\sqrt[3]{x_1^3+x_2^3+\cdots+x_n^3}.$$

1906. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*
Let AP bisect angle A of triangle ABC, with P on BC. Let Q be the point on segment BC such that $BQ = CP$. Prove that $(AQ)^2 = (AP)^2 + (b-c)^2$.

1907. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*
Find the largest constant k such that

$$\frac{kabc}{a+b+c} \le (a+b)^2 + (a+b+4c)^2$$

for all $a, b, c > 0$.

1908. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*
In triangle ABC the feet of the perpendiculars from A, B, C onto BC, CA, AB are denoted by D, E, F respectively. H is the orthocentre. The triangle is such that all of $AH-HD$, $BH-HE$, $CH-HF$ are positive. K is an internal point of ABC and L, M, N are the feet of the perpendiculars from K onto BC, CA, AB respectively. Prove that AL, BM, CN are concurrent if $KL : KM : KN$ is equal to

(i) $AH-HD : BH-HE : CH-HF$; (ii) $\dfrac{1}{AH-HD} : \dfrac{1}{BH-HE} : \dfrac{1}{CH-HF}$.

1909. *Proposed by Charles R. Diminnie, Saint Bonaventure University, Saint Bonaventure, New York.*
Solve the recurrence

$$p_0 = 1, \quad p_{n+1} = 5p_n(5p_n^4 - 5p_n^2 + 1)$$

for p_n in terms of n. [This problem was inspired by *Crux* 1809 which is solved in this issue.]

1910. *Proposed by Jisho Kotani, Akita, Japan.*
The octahedron $ABCDEF$ is inscribed in a sphere so that the three diagonals AF, BD, CE meet at a point, and the centroids of the six (triangular) faces of the octahedron are also inscribed in a sphere. Show that

(i) the orthocenters of the six faces are inscribed in a sphere;

(ii) $(AB\cdot DF + AD\cdot BF)(AC\cdot EF + AE\cdot CF)(BC\cdot DE + CD\cdot BE) = 36V^2$, where V is the volume of $ABCDEF$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1730. [1992: 76; 1993: 81] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that
$$\sum bc(s-a)^2 \ge \frac{sabc}{2},$$
where a, b, c, s are the sides and semiperimeter of a triangle, and the sum is cyclic over the sides.

III. *Comment by Ji Chen, Ningbo University, China.*

In his published solution [1993: 82], Murray Klamkin conjectured that
$$\sum_{i=1}^{m} \frac{x_i^n}{1-x_i} \ge \frac{m^{2-n}}{m-1} \tag{1}$$
whenever $x_1 + \cdots + x_m = 1$, $x_i \ge 0$, and $0 < n < 1$. (He proved it for all other real values of n.) We will show that (1) holds for
$$\frac{m-2}{m-1} \le n \le 2.$$
For $m = 3$, this gives (1) for $1/2 \le n \le 2$; $n = 2$ will solve the original problem again.

By Cauchy's inequality,
$$\sum_{i=1}^{m} \frac{x_i^n}{1-x_i} \cdot \sum_{i=1}^{m} x_i^{2-n}(1-x_i) \ge \left(\sum_{i=1}^{m} x_i\right)^2 = 1,$$
so we only need to prove
$$\sum_{i=1}^{m} x_i^{2-n}(1-x_i) \le \frac{m-1}{m^{2-n}},$$
i.e.
$$\frac{m-1}{m^{2-n}} + \sum_{i=1}^{m} x_i^{3-n} \ge \sum_{i=1}^{m} x_i^{2-n}. \tag{2}$$
By the A.M.–G.M. inequality,
$$\frac{(m-1)x_i}{m^{2-n}} + x_i^{3-n} \ge m\left(\frac{x_i}{m^{2-n}}\right)^{(m-1)/m} x_i^{(3-n)/m} = m^{[n(m-1)-(m-2)]/m} x_i^{(2-n+m)/m},$$
so (2) is implied by
$$\sum_{i=1}^{m} x_i^{(2-n+m)/m} \ge \left(\frac{1}{m}\right)^{[n(m-1)-(m-2)]/m} \sum_{i=1}^{m} x_i^{2-n},$$

that is,

$$[M_{(2-n+m)/m}(x_i)]^{(2-n+m)/m} \geq [M_1(x_i)]^{[n(m-1)-(m-2)]/m}[M_{2-n}(x_i)]^{2-n}, \tag{3}$$

where $M_k(x_i) = (\frac{1}{m}\sum_{i=1}^m x_i^k)^{1/k}$ is the kth mean of the x_i's. Since

$$\frac{2-n+m}{m} = \frac{n(m-1)-(m-2)}{m} + 2 - n,$$

we have

$$[M_{(2-n+m)/m}(x_i)]^{(2-n+m)/m} = [M_{(2-n+m)/m}(x_i)]^{[n(m-1)-(m-2)]/m}[M_{(2-n+m)/m}(x_i)]^{2-n},$$

and (3) would follow from the known fact

$$a \geq b \quad \Rightarrow \quad M_a(x) \geq M_b(x)$$

provided that

$$\frac{2-n+m}{m} \geq 1 \qquad \text{and} \qquad \frac{2-n+m}{m} \geq 2-n.$$

But the first inequality is equivalent to $n \leq 2$, and the second is equivalent to $n \geq (m-2)/(m-1)$. Hence the result is true.

[*Editor's note.* This still leaves (1) open for $0 < n < (m-2)/(m-1)$.]

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1809. [1993: 16] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Solve the recurrence $p_{n+1} = 5p_n^3 - 3p_n$ for $n \geq 0$, where $p_0 = 1$.

Solution by Tim Cross, Wolverley High School, Kidderminster, U.K.

Note that $p_0 = 1$, $p_1 = 2$, $p_2 = 34$ and $p_3 = 196418$ which suggests that

$$p_n = F(3^n)$$

where $F(n)$ is the nth Fibonacci number.

Assume for some $n \geq 3$ that $p_n = F(3^n)$. Now by Binet's formula

$$F(3^n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{3^n} - \left(\frac{1-\sqrt{5}}{2}\right)^{3^n}\right\}.$$

Write this as $(\omega^m - \phi^m)/\sqrt{5}$ where $\omega = (1+\sqrt{5})/2$, $\phi = (1-\sqrt{5})/2$ and $m = 3^n$. Then

$$\begin{aligned} 5p_n^3 &= 5\cdot\frac{1}{5\sqrt{5}}(\omega^{3m} - 3\omega^{2m}\phi^m + 3\omega^m\phi^{2m} - \phi^{3m}) \\ &= \frac{1}{\sqrt{5}}(\omega^{3m} - \phi^{3m}) - 3\omega^m\phi^m\frac{1}{\sqrt{5}}(\omega^m - \phi^m). \end{aligned}$$

Now

$$(\omega\phi)^m = \left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^m = (-1)^m = -1,$$

since m is odd. Therefore,

$$\begin{aligned} 5p_n^3 &= \frac{1}{\sqrt{5}}(\omega^{3m} - \phi^{3m}) + 3\frac{1}{\sqrt{5}}(\omega^m - \phi^m) \\ &= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{3^{n+1}} - \left(\frac{1-\sqrt{5}}{2}\right)^{3^{n+1}}\right\} + 3p_n = F(3^{n+1}) + 3p_n, \end{aligned}$$

and so we have

$$p_{n+1} = 5p_n^3 - 3p_n = F(3^{n+1}),$$

and the result now follows by induction.

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; C. R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; SHAILESH SHIRALI, Rishi Valley School, India; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Several people solved the problem by putting $p_n = (2/\sqrt{5})\cosh x_n$ *and noting that* $\cosh 3x = 4\cosh^3 x - 3\cosh x$. *Not all the solvers who used this method found the connection with the Fibonacci numbers, though. Shirali and Williams solved the more general question* $p_{n+1} = ap_n^3 - 3p_n$, $p_0 = 1$.

For a similar problem, see Crux 1909, *this issue.*

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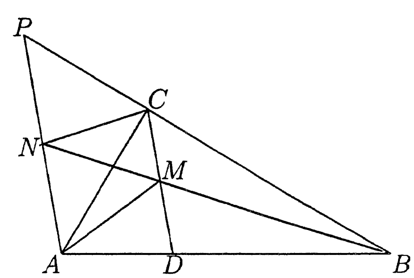
1812. [1993: 48] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a right-angled triangle with the right angle at C. Let D be a point on side AB, and let M be the midpoint of CD. Suppose that $\angle AMD = \angle BMD$. Prove that

$$\angle ACD : \angle BCD = \angle CDA : \angle CDB.$$

I. *Solution by the proposer.*



Let P be a point on BC produced such that $AP \| CD$, and let N be the intersection of AP with BM. Then as $CM = MD$, we have $PN = NA$. Because $\angle PCA = 90°$, we get $CN = PN = NA$. Therefore we have $\angle ACM = \angle CAN = \angle NCA$, so that

$$\angle NCM = 2\angle ACM. \qquad (1)$$

Because $\angle MAN = \angle AMD = \angle BMD = \angle MNA$, we have $MA = MN$. Now from $MD = MC$, $MA = MN$ and $\angle AMD = \angle BMD = \angle NMC$ we get $\Delta MAD \equiv \Delta MNC$, therefore

$$\angle MDA = \angle MCN. \tag{2}$$

From (1) and (2) we obtain

$$\angle CDA = 2\angle ACD. \tag{3}$$

Hence

$$\angle CDB = 180° - \angle CDA = 2(90° - \angle ACD) = 2\angle BCD. \tag{4}$$

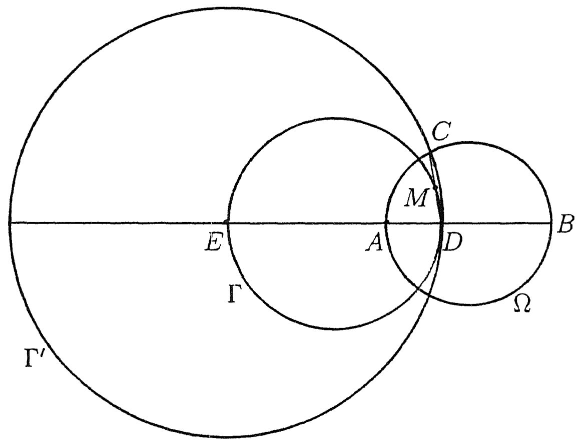
From (3) and (4) we obtain the result.

II. *Solution by Shailesh Shirali, Rishi Valley School, India.*

We shall proceed in the reverse direction (keeping in mind Abel's injunction: "Always invert!"). More specifically, we shall ask how one might actually go about constructing a configuration of the type described in the problem. Here is one possible way.

Let the segment AB be given and let D be any point on AB. Let Γ be the Apollonius locus

$$\Gamma = \left\{P : \frac{PA}{PB} = \frac{DA}{DB}\right\}.$$



This is a circle passing through D. If E is the point on line AB for which EA/EB equals DA/DB, with $E \neq D$, then DE is a diameter of Γ.

An important comment needs to be made here: if Ω denotes the circle on AB as diameter, then the circles Γ, Ω bear a symmetrical relation with respect to each other. in the sense that Ω is also the locus $\{P : PD/PE = AD/AE\}$. This is true because of the relation $BD/BE = AD/AE$.

Now consider the enlargement map that has center D and scale factor 2; when applied to circle Γ, the result is a circle Γ' which has center E and radius $|DE|$. Let Γ' intersect Ω at points C, C'. Then $\angle ACB$ is a right angle and the midpoint M of segment CD lies on Γ (by construction). It follows that $MA/MB = DA/DB$ and therefore that the line MD bisects $\angle AMB$. We have now recovered the original configuration! It should be clear from the development above that the original positioning of the points A, D, B uniquely fixes the whole diagram.

From the comment made on the symmetric relation between Γ and Ω, we have that $CE/CD = AE/AD$; therefore line CA bisects $\angle ECD$ and so $\angle ECD = 2\angle ACD$. Next, $\angle ECD = \angle EDC$ since C, D lie on the circle Γ' whose center is E, and therefore $\angle ADC = 2\angle ACD$. Finally, since $\angle ADB$ is twice $\angle ACB$, we have that

$$\angle ADC : \angle BDC = \angle ACD : \angle BCD.$$

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; and D.J. SMEENK, Zaltbommel, The Netherlands.

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1813*. [1993: 49] *Proposed by D.N. Verma, Bombay, India.*

Suppose that $a_1 > a_2 > a_3$ and $r_1 > r_2 > r_3$ are positive real numbers. Prove that the determinant

$$\begin{vmatrix} a_1^{r_1} & a_1^{r_2} & a_1^{r_3} \\ a_2^{r_1} & a_2^{r_2} & a_2^{r_3} \\ a_3^{r_1} & a_3^{r_2} & a_3^{r_3} \end{vmatrix}$$

is positive.

I. *Solution by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Let $s_1 = r_1 - r_3$, $s_2 = r_2 - r_3$, so $s_1 > s_2 > 0$. Further define $s_1 = ks_2$ with $k > 1$. Let $a_1^{s_2} = x$, $a_2^{s_2} = y$, $a_3^{s_2} = z$, so $x > y > z > 0$. Then the given determinant is

$$a_1^{r_3}a_2^{r_3}a_3^{r_3}\begin{vmatrix} a_1^{s_1} & a_1^{s_2} & 1 \\ a_2^{s_1} & a_2^{s_2} & 1 \\ a_3^{s_1} & a_3^{s_2} & 1 \end{vmatrix} = a_1^{r_3}a_2^{r_3}a_3^{r_3}\begin{vmatrix} x^k & x & 1 \\ y^k & y & 1 \\ z^k & z & 1 \end{vmatrix},$$

so to answer the question it is sufficient to prove

$$x^k(y-z) + y^k(z-x) + z^k(x-y) > 0$$

for $x > y > z > 0$ and $k > 1$. Now by the (weighted) power means inequality we have

$$\frac{(y-z)x^k + (x-y)z^k}{x-z} > \left(\frac{(y-z)x + (x-y)z}{x-z}\right)^k = y^k.$$

This establishes the result.

II. *Generalization by Len Bos, University of Calgary.*

[*Editor's note.* The proposer's original problem was for an $n \times n$ determinant. Below is colleague Len Bos's proof of this general problem.]

Suppose that $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$, and let A be the $n \times n$ matrix whose i,j entry is $e^{x_iy_j}$. Then we prove that $\det A > 0$. [By putting $a_i = e^{x_i}$ and $r_i = y_i$, one obtains the obvious extension of the original problem to an $n \times n$ determinant.]

To do this we first show that $\det A \neq 0$, by induction on n. This is clearly true for $n = 1$. For $n > 1$, if $A\mathbf{v} = \mathbf{0}$ for some vector $\mathbf{v} = (v_1, \ldots, v_n)$, then $f(x_i) = 0$ for all $1 \le i \le n$, where

$$f(x) = \sum_{j=1}^{n} v_j e^{xy_j}.$$

Thus $g(x_i) = 0$ for all $1 \le i \le n$, where

$$g(x) = v_1 + \sum_{j=2}^{n} v_j e^{x(y_j - y_1)}.$$

By Rolle's theorem, for each $i \in \{2, \ldots, n\}$ $g'(u_i) = 0$ for some $x_{i-1} < u_i < x_i$, and so $u_2 < u_3 < \cdots < u_n$. That is, $h(u_i) = 0$ for $2 \le i \le n$, where

$$h(x) = g'(x) = \sum_{j=2}^{n} v_j(y_j - y_1)e^{x(y_j - y_1)} = \sum_{j=2}^{n} w_j e^{xz_j}$$

($w_j = v_j(y_j - y_1)$, $z_j = y_j - y_1$). Since $u_2 < u_3 < \cdots < u_n$ and $z_2 < z_3 < \cdots < z_n$, by induction the vector $\mathbf{w} = (w_2, \ldots, w_n)$ must be $\mathbf{0}$. Since $y_j - y_1 \ne 0$, this means $v_2 = v_3 = \cdots = v_n = 0$, so $g \equiv v_1$ and thus $v_1 = 0$. Hence $\mathbf{v} = \mathbf{0}$ and so $\det A \ne 0$.

Now since the classical Vandermonde determinant

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix}$$

is positive (where $0 < a_1 < a_2 < \cdots < a_n$), and is of the form $A = (e^{x_i y_j})$ (put $x_i = \ln a_i$ and $y_i = i - 1$), by continuity $\det A$ for all A's must be positive.

Incidentally, this fact is equivalent to the known result that e^{xy} is "strictly totally positive", which is mentioned on page 15 of S. Karlin, *Total Positivity*, Stanford University Press, 1968.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Albany, California; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARIA JESÚS VILLAR RUBIO, I.B. Torres Quevedo, Santander, Spain; and CHRIS WILDHAGEN, Rotterdam, The Netherlands. There was one incorrect solution sent in.

Klamkin located the generalization (solution II) in a 1950 manuscript of F.P. Gantmacher and M.G. Krein.

Janous remarks (and the above solutions show) that the r_i's need not be positive.

*   *   *   *   *

1814. [1993: 49] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Given are the fixed line ℓ with two fixed points A and B on it, and a fixed angle φ. Determine the locus of the point C with the following property: the angle between ℓ and the Euler line of ΔABC equals φ.

Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia. (somewhat modified)

We introduce Cartesian coordinates. Let $A = (-1, 0)$ and $B = (1, 0)$, so that the y-axis is the perpendicular bisector of AB. If C has coordinates (x, y) then G, the centroid of ΔABC, has coordinates $(x/3, y/3)$, while the perpendicular bisector of AC meets the y-axis in the circumcentre $O = (0, (y^2 + x^2 - 1)/2y))$. The slope of the Euler line OG is therefore

$$\frac{y^2 + 3x^2 - 3}{-2xy} = t,$$

where $t = \tan\varphi$, the tangent of the fixed angle between $\ell = AB$ and the Euler line. [*Editor's comment.* Ardila and one other solver defined φ to be the *acute* angle between the two lines; we are instead using the convention that φ is the angle directed from ℓ to the Euler line, so that as φ runs from $-90°$ to $90°$, t goes from $-\infty$ to ∞.] Thus, the variable vertex C satisfies the equation

$$3x^2 + 2txy + y^2 = 3.$$

Conversely, C can be any point that satisfies this equation as long as the Euler line is well defined: C is a point off ℓ and is different from the apex $(0, \pm\sqrt{3})$ of an equilateral triangle whose base is AB.

The analysis from this point is routine and can be left to the student. Oh no! That's me. Well then,

- for $\varphi = 90°$, $t = \infty$ and the locus is $x = 0$ (the y-axis except for three missing points);
- for $90° > |\varphi| > 60°$, $\infty > t^2 > 3$ and the locus is a hyperbola (with four missing points);
- for $|\varphi| = 60°$, $t^2 = 3$ and the locus is a pair of lines $y = -t(x \pm 1)$ (with four missing points); the lines, one through A and one through B, make an angle of $-\varphi$ with AB. [*Editor's comment.* Solvers Penning and Shirali point out that this case follows from the proposer's problem *Crux* 1673 [1992: 218–219]; see also his problem (proposed with J.T. Groenman) #1232 of *Mathematics Magazine* 60:1 (February 1987) 43–45.]
- for $|\varphi| < 60°$, $t^2 < 3$ and the locus is an ellipse (with four missing points).

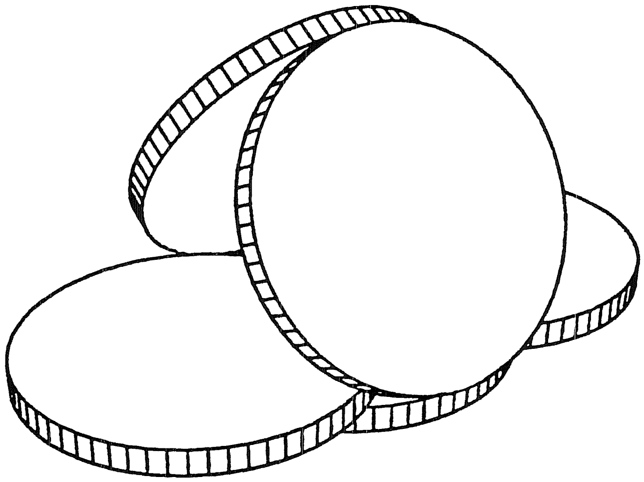
Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; L. J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer.

Dou summarizes the answer by saying that for fixed angle φ the locus is a conic of the pencil determined by A, B, $(0, \pm\sqrt{3})$, minus the forbidden points.

* * * * *

1815. [1993: 49] *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

An old puzzle (see *Mathematical Puzzles and Diversions*, Martin Gardner, Simon & Schuster, New York, 1959, p. 114, or *Puzzlegrams*, Simon & Schuster/Fireside, New York, 1989, p. 171) asks that five congruent coins be placed in space so that each touches the other four. The solution often given is as illustrated below: one coin supports two others, which meet over the center of the bottom coin, with two tilted coins forming the sides of the tent-like figure.



Show that this solution is *invalid* if the coins are nickels. Assume (despite the picture) that the nickels are ordinary circular cylinders with diameter to height ratio of 11 to 1.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

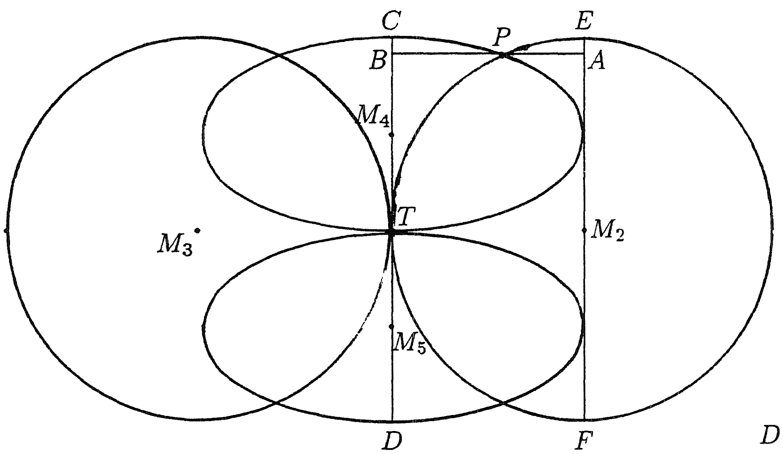


Figure 1 (overhead view)

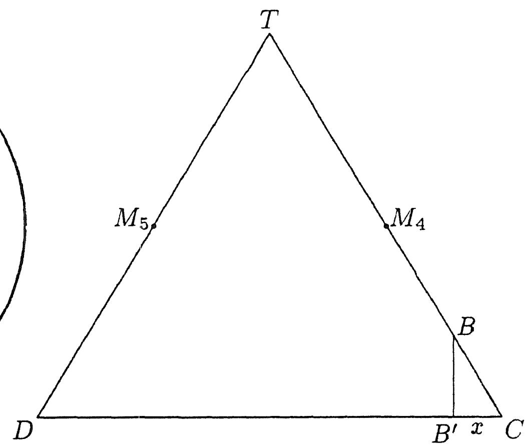


Figure 2 (side view)

Figure 1 is a ground plan (projection down) of the coins without the first (bottom) coin, whose diameter is CD. M_2 and M_3 are the centers of the next two coins, and M_4 and M_5 are the centers of the upper (slanted) coins. [Note that coins 4 and 5 become ellipses when projected down; actually the projections of their bottom surfaces are shown. Also, for the optimum situation it is assumed that coins 4 and 5 just touch the edge of coin 1 at antipodal points C and D.—*Ed.*]

Let P be the point of touching between coins 2 and 4, T the point of touching between coins 4 and 5, and let $r = 1$ be the radius of the coins. Then P lies on a horizontal line $AB \| M_2M_3$, AB having length 1, with the segment AP lying on (the top surface of) coin 2 and the segment PB lying on (the bottom surface of) coin 4. The diameters TC and TD of coins 4 and 5 are the sides of an equilateral triangle CDT with B on the segment CT (see Figure 2).

Now in coin 2, $M_2E = 1$ is the radius with A lying on M_2E. So, letting $x = AE$, we have [since ΔEPF is right-angled]

$$(AP)^2 = x(2 - x). \tag{1}$$

In coin 4, $M_4C = 1$ is the radius with B lying on M_4C. Note that the projection $B'C$ of BC has length x so that $BC = 2x$ (Figure 2). Thus [since ΔPCT is right-angled]

$$(PB)^2 = 2x(2 - 2x). \tag{2}$$

Since $AP + PB = 1$, we have from (1) and (2) that

$$2x(2-2x) = (PB)^2 = (1 - AP)^2 = \left(1 - \sqrt{x(2-x)}\right)^2 = 1 - 2\sqrt{x(2-x)} + x(2-x),$$

so

$$4x(2-x) = (3x^2 - 2x + 1)^2 = 9x^4 - 12x^3 + 10x^2 - 4x + 1$$

or

$$0 = 9x^4 - 12x^3 + 14x^2 - 12x + 1 = (x-1)(9x^3 - 3x^2 + 11x - 1).$$

Since $x = 1$ is not a solution, $9x^3 - 3x^2 + 11x - 1 = 0$ which has the real solution $x \approx 0.0926$. Thus the maximum thickness of the coins is

$$BB' = x\sqrt{3} \approx 0.16038,$$

which means a diameter to height ratio of

$$\frac{2}{BB'} \approx 12.47.$$

Therefore the solution of the five coins problem is invalid for a coin with diameter to height ratio smaller than 12.47 to 1.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; PAUL PENNING, Delft, The Netherlands; and the proposer.

The proposer notes that all Canadian and U.S. coins except nickels are sufficiently thin for the construction to work. As partial evidence he includes a photograph of five pennies forming the configuration, which he assembled easily using superglue.

*   *   *   *   *

1818. [1993: 50] *Proposed by Ed Barbeau, University of Toronto.*
Prove that, for $0 \le x \le 1$ and a positive integer k,

$$(1+x)^k[x+(1-x)^{k+1}] \ge 1.$$

I. *Solution by Panos E. Tsaoussoglou, Athens, Greece.*
The proof is by induction on k.
Let $k=1$; then

$$(1+x)[x+(1-x)^2] = (1+x)(1-x+x^2) = 1+x^3 \ge 1,$$

and thus the inequality is true for $k=1$.

Assume that the inequality holds for some $k=n\ge 1$, i.e., $(1+x)^n[x+(1-x)^{n+1}] \ge 1$. It is sufficient to prove that

$$(1+x)^{n+1}[x+(1-x)^{n+2}] - (1+x)^n[x+(1-x)^{n+1}] \ge 0,$$

or equivalently that

$$x(1+x)^{n+1} + (1+x)^{n+1}(1-x)^{n+2} - x(1+x)^n + (1+x)^n(1-x)^{n+1} \ge 0.$$

However, the left hand side reduces to

$$\begin{aligned} &x(1+x)^n(1+x-1) + (1+x)^n(1-x)^{n+1}(1-x^2-1) \\ &\qquad = x^2(1+x)^n - x^2(1+x)^n(1-x)^{n+1} = x^2(1+x)^n[1-(1-x)^{n+1}]. \end{aligned}$$

Thus we have to show that

$$x^2(1+x)^n[1-(1-x)^{n+1}] \ge 0.$$

But this inequality is true since $1 \ge x \ge 0$ and $1 \ge (1-x)^{n+1}$. Therefore, the given inequality is true for all k.

II. *Solution by Chris Wildhagen, Rotterdam, The Netherlands.*
We have

$$\begin{aligned} (1+x)^k[x+(1-x)^{k+1}] &= x(1+x)^k + (1-x)(1-x^2)^k \\ &\ge [x(1+x)+(1-x)(1-x^2)]^k \qquad (1) \\ &= (1+x^3)^k \ge 1, \end{aligned}$$

where for (1) the convexity of the function $t \mapsto t^k$ $(k \ge 1)$ on the interval $[0,\infty)$ is used. [*Editor's note*: (1) uses Jensen's inequality; we could also use the fact that the (weighted) kth power mean for $k>1$ is greater than the arithmetic mean.]

III. *Solution by Murray S. Klamkin, University of Alberta.*
We show that the inequality is true for $0 \le x \le 1$ and for any real $k \ge 0$. Let

$$F(x) = (1+x)^k[x+(1-x)^{k+1}].$$

Since $F(0) = 1$, it suffices to show that $F(x)$ is nondecreasing in $[0,1]$, i.e., that $F'(x) \geq 0$ for $0 \leq x \leq 1$. Here

$$\begin{aligned} F'(x) &= k(1+x)^{k-1}[x + (1-x)^{k+1}] + (1+x)^k[1-(k+1)(1-x)^k] \\ &= (1+x)^{k-1}[xk(1-(1-x)^k) + x + 1 - (x(k+1)+1)(1-x)^k]. \end{aligned}$$

Since $1-(1-x)^k \geq 0$ for $0 < x < 1$ and $k \geq 0$, it now suffices to show that

$$G(x) \equiv x + 1 - [x(k+1)+1](1-x)^k \geq 0. \qquad (2)$$

Since $G(0) = 0$, we need only show that $G'(x) \geq 0$ for $0 < x < 1$, or that

$$1 \geq (k+1)(1-x)^k - k[x(k+1)+1](1-x)^{k-1} = (1-x)^{k-1}[1 - x(k+1)^2].$$

This is clearly valid for all $k \geq 1$. We now show that (2) is valid for $0 \leq k < 1$. The separate cases $k = 0$ and $x = 0$ are obviously valid. Now we use an extended form of Bernoulli's inequality:

$$(1-x)^k \leq 1 - kx \quad \text{for} \quad 0 < k < 1 \quad \text{and} \quad 0 < x < 1. \qquad (3)$$

[For instance, let $h(x) = (1-x)^k + kx$; then since $h(0) = 1$ and

$$h'(x) = -k(1-x)^{k-1} + k = k\left(1 - \frac{1}{(1-x)^{1-k}}\right) < 0$$

for $0 < k < 1$ and $0 < x < 1$, we get $h(x) \leq 1$ for $0 < x < 1$, which is (3).—*Ed.*] From (3) we finally have that

$$G(x) \geq x + 1 - [x(k+1)+1](1-kx) = x^2k(k+1) \geq 0.$$

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; TIM CROSS, Wolverley High School, Kidderminster, U.K.; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; F. J. FLANIGAN, San Jose State University, California; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; KEE-WAI LAU, Hong Kong; NICK LORD, Tonbridge School, Kent, England; PAVLOS MARAGOUDAKIS, Pireas, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. There was one incorrect solution sent in.

Bradley and Janous had the same solution as Wildhagen. These and some other solutions are all valid for arbitrary real $k \geq 1$, though most of the solvers didn't point that out. Interestingly, these solutions do not appear to extend to $0 < k < 1$. Even Klamkin, the only solver to get this extension, needed two cases.

* * * * *

1819. [1993: 50] *Proposed by Joaquín Gómez Rey, I.B. Luis Bunuel, Alcorcón, Madrid, Spain.*

An urn contains n balls numbered from 1 to n. We draw n balls at random, *with replacement* after each ball is drawn. What is the probability that ball 1 will be drawn an odd number of times, and what is the limit of this probability as $n \to \infty$?

Solution by Nick Lord, Tonbridge School, Kent, England.

The number of times ball 1 is drawn follows a binomial distribution. So the probability that ball 1 is drawn an odd number of times is

$$\sum_{\substack{1 \le r \le n \\ r \text{ odd}}} \binom{n}{r} \left(\frac{1}{n}\right)^r \left(1 - \frac{1}{n}\right)^{n-r}.$$

But

$$\sum_{\substack{1 \le r \le n \\ r \text{ odd}}} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}[(y+x)^n - (y-x)^n];$$

thus the required answer is

$$\sum_{\substack{1 \le r \le n \\ r \text{ odd}}} \binom{n}{r} \left(\frac{1}{n}\right)^r \left(1 - \frac{1}{n}\right)^{n-r} = \frac{1}{2}\left[1 - \left(1 - \frac{2}{n}\right)^n\right],$$

which approaches

$$\frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$

as $n \to \infty$, since $(1 - 2/n)^n \to e^{-2}$ as $n \to \infty$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; DAVID HANKIN, Brooklyn, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria: KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; DAVID G. POOLE, Trent University, Peterborough, Ontario; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; SHAILESH SHIRALI, Rishi Valley School, India; CHRIS WILDHAGEN, Rotterdam. The Netherlands: and the proposer.

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1820*. [1993: 50] *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let O be the point of intersection of the diagonals AC and BD of the quadrangle $ABCD$. Prove that the orthocenters of the four triangles OAB, OBC, OCD, ODA are the vertices of a parallelogram that is similar to the figure formed by the centroids of these four triangles. What if "centroids" is replaced by "circumcenters"?

Preliminary comments by the editor (Chris Fisher).

We shall let H_i $(i = 1,2,3,4)$ be the orthocenters of the triangles in the order OAB, OBC, OCD, ODA, and let G_i be their centroids and O_i be their circumcenters. Toshio Seimiya reminds us that *Crux* 1670 [1992: 216] called for a proof that *in each case the given points form the vertices of a parallelogram whose angles equal the supplementary angles between AC and BD.* (Rather than look up the solution it is easier to prove for oneself that the sides of the parallelogram formed by the G_i are parallel to AC or BD, while the sides of the other two parallelograms are perpendicular to AC or BD.) The novelty here is to prove that, moreover, these three parallelograms are similar. Since we know that they have the same set of angles, it remains to show either (a) that corresponding diagonals form the same angle, or (b) that corresponding sides are proportional. Our solvers provided examples of each approach.

Part (a). $G_1G_2G_3G_4$ is similar to $H_1H_2H_3H_4$. *Solution by Shailesh Shirali, Rishi Valley School, India.*

First note that the quadrangle formed by the midpoints M_i of the sides of $ABCD$ (in the order M_1 on AB, etc.) is a homothetic version of the parallelogram $G_1G_2G_3G_4$ (center O, scale factor 2). We shall show that the midpoint figure is similar to the parallelogram of orthocenters.

Consider the circles Γ_1, Γ_2, Γ_3 constructed on the sides AB, BC, CD as diameters. The radical axis of Γ_1 and Γ_2 is the common chord BH_1H_2; that of Γ_2 and Γ_3 is the line CH_2H_3. Thus, the common point of these two lines, namely H_2, lies on the radical axis of Γ_1 and Γ_3. Arguing likewise using the circles on sides CD, DA, AB as diameters, we see that the radical axis of Γ_1 and Γ_3 is the line H_2H_4, which is consequently perpendicular to the line of centers of these two circles, namely M_1M_3. Similarly, $H_1H_3 \perp M_2M_4$, so that corresponding diagonals of the two parallelograms are perpendicular to one another, and they enclose equal angles, as desired.

Part (b). $G_1G_2G_3G_4$ is similar to $O_1O_2O_3O_4$. *Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Since $G_1G_2 : AC = 1 : 3 = G_2G_3 : BD$, we need only show that $O_1O_2 : O_2O_3 = AC : BD$. But we have Area$(O_1O_2O_3O_4) = O_1O_2 \times N_2N_4 = O_2O_3 \times N_1N_3$, where the N_i are the midpoints of OA, OB, OC, and OD (i.e., the points where the sides $O_{i-1}O_i$ meet the diagonals of $ABCD$). So

$$\frac{O_1O_2}{O_2O_3} = \frac{N_1N_3}{N_2N_4} = \frac{AC/2}{BD/2} = \frac{AC}{BD},$$

which is precisely what we wished to prove.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; C. J. BRADLEY, Clifton College, Bristol, U.K.; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. One other submission was incomplete.

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Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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