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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

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IT'S ELEMENTARY (COMBINATORICS) III

William Moser

[*Editor's note.* This is the last of a series of three articles by Professor Moser. For the other two see the last two issues.]

4. Rectangular arrays The multinomial coefficient

$$\binom{n}{a_1, a_2, \ldots, a_r} = n!/a_1!\, a_2! \cdots a_r!$$
$$= \binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3}\cdots\binom{n-a_1-a_2-\cdots-a_{r-2}}{a_{r-1}},$$

where $n = a_1 + a_2 + \cdots + a_r$, counts the number of linear displays of a_1 1's, a_2 2's, ..., a_r r's.

Now consider $0,1$ matrices $(a_{i,j})$ of size $m \times n$, i.e.,

$$a_{i,j} \in \{0,1\} \qquad i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n.$$

Of course, there are 2^{mn} such matrices, and $\binom{mn}{k}$ of them have precisely k entries 1, i.e.,

$$\sum_{i,j} a_{i,j} = k.$$

Among these, the number which have (for given integers $a_0, a_1, a_2, \ldots, a_m$) a_i columns whose entries sum to i is

$$\binom{n}{a_0, a_1, a_2, \ldots, a_m}\binom{m}{0}^{a_0}\binom{m}{1}^{a_1}\binom{m}{2}^{a_2}\cdots\binom{m}{m}^{a_m}.$$

It follows that

$$\binom{mn}{k} = \sum \binom{n}{a_0, a_1, a_2, \ldots, a_m}\binom{m}{0}^{a_0}\binom{m}{1}^{a_1}\binom{m}{2}^{a_2}\cdots\binom{m}{m}^{a_m},$$

where the sum is taken over $(m+1)$-tuples $(a_0, a_1, a_2, \ldots, a_m)$ satisfying

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_m = n, \qquad a_1 + 2a_2 + 3a_3 + \cdots + ma_m = k.$$

Several special cases reduce nicely. Taking $m = 2$, we have

$$\binom{2n}{k} = \sum_{\substack{a_0+a_1+a_2=n \\ a_1+2a_2=k}} \binom{n}{a_0, a_1, a_2} 2^{a_1} = \sum_{\substack{a_0+a_1+a_2=n \\ a_1+2a_2=k}} \binom{n}{a_0}\binom{n-a_0}{a_1} 2^{k-2a_2},$$

and taking $i = a_2$, so that $a_1 = k - 2i$ and $a_0 = n - k + i$,

$$\begin{aligned}\binom{2n}{k} &= \sum_{i=0} \binom{n}{n-k+i}\binom{k-i}{k-2i}2^{k-2i} = \sum_{i=0}\binom{n}{k-i}\binom{k-i}{i}2^{k-2i} \\ &= \sum_{i=0}\binom{n}{k-2i}\binom{n-k+2i}{i}2^{k-2i}. \qquad (7)\end{aligned}$$

Identity (7) with k replaced by $k-1$ is

$$\binom{2n}{k-1} = \sum_{i=0}\binom{n}{k-(2i+1)}\binom{n-k+(2i+1)}{i}2^{k-(2i+1)}. \qquad (8)$$

Adding (7) and (8) gives the companion to (7):

$$\binom{2n+1}{k} = \sum_{i=0}\binom{n}{k-i}\binom{n-k+i}{\lfloor i/2\rfloor}2^{k-i}. \qquad (9)$$

The special case $k = n$ of (7) is the identity

$$\binom{2n}{n} = \sum_{i=0}\binom{n}{n-i}\binom{n-i}{i}2^{n-2i} = \sum_{i=0}\binom{n}{2i}\binom{2i}{i}2^{n-2i}$$

of Gould (1959, identity 3.99), while the special case $k = n$ of (9) is the identity

$$\binom{2n+1}{n} = \sum_{i=0}^{n}\binom{n}{i}2^{n-i}\binom{i}{\lfloor i/2\rfloor}$$

of Gonciulea (1988).

References

H.W. Gould (1959). *Combinatorial Identities.* West Virginia University.

N. Gonciulea (1988). Problem E 3258. *Amer. Math. Monthly* **95**, p. 259.

Department of Mathematics and Statistics
McGill University
Montreal, P.Q. H3A 2K6

* * * * *

THE OLYMPIAD CORNER

No. 141

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year has passed. This time last year I wrote that we were settling into the use of LaTeX and indeed, thanks to the speed and skill of Joanne Longworth, the autumn numbers appeared, some even in the month they were supposed to! Let us hope that in 1993 we will establish a routine so that the eager readers receive their copies promptly.

It is also time to thank those who have contributed problem sets, solutions, and comments. Without the faithful readers this Corner would be a vastly poorer effort. Among the contributors whose work was mentioned last year were:

Miguel Amengual Covas	Walther Janous	John Morvay
Seung-Jin Bang	O. Johnson	Bob Prielipp
Ed Barbeau	Murray S. Klamkin	M. Selby
Margherita Barile	Lászlo Köszegi	Zun Shan
Francisco Bellot Rosado	Marcin E. Kuczma	Huang Tu Sheng
Xiong Bin	Hans Lausch	Shailesh Shirali
William Y.C. Chen	Andy Liu	D.J. Smeenk
P.H. Cheung	Maria A. López Chamorro	David Vaughan
Hans Engelhaupt	J. Lou	Charlton Wang
George Evagelopoulos	Pavlos Maragoudakis	E.T.H. Wang
Barry Ferguson	Beatriz Margolis	Wan-Di Wei
Georg Gunther	Stewart Metchette	Chris Wildhagen
R.K. Guy	Leroy F. Meyers	C.C. Yang
Stephen D. Hnidei	Walter Mientka	

Thank you all (and anyone I've left out by accident).

* * *

It has been suggested that devoting part of the Corner to contests at the "pre-Olympiad" level would be of interest to the readers, and especially useful to those "getting started", either as problem solvers themselves, or coaches of students. We have access to some material which is appropriate, but mostly from Canadian sources. I would welcome suitable material from all our readers in order to give this aspect of the Corner an international scope and appeal. To get things going we give the problems of Part II of the 1991–1992 Alberta High School Mathematics Contest, written in February 1992. This contest is primarily for students in Grades XI and XII, and in order to be invited to write it the student must have scored reasonably well on Part I, which is a multiple-choice exam written in the fall. Results of the contest assist in the selection of candidates for the Canadian Mathematics Olympiad.

1991–92 A.H.S.M.C. Part II

1. The Committee to Halt Excessive Amount of Photocopying (CHEAP) is itself accused of over-expenditure in photocopying, even though it never makes more than one copy of anything. The new committee set up to investigate this accusation makes, for each of its 13 members, a photocopy of everything CHEAP has photocopied, so that it can study whether the expenditure has been justified. Each committee is charged 7 cents per page for the first 2000 pages and 5 cents per page thereafter. It turns out that the photocopying expenditure of the new committee is 10 times that of CHEAP. How many photocopies did CHEAP make? Find all possible solutions.

2. The base of a tub is a square with sides of length 1 metre. It contains water 3 centimetres deep. A heavy rectangular block is placed in the tub three times. Each time, the face that rests on the bottom of the tub has a different area. When this is done, the water in the tub ends up being 4 centimetres, 5 centimetres and 6 centimetres deep. Find the dimensions of the block.

3. A positive integer is said to have the 32-property if the sum of the five remainders obtained when it is divided by some five consecutive positive integers is 32. For example, 24 has the 32-property since when 24 is divided by 11, 12, 13, 14 and 15, the respective remainders 2, 0, 11, 10 and 9 add up to 32.

(a) Verify that 26 has the 32-property.
(b) Determine the smallest positive integer with the 32-property.
(c) Prove that there are infinitely many positive integers with the 32-property.

4. Suppose x, y and z are real numbers which satisfy the equation $ax+by+cz=0$, where a, b and c are given positive numbers.

(a) Prove that $x^2+y^2+z^2 \geq 2xy+2yz+2xz$.
(b) Determine when equality holds in (a).

5. $ABCD$ is a square piece of paper with sides of length 1 metre. A quarter-circle is drawn from B to D with centre A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF could have.

*　　　*　　　*

The Olympiad problems we give this issue are those of the British Mathematical Olympiad, written Wednesday, 16 January 1991. Thanks go to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for collecting materials and forwarding them to me.

BRITISH MATHEMATICAL OLYMPIAD

1. Prove that the number

$$3^n + 2 \times 17^n$$

where n is a non-negative integer, is never a perfect square.

2. Find all positive integers k such that the polynomial $x^{2k+1}+x+1$ is divisible by the polynomial x^k+x+1. For each such k specify the integers n such that x^n+x+1 is divisible by x^k+x+1.

3. $ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E. Prove that if AC is perpendicular to BD then

$$EA^2+EB^2+EC^2+ED^2=4r^2. \qquad (*)$$

Is it true that if $(*)$ holds then AC is perpendicular to BD? Give a reason for your answer.

4. Find, with proof, the minimum value of $(x+y)(y+z)$ where x, y, z are positive real numbers satisfying the condition $xyz(x+y+z)=1$.

5. Find the number of permutations (arrangements) p_1,p_2,p_3,p_4,p_5,p_6 of 1, 2, 3, 4, 5, 6 with the property: for no integer n, $1\le n\le 5$, do $p_1,p_2,\ldots,p_n$ form a permutation of $1,2,\ldots,n$.

6. Show that if x and y are positive integers such that x^2+y^2-x is divisible by $2xy$ then x is a perfect square.

7. A ladder of length l rests against a vertical wall. Suppose that there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find *explicitly*, in terms of l and d, the height h from the ground that the ladder reaches up the wall.

* * *

We continue the column with solutions sent in by the readers to problems proposed but not used at the 31st I.M.O. in China (see the September and October 1991 columns).

4. [1991: 196] *Proposed by France.*

Given ΔABC with no side equal to another side, let G, K and H be its centroid, incentre and orthocentre, respectively. Prove that $\angle GKH>90°$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Murray S. Klamkin, University of Alberta. We use Klamkin's solution and comment.

The solution employs vectors. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be vectors from the circumcentre to the respective vertices A, B, C. Then $\mathbf{H}=\mathbf{A}+\mathbf{B}+\mathbf{C}$, $\mathbf{G}=(\mathbf{A}+\mathbf{B}+\mathbf{C})/3$, $\mathbf{K}=(a\mathbf{A}+b\mathbf{B}+c\mathbf{C})/(a+b+c)$. All we need to show is that

$$(\mathbf{G}-\mathbf{K})\cdot(\mathbf{H}-\mathbf{K})=\mathbf{G}\cdot\mathbf{H}+\mathbf{K}\cdot\mathbf{K}-\mathbf{K}\cdot(\mathbf{G}+\mathbf{H})<0. \qquad (1)$$

We now evaluate the various dot products using $\mathbf{A}\cdot\mathbf{A}=R^2$, $2\mathbf{B}\cdot\mathbf{C}=2R^2-a^2$, etc., where R is the circumradius, and a, b, c are the lengths of the sides opposite A, B, C, respectively. Hence

$$\mathbf{G}\cdot\mathbf{H}=\frac{\mathbf{A}^2+\mathbf{B}^2+\mathbf{C}^2+2\mathbf{B}\cdot\mathbf{C}+2\mathbf{C}\cdot\mathbf{A}+2\mathbf{A}\cdot\mathbf{B}}{3}=\frac{3R^2-(a^2+b^2+c^2)}{3}\ ,$$

$$\mathbf{K}\cdot\mathbf{K} = R^2 - \frac{abc}{a+b+c},$$

$$\mathbf{K}\cdot(\mathbf{G}+\mathbf{H}) = 4R^2 - \frac{2[a^2(b+c)+b^2(c+a)+c^2(a+b)]}{3(a+b+c)}.$$

Thus (1) is equivalent to

$$(a+b+c)(a^2+b^2+c^2)+3abc > 2[a^2(b+c)+b^2(c+a)+c^2(a+b)],$$

and this is the special case $n=1$ of Schur's inequality (see [1], [2], or [1991: 50])

$$a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b) > 0.$$

The left hand side is zero only for an equilateral triangle or for a degenerate triangle with one side 0.

This is a known result, for in [3], it is shown that the incenter lies inside the circle on diameter GH, and the three excentres lie outside of it.

References:

[1] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 119.

[2] M.S. Klamkin, *International Mathematical Olympiads 1978–1985*, M.A.A., Washington, D.C., 1986, p. 50.

[3] A.P. Guinand, Euler lines, tritangent centers and their triangles, *American Math. Monthly* **91** (1984) pp. 290–300.

5. [1991: 196] *Proposed by Greece.*

Let $f(0)=f(1)=0$ and

$$f(n+2) = 4^{n+2}f(n+1) - 16^{n+1}f(n) + n2^{n^2},$$

$n=0,1,2,\ldots$. Show that the numbers $f(1989)$, $f(1990)$ and $f(1991)$ are divisible by 13.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Since

$$f(n+2)-2^{n+3}f(n+1) = 2^{2n+3}[f(n+1)-2^{2n+1}f(n)] + n2^{n^2}, \quad n=0,1,2,\ldots,$$

using induction on n, we have

$$f(n+1)-2^{2n+1}f(n) = \sum_{k=1}^{n-1}(n-k)2^{(n-1)^2+4(k-1)} \quad \text{for } n\ge 2.$$

Now from

$$\frac{f(n+1)}{2^{(n+1)^2}} = \frac{f(n)}{2^{n^2}} + 2^{-4n-4}\sum_{k=1}^{n-1}(n-k)2^{4k} \quad (n\ge 2),$$

we obtain

$$\frac{f(n)}{2^{n^2}} = \frac{f(2)}{2^4} + \sum_{l=2}^{n-1}2^{-4l-4}\sum_{k=1}^{l-1}(l-k)2^{4k},$$

for $n \geq 3$. It follows that

$$\begin{aligned}
f(n) &= 2^{n^2} \sum_{l=2}^{n-1} \sum_{k=1}^{l-1} (l-k) 2^{4k-4l-4} = 2^{n^2} \sum_{l=2}^{n-1} \sum_{j=1}^{l-1} j 2^{-4j-4} \qquad (j = l-k) \\
&= 2^{n^2-4} \sum_{j=1}^{n-2} \sum_{l=j+1}^{n-1} j 2^{-4j} = 2^{n^2-4} \sum_{j=1}^{n-2} j(n-j-1) 2^{-4j} \\
&= 2^{(n-2)^2-8} \sum_{l=2}^{n-1} (n-l)(l-1) 2^{4l} \qquad (n-j=l).
\end{aligned}$$

Now we have

$$f(n) = 2^{(n-2)^2-8} \sum_{l=2}^{n-1} (n-l)(l-1) 2^{4l} \quad (n \geq 3).$$

Note that

$$f(1989) = 2^{(1987)^2-8} \sum_{l=2}^{1988} (1989-l)(l-1) 16^l$$

and

$$f(1990) = 2^{(1988)^2-8} \sum_{l=2}^{1989} (1990-l)(l-1) 16^l.$$

Since $1989 = 13 \cdot 153$ and $16 \equiv 3 \bmod 13$, we see that $f(1989)$ and $f(1990) \equiv 0 \bmod 13$ just in case

$$\sum_{l=2}^{1988} l(l-1) 3^l \quad \text{and} \quad \sum_{l=2}^{1989} (l-1)^2 3^l \equiv 0 \bmod 13$$

respectively. Now note that $3^3 \equiv 1 \bmod 13$,

$$(1-x)^3 \sum_{k=1}^{n} k(k+1) x^k = 2x - (n+1)(n+2)x^{n+1} + 2n(n+2)x^{n+2} - n(n+1)x^{n+3},$$

and

$$(1-x)^3 \sum_{k=1}^{n} k^2 x^k = x(1+x) - x^{n+1}[(n+1)^2 - (2n(n+1)-1)x + n^2 x]$$

(see A.P. Prudnikov, Yu A. Brychkov, O.I. Marichev, *Integral and Series*, Vol. 1, Gordon and Breach, 1986, p. 604). It follows that

$$5 \sum_{l=2}^{1988} l(l-1) 3^l \equiv 3[6 - 1987 \cdot 1988 \cdot 3^{1990}] \bmod 13.$$

But $6 - 1987 \cdot 1988 \cdot 3^{1990} \equiv 6 - (-2)(-1) \cdot 3 \equiv 0 \bmod 13$, so $f(1989) \equiv 0 \bmod 13$. Also we obtain that

$$5 \sum_{l=2}^{1989} (l-1)^2 3^l \equiv 3[3 \cdot 4 - 3^{1989}(3 + (1988)^2 3^2)] \bmod 13.$$

But

$$3 \cdot 4 - 3^{1989}(3 + (1988)^2 3^2) \equiv 12 - (3 + (-1)^2 \cdot 3^2) \equiv 0 \bmod 13.$$

Thus $f(1990) \equiv 0 \bmod 13$. Let $f(1989) = 13L$ and $f(1990) = 13M$. Then

$$f(1991) = 13(4^{1991}M - 16^{1990}L) + 1989\ 2^{1989^2} \equiv 0 \bmod 13.$$

This completes the proof.

6. [1991: 196] *Proposed by Hungary.*

For a given positive integer k, denote the square of the sum of its digits by $f_1(k)$ and let $f_{n+1}(k) = f_1(f_n(k))$. Determine the value of $f_{1991}(2^{1990})$.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Let $a_n = f_n(2^{1990})$. Since

$$b_l 10^l + b_{l-1}10^{l-1} + \cdots + b_1 10 + b_0 \equiv b_l + b_{l-1} + \cdots + b_0 \bmod 9,$$

we have $f_1(k) \equiv k^2 \bmod 9$, and $a_n \equiv a_{n-1}^2 \bmod 9$, $n > 1$. This leads to $a_1 \equiv 4$, $a_2 \equiv 7 \bmod 9$. In fact $\{a_n \bmod 9\}$ is a pure modulo period sequence, with period 2. It follows that $a_{2m-1} \equiv 4$, $a_{2m} \equiv 7 \bmod 9$, for $m \geq 1$. Since $2^{1990} < 10^{600}$ we have $a_1 \leq (9 \times 600)^2 < 10^8$, $a_2 < (9 \times 8)^2 < 10^4$, $a_3 < (9 \times 4)^2 = 1296$, $a_4 \leq (9 \times 3)^2 = 729$, and $a_n \leq 729$ for all $n \geq 4$. From $9 + 9 + 9 > 21$, the sum of the digits of a_4 is 7 or 14. It follows that $a_5 = 49$ or 196. But $a_5 \equiv 4 \bmod 9$ implies $a_5 = 49$. Note that $a_6 = 169$, $a_7 = 256$, $a_8 = 169$, $a_9 = 256, \ldots$. Therefore $f_{1991}(2^{1990}) = 256$.

8. [1991: 197] *Proposed by Ireland.*

Let ABC be a triangle and ℓ the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line ℓ at E. Let the internal bisector of the angle at B meet the side AC at F and the line ℓ at G. If $GF = DE$ prove that $AC = BC$.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

It is well known that

$$BF = \frac{1}{c+a}\sqrt{ca((c+a)^2 - b^2)} \quad \text{and} \quad AD = \frac{1}{b+c}\sqrt{bc((b+c)^2 - a^2)}.$$

Since

$$BG = \frac{a \sin B}{\sin(B/2)} = 2a \cos\frac{B}{2} = \frac{1}{c}\sqrt{ac((a+c)^2 - b^2)}, \quad AE = \frac{1}{c}\sqrt{bc((c+a)^2 - b^2)},$$

we have

$$\frac{\sqrt{ca((a+c)^2 - b^2)}}{c} - \frac{\sqrt{ca((c+a)^2 - b^2)}}{c+a} = \frac{\sqrt{bc((b+c)^2 - a^2)}}{c} - \frac{\sqrt{bc((b+c)^2 - a^2)}}{b+c}.$$

It follows that

$$((a+c)^2 - b^2)a^3(b+c)^2 = ((b+c)^2 - a^2)b^3(a+c)^2$$

and

$$
\begin{aligned}
&(a^3-b^3)c^4+2(a+b)(a^3-b^3)c^3+[a^4(a+4b)-b^4(b+4a)]c^2\\
&\qquad+2ab[a^2(a^2+ab-b^2)-b^2(b^2+ab-a^2)]c+a^2b^2[a(a^2-b^2)-b(b^2-a^2)]\\
&\quad=(a-b)f(a,b,c)=0
\end{aligned}
$$

where $f(a,b,c)>0$ for all positive a, b, c. This implies that $a=b$, that is, $AC=BC$, completing the proof.

10. [1991: 197] *Proposed by Mexico.*

Determine for which positive integers k the set $X=\{1990,1991,1992,\ldots,1990+k\}$ can be partitioned into two disjoint subsets A and B such that the sum of the elements of A is equal to the sum of the elements of B.

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

Since the total of all elements of X is $1990(k+1)+(k(k+1))/2$, we see that the sums of all elements of A and B are $995(k+1)+(k(k+1))/4$. Now we have $k(k+1)$ is a multiple of 4, because $995(k+1)+(k(k+1))/4$ is an integer.

Case I. $k+1$ is a multiple of 4.

Since $k=4m-1$ for some integer m, the number of elements in X is m. Note that each sum of the $2m$ sets

$$\{1990,1990+4m-1\},\{1990+1,1990+4m-2\},\ldots,\{1990+2m-1,1990+2m\}$$

is $1990\times 2+4m-1$. Then

$$A=\{1990,1990+4m-1,1990+1,190+4m-2,\ldots,1990+m-1,190+3m\}$$

and

$$B=\{1990+m,1990+3m-1,1990+m+1,1990+3m-2,\ldots,1990+2m-1,1990+2m\}$$

are as required for X.

Case II. k is a multiple of 4.

The number of all elements in X is $4m+1$. Let the numbers of elements in A and B be $2m+1+j$, $2m-j$ $(j=0,1,2,\ldots)$, respectively. Then the sums of A and B are

$$1990(2m+1+j)+\sum_{i=1}^{2m+1+j}a_i \quad\text{and}\quad 1990(2m-j)+\sum_{i=1}^{2m-j}b_i,$$

where $\{a_i\}_{i=1}^{2m+1+j}$ and $\{b_i\}_{i=1}^{2m-j}$ partition the set $\{1,2,\ldots,4m\}$. It follows that

$$\sum a_i+\sum b_i=2m(4m+1)\quad\text{and}\quad \sum b_i-\sum a_i=1990(2j+1).$$

From the system we have

$$\sum a_i=m(4m+1)-995(2j+1),\qquad \sum b_i=m(4m+1)+995(2j+1).$$

Meanwhile from $\sum a_i \geq 1 + \cdots + 2m = m(2m+1)$, we have $m \geq 23$, that is, $k \geq 92$.

From this point on, using induction on m, we construct suitable A and B.

Let $m = 23$ ($k = 92$). Since $\sum a_i = 23 \cdot 93 - 995 = 1144$, $1 + 2 + \cdots + 2m = 23 \cdot 47 = 1081$, and $1144 - 1081 = 63$, we take

$$A = \{1990, 1990 + 64, 1990 + 2, \ldots, 1990 + 46\}$$

and

$$B = \{1990 + 1, 1990 + 47, \ldots, 1990 + 63, 1990 + 65, \ldots, 1990 + 92\}.$$

Now suppose the assertion holds for m, and that A' and B' are suitable subsets of X' where $X' = \{1990, 1991, \ldots, 1991 + 4m\}$. Note that

$$X = X' \cup \{1990 + 4m + 1, 1990 + 4m + 4, 1990 + 4m + 2, 1990 + 4m + 3\}.$$

Let

$$A = A' \cup \{1990 + 4m + 1, 1990 + 4m + 4\}, \quad B = B' \cup \{1990 + 4m + 2, 1990 + 4m + 3\}.$$

Then A and B are as required for X, and the assertion holds for $m + 1$.

Answer: $k \equiv 3 \bmod 4$ or $k \equiv 0 \bmod 4$, $k \geq 92$.

*

5. [1991: 226] *Proposed by Poland.*

Prove that every integer $k > 1$ has a multiple which is less than k^4 that can be written in the decimal system with at most four different digits.

Solution by Gillian Nonay and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Clearly we may assume that $k > 10^4$. Suppose k^4 has t digits and let S denote the set of all integers consisting of at most $t - 1$ digits, each of which is either 0 or 1. Thus

$$|S| = 2^{t-1} > 2^{4\log_{10} k - 1} = \frac{1}{2}(2^{(4\ln k)/(\ln 10)}) = \frac{1}{2}(k^{(4\ln 2)/(\ln 10)}) > k.$$

The last inequality is easily checked for all $k \geq 30$. By the pigeon hole principle there exist two integers in S, say m and n with $m > n$, that leave the same remainder on division by k. Then $m - n$ is divisible by k. Since the digits of m and n are either 0 or 1, it is readily seen that $m - n$ consists entirely of digits from the set $\{0, 1, 8, 9\}$.

8. [1991: 226] *Proposed by Thailand.*

Let a, b, c and d be non-negative real numbers such that $ab + bc + cd + da = 1$. Show that

$$\frac{a^3}{b+c+d} + \frac{b^3}{c+d+a} + \frac{c^3}{d+a+b} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; by Murray S. Klamkin, University of Alberta; and by Pavlos Maragoudakis, student, University of Athens, Greece. We present Klamkin's generalization and solution.

We will show that if $a_1, a_2, \ldots, a_n$ are non-negative real numbers with sum s and such that $a_1a_2 + a_2a_3 + \cdots + a_na_1 = k^2$, then for $r = 0, 1, 2, 3$,

$$s_n = \frac{a_1^r}{s-a_1} + \frac{a_2^r}{s-a_2} + \cdots + \frac{a_n^r}{s-a_n} \geq \frac{(2k)^{r-1}}{(n-1)n^{r-2}} \ .$$

Proof. The function $x^r/(s-x)$ is convex in x for $s > x \geq 0$ and $r = 0, 1, 2, 3$. Hence by Jensen's inequality

$$s_n \geq \frac{n(s/n)^r}{s - s/n} = \frac{s^{r-1}}{(n-1)n^{r-2}} \ .$$

It is also known [1985: 284] that

$$(a_1 + a_2 + \cdots + a_n)^2 \geq 4(a_1a_2 + \cdots + a_na_1) = 4k^2 \tag{1}$$

so that

$$s_n \geq \frac{(2k)^{r-1}}{(n-1)n^{r-2}} \ .$$

This inequality is sharp for $n = 4$ and in this case there is equality if and only if $a_1 = a_2 = a_3 = a_4$. The inequality is not sharp for $n > 4$ and this is because of (1) being sharp only when two consecutive a_i's are equal and the rest are zero.

The proposed inequality corresponds to the special case $n = 4$, $r = 3$, and $k = 1$.

[*Editor's note.* Bang points out that Klamkin further generalizes the question in *Crux* 1662 (for solution see [1992: 186]).]

10. [1991: 226] *Proposed by the U.S.S.R.*

Find all positive integers n for which every positive integer whose decimal representation has $n-1$ digits 1 and one digit 7 is prime.

Solutions by Bob Prielipp, University of Wisconsin–Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Say that an integer s is admissible if its decimal representation consists of a number of 1's and one 7. We show that the only values of n for which all n-digit admissible numbers are prime are $n = 1$ and $n = 2$. Since 7, 17, and 71 are prime, it remains to show there is an n-digit admissible number which is *not* prime for $n \geq 3$. This is clear if $n = 3k$ since the sum of the digits is $3k - 1 + 7 = 3k + 6$ which is divisible by 3 so that *no* $3k$-digit admissible number is prime. It remains to consider the cases when $n > 3$ and $n \equiv 1$, 2, 4 or 5 mod 6.

We invoke the following divisibility criterion, which can be found in *Elementary Number Theory and its Applications* (2nd edition) by Kenneth Rosen, p. 149. It amounts to observing that $1000 \equiv -1$ mod 7 (and mod 13).

Theorem. An integer q is divisible by 7 (respectively by 13) just in case the integer formed by successively subtracting and adding the 3-digit integers with (decimal expansion) formed from successive blocks of 3 decimal digits of q, grouping from the right most digit, is divisible by 7 (respectively 13).

Case 1. $n = 6k + 1$, $k \geq 1$. Then $711 \cdots 1$ is divisible by 7 and is thus composite.

Case 2. $n = 6k + 2$, $k \geq 1$. Observe first that 11111711 is divisible by 13 since $711 - 111 + 11 = 611 = 13 \times 47$. Hence the admissible number formed by adding $6(k-1)$ 1's to the right of 1111171 is also divisible by 13.

Case 3. $n = 6k + 4$, $k \geq 0$. Observe that 7111 is divisible by 13 since $111 - 7 = 104 = 13 \times 8$. Adding $6k$ 1's to the right of 7111, the result is also divisible by 13.

Case 4. $n = 6k+5$, $k \geq 0$. Observe that 11711 is divisible by 7 since $711-11 = 700$. Hence the admissible integer formed by adding $6k$ 1's to the right of 11711 is also divisible by 7.

This completes the proof.

11. [1991: 226] *Proposed by the U.S.S.R.*

Prove that on a coordinate plane it is impossible to draw a closed broken line such that

(1) the coordinates of each vertex are rational;
(2) the length of each edge is 1; *and*
(3) the line has an odd number of vertices.

Solution by Weixuan Li, University of Ottawa, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points on the plane with rational coordinates such that the length PQ is one. For $i = 1, 2$ let d_i denote the smallest positive common denominator of x_i and y_i. Then we can write $x_i = a_i/d_i$, $y_i = b_i/d_i$, where a_i, b_i and $d_i > 0$ are integers. Let d denote the smallest common denominator of $x_2 - x_1$ and $y_2 - y_1$. Then we can write $x_2 - x_1 = x/d$ and $y_2 - y_1 = y/d$, where x, y and $d > 0$ are integers. Now we claim that

(i) both d and $x + y$ are odd;
(ii) if d_1 is odd, then d_2 is odd; and
(iii) if d_1 is odd, then $a_1 + b_1$ and $a_2 + b_2$ have opposite parities.

To verify the claim, first observe that since $(x_2 - x_1)^2 + (y_2 - y_1)^2 = 1$ we have $x^2 + y^2 = d^2$. From known results on Pythagorean triples, we get $x = m^2 - n^2$, $y = 2mn$ (or $x = 2mn$, $y = m^2 - n^2$) and $d = m^2 + n^2$ for some natural numbers m, n. We may suppose the former, from which x and d clearly have the same parity. If d is even, then both x and y are even, so that x/d and y/d can be reduced simultaneously, contradicting the choice of d. Hence d is odd which implies that x and hence $x + y$ is also odd. This establishes (i).

To see (ii), note that

$$\frac{a_2}{d_2} = x_1 + (x_2 - x_1) = \frac{a_1}{d_1} + \frac{x}{d} = \frac{a_1 d + x d_1}{d d_1},$$

$$\frac{b_2}{d_2} = y_1 + (y_2 - y_1) = \frac{b_1}{d_1} + \frac{y}{d} = \frac{b_1 d + y d_1}{d d_1},$$

so $d_2 | a_2 d d_1$ and $d_2 | b_2 d d_1$. Since either $(d_2, a_2) = 1$ or $(d_2, b_2) = 1$ we must have $d_2 | d d_1$. Since d is odd, (ii) follows.

To see (iii) suppose d_1 is odd. Note that

$$\frac{a_2+b_2}{d_2} - \frac{a_1+b_1}{d_1} = (x_2+y_2)-(x_1+y_1) = \frac{x+y}{d}\ ,$$

which is odd by (i). Since d_2 is odd by (ii), we readily see that a_1+b_1 and a_2+b_2 must have opposite parities.

To prove the given problem, suppose $P_1, P_2, \ldots, P_k$ are points with rational coordinates such that $P_iP_{i+1} = 1$, for all $i = 1, 2, \ldots, k$ $(P_{k+1} = P_1)$. Let d_i denote the smallest positive common denominator of the coordinates of P_i so that $P_i \equiv (a_i/d_i,\ b_i/d_i)$ where a_i, b_i and $d_i > 0$ are integers. Without loss of generality we may assume $P_1 = (0/1,\ 0/1)$ is the origin. Since $d_1 = 1$ is odd and $a_1+b_1 = 0$ is even, both d_2 and a_2+d_2 are odd by (ii) and (iii). Moving from P_i to P_{i+1} and repeatedly applying (ii) and (iii) we see that a_i+b_i is even if and only if i is odd. Since $a_{k+1}+b_{k+1} = a_1+b_1$ is even, $k+1$ must be odd. Therefore k must be even.

[*Editor's note.* Seung-Jin Bang, Seoul, Republic of Korea, sent an article by himself and another Korean mathematician from the Newsletter of the Korean Mathematical Society (No. 27, 28) in which a generalization of this problem is considered.]

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That completes our file of solutions received to problems proposed but not used for the 31st I.M.O. in China. Send me your nice solutions as well as contests!

*   *   *   *   *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

New Books of Puzzles, by Jerry Slocum and Jack Botermans. Published by W. H. Freeman and Company, New York, 1992, ISBN 0-7167-2356-5, hardcover, 128 pages, US$19.95. *Reviewed by Andy Liu.*

This is a book every puzzle lover should have. It gives full description of 101 puzzles, both classic and modern. These are all manipulative or mechanical puzzles, illustrated with precise drawings or colour photographs by Jack Botermans.

Some of these puzzles are well-known, such as the tangrams, Sam Loyd's pony puzzle and the Tower of Hanoi. The illustrations are not of readily available commercial products, but of antique copies in the outstanding collection of puzzles of Jerry Slocum, a distinguished puzzle historian.

Other puzzles are perhaps not as well-known outside the puzzlist circle as they should be. There is David Klarner's 25Y puzzle, which consists of 25 copies of the Y-shaped pentacube, to be assembled into a 5 by 5 by 5 cube. Nine sets of dice, each consisting of three glued together in the form of the L-shaped tricube but in different number patterns, constitute the Diabolical Die puzzle of Wil Strijbos, a famed Dutch inventor. The object is to assemble a 3 by 3 by 3 cube, so that each outside face has only one kind of number, and the six faces are numbered as in an ordinary die. There is also Bill Cutler's Splitting Headache, a very difficult variation of the Soma Cube which utilizes half-cubes.

The book also contains quite a number of puzzles the reviewer has not heard of before, including the Magic Number puzzle, the solution of which was the combination to the safe of a major bank!

Among other puzzles described are dissection puzzles, geometrical vanishes, secret boxes, burrs, rope puzzles, wire puzzles as well as solitaires. The book concludes with a challenge to the readers, to explain how Gary Foshee's Arrow through the Bottle puzzle is made.

This book is a sequel to the highly successful *Puzzles Old and New* by the same authors (see my Mini-Review on [1991: 76]). As in the earlier volume, the current one contains detailed instructions on how to make most of the puzzles described, as well as how to solve some of them. The switch to a major publisher in popular mathematics will bring this superbly produced book to an even wider audience. Both volumes are highly recommended.

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PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1801. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to O. Bottema.)

If A_1, A_2, A_3 are angles of a triangle, prove that

$$\sum(1+8\cos A_1\sin A_2\sin A_3)^2\sin A_1 \ge 64\sin A_1\sin A_2\sin A_3,$$

where the summation is cyclic over the indices 1, 2, 3.

1802. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove that, for any real numbers x and y,

$$x^4 + y^4 + (x^2 + 1)(y^2 + 1) \geq x^3(1 + y) + y^3(1 + x) + x + y,$$

and determine when equality holds.

1803. *Proposed by Grant Reinhardt, Vernon, B.C.*

Let a be a real number and n and m be positive integers. Let A be the $m \times m$ matrix with every entry equal to a. Find the common entry for the matrix A^n.

1804. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

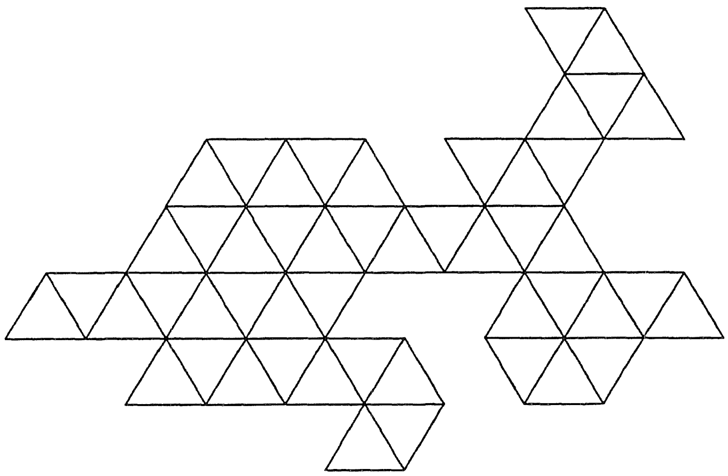
ABC is a right-angled triangle with the right angle at C, and the internal bisectors of $\angle A$ and $\angle B$ meet BC and AC at D and E respectively. Let L, M and N be the midpoints of AD, AB and BE respectively. Let $X = LM \cap BE$, $Y = MN \cap AD$, and $Z = NL \cap DE$. Prove that X, Y and Z are collinear.

1805. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Note that $1805 = 19^2 \times 5$, that is, when you divide 1805 by its last two digits, the result is the square of one more than its first two digits. Find the next (four-digit) number with this property.

1806. *Proposed by Andy Liu, University of Alberta.*

Dissect the figure into three pieces which can be reassembled into an equilateral triangle. [*Warning:* the dissecting lines need not lie along the grid lines!]



1807. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

The infinite series $a_0 + a_1 + a_2 + \cdots$ is generated recursively by: $a_0 = 5678$ and for $n \geq 1$,

$$a_n = \sin(a_0 + \cdots + a_{n-1})$$

(where the quantity $a_0 + \cdots + a_{n-1}$ is considered to be in radians). Show that the series converges and find its sum.

1808. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Three congruent circles that pass through a common point meet again in points A, B, C. $A'B'C'$ is the triangle containing the three circles and whose sides are each tangent to two of the circles. Prove that $[A'B'C'] \geq 9[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

1809. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*

Solve the recurrence $p_{n+1} = 5p_n^3 - 3p_n$ for $n \geq 0$, where $p_0 = 1$.

1810. *Proposed by C.J. Bradley, Clifton College, Bristol, England.*

(a) Find all natural numbers n so that

$$S_n = 9 + 17 + 25 + \cdots + (8n + 1)$$

is a perfect square.

(b) Find all natural numbers n so that

$$T_n = 5 + 11 + 17 + \cdots + (6n - 1)$$

is a perfect square.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1607. [1991: 15; 1992: 57] *Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.*

Find a triangle such that the length of one of its internal angle bisectors (measured from the vertex to the opposite side) equals the length of the external bisector of one of the other angles.

IV. *Comment by Diane and Roy Dowling, University of Manitoba, Winnipeg.*

Suppose that A, B and C are rational in degrees, and that the internal bisector of A equals the external bisector of B. Then Festraets–Hamoir [1992: 60] has shown that

$$\cos\frac{A}{2}\sin\frac{B}{2} = \cos^2\frac{C}{2}\ .$$

This may be written (since $A + B + C = 180°$)

$$\cos(180° - (A + B)) + \cos\left(90° + \frac{A}{2} + \frac{B}{2}\right) + \cos\left(90° + \frac{B}{2} - \frac{A}{2}\right) + 1 = 0,$$

where each of the angles

$$180° - (A + B), \quad 90° + \frac{A}{2} + \frac{B}{2}, \quad 90° + \frac{B}{2} - \frac{A}{2}$$

is rational in degrees and lies between 0° and 180°. This last equation may be written

$$\cos X + \cos\left(180° - \frac{X}{2}\right) + \cos Y + 1 = 0 \tag{1}$$

where

$$X = 180° - (A+B), \quad Y = 90° + \frac{B}{2} - \frac{A}{2}\,.$$

Gordan (*Mathematische Annalen* 12 (1877), p. 35) has shown that, for U, V and W rational in degrees with $0° < U \le V \le W < 180°$, the equation

$$\cos U + \cos V + \cos W + 1 = 0$$

has only two solutions. They are:

$$U = 72°, \quad V = 120°, \quad W = 144°$$

and

$$U = 90°, \quad V = 120°, \quad W = 120°$$

(see also our solution to *Crux* 1632 [1992: 118]). This result tells us that the equation (1) (for X and Y rational and between 0° and 180°) has only two solutions: $X = 72°$, $Y = 120°$ and $X = 120°$, $Y = 90°$. Solving

$$180° - (A+B) = 72°, \quad 90° + \frac{B}{2} - \frac{A}{2} = 120°$$

we find

$$A = 24°, \quad B = 84°, \quad \text{and consequently } C = 72°. \tag{2}$$

Solving

$$180° - (A+B) = 120°, \quad 90° + \frac{B}{2} - \frac{A}{2} = 90°$$

we find

$$A = 30°, \quad B = 30°, \quad \text{and consequently } C = 120°. \tag{3}$$

(2) and (3) were given in the published solutions [1992: 58–60]. This answers the editor's question on [1992: 60]: there are no other integral solutions. [The Dowlings also point out *Crux* 365 [1979: 114], in which the analogous problem involving two external bisectors is discussed. — *Ed.*]

* * * * *

1654*. [1991: 171; 1992: 158] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\sum \frac{x}{x + \sqrt{(x+y)(x+z)}} \le 1,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

II. *Solution by Ji Chen and Jia-Zhi Lian, Ningbo University, China.*
By Hölder's inequality,

$$\begin{aligned}\sum \frac{x}{x+\sqrt{(x+y)(x+z)}} &= \sum \frac{x}{x+\sqrt{(x+z)(y+x)}} \\ &\le \sum \frac{x}{x+\sqrt{xy}+\sqrt{zx}} \\ &= \sum \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 1.\end{aligned}$$

It is easy to generalize this to

$$\sum_{i=1}^{n} \frac{x_i}{x_i + \left(\prod_{j\neq i}(s-x_j)\right)^{1/(n-1)}} \le 1,$$

where $x_1, \ldots, x_n > 0$ and $s = x_1 + \cdots + x_n$. For example, for $n = 4$ we get

$$\begin{aligned}\sum \frac{x_1}{x_1 + \sqrt[3]{(x_1+x_3+x_4)(x_2+x_1+x_4)(x_2+x_3+x_1)}} &\le \sum \frac{x_1}{x_1+\sqrt[3]{x_1x_2^2}+\sqrt[3]{x_1x_3^2}+\sqrt[3]{x_1x_4^2}} \\ &= \sum \frac{\sqrt[3]{x_1^2}}{\sqrt[3]{x_1^2}+\sqrt[3]{x_2^2}+\sqrt[3]{x_3^2}+\sqrt[3]{x_4^2}} = 1.\end{aligned}$$

Finally, here is a conjecture by Li-Xin Pan, a student at Ningbo University, which would also generalize the given inequality: *for* $n \ge 2$ *and* $x_1, \ldots, x_n > 0$,

$$\sum_{i=1}^{n} \frac{x_i}{x_i + \prod_{j\neq i}(x_i+x_j)^{1/(n-1)}} \le \frac{n}{3}\ .$$

* * * * *

1707. [1992: 13] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*
What is the largest integer m for which an $m \times m$ square can be cut up into 7 rectangles whose dimensions are $1, 2, \ldots, 14$ in some order?

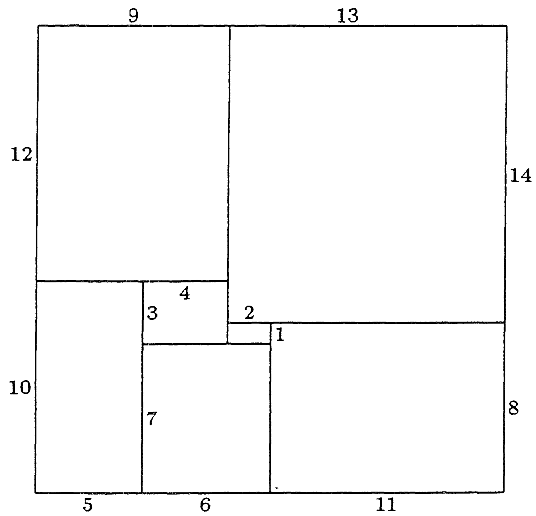
Solution by Christopher A. Jepsen, student, Grinnell College, Grinnell, Iowa.
For any $n > 0$, let $a_1, a_2, \ldots, a_{2n}$ be an arrangement of the integers $1, 2, \ldots, 2n$. It is easily shown by induction on n that

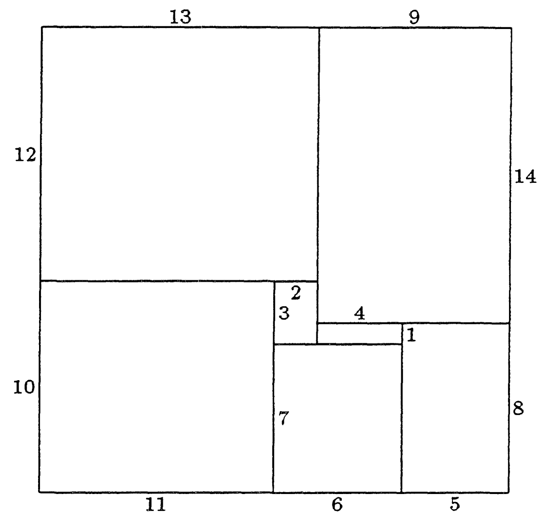
$$a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n} \le 1\cdot 2 + 3\cdot 4 + \cdots + (2n-1)2n.$$

[*Editor's note.* In fact, as one solver pointed out, this is just *Crux* 1630 [1992: 114]!] It follows that the maximum area of n rectangles with side lengths $1, 2, \ldots, 2n$ is $1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1)2n$. Therefore the maximum side length of a square dissected into n rectangles with side lengths $1, 2, \ldots, 2n$ is the largest integer m such that

$$m \leq \sqrt{1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1)2n}\ .$$

If $n = 7$, $m = 22$. Here are two solutions for dissecting a 22×22 square into 7 rectangles with side lengths $1, 2, \ldots, 14$:





Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

Guy and the proposer give solutions for analogous problems when the number of rectangles is different than seven. Guy gives an example of a 13×13 *square cut up into five rectangles of side lengths* $1, 2, \ldots, 10$. *(Guy also shows that for the* 6*-rectangle problem there are no square solutions at all.) The proposer found a* 27×27 *square dissected into* 8 *rectangles of side lengths* $1, 2, \ldots, 16$, *and also a* 32×32 *square dissected into* 9 *rectangles of side lengths* $1, 2, \ldots, 18$. *In all these cases the square is provably maximal as above. Readers are invited to discover such examples for themselves and to find maximal-sized squares for even larger numbers of rectangles.*

Engelhaupt and Guy also investigate smaller *squares composed of* 7 *rectangles with side lengths* $1, 2, \ldots, 14$, *with conflicting reports as to which sizes are possible.*

For an interesting discussion on how to find 5*-rectangle solutions forming different-sized squares, see Ruth Tancrede's letter on p.* 398 *of the May* 1992 *issue of* Mathematics Teacher.

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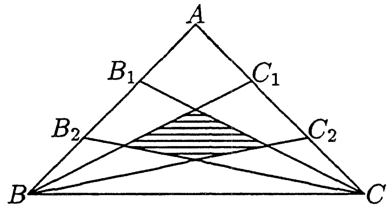
1708. [1992: 13] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

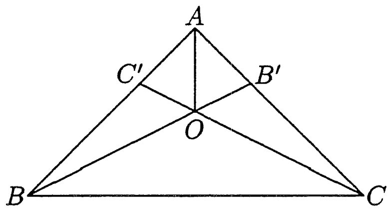
ABC is a triangle of area 1, and B_1, B_2 and C_1, C_2 are the points of trisection of edges AB and AC respectively. Find the area of the quadrilateral formed by the four lines CB_1, CB_2, BC_1, BC_2.



Solution by David Hankin, Brooklyn, N.Y.

Lemma. If ΔABC has B' on CA and C' on AB, and $BB' \cap CC' = O$, then

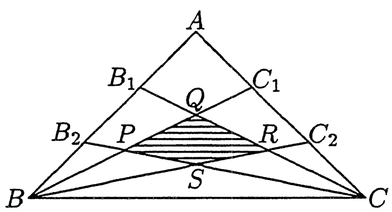
$$\frac{BO}{OB'} = \frac{BC'}{C'A}\left(\frac{AB'}{B'C} + 1\right).$$



Proof. Letting $[XYZ]$ denote the area of ΔXYZ,

$$\begin{aligned}\frac{BO}{OB'} = \frac{[ABO]}{[AB'O]} = \frac{[CBO]}{[CB'O]} &= \frac{[ABO]+[CBO]}{[ACO]} \\ = \frac{[CBO]}{[ACO]}\left(\frac{[ABO]}{[CBO]} + 1\right) &= \frac{BC'}{C'A}\left(\frac{AB'}{B'C} + 1\right). \quad \square\end{aligned}$$

Now call the shaded quadrilateral $PQRS$ as shown. From the lemma,



$$\frac{BP}{PC_1} = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}$$

and

$$\frac{BQ}{QC_1} = 2\left(\frac{1}{2} + 1\right) = 3,$$

so $BP = \frac{3}{7}BC_1$, $QC_1 = \frac{1}{4}BC_1$, and $PQ = (1 - \frac{3}{7} - \frac{1}{4})BC_1 = \frac{9}{28}BC_1$. Thus

$$BP : PQ : QC_1 = 12 : 9 : 7.$$

Similarly,

$$\frac{BS}{SC_2} = \frac{1}{2}(2+1) = \frac{3}{2} \quad \text{and} \quad \frac{BR}{RC_2} = 2(2+1) = 6,$$

so $BS = \frac{3}{5}BC_2$, $RC_2 = \frac{1}{7}BC_2$, and $RS = (1 - \frac{3}{5} - \frac{1}{7})BC_2 = \frac{9}{35}BC_2$. Thus

$$BS : SR : RC_2 = 21 : 9 : 5.$$

Now

$$[BPS] = \frac{BP}{BQ} \cdot \frac{BS}{BR}[BQR] = \frac{12}{21} \cdot \frac{21}{30}[BQR] = \frac{2}{5}[BQR],$$

so

$$[PQRS] = \frac{3}{5}[BQR] = \frac{3}{5} \cdot \frac{BQ}{BC_1} \cdot \frac{BR}{BC_2}[BC_1C_2] = \frac{3}{5} \cdot \frac{3}{4} \cdot \frac{6}{7} \cdot \frac{1}{3} = \frac{9}{70}.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; BENO ARBEL, Tel-Aviv University, Israel; SAM BAETHGE, Science Academy, Austin, Texas; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JULIO CESAR DE LA YNCERA, Gaithersburg, Maryland; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia (two solutions); FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, The Netherlands; A.W. WALKER, Toronto, Ontario; and the proposer.

Higham points out that his article "Fermat, last problem" in Mathematical Mayhem *Volume* 3, *Issue* 5 (1991) *generalizes the problem to the case when AB and AC are each divided into n equal parts and the corresponding lines drawn. He finds the areas of all* n^2 *regions formed. The article, written when Higham edited* Mayhem, *was suggested by problem* 25 *of the* 1982 *Fermat Contest, which was to find the area of quadrilateral* B_1B_2PQ *in the above diagram.*

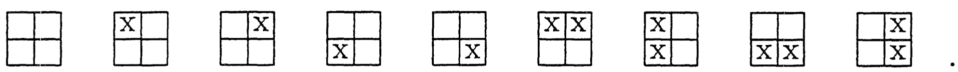
Janous and Konečný gave generalizations. de la Yncera sent in a related problem which he had proposed in 1989 *for a Cuban mathematical contest.*

The problem came from an undated sangaku *(Japanese temple geometry problem).*

*   *   *   *   *

1709. [1992: 13] *Proposed by Bill Sands, University of Calgary.*

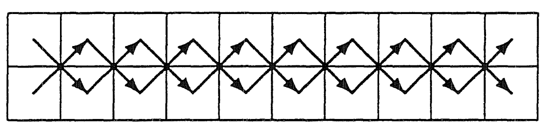
Find the number of ways to choose cells from a $2 \times n$ "chessboard" so that no two chosen cells are next to each other **diagonally** (one way is to choose no cells at all). For example, for $n = 2$ the number of ways is 9, namely



Solution by Steven Robertson, student, University of Calgary.

The required number is f_{n+2}^2, where the f_n's are the Fibonacci numbers.

In the nth column, whether the top square can be chosen depends only on the bottom square in the $(n-1)$st column; similarly for the other two squares in these columns. Thus the chessboard can be separated into two sequences of squares:



i.e., the black squares and the white squares, which are independent of each other. Thus the problem is equivalent to finding the number of ways of choosing two subsets of a row of n (white or black) squares, with the condition in each subset that if the kth square is chosen then the $(k+1)$st square cannot be chosen. The number of ways of choosing each such subset is just the Fibonacci number f_{n+2} ($f_1 = f_2 = 1,\ f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$).

Also solved by ALBRECHT BEUTELSPACHER, J. CHRIS FISHER, and ROBERT E. JAMISON, Clemson University, Clemson, South Carolina; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; CURTIS COOPER and ROBERT E. KENNEDY, Central Missouri State University, Warrensburg; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; ZHI-CHENG GAO, Carleton University, Ottawa, and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; RICHARD K. GUY, University of Calgary; DAVID HANKIN, Brooklyn, N.Y.; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CHARLES H. JEPSEN, Grinnell College, Grinnell, Iowa; NEVEN JURIĆ, Zagreb, Croatia; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; ANDY LIU, University of Alberta; W. MOSER, McGill University; P. PENNING, Delft, The Netherlands; DAVID SINGMASTER, South Bank Polytechnic, London, England; DAVID R. STONE, Georgia Southern University, Statesboro; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One reader sent in the correct answer without proof.

About half a dozen solvers found the short elegant solution given above. The other solutions (including the proposer's!) were more complicated.

Guy, Jurić and Liu looked at the same problem on a $3 \times n$ board, with less striking results. Singmaster considered forbidding other types of adjacencies (horizontal, vertical) instead of, or as well as, diagonal adjacency. Moser solved the problem on the circular $2 \times n$ board, obtaining a nice answer in terms of Lucas numbers which is left to the reader to discover!

The result (used in the above proof), that a row of n squares contains f_{n+2} subsets having no adjacent squares, is fairly well known, and can easily be proved by recurrence.

* * * * *

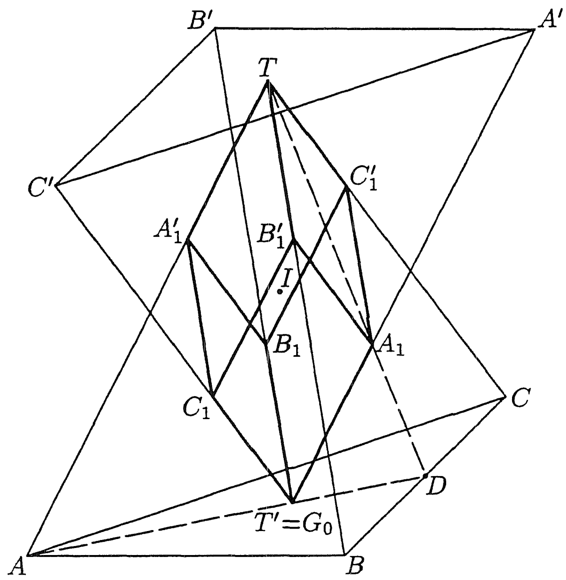
1710. [1992: 13] *Proposed by P. Penning, Delft, The Netherlands.*

A tetrahedron $TABC$ of volume 1 has top T and base an equilateral triangle ABC. The projection T' of T onto the base is the centre of ABC. Point I is the midpoint of TT'. A congruent tetrahedron $T'A'B'C'$ is generated by reflecting the original one through I (so $\overrightarrow{AI}=\overrightarrow{IA'}$, etc.). Find the volume that the two tetrahedra have in common.

Combination of solutions by Iliya Bluskov, Technical University, Gabrovo, Bulgaria, and Jordi Dou, Barcelona, Spain.

The answer is 2/9. We shall prove this with weaker requirements.

Let $\mathcal{T}$ be *any* tetrahedron $TABC$ of volume 1, G_0 the centroid of ABC, I the midpoint of TG_0. Let S be the symmetry with centre I, and put $S(A) = A'$, etc. (Thus $T' = G_0$.) Let $\mathcal{T}' = T'A'B'C'$. Clearly $\Pi = \mathcal{T} \cap \mathcal{T}'$ is a parallelepiped.



Let D be the midpoint of BC. Because $T'A' \| AT$ and A, T', D are collinear, $T'A'$ must intersect TD, in point A_1 say. Moreover

$$\frac{DA_1}{DT} = \frac{DT'}{DA} = \frac{1}{3},$$

so A_1 must be the centroid of ΔTBC. Thus

$$\frac{T'A_1}{T'A'} = \frac{DA_1}{DT} = \frac{1}{3},$$

so $T'A_1 = \frac{1}{3}TA$. Analogously, the other corners B_1, C_1, etc. of Π are the centroids of triangles TCA, TAB, etc., and so $T'B_1 = \frac{1}{3}TB$, $T'C_1 = \frac{1}{3}TC$. Thus the required volume of Π is 1/27 of the volume of the parallelepiped generated by the vectors $\overrightarrow{TA}$, $\overrightarrow{TB}$, $\overrightarrow{TC}$, which in turn is 6 times the volume of the tetrahedron $TABC$, which is 1. Hence

$$\text{volume } (\Pi) = \frac{1}{27} \cdot 6 \cdot 1 = \frac{2}{9}.$$

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia; MARCIN E. KUCZMA, Warszawa, Poland; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Janous' solution was similar to the above.

Bluskov located the problem (slightly generalized) as 3.28 *in V.V. Prasolov & I.F. Sharigin,* Problems in Space Geometry, *Moscow,* 1989.

The proposer's original problem had a pyramid with a regular n-gon as base, but the editor decided to present only the special case $n = 3$. The proposer also let $n \to \infty$, which amounts to asking for the intersection of two cones, and obtains the volume 1/4 (which readers may like to check for themselves).

* * * * *

1711*. [1992: 43] *Combination of independent proposals by Herta T. Freitag, Roanoke, Virginia, and by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < r < 1$, let $x_1, x_2, \ldots, x_k$ be fixed positive numbers, and define

$$x_{n+k+1} = \sum_{i=1}^{k} x_{n+i}^r$$

for $n \geq 0$. Show that the sequence $\{x_n\}$ converges and determine its limit.

Solution by H.L. Abbott, University of Alberta.

Let L be defined by $L^{1-r} = k$. Let

$$m = \min\{x_1, x_2, \ldots, x_k, L\}, \quad M = \max\{x_1, x_2, \ldots, x_k, L\}.$$

We show first by induction that $m \leq x_n \leq M$ for all $n \geq 1$. It is valid for $1 \leq n \leq k$. For $n > k$ we have, since $0 < r < 1$,

$$x_n = x_{n-1}^r + x_{n-2}^r + \cdots + x_{n-k}^r \geq km^r = L^{1-r}m^r \geq m^{1-r}m^r = m.$$

Similarly, one finds that $x_n \leq M$. Thus $\{x_n\}$ is bounded. Let $A = \liminf x_n$ and $B = \limsup x_n$. Then

$$A = \liminf(x_{n-1}^r + x_{n-2}^r + \cdots + x_{n-k}^r) \geq kA^r$$

from which we get $A^{1-r} \geq k$. It follows that $A \geq L$. Similarly, $B \leq L$ so that $A = B = L$. Thus $\lim x_n$ exists and equals $L = k^{1/(1-r)}$.

Also solved by IAN GOLDBERG, student, University of Waterloo; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the second proposer (Janous).

Janous and Klamkin gave generalizations in which the right side of the given recurrence was replaced by a more general function.

A similar problem was presented by Freitag at a mathematics meeting in Orono, Maine in August 1991, *and appeared (in a special case) in the August* 1991 AMS Abstracts, *p.* 345.

* * * * *

1712. [1992: 44] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the minimum value of

$$\frac{16\sin^2(A/2)\sin^2(B/2)\sin^2(C/2) + 1}{\tan(A/2)\tan(B/2)\tan(C/2)}$$

where A, B, C are the angles of a triangle.

Solution by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.

Using

$$\tan\frac{A}{2} = \frac{r}{s-a} = \sqrt{\frac{(rs)^2}{s^2(s-a)^2}} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \text{etc.}$$

and

$$\sin\frac{A}{2} = \sqrt{\frac{\tan^2(A/2)}{1+\tan^2(A/2)}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)+(s-b)(s-c)}}$$
$$= \sqrt{\frac{(s-b)(s-c)}{s(2s-a-b-c)+bc}} = \sqrt{\frac{(s-b)(s-c)}{bc}}\,, \quad \text{etc.,}$$

where a, b, c, r, s denote the sides, inradius and semiperimeter of the triangle, we have by the A.M.–G.M. inequality

$$\frac{16\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)+1}{\tan(A/2)\tan(B/2)\tan(C/2)}$$
$$= \frac{16[s(s-a)(s-b)(s-c)]^{3/2}}{(abc)^2} + \frac{s\sqrt{s}}{[(s-a)(s-b)(s-c)]^{1/2}}$$
$$= \frac{16[s\prod(s-a)]^{3/2}}{(abc)^2} + \frac{1}{4}\frac{s\sqrt{s}}{\prod\sqrt{s-a}} + \frac{1}{4}\frac{s\sqrt{s}}{\prod\sqrt{s-a}} + \frac{1}{4}\frac{s\sqrt{s}}{\prod\sqrt{s-a}} + \frac{1}{4}\frac{s\sqrt{s}}{\prod\sqrt{s-a}}$$
$$\geq 5\left(\frac{16[s\prod(s-a)]^{3/2}}{(abc)^2}\left(\frac{1}{4}\frac{s\sqrt{s}}{\prod\sqrt{s-a}}\right)^4\right)^{1/5} = 5\left(\frac{1}{4^5}\sqrt{\frac{s^3}{\prod(s-a)}}\cdot\frac{4^3s^6}{(abc)^2}\right)^{1/5}$$
$$= 5\left(\frac{1}{4^5}\sqrt{\frac{[(s-a)+(s-b)+(s-c)]^3}{\prod(s-a)}}\cdot\left(\frac{(a+b+c)^3}{abc}\right)^2\right)^{1/5}$$
$$\geq 5\cdot\frac{1}{4}(3^{3/2}\cdot 3^6)^{1/5} = \frac{5}{4}\cdot 3^{3/2} = \frac{15\sqrt{3}}{4}\,,$$

equality holding when $a = b = c$. Thus the minimum value is $15\sqrt{3}/4$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; and the proposer. One incorrect solution was received.

Janous rewrites the inequality arising from the problem as

$$\frac{r}{R} + \frac{R}{r} \geq \frac{15\sqrt{3}}{4}\cdot\frac{R}{s}\,, \tag{1}$$

where R is the circumradius of the triangle, and notes that $R/s \geq 2/3^{3/2}$ (item 5.3 *of Bottema et al,* Geometric Inequalities*) then implies*

$$\frac{r}{R} + \frac{R}{r} \geq \frac{5}{2}\,,$$

which is item 3.1, *page* 165 *of Mitrinović, Pečarić, Volenec,* Recent Advances in Geometric Inequalities.

The proposer also verifies (1)*, but in the equivalent form*

$$\frac{F}{R^2}+\frac{F}{r^2}\ge\frac{15\sqrt{3}}{4}\ ,$$

where F is the area of the triangle. This inequality says that the sum of the reciprocals of the areas of the circumcircle and the incircle of a triangle of fixed area is minimized when the triangle is equilateral.

Kuczma calculates that the 16 *in the given expression can be replaced by any number between* 0 *and* 128/7*, and the resulting expression will still attain its minimum when the triangle is equilateral.*

* * * * *

1713. [1992: 44] *Proposed by Jeremy Bem, student, Ithaca H.S., Ithaca, N.Y.*

For a fixed positive integer n, let K be the area of the region

$$\left\{z:\sum_{k=1}^{n}\left|\frac{1}{z-k}\right|\ge 1\right\}$$

in the complex plane. Prove that $K\ge\pi(11n^2+1)/12$.

Solution by Marcin E. Kuczma, Warszawa, Poland.

A point $z=x+iy$ lies in the region considered if and only if the harmonic mean of the n numbers $|z-1|,\ldots,|z-n|$ is not greater than n. Therefore the given region contains the region D defined by the analogous condition, with harmonic mean replaced (wastefully!) by quadratic mean, the latter equal to

$$\begin{aligned}\left(\frac{1}{n}\sum_{k=1}^{n}|z-k|^2\right)^{1/2} &= \left(\frac{1}{n}\sum_{k=1}^{n}((x-k)^2+y^2)\right)^{1/2} = \left(x^2-\frac{2x}{n}\sum_{k=1}^{n}k+\frac{1}{n}\sum_{k=1}^{n}k^2+y^2\right)^{1/2}\\ &= \left(x^2-(n+1)x+\frac{(n+1)(2n+1)}{6}+y^2\right)^{1/2}\\ &= \left(\left(x-\frac{n+1}{2}\right)^2+y^2+\frac{n^2-1}{12}\right)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}D &= \left\{x+iy:\left(x-\frac{n+1}{2}\right)^2+y^2+\frac{n^2-1}{12}\le n^2\right\}\\ &= \left\{x+iy:\left(x-\frac{n+1}{2}\right)^2+y^2\le\frac{11n^2+1}{12}\right\},\end{aligned}$$

which is a disc of radius $\sqrt{(11n^2+1)/12}$, and the claim follows.

Also solved by KEE-WAI LAU, Hong Kong; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer (whose solution was similar to Kuczma's).

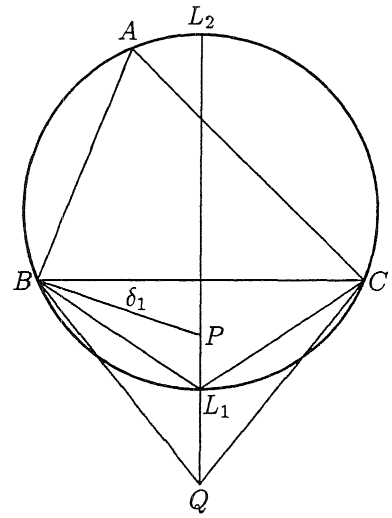
* * * * *

1714. [1992: 44] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let P and Q be two points lying in the interior of $\angle BAC$ of ΔABC, such that the line PQ is the perpendicular bisector of BC, and such that $\angle ABP + \angle ACQ = 180°$. Prove that $\angle BAP = \angle CAQ$.

I. *Solution by Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.*

Suppose that PQ intersects the circumcircle at L_1 (below BC) and L_2 (above BC). Let $\angle CBP = \delta_1$ and $\angle BCQ = \delta_2$. As $B + \delta_1 + C + \delta_2 = 180°$ (given), $\delta_1 + \delta_2 = A$, and so we can put



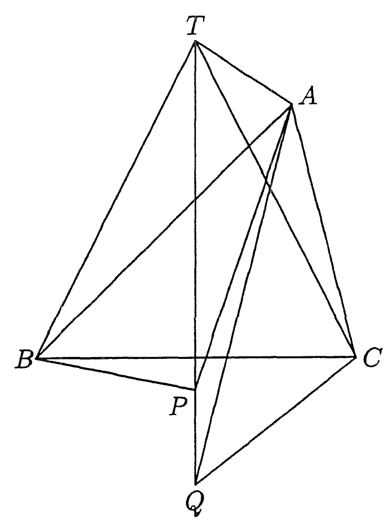
$$\delta_1 = \frac{A}{2} - \delta, \quad \delta_2 = \frac{A}{2} + \delta.$$

Therefore BL_1 is the internal bisector of $\angle PBQ$ [since $A + \angle BL_1C = 180°$, $\angle CBL_1 = \angle BCL_1 = A/2$; thus $\angle L_1BQ = \angle L_1CQ = \delta_2 - A/2 = \delta = A/2 - \delta_1 = \angle PBL_1$ — *Ed.*], which means that $PB/QB = PL_1/QL_1$. Thus B (and also C) lies on the locus of all points X such that $PX/QX = PL_1/QL_1$, which is a circle, and must therefore be the circumcircle of BCL_1, that is, the circumcircle of ABC. Therefore AL_1 is the bisector of $\angle PAQ$, and we are done (since AL_1 also bisects $\angle BAC$).

[*Editor's note.* If P lies above the line BC, there are only minor changes in the argument.]

II. *Solution by the proposer.*

Let T be the point of intersection of PQ with the circumcircle of ΔACQ, other than Q. Then we get



$$\angle ATP = \angle ATQ = 180° - \angle ACQ = \angle ABP \qquad (1)$$

and

$$\angle CAQ = \angle CTQ = \angle CTP. \qquad (2)$$

From (1), A, T, B, P are concyclic. Therefore we get

$$\angle BAP = \angle BTP. \qquad (3)$$

As PQ is the perpendicular bisector of BC, we have

$$\angle BTP = \angle CTP. \qquad (4)$$

From (2), (3) and (4), we obtain $\angle BAP = \angle CAQ$.

[*Editor's note.* If $\angle B < \angle C$, there are only minor changes in the argument.]

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. J. BRADLEY, Clifton College, Bristol, England; JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; ANDY LIU, University of Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and JOSÉ YUSTY PITA, Madrid, Spain.

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1715. [1992: 44] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Evaluate the sum

$$\sum_{k=0}^{n-2} \frac{1}{k!}\left(\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n-k}\frac{1}{(n-k)!}\right)$$

for $n \geq 2$.

Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

Let

$$\begin{aligned} f(n) &= \sum_{k=0}^{n-2} \frac{1}{k!}\left(\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n-k}\frac{1}{(n-k)!}\right) \\ &= \frac{1}{0!2!}-\frac{1}{0!3!}+\cdots+(-1)^{n-1}\frac{1}{0!(n-1)!}+(-1)^{n}\frac{1}{0!n!} \\ &+ \frac{1}{1!2!}-\frac{1}{1!3!}+\cdots+(-1)^{n-1}\frac{1}{1!(n-1)!} \\ &+ \cdots \\ &+ \frac{1}{(n-3)!2!}-\frac{1}{(n-3)!3!} \\ &+ \frac{1}{(n-2)!2!}\,. \end{aligned}$$

Summing the terms along the upward-sloping diagonals gives

$$\begin{aligned} f(n) &= \sum_{k=2}^{n}\left(\sum_{i=2}^{k}(-1)^i\frac{1}{(k-i)!i!}\right) = \sum_{k=2}^{n}\frac{1}{k!}\sum_{i=2}^{k}(-1)^i\binom{k}{i} \\ &= \sum_{k=2}^{n}\frac{1}{k!}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}-\binom{k}{0}+\binom{k}{1}\right). \end{aligned}$$

Now the first term (the summation) within the large brackets is zero, since it is the expansion of $(1+x)^k$ with $x=-1$. Therefore (by telescoping)

$$f(n) = \sum_{k=2}^{n}\left(-\frac{1}{k!}+\frac{1}{(k-1)!}\right) = 1-\frac{1}{n!}\,.$$

Also solved by H.L. ABBOTT, University of Alberta; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, England; IAN GOLDBERG, student, University of Waterloo; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario (two solutions); CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The ideas in the above solution were used by several other solvers. Induction was another popular technique.

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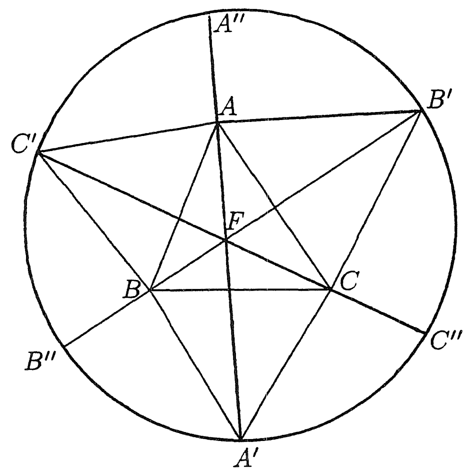
1716. [1992: 44] *Proposed by Jordi Dou, Barcelona, Spain.*

Equilateral triangles $A'BC, B'CA, C'AB$ are erected outward on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the other intersections of Ω with the lines $A'A, B'B, C'C$, respectively. Prove that $AA'' + BB'' + CC'' = AA'$. [It is known that AA', BB' and CC' are concurrent; e.g., see [1991: 308].]

Solution by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.



Observe that a rotation of 60° takes $\Delta ACC'$ into $\Delta AB'B$ and it follows that $CC' = BB'$ and $\angle BFC' = 60°$. By symmetry $AA' = BB' = CC'$ and AA', BB', CC' meet at angles of 60°. Quadrilateral $AFBC'$ is cyclic, so by Ptolemy's theorem we have

$$FC' \cdot c = AF \cdot c + FB \cdot c$$

(where $AB = c$), or $FC' = FA + FB$. Similarly,

$$FA' = FB + FC, \quad FB' = FC + FA, \quad FC' = FA + FB. \tag{1}$$

[Thus $FA + FB + FC = FA + FA' = AA'$.] Next we have

$$FA' + FB' + FC' = FA'' + FB'' + FC'' \tag{2}$$

from problem 7 on [1992: 234]. Using (1) we can write

$$FA' + FB' + FC' = 2(FA + FB + FC), \tag{3}$$

while

$$FA'' + FB'' + FC'' = FA + FB + FC + AA'' + BB'' + CC'',$$

which implies with (2), (3) and (1) that

$$AA'' + BB'' + CC'' = FA + FB + FC = AA'.$$

[*Editor's note.* Heuver also gave a proof of (2). Minor changes are required in the above solution if one of the angles of ΔABC is greater than 120° (so that F lies outside ΔABC).]

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; IAN GOLDBERG, student, University of Waterloo; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Many solvers used the earlier Olympiad Corner problem on [1992: 234], *and/or mentioned the related problem* Crux 1658 [1992: 181]. *One could also point out the even more related* Crux 1660 [1992: 184] *and doubtless several others. References for the properties of the Fermat point F abound; e.g., Bottema's* Elementaire Meetkunde *(p. 137, 138) and Coxeter's* Introduction to Geometry *(p. 21, 22) were given by readers.*

* * * * *

1717. [1992: 44] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

For each positive integer n, let $f(n)$ denote the number of ordered pairs (x, y) of nonnegative integers such that $n = x^2 - y^2$. For example, $f(9) = 2$ as $9 = 3^2 - 0^2 = 5^2 - 4^2$ are the only representations. Find a formula for $f(n)$.

Solution by R.P. Sealy, Mount Allison University, Sackville, New Brunswick.

The formula for $f(n)$ is given by:

(case 1) $f(n) = 0$ if n is congruent to 2 mod 4,
(case 2) $f(n) = [(\tau(n) + 1)/2]$ if n is congruent to 1 or 3 mod 4,
(case 3) $f(n) = [(\tau(n/4) + 1)/2]$ if n is congruent to 0 mod 4.

Here [ ] is the greatest integer function and $\tau(n)$ is the number of divisors of n.

Case 1. We first note that a square is congruent to 0 or 1 mod 4. Hence the difference of two squares is congruent to 0, 1 or 3 mod 4. Therefore an integer which is congruent to 2 mod 4 cannot be written as the difference of two squares.

Otherwise, if

$$n = x^2 - y^2 = (x + y)(x - y) = ab$$

with $a \geq b$, we can solve the system $x + y = a$, $x - y = b$ to obtain

$$x = \frac{a + b}{2}, \quad y = \frac{a - b}{2} .$$

Since x and y are integers, we conclude that n can be written as the difference of two squares if and only if n can be written as the product of two factors, a and b, of the same parity. The question becomes: in how many ways can this be done with $a \geq b$?

Note that a can equal b if and only if n is a square. List the divisors of n: $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$. The number of divisors is even if and only if n is not a square. (Each divisor has a "companion" if and only if n is not a square.)

Case 2. If n is odd, then all of its divisors are odd. Therefore there are $\tau(n)/2$ choices for the divisor a if n is not a square, and $(\tau(n)-1)/2+1=(\tau(n)+1)/2$ choices for the divisor a if n is a square. The expression $[(\tau(n)+1)/2]$ takes care of both cases.

Case 3. If n is congruent to 0 mod 4 and if $n=ab$, then at least one of the divisors must be even. Since a and b are of the same parity, both must be even. Then in this case we have a pair (a,b), both even, of divisors of n if and only if the pair $(a/2,b/2)$ are integer factors of $n/4$. The method of Case 2 applied to the integer $n/4$ gives the answer.

Also solved by H.L. ABBOTT, University of Alberta; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; C.J. BRADLEY, Clifton College, Bristol, England; IAN GOLDBERG, student, University of Waterloo; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, State University of New York, Oneonta; P. PENNING, Delft, The Netherlands; LAWRENCE SOMER, Catholic University of America, Washington, D.C.; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer. Three incorrect solutions were received.

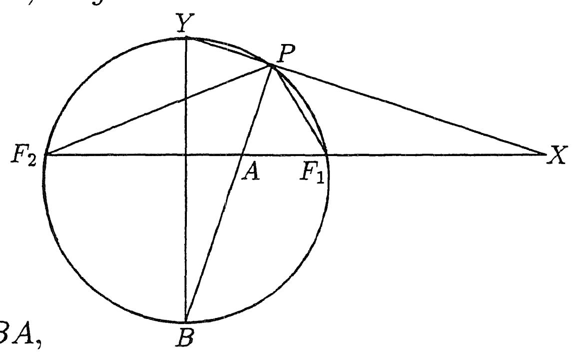
* * * * *

1718. [1992: 44] *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*

Let $\mathcal{C}$ be a central conic with foci F_1 and F_2, and let X and Y be the points where the tangent to $\mathcal{C}$ at a point P on $\mathcal{C}$ meets the axes (extended) of $\mathcal{C}$. Prove that

$$PX\cdot PY=PF_1\cdot PF_2.$$

Solution by C. Festraets-Hamoir, Brussels, Belgium.



Soient A et B les points où la normale en P à l'ellipse $\mathcal{C}$ rencontre les axes de cette conique. BY est la médiatrice de F_1F_2, et la tangente et la normale en P sont les bissectrices de l'angle $\angle F_1PF_2$; donc BY est un diamètre du cercle circonscrit au triangle F_1PF_2.

$$\angle PXA=90°-\angle PAX=90°-\angle F_2AB=\angle YBA,$$

donc $\Delta BPY\sim\Delta XPA$ et, par conséquence, $BP/XP=PY/PA$, d'où

$$PX\cdot PY=PA\cdot PB. \qquad (1)$$

Également, les triangles $\Delta BPF_2\sim\Delta F_1PA$, car $\angle F_2PB=\angle APF_1$ et $\angle F_2BP=\angle F_2F_1P$, d'où $BP/F_1P=PF_2/PA$, et

$$PA\cdot PB=PF_1\cdot PF_2. \qquad (2)$$

De (1) et (2), on a bien $PX\cdot PY=PF_1\cdot PF_2$.

Editor's Comment. Of the solvers listed below, both Seimiya and Penning used an argument similar to the featured solution; they added that by interchanging the words "normal" and "tangent" (and consequently switching X with A and Y with B) the proof for an ellipse is converted into a proof for the hyperbola.

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain (two solutions); JORDI DOU, Barcelona, Spain; IAN GOLDBERG, student, University of Waterloo; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The submitted solutions came in two varieties: one type used similar triangles as in the featured solution, the others used coordinates. Several of the latter were also quite elegant. Bellot sent in one of each type.

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OENE BOTTEMA

An eminent geometer, very familiar to readers of *Crux Mathematicorum*, has been lost to us with the death of O. Bottema on November 30, 1992, at the age of 90 years.

Professor Bottema was a frequent contributor to *Crux* up until 1987, publishing seven articles as well as numerous problems and solutions. But he is probably best known to most readers from the book *Geometric Inequalities*, coauthored with Djordjević, Janić, Mitrinović, and Vasić in 1968. As most of the ubiquitous references in *Crux* to this influential book abbreviated the list of authors to "Bottema et al", the book came to be called, by many *Crux* readers expert in the field, and occasionally in the pages of *Crux*, fondly as "Bottema's bible"! *Geometric Inequalities* easily ranks (with Johnson's *Advanced Euclidean Geometry*) as the reference book most frequently consulted by this editor in some seven years in the post. (My own copy I treasure, as a gift from Léo Sauvé.)

Professor Bottema's list of mathematical publications, according to a commemorative article in the November 1987 issue of *Nieuw Archief voor Wiskunde*, consisted of an impressive 447 articles and 10 books.

The above information was thoughtfully supplied by *Crux* faithfuls P. Penning and D.J. Smeenk. To the first of these, Professor Bottema was a neighbour; to the second, a teacher. However, as is clear from their letters, to both he was also a friend. Obviously the loss is great, to *Crux* readers, to mathematics and beyond.

* * * * *

Short articles intended for publication should be sent to Dr. D. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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