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CONTENTS

RELATIONS AMONG SEGMENTS OF CONCURRENT CEVIANS OF A SIMPLEX

MURRAY S. KLAMKIN

In [1], Satterly had given the following interesting set of identities relating the segments of the altitudes AK, BL, CM of a triangle ABC which concur at orthocenter H and which when extended cut the circumcircle at K', L', M', respectively:

$$\overline{HK}/\overline{AK} + \overline{HL}/\overline{BL} + \overline{HM}/\overline{CM} = 1, \tag{1}$$

$$\overline{AH}/\overline{AK} + \overline{BH}/\overline{BL} + \overline{CH}/\overline{CM} = 2, \tag{2}$$

$$\overline{AK}/\overline{AK} + \overline{BL}/\overline{BL} + \overline{CM}/\overline{CM} = 3, \tag{3}$$

$$\overline{AK'}/\overline{AK} + \overline{BL'}/\overline{BL} + \overline{CM'}/\overline{CM} = 4, \tag{4}$$

$$(\overline{AK'}/\overline{AK})^2 + (\overline{BL'}/\overline{BL})^2 + (\overline{CM'}/\overline{CM})^2 = 5 + (\Sigma \tan^2 A)/(\Sigma \tan A)^2. \tag{5}$$

We will show that these identities generalize to arbitrary concurrent cevians in triangles as well as for simplexes. A natural tool for this is barycentric coordinates [2].

Let $A_0, A_1, \ldots, A_n$ be the vertices of an n-dimensional simplex and P an arbitrary point. Also let the lines A_jP intersect the $(n-1)$-dimensional faces opposite A_j in the points K_j and finally let the points K_j' lie on A_jK_j extended such that $\overline{PK_j} = \overline{K_jK_j'}$.

Using barycentric coordinates, P will have a vector representation $\mathbf{P} = \sum_{j=0}^{n} x_j\mathbf{A}_j$ where $\sum_{j=0}^{n} x_j = 1$. It follows easily that the vector representation for the K_j is given by

$$\mathbf{K}_j = \frac{\mathbf{P} - x_j\mathbf{A}_j}{1 - x_j}.$$

Therefore

$$\mathbf{A}_i - \mathbf{K}_i = \mathbf{A}_i - \frac{\mathbf{P} - x_i\mathbf{A}_i}{1 - x_i} = \frac{\mathbf{A}_i - \mathbf{P}}{1 - x_i},$$

whence

$$\overline{A_iK_i} = \frac{\overline{A_iP}}{1 - x_i},$$

$$\overline{PK_i} = \overline{A_iK_i} - \overline{A_iP} = \frac{x_i\overline{A_iP}}{1 - x_i},$$

and so

$$\overline{A_iK_i'} = \overline{A_iK_i} + \overline{PK_i} = \frac{(1 + x_i)\overline{A_iP}}{1 - x_i}\ .$$

It now follows that

$$\sum_{i=0}^{n} \overline{PK_i}/\overline{A_iK_i} = \sum_{i=0}^{n} x_i = 1, \tag{1'}$$

$$\sum_{i=0}^{n} \overline{A_iP}/\overline{A_iK_i} = \sum_{i=0}^{n} (1 - x_i) = n, \tag{2'}$$

$$\sum_{i=0}^{n} \overline{A_iK_i}/\overline{A_iK_i} = n + 1, \tag{3'}$$

$$\sum_{i=0}^{n} \overline{A_iK_i'}/\overline{A_iK_i} = \sum_{i=0}^{n} (1 + x_i) = n + 2, \tag{4'}$$

$$\sum_{i=0}^{n} (\overline{A_iK_i'}/\overline{A_iK_i})^2 = \sum_{i=0}^{n} (1 + x_i)^2 = n + 3 + \sum_{i=0}^{n} x_i^2\ . \tag{5'}$$

For the case $n = 2$, (1)'-(4)' give the same sums (1), (2), (3), (4) as Satterly. However, here point P is arbitrary. If P is the orthocenter (for $n = 2$), $x_1 = \cot B \cot C$, etc. and (5)' also gives the same sum as Satterly.

In general, it is to be noted that if P is outside the simplex, then our distances are signed. In this case, for example, if PK_i and A_iK_i have opposite orientations, the ratio of their lengths is negative. Also, it is possible for a distance to become infinite. In these cases one must use a limit procedure. For example if $n = 2$ and P has coordinates $(-1,1,1)$, A_2P is parallel to A_1A_3 and A_3P is parallel to A_1A_2 so that $\overline{A_2K_2} = \overline{A_3K_3} = \infty$. However, if we use a limit procedure, identity (1)' becomes $-1 + 1 + 1 = 1$.

References.

[1] J. Satterly, Relations between the portions of the altitudes of a plane triangle, *Math. Gazette* 46 (1962) 50-51.

[2] H.S.M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1969, 216-220.

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THE OLYMPIAD CORNER: 89

R.E. WOODROW

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We continue this month with more of the problems proposed for, but not used at, the 28th I.M.O. in Havana. Thanks again go to Bruce Shawyer for transmitting the problems to me. Errors in translation and that arise because of editing the problems for this column are my own.

Australia 1. A perfect shuffle of a pack of cards, in the order $1,2,3,\ldots,2n$, changes the order to $n + 1,1,n + 2,2,\ldots,n - 1,2n$, i.e. the cards originally in the first n positions have been moved to the places $2,4,6,\ldots,2n$, while the remaining n cards, in their original order, fill the odd positions $1,3,5,\ldots,2n - 1$. Determine those n for which a sequence of perfect shuffles beginning with the order $1,2,\ldots,2n$ will return the deck to that state.

Australia 2. Let K_1, K_2, and K_3 be three intersecting circles which have centres O_1, O_2, and O_3, respectively, and which intersect in point P. Suppose also that K_1 and K_2 intersect again at A, K_2 intersects K_3 again at B, and K_3

meets K_1 again at C. Let X be an arbitrary point on K_1. Join X to A to meet K_2 again in Y. Also join X to C to meet K_3 again in Z.

(i) Show that Z, B, and Y are collinear.

(ii) Show that the area of triangle XYZ is less than or equal to four times the area of triangle $O_1O_2O_3$.

Belgium 1. Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

Finland 1. Let $S \subset [0,1]$ be a set of 5 points with $\{0,1\} \subset S$. Let f be a continuous real valued function mapping $[0,1]$ to itself such that f is linear on each subinterval I of $[0,1]$ with endpoints (but no interior points) in S. One wants to compute, with the aid of a computer, the extreme values of

$$g(x,t) = \frac{f(x+t) - f(x)}{f(x) - f(x-t)}$$

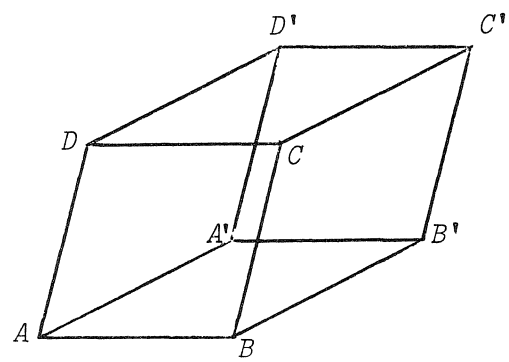
for $[x-t, x+t] \subset [0,1]$. What is the number of pairs (x,t) for which one must compute $g(x,t)$?

France 1. Let a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be nine positive real numbers. Let

$$S_1 = a_1b_2c_3, \qquad S_2 = a_2b_3c_1, \qquad S_3 = a_3b_1c_2,$$
$$T_1 = a_1b_3c_2, \qquad T_2 = a_2b_1c_3, \qquad T_3 = a_3b_2c_1.$$

Suppose that the set $\{S_1, S_2, S_3, T_1, T_2, T_3\}$ has at most two elements. Prove that $S_1 + S_2 + S_3 = T_1 + T_2 + T_3$.

France 2. Let $ABCDA'B'C'D'$ be an arbitrary parallelepiped as shown.



Establish the inequality

$$AB' + AD' + AC \leq AB + AD + AA' + AC'.$$

When is there equality?

Great Britain 1. Find, with proof, the point P in the interior of an acute-angled triangle ABC for which $(BL)^2 + (CM)^2 + (AN)^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P onto BC, CA, AB, respectively.

Great Britain 2. Prove that if x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$ then $x + y + z \leq xyz + 2$.

Greece 1. Construct a triangle ABC given its side $a = BC$, its circumradius R $(2R \geq a)$, and the difference $c^{-1} - b^{-1}$ where $c = AB$ and $b = AC$.

Greece 2. Show that if a, b, c are the lengths of the sides of a triangle and if $2s = a + b + c$, then

$$\frac{a^n}{b+c} + \frac{b^n}{a+c} + \frac{c^n}{a+b} \geq (2/3)^{n-2} s^{n-1}, \qquad n \geq 1.$$

Holland 1. Let P be an interior point of triangle ABC. Let the lines ℓ, m, and n be drawn (through P) perpendicular to AP, BP, and CP, respectively. Prove that if ℓ intersects the line BC in Q, m intersects the line AC in R, and n intersects the line AB in S, then the points Q, R, and S are collinear.

Hungary 1. (i) Suppose that $\gcd(m,k) = 1$. Prove that there are integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_k$ such that each product $a_i b_j$ $(i = 1,2,\ldots,m,\ j = 1,2,\ldots,k)$ gives a different residue when divided by mk.

(ii) Assume $\gcd(m,k) > 1$. Prove that for any integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_k$ there must be two products $a_i b_j$ and $a_s b_t$ $((i,j) \neq (s,t))$ which give the same residue when divided by mk.

Iceland 1. Five distinct numbers are drawn successively and at random from the set $\{1,\ldots,n\}$. Show that the probability that the first three numbers drawn, as well as all five numbers, can be arranged to form an arithmetic progression, is greater than $6/(n-2)^3$.

Morocco 1. Let C be a fixed circle in the plane and let ℓ be a fixed line intersecting C. Suppose that $P_1, P_2, \ldots, P_{2n}$ are fixed points on ℓ such that there is some polygon $A_1, \ldots, A_{2n}$ with vertices on C having P_i on the segment $A_i A_{i+1}$ for $i < 2n$ and P_{2n} on $A_{2n} A_1$. Show that if $V_1, \ldots, V_{2n}$ is any polygon with vertices on C having P_i on the segment $V_i V_{i+1}$ for $i < 2n$, then P_{2n} lies on $V_{2n} V_1$.

Poland 1. Let F be a one-to-one mapping of the plane into itself which maps closed rectangles into closed rectangles. Show that F maps squares into squares. Continuity of f is not assumed.

Rumania 1. The bisectors of the angles B and C of a triangle ABC intersect the opposite sides in B' and C', respectively. Prove that the straight line $B'C'$ cuts the inscribed circle.

<u>Rumania 2</u>. For any integer $r \geq 1$, determine the smallest integer $h(r) \geq 1$ such that for any partition of the set $\{1,2,\ldots,h(r)\}$ in r classes, there are integers $a \geq 0$, $1 \leq x \leq y \leq h(r)$ so that

$$a + x, \quad a + y, \quad \text{and} \quad a + x + y$$

belong to the same class.

<u>Spain 1</u>. Knowing that a_{11}, a_{22} are real numbers and that x_1, x_2, a_{12}, b_1, b_2 are complex numbers satisfying $a_{11}a_{22} = a_{12}\overline{a_{12}}$ (where $\overline{z}$ is the complex conjugate of z), consider the system of equations

$$\overline{x_1}(a_{11}x_1 + a_{12}x_2) = b_1$$
$$\overline{x_2}(\overline{a_{12}}x_1 + a_{22}x_2) = b_2.$$

(i) Give condition(s) for the system to be compatible.

(ii) Give condition(s) that the principal angle for x_1 is greater than that for x_2 by 90°.

<u>U.S.A. 1</u>. The *runs* of a decimal number $d_1d_2\ldots d_n$ are its maximal increasing or decreasing blocks of digits (here each d_i is a digit and we allow $d_1 = 0$). Thus 024379 has three runs: 024, 43 and 379. Determine the average number of runs for a decimal number in the set $\{d_1d_2\ldots d_n \mid d_k \neq d_{k+1}, k = 1,2,\ldots,n-1\}$ for any $n \geq 2$.

<u>U.S.A. 2</u>. At a party attended by n couples, each person is at any time in some conversational grouping (called a clique). A person and his or her spouse are never in the same clique, but every other pair of persons takes part in the same clique exactly once. Prove: If the total number of cliques formed during the party is k and if $n \geq 4$ then $k \geq 2n$.

<u>Vietnam 1</u>. Let $\{a_k\}$ be positive reals such that $a_1 \geq 1$ and $a_{k+1} - a_k \geq 1$ $(k = 1,2,\ldots)$. Show that

$$\sum_{k=1}^{n} \frac{1}{a_{k+1}\sqrt[1987]{a_k}} < 1987 \quad \text{for all} \quad n \geq 1.$$

<u>Vietnam 2</u>. Suppose that $\{a_k\}$ and $\{b_k\}$ are two sequences of positive numbers such that

(i) $a_k < b_k$

and

(ii) $\cos a_kx + \cos b_kx \geq -1/k$

for all reals x and all $k = 1,2,\ldots$. Show that $\lim_{k\to\infty} a_k/b_k$ exists and determine its value.

<u>West Germany 1</u>. Let S be a set of n elements. We denote the number of all permutations of S which have exactly k fixed points by $P_n(k)$. Prove

$$\sum_{k=0}^{n} (k-1)^2 \cdot P_n(k) = n!.$$

(Compare with I.M.O. problem #1 [1987: 210]!)

<u>Yugoslavia 1</u>. Prove that for each $k = 2,3,4,\ldots$ there exists an irrational number r such that

$$[r^m] \equiv -1 \bmod k$$

for every natural number m, where $[x]$ is the greatest integer $\le x$.

* * *

Next, we give solutions submitted for problems that have appeared in this column. We continue with solutions submitted in response to the challenge issued earlier this year.

<u>3</u>. [1984: 214] *1983 Swedish Mathematical Contest (Final Round).*

Prove that if there exist n positive integers $x_1,x_2,\ldots,x_n$ satisfying the n equations

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_{k-1} + 2x_k - x_{k+1} = 1, \qquad k = 2,3,\ldots,n-1 \\ -x_{n-1} + 2x_n = 1 \end{cases}$$

then n is even.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

The solution of the inhomogeneous difference equation $-x_{k-1} + 2x_k - x_{k+1} = 1$ is the sum of the general solution to the homogeneous equation $-x_{k-1} + 2x_k - x_{k+1} = 0$ and any particular solution of the inhomogeneous equation. By inspection $x_k = -k^2/2$ is such a particular solution.

The roots of the characteristic equation of the homogeneous equation $\lambda^2 - 2\lambda + 1 = 0$ are both equal to 1, so the general solution to $-x_{k-1} + 2x_k - x_{k+1} = 0$ is $x_k = a + bk$. Thus the general solution to $-x_{k-1} + 2x_k - x_{k+1} = 1$ is

$$x_k = a + bk - k^2/2.$$

Setting $k = 1$ and $k = 2$ gives

$$a + b = 1/2 + x_1,$$
$$a + 2b = 2 + x_2 = 2 + (2x_1 - 1).$$

Hence $b = x_1 + 1/2$, $a = 0$ and

$$x_k = (x_1 + 1/2)k - k^2/2, \qquad 1 \le k \le n.$$

In particular,

$$\begin{aligned} 1 &= -x_{n-1} + 2x_n \\ &= -[(x_1 + 1/2)(n - 1) - (n - 1)^2/2] + 2[(x_1 + 1/2)n - n^2/2] \\ &= (n + 1)x_1 - n^2/2 - n/2 + 1 \end{aligned}$$

or

$$(n + 1)x_1 = n^2/2 + n/2 = n(n + 1)/2.$$

Therefore $x_1 = n/2$. Since x_1 is an integer by assumption, n is even.

*

M858. [1984: 284] *Problems from KVANT, April 1984, proposed by P.B. Gusyatnikov.*

The angles α, β, and γ of a triangle satisfy the relation

$$\sin^2\alpha + \sin^2\beta = \sin\gamma.$$

(a) Find α, β, γ if the triangle is isosceles (consider all possible cases).

(b) Can the triangle have only acute angles?

(c) What values can the largest angle of the triangle assume?

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

(a) Since the given equation is symmetric with respect to α and β, we need only consider the cases $\alpha = \beta$ and $\alpha = \gamma$.

(i) $\alpha = \beta$; then $\gamma = \pi - 2\alpha$, and the equation becomes

$$2\sin^2\alpha = \sin(\pi - 2\alpha) = 2\sin\alpha\cos\alpha,$$

or

$$2\sin\alpha(\sin\alpha - \cos\alpha) = 0.$$

Thus $\sin\alpha = 0$, in which case $(\alpha,\beta,\gamma) = (0,0,\pi)$ (a degenerate case), or $\sin\alpha = \cos\alpha$, in which case $(\alpha,\beta,\gamma) = (\pi/4,\pi/4,\pi/2)$.

(ii) $\alpha = \gamma$. Then $\beta = \pi - 2\alpha$ and

$$\sin^2\alpha + \sin^2(\pi - 2\alpha) = \sin\alpha$$

or

$$\sin\alpha = \sin^2\alpha + 4\sin^2\alpha\cos^2\alpha = \sin^2\alpha(5 - 4\sin^2\alpha).$$

This becomes

$$\sin\alpha(4\sin^3\alpha - 5\sin\alpha + 1) = 0$$

or

$$\sin\alpha(\sin\alpha - 1)(4\sin^2\alpha + 4\sin\alpha - 1) = 0.$$

This gives $\sin\alpha = 0,1,(-1+\sqrt{2})/2,(-1-\sqrt{2})/2$ as possibilities. As α is not obtuse this leaves the degenerate cases $(0,\pi,0)$, $(\pi/2,0,\pi/2)$ (corresponding to parallel sides intersecting at infinity), and

$$\left(\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\pi - 2\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right]\right).$$

Taking into account the symmetry of α and β this gives nondegenerate solutions

$$(\pi/4,\pi/4,\pi/2),$$

$$\left(\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\pi - 2\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right]\right),$$

$$\left(\pi - 2\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right],\sin^{-1}\left[\frac{\sqrt{2}-1}{2}\right]\right).$$

(b), (c) Using trigonometric identities, we write

$$\begin{aligned}\sin\gamma &= \sin^2\alpha + \sin^2\beta\\ &= (1-\cos 2\alpha)/2 + (1-\cos 2\beta)/2\\ &= 1 - (\cos 2\alpha + \cos 2\beta)/2\\ &= 1 - \cos(\alpha+\beta)\cos(\alpha-\beta)\\ &= 1 + \cos\gamma\cos(\alpha-\beta).\end{aligned}$$

Suppose $\gamma \ge \alpha$, $\gamma \ge \beta$ and $\gamma < \pi/2$. Then

$$\cos(\alpha-\beta) = \frac{\sin\gamma - 1}{\cos\gamma} < 0.$$

Thus $|\alpha-\beta| > \pi/2$. Assuming without loss that $\alpha \ge \beta$ we have $\alpha \ge \alpha - \beta = |\alpha-\beta| > \pi/2 > \gamma$, a contradiction. Thus if γ is the largest angle then $\gamma \ge \pi/2$.

Suppose γ is not the largest angle. By symmetry, we may assume $\alpha \ge \beta,\gamma$. Suppose $\alpha < \pi/2$. Now

$$\sin^2\alpha = \sin\gamma - \sin^2\beta \le 1 - \sin^2\beta = \cos^2\beta = \sin^2(\pi/2-\beta).$$

Since $0 \le \alpha$ and $\pi/2 - \beta \le \pi/2$, this simplifies to $\sin\alpha \le \sin(\pi/2-\beta)$ or $\alpha \le \pi/2 - \beta$, and $\alpha + \beta \le \pi/2$. Then $\gamma \ge \pi/2 > \alpha$, a contradiction. Therefore, the largest angle must be at least $\pi/2$, answering (b).

Now let α satisfy $\pi/2 \le \alpha \le \pi$. The function $f(x) = \sin(\alpha+x) - \sin^2 x$ is continuous and decreasing on $[0,\pi-\alpha]$, with $f(0) = \sin\alpha$, $f(\pi-\alpha) = -\sin^2\alpha$. Since $-\sin^2\alpha \le \sin^2\alpha \le \sin\alpha$ there is x in $[0,\pi-\alpha]$ with $f(x) = \sin^2\alpha$. Set $\beta = x$ and $\gamma = \pi - \alpha - \beta$. Now $0 \le \beta$, $\gamma \le \pi - \alpha \le \alpha$, and $\sin^2\alpha + \sin^2\beta = \sin\gamma$. Hence the largest angle can be anything in the interval $[\pi/2,\pi)$.

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M859. [1984: 284] *Problems from KVANT, April 1984, proposed by V.P. Pikulin.*

Find the least positive number a such that any quadratic trinomial $f(x)$ for which $|f(x)| \le 1$ whenever $0 \le x \le 1$ satisfies $|f'(1)| \le a$.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Suppose $f(x) \equiv ax^2 + bx + c$ satisfies $|f(x)| \le 1$ when $0 \le x \le 1$. Then

$$\begin{aligned}|f'(1)| &= |2a + b| = |c + 3(a + b + c) - 4(a/4 + b/2 + c)| \\ &= |f(0) + 3f(1) - 4f(1/2)| \le |f(0)| + 3|f(1)| + 4|f(1/2)| \\ &\le 1 + 3 + 4 = 8.\end{aligned}$$

Conversely, $f(x) \equiv 8x^2 - 8x + 1 \equiv 8(x - 1/2)^2 - 1$ has $|f(x)| \le 1$ for x in $[0,1]$ and has $|f'(1)| = 8$. Thus the upper bound sought is 8.

*

1. [1984: 311] *7th Austrian-Polish Mathematical Competition (Poznan, Poland). First Day: July 4, 1984.*

In a given tetrahedron, the foot of the altitude issued from each vertex coincides with the incenter of the opposite face. Prove that this tetrahedron is regular.

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Label the vertices A, B, C, D. Let $\theta_1 = \angle ADB$, $\theta_2 = \angle BDC$, $\theta_3 = \angle CDA$, $\overline{DA} = a$, $\overline{DB} = b$, $\overline{DC} = c$, $\overline{AB} = c'$. Now the incenter of a triangle XYZ is located at the head of the vector

$$\frac{x\overrightarrow{OX} + y\overrightarrow{OY} + z\overrightarrow{OZ}}{x + y + z}$$

for an arbitrary origin O, where x, y, z are the lengths of the sides opposite X, Y, Z, respectively. For this problem, for a point P, let $\mathbf{P}$ denote the vector to P from D as origin. Then the incenter of triangle DAB has position vector

$$\frac{c'\mathbf{0} + b\mathbf{A} + a\mathbf{B}}{a + b + c'} = \frac{b\mathbf{A} + a\mathbf{B}}{a + b + c'} .$$

Since this is the foot of the altitude from C onto triangle DAB, we have

$$\mathbf{A}\cdot\left[\mathbf{C} - \frac{b\mathbf{A} + a\mathbf{B}}{a + b + c'}\right] = 0 = \mathbf{B}\cdot\left[\mathbf{C} - \frac{b\mathbf{A} + a\mathbf{B}}{a + b + c'}\right] .$$

Expanding this gives

$$ac \cos \theta_3 - \frac{a^2b}{a + b + c'}(1 + \cos \theta_1) = 0 = bc \cos \theta_2 - \frac{ab^2}{a + b + c'}(1 + \cos \theta_1).$$

Thus

$$\cos \theta_2 = \cos \theta_3 = \frac{ab(1 + \cos \theta_1)}{c(a + b + c')} .$$

By symmetry we find $\cos\theta_1 = \cos\theta_2 = \cos\theta_3$, and since $0 < \theta_i < \pi$ we conclude $\theta_1 = \theta_2 = \theta_3 = \theta_D$, say. Again, by symmetry all of the face angles at each vertex are equal. Call these angles θ_A, θ_B, θ_C, θ_D, where θ_A is the common face angle at A, etc. Then

$$\theta_A + \theta_B + \theta_C = \pi = \theta_A + \theta_B + \theta_D = \theta_A + \theta_C + \theta_D = \theta_B + \theta_C + \theta_D.$$

Solving, we get that $\theta_A = \theta_B = \theta_C = \theta_D = \pi/3$. Therefore, the tetrahedron is regular.

*

2. [1984: 311] *7th Austrian-Polish Mathematical Competition (Poznan, Poland). First Day: July 4, 1984.*

Let A be the set of all natural numbers between 1000 and 9999 which contain precisely two different digits, both different from zero. Interchanging the two digits yields for every $n \in A$ another number $f(n) \in A$ (e.g., $f(3111) = 1333$). Determine $n \in A$, with $n > f(n)$, such that $\gcd(n,f(n))$ is a maximum. (No calculators are to be used.)

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Each $n \in A$ is of one of the following forms, for some distinct nonzero digits a, b.

	n	$n - f(n)$
(i)	$aaab$	$1109(a - b)$
(ii)	$aaba$	$1091(a - b)$
(iii)	$abaa$	$911(a - b)$
(iv)	$baaa$	$7\cdot127(b - a)$
(v)	$aabb$	$3^2\cdot11^2(a - b)$
(vi)	$abba$	$3^4\cdot11(a - b)$
(vii)	$abab$	$3^2\cdot101(a - b)$

We consider several cases. Note that in every case $n + f(n) = 11\cdot101(a + b)$.

I. Both a,b odd. Then

$$\gcd(n,f(n)) = \gcd\left[\frac{n + f(n)}{2},\frac{n - f(n)}{2}\right] = \gcd\left[11\cdot101\cdot\frac{a + b}{2},\frac{n - f(n)}{2}\right].$$

If n is of the form (i), (ii) or (iii) the fact that 1109, 1091 and 911 are prime together with the obvious equality $\gcd(11\cdot101,\frac{a - b}{2}) = 1$ shows that

$$\gcd(n,f(n)) = \gcd\left[\frac{a + b}{2},\frac{a - b}{2}\right] \le 4.$$

If (iv) holds, we use

$$\gcd(127,\frac{a+b}{2}) = \gcd(11\cdot 101,\frac{a-b}{2}) = 1$$

to see that

$$\gcd(n,f(n)) = \gcd\left[\frac{a+b}{2},7\cdot\frac{a-b}{2}\right] \le 8.$$

In a similar fashion, we see that if n is of form (v) or (vi)

$$\gcd(n,f(n)) = 11\cdot\gcd\left[\frac{a+b}{2},\frac{9(a-b)}{2}\right] \le 88.$$

If (vii) holds,

$$\gcd(n,f(n)) = 101\cdot\gcd\left[\frac{a+b}{2},\frac{9(a-b)}{2}\right] \le 303,$$

where one sees that this last gcd is at most 3 by exhausting the small set of possibilities.

II. a odd, b even (or b odd, a even).

Here $\gcd(n,f(n)) = \gcd(n+f(n),n-f(n))$. As in case I, this gives $\gcd(n,f(n)) \le 8$ if (i), (ii) or (iii) hold. If (v) holds, we see that

$$\gcd(n,f(n)) \le a+b \le 17.$$

If (v) or (vi) holds, then

$$\gcd(n,f(n)) = 11\cdot\gcd(a+b,\frac{n-f(n)}{11}) \le 187.$$

If (vii) holds,

$$\gcd(n,f(n)) = 101\cdot\gcd(a+b,9(a-b)) \le 909.$$

III. a, b even.

If neither a nor b is divisible by 4 then

$$\gcd(n,f(n)) = 2\cdot\gcd(n/2,f(n)/2) \le 606$$

(by case I). If exactly one of a, b is divisible by 4,

$$\gcd(n,f(n)) = 2\cdot\gcd(n/2,f(n)/2) \le 374$$

if n is not of form (vii) (by case II). If n is of the form (vii),

$$\gcd(n,f(n)) = 202\cdot\gcd\left[\frac{a+b}{2},\frac{9(a-b)}{2}\right] \le 606.$$

The only case left is when $4|a$ and $4|b$, i.e. when $(a,b) = (8,4)$ or $(4,8)$. But then

$$\gcd(n,f(n)) = 4\cdot\gcd(n/4,f(n)/4) \le 4\cdot 187 = 748$$

unless n is of form (vii). In that case,

$$\gcd(n,f(n)) = \gcd(8484,4848) = \gcd(2^2\cdot 3\cdot 7\cdot 101,2^4\cdot 3\cdot 101) = 1212.$$

Thus $n = 8484$ satisfies $n > f(n)$ and maximizes $\gcd(n,f(n))$ for $n \in A$.

*

4. [1984: 311] *7th Austrian-Polish Mathematical Competition (Poznan, Poland). Second day: July 5, 1984.*

If $A_1A_2...A_7$ is a regular heptagon with circumcircle C and P is a point on minor arc A_7A_1, prove that

$$\overline{PA_1} + \overline{PA_3} + \overline{PA_5} + \overline{PA_7} = \overline{PA_2} + \overline{PA_4} + \overline{PA_6}.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

Let a_1 denote the length of a side of the heptagon, let a_2 denote the length of any chord A_iA_{i+2}, and let a_3 denote the length of any chord A_iA_{i+3}.

We apply Ptolemy's theorem to the cyclic quadrilaterals $PA_1A_2A_3$, $PA_5A_6A_7$, $PA_2A_4A_6$ and $A_1A_3A_6A_7$ to obtain

(i) $a_1\overline{PA_1} + a_1\overline{PA_3} = a_2\overline{PA_2}$

(ii) $a_1\overline{PA_5} + a_1\overline{PA_7} = a_2\overline{PA_6}$

(iii) $a_2\overline{PA_2} + a_2\overline{PA_6} = a_3\overline{PA_4}$

(iv) $a_1a_3 + a_1a_2 = a_2a_3$.

By (iv), $a_2/a_3 = a_2/a_1 - 1$; then (iii) gives

$$\overline{PA_4} = \frac{a_2}{a_3}(\overline{PA_2} + \overline{PA_6}) = (\frac{a_2}{a_1} - 1)(\overline{PA_2} + \overline{PA_6}).$$

This with (i) and (ii) gives

$$\begin{aligned}\overline{PA_2} + \overline{PA_4} + \overline{PA_6} &= \frac{a_2}{a_1}\cdot\overline{PA_2} + \frac{a_2}{a_1}\cdot\overline{PA_6} \\ &= \overline{PA_1} + \overline{PA_3} + \overline{PA_5} + \overline{PA_7}\end{aligned}$$

as desired.

*

9. [1984: 312] *7th Austrian-Polish Mathematical Competition (Poznan, Poland). Second day: July 5, 1984.*

Determine all real-valued functions f defined on the set $\mathbb{Q}$ of rational numbers and satisfying, for all $x,y \in \mathbb{Q}$,

$$f(x + y) = f(x)f(y) - f(xy) + 1.$$

Solution by Daniel Ropp, Washington University, St. Louis, MO, U.S.A.

In the given equation we set $(x,y) = (0,0),(-1,1),(-1,-1),(-2,1)$ to obtain the following equations:

(i) $f(0) = (f(0))^2 - f(0) + 1$

(ii) $f(0) = f(1)f(-1) - f(-1) + 1$

(iii) $f(-2) = (f(-1))^2 - f(1) + 1$

(iv) $f(-1) = f(-2)f(1) - f(-2) + 1$.

By (i), $f(0) = 1$; then (ii) gives $f(-1) = 0$ or $f(1) = 1$. If $f(1) = 1$, we set $(x,y) = (z - 1,1)$ in the functional equation to obtain

$$f(z) = f(z - 1) - f(z - 1) + 1 \qquad \text{or} \qquad f(z) \equiv 1.$$

So we assume $f(-1) = 0$. Solving (iii) and (iv) simultaneously yields $f(1) = 0$ or $f(1) = 2$.

Suppose $f(1) = 0$. Substituting $(x,y) = (x,1),(1/2,-1),(1/2,2)$ in the functional equation, we obtain

(v) $f(x + 1) = 1 - f(x)$

(vi) $f(-1/2) = -f(-1/2) + 1$

(vii) $f(5/2) = f(2)f(1/2) + 1$.

Now, (vi) implies $f(-1/2) = 1/2$, and (v) gives $f(1/2) = f(3/2) = f(5/2) = 1/2$. Next (vii) gives that $f(2) = -1$. This with (v) gives $f(1) = 2$, giving a contradiction to the assumption that $f(1) = 0$.

Thus $f(1) = 2$. Setting $y = 1$ in the original functional equation, we have

(viii) $f(x + 1) = f(x) + 1$.

By an easy induction, $f(n) = n + 1$ for $n \in \mathbb{Z}$ and $f(x + n) = f(x) = n$ for $x \in \mathbb{Q}$, $n \in \mathbb{Z}$. For $p,q \in \mathbb{Z}$ set $x = p/q$, $y = q$ in the functional equation to obtain

$$f(p/q) + q = f(p/q + q) = f(p/q)(q + 1) - (p + 1) + 1,$$

or

$$f(p/q) = p/q + 1.$$

Thus $f(x) = x + 1$ on $\mathbb{Q}$. It is easy to check that both $f(x) \equiv 1$ and $f(x) = x + 1$ are solutions of the functional equation.

* * *

With these solutions submitted this summer we have exhausted the backlog of solutions to problems posed up to 1984 (Volume 10). According to my records the problems that remain unanswered in this column are the following:

Volume 7 (1981)

[1981: 15-17] 8, 9, 13, 15

[1981: 46] 1, 2

[1981: 75-78] all problems except #4 on p.77

[1981: 114, 115] B1, B6

[1981: 236, 237] 1, 13

[1981: 268, 269] 1, 5, 6

Volume 8 (1982)

[1982: 100] 7

[1982: 134] 6

[1982: 134, 135] 4, 5

[1982: 237] 1, 3-6

[1982: 238, 239] all problems

[1982: 300-302] all student proposals

Volume 9 (1983)	Volume 10 (1984)
[1983: 107, 108] 1, 2, 3	[1984: 9-13] 1-8
[1983: 108, 109] 1, 2, 3, 5, 6	[1984: 74] 1, 3, 4
[1983: 237, 238] F2412, F2414, P375, P376	[1984: 75] Gy2147-Gy2149
[1983: 269, 270] M796, M797, M800	[1984: 107] 4, 5
[1983: 305] 16	[1984: 108] 2-5
	[1984: 182] 4
	[1984: 214] 5
	[1984: 215] 2
	[1984: 283] 3, 4
	[1984: 283, 284] M856, M860
	[1984: 312] 5-8

We welcome your solutions to these problems for consideration for use in this column.

* * *

We now turn to solutions to problems given in Volume 11 (1985). Here we shall have over the next several issues a mixture of solutions submitted in response to our challenge and solutions submitted some time ago. First a recent submission.

1. [1985: 238] *1984 Bulgarian Mathematical Olympiad, First Day.*

Determine all nonnegative integer triples (x,y,z) such that

$$5^x \cdot 7^y + 4 = 3^z.$$

Solution by John Morvay, Dallas, Texas.

Clearly $z \geq 2$. Now $5^x \cdot 7^y + 4 \equiv 0 \bmod 3$ gives $2^x + 1 \equiv 0 \bmod 3$ (since $7 \equiv 4 \equiv 1 \bmod 3$). This means x must be odd, in particular $x \geq 1$. But then we see that $4 \equiv 3^z \bmod 5$. This entails that $z \equiv 2 \bmod 4$; in particular, z is even. If $z = 2$ we have $5^x \cdot 7^y = 5$ giving $(1,0,2)$ as a solution. We shall show that this is the only solution by showing that $z \geq 4$ is impossible. Now $5^x \cdot 7^y = 3^z - 4 = (3^{z/2} - 2)(3^{z/2} + 2)$. (Recall z is even!) Neither 5 nor 7 can divide both $3^{z/2} - 2$ and $3^{z/2} + 2$. Thus $5^x = 3^{z/2} - 2$ and $7^y = 3^{z/2} + 2$ or $5^x = 3^{z/2} + 2$ and $7^y = 3^{z/2} - 2$. For $z \geq 4$ this is impossible unless $x \geq 2$. From the factoring notice also that 5^x and 7^y must differ by 4. For $x \geq 2$ the last two digits of 5^x are 25. The last two digits of 7^y are 01, 07, 43 or 49. No difference gives 4 (mod 100). Therefore the only solution is (1,0,2).

* * *

New contest problems, as well as your solutions to problems past and present, are welcome for consideration in the Olympiad Corner. Please send them directly to R.E. Woodrow.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1988, although solutions received after that date will also be considered until the time when a solution is published.

<u>1281</u>*. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find the area of the largest triangle whose vertices lie in or on a unit n-dimensional cube.

<u>1282</u>. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle, I the incenter, and A', B', C' the intersections of AI, BI, CI with the circumcircle. Show that

$$IA' + IB' + IC' - (IA + IB + IC) \le 2(R - 2r)$$

where R and r are the circumradius and inradius of ΔABC.

<u>1283</u>. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Show that the polynomial

$$P(x,y,z) = (x^2 + y^2 + z^2)^3 - (x^3 + y^3 + z^3)^2 - (x^2y + y^2z + z^2x)^2 - (xy^2 + yz^2 + zx^2)^2$$

is nonnegative for all real x, y, z.

<u>1284</u>. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3A_4$ be a cyclic quadrilateral with $\overline{A_1A_2} = a_1$, $\overline{A_2A_3} = a_2$, $\overline{A_3A_4} = a_3$, $\overline{A_4A_1} = a_4$. Let ρ_1 be the radius of the circle outside the quadrilateral, tangent to the segment A_1A_2 and the extended lines A_2A_3 and A_4A_1. Define ρ_2, ρ_3, ρ_4 analogously. Prove that

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_4} \geq \frac{8}{\sqrt[4]{a_1a_2a_3a_4}} .$$

When does equality hold?

<u>1285</u>. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

I is the incenter of a triangle ABC and I_1 is the excenter opposite A. Lines through I and I_1 parallel to BC meet AB at P, S and AC at Q, R respectively.

(a) Show that the trapezium $PQRS$ has an inscribed circle.

(b) Find the length of BC in terms of the lengths of PQ and RS.

<u>1286</u>. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\Pi\left\{\frac{x(x + y + z)}{(x + y)(x + z)}\right\}^x \leq \left[\frac{(\Sigma yz)^2}{4xyz(x + y + z)}\right]^{x+y+z} ,$$

where Π and Σ are to be understood cyclically.

<u>1287</u>. *Proposed by Leroy F. Meyers, The Ohio State University.*

Find all differentiable functions f such that $f'(x) = f(3) + f(6)$ for all real x.

<u>1288</u>. *Proposed by Len Bos, University of Calgary, Calgary, Alberta.*

Show that for $x_1, x_2, \ldots, x_n > 0$,

$$n(x_1^n + x_2^n + \ldots + x_n^n) \geq (x_1 + x_2 + \ldots + x_n)(x_1^{n-1} + x_2^{n-1} + \ldots + x_n^{n-1}).$$

<u>1289</u>. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

"To reward you for slaying the dragon", the Queen said to Sir George, "I grant you all the land you can walk around in a day."

She pointed to a pile of wooden stakes. "Take some of these stakes with you", she continued. "Pound them into the ground along the way, and be back at your starting point in 24 hours. All the land in the convex hull of your stakes will then be yours." (The Queen had read a little mathematics.)

Assume that it takes Sir George 1 minute to pound in a stake, and that he walks at constant speed between stakes. How many stakes should he use, to get as much land as possible?

1290. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

The triangles $B_1B_2B_3$ and $C_1C_2C_3$ are homothetic and each of them is in perspective with the triangle $A_1A_2A_3$ (vertices with the same index correspond). D_i $(i = 1,2,3)$ is the midpoint of the segment B_iC_i. Prove that the triangles $A_1A_2A_3$ and $D_1D_2D_3$ are in perspective.

*   *   *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

559. [1980: 184: 1981: 192] *Proposed by Charles W. Trigg, San Diego, California.*

Are there any positive integers k such that the expansion of 2^k in the decimal system terminates with k?

III. *Solution by Daniel B. Shapiro, Ohio State University, Columbus, Ohio.*

If b, n, k_n are positive integers with k_n having exactly n decimal digits (allowing initial zeros), then k_n is *expomorphic to base b* if

$$b^{k_n} \equiv k_n \pmod{10^n}.$$

That is, the decimal expansion of b^{k_n} terminates in k_n. This terminology was introduced in A. Wayne's solution [1981: 192].

The following theorem settles the conjecture on [1981: 192]. All the variables considered are positive integers.

Theorem. Let b, n be given where $n \geq 2$ and $10 \nmid b$. Then

(a) there is exactly one n-digit number k_n expomorphic to base b;

(b) $k_{n+1} \equiv b^{k_n} \pmod{10^{n+1}}$.

This result then shows how to compute the number k_n inductively. Several numerical examples are listed in [1981: 192]. General congruence problems of this type are considered in [1]. We prove the theorem using some of the ideas found there.

Lemma. Let b be fixed. If $r \equiv s \pmod{10^n}$ and $r,s > n$ then $b^r \equiv b^s \pmod{10^{n+1}}$. If b is coprime to 10, the inequalities on r, s are unnecessary.

Proof. Let $\psi(m)$ denote the smallest positive integer e such that $k^e \equiv 1 \pmod{m}$ for every k coprime to m. That is, $\psi(m)$ is the lcm (least common multiple) of the orders of the elements of the multiplicative group of units of the integers modulo m. ψ and Euler's phi function φ are related as follows: $\psi(p^m) = \varphi(p^m)$ for odd primes p and nonnegative integers m; $\psi(2) = \varphi(2)$; $\psi(4) = \varphi(4)$; but $\psi(2^m) = \varphi(2^m)/2$ for $m \geq 3$.

It is well known that

(i) $\psi(\mathrm{lcm}\{a,b\}) = \mathrm{lcm}\{\psi(a),\psi(b)\}$;

(ii) $\psi(p^m) = p^{m-1}(p-1)$ for an odd prime p;

(iii) $\psi(2) = 1$, $\psi(4) = 2$, and $\psi(2^m) = 2^{m-2}$ whenever $m \geq 3$.

For example see [2], pp.106-107 of [3], or Theorems 6.2.2 and 6.3.1 of [4]. In particular it follows that if $n \geq 2$ then

$$\psi(10^{n+1}) = \mathrm{lcm}\{\psi(5^{n+1}),\psi(2^{n+1})\} = \mathrm{lcm}\{4 \cdot 5^n, 2^{n-1}\},$$

and thus $\psi(10^{n+1})$ divides 10^n.

Since $r \equiv s \pmod{10^n}$ we get $r \equiv s \pmod{\psi(10^{n+1})}$. If b is coprime to 10 then $b^r \equiv b^s \pmod{10^{n+1}}$, with no further restriction needed on r, s. Suppose $2|b$ and b is coprime to 5. Then

$$r \equiv s \bmod \psi(5^{n+1}) \Rightarrow b^r \equiv b^s \bmod 5^{n+1}$$

as before, and since $\psi(5^{n+1})|\psi(10^{n+1})$ we have

$$r \equiv s \bmod \psi(10^{n+1}) \Rightarrow b^r \equiv b^s \bmod 5^{n+1}. \tag{1}$$

Also, since $r,s > n$ we get

$$b^r \equiv 0 \equiv b^s \pmod{2^{n+1}}. \tag{2}$$

Gluing (1) and (2) together gives the result. Similar arguments work if $5|b$ and b is odd. Finally if $10|b$ the lemma is obvious. □

Proof of the theorem. We begin with (b). Fix n and suppose that k is expomorphic of n digits, so that $b^k \equiv k \pmod{10^n}$. If b is coprime to 10, the lemma implies

$$b^{b^k} \equiv b^k \pmod{10^{n+1}}. \tag{3}$$

Consequently $x = b^k$ satisfies $b^x = x \pmod{10^{n+1}}$ so the least positive residue of $b^k \pmod{10^{n+1}}$ is expomorphic of $n + 1$ digits. Next suppose $2|b$. From $b^k \equiv k \pmod{2^n}$ we argue that $k \equiv 0 \pmod{2^n}$. Then $k \geq 2^n > n$ and the lemma applies, giving (3) again. When $5|b$ a similar argument works.

To prove (a), suppose k is expomorphic of n digits: $b^k \equiv k \pmod{10^n}$. By part (b) above it follows that

$$b^{b^k} \equiv b^k \pmod{10^{n+1}},$$

so that

$$b^{b^k} \equiv b^k \equiv k \pmod{10^n}.$$

Repeating this process yields

$$k \equiv b^k \equiv b^{b^k} \equiv \ldots \equiv b^{b^{b^{\cdot^{\cdot^{\cdot}}}}} \pmod{10^n}.$$

Such an "iterated exponent" always becomes stable after a few steps, at a value (mod 10^n) independent of k. This claim is proved in detail in [1]. This proves the uniqueness.

Remark. As pointed out by L.F. Meyers, the second solution [1981: 193], by K.M. Wilke, contains the following gap. From equations (2) and (3) of that solution it follows that

$$2^{\phi(5^n)}(E_n - 1) \equiv 0 \pmod{10^n}.$$

Here E_n is the n-digit even automorphic number. Therefore

$$2^n(E_n - 1) \equiv 0 \pmod{10^n},$$

and the exponent n is the best possible. One can conclude that

$$k_n(E_n - 1) \equiv 0 \pmod{10^n}$$

only if it is known that $k_n \geq n$. Some further argument is needed to prove this inequality.

[Editor's note: In a 1981 letter, Meyers in fact noted that the gap can be closed by replacing the phrase "it follows from (2) and (3) that" on [1981: 193] by "it follows from $2^{k_n} > k_n$ that $2^{k_n} \geq 10^n > 2^n$, so that $k_n > n$. Hence (2) and (3) may be applied to yield".]

References:

[1] D.B. Shapiro and S.D. Shapiro, Iterated exponents in number theory, preprint (submitted to *Amer. Math. Monthly*).

[2] R.D. Carmichael, Note on a new number theory function, *Bull. Amer. Math. Soc.* 16 (1910) 232-238.

[3] I.M. Vinogradov, *Elements of Number Theory*, Dover, 1954.

[4] H.N. Shapiro, *Introduction to the Theory of Numbers*, John Wiley and Sons, 1983.

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<u>1139</u>. [1986: 79; 1987: 231] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let ABC be a triangle and let A', B', C' be the touch points of the nine-point circle with the A-excircle, B-excircle, and C-excircle, respectively. Prove that AA', BB', CC' concur in a point F', and that F' is collinear with the centers of the incircle and nine-point circle.

Editor's comment.

A further two solutions (translated and forwarded by Hidetosi Fukagawa) have been received from TOSIO SEIMIYA, Kawasaki, Japan.

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<u>1157</u>*. [1986: 140] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find all triples of positive integers (r,s,t), $r \le s,t$, for which $(rs + r + 1)(st + s + 1)(tr + t + 1)$ is divisible by $(rst - 1)^2$. This problem was suggested by Routh's Theorem (see *Crux* [1981: 199]).

Solution by G. Szekeres, Sydney, Australia.

Let d be the g.c.d. of $rs + r + 1$ and $rst - 1$. Then

$$d|t(rs + r + 1) - (rst - 1) = tr + t + 1,$$

and thus similarly $d|st + s + 1$. By symmetry, d is the g.c.d. of $rst - 1$ and any of the three factors in $(rs + r + 1)(st + s + 1)(tr + t + 1)$. Set

$$rs + r + 1 = dh, \quad st + s + 1 = dk, \quad tr + t + 1 = dm, \quad rst - 1 = dn;$$

then $(h,n) = (k,n) = (m,n) = 1$. By assumption d^3hkm is divisible by d^2n^2, hence d is divisible by n^2, $d = en^2$, and

$$rs + r + 1 = n^2he,$$
$$st + s + 1 = n^2ke,$$
$$tr + t + 1 = n^2me,$$
$$rst - 1 = n^3e.$$

From the first of these equations,

$$trs + tr + t = n^2hte,$$

hence from the third and last equations,

$$n^2me + n^3e = n^2hte,$$

giving

$$t = \frac{n + m}{h}.$$

Similarly,

$$r = \frac{n + h}{k}, \quad s = \frac{n + k}{m}, \quad t = \frac{n + m}{h}. \tag{1}$$

Substituting into $rst - 1 = n^3e$, we get

$$\frac{(n + m)(n + h)(n + k)}{hkm} - 1 = n^3e,$$

hence

$$n^2 + n(h + k + m) + hk + hm + km = n^2hkme. \tag{2}$$

So we want to determine all positive integer solutions of this equation subject to

$$h|n + m, \quad k|n + h, \quad m|n + k. \tag{3}$$

Clearly the set of solutions is invariant to cyclic permutations of h, k, m and we may assume $h \leq k,m$.

Suppose first that $h = m = 1$; then the conditions are

$$kn^2|n^2 + n(k + 2) + 2k + 1, \tag{4}$$

$$k|n + 1. \tag{5}$$

The first condition requires that

$$n|2k + 1, \tag{6}$$

hence

$$kn|(n + 1)(2k + 1),$$

$$kn|2k + n + 1,$$

and in particular

$$kn \leq 2k + n + 1,$$

$$(k - 1)(n - 2) \leq 3. \tag{7}$$

Now if $k = 1$ then $n|3$ by (6), and if $k > 1$ then $n \leq 5$ by (7). These and (5), (6) supply the following possibilities:

(i) $k = 1$, $n = 1$,

(ii) $k = 1$, $n = 3$,

(iii) $k = 2$, $n = 1$,

(iv) $k = 2$, $n = 5$,

(v) $k = 4$, $n = 3$.

Of these, (ii) does not satisfy (4); the others do. So from (1) we obtain the following four solutions (r,s,t): $(2,2,2)$, $(1,3,2)$, $(3,7,6)$, $(1,7,4)$.

If $h = k = 1$, we just obtain cyclic permutations of the above solutions. Thus suppose next that $h \geq 1$, $k \geq 2$, $m \geq 2$. From equation (2) we have that

$$n^2hkm \leq n^2 + n(h + k + m) + hk + hm + km, \tag{8}$$

hence

$$n^2 \leq \frac{n^2}{hkm} + n\left[\frac{1}{hk} + \frac{1}{hm} + \frac{1}{km}\right] + \frac{1}{h} + \frac{1}{k} + \frac{1}{m} \leq \frac{n^2}{4} + \frac{5n}{4} + 2,$$

which only admits $n = 1$ or $n = 2$.

If $h = 1$, we have from (3) that $k|n + 1$ and $m|n + k$, and the only possible solutions are $n = 1$, $k = 2$, $m = 3$ and $n = 2$, $k = 3$, $m = 5$. The second is ruled out by (2) because $hk + hm + km$ is odd, and the first gives $(r,s,t) = (1,1,4)$.

Finally if $h \geq 2$, $k \geq 2$, $m \geq 2$, then only $n = 1$ is admitted by (8). But from (3),

$$h|m + 1, \quad k|h + 1, \quad m|k + 1,$$

and $h = 2$ is ruled out since then m must be odd, k must be 3 and this contradicts $m|k + 1$. Also $h > 3$ is ruled out because then $h \geq 4$, $k \geq 4$, $m \geq 4$ and this contradicts

$$2hkm \leq (h + 1)(k + 1)(m + 1),$$

a consequence of (8) with $n = 1$. So let $h = 3$, $k \geq 3$, $m \geq 3$. But then from (3)

$$3|m + 1, \quad k|4, \quad m|k + 1$$

giving $(h,k,m) = (3,4,5)$ and $(r,s,t) = (1,1,2)$.

The following table gives the complete set of solutions:

r	s	t	rs + r + 1	st + s + 1	tr + t + 1	rst − 1
1	1	2	3	4	5	1
1	1	4	3	6	9	3
1	3	2	5	10	5	5
1	7	4	9	36	9	27
2	2	2	7	7	7	7
3	7	6	25	50	25	125

The proposer found all the solutions by computer, but didn't prove there were no others. One other reader submitted an incomplete list of solutions.

* * *

<u>1158</u>. [1986: 140] *Proposed by Svetoslav Bilchev, Technical University, Russe, Bulgaria.*

Prove that

$$\sum \frac{1}{(\sqrt{2} + 1)\cos A/8 - \sin A/8} \geq \sqrt{6 - 3\sqrt{2}}$$

where the sum is cyclic over the angles A, B, C of a triangle. When does equality occur?

Solution by Vedula N. Murty, Penn State University, Middletown, Pennsylvania.

We have

$$\cos\frac{3\pi}{8} = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \qquad \text{and} \qquad \sin\frac{3\pi}{8} = \frac{\sqrt{2} + 1}{\sqrt{4 + 2\sqrt{2}}},$$

and thus

$$\begin{aligned}(\sqrt{2} + 1)\cos A/8 - \sin A/8 &= \sqrt{4 + 2\sqrt{2}}(\sin 3\pi/8 \cos A/8 - \cos 3\pi/8 \sin A/8) \\ &= \sqrt{4 + 2\sqrt{2}}\,\sin\left[\frac{3\pi - A}{8}\right].\end{aligned}$$

Therefore

$$\sum \frac{1}{(\sqrt{2} + 1)\cos A/8 - \sin A/8} = \frac{1}{\sqrt{4 + 2\sqrt{2}}}\sum \csc\left[\frac{3\pi - A}{8}\right].$$

Let

$$\alpha = \frac{3\pi - A}{8}, \quad \beta = \frac{3\pi - B}{8}, \quad \gamma = \frac{3\pi - C}{8};$$

then $\alpha + \beta + \gamma = \pi$, hence α, β, γ are the angles of a triangle. Using the known inequality

$$\Sigma \csc \alpha \geq 2\sqrt{3}$$

(see 2.49 of Bottema et al, *Geometric Inequalities*), equality being attained if and only if $\alpha = \beta = \gamma = \pi/3$, we have

$$\sum \frac{1}{(\sqrt{2} + 1)\cos A/8 - \sin A/8} \geq \frac{2\sqrt{3}}{\sqrt{4 + 2\sqrt{2}}} = \frac{2\sqrt{3}\sqrt{4 - 2\sqrt{2}}}{\sqrt{8}} = \sqrt{6 - 3\sqrt{2}},$$

with equality if and only if $A = B = C = \pi/3$.

Also solved by J.T. GROENMAN, *Arnhem, The Netherlands;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* KEE-WAI LAU, *Hong Kong; and the proposer.*

* * *

1159. [1986: 140] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and P some interior point with distances $AP = x_1$, $BP = x_2$, $CP = x_3$. Show that

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \geq 8F,$$

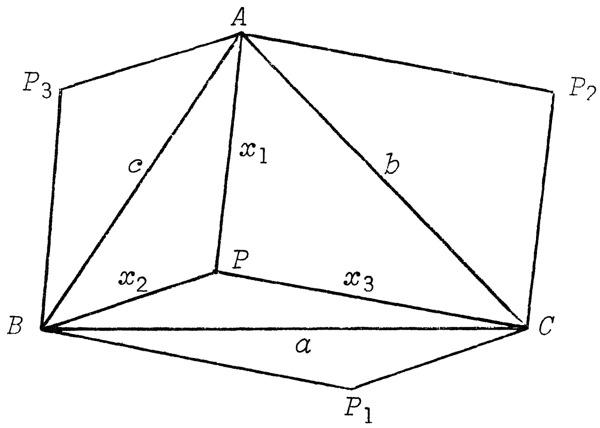
where a, b, c are the sides of ΔABC and F is its area.

Solution by G.R. Veldkamp, De Bilt, The Netherlands.

We use the following well-known theorem: for any simple plane quadrangle $WXYZ$, the inequalities

$$\overline{WX}\cdot\overline{YZ} + \overline{XY}\cdot\overline{WZ} \geq \overline{WY}\cdot\overline{XZ} \geq 2\ \text{area}(WXYZ)$$

hold, with equality if and only if $WXYZ$ is cyclic and $WY \perp XZ$, respectively.



Let P_1, P_2, P_3 be the mirror images of P into the midpoints of BC, CA, AB respectively. Denoting the areas of ΔPBC, ΔPCA, ΔPAB in this order by F_1, F_2, F_3, we apply the above theorem to the quadrangle ABP_1C and obtain

$$bx_3 + cx_2 \geq a \cdot \overline{AP_1} \geq 2(F + F_1), \qquad (1)$$

with equality if and only if P_1 is on the circumcircle of ΔABC and the orthocenter H is on AP_1, that is if and only if $\overline{CH} = x_2$ and $\overline{BH} = x_3$, or equivalently if and only if H and P are mirror images into the perpendicular bisector of BC.

By cyclic permutation we obtain

$$cx_1 + ax_3 \geq 2(F + F_2), \qquad (2)$$

with equality if and only if H and P are mirror images into the perpendicular bisector of CA, and

$$ax_2 + bx_1 \geq 2(F + F_3), \qquad (3)$$

with equality if and only if H and P are mirror images into the perpendicular bisector of AB. Hence by addition of (1), (2), and (3),

$$(b + c)x_1 + (c + a)x_2 + (a + b)x_3 \geq 6F + 2(F_1 + F_2 + F_3) = 8F,$$

with equality if and only if the triangle is equilateral and P coincides with its center.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; and the proposer.

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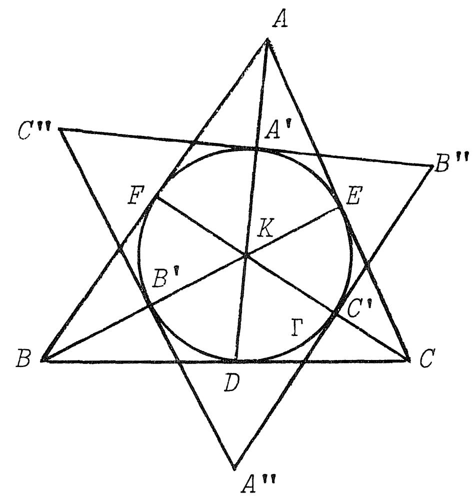
<u>1160</u>. [1986: 140] *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Let A', B', C' be the first points of intersection of the angle bisectors of a triangle ABC with its incircle Γ. The tangents to Γ at A', B', C' form a triangle $A''B''C''$. Prove that the lines AA'', BB'', CC'' are concurrent.

Solution by G.R. Veldkamp, De Bilt, The Netherlands.

Let D, E, and F be the points of contact of Γ with BC, CA, and AB respectively. Then it is immediately clear that DA', EB', and FC' are the



inner bisectors of $ADEF$ (since arcs $A'F$ and $A'E$ are equal, etc.). Their common point K is therefore the incentre of this triangle. It is also clear that DA', EB', and FC' are in this order the polars with respect to Γ of the points $BC \cap B''C''$, $CA \cap C''A''$, and $AB \cap A''B''$. These points are therefore on the same line k, the polar line of K with respect to Γ. Applying Desargues' theorem, we conclude that AA'', BB'', and CC'' are concurrent.

Also solved by AAGE BONDESEN, Royal Danish School of Educational Studies, Copenhagen, Denmark; JORDI DOU, Barcelona, Spain; DAN SOKOLOWSKY, Williamsburg, Virginia; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

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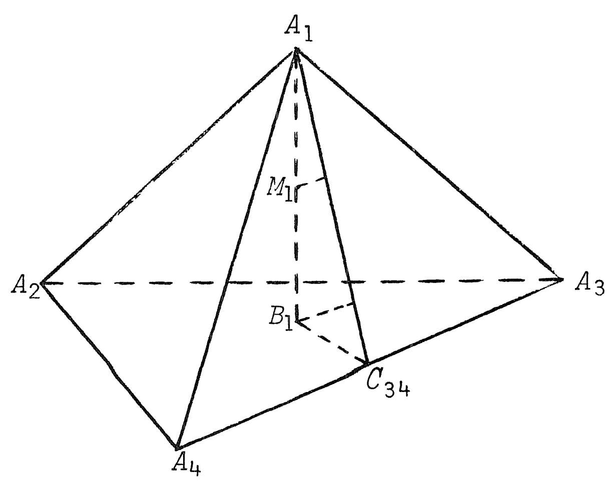
1161. [1986: 177] *Proposed by O. Bottema, Delft, The Netherlands; J.T. Groenman, Arnhem, The Netherlands; and D.J. Smeenk, Zaltbommel, The Netherlands.* (Dedicated to Léo Sauvé.)

The dihedral angle on the edge A_iA_j of a given tetrahedron $A_1A_2A_3A_4$ is denoted by α_{ij} $(= \alpha_{ji})$, for $1 \le i,\ j \le 4,\ i \ne j$. Determine a necessary and sufficient condition on the α_{ij} for the midpoints of the four altitudes of the tetrahedron to be coplanar.

Solution by the proposers.



Let B_1 and C_{34} be the projections of A_1 on the plane $A_2A_3A_4$ and on the edge A_3A_4 respectively. Then $\angle B_1C_{34}A_1 = \alpha_{34}$. Let furthermore $A_1B_1 = h_1$ and M_1 the midpoint of A_1B_1. Then the distance of B_1 to $A_1A_3A_4$ is $h_1\cos\alpha_{34}$ and that of M_1 is therefore $1/2\ h_1 \cos\alpha_{34}$. In the same way we conclude that the distance of M_1 to $A_1A_4A_2$ and to $A_1A_2A_3$ are $1/2\ h_1\cos\alpha_{42}$ and $1/2\ h_1\cos\alpha_{23}$ respectively. And as $M_1B_1 = 1/2\ h_1$ the homogeneous distance coordinates of M_1 with respect to the tetrahedron are seen to be

$$(1\ ,\ \cos\alpha_{34}\ ,\ \cos\alpha_{42}\ ,\ \cos\alpha_{23}).$$

Hence a necessary and sufficient condition for the four midpoints to be coplanar reads

$$\begin{vmatrix} 1 & \cos\alpha_{34} & \cos\alpha_{42} & \cos\alpha_{23} \\ \cos\alpha_{34} & 1 & \cos\alpha_{41} & \cos\alpha_{13} \\ \cos\alpha_{24} & \cos\alpha_{41} & 1 & \cos\alpha_{12} \\ \cos\alpha_{23} & \cos\alpha_{31} & \cos\alpha_{12} & 1 \end{vmatrix} = 0. \tag{1}$$

This condition, which solves the problem, may be simplified if we keep in mind that the α_{ij} in an arbitrary tetrahedron are not independent. Indeed, if we denote the area of the face $A_2A_3A_4$ by F_1, etc., and if we project $A_1A_2A_3$, $A_1A_3A_4$, and $A_1A_4A_2$ onto $A_2A_3A_4$, we obtain

$$F_1 = F_2\cos\alpha_{34} + F_3\cos\alpha_{42} + F_4\cos\alpha_{23}$$

and three analogous relations. The F_i satisfy four homogeneous linear equations, which implies

$$\begin{vmatrix} 1 & -\cos\alpha_{34} & -\cos\alpha_{42} & -\cos\alpha_{23} \\ -\cos\alpha_{34} & 1 & -\cos\alpha_{41} & -\cos\alpha_{13} \\ -\cos\alpha_{24} & -\cos\alpha_{41} & 1 & -\cos\alpha_{12} \\ -\cos\alpha_{23} & -\cos\alpha_{31} & -\cos\alpha_{12} & 1 \end{vmatrix} = 0. \tag{2}$$

Note that (2) is obtained from (1) by replacing $\cos\alpha_{ij}$ by $-\cos\alpha_{ij}$ for all i and j. If we develop the determinant D_1 of (1) and order the terms according to the degrees of the cosines we obtain

$$D_1 = 1 + P_2 + P_3 + P_4$$

where

$$P_2 = -\cos^2\alpha_{12} - \cos^2\alpha_{34} - \cos^2\alpha_{13} - \cos^2\alpha_{42} - \cos^2\alpha_{14} - \cos^2\alpha_{23},$$

$$P_3 = 2(\cos\alpha_{12}\cos\alpha_{13}\cos\alpha_{14} + \cos\alpha_{23}\cos\alpha_{24}\cos\alpha_{21} + \cos\alpha_{34}\cos\alpha_{31}\cos\alpha_{32} + \cos\alpha_{41}\cos\alpha_{42}\cos\alpha_{43}),$$

$$P_4 = \cos^2\alpha_{12}\cos^2\alpha_{34} + \cos^2\alpha_{13}\cos^2\alpha_{42} + \cos^2\alpha_{14}\cos^2\alpha_{23} - 2(\cos\alpha_{12}\cos\alpha_{34}\cos\alpha_{13}\cos\alpha_{42} + \cos\alpha_{13}\cos\alpha_{42}\cos\alpha_{14}\cos\alpha_{23} + \cos\alpha_{14}\cos\alpha_{23}\cos\alpha_{12}\cos\alpha_{34}).$$

Hence, if D_2 is the determinant of (2) it follows that

$$D_2 = 1 + P_2 - P_3 + P_4$$

and therefore $D_1 - D_2 = 2P_3$. Our final result reads: *the four midpoints are coplanar if and only if*

$$\cos\alpha_{12}\cos\alpha_{13}\cos\alpha_{14} + \cos\alpha_{23}\cos\alpha_{24}\cos\alpha_{21} + \cos\alpha_{34}\cos\alpha_{31}\cos\alpha_{32} + \cos\alpha_{41}\cos\alpha_{42}\cos\alpha_{43} = 0. \tag{3}$$

Any term on the left-hand side corresponds to a triple of dihedral angles at three edges through a vertex.

The set of tetrahedra with the said property is not empty: the case $\alpha_{12} = \alpha_{34} = \pi/2$ satisfies (3). Another example is $\alpha_{12} = \alpha_{13} = \alpha_{14} = \pi/2$, the rectangular tetrahedron.

Remark. The analogous problem in the plane is much simpler. If the angles of a triangle are α, β, γ it may be proved (by the same method as above, or otherwise) that the midpoints of the three altitudes are collinear if and only if $\cos\alpha\cos\beta\cos\gamma = 0$, which means that the triangle is a right triangle.

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<u>1162</u>. [1986: 178] *Proposed by G. Tsintsifas, Thessaloniki, Greece.* (Dedicated to Léo Sauvé.)

Let $G = \{A_1, A_2, \ldots, A_{n+1}\}$ be a point set of diameter D (that is, $\max A_iA_j = D$) in E^n. Prove that G can be contained in a slab of width d, where

$$d \le \begin{cases} 2D/\sqrt{2n+2} & \text{for } n \text{ odd} \\ D\cdot\sqrt{\dfrac{2(n+1)}{n(n+2)}} & \text{for } n \text{ even.} \end{cases}$$

(A *slab* is a closed connected region in E^n bounded by two parallel hyperplanes. Its *width* is the distance between these hyperplanes.)

Solution by the proposer.

The problem is obvious if G lies in a k-plane with $k \le n - 1$. Assuming that $A_1, A_2, \ldots, A_{n+1}$ are the vertices of an n-simplex, its circumradius R and its inradius r satisfy the inequalities

$$R \le D\sqrt{\frac{n}{2n+2}}, \quad r \ge \frac{d}{2\sqrt{n}} \quad \text{for } n \text{ odd}$$

$$R \le D\sqrt{\frac{n}{2n+2}}, \quad r \ge \frac{d\sqrt{n+2}}{2(n+1)} \quad \text{for } n \text{ even}$$

(see pp.112-113 of H.G. Eggleston, *Convexity*, Cambridge University Press, 1966). But it is also known that

$$R \ge nr$$

(see, e.g., M.S. Klamkin and G. Tsintsifas, The circumradius-inradius inequality for a simplex, *Math. Magazine* 52 (1979) 20-22). The result follows.

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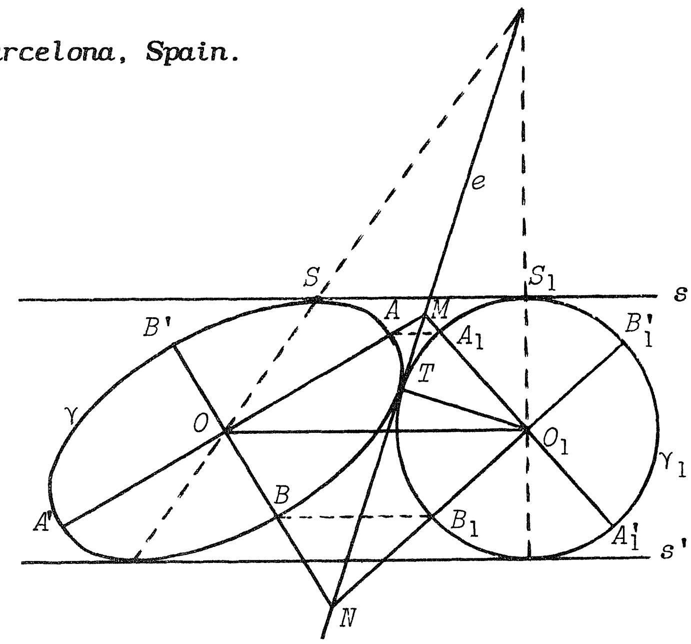
1163. [1986: 178] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.* (Dedicated to Léo Sauvé.)

Draw two parallel tangent lines to an ellipse with semiaxes a and b, and draw a circle tangent to the ellipse (externally) and to both lines. Prove that the center of the circle is at distance $a + b$ from the center of the ellipse.

I. *Solution by Jordi Dou, Barcelona, Spain.*



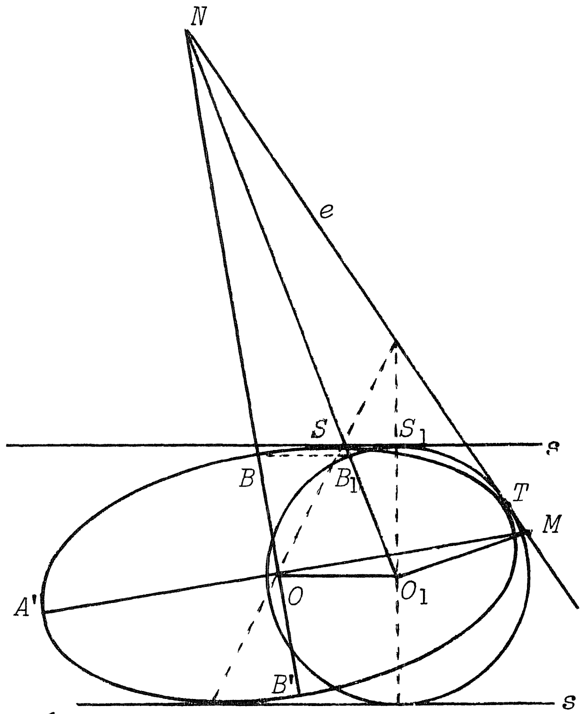
Ellipse γ (center O, axes AA', BB') and circle γ_1 (center O_1) are tangent to parallel lines s, s' and mutually tangent at point T. Then between γ and γ_1 there is a homologic affinity [affine transformation] H whose axis e is the common tangent through T and whose center is the point at infinity $s \cap s'$. [So H fixes each point on e and, for all points X not on e, the line $XH(X)$ is parallel to s and s'.] We have that $H(\gamma) = \gamma_1$ and $H(O) = O_1$, and we put $H(A) = A_1$, $H(B) = B_1$, $H(A') = A_1'$, $H(B') = B_1'$. Then A_1A_1' and B_1B_1' are perpendicular because AA' and BB' are conjugate diameters of the ellipse. Put $M = AA' \cap A_1A_1'$ and $N = BB' \cap B_1B_1'$, so that M and N lie on the axis e.

Now quadrilateral $MONO_1$ is cyclic, since O and O_1 are right angles, and so

$$MO\cdot NO_1 + MO_1\cdot NO = MN\cdot OO_1$$

by Ptolemy's theorem. Thus

$$\begin{aligned} a + b = OA + OB &= O_1A_1\cdot\frac{MO}{MO_1} + O_1B_1\cdot\frac{NO}{NO_1} \\ &= O_1T\cdot\frac{MO\cdot NO_1 + NO\cdot MO_1}{MO_1\cdot NO_1} \\ &= \frac{O_1T\cdot MN\cdot OO_1}{MO_1\cdot NO_1} = OO_1 . \end{aligned}$$



When the ellipse and circle are *internally* tangent, then O, O_1 lie in the same semiplane of MN, and in fact lie on a semicircle with MN as diameter. From the cyclic quadrilateral $MNOO_1$ we obtain

$$MO \cdot NO_1 - NO \cdot MO_1 = MN \cdot OO_1.$$

Thus the above method will yield

$$OO_1 = a - b.$$

II. *Solution by Kee-wai Lau, Hong Kong.*

Let the equation of the ellipse be

$$x^2/a^2 + y^2/b^2 = 1.$$

The result is obvious if the parallel tangent lines are parallel to one of the axes. Let the slope of the tangents be $\tan\theta$. It is well known that the equation of the tangent lines is given by

$$y = x\tan\theta \pm \sqrt{b^2 + a^2\tan^2\theta}\ .$$

Thus the perpendicular distance between the tangent lines is

$$2\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}\ .$$

To prove that the center of the circle is at distance $a + b$ from the center of the ellipse, we claim that the circle with center

$$((a + b)\cos\theta\ ,\ (a + b)\sin\theta)$$

and radius

$$\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$$

touches the ellipse at the point

$$P = (a\cos\theta\ ,\ b\sin\theta).$$

The equation of the circle is

$$(x - (a + b)\cos\theta)^2 + (y - (a + b)\sin\theta)^2 = b^2\cos^2\theta + a^2\sin^2\theta.$$

Obviously the point P belongs to both the circle and the ellipse. Now both the slope of the tangent to the ellipse at P and the slope of the tangent to the circle at P are equal to $-b/a\cot\theta$. It follows that these tangents coincide. This shows that in fact the circle and ellipse touch each other.

Also solved by FRANCISCO BELLOT ROSADO, *Valladolid, Spain*; K. CAPELL, *University of Queensland, St. Lucia, Australia*; A.D.D. CRAIK, *University of St. Andrews, St. Andrews, Scotland*; C. FESTRAETS-HAMOIR, *Brussels, Belgium*; J.T. GROENMAN, *Arnhem, The Netherlands*; RICHARD I. HESS, *Rancho Palos Verdes, California*; SHIKO IWATA, *Gifu, Japan*; and D.J. SMEENK, *Zaltbommel, The*

Netherlands.

This beautiful problem, which the proposer took from a Japanese mathematical wooden tablet hung in 1911, has attracted a lot of interest and several solutions, one of which arrived here even before the problem had been published in Crux! The gorgeous geometric solution of Jordi Dou which appears above would be very hard to surpass. But I have also included the best of the analytic solutions received, that of Lau.

The case when the ellipse and circle are internally tangent was also considered by Capell, Craik, and Iwata.

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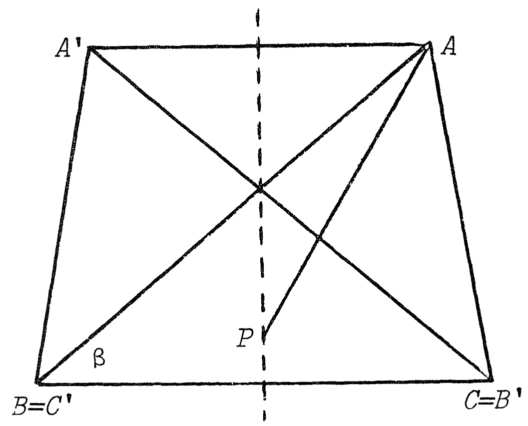
<u>1164</u>. [1986: 178] *Proposed by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.* (Dedicated to Léo Sauvé.)

In ΔABC, $\angle C = 2\angle B$, and a point P in the interior of ΔABC satisfies $AP = AC$ and $PB = PC$. Show that AP trisects $\angle A$.

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*



Let $\angle B = \beta$. We draw the reflection of ΔABC through the line of symmetry of side BC, thus getting $\Delta A'B'C'$ (with $C' = B$ and $B' = C$). Then

$$\angle A'CB = \angle ABC = \beta,$$

so $A'C$ bisects $\angle C$. As the trapezoid $BCAA'$ is isosceles, also $\angle ABA' = \beta$. As $BC \| AA'$, also $\angle CA'A = \beta = \angle A'AB$. Hence $\Delta AA'C$ is isosceles, whence $AA' = AC = AP = A'P$, i.e. $\Delta APA'$ is equilateral. Finally,

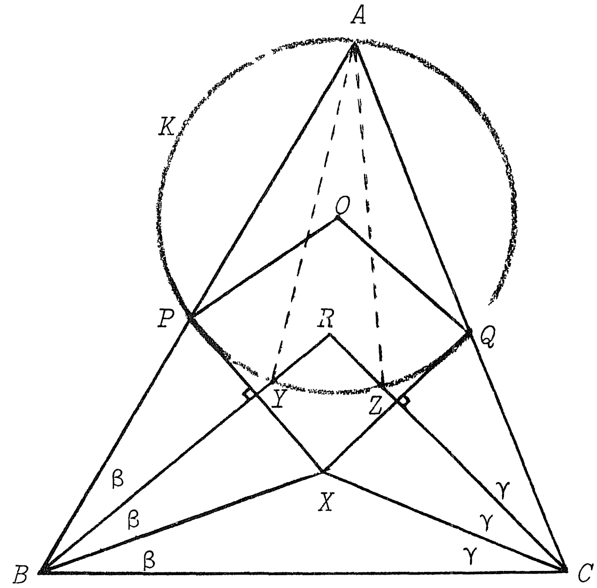
$$\angle PAB = \angle PAA' - \angle BAA' = 60^\circ - \beta = (180^\circ - 3\beta)/3$$
$$= (180^\circ - \angle B - \angle C)/3 = \angle A/3,$$

so PA trisects $\angle A$.

II. *Comment by the proposer.*

In December 1977, as part of an all-Morley issue, Léo published an article of mine called "An elementary geometric proof of the Morley Trisector Theorem" [1977: 291]. In that proof I used a lemma similar in spirit to the above proposal, but which required a case by case analysis, as I pointed out in a letter to the Editor [1978: 33]. Some years later I realized that the present proposal would have made a much better lemma for that purpose. Here is how the complete argument now goes.

We wish to prove the Morley Trisector Theorem, which states that the intersections of adjacent trisectors of the interior angles of a triangle are the vertices of an equilateral triangle.



In the figure, BX and BR are the trisectors of $\angle B$ of ΔABC, CX and CR are the trisectors of $\angle C$. X is then the incenter of ΔBRC, hence is at a common distance r from BR and CR. Reflect X about BR to P (on AB) and about CR to Q (on AC). Then $XP = XQ = 2r$, while BR and CR are the perpendicular bisectors of PX and QX respectively.

Let $\angle A = 3\alpha$, $\angle B = 3\beta$, $\angle C = 3\gamma$, and let O be the center of the circumcircle K of ΔAPQ. Then $\angle POQ = 2\angle A = 6\alpha$. We also have

$$\angle BXP = 90^\circ - \beta,\ \angle CXQ = 90^\circ - \gamma,\ \angle BXC = 180^\circ - \beta - \gamma,$$

and thus $\angle PXQ = 2(\beta + \gamma) = 120^\circ - 2\alpha$. Since $OP = OQ$ and $XP = XQ$, $\Delta OPX \equiv \Delta OQX$, so $\angle OPX = \angle OQX$, whence

$$\angle OPX = \angle OQX = (360^\circ - \angle POQ - \angle PXQ)/2 = 120^\circ - 2\alpha = \angle PXQ.$$

In particular, since $\angle OXP = \angle OXQ$, we get

$$\angle OQX = 2\angle OXQ. \qquad (1)$$

Also, letting Y and Z be the first intersection of BR and CR, respectively, with the circumcircle K, we have

$$ZX = ZQ, \quad OZ = OQ. \qquad (2)$$

Now (1) and (2) show that the hypotheses of *Crux* 1164 apply to ΔOXQ and Z (with Z playing the role of P and O the role of A), whence we conclude that

$$\angle ZOX = 1/3\ \angle XOQ = 1/6\ \angle POQ = \alpha,$$

and hence

$$\angle ZAQ = 1/2\ \angle ZOQ = \angle ZOX = \alpha,$$

so AZ trisects $\angle A$. In the same way we can show that AY trisects $\angle A$, and it remains to show that ΔXYZ is equilateral. Note that

$$\angle ZOQ = 2\alpha = \angle POY$$

and from $\angle POQ = 6\alpha$ also $\angle YOZ = 2\alpha$, hence

$$PY = YZ = QZ.$$

But $PY = YX$ and $QZ = ZX$ imply that ΔXYZ is equilateral.

Also solved by JORDI DOU, *Barcelona, Spain;* C. FESTRAETS-HAMOIR, *Brussels, Belgium;* J.T. GROENMAN, *Arnhem, The Netherlands;* KEE-WAI LAU, *Hong Kong;* D.J. SMEENK, *Zaltbommel, The Netherlands;* GEORGE TSINTSIFAS, *Thessaloniki, Greece; and the proposer. There was also one incorrect solution. Three solvers, including the proposer, sent in the same solution as Janous.*

The proposer notes (and the above solution of Janous shows) that the point P need not be inside ΔABC*, but only in the interior of* $\angle A$.

Dou notes that $\angle ABP = 30^{\circ}$ *and that the angle between* BC *and* PA *is* 60°. *(Proof?)*

* * *

<u>1165</u>*. [1986: 178] *Proposed by* M.S. *Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

For fixed $n \geq 5$, consider an n-gon P imbedded in a unit cube.

(i) Determine the maximum perimeter of P if n is odd.

(ii) Determine the maximum perimeter of P if it is convex (which implies it is planar).

(iii) Determine the maximum volume of the convex hull of P if also $n < 8$.

Editor's comment.

There have been no solutions submitted for this problem.

* * *

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