Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing. The copyright of all published material is vested in the Applied Probability Trust.

From the Editor

2014 and all that

No doubt the first thing you do on the first day of each year is to factorise the number of that year into its prime factors. Thus,

$$2014 = 2 \times 19 \times 53.$$

Unfortunately you have been beaten to it by the English mathematician John Pell (1611–1685) who factorised the first 100 000 numbers into their prime factors, a labour of love if ever there was one. Today, John Pell's name is associated with equations of the form $x^2 - dy^2 = 1$, where d is a perfect integer which is not a perfect square. These are to be solved in integers. There is no evidence that Pell ever considered such equations. They were wrongly attributed to him by Euler, and should more correctly be known as Fermat equations. But no matter.

One of our frequent contributors, Jonny Griffiths, has gone further with the current year. Trying to find a quick numerical problem to set his students, he wrote down as his first few numbers: 2014, 20142014, 201420142014, ... When he tried factorising these, they seemed to be remarkably reluctant to have square factors, so he came up with a conjecture: 'if we define 2014_n as n lots of 2014 concatenated, none of them is divisible by a perfect square bigger than 1'. The conjecture holds for $n = 1, \ldots, 8$, but then

$$\begin{aligned} 2014_9 &= 2014(1 + 10^4 + 10^8 + \cdots + 10^{32}) \\ &\equiv 2014(1^0 + 1^4 + 1^8 + \cdots + 1^{32}) \pmod 9 \\ &\equiv 0 \pmod 9, \end{aligned}$$

so that 3^2 divides 2014_9. By the same token, 2014_{9k} is divisible by 3^2.

Clearly, the number 2014 is something of a red herring – the crucial things are the numbers

$$10^0 + 10^4 + 10^8 + \cdots + 10^{4n}.$$

Jonny has not yet found any of these to be prime, although there seems to be no obvious reason why they are guaranteed to be composite. However,

$$10^0 + 10^4 + 10^8 + \cdots + 10^{80}$$

is divisible by 7^2. This can be seen by judicious use of modular arithmetic and a calculator. Thus,

$$\begin{aligned} 10^0 + 10^4 + 10^8 + \cdots + 10^{80} &= 100^0 + 100^2 + 100^4 + \cdots + 100^{40} \\ &\equiv 2^0 + 2^2 + 2^4 + 2^8 + \cdots + 2^{40} \pmod{49} \\ &\equiv \frac{2^{42} - 1}{3} \pmod{49} \end{aligned}$$

and

$$2^{42} - 1 \equiv 64^7 - 1 \equiv 15^7 - 1 \equiv 0 \pmod{49}.$$

The number 2014_{21} factorises as

$$3 \times 7^2 \times 13 \times 29 \times 37 \times 43 \times 127 \times 239 \times 281 \times 1\,933$$
$$\times\, 2\,689 \times 4\,649 \times 9\,901 \times 226\,549 \times 459\,691 \times 909\,091$$
$$\times\, 10\,838\,689 \times 121\,499\,449 \times 4\,458\,192\,223\,320\,340\,849.$$

Jonny suggests readers might find it fun to tackle these numbers with Wolfram Alpha (see http://m.wolframalpha.com/).

On a lighter note, perhaps it is time for a partial resurrection of our annual puzzle. The idea is to see how far you can get expressing numbers using the digits of the year once and once only in order, using the operations $+$, $-$, $\times$, $\div$, factorial, square root, and concatenation. Thus, to get you started, $1 = -2 + 0 - 1 + 4$. In past years we have suggested trying to get to 100, which may be too optimistic this year, but you might get into the 30s. Don't forget that $0! = 1$. Worth trying whilst you are waiting for someone, or at the dentist!

Two appearances of e

$$\cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \cdots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}$$

(S. Ramanujan, 1913)

$$x^{e-2} = \sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\cdots}}}}$$

(J. R. Fielding, 2002)

References

1 http://gauravtiwari.org/wp-content/uploads/2013/10/ramanujan_update_1310.pdf.

2 E. W. Weisstein, 'Nested radical'. From *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/NestedRadical.html.

Eleia, Greece **Spiros P. Andriopoulos**

Correction

David Benjamin has emailed to point out an error in the small item 'Reversing digits' (Volume 46, Number 2, p. 55). The difference between an odd number n and $R(n)$, the number obtained by reversing the digits of n, is always divisible by 9. In order to be divisible by 18, as claimed, the first and last digits of n need to have the same parity, i.e. to be both even or both odd. We apologize for this error.

A Filigree Sphere

RABE VON RANDOW

We analyse the geometric properties of an artistic polyhedral structure in several different ways.

1. Introduction

One frequently comes across 'polyhedrally shaped' wooden or plastic models made of a number of equal rods. These are almost always based on the dodecahedron or the icosahedron, or other closely related polyhedrons, and are usually variations of a small number of basic shapes. There are also some *tensegrity* toys among these. Tensegrity is a mechanically stable structural principle using isolated rods in compression, whose ends are connected by cables in tension. The term was coined by Buckminster Fuller (1895–1983) as a contraction of 'tensional integrity'. Fuller was best known for his geodesic domes, but he also devised a tensegrity icosahedron. Many more symmetrical tensegrity models and several amazing architectural tensegrity structures have been created since, which can be viewed on the internet. Tensegrity principles have also been applied to biological structures, which has led to the field of biotensegrity.

Very occasionally, however, one sees a shape which is in a totally different category – it takes your breath away, it overwhelms you, and you cannot stop looking at it and wondering what it is. Such a shape was created by the Swedish artist Lars Englund, born in 1933 in Stockholm. He called it *Borderline* and constructed several variations of it. Figure 1 shows the most rarified, minimalistic version. It is made of 30 equal pieces of wire shaped as shown in figure 2. This beautiful structure is in the Arithmeum Collection and hangs in the Institute for Discrete Mathematics of the University of Bonn, Germany.

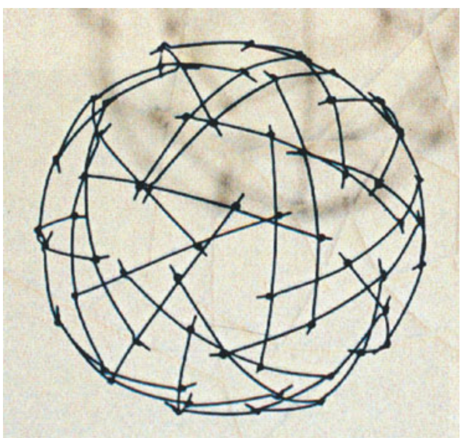

Figure 1 Lars Englund's *Borderline* structure.

Figure 2 Wire piece.

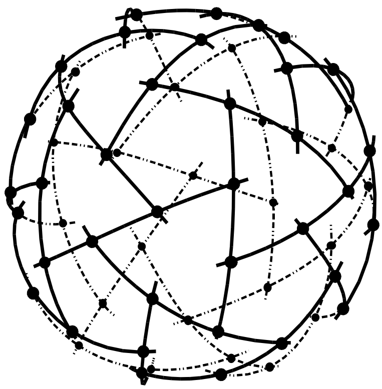

Figure 3 Drawing of Englund's *Borderline* structure.

2. The geometry of *Borderline*

Each of the 30 pieces of wire is connected to four others: at each end and also one third and two thirds of the way along. Moreover, the ends of the pieces touch other pieces at 'internal' points only (the hooks go through the loops), so that there is a 'T-junction' at every meeting point.

If we look at this structure more closely (figure 3), we see that the 30 wire pieces form 12 pentagons and 20 triangles. Taking one third of the length of a wire piece as our unit of length, we see that the edges of the pentagons have length 1 and the edges of the triangles have length 2. The number of vertices is easily seen to be $(30 \times 4)/2 = 60$. The structure shows great regularity, but there is one particular symmetry property which it does not share with the Platonic (or regular) polyhedrons: it is different from its mirror image, i.e. it has two *enantiomorphic* versions. So it is not a Platonic polyhedron. Nor is it an Archimedean (or semi-regular) polyhedron, as each vertex of a pentagon lies halfway along the edge of a triangle. It looks as if as many edges as possible have been removed from some polyhedron without the structure collapsing!

Following this line of thought, it is rather obvious where to insert edges: in each of the 20 triangles with edge length 2 we inscribe a triangle with unit edge length whose vertices are at the midpoints of the edges of the given triangle. Thus, each of the 20 original triangles now consists of four smaller triangles and all edges of the structure now have length 1. This is a polyhedron with 12 pentagonal faces as before, $20 \times 4 = 80$ triangular faces, $(30 \times 3) + (20 \times 3) = 150$ edges, and 60 vertices as before. Is it one of the known polyhedrons? Yes, it is the *snub dodecahedron* (figure 4), one of the thirteen Archimedean polyhedrons.

This is, however, not the only way to obtain Englund's *Borderline* structure – two others come to mind. Let *Method 1* be the process of inserting edges which was discussed in the previous paragraph; we now present two alternative methods.

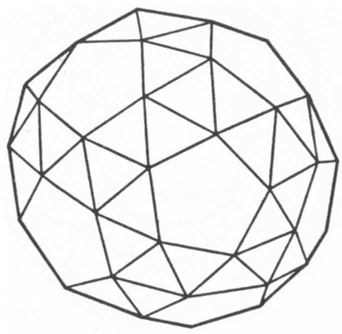

Figure 4 Snub dodecahedron.

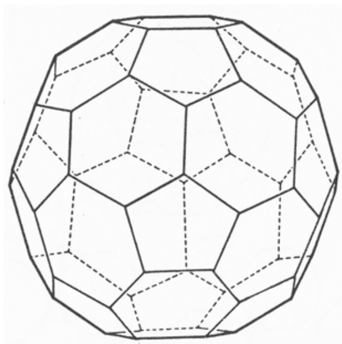

Figure 5 Truncated icosahedron.

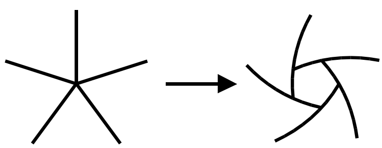

Figure 6 Separation step.

Method 2. Another way of getting rid of the problem of the vertices of the pentagons lying halfway along an edge of a triangle is to consider the triangles to be distorted hexagons with unit edge length. If we bend these hexagons until they are regular, the result is a polyhedron with 12 pentagonal faces, 20 hexagonal faces, $30 \times 3 = 90$ edges, and 60 vertices. Is this one of the known polyhedrons? Yes, it is the *truncated icosahedron* (figure 5), another one of the thirteen Archimedean polyhedrons.

Method 3. The fact that Englund's *Borderline* structure has 20 triangles and 30 wire pieces shows that it is closely related to the icosahedron, which has 20 triangular faces and 30 edges. Indeed, let us imagine that the loops in the wire pieces are small rings to which the ends of the wire pieces are attached. If we then let each ring slide towards the adjacent end of its wire piece, we will see the pentagons shrink to a point. Clearly, the resulting polyhedron is an *icosahedron* with edge length 3. The reverse operation (to be applied at every vertex of the icosahedron) is depicted in figure 6. Note that it results in a structure like a Catherine wheel which could just as well have been drawn to rotate in the other direction. This reveals why the structure comes in two enantiomorphic versions.

3. Are there any others?

Having thus elucidated the polyhedral nature of Englund's *Borderline* structure, it is only natural to wonder if there are other similar structures. We will pursue two quite distinct lines of thought.

3.1. First direction

If we look carefully at the way Englund's *Borderline* structure is constructed (figure 3), we could hit upon the following idea: why not, for each wire piece, interchange the two wire pieces that are attached to its two internal points? In other words: for each wire piece, pull the two hooks out of their loops, switch them and insert them again. Clearly, the result is a very similar shape, and yet it is quite different: it still has 12 pentagons and 20 triangles, but their edge lengths have been interchanged. Figure 7 depicts the net for this shape, which shows all its faces except for one pentagon, which is the region outside the net (i.e. it is antipodal to the central pentagon).

If we apply *Method 1* to this shape, we must, in each of the 12 pentagons with edge length 2, inscribe a pentagon whose vertices are at the midpoints of the edges of the given pentagon. Thus, each of the 12 original pentagons with edge length 2 now consists of a pentagon plus five triangles, all with unit edge length (after a bit of distortion to get unit edge lengths). This yields a polyhedron with 12 pentagonal faces as before, $20 + (12 \times 5) = 80$ triangular faces, $(30 \times 3) + (12 \times 5) = 150$ edges, and 60 vertices. This is the same polyhedron as before, namely the snub dodecahedron! In other words, applying *Method 1* to Englund's *Borderline* structure and to the shape obtained above (by switching the two wire pieces attached to each wire piece) yields the snub dodecahedron in both cases.

Applying *Method 2* to this shape (i.e. the one obtained above by switching the two wire pieces attached to each wire piece) involves bending the pentagons into decagons. This clearly results in a *truncated dodecahedron* (figure 8), again one of the thirteen Archimedean polyhedrons.

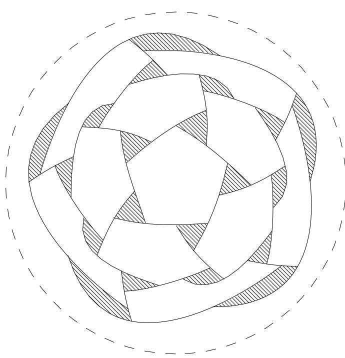

Figure 7 Net for shape based on a dodecahedron.

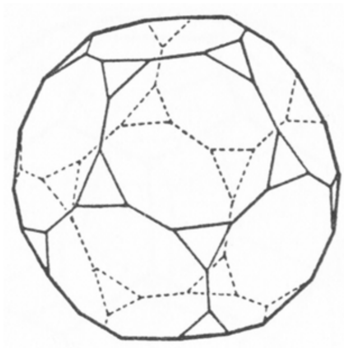

Figure 8 Truncated dodecahedron.

Note that the only difference in the parameters of the truncated dodecahedron and the truncated icosahedron (obtained by applying *Method 2* to Englund's *Borderline* structure) is that the former has 12 decagons and 20 triangles, while the latter has 12 pentagons and 20 hexagons. It might be worth mentioning here that the *truncated* Platonic polyhedrons are obtained by cutting off the corners in a symmetric way.

Finally, applying *Method 3* to this shape involves shrinking the triangles to points, which results in a *dodecahedron* with edge length 3. Note that the dodecahedron and the icosahedron (obtained by applying *Method 3* to Englund's *Borderline* structure) are duals.

3.2. Second direction

We have seen that Englund's *Borderline* structure is closely connected with the dual pair of regular Platonic polyhedrons: the icosahedron and dodecahedron. How about the dual pair octahedron and cube and the self-dual tetrahedron? We can, in fact, treat these cases as above and obtain similar structures.

We will treat the octahedron and cube case first. Clearly, we need only deal with the octahedron, as the cube, being the dual of the octahedron, will follow as above for the dodecahedron. There is, however, a significant difference: earlier we *had* a structure to which to apply our three

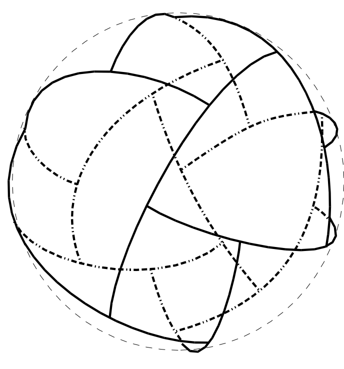

Figure 9 Shape based on an octahedron.

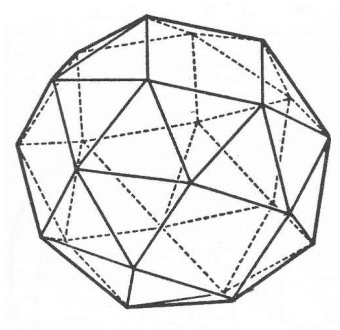

Figure 10 Snub cube.

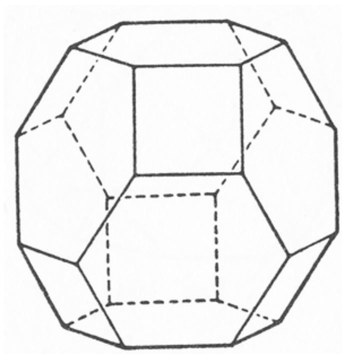

Figure 11 Truncated octahedron.

methods, but here we wish to *find* the structure. So we cannot apply *Method 1* or *Method 2*, but we *can* apply *Method 3* to the octahedron and get the new structure. The operation shown in figure 6 is the same except that now four edges meet at a vertex. An octahedron has eight triangular faces, twelve edges, and six vertices. Taking an octahedron with edge length 3, the operation yields the following structure: it consists of twelve wire pieces and has six squares with edge length 1 and eight triangles with edge length 2, and the number of vertices is $6 \times 4 = 24$ (figure 9). Now that we have our required shape, we can apply *Method 1* and *Method 2*.

Applying *Method 1* to this new shape (figure 9), we see that inscribing triangles leads to a polyhedron with 24 vertices, $(12 \times 3) + (8 \times 3) = 60$ edges, six squares, and $8 \times 4 = 32$ triangles, namely a *snub cube* (figure 10), also an Archimedean polyhedron. Similarly, *Method 2* (bending the triangles into hexagons) yields a polyhedron with eight hexagons and six squares, $12 \times 3 = 36$ edges, and 24 vertices, namely a *truncated octahedron* (figure 11), yet another Archimedean polyhedron. This is what we expected, since we started with the octahedron.

The results for the cube are now clear: we proceed as in Section 3.1. The polyhedrons which turn up here are, of course, the *truncated cube* (an Archimedean polyhedron, see figure 12) and the snub cube (figure 10). Again, the only difference in the parameters of the truncated cube and the truncated octahedron obtained above is that the former has eight triangles and six octagons, while the latter has eight hexagons and six squares. Figure 13 depicts the net for the shape

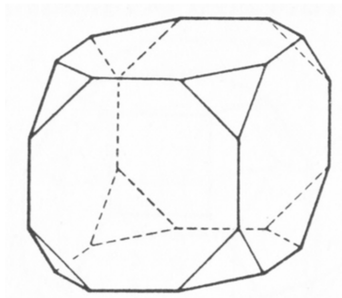

Figure 12 Truncated cube.

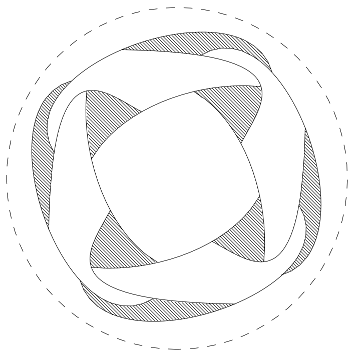

Figure 13 Net for shape based on a cube.

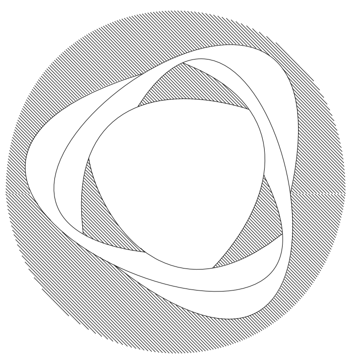

Figure 14 Net for shape based on a tetrahedron.

obtained here (twelve wire pieces, six large squares, and eight small triangles). It shows all its faces except for one square, which is the region outside the net (i.e. it is antipodal to the central square).

We will briefly compile the results for the self-dual tetrahedron. *Method 3* produces a structure with six wire pieces, four large and four small triangles, and $4 \times 3 = 12$ vertices.

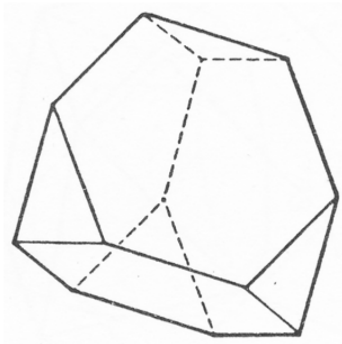

Figure 15 Truncated tetrahedron.

Figure 14 depicts the net for this shape. Note that there are four triangles with edge length 2 and four triangles with edge length 1. The latter are hatched. Note also that the outer region of the net is hatched, showing that the triangle it depicts is different from the central triangle. All the earlier shapes have the property that antipodal faces are the same. This is reflected in the truncated Platonic polyhedrons: the only one whose antipodal faces are not the same is the truncated tetrahedron. As there are comparatively few vertices, this shape is rather peaked and does not appear spherical.

Starting with this shape, *Method 1* (inscribed triangles) leads to a regular polyhedron with 12 vertices, $(6\times 3)+(4\times 3) = 30$ edges, and $4+(4\times 4) = 20$ triangles, namely an *icosahedron*. Similarly, *Method 2* (bending the large triangles into hexagons) yields a polyhedron with four hexagons and four triangles, $6\times 3 = 18$ edges, and 12 vertices, namely a *truncated tetrahedron* (an Archimedean polyhedron, see figure 15). This is what we expected.

We have discussed five shapes, one for each regular Platonic polyhedron, and have explained these in terms of transitions summarized as follows:

$$\begin{aligned}
\text{icosahedron} &\rightarrow \text{truncated icosahedron} \rightarrow \text{snub dodecahedron},\\
\text{dodecahedron} &\rightarrow \text{truncated dodecahedron} \rightarrow \text{snub dodecahedron},\\
\text{octahedron} &\rightarrow \text{truncated octahedron} \rightarrow \text{snub cube},\\
\text{cube} &\rightarrow \text{truncated cube} \rightarrow \text{snub cube},\\
\text{tetrahedron} &\rightarrow \text{truncated tetrahedron} \rightarrow \text{icosahedron}.
\end{aligned}$$

The author hopes that readers may be motivated to construct one or other of these shapes.

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Measuring the Accuracy of an Ancient Area Formula

ERIK TOU

In the ancient world, geometers were concerned primarily with mensuration (the practice of accurate measurement), with the most obvious applications being in construction and surveying. One well-known formula from this time (appearing most famously in the Temple of Horus in Egypt, c. 237–57 BC) purports to give the area of a general quadrilateral by averaging the lengths of opposite pairs of sides and then multiplying the averages. While this formula is erroneous, it produces highly accurate results when the quadrilateral is nearly rectangular. We examine the relative accuracy of this area formula, including: (i) different methods of finding the exact area, (ii) how to find the interior angle that minimizes the error of the formula, and (iii) how significantly the error varies as the interior angle varies from the ideal. In the end, these observations lead to an improved version of the formula that relies only on the side-lengths of a quadrilateral.

1. Introduction

Imagine yourself living in ancient Egypt, about 2 000 years ago. You are a surveyor, and one of your tasks is to measure the size of the various farm fields in the area, so that the local authorities can tax the farmers. All of the fields under your jurisdiction are quadrilaterals; many are very nearly rectangular, but some of them have a skewed shape. How do you complete this task? Also, assuming that your measurements are accurate, what data do you need to find the exact area of a field?

This scenario is an example of *mensuration*, the practice of accurate measurement. Over time, builders and surveyors in the ancient world developed a collection of mathematical formulas to expedite the calculation process. These formulas were not justified by rigorous proof, but rather by experience. If a formula provided a measure of area or volume that was indistinguishable from the true value, then it was considered correct.

First, let us consider some of the background on this problem. Since the sources here are scarce, archaeologists and historians have based their conclusions on a handful of well-known sources. One novel source of information is the Temple of Horus in Edfu, Egypt (see figure 1) whose inscriptions were first published in the West by Lepsius (see reference 1, p. 75ff.) in 1855 (see figure 2). (An impressive collection of information on the temple has been complied and made available online by the Edfu Project: http://www.edfu-projekt.gwdg.de/.) At the dedication of the temple, several tracts of land were dedicated to Horus and donated to the temple; both the dimensions and area of each field are given in the temple inscriptions. As recorded by Thomas Heath in his landmark text *A History of Greek Mathematics* (see reference 2, p. 124),

> From so much of these inscriptions as were published by Lepsius we gather that $\frac{1}{2}(a+c)\cdot\frac{1}{2}(b+d)$ was a formula for the area of a quadrilateral the sides of which are in order a, b, c, d.

Figure 1 Temple of Horus in Edfu, Egypt.

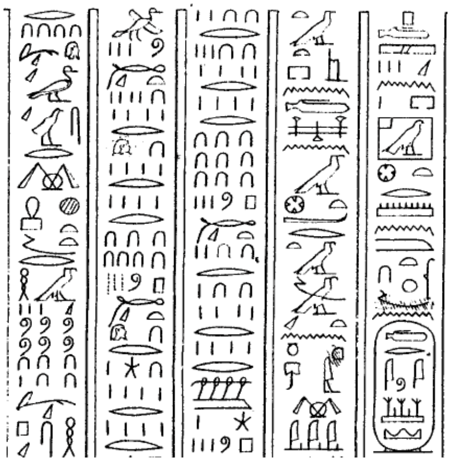

Figure 2 Inscriptions at the Temple of Horus in Edfu, Egypt.

In other words: average the lengths of opposite pairs of sides and then multiply the averages. (We call this formula the *surveyor's formula*, in keeping with Gupta (see reference 3).) This formula appears in many different cultures over the course of many centuries, so it is not unique to the Egyptians (see reference 3 for a comprehensive list of appearances of the surveyor's formula).

In a sense, the surveyor's formula 'forces' the quadrilateral to be a rectangle with side-lengths of $(a+c)/2$ and $(b+d)/2$, and then calculates the area of the rectangle (see figure 3). However, it is easy to see that this formula is incorrect. For example, choose any nonrectangular parallelogram and you will find the formula overestimates its area. This observation is nothing new: as Heath noted, 'It is remarkable enough that the use of a formula so inaccurate should have lasted till 200 years after Euclid had lived and taught in Egypt' (see reference 2, p. 124). Furthermore, it is known that this formula will never underestimate the true area of the quadrilateral (see reference 3, p. 55, or reference 4, p. 302, for a proof of this fact).

But let us return to your task as an ancient Egyptian surveyor. Your goal is to use a formula that is both accurate (as far as you are able to tell) and effective (it is quick and easy to use)

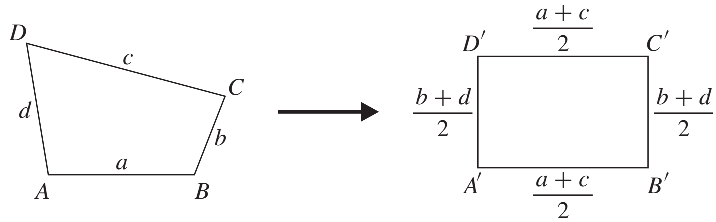

Figure 3 The 'forcing' of a quadrilateral into a rectangular shape.

for the farm fields under your jurisdiction. Our goal in this article is to assess the relative accuracy of one such formula – the surveyor's formula. After reviewing the various formulas for calculating the *exact* area of a quadrilateral, we use numerical methods to determine the relative error of the surveyor's formula. Then we use these methods to analyse some specific examples, two of which are themselves taken from the Temple of Horus. Lastly, we generate some more comprehensive data that point toward a more accurate formula.

2. Orienting the quadrilateral

As a surveyor, your task is relatively simple: first choose a corner of the field, and then walk around the perimeter while making note of the side-lengths. Geometrically, this is equivalent to choosing a corner (one of A, B, C, or D) and an *orientation* (clockwise or counterclockwise). With this in mind, a generic quadrilateral $ABCD$ will be labelled counterclockwise so that AB (taken to be the base) has length a, BC has length b, CD has length c, and DA has length d; this is implicit in figure 3.

For you, this information is quite enough: once the side-lengths are known, you can apply the averaging technique of the surveyor's formula and report the result to your supervisors. However, the side-lengths alone are not sufficient to produce a well-defined figure: one may change the interior angles of a quadrilateral while leaving the side-lengths fixed. Thus, any *exact* calculation of area requires at least one more piece of information. Typically, there are two ways to do this: the first is to take the measure of one interior angle, while the other is to take the average of a pair of opposite interior angles.

While we will decline to use the angle-averaging method in the end, it will aid in our analysis of the single-area formula, so let us first examine the formula that it produces. To do this, first relabel the angles $\angle B$ and $\angle D$ as θ and φ, respectively (see figure 4). Letting $\psi = \frac{1}{2}(\theta + \varphi)$,

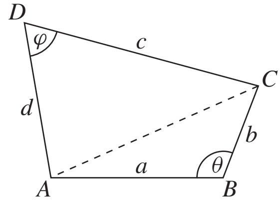

Figure 4 A quadrilateral with all vertices and sides labelled, along with two opposite angles.

the formula for the exact area is

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\psi}, \tag{1}$$

where $s = \frac{1}{2}(a+b+c+d)$ is the semiperimeter (see reference 3, p. 53). It is easy to see that the area is maximized when ψ is a right angle, so that $\cos\psi = 0$. In this case, the formula is identical to Brahmagupta's formula (see reference 5, p. 29, or reference 6, p. 52) for the area of a cyclic quadrilateral.

While this formula is elegant, and its maximum value is easily calculated, it is not the most practical way to calculate the area. From your perspective as a surveyor, it would be simpler to take the measure of a single interior angle. However, even this task is ambiguous: any one of the four interior angles could give rise to a valid formula. To eliminate this ambiguity, we need to establish an orientation to be used for all quadrilaterals we encounter.

Definition 1 For any quadrilateral, we label the vertices A, B, C, and D in a counterclockwise fashion (also labelling the corresponding side-lengths as $AB = a$, $BC = b$, $CD = c$, and $DA = d$) so that $a+b \le c+d$ and $a+d \le b+c$. We will refer to this as the *standard orientation* of a quadrilateral.

This orientation is easy to accomplish. The two inequalities can be written as $c-a \ge b-d$ and $c-a \ge d-b$, so we may choose the pair of opposite sides with the greater difference between them and label them as a and c with $c \ge a$.

3. The formula for exact area

The simplest way to calculate the exact area of a quadrilateral (using the standard orientation with additional labelling as given in figure 4) is to slice it along the diagonal AC and then add the areas of the resultant triangles as follows:

$$\text{Area} = \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\varphi. \tag{2}$$

Of course, this depends on two interior angles instead of one. We eliminate φ with the following result.

Theorem 1 *Given a quadrilateral $ABCD$ with the standard orientation, its exact area may be obtained from the formula*

$$A(\theta) = \tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sqrt{1-(L+M\cos\theta)^2}, \tag{3}$$

where $L = (c^2+d^2-a^2-b^2)/2cd$, $M = ab/cd$, and θ is the measure of the angle B.

The standard orientation is necessary in order to assign unambiguous labels to the four sides and the angle θ. The truth of the theorem follows readily from the law of cosines, since the two triangles ABC and ACD share the common side AC. Thus, we may use the relation between $\cos\theta$ and $\cos\varphi$ to express $\sin\varphi$ in terms of $\cos\theta$. Next, we want to know how accurate the surveyor's formula will be in practice. Treating the side-lengths as fixed, it is easy to find the minimum possible error: simply maximize the function $A(\theta)$.

Theorem 2 *The single-variable formula $A(\theta)$ is maximized when*

$$\theta = \theta_0 = \arccos\left(\frac{-L}{M+1}\right).$$

This result follows from the fact that the angle-averaging area formula (1) is maximized when $\varphi + \theta = \pi$, which provides the relation $\cos\varphi = -\cos\theta$. Then, taking θ_0 and φ_0 to be the angles at which the area is maximized, the proof of the single-angle formula (3) tells us that

$$-\cos\theta_0 = \cos\varphi_0 = L + M\cos\theta_0,$$

from which we obtain $\cos\theta_0 = -L/(M+1)$.

4. Measures of average

Since (3) treats area as a function of the single, continuous variable θ, integration is the simplest way to calculate the average area over any interval $[\theta_1, \theta_2]$:

$$\text{Average area over } [\theta_1, \theta_2] = \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} A(\theta)\,\mathrm{d}\theta. \tag{4}$$

We also define the relative error as a function of θ:

$$R(\theta) = \frac{|E - A(\theta)|}{A(\theta)} = \left|\frac{E}{A(\theta)} - 1\right|,$$

where E denotes the estimated area given by the surveyor's formula. (Since the surveyor's formula depends only on the side-lengths, it follows that E is constant with regard to θ.) Lastly, the average relative error for an interval $[\theta_1, \theta_2]$ is obtained by integrating $R(\theta)$:

$$\frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} R(\theta)\,\mathrm{d}\theta = \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2}\left|\frac{E}{A(\theta)} - 1\right|\mathrm{d}\theta. \tag{5}$$

In order to apply these formulas, we need to find the minimum and maximum values of θ for a given set of side-lengths. This is where the standard orientation will help us: since $b + c \geq a + d$, minimizing θ will produce a triangle with side-lengths $a+d$, b, and c. Again using the law of cosines, we find that $\cos\theta = ((a+d)^2 + b^2 - c^2)/2(a+d)b$ in this case (let θ_1 denote this angle). Since $c + d \geq a + b$, maximizing θ will produce a triangle with sides $a+b$, c, and d, and $\theta_2 = \pi$. With these bounds, we can now apply (4) and (5) to obtain the average area and average relative error over all possible values of θ (see figure 5).

Example 1 We have $(a, b, c, d) = (4, 6, 7, 5)$. Let us return to your surveying task: you come upon a field with side-lengths 4, 6, 7, and 5. After measuring these side-lengths, you apply the

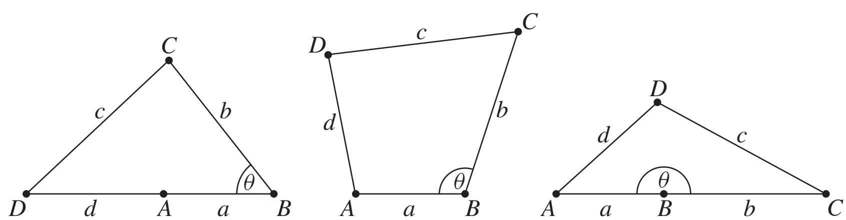

Figure 5 Finding the minimum (left) and maximum (right) values of θ for a quadrilateral $ABCD$ (centre).

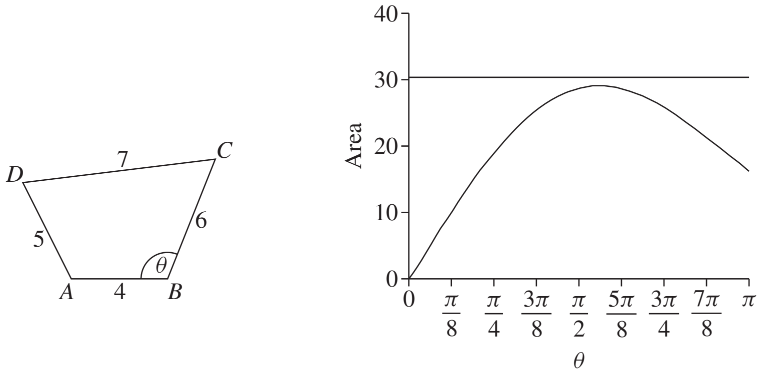

Figure 6 The quadrilateral $(4, 6, 7, 5)$ and a graph comparing $A(\theta)$ and E. Very large or small values of θ will produce highly erroneous estimates.

surveyor's formula to produce an area estimate of $\frac{1}{2}(4+7) \cdot \frac{1}{2}(6+5) = 30.25$. Unbeknownst to you, the average area for this set of side-lengths (taken over the full range of possible values of θ) is approximately 24.906 and the average relative error is approximately 24.498%. (For consistency, all calculations will be displayed at five significant figures.) However, you may not notice the error in the surveyor's formula if the value of θ for your particular field is not extremely large or small (see figure 6).

To apprehend this phenomenon more clearly, consider instead some more median intervals for θ. The easiest way to do this is to take the maximum-area angle θ_0 and then use intervals centred on this value. In this example, θ_0 is approximately $100.75°$ (≈ 1.7583 radians) and the relative errors are shown in table 1.

While it is possible that you may notice a 5% relative error in your calculation, it is also possible that this error would pass unnoticed. We may also guess that the relative proportions of the four sides have a bearing on the relative error. Indeed, this seems to be the case, as the following example demonstrates.

Example 2 We have $(15, 3.5, 16, 4)$. This is a more 'evenly balanced' quadrilateral than the previous one, in that opposite pairs of sides are more nearly equal. As a side note, this quadrilateral matches the side-lengths of a field described on the Temple of Horus. (See references 1 and 2 for more detail on this fact.)

Table 1

Deviation from θ_0	Average relative error
$\pm 2°$	4.3875%
$\pm 4°$	4.4327%
$\pm 6°$	4.5082%
$\pm 8°$	4.6140%
$\pm 10°$	4.7506%

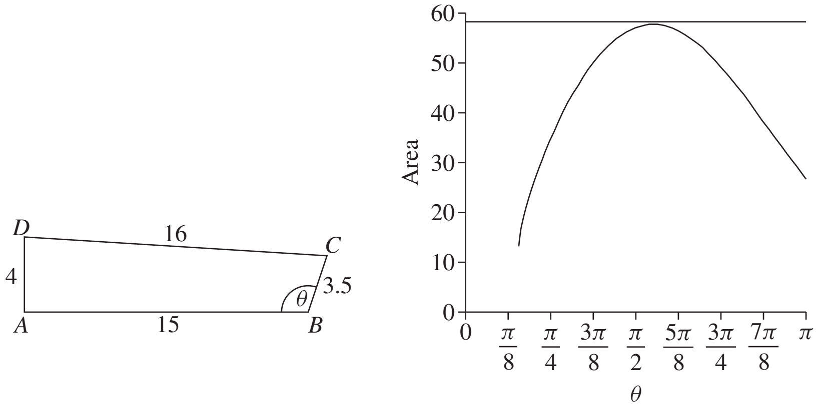

Figure 7 The quadrilateral $(15, 3.5, 16, 4)$ and the graphs of $A(\theta)$ and E.

Table 2

Deviation from θ_0	Average relative error
$\pm 2°$	0.93124%
$\pm 4°$	0.98289%
$\pm 6°$	1.0691%
$\pm 8°$	1.1902%
$\pm 10°$	1.3464%

Here, your calculation gives $\frac{1}{2}(15+16) \cdot \frac{1}{2}(3.5+4) = 58.125$ for the area. The average area (again taken over all possible θ values) is 45.060, while the average relative error is 39.131% (see figure 7). In this case $\theta_0 \approx 98.577°$ (about 1.7205 radians), and the relative errors are shown in table 2. The formula is more accurate (by a factor of four) than the one in example 1. It is far less likely that you would notice anything amiss with the surveyor's formula in this case.

5. Measures of unevenness

Of course, there are several issues with this analysis. One is that you and your fellow surveyors do not have access to a wide range of farm fields to analyze, and most of the fields you encounter are roughly rectangular by design. Another is that the concept of 'unevenness' is a nebulous one, and many different definitions could be used. To resolve this second issue, we choose to measure unevenness in the following way.

Definition 2 Given a quadrilateral $ABCD$ with the standard orientation, define the unevenness measure μ as the difference between θ_0 (measured in radians) and $\pi/2$, i.e.

$$\mu = \left|\theta_0 - \frac{\pi}{2}\right|.$$

Since $0 \le \theta_0 \le \pi$, it follows that $0 \le \mu \le \pi/2$.

Table 3

Sides	(4, 6, 7, 5)	(15, 3.5, 16, 4)	(22, 4, 23, 4)	(10, 4.5, 10.5, 4)
μ	0.18754	0.14970	0.12533	0.034490
Deviation	Average relative error			
$\pm 2°$	4.3875%	0.93124%	0.81042%	0.22511%
$\pm 4°$	4.4327%	0.98289%	0.87018%	0.29071%
$\pm 6°$	4.5082%	1.0691%	0.97002%	0.40034%
$\pm 8°$	4.6140%	1.1902%	1.1103%	0.55448%
$\pm 10°$	4.7506%	1.3464%	1.2916%	0.75377%

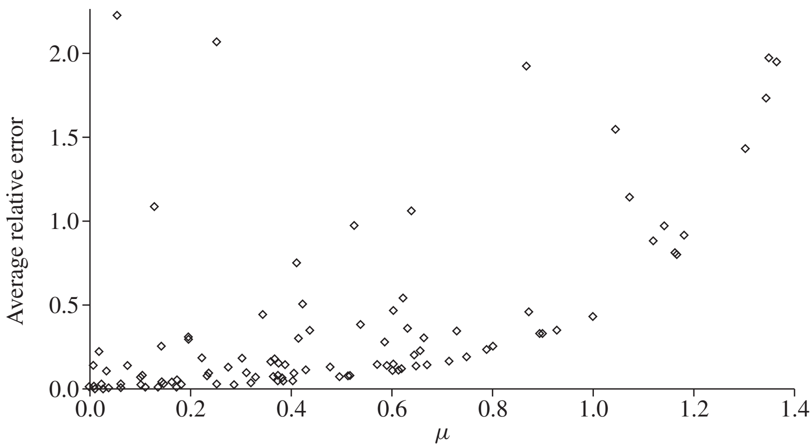

Figure 8 Average relative error within $6°$ of maximum-area angle θ_0 plotted against μ for 100 randomly generated quadrilaterals using the surveyor's formula.

In this way, a quadrilateral with $\mu = 0$ is one for which the 'ideal' angle is a right angle. Furthermore, since $\theta = \theta_0$ implies that θ and φ are supplementary, a quadrilateral with $\mu = 0$ is one that can be decomposed into two right-angled triangles by slicing along the diagonal AC.

Next we add two quadrilaterals to the list: $(22, 4, 23, 4)$ (this quadrilateral also appears on the Temple of Horus in Edfu) and $(10, 4.5, 10.5, 4)$. We now compare unevenness to average relative error for all four quadrilatrals (see table 3). It is easy to see that a decrease in μ corresponds to a decrease in average relative error.

Next, we repeat this analysis for a larger set of quadrilaterals. Specifically, 100 quadrilaterals were generated randomly (all sides were restricted to an integer length between 1 and 40, and the fourth side-length was chosen with the restriction that it be smaller than the sum of the previous three) and the average relative errors were computed for an interval deviating from θ_0 by $\pm 6°$. When these errors are plotted against μ, the chart in figure 8 is obtained.

Here, the trend is much clearer. It is interesting to note the presence of three outliers in the upper-left part of the chart; these are the quadrilaterals $(1, 27, 27, 4)$, $(2, 19, 20, 2)$, and $(1, 25, 23, 7)$. In each of these cases, the two longest sides are adjacent to each other, and greatly exceed the lengths of the two smaller sides (i.e. these are kite-shaped quadrilaterals). The upper-centre outlier is $(5, 29, 38, 7)$.

Lastly, when $\mu = 0$, finding the difference between the surveyor's formula and the actual area is a matter of halving the difference of pairs of opposite sides (called the *difference average* to distinguish it from the common notion of an average).

Theorem 3 *Let $ABCD$ be a quadrilateral with the standard orientation. The estimated area given by the surveyor's formula exceeds the actual area by at least the product of the difference averages of opposite pairs of sides, i.e.*

$$E - A(\theta_0) \geq \tfrac{1}{2}|a-c|\tfrac{1}{2}|b-d|.$$

Proof First, from (2) we know that slicing the quadrilateral along the diagonal AC produces two triangles, from which it follows that Area $= \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\varphi \leq \frac{1}{2}ab + \frac{1}{2}cd$. Alternatively, if we slice the quadrilateral along BD we have Area $\leq \frac{1}{2}ad + \frac{1}{2}bc$.

Next, note that it suffices to show that $A(\theta_0) + \frac{1}{2}|a-c|\frac{1}{2}|b-d| \leq \frac{1}{2}(a+c)\frac{1}{2}(b+d)$. Recalling that the standard orientation dictates that $c \geq a$, we can see that

$$\begin{aligned} A(\theta_0) + \tfrac{1}{2}|a-c|\tfrac{1}{2}|b-d| &\leq \tfrac{1}{2}ab + \tfrac{1}{2}cd + \tfrac{1}{4}(c-a)(b-d) \\ &= \tfrac{1}{2}(a+c)\tfrac{1}{2}(b+d), \end{aligned}$$

provided that $b \geq d$. In the case $d \geq b$, we merely use the other inequality from above, i.e.

$$\begin{aligned} A(\theta_0) + \tfrac{1}{2}|a-c|\tfrac{1}{2}|b-d| &\leq \tfrac{1}{2}ad + \tfrac{1}{2}bc + \tfrac{1}{4}(c-a)(d-b) \\ &= \tfrac{1}{2}(a+c)\tfrac{1}{2}(b+d). \end{aligned}$$

6. Conclusion

The surveyor's formula, $\frac{1}{2}(a+c)\frac{1}{2}(b+d)$, was in widespread use in many cultures around the world over the course of many centuries. It is conceivable that a few perceptive individuals recognized that the formula was incorrect in some cases, and that it was only applicable when the quadrilateral in question was roughly rectangular. However, no evidence of this has been found (or is likely to be found). Nevertheless, the current analysis has led to the conclusion that errors in calculation are indeed small enough not to have been noticed, provided that the unevenness of the quadrilateral is sufficiently small.

Furthermore, theorem 3 suggests that an improvement can be made by subtracting the product of the difference averages from the estimate given by the surveyor's formula as follows:

$$\tfrac{1}{2}(a+c)\tfrac{1}{2}(b+d) - \tfrac{1}{2}|a-c|\tfrac{1}{2}|b-d|. \tag{6}$$

Theorem 3 assures us that this will always be greater than or equal to the actual area. Returning once again to your task as an ancient Egyptian surveyor, this new formula would not unnecessarily complicate your work: after measuring the four sides, it would only be necessary to subtract an additional term when doing the calculation. Furthermore, this formula can be reduced to $\frac{1}{2}(ab+cd)$ when $b \geq d$, or to $\frac{1}{2}(ad+bc)$ when $d \geq b$. (Interestingly, each of these formulas appears in reference 3, p. 55, though with a different purpose in mind.) This version of the formula only requires you to multiply the lengths of two adjacent sides and

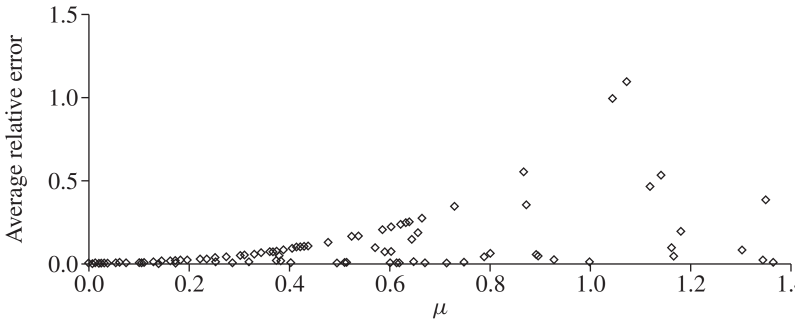

Figure 9 Average relative error within 6° of maximum-area angle θ_0 plotted against μ for 100 randomly generated quadrilaterals using the improved surveyor's formula (6).

average the resulting numbers. Since $a \leq c$ under the standard orientation, the reduced version of (6) can be summarized as follows.

> *Improved surveyor's formula*: Given the two pairs of opposite sides, separately multiply the smaller from each pair by the larger from the other pair. Then take the average of these two products.

This formula still employs the averaging technique of the surveyor's formula, but relies instead on the average of two products (as opposed to the product of two averages). Moreover, it does not rely on the particular orientation of the quadrilateral. The relative error data for the same 100 quadrilaterals makes it clear that (6) really is an improvement (see figure 9).

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A Criterion for the Numbers of Positive and Negative Solutions of a Cubic Equation

JINGCHENG TONG and SIDNEY KUNG

Using the derivative of a function, we develop a new criterion for the numbers of positive and negative solutions of a cubic equation.

It is well known that a cubic equation $X^3+aX^2+bX+c=0$ can be reduced to $x^3+px+q=0$ by the substitution $X=x-a/3$. The types of solutions can be determined by the discriminant $D=27q^2+4p^3$. If $D>0$, there is one real solution. If $D\leq 0$, there are three real solutions (a solution of multiplicity k is counted as k solutions). In this article we use the derivative of a function to establish not only these results about the number of real solutions but also the signs of these solutions.

Since the discussion of solutions for $p=0$ or $q=0$ is trivial, we assume that $pq\neq 0$.

Criterion 1 *Let $x^3+px+q=0$ be a cubic equation with real coefficients p and q.*

(a) *If $p>0$, then there is only one real solution, negative for $q>0$ and positive for $q<0$.*

(b) *If $p<0$, then*

- (i) *for $q>(2\sqrt{3}/9)(-p)^{3/2}$, there is only one real solution which is negative,*
- (ii) *for $q<-(2\sqrt{3}/9)(-p)^{3/2}$, there is only one real solution which is positive,*
- (iii) *for $0<q\leq(2\sqrt{3}/9)(-p)^{3/2}$, there are three real solutions, one negative and two positive (equal if $q=(2\sqrt{3}/9)(-p)^{3/2}$),*
- (iv) *for $0>q\geq-(2\sqrt{3}/9)(-p)^{3/2}$, there are three real solutions, one positive and two negative (equal if $q=-(2\sqrt{3}/9)(-p)^{3/2}$).*

Proof We rewrite $x^3+px+q=0$ as

$$x^3=-px-q. \tag{1}$$

Finding the solutions of (1) is equivalent to finding the x-coordinates of the intersection points of the cubic curve $y=x^3$ and the line $y=-px-q$ with slope $-p$ and y-intercept $-q$.

If $p>0$, the conclusion of part (a) is immediate (see figure 1(a)).

If $p<0$, then $-p>0$. On the graph of $y=x^3$, there are two points S and T at which the tangent lines are parallel to $y=-px-q$. The slope of a tangent line of the curve $y=x^3$ at the point (x,y) is $3x^2$, which should be equal to that of the line $y=-px-q$. By letting $3x^2=-p$ we get

$$S=\left(-\left(-\frac{p}{3}\right)^{1/2},-\left(-\frac{p^3}{27}\right)^{1/2}\right),\qquad T=\left(\left(-\frac{p}{3}\right)^{1/2},\left(-\frac{p^3}{27}\right)^{1/2}\right).$$

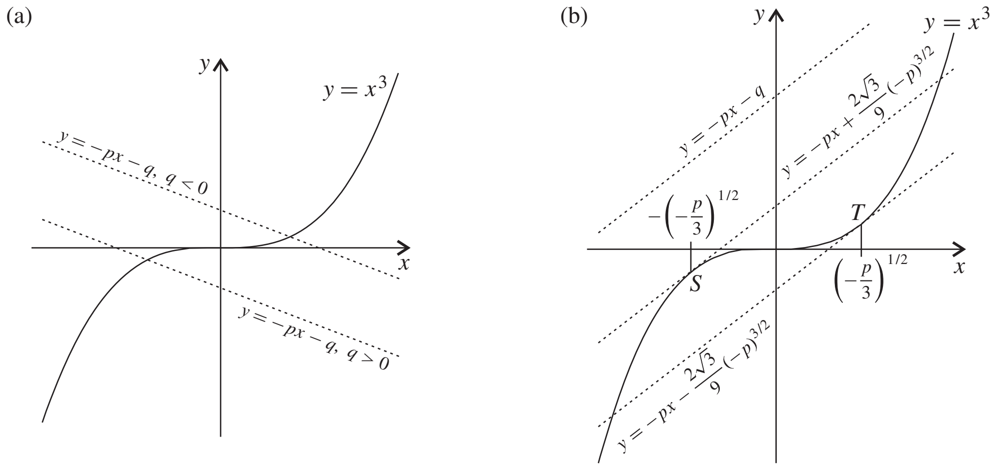

Figure 1

The equations of the two tangent lines parallel to $y = -px - q$ and passing through S and T are

$$y = -px + \frac{2\sqrt{3}}{9}(-p)^{3/2} \quad \text{and} \quad y = -px - \frac{2\sqrt{3}}{9}(-p)^{3/2},$$

respectively. Thus, if $q > (2\sqrt{3}/9)(-p)^{3/2}$, the line $y = -px - q$ is below the two tangent lines, resulting in just one real solution which is negative, as in part (i) (see figure 1(b)). If $q < -(2\sqrt{3}/9)(-p)^{3/2}$, the line $y = -px - q$ is above the two tangent lines, resulting in just one real solution which is positive, as in part (ii).

Now, if $0 < q \leq (2\sqrt{3}/9)(-p)^{3/2}$, then $y = -px - q$ is between $y = -px$ and the lower tangent line and there are three real solutions, one negative and two positive (equal if $q = (2\sqrt{3}/9)(-p)^{3/2}$), as in part (iii). Finally, if $0 > q \geq -(2\sqrt{3}/9)(-p)^{3/2}$, then $y = -px - q$ is between $y = -px$ and the upper tangent line. There are three real solutions, two negative (and equal if $q = -(2\sqrt{3}/9)(-p)^{\sqrt{3}/2}$), and one positive, as in part (iv).

Readers may use criterion 1 to verify each of the following results.

- $x^3 + x - 3 = 0,\ p > 0,\ q < 0$, criterion 1(a). Only one real solution which is positive (lying between $x = 1$ and $x = 1.5$).
- $x^3 - x + 2 = 0,\ p < 0$, criterion 1(b)(i). Only one real solution which is negative (lying between $x = -2$ and $x = -1$).
- $x^3 - x + \frac{1}{4} = 0,\ p < 0$, criterion 1(b)(iii). Two positive and one negative real solutions.
- $x^3 - 2x - \frac{1}{2} = 0,\ p < 0$, criterion 1(b)(iv). Two negative and one positive real solutions.

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Finding Arbitrarily Many Integer Solutions to the Equation $k^2 - 5 = t^2 - 5u^2$

JONNY GRIFFITHS

As we vary k, is there an upper limit on the number of solutions (t, u) for the equation $k^2 - 5 = t^2 - 5u^2$? This article asserts that there is not, and employs an unusual argument to support this claim, one that could be of wider significance to those working on problems in number theory.

Not long ago, I was trying to prove the following conjecture.

Conjecture 1 *Define a Hikorski Triple (or HT) (see reference 1) as* (a, b, c) *where* a, b, *and* $c = (ab + 1)/(a + b)$ *are all natural numbers. Show that the number of HTs whose three elements add to* k *is unbounded as* $k \to \infty$.

This reduces to the following conjecture.

Conjecture 2 *Show that, as we vary* k, *the equation* $k^2 - 5 = t^2 - 5u^2$ *can have arbitrarily many integer solutions for* t *and* u.

Or to put this another way, show that, given any $n \in \mathbb{N}$, we can find a $k \in \mathbb{N}$ so that $k^2 - 5 = t^2 - 5u^2$ has more than n integer solutions (t, u).

A 'proof' of conjecture 2 does emerge, but not one that would pass any strict tests of rigour. It does, however, utilise a neat trick that I hope might be of interest to *Mathematical Spectrum* readers.

Preliminary computer searches suggest that 3230 will prove to be a fruitful choice for k, one where many (t, u) solutions are possible. In this case, $k^2 - 5 = 10\,432\,895 = 5 \cdot 11 \cdot 29 \cdot 31 \cdot 211$. Now,

$$5 \cdot 11 \cdot 29 \cdot 31 \cdot 211 = (5^2 - 5 \times 2^2)(4^2 - 5 \times 1^2)(7^2 - 5 \times 2^2)(6^2 - 5 \times 1^2)(16^2 - 5 \times 3^2).$$

This suggests that if p is a prime dividing $k^2 - 5$, then p can always be written as $x^2 - 5y^2$. Are there primes that cannot be expressed as $x^2 - 5y^2$ for some integers x and y? The prime 2 cannot be so expressed, since $2 = x^2 - 5y^2$ implies that $2 \equiv x^2 \pmod 5$, which implies that 2 is a quadratic residue modulo 5, which is untrue. Noting that $5 = 5^2 - 5 \times 2^2$, take now an odd prime p that is not equal to 5. If $p = x^2 - 5y^2$ then $p \equiv x^2 \pmod 5$, which means $\left(\frac{p}{5}\right) = 1$, where $\left(\frac{a}{b}\right)$ is the Legendre symbol. Now, p is congruent to 1, 2, 3, or 4 modulo 5, and 1 and 4 are squares, so 1 and 4 are quadratic residues modulo 5, while 2 and 3 are not. This gives us $p = 5(2m) + 1$ or $p = 5(2m + 1) + 4$, since p is odd, and so p must be of the form $10m + 1$ or $10m + 9$.

In fact, the implication can be reversed, although not straightforwardly; we can quote the following result.

Theorem 1 *For an odd prime* $p \neq 5$, p *can be expressed as* $x^2 - 5y^2$ *if and only if* $p = 10m + 1$ *or* $p = 10m + 9$.

This is a special case of a much larger result concerning the representation of primes by quadratic forms (see reference 2).

We now turn to the following result.

Theorem 2 *If $p \neq 2$ is a prime dividing $k^2 - 5$, then p can be written as $x^2 - 5y^2$.*

Proof We know 5 can be so expressed, so suppose that $p \neq 5$. Then $k^2 \equiv 5 \pmod{p}$, so $\left(\frac{5}{p}\right) = 1$. Now by the theorem of quadratic reciprocity, $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$, so p is congruent to 1 or 4 modulo 5, so p is congruent to 1 or 9 modulo 10 (since p is odd). Hence, by theorem 1, p can be expressed as $x^2 - 5y^2$.

Now, writing $k^2 - 5$ for k even as a product of primes, all of which are of the form $x^2 - 5y^2$, we can see that they each factorise into $(x + y\sqrt{5})(x - y\sqrt{5})$. We can now write out the full factorisation of $k^2 - 5$ as

$$\prod_i (x_i + y_i\sqrt{5}) \prod_i (x_i - y_i\sqrt{5}).$$

The first product simplifies to $\alpha + \beta\sqrt{5}$, while the second product becomes $\alpha - \beta\sqrt{5}$. Thus, for example,

$$\begin{aligned} 3\,230^2 - 5 &= 5 \cdot 11 \cdot 29 \cdot 31 \cdot 211 \\ &= (5^2 - 5 \times 2^2)(4^2 - 5 \times 1^2)(7^2 - 5 \times 2^2)(6^2 - 5 \times 1^2)(16^2 - 5 \times 3^2) \\ &= (5 + 2\sqrt{5})(4 + 1\sqrt{5})(7 + 2\sqrt{5})(6 + 1\sqrt{5})(16 + 3\sqrt{5}) \\ &\quad \times (5 - 2\sqrt{5})(4 - 1\sqrt{5})(7 - 2\sqrt{5})(6 - 1\sqrt{5})(16 - 3\sqrt{5}) \\ &= 63\,410^2 - 5 \times 28\,321^2. \end{aligned}$$

Here is the neat trick: if we exchange a set of plus signs in the first product for the corresponding set of minus signs in the second, the pair of products becomes $(\alpha'\sqrt{5} + \beta')(\alpha'\sqrt{5} - \beta') = k^2 - 5$. Thus, for example,

$$\begin{aligned} 3\,230^2 - 5 &= 5 \cdot 11 \cdot 29 \cdot 31 \cdot 211 \\ &= (5 + 2\sqrt{5})(4 + 1\sqrt{5})(7 + 2\sqrt{5})(6 + 1\sqrt{5})(16 - 3\sqrt{5}) \\ &\quad \times (5 - 2\sqrt{5})(4 - 1\sqrt{5})(7 - 2\sqrt{5})(6 - 1\sqrt{5})(16 + 3\sqrt{5}) \\ &= 26\,030^2 - 5 \times 11\,551^2 \\ &= (5 + 2\sqrt{5})(4 + 1\sqrt{5})(7 + 2\sqrt{5})(6 - 1\sqrt{5})(16 - 3\sqrt{5}) \\ &\quad \times (5 - 2\sqrt{5})(4 - 1\sqrt{5})(7 - 2\sqrt{5})(6 + 1\sqrt{5})(16 + 3\sqrt{5}) \\ &= 12\,070^2 - 5 \times 5\,201^2. \end{aligned}$$

The full set of resulting values for t and u is given in table 1.

A computer search tells us that these are not the only possibilities. It seems that predicting the number of solutions for (t, u) from the starting k is not an exact science, but we can certainly say that the more prime factors we have, the more (t, u) pairs we are likely to find.

So one thing remains: can we always find a value for k such that $k^2 - 5$ has arbitrarily many prime factors? There is a helpful identity here: two integers of the form $x^2 - 5y^2$ always

Table 1 Possible values for (t, u) when $k = 3\,230$.

t	u	t	u
63 410	28 321	3 230	1
26 030	11 551	3 250	161
29 050	12 911	4 610	1 471
14 150	6 161	8 510	3 521
18 070	7 951	7 690	3 121
4 270	1 249	6 790	2 671
3 610	721	6 170	2 351
4 130	1 151	12 070	5 201

multiply to an integer of the same shape, since

$$\begin{aligned}(x_1^2 - 5y_1^2)(x_2^2 - 5y_2^2) &= (x_1 + y_1\sqrt{5})(x_2 + y_2\sqrt{5})(x_1 - y_1\sqrt{5})(x_2 - y_2\sqrt{5}) \\ &\equiv (x_1x_2 + 5y_1y_2)^2 - 5(x_1y_2 + x_2y_1)^2.\end{aligned}$$

Given this, we can multiply together arbitrarily many distinct odd prime factors, say $p_1, p_2, \ldots, p_m$, each of the form $x^2 - 5y^2$ (primes ending in 1 or 9) to give a number of the form $X^2 - 5Y^2$. So we have

$$X^2 - 5Y^2 \equiv 0 \pmod{p_1p_2\cdots p_m},$$

and so $X^2 \equiv 5Y^2 \pmod{p_1p_2\cdots p_m}$, and so $(XY^{-1})^2 \equiv 5 \pmod{p_1p_2\cdots p_m}$, and $p_1p_2\cdots p_m|((XY^{-1})^2 - 5)$. (We know that Y^{-1} exists, since $\gcd(Y, p_1p_2\cdots p_m) = 1$, because if $p_i|Y$, then $p_i|X$, and $p_i^2|X^2 - 5Y^2$, which contradicts the fact that the p_i are distinct.) Thus, we have a number of the form $k^2 - 5$ that has arbitrarily many prime factors of the desired form, which lends support to (but does not prove) conjecture 2.

I hope that this method deserves wider attention.

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Revisiting Euler's Theorem on Odd Perfect Numbers

M. A. NYBLOM

A slightly alternative proof is presented which establishes a classic result of Euler on odd perfect numbers. The argument depends on nothing more than a simple factorization of polynomials of the form $1 + p + \cdots + p^n$, together with some elementary algebraic identities.

1. Introduction

A *perfect number* is a positive integer N having the property that it is equal to the sum of all its divisors less than N. The first two even perfect numbers to be identified were

$$6 = 1 + 2 + 3 \quad \text{and} \quad 28 = 1 + 2 + 4 + 7 + 14,$$

while the next two largest are 496 and 8128, which with some effort the reader may care to verify. By studying these first four perfect numbers, Euclid, in Proposition 36 of Book IX of *Elements* (see reference 1), observed that if $2^m - 1$ is a prime number, for some positive integer m, then $2^{m-1}(2^m - 1)$ would be an even perfect number. It would be some two thousand years later when Euler proved that every even perfect number must necessarily be of the form $2^{m-1}(2^m - 1)$, where $2^m - 1$ is a prime number. Despite Euler's characterization, we do not know whether there are infinitely many even perfect numbers, as the infinitude of the Mersenne primes (i.e. primes of the form $2^m - 1$) to this date is an open problem. In stark contrast, one of the oldest open problems in number theory and indeed of all mathematics is the question of whether odd perfect numbers exist at all.

It is known that if an odd perfect number exists, then it must have at least nine distinct prime factors, a result proved recently by Nielsen (see reference 2). The earliest result on the number of distinct prime factors necessary for an odd perfect number to exist can be traced to J. J. Sylvester, who proved in 1888 that an odd perfect number must have at least two distinct prime factors. In Sylvester's proof (see reference 3, pp. 589–590), which is both short and elegant, we see for the first time the employment of the so-called abundancy index $\sigma(N)/N$ of an integer N, where $\sigma(N)$ is the sum of the divisors of N, which now includes N as a divisor of itself. Thus, a positive integer N is perfect if and only if $\sigma(N)/N = 2$. Sylvester's argument entailed showing that $\sigma(N)/N < 2$ when $N = p^r$ for an odd prime number p. Recalling that $\sigma(p^r) = 1 + p + p^2 + \cdots + p^r$, observe, as $p \geq 3$,

$$\frac{\sigma(p^r)}{p^r} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^r} \leq 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^r} < \frac{1}{1 - \frac{1}{3}} = \frac{3}{2};$$

consequently, $N = p^r$ cannot be an odd perfect number. Using a variation of the previous argument together with the fact that the arithmetic function $\sigma(\cdot)$ is multiplicative, in that $\sigma(mn) = \sigma(m)\sigma(n)$, for positive integers whose greatest common divisor is one, Sylvester also showed that $N = p^r q^s$ for distinct odd primes p, q could not be a perfect number as follows. Considering

$$\frac{\sigma(p^r q^s)}{p^r q^s} = \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^r}\right)\left(1 + \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^s}\right)$$
$$\leq \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^r}\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^s}\right),$$

noting here that $3 \leq p$ and $5 \leq q$ as p, q are odd primes with $p < q$. Now, bounding each of the finite geometric series above by their infinite geometric series sums, namely $\frac{3}{2}$ and $\frac{5}{4}$ respectively, we arrive at $\sigma(N)/N < 2$. Thus, an odd perfect number must have at least three distinct prime factors.

As in the case of the even perfect numbers, Euler gave the following result about odd perfect numbers, namely that if an odd number N is perfect, then necessarily $N = p^s q^2$ for an odd prime p and an odd integer q with $p, s \equiv 1 \pmod 4$ and $\gcd(p, q) = 1$. Most presentations of Euler's proof rely on the use of the calculus of modular arithmetic. In this article, we shall present an alternative proof that avoids the use of modular arithmetic calculations, by the employment of an argument using nothing more than a simple factorization of the polynomial $\sigma(p^r) = 1 + p + \cdots + p^r$, together with some elementary algebraic identities.

2. The alternative proof

In the ensuing proof of Euler's theorem we shall make use of the following simple fact: for an odd prime p and a positive integer r, the finite geometric sum $\sigma(p^r)$ is odd when r is even and *vice versa*.

Theorem 1 *If a positive odd integer N is perfect, then necessarily $N = p^s q^2$, for an odd prime p and an odd integer q with $p, s \equiv 1 \pmod 4$ and $\gcd(p, q) = 1$.*

Proof Assume that a positive odd integer N is perfect and so has three or more distinct prime factors, that is suppose that

$$N = \prod_{i=1}^{n} p_i^{n_i},$$

with $n \geq 3$, satisfies

$$2N = \sigma(N) = \prod_{i=1}^{n} \sigma(p_i^{n_i}).$$

Recalling that each $\sigma(p_i^{n_i})$ is odd when n_i is even and *vice versa*, we deduce, as the largest power of 2 to divide $2N$ is 2^1, that there must exist a unique $j \in \{1, 2, \ldots, n\}$ such that n_j is odd. By setting $p_j^{n_j} = p^s$ and $q = \prod_{i=1,\, i \neq j}^{n} p_i^{n_i/2}$, we find that $N = p^s q^2$ and, as N is perfect, then $2N = \sigma(p^s)\sigma(q^2)$ since clearly $\gcd(p, q) = 1$. Now, if $s \equiv 3 \pmod 4$, say $s = 4m - 1$ for some positive integer m, then

$$\sigma(p^{4m-1}) = 1 + p + \cdots + p^{4m-1} = (1 + p^{2m})(1 + p + \cdots + p^{2m-1}).$$

As p is odd, we deduce from this factorization that $\sigma(p^s)$ is divisible by 2^2, which produces a contradiction as the largest power of 2 to divide the left-hand side of the assumed equality $2N = \sigma(p^s)\sigma(q^2)$ is 2^1. Hence, $s \equiv 1 \pmod 4$. Finally, to conclude that $p \equiv 1 \pmod 4$

assume the contrary and set $p = 4k + 3$ and $s = 4r + 1$, for some nonnegative integers k, r. Replacing $\sigma(p^s)$ by $(p^{s+1} - 1)/(p - 1)$ in $2N = \sigma(p^s)\sigma(q^2)$, we obtain that the largest power of 2 to divide the left-hand side of

$$2(p-1)p^s q^2 = (p^{s+1} - 1)\sigma(q^2) \tag{1}$$

is 2^2, as $2(p-1)p^s q^2 = 2^2(2k+1)p^s q^2$. By recalling the identity

$$a^{2r+1} + 1 = (a+1)(a^{2r} - a^{2r-1} + a^{2r-2} - \cdots - a + 1),$$

we see that $(a+1)|(a^{2r+1} + 1)$, and so $4|((4k+3)^{2r+1} + 1)$, while clearly

$$2|((4k+3)^{2r+1} - 1).$$

Thus, as

$$p^{s+1} - 1 = ((4k+3)^{2r+1} - 1)((4k+3)^{2r+1} + 1),$$

the right-hand side of (1) is divisible by 2^3, a clear contradiction. Hence, $N = p^s q^2$ with $p, s \equiv 1 \pmod 4$ and $\gcd(p, q) = 1$, as required.

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Prime numbers in Pythagorean triples

If p is a prime number greater than 2, then it belongs to the Pythagorean triple

$$\left(p, \frac{p^2 - 1}{2}, \frac{p^2 + 1}{2}\right).$$

For example,

$$(3, 4, 5), \quad (5, 12, 13), \quad (7, 24, 25), \quad (11, 60, 61), \quad (13, 84, 85)$$

are Pythagorean triples. Also, $(3, 5)$, $(5, 7)$, $(11, 13)$ are twin primes and

$$12 - 4 = 3 + 5, \qquad 24 - 12 = 5 + 7, \qquad 84 - 60 = 11 + 13.$$

Such will be true for all pairs of twin primes.

Tienen, Belgium **Guido Lasters**

What is the Next Number in this Sequence?

PEE CHOON TOH and ENG GUAN TAY

Given a sequence of numbers, or a sequence, in short, that is generated by an unknown polynomial or by a first-order recurrence relation, we describe an easy way to write down the polynomial or the recurrence relation by computing successive differences of the terms of the sequence.

What is the next number in each of the following sequences?

$$\begin{aligned} &\text{S1}: \ 1, 3, 8, 19, 42, 89, \ldots \\ &\text{S2}: \ 23, 48, 84, 133, \ldots \\ &\text{S3}: \ 7, 27, 58, 102, \ldots \end{aligned}$$

One may immediately protest that it is impossible to uniquely determine a sequence based only on its first few terms. R. K. Guy called this the ‘strong law of small numbers’ (see reference 1). Furthermore, it is well known that, given any n numbers, we can construct infinitely many polynomials $f(x)$ of degree n that evaluate to these n numbers when x takes on the value of the integers from 0 to $n-1$. As a result, C. E. Linderholm (see reference 2, p. 97) even suggested (with tongue firmly in cheek) that one should answer ‘19’ to every ‘find the next term’ question.

Yet such questions continue to appear in IQ tests (the sequences S1, S2, and S3 are, respectively, puzzle 43 (from Test 1), puzzle 15 (from Test 6), and puzzle 17 (from Test 8), taken from reference 3, a book of IQ puzzles written by two UK Mensa puzzle editors). More importantly, they also appear in the school syllabus for the purpose of teaching number patterns and algebra. We must acknowledge, however, that such number pattern questions have been useful as training in recognising patterns and for the appreciation of mathematics. What is needed in such texts is an insertion somewhere that such questions as they stand are ‘nonsense’ if there is no assumption about the sequence. We should also add that a sequence is only defined once *all* the terms are given, either explicitly (which in this case is no fun because then there would be no problem) or in some other way (for example, stating that the numbers in the sequence are the consecutive digits of the decimal representation of π).

One way to avoid ambiguity in such number pattern problems is to restrict ourselves to only sequences generated by polynomials of degree k, and when posing the question, to give at least $k+1$ terms. With that assumption, the standard technique to make sense of a complicated number pattern is the method of calculating successive differences. Take the sequence S2.

$$\begin{array}{lc} \text{S2} & : \ 23 \quad 48 \quad 84 \quad 133 \\ \text{1st successive difference} & : \quad 25 \quad 36 \quad 49 \\ \text{2nd successive difference} & : \quad 11 \quad 13 \\ \text{3rd successive difference} & : \quad 2 \end{array}$$

Here, the numbers in each new row are the difference of the two corresponding numbers of the previous row. Formally, if $a(n)$ denotes a sequence, then $a_i(n)$, the i th successive difference of $a(n)$, is defined recursively by setting $a_0(n) = a(n)$ and

$$a_i(n) = a_{i-1}(n+1) - a_{i-1}(n) \quad \text{for integers } n \geq 0, i \geq 1,$$

Most people would be able to recognize that the first successive differences of S2 are square numbers starting from 5^2, and so the fifth term in S2 should be $133 + 8^2 = 197$, which is in agreement with the solution provided in reference 3.

S3 might provide more of a challenge since the patterns that emerge are slightly harder to decipher.

$$\begin{array}{lccccccccc}
\text{S3} & : & 7 & & 27 & & 58 & & 102 \\
a_1(n) & : & & 20 & & 31 & & 44 & \\
a_2(n) & : & & & 11 & & 13 & & \\
a_3(n) & : & & & & 2 & & &
\end{array}$$

Reference 3 gives the next term as 161 with the explanation 'the differences between the numbers are consecutive squares less 5'. So, intrinsically, S2 and S3 are similar and one telling sign is that they share the same second and third successive differences. In fact, both S2 and S3 are generated by polynomials of degree 3.

It is fairly well known that the kth successive difference of a sequence generated by a polynomial of degree k would be a constant sequence. So if we are faced with a number pattern generated by a polynomial, we can simply compute successive differences until we arrive at a constant sequence which informs us of the value of k. After that, we can recover the $k+1$ coefficients of the unknown polynomial with some (possibly messy) algebra. What is perhaps not so well known is that there is a very easy way to write down the polynomial in question, if we allow ourselves some fancier notation.

Theorem 1 *If $f(x)$ is a polynomial of degree k and the sequence $a(n)$ is defined to be the value of $f(n)$, then the kth successive difference $a_k(n)$ is a constant for all $n \geq 0$. Furthermore,*

$$f(n) = \sum_{i=0}^{k} a_i(0)\binom{n}{i},$$

where $a_i(0)$ is the first term of the ith successive difference of $a(n)$.

We remark that $\binom{n}{i}$ is the binomial coefficient defined by

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!},$$

where we assume that $\binom{n}{i} = 0$ if $i > n$. Furthermore, $\binom{n}{i}$ can be viewed as a polynomial of degree i in n.

This theorem can be found in reference 4 (pp. 36–38), reference 5 (p. 279), and also in standard numerical analysis textbooks disguised as a special case of Newton's divided difference formula. (See, for example, reference 6, p. 123.) However, the proof requires nothing more than the principle of mathematical induction and the following combinatorial identity (see reference 7, p. 79):

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}.$$

We illustrate theorem 1 by writing down the generating polynomials for S2 and S3:

$$23\binom{n}{0}+25\binom{n}{1}+11\binom{n}{2}+2\binom{n}{3}=\frac{2n^3+27n^2+121n+138}{6},$$
$$7\binom{n}{0}+20\binom{n}{1}+11\binom{n}{2}+2\binom{n}{3}=\frac{2n^3+27n^2+91n+42}{6}.$$

Notationally, there is nothing superior about our usual way of writing polynomials that appear on the right-hand side of the two equations above. In fact, from these expressions, it is not easy to see that these polynomials are always integer-valued when n is an integer.

We now prove theorem 1 by induction on k, the degree of $f(x)$.

Proof of theorem 1 If $k=1$, i.e. $f(n)=an+b$, then it is an arithmetic progression and the first successive difference would be the constant sequence consisting of all a, and

$$a_0(0)\binom{n}{0}+a_1(0)\binom{n}{1}=b+an=f(n).$$

Assume that theorem 1 holds for some $k\geq 1$ and define

$$h(x)=f(x+1)-f(x).$$

Then

$$\begin{aligned}\sum_{j=0}^{n-1}h(j)&=\sum_{j=0}^{n-1}f(j+1)-f(j)\\&=f(n)-f(0).\end{aligned}$$

On the other hand, by its definition, $h(x)$ is a polynomial of degree $k-1$. From the induction hypothesis, we have

$$h(n)=\sum_{\ell=0}^{k-1}b_\ell(0)\binom{n}{\ell},$$

where $b_\ell(0)$ is the first term of the ℓth successive difference of $h(n)$. This means that the $(k-1)$th successive differences of $h(n)$ are constant, so the kth successive differences of $f(n)$ are also constant. We also have $b_\ell(0)=a_{\ell+1}(0)$, the first term of the $(\ell+1)$th successive difference of $f(n)$. Thus,

$$\begin{aligned}f(n)&=f(0)+\sum_{j=0}^{n-1}h(j)\\&=f(0)+\sum_{j=0}^{n-1}\sum_{\ell=0}^{k-1}b_\ell(0)\binom{j}{\ell}\\&=f(0)+\sum_{j=0}^{n-1}\sum_{\ell=0}^{k-1}a_{\ell+1}(0)\binom{j}{\ell}\end{aligned}$$

$$= f(0) + \sum_{\ell=0}^{k-1} a_{\ell+1}(0) \sum_{j=\ell}^{n-1} \binom{j}{\ell}$$

$$= f(0) + \sum_{\ell=0}^{k-1} a_{\ell+1}(0) \binom{n-1+1}{\ell+1}$$

$$= a_0(0)\binom{n}{0} + \sum_{i=1}^{k} a_i(0)\binom{n}{i},$$

which proves the inductive step.

What about the sequence S1? It turns out that this sequence is not generated by a polynomial, so theorem 1 does not apply. But let us take a look at the successive differences anyway.

S1	:	1		3		8		19		42		89
$a_1(n)$	:		2		5		11		23		47	
$a_2(n)$	:			3		6		12		24		
$a_3(n)$	:				3		6		12			
$a_4(n)$	:					3		6				
$a_5(n)$	:						3					

Note that $a_2(n)$ and $a_3(n)$, the second and third successive differences, *appear* to be identical. Furthermore, the successive ratios of $a_2(n)$ appear to be a constant, i.e. 6 is twice 3, 12 is twice 6, and 24 is twice 12. There is a 'pattern' for further exploration.

Indeed, we now consider sequences $a(n)$ that are generated by a linear first-order recurrence relation of the form

$$a(n) = ma(n-1) + f(n), \tag{1}$$

where m is a constant and $f(n)$ is a polynomial in n of degree at most k. If an initial value for $a(0)$ is given, such a relation can be shown (see reference 7, Chapter 6) to possess the unique solution $a(n) = pm^n + r(n)$ for some constant p and some polynomial $r(n)$. This gives rise to the following generalization.

Theorem 2 *Let $g(x) = pm^x$ and $f(x)$ be a polynomial of degree k. If we define the sequence $a(n)$ as the value of $g(n) + f(n)$, then the i th successive difference of $a_i(n)$ is given by $p(m-1)^i m^n$ for all $i \geq k+1$ and $n \geq 0$. Furthermore,*

$$a(n) = \sum_{i=0}^{n} a_i(0)\binom{n}{i}.$$

Proof We first note that if $m = 0$ then

$$a(n) = 0 + f(n) = \sum_{i=0}^{k} a_i(0)\binom{n}{i} = \sum_{i=0}^{n} a_i(0)\binom{n}{i},$$

since $a_i(0) = 0$ for all $i \geq k+1$.

Now let $g_i(n)$ and $f_i(n)$ be the i th successive differences of the sequences $g(n)$ and $f(n)$, respectively. It is easy to show by induction that $g_i(n) = p(m-1)^i m^n$.

Since the operation of computing successive differences is linear, we have

$$a_i(n) = g_i(n) + f_i(n) = p(m-1)^i m^n,$$

for all $i \geq k+1$. Now,

$$\begin{aligned}
\sum_{i=0}^{n} a_i(0)\binom{n}{i} &= \sum_{i=0}^{n} g_i(0)\binom{n}{i} + \sum_{i=0}^{k} f_i(0)\binom{n}{i} \\
&= \sum_{i=0}^{n} p(m-1)^i m^0 \binom{n}{i} + f(n) \quad \text{(from theorem 1)} \\
&= p\sum_{i=0}^{n} (m-1)^i \binom{n}{i} + f(n) \\
&= p((m-1)+1)^n + f(n) \\
&= g(n) + f(n) \\
&= a(n).
\end{aligned}$$

Suppose that we are told that the sequence S1 is generated by a recurrence relation of the form (1) where the degree of the polynomial $f(x)$ is not larger than 1. We can then assume that $a_i(0) = 3$ for all $i \geq 2$, so

$$\begin{aligned}
a(n) &= 1 + 2n + \sum_{i=2}^{n} 3\binom{n}{i} \\
&= -2 - n + 3\sum_{i=0}^{n} \binom{n}{i} \\
&= 3 \cdot 2^n - n - 2.
\end{aligned}$$

To answer our original question, the next term of the sequence is $a(6) = 3 \cdot 2^6 - 6 - 2 = 184$. We can also recover the recurrence relation by writing $a_i(0)$ as $3 \cdot (2-1)^i$ for all $i \geq 2$. Thus, $m = 2$ and

$$\begin{aligned}
f(n) &= a(n) - 2a(n-1) \\
&= 3 \cdot 2^n - n - 2 - 2(3 \cdot 2^{n-1} - (n-1) - 2) \\
&= n.
\end{aligned}$$

In other words,

$$a(n) = 2a(n-1) + n,$$

agreeing with reference 3 which explained that 'each of the numbers is doubled and 1, 2, 3, 4, 5, 6 is added in turn'.

Theorem 2 now allows us to handle all sequences generated by recurrence relations of the form (1), as long as we are given at least $k+3$ terms of the sequence. The two extra terms are necessary for us to compute the constant m. For example, suppose that S4 is as follows.

$$\begin{array}{lclllll} \text{S4} & : & 1 & 7 & 19 & 45 & 109 \\ a_1(n) & : & & 6 & 12 & 26 & 64 \\ a_2(n) & : & & & 6 & 14 & 38 \\ a_3(n) & : & & & & 8 & 24 \\ a_4(n) & : & & & & & 16 \end{array}$$

Working on the assumption that if we are given five terms, k must not exceed 2, we have $a_4(0) = p(m-1)^4$ and $a_3(0) = p(m-1)^3$. Thus, $m - 1 = \frac{16}{8} = 2$ which implies $p = 1$ and $a_i(0) = 2^i$ for all $i \geq 3$. Therefore,

$$\begin{aligned} a(n) &= 1 + 6n + 6\binom{n}{2} + \sum_{i=3}^{n} 2^i \binom{n}{i} \\ &= 4n + 2\binom{n}{2} + \sum_{i=0}^{n} 2^i \binom{n}{i} \\ &= 3^n + n^2 + 3n. \end{aligned}$$

A straightforward computation shows that the recurrence relation is

$$a(n) = 3a(n-1) - 2n^2 + 6.$$

Let us end with the following well known combinatorial problem.

> Given a circle with $n+1$ arbitrary points on its circumference, if every possible pair of points is joined by a straight line, what is the maximal number of regions that the circle can be partitioned into?

Working out the first five terms of this sequence and observing that they are, respectively, 1, 2, 4, 8, and 16, we are tempted to jump to the conclusion that this sequence is generated by 2^n. We certainly should not do this. To be sure of the generating function, we need the additional knowledge that this sequence is generated by a polynomial of degree 4. With this fact in hand, plus the first five terms, we obtain the following.

$$\begin{array}{lclllll} a(n) & : & 1 & 2 & 4 & 8 & 16 \\ a_1(n) & : & & 1 & 2 & 4 & 8 \\ a_2(n) & : & & & 1 & 2 & 4 \\ a_3(n) & : & & & & 1 & 2 \\ a_4(n) & : & & & & & 1 \end{array}$$

Theorem 1 then tells us that the generating function is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4}.$$

A geometric explanation for this sequence can be found in reference 1, example 5.

For further reading, see reference 8 where I. Stewart weaves an interesting tale about sequences that look like the Fibonacci numbers. Also, in reference 9 E. M. Whitton describes how one may extend a known sequence by adding a next term of any arbitrary value.

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A magic square based on Srinivasa Ramanujan's date of birth (22 December 1887)

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

Delhi University, India **Vinod Tyagi**

Solve the equation

$$3^3\sqrt{x-1} = (\sqrt{x})!$$

Palestine Polytechnic University **Muneer Jebreel Karama**

A Golden Maximum

PRITHWIJIT DE

In many problems the *golden ratio* or quantities related to it appear in the solution. This short article extends the list of such problems.

Throughout this article we use the following notation: $[PQR]$ denotes the area of triangle PQR.

Many mathematical enthusiasts are fascinated by the appearance of the *golden ratio* in the solution to a problem. This article is dedicated to them. Before we describe the problem that leads to the *golden answer*, some context has to be provided.

Consider a triangle ABC. Does there exist a line L which cuts AB and AC at points X and Y, respectively, such that the triangle is divided into two regions of equal area? The answer is yes. Choose X and Y such that $AX = AB/\sqrt{2}$ and $AY = AC/\sqrt{2}$. Then the line segment XY divides the triangle into two regions of equal area. The Euclidean construction goes like this.

1. Draw a semicircular arc $S_1(S_2)$ on $AB(AC)$.
2. Draw the perpendicular bisector of $AB(AC)$.
3. Let it intersect $S_1(S_2)$ at $U(V)$.
4. With A as centre and $AU(AV)$ as radius draw an arc intersecting $AB(AC)$ at $X(Y)$.
5. Join X and Y.

Now we may ask the following question: what is the length of the shortest segment with endpoints on AB and AC which bisects the area of the triangle?

Let a line L intersect the sides AB and AC at X and Y, respectively, and divide the triangle into two regions of equal area. Suppose that the length of the segment intercepted between the sides AB and AC is l. Then we obtain

$$l^2 = x^2 + y^2 - 2xy\cos A,$$
$$bc = 2xy,$$

where $x = AX$, $y = AY$, $b = AC$, and $c = AB$. But $x^2 + y^2 \geq 2xy = bc$. Thus, $l \geq \sqrt{2bc}\sin(A/2)$ with equality if and only if $x = y = \sqrt{bc/2}$. Therefore, the shortest length of the segment is $l_{\min} = \sqrt{2bc}\sin(A/2)$.

Now we are ready to describe the *golden problem*. Consider a fixed line segment BC. Let A be a moving point in the plane containing BC such that ABC is a nondegenerate triangle and $AB \cdot AC$ is a positive constant. For every admissible position of A obtain the line segment of shortest length which intersects AB and AC at X and Y, respectively, and bisects the area of ABC. Now evaluate the inradius of AXY. Does it ever attain a maximum or a minimum as A moves along its locus?

The inradius of any triangle is the radius of its incircle. The centre of this circle is called the *incentre* and is the point of concurrency of the internal bisectors of the angles of the triangle.

The incentre is equidistant from the sides of the triangle. Let I be the incentre of AXY and $r(A)$ be its inradius. We write $r(A)$ instead of r to emphasize the fact that the inradius changes with the angle at the vertex A. Then we have

$$[AXY] = [AIX] + [IXY] + [AIY] = \tfrac{1}{2}[r(A)x + r(A)y + r(A)l_{\min}].$$

But $[AXY] = \frac{1}{2}[ABC] = \frac{1}{4}bc\sin A$. Thus,

$$r(A) = \frac{\frac{1}{2}bc\sin A}{x + y + l_{\min}} = \frac{\frac{1}{2}bc\sin A}{\sqrt{bc/2} + \sqrt{bc/2} + \sqrt{2bc}\sin(A/2)}.$$

Upon simplification we obtain

$$r(A) = \sqrt{\frac{bc}{8}}\,\frac{\sin A}{1 + \sin(A/2)},$$

and its derivative

$$r'(A) = \sqrt{\frac{bc}{8}}\left(\frac{1 - \sin(A/2) - \sin^2(A/2)}{1 + \sin(A/2)}\right).$$

As $\sin(A/2) \in (0, 1)$ the only admissible value of $\sin(A/2)$ for which $r'(A)$ vanishes is $(\sqrt{5} - 1)/2$, and this is $1/\phi$, where ϕ is the golden ratio. Note that the derivative of $r'(A)$ is

$$r''(A) = -\frac{1}{2}\sqrt{\frac{bc}{8}}\cos\left(\frac{A}{2}\right)\left(\frac{1}{(1 + \sin(A/2))^2} + 1\right),$$

which is negative. Thus, $r(A)$ attains a maximum at $A = 2\sin^{-1}(1/\phi)$, an angle that is related to the golden ratio.

Prithwijit De *is a faculty member at Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, Mumbai, India. He holds a PhD in Statistics from University College Cork, Ireland. He enjoys mathematical problem solving and recreational mathematics.*

More infinite roots

We have

$$1 = \sqrt{2 - \sqrt{2 - \sqrt{2 - \cdots}}}, \qquad 2 = \sqrt{6 - \sqrt{6 - \sqrt{6 - \cdots}}}.$$

Generally,

$$n = \sqrt{m - \sqrt{m - \sqrt{m - \cdots}}},$$

where $m = n^2 + n$.

Las Palmas de Gran Canaria, Spain **Ángel Plaza**

The Central Binomial Coefficient in $(a+b)^{2^n} \pmod 8$

THOMAS KOSHY and ZHENGUANG GAO

We compute the central binomial coefficient in the binomial expansion of $(a+b)^{2^n} \pmod 8$. Using this result, we then evaluate $C_{2^k} \pmod 8$, where C_n denotes the nth Catalan number.

Catalan numbers

The Belgian mathematician Eugène Charles Catalan (1814–1894) found an interesting family of positive integers, now called *Catalan numbers*, in his study of the well-known parenthesization problem in 1838: find the number of correctly parenthesized sequences that can be formed with n pairs of left and right parentheses (see references 1 and 2). Catalan numbers are so ubiquitous that they pop up in numerous unexpected places (see references 1, 2, and 3).

Catalan numbers C_n are often defined explicitly (see references 1 and 2) as follows:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0.$$

This formula shows that the central binomial coefficients $\binom{2n}{n}$ play a significant role in the study of Catalan numbers.

Catalan numbers can be defined recursively as well:

$$\begin{aligned} C_0 &= 1, \\ C_n &= \frac{4n-2}{n+1} C_{n-1}, \qquad n \geq 1. \end{aligned} \tag{1}$$

This recurrence was discovered by the greatest Swiss mathematician Leonhard Euler (1707–1783) in his study of triangulations of convex n-gons around in 1761 (see references 1 and 2). Table 1 shows the Catalan numbers C_n, where $0 \leq n \leq 10$.

Next we compute the central binomial coefficient in the expansion of $(a+b)^{2^n} \pmod 8$. To this end, we let $f(a,b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = (a+b)^{2^2}$. Then we obtain

$$\begin{aligned} f^2(a,b) &\equiv a^8 + 4a^6b^2 + 6a^4b^4 + 4a^2b^6 + b^8 \pmod 8 \\ &\equiv f(a^2, b^2) \pmod 8, \\ (a+b)^{2^3} &\equiv (a^2+b^2)^{2^2} \pmod 8. \end{aligned}$$

More generally, we have the following result. We will establish it using induction on n.

Theorem 1 *Let $n \geq 1$. Then we have*

$$f^{2^n}(a,b) \equiv f(a^{2^n}, b^{2^n}) \pmod 8. \tag{2}$$

Table 1 The first eleven Catalan numbers.

n	C_n
0	1
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1 430
9	4 862
10	16 796

Proof Clearly, (2) holds when $n = 1$. Now assume that it works for an arbitrary integer $n \geq 1$. Then we have

$$\begin{aligned} f^{2^{n+1}}(a, b) &= [f^{2^n}(a, b)]^2 \\ &\equiv [f(a^{2^n}, b^{2^n})]^2 \pmod 8 \\ &\equiv f[(a^{2^n})^2, (b^{2^n})^2] \pmod 8 \\ &\equiv f(a^{2^{n+1}}, b^{2^{n+1}}) \pmod 8. \end{aligned}$$

Thus, by induction, the congruence is true for all $n \geq 1$.

As an example, we have

$$\begin{aligned} (a+b)^{2^5} &= f(a, b)^{2^3} \\ &\equiv f(a^{2^3}, b^{2^3}) \pmod 8 \\ &\equiv a^{2^5} + 4a^{3\cdot 2^3}b^{2^3} + 6a^{2^4}b^{2^4} + 4a^{2^3}b^{3\cdot 2^3} + b^{2^5} \pmod 8 \\ &\equiv a^{32} + 4a^{24}b^8 + 6a^{16}b^{16} + 4a^8b^{24} + b^{32} \pmod 8. \end{aligned}$$

The next result follows immediately from theorem 1.

Corollary 1 *The central binomial coefficient in* $(a+b)^{2^n}$ *modulo* 8 *is* 6, *where* $n \geq 2$.

Proof Since

$$\begin{aligned} (a+b)^{2^n} &= f(a, b)^{2^{n-2}} \\ &\equiv f(a^{2^{n-2}}, b^{2^{n-2}}) \\ &\equiv a^{2^n} + 4a^{3\cdot 2^{n-2}}b^{2^{n-2}} + 6a^{2^{n-1}}b^{2^{n-1}} + 4a^{2^{n-2}}b^{3\cdot 2^{n-2}} + b^{2^n} \pmod 8, \end{aligned}$$

the result follows.

It follows from corollary 1 that $\binom{2^n}{2^{n-1}} \equiv 6 \pmod 8$, where $n \geq 2$. For example, $\binom{32}{16} = 601\,080\,390 \equiv 6 \pmod 8$.

The next result also follows from theorem 1; it also follows from the proof of corollary 1.

Corollary 2 *Let $n \geq 2$ be an integer and $0 \leq r \leq 2^n$. Then we have*

$$\binom{2^n}{r} \equiv \begin{cases} 1 \pmod 8 & \textit{if } r = 0, 4 \cdot 2^{n-2}, \\ 4 \pmod 8 & \textit{if } r = 1 \cdot 2^{n-2}, 3 \cdot 2^{n-2}, \\ 6 \pmod 8 & \textit{if } r = 2 \cdot 2^{n-2}, \\ 0 \pmod 8 & \textit{otherwise.} \end{cases}$$

As an example, $\binom{2^6}{7 \cdot 2^3} = \binom{64}{56} = 66\,392\,480\,520 \equiv 0 \pmod 8$, whereas $\binom{2^5}{3 \cdot 2^3} = \binom{32}{24} = 10\,518\,300 \equiv 4 \pmod 8$.

The next result follows by corollary 1 and the explicit formula for C_n.

Corollary 3 $C_{2^k} \equiv 6 \pmod 8$, *where $k \geq 2$.*

Proof Since $C_4 \equiv 6 \pmod 8$, the result is true for $k = 2$. When $k \geq 3$, $2^k + 1 \equiv 1 \pmod 8$; we then have

$$\begin{aligned} C_{2^k} &= \frac{1}{2^k + 1}\binom{2^{k+1}}{2^k} \\ &\equiv 1 \cdot 6 \pmod 8 \\ &\equiv 6 \pmod 8, \end{aligned}$$

as required.

As an example, $C_8 = 1430 \equiv 6 \pmod 8$ and $C_{16} = 35\,357\,670 \equiv 6 \pmod 8$.

Euler's recurrence (1), coupled with corollary 3, can be employed to establish the next result.

Corollary 4 *Let $k \geq 0$. Then* $C_{2^k-1} \equiv 1 \pmod 4$.

Proof Let $k \geq 3$. By Euler's recurrence, we have

$$\begin{gathered} C_{2^k} = \frac{4 \cdot 2^k - 2}{(2^k - 1) + 2} C_{2^k-1} = \frac{4 \cdot 2^k - 2}{2^k + 1} C_{2^k-1}, \\ (2^k + 1) C_{2^k} = (2^{k+2} - 2) C_{2^k-1}, \\ (0 + 1) \cdot 6 \equiv (0 - 2) C_{2^k-1} \pmod 8, \\ 6 \equiv -2 C_{2^k-1} \pmod 8, \\ C_{2^k-1} \equiv 1 \pmod 4. \end{gathered}$$

Clearly, $C_0 = C_1 \equiv 1 \pmod 4$ and $C_3 = 5 \equiv 1 \pmod 4$. So $C_{2^k-1} \equiv 1 \pmod 4$ for every nonnegative integer k.

As an example, $C_{2^5-1} = C_{31} = 14\,544\,636\,039\,226\,909 \equiv 1 \pmod 4$.

Finally, it follows from corollary 4 that C_n is odd if $n = 2^k - 1$, where $k \geq 0$. In fact, it is well known that C_n is odd if and only if $n = 2^k - 1$, where $k \geq 0$ (see references 1 and 4).

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Letters to the Editor

Dear Editor,

On repdigit numbers as a sum of certain figurate numbers

Let $\mathrm{S}_a = a^2$, $\mathrm{T}_b = b(b+1)/2$, and $\mathrm{P}_c = c(3c-1)/2$ be the sequences of square, triangular, and pentagonal numbers, respectively, for all positive integers a, b, c. I have discovered some nice relations connecting these figurate numbers to repdigit numbers, obtained when I asked *which repdigit numbers are a sum of a square and a triangular number and a pentagonal number?* A MAPLE® program gave the following successful results:

$$\begin{gathered}
\mathrm{S}_4 + \mathrm{T}_1 + \mathrm{P}_2 = 16 + 1 + 5 = 22, \\
\mathrm{S}_6 + \mathrm{T}_4 + \mathrm{P}_{14} = 36 + 10 + 287 = 333, \\
\mathrm{S}_{16} + \mathrm{T}_1 + \mathrm{P}_{53} = 256 + 1 + 4\,187 = 4\,444, \\
\mathrm{S}_{93} + \mathrm{T}_1 + \mathrm{P}_{177} = 8\,649 + 1 + 46\,905 = 55\,555, \\
\mathrm{S}_{671} + \mathrm{T}_5 + \mathrm{P}_{380} = 450\,241 + 15 + 216\,410 = 666\,666, \\
\mathrm{S}_{1987} + \mathrm{T}_1 + \mathrm{P}_{1598} = 3\,948\,169 + 1 + 3\,829\,607 = 7\,777\,777.
\end{gathered}$$

Each of these equations represents the particular repdigit number result in the desired form. These representations are by no means unique. For example,

$$\begin{aligned} \mathrm{S}_{11} + \mathrm{T}_8 + \mathrm{P}_{11} &= 121 + 36 + 176 = 333, \\ \mathrm{S}_{14} + \mathrm{T}_9 + \mathrm{P}_8 &= 196 + 45 + 92 = 333, \\ \mathrm{S}_1 + \mathrm{T}_9 + \mathrm{P}_{14} &= 1 + 45 + 287 = 333, \\ \mathrm{S}_{16} + \mathrm{T}_{10} + \mathrm{P}_4 &= 256 + 55 + 22 = 333. \end{aligned}$$

There seems to be no discernible patterns among the subscripts. More disappointingly, the number 88 888 888 cannot be so represented, which I checked by brute force using a MAPLE program. That is,

$$88\,888\,888 = a^2 + \frac{b(b+1)}{2} + \frac{c(3c-1)}{2}$$

is impossible in positive integers a, b, c.

Here are some questions to explore.

- Can 999 999 999 be so represented?
- For digit 2 I found only 22 representable in this way, for digit 3 only 333, for 4 only 4 444, for 5 only 55 555, for 6 only 666 666, and for 7 only 7 777 777. Is it true that no other string of digits $222\ldots22$ other than 22 itself can be so represented, and no other string of digits $333\ldots33$ except 333, and so on?
- Can a reader supply an elegant proof, other than brute force, perhaps involving congruences, for the number 88 888 888 not being representable in this way?

Yours sincerely,

Tom Moore
(Bridgewater State University
Bridgewater, MA 02325
USA)

Dear Editor,

A cute inequality with cubes

I would like to remind the readers of the following inequality: for n distinct positive integers $a_1, a_2, \ldots, a_n$,

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$

Proof We can prove this by induction. We assume without loss of generality that $a_1 < a_2 < \cdots < a_n$. Since $a_1 \geq 1$, the inequality is valid for $n = 1$. We suppose that the inequality is valid for $n = k$, i.e.

$$a_1^3 + a_2^3 + \cdots + a_k^3 \geq (a_1 + a_2 + \cdots + a_k)^2. \tag{1}$$

We shall prove the inequality for $n = k + 1$.

Since $a_{k+1} \geq a_k + 1$,

$$\frac{(a_{k+1} - 1)a_{k+1}}{2} \geq \frac{a_k(a_k + 1)}{2} = 1 + 2 + \cdots + a_k \geq a_1 + a_2 + \cdots + a_k.$$

Multiplying the last inequality by $2a_{k+1}$ gives

$$(a_{k+1}^2 - a_{k+1})a_{k+1} \geq 2(a_1 + a_2 + \cdots + a_k)a_{k+1},$$

or

$$a_{k+1}^3 \geq 2(a_1 + a_2 + \cdots + a_k)a_{k+1} + a_{k+1}^2. \tag{2}$$

Adding (1) and (2) we have

$$a_1^3 + a_2^3 + \cdots + a_{k+1}^3 \geq (a_1 + a_2 + \cdots + a_{k+1})^2,$$

which proves the inductive step.

Yours sincerely,

Spiros P. Andriopoulos
(Third High School of Amaliada
Eleia
Greece)

Dear Editor,

Goldbach $\Longrightarrow$ Bertrand

The observation that Goldbach $\Longrightarrow$ Bertrand, proved neatly in Guido Lasters' letter in Volume 46, Number 2, p. 91, was proved by me in *The American Mathematical Monthly* (June–July 2005, p. 492). This result was also submitted (independently) by Yoshihiro Tanaka, Hokkaido University, Japan.

Yours sincerely,

Henry J. Ricardo
(33 Knutsen Knoll
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Dear Editor,

The equation $x^2 + y^2 = z^n$

In Volume 43, Number 2, p. 89, we showed how to find integer solutions of the equation $x^2 + y^2 = z^3$. The same method can be used for the equation $x^2 + y^2 = z^n$. Consider, for example, $x^2 + y^2 = z^5$. If $z = a^2 + b^2$, then

$$\begin{aligned} z^5 &= (a^2 + b^2)^5 \\ &= (a + \mathrm{i}b)^5(a - \mathrm{i}b)^5 \\ &= (a^5 + 5\mathrm{i}a^4b - 10a^3b^2 - 10\mathrm{i}a^2b^3 + 5ab^4 + \mathrm{i}b^5) \times \text{(the conjugate of this expression)} \\ &= (a^5 - 10a^3b^2 + 5ab^4)^2 + (b^5 - 10a^2b^3 + 5a^4b)^2. \end{aligned}$$

For example, $a = 3$, $b = 1$ gives $12^2 + 316^2 = 10^5$.

We can use this solution to find rational solutions of the equation $x^2 + y^2 = 2$ in which x, y have five decimal digits. Multiplying through by 10^{10}, we want positive integer solutions X, Y of the equation $X^2 + Y^2 = 2(10^5)^2$. Following my letter in Volume 44, Number 2, p. 89, we use integers s, t such that $s^2 + t^2 = 10^5$, i.e. $s = 316$ and $t = 12$. Then, from my letter in Volume 43, Number 2, pp. 90–91,

$$X = s^2 - t^2 + 2st = 107\,296 \quad \text{and} \quad Y = |s^2 - t^2 - 2st| = 92\,128.$$

This gives $(1.072\,96)^2 + (0.921\,28)^2 = 2$.

Yours sincerely,

Abbas Rouhol Amini
(Sirjan
Iran)

Dear Editor,

A continued fraction linked to Pascal's triangle

I have considered the continued fraction

$$[1; 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, \ldots]$$

whose successive terms are the coefficients of Pascal's triangle. The ninth convergent is $\frac{475}{291} = 1.632\,302\,405\,498\,281\,8\ldots$, with reciprocal $0.612\,315\,789\,473\,684\,162\ldots$. The 39th convergent begins $1.632\,296\,520\,591\,951\,7$, with reciprocal beginning $0.612\,633\,787\,663\,377\,722\,5$. Curiously, the last two numbers differ from $\sqrt{\mathrm{e} - 1/\mathrm{e}^3}$ and $\sqrt{1/\mathrm{e}}$ by $0.001\,256\,28\ldots$ and $0.006\,103\,27\ldots$, respectively.

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

46.9 Prove that every number with $6k$ digits, where k is a natural number, which has a repeating block of two or three digits (as, for example, 292 929 and 157 157), is divisible by 13. Prove also that every number with $12k$ digits which has a repeating block of four digits (as, for example, 262 726 272 627) is also divisible by 13.

(Submitted by Rajenki Das (aged 19), India)

46.10 Given an acute-angled triangle $\triangle$, it is a fact that there is a unique tetrahedron T whose four faces are congruent to $\triangle$. Label the lengths of the sides of $\triangle$ by a, b, c. Prove that the volume of T is

$$\sqrt{\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}{72}}.$$

(Submitted by Fionntan Roukema, University of Sheffield)

46.11 The three points A, B, C are arbitrary distinct points in the plane. The following are equilateral triangles:

$$\begin{aligned}
&ABD \text{ with } A, B, D \text{ anticlockwise,}\\
&BCE \text{ with } B, C, E \text{ anticlockwise,}\\
&DEF \text{ with } D, E, F \text{ clockwise,}\\
&ABG \text{ with } A, B, G \text{ clockwise,}\\
&BCH \text{ with } B, C, H \text{ clockwise.}
\end{aligned}$$

Prove that GHF is an equilateral triangle with G, H, F anticlockwise.

(Submitted by Guido Lasters, Tienen, Belgium)

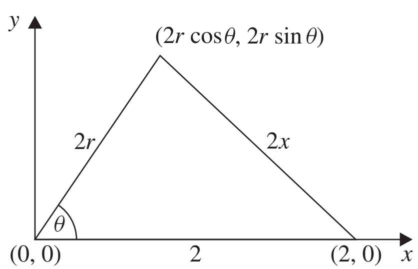
46.12 In a triangle ABC, the medians BD and CE intersect at right angles. Show that $\cos A \geq \frac{4}{5}$. When does equality occur?

(Submitted by K. S. Bhanu and M. N. Deshpande, Nagpur, India)

Solutions to Problems in Volume 46 Number 1

46.1 A uniform nondegenerate triangular lamina ABC has its centre of mass at G. The lamina is now removed and A, B, and C are joined by uniform rods to create a triangle. The centre of mass is now at H. If the point G coincides with the point H, show that ABC is an equilateral triangle.

Solution by Jonny Griffiths, who proposed the problem



Choose axes and scale so that the triangle is as shown. The centre of mass of the lamina is at

$$\frac{1}{3}\left(\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}2r\cos\theta\\2r\sin\theta\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{3}(2+2r\cos\theta)\\\frac{2}{3}r\sin\theta\end{bmatrix}.$$

For the three rods, the centre of mass is at

$$\frac{1}{2+2r+2x}\left(2\begin{bmatrix}1\\0\end{bmatrix}+2r\begin{bmatrix}r\cos\theta\\r\sin\theta\end{bmatrix}+2x\begin{bmatrix}r\cos\theta+1\\r\sin\theta\end{bmatrix}\right)$$
$$=\begin{bmatrix}(1+r^2\cos\theta+x(r\cos\theta+1))/(1+r+x)\\(r^2\sin\theta+xr\sin\theta)/(1+r+x)\end{bmatrix}.$$

Equating the x-coordinates of the two centres of mass, we have

$$\frac{1}{3}(2+2r\cos\theta)=\frac{1+r^2\cos\theta+x(r\cos\theta+1)}{1+r+x},$$

which simplifies to

$$2r\cos\theta+2r-r^2\cos\theta-1=x(r\cos\theta+1). \tag{1}$$

Equating the y-coordinates, we have

$$\frac{2}{3}r\sin\theta=\frac{r^2\sin\theta+xr\sin\theta}{1+r+x},$$

which simplifies to

$$2-r=x,$$

assuming that the triangle is nondegenerate so that $r\neq 0$ and $\sin\theta\neq 0$. If we substitute for x in (1) and simplify, we obtain $r=1$ and so $x=1$. Thus, the triangle is equilateral.

46.2 For a given natural number n, how many irrational numbers of the form $\sqrt{m}$ are there smaller than n, where m is a natural number?

Solution by Bablu Chandra Dey, who proposed the problem

We have $\sqrt{m}<n$ if and only if $m<n^2$ and $\sqrt{m}$ is irrational if and only if m is not a perfect square, so m must be one of $1,2,\ldots,n^2$ excluding $1^2,2^2,\ldots,n^2$ and the answer is n^2-n.

Also solved by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece.

46.3 For positive real numbers a, b, c with $a\le b\le c$, prove that

(i) $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}\ge\dfrac{a}{c}+\dfrac{c}{b}+\dfrac{b}{a}$,

(ii) $b^2(c^2-a^2)(c-a)^2\ge a^2(c^2-b^2)(c-b)^2+c^2(b^2-a^2)(b-a)^2$.

Solution by Subramanyam Durbha, Norristown, PA, USA

For part (i), we have

$$abc=(\text{LHS}-\text{RHS})=(b-a)(c-a)(c-b)\ge 0.$$

For part (ii), we have

$$\text{LHS}=b^2(c^2-b^2+b^2-a^2)(c-a)^2.$$

Now,

$$b^2(c^2-b^2)(c-a)^2\ge a^2(c^2-b^2)(c-b)^2,$$

since $0 < a \le b \le c$, and

$$b^2(b^2 - a^2)(c - a)^2 \ge c^2(b^2 - a^2)(b - a)^2,$$

since $b(c - a) \ge c(b - a)$. If we add these inequalities, we obtain the result.

Also solved by Henry Ricardo, New York Math Circle.

An alternative solution to part (ii), given by Spiros Andriopoulos, who proposed the problem, is to square both sides of (i) to give, after some rearrangement,

$$\left(\frac{a}{b} - 1\right)^2 + \left(\frac{b}{c} - 1\right)^2 + \left(\frac{c}{a} - 1\right)^2 \ge \left(\frac{a}{c} - 1\right)^2 + \left(\frac{c}{b} - 1\right)^2 + \left(\frac{b}{a} - 1\right)^2;$$

whence,

$$c^2a^2(a-b)^2 + a^2b^2(b-c)^2 + b^2c^2(c-a)^2 \ge a^2b^2(a-c)^2 + c^2a^2(c-b)^2 + b^2c^2(b-a)^2,$$

which rearranges to give the result.

46.4 If $P(x)$, $Q(x)$, and $R(x)$ are polynomials such that

$$xP(x^3) + Q(x^3) = (x^2 + x + 1)R(x),$$

prove that $x - 1$ is a factor of $P(x)$.

Solution by Dionisios Andriopoulos, Amaliada, Eleia, Greece

Denote by 1, ω, ω^2 the complex roots of unity. Substituting $x = \omega$ and $x = \omega^2$, we obtain

$$\omega P(1) + Q(1) = (\omega^2 + \omega + 1)R(\omega) = 0$$

and

$$\omega^2 P(1) + Q(1) = (\omega + \omega^2 + 1)R(\omega^2) = 0,$$

since $0 = \omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$, so that $\omega^2 + \omega + 1 = 0$. Hence,

$$(\omega - \omega^2)P(1) = 0,$$

so that $P(1) = 0$ and $x - 1$ is a factor of $P(x)$.

An alternative solution is to write the equation as

$$x(p_0 + p_1x^3 + p_2x^6 + \cdots) + (q_0 + q_1x^3 + q_2x^6 + \cdots) = (x^2 + x + 1)(r_0 + r_1x + r_2x^2 + \cdots)$$

and equate coefficients to give

$$\begin{gathered} q_0 = r_0, \qquad p_0 = r_0 + r_1, \qquad 0 = r_0 + r_1 + r_2, \\ q_1 = r_1 + r_2 + r_3, \qquad p_1 = r_2 + r_3 + r_4, \qquad 0 = r_3 + r_4 + r_5, \end{gathered}$$

and so on, so that

$$r_0 + r_1 + r_2 + r_3 + r_4 + r_5 + \cdots = 0$$

and

$$p_0 + p_1 + p_2 + \cdots = r_0 + r_1 + r_2 + r_3 + r_4 + r_5 + \cdots = 0,$$

so that $P(1) = 0$.

Note also that $Q(1) = 0$ and $R(1) = 0$, so that $x - 1$ is also a factor of $Q(x)$ and $R(x)$.

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