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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
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An Unexpected Appearance of π

L. SHORT AND J. P. MELVILLE, *Napier University*

The authors are both lecturers in the mathematics department at Napier University, Edinburgh. Dr Melville has research interests in solar physics and Dr Short in how mathematical ideas can best be taught.

1. A geometrical problem

MacKinnon (see reference 2) discusses a situation in which e unexpectedly appears. A recent problem in the *American Mathematical Monthly* (see reference 3) provides an interesting geometrical situation in which π surprisingly appears. The problem is the following.

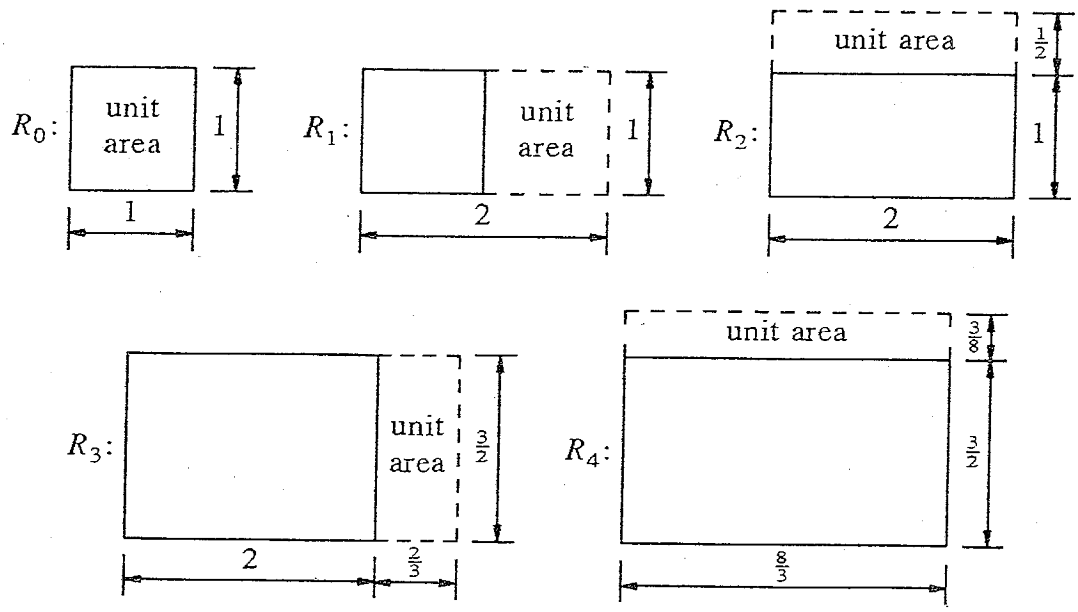


Figure 1

A sequence $\{R_i\}$ of rectangles is generated as in figure 1. Begin with a unit square R_0 and adjoin a rectangle, of unit area, on one *side* (say the right) to produce R_1; to R_1 adjoin a rectangle, of unit area, on *top* to produce R_2. Repeat this process of adjoining a unit rectangle alternately on the side and on top of the previous rectangle. The problem is to show that the sequence of ratios of the length to the height of these rectangles converges and to find the limit.

2. Some numerical evidence

If we denote by l_k and h_k the length and height, respectively, of R_k, and set

$$C_k = \frac{l_k}{h_k}, \tag{1}$$

then we wish to prove that $\lim_{k\to\infty} C_k$ exists and to determine its value. From the conditions of the problem we obtain the following equations:

$$h_k l_k = k \qquad \text{(for all } k \geqslant 1\text{)} \tag{2}$$

$$l_k = l_{k-1}, \qquad h_k = h_{k-1} + \frac{1}{l_{k-1}} \qquad (k \text{ odd}), \tag{3}$$

$$h_k = h_{k-1}, \qquad l_k = l_{k-1} + \frac{1}{h_{k-1}} \qquad (k \text{ even}). \tag{4}$$

We can use (3) and (4) to compile table 1; although the sequence $\{C_k\}$ does appear to approach a limit, convergence is very slow and the precise limiting value is not clear. If a programmable calculator is available, the first few hundred ratios can easily be evaluated, and indicate the value of $\frac{1}{2}\pi$ as limit. An interesting alternative is to *average* the first few terms of

Table 1

k	$C_k = l_k/h_k$	$C'_k = \frac{1}{2}(C_k + C_{k+1})$	$C''_k = \frac{1}{2}(C'_k + C'_{k+1})$
1	1	1.5	1.583 334
2	2	1.666 667	1.611 112
3	1.333 333	1.555 556	1.577 778
4	1.777 778	1.6	1.582 223
5	1.422 222	1.564 445	1.574 604
6	1.706 667	1.584 762	1.576 055
7	1.462 857	1.567 347	1.573 152
8	1.671 837	1.578 957	1.573 797
9	1.486 077	1.568 637	1.572 39
10	1.651 197	1.576 143	1.572 732
11	1.501 088	1.569 320	1.571 944
12	1.637 551	1.574 568	1.572 146
13	1.511 585	1.569 723	1.571 661
14	1.627 861	1.573 599	1.571 791
15	1.519 337	1.569 982	1.571 472
16	1.620 626	1.572 961	1.571 559
17	1.525 295	1.570 157	1.571 338
18	1.615 018	1.572 518	1.571 400
19	1.530 017	1.570 281	1.571 240
20	1.610 544	1.572 198	
21	1.533 852		
501	1.570 011 909	1.570 796 131	
502	1.571 580 353		

the sequence $\{C_k\}$ to obtain a better indication of the limit. This procedure suggests itself since our sequence seems to split into two subsequences, one increasing and one decreasing; averaging should provide a more rapidly converging sequence.

3. Wallis' product—a proof of convergence

To prove that the limit is indeed $\frac{1}{2}\pi$ we have, using (1)–(4), for k odd,

$$C_{k+1} = \frac{l_{k+1}}{h_{k+1}} = \frac{l_k + \dfrac{1}{h_k}}{h_k} = C_k + \frac{1}{h_k^2} = C_k + \frac{1}{h_k \dfrac{k}{l_k}} = C_k + \frac{1}{k} C_k,$$

i.e.

$$C_{k+1} = \frac{k+1}{k} C_k \quad (k \text{ odd}). \tag{5}$$

Further,

$$C_k = \frac{l_k}{h_k} = \frac{l_{k-1}}{h_{k-1} + \dfrac{1}{l_{k-1}}} = \frac{l_{k-1}}{h_{k-1}\left(1 + \dfrac{1}{h_{k-1} l_{k-1}}\right)} = \frac{C_{k-1}}{1 + \dfrac{1}{k-1}},$$

i.e.

$$C_k = \frac{k-1}{k} C_{k-1} \quad (k \text{ odd}). \tag{6}$$

Although (5) and (6) represent non-linear difference equations, we can solve them explicitly by noting the pattern that emerges when we evaluate the first few terms. Using the starting value

$$C_2 = \frac{l_2}{h_2} = \frac{2}{1}, \tag{7}$$

we easily obtain

$$C_3 = \tfrac{2}{3} C_2 = \tfrac{2}{1} \times \tfrac{2}{3}, \qquad C_4 = \tfrac{4}{3} C_3 = \tfrac{2}{1} \times \tfrac{2}{3} \times \tfrac{4}{3},$$

$$C_5 = \tfrac{4}{5} C_4 = \tfrac{2}{1} \times \tfrac{2}{3} \times \tfrac{4}{3} \times \tfrac{4}{5}, \qquad C_6 = \tfrac{6}{5} C_5 = \tfrac{2}{1} \times \tfrac{2}{3} \times \tfrac{4}{3} \times \tfrac{4}{5} \times \tfrac{6}{5},$$

with, in general

$$\begin{aligned} C_{2n} &= \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \cdots \times \frac{2n}{2n-1}, \\ C_{2n+1} &= \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \cdots \times \frac{2n}{2n+1}. \end{aligned} \tag{8}$$

We can recognise (8) as the Wallis product for $\frac{1}{2}\pi$ (e.g. see reference 4):

$$\tfrac{1}{2}\pi = \tfrac{2}{1}\times\tfrac{2}{3}\times\tfrac{4}{3}\times\tfrac{4}{5}\times\tfrac{6}{5}\times\tfrac{6}{7}\times\cdots. \tag{9}$$

Hence

$$\lim_{n\to\infty} C_n = \tfrac{1}{2}\pi. \tag{10}$$

4. Some comments

From a *geometrical* point of view it is far from clear *why* π appears in the solution. In fact, the problem is quite a special one in that, if we alter the areas of adjoined rectangles, the limit is destroyed. For example, if we start with a unit square but add rectangles 'lengthwise' each of area A and 'heightwise' of area B, then (2)–(4) are replaced by

$$h_k l_k = \begin{cases} 1+\tfrac{1}{2}kA+(\tfrac{1}{2}k-1)B & (k \text{ even}), \\ 1+\tfrac{1}{2}(k-1)A+\tfrac{1}{2}(k-1)B & (k \text{ odd}), \end{cases} \tag{11}$$

$$l_k = l_{k-1}, \qquad h_k = h_{k-1}+\frac{B}{l_{k-1}} \qquad (k \text{ odd}), \tag{12}$$

$$h_k = h_{k-1}, \qquad l_k = l_{k-1}+\frac{A}{h_{k-1}} \qquad (k \text{ even}). \tag{13}$$

This leads to the equivalence relations, for k odd,

$$C_{k+1} = \left(1+\frac{A}{1+\tfrac{1}{2}(k-1)(A+B)}\right)C_k, \tag{14}$$

$$C_k = \frac{C_{k-1}}{1+\dfrac{B}{1+\tfrac{1}{2}(k-1)A+\tfrac{1}{2}(k-3)B}}. \tag{15}$$

We may show quite generally from (14) and (15) that

$$\lim_{n\to\infty} C_n = \begin{cases} 0 & (A < B), \\ \infty & (A > B), \\ L_A & (A = B), \end{cases} \tag{16}$$

where, in the case $A = B$, the limit L_A is finite and positive, but its precise value depends on that of A. Indeed, it is only when $A = B = 1$ that we can identify a 'simple' limiting value. We can see why (16) is true by considering a couple of particular cases.

Case 1 ($A = 1$, $B = 2$). Here (14) and (15) become

$$C_{k+1} = \frac{3k+1}{3k-1}C_k; \qquad C_k = \frac{3k-5}{3k-1}C_{k-1} \qquad (k \text{ odd}).$$

Then, again for k odd,

$$C_{k+1} = \frac{3k+1}{3k-1}\frac{3k-5}{3k-1}C_{k-1} = \left(1-\frac{6(k+1)}{(3k-1)^2}\right)C_{k-1},$$
$$C_{k+2} = \frac{3k+1}{3k+5}\frac{3k+1}{3k-1}C_k = \left(1-\frac{6(k-1)}{(3k+5)(3k-1)}\right)C_k.$$

Hence both subsequences $\{C_{2n}\}$ and $\{C_{2n+1}\}$ are decreasing. Since the terms are all positive, it follows that the two sequences converge, and since $C_{2n}/C_{2n+1} \to 1$ as $n \to \infty$ the limits are equal. Thus $\{C_n\}$ converges. More sophisticated analysis shows that the limit is, in fact, 0. A similar analysis for the case $A = 2$ and $B = 1$ will show that both subsequences are increasing, but divergence to infinity is more difficult to prove.

Case 2 ($A = 3$, $B = 3$). Here (14) and (15) yield

$$C_{k+1} = \frac{3k+1}{3k-2}C_k; \qquad C_k = \frac{3k-5}{3k-2}C_{k-1} \qquad (k \text{ odd}). \tag{17}$$

Then, for k odd,

$$C_{k+1} = \frac{3k+1}{3k-2}\frac{3k-5}{3k-2}C_{k-1} = \left(1-\frac{9}{(3k-2)^2}\right)C_{k-1},$$
$$C_{k+2} = \frac{3k+1}{3k+4}\frac{3k+1}{3k-2}C_k = \left(1+\frac{9}{(3k+4)(3k-2)}\right)C_k.$$

The subsequence $\{C_{2n}\}$ is thus decreasing and $\{C_{2n+1}\}$ is increasing. Hence, as in case 1, $\{C_{2n}\}$ converges. But $C_{2n}/C_{2n+1} \to 1$ as $n \to \infty$ and so $\{C_{2n+1}\}$ also converges and has the same limit.

The balance of terms in (9) is thus very delicate. If we make the numerators increase 'too rapidly' compared to the denominators, the product diverges; vice versa, and the product tends to zero. Making the numerators and denominators increase 'at the same rate', i.e. when $A = B$, ensures the limit is preserved. However, the precise limiting value is not so easily determined. For example, when $A = B = 3$ we can use (17) to obtain

$$\begin{aligned} C_{2n} &= \frac{4}{1}\times\frac{4}{7}\times\frac{10}{7}\times\frac{10}{13}\times\frac{16}{13}\times\frac{16}{19}\times\frac{22}{19}\times\cdots\times\frac{6n-2}{6n-5}, \\ C_{2n+1} &= \frac{4}{1}\times\frac{4}{7}\times\frac{10}{7}\times\frac{10}{13}\times\frac{16}{13}\times\frac{16}{19}\times\frac{22}{19}\times\cdots\times\frac{6n-2}{6n+1}. \end{aligned} \tag{18}$$

The products may be shown to converge (to the same limit), but they do so very slowly. This raises the question of accelerating the convergence, as in the averaging technique employed in table 1. Many techniques are available (see reference 1) and the reader may care to show that numerical methods indicate the limit

$$L_3 = 2.816251796$$

for the sequences (18). (Richardson extrapolation and the Shanks transformation are both powerful, but simple, acceleration techniques in the present situation; see reference 5.)

References

1. C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
2. N. MacKinnon, 'Cayley's theorem and a method for approximating e', *Mathematical Gazette* **72** (1988), 293–297.
3. G. Myerson, Problem E3414, *American Mathematical Monthly* **97** (1990), 917.
4. D. B. Scott and S. R. Tims, *Mathematical Analysis—An Introduction* (Cambridge University Press, 1966).
5. L. Short, Some generalisations of the Wallis product. Napier University Preprint.

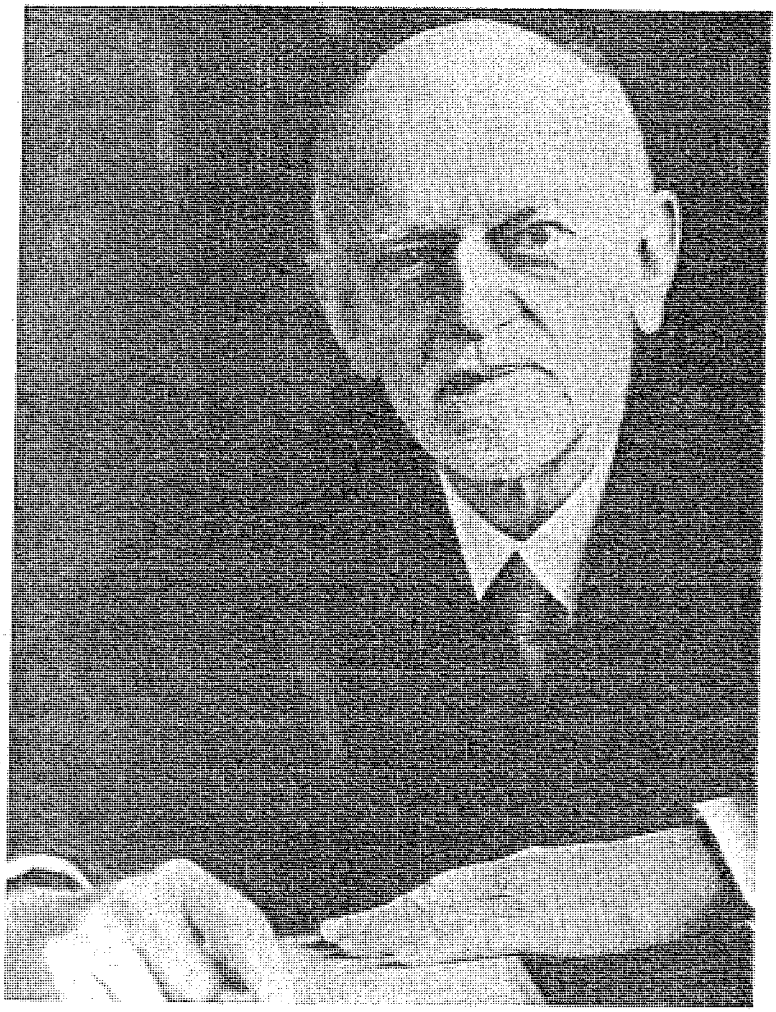
David Hilbert

B. H. NEUMANN, *The Australian National University, Canberra*

Bernhard Hermann Neumann, born in 1909 in Berlin, Germany, is well known for having been the husband of one of the greatest female mathematicians of this century, Hanna Neumann (1914–1971), with whom he produced much joint research and five children, of whom two are very well-known mathematicians in their own right.

14 February 1993 marks the 50th anniversary of the death of one of the most remarkable mathematicians of modern times. David Hilbert (1862–1943) has given his name to a famous series of problems, to the Hilbert Nullstellensatz, to the Hilbert invariant integral, to the Hilbert transform, to the Hilbert inequality, to the Hilbert axiom and to Hilbert space (so named by John von Neumann (1903–1957), no relative of mine, but one of my greatly admired university teachers); and, most recently, to the David Hilbert International Awards of the World Federation of National Mathematics Competitions, awarded first in 1991; and this list is not exhaustive!

It is impossible to give in a few pages an adequate picture of one so versatile, so creative, of such great influence on generations of mathematicians. Thus a thumb-nail sketch will have to do. David Hilbert was born near Königsberg, the capital of East Prussia, into an upper-middle-class family. Königsberg was a famous and important merchant city near the Baltic Sea, with a respected old university. Its seven river bridges became the object of one of Leonhard Euler's (1701–1783) best-known theorems. (Since the end of the Second World War Königsberg has become Kaliningrad, in what was, until recently, the USSR.)



David Hilbert, 1862–1943
(Reproduced by permission of Springer-Verlag, Heidelberg, FRG)

David's father, a judge, soon moved into Königsberg itself, and David grew up there. While he was a boy, Prussia became part, and indeed the leader, of the newly unified Germany (this was 1871, not 1990). David went to school and to university in Königsberg, except for one semester spent at the University of Heidelberg, and finished with the degree of Doctor of Philosophy just after his 23rd birthday. Later in the same year, he took the *Staatsexamen*, the state examination that qualified him for school teaching, as a sort of insurance against being unsuccessful in an academic career. He need not have worried.

After his doctorate he travelled to Leipzig to learn from Felix Klein (1849–1925), then, on Klein's advice, to Paris to learn from Charles Hermite (1822–1901), Camille Jordan (1838–1922), Henri Poincaré (1854–1912), Émile Picard (1856–1941), and to immerse himself in the vibrant mathematical life of Paris. He then returned to Königsberg for his *Habilitation*, the higher examination that gave him the right to lecture, though no assured income. He was by then 24 years old. The (then only) Professor of Mathematics at Königsberg was Adolf Hurwitz (1859–1919); and a close friend was his former fellow student Hermann Minkowski (1864–1909).

After lecturing for two years to minute classes of students, he went travelling again, planning to visit 21 prominent mathematicians. The first of them was Paul Gordan (1837–1912), then the Grand Old Man of invariant theory, to which Hilbert's work had so far been dedicated. Gordan had long before formulated the so-called finite basis problem of invariant theory, which had resisted all attempts, by many fine mathematicians, to solve it, though Gordan himself had early found a solution in a very special case. The problem captured Hilbert's imagination, and on his return to Königsberg he found a solution by a completely novel method. The method was non-constructive, and thus appeared weird to many of his contemporaries; but when, a few years later, at the age of 30, he improved his method to construct the finite bases, it quickly established his fame.

The next few years saw many changes: Hilbert became an *Ausserordentlicher Professor* (which gave him a salary and a title of highest value in Germany), got married, gave simplified proofs of the transcendence of e (first proved by Hermite) and π (first proved by Ferdinand Lindemann (1852–1939)), started work in other aspects of number theory, and on Lindemann's translation from Königsberg, where he had briefly succeeded Hurwitz, to Munich, he became full Professor; and after a while he was able to attract his friend Minkowski back to Königsberg. The newly founded Deutsche Mathematiker-Vereinigung (the German Mathematical Society) had asked him and Minkowski to prepare an up-to-date report on the state of the theory of numbers, and the resulting *Zahlbericht* (1897), which is Hilbert's contribution, became a classic, and greatly enhanced his reputation. However, even before that, in 1895, he had been invited by Felix Klein to fill a chair in Göttingen, an invitation he could not refuse: Göttingen had been, since the time of Carl Friedrich Gauss (1777–1855), the summit of German mathematical life, and remained so until Hitler drove some of its best talent into exile in the 1930s.

Soon after the publication of the *Zahlbericht*, Hilbert turned his attention to the axiomatic foundations of geometry, and his book *Grundlagen der Geometrie* (available in English as *Foundations of Geometry*; I have extensively used the 7th German edition of 1930) again became an immediate great success. Hilbert's interest in the foundations of geometry was carried over well beyond his death by his pupils and their pupils.

Hilbert then turned his attention to other areas of mathematics, and made numerous important contributions; but the outstanding event of that time was the second International Congress of Mathematicians, held in Paris in the summer of 1900. (The first had been at Chicago in 1897, to mark the opening of the University of Chicago; Hilbert had not attended that one.) Hilbert had been invited to give one of the main lectures at the

Paris Congress, and this took the form of a list and discussion of 23 mathematical problems that Hilbert thought of as important and as giving a direction to mathematical research in the new century. These 23 problems were indeed to have a fundamental influence on the development of mathematics, extending to this day.

The first problems dealt with the logical foundations of mathematics. Hilbert was an inveterate optimist, and he was convinced that every mathematical problem must have a solution, and that the solution would eventually be found. It therefore came as a great shock to him, as indeed to the mathematical world at large, that Kurt Gödel (1906–1978) showed, in 1931, that if an axiom system for arithmetic is consistent (that is to say, does not lead to contradictions), then there is a proposition formulated in the system that can be neither proved nor disproved within that system.

I have now skipped three decades of Hilbert's life. They were full of new ideas, new directions of research, many new pupils, many new honours. The First World War came and passed. Hilbert's interests spanned the whole of mathematics from the foundations to mathematical physics and mathematical genetics. Indeed the one lecture of Hilbert's that I remember hearing (at a mathematical colloquium at the University of Berlin, about 1930) began dramatically: 'Klein ist die Fliege *Drosophila*, aber gross ist ihre Bedeutung für die Vererbungslehre'. ('Small as the fly *Drosophila* may be, its importance in genetics is large.') The fruit fly *Drosophila melanogaster* is much used by geneticists because of its large chromosomes and its short generation span.

The last years of Hilbert saw Hitler and the Nazis come to power in Germany, and destroy the mathematical glory of Göttingen. They were bitter years for the ageing Hilbert, with his health deteriorating, his friends and pupils gone; he died in the middle of the Second World War. Yet he is immortal; in his works, in his pupils and their pupils, in the direction he has given to the mathematics of this century. *We shall not see his like again.*

Further reading

Constance Reid, *Hilbert* (Springer-Verlag, Berlin, 1970). I have made much use of this very readable account of Hilbert's life and times.

Prizes for Student Contributors

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

Knot Guilty

KEITH AUSTIN, *University of Sheffield*

Keith Austin teaches mathematics at Sheffield. A course on knots by one of his colleagues, John Greenlees, sowed the seeds for this article.

'Fill the bag with money', shouted the masked man as he waved his gun at the bank cashier.

'That is not a very good quality picture', said Susan to her father, Police Inspector Gray, as he sat watching the video he had brought home to study.

'It was good enough to enable us to clear one of our two suspects. From the pictures and some measurements taken at the bank we were able to calculate that the robber was 5 feet 10 inches tall. One of our suspects was 5 feet 9 inches, and so he was innocent.'

'That is because height, for adults, does not change; it is an invariant of the person', replied Susan. 'I have just been working with invariants in my maths homework.'

'However, the other suspect is 5 feet 10 inches tall.'

'But', interrupted Susan, 'that does not prove he is guilty; there are plenty of people who are 5 feet 10 inches tall; the invariant only works to prove innocence.'

'Agreed. But there is also the evidence of the knotted rope. Look, you can see it on the video, hanging round his arm. Well, when we found our second suspect he also had a knotted rope round his arm.'

'Were they the same knot?' asked Susan.

'They don't look the same. Here, see for yourself, I have drawings of both of them (figure 1).

robber's knot suspect's knot

Figure 1

'The question we were trying to answer at the police station is whether one could be moved around to look like the other. We couldn't do it, but that doesn't prove they are different.'

'What you want', explained Susan, 'is an invariant of each of the knots, some measure like height which does not change however you move the knots around. Then if the knots have different "heights", that is, different invariants, then they cannot be the same.

'We have been shown an invariant for knots that was only discovered in 1987.'

'What is the name of the mathematician who found it? I'll bet it's a long strange-sounding name.'

'Jones! Now to work out the invariants we need pencil and paper. Let's start with the robber's knot. First we mark each crossover point with As and Bs using the rule in figure 2; then we get figure 3.' (The crossovers have also been numbered 1, 2, 3 for reference.)

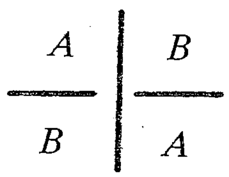


Figure 2

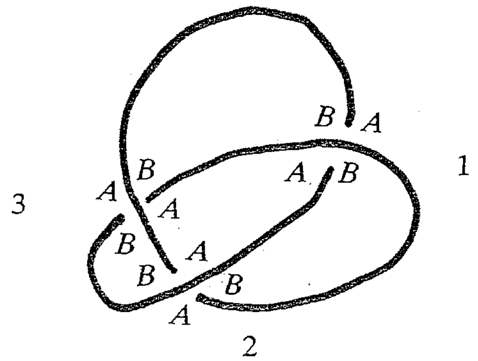


Figure 3

'Now each crossover can be split the A way or the B way (figure 4).

Figure 4

'We consider all possible combinations of As and Bs for our knot, which gives eight possibilities: we show them in figure 5. For each possibility we draw the picture we get if we make the splits; the picture will be a number of loops, and we count these.

'Next we do some algebra with a number A; we don't need to know its value, except that it is not zero. We also use A^{-1}, which is $1/A$, so $AA^{-1} = 1$.

'Now each row of figure 5 gives us an expression as follows.

'*Step 1*. Replace B by A^{-1} and multiply the three terms together, and subtract 1 from the number of loops.

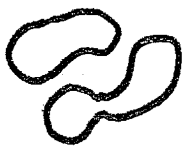
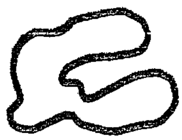
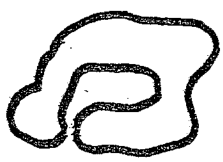
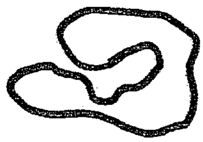
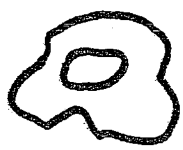
Splitting at each crossover 1 2 3	Picture we get	Number of loops
$A \quad A \quad A$		3
$A \quad A \quad B$		2
$A \quad B \quad A$		2
$A \quad B \quad B$		1
$B \quad A \quad A$		2
$B \quad A \quad B$		1
$B \quad B \quad A$		1
$B \quad B \quad B$		2

Figure 5

$AAA, 3 \rightarrow A^3, 2;$ $\qquad AAB, 2 \rightarrow AAA^{-1}, 1 = A, 1;$

$ABA, 2 \rightarrow AA^{-1}A, 1 = A, 1;$ $\qquad ABB, 1 \rightarrow AA^{-1}A^{-1}, 0 = A^{-1}, 0;$

$BAA, 2 \rightarrow A^{-1}AA, 1 = A, 1;$ $\qquad BAB, 1 \rightarrow A^{-1}AA^{-1}, 0 = A^{-1}, 0;$

$BBA, 1 \rightarrow A^{-1}A^{-1}A, 0 = A^{-1}, 0;$ $\qquad BBB, 2 \rightarrow A^{-1}A^{-1}A^{-1}, 1 = A^{-3}, 1.$

'*Step 2*. Multiply the A term by $-(A^2+A^{-2})$ raised to the power of the number term; remember $[-(A^2+A^{-2})]^0 = 1$.

$$A^3, 2 \rightarrow A^3(A^2+A^{-2})^2 = A^3(A^4+2+A^{-4}) = A^7+2A^3+A^{-1};$$

$$A, 1 \to -A(A^2+A^{-2}) = -A^3-A^{-1}; \qquad A^{-1}, 0 \to A^{-1};$$
$$A^{-3}, 1 \to -A^{-3}(A^2+A^{-2}) = -A^{-1}-A^{-5}.$$

'Then we add all eight expressions together to give

$$A^7+2A^3+A^{-1}+3(-A^3-A^{-1})+3A^{-1}-A^{-1}-A^{-5} = A^7-A^3-A^{-5}.$$

'Finally we trace round the knot with an arrow; we can go in either direction—it does not matter which. Each crossover point is given a value of $+1$ or -1 according to the rule in figure 6, so we get figure 7. We then add up all the numbers to get their total T, and work out $(-A)^{-3T}$. In this case T is -3 so we get $(-A)^9 = -A^9$.

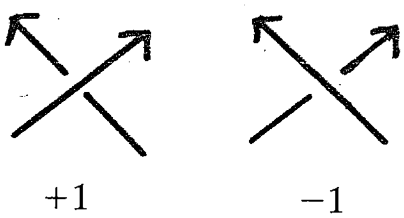


Figure 6

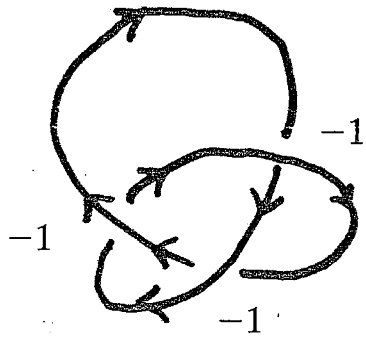


Figure 7

'Our invariant is obtained by multiplying the two expressions we have, to give

$$(-A^9)(A^7-A^3-A^{-5}) = -A^{16}+A^{12}+A^4.$$

This is the invariant for the robber's knot. Now we do the same calculation for the suspect's knot.'

Susan and her father worked away for some time until they announced, 'The suspect's knot has invariant

$$A^8-A^4+1-A^{-4}+A^{-8}.\text{'}$$

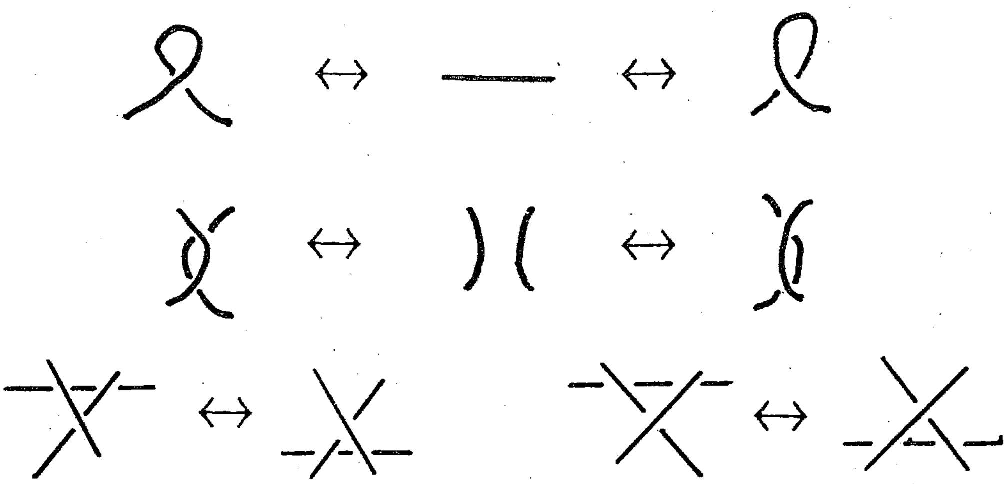


Figure 8

'So the suspect is not the robber', cried Susan with delight.

'You're right. But tell me, how did your teacher prove to you that the invariant does not change as we move the knot about?'

'Well, there are only three types of move you can make; they are shown in figure 8. The proof involves checking that these moves do not change the invariant.

'One word of warning. Remember that if the robber and the suspect have the same height it does not mean the suspect is the robber. So with knots: if two knots have the same invariant it does not mean that one can be moved into the other. For example, the two knots in figure 9 have the same invariant but one cannot be moved into the other.'

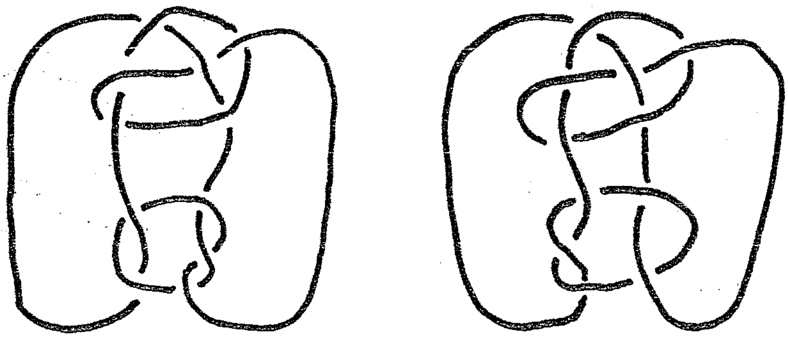


Figure 9

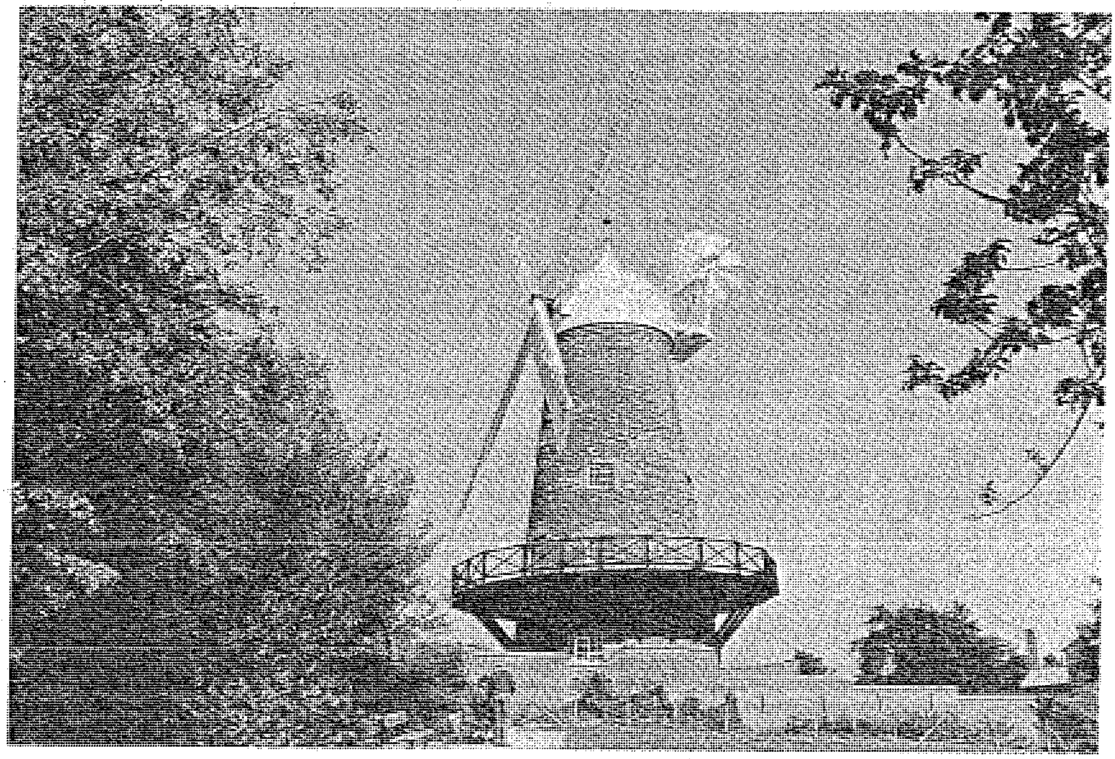
'You seem very interested in our suspect, Susan. Could it be because he is young, male and, due to your detective work, free?'

'Don't be silly, Dad. You sound as if I were going to tie the knot!'

George Green to be Commemorated in Westminster Abbey

The Dean of Westminster has accepted a proposal to place a memorial slab to George Green in the Abbey. It is planned that this should be unveiled in July 1993 at the time of the bicentenary of his birth.

George Green (1793–1841) was a pioneer in the application of mathematics to physical problems. He was a miller who lived in Nottingham nearly all his life and had very little formal education until he had completed most of his best work. Then, at the age of 40, he went to Caius College, Cambridge, to read for a degree in mathematics and became a fellow of his college. Partly as a result of his unusual circumstances he received little public recognition in his lifetime, and it was William Thomson (Lord Kelvin) who first recognised the value of his work and gave it



Green's Mill and Science Centre at Sneinton, Nottingham

wide publicity. His work has had a great influence and nowadays he is remembered principally for Green's theorem in vector analysis, Green's tensor (or the Cauchy–Green tensor) in elasticity theory and above all for Green's functions for solving differential equations. The Green's function technique has been very widely applied to equations arising in classical physics and engineering and in recent years has been adapted to quantum-mechanical problems in areas as diverse as nuclear physics, quantum electrodynamics and superconductivity.

Green is buried in Nottingham, where his windmill has recently been fully restored.

The proposal to commemorate him in the Abbey was made by Professor L. J. Challis, Professor A. J. M. Spencer, Professor K. W. H. Stevens and Dr F. W. Sheard of Nottingham University, with the strong support of the President of the Royal Society, Sir Michael Atiyah, Professor Sir Sam Edwards, Cambridge University, Professor Sir Roger Elliott, Oxford University, and Sir James Lighthill, former Provost of University College, London.

In the Abbey Green's name will join those of other leading nineteenth-century scientists—Faraday, Joule, Kelvin, Maxwell and Stokes.

For further information contact Professor L. J. Challis, Physics Department, University of Nottingham, Nottingham, NG7 2RD.

Editor's note. See the article on George Green in Volume 20, pages 45–52.

Improving Convergence in Iterations

JOHN MOONEY, *Glasgow Polytechnic*

The author obtained his B.Sc. and M.Litt. degrees at the University of Glasgow and his Ph.D. at the University of Strathclyde. He is currently a Senior Lecturer in mathematics at Glasgow. His research interests are in non-linear differential equations and numerical methods. The Scottish hills and conservation issues consume a good part of his spare time.

1. Iteration diagrams

Convergence of iteration sequences may be illustrated by means of iteration diagrams. These diagrams can also suggest methods for speeding up and extending the range of convergence of the iterations.

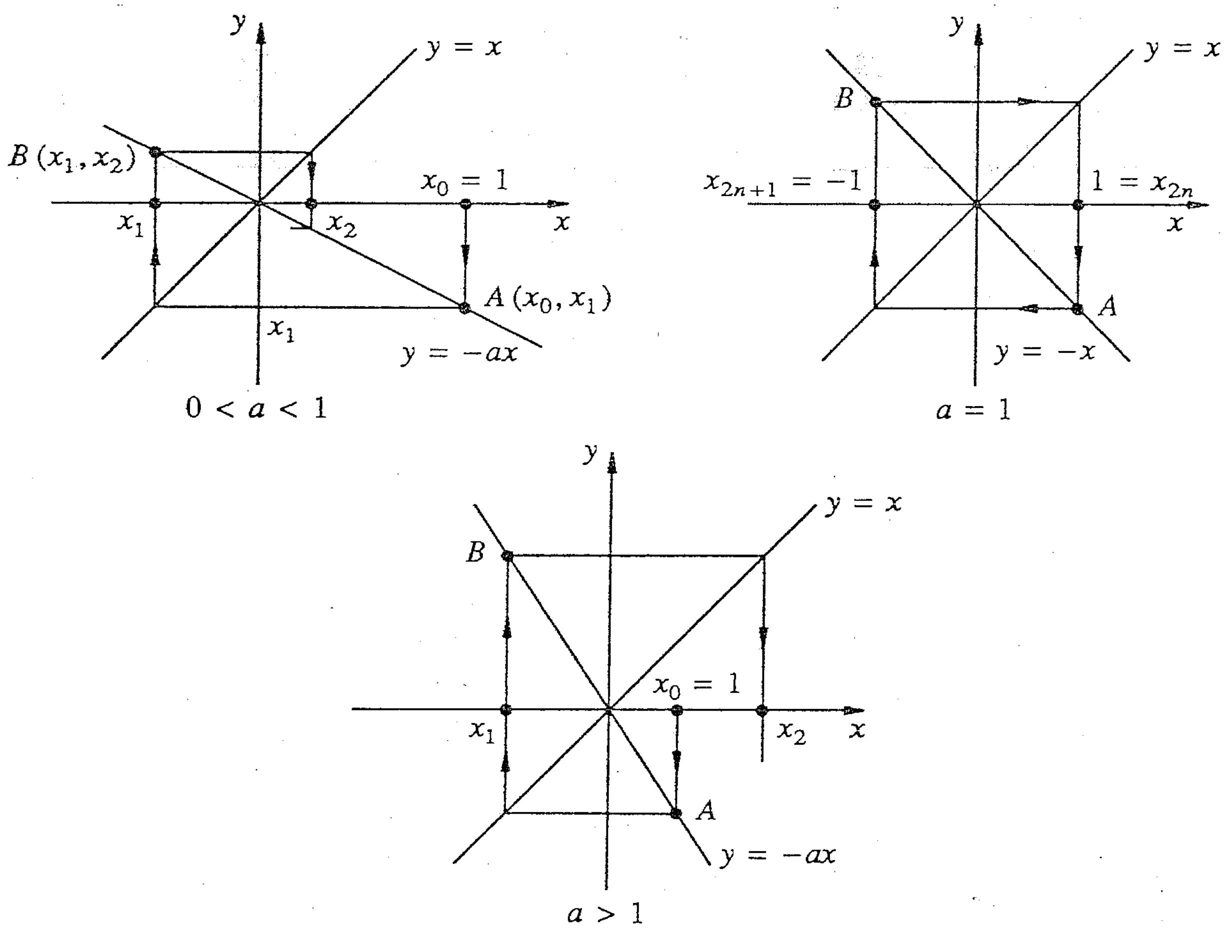


Figure 1

Consider the sequence $1, -a, a^2, -a^3, \ldots$ (with $a > 0$) which is generated by the iteration $x_{n+1} = -ax_n$, with $x_0 = 1$. The sequence converges to zero for $0 < a < 1$, jumps between 1 and -1 for $a = 1$, and goes through ever-increasing jumps for $a > 1$. When the sequence converges, it converges to the solution of $x = -ax$, i.e. to the value of x where the line $y = -ax$ intersects the line $y = x$. The behaviour of the iterates is shown in figure 1 in which (x_n, x_{n+1}) are plotted for successive values of n.

However, if we use the iterations to help approximate the solution of the equation $x = -ax$, then the convergence is fast only when a is close to 0, and there is no convergence at all for $a \geqslant 1$.

The convergence rate is *linear* since the function $y = -ax$ is a *linear function*; the slow convergence can be seen by iterating with $a = 0.9$, say. To improve matters, we use the first two iterates x_1 and x_2 in the sequence $x_{n+1} = -ax_n$, and consider the points A (x_0, x_1) and B (x_1, x_2) in figure 1. We form an improved iterate X_1 by choosing X_1 to be the value of x at which the line AB intersects the line $y = x$. It is clear from figure 1 that this value will be $X_1 = 0$, giving the solution of the linear equation $x = -ax$ immediately for all a satisfying $a > 0$. The reader may care to sketch the corresponding diagrams for $a < 0$ and show that $X_1 = 0$ for negative a also.

If we now consider the 'nearly linear' function $y = g(x) = -ax + \epsilon x^2$ (where $\epsilon > 0$ is small), then the iterations $x_{n+1} = g(x_n)$ will also converge to 0 (with linear convergence) for $0 < a < 1$ and diverge for $a > 1$, since $|g'(0)| = a$. Similar functions have been considered previously in *Mathematical Spectrum* by Dermot Roaf (reference 1). If we choose $\epsilon = 0.1$, $a = 0.9$ and $x_0 = 1$, then we obtain 1, -0.8, 0.784, -0.644, etc. The line AB, with A $(1, -0.8)$ and B $(-0.8, 0.784)$, will meet the line $y = x$ at the point $x = X_1 \approx 0.043$, which is a significant improvement on $x_2 = 0.784$. The procedure can be repeated by considering $x_0 = 0.043$ in $x_{n+1} = g(x_n)$ and constructing another line AB to produce the next improved iterate X_2.

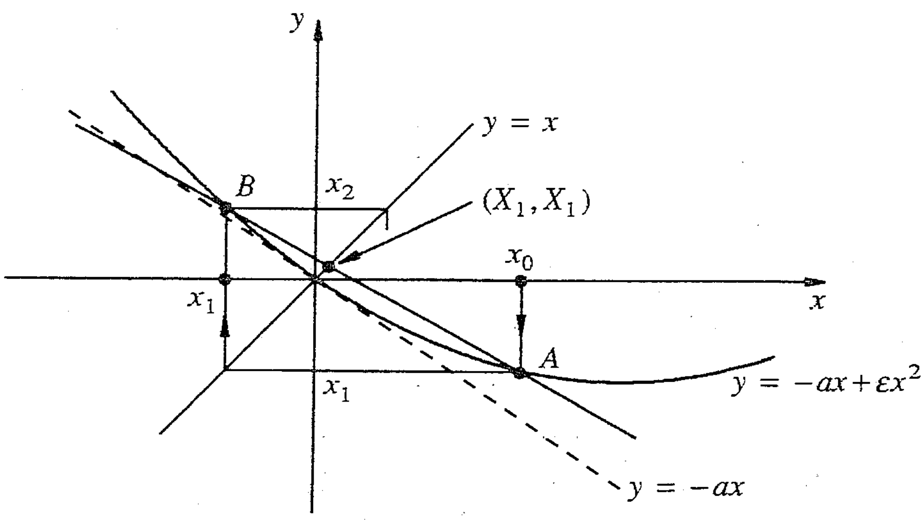


Figure 2

2. Steffenson iterates

We call the resulting sequence of iterates X_1, X_2, etc., a Steffenson (improved) iteration sequence since it is equivalent to a sequence introduced by J. F. Steffenson in 1933 (reference 2). To show this, we first obtain a formula for X_1 in terms of x_0, x_1 and x_2.

The line joining $A\ (x_0, x_1)$ and $B\ (x_1, x_2)$ has equation

$$y - x_1 = \frac{x_2 - x_1}{x_1 - x_0}(x - x_0).$$

Its intersection with the line $y = x$ satisfies the equation

$$(x - x_1)(x_1 - x_0) = (x_2 - x_1)(x - x_0).$$

This gives

$$X_1 = x = \frac{x_0 x_2 - x_1^2}{x_2 - 2x_1 + x_0}, \tag{1}$$

where x_0, x_1 and x_2 are the first three iterates in the iteration scheme $x_{n+1} = g(x_n)$, i.e.

$$X_1 = \frac{x_0 g(g(x_0)) - [g(x_0)]^2}{g(g(x_0)) - 2g(x_0) + x_0}. \tag{2}$$

We can write this as $X_1 = h(x_0)$ and define the Steffenson iteration sequence $X_{n+1} = h(X_n)$ for $n \in \mathbb{Z}$, where

$$X_{n+1} = \frac{X_n g(g(X_n)) - [g(X_n)]^2}{g(g(X_n)) - 2g(X_n) + X_n} = h(X_n). \tag{3}$$

A calculator can be used to work out the first few iterates because the method often converges quickly. For more complicated functions, a simple program on a microcomputer would be more suitable. Consider

(a) $g(x) = -3x$, with $x_0 = 1$;
(b) $g(x) = -0.9x + 0.1x^2$, with $x_0 = 1$;
(c) $g(x) = -5\sin x$, with $x_0 = 0.3$ and $x_0 = 4$;
(d) $g(x) = x^3 + 0.1$, with $x_0 = 0$, $x_0 = 1$ and $x_0 = -1$.

Dermot Roaf considered similar functions in his article and discussed both linear and higher-order convergence rates. Note that the sequence $x_{n+1} = g(x_n)$ is convergent to a fixed point of g only in (b) and in (d) for $x_0 = 0$. The Steffenson iterates obtained using (3) are

(a) $X_1 = 0$;
(b) 0.042 5532, 0.000 0855, 0;
(c) −0.083 7813, 0.001 6479, 0.000 0003 with $x_0 = 0.3$,
4.081 5066, 4.103 0437, 4.104 5757 with $x_0 = 4$;

(d) 0.101 0101, 0.101 0312 with $x_0 = 0$,
0.956 71, 0.946 1802, 0.945 6319 with $x_0 = 1$,
$-1.052\,3256$, $-1.046\,8215$, $-1.046\,6805$ with $x_0 = -1$.

The Steffenson iterates give the sharpest improvement in the examples (a), (b) and (c), where there is linear convergence. To compare the order of convergence, we can use expression (1) and let $x_0 = e_0$. If the original sequence has linear convergence, then we may write $x_{n+1} \approx kx_n + lx_n^2$. In expression (1), this gives

$$X_1 = \frac{e_0(k^2e_0 + kle_0^2 + lk^2e_0^2) - (k^2e_0^2 + 2kle_0^3)}{e_0 + k^2e_0 - 2ke_0} + O(e_0^3),$$

i.e.

$$X_1 = \frac{kle_0^2}{k-1} + O(e_0^3).$$

It follows that the convergence rate of the Steffenson iterates will be at least quadratic (cubic if $l = 0$) when the original sequence is linearly convergent. Also, if the original sequence has quadratic convergence ($k = 0$), the Steffenson scheme will have at least cubic convergence. However, since two iterates in the original sequence are required to produce one Steffenson iterate, it is generally inefficient to apply the Steffenson method to speed up the convergence of quadratically convergent sequences.

Finally, consider $g_1(x) = 1 - \frac{1}{2}\sin x$ with $x_0 = 0.7$ and $g_2(x) = \frac{1}{2}e^{-x}$ with $x_0 = 0$ (g_1 and g_2 are linearly convergent sequences). For g_1, the iterates are 0.7, 0.684 0479, 0.684 0364; the original sequence has 0.684 0905 as the sixth iterate and the limit is 0.684 0367. For g_2, the iterates are 0, 0.358 8166, 0.351 7353; the original sequence has 0.351 7323 as the thirteenth iterate and the limit is 0.351 7337. However, some caution is needed. As one approaches the fixed point, the Steffenson iterates approach 0/0 and become unstable numerically. Consequently, if accuracy of more than five or six decimal places is required, the method will require a computer. Alternatively, the Newton–Raphson method is very appropriate when close to a fixed point.

References

1. D. Roaf, 'Iteration', *Mathematical Spectrum* **20** (1987/88) 65–70.
2. J. F. Steffenson, 'Remarks on iteration', *Skand. Aktuar. Tidskr.* **16** (1933), 64–72.

Flyaway

DYLAN GOW, *Trinity College, Cambridge*

The author is an undergraduate at Trinity College, reading mathematics. His favourite pastimes are playing the piano and teasing his younger sister with puzzles.

I was reading *Flyaway* by Desmond Bagley when the following problem occurred to me.

> A man wants to cross a straight desert track in his Land-Rover and return to base. He has unlimited fuel at base, but the amount he can carry is limited by the capacity of the vehicle's tank. Can he do this? If so, what is the optimal solution?

It does not take long to see that the traveller can get as far as he likes, provided he can leave fuel at points along his route. The question is, what is the most efficient way in terms of fuel used to reach a given distance along the track? I do not know the answer to this, but here is my solution.

Denote by d the distance that can be covered by the Land-Rover using one tank of fuel. Denote by P_0 the initial point and by $P_1, P_2, P_3, \ldots$ the points distant $\frac{1}{4}d, \frac{1}{2}d, \frac{3}{4}d, \ldots$ from base along the track. We first show that half a tank of fuel can be deposited at each point P_i, and the return journey made to base. To deposit half a tank of fuel at P_n $(n \geqslant 1)$, suppose inductively that this has been done at $P_1, P_2, \ldots, P_{n-1}$. Now set out from P_0 with a full tank, and, on arrival at each P_i, fill up with fuel, leaving a quarter of a tankful behind at each point. On reaching P_n, leave half a tank of fuel and return to base, at each P_i filling up with the quarter tankful of fuel remaining there. This procedure is illustrated; figure 1 shows how half a tankful of fuel is deposited at P_3.

If F_i denotes the number of tanks of fuel used by this method to deposit half a tank of fuel at P_i and return to base, then

$$F_1 = 1, \qquad F_n = F_1 + F_2 + \cdots + F_{n-1} + 1 \quad (n > 1).$$

We prove inductively that $F_n = 2^{n-1}$ for $n \geqslant 1$. This is true when $n = 1$. Let $n > 1$, and assume that it is true for $F_1, \ldots, F_{n-1}$. Then

$$F_n = 1 + 2 + 2^2 + \cdots + 2^{n-2} + 1 = 2^{n-1}.$$

To reach P_n and return to base, the Land-Rover deposits half a tank of fuel at $P_1, P_2, \ldots, P_{n-2}$ in the manner described. On the last outward journey, it sets out with a full tank and fills up at each P_i. With a full tank at

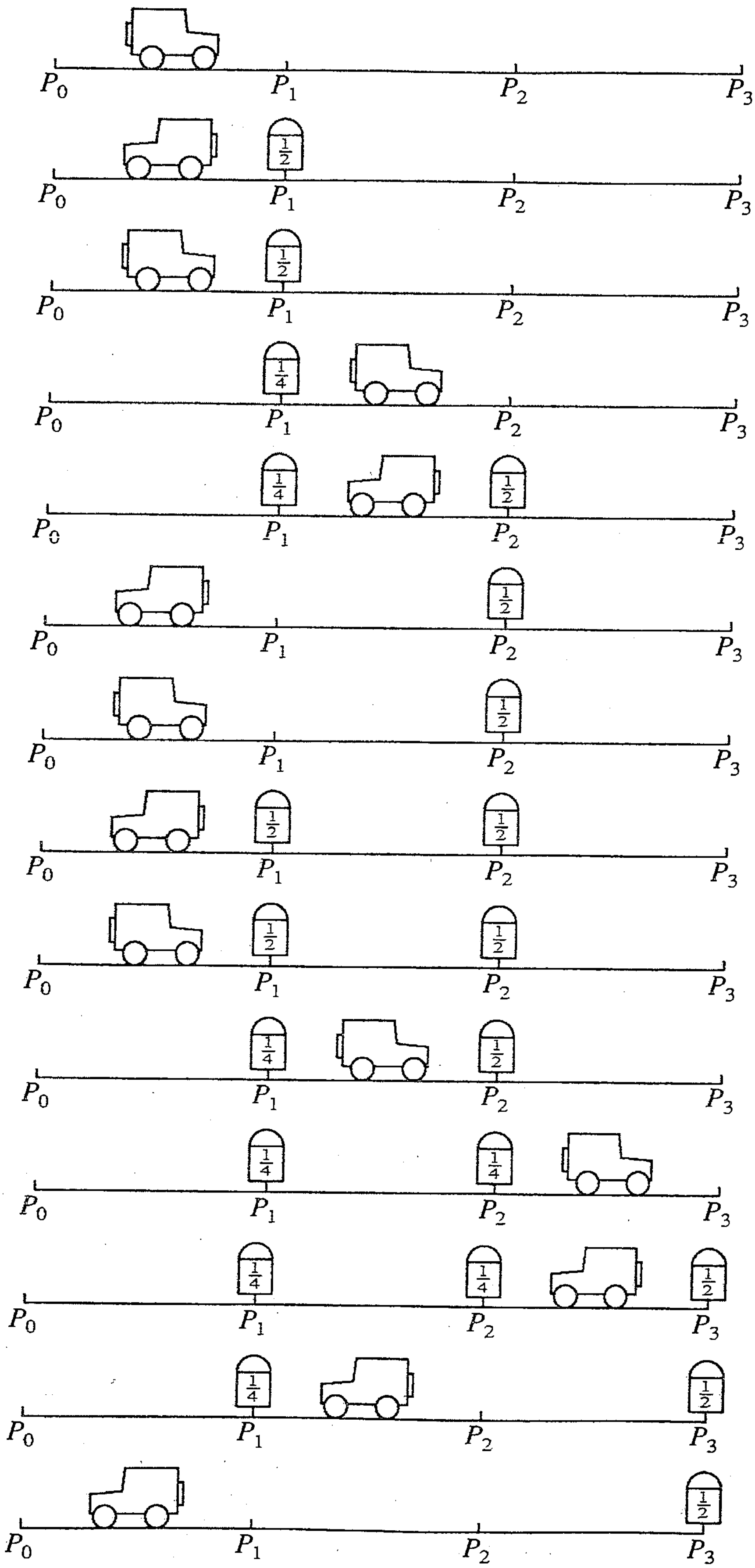


Figure 1. How to deposit half a tank of fuel at P_3

P_{n-2} it can travel to P_n and back to P_{n-2} and from there to $P_{n-3}, P_{n-4}, \ldots, P_1$ and back to base, filling up with fuel at each P_i. Hence, to cover a range of $\frac{1}{4}nd$ and return to base it needs

$$F_1 + F_2 + \cdots + F_{n-2} + 1 = F_{n-1} = 2^{n-2}$$

tanks of fuel.

Suppose that the range to be covered is D, returning to base. If n is the smallest integer such that $\frac{1}{4}nd \geqslant D$, then the number of tanks of fuel used is

$$2^{n-2} - \frac{2}{d}\left(\frac{nd}{4} - D\right) = 2^{\{4D/d\}-2} - \frac{1}{2}\left\{\frac{4D}{d}\right\} + \frac{2D}{d},$$

where $\{4D/d\}$ denotes the smallest integer greater than or equal to $4D/d$.

Can anyone do better? Out of interest, I worked on a second method in which the Land-Rover dumps all the fuel needed at P_1, then proceeds to dump fuel from P_1 at P_2, and so on until the destination is reached. The fuel requirement comes out the same.

Two further questions:

1. What is the most fuel-efficient method with an unlimited number of Land-Rovers?
2. What if the Land-Rover(s) need not return to base?

Computer Column

MIKE PIFF

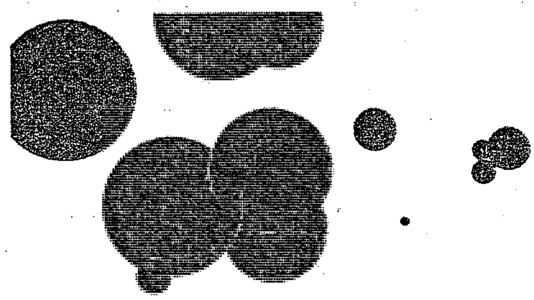
Flood-filling

Having seen in the last two columns how to draw lines and circles, we now progress to drawing solid objects. We will assume that a boundary has been drawn on the screen, say a polygon or a circle, and a point is specified interior to that boundary. Of course, for a circle the centre will do. The idea is then to let colour run from that point as far as it can within the specified boundary. Our circle will be converted into a disk, for example.

If in addition we specify that the colour should not run outside the screen, we are allowed to take the starting point as exterior to the polygon or circle and produce a screen with a hole in it.

We have to decide what we intend to do if part of the screen is already coloured. There are several possibilities. One is to ignore previously coloured regions, but just to colour up to the new boundary. This gives a way of obliterating 'hidden surfaces'.

Although a flood-fill algorithm can be written in about five lines—colour current point and then those to the north, south, east and west of it if they are not already coloured—the degree of recursion involved is such that we generally run out of stack space. The following algorithm cuts down the amount of recursion involved by replacing it partly by iteration. We also introduce another graphics primitive for the PC, to inquire about the colour of a specified point.



```
CONST
  inqfn=13;
PROCEDURE GetPixel(column,
  row:INTEGER):INTEGER;
BEGIN
  IF (column≥0)AND
    (column≤ScreenCols)AND
     (row≥0)AND
      (row≤ScreenRows) THEN
    CX:=column; DX:=row;
    AX:=inqfn×high; BX:=high×Page;
    Trap(GraphInt);
    RETURN AX MOD high;
  ELSE
    RETURN -1;
  END;
END GetPixel;
PROCEDURE FloodFill(x,y,colour:INTEGER);
VAR
  i, low, high:INTEGER;
BEGIN
  low:=x; high:=x;
  WHILE (GetPixel(low,y)≠colour) AND
   (low≥0) DO DEC(low); END;
  INC(low);
  WHILE (GetPixel(high,y)≠colour) AND
   (high≤ScreenCols) DO INC(high); END;
  DEC(high);
  FOR i:=low TO high DO
    PutPixel(i,y,colour);
  END;
  x:=low; INC(y);
  IF y≤ScreenRows THEN
    REPEAT
      IF GetPixel(x,y)≠colour THEN
        FloodFill(x,y,colour);
        WHILE (GetPixel(x,y)=colour) AND
         (x<high) DO INC(x); END;
      ELSE
        INC(x);
      END;
    UNTIL x≥high;
  END;
  x:=low; DEC(y); DEC(y);
  IF y≥0 THEN
    REPEAT
      IF GetPixel(x,y)≠colour THEN
        FloodFill(x,y,colour);
        WHILE (GetPixel(x,y)=colour) AND
         (x<high) DO INC(x); END;
      ELSE
        INC(x);
      END;
    UNTIL x≥high;
  END;
END FloodFill;
```

The 1993 Puzzle

It is time once again for our annual puzzle: can you express the numbers 1 to 100 in terms of the digits of the year in order, using only +, −, ×, ÷, √, ! and concatenation (i.e. constructing the number 19 from 1 and 9, for example)?

A Sum of Binomial Coefficients

P. GLAISTER, *University of Reading*

The author is currently a lecturer in mathematics at Reading University and has recently become interested in making the field of mathematics more widely understood by a population that is, on the whole, frightened of mathematics.

Recently, while looking at ways of introducing infinite binomial expansions, I discovered the following result on the sum of binomial coefficients.

Since

$$(1-x)^{-1} = \{(1-x)^{-1/2}\}^2,$$

then

$$\begin{aligned}\sum_{r=0}^{\infty} x^r &= \left(\sum_{i=0}^{\infty} \frac{x^i}{2^{2i}}\binom{2i}{i}\right)^2 \\ &= \sum_{r=0}^{\infty} x^r \left(\sum_{i=0}^{r} \frac{1}{2^{2i}}\binom{2i}{i}\frac{1}{2^{2(r-i)}}\binom{2(r-i)}{r-i}\right) \\ &= \sum_{r=0}^{\infty} \frac{x^r}{2^{2r}}\left(\sum_{i=0}^{r}\binom{2i}{i}\binom{2(r-i)}{r-i}\right),\end{aligned}$$

so that

$$\sum_{i=0}^{r}\binom{2i}{i}\binom{2(r-i)}{r-i} = 2^{2r} \quad (r \geqslant 0).$$

For example, with $r = 2$ we have

$$\binom{0}{0}\binom{4}{2}+\binom{2}{1}\binom{2}{1}+\binom{4}{2}\binom{0}{0} = 2^4,$$

or with $r = 5$

$$\binom{0}{0}\binom{10}{5}+\binom{2}{1}\binom{8}{4}+\binom{4}{2}\binom{6}{3}+\binom{6}{3}\binom{4}{2}+\binom{8}{4}\binom{2}{1}+\binom{10}{5}\binom{0}{0} = 2^{10}.$$

Readers may like to investigate other ways of proving this result. For similar results of this kind readers should consult John Riordan's comprehensive book *Combinatorial Identities* (Wiley, New York, 1968, reprinted by Krieger, Melbourne, Florida, 1979).

Letter to the Editor

Dear Editor,

Bernoulli numbers

Your contributor Joseph McLean (Volume 25, Number 1, page 9) raises the question of whether the Bernoulli number ${}^{155\,924}B_{11}$ is prime. In fact it is divisible by $2\times3^2\times5$. This completes the proof that ${}^{n}B_{11}$ is never prime.

Yours sincerely,
JEREMY BYGOTT
(23 Capel Close, Oxford OX2 7LA)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

25.7 (Submitted by Ahmet Özban, Karadeniz Technical University, Turkey)
For a natural number n, $S(n)$ is obtained by adding the digits of n and taking the number between 1 and 9 to which this sum is congruent modulo 9. (Thus $S(18) = 9$, $S(28) = 1$, $S(38) = 2$.) Determine

1. $S(2^p)$ when $p > 3$ is prime;
2. $\displaystyle\sum_{k=1}^{\infty} \frac{S(2^{k-1})}{2^k}$.

25.8 (Submitted by Glenn Vickers, University of Sheffield)
Show that the simultaneous equations

$$x^2+y = a, \qquad x+y^2 = b,$$

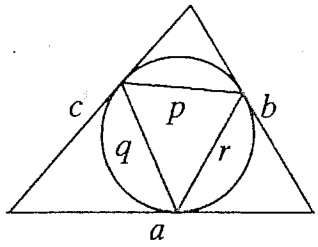
where a and b are fixed, can have at most one solution in which x and y are integers when $a \neq b$. Show also that, if there is a repeated (double) solution with $x = \alpha$ and $y = \beta$, then $4\alpha\beta = 1$, and find a and b if the equations have a triple solution. You might also like to consider the simultaneous equations

$$x^n+y = a, \qquad x+y^n = b,$$

for $n > 2$. (This problem was prompted by David Singmaster's letter about a problem of Ramanujan's in Volume 25 Number 1 page 26.)

25.9 (Submitted by Alan Douglas and Glenn Vickers, University of Sheffield)
Show that

$$\frac{p^2}{a(b+c-a)} = \frac{q^2}{b(c+a-b)} = \frac{r^2}{c(a+b-c)}.$$



Solutions to Problems in Volume 25 Number 1

25.1 Find a formula for the finite product

$$\prod_{r=1}^{n} (4r^2+4r-3).$$

Solution by Khalid Khan (London School of Economics)
The product is equal to

$$4^n \prod_{r=1}^{n} (r-\tfrac{1}{2}) \prod_{r=1}^{n} (r+\tfrac{3}{2}).$$

Now

$$\prod_{r=1}^{n} (r-\tfrac{1}{2}) = \frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\cdots\times\frac{(2n-1)}{2}$$
$$= \frac{1}{2^n}\,\frac{1\times2\times3\times\cdots\times2n}{2\times4\times6\times\cdots\times2n}$$
$$= \frac{(2n)!}{4^n n!}.$$

Also

$$\prod_{r=1}^{n} (r+\tfrac{3}{2}) = \frac{4}{3}\prod_{r=1}^{n+2} (r-\tfrac{1}{2}) = \frac{4}{3}\,\frac{(2n+4)!}{4^{n+2}(n+2)!}.$$

Hence the product is equal to

$$\frac{(2n)!\,(2n+4)!}{3\times4^{n+1}n!\,(n+2)!}.$$

Also solved by Clare Phethean (Gresham's School, Holt), Jeremy Bygott (Oxford), John de Sa (Royal Grammar School, Newcastle upon Tyne), Mark Blyth (University of Bristol) and David Brackin (St Catherine's College, Cambridge).

25.2 $(10^n)^3$ written in full ends with $3n$ 0's. Thus a cube can end with an unlimited run of 0's. What is the case for the other digits 1–9?

Solution

$1^3 = 1$, $7^3 = 343$, $3^3 = 27$, $9^3 = 729$, so we can find cubes which end in 1, 3, 7 and 9. Now let a be one of these four numbers, and suppose that, for some $r \geqslant 1$, there is a number B such that B^3 ends in r a's, i.e. $B^3 \equiv aa\ldots a \pmod{10^r}$ (r a's). We shall find a number whose cube ends in $r+1$ a's. It will then follow by induction that we can obtain cubes ending in an unlimited run of a's. We look for a number of the form $10^r A+B$. Now

$$(10^r A+B)^3 = (10^r A)^3+3(10^r A)^2 B+3(10^r A) B^2+B^3.$$

Denote the 10^r-digit of B^3 by b. Then we want

$$3AB^2+b \equiv a \pmod{10},$$

or

$$3Ab_0^2+b \equiv a \pmod{10},$$

where b_0 is the units digit of B. Now $b_0^3 \equiv a \pmod{10}$ and $(a,10) = 1$, so that $(b_0,10) = 1$. Hence there is an A satisfying the required congruence, so we can find a number whose cube ends in $r+1$ a's. Thus we can obtain unlimited runs of 1's, 3's, 7's and 9's. Given A_r such that

$$A_r^3 \equiv 11\ldots1 \pmod{10^r} \ (r\ 1\text{'s}),$$

then

$$(2Ar)^3 \equiv 88\ldots8 \pmod{10^r} \ (r\ 8\text{'s}),$$

so that we can also obtain unlimited runs of 8's. Now $8^3 = 512$, $4^3 = 64$, $5^3 = 125$ and $6^3 = 216$. Any number whose cube ends in 22 must be of the form $10A+8$. If

$$(10A+8)^3 \equiv 22 \pmod{100},$$

then

$$\begin{aligned} 10^3A^3+3(10A)^2 8+3(10A)8^2+8^3 &\equiv 22 \pmod{100} \\ 1920A+512 &\equiv 22 \pmod{100}, \\ 1920A &\equiv -490 \pmod{100}, \\ 192A &\equiv -49 \pmod{10}, \\ 2A &\equiv 1 \pmod{10}, \end{aligned}$$

and there is no such A. Hence we cannot extend the run of 2's beyond one. Similar arguments show that we cannot extend runs of 5's and 6's beyond one. Now $14^3 = 2744$, $64^3 = 262\,144$. To extend to 444, the only possibilities are $100A+14$ or $100A+64$. Now

$$(100A+14)^3 \equiv 444 \pmod{1000}$$

reduces to

$$\begin{aligned} 58\,800A+2744 &\equiv 444 \pmod{1000}, \\ 588A &\equiv -23 \pmod{10}, \\ 8A &\equiv 7 \pmod{10}, \end{aligned}$$

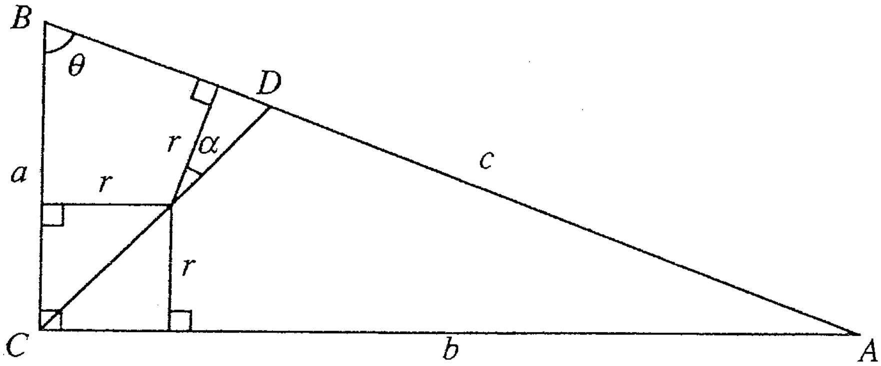
and this has no solution.

A similar argument can be given for $100A+64$, which shows that we cannot extend a run of 4's beyond 44.

Solved by Jeremy Bygott. Almost complete solutions were given by David Brackin (St Catherine's College, Cambridge) and Khalid Khan.

25.3 Let T and T' be right-angled triangles and denote by R, r and R', r' the radii of the circles circumscribing and inscribing T and T' respectively. If $R/R' = r/r'$, prove that T and T' are similar triangles.

Solution by Khalid Khan



Consider a right-angled triangle T as shown. It is easily shown that the bisector CD of the right angle has length $l = \sqrt{2}ab/(a+b)$, so that

$$l = \frac{\sqrt{2}c\cos\theta\sin\theta}{\cos\theta+\sin\theta}.$$

Also,

$$\begin{aligned} l &= r\sqrt{2}+r\sec\alpha \\ &= r\sqrt{2}+r\sec(\theta-\tfrac{1}{4}\pi) \\ &= r\sqrt{2}+\frac{r\sqrt{2}}{\cos\theta+\sin\theta} \\ &= \frac{r\sqrt{2}(\cos\theta+\sin\theta+1)}{\cos\theta+\sin\theta}. \end{aligned}$$

Hence

$$r(\cos\theta+\sin\theta+1) = c\cos\theta\sin\theta,$$

so that

$$r = \tfrac{1}{2}c(\cos\theta+\sin\theta-1).$$

Since the centre of the circumcircle of T is at the midpoint of AB, this gives

$$r = R(\cos\theta+\sin\theta-1).$$

Now

$$\frac{R}{r} = \frac{R'}{r'},$$

so that

$$\cos\theta+\sin\theta-1 = \cos\theta'+\sin\theta'-1,$$

where θ' is the corresponding angle of T'. Thus

$$\cos\theta-\cos\theta' = \sin\theta'-\sin\theta,$$

$$-2\sin\tfrac{1}{2}(\theta+\theta')\sin\tfrac{1}{2}(\theta-\theta') = 2\cos\tfrac{1}{2}(\theta'+\theta)\sin\tfrac{1}{2}(\theta'-\theta),$$

so that either $\theta = \theta'$ or $\theta+\theta' = \frac{1}{2}\pi$. Hence T and T' are similar triangles.

Also solved by Jeremy Bygott.

Reviews

Fearful Symmetry: Is God a Geometer? By IAN STEWART AND MARTIN GOLUBITSKY. Blackwell, Oxford, 1992. Pp. xix + 287. Hardback £16.95 (ISBN 0-631-18251-9).

This book is about what happens to situations which possess symmetry when that symmetry breaks down. For example, an infinite flat pond has an immense amount of symmetry (although it appears rather uninteresting). If that symmetry is broken, for example by dropping in a pebble, then the symmetry is not wholly destroyed but it is radically reduced (and the resulting pattern becomes rather more interesting). The authors show how these ideas can be used to analyse a wide range of phenomena: fluid flow, crystals, development of embryos, mammalian motion, the structure of elementary particles and the structure of the universe, amongst others. Such systems may start with a considerable amount of symmetry but as they evolve that symmetry may become unstable and break down. However, the subsequent patterns resulting from the breakdown will retain as symmetries a subset of the original ones. This means that you can obtain sensible predictions about the evolution of a system even if a model you are using is not that accurate or if it is so complex that not even a supercomputer can process it.

I found the book very interesting but largely descriptive and rather lacking in depth, perhaps inevitably, given the complexity of much of the underlying mathematics and science. However, many of the effects mentioned can be easily observed, for example the motion of quadrupeds. I was surprised here to see that the authors omitted a discussion of human quadruped motion, which is actually not uncommon, especially in very young humans. Readers may like to attempt to classify their own quadruped motion according to the scheme outlined in Chapter 8. For me, the most interesting chapter was the final one, with its discussion on the extent to which symmetry 'really' exists in nature or whether much of it is a consequence of how we humans like to view things. This parallels present debate in quantum mechanics on whether the physical world is shaped in some sense by our perception of it.

It is important also to point out (as the authors often do) that symmetry breaking does not always provide an explanation. For example, corn circles—'ripples' on the 'pools' which are corn fields—have proved to be a hoax, and the symmetry of dew drops on a spider's web need not be entirely due to symmetry breaking (see *Scientific American*, March 1992, for a fascinating analysis of spiders' webs). Similarly, the cosmic-string theory of symmetry breaking for the creation of galaxies, mentioned at the end of Chapter 6, has recently run into trouble (see *Scientific American*, July 1992).

I would certainly recommend this as a book worth reading, but personally I would prefer to borrow it from the library rather than feel I needed it on my shelves.

Oakham School G. N. THWAITES

The Most Beautiful Mathematical Formulas. By LIONEL SALEM, FRÉDÉRIC TESTARD AND CORALIE SALEM, translated by JAMES D. WUEST. Wiley, Chichester, 1992. Pp. xiii + 141. Hardback £12.95 (ISBN 0-471-55276-3).

This English translation of the French *Les Plus Belles Formules Mathématiques* claims to be 'an instructive unwaveringly playful romp through the 49 most interesting, useful and quirky mathematical formulas of all time ... with amusing cartoons and packed with whimsical everyday problems from the mundane to the earth-shattering ... short thoroughly entertaining chapters ... the only prerequisites are high-school algebra and geometry, and a desire to have fun'.

The volume is not only physically slender, but also thin on ideas, interest, humour and historical accuracy. Consider as an example Chapter 3, which is devoted to that *most interesting, useful and quirky mathematical formula*: the area of a rectangle is the product of its sides. It is indeed short, six brief sentences taking all of twenty seconds to read. The opening paragraph tells of the *Trifolium giganteum*, a bizarre mutation of the four-leaved clover, which only grows when it occupies a space of 1 metre by 1 metre! The second and concluding paragraph addresses the problem of a farmer who wishes to grow these curious plants on a rectangular field of 5 metres by 7 metres. It is calculated that he can grow $5\times7 = 35$ of them, *without wasting the least bit of the field*. The amusing cartoon accompanying this whimsical everyday problem shows the farmer planting one of the 35 *Trifolia gigantea*. Included among the remaining 48 chapter headings are: *the angle at the centre of a regular tetrahedron is 109°28′* (interesting?); *Fermat's Last Theorem* (useful?); *the area of a triangle is one half the product of its base and height* (quirky?); $\pi \approx 355/133$ (a formula?); $2^m\times2^n = 2^{m+n}$ (whimsical?); and *Goldbach's Conjecture* (everyday?). The authors admit to taking liberties with the historical truth, but compensate for this by including an 'annex' at the end of the book to 'set the record straight'! Of the many titles on popular mathematics currently available this is not the best, nor the second best, nor the third best, nor .... That the publishers share the same quirky, whimsical sense of humour as the authors is shown by the price of the book, £12.95.

University of Sheffield R. J. WEBSTER

The Crest of the Peacock: Non-European Roots of Mathematics. By GEORGE GHEVERGHESE JOSEPH. Penguin, London, 1992. Pp. xv + 371. Paperback £8.99 (ISBN 0-14-012529-9).

In this enlightening book, George Gheverghese Joseph sets out to dispel the classical Eurocentric view of the origins of mathematics, namely that Greece and subsequently Europe and her cultural dependencies were the primary channels of mathematical development. For instance, Joseph provides evidence that the Babylonians were using Pythagoras's theorem over a thousand years before him and that the triangle of numbers now attributed to Blaise Pascal was in use in China as early as 1100 AD.

Joseph, then, takes us on a journey through the rich mathematical achievements of the Egyptians, Babylonians, Chinese, Indians and Arabs. After a brief historical sketch of each, we are introduced to their number systems and their progress in the mathematical field. Chronological tables of events are provided to help keep track. Problems from varying ancient manuscripts are neatly presented with their solutions, and occasionally the reader is invited to enter into the spirit of things and solve, for example, a few problems from the Chinese mathematical text, the *Chiu Chang* (ca. 300 BC to 200 AD). Wouldn't it be nice, though, if modern examiners had the charm of the ancient Indians: 'O beautiful maiden, with beaming eyes, tell me, what number multiplied by 3 ...'?

Joseph's meticulous style makes this both an entertaining and informative read. Also, subject and name indexes at the back make reference easy. This is an important book with something significant to say. It is well worth reading.

Sixth Form, Gresham's School, Holt MARK BLYTH

Beyond Numeracy. By JOHN ALLEN PAULOS. Penguin, London, 1992. Pp. 285. Paperback £6.99 (ISBN 0-14-014574-5).

This book is the successor to Paulos's *Innumeracy* (reviewed in *Mathematical Spectrum* Volume 23, Number 2). The introduction begins: 'This book is in part a dictionary, in part a collection of short mathematical essays, and in part the ruminations of a numbers man'.

It contains 73 entries which range from algebra to Zeno's arrow. These include summaries of many disciplines (calculus, trigonometry, topology), biographical and historical notes (Gödel, Pythagoras, non-Euclidean geometry), mathematical folklore (infinite sets, Platonic solids, QED), new areas (chaos and fractals, recursion, complexity), classical ones (conic sections, mathematical induction, prime numbers), a chronological list of the 'top 40' mathematicians from ancient times up to the early 20th century (the author's inverted commas), and a list of suggested reading.

(GCSE and A-level students should be alerted to the only error I spotted—the 0th power of anything is 1, on page 117.)

Having read this book twice, so far, I feel I must try to convince as many people as possible to read it: mainly because it has charm, and also because it is such a refreshing book. Professor Paulos makes his exercises fascinating, and the methods of solution even more so. I am tempted to include several typical examples but I must restrain myself to two.

1. The population of a certain animal species is given by the formula $X' = RX(1-X)$, where X' is the population one year, X is the population the preceding year and R is a parameter. For simplicity we take X and X' to be numbers between 0 and 1, the true population being 1000 000 times these values. What happens to the population of this species as R varies between 0 and 4?
2 A secretary randomly scrambles 50 different letters and the addressed envelopes into which they are to be stuffed. If he or she haphazardly fills the envelopes with the letters, what is the probability that at least one letter is placed in the correctly addressed envelope?

And from the publishers: 'Professor Paulos's quirky, humorous and informative approach makes *Beyond Numeracy* an irresistible introduction to new ways of seeing the world'. And this is not just publisher's exaggeration—it is the truth.

Medical School, University of Newcastle upon Tyne GREGORY D. ECONOMIDES

Understanding Einstein's Theories of Relativity. By STAN GIBILISCO. Dover, New York, 1991. Pp. viii + 200. Paperback £6.25 (ISBN 0-486-26659-1).

I'm afraid my heart sank before I had read a word: above each chapter is a picture of a stationary rocket and a 'contracted' moving rocket. This is not what you would actually see ....

The book fits into the Special Relativity, General Relativity, Cosmology format, with additional chapters on space travel and hyperspace. It is intended for the novice 'with a minimum of high school maths'. The early chapters fail to present a coherent model of flat space-time. Confusion follows from not defining simultaneity fully, and we end up with some futile discussion about clocks in hyperspace. Blatant error reaches its peak on page 34: 'We must conclude that the light flashes will not arrive simultaneously at *D*'. Note also on page 65 that the period is proportional to the square root of the mass. The hyperspace chapter creates more confusion: 'The universe is a 4-sphere' is very glib. The large-scale structure of space may be a 4-sphere. However, general space-time cannot be embedded in less than a 10-dimensional Euclidean space. The discussion of rotation in the chapter on General Relativity is good. Sadly, though, I could quote many more inaccuracies.

Despite its low price, I would not recommend this book to anyone. Einstein's popular book *Relativity* still remains for me the most philosophically satisfying introduction to this field.

Undergraduate at Trinity College, Cambridge DYLAN GOW

Other books received

Advanced Level Mathematics. By R. C. SOLOMON. DP Publications Ltd, London, 1992, second edition. Pp. xvii + 599. £8.95 (ISBN 1-870941-24-4).

Mental Methods in Mathematics. By WILL CONNOLLY ET AL. The Mathematical Association, Leicester, 1992. Pp. 72. £8.50 (ISBN 0-906588-27-8).

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