Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

Volume 47 2014/2015 Number 1

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

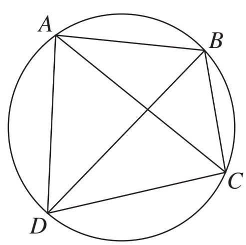
Ptolemy through the looking-glass

Figure 1 shows *Ptolemy's theorem* for a cyclic quadrilateral. It goes back to Claudius Ptolemy, who worked in Alexandria in the second century AD and whose most famous work is the *Almagest* (see reference 1 for a modern translation). Readers will find proofs of Ptolemy's theorem on the internet, some of them anachronistic, using, for example, trigonometry or complex numbers.

The late William Wynne Willson constructed a proof of Ptolemy's theorem using simple ideas of transformation. His book, *Ptolemy Through the Looking-Glass* (see reference 2), is the result. He sadly died before he had finished writing the book, and his friend and colleague Geoff Wain, aided by Douglas Quadling, completed it.

William Wynne Willson was evidently a gifted and sympathetic teacher. Readers are led painlessly through the steps, with fantastic multi-coloured diagrams which will only cause problems to colour-blind readers like myself. Some readers may find the approach too painstaking, but this reader was enchanted by an exposition that unfolded at a gentle pace. It causes a smile of pleasure as the theorem gradually emerges. If only all expositors were as considerate to their readers! This is not Ptolemy's proof, but it is a proof that Ptolemy would have understood. The book is more than a proof of a single theorem; it is a demonstration of the power of transformations.

There is a delightful surprise in the crucial appearance of the *Simson line* (see figure 2). Take any triangle and any point P on its circumcircle. Drop perpendiculars from P to its three sides (produced if necessary). Then the feet of the perpendiculars lie on a straight line, called the *Simson line* of P. It is fascinating to see how the Simson line varies as P moves round the circumcircle of the triangle (which Willson showed in reference 2, figure 10.6). Willson then discussed a variant of the Simson line. If P is reflected in the three sides of the triangle, the resulting three points again lie on a straight line, which Willson called the *Wallace line* of P (see figure 3) after William Wallace, the Scottish mathematician who in 1799 first introduced the Simson line, perhaps, wrote Willson, as a tribute to his compatriot Robert Simson who died in 1768 and whose writings do not include any reference to the Simson line by this or any other name. The proof of Wallace's (or Simson's?) result comes out painlessly in Willson's approach.



$$AB \times CD + AD \times BC = AC \times BD$$

Figure 1 Ptolemy's theorem.

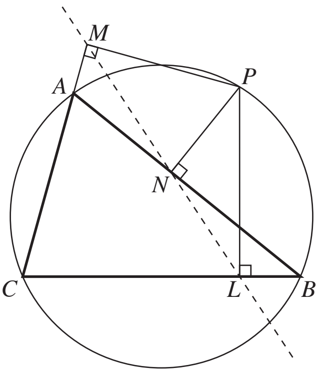


Figure 2 The Simson line of P (dashed line).

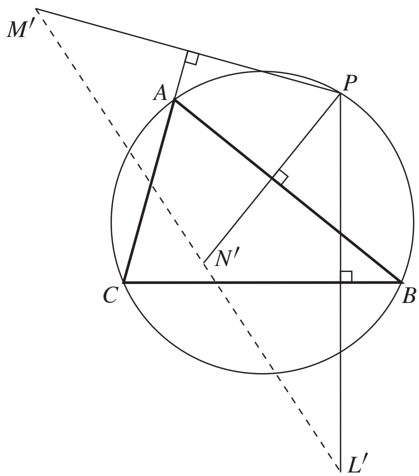


Figure 3 The Wallace line of P (dashed line).

All in all, Willson's book is a delightful presentation of a too-little-known theorem. Douglas Quadling contributes an 'Afterword' about life after Ptolemy. As our own simpler afterwords, there is a delightful use of Ptolemy's theorem in the letter from Abbas Rouhol Amini on pages 41–42 of this issue. For your own application, try deducing the famous Pythagoras' theorem on right-angled triangles from Ptolemy's theorem.

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Are Most Triangles Obtuse?

SETH ZIMMERMAN

What is the probability that one of the angles of a 'randomly' chosen triangle is obtuse? This question was first investigated in the nineteenth century by Lewis Carroll, who believed he had found the answer. We first show that the question is not well defined, and survey several of its traditional interpretations. We then look at the question from a fresh perspective, using computer simulations to obtain results which are different from any of those previously found.

1. Introduction

Try a brief experiment. Picture a triangle—just one single triangle. Hold it clearly in your mind for an instant, then note if is it acute, right-angled, or obtuse? Most likely it is acute. When 24 nonmathematicians were informally asked to sketch a triangle, all but two of the triangles drawn were acute. None was obtuse. Given this, how would we answer the following question: what is the probability that a random triangle is obtuse? A quick, uncalculated guess, consistent with the above experiments, would be low, certainly below $\frac{1}{2}$. But some consideration would reveal that the question is actually ill-defined and open to a variety of interpretations.

2. Three traditional interpretations

2.1. Lewis Carroll's version

Recent interest in the question largely stems from the night-time musings of the mathematician Charles Dodgson (also known as Lewis Carroll). Here is how he put the question, a bit less loosely than we posed it above (see references 1 (p. 87), 2, 3, and 4).

> Three points are taken at random on an infinite plane. Find the chance of them being the vertices of an obtuse-angled triangle.

Although this specifies that 'random' triangles are to be selected by choosing three vertices, a modern student of probability will catch the difficulty still remaining. We cannot simply choose a point at random from an infinite plane. While nothing prohibits Carroll from taking the full $x-y$ plane for his sample space, he hasn't defined a probability density function that integrates to 1 over the entire space. For an introduction to such functions, see any standard calculus text (for example, reference 5). Unaware of these matters in the nineteenth century, Carroll gave what he thought was the correct answer—a transcendental number close to 0.639 38—a value that was really the solution to the following different, more precise question.

> Given two arbitrary points A and B, a triangle is constructed by choosing a third point at random from the region for which AB is the triangle's *longest* side. What is the probability that this triangle is obtuse?

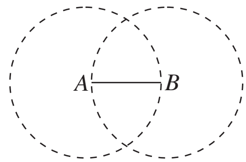


Figure 1 Assume that AB is the longest side of a random triangle.

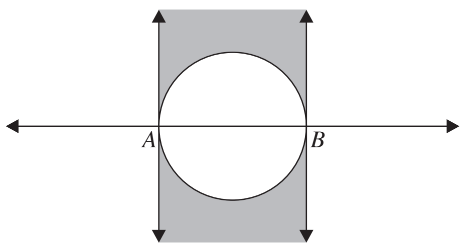


Figure 2 AB is one side of a ‘random’ triangle. If the third point lies outside of the shaded region the triangle will be obtuse.

Assisted by figure 1, in which the third point of the triangle must reside within the intersection of the two circles, readers might enjoy the challenge of deriving Carroll’s answer to this well-defined question, $3\pi/(8\pi - 6\sqrt{3})$. A detailed derivation is given in reference 6, p. 14.

We might then return to Carroll’s own incomplete formulation of the problem and consider figure 2. Where is the flaw in the following argument? AB is one side of a triangle. Given that Carroll is using the entire $x-y$ plane, the length of AB should not matter. The triangle’s third vertex can lie anywhere in the plane except on AB extended. To the naive eye it appears that the area in which this produces an obtuse triangle is much larger than the area in which it produces an acute one, indeed, ‘infinitely’ larger. Thus, the probability of an obtuse triangle is 1—we seem to have demonstrated that just about every triangle one can sketch is obtuse!

Readers interested in researching Carroll’s approach more deeply will find a comprehensive study of the contradictions in his reasoning useful (see reference 3).

2.2. The version on a rectangular sample space

If we follow Carroll’s procedure of selecting three points at random but limit our sample space to a rectangle, such as a sheet of paper, then the question can be stated as follows.

> Choose three points at random in an arbitrary rectangle. What is the probability that the triangle thus formed is obtuse?

Note that all rectangles with the same ratio of width to length will have the same answer to the above question. We can therefore limit ourselves to a sample space consisting of a rectangle whose sides are 1 and L. If no region of the rectangle is more likely to contain a chosen point than any other region of equal size, then the probability density function will simply be the reciprocal of the rectangle’s area, namely $p(x, y) = 1/L$. This integrates to 1 over the entire sample space in a uniform distribution. A derivation of the probability of obtuseness as a function of L is beyond the scope of this article, but is presented in detail elsewhere (see

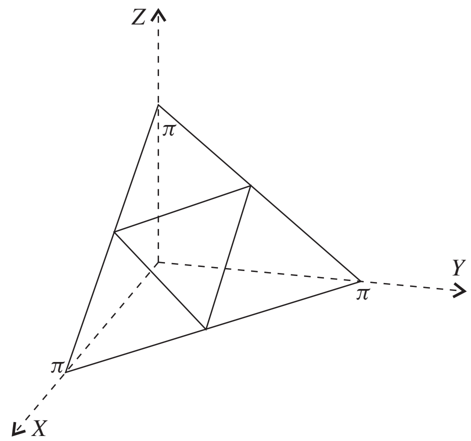


Figure 3 Three angles chosen to produce a 'random' triangle.

reference 7). According to this formula (given in appendix A, the probability of a random triangle's being obtuse in a square is 0.725 21, to five decimal places. The probability of a triangle's being obtuse in a rectangle whose sides are in the ratio 1:2 is 0.798 37. These results have been confirmed to within 0.1% by the author and others using Monte Carlo methods (see reference 1, p. 87). These procedures are discussed in section 3.2.

2.3. The angular version

An appealing version of the question occurs when we stipulate that angles be chosen rather than sides or vertices. If x, y, and z are the angles, the sample space is defined by $x + y + z = \pi$, $x, y, z > 0$. Drawn on a three-dimensional coordinate system, this linear equation represents a plane, restricted by the inequalities to the octant in which all three variables are positive. Thus, it is represented in figure 3 by the larger equilateral triangle, excluding the boundary. If we assume a uniform distribution, then since the area of the larger triangle is $(\sqrt{3}/2)\pi^2$, the probability density function will be its reciprocal, $p(x, y, z) = 2/\sqrt{3}\pi^2$. This clearly integrates to 1 over the entire sample space, its constancy implying that the probability of any region of the triangle is proportional to its area. Only points in the smaller inner triangle produce acute triangles, where $x, y, z < \pi/2$. Thus, if we choose 'random' triangles by this angular procedure, the probability of generating an obtuse triangle is exactly $\frac{3}{4}$. Five different proofs are offered for the result $\frac{3}{4}$ in reference 8. Our particular version is equivalent to the well-known 'broken stick' problem (see reference 6, p. 14).

3. A new approach: experimenting with a normal distribution

The indefiniteness of the obtuse angle question lends itself to a variety of interpretations, often very different from the popular ones surveyed above. Some, like the version we investigate in sections 3.1 and 3.2, introduce new methods, such as the use of simulation. Other versions will sharpen our sense of what constitutes mathematical clarity and give us the opportunity to apply mathematical techniques to our own studies.

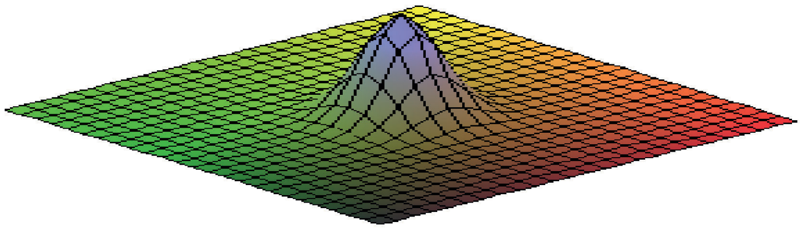


Figure 4 A sketch of the joint probability density function $\exp(-\pi r^2) = \exp(-\pi(x^2 + y^2))$.

3.1. The probability density function

Let us imagine someone sitting at a desk before a tilted rectangular drafting board, looking perpendicularly down at its centre and choosing points with the tip of a pencil. Let us also assume that this person is more inclined to choose points near the centre rather than those farther out toward the edges, and favours no particular direction out from the centre. The empirical distribution of thousands of such choices clearly demands an experiment beyond any mathematician's time or funding, although conceivably some software company might find commercial benefit in such research. Thus, we can choose an arbitrary distribution for the model we are creating without fear of being disproved. Placing the origin of an $x-y$ or $r-\theta$ plane at the centre of the board, it would be convenient if the probability density function of the points along any line through the origin were assumed to be a standard normal distribution. (For convenience we will drop 'probability' and write simply 'density function'.) However, this is not exactly possible, since it is the entire board upon which the density function must integrate to 1, not on each individual line through the origin. We can rectify this in two steps. First, instead of the board, we consider the entire plane to be the domain or sample space of the density function. As we will see in section 3.2, this has a negligible effect upon the computed results. Second, we construct a joint density function of the form $p(r,\theta) = a\exp(-br^2)$, $a, b > 0$, such that $p(r,\theta)$ integrates to 1 over the entire plane. Note that by its definition $p(r,\theta)$ is independent of θ, corresponding to the person's lack of directional bias. To find a and b we integrate $p(r,\theta)$ over the plane and set the result equal to 1, i.e.

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} a\exp(-br^2)r\,\mathrm{d}r\,\mathrm{d}\theta &= 2\pi\int_0^{\infty} a\exp(-br^2)r\,\mathrm{d}r \\ &= -2\pi\left[\frac{a}{2b}\exp(-br^2)\right]_0^{\infty} \\ &= \pi\frac{a}{b} \\ &= 1.\end{aligned}$$

This yields an infinite collection of potential density functions, producing an infinite collection of similar models. For simplicity, we let $a = 1$, so that the density function for our model is $p(r,\theta) = \exp(-\pi r^2)$, plotted without scale in figure 4.

As an aside, any vertical cross-section parallel to either the x- or y-axis seems to resemble a normal curve, i.e. a function that, multiplied by an appropriate constant, would represent a normal distribution. For convenience, call this 'normal-shaped'. We can confirm this visual observation if we write the function in rectangular form as follows:

$$\exp(-\pi r^2) = \exp(-\pi(x^2 + y^2)) = \exp(-\pi x^2)\exp(-\pi y^2).$$

Setting x equal to a constant produces a normal-shaped function in the y direction, and similarly if we let y be a constant. Further, by the rotational symmetry of $\exp(-\pi r^2)$, a normal-shaped cross-section must be produced by any vertical slice, whether parallel to an axis or not.

Before we inspect a simulation of this circular model, we can create a variant of it for comparison. The person with the pencil could be sitting at an ordinary horizontal desk, his/her eyes just above the closer edge. Instead of choosing points from a circular sample space, the person now chooses points in a semicircle whose radius is the stretch of his/her arm. If we proceed as we did for a circle, the density function for this semicircle model becomes $p_s(r, \theta) = \exp(-(\pi/2)r^2)$. We will compare the two models in section 3.3.

3.2. Computer simulations

It is not always easy to derive a general formula, such as the one in appendix A. Even the exact numerical solution to a particular case can be difficult or even impossible to obtain. In such a situation we can sometimes resort to a Monte Carlo simulation. This consists of producing a sufficiently large number of appropriate examples so that we accept the resulting statistics as an adequate approximation to the theoretical solution. In our case, for each sample space and probability density function that we assumed, we produced four million triangles, the coordinates of whose vertices were chosen by the pseudo-random number generators of our computer program. We then accepted whatever fraction of these triangles were obtuse as a good approximation of the probability of obtuseness.

There is something else of significance to note, however, which applies to many simulations besides ours. A computer cannot choose a point from a continuum of points, or an angle from every possible angle between 0 and 2π. It can choose from only a finite number of points whose rational coordinates are specified to a finite number of decimal places. This is the nature of computer programs. (Indeed, any nonelectronic method we might imagine would also reduce to a selection from a finite collection of points. Ultimately, physicists would insist that we cannot measure to any finer accuracy than the Planck length.) To deal with this digital reality we have to replace our smooth sample space with a fine grid whose intersections are the only points at which a vertex is permitted to exist. In all of our examples the grid upon which vertices were selected measured 36 000 by 36 000, a mesh at least 28 times finer, linearly, than the pixels of an average computer monitor.

3.3. Simulation results

Whatever the sample space—rectangle, circle, semicircle, the entire plane—different probability density functions will produce different results. As our main objective is to illustrate rather than research, we limit ourselves to the circle and semicircle for our sample spaces, using four different probability density functions. Triangles are determined by randomly selecting three vertex points, each triangle then being tested for obtuseness by the Pythagorean inequality. That is, if the triangle whose sides are a, b, and c satisfies $a^2 + b^2 < c^2$ the triangle is obtuse, but if $a^2 + b^2 > c^2$ the triangle is acute.

In table 1, each probability of obtuseness is the result of inspecting four million triangles. These were collected in 40 groups of 100 000, with the standard deviation of the resulting 40 probabilities shown in the column labelled σ. All work was done using MAPLE™ 10. We now interpret each row of table 1.

Table 1 Probability of obtuseness if the sample space is a circle or semicircle, under four different assumptions.

	Space	Density function	Circle		Semicircle	
			Probability	σ	Probability	σ
(i)	x, y	uniform	0.719 96	0.001 44	0.794 10	0.001 17
(ii)	r, θ	uniform	0.795 39	0.001 18	0.813 33	0.001 00
(iii)	r, θ	normal (thinning)	0.788 55	0.001 23	0.800 87	0.001 11
(iv)	r, θ	normal (adjusted)	0.753 08	0.001 35	0.792 72	0.002 82

(i) The most common selection procedure in the literature is that of choosing rectangular coordinates x and y, with a uniform density function. This means that each of the $36\,000 \times 36\,000$ points on our grid that lie within the circle has the same probability of being chosen as a vertex. A simplified picture of this is shown in figure 5(a) for a circle drawn on rectangular graph paper. Note that the size of the circle is not indicated and indeed has no significance, since the same grid would be superimposed on any circle. For a circle under these assumptions, the probability of a triangle being obtuse is 0.719 96, and if the same procedure is done in a semicircle, the probability is 0.794 10.

(ii) This is the simplest of the three variations using polar coordinates. We assume independent uniform density functions for the radius and angle. We might first choose the radial coordinate from the 36 000 evenly spaced choices, then the angle θ ($0 \leq \theta \leq 2\pi$ for a circle, $0 \leq \theta \leq \pi$ for a semicircle), again from 36 000 evenly spaced choices. Or we could choose them in reverse order, since they are independent. In figure 5(b) this is represented by the points of intersection of the solid rays and circles, with each intersection equally likely to be selected. Given that the density of intersections diminishes with distance from the origin, this might correspond to our sitting at a desk and reaching out equally in any direction, perceiving distance between points more coarsely the farther away they are. As in row (i) the size of the circle is irrelevant.

(iii) This is also represented by the solid rays and circles in figure 5(b), but now the normal density function discussed in section 3.1 is assumed. For this example there must be a specific size to the circle, for the normal density function extends to infinity in all directions and different size circles create slightly different sample spaces. A radius of 4

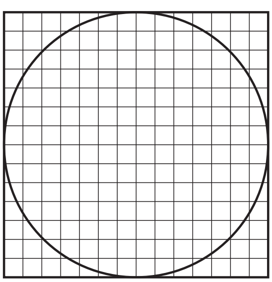
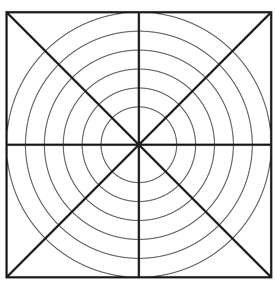
(a)  (b) 

Figure 5 (a) A rectangular grid, (b) a radial grid.

will suffice, because the total probability of choosing a point beyond radius 4 is negligibly small. That is,

$$\int_0^{2\pi} \int_4^{\infty} \exp(-\pi r^2) r \, \mathrm{d}r \, \mathrm{d}\theta = \exp(-16\pi) < 1.5 \times 10^{-22}.$$

(We obtain a similar result with the semicircle.) Angular directions would again be equally likely, so that, as in row (ii), the intersections thin out with distance from the origin. But in this example, the normal density function diminishes with distance from the centre, perhaps reflecting the person's exponentially diminishing inclination to stretch out his/her arm to mark a point. (We remind ourselves that this is all hypothetical, and that other models may be more realistic.) The average probability of 0.800 87 for a semicircle is too close to 0.8 to ignore. It is possible that exactly four out of five triangles chosen under these assumptions are obtuse, but attempts to prove this have so far been unsuccessful. It remains an open hypothesis that might very well have a simple proof.

(iv) As in row (iii), this would correspond to an inclination to stretch that diminishes with distance. However, the program has been adjusted so that instead of the density of points thinning with greater radius, the density stays approximately the same. This would reflect an ability to draw points with equal fineness anywhere on the page.

Looking at table 1 as a whole, we note that in all four cases the probability of obtuseness is greater for the semicircle than the circle. This agrees with the naive expectation that the 'flatter' of the two sample spaces would lend itself to obtuse triangles more readily. It is surprising to note how little influence the density function has on the probability of obtuseness when the sample space is a semicircle. This is not so for the circle.

4. Conclusion

The obtuse triangle question, with its rich potential for investigation, deserves to be better known. A study of the above interpretations, from Carroll's onward, can offer readers interesting applications of concepts such as sample space, density function, simulation, and, above all, randomness. They should appreciate more deeply the value of definitional clarity. More ambitiously, the article might prepare readers for the challenging task of postulating their own sample space and density function, programming a simulation, gathering the results, and presenting them.

Appendix

A. The rectangular formula

For a rectangle whose sides are $1 \times L$, the probability of a triangle being obtuse is

$$\mathrm{P}(L) = \frac{1}{3} + \frac{47}{300}\left(L^2 + \frac{1}{L^2}\right) + \frac{\pi}{80}\left(L^3 + \frac{1}{L^3}\right) - \frac{\ln(L)}{5}\left(L^2 - \frac{1}{L^2}\right), \qquad 1 \le L \le 2,$$

$$\begin{aligned}\mathrm{P}(L) = {} & \frac{1}{3} + \frac{1}{L^2}\left(\frac{\pi}{80L} + \frac{47}{300} + \frac{\ln(L)}{5}\right) + \frac{47L^2}{300} - \frac{L^2\ln(L)}{5} + \frac{L^3}{40}\arcsin\left(\frac{2}{L}\right) \\ & + \left(\frac{L^2}{10} - \frac{3}{5L^2}\right)\ln\left(\frac{L+\sqrt{L^2-4}}{L-\sqrt{L^2-4}}\right) + \frac{L\sqrt{L^2-4}}{150}\left(-31 + \frac{63}{L^2} + \frac{64}{L^4}\right), \qquad L \ge 2.\end{aligned}$$

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Magic squares of odd size

4	9	2
3	5	7
8	1	6

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Can you continue this pattern?

U.A.E. **Murali Velayudhan**

Using Triangular Numbers as 'Steps' when Constructing Quadratic and Cubic Sequences

KAREN HEINZ and THOMAS E. SHOWN

Finding a pattern in a sequence of numbers and describing that pattern recursively and with an explicit formula are common mathematical activities. Rather than starting with a sequence and then analysing it to discover patterns, we started with particular patterns then constructed sequences that were built from those patterns. As we engaged in geometric and algebraic explorations of that process, we were fascinated by ways in which we could reveal the triangular numbers as 'steps' in the construction of quadratic and cubic sequences. In this article, we share highlights of those explorations, which include using triangular numbers to derive formulas for quadratic and cubic sequences and constructing a 2-dimensional geometric proof of the formula for the *n*th tetrahedral number.

Examining a linear sequence

As a foundation for our exploration of quadratic and cubic sequences, we note that there is a common difference between consecutive terms of a linear sequence. For example, figure 1 is a geometric representation of a sequence that begins with 4 and has a common difference of 3. Note that the right-most column in each term is the amount of increase from the previous term and that this amount, 3, is the same for each term. By examining the representation for the nth term, we can derive a formula that models this sequence, namely $f(n) = 4+3(n-1) = 3n+1$, where n is the term number and $f(n)$ is the term (e.g. $f(3) = 10$).

Constructing sequences with a common second difference

We can construct a well-known sequence with a common second difference (where the second difference is the difference between the differences of the terms) by starting with 1 as the first term, adding 3 to 1 to get the second term of 4, then continuing to add successive odd numbers to each term thereafter to generate the next term. The resultant sequence is $1, 4, 9, 16, 25, 36, \ldots$. Adding successive odd numbers to each term means that the increase that produced any kth

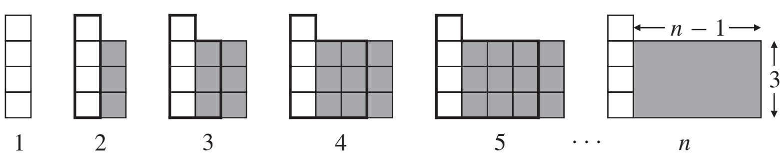


Figure 1 A geometric representation of the linear sequence $4, 7, 10, 13, 16, \ldots, 3n+1$.

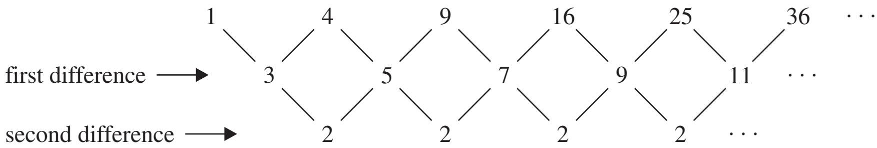


Figure 2 First and second differences for the sequence $1, 4, 9, 16, 25, 36, \ldots$.

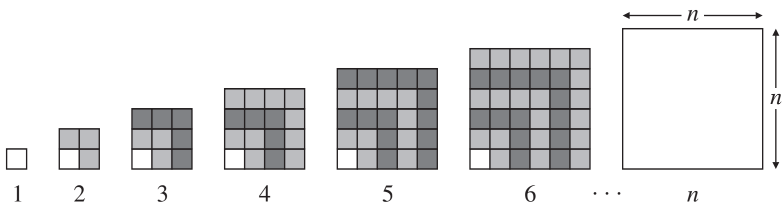


Figure 3 A geometric representation of $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

term is 2 more than the increase that produced the $(k - 1)$th term. Thus, 2 is the common second difference, as indicated in figure 2. This sequence is often depicted geometrically, as in figure 3, to illustrate that it is the perfect squares sequence.

Deriving an alternative expression for n^2 in terms of triangular numbers

In figure 4, we constructed this same sequence, i.e. we began with 1, added 3 to obtain a second term of 4, and then generated each consecutive term by growing by 2 the amount of increase from one term to the next. However, rather than adjoining the new unit squares for each term so as to build a larger square, we stacked all of them in the right-most column. Thus, the right-most column in any term is two units taller than the right-most column of the previous term. Furthermore, we highlighted and kept track of the components (1, 2s, 3s) in each term. By doing so, we observed the following about each term.

- $f(1) = 1$ appears once.
- $f(2) - f(1) = 3$ appears $n - 1$ times.

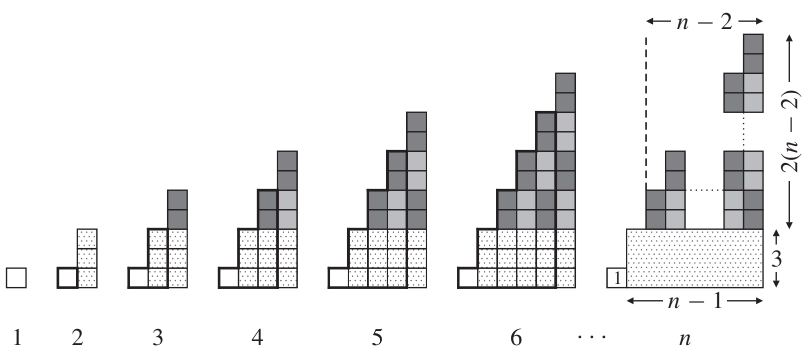
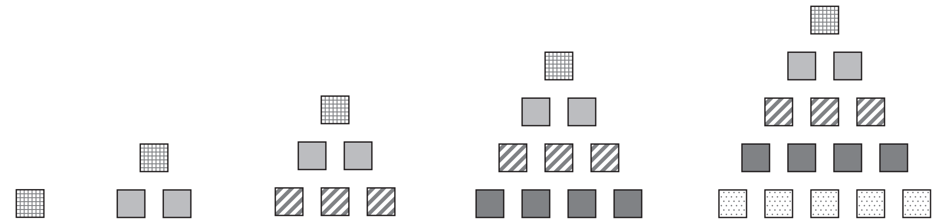


Figure 4 Tracking components in the terms of the sequence $1, 4, 9, 16, 25, 36, \ldots$.



$T_1 = 1$ $T_2 = 1 + 2 = 3$ $T_3 = 1 + 2 + 3 = 6$ $T_4 = 1 + 2 + 3 + 4 = 10$ $T_5 = 1 + 2 + 3 + 4 + 5 = 15$

Figure 5 A geometric representation of the first five triangular numbers.

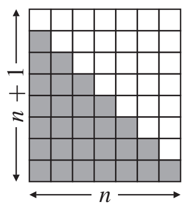


Figure 6 A rectangular arrangement of two copies of T_n for $n = 7$.

- The common second difference, $d_2 = 2$, appears $1+2+3+\cdots+(n-2)$ times. Thus, recalling that the nth triangular number, T_n, is $1 + 2 + 3 + 4 + \cdots + n$ (see figure 5), we can say that the common second difference of 2 appears T_{n-2} times in the nth term, for $n \geq 3$.

Using these observations, we obtain the following expression for $f(n)$ in terms of triangular numbers:

$$f(n) = 1 + 3(n-1) + 2\mathrm{T}_{n-2}. \tag{1}$$

Next, we use (1) and a formula for the nth triangular number to prove that the sequence we generated is indeed the sequence of square numbers. First, we derive the formula for the nth triangular number by arranging two copies of T_n so as to form an n by $(n+1)$ rectangle, as is commonly done (see figure 6). (Note that figure 6 also highlights how triangular numbers can be represented to resemble a staircase, or steps.) The area of the rectangle is $n(n+1)$; therefore, the area of the nth triangular number is half of that. Thus, $\mathrm{T}_n = n(n+1)/2$. By using this formula in (1), we show that $f(n) = n^2$ as follows:

$$\begin{aligned} f(n) &= 1 + 3(n-1) + 2\mathrm{T}_{n-2} \\ &= 1 + 3(n-1) + 2\frac{(n-2)(n-1)}{2} \\ &= 1 + 3n - 3 + n^2 - 3n + 2 \\ &= n^2. \end{aligned}$$

Deriving an explicit formula for any quadratic sequence

We can repeat the process we used to construct the n^2 sequence to derive a general formula for any sequence that has a common second difference. As illustrated in figure 7, we start with

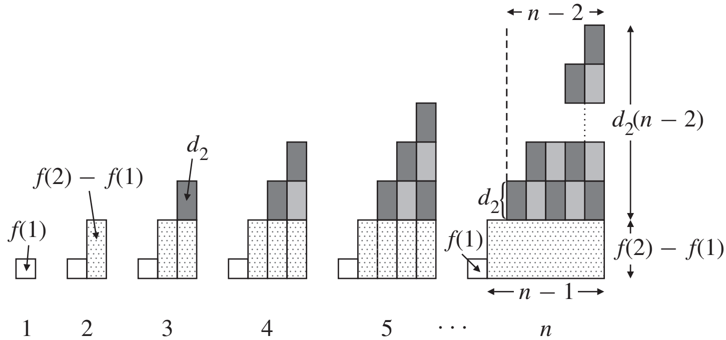


Figure 7 Tracking components in the terms of a sequence with a common second difference.

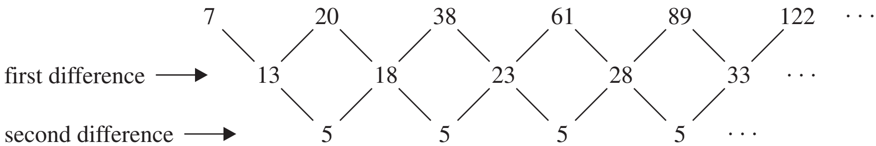


Figure 8 The first and second differences of the sequence 7, 20, 38, 61, 89, 122, ...

some number, $f(1)$, then generate the second term by adding some amount to $f(1)$. We can represent that increase as $f(2) - f(1)$. The third term, $f(3)$, is obtained by starting with $f(2)$ and adding the same increase as from $f(1)$ to $f(2)$ plus an additional amount, which can be represented by $(f(2) - f(1)) + d_2$, where d_2 is the common second difference. We construct each term thereafter by making the increase from one term to the next grow by the same amount, d_2. The resultant formula, which is quadratic in n, is

$$f(n) = f(1) + (f(2) - f(1))(n-1) + d_2\mathrm{T}_{n-2}. \tag{2}$$

We can use (2) to derive an explicit formula for any sequence that has a common second difference. For example, consider the sequence in which we start with 7, add 13, and then obtain each consecutive term by adding to each term a number that is 5 more than the previous amount we added. As shown in figure 8, this process will produce the sequence 7, 20, 38, 61, 89, 122, ... This sequence has a common second difference, so we can represent it with a quadratic expression. Substituting the values of $d_2 = 5$, $f(2) - f(1) = 13$, and $f(1) = 7$ into (2) gives

$$f(n) = 7 + 13(n-1) + 5\frac{(n-2)(n-1)}{2} = \frac{5}{2}n^2 + \frac{11}{2}n - 1.$$

Exploring sequences with a common third difference

We also derived an explicit formula for cubic sequences (i.e. those with a common third difference) by again keeping track of the components of each term as we constructed the sequences. For example, we began with $f(1) = 4$ and chose 3 as the first term of the first difference, which is represented by $f(2) - f(1)$. We then chose 2 as the first term of the

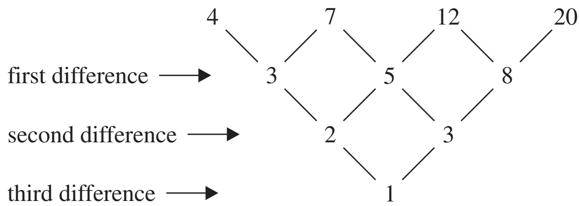


Figure 9 Generating the first four terms of a sequence with a common third difference.

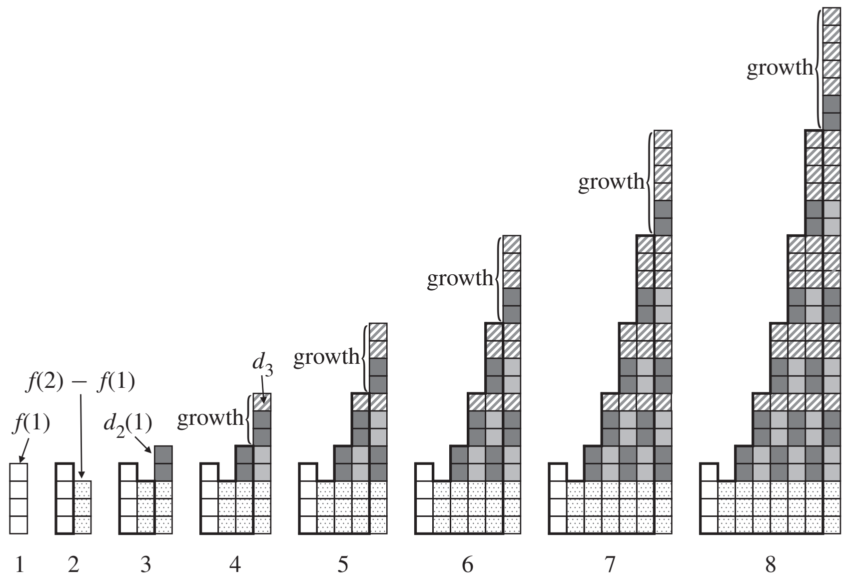


Figure 10 Tracking components in the terms of a sequence with a common third difference.

second difference, denoted by $d_2(1)$, and 1 as the common third difference, d_3. As shown in figures 9 and 10, we used these values to generate the first four terms of the sequence. We found additional terms, as represented geometrically in figure 10, using the fact that in a cubic sequence, the amount of increase from one term to the next exhibits quadratic growth. In other words, for a cubic sequence, the sequence of the first differences is a quadratic sequence. (Also, as we illustrate in the next paragraph, the sequence of the second differences is linear.)

Again, as shown in figure 10, we placed the squares representing the increase from the previous term in the right-most column of each term. Recall that in linear sequences the increase from one term to the next remains constant and that in quadratic sequences it grows by a constant amount. Now, for a cubic sequence, the growth of the increase is quadratic, i.e. the growth of the increase grows by a constant amount, namely d_3. This quadratic growth in the increase is illustrated in the portion of the right-most column in each term of figure 10 that we labelled as *growth*. Observe that from one term to the next, those growth portions increase by $d_3 = 1$. Also note that, if we look only at the diagonally-striped (d_3) squares in each growth portion, we see the linearity of the second differences sequence. That is, there are 1, then 2, then 3, then 4, …, then $n - 3$ diagonally-striped squares in the growth portions of terms 4 to n.

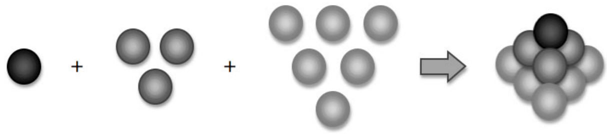


Figure 11 A geometric representation of $1+3+6=\mathrm{Te}_3$, the third tetrahedral number.

By analysing the patterns in figure 10, we note the following about each term.

- $f(1)=4$ appears once.
- $f(2)-f(1)=3$ appears $n-1$ times.
- $d_2(1)=2$ appears T_{n-2} times.
- $d_3=1$ appears $1+3+6+10+15+\cdots$ times. To see this, we can look at term 8 and find the total number of diagonally-striped squares in each column, from left to right, which are 1, 3, 6, 10, and 15 diagonally-striped squares. We can understand how this pattern builds by looking at the individual columns, from the bottom to the top, to see groups of 1, then 2 more, then 3 more, then 4 more, then 5 more diagonally-striped squares in each consecutive column. Thus, the number of diagonally-striped squares in the columns of one term, moving from left to right, are consecutive triangular numbers, starting with 1 and ending with $n-3$. This means that the total number of diagonally-striped squares in the nth term is the sum of the first $n-3$ triangular numbers. This sum is known as the $(n-3)$th tetrahedral number, Te_{n-3}, because a 3-dimensional representation of the sum of the triangular numbers is a tetrahedron (see figure 11).

These observations enable us to write the following expression for $f(n)$:

$$f(n)=4+3(n-1)+2\mathrm{T}_{n-2}+1(\mathrm{Te}_{n-3}). \tag{3}$$

Using a 2-dimensional geometric approach to derive the formula for Te_n

If we have a formula for the nth tetrahedral number, Te_n, we can use it in (3) to find an expression for $f(n)$. Others have developed a 3-dimensional geometric proof of the formula for Te_n. We created a 2-dimensional geometric proof by using the stacking technique that we used in figure 10, i.e. we isolated the diagonally-striped squares. In figure 12, we show the resultant 2-dimensional representation of the seventh tetrahedral number. We then arranged three copies of that representation into a rectangle with a base of $n+2$ and a height of $n(n+1)/2$, the nth triangular number (see figure 13). Note that the sum of $1+(1+2)+(1+2+3)+(1+2+3+4)+(1+2+3+4+5)+(1+2+3+4+5+6)+(1+2+3+4+5+6+7)$, which produces the seventh tetrahedral number, appears in each of the three types of shading, indicating that the rectangle is made up of three objects of equal area. The area of the entire rectangle is $n(n+1)(n+2)/2$; therefore, the area of one of the three objects – the nth tetrahedral number – is one-third of that. Thus, $\mathrm{Te}_n=n(n+1)(n+2)/6$.

We now return to (3) and use this formula for Te_n to derive an expression for $f(n)$:

$$f(n)=4+3(n-1)+2\frac{(n-2)(n-1)}{2}+\frac{(n-3)(n-2)(n-1)}{6}=\frac{1}{6}n^3+\frac{11}{6}n+2.$$

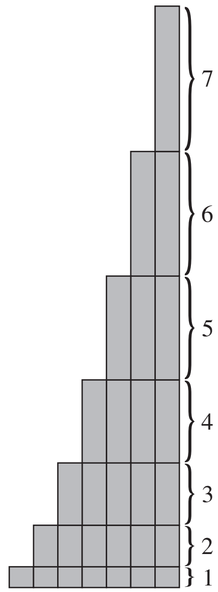


Figure 12 A 2-dimensional representation of Te_n for $n = 7$.

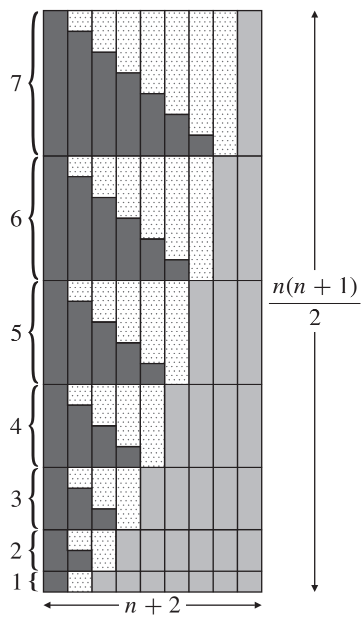


Figure 13 A rectangular arrangement of three copies of Te_n for $n = 7$.

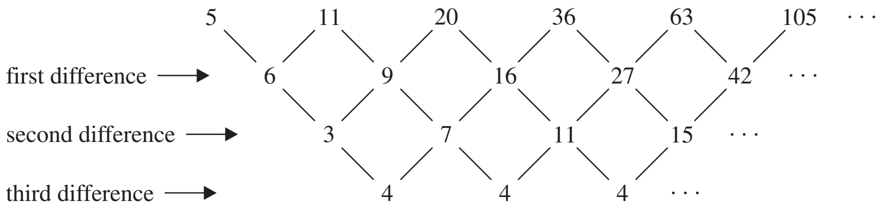


Figure 14 The first, second, and third differences for the sequence 5, 11, 20, 36, 63, 105, ...

Deriving an explicit formula for any cubic sequence

Now, to generalize, we refer to our previous list of observations regarding figure 10 and insert the variables $f(1)$, $f(2)-f(1)$, $d_2(1)$, and d_3 in place of the specific numbers in (3) to derive the following formula:

$$f(n) = f(1) + (f(2) - f(1))(n-1) + d_2(1)\mathrm{T}_{n-2} + d_3\mathrm{Te}_{n-3}. \tag{4}$$

Now we can use (4) to find an explicit formula for any sequence with a common third difference. As a final example to bring our explorations to a close, consider a sequence with a common third difference: 5, 11, 20, 36, 63, 105, ... As indicated in figure 14, $f(1) = 5$, $f(2) - f(1) = 6$, $d_2(1) = 3$, and $d_3 = 4$. By substituting these values into (4) and simplifying, we obtain the desired formula, i.e.

$$f(n) = 5 + 6(n-1) + 3\frac{(n-2)(n-1)}{2} + 4\frac{(n-3)(n-2)(n-1)}{6} = \frac{2}{3}n^3 - \frac{5}{2}n^2 + \frac{53}{6}n - 2.$$

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A magic square based on Mahatma Gandhi's date of birth (2 October 1869)

2	10	18	69
68	19	9	3
13	5	66	15
16	65	6	12

The magic sum for this square is 99. Is there a general method to obtain such magic squares?

University of Delhi **Subhash Garg and Vinod Tyagi**

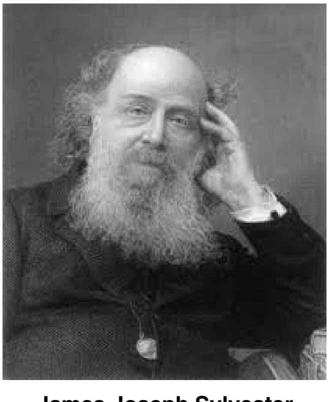
James Joseph Sylvester (1814–1897): The Bicentenary Anniversary of his Birth

SCOTT H. BROWN

James Joseph Sylvester was one of England's most notable mathematicians of the 19th century. He is best known for his significant contributions to the theory of algebraic invariants with Arthur Cayley. Sylvester also made advancements in the theory of partitions, number theory, and the theory of equations. He established the United States' 'first leading research-level programme' in mathematics at the Johns Hopkins University. Sylvester initiated the development of and became the first editor of the *American Journal of Mathematics*. He was a lawyer, actuary, and had a passion for poetry, music, and posing problems.

Sylvester was born in London on 3 September 1814. During his early education at the London Private Boarding School for young Jewish men, his mathematical abilities were quickly recognized by his teacher, who notified Olinthus Gregory the well-known mathematician. With Gregory's assistance and direction, Sylvester entered London University in 1828. There he received extensive instruction in advanced mathematics from Augustus De Morgan. However, after stabbing a student with a table-knife he had to leave the university. In February 1829 he entered the Royal Institution School in Liverpool. There he proved he was the top student in mathematics, when in February 1830 he won first prize in the subject. On account of his religion and intellectual abilities, the students did not like Sylvester and often picked fights with him. He would also question the abilities of some of his professors, which often resulted in his being caned. Sylvester could no longer stand the humiliation and left the school after a year and a half.

He continued his education at St. John's College, Cambridge, in 1831. However, he left the college at the end of 1833 due to health problems and would not return until January 1836. During this period he completed two mathematical papers. Both papers were under the



James Joseph Sylvester

authorship as 'Member of the University'. The first, entitled 'Collection of examples on the integral calculus', was published in 1835. The collection was a study guide which consisted of 135 integral problems and detailed solutions by Sylvester. The problems ranged from simple problems such as solving $\mathrm{d}u/\mathrm{d}x = ax^4$ to very difficult integrals such as evaluating

$$\int_x \frac{1}{(1-x)^2} \log x \, \mathrm{d}x.$$

His second paper, entitled 'A supplement of Newton's first section containing a rigid demonstration of the fifth lemma, and the general theory of the equality and proportions of linear magnitude', was published in 1836. In this paper, he first showed the shortfalls of the definitions of a straight line by Euclid and Legendre, followed by giving his 'correct' definition. Next, he discussed Newton's lack of justification for the fifth lemma and proceeded to give his proof, which he believed to be a 'logical foundation of a well-defined concept of a straight line' (see reference 12).

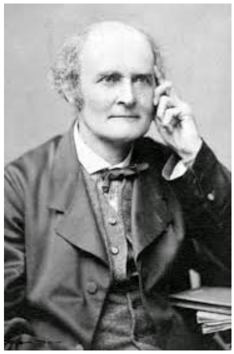
After returning to St. John's in 1836 he attended lectures in chemistry and mathematics. The next year he competed in the highly prestigious mathematics competition the 'Tripos' and achieved the rank of 'Second Wrangler'. Sylvester, as one of the top six wranglers, was now in an elite group of individuals, many of whom would become famous and have highly distinguished careers. However, due to his religious convictions he would not sign the Thirty Nine Articles of the Church of England, and was not allowed to obtain a degree from St. John's or participate in their competition for Smith's mathematical prizes.

In the autumn of 1837, Sylvester was appointed to the position of Professor of Natural Philosophy at University College London. Sylvester found the university quite different since his days as a student. He was now a colleague of some of his past Professors such as De Morgan. Prior to classes starting, he served as an examiner for the 'Flaherty Scholarship in Mathematics and Physics'. The courses he taught were in the field of Newtonian Mechanics. He soon found the university's physical laboratory was inadequate and he quickly became discontented with the teaching requirements. While at the university he was elected in April 1839 as a fellow of The Royal Society of London.

Sylvester did not enjoy the lecturing aspect of this position, so he spent as much time as possible pursuing his research. He published a paper in mathematical physics, entitled 'Analytical development of Fresnel's optical theory of crystals', based on Augustin Fresnel's work involving crystals that produce a double refraction. While Fresnel gave explanations of the double refraction based on a wave model, there was a need to simplify the mathematics involved. Sylvester's paper provided an original method of simplifying the mathematics, using both algebra and geometry.

He next wrote three significant papers on the theory of elimination, which were published in 1839 and 1840 in the *Philosophical Magazine*. In these papers, Sylvester first considered two homogenous polynomials in two variables and discussed the possible existence of a condition so that the two polynomials would have a common root. He then developed what would be called the 'dialytic method' for finding when the polynomials had a common root. This work illustrated the special insight that Sylvester would have as a pure algebraist.

At this point in his career, he decided to search for a teaching position in Pure Mathematics, but still be able to dedicate time towards his research. In 1841 he resigned from London University, and later in the year he received a BA and MA from Trinity College, Dublin. After he received his degrees, Sylvester accepted a position as Professor of Mathematics at the University of Virginia in November 1841. He was very impressed with the 'Jeffersonian' look



Arthur Cayley

of the university and the warm welcome he received from the administration. The courses he taught were arithmetic, algebra, geometry, and trigonometry, part of the standard curriculum in the United States. He immediately found several of the students to be rude and had serious discipline problems. After a fight with two students, which resulted in Sylvester injuring one of them, he found there was a lack of support from the administration in his favour. Sylvester found it advisable to leave his position in February 1842.

He travelled to New York City to visit his brothers. During his visit he met the well-known mathematician, Benjamin Peirce, Professor at Harvard. Sylvester would later establish the *American Journal of Mathematics*, and was Editor in Chief with the cooperation of Peirce in Mechanics. Through Peirce, he was introduced to the influential community of New York City. However, he searched for employment at Harvard, Columbia, and other institutions, which proved to be unsuccessful.

Sylvester returned to England in November 1843. During the next year he would re-establish himself in Britain's mathematics community by writing on such subjects as combinatorics. His paper 'Elementary researches in the analysis of combinatorial aggregation' published by the *Philosophical Magazine* provided an analysis and solutions to an arrangement problem. In 1844, he accepted a position with an actuarial company having the goal to qualify himself to practise 'conveyancing' or more simply drawing up deeds for transferring the title to property. He began his studies at the Inner Temple in 1846 to prepare for the Bar, which he passed in 1850 (see reference 12).

As a student, Sylvester met Arthur Cayley. Although their personalities and mathematical interests were different in many respects, the two would establish a life-long friendship. Prior to Sylvester and Cayley's collaboration on invariant theory, the concept had already been researched by such mathematicians as Gauss, Boole, and Hesse. Cayley had already published an article in 1845, which discussed calculating the algebraic relations among coefficients of higher degrees that met the conditions set forth by Boole. Cayley would come up with the term 'invariants' for such algebraic relations. By 1851, Sylvester and Cayley had set forth the mathematical language and fundamental concepts relating to the theory of algebraic invariants. Sylvester introduced this language and these concepts in the publication 'On the general theory of associated algebraic forms' that same year. Together they would go on to make significant contributions to this theory.

After balancing his actuarial duties and his mathematics research for a decade, he was ready to return to academics. He applied for the Professorship of Mathematics at the Royal Military

Academy (RMA) in Woolwich, but was denied. He applied again in the summer of 1855 and the RMA hired him for an examiner position and he was eventually appointed to a Professorship of Mathematics in the autumn. Over the next few years, Sylvester found the time apart from his duties of teaching and as an editor of *Quarterly Journal of Pure and Applied Mathematics* to conduct research. He continued his work on the theory of invariants, which led to his contribution to number theory.

In 1855, Cayley sent a letter to Sylvester explaining how he implemented the partitioning of numbers in his work on the theory of algebraic invariants. Sylvester had also been corresponding with Thomas Kirkman, relating to partitions in 1856. Inspired by their ideas, Sylvester produced two papers. His first paper was published in 1858 entitled 'Note on the equation in numbers of the first degree between any number of variables with positive coefficients', and provided an 'analytical solution of the problem, which was to count the solutions in positive integers of the indeterminate equation: $ax + by + cz + \cdots + ld = m$'.

In the same year, his paper 'On the problems of virgins, and the general theory of compound partition' examined Euler's application of the method of counting the number of solutions to a pair of indeterminate equations, and provided a partitioning method that could be generalized to systems of n such equations. Sylvester outlined his theory on compound partitioning in a series of seven lectures that he gave in 1859 at King's College London (see references 8 and 12).

During the early to mid 1860s, Sylvester was able to be fairly productive and achieved 'International Status' in mathematics when he was elected in 1863 to the French Academy of Sciences. The following year he published his paper 'On an elementary proof and generalization of Newton's hitherto undemonstrated rule for the discovery of imaginary roots'. In this paper he established the validity of Newton's rule for enumerating imaginary roots of algebraic equations. This is considered to be Sylvester's most significant contribution in mathematics.

Sylvester's productivity in mathematics began to decrease as he shifted his interest to educational reform at the Academy and in England. Then in September 1870 he was forced to retire from the Academy on half pay. He returned to London and continued his interest in poetry and published his book 'Laws of verse'. He was also fond of music and studied voice and sang publicly.

In 1876, his life would change in a positive way. Sylvester was appointed as the first Professor of Mathematics at the newly established Johns Hopkins University. This job was a great catalyst for rejuvenating Sylvester's interest in teaching, research, and as an editor of an international mathematics journal. He was afforded the opportunity to develop a first-rate graduate programme and had several gifted students, to include the notable mathematician George Halstead.

During his tenure at John Hopkins, Sylvester immersed himself in his earlier work on the theory of algebraic invariants. He also combined his teaching and research by involving his graduate students in his work on the theory of partitions. As a result, Sylvester and his students together published an article on this subject in the *American Journal of Mathematics*. This journal, which was initiated and established by Sylvester in 1878, was one of his major contributions to the international mathematics community.

Sylvester's interests in mathematics included Egyptian fractions. In 1880 he published the paper 'On a point in the theory of vulgar fractions', proving that any proper fraction of the form a/b could be written as the sum of distinct unit fractions. In his paper he named the unit fractions as fractional *sorites* and introduced an algorithm to find each of these *sorites*. For example, given $\frac{2}{9}$, the first step is to take the reciprocal of $\frac{2}{9}$, which is $\frac{9}{2}$. Then find the first integer greater than or equal to this fraction, which in this case is 5. Now, take the reciprocal of 5 and we have our first unit fraction, $\frac{1}{5}$. Next, perform the subtraction $\frac{2}{9} - \frac{1}{5} = \frac{1}{45}$. Since $\frac{1}{45}$ is a unit fraction, we have $\frac{2}{9} = \frac{1}{5} + \frac{1}{45}$.

Sylvester continued to make significant advances in the theory of algebraic invariants and the theory of equations and combinatorics while at Johns Hopkins University. However, the duress of teaching and supervising the leading research programmes in America began taking a toll on Sylvester. After seven years at the university, he looked for a change of pace and accepted an appointment as the Savilian Professor of Geometry at Oxford.

He soon found that he was not able to recreate at Oxford the success and enjoyment of teaching and research that he had had with his students at Johns Hopkins. Subsequently, he settled into his duties as Chair and the daily routine of teaching. While at Oxford, Sylvester was recognized by the London Mathematical Society for his contributions to mathematics. The society awarded him the De Morgan Medal in 1887.

By this time his health had deteriorated and in 1894 he retired from his position at Oxford. He then moved to London and continued his passion for conducting research in mathematics. During his final year he was working on compound partitions and the Goldbach–Euler conjecture. He passed away at the age of eighty-three on 15 March 1897. Sylvester remained a bachelor throughout his entire life. The Royal Society of London established a fund and a medal in honour of Sylvester's work in mathematics. The first Sylvester award of a medal and money was presented to Henri Poincaré in 1901 (see reference 8).

As a final note, Sylvester enjoyed proposing problems in the journal *The Educational Times*. Sylvester's first problem in *The Educational Times* appeared in 1863 and his final problem appeared in 1897. The following is an example of a problem proposed in *The Educational Times* (see reference 9).

> **Problem 3305** Find the chance that, if three points be taken at random inside a circle, any two of them shall be nearer to one another than the remaining one to the centre.

He also proposed problems in the United States journal *Mathematical Visitor*, which was published from 1877 to 1896. Most of his problems were rather difficult, as the example given demonstrates, so, in both journals, only a few solutions were received.

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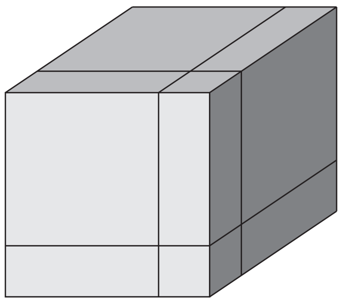
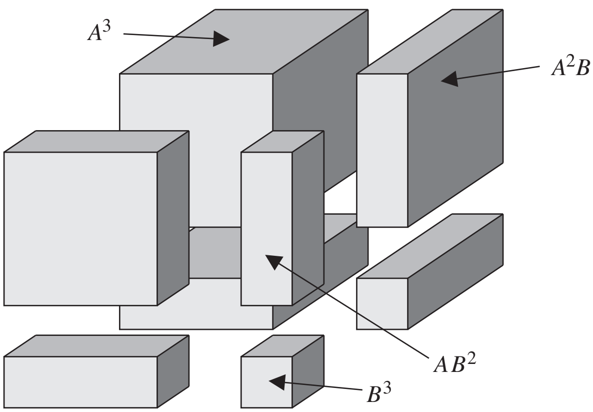
$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$



$(A + B)^3$



Delaware State University **Lyndon Barton**

The Staircase Problem and Fibonacci Numbers

SIMON ZASLAVSKY and ROGER KHAZAN

The *staircase problem* is a well-known folklore problem. It is often solved recursively, and is a standard example for presenting the Fibonacci numbers. The staircase problem can also be solved combinatorially, thereby yielding an explicit formula for the Fibonacci numbers.

The staircase problem

The staircase problem can be expressed as follows.

> Given a staircase with k steps, in how many different ways can you climb the staircase if you can take either one or two steps at a time?

Let $s(k)$ denote the number of different ways to climb k steps. We shall adopt the convention that a staircase with no steps can be climbed in one way; so $s(0) = 1$. A single step can be climbed uniquely, i.e. $s(1) = 1$. Two steps can be climbed in two ways: one way is to take two single steps, the other is to take one double step. Hence, $s(2) = 2$. Figure 1 illustrates all five ways that you can climb a staircase that has four steps.

Recursive solution The standard solution to the staircase problem is recursive. The very last step that reaches the kth step can be made either by taking a single step from the $(k-1)$th step or a double step from the $(k-2)$th step. Thus, the number of ways to climb k steps is equal to the number of ways to climb $k-1$ steps followed by a single step plus the number of ways to climb $k-2$ steps followed by a double step. In other words, the formula for $s(k)$ can be written recursively as follows:

$$s(k) = \begin{cases} 1 & \text{for } k = 0 \text{ and } k = 1, \\ s(k-1) + s(k-2) & \text{for } k > 1. \end{cases}$$

The *Fibonacci sequence* $f(n) = 1, 1, 2, 3, 5, 8, \ldots$ is the sequence starting with $1, 1$ in which every succeeding term is equal to the sum of the two preceding terms, i.e.

$$f_n = \begin{cases} 1 & \text{for } n = 1 \text{ and } n = 2, \\ f_{n-1} + f_{n-2} & \text{for } n \geq 3. \end{cases}$$

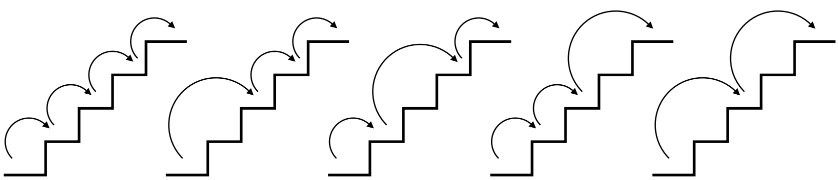


Figure 1 All five different ways to climb a four-step staircase using single and double steps.

Hence, using the recursive solution to the staircase problem, we get

$$f_n = s(n-1). \tag{1}$$

Combinatorial solution The staircase problem can also be solved combinatorially as the problem of arranging single and double steps on a k-step staircase.

Denote by $r(k, m)$ the number of different ways to climb k steps so that the climb includes exactly m double steps. Thus, $m \leq \lfloor k/2 \rfloor$, the integer part of $k/2$. The total number, $s(k)$, of ways you can climb the staircase can then be expressed as the following sum:

$$s(k) = \sum_{m=0}^{\lfloor k/2 \rfloor} r(k, m).$$

That is, the number of ways to climb a staircase is the number of ways to climb with no double steps, plus the number of ways to climb with one double step, plus the number of ways to climb with two double steps, etc.

But what is $r(k, m)$? When a climb of a k-step staircase includes m double steps, the total number of steps taken is $k - m$. Hence, $r(k, m)$ is the binomial coefficient $\binom{k-m}{m}$, and the staircase problem has the following solution:

$$s(k) = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k-m}{m}. \tag{2}$$

By combining (1) and (2), we derive the following explicit formula for the nth Fibonacci number:

$$f_n = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-m-1}{m}, \quad \text{for } n \geq 1.$$

This formula goes back to Edouard Lucas in 1876 (see reference 1, exercise 23, p. 277). To illustrate, consider the case of $n = 6$:

$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8,$$

which is indeed the sixth Fibonacci number.

Reference

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Simon Zaslavsky *(1948–2013) A man of great integrity, insatiable curiosity, love of learning, and an enviable ability to stick with an interesting puzzle until he solved it, whether it took a few minutes or a few weeks. Simon's virtues were many.*

Roger Khazan *A man who strives to have Simon's biography.*

A Mathematical Meander from Zeckendorf to Trigonometry

MARTIN GRIFFITHS, CHRISTOPHER BROWN
and HENRY SEATON

In this article we take a mathematical stroll that leads us from number-theoretic to trigonometric to analytic ideas. During the course of our journey, we encounter Zeckendorf representations, the inverse cosine function, and even a Fibonacci-related infinite series for π.

1. Introduction

We take for granted the decimal representation of positive integers since we are so familiar with it. Many will also be well acquainted with the method of representing the positive integers in binary. It is not quite so well known, however, that for each positive integer n there also exists, subject to two restrictions, a unique way of representing it as a sum of distinct Fibonacci numbers. This is known as the *Zeckendorf representation* of n, and will be discussed in section 2.

This article describes the fruits of a mathematical meander undertaken jointly by a teacher and two high-school students. It came about after we had carried out some simple explorations involving the Zeckendorf representations of the terms of a particular sequence of numbers. Our findings led to a Fibonacci identity, which in turn lent itself nicely to the derivation of a relation linking certain expressions in the Fibonacci numbers by way of the inverse cosine function. Finally, we used these ideas to derive a Fibonacci-related family of infinite series representations for π. This is a good example of how mathematical exploration can lead to the most unexpected places.

2. Zeckendorf representations

Zeckendorf's theorem, named after the Belgian Edouard Zeckendorf, is a result concerning the possibility of writing positive integers as a sum of distinct Fibonacci numbers. Despite the simplicity of this theorem, it was not published until 1972 (see reference 1), although it is thought that Zeckendorf had actually proved it as early as 1939. It is interesting to note that Zeckendorf was a medical doctor who dabbled in mathematics in his spare time.

The nth *Fibonacci number* F_n may be defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, where $F_1 = F_2 = 1$ (see references 2 and 3). Zeckendorf's theorem states that every $n \in \mathbb{N}$ can be represented in a unique way as the sum of one or more distinct Fibonacci numbers, excluding F_1, in such a way that the sum does not include any two consecutive Fibonacci numbers (see reference 2 and http://en.wikipedia.org/wiki/Zeckendorf's_theorem). Without any restrictions, the representations are not necessarily unique. For example,

$$F_4 + F_5 + F_8 = 3 + 5 + 21 = 29 = 8 + 21 = F_6 + F_8.$$

The representation on the left-hand side is not the Zeckendorf representation of 29 since it contains consecutive Fibonacci numbers. However, there are no consecutive Fibonacci numbers on the right-hand side, so this is the Zeckendorf representation of 29.

We outline here a proof of Zeckendorf's theorem. First, it is shown by induction that every positive integer k does indeed have a Zeckendorf representation. By way of initialisation, $1 = F_2$, so let us now assume that the statement is true for all $k \leq m$ for some $m \geq 1$. If $m+1$ happens to be a Fibonacci number, then we are finished. Otherwise, there exists some $i \in \mathbb{N}$ such that $F_i < m+1 < F_{i+1}$, in which case the integer $j = m+1-F_i$ has a Zeckendorf representation by the inductive hypothesis. Since $j + F_i = m+1 < F_{i+1}$, it follows that $j < F_{i-1}$, which in turn implies that F_{i-1} does not appear in the Zeckendorf representation of j. From this we see that $m+1$ has a Zeckendorf representation comprising that of j with the additional term F_i, as required. Second, the uniqueness of the representation follows from a fairly well-known result which states that the Zeckendorf representation of any integer j necessarily includes the largest Fibonacci number not exceeding j (see reference 2 and http://en.wikipedia.org/wiki/Zeckendorf's_theorem).

In this article we shall also have cause to use a result known as *Binet's formula* (see references 2 and 4) for the nth Fibonacci number. This is given by

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n), \tag{1}$$

where $\phi = (1+\sqrt{5})/2$ is known as the *golden ratio* and $\hat{\phi} = -1/\phi$. This will be utilised to prove two Fibonacci identities, which in turn allow us to derive some of the results given here.

3. Initial explorations

The Zeckendorf representations of some sequences of positive integers follow easily-defined rules whilst those of others would appear not to possess any discernable patterns. To keep things relatively simple, we will only be considering here sequences of positive integers whose terms are functions of Fibonacci numbers. The trivial case is the sequence of Fibonacci numbers itself, starting at F_2 and denoted by $(F_n)_{n\geq 2}$, as the numbers in this sequence are their own Zeckendorf representations. Next, we might consider $(2F_n)_{n\geq 2}$. As is easily checked, we have $2F_2 = F_3$, $2F_3 = F_2 + F_4$, and $2F_n = F_{n-2} + F_{n+1}$ for $n \geq 4$, which follows from $F_{n+1} - F_n = F_{n-1} = F_n - F_{n-2}$. The sequence we study here is $(F_n^2)_{n\geq 2}$, which is a little more interesting.

Some initial calculations showed that the Zeckendorf representations of $(F_n^2)_{n\geq 2}$ had slightly different forms, depending on whether n was odd or even. In fact, it appeared that

$$F_{2n}^2 = F_2 + F_6 + F_{10} + \cdots + F_{4n-2} = \sum_{k=1}^{n} F_{4k-2}, \tag{2}$$

$$F_{2n+1}^2 = F_2 + F_4 + F_8 + \cdots + F_{4n} = F_2 + \sum_{k=1}^{n} F_{4k}. \tag{3}$$

In order to show that these results are true, we looked for a relation involving the squares of the Fibonacci numbers and the even-numbered Fibonacci numbers. It did not take long to discover that

$$F_n^2 = F_{2(n-1)} + F_{n-2}^2, \tag{4}$$

which, as is now demonstrated, may be proved using Binet's formula (1).

First,

$$\begin{aligned} F_n + F_{n-2} &= \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n) + \frac{1}{\sqrt{5}}(\phi^{n-2} - \hat{\phi}^{n-2}) \\ &= \frac{1}{\sqrt{5}}(\phi^{n-1}(\phi - \hat{\phi}) + \hat{\phi}^{n-1}(\phi - \hat{\phi})) \\ &= \phi^{n-1} + \hat{\phi}^{n-1}. \end{aligned}$$

We may use this result and Binet's formula once more to obtain

$$\begin{aligned} F_{n-1}(F_n + F_{n-2}) &= \frac{1}{\sqrt{5}}(\phi^{n-1} - \hat{\phi}^{n-1})(\phi^{n-1} + \hat{\phi}^{n-1}) \\ &= \frac{1}{\sqrt{5}}(\phi^{2(n-1)} - \hat{\phi}^{2(n-1)}) \\ &= F_{2(n-1)}. \end{aligned}$$

It then follows that,

$$F_{2(n-1)} = (F_n - F_{n-2})(F_n + F_{n-2}) = F_n^2 - F_{n-2}^2,$$

which may be rearranged to give (4). This result does indeed imply, via induction, the truth of (2) and (3).

Taking things a little further, we have, from (2) and (3),

$$F_{2n}^2 + F_{2n+1}^2 = 2F_2 + F_4 + F_6 + \cdots + F_{4n} = F_2 + \sum_{k=1}^{2n} F_{2k};$$

hence,

$$F_{2n}^2 + F_{2n+1}^2 - 1 = \sum_{k=1}^{2n} F_{2k}.$$

For ease of notation, we set

$$S_n = \sum_{k=1}^{2n} F_{2k},$$

giving us the identity

$$F_{2n}^2 + F_{2n+1}^2 - 1 = S_n. \tag{5}$$

We also explored the possibility of finding similar patterns in the Zeckendorf representions of the terms of the sequences $(F_n^3)_{n\geq 2}$ and $(F_n^4)_{n\geq 2}$, but, interestingly enough, none could be found.

4. Some trigonometry

One of us could not help noticing that if θ_1 is such that

$$2F_{2n}F_{2n+1}\cos\theta_1 = 1,$$

then (5) is a statement of the cosine rule for a triangle with side lengths F_{2n}, F_{2n+1}, and $\sqrt{S_n}$, where θ_1 is the angle between the first two sides. For this to be the case, we require

$$\theta_1 = \cos^{-1}\left(\frac{1}{2F_{2n}F_{2n+1}}\right). \tag{6}$$

Next, let θ_2 and θ_3 be the angle between the sides of length F_{2n} and $\sqrt{S_n}$, and F_{2n+1} and $\sqrt{S_n}$, respectively. By the sine rule we have

$$\frac{\sin\theta_2}{F_{2n+1}} = \frac{\sin\theta_1}{\sqrt{S_n}}.$$

After some rearrangement, using the result $\sin(\cos^{-1} x) = \sqrt{1-x^2}$ (see http://en.wikipedia.org/wiki/Inverse_trigonometric_functions) along with (6), this leads to the result

$$\sin\theta_2 = \frac{1}{\sqrt{S_n}}\sqrt{F_{2n+1}^2 - \frac{1}{4F_{2n}^2}}.$$

However,

$$\begin{aligned}
F_{4n+1} - S_n &= F_{4n+1} - (F_{4n} + F_{4n-2} + \cdots + F_2) \\
&= F_{4n-1} - (F_{4n-2} + F_{4n-4} + \cdots + F_2) \\
&\ \ \vdots \\
&= F_3 - F_2 \\
&= 1,
\end{aligned}$$

so that $S_n = F_{4n+1} - 1$. Hence,

$$\begin{aligned}
\theta_2 &= \sin^{-1}\left(\frac{1}{2F_{2n}}\sqrt{\frac{4F_{2n}^2F_{2n+1}^2 - 1}{F_{4n+1}-1}}\right) \\
&= \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2F_{2n}}\sqrt{\frac{4F_{2n}^2F_{2n+1}^2 - 1}{F_{4n+1}-1}}\right).
\end{aligned} \tag{7}$$

Similarly, we may obtain

$$\theta_3 = \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{2F_{2n+1}}\sqrt{\frac{4F_{2n}^2F_{2n+1}^2 - 1}{F_{4n+1}-1}}\right). \tag{8}$$

Then, as $\theta_1 + \theta_2 + \theta_3 = \pi$, it is the case that (6), (7), and (8) result in the following relation:

$$\cos^{-1}\left(\frac{1}{2F_{2n}F_{2n+1}}\right) = \cos^{-1}\left(\frac{f(n)}{2F_{2n}}\right) + \cos^{-1}\left(\frac{f(n)}{2F_{2n+1}}\right),$$

where

$$f(n) = \sqrt{\frac{4F_{2n}^2F_{2n+1}^2 - 1}{F_{4n+1}-1}}.$$

This identity does not appear on the authoritative Fibonacci website (see reference 4) or on any other lists of Fibonacci-related results we could find.

A slightly easier, though related, result to prove is

$$\cos^{-1}\left(\frac{F_{2n}}{\sqrt{F_{4n+1}}}\right) + \cos^{-1}\left(\frac{F_{2n+1}}{\sqrt{F_{4n+1}}}\right) = \frac{\pi}{2}, \tag{9}$$

which follows from the identity $F_{2n}^2 + F_{2n+1}^2 = F_{4n+1}$, a result that can be proved using Binet's formula (1). This allows us to take things a little further in that we are able to use it to obtain an infinite family of infinite series for π involving the Fibonacci numbers. This may be achieved by using the power series for the inverse cosine function (see http://en.wikipedia.org/wiki/Inverse_trigonometric_functions) given by

$$\begin{aligned}\cos^{-1} x &= \frac{\pi}{2} - \sin^{-1} x \\ &= \frac{\pi}{2} - x - \left(\frac{1}{2}\right)\frac{x^3}{3} - \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{x^5}{5} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{x^7}{7} - \cdots \\ &= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{4^k(2k+1)},\end{aligned}$$

which, in conjunction with (9), leads to

$$\pi = \frac{2}{\sqrt{F_{4n+1}}} \sum_{k=0}^{\infty} \frac{\binom{2k}{k}[F_{2n}^{2k+1} + F_{2n+1}^{2k+1}]}{(4F_{4n+1})^k(2k+1)},$$

noting that this is valid for any fixed $n \in \mathbb{N}$.

Incidentally, since $2/\sqrt{F_{4n+1}}$ is algebraic, and the product of any two algebraic numbers is also algebraic, the fact that π is transcendental implies that the sum of the above infinite series is also transcendental.

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The Pappus Chain Theorem and Its Applications to Geometry

SEUNG HEON LEE

Professor Harold P. Boas, of Texas A&M University, USA, suggested, in the article 'Reflections on the arbelos', a Challenge problem related to arbelos. In solving the Challenge problem, I have found interesting features of the method of circle inversion and the Pappus chain theorem. This article solves the Challenge problem and studies the Pappus chain theorem and its applications to geometry.

1. Introduction

We begin with the Challenge problem posed by Boas in reference 1.

Challenge problem As in figure 1, let a large semicircle embrace two smaller semicircles, with their diameters being collinear. Draw a line segment vertical to AB upward starting from the point C, assuming that AB is horizontal. A circle is simultaneously tangent to the large semicircle, the right-hand small semicircle, and the vertical line. Draw a slant line that passes through the centre of the inscribed circle and the tangent point between the circle and the right-hand small semicircle (F). Let E be the point where the slant line meets the vertical line. Prove that $\overline{AC} = \overline{EF}$.

In section 2, we present a brief history of arbelos, the *method of circle inversion* (MOCI), and the Pappus chain theorem and its applications, as preliminaries. In section 3 we give two different proofs for the Challenge problem: one utilizing the MOCI and the other based on conventional algebraic and geometric methods. In section 4, interesting extra properties related to the Challenge problem are discussed.

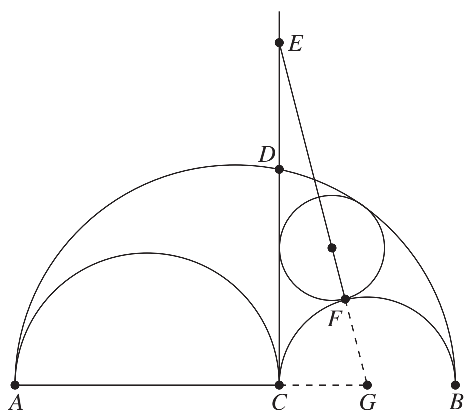

Figure 1 A Challenge problem suggested in reference 1: prove that $\overline{AC} = \overline{EF}$.

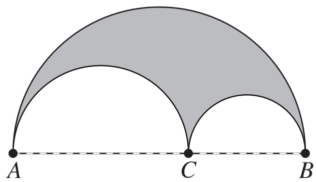

Figure 2 The area bounded by three semicircles is called an arbelos.

2. Preliminaries

This section presents a brief review of arbelos, the method of circle inversion, and the Pappus chain theorem.

2.1. Arbelos

The arbelos is a classical geometric shape bounded by three mutually tangential semicircles with collinear diameters, as shown in figure 2. In Archimedes' *Book of Lemmas* (c. 250), he introduced a figure that, due to its shape, has historically been known as the *shoemaker's knife* or *arbelos*. If, in a given semicircle with radius R and diameter $\overline{AB}$, two semicircles with radii r_1 and r_2, where $r_1 \neq r_2$ and $r_1 + r_2 = R$, are constructed so that they meet at point C on the line segment AB, then the region bounded by the three circumferences is called an *arbelos*. Archimedes was fascinated by the mathematical properties of the arbelos.

2.2. The method of circle inversion (MOCI)

Let the inverting circle (the circle of inversion) be centred at O and have radius γ. Let P be a point. Then its inversion P' is defined such that

- $\overline{OP} \cdot \overline{OP'} = \gamma^2$, and
- P and P' are on the same half line from O.

It is clear to see that every point on the inverting circle is inverted onto itself, which defines the set of fixed points. Also, if P' is the inversion of P, then P is the inversion of P' so that, if you invert twice, you get back to where you were before.

Circle inversion is sometimes called *reflection in a circle*. The MOCI is a generalization of 'finding reciprocals' of nonnegative real numbers. Given a real number $p \geq 0$, let p' denote its reciprocal. Then we have

$$p \cdot p' = 1^2 = 1.$$

Thus, $p' = 1/p$, with a convention that the reciprocal of 0 is ∞. The reciprocal of 1 becomes 1 itself, the multiplicative identity, which is the only fixed point of the reciprocal operation for nonnegative real numbers. On the other hand, the inverting circle itself is the set of fixed points for circle inversion.

Here we collect properties (see references 2 and 3) that are useful to prove the Pappus chain theorem and the Challenge problem as well.

P_1. A line not passing through the centre of the inverting circle inverts to a circle going through the centre of the inverting circle, and conversely.

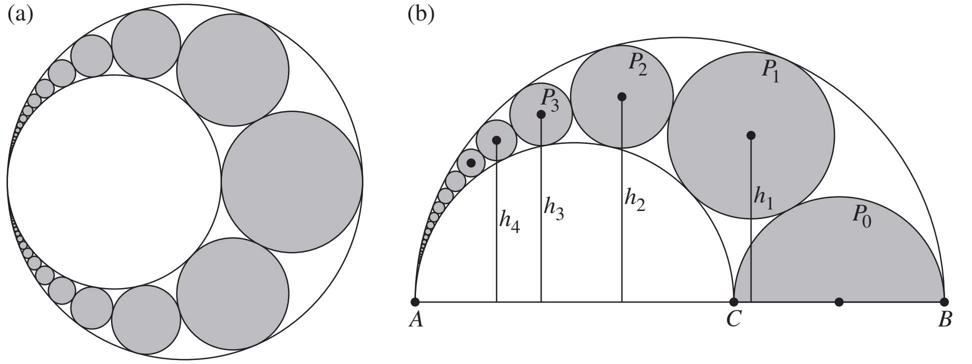

Figure 3 (a) A Pappus chain and (b) its upper half.

P_2. A line passing through the centre of the inverting circle inverts to itself. Points outside the inverting circle invert inside, and points inside the inverting circle invert outside.

P_3. A circle not going through the centre of the inverting circle will invert to a circle not passing through the centre of the inverting circle.

P_4. The angle at which two oriented curves intersect has the same magnitude as the angle at which the inverted curves intersect, but the opposite sense (see reference 1).

2.3. The Pappus chain theorem

The Pappus chain was introduced by Pappus of Alexandria in the third century AD, as shown in figure 3(a). We end the section with the Pappus chain theorem and its proof, which is useful for solving our Challenge problem.

Pappus chain theorem *See figure 3(b). The height, h_n, of the centre of the nth inscribed circle, P_n, above the line segment AB is equal to n times the diameter of P_n. That is,*

$$h_n = n \cdot d_n, \tag{1}$$

where d_n is the diameter of P_n.

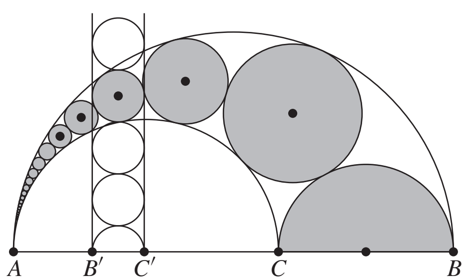

Figure 4 A Pappus chain and its transformed circles.

Proof We will present a proof that utilizes the MOCI (see http://en.wikipedia.org/wiki/Pappus_chain). Choose the inverting circle to be centred at the tangent point A and intersect the nth circle P_n perpendicularly, so that P_n inverts to itself. The two arbelos circles, $\widetilde{AB}$ and $\widetilde{AC}$, are transformed into two parallel vertical lines passing B' and C' that are tangent to and sandwiching the nth circle. Hence, the other circles of the Pappus chain are transformed similarly into sandwiched circles of the same diameter, as shown in figure 4. The initial circle P_0 and the final circle P_n each contribute $d_n/2$ to the height h_n, whereas the circles $P_1 - P_{n-1}$ each contribute d_n. Adding these contributions together yields (1).

3. Solutions of the Challenge problem

This section solves the Challenge problem in two different ways: one with the method of circle inversion and the other using a traditional algebraic/geometric method.

3.1. Method of circle inversion

In figure 5(a), the point G must be the centre of the small semicircle on the right-hand side. Thus, proving $\overline{AC} = \overline{EF}$ is equivalent to proving

$$\overline{AG} = \overline{EG}.$$

Let us put in a line segment AF. Then ΔAGF is congruent to ΔEGC, provided that $\angle AFG$ is a right angle. Thus, the problem is reduced to proving that

$$\angle AFG = 90°. \tag{2}$$

In order to prove (2), we first use the MOCI in this subsection. As in the proof of the Pappus chain theorem, choose an inverting circle that is centred at A and intersects the inscribed circle perpendicularly. So, the inscribed circle inverts to itself. Note that the line segment AB inverts to itself, by property P_2. The semicircle $\widetilde{ADB}$ passes through the centre of the inverting circle and intersects the segment AB perpendicularly. Thus, by properties P_1 and P_4, the semicircle $\widetilde{ADB}$ must invert to a vertical line. Furthermore, since it is tangent to the inscribed circle, its inversion much be tangent to the inscribed circle. This implies that the semicircle $\widetilde{ADB}$ inverts to the vertical line CD. It also implies that the point B inverts to C, the point C inverts to B, and therefore the semicircle $\widetilde{CFB}$ inverts to itself. Let H be the lower of the

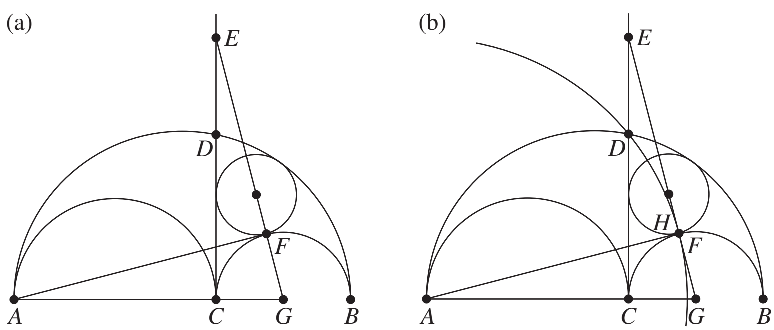

Figure 5 Proof by the method of circle inversion.

points where the inverting circle intersects the inscribed circle, which is a fixed point for the inversion. However, since $\widetilde{CFB}$ inverts to itself, the intersection of the inscribed circle and the right small semicircle, F, must also be a fixed point. This implies that $F = H$ and therefore AF is perpendicular to EG. This proves (2).

3.2. Traditional algebraic/geometric method

For simplicity, but without loss of generality, we assume that

$$A(0,0), \quad B(1,0), \quad C(p,0), \qquad 0 < p < 1.$$

Let the inscribed circle have centre $K(a,b)$ and radius r. Then it can be expressed as

$$(x-a)^2 + (y-b) = r^2. \tag{3}$$

In this subsection, we will try to determine the quantities a, b, and r as functions of p.

Let $(x, y) = (p, b)$, the point on the inscribed circle intersecting the vertical line. Then (3) reads as $(p-a)^2 = r^2$ and, therefore, (because $a > p$) we have

$$a = p + r. \tag{4}$$

Let L be the vertical projection of the centre $K(a,b)$ onto AB (see figure 6). Then ΔKLG is a right-angled triangle, so the sides satisfy Pythagoras' theorem, i.e.

$$\left(\frac{1+p}{2} - a\right)^2 + b^2 = \left(\frac{1-p}{2} + r\right)^2,$$

which can be simplified to

$$b^2 = 2r(1-p). \tag{5}$$

Let M denote the centre of the outer semicircle, i.e. $M(\frac{1}{2}, 0)$. Then

$$\overline{MK} = \sqrt{(a-\tfrac{1}{2})^2 + b^2};$$

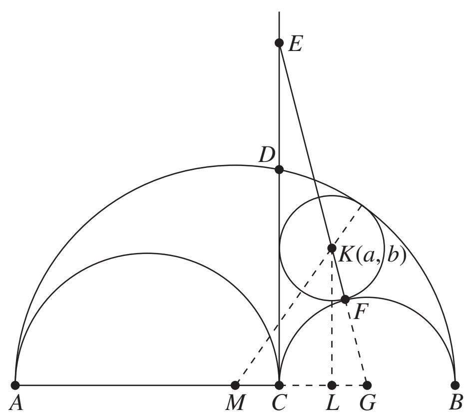

Figure 6 Proof by the traditional algebraic/geometric method.

therefore,

$$\sqrt{(a-\tfrac{1}{2})^2+b^2}+r=\tfrac{1}{2},$$

from which we obtain

$$b^2=(a-r)(1-a-r)=p(1-a-r). \tag{6}$$

So far, we have found three equations: (4), (5), and (6), in order to determine the three unknowns (a, b, r). Equations (5) and (6) imply

$$2r(1-p)=p(1-a-r).$$

Utilizing (4), the above equation can be simplified to

$$r=\frac{p(1-p)}{2}. \tag{7}$$

A combination of (4) and (7) gives

$$a=\frac{p(3-p)}{2}.$$

Finally, it follows from (5) and (7) that

$$b=(1-p)\sqrt{p}.$$

Using the fact that the line EF passes through $K(a, b)$ and $G((1+p)/2, 0)$, we can find the coordinates of E, which are $(p, \sqrt{p})$. Since

$$(a-p)^2+(b-\sqrt{p})^2=\frac{p^2(1+p)^2}{4},$$

we obtain

$$\sqrt{(a-p)^2+(b-\sqrt{p})^2}+r=\frac{p(1+p)}{2}+\frac{p(1-p)}{2}=p,$$

which proves that $\overline{AC}=\overline{EF}$.

4. Extra properties

4.1. Radii and curvatures

In the previous section, we have found the radius of the inscribed circle r, as a function of p; see (7). Let R_l and R_r be the radii of the left-hand small and the right-hand small semicircles, respectively. Then we have

$$R_\mathrm{l}=\frac{p}{2}, \qquad R_\mathrm{r}=\frac{1-p}{2}.$$

The ratios of radii read

$$\frac{r}{R_\mathrm{l}}=\frac{p(1-p)/2}{p/2}=1-p, \qquad \frac{r}{R_\mathrm{r}}=\frac{p(1-p)/2}{(1-p)/2}=p.$$

From the above equations, we can see that

- the ratio r/R_l is decreasing linearly from 1 to 0, as p grows from 0 to 1, while the ratio r/R_l is increasing linearly from 0 to 1, and
- the sum of the ratios is always 1, i.e. for all $0 < p < 1$,

$$\frac{r}{R_\mathrm{l}} + \frac{r}{R_\mathrm{r}} = 1,$$

or, equivalently,

$$\frac{1}{R_\mathrm{l}} + \frac{1}{R_\mathrm{r}} = \frac{1}{r}.$$

Thus, the sum of the curvatures of two small semicircles is the same as the curvature of the inscribed circle.

4.2. Traces of points

At some point in this project, I wondered if the point E could either be inside or touch the big semicircle in particular when p is small. However, I figured out that it never would happen when $p \neq 0$. To prove it mathematically, consider the equation of the big semicircle

$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}, \qquad y \geq 0. \tag{8}$$

Recall that the coordinates of E are $(p, \sqrt{p})$. Thus, for $(x, y) = (p, \sqrt{p})$, the left-hand side of (8) reads

$$(p - \tfrac{1}{2})^2 + (\sqrt{p})^2 = p^2 + \tfrac{1}{4},$$

which is larger than the right-hand side of (8) when $p \neq 0$. This proves that the point E always resides outside the big semicircle.

As p varies, $E(p, \sqrt{p})$ lies on the parabola with equation

$$y = \sqrt{x}.$$

Since $E(p, \sqrt{p})$, $G((1+p)/2, 0)$, $\overline{EF} = p$, and $\overline{FG} = (1-p)/2$, we have

$$F = \frac{[(1-p)/2] \cdot E + p \cdot G}{[(1-p)/2] + p},$$

which can be simplified to

$$F = \left(\frac{2p}{1+p}, \frac{1-p}{1+p}\sqrt{p} \right).$$

The elimination of the parameter p from F gives that F lies on the curve with equation

$$y = (1-x)\sqrt{\frac{x}{2-x}}.$$

Figure 7 depicts traces of E, F, and K, for which I have implemented a small script in MAPLE®. As can be seen from figure 7, the trace of E is always outside of the big semicircle. It is interesting to find that the trace of F looks like a typical trajectory of top-spin tennis balls. This visual observation has motivated me to consider an exciting project on the trajectory of tennis balls, which requires both mathematical modelling and numerical simulation.

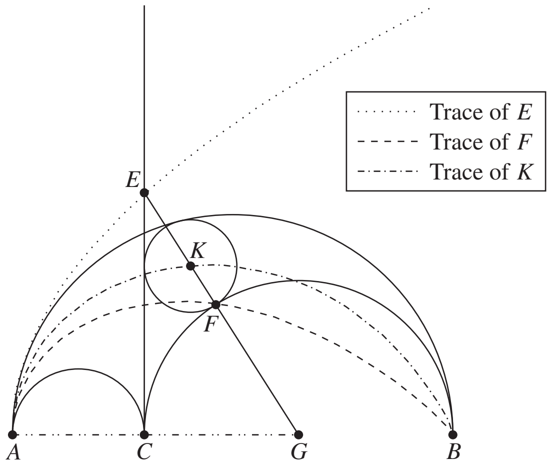

Figure 7 Traces of the points E, F, and K, where K is the centre of the circle.

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***Seung Heon Lee** was born in Seoul, South Korea, in 1998 and has been educated in the USA since September 2009. He is currently a 10th grader at Starkville Academy, Starkville, MS. His interests include solving challenging problems in mathematics and the development of App games for mobile phones. Seung Heon deeply appreciates the help and encouragement of Dr Seongjai Kim, Professor of Mathematics at Mississippi State University, during this project. Seung Heon also expresses sincere thanks to the Editor of Mathematical Spectrum; the Editor's suggestions improved the presentation of this article.*

$a^2 + b^2 + c^2 = d^2$

There are infinitely many Pythagorean triples (a, b, c) of positive integers such that $a^2+b^2 = c^2$. There are also infinitely many examples of positive integers (a, b, c, d) such that $a^2 + b^2 + c^2 = d^2$. In fact, for every positive integer n,

$$n^2 + (n+1)^2 + (n(n+1))^2 = (n(n+1)+1)^2.$$

For example, $2^2 + 3^2 + 6^2 = 7^2$, $3^2 + 4^2 + 12^2 = 13^2$, and $4^2 + 5^2 + 20^2 = 21^2$.

Kolkata, India **Bablu Chandra Dey**

The Infinity of Triangular Oblong Numbers

TOM MOORE

The product of two consecutive integers results in what is called an *oblong number* or *pronic number*. For $a = 1, 2, 3, 4, \ldots$ the corresponding oblong numbers $a(a+1)$ are $2, 6, 12, 20, \ldots$. We are interested in which triangular numbers $T_n = n(n+1)/2$, where n is a positive integer, are also oblong numbers. The first few of these are

$$\begin{aligned} T_3 &= 6 = 2 \cdot 3, \\ T_{20} &= 210 = 14 \cdot 15, \\ T_{119} &= 7\,140 = 84 \cdot 85, \\ T_{696} &= 242\,556 = 492 \cdot 493, \\ T_{4059} &= 8\,239\,770 = 2870 \cdot 2871. \end{aligned}$$

In general, these examples have the form $T_{x_n} = y_n(y_n + 1)$.

Now, the sequence of subscripts x_n satisfies the following linear nonhomogeneous recurrence relation:

$$x_n = 6x_{n-1} - x_{n-2} + 2, \qquad x_0 = 0,\ x_1 = 3.$$

In addition, the first factors y_n, above, that reveal the oblong character of these numbers, satisfy *the very same recurrence relation*, $y_n = 6y_{n-1} - y_{n-2} + 2$, but with different initial terms, $y_0 = 0,\ y_1 = 2$.

We will show that $x_n(x_n + 1) = 2y_n(y_n + 1)$, for all $n \geq 1$, which establishes the infinity of such numbers. To do this we took advantage of the online recurrence solver PURRS (see http://www.cs.unipr.it/purrs/) as well as the software program MAPLE®. The former produced these explicit solutions for our sequences:

$$\begin{aligned} x_n &= -\frac{1}{2} - \frac{\sqrt{2}(3-2\sqrt{2})^n}{4} + \frac{(3+2\sqrt{2})^n}{4} + \frac{(3+2\sqrt{2})^n\sqrt{2}}{4} + \frac{(3-2\sqrt{2})^n}{4}, \\ y_n &= -\frac{1}{2} - \frac{\sqrt{2}(3-2\sqrt{2})^n}{8} + \frac{(3+2\sqrt{2})^n}{4} + \frac{(3+2\sqrt{2})^n\sqrt{2}}{8} + \frac{(3-2\sqrt{2})^n}{4}. \end{aligned}$$

In MAPLE we separately multiplied each of the expressions $x_n(x_n + 1)$ and $2y_n(y_n + 1)$ and, happily, found both answers agreed; they are equal to

$$-\frac{3}{8} - \frac{\sqrt{2}(17-12\sqrt{2})^n}{8} + \frac{3(17+12\sqrt{2})^n}{16} + \frac{(17+12\sqrt{2})^n\sqrt{2}}{8} + \frac{3(17-12\sqrt{2})^n}{16}.$$

This establishes our goal, to show that there are infinitely many such numbers. In Problem 47.4 on page 44 of this issue, readers are invited to pursue two other aspects of these triangular oblong numbers.

Tom Moore *has delighted in teaching courses in abstract algebra and number theory over a career that spans 44 years, all spent at Bridgewater State University (BSU) in southeastern Massachusetts, USA. He has been an active member of the Mathematics Association of America (MAA) and, with his colleagues, has hosted the annual fall meeting of the Northeastern Section of this organization at BSU in 2001 (recreational mathematics) and 2012 (the art of problem posing and problem solving). He is also the recipient of the MAA's Award for Distinguished College or University Teaching.*

Letters to the Editor

Dear Editor,

Lazar Karno's theorem

Readers may be interested in the following question. The points A', B', C' are the feet of the perpendiculars from the circumcentre O of a triangle ABC to its sides (see figure 1). We ask: what is $OA' + OB' + OC'$?

The quadrilateral $OB'AC'$ is cyclic so, by Ptolemy's theorem (see 'From the Editor' on pages 1–2 of this issue), we have $OB' \times (c/2) + OC' \times (b/2) = OA \times B'C' = R \times (a/2)$, where R is the radius of the circumcircle. Similarly, from the cyclic quadrilaterals $OC'BA'$ and $OA'CB'$,

$$OC' \times \frac{a}{2} + OA' \times \frac{c}{2} = R \times \frac{b}{2}, \qquad OA' \times \frac{b}{2} + OB' \times \frac{a}{2} = R \times \frac{c}{2}.$$

Also,

$$\text{Area}\triangle ABC = \tfrac{1}{2}OA' \times a + \tfrac{1}{2}OB' \times b + \tfrac{1}{2}OC' \times c = \tfrac{1}{2}r(a+b+c),$$

where r is the radius of the inscribed circle of the triangle. If we add these four equations we obtain

$$\tfrac{1}{2}(OA' + OB' + OC')(a+b+c) = \tfrac{1}{2}(R+r)(a+b+c),$$

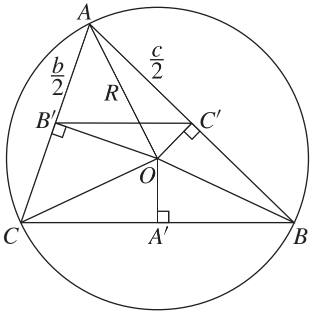

Figure 1

from which

$$OA' + OB' + OC' = R + r,$$

which is Lazar Karno's theorem.

Yours sincerely,
Abbas Rouhol Amini
(10 Shahid Azam Alley
Sirjan
Iran)

Dear Editor,

The centre of the Euler circle

For a triangle ABC, let G denote its centroid, H its orthocentre, and O its circumcentre. Then G, H, and O are collinear. The straight line through G, H, and O is called the *Euler line* of the triangle. Moreover, G lies between O and H and $OG : GH = 1 : 2$. The midpoints of the sides of the triangle, the feet of the three perpendiculars from the vertices to the opposite sides, and the midpoints of AH, BH, and CH lie on a circle called the *Euler circle* or *nine-point circle* of the triangle. Its centre N lies on the Euler line midway between H and O. All this is well-known. If we use vectors or complex numbers to denote the various points relative to some rectangular axes (with $\boldsymbol{a}$ representing A, and so on), then $\boldsymbol{g} = (2\boldsymbol{o} + \boldsymbol{h})/3$ and $\boldsymbol{n} = (\boldsymbol{o} + \boldsymbol{h})/2$; whence, $\boldsymbol{n} = (3\boldsymbol{g} + \boldsymbol{h})/4 = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{h})/4$, so that N, the centre of the Euler circle, is the centroid of the four points A, B, C, and the orthocentre H.

Yours sincerely,
Guido Lasters
(Tienen
Belgium)

Dear Editor,

The history of a famous inequality

The Editor of *Octogon Mathematical Magazine*, Mihály Bencze, presented the inequality

$$\frac{2ab}{a+b} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2},$$

for any different real numbers $a, b > 0$ (see reference 1). This is a consequence of the inequality

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}, \tag{1}$$

since $\sqrt{ab} \geq 2ab/(a+b)$. The inequality (1) is known as the arithmetic-logarithmic-geometric mean inequality.

Although an elementary proof of the right-hand side of (1) exists (see, for example, reference 2), the proof of (1) is an application of the Hermite–Hadamard inequality (see references 3 and 4), namely that, for any convex function $f(x)$ on an interval $[a, b]$, the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

If the function $f(x)$ is strictly convex (see reference 5) on $[a, b]$ the above inequality becomes

$$f\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{2}$$

Since the exponential function $f(x) = \mathrm{e}^x$ is a strictly convex function on $\mathbb{R}$, substituting $f(x) = \mathrm{e}^x$ in (2) we get $\mathrm{e}^{(a+b)/2} < (\mathrm{e}^b - \mathrm{e}^a)/(b-a) < (\mathrm{e}^a + \mathrm{e}^b)/2$. Substituting $\mathrm{e}^b = x$ and $\mathrm{e}^a = y$ in the last inequality, we get

$$\sqrt{xy} < \frac{x-y}{\ln x - \ln y} < \frac{x+y}{2},$$

as required. Some authors (see reference 6) mention the entire inequality (1) as a result of Alexandru Lupaş, who died in 2007 in Romania.

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Yours sincerely,

Spiros P. Andriopoulos
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Greece)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

47.1 Let f be a continuous positive function on the interval $[0, 1]$. Prove that

(i)

$$\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x = \frac{1}{2},$$

(ii)
$$\int_0^1 \frac{f(x)}{f(1-x)}\,\mathrm{d}x \geq 1,$$

(iii) if $f''(x) \geq 0$ for all $x \in [0, 1]$ then
$$\int_0^1 f(x)\,\mathrm{d}x \geq f(\tfrac{1}{2}),$$

and if $f''(x) \leq 0$ for all $x \in [0, 1]$ then the inequality is reversed.

(Submitted by Spiros P. Andriopoulos, Third High School of Amaliada, Eleia, Greece, and dedicated to Bob Bertuello)

47.2 A circle of radius n touches the x-axis and the line $y = nx$. Determine the locus of the centre of the circle as n varies.

(Submitted by Jonny Griffiths, Paston College, Norfolk, UK)

47.3 For positive real numbers a, b, c, prove that
$$(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9,$$

and generalize.

(Submitted by Mihaly Bencze, Brasov, Romania)

47.4 Prove that a number which is both triangular and oblong is divisible by 6 and has units digit 0 or 6.

(Submitted by Tom Moore, Bridgewater State University, USA – see Tom Moore's article on pages 40–41 of this issue)

Solutions to Problems in Volume 46 Number 2

46.5 Find all isosceles triangles such that the length of each side is numerically equal to the square of the secant of the angle opposite.

Solution by Herb Bailey and William Gosnell, who proposed the problem

In figure 1, $b = \sec^2\theta$ and
$$a = \sec^2\phi = \frac{1}{\cos^2\phi} = \frac{2}{1+\cos 2\phi} = \frac{2}{1+\cos(\pi-\theta)} = \frac{2}{1-\cos\theta}.$$

Also,
$$\frac{b}{2} = a\sin\frac{\theta}{2},$$

so that
$$\frac{b^2}{4} = a^2\sin^2\frac{\theta}{2} = \frac{a^2}{2}(1-\cos\theta).$$

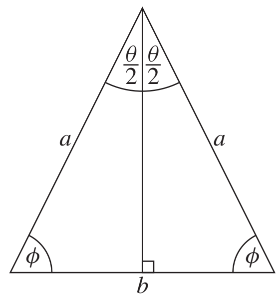

Figure 1

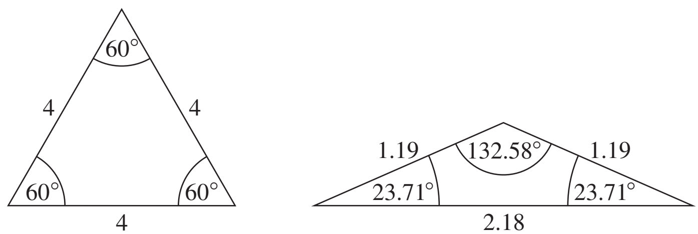

Figure 2

Hence,

$$\frac{1}{4\cos^4\theta} = \frac{2}{1-\cos\theta},$$

so that

$$8\cos^4\theta + \cos\theta - 1 = 0,$$

or

$$(2\cos\theta - 1)(4\cos^3\theta + 2\cos^2\theta + \cos\theta + 1) = 0;$$

whence,

$$\cos\theta = \tfrac{1}{2} \quad \text{or} \quad \cos\theta \simeq -0.6766,$$

i.e.

$$\theta = 60^\circ \quad \text{or} \quad \theta \simeq 132.58^\circ,$$

which gives the two isosceles triangles shown in figure 2.

46.6 A sheet of paper is 11 cm $\times$ 8 cm. It is folded so that the bottom right-hand corner touches the left-hand edge to form a crease of length c cm. What is the minimum value of c?

Solution by Bob Bertuello, who proposed the problem

From figure 3,

$$\cos\alpha = \frac{\sqrt{x^2-(8-x)^2}}{x} = \frac{8}{y},$$

so that

$$y = \frac{2x}{\sqrt{x-4}}.$$

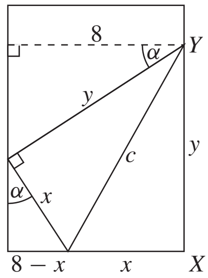

Figure 3

Now,

$$c^2 = x^2 + y^2 = x^2 + \frac{4x^2}{x-4} = \frac{x^3}{x-4}.$$

Thus,

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{3/2}}{\sqrt{x-4}}\right) = \frac{(x-6)\sqrt{x}}{(x-4)^{3/2}} = 0, \quad \text{when } x = 6.$$

Note that, when $x = 6$, $y = 6\sqrt{2} < 11$ and that $\mathrm{d}c/\mathrm{d}x < 0$ when $4 < x < 6$ and $\mathrm{d}c/\mathrm{d}x > 0$ when $x > 6$. Hence, c is minimum when $x = 6$, so the minimum value of c is $6\sqrt{3}$.

46.7 If four fair six-sided dice are thrown, what is the probability that the numbers add up to a prime?

Solution by Chris Caldwell, who proposed the problem

The possible outcomes range from 4 to 24, and the primes in this range are precisely those numbers which are congruent to $1 \pmod 6$ or $5 \pmod 6$. Denote by x the sum of the first three throws and by y the last throw. We require $x + y \equiv 1 \pmod 6$ or $x + y \equiv 5 \pmod 6$, i.e. $y \equiv 1 - x \pmod 6$ or $y \equiv 5 - x \pmod 6$. Table 1 gives the possibilities. Thus, no matter what the sum is after the first three throws, two out of the possible six throws of the last die will give a prime sum, so the probability of a prime sum is $\frac{1}{3}$.

46.8 If $\boldsymbol{A}$, $\boldsymbol{B}$ are 3×3 matrices such that $\det \boldsymbol{A} = \det \boldsymbol{B} = 3$ and $\det(\boldsymbol{A} + \boldsymbol{B}) = -1$, how are $\det(3\boldsymbol{A} + 2\boldsymbol{B})$ and $\det(2\boldsymbol{A} + 3\boldsymbol{B})$ related?

Table 1

$x \pmod 6$	y
0	1, 5
1	6, 4
2	5, 3
3	4, 2
4	3, 1
5	2, 6

Solution by Marcel Chirita, who proposed the problem

We can write

$$\det(x\boldsymbol{A} + y\boldsymbol{B}) = x^3(\det \boldsymbol{A}) + x^2ya + xy^2b + y^3(\det \boldsymbol{B}).$$

Thus,

$$\begin{aligned}\det(3\boldsymbol{A} + 2\boldsymbol{B}) &= 27(\det \boldsymbol{A}) + 18a + 12b + 8\det \boldsymbol{B}\\ &= 105 + 18a + 12b,\\ \det(2\boldsymbol{A} + 3\boldsymbol{B}) &= 8(\det \boldsymbol{A}) + 12a + 18b + 27(\det \boldsymbol{B})\\ &= 105 + 12a + 18b,\end{aligned}$$

and

$$-1 = \det(\boldsymbol{A} + \boldsymbol{B}) = \det \boldsymbol{A} + a + b + \det \boldsymbol{B} = 6 + a + b,$$

so that $a + b = -7$. Hence,

$$\det(3\boldsymbol{A} + 2\boldsymbol{B}) + \det(2\boldsymbol{A} + 3\boldsymbol{B}) = 210 + 30a + 30b = 210 - 30 \times 7 = 0$$

and $\det(3\boldsymbol{A} + 2\boldsymbol{B}) = -\det(2\boldsymbol{A} + 3\boldsymbol{B})$.

Reviews

Beautiful Geometry. By Eli Maor and Eugen Jost. Princeton University Press, 2014. Hardback, 208 pages, £19.95 (ISBN 9781400848331).

This book consists of 51 largely independent chapters, each three to four pages in length, on mathematical topics ranging across elementary geometry and number theory. Each topic is presented in a historical context by Eli Maor and interpreted by the artist Eugen Jost with computer illustrations or acrylics on canvas. The chapters are mainly on material not normally covered in school or university mathematics syllabi, but should be interesting and accessible to anyone with a modest understanding of school mathematics (no calculus necessary).

The chapters consist of an eclectic mix of topics across geometry and number theory. Chapter titles include ‘Triangles of equal area’, ‘Eleven’, ‘The 17-Sided regular polygon’, ‘$\frac{3}{3} = \frac{4}{4}$’, ‘The Reuleaux triangle’, and ‘Beyond infinity’. The presentation is enthusiastic, but, for my taste, the mathematical emphasis is often in the wrong place.

I liked the artwork; the illustrations are beautiful and the process of reconciling the mathematics with the artistic interpretation is enjoyable. Often the artwork presents illustrations that underline the absoluteness of mathematical statements. In particular, many of the illustrations follow a common theme of presenting the geometric pictures associated to theorems for differing values of a parameter.

Overall, I feel this book could be an enjoyable talking point and should find a happy place on a coffee table for mathematicians and nonmathematicians alike.

University of Sheffield **Fionntan Roukema**

Mathematical Spectrum

Volume 47 2014/2015 Number 1

ISSN 0025-5653

http://ms.appliedprobability.org

Published by the Applied Probability Trust

Printed by Berforts Information Press Ltd, Stevenage, Hertfordshire, UK