MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

Volume 18 1985/86 Number 1

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 18 of *Mathematical Spectrum* will consist of three issues, of which this is the first. The second will be published in January 1986 and the third in May 1986.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH

Geographical and Population Centres of Nations

A. TAN, *Alabama Agricultural and Mechanical University*

The author is a faculty member of Alabama A & M University. He has frequently published articles in applied and educational mathematics, including one in Volume 17 Number 2 of *Mathematical Spectrum*.

The centres of area and population are important concepts in the planning, development and reconstruction of a nation. The centre of area, i.e. the geographical centre, may be defined as the point upon which the nation would balance under gravity if it were a plane of uniform density (reference 1). Physically speaking, we can think of it as the centre of gravity of the nation's plane. The centre of population, on the other hand, may be regarded as the point upon which the nation would balance if it were a rigid but weightless plane with its population distributed thereon with each individual having equal weight (reference 1).

Both centres are susceptible to change. The geographical centre may change due to addition or cessation of territory, or a change of sovereignty due to annexation, merger or reunification through conflict, conquest or treaty. These changes could be abrupt. However, during the past decades, international boundaries have become more stabilised, and radical changes have been few. The population centre necessarily changes with every change of the geographical centre. In addition, it also changes with the migration of the population, whether immigration, emigration or internal migration. Theoretically speaking, every movement of an individual alters the centre of population, however small it may be. As a result, the population centre of a nation changes constantly, even though its geographical centre may be stationary. In modern times, this change has been accelerated by industrialisation, urbanisation and development.

The centre of area may be easily determined by cutting out the outline of the nation from a map (preferably not a valuable one!) and balancing it on a small circular area, such as the end of a pencil. This method is suitable for most nations which consist of one block of land, but becomes impracticable for disconnected ones, such as those having many islands. Determination of the population centre, on the other hand, is more complex and requires detailed demographic data. The population centre of the continental United States, excluding Alaska and Hawaii, has been determined by the Census Bureau from each of its censuses.

Figure 1 shows both the geographical and population centres of the United States (excluding Alaska and Hawaii) every ten years since 1790. The population centres are taken from the U.S. Bureau of the Census (reference 2). The geographical centres are determined by the method outlined

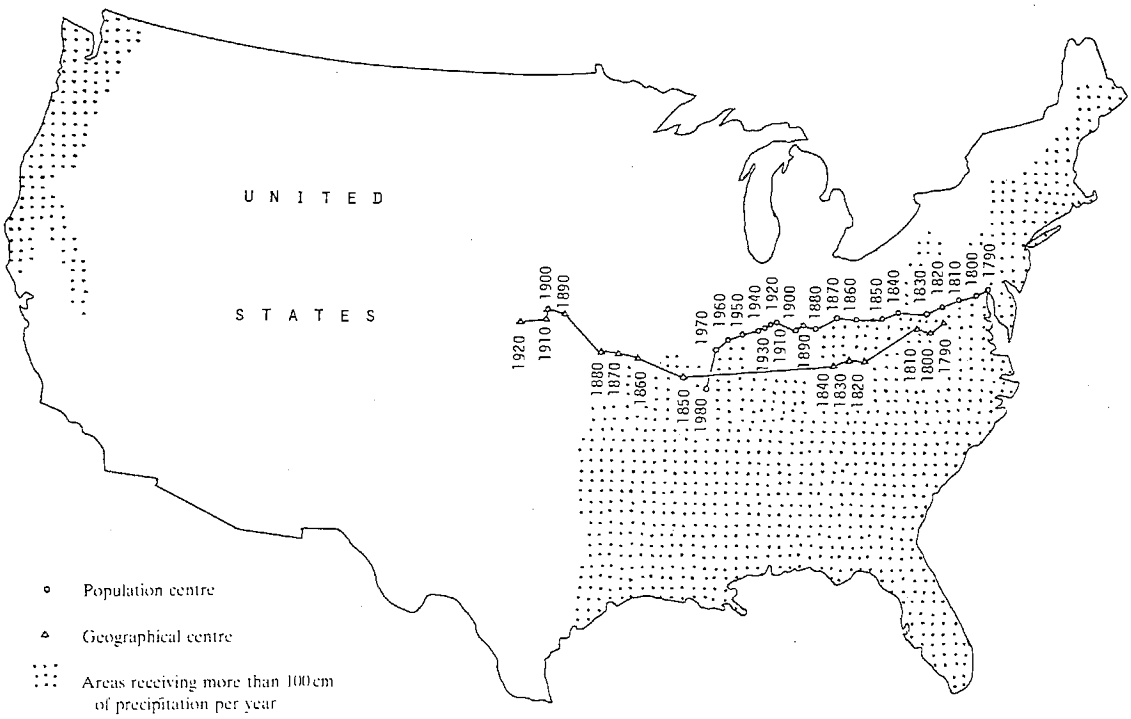


Figure 1

above from the maps of the *National Atlas of the United States* (reference 3). The steady westward movement of both centres reflects the growth history of the United States. Although the westward drift of the population centre is well known, its relation to the parallel westward movement of the geographical centre is seldom mentioned. Whereas the geographical centre came to a halt in 1920 with the incorporation of the last state (again excluding Hawaii and Alaska), the population centre continues to drift, the latest tendency being southward. The shaded parts mark the areas which receive more than 40 inches of precipitation annually. Since the water-rich areas lie mostly in the south-east, it is unlikely that the population centre will ever move to the west of the geographical centre. Indeed, the latest tendency might indicate that the steady westward trend may be already slowing down.

Some interesting observations may be made on geographical and population centres for other nations. Ideally speaking, it may be well to have the capital of a nation located near one of its centres, for administrative convenience. We find that the capital of Spain (Madrid) is located at the geographical centre, whereas the capital of Egypt (Cairo) is situated at the population centre. Of course, this privilege is not bestowed upon all countries. Both the geographical centre and the population centre of Norway lie inside Sweden, with which Norway was once united. The centres of Vietnam lie inside Laos, which Vietnam currently dominates. And Somalia's centres lie in the Ogaden region of Ethiopia, which Somalia claims as her own (cf. reference 4).

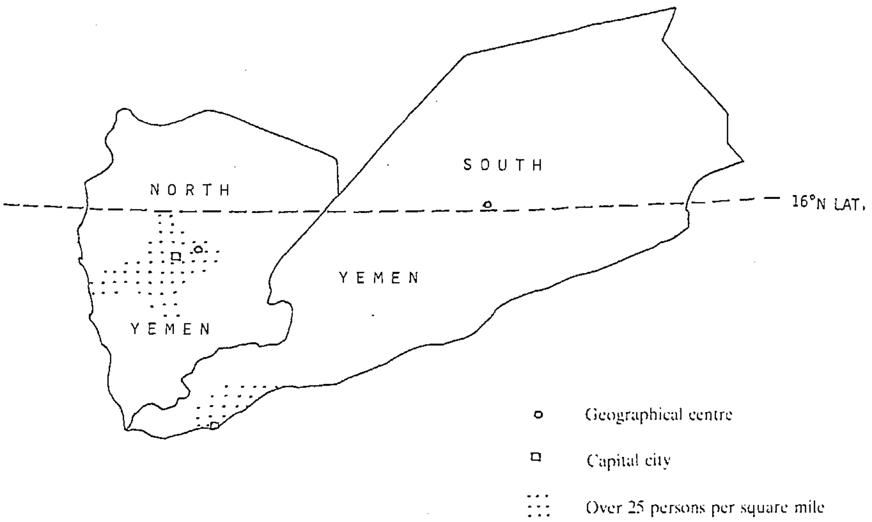


Figure 2

Finally, which of these centres is more important in the nomenclature of a divided nation? Is East Germany 'east' because its geographical centre lies east of that of West Germany, or is it because of its population centre? To answer the question, let us examine the interesting case of the Yemens (cf. figure 2, drawn from reference 5). The geographical centre of Yemen (also called North Yemen: officially, the Yemen Arab Republic) is located at 15.5°N latitude, whereas that of South Yemen (officially, the People's Democratic Republic of Yemen) is located at 16.1°N latitude. Geographically speaking, South Yemen lies north of North Yemen! The confusion can be cleared up if we consider the population centres of the two nations. South Yemen's population is concentrated near the capital city of Adan (formerly Aden), which is south of the population centre of Yemen. Hence South Yemen is 'south' because of her population centre, not because of her location: a definite indication of the importance of the population centre.

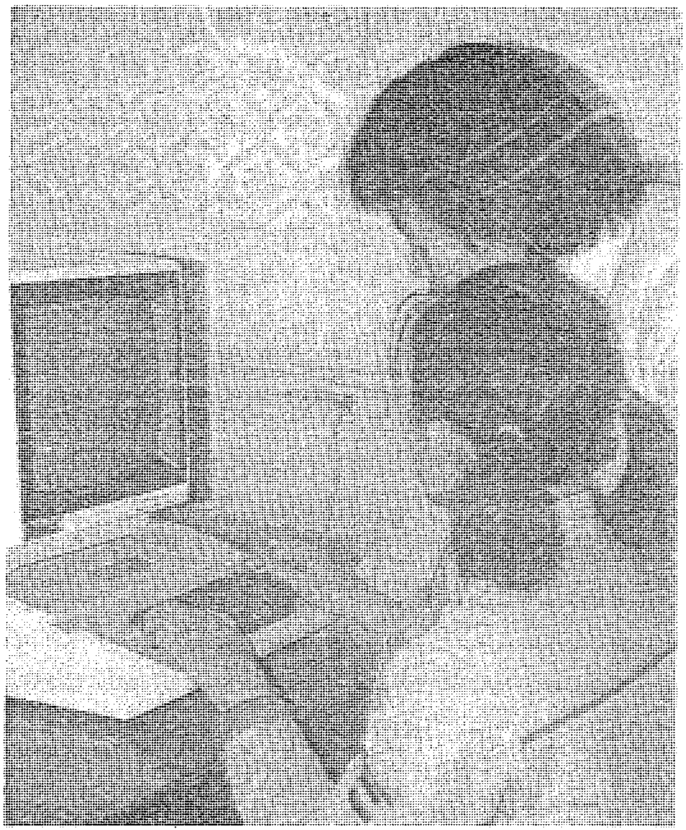
References

1. J. H. Hayford, *J. Amer. Statist. Assoc.*, **8** (1902), 47–58.
2. *Census of Population: 1980* (U.S. Bureau of the Census, Washington DC, 1981).
3. *The National Atlas of the United States* (U.S. Department of the Interior, Washington DC, 1970), 140–141.
4. *Border and Territorial Disputes*, Alan J. Day ed. (Gale Research Co., Detroit, 1982), 114–119.
5. *The National Geographic Atlas of the World* (National Geographic Society, Washington DC, 1981), 182–183.

School Mathematics in China

LIU ZHIQING, *Beijing, People's Republic of China*

Zhiqing is a frequent contributor to *Spectrum*. He was a pupil at No. 4 Middle School, Beijing (which used to be called Peking), and is now studying computing at Qinghua University in Beijing. We asked Zhiqing to write us a short article on mathematics in Chinese schools. You may be as confused as we were about Zhiqing's names; Liu is his surname!



A student in China spends six years in a middle school, which is a secondary or high school. Every year he will have mathematics lessons; mathematics and Chinese are regarded as the most important lessons at a middle school. There are five or six mathematics lessons, each of 45 minutes, each week.

At middle school, we learn most of the traditional elementary mathematics such as elementary algebra, geometry and trigonometry. We also learn some more modern topics, such as the beginnings of set theory, and some more advanced topics such as fractions, limits, probability, statistics and differentiation.

A few schools, my own included, are beginning to have lessons in computing for some good students. As the price of computers in China is very high, most middle schools do not have the money to buy computers for their students. Most computers used in middle schools are microcomputers such as Apple-II-plus, TRS-80 or the IBM-PC. In computing lessons, we learn something about computers and how to program them using BASIC. I think this is very good, but unfortunately only a few students have such a chance.

In most middle schools, the method of teaching is still lecturing by the teachers, but, in some good schools, there is the opportunity for discussion, and this is growing.

For students who excel in mathematics, schools have special mathematics groups where more advanced subjects are taught. We were encouraged to write short articles on mathematical topics whenever we had an idea which we thought would be useful to share with others.

A National Mathematics Contest has been held in China every year since 1978. Students in high school sit the test in their own provinces. Besides this, students also try some of the American mathematics contests. These give added opportunities for good students.

Most students in middle school would like to have the chance to go on to study at a college, but this is not very easy in China at the moment, and we have to pass a difficult examination to get there. To prepare for this, we have to do a lot of extra reading and exercises, which leaves no time for anything else.

China is in the process of changing, not only in its economy but also in such things as mathematics teaching in schools. Mathematics textbooks, teaching methods and college entrance examinations are all changing, I hope for the better.

That was the year that was (1)

Did you know that this is the year

$$\cot\left(\tfrac{1}{4}\pi - 3\tan^{-1}\tfrac{1}{4} - \tan^{-1}\tfrac{1}{20}\right)?$$

JOHN MACNEILL
The Royal Wolverhampton School

Stable Matching Problems

ROBERT W. IRVING, *University of Salford*

The author is a lecturer in the Department of Mathematics and Computer Science at Salford. Although his background is in pure mathematics, his main teaching and research interests now lie in computer science, particularly in the analysis of algorithms and in complexity theory. His leisure activities include hill-walking—in his native Scotland when possible.

1. Introduction

A number of interesting problems in combinatorial mathematics are of the following general form: given two disjoint finite sets of equal size (say of men and women), or, alternatively, a single finite set of even size, how can the members of the two sets, or of the single set, be 'paired off' in order to meet some specified criterion, or to maximise some overall measure of 'compatibility'? Any such set of pairs, in which each man is partnered by a unique woman, or, in the single-set case, in which everyone has a unique partner, is called a (complete) *matching*.

Among these problems are two based on the ordered *preferences* of individuals and a *stability* criterion.

In the *stable marriage problem*, each person ranks the members of the opposite sex strictly in order of preference. A *stable matching* is a matching with the property that no man and woman who are *not* partners both prefer each other to their actual partners in the matching. The existence of such a man and woman would clearly cause instability, as the two 'marriages' concerned would be likely to break down.

The corresponding problem for a single set (of even size) in which each member ranks all of the others strictly in order of preference, and a stable matching is similarly defined, is known as the *stable room-mates problem*, from its possible interpretation in terms of pairing off individuals for room-sharing purposes.

A number of applications of these stable matching problems have been suggested and investigated. Indeed, the problems were first posed (see reference 1) in the context of assigning applicants to colleges, taking into account the preferences of both applicants and colleges. Other areas of application have included matching supervisors with students for project work, the allocation of houses on a local council list and the assignment of house jobs in a hospital. Of course, in practice, it is unrealistic to expect a complete ranking in each preference list, so that the mathematical model has to be extended to incorporate the situation where not all preferences are strict. Here, however, we shall restrict our discussion to theoretical aspects of the problems.

Example 1. The following 'preference matrices' contain, in their rows, the preference lists of men and women, respectively, and constitute an instance of the stable marriage problem of size 6:

1:	4	1	5	2	6	3		1:	6	1	5	3	4	2
2:	3	4	2	6	1	5		2:	4	6	2	1	3	5
3:	1	4	6	2	5	3		3:	5	4	3	6	1	2
4:	1	5	3	4	6	2		4:	4	3	1	6	5	2
5:	4	2	3	6	5	1		5:	3	2	6	5	4	1
6:	6	2	3	4	1	5		6:	1	6	2	5	3	4

male preferences   female preferences

It can be verified that the matching

$$\{(1,1),\ (2,2),\ (3,5),\ (4,4),\ (5,3),\ (6,6)\}$$

is stable. Here, as throughout, a matching is represented as a set of male/female pairs. On the other hand, the matching

$$\{(1,4),\ (2,3),\ (3,6),\ (4,1),\ (5,5),\ (2,2)\}$$

is not stable; man 5 and woman 3, for instance, cause instability.

Example 2. The rows of the following matrix are the preference lists in a stable room-mates instance of size 8:

1:	3	8	7	4	5	2	6		5:	4	1	2	8	7	6	3
2:	6	8	5	4	3	1	7		6:	4	5	1	2	8	3	7
3:	8	5	2	1	6	7	4		7:	5	3	1	4	8	2	6
4:	7	1	8	3	6	5	2		8:	7	2	6	4	5	3	1

It can be verified that the matching

$$\{(1,3),\ (2,8),\ (4,6),\ (5,7)\}$$

is stable, whereas, for instance, the matching

$$\{(1,3),\ (2,8),\ (4,7),\ (5,6)\}$$

is not, due to persons 5 and 7. (In the case of a room-mates instance of size n, a matching is represented as a set of $\frac{1}{2}n$ pairs.)

2. Solving instances of the stable marriage problem

For a given instance of the stable marriage problem, the questions arise as to how we determine whether a stable matching exists, and how such a stable matching may be found. One possibility is to carry out an 'exhaustive search', in which all possible matchings are generated in sequence and tested for stability. However, it should be clear that, for an instance of size n, the total number of matchings is $n!$ and, because of the rapid growth of the factorial function, this method is computationally impracticable for all but the smallest values of n.

Gale and Shapley (see reference 1) were the first to treat this stable matching problem mathematically and they presented an algorithm which, for a situation involving sets of size n, is *guaranteed* to find a stable matching in a number of steps that is bounded by a constant times n^2. (The algorithm is said to have '$O(n^2)$ time complexity'.) It follows, perhaps rather surprisingly, that at least one stable matching exists for *every* instance of the stable marriage problem.

This fundamental algorithm is based on a sequence of 'proposals' from the members of one sex to the members of the other. In keeping with the times, we shall allow the women the privilege of making the proposals in our description of the algorithm.

The algorithm. When a man receives a proposal, he does one of two things:

(i) if it is the best proposal he has so far received (measured by position in his preference list), then he agrees to hold it for consideration, simultaneously rejecting any poorer proposal currently held;

(ii) if he already holds a better proposal, then he rejects the new proposal at once.

Each woman, in turn, proposes to the men on her preference list in order, pausing when a promise of consideration is received, but continuing on any immediate or subsequent rejection.

Example 1. For the preference matrices specified earlier, the proposal sequence (from women to men) would proceed as follows:

woman 1 proposes to man 6; man 6 holds woman 1;

woman 2 proposes to man 4; man 4 holds woman 2;

woman 3 proposes to man 5; man 5 holds woman 3;

woman 4 proposes to man 4; man 4 holds woman 4 and rejects woman 2;

woman 2 proposes to man 6; man 6 holds woman 2 and rejects woman 1;

woman 1 proposes to man 1; man 1 holds woman 1;

woman 5 proposes to man 3; man 3 holds woman 5;

woman 6 proposes to man 1; man 1 rejects woman 6;

woman 6 proposes to man 6; man 6 holds woman 6 and rejects woman 2;

woman 2 proposes to man 2; man 2 holds woman 2.

We claim that this proposal sequence ends with every man holding a proposal, and further, that the set of proposals under consideration when the algorithm terminates constitutes a stable matching.

To see that the algorithm cannot terminate until every man holds a proposal, we suppose the contrary. Then, since no man can simultaneously hold two proposals, there must be a woman who has been rejected by every man to whom she has proposed. If she has not proposed to all the men, then the algorithm has not, after all, terminated. But if she has, then the man with no proposal must have rejected her in favour of a better proposal, and once a man has a proposal, he is never subsequently without one. This gives a contradiction, so at least the algorithm, on termination, specifies a matching.

To prove stability, we suppose that man x and woman Y prefer each other to their partners, woman X and man y respectively. Then woman Y must have proposed to man x and must have been rejected. This could have happened only because man x received a better proposal, so the proposal that he finally holds must also be from someone better than Y, giving a contradiction.

The stable matching found by the above algorithm is known as the *female optimal solution*, for no woman can do better in any stable matching than she does in this particular one. A proof of this fact may be found in reference 1. If the roles of men and women are exchanged, then an analogous algorithm, involving proposals from men to women, will yield the *male optimal solution*, which similarly favours the men.

If the male and female optimal solutions should happen to coincide, then it is easy to see that they constitute the *unique* stable matching for the particular problem. Otherwise, as in the case of our illustrative example, there may be further stable matchings lying 'between' these two extremes.

Example 1. Our chosen example of size 6 has a total of 5 stable matchings, namely

(i) $\{(1,1), (2,2), (3,5), (4,4), (5,3), (6,6)\}$ (female optimal)

(ii) $\{(1,1), (2,2), (3,4), (4,5), (5,3), (6,6)\}$

(iii) $\{(1,1), (2,3), (3,4), (4,5), (5,2), (6,6)\}$

(iv) $\{(1,4), (2,2), (3,1), (4,5), (5,3), (6,6)\}$

(v) $\{(1,4), (2,3), (3,1), (4,5), (5,2), (6,6)\}$ (male optimal)

3. Solving instances of the stable room-mates problem

Gale and Shapley (reference 1) observed that, quite unlike the case of the stable marriage problem, there exist instances of the stable room-mates problem for which no stable matching is possible. For instance, in the following instance of size 4, it is not hard to see that the partnership involving person 4 is bound to cause instability:

1:	2	3	4
2:	3	1	4
3:	1	2	4
4:	arbitrary		

Until recently, there was no known efficient algorithm to find a stable matching, if one exists, for a stable room-mates instance, and otherwise to establish non-existence. (Of course, exhaustive search, or any of its refinements such as a 'backtrack' approach, could be used, but this would lead to an algorithm requiring exponentially many steps to solve some instances of the problem, since the total number of matchings is

$$(n-1)(n-3)\times\ldots\times 3\times 1 = \frac{n!}{(\frac{1}{2}n)!\times 2^{n/2}}\,.\;)$$

Recently, an $O(n^2)$ algorithm for the stable room-mates problem has been discovered (see reference 2), an algorithm that builds on the 'proposal sequence' method of Gale and Shapley. Once again, we shall describe the algorithm with reference to its effect in a particular case—namely that of example 2 introduced in section 1.

The algorithm—first phase. Consider the outcome if a sequence of proposals similar to that of the Gale–Shapley algorithm is carried through for a stable room-mates instance. Since just a single set is involved, each person, as well as making proposals, can expect also to receive proposals.

Example 2. Consider the instance of size 8 introduced earlier. The proposal sequence proceeds as follows:

1 proposes to 3; 3 holds 1;

2 proposes to 6; 6 holds 2;

3 proposes to 8; 8 holds 3;

4 proposes to 7; 7 holds 4;

5 proposes to 4; 4 holds 5;

6 proposes to 4; 4 holds 6 and rejects 5;
5 proposes to 1; 1 holds 5;

7 proposes to 5; 5 holds 7;

8 proposes to 7; 7 rejects 8;
8 proposes to 2; 2 holds 8.

In the example, the sequence ends at this point with everyone holding a proposal, but, of course, the proposals held do not constitute a matching.

In general, there are just two other possible outcomes after this first phase of the algorithm:

(i) the initial sequence of proposals and rejections may indeed lead to a matching, in which case it can be proved, essentially as in the case of the stable marriage algorithm, that the matching is stable;

(ii) the sequence may end with all but one person holding a proposal, that person having been rejected by all of the others—this happens, for example, in the instance of size 4 mentioned earlier. In the event of this outcome, it is not hard to show that no stable matching is possible.

However, the case illustrated by example 2 above still has to be resolved by further analysis.

The following two facts concerning the first phase of the algorithm are proved in reference 2:

(a) if x has proposed to y, then there is no stable matching in which x has a better partner than y;

(b) if y has received a proposal from x, then there is no stable matching in which y has a worse partner than x.

These facts allow us to remove certain impossible pairings from the preference matrix, and to form reduced preference lists that we shall call the *shortlists*. In example 2, the shortlists are as shown:

1:	3	7	4	5	5:	1	8	7	
2:	6	8			6:	4	2		
3:	8	1			7:	5	1	4	
4:	7	1	8	6	8:	2	4	5	3

For instance, 6 is absent from 5's shortlist, and 5 from that of 6, because 5 received a proposal from someone better than 6.

The algorithm—second phase. To understand the second phase of the algorithm, we consider the consequences if, in order to break the 'deadlock', one person were to be rejected by the first person on his shortlist—say 1 were to be rejected by 3 in our example. Then 1 can do no better than have 7 as a partner, so 1 proposes to 7, and 7, as a consequence, need settle for no worse partner than 1, and so rejects 4. Hence, using ! as an abbreviation for 'is rejected by', we have the implication

$$1!3 \Rightarrow 4!7.$$

Similarly, this causes 4 to propose to 1 and 1 to reject 5. Continuing in this way, we get the chain of implications

$$1!3 \Rightarrow 4!7 \Rightarrow 5!1 \Rightarrow 3!8 \Rightarrow 5!1$$

which, eventually, must cycle, as we see in this example. We shall call such a cycle of pairs a *mutually dependent cycle*, (or MD-cycle, for short), for it should not be hard to see that either all the pairs in such a cycle are room-mates in a stable matching, or none of them are.

The crux of phase 2 of the algorithm is the following theorem, which we do not prove here.

Theorem (see reference 2). For a stable room-mates instance, if a stable matching exists in which all the pairs in an MD-cycle are partners, then there is also a stable matching in which none of them are partners.

As a consequence of this theorem, if we are seeking just one stable matching, then we may as well force apart all the pairs of an MD-cycle. This causes further reduction of the shortlists, as we now demonstrate in the case of our example. If we force 1 to reject 5 and 8 to reject 3, this means that 5 proposes to 8 and 3 to 1 giving the further reduced lists

1:	3			5:	8	7	
2:	6	8		6:	4	2	
3:	1			7:	5	4	
4:	7	8	6	8:	2	4	5

This process of further reducing the preference lists by forcing apart the pairs of an MD-cycle is referred to as *eliminating* the cycle.

Now, the theorem concerning MD-cycles applies not just to the so-called shortlists obtained after the first phase of the algorithm, but also to any further reduced lists obtained by eliminating one or more MD-cycles. So, if necessary, we can repeatedly locate and eliminate MD-cycles until there are none left. If, during the elimination of some MD-cycle, one or more of the lists becomes empty, then it follows from our theorem that no stable matching can exist. Otherwise, as long as some list has more than one entry, there is bound to be an MD-cycle (which can be found just as in the example above) so that, unless some list becomes empty, MD-cycle elimination can continue until all lists contain only one entry. If this situation is reached, it can be shown that these final reduced lists specify a stable matching.

Pursuing example 2, starting with 2!6, we obtain

$$2!6 \Rightarrow 5!8 \Rightarrow 4!7 \Rightarrow 5!8$$

and eliminating the MD-cycle $4!7 \Leftrightarrow 5!8$, we obtain the further reduced lists

1:	3		5:	7	
2:	6	8	6:	4	2
3:	1		7:	5	
4:	8	6	8:	2	4

Finally, if we start again with 2!6, we obtain

$$2!6 \Rightarrow 4!8 \Rightarrow 2!6$$

and we end up with

1:	3	5:	7
2:	8	6:	4
3:	1	7:	5
4:	6	8:	2

which specifies a stable matching.

Example 3. As an illustration of the other possible outcome of the second phase of the algorithm, we resolve the following room-mates instance of size 6:

1:	2	6	4	3	5	4:	5	2	3	6	1
2:	3	5	1	6	4	5:	6	1	3	4	2
3:	1	6	2	5	4	6:	4	2	5	1	3

The first phase of the algorithm involves no rejections at all, but leads to the shortlists

1:	2	3	4:	5	6
2:	3	1	5:	6	4
3:	1	2	6:	4	5

The MD-cycle

$$1!2 \Rightarrow 2!3 \Rightarrow 3!1 \Rightarrow 1!2$$

leads to empty lists for 1, 2 and 3, and the conclusion that no stable matching is possible.

A full implementation of the algorithm, with a proof of its $O(n^2)$ complexity, appears in reference 2.

An unsolved problem. We have seen an instance of the stable room-mates problem that does admit a stable matching, and others that do not. One of the interesting unresolved issues concerns the proportion p_n of instances of size n for which a stable matching exists. It is known that $p_4 = 0.963$ [1248 out of the $1296 = (3!)^4$ instances possess stable matchings], but no other exact values of p_n are known. Empirical evidence suggests that p_n decreases as n increases—for example, of 200 randomly generated problem instances of size 100, only 124 had a stable matching. However, it is not even known whether $\lim p_n$ as $n \to \infty$ exists, and if it does, whether it is positive or zero. Any theoretical analysis leading to a better understanding of the behaviour of p_n would be of considerable interest.

Finally, the book by Knuth (reference 3) provides a very readable account (in French) of a wide range of issues associated with these stable matching problems and some of their fascinating links with other combinatorial and algorithmic problems.

References

1. D. Gale and L. S. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* **69** (1962), 9–15.
2. R. W. Irving, An efficient algorithm for the stable room-mates problem, *J. Algorithms*. To appear.
3. D. E. Knuth, *Marriages Stables* (Les Presses de l'Université de Montréal, Montréal, 1976).

Folding the Perfect Corner

Every day 1200 American Airlines jets crisscross the U.S., Mexico, Canada and the Caribbean, stopping in 110 cities and bearing over 80 000 passengers. More than 4000 pilots, copilots, flight personnel, maintenance workers and baggage carriers are shuffled among the flights; a total of 3.6 million gallons of high-octane fuel is burned. Nuts, bolts, altimeters, landing gears and the like must be checked at each destination. And while performing these scheduling gymnastics, the company must keep a close eye on costs, projected revenue and profits.

Like American Airlines, thousands of companies must routinely untangle the myriad variables that complicate the efficient distribution of their resources. Solving such monstrous problems requires the use of an abstruse branch of mathematics known as linear programming. It is the kind of maths that has frustrated theoreticians for years, and even the fastest and most powerful computers have had great difficulty juggling the bits and pieces of data. Now Narendra Karmarkar, a 28-year-old Indian-born mathematician at Bell Laboratories in Murray Hill, N.J., after only a year's work has cracked the puzzle of linear programming by devising a new algorithm, a step-by-step mathematical formula. He has translated the procedure into a program that should allow computers to track a greater combination of tasks than ever before and in a fraction of the time.

Unlike most advances in theoretical mathematics, Karmarkar's work will have an immediate and major impact on the real world. 'Breakthrough is one of the most abused words in science', says Ronald Graham, director of mathematical sciences at Bell Labs. 'But this is one situation where it is truly appropriate'.

Before the Karmarkar method, linear equations could be solved only in a cumbersome fashion, ironically known as the simplex method, devised by mathematician George Dantzig in 1947. Problems are conceived of as giant geodesic domes with thousands of sides. Each corner of a facet on the dome represents a possible solution to the equation. Using the simplex method, the computer scours the surface of the dome millions of times to pinpoint the corner with the most likely solution. But the method is slow, and it works only when there are merely a few thousand variables to sort through. Says Karmarkar: 'Once you get above 15 000 or 20 000 variables the method sort of runs out of steam'.

Karmarkar's technique does not attempt to calculate the location of every solution but takes a circuitous route, eliminating groups of combinations without actually considering them, all the time changing the shape of the dome. The mathematician compares this search to origami, the Japanese art of paper folding: the pieces of paper are creased and shaped until the perfect corner—the long-sought solution—is in the centre of the figure.

When the computer program becomes available to commercial users, American Airlines will be far from the only customer waiting in line. Bell Labs' parent company, A T & T, will probably employ the algorithm to route millions of telephone calls through hundreds of thousands of cities and towns more efficiently and profitably. Exxon has expressed interest in Karmarkar's program to help improve its allocation of supplies of crude oil among various refineries. For many large companies, says Graham, finding the best solution, as opposed to one that is merely workable can mean the difference between a good balance sheet and a mediocre one. *By Natalie Angler. Reported by Peter Stoler/New York.*

Family Planning — A Probabilistic Approach

D. J. COLWELL AND J. R. GILLETT, *North Staffordshire Polytechnic*

The authors are lecturers in mathematics at the North Staffordshire Polytechnic. They are both involved with teaching mathematics and statistics to students on engineering, science and economics degree courses.

People's views on the size and make-up of their ideal family vary considerably, and the views that they have are often very strongly held. A couple that let it be known to their relatives or friends they would like to have children will often be inundated with well-meaning advice and comments on the composition of their family-to-be. Even if they escape such comments so far as the first child is concerned, comments can virtually be guaranteed after the birth of a second child.

For instance, couples whose first child is a boy will often be commiserated with, rather than congratulated, if their second child is not a girl. Similarly, in spite of the move towards equality of the sexes, it is not uncommon in many countries to hear of the birth of a girl being greeted with dismay, while the birth of a boy is greeted with celebration.

Should a couple have particularly strong views on how many boys and/or girls they would like, they could turn to probability theory for advice on how many children they must expect to have before they reach their ideal.

A possible probabilistic approach is to model the birth of successive children in a family by a sequence of independent Bernoulli trials in which the outcomes are B (a boy), with probability p (> 0) say, and G (a girl) with probability $q = 1-p$. Then if a couple would like at least m boys and n girls, the problem becomes that of determining the expected number of trials, denoted by $N(m,n)$, needed to attain the required m B's and n G's.

The m B's and n G's will be reached from one of two possible situations, namely $m-1$ B's and at least n G's or at least m B's and $n-1$ G's. Probabilities associated with each of these possibilities are analysed in detail in part 1 of the appendix, and lead to the recurrence relation

$$N(m,n) = 1+pN(m-1,n)+qN(m,n-1) \quad (m,n \geqslant 1) \tag{1}$$

for the expected size of family with m B's and n G's.

We cannot initiate the recurrence relation until we know the values of $N(1,0)$ and $N(0,1)$. Further, it should be noted that we cannot deduce from (1) the values of $N(m,0)$ ($m > 1$) and $N(0,n)$ ($n > 1$).

The value of $N(m,0)$ ($m \geqslant 1$), being the expected number of trials needed to attain m B's, may be deduced from a standard probability distribution — the negative binomial distribution. As shown in part 2 of the appendix,

$$N(m,0) = \frac{m}{p} \quad (m \geqslant 1). \tag{2}$$

Similarly

$$N(0,n) = \frac{n}{q} \quad (n \geqslant 1). \tag{3}$$

All that we need to do now is estimate the value of p. Guidance on this probability can be obtained from *The Annual Abstract of Statistics*, published by The Central Statistical Office. This abstract indicates that over the years 1976–82, the most recent years for which information is available, the value of p has averaged 0.5138 (approximately, 1057 males born per 1000 females), with only very small variations in any one year.

If we take $p = 0.5138$, and thus $q = 1-p = 0.4862$, equations (1), (2) and (3) may be used to obtain the values for $N(m,n)$ given in table 1.

Table 1
Expected size $N(m,n)$ of family with m boys and n girls

		Values of n (girls)					
	$N(m,n)$	0	1	2	3	4	5
	0	0	2.0568	4.1135	6.1703	8.2271	10.2838
	1	1.9463	3.0031	4.5736	6.3940	8.3358	10.3367
Values of	2	3.8926	4.4355	5.5065	6.9625	8.6681	10.5254
m (boys)	3	5.8389	6.1178	6.8037	7.88529	9.2875	10.9236
	4	7.7851	7.9285	8.3488	9.1107	10.2015	11.5725
	5	9.7314	9.8051	10.0568	10.5707	11.3810	12.4794

Thus, for example, the expected number of children needed to obtain a family of one boy and one girl is 3.0031—in practice, three children. On the other hand a much larger family (10 children) is required to obtain four boys and four girls.

Appendix

1. *Recurrence relation for the expected value $N(m,n)$*

Assuming $m,n \geqslant 1$, suppose the goal of at least m B's and n G's is reached with the $(m+n+u)$th child. Then we either have a sequence including $m+u$ B's and $n-1$ G's, followed by a G, or a sequence including $m-1$ B's and $n+u$ G's, followed by a B. The probability of the former is

$$\frac{(m+n+u-1)!}{(n-1)!\,(m+u)!}p^{m+u}q^{n} \quad (u = 0,1,\ldots),$$

whereas the probability of the latter is

$$\frac{(m+n+u-1)!}{(m-1)!\,(n+u)!}p^m q^{n+u} \quad (u = 0,1,\ldots).$$

Hence

$$N(m,n) = \sum_{u=0}^{\infty} (m+n+u)\left(\frac{(m+n+u-1)!}{(m-n)!\,(n+u)!}p^m q^{n+u} + \frac{(m+n+u-1)!}{(n-1)!\,(m+u)!}p^{m+u}q^n\right).$$

If we write

$$m+n+u = 1+(n+u)+(m-1)$$

in the first of the general terms of this series and

$$m+n+u = 1+(m+u)+(n-1)$$

in the second, we may deduce that

$$\begin{aligned}
N(m,n) = \sum_{u=0}^{\infty} &\left(\frac{(m+n+u-1)!}{(m-1)!\,(n+u)!}p^m q^{n+u} + \frac{(m+n+u-1)!}{(n-1)!\,(m+u)!}p^{m+u}q^n\right) \\
&+p\sum_{u=0}^{\infty}(m+n+u-1)\left(\frac{(m+n+u-2)!}{(m-2)!\,(n+u)!}p^{m-1}q^{n+u} + \frac{(m+n+u-2)!}{(n-1)!\,(m+u+1)!}p^{m+u-1}q^n\right) \\
&+q\sum_{u=0}^{\infty}(m+n+u-1)\left(\frac{(m+n+u-2)!}{(m-1)!\,(n+u-1)!}p^m q^{n+u-1} + \frac{(m+n+u-2)!}{(n-2)!\,(m+u)!}p^{m+u}q^{n-1}\right)
\end{aligned}$$

or, since the first term here sums the probabilities associated with each possible value of u,

$$N(m,n) = 1+pN(m-1,n)+qN(m,n-1).$$

2. *The negative binomial distribution*

Let X be the random variable denoting the number of trials needed for the occurrence of the kth success in a sequence of independent Bernoulli trials, where the probability of success is p. Then

$$P(X = x) = \frac{(x-1)!}{(k-1)!\,(x-k)!}p^k(q)^{x-k} \quad (x = k, k+1,\ldots;\; q = 1-p)$$

and X is said to have a negative binomial distribution.

The probability generating function of X is

$$G(t) = \sum_{x=k}^{\infty} \frac{(x-1)!}{(k-1)!\,(x-k)!} p^k (q)^{x-k} t^x$$

$$= \frac{(pt)^k}{(1-qt)^k}$$

and the mean of X is given by $G'(1)$. Hence, since

$$G'(t) = \frac{kp^k t^{k-1}}{(1-qt)^k} + \frac{(pt)^k qk}{(1-qt)^{k+1}},$$

the mean of X is

$$G'(1) = \frac{k}{p}.$$

Thus, in the notation of this article,

$$N(m,0) = \frac{m}{p} \quad \text{and} \quad N(0,n) = \frac{n}{q}.$$

That was the year that was (2)

Three positive integers x, y, z are said to form a *Pythagorean triple* if $x^2 + y^2 = z^2$. Did you know that the consecutive years 1984, 1985 are part of a Pythagorean triple, that the last time this happened was in the years 1860, 1861, and that the next time will be in the years 2112, 2113?

War Games

RICHARD I. LOEBL, *Wayne State University*

The author received his bachelor's degree from Harvard and his Ph.D. from Berkeley, and has been at Wayne since 1973. His primary research interests are in pure mathematics; his excursions into probability are usually by way of recreation!

1. War

I was asked by a friend to answer a question about a game called 'War'. In this game, the first player throws three dice and chooses the highest two; the second player throws the usual two dice. The dice are then matched up, top against top, second against second, with ties going to the second player. The question asked was: what are the respective probabilities that the first player wins both matchups, loses both matchups, or splits the matchups?

The question can be answered by straightforward enumeration, and the answer is not particularly enlightening. However, a more mathematical variant of the game suggested itself. Suppose that, instead of dice, the players picked positive integers at random; in this case, what are the respective probabilities? In particular, since ties are extremely unlikely in the new game, the second play loses his only advantage, so that initially the first player appears to have a three to two edge: is this so?

It is possible to extend this 'Infinite War' to the case in which both players may choose from among k and l 'dice' respectively, and where players may match their top three 'dice'. (Readers interested in this extension should get in touch with the author.)

The proof of our facts is suggested by a series of statements: the ambitious student may wish to verify these statements. Furthermore, the description given in section 3 also lends itself to computer experimentation, as we shall see at the end of the article.

2. A simple version of the game

Let us begin with the basic game. Suppose I roll $\{6, 3, 4\}$ with my three dice and my opponent rolls $\{4, 4\}$ with his pair of dice. The matchups are my 6 against his 4 (I win!) and my 4 against his other 4 (I lose since ties go to my opponent). Thus we split the matchups.

The case of standard six-sided dice can be analysed by straightforward calculations. To illustrate these calculations, let us examine an even simpler case, namely where the dice are two-sided (i.e. coins). Our possible throws are $\{(2,2,2); (2,2,1); (2,1,2); (2,1,1); (1,2,2); (1,2,1); (1,1,2); (1,1,1)\}$. We end with the pairs $(2,2)$, $(2,1)$ and $(1,1)$ with probabilities $\frac{4}{8}$, $\frac{3}{8}$ and $\frac{1}{8}$, respectively. Our opponent throws $(2,2)$, $(2,1)$ and $(1,1)$ with probabilities

$\frac{1}{4}$, $\frac{2}{4}$ and $\frac{1}{4}$, respectively. Table 1 summarizes the results, with W denoting a win and L a loss. We see that we have a double win WW with probability $\frac{4}{32}$, a double loss LL with probability $\frac{17}{32}$, and a split LW or WL with probability $\frac{11}{32}$.

Table 1.

	Result			Probability		
	22	21	11			
22	LL	LW	WW	$\frac{4}{32}$	$\frac{8}{32}$	$\frac{4}{32}$
21	LL	LL	WL	$\frac{3}{32}$	$\frac{6}{32}$	$\frac{3}{32}$
11	LL	LL	LL	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{1}{32}$

This procedure will handle the case of six-sided dice, although the arithmetic is tedious. It is not difficult to program a computer to do the calculations, and it is quite instructive to do so. My calculations give 0.37 as the probability of a double win, 0.29 as the probability of a double loss and 0.34 as the probability of a split.

Obviously, we could continue this procedure as long as our patience lasted. Let us, however, now consider the more mathematical version of the game, where the players pick positive integers at random; in this case what are the respective probabilities? Note that, as we pass from two-sided to six-sided dice, the second player begins to lose his advantage as ties become less likely.

3. The war game with integers

A uniform probability density on an infinite discrete set is impossible, so our approach will be to choose an integer from the set $\{1,2,\ldots,N\}$, with equal probabilities $1/N$, and then take the limit as $N \to +\infty$. We shall analyse in some detail the basic game where the first player takes the best two out of three; analogous results hold for the more general case where the first player takes the best two out of k and the second player takes the best two out of l.

Statement 1, which follows, will be used in the rest of our analysis. We assume that the second player chooses numbers from the set $\{1,2,\ldots,N\}$. Remember also that ties are won by the second player.

Statement 1. If the first player's two numbers are (m,n) with $m \geqslant n$, then:

(*a*) *the probability that the first player wins both is* $(2m-n-1)(n-1)/N^2$;

(*b*) *the probability that the first player loses both is* $(N-m+1)(N+m-2n+1)/N^2$;

(*c*) *the probability that a split occurs is* $[(m-n)^2+2(n-1)(N-m+1)]/N^2$.

For example, with $N = 10$, we have that, if the first player's two numbers are $(7,4)$, then the probability he wins both is $\frac{27}{100}$, the probability he loses both is $\frac{40}{100}$ since ties go to the second player, and the probability of a split is $\frac{33}{100}$.

Statement 2. If the first player chooses the best two of three numbers from the set $\{1,2,\ldots,N\}$, *then:*
(*a*) *the probability of the ordered pair* (m,n) *with* $m > n$ *is* $(6n-3)/N^3$;
(*b*) *the probability of the pair* (n,n) *is* $(3n-2)/N^3$.

Again with $N = 10$, the probability that the first player's pair will be $(7,4)$ is $\frac{21}{1000}$, which is the same as for the pairs $(10,4)$, $(9,4)$, $(8,4)$, $(6,4)$ and $(5,4)$. The doubles pair $(4,4)$ has probability $\frac{10}{1000}$.

Statement 3. If the first player takes the best two out of three from $\{1,2,\ldots,N\}$, *and the second player just takes two, then:*
(*a*) *the probability that the first player wins both matchups is*

$$\frac{1}{N^3}\frac{1}{N^2}\left(\sum_{m=2}^{N}\sum_{n=1}^{m-1}(2m-n-1)(n-1)(6n-3)+\sum_{n=1}^{N}(n-1)^2(3n-2)\right);$$

(*b*) *the probability that the first player loses both matchups is*

$$\frac{1}{N^3}\frac{1}{N^2}\left(\sum_{m=2}^{N}\sum_{n=1}^{m-1}(N-m+1)(N+m-2n+1)(6n-3)\right.$$
$$\left.+\sum_{n=1}^{N}(N-n+1)^2(3n-2)\right);$$

(*c*) *the probability of splitting the matchups is*

$$\frac{1}{N^3}\frac{1}{N^2}\left(\sum_{m=2}^{N}\sum_{n=1}^{m-1}[(m-n)^2+2(n-1)(N-m+1)](6n-3)\right.$$
$$\left.+\sum_{n=1}^{N}2(n-1)(N-n+1)(3n-2)\right).$$

Proof. The first summand accounts for the first player's having the pair (m,n) with $m > n$; the second summand accounts for the pair (n,n). The summands are computed by multiplying the results given in statements 1 and 2.

We illustrate the computations by evaluating the sums in statement 3(a). The second sum yields $(9N^4-14N^3+3N^2+2N)/12N^5$. Taking the limit as $N \to +\infty$, the final contribution of the second sum is 0; in other words, in the limit 'doubles' make no contribution.

In the first sum, the inner summation gives

$$\tfrac{1}{2}(5m^4-22m^3+34m^2-23m+6).$$

When $m = 1$, this expression is 0, so we change the lower limit of the outer summation from 2 to 1. Thus we must find

$$\sum_{m=1}^{N} \tfrac{1}{2}(5m^4-22m^3+34m^2-23m+6).$$

However, before finding this sum, notice that, after evaluating it, we must divide by N^5 and take the limit as $N \to +\infty$. We remind the reader of the following fact.

Statement 4. $\sum_{i=1}^{M} i^k = \dfrac{M^{k+1}}{k+1} + p(M)$, *where $p(M)$ is a polynomial of degree at most k.*

Sketch of proof.

$$(M+1)^{k+1} = \sum_{i=0}^{M} [(i+1)^{k+1}-i^{k+1}] = \sum_{i=0}^{M} [(i^{k+1}+(k+1)i^k+\ldots)-i^{k+1}]$$
$$= (k+1)\sum_{i=0}^{M} [i^k+\ldots].$$

Now expand the left-hand side and use induction.

Using this fact, we see that the only term of

$$\sum_{m=1}^{N} \tfrac{1}{2}(5m^4-22m^3+34m^2-23m+6)$$

that will contribute to the limit is the highest-order term. We then compute

$$\sum_{m=1}^{N} \tfrac{5}{2}m^4 = \tfrac{5}{2}(\tfrac{1}{5}N^5+\text{lower terms});$$

dividing by N^5 and taking the limit as $N \to +\infty$, we obtain the final answer $\frac{1}{2}$. We summarize the results.

Statement 5. If the first player takes the best two out of three integers picked, then as $N \to +\infty$ the probability that the first player wins both match-ups is $\frac{1}{2}$, loses both is $\frac{1}{5}$, and splits is $\frac{3}{10}$.

For the case when $N = 10$, the exact answers are 0.423 60 for a double win, 0.253 33 for a double loss and 0.323 07 for a split. The basic reason that the chances of a loss are greater is the possibility of ties (won by the second player) which becomes negligible as $N \to +\infty$.

Note that the expected number of wins for the first player is $2(\frac{1}{2})+1(\frac{3}{10}) = \frac{13}{10}$, and that the expected number of losses is $2(\frac{1}{5})+1(\frac{3}{10}) = \frac{7}{10}$. Therefore, the first has a 13 to 7 advantage, much better than the naive prediction of a 3 to 2 advantage. We cannot explain this phenomenon. (For the case $N = 10$, the ratio is 1.17027/0.82973, or 2.8 to 2. For the case $N = 6$, the ratio is 1.17 to 1, so that the passage from $N = 6$ to $N = 10$ already shows a substantial improvement for the first player. The cases $N \leqslant 4$ even favoured the second player.)

4. Concluding remarks

As mentioned in section 1, the methods used can be applied to the case when the first player takes the best two of k, and the second player takes the best two of l integers. It is fairly straightforward to program these games, especially as described in section 3. The analysis given there works not only for integers but also for real numbers. That is, the same formulae apply if the numbers used are random numbers (say between 0 and 1, as generated by an Apple computer). The reader may wish to confirm experimentally the results for two or three matchups, and may wish to determine empirically the results for four matchups.

Acknowledgement

Thanks are due to Leonard Dor and Peter Malcolmson for helpful conversations concerning these questions. I also wish to thank Alfred Gruber for bringing the problem to my attention. The games described are similar to the game known as 'Risk', and of course to the childhood card matching game called 'War'.

That was the year that was (3)

Last year we asked readers to try to construct the numbers 1 to 100 using the digits of the year 1984 in their correct order, using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$ and ! [e.g. $21 = 19+(8\div4)$, $49 = -1+((\sqrt{9})!\times8)+\sqrt{4}$]. Only two numbers eluded readers, namely 63 and 83. Subscribers to *Spectrum* must have wondered when we would ask the inevitable question. Well here it is! Try the same thing with the year 1985. Readers who are successful with more than 92 of the numbers are invited to send in their solutions. (The editor is stuck with 8 of the numbers!)

Computer Column

MIKE PIFF

A finite rod of length 1 has its ends kept at temperature zero, and initially is given a temperature distribution $U = \sin(\text{pi}*x)$. For large values of m and n, the temperature at the point i/n at the time j/m is given approximately by

$$U[i,j] = U[i,j-1]+r*(U[i-1,j-1]-2*U[i,j-1]+U[i+1,j-1])$$

for $i = 1,\dots,n-1$, and, of course,

$$U[0,j] = U[n,j] = 0, \quad \text{for } j = 1,\dots,m-1,$$
$$U[i,0] = \sin(\text{pi}*i/n), \quad \text{for } i = 1,\dots,n-1,$$

where $r = n*n/m$.

Thus, for $n = 10$ and $m = 300$, say, we can calculate the temperature changes over the first second of time by keeping an array $U[10,300]$, calculating the entries $U[i,0]$, then using these to calculate all entries $U[i,1]$, $U[i,2]$, etc, successively. Write a program to calculate and display these values to give a moving picture of the graph of U against x. Note that your BASIC may not allow zero indices, so you may have to let i run from 1 to 11, to represent $x = (i-1)/n$, and similarly for j.

If $n = 100$ and $m = 30\,000$, you will almost certainly find that $U[100,30\,000]$ is inadmissible. How can you get away with just using two or even one one-dimensional arrays $U[100]$?

What we are doing in the above is to solve the partial differential equation for heat conduction by a simple finite-difference technique. The initial distribution above is completely arbitrary. You may like to experiment with various other distributions of heat. For example, try

$$U[n/2,0] = 10, \qquad U[i,0] = 0 \text{ for } i <> n/2.$$

Another easy change is to take, say, $U[0,j] = 0$, $U[n,j] = 10$ and initial distribution

$$U[i,0] = 10*i/n+i*(i-n)*f(i)$$

for some chosen function $f(i)$. What is the result?

Note that the above method only works for $r < 0.5$.

Letters to the Editor

Dear Editor,

Dividing by x

In Volume 17 Number 3, page 92, you gave Ruth Lawrence's solution to L.B. Dutta's problem on dividing by 11. It is a simple matter to generalize Ruth's solution to prove the following result: if a number m is expressed to an arbitrary base $x-1$ as $a_n a_{n-1} \ldots a_0$, so that

$$m = a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots + a_n(x-1)^n,$$

then m has the same remainder on division by x as does

$$a_0 - a_1 + a_2 - a_3 + a_4 - \ldots .$$

Yours sincerely,

ANDREA CARLSON
(Freshman Member of St. Olaf College
Problem Solving Class,
Northfield, Minnesota.)

Dear Editor,

Dividing by 7 or 13

Your little item on page 94 of Volume 17 Number 3 was rather intriguing and interesting as tests for divisibility by 7 are hard to come by, even if the test itself *does* require division by 7! (no, not the factorial variety). However, I have devised an alternative method, which may be considered simpler.

Divide the number N by 1000 to give quotient q_1 and remainder r_1. Then divide q_1 by 1000 to give quotient q_2 and remainder r_2 etc., until a zero quotient occurs. This is equivalent to putting commas in the number N every 3 digits from the right and calling the right-hand group r_1, the next r_2, etc. Then both N and $r_1 - r_2 + r_3 - r_4 + \ldots$ have the same remainder when divided by either 7 or 13 or indeed 11, 77, 91, 143, not forgetting 1001 itself. Perhaps readers can work out why this is so. As a clue,

$$1001 = (7 \times 11 \times 13) - 1.$$

Yours sincerely,

BOB BERTUELLO
(12, Pinewood Road,
Midsomer Norton,
Bath, Avon BA3 2RG.)

Dear Editor,

Consecutive positive cubes

Can any reader provide a simple proof that the difference between two consecutive positive cubes can never be a cube [i.e. $(n+1)^3-n^3 \neq k^3$ when n and k are positive integers]?

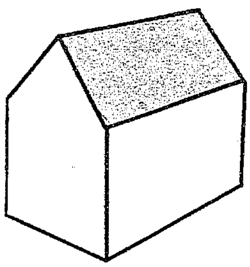
Yours sincerely,
PETER DERLIEN
(50, Filey Street,
Sheffield S10 2FG.)

Problems and Solutions

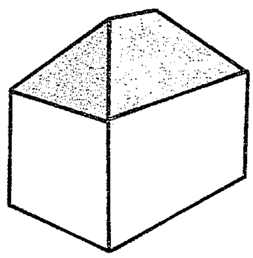
Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

18.1. (Submitted by Mary Crowley, Dalhousie University, Canada)
Two houses have the same dimensions for their rectangular bases. One is gabled and the other has a cottage roof, and the slopes of their roofs are the same. Which is more economical on roof-felting?



gabled roof



cottage roof

18.2. (Submitted by T. G. Dale, Perth)
Which of e^{π} and π^{e} is larger? (Calculators are not allowed!)

18.3. (Submitted by Louis Funar, University of Craiova, Romania)
Prove that there exist integers a and b such that $|a|, |b| \leqslant 10^7$ and

$$0 < |a+b\sqrt{2}| < 3\times10^{-7}.$$

Solutions to Problems in Volume 17 Number 2

17.4. Determine $(9+4\sqrt{5})^{1/3}$ as a quadratic surd (i.e. in the form $a+b\sqrt{c}$, where a, b and c are rational numbers).

Solution 1 by Richard Dobbs (Magdalen College, Oxford)

Write $x_1 = (9+4\sqrt{5})^{1/3}$ and $x_2 = (9-4\sqrt{5})^{1/3}$. Then $x_1x_2 = 1$. Put $x_1+x_2 = s$. Then

$$s^3 = x_1^3+3x_1^2x_2+3x_1x_2^2+x_1^3 = 18+3s,$$

so that

$$s^3-3s-18 = 0,$$

that is,

$$(s-3)(s^2+3s+6) = 0.$$

Now s^2+3s+6 has no real roots: so $s = 3$. Thus x_1 and x_2 are the roots of the polynomial x^2-3x+1: so

$$x_1 = \tfrac{1}{2}(3+\sqrt{5}).$$

Solution 2 by Adrian Hill (The Royal Grammar School, High Wycombe)

Put $y = 9+4\sqrt{5}$. Then $y^2-18y+1 = 0$. Put $x = (9+4\sqrt{5})^{1/3}$. Then $x^3 = y$, so $x^6-18x^3+1 = 0$. This factorizes to give

$$(x^2-3x+1)(x^4+3x^3+8x^2+3x+1) = 0.$$

The quartic factor has no positive solutions, so that

$$x^2-3x+1 = 0$$

and

$$x = \tfrac{1}{2}(3\pm\sqrt{5}).$$

Since $\sqrt{5} > 2$, $3-\sqrt{5} < 1$. Hence $x = \tfrac{1}{2}(3+\sqrt{5})$.

Also solved by Malcolm Smithers (The Open University), Ruth Lawrence (St. Hugh's College, Oxford), Michael McQuillan (University of Glasgow) and David Perkins (Chatham House Grammar School, Ramsgate).

17.5. Prove that, when p and q are prime numbers greater than 5, then p^4-q^4 is always divisible by 10.

We received solutions to this problem from Richard Dobbs (Magdalen College, Oxford), Deepa Gumaste (New Ash Green Middle School, Dartford), G. M. Hamilton-Hopkins (The Open University), Malcolm Smithers (The Open University), Ken Lee (The Open University), Stephen Lewis (Wootton Upper School, Bedford), Ruth Lawrence (St. Hugh's College, Oxford), Michael McQuillan (University of Glasgow), David Perkins (Chatham House Grammar School, Ramsgate), Erik Cleven, Jim Swift and Steve Morics (St. Olaf Problem Solving Group, Northfield, USA). Brian Taylor (Lancashire Polytechnic) proved the improved result that p^4-q^4 is always divisible by 40, but Adrian Hill (The Royal Grammar School, High Wycombe) and Guy Willard (Haberdasher' Ashe's School, Elstree) went one better, and proved that p^4-q^4 is always divisible by 80. Their proofs were essentially the same, as follows.

The numbers p and q end in 1, 3, 7 and 9, so p^4 and q^4 both end in 1 because 1^4, 3^4, 7^4 and 9^4 all end in 1. Hence p^4-q^4 must end in 0 and p^4-q^4 is divisible by 10. Next,

$$p^4-q^4 = (p^2-q^2)(p^2+q^2).$$

Put $p = 2m+1$, $q = 2n+1$, where m and n are integers. Then

$$p^2-q^2 = (p-q)(p+q) = 4(m-n)(m+n+1).$$

Now $m-n$ and $m+n$ are either both even or both odd, so one of $m-n$ and $m+n+1$ is even. Hence p^2-q^2 is divisible by 8. Since p^2+q^2 is even, this means that p^4-q^4 is divisible by 16. Since we already know that it is divisible by 10 and so 5, it must therefore be divisible by 80.

We note that nowhere has the proof used the fact that p and q are prime, only that they are odd and not divisible by 5.

17.6. Prove that, for $0 \leqslant x \leqslant \frac{1}{2}\pi$,

$$\tanh x \geqslant \sin(\sin x) \geqslant \sin x \cos x,$$

and the equality occurs if and only if $x = 0$.

Solution by Michael McQuillan (University of Glasgow)

We first note that, for $0 < x \leqslant \frac{1}{2}\pi$, $x > \sin x$, so that

$$\cos x < \cos(\sin x) \quad \text{for } 0 < x \leqslant \tfrac{1}{2}\pi. \tag{*}$$

Now consider the function f defined by

$$f(x) = x-\tanh^{-1}\{\sin(\sin x)\} \quad \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}\pi.$$

Now

$$f'(x) = 1-\frac{\cos(\sin x)\cos x}{\cos^2(\sin x)} = 1-\frac{\cos x}{\cos(\sin x)} > 0 \quad \text{for } 0 < x \leqslant \tfrac{1}{2}\pi,$$

so that f is strictly increasing for $0 < x \leqslant \frac{1}{2}\pi$. Now $f(0) = 0$, so that $f(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, with $f(x) = 0$ if and only if $x = 0$. Thus

$$\tanh x \geqslant \sin(\sin x),$$

with equality if and only if $x = 0$.

Now consider the function g defined by

$$g(x) = \sin(\sin x)-\sin x \cos x \quad \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}\pi.$$

Then

$$\begin{aligned} g'(x) &= \cos(\sin x)\cos x-\cos^2 x+\sin^2 x \\ &\geqslant \sin^2 x, \quad \text{from } (*), \\ &> 0 \quad \text{for } 0 < x \leqslant \tfrac{1}{2}\pi. \end{aligned}$$

Thus g is strictly increasing for $0 < x \leqslant \frac{1}{2}\pi$ and $g(0) = 0$, so that $g(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ with $g(x) = 0$ if and only if $x = 0$, i.e.,

$$\sin(\sin x) \geqslant \sin x \cos x$$

with equality if and only if $x = 0$.

Also solved by Ruth Lawrence (St. Hugh's College, Oxford).

Book Reviews

Microchip Mathematics: Number Theory for Computer Users. By KEITH DEVLIN. Shiva Publishing Limited, Nantwich, Cheshire, 1984. Pp. vii+205. Paperback £12·95.

The first chapter of the book deals with prime numbers. In an easy-to-read fashion it leads up to conjectures on the subject; examples of recent results related to their resolution are given.

The second chapter deals with the 'basic concepts' of number theory, spread over forty pages. The reader is introduced first to mathematical induction and then to the Euclidean algorithm and its efficiency and applications. Apart from one page describing the practical use of the numerous lemmas, corollaries and theorems for the computer user, the chapter is pure maths (pun intended) throughout. Although the preface suggests the reader could omit the proofs, it was sometimes worth reading through them in order to gain a more thorough understanding of the section.

The third chapter deals with congruencies, and I found this the hardest and most complicated part of the book. I think the computer user with no strong mathematical grounding will be in a sticky position, as there is a quite overwhelming stream of theorems about congruencies which are set out in textbook form. However, the second section of the chapter, which deals with modular arithmetic, sets out the maths in plain English, and I think this will be much clearer for computer users as well as for people who have not come across number theory before.

In the final chapter, prime numbers are discussed in an interesting way, giving the reader some idea of the work currently being done in number theory. For example, there is a section on the use of number theory in public-key cryptography.

At the end of each chapter, in textbook tradition, there are exercises—maths exercises and computer exercises. I was able to solve quite a few of the latter on my ZX81, so they are quite reasonable questions.

Dr Devlin states that his book is written for undergraduates and the 'keen but largely untrained "amateur"'. I am an A-level maths student, and I found the book hard going. So, unless the amateur has an A-level background, he or she will probably have a tough time with this book.

The book is rather expensive but ought to find its way into the sixth-form library, so that students can be introduced to an area of maths they have probably not met before.

The Sixth Form, Tapton School, Sheffield PETER AUSTIN

Mathematics for Everyman. By LAURIE BUXTON. J. M. Dent & Sons Ltd, London, 1984. Pp. 270. £12·95 (hardback); £3·95 (paperback).

I well remember my first chemistry lesson at age 11. The class gathered round and the master gave a lively 40-minute demonstration of chemical spectaculars: explosions, conflagrations, effervescences, changes of colour, liquids changing into solids ... and we watched enthralled. This book is a little like that—a sequence of mathematical conjuring tricks, thought it is only fair to say that in many cases the dénouement is revealed in the appendix. Perhaps the spirit of this book may be summarised by a quotation (page 163): 'The geometry of the parabola is pure mathematics. Dropping cannon balls on people is applied mathematics.'

Much of the material is sound mathematics and, while there are places where it slips into murky waters, there are also examples of pearls of mathematical wisdom and truths expressed as well as I have seen anywhere. I would quarrel to some extent with the order of presentation; for example, would not Chapter 12 have been a splendid way of starting, to arouse interest, particularly as no previous knowledge is required?

I have some criticisms of both subject matter and style. On the question of subject matter, the target readership implied by the title is 'everyman'. I hardly think that everyman—unless he is a very special breed of everyman(!)—is going to have the slightest interest in the somewhat prolix excursions into philosophy (e.g. pages 223 ff., 254 ff.). Moreover, some of the problems and investigations which are 'left to the reader' are of mind-bending difficulty or complexity (e.g. page 84, lines 18–20; page 252, the final four questions). On the question of style, the author shows that he can write well, both in a formal textbook style as well as in a literary style suitable for this type of book. But why does he so frequently have to lapse into slang and colloquialisms which reduce the level to that of a mathematical 'chat-show' (for the 'kids')? One can forgive 'dots' for points and 'tricks' for formulae. He does not shrink from using the correct terms 'commutative', 'associative' and 'distributive', yet does shrink from calling the 'top point' of a cone by its proper name 'apex'. The excesses of colloquialism are legion: 'overkill', 'slippery' (of the quintic), 'we are in business', 'Bingo!', a 'chunk of the curve', 'lump it up', and so on. There are those of course who will claim that this easy conversational style is colourful and makes a welcome change. The reference (page 36) to extramarital sex (which, unbelievably, is listed in the index) is in very questionable taste.

I noted several errors, in particular some serious ones occurring in the lower half of page 124; and on page 238, (2) is an example of 'derangements' and the answer should be $1/2!-1/3!+1/4!-1/5!+\dots+1/52! \simeq 1/e \simeq 0.36788$. There are a number of minor omissions and missed opportunities. Incidentally, 'the man' on page 220 was Pontius Pilate!

I found Chapters 7, 11 and 12 the most successful and Chapters 8, 9 and 10 the poorest.

This would be a good book if, in subsequent editions, some of the shortcomings mentioned could be amended.

Formerly Head of the Mathematics Department
The Royal Grammar School
Newcastle upon Tyne

F. J. BUDDEN

Mathematics: People, Problems, Results, Volumes 1–3. Edited by DOUGLAS M. CAMPBELL and JOHN C. HIGGINS. Wadsworth International, Belmont, California, 1984. Pp. xvi+304, iv+275, iv+292. Price for the set $53·95 (hardback), $37·75 (paperback).

If a television producer decided to make a programme on mathematics, for a general audience, then these three volumes would make an ideal source of material. To quote from the preface: 'They (the volumes) are an introduction to the spirit of mathematics. The purpose of this anthology is to give the non-mathematician some insight into the nature of mathematics and those who create it.' The anthology consists of over ninety articles, published mainly in the latter half of this century. The authors include Jacob Bronowski, John von Neumann, David Hilbert, Donald E. Knuth, Douglas R. Hofstadter, Le Corbusier and George Pólya. The articles fall under such headings as history, the development and nature of mathematics, real mathematics ('The essays in this part are all quite readable if the reader is willing to make a modest investment of time and thought. The key word is *investment*.'), foundations and philosophy, computers, mathematics in art and nature, uses—statistics and applied mathematics, sociology and education.

To quote again: 'If this anthology has a central purpose, it is to help the reader answer the question "why mathematics?... Why ought we to care?...". But to understand why mathematics exists and why it is perpetuated, one must know something of its history and of the lives of the mathematicians and their productions.... Our intent is to allow the reader to examine mathematics from as many different viewpoints as our space permits... We would not presume to tell readers what mathematics is. Let them read the articles, listen to the voices, and form their own opinion.' For example, one question which the reader of these volumes will be more qualified to discuss is the objective/subjective dichotomy in mathematics; would mathematics have existed if there had been no human beings, or is it a creation of the mathematician's mind? Incidentally, note how this question reflects similar ones in other subjects. In theology, we can ask whether God exists independently of the human mind or is He just a creation of man which embodies his highest ideals? In literary criticism, should we consider just the text, or the text and the reader? These volumes will allow the reader to appreciate the usefulness, the importance, the interest, the attractiveness and the truth of mathematics. Hence I can strongly recommend that they should be in the library of your school, college or university. To give just one more quotation from the many which catch the eye and challenge the mind: 'You are proposing to give a precise definition of logical correctness which is to be the same as my vague intuitive feeling for logical correctness. How do you intend to show that they are the same?... The average mathematician should not forget that intuition is the final authority.' These volumes tell us something important that we may not have realised. *Mathematics has style!*

University of Sheffield

KEITH AUSTIN

CONTENTS

ISSN 0025-5653

PRICES (*postage included*)

Prices for Volume 18 (Issues Nos. 1, 2 and 3)

Subscribers in	*Price per volume*
North, Central and South America	US\$10.00 or £8.00
Australia	\$A11.00 or £8.00
Britain, Europe and all other countries	£4.00

(Note: These overseas prices apply even if the order is placed by an agent in Britain.)

A discount of 10% will be allowed on all orders for five or more copies of Volume 18 sent to the same address.

Back issues

All back issues are available; information concerning prices and a list of the articles in Volumes 1 to 17 may be obtained from the Editor.

Enquiries about rates, subscriptions and advertisements should be directed to:

Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Published by the Applied Probability Trust
Typeset by the University of Nottingham
Printed in England by Galliard (Printers) Ltd, Great Yarmouth