Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 6, No. 6

June - July 1980

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin

The assistance of the publisher and the support of the Canadian Mathematical Olympiad Committee, the Carleton University Mathematics Department, the Ottawa Valley Education Liaison Council, and the University of Ottawa Mathematics Department are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $10.00. Back issues: $1.00 each. Bound volumes with index: Vols. 1&2 (combined), $10.00; Vols. 3,4,5, $10.00 each. Cheques and money orders, payable to CRUX MATHEMATICORUM (in US funds from outside Canada), should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

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PROPRIÉTÉS GÉOMÉTRIQUES ET APPROXIMATION DES FONCTIONS: II

J.G. DHOMBRES

Les résultats acquis dans un article précédent [1980: 132] sont utilisés pour démontrer un élégant résultat d'analyse harmonique.

3. *Un exemple de meilleure approximation.*

Nous allons nous placer sur l'ensemble E des fonctions définies sur l'axe réel, à valeurs complexes, périodiques de période 2π, continues ou possédant un nombre fini de discontinuités de première espèce sur l'intervalle d'une période.

Sur cet espace E, nous construisons une norme intégrale en posant

$$\|f\| = \frac{1}{2\pi}\int_{-\pi}^{+\pi}|f(t)|dt.$$

Introduisons tout de suite un outil dont nous aurons besoin sur E, à savoir le coefficient de Fourier $c_n(f)$ d'ordre n d'une fonction f de E. Par définition, on a pour tout entier relatif n

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)e^{-int}dt.$$

Soit alors Λ un sous-ensemble non vide et propre de l'ensemble des entiers relatifs. On prend pour $F(\Lambda)$ le sous-ensemble des f de E telles que $c_n(f) = 0$ pour tout n n'appartenant pas à Λ. On laisse au lecteur le soin de montrer que $F(\Lambda)$ est un sous-espace fermé propre de E. Notre propos est d'approximer un élément quelconque de E par $F(\Lambda)$. Nous allons un peu spécialiser cette étude en remarquant que si f prend des valeurs réelles et $c_n(f) = 0$ alors $c_{-n}(f) = 0$ également.

Soit alors Λ un sous-ensemble de l'ensemble des entiers positifs ou nuls et soit $S(\Lambda)$ la réunion de Λ et de $-\Lambda$. On dispose du théorème suivant (cf. [3]), surprenant au premier abord:

THÉORÈME 5. *Soit Λ un sous-ensemble des entiers positifs ou nuls. Pour tout f de E il existe une unique meilleure approximation par les éléments de $F(S(\Lambda))$ si et seulement si Λ est une progression arithmétique infinie de raison impaire et débutant à* 0.

A) On commence par prouver que la condition donnée au Théorème 5 est nécessaire. On procède en quatre étapes. Dans les trois premières étapes, Λ peut désigner un sous-ensemble propre non vide de l'ensemble des entiers relatifs.

Démonstration.

1ère étape: *Caractérisation de certaines formes linéaires sur E.*

Pour pouvoir utiliser le Théorème 4, il convient d'abord de caractériser les formes linéaires complexes sur E muni de sa norme intégrale, et qui satisfont l'égalité (1) du Théorème 4. Soit g une fonction définie sur R et à valeurs complexes, périodique et de période 2π, que nous supposerons bornée. L'application définie sur E, à valeurs complexes, donnée par l'expression

$$L_g(f) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)g(t)\,dt,$$

est linéaire dès que fg est intégrable sur $[-\pi,+\pi]$ pour tout f de E. En outre on dispose de la majoration

$$\|L_g(f)\| \leq \frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(t)|\ |g(t)|dt \leq (\sup_{t\in R}|g(t)|)\|f\|.$$

Il est donc clair que

$$\|L_g\| = \sup_{\substack{\|f\|\leq 1\\ f\in E}} |L_g(f)| \leq \sup_{t\in R}|g(t)| = \|g\|_\infty.$$

De fait, on a même $\|L_g\| = \|g\|_\infty$. Nous nous contenterons de démontrer ce point dans le cas où g est continue et prend des valeurs réelles. Il existe alors un point x_0 de $[-\pi,+\pi]$ où $|g(x_0)| = \sup_{t\in R}|g(t)|$ d'après la propriété bien connue des fonctions continues. Donc, pour tout $\varepsilon > 0$, on peut trouver un $\eta > 0$ tel que pour tout x satisfaisant $|x - x_0| < \eta$ on ait $|g(x)| \geq |g(x_0)|(1-\varepsilon)$.

Puisque, pour une fonction périodique h et de période 2π, on a

$$\int_{-\pi}^{+\pi} h(t)\,dt = \int_{-\pi+x_0}^{+\pi+x_0} h(t)\,dt,$$

on peut toujours supposer que $x_0 \in]-\pi,+\pi[$ et prendre η assez petit de sorte que $]x_0-\eta,x_0+\eta[\subset [-\pi,+\pi]$. On définit alors une fonction f sur $[-\pi,+\pi]$, que l'on prolonge par périodicité 2π, en posant

$$\begin{cases} f(t) = 0 \quad \text{si} \quad t \notin]x_0-\eta,x_0+\eta[, \\ f(x_0) = \dfrac{2\pi}{\eta}, \\ \text{et } f \text{ continue et linéaire par morceaux.} \end{cases}$$

On calcule que $\|f\| = 1$ et en outre on minore $L_g(f)$ selon

$$|L_g(f)| = \frac{1}{2\pi}\left|\int_{x_0-\eta}^{x_0+\eta} f(t)g(t)\,dt\right|$$
$$\geq |g(x_0)|(1-\varepsilon) = \sup_{t\in R} |g(t)|(1-\varepsilon).$$

Comme cette inégalité reste vraie pour tout $\varepsilon > 0$, on a l'inégalité inverse cherchée $\|L_g\| \geq \|g\|$. D'où l'on déduit $\|L_g\| = \|g\|_\infty$.

Ce que nous venons de voir montre que toute fonction g de borne supérieure égale à 1 et 2π-périodique (telle que fg soit intégrable) fournit une forme linéaire L_g sur E satisfaisant la relation $\|L_g\| = 1$. On constate que $L_{g_1} = L_{g_2}$ implique $g_1 = g_2$, comme il est bien facile de le voir du moins dès que g_1 et g_2 sont continues. Nous admettrons[1] la réciproque, a savoir que toute forme linéaire L sur E qui vérifie l'égalité (1) du Théorème 4, soit $\|L\| = 1$, provient ainsi d'une fonction g dont le module a une borne supérieure égale à 1 sur l'axe réel, 2π-périodique (et telle que fg soit intégrable pour tout f de E). Cela revient à dire $L = L_g$.

2ème étape: Détermination de g.

Rappelons que nous voulons établir une condition nécessaire. Soit f un élément de E et Pf sa meilleure approximation par $F(\Lambda)$. Ici, pour commencer, Λ est un sous-ensemble propre non vide de l'ensemble des entiers relatifs. D'après le Théorème 4, on dispose donc pour toute fonction f de E d'une fonction g telle que

$$\sup_{t\in R} |g(t)| = 1, \tag{1}$$

$$L_g(f) = \|f - Pf\|, \tag{2}$$

et

$$L_g(h) = 0 \text{ pour tout } h \text{ de } F(\Lambda). \tag{3}$$

Réécrivons (2) sous une forme intégrale en tenant compte de (3):

$$\begin{aligned}\int_{-\pi}^{+\pi} f(t)g(t)dt &= \int_{-\pi}^{+\pi} (f(t) - Pf(t))g(t)dt \\ &= \int_{-\pi}^{+\pi} |f(t) - Pf(t)|dt;\end{aligned} \tag{2 bis}$$

ce qui s'écrit encore

$$\int_{-\pi}^{+\pi} [|f(t) - Pf(t)| - (f(t) - Pf(t))g(t)]dt = 0.$$

Mais on a $|f(t) - Pf(t)g(t)| \leq |f(t) - Pf(t)|$ en tous points t de $[-\pi,+\pi]$. Si g

---

[1] Cette réciproque exige la notion d'intégrale de Lebesgue. On pourra consulter [4] et [5] entre autres multiples références.

était continue, nous savons bien que ces deux relations impliquent

$$|f(t) - Pf(t)| = (f(t) - Pf(t))g(t). \tag{4}$$

Nous admettrons que cette relation reste vraie en général (quitte éventuellement à modifier g quelque peu). Cette dernière égalité (4) détermine g en tous points où $f - Pf$ ne s'annule pas.

Montrons maintenant que *Pf est une meilleure approximation de f dans $F(\Lambda)$ si et seulement si l'on a une fonction g satisfaisant* (1), (4) *et satisfaisant également la relation*

$$c_n(g) = 0 \text{ pour tout } n \text{ de } -\Lambda. \tag{5}$$

Il suffit de montrer que (5) équivaut à (3). D'une part si l'on a (3), puisque e^{-int} appartient à $F(\Lambda)$ si n appartient à $-\Lambda$, on a bien (5). Réciproquement si l'on a (5), on a encore

$$\int_{-\pi}^{+\pi} g(t)h(t)dt = 0$$

pour toute somme finie de la forme

$$h(t) = \sum_{k\in\Lambda} c_k e^{ikt} \tag{6}$$

où les c_k sont complexes. Nous admettrons (théorème de Féjer) que toute h de $F(\Lambda)$ s'obtient comme limite, au sens de la norme intégrale de fonctions du type précédent (6). On en déduit (3) par passage à la limite.

3ème étape: Invariance de la propriété par translation.

Λ désigne encore un sous-ensemble non vide et propre de l'ensemble des entiers relatifs tel que tout f de E ait une unique meilleure approximation Pf par $F(\Lambda)$.

Soit n_0 un entier relatif quelconque et désignons par Λ_0 le sous-ensemble $n_0 + \Lambda$. Montrons que tout f de E possède une meilleure approximation par $F(\Lambda_0)$. En effet, soit $f \in E$. Par hypothèse, il existe une unique approximation dans $F(\Lambda)$ de $f(t)e^{-in_0t}$. Grâce au Théorème 4, on dispose donc de g telle que

$$\begin{cases} \sup\limits_{t\in R}|g(t)| = 1, \\ |f(t)e^{-in_0t} - P(f(t)e^{-in_0t})| = \left(f(t)e^{-in_0t} - P(f(t)e^{-in_0t})\right)g(t), \\ c_n(g) = 0 \text{ pour tout } n \text{ de } -\Lambda. \end{cases}$$

Par suite, $f_0(t) = e^{in_0t}P(f(t)e^{-in_0t})$ appartient à $F(\Lambda_0)$ puisque, pour tout h de E, on a

$$c_n(h(t)e^{in_0t}) = c_{n-n_0}(h(t)).$$

Posons $g_0(t) = e^{-in_0t}g(t)$. Il vient les trois relations

$$\begin{cases} \sup\limits_{t\in R}|g_0(t)| = 1, \\ |f(t)-f_0(t)| = (f(t)-f_0(t))g_0(t), \\ c_n(g_0) = 0 \text{ pour tout } n \text{ de } -\Lambda_0. \end{cases}$$

Grâce au Théorème 4 une nouvelle fois, on constate que $f_0 \in F(\Lambda_0)$ est une meilleure approximation de f par $F(\Lambda_0)$. L'unicité d'une telle meilleure approximation provient de ce que le passage de Λ à Λ_0 est reversible par une opération de même nature.

La propriété établie à la 3ème étape se démontre simplement grâce à l'utilisation de l'exponentielle complexe. C'est uniquement pour cette raison de simplicité que nous nous sommes imposés de prendre le corps des complexes C comme corps de base des espaces vectoriels considérés au lieu de se restreindre à R.

4ème étape: Λ désigne maintenant un sous-ensemble non vide et propre de l'ensemble des entiers positifs ou nuls. Nous supposons que tout f de E a une unique meilleure approximation par $F(S(\Lambda))$. Rappelons que $S(\Lambda) = \Lambda \cup (-\Lambda)$.

(a) Commençons par supposer que 0 n'appartient pas à Λ. Soit $f \in E$ et $f(t) \geq 0$ pour tout t de R. On a nécessairement $Pf = 0$ car on peut prendre $g(t) \equiv 1$ et on a ainsi une fonction g satisfaisant (1), (4), et enfin (5).

Soit alors n un élément de Λ. Considérons les deux fonctions $f_1(t) \equiv 1$ et $f_2(t) = 1 - \sin nt$. Ces deux fonctions sont positives et $f_2 - f_1$ est un élément non nul de $F(S(\Lambda))$. Par suite $Pf_1 = Pf_2 = 0$ et on dispose alors des inégalités suivantes contradictoires puisqu'il y a unicité supposée de la meilleure approximation:

$$\|f_1\| = \|f_1 - Pf_1\| < \|f_1 - (f_1 - f_2)\| = \|f_2\|$$

et

$$\|f_2\| = \|f_2 - Pf_2\| < \|f_2 - (f_2 - f_1)\| = \|f_1\|.$$

Par suite 0 appartient toujours à l'ensemble Λ.

(b) Supposons maintenant que n_1 et n_2 soient deux éléments de Λ. Montrons que si $\frac{1}{2}(n_1 + n_2)$ est un nombre entier, alors $(n_1 + n_2)/2$ et $|n_1 - n_2|/2$ appartiennent à Λ.

En utilisant $S(\Lambda)$, il suffit de montrer que si n_1' et n_2' sont dans $S(\Lambda)$ et si $(n_1' + n_2')/2$ est un entier relatif, alors $k = (n_1' + n_2')/2$ appartient à $S(\Lambda)$. On utilise alors l'ensemble $\Lambda_0 = S(\Lambda) - k$ qui, d'après la 3ème étape, possède la même propriété d'unicité que $S(\Lambda)$. Selon la même démarche qu'en (a), on aboutit à une contradiction si $0 \notin \Lambda_0$, donc si $k \notin S(\Lambda)$.

(c) Soient λ et λ' deux éléments successifs et distincts de l'ensemble Λ, c'est-à-dire $\lambda < \lambda'$, $\lambda, \lambda' \in \Lambda$ et il n'y a aucun élément de Λ entre λ et λ'.

Si $\lambda' - \lambda$ est un nombre entier pair, le point milieu $(\lambda + \lambda')/2$ est un entier qui appartient à Λ grâce à (b). Donc il y a contradiction. On doit donc toujours avoir un intervalle de longueur impaire entre deux entiers successifs de Λ.

(d) Montrons que Λ est un ensemble infini. Par l'absurde, supposons que tout λ de Λ satisfasse $0 \leq \lambda < n$. Considérons alors la fonction $f(t)$ définie sur l'axe réel par

$$\begin{aligned} f(t) &= +1 \text{ si } \sin nt > 0, \\ &= 0 \text{ si } \sin nt = 0, \\ &= -1 \text{ si } \sin nt < 0. \end{aligned}$$

Cette fonction est périodique intégrable et de période $2\pi/n$. Par suite, supposons que k soit un entier relatif, non multiple de n. On a $c_k(f) = 0$ puisque

$$\begin{aligned} c_k(f) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t) e^{-ikt} dt = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t + \frac{2\pi}{n}) e^{-ikt} dt \\ &= (\frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t) e^{-ikt} dt) e^{2ik\pi/n}. \end{aligned}$$

Soit $(1 - e^{2ik\pi/n}) c_k(f) = 0$ et $e^{2ik\pi/n} \neq 1$ par hypothèse. Donc $c_k(f) = 0$. En outre $c_0(f) = 0$ puisque f est une fonction impaire. On déduit finalement que $c_k(f) = c_{-k}(f) = 0$ pour tout k de Λ. Naturellement

$$\sup_{t \in R} |f(t)| = 1.$$

Grâce à la relation $f(t) f(t) = |f(t)|$ on constate que 0 est l'unique meilleure approximation de f par $F(S(\Lambda))$ puisqu'on peut appliquer la 2ème étape avec $g(t) = f(t)$ et $Pf = 0$.

Considérons maintenant la relation $g(t)(f(t) - 1) = |f(t) - 1|$ où $g(t) = f(t)$ si $f(t) \neq 0$ et $g(t) = +1$ si $f(t) = 0$. Il est facile de noter que $c_k(g) = c_{-k}(g) = 0$ pour tout k de Λ et

$$\sup_{t \in R} |g(t)| = +1.$$

Donc, encore grâce à la 2ème étape, on note que 0 est l'unique meilleure approximation de $f(t) - 1$ par $F(S(\Lambda))$.

La contradiction provient maintenant, comme lors de l'étape (a), de ce que $f(t) - (f(t) - 1) = 1$ est un élément de $F(S(\Lambda))$.

(e) Enfin, terminons la démonstration de la nécessité du Théorème 5 avec les notations de (c) notamment.

Ici $0 \in \Lambda$. Supposons que λ soit le premier élément non nul de Λ. On a, grâce à (c), $\lambda = 2k+1$. Soit λ' le premier élément de Λ à droite de λ, lequel existe grâce à (d). On a, grâce à (c), $\lambda' = \lambda + 2k' + 1$. Par suite $\lambda' = 2(k+k') + 2$ et donc $k + k' + 1$ doit appartenir à Λ et ne peut que coïncider avec λ, ce qui implique $k' = k$. Puisque Λ est infini, on conclut de proche en proche que Λ est une progression arithmétique infinie de raison $2k+1$ débutant en 0.

B) Passons maintenant à la démonstration de la suffisance de la condition donnée au Théorème 5. Soit donc Λ l'ensemble des nombres entiers de la forme $(2k+1)n$ où n parcourt les entiers positifs ou nuls et où k est un nombre entier fixé, positif ou nul. Soit alors $f \in E$. Nous cherchons $Pf \in F(S(\Lambda))$ telle que Pf soit l'unique meilleure approximation de f par $F(S(\Lambda))$. Comme à la deuxième étape, nous admettrons (théorème de Féjer) que tout élément g de $F(S(\Lambda))$ est limite au sens de la norme intégrale de combinaisons linéaires finies d'exponentielles de la forme e^{imx} où $m \in S(\Lambda)$. Ces exponentielles possèdent toutes $2\pi/(2k+1)$ comme période, ainsi en est-il de tout g de $F(S(\Lambda))$. Calculons alors $\|f - Pf\|$ selon

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t) - Pf(t)|\,dt = \frac{1}{2\pi}\sum_{m=0}^{m=2k}\int_{2\pi m/(2k+1)}^{(m+1)(2\pi/(2k+1))} |f(t) - Pf(t)|\,dt$$

$$= \frac{1}{2\pi}\sum_{m=0}^{m=2k}\int_0^{2\pi/(2k+1)} \left|f\left(m\frac{2\pi}{2k+1} + t\right) - Pf(t)\right|dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi/(2k+1)} \left\{\sum_{m=0}^{m=2k} |f_m(t) - Pf(t)|\right\}dt.$$

Nous pouvons essayer de déterminer, pour chaque t fixé de $[0, 2\pi/(2k+1)]$, un élément $Pf(t)$ tel que

$$\sum_{m=0}^{m=2k} |f_m(t) - Pf(t)|$$

soit minimal.

Il s'agit d'un problème de géométrie plane: à $2k+1$ points donnés $z_0, z_1, \ldots, z_{2k}$ du plan, associer un unique point Z de ce plan tel que

$$\sum_{m=0}^{m=2k} |z_m - Z|$$

soit minimal.

Si l'on peut résoudre ce problème géométrique, et si Z dépend continûment des $2k+1$ points, il est alors bien clair que toute f de E possède une unique meilleure approximation par $F(S(\Lambda))$, ce qui terminerait la démonstration du Théorème 5.

L'existence du point Z n'est guère difficile à établir puisque la fonction définie sur C, le corps des complexes, par

$$z \to \sum_{m=0}^{2k} |z - z_m| = f(z)$$

est continue: on a même l'inégalité déduite de l'inégalité triangulaire

$$|f(z) - f(z')| \leq (2k+1)|z - z'|.$$

En outre

$$\lim_{|z|\to+\infty} f(z) = +\infty.$$

Par suite la borne inférieure

$$\inf_{z\in C} f(z)$$

est égale à la borne inférieure

$$\inf_{|z|\leq R} f(z)$$

pour un R assez grand. Mais comme le disque fermé de centre 0 et de rayon R est compact, la fonction continue f atteint son minimum en au moins un point Z_0 de ce disque

$$\inf_{z\in C} f(z) = f(Z_0).$$

L'unicité du point Z exige une analyse un peu plus étoffée. Appelons K le sous-ensemble non vide des points de C où f atteint son minimum. C'est évidemment un sous-ensemble fermé et borné de C, donc compact. Plus intéressante est la propriété de convexité de K:

Si Z et Z' sont dans K, $\lambda Z + (1-\lambda)Z'$ est encore dans K et ce pour tout λ satisfaisant $0 \leq \lambda \leq 1$. Géométriquement, le segment joignant Z à Z' appartient tout entier à K. En effet

$$f(\lambda Z + (1-\lambda)Z') = \sum_{m=0}^{2k} |\lambda(Z - z_m) + (1-\lambda)(Z' - z_m)|.$$

Soit

$$f(\lambda Z + (1-\lambda)Z') \leq \lambda \sum_{m=0}^{2k} |Z - z_m| + (1-\lambda) \sum_{m=0}^{n} |Z' - z_m|$$

$$\leq \lambda f(Z) + (1-\lambda) f(Z')$$

$$\leq (\lambda + 1 - \lambda) f(Z_0) = f(Z_0).$$

L'inégalité stricte ne pouvant avoir lieu, toutes les inégalités précédentes deviennent des égalités. En particulier pour tout m variant entre 0 et $2k$,

$$|\lambda(Z - z_m) + (1-\lambda)(Z' - z_m)| = |\lambda(Z - z_m)| + |(1-\lambda)(Z' - z_m)|.$$

Par suite $(Z - z_m)$ *et* $(Z' - z_m)$ *sont sur une même demi-droite, c'est-à-dire qu'il existe des nombres* α_m, β_m, *positifs ou nuls et*

$$\alpha_m (Z - z_m) = \beta_m (Z' - z_m) \qquad \textit{avec} \qquad \alpha_m + \beta_m > 0.$$

Soit

$$\alpha_m Z - \beta_m Z' = (\alpha_m - \beta_m) z_m .$$

S'il existe un indice pour lequel $\alpha_m = \beta_m$ on déduit aussitôt $Z = Z'$. On peut alors supposer que $\alpha_m \neq \beta_m$ pour tout $m = 0,1,\dots,2k$, auquel cas tous les points z_m doivent être situés sur la droite passant par Z et Z', et on est ramené à un problème rectiligne analogue. On note alors que $Z - z_m$ et $Z' - z_m$ ont même signe c'est-à-dire qu'il existe des $\varepsilon_m (= \pm 1)$ tels que

$$|Z - z_m| = \varepsilon_m (Z - z_m) \qquad \text{et} \qquad |Z' - z_m| = \varepsilon_m (Z' - z_m).$$

D'où

$$f(Z_0) = \sum_{m=0}^{2k} |Z - z_m| = \sum_{m=0}^{2k} \varepsilon_m (Z - z_m) = \sum_{m=0}^{2k} \varepsilon_m (Z' - z_m) = \sum_{m=0}^{2k} |Z' - z_m| = f(Z_0)$$

et, par l'égalité centrale,

$$(Z - Z') \sum_{m=0}^{2k} \varepsilon_m = 0.$$

On note que $\Sigma_{m=0}^{2k} \varepsilon_m$ est toujours différent de 0. Par suite $Z = Z'$.

La méthode utilisée conduit évidemment à un résultat plus précis que celui dont nous avons réellement besoin. Pour l'étude de tels minimums, et quelques généralisations, cf. [6].

Montrons enfin que le point Z dépend continûment des points z_m $(m = 0,1,\dots,2k)$. On peut raisonner par l'absurde. Soit donc $z_m^{(n)}$ une suite de points et $\lim_{n\to\infty} z_m^{(n)} = z_m$. Soit $Z^{(n)}$ le point assurant le minimum de

$$\sum_{m=0}^{2k} |z - z_m^{(n)}|$$

et Z le point assurant le minimum de

$$\sum_{m=0}^{2k} |z - z_m|.$$

Les $Z^{(n)}$ restent dans un sous-ensemble borné. Si donc l'on suppose que $Z^{(n)}$ ne converge pas vers Z, alors il existe une sous-suite, encore notée $Z^{(n)}$ pour simplifier les notations, et un point Z' ($Z' \neq Z$) tel que $\lim_{n\to\infty} Z^{(n)} = Z'$.

Dès lors, pour tout z de C, on dispose de l'inégalité au sens large

$$f(Z^{(n)}) \leq \sum_{m=0}^{2k} |z - z_m^{(n)}|.$$

En passant à la limite en n, puisque f est continue,

$$f(Z') \leq \sum_{m=0}^{2k} |z - z_m|.$$

Cette inégalité contredit l'unicité de Z et termine la démonstration.

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(Pour les références [1]-[4], voir [1980: 140].)

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U.E.R. de Mathématiques, Université de Nantes, 2 chemin de la Houssinière, BP 1044, 44037 Nantes Cedex, France.

* * *

A PANDIAGONAL FIFTH-ORDER PRIME MAGIC SQUARE

11	43	83	127	131
113	31	137	41	73
167	71	103	17	37
7	23	67	197	101
97	227	5	13	53

This pandiagonal fifth-order magic square is composed of distinct primes. The magic constant is 395, the smallest possible for a magic square of this type.

ALLAN WM. JOHNSON JR.,
Washington, D.C.

TWO POLYHEDRAL PROBLEMS

The following two problems show that one must be wary inducing results by intuition or by analogy.

Problem 1.

Of two regular polygons inscribed in the same circle, that with the greater number of sides has the greater area. Is it true that of two regular polyhedra inscribed in the same sphere, that with the greater number of faces has the greater volume?

Problem 2.

Of two regular polygons circumscribed about the same circle, that with the greater number of sides has the lesser area. Is it true that of two regular polyhedra circumscribed about the same sphere, that with the greater number of faces has the lesser volume?

HOWARD EVES,
University of Maine.

Solutions to these problems appear on page 182 in this issue.

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THE OLYMPIAD CORNER: 16

MURRAY S. KLAMKIN

We present this month the questions asked at the Ninth U.S.A. Mathematical Olympiad on May 6, 1980 and, through the courtesy of Willie S.M. Yong of South Wales, those asked at the Sixteenth British Mathematical Olympiad on March 13, 1980. Solutions to the British Olympiad problems will appear here in the next issue. For the U.S.A. Olympiad, the solutions (along with those of the Twenty-second International Mathematical Olympiad, which has yet to take place) will appear later this year in a booklet compiled by Samuel L. Greitzer and available (at 50¢ per copy) from

Dr. Walter E. Mientka,
Executive Director,
MAA Committee on H.S. Contests,
917 Oldfather Hall,
University of Nebraska,
Lincoln, Nebraska 68588.

*

THE NINTH U.S.A. MATHEMATICAL OLYMPIAD

May 6, 1980 - 3½ hours

1. A two-pan balance is inaccurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A, B, and C are each weighed separately. When placed on the left-hand pan, they are balanced by weights A_1, B_1, C_1, respectively. When A and B are placed on the right-hand pan, they are balanced by A_2 and B_2, respectively. Determine the true weight of C in terms of A_1, B_1, C_1, A_2, and B_2.

2. Determine the maximum number of different three-term arithmetic progressions which can be chosen from a sequence of n real numbers $a_1 < a_2 < \ldots < a_n$.

3. Let

$$F_r = x^r \sin(rA) + y^r \sin(rB) + z^r \sin(rC),$$

where x, y, z, A, B, C are real and $A+B+C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

4. The inscribed sphere of a given tetrahedron touches each of the four faces of the tetrahedron at their respective centroids. Prove that the tetrahedron is regular.

5. If $1 \geq a,b,c \geq 0$, prove that

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

*

THE SIXTEENTH BRITISH MATHEMATICAL OLYMPIAD

March 13, 1980 - 3½ hours

1. Prove that the equation $x^n + y^n = z^n$, where n is an integer greater than 1, has no solution in integers x,y,z with $0 < x \leq n$, $0 < y \leq n$.

2. Find a set $S = \{n\}$ of 7 consecutive positive integers for which a polynomial $P(x)$ of the 5th degree exists with the following properties:

(a) all the coefficients in $P(x)$ are integers;

(b) $P(n) = n$ for 5 members of S, including the least and greatest;

(c) $P(n) = 0$ for one member of S.

3. On the diameter AB bounding a semicircular region there are two points P and Q, and on the semicircular arc there are two points R and S such that

PQRS is a square. C is a point on the semicircular arc such that the areas of the triangle ABC and the square PQRS are equal.

Prove that a straight line passing through one of the points R and S and through one of the points A and B cuts a side of the square at the incentre of the triangle.

4. Find the set of real numbers a_0 for which the infinite sequence $\{a_n\}$ of real numbers defined by

$$a_{n+1} = 2^n - 3a_n, \qquad n = 0,1,2,\ldots$$

is strictly increasing, that is, $a_n < a_{n+1}$ for $n \geq 0$.

5. In a party of ten persons, among any three persons there are at least two who do not know each other. Prove that at the party there are four persons none of whom knows another of the four.

*

SOLUTIONS TO PRACTICE SET 13

13-1. In n-dimensional Euclidean space E^n, determine the least and greatest distances between the point $A = (a_1, a_2, \ldots, a_n)$ and the n-dimensional rectangular parallelepiped whose vertices are $(\pm v_1, \pm v_2, \ldots, \pm v_n)$ with $v_i > 0$.

(Some may find it helpful first to do the problem in E^3 or even in E^2.)

Solution.

By symmetry, the desired distances will not change if we replace a_i by $|a_i|$. Consequently, we may assume that $a_i \geq 0$ without loss of generality.

Case 1. The point A *lies on the parallelepiped.* Here one or more of the a_i's are equal to the corresponding v_i's, while $a_i < v_i$ for the remaining a_i's. The least distance, D_{min}, is obviously zero. The furthest point of the figure from point A is the vertex $(-v_1, -v_2, \ldots, -v_n)$. Thus the greatest distance, D_{max}, is given by

$$D^2_{\text{max}} = (a_1 + v_1)^2 + (a_2 + v_2)^2 + \ldots + (a_n + v_n)^2. \tag{1}$$

Case 2. The point A *lies in the interior of the figure.* Here $a_i < v_i$ for all i. The least distance is then given by

$$D_{\text{min}} = \min_{1 \leq i \leq n} (v_i - a_i),$$

this being the distance from A to the point

$$P = (a_1, a_2, \ldots, a_{r-1}, v_r, a_{r+1}, \ldots, a_n),$$

where r is any one of the values of i such that $v_i - a_i$ is a minimum. The furthest point from A is again $(-v_1, -v_2, \ldots, -v_n)$ and the corresponding greatest distance is again given by (1).

Case 3. *The point* A *lies outside the figure*. Here there is at least one i such that $a_i > v_i$. If the point $X = (x_1, x_2, \ldots, x_n)$ on the figure is nearest to A, then we must have

$$x_i = \begin{cases} a_i, & \text{if } 0 \le a_i \le v_i, \\ v_i, & \text{if } a_i > v_i. \end{cases}$$

This can conveniently be written as a single equation with the Heaviside unit function H defined by $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \ge 0$:

$$x_i = a_i + (v_i - a_i)H(a_i - v_i),$$

from which

$$D^2_{\min} = \sum_{i=1}^{n} (v_i - a_i)^2 H(a_i - v_i).$$

The greatest distance is again given by (1).

13-2. If A,B,C are positive angles whose sum does not exceed π, and such that the sum of any two of the angles is greater than the third, show that

(a) there exists a triangle with sides $\sin A$, $\sin B$, $\sin C$;

(b) $4 + \sum \sin^2 A \csc^2 B \csc^2 C \le 2 \sum \csc^2 A$ (cyclic sums).

Solution.

(a) We have

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} > 2\sin\frac{A+B}{2}\cos\frac{A+B}{2} = \sin(A+B)$$

and

$$\sin(A+B) - \sin C = 2\cos\frac{A+B+C}{2}\sin\frac{A+B-C}{2} \ge 0.$$

Thus $\sin A + \sin B > \sin C$, and the other two triangle inequalities are proved in the same way.

Alternatively, consider a tetrahedron O-PQR with face angles 2A,2B,2C at vertex O and $OP = OQ = OR = \frac{1}{2}$. Such a tetrahedron exists since $0 < A + B + C \le \pi$ and $B + C > A$, etc. Then it is easy to show that triangle PQR has sides $\sin A$, $\sin B$, $\sin C$.

(b) The inequality to be proved is equivalent to

$$\frac{4\,\Pi\sin^2 A}{2\,\Sigma\sin^2 B\sin^2 C-\Sigma\sin^4 A}\le 1, \tag{1}$$

where the sums and product are cyclic over A,B,C. We recall the familiar relations

$$abc = 4\Delta R, \qquad 16\Delta^2 = 2\,\Sigma\, b^2c^2 - \Sigma a^4$$

for a triangle with sides a,b,c, area Δ, and circumradius R. If we apply these relations to the triangle PQR of part (a), then (1) is equivalent to

$$\frac{4a^2b^2c^2}{16\Delta^2} = 4R^2 \le 1 \qquad \text{or} \qquad R\le \tfrac{1}{2}, \tag{2}$$

and this is clearly true. Equality occurs in (2) in the degenerate case when O is the circumcenter of triangle PQR, that is, when $A+B+C=\pi$.

13-3. (a) If $0\le x_i\le a$, $i=1,2,\ldots,n$, determine the maximum value of

$$A \equiv \sum_{i=1}^{n} x_i - \sum_{1\le i<j\le n} x_i x_j.$$

(b) If $0\le x_i\le 1$, $i=1,2,\ldots,n$ and $x_{n+1}=x_1$, determine the maximum value of

$$B_n \equiv \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i x_{i+1}.$$

Solution.

We assume throughout that $n\ge 2$.

(a) Since A is linear in each x_i, the maximum value is taken on for the extreme values of each x_i. By letting k of the x_i's $=a$ and the remaining $n-k$ of the x_i's $=0$ we obtain the following set of possible extreme values:

$$\begin{aligned}
M_1 &= a && \text{for } k=1,\\
M_2 &= 2a-\binom{2}{2}a^2 && \text{for } k=2,\\
M_3 &= 3a-\binom{3}{2}a^2 && \text{for } k=3,\\
&\vdots && \qquad\vdots\\
M_n &= na-\binom{n}{2}a^2 && \text{for } k=n.
\end{aligned}$$

It now follows easily by comparing M_i with M_{i+1} that

$$A_{\max} = \begin{cases} a & \text{if } a \geq 1, \\ 2a - \binom{2}{2}a^2 & \text{if } 1 > a \geq \frac{1}{2}, \\ 3a - \binom{3}{2}a^2 & \text{if } \frac{1}{2} > a \geq \frac{1}{3}, \\ \vdots & \vdots \\ na - \binom{n}{2}a^2 & \text{if } \frac{1}{n-1} > a. \end{cases}$$

(b) We show by induction that

$$\max B_n = \left[\frac{n}{2}\right], \qquad n = 2, 3, 4, \ldots, \tag{1}$$

where the square brackets denote the greatest integer function.

For $n = 2$, we have

$$B_2 = x_1 + x_2 - (x_1x_2 + x_2x_1) = 1 - x_1x_2 - (1 - x_1)(1 - x_2) \leq 1,$$

and $\max B_2 = 1$ is attained when one of x_1, x_2 is 0 and the other is 1; so (1) holds for $n = 2$. And it follows from part (a) with $a = 1$ that $\max B_3 = 1$, a value attained at least once (when one of the x_i's is 1 and the other x_i's are 0); so (1) holds also for $n = 3$.

Suppose now that (1) holds for some $n = k \geq 2$; then, for any choice of $k + 2$ numbers x_i, we have

$$\begin{aligned} B_{k+2} &= B_k + x_{k+1} + x_{k+2} + x_kx_1 - x_kx_{k+1} - x_{k+1}x_{k+2} - x_{k+2}x_1 \\ &\leq \left[\frac{k}{2}\right] + a + b + cd - ca - ab - bd, \end{aligned}$$

where for convenience we write $x_{k+1} = a$, $x_{k+2} = b$, $x_k = c$, $x_1 = d$. The inequality

$$B_{k+2} \leq \left[\frac{k+2}{2}\right] = \left[\frac{k}{2}\right] + 1 \tag{2}$$

will then follow from $a + b + cd - ca - ab - bd \leq 1$ or

$$(1 - a)(1 - b) + bd + ca \geq cd. \tag{3}$$

This follows immediately if $b \geq c$ or $a \geq d$, so assume $b < c$ and $a < d$. Inequality (3) is then still valid since it is equivalent to

$$(1 - a)(1 - b) + ab \geq (d - a)(c - b).$$

Finally, we must show that equality can occur in (2). For this, just let $x_i = 1$ for all odd i and $x_i = 0$ for all even i. □

As an extension of this problem, consider the constraints $0 \leq x_i \leq a$ instead of

$0 \le x_i \le 1$.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

SOLUTIONS TO "TWO POLYHEDRAL PROBLEMS" (see page 176)

Solution to Problem 1. It *is not* true. The regular dodecahedron (a solid having 12 faces) has a greater volume than the regular icosahedron (a solid having 20 faces), and the regular hexahedron or cube (a solid having 6 faces) has a greater volume than the regular octahedron (a solid having 8 faces). These unanticipated results are easily shown from tabular information about the regular polyhedra, as given, for example, in the *CRC Standard Mathematical Tables*, 24th edition, pp. 14-15.

Solution to Problem 2. It *is* true, as can be shown from the tabular information cited in the solution to Problem 1. It is interesting that a regular dodecahedron and a regular icosahedron inscribed in the same sphere have a common inscribed sphere. The same is true of a cube and a regular octahedron inscribed in the same sphere. Also, the circumcircles of the faces of a regular dodecahedron and a regular icosahedron inscribed in the same sphere are equal, and the same is true of a cube and a regular octahedron inscribed in the same sphere.

HOWARD EVES

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A MATHEMATICS CALENDAR

The Department of Mathematics and Statistics of Ottawa's Carleton University has just published a mathematics calendar for the academic year 1980-1981. It runs from July 1980 to June 1981. It will be of interest to any individual who is intrigued by the complexity and rare functional beauty of the mathematical world.

The calendar measures 17×11 inches when opened. It opens to present a topic in mathematics on each upper leaf, with the calendar month and a "problem of the month" below. In addition, items of mathematical interest are highlighted at various days of the year.

Copies at $3.00 each (cheques payable to Carleton University) can be ordered from: The Mathematics Calendar, Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

551. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Here is another alphametical Cook's tour for which no visas are needed:

$$\begin{array}{r} \text{FRANCE} \\ \text{GREECE} \\ \hline \text{FINLAND} \end{array}.$$

552. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

Given positive constants a,b,c and nonnegative real variables x,y,z subject to the constraint $x+y+z=\pi$, find the maximum value of

$$f(x,y,z) \equiv a\cos x + b\cos y + c\cos z.$$

553. *Proposed by Leroy F. Meyers, The Ohio State University.*

Let X be an angle strictly between 0° and 45°. Show that 2X is an acute angle of a Pythagorean triangle (a right triangle in which all sides have integer lengths) if and only if X is the smallest angle of a right triangle in which the legs (but not necessarily the hypotenuse) have integer lengths.

554. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

A sequence of triangles $\{\Delta_0, \Delta_1, \Delta_2, \ldots\}$ is defined as follows: Δ_0 is a given triangle and, for each triangle Δ_n in the sequence, the vertices of Δ_{n+1} are the points of contact of the incircle of Δ_n with its sides. Prove that Δ_n "tends to" an equilateral triangle as $n\to\infty$.

555. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

An *n-persistent number* is an integer such that, when multiplied by $1, 2, \ldots, n$, each product contains each of the ten digits at least once. (A *persistent number* is one that is n-persistent for each $n = 1, 2, 3, \ldots$. It is known [1979: 163] that there are no persistent numbers. The number

$$N = 526315789473684210$$

given there is n-persistent for $1 \leq n \leq 18$, but is not 19-persistent.)

(a) Find a 19-persistent number.

(b)* Prove or disprove: for each n, there exist n-persistent numbers.

556.* *Proposed by Paul Erdős, Mathematical Institute, Hungarian Academy of Sciences.*

Every baby knows that

$$\frac{(n+1)(n+2)\ldots(2n)}{n(n-1)\ldots 2.1}$$

is an integer. Prove that for every k there is an integer n for which

$$\frac{(n+1)(n+2)\ldots(2n-k)}{n(n-1)\ldots(n-k+1)} \qquad (1)$$

is an integer. Furthermore, show that if (1) is an integer, then $k = o(n)$, that is, $k/n \to 0$.

557. *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

For the geometric progression 2, 14, 98, 686, 4802, we have

$$(2 + 14 + 98 + 686 + 4802)(2 - 14 + 98 - 686 + 4802) = 2^2 + 14^2 + 98^2 + 686^2 + 4802^2.$$

Prove that infinitely many geometric progressions have this property.

558. *Proposed by Andy Liu, University of Regina.*

(a) Find all n such that an $n \times n$ square can be tiled with L-tetrominoes.

(b) What if X-pentominoes are also available, in addition to L-tetrominoes?

(The term PENTOMINOES has been, since 15 April 1975, a registered trademark of Solomon W. Golomb.)

559. *Proposed by Charles W. Trigg, San Diego, California.*

Are there any positive integers k such that the expansion of 2^k in the decimal system terminates with k?

560. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

Take a complete quadrilateral. On each of the three diagonals as diameter, draw a circle. Prove that these three circles are coaxal.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

464. [1979: 200] *Proposed by J. Chris Fisher and E.L. Koh, University of Regina.*

(a) If the two squares ABCD and AB'C'D' have vertex A in common and are taken with the same orientation, then the centers of the squares together with the midpoints of BD' and B'D are the vertices of a square.

(b) What is the analogous theorem for regular n-gons?

I. *Solution by Leroy F. Meyers, The Ohio State University.*

The problem can be considerably generalized. In the following theorem, the polygons need not be regular, nor convex, nor even simple. In fact, they may be degenerate.

THEOREM. Let $A_1A_2\ldots A_n$ *and* $A_1'A_2'\ldots A_n'$ *be similar and similarly oriented polygons in the plane. Let* $B_1, B_2, \ldots, B_n$ *be points in the plane such that the (possibly degenerate) triangles* $A_iA_i'B_i$ *for* $i=1,2,\ldots,n$ *are similar and similarly oriented. Then* $B_1B_2\ldots B_n$ *and* $A_1A_2\ldots A_n$ *are similar and similarly oriented.*

Proof. We imbed the configuration in a complex plane and we identify the points A_i, A_i', and B_i with their affixes. The similarity of $A_1A_2\ldots A_n$ to $A_1'A_2'\ldots A_n'$ is equivalent to the existence of a complex constant α such that

$$A_i' - A_1' = \alpha(A_i - A_1), \qquad i=1,2,\ldots,n.$$

The similarity of the triangles $A_iA_i'B_i$ is equivalent to the existence of a complex constant β such that $B_i - A_i = \beta(A_i' - A_i)$ or

$$B_i = (1-\beta)A_i + \beta A_i', \qquad i=1,2,\ldots,n.$$

Hence

$$B_i - B_1 = (1-\beta)(A_i - A_1) + \beta(A_i' - A_1') = (1-\beta+\alpha\beta)(A_i - A_1), \qquad i=1,2,\ldots,n.$$

Since $1-\beta+\alpha\beta$ is constant, $B_1B_2\ldots B_n$ is similar and similarly oriented to $A_1A_2\ldots A_n$. □

One answer to part (b) of our problem is the special case in which $A_1A_2\ldots A_n$ is a regular polygon and $\beta=\frac{1}{2}$. Part (a) is a special case of *this* special case for $n=4$. Observe here that then $B_1B_2B_3B_4$ is a square whether or not the two given squares have a common vertex. But if $A_1=A_3'$, then B_1 and B_3 are the centers of the given squares. We then obtain the very special case of part (a) by the identification

$$A_1A_2A_3A_4 = ABCD \quad \text{and} \quad A_1'A_2'A_3'A_4' = C'D'AB'.$$

II. *Solution and comment by M.S. Klamkin, University of Alberta.*

More general results are already known and these are the subject of a joint incomplete paper of myself and Leon Bankoff. For earlier references to related results, see [1] and [2]. One general result is the following:

Let $A_i,B_i,C_i,\ldots$, $i=1,2,\ldots,m$, be complex numbers denoting the vertices of a set of n directly similar m-gons, all lying in the same plane. Also, let there be weights $\alpha\rho_i$, $\beta\rho_i$, $\gamma\rho_i$, etc., at vertices A_i,B_i,C_i, etc., respectively, for $i=1,2,\ldots,m$. Finally, let Φ_i denote the centroid of the weights at vertex i of A_i,B_i,C_i, etc. Then, for $i=1,2,\ldots,m$, the Φ_i are also vertices of an m-gon directly similar to the m-gons A_i,B_i,C_i, etc.

Proof. The centroids Φ_i are given by

$$\Phi_i = \frac{\alpha A_i + \beta B_i + \gamma C_i + \ldots}{\alpha+\beta+\gamma+\ldots}, \qquad i=1,2,\ldots,m. \tag{1}$$

By similarity, for $r=2,3,\ldots$,

$$A_{r+1}-A_r = z_r(A_r-A_{r-1}),\ B_{r+1}-B_r = z_r(B_r-B_{r-1}),\ C_{r+1}-C_r = z_r(C_r-C_{r-1}),\ \text{etc.}$$

Then by (1),

$$\Phi_{r+1}-\Phi_r = z_r(\Phi_r-\Phi_{r-1})$$

and thus the polygon of vertices Φ_i is directly similar to the polygons of vertices A_i,B_i,C_i, etc. □

Note that part (a) of the proposed problem just corresponds to the special case $m=4$, $n=2$, $\alpha=\beta=\gamma=\ldots$.

Also solved by LEON BANKOFF, Los Angeles, California; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; HOWARD EVES, University of Maine; J. CHRIS FISHER, University of Regina (in addition to the joint solution with his coproposer);

JACK GARFUNKEL, Flushing, N.Y.; G.P. HENDERSON, Campbellcroft, Ontario: J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University (second solution); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposers.

Editor's comment.

Part (a) of the problem, as given with the red herring stipulation of a common vertex, can easily be proved synthetically. I outline a proof given by several solvers (make a figure!):

Let P and Q be the centres of ABCD and AB'C'D', and R and S the midpoints of BD' and B'D. We see that PS and QR are both parallel to and equal to one-half of BB', and that PR and QS are both parallel to and equal to one-half of DD'. Since a rotation of 90° with centre A takes BB' into DD', it follows that BB' is equal and perpendicular to DD', and hence that PSQR is a square. □

The proposers mentioned that proofs of part (a) (different from the above) are given in Finsler and Hadwiger [3] and in Neumann [4]. As their answer to part (b), they gave the following theorem (which will also appear in the forthcoming joint paper [5] by one of the proposers):

For any $n \geq 3$ the set of regular, convex, counterclockwise-oriented n-gons in the Euclidean plane is a two-dimensional vector space over the complex numbers.

Part (a) easily follows from this theorem for if the vertices are properly associated, with ABCD $\longleftrightarrow$ C'D'AB', then the new figure is half the sum of the two squares and is therefore a square itself.

Finally, as a generalization of part (b), Eves gave without reference the following theorem of H. Van Aubel: *If the lines joining corresponding points of two directly similar figures are divided in the same ratio, the points of section form a figure directly similar to the given ones.*

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5. J.C. Fisher, D. Ruoff, and J. Shilleto, "Polygons and Polynomials," to appear in the *Proceedings of the Coxeter Symposium (May 1979)*.

465. [1979: 200] *Proposed by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.*

For positive integer n, let $\sigma(n)$ = the sum of the divisors of n and $\tau(n)$ = the number of divisors of n. Show that if $\sigma(n)$ is a prime then $\tau(n)$ is a prime.

Solution by Andy Liu, University of Regina.

By hypothesis $\sigma(n)$ is prime, which implies $n > 1$ since $\sigma(1) = 1$. The arithmetic function σ is multiplicative; so we may assume that the canonical factorization of n is $n = p^{\alpha}$ for some prime p and positive integer α, for otherwise $\sigma(n)$ would be composite. Thus we have

$$\sigma(n) = \frac{p^{\alpha+1} - 1}{p - 1} \quad \text{and} \quad \tau(n) = \alpha + 1.$$

Suppose $\tau(n)$ is composite, say $\alpha + 1 = rs$, with $r > 1$ and $s > 1$. Then

$$\sigma(n) = \frac{p^{rs} - 1}{p - 1} = \frac{(p^r)^s - 1}{p^r - 1} \cdot \frac{p^r - 1}{p - 1} \tag{1}$$

and $\sigma(n)$ is composite, for each of the two factors on the right in (1) is an integer greater than 1. This contradicts the hypothesis, so $\tau(n)$ is prime.

Also solved by MICHAEL ABRAMSON, 14, student, Benjamin N. Cardozo H.S., Bayside, N.Y.; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; CLAYTON W. DODGE, University of Maine at Orono; HOWARD EVES, University of Maine; ALLAN WM. JOHNSON JR., Washinton, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ROBERT and JUDITH KNAPP, Herkimer, N.Y.; VIKTORS LINIS, University of Ottawa; LEROY F. MEYERS, The Ohio State University; GREGG PATRUNO, student, Stuyvesant H.S., New York; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer. In addition, two badly flawed solutions were received.

Editor's comment.

One of the flawed solutions purported to show that if $\sigma(p^{\alpha})$ is prime for some prime $p \neq 2$, then necessarily $\alpha = 2$ and $\tau(p^{\alpha}) = 3$. This is contradicted by

$$\sigma(3^6) = \text{the prime } 1093 \quad \text{and} \quad \tau(3^6) = 7.$$

The other contained a flaw of heroic proportions: the solver convinced himself that if $\sigma(n)$ is prime, then n is also prime and $\tau(n) = 2$, which deserves to be followed by a non-factorial!. We all have our bad moments.

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466. [1979: 200] *Proposé par Roger Fischler, Université Carleton, Ottawa.*

Soient AB et BC deux arcs d'un cercle tels que arc AB > arc BC, et soit D le point de milieu de l'arc AC (voir la figure 1). Si DE ⊥ AB, montrer que AE = EB + BC. (Ce théorème est attribué à Archimède.)

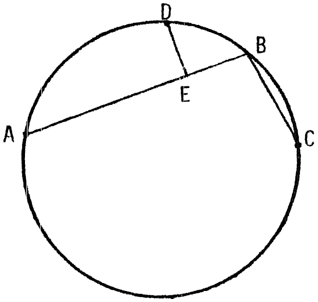

Figure 1

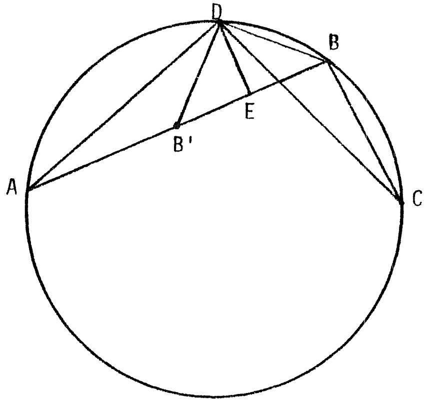

Figure 2

I. *Solution by Gregg Patruno, student, Stuyvesant H.S., New York.*

Since DA = DC and $\angle$DAB = $\angle$DCB (see Figure 2), a rotation of ΔDBC around D takes C into A and B into a point B' on AB. Now ΔDBB' is isosceles, so EB = EB' and

$$AE = EB' + AB' = EB + BC.$$

II. *Comment by Howard Eves, University of Maine.*

This theorem, known as the "Theorem on the Broken Chord," is of historical interest, for it has been credited by Arabic scholars to Archimedes, and the attribution could well be correct since many of the works of Archimedes have become lost to us. Tropfke [7] has suggested that this theorem may have served Archimedes as a formula analogous to our

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

To show the correspondence, one has merely to set arc AD = $2x$ and arc DB = $2y$, and then successively show that, in a unit circle,

$$AD = 2\sin x, \quad DB = 2\sin y,$$

$$BC = 2\sin(x-y), \quad AE = 2\sin x\cos y, \quad EB = 2\cos x\sin y.$$

One can also use the theorem to obtain the identity

$$\sin(x+y) = \sin x\cos y + \cos x\sin y.$$

In short, Archimedes may have found the Theorem on the Broken Chord a useful tool in his work in astronomy.

Also solved by MICHAEL ABRAMSON, 14, student, Benjamin N. Cardozo H.S., Bayside, N.Y.; HAYO AHLBURG, Benidorm, Alicante, Spain; LEON BANKOFF, Los Angeles, California and CHARLES W. TRIGG, San Diego, California (jointly - 7 solutions); W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MILTON and GAIL EISNER, Richmond, Virginia (jointly); HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; G.C. GIRI, Midnapore College, West Bengal, India; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; JOE KONHAUSER, Macalester College, Saint Paul, Minnesota; VIKTORS LINIS, University of Ottawa; ANDY LIU, University of Regina; L.F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A. WINTERINK, Albuquerque Technical-Vocational Institute, New Mexico; and the proposer.

Editor's comment.

According to Boyer [3] and Eves [4], this theorem was attributed to Archimedes by (unidentified) Arabic scholars. According to the proposer, Suter [6] attributes the attribution to Abu'l Raihan al-Biruni (973-1048) and goes on to give 13 proofs of the theorem. Avishalom [1] stated the theorem without reference or proof, and this caused Bankoff to offer it as a challenge problem [2], for which solutions by Bankoff, Konhauser, and Trigg were later published [5].

Of all the solutions the editor has seen, both in and out of the literature, none, in his opinion, rivaled the classic simplicity of our solution I.

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467, [1979: 200] *Proposed by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

Let $n_1,\ldots, n_k$ be given positive integers and form the vectors $(d_1,\ldots, d_k)$ where, for each $i=1,\ldots, k$, d_i is a divisor of n_i. Letting $\tau(d)$ = the number of divisors of d, the number of these vectors is $\tau(n_1)\tau(n_2)\ldots\tau(n_k)$. How many of these have the property that their components are relatively prime in pairs?

Solution by L.F. Meyers, The Ohio State University.

Let $e_p(n)$ be the exponent of the prime p in the canonical (or prime-power) factorization of n. Then

$$\tau(n) = \prod_p (e_p(n)+1).$$

(Here the product is extended over all primes p, and makes sense since all but finitely many of the factors are 1.)

The components of the vector $(d_1,d_2,\ldots,d_k)$ are relatively prime in pairs if and only if any prime p occurs as a factor of at most one of the components. If d_i is such a component, then $e_p(d_i)$ can be any integer from 1 through $e_p(n_i)$, inclusively. Since p need not divide any component, the total number of distinct ways in which p can occur in any acceptable vector is

$$e_p(n_1)+e_p(n_2)+\ldots+e_p(n_k)+1 = e_p(N)+1,$$

where $N=n_1n_2\ldots n_k$. Hence the total number of acceptable vectors is

$$\prod_p (e_p(N)+1) = \tau(N) = \tau(n_1n_2\ldots n_k).$$

Also solved by GREGG PATRUNO, student, Stuyvesant H.S., New York; and the proposer.

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468. [1979: 200] *Proposed by Viktors Linis, University of Ottawa.*

(a) Prove that the equation

$$a_1x^{k_1}+a_2x^{k_2}+\ldots+a_nx^{k_n}-1 = 0,$$

where $a_1,\ldots,a_n$ are real and $k_1,\ldots,k_n$ are natural numbers, has at most n positive roots.

(b) Prove that the equation

$$ax^k(x+1)^p+bx^l(x+1)^q+cx^m(x+1)^r-1 = 0, \tag{1}$$

where a, b, c are real and k, l, m, p, q, r are natural numbers, has at most 14 positive roots.

Editor's comment.

Our proposer found this problem in the Russian journal *Kvant* (No. 4, 1977, p. 30, Problem M439), where it was proposed by A. Kušnarenko, and a solution by L. Limanov was published in a later issue (No. 1, 1978, pp. 31-33). Part (b) of Limanov's solution, which is given below, was edited from a translation supplied by V. Linis.

I. *Solution of part* (a) *by Friend H. Kierstead, Cuyahoga Falls, Ohio.*

The equation has $n+1$ terms, so it can have at most n changes of sign; hence, by Descartes' Rule of Signs, it has at most n positive roots.

II. *Solution of part* (b) *by L. Limanov.*

We will need the following fact, which is easily checked directly: if $P_m(x)$ is a polynomial of degree m and u,v are arbitrary real numbers, then

$$x^u(x+1)^vP_m(x)$$

has a derivative of the form

$$x^{u-1}(x+1)^{v-1}P_{m+1}(x).$$

Suppose the given equation (1) has more than 14 positive roots. Then the equation obtained by differentiating (1) has more than 13 positive roots and is of the form

$$x^{k-1}(x+1)^{p-1}A_1(x)+x^{l-1}(x+1)^{q-1}B_1(x)+x^{m-1}(x+1)^{r-1}C_1(x) = 0, \tag{2}$$

where $A_i(x)$, $B_i(x)$, $C_i(x)$ are polynomials of degree i. If we divide (2) by $x^{m-1}(x+1)^{r-1}$, the resulting equation

$$x^{k-m}(x+1)^{p-r}A_1(x)+x^{l-m}(x+1)^{q-r}B_1(x)+C_1(x) = 0 \tag{3}$$

has the same *positive* roots as (2). After two differentiations, (3) becomes

$$x^{k-m-2}(x+1)^{p-r-2}A_3(x)+x^{l-m-2}(x+1)^{q-r-2}B_3(x) = 0, \tag{4}$$

which has more than 11 positive roots. We repeat the process. Dividing (4) by $x^{l-m-2}(x+1)^{q-r-2}$ yields the equation

$$x^{k-l}(x+1)^{p-q}A_3(x)+B_3(x) = 0, \tag{5}$$

which has the same positive roots as (4). Finally, differentiating (5) four times yields the equation

$$x^{k-l-4}(x+1)^{p-q-4}A_7(x) = 0,$$

which has more than 7 positive roots. This is clearly a contradiction.

Also partially solved (part (a) only) by MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

Limanov's solution was presumably the best of those submitted to the editors of *Kvant*. His proof of part (b) is admirable, but his proof of part (a) was a proper horror, nearly as long as that of part (b). The Russians have apparently never heard of Descartes' Rule of Signs; otherwise they would surely have put Descartes before the horrors.

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469. [1979: 200] *Proposed by Gali Salvatore, Perkins, Québec.*

Of the conics represented by the equations

$$\pm x^2 \pm 2xy \pm y^2 \pm 2x \pm 2y \pm 1 = 0,$$

how many are proper (nondegenerate)?

Solution by the proposer.

Since two quadratic forms with opposite signs give rise to the same conic, there are only 32 distinct conics to investigate. These are given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + 1 = 0, \tag{1}$$

where each of $a,b,f,g,h = \pm 1$. The proper conics are those for which $\Delta \neq 0$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & 1 \end{vmatrix}$$

or, as $f^2 = g^2 = h^2 = 1$,

$$\Delta = (a-1)(b-1) + 2(fgh-1).$$

We have $fgh = 1$ for 16 of the 32 conics and $fgh = -1$ for the remaining 16.

Case 1: $fgh = 1$. Here

$$\Delta \neq 0 \iff a = b = -1.$$

This occurs exactly four times, so we have 4 proper (and 12 degenerate) conics.

Case 2: $fgh = -1$. Here

$$\Delta = 0 \iff a = b = -1.$$

This occurs exactly four times, so we have 4 degenerate (and 12 proper) conics.

To sum up, there are 16 proper and 16 degenerate conics.

A conic of the form (1) is a hyperbola, a parabola, or an ellipse according as $ab - h^2$ is negative, zero, or positive. Since here

$$ab - h^2 = ab - 1 = 0 \text{ or } -2$$

according as a,b are equal or unequal, the 4 proper conics of Case 1 are all parabolas; and of the 12 proper conics of Case 2, 4 are parabolas and 8 are hyperbolas. So our 16 proper conics are very equitably divided into 8 parabolas and 8 hyperbolas. There is order in the universe.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; HOWARD EVES, University of Maine; G.C. GIRI, Midnapore College, West Bengal, India; F.G.B. MASKELL, Algonquin College, Ottawa; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas. One incorrect solution was received.

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470. [1979: 201] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Construct an integral square of eleven decimal digits such that, if each digit is increased by unity, the resulting integer is a square. (A four-digit example is $45^2 = 2025 \rightarrow 3136 = 56^2$.)

Solution by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.

Let n^2 and m^2 be the required and the augmented squares, respectively. Then

$$m^2 - n^2 = (m-n)(m+n) = R_{11},$$

where R_{11} is the repunit 11,111,111,111. It is known [1] that the prime factorization of R_{11} is $21649 \cdot 513239$. So a solution (m,n), if there is one, must come from one of the systems

$$\begin{matrix} m+n = R_{11} & & m+n = 513239 \\ & \text{or} & \\ m-n = 1 & & m-n = 21649. \end{matrix}$$

The first system does not give values of m and n which have 11-digit squares, and the second gives $(m,n) = (267444,\ 245795)$, from which we get the unique solution to our problem, because here m^2 and n^2 both have 11 digits and n^2 does not include the digit 9:

$$n^2 = 245795^2 = 60415182025,$$

$$m^2 = 267444^2 = 71526293136.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; CLAYTON W. DODGE, University of

Maine at Orono; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; L.F. MEYERS, The Ohio State University; GREGG PATRUNO, student, Stuyvesant H.S., New York; BOB PRIELIPP, The University of Wisconsin-Oshkosh; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer. One incorrect solution was received.

Editor's comment.

Known factors of repunits can also be found in Beiler [2].

The proposer noted that the example given in the proposal is a special case of the following rule, which Dickson [3] credits to R. Goormaghtigh: if $A = 55\ldots56$ and $B = 44\ldots45$ (each to n digits), then $A^2 - B^2 = R_{2n}$. Goormaghtigh also gave the following examples for 5- and 7-digit squares:

$$\begin{array}{l} n^2 = 115^2 = 13225 \\ m^2 = 156^2 = 24336 \end{array} \quad \text{and} \quad \begin{array}{l} n^2 = 2205^2 = 4862025 \\ m^2 = 2444^2 = 5973136 \end{array}.$$

The repunit R_{11} has a curious property which it is almost certain has never been noticed before. Gali Salvatore, who discovered it and communicated it to the editor, wishes to dedicate it to that prince of digit delvers, Charles W. Trigg.

Let p_n be the nth prime. The factors of R_{11} are

$$21649 = p_{2431} \quad \text{and} \quad 513239 = p_{42536}.$$

The ranks of the factors of R_{11} in the sequence of primes, 2431 *and* 42536*, are permutations of the consecutive digits* 1 *to* 4 *and* 2 *to* 6*, respectively.*

Knowledge of this fact will not bring peace to the Middle East, but it should make Trigg very happy.

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471. [1979: 228] *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

Restore the digits in this multiplication, based on a sign in a milliner's shop window:

$$\begin{array}{r} WE \\ \underline{DO} \\ TAM \\ \underline{HAT} \\ TRIM \end{array}$$

Solution by the proposer.

Clearly, none of W, D, T, H, E, O, A, R can be zero; so either $M = 0$ or $I = 0$.

Suppose $M = 0$. Then $O = 5$ and E is even. It follows that T is even and must equal 2 or 4. If $T = 2$, then $H = 1$ and there is no value for R. If $T = 4$, then $A = 7$, $I = 1$, and $R = 2$, so that $TRIM = 4210$. But this integer does not factor into two two-digit integers. Thus $M \neq 0$.

Therefore $I = 0$, whence $R = 1$ and only W or H can be 5. If $H = 5$, then $T = 6$ and TRIM is one or more of 6102, 6103, 6107, 6108, or 6109. But none of these is factorable into two two-digit integers. So $W = 5$.

Then (T, H, A) = (3, 2, 7) or (4, 3, 6). The first of these leads to $O = 6$, $M = 4$ or 8, with $E = 9$ and $D = 3 = T$. Hence (T, H, A) = (4, 3, 6). Finally, $O = 9$, $E = 2$, $M = 8$, and $D = 7$.

The solution is, uniquely:

$$\begin{array}{r} 52 \\ \underline{79} \\ 468 \\ \underline{364} \\ 4108. \end{array}$$

Also solved by MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, J.F. Kennedy H.S., Bronx, N.Y.; CHARLES W. TRIGG, San Diego, California; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; and KENNETH M. WILKE, Topeka, Kansas.

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472. [1979: 228] *Proposé par Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Espagne.*

Construire un triangle connaissant le côté b, le rayon R du cercle circonscrit, et tel que la droite qui joint les centres des cercles inscrit et circonscrit soit parallèle au côté a.

Solution by Howard Eves, University of Maine.

We will use the following theorem of Carnot [1], for which a proof can be found in Altshiller Court [2]:

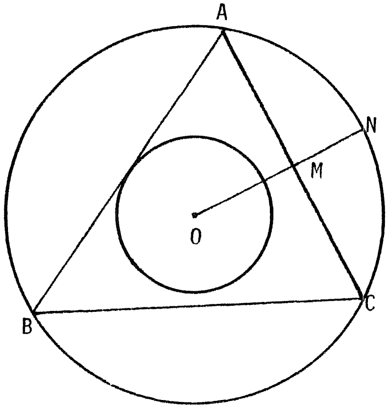
The sum of the distances of the circumcenter of a triangle from the three sides of the triangle is equal to the circumradius increased by the inradius.

For a triangle ABC with circum- and in-radii R and r, Carnot's Theorem says that

$$R\cos A + R\cos B + R\cos C = R + r. \qquad (1)$$

Suppose a solution triangle exists. Then $R\cos A = r$ and (1) yields $\cos B + \cos C = 1$. Thus B is acute and $b < 2R$.

Conversely, we show there is a unique solution triangle if $b < 2R$. Draw a circle $O(R)$ and mark off in it a chord AC of length b (see figure). Draw the radius OMN perpendicular to AC to cut AC in M and circle $O(R)$ in N. Draw circle O(MN), and then draw the chord AB of circle $O(R)$ tangent to circle O(MN) and such that O lies within angle BAC. Triangle ABC is the sought triangle.



The proof follows easily from Carnot's Theorem.

Also solved by LEON BANKOFF, Los Angeles, California; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; FRANCISCO HERRERO RUIZ, Madrid, Spain; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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473.* [1979: 229] *Proposed by A. Liu, University of Regina.*

The set of all positive integers is partitioned into the (disjoint) subsets $T_1, T_2, T_3, \ldots$ as follows: for each positive integer m, we have $m \in T_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the subsets. Prove that each T_k is finite.

(This is a variant of Crux 342* [1978: 133, 297].)

Editor's comment.

No solution was received for this problem, which therefore remains wide open. As the proposer noted, the problem is closely related to Crux 342* [1978: 133, 297], which is also unsolved. We have received for No. 342 two partial solutions, each of which shows that the proposed theorem is true for all odd n but draws no conclusion for even n. We make a final appeal to readers to find a complete solution to either or both of these problems. Failing that, a partial solution to No. 342 will be published in a few months.

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474. [1979: 229] *Proposed by James Propp, Harvard College, Cambridge, Massachusetts.*

Suppose (s_n) is a monotone increasing sequence of natural numbers satisfying $s_{s_n} = 3n$ for all n. Determine all possible values of s_{1979}.

Solution by Gregg Patruno, student, Stuyvesant H.S., New York.

We use functional notation for typographical convenience. Suppose such a sequence s exists. Since

$$s(m) = s(n) \implies 3m = s(s(m)) = s(s(n)) = 3n \implies m = n,$$

s is injective and hence strictly increasing. If $s(n) \le n$ for some n, then $3n = s(s(n)) \le s(n) \le n$, a contradiction; hence for all positive integers n we have

$$n < s(n) < s(s(n)) = 3n.$$

In particular, $1 < s(1) < s(s(1)) = 3$, from which $s(1) = 2$ and $s(2) = 3$. With the help of $s(3n) = s(s(s(n))) = 3s(n)$, easy inductions show that

$$s(3^k) = 2 \cdot 3^k \quad \text{and} \quad s(2 \cdot 3^k) = 3^{k+1}, \qquad k = 0,1,2,\ldots .$$

Since each of the sets $A_k = \{n \mid 3^k \le n < 2 \cdot 3^k\}$ and $B_k = \{n \mid 2 \cdot 3^k \le n < 3^{k+1}\}$ contains exactly 3^k distinct elements, we have

$$s(A_k) = B_k \quad \text{and} \quad s(B_k) = s(s(A_k)) = 3A_k \equiv \{3j \mid j \in A_k\}.$$

More precisely,

$$s(n) = \begin{cases} n + 3^k, & \text{if } \quad 3^k \le n < 2 \cdot 3^k, \\ 3(n-3^k), & \text{if } 2 \cdot 3^k \le n < 3^{k+1}, \end{cases} \qquad k = 0,1,2,\ldots . \quad (1)$$

Conversely, it is easy to verify that (1) satisfies the conditions of the problem, so it is the one and only sequence defined by the problem.

Now $2 \cdot 3^6 < 1979 < 3^7$, so we have uniquely $s(1979) = 3(1979 - 3^6) = 3750$.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; FRANCISCO HERRERO RUIZ, Madrid, Spain; JAN VAN DE CRAATS, Leiden University, The Netherlands; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; and the proposer. Incomplete solutions were submitted by W.J. BLUNDON, Memorial University of Newfoundland; and CLAYTON W. DODGE, University of Maine at Orono.

Editor's comment. As a lagniappe, Dou calculated that $s(1)+s(2)+\ldots+s(1979) = 3342077$. This sum is a prime, the 239704th prime, in fact.

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