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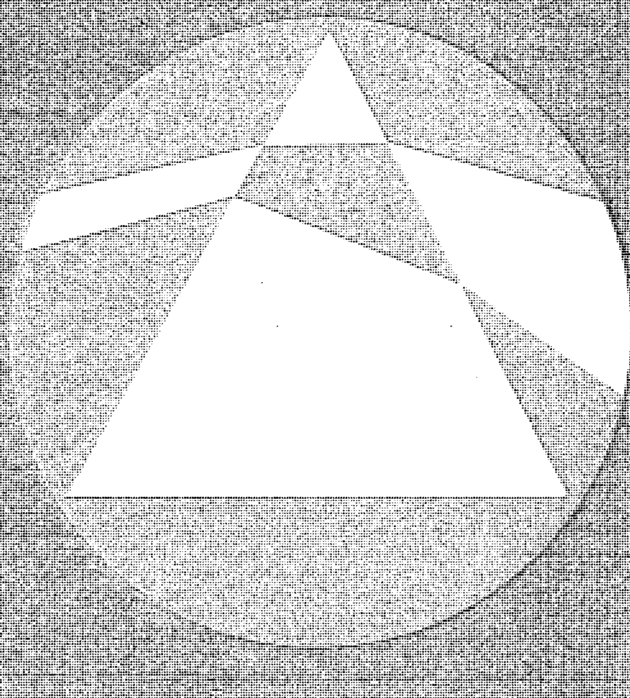

- How long is a spoke?
- Card shuffling
- Chaotic music

A magazine for students and teachers of mathematics
in schools, colleges and universities

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Get Your Spoke in with the Cosine Rule

W. M. PICKERING

How long is the spoke of a bicycle wheel? This is not a variant of the question 'How long is a piece of string', as this article shows.

Introduction

Being the only mathematically trained member of my local golf club I am occasionally asked to expound (informally!) on some problem or other which is thought to involve mathematics. Such problems are often vaguely posed and of considerable complexity and on several occasions I have had to resort to the promise, not always fulfilled, of 'thinking about it' some time in the future. When asked about the possibility of a formula for the length of spokes on a bicycle wheel, my instinctive reaction was that there must be one somewhere in the literature. However, the immediate attraction of being able to demonstrate a practical application of mathematics to a largely non-scientific but interested audience proved to be extremely compelling. Thus I was 'hooked' and felt duty bound to produce the goods. Subsequently it came as little surprise to discover that the formula was known to some of the enthusiasts, retailers and manufacturers in the cycling world.

The question, as posed, was to determine the length of spoke if the spokes are required to cross once, twice or more times given the dimensions of the rim and hub and the total number of spokes in the wheel. By spokes crossing, for example three times, we mean that each spoke crosses three other spokes.

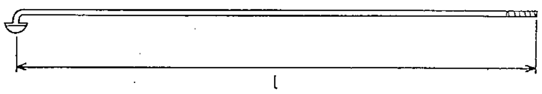

Figure 1. The length of a plain spoke

For a given configuration all spokes are of the same length and what is regarded as the length, l, of a spoke is indicated in figure 1.

The choice between arrangements of spokes with one, two or more crossings is largely determined by wheel rigidity requirements in the radial and lateral directions. Here lateral means normal to the plane through the centre-line of the rim. Lateral stiffness of the wheel increases as the number of crosses increases; radial spoking (zero crosses) produces a high radial stiffness but a relatively low lateral stiffness. For everyday purposes spokes which cross two, three or four times are in common usage and some details of the practical techniques of wheel building are given, for example, by van der Plas (reference 1).

Derivation of the spoke length formula

We begin by considering the spokes on one side of the wheel and the related problem in the plane through the centre-line of the rim. Figure 2 shows the geometry of this configuration, and the projection of the hub flange on to this plane has been enlarged for reasons of clarity. Clearly it is possible to consider each side of the wheel separately whether or not the wheel is symmetrical with respect to this plane. A rear wheel, for example, is often dished on one side to allow space for gears or other equipment and consequently the spoke lengths on each side of the wheel are not the same. Usually, however, identical patterns of spokes are used on each side of the wheel.

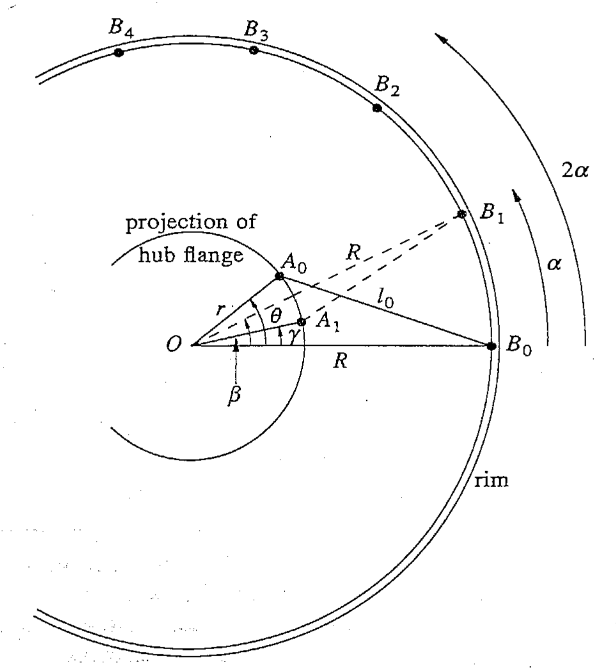

Figure 2 Geometry of the problem in a plane through the rim centre-line

The (internal) rim radius is denoted by R, the effective radius of the hub flange (to the centre of the spoke holes) is denoted by r and spoke holes in the hub flange and rim are indicated, respectively, by the points A_i, B_i $(i = 0, 1, 2, \ldots)$. Figure 2 shows only the rim holes which are used by the spokes from one hub flange. The angular spacing of the spoke holes in the hub flange is α so that, for example, $\angle A_0OA_1 = \alpha$ and the angular spacing of the rim holes shown in figure 2 is also equal to α. On an actual wheel rim the angular spacing of the spoke holes is $\frac{1}{2}\alpha$ and spokes from each hub flange go into alternate holes in the rim. The length l_0 denotes the distance A_0B_0 and the angles β, γ and θ are measured positively in the usual anticlockwise sense as shown in figure 2. It is clear that

$$\alpha = \frac{2\pi}{N}, \tag{1}$$

where N is the number of holes in one hub flange.

We next consider how the various spoke patterns may be constructed, and it should be noted that the following explanation may differ from the way in which real wheels are built up in a workshop. Apart from the special case of radial spokes, we shall construct our spoke patterns by considering pairs of spokes which are in adjacent holes in the hub flange. Hence N must be an even integer and the total number of spokes, M, in the wheel is given by

$$M = 2N, \tag{2}$$

where M must therefore be divisible by 4.

It may be intuitively clear to some readers that, for all the spokes (on one side of the wheel) to be of the same length, it is essential that the hub flange holes and rim holes must be radially aligned. This does indeed turn out to be the case, but is not assumed at this stage. Thus, regarding the line OB_0 as fixed (suppose the rim is clamped in this position), we have to determine the angle θ as part of the problem, where θ may be interpreted as the angle through which the hub must be turned, relative to the rim, to allow the construction of the desired spoke pattern. In the following discussion it is assumed that there is *always* a spoke from A_0 to B_0.

Each spoke of a pair of spokes is arranged to cross the other spoke of that pair, as illustrated by the lines A_0B_0 and A_1B_1 in figure 2. We will regard A_0B_0 as the 'left-hand' spoke and A_1B_1 as the 'right-hand' spoke of the pair. Depending on the number of crossings required, each spoke of such a pair may or may not cross spokes from other similar pairs. Figure 2 shows the case where the spokes cross once, and we note that $\angle B_1OB_0 = \alpha$. This angle is the angular displacement on the rim of the spokes from A_0 and A_1 and is also equal to β as indicated by figure 2. Thus, for this configuration, $\beta = \alpha$. If two crosses are required the 'right-hand' spoke from A_1 is taken to B_3 (so that $\angle B_3OB_0 = 3\alpha$) and the 'left-hand' spoke of the next pair of spokes (situated anticlockwise from A_0) is taken to B_2. The spoke to B_1 comes from the 'right-hand' spoke of the pair next to A_0A_1 situated clockwise from A_1. For this configuration the angle β is defined as $\angle B_3OB_0$ and the angular displacement on the rim of spokes from A_0 and A_1 is, as before, β, where now $\beta = 3\alpha$. A rough sketch and a little thought leads to the conclusion that to construct a pattern with three crosses the spokes from A_0 and A_1 should cross and be fixed in the rim at points separated by angular displacement 5α, and similarly for all other such pairs of spokes. The reader should be sufficiently convinced by now to be able to propose that, in general, the left-hand and right-hand spokes from A_0 and A_1 should cross and be taken to points on the rim separated by angular displacement $n\alpha$ (n odd) in order to produce a spoke pattern with $\frac{1}{2}(n+1)$ crosses. Thus, in general,

$$\beta = n\alpha, \tag{3}$$

and, for all spoke patterns, it is clear that

$$\theta - \gamma = \alpha. \tag{4}$$

Having indicated how the general pattern may be constructed, we now turn our attention to deriving a formula for l_0. Once l_0 is obtained it is easy to calculate the spoke length, l, from the formula

$$l^2 = l_0^2 + h^2, \tag{5}$$

where $h = h_1$ or $h = h_2$ $(h_1 + h_2 = w)$ and w is the hub width as shown in figure 3. For a symmetrical wheel, $h_1 = h_2 = \frac{1}{2}w$.

Using the cosine rule on triangle OB_0A_0 of figure 2, we find that

$$l_0^2 = r^2 + R^2 - 2rR\cos\theta, \tag{6}$$

and it should be noted that this equation is the same for all spoke patterns. Considering the one-cross pattern shown in figure 2 and the triangle OA_1B_1 we find that

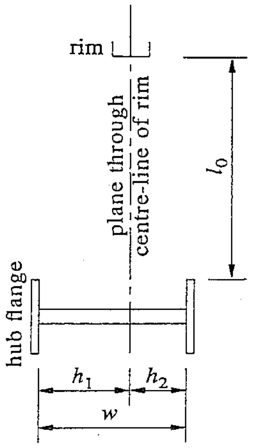

Figure 3. Cross-section through hub and rim

$$(A_1B_1)^2 = r^2+R^2-2rR\cos(\beta-\gamma), \quad (7)$$

where $\beta = \alpha$. Thus for this case, since we require that $A_1B_1 = l_0$, we obtain

$$\cos\theta = \cos(\beta-\gamma). \quad (8)$$

For a two-cross pattern the equation corresponding to (7) is the same as (7) but with A_1B_1 replaced by A_1B_3 and $\beta = 3\alpha$. The requirement that $A_1B_3 = l_0$ leads again to equation (8). Similar arguments may be applied for other spoke patterns and it should be clear that equation (8) is valid in general, that is with β given by (3). Hence from (3), (4) and (8) we obtain

$$\cos\theta = \cos[(n+1)\alpha-\theta], \quad (9)$$

and the appropriate solution of this equation may be written in the form

$$\theta = k\alpha, \quad (10)$$

where

$$k = \tfrac{1}{2}(n+1). \quad (11)$$

It should be clear that k is the number of crosses and relations (10) and (11) show that, in order to construct the spoke patterns as described, the appropriate values of γ are given by $\gamma = (k-1)\alpha$. Making use of (1), (2), (5), (6) and (10) we find that the spoke length, l, for a pattern of $k = 1, 2, 3, \ldots$ crosses may be written as

$$l = \sqrt{h^2+r^2+R^2-2rR\cos\frac{4k\pi}{M}} \quad (12)$$

and it is easily seen that this equation also gives the correct result for the radial case, $k = 0$.

Numerical results and comparison with measurement

As a practical test of equation (12), five different wheels with various spoke patterns were measured. The data obtained are summarised in table 1. This table also shows the spoke length computed via (12). It should be emphasised that all the measurements were made accurate to about ± 1 mm using a standard steel tape-measure, and no special equipment was used. It may come as a surprise to anyone unfamiliar with cycle-wheel design to discover that the internal rim diameters of wheels can vary for the same overall size (diameter) of wheel. The term wheel size is normally taken to include the tyre whose (approximately circular) cross-section depends on the rim width. For example, for a 66-cm (26-inch) wheel the rim diameter can vary by almost 4 cm (1.5 inches), depending on the rim width and hence the size of tyre fitted. All such rims are normally referred to (in the UK) as 26-inch rims. For the currently popular mountain bikes the situation is reversed: rim diameters are 'fixed' and tyres of differing sections may be fitted to the same size rim.

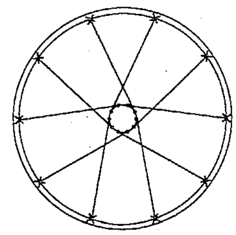

Figure 4. Wheel 1: 16-inch front wheel with 20 spokes which cross twice

Wheel 1 was a 40-cm (16-inch) front wheel with 20 spokes and the two-cross spoke pattern for this wheel is illustrated in figure 4. Wheel 2 was a standard 26-inch wheel with 36 spokes arranged in a three-cross pattern as shown in figure 5. Wheels 3 and 4 were also 26-inch size, with respectively 36 and 40 spokes arranged in a four-cross pattern. Figure 6 shows wheel 4 which has a large hub (actually a 'dynahub' for the generation of electrical power). This wheel was also very slightly dished but this has not been taken into account in the calculation. For wheels 1–4 we assumed that $h = \frac{1}{2}w$ in equation (12). Wheel 5 was a 69-cm (27-inch) rear wheel with significant dishing on one side to allow for gearing, and it is notable that the spoke lengths on each side differ by only 2 or 3 mm.

Table 1. Measurements taken from five different wheels and the corresponding values of l calculated from (12). For wheels 1–4, $h_1 = h_2 = \frac{1}{2}w$. (All lengths are in mm and measurements are accurate to approximately ± 1 mm.)

Wheel number	Internal rim diameter ($2R$)	Hub width (w)	Hub diameter ($2r$)	Number of spokes (M)	Number of crosses (k)	Measured spoke length	l calculated from (12) (nearest mm)
1	302	73	40	20	2	150	151
2	554	72	52	36	3	266	267
3	549	58	43	36	4	272	273
4	588	66	109	40	4	284	284
5	613	$h_1 = 45$ $h_2 = 15$	52	36	3	297 295	298 295

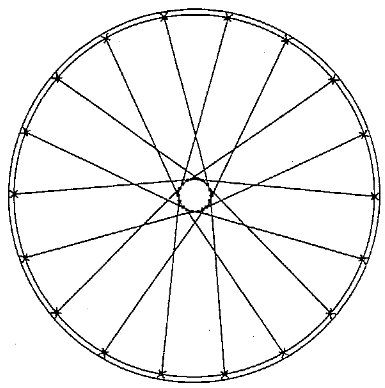

Figure 5. Wheel 2: 26-inch wheel with 36 spokes which cross three times

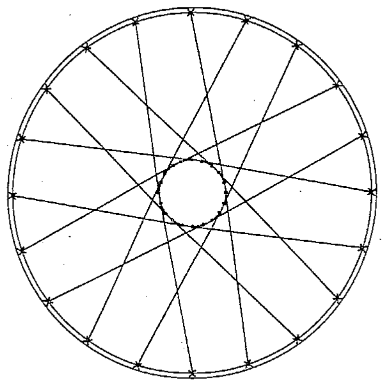

Figure 6. Wheel 4: 26-inch wheel with large hub and 40 spokes which cross four times

There is clearly very good agreement between the measured and calculated spoke lengths for all the cases considered. The largest discrepancy between measured and calculated spoke lengths is 1 mm, which is sufficiently good for practical purposes.

Discussion

Various handbooks and workshop manuals are available in the cycle trade. For example, van der Plas (reference 1, page 87) quotes a formula for spoke length which is easily shown to be the same as equation (12) (apart from an obvious printing error and the fact that r, the effective hub radius, is measured to the outer edge of the hub flange holes). Sutherland (reference 3, Appendix page 16-4) quotes a similar formula and he has also published (reference 4) a 'slide-rule' calculator for which the required value of r is defined as in (12), whereas R is measured to the outer edge of the rim and a correction is applied to allow for the shape and size of the rim. Rim cross-sections vary considerably and are rarely of the idealised form shown in figure 3. By comparison it is a very straightforward matter to measure (sufficiently accurately) the quantities required for the evaluation of (12) and the use of this form of data does not require the subsequent application of corrections for rim shape and size.

Practical wheel builders often rely solely on many years of experience to select the correct spoke length for a given configuration, and only occasionally get it wrong. In such a case the wheel has to be re-built from the beginning. Such time wasting can be avoided, however, if the formula is stored, say, on a programmable calculator, for it is then a trivial matter to determine the appropriate spoke length once the data have been obtained.

For any reader interested in the early development of bicycles and tricycles the book by Sharp (reference 2) makes fascinating reading. This large treatise covers a very wide area including the design, construction, statics and dynamics of bicycles and tricycles. The book by Whitt and Wilson (reference 5) provides an excellent, more modern, account and contains a large number of references.

Acknowledgements

I am indebted to Mr D. Butterworth of A. E. Butterworth (Cycles), Sheffield, who originally fired my interest in this problem and who also supplied the data given in Table 1. I also extend my thanks to Dr T. B. Smith of the Physics Department, Open University, Milton Keynes, for drawing my attention to the book by van der Plas, to Mr E. Gilbert of Eric Gilbert Carpets, Sheffield, for the extended loan of Sharp's treatise, and to Dr D. W. Windle, Department of Applied and Computational Mathematics, University of Sheffield, for his helpful comments on an earlier draft of this article.

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Card Shuffling for Beginners

P. GLAISTER

Riffle shuffles, down-shuffles, up-shuffles—mathematics for those with a passion for cards.

One way to shuffle a pack of cards is to cut them into 2 equal piles, A and B say, then re-assemble by taking one card from A followed by one from B, then one again from A and so on. This type of shuffle is sometimes called a *riffle*: a comprehensive discussion of it by the well-known author Martin Gardner appears in reference 1. In this article we extend, and then analyse, the riffle shuffle by dividing a pack into more than 2 piles. We begin with a brief description of the problem.

If we label the cards uniquely by numbering them $1, 2, \ldots$, then for a pack of 4 cards one shuffle can be represented diagrammatically as shown in figure 1, or more simply as shown in figure 2. If we shuffle again, we can see from figure 3 that we get back to the beginning. Thus, after 2 shuffles of a pack of 4 cards we reproduce our original pattern.

$$\begin{array}{cccc} & A & B & \\ 1 & 1 & 3 & 1 \\ 2 & 2 & 4 & 3 \\ 3 & & & 2 \\ 4 & & & 4 \end{array}$$

Figure 1

$$\begin{array}{ccc} 1 & & 1 \\ \underline{2} & & 3 \\ 3 & & 2 \\ 4 & \xrightarrow[\text{one shuffle}]{} & 4 \end{array}$$

Figure 2

$$\begin{array}{ccccc} 1 & & 1 & & 1 \\ \underline{2} & & \underline{3} & & 2 \\ 3 & & 2 & & 3 \\ 4 & \xrightarrow[\text{one shuffle}]{} & 4 & \xrightarrow[\text{one shuffle}]{} & 4 \end{array}$$

Figure 3

Since we are taking a card from the top pile and working down, we call this a *down-shuffle*, or *top-shuffle*. An obvious alternative is to take a card from the bottom pile and work upwards, and we call this an *up-shuffle*, or *bottom-shuffle*. Thus for a pack of 4 cards we see that 4 up-shuffles are required to restore the order (figure 4), as compared with 2 down-shuffles. What happens in the case of 6 cards? It is a simple matter to verify that 4 down-shuffles and only 3 up-shuffles are required to restore the order. In this case, however, we could split the pack into 3 piles each of 2 cards instead of 2 piles each of 3 cards. For this division we see that again 4 down-shuffles are required (figure 5), but 6 up-shuffles are required to restore the order (figure 6).

For larger packs there is very often a greater choice for the size of pile: for instance a 24-card pack can be down-shuffled and up-shuffled using 2, 3, 4, 6, 8 and 12 piles each of 12, 8, 6, 4, 3 and 2 cards, respectively. Actual shuffling now becomes a little tedious, and two alternative approaches can be followed. The first uses a computer program to simulate the shuffling process, and we leave readers to follow this one up. Before we investigate the second approach, we make a few observations on the results shown in table 1.

(a) It is clearly not always the case that the number of down-shuffles is the same for any division.

$$\begin{array}{ccccccccc} 1 & & 3 & & 4 & & 2 & & 1 \\ \underline{2} & & \underline{1} & & \underline{3} & & \underline{4} & & 2 \\ 3 & & 4 & & 2 & & 1 & & 3 \\ 4 & \xrightarrow[\text{one shuffle}]{} & 2 & \xrightarrow[\text{one shuffle}]{} & 1 & \xrightarrow[\text{one shuffle}]{} & 3 & \xrightarrow[\text{one shuffle}]{} & 4 \end{array}$$

Figure 4

$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 1 \\ \underline{2} & & \underline{3} & & \underline{5} & & \underline{4} & & 2 \\ 3 & & 5 & & 4 & & 2 & & 3 \\ \underline{4} & & \underline{2} & & \underline{3} & & \underline{5} & & 4 \\ 5 & & 4 & & 2 & & 3 & & 5 \\ 6 & \xrightarrow[\text{one shuffle}]{} & 6 & \xrightarrow[\text{one shuffle}]{} & 6 & \xrightarrow[\text{one shuffle}]{} & 6 & \xrightarrow[\text{one shuffle}]{} & 6 \end{array}$$

Figure 5

$$\begin{array}{ccccccccccccc} 1 & & 5 & & 4 & & 6 & & 2 & & 3 & & 1 \\ \underline{2} & & \underline{3} & & \underline{1} & & \underline{5} & & \underline{4} & & \underline{6} & & 2 \\ 3 & & 1 & & 5 & & 4 & & 6 & & 2 & & 3 \\ \underline{4} & & \underline{6} & & \underline{2} & & \underline{3} & & \underline{1} & & \underline{5} & & 4 \\ 5 & & 4 & & 6 & & 2 & & 3 & & 1 & & 5 \\ 6 & \xrightarrow[\text{one shuffle}]{} & 2 & \xrightarrow[\text{one shuffle}]{} & 3 & \xrightarrow[\text{one shuffle}]{} & 1 & \xrightarrow[\text{one shuffle}]{} & 5 & \xrightarrow[\text{one shuffle}]{} & 4 & \xrightarrow[\text{one shuffle}]{} & 6 \end{array}$$

Figure 6

Table 1

nm cards	n piles	each of m cards	Down-shuffles	Up-shuffles
4	2	2	2	4
6	2	3	4	3
	3	2	4	6
8	2	4	3	6
	4	2	3	3
10	2	5	6	10
	5	2	6	5
12	2	6	10	12
	3	4	5	3
	4	3	5	6
	6	2	10	12
14	2	7	12	4
	7	2	12	4
16	2	8	4	8
	4	4	2	4
	8	2	4	8
18	2	9	8	18
	9	2	8	9
20	2	10	18	6
	4	5	9	3
	5	4	9	6
	10	2	18	6
22	2	11	6	11
	11	2	6	22
24	2	12	11	20
	3	8	11	20
	4	6	11	10
	6	4	11	5
	8	3	11	20
	12	2	11	20
36	2	18	12	36
	3	12	12	18
	4	9	6	18
	6	6	2	4
	9	4	6	9
	12	3	12	9
	18	2	12	36

(b) The number of up-shuffles does not always exceed the number of down-shuffles.

(c) The number of down-shuffles required with n piles each of m cards is the same as the number of down-shuffles required with m piles each of n cards, but this is not true, in general, for up-shuffles.

(d) The number of down-shuffles (for a pack of $2M$ cards) with 2 piles each of M cards is the same as the number of up-shuffles (for a pack of $2M-2$ cards) with 2 piles each of $M-1$ cards.

(e) The number of down-shuffles and up-shuffles with N piles each of N cards are 2 and 4, respectively.

We prove conjectures (c), (d) and (e) later on. My particular favourite is a pack of 24 cards, where 11 down-shuffles are required, regardless of which of the 6 possible divisions is chosen.

Our aim is to find a simple way of determining the number of down-shuffles and up-shuffles required for any (equal) division of a pack of cards of any size (but not prime, of course); in particular, for the standard pack of 52 cards.

Suppose that a pack of nm cards has down-shuffles or up-shuffles performed by dividing into n piles each of m cards.

Down-shuffles

After one down-shuffle a card originally in position $i+(j-1)m$ moves to position $j+(i-1)n$ $(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$ and, since

$$n\{i+(j-1)m\}-(n-1)-\{j+(i-1)n\} = (j-1)(nm-1),$$

a card in position a_0 moves to position a_1, where

$$a_1 \equiv na_0-(n-1) \pmod{nm-1}.$$

After k down-shuffles the card originally in position a_0 has moved to position a_k where

$$a_k \equiv n^k a_0-(n^k-1) \pmod{nm-1},$$

by repeated application of the formula above. The pack is restored to the original order after the smallest number of shuffles k for which $a_k = a_0$ for all a_0, and thus

$$a_0 \equiv n^k a_0-(n^k-1) \pmod{nm-1},$$

i.e.

$$(a_0-1)n^k \equiv a_0-1 \pmod{nm-1},$$

and hence k is the least positive integer for which $n^k \equiv 1 \pmod{nm-1}$. Conversely, if k is the least positive integer for which $n^k \equiv 1 \pmod{nm-1}$, then $a_k \equiv a_0 \pmod{nm-1}$ for all a_0, and thus $a_k = a_0$ for all a_0, except possibly for the top and bottom cards, which may have changed. However, for down-shuffles the top and bottom cards remain fixed, and thus k is the smallest number of shuffles for which $a_k = a_0$ for all a_0, so that the original order is restored.

Up-shuffles

Similarly, after one up-shuffle a card originally in position $i+(n-j)m$ moves to position $j+(i-1)n$ $(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$ and, since

$$n\{i+(n-j)m\}-\{j+(i-1)n\} = (n-j)(nm+1),$$

a card in position a_0 moves to position a_1, where $a_1 \equiv na_0 \pmod{nm+1}$. After k up-shuffles the card originally in position a_0 has moved to position a_k, where $a_k \equiv n^k a_0 \pmod{nm+1}$. The pack is restored to the original order after the smallest number of shuffles k for which $a_k = a_0$ for all a_0, and thus $a_0 \equiv n^k a_0 \pmod{nm+1}$, and hence k is the least positive integer for which $n^k \equiv 1 \pmod{nm+1}$. Conversely, if k is the least positive integer for which $n^k \equiv 1 \pmod{nm+1}$, then $a_k \equiv a_0 \pmod{nm+1}$, and thus $a_k = a_0$ for all a_0. Thus k is the least number of shuffles for which the original order is restored.

Summarising, we see that the number of down-shuffles of nm cards divided into n piles of m cards is given by the smallest positive integer k satisfying $n^k \equiv 1 \pmod{nm-1}$, and for up-shuffles this is replaced by $n^k \equiv 1 \pmod{(nm+1)}$. Readers should check these against the results given in table 1.

We now prove conjectures (c)–(e) above.

Proof of conjecture (c)

Since $nm \equiv 1 \pmod{(nm-1)}$, then $(nm)^k \equiv 1 \pmod{(nm-1)}$ and $n^k \equiv 1 \pmod{(nm-1)}$ if and only if $m^k \equiv 1 \pmod{(nm-1)}$. Thus the number of down-shuffles required with n piles each of m cards is the same as the number of down-shuffles required with m piles each of n cards.

Proof of conjecture (d)

The number of down-shuffles (for a pack of $2M$ cards) with 2 piles each of M cards is the smallest integer k for which $2^k \equiv 1 \pmod{(2M-1)}$. Since $2M-1 = 2(M-1)+1$, k is the least integer for which $2^k \equiv 1 \pmod{(2[M-1]+1)}$ and this is the number of up-shuffles (for a pack of $2M-2$ cards) with 2 piles each of $M-1$ cards.

Proof of conjecture (e)

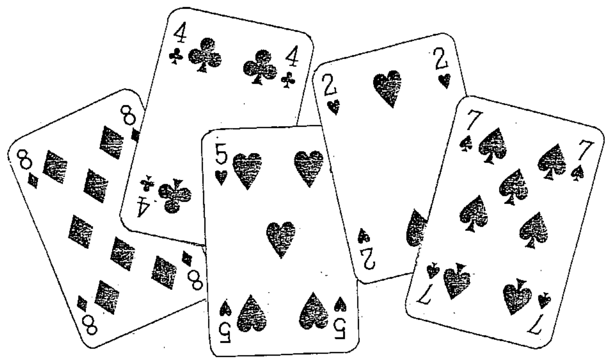
Since $N^2 \equiv 1 \pmod{(N^2-1)}$ then 2 down-shuffles are required for a pack of N^2 cards with N piles each of N cards. This is analogous to transposing an $N \times N$ matrix where, after 2 transposes, the original matrix is obtained. Similarly, since $N^4-1 = (N^2-1)(N^2+1)$ then $N^4 \equiv 1 \pmod{(N^2+1)}$, so that 4 up-shuffles are required for a pack of N^2 cards with N piles each of N cards.

Finally, the missing results from table 1 are given in table 2. Thus only 4 down-shuffles of a pack of 52 cards divided into 4 piles each of 13 cards, i.e. 4 suits, will restore the original order. Curiously, for a 51-card pack, 20 down-shuffles are required for 3 piles of 17 cards or 17 piles of 3 cards, but only 6 up-shuffles are required for the same divisions. There are clearly many interesting variations for the reader to follow up.

Table 2. Results missing from table 1.

Cards	Piles	No. of cards	Up-shuffles	Down-shuffles
52	2	26	8	52
	4	13	4	26
	13	4	4	13
	26	2	8	52

Now, where is that pack of cards!



Reference

1. M. Gardner, *The Mathematical Carnival* (Alfred Knopf, New York, 1975).

__Paul Glaister__ is currently a lecturer in mathematics at the University of Reading. Although his list of hobbies does not include playing cards, a day rarely passes without his small daughter insisting that they play a game of 'pairs' with a pack of Postman Pat cards. What he finds particularly infuriating is that she always wins. He intends getting his own back soon by teaching her about modulo arithmetic.

Langley's adventitious angles

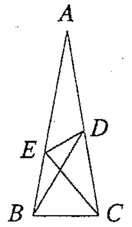
S. Maloney, Head of Mathematics at Hammersmith School in London, recently came across this question in an excellent book entitled *The Penguin Dictionary of Curious and Interesting Puzzles*. It looks trivial, but Mr Maloney and his colleagues found it deceptively difficult and he thinks it deserves a wider audience. It's called 'Langley's adventitious angles', but we don't know why.

$AB = AC$

$\angle BAC = 20°$

$\angle DBC = 60°$

$\angle ECB = 50°$



Find $\angle EDB$.

Hazel Perfect

Dr Perfect has served as an editor of *Mathematical Spectrum* since 1976 and during that time has read literally hundreds of contributions. This she has always done with great care and understanding. She has now decided to step down as an editor. The other editors would like to express their deep appreciation for all that she has done: her help will be greatly missed. Fortunately, her contributions to *Mathematical Spectrum* are not over as she has agreed to serve on the advisory board. Also, she has written a biographical article on the great German mathematician Georg Cantor which is awaiting publication and is working on one on the neglected British mathematician T. P. Kirkman, which we await with considerable interest.

The Editor

Laplace Transforms in Theory and Practice

BRIAN H. DENTON

Maybe you feel intimidated by the very words 'Laplace transform'. If so, this article is designed to dispel your fears.

Laplace transforms are usually first studied in undergraduate courses especially by budding engineers. Since they have quite unjustifiably acquired a rather intimidating reputation, this article is intended to show that they are not only useful, but basically also quite easy. The principal application of Laplace transforms is to the solution of differential equations which arise in mathematics, science, engineering and other areas, and two examples will illustrate the simplicity of the method.

Pierre-Simon, Marquis de Laplace, was born in 1749. His major mathematical works were the *Traité de méchanique céleste*, published in five volumes between 1799 and 1825, and the *Théorie analytique de probabilités*, published in 1812. The Laplace transform is only one of his many substantial contributions to mathematics.

Our starting point is a function $f(t)$ which is defined for $t \geqslant 0$. Next we need to be able to handle an 'infinite integral' of the form

$$I = \int_0^\infty f(t)\,\mathrm{d}t.$$

This integral is defined as

$$\lim_{T\to\infty}\int_0^T f(t)\,\mathrm{d}t,$$

if this limit exists. If f has the antiderivative (or primitive) g, which means that $g'(t) = f(t)$, then

$$\int_0^T f(t)\,\mathrm{d}t = g(T) - g(0)$$

and the definition of I is equivalent to

$$I = \lim_{T\to\infty} g(T) - g(0).$$

So now let us evaluate the integral

$$I = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t,$$

where s is a positive constant. We have

$$I = \frac{\mathrm{e}^{-st}}{-s}\bigg|_0^\infty = \frac{1}{-s}\Big(\lim_{T\to\infty}\mathrm{e}^{-sT}\Big) - \frac{1}{-s}.$$

Hence

$$I = \frac{1}{s}. \tag{1}$$

We have just evaluated the Laplace transform of 1, without too much pain!

There is no point in being mysterious; we might as well write down the definition of the Laplace transform of any function $f(t)$, which we denote as $\mathscr{L}[f(t)]$. It is, when the integral exists,

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{2}$$

This integral is, in fact, a function of s and we therefore also denote it by $F(s)$. Thus, by (1), $F(s) = \mathscr{L}[1] = 1/s$ $(s > 0)$.

It can be shown that, if the integral (2) exists for $s = s_0$, say, then it exists also for $s > s_0$. Hence a Laplace transform $F(s)$, if it exists anywhere, exists on some interval (c, ∞).

Next we evaluate $\mathscr{L}[t]$ using integration by parts, our knowledge of $\mathscr{L}[1]$ and the fact that, if $s > 0$, $T\mathrm{e}^{-sT} \to 0$ as $T \to \infty$. For $s > 0$,

$$\begin{aligned}\mathscr{L}[t] &= \int_0^\infty t\mathrm{e}^{-st}\,\mathrm{d}t \\ &= \frac{t\mathrm{e}^{-st}}{-s}\bigg|_0^\infty + \int_0^\infty \frac{\mathrm{e}^{-st}}{s}\,\mathrm{d}t \\ &= 0 + \frac{1}{s}\mathscr{L}[1] \\ &= \frac{1}{s^2}.\end{aligned}$$

So

$$\mathscr{L}[t] = \frac{1}{s^2} \quad (s > 0).$$

By means of a similar technique we can now deduce that

$$\mathscr{L}[t^2] = \frac{2}{s^3} \quad (s > 0).$$

However, it is easy to prove by induction that, for all $n \geqslant 0$,

$$\mathscr{L}[t^n] = \frac{n!}{s^{n+1}} \quad (s > 0), \tag{3}$$

where 0! is, as usual, defined to be 1. The formula has already been proved for $n = 0$. So assume that it holds for $n = k$ ($\geqslant 0$). Then, since $T^{k+1}e^{-sT} \to 0$ as $T \to \infty$,

$$\begin{aligned}\mathscr{L}[t^{k+1}] &= \int_0^\infty t^{k+1}e^{-st}\,dt \\ &= t^{k+1}\frac{e^{-st}}{-s}\bigg|_0^\infty + \int_0^\infty \frac{k+1}{s}e^{-st}t^k\,dt \\ &= 0+\frac{k+1}{s}\mathscr{L}[t^k] \\ &= \frac{(k+1)k!}{s\cdot s^{k+1}} \\ &= \frac{(k+1)!}{s^{k+2}}.\end{aligned}$$

Hence (3) holds for $n = k+1$ and so, by induction, the formula holds for all $n \geqslant 0$.

It follows at once from the linear property of integrals that

$$\mathscr{L}[cf(t)] = c\mathscr{L}[f(t)]$$

when c is any constant, and that

$$\mathscr{L}[f(t)+g(t)] = \mathscr{L}[f(t)]+\mathscr{L}[g(t)].$$

This means that we can now calculate the Laplace transform of any polynomial.

Another simple Laplace transform is that of e^{at}. When $s > a$,

$$\begin{aligned}\int_0^\infty e^{at}e^{-st}\,dt &= \int_0^\infty e^{(a-s)t}\,dt \\ &= \frac{e^{(a-s)t}}{a-s}\bigg|_0^\infty \\ &= \frac{-1}{a-s} \\ &= \frac{1}{s-a}.\end{aligned}$$

So

$$\mathscr{L}[e^{at}] = \frac{1}{s-a} \quad (s > a).$$

We shall see that, the larger our store of Laplace transforms is, the greater is the variety of differential equations that we can tackle. Two more transforms that are not difficult to obtain are those of $\cos at$ and $\sin at$.

For any $s > 0$, two integrations by parts give

$$\begin{aligned}I &= \int_0^\infty \sin at\, e^{-st}\,dt \\ &= \sin at\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty a\cos at\frac{e^{-st}}{-s}\,dt \\ &= \frac{a}{s}\int_0^\infty \cos at\, e^{-st}\,dt \qquad (4) \\ &= \frac{a}{s}\left(\cos at\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty -a\sin at\frac{e^{-st}}{-s}\,dt\right) \\ &= \frac{a}{s}\left(\frac{1}{s}-\frac{a}{s}I\right).\end{aligned}$$

Hence

$$\left(1+\frac{a^2}{s^2}\right)I = \frac{a}{s^2},$$

i.e.

$$I = \frac{a}{s^2+a^2}.$$

Then, by (4), if $a \neq 0$,

$$\int_0^\infty \cos at\, e^{-st}\,dt = \frac{s}{s^2+a^2}.$$

This formula actually holds for $a = 0$ also because in that case it reduces to

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s}.$$

We have therefore shown that, for all a,

$$\mathscr{L}[\cos at] = \frac{s}{s^2+a^2}, \qquad \mathscr{L}[\sin at] = \frac{a}{s^2+a^2} \quad (s > 0).$$

Finally, before we can solve differential equations using Laplace transforms we need to know the transforms of f' and f'' and derivatives of higher orders.

Suppose that $F(s) = \mathscr{L}[f(t)]$ exists for $s > c$. Then, assuming that $f(T)e^{-sT} \to 0$ as $T \to \infty$, we have, for $s > c$,

$$\begin{aligned}\mathscr{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\,dt \\ &= f(t)e^{-st}\Big|_0^\infty + \int_0^\infty sf(t)e^{-st}\,dt \\ &= -f(0)+s\mathscr{L}[f(t)] \qquad (5) \\ &= sF(s)-f(0).\end{aligned}$$

Hence we have the result

$$\mathscr{L}[f'(t)] = sF(s)-f(0) \quad (s > c).$$

If also $f'(T)e^{-sT} \to 0$ as $T \to \infty$, then, when $c > 0$, we can twice apply (5), first with f replaced by f', to obtain

$$\begin{aligned}\mathscr{L}[f''(t)] &= -f'(0)+s\mathscr{L}[f'(t)] \\ &= -f'(0)+s^2F(s)-sf(0).\end{aligned}$$

So

$$\mathscr{L}[f''(t)] = s^2F(s)-sf(0)-f'(0) \quad (s > c).$$

The Laplace tranforms of higher-order derivatives can be obtained in the same way. These results are summarised in table 1. The last entry in the table has not been proved in this article, and is left as an exercise for the reader. Here $s > c+b$, when $F(s)$ is defined for $s > c$. Clearly other transforms now follow such as those of $e^{bt}t^n$, $e^{bt}\cos at$ and $e^{bt}\sin at$.

Table 1. Some Laplace transforms.

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$e^{bt}f(t)$	$F(s-b)$

We will now apply our theory to a couple of practical examples. First a traditional problem from mechanics, stated in a traditional way!

Problem 1. A particle of unit mass is moving along the x axis and at the point x the force acting on it is $-4x$. If it is initially at rest at the point $x = 20$, find the position of the particle at any time.

Solution. The equation of motion is

$$\frac{d^2x}{dt^2} = -4x,$$

i.e.

$$\frac{d^2x}{dt^2}+4x = 0. \tag{6}$$

Furthermore we are given the initial conditions $x(0) = 20$ and $x'(0) = 0$.

We now take Laplace transforms of each side of the equation. From table 1, with $f(t)$ and $F(s)$ replaced by $x(t)$ and $X(s)$, respectively, we get

$$s^2X(s)-sx(0)-x'(0)+4X(s) = 0.$$

In view of the initial conditions this equation is

$$X(s)(s^2+4)-20s = 0,$$

so that

$$X(s) = \frac{20s}{s^2+4}. \tag{7}$$

It is therefore a matter of finding the function $x(t)$ which has the Laplace transform (7). But table 1 shows that

$$\mathcal{L}[\cos 2t] = \frac{s}{s^2+4},$$

and so the solution of our problem is

$$x(t) = 20\cos 2t.$$

The usual elementary method for solving the differential equation (6) with the given initial conditions operates in two stages: first the general solution of (6) is found and this involves two arbitrary constants; the second stage is to obtain the particular values of these constants which ensure that the initial conditions are satisfied. In contrast, the Laplace transform method saves time because it solves (6) together with the initial conditions at one fell swoop.

Another less simple example is more typical of the situation that might be met in practice. The differential equation below could, for instance, arise in electrical engineering. Some of the details of the solution are left to the reader.

Problem 2. Solve the differential equation

$$\frac{d^2x}{dt^2}-2\frac{dx}{dt} = e^x(x-3),$$

with the initial conditions $x(0) = 2 = x'(0)$.

Solution. As before, we take Laplace transforms of both sides, using the table of transforms. We obtain

$$s^2X(s)-sx(0)-x'(0)-2sX(s)+2x(0) = \mathcal{L}[e^x(x-3)]. \tag{8}$$

To evaluate the right-hand side we first note that

$$\mathcal{L}[x-3] = \frac{1}{s^2}-\frac{3}{s} = F(s),$$

say, and so, from the last entry in table 1,

$$\mathcal{L}[e^x(x-3)] = F(s-1) = \frac{1}{(s-1)^2}-\frac{3}{s-1} = \frac{4-3s}{(s-1)^2}.$$

Now returning to (8) and putting in our initial conditions we have

$$s(s-2)X(s) = 2s-2+\frac{4-3s}{(s-1)^2} = \frac{2(s-1)^3+(4-3s)}{(s-1)^2} = \frac{(s-2)(2s^2-2s-1)}{(s-1)^2}.$$

Hence

$$X(s) = \frac{2s^2-2s-1}{s(s-1)^2},$$

and in partial fractions

$$X(s) = \frac{3}{s-1}-\frac{1}{(s-1)^2}-\frac{1}{s}.$$

Table 1 is again used to find $x(t)$; the last entry in the table is needed for the second term. We obtain

$$x(t) = 3e^t-te^t-1$$

as the solution with the conditions given at the start of the problem.

The Laplace transform not only helps to solve ordinary differential equations of the type we have just considered; it is also a powerful tool for the solution of partial differential equations. Moreover it is a fascinating subject in its own right and it thus caters for the pure mathematician as well as for people mainly interested in the applications of mathematics. □

__Brian Denton__ lectures in mathematics at Liverpool John Moores University. He writes: 'I have always been interested in maths education and the teaching of mathematics to all age groups. I am a keen ornithologist, and I also enjoy good wines and good books. I am a member of so many societies, it's a wonder I find time to do any work!'

Chaotic Music on the BBC Micro or IBM PC

PETER DIXON

How to program your PC to make music.

One of the simplest examples in chaos theory is provided by the so-called *logistic equation*

$$x_{n+1} = \mu x_n(1-x_n). \qquad (1)$$

Here, μ is a number between 1 and 4; think of it as a constant. The number x_1 is given (and it is always strictly between 0 and 1). Equation (1) holds for $n = 1, 2, 3, \ldots$, so it defines $x_2, x_3, \ldots$ 'by iteration'.

The iteration works as follows: we are told what x_1 is; then (1) with $n = 1$ is

$$x_2 = \mu x_1(1-x_1),$$

which tells us what x_2 is. Then (1) with $n = 2$ is

$$x_3 = \mu x_2(1-x_2),$$

which gives us x_3 and so on.

The following program, in BBC Basic, prints out the sequence $x_1, x_2, x_3, \ldots$.

```
10 INPUT "MU ="; MU
20 X = 0.4137
30 PRINT X
40 PITCH = 41 + 96*X
50 SOUND 1,-15,PITCH,5
60 X = MU*X*(1-X)
70 GOTO 30
```

Let us see how it works. First, line 10 inputs your choice of the parameter μ, assigning it the name 'MU'. This remains unchanged for the rest of the program. Line 20 sets the variable X initially to the given number x_1. I have chosen a specific value for x_1, but my choice is quite random—any number strictly between 0 and 1 will do. You could write a line like line 10 allowing you to input your own choice of x_1 each time you run the program, but the value used makes so little difference to the ultimate behaviour of the solution that it becomes boring to have to input it each time.

The variable X starts by taking the value x_1. It is then printed (by line 30). At line 60, it is changed to x_2; think of this line as saying

$$\begin{aligned}&(\text{new value of } X)\\ &\quad = \mu(\text{old value of } X)[1-(\text{old value of } X)].\end{aligned}$$

Line 70 then takes us back to line 30, where the number x_2 is printed. The second encounter with line 60 changes the value of X to x_3, and so on.

I have not mentioned lines 40 and 50. These are the ones which convert mathematics into music! A short calculation will show that, because the starting value x_1 of X lies between 0 and 1, every subsequent value of X will also lie in that interval. (This calculation uses the fact that $1 \leqslant \mu \leqslant 4$.) We convert numbers between 0 and 1 into musical notes between the A below middle C and the A two octaves higher. Line 40 interprets X as a pitch: 41 represents the A below middle C and 96 represents 2 octaves (one semitone equals 4). Line 50 creates the sound on channel 1, volume -15 (this is classroom volume; try -5 for individual work), at the chosen pitch, for $\frac{5}{20}$ of a second. (For a PC, running MS-DOS QBasic, these lines should be replaced by

```
40 PITCH = 220 * 4 ^ X
50 SOUND PITCH, 4
```

The pitch is measured in Hertz and each note sounds for 4/18.2 seconds.)

To keep the program simple, I have not incorporated any means of stopping it; the program just continues indefinitely until the ESCAPE key is pressed. (Why bother to program a break mechanism when the writers of the BASIC interpreter have done it for you?)

With this program, we can study the behaviour of solutions for various values of the parameter μ. The behaviour you might expect is that the sequence $x_1, x_2, x_3, \ldots$ converges to a number x which is such that $x = \mu x(1-x)$. For values of μ between 1 and 3, this is exactly what happens. Try $\mu = 2.5$: convergence is so fast that before you can tell what has happened, the values become so close to such an x as to be indistinguishable from it. After a *very* short time, you hear a continuous tone. Try $\mu = 2.9$: the same thing happens, but convergence is slower and you get a chance to appreciate what is going on.

Now go above 3: try $\mu = 3.3$. A surprising thing happens: instead of tending to a fixed number, the sequence tends to an oscillation between two numbers. The musical interpretation shows this clearly.

Go further: try $\mu = 3.5$. You find convergence to a 4-cycle. (The exact value of μ at which the 2-cycle ended was $\mu = 1+\sqrt{6}$.) Try $\mu = 3.56$ and listen carefully—it converges to an 8-cycle, not a 4-cycle.

This 'cascade of period-doubling bifurcations' continues up to $\mu = 3.569\,946\ldots$. Between there and $\mu = 4$ there are many values where the resulting sequence is best described as 'chaotic'—try $\mu = 3.6$, 3.7, 3.8 or 3.9, for instance. However, there are 'windows of periodicity'. Try values of μ between 3.7382 and 3.7411 or between 3.8285 and 3.8415, but be prepared to wait 30 seconds or more for the sequence to settle down. Look too at values just below these windows, $\mu = 3.8284$, for example.

The mathematics of this equation is only just being worked out. Until recently it was a famous unsolved problem whether there are 'windows of chaos': i.e. whether there exist numbers a and b with $1 \leqslant a < b \leqslant 4$ such that every μ with $a \leqslant \mu \leqslant b$ gives rise to a non-periodic sequence. The expected result, that such windows do not exist, was announced by Professor G. Swiatek in 1992, but the details of the proof have yet to be published. It is clear from his lectures on the subject that the complete proof is long and complicated! A more subtle question, which does not follow from Swiatek's result, is whether or not the total length of the periodic regions between 1 and 4 is equal to 3.

All this is closely related to the shape of the Mandelbrot set near the real axis. But that is another, more complex, story!

□

__Peter Dixon__ is a reader in pure mathematics at the University of Sheffield. He does research on Banach algebra theory—the algebra of infinite matrices—and on the geometry of infinite-dimensional spaces. His lecture course for third-year undergraduates on chaos and fractals opens with distinctive theme music.

A Partial Proof of Fermat's Last Theorem

JOHN SHERRILL

A proof of Fermat's last theorem has recently been announced, although apparently it is not yet complete (see Volume 26 Numbers 3 and 4). This is beyond the understanding of all but a very few mathematicians. Here is a simple proof of a special case.

Fermat's last theorem is the conjecture that the equation

$$a^n = b^n + c^n$$

has no solution in natural numbers a, b, c and n when $n > 2$. This article will show that there is no possible solution when a is prime or a power of a prime.

Fermat himself proved the case of $n = 4$ and thereby proved all cases in which n is a multiple of 4. Since all other integers greater than 2 are divisible by an odd prime, it is sufficient to assume that n is an odd prime.

The relationships between a, b and c are: $a > b$, $a > c$, and $\frac{1}{2}(b+c) < a < b+c < 2a$. The identity

$$\begin{aligned} a^n &= b^n + c^n \\ &= b^n - (-c)^n \\ &= (b+c)[b^{n-1} + b^{n-2}(-c) + \cdots + b(-c)^{n-2} + (-c)^{n-1}] \end{aligned}$$

shows that $b+c$ divides a^n. Since $b+c > a$, there exists a natural number y such that $b+c = a+y$. If a is either prime or a power of a prime, there exist natural numbers p and m, where p is prime, such that $p^m = a$, $p^{mn} = b^n + c^n$ and $p^m + y$ divides p^{mn}. Since p is the only prime divisor of p^{mn}, any divisor of the form $p^m + y$ must equal p^k, where the natural number k satisfies $m < k \leqslant mn$. Thus $p^m + y = p^k$, $y = p^k - p^m = p^m(p^{k-m} - 1)$ and a divides y. But if a divides y then $a \leqslant y$, which is a contradiction because $a > \frac{1}{2}(a+y)$ and so $a > y$.

Consequently the equation $a^n = b^n + c^n$ has no solution in natural numbers when a is prime or a power of a prime and $n > 2$.

Acknowledgement

For his assistance during the evolution of this proof I wish to thank Professor Emeritus Evan K. Jobe of the Texas Tech University Department of Philosophy. □

__John Sherrill__'s background is in the humanities, although he minored in mathematics as a student.

Simpson's Rule for Numerical Integration

FRANK CHORLTON

Many readers will have used Simpson's rule for finding an approximate value of an integral. But where does it come from? Read on.

Introduction

Many *Spectrum* readers will be acquainted with Simpson's rule for obtaining approximate values of definite integrals of finite integration range. This celebrated rule is still applicable in many areas even in an increasingly computer-oriented age. Its origins stem from Thomas Simpson (1710–1761).

The development here utilises the simple form of the mean value theorem and also its Cauchy variant. It differs from the more usual treatments, for instance in references 2 and 3. We therefore start with an illustrative enunciation of the elementary form of the mean value theorem, derive the Cauchy form from it and then proceed to obtain Simpson's rule for two adjacent strips of equal breadth. Finally, the more extended form of the rule is obtained and its application illustrated by means of a worked example.

Mean value theorems

For the benefit of readers unfamiliar with notation in mathematical analysis, or who wish to be reminded of it, we first make the following citations:

$$x \text{ in } [a,b] \text{ means } a \leqslant x \leqslant b;$$

$$x \text{ in } (a,b) \text{ means } a < x < b.$$

Figure 1 shows the sketch of a function $y = f(x)$ which is differentiable throughout $[a,b]$. Figure 2 illustrates a function $y = g(x)$ which fails to be differentiable at just one point in $[a,b]$. Figure 1 indicates the existence of a point C on the curve between A and B such that the tangent to $y = f(x)$ at C is parallel to the chord AB. If $x = c$ at C, the figure strongly suggests that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

On the other hand, no corresponding property seems to derive from figure 2. On the basis of this geometrical intuition we can now formulate the *first mean value theorem*: Suppose f is differentiable at each point of $[a,b]$. Then there is a point c in (a,b) such that

$$f(b)-f(a) = (b-a)f'(c).$$

It is important to note that the theorem says it is possible to choose a suitable c which is not a or b, though in particular cases $f'(a)$ or $f'(b)$ may be equal to $[f(b)-f(a)]/(b-a)$, e.g. when f is constant. This last example also shows that c need not be unique.

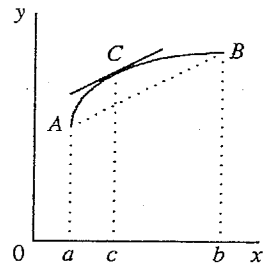


Figure 1

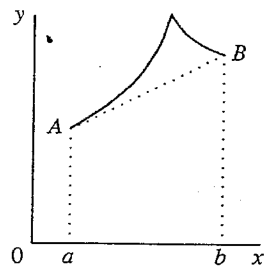


Figure 2

Rigorous treatments of this result are found in many analysis books, e.g. in reference 1.

Let us now apply the first mean value theorem to the function $f(x)$ having the particular form

$$f(x) = [F(b)-F(a)]G(x)-[G(b)-G(a)]F(x),$$

where $F(x)$ and $G(x)$ are both differentiable in the interval $[a,b]$ and $G'(x) \neq 0$ in (a,b). Then

$$f(a) = F(b)G(a)-G(b)F(a) = f(b).$$

Since $f(a) = f(b)$, the first mean value theorem shows that there exists c in (a,b) such that

$$f'(c) = 0$$

or

$$[F(b)-F(a)]G'(c) = [G(b)-G(a)]F'(c).$$

As $G'(x) \neq 0$ in (a,b), so that $G'(c) \neq 0$ and $G(b)-G(a) \neq 0$,

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)},$$

which is the *Cauchy form of the mean value theorem.* An interesting feature is that this form cannot be obtained by applying the first form separately to $F(x)$ and $G(x)$; for the first form only proves the existence of c_1 and c_2 in (a,b) such that $F(b)-F(a) = (b-a)F'(c_1)$ and $G(b)-G(a) = (b-a)G'(c_2)$ and there is no guarantee that $c_1 = c_2$.

Derivation of the simple form of Simpson's rule

Our first problem is to approximate the value of the definite integral

$$I(h) = \int_{-h}^{h} f(x)\,dx,$$

where h is positive and small and f is assumed to be four times differentiable with f^{iv} continuous. It is reasonable to take an approximate form

$$h\{\lambda f(-h)+\mu f(0)+\nu f(h)\}$$

for $I(h)$, where λ, μ and ν are numerical constants to be determined. The difference between $I(h)$ and this assumed approximation defines a remainder function $R(h)$ as

$$R(h) = I(h)-h\{\lambda f(-h)+\mu f(0)+\nu f(h)\}.$$

In what follows the first three derivatives of $R(h)$ will be made to vanish at $h = 0$ so that $R(h)$ is as small as possible. These conditions will determine λ, μ and ν.

The requirement that $R'(h) = 0$ involves first finding $I'(h)$, using the fundamental theorem of the integral calculus. The latter says that

$$\frac{\mathrm{d}}{\mathrm{d}h}\int_0^h f(x)\,\mathrm{d}x = f(h),$$

which expresses the converse nature of the differentiation and integration processes. It follows at once that

$$\frac{\mathrm{d}}{\mathrm{d}h}\int_{-h}^0 f(x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}(-h)}\int_0^{-h} f(x)\,\mathrm{d}x = f(-h).$$

Putting these results together, we have

$$\frac{\mathrm{d}}{\mathrm{d}h}\int_{-h}^h f(x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}h}\left(\int_{-h}^0 + \int_0^h\right)f(x)\,\mathrm{d}x = f(h)+f(-h).$$

Now

$$R'(h) = (1-\nu)f(h)-\mu f(0)+(1-\lambda)f(-h) + h\{\lambda f'(-h)-\nu f'(h)\}.$$

Hence the condition $R'(0) = 0$ requires $\lambda+\mu+\nu = 2$ and so

$$R'(h) = (1-\nu)f(h)-(2-\lambda-\nu)f(0)+(1-\lambda)f(-h) + h\{\lambda f'(-h)-\nu f'(h)\}.$$

Next

$$R''(h) = (1-2\nu)f'(h)-(1-2\lambda)f'(-h) - h\{\nu f''(h)+\lambda f''(-h)\}.$$

If $R''(0) = 0$ then $\lambda = \nu$ and the last form simplifies to

$$R''(h) = (1-2\lambda)\{f'(h)-f'(-h)\}-\lambda h\{f''(h)+f''(-h)\}.$$

The next stage of differentiation shows

$$R'''(h) = (1-3\lambda)\{f''(h)+f''(-h)\} -\lambda h\{f'''(h)-f'''(-h)\}.$$

If we make $R'''(0) = 0$ then $\lambda = \frac{1}{3}$, so that

$$R'''(h) = -\tfrac{1}{3}h\{f'''(h)-f'''(-h)\}. \qquad (1)$$

We now show that $R(h)$ is so small that $R(h)/h^5$ tends to a finite limit as $h \to 0$ (so that $R(h)/h^n \to 0$ as $h \to 0$ when $n < 5$). Remembering that $R(0) = R'(0) = R''(0) = 0$ and applying Cauchy's form of the mean value theorem three times, we see that there are numbers h_1, h_2 and h_3 such that $0 < h_3 < h_2 < h_1 < h$ and

$$\frac{R(h)}{h^5} = \frac{R(h)-R(0)}{h^5-0^5} = \frac{R'(h_1)}{5h_1^4} = \frac{R'(h_1)-R'(0)}{5(h_1^4-0^4)} = \frac{R''(h_2)}{20h_2^3} = \frac{R''(h_2)-R''(0)}{20(h_2^3-0^3)} = \frac{R'''(h_3)}{60h_3^2}.$$

Using the form (1) for $R'''(h)$ with $h_3 = h$, we now obtain

$$\frac{R(h)}{h^5} = -\frac{f'''(h_3)-f'''(-h_3)}{180h_3} = -\frac{f'''(h_3)-f'''(-h_3)}{90[h_3-(-h_3)]}.$$

Applying the first mean value theorem over $(-h_3, h_3)$, we see that there exists h_4 such that $-h < -h_3 < h_4 < h_3 < h$ and

$$\frac{R(h)}{h^5} = -\tfrac{1}{90}f^{\text{iv}}(h_4).$$

Thus $R(h)/h^5 \to -f^{\text{iv}}(0)/90$ as $h \to 0$.

In view of the definition of $R(h)$ (with $3\lambda = 3\nu = 1$ and $3\mu = 4$) the last form of $R(h)/h^5$ shows that

$$\int_{-h}^h f(x)\,\mathrm{d}x = \tfrac{1}{3}h\{f(-h)+4f(0)+f(h)\}-\tfrac{1}{90}h^5f^{\text{iv}}(h_4),$$

where $-h < h_4 < h$. If f has a continuous fourth derivative in $[a-h, a+h]$, then the equivalent statement becomes

$$\int_{a-h}^{a+h} f(x)\,\mathrm{d}x = \tfrac{1}{3}h\{f(a-h)+4f(a)+f(a+h)\} -\tfrac{1}{90}h^5f^{\text{iv}}(c),$$

where $a-h < c < a+h$. We observe that for any polynomial form of f having degree less than 4, $f^{\text{iv}}(c) = 0$ and the Simpson approximation for the definite integral over $(a-h, a+h)$ is exact.

Extended form of Simpson's rule

For completion we now cite the extended form of the last result that is actually used in practice for integration over the range $[a,b]$. If $2n$ equally spaced strips are used, requiring $2n+1$ ordinates, then the spacing between any two neighbouring ordinates is $h = (b-a)/2n$. If $|f^{\text{iv}}(x)| \leqslant M$ in $[a,b]$ and if $a_k = a+kh$ $(k = 0,1,\ldots,2n)$, then application of the simple formula over the n strips $[a_{2m}, a_{2m+2}]$, i.e

$$[a_{2m+1}-h, a_{2m+1}+h] \quad (m = 0,1,\ldots,n-1),$$

gives

$$\left|\int_a^b f(x)\,\mathrm{d}x-\frac{b-a}{6n}\{f(a)+4[f_1+f_3+\cdots+f_{2n-1}] +2[f_2+f_4+\cdots f_{2n+2}]+f(b)\}\right| \leqslant \frac{(b-a)^5M}{2880n^4}, \qquad (2)$$

where $f_k = f(a_k)$ $(k = 0,1,\ldots,2n)$.

As an example we approximate to

$$\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2$$

by use of Simpson's rule. Taking $n = 3$, so that the points of division a_k ($k = 0, 1, \ldots, 6$) are

$$1,\ 1\tfrac{1}{6},\ 1\tfrac{1}{3},\ 1\tfrac{1}{2},\ 1\tfrac{2}{3},\ 1\tfrac{5}{6},\ 2,$$

we obtain

$$\tfrac{1}{18}\{f_0+4(f_1+f_3+f_5)+2(f_2+f_4)+f_6\}$$

as an approximation to the integral, where $f_k = 1/a_k$. Hence our approximation to $\ln 2$ is 0.693 170, correct to six decimal places. In fact, the value of $\ln 2$, correct to six decimal places, is 0.693 147, so that our error is only 0.000 023.

With $f(x) = 1/x$ we have $f^{\mathrm{iv}}(x) = 24/x^5$, so that $|f^{\mathrm{iv}}(x)| \leqslant 24$ in the interval $[1,2]$. Thus the bound for the error given by (2) is

$$\frac{24}{2880\times 3^4} = \frac{1}{9720},$$

which is less than 0.000 103; and the actual error turned out to be a fifth of this.

References

1. J. C. Burkhill, *A First Course in Mathematical Analysis*, pp. 73–75 (Cambridge University Press, 1962).
2. R. Courant, *Differential and Integral Calculus*, pp. 344–345 (Blackie, Glasgow, 1937).
3. G. H. Hardy, *A Course of Pure Mathematics*, 9th edn., pp. 328–330 (Cambridge University Press, 1944). □

***Frank Chorlton** retired from his position as Senior Lecturer in Mathematics at Aston University, Birmingham, in July 1982, but still does a modicum of part-time teaching there. He has published books and papers on mathematics. His other main interests are musical: he is a member of two choirs and a Bach devotee.*

Extensions of the 1994 Problem

MIKE WENBLE

Every year in recent volumes of *Mathematical Spectrum* we have posed the puzzle of expressing the numbers 1 to 100 in terms of the digits of the year in order using certain operations. This article considers how the problem can be generalised.

1994 is the tenth anniversary of the annual problem set in *Mathematical Spectrum* to generate the integers from 1 to 100 using only the digits of the new year in order and the arithmetic operators including $\sqrt{}$, ! and concatenation (see Volume 26 Number 3, page 73). This usually seems to provoke considerable interest, and it is a natural move to try to generalise the problem. This note is the first step in this direction.

We start with some notation. Let $S_n(i,j)$ be the set of integers between 1 and n inclusive that can be formed by the digits i and j ($1 \leqslant i, j \leqslant 9$) and the arithmetic operators including $\sqrt{}$, ! and concatenation. Then, with an obvious extension to the notation, the initial problem can be concisely re-stated as determining whether the set $S_{100}(1,9,9,x)$ is complete.

Table 1 displays the number of elements in $S_{100}(i,j)$ for all combinations of i and j. (For clarity, we will drop the subscript throughout the remainder of this note when $n = 100$.) For example,

$$S(9,4) = \{1,2,3,4,5,6,7,8,10,11,12,13,15,18,21,24,27,30,33,36,72,94\}.$$

Table 1

i \ j	1	2	3	4	5	6	7	8	9	Total
1	3	5	8	10	7	7	5	6	11	62
2	5	5	9	9	6	6	6	7	12	65
3	8	10	9	17	8	7	9	8	13	89
4	10	11	16	15	12	9	10	10	21	114
5	7	6	9	15	6	5	4	6	13	71
6	4	8	9	11	5	5	4	5	13	64
7	4	6	10	10	5	5	5	4	12	61
8	6	6	8	9	5	5	5	8	11	63
9	11	13	12	22	12	11	12	14	14	121
Total	58	70	90	118	66	60	60	68	120	710

One representation of $S(9,4)$—there are others—is:

$\sqrt{9}-\sqrt{4}$; $\sqrt{9}!-4$; $\sqrt{9}!/\sqrt{4}$; $\sqrt{9}!-\sqrt{4}$; $9-4$;
$\sqrt{9}\times\sqrt{4}$; $9-\sqrt{4}$; $\sqrt{9}!+\sqrt{4}$; $\sqrt{9}!+4$; $9+\sqrt{4}$;
$\sqrt{9}\times 4$; $9+4$; $-9+4!$; $9\times\sqrt{4}$; $-\sqrt{9}+4!$;
$\sqrt{9}!\times 4$; $\sqrt{9}+4!$; $\sqrt{9}!+4!$; $9+4!$; 9×4;
$\sqrt{9}\times 4!$; 94.

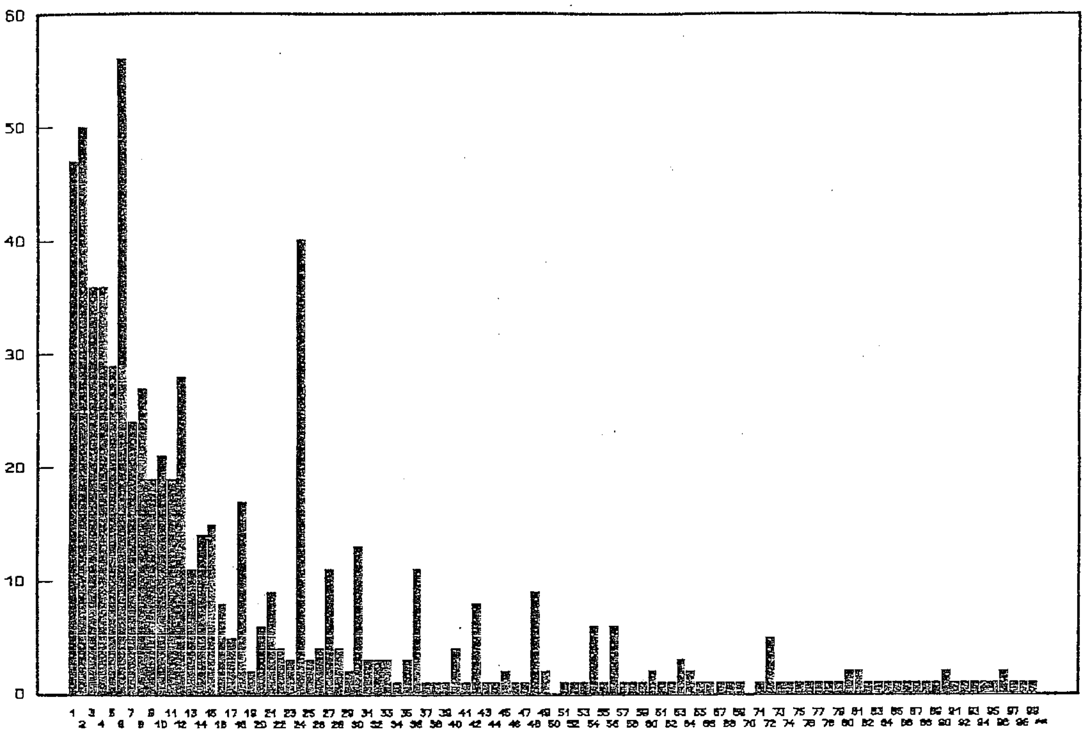


Figure 1. Frequency distribution

The row and column totals demonstrate that 9 and 4, because they are perfect squares, are easily the most 'flexible' digits in constructing these sets, and, unsurprisingly, 1 is the least helpful. As can be seen, the average set size is a little over $8\frac{1}{2}$.

It will be observed from the above table that $S(i,j) = S(j,i)$ on 13 occasions, that $|S(i,j)-S(j,i)| = 1$ on 16 occasions, that $|S(i,j)-S(j,i)| = 2$ on 5 occasions and that $|S(i,j)-S(j,i)| = 3$ on 3 occasions.

Figure 1 indicates the frequency with which the different integers occur in the sets $S(i,j)$. The most commonly occurring integer is 6, which occurs in 55 of the 81 sets $S(i,j)$. The least common integers are 50, 70 and 100, which cannot be constructed with any pair of digits. The following integers occur just once, and in all cases are formed by concatenation:

34, 37, 38, 39, 41, 43, 44, 46, 47, 51, 52, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 68, 69, 71, 73, 74, 75, 76, 77, 78, 79, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92, 93, 94, 95, 97, 98 and 99.

Interestingly, the integer 14 occurs in 14 of the $S(i,j)$ and the integer 15 in 15 of the $S(i,j)$.

Readers are invited first to verify the above table, as there does not seem to be a more systematic approach than trial and error, and it is possible that, despite thorough review, there are some omissions: and then to extend the analysis to the sets $S_n(i,j,k)$ and beyond. □

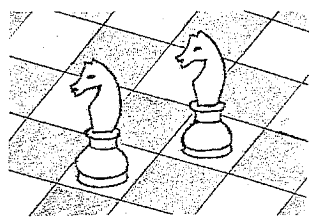
__Mike Wenble__ graduated from the University of Sheffield in 1975. After spells as an operational researcher and a statistician for various organisations, he joined the Mars group of companies 13 years ago and is at present UK Market Planning Manager for Mars Confectionery. He lives in a village near Henley-on-Thames, where he is a parish councillor and divides his spare time between training for half-marathons and practising juggling (although not simultaneously!).

Non-challenging rooks

In how many ways can n non-challenging rooks be placed on

- the black squares;
- the white squares

of an $n \times n$ chessboard?



Mathematics in the Classroom

This is the first of what we hope will be a regular feature in *Mathematical Spectrum*. The aim is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

Making the most of graphic calculators

We are always being urged to introduce more IT into our mathematics teaching at 'A' level but, although we have a well-equipped computer room in college, by the time the six sets studying Computer Studies have been accommodated, there is no time left for other teachers to book the facilities for their sets of mathematicians. A shortage of resources is nothing new in teaching, but this particular problem can in some ways be overcome now that a set of graphic calculators can be purchased for a class just about as cheaply as a single computer. Having acquired sufficient of these calculators for two sets, and being aware that many students own their own, we now wonder how they can best be used in the teaching of A-level mathematics. The following describes some of the ways in which we find them helpful.

Transformations of functions

Students very quickly see the effect of transforming a function if they plot the graph of the function and then superimpose various transformations. For example, getting the calculator to draw $y = x^2$, followed by $y = (x-3)^2$ and $y = x^2-3$; and then $y = 2x^2$ followed by $y = 2(x-3)^2$ and $y = 2x^2-3$, clearly demonstrates the effect of these translations and stretches.

This can be followed up by investigating the relationships (in terms of translations and stretches) that occur in the double-angle formulae. The graph of $y = \cos 2x$, for instance, can be related to the graph of $y = \cos^2 x$ and the student encouraged to work out the transformations that take the latter function into the former.

Exploring polar curves

Some polar curves can take a long time to sketch, especially without the use of polar graph paper, but with a graphic calculator this exercise becomes quick and easy. Drawing cardioids such as $r = 4(1+\cos\theta)$, or the flower $r = 4\sin 5\theta$, can help the student predict what sort of curves will result from drawing $r = a+b\cos\theta$ and $r = a+b\sin\theta$ for various values of a and b and positive as well as negative values of θ. They will then confidently set about drawing flowers with, say, seven petals.

Manipulating matrices

Matrix multiplication and inversion can be fraught with arithmetical errors that frustrate the student, especially when they cannot be readily located. However, graphic calculators enable these operations to be checked and the elusive errors found. This does not really help students to develop insight into matrices, but it does enable these calculators to be used for arithmetic in the same way that students have been trained to use ordinary calculators, but in an area where the ordinary calculator is ineffective.

Locating the roots of polynomials

A typical examination question on this topic might ask for the roots of the equation $6x^3-7x^2-9x-2 = 0$. Use of the factor theorem will confirm that an obvious root is $x = 2$. On drawing the graph for a range of $-10 \leqslant x \leqslant 10$, $-10 \leqslant y \leqslant 10$, this is confirmed, but it looks as if the curve could just touch the x axis somewhere in the range $-1 \leqslant x \leqslant 0$. Zooming in on this part of the curve, it becomes clear that the graph actually cuts the x axis and the points of intersection can be located at $x = -\frac{1}{2}$ and $x = -\frac{1}{3}$, thus completing the solution of the problem.

A final comment

In my experience, students are not always eager to spend time on mastering this sort of calculator, especially as it can seem harder to do the simpler functions for which they are quite used to using a calculator. More access to this type of calculator will no doubt overcome some of this reticence, which is why more tried and tested exercises or ideas for their use would be very welcome, particularly if they develop insight into areas where basic concepts have previously proved difficult to convey to the average student. If anyone has such material which they have found useful in their teaching, it would be good to hear from them.

Carol Nixon □

***Carol Nixon** is head of the statistics section in the mathematics department at Solihull Sixth Form College. The department has thirteen full-time and two part-time members of staff, and in the 1993/94 academic year there were almost 600 A-level students.*

Computer Column

Autostereopsis

The idea of Julesz stereo patterns has been around for some time. The innovation with autostereo patterns is that the two images are combined into one compound image, and with some patience the viewer's eyes can learn to recombine these two images into a stereo pattern.

Suppose that point (x, y) of the image is to be perceived at distance d. The idea is that in the autostereo image, points $(x-s(d), y)$ and $(x+s(d), y)$ are coloured the same, where s is an increasing function of d. Each row of the image is constructed separately subject to these restrictions, and equivalenced points are all coloured the same with a randomly chosen colour. The algorithm is extremely simple, and I teach the technique to first year undergraduates, as that of finding equivalence classes. It is then up to the observer to stare, cross his/her eyes and wait for the image to spring out.

The following program draws a sphere. If you have e-mail access, you can get the latest version of the `Graphics` and `Utils` modules from the HENSA archive `micros.hensa.ac.uk` if you require them. This archive also stocks the FST Modula-2 compiler.

```
MODULE stereo;
    (*Writes to the screen.*)
FROM Graphics IMPORT CGA, EGAMono,
  EGA, MCGA, VGA, VGA2, VGA256,
  BeginGraph, EndGraph,
  PutPixel, ScreenRows, ScreenCols, MaxColours;
FROM Utils IMPORT rand, KeyPressed;
FROM MathLib0 IMPORT sqrt, arctan;
FROM InOut IMPORT Read, WriteString, WriteLn;
CONST
  copyright = 'Copyright␣(C)␣1994␣Mike␣Piff';
TYPE
  Funct2 = PROCEDURE(REAL,REAL):REAL;
VAR
  same, colour:ARRAY[0..2000]OF INTEGER;
  Ground:REAL;
PROCEDURE modulus(x,y:REAL):REAL;
  BEGIN
    RETURN sqrt(x×x+y×y);
  END modulus;
PROCEDURE distance(x,y,a,b:REAL):REAL;
BEGIN
  RETURN modulus(x-a,y-b);
END distance;
PROCEDURE ran():INTEGER;
BEGIN
  RETURN TRUNC(rand()×FLOAT(MaxColours+1));
END ran;
PROCEDURE Sphere(x,y:REAL):REAL;
VAR
  r,height,radius:REAL;
BEGIN
  radius := 0.45×FLOAT(ScreenRows);
  r := distance(x,y,FLOAT(ScreenCols DIV 2),
    FLOAT(ScreenRows DIV 2));
  IF r ≤ radius THEN
    height := 5.0+sqrt(radius×radius-r×r)/10.0;
  ELSE
    height := 0.0;
  END;
  RETURN Ground-height;
END Sphere;
PROCEDURE Draw3D(z:Funct2);
VAR
  x,y,sep,i,j,k,l,s:INTEGER;
  PROCEDURE ancestor(i:INTEGER):INTEGER;
  BEGIN
    WHILE same[i] ≠ i DO i := same[i];END;
    RETURN i;
  END ancestor;
BEGIN
  FOR y := 0 TO ScreenRows-1 DO
  IF NOT KeyPressed() THEN
    FOR x := 0 TO ScreenCols-1 DO same[x] := x; END;
```

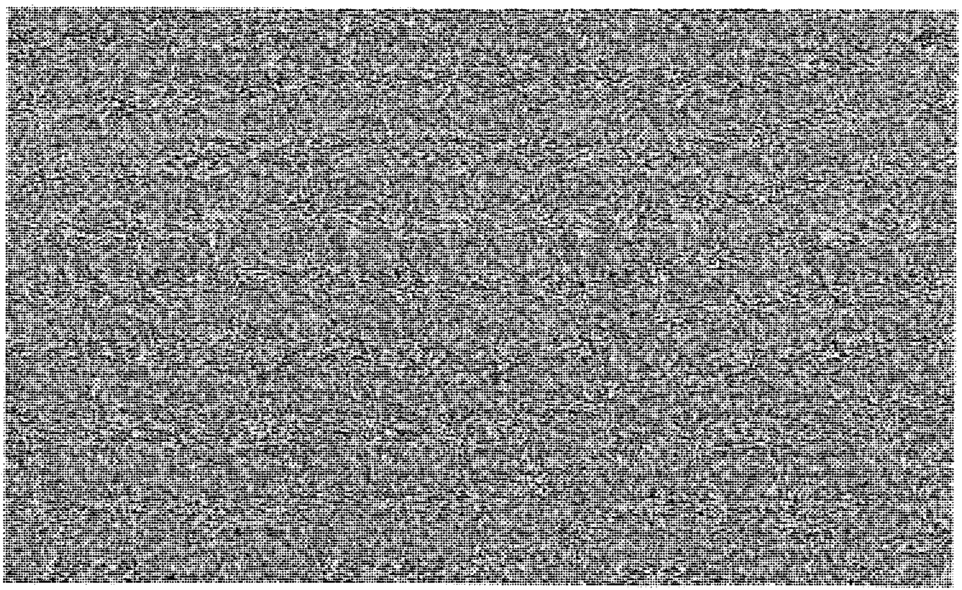


Figure 1. The picture of the sphere

```
FOR x := 0 TO ScreenCols-1 DO
  sep := TRUNC(z(FLOAT(x),FLOAT(y)));
  IF ODD(sep) & ODD(y) THEN
    i := x-((sep+1) DIV 2);
  ELSE
    i := x-(sep DIV 2);
  END;
  j := i+sep;
  IF (i >= 0) lineup (j < ScreenCols) THEN
    k := i; l := j;
    k := ancestor(k); l := ancestor(l);
    IF k < l THEN
      same[k] := l;  same[i] := l;  same[j] := l;
    ELSE
      same[l] := k;  same[i] := k;  same[j] := k;
    END;
  END;
END;
FOR x := ScreenCols-1 TO 0 BY -1 DO
  IF (same[x] = x) THEN
    colour[x] := ran();
      ELSE colour[x] := colour[ancestor(x)];
      END;
      PutPixel(x,y,colour[x]);
    END;
  END; END;
END Draw3D;
PROCEDURE Draw(z:Funct2);
VAR ch:CHAR;
  BEGIN
    Draw3D(z);
    REPEAT Read(ch); UNTIL (ch = 'q');
  END Draw;
BEGIN
  WriteString(copyright); WriteLn;
  BeginGraph(VGA);
  Ground := FLOAT(ScreenCols)×0.15;
  (*Half distance between eyes in pixels*)
  Draw(Sphere);
  EndGraph;
END stereo.
```

Mike Piff □

The Markov inequality

One of the simplest and most useful inequalities in statistics is the Markov inequality for positive random variables. This states that, for a random variable $X \geqslant 0$,

$$\Pr\{X \geqslant a\} \leqslant \frac{E(X)}{a} \quad (a > 0).$$

The proof is simple; for X a continuous random variable we need only write

$$\begin{aligned} E(X) &= \int_0^\infty x f(x)\,\mathrm{d}x \\ &\geqslant \int_a^\infty x f(x)\,\mathrm{d}x \\ &\geqslant a \int_a^\infty f(x)\,\mathrm{d}x \\ &= a \Pr\{X \geqslant a\}, \end{aligned}$$

where $f(x)$ is the probability density function of X. For X discrete, a similar argument holds.

As an example, take X to be exponentially distributed with probability density function $\lambda \mathrm{e}^{-\lambda x}$ and $E(X) = \lambda^{-1}$. For $a > 0$, we have the exact result

$$\Pr\{X \geqslant a\} = \int_a^\infty \lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x = \mathrm{e}^{-\lambda a}.$$

The Markov inequality gives

$$\Pr\{X \geqslant a\} \leqslant (\lambda a)^{-1}.$$

We note that $\mathrm{e}^{-\lambda a} \leqslant (\lambda a)^{-1}$ for all λa; for $x\mathrm{e}^{-x}$ is always less than 1, since it achieves its maximum of $\mathrm{e}^{-1} < 1$ at $x = 1$.

J. Gani

Andrei Andreevich Markov was born in 1856 in Ryazan, Russia and died in 1922 in St Petersburg. He taught mathematics at the University of St Petersburg, where he carried out research in number theory, analysis, probability and statistics. He is perhaps best known as the originator of Markov chains; initially these appeared in a paper of his in 1906, studying the law of large numbers for dependent random variables. The Markov inequality first occurred in his textbook *The Calculus of Probabilities* in 1913. For further details, the reader is referred to Seneta's article on Markov in the *Encyclopedia of Statistical Sciences*, Volume 5, 247–249 (1985).

Proving the obvious

As an undergraduate at the University of Glasgow, I was told not to describe a statement as 'obviously true' unless a proof springs readily to mind. 'I don't see how it could be false' is not a proof, I was told, but a confession of failure of the imagination.

You may know the story of the maths lecturer who in mid-lecture used the phrase, 'this is obvious'. He then went away to think about it for half an hour, came back, said 'Yes, it *is* obvious', and carried on with his lecture!

Readers may like to send in 'obvious statements'. Here are a couple which readers are challenged to prove.

(a) If two straight lines are drawn in a plane, none of the regions into which the lines divide the plane has finite non-zero area.
(b) A plane through the centre point of a cube divides the cube into two congruent parts.

John MacNeill
(Analyst Programmer,
University of Warwick)

Letters to the Editor

Dear Editor,

Primorial, factorial and multifactorial primes

I was most interested in the discovery that there are very few prime numbers of the form $n!!+1$. (See *Mathematical Spectrum* Volume 26 Number 1, pages 1–7.) The authors show that, when $n = 2(p-1)$ (p an odd prime number), $n!!+1$ cannot be prime. The proof involves the factorisation of $n!!$ as $2^{\frac{1}{2}n}\times(\frac{1}{2}n)!$. If $n = 2p-2$ (p an odd prime), each factor is congruent to 1 or -1 modulo a prime number (p), one being $+1$ and the other -1. Therefore, I wonder if it could be that $+1$ and -1 occur often with two numbers 2^r and $r!$, not just in the known case where r is one less than a prime number. Perhaps it would be worth doing tests on 2^r and $r!$ with a computer.

The authors feel that eliminating numbers of the form $2(p-1)$ (p an odd prime) for $n!!+1$ prime is not particularly significant, presumably because only a minority of odd numbers are prime. However, I noted that many of the n's such that $n!!-1$ is prime are with $n = 2p$, p a prime number. This shows that the number of primes produced by a particular class of number is not necessarily proportional to the density of that type of number.

I would mention one other interesting point about this. It has been said that there are many cases, more than might be expected *a priori*, of successive prime numbers differing by 2. This is clearly not so for $n!!\pm 1$ as so few primes are of the form $n!!+1$.

Yours sincerely,

ANTHONY PEPPER
(9 Rushmore Close, Bow Brickhill,
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Note from the editors. The question here seems to be whether there exist values of r (other than $q-1$) and primes q for which

$$2^r \equiv 1 \pmod q \quad \text{and} \quad r! \equiv -1 \pmod q.$$

We must have $r < q$ (otherwise $r! \equiv 0$). Let h denote the order of 2 (mod q), i.e. the least positive integer such that $2^h \equiv 1 \pmod q$. Then $h\,|\,(q-1)$ and r must be a multiple of h.

One set of examples is when $r = \frac{1}{2}(q-1)$. Dirichlet evaluated $\{\frac{1}{2}(q-1)\}! \pmod q$ (see Hardy and Wright, *An Introduction to the Theory of Numbers*, 5th edition, section 7.7, Oxford University Press). We need $q \equiv -1 \pmod 4$ and then $r! \equiv (-1)^{\nu} \pmod q$, where ν is the number of quadratic non-residues less than $\frac{1}{2}q$. Thus we need $q \equiv -1 \pmod 4$ and ν to be odd. Now

$$2^{\frac{1}{2}(q-1)} \equiv 1 \pmod q$$

when q is a quadratic residue (mod q) (using Euler's criterion), so we need $q \equiv \pm 1 \pmod 8$. Thus for solutions with $r = \frac{1}{2}(q-1)$ we must have $q \equiv -1 \pmod 8$. Computations yield the following examples with $q < 200$:

q: 7, 47, 79, 103, 127, 191, 199.

Two particular sequences of primes seem particularly rich in examples.

(a) *Primes of the forms* $40k-1$:

k	2	5	12	15	21
q	79	199	479	599	839

(The values $k = 6, 9, 11, 18$ and 23 give primes with $2^r \equiv +1 \pmod q$.)

(b) *Primes of the form* $40k+7$:

k	0	1	3	9	15	18	22
q	7	47	127	367	607	727	887

(The values $k = 4, 12, 16$ and 24 give primes q with $2^r \equiv +1 \pmod q$, which suggests $4\,|\,k$ is of importance.) The cases $k = 0$ and 3 are of interest since then q is a Mersenne prime. For a Mersenne prime $q = 2^p-1$, with p prime, the order of 2 (mod q) is p, which is small compared to q. Thus there will be many multiples $r = cp$ for which $2^r \equiv 1 \pmod p$, and these may give further examples where $r! \equiv -1 \pmod q$ with r relatively small.

Dear Editor,

The Smarandache function

With regard to the question on the Smarandache function posed by J. Rodriguez in Volume 26 Number 3 page 84, it is possible to construct an increasing sequence of any (finite) length whose Smarandache values are strictly decreasing. One such construction is as follows. Let p be a sufficiently large even integer. Then consider the sequence

$$\begin{aligned}
r_0 &= p(p-1)(p-2)(p-4)(p-6)\cdots 6\times 4\times 2\\
r_1 &= (p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)\\
&\qquad \times(p-10)\times\cdots\times 6\times 4\times 2\\
r_2 &= (p-4)(p-5)(p-6)(p-7)(p-8)(p-9)(p-10)\\
&\qquad \times(p-11)(p-12)(p-13)(p-14)\\
&\qquad \times(p-16)\times\cdots\times 6\times 4\times 2\\
\ldots &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
r_k &= \prod_{i=k}^{p/2-1}(p-2i)\prod_{i=k}^{3k}[p-(2i+1)].
\end{aligned}$$

Note that every term r_k contains all the even numbers from $p-2k$ down to 2. Every time two factors are removed from the front of the product, three extra odd factors are included in order to ensure that the sequence is indeed increasing.

Now $S(r_k) = p-2k$. We prove this as follows. Since r_k contains every even number up to $p-2k$, it must be the case that $S(r_k) \geqslant p-2k$. Further, since r_k contains only distinct integer factors not greater than $p-2k$, it must also be the case that $S(r_k) \leqslant p-2k$. We have proved that the Smarandache values of the sequence are indeed decreasing. Now all that remains to be shown is that the sequence itself is increasing. This relies on the terms we add at each stage being larger than the terms we delete, i.e.

$$[p-(6k-3)][p-(6k-1)][p-(6k+1)] > [p-(2k-2)][p-(2k-1)],$$

which is clearly true if p is large enough. This will give us a sequence of about $\frac{1}{6}p$ in length and, since the even numbers are unbounded in size, so is the length of such sequences.

Yours sincerely,
KHALID KHAN
(Student at London
School of Economics)

Dear Editor,

The Smarandache function

In Volume 26 Number 3 page 84, J. Rodriguez asked whether there is a bound on the length of an increasing sequence $x_1, x_2, x_3, \ldots$ such that the sequence $S(x_1), S(x_2), S(x_3), \ldots$ is strictly decreasing. Here, S is the Smarandache function, so that $S(m)$ is the smallest positive integer such that $m \mid S(m)!$. We show here that, for every positive integer n, there exists such a sequence of length n, so there is no such bound.

Denote the sequence of prime numbers by $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, p_4, p_5, $\ldots$, and put $x_k = 2^k p_{2n-k}$ for $k = 0, 1, 2, \ldots, n-1$. It follows from Bertrand's postulate that $p_{2n-k} < 2p_{2n-k-1}$, from which $x_k < x_{k+1}$. Thus $x_0, x_1, \ldots, x_{n-1}$ is a strictly increasing sequence of length n. Now $p_{k+1} - p_k \geqslant 2$ for $k \geqslant 2$, so

$$p_n - p_2 = \sum_{k=2}^{n-1} (p_{k+1} - p_k) \geqslant 2(n-1),$$

whence $p_n > 2(n-1)$. Thus, for $0 \leqslant k \leqslant n-1$,

$$p_{2n-k} > p_n > 2(n-1) \geqslant 2k$$

and

$$S(x_k) = S(2^k p_{2n-k}) = p_{2n-k}.$$

Thus

$$S(x_0) > S(x_1) > S(x_2) > \ldots > S(x_{n-1}).$$

Yours sincerely,
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Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

27.1 Prove that there are infinitely many odd integers n such that no term of the infinite sequence

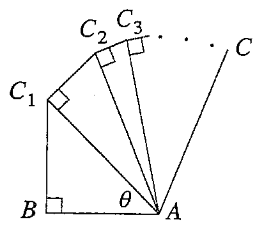
$$1994^{1994}+1, \quad 1994^{1994^{1994}}+1, \quad 1994^{1994^{1994^{1994}}}+1, \quad \ldots$$

is divisible by n.
(Submitted by Farshid Arjomandi, University of California at Santa Barbara.)

27.2 Imagine 1994 trees standing in a circle, with a starling in each tree. Every few minutes, two of the birds fly in opposite directions round the circle to the neighbouring tree. Show that at no time are all the birds in the same tree.
(Submitted by Farshid Arjomandi.)

27.3 In the figure, the sequence of angles is $\theta, \theta/2, \theta/2^2, \theta/2^3, \ldots, \theta/2^n$, and C is the limiting point of the sequence $C_1, C_2, C_3, \ldots$. Express AC in terms of AB and θ.

$$\angle BAC_1 = \theta$$
$$\angle C_1AC_2 = \theta/2$$
$$\angle C_2AC_3 = \theta/2^2$$
$$\ldots = \ldots$$
$$\angle C_iAC_{i+1} = \theta/2^i$$



(Submitted by Sajedul Karim, Dhaka.)

27.4 Determine, for all positive integral values of k, the behaviour as $x \to 0+$ (i.e. as x tends to 0 through positive values) of

$$\frac{1}{\sin^k x} - \frac{1}{x^k}.$$

(Submitted by S. M. R. H. Mousavi, Tehran.)

Solutions to Problems in Volume 26 Number 3

26.7 Curves $C_1, \ldots, C_n$ in the plane have equations $r_1 = r_1(\theta), \ldots, r_n = r_n(\theta)$ in polar form, and

$$r_1(\theta) r_2(\theta) \cdots r_n(\theta) = \text{constant}.$$

A straight line l from the origin cuts the curves at points $P_1, \ldots, P_n$, respectively, and the angle between l and the normal to the curve C_i at P_i is α_i, measured so that it is positive if the normal is on one side of l and negative if it is on the other side. Prove that

$$\sum_{i=1}^{n} \tan \alpha_i = 0.$$

Solution
Taking logarithms, we have

$$\sum_{i=1}^{n} \log r_i = \text{constant},$$

and differentiating we have

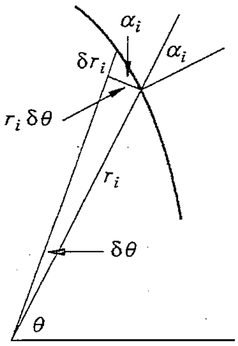
$$\sum_{i=1}^{n} \frac{1}{r_i}\frac{dr_i}{d\theta} = 0.$$

But

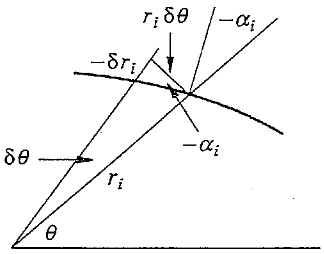
$$\tan\alpha_i = \lim_{\delta\theta\to 0}\frac{\delta r_i}{r_i\,\delta\theta} = \frac{1}{r_i}\frac{dr_i}{d\theta},$$

so

$$\sum_{i=1}^{n} \tan\alpha_i = 0.$$



α_i and δr_i positive



α_i and δr_i negative

Also solved by Khalid Khan (London School of Economics), David Johansen (University of Cambridge) and Mark Blyth (University of Bristol).

26.8 For a positive integer n, $S(n)$ is defined as the smallest positive integer such that n divides $S(n)!$. Find an infinite strictly increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ such that no three terms in the sequence $S(n_1), S(n_2), S(n_3), \ldots$ are increasing or decreasing.

Solution by Khalid Khan
Let $p_1, p_2, \ldots$ be any infinite strictly increasing sequence of prime numbers with $p_1 > 3$. Now $S(p_i) = p_i$ and $S(p_i+1) \leqslant \frac{1}{2}(p_i+1) < p_i = S(p_i)$. Also, $S(p_i+1) \leqslant \frac{1}{2}(p_i+1) < p_i < p_{i+1} = S(p_{i+1})$. Therefore the sequence $p_1, p_1+1, p_2, p_2+1, p_3, p_3+1, \ldots$ has the required property.
Also solved by David Johansen and Polly Shaw (Dame Allan's Girls' School, Newcastle upon Tyne).

26.9 Find all three-digit numbers which are equal to the sum of their hundreds digit, the square of their tens digit and the cube of their units digit.

Solution by Khalid Khan
We require integers a, b and c between 0 and 9 with $a \neq 0$ and

$$10^2a + 10b + c = a + b^2 + c^3.$$

Since $a, b \leqslant 9$, we have $a + b^2 \leqslant 9 + 81 = 90$, so $a = c_3$ or $a = c_3 + 1$, where c_3 is the hundreds digit of c^3. If $c \leqslant 2$, then $a + b^2 + c^3 \leqslant 98$, and we do not have a 3-digit number, so we must have $c \geqslant 3$. Also,

$$b^2 \equiv c - a - c_1 \pmod{10},$$

where c_1 is the units digit of c^3. We now have the possible values shown in table 1. Of these possible values, only four, 135, 175, 518 and 598, are valid.
Also solved by David Johansen, Mario Bordogna (Allegheny College, Meadville, Pennsylvania) and Gregory Notch (Allegheny College).

Table 1

c	c_3	a	c_1	$c-a-c_1$ (mod 10)	b
3	0	1	7	5	5
4	0	1	4	9	3, 7
5	1	1, 2	5	9, 8	3, 7
6	2	2, 3	6	8, 7	-
7	3	3, 4	3	1, 0	1, 9, 0
8	5	5, 6	2	1, 0	1, 9, 0
9	7	7, 8	9	3, 2	-

An infinite exponential

I have recently come across an oddity in *Mathematical Entertainments* by M. H. Greenblatt (Allen and Unwin, London, 1965) which greatly intrigued me and which might interest readers. When $x > 0$, write

$$S_0(x) = x, \quad S_1(x) = x^x, \quad S_2(x) = x^{x^x}, \quad \ldots,$$

and generally

$$S_n(x) = x^{x^{\cdot^{\cdot^{x}}}},$$

with n iterations. One might be forgiven for thinking that, for all $x > 1$, $S_n(x)$ diverges as $n \to \infty$ and, for all $x < 1$, $S_n(x) \to 0$. This is not so. For example, $S_n(1.2) \to 1.258\ldots$ and $S_n(0.5) \to 0.641\ldots$. Greenblatt says that $S_n(x)$ converges for $0 < x \leqslant e^{1/e}$, that $S_n(e^{1/e}) \to e$ and that $S_n(x)$ diverges for $x > e^{1/e}$. He does not give a proof and neither could I find one.
Editor: This problem is also discussed in Bromwich's *Infinite Series* (Macmillan, London, 1949). It goes back to Eisenstein (1844) and Seidel (1870): references are given.

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Reviews

Waves. By ALAN DAVIES. Macmillan, Basingstoke, 1992. Paperback £8.50 (ISBN 0-333-54112-X).

In the world around us and throughout nature there are many phenomena, apparently unconnected, but which may all be explained in terms of wave motion; the link between a crowd's Mexican wave and the way light pours into a room on a sunny day may seem tenuous, but not so with wave theory.

After an illuminating introduction showing different examples of waves and how they may be classified, Davies guides the reader through from the simplest spring–mass system to the complex idea that a violin and piano have different waveforms even when playing the same note. The mathematics which forms the foundation of these physical occurrences is introduced hand in hand with the system to which it relates, together with clear diagrams to explain them both. Initially simple harmonic motion is explored and then more degrees of freedom are added, culminating in the use of Fourier's theorem for harmonic analysis.

As a musician, I particularly enjoyed the attention paid to sound waves. The explanations of such qualities as timbre and loudness were pleasing in their own right, as well as a useful aid in understanding properties of other waves. I found the exploration of the musical scale in terms of note frequencies both novel and revealing.

PFTAs (Pauses For Thought and Action) appear throughout the text, taking the form of short answered questions which challenge the reader to understand the preceding concept. This is much more friendly and allows quicker progress than the daunting array of exercises which are common in other books.

The book closes by briefly covering the more complicated general properties of waves and their underlying mathematics. This book, requiring only basic mathematical knowledge, would be ideal for a sixth-form student wishing to advance his/her understanding of this area as well as a useful reference for undergraduates.

Student, St. Catharine's College, Cambridge — DAVID BRACKIN

MEI: Statistics Book 2. By ANTHONY ECCLES, NIGEL GREEN AND ROGER PORKESS. Hodder and Stoughton, London, 1993. Pp. iv + 138. Paperback £6.99 (ISBN 0-340-57300-7).

This is the second in a series of books written to support the statistics components of the new MEI Structured Mathematics scheme, with four further books yet to appear to form the complete set. Stylistically it is very much in the same vein as the first book in the series, which was warmly reviewed in Volume 26 Number 2 of *Mathematical Spectrum*. The authors continue to be very successful in their use of articles from a fictional newspaper as illustrations of how to evaluate information that reaches us via the media, and their easy-reading style encourages the belief that this book is, like the first, very accessible to post-GCSE students.

The book continues its exploration of discrete random variables in the opening two chapters where it progresses to a consideration of the Poisson distribution. Then, for something completely different, there follows a chapter on the normal distribution. The final chapter is devoted to bivariate data and considers aspects of correlation and regression. All the unpleasant-looking proofs, such as mean and variance of binomial and Poisson distributions, have been consigned to an appendix at the back where they can happily be ignored by the less mathematical reader or avidly devoured by the enthusiast—a rather satisfactory arrangement.

I enjoyed the first book and was no less enamoured of this second. The significance of the subtle change from the highlighted 'Key Points' at the ends of chapters in Book 1, to the unhighlighted 'Summaries' of Book 2 eludes me, but perhaps that is as it should be. I was certainly fascinated by the historical notes that crop up at appropriate points, but surely there are easier ways to develop the mind of the aspiring student than hanging him or her from a hook on the wall, hands tied, to prevent an escape to mischief—treatment that we learn was meted out to a young Poisson by his nanny, and which may have been responsible for his interest in the pendulum!

My copy of the first book is well thumbed; I am sure that it will not be long before the second book is in the same happy state.

Solihull Sixth Form College — CAROL NIXON

MEI: Statistics Book 3. By ANTHONY ECCLES, NIGEL GREEN AND ROGER PORKESS. Hodder and Stoughton, London, 1993. Pp. 135. Paperback £6.99 (ISBN 0-340-57863-7).

This is the next book in the series. These books for students are written to go with the MEI Structured Mathematics Advanced Level course. They are designed for classroom and independent study, and adopt an active approach.

The Search for E. T. Bell. By CONSTANCE REID. Mathematical Association of America, Washington DC, 1993. Pp. 384. Hardback $35.00 (ISBN 0-88385-508-9).

The name of E. T. Bell is known to generations of mathematicians, both students and teachers, as the author of 'Men of Mathematics', which must be the most widely read book on the history of mathematics and the men who invented it. (The women come on the scene after the period covered by the book!) The author of this biography tells how one H. F. Bohnenblust read 'Men of Mathematics' aloud to his non-mathematical bride on their honeymoon. (She doesn't say whether the marriage lasted!)

But who was E. T. Bell? This is the question that Constance Reid set out to answer, having previously written biographies of such greats as Hilbert and Courant. The task proved to be something of a detective operation, as E. T. Bell seems to have been very successful at concealing his early life and, even in his later life, it was hard to discover the real man.

Be warned that the title of the book is 'The Search for E. T. Bell'. Especially in the early parts, the central figure is not E. T. Bell at all but the author, as she did detective work worthy of Miss Marple to try to discover the details of Bell's life.

And what of the man himself, the author of numerous science fiction stories written under the name of John Taine, a poet who consistently failed to get his poetry published or recognized, a creative mathematician of high order? He seems to have aroused conflicting reactions amongst those who met him. Clearly he could be infuriating.

For most of us, E. T. Bell' abiding legacy is his 'Men of Mathematics'. I use it continually in my teaching, although I am always uncertain how much of it to believe. Constance Reid quotes A. W. Tucker's verdict on it: 'almost fiction'. It is infuriating at times, but always stimulating and entertaining. Which probably describes the man behind the book. Perhaps, after all, that is the best way to discover who E. T. Bell really was.

University of Sheffield DAVID SHARPE

Essays in Humanistic Mathematics. Edited by ALVIN M. WHITE. Mathematical Association of America, Washington DC, 1993. Pp. xii+212. Paperback $24.00 (ISBN 0-88385-089-3).

This book presents mathematics as an intellectual discipline with a human perspective and a history that matters and, as such, is most welcome. It consists of five parts.

Part I: Introduction to humanistic mathematics—three essays (Philip J. Davis, Thomas Tymoczko, Reuben Hersh).

Part II: Mathematics in the world—five essays (Hardy Grant, Jack V. Wales, Ubiratan D'Ambrosio, Harald M. Ness, Jr., Robert Osserman).

Part III: Inner life of mathematics—five essays (Nelson D. Goodman, Thomas Tymoczko, Philip J. Davis, Gian-Carlo Rota, Gian-Carlo Rota).

Part IV: Teaching and learning experiences—five essays (Stephen I. Brown, Raffaella Borasi, Larry Copes, Dorothy Buerk and Jackie Szablewski, Robert B. Davis).

Part V: Contemporary views of old mathematics—four essays (William Dunham, Elena Anne Marchisotto, Peter Flusser, Abe Shenitzer).

The essays in this volume, each lucidly written, illustrate and help to define humanistic mathematics. Each essay is independent, but a unifying theme emerges—mathematics; mathematics free of anxiety and boredom. We hear the voices not only of the teachers, but of the students as well.

As a final note to his essay 'Rational and irrational: music and mathematics', Robert Osserman points out that it is appropriate to let Galileo have the last word. Writing in *Two New Sciences*, he pointed out that too much consonance strikes us as 'too bland, and lacks fire'. What we seek is just the right amount of dissonance. 'This produces a tickling and teasing of the cartilage of the eardrum, so that the sweetness is tempered by a sprinkling of sharpness, giving the impression of being simultaneously sweetly kissed and bitten.' The message of each essay in this volume has had a similar effect on me.

However, my highly enthusiastic response to this work has to be tempered by a suspicion that the same effect may not be felt by managers of mathematical education in schools and colleges; in some institutions, the number of hours given to mathematics is now only 60% of the figure six years ago, and this is not the end. Efficiency targets and flexible contracts are great forces opposing the humanistic mathematics movement. Another message of this book is that of hope and liberation from routine, and I certainly recommend it to promote the health of mathematical education.

Medical School, GREGORY D. ECONOMIDES
University of Newcastle upon Tyne

The Broken Dice. By IVAR ECKELAND, translated by CAROL VOLK. Chicago University Press, 1993. Pp. 183. Hardback £15.95 (ISBN 0-226-19911-6).

This book begins by ascertaining the existence of chance in the physical world, and then explains how it affects everything. It introduces the reader to essential mathematics for chaos theory such as strange attractors, entropy and exponential instability, and then explores how they affect the unpredictability of the weather and even the sun's eclipses.

However, the book is not purely a mathematics book—it has many strands, one of which is strongly artistic. Ekeland punctuates the book with stories from the 'ancient saga' of King Olaf Harraldsson of Norway, who ruled in the eleventh century. It gains a highly cosmopolitan feel by drawing from sources such as the Bible, Shakespeare, Goethe and other writers. Two strange attractors are compared to patterns on ancient artifacts. By all these references and comparisons, he shows that the mathematics mathematicians work hard to create is already in the hearts of artists.

In another strand Ekeland discussed humanity and probability—the difficulty in calculating probabilities of accidents in power stations and in disposing of nuclear waste, and our attitude to risk.

An attractive (to my mind) aspect of this book is that it is nearly all prose—there are very few of those alarming calculations and proofs that weigh on the conscience if neglected. There are places where descriptions must be read twice to secure understanding, but generally these are sparse and necessary, as well as precise and limpid. This makes the book accessible to anyone with interest and perseverance, but mathematical stamina and previous knowledge about chaos are not necessary.

Most importantly, I found the book genuinely interesting. It answered in print many questions about which I had half wondered before. It inspires the reader to think, as hard conclusions are not generally drawn from the sub-themes which orbit the main theme: how chance affects everything.

However, I spotted a few mistakes, the worst of which was that a certain message of a billion bits in a random binary series appears with a frequency of $(\frac{1}{2})^{1000\,000\,000} \times 999\,999\,999!$.

This book is written in a lofty style and comparisons are occasionally stretched. Yet it is not pretentious—it is a fascinating book, an impressive fusion of art and mathematics.

Dame Allan's Girls' School, POLLY SHAW
Newcastle upon Tyne □

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