PI MU EPSILON JOURNAL

VOLUME 7 SPRING 1983 NUMBER 8

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VOLUME 7 **SPRING 1983** NUMBER **8**

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PI MU EPSILON JOURNAL
THE OFFICIAL PUBLICATION
OF THE HONORARY MATHEMATICAL FRATERNITY

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PI MU EPSILON JOURNAL is published at the South Dakota School of Mines and Technology twice a year—Fall and Spring. One volume consists of five years (10 issues) beginning with the Fall 19x4 or Fall 19x9 issue, starting in 1949. For rates, see inside back cover.

Faculty Advisor
Professor Richard Andree

INVESTIGATIONS ON MAXFIELD'S THEOREM

by Laura Southard
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Various properties of factorials have fascinated recreational mathematicians for many years. John E. Maxfield proved the following theorem on factorials in 1970[1]:

Theorem 1: If A is any positive integer having m digits, there exists a positive integer N such that the first m digits of N! constitute the integer A.

Although this theorem has already been proved, it still provides food for thought. The purpose of this article is to report the results of an investigation into the smallest N that meets the criteria given in Theorem 1 for various values of A.

A FORTRAN computer program was written to find the smallest N that meets Maxfield's criteria for $A = 1$ to 999. The program was 41 lines long and required 2.0 seconds of execution time on a VAX computer. The entire output of the program for A = 1 to 999 can be ordered from the Editor for the cost of reproduction ($1.00).

Several interesting facts were found by studying the output from the program. For A = 1 to 8, the smallest value of N is small enough to be calculated on a hand calculator easily. For $A = 9$, the smallest value of N is 96. $96! = 9.91678E+149$. This is obviously much too large to calculate without the aid of a computer.

While the smallest value of N for $A = 9$ is 96, 60 of the smallest values of N are less than or equal to 96 for $A = 1$ to 99. The distribution of these values of A is given in Table 1.

TABLE 1

A	Number of N's that are less than or equal to 96
1-9	9
10-19	10
20-29	8
30-39	7
40-49	4
50-59	6

60-69	4
70-79	2
80-89	9
90-99	1

The largest value of N that was calculated for $A = 1$ to 99 was 716 for $A = 97$. For $A = 1$ to 99, 63 of the smallest values of N were less than 100, 79 of the N's were less than 200, and 95 of the N's were less than 500. While the smallest value of N for $A = 97$ is 716, 539 of the smallest values of N are less than or equal to 716 for $A = 1$ to 999. The distribution of these 539 values of A is given in Table 2.

TABLE 2

A	Number of N's that are less than or equal to 716
1-99	99
100-199	83
200-299	75
300-399	67
400-499	42
500-599	38
600-699	48
700-799	29
800-899	29
900-999	29

The distribution in Table 2 is substantially different than the distribution in Table 1.

The largest value of N that was calculated for $A = 1$ to 999 was 12745 for $A = 841$. For $A = 1$ to 999, 821 of the smallest values of N were less than 2000, 965 of the N's were less than 5000, and 995 of the N's were less than 10,000.

The smallest values of N for values of A that are perfect squares are given in Table 3.

TABLE 3

A	N	N factorial
1	1	1.000000e+ 0
4	8	4.032000e+ 4
9	96	9.916780e+ 149
16	89	16.507956e+ 135
25	23	25.852024e+ 21
36	9	36.288002e+ 4
49	129	49.745037e+ 216
64	18	64.023750e+ 14
81	40	81.591545e+ 46
100	197	100.078415e+ 366
121	19	121.645126e+ 15
144	109	144.385956e+ 174
169	239	169.495331e+ 464
196	786	196.565201e+ 1934
225	590	225.694458e+ 1378
256	304	256.267303e+ 622
289	887	289.827087e+ 2229
324	853	324.293488e+ 2129
361	719	361.280975e+ 1741
400	1155	400.386536e+ 3035
441	294	441.493835e+ 597
484	1401	484.836578e+ 3799
529	1023	529.155396e+ 2634
576	243	576.511169e+ 473
625	3152	625.057922e+ 9658
676	11376	676.960144e+41200
729	2720	729.152344e+ 8160
784	1591	784.666382e+ 4402
841	12745	841.102356e+46787
900	166	900.369019e+ 295
961	97	961.927856e+ 149

No significant patterns were found in Table 3 or in a similar table for values of A that are prime.

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Faculty Advisor
Professor Theodore *Phillips*

VOLUME OF AN n-DIMENSIONAL UNIT SPHERE

by Ravi Salgia
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Many interesting and useful examples of Dirichlet integrals occur and one of them is concerned with the evaluation of volume of certain closed surfaces for various dimensions. Consider the unit circle and its interior, $x_1^2 + x_2^2 \leqq 1$, where x_1 and x_2 are Cartesian coordinates (see Figure 1). The area, which will be referred to as volume in two dimensions or V_2, enclosed by the circle is determined as follows:

$$V_2 = 2^2 \iint dx_1 dx_2$$

over the quadrant bounded by the curve

$$x_1^2 + x_2^2 \leqq 1$$

and the coordinate axes. From simple integration, V_2 can be shown to equal π.

Similarly, for a unit sphere in three dimensional Euclidean space, the volume (V_3) is determined by:

$$V_3 = 2^3 \iiint dx_1 dx_2 dx_3$$

over the octant bounded by the surface

$$x_1^2 + x_2^2 + x_3^2 \leqq 1$$

and the coordinate planes. By evaluating this integral, V_3 is $4\pi/3$.

In generalizing the previous results, the volume of an n-dimensional unit sphere--which is analogous to volume in three dimensions and area in two dimensions--can be calculated. If V_n represents the volume of an n-dimensional unit sphere then V_n can be evaluated by using an n-tuple integral and

$$V_n = 2^n \iint \cdots \int dx_1 dx_2 \ldots dx_n,$$

where x_n represents the nth axis, and the integral is bounded by the curve

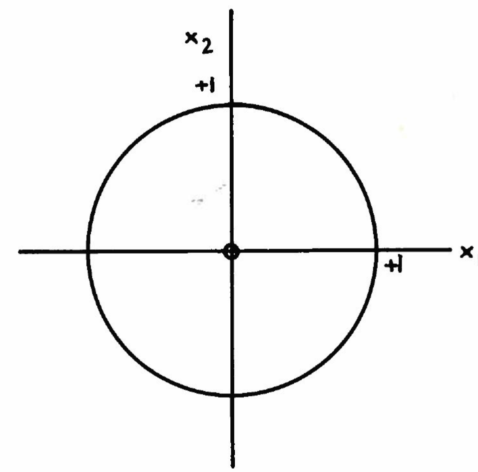

Figure 1.

$$x_1^2 + x_2^2 + \cdot \quad + x_n^2 \leqq 1$$

and the appropriate coordinate planes. It is shown in reference 3, using Dirichlet integral in n-dimensional Euclidean space, that the volume of an n-dimensional unit sphere is

$$V_n = 2^n \left[\frac{[\Gamma(\frac{1}{2})]^n}{2^n \Gamma(\frac{n}{2} + 1)} \right] = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \quad , \tag{1}$$

where n is a natural number greater than one.

The Dirichlet integral, D_n , is defined as

$$D_n = \iint \cdots \int x_1^{m_1 - 1} x_2^{m_2 - 1} \cdots x_n^{m_n - 1} dx_1 dx_2 \ldots dx_n ,$$

bounded by the surface

$$\sum_{i=1}^{n} \left(\frac{x_i}{a_i} \right)^{p_i} \leqq k$$

and the appropriate coordinate planes, where a_i, m_i , p_i $(i=1, \ldots, n)$, and k are all constants greater than zero. By using the substitution $x_i = a_i (k\xi_i)^{1/p_i}$ in the above expression, it is seen that

$$D_n = \left[k \sum_{i=1}^{n} \frac{m_i}{p_i} \right] \frac{\prod_{i=1}^{n} \frac{a_i^{m_i}}{p_i} \Gamma\left(\frac{m_i}{p_i}\right)}{\Gamma\left[1 + \sum_{i=1}^{n} \frac{m_i}{p_i}\right]}$$

It is easily seen that if one substitutes the values $a_i = m_i = k = 1$, and $p_i = 2$ $(i = 1, 2, \ldots, n)$ in the above expression for D_n, equation (1) is obtained.

Note that in equation (1), Γ is the well known Gamma Function--which for a non-zero positive number n is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx .$$

When $n = 1$, $\Gamma(1) = \int_0^\infty x^{n-1} e^{-x}\, dx = 1$;

When $n = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx = \sqrt{\pi}$; and

When $n = j + 1$, $\Gamma(j + 1) = j\,\Gamma(j)$.

From equation (1), V_n can be calculated numerically, with relative ease, for all natural numbers n greater than one. Some of these values have been calculated and are summarized in Table 1. From Table 1, it can be seen that for $n = 5$, V_n attains a maximum. To see that this is the only maximum for all $n > 1$, see the theorem which follows.

TABLE 1

n	2	3	4	5	6
V_n	π	$\frac{4}{3}\pi$	$\frac{\pi^2}{2}$	$2^3\pi^2/(5\cdot 3)$	$\pi^3/3!$
Approximate V_n	3.142	4.189	4.935	5.264	5.168

n	7	8	9	10
V_n	$2^4\pi^3/(7\cdot 5\cdot 3)$	$\pi^4/4!$	$2^5\pi^4/(9\cdot 7\cdot 5\cdot 3)$	$\pi^5/5!$
Approximate V_n	4.725	4.059	3.299	2.550

Theorem. Let $V_n = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$, where n is a natural number > 1.

Then for $n = 5$, V_n attains a maximum; that is, $V_5 > V_j$ for all natural numbers j greater than one and not-equal to five.

Proof. The proof will be divided into three parts; the first part uses a simple observation and the last two are based on the principle of induction.

(a) It can be seen from Table 1 that

$V_5 > V_4 > V_3 > V_2$, or

$V_5 > V_j$ for $j = 2, 3, 4$.

(b) In this part, we shall prove that

$V_5 > V_6$, $V_7 > V_8$, ... , $V_{2k+1} > V_{2k+2}$, ...

To verify that $V_{2k+1} > V_{2k+2}$ for each natural number $k > 1$, the principle of induction can be utilized.

When $k = 2$, it is certainly true that $V_5 > V_6$ (see Table 1).

Assuming $V_{2k+1} > V_{2k+2}$, it can be proved that $V_{2(k+1)+1} > V_{2(k+1)+2}$. Notice that

$$V_{2k+1} = \frac{\sqrt{\pi^{2k+1}}}{\Gamma\left(\frac{2k+1}{2} + 1\right)}$$

and

$$V_{2k+2} = \frac{\sqrt{\pi^{2k+2}}}{\Gamma\left(\frac{2k+2}{2} + 1\right)}$$

where $k = 2, 3, 4, \ldots$. It is given that $V_{2k+1} > V_{2k+2}$, so

$$\frac{V_{2k+1}}{V_{2k+2}} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2k+2}{2} + 1\right)}{\left(\frac{2k+1}{2} + 1\right)} > 1 .$$

Now, note that

$$\frac{V_{2k+3}}{V_{2k+4}} = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{2k+4}{2}+1\right)}{\Gamma\left(\frac{2k+3}{2}+1\right)} = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{2k+2}{2}+1+1\right)}{\Gamma\left(\frac{2k+1}{2}+1+1\right)}$$

$$= \frac{1}{\sqrt{\pi}}\,\frac{\frac{2k+2}{2}+1}{\frac{2k+1}{2}+1} = \frac{\Gamma\left(\frac{2k+2}{2}+1\right)}{\Gamma\left(\frac{2k+1}{2}+1\right)} = \frac{2k+4}{2k+3}\,\frac{V_{2k+1}}{V_{2k+2}}$$

which is certainly greater than 1 ; this is true since $\frac{2k+4}{2k+3} > 1$ for all $k > 0$ and $\frac{V_{2k+1}}{F_{2k+2}} > 1$ is given.

Thus, from induction, $V_{2k+1} > V_{2k+2}$ for each natural number $k > 1$.

(c) In this part, we shall prove that

$$V_6 > V_7 \ , \ V_8 > V_9 \ , \ \ldots \ , V_{2k} > V_{2k+1} \ , \ \ldots \ .$$

To verify that $V_{2k} > V_{2k+1}$ for each natural number $k > 2$, the same types of arguments as in part (b), using the principle of induction, can be utilized. It can thus be shown that the above is true.

Combining the results of parts (b) and (c), it is obtained that $V_5 > V_6 > V_7 > \ldots$. With the previous result from part (a) and the preceding statement, it can be concluded that $V_5 > V_j$ for each natural number j greater than one and not equal to five.

As a further interesting example, consider what the volume of the n-dimensional unit sphere would be as n becomes very large,

$$\lim_{k\to\infty} V_{2k} = \lim_{k\to\infty} \frac{2k/2}{\Gamma\left(\frac{2k}{2}+1\right)} = \lim_{k\to\infty} \frac{\pi^k}{k!}$$

$$= \lim_{k\to\infty} \exp\left[\ln\left(\frac{\pi^k}{k!}\right)\right]$$

$$= \lim_{k\to\infty} \exp\,(k \ln \pi - \ln k!) \ .$$

By Stirling's approximation for large k,

$$\ln k! \approx k \ln k - k \ .$$

So,

$$k \ln \pi - \ln k! \cong k \ln \pi - k \ln k + k = k \ln \frac{\pi}{k} + k$$

$$= k[\ln \frac{\pi}{k} + \ln e] = k \ln \frac{\pi e}{k} \ .$$

Thus,

$$\lim_{k\to\infty} V_{2k} = \lim_{k\to\infty} \exp\left(k \ln \frac{\pi e}{k}\right) = \lim_{k\to\infty}\left(\frac{\pi e}{k}\right)^k = 0 \ .$$

Therefore, as n becomes a very large number, V_n approaches zero.

The fact that the volume of a five dimensional unit sphere is a maximum with respect to the volume of any n-dimensional unit sphere, and the volume, as n becomes very large, approaches zero, proves very interesting. The same types of arguments, like those worked in this article, can be carried out for any bounded surface; and with the help of Dirichlet integrals, interesting and fascinating properties can be determined.

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This paper was written under the direction of *Professor Theodore G. Phillips*.

THE DIRICHLET PROBLEM: A MATHEMATICAL DEVELOPMENT

by John Goulet
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Introduction.

Most students of advanced mathematics have encountered The Dirichlet 'Problem. It occurs in different forms in courses in partial differential equations, the calculus of variations, complex analysis, vector calculus, and in many areas of physics. The problem is worthy of attention because in virtually all of these cases it appears in quite different forms with entirely different techniques of solution.

The problem has fascinated the world's finest mathematicians from Gauss to Poincare, continually providing challenging questions as we'll as answers to an ever growing vaxiety of problems.

It is the aim of this article to acquaint the reader with various mathematical contexts in which the Dirichlet Problem arises, and then to outline its historical development.

In this section, we briefly examine several of the contexts in which the Dirichlet Problem arises.

First, the Dirichlet Problem may be considered as a boundary value problem of partial differential equations. Suppose D is a bounded domain in R_n and Γ its piecewise smooth boundary. Then one must find a scalar function u such that

$$(1) \quad \nabla^2 u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0 \text{ in } D$$

and (2) $$u|_\Gamma = f \text{ (given)}.$$

Equation (1) is called Laplace's equation, and any function satisfying it is called harmonic.

A second type of problem is the following: Let $\bar{v}$ denote a vector field in a finite bounded domain D in R_n. The problem is to find a scalar function u such that

$$(3) \quad \nabla u = \bar{v} \text{ in } D$$

and (4) $$u|_\Gamma = f \text{ (given)},$$

where Γ is once again the piecewise smooth boundary of D. A function u satisfying (3) is called a potential of $\bar{v}$ (∇ denotes the gradient here).

The problem just outlined is-a generalization of problems which arise in gravitational theory and electromagnetic theory. These problems may be characterized in the following way. In R_n, let $\bar{x}$ and $\bar{\xi}$ be two points, $\bar{r} = \bar{x} - \bar{\xi}$, $r = |\bar{r}| = |\bar{x} - \bar{\xi}|$, and D a finite domain. If the vector field $\bar{F}$ is defined by

$$(5) \quad \bar{F}(\bar{x}) = \int_D p(\bar{\xi}) \frac{\bar{r}}{r^n} dv ;$$

the density p is a non-negative function in D and zero outside D , then the problem is, again, to find a scalar function u such that

$$(6) \quad \nabla u = \bar{F}$$

and (7) $$u|_\Gamma = f \text{ (given)}.$$

How are these latter two problems related to the first Dirichlet Problem, stated in equations (1) and (2)? For a vector field of the form (5), one may show by direct calculation that if $\bar{x}$ is outside D, then

$$\nabla \cdot \bar{F} = 0.$$

By substitution from (6), one has $\nabla \cdot \nabla u = 0$ or $\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0$, which is Laplace's equation. On the other hand, if u satisfies Laplace's equation and equals f on Γ , does $\nabla u = \bar{F}$? Using uniqueness theorems for vector fields, one does indeed have $\nabla u = \bar{F}$ (see [5], chapter 8). Thus the first Dirichlet Problem and the second formulation are equivalent.

Another context in which the Dirichlet Problem arises is the following "energy method" approach. Suppose D is again a bounded domain in R_n and u is continuously differentiable in D . If one forms the quadratic functional

$$(8) \quad L(u) = \int_D \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_n}\right)^2\right] dV$$

$$= \int_D |\nabla \cdot u|^2 dV$$

then one may formulate the following problem: find a function u that

(9) minimizes the functional $L(u)$, and

(10) equals a given function f on the boundary of D.

An integral of the form (8) is called a Dirichlet or energy integral, and may represent the potential energy of a system. This is a problem in the calculus of variations; the minimization of a functional subject to prescribed boundary conditions. The exact requirements on D and its boundary were the subject of much work, and are discussed later.

How is this problem related to our first Dirichlet Problem? First, if u is a solution of (9) and (10), one can show $\nabla^2 u = 0$ in D. Alternatively, one can show that if u satisfies Laplace's equation (1) in D and if $u = f$ on Γ, then u minimizes (8), the Dirichlet Integral. The details may be found in [9], pages 135-9. The two problems are thus equivalent.

Next, in the realm of complex variables, suppose one is dealing with a multiple valued, complex, analytic mapping. A classic example is the complex function Log Z. In order to obtain a single-valued mapping, and hence a function, one makes copies ("branches") of the domain (the complex plane), the result being a Riemann surface. These are then "patched together" along branch "cuts." The number of copies depends on the mapping in question; Log Z requires infinitely many, whereas $Z^{\frac{1}{2}}$ requires only two. The result is that the properly defined mapping on this new domain, the Riemann surface, is single valued. It is also desired that the function retain its analyticity. This is clear within each branch, but unclear along the branch cuts. Riemann was faced with this problem. If we think of having two functions, one defined on each branch, and regard one of them as fixed, then the other must be chosen so that

(11) it agrees with the first function on the branch cut, and

(12) it is analytic.

If we denote the second function by $g = u + iv$, the first by f, the cut by Γ, and g's domain by D, then g must satisfy the Cauchy-Riemann equations in D to be analytic, and equal f on Γ. The Cauchy-Riemann equations are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. By taking second derivatives and assuming equality of mixed partial derivatives, one obtains

$$(12) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus the real and imaginary parts of an analytic, complex function satisfy Laplace's equation, and the problem just stated is equivalent to two Dirichlet Problems. In his doctoral thesis, Riemann used the Dirichlet integral method to attack the problem.

Finally, we note that Dirichlet Problems are the source of several well known "special functions" of-mathematical physics. For example, in R_3, if spherical coordinates and separation of variable are used, the ordinary differential equation for the θ variable (after the substitution $x = \cos\theta$) is

$$\frac{d}{dx}\left((1-x^2)\frac{d\theta}{dx}\right) + n(n+1)\,\theta = 0,$$

which is Legendre's equation. The $n(n+1)$ parameter is a separation constant, and solutions are the Legendre polynomials, $P_n(x)$, if n is a nonnegative integer.

If a change to cylindrical coordinates is made and separation of variables used, the ordinary differential equation for the r variable is

$$r^2 R'' + rR' + (\lambda^2 r^2 - \nu^2)R = 0,$$

where the parameters λ and ν are separation constants chosen so as to yield physically reasonable solutions. The series solutions obtained are $J_\nu(\lambda r)$ and $J_{-\nu}(\lambda r)$, the Bessel functions of order ν and $-\nu$, respectively, in the case where ν is not an integer. If ν is an integer, other special functions must be used in place of $J_{-\nu}(\lambda r)$ in order that two linearly independent solutions result.

Historical Development of the Dirichlet Problem.

In examining the historical development of the Dirichlet Problem, several approaches are possible: the chronological development, the biographical development, or the geographical development. With regard to the Dirichlet Problem, we shall use the geographical, as there were three places (England, France and Germany) where separate developments took place, each having a unique approach to the problem.

In England, the first work of significance was done by George Green (1793-1841). Green's primary areas of interest were electricity and fluid mechanics. The reader is referred to [4] for a complete survey. In studying these areas, Green was led to a number of mathematical results, most of which bear his name and all of which have proven indispensable to the study of potential theory. In his 1828 booklet, "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," he introduces what are now referred to as Green's theorems of potential theory, as well as the so-called Green's Function method of solution of the Dirichlet Problem. In his 1833 paper, "Laws of the Equilibrium of Fluids," he considers the gravitational potential of fluids of ellipsoidal shape, using many of the mathematical techniques developed in the 1828 paper. It is interesting to note that the 1833 paper is generalized to n dimensions, as opposed to two or three dimensions. It should also be noted that many of Green's proofs are not mathematically rigorous, being based in part upon physical arguments. Two examples are his conclusion that a minimum exists for the energy integral (later called the Dirichlet integral),

$$\int_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dV,$$

where u is a real-valued function to be determined, and that a Green's function exists for a given region in space. Though lacking in mathematical rigor, his work partially rejuvenated mathematical analysis in England, which had produced little since the work of Newton (1642-1727). Following in his steps were Kelvin, Stokes, Rayleigh and Maxwell.

Sir William Thomson, or Lord Kelvin (1824-1907), continued British interest in mathematical physics. Thomson saw fit to call a function "harmonic" if it satisfied the potential equation. His chief tool was again the energy integral. Using the calculus of variations, he thought he had established the existence of a minimum for it and hence the existence of a solution to the Dirichlet Problem. Published in 1847, the result was called "Thomson's Principle" in England.

In France, Pierre-Simon de Laplace (1749-1827) spent much of his life working on celestial mechanics, although he had many other scientific interests. In 1792, he published a paper concerning the gravitational potential due to a spherical mass. In it he uses and solves the potential equation in spherical coordinates, employing series techniques and Legendre polynomials. Today his solutions are called spherical harmonics, so named by Kelvin. During that period in France, Joseph Fourier (1768-1830) was studying the theory of heat conduction. He derived the "heat equation,"

$$\nabla^2 T = k^2 \frac{\partial T}{\partial t},$$

and initiated the trigonmetric, or Fourier Series technique to study it in rectangular coordinates. These results are summarized in his classic "Theorie Analytique de la Chaleur," published in 1822. Although correct, his results were not rigorous and served to stimulate much activity in mathematical analysis in the remainder of the 19th century. He also considered the steady-state heat equation, which is Laplace's equation. The term "steady-state" means that $\frac{\partial T}{\partial t} = 0$; that is, the temperature, T, does not vary with time. Further, let us assume that T is prescribed on a two dimensional boundary, Γ. The problem of heat conduction now becomes

$$\nabla^2 T = 0,$$

$$T|_\Gamma \text{ given,}$$

another Dirichlet Problem.

Fourier studied the above problem in the case where Γ is a rectangle. We shall assume that Γ is the boundary of the region where $0 \le x \le 1$, $0 \le y \le 1$. Also, we assume that T must satisfy

$$T(0,y) = T(1,y) = 0$$

$$T(x,0) = 0$$

$$T(x,1) = f(x) \quad \text{(given).}$$

In rectangular coordinates, Laplace's equation becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 .$$

Fourier assumed $T(x,y) = X(x)\cdot Y(y)$, where X and Y are to be found. This is the method of separation of variables, which D'Alembert first used in 1752. Using this, Fourier obtained the following equations:

$$X'' - \lambda X = 0 \qquad X(0) = X(1) = 0$$

$$Y'' + \lambda Y = 0 \qquad Y(0) = 0$$

$$T(x,1) = f(x),$$

where λ is a separation constant. This system can be shown to have the solution

$$\lambda = -n^2\pi^2 \qquad n = 0, 1, 2, \cdots$$

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin n\pi x \sinh n\pi y,$$

where

$$C_n = \frac{1}{\sin h n\pi} \int_0^1 f(x) \sin n\pi x dx .$$

Here, C_n was obtained by finding the Fourier Sine Series of $f(x)$. The time varying problem, where $\partial T/\partial t \neq 0$ is solved by similar means. See [13], for example.

Gabriel Lamé (1795-1870), at the Ecole Polytechnic in Paris, was also interested in solving the steady-state heat equation in other coordinate systems. This is because if the boundary, Γ, is other than rectangular, Cartesian coordinates may very well be inadequate. To illustrate this, suppose once again that we are dealing with two dimensions and that Γ is a circle, say of radius one, with T prescribed on it. In Cartesian coordinates, the problem is not solvable. In polar coordinates, however, it becomes

$$r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0$$

$$T(1,\theta) = f(\theta) \quad \text{(given)} \quad 0 \leq \theta \leq 2\pi .$$

The partial differential equation can be solved by assuming $T(r,\theta) = R(r)\cdot\Theta(\theta)$. This yields two ordinary differential equations after separation, namely,

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0$$

with conditions

$$\Theta(0) = \Theta(2\pi)$$

$$\Theta'(0) = \Theta'(2\pi),$$

so that the solution will be continuously differentiable, and

$$T(1,\theta) = f(\theta).$$

The solution is then found to be

$$T(r,\theta) = \sum_{n=0}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta$$

are found by Fourier Series techniques.

Lamé sought to solve Dirichlet Problems for any boundary by employing a suitable coordinate system, so that the boundary surface took on the simple form X_i = constant, where X_i is a variable in the desired coordinate system. Having found such a coordinate system for a given problem, he hoped then to separate Laplace's equation as we have done in the above examples, and solve the Dirichlet Problem. However, he eventually came to realize this separation is not always possible. Today, we know that Laplace's equation is separable in eleven coordinate systems. Lamé's work is summarized in his 1859 book Lectures on Curvilinear Coordinates.

Finally, we come to Germany, whose mathematical center was Berlin, and to a lesser extent, Göttingen. In 1828, P. L. Dirichlet (1805-1859) was appointed as a professor in Berlin. A graduate of the University of Cologne, he had taught in France and studied with Fourier before returning to Germany. Although chiefly remembered for his work in number theory, he published papers concerning Fourier Series, fluid mechanics and potential theory "Ueber einen neuen Ausdruck zur Bestimung der Dichtigkeit einer unendlich dunnen Kugelschale wenn der Werth des Potentials derselben in jedem Punkte ihrer oberfläche gegeben ist." In part because of his interest in number theory and mathematical physics, he was appointed chairman of Göttingen in 1855, succeeding K. F. Gauss (1777-1855). His 1850 paper solved the potential equation as Green and Thomson did, by means of minimizing the energy integral. Although only at Göttingen for four years, Dirichlet taught and influenced Behrnard Riemann (1826-1866). Riemann named the technique of minimizing the energy integral the Dirichlet Principle, and the boundary value problem the Dirichlet Problem. In his doctoral thesis ("Grundlagen fur eine allgeneine Theorie der Functionen einer veranderlichen complexen Grösse") which introduced and developed the Riemann surface approach to multiple-valued, complex mappings, he frequently assumed and used the Dirichlet Principle whenever needed.

During this period, analysis was coming under greater scrutiny with regard to its rigorous validity. In Germany, Karl Weierstrass (1815-1897) made many contributions to the foundations of analysis, including a number of startling and thought provoking counterexamples to previously accepted results. In 1879, while at the University of Berlin, he showed that under the current conditions used with the Dirichlet Principle, that there is not always a continuously differentiable function that minimizes the Dirichlet Integral.

This result caused a great deal of discussion and disappointment in the mathematical community; a method that had been successfully and frequently used for years was possibly invalid. In particular, this left a large logical loophole in the dissertation of Riemann, a fact he was well aware of, and which he attempted to rectify. In a later paper on minimal surfaces, he attempted to establish the existence of the desired function by geometric arguments but his arguments fell short of sufficient generality. Carl Neumann, another mathematical physicist of the era, was saddened that a theory "which was so beautiful and could be utilized so much in the future, has forever sunk from sight."

This result of Weierstrass' did not mean that the Dirichlet Problem was to go ignored, however. As we have seen in the first part of this article, the problem can be studied from several perspectives. Herman Schwarz (1843-1921), a student of Weierstrass' at Berlin, Carl Neumann, and Henri Poincaré (1854-1912) at the University of Nice in France, all gave existence proofs by attacking Laplace's equation.

David Hilbert (1862-1943) was not, however, convinced that the Dirichlet Problem could not be solved using the Dirichlet Integral. He suspected that the assumptions underlying the calculus of variations were at fault, not the method of minimizing the Dirichlet Integral. The problem, we recall, was in showing that the function minimizing the Dirichlet Integral was contained in the set of admissible functions. As this is a limiting process, it refers to what we call "completeness," which is fundamental to the notion of Hilbert Spaces. It is therefore quite conceivable that Hilbert would correctly solve the problem, and he did, in 1899. (see "Uber das Dirichletsche Prinzip," Jahresbericht der Vereinignng, 1900). A modern and elegant treatment of the applications of variational techniques to partial differential equations is given in chapter 4 and includes the Dirichlet Problem as a special case. It is interesting to

note that Hilbert's student and later colleague, Richard Courant, (1888-1972) became even more interested in the problem, chose it as the subject of his doctoral thesis ("On the application of Dirichlet's Principle to the problems of conformal mapping," 1910), and eventually made it the subject of a book, [2].

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DESCARTES: PHILOSOPHER OR MATHEMATICIAN?

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Rene Descartes, the seventeenth century French thinker, had a profound effect on two disciplines: Philosophy and Mathematics. It was Descartes' methodic doubt that led him to the thinking self as a philosophic starting point. This "methodic doubt" was adopted by later philosophers as well, takingphilosophyin an etirely new direction. As a result, Descartes is often considered to be the father of modern philosophy. Moreover, Descartes was also an influential mathematician. It was Descartes who combined algebra and geometry to come up with what is now known as analytic geometry. Analytic geometry and the accompanying Cartesian coordinate system were two of the necessary precursors to the discovery of the Calculus.

It is exceptional for one person to have such a profound effect on two quite distinct disciplines. The question that occurred to me is, "What is the common thread between Descartes' contributions to philosophy and his contributions to mathematics?" It seems like a reasonable question: every discovery takes place in a context, and the context of these two discoveries includes the same person. What is the common element?

In order to discover the common element in Descartes' many intellectual contributions, it is helpful to learn a little about Descartes' background. Descartes finished the regular university course at the University of Paris at the beginning of the seventeenth century. After reflecting on his years at the University, he became quite frustrated. Despite the fact that he had studied for most of his life, he felt that he was <u>certain</u> of nothing. Philosophy was the discipline which frustrated him the most. In his work entitled "A Discourse on Method", Descartes wrote:

> Despite the fact that philosophy has been cultivated by the best minds that have ever lived, nevertheless no

> single thing is to be found in it which is not subject to dispute, and in consequence which is not dubious. [1, p. 86].

In other words, it seemed to Descartes that philosophical reflection had hardly advanced beyond the first philosophical queries of the earliest Greek philosophers, and consequently that philosophers needed to start all over again at the very beginning and to do it right this time.

Unlike philosophy, there did seem to be a sure body of knowledge in mathematics. Descartes wrote, "Most of all I was delighted with mathematics because of the certainty of its demonstration and the evidence of its reasoning." [1, p. 85]. As a matter of fact, mathematics was the only discipline in which Descartes found certainty. Therefore, he decided to generalize the method of mathematics, namely the method of starting with unquestionable axioms and proceeding logically from the simple axioms to more complex theorems. The Cartesian method consisted of four principles:

1) to accept nothing as true which is not so clear and distinct that all doubt is excluded;
2) to divide large problems into smaller ones;
3) to proceed from the simple to the complex;
4) to enumerate and review the steps of your deductive reasoning so thoroughly that no error can be admitted.

After arriving at this method, Descartes intended to apply it to all disciplines, beginning with philosophy.

It was the application of the Cartesian method to philosophical reflection which resulted in the unique Cartesian starting point. Recall that the first principle of the Cartesian method is "to accept nothing as true which is not so clear and distinct that all doubt is excluded." According to Descartes, everything is subject to doubt, including sense data. Everything, that is, with the exception of one thing, the existence of the doubter. The doubter, my self, must exist or else doubt itself would be impossible. It is this reflective process which resulted in the famous Cartesian assertion, "cogito, ergo sum" (which means, "I think, therefore I am"). Arguing logically from this unquestionable starting point, Descartes arrived at the existence of God and at a certain knowledge of the physical world.

Obviously, the next step is to show how Descartes applied his method to mathematics in order to come up with his coordinate geometry. But first, let us consider a few questions. If Descartes was so delighted with mathematics, what need was there to apply his method to the discipline? Secondly, if the Cartesian method was abstracted from mathematics, how could it be reapplied to mathematics?

In response to the first question, while Descartes was "delighted with mathematics because of the certainty of its demonstrations and the evidence of its reasonings," he was convinced that there was enormous potential for much further progress in the field. In Descartes' time mathematics consisted essentially of Euclidean geometry with algebraic appendages. Euclidean geometry confines itself primarily to figures formed by straight lines and circles, so in order to explain the baffling physical phenomena of his day in mathematical terms, phenomena such as the elliptical path of the planets or the parabolic path of a cannonball, Descartes needed to come up with a way of dealing efficiently with curves such as ellipses and parabolas. Descartes decided, therefore, that it would be nice to establish a general procedure which one could follow when dealing with ellipses, parabolas, and the like, and he set out to accomplish just that.

Now that we have seen why there was a need for Descartes to apply this "method" to mathematics, we must explain how he applied his method to the very discipline from which he abstracted it. Recall the Cartesian method. The first principle of the Cartesian method is to accept nothing as true which is not so clear and distinct thatalldoubt is excluded. In Descartes' opinion, Euclidean geometry and the algebra of his day were undoubtedly true; thus, they formed the foundation for his coordinate geometry. The third principle of his method is to proceed from the simple to the complex. Obviously, a straight line is a simpler geometric figure than a curve. Descartes' method suggested, therefore, that it should be possible to generate a procedure for dealing with curves using what he knew about straight lines. The Cartesian method could only take him this far. At this point Descartes had to discover a way of dealing with curves based upon what he knew about straight lines.

Descartes made a start towards this discovery through a creative way of envisioning curves. Imagine a curve to be the path formed by the endpoint E of the line segment $\overline{EF}$ (see figure 1 below). As the vertical line segment $\overline{EF}$ moves towards or away from the fixed point 0,

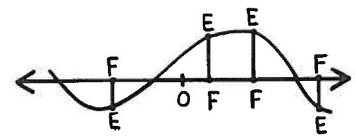

Figure 1

the line segment shortens, lengthens, or even changes directions, depending upon the path of the curve. The line segment's one endpoint E follows the path of the given curve, whereas its other endpoint F always remains on the same horizontal line. Once Descartes discovered this way of envisioning curves, he was one creative insight away from establishing a general procedure for dealing with any geometric figure.

The second step in establishing a procedure for dealing with geometric figures involved algebra. Descartes discovered that it was possible to compare the ever-changing position of the point E with the constant position of the origin (some fixed point 0) using algebra (see figure 2 below). The distance from the origin to the vertical line segment $\overline{EF}$, he called x. The distance from the horizontal line $\overline{OF}$ to the

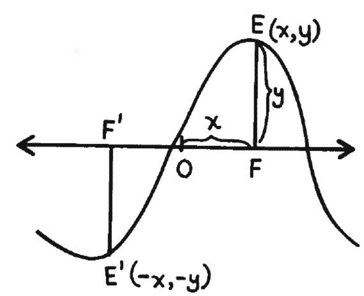

Figure 2

endpoint E, he called y. Thus, the position of each point E could be represented by an (x,y) coordinate. To avoid the type of confusion engendered by points equi-distant from the origin (as exhibited), for example, by points E and E' in figure 2), Descartes arbitrarily decided that the x-coordinate of a point to the left of the origin would be negative whereas the x-coordinate of a point to the right of the origin would be positive. Likewise, he decided that the y-coordinate of a point below the origin would be negative whereas the y-coordinate of a point above the origin would be positive. Upon establishing this coordinate system, Descartes discovered that there is one algebraic equation which can describe the relationship between the x and y coordinates for every point E on the given curve. Thus, his general procedure for dealing with curves was complete.

To summarize, there are three basic insights which resulted in Descartes' discovery of coordinate geometry. First, he had to come up with a creative way of envisioning curves, namely as a series of points formed by a "magic" line segment. Secondly, he had to discover how to relate each point of the curve to a fixed point. Thirdly, he had to discover that there was a unique algebraic equation associated with each unique curve, an equation which described the relationship between the x and y coordinates for every point on that curve. The Cartesian method led Descartes to the doorstep of coordinate geometry; but as in all mathematical discoveries, creative insight was needed to open the door.

In conclusion, we see that the Cartesian method played a part in Descartes' mathematics as well as his philosophy. There are other similarities as well. Just as Descartes' philosophy is no longer based on sense data, so Descartes' mathematics is no longer dependent on the sensible figure. As Morris Kline wrote in his *Mathematics in Western Culture*, in Descartes' coordinate geometry "the mind has replaced the eye." 2, p. 177 . The curve is no longer represented by a sensisble figure; it is represented by an algebraic equation. This algebraic geometry formed the foundation for modern mathematics, making more abstract mathematics possible.

But back to the title of this paper. Is Descartes primarily a mathematician or a philosopher? I would venture to say that Descartes was a mathematician at heart who took philosophy very seriously. He was a man who was frustrated with the lack of true knowledge in a world which acted as if it knew it all, and using the method of mathematics, Descartes tried to advance many of the various disciplines. In retrospect, however, while Descartes did make a profound contribution to

philosophical thought, it was mathematics itself which benefitted the most from the mathematical "Cartesian method".

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A THEOREM OF SIMILAR TRIANGLES

by *Michael Eisenstein*
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Theorem: Let AABC be any triangle.

Let $O = (p,q)$ be any interior point of AABC.

Let K be the point of intersection of the medians of AAOB.

Let L be the point of intersection of the medians of ACOB.

Let M be the point of intersection of the medians of AAOC.

Then ALMK is similar to AABC, the ratio of a side of ALMK to the corresponding side of AABC is 1/3, and the corresponding sides are parallel.

Proof. We refer to the drawing in Figure 1.

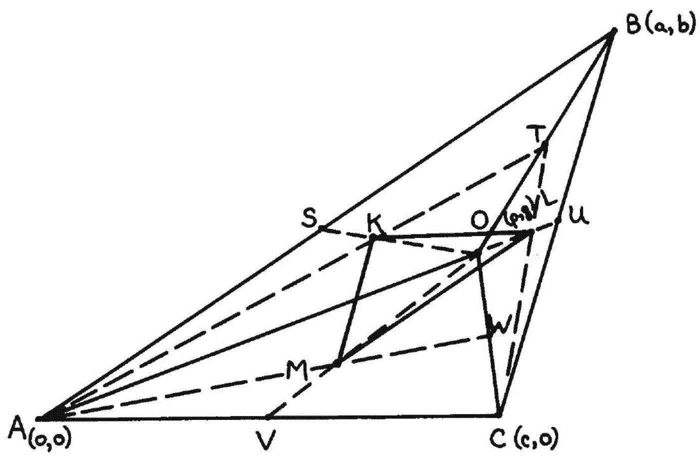

Figure 1.

We find the coordinates of the points K, L, and M. In any triangle the three medians intersect at a common point. Therefore, it suffices to find the point of intersection of any two medians in each triangle.

In each triangle we find the equations of the lines through two

medians. Setting them equal to each other, we find the point of intersection. Let $A = (0,0)$, $B = (a,b)$, $C = (c,o)$ where a, b, a are positive real numbers.

I. We consider ΔAOB.

Let S be the midpoint of $\overline{AB}$.

Then $S = \left(\frac{a+0}{2}, \frac{b+o}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$ and $\overline{OS}$ is a median of MOB.

The slope of $\overline{OS} = \dfrac{\frac{b}{2} - q}{\frac{a}{2} - p}$ and with the point $\left(\frac{a}{2}, \frac{b}{2}\right)$, the equation of the line through $\overline{OS}$ is

$$y - \frac{b}{2} = \frac{b-2q}{a-2p}\left(x - \frac{a}{2}\right) \qquad (1)$$

Let T be the midpoint of $\overline{OB}$. Then $T = \left(\frac{a+p}{2}, \frac{b+q}{2}\right)$ and $\overline{AT}$ is a median of AAOB.

The slope of $\overline{AT}$ is $\frac{b+q}{a+p}$ and the equation of the line through $\overline{AT}$ is

$$y = \frac{b+q}{a+p}\,x\,. \qquad (2)$$

Solving (1) and (2) above for x and y, we have $x = \frac{a+p}{3}$, $y = \frac{b+q}{3}$ and the point of intersection $k = \left(\frac{a+p}{3}, \frac{b+q}{3}\right)$.

II. We consider ABOC.

Let U be the midpoint of $\overline{BC}$. Then $u = \left(\frac{a+c}{2}, \frac{b}{2}\right)$ and $\overline{OU}$ is a median of ΔBOC.

The slope of $\overline{OU} = \dfrac{b-2q}{a+c-2p}$. With the point (p,q), the equation of the line through $\overline{OU}$ is

$$y = \left(\frac{b-2q}{a+c-2p}\right)(x-p) + q. \qquad (3)$$

Let $T = \left(\frac{a+p}{2}, \frac{b+q}{2}\right)$. Then $\overline{CT}$ is a median of ΔBOC.

The slope of $\overline{CT} = \dfrac{b+q}{a+p-2c}$. With the point (c,o), the equation of the line through the median $\overline{CT}$ is

$$y = \left(\frac{b+q}{a+p-2c}\right)(x-c)\,. \qquad (4)$$

Solving (3) and (4) above for x and y, we have $x = \frac{a+p+c}{3}$, $y = \frac{b+q}{3}$ and the point of intersection $L = \left(\frac{a+p+c}{3}, \frac{b+q}{3}\right)$.

III. We consider ΔAOC.

Let V be the midpoint of A?. Then $V = \left(\frac{c}{2}, 0\right)$ and $\overline{OV}$ is a median of ΔAOC. The slope of $\overline{OV} = \frac{q}{p-\frac{c}{2}}$ and with the point $\left(\frac{c}{2}, 0\right)$, the equation of the line through $\overline{OV}$ is

$$y = \frac{2q}{2p-c}x - \frac{qc}{2p-c} \quad . \tag{5}$$

Let W be the midpoint of $\overline{OC}$. Then $W = \left(\frac{p+c}{2}, \frac{q}{2}\right)$ and $\overline{AW}$ is a median of ΔAOC. The slope of $\overline{AW} = \frac{q}{}$. With the point $A(0,0)$, the equation of the line through AW is

$$y = \frac{q}{p+c}x. \tag{6}$$

Solving (5) and (6) above for x and y, we have $x = \frac{p+c}{3}$, $y = \frac{q}{3}$ and the point of intersection $M = \left(\frac{p+c}{3}, \frac{q}{3}\right)$.

We have from I, II, III above:

$$K = \left(\frac{a+p}{3}, \frac{b+q}{3}\right), \quad L = \left(\frac{a+p+c}{3}, \frac{b+q}{3}\right), \quad M = \left(\frac{p+c}{3}, \frac{q}{3}\right) \quad .$$

Then

$$KL = \sqrt{\left(\frac{c}{3}\right)^2 + 0} \qquad = \frac{c}{3} = \frac{1}{3}AC \ .$$

$$LM = \sqrt{\left(\frac{a}{3}\right)^2 + \left(\frac{b}{3}\right)^2} \qquad = \frac{1}{3}\sqrt{a^2 + b^2} = \frac{1}{3}AB \ .$$

$$MK = \sqrt{\left(\frac{a-c}{3}\right)^2 + \left(\frac{b}{3}\right)^2} \qquad = \frac{1}{3}\sqrt{(a-c)^2 + b^2} = \frac{1}{3}BC \ .$$

Therefore ΔLMK is similar to ΔABC and the ratio of the corresponding sides is $\frac{1}{3}$.

Slope of $\overline{KL} = \dfrac{0}{\frac{a+p+c}{3} - \frac{a+p+c}{3}} = 0$ = slope of $\overline{AC}$.

Therefore, $\overline{KL} \,||\, \overline{AC}$.

Slope of $\overline{LM} = \dfrac{\frac{b}{3}}{\frac{a}{3}} = \dfrac{b}{a}$ = slope of $\overline{AB}$.

Therefore $\overline{LM} \,||\, \overline{AB}$.

Slope of $\overline{KM} = \dfrac{\frac{b}{3}}{\frac{a-c}{3}} = \dfrac{b}{a-c}$ = slope of $\overline{BC}$.

Therefore, $\overline{KM} \,||\, \overline{BC}$.

Alternate Proof.

We first establish:

Theorem A. Given MBC, let K be the point of intersection of the medians. Then the length of the line from a vertex to K = $\frac{2}{3}$ of the length of the median from that vertex.

Proof.

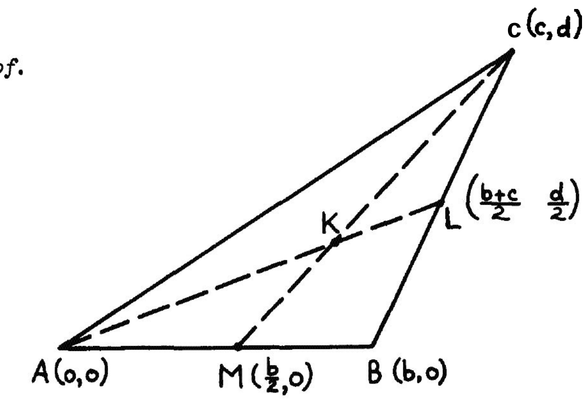

Figure 2.

Consider Figure 2. Let M be the midpoint of $\overline{AB}$. Then $M = \left(\frac{b}{2}, 0\right)$ and $\overline{CM}$ is a median of ΔABC.

The slope of $\overline{CM}$ is $\dfrac{d}{c-\frac{b}{2}}$ and with the point $\left(\frac{b}{2}, 0\right)$, the equation of the line through $\overline{CM}$ is

$$y = \frac{2d}{2c-b}x - \frac{db}{2c-b} \quad . \tag{7}$$

Let L be the midpoint of $\overline{BC}$. Then $L = \left(\frac{b+c}{2}, \frac{d}{2}\right)$ and $\overline{AL}$ is a median of ΔABC.

The slope of $\overline{AL}$ is $\frac{d}{b+c}$ and with the point $A(0,0)$, the equation at the line through $\overline{AL}$ is

$$y = \frac{d}{b+c}x. \tag{8}$$

To get the point of intersection we equate (7) and (8) and have:

$$\frac{2d}{2c-b}x - \frac{db}{2c-b} = \frac{d}{b+c}x .$$

$$\frac{2db + 2dc - 2cd + db}{(2c-b)(b+c)}x = \frac{db}{2c-b}$$

Therefore $X = \dfrac{b+c}{3}$

$$y = \frac{d}{3}$$

So $K = \left(\dfrac{b+c}{3}, \dfrac{d}{3}\right)$

Then $AK = \sqrt{\left(\dfrac{b+c}{3}\right)^2 + \left(\dfrac{d}{3}\right)^2} = \dfrac{1}{3}\sqrt{(b+c)^2 + d^2}$

$AL = \sqrt{\left(\dfrac{b+c}{2}\right)^2 + \left(\dfrac{d}{2}\right)^2} = \dfrac{1}{2}\sqrt{(b+c)^2 + d^2}$

Therefore $AK = \dfrac{2}{3}AL$. Similarly for the other two medians.

To complete the solution, we now refer to the Figure 1 above. As K is the point of intersection of the medians of ΔADB,

$$KT = \frac{1}{3}AT \text{ from Theorem A above.}$$

Similarly $LT = \dfrac{1}{3}CT$.

Therefore, ΔKTL is similar to ΔATC, as we have one congruent angle and the corresponding adjacent sides in proportion.

Thus

$$KL = \frac{1}{3}AC$$

As $\angle TKL \approx \angle TAC$ we have $\overline{KL} \parallel \overline{AC}$.

Similar arguments for $\overline{LM}$ and $\overline{KM}$ complete the proof.

A SIMPLE MODEL FOR TWO INTERACTING SPECIES AND THE PRINCIPLE OF COMPETITIVE EXCLUSION

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1. *Introduction*

The theory of competition between two species living in the same environment was first published by Charles Darwin in 1859 [6, p. 1295]. Volterra formally developed a mathematical model and hypothesis in 1931 which was verified experimentally by Gause in 1934 and 1935 [3] and [9, Chap. 8]. Since that time there has been considerable controversy over the validity of Volterra's equations and the various modified forms of Gause's principle of competitive exclusion.

In its simplest form, the principle states that two species that make their living in identical ways cannot coexist in the same environment. Then how different must the species be to coexist in an equilibrium community? Eventually, after many diverse versions, the following generally accepted statement was developed:

Principle of Competitive **Exclusion:** "Two species competing for limited resources can only coexist if they inhibit the growth of the competing species less than their own growth" [1, p. 89]. We will demonstrate the validity of this principle by first deriving a system of equations for two competing species and then showing that the behavior of its solution supports the competititve exclusion principle.

2. *Derivation of Equations*

Consider a population model for one species where the growth rate (the difference between the birth and death rate) is a constant. Let $N(t)$ be the size of an isolated population at time t and let r be its growth rate. If at some initial time t_0 the population is N_0, then the rate of change of the population $dN(t)/dt$ is the growth rate times the size of the population. So we have a linear differential equation, investigated by the British economist Malthus around 1800 [5, p. 125]:

$$dN(t)/dt = rN(t).$$

With the initial condition $N(t_0) = N_0$, the solution to this equation is

$$N(t) = N_0 e^{r(t-t_0)}.$$

If $r > 0$, as it would be in an unlimited environment, the population grows exponentially with time. Now let us restrict the species to a microcosm, a representative of the total environment, but with limited space and food.

Experimentally, it has been shown that the growth rate diminishes as the population density increases [5, p. 151]. Also, the probability that two members of the species will encounter each other is proportional to N^2 [2, p. 29]. So if we add a competition term of $-bN^2$ to the Malthusian equation, where b is a positive constant, we have:

$$dN/dt = rN - bN^2.$$

This equation was first investigated by Verhulst, a Dutch biologist, in 1837 and is known as the <u>logistic equation</u> for population growth [5, p. 153]. In this equation, r is the uninhibited growth rate and b represents the effect of crowding, where r and b are positive constants.

We are now interested in an equilibrium population, when $dN/dt = 0$. By examination, we see that there are two equilibrium populations, when

$$N = 0 \quad \text{or} \quad N = r/b.$$

When $N = 0$, the solution is trivial, but when the population is $N = r/b$, this is called the <u>saturation population</u> and is the largest population a species can sustain in a microcosm without loss. The logistic equation is separable and solving [5, pp. 159-160] we have

$$N(t) = \frac{r/b}{1+\left(\frac{r-bN_0}{bN_0}\right)e^{-r(t-t_0)}}.$$

When $t \to \infty$, then $N(t) \to r/b$. Hence, the popluation approaches the saturation population, regardless of its initial value (N_0 # 0). Notice if the initial population is less than the saturation population, then $dN/dt > 0$ and so the population increases; but if it is greater than the saturation population, then $dN/dt < 0$ and so the population decreases. We shall call this a <u>stable equilibrium population</u>, since as time increases, all solutions near the saturation population stay near it.

It will now be to our advantage to express the logistic equation in terms of the saturation population, r/b. Let $K = r/b$ and notice K is always positive. We have:

$$dN/dt = rN - bN^2 = rN(1-\tfrac{b}{r}N) = rN(1-\tfrac{1}{K}N) = rN(\tfrac{K-N}{K}).$$

Since K represents the available number of spaces in the microcosm for the species, then $(K-N)/K$ is the number of vacant spaces relative to any N. Hence, for each member of the species added to the population, one more place is occupied and the growth rate is reduced by the constant factor $1/K$.

To derive Volterra's equations for two competing species first consider the logistic equations for species 1 and 2 in the absence of the other. For species 1, when $N_2 = 0$, we have

$$dN_1/dt = r_1 N_1((K_1-N_1)/K_1)$$

and for species 2, when $N_1 = 0$, we have

$$dN_2/dt = r_2 N_2((K_2-N_2)/K_2).$$

If these two species compete for the same food and space, we can assume that each individual of one species inhibits the other species' growth rate by a constant factor. Assume that each member of species 2 reduces the growth rate of species 1 by the constant α/K_1. Similarly, assume the growth rate of species 2 is decreased by the constant β/K_2 for each member of species 1. Adding these competition factors to the respective logistic equations gives us <u>Volterra's equations</u>:

$$dN_1/dt = r_1 N_1((K_1-N_1-\alpha N_2)/K_1)$$

$$dN_2/dt = r_2 N_2((K_2-N_2-\beta N_1)/K_2).$$

The positive constants a and β are called the <u>coefficients of competition</u> and indicate the influence of each species on the other. So, one individual of species 1 has an inhibitory effect of $1/K_1$ on its own growth rate and an inhibitory effect of β/K_2 on species 2 growth rate. Similarly, each member of species 2 inhibits its own growth rate by $1/K_2$ and inhibits species 1 growth rate by α/K_1 [9, Chap. 7].

3. <u>*Behavior of Solutions* of, *Volterra's* Equations,</u>

Notice that the variable t does not appear explicitly in the right-hand members of this system. This type of system is called <u>time-invariant</u> or <u>autonomous</u>. If we regard t as a parameter, we can examine the population changes in time using N_1 and N_2 as the axes of our coordinate; system. The $N_1 - N_2$ plane is called a phase plane and the solution curves, called trajectories, are depicted with arrows to indicate how the populations change with time [7, Chap. 8].

First we would like to insure that neither population is ever less than zero, so we must show that we are concerned only with the first quadrant and nonnegative axes of the $N_1 - N_2$ phase plane. Considering each species in the absence of the other, we then have

$$dN_1/dt = r_1N_1((K_1-N_1)/K_1), \quad N_2 = 0,$$

which is logistic equation for species 1. Let us examine the phase plane solution. We have with $N_2 = 0$

$$dN_1/dt = 0 \text{ if } N_1 = 0,$$

which implies (0,0) is an equilibrium population;

$$dN_1/dt > 0 \text{ if } 0 < N_1 < K_1,$$

which implies the N_1 population is increasing if $0 < N_1 < K_1$ and $N_2 = 0$;

$$dN_1/dt = 0 \text{ if } N_1 = K_1$$

which implies $(K_1,0)$ is an equilibrium population;

$$dN_1/dt < 0 \text{ if } K_1 < N_1,$$

which implies the N_1 population is decreasing if $K_1 < N_1$ and $N_2 = 0$.

This is sketched on the phase plane as in Fig. 1. Similarly, we can sketch the solution of the logistic equation for species 2, when $N_1 = 0$ on the N_2 axis. If we take the union of these trajectories, we see that the origin and the positive axes are covered as sketched in Fig. 2. From fundamental existence and uniqueness theory for ordinary differential equations, we know that given any set of initial conditions, an autonomous system has no two solutions passing through the same point [2, p. 391]. Hence, trajectories in the phase plane may never cross

So if we start off with initial population in the first quadrant of the phase plane, the solution of the system must remain in the first quadrant.

We now would like to consider the equilibrium populations of the

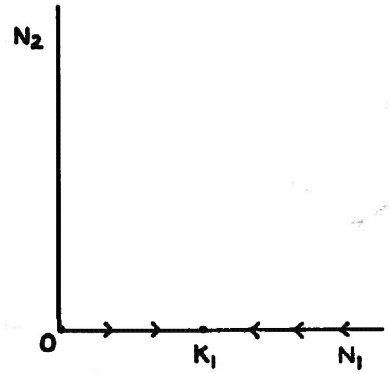

Figure 1

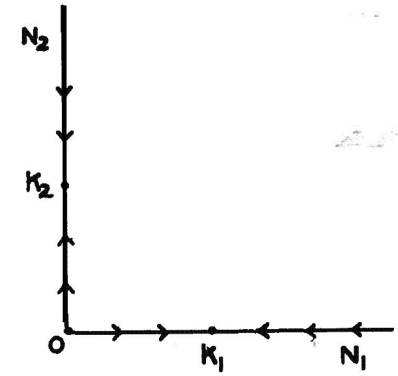

Figure 2

Volterra system, when $dN_1/dt = dN_2/dt = 0$. Looking at

$$dN_1/dt = r_1N_1((K_1-N_1-\alpha N_2)/K_1),$$

we know $dN_1/dt = 0$ if and only if $N_1 = 0$ or $K_1 - N1 - \alpha N_2 = 0$. If $N_1 = 0$ we have already seen what happens. Consider the line $K_1 - N_1 - \alpha N2 = 0$ or in slope-intercept form: $N_2 = -N_1/\alpha + K_1/\alpha$. Notice above this line $(N_2 > -N_1/\alpha + K_1/\alpha)$ that $dN_1/dt < 0$; hence the N_1 population is decreasing. Then below the line we have $dN_1/dt > 0$, and the N_1 population is increasing. This is sketched on the phase plane in Fig. 3 with arrows indicating change of the N1 population only.

If we now look at

$$dN_2/dt = r_2N_2((K_2-N_2-\beta N_1)/K_2),$$

then we know $dN_2/dt = 0$ if and only if $N_2 = 0$ or $K_2 - N_2 - \beta N_1 = 0$. We have already observed what happens when $N_2 = 0$, so lets consider the line $N2 = -\beta N_1 + K_2$. As with the first equation notice that above this line $dN_2/dt < 0$ and the N_2 population is decreasing. Also below the line $dN_2/dt > 0$ and the N_2 population is increasing. In Fig. 4, the phase plane is sketched with arrows indicating change of the N_2 population only.

If we eliminate the parameter t in Volterra's equations we have:

$$dN_2/dN_1 = (r_2N_2K_1(K_2-N_2-\beta N_1))/(r_1N_1K_2(K_1-N_1-\alpha N_2)).$$

From this equation we can see that we can determine the slope of the trajectories. When $K_2 - N_2 - \beta N_1 = 0$, then $dN2/dN_1 = 0$; hence the trajectories have slope 0. When $K_1 - N_1 - \alpha N_2 = 0$, then dN_2/dN_1 is undefined,

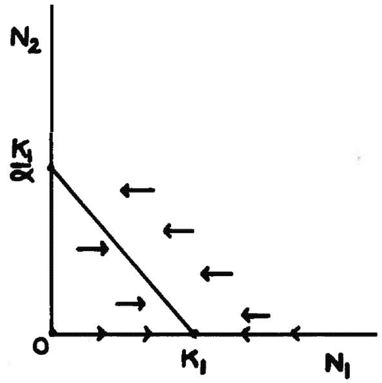

Figure 3

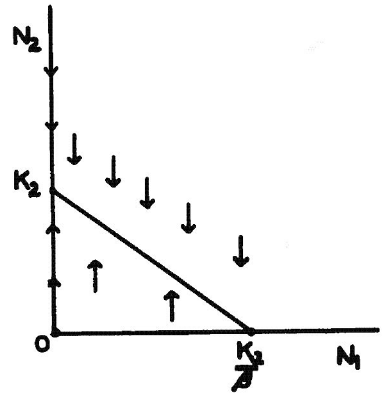

Figure 4

hence the trajectories have no slope. Since the slope is constant along these lines, they are called <u>isoclines</u>. Adding dashes to indicate slope we now represent in Fig. 5 the change of the N population, and in Fig. 6 the change of the N_2 population.

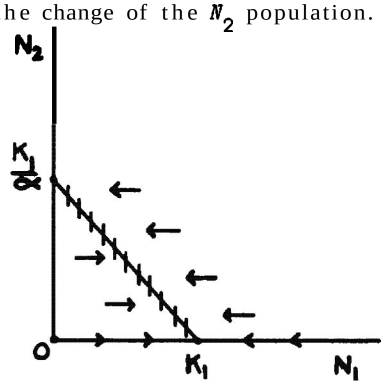

Figure 5

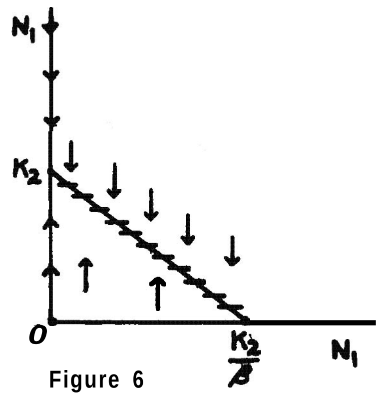

Figure 6

We are now ready to determine those values of a and β for which two species can coexist. There are four possible cases which are presented as (1), (2), (3) and (4) below [9, Chap. 7]:

(1) $$1/K_1 < \beta/K_2 \quad \text{and} \quad 1/K_2 > \alpha/K_1.$$

Species 1 inhibits its own grwoth rate less than it inhibits species 2 growth rate and species 2 inhibits its own growth rate more than it inhibits species 1 growth rate.

(2) $$1/K_1 > \beta/K_2 \quad \text{and} \quad 1/K_2 < \alpha/K_1.$$

The converse of Case 1, both species inhibit species 1 growth rate more

than they inhibit species 2 growth rate.

(3) $$1/K_1 < \beta/K_2 \quad \text{and} \quad 1/K_2 < \alpha/K_1.$$

Each species inhibits the other species growth rate more than its own.

(4) $$1/K_1 > \beta/K_2 \quad \text{and} \quad 1/K_2 > \alpha/K_1.$$

Each species inhibits its own growth rate more than that of the other species. According to the principle of competitive exclusion we would expect a stable coexistence between the species only with Case 4.

<u>*Case 1.*</u> We want $1/K_1 < \beta/K_2$ and $1/K_2 > \alpha/K_1$. For sketching the phase plane we restate these as $K_2/\beta < K_1$ and $K_2 < K_1/\alpha$. Plotting the N_1 and N_2 isoclines in Fig. 7, we note the direction each population must go in each region. Possible equilibrium populations are marked •. The solution curves for Case 1 are sketched in Fig. 8. Checking our equilibrium points for stability we see that (0,0) is unstable, since no solution stays close to it. The population $(0,K_2)$ is also unstable, since there are trajectories which pass arbitrarily close to, but do not stay close to $(0,K_2)$. But we see for the population $(k_1,0)$ that any initial population with both species competing eventually ends up at $(k_1,0)$. Hence, for Case 1, we have that if we begin with both species competing eventually species 2 will become extinct, and species 1 will be at its saturation population.

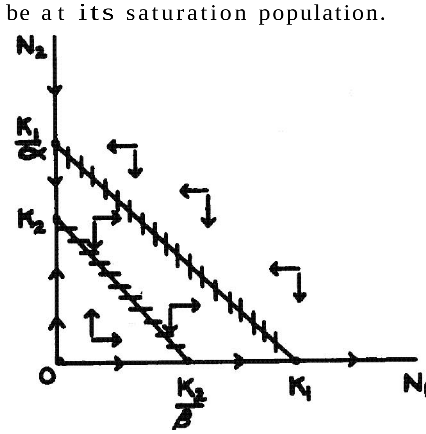

Figure 7

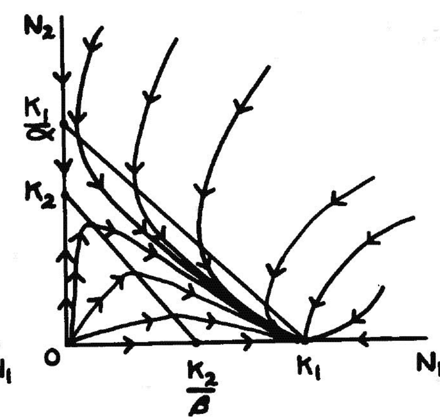

Figure 8

<u>*Case 2.*</u> We want $1/K_1 > \beta/K_2$ and $1/K_2 < \alpha/K_1$, or $K_1 < K_2/\beta$ and $K_1/\alpha < K_2$. This is very similar to Case 1 except that the N_2 isocline

is now above the N_1 isocline. The two isoclines and general direction of the trajectories are shown in Fig. 9. Figure 10 has the solution curves. Again equilibrium populations are marked with • . As we would expect, we have two unstable equilibrium populations, at (0,0) and $(K_1,0)$. The only stable equilibrium is at $(0,K_2)$. Hence, for Case 2, we have that for an initial population including both species, only species 2 will survive.

Notice that Cases 1 and 2 can be used to demonstrate that two nearly identical species cannot coexist. If the two species are very similar, an individual of either species has about the same inhibitory effect on each species growth rate. Hence α and β are close to 1. If we assume $\alpha = \beta = 1$, then species 1 and 2 are inhibited by $1/K_1$ and $1/K_2$ respectively for each member of either species added to the microcosm. Case 1 then reduces to $K_2 < K_1$, in which only species 1 survived and Case 2 reduces to $K_1 < K_2$, where only species 2 remained. Therefore, if two species are very similar then the one which has the greater saturation population in the microcosm will survive and the other will become extinct.

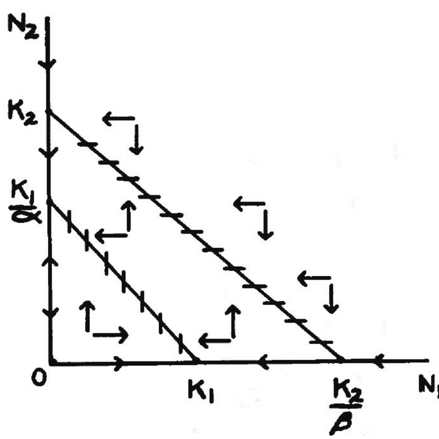

Figure 9

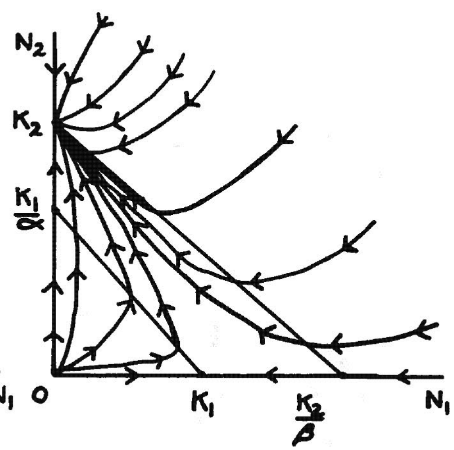

Figure 10

Case 3. We want $1/K_1 < \beta/K_2$ and $1/K_2 < \alpha/K_1$, which implies $K_2/\beta < K_1$ and $K_1/\alpha < K_2$. Plotting the N_1 and N_2 isoclines and direction arrows in Fig. 11, we notice the isoclines intersect, which implies another equilibrium population. The solution curves for Case 3 are sketched in Fig. 12. As in the previous two cases we see that the

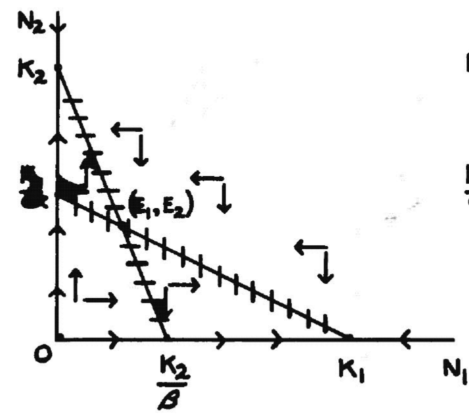

Figure 11

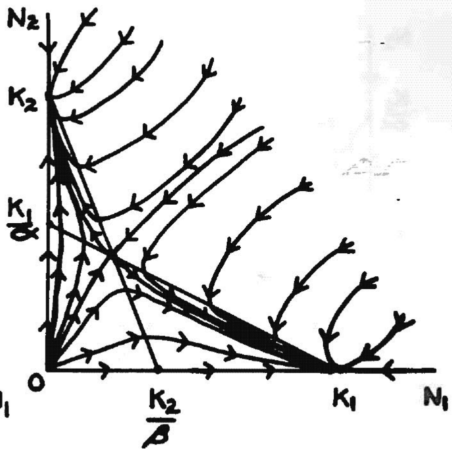

Figure 12

origin is again unstable. Looking at $(K_1,0)$ we see that all trajectories that get close to it approach it. Hence the equilibrium population $(K_1,0)$ is stable. Similarly we see that $(0,K_2)$ is also a stable equilibrium population. Let the intersection of the two isoclines be denoted by (E_1,E_2). We can see that the motion of some trajectories approach (E_1,E_2) while others move away from it. Hence the only equilibrium population with the species coexisting is unstable. Therefore, depending on the initial populations of the two species, only one species will survive.

Case 4. We want $1/K_1 > \beta/K_2$ and $1/K_2 > \alpha/K_1$, which says $K_1 < K_2/\beta$ and $K_2 < K_1/\alpha$. As in Case 3, we notice a fourth equilibrium population when we draw the N_1 and N_2 isoclines in the phase plane. We will represent this equilibrium population by (S_1, S_2). Again arrows indicate the change of each population in each region, as sketched in Fig. 13. Also the trajectories for Case 4 are sketched in Fig. 14. Again the origin is unstable. For the two equilibrium populations $(K_1,0)$ and $(0,K_2)$ we note that there are trajectories which move arbitrarily close to these populations but are in motion away from them. Hence both populations corresponding to the extinction of one species are unstable.

Examining the equilibrium population (S_1, S_2) we have all the trajectories near it moving toward it. Hence the only stable equilibrium population in this case involves the two species coexisting. Then regardless of the initial values of the two species (except not equal to 0), we have the populations tending toward a stable coexistence.

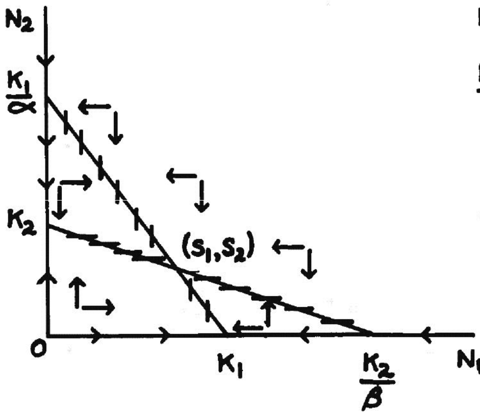

Figure 13

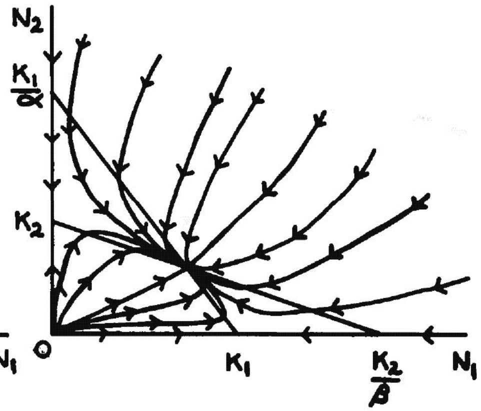

Figure 14

Hence we have shown that the only conditions under which both species survive is if each individual is more detrimental to its own species than to the other [4, p. 356].

We must realize that many innate and environmental factors are simplified or ignored in the derivation of Volterra's model. Because of this, the principle of competitive exclusion is virtually impossible to test empirically because the hypotheses are not met. For additional ideas in population theory see Levin's summary [8] including his extensive bibliography.

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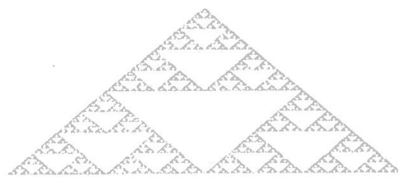
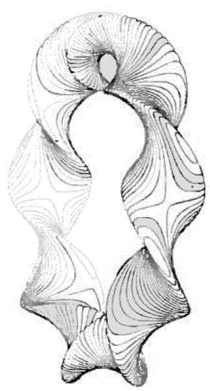
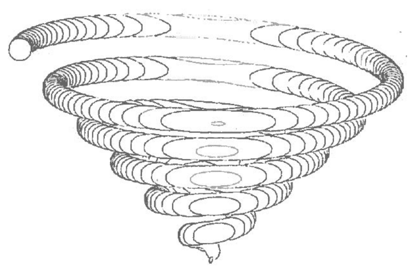
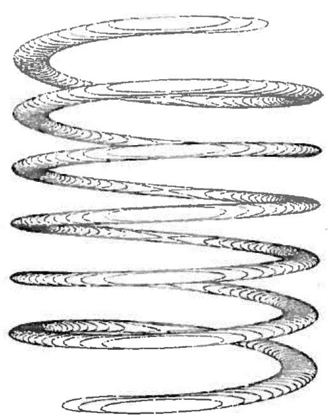
Computer Art

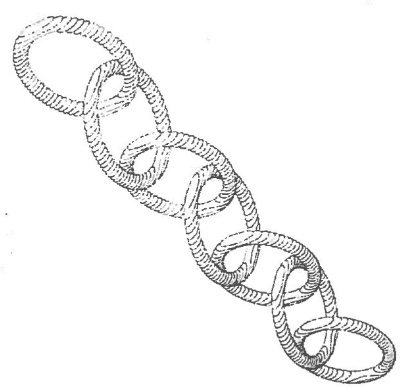
by Gheg Kline, Neal Thompson, David O'Connor
South Dakota School of Mines and Technology











PUZZLE SECTION

Edited by
David Ballew

This Department is for the enjoyment of those readers who are addicted to working crossword puzzles or who find an occasional mathematical puzzle attractive. We consider mathematical puzzles to be problems whose solutions consist of answers immediately recognizable as correct by simple observation and requiring little formal proof. Material submitted and not used here will be sent to the Problems Editor if deemed appropriate for that Department.

Address all proposed puzzles and puzzle solutions to Professor Joseph Konhauser, Department of Mathematics, Macalester College, St. Paul, Minnesota, 55105. Deadlines for puzzles appearing in the Fall Issue will be the next February 15, and for puzzles appearing in the Spring Issue will be the next September 15.

Mathacrostic No. 16

Submitted by Theodor Kaufman, M.D.
Nassau Hospital, Mineola, L. I., New York 11501

Like the preceding puzzles, this puzzle (on the following two pages) is a keyed anagram. The 207 letters to be entered in the diagram in the numbered spaces will be identical with those in the 25 keyed words at matching numbers, and the key numbers have been entered in the diagram to assist in constructing your solution. When completed, the initial letters will give a famous author and the title of his book; the diagram will be a quotation from that book. (See an example solution in the solutions section of this Department.)

DEFINITIONS — **WORDS**

A. a muscle which partially twists on its axis when tensing — 21 67 90 205 95 191 28

B. an accompanying part of semi-independent melodic character — 63 ⌣152⌣ 6 141 17 79 161 101

C. road; mosaic; beetle or plant — 125 129 198 111 159 134 53

D. giddy, volatile, heedless; this puzzle perhaps — 156 183 200 3 71 168 97 9 146 107 27

E. a possibility (comp.) — 148 100 164 106 144 73 133

F. exculpation — 22 179 24 7 126 105 99 147 192

G. a dwarf male in botany — 54 169 62 77 194 102 174 180 44

H. daisy; dam or mountain — 190 91 120 189 14 173

I. arch; crab; curve; kidney or footwear — 132 118 82 35 167 108 69 127 47

J. burial rites — 186 57 4 197 48 25 86 45 145

K. what Sammy Glick did — 34 29 19

L. "He that sleeps feels not the ___ " Shak - Cymbeline — 30 13 33 8 162 201 124 2 16

M. if you're serving, this is not too good (comp.) — 75 135 193 46 153 60

N. mutually destructive — 52 158 177 165 87 143 93 204 78 187 42

O. habituate — 137 96 113 84 49 18 39 66

P. John D. Rockefeller's grandchild perhaps — 136 ⌣56⌣ 76 65

Q. not regularly (3 wds.) — 10 130 119 154 123 40 103 80

R. detached mass of loosely fibrous structure like a shredded tuft of wool — 138 206 115 59 155 85 196 20

S. uniformly — 142 171 182 55 195 23 74

T. way and shark (2 wds.) — 43 112 92 176 184 64 15 31 1 70 89

U. passively compliant — 50 140 181 110 178 149 207 32 36 128 117

V. talkativeness — 170 116 109 122 26 98 203 68 151

W. the part of the Eucharistic Service just before the bread and wine are consecrated — 38 121 104 88 114 51 5 175 199

X. Q. How far is it to the nearest phone?
A. Oh, 1 - 2 miles up the road. — 139 166 83 58 72 37 131 188 12 172 41 157

Y. hypophosphatemia; hemophilia and rape (comp.) — 81 160 94 150 185 11 61 163 202

1 T	2 L	3 D		4 J	5 W	6 B	7 F	8 L	9 D	10 Q	11 Y		12 X
13 L		14 H	15 T	16 L		17 B	18 O	19 K	20 R	21 A	22 F	23 S	
24 F	25 J	26 V	27 D	28 A	29 K	30 L	31 T	32 U		33 L	34 K		35 I
36 U	37 X	38 W	39 O	40 Q	–	41 X	42 N	43 T	44 G	45 J	46 M		47 I
48 J	49 O	50 U	51 W	52 N	53 C	54 G		55 S	56 P	+	57 J	58 X	+
59 R	=	60 M		61 Y	62 G	63 B	64 T	65 P		66 O	67 A		68 V
69 I	70 T		71 D	72 X	73 E	74 S	75 M	76 P	77 G	78 N	79 B	80 Q	81 Y
	82 I	83 X	84 O	85 R	86 J	87 N	88 W	89 T		90 A	91 H	92 T	
93 N	94 Y	95 A	96 O	97 D	98 V	99 F	100 E	101 B	102 G		103 Q	104 W	
105 F	106 E	107 D		108 I	109 V	110 U	111 C	112 T	113 O		114 W	115 R	116 V
117 U		118 I	119 Q		120 H		121 W	122 V	123 Q	124 L	125 C	126 F	127 I
128 U		129 C	130 Q		131 X	132 I	133 E		134 C	135 M	136 P	137 O	138 R
139 X	140 U	141 B	142 S	143 N	144 E	145 J		146 D	147 F	148 E	149 U	150 Y	151 V
	152 B	–	153 M	154 Q	155 R		156 D	157 X	158 N	159 C	160 Y		161 B
162 L	163 Y		164 E	165 N	166 X	167 I	168 D	169 G	170 V		171 S	172 X	173 H
174 G	175 W	176 T	177 N	178 U	179 F		180 G	181 U	182 S	183 D	184 T	185 Y	186 J
187 N		188 X	189 H		190 H	191 A	192 F	193 M	194 G	195 S	196 R	197 J	
198 C	199 W		200 D	201 L	202 Y	203 V	204 N	205 A	206 R	207 U			

SOLUTIONS

Mathacrostic No. 15. (See Fall 1982 Issue) (*Proposed by Joseph D. E. Konhauser*)

Definitions and Key:

A. Bowditch
B. Uintaite
C. Horned sphere
D. Lotions
E. Effervesce
F. Rheotaxis
G. Graffito
H. Aliquot part
I. Unguent
J. Sawtooth wave
K. Sassafras
L. Attenuate
M. Brewster
N. Invertase
O. octahedron
P. God does not play dice
Q. Rhinitis
R. Atavism
S. Penrose staircase
T. Heath hen
U. Injective
V. Chebyshev
W. Arcturus
X. Lebesgue
Y. Snowflake curve
Z. Twistor
a. Unicorn
b. Dustbin
c. Yeti

First Letters: Buhler Gauss: A Biographical Study

Quotation: *H A attitude towards his students has been overshadowed by his reputation as a strict, even unfair, critic of the work of others. HA private judgments, particularly of colleagues, were often quite arbitrary and inconsistent. One perceives--sit venia verbo--the extravagance of the genius who cannot be sure what scales to use.*

Solved by: Jeaneatte Bickley, Webster Grove High School, Missouri; Rod Chaumont and Jim Stanfield, Offutt Air Force Base, Omaha; Charles R. Diminnie, St. Bonaventure University; Victor G. Feser, Mary College, Bismarck; Robert C. Gebhardt, New Jersey; Joel K. Haack, Oklahoma State University; Theodor Kaufman, Brooklyn; Roger Kuehl, Kansas City; Henry S. Lieberman, John Hancock Mutual; Eric C. Nummela, New England College; Bob Prielipp, University of Wisconsin-Oshkosh; Sister Stephanie Sloyan, Georgian Court College; Allan Tuchman, University of Illinois; The Proposer and The Editor.

PUZZLES FOR SOLUTION

1. *Proposed by Jobeph Konhauser, Macalester College, St. Paul, Minnesota.*

In the square array each letter represents one of the digits 0 through 9. Determine the correspondence, given that

1. ABC and CBD are primes,
2. BBC and CDF are perfect squares, and
3. ACE and ECF are perfect cubes.

A	B	C
C	B	D
E	C	F

2. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

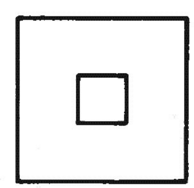
Locate eight points in a plane so that the perpendicular bisector of the line segment joining any two of the points passes through exactly two of the others.

3. *Proposed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

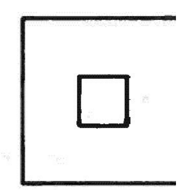
In the equal products 4 x NUMBER = 9 x BERNUM each letter represents one of the integers 0 through 9. Determine the correspondence.

4. *Phopobed by Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

The top and front views of a solid object are given. Draw the side view.



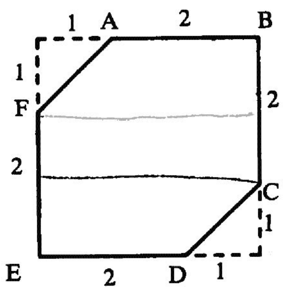
Top View



Front View

5. *Proposed bq Joseph Konhauser, Macalester College, St. Paul, Minnesota.*

Dissect the hexagon ABCDEF into three pieces which can be reassembled to form a square.



6. *Phopobed bq Jobeph Konhauser, Macalester College, St. Paul, Minnesota.*

Five unmarked opaque capsules contain equal amounts of sugar. A small amount of sugar is transferred from one of the capsules into another. Is it possible to isolate both the light and the heavy capsules using an uncalibrated equal-arm balance just three times?

7. *Proposed by Jobeph Konhauser, Macalester College, St. Paul, Minnesota.*

Cross out eleven of the integers from the array below so that no three of the remaining nine are in arithmetic progression.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

They said it couldn't be done
So he went right to it.
He took that thing that couldn't
be done ....
And couldn't do it.

CHAPTER REPORTS

<u>KANSAS GAMMA (Wichita State University)</u> heard *Prof. D. V. Chopra* speak on the *"History of Pi Mu Epsilon"; Prof. Monte Zerger* of Friends University on *"Mathematics Side Show"*; and *Prof. Jack R. Porter,* University of Kansas on *" Some Topological Concepts".*

<u>MINNESOTA ZETA (Saint Mary's College)</u> heard *"A Nonarchimedean Extension of R" by Gheg Force; "The MU-Puzzle: A Gateway to Artificial Intelligence" by Paul Froeschl; "Career Opportunities at National Security Agency" by Stan Hanson; "The Birthday Problem" by Dr. William A. Haltman; "Regenerative Simulation" bq Gerald Karel; "Conditional Probabilities and Paradoxes" by Peter Christenson; "Some Mathematics for Marathoning" by Vh. Richard Jarvinen; "Career Opportunities in Mathematics" by a representative from the Peach Corps; "An Introduction to Queuing Theory" by Gerald Stanczak;* and *"What is the Calculus of Variations?" by Michelle M. Kust.*

<u>MISSOURI GAMMA (St. Louis University)</u> presented **The James W. Garneau Mathematics Award** to *Leona Martens;* **The Francis Regan Scholarship** to *Eric Fiore;* **The Missouri Gamma Undergraduate Award** to *Mary Perry* and *Kim Wick;* **The Missouri Gamma Graduate Award** to *Martha Hasting;* **The John J. Andrews Graduate Service Award** to *Glen Wurglitz, Ch;* and **Beradino Family Fraternityship Award** to *James Goeke, S.J.* and *Martha Hastings.* The winners of **The Pi Mu Epsilon Contests** were *Yoshimura Nobuhiro (Senior Contest)* and *Sanjay Jain (Junior Contest). Mr. Robert Emnett* of McDonnell Douglas Automation Company presented a talk on *"Career Opportunities Involving Applied Mathematics"* and *Prof. Edward Spitznagel* gave the *"James E. Case, S.J. Memorial Lecture"* on *"Mathematics and the Law"* and *"The Use of Bayes' Rule in Paternity Testing. Prof. Edward Spitznagel* is from Washington University.

<u>NEW YORK PHI (State University College at Potsdam)</u> The winners of the **Pi Mu Epsilon Senior Award** were *Lydia P. Hardy* and *Nancq Offslager.* Winners are chosen by the members of the Chapter who have not yet graduated, on the basis of outstanding achievement in mathematics and for the promotion of scholarly activity in mathematics.

RING — GRIN
GROUP —
SYMMETRY —
INVERSE —

People Gones

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, Math Dept., University of Maim, Orono, ME 04469. Please submit each proposal and solution on a separate sheet (one side only) properly identified with m e and address. Solutions to problems in this issue should be mailed by December 15, 1983.

Problems for Solution

534. *Proposed by Charles W. Trigg, San Diego, California.*

Find the mathematical term that is the anagram of each of the following words and phrases: (1) RITES OF, (2) NILE GETS MEN, (3) PANTS GONE, (4) IRAN CLAD, (5) COVERT, (6) CLERIC, (7) GRABS ALE, (8) IRON LAD, (9) TRIED A VIVE, (10) HAG, NO SEX, (11) ALTERING, (12) RELATING.

PENTAGONS VECTOR CIRCLE DERIVATIVE HEXAGONS INTEGRAL TRIANGLE

535. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

In the small hamlet of Abacinia, two base systems are in common use. Also, everyone speaks the truth. One resident said, "26 people use my base, base 10, and only 22 people speak base 14." Another said, "Of the 25 residents, 13 are bilingual and 1 is illiterate." How many residents are there?

536. *Proposed by Martha Matticks, Veazie, Maine.*

A recent alphametric in *Crux Mathematicorum* [1982: 77, problem 721] asks one to show that, in base ten,

14

16 1L
11 17
12 18
13 19
14 110
15

TRIGG is three times WRONG.

In defense of the Dean of Numbers, solve these alphametrics independently of each other:

(a) TRIGG x 3 = RIGHT in base ten where the digit 3 can be reused.

(b) TRIGG = 3 x RIGHT in base ten where the digit 3 can be reused, and

(c) TRIGG x 7 = RIGHT in base seventeen.

537. *Proposed by Charles W. Trigg, San Diego, California.*

Find the unique four-digit integer in the decimal system that can be converted into its equivalent in the septenary system (base 7) by interchanging the left hand and the right hand digit pairs.

538. *Proposed by Emmanuel, O. C. Imonitie, Northwest Missouri State University, Maryville.*

The roots of $ax^2 + bx + c = 0$, where none of the coefficients a, b, and c is zero, are α and β. The roots of $a^2x^2 + b^2x + c^2 = 0$ are 2α and 2β. Show that the equation whose roots are $n\alpha$ and $n\beta$ is $x^2 + 2nx + 4n^2 = 0$.

539. *Proposed by Hao-Nhien Q. Vu, Purdue University, West Lafayette, Indiana.*

Find a quadratic equation with integral coefficients that has cos 72° and cos 144° as roots.

*Does there exist such a quadratic with roots sin 72° and sin 144°?

540. *Proposed by M. S. Klamkin, University of Alberta, Edmonton, Canada.*

If the radii r_1, r_2, r_3 of the three escribed circles of a given triangle $A_1A_2A_3$ satisfy the equation,

$$\left(\frac{r_1}{r_2} - 1\right)\left(\frac{r_1}{r_3} - 1\right) = 2,$$

determine which of the angles A_1, A_2, A_3 is the largest.

541. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

A line meets the boundary of an annulus A_1 (the ring between two concentric circles) in four points P, Q, R, S with R and S between P and Q. A second annulus A_2 is constructed by drawing circles on PQ and RS

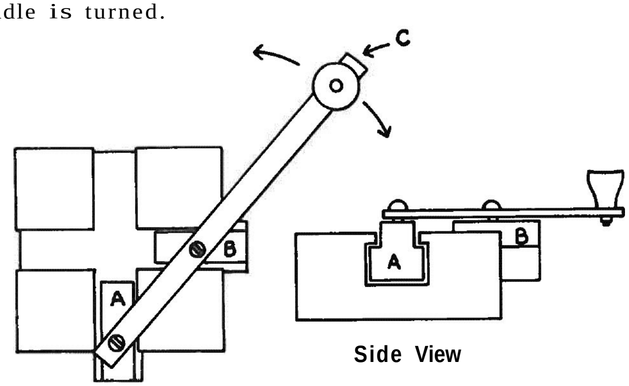
as diameters. Find the relationship between the areas of A_1 and A_2.

542. *Proposed by Herbert R. Bailey, Rose Polytechnic Institute, Terre Haute, Indiana.*

A circle of unit radius is to be covered by three circles of equal radii. Find the minimum radius required.

543. *Proposed by Dominic C. Milioto, Southeastern Louisiana University, Hammond.*

A linkage device, shown in the figure, consists of a wood block with two tracks cut perpendicular to one another and crossing at the center of the block. Riding within the tracks are two small skids A and B, joined together by a long handle. As the handle is turned, the skids move within their respective tracks: A up and down and B from side to side. Describe the curve generated by point C (at the end of the handle) as the handle is turned.



544. Proposed by Jack Garfunkel, Flushing, New York.

Show that a quadrilateral ABCD with sides AD = BC = s and $\measuredangle A + \measuredangle B = 120^o$ has maximum area if it is an isosceles trapezoid. A solution without calculus is preferred.

545. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let f_n denote the nth Fibonacci number ($f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$ for n a positive integer). Find a formula for f_{mn} in terms of f_m and f_n (only).

546. *Proposed by Robert C. Gebhardt, Parsippany, New Jersey.*

Show that the square of the sum of the squares of four integers can be expressed as the sum of the squares of three integers, as in $(2^2 + 3^2 + 4^2 + 5^2)^2 = 14^2 + 28^2 + 44^2$.

Solutions

510. [Spring 1982] *Proposed by Charles W. Trigg, San Diego, California.*

A hexagonal number has the form $2n^2 - n$. In base nine, show that the hexagonal number corresponding to an n that ends in 7 terminates in 11.

Editor's Comment. We, that is, I, goofed. This is problem 415, which appeared on Page 620 of the Spring 1979 issue, as pointed out by the proposer.

Solutions were submitted by WALTER BLUMBERG, PETER JOHN DOMBROWSKY, VICTOR G. FESER, JOHN M. HOWELL, HENRY S. LIEBERMAN, BOB PRIELIPP, STANLEY RABINOWITZ, DOUGLAS F. RALL, HARRY SEDINGER, WADE H. SHERARD, KEVIN THEALL, KENNETH M. WILKE, *and the* PROPOSER.

511. [Spring 1982] *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, New York.*

If $\alpha > 0$ and $\beta \ge 1$, prove that

(A) $\left(\frac{e}{\alpha}\right)^{\beta\alpha^\beta} \le e^{e^\beta-\alpha^\beta}$ and

(b) $\left(\frac{\alpha}{e}\right)^{\beta e^\beta} \le e^{\alpha^\beta-e^\beta}$

Solution by Walter Blumberg, Coral Gables, Florida.

Consider the well known inequality for all real x

(*) $$x \le e^{x-1},$$

with equality if and only if $x = 1$. The restriction $\beta \ge 1$ can be removed; that is, we let β be any real number. Now substitute $x = (e/\alpha)^\beta$ in inequality (*) and raise each side of the result to the α^β power. We get

$$[(e/\alpha)^\beta]^{\alpha^\beta} \le [e^{(e/\alpha)^\beta-1}]^{\alpha^\beta},$$

which yields part (A). To prove part (b) substitute $x = (\alpha/e)^\beta$ in (*) and raise each side of the result to the e^β power. In both (A) and (b) if $\beta = 0$, equality holds; if $\beta \ne 0$, then equality holds if and only if $\alpha = e$.

Also solved by PETER JOHN DOMBROWSKY, J. DOUGLAS FAIRES, DAVID INY, TIM KEARNS, ROBERT MEGGINSON, *and the* PROPOSERS.

512. [Spring 1982] *Proposed by Jack Garfunkel, Flushing, New York.*

Denote the number of ways a positive integer n can be partitioned into 3 positive integers by $P_3(n)$. Thus, for example, $P_3(7) = 4$, since we have

$1 + 1 + 5$, $1 + 2 + 4$, $1 + 3 + 3$, and $2 + 2 + 3$ each equal to 7.

Prove the following: If a, b, c are positive integers and $a^2 + b^2 = c^2$, then

$$P_3(a) + P_3(b) = P_3(c).$$

Solution by David Tng, Rensselaer Polytechnic Institute, Troy, New York.

Denote the number of ways a positive integer n can be partitioned into two positive integers by $P_2(n)$. Then $P_2(n) = [n/2]$, where the brackets indicate the greatest integer function. To find $P_3(n)$ we observe that there are $P_2(n-1)$ sums starting with the addend 1, $P_2(n-4)$ sums starting with the addend 2, and in general $P_2(n-3k+2)$ sums starting with the addend k. Then

$$P_3(n) = \sum_{k=1}^{[n/3]} P_2(n - 3k + 2).$$

By noting that the odd terms and the even terms form arithmetic progression we obtain

$$P_3(6n \pm 2) = 3n^2 \pm 2n, \quad P_3(6n \pm 1) = 3n^2 \pm n,$$

$$P_3(6n) = 3n^2, \text{ and } P_3(6n + 3) = 3n^2 + 3n + 1.$$

Therefore $P_3(k) = k^2/12 - 1/3$, or $= k^2/12$, or $= k^2/12 + 1/4$, so

$$\frac{a^2+b^2-c^2}{12} - \frac{1}{3} - \frac{1}{3} - \frac{1}{4} \leq P_3(a) + P_3(b) - P_3(c) \leq \frac{a^2+b^2-c^2}{12} + \frac{1}{4} + \frac{1}{4} + \frac{1}{3}.$$

Given that $a^2 + b^2 = c^2$, this inequality reduces to

$$-\frac{11}{12} \leq P_3(a) + P_3(b) - P_3(c) \leq \frac{10}{12}.$$

Since $P_3(n)$ is an integer for each n, it follows that

$$P_3(a) + P_3(b) = P_3(c).$$

This method of proof can be extended to establish the following

generalizations:

1) If $|a^2 + b^2 - c^2| > 12$, then $P_3(a) + P_3(b) \neq P_3(c)$;

2) If $a^2 + b^2 = c^2 + 1$, then $P_3(a) + P_3(b) = P_3(c)$;

3) If a, b, c, d are all even positive integers and if $(a^3 + 3a^2) + (b^3 + 3b^2) + (c^3 + 3c^2) = (d^3 + 3d^2)$, then $P_4(a) + P_4(b) + P_4(c) = P_4(d)$; and

4) If a, b, c, d are all odd positive integers and if

$$(a^3 + 3a^2 - 32a) + (b^3 + 3b^2 - 32b) + (c^3 + 3c^2 - 32c) = (d^3 + 3d^2 - 32d), \text{ then}$$

$$P_4(a) + P_4(b) + P_4(c) = P_4(d).$$

Generalization (1) states that (a, b, c) has to be "close" to a Pythagorean triple for the desired equality to hold; generalization (2) shows such a case. It would be interesting to see what patterns are found for $P_5(n)$, $P_6(n)$, and so forth.

Also solved by WALTER BLUMBERG, JOHN OMAN *and* BOB PRIELIPP (*jointly*), *and the* PROPOSER.

513. [Spring 1982] *Proposed by Ronald E. Shiffler, Georgia State University, Atlanta, Georgia.*

Our old friend, Prof. Euclide Pasquale Bombasto Umbugio, eminent retired numerologist from Guayazuela, has been delving into statistics of late in an effort to prove that his retirement salary is so laughably low that he should be given food stamps in addition to his good conduct pass to the 1986 baton twirlers semifinals. He has checked several distributions involving real numbers and in every case the average deviation $(a.d.)$ is less than or equal to the standard deviation σ, where

$$a.d. = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}| \quad \text{and} \quad \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

Of course, $\bar{x}$ is the data mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

He conjectures that $a.d. \leq \sigma$ is always true. Help the professor to prove his conjecture.

Solution by ***Tim Kearns, Catharpin, Virginia***

Let $c_i = |x_i - \bar{x}|$. Then $\sum_{i=1}^{n} c_i = n\bar{c}$. Also $a.d. \le \sigma$ if and only if $(a.d.)^2 \le \sigma^2$, which is true if and only if

$$\frac{1}{n^2}\left(\sum_{i=1}^{n} c_i\right)^2 \le \frac{1}{n}\sum_{i=1}^{n} c_i^2,$$

$$(n^2\,\bar{c}^2) \le n\sum_{i=1}^{n} c_i^2,$$

$$n\sum_{i=1}^{n} c_i^2 - n^2\,\bar{c}^2 \ge 0, \qquad \text{or} \qquad n\sum_{i=1}^{n} (c_i - \bar{c})^2 \ge 0,$$

which is, of course, always true. The last two forms are equivalent expressions for $n^2\sigma^2$, as given in any elementary statistics text.

Also solved by **WALTER BLUMBERG, PETER JOHN DOMBROWSKY, MARK EVANS, DAVID INY, ROBERT E. LaBARRE, DOUG MATLOCK, BOB PRIELIPP, KEVIN THEALL,** *and the* **PROPOSER.**

Given a set of positive numbers, it is well known that their harmonic mean $\le$ their geometric mean $\le$ their arithmetic mean $\le$ their root-mean-square. The November 1977 *Mathematics Magazine* (vol. 50, p. 277) shows these inequalities for two numbers in a concise geometric figure. This problem demonstrates the last of these inequalities.

514. [Spring 1982] *Proposed by* *Raymond E. Spaulding,* ***Radford University, Radford, Virginia.***

Let $A_1A_2A_3 \ldots A_n$ be a regular polygon where $A_{n+j} = A_j$ and $A_iA_{i+1} = 1$. Let B_i be a point on the segment A_iA_{i+1} where $A_iB_i = x$. Let C_i be the point where A_iB_{i+1} intersects $A_{i+1}B_{i+2}$. Find the area of a regular polygon $C_1C_2C_3 \ldots C_n$ in terms of n and x.

Solution by ***Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.***

Let $A_2B_3 = k$, $A_3C_2 = A_2C_1 = y$, and $C_2B_3 = z$. By symmetry, $\angle B_3A_2A_3 = \angle C_2A_3B_3$, so triangles $B_3A_3A_2$ and $B_3C_2A_3$ are similar (by angle-angle). Thus

$$\frac{x}{1} = \frac{z}{y} \text{ and } \frac{k}{1} = \frac{x}{y}, \qquad \text{so} \qquad y = \frac{x}{k} \text{ and } z = \frac{x^2}{k}$$

and

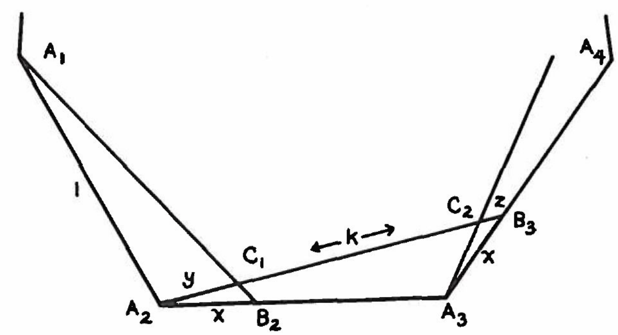
$$C_1C_2 = A_2B_3 - A_2C_1 - C_2B_3 = k - \frac{x}{k} - \frac{x^2}{k} = \frac{k^2 - x - x^2}{k}$$

Applying the law of cosines to triangle $B_3A_2A_3$, we find that

$$k^2 = 1 + x^2 + 2x\cos\theta$$

where $\theta = 2\pi/n$ is the exterior angle of the regular n-gon. The area of a regular n-gon of side s is $ns^2/(4\tan(\theta/2))$, so the area of the regular n-gon with side C_1C_2 is, where $\theta = 2\pi/n$,

$$\frac{n(k^2 - x - x^2)^2}{4k^2\tan(\theta/2)} = \frac{n(1 + 2x\cos\theta - x)^2}{4(1 + 2x\cos\theta + x^2)\tan(\theta/2)}$$



Also solved by **WALTER BLUMBERG** *and the* **PROPOSER.**

*515. [Spring 1982] *Proposed by Jack* ***Garfunkel, Flushing, New York.***

Given a sequence of concentric circles with a triangle ABC circumscribing the outermost circle. Tangent lines are drawn from each vertex of ABC to the next inner circle, forming the sides of triangle A', B', C'. Tangents are now drawn from vertices A', B', C' to the next inner circle and they are the sides of triangle A'', B'', C'', and so on. Prove that the angles of triangle $A^{(n)}B^{(n)}C^{(n)}$ approach $\pi/3$.

1. *Disproof by David* ***Iny, Rensselaer Polytechnic Institute, Troy, New York.***

It is clear that the angles of triangle $A^{(n+1)}B^{(n+1)}C^{(n+1)}$ can be made arbitrarily close to those of triangle $A^{(n)}B^{(n)}C^{(n)}$ by choosing the difference in the radii of the appropriate circles small enough. Thus it is possible to choose a sequence of decreasing concentric circles such that the angles of successive triangles never vary from those of triangle

ABC by more than any given $\varepsilon > 0$.

2. *Proof by Morris Katz, Macwahoc, Maine.*

Let the concentric circles have center I, let S and T be the points of tangency of sides AB and $A'B'$ with their respective incircles and let those circles have radii R and r respectively, as shown in the figure. Now

$$\sin(A/2) = \sin SAI = R/AI \text{ and } \sin TAI = r/AI,$$

$$\frac{\angle TAI}{A/2} < \frac{\sin TAI}{\sin(A/2)} = \frac{r}{R}$$

since the sine function is increasing and concave downward in the range from 0 to $\pi/2$. Similarly

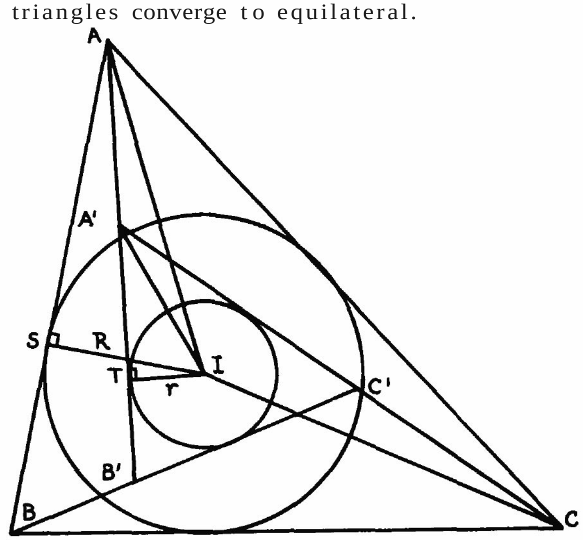
$$\frac{\angle A'CI}{C/2} < \frac{r}{R}.$$

The closer r/R is to 1, the more nearly equal these fractions are. From triangle $A'AC$ we have that $\angle B'A'C' = \angle A'AC + \angle A'CA$, so that

$$\angle A' = \angle B'A'C' = \angle A'AI + \angle IAC + \angle ICA - \angle ICA'$$

$$\approx \frac{r}{R}\cdot\frac{A}{2} + \frac{A}{2} + \frac{C}{2} - \frac{r}{R}\cdot\frac{C}{2} = \frac{R+r}{R}\cdot\frac{A}{2} + \frac{R-r}{R}\cdot\frac{C}{2}.$$

This sequence converges provided $r/R < u < 1$ for some fixed u. But then the sequence of radii converges to 0. Conversely, if the radii converge to 0, then the triangles converge to equilateral.



516. [Spring 1982] *Proposed by J. L. Brenner, Palo Alto, California.*

Prove, for a, b, c positive, that $\frac{1}{3}(a+b+c) \geq \sqrt{\frac{1}{3}(ab+bc+ca)}$ with equality if and only if $a = b = c$. Does this generalize to

$$\frac{1}{4}(a+b+c+d) \geq \sqrt{\frac{1}{4}(ab+bc+cd+da)}\ ?$$

Solution by M. S. Klamkin, University of Alberta, Edmonton, Canada.

The first inequality is known. By squaring, it reduces to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0.$$

More generally,

$$T_1/\binom{n}{1} \geq \{T_2/\binom{n}{2}\}^{1/2} \geq \{T_3/\binom{n}{3}\}^{1/3} \geq \dots \geq \{T_n/\binom{n}{n}\}^{1/n}$$

where the T_r are the elementary symmetric functions, i.e.

$$(x+a_1)(x+a_2)\dots(x+a_n) \equiv x^n + T_1x^{n-1} + T_2x^{n-2} + \dots + T_n.$$

The above are the Maclaurin inequalities, the first of which gives the extension of the first proposed inequality to n variables. For four variables it states

$$\frac{a+b+c+d}{4} \geq \sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}}$$

The second inequality is valid for all a, b, c, $d \geq 0$, for by squaring, it reduces to

$$(a-b+c-d)^2 \geq 0.$$

More generally, if $x_i \geq 0$ and $n \geq 4$, then we have

$$(x_1 + x_2 + \dots + x_n)^2 \geq 4(x_1x_2 + x_2x_3 + \dots + x_nx_1),$$

which can be proved by Mathematical induction.

Also solved by WALTER BLUMBERG, LOUIS H. CAIROLI, PETER JOHN DOMBROWSKY, MARK EVANS, VICTOR G. FESER, TAGHI REZAY GARACANI, DAVID INY, RALPH KING, HENRY S. LIEBERMAN, DOUG MATLOCK, BOB PRIELIPP, STANLEY RABINOWITZ, HARRY SEDINGER, KEVIN THEALL, *and* the PROPOSER.

517. [Spring 1982] *Proposed by Charles W. Trigg, San Diego, California.*

The nine non-zero digits are arranged to form three three-digit primes with a sum that is divisible by 11. Find the primes and their sum.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

There are 83 three-digit primes that have distinct non-zero digits. Since primes greater than 5 terminate only in the digits 1, 3, 7, or 9, any primes that contain three of these digits can be eliminated. The remaining primes can be listed in columns headed by $11k + n$ for $n = 1, 2, \ldots, 10$, that is, according to their remainders when divided by 11. Using these lists, a hand calculator, the fact that $1 + 2 + \ldots + 9 = 45$, eighteen solutions to the given problem were found. As a check a BASIC program was constructed and run, yielding the same results:

primes			sum	primes			sum
683	947	251	1881	683	257	149	1089
683	257	941	1881	683	521	479	1683
947	653	281	1881	431	587	269	1287
563	827	491	1881	827	461	593	1881
641	257	389	1287	641	257	983	1881
389	251	647	1287	587	239	461	1287
983	251	647	1881	467	821	593	1881
281	347	659	1287	281	743	659	1683
281	359	647	1287	281	953	647	1881

Also solved by WALTER BLUMBERG, ROBERT C. GEBHARDT, *and* STANLEY RABINOWITZ. *Partial solutions were found by* LOUIS H. CAIROLI (*1 solution*), VICTOR G. FESER (*12*), DAVID INY (*17*), HENRY S. LIEBERMAN (*7*), KEVIN THEALL (*1*), KENNETH M. WILKE (*14*), *and the* PROPOSER (*8*).

518. [Spring 1982] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

A baseball player gets a hit and observes that his batting average rises by exactly 10 points, i.e., by .010, and no rounding is necessary at all, where batting average is ratio of number of hits to times at bat (excluding walks, etc.). If this is not the player's first hit, how many hits does he now have?

Solution by Robert E. LaBarre, United Technologies Research Center, East Hartford, Connecticut.

Let y and x be the numbers of current hits and at bats, respectively. Then we have

$$\frac{y}{x} = \frac{y-1}{x-1} + .01,$$

which reduces to

$$y = -.01x^2 + 1.01x.$$

The graph of this equation is a parabola with vertex at (50.5, 25.5025) and re-intercepts at (0, 0) and (101, 0). Since it is symmetric about the line $x = 50.5$, we need try only $x = 1, 2, \ldots, 50$ and then use symmetry to find a second solution. Using a personal computer to perform the drudgery, we find (25, 19) is a solution, so (76, 19) is the second solution. Thus, in either case, he now has 19 hits.

Also solved by JEANETTE BICKLEY, WALTER BLUMBERG, MARTIN BROWN, LOUIS H. CAIROLI, PETER JOHN DOMBROWSKY, MARK EVANS, VICTOR G. FESER, ROBERT C. GEBHARDT, JOHN M. HOWELL, DAVID INY, RALPH KING, HENRY S. LIEBERMAN, BOB PRIELIPP, STANLEY RABINOWITZ, DOUGLAS F. RALL, HARRY SEDINGER, KEVIN THEALL, CHARLES W. TRIGG, TIMMY TUCKER, KENNETH M. WILKE, *and the* PROPOSER.

519. [Spring 1982] *Proposed by Charles W. Trigg, San Diego, California.*

Solve the equation

$$3^{2x} - (34)15^{x-1} + 5^{2x} = 0.$$

Solution by Peter John Dombrowsky, University of Texas, Austin.

We have

$$3^{2x} - \frac{34}{15} \cdot 3^x \cdot 5^x + 5^{2x} = 0,$$

which factors into

$$(3^x - \frac{5}{3} \cdot 5^x)(3^x - \frac{3}{5} \cdot 5^x) = 0.$$

Then we have

$$3^{x+1} - 5^{x+1} = 0 \quad \text{and} \quad 3^{x-1} - 5^{x-1} = 0.$$

Now $3^n = 5^n$ has a solution only when $n = 0$. Hence $x = \pm 1$.

Also solved by JEANETTE BICKLEY, WALTER BLUMBERG, LOUIS H. CAIROLI, FREDERICK C. DAY, MARK EVANS, J. DOUGLAS FAIRES, VICTOR G. FESER, ROBERT C. GEBHARDT, EMMANUEL, O.C. IMONITIE, DAVID INY, TIM KEARNS, RALPH KING, JEAN LANE, HENRY S. LIEBERMAN, D.C. MILIOTO, BOB PRIELIPP, JOHN PUTZ, STANLEY RABINOWITZ, HARRY SEDINGER, WADE H. SHERARD, KEVIN THEALL, HAO-NHIEN Q.VU, KENNETH M. WILKE, *and the* PROPOSER.

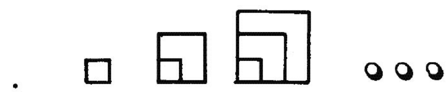
520. [Spring 1982] *Proposed by Chuck Allison, Huntington Beach, California.*

The following diagrams describe the first few polygonal or k-gonal numbers:

Triangular; $k = 3$: $P(n, 3) = \frac{n(n+1)}{2}$

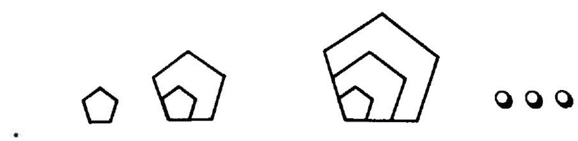
1 3 6 10 15 21 ...

Square; $k = 4$: $P(n, 4) = n^2$



1 4 9 16 25 36 ...

Pentagonal; $k = 5$: $P(n, 5) = \frac{n(3n-1)}{2}$



1 5 12 22 35 51 ...

where the numbers represent the number of dots shown, and each figure is an extension of its predecessor. The nth number of each sequence is given by the above formulas. Find a general formula for the nth k-gonal number $P(n, k)$.

Solution by Harry Sedinger, St. Bonaventure University, New York.

The k-gonal numbers are the partial sums of the series

$$1 + a_2 + a + \cdots,$$

where $a_m = a_{m-1} + k - 2$. Thus the nth k-gonal number is

$$1 + (1 + k - 2) + (1 + 2(k-2)) + \cdots + (1 + (n-1)(k-2))$$
$$= n + (k-2)(1 + 2 + \cdots + (n-1))$$
$$= n + (k-2)\frac{(n-1)n}{2} = \frac{n}{2}[(n-1)(k-2) + 2].$$

Also solved by LOUIS H. CAIROLI, PETER JOHN DOMBROWSKY, MARK EVANS, VICTOR G. FESER, JOHN M. HOWELL, ROBERT E. LaBARRE, HENRY S. LIEBERMAN, BOB PRIELIPP, JOHN PUTZ, STANLEY RABINOWITZ, KEVIN THEALL, KENNETH M. WILKE (2 *solutions*), *and the* PROPOSER.

521. [Spring 1982] *Proposed by Morris Katz, Macwahoc, Maine.*

I was told, when I first saw that alphametic, that a particular value for K produced a unique solution, but I have forgotten what that value is. So find the unique solution where *DAILY* is prime.

$$\begin{array}{r} WE \\ DO \\ WEE \\ \underline{WORK} \\ DAILY \end{array}$$

1. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

The given alphametic yields the following relations:

$$2E + O + K = Y + 10c_1, \quad c_1 + W + D + E + R = L + 10c_2,$$
$$c_2 + W + O = I + 10c_3, \quad c_3 + W = A + c_4, \quad c_4 = D.$$

Hence we have $D = 1$, $A = 0$, $W = 9$, $Y = 3$ or 7, and the third relation implies $c_2 = 2$ and $O = I - 1$. Now trial of some 22 cases produces the unique solution

$$\begin{array}{r} 96 \\ 13 \\ 966 \\ \underline{9382} \\ 10457 \end{array}.$$

2. *Comment by Charles W. Trigg, San Diego, California.*

The unique solution with an odd *DAILY* is that given above. Similar searches for an even *DAILY* yield two solutions:

$$\begin{array}{r} 97 \\ 12 \\ 977 \\ \underline{9268} \\ 10354 \end{array} \quad \text{and} \quad \begin{array}{r} 97 \\ 14 \\ 977 \\ \underline{9438} \\ 10526 \end{array}.$$

Each of these could be made unique by imposing on the first the condition that the units' digit of *DAILY* is also its digital root; and on the second, that the sum of the extreme digits of *DAILY* equals the sum of its three other digits.

We note that there is a unique solution with $K = 2$, two solutions with $K = 8$, and no other solutions with other values of K.

Also solved by WALTER BLUMBERG, VICTOR G. FESER, DAVID INY, CHARLES W. TRIGG, *and the* PROPOSER.

During 1983-84, we will continue our National Paper Competition. Every paper written by an undergraduate or a graduate student who has not received a Master's Degree at the tine of submission is eligible. The winners for 1981-82 are:

FIRST PRIZE ($200)	Bradley Strand, "Vector Subspaces of Magic Squares", Department of Mathematics, Carlton College, Northfield, MN, 55057 (See the Fall 1982 Issue)
SECOND PRIZE ($100)	Karen Cunningham, "A Simple Model for Two Interacting Species and the Principle Competitive Exclusion", Univ. of Texas at Arlington (See this Issue of the Journal)
THIRD PRIZE ($50)	Ravi Salgia, "Volume of an N-Dimensional Unit Sphere", Loyola University/Chicago, (See this Issue of the Journal)

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