EDITORIAL

Václav Linek

It has come to our attention that some problems appearing in ***CRUX with MAYHEM*** have been submitted to other places; in particular some ***CRUX with MAYHEM*** problems have appeared on certain problem-solving websites. While the trend of online problem solving is well established and quite popular, it is nevertheless still the case that ***CRUX with MAYHEM*** is a *traditional print journal*. This means that submissions to ***CRUX with MAYHEM*** should not be submitted elsewhere and should not have already appeared elsewhere (although some rare exceptions may be permitted for older problems that may not be that well known today). This includes problem proposals, solutions, and articles: if you submit them to ***CRUX with MAYHEM***, then we ask that you do not submit them anywhere else!

I would like to point out that material that is submitted to ***CRUX with MAYHEM*** receives the attention of the Editorial Board, whereas many online sources do not provide this kind of attention. For instance, problems and solutions featured in ***CRUX with MAYHEM*** are edited and typeset with care, and we do our best to filter the problem proposals we receive.

Sometimes it takes a while to process your material. For example, we have a backlog of problem proposals to process because only 100 four-digit problems are published per year, whereas we receive considerably more than 100 problem proposals a year. If the wait is too long, then (if we have not already used your problem) you can inform us that you would like to submit your problem elsewhere and we will then remove it from the queue. The same applies for other materials.

If it comes to our attention that some duplication has occurred, then we may not publish the duplicated material.

Having said this, it is obvious that our journal has changed with time. Indeed, "***Crux***" was originally founded as ***Eureka*** by Léo Sauvé in 1975, and it did not include book reviews, contributor profiles, or ***Mayhem*** at that time, and it was certainly not posted on the internet because the internet did not exist then! Moving forward, it is inevitable that ***CRUX with MAYHEM*** will change yet again.

Finally, I am happy to report that work has already started on the Jim Totten special issue slated for September. I thank you, the readers, for your patience with the delays of the last eight months. With time available for preparation this summer (unlike last summer) circumstances are favourable for autumn to go according to schedule.

SKOLIAD No. 117

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by ***1 November, 2009***. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Calgary Mathematical Association Junior High School Mathematics Contest, Part B, 2008. Our thanks go to Joanne Canape of the University of Calgary, Calgary, Alberta, for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, University College of St. Boniface, Winnipeg, MB for translating this contest.

Concours de l'Association mathématique de Calgary Niveau présecondaire

Ronde finale, partie B, 2008

1. Richard a besoin de se rendre en taxi de sa maison à un parc pas très loin. Deux compagnies de taxi offrent leurs services. La première facture ses clients à un taux fixe de 10,00$, auquel s'ajoute un taux variable de 0,50$ pour chaque kilomètre du trajet, tandis que la deuxième a un taux fixe de 4,00$ et un taux variable de 0,80$ le kilomètre. Richard constate que le coût est le même, quelle que soit la compagnie choisie. Quelle est la distance en kilomètres de sa maison au parc ?

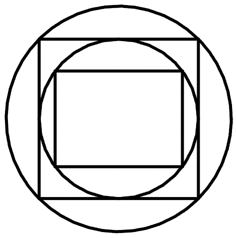
2. Un poste de radio lance un concours où chacun des gagnants pourra assister à deux matchs des Canadiens de Montréal, puis d'y amener un ami à chacun de ces deux matchs, soit un ami aux deux matchs, soit deux amis, chacun à un match différent. La chance a fait que les amis Alice, Bertrand, Carole, David et Evelyne ont tous été déclarés gagnants au concours. Montrer comment chacun de ces gagnants peut choisir ses amis parmi le groupe, de façon à ce que chaque paire d'amis parmi les cinq assiste à au moins un match ensemble.

3. Deux tests sont administrés à un groupe d'étudiants. Chaque étudiant reçoit un score pour chacun des deux tests, ce score étant un entier non négatif au plus égal à 10. Adrien remarque que, pour le premier test, seulement un étudiant a reçu un score plus élevé que lui et que personne n'a égalé son score ; il en est de même pour le deuxième test. Après que le professeur a affiché les scores moyens des deux tests, Adrien constate qu'il y a plus qu'un étudiant avec un score moyen plus élevé que le sien.

(a) Donner un exemple concret, incluant tous les scores, pour montrer que ceci est possible.

(b) Quel est le plus grand nombre d'étudiants pouvant avoir un score moyen plus élevé que celui d'Adrien ? Expliquer clairement pourquoi votre réponse est correcte.

4. On commence par tracer un rectangle de taille 6 cm par 8 cm. Ensuite, on trace le cercle circonscrit de ce rectangle, puis un carré circonscrit autour du cercle. Enfin, on trace le cercle circonscrit du carré tout juste construit. Déterminer la surface de ce dernier cercle en cm^2.



5. Soit A un entier positif à deux chiffres décimaux, ne contenant aucun zéro et soit B un entier positif à trois chiffres décimaux. Si $A\%$ de B donne 400, déterminer toutes les valeurs possibles de A et B.

6. Déterminer un rectangle ayant les deux propriétés suivantes : (i) son périmètre est un entier impair ; et (ii) aucun de ses côtés n'est entier.

Déterminer maintenant un rectangle ayant les deux propriétés suivantes : (i) sa surface est un entier pair ; et (ii) aucun de ses côtés n'est entier.

Enfin, déterminer un quadrilatère, pas nécessairement rectangulaire, ayant les trois propriétés suivantes : (i) son périmètre est un entier positif ; (ii) sa surface est un entier positif ; et (iii) aucun de ses côtés n'est entier.

Calgary Mathematical Association Junior High School Mathematics Contest Final Round, Part B, 2008

1. Richard needs to go from his house to the park by taking a taxi. There are two taxi companies available. The first taxi company charges an initial cost of \$10.00, plus \$0.50 for each kilometre travelled. The second taxi company charges an initial cost of \$4.00, plus \$0.80 for each kilometre travelled. Richard realises that the cost to go to the park is the same regardless of which taxi company he chooses. What is the distance in km from his house to the park?

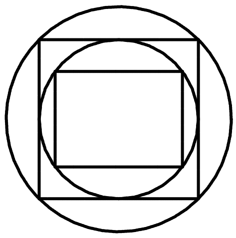
2. A radio station runs a contest in which each winner will get to attend two Calgary Flames playoff games and to take one guest to each game. The winner does not have to take the same guest to the two games. Luckily, five school friends Alice, Bob, Carol, David, and Eva are all winners of this contest. Show how each winner can choose two others from this group to be his or her guests, so that each pair of the five friends gets to go to at least one playoff game together.

3. A class was given two tests. In each test each student was given a non-negative integer score with a maximum possible score of 10. Adrian noticed that in each test, only one student scored higher than he did and nobody got the same score as he did. But then the teacher posted the averages of the two scores for each student, and now there was more than one student with an average score higher than Adrian.

(a) Give an example (using exact scores) to show that this could happen.

(b) What is the largest possible number of students whose average score could be higher than Adrian's average score? Explain clearly why your answer is correct.

4. A rectangle with dimensions 6 cm by 8 cm is drawn. A circle is drawn circumscribing this rectangle. A square is drawn circumscribing this circle. A second circle is drawn that circumscribes this square. What is the area in cm^2 of the bigger circle?



5. If A is a two-digit positive integer that does not contain zero as a digit, B is a three-digit positive integer, and $A\%$ of B is 400, find all possible values of A and B.

6. Find a rectangle with the following two properties: (i) its perimeter is an odd integer; and (ii) none of its sides is an integer.

Next, find a rectangle with the following two properties: (i) its area is an even integer; and (ii) none of its sides is an integer.

Finally, find a quadrilateral (not necessarily a rectangle) with the following three properties: (i) its perimeter is a positive integer; (ii) its area is a positive integer; and (iii) none of its sides is an integer.

We now give solutions to the selected questions of the 2007 Christopher Newport University Mathematics Contest in Skoliad 111 [2008 : 257–259].

1. Find the midpoint of the domain of the function $f(x) = \sqrt{4 - \sqrt{2x+5}}$.

(A) $\frac{1}{4}$ (B) $\frac{3}{2}$ (C) $\frac{2}{3}$ (D) $\frac{-2}{5}$

Solution by an unknown solver.

The domain of the function f is the set of real numbers, x, such that both $4 - \sqrt{2x+5} \geq 0$ and $2x + 5 \geq 0$. The latter inequality is the same as $2x \geq -5$, hence $x \geq -\frac{5}{2}$.

The first inequality is the same as $4 \geq \sqrt{2x+5}$, so $16 \geq 2x+5$. Thus, $11 \geq 2x$ and $\frac{11}{2} \geq x$. Hence the domain of f is the interval $\left[-\frac{5}{2}, \frac{11}{2}\right]$. The midpoint of this interval is $\frac{1}{2}\left(-\frac{5}{2}+\frac{11}{2}\right) = \frac{3}{2}$, so the answer is (B).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

2. The sum $a+b$, the product ab, and the difference a^2-b^2 for two positive numbers a and b is the same nonzero number. What is b?

(A) 2 (B) $\dfrac{1+\sqrt{5}}{2}$ (C) $\sqrt{5}$ (D) $\dfrac{3-\sqrt{5}}{3}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

If $ab = a+b$ and $ab = a^2-b^2$, then

$$1 \;=\; \frac{ab}{ab} \;=\; \frac{a^2-b^2}{a+b} \;=\; \frac{(a-b)(a+b)}{a+b} \;=\; a-b\,,$$

hence $a = b+1$. Since $ab = a+b$, it follows that $(b+1)b = (b+1)+b$, so $b^2-b-1=0$. The quadratic formula yields that $b = \dfrac{1\pm\sqrt{5}}{2}$. Since b is given to be positive, $b = \dfrac{1+\sqrt{5}}{2}$, and the answer is (B).

Also solved by LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

3. Let $f(x)$ be a quadratic polynomial with $f(3) = 15$ and $f(-3) = 9$. Find the coefficient of x in $f(x)$.

(A) 2 (B) 3 (C) 1 (D) -2

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Since $f(x)$ is a quadratic polynomial, $f(x) = ax^2+bx+c$ for some coefficients a, b, and c. Now

$$\begin{array}{rcccl} 15 & = & f(3) & = & 9a+3b+c \\ 9 & = & f(-3) & = & 9a-3b+c \end{array}$$

Subtracting the second equation from the first yields $6 = 6b$, so $b = 1$, and the answer is (C).

Also solved by LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON. One correct solution was submitted by an unknown solver.

4. A pair of fair dice is cast. What is the probability that the sum of the numbers falling uppermost is 7 or 11 if it is known that one of the numbers is a 5?

(A) $\frac{2}{9}$ (B) $\frac{7}{36}$ (C) $\frac{1}{9}$ (D) $\frac{4}{11}$

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

One can roll at least one 5 in eleven ways:

First die	1	2	3	4	5	6	5	5	5	5	5
Second die	5	5	5	5	5	5	1	2	3	4	6
Sum	6	7	8	9	10	11	6	7	8	9	11

Of these the sum is 7 or 11 in four cases, so the probability is $\frac{4}{11}$ and the answer is (D).

One may also use the formula for conditional probability,

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)},$$

where $P(A|B)$ is the probability that A happens given that B did happen. In the case of rolling the dice, A is "the sum is 7 or 11" while B is "at least one 5". The probability of rolling at least one 5 with two dice is $\frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6}$ or $\frac{11}{36}$. When rolling two dice, one may get a sum of 7 or 11 and *at least one 5 in four ways, $2+5$, $5+2$, $5+6$, and $6+5$, so $P(A \text{ and } B) = \frac{4}{36}$. The formula above now yields $P(A|B) = \frac{4/36}{11/36} = \frac{4}{11}$, as before.*

5. The number $\sqrt{24+\sqrt{572}}$ can be written in the form $\sqrt{a}+\sqrt{b}$, where a and b are whole numbers and $b > a$. What is the value $b - a$?

(A) 4 (B) -2 (C) 2 (D) 3

Solution by an unknown solver.

Note that $\sqrt{572} = \sqrt{2^2 \cdot 11 \cdot 13} = 2\sqrt{11}\sqrt{13}$. Thus,

$$\begin{aligned} \sqrt{24+\sqrt{572}} &= \sqrt{11+13+2\sqrt{11}\sqrt{13}} \\ &= \sqrt{\sqrt{11}^2 + 2\sqrt{11}\sqrt{13} + \sqrt{13}^2} \\ &= \sqrt{\left(\sqrt{11}+\sqrt{13}\right)^2} \\ &= \sqrt{11}+\sqrt{13}\,. \end{aligned}$$

Hence $b - a = 13 - 11 = 2$, and the answer is (C).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

6. Let $x = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cdots}}}}$ be the indicated continued fraction. Which one of the following is equal to x?

(A) $\dfrac{\sqrt{15}+1}{2}$ (B) $\dfrac{\sqrt{2}+1}{3}$ (C) $\dfrac{-3+\sqrt{15}}{2}$ (D) $\dfrac{-\sqrt{15}-3}{2}$

Solution by an unknown solver.

Note that

$$x = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cdots}}}} = \cfrac{1}{2+\cfrac{1}{3+x}}.$$

Therefore

$$x = \cfrac{1}{\cfrac{2(3+x)+1}{3+x}} = \frac{3+x}{7+2x}.$$

Hence $7x + 2x^2 = 3 + x$, so $2x^2 + 6x - 3 = 0$, and the quadratic formula yields

$$x = \frac{-6 \pm \sqrt{60}}{4} = \frac{-6 \pm 2\sqrt{15}}{4} = \frac{-3 \pm \sqrt{15}}{2}.$$

Since x is clearly positive, $x = \dfrac{-3+\sqrt{15}}{2}$ and the answer is (C).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

7. If $f\left(\sqrt{\dfrac{1+x}{1-x}}\right) = 5x$, find $f(2)$.

(A) -15 (B) $15\sqrt{-1}$ (C) 3 (D) -4

Solution by an unknown solver.

If $\sqrt{\dfrac{1+x}{1-x}} = 2$, then $\dfrac{1+x}{1-x} = 4$, so $1 + x = 4 - 4x$, whence $x = \frac{3}{5}$. Substituting $x = \frac{3}{5}$ into the given equation now yields $f(2) = 5 \cdot \frac{3}{5} = 3$, and the answer is (C).

Also solved by JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and LUYUN ZHONG-QIAO, Columbia International College, Hamilton, ON.

As you will have noticed, the identity of one of our solvers was lost in the transition of Skoliad editors. If you submit solutions on paper, or scans of paper solutions, we request that you write your name and affiliation on each sheet.

This issue's prize of one copy of ***Crux with Mayhem*** for the best solutions goes to Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON. We would very much appreciate receiving more solutions from our pre-university readers. Solutions to just some of the problems are also welcome.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga) and Eric Robert (Leo Hayes High School, Fredericton).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **31 août 2009**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

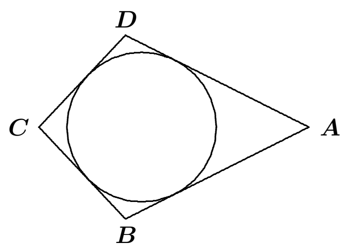
La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M394. *Proposé par l'Équipe de Mayhem.*

Les nombres a, b, c, d et e sont cinq entiers successifs. Montrer que la différence entre la moyenne des carrés de c et e et celle des carrés de a et c est égale à quatre fois c.

M395. *Proposé par l'Équipe de Mayhem.*

Le quadrilatère $ABCD$ est tel que chacun de ses côtés est tangent à un cercle donné, comme dans la figure ci-contre. Si $AB = AD$, montrer que $BC = CD$.



M396. *Proposé par l'Équipe de Mayhem.*

Dans un rectangle $ABCD$ de côtés $AB = 8$ et $BC = 6$, on inscrit respectivement deux cercles de centre O_1 et O_2 dans les triangles ABD et BCD. Trouver la distance entre O_1 et O_2.

M397. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Trouver toutes les paires (x, y) d'entiers tels que

$$x^4 - x + 1 = y^2 .$$

M398. *Proposé par l'Équipe de Mayhem.*

(a) Soit r, s et t les racines de l'équation cubique $w^3 - bw^2 + cw - d = 0$. Déterminer b, c et d en termes de r, s et t.

(b) On suppose que a est un nombre réel. Déterminer toutes les solutions du système d'équations

$$\begin{aligned} x + y + z &= a\,, \\ xy + yz + zx &= -1\,, \\ xyz &= -a\,. \end{aligned}$$

M399. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Trouver tous les triplets (a, b, c) d'entiers positifs tels que $\dfrac{3ab-1}{abc+1}$ soit un entier positif.

M400. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Supposons que a, b et c sont trois nombres réels positifs. Supposons de plus que $a^n + b^n = c^n$ pour un certain entier positif n avec $n \geq 2$. Montrer que si k est un entier positif avec $1 \leq k < n$, alors a^k, b^k et c^k sont les longueurs des côtés d'un triangle.

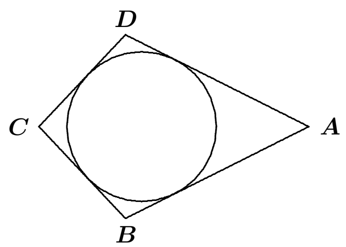
. . . .

M394. *Proposed by the Mayhem Staff.*

The numbers a, b, c, d, and e are five consecutive integers, in that order. Prove that the difference between the average of the squares of c and e and the average of the squares of a and c is equal to four times c.

M395. *Proposed by the Mayhem Staff.*

The quadrilateral $ABCD$ is such that each of its sides is tangent to a given circle, as shown. If $AB = AD$, prove that $BC = CD$.



M396. *Proposed by the Mayhem Staff.*

The rectangle $ABCD$ has side lengths $AB = 8$ and $BC = 6$. Circles with centres O_1 and O_2 are inscribed in triangles ABD and BCD. Determine the distance between O_1 and O_2.

M397. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Determine all pairs (x, y) of integers such that

$$x^4 - x + 1 \;=\; y^2\,.$$

M398. *Proposed by the Mayhem Staff.*

(a) The cubic equation $w^3 - bw^2 + cw - d = 0$ has roots r, s, and t. Determine b, c, and d in terms of r, s, and t.

(b) Suppose that a is a real number. Determine all solutions to the system of equations

$$\begin{aligned} x + y + z &= a\,, \\ xy + yz + zx &= -1\,, \\ xyz &= -a\,. \end{aligned}$$

M399. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all triples (a, b, c) of positive integers for which $\dfrac{3ab-1}{abc+1}$ is a positive integer.

M400. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that a, b, and c are positive real numbers. In addition, suppose that $a^n + b^n = c^n$ for some positive integer n with $n \geq 2$. Prove that if k is a positive integer with $1 \leq k < n$, then a^k, b^k, and c^k are the side lengths of a triangle.

Mayhem Solutions

M357. *Proposed by the Mayhem Staff.*

Determine all real numbers x that satisfy $3^{2x+2} + 3 = 3^x + 3^{x+3}$.

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

The equation $3^{2x+2} + 3 = 3^x + 3^{x+3}$ can be written in the form $3^2(3^x)^2 + 3 = 3^x + 3^3 3^x$. Substituting $k = 3^x$, we obtain the equivalent quadratic equations $9k^2 + 3 = k + 27k$ and $9k^2 - 28k + 3 = 0$.

Continuing to solve for k by factoring, we obtain $(k-3)(9k-1) = 0$. Hence, $k = 3$ or $k = \frac{1}{9}$.

Substituting $k = 3^x$, we have $3^x = 3$ or $3^x = \frac{1}{9}$, so $x = 1$ or $x = -2$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and CARLIN WHITE, student, California State University, Fresno, CA, USA. There were two incorrect solutions submitted.

M358. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

How many integers in the list 1, 2008, 2008^2, $\dots$, 2008^{2009} are simultaneously perfect squares and perfect cubes?

Solution by Luis De Sousa, student, IST-UTL, Lisbon, Portugal, modified by the editor.

The integer 2008 has prime factorization $2^3 \cdot 251$. Thus, 2008^k has prime factorization $\left(2^3 \cdot 251\right)^k = 2^{3k} \cdot 251^k$. For a positive integer greater than 1 to be a perfect square, its prime factorization must include only even exponents; for a positive integer greater than 1 to be a perfect cube, its prime factorization must include only exponents divisible by 3.

The exponents $3k$ and k are both even if and only if k is even. The exponents $3k$ and k are both multiples of 3 if and only if k is a multiple of 3.

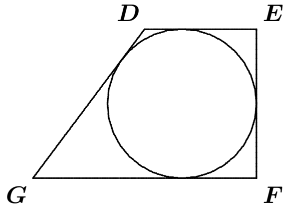
Therefore, the powers of 2008 which are perfect squares are the ones with even exponents. Also, the powers of 2008 which are perfect cubes are those with exponents that are multiples of 3. Hence, the powers of 2008 which are perfect squares and perfect cubes are those whose exponents are both multiples of 2 and of 3. In other words, they are those whose exponents are multiples of 6.

The largest multiple of 6 less than 2009 is $2004 = 6 \cdot 334$, so there are 334 multiples of 6 in the list 1, 2, 3, $\dots$, 2008, 2009. This tells us that 334 of the integers 2008^1, 2008^2, $\dots$, 2008^{2009} are simultaneously perfect squares and perfect cubes. The integer 1 is also in the list and is both a perfect square and a perfect cube. Therefore, there are $334 + 1 = 335$ integers in the list that are simultaneously perfect squares and perfect cubes.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JACLYN CHANG, student, Western Canada High School, Calgary, AB; CHANTHOEUN CHAP and JUSTIN HENDERSHOTT, students, California State University, Fresno, CA; USA; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; GEORGE TSAPAKIDIS, Agrinio, Greece; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

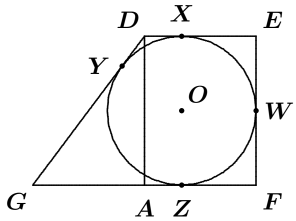
M359. *Proposed by the Mayhem Staff.*

A trapezoid $DEFG$ is circumscribed about a circle of radius 2, as shown in the diagram. The side DE has length 3 and there are right angles at E and F. Determine the area of the trapezoid.



Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Let O be the centre of the circle. Let A be the point of intersection of the perpendicular from D to GF. Let the points of tangency of EF, DE, GD, and GF to the circle be W, X, Y, and Z, respectively.



We twice use the fact that tangents to the same circle from the same point have equal lengths. Let $GZ = x$. Then $GY = x$ by equal tangents.

Since OX and OW are perpendicular to DE and EF, respectively, and the trapezoid has a right angle at E, then $OXEW$ is a square.

Since the radius of the circle is 2, then $OW = 2$, so $XE = 2$. Since $DE = 3$, then $DX = DE - XE = 1$. By equal tangents, $DY = 1$.

Now $DEFA$ is also a rectangle since it has three right angles (at E, F, and A) hence the fourth angle is also a right angle. Also, XZ is parallel to EF. Thus, $AZ = DX = 1$.

We know that $DA = XO + OZ = 2 + 2 = 4$, $GD = GY + YD = x + 1$, and $GA = GZ - AZ = x - 1$. By the Pythagorean Theorem,

$$\begin{aligned} GD^2 &= GA^2 + AD^2\,; \\ (x+1)^2 &= (x-1)^2 + 4^2\,; \\ x^2 + 2x + 1 &= x^2 - 2x + 1 + 16\,; \end{aligned}$$

hence $4x = 16$ and $x = 4$. Therefore, $GF = GZ + ZF = 4 + 2 = 6$, and so the area of the trapezoid is $\frac{1}{2}(GF + DE)(DA)$ or $\frac{1}{2}(6 + 3)(4) = 18$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; SERDAR ALTUNTAS, student, University of Karlsruhe, Karlsruhe, Germany; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; JORDAN CRIST, student, Auburn University at Montgomery, Montgomery, AL, USA; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; JOSH GUZMAN and AMANDA PERCHES, students, California State University, Fresno, CA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and GEORGE TSAPAKIDIS, Agrinio, Greece. There was one incorrect and one incomplete solution submitted.

M360. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all positive integers x that satisfy

$$3^x = x^3 + 3x^2 + 2x + 1 .$$

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

We will show that there are no such positive integers.

Suppose that there was a positive integer solution x. We rewrite the equation in the form

$$3^x = x\left(x^2 + 3x + 2\right) + 1 = x(x+1)(x+2) + 1 .$$

Since $x > 0$, the left side is divisible by 3. Therefore, the right side should be divisible by 3 as well.

But $x(x+1)(x+2)$ is divisible by 3, since it is the product of three consecutive integers. Thus $x(x+1)(x+2)+1$ gives a remainder of 1 when divided by 3, so is not divisible by 3. This is a contradiction, so there are no positive integer solutions.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; GEORGE TSAPAKIDIS, Agrinio, Greece; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were two incomplete solutions submitted.

M361. *Proposed by George Tsapakidis, Agrinio, Greece.*

Let a, b, and c be positive real numbers. Prove that

$$ab(a+b-c) + bc(b+c-a) + ca(c+a-b) \geq 3abc .$$

Solution by Shamil Asgarli, student, Burnaby South Secondary School, Burnaby, BC.

First, by the Arithmetic Mean–Geometric Mean Inequality, we find that

$$\begin{aligned} &\frac{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2}{6} \\ &\qquad \geq \sqrt[6]{\left(a^2b\right)\left(ab^2\right)\left(b^2c\right)\left(bc^2\right)\left(c^2a\right)\left(ca^2\right)} \\ &\qquad = \sqrt[6]{a^6b^6c^6} = abc . \end{aligned}$$

It follows that

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 6abc .$$

Rearranging the last expression, we find that

$$a^2b + ab^2 - abc + b^2c + bc^2 - abc + c^2a + ca^2 - abc \;\geq\; 3abc\,,$$

which is equivalent to

$$ab(a+b-c) + bc(b+c-a) + ca(c+a-b) \;\geq\; 3abc\,,$$

as required.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; COURTIS G. CHRYSSOSTOMOS, Larissa, Greece; LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; ANA GUTIERREZ and MARYLOV INSISIENGMAY, students, California State University, Fresno, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M362. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that

$$\big||x_1| + |x_2| + \cdots + |x_n| \;-\; |x_1 + x_2 + \cdots + x_n|\big| \;=\; 2\,.$$

Prove that at least one of $x_1, x_2, \ldots, x_n$ equals 1 or -1.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

Without loss of generality, assume that $x_1, x_2, \ldots, x_k$ are positive and $x_{k+1}, x_{k+2}, \ldots, x_n$ are not positive (if this is not the case, then we can rearrange the numbers to make it so). There are now two cases to consider.

Case 1 We have $x_1 + x_2 + x_3 + \cdots + x_n \geq 0$. Then

$$\begin{aligned}
|x_1 + x_2 + x_3 + \cdots + x_n| &= x_1 + x_2 + \cdots + x_n \\
&= x_1 + x_2 + \cdots + x_k - (-x_{k+1}) - (-x_{k+2}) - \cdots - (-x_n) \\
&= |x_1| + |x_2| + \cdots + |x_k| - \big(|x_{k+1}| + |x_{k+2}| + \cdots + |x_n|\big)\,.
\end{aligned}$$

From this, we have

$$\begin{aligned}
2 &= \big||x_1| + |x_2| + \cdots + |x_n| - |x_1 + x_2 + \cdots + x_n|\big| \\
&= \Big||x_1| + |x_2| + \cdots + |x_n| - \Big(|x_1| + |x_2| + \cdots + |x_k| \\
&\qquad - \big(|x_{k+1}| + |x_{k+2}| + \cdots + |x_n|\big)\Big)\Big| \\
&= \big|2\big(|x_{k+1}| + |x_{k+2}| + \cdots + |x_n|\big)\big| \\
&= 2\big(|x_{k+1}| + |x_{k+2}| + \cdots + |x_n|\big)\,.
\end{aligned}$$

It follows that $|x_{k+1}| + |x_{k+2}| + \cdots + |x_n| = 1$. Since the sum consists of nonnegative integers, one term in the sum is equal to 1 (and the rest are 0), hence one of the (nonpositive) numbers x_{k+1}, x_{k+2}, ..., x_n equals -1.

Case 2 We have $x_1 + x_2 + x_3 + \cdots + x_n \leq 0$. In a similar way to Case 1, we see that

$$\begin{aligned} & |x_1 + x_2 + x_3 + \cdots + x_n| \\ & \quad = \; -(x_1 + x_2 + x_3 + \cdots + x_n) \\ & \quad = \; -x_1 - x_2 - \cdots - x_k + (-x_{k+1}) + (-x_{k+2}) + \cdots + (-x_n) \\ & \quad = \; |x_{k+1}| + |x_{k+2}| + \cdots + |x_n| - \big(|x_1| + |x_2| + \cdots + |x_k|\big)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} 2 &= \big||x_1| + |x_2| + \cdots + |x_n| - |x_1 + x_2 + \cdots + x_n|\big| \\ &= \Big||x_1| + |x_2| + \cdots + |x_n| - \Big(|x_{k+1}| + |x_{k+2}| + \cdots + |x_n| \\ &\qquad - \big(|x_1| + |x_2| + \cdots + |x_k|\big)\Big)\Big| \\ &= \big|2\big(|x_1| + |x_2| + \cdots + |x_k|\big)\big| \;=\; 2\big(|x_1| + |x_2| + \cdots + |x_k|\big)\,. \end{aligned}$$

Therefore, $|x_1| + |x_2| + \cdots + |x_k| = 1$. The sum consists of positive integers, so it has exactly one term, which is equal to 1. Hence, $k = 1$ and $x_1 = 1$.

By the two cases, one of x_1, x_2, ..., x_n is equal to 1 or -1.

Also solved by LUIS DE SOUSA, student, IST-UTL, Lisbon, Portugal; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ALEX SONG, student, Elizabeth Ziegler Public School, Waterloo, ON; and GEORGE TSAPAKIDIS, Agrinio, Greece. There was one incomplete solution submitted.

Case 2 can be avoided by replacing x_1, x_2, ..., x_n with $-x_1$, $-x_2$, ..., $-x_n$.

Problem of the Month

Ian VanderBurgh

Contest problems involving complex numbers don't appear very often...

Problem (2007 American Invitational Mathematics Challenge A) The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

...and when they do, they're often easier than they look. The one piece of information that we need to remember is that the imaginary part of a complex number is the coefficient of i. As the famous slogan (almost) says, just calculate it!

Solution Since $z = 9 + bi$, then

$$\begin{aligned} z^2 &= (9+bi)^2 = (9+bi)(9+bi) = 81 + 18bi + b^2 i^2 \\ &= 81 + 18bi - b^2 = \left(81 - b^2\right) + (18b)i \,, \end{aligned}$$

which gives

$$\begin{aligned} z^3 &= (9+bi)^3 \;=\; (9+bi)^2(9+bi) \\ &= \left[\left(81 - b^2\right) + (18b)i\right](9+bi) \\ &= \left(729 - 9b^2\right) + \left(81b - b^3\right)i + (162b)i + \left(18b^2\right)i^2 \\ &= \left(729 - 9b^2\right) + \left(81b - b^3\right)i + (162b)i - \left(18b^2\right) \\ &= \left(729 - 27b^2\right) + \left(243b - b^3\right)i \,. \end{aligned}$$

The imaginary part of z^2 is $18b$ and the imaginary part of z^3 is $243b - b^3$. If these are equal, then $243b - b^3 = 18b$, or $b^3 - 225b = 0$, or $b(b^2 - 225) = 0$, or $b(b-15)(b+15) = 0$. Since $b > 0$, then $b = 15$. ∎

At this point, you're all probably relieved. Finally, a Problem of the Month column with a short solution. But you can't get rid of me that easily!

I was actually a bit surprised by the answer $b = 15$, so decided to try to answer the question for a general complex number $z = a + bi$. I did this for two reasons. First, I was going to use it to verify my answer. Second, I was curious how the "15" appeared.

For $z = a + bi$,

$$\begin{aligned} z^2 &= (a+bi)^2 = (a+bi)(a+bi) = a^2 + 2abi + b^2 i^2 \\ &= a^2 + 2abi - b^2 = \left(a^2 - b^2\right) + (2ab)i \end{aligned}$$

and

$$\begin{aligned} z^3 &= (a+bi)^3 \;=\; (a+bi)^2(a+bi) \\ &= \left[\left(a^2 - b^2\right) + (2ab)i\right](a+bi) \\ &= \left(a^3 - ab^2\right) + \left(a^2 b - b^3\right)i + \left(2a^2 b\right)i + \left(2ab^2\right)i^2 \\ &= \left(a^3 - ab^2\right) + \left(a^2 b - b^3\right)i + \left(2a^2 b\right)i - \left(2ab^2\right) \\ &= \left(a^3 - 3ab^2\right) + \left(3a^2 b - b^3\right)i \,. \end{aligned}$$

In this general case, the imaginary part of z^2 is $2ab$ and the imaginary part of z^3 is $3a^2b - b^3$. If these are equal, we have

$$\begin{aligned} 2ab &= 3a^2 b - b^3 \,, \\ b^3 - 3a^2 b + 2ab &= 0 \,, \\ b\left[b^2 - \left(3a^2 - 2a\right)\right] &= 0 \,. \end{aligned}$$

Thus, $b = 0$ or $b = \pm\sqrt{3a^2 - 2a}$. However $b > 0$, so $b = \sqrt{3a^2 - 2a}$. Can you see why $3a^2 - 2a$ is not negative if a is an integer? In the original question, $a = 9$, so that $b = \sqrt{3(9^2) - 2(9)} = \sqrt{225} = 15$. This addresses my first concern. It also gives somewhat of an answer to the second one as well – the 15 doesn't appear in any really obvious way. (In other words, we shouldn't have been able to guess "15" by inspection right off the bat.)

But wait, there's more! It's often interesting to think about what other questions we could ask. One question that immediately came to my mind was whether or not it was surprising that we got an integer answer for b with $a = 9$. (Given that this problem appears on the AIME, in which every answer is an integer, it shouldn't be that surprising.) Put a bit differently, though, for a random positive integer a, is $b = \sqrt{3a^2 - 2a}$ always an integer or is $a = 9$ somewhat special? If $a = 1$, we get $b = \sqrt{3(1^2) - 2(1)} = \sqrt{1} = 1$, which is an integer; if $a = 10$, we get $b = \sqrt{3(10^2) - 2(10)} = \sqrt{280}$, which is not an integer.

Next question: For what positive integer values of a is $b = \sqrt{3a^2 - 2a}$ a positive integer?

This is a fair bit harder, but is worth looking at. We'll take what seems initially like a backwards approach by starting with a positive integer b and then trying to find the corresponding values of a that are actually integers.

Suppose that $b = \sqrt{3a^2 - 2a}$ is a positive integer. Then $3a^2 - 2a = b^2$ and so $3a^2 - 2a - b^2 = 0$. While it may seem a bit unnatural at this point, a good approach is to apply the quadratic formula to solve for a in terms of b. When we do this, we obtain

$$a = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-b^2)}}{2(3)} = \frac{2 \pm \sqrt{4 + 12b^2}}{6} = \frac{1 \pm \sqrt{1 + 3b^2}}{3}.$$

Remember that we're trying to determine the positive integers b that give positive integer values for a. (Actually, we're really interested in the other direction, but this is the approach that we're taking.)

Since we want a to be positive, which sign do we take? We want the "$+$". Can you see why?

So we focus on the case $a = \dfrac{1 + \sqrt{1 + 3b^2}}{3}$. For a to be an integer, what needs to be true? Certainly, $\sqrt{1 + 3b^2}$ needs to be a positive integer, though this is actually not quite enough since $\sqrt{1 + 3b^2}$ needs to be an integer *and* we want $1 + \sqrt{1 + 3b^2}$ to be divisible by 3.

Let's focus on the first issue. Suppose that $\sqrt{1 + 3b^2} = m$ for some positive integer m. In this case, $1 + 3b^2 = m^2$ or $m^2 - 3b^2 = 1$. Do you recognize this equation? This is an example of a *Pell Equation*.

Oh dear – this keeps getting more complicated. Let's summarize for a second to see where we are. We are trying to find positive integers a for which $b = \sqrt{3a^2 - 2a}$ is a positive integer. We've now converted this problem into finding positive integer solutions to the Pell Equation $m^2 - 3b^2 = 1$. If we have a solution (m, b) to this equation, we take the value of b and it is a candidate to give a value of a that works.

Finding all of the solutions to such a Pell Equation is a bit beyond the scope of this column. If you've never seen how to find the positive integer solutions to $m^2 - 3b^2 = 1$, here's a quick summary:

Step 1 Find the smallest pair of positive integers that is a solution. You can do this by starting with $b = 1$ and increasing b until you find a solution. Here, we're lucky and $b = 1$ gives $m = 2$, so $(m, b) = (2, 1)$ is the smallest positive integer solution.

Step 2 Starting with a solution (m, b), we can get another solution (M, B) by calculating $(M, B) = (2m + 3b, m + 2b)$. By this method we can get an infinite number of positive integer solutions to the equation $m^2 - 3b^2 = 1$. The pair $(2, 1)$ leads to $(7, 4)$ which leads to $(26, 15)$ which leads to $(97, 56)$ which yields $(362, 209)$, and so forth. Those of you who are more enterprising could try to write down some kind of closed form solution. (As an additional challenge, try using the relation between (M, B) and (m, b) to solve for m and b in terms of M and B. If you can do this, then you can use a larger solution to generate smaller ones.)

This method gives us a list of values of b that we might try. At the very least, these values of b make $\sqrt{1 + 3b^2}$ an integer, though they might not make a an integer because of the extra condition. Let's try the first few:

- If $b = 1$, then $a = \frac{1 + \sqrt{1 + 3b^2}}{3} = \frac{1 + \sqrt{1 + 3(1^2)}}{3} = 1$, which is an integer.
- If $b = 4$, then $a = \frac{1 + \sqrt{1 + 3b^2}}{3} = \frac{1 + \sqrt{1 + 3(4^2)}}{3} = \frac{8}{3}$, which is not an integer.
- If $b = 15$, then we've already seen that $a = 9$.
- If $b = 56$, then $a = \frac{1 + \sqrt{1 + 3b^2}}{3} = \frac{1 + \sqrt{1 + 3(56^2)}}{3} = \frac{98}{3}$, which is not an integer.
- If $b = 209$, then $a = \frac{1 + \sqrt{1 + 3b^2}}{3} = \frac{1 + \sqrt{1 + 3(209^2)}}{3} = 121$, which is an integer.

At a quick glance, it appears that every other value of b in this sequence gives an integer value of a, but we have by no means actually proven this. Certainly, every value of b that we get out will give us an integer value for $\sqrt{1 + 3b^2}$, but this does not guarantee us integer values for a.

Let's wrap up what we've done. We started out with a problem involving complex numbers. We then talked about some of the structure of the problem and tried to do some investigation into what was happening "behind the scenes". We've found two more values of a that would give us integer values of b. It looks as if this method might lead to many more such values, though we haven't proven this.

So we started with a problem on complex numbers that looked scary, but wasn't actually that complicated. This not-so-complicated problem led us down some pretty unexpected paths. There's lots more investigating that could be done here too, for instance looking at when the imaginary parts of z^2 and z^4 might be equal.

THE OLYMPIAD CORNER

No. 278

R.E. Woodrow

We start this number with problems of the Hungarian Mathematical Olympiad 2005-2006, National Olympiad, Grades 11-12, Second Round and Final Round. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

Hungarian Mathematical Olympiad 2005–2006 National Olympiad, Grades 11-12

Second Round

1. Find the positive values of x that satisfy

$$x^{(2\sin x - \cos 2x)} \;<\; \frac{1}{x}\,.$$

2. For $f(x) = ax^2 - bx + c$ we know that $0 < |a| < 1$, $f(a) = -b$, and $f(b) = -a$. Prove that $|c| < 3$.

3. The convex quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD$. The midpoints of BC and AD are E and F (respectively), $O = AC \cap BD$, and G and H are the perpendicular projections of O on the lines AB and CD (respectively). Prove that the lines EF and GH are perpendicular.

4. Let a, b, c, d, and n be integers such that $n|ab$, $n|(bc+ad)$, and $n|bd$. Prove that $n|bc$ and $n|ad$.

Final Round

1. Define the function $t(n)$ on the nonnegative integers by $t(0) = t(1) = 0$, $t(2) = 1$, and for $n > 2$ let $t(n)$ be the smallest positive integer which does not divide n. Let $T(n) = t\big(t(t(n))\big)$. Find the value of S if

$$S \;=\; T(1) + T(2) + T(3) + \cdots + T(2006)\,.$$

2. Let A and B be two vertices of a tree with 2006 edges. We move along the edges starting from A and would like to get to B without ever turning back. At any vertex we choose the next edge among all possible edges (except the one on which we arrived) with equal probability.

Over all possible trees and choices of vertices A and B, find the minimum probability of getting from A to B.

3. A unit circle k with centre K and a line e are given in the plane. The perpendicular from K to e intersects e in point O and $KO = 2$. Let $\mathcal{H}$ be the set of all circles centred on e and externally tangent to K.

Prove that there is a point P in the plane and an angle $\alpha > 0$ such that $\angle APB = \alpha$ for any circle in $\mathcal{H}$ with diameter AB on e. Determine α and the location of P.

Next we give the first round of the Hungarian Mathematical Olympiad 2005–2006 for Specialized Mathematical Classes. Thanks again go to Robert Morewood, Canadian Team Leader to the 47^{th} IMO in Slovenia, for collecting it for our use.

Hungarian Mathematical Olympiad 2005–2006
Specialized Mathematical Classes, First Round

1. Is it true that there are infinitely many palindromes in the arithmetic progression $7k + 3$, $k = 0, 1, 2, \ldots$? (A number is a palindrome if reversing its digits yields the same number, for example, 12321 is a palindrome.)

2. We have finitely many (but at least two) numbers of the form $\frac{1}{2^k}$ whose sum is at most 1. Prove that they can be divided into two groups such that the sum of the numbers in each group is at most $\frac{1}{2}$.

3. The interval $[0, 1]$ is divided by 999 red points into 1000 equal parts and by 1110 blue points into 1111 equal parts. Find the minimum distance between a red point and a blue point. How many pairs of blue and red points achieve this minimum distance?

4. A tetrahedron has at least four edges each at most 1 unit in length. Determine the maximum possible volume of the tetrahedron.

5. Let k be a circle with centre O and let AB be a chord of k whose midpoint, M, is distinct from O. The ray from O through M meets k at R. Let P be a point on the minor arc AR of k, let PM meet k again at Q, and let AB meet QR at S. Which segment is longer, RS or PM?

The next problem set is the 2005 Kürschák Competition (a Hungarian competition). Thanks again go to Robert Morewood, Team Leader to the 47^{th} IMO in Slovenia, for collecting it for our use.

2005 Kürschák Competition

1. Let $N > 1$ and assume that the sum of the nonnegative real numbers $a_1, a_2, \ldots, a_N$ is at most 500. Prove that there exists an integer $k \geq 1$ and there exist integers $1 = n_0 < n_1 < \cdots < n_k = N$ such that

$$\sum_{i=1}^{k} n_i a_{n_{i-1}} < 2005 .$$

2. Ann and Bob are playing tennis. The winner of a match is the player who is the first to win at least four games, being at least two games ahead of his or her opponent. Ann wins a game with probability $p \leq \frac{1}{2}$ independently of the outcome of the previous games. Prove that Ann wins the match with probability at most $2p^2$.

3. A tower is built using dominos of size 2×1. The first level of the tower is a 10×11 rectangle consisting of 55 dominos, and each subsequent level is also a 10×11 rectangle consisting of 55 dominos that exactly covers the previous level. The tower thus built is called *rigid* if above each internal point of the first 10×11 rectangle which is not a gridpoint, there is an internal point of some domino of the tower. What is the minimum number of levels a rigid tower may have?

Next we give the problems of the 8th Hong Kong (China) Mathematical Olympiad written on December 3, 2005. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for us.

8th Hong Kong (China) Mathematical Olympiad

1. On a planet there are $3 \cdot 2005!$ aliens and 2005 languages. Each pair of aliens communicate with each other in exactly one language. Show that there are 3 aliens who communicate with each other in one common language.

2. Suppose that there are $4n$ line segments of unit length inside a circle of radius n. Given a straight line ℓ, prove that there exists a straight line ℓ' that is either parallel to or perpendicular to ℓ and such that ℓ' intersects at least two of the given line segments.

3. Let a, b, c, and d be positive real numbers such that $a + b + c + d = 1$. Prove that $6(a^3 + b^3 + c^3 + d^3) \geq (a^2 + b^2 + c^2 + d^2) + \frac{1}{8}$.

4. Show that there exist infinitely many squarefree positive integers n that divide $2005^n - 1$. (An integer is *squarefree* if it has no factor of the form d^2 for an integer $d > 1$.)

Next we give the two Hong Kong Team Selection Tests for the International Mathematical Olympiad 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for the *Corner*.

Hong Kong Team Selection Test 1

1. Find the integer solutions of the equation $7(x+y) = 3(x^2 - xy + y^2)$.

2. The function $f(x,y)$, defined for nonnegative integers x and y, satisfies

(a) $f(0,y) = y+1$,

(b) $f(x+1,0) = f(x,1)$, and

(c) $f(x+1,y+1) = f\big(x, f(x+1,y)\big)$.

Find $f(3,2005)$ and $f(4,2005)$.

3. In triangle ABC, the altitude, angle bisector, and median from C divide $\angle C$ into four equal angles. Find $\angle B$.

4. Let x, y, and z be positive real numbers such that $x+y+z=1$. For a positive integer n, let $S_n = x^n + y^n + z^n$. Also, let $P = S_2 S_{2005}$ and $Q = S_3 S_{2004}$.

(a) Find the smallest possible value of Q.

(b) If x, y, and z are distinct, determine which of P or Q is the larger.

5. Finitely many points lie in a plane such that the area of the triangle formed by any three of them is less than 1. Show that all of the points lie inside or on the boundary of a triangle with area less than 4.

6. Find 2^{2006} positive integers such that

(a) each positive integer has 2^{2005} digits;

(b) each digit of each positive integer is a 7 or an 8;

(c) any two positive integers have at most half of their digits in common.

Hong Kong Team Selection Test 2

1. Let $ABCD$ be a cyclic quadrilateral. Show that the orthocentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$ are the vertices of a quadrilateral congruent to $ABCD$ and show that the centroids of the same triangles are the vertices of a cyclic quadrilateral.

2. Let $ABCD$ be a cyclic quadrilateral with $BC = CD$. The diagonals AC and BD intersect at E. Let X, Y, Z, and W be the incentres of $\triangle ABE$, $\triangle ADE$, $\triangle ABC$, and $\triangle ADC$, respectively. Show that X, Y, Z, and W are concyclic if and only if $AB = AD$.

3. Points A and B lie in a plane and ℓ is a line in that plane passing through A but not through B. The point C moves from A toward infinity along a half-line of ℓ. The incircle of $\triangle ABC$ touches BC at D and AC at E. Show that the line DE passes though a fixed point.

4. Let $\triangle ABC$ have circumradius R. Let $AB = c$, $BC = a$, $CA = b$, and let k_1 and k_2 be the circles with diameters CA and CB, respectively. Let k be the circle of radius r which is tangent to k_1, k_2, and the line AB.

(a) Express r in terms of a, b, c, and R.

(b) Find $\angle C$ if $r = \frac{1}{4}R$.

Next we give the 20th Nordic Mathematical Contest written on March 30, 2006. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for obtaining it for the *Corner*.

20th Nordic Mathematical Contest
March 30, 2006

1. Let B and C be points on two given rays from the same point A, such that $AB + AC$ is constant. Prove that there exists a point D distinct from point A such that the circumcircles of the triangles ABC pass through D for all choices of B and C subject to the given constraint.

2. The real numbers x, y, and z are not all equal and satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k\,.$$

Determine all possible values of k.

3. The sequence $\{a_n\}$ of positive integers is defined by $a_0 = m$ and the recursion $a_{n+1} = a_n^5 + 487$ for all $n \geq 0$. Determine all values of m for which the sequence contains as many square numbers as possible.

4. The squares of a 100×100 chessboard are coloured with 100 different colours. Each square is painted with one colour only and each colour is used exactly 100 times. Show that there exists a row or a column on the chessboard such that at least 10 different colours are used to paint its squares.

As three sets of problems for your problem solving pleasure over the break we give the questions of the 9th, the 10th, and the 11th form of the fifth (final) round of the 2005–2006, XXXII Russian Mathematical Olympiad. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for obtaining them for our use. We remark that the odd-numbered questions of the 10th form correspond to the odd-numbered questions on the 9th form, hence they are omitted.

XXXII Russian Mathematical Olympiad 2005–2006
Final Round, 9th form
First Day

1. A square board 15×15 is divided into 15^2 unit squares. Some pairs of centres of neighbouring (along a side) cells are connected by segments so that these segments form a closed broken line that does not intersect itself and that is symmetric with respect to one of the diagonals. Prove that the length of the broken line is at most 200 units.

2. Show that there exist four integers a, b, c, and d whose absolute values are greater than $1\,000\,000$ and which satisfy

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \;=\; \frac{1}{abcd}\,.$$

3. On a circle 2006 points are given. Peter colours each of these points with one of 17 colours. After that Michael connects these points by segments so that two endpoints of each segment have the same colour and the segments do not intersect (in particular, the segments do not have common endpoints). Michael wants to draw as many segments as possible, but Peter tries to hinder him. Find the greatest number of segments Michael can draw regardless of Peter's colouring.

4. A circle ω touches the circumcircle of a triangle ABC at A, intersects side AB at K, and intersects side BC. A tangent CL to ω, with L on ω, is such that the segment KL intersects side BC at T. Prove that the length of BT equals the length of the tangent from B to ω.

9th form
Second Day

5. Let a_1, a_2, $\dots$, a_{10} be positive integers such that $a_1 < a_2 < \cdots < a_{10}$. Let b_k be the greatest divisor of a_k such that $b_k < a_k$. If $b_1 > b_2 > \cdots > b_{10}$, prove that $a_{10} > 500$.

6. The points P, Q, and R lie on the sides AB, BC, and CA of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at A and C meet the respective lines RP and RQ at X and Y. Prove that $RX = RY$.

7. A 100×100 square board is cut into dominos (that is, into 2×1 rectangles). Two players play a game. At each turn, a player may glue together any two adjacent squares if there is a cut between them. A player loses if he or she reconnects the board (thus allowing the board to be lifted by a corner without it falling apart). Who has a winning strategy, the first player or the second player?

8. A quadratic polynomial $f(x) = x^2 + ax + b$ is given. Suppose that the equation $f\big(f(x)\big) = 0$ has four distinct real roots and that the sum of two of them is equal to -1. Prove that $b \le -\frac{1}{4}$.

10th form
First Day

2. Assume that the sum of the cubes of three consecutive positive integers is a cube of some positive integer. Prove that the middle number of these three numbers is divisible by 4.

4. Let ABC be an isosceles triangle with $AB = AC$. Let ω be a circle touching the sides AB and AC and meeting the side BC at K and L. The segment AK meets ω for the second time at M. The points P and Q are the reflections of K with respect to B and C, respectively. Prove that the circumcircle of $\triangle PMQ$ is tangent to ω.

10th form
Second Day

6. Let K and L be points lying on the arcs AB and BC of the circumcircle of $\triangle ABC$, respectively, so that the lines KL and AC are parallel. Prove that the incentres of $\triangle ABK$ and $\triangle CBL$ are equidistant from the midpoint of the arc ABC.

8. A 3000×3000 square is divided into dominos (that is, into 2×1 rectangles). Prove that one can paint the dominos with three colours such that each colour is used equally often and each piece shares a side with no more than two pieces of the same colour.

11th form
First Day

1. Prove that $\sin\sqrt{x} < \sqrt{\sin x}$ whenever $0 < x < \frac{\pi}{2}$.

2. Assume that the sum and the product of some pure periodic decimal fractions is a pure decimal fraction with period T. Prove that each of the original fractions has a period not exceeding T.

3. Consider a 49×69 grid. All the $50 \cdot 70$ gridpoints are marked. Two players take turns connecting pairs of gridpoints by a segment so that no gridpoint is the endpoint of two segments. The segments are drawn until there are no free gridpoints. The first player wins if, after this process, he can orient each of the segments so that the sum of all vectors obtained is equal to the null vector; otherwise the second player wins. Which player has a winning strategy?

4. The angle bisectors BB_1 and CC_1 of $\triangle ABC$ (with B_1 on AC and C_1 on AB) meet at I. The line B_1C_1 meets the circumcircle of $\triangle ABC$ at M and N. Prove that the circumradius of $\triangle MIN$ is twice the circumradius of $\triangle ABC$.

11th form
Second Day

5. The sequences of positive numbers $\{x_n\}$ and $\{y_n\}$ satisfy

$$\begin{aligned} x_{n+2} &= x_n + x_{n+1}^2 , \\ y_{n+2} &= y_n^2 + y_{n+1} , \end{aligned}$$

for all positive integers n. Prove that if x_1, x_2, y_1, and y_2 are each greater than 1, then $x_n > y_n$ for some positive integer n.

6. Let $SABC$ be a tetrahedron. The incircle of $\triangle ABC$ has incentre I, and touches AB, BC, CA at D, E, F, respectively. The points A', B', C' lie on the edges SA, SB, SC, respectively, so that $AA' = AD$, $BB' = BE$, and $CC' = CF$. Let SS' be a diameter of the circumsphere of $SABC$. Suppose that SI is an altitude of the pyramid. Prove that S' is equidistant from A', B', and C'.

7. The polynomial $(x+1)^n - 1$ is divisible by a polynomial

$$P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1x + c_0$$

of even degree k such that c_0, c_1, $\ldots$, c_{k-1} are odd integers. Prove that n is divisible by $k+1$.

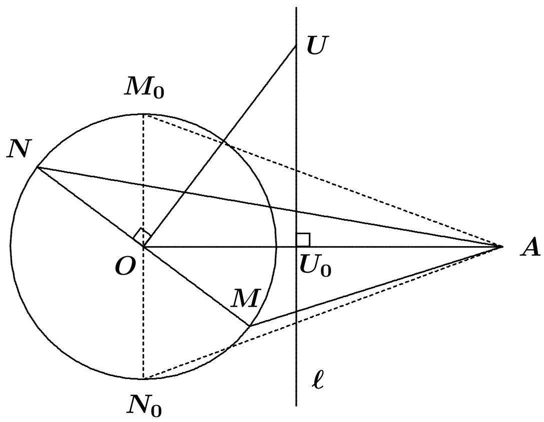
8. A group of pioneers has arrived at summer camp. Each pioneer has at least 50 and at most 100 friends among the others. Prove that one can distribute field caps that come in 1331 different colors one each to the pioneers so that the friends of each pioneer have caps of at least 20 different colors.

Now we return to the file of solutions from our readers to problems in the May 2008 number of the *Corner* and the XIX Olimpiada Iberoamericana de Matematicas, given at [2008 : 214].

2. Let A be a fixed exterior point with respect to a given circle with centre O and radius r. Let M be a point on the circle and let N be diametrically opposite to M with respect to O. Find the locus of the centres of the circles passing through A, M, and N, as the point M is varied on the circle.

Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comǎneşti, Romania. We give Bataille's write up.

Let U be the circumcentre of $\triangle AMN$. Since O is the midpoint of MN, UO is orthogonal to MN and $UM^2 = UO^2 + r^2$. But $UM = UA$, so we have $UA^2 - UO^2 = r^2$. It follows that U is on a line ℓ perpendicular to AO. More precisely, if M_0N_0 is the diameter perpendicular to AO and U_0 is the circumcentre of $\triangle AM_0N_0$, then ℓ is the perpendicular to AO through U_0. Note that $U_0 \neq O$.



Conversely, Let U be any point on ℓ (so that $UA^2 - UO^2 = r^2$) and let MN be the diameter perpendicular to UO. Then $UO^2 + r^2 = UM^2 = UN^2$ so that $UM = UN = UA$ and U is the circumcentre of $\triangle AMN$ (note that A, M, and N are not collinear because $U_0 \neq O$).

3. Let n and k be positive integers such that n is odd or n and k are even. Prove there exist two integers a and b such that $\gcd(a, n) = \gcd(b, n) = 1$ and $k = a + b$.

Solution by Michel Bataille, Rouen, France.

If $n = 1$, we take $a = k - 1$ and $b = 1$; so we suppose that $n > 1$.

We first consider the case where n is odd. Let the prime divisors of n be $p_1, p_2, \ldots, p_r$. Note that each p_j is odd and does not divide both $k - 1$ and $k + 1$ (otherwise p_j divides $(k + 1) - (k - 1) = 2$). Let $a_j \in \{k - 1, k + 1\}$ be such that p_j does not divide a_j. By the Chinese Remainder Theorem, there exists an integer a such that $a \equiv a_j \pmod{p_j}$ for each j. We take $b = k - a$. Then, $ab \equiv a_j(k - a_j) \pmod{p_j}$ with $a_j \not\equiv 0 \pmod{p_j}$ and $k - a_j$ is 1 or -1. Hence, $ab \not\equiv 0 \pmod{p_j}$ for each j. It follows that $\gcd(a, n) = \gcd(b, n) = 1$ and $k = a + b$.

We consider next the case where both n and k are even.

If $n = 2^m$ for some positive integer m, then (since k is even) $a = k - 1$ and $b = 1$ satisfy the requirements.

Otherwise, we denote by $p_1, p_2, \ldots, p_r$ the odd prime divisors of n and let $a_j \in \{k-1, k+1\}$ be such that p_j does not divide a_j. The Chinese Remainder Theorem provides an integer a such that $a \equiv a_j \pmod{p_j}$ for each j and $a \equiv 1 \pmod{2}$. Setting $b = k - a$ (note that b is odd), it is readily seen that $\gcd(a, n) = \gcd(b, n) = 1$ and $k = a + b$.

4. Find all pairs of positive integers (a, b) such that a and b each have two digits, and such that $100a + b$ and $201a + b$ are perfect squares with four digits each.

Solution by Titu Zvonaru, Comăneşti, Romania

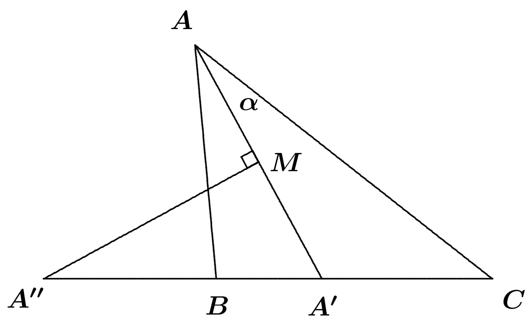
Let x, y be positive integers such that $100a+b = x^2$ and $201a+b = y^2$. Since x^2 and y^2 have four digits each, we have $32 \leq x, y \leq 99$. Subtracting the two equations yields

$$101a \;=\; y^2 - x^2 \;=\; (y-x)(y+x)\,.$$

Because 101 is prime, $y - x < 101$, and $y + x < 2 \cdot 101$, it follows that $y + x = 101$ and $y - x = a$. Therefore, $y = 101 - x$, $a = 101 - 2x$, and $b = x^2 + 200x - 10100$. We have $x^2 + 200x - 10100 > 9$ since b has two digits, hence $x > -100 + \sqrt{20109} > 41$. On the other hand, since $x^2 + 200x - 10100 \leq 99$, we have $x \leq -100 + \sqrt{20199} < 43$. We now deduce that $x = 42$, $y = 59$, $a = 17$, and $b = 64$ is the unique solution to the problem.

5. In a scalene triangle ABC, the interior bisectors of the angles A, B, and C meet the opposite sides at points A', B', and C' respectively. Let A'' be the intersection of BC with the perpendicular bisector of AA', let B'' be the intersection of AC with the perpendicular bisector of BB', and let C'' be the intersection of AB with the perpendicular bisector of CC'. Prove that A'', B'', and C'' are collinear.

Solution by Titu Zvonaru, Comăneşti, Romania



Let $a = BC$, $b = CA$, and $c = AB$. Let M be the midpoint of AA'. By the Bisector Theorem we have $A'C = \dfrac{ab}{b+c}$, $BA' = \dfrac{ac}{b+c}$, and (with $\alpha = \dfrac{A}{2}$) it is known that $AA' = \dfrac{2bc\cos\alpha}{b+c}$. Since $\angle MA'A'' = \alpha + C$, in the triangle $A''A'M$ we obtain

$$A'A'' = \frac{MA'}{\cos(\alpha+B)} = \frac{AA'}{2\cos(\alpha+C)} = \frac{bc\cos\alpha}{(b+c)\cos(\alpha+C)}\,.$$

Using the Law of Sines we obtain:

$$\begin{aligned}\frac{A''B}{A''C} &= \frac{A'A''-A'B}{A'A''+A'C} = \frac{bc\cos\alpha - ac\cos(\alpha+C)}{bc\cos\alpha + ab\cos(\alpha+C)}\\ &= \frac{c}{b}\cdot\frac{\sin B\cos\alpha - \sin A\cos(\alpha+C)}{\sin C\cos\alpha + \sin A\cos(\alpha+C)}\\ &= \frac{c}{b}\cdot\frac{\sin B\cos\alpha - 2\sin\alpha\cos\alpha\cos(\alpha+C)}{\sin C\cos\alpha + 2\sin\alpha\cos\alpha\cos(\alpha+C)}\,.\end{aligned}$$

Since $\cos\alpha \neq 0$, we obtain

$$\begin{aligned}\frac{A''B}{A''C} &= \frac{c}{b}\cdot\frac{\sin B - 2\sin\alpha\cos(\alpha+C)}{\sin C + 2\sin\alpha\cos(\alpha+C)}\\ &= \frac{c}{b}\cdot\frac{\sin B - \sin(\alpha+\alpha+C) - \sin(\alpha-\alpha-C)}{\sin C + \sin(\alpha+\alpha+C) + \sin(\alpha-\alpha-C)}\\ &= \frac{c}{b}\cdot\frac{\sin B - \sin(A+C) + \sin C}{\sin C + \sin(A+C) - \sin C}\\ &= \frac{c}{b}\cdot\frac{\sin B - \sin B + \sin C}{\sin C + \sin B - \sin C} = \frac{c}{b}\cdot\frac{\sin C}{\sin B}\,,\end{aligned}$$

hence $\frac{A''B}{A''C} = \frac{c^2}{b^2}$.

By the converse of Menelaus' theorem, it follows that A'', B'', and C'' are collinear.

Next we look at solutions from our files to problems of the Swedish Mathematical Contest 2004/2005 Qualification Round given at [2008 : 215].

1. The cities A, B, C, D, and E are connected by straight roads (more than two cities may lie on the same road). The distance from A to B, and from C to D, is 3 km. The distance from B to D is 1 km, from A to C it is 5 km, from D to E it is 4 km, and finally, from A to E it is 8 km. Determine the distance from C to E.

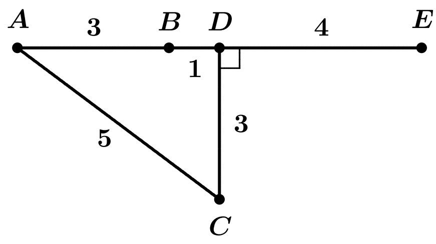
Solved by Titu Zvonaru, Comănești, Romania.

Since $AB + BD + DE = 3 + 1 + 4 = 8 = AE$, it follows that the cities A, B, D, and E are collinear (on the same road).

We have

$$\begin{aligned} AC^2 &= 5^2 = 3^2 + 4^2 \\ &= CD^2 + AD^2 , \end{aligned}$$

hence, by the converse of the Pythagorean Theorem, CD is perpendicular to AD. By the Pythagorean Theorem, we now obtain $CE = 5$.



2. Linda writes the four positive integers a, b, c, and d on a piece of paper. Since she is amused by arithmetic, she adds the numbers in pairs, obtaining the sums $a + b$, $a + c$, $\ldots$, $c + d$, but she forgets to write down one of the possible sums. The five sums she obtains are 7, 11, 12, 18, and 23. Which sum did Linda forget? What are the positive integers a, b, c, d?

Solved by Jean-David Houle, student, McGill University, Montreal, QC; John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Titu Zvonaru, Comăneşti, Romania. We give Houle's solution, modified by the editor.

Without loss of generality, let the unknown sum be $c + d$. Then we know that $3(a + b) + 2(c + d) = 7 + 11 + 12 + 18 + 23 = 71$. Since 71 is odd, $a + b$ is also odd, hence $a + b$ is 7, 11, or 23 and then $c + d$ is 25, 19, or 1, respectively. Obviously $c + d = 1$ is impossible, so $a + b + c + d$ is either $7 + 25 = 32$ or $11 + 19 = 30$. Since two numbers in $\{11, 12, 18, 23\}$ must also add up to $a + b + c + d$ but no two from this set add up to 32, it follows that $a + b = 11$ and that Linda forgot the sum $c + d = 19$.

Now, $a + b = 11$ and 12 is the sum of a number from $\{a, b\}$ and a number from $\{c, d\}$, so we may take $c = a + 1$. Then $a + c = 2a + 1$ is either 7 or 23, so $a = 3$ or $a = 11$. However, $a + b = 11$, hence $a \neq 11$. We deduce that $(a, b, c, d) = (3, 8, 4, 15)$, which yields all of the required sums.

3. Determine the greatest and the least value of

$$\frac{mn}{(m+n)^2} ,$$

if m and n are positive integers, each not greater than 2004.

Solved by George Apostolopoulos, Messolonghi, Greece; Jean-David Houle, student, McGill University, Montreal, QC; Pavlos Maragoudakis, Pireas, Greece; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give Zvonaru's write up.

By the AM–GM Inequality we have $\dfrac{mn}{(m+n)^2} \leq \dfrac{mn}{4mn} = \dfrac{1}{4}$, hence the greatest value is $\dfrac{1}{4}$, achieved when $m = n$.

Let $t = \frac{m}{n}$. Since $1 \le m,\ n \le 2004$, then $\frac{1}{2004} \le t \le 2004$. We have

$$\begin{aligned}
& \frac{mn}{(m+n)^2} \ \ge\ \frac{2004}{2005^2} \\
\iff\ & \frac{t}{(t+1)^2} \ \ge\ \frac{2004}{(2004+1)^2} \\
\iff\ & 2004t^2 + 4008t + 2004 - 2004^2 t - 4008t - t \ \le\ 0 \\
\iff\ & 2004t^2 - t - 2004^2 t + 2004 \ \le\ 0 \\
\iff\ & t(2004t - 1) - 2004(2004t - 1) \ \le\ 0 \\
\iff\ & (t - 2004)(2004t - 1) \le 0\,,
\end{aligned}$$

and the last inequality is true since $\frac{1}{2004} \le t \le 2004$. Hence the least value is $\frac{2004}{2005^2}$, achieved when $(m,n) = (2004, 1)$ or $(m,n) = (1, 2004)$.

4. Let k and n be integers with $1 < k < n$. If a set of n real numbers has the property that the mean value of any k of them is an integer, show that all n numbers are integers.

Solved by Jean-David Houle, student, McGill University, Montreal, QC; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's write up.

Let $A = \{a_1,\ a_2,\ \dots,\ a_n\}$ be a set of real numbers satisfying the hypotheses. There are integers $z_2,\ z_2,\ \dots,\ z_{k+1}$ such that

$$\begin{aligned}
a_1 + a_2 + \cdots + a_k &= kz_{k+1} \\
a_1 + a_2 + \cdots + a_{k-1} + a_{k+1} &= kz_k \\
\vdots\qquad\qquad & \qquad \vdots \\
a_1 + a_2 + a_4 + \cdots + a_k + a_{k+1} &= kz_3 \\
a_1 + a_3 + a_4 + \cdots + a_k + a_{k+1} &= kz_2
\end{aligned}$$

Adding these equations we obtain

$$ka_1 + (k-1)(a_2 + a_3 + \cdots + a_{k+1}) \ =\ k(z_2 + z_3 + \cdots + z_{k+1})\,.$$

Now, there is an integer z_1 such that $a_2 + a_3 + \cdots + a_{k+1} = kz_1$, hence $a_1 = z_2 + z_3 + \cdots + z_{k+1} - (k-1)z_1$ and a_1 is an integer.

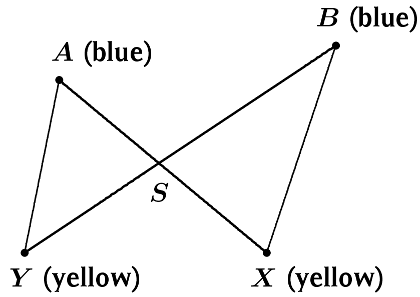
It follows that $a_1,\ a_2,\ \dots,\ a_n$ are integers.

6. Let $2n$ (where $n \ge 1$) points lie in the plane so that no straight line contains more than two of them. Paint n of the points blue and paint the other n points yellow. Show that there are n segments, each with one blue endpoint and one yellow endpoint, such that each of the $2n$ points is an endpoint of one of the n segments and none of the segments have a point in common.

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give Geupel's solution.

There are finitely many (namely $n!$) bijective correspondences between the blue and the yellow points. Hence, there is at least one correspondence, Γ, where the sum, $s(\Gamma)$, of the Euclidean lengths of its n segments has minimum value. We will prove that each such Γ has the desired property.

Assume the contrary. Then, there are two blue points, say A and B, and two yellow points, say X and Y, such that the segments AX and BY are drawn and intersect in a point, say S. We construct another correspondence Γ' from Γ by removing the segments AX and BY and adding the segments AY and BX [*Ed.: note that by the hypotheses AY and BX contain no coloured points except their endpoints*]. The correspondence Γ' is also bijective. By the triangle inequality,



$$\begin{aligned} s(\Gamma') &= s(\Gamma) + AY + BX - AX - BY \\ &= s(\Gamma) + (AY - AS - YS) + (BX - SX - SB) \ < \ s(\Gamma)\,, \end{aligned}$$

contradicting the fact that $s(\Gamma)$ has minimum value. The proof is complete.

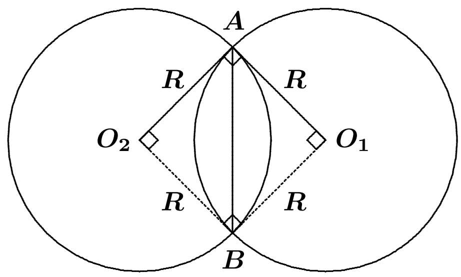
Next we look at solutions to problems of the Swedish Mathematical Contest 2004/2005 Final Round, given at [2008 : 216].

1. Two circles in the plane of the same radius R intersect at a right angle. How large is the area of the region which lies inside both circles?

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give Apostolopoulos' version.

Since $\angle O_1AO_2 = 90°$, half the area of the lens-shaped region of intersection equals the area of a quarter circle minus the area of $\triangle AO_2B$. Hence, the required area is

$$2\left(\frac{1}{4}\pi R^2 - \frac{1}{2}R^2\right) = \frac{\pi - 2}{2}\cdot R^2.$$



[*Ed.: Similarly, if the circles intersect at θ radians, then the area between them is $(\theta - \sin\theta)R^2$.*]

3. The function f satisfies $f(x) + xf(1-x) = x^2$ for all real numbers x. Determine the function f.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write up.

The only such function is $f(x) = -x + 2 + \dfrac{2x-2}{x^2-x+1}$.

Replacing x by $1-x$ in the identity for f, we obtain

$$f(1-x) + (1-x)f(x) \;=\; (1-x)^2\,. \tag{1}$$

Multiplying each side of (1) by x, we have

$$xf(1-x) + x(1-x)f(x) \;=\; x(1-x)^2\,. \tag{2}$$

Subtracting the identity from (2) yields $\left(x - x^2 - 1\right) f(x) = x - 3x^2 + x^3$. Hence,

$$f(x) \;=\; \frac{-x^3+3x^2-x}{x^2-x+1} \;=\; -x+2+\frac{2x-2}{x^2-x+1}\,. \tag{3}$$

Conversely, if $f(x)$ is given by (3), then

$$\begin{aligned} f(1-x) &= -(1-x)+2+\frac{2(1-x)-2}{(1-x)^2-(1-x)+1} \\ &= x+1-\frac{2x}{x^2-x+1}\,; \\ f(x)+xf(x-1) &= -x+2+x^2+x-\frac{2x^2-2x+2}{x^2-x+1} \\ &= x^2+2-2 \;=\; x^2\,, \end{aligned}$$

and the proof is complete.

4. If $\tan v = 2v$ and $0 < v < \frac{\pi}{2}$, then does it follow that $\sin v < \frac{20}{21}$?

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bataille's write up.

Yes, it does. To prove this, we introduce the function $\phi(x) = \tan x - 2x$ for $x \in \left(0, \dfrac{\pi}{2}\right)$, whose derivative is $\phi'(x) = \sec^2 x - 2$. Note that $\phi(0) = 0$, that ϕ decreases on $\left(0, \dfrac{\pi}{4}\right)$ and increases on $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, and that $\phi(x) \to +\infty$ as $x \to \dfrac{\pi}{2}^-$. Therefore, ϕ has a unique root $v \in \left(0, \dfrac{\pi}{2}\right)$ and $\phi(x) < 0$ for $x \in (0, v)$ and $\phi(x) > 0$ for $x \in \left(v, \dfrac{\pi}{2}\right)$.

This said, we have

$$\sin^2 v \;=\; 1-\frac{1}{1+\tan^2 v} \;=\; 1-\frac{1}{1+4v^2}\,.$$

It follows that $\sin v < \frac{20}{21}$ is equivalent to $v < \frac{10}{\sqrt{41}}$, which in turn is equivalent to $\phi\left(\frac{10}{\sqrt{41}}\right) > 0$.

Now, $\frac{10}{\sqrt{41}} > \frac{5\pi}{12}$, so that $\tan\left(\frac{10}{\sqrt{41}}\right) > \tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}$. Also, $\frac{20}{\sqrt{41}} < \frac{10}{3} < 2+\sqrt{3}$, so we conclude that $\phi\left(\frac{10}{\sqrt{41}}\right) > 0$, as desired.

5. A square of integer side n, where $n \geq 2$, is divided into n^2 squares of side 1. Next, $n-1$ straight lines are drawn so that the interior of each of the small squares (the boundary not being included in the interior) is intersected by at least one of the straight lines.

(a) Give an example which shows that this can be achieved for some $n \geq 2$.

(b) Show that among the $n-1$ straight lines there are two lines which intersect in the interior of the square of side n.

Solution by Oliver Geupel, Brühl, NRW, Germany.

We use Cartesian coordinates with $(0,0)$, $(n,0)$, (n,n), and $(0,n)$ being the corners of the big square S_n. For $0 \leq i,\, j < n$, let $S(i,j)$ denote the unit square whose southwest corner is the point (i,j).

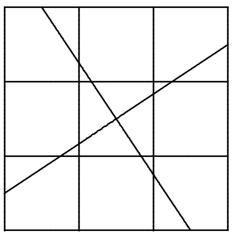


Figure A

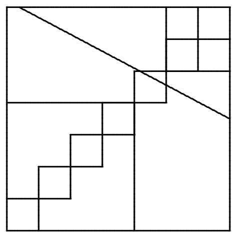


Figure B

For part (a) we give an example with $n=3$. Let g be the line through $\left(0,\frac{1}{2}\right)$ and $\left(3,\frac{5}{2}\right)$, and let h be the line through $\left(\frac{1}{2},3\right)$ and $\left(\frac{5}{2},0\right)$. The line g intersects (see Figure A) the interiors of $S(0,0)$, $S(0,1)$, $S(1,1)$, $S(2,1)$, and $S(2,2)$, while the line h intersects the interiors of the remaining squares $S(1,0)$, $S(2,0)$, $S(0,2)$, and $S(1,2)$.

For part (b), it is suffices to prove the following statement for all $n \geq 1$:

If m straight lines are drawn so that (a) the interior of each $S(i,j)$ is intersected by at least one of the lines, and (b) no two of the lines intersect in the interior of S_n, then $m \geq n$.

The proof is by Mathematical Induction. The claim is trivial for $n=1$. Assume that it is valid for each integer n with $1 \leq n < N$. We prove it for $n = N$.

Assume that m lines ℓ_1, ℓ_2, $\ldots$, ℓ_m are drawn so that (a) and (b) hold for S_N. We select the appropriate lattice point of S_N as the origin of a coordinate system so that the inclination ϕ of ℓ_1 satisfies $-90^\circ \le \phi < 0$.

The line ℓ_1 intersects the interior of at most one of the diagonal squares $S(k,k)$, where $0 \le k < N$, thus dividing S_N in a southwest segment P containing p of the $S(k,k)$, and a northeast segment Q containing q of the $S(k,k)$, where $p+q \ge N-1$. Therefore, P contains a square of side p, and Q contains a square of side q. By the induction hypothesis, at least p and q of the lines ℓ_k intersect P and Q, respectively. (This remains true if p or q is zero.) By (b), ℓ_1 does not intersect any other line in the interior of S_N. Therefore, the p lines intersecting P are distinct from the q lines intersecting Q. We conclude that $m \ge 1+p+q \ge N$, which completes the induction.

Now we turn to solutions to the problems of the Abel Competition 2004-2005 given at [2008 : 216-217].

1. (a) A positive integer m is *triangular* if $m = 1+2+\cdots+n$ for some integer $n > 0$. Show that m is triangular if $8m+1$ is a perfect square.

(b) The base of a pyramid is a right-angled triangle with sides of integer lengths. The height of the pyramid is also an integer. Show that the volume of the pyramid is an even integer.

Solution by Titu Zvonaru, Comăneşti, Romania.

(a) We have $1+2+\cdots+n = \dfrac{n(n+1)}{2}$, so all triangular numbers are of this form. Suppose that $8m+1$ is an odd perfect square. Then the equation $\dfrac{n(n+1)}{2} = m$ has a solution if and only if $n^2+n-2m = 0$ has a positive integer root. However, since $8m+1$ is an odd perfect square, $n = \dfrac{-1+\sqrt{1+8m}}{2}$ is a positive integer root of the last equation; hence m is triangular.

(b) Since the base of the pyramid is a right-angled triangle with sides of integer lengths, the legs of this triangle are $k \cdot 2mn$ and $k(m^2-n^2)$, where m, n, and k are positive integers and $m > n > 0$. If the height of the pyramid is h, then its volume is $V = \frac{1}{3}k^2mn(m-n)(m+n)h$.

It suffices to show that $V' = mn(m^2-n^2) \equiv 0 \pmod{6}$ in order to show that V is even.

If m or n is even, then mn is even, and if both m and n are odd, then m^2-n^2 is even. Hence, V' is even.

Similarly, if m or n is divisible by 3, then mn is divisible by 3, and if both m and n are not divisible by 3, then m^2-n^2 is divisible by 3 (since $x^2 \equiv 1 \pmod{3}$ for $x \not\equiv 0 \pmod{3}$). Hence, V' is divisible by 3.

It follows that the volume of the pyramid is an even integer.

2. (a) There are nine small fish in a cubic aquarium with a side of 2 metres, entirely filled with water. Show that at any given time there can be found two fish whose distance apart is not greater than $\sqrt{3}$ metres.

(b) Let A be the set of all points in space with integer coordinates. Show that for any nine points in A, there are at least two points among them such that the midpoint of the segment joining the two is also a point in A.

Solution by Titu Zvonaru, Comăneşti, Romania.

(a) Divide the aquarium into 8 cubes each 1 metre on a side. There are at least two fish in one of the cubes by the Pigeonhole Principle, and they are no further apart than the diagonal of that cube, which is $\sqrt{3}$ metres.

(b) Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points in the set A. The midpoint of the segment joining P_1 to P_2 is $M\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \frac{x_3+y_3}{2}\right)$ and M is in the set A if and only if the parities of corresponding coordinates of P_1 and P_2 agree.

For any nine points in A, by the Pigeonhole Principle, at least five points agree in the parity of the first coordinate. Among these five points, at least three agree in the parity of the second coordinate, and among these three points, at least two points agree in the parity of the third coordinate. Thus, we obtain at least two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ whose midpoint is in A.

3. (a) Let $\triangle ABC$ be isosceles with $AB = AC$, and let D be the midpoint of BC. The points P and Q lie respectively on the segments AD and AB such that $PQ = PC$ and $Q \neq B$. Show that $\angle PQC = \frac{1}{2}\angle BAC$.

(b) Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Let F, G, and H be points on the segments AD, CA, and DC respectively such that $DFGH$ is a parallelogram. Show that $\triangle BHF$ is equilateral.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; D.J. Smeenk, Zaltbommel, the Netherlands; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.

(a) Since $PB = PC$, we have $PB = PC = PQ$. If the base angles in isosceles triangles PQB, PBC, and PCQ are x, y, and z, respectively, then in $\triangle BCQ$ we have $(x+y)+(y+z)+(z+x) = 180^\circ$, hence $z = 90^\circ-(x+y)$. Thus, $\angle PQC = z = 90^\circ - (x+y) = 90^\circ - \angle ABD = \angle BAD = \frac{1}{2}\angle BAC$.

(b) Since $GH \| AD$, we have $\angle CGH = \angle CAD$. However, $ABCD$ is a rhombus, so $\angle CAD = \angle ACD$. Therefore, $\angle CGH = \angle ACD = \angle GCH$, and hence $CH = GH = FD$. Since $\triangle ABD$ and $\triangle BCD$ are equilateral, we also have $BC = BD$ and $\angle BCH = \angle BDF$. Thus, $\triangle BCH$ and $\triangle BDF$ are congruent with (i) $BH = BF$, and (ii) $\angle CHB = \angle DFB$. By (i), $\triangle BFH$ is isosceles; by (ii) $DFBH$ is a cyclic quadrilateral, so that $\angle FBH = 180^\circ - \angle HDF = 180^\circ - (60^\circ + 60^\circ) = 60^\circ$. We conclude that $\triangle BHF$ is equilateral, as desired.

Figure A

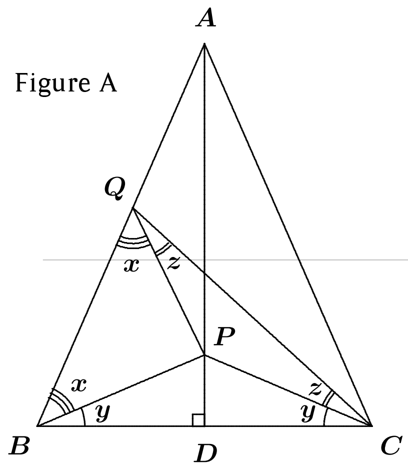
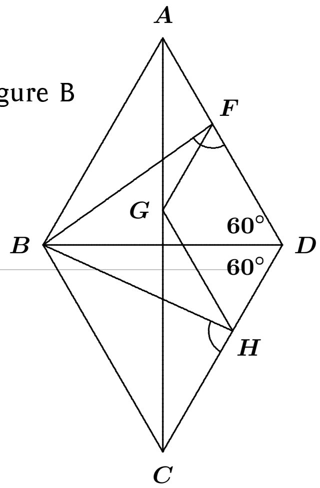


Figure B



4. (a) Let a, b, and c be positive real numbers. Prove that

$$(a+b)(a+c) \;\geq\; 2\sqrt{abc(a+b+c)}\,.$$

(b) Let a, b, and c be real numbers such that $ab+bc+ca > a+b+c > 0$. Prove that $a+b+c \geq 3$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; Pavlos Maragoudakis, Pireas, Greece; and Titu Zvonaru, Comănești, Romania. We use Bataille's solution.

(a) Equivalently, we show that $L = (a+b)^2(a+c)^2 - 4abc(a+b+c)$ is nonnegative. We have that $(a+b)^2(a+c)^2 = \big(a^2+(ab+bc+ca)\big)^2$ and also $(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$. As a result,

$$\begin{aligned} L &= a^2(a^2+b^2+c^2)+b^2c^2+2a^2(ab+bc+ca)-2abc(a+b+c) \\ &= a^2(a+b+c)^2-2abc(a+b+c)+b^2c^2 \\ &= \big(a(a+b+c)-bc\big)^2\,. \end{aligned}$$

Thus, $L \geq 0$ and the inequality is proved.

(b) Let $s = a+b+c$. We are given that $ab+bc+ca > s$, and s is positive, hence we obtain $s(2ab+2bc+2ca) > 2s^2$.

From the known inequality $a^2+b^2+c^2 \geq ab+bc+ca$ and the hypothesis, we obtain $s\big(a^2+b^2+c^2\big) \geq s(ab+bc+ca) > s^2$.

Since $s^2 = (a+b+c)^2 = \big(a^2+b^2+c^2\big)+(2ab+2bc+2ca)$, adding across the two inequalities we obtained yields $s^3 = s\cdot s^2 > 2s^2+s^2 = 3s^2$, and $s > 3$ follows immediately.

That completes the *Corner* for the May Crux. Send me your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

The Möbius Strip: Dr. August Möbius's Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology
By Clifford Pickover, Thunder's Mouth Press, New York, 2006
ISBN 978-0-7394-7542-3, softcover, 244+xxiii pages, US$15.95
Reviewed by ***Edward J. Barbeau****, University of Toronto, Toronto, ON*

Quoth mother of four year old Pete:
"You must not cross Möbius Street."
But an easy walk,
Once round the block,
Allowed him to manage the feat.

[A winning limerick in a contest sponsored by the author.]

What a wealth of connections and divergent thinking there is in this book! Clifford A. Pickover is employed at the IBM Thomas J. Watson Research Center in Yorktown, New York. Although his doctorate from Yale is in molecular biology and biochemistry, where he studied X-ray scattering and protein structure, his interest and knowledge is eclectic. His over thirty-five books include not only a number of popularizations of science (including mathematics), but also some science fiction. He is the author of over 200 papers and the holder of 30 patents "mostly concerned with novel features for computers". His website `http://www.pickover.com` is worth a visit.

This book is a rich feast that could either be read through or sampled and returned to frequently. One idea leads to another, and the text is punctuated with puzzles, games, literary quotes, vignettes, diagrams of inventions, and pictures of works of art. Of course, the author has to introduce us to August Ferdinand Möbius (1790-1868) himself, who studied astronomy under Gauss and later directed the Observatory in Leipzig in 1848. We learn about his times, his distinguished ancestry (including Martin Luther on his mother's side) and progeny, and see a picture of his skull, reproduced from a book by his grandson, a neurologist who made a phrenological study of mathematicians' heads to see if there was a connection between bumps and mathematical ability.

The book opens with an introduction to the Möbius Strip and its singular properties, which leads to a discussion of knots and their classification. A chapter deals with various inventions inspired by the strip, and then we begin to explore topology, higher dimensions, turning spheres and doughnuts inside out, the Klein Bottle and the Alexander Horned Sphere. However, the name Möbius is familiar to mathematicians in another context, through his number theoretic function. This provides the opportunity for a detour to Mertens' conjecture, series involving π, and the Riemann zeta function.

Then we enter the world of the imagination, of literature, art, cosmology, and philosophy. No summary can do justice to this part of the book, but the reader may be challenged and fascinated by the author's speculations on multiple universes and artificial life. In all, this is a rich and rewarding journey.

The Contest Problem Book IX: American Mathematics Competitions (AMC 12) 2001-2007
Compiled and edited by David Wells and J. Douglas Faires,
Mathematical Association of America, 2008
ISBN 978-0-88385-826-4, softcover, 214+xiv pages, US$49.95
Reviewed by ***John Grant McLoughlin****, University of New Brunswick, Fredericton, NB*

Another collection of contest problems has been prepared by the MAA. The high quality of this collection of problems, based on the standard grade 12 curriculum, is not surprising as MAA problem books generally feature a healthy range of problems organized in a "resourceful" manner. This book is potentially valuable for teachers working with students in high school math contest venues. The book consists of 13 contest papers with 25 questions each. The detailed solutions to these 325 problems are a practical asset to any teacher or individual wishing to work with such a collection.

The opening set of problems is the 2001 contest that appears under the heading "AMC 12 Problems". All subsequent years (2002-2007) feature two sets of problems, AMC 12A and AMC 12B. Each problem set is a stand-alone contest of 25 multiple choice problems. The format was changed in 2002 to permit two separate offerings of the same contest. AMC 12A and AMC 12B are offered just two weeks apart from one another for different students. Hence, the problem setters are required to ensure that the difficulty level and the balance of topics be consistent within these two papers while not providing an unfair advantage to those who write the 12B contest two weeks after the 12A contest has been completed. It is noted that calculus is not required to solve any of the problems, as calculus is not a standard grade 12 topic.

The preface is an honest and contextually valuable reflection on the challenges, including pitfalls, of putting together a contest. Wells and Faires offer helpful insight to people who have never tried creating such papers. My own experiences in various settings are consistent with those reflected here, acknowledging the tensions between concise problem statements and detail, among other issues. The informative preface is one of three features that strike me as being exceptional about this resource. Familiarity with other publications in this series has reinforced my appreciation and expectation for a second noteworthy feature: the "Problem Difficulty" summary for each paper. Here the authors present the distribution of student responses (% A, B, C, D, E, and Omitted) for each problem while clearly identifying the correct response. These tables telling the popularity of some wrong choices and/or the low level of correct responses inform me as to misconceptions or troublesome areas of mathematics. Insight into such matters has been valuable to me as a mathematician working in a Faculty of Education with teachers at different levels. Finally, the third outstanding feature is the Index of Problems that identifies areas, such as Algebra, with many subheadings (e.g., AM–GM Inequality; parametric equations of lines; sum and product of roots) each containing a listing of the specific problem numbers that apply to the topic. In the case of Geometry, the subtopics are further categorized under six broad areas: circles, coordinates, polygons, quadrilaterals, solids, and triangles. Any teacher looking for problems on a particular topic can work from here – always a good strategy to consider.

While applauding the merits of the book, it is noted here that its cost seems to be the lone disincentive to acquiring it as an individual resource. The book's value would be enhanced as a shared resource through a library, math department, or common browsing area.

A Taste of Mathematics, Volume IX, The CAUT Problems
By Edward Barbeau, Canadian Mathematical Society, 2009
ISBN 978-0-919558-21-2, softcover, 59+ii pages, US$15.00
Reviewed by ***Amar Sodhi****, Sir Wilfred Grenfell College, Corner Brook, NL*

The *CAUT Bulletin* is a monthly newspaper published by the Canadian Association of University Teachers. This periodical contains articles of interest to both tenured and non-tenured faculty and, by virtue of having a large section devoted to job listings, is invaluable to anyone who is searching for an academic position at a Canadian university. It is therefore a pleasant surprise to find that a paper which eschews both a crossword puzzle and a Sudoku nevertheless includes a mathematical brainteaser.

The "Homework!" is set by Edward Barbeau, a mathematician well known (among other things) for his work with the Canadian Mathematical Olympiad. However, it is his ability to convey mathematics to a general audience which comes to the forefront in *The CAUT Problems*. This collection of forty-two puzzles, which have no doubt intrigued many a scholar of the humanities or social sciences, are bound to please a person who enjoys tackling a problem which requires thought, but no more than basic skills in algebra.

Undoubtedly *The CAUT Problems* is a valuable resource for teachers who are searching for nifty posers to give to their keenest math students. The bonus chapter on card tricks may even allow a teacher to perform a little magic in the classroom. As with other volumes in the ATOM series, this book is well suited as a prize in a high school mathematics event.

However, unlike the other volumes in the ATOM series, this is a book which a student can enjoy sharing with grandparents during a summer vacation. I should give a sample problem here, but my teenage daughter has run off with the book again!

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er novembre 2009**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3419. Correction. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Soit a, b et c trois nombres réels positifs.

(a) Montrer que $\displaystyle\sum_{\text{cyclique}} \sqrt{\frac{a^2+4bc}{b^2+c^2}} \geq 2+\sqrt{2}$.

(b) Montrer que $\displaystyle\sum_{\text{cyclique}} \sqrt[3]{\frac{a^2+bc}{b^2+c^2}} \geq 2+\frac{1}{\sqrt[3]{2}}$.

3424. Correction. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Pour un entier positif m, soit σ la permutation de $\{0, 1, 2, \ldots, 2m\}$ définie par $\sigma(2i) = i$ pour $i = 0, 1, 2, \ldots, m$ et $\sigma(2i-1) = m+i$ pour $i = 1, 2, \ldots, m$. Montrer qu'il existe un entier positif k tel que $\sigma^k = \sigma$ et $1 < k \leq 2m+1$.

3439. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

On suppose que dans le triangle ABC la hauteur $h_a = AD$ est égale à $a = BC$. Soit H l'orthocentre, M le point milieu de BC et E le point milieu de AD. Montrer que $HM = HE$. Quand la réciproque est-elle valide ?

3440. *Proposé par Hidetoshi Fukugawa, Kani, Gifu, Japon.*

Sur une table, on a N pièces de monnaie, toutes de même dimension. Ces N pièces de monnaie peuvent être arrangées aussi bien en un carré qu'en un triangle équilatéral. Trouver N.

3441★. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $ABCD$ un quadrilatère convexe et P un point dans l'intérieur de $ABCD$ tel que $PA = \frac{AB}{\sqrt{2}}$, $PB = \frac{BC}{\sqrt{2}}$, $PC = \frac{CD}{\sqrt{2}}$, et $PD = \frac{DA}{\sqrt{2}}$. Confirmer ou infirmer que $ABCD$ est un carré.

3442. *Proposé par Iyoung Michelle Jung, étudiante, Collège de Langues Étrangères Hanyoung, Séoul, Corée du Sud et Sung Soo Kim, Université Hanyang, Séoul, Corée du Sud.*

Soit C un cône circulaire droit et D un disque de rayon fixé situé dans la base du cône C. Montrer que l'aire A de la partie du cône située directement au-dessus de D est indépendante de la position du disque D.

3443. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $a+b+c = 3$. Montrer que

$$\sum_{\text{cyclique}} \frac{a^2(b+1)}{a+b+ab} \geq 2\,.$$

3444. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a, b et c trois nombres réels positifs tels que $a+b+c = 1$. Montrer que

$$\sum_{\text{cyclique}} \frac{ab}{3a^2+2b+3} \leq \frac{1}{12}\,.$$

3445. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine, à la mémoire de Murray S. Klamkin.*

Soit a, b et c trois nombres réels non négatifs tels que $ab+bc+ac = 1$. Montrer que

$$\text{(a)} \sum_{\text{cyclic}} \frac{a}{1+bc} \geq \frac{3\sqrt{3}}{4}\,; \qquad \text{(b)} \sum_{\text{cyclic}} \frac{a^2}{1+a} \geq \frac{\sqrt{3}}{\sqrt{3}+1}\,.$$

3446. *Proposed by Mihály Bencze, Brasov, Roumanie.*

Pour tout entier positif n montrer que

$$\left\lfloor \sqrt{n^2-n+1} + \sqrt{n^2+n+1} \right\rfloor + \left\lfloor \sqrt{n^2+n} + \sqrt{n^2+3n+2} \right\rfloor = \left\lfloor \sqrt{4n^2+3} \right\rfloor + \left\lfloor \sqrt{4n^2+8n+3} \right\rfloor\,,$$

où $\lfloor x \rfloor$ désigne le plus grand entier ne dépassant pas x.

3447. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un nombre entier positif. Montrer que

$$\frac{2}{n!(n+2)!} < \prod_{k=1}^{n} \left(\sqrt[k+1]{\frac{k+1}{k}} - 1 \right) < \frac{1}{(n+1)(n!)^2}\,.$$

3448. *Proposé par José Luis Díaz-Barrero and Miquel Grau-Sánchez, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit F_n le n^{e} nombre de Fibonacci, c-à-d, $F_0 = 0$, $F_1 = 1$ et $F_n = F_{n-1} + F_{n-2}$ pour $n \geq 2$. Montrer qu'on a

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \ \geq\ 4S\left(\sum_{k=1}^{n+2} F_k^2 - F_{n+1}^2\right)^{1/2}$$

pour tout triangle ABC dont a, b, c sont les longueurs respectives des côtés et S sa surface.

3449. *Proposé par un proposeur anonyme.*

Soit $ABCD$ un carré de côté unité, M et N les milieux respectifs de AB et CD. Y a-t-il un point P sur MN tel que les longueurs AP et PC sont toutes les deux des nombres rationnels ?

3450. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit ABC un triangle avec r comme rayon de son cercle inscrit, r_a, r_b et r_c les rayons respectifs de ses cercles exinscrits, et h_a, h_b, h_c ses hauteurs respectives. Montrer que

$$\frac{h_a + 2r_a}{r + r_a} + \frac{h_b + 2r_b}{r + r_b} + \frac{h_c + 2r_c}{r + r_c} \ \geq\ \frac{27}{4}\,.$$

....

3419. Correction. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

Let a, b, and c be positive real numbers.

(a) Prove that $\displaystyle\sum_{\text{cyclic}} \sqrt{\frac{a^2 + 4bc}{b^2 + c^2}} \ \geq\ 2 + \sqrt{2}$.

(b) Prove that $\displaystyle\sum_{\text{cyclic}} \sqrt[3]{\frac{a^2 + bc}{b^2 + c^2}} \ \geq\ 2 + \frac{1}{\sqrt[3]{2}}$.

3424. Correction. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

For a positive integer m, let σ be the permutation of $\{0, 1, 2, \ldots, 2m\}$ defined by $\sigma(2i) = i$ for each $i = 0, 1, 2, \ldots, m$ and $\sigma(2i - 1) = m + i$ for each $i = 1, 2, \ldots, m$. Prove that there exists a positive integer k such that $\sigma^k = \sigma$ and $1 < k \leq 2m + 1$.

3439. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In triangle ABC suppose that the altitude $h_a = AD$ equals $a = BC$. Let the orthocentre be H, M be the midpoint of BC, and E be the midpoint of AD. Prove that $HM = HE$. When does the converse hold?

3440. *Proposed by Hidetoshi Fukugawa, Kani, Gifu, Japan.*

There are N coins on a table all of the same size. These N coins can be arranged in a square and they can also be arranged into an equilateral triangle. Find N.

3441★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a convex quadrilateral and let P be a point in the interior of $ABCD$ such that $PA = \frac{AB}{\sqrt{2}}$, $PB = \frac{BC}{\sqrt{2}}$, $PC = \frac{CD}{\sqrt{2}}$, and $PD = \frac{DA}{\sqrt{2}}$. Prove or disprove that $ABCD$ is a square.

3442. *Proposed by Iyoung Michelle Jung, student, Hanyoung Foreign Language High School, Seoul, South Korea and Sung Soo Kim, Hanyang University, Seoul, South Korea.*

Let C be a right circular cone and let D be a disk of fixed radius lying within the base of the cone C. Prove that if A is the area of that part of the cone lying directly above D, then A is independent of the position of the disk D.

3443. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c=3$. Prove that

$$\sum_{\text{cyclic}} \frac{a^2(b+1)}{a+b+ab} \;\geq\; 2\,.$$

3444. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c=1$. Prove that

$$\sum_{\text{cyclic}} \frac{ab}{3a^2+2b+3} \;\leq\; \frac{1}{12}\,.$$

3445. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, in memory of Murray S. Klamkin.*

Let a, b, and c be nonnegative real numbers such that $ab+bc+ac=1$. Prove that

$$\text{(a)}\ \sum_{\text{cyclic}} \frac{a}{1+bc} \;\geq\; \frac{3\sqrt{3}}{4}\,; \qquad \text{(b)}\ \sum_{\text{cyclic}} \frac{a^2}{1+a} \;\geq\; \frac{\sqrt{3}}{\sqrt{3}+1}\,.$$

3446. *Proposed by Mihály Bencze, Brasov, Romania.*

For any positive integer n prove that

$$\left\lfloor\sqrt{n^2-n+1}\;+\;\sqrt{n^2+n+1}\right\rfloor\;+\;\left\lfloor\sqrt{n^2+n}\;+\;\sqrt{n^2+3n+2}\right\rfloor$$
$$=\;\left\lfloor\sqrt{4n^2+3}\right\rfloor\;+\;\left\lfloor\sqrt{4n^2+8n+3}\right\rfloor\;,$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

3447. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. Prove that

$$\frac{2}{n!(n+2)!}\;<\;\prod_{k=1}^{n}\left(\sqrt[k+1]{\frac{k+1}{k}}-1\right)\;<\;\frac{1}{(n+1)(n!)^2}\,.$$

3448. *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let F_n be the n^{th} Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$a^2F_n+b^2F_{n+1}+c^2F_{n+2}\;\geq\;4S\left(\sum_{k=1}^{n+2}F_k^2-F_{n+1}^2\right)^{1/2}$$

holds for any triangle ABC, where a, b, c, and S are the side lengths and area of the triangle, respectively.

3449. *Proposed by an anonymous proposer.*

Let $ABCD$ be a unit square, M the midpoint of AB, and N the midpoint of CD. Is there a point P on MN such that the lengths of AP and PC are both rational numbers?

3450. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let $\triangle ABC$ have inradius r, exradii r_a, r_b, r_c, and altitudes h_a, h_b, h_c. Prove that

$$\frac{h_a+2r_a}{r+r_a}\;+\;\frac{h_b+2r_b}{r+r_b}\;+\;\frac{h_c+2r_c}{r+r_c}\;\geq\;\frac{27}{4}\,.$$

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3330. [2008 : 171, 174] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let n be a natural number, let r be a real number, and let $a_1, a_2, \ldots, a_n$ be positive real numbers satisfying $\prod\limits_{k=1}^{n} a_k = r^n$; prove that

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+r)^3}\,,$$

(a) for $n = 2$ if and only if $r \geq \frac{1}{3}$;

(b) for $n = 3$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(c) for $n = 4$ if $r \geq \frac{1}{\sqrt[3]{4}}$;

(d) for $n \geq 5$ if and only if $r \geq \sqrt[3]{n} - 1$.

Solution to parts (a)-(c) *by Oliver Geupel, Brühl, NRW, Germany, solution to part* (d) *by the proposer.*

(a) The statement is not correct in the strict sense, because for each $r > 0$ the inequality is satisfied by $a_1 = a_2 = r$ (and similarly for part (d)). We prove instead that for $r > 0$, the inequality

$$\frac{1}{(1+a)^3} + \frac{1}{(1+b)^3} \;\geq\; \frac{2}{(1+r)^3}\,, \tag{1}$$

holds for all positive real numbers a and b satisfying $ab = r^2$ if and only if $r \geq \dfrac{1}{3}$.

If $ab \geq \dfrac{1}{9}$, then the inequality (1) follows from the result given in ***CRUX with Mayhem***, problem 3319 (solution at [2009 : 121-122]).

Conversely, suppose that $r < \dfrac{1}{3}$. Let $f : [0,\infty) \to \mathbb{R}$ be given by

$$f(x) \;=\; \frac{1}{(1+x)^3} + \frac{x^3}{\left(x+r^2\right)^3}\,.$$

We have $f''(r) = \dfrac{6(3r-1)}{r(1+r)^5} < 0$ and f'' is continuous. It follows that there exists an $x_0 > 0$ such that $f(x_0) < f(r) = \dfrac{2}{(1+r)^3}$. We conclude that

$a = x_0$ and $b = \dfrac{r^2}{a}$ violate the inequality (1). This completes the proof of part (a). Equality holds if and only if $a = b = r$.

(b) We prove the result under the less restrictive condition $r \geq 0.47$. Without loss of generality, let $a_3 \leq r$ and put $x = \sqrt{a_1 a_2}$. Then $x \geq r$, and by part (a) we have

$$\frac{1}{(1+a_1)^3} + \frac{1}{(1+a_2)^3} \;\geq\; \frac{2}{(1+x)^3}\,.$$

It therefore suffices to show that

$$\frac{2}{(1+x)^3} + \frac{x^6}{\left(x^2+r^3\right)^3} \;\geq\; \frac{3}{(1+r)^3}\,.$$

Clearing denominators and rearranging terms in this last inequality, we find that it is equivalent to

$$(x-r)^2 \sum_{k=0}^{7} p_k(r) x^k \;\geq\; 0\,,$$

where

$$\begin{aligned}
p_0(r) &= r^7\left(2r^3+6r^2+6r-1\right)\\
p_1(r) &= r^6\left(4r^3+12r^2+3r-2\right)\\
p_2(r) &= r^4\left(6r^4+15r^3+18r^2+15r-3\right)\\
p_3(r) &= r^3\left(5r^4+18r^3+33r^2+5r-6\right)\\
p_4(r) &= r\left(4r^5+21r^4+27r^3+13r^2+9r-3\right)\\
p_5(r) &= r\left(3r^4+15r^3+21r^2+21r-3\right)-6\\
p_6(r) &= 2r^4+9r^3+15r^2+5r-6\\
p_7(r) &= r^3+3r^2+3r-2
\end{aligned}$$

It suffices to prove that $p_k(r) > 0$ for $r \geq 0.47$ and $0 \leq k \leq 7$. Using a calculator, we verify that $p_k(0.47) > 0$ for each k. Moreover, the polynomials $p_k(r)$ are increasing functions for real arguments $r \geq 0.47$. This completes the proof. Equality holds if and only if $a_1 = a_2 = a_3 = r$.

(c) We prove the result under the weaker condition $r \geq 0.59$. Without loss of generality, let $a_4 \leq r$ and put $x = \sqrt[3]{a_1 a_2 a_3}$. Then $x \geq r$, and by part (b) we have

$$\frac{1}{(1+a_1)^3} + \frac{1}{(1+a_2)^3} + \frac{1}{(1+a_3)^3} \;\geq\; \frac{3}{(1+x)^3}\,.$$

It therefore suffices to show that

$$\frac{3}{(1+x)^3} + \frac{x^9}{\left(x^3+r^4\right)^3} \;\geq\; \frac{4}{(1+r)^3}\,.$$

Clearing denominators and rearranging terms in this last inequality, we find that it is equivalent to

$$(x-r)^2 \sum_{k=0}^{10} q_k(r)x^k \ \geq \ 0\,,$$

where

$$\begin{aligned}
q_0(r) &= r^{10}\bigl(3r^3+9r^2+9r-1\bigr)\\
q_1(r) &= r^9\bigl(6r^3+18r^2+6r-2\bigr)\\
q_2(r) &= r^8\bigl(9r^3+15r^2+3r-3\bigr)\\
q_3(r) &= r^6\bigl(8r^4+21r^3+27r^2+23r-3\bigr)\\
q_4(r) &= r^5\bigl(7r^4+27r^3+51r^2+13r-6\bigr)\\
q_5(r) &= r^4\bigl(6r^4+33r^3+39r^2+3r-9\bigr)\\
q_6(r) &= r^2\bigl(5r^5+27r^4+36r^3+20r^2+15r-3\bigr)\\
q_7(r) &= r\bigl(4r^5+21r^4+33r^3+37r^2+3r-6\bigr)\\
q_8(r) &= r\bigl(3r^4+15r^3+30r^2+18r-9\bigr)-9\\
q_9(r) &= 2r^4+9r^3+15r^2+3r-9\\
q_{10}(r) &= r^3+3r^2+3r-3
\end{aligned}$$

It suffices to prove that $q_k(r) > 0$ for $r \geq 0.59$ and $0 \leq k \leq 10$. Using a calculator, we verify that $q_k(0.59) > 0$ for each k. Moreover, the polynomials $q_k(r)$ are increasing functions for real arguments $r \geq 0.59$. This completes the proof. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = r$.

(d) Suppose that r is such that the inequality holds for all $a_1, a_2, \ldots, a_n$ subject to the given constraint. Let x be a positive real number, let $a_i = x$ for $i = 1, 2, \ldots, n-1$, and let $a_n = \frac{r^n}{x^{n-1}}$. Then

$$\frac{n-1}{(1+x)^3} + \frac{x^{3n-3}}{\bigl(x^{n-1}+r^n\bigr)^3} \geq \frac{n}{(1+r)^3}$$

holds for all $x > 0$. Taking the limit as $x \to \infty$ yields $\dfrac{n}{(1+r)^3} \leq 1$, hence $r \geq \sqrt[3]{n}-1$.

Conversely, suppose that $r \geq \sqrt[3]{n}-1$. We will prove by Mathematical Induction that if $n \geq 4$, $r \geq \max\left\{\frac{1}{\sqrt[3]{4}}, \sqrt[3]{n}-1\right\}$, and $a_1, a_2, \ldots, a_n$ satisfy the given constraint, then $\sum\limits_{k=1}^{n} \dfrac{1}{(1+a_k)^3} \geq \dfrac{n}{(1+r)^3}$.

Note that the statement is true for $n = 4$ by part (c).

Now suppose that the statement is true for some $n \geq 4$ and that $r \geq \max\left\{\frac{1}{\sqrt[3]{4}}, \sqrt[3]{n+1}-1\right\} = \sqrt[3]{n+1}-1$, and let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers such that $a_1a_2\cdots a_{n+1} = r^{n+1}$. By symmetry, we may assume that $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$. Let $x = \sqrt[n]{a_1 a_2 \cdots a_n}$, then $x \geq a_{n+1} = \dfrac{r^{n+1}}{x^n}$ and $x^{n+1} \geq r^{n+1}$, so that $x \geq r \geq \sqrt[3]{n+1} - 1 > \frac{1}{\sqrt[3]{4}}$. By induction, we have $\sum\limits_{k=1}^{n} \dfrac{1}{(1+a_k)^3} \geq \dfrac{n}{(1+x)^3}$, hence

$$\sum_{k=1}^{n} \frac{1}{(1+a_k)^3} \;\geq\; \frac{n}{(1+x)^3} + \frac{x^{3n}}{\left(x^n + r^{n+1}\right)^3}\,.$$

Let $h(x)$ be the function of x on the right side of the above inequality for $x \geq r$. After some (tedious) calculations we find that

$$\begin{aligned} h'(x) &= \frac{3n\left(x^{n+1} - r^{n+1}\right)P(x)}{(1+x)^4\left(x^n + r^{n+1}\right)^4}\,; \\ P(x) &= 6x^{2n}r^{n+1} + 4x^{2n+1}r^{n+1} + 4x^n r^{2n+2} \\ &\qquad + x^{2n+2}r^{n+1} + x^{n+1}r^{2n+2} + r^{3n+3} - x^{3n-1}\,. \end{aligned}$$

Now $P(r) = r^{3n-1}(r+1)^3(3r-1) > 0$, since $r > \frac{1}{\sqrt[3]{4}} > \frac{1}{3}$, and by degree considerations $P(x) \to -\infty$ as $x \to \infty$, hence $P(x)$ has exactly one root $x_0 \in [0,\infty)$. [*Ed.: note that for positive x and positive C_1, C_2, ..., C_n, the function* $\dfrac{C_n}{x^n} + \dfrac{C_{n-1}}{x^{n-1}} + \cdots + \dfrac{C_1}{x} + C_0$ *is decreasing, and* $\dfrac{P(x)}{x^{3n-1}}$ *is of this form.*] So, $P(x) > 0$ for $x \in [r, x_0)$ and $P(x) < 0$ for $x \in (x_0, \infty)$. Hence, $h(x)$ is increasing on $[r, x_0)$ and decreasing on (x_0, ∞). Thus,

$$\min_{x \in [r, x_0]} h(x) \;=\; h(r) \;=\; \frac{n}{(1+r)^3} + \frac{r^{3n}}{\left(r^n + r^{n+1}\right)^3} \;=\; \frac{n+1}{(1+r)^3}$$

and for any $x \in [x_0, \infty)$ we have

$$h(x) \;>\; \lim_{x\to\infty} h(x) \;=\; 1 \;\geq\; \frac{n+1}{(1+r)^3} \;=\; h(r)\,.$$

Therefore, the minimum value of $h(x)$ on $[r,\infty)$ is $h(r) = \dfrac{n+1}{(1+r)^3}$, which completes the induction step and the proof.

Also solved by the proposer (parts (a)-(c)). *There was one incomplete solution submitted.*

*The proposer leaves **Crux** readers with the problem of determining the minimum values of r for which parts* (b) *and* (c) *hold.*

3338. [2008 : 239, 242] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A convex cyclic quadrilateral $ABCD$ has an incircle with centre I. Let P be the intersection of AC and BD. Prove that $AP : CP = AI^2 : CI^2$.

Solution by Michel Bataille, Rouen, France.

It has been shown (Crux problem 2027, [1995 : 90; 1996 : 94, 95]) that

$$\frac{AP}{CP} = \frac{1 - \cos\angle BCD}{1 - \cos\angle BAD}.$$

Set $\theta = \angle BAD$; then $\angle BCD = \pi - \theta$, because the quadrilateral $ABCD$ is cyclic. Thus

$$\frac{AP}{CP} = \frac{1+\cos\theta}{1-\cos\theta} = \cot^2\frac{\theta}{2}. \tag{1}$$

On the other hand, if r denotes the radius of the incircle, then

$$AI = \frac{r}{\sin\frac{\theta}{2}} \quad \text{and} \quad CI = \frac{r}{\sin\left(90^\circ - \frac{\theta}{2}\right)} = \frac{r}{\cos\frac{\theta}{2}},$$

so that

$$\frac{AI^2}{CI^2} = \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \cot^2\frac{\theta}{2}. \tag{2}$$

The desired result follows from (1) and (2).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; SON HONG TA, Hanoi, Vietnam; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3339. [2008 : 239, 242] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let Γ_1 and Γ_2 be two nonintersecting circles each lying in the exterior of the other. Let ℓ_1 and ℓ_2 be the common external tangents to Γ_1 and Γ_2. Let ℓ_1 meet Γ_1 and Γ_2 at A and B, respectively, and let ℓ_2 meet Γ_1 and Γ_2 at C and D, respectively. Let M and N be the midpoints of AB and CD, respectively, and let P and Q be the intersections of NA and NB with Γ_1 and Γ_2, respectively, different from A and B. Prove that CP, DQ, and MN are concurrent.

I. *Composite of similar solutions by George Apostolopoulos, Messolonghi, Greece and by John G. Heuver, Grande Prairie, AB.*

We shall see that the result holds, more generally, for any two circles that have common external tangents, whether or not they intersect. Let C' and D' be the points where CP and DQ meet MN. We are to prove that $C' = D'$. Because of the symmetry about the line of centres of the two circles, $AMNC$ and $BMND$ are isosceles trapezoids; thus, $AC||MN$ and

$MN||BD$ and $\angle AMN = \angle MNC = \angle C'NC$. Moreover, because NC is tangent to Γ_1 while CP is a chord, we have

$$\angle C'CN = \angle PCN = \angle CAP = \angle CAN = \angle ANM,$$

whence $\triangle CNC' \sim \triangle NMA$. Consequently, $\dfrac{CN}{NM} = \dfrac{NC'}{MA}$, or

$$NC' = \frac{CN \cdot MA}{NM}.$$

By the analogous argument the triangles DND' and NMB are similar, so that

$$ND' = \frac{DN \cdot MB}{NM}.$$

Because $CN = DN$ and $MA = MB$, it follows that $NC' = ND'$; furthermore, since C' and D' lie on the same side of N on the line MN, the points C' and D' coincide, as desired.

II. *Solution by Daniel Reisz, Auxerre, France.*

La droite MN est l'axe radical—lieu des points ayant même puissance—des deux cercles Γ_1 et Γ_2. Il suffit donc de montrer que le point K intersection des droites CP et DQ a même puissance par rapport aux deux cercles pour pouvoir conclure que K se situe aussi sur la droite MN. De $NP \cdot NA = NQ \cdot NB$, c'est à dire de $\dfrac{NP}{NQ} = \dfrac{NB}{NA}$, on déduit la similitude des triangles NPQ et NBA, et donc l'égalité d'angles

$$\angle NPQ = \angle NBA = \angle BDQ = \angle BDK.$$

Montrons maintenant que les triangles KPQ et KDC sont semblables. Ils ont déjà un angle commun en K. Par ailleurs : (i) $\angle BDC$ et $\angle DCA$ sont supplémentaires (à cause de $AC||BD$), (ii) $\angle DCA$ et $\angle CPA$ sont supplémentaires (CN est la tangente à Γ_1 en C, CA est une corde) et (iii) $\angle CPA = \angle KPN$.
Donc, $\angle BDC = \angle KPN$. Or on sait déjà que $\angle NPQ = \angle NBA = \angle BDQ$, donc par soustraction, $\angle KPQ = \angle KDC$ d'où la similitude de nos deux triangles, ce qui permet d'écrire $\dfrac{KP}{KQ} = \dfrac{KD}{KC}$; c'est à dire

$$KP \cdot KC = KD \cdot KQ.$$

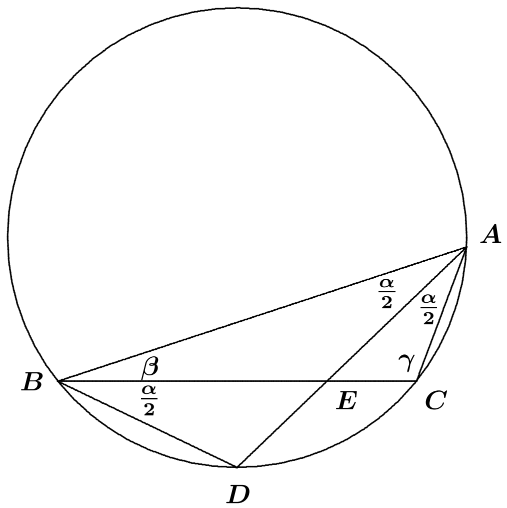
Le point K a donc même puissance par rapport aux deux cercles et se trouve donc sur la droite MN.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3340. [2008 : 239, 242] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The bisector of $\angle BAC$ intersects the circumcircle of $\triangle ABC$ at a second point D. Suppose that $AB^2 + AC^2 = 2AD^2$. Prove that the angle of intersection of AD and BC is 45°.

Similar solutions by D.J. Smeenk, Zaltbommel, the Netherlands and Peter Y. Woo, Biola University, La Mirada, CA, USA.



Let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$, and let R be the circumradius. Using the Law of Sines for triangles ABC and ABD, we have

$$\frac{AB}{\sin\gamma} = \frac{AC}{\sin\beta} = \frac{AD}{\sin\left(\frac{\alpha}{2}+\beta\right)} = 2R\,,$$

so that we can rewrite the condition $AB^2+AC^2 = 2AD^2$ as

$$\sin^2\gamma + \sin^2\beta = 2\sin^2\left(\frac{\alpha}{2}+\beta\right).$$

This yields successively

$$\begin{aligned}
\cos 2\beta + \cos 2\gamma &= 2\cos(\alpha + 2\beta)\,,\\
\cos 2\beta + \cos 2\gamma &= -2\cos(\beta - \gamma)\,,\\
\cos(\beta+\gamma)\cos(\beta-\gamma) &= -\cos(\beta-\gamma)\,,\\
\cos(\beta-\gamma)\big[\cos(\beta+\gamma)+1\big] &= 0\,.
\end{aligned}$$

Since $\cos(\beta+\gamma) \neq -1$, it follows that $\cos(\beta-\gamma) = 0$. We can assume, without loss of generality, that $\beta \leq \gamma$. Then $\beta - \gamma = -\frac{\pi}{2}$. It follows that $\alpha + 2\beta = \frac{\pi}{2}$, since $\alpha+\beta+\gamma = \pi$, and therefore, $\angle AEC = \frac{\alpha}{2} + \beta = \frac{\pi}{4}$, as claimed.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; BOB SERKEY, Leonia, NJ, USA; SON HONG TA, Hanoi, Vietnam; TITU ZVONARU, Comănești, Romania; and the proposer. There were two incorrect solutions submitted.

3341. [2008 : 240, 242] *Proposed by Arkady Alt, San Jose, CA, USA.*

For any triangle ABC with sides of lengths a, b, and c, prove that $\sqrt{3}(R_a + R_b + R_c) \le a + b + c$, where R_a, R_b, and R_c are the distances from the incentre of $\triangle ABC$ to the vertices A, B, and C, respectively.

Solution by George Apostolopoulos, Messolonghi, Greece.

Let s be the semiperimeter of triangle ABC. We have

$$R_a = \frac{s-a}{\cos\frac{A}{2}} = \frac{s-a}{\sqrt{\frac{s(s-a)}{bc}}} = \frac{\sqrt{bc}\sqrt{s-a}}{\sqrt{s}} = \sqrt{bc}\sqrt{1-\frac{a}{s}}\,.$$

Similarly,

$$R_b = \sqrt{ca}\sqrt{1-\frac{b}{s}} \quad \text{and} \quad R_c = \sqrt{ab}\sqrt{1-\frac{c}{s}}\,.$$

Using the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned}(R_a+R_b+R_c)^2 &= \left(\sqrt{bc}\sqrt{1-\frac{a}{s}}+\sqrt{ca}\sqrt{1-\frac{b}{s}}+\sqrt{ab}\sqrt{1-\frac{c}{s}}\right)^2\\ &\le (bc+ca+ab)\left(1-\frac{a}{s}+1-\frac{b}{s}+1-\frac{c}{s}\right)\\ &= (ab+bc+ca)\left(3-\frac{2s}{s}\right) = ab+bc+ca\,,\end{aligned}$$

or

$$\sqrt{3}(R_a+R_b+R_c) \le \sqrt{3(ab+bc+ca)}\,.$$

It suffices to show that

$$\sqrt{3(ab+bc+ca)} \le a+b+c\,.$$

The last inequality is equivalent to $a^2+b^2+c^2 \ge ab+bc+ca$, which is well known and easy to prove. This completes the solution.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3342. [2008 : 240, 242] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let r and R be the inradius and circumradius of $\triangle ABC$, respectively. Prove that

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; 1+\frac{r}{R}\,.$$

Similar solutions by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Oliver Geupel, Brühl, NRW, Germany; and Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

It is well known that $\cos A+\cos B+\cos C = 1+\frac{r}{R}$. Thus, it suffices to show that

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; \sum_{\text{cyclic}}\cos A\,.$$

For positive x we have $2 \le x+\frac{1}{x}$. Taking $x=\dfrac{\cos(A/2)}{\cos(B/2)}$ and multiplying by $\sin\dfrac{A}{2}\sin\dfrac{B}{2}$, we obtain

$$2\sin\frac{A}{2}\sin\frac{B}{2} \;\le\; \frac{1}{2}\left(\sin A\tan\frac{B}{2}+\sin B\tan\frac{A}{2}\right)\,,$$

hence,

$$2\sum_{\text{cyclic}} \sin\frac{A}{2}\sin\frac{B}{2} \;\le\; \frac{1}{2}\sum_{\text{cyclic}}\tan\frac{A}{2}(\sin B+\sin C)\,.$$

By sum to product formulas, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{\text{cyclic}}\tan\frac{A}{2}(\sin B+\sin C)\\
&\qquad= \sum_{\text{cyclic}}\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\sin\frac{B+C}{2}\cos\frac{B-C}{2}\\
&\qquad= \sum_{\text{cyclic}}\cos\frac{B+C}{2}\cos\frac{B-C}{2} \;=\; \sum_{\text{cyclic}}\cos A\,,
\end{aligned}$$

as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; and the proposer. There was one incomplete solution submitted.

3343. [2008 : 240, 242] *Proposed by Stan Wagon, Macalester College, St. Paul, MN, USA.*

If the factorials are deleted in the Maclaurin series for $\sin x$, one obtains the series for $\arctan x$. Suppose instead that one alternates factorials in the series. Does the resulting series have a closed form? That is, can one find an elementary expression for the function whose Maclaurin series is

$$x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7} + \frac{x^9}{9!} - \frac{x^{11}}{11} + \cdots\,?$$

Essentially the same solution by Michel Bataille, Rouen, France; Richard I. Hess, Rancho Palos Verdes, CA, USA; Václav Konečný, Big Rapids, MI, USA; Ralph Lozano, student, Missouri State University, Missouri, USA; Xavier Ros, student, Universitat Politècnica de Catalunya, Barcelona, Spain; Digby Smith, Mount Royal College, Calgary, AB; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $f(x)$ denote the given series. The following Maclaurin series are all well known:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,, && -\infty < x < \infty\,; && (1)\\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\,, && -\infty < x < \infty\,; && (2)\\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,, && -1 < x < 1\,; && (3)\\
\tanh^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\,, && -1 < x < 1\,. && (4)
\end{aligned}$$

From (1) and (2) we have

$$\frac{1}{2}(\sin x + \sinh x) = x + \frac{x^5}{5!} + \frac{x^9}{9!} + \cdots\,. \qquad (5)$$

From (3) and (4) we have

$$\frac{1}{2}\left(\tan^{-1} x - \tanh^{-1} x\right) = -\left(\frac{x^3}{3} + \frac{x^7}{7} + \frac{x^{11}}{11} + \cdots\right)\,. \qquad (6)$$

From (5) and (6) we conclude that

$$f(x) = \frac{1}{2}\left(\sin x + \sinh x + \tan^{-1} x - \tanh^{-1} x\right)\,, \qquad -1 < x < 1\,.$$

Also solved by CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DOUGLASS L. GRANT, Cape Breton University, Sydney, NS; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOHN KLASSEN, Champlain Regional College – St. Lawrence, Ste. Foy, QC; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; and the proposer.

With the exception of the proposer, all of these solvers gave the same answer; namely $\frac{1}{2}\left[\sin x + \sinh x + \tan^{-1} x - \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$ *which is, of course, equivalent to the answer given above.*

3344. [2008 : 240, 242] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let n be a positive integer, $n \geq 4$, and let a_1, a_2, ..., a_n be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \;\geq\; \frac{3}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2 - n\right).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

We will use the following theorem.

Theorem (Right Convex Function Theorem) Let $f(u)$ be a function on an interval $I \subset \mathbb{R}$, which is convex for $u \geq s$, $s \in I$. If $f(x)+(n-1)f(y) \geq nf(s)$ for all x, $y \in I$ such that $x \leq s \leq y$ and $x + (n-1)y = ns$, then

$$f(x_1) + f(x_2) + \cdots + f(x_n) \;\geq\; nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

for all x_1, x_2, ..., $x_n \in I$ such that $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \geq s$. ■

The desired inequality may be rewritten as

$$f(-a_1) + f(-a_2) + \cdots + f(-a_n) \;\geq\; nf\left(-\frac{a_1 + a_2 + \cdots + a_n}{n}\right),$$

where $f(u) = -3u^2 - \dfrac{n}{u}$ for $-n < u < 0$. Since $f''(u) = -\dfrac{2n}{u^3} - 6 > 0$ for $u \geq -1$, the function f is convex for $u \geq s = -1$. It suffices to prove that

$$f(x) + (n-1)f(y) \;\geq\; nf(-1) \tag{1}$$

whenever $-n < x \leq -1 \leq y < 0$ and $x + (n-1)y = -n$.

Substituting $x = -n - (n-1)y$ in (1), we obtain the inequality

$$-3\big(n + (n-1)y\big)^2 + \frac{n}{n + (n-1)y} + (n-1)\left(-3y^2 - \frac{n}{y}\right) \;\geq\; n(n-3)$$

for $-1 \leq y < 0$. After clearing denominators and rearranging terms, it becomes $n(n-1)(y+1)^2\big[(3(n-1)y^2 + 3ny + n\big] \geq 0$, or equivalently

$$n(n-1)(y+1)^2\left[\left(y + \frac{n}{2(n-1)}\right)^2 + \frac{n(n-4)}{12(n-1)^2}\right] \;\geq\; 0,$$

which is clearly true.

Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer

For further applications of the Right Convex Function Theorem, the interested reader is referred to the article "The Proof of Three Open Inequalities" by V. Cîrtoaje, [2008 : 231-238].

3345. [2008 : 240, 243] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, c, and d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{1+b^2c}+\frac{b}{1+c^2d}+\frac{c}{1+d^2a}+\frac{d}{1+a^2b} \;\geq\; 2\,.$$

Solution by Dung Nguyen Manh, High School of HUS, Hanoi, Vietnam.

By the AM–GM Inequality, we have

$$\begin{aligned}\frac{a}{1+b^2c} &= a-\frac{ab^2c}{1+b^2c} \;\geq\; a-\frac{ab^2c}{2b\sqrt{c}} \;=\; a-\frac{ab\sqrt{c}}{2} \\ &= a-\frac{b\sqrt{a\cdot ac}}{2} \;\geq\; a-\frac{b(a+ac)}{4}\,.\end{aligned}$$

Taking the cyclic sum across the inequality derived above and using the relation $a+b+c+d=4$, we obtain

$$\sum_{\text{cyclic}}\frac{a}{1+b^2c} \;\geq\; 4 \;-\; \frac{1}{4}\sum_{\text{cyclic}} ab \;-\; \frac{1}{4}\sum_{\text{cyclic}} abc\,. \tag{1}$$

Applying the AM–GM Inequality again, we have

$$ab+bc+cd+da \;=\; (a+c)(b+d) \;\leq\; \frac{(a+b+c+d)^2}{4} \;=\; 4 \tag{2}$$

and

$$\begin{aligned}abc+bcd+cda+dab &= ab(c+d)+cd(a+b) \\ &\leq \frac{(a+b)^2(c+d)}{4}+\frac{(c+d)^2(a+b)}{4} \\ &= \frac{(a+b)(c+d)}{4}(a+b+c+d) \\ &= (a+b)(c+d) \;\leq\; \frac{(a+b+c+d)^2}{4} \;=\; 4\,.\end{aligned} \tag{3}$$

From inequalities (1), (2), and (3) we then have

$$\sum_{\text{cyclic}}\frac{a}{1+b^2c} \;\geq\; 4-\frac{1}{4}(4+4) \;=\; 2\,.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; TRAN THANH NAM, Tomsk Polytechnic University, Tomsk, Russia; and the proposer. A solution was received that appears to be correct, but due to its length and complexity could not be verified in the available time.

3346. [2008 : 240, 243] *Proposed by Bin Zhao, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China.*

Given triangle ABC, prove that

$$\pi \sum_{\text{cyclic}} \frac{1}{A} \;\geq\; \left(\sum_{\text{cyclic}} \sin\frac{A}{2}\right)\left(\sum_{\text{cyclic}} \csc\frac{A}{2}\right).$$

Solution by Michel Bataille, Rouen, France.

Note that $\pi = A+B+C$ and set $x = \frac{A}{2}$, $y = \frac{B}{2}$, and $z = \frac{C}{2}$ (so that x, y, $z \in \left(0, \frac{\pi}{2}\right)$), the inequality is then rewritten as

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \;\geq\; (\sin x+\sin y+\sin z)\left(\frac{1}{\sin x}+\frac{1}{\sin y}+\frac{1}{\sin z}\right)$$

or

$$\left(\frac{x}{y}-\frac{\sin x}{\sin y}+\frac{y}{x}-\frac{\sin y}{\sin x}\right) + \left(\frac{y}{z}-\frac{\sin y}{\sin z}+\frac{z}{y}-\frac{\sin z}{\sin y}\right) + \left(\frac{z}{x}-\frac{\sin z}{\sin x}+\frac{x}{z}-\frac{\sin x}{\sin z}\right) \;\geq\; 0\,,$$

that is,

$$\frac{\big(f(x)-f(y)\big)\big(g(y)-g(x)\big)}{\sin x \sin y} + \frac{\big(f(y)-f(z)\big)\big(g(z)-g(y)\big)}{\sin y \sin z} + \frac{\big(f(z)-f(x)\big)\big(g(x)-g(z)\big)}{\sin z \sin x} \;\geq\; 0\,,$$

where the functions f and g are defined by $f(u) = \frac{\sin u}{u}$ and $g(u) = u\sin u$.

Now, f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and g is increasing on $\left(0, \frac{\pi}{2}\right)$. Thus, $\big(f(u)-f(v)\big)\big(g(v)-g(u)\big) \geq 0$ whenever u, $v \in \left(0, \frac{\pi}{2}\right)$ and so the left side of (1) is the sum of three nonnegative terms. The inequality (1) follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3347. [2008 : 241, 243] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $A_1A_2A_3A_4$ be a convex quadrilateral. Let B_i be a point on A_iA_{i+1} for $i \in \{1, 2, 3, 4\}$, where the subscripts are taken modulo 4, such that

$$\frac{B_1A_1}{B_1A_2} = \frac{B_3A_4}{B_3A_3} = \frac{A_1A_4}{A_2A_3} \quad\text{and}\quad \frac{B_2A_2}{B_2A_3} = \frac{B_4A_1}{B_4A_4} = \frac{A_1A_2}{A_3A_4}\,.$$

Prove that $B_1B_3 \perp B_2B_4$ if and only if $A_1A_2A_3A_4$ is a cyclic quadrilateral.

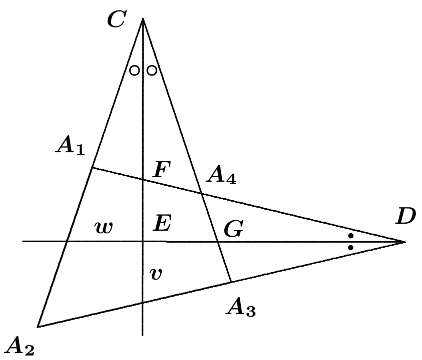
Solution by Oliver Geupel, Brühl, NRW, Germany.

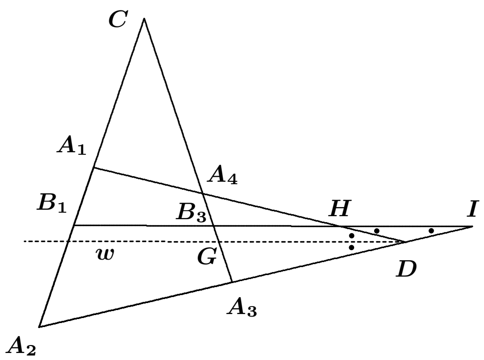
If $A_1A_2 \nparallel A_3A_4$, then let $C = A_1A_2 \cap A_3A_4$ and v be the bisector of $\angle A_1CA_4$; otherwise, if $A_1A_2 \parallel A_3A_4$ define v to be the midline between the two parallel lines. Similarly, if $A_1A_4 \nparallel A_2A_3$, then let $D = A_2A_3 \cap A_1A_4$ and w be the bisector of $\angle A_3DA_4$; otherwise, if $A_1A_4 \parallel A_2A_3$ define w to be their midline. The desired result follows readily from the following two lemmas:

Lemma 1. The quadrilateral $A_1A_2A_3A_4$ is cyclic if and only if $v \perp w$.

Lemma 2. It is the case that $B_2B_4 \parallel v$ and $B_1B_3 \parallel w$.

Proof of Lemma 1. Assume that the figure has been labeled so that C and A_2A_3 are on opposite sides of line A_1A_4 while D and A_1A_2 are on opposite





sides of A_3A_4; denote the angles at the vertices A_i by α_i, and define

$$E = v \cap w\,, \quad F = v \cap A_1A_4\,, \quad \text{and} \quad G = w \cap A_3A_4\,.$$

We will first prove that $\angle(v, A_1A_4) = 90^\circ + \frac{\alpha_1 - \alpha_4}{2}$. If $A_1A_2 \parallel A_3A_4$ then $\alpha_4 = 180^\circ - \alpha_1$; therefore $\angle EFA_4 = \alpha_1 = 90^\circ + \frac{\alpha_1 - \alpha_4}{2}$, as claimed. If $A_1A_2 \nparallel A_3A_4$, note that $\angle CA_1A_4 = 180^\circ - \alpha_1$, $\angle CA_4F = 180^\circ - \alpha_4$, and $\angle A_1CE = \angle A_4CF = \frac{\alpha_1 + \alpha_4}{2} - 90^\circ$. Hence,

$$\begin{aligned} \angle A_4FE &= \angle A_4CF + \angle CA_4F = \left(\frac{\alpha_1 + \alpha_4}{2} - 90^\circ\right) + (180^\circ - \alpha_4) \\ &= 90^\circ + \frac{\alpha_1 - \alpha_4}{2}\,, \end{aligned}$$

which is the desired angle between v and A_1A_4. Similarly, we have that $\angle A_4GE = 90^\circ + \frac{\alpha_3 - \alpha_4}{2}$.

Inspecting the angles in quadrilateral EGA_4F, we see that

$$\begin{aligned}\angle(v,w) &= \angle FEG \\ &= 360^\circ - \left(90^\circ + \frac{\alpha_1-\alpha_4}{2}\right) - \alpha_4 - \left(90^\circ + \frac{\alpha_3-\alpha_4}{2}\right) \\ &= 180^\circ - \frac{\alpha_1+\alpha_3}{2}\,.\end{aligned}$$

Hence, $\angle(v,w) = 90^\circ$ is equivalent to $\alpha_1+\alpha_3 = 180^\circ$ which, in turn, is equivalent to $A_1A_2A_3A_4$ being cyclic.

Proof of Lemma 2. If $A_1A_4 \parallel A_2A_3$ then, because parallels cut proportional segments from transversals, $B_1B_3 \parallel A_1A_4 \parallel w$. It remains to consider the case $A_1A_4 \not\parallel A_2A_3$. Let H and I be the points where the line B_1B_3 meets A_1A_4 and A_2A_3, respectively. We now apply the hypothesis and Menelaus' theorem to triangles A_1A_2D and A_3A_4D and transversal B_1B_3 to obtain

$$\begin{aligned}\frac{A_1A_4\cdot IA_2\cdot HD}{A_2A_3\cdot ID\cdot HA_1} &= \frac{B_1A_1\cdot IA_2\cdot HD}{B_1A_2\cdot ID\cdot HA_1} \\ = 1 = \frac{B_3A_4\cdot IA_3\cdot HD}{B_3A_3\cdot ID\cdot HA_4} &= \frac{A_1A_4\cdot IA_3\cdot HD}{A_2A_3\cdot ID\cdot HA_4}\,.\end{aligned} \tag{1}$$

By comparing the first and last fractions in (1), we see that

$$\frac{IA_3}{HA_4} = \frac{IA_2}{HA_1} = \frac{IA_3+A_3A_2}{HA_4+A_4A_1}\,;$$

whence, $\dfrac{A_3A_2}{A_4A_1} = \dfrac{IA_3}{HA_4}$. We substitute this last fraction into the final fraction appearing in (1) and deduce that $\dfrac{HD}{ID} = 1$. That is, $HD = ID$. It follows that

$$\angle GDH = \frac{\angle A_2DH}{2} = \frac{\angle DIH + \angle DHI}{2} = \angle DHI\,.$$

Consequently, $w = GD \parallel HI = B_1B_3$. Analogously, $v \parallel B_2B_4$.

Also solved by MICHEL BATAILLE, Rouen, France; KHANH BAO NGUYEN, High School for Gifted Students, Hanoi University of Education, Hanoi, Vietnam; and PETER Y. WOO, Biola University, La Mirada, CA, USA. There was one incomplete submission.

*Lemma 1 is readily found as a problem in textbooks and on contests. It has appeared more than once in **CRUX**; see* [1980 : 226-230] *for an alternative proof, several references, and a related result. Geupel found the "*if*" part of our problem on the Mathlinks website,* `http://www.mathlinks.ro/viewtopic.php?t=200396`. *The proof there is very easy: Under the assumption that the given quadrilateral is cyclic, the diagonals of the quadrilateral divide its interior into two pairs of similar triangles, and the lines B_1B_3 and B_2B_4 are the bisectors of the angles formed by those diagonals. It seems as if the converse is not so easily proved.*

3348. [2008 : 241, 243] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be an acute-angled triangle. Let A_1, B_1, and C_1 be points on the sides BC, CA, and AB, respectively, such that the angles $\angle AC_1B_1$, $\angle BC_1A_1$, and $\angle ACB$ are all equal. Let M, N, and P be the circumcentres of $\triangle AC_1B_1$, $\triangle BA_1C_1$, and $\triangle CB_1A_1$, respectively. Prove that AM, BN, and CP are concurrent if and only if AA_1, BB_1, and CC_1 are altitudes of $\triangle ABC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, revised by the editor.

Denote the angles of $\triangle ABC$ by $\angle A$, $\angle B$, $\angle C$, and let AA', BB', and CC' be the altitudes. The given angle assumptions imply that the triangles A_1BC_1 and AB_1C_1 are both similar to $\triangle ABC$. Because the angle at the circumcentre N that is subtended by the side BC_1 of $\triangle A_1BC_1$ is twice the angle at A_1 (which equals $\angle A$), we deduce that $\angle NBC_1 = 90^\circ - \angle A$ for any position of C_1 between B and A. Moreover, since $\angle B'BA = 90^\circ - \angle A$, we know that N must lie on the altitude BB'. Similarly, M must lie on the altitude CC'. Because the lines AM and BN therefore meet at the orthocentre H of $\triangle ABC$, when the three lines AM, BN, and CP are concurrent, their common point must be H. Our problem is thus reduced to proving that *CP passes through H if and only if $C_1 = C'$* (the foot of the altitude from C to AB).

When $C_1 = C'$, it is a standard result that $\angle BC'A' = \angle AC'B' = \angle C$, whence $A_1 = A'$ and $B_1 = B'$. As a consequence, $\angle A_1B_1C = \angle B$ and, as we argued earlier, $\angle A_1CP = 90^\circ - \angle B$ so that P is on the altitude $CC_1 = CH$, as desired.

We now prove that conversely, when $C_1 \neq C'$, then H is not on the line CP. When C_1 is taken between B and C' then (because by our angle assumption $C_1B_1 \parallel C'B'$) B_1 must lie beyond B' on the halfline from A in the direction of C. Similarly A_1 lies beyond A' on the halfline from C in the direction of B. It follows that $\angle A_1B_1C > \angle A'B'C$; whence

$$\angle A_1CP \;=\; 90^\circ - \angle A_1B_1C < 90^\circ - \angle A'B'C \;=\; \angle A'CH\,,$$

and we conclude that H is not on the line CP. A similar argument establishes that H is not on the line CP when C_1 is taken between C' and A.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3349. [2008 : 241, 243] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, and c be positive real numbers. Show that

$$6\prod_{\text{cyclic}}\frac{a^3+1}{a^2+1} \;\geq\; \max\left\{\sum_{\text{cyclic}}\frac{a(1+bc)(a^2+1)}{a^3+1}\,,\;\sum_{\text{cyclic}}\frac{ab(1+c)(a^2b^2+1)}{a^3b^3+1}\right\}\,.$$

Solution by Tran Thanh Nam, Tomsk Polytechnic University, Tomsk, Russia.

We first prove a lemma.

Lemma If t is a positive real number, then

$$\text{(a)}\ \frac{t^3+1}{t^2+1} \geq \sqrt{t^2-t+1} \geq \sqrt[4]{\frac{t^4+1}{2}}\,, \qquad \text{(b)}\ \frac{t+1}{2} \geq \frac{t(t^2+1)}{t^3+1}\,.$$

Proof: After squaring, collecting like terms, and factoring, the inequality $\dfrac{t^3+1}{t^2+1} \geq \sqrt{t^2-t+1}$ of part (a) reduces to $t(t-1)^2 \geq 0$, which is true. Similarly, the inequality $\sqrt{t^2-t+1} \geq \sqrt[4]{\dfrac{t^4+1}{2}}$ reduces to $(t-1)^4 \geq 0$, which is true.

Collecting the fractions in inequality (b) on one side and factoring reduces it to $(t-1)^2(t^2+t+1)$, which is true. ∎

We now proceed to prove that $6\prod\limits_{\text{cyclic}} \dfrac{a^3+1}{a^2+1} \geq \sum\limits_{\text{cyclic}} \dfrac{a(1+bc)(a^2+1)}{a^3+1}$. By part (a) of the Lemma, we have

$$\frac{(b^3+1)(c^3+1)}{(b^2+1)(c^2+1)} \geq \sqrt[4]{\frac{(b^4+1)(c^4+1)}{4}} \geq \sqrt{\frac{b^2c^2+1}{2}} \geq \frac{bc+1}{2}$$

and

$$\left(\frac{a^3+1}{a^2+1}\right)^2 \geq \sqrt{\frac{a^4+1}{2}} \geq a\,.$$

Hence,

$$2\prod_{\text{cyclic}} \frac{a^3+1}{a^2+1} \geq \frac{a(1+bc)(a^2+1)}{a^3+1}\,.$$

Adding across the cyclic permutations of the last inequality gives the desired result.

We next proceed to prove that $6\prod\limits_{\text{cyclic}} \dfrac{a^3+1}{a^2+1} \geq \sum\limits_{\text{cyclic}} \dfrac{ab(1+c)(a^2b^2+1)}{a^3b^3+1}$.

By parts (a) and (b) of the Lemma, we have

$$\frac{(a^3+1)(b^3+1)}{(a^2+1)(b^2+1)} \geq \frac{ab+1}{2} \geq \frac{ab(a^2b^2+1)}{a^3b^3+1}$$

and also $\dfrac{c^3+1}{c^2+1} \geq \dfrac{1+c}{2}$ (as it is equivalent to $(c+1)(c-1)^2 \geq 0$). Hence,

$$2\prod_{\text{cyclic}} \frac{a^3+1}{a^2+1} \geq \frac{ab(1+c)(a^2b^2+1)}{a^3b^3+1}\,.$$

Adding across the cyclic permutations of this last inequality gives the second desired result.

The overall inequality follows from the two results we have obtained.

Also solved by Arkady Alt, San Jose, CA, USA; and the proposer.

3350. [2008 : 241, 243] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, and z be positive real numbers such that $x+y+z=1$. Prove that
$$\frac{yz}{1+x}+\frac{zx}{1+y}+\frac{xy}{1+z} \;\leq\; \frac{1}{4}\,.$$

I. *Similar solutions by George Apostolopoulos, Messolonghi, Greece; Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam; Khanh Bao Nguyen, High School for Gifted Students, Hanoi University of Education, Hanoi, Vietnam; Babis Stergiou, Chalkida, Greece; Son Hong Ta, Hanoi, Vietnam; and Titu Zvonaru, Comăneşti, Romania.*

For positive real numbers a, b, and c we have that $(b+c)^2 \geq 4bc$, hence $\frac{a}{b+c} \leq \frac{a}{4}\left(\frac{1}{c}+\frac{1}{b}\right)$. Thus,
$$\begin{aligned}\sum_{\text{cyclic}}\frac{yz}{1+x} &= \sum_{\text{cyclic}}\frac{yz}{(x+y)+(z+x)} \leq \sum_{\text{cyclic}}\frac{yz}{4}\left(\frac{1}{x+y}+\frac{1}{z+x}\right)\\ &= \sum_{\text{cyclic}}\frac{xy+zx}{4(y+z)} = \sum_{\text{cyclic}}\frac{x}{4}=\frac{1}{4}\,.\end{aligned}$$

Equality holds if and only if $x=y=z=\frac{1}{3}$.

II. *Solution by Arkady Alt, San Jose, CA, USA, condensed by the editor.*

Let $e_1=x+y+z$, $e_2=xy+yz+zx$, $e_3=xyz$, and $S=\sum' \frac{xy}{kz+1}$ ($\sum'$ denotes a cyclic sum over x, y, z). We will prove that if k, x, y, z are positive real numbers and $e_1=1$, then $S\leq \max\left\{\frac{1}{4},\ \frac{1}{k+3}\right\}$.

Let $k\in(0,1]$. Since $S=e_3\sum'\left(\frac{1}{z}-\frac{k}{kz+1}\right)=e_2-ke_3\sum'\frac{1}{kz+1}$ and $\sum'\frac{1}{kz+1}\geq 9\left(\sum'(kz+1)\right)^{-1}=\frac{9}{k+3}$, it follows that $S\leq e_2-\frac{9ke_3}{k+3}$. It therefore suffices to prove that $e_2-\frac{9ke_3}{k+3}\leq\frac{1}{k+3}$, which is equivalent to
$$(k+3)e_2-9ke_3 \;\leq\; 1\,. \tag{1}$$

The inequality (1) follows from the two inequalities
$$\begin{aligned} e_2 &\geq 9e_3\,, &(2)\\ 4e_2 &\leq 1+9e_3\,, &(3)\end{aligned}$$

since $1-(k+3)e_2+9ke_3 = (1-4e_2+9e_3)+(1-k)(e_2-9e_3)$ and $k \le 1$. Now, (2) follows from $3\sqrt[3]{e_3} \le e_1 = 1$ and $3\sqrt[3]{e_3^2} \le e_2$, and these follow from the AM–GM Inequality, while (3) follows by rewriting the Schur Inequality modulo the relation $e_1 = 1$; that is, one rewrites $\sum' x(x-y)(x-z) \ge 0$ as $2e_1^3 - 6e_1e_2 - e_1^2 + 2e_2 + 9e_3 \ge 0$ and puts $e_1 = 1$. This completes the proof of the inequality for $k \in (0,1]$.

Note that if $k = 1$, then we obtain the original inequality to be proved, while if $k > 1$ then $S = S(k) < S(1) \le \frac{1}{4} = \max\left\{\frac{1}{4},\ \frac{1}{k+3}\right\}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; REBECCA EVERDING and JENNIFER PAJDA, students, Southeast Missouri State University, Cape Girardeau, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; WEI-DONG, Weihai Vocational College, Weihai, Shandong Province, China; NGUYEN THANH LIEM, Tran Hung Dao High School, Phan Thiet, Vietnam; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; D.P. MEHENDALE (Dept. of Electronics) and M.R. MODAK, (formerly of Dept. Mathematics), S. P. College, Pune, India; TRAN THANH NAM, Tomsk Polytechnic University, Tomsk, Russia; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

The following solvers submitted multiple solutions: Alt (5 solutions), *Apostolopoulos* (3 solutions) *and Cao* (2 solutions).

Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina indicated that since $x+y+z=1$, our problem appears as problem 35 (solved on pp. 48-49) in the book Old and New Inequalities *by T. Andreescu, V. Cîrtoaje, G. Dospinescu, and M. Lascu; GIL Publishing House.*