Mathematical Spectrum

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- Antimagic graphs
- The golden section
- A series of intrigues
- Fermat's method of factorization

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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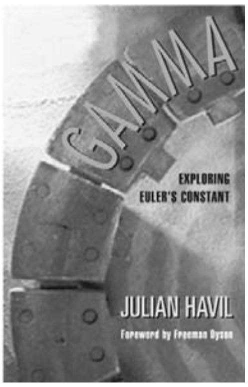
Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing. The copyright of all published material is vested in the Applied Probability Trust.

From the Editor

Gamma



Sometimes we receive books for review which, although somewhat difficult for our student readers, are nevertheless worth an airing in *Mathematical Spectrum*. This book is one such.

If you were asked to give a famous real number, you might reply with $\sqrt{2}$ or π or e — but γ?

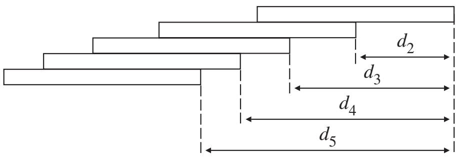


Figure 1.

Let's begin with the problem of overhanging planks (or playing cards, as in the book). You have an unlimited supply of equal planks and wish to make a pile which extends outwards as far as possible without toppling over. How far can you reach? Suppose we have n planks each of length 2 metres (say); figure 1 shows five. We measure the distances from the extreme point of the top plank to the ends of the lower planks as shown, numbering from the top (so that $d_1 = 0$). To avoid the pile's toppling over, the centre of gravity of the planks above any given plank must lie within that plank (the author's explanation on page 133 seems incorrect here). Maximum overhang will occur when this just happens. Try putting on the $(r+1)$th plank. Taking moments about the right-hand edge of the new plank,

$$(d_1+1)+(d_2+1)+(d_3+1)+\cdots+(d_r+1)=rd_{r+1}$$

(taking the planks of unit weight), so that

$$rd_{r+1}=r+d_1+d_2+\cdots+d_r\,,$$

and this must be true for $r=1,2,\ldots,n-1$. Thus,

$$(r-1)d_r=(r-1)+d_1+d_2+\cdots+d_{r-1}\,.$$

Hence,

$$rd_{r+1}-(r-1)d_r=1+d_r\,,$$

so that

$$d_{r+1}=d_r+\frac{1}{r}\quad\text{for } r=1,2,\ldots,n-1\,.$$

It follows that

$$d_n=\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{n-1}\,.$$

How big can we make this?

$$\begin{aligned}\tfrac{1}{3}+\tfrac{1}{4}&>\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2}\,,\\ \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}&>\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}=\tfrac{1}{2}\,,\\ \tfrac{1}{9}+\tfrac{1}{10}+\cdots+\tfrac{1}{16}&>8\times\tfrac{1}{16}=\tfrac{1}{2}\,,\end{aligned}$$

and so on. So we can make d_n as big as we like if n is large enough, and so the overlap can be made as large as you like.

This introduces the famous *harmonic series*

$$\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\cdots\,,$$

which is divergent. Mind you, it increases very slowly indeed as more terms are added. The sum of the first 100 terms is $5.187\ldots$, the sum of the first 1000 is $7.486\ldots$, the sum of the first million is $14.392\ldots$. The number of terms that you need to reach 100 is a 44-digit number. Yet it is divergent.

Come to think of it, you might be able to think of another function which increases very slowly. What about logarithms? Using base 10,

$$\log 10=1\,,\qquad \log 100=2\,,\qquad \log 1000=3\,,$$

and so on. Natural logarithms, to base e, behave similarly.

Consider the graph of $y=1/x$ in figure 2. Looking at the area under the curve from 1 to n, we have

$$\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{n}<\int_1^n\frac{1}{x}\,\mathrm{d}x<1+\tfrac{1}{2}+\cdots+\frac{1}{n-1}\,,$$

***Mathematical Spectrum* Awards for Volume 35**

Prizes have been awarded to the following student readers for contributions in Volume 35:

Yunpeng Li for the article 'Suppose Snow White agreed to take part as well' (with Paul Belcher);

Farshid Arjomandi for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

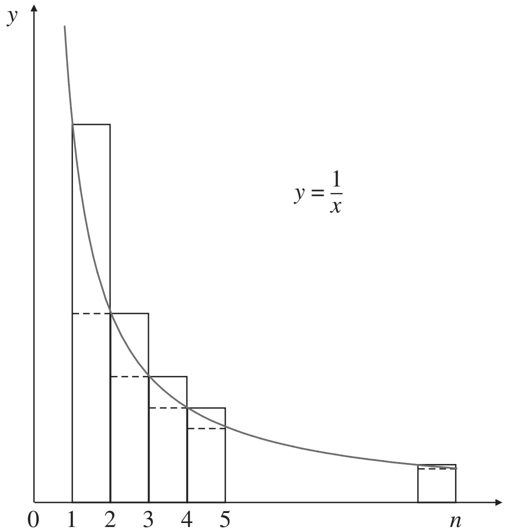


Figure 2.

so

$$H_n - 1 < \ln n < H_n - \frac{1}{n},$$

where

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{n}.$$

Hence,

$$\frac{1}{n} < H_n - \ln n < 1, \qquad (1)$$

so it does look as though the harmonic series and $\ln n$ are close. In fact, $H_n - \ln n$ tends to a constant as $n \to \infty$, and this constant is γ, the subject of the book, called *Euler's constant* after the great 18th century Swiss mathematician Leonhard Euler, who first introduced it.

Euler's constant remains a mystery to this day. How big is it? We can see from (1) that it is between 0 and 1. Is it a fraction like $\frac{1}{2}$ or $\frac{57}{100}$? No-one knows. It begins

$$\gamma = 0.577215\ldots,$$

and has been calculated to 108 million decimal places, believe it or not. In case you want to know, its millionth digit is 9.

There is some fascinating mathematics here, and you can delve into this book as far as you would like to go. It does shoot off into the stratosphere towards the end, with a discussion of the prime number theorem and the Riemann hypothesis — no proofs!

But why is it called the *harmonic series*? What has it got to do with music? Page 123 will tell you.

Coincidentally, our most assiduous correspondent has sent us a problem which seems to be connected to Euler's constant — see Problem 36.8. There's a clue to get you going!

Gamma. By Julian Havil. Princeton University Press, 2003. Pp. xiv+266. Hardback £19.95 (ISBN 0-691-09983-9).

Notes on Pascal's triangle

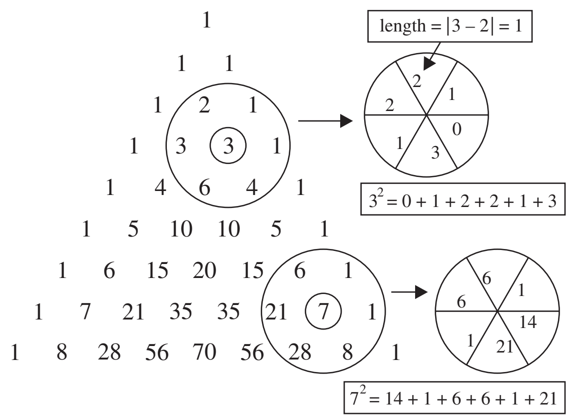
(a)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

$$\begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 6 \\ 1 & 5 & 10 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \\ 1 & 6 & 15 & 20 \\ 1 & 7 & 21 & 35 \end{vmatrix} = 1$$

(b)



Can you generalize these?

Seyamack Jafari
Razi Petrochemical Complex,
Bandar Imam, Khozestan, Iran.

An Arithmetic Problem and Special Antimagic Graphs

GERD WALTHER

1. Introduction

In a recent issue of *Mathematical Spectrum* (Volume 34, No. 3, p. 69) Hassan Shah Ali posed the following problem:

34.9 Find all natural numbers $n > 2$ for which there exists a permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ such that $\{a_1+a_2, a_2+a_3, \ldots, a_{n-1}+a_n, a_n+a_1\}$ forms a set of n consecutive natural numbers.

We give a solution of this problem in the context of graph labelling, thereby opening up a field of interesting investigations.

2. Antimagic and (a, d)-antimagic graphs

Definition 1. A *graph* G is a pair of sets (V, E) where V is not empty and E is a set of unordered pairs of elements of V. The elements of V are called *vertices* of G and the elements of E are called the *edges* of G.

In this article we will only consider graphs with finite vertex set V. Usually the vertices of a graph are represented by points in the plane and the edges by a straight or curved line connecting two vertices in the plane.

Graphs which are not split up into two or more 'pieces' are *connected*. Below, *graph* means *finite connected graph*. A graph G with at least one edge is called a *tree* if removal of any edge splits G into two pieces. Suppose that $G = (V, E)$ is a graph with n vertices and m edges. According to Hartsfield and Ringel (reference 1), G is called *antimagic* if the edges of G can be labelled by the integers $1, 2, 3, \ldots, m$ so that the sum of the labels at any given vertex is different from the sum of the labels at any other vertex. This condition means that no two vertices of G have the same sum.

Hartsfield and Ringel conjectured that every connected graph other than the graph with two vertices and one edge (which we will call K_2) is antimagic. However, graph theorists have not even succeed in proving the weaker conjecture that every tree different from K_2 is antimagic.

Some years ago, R. Bodendiek and the author tried to attack that problem by first of all investigating a much more restricted version of antimagic graphs (see references 2–4).

Definition 2. Let $G = (V, E)$ be a graph with n vertices and m edges and let a, d be positive integers. If the edges of G can be labelled by the integers $1, 2, \ldots, m$ so that the numbers resulting from the sums of the labels at vertices of G form an arithmetic progression $a,\ a+d, \ldots, a+(n-1)d$, then G is called (a, d)-antimagic. The corresponding edge labelling of G is called an (a, d)-antimagic labelling of G.

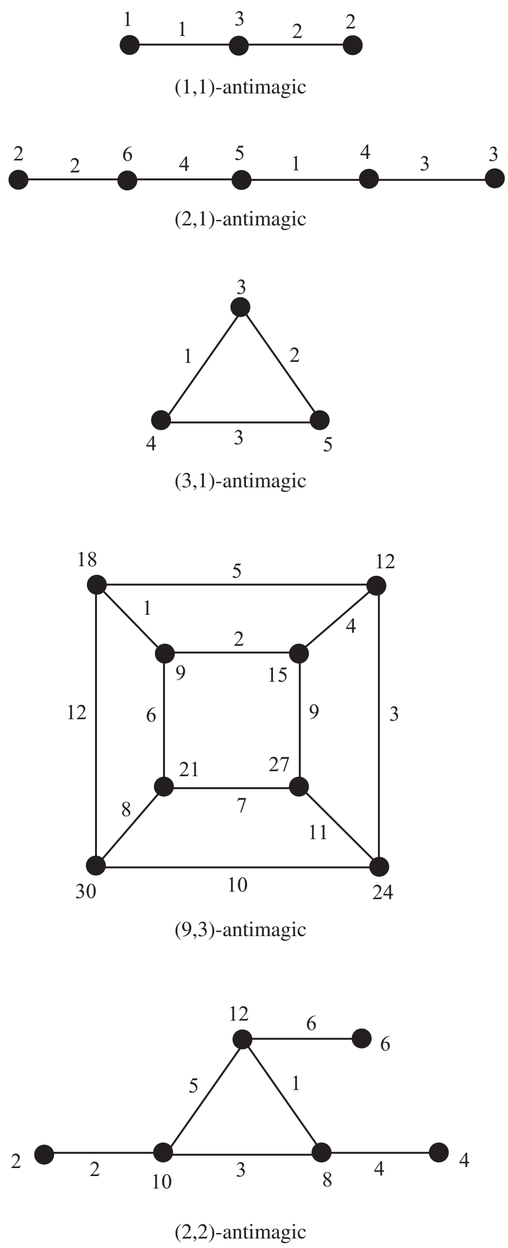


Figure 1. Examples of (a, d)-antimagic graphs.

Figure 1 shows examples of (a, d)-antimagic graphs.

Obviously, every (a, d)-antimagic graph is antimagic. Before we look at the converse statement, we prove a necessary condition for (a, d)-antimagic graphs.

3. A necessary condition for (a, d)-antimagic graphs

Theorem 1. *Let $G = (V, E)$ be a graph with n vertices and m edges. If G is (a, d)-antimagic for suitable $a, d \in \mathbb{N}$, then*

$$2m(m + 1) = (2a + (n - 1)d)n\,. \qquad (1)$$

If μ is the minimum vertex degree of G, then

$$a \geq 1 + 2 + \cdots + \mu\,.$$

Proof. If $G = (V, E)$ is (a, d)-antimagic for suitable $a, d \in \mathbb{N}$, then twice the sum of the edge labels $1, 2, \ldots, m$ is equal to the sum of the induced vertex labels $a, a+d, \ldots, a + (n - 1)$; thus

$$2(1 + 2 + \cdots + m) = a + (a + d) + \cdots + a + (n - 1)d\,.$$

A simple computation yields

$$2m(m + 1) = (2a + (n - 1)d)n\,.$$

The second statement is obvious.

Compared to antimagic graphs, the concept of (a, d)-antimagic graphs has the advantage of leading immediately to a necessary criterion in the language of arithmetic. A graph G with n vertices and m edges has no (a, d)-antimagic labelling for any $a, d \in \mathbb{N}$ if the diophantine equation

$$2an + n(n - 1)d = 2m(m + 1) \qquad (2)$$

resulting from (1) has no solution in positive integers a, d.

As an application, we try to find out whether the cycle graph C_4 with 4 vertices and 4 edges and the path P_4 with 4 vertices and 3 edges is (a, d)-antimagic.

For P_4 we get from (2) the equation $2a+3d = 6$, which obviously has no solutions in positive integers. Hence, P_4 and any other graph with 4 vertices and 3 edges cannot be (a, d)-antimagic for any (a, d).

The equation for C_4 is $2a + 3d = 10$, with the unique solution $a = 2, d = 2$ in positive integers. Since the minimum vertex degree is 2, we have $a \geq 1 + 2$. Consequently, C_4 is not (a, d)-antimagic for any (a, d).

It is easy, however, to show that P_4 and C_4 are antimagic graphs.

4. (a, d)-antimagic cycles and the solution of problem 34.9

After warming up with antimagic graphs we come to a graph-theoretical solution of Hassan Shah Ali's problem stated above.

Theorem 2. *Every cycle C_{2k+1}, $k \in \mathbb{N}$, with $2k+1$ vertices and $2k + 1$ edges is $(k + 2, 1)$-antimagic.*

Before we prove this theorem we try to understand what this statement means with respect to problem 34.9.

For $k \geq 1$, the $2k + 1$ integers $1, 2, \ldots, 2k + 1$ can be assigned to the $2k + 1$ edges of C_{2k+1} so that the numbers corresponding to the sums of the labels at the vertices of C_{2k+1} form an arithmetic progression $k+2, k+3, \ldots, 3k + 2$.

Since the number at any vertex of C_{2k+1} is the sum of the labels of the two adjacent edges, we get the following partial answer to the above problem: for each odd number $n = 2k+1 \geq 3$ there exists a permutation $a_1, \ldots, a_n$ of $1, 2, \ldots, n$ such that $\{a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + a_1\}$ forms a set of n consecutive natural numbers.

The following proof of theorem 2 is constructive in the sense that the permutation $a_1, \ldots, a_n$ of $1, \ldots, n$ is explicitly constructed.

Proof. Suppose that C_{2k+1}, $k \in \mathbb{N}$, is (a, d)-antimagic. Since C_{2k+1} has $2k + 1$ vertices and $2k + 1$ edges, from (2) we get the diophantine equation

$$a + kd = 2(k + 1)\,, \qquad (3)$$

which is solvable.

As a special solution of (3) we immediately guess $a_0 = 2$, $d_0 = 2$. From elementary number theory we know that each solution of (3) can be represented in parametric form:

$$a = a_0 + \lambda k\,, \quad d = d_0 - \lambda\,, \quad \text{where } \lambda \in \mathbb{Z}\,. \qquad (4)$$

Only $\lambda = 1$, $\lambda = 0$, $\lambda = -1$ lead to non-negative values for a and d. But $a = 2$, $d = 2$ and $a = 2 - k$, $d = 3$ can be discarded because $a \geq 3$. Hence, the only solution of (3) in natural numbers is $a = 2 + k$, $d = 1$.

We obtain a $(k + 2, 1)$-antimagic labelling of C_{2k+1}, $k \geq 1$, as shown in figure 2.

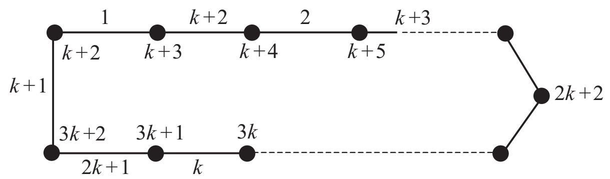


Figure 2. $(k + 2, 1)$-antimagic labelling of C_{2k+1}.

The edge labelling of C_{2k+1} can be described in the following way. Choose an edge and label it with 1, jump over the adjacent edge and label the next edge with 2 etc., until the last edge of C_{2k+1} is labelled with $2k + 1$. Starting with the vertex incident with the edges labelled with $k + 1$ and 1 and walking clockwise around the cycle C_{2k+1}, the vertex numbers are the sequence of integers $a := k + 2, a + 1, a + 2, \ldots, a + 2k$.

Let us now consider cycles C_{2k}, $k \geq 2$, with an even number of vertices and edges. In a similar manner as in theorem 2 we can prove the following theorem.

Theorem 3. *None of the cycles C_{2k}, $k \geq 2$, is (a, d)-antimagic for any a, d.*

Proof. As in the odd case, we show that the corresponding diophantine equation

$$2a + (2k - 1)d = 2(1 + 2k)$$

has $a = 2, d = 2$ as its unique solution in natural numbers. In a cycle, however, $a \geq 3$, which contradicts $a = 2$.

Let us come back to problem 34.9. Given $n \geq 3$, each permutation $a_1, \ldots, a_n$ of $1, 2, \ldots, n$ which satisfies the other condition of the problem obviously corresponds to an $(a, 1)$-antimagic labelling of a cycle C_n for some a and vice versa. Theorems 2 and 3 show that the odd numbers $n \geq 3$ are solutions of the problem and are the only solutions.

5. Investigating (a, d)-antimagic graphs

Now we come back to (a, d)-antimagic graphs and propose a couple of investigations.

1. Figure 1 shows (a, d)-antimagic labellings of P_3, P_5, the paths with 3 and 5 vertices respectively. Which of the paths P_n, $n > 1$, are (a, d)-antimagic?
2. The cube graph in figure 1 has a $(9, 3)$-antimagic labelling. Determine all (a, d)-antimagic labellings of this graph.
3. Show that the complete graph K_4 with 4 vertices and 6 edges is not (a, d)-antimagic for any a, d. Find a complete graph K_n, $n > 4$, which is (a, d)-antimagic for some a, d.
4. Find trees which are not paths and which are (a, d)-antimagic.
5. Let C_n, $n \geq 2$, be a cycle with n vertices. Paste an edge at each vertex of C_n. The resulting graph is a *crown* Cr_n with $2n$ vertices and $2n$ edges. Which of the crowns are (a, d)-antimagic?
6. Investigate whether graphs or classes of graphs that you know are (a, d)-antimagic.
7. Is there a ladder graph which is (a, d)-antimagic? (See figure 3.)

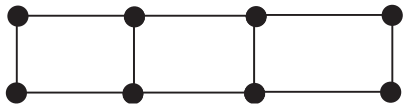


Figure 3. Ladder graph.

A big problem remains unsolved: determine the class of all (a, d)-antimagic graphs.

References

1. N. Hartsfield and R. Ringel, *Pearls in Graph Theory* (Academic Press, San Diego, 1990).
2. R. Bodendiek and G. Walther, Arithmetisch-antimagische Numerierungen, in *Graphentheorie III*, eds K. Wagner and R. Bodendiek (Bibliographisches Institut, Mannheim, 1993), pp. 260–270.
3. R. Bodendiek and G. Walther, On number theoretical methods in graph labellings, in *General Algebra and Discrete Mathematics*, eds K. Deneke and O. Lüders (Heldermann Verlag, Lemgo, 1995), pp. 3–25.
4. R. Bodendiek and G. Walther, On (a, d)-antimagic graphs, *Mitteilungen der Mathematischen Gesellschaft Hamburg* **17** (1998), pp. 85–99.

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Change of Base and the Golden Section

BOB BERTUELLO and GUIDO LASTERS

This article deals with changes of base from base 10 to other bases, i.e. to integer, rational or irrational bases, and concludes with an unexpected appearance of φ, the golden section. Numbers in a base other than 10 that need to be converted can easily be dealt with by first converting them to base 10.

We start with a simple example, to convert from an integer base to another integer base. Bases are represented by a suffix to a number in brackets, except that base 10 numbers are shown with no brackets or suffix.

Example 1. Convert 65 to base 7. To do this, we divide repeatedly by 7 and note the remainders in reverse order. That is, we proceed thus:

$$\begin{array}{rl} 7\,\overline{)65} & \\ 7\,\overline{)\ 9} & \text{rem } 2 \\ 7\,\overline{)\ 1} & \text{rem } 2 \\ 0 & \text{rem } 1 \end{array}$$

Therefore, $65 = (122)_7$.

If a decimal part is involved, we must proceed differently.

Example 2. Convert 0.63 to base 5. To do this, we repeatedly multiply the decimal part only by the new base and note the sequence of integer parts:

$$\begin{aligned} &0.63\\ 0.63\times 5 &= 3.15\\ 0.15\times 5 &= 0.75\\ 0.75\times 5 &= 3.75\\ 0.75\times 5 &= 3.75\\ &\vdots \end{aligned}$$

Therefore, $0.63 = (0.30\underline{3})_5$, where the underlined digit repeats.

In cases such as this, where a terminating decimal fraction is converted to an integer base, the result always either terminates or contains a repeating block.

If the rational number to be converted is written as a vulgar fraction, it may be more convenient to use the following division algorithm.

Example 3. Convert $\frac{5}{7}$ to base 9. One way to do this is to write $\frac{5}{7}$ as a sum of powers of $\frac{1}{9}$ multiplied by coefficients:

$$\begin{aligned} \frac{5}{7} &= \frac{1}{9}\left(\frac{5}{7}\cdot 9\right) = \frac{1}{9}\left(6+\frac{3}{7}\right)\\ &= \frac{1}{9}\cdot 6+\frac{1}{9^2}\left(\frac{3}{7}\cdot 9\right) = \frac{1}{9}\cdot 6+\frac{1}{9^2}\left(3+\frac{6}{7}\right)\\ &= \frac{1}{9}\cdot 6+\frac{1}{9^2}\cdot 3+\frac{1}{9^3}\left(\frac{6}{7}\cdot 9\right)\\ &= \frac{1}{9}\cdot 6+\frac{1}{9^2}\cdot 3+\frac{1}{9^3}\cdot 7+\frac{1}{9^4}\left(\frac{5}{7}\cdot 9\right)\\ &= \frac{1}{9}\cdot 6+\frac{1}{9^2}\cdot 3+\frac{1}{9^3}\cdot 7+\frac{1}{9^4}\cdot 6+\cdots, \end{aligned}$$

and the sequence 6, 3, 7 of coefficients repeats. Therefore, $\frac{5}{7} = (0.\underline{637})_9$.

An alternative, and faster, method for the above is the following: repeatedly divide by the denominator, 7, and multiply each remainder by the base, 9:

$$\begin{array}{ll} 7\,\overline{\lceil 5} & \\ \quad 0 & \text{rem } 5\\ & \swarrow \times 9\\ 7\,\overline{\lceil 45} & \\ \quad 6 & \text{rem } 3\\ & \swarrow \times 9\\ 7\,\overline{\lceil 27} & \\ \quad 3 & \text{rem } 6\\ & \swarrow \times 9\\ 7\,\overline{\lceil 54} & \\ \quad 7 & \text{rem } 5\\ & \swarrow \times 9\\ 7\,\overline{\lceil 45} & \cdots \end{array}$$

We again obtain the repeating sequence 6, 3, 7.

It is of interest to note that, when converted to an integer base, a rational n/p with p prime will have at most $p-1$ repeated digits, whatever the base. For example,

$$\begin{aligned} \tfrac{1}{5} &= (0.\underline{4, 13, 18, 9})_{23},\\ \tfrac{5}{7} &= (0.\underline{37, 7, 22, 14, 44, 29})_{52},\\ \tfrac{4}{7} &= (0.\underline{7, 5})_{13} = (0.\underline{6, 3, 1})_{11}. \end{aligned}$$

We now consider conversion to rational bases. This may be done in the following way. To convert a number n to base b, first find the highest required power p of the base, that is, find p such that $b^p \le n$, so

$$p = \left\lfloor \frac{\log n}{\log b} \right\rfloor$$

(where $\lfloor\cdot\rfloor$ is the integer-part function). The required number is then

$$a_1 b^p + a_2 b^{p-1} + \cdots + a_p b + a_{p+1} + a_{p+2} b^{-1} + \cdots,$$

where

$$\begin{aligned} a_1 &= \left\lfloor \frac{n}{b^p} \right\rfloor,\\ a_i &= \lfloor d_{i-1} b \rfloor \quad \text{for } i>1, \end{aligned}$$

with

$$\begin{aligned} d_1 &= \frac{n}{b^p} - \left\lfloor \frac{n}{b^p} \right\rfloor\\ &= \text{the decimal part of } \frac{n}{b^p},\\ d_i &= d_{i-1}b - \lfloor d_{i-1} b\rfloor\\ &= \text{the decimal part of } d_{i-1}b \quad \text{for } i>1. \end{aligned}$$

Example 4. Convert 3178.22 to base 5.2. Here, $p = \lfloor \log 3178.22/\log 5.2 \rfloor = 4$,

$$\begin{array}{ll} a_1 = \left\lfloor \dfrac{3178.22}{5.2^4} \right\rfloor = 4, & d_1 = 0.346809\ldots,\\ a_2 = \lfloor 0.346809\times 5.2\rfloor = 1, & d_2 = 0.803408\ldots,\\ a_3 = \lfloor 0.803408\times 5.2\rfloor = 4, & d_3 = 0.177722\ldots,\\ a_4 = \lfloor 0.177722\times 5.2\rfloor = 0, & d_4 = 0.924154\ldots,\\ a_5 = \lfloor 0.924154\times 5.2\rfloor = 4, & d_5 = 0.805600\ldots,\\ a_6 = \lfloor 0.805600\times 5.2\rfloor = 4, & d_6 = 0.189120\ldots,\\ a_7 = \lfloor 0.189120\times 5.2\rfloor = 0, & d_7 = 0.983424\ldots,\\ a_8 = \lfloor 0.983424\times 5.2\rfloor = 5 & \text{etc.} \end{array}$$

Therefore, $3178.22 = (41404.405\ldots)_{5.2}$.

We now look at some particular cases. Generally, rationals converted to a rational base do not seem to repeat. For example,

$$\begin{aligned} \tfrac{1}{5} &= (0.10102101504\ldots)_{5.2},\\ \tfrac{1}{2} &= (0.2303112044311\ldots)_{5.2}. \end{aligned}$$

However, there are examples that *do* repeat with base 5.2:

$$\frac{120}{217} = (0.\underline{24})_{5.2},$$
$$\frac{1685}{5817} = (0.\underline{123})_{5.2}.$$

Now, in an irrational base,

$$\tfrac{1}{5} = (0.00001000001000000001\ldots)_{\sqrt{2}}$$

does not repeat, but

$$\tfrac{1}{3} = (0.\underline{0001})_{\sqrt{2}}$$

does. Thus, a fraction can be represented as a repeating decimal in an irrational base.

Now consider $\frac{1}{5}$ converted to base $\frac{3}{2}$. Using one of our previous methods, we obtain

$$\tfrac{1}{5} = (0.000100000000000010000001001\ldots)_{1.5},$$

that is,

$$\tfrac{1}{5} = (\tfrac{2}{3})^4 + (\tfrac{2}{3})^{15} + (\tfrac{2}{3})^{22} + (\tfrac{2}{3})^{25} + \cdots .$$

We wondered if there could be two or more adjacent 1s in this expansion, but, looking at the above, it did not look very promising. However, the subsequent search resulted in the following.

We discovered that $\frac{1}{5}$ could be converted to base $\frac{3}{2}$ in a different way. Since

$$(0.000011)_{1.5} = 0.21947\ldots,$$
$$(0.0000011)_{1.5} = 0.146319\ldots,$$

we can conclude that $\frac{1}{5} = 0.2$ lies between these two values.

We then tried the following:

$$(0.00000111)_{1.5} = 0.18533\ldots,$$
$$(0.000001111)_{1.5} = 0.21134\ldots,$$
$$(0.0000011101)_{1.5} = 0.202679\ldots,$$
$$(0.00000111001)_{1.5} = 0.196898\ldots,$$
$$(0.00000111001001)_{1.5} = 0.200324\ldots .$$

This last term is now close to $\frac{1}{5}$ and has three adjacent 1s. Now, the mystifying thing is that, usually, when working with positive integer bases,

$$0.000011 < 0.0001.$$

But here we have

$$(0.000011)_{1.5} = 0.21947 > (0.0001)_{1.5} = 0.1975\ldots .$$

After some head-scratching, the reason was found to be this:

$$(\tfrac{2}{3})^5 + (\tfrac{2}{3})^6 > (\tfrac{2}{3})^4,$$

that is,

$$(\tfrac{2}{3})(\tfrac{2}{3})^4 + (\tfrac{2}{3})^2(\tfrac{2}{3})^4 > (\tfrac{2}{3})^4.$$

Dividing by $(\frac{2}{3})^4$, we get

$$\tfrac{2}{3} + \tfrac{4}{9} > 1,$$

which is correct.

So, where is the dividing line between the expected and the unexpected?

Let c be the critical base that divides these two cases. Then, for the above case, we have

$$\left(\frac{1}{c}\right)^5 + \left(\frac{1}{c}\right)^6 > \left(\frac{1}{c}\right)^4,$$
$$\left(\frac{1}{c}\right)\left(\frac{1}{c}\right)^4 + \left(\frac{1}{c}\right)^2\left(\frac{1}{c}\right)^4 > \left(\frac{1}{c}\right)^4,$$

or

$$\frac{1}{c} + \left(\frac{1}{c}\right)^2 \geq 1.$$

Whence $c^2 - c - 1 \leq 0$. Solving this, we find that $c < (1 + \sqrt{5})/2$, ignoring the negative value. Thus, c is the golden ratio (or golden section), $\varphi = 1.618033988\ldots$ (see remark 1 and reference 1). Hence,

$$(0.01)_b > (0.0011)_b \text{ in any base } b > \varphi,$$
$$(0.01)_b < (0.0011)_b \text{ in any base } b < \varphi,$$
$$(0.01)_\varphi = (0.0011)_\varphi \text{ in base } \varphi$$

since

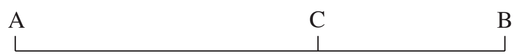
$$(0.01)_\varphi = \frac{1}{\varphi^2} = 0.381966\ldots$$

and

$$(0.0011)_\varphi = \frac{1}{\varphi^3} + \frac{1}{\varphi^4} = 0.381966\ldots .$$

This is quite extraordinary. There are many areas where φ may be encountered but we have never found it in the area of change of base where the base is a rational or an irrational number greater than 1.

Remark 1. The golden section is generally defined by the division of a straight line, as follows:



Let the line AB be divided into two segments by the point C. If AB/AC = AC/CB, then C is the golden cut or golden section of AB.

Of the many properties of φ, one of the best known and fascinating is that the ratio of successive terms of the Fibonacci sequence get ever closer to the golden section as the numbers of the sequence increase. Recall that the Fibonacci sequence is generated by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$. The sequence begins $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ and $\frac{144}{89} = 1.6179775\ldots$.

Remark 2. So far, all the bases considered have been greater than 1. But we can ask whether it is feasible to construct a number system with a positive base less than 1.

Suppose that we decide on 0.5 as a base and limit the digits of base 0.5 to 0 and 1. We then obtain a number system that looks like a reverse (or mirror image) of the binary system (but with the decimal (radix) point displaced one place to the left).

We would then obtain the following results:

$$(1)_{0.5} = 1, \qquad (0.1)_{0.5} = \frac{1}{0.5} = 2,$$

$$(10)_{0.5} = 1 \times 0.5 = 0.5, \qquad (0.01)_{0.5} = \frac{1}{0.5^2} = 4,$$

$$(100)_{0.5} = 1 \times 0.5^2 = 0.25, \qquad (0.001)_{0.5} = \frac{1}{0.5^3} = 8.$$

As examples, we have

$$15.75 = (111.111)_{0.5}$$

whereas, in binary,

$$15.75 = (1111.11)_2$$

and

$$1.5 = (11)_{0.5}$$

whereas, in binary,

$$1.5 = (1.1)_2.$$

So, it would appear that, as just defined, a base less than 1 could form a feasible number system.

Remark 3. A question that requires an answer is this: if a rational is converted to a rational base, is it generally true that the answer is also rational, i.e. will it either terminate or include a repeating block?

Readers may like to prove some of the assertions made in this article.

Reference

1. H. E. Huntley, *The Divine Proportion: a Study in Mathematical Beauty* (Dover, New York, 1970).

Bob Bertuello *worked in the Singapore Dockyard Design Offices (with log tables and slide rules) before working on computer-aided ship design in Bath, where he advanced to the use of a Sirius Computer (complete with 5-hole paper tape and programming in Sirius Autocode). He retired in 1989. His long-suffering wife says, 'he's nuts on numbers'.*

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A Series of Intrigues or Just Some Intriguing Series?

P. GLAISTER

In my recent article 'Squaring up to factorials' (reference 1) I began by inviting readers to seek series $a_1 + a_2 + a_3 + \cdots$ having the property

$$\frac{1}{a_1 + a_2 + a_3 + \cdots} = a_1 - a_2 + a_3 - \cdots$$

other than the well-known example for e, i.e.

$$\begin{aligned}\frac{1}{\mathrm{e}} &= \frac{1}{\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots} \\ &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \mathrm{e}^{-1},\end{aligned}$$

which is based on the series for the exponential function

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1}$$

I then completely forgot about it! Until recently, that is, when I noticed a similar relationship in an earlier article of mine (reference 2) which is based on the two series

$$\begin{aligned}\int_0^1 x^x \, \mathrm{d}x &= \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots, \\ \int_0^1 x^{-x} \, \mathrm{d}x &= \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots,\end{aligned} \tag{2}$$

namely

$$\begin{aligned}\int_0^1 f(x) \, \mathrm{d}x &= a_1 + a_2 + a_3 + \cdots, \\ \int_0^1 \frac{1}{f(x)} \, \mathrm{d}x &= a_1 - a_2 + a_3 - \cdots,\end{aligned} \tag{3}$$

which holds for $f(x) = x^{-x}$ and $a_i = 1/i^i$. This property says that where a particular series can be expressed in terms of an integral of some function f, the corresponding alternating series is the integral of the reciprocal of f. Naturally I then began a frantic search for other examples.

One of the more obvious examples takes us back to the series for the exponential function e^x since, from (1),

$$\int_0^1 \mathrm{e}^x \, \mathrm{d}x = \mathrm{e} - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots ,$$
$$\int_0^1 \mathrm{e}^{-x} \, \mathrm{d}x = 1 - \mathrm{e}^{-1} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots ,$$

so that (3) holds for $f(x) = \mathrm{e}^x$ and $a_i = 1/i!$.

My next thought was the generalisation of (2) based on the following expansion in reference 2:

$$\int_0^1 x^{\alpha x+\beta} \, \mathrm{d}x = \frac{1}{(\beta+1)^1} - \frac{\alpha}{(\beta+2)^2} + \frac{\alpha^2}{(\beta+3)^3} - \frac{\alpha^3}{(\beta+4)^4} + \cdots . \qquad (4)$$

Unfortunately, the property in (3) only works in the case $\beta = 0$, i.e.

$$\int_0^1 x^{-\alpha x} \, \mathrm{d}x = \frac{1}{1^1} + \frac{\alpha}{2^2} + \frac{\alpha^2}{3^3} + \frac{\alpha^3}{4^4} + \cdots ,$$
$$\int_0^1 x^{\alpha x} \, \mathrm{d}x = \frac{1}{1^1} - \frac{\alpha}{2^2} + \frac{\alpha^2}{3^3} - \frac{\alpha^3}{4^4} + \cdots ,$$

so that $f(x) = x^{-\alpha x}$ and $a_i = \alpha^{i-1}/i^i$.

So that would appear to be that, or so I thought, until I tried a substitution in (4). With $x = X^\delta$, where $\delta > 0$, the left-hand side of (4) becomes

$$\delta \int_0^1 X^{\delta\alpha X^\delta + \delta(\beta+1)-1} \, \mathrm{d}X = \frac{1}{\beta+1} \int_0^1 X^{(\alpha/(\beta+1))X^{1/(\beta+1)}} \, \mathrm{d}X \qquad (5)$$

by choosing $\delta = 1/(\beta+1)$. If we now set $\alpha/(\beta+1) = p$ and $1/(\beta+1) = q$ in the right-hand side of (4) and use (5), then after rearrangement we find that

$$\int_0^1 X^{pX^q} \, \mathrm{d}X = 1 - \frac{p}{(1+q)^2} + \frac{p^2}{(1+2q)^3} - \frac{p^3}{(1+3q)^4} + \cdots . \qquad (6)$$

The expression in (6) shows that (3) holds for $f(x) = x^{-px^q}$, $a_i = p^{i-1}/(1+(i-1)q)^i$ as another solution to our problem. As a bonus we obtain some other interesting integral-series relationships from (6), such as

$$\int_0^1 X^{-X^2} \, \mathrm{d}X = \frac{1}{1^1} + \frac{1}{3^2} + \frac{1}{5^3} + \cdots$$

and

$$\int_0^1 X^{X^2} \, \mathrm{d}X = \frac{1}{1^1} - \frac{1}{3^2} + \frac{1}{5^3} - \cdots ,$$

with $p = \pm 1$, $q = 2$, as well as being an example of the property in (3). Further, substituting $p = a/c$, $q = b/c$ in (6) yields

$$\frac{1}{c} \int_0^1 X^{(a/c)X^{b/c}} \, \mathrm{d}X = \frac{1}{c} - \frac{a}{(c+b)^2} + \frac{a^2}{(c+2b)^3} - \frac{a^3}{(c+3b)^4} + \cdots , \qquad (7)$$

which is a generalisation of (6) and an example of (3) by replacing 'a' by '$-a$'. Moreover, substituting $x = X^{b/c}$ in (7) gives

$$\frac{1}{b} \int_0^1 x^{(a/b)x + c/b - 1} \, \mathrm{d}x = \frac{1}{c} - \frac{a}{(c+b)^2} + \frac{a^2}{(c+2b)^3} - \frac{a^3}{(c+3b)^4} + \cdots , \qquad (8)$$

which, although not an example of (3), does give series such as

$$1 - \tfrac{1}{2} \int_0^1 x^{(1/2)x+1} \, \mathrm{d}x = \frac{1}{2^0} - \frac{1}{4^1} + \frac{1}{6^2} - \cdots ,$$

with $a = 1$, $b = 2$ and $c = 4$, or, for a more complicated example with $a = -1$ and $b = 2$,

$$\begin{aligned}
\frac{1}{2^2} + \frac{1}{4^3} + \frac{1}{6^4} + \cdots
&= \lim_{c\to 0} \left(\frac{1}{(c+2)^2} + \frac{1}{(c+4)^3} + \frac{1}{(c+6)^4} + \cdots \right) \\
&= \lim_{c\to 0} \left(\tfrac{1}{2} \int_0^1 x^{(-1/2)x + c/2 - 1} \, \mathrm{d}x - \frac{1}{c} \right) \\
&= \lim_{c\to 0} \left(\tfrac{1}{2} \int_0^1 (x^{(-1/2)x + c/2 - 1} - x^{c/2-1}) \, \mathrm{d}x \right) \\
&= \lim_{c\to 0} \left(\tfrac{1}{2} \int_0^1 (x^{(-1/2)x} - 1) x^{c/2-1} \, \mathrm{d}x \right) \\
&= \tfrac{1}{2} \int_0^1 \frac{x^{(-1/2)x} - 1}{x} \, \mathrm{d}x
\end{aligned}$$

using (8), i.e.

$$\tfrac{1}{2} \int_0^1 \frac{x^{(-1/2)x} - 1}{x} \, \mathrm{d}x = \frac{1}{2^2} + \frac{1}{4^3} + \frac{1}{6^4} + \cdots .$$

Returning to the original problem, the reason why $f(x) = x^{-x}$ satisfies (3) is that this is essentially an exponential function, i.e. $x^{\pm x} = \mathrm{e}^{\ln x^{\pm x}} = \mathrm{e}^{\pm x \ln x}$, and is a special case of the obvious general solution

$$\begin{aligned}\int_0^1 e^{g(x)}\,dx &= \int_0^1 \Big(1 + g(x) + \frac{g(x)^2}{2!} + \cdots\Big)\,dx \\ &= \int_0^1 1\,dx + \int_0^1 g(x)\,dx + \frac{1}{2!}\int_0^1 g(x)^2\,dx + \cdots ,\\ \int_0^1 \frac{1}{e^{g(x)}}\,dx &= \int_0^1 e^{-g(x)}\,dx \\ &= \int_0^1 1\,dx - \int_0^1 g(x)\,dx + \frac{1}{2!}\int_0^1 g(x)^2\,dx - \cdots ,\end{aligned} \tag{9}$$

with

$$g(x) = x\ln x.$$

It can also be shown that (6) is a special case of (9) with

$$g(x) = px^q \ln x,$$

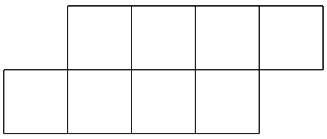
which we leave readers to struggle with. Further results can be found by taking other functions $g(x)$ in the form of a logarithm, although the individual terms on the right-hand side of (9) can also be quite tricky to determine.

Any other examples?

References

1. P. Glaister, Squaring up to factorials, *Math. Spectrum* **34** (2001/02), pp. 53–55.
2. P. Glaister, To infinity and beyond — further mathematical journeys, *Math. Spectrum* **34** (2001/02), pp. 26–29.

The author lectures in mathematics at Reading University. He never ceases to be amazed by the wealth of interesting mathematics problems that his children are set for homework. Many of these are combinatorial in nature, which is definitely not one of his strong points! The most recent one was 'How many arrangements of the numbers 1 to 8 can be made on the grid below so that consecutive numbers are not adjacent to one another, either vertically, horizontally or diagonally?' There are obviously many variations on this puzzle.



Fermat's Method of Factorization

XIANG GUI

Although it is an easy computational exercise to multiply two positive integers, it is quite a different matter to go in the opposite direction and factorize a number. Modern cryptography relies on this. We describe here a method of factorization which Pierre Fermat developed in the 17th century. For an odd composite number $n = ab$ (where a, b are integers and $1 < b < a < n$), we can express n as the difference of two squares thus:

$$n = \Big(\frac{a+b}{2}\Big)^2 - \Big(\frac{a-b}{2}\Big)^2.$$

Note that a and b will both be odd, so that $(a+b)/2$ and $(a-b)/2$ are integers. Also $(a+b)/2 - (a-b)/2 = b > 1$. (We need not concern ourselves with even positive integers n, which can easily be factorized as a product of two smaller positive integers!) Conversely, if we can express n in the form

$$n = x^2 - y^2, \tag{1}$$

where x and y are positive integers which differ by more than 1, then

$$n = (x+y)(x-y)$$

and we have factorized n. (The expression

$$n = \Big(\frac{n+1}{2}\Big)^2 - \Big(\frac{n-1}{2}\Big)^2$$

is no help in the attempt to factorize n.) Thus, the aim is to try to express n as a difference of two squares which differ by more than 1. If we rewrite (1) in the form

$$y^2 = x^2 - n,$$

then we are looking for a positive integer x greater than $\sqrt{n}$ such that $x^2 - n$ is a perfect square. If k is the smallest positive integer greater than or equal to $\sqrt{n}$, then the procedure is to look at $k, k+1, k+2, \ldots$ in turn until we find such an x. Provided that n has two factors of approximately the same size, and hence close to $\sqrt{n}$, this method gives a solution fairly quickly. Fermat used it to obtain the factorization

$$2\,027\,651\,281 = 44\,021 \times 46\,061\,. \qquad (2)$$

We can halve the number of integers that need to be checked. If $n \equiv 1 \pmod 4$, then we can see from (1) that x must be odd (and y even); if $n \equiv 3 \pmod 4$, then x must be even (and y odd). This is because an odd square is congruent to $1 \pmod 4$ and an even square is congruent to $0 \pmod 4$. So we need only consider alternate numbers in looking for x. For example, if $n = 8051$, then $n \equiv 3 \pmod 4$ and $\sqrt{8051} = 89.7\ldots$, so we need only try $x = 90, 92, 94, \ldots$. In fact $90^2 - 8051 = 7^2$, so $8051 = 97 \times 83$. In Fermat's example,

$$2\,027\,651\,281 \equiv 1 \pmod 4$$

and

$$\sqrt{2\,027\,651\,281} = 45\,029.4\ldots\,,$$

so we look at the values $k = 45\,031, 45\,033, \ldots$. We quickly reach $k = 45\,041$, and $45\,041^2 - 2\,027\,651\,281 = 1020^2$, so $x = 45\,041$ and $y = 1020$, giving the factorization (2).

__Xiang Gui__ is currently a sixth-form student at Borden Grammar School, Kent. He became fascinated with number theory after reading about Fermat's last theorem.

From Two- to Three-Dimensional Transformations with DERIVE™ 5

P. SCHOFIELD

Extending two-dimensional linear transformations to their three-dimensional counterparts with the aid of a users' file of DERIVE 5.06.

There are many excellent mathematical IT programs (for educational use) that can deal with the topic of two-dimensional linear transformations. For example, Autograph, Cabri II, Geometer's Sketchpad, Omnigraph. However, apart from special cases, none of these expands the topic systematically beyond two dimensions. DERIVE has the advantage of having both two- and three-dimensional plot windows for graphical interpretation of algebraic expressions. This makes it an excellent package for attempting an expansion of the common two-dimensional linear transformations into their three-dimensional counterparts. Over the past two years I have contributed to the teaching of a module on linear algebra and transformation geometry for college students following primary BA degree and secondary PGCE courses with a specialist mathematical content. In particular, with the help of these students, I have explored how to use DERIVE to interpret geometrically both two- and three-dimensional linear transformations. These ideas can currently be found in a users' file of DERIVE 5.06 '2D-&3D-Transformations.dfw' which can be downloaded from the DERIVE web page (reference 1).

1. General-purpose transformers

A general-purpose transformer is an instruction that will process expressions for plotting in both the 2D- and 3D-plot windows (depending upon the information in its argument places). In addition, it can be applied to many types of objects that can be plotted by DERIVE. It can process single objects or a list of such objects. General-purpose transformers provide a significant boost to the power of DERIVE for performing linear transformation activities in two and three dimensions. A linear transformation can be thought of as acting on a point, an object, or the whole of the domain space. With their ability to process whole collections of objects (of various types), general-purpose transformers are ideal tools for illustrating these aspects of transformations in two or three dimensions.

As well as the general-purpose transformers, 2D-&3D-Transformations contains a number of instructions for constructing and plotting a collection of objects (to transform). These include the following laminas: square, triangle, parallelogram, regular polygon, circle; and the three-dimensional closed surfaces: cube, cuboid, tetrahedron, pyramid, parallelepiped, cone, double cone, sphere, spheroid.

2. Specifications and coding

The general matrix–vector form of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ (about the origin) is $T\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{x}$ is an $n \times 1$ column vector. The instruction **TRAN($\boldsymbol{o}$,$\boldsymbol{a}$),** where $\boldsymbol{o}$ is the coding for some DERIVE object(s) and $\boldsymbol{a}$ is a 2×2 (for two dimensions) or 3×3 (for three dimensions) matrix, plots the image of $\boldsymbol{o}$ under $\boldsymbol{a}$. (If you prefer the row-vector–matrix specification, then use **TRAN_R($\boldsymbol{o}$,$\boldsymbol{a}$)**.) Vector displacements are carried out using **DIS** $(\boldsymbol{o},\boldsymbol{d})$, where $\boldsymbol{d}$ can be a two- or three-dimensional row or column vector, as appropriate.

There are a number of special types of transformations:

(i) **STR**$(\boldsymbol{o},\boldsymbol{d})$, where $\boldsymbol{d}$ is a row or column vector specifying a two- or three-way stretch. (There are two special cases: **PROJ**$(\boldsymbol{o},\boldsymbol{s})$, a central similarity projection with scale factor $\boldsymbol{s}$; and **INV**$(\boldsymbol{o})$, a central inversion.)

(ii) **ROT**$(\boldsymbol{o},\boldsymbol{\alpha})$ rotates $\boldsymbol{o}$ about the origin through an angle $\boldsymbol{\alpha}$ (in two dimensions). **ROT**$(\boldsymbol{o},\boldsymbol{d},\boldsymbol{\alpha})$ (in three dimensions) rotates $\boldsymbol{o}$ (through an angle $\boldsymbol{\alpha}$) about an axis through the origin with direction ratios specified by the row or column vector $\boldsymbol{d}$.

(iii) **REF**$(\boldsymbol{o},\boldsymbol{\alpha})$ reflects $\boldsymbol{o}$ in an axis through the origin making an angle $\boldsymbol{\alpha}$ with the positive direction of the x-axis (in two dimensions).

(iv) **REF**$(\boldsymbol{o},\boldsymbol{d})$ (in three dimensions) reflects $\boldsymbol{o}$ in a plane containing the origin whose normal has direction ratios specified by the row or column vector $\boldsymbol{d}$.

(v) **SHEAR**$(\boldsymbol{o},\boldsymbol{d})$ shears $\boldsymbol{o}$ in the directions of the standard axes with factors specified by a vector $\boldsymbol{d}$.

Prefixing any of the transformers by '**A**' forms a corresponding instruction (**ASTR**, **AROT**, etc.) that will plot both $\boldsymbol{o}$ and its image(s).

3. Examples

The users' file 2D-&3D-Transformations.dfw can be used to illustrate many aspects of two- and three-dimensional linear tranformation algebra and geometry.

Open the file 2D-&3D-Transformations.dfw and use rectangular coordinates in DERIVE's 2D- and 3D-plot windows together with the following settings: in two dimensions, Options> Display>Points>Connect(YES), Small and Options>Approximate Before Plotting (ON); in three dimensions, Options>Approximate Before Plotting (ON).

Example 1. (*Transforming a square and cube.*) The matrix $\boldsymbol{A}$ of a linear transformation is closely linked to the way it acts on the standard unit vectors. In fact, these vectors are mapped onto vectors corresponding to the columns of $\boldsymbol{A}$. In 2D-&3D-Transformations the unit square and unit cube are constructed with sides along the standard unit vectors, and so 2D-plot

TRAN(UNIT_SQ2,[2,1;-2,2])

for figure 1, and 3D-plot

TRAN(UNIT_CUBE,[1,1,1;-2,0,4;3,-3,2])

for figure 2. Using the 2D-plot setting Options>Display> Grids>Points (ON), it is clear, in figure 1, that the sides of the transformed square (anchored at the origin) lie along vectors

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In figure 2, grids have also been constructed in the coordinate planes by plotting **GRID_LINES(-5,5)**. The 3D-plot window can now be manipulated to verify that the three sides of the transformed cube (incident with the origin) are along vectors corresponding to the columns of the matrix. (I should point out that the DERIVE graphics are in colour.)

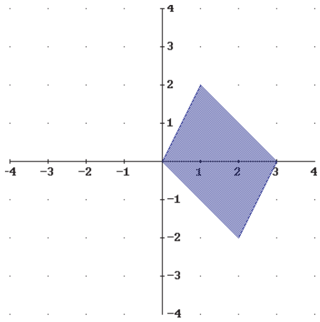


Figure 1.

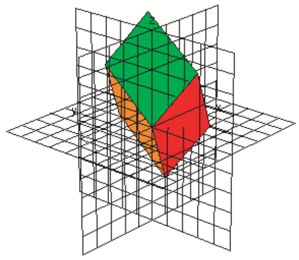


Figure 2.

Another feature of this example is that the area of the parallelogram and the volume of the parallelepiped (in terms of standard units) are equal in value to the determinants of the corresponding transformation matrices. This can also be investigated. (If calculating the volume of a parallelepiped is too daunting a task, try something simpler like a cuboid.)

Example 2. (*Direct and opposite transformations.*) The determinant of the matrix $\boldsymbol{A}$ of a linear transformation $T\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$ characterises many of its properties. For example, if $|\boldsymbol{A}| > 0$, T is called *direct*; if $|\boldsymbol{A}| < 0$, T is called *opposite*. It should be quite clear that rotations are direct and reflections are opposite transformations. This can also be confirmed by asking for the determinant of the transformer applied to the identity matrix; for example, by simplifying

DET(ROT(IDENTITY_MATRIX(2),35deg)) $= 1$

and

DET(REF(IDENTITY_MATRIX(3),[1,1,1])) $= -1$.

(These instructions calculate the determinant of the transpose, but this is equal to the determinant of the original matrix.) Also, displacements are direct. Which of the other special types of transformations are direct and which are opposite?

To interpret these properties geometrically, in two dimensions we can think of an opposite transformation reflecting or 'turning an object over' to form its image, whereas a direct transformation does not. For example, 2D-plotting **AINV(TRIANGLE[0,1;3,1;4,3])** is an illustration of the triangular lamina with vertices at $(0,1)$, $(3,1)$ and $(4,3)$ plus a direct centrally inverted image (see figure 3). **AREF(TRIANGLE[0,1;3,1;4,3],-pi/4)** plots and transforms the same triangle lamina with an opposite reflected image (see figure 4).

In DERIVE's 3D-plot window it is possible to select a colour scheme to identify direct and opposite images! Start by selecting Options>Change Plot Colors (OFF). Suppose we take as our initial object the cube of side 2 with one corner at (1,1,0.5) which can be drawn by 3D-plotting

CUBE(2,[1,1,0.5]). Using Insert>Plot>Scheme - Auto Plot Color; Apply parameters to rest of plot list (ON), the cube will appear to be plotted in a single colour. However, this is not the case, since the cube is hollow and its inside will be a different colour (this can be revealed by selecting and deleting a face of the cube). Now, 3D-plot the cube together with a direct and an opposite image using, for example, **ASTR(CUBE(2,[1,1,0.5]),[-1,-1,2],[-1,-1,-2])** (see figure 5). It should be clear that the inside and outside colours of the opposite image have been swapped over, whereas the colours have stayed the same in the direct image. In the opposite image, as well as being stretched, it appears as if the cube has been turned inside out. (Can you describe how the colour change in the cube's opposite image has come about?)

Try using the 'Auto plot Color' scheme with direct and opposite transformations applied to some of the other three-dimensional objects.

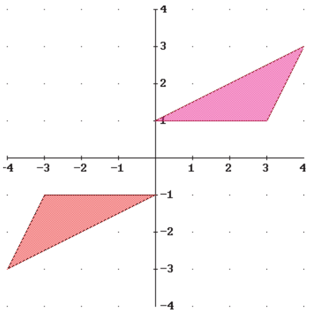


Figure 3.

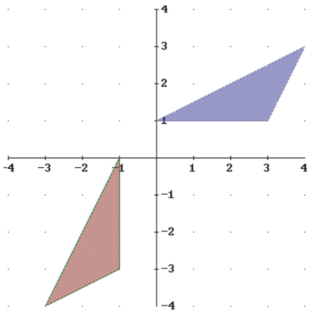


Figure 4.

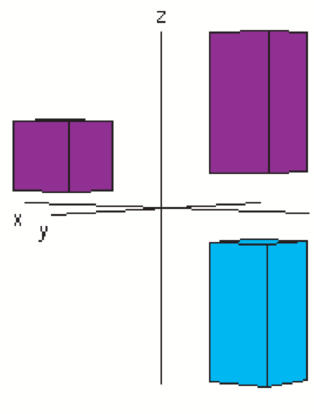


Figure 5. Direct and opposite images of a cube.

Example 3. (*Singular transformations and shadows.*) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with matrix $\boldsymbol{A}$ is called singular if $|\boldsymbol{A}| = 0$, and non-singular if $|A| \neq 0$. If T is singular, then it will collapse the domain space onto some proper subspace of $\mathbb{R}^n$. This can be illustrated for both $n = 2$ and $n = 3$. In two dimensions, suppose we set up the cyclic group pattern of parallelogram laminas (see figure 6) by plotting:

palpat:=VECTOR(ROT(PARALLELOGRAM[0,0; 2,0;2,1],α), α, 0, 2pi, pi/5).

A singular two-dimensional matrix can be constructed by making one column a multiple of the other. For example,

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}.$$

2D-plotting **TRAN(palpat,[1,-2;-2,4])** will generate the image of **palpat** (figure 7), which has been collapsed into a straight line subspace of $\mathbb{R}^2$.

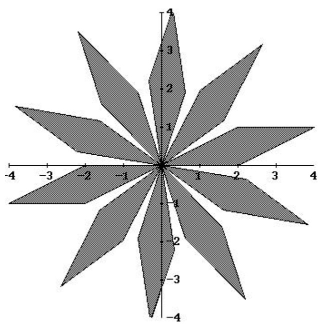


Figure 6.

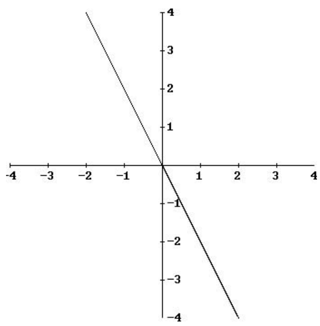


Figure 7.

I leave it as an exercise to calculate the equation of the straight line in figure 7. (I must own up to a bit of 'sleight of hand' here! It is only the outlines of the parallelogram laminas that have been transformed by the singular matrix. The areas (fillers) have to be transformed by applying the matrix inverse to change the variables involved in Boolean expressions. Since singular matrices do not have inverses, it not possible to transform the fillers in this way.)

In three dimensions we shall take as our first object the double cone in the first octant obtained by plotting **DCONE_(2,[3,3,3])**. It is possible to project shadow images of the double-cone onto each of the coordinate planes by plotting **ASTR(DCONE_(2,[3,3,3]),[1,1,0],[1,0,1],[0,1,1])** (see figure 8). Another method of constructing a three-dimensional singular matrix is to select linearly dependent columns. For example,

$$\begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0 & -0.5 \\ 1 & 0.5 & -1 \end{bmatrix}.$$

This can be illustrated by plotting

ATRAN(DCONE_(2,[3,3,3]),[.5,.5,-.5;.5,0,-.5;1,.5,-1])

(see figure 9). It should be quite clear that the whole of the shadow image of **DCONE** lies in a plane.

Also plotted is

ATRAN(SPHERE_(2,[-3,-3,-3]),[.5,.5,-.5;.5,0,-.5;1,.5,-1])

to illustrate that the singular transformation of the double cone is not just a 'one off'.

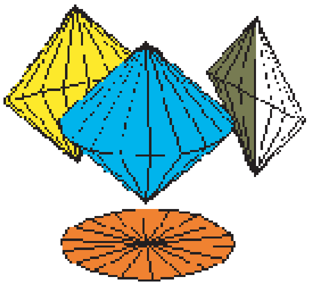


Figure 8.

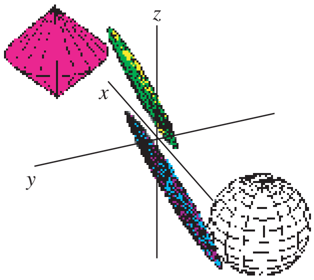


Figure 9.

In figure 9, the plane involved could also have been included by plotting $z = x + y$.

Example 4. (*Composition of transformations.*) The ability to compose transformations (one transformation followed by another) is a fundamental property, and so I have deliberately used short words for the transformers in an attempt to interpret this.

An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ can be defined in terms of a matrix–vector equation of the form $Tx = Ax + d$, where A is an $n \times n$ matrix, x is an $n \times 1$ column vector and d is a fixed $n \times 1$ column displacement vector. This is a composition of a central transformation followed by a displacement and so we can define a new transformation **aff** as follows: **aff(*o*,*a*,*d*):=DIS(TRAN(*o*,*a*),*d*)**. With this definition, it is now possible to answer questions such as: in two dimensions, which affine transformation will plot **UNIT_SQ2** (with sides along the two-dimensional unit vectors) onto **PARALLELOGRAM[1,1;-3,2;2,-3]**? Or in three dimensions, which affine transformation will plot **UNIT_TETRAHEDRON** (with sides along the unit vectors) onto **TETRAHEDRON[-1,0,1;2,3,0;3,-3,1;1,2,3]** (this is a tetrahedron with anchor vertex $(-1, 0, 1)$ and adjacent vertices at $(2, 3, 0), (3, -3, 1)$ and $(1, 2, 3)$)?

I will sketch out the answer to the three-dimensional question. **TETRAHEDRON[-1,0,1;2,3,0;3,-3,1;1,2,3]** has adjacent vector sides

$$\begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

The required affine transformation uses these for the columns of the matrix and

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for the displacement. Hence, using the above definition of **aff**, the 3D-plot is:

aff(UNIT_TETRAHEDRON, [3,4,2;3,-3,2;-1,0,2],[-1;0;1])

(Note that the displacement has been entered as a column.) This can be compared with a 3D-plot of **TETRAHEDRON [-1,0,1;2,3,0;3,-3,1;1,2,3]** to check that the answer is correct.

Another form of composition in three dimensions can be used to set up a rotation about an axis through a point other than the origin. This involves displacing this point to the origin, carrying out a corresponding rotation, then displacing the origin back to the point. Thus, define

rotp(*o*, *p*, *d*, *α*):=DIS(ROT(DIS(*o*, -*p*, *d*, *α*), *p*)

where d is the direction ratio of the axis through the point with coordinates p. A demonstration of this composition can be formed in combination with DERIVE's **VECTOR** instruction. For example, 3D-plot

VECTOR(rotp([*s*, *t*, *s*^2 + *t*^2],[0,0,-1],[1,1,1],*α*), *α*,90deg,360deg,90deg).

The axis of rotation in figure 10 has been plotted using **[0,0,-1] + *s* [1,1,1]**.

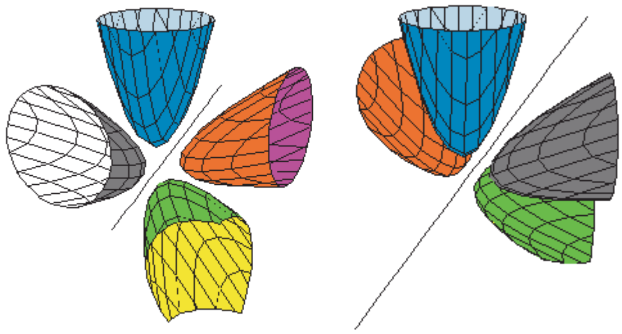


Figure 10. Two views of the rotations.

Example 5. (*Isometric transformations and symmetry.*) An isometric transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one which preserves distance and angle. That is, one which maps objects onto congruent images. Necessarily, if $Tx = Ax + d$ is an isometry, then $|A| = \pm 1$. However, we need more than this. The matrix A has to be orthogonal (the columns of A have to form a mutually orthogonal set of unit vectors). Clearly, in two or three dimensions, a rotation is a direct isometry and a single reflection is an opposite isometry. Also, the class of isometries on $\mathbb{R}^n$ (n fixed) is closed under composition.

For example, in the engine room of 2D-&3D-Transformations, a composition of isometries has been used to reflect an object in a general plane containing the origin. First, construct an orthogonal matrix A that will transform k (the unit vector pointing in the direction of the z-axis) onto the unit normal vector n of the plane. Next, use the composition

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} A^{-1}$$

to isometrically transform n to k, reflect in the Oxy plane, then transform k back to n. To construct a suitable matrix A, suppose that $n = [a, b, c]$. Then, if $a \neq 0$ or $b \neq 0$ the vector $m = [b, -a, 0]$ is orthogonal to n, and so A can be constructed as

[SIGNCROSS(*m*,*n*),SIGN*m*,SIGN*n*]´

(**SIGN** converts to unit vectors, ´ transposes); if $a = b = 0$, then $\boldsymbol{A}$ can simply be the 3×3 identity matrix.

You can code this in DERIVE using:

orthog(*n*):=if (*n* ↓ 1=0 AND *n* ↓2=0,
IDENTITY_MATRIX(3),
[SIGNCROSS([*n* ↓2,-*n* ↓1,0],*n*),
SIGN[*n* ↓2,-*n* ↓1,0],SIGN(*n*)])´
reflect(*o*,*n*):=TRAN(*o*,orthog(*n*)[1,0,0;0,1,0;0,0,-1]
orthog(*n*)^-1).

To test this, enter the above code and 3D-plot in turn

3UNIT_PYRAMID
reflect(3UNIT_PYRAMID,[1,1,1])
$$x + y + z = 0$$

to obtain figure 11, which shows the plane together with the pyramid and its reflection.

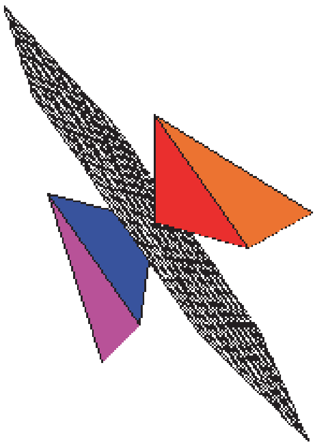


Figure 11. A pyramid reflected in $x + y + z = 0$.

Isometric transformations also play an important role in the construction of symmetry groups. A pattern (or partial pattern) possessing the group symmetries can be generated by taking any basic object and transforming and copying it through the isometries of the group. For example, in two dimensions, cyclic group patterns of order n and dihedral group patterns of order $2n$ can be formed by:

cyclic(o, n):=VECTOR(ROT(o,2pi*α*/n),*α*0,n-1)
dihedral(o,n):=cyclic(AREF(o,0),n).

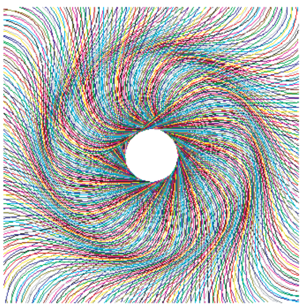


Figure 12.

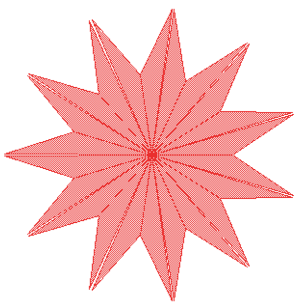


Figure 13.

Set up these assignments, and 2D-plot the following for figures 12 and 13:

cyclic(y = sin x + 1, 200)
dihedral(triangle[0,0;2,0;3.5,1],11).

By adding displacements, strip-repeating and plane-filling symmetry group patterns can also be generated. See figures 14 and 15. An excellent introduction to symmetry groups can be found in reference 2, or consult the DERIVE 5.06 users' file 'Patts2D.dfw'.

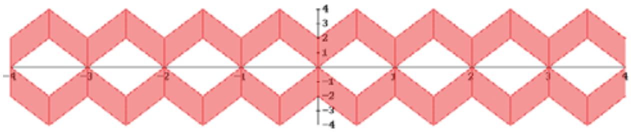


Figure 14. A strip-repeating group pattern.

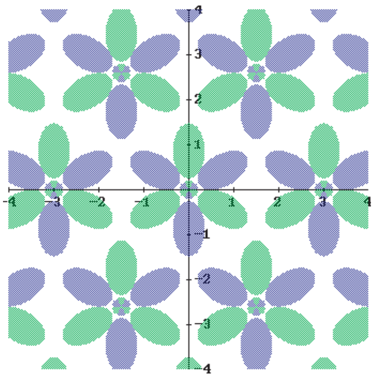


Figure 15. A plane-filling group pattern.

Three-dimensional counterparts of generators for cyclic and dihedral groups can be constructed by, for example,

cyc3(o,n):=VECTOR(ROT(o,[0,0,1],2pi*α*/n),*α*,0,n-1)
dihe31(o,n):=cyc3(AREF(o,[0,0,1]),n).

However, what about the three-dimensional groups associated with the regular tetrahedron, cube (regular octahedron) and regular dodecahedron (regular icosahedron)? A framework for the isometries of these groups can be constructed by building up the corresponding polyhedron from a single face. To do this, we require the radius of the inner sphere of a regular polyhedron whose regular polygon faces are inscribed in a circle (of radius 2 units, say — see table 1). Once this has been calculated, the polyhedron can be put together by general-purpose rotations.

Table 1.

Polyhedron	Radius of inner sphere (3 decimal places)
Regular tetrahedron	0.707
Cube	1.414
Regular dodecahedron	2.618

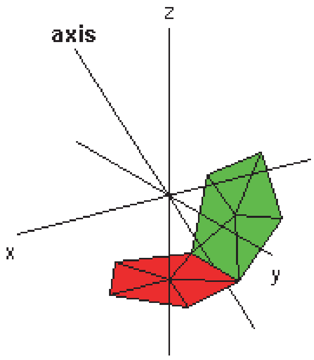


Figure 16.

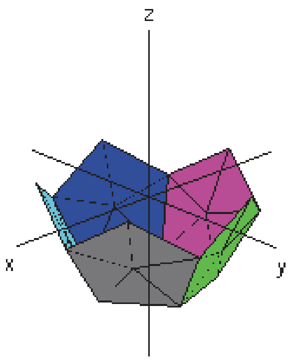


Figure 17.

I shall do this for the dodecahedron (the cube and tetrahedron are similar). Start by carefully placing a regular pentagon face (inscribed in a circle of radius 2) in position using **pentface:=RPOLY(5,2,[0,0,-2.618])**. This face can now be rotated a half turn about an axis through the origin and the mid-point of one of its edges using **dod_(*f*):=ROT(*f* , [-1.618,0,-2.618],pi)**. This constructs the correct dihedral angle between the two faces (see figure 16). The rest of the dodecahedron can be put together in two halves using:

dodec_(*f*):=[*f*,AROT(dod_(*f*), [0,0,1],2pi/5,4pi/5,6pi/5,8pi/5)]

(see figure 17) and

dodecahedron(*f*):=AROT(dodec_(*f*),[0,pi/5,0],pi).

The complete dodecahedron of figure 18 can be drawn by 3D-plotting **dodecahedron(pentface)**.

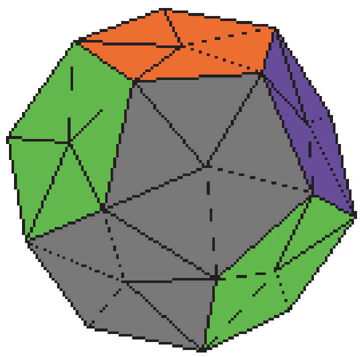


Figure 18. A complete dodecahedron.

We could replace **pentface** by any object (with a suitable five-fold symmetry) to generate other three-dimensional designs based on the dodecahedron group. Figure 19 illustrates some possibilities. To see how all five of the convex regular polyhedrons can be constructed in their facial, skeletal and stellated forms, consult the DERIVE 5.06 users' file 'Regular Polyhedrons.dfw'.

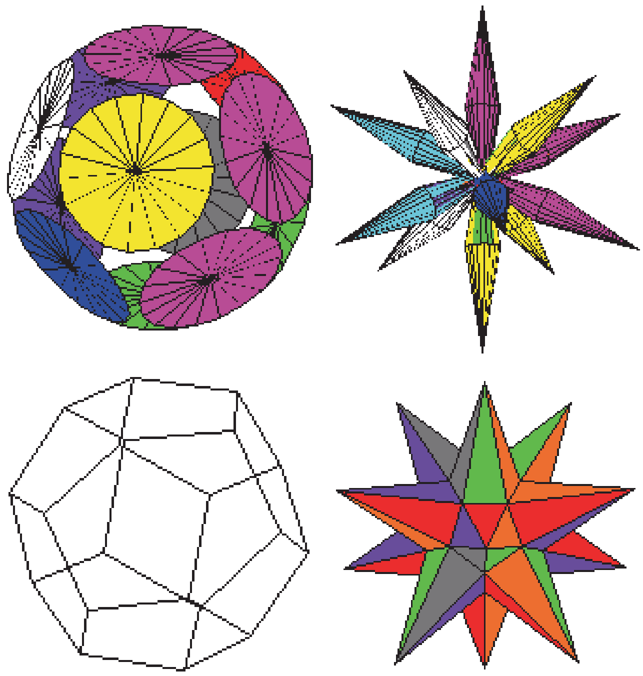


Figure 19. Some dodecahedron group objects.

4. Conclusion

The applications of 2D-&3D-Transformations are so rich and varied that I am having considerable difficulty in deciding what to put in and what to leave out of the college degree course module. The introduction of general-purpose transformers is the start of a journey into some interesting plotting techniques using DERIVE. They are beginning to fully utilize the opportunities provided by the powerful facilities of the 2D- and 3D-plot windows. If you are interested in more workshop activities with general-purpose transformers, consult reference 3.

References

1. DERIVE Web site: http://www.derive.com/.
2. A. Bell and T. Fletcher, *Symmetry Groups* (Association of Teachers of Mathematics, Derby, 1964).
3. P. Schofield, Some general-purpose tools for carrying out 2D- and 3D-linear transformation geometry plots with DERIVE 5 (workshop) in: *Proc. 5th Internat. DERIVE & TI-89/92 Conf.*, Vienna, 2002 (bk teachware, Hagenburg , 2002 (CD-ROM)).

The author has recently retired from the post of Head of Mathematics at Trinity and All Saints College, University of Leeds, where he has taught mathematics for over 30 years. Dr Schofield continues to teach part-time at the College and to make extensive use of DERIVE in working with students.

Mathematics in the Classroom

A magical mistake

In this note, I would like to share a pedagogical observation that I have made during the past eight years of teaching calculus. This observation has probably been made by other teachers of calculus around the world, but is not well documented.

A fairly common situation in mathematical problem solving is to arrive at the absolutely correct answer by a fundamentally wrong method. The observation I refer to is an instance of this situation, namely, when the students happen to get the right answer for the definite integral $\int_a^b f(x)\,\mathrm{d}x$ simply by substituting the upper and the lower bound for x in $f(x)$ (rather than in the anti-derivative $F(x)$).

In this note, I will cite a few examples (or, rather, families of examples) when the wrong method gives the right answer. That is, functions f such that

$$f(b) - f(a) = F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x\,.$$

I will start with a few algebraic examples. Consider

$$\begin{aligned}\int_{-a}^{n+1-a} k(x+a)^n\,\mathrm{d}x &= k\Big[\frac{(x+a)^{n+1}}{n+1}\Big]_{-a}^{n+1-a}\\ &= k\frac{(n+1)^{n+1}}{n+1} = k(n+1)^n\,,\end{aligned}$$

which is also equal to $\big[k(x+a)^n\big]_{-a}^{n+1-a} = k(n+1)^n$. Many of the actual examples I have observed in my class are specific instances of this, e.g. $\int_0^2 3x\,\mathrm{d}x$ and $\int_{-3}^1 (x+3)^3\,\mathrm{d}x$. Further examples are $\int_{-1}^2 (3x^2+4x+2)\,\mathrm{d}x$ and $\int_{-1}^1 (3x^2+2x+1)\,\mathrm{d}x$. Readers are invited to generalize these findings. By using this magical method, I also found the correct answer for the integral of the rational expression $\int_0^1 (x/(1+x^2)^2)\,\mathrm{d}x$.

Other families of algebraic examples relate to the greatest integers (or floor) function $\lfloor x\rfloor$, the absolute value function $|x|$, or a combination of these. Here are a few examples.

Example 1. The integral $\int_{-n}^{n+3} \lfloor x\rfloor\,\mathrm{d}x$, where n is a positive integer. One of the correct methods of computing this integral is by plotting the function $\lfloor x\rfloor$ and summing up the rectangular signed area under this from $-n$ to $n+3$. The answer is

$$\begin{aligned}(-n) + (-n+1) + \cdots + (-2) + (-1) + 0 + (1)\\ + \cdots + (n) + (n+1) + (n+2)\\ = (n+1) + (n+2)\\ = 2n+3\,.\end{aligned}$$

We can get the same answer by simply substituting the upper and lower limits in the integral: $\lfloor n+3\rfloor - \lfloor -n\rfloor = n+3+n = 2n+3$. Simple variations of this example are $\int_{-n}^{n+1} \lfloor x+1\rfloor\,\mathrm{d}x$ and $\int_{-n}^{n+5} \lfloor x-1\rfloor\,\mathrm{d}x$. Similar functions also work, e.g.

$$\int_0^1 \lfloor x^2 - x + 1\rfloor\,\mathrm{d}x\,.$$

Example 2. The integral $\int_{-a}^{2+a} (|x|/x)\,\mathrm{d}x$, where a is a positive real number. Here again we can compute the integral completely by the area method. The signed area from $-a$ to 0 is $a\times(-1) = -a$ and from 0 to $2+a$ is $(2+a)(1) = 2+a$. Thus, the value of the integral is $-a+2+a = 2$. This is also the value obtained by substituting the upper and lower limits directly into the function:

$$\frac{|2+a|}{2+a} - \frac{|-a|}{-a} = 1 - (-1) = 2\,.$$

Another example involving the modulus function is equally interesting:

$$\int_{-2n}^{2n} (|x| - n)\,\mathrm{d}x = [|x| - n]_{-2n}^{2n} = 0\,.$$

Some variations of the above algebraic examples are:

$$\int_{-2}^{2} \frac{x+|x|}{2}\,\mathrm{d}x\,,$$

$$\int_{-3}^{3} (|x| + \lfloor x\rfloor)\,\mathrm{d}x\,,$$

$$\int_{1}^{5} (|x-1| - |x-3|)\,\mathrm{d}x\,.$$

Next, we consider some trigonometric examples. A simple one is $\int_a^b \sin x\,\mathrm{d}x$, where $a+b = \pi/2$. Indeed,

$$\int_a^b \sin x\,\mathrm{d}x = [-\cos x]_a^b = \cos a - \cos b\,.$$

On the other hand, direct substitution of upper and lower limits gives

$$\sin b - \sin a = \sin\Big(\frac{\pi}{2} - a\Big) - \sin\Big(\frac{\pi}{2} - b\Big) = \cos a - \cos b\,.$$

Another example is $\int_0^{2\pi} \sin^3 x\,\mathrm{d}x$. Further examples involving $\sin x$ and $\cos x$ are

$$\int_{-\pi/4}^{\pi/4} (\sin x + \cos x)\,\mathrm{d}x$$

and

$$\int_0^{\pi} (\sin^2 x\cos x)\,\mathrm{d}x\,.$$

Let's now look at a few examples from the logarithm family. Consider

$$\begin{aligned}\int_0^{\mathrm{e}-1} \log(1+x)\,\mathrm{d}x &= [(x+1)\log(1+x) - x]_0^{\mathrm{e}-1}\\ &= \mathrm{e}\log\mathrm{e} - (\mathrm{e}-1) - 0\\ &= \mathrm{e} - (\mathrm{e}-1)\\ &= 1\,.\end{aligned}$$

We also get the same answer by direct substitution:

$$\log(1+\mathrm{e}-1)-\log(1+0)=\log\mathrm{e}-\log 1=1\,.$$

Further examples where we can get the right answers by directly substituting the upper and lower limits are

$$\int_{-a}^{a} x(x-a)(x+a)\,\mathrm{d}x\,,$$

$$\int_{-a}^{a} x^3\sqrt{a^2-x^2}\,\mathrm{d}x\,.$$

These can be generalized as $\int_{-a}^{a} f(x)\,\mathrm{d}x$, where $f(x)$ is an odd function and $f(a)=f(-a)=0$.

Finally, let me mention two examples that are more cryptic than the ones above but which also seem to fall under the spell of this 'magical method' of computing the definite integral $\int_a^b f(x)\,\mathrm{d}x$ as $f(b)-f(a)$:

$$\int_0^{\pi} \frac{x}{1+\sin x}\,\mathrm{d}x\,,\qquad \int_0^{2\pi} \mathrm{e}^{\sin^2 nx}\tan nx\,\mathrm{d}x\,.$$

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Anand Kumar

Computer Column

The future of computing

Computers are getting ever more powerful, so fast that today's cutting-edge PC will be next year's doorstop. For years, their speed has been doubling every couple of years or so — a rule known as Moore's law. The law was first proposed by Intel's Gordon Moore in 1965, and has held true to this day. This rate of progress has been going on so long that we have come to expect it, but can it continue?

The answer (perhaps not surprisingly) is no. Computer processors are made up of millions of tiny transistors, etched into a square of silicon only a few centimetres across. To make them faster, manufacturers will have to make these transistors even smaller and pack them in even more tightly; not only is this increasingly difficult (and expensive), but there is a limit to how small the transistors can be. When each one is made up of only a handful of atoms, and the 'currents' between them consist of one or two electrons, we will not be able to go any further down this road. This will happen sooner than you might think: if we manage to keep going according to Moore's law, we will reach this point in about 2010.

Where will we go from there? The race is on to develop the 'next big thing', a device to pick up where silicon chips leave off. In principle, the range of possibilities is vast: everything from billiard balls to tanks of liquid can perform computation, just not particularly efficiently. After all, any physical system starts from an initial condition (the 'input') and evolves to a different state at a later time (the 'output'). The tricky part is arranging for the evolution of the system to do something useful, and in a reliable way.

At the moment, there are two serious challengers in the race: DNA computers and quantum computers. Both face stiff technical challenges before they can be turned into practical devices, but both also have some powerful features. Of the two, quantum computers are by far the most difficult to implement, but they also hold out the prospect of a greater reward. DNA computing, by contrast, is probably best seen as a stop-gap measure, for reasons that we will now see. (A discussion of quantum computers will have to wait until next time; all good series need a cliffhanger from time to time!)

DNA computers

A strand of DNA consists of a sequence of 'bases', which come in four types: A, C, G and T. Each base can only bond with one other, so A and T can bond together, as can C and G, but not A and G, for example. Our genetic structure is encoded in a pair of DNA strands, bonded together (the famous 'double helix'), but the bonding abilities of the bases mean that each strand is essentially a mirror image of the other: where one has an A, the other must have a T, and so on. (This turns out to be crucial in the process by which DNA reproduces.)

Leonard Adleman realised that this ability of DNA strands to seek out mirror-image strands to bond with could be put to a quite different use: as part of a computer. The first problem he tackled was a variant of the travelling salesman problem (TSP). Attentive readers may remember this problem from last issue's Computer Column, but briefly it is this: a salesman has to travel round a list of clients in different cities, visiting each client once and returning home afterwards. The goal is to find the shortest possible route.

Adleman's solution was to prepare a set of DNA strands representing all the possible legs of the journey, with a characteristic sequence of bases at one end of each strand to represent the city at the start of the leg and another at the other end to represent the city at the end of the leg. The other important feature of the strands was that the sequence for a given city as the start of a leg was the mirror image of the sequence for the same city as the end of a leg.

Large numbers of all these strands were then mixed together and allowed to bond. By design, the only bonds that could form were between, say, strands representing the trips $a \to b$ and $b \to c$ — thanks to the mirror-image structure of the strands — giving a strand representing $a \to b \to c$. The end result of the bonding process was a set of strands representing random tours round a number of different cities.

The next step was to filter out all the strands not including all the city sequences (by adding 'markers' which bonded to these sequences), leaving just tours which took in all the cities. The shortest of these strands was then the answer he was looking for.

The advantages of this method are that the basic units — the DNA strands — are very small and easy to produce in bulk, and that they react in parallel. The number of strands in a feasible device would be of the order of Avogadro's number, i.e. about 10^{23}, which is large even compared to the number of neurons in the brain (10^{11}), never mind the 'mere' 42 million transistors in a Pentium® 4 chip. On the other hand, the procedures for first preparing the strands and later filtering them are difficult and slow. In addition, there is no guarantee that the method will find an answer; with enough strands, it becomes quite likely, but it is never certain. In the end, this means that the DNA method is not very practical for the TSP: there need to be at least as many strands as there are possible routes, of which an N-city problem has $(N-1)!/2$, even if we only count the routes that visit each city exactly once. For any problem with more than about 25 cities, this makes finding the optimal route pretty unlikely. In addition, most of the routes that the DNA strands generate will wander around aimlessly, visiting some cities several times, so even 25-city problems will probably prove to be impossible.

Having said all this, of course, the poor performance of Adleman's DNA computer is more a reflection of the difficulty of the problem than anything else. As compared to a silicon computer with parts of a similar size, it is much less organized (and would not be very efficient), but it would be much cheaper to develop and build.

A more serious limitation of this approach is that it has to be adapted to each new problem: the ability to solve the TSP is built in to the 'hardware'. To get around this, we would need to find a way of converting more general algorithms into DNA manipulation steps, and it is not yet clear how this could be done. If it could, of course, and all the steps could be put under the control of a conventional computer, we could then effectively have a 'DNA co-processor'.

The usefulness of the DNA approach hinges on how difficult it proves to be to shrink conventional computer components down to molecular sizes: if this can be done, DNA computers would be inefficient by comparison; otherwise, they could be a useful way of squeezing out some more power.

There are two main difficulties in reducing the size of conventional computer components to the level required to compete with DNA computers: etching them into silicon requires very high energy laser beams (to prevent diffraction giving fuzzy results), and quantum effects become important in their operation. These would need to be suppressed if we wanted simply to replicate the behaviour of current computers. On the other hand, we could choose to go the other way and try to make quantum mechanics work for us instead. To hear more about that, however, tune in next time!

Websites

1. Moore's law:
   http://www.intel.com/research/silicon/mooreslaw.htm
2. Leonard Adleman's home page:
   http://www.usc.edu/dept/molecular-science/fm-adleman.htm

Peter Mattsson

Letters to the Editor

Dear Editor,

New variants of famous problems in number theory

The Xor product of two binary expansions of positive integers, $a = a_1a_2a_3\ldots$ and $b = b_1b_2b_3\ldots$ is defined to be $c = c_1c_2c_3\ldots$, where $c_i = 1$ if $b_i \neq a_i$ and $c_i = 0$ if $b_i = a_i$. We write $c = a \operatorname{Xor} b$.

Fermat's last theorem

This is the conjecture of Pierre Fermat that, for positive integers x, y, z, n, the equation

$$x^n + y^n = z^n$$

has no solutions in positive integers for $n \geq 3$. This was proved by Andrew Wiles (see reference 1) after over 300 years. For $n = 2$, the solutions x, y, z are Pythagorean triples, and a description of a complete solution is given in references 2 and 3.

Consider the equation

$$x^n \operatorname{Xor} y^n = z^n, \tag{1}$$

with $x \neq y$. As with Fermat's equation, equation (1) has solutions for $n = 2$. For example, $3^2 \operatorname{Xor} 4^2 = 5^2$. Then, from properties of Xor, the value of z can be swapped with x or y to give another solution. Hence $3^2 \operatorname{Xor} 5^2 = 4^2$ is also a solution. Other solutions for $n = 2$ are $(12,\ 16,\ 20)$, $(24,\ 32,\ 40)$ and $(48,\ 64,\ 80)$. But not all Pythagorean triples are solutions, for example $(5,\ 12,\ 13)$, $(8,\ 15,\ 17)$. My first conjecture is that, for $n = 2$, there are no solutions x, y, z of (1) in which x, y, z are not powers of 2.

Now consider $n \geq 3$. I have verified using MATLAB® that there are no solutions of (1) for $1 \leq x,\ y \leq 15$ and $n = 3$.

Hence, my conjecture in this case is that there are no positive integer solutions x, y, z of equation (1) for $n \geq 3$.

Goldbach's conjecture

I now present conjectures relating integers and prime numbers.

In two letters addressed to Euler in 1742, Goldbach proposes his famous conjecture that every even number greater than 2 is the sum of two primes; see reference 4. This conjecture has remained unproven. There has been a \$1 million prize offered for the solution of this problem; see reference 5. I conjecture that every non-negative even number is the Xor product of two prime numbers. Some examples are $5 \operatorname{Xor} 7 = 2$, $3 \operatorname{Xor} 7 = 4$, $3 \operatorname{Xor} 5 = 6$. I have verified this up to 1000 using MATLAB. Goldbach's conjecture, if true, trivially implies Goldbach's weak conjecture, which is that every odd number greater than 5 is the sum of three primes. It is known that the extended Riemann hypothesis implies this weak conjecture; see reference 6.

I conjecture that every positive odd number is the Xor product of three odd primes, and that every even number is the Xor product of 2 and two primes. Again, using MATLAB I have verified these conjectures up to 127 and 126 (using all the primes less than 92 and none greater than 92).

It may be asked if the extended Riemann hypothesis implies any of the new conjectures. I have observed for the variant of Goldbach's conjecture that the Xor product of primes is never equal to their sum, up to 1000.

Dirichlet's theorem

This theorem states that, if $\gcd(a, d) = 1$, then there are infinitely many primes in the arithmetic progression $an + d$; see reference 7.

I conjecture that, if $\gcd(a, d) = 1$, then there are infinitely many primes in the sequence $an \operatorname{Xor} d$.

Using MATLAB, I have found the first 80 primes in the sequence $an \operatorname{Xor} d$ for $(a, d) = (3, 7)$, $(41, 17)$ and $(67, 101)$.

References

1. A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. Math.* **141** (1995), pp. 433–551.
2. G. Lasters and D. Sharpe, Pythagorean quadruples, *Math. Spectrum* **34** (2001/2002), p. 38.
3. M. Jebreel, Pythagorean triples (letter to the Editor), *Math. Spectrum* **34** (2001/2002), p. 68.
4. G. Tenebaum and M. France, *The Prime Numbers and Their Distribution* (American Mathematical Society, Providence, RI, 2000), p. 108.
5. I. Peterson, Goldbach's prime pairs, *Science News* **158**. Available at http://www.sciencenews.org/20000819/mathtrek.asp
6. Goldbach's weak conjecture, Wikipedia, the free encyclopedia, available at http://www.wikipedia.org/wiki/Weak_ conjecture_of_Goldbach
7. S. Gueron and R. Tessler, Infinitely many primes in arithmetic progression, *Math. Gazette* **86** (2002), pp. 110–114.

Yours sincerely,

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Dear Editor,

Looking for patterns

One of the numerical patterns which has intrigued Abbas Rooholamine (see 'From the Editor' in Volume 35, Number 2) is

$$3^3 + 4^3 + 5^3 + 5^3 + 6^3 + 7^3 = (3 + 4 + 5 + 5 + 6 + 7)^2.$$

If this is a simple illustration of a general result, then it might be possible to find integers m, p and n such that

$$\sum_{r=m}^{p} r^3 + \sum_{r=p}^{n} r^3 = \left[\sum_{r=m}^{p} r + \sum_{r=p}^{n} r\right]^2,$$

where $p - m = k = n - p$ for some constant k.

Using the formulae for the sums of integers and their cubes, we have

$$p^3 + \frac{n^2(n+1)^2}{4} - \frac{m^2(m-1)^2}{4} = \left[p + \frac{n(n+1)}{2} - \frac{m(m-1)}{2}\right]^2.$$

Simple manipulation reduces this equation to

$$p^2 - p = (2p + m - m^2)(n - m + 1),$$

which on changing the variables to m and k gives

$$2m^2 - 4m - 3k = 0,$$

with the solutions

$$m = \frac{4 \pm \sqrt{8(2 + 3k)}}{4}.$$

For m to be an integer, put

$$2 + 3k = 2u^2,$$

where u is a positive integer. Hence,

$$m = 1 + u,$$

as m cannot be zero or negative. Also

$$k = \frac{2(u-1)(u+1)}{3}.$$

For k to be an integer, either

$$u + 1 = 3s,$$

where $s = 1, 2, 3, \ldots$, with

$$m = 3s, \qquad p = s(2m - 1) \quad \text{and} \quad n = 2p - m$$

or

$$u - 1 = 3s$$

with

$$m = 3s + 2, \qquad p = m(2s + 1) \quad \text{and} \quad n = 2p - m.$$

So there are two distinct families of integers which satisfy the general relationship. The quoted result is given by the first family when $s = 1$, which in the second family gives

$$\begin{aligned}&5^3 + 6^3 + \cdots + 14^3 + 15^3 + 15^3 + 16^3 + \cdots + 24^3 + 25^3\\ &= 108\,900\\ &= (5 + 6 + \cdots + 14 + 15 + 15 + 16 + \cdots + 24 + 25)^2.\end{aligned}$$

The subsequent statements become steadily more complicated.

Yours sincerely,
ROBERT J. CLARKE
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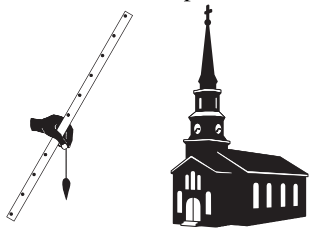
Steeplejack

How could you estimate the height of a church steeple? You are afraid of heights and have only got a metre rule and a short plumb line.



Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

36.5 P is a point on an ellipse. Prove that every chord of the ellipse parallel to the tangent at P has its midpoint on the line OP, where O is the centre of the ellipse. Do the same for a hyperbola.

(Submitted by Guido Lasters, Tienen, Belgium, and Hugo Staelens, Eeklo, Belgium)

36.6 Find a finite sequence of integers whose sum is 2004 such that the sum of every four consecutive numbers in the sequence is negative.

(Submitted by Abbas Rooholaminy, Iran)

36.7 Let $x_1, \dots, x_n$ $(n \geq 2)$ be real numbers such that $x_1 + \cdots + x_n = 1$. Determine the maximum value of $\sum_{1\leq i<j\leq n} x_i x_j$ and also when this maximum value is attained.

(Submitted by H. A. Shah Ali, Tehran)

36.8 The sequence (u_n) has nth term

$$u_n = \mathrm{e}^{1/(10^n+1)}\mathrm{e}^{1/(10^n+2)}\mathrm{e}^{1/(10^n+3)}\cdots\mathrm{e}^{1/(2\times 10^n)}.$$

What is its limit?

(Submitted by Guido Lasters, Tienen, Belgium)

Solutions to Problems in Volume 35 Number 3

35.9 *The box problem extended.* Given a square sheet of card, side 1 m, what is the maximum volume of the lid-less box obtained by cutting four equal squares from the corners of the sheet and folding up the four flaps? The box-maker uses the four cut-out squares and cuts boxes from these in the same proportions as the original box. And so on. Which value of x gives the maximum volume of all the boxes now, and by how much does the maximum volume increase?

Solution by Jonny Griffiths, who proposed the problem

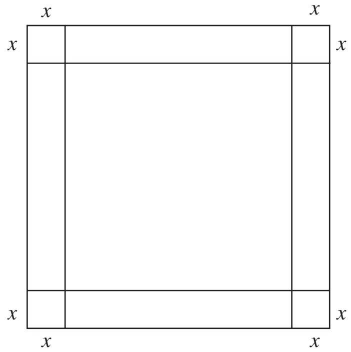
The volume of the box is V, where

$$V = (1 - 2x)^2 x,$$

so

$$\frac{\mathrm{d}V}{\mathrm{d}x} = (1 - 2x)^2 + 2(1 - 2x)(-2)x,$$

and this is zero when $x = \frac{1}{2}$ (in which case $V = 0$) or when $1 - 2x - 4x = 0$, i.e. when $x = \frac{1}{6}$. Hence the maximum volume is $(1 - \frac{1}{3})^2\frac{1}{6} = \frac{2}{27}\ \mathrm{m}^3$ or approximately $74.07\ \mathrm{cm}^3$.



Each of the four corner squares gives a box of volume $x^3\times(1-2x)^2x$, so the four corners will give a combined volume of $4x^4(1-2x)^2$. The sixteen cubes made from the corners of these will give a combined volume of $16x^7(1-2x)^2$. And so on. The total volume is now

$$\begin{aligned}V &= x(1-2x)^2+4x^4(1-2x)^2+16x^7(1-2x)^2+\cdots\\ &= x(1-2x)^2(1+4x^3+4^2x^6+4^3x^9+\cdots)\\ &= \frac{x(1-2x)^2}{1-4x^3},\end{aligned}$$

so

$$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{\begin{array}{c}(1-4x^3)[(1-2x)^2+2(1-2x)(-2)x]\\ -x(1-2x)^2(-12x^2)\end{array}}{(1-4x^3)^2},$$

and this is zero when $x=\frac{1}{2}$ (in which case $V=0$) or when

$$(1-4x^3)(1-6x)+12x^3(1-2x)=0,$$

that is,

$$1-6x+8x^3=0,$$

and this has approximate solution $x=0.1736$ (its exact solution is $\cos 80°$), giving $V\approx 75.56\ \mathrm{cm}^3$, an increase of approximately $1.5\ \mathrm{cm}^3$.

Jonny Griffiths asks whether this volume can be increased if we remove the restriction that the offcuts are in the same proportions as the original.

35.10 If $x,y,z,t>1$, prove that

$$\frac{1}{\log_x y^3z}+\frac{1}{\log_y z^3t}+\frac{1}{\log_z t^3x}+\frac{1}{\log_t x^3y}\geq 1.$$

When does equality occur?

Solution by Mihály Bencze, who proposed the problem

We have

$$\frac{1}{\log_x y^3z}=\frac{1}{3\log_x y+\log_x z}=\frac{\ln x}{3\ln y+\ln z},$$

so, with $a=\ln x$, $b=\ln y$, $c=\ln z$, $d=\ln t$, the expression is

$$\frac{a}{3b+c}+\frac{b}{3c+d}+\frac{c}{3d+a}+\frac{d}{3a+b}=E\quad(\text{say}).$$

Put

$$a_1=\sqrt{\frac{a}{3b+c}},\quad b_1=\sqrt{a(3b+c)},$$

with similar expression for a_2,a_3,a_4 and b_2,b_3,b_4. By Cauchy's inequality,

$$\begin{aligned}&(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2)\\ &\qquad\geq(a_1b_1+a_2b_2+a_3b_3+a_4b_4)^2,\end{aligned}$$

that is,

$$E\big(\textstyle\sum a(3b+c)\big)\geq(\sum a)^2.$$

But

$$\begin{aligned}&\big(\textstyle\sum a\big)^2-\sum a(3b+c)\\ &\quad=(a+b+c+d)^2\\ &\qquad-3(ab+bc+cd+da)-2(ac+bd)\\ &\quad=a^2+b^2+c^2+d^2-ab-bc-cd-da\\ &\quad=\tfrac{1}{2}[(a-b)^2+(b-c)^2+(c-d)^2+(d-a)^2]\\ &\quad\geq 0,\end{aligned}$$

so that $E\geq 1$. For equality to occur, we must have $a=b=c=d$, that is, $x=y=z=t$, and equality does occur in this case.

35.11 Let $a_1,\ldots,a_n$ $(n\geq 2)$ be distinct real numbers and let m be a positive integer smaller than n. Prove that

$$\sum_{k=1}^{n}\frac{a_k^m}{\prod_{j\neq k}(a_k-a_j)}=\begin{cases}1 & \text{if } m=n-1,\\ 0 & \text{if } m<n-1.\end{cases}$$

Solution By Lagrange's interpolation formula, the unique polynomial of degree less than n which takes the value a_k^m at a_k for $k=1,\ldots,n$ is

$$\sum_{k=1}^{n}a_k^m\frac{\prod_{j\neq k}(x-a_j)}{\prod_{j\neq k}(a_k-a_j)}.$$

This is just x^m. If we equate coefficients of x^{m-1}, we obtain the result.

35.12 Prove that

$$\begin{vmatrix}a_1&b_1&c_1&d_1\\a_2&b_2&c_2&d_2\\a_3&b_3&c_3&d_3\\a_4&b_4&c_4&d_4\end{vmatrix}=\frac{1}{a_2a_3}\begin{vmatrix}\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}&\begin{vmatrix}a_1&c_1\\a_2&c_2\end{vmatrix}&\begin{vmatrix}a_1&d_1\\a_2&d_2\end{vmatrix}\\[2ex] \begin{vmatrix}a_2&b_2\\a_3&b_3\end{vmatrix}&\begin{vmatrix}a_2&c_2\\a_3&c_3\end{vmatrix}&\begin{vmatrix}a_2&d_2\\a_3&d_3\end{vmatrix}\\[2ex] \begin{vmatrix}a_3&b_3\\a_4&b_4\end{vmatrix}&\begin{vmatrix}a_3&c_3\\a_4&c_4\end{vmatrix}&\begin{vmatrix}a_3&d_3\\a_4&d_4\end{vmatrix}\end{vmatrix},$$

where $a_2,a_3\neq 0$, and extend this to the determinant of a 5×5 matrix.

Solution With $a_2,a_3\neq 0$,

$$\begin{vmatrix}a_1&b_1&c_1&d_1\\a_2&b_2&c_2&d_2\\a_3&b_3&c_3&d_3\\a_4&b_4&c_4&d_4\end{vmatrix}=\begin{vmatrix}0 & b_1-\dfrac{b_2a_1}{a_2} & c_1-\dfrac{c_2a_1}{a_2} & d_1-\dfrac{d_2a_1}{a_2}\\ a_2 & b_2 & c_2 & d_2\\ 0 & b_3-\dfrac{b_2a_3}{a_2} & c_3-\dfrac{c_2a_3}{a_2} & d_3-\dfrac{d_2a_3}{a_2}\\ 0 & b_4-\dfrac{b_3a_4}{a_3} & c_4-\dfrac{c_3a_4}{a_3} & d_4-\dfrac{d_3a_4}{a_3}\end{vmatrix}$$

$$=-\frac{1}{a_2a_3}\begin{vmatrix}-\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}&-\begin{vmatrix}a_1&c_1\\a_2&c_2\end{vmatrix}&-\begin{vmatrix}a_1&d_1\\a_2&d_2\end{vmatrix}\\[2ex] \begin{vmatrix}a_2&b_2\\a_3&b_3\end{vmatrix}&\begin{vmatrix}a_2&c_2\\a_3&c_3\end{vmatrix}&\begin{vmatrix}a_2&d_2\\a_3&d_3\end{vmatrix}\\[2ex] \begin{vmatrix}a_3&b_3\\a_4&b_4\end{vmatrix}&\begin{vmatrix}a_3&c_3\\a_4&c_4\end{vmatrix}&\begin{vmatrix}a_3&d_3\\a_4&d_4\end{vmatrix}\end{vmatrix}.$$

The result follows by taking a minus sign out of the first row. We leave to readers the case of a 5×5 matrix.

Reviews

The Mathematics of Oz: Mental Gymnastics from Beyond the Edge. By CLIFFORD A. PICKOVER. Cambridge University Press, 2002. Pp. 368. Hardback £21.95 (ISBN 0-521-01678-9).

'Just when Dorothy thought her travelling days were over and that she would never return to the land of Oz, a dark monolith appears on her native Kansas prairie and whisks her away on an even wilder adventure.' Thus begins the back story to Pickover's imaginatively written book, which brims full of mathematical puzzles, interspersed with the story of Dorothy's adventures.

This book contains 108 puzzles, of varying difficulties — the author's four-star rating system runs (according to him) from 'challenging' to 'outrageously difficult: probably impossible …'. The easier problems are MENSA-type challenges, along the lines of exotic mazes, pattern-spotting, and variants on magic squares. Further up the scale, there are some fine conceptual problems. For instance, what is the expected ratio between the lengths of the two shards of a bone, broken at a random spot along its length? Also, if I have a sphere containing two billion points, can I cleverly introduce a dividing wall that will have precisely a billion of the points on each side? A-level students should have no trouble understanding the maths being used. The bias is towards pure maths; puzzles, rather than calculations.

Some of the problems are outrageously difficult; for instance unsolved questions relating to the digits of π and e, and the shape of the graph of x^x in the imaginary plane. However, there are not too many such problems: they provide interesting variety, and should not put anyone off. Helpfully, the last third of the book is devoted to very detailed discussions and solutions of all the problems; references, diagrams, and even some computer programs are given. This section is packed with information, and will be very useful to teachers, as well as general readers.

One puzzle that deserves special mention is 'Cosmic Call'. The author reproduces parts of a message which was beamed towards nearby stars by two Canadian physicists in the hope of communicating with extraterrestrials. The message is designed to be understood with no prior information about humans and their language. It is fantastic fun to decipher, and very illuminating.

One of the joys of this book is Pickover's style in his spritely, tongue-in-cheek narrative of Dorothy and her captor Dr Oz. The latter is an octopus-like alien who initially shows no mercy to Dorothy (or her dog Toto) and only spares the rest of the human race as long as Dorothy can keep solving his puzzles. Their travels are gloriously illustrated in Pickover's prose and in the cartoons sprinkled throughout the book. Each problem is prefaced by a stimulating quotation. Overall, this is an entertaining and exciting piece of work.

Student, Queens' College, Cambridge — WILL DONOVAN

Duelling Idiots and Other Probability Puzzlers. By PAUL J. NAHIN. Princeton University Press, 2002. Pp. 269. Paperback £13.95 (ISBN 0-691-10286-4).

What is the mean number of times that you need to press the random number button on your calculator in order to generate a total greater than 1? The surprising, but very satisfying answer is $e = 2.71828\ldots$! This gem occurs as part of Problem 5 in this entertaining collection of 21 such 'probability puzzlers' and, if it has whetted your appetite, you are likely to enjoy this book. The problems were originally presented by Nahin as challenges to his undergraduate electrical engineering students: they come with scene-setting commentaries, detailed solutions, and print-outs of MATHLAB® programs which deal with the computational aspects. The latter may surprise some readers, but one of the author's main messages is that real problems need real solutions in real time, so that a computer simulation may be as effective and efficient a way of getting a handle on a solution as an analytical solution. Indeed, it may be the *only* way of approaching a solution for, as Nahin ruefully observes in his preface, 'No matter how smart you are, there will always be a problem harder than one you can solve analytically.'

Problem 10 is a good example of a 'harder than it looks' problem: if a chess player scores 1 point for a win with probability p and $\frac{1}{2}$ point for a draw with probability q, what is the probability that the player wins or that he or she ties a match consisting of $2n$ games? Problem 15 is a fine example of one with a counter-intuitive conclusion: if a couple decide to stop their family after they have a child of the same sex as their first child, then the expected size of the family is independent of the (fixed) probability that a child is a boy.

Some readers may find Nahin's style rather chatty and discursive, but his selection of problems is interesting and is presented in a stimulating, original manner. I recommend this book to anyone wondering whether there are any other surprises in probability theory to rank alongside the birthday problem and Buffon's needle experiment.

Tonbridge School, Kent — NICK LORD

150 Puzzles. By ROD MARSHALL AND TIM SOLE. The Faculty and Institute of Actuaries, Edinburgh and London, 1998. Pp. 83. Paperback £5.50 (ISBN 0-901066-58-3).

This book, published in 1998 to celebrate the 150th anniversary of the founding of the Institute of Actuaries, contains a wide variety of puzzles to engage, intrigue and confound the mind. They vary in nature from strictly mathematical puzzles to logical puzzles, yet are united by an underlying theme, namely that of solving problems by thinking about them in abstract and unconventional ways.

The problems are presented in an unambiguous and clear fashion, allowing time to be spent solving the puzzle rather than deciphering the question itself, and are couched in such terminology as to make them readily accessible for anyone wishing to consider them, although in some cases specialist mathematical knowledge may be of use. The solutions are clearly presented and are arranged, in a novel fashion, by the last digits of the question, so that if you are attempting to solve the problems in a roughly linear fashion, the possibility of observing the answer to the next puzzle due to its proximity to the answer which you are engaged in is eliminated.

The nature of this book is more about problem solving than mathematics itself, the mathematics often being secondary in the solutions to many of the puzzles contained, so that those wishing to learn more mathematics would be well advised not to buy it. However, for those who enjoy puzzling, it would be a welcome addition to the collection.

Student, Berkhamsted Collegiate School PAUL JEFFERYS

Chance in Biology: Using Probability to Explore Nature. By MARK DENNY AND STEVEN GAINES. Princeton University Press, 2002. Pp. xiii+291. Paperback $29.95 (ISBN 0-691-09494-2).

This book is an attempt to make the application of probability theory to biology accessible to students as well as intelligent laymen; it largely succeeds in its objective. The authors are at pains to point out that their book does not deal with inferential statistics.

Chapter 1, 'The Nature of Chance', opens with a description of the properties of spider's silk, and gives reasons for its great strength. Chance, determinism and chaos are discussed, and the chapter ends with a brief account of the book's contents.

Chapter 2, 'Rules of Disorder', provides a summary of probability theory. It includes the basic laws of probability, and an account of Bayes' theorem with an illustration from AIDS testing. The notion of a probability distribution is outlined.

Chapter 3, 'Discrete Patterns of Disorder', introduces the concept of a discrete random variable and defines its expectation and variance. Bernoulli trials, each resulting in a 0 or 1 are considered, and lead to the binomial distribution; permutations and combinations are briefly mentioned. The geometric distribution is discussed, together with the waiting time to a success.

Chapter 4, 'Continuous Patterns of Disorder', considers continuous random variables. The uniform distribution is described, and the concepts of cumulative as well as probability distribution functions are introduced. The expectation and variance of a continuous random variable is defined, and the shape of a distribution discussed. The normal distribution and the central limit theorem are outlined, and the standard error is defined. The chapter ends with some elements of statistics, followed by two mathematical appendices.

Chapter 5, 'Random Walks', considers the motion of molecules and the rules for random walks. The average and variance of molecular motion are discussed, and the role of diffusion in the real world is mentioned. The biology of diffusion is outlined, and Fick's equation derived; an example of its use is given. The chapter ends with an account of receptors and channels.

Chapter 6, 'More Random Walks', continues the study of molecular diffusion and the collision of molecules against barriers. The cases of two absorbing walls and one reflecting wall are analysed, and an example of the turbulent mixing of plankton is studied. Genetic diffusion and the concept of fixation are introduced. The chapter ends with an account of the biology of elastic materials, and the limits of energy storage. Random walks in three dimensions and random protein configurations are briefly mentioned.

Chapter 7, 'The Statistics of Extremes', begins with an account of Rayleigh's 'Cocktail Party Problem', and proceeds to calculate the maximum amount of sound generated at such a party. Ocean waves are the next object of study, leading to the statistics of extremes, and Gumbel's three types of asymptotic extreme value distributions. Other examples concerned with life and death in Rhode Island, and baseball statistics are presented.

Chapter 8, 'Noise and Perception', considers the concept of noise, and offers an example of vision in dim light or with fuzzy images. The Poisson distribution is derived, and Bayes' formula is discussed in the context of the design of retinal rods. Membrane potential in rod cells is considered, and is followed by a study of noise and hearing, with fluctuations in pressure and velocity. The chapter ends with an account of stochastic resonance and nonlinear systems, followed by an appendix.

Chapter 9, 'The Answers', provides answers to the problems posed in earlier chapters.

There are four pages of cited literature, and three indices: a symbol index of five pages, an author index of two pages and a subject index of six pages. Most chapters end with a summary and problems. I found the book to be an excellent read, though probably better suited to students with an interest in biology. It merits a place on the library shelf of every school and college, and is sufficiently inexpensive to be afforded by individuals: it will amply repay its reading.

Australian National University JOE GANI

Other books received

Mathematical Analysis. Edited by ALLADI SITARAM AND VISHWAMBHAR PATI. Universities Press (India), Hyderabad, 2001 Pp. 142. Paperback £15.95 (ISBN 81-7371-291-3).

The Geometry of Numbers. By C. D. OLDS, ANNELI LAX AND GIULIANA DAVIDOFF. MAA, Washington, DC, 2001. Pp. 168. Paperback $24.95 (ISBN 0-88385-643-3).

Ordinary Differential Equations. By DAVID SÁNCHEZ. MAA, Washington, DC, 2002. Pp. 132. Paperback $28.95 (ISBN 0-88385-723-5).

Innovations in Teaching Abstract Algebra. Edited by ALLEN C. HIBBARD AND ELLEN J. MAYCOCK. MAA, Washington, DC, 2002. Pp. 136. Paperback $27.95 (ISBN 0-88385-171-7).

Mathematical Structures for Computer Science. By JUDITH L. GERSTING. Freeman, New York, 2002. Pp. 729. Hardback £41.99 (ISBN 0-7167-4358-2).

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