Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 19 #9

November
novembre
1993

<u>*CONTENTS / TABLE DES MATIÈRES*</u>

Canadian Mathematical Society   Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

SUBSCRIPTION INFORMATION

Crux Mathematicorum is published monthly (except July and August). The subscription rates for ten issues are detailed on the inside back cover. Information on Crux Mathematicorum back issues is also provided on the inside back cover.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum

ACKNOWLEDGEMENTS

The support of the Department of Mathematics & Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

REMERCIEMENTS

Nous rendons hommage à l'appui du département de mathématique et statistique de l'Université de Calgary et du département de mathématiques de l'Université d'Ottawa.

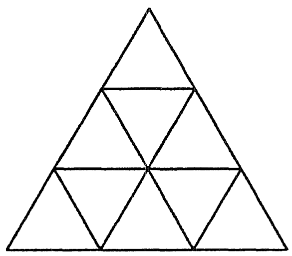
Printed at / imprimé à: Ottawa Laser Copy

ISSN 0705-0348 Publication Mail Registration No. 5432

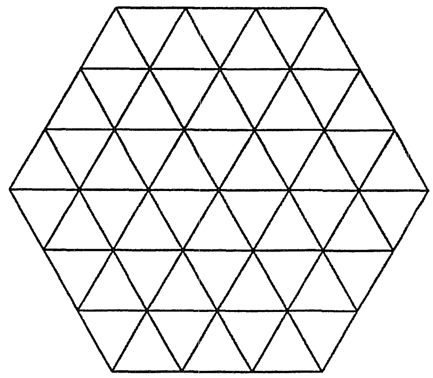
ENUMERATING 3-, 4-, 6-GONS WITH VERTICES AT LATTICE POINTS: PART I

Y. Arzoumanian and W. O. J. Moser

[*Editor's note.* This is the first part of the article. The second (final) Part II will appear in the next issue of *Crux*.]



T_3



H_3

The figures above indicate without further elaboration what is meant by the lattices (points and lines) T_n and H_n. In this two-part paper we solve a variety of problems concerning counts of 3-, 4- and 6-gons with vertices at lattice points. Some of the problems seem to be new, e.g., we have not seen problems 2b, 5b, 7b, 10 or 11 in the literature. Other problems have appeared several times and have been solved by summations, recurrences and generating series. We shall not attempt to detail such "history", although it would be an interesting project to see how some problems are independently rediscovered many times. Our aim is to solve these old problems and several new ones in the same particularly simple way using *only* the elementary

Lemma. $\underbrace{a_1} + \underbrace{a_2} + \cdots + \underbrace{a_k} \le n$ has $\binom{n+k}{k}$ solutions $(a_1, a_2, \ldots, a_k)$, where $a_1, \ldots, a_k$ are non-negative integers.

For convenience and emphasis we have adopted the notation $\underbrace{x}$ to indicate that x is an integer and $x \ge 0$; and of course, for integers $n \ge 1$, $k \ge 0$,

$$\binom{n}{k} = \begin{cases} n!/k!(n-k)! & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

is the number of subsets of size k of a set of size n.

To prove the Lemma, note that for each such sum there corresponds a linear arrangement of n 1's and k "strokes":

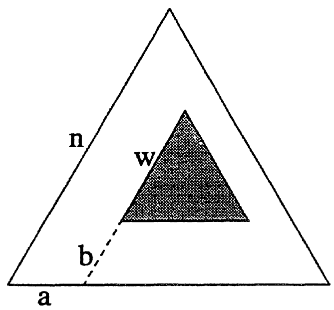
$$\overbrace{1\,1\,\cdots\,1}^{a_1\text{ 1's}}/\overbrace{1\,1\,\cdots\,1}^{a_2\text{ 1's}}/\overbrace{1\,1\,\cdots\,1}^{a_3\text{ 1's}}/\cdots/\overbrace{1\,1\,\cdots\,1}^{a_k\text{ 1's}}/\ \overbrace{1\,1\,\cdots\,1}^{n-(a_1+\cdots+a_k)\text{ 1's}} \qquad \begin{matrix} k \text{ strokes } / \\ n \text{ ones } 1 \end{matrix}$$

All such arrangements are obtained by starting with $n+k$ 1's in a row, choosing k of them in $\binom{n+k}{k}$ ways and then elongating the chosen 1's into strokes.

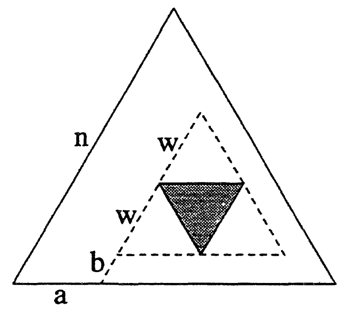
Problem 1. (a) How many equilateral triangles have vertices at the lattice points of T_n and sides on lattice lines?

(b) How many equilateral triangles have vertices at lattice points of T_n? (The sides need not lie on lattice lines.)

Solution. (a) There are two types of triangles.



Type I



Type II

To each Type I triangle there corresponds a 3-tuple (a, b, w) of integers satisfying $\underbrace{a}+\underbrace{b}+w \leq n$ with $w \geq 1$, or $\underbrace{a}+\underbrace{b}+\underbrace{w-1} \leq n-1$, and hence (by the Lemma) there are

$$\binom{n-1+3}{3} = \binom{n+2}{3}$$

such triangles.

For each Type II triangle the 3-tuple (a, b, w) satisfies

$$\underbrace{a}+\underbrace{b}+2(\underbrace{w-1}) \leq n-2. \tag{1}$$

We count these in four subsets $S(i,j)$, $i,j \in \{0,1\}$, in which $a \equiv i \bmod 2$, $b \equiv j \bmod 2$. In $S(i,j)$,

$$\underbrace{\frac{a-i}{2}}+\underbrace{\frac{b-j}{2}}+\underbrace{w-1} \leq \left\lfloor \frac{n-(i+j)}{2} \right\rfloor - 1$$

from (1), so by the Lemma $S(i,j)$ contains

$$\binom{\lfloor\frac{n-(i+j)}{2}\rfloor - 1 + 3}{3} = \binom{\lfloor\frac{n-(i+j)}{2}\rfloor + 2}{3}$$

3-tuples, and hence the number of Type II triangles is

$$\begin{aligned} t_{\text{II}}(n) &= \binom{\lfloor\frac{n}{2}\rfloor+2}{3} + 2\binom{\lfloor\frac{n-1}{2}\rfloor+2}{3} + \binom{\lfloor\frac{n-2}{2}\rfloor+2}{3} \\ &= \begin{cases} \frac{1}{24}n(n+2)(2n-1) & \text{if } n \text{ is even,} \\ \frac{1}{24}(n-1)(n+1)(2n+3) & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{2}$$

Altogether, the number of triangles is

$$\binom{n+2}{3} + t_{\text{II}}(n) = \begin{cases} \frac{1}{8}n(n+2)(2n+1) & \text{if } n \text{ is even,} \\ \frac{1}{8}(n+1)(2n^2+3n-1) & \text{if } n \text{ is odd.} \end{cases}$$

For other solutions see Carlitz & Scoville (1974), Cormier & Eggleton (1976), Garstang (1986), Gerrish (1970), Halsall (1962), Larson (1989), Martin (1971), Moon & Pullman (1973), Prielipp & Kuenzi (1974), and Smiley (1993).

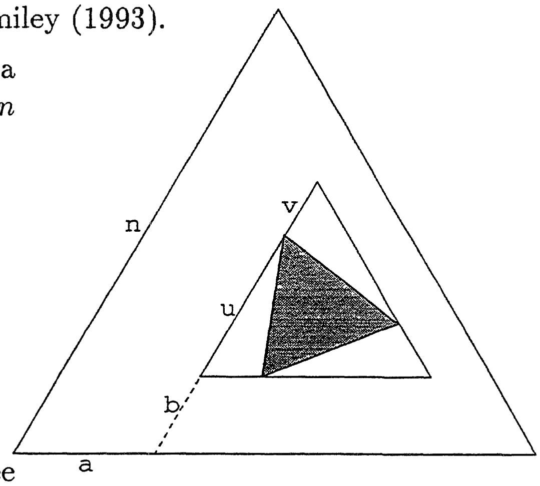
(b) To each triangle there corresponds a 4-tuple (a, b, u, v) satisfying $\underbrace{a} + \underbrace{b} + \underbrace{u} + v \leq n$ where $v \geq 1$, i.e.

$$\underbrace{a} + \underbrace{b} + \underbrace{u} + \underbrace{v-1} \leq n-1,$$

so by the Lemma there are

$$\binom{n-1+4}{4} = \binom{n+3}{4}$$

triangles. This is related to a problem in *Crux* (see Janous (1986), Moser (1987)).



Problem 2. (a) How many regular hexagons have vertices at lattice points of T_n and sides on lattice lines? (See Gibbs (1977).)

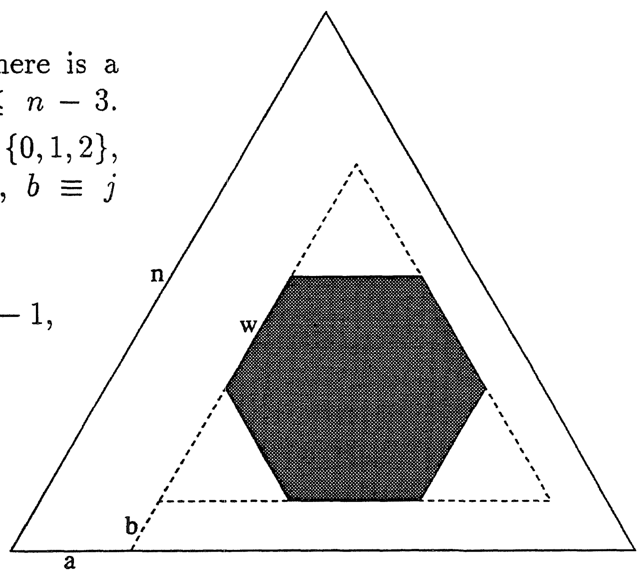
(b) How many regular hexagons have vertices at lattice points of T_n? (The sides need not lie on lattice lines.)

Solution. (a) For each hexagon there is a (a, b, w) satisfying $\underbrace{a} + \underbrace{b} + 3(\underbrace{w-1}) \leq n-3$. We count these in nine subsets $S(i,j)$, $i, j \in \{0,1,2\}$, containing the 3-tuples with $a \equiv i \bmod 3$, $b \equiv j \bmod 3$. In $S(i,j)$,

$$\underbrace{\frac{a-i}{3}} + \underbrace{\frac{b-j}{3}} + \underbrace{w-1} \leq \left\lfloor \frac{n-(i+j)}{3} \right\rfloor - 1,$$

so by the Lemma $S(i,j)$ contains

$$\binom{\left\lfloor \frac{n-(i+j)}{3} \right\rfloor - 1 + 3}{3} = \binom{\left\lfloor \frac{n-(i+j)}{3} \right\rfloor + 2}{3}$$



3-tuples, and the total number of hexagons is

$$\binom{\left\lfloor \frac{n}{3} \right\rfloor + 2}{3} + 2\binom{\left\lfloor \frac{n-1}{3} \right\rfloor + 2}{3} + 3\binom{\left\lfloor \frac{n-2}{3} \right\rfloor + 2}{3} + 2\binom{\left\lfloor \frac{n-3}{3} \right\rfloor + 2}{3} + \binom{\left\lfloor \frac{n-4}{3} \right\rfloor + 2}{3}$$

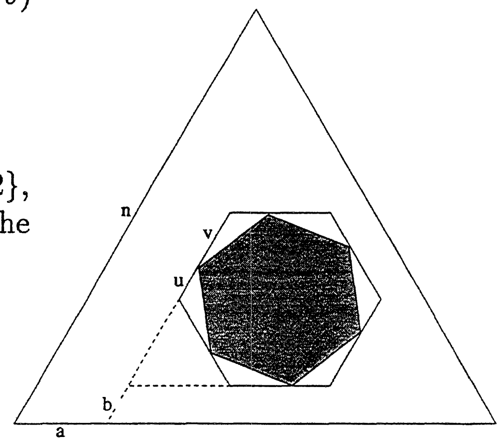
$$= \begin{cases} \frac{1}{18} n(n^2-3) & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{18}(n-1)^2(n+2) & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{18}(n-2)(n+1)^2 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

(b) For each hexagon there is a 4-tuple (a, b, u, v) satisfying

$$\underbrace{a}_{} + \underbrace{b}_{} + \underbrace{3u}_{} + \underbrace{3(v-1)}_{} \leq n-3.$$

We count these in nine subsets $S(i,j)$, $i, j \in \{0,1,2\}$, with $a \equiv i \bmod 3$, $b \equiv j \bmod 3$, and find that the number of hexagons is

$$\begin{array}{ll} \frac{1}{2^3 3^3} n(n+3)(n^2+3n-6) & \text{if } n \equiv 0 \bmod 3, \\ \frac{1}{2^3 3^3}(n-1)n(n+2)(n+5) & \text{if } n \equiv 1 \bmod 3, \\ \frac{1}{2^3 3^3}(n-2)(n+1)(n+3)(n+4) & \text{if } n \equiv 2 \bmod 3. \end{array}$$

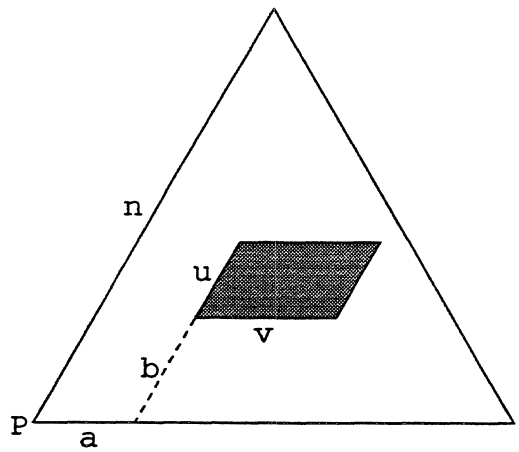


Problem 3. How many parallelograms have vertices at lattice points of T_n and sides along lattice lines?

Solution. Let P be a vertex of the "big" triangle. A parallelogram with sides parallel to the two lattice lines through P corresponds to a 4-tuple (a, b, u, v) satisfying

$$\underbrace{a}_{} + \underbrace{b}_{} + \underbrace{u-1}_{} + \underbrace{v-1}_{} \leq n-2,$$

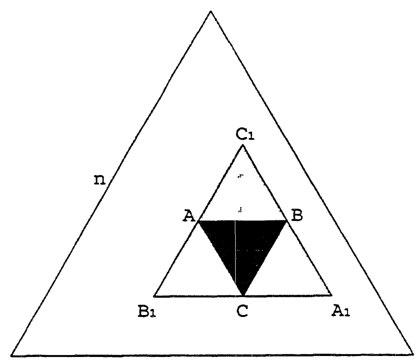
so by the Lemma there are $\binom{n-2+4}{4}$ such 4-tuples and $3\binom{n+2}{4}$ parallelograms altogether. (cf. Levy (1978))



Problem 4. How many rhombi have vertices at points of T_n and sides along lines of T_n?

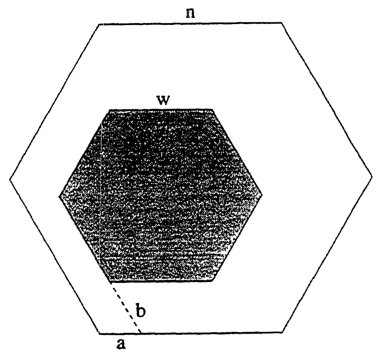
Solution. For each Type II triangle of Problem 1(a), for example ABC, there are 3 rhombi, namely AC_1BC, BA_1CA, CB_1AB. The number of rhombi is (see (2))

$$3t_{\text{II}}(n) = \begin{cases} \frac{1}{8}n(n+2)(2n-1) & \text{if } n \text{ is even}, \\ \frac{1}{8}(n-1)(n+1)(2n+3) & \text{if } n \text{ is odd}. \end{cases}$$



Problem 5. (a) How many regular hexagons have vertices at lattice points of H_n and sides along lattice lines?

(b) How many regular hexagons have vertices at lattice points of H_n? (The sides need not lie on lattice lines.)



Solution. (a) To each hexagon there is a triple (a, b, w) satisfying

$$\underbrace{a} + 2w \le 2n, \quad w \ge 1,$$

$$\underbrace{b} + 2w \le 2n, \quad |a - b| \le n - w,$$

which we count in 3 subsets according to $a < b$, $a = b$, $b < a$.

In the first subset the triple satisfies

$$\underbrace{b-a-1} + \underbrace{a} + 2\underbrace{(w-1)} \le 2n-3, \quad b - a \le n - w.$$

Here we pick up the triples satisfying

$$\underbrace{b-a-1} + \underbrace{a} + 2\underbrace{(w-1)} \le 2n-3$$

(this is (1) with n replaced by $2n-1$, so we have

$$\frac{1}{24}(2n-1-1)(2n-1+1)(4n-2+3) = \frac{1}{6}(n-1)n(4n+1)$$

triples), reduced by those triples which satisfy

$$\underbrace{b-a-1} + \underbrace{a} + 2\underbrace{(w-1)} \le 2n-3, \quad b - a > n - w,$$

equivalently

$$\underbrace{(b-a-1)-(n-w)} + \underbrace{a} + \underbrace{w-1} \le n-2;$$

by the Lemma we have here $\binom{n-2+3}{3} = \frac{1}{6}(n+1)n(n-1)$ triples. Apparently the first subset (in which $a < b$) has size

$$\frac{1}{6}(n-1)n(4n+1) - \frac{1}{6}(n+1)n(n-1) = \frac{1}{2}n^2(n-1).$$

In the second subset (where $a = b$) we have

$$\underbrace{a} + 2\underbrace{(w-1)} \le 2n-2;$$

if a is even there are $\binom{n+1}{2}$ triples, and if a is odd there are $\binom{n}{2}$ triples. Of course the third subset has the same size as the first subset, so the answer to (a) is

$$n^2(n-1) + \binom{n+1}{2} + \binom{n}{2} = n^3.$$

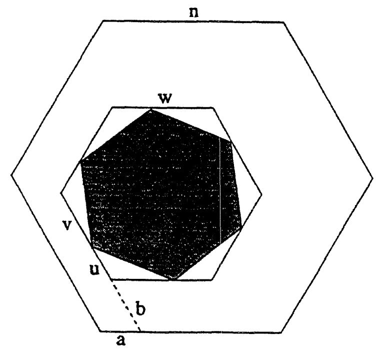
(b) To each hexagon there corresponds a 4-tuple (a, b, u, v) satisfying

$$a + 2(u+v) \leq 2n, \quad u \geq 0,\ v \geq 1,$$

$$b + 2(u+v) \leq 2n, \quad |a-b| \leq n - (u+v),$$

and we are led (details left to the reader) to

$$\binom{n+3}{4} + 4\binom{n+2}{4} + \binom{n+1}{4}.$$



References

L. Carlitz & R. Scoville (1974), Solution to Problem 889, *Mathematics Magazine* **47**, 290–291.

R. J. Cormier & R. B. Eggleton (1976), Counting by correspondence, *Mathematics Magazine* **49**, 181–186.

R. H. Garstang (1986), Triangles in a triangle, *Mathematical Gazette* **70**, 288–289.

F. Gerrish (1970), How many triangles?, *Mathematical Gazette* **54**, 241–246.

R. A. Gibbs (1977), Solution to Problem 975, *Mathematics Magazine* **50**, 266.

J. Halsall (1962), Classroom note: An interesting series. *Mathematical Gazette* **46**, 55–56.

W. Janous (1986), Partial Solution to Problem 976, *Crux Mathematicorum* **12**, 145–147.

M. E. Larson (1989), The eternal triangle — a history of a counting problem. *College Math. J.* **20**, 370–384.

J. I. Levy (1978), Solution of Problem 1001, *Mathematics Magazine* **51**, 246.

B. W. Martin (1971), Mathematical Notes 3316. How many triangles? *Mathematical Gazette* **55**, 438–441.

J. W. Moon & N. J. Pullman (1973), The number of triangles in a triangular lattice, *Delta* **3**, 28–31.

W. Moser (1987), Comment on problem 976, *Crux Mathematicorum* **13**, 16–17.

B. Prielipp & N. J. Kuenzi (1974), Solution to Problem 889, *Mathematics Magazine* **47**, 289–291.

L. M. Smiley (1993), A quick solution of triangle counting, *Mathematics Magazine* **66**, 40.

Department of Mathematics and Statistics
McGill University
Montréal, PQ, Canada H3A 2K6

* * * * *

THE OLYMPIAD CORNER

No. 149

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this number by giving the remaining problems that were proposed to the jury but not used at the 33rd International Mathematical Olympiad in Moscow, Russia in July 1992. My thanks to Georg Gunther, the Canadian Team Leader, for sending me the material.

8. *Proposed by India.*

Circles G, G_1, G_2 are three circles related to each other as follows: circles G_1 and G_2 are externally tangent to one another at a point W and both these circles are internally tangent to the circle G. Points A, B, C are located on the circle G as follows: line BC is a direct common tangent to the pair of circles G_1 and G_2, and line WA is the transverse common tangent at W to G_1 and G_2, with W and A lying on the same side of the line BC.

Problem: Prove that W is the incenter of the triangle ABC.

9. *Proposed by Japan.*

In a triangle ABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB, respectively. Determine the angles $\angle A$, $\angle B$, $\angle C$, if

$$\angle BDE = 24^\circ, \quad \angle CED = 18^\circ.$$

10. *Proposed by the Netherlands.*

Let f, g and a be polynomials with real coefficients, f and g in one variable and a in two variables. Suppose

$$f(x) - f(y) = a(x,y)(g(x) - g(y)) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Prove that there exists a polynomial h with $f(x) = h(g(x))$ for all $x \in \mathbb{R}$.

11. *Proposed by Poland.*

For any positive integer x define

$$g(x) = \text{ greatest odd divisor of } x,$$

$$f(x) = \begin{cases} x/2 + x/g(x), & \text{if } x \text{ is even;} \\ 2^{(x+1)/2}, & \text{if } x \text{ is odd.} \end{cases}$$

Construct the sequence $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the number 1992 appears in this sequence, determine the least n such that $x_n = 1992$, and find out whether n is unique.

12. *Proposed by South Korea.*

Prove that $n = \dfrac{5^{125} - 1}{5^{25} - 1}$ is a composite number.

13. *Proposed by Sweden.*

Let $\alpha(n)$ be the number of digits equal to one in the dyadic representation of a positive integer n. Prove that

(a) the inequality $\alpha(n^2) \leq \frac{1}{2}\alpha(n)(\alpha(n) + 1)$ holds,

(b) the above inequality is equality for infinitely many positive integers, and

(c) there exists a sequence $(n_i)_1^\infty$, such that $\alpha(n_i^2)/\alpha(n_i) \to 0$ as $i \to \infty$.

14. *Proposed by the United States of America.*

Let $[x]$ denote the greatest integer less than or equal to x. Pick any x_1 in $[0, 1]$ and define a sequence $x_1, x_2, x_3, \ldots$ by $x_{n+1} = 0$ if $x_n = 0$ and $x_{n+1} = 1/x_n - [1/x_n]$ otherwise. Prove that

$$x_1 + x_2 + \cdots + x_n < F_1/F_2 + F_2/F_3 + \cdots + F_n/F_{n+1},$$

where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

* * *

The next item is the 34th I.M.O. which was held in Istanbul, Turkey, July 18–19, 1993. My sources this year are Professor Georg Gunther, the Canadian Team leader; Professor Cecil Rousseau, the United States of America Team leader; Wei-Hua Huang of the U.S. Team; and (via Andy Liu of Edmonton) Professor P.H. Cheung of Hong Kong. I hope I have made no serious mistakes in compiling this report from the information they sent. Many thanks to those who sent in information.

This year a total of 412 students from 73 countries participated officially, with Mongolia participating unofficially. This record number of students (and countries) makes the organizational headaches for the host country expand exponentially.

The contest is officially an individual competition and the six problems were assigned equal weights of seven points each (the same as the last 12 I.M.O.'s) for a maximum possible individual score of 42 (and a total possible of 252 for a national team of six students). For comparisons see the last 12 I.M.O. reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193], [1989: 193], [1990: 193], [1991: 257] and [1992: 263].

This year there were only two perfect scores, one from the Chinese team, and one from Taiwan. The third score was 39, by a student from Germany, and all other scores were less than 38. The median score was 10. The jury awarded a first prize (Gold) to the thirty-five students who scored 30 or more on the test. Second (Silver) prizes went to the sixty-six students with scores from 20 to 29, and third (Bronze) prizes went to the ninety-six students who scored from 11 to 19 points. Any student who did not receive a medal, but who got 7 on one problem, was awarded Honourable Mention.

Normally I would now list the Gold Medal winners for the record, but none of my sources provided the names of the thirty-five winners. I am trying to obtain this listing and will honour the winners later if I can. Congratulations, anyway.

Next we give the problems from this year's I.M.O. Competition. Solutions to these problems, along with those of the 1993 U.S.A. Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads* 1993, which may be obtained for a small charge from: Dr. W.E. Mientka, Executive Director, M.A.A. Committee on H.S. Contests, 917 Oldfather Hall, University of Nebraska, Lincoln, Nebraska, U.S.A. 68588.

34th INTERNATIONAL MATHEMATICAL OLYMPIAD

Istanbul, Turkey

First Day – July 18, 1993 (4.5 hours)

1. Let $f(x) = x^n + 5x^{n-1} + 3$ where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two polynomials, each of which has all its coefficients integers and degree at least 1.

2. Let D be a point inside the acute-angled triangle ABC such that $\angle ADB = \angle ACB + 90^\circ$ and $AC \times BD = AD \times BC$.

(a) Calculate the value of the ratio $\dfrac{AB \times CD}{AC \times BD}$.

(b) Prove that the tangents at C to the circumcircles of the triangles ACD and BCD are perpendicular.

3. On an infinite chessboard, a game is played as follows. At the start n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed.

Find those values of n for which the game can end with only one piece remaining on the board.

Second Day – July 19, 1993 (4.5 hours)

4. For three points P, Q, R in the plane, we define $m(PQR)$ to be the minimum of the lengths of the altitudes of the triangle PQR (where $m(PQR) = 0$ when P, Q, R are collinear).

Let A, B, C be given points in the plane. Prove that, for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine whether or not there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} & & f(1) &= 2, & & \\ & & f(f(n)) &= f(n) + n & &\text{for all } n \text{ in } \mathbb{N}, \\ &\text{and} & f(n) &< f(n+1) & &\text{for all } n \text{ in } \mathbb{N}. \end{aligned}$$

6. Let $n > 1$ be an integer. There are n lamps $L_0, L_1, \ldots, L_{n-1}$ arranged in a circle. Each lamp is either ON or OFF. A sequence of steps $S_0, S_1, \ldots, S_i, \ldots$ is carried out. Step S_j affects the state of L_j only (leaving the state of all other lamps unaltered) as follows:

if L_{j-1} is ON, S_j changes the state of L_j from ON to OFF or from OFF to ON;
if L_{j-1} is OFF, S_j leaves the state of L_j unchanged. The lamps are labelled mod n, that is,

$$L_{-1} = L_{n-1}, \quad L_0 = L_n, \quad L_1 = L_{n+1}, \quad \text{etc.}\ .$$

Initially all lamps are ON. Show that

(a) there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are ON again;

(b) if n has the form 2^k then all the lamps are ON after $n^2 - 1$ steps;

(c) if n has the form $2^k + 1$ then all the lamps are ON after $n^2 - n + 1$ steps.

*

As the I.M.O. is officially an individual event, the compilation of team scores is unofficial, if inevitable. These totals (with possible errors!) are given in the following table. It is worth noting the 26 point spread between China and Germany. Congratulations to the Chinese Team.

	Country	Score	Gold	Silver	Bronze
1.	China	215	6	-	-
2.	Germany	189	4	2	-
3.	Bulgaria	178	2	4	-
4.	Russia	177	4	1	1
5.	Taiwan	162	1	4	1
6.	Iran	153	2	3	1
7.	U.S.A.	151	2	2	2
8.	Hungary	143	3	1	2
9.	Vietnam	138	1	4	1
10.	Czech Republic	132	1	2	3
11.	Romania	128	1	2	3
12.	Slovak Republic	126	1	3	1
13.	Australia	125	1	2	2
14.	U.K.	118	-	3	3
15.–16.	India	116	-	4	1
15.–16.	Republic of Korea	116	-	3	3
17.	France	115	2	1	1
18.–19.	Canada	113	1	1	3
18.–19.	Israel	113	1	2	2
20.	Japan	98	-	2	3
21.	Ukraine	96	-	2	3
22.	Austria	87	-	1	3
23.	Italy	86	1	-	3

	Country	Score	Gold	Silver	Bronze
24.	Turkey	81	-	1	2
25.	Kazakhstan	80	-	1	3
26.–27.	Colombia	79	-	-	4
26.–27.	Georgia	79	-	1	3
28.–29.	Armenia	78	-	1	-
28.–29.	Poland	78	-	2	1
30.	Singapore	75	-	1	3
31.	Latvia	73	-	2	1
32.	Denmark	72	-	1	3
33.	Hong Kong	70	-	-	4
34.	Brazil	60	-	1	5
35.	The Netherlands	58	-	-	1
36.	Cuba	56	-	1	1
37.	Belgium	55	-	-	1
38.	Byelorussia (Team of 4)	54	-	1	1
39.	Sweden	51	-	1	1
40.	Morocco	49	-	-	1
41.	Thailand	47	-	-	2
42.–43.	Argentina	46	-	1	1
42.–43.	Switzerland (Team of 4)	46	-	1	1
44.	Norway (Team of 5)	44	-	-	2
45.–47.	New Zealand	43	-	-	2
45.–47.	Slovenia (Team of 5)	43	-	-	2
45.–47.	Spain	43	-	1	1
48.	Macedonia (Team of 4)	42	-	-	3
49.	Lithuania	41	-	-	-
50.	Ireland	39	-	-	1
51.	Portugal	35	-	-	1
52.–54.	Azerbaijan (Team of 5)	33	-	-	1
52.–54.	Finland	33	-	-	-
52.–54.	The Philippines	33	-	-	1
55.	Croatia	32	-	-	1
56.–57.	Estonia	31	-	-	1
56.–57.	Mongolia	31	-	-	1
58.–59.	South Africa	30	-	-	-
58.–59.	Trinidad and Tobago	30	-	-	-
60.	Moldavia	29	-	-	-
61.	Kirgistan (Team of 5)	28	-	-	-
62.–63.	Macau	24	-	-	-
62.–63.	Mexico	24	-	-	1
64.	Iceland (Team of 4)	23	-	-	-
65.	Luxembourg (Team of 1)	20	-	1	0
66.	Albania	18	-	-	-
67.	North Cyprus	17	-	-	-
68.–69.	Brunei	16	-	-	-
68.–69.	Kuwait	16	-	-	-
70.	Indonesia	15	-	-	-
71.	Bosnia-Hercegovina (Team of 2)	14	-	-	1
72.	Algeria (Team of 5)	9	-	-	-
73.	Turkmenistan (Team of 2)	9	-	-	-

This year the Canadian Team climbed back to a tie with Israel for 17th place. The team members and scores were:

Ka-Ping Yee	34	Gold
Naoki Sato	22	Silver
Edward Leung	19	Bronze
Alex Lee	18	Bronze
Kevin Purbhoo	16	Bronze
Peter Dukes	4	

The Team Leader was Dr. Georg Gunther (Sir Wilfred Grenfell College, Corner Brook, Newfoundland), and Mr. Ravi Vakil, now at Harvard University, was Deputy Leader. They were asked to lead (pro tem) the team from Bosnia-Hercegovina when the main part of its team was stuck in Sarajevo.

The United States of America Team slipped a bit from 2nd place (in Moscow) to 7th place in Istanbul. Its team members were:

Lenny Ng	37	Gold
Andrew Dittmer	33	Gold
Wei-Hua Huang	23	Silver
Stephen Wang	22	Silver
Jeremy Bem	18	Bronze
Tim Chklovski	18	Bronze

The team leader was Prof. Cecil Rousseau of Memphis State University, and the deputy was Prof. Anne Hudson from Savannah, Georgia.

* * *

I fell in the trap! In the last issue of the Corner we gave a "solution" to a problem proposed by the Soviet Union at Sigtuna, Sweden. I must not have been reading critically because I missed a fatal flaw in the argument. Fortunately, Stan Wagon had thought about the problem and spotted the error. Below we give his correction. Mea culpa.

8. [1992: 196; 1993: 228] *Proposed by the U.S.S.R.*

Let a_n be the last nonzero digit in the decimal representation of the number $n!$. Does the sequence become periodic after a finite number of terms?

CORRECTION by Stan Wagon, Macalester College, St. Paul, Minnesota.

Let us say that integers m and n are equivalent, $m \sim n$, if they have the same rightmost nonzero digit. It is **not** generally true that if $a \sim b$, then $ac \sim bc$, for example, $2 \sim 12$ but $2 * 5 = 10$ while $12 * 5 = 60$. But if none of the rightmost digits is 5, then no zeros can be introduced and we get the following useful and easy-to-prove assertion.

Lemma 1. If 5 is not the rightmost digit of a, b, or c, then $a \sim b$ implies $ac \sim bc$.

Now, Lemma 1 leads us to a more subtle fact. Recall that there are more 2's than 5's in $n!$ provided $n > 1$, and so the rightmost nonzero digit of $n!$ is one of 1, 2, 4, 6, or 8.

Lemma 2. $(5n)! \sim 2^n n!$.
Proof.

$$\begin{aligned}
(5n)! &= 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdots(5n-1)\cdot 5n\\
&= 1\cdot 1\cdot 3\cdot 4\cdot 10\cdot 3\cdot 7\cdot 8\cdot 9\cdot 20\cdot 11\cdot 6\cdot 13\cdot 14\cdot 30\cdot 8\cdot 17\cdot 18\cdot 19\cdot 40\\
&\qquad\qquad\cdots\frac{1}{2}(5n-4)(5n-3)(5n-2)(5n-1)\cdot 10n\\
&\sim 1\cdot 1\cdot 3\cdot 4\cdot 1\cdot 1\cdot 3\cdot 7\cdot 8\cdot 9\cdot 2\cdot 11\cdot 6\cdot 13\cdot 14\cdot 3\\
&\qquad\qquad\cdots\frac{1}{2}(5n-4)(5n-3)(5n-2)(5n-1)n\\
&= (1\cdot 1\cdot 3\cdot 4)(3\cdot 7\cdot 8\cdot 9)(11\cdot 6\cdot 13\cdot 14)\cdot\frac{1}{2}(16\cdot 17\cdot 18\cdot 19)\cdot\frac{1}{2}(21\cdot 22\cdot 23\cdot 24)\\
&\qquad\qquad\cdots\frac{1}{2}(5n-4)(5n-3)(5n-2)(5n-1)\cdot n!\\
&\sim 2\cdot 2\cdot 2\cdots 2\cdot n!\\
&= 2^n\cdot n!
\end{aligned}$$

(by Lemma 1, since the rightmost nonzero digit of $n!$ is never 5).

Now suppose that, beyond n, the sequence is periodic of period p. This statement is true if p is replaced by any multiple of p and if n is increased, so we may assume that p is divisible by 4, and n has the form $kp-1 = 4j+3$, with $j \geq 1$. Then, periodicity implies that $n! \sim (n+4(n+1))! = (5n+4)!$.

But we claim that $(5n+4)! \sim 2\cdot n!$, which is a contradiction since $2\cdot\{1,2,4,6,8\} = \{2,4,8,12,16\}$. The claim follows from the lemmas:

$$\begin{aligned}
(5n+4)! &= (5n)!(5n+1)(5n+2)(5n+3)(5n+4)\\
&\sim 2^n n!\cdot 4 = 4\cdot 2^{kp-1}\cdot n! = 4\cdot 2^{4j}\cdot 8\cdot n!\\
&\sim 4\cdot 6\cdot 8\cdot n! \sim 2n!
\end{aligned}$$

since any power of 16 ends in a 6.

So, contrary to the "solution" published last month the sequence a_n is **not** eventually periodic.

* * *

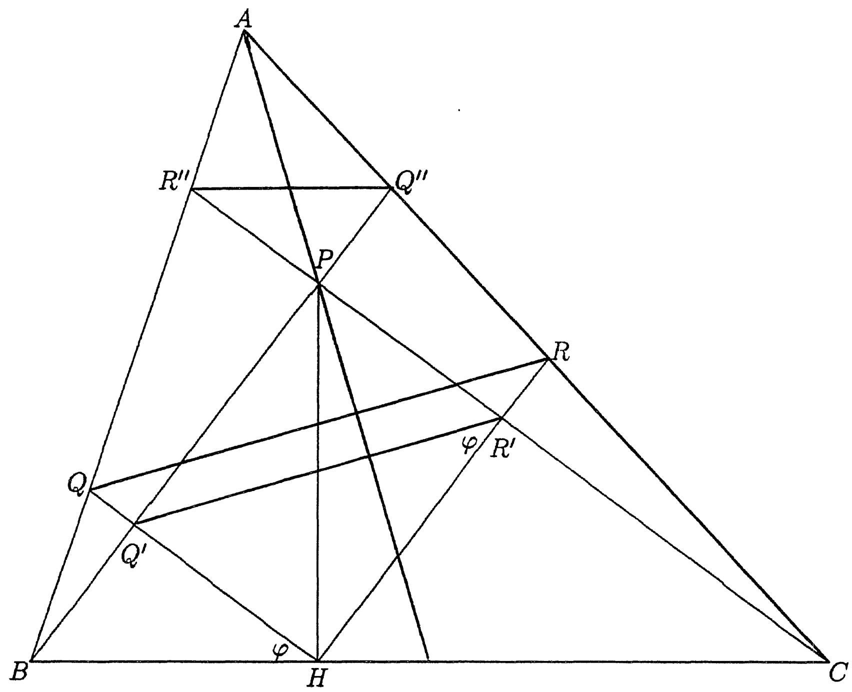
To finish the column we turn to solutions sent in by the readers to problems proposed but not used at the 32nd I.M.O. at Sigtuna, Sweden [1992: 225].

13. *Proposed by Japan.*

For an acute triangle ABC, M is the midpoint of the segment BC, P is a point on the segment AM such that $PM = BM$, H is the foot of the perpendicular line from P to BC, Q is the point of intersection of the segment AB and the line passing through H that is perpendicular to PB, and finally R is the point of intersection of the segment AC and the line passing through H that is perpendicular to PC. Show that the circumcircle of ΔQHR is tangent to the side BC at the point H.

Solutions by George Evagelopoulos, Athens, Greece; and by D.J. Smeenk, Zaltbommel, The Netherlands. We give Smeenk's solution.

Let HQ intersect PB in Q' and let HR intersect PC in R'. Now $BM = MC = MP$ so P lies on the circle centred at M with diameter BC. Thus $\angle Q'PR' = 90°$ and the quadrilateral $HR'PQ'$ is a rectangle with its diagonal $PH \perp BC$.



It follows easily that $\angle BHQ = \angle HR'Q' = \varphi$, say, we are to show that $\angle HRQ = \varphi$; that means that $QR \| Q'R'$.

Let the production of PB meet AC in Q'' and let the production of CP meet AB in R''. Now $\Delta PQ''R''$ and ΔPBC are similar and

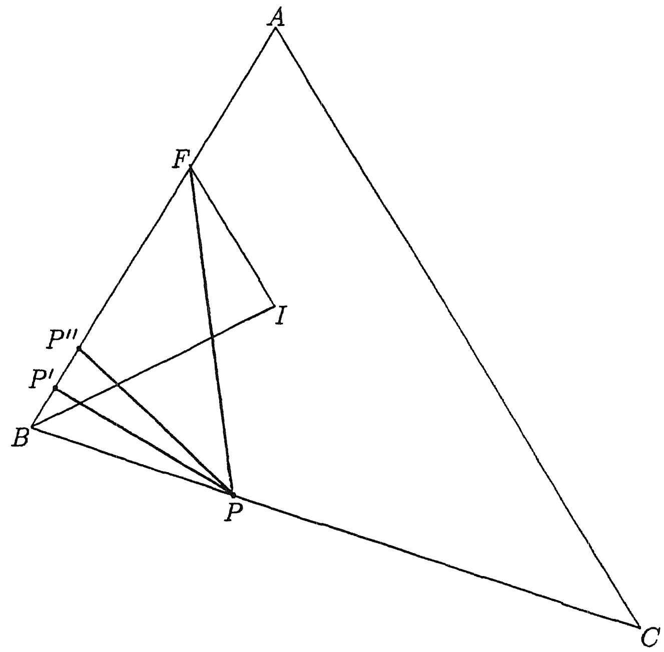
$$PR'' = \lambda PC, \; PQ'' = \lambda PB \quad \text{with} \quad \lambda > 0. \tag{1}$$

From ΔBCP and $\Delta HPQ'$ we have $BQ' : BP = HQ' : CP$ and from $\Delta BP''P$ we obtain $BQ' : BP = QQ' : R''P$. From this it follows that $HQ' : QQ' = CP : R''P$ and, with (1), $HQ' : QQ' = 1 : \lambda$. Similarly $HR' : RR' = 1 : \lambda$. Therfore, we conclude that $HQ' : QQ' = HR' : RR'$ and therefore $QR \| Q'R'$ as required.

14. *Proposed by Spain.*

In the triangle ABC, with $\angle A = 60°$, a parallel IF to AC is drawn through the incenter I of the triangle, where F lies on the side AB. The point P on the side BC is such that $3BP = BC$. Show that $\angle BFP = \angle B/2$.

Solutions by George Evagelopoulos, Athens, Greece; and by D.J. Smeenk, Zaltbommel, The Netherlands. We use Smeenk's solution.



We know that $BI = 4R\sin\frac{1}{2}\alpha\sin\frac{1}{2}\gamma = 2R\sin\frac{1}{2}\gamma$ since $\alpha = 60°$. Also, since $\alpha = 60°$ we have $\frac{1}{2}\beta + \frac{1}{2}\gamma = 60°$ and so we have $BI = 2R\sin(60° - \frac{1}{2}\beta)$. Applying the law of sines in ΔBIF we see that $BI : BF = \sin 60° : \sin(60° + \frac{1}{2}\beta)$, so that

$$2R\sin(60° - \frac{1}{2}\beta) : BF = \sin 60° : \sin(60° + \frac{1}{2}\beta).$$

After some simplification

$$BF = \frac{2}{3}R\sqrt{3}\left[\cos\beta + \frac{1}{2}\right] = \frac{2}{3}a\cos\beta + \frac{1}{3}a \qquad (1)$$

since $a = R\sqrt{3}$.

Now let P' be on AB so that $PP' \perp AB$. Then $BP' = \frac{1}{3}a\cos\beta$. Let P'' lie on AB so that P' is the midpoint of BP''.

Now $BP'' = \frac{2}{3}a\cos\beta$ and $PP'' = PB = \frac{1}{3}a$. From (1) it follows that $P''F = \frac{1}{3}a = P''P$. Now since $\Delta PBP''$ is isosceles ($PB = PP''$) we get $\angle BP''P = \beta$. Also, since $\Delta P''FP$ is isosceles ($P''F = P''P$) we have $\angle P''FP = \angle P''PF = \frac{1}{2}\angle BP''P = \frac{1}{2}\beta$, as required.

* * *

We will continue with readers' solutions to these proposed I.M.O. problems next issue, but this is all the space we have this issue. Send me your nice solutions and contests.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1881. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $BC > CA > AB$. D is a point on side BC, and E is a point on BA produced beyond A, so that $BD = BE = CA$. Let P be a point on side AC such that E, B, D, P are concyclic, and let Q be the second intersection of BP with the circumcircle of ΔABC. Prove that $AQ + CQ = BP$.

1882. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Arthur tosses a fair coin until he obtains two heads in succession. Betty tosses another fair coin until she obtains a head and a tail in succession, with the head coming immediately prior to the tail.

(i) What is the average number of tosses each has to make?

(ii) What is the probability that Betty makes *fewer* tosses than Arthur (rather than the same number or more than Arthur)?

1883. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle and construct the circles with the sides AB, BC, CA as diameters. $A'B'C'$ is the triangle containing these three circles and whose sides are each tangent to two of the circles. Prove that

$$[A'B'C'] \geq \left(\frac{13}{4} + \sqrt{3}\right)[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

1884. *Proposed by Ian Affleck, student, University of Regina.*

(a) Let $v(n)$ be the set of integers that result from "truncations" of the positive integer n; for example $v(135) = \{1, 3, 5, 13, 35, 135\}$. Call n a *v-prime* if every number in $v(n)$ is a prime or 1, so that 173 is v-prime for example. Find all v-primes.

(b)* Let $t(n)$ be the set of integers that result from a *single* truncation of n; for example, $t(1806) = \{1, 18, 180, 1806, 806, 6\}$. Define *t-prime* analogously to v-prime. How many t-primes are there?

1885. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

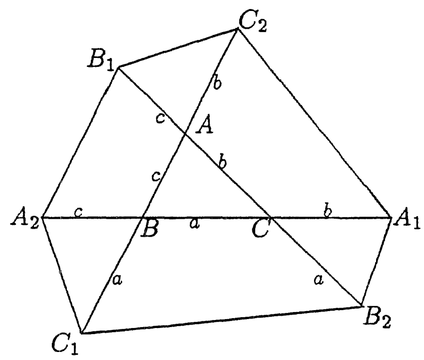
Circles $\mathcal{C}_1$ and $\mathcal{C}_2$, with centres O_1 and O_2, intersect in A and B. A line ℓ through A intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ for the second time in C and D respectively. CO_1 and DO_2 intersect in P, and the line m through P perpendicular to CD intersects AB in Q. Show that P, D, Q, C and B are concyclic.

1886. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all integers $n > 1$ such that $\{1!, 2!, \ldots, n!\}$ is a complete set of residues modulo n. (This problem was inspired by problem 1424 in the June 1993 issue of *Mathematics Magazine*.)

1887. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

Given an acute triangle ABC, form the hexagon $A_1C_2B_1A_2C_1B_2$ as shown, where



$$BC = BC_1 = CB_2,$$

$$CA = CA_1 = AC_2,$$

and

$$AB = AB_1 = BA_2.$$

Prove that the area of the hexagon is at least 13 times the area of ΔABC, with equality when ABC is equilateral.

1888. *Proposed by Erich Friedman, Stetson University, DeLand, Florida.*

Prove that for every finite set A of positive integers, there exists a finite set B of positive integers so that $B \supseteq A$ and

$$\prod_{x \in B} x = \sum_{x \in B} x^2.$$

1889. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

I is the incenter of ΔABC. IH parallel to AB meets BC at H, and IK parallel to BC meets AC at K. Assume that $C = 12°$ and that the quadrilateral $ABHK$ is cyclic. Find angles A and B.

1890. *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let n be a positive integer and let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad g(x) = \frac{1}{a_n}x^n + \frac{1}{a_{n-1}}x^{n-1} + \cdots + \frac{1}{a_1}x + \frac{1}{a_0},$$

where the a_i's are nonzero real numbers. Prove that

$$f(g(1))g(f(1)) \geq 4.$$

When does equality hold?

* * * * *

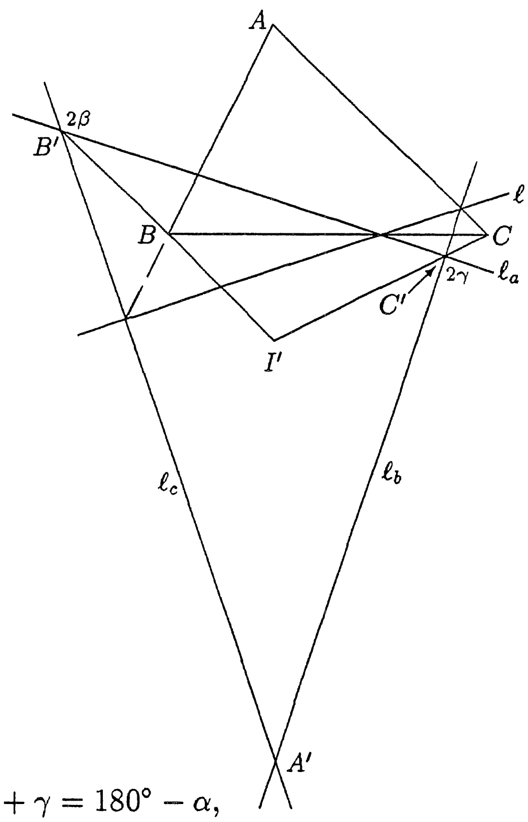
SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1787. [1992: 275] *Proposed by Isao Ashiba, Tokyo, Japan.*

ABC is an acute triangle and ℓ is a line in the same plane. Let ℓ_a be the line symmetric to ℓ with respect to line BC, and similarly define lines ℓ_b and ℓ_c. Let $A' = \ell_b \cap \ell_c$, $B' = \ell_c \cap \ell_a$, $C' = \ell_a \cap \ell_b$. Show that the incenter of $\Delta A'B'C'$ lies on the circumcircle of ΔABC.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.*



Suppose triangle ABC is counter-clockwise oriented and has angles α, β, γ. Let s_a, s_b, s_c be the axial symmetries [reflections] with respect to lines BC, CA, AB respectively. Then

$$\ell_a \xrightarrow{s_a} \ell \xrightarrow{s_c} \ell_c;$$

the product $s_a \circ s_c$ of these symmetries is a rotation with center B and amplitude 2β in the counterclockwise sense [e.g. see Proposition 3, page 50 of I.M. Yaglom, *Geometric Transformations I*, MAA, 1962]. Because B is equidistant from ℓ_a and its image ℓ_c, BB' is the bisector of the angle between ℓ_a and ℓ_c. Analogously, by the rotation with center C and amplitude 2γ, $\ell_b \to \ell_a$, and CC' is the bisector of the angle between ℓ_a and ℓ_b. In the same way, AA' is the bisector of the angle between ℓ_b and ℓ_c. So $AA' \cap BB' \cap CC'$ is the incenter I' of the triangle $A'B'C'$. As a consequence, $\angle C'B'I' = 90° - \beta$ and $\angle B'C'I' = 90° - \gamma$, and thus

$$\angle BI'C = \angle B'I'C' = \beta + \gamma = 180° - \alpha,$$

so I' belongs to the circumcircle of ABC as claimed.

II. *Comment by the Editor (in this case, mostly Chris Fisher).*

Two other solvers, John Heuver and Toshio Seimiya, point out that if ℓ passes through the orthocentre of ΔABC, then $A'B'C'$ degenerates to a point on the circumcircle of ABC; Heuver refers to Problem 60, pages 76 and 148 of I.M. Yaglom, *Geometric Transformations II*, MAA, 1962.

Heuver and Seimiya also both observe that if ABC were obtuse, then an **excentre** of $\Delta A'B'C'$ would lie on the circumcircle of ABC, as may be suggested by the above proof.

When the editor confessed that it wasn't clear to him why, for ABC acute, every line ℓ should result in the intersection $A'A \cap B'B \cap C'C$ being the incentre of $A'B'C'$, and never an excentre, Chris Fisher contributed the following explanation.

Yaglom [in Problem 60 of *Geometric Transformations II*, mentioned above] says essentially that when ABC has acute angles then triangle $A'B'C'$ has angles $180° - 2A$, $180° - 2B$, and $180° - 2C$ (independent of the choice of ℓ). Should the angle A, say, be $90°$ then ℓ_b and ℓ_c would be parallel (with A' undefined). Were A obtuse, then $180° - 2A$ would be negative and the angles used in triangle $A'B'C'$ would be $2A - 180°$, $2B$ and $2C$: since all lines ℓ lead to similar triangles, an easy way to confirm this claim is to let ℓ be parallel to the line AB; more interestingly, let ℓ be the line joining the feet of two of the altitudes of triangle ABC, in which case $A'B'C'$ is congruent to the "pedal triangle" (whose vertices are the feet of the altitudes; see the fascinating article by J.C. Alexander, "The symbolic dynamics of the sequence of pedal triangles", *Mathematics Magazine* **66**:3 (June, 1993), pp. 147–158, especially p. 148). Thus, as Heuver and Seimiya both observed, if the angle at A were obtuse, then the excentre of triangle $A'B'C'$ opposite A' would lie on the circumcircle of ABC.

Also solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Festraets-Hamoir's solution is similar to the above. Bellot and López, and also Hut, note that one can use Desargues' Theorem to prove that triangles ABC and A'B'C' have a centre of perspective, since ℓ is by definition an axis of perspective.

* * * * *

1795. [1992: 304] *Proposed by Hayo Ahlburg, Benidorm, Spain.*

A triangle has the angles $A < B < C$. Angle D is then defined by $\tan A + \tan B + \tan C = \tan D$. Find a triangle for which A, B, C, D are in arithmetic progression.

Solution by C. Festraets-Hamoir, Brussels, Belgium.

Si A, B, C, D sont en progression arithmétique, posons

$$B = A + \alpha, \qquad C = A + 2\alpha, \qquad D = A + 3\alpha.$$

Dès lors

$$A + B + C = 3A + 3\alpha = 180° \iff B = A + \alpha = 60°.$$

Et la relation

$$\tan A + \tan B + \tan C = \tan D$$

s'écrit

$$\tan(60° - \alpha) + \tan 60° + \tan(60° + \alpha) = \tan(60° + 2\alpha)$$

ou encore

$$\tan(60° - \alpha) \cdot \tan 60° \cdot \tan(60° + \alpha) = \tan(60° + 2\alpha),$$

$$\frac{\tan 60^\circ - \tan\alpha}{1+\tan 60^\circ \tan\alpha}\cdot \tan 60^\circ \cdot \frac{\tan 60^\circ + \tan\alpha}{1-\tan 60^\circ \tan\alpha} = \frac{\tan 60^\circ + \tan 2\alpha}{1-\tan 60^\circ \tan 2\alpha}\,,$$

et

$$\begin{aligned}\frac{\sqrt{3}(\sqrt{3}-\tan\alpha)(\sqrt{3}+\tan\alpha)}{1-3\tan^2\alpha} &= \frac{\sqrt{3}+\dfrac{2\tan\alpha}{1-\tan^2\alpha}}{1-\sqrt{3}\cdot\dfrac{2\tan\alpha}{1-\tan^2\alpha}} = \frac{\sqrt{3}-\sqrt{3}\tan^2\alpha+2\tan\alpha}{1-\tan^2\alpha-2\sqrt{3}\tan\alpha}\\ &= \frac{(\sqrt{3}-\tan\alpha)(\sqrt{3}\tan\alpha+1)}{1-2\sqrt{3}\tan\alpha-\tan^2\alpha}\,.\end{aligned}$$

Si $\tan\alpha = \sqrt{3}$, $\alpha = 60^\circ$ et

$$A = 0^\circ, \quad B = 60^\circ, \quad C = 120^\circ, \quad D = 180^\circ;$$

mais on ne peut pas vraiment dire que ABC soit un triangle!

Donc l'équation précédente simplifiée s'écrit alors

$$\sqrt{3}(\sqrt{3}+\tan\alpha)(1-2\sqrt{3}\tan\alpha-\tan^2\alpha) = (1-3\tan^2\alpha)(\sqrt{3}\tan\alpha+1)$$

ou encore

$$0 = 2\sqrt{3}\tan^3\alpha - 6\tan^2\alpha - 6\sqrt{3}\tan\alpha + 2 = 2\sqrt{3}(\tan^3\alpha - 3\tan\alpha) - 2(3\tan^2\alpha - 1),$$

$$\frac{1}{\sqrt{3}} = \frac{\tan^3\alpha - 3\tan\alpha}{3\tan^2\alpha - 1} = \tan 3\alpha,$$

d'où $3\alpha = 30^\circ$, $\alpha = 10^\circ$, et

$$A = \mathbf{50^\circ}, \quad B = \mathbf{60^\circ}, \quad C = \mathbf{70^\circ}, \quad D = \mathbf{80^\circ}.$$

Also solved by ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Albany, California; C. J. BRADLEY, Clifton College, Bristol, U.K.; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD K. GUY, University of Calgary; RICHARD I. HESS, Rancho Palos Verdes, California; JUN-HUA HUANG, The 4th Middle School of Nanxian, Hunan, China; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; PAUL PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; ALBERT W. WALKER, Toronto, Ontario; KENNETH M. WILKE, Topeka, Kansas; PAUL YIU, Florida Atlantic University, Boca Raton; and the proposer. Two other readers approximated the solution. There was also one incorrect answer sent in.

* * * * *

1796. [1992: 304] *Proposed by Ji Chen, Ningbo University, China.*
If A, B, C are the angles of a triangle, prove that

$$\sum \sin B \sin C \leq 3 \sum \sin(B/2)\sin(C/2),$$

where the sums are cyclic.

I. *Solution by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
The proposed inequality is equivalent to

$$4\sum \frac{\cos(B/2)\cos(C/2)}{\sin(A/2)} \leq 3\sum \frac{1}{\sin(A/2)} \,. \tag{1}$$

Suppose (without loss of generality) that $A \geq B \geq C$. Then

$$\cos\frac{B}{2}\cos\frac{C}{2} \geq \cos\frac{C}{2}\cos\frac{A}{2} \geq \cos\frac{A}{2}\cos\frac{B}{2} \quad \text{and} \quad \frac{1}{\sin(A/2)} \leq \frac{1}{\sin(B/2)} \leq \frac{1}{\sin(C/2)} \,.$$

So by Chebyshev's inequality,

$$\begin{aligned} 4\sum \frac{\cos(B/2)\cos(C/2)}{\sin(A/2)} &\leq \frac{4}{3}\left(\sum \cos\frac{B}{2}\cos\frac{C}{2}\right)\left(\sum \frac{1}{\sin(A/2)}\right) \\ &\leq \frac{4}{3}\left(\sum \cos^2\frac{A}{2}\right)\left(\sum \frac{1}{\sin(A/2)}\right) \leq 3\sum \frac{1}{\sin(A/2)} \,, \end{aligned}$$

which is (1). [*Editor's note*: the last two inequalities respectively use the familiar

$$xy + yz + zx \leq x^2 + y^2 + z^2,$$

which appeared in *Crux* just last month [1993: 237], and item 2.29 of Bottema et al, *Geometric Inequalities.*]

II. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
This inequality is indeed *not new*! In D.S. Mitrinović, J.E. Pečarić, V. Volenec, Ji Chen: "Addenda to the Monograph *Recent Advances in Geometric Inequalities*, Part I", *Journal of Ningbo University* Vol. **4**, No. 2 (December 1991), p. 85, item 10.11, it is attributed to Hua-Ming Su, Several chains of trigonometric inequalities (Chinese), *Teaching and Studies (Secondary Math., Jinhua)* 1987, No. 3, p. 13–14.

Also solved by VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; and the proposer, who also refers to the Chinese paper of Hua-Ming Su, but remarks that the proposed inequality is stated there without proof. The proposer's proof reduces the inequality to problem E3146 *of the* Amer. Math. Monthly, *solution in Vol.* 95 (1988) 767–769, *which coincidentally was proposed by Walther Janous!*

*　　　*　　　*　　　*　　　*

1797. [1992: 305] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Solve the equation $2^x - 5 = 11^y$ in positive integers.

Solution by Kee-Wai Lau, Hong Kong.

The equation has the obvious solution $x = 4$, $y = 1$. We show that there are no other positive integer solutions.

Let $s = x - 4$ and $t = y - 1$. Then the equation can be rewritten as

$$16(2^s - 1) = 11(11^t - 1). \tag{1}$$

We need only show that (1) has no solutions in positive integers s and t. Since 11 and 16 are coprime, by (1) we have

$$2^s - 1 = 11k \tag{2}$$

and

$$11^t - 1 = 16k, \tag{3}$$

where k is a positive integer.

Now 2 is a primitive root mod 11. [Recall this means that the smallest positive integer power of 2 which is congruent to 1 modulo 11 is the one given by Fermat's theorem; i.e., $2^{10} = 1 \bmod 11$.—*Ed.*] By (2) we have $10|s$. Also, from (3) we have $k \equiv 0 \bmod 5$. Since 2 is a primitive root mod 5, by (2) we have $4|s$. Therefore we have $20|s$. It follows that

$$2^s - 1 = (\text{an integer}) \cdot (2^{20} - 1) \equiv 0 \bmod 41$$

[note $2^{10} = 1024 \equiv -1 \bmod 41$ and so $2^{20} \equiv 1 \bmod 41$], and by (2) we have $41|k$. Since 11 is a primitive root mod 41 [$11^2 \equiv -2 \bmod 41$ so $11^{20} \equiv (-2)^{10} = 1024 \equiv -1 \bmod 41$ and $11^8 \equiv (-2)^4 \equiv 16 \bmod 41$, and this does it.—*Ed.*], by (3) we have $40|t$. Hence

$$11^t - 1 = (\text{an integer}) \cdot (11^{40} - 1) \equiv 0 \bmod 32.$$

[note $11^2 \equiv -7 \bmod 32$, so $11^4 \equiv 49 \equiv -15 \bmod 32$, so $11^8 \equiv 225 \equiv 1 \bmod 32$ and thus certainly $11^{40} \equiv 1 \bmod 32$]. Thus by (3) k is even. However by (2) k is odd. This shows that (2) and (3) have no solutions in positive integers and this completes the solution of the problem.

Also solved by MANUEL BENITO MUÑOZ and EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; J.L. BRENNER, Palo Alto, California, and LORRAINE L. FOSTER, California State University, Northridge; RICHARD K. GUY, University of Calgary; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer. Three other readers sent in the solution ($x = 4$, $y = 1$) either with no proof or an incorrect proof that it is unique. A further reader sent in a comment.

Brenner and Foster refer to their paper "Exponential Diophantine equations" in the Pacific Journal of Math. 101 (1982), *pp.* 263–301, *for many solutions to similar problems.*

* * * * *

1798. [1992: 305] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Show that there is a three-term arithmetic progression a_1, a_2, a_3 of positive integers so that $a_1a_2a_3 = x^{1992} - y^{1992}$ for distinct positive integers x, y.

Solution by Ignatus, Maputo, Mozambique.

Let

$$a_1 = 2^{1992} - 1, \qquad a_2 = 2^{1992}, \qquad a_3 = 2^{1992} + 1,$$

an arithmetic progression. Then

$$a_1 a_2 a_3 = (2^{1992} - 1)2^{1992}(2^{1992} + 1) = 2^{1992}(4^{1992} - 1) = 8^{1992} - 2^{1992}$$

as required.

Also solved by H.L. ABBOTT, University of Alberta; SAM BAETHGE, Science Academy, Austin, Texas; RICHARD I. HESS, Rancho Palos Verdes, California; JOSEPH LING, University of Calgary; P. PENNING, Delft, The Netherlands; and the proposer. Two other readers sent in comments without solutions.

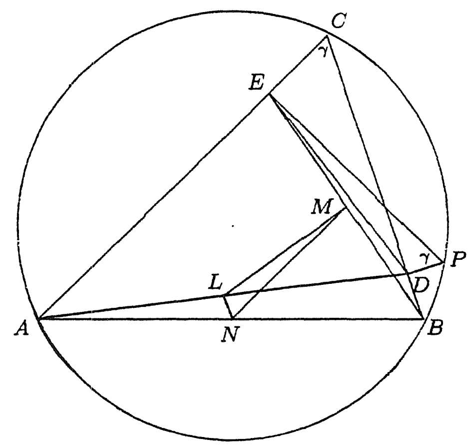
Most solvers (including the proposer) noted that 1992 *could be replaced by any positive integer, as can be seen from the above proof.*

* * * * *

1799. [1992: 305] *Proposed by Shiko Iwata, Gifu, Japan.*

Let P be on the circumcircle of triangle ABC. D and E are the feet of the perpendiculars from P to BC and CA, respectively. L and M are the midpoints of AD and BE, respectively. Show that $DE \perp LM$.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.



Since $\angle CAP = \angle CBP$ and $\angle AEP = \angle BDP$ $(= 90°)$, we have $\Delta AEP \sim \Delta BDP$ and so

$$AE : BD = EP : DP. \qquad (1)$$

Let N be the midpoint of AB. Then

$$NL \| BD \qquad \text{and} \qquad NM \| AE, \qquad (2)$$

so

$$NL = \frac{1}{2}BD \qquad \text{and} \qquad NM = \frac{1}{2}AE. \qquad (3)$$

From (1) and (3),

$$NM : NL = EP : DP. \qquad (4)$$

Since $EP \perp AE$ and $DP \perp BD$, we have also from (2) that

$$EP \perp NM \qquad \text{and} \qquad DP \perp NL. \qquad (5)$$

So by (2),

$$\angle LNM = \angle DPE \ (= \gamma). \qquad (6)$$

By (4) and (6), $\Delta NLM \sim \Delta PDE$. Thus by (5) it follows that $LM \perp DE$.

[*Editor's note*: the proof works with slight changes if P lies on the arc AB not containing C.]

Also solved by SEUNG-JIN BANG, Albany, California; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York, N.Y.; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Seimiya's solution is quite similar to Smeenk's. Most other solutions use analytic geometry, although those of Penning and Rausen (which are similar), and of Heuver, make use of the fact that DE is a Simson (or Wallace) line of the triangle. For instance, Heuver shows that the radical axis of the circles with diameters BE and AD is parallel to DE, from which the result follows immediately.

* * * * *

1800. [1992: 305] *Proposed by Weixuan Li, University of Ottawa, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Call a permutation π on $\{1,2,\ldots,n\}$ an *equidistance permutation* if there is a constant $c \neq 0$ such that $|\pi(i) - i| = c$ for all $i \in \{1,2,\ldots,n\}$. Find the number of equidistance permutations for $n = 1800$.

Solution by R.P. Sealy, Mount Allison University, Sackville, New Brunswick.

The answer is **27**; in general the answer is $\tau(n/2)$, the number of divisors of $n/2$ if n is even (and 0 if n is odd).

Call a constant c corresponding to an equidistance permutation π on $\{1,2,\ldots,n\}$ *admissible.* The proof then follows from two lemmas.

Lemma 1. *For fixed n and each admissible c there exists a unique permutation $\pi = \pi_c$ on $\{1,2,\ldots,n\}$ which is an equidistance permutation.*

Proof: We show that π must be the following product of disjoint 2-cycles. It is clear that $\pi(1) = 1 + c$. Then $\pi(1+c)$ must be 1 or $2c+1$. But $\pi(1+c) = 2c+1$ leads to a contradiction in finding $\pi^{-1}(1)$, so $\pi(1+c)$ must be 1. Similarly, for $c > 1$, $\pi(2) = 2 + c$ and $\pi(2+c) = 2$. Continuing in this fashion we get $\pi(c) = 2c$ and $\pi(2c) = c$. Remove the block $\{1,2,\ldots,2c\}$ and repeat the process beginning from the left with $\pi(2c+1) = 3c+1$. □

Lemma 2. *A positive integer c is admissible if and only if $2c$ divides n.*

Proof: If c is admissible, it follows from the construction of π_c in the proof of Lemma 1 that n is a multiple of $2c$. On the other hand, if n is a multiple of $2c$, it is clear that the construction in the proof of Lemma 1 can be carried out. □

Since $n/2 = 1800/2 = 900 = 2^2 \cdot 3^2 \cdot 5^2$, $\tau(900) = 3 \cdot 3 \cdot 3 = 27$ and thus there are 27 equidistance permutations for $n = 1800$.

Also solved by MARGHERITA BARILE, student, Universität Essen, Germany; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I.

HESS, Rancho Palos Verdes, California; JOSEPH LING, University of Calgary; LAMARR WIDMER, Messiah College, Grantham, Pennsylvania; and the proposers.

The proposers' original problem asked for the number of equidistance permutations for arbitrary n, and essentially all solvers answered this question. The problem was inspired by problem A55 *proposed by Peter Jong in* Mathematical Mayhem *in* 1991 *(solution in Volume* 4, *Issue* 4 *(*1992*), pages* 29–30*). Jong's problem asked to show that an equidistance permutation exists if and only if n is even.*

* * * * *

1801. [1993: 14] *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to O. Bottema.)

If A_1, A_2, A_3 are angles of a triangle, prove that

$$\sum(1+8\cos A_1\sin A_2\sin A_3)^2\sin A_1 \geq 64\sin A_1\sin A_2\sin A_3,$$

where the summation is cyclic over the indices 1, 2, 3.

Solution by Kee-Wai Lau, Hong Kong.

It suffices to show that

$$\sum(1+8\cos A_1\sin A_2\sin A_3)^2\csc A_2\csc A_3 \geq 64,$$

or

$$\sum\csc A_2\csc A_3+16\sum\cos A_1+64\sum\sin A_2\sin A_3-64\left(\prod\sin A_1\right)\sum\sin A_1 \geq 64. \quad (1)$$

Denote by R, r and s respectively the circumradius, inradius and semiperimeter of the triangle. It is known that

$$\sum\csc A_2\csc A_3=\frac{2R}{r}\,,\qquad \sum\sin A_2\sin A_3=\frac{s^2+4Rr+r^2}{4R^2}\,,$$

$$\sum\cos A_1=1+\frac{r}{R}\,,\qquad \sum\sin A_1=\frac{s}{R}\,,\qquad \prod\sin A_1=\frac{sr}{2R^2}$$

(e.g., see pp. 55–56 of [1]). Therefore (1) is equivalent to

$$\frac{2R}{r}+16\left(1+\frac{r}{R}\right)+\frac{16(s^2+4Rr+r^2)}{R^2}-\frac{32s^2r}{R^3}\geq 64,$$

or

$$R^4-24rR^3+40r^2R^2+8r(s^2+r^2)R-16s^2r^2\geq 0,$$

or

$$(R-2r)(R^3-22rR^2-4r^2R+8rs^2)\geq 0.$$

Since $R\geq 2r$ it remains to show that

$$R^3-22rR^2-4r^2R+8rs^2\geq 0. \quad (2)$$

It is known (see solution IV to *Crux* 653 [1982: 191], or p. 2 of [1]) that

$$s^2 \ge 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr}\,.$$

Hence to prove (2) we only need to show that

$$R^3 - 6R^2r + 76Rr^2 - 8r^3 \ge 16r(R-2r)\sqrt{R^2-2Rr}\,. \tag{3}$$

Now for any $t \ge 0$ we have [by the A.M.–G.M. inequality]

$$t^3 + 64(t+2) \ge 2\sqrt{64(t+2)t^3} = 16t\sqrt{t^2+2t}\,.$$

By substituting $t = (R/r) - 2$ we obtain (3) easily and this completes the solution of the problem.

Reference:

[1] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

The proposer proved more generally that if R_1, R_2, R_3 are the distances of a point P in the plane of the triangle from its vertices, then

$$\sum (R_2^2 + R_3^2 - R_1^2 + 8R^2\cos A_1 \sin A_2 \sin A_3)^2 R_1 \sin A_1 \ge 64R^2 R_1R_2R_3 \sin A_1 \sin A_2 \sin A_3.$$

His proof uses vectors. Letting P be the circumcentre produces the proposed inequality.

* * * * *

1802. [1993: 15] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that, for any real numbers x and y,

$$x^4 + y^4 + (x^2+1)(y^2+1) \ge x^3(1+y) + y^3(1+x) + x + y,$$

and determine when equality holds.

Solution by Yi-Ming Ding, student, Ningbo University, China.

The result follows from

$$\begin{aligned}
&2[x^4+y^4+(1+x^2)(1+y^2)] - 2[x^3(1+y)+y^3(1+x)+x+y] \qquad (1)\\
&\quad = 2[(x^4+y^4)-(x^3y+xy^3)] + (1+x^2)(1+y^2-2x) + (1+y^2)(1+x^2-2y)\\
&\quad = 2(x^3-y^3)(x-y) + (1+x^2)(1-x)^2 + (1+y^2)(1-y)^2 - (x^2-y^2)^2\\
&\quad = (x-y)^2[2(x^2+xy+y^2)-(x+y)^2] + (1+x^2)(1-x)^2 + (1+y^2)(1-y)^2\\
&\quad = (x-y)^2(x^2+y^2) + (1+x^2)(1-x)^2 + (1+y^2)(1-y)^2 \ge 0.
\end{aligned}$$

Equality holds if and only if $x = y = 1$.

Also solved by H.L. ABBOTT, University of Alberta; SEUNG-JIN BANG, Albany, California; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; JOSEPH LING, University of Calgary; BEATRIZ MARGOLIS, Paris, France; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Bang also mentions that the polynomial (1) *is irreducible (i.e., cannot be factored nontrivially) over the real numbers.*

* * * * *

1803. [1993: 15] *Proposed by Grant Reinhardt, Vernon, B.C.*

Let a be a real number and n and m be positive integers. Let A be the $m \times m$ matrix with every entry equal to a. Find the common entry for the matrix A^n.

Solution by Woonghee Tim Huh, student, Sir Robert Borden H.S., Nepean, Ontario.

Let

$$A = \left.\overbrace{\begin{bmatrix} a & a & \dots & a \\ a & a & \dots & a \\ \vdots & \vdots & & \vdots \\ a & a & \dots & a \end{bmatrix}}^{m}\right\} m$$

Claim. the common entry for A^n is $m^{n-1}a^n$.

Use induction to prove it. When $n = 1$, the common entry is $a = m^0 a^1$ as claimed. Assume the claim is true for $n = k$, that is, the common entry for A^k is $m^{k-1}a^k$. For $n = k+1$, $A^{k+1} = A^k A$, and the common entry is then

$$(\text{the common entry for } A^k) \times (\text{the common entry for } A) \times (\text{no. of terms added})$$

$$= m^{k-1}a^k \times a \times m = m^k a^{k+1} = m^{(k+1)-1}a^{k+1}.$$

This proves the claim.

Also solved by IAN AFFLECK, student, University of Regina; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Albany, California; MARGHERITA BARILE, student, Universität Essen, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATTEO FERRANTI, student, University of Bologna, Italy; RICHARD I. HESS, Rancho Palos Verdes, *California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; MURRAY S. KLAMKIN, University of Alberta; JOSEPH LING, University of Calgary; HENRY J. RICARDO, Tappan, New York; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer. There was one incorrect answer sent in.*

* * * * *

1804. [1993: 15] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

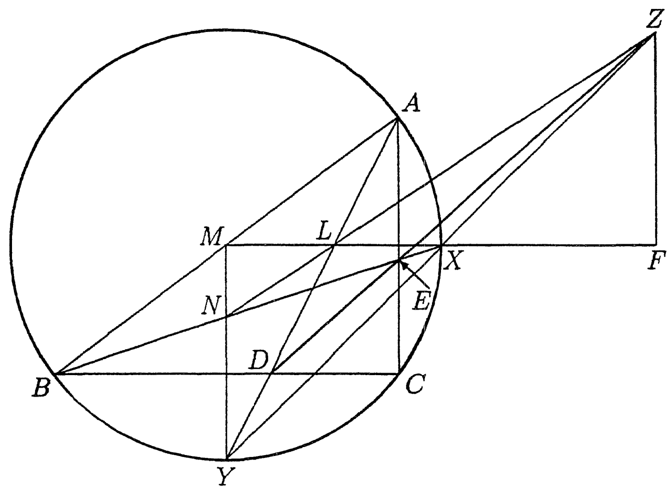
ABC is a right-angled triangle with the right angle at C, and the internal bisectors of $\angle A$ and $\angle B$ meet BC and AC at D and E respectively. Let L, M and N be the midpoints of AD, AB and BE respectively. Let $X = LM \cap BE$, $Y = MN \cap AD$, and $Z = NL \cap DE$. Prove that X, Y and Z are collinear.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

As in any right-angled triangle, the midpoint M of the side opposite the right angle is the circumcenter of ABC. Since

$$BM : BA = 1 : 2 = BN : BE,$$

MN is parallel to AC and passes through the midpoint of BC. Thus MN is the perpendicular bisector of BC; analogously ML is the perpendicular bisector of AC, and thus $MX \perp MY$. Since in any triangle the internal bisector of an angle and the perpendicular bisector of the opposite side meet at a point on the circumcircle, X and Y lie on the circumcircle, hence $MX = MY$.



According to Menelaos' theorem, X, Y and Z are collinear if

$$\frac{LX}{MX} \cdot \frac{MY}{NY} \cdot \frac{NZ}{LZ} = 1.$$

Let F be the foot of the perpendicular from Z to ML. Then $MX = MY$ implies $XF = ZF$, and so

$$ML : (LX + ZF) = ML : LF = MN : ZF,$$

thus $ML \cdot ZF = MN \cdot LX + MN \cdot ZF$, or

$$ZF = \frac{MN \cdot LX}{ML - MN} \, .$$

Hence

$$\frac{NZ}{LZ} = \frac{MN + ZF}{ZF} = \frac{ML - MN}{LX} + 1 = \frac{ML + LX - MN}{LX}$$
$$= \frac{MX - MN}{LX} = \frac{MY - MN}{LX} = \frac{NY}{LX} .$$

Thus

$$\frac{LX}{MX} \cdot \frac{MY}{NY} \cdot \frac{NZ}{LZ} = \frac{LX}{NY} \cdot \frac{NY}{LX} = 1.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SEUNG-JIN BANG, Albany, California; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. J. BRADLEY, Clifton College, Bristol, U.K.; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen; TIM CROSS, Wolverley High School, Kidderminster, U.K.; JORDI DOU, Barcelona, Spain (two solutions); MATTEO FERRANTI, student, University of Bologna, Italy; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; P. PENNING, Delft, The Netherlands; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

* * * * *

1805. [1993: 15] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Note that $1805 = 19^2 \times 5$, that is, when you divide 1805 by its last two digits, the result is the square of one more than its first two digits. Find the next (four-digit) number with this property.

Solution by Ian Affleck, student, University of Regina.

The question asks for a 4-digit number, say $100c_1 + c_2$, where $10 \le c_1 \le 99$ and $1 \le c_2 \le 99$, such that

$$100c_1 + c_2 = (c_1 + 1)^2 c_2 = c_1^2 c_2 + 2c_1c_2 + c_2,$$

that is,

$$100c_1 = c_1c_2(c_1 + 2)$$

or

$$c_2 = \frac{100}{c_1 + 2} .$$

This equation is solved whenever $(c_1 + 2) \mid 100$, $10 \le c_1 \le 99$. So $c_1 \in \{18, 23, 48, 98\}$, and the respective values of c_2 are $5, 4, 2, 1$. So the 4-digit numbers with the required property are

$$\{1805, 2304, 4802, 9801\},$$

i.e., the next one is **2304**.

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Albany, California; MARGHERITA BARILE, student, Universität Essen, Germany; C. J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, Wolverley High School, Kidderminster, U.K.; CHARLES R. DIMINNIE, St. Bonaventure University, St. Bonaventure, New York; KEITH EKBLAW, Walla Walla Community College, Walla Walla, Washington; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; MATTEO FERRANTI, student, University of Bologna, Italy; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; WOONGHEE TIM HUH, student, Sir Robert Borden H.S., Nepean, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; JOSEPH LING, University of Calgary; C.S. METCHETTE, Culver City, California; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, Warszawa, Poland; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHARLTON WANG, student, Waterloo Collegiate Institute, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * * * *

DID YOU KNOW...

— that 6 weeks is 10! seconds?

I heard this from Keith Lloyd, who heard it from Peter McMullen. (Sol Golomb observes that it is easy to see if you know that the fifth Catalan number $\frac{1}{6}\binom{10}{5}$ is 42.) This inspires some other useful information. In the old British monetary system, 5! farthings are half-a-crown and 7! farthings are 5 guineas. Moreover, 11! inches are 15 paces, if you're wearing 7-league boots.

— Richard K. Guy

* * * * *

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

BACK ISSUES AND SUBSCRIPTION INFORMATION
INFORMATION SUR L'ABONNEMENT ET LES NUMÉROS PRÉCÉDENTS

1994 (Volume 20) SUBSCRIPTIONS / ABONNEMENTS 1994 (volume 20)
(10 issues per volume / 10 numéros par volume)

Regular / Régulier	\$46.00
CMS Members / membres de la SMC	\$23.00

These subscription rates include shipping and handling
Les frais de manutentions sont inclus dans ces tarifs
CANADIAN ADDRESSES ADD 7% GST / ADRESSES CANADIENNES, AJOUTER 7% DE TPS)

BACK ISSUES / NUMÉROS ANTÉRIEURS

Volume	\$50.00
Issue / Numéro	\$ 6.00

BOUND VOLUMES / VOLUMES RELIÉS
(available / disponible: 1&2, 3, 7, 8, 9)

Volume	\$10.00 each/chacun

FOR BACK ISSUES AND BOUND VOLUMES ONLY
POUR NUMÉROS ANTÉRIEURS ET VOLUMES RELIÉS SEULEMENT
Canadian addresses, please add 7% GST (#R118 833 979).
Foreign addresses, please add 15% shipping and handling charges.

Les adresses canadiennes, veuillez ajouter 7% de TPS (#R118 833 979).
Les adresses étrangères, veuillez ajouter 15% pour les frais d'envois.

CANADIAN MATHEMATICAL SOCIETY
SOCIÉTÉ MATHÉMATIQUE DU CANADA
577 King Edward
P.O. Box 450, Station A
Ottawa, Ontario K1N 6N5 CANADA
Tel: (613) 564-2223, Fax: (613) 565-1539

ORDER FORM / BON DE COMMANDE

CRUX MATHEMATICORUM

Subscription / Abonnement	______	X	\$46.00	=	________(1)
Back issues / Numéros antérieurs	______	X	\$ 6.00	=	________(2)
	______	X	\$50.00	=	________(3)
Bound volumes / Volumes reliés	______	X	\$10.00	=	________(4)
					________ (1+2+3+4)

Canadian addresses must add GST: 7% of total of lines 1 to 4 (# R118 833 979)
Les adresses canadiennes: veuillez ajouter 7% de TPS sur la somme des lignes 1 à 4 (# R118 833 979) ________

Foreign addresses must add Shipping and Handling: 15% of total of lines 2, 3 and 4
Les adresses étrangères: veuillez ajouter 15% pour les frais de manutention (15% de la somme des lignes 2, 3 et 4) ________

TOTAL \$ ________

☐ Cheque enclosed / chèque inclus (micro-encoded cheques only chèques encodés seulement)

☐ Invoice / Facture

☐ Please charge / Portez à mon compte ☐ VISA ☐ MasterCard

____
Account no. / Numéro de compte

____
Expiry date / date d'expiration

____
Signature

MAILING ADDRESS / ADRESSE POSTALE

Name / Nom:
Address / Adresse:
City, Province / Ville, Province:
Postal Code, Country / Code postal, pays:

CRUX MATHEMATICORUM

Volume 19 #10

December
décembre
1993

CONTENTS / TABLE DES MATIÈRES

Canadian Mathematical Society

Société mathématique du Canada