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GENERAL INFORMATION

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A GENERALIZATION OF LENHARD'S INEQUALITY

Ji Chen and Zhen Wang

In 1937, Barrow [1] sharpened the Erdős–Mordell inequality to:

Twice the sum of the bisectors of the three angles formed by joining any point inside a triangle to the vertices is less than or equal to the sum of the distances of the point from the vertices.

Twenty–four years later, Lenhard [2] generalized Barrow's inequality to convex polygons:

Let $A_1A_2\cdots A_n$ be a given convex n–gon and P an interior point. Put $PA_i = R_i$ and let w_i be the length of the angle bisector from P of ΔA_iPA_{i+1}, $i = 1,\ldots,n$ ($A_{n+1} \equiv A_1$). Then

$$R_1 + R_2 + \cdots + R_n \geq \sec\frac{\pi}{n}(w_1 + w_2 + \cdots + w_n). \tag{1}$$

In this note we generalize (1) to the following

THEOREM. *For* $0 \leq k \leq 1$,

$$\sum_{i=1}^{n} R_i^k \geq \sec^k\frac{\pi}{n}\sum_{i=1}^{n} w_i^k.$$

Proof. Let $\alpha_i = \angle A_iPA_{i+1}$, $i = 1,\ldots,n$ ($A_{n+1} \equiv A_1$). For a point O in the plane, we consider a polygon $B_1B_2\cdots B_n$ such that

$$OB_i = R_i^{k/2} \quad (\equiv R_i')$$

and

$$\angle B_iOB_{i+1} = \frac{\alpha_i}{2} + \frac{\pi}{n} \quad (\equiv \beta_i)$$

for $i = 1,\ldots,n$ (and $B_{n+1} \equiv B_1$). Then

$$\begin{aligned} 2\sum_{i=1}^{n} R_i'^2 - 2\sum_{i=1}^{n} R_i'R_{i+1}'\cos\beta_i &= \sum_{i=1}^{n}(R_i'^2 + R_{i+1}'^2 - 2R_i'R_{i+1}'\cos\beta_i) \\ &= \sum_{i=1}^{n}(B_iB_{i+1})^2. \end{aligned} \tag{2}$$

By the power mean inequality and the fact that the regular n–gon has maximum area among all n–gons of a given perimeter,

$$\sum_{i=1}^{n}(B_iB_{i+1})^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} B_iB_{i+1}\right)^2$$

$$\geq 4 \tan \frac{\pi}{n} \cdot \text{Area}(B_1 B_2 \cdots B_n)$$

$$= 2 \tan \frac{\pi}{n} \sum_{i=1}^{n} R_i' R_{i+1}' \sin \beta_i \ . \tag{3}$$

Thus by (2) and (3)

$$\sum_{i=1}^{n} R_i^k = \sum_{i=1}^{n} R_i'^2$$

$$\geq \sum_{i=1}^{n} R_i' R_{i+1}' (\cos \beta_i + \tan \frac{\pi}{n} \sin \beta_i)$$

$$= \sec \frac{\pi}{n} \sum_{i=1}^{n} R_i' R_{i+1}' \cos(\beta_i - \frac{\pi}{n})$$

$$= \sec \frac{\pi}{n} \sum_{i=1}^{n} R_i^{k/2} R_{i+1}^{k/2} \cos \frac{\alpha_i}{2} \ . \tag{4}$$

Since

$$\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i x_{i+1}$$

for any positive reals x_i,

$$\sum_{i=1}^{n} R_i^k \geq \sum_{i=1}^{n} R_i^{k/2} R_{i+1}^{k/2} \ . \tag{5}$$

Also, from

$$(R_i + R_{i+1})^2 \geq 4 R_i R_{i+1}$$

we have

$$R_i^{k/2} R_{i+1}^{k/2} \geq \left(\frac{2 R_i R_{i+1}}{R_i + R_{i+1}} \right)^k . \tag{6}$$

Finally, by the weighted A.M.–G.M. inequality,

$$k\left(\sec \frac{\pi}{n} \cos \frac{\alpha_i}{2}\right) + (1 - k) \geq \left(\sec \frac{\pi}{n} \cos \frac{\alpha_i}{2}\right)^k \cdot 1^{1-k} = \left(\sec \frac{\pi}{n} \cos \frac{\alpha_i}{2}\right)^k . \tag{7}$$

So we have, from (4)–(7),

$$\sum_{i=1}^{n} R_i^k = k \sum_{i=1}^{n} R_i^k + (1 - k) \sum_{i=1}^{n} R_i^k$$

$$\geq k \sec \frac{\pi}{n} \sum_{i=1}^{n} R_i^{k/2} R_{i+1}^{k/2} \cos \frac{\alpha_i}{2} + (1-k) \sum_{i=1}^{n} R_i^{k/2} R_{i+1}^{k/2}$$

$$= \sum_{i=1}^{n} R_i^{k/2} R_{i+1}^{k/2} \left(k \sec \frac{\pi}{n} \cos \frac{\alpha_i}{2} + 1 - k \right)$$

$$\geq \sum_{i=1}^{n} \left(\frac{2R_i R_{i+1}}{R_i + R_{i+1}} \right)^k \left(\sec \frac{\pi}{n} \cos \frac{\alpha_i}{2} \right)^k$$

$$= \sec^k \frac{\pi}{n} \sum_{i=1}^{n} \left(\frac{2R_i R_{i+1}}{R_i + R_{i+1}} \cos \frac{\alpha_i}{2} \right)^k$$

$$= \sec^k \frac{\pi}{n} \sum_{i=1}^{n} w_i^k ,$$

which was to be proved. □

References:

[1] D.F. Barrow, Advanced Problem 3740, *Amer. Math. Monthly* 44 (1937) 252–254.

[2] H.–Chr. Lenhard, Verallgemeinerung und Verschärfung der Erdős–Mordellschen Ungleichung für Polygone, *Arch. Math.* 12 (1961) 311–314.

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THE OLYMPIAD CORNER
No. 109
R.E. WOODROW

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We begin this month by giving the remaining 14 problems that were proposed to the jury but not used for the 30th I.M.O. at Braunschweig, West Germany. (One problem has been left out because it was the same as *Crux* 1119 [1987: 258].) My thanks to Professor Bruce Shawyer of Memorial University who sent the problems to me.

12. *Proposed by Australia.*

Ali Baba the carpet merchant has a rectangular piece of carpet whose dimensions are unknown. Unfortunately, his tape measure is broken and he has no other measuring instruments. However, he finds that if he lays it flat on the floor of either of his storerooms, then each corner of the carpet touches a different wall of that room. He knows that the sides of the rooms are integral numbers of feet, and that his two storerooms have the same (unknown) length, but widths of 38 and 55 feet respectively. What are the carpet's dimensions? [*Editor's note*: This is very similar to the other Australian problem, given last month.]

13. *Proposed by France.*

For each nonnegative integer n we write

$$(1 + 4\sqrt[3]{2} - 4\sqrt[3]{4})^n$$

in the unique form

$$(1 + 4\sqrt[3]{2} - 4\sqrt[3]{4})^n = a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4} ,$$

with a_n, b_n, and c_n integers. Show that $c_n = 0$ implies $n = 0$.

14. *Proposed by Hungary.*

At n distinct points of a circle–shaped race course there are n cars ready to start. They cover the circle in an hour. Hearing the signal each of them selects a direction and starts immediately. If two cars meet both of them change directions and go on without loss of speed. Show that at a certain moment each car will be at its starting point.

15. *Proposed by Mongolia.*

Let $A_1A_2...A_n$ be a convex polygon of area S. Let M be a point in the plane. Determine the area of the polygon $M_1M_2...M_n$ where M_i is the image of M under the rotation $R^{\alpha}_{A_i}$ $(i = 1,2,...,n)$.

16. *Proposed by Mongolia.*

In each square of a rectangular $m \times n$ chess board is written a natural number. One may add the same integer k to all adjacent numbers of a square, provided that the resulting numbers are all non–negative. (Two squares are called adjacent if they share a common edge.) Find the necessary and sufficient condition that after finitely many such operations all numbers on the board are zero.

17. *Proposed by the Netherlands.*

Prove that the intersection of a plane and a regular tetrahedron can be an obtuse–angled triangle and that the obtuse angle in any such triangle is always smaller than 120°.

18. *Proposed by Poland.*

For points A_1, A_2, A_3, A_4, A_5 on the sphere of radius 1, what is the maximum value that

$$\min_{1 \leq i < j \leq 5} A_i A_j$$

can take? Determine all configurations for which this maximum is attained. Determine the diameter of any set $\{A_1,\ldots,A_5\}$ for which the maximum is attained.

19. *Proposed by the Republic of Korea.*

Let a and b be integers which are not perfect squares. Prove that if

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$

has a non–trivial solution in integers, then so does

$$x^2 - ay^2 - bz^2 = 0 .$$

20. *Proposed by the Republic of Korea.*

Let n be a positive integer and let a and b be given real numbers. Let $x_0,x_1,\ldots,x_n$ be real variables which satisfy

$$\sum_{i=0}^{n} x_i = a , \qquad \sum_{i=0}^{n} x_i^2 = b .$$

Determine the range of x_0.

21. *Proposed by Romania.*

Let m be a positive odd integer, $m > 2$. Find the smallest positive integer n such that 2^{1989} divides $m^n - 1$.

22. *Proposed by Romania.*

Consider points O, A_1, A_2, A_3, A_4 in the plane such that the area $\sigma(OA_iA_j) \geq 1$ for all $1 \leq i < j \leq 4$. Prove that there are i_0, j_0 such that $\sigma(OA_{i_0}A_{j_0}) \geq \sqrt{2}$.

23. *Proposed by Romania.*

One hundred fifty–five birds are sitting on a circle C. Two birds P_i, P_j are mutually visible if the angle of the smaller arc P_iP_j that they subtend is at most 10°. Find the smallest number of mutually visible pairs of birds. (One allows several birds to simultaneously occupy a point of C.)

24. *Proposed by Sweden.*

Let $a_1 \geq a_2 \geq a_3$ be given positive numbers and let $N(a_1,a_2,a_3)$ be the number of solutions (x_1,x_2,x_3) in positive integers of the equation

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} = 1 .$$

Prove that $N(a_1,a_2,a_3) \leq 6a_1a_2(3 + \ln(2a_1))$.

25. *Proposed by the United States of America.*

Vertex A of the acute triangle ABC is equidistant from the circumcenter O and the orthocenter H. Determine all possible values for the measure of angle A.

* * *

Next we give some solutions from the files to problems posed in the February 1988 number.

1. [1988: 34] *18th Austrian Mathematics Olympiad, 2nd Round.*

Show that $\sin x + \cos x \leq 0$ implies that $\sin^{1987}x + \cos^{1987}x \leq 0$, as well.

Solution by M.A. Selby, Department of Mathematics, The University of Windsor.

Since the equation $a^{1987} = -b^{1987}$ has only one real root $a = -b$, we have

$$a^{1987} + b^{1987} = (a + b)Q(a,b) ,$$

where

$$Q(a,b) = \sum_{j=0}^{1986} a^{1986-j}b^j(-1)^j$$

has no real roots and therefore $Q(a,b) > 0$ for all a and b with $a^2 + b^2 > 0$. Since $\sin^2x + \cos^2x = 1$,

$$\sin^{1987}x + \cos^{1987}x = (\sin x + \cos x)Q(\sin x,\cos x)$$

where $Q(\sin x,\cos x) > 0$ for all x.

We now have $\sin x + \cos x \leq 0$ if and only if

$$(\sin x + \cos x)Q(\sin x,\cos x) \leq 0,$$

or $\sin^{1987}x + \cos^{1987}x \leq 0$. Notice that this proof works with 1987 replaced by any odd positive integer.

Alternate solution by Edward T.H. Wang, Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario.

We show that $\sin x + \cos x \leq 0$ implies $\sin^n x + \cos^n x \leq 0$ for all odd positive integers n.

Since $\sin(x + \pi/4) = (\sin x + \cos x)/\sqrt{2} \leq 0$ we have

$$(2k + 1)\pi \leq x + \pi/4 \leq (2k + 2)\pi,$$

or

$$(2k + 3/4)\pi \leq x \leq (2k + 7/4)\pi$$

where k denotes an integer. This leads to three possibilities:

(i) $(2k + 3/4)\pi \leq x \leq (2k + 1)\pi$.

Then $\sin x \geq 0$, $\cos x \leq 0$, and $\sin x \leq -\cos x$ implies that $\sin^n x \leq -\cos^n x$, i.e. $\sin^n x + \cos^n x \leq 0$.

(ii) $(2k + 1)\pi \leq x \leq (2k + 3/2)\pi$.

Then $\sin x \leq 0$ and $\cos x \leq 0$, and obviously $\sin^n x + \cos^n x \leq 0$.

(iii) $(2k + 3/2)\pi \leq x \leq (2k + 7/4)\pi$.

Then $\sin x \leq 0$, $\cos x \geq 0$, and $\cos x \leq -\sin x$ implies that $\cos^n n \leq -\sin^n x$, and the inequality follows.

[*Editor's note.* Seung–Jin Bang, Seoul, Korea and Bob Prielipp, University of Wisconsin, Oshkosh point out that $f(x) = x^n$ is strictly increasing for odd positive n, since $f'(x) = nx^{n-1}$. Using this, the argument given in the alternate solution is easily shortened to one line.]

2. [1988: 34] *18th Austrian Mathematics Olympiad, 2nd Round.*

The solutions x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$ (a real, $a \neq 0$) also satisfy

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8 .$$

Determine x_1, x_2, x_3.

Solutions by George Evagelopoulos, Law student, Athens, Greece; Seung–Jin Bang, Seoul, Korea; the late J.T. Groenman, of Arnhem, The Netherlands; Bob Prielipp, The University of Wisconsin, Oshkosh; M.A. Selby, Department of Mathematics, The University of Windsor; and by Edward T.H. Wang, Department of Mathematics, Wilfrid Laurier University, Waterloo.

Since x_1, x_2 and x_3 are the solutions of the equation $x^3 + ax + a = 0$, we have

$$x_i^3 = -a - ax_i , \qquad i = 1,2,3 , \tag{1}$$

$$x_1x_2x_3 = -a , \tag{2}$$

$$x_1x_2 + x_1x_3 + x_2x_3 = a\ , \tag{3}$$
$$x_1 + x_2 + x_3 = 0\ . \tag{4}$$

From (1), $x_i \neq 0$ for $i = 1,2,3$ and so the given equation is equivalent to

$$x_1^3x_3 + x_2^3x_1 + x_3^3x_2 = -8x_1x_2x_3\ .$$

From (1) and (2)

$$(-a - ax_1)x_3 + (-a - ax_2)x_1 + (-a - ax_3)x_2 = -8(-a)\ .$$

Thus

$$(1 + x_1)x_3 + (1 + x_2)x_1 + (1 + x_3)x_2 = -8\ .$$

It follows from (3) and (4) that

$$-8 = (x_1 + x_2 + x_3) + (x_1x_2 + x_1x_3 + x_2x_3) = 0 + a\ ,$$

making $a = -8$. Now the roots of

$$(x + 2)(x^2 - 2x - 4) = x^3 - 8x - 8 = 0$$

are -2, $1 + \sqrt{5}$, and $1 - \sqrt{5}$.

3. [1988: 34] *18th Austrian Mathematics Olympiad, 2nd Round.*

Determine all sequences $x_0,x_1,\ldots$ of real numbers satisfying $0 < x_0 \le 1$ and

$$0 < x_{n+1} \le 2 - \frac{1}{x_n}\ , \qquad n \ge 0\ .$$

Solutions by Seung–Jin Bang, Seoul, Korea and by M.A. Selby, Department of Mathematics, The University of Windsor.

We show that the only sequence with these properties is $x_k = 1$ for all $k \ge 0$.

Since $0 < 2 - 1/x_n$ for $n \ge 0$ we must have $x_n > 1/2$ for all $n \ge 0$. Suppose $0 < x_j < 1$ for some j. Then

$$x_{n+1} - x_n \le 2 - 1/x_n - x_n \le 0$$

for all n, with the last inequality strict whenever $x_n < 1$. Hence for $n \ge j$, $x_{n+1} < x_n$. That is, the sequence is monotone decreasing for $n \ge j$ and is bounded below (by 1/2). Let

$$\lim_{n\to\infty} x_n = L \ge 1/2.$$

Since $x_{n+1} \le 2 - 1/x_n$, $L \le 2 - 1/L$, or $(L - 1)^2 \le 0$. We conclude that $L = 1$, but this contradicts $L < x_n < x_j < 1$, for $n > j$. Therefore $x_j = 1$ for all j.

*

2. [1988: 34] *18th Austrian Mathematics Olympiad, Final Round.*

Determine the number of all sequences $(x_1,\ldots,x_n)$, with $x_i \in \{a,b,c\}$ for $1 \le i \le n$, that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $1 \le i \le n - 1$.

Solutions by Len Bos, Department of Mathematics and Statistics, The University of Calgary, and by Edward T.H. Wang, Department of Mathematics, Wilfrid Laurier University, Waterloo.

Let f_n be the number of sequences $(x_1,\ldots,x_n)$ satisfying the given conditions. Clearly $f_2 = 0$, while $f_3 = 2 = f_4$. Consider now such a sequence $(a,x_2,\ldots,x_{n-1},a)$. Suppose that the second a from the left occurs at x_i. Then $i \neq 2$ and $i \neq n-1$. If $i = n$ there are two possibilities, $(a,b,c,b,c,\ldots,a)$ and $(a,c,b,c,b,\ldots,a)$. Suppose next that $2 < i < n-1$. As $x_2,\cdots,x_{i-1}$ must alternate b's and c's, the number of such sequences is $2f_{n-i+1}$. From this we obtain

$$f_n = 2(f_{n-2} + f_{n-3} + \cdots + f_3) + 2$$

and $f_5 = 6$. For $n \geq 6$ we have

$$f_{n-1} = 2(f_{n-3} + \cdots + f_3) + 2$$

and on subtraction we obtain the two term recurrence

$$f_n = f_{n-1} + 2f_{n-2} \,. \tag{*}$$

The characteristic equation of this recurrence is $\lambda^2 - \lambda - 2 = 0$ with roots 2 and –1. Thus the general solution of (*) is

$$f_n = c_1 2^n + c_2(-1)^n.$$

Employing the initial conditions $f_3 = 2 = f_4$ we find

$$f_n = \frac{2}{3}[2^{n-2} + (-1)^{n+1}].$$

Alternate solution by Curtis Cooper, Central Missouri State University.

Let a_n be the number of all sequences $(x_1,\ldots,x_n)$ with $x_i \in \{a,b,c\}$ for $1 \leq i \leq n$ that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $1 \leq i \leq n-1$. Let b_n and c_n be the number with the condition $x_1 = x_n = a$ replaced by $x_1 = a$, $x_n = b$ and $x_1 = a$, $x_n = c$ respectively. These triples satisfy the following recurrence relation:

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} .$$

Solving this relation, we obtain

$$a_n = \frac{1}{6}2^n - \frac{2}{3}(-1)^n \,.$$

[*Editor's Note*: While shorter, the "solving" leads us into eigenvalues and may be beyond the usual methods for this sort of problem.]

4. [1988: 34] *18th Austrian Mathematics Olympiad, Final Round.*
Determine all triples (x,y,z) of natural numbers satisfying

$$2xz = y^2 \qquad \text{and} \qquad x + z = 1987.$$

Solution by Stewart Metchette, Culver City, California.

Note that 1987 is prime and y is even. Also $x \neq z$, for $2x = 1987$ is impossible. In fact x and z can have no common factor. From $2xz = y^2$ it follows that one of x and z is a square and the other is twice a square. Without loss set $x = 2a^2$, $z = b^2$ where $\gcd(a,b) = 1$, $a \neq b$. Then $x + z = 1987$ becomes

$$2a^2 + b^2 = 1987 ,$$

and we can replace 1987 by any prime p to have

$$2a^2 + b^2 = p . \tag{1}$$

It follows from classical number theoretic arguments that (1) has a solution (and it is unique) just in case p is congruent to 1 or 3 mod 8. Now 1987 is congruent to 3 mod 8 and so there is a unique solution. As 1987 is odd b^2 is odd and $2a^2 + 1 \equiv 3 \bmod 8$. This gives $a^2 \equiv 1 \bmod 8$, and a^2 is odd.

From $b^2 = 1987 - 2a^2$ we have $2a^2 < 1987$, so $a \leq 31.5$. For $a = 31$ and 29, $1987 - 2a^2$ is not a square. However for $a = 27 = 3^3$, $b^2 = 529$ and $b = 23$. Thus the unique solution is

$$x = 2a^2 = 2\cdot 3^6 = 1458, \quad z = b^2 = 23^2 = 529, \quad \text{and} \quad y = \sqrt{2xz} = 2\cdot 3^3\cdot 23 = 1242.$$

References:

[1] L.E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, New York, 1957, p. 76–77.

[2] T. Nagell, *Introduction to Number Theory*, Wiley, New York, 1951, p. 188.

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<u>3</u>. [1988: 35] *10th Austrian–Polish Mathematics Competition.*

Let $\mathbb{R}$ be the set of real numbers and $f\colon \mathbb{R} \to \mathbb{R}$ a function with

$$f(x + 1) = f(x) + 1 , \qquad x \in \mathbb{R} .$$

Let $a \in \mathbb{R}$ be given, and define a sequence $x_0, x_1, \ldots$ by $x_0 = a$ and

$$x_{n+1} = f(x_n) , \qquad n \geq 0.$$

Assume that there is a positive natural number m such that the difference $x_m - x_0$ is a whole number k. Prove that the limit $\lim\limits_{n\to\infty} \frac{x_n}{n}$ exists, and evaluate it.

Solutions by Seung–Jin Bang, Seoul, Korea, and by L.P. Bos, Department of Mathematics and Statistics, The University of Calgary.

It is easy to see that $f(x + k) = f(x) + k$ for $x \in \mathbb{R}$.

Claim. $x_{n+m} = x_n + k$ for $n = 0,1,2,\ldots$.

Proof by induction. The case $n = 0$ is by the assumption on m and k, and

$$x_{(n+1)+m} = f(x_{n+m}) = f(x_n + k) = f(x_n) + k = x_{n+1} + k. \qquad \square$$

We next establish the following claim.

Claim. $x_n = x_t + \left(\frac{n-t}{m}\right)k$ if $n \equiv t \bmod m$, $0 \le t \le m-1$.

Proof. If $n \equiv t \bmod m$, then $n = \frac{n-t}{m}\cdot m + t$. Thus

$$\begin{aligned} x_n = x_{t+m((n-t)/m)} &= x_{t+m((n-t)/m - 1)} + k \\ &= x_{t+m((n-t)/m - 2)} + 2k \\ &= \cdots \\ &= x_t + \left(\frac{n-t}{m}\right)k . \qquad \square \end{aligned}$$

Hence

$$\frac{x_n}{n} = \frac{x_t}{n} + \left(\frac{n-t}{n}\right)\frac{k}{m}$$

for $n \equiv t \bmod m$, $0 \le t \le m-1$. It follows that $\lim\limits_{n\to\infty} \frac{x_n}{n} = \frac{k}{m}$.

4. [1988: 35] *10th Austrian–Polish Mathematics Competition.*

Does the set $\{1,2,\ldots,3000\}$ contain a subset A of 2000 elements such that $x \in A$ implies $2x \notin A$?

Solutions by Curtis Cooper, Central Missouri State University, and by John Morvay, Dallas, Texas.

The answer is no. To see this, partition $\{1,2,\ldots,3000\}$ into equivalence classes as follows:

$\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048\}$,
$\{3, 6, 12, 24, 48, 96, 192, 384, 768, 1536\}$,
$\{5, 10, 20, 40, 80, 160, 320, 640, 1280, 2560\}$,
$\{7, 14, 28, 56, 112, 224, 448, 896, 1792\}$,
$\vdots$
$\{749, 1498, 2996\}$,
$\{751, 1502\}, \{753, 1506\}, \cdots, \{1499, 2998\}$,
$\{1501\}, \{1503\}, \cdots, \{2999\}$.

Let A be a subset of $\{1,2,\ldots,3000\}$ such that $x \in A$ implies $2x \notin A$. Then A can contain at most 6 elements from the equivalence class containing 1. Similarly A can contain at most 5 elements from the class containing 3. The same can be said about the classes containing 5, 7, 9 and 11. Continuing this process we finally see that A can contain at most 1 element from the equivalence class containing 2999. Thus A contains at most

$$1\cdot 6 + 5\cdot 5 + 17\cdot 4 + 71\cdot 3 + 281\cdot 2 + 1125\cdot 1 = 1999$$

elements. Thus no subset A of $\{1,2,...,3000\}$ such that $x \in A$ implies $2x \notin A$ can contain 2000 elements.

* * *

This completes the solutions received to problems from the February 1988 Corner. Please send me your solutions and contests.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1990,** *although solutions received after that date will also be considered until the time when a solution is published.*

<u>1481</u>. *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let A, B, C be points on a fixed circle with B, C fixed and A variable. Points D and E are on segments BA and CA, respectively, so that $\overline{BD} = m$ and $\overline{CE} = n$ where m and n are constants. Points P and Q are on BC and DE, respectively, so that

$$\overline{BP} : \overline{PC} = \overline{DQ} : \overline{QE} = k,$$

also a constant. Prove that the length of PQ is a constant. (This is not a new problem. A reference will be given when the solution is published.)

<u>1482</u>. *Proposed by M.S. Klamkin, University of Alberta.*

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are vectors such that

$$|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = |\mathbf{A} + \mathbf{B} + \mathbf{C}|,$$

prove that

$$|\mathbf{B} \times \mathbf{C}| = |\mathbf{A} \times (\mathbf{B} + \mathbf{C})|.$$

1483. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$, and so that $A'B'C'$ and ABC are directly similar.

(a) Show that, if the centroids G, G' of the triangles coincide, then either the triangles are equilateral or A', B', C' are the midpoints of the sides of ΔABC.

(b) Show that if either the circumcenters O, O' or the incenters I, I' of the triangles coincide, then the triangles are equilateral.

1484. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < r,s,t \le 1$ be fixed. Show that the relation

$$r \cot rA = s \cot sB = t \cot tC$$

holds for exactly one triangle ABC, and that this triangle maximizes the expression

$$\sin rA \sin sB \sin tC$$

over all triangles ABC.

1485. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

From a deck of 52 cards, 13 are chosen. Replace one of them by one of the remaining 39 cards. Continue the process until the initial set of 13 cards reappears. Is it possible that all the $\binom{52}{13}$ combinations appear on the way, each exactly once?

1486. *Proposed by Jordi Dou, Barcelona, Spain.*

Given three triangles T_1, T_2, T_3 and three points P_1, P_2, P_3, construct points X_1, X_2, X_3 such that the triangles $X_2X_3P_1$, $X_3X_1P_2$, $X_1X_2P_3$ are directly similar to T_1, T_2, T_3, respectively.

1487. *Proposed by Kee–Wai Lau, Hong Kong.*

Prove the inequality

$$x + \sin x \ge 2 \log(1 + x)$$

for $x > -1$.

1488. *Proposed by Avinoam Freedman, Teaneck, New Jersey.*

Prove that in any acute triangle, the sum of the circumradius and the inradius is less than the length of the second–longest side.

1489. *Proposed by M. Selby, University of Windsor.*

Let

$$A_n = (7 + 4\sqrt{3})^n ,$$

where n is a positive integer. Find a simple expression for $1 + [A_n] - A_n$, where $[x]$ is the greatest integer less than or equal to x.

1490*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

This was suggested by Walther Janous' problem *Crux* 1366 [1989: 271]. Find the smallest constant k such that

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}} \le k\sqrt{x+y+z}$$

for all positive x, y, z.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1286. [1987: 290; 1988: 310] *Proposed by Walther Janous, Ursulinen–gymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\prod\left[\frac{x(x+y+z)}{(x+y)(x+z)}\right]^x \le \left[\frac{\left(\sum yz\right)^2}{4xyz(x+y+z)}\right]^{x+y+z},$$

where $\prod$ and $\sum$ are to be understood cyclically.

II. *Solution by the proposer.*

It was shown in *Crux* 908 [1985: 93] that

$$\prod \sin^x A \le \prod\left[\frac{x(x+y+z)}{(x+y)(x+z)}\right]^{x/z}, \tag{1}$$

where A, B, C are the angles of a triangle and x, y, z are positive real numbers, and that for any x, y, $z > 0$ equality in (1) is attained for certain angles A, B, C depending on x, y, z. On the other hand, by the general geometric–arithmetic inequality,

$$\prod \sin^x A \le \left(\frac{\sum x\sin A}{x+y+z}\right)^{x+y+z}. \tag{2}$$

Using Murray Klamkin's result

$$\sum x\sin A \le \frac{1}{2}\sqrt{\frac{x+y+z}{xyz}}\sum yz$$

(M.S. Klamkin, On a triangle inequality, *Crux* [1984: 139–140]) we infer from (2) that

$$\prod \sin^x A \le \left(\frac{\sum yz}{2\sqrt{xyz(x+y+z)}}\right)^{x+y+z}.$$

Therefore, letting A, B, C be the angles giving equality in (1), we obtain the given inequality.

* * *

1366*. [1988: 202] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \ge \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}}$$

for all positive real numbers x, y, z.

I. *Solution by G.P. Henderson, Campbellcroft, Ontario.*

We will prove that the inequality is true. Set $\sqrt{x} = a$, $\sqrt{y} = b$, $\sqrt{z} = c$. Then we are to show that if $a, b, c \ge 0$,

$$\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{b^2+c^2}} + \frac{c^2}{\sqrt{c^2+a^2}} \ge \frac{a+b+c}{\sqrt{2}}. \tag{1}$$

By cyclic permutation of a, b, c we can ensure that $a \ge b, c$. By interchanging b and c if necessary, we can assume that $a \ge b \ge c$. But then in addition to (1) we must prove that

$$\frac{a^2}{\sqrt{a^2+c^2}} + \frac{c^2}{\sqrt{c^2+b^2}} + \frac{b^2}{\sqrt{b^2+a^2}} \ge \frac{a+b+c}{\sqrt{2}}. \tag{2}$$

We have

$$2(a^2+b^2)\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{1}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{a^2+b^2}} - \frac{1}{\sqrt{2}}\right) = 2(a^2+b^2)\left(\frac{a^2}{a^2+b^2} - \frac{1}{2}\right)$$
$$= a^2 - b^2$$

and hence

$$\frac{a^2}{\sqrt{a^2+b^2}} - \frac{a}{\sqrt{2}} = \frac{a(a-b)(a+b)}{2(a^2+b^2)\left(\dfrac{a}{\sqrt{a^2+b^2}} + \dfrac{1}{\sqrt{2}}\right)}$$
$$= \frac{a(a-b)(a+b)}{2a\sqrt{a^2+b^2} + \sqrt{2}(a^2+b^2)}$$

$$= (a - b)f\left(\frac{b}{a}\right),$$

where

$$f(t) = \frac{t+1}{2\sqrt{t^2+1} + \sqrt{2}(t^2+1)}.$$

Thus (1) can be written

$$(a-b)f\left(\frac{b}{a}\right) + (b-c)f\left(\frac{c}{b}\right) \geq (a-c)f\left(\frac{a}{c}\right), \tag{3}$$

and (2) becomes

$$(a-c)f\left(\frac{c}{a}\right) \geq (b-c)f\left(\frac{b}{c}\right) + (a-b)f\left(\frac{a}{b}\right). \tag{4}$$

Now

$$f'(t) = \frac{2\sqrt{t^2+1} + \sqrt{2}(t^2+1) - (t+1)\left(\dfrac{2t}{\sqrt{t^2+1}} + 2\sqrt{2}t\right)}{\left(2\sqrt{t^2+1} + \sqrt{2}(t^2+1)\right)^2}$$

so that $f'(t)$ has the same sign as

$$\frac{2(1-t)}{\sqrt{t^2+1}} - \sqrt{2}(t^2+2t-1).$$

This is a decreasing function for $0 \leq t \leq 1$. It is positive at $t = 0$ and negative for $t \geq 1$. Therefore, starting with

$$f(0) = \frac{1}{2+\sqrt{2}} = 1 - \frac{\sqrt{2}}{2},$$

$f(t)$ increases to a maximum then steadily decreases. The maximum occurs in $0 < t < 1$.

From the definition of f,

$$\frac{1}{f(1/t)} - \frac{1}{f(t)} = \frac{2\sqrt{\dfrac{1}{t^2}+1} + \sqrt{2}\left(\dfrac{1}{t^2}+1\right)}{\dfrac{1}{t}+1} - \frac{2\sqrt{t^2+1} + \sqrt{2}(t^2+1)}{t+1}$$

$$= \frac{2t\sqrt{t^2+1} + \sqrt{2}(t^2+1)}{t(t+1)} - \frac{2\sqrt{t^2+1} + \sqrt{2}(t^2+1)}{t+1}$$

$$= \frac{\sqrt{2}(t^2+1)(1-t)}{t(t+1)}.$$

Therefore, if $0 < t \leq 1$,

$$f\left(\frac{1}{t}\right) \leq f(t).$$

Since $a \geq b \geq c$, the left side of (3) is greater than or equal to

$$(a-b)f\left(\frac{a}{b}\right) + (b-c)f\left(\frac{b}{c}\right) \geq (a-b)f\left(\frac{a}{c}\right) + (b-c)f\left(\frac{a}{c}\right)$$
$$= (a-c)f\left(\frac{a}{c}\right).$$

Therefore (3) and (1) are true.

To prove (4) we consider two cases. First, assume

$$f\left(\frac{c}{a}\right) \geq f(1).$$

Then the left side of (4) is greater than or equal to

$$(a-c)f(1) = (b-c)f(1) + (a-b)f(1)$$
$$\geq (b-c)f\left(\frac{b}{c}\right) + (a-b)f\left(\frac{a}{b}\right).$$

Finally we have the case

$$f\left(\frac{c}{a}\right) < f(1).$$

In (2), set $u = b/a$ and $v = c/a$. We have $0 \leq v \leq u \leq 1$, $f(v) < f(1)$ and we are to prove that

$$\frac{1}{\sqrt{v^2+1}} + \frac{v^2}{\sqrt{u^2+v^2}} + \frac{u^2}{\sqrt{u^2+1}} \geq \frac{1+u+v}{\sqrt{2}}. \tag{5}$$

Solving the equation

$$f(t) = f(1) = \frac{\sqrt{2}}{4},$$

where $0 < t < 1$, we find

$$\frac{t+1}{2\sqrt{t^2+1} + \sqrt{2}(t^2+1)} = \frac{\sqrt{2}}{4},$$
$$2t + 1 - t^2 = \sqrt{2}\sqrt{t^2+1},$$
$$t^4 - 4t^3 + 4t - 1 = 0,$$
$$(t-1)(t+1)(t^2-4t+1) = 0,$$

and thus $t = 2 - \sqrt{3}$, so the condition $f(v) < f(1)$ is equivalent to $v < 2 - \sqrt{3}$.

The left side of (5) exceeds the right side by at least

$$\frac{1}{\sqrt{v^2+1}} + \frac{v^2}{\sqrt{v^2+1}} + \frac{u^2}{\sqrt{u^2+1}} - \frac{1+u+v}{\sqrt{2}}$$
$$= \sqrt{v^2+1} - \frac{v}{\sqrt{2}} + \frac{u^2}{\sqrt{u^2+1}} - \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}}, \tag{6}$$

so we must show that this is nonnegative. Since

$$\sqrt{v^2+1} - \frac{v}{\sqrt{2}} \tag{7}$$

is a decreasing function for $0 \leq v \leq 1$, and since

$$v < t = 2 - \sqrt{3} = \frac{(\sqrt{3}-1)^2}{2} \qquad \text{and} \qquad t^2 + 1 = 4t\ ,$$

we can replace the terms (7) in (6) by

$$\sqrt{t^2+1} - \frac{t}{\sqrt{2}} = 2\sqrt{t} - \frac{t\sqrt{2}}{2} = \sqrt{2}(\sqrt{3}-1) - \frac{(2-\sqrt{3})\sqrt{2}}{2}$$

$$= \frac{3\sqrt{6} - 4\sqrt{2}}{2}\ .$$

Hence (2) is proved if we show that for $0 \leq u \leq 1$,

$$g(u) = \frac{u}{\sqrt{2}} - \frac{u^2}{\sqrt{u^2+1}} \leq \frac{3\sqrt{6} - 5\sqrt{2}}{2}\ .$$

For $u \geq 0$,

$$g'(u) = \frac{1}{\sqrt{2}} - \frac{u^3 + 2u}{(u^2+1)^{3/2}}$$

has the same sign as

$$(u^2+1)^3 - 2(u^3+2u)^2 = -(u^6 + 5u^4 + 5u^2 - 1). \tag{8}$$

This is positive at $u = 0$, decreases as u increases and is negative at $u = 1$. Therefore g' has a unique zero, u_0, between 0 and 1 and changes sign from positive to negative at this point. Hence g has a maximum at u_0. The expression (8) is positive at $u = 0.4$ and negative at $u = 0.5$. Hence $0.4 < u_0 < 0.5$. The tangent to the graph of g at 0.4 is above the curve. Therefore the maximum value of g does not exceed

$$g(0.4) + (u_0 - 0.4)g'(0.4) < g(0.4) + (0.1)g'(0.4) < \frac{3\sqrt{6} - 5\sqrt{2}}{2}\ .$$

II. *Partial solution by Emilio Fernandez Moral, I.B. Sagasta, Logrono, Spain.*

We can prove that the inequality holds when $x \geq y \geq z$ (or for the circular permutations $y \geq z \geq x$ and $z \geq x \geq y$).

From the means inequality

$$\frac{x+y}{2} \geq \sqrt{xy}$$

we deduce

$$x + y \geq \frac{x+y}{2} + \sqrt{xy} = \left(\frac{\sqrt{x}+\sqrt{y}}{\sqrt{2}}\right)^2$$

and thus

$$\sqrt{x+y} \geq \frac{\sqrt{x}+\sqrt{y}}{\sqrt{2}}.$$

Similarly,

$$\sqrt{y+z} \geq \frac{\sqrt{y}+\sqrt{z}}{\sqrt{2}}, \qquad \sqrt{z+x} \geq \frac{\sqrt{z}+\sqrt{x}}{\sqrt{2}},$$

and consequently

$$\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x} \geq \frac{2}{\sqrt{2}}(\sqrt{x}+\sqrt{y}+\sqrt{z}).$$

Thus

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}} \geq \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}}+\frac{1}{2}\left(\frac{x-y}{\sqrt{x+y}}+\frac{y-z}{\sqrt{y+z}}+\frac{z-x}{\sqrt{z+x}}\right),$$

so it suffices to prove that, for $x \geq y \geq z > 0$,

$$\frac{x-y}{\sqrt{x+y}}+\frac{y-z}{\sqrt{y+z}} \geq \frac{x-z}{\sqrt{x+z}}.$$

Squaring both sides and rearranging gives the equivalent inequality

$$\frac{2(x-y)(y-z)}{\sqrt{(x+y)(y+z)}} \geq \frac{(x-z)^2}{x+z}-\frac{(x-y)^2}{x+y}-\frac{(y-z)^2}{y+z}. \tag{9}$$

Using

$$(x-z)^2 = (x-y)^2+(y-z)^2+2(x-y)(y-z)$$

in (9) yields

$$\frac{2(x-y)(y-z)}{\sqrt{(x+y)(y+z)}} \geq \frac{(x-y)^2(y-z)}{(x+y)(x+z)}+\frac{(y-z)^2(y-x)}{(x+z)(y+z)}+\frac{2(x-y)(y-z)}{x+z}. \tag{10}$$

If $(x-y)(y-z) = 0$ both sides of (10) are zero. Otherwise, dividing by $(x-y)(y-z) > 0$, (10) becomes

$$\begin{aligned}\frac{2}{\sqrt{(x+y)(y+z)}} &\geq \frac{x-y}{(x+y)(x+z)}-\frac{y-z}{(x+z)(y+z)}+\frac{2}{x+z}\\ &= \frac{(x-y)(y+z)-(y-z)(x+y)+2(x+y)(y+z)}{(x+z)(x+y)(y+z)}\\ &= \frac{2(xy+2xz+yz)}{(x+z)(x+y)(y+z)},\end{aligned}$$

or equivalently

$$(x+z)\sqrt{(x+y)(y+z)} \geq xy+2xz+yz.$$

Squaring another time gives

$$(x+z)^2(x+y)(y+z) \geq (xy+2xz+yz)^2 = (y(x+z)+2xz)^2,$$

$$(x+z)^2(xy+xz+yz) \geq 4xyz(x+z)+4x^2z^2 = 4xz(xy+xz+yz),$$

and finally

$$(x-z)^2(xy+xz+yz) \geq 0,$$

which is true.

III. *Partial generalization by David Vaughan, Wilfrid Laurier University.*

We show that for $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, $n \geq 2$,

$$\frac{x_1}{\sqrt{x_1 + x_2}} + \frac{x_2}{\sqrt{x_2 + x_3}} + \cdots + \frac{x_n}{\sqrt{x_n + x_1}} \geq \frac{1}{\sqrt{2}}(\sqrt{x_1} + \cdots + \sqrt{x_n}) .$$

For $n = 2$ the inequality is

$$\frac{x_1}{\sqrt{x_1 + x_2}} + \frac{x_2}{\sqrt{x_2 + x_1}} \geq \frac{1}{\sqrt{2}}(\sqrt{x_1} + \sqrt{x_2})$$

or

$$\sqrt{x_1 + x_2} \geq \frac{1}{\sqrt{2}}(\sqrt{x_1} + \sqrt{x_2}) .$$

But on squaring, this is equivalent to

$$x_1 + x_2 \geq \frac{1}{2}(x_1 + x_2 + 2\sqrt{x_1 x_2})$$

which is the same as

$$\frac{1}{2}(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$$

which is true.

Now assume the inequality is true for $n \geq 2$. We wish to show this implies it is true for $n + 1$. Without loss of generality we may take $x_{n+1} = 1$. Then

$$\begin{aligned}\frac{x_1}{\sqrt{x_1 + x_2}} + \cdots + \frac{x_n}{\sqrt{x_n + x_{n+1}}} + \frac{x_{n+1}}{\sqrt{x_{n+1} + x_1}} &= \frac{x_1}{\sqrt{x_1 + x_2}} + \cdots + \frac{x_{n-1}}{\sqrt{x_{n-1} + x_n}} \\ &\quad + \frac{x_n}{\sqrt{x_n + x_1}} - \frac{x_n}{\sqrt{x_n + x_1}} + \frac{x_n}{\sqrt{x_n + 1}} + \frac{1}{\sqrt{x_1 + 1}} \\ &\geq \frac{1}{\sqrt{2}}(\sqrt{x_1} + \cdots + \sqrt{x_n}) - \frac{x_n}{\sqrt{x_n + x_1}} + \frac{x_n}{\sqrt{x_n + 1}} + \frac{1}{\sqrt{x_1 + 1}} ,\end{aligned}$$

and the result will follow if we can show

$$\frac{-x_n}{\sqrt{x_n + x_1}} + \frac{x_n}{\sqrt{x_n + 1}} + \frac{1}{\sqrt{x_1 + 1}} \geq \frac{1}{\sqrt{2}}$$

for $1 \leq x_n \leq x_1$. Set

$$f(x,y) = \frac{-y}{\sqrt{y + x}} + \frac{y}{\sqrt{y + 1}} + \frac{1}{\sqrt{x + 1}} - \frac{1}{\sqrt{2}}$$

for $1 \leq y \leq x$. We need to show that $f(x_1,x_n) \geq 0$. Now

$$\frac{\partial f}{\partial x} = \frac{y}{2(x + y)^{3/2}} - \frac{1}{2(x + 1)^{3/2}} = \frac{y(x + 1)^{3/2} - (x + y)^{3/2}}{2(x + y)^{3/2}(x + 1)^{3/2}} ,$$

so that $\partial f/\partial x \geq 0$ for $x \geq y \geq 1$ is equivalent to

$$y(x + 1)^{3/2} \geq (x + y)^{3/2} ,$$

$$y^2(x + 1)^3 \geq (x + y)^3 ,$$

$$(y^2 - 1)x^3 + (y^2 - y)3x^2 + (y^2 - y^3) \geq 0 ,$$

which is true for $y = 1$ and equivalent for $y > 1$ to

$$(y + 1)x^3 + 3yx^2 - y^2 \geq 0 ,$$

which is obviously true for $x \geq y \geq 1$. Thus f increases as a function of x, so we need only check that $f(x,x) \geq 0$ for $x \geq 1$. But

$$\begin{aligned} f(x,x) &= \frac{-x}{\sqrt{2x}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{2}} \\ &= \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{x+1}} - \frac{\sqrt{x}+1}{\sqrt{2}} \\ &\geq \frac{1}{\sqrt{2}}(\sqrt{x}+1) - \frac{\sqrt{x}+1}{\sqrt{2}} \geq 0 \end{aligned}$$

by the case $n = 2$.

IV. *Partial "solution" by N. Withheld.*

We prove the inequality for $0 < x \leq y \leq z$. By symmetry this also does it for $y \leq z \leq x$ and for $z \leq x \leq y$.

We may rewrite the inequality as

$$\left(\frac{x}{\sqrt{x+y}} - \frac{x}{\sqrt{x+z}}\right) + \frac{y}{\sqrt{y+z}} + \sqrt{x+z} \geq \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}} .$$

Since clearly

$$\frac{x}{\sqrt{x+y}} - \frac{x}{\sqrt{x+z}} \geq 0 ,$$

it is enough to prove

$$F(x,y,z) = \frac{y}{\sqrt{y+z}} + \sqrt{x+z} - \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}} \geq 0$$

for $0 < x \leq y \leq z$. Since

$$\frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x+z}} - \frac{1}{2\sqrt{2x}} \leq 0$$

for $0 < x \leq z$, the minimum for F will occur when $x = z$ and thus when $x = y = z$. Since

$$F(x,x,x) = \frac{x}{\sqrt{2x}} + \sqrt{2x} - \frac{3\sqrt{x}}{\sqrt{2}} = 0 ,$$

we are done.

V. *Editor's comments.*

This very interesting problem, although solved in Solution I by Henderson, leaves much still to be done. Solution II handles one case in a reasonably nice manner (and even without calculus). "Solution" IV is an attempt to do the other case, which nearly succeeds; readers are invited to find the mistake, and correct it if possible. Better yet, find a proof of this case that does not use calculus. Better yet, find a unified proof (without calculus) of the entire problem!

Solution III is a special case of one possible generalization to n variables. Is it true for all positive $x_1,x_2,\ldots,x_n$? Another generalization, analogous to the proposer's companion problem *Crux* 1356 [1989: 241], might be

$$\frac{x_1}{\sqrt{1-x_2}} + \frac{x_2}{\sqrt{1-x_3}} + \cdots + \frac{x_n}{\sqrt{1-x_1}} \geq \frac{\sqrt{x_1} + \cdots + \sqrt{x_n}}{\sqrt{n-1}},$$

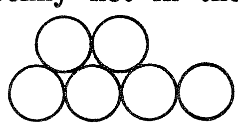
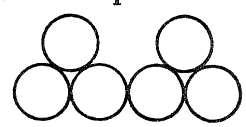
where $x_1,\ldots,x_n > 0$ satisfy $x_1 + \cdots + x_n = 1$.

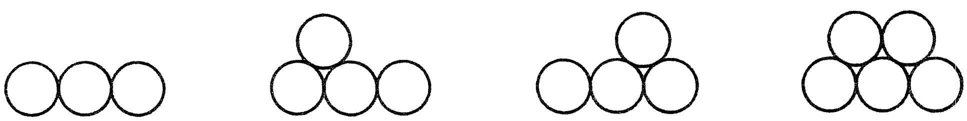
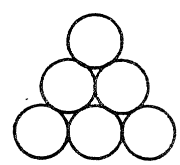
This problem suggested to Jack Garfunkel a similar problem, which appears in this issue as *Crux* 1490.

There were three incorrect solutions received.

* * *

1367. [1988: 202] *Proposed by Richard K. Guy, University of Calgary.*

Consider arrangements of pennies in rows in which the pennies in any row are contiguous, and each penny not in the bottom row touches two pennies in the row below. For example,  is allowed, but  isn't. How many arrangements are there with n pennies in the bottom row? To illustrate, there are five arrangements with $n = 3$, namely

I. *Solution by Colin Springer, student, University of Waterloo.*

We claim that the desired number of arrangements takes on alternate values in the usual Fibonacci series, and proceeds: 1,2,5,13,34,89,... .

To prove this we work with two sequences. Let

a_n = the number of legal arrangements with n pennies in the bottom row

(this is the desired function), and

b_n = the number of legal arrangements with *at most* n pennies in the bottom row.

Clearly

$$b_n = a_n + b_{n-1},$$

since an arrangement with at most n pennies on the bottom row either has n pennies there, or has fewer.

Now consider the second row from the bottom in each of the patterns counted by a_{n+1}. If there is a penny in the left–most position here, there are b_n arrangements for the second row up, by placing each of the b_n patterns on top, left–justified. If the left–most position of the second row is vacant, then there are a_n arrangements: for each pattern in a_n, add a penny to the left of the bottom row. Thus we see that

$$a_{n+1} = b_n + a_n \ .$$

From these equations, and the fact that $a_1 = b_1 = 1$, we conclude that the sequence $a_1,b_1,a_2,b_2,\ldots$ is the Fibonacci sequence $1,1,2,3,5,\ldots$. Thus it follows that

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-1}.$$

II. *Editor's comments.*

This was the nicest solution received. Nearly all others began by deriving the recurrence

$$f_n = f_{n-1} + 2f_{n-2} + 3f_{n-3} + \cdots + (n-1)f_1 + 1$$

for the number f_n of arrangements with n pennies in the bottom row, and then showed that the alternate Fibonacci numbers satisfied this recurrence.

These arrangements of pennies were first considered by Auluck [1]. He counted such arrangements having a *total* of n pennies. Readers who do likewise will arrive at the sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots\ ;$$

and who among you would dream that the next term isn't 13? (Certainly the editor is enough of an innocent to place a small bet on it.) Well, it's 12!

The analogous problem where arrangements have n pennies in the bottom row and may have *noncontiguous* rows above it results in the Catalan numbers $1,2,5,14,42,\ldots$. For more information, see the recent paper of Odlyzko and Wilf [3]. This problem was first proposed by Jim Propp, and turned up as example 34 in Richard K. Guy's excellent article on the "strong law of small numbers" [2]. (Incidentally, this article has won for Professor Guy the 1989 Lester R. Ford award of the Mathematical Association of America.)

References:

[1] F.C. Auluck, On some new types of partitions associated with generalized Ferrers graphs, *Proc. Cambridge Phil. Soc.* **47** (1951) 679–686.

[2] R.K. Guy, The strong law of small numbers, *Amer. Math. Monthly* **95** (1988) 697–712.

[3] A.M. Odlyzko and H.S. Wilf, The editor's corner: n coins in a fountain, *Amer. Math. Monthly* **95** (1988) 840–843.

Also solved by HAYO AHLBURG, Benidorm, Spain; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; BRIAN CALVERT, Brock University, St. Catharines, Ontario; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; GUO–GANG GAO, Université de Montréal; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; SAM MALTBY, student, Calgary; P. PENNING, Delft, The Netherlands; HUME SMITH, Acadia University; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; and the proposer. A further reader gave an unsimplified answer in the form of an $n \times n$ determinant. There was also one incorrect solution submitted.

* * *

1369. [1988: 203] *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

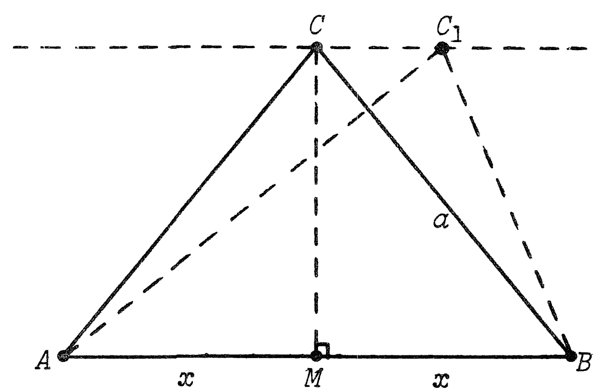
The perimeter of a triangle is 24 cm and its area is 24 cm^2. Find the maximal length of a side and write it in a simple form.

Solution by Hans Engelhaupt, Gundelsheim, Federal Republic of Germany.

If the length of side AB and the area of a triangle ABC are fixed, then the perimeter is minimal for $\overline{AC} = \overline{BC}$.

Therefore in the given problem, to get $\overline{AB} = c = 2x$ maximal the triangle must be isosceles. Thus $a + x = 12$ and

$$x\sqrt{a^2 - x^2} = 24 ,$$



from which one gets

$$24^2 = x^2(a - x)(a + x) = 12x^2(12 - 2x) ,$$
$$x^3 - 6x^2 + 24 = 0 ,$$

and, with $x = z + 2$,

$$z^3 - 12z + 8 = 0 .$$

Thus [since $2x \geq 8$ so that $z \geq 2$] the solution is

$$z = 4 \cos 40^\circ,$$

as proved by

$$z^3 - 12z + 8 = 16(4\cos^3 40^\circ - 3\cos 40^\circ) + 8$$
$$= 16\cos 120^\circ + 8 = 0 .$$

Thus the maximum length of a side is

$$2x = 4 + 8\cos 40^\circ \approx 10.12835 .$$

Also solved by JORDI DOU, Barcelona, Spain; EMILIO FERNANDEZ MORAL, I.B. Sagasta, Logrono, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Murty and Wang note that the problem seems to have been suggested by the 6–8–10 *right triangle.*

* * *

1370. [1988: 203] *Proposed by Peter Watson–Hurthig, Columbia College, Burnaby, British Columbia.*

Let $L(n)$ be the number of steps required to go from n to 1 in the Collatz sequence

$$C_1(n) = n\ ,\ C_{k+1}(n) = \begin{cases} 3C_k(n) + 1 & \text{if } C_k(n) \text{ is odd,} \\ C_k(n)/2 & \text{if } C_k(n) \text{ is even .} \end{cases}$$

It is notoriously unknown whether $L(n)$ exists for all positive integers n. Show that there exist infinitely many n such that

$$L(n) = L(n+1) = L(n+2).$$

I. *Solution by Emilio Fernandez Moral, I.B. Sagasta, Logrono, Spain.*

From

$$\left.\begin{array}{l} 8k+4 \to 4k+2 \to 2k+1 \\ 8k+5 \to 24k+16 \to 12k+8 \end{array}\right\} \to 6k+4$$

i.e. the fusion in the Collatz graph of the orbits of the numbers $8k + 4$ and $8k + 5$ after the same number of steps (see [1]), and the observation that

$$8k+6 \to 4k+3 \to 12k+10 \to 6k+5,$$

also in four steps, we see that with $k = 4h$ the consecutive numbers $6k + 4$ and

$6k + 5$ will themselves fuse in an equal number of steps and thus the triple

$$\{\ 32k + 4\ ,\ 32k + 5\ ,\ 32k + 6\ \}$$

for $k \geq 1$ will always fuse after the same number of steps:

$$\left.\begin{array}{l} \left.\begin{array}{l} 32k{+}4 \to 16k{+}2 \to 8k{+}1 \\ 32k{+}5 \to 96k{+}16 \to 48k{+}8 \end{array}\right\} \to 24k{+}4 \to 12k{+}2 \to 6k{+}1 \\ \\ 32k{+}6 \to 16k{+}3 \to 48k{+}10 \to 24k{+}5 \to 72k{+}16 \to 36k{+}8 \end{array}\right\} \to 18k{+}4 \to 9k{+}2\ .$$

Then

$$L(32k + 4) = L(32k + 5) = L(32k + 6) = 7 + L(9k + 2)$$

whenever $L(9k + 2) < \infty$. But $L(9k + 2) < \infty$ clearly, whenever $9k + 2$ is a power of 2. Now for every natural number n,

$$2^{6n+1} \equiv 2 \bmod 9\ ,$$

which implies that for every $n = 1,2,3,\ldots$, and taking

$$k = \frac{2^{6n+1} - 2}{9}\ ,$$

we have

$$L(32k + 4) = L(32k + 5) = L(32k + 6) = L(2^{6n+1}) + 7 = 6n + 9.$$

Reference:

[1] J.C. Lagarias, The $3x + 1$ problem and its generalizations, *American Mathematical Monthly*, 92 (1985) 3–23.

II. *Editor's comments.*

Despite the broad hint dropped by the editor in pointing out (in the statement of the problem) that $L(n)$ was not known always to exist, of all the solutions (listed below) received for this problem only those of Fernandez Moral, Maltby (which was similar) and the proposer actually answered the question; that is, showed that $L(n) = L(n + 1) = L(n + 2)$ *as finite integers* for infinitely many n!

Meyers found that for every r the five consecutive integers

$$256r + 98\ ,\ 256r + 99\ ,\ 256r + 100\ ,\ 256r + 101\ ,\ 256r + 102$$

produce Collatz sequences of equal length, merging at $54r + 22$ after 10 steps. As above, one can then find infinitely many values of r for which $54r + 22$ is a power of 2; since $2^{18} \equiv 1 \bmod 27$ and $2^{13} \equiv 11 \bmod 27$,

$$r = \frac{2^{18n+13} - 11}{27}$$

works for all $n \geq 0$.

Penning found the 17–tuple 7083 through 7099, all of which took 57 steps to reach 1. (The same example is given in Lagarias' paper referred to above, and is the longest known consecutive string of integers with equal L–values.) Penning notes that of the 57 steps it takes these numbers to reach 1, 40 are divisions by 2, and hence the consecutive integers

$$7083 + r\cdot 2^{40}\ , \ \ldots\ , \ 7099 + r\cdot 2^{40}$$

will all reach the common number $1 + r\cdot 3^{17}$ after 57 steps, for every $r \geq 0$. Since

$$2^{3^{16}\cdot 2} = 2^{\phi(3^{17})} \equiv 1 \bmod 3^{17}$$

we may let

$$r = \frac{2^{3^{16}\cdot 2n} - 1}{3^{17}}$$

for any $n \geq 1$, and the above common value will then be a power of 2. This shows that there are infinitely many sets of 17 consecutive integers with the same finite L–value. Several readers suggest that there likely are arbitrarily long sets of consecutive integers with equal L–values. Can anyone at least beat the record of 17?

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; SAM MALTBY, student, University of Waterloo; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

* * *

<u>1371</u>*. [1988: 234] *Proposed by Murray S. Klamkin, University of Alberta.*

In *Mathematics Gazette* **68** (1984) 222, P. Stanbury noted the close approximation $\pi^9/e^8 \approx 9.999838813 \approx 10$. Are there positive integers l, m such that π^l/e^m is closer to a positive integer than for the case given? (See *Crux* 1213 [1988: 116] for a related problem.)

Solution by Colin Springer, student, University of Waterloo.

We shall show that, using positive integers l, m, π^l/e^m may be made arbitrarily close to 1.

To do this we claim that $l \ln \pi - m$ may be made arbitrarily close to 0 for positive integers l, m. For let n be any positive integer and define

$$x_k = k \ln \pi - [k \ln \pi]\ , \qquad k = 1,2,\ldots,n + 1$$

(here $[x]$ denotes the greatest integer less than or equal to x). Then $0 \le x_k < 1$ for $k = 1,2,\ldots,n + 1$, so by the pigeonhole principle some two x_i's lie within $1/n$ of each other, say

$$|x_b - x_a| = |(b - a)\ln \pi - ([b \ln \pi] - [a \ln \pi])| < \frac{1}{n},$$

where $a < b$. Since both $b - a = l$ and $[b \ln \pi] - [a \ln \pi] = m$ are positive integers, by making n arbitrarily large the claim follows. But now

$$e^{l \ln \pi - m} = \frac{\pi^l}{e^m}$$

is arbitrarily close to 1.

Also solved by HARVEY ABBOTT, University of Alberta; and RICHARD I. HESS, Rancho Palos Verdes, California.

Hess gave several examples of values l, m for which π^l/e^m is closer to an integer than for the given values, the smallest being $l = 599$, $m = 685$ where

$$\frac{\pi^l}{e^m} \approx 2.0001088 .$$

The proposer also asks how close π^{m+1}/e^m can come to an integer, for $m > 8$.

* * *

1372. [1988: 234] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcentre O and median point G, and the lines AG and BG intersect the circumcircle again at A_1 and B_1 respectively. Suppose the points A, B, O and G are concyclic. Show that

(a) $AA_1 = BB_1$;

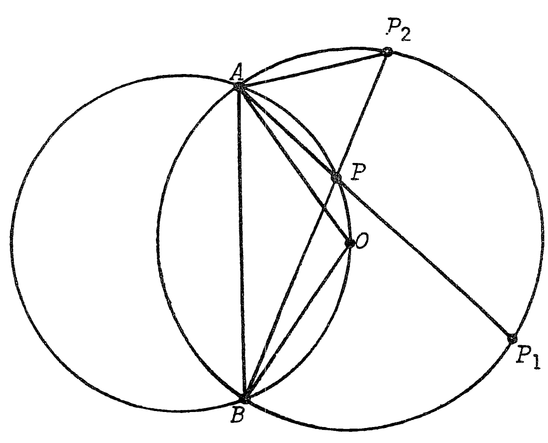
(b) ΔABC is acute angled.

I. *Solution to* (a) *by R.H. Eddy, Memorial University of Newfoundland.*

More generally, let P be any point on the arc BOA and P_1, P_2 the intersections of AP and BP with the circumcircle of any given triangle ABC. Now

$$\angle BPA = \angle BOA = 2\gamma,$$

where γ is the measure of $\angle BCA$. Since also $\angle BP_2A = \gamma$ and $\angle APP_2 = 180 - 2\gamma$, it follows that $\angle P_2AP = \gamma$ and $AP = P_2P$. Similarly $BP = P_1P$, and the result follows.

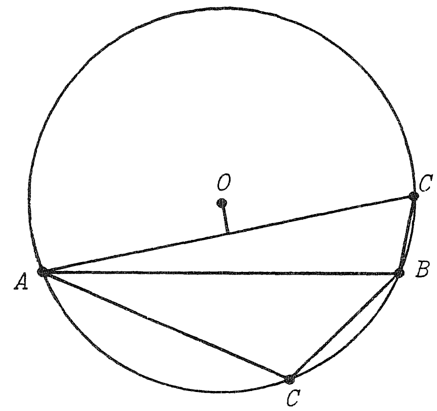


When $P = A$, we get the interesting result that the chord AB is equal to the tangent segment AP_1 to the circle AOB at A. [*Editor's note*: Eddy has submitted this result as a "Proof without Words" to *Mathematics Magazine*.]

II. *Combined solution to* (b) *by George Tsintsifas, Thessaloniki, Greece, and the editor.*

[*Editor's note*: The solutions to (b) of this problem were all either too complicated or not clear enough, in the editor's opinion. He therefore modified the best of these, by Tsintsifas (who also solved part (a)), to reach the solution below. Any inadequacies in this solution can be blamed on the editor!]

If $C > 90^\circ$ then circle AOB contains ΔABC in its interior, so cannot contain the median point of ΔABC, a contradiction. On the other hand, if $B > 90^\circ$, say, then O lies outside ΔABC, so the midpoint of AC lies inside circle AOB (since it lies on the circle with diameter OA). Thus the median point lies inside circle AOB, a contradiction. Therefore ΔABC is acute.



Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; P. PENNING, Delft, The Netherlands; and the proposer.

The proposer proved the additional results that the sides a, b, c and angle γ of ΔABC satisfy

$$c^4 = a^4 - a^2b^2 + b^4,$$

$$a > c > b \quad \text{or} \quad a < c < b$$

(both also noted by Groenman), and

$$60^\circ < \gamma < 90^\circ$$

(also observed by Dou).

* * *

1373. [1988: 234] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Evaluate

$$\sum_{n=1}^{\infty} \frac{9^n - 6^{n-1}}{(3^n - 2^n)(3^{n+1} - 2^{n+1})(3^{n+2} - 2^{n+2})}.$$

Solution by Hans Engelhaupt, Gundelsheim, Federal Republic of Germany.

Since

$$\frac{9^n - 6^n}{(3^n-2^n)(3^{n+1}-2^{n+1})(3^{n+2}-2^{n+2})} = \frac{3^{n-1}/2}{(3^n-2^n)(3^{n+1}-2^{n+1})} - \frac{3^n/2}{(3^{n+1}-2^{n+1})(3^{n+2}-2^{n+2})}$$

$$= a_n - a_{n+1} ,$$

where

$$a_n = \frac{3^{n-1}/2}{(3^n - 2^n)(3^{n+1} - 2^{n+1})} ,$$

and since the given series is absolutely convergent, its sum is

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 = \frac{1}{10} .$$

Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; JÖRG HÄRTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University; M. SELBY, University of Windsor; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; DAVID VAUGHAN, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Two readers mentioned the similar but easier problem A–2 *on the 1984 Putnam Examination. Several readers gave generalizations.*

*　　　　　*　　　　　*

1375. [1988: 235] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate

$$\int_0^{\pi/4} \frac{e^{\sec x}\sin(x + \pi/4)}{\cos x(1 - \sin x)}\, dx .$$

Solution by Seung–Jin Bang, AJOU University, Suwon, Republic of Korea.

Note that

$$\frac{\sin(x + \pi/4)}{\cos x(1 - \sin x)} = \frac{1}{\sqrt{2}} \frac{(\sin x + \cos x)(1 + \sin x)}{\cos^3 x}$$

$$= \frac{1}{\sqrt{2}} \frac{\sin x(1 + \cos x) + (\cos x + \sin^2 x)}{\cos^3 x}$$

$$= \frac{1}{\sqrt{2}}[(\sec x + 1)\sec x \tan x + (\sec^2 x + \sec x \tan^2 x)]$$

and

$$(e^{\sec x}\tan x)' = e^{\sec x}(\sec^2 x + \sec x \tan^2 x).$$

Now we see that, putting $u = \sec x$,

$$\int_0^{\pi/4} \frac{e^{\sec x}\sin(x + \pi/4)}{\cos x(1 - \sin x)}\,dx = \frac{1}{\sqrt{2}}\int_1^{\sqrt{2}} e^u(u + 1)du + \left[\frac{1}{\sqrt{2}}e^{\sec x}\tan x\right]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}}\Big[ue^u\Big]_1^{\sqrt{2}} + \frac{1}{\sqrt{2}}e^{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{2}e^{\sqrt{2}} - e\right) + \frac{1}{\sqrt{2}}e^{\sqrt{2}}$$

$$= \left(1 + \frac{1}{\sqrt{2}}\right)e^{\sqrt{2}} - \frac{e}{\sqrt{2}} .$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; and the proposer. There was one incorrect solution submitted.

* * *

1376. [1988: 235] *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let $ABCD$ be a quadrilateral with an inscribed circle of radius r and a circumscribed circle of radius R. Let $AC = p$ and $BD = q$ be the diagonals. Prove that

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = 1 .$$

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In the course of solving *Crux* 1203 [1988: 91], Murray Klamkin proved (line (5) on [1988: 92]) that

$$pq = 2r\left(r + \sqrt{r^2 + 4R^2} \right) .$$

Since the stated relation, which may also be written

$$(pq)^2 - 4r^2(pq) - 16R^2r^2 = 0 ,$$

has the solutions

$$pq = 2r\left(r \pm \sqrt{r^2 + 4R^2} \right) ,$$

we're done.

[*Editor's note*: The editor is sorry that this problem falls out so easily using an earlier *Crux* solution. (Solvers Klamkin and Prielipp also did it this way.) On the other hand, the above relation is nice and deserves to be written down somewhere! And the solvers (and there were a good number) seemed to have fun. But so they won't feel their time was completely wasted, here is another solution.]

II. *Solution by D.J. Smeenk, Zaltbommel, The Netherlands.*

We denote $AB = a$, $BC = b$, $CD = c$, $DA = d$, and F the area of $ABCD$. We assume that it is well known that

$$pq = ac + bd ,$$
$$a + c = b + d ,$$
$$F = \sqrt{abcd} ,$$
$$R = \frac{\sqrt{(ab + cd)(ac + bd)(ad + bc)}}{4F} ,$$
$$r = \frac{F}{a + c} .$$

Then

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = \frac{(ac + bd)(a + c)^2}{4abcd} - \frac{(ab + cd)(ad + bc)}{4abcd}$$

$$= \frac{ac(a + c)^2 - ac(b^2 - 2bd + d^2)}{4abcd}$$

$$= \frac{(b + d)^2 - (b - d)^2}{4bd} = 1 .$$

Also solved by RAUL F.W. AGOSTINO, Rio de Janeiro, Brazil; WILSON DA COSTA AREIAS, Rio de Janeiro, Brazil; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SVETOSLAV J. BILCHEV, Technical University, Russe, Bulgaria; R.H. EDDY, Memorial University of Newfoundland; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Eddy points out that the given relation, multiplied out as in Solution I, *appears (with a typo) on p.* 49 *of his and G.C.W. Sabin's paper "Inequalities in R, r, s for the inscribed circumscribed quadrilateral",* Nieuw Archief voor Wiskunde **6** (1988) 47–56.

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