MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

Volume 20 1987/88 Number 2

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 20 of *Mathematical Spectrum* will consist of three issues, of which this is the second. The first was published in September 1987 and the third will appear in May 1988.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

The Editorial Committee welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. Students are encouraged to send in contributions. All correspondence about the contents should be sent to:

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Mathematical Spectrum Awards for Volume 19

The editors have made three awards this year. Two prizes for articles go to Simon Johnson for 'Approximating $\sqrt{n}$' (pages 37–40) and Anthony Quas for 'Ratio Derivatives' (pages 72–75). The third goes to Gregory Economides for a problem submitted and solutions to problems (pages 92 and 93).

Readers are reminded that awards of up to £30 are available for articles and of up to £15 for letters, solutions to problems, or other items. To qualify, contributors must be students at school, college or university.

Pascal's Triangle—and Bernoulli's and Vieta's

A. W. F. EDWARDS, *Gonville & Caius College, Cambridge*

The author is Reader in Mathematical Biology at Cambridge and a Fellow of Gonville & Caius College. His book *Pascal's Arithmetical Triangle* was recently published by Charles Griffin & Son, London. His interest in binomial coefficients was kindled many years ago when he was a research student in genetics and has been maintained ever since.

Everybody knows Pascal's arithmetical triangle—or do they? Contrary to popular belief it is *not* the arithmetical triangle in its common form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

but in the form

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & \cdot \\
1 & 2 & 3 & 4 & \cdot & \\
1 & 3 & 6 & \cdot & & \\
1 & 4 & \cdot & & & \\
1 & \cdot & & & & \\
\cdot & & & & &
\end{array}$$

which Blaise Pascal (1623–1662) used in his book *Traité du Triangle Arithmétique*, published posthumously in 1665. In this form one can readily see the series of *figurate numbers* developing, the numbers in each row being formed by summing the numbers in the previous row starting with an initial row of ones. The numbers in the third row, 1 3 6 10 15 ..., are known as the *triangular numbers* because they are the numbers which can be represented by triangular arrays of dots, whilst those in the fourth row, 1 4 10 20 35 ..., are the *tetrahedral numbers*, representable by tetrahedral arrays of dots in three dimensions (see figure 2 of reference 2).

In the first of the above forms the numbers are more readily seen to be the *binomial numbers*, the coefficients in the expansion of the binomial $(x+y)^n$ for $n = 0, 1, 2, \ldots$. Both forms, incidentally, are very much older than Pascal (as he would have been the first to acknowledge). The figurate numbers originated with the Pythagoreans and the binomial numbers are found in Chinese and Arabic mathematics.

However, it was neither the figurate nor the binomial numbers which primarily concerned Pascal, but the *combinatorial numbers*, for he understood (as the Hindus had done before him) that the number of ways of choosing r things from n different things is

$$^{n}C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

and that these combinatorial numbers are identical to the figurate and binomial numbers. It is this identity which makes the arithmetical triangle so rich in elementary relations amongst its numbers, and which so fascinated Pascal that he wrote a whole book based on it. The book is full of novel arguments. It uses proof by induction in recognizably modern form, and introduces the probability notion of *expectation* for the first time. It contains the first rigorous proof of the binomial theorem for positive integral index, and advances an ingenious combinatorial argument for the identity

$$^{n+1}C_{r+1} = {}^{n}C_r + {}^{n}C_{r+1}.$$

Consider combinations of $r+1$ of the $n+1$ things. Take any particular one of the $n+1$ things: $^{n}C_r$ gives the number of combinations that contain it, whilst $^{n}C_{r+1}$ gives the number that exclude it, the two numbers together giving the total.

Pascal's interest in the arithmetical triangle arose because he had shown its numbers to be involved in the solution of the 'problem of points', a famous gambling problem of the day which asked the question: how should the stake money be divided between two players in an interrupted game? Suppose the game consists of tossing a fair penny, player A to count heads as points in his favour and player B to count tails, the winner to be the first to score an agreed number of points. How should the total stake be divided if the game is interrupted when A still needs a points to win, and B still needs b?

To find the solution, first note that, had the game been continued, at most $a+b-1$ more tosses would have been needed to conclude it, and these would have contained $0, 1, 2, \ldots, a+b-1$ heads with probabilities given by expanding the binomial $(\frac{1}{2}+\frac{1}{2})^{a+b-1}$. Secondly, note that when there are $0, 1, 2 \ldots, a-1$ heads *in any order* B wins the game, but when there are $a, a+1, a+2, \ldots, a+b-1$ heads *in any order* A wins the game. It is remarkable that the order of occurrence of the heads and tails does not matter, but only the total numbers of each (a fact unique to the two-player version of the game). The reason is that, in this 'imaginary' continuation of $a+b-1$ tosses, even if A only scores the a heads he needs to win, B, playing on to the bitter end, could only score $b-1$, one short, and, if A scores more than a heads, B is even further away from winning.

Thus the total stake should be divided in the ratio (chance that B would have won : chance that A would have won) = (sum of the first a terms : sum of the last b terms on expanding the binomial). Both Pascal and Fermat, with whom he corresponded, obtained the solution in this way, but then Pascal, in his *Traité du Triangle Arithmétique*, proved the result by applying mathematical induction to each player's expectation of the total stake (the full argument is given in chapter 7 of reference 1).

We see from this solution that the *partial sum of the binomial coefficients* for $n = a+b-1$ is involved. Pascal's form of the arithmetical triangle of binomial coefficients can easily be turned into the corresponding triangle for the partial sums by adding cumulatively along the rising diagonals to obtain

$$\begin{array}{cccccc}
1 & 2 & 4 & 8 & 16 & \cdot \\
1 & 3 & 7 & 15 & \cdot & \\
1 & 4 & 11 & \cdot & & \\
1 & 5 & \cdot & & & \\
1 & \cdot & & & & \\
\cdot & & & & &
\end{array}$$

The reader should extend this triangle for himself, when he will discover that it has many of the properties of Pascal's triangle. Thus the numbers in each row are still obtained by summing the numbers in the previous row, but now the first row consists of the powers of two! Moreover, each number is still formed by adding together the two numbers immediately above and to the left.

This tabulation of the solution to the problem of points was given posthumously by James Bernoulli in his *Ars Conjectandi* of 1713: let us therefore call it *Bernoulli's triangle*. Just as we recognize the rising diagonals of Pascal's triangle as generating the successive terms of the binomial distribution of statistics (for probability of success at each trial equal to $\frac{1}{2}$), we recognize the rising diagonals of Bernoulli's triangle as generating the corresponding terms of the *cumulative* binomial distribution. And, of course, the last term is always a power of 2 because binomial coefficients always sum to a power of 2. The numbers in the descending diagonal starting with the second 1 in the first column are the powers of 4 since they correspond to half the sum for the n-odd cases. As we expand Bernoulli's triangle we approximate the curve of the cumulative distribution function of the normal distribution.

If we write the arithmetical triangle in yet another form (due to Montmort in 1708)

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{matrix}$$

we obtain a square matrix (with as many rows and columns as we like) which has entertaining properties. Try inverting it. Give the elements in the even-numbered rows negative signs and then try squaring it. Better still, try multiplying it by its transpose.

Naturally, once the idea has occurred of creating further number-triangles using the addition property of Pascal's triangle but starting with a first row of numbers other than 1, endless different triangles can be constructed besides Bernoulli's. Thus we might start with the row 2 1 1 1 ..., and each number in the resulting triangle is then the sum of two combinatorial numbers: not counting the first row and column, the element in the pth row and rth column is

$$\binom{p+r}{r} + \binom{p+r-1}{r}.$$

$$\begin{matrix} 2 & 1 & 1 & 1 & 1 & \cdot \\ 2 & 3 & 4 & 5 & 6 & \cdot \\ 2 & 5 & 9 & 14 & 20 & \cdot \\ 2 & 7 & 16 & 30 & 50 & \cdot \\ 2 & 9 & 25 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Lest you might think this number-triangle new, consider how you would express $2\cos\frac{1}{2}nx$ in terms of $2\cos\frac{1}{2}x$ for successive values of n; Vieta did just that in about 1591, well before the time of Pascal, and out came this triangle!

For let $u_n = 2\cos\frac{1}{2}nx$ and prove $u_{n+1} = u_1 u_n - u_{n-1}$ $(n = 1, 2, 3, \ldots)$ by using $\cos(A+B) = 2\cos A\cos B - \cos(A-B)$. Then

$$u_0 = 2, \qquad u_1 = u_1, \qquad u_2 = u_1^2 - 2, \qquad u_3 = u_1^3 - 3u_1, \qquad \ldots,$$

and the coefficients of the successive expressions for u_n in terms of u_1 may be found from this *Vieta's triangle*. The first row gives successively the coefficient of the highest power of u_1 in each expression, the second row minus the coefficient of the next highest power, and so on (the signs alternating). Why not work out a few? As Vieta remarked, the numbers are generated like the figurate numbers but 'not starting with unity, as in the generation of powers, but starting with two'.

References

1. A. W. F. Edwards, *Pascal's Arithmetical Triangle* (Griffin, London, 1987).
2. Liu Zhiqing, Pascal's pyramid, *Mathematical Spectrum* **17** (1984/85), 1–3.

You Must Remember This

KEITH AUSTIN, *University of Sheffield*

Keith Austin is a lecturer in pure mathematics at the University of Sheffield. He is concerned about people's attitude to the subject of mathematics. Most people who are not mathematicians view it from afar with awe, and declare their unworthiness to approach such a wondrous and mysterious subject. This attitude is encouraged by the mathematicians, the high priests of the subject, who often appear to carry out their duties in an obsequious manner.

Keith hopes that this article will encourage a more balanced and sensible respect for mathematics. For, although the mathematics plays a vital role in the article, it is denied entire domination, and parts of the story are allowed to proceed independently of the mathematics.

Look out! Don't hit the green hippopotamus!

I opened my eyes. I was lying in bed and a nurse was sitting by the window. She smiled when she saw I was awake.

'How are you feeling?'

'Fine. But where am I?'

'Professor Westholm's clinic just outside the village of Trevetho. You were found on the beach at Trevetho. Don't you remember?'

'I can't remember a thing. Where is Trevetho?'

'Cornwall. It's a lovely old fishing village with towering cliffs on either side. The clinic is on the top of one of the cliffs.'

Just then the door opened.

'Professor Westholm, the patient has woken up.'

The professor introduced himself and gave me a thorough examination.

'Physically, you appear to be as fit as a fiddle, but you say you can remember nothing of your past or how you came to be here. There were no identifying papers on you when you were found.'

'Can you help me regain my memory?'

'I hope so. By coincidence my work at the clinic is concerned with memory, and so I should welcome the chance to work on your case. That is if you don't mind staying here.'

'I have nowhere else to go, so that's fine by me.'

'Right then. We will make a start in the morning.'

After breakfast, the nurse took me to the professor's consulting room which faced out over the rolling Atlantic.

'In my work I am trying to use the new developments in computing to help me understand the human memory. The structure of a computer resembles the human brain, with its nerve cells and links between them. However, the brain is ever changing as new links grow to reflect the way our mind develops. Thus the pattern of links in your brain indicates your personality—the inner you.'

'So if the pattern is reproduced in some way then you have my personality without my body. It sounds very spooky.'

'The question of ghosts is not a laughing matter to the people in these parts.'

The professor took me to the window and pointed along the cliff top to a ruined house.

'Not even the bravest of the villagers would venture there alone at night.'

'I'm sorry. Can we return to the business in hand?'

'I will begin with a question. A bus leaves the terminus with 17 people. At the first stop, 6 get off and 10 get on. At the next, 3 get off and 7 get on. At the next, 8 get off and 15 get on. At the next 16 get off and 3 get on. The bus then reaches its destination. How many ... times did the bus stop?'

'I was so busy counting people on and off that I cannot remember.'

'If I had told you the question before I started then you would have had no problem. For the next question I will read out a string of letters and then I want you to tell me if it contains 3 *a*'s or 2 *b*'s in a row.

'The string is *abaababababbaba*.'

'Yes.'

'Good. What was the string?'

'I don't know. I only remembered what I needed to.'

'Correct. Tell me what you did remember.'

'At each stage I remembered whether the answer was "Yes" or "No" if you stopped at that point. Also, if the answer was "No", then I remembered an appropriate amount of information about what had gone before, namely, one of the following four:

> nothing (at the start);
> the string to the present point ends with a but not with aa;
> the string to the present point ends with aa;
> the string to the present point ends with b.

Each time you told me another letter, I had to decide whether to change what I was remembering and, if necessary, to make the change. For example, if my memory was

> No; the string to the present point ends with aa, and you said b,

then I should change my memory to

> No; the string to the present point ends with b.'

'Fine. Now we will draw a diagram of the way your memory was working (figure 1). First we draw a circle for each of your memories. I have abbreviated your descriptions of what you remembered, e.g. $\dots b$ means

> the string to the present point ends with b.

Next we add arrows to indicate how you changed your memory when I told you a letter. There is an a-arrow and a b-arrow going out from each circle. This gives us the complete memory diagram, as we will call it.

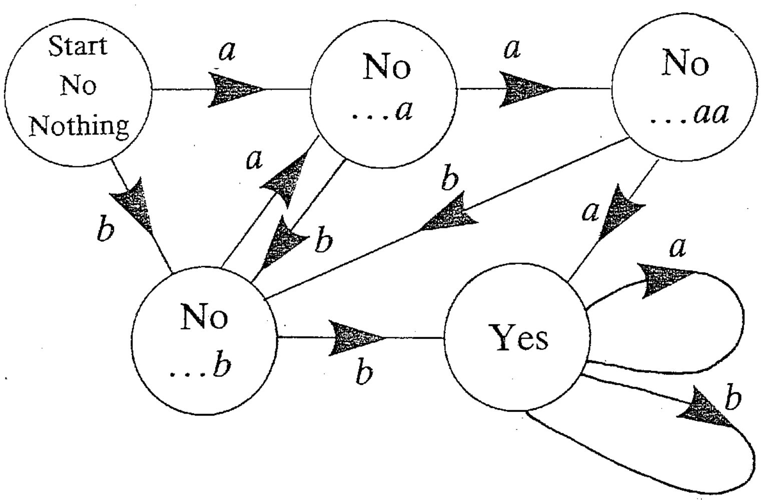


Figure 1

'We will now try the same question as before—3 a's or 2 b's—but with a new string. However, this time, instead of using your memory, I want you to use the memory diagram.

‘Place your finger in the start circle and, when I read out a letter, move your finger along the outgoing arrow with that letter, to the circle it goes to. When I read out another letter, move on to another circle, and so on until I finish. If your finger ends in the “Yes” circle then that is the answer, otherwise the answer is “No”.’

The memory diagram worked as the professor had said. As we were finishing for lunch he remarked, ‘We will have another session tomorrow. Before then, I should like you to draw a memory diagram for the question: Does the string contain *aba* ?’

That evening after dinner I walked out of the French windows onto the cliff top and stood looking down at the waves breaking on the rocks many hundreds of feet below. Slowly, as I watched the water surging to and fro, the cliffs began to move in sympathy until the ground beneath my feet threatened to hurl me over the edge. I felt unable to escape, when suddenly a hand gripped my wrist and pulled me away from the edge and helped me to sit down.

As I recovered I saw that my rescuer was a tall slim girl whom I recognized as one of the domestic staff at the clinic. I began to thank her but she said nothing and quietly slipped away when she saw that I was safe.

‘You are not yet back to full strength’ explained the professor. ‘You will have to be more careful of our cliffs. It was a good job Christine saw what was happening.’

I asked the professor why she had not answered me.

‘Christine is a Trevetho girl and she came to work at the clinic four years ago. About a year and a half ago she vanished for twenty-four hours. When she returned she had lost her voice and her memory of those missing hours. I was happy for her to continue working here and it gave me the chance to try to cure her. There is nothing physically wrong with her; her loss of voice is purely psychological, caused by whatever happened on that fateful day.’

I did not mention it to the professor, but I wondered if it was purely by chance that Christine had been on the cliff top, for I noticed her looking at me a number of times as if she wanted to tell me something.

The next day my session with the professor was interrupted by a call from the nearby caves where a party of visitors had got lost. When we arrived we were shown a map of the caverns (the circles) and connecting tunnels (the arrows) (figure 2).

‘Each cavern has two exit tunnels, labelled *a* and *b*’ explained the cave manager. ‘We are in touch with the party but we do not know which cavern they are in.’

The professor studied the map for some time and then said the party should take exit *a* then, at the next cavern, exit *b*, then *a*, then *a*, then *b*, and finally *a*.

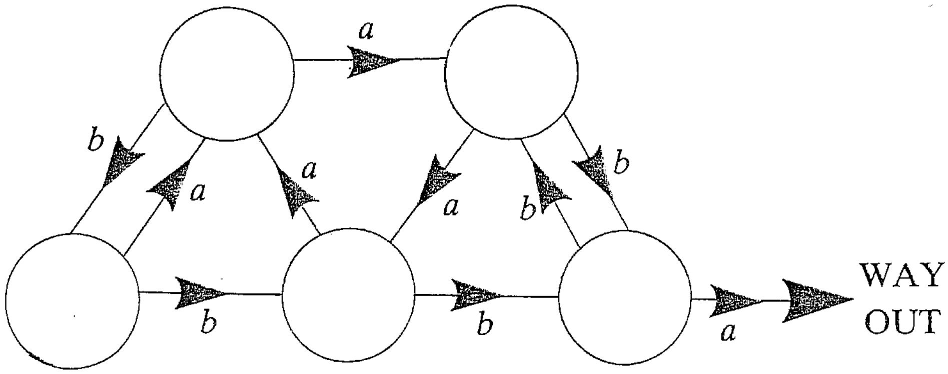


Figure 2

After about half an hour a great cheer went up from the anxious crowd as the party emerged into the daylight.

On the way back to the clinic I asked the professor how he had selected the sequence of *a*'s and *b*'s for the rescue.

'I imagined one person in each cavern. I selected *a* to rescue one of them and worked out where they all moved to. Then I rescued another two by *ba* and so on until they were all out.'

That evening I went into the music room which looked out to the west and so allowed me to watch the setting sun. I sat down at the piano and soon the sound of the waves crashing against the rocks moved me to play the Cornish Rhapsody. The music drove all sense of time from my mind and it was dark when I had finished. Just then a light shone, for an instant, some way along the cliff top. I felt like some exercise and so I decided to investigate.

As I drew nearer to the spot where the light had been I realized it had come from the ruined house that the villagers believed was haunted. However, I made my way inside and I was stumbling around in the dark when I realized that I was not alone. Someone or something was coming towards me. In terror I backed away until I found myself against an open window with the cliffs below me. Nearer the figure came until suddenly the moon came from behind a cloud.

'Christine' I gasped in relief.

She gave me time to recover and then she held out her hand and indicated, with a toss of her head, that I should go with her. We made our way along various cliff paths until we were about a mile from the village and nearly down to the water's edge. Here Christine stopped and pointed into the water. By the light of the moon I could just make out something green beneath the waves. It seemed to be familiar and so I decided, as it was not at a great depth, to take a swim and investigate. It turned out to be a car with the driver's door open. I reached inside and took out a jacket which was lying on the passenger's seat. Back on dry land I started to go through papers in the pockets.

'It is your car, isn't it?'

'Yes, I believe it is ...' I replied, and then I stopped in surprise and looked up; it was the first time I had seen Christine smile.

'Because she saw you had recovered the past you had lost,' said the professor, 'and as she had helped you to do this, she experienced a great relief, which allowed her to regain her voice. When the police recovered your car they also found a second one. Unfortunately its occupants had not escaped as you had. From police records the car disappeared at the same time as Christine lost her voice. She is still not clear about what happened. However, we can assume she saw the car go over the cliff and into the water but was unable to help. Then a similar thing must have happened with you, but you managed to escape.

'I understand there were enough personal belongings in the car to enable you to find out who you are. So you will be leaving us in the morning.

'Before you go, I should like to say a little more about memory diagrams.

'Just as the process by which you lost your memory has been reversed, so we can reverse the main problem we have considered. Instead of going from pattern to memory diagram, we can go the other way. Thus we are given a memory diagram and then asked, what is the pattern associated with it, that is, which strings of letters take us from Start to Yes?

'Why not call in at your local library and see what books they have on computing science. The topic you want is finite-state machines or finite automata, particularly the theorem of S. C. Kleene.'

I still wanted to discover why I had driven over the cliff. When I saw my car—a green Allegro—I recalled that it had always reminded me of a green hippopotamus, but that did not explain everything. At the police station I asked if there had been any incidents on the cliff road at the point where my car had been found, on the night of my crash. They told me that the local builder's lorry had broken down there, and so I decided to visit his yard. He showed me the lorry but he could not help.

As I was leaving I asked him what the wooden frame on the side was for. He explained it was for carrying glass.

'Was there glass in it when you broke down?'

'Yes, I believe there was.'

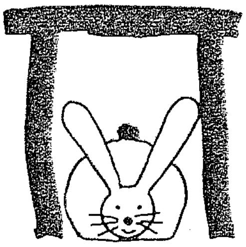
'That's it. It was dark and when I reached your lorry I saw my own car reflected in your glass, and drove off the road as I thought it was another car coming towards me.'

The next morning I set off from the clinic in my car. Up on the cliff top above me Christine waved and set off, pretending to race me. As she ran, laughing and full of the joy of life, with the seagulls wheeling overhead and the waves breaking below, it made a scene that I hope never to lose from my memory.

Pulling Pi out of the Hat

KEITH DEVLIN, *University of Lancaster*

This article is reprinted from Keith Devlin's regular column in the *Guardian*, 5 February 1987, by kind permission. Keith Devlin is Reader in Mathematics at the University of Lancaster.



Rabbit pi. Familiar in the dining room, perhaps, but in mathematics? Well, it is now, thanks to a recent result of the Soviet mathematician Yuri Matiyasevich, which shows that there is an intimate (though sophisticated) connection between an old problem of Fibonacci, concerning rabbits, and the mathematical constant π.

The so-called Fibonacci sequence gets its name from the thirteenth-century Italian mathematician Leonardo of Pisa, who wrote under the name of Fibonacci (from the Latin Filius Bonacci, meaning son of Bonacci). It was Fibonacci's book *Liber Abaci* that introduced into Europe the powerful Hindu-Arabic decimal notation that we use today. It also presented mankind with the following problem.

A man puts one pair of rabbits in a certain place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive?

It does not take long to figure out that the number of pairs of rabbits present each month is given by the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Turning the problem into a piece of general mathematics, the Fibonacci sequence is defined by the iterative formula

$$u_0 = 0, \qquad u_1 = 1, \qquad u_{n+1} = u_n + u_{n-1} \quad (n \geqslant 1).$$

So $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, and so on, giving the rabbits sequence.

The Fibonacci sequence crops up in many parts of mathematics and computer science: for instance, in the study of algorithm efficiency and in database design. But it is not at all apparent that it is connected with that ubiquitous mathematical constant π, the number obtained by dividing the circumference of any circle by its diameter.

It has long been known that it is impossible to give an exact value of π as a decimal—its decimal representation would be infinite. To the first five places of decimals it is 3.141 59. Recent computational work using a CRAY supercomputer belonging to NASA produced the first 29 million decimal places. But if calculating its exact value is out of the question, there is no shortage of formulae that yield π as their answer. For instance,

$$4(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\ldots) = \pi,$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \frac{\pi^2}{6}.$$

In probability theory you find π making an appearance as well. Suppose you have a board ruled with parallel lines one inch apart, and you throw a one-inch matchstick onto the board, then the probability that the matchstick ends up touching a line is exactly $2/\pi$.

So what is the connection between π and the Fibonacci sequence?

Well, suppose you start out with the Fibonacci numbers $u_1, u_2, u_3, \ldots$ up to some point u_m. Multiply them all together to give a single number, P_m. When you have done that, calculate the least common multiple $\operatorname{lcm}(m)$ of those same Fibonacci numbers u_1 through to u_m. Then work out the quotient

$$\frac{6\log P_m}{\log \operatorname{lcm}(m)}.$$

Now take the square root of this quantity and call it Z_m. Then, if you were to do this for increasing values of m, the numbers Z_m you got would get closer and closer to π. By calculating the number Z_m for sufficiently large m you can calculate π to however many decimal places you required—in principle, at least. Not that this is intended as a new method for calculating π; rather, it demonstrates that, however unlikely it may seem that Fibonacci's rabbits ever got near to π, and however different the Fibonacci sequence may seem mathematically from π, the two notions are nevertheless connected. And that is what is surprising.

For those giving the method a try, the value $m = 8$ gives:

$$P_m = 1\times1\times2\times3\times5\times8\times13\times21 = 65\,520, \qquad \operatorname{lcm}(m) = 10\,920.$$

Taking logs to base 10 (it does not matter which base),

$$\frac{6\log 65\,520}{\log 10\,920} = \frac{6\times4.816}{4.038} = 7.156.$$

Taking the square root gives $Z_m = 2.676$. This is not very close to π yet, but why not see what happens for significantly larger values of m. Using a computer, it's as easy as pie.

Reference

The American Mathematical Monthly **93** (1986), 631–635.

George Green—Miller, Mathematician and Physicist

L. J. CHALLIS, *University of Nottingham*

Lawrie Challis is Professor of Low Temperature Physics at the University of Nottingham. His main interests are experimental studies of electrons in semiconductors using very high frequency sound at low temperatures. He is also chairman of the George Green Memorial Fund which has contributed to the restoration of Green's windmill, now open to the public and grinding corn again.

Introduction

George Green (1793–1841) was a pioneer in the application of mathematics to physical problems. He was a miller who lived in Nottingham virtually all his life and had very little formal education until he had completed most of his best work. Partly as a result of his unusual circumstances he received little public recognition in his lifetime, and it was William Thomson (Lord Kelvin) who first recognised the value of his work and gave it wide publicity. His work has had great influence and nowadays he is remembered principally for Green's theorem in vector analysis, Green's tensor (or the Cauchy–Green tensor) in elasticity theory and above all for Green's functions for solving differential equations. The Green's function technique has been very widely applied to equations arising in classical physics and engineering and in recent years has been adapted to quantum mechanical problems in areas as diverse as nuclear physics, quantum electrodynamics and superconductivity. Yet until quite recently Green was a shadowy figure whose name was practically unknown to most non-scientists even in his home town. All this has now changed and, with the help of many organisations and of interested people from many countries, the City of Nottingham has created a splendid and living memorial to George Green. They have restored his ruined windmill to full working order and built a Science Centre illustrating some of his many interests.

George Green's life

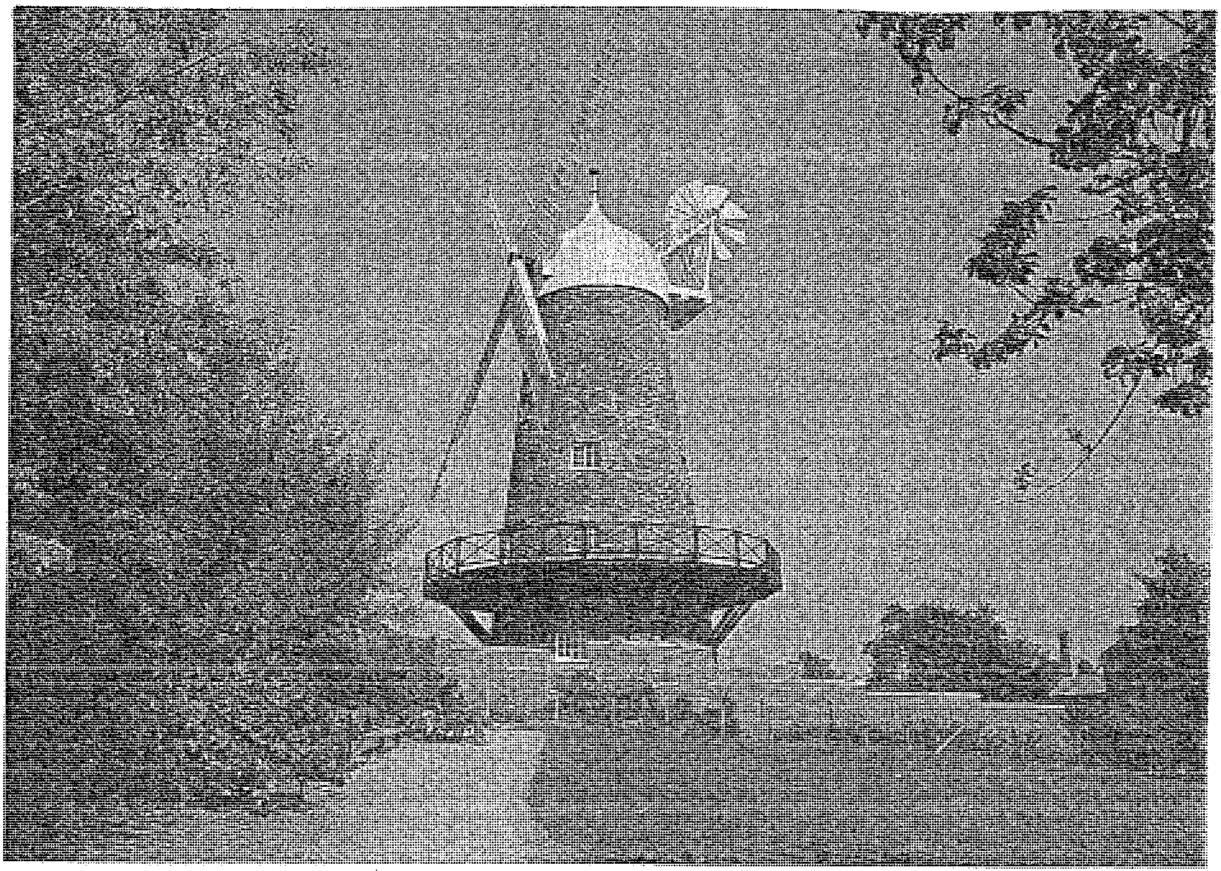
George Green was born in Nottingham on 13 July 1793. For generations his ancestors had been farmers in the village of Saxondale just a few miles from Nottingham, but his father, the youngest of three sons, had been sent there in 1774 to be apprenticed to a baker in Nottingham. In time he bought his own bakery and prospered, acquiring both land and property, which he rented out, as well as a warehouse on the banks of the River Lean which he used to store grain before sending it to be milled into flour for his bakery. When George was eight, he was sent to Robert Goodacre's Academy. His schooling lasted only four terms, however, and then he left to help in the bakery. He was lucky in that his father sent him to that particular school, as Robert Goodacre was an enthusiastic science teacher. Indeed, later in his life, Goodacre became a popular speaker on astronomy, lecturing throughout the British Isles and America. So George Green would have acquired a taste for science, although it is doubtful whether his teacher could have stretched him very far in mathematics; Goodacre had had no formal training and had been apprenticed to a tailor before becoming a schoolmaster.

So at the age of nine, George Green had received all the formal education he was going to get until he was 40! There were of course bookshops in Nottingham where he could buy textbooks and encylopedias, but there were as yet no libraries. It is possible that he may have received some guidance in his reading from one of the graduate mathematicians living in Nottingham.

When Green was 14, his father built a windmill at Sneinton, then a separate village a mile or so from Nottingham. It was a fine five-storey brick tower mill with stables for eight horses and storage for hay and corn. Milling was a skilled trade and he employed a foreman-manager, William Smith, who lived in a cottage attached to the side of the mill. The mill could not easily be worked single-handed; George helped William Smith, and so learned to operate the mill himself. This must have been an exciting change from the bakery for a boy of 14, and it would have been a strenuous, mostly outdoor life. When there was wind enough, he would have worked long hours even at night trimming the sails, maintaining the inflow of grain and taking away the milled flour or cattle fodder. As the Greens were still living in Nottingham, it seems likely that George might have stayed overnight with the Smiths or perhaps slept in the mill rather than walking back through the dark and probably dangerous streets of overcrowded Nottingham. It seems likely too that during this time he would have spent some of the calm days studying mathematics while waiting for the wind to come. Certainly his youngest daughter, Clara, who lived until 1919, told Professor Granger of University College, Nottingham, that her father used the top floor of the mill as a study.

When George was 24, he and his parents moved to a five-bedroomed house they built next to the mill and a few years later he joined the recently-opened Nottingham Subscription Library. This soon became the centre of intellectual life in Nottingham. It contained a modest collection of mathematical and scientific textbooks, and, of great importance, it took the important British scientific journals. These usually also included the titles and abstracts of papers from foreign journals, so that Green would have been able to follow what was being done elsewhere. In principle, he could then have written to the authors asking for copies of their papers.

In 1828, at the age of 35, George Green published his first paper 'An Essay on the Application of Mathematical Analysis to the Theories of



Green's Mill and Science Centre at Sneinton, Nottingham

Electricity and Magnetism'. It was a major work of striking originality. He invented completely new mathematical techniques to solve the problems that arose in the analysis and it would have had an immediate and profound effect had it been read by others working in the field. Tragically, it did not have this effect until some years after his death. He was advised that as he had had no formal training and his social position was modest, he could not presume to send his paper to a scientific journal! So instead he had it printed privately in Nottingham and presumably distributed a few copies to other mathematicians and physicists working in Britain. It had virtually no impact; hardly anyone in Britain had worked in this field for 50 years. British mathematicians were interested in mechanics, optics, astronomy, planetary motion and hydrodynamics; Green's inspiration came from France, from Laplace and Poisson, but nobody there appears to have seen his paper. This lack of response must have depressed Green, but he soon started work on a second paper. He received valuable encouragement from Sir Edward Bromhead, an influential Cambridge mathematics graduate who lived in Lincolnshire and clearly realised Green's exceptional ability. Green turned to areas of much more interest to British mathematical physicists and, with Bromhead's connections as his passport, he started to publish papers in the scientific journals. His family life also changed considerably about this time, when his father died. His mother had died some years before, so Green became a rather wealthy man. He stopped milling, leasing out his mill, and then in 1833, at the age of 40, he realised what may have been his ambition for many years. With Bromhead's help he entered Cambridge as an undergraduate to read for a degree in mathematics. He took his degree in 1837 and shortly afterwards was elected to a fellowship at his college, Gonville and Caius.

Sadly, he held the post for only two years before becoming ill and returning to Nottingham where he died in 1841. Sadly too, the full value of his work was not appreciated until after his death. Indeed, the only obituary that appeared was in a local Nottingham paper. It was three sentences long and ends 'Had his life been prolonged, he might have stood eminently high as a mathematician'.

Green's mathematics

Green's mathematics was nearly all devised to solve very general physical problems. His first interest was in electrostatics. The inverse-square law had recently been established experimentally, and he wanted to calculate how this determined the distribution of charge on the surfaces of conductors. He made great use of the electrical potential, and gave it that name, and one of the theorems that he proved in this work became famous and is known as Green's theorem. It relates the properties of mathematical functions at the surfaces of a closed volume to other properties inside. In its usual form, the theorem involves two functions, but it readily simplifies to what is often called the divergence theorem or Gauss's theorem. (Many early textbooks also called this form Green's theorem as well, presumably to emphasise his claims to precedence.)

To illustrate the theorem, we consider gas leaking from holes in the walls of a gas cylinder. The mass leaving per second per unit area equals the product of the density of the gas and its velocity at each hole. So we can find the total loss rate by integrating this over all the holes. (The integral can in fact be carried out over the whole surface since the contribution from the rest is zero.) But this loss rate from the surface must equal the sum of the masses leaving per second from all the small volume elements $\mathrm{d}V$ inside the surface and this can be found by integrating a particular function over the whole volume V. The function is the result of a differential operator ∇ called the divergence acting on the product of density and velocity of the gas at the element $\mathrm{d}V$. The theorem relating the integral over the surface to the integral over the volume inside is useful in many branches of physics. For example, in electrostatics, a development closely related to this links the electric flux leaving a surface to the total charge inside it.

Another powerful technique invented by Green is used for solving differential equations. It involves what are now called Green's functions,

$G(x,x')$. If we have a differential equation $\mathrm{L}y = F(x)$, where L is a linear differential operator, then the solution can be written

$$y(x) = \int_0^x G(x,x')F(x')\,\mathrm{d}x'.$$

To see this, consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + ky = F(x).$$

This can be solved by the standard integrating factor technique to give

$$y = \exp(-kx)\int_0^x \exp(kx')F(x')\,\mathrm{d}x' = \int_0^x \exp[-k(x-x')]F(x')\,\mathrm{d}x'$$

so that $G(x,x') = \exp[-k(x-x')]$.

To understand the meaning of this in a physical situation, consider the motion of a unit mass (initially at rest) following the application of a time-varying force $F(t)$. If the motion is damped by a force $-kv$ then, from Newton's law,

$$\frac{\mathrm{d}v}{\mathrm{d}t} + kv = F(t)$$

with solution

$$v(t) = \int_0^t \exp[-k(t-t')]F(t')\,\mathrm{d}t'$$

as before. This can be interpreted by visualising the time-varying force $F(t)$ as a rapid sequence of sharp blows each acting for a short time $\mathrm{d}t'$ and causing a change of momentum or impulse $F(t')\,\mathrm{d}t'$. Thus the velocity $v(t)$ at time t is the sum (or integral) of the effects of all blows from $t' = 0$ to $t' = t$. The velocity at time t due to a single impulse (with unit change of momentum) applied at time t' is called the Green's function

$$G(t,t') = \exp[-k(t-t')].$$

This technique may be applied to other more complicated systems. In an electrical circuit the Green's function is the current due to an applied voltage pulse. In electrostatics the Green's function is the potential due to a charge applied at a particular point in space. In general the Green's function is the response of a system to a stimulus applied at a particular point in space or time. This concept has been readily adapted to quantum physics where the applied stimulus is the injection of a quantum of energy. It is in the quantum domain that the application of Green's functions to physical problems has grown most spectacularly in the past few decades.

Green also did very original work on elasticity, where he is remembered by Green's tensor. The elastic properties of an isotropic solid are rather simple. If stress is applied, all the strains can be worked out from the magnitude and direction of the stress and just two elastic moduli (the bulk modulus and the rigidity modulus). But in a crystal the elastic properties can vary considerably from one direction to another. Green showed that in the most general case 21 different moduli are needed to describe the strain. He also showed how symmetry can reduce this number. He became involved in this problem because he was interested in the 'aether'. At that time scientists believed that a real medium, the aether, existed everywhere. In outer space it was needed to carry vibrations of the light coming to us from the stars. Fresnel had shown that light was a transverse wave so the aether had to be a solid (!) since gases and liquids could support only longitudinal waves. So Green started to analyse the properties of waves in solids and this brought him immediately to consider their elastic properties. He also calculated how much of a wave was reflected and how much was transmitted at an interface and explained the phenomenon of total internal reflection. In this work he was also the first to write down the principle of the conservation of energy, which had still to be established experimentally. His later work also contained a number of mathematical 'firsts', including for example his work on hydrodynamics where he devised a powerful approximate method for solving differential equations which reappeared over a century later as the Wentzel–Kramers–Brillouin (WKB) method. He was also the first to state Dirichlet's principle, although Riemann gave it the name it usually bears.

Recognition

We have seen that, during his life, very little recognition was paid to Green's exceptional ability. His major work on electricity and magnetism was largely overlooked in Britain and unknown elsewhere. His contributions in other fields which were published between 1835 and 1839 were better known to his contemporaries but their true value was not appreciated until later. His standing is reflected by the lack of any obituary other than the grudging comment in a Nottingham paper which was quoted earlier.

Happily, though, that is not the end of the story. In the year Green died, William Thomson (later Lord Kelvin) went to Cambridge at the age of 17. He already had a mathematics degree from the University of Glasgow, where his father was Professor of Mathematics, and in the same year he published a paper on electrostatics. His interest in the theory of electricity continued and in 1845, just after he had taken his Cambridge degree, he arranged to go to Paris for a few months to work with French scientists active in this field. He had learned of Green's essay but, not surprisingly, had been unable to find a copy in any of the bookshops. Fortunately his tutor, Hopkins, had a copy which Thompson read on the way to Paris. His astonishment at what

Green had accomplished is recalled in a letter he wrote to Sir Joseph Larmor shortly before his death in 1907. In it he describes very vividly how he read the essay on top of a stagecoach and how he showed it to the French mathematicians as soon as he arrived. They then discovered that Green had already done much of what they had thought was their original work!

Thomson was greatly influenced by Green and adopted many of his techniques, as did Stokes. To quote Sir Edmund Whittaker's authoritative history of *Theories of Aether and Electricity*—'It is no exaggeration to describe Green as the real founder of that "Cambridge school" of natural philosophers of whom Kelvin, Stokes, Lord Rayleigh and Clerk Maxwell were the most illustrious members in the latter half of the nineteenth century'. The significance of his work on elasticity is described in Love's *Mathematical Theory of Elasticity* which speaks of 'the revolution which Green effected in the elements of the theory'.

So practising scientists have no doubt of the importance of Green's contribution. But what of the world outside? Even in Nottingham Green was an obscure figure until recently, despite attempts in the 1920s to learn more about him and efforts in the 1930s by the British Association which led to his grave being restored. The brick tower of his windmill still stood in Sneinton, though it was a sad sight: it had been totally burned out following a fire in 1947 during a brief existence as a polish factory. The wooden sails and balcony had gone long before, although in 1921 the then owner, Oliver Hind, had clad the rotting wooden cap (roof) with copper to make it watertight. At about the same time, the Holbrook bequest had placed a plaque on the side of the mill to record its association with Green, although sadly this disappeared in 1969. The family house still stands and, thanks to the present owner, this also has a plaque.

In 1974, there was a rumour that the mill might be knocked down to make way for a by-pass. The timing could not have been less appropriate. Green's functions were being very widely used throughout the world, and this was hardly the moment for such destruction. So the George Green Memorial Fund was created with the aim of restoring the mill to provide a living and educational memorial to George Green. The City of Nottingham was immensely supportive of the idea and finally in July 1985 the mill was 'opened' together with a Science Centre built on the foundations of the stables and storage areas. It is now grinding corn under the direction of the miller, David Bent. The Science Centre which was opened appropriately by Sir Sam Edwards, Cavendish Professor of Physics at Cambridge and Fellow of Gonville and Caius, contains a number of working models illustrating Green's interest in electricity and magnetism, optics and elasticity. Green's theorem is illustrated by a fountain sculpted following a national competition, by Mr Ron Haselden, and Green's functions are presently illustrated by a computer game (although suggestions for a working exhibit would be gratefully received). The mill and science centre attract large numbers of visitors including school-parties and is a splendid sight on the Nottingham skyline.

Many people and organisations have contributed to this project. It was effectively launched in 1974 by a telegram to the Lord Mayor of Nottingham from the 500 delegates at an International Physics Conference in Budapest and it was appropriate that a telegram of thanks could be sent from a similar conference, also in Budapest, in the autumn of 1985. The national scientific societies have been very supportive and the Institute of Physics can be singled out for its generous donation towards the scientific exhibits. Altogether the George Green Memorial Fund raised nearly £40 000 towards the project, including the purchase of the mill in 1979, but the major contributor by far was the City of Nottingham. The success of these efforts and the publicity the project has received throughout the world has ensured that George Green has at last had the recognition he was so sadly denied in his lifetime.

Acknowledgements

The information of Green's life comes with many thanks from the writings of the late Gwynneth Green, David Phillips, Freda Wilkins-Jones and Mary Cannell. I am also very grateful to my colleagues, particularly Fred Sheard, for discussions on Green's contribution to science.

References and further information

Green's Mill and Science Centre is open, without charge, from Tuesday to Sunday. Further details can be obtained from Mr D. Plowman, Keeper, Green's Mill and Science Centre, Belvoir Hill, Sneinton, Nottingham, NG2 4LF. Telephone (0602) 503635.

Further information about Green's life may be found in *George Green, Miller, Snienton*, ed. D. Phillips (1976). It is now out of print but available in a number of libraries. A new book on Green by D. M. Cannell should be available within the next year. Further information about his mathematics appears in an article 'The Work, and Significance of George Green, the miller mathematician', by J. E. G. Farina, *Bulletin of the Institute of Mathematics and its Applications* **12** (1976) , 98–105.

A handsome hardback facsimile edition of Green's essay may be purchased from Professor Challis, Physics Department, Nottingham University, Nottingham, NG7 2RD, price £12, including postage (cheque to George Green Memorial Fund). The edition is one of a limited edition of 1000 prepared in 1958 for Professor Ekelof, Chalmers Institute, Gothenburg, who generously donated copies to the Fund.

1988

Here is our annual puzzle. The aim is to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\surd$, ! and concatenation (i.e. forming 19 from 1 and 9, for example). Thus

$$1 = -1+\sqrt{9}-(8\div 8),$$

but $1 = 1^9\times(8\div 8)$ is not allowed. With the help from a reader, Eddie Kent, the editorial office managed all but eleven of the numbers. See how you get on!

Computer Column

MIKE PIFF

Predator and prey

Volterra's model for the population x of a prey and y of a predator at time t is

$$\dot{x} = Ax-Cxy-Ex^2, \qquad \dot{y} = -By+Dxy-Fy^2.$$

With $C = D = 0$, we have independent populations, the prey rising to a stable population and the predator dying out. With C and D non-zero, but $E = F = 0$, the populations interact in a fascinating way. Depending upon the initial population of predator and prey, over time, both populations oscillate regularly, with a different-sized oscillation for each initial population. Finally, with E and F also non-zero (insert line 15 in the program), there is a steady distribution of predator and prey, reached from any initial state of the two populations.

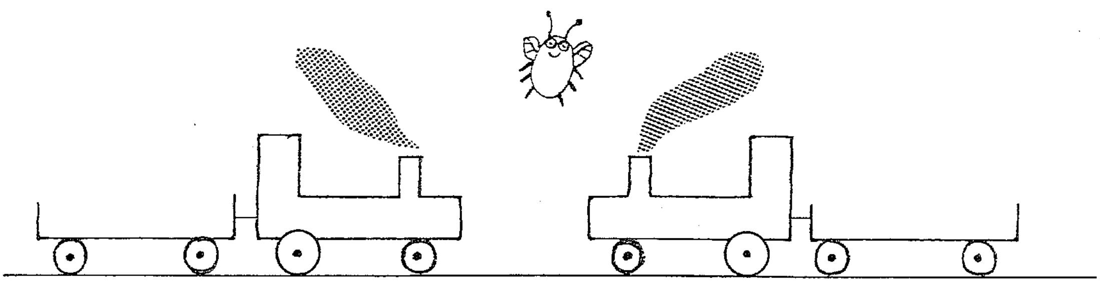
These effects can be simulated easily with the BBC BASIC program overleaf. You might also like to make the model more realistic by allowing x and y to take only integer values: try inserting

```
185   X = INT(X):  Y = INT(Y)
```

to see the effect of this. Also, try changing the parameters in line 10 to various different values, but do not change their signs or the model will not be of a predator and prey! For example, $C < 0$ would represent cooperating species.

```
10 A=500:C=2:B=900:D=3:E=0:F=0
15 REM E=1E-1:F=1E-1
20 MODE1
30 INPUT "X=",X
40 IF X<0 THEN X=0
50 INPUT "Y=",Y
60 IF Y<0 THEN Y=0
70 PROCINIT
80 MOVE X,Y
90 PROCBLOB(X,Y)
100 PROCDISPLAY(X,Y)
110 DT=1E-4
120 GCOL0,2
130 OLDTIME=TIME
140 REPEAT
150 XOLD=X:YOLD=Y
160 PROCRK
170 PROCBLOB(XOLD,YOLD)
180 PROCDISPLAY(XOLD,YOLD)
190 DRAW X,Y
200 PROCBLOB(X,Y)
210 PROCDISPLAY(X,Y)
220 REPEAT:UNTIL TIME>(OLDTIME+50)
230 OLDTIME=TIME
240 UNTIL FALSE
250 END
260 DEF FNF(X,Y)=X*(A-C*Y-E*X)
270 DEF FNG(X,Y)=Y*(-B+D*X-F*Y)
280 DEF PROCRK
290 LOCAL K1,K2,K3,K4,L1,L2,L3,L4
300 K1=DT*FNF(X,Y)
310 L1=DT*FNG(X,Y)
320 K2=DT*FNF(X+K1/2,Y+L1/2)
330 L2=DT*FNG(X+K1/2,Y+L1/2)
340 K3=DT*FNF(X+K2/2,Y+L2/2)
350 L3=DT*FNG(X+K2/2,Y+L2/2)
360 K4=DT*FNF(X+K3,Y+L3)
370 L4=DT*FNG(X+K3,Y+L3)
380 X=X+(K1+K4+2*(K2+K3))/6
390 Y=Y+(L1+L4+2*(L2+L3))/6
400 ENDPROC
410 DEF PROCBLOB(X,Y)
420 GCOL3,3
430 MOVE X-16,Y+16
440 PRINT "+"
450 MOVE X,Y
460 GCOL0,2
470 ENDPROC
480 DEF PROCINIT
490 LOCAL I,I$
500 VDU19,1,1,0,0,0
510 VDU19,3,6,0,0,0
520 VDU23,1,0;0;0;0;
530 VDU5:CLG
540 VDU29,100;100;
550 GCOL0,1
560 MOVE -100,0:DRAW 1200,0:MOVE 0,-100:DRAW 0,1000
570 FOR I=100 TO 900 STEP 200
580 I$=STR$(I)
590 MOVE I,-10:DRAW I,10:MOVE I,0:PRINT I$
600 MOVE -10,I:DRAW 10,I:MOVE 0,I:PRINT I$
610 NEXT I
620 MOVE 800,-50:PRINT "x":MOVE -50,800:PRINT "y"
630 ENDPROC
640 DEF PROCDISPLAY(X,Y)
650 LOCAL I$
660 I$="X="+STR$(INT(X))+" Y="+STR$(INT(Y))+"      "
670 MOVE 700,800
680 GCOL3,1
690 PRINT I$
700 MOVE X,Y
710 GCOL0,2
720 ENDPROC
```

The Trains and the Fly



Two trains A and B are moving at 40 km/h and 60 km/h, respectively, towards one another from 100 km apart on the same track. A fly starts at train A and flies to B, then back to A, etc. How far does he fly before the trains collide if he flies at 50 km/h?

David Singmaster
Polytechnic of the South Bank,
London SE1 0AA.

Letters to the Editor

Dear Editor,

Smith numbers

Please allow me to comment on the note on Smith numbers on page 71 of Volume 19 Number 3 of *Mathematical Spectrum*.

A. Wilansky and all other writers on the subject define a Smith number, or a Smith, as a *composite* number the sum of whose digits is the same as the sum of the digits in its representation as a product of primes. Of all natural numbers less than 1000 000, only 29 928 of them, or 3%, are Smiths.

In answer to your correspondent's first question, W. L. McDaniel has shown that there is indeed an infinity of Smiths. In fact, Smith numbers that are multiples of any prime exist.

The largest known Smith, which I have recently computed, has 10 694 685 decimal digits; it is

$$(10^{1031}-1)(10^{4594}+3\times 10^{2297}+1)^{1476}\times 10^{3913\,210}.$$

The middle factor, whose exponent is 1476, is a palindromic prime recently discovered by H. Dubner. I shall be glad to explain to interested readers why it is a Smith and how it can be determined without examining thousands of digits.

With regard to the second question, I am not aware of any past or current research on perfect Smith numbers, although related questions about which powers of 2 and which Mersenne numbers are Smiths have arisen. P. Costello of Eastern Kentucky University has computed Smith numbers of the form $a\times M\times 10^b$, where M is any one of the known Mersenne primes.

Yours sincerely,

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Delray Beach, Florida 33484,
USA

References

1. S. Yates, Special sets of Smith numbers, *Mathematics Magazine*, Vol. 59 No. 5 (1986), 293–296.
2. S. Yates, Smith numbers congruent to 4 (mod 9), *Journal of Recreational Mathematics*, Vol. 19 No. 2 (1987), 139–141.
3. S. Yates, How odd the Smiths are!, to appear.

These articles contain references to other articles on Smith numbers.

Dear Editor,

$$x^n + y^n = z^m$$

As a result of Joseph McLean's correspondence (*Mathematical Spectrum* Volume 18 Number 3 page 88 and Volume 19 Number 3 page 91), I have been looking for primitive solutions to the equation $x^n + y^n = z^m$ for positive integers x, y, z, n and m. A primitive solution is one where $\mathrm{hcf}(x, y, z) = 1$.

If $n = m = 2$, then x, y and z form a Pythagorean triple. Let s and t be any positive integers such that $\mathrm{hcf}(s, t) = 1$ and one of s and t is even and one is odd. Then $x = 2st$, $y = |s^2 - t^2|$ and $z = s^2 + t^2$ will generate all primitive solutions. (See reference 1 page 245.)

If $n = 2$ and $m = 3$, then let s and t be as before. It seems that $x = |s^3 - 3st^2|$, $y = |t^3 - 3s^2t|$ and $z = s^2 + t^2$ generates all primitive solutions to $x^2 + y^2 = z^3$. It is easy to verify that this does generate solutions and easy to prove that they are primitive. I have been unable to prove that it generates all solutions.

If $n = 2$ and $m = 4$, then let s and t be as before. I conjecture that $x = |4s^3t - 4st^3|$, $y = |s^4 + t^4 - 6s^2t^2|$ and $z = s^2 + t^2$ generates all primitive solutions to $x^2 + y^2 = z^4$. Again, it is easy to verify that it generates primitive solutions, but I cannot prove that it generates all solutions.

If $n = 2$ and m is any positive integer, I conjecture that the following procedure will create an algorithm that will generate all primitive solutions to $x^2 + y^2 = z^m$:

(1) Write down the binomial expansion of $(s + t)^m$ with the terms written in the conventional order.

(2) Let $x = |$(the first term $-$ the third term $+$ the fifth term $- \ldots$)$|$.

(3) Let $y = |$(the second term $-$ the fourth term $+$ the sixth term $- \ldots$)$|$.

(4) Let $z = s^2 + t^2$.

Then s and t as before will generate all primitive solutions.

Another way of looking at the above is to consider the complex number $w = s + it$, where s and t are as above. Then $z = |w|^2$, $x =$ the real part of w^m and $y =$ the imaginary part of w^m generates all primitive solutions.

I have conducted a fairly extensive computer search with x, y and z up to 65 000 and failed to find any counterexamples to the above.

Table

Integer solutions to $x^3 + y^3 = z^2$ (with $\mathrm{hcf}(x, y, z) = 1$)

x	y	z
2	1	3
37	11	228
65	56	671
112	57	1 261
312	217	6 371
433	242	9 765
877	851	35 928
1064	305	35 113
1177	23	40 380
1201	122	41 643

I have also found a number of solutions to $x^3+y^3=z^2$. These do not fall into any pattern that I can recognise, but I am still trying (see the table).

However, the most surprising result of my computer searches is that I have failed to find any primitive solutions to the equation when $m>2$ and $n>2$. Could it be that there are none? If this is so, then Fermat's conjecture follows as the special case $n=m$, since the existence of a non-primitive solution implies the existence of a primitive one.

Yours sincerely,
MIKE SWAIN
157 Old Woosehill Lane,
Wokingham,
Berkshire, RG11 2UN

Some of the questions raised by Mike Swain in his interesting letter are answered in reference 2—see pages 122 and 235. Ed.

References

1. D. M. Burton, *Elementary Number Theory*, Revised printing, (Allyn and Bacon, 1980).
2. L. J. Mordell, *Diophantine Equations* (Academic Press, 1969).

Dear Editor,

Approximating $n^{1/3}$

In his article 'Approximating $\sqrt{n}$' in *Mathematical Spectrum* Volume 19 Number 2 page 40, Simon Johnson suggested that the iterative formula

$$z \to I(z) = \frac{z+n}{z^2+1}$$

applied to the rational number z (there written a/b) converges to $n^{1/3}$. We have discovered that this iteration converges (necessarily to $n^{1/3}$) if $0<n\leqslant 2^{3/2}$ whatever rational number we begin with, but that, if $n>2^{3/2}$, it only converges for certain unusual initial values of z.

The original iteration was proposed as a method of approximating cube roots of integers. There is not much scope for this because $3>2^{3/2}$. The convergence is also rather slow. A better method might be to replace the n in the expression for $I(z)$ by $1/n$. The result is an iterative formula that converges to $n^{-1/3}$ for all $n\geqslant 2^{-3/2}$. This convergence appears to be more rapid. Having approximated $n^{-1/3}$, we can reciprocate to give an approximation to $n^{1/3}$.

In references 1 and 2, Graham Hoare and Christopher Bradley have used the iteration

$$J(z) = \frac{z(z^3+2n)}{2z^3+n}.$$

This converges rapidly to $n^{1/3}$, provided that the initial z is a sufficiently good approximation to $n^{1/3}$, and this generalizes to $n^{1/p}$.

As an example, consider $10^{1/3}$. If we put $n = \frac{1}{10}$ and take $\frac{1}{2}$ as initial value, then $I^6(z) = 0.464\,50$ and $1/I^6(z) = 2.152\,85$. A calculator gives $10^{1/3}$ as $2.154\,43$, so we have two-decimal-place accuracy after six iterations. If we take 2 as initial value and use J, we obtain $J^2(z) = 2.154\,43$, and we have five-decimal-place accuracy after two iterations.

Yours sincerely,

N. M. IRVING, I. M. RICHARDS
AND K. SOWLEY, Weymouth College.

References

1. G. Hoare, Rational approximations to $a^{1/p}$—an investigation, *Mathematical Gazette* **67** (1983), 223–226.
2. C. Bradley, A theorem concerning rational approximation to $a^{1/p}$, *Mathematical Gazette* **67** (1983), 226–227.

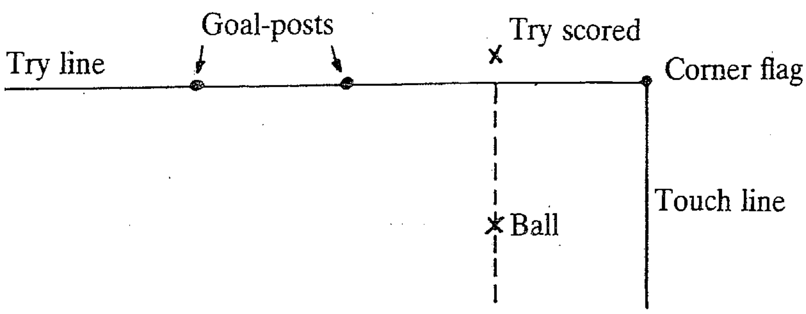
Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

20.5. (Submitted by Seung Jin Bang, Seoul, Korea)
When does the polynomial ax^2+bx+c, with a, b and c real numbers and $a > 0$, have at least one root between 0 and 1?

20.6. (Submitted by Arthur Pounder, Manchester)
In the sport of rugby, when a try has been scored a conversion is attempted by a goal-kicker who must score by kicking the ball over a bar between two goal-posts. Before attempting the conversion, the kicker must place the ball at any point in a line parallel to the touchline passing through the point where the try was touched down, i.e. the ball must be placed somewhere along the dotted line shown in the figure. Given that a try has been scored outside the goal-posts, where should the kicker place the ball?



20.7. (Submitted by Malcolm Smithers, a student at the Open University) Denote by $C(n, k)$ the list of natural numbers which can be written in binary form using k 1s and $n-k$ 0s, arranged in increasing order. In $C(14, 8)$, where does 10101110101001 occur in the list, and what is the 19 187th term in $C(19, 11)$?

20.8. (Submitted by Stanley Rabinowitz, Massachusetts, USA) Let P be a point on side BC of triangle ABC. If $(AP)^2 = (AB)(AC)-(PB)(PC)$, prove that either $AB = AC$ or that AP bisects angle BAC.
(Readers might also like to consider the converse—Ed.)

Solutions to Problems in Volume 19 Number 3

19.8 What are the last two digits of 7^{1987}?

Solution by Amites Sarkar (aged 14, Winchester College)

$7^4 = 2401 \equiv 1 \pmod{100}$, so $7^{4n} \equiv 1 \pmod{100}$. Also $7^3 = 343 \equiv 43 \pmod{100}$. Now $7^{1987} = 7^{4\times 496}7^3 \equiv 1\times 43 \pmod{100}$, so the last two digits of 7^{1987} are 43.

Also solved by Jonathan Green and Ian Moss (City of Stoke-on-Trent Sixth Form College), Nicholas O'Shea (Gresham's School, Holt), Keith Gordon (The Haberdashers' Aske's School, Elstree), Eddie Cheng (Memorial University of Newfoundland), Jean Corriveau (McMaster University, Canada), Adrian Hill (Trinity College, Cambridge) and William Croghan (Mount Mary College, Cedar Rapids, Iowa, USA).

19.9 Show that $1/\pi = \frac{1}{4}\tan\frac{1}{4}\pi + \frac{1}{8}\tan\frac{1}{8}\pi + \frac{1}{16}\tan\frac{1}{16}\pi + \dots$.

Solution by Gregory Economides (Royal Grammar School, Newcastle upon Tyne)

We have

$$\cot\theta - 2\cot 2\theta = \cot\theta - \frac{2(\cot^2\theta - 1)}{2\cot\theta} = \tan\theta.$$

Put

$$S_n = \sum_{r=0}^{n} \frac{1}{2^r}\tan\frac{\theta}{2^r}.$$

Then

$$\begin{aligned} S_n &= (\cot\theta - 2\cot 2\theta) + \frac{1}{2}\left(\cot\frac{\theta}{2} - 2\cot\theta\right) + \frac{1}{2^2}\left(\cot\frac{\theta}{2^2} - 2\cot\frac{\theta}{2}\right) \\ &\qquad + \dots + \frac{1}{2^n}\left(\cot\frac{\theta}{2^n} - 2\cot\frac{\theta}{2^{n-1}}\right) \\ &= \frac{1}{2^n}\cot\frac{\theta}{2^n} - 2\cot 2\theta. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} S_n = \frac{1}{\theta}\lim_{n\to\infty}\frac{\dfrac{\theta}{2^n}}{\sin\dfrac{\theta}{2^n}}\cos\frac{\theta}{2^n} - 2\cot 2\theta = \frac{1}{\theta} - 2\cot 2\theta.$$

Hence

$$\sum_{r=0}^{\infty} \frac{1}{2^r} \tan\left(\frac{1}{2^r}\frac{\pi}{4}\right) = \frac{4}{\pi} - 2\cot\frac{\pi}{2} = \frac{4}{\pi}.$$

If we divide by 4, the result follows.

Also solved by Adrian Hill, Trinity College, Cambridge.

A. J. Douglas and G. T. Vickers, who originally proposed the problem, tell us that the first 15 terms in the sum give 9-figure accuracy to $1/\pi$.

19.10 Prove that $[x]+[2x+2y] \geqslant [2x]+[y]+[x+y]$, where $[a]$ denotes the integer part of the real number a.

Solution by Jean Corriveau (McMaster University, Canada)

Put $x = k+\alpha$ and $y = m+\beta$, where $k, m \in \mathbb{Z}$ and $0 \leqslant \alpha, \beta < 1$. Then

$$\begin{aligned}
&[x]+[2x+2y]-[2x]-[y]-[x+y] \\
&\quad = k+2k+2m+[2\alpha+2\beta]-2k-[2\alpha]-m-k-m-[\alpha+\beta] \\
&\quad = [2\alpha+2\beta]-[2\alpha]-[\alpha+\beta]. \qquad (*)
\end{aligned}$$

Now $[\alpha+\beta]$ is 0 or 1. If $[\alpha+\beta] = 0$, then (*) is non-negative. If $[\alpha+\beta] = 1$, then $\alpha+\beta \geqslant 1$, so that $[2\alpha+2\beta] \geqslant 2$ and $[2\alpha] \leqslant 1$, and again (*) is non-negative.

Also solved by Keith Gordon (The Haberdashers' Aske's School, Elstree), Gregory Economides (Royal Grammar School, Newcastle upon Tyne), Adrian Hill (Trinity College, Cambridge) and Amites Sarkar (Winchester College).

19.11 Let a, b and c be real numbers with $0 < a, b, c < \frac{1}{2}$ and $a+b+c = 1$. Prove that

$$\sqrt{a(1-2a)}+\sqrt{b(1-2b)} > \sqrt{c(1-2c)}.$$

Solution by Gregory Economides (Royal Grammar School, Newcastle upon Tyne)

We first remark that, for positive numbers x and y, $\sqrt{x}+\sqrt{y} > \sqrt{x+y}$. This is so because

$$(\sqrt{x}+\sqrt{y})^2 = x+2\sqrt{xy}+y > x+y.$$

Now $a(1-2a), b(1-2b), c(1-2c) > 0$, so that

$$\begin{aligned}
\sqrt{a(1-2a)}+\sqrt{b(1-2b)} &> \sqrt{a(1-2a)+b(1-2b)} \\
&= \sqrt{a+b-2[(a+b)^2-2ab]} \\
&= \sqrt{1-c-2[(1-c)^2-2ab]} \\
&= \sqrt{c(1-2c)+2c-1+4ab} \\
&= \sqrt{c(1-2c)+2[1-(a+b)]-1+4ab} \\
&= \sqrt{c(1-2c)+(1-2a)(1-2b)} \\
&> \sqrt{c(1-2c)}.
\end{aligned}$$

Also solved by Jean Corriveau (McMaster University, Canada) and Adrian Hill (Trinity College, Cambridge).

19.12 For which positive integers n is the standard deviation of n consecutive integers rational?

Solution This is essentially the solution sent in by Amites Sarkar (Winchester College), Keith Gordon (The Haberdashers' Aske's School, Elstree), Gregory Economides (Royal Grammar School, Newcastle upon Tyne) and Adrian Hill (Trinity College, Cambridge).

The mean, μ, of the n consecutive integers $m+1, m+2, \ldots, m+n$ is given by

$$\mu = \frac{1}{n}\sum_{r=1}^{n}(m+r) = m+\tfrac{1}{2}(n+1),$$

and the standard deviation, σ, is given by

$$\sigma^2 = \frac{1}{n}\sum_{r=1}^{n}(m+r-\mu)^2 = \tfrac{1}{12}(n^2-1).$$

Thus $3t^2 = n^2-1$, where $t = 2\sigma$. Write $t = p/q$, where p and q are positive integers. Then $3p^2 = q^2(n^2-1)$. Since $3p^2$ has an odd number of threes in its prime factorization, so too has $q^2(n^2-1)$ and therefore 3 divides n^2-1. Hence q divides p and t is an integer. Thus we require solutions in integers of the 'Pell equation'

$$n^2-3t^2 = 1.$$

The smallest positive solution is given by $n = 2$, $t = 1$, and the required integers n are given by

$$n = \tfrac{1}{2}[(2+\sqrt{3})^s+(2-\sqrt{3})^s]$$

for $s = 0, 1, 2, \ldots$. The first nine integers are 1, 2, 7, 26, 97, 362, 1353, 5042 and 18 817.

Book Reviews

Littlewood's Miscellany. Edited by BÉLA BOLLOBÁS. Cambridge University Press, Cambridge. Pp. 200. £17·50 hardback, £5·95 paperback.

John Edensor Littlewood (1885–1977) was undoubtedly the greatest analyst of his time. Together with G. H. Hardy (1877–1947), his collaborator for 35 years, he created the modern British school of analysis which, between the wars, was pre-eminent in the world. He was less flamboyant than Hardy, but his personality was equally forceful and, moreover, he possessed a most extraordinary vitality, physical as well as mental. He was still rock climbing at the age of 80 and his last research paper appeared when he was 87. Both Littlewood's father and paternal grandfather read mathematics at Cambridge. His grandfather was bracketed 35th Wrangler†, his father was ninth Wrangler and he himself was bracketed Senior Wrangler (but the

†A Wrangler is a student in the First Class of the Cambridge Mathematical Tripos. Until 1910 Wranglers were listed in order of merit.

sequence was broken with a non-mathematical daughter). Soon after graduating Littlewood went to Manchester for three years as a lecturer. He returned to Cambridge in 1910 and there he remained until he died except for a period during the First World War when he was in the Royal Artillery and worked in ballistics. A life as settled as this might appear to offer little scope for reminiscences, but Littlewood's mind was so sharp and his interests were so wide that his observations on a multitude of topics make fascinating reading. These were added to a very large collection of mathematical material ranging from penetrating remarks a few lines long to full-scale essays, to form the forerunner of this book, *A Mathematician's Miscellany* (Methuen, 1953). The present book contains, in addition, everything that Littlewood collected for a second edition and also a long biographical essay by Béla Bollobás (Reader in Pure Mathematics in Cambridge) who, with his wife, was very close to Littlewood during the last eight years of his life.

The mathematical part of the book begins with a chapter entitled 'Mathematics with Minimum Raw Material'. Some of this needs nothing but logical reasoning. For instance: Three ladies in a railway carriage all have dirty faces and are laughing. Suddenly one realises the *she* must have a dirty face. Why? At the other extreme are theorems involving series and integrals.

A chapter on the Mathematical Tripos debunks some of the questions that had been set over the years. It also tells us that there used to be 18(!) three-hour papers and that, one year in the 1880s, the Senior Wrangler got 16 368 marks (just under half the maximum) and the bottom candidate (number ninety odd) 247. This chapter might well be read in conjunction with that on his own mathematical education from schoolboy to assured researcher.

In the chapter on large numbers Littlewood tackles the problem of the succinct representation of large numbers by defining the very quickly increasing sequence

$$N_1 = 10^{10}, \quad N_2 = 10^{10^{10}}, \quad \ldots, \quad N_n = 10^{N_{n-1}}, \quad \ldots$$

He also denotes, for instance, $10^{10^{4.7}}$ by $N_{1.47}$. With this notation he shows that there are not more than $N_{2.185}$ games of chess. However, he also dispels some myths. Thus, in 1947, the popular press went into spasms of excitement on learning that two batsmen had each scored 1111 runs for an average of 44.44. Littlewood demonstrates that the chance of this happening is 1 in 10^4 and comments 'a modest degree of surprise is legitimate'.

There are hundreds of brief anecdotes and pithy remarks. One of my favourites concerns a report he wrote for the Ballistics Office. The last sentence of his manuscript 'Thus σ should be made as small as possible' was omitted from the printed minute. Instead there was a little speck which turned out to be a tiny letter σ. Now here is a lesson in economics: when Littlewood's college, Trinity, once had some cheapish, not very good wine, it would not sell. So they doubled the price and it was all bought up. As a final example I might mention that, ever since 1953, I have been unable to use a construction such as 'I should like to thank Mr Smith' because Littlewood's rejoinder would be 'Then why not do so?'.

One can open this book almost at random and start reading. The difficulty is stopping.

University of Sheffield — H. BURKILL

The Philosophy of Mathematics: An Introductory Essay. By STEPHAN KOERNER. Dover Publications, New York, 1986. Pp. 198. £5·35.

As the title suggests, this book is about the philosophy of mathematics and is aimed at the non-philosopher who is interested in mathematics.

The author has examined three different philosophies by looking at how each sees the nature of pure mathematics and the relationship between pure and applied mathematics, and how it deals with infinities. The three philosophies he has examined are *logicism*, which asserts that mathematics is a branch of logic; *formalism*, which bases mathematics on the counting of strokes on a piece of paper or the manipulation of strings of symbols; and *intuitionism*, which asserts that the only valid mathematical arguments are those which are in a particular sense intuitive.

The author has decided that each of these three philosophies is unsatisfactory in some way. He concludes that mathematics is neither a branch of logic, nor is it directly related to perception (whether of strings of symbols or of objects in the intuition); though after an act of translation it may be applicable to the world of sense experience. He decides that pure mathematics is autonomous, though he does not say what he thinks it is about. Applied mathematics he believes to be the mathematical discussion of idealizations of real-world situations, and the results of a piece of applied mathematics are to be justified by observation. In his system the counting of strokes on a piece of paper, for example, is applied mathematics.

I am not competent to criticize this book as a piece of philosophy; its stated purpose is to be an introduction, and as such it is admirable. Its style, while a little formal and reminding me of the style of many textbooks on mathematics, is still readable and on the whole clear. This book certainly aroused my interest in the subject. No previous knowledge is assumed: the small amount of actual mathematics is explained, so anyone with a mathematical bent should be able to read this book and enjoy it, though some knowledge of formal logic would be an advantage.

St. Anne's College, Oxford ROBIN KNIGHT

Turtle Geometry. By HAROLD ABELSON and ANDREA DI SESSA. MIT Press, 1980; new paperback edition, 1986. Pp. xx + 477. £10·95.

This book is not for the timid. You will need a lot of ingenuity to translate the programming fragments in a turtle-like language appearing here into working programs, but the experience could be quite rewarding. I feel strongly that a PASCAL or BASIC version of the book would have been more worthwhile. After all, life is short, but not *that* short; even FORTRAN is still around! As it is, the language used is ... well, think of all the worst features of COBOL, ALGOL 68, ADA, BASIC, PASCAL, MIX, WRITTEN ENGLISH, and there you just about have it.

The material covered varies between random motion and the differential geometry of space-time, suggesting a rather ambitious range. True, there is some mathematics there, but any aspiring mathematician would find some of the authors' arguments loose, to say the least. However, the intention of the book is not to lead the student through the usual catalogue of theorems and proofs, but to give him the chance to experiment using the computer, and discover mathematics for himself—in particular, in this book, to discover geometry. He is more likely to learn something about computer-simulated motion from this book, but will have to do most of the programming himself. The mathematics, unfortunately, will just have to be learnt the hard way. Although a computer simulation can sometimes be more useful than a static diagram to illustrate some tricky point, there are limits to what one can do with such simulations, since much of mathematics exists independently of diagrams or animated illustrations.

For those who are not primarily mathematicians, who wish to have a feeling for some of the objects mathematicians have studied, without worrying about the details, and who are reasonably competent programmers, this book might be worth looking at.

University of Sheffield — MIKE PIFF

Topics in Applied Probability. By PETER W. JONES and PETER SMITH. Keele Mathematical Publications, University of Keele, Staffs. ST5 5BG, England. Pp. vi + 124. £2.

This booklet begins with two chapters giving basic ideas on probability and on conditional probability and independence. It continues with a third chapter on some elementary probability models leading to the binomial and Poisson distributions. The final fourth chapter on Markov processes considers the gambler's ruin problem, random walks and simple Markov chains. There are problems included in each chapter, to which answers are provided at the end of the book. A novel feature is the sixteen full programs in BBC BASIC; these are listed at the end of the book, as are also seven reference books. The booklet is written in an easily readable style, and could usefully serve as a first introduction to the study of probability and its applications.

University of California at Santa Barbara — J. GANI

Detailed Solutions to A-level Questions: 1. Pure Mathematics. 2. Applied Mathematics. By D. MURRAY and D. BENJAMIN. Stanley Thornes (Publishers) Ltd, 1986. Pp. v + 126, vi + 122. £2·75 each.

In order to use these solutions it is necessary to buy two books of London University Papers (at £1·25 each), and it seems a pity therefore that the questions are not incorporated. It is not entirely clear how the books are to be used: whether the students are to be given them prior to working out the examination papers for themselves. If so, it will need a very strong-minded candidate not to look at the solutions before a real effort has been made to work the questions.

As would be expected from two experienced mathematics masters, the solutions are in general sound. However, some of them are unnecessarily long for able candidates and rather daunting for the less able. In particular, for questions involving inequalities, an algebraic solution is generally quick and simple, and complete and accurate sketches of functions are unnecessary. Also I do not like the practice of splitting dy and dx when writing out the solution of a differential equation with separable variables since dy/dx is not a fraction.

The two books could prove useful to candidates working alone or where teaching is minimal because of staff shortages.

Formerly of Wycombe Abbey School — M. V. BILSBORROW

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ISSN 0025-5653

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Published by the Applied Probability Trust
Typeset by the University of Nottingham
Printed in England by Galliard (Printers) Ltd, Great Yarmouth