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Relativity: from the Special to the General Theory

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1. Introduction

This year, 1979, is the centenary of the birth of Albert Einstein, the man responsible for two of the most important and spectacular advances in physics. First, in 1905 he propounded the Theory of Special Relativity, abolishing at a stroke the aether and introducing radical new ideas about space and time. Undergraduates now studying physics quickly become familiar with the Lorentz transformation between two observers moving uniformly with respect to each other, with the contraction of moving rods and with the behaviour of moving clocks.

Then a decade later came the second thunderbolt from Einstein in the shape of the Theory of General Relativity, a radically new approach to the study of gravitation. Here perhaps some undergraduates are less sure of their ground when discussing the behaviour of the planet Mercury using general relativity, or when considering the bending of light near the Sun, or when thinking about the red-shift. The derivation of these results from the concepts of a curvature in space-time is mathematically more demanding, and the basic ideas of a curved space-time rather than a flat space-time are more subtle and require more effort and imagination for their assimilation.

The content of this note may perhaps encourage those who feel themselves on firm ground when considering special relativity to appreciate that there is a solid reason to make the jump from a study of flat geometry to a study of curved spaces, and perhaps to follow up by reading some of the many textbooks on general relativity.

Einstein spent the earlier part of his life in Germany and Switzerland before moving to the United States in the politically difficult times of the thirties. He also had a connection with Oxford, and with Christ Church in particular, a fact which is not often recorded, for he was from 1931 until 1936 a Research Student (the College name for what at other colleges would be Research Fellow) and a member of the Governing Body of Christ Church. Although not required to be resident, he did live in the college for a short time and gave lectures on relativity to members of the University. A blackboard on which he wrote has been carefully preserved in the Museum of the History of Science at Oxford. A member of the college in rooms near those occupied by Einstein still recalls his frequent violin-playing in the intervals between his periods of concentration on physics.

In 1934 he wrote to the College suggesting, since he would not be in Oxford for that year, that he was not really entitled to his stipend, and expressing the hope that the College would devote the money towards helping some of his colleagues who were being forced into exile at that time. The College agreed to this and did so for the remainder of the tenure of his Studentship.

While Einstein was at Oxford, an undergraduate who later became a Research Student of Christ Church was detailed by his tutor to entertain Einstein by taking him sailing. On the way back from the river, Einstein's well-known and recognisable figure attracted a natural interest from people, an interest which Einstein modestly put down to the friendly feelings which the English had for a foreigner. His sailing companion still remembers, with a twinge of conscience, how on their return he was reprimanded both by his tutor and by Mrs Einstein for allowing Einstein to get his socks wet.

2. Flat and curved 2-space

The axioms of Euclidean geometry are enormously powerful and yield a treasure trove of interesting mathematical results, achievable either by using axiomatic methods or by employing coordinates. It is a treasure trove which also has immediate application. But it is rewarding to think about the geometry of a space like that of a field or a mountain in the way that a surveyor might approach it. We suppose that he has been equipped with a metre stick and that by laborious trial and error methods he is able to measure the shortest distance between any two recognisable points of the space. He will define the straight line between the two points as the path associated with the shortest distance.

The surveyor might now want to go further and try to set up a coordinate system, attributing to each point P a pair of coordinates (x, y). This he could do by the process illustrated in Figure 1. Having selected an origin O and some base-line OX, he could then, by continuing in this laborious way, seek out the path of shortest distance from a point P to the base-line, obtaining PA, and then measuring x, the distance OA, and y, the distance AP. The point P is then given the coordinates (x, y).

The surveyor, if he were inquisitive, would then go on to measure the distance between the point $P(x, y)$, and some other point $Q(x + dx, y + dy)$ in the neighbourhood of P. This distance ds may or may not be related to dx and dy in the simple manner

$$ds^2 = dx^2 + dy^2.$$

For example, if we had asked the surveyor to map out and make measurements on a sphere of radius R, restricting himself to measurements on the surface of the sphere,

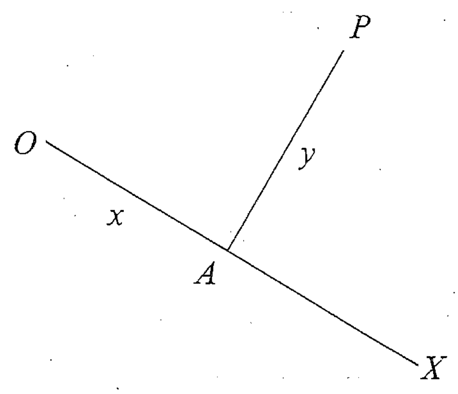

Figure 1. Definition of the coordinates (x, y).

he might have taken an origin on the equator at longitude zero, he might have chosen his base-line OX along the equator and then in that case he would have verified the relationship

$$ds^2 = \cos^2 \frac{y}{R} dx^2 + dy^2,$$

because in this case we can see quickly that x/R is a measure of longitude and y/R is a measure of latitude on the sphere.

The most that the surveyor can expect, as long as the space is not too odd, is that the relationship for the surface which he is given will be

$$ds^2 = E^2(x, y)\, dx^2 + dy^2,$$

where $E(x, y)$ is some function of the coordinates. It is the job of the surveyor to make measurements sufficient to determine this function $E(x, y)$, or in other words to determine the geometry or the metric of the space.

There is no *a priori* reason why the space being investigated should have the very special property $E(x, y) = 1$ everywhere. If it does, then the surveyor will say that the space is flat or Euclidean; if $E(x, y) \neq 1$, then he will say that the space is curved or non-Euclidean.

The same approach, in this open-minded manner, can be made to the space of three dimensions surrounding us, and such measurements show that this space is Euclidean or very nearly so. It is this Euclidean model of 3-space which lies at the basis of Newtonian dynamics, coupled with the Newtonian concept of time. It is this space-time which is the stage for Newtonian dynamics, a stage on which particles and rigid bodies play their part, moving under forces which are postulated (like, for example, the Newtonian force of gravitation). On such a stage, a particle moving under no force proceeds uniformly along the straight lines of the space. The reader who is familiar with the motion of the symmetrical top, moving under no forces about its vertex, may find it amusing to consider the motion of tiny spinning discs, moving freely on, but constrained to, the smooth surface of a sphere. It can be quickly shown that such 'particles' do not move on the 'straight lines' or great circles or geodesies of the sphere, although they still move on circles.

The model of Euclidean 3-space and Newtonian time is of great value for a very large range of dynamical phenomena.

3. Special relativity

The Euclidean model of space-time was shattered by Einstein in 1905 when he propounded the Theory of Special Relativity. Space-time for him became the space of events, parametrised by four coordinates x, y, z and t, with a metric given by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

where c is the velocity of light, a constant for all observers moving uniformly relatively to each other equipped with similar metre sticks and clocks. Not only now is the velocity of light the same for all observers, but all the laws of physics take the same form for all such observers, and there is no way in which any observer can be singled out as a special case and thought to be, in some sense, at rest.

This new space-time model became the stage on which particles and the electromagnetic field played their parts. The Lorentz transformation is the connecting link between the coordinates of an event measured by one observer and the coordinates of the same event measured by another observer. The space-time interval ds between two events is measured by a clock passing through these two events. One of the casualties of the new theory was the notion of two events being simultaneous. Although it is sensible to talk of one observer measuring two events as simultaneous, another observer moving with respect to the first would not measure the events to be simultaneous.

The Theory of Special Relativity is enormously successful, containing the older Newtonian dynamics as an approximation, and is able to cope with situations involving particles moving at speeds comparable with the velocity of light. Besides this, the phenomenon of electromagnetism can be elegantly expressed in relativistic form.

But this model of space-time with the metric given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is basically as 'flat' and almost as simple as the model of Euclidean 3-space used in Newtonian dynamics, and it was only in the decade following 1905, when Einstein began speculating about accelerated frames of reference and about gravitation, that this 'flatness' of the new stage was questioned and finally destroyed.

4. Gravitation

The Newtonian theory of gravitation is contained in the famous inverse square law of attraction, postulated by Newton. It is a simple theory and accounted beautifully for the motion of the planets. But, like the Coulomb inverse square law of electrostatics, the Newtonian concept involves the idea of an instantaneous action at a distance and so does not fit in at all with special relativity. Now the Coulomb law is integrated into the larger scheme of electromagnetism, in which the notion of instantaneous action at a distance is replaced by the notion of the action of one charge on another occurring through the electric and magnetic fields, each satisfying a wave equation involving a finite wave velocity.

Perhaps if there had been a long delay between the introduction of special relativity in 1905 and that of general relativity, just a decade later, then there might have been many serious attempts to retain the flat stage of special relativity and special efforts might have been made to modify the Newtonian inverse square law along the lines of the transformation of the Coulomb law into the equations of electromagnetism. We might have had strong supporters for special relativistic gravitation and have become accustomed to thinking about the gravitational field as yet another actor on the flat stage of space-time constructed so ingeniously by Einstein in 1905. Indeed, it is possible to introduce a field playing such a role in the form of a scalar field produced by matter, say, and acting on other matter, in the way in which we think of the electromagnetic field produced by charge and acting on other charge.

It is possible to consider the motion of the planets in such a theory and to show that there are small discrepancies from the elliptic orbits of the Newtonian theory. Such a discrepancy had been observed in the behaviour of the planet Mercury, but the magnitude of the effect deduced from a simple scalar theory does not agree with the observed magnitude.

But had Einstein's deliberations delayed longer, we might have had refinements of the simple scalar theory or theories involving more complicated fields which would have been more in accord with experiment. However, instead of speculations and theories being developed extensively along these lines, Einstein made the important step of departing from the concept of the flat space-time model which he had introduced with special relativity in 1905. From the consideration of frames of reference falling freely in the Earth's gravitational field and from his conviction that he should seek a theory which would cope with observers with acceleration as well as those moving uniformly with respect to each other, Einstein argued that not only was matter affected by other matter through the gravitational field, but that light was also affected by, for example, the gravitational field of the Sun.

It is possible that, had this effect been observed before Einstein had produced his Theory of General Relativity, then there might have been attempts to understand the bending of light in terms of some interaction between the gravitational field and the electromagnetic field. After all, having introduced the two actors of gravitation and electromagnetism on to the flat space-time stage, it would not be supposing too much to think of them interacting with each other. From time to time, attempts are made to construct models of this sort, models which can be adjusted to yield the measured results both for the bending of light and for the red-shift in the behaviour of atomic clocks at points near a gravitating body.

But such a programme has one fundamental difficulty associated with it, a difficulty which Einstein did not have to meet in his Theory of General Relativity. It is a difficulty of a fundamental sort which we can illustrate in the next section in an elementary way.

5. The geometry of a hot-plate

We suppose that an observer E, equipped with a metre stick, has investigated the geometry of some material surface S and has found the geometry to be Euclidean. He has set up a coordinate system in the manner described in Section 2 and found that the metric has the form

$$ds^2 = dx^2 + dy^2.$$

We now suppose that we heat the surface S differentially, so that the further out from some origin O, the hotter it gets. But the observer E, with a metre stick impervious to changes of temperature, still verifies that the space is Euclidean.

We now introduce a second observer N, to whom we have given a metre stick which is very susceptible to changes of temperature, so that, at a distance from O, his metre stick extends and covers a length interval of the surface S which, if measured by E, would be considerably larger than one metre. We go further and arrange the temperature distribution so that N's metre stick occupies the interval AB of the hot-plate S, where AB is the projection, on to the hot-plate, from the centre C of a sphere of radius R, touching the hot-plate at O, of $A'B'$, which is an interval of arc of the sphere of one metre measured by E. This situation is illustrated in Figure 2.

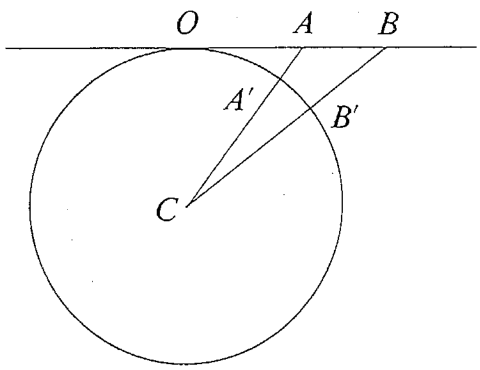

Figure 2. The behaviour of N's metre stick on the hot-plate.

Now N will not think that the hot-plate is flat according to the measurements which he makes. He will not find that the metric takes the simple form found by E. For if N sets up a system of polar coordinates (r, θ) based on O as origin, then the expression which he will find for the metric is

$$ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\theta^2,$$

whereas E, had he used polar coordinates, would have obtained

$$ds^2 = dr^2 + r^2 d\theta^2.$$

More strikingly, N will find that no point on the hot-plate is further than $\frac{1}{2}\pi R$ from O, and that the ratio of the circumference to radius of a circle centre O will decrease from nearly 2π for a small circle down to a value just above 4 for a very large circle.

The observer N measures the geometry and deduces that the hot-plate is non-Euclidean.

Now if N and E get together to discuss things, then, probably after some experiment, they will hit on the notion of temperature and on the differential behaviour of their metre sticks. But, if N never knows of the existence of E, then he will just describe the geometry of the plate as non-Euclidean and might state that the geometry is similar to that kind of non-Euclidean geometry which is valid for a sphere embedded in Euclidean 3-space. As far as N is concerned, he has done his job of determining the metric with the resources available to him, namely his rather inferior metre stick.

Although this is a simple example, it illustrates a very significant feature when we come in the next section to consider the real world around us and try to determine its geometry with the resources available to us.

6. All-pervading gravitation

In the real world around us, it appears that gravitation affects everything. One piece of matter is influenced by another, the paths of light rays are affected by passage near matter, and clocks likewise are influenced by neighbouring matter. When this is realised, we can see that it is very unwise to try to think of a background space-time or stage parametrised by Cartesian coordinates x, y and z and some time parameter t, and having the simple form of metric or geometry characteristic of special relativity. In the presence of matter, we do not have the measuring instruments to map out the background or stage. We are in the position in which the observer N of Section 5 found himself, and there is no super-observer E to whom we can turn for help and who might have methods of parametrising the fictitious flat background. There is no such super-observer E because there are no physical phenomena which are unaffected by gravitation. We are left then with a situation similar to that in which the observer N was placed, and so we are encouraged to accept the prospect of the curved space-time of Einstein and to make it our business to determine as much as possible about the geometry and about its curvature.

From this initial step to a fuller understanding of general relativity is a fascinating journey through Einstein's equations, the Schwarzschild metric, the expanding universe and other intriguing matters. There are many texts, with different degrees of mathematical complexity, which chart the course. One suitable treatment can be found in W. Rindler, *Special Relativity* (Oliver and Boyd, 1960), but the reader will no doubt find other books useful and informative guides for the study of this most elegant and important subject, a towering monument to the genius of Einstein.

The Schoolgirls Problem

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Kirkman's problem

In 1850, the Rev. T. P. Kirkman posed the following problem. 'Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.' (Reference 1.)

It is fairly easy to solve Kirkman's problem by simple trial and error. If we denote the fifteen girls by the letters $A, B, C, \ldots, N, O$, then one solution is:

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
ABC	ADE	AFG	AHI	AJK	ALM	ANO
DHL	BHJ	BLN	BMO	BDF	BEG	BIK
EKN	CLO	CHK	CEF	CMN	CIJ	CDG
FIO	FKM	DIM	DJN	EIL	DKO	EHM
GJM	GIN	EJO	GKL	GHO	FHN	FJL.

In order to verify that this is indeed a solution to the problem, we have to check that each girl is listed once every day, and that each pair of girls is listed just once in the same triple. Perhaps the best way of understanding the problem is to construct a solution for oneself, and the reader is urged to do this.

Kirkman's problem soon became famous, and mathematicians began to analyse and generalise it. In this article I shall explain the ideas involved, and point out that, even today, we do not know the answers to questions closely related to Kirkman's.

The general problem

Suppose that there are n girls ($n = 15$ in the original problem); then the general form of the question can be divided into two parts. First, we have to find a set of triples of the n girls, with the property that each pair appears just once in the same triple. We shall use the symbol S_n to denote such a system. The $7 \times 5 = 35$ triples in our solution to Kirkman's problem provide an example of a system S_{15}, but it is easy to see that a system S_n does not exist for some values of n. For example, if we attempt to construct an S_4, using A, B, C, D to denote the four girls, we soon run in to difficulties: there is no loss of generality in supposing that ABC is one triple, but any other triple we suggest will repeat one of the pairs AB, AC, BC, while one of the pairs AD, BD, CD will not yet be listed in a triple. Thus an S_4 is impossible!

Our first problem is to discover for which values of n a system S_n is possible. We shall see that there is a simple criterion, resulting from quite basic counting arguments. How many times does a particular girl (let us call her Ann) go for a walk? There are $n - 1$ other girls, and each of them walks in the same triple as Ann exactly once; also Ann has two companions on each walk. Hence the $n - 1$ girls accompany Ann two at a time, and Ann walks $\frac{1}{2}(n - 1)$ times altogether. Since this must be an integer (whole number), we see at once that n must be an odd number if a system S_n is possible. For reasons which will appear very shortly, we shall write this condition in the form

$$n = 6k + l, \qquad \text{where } k \text{ is an integer and } l = 1, 3, \text{ or } 5.$$

How many triples are there in a system S_n? Suppose the answer is t. Then each triple contains three pairs of girls (ABC contains AB, AC, and BC), so that there are $3t$ pairs in all. But according to the conditions, this must account for all possible pairs of the n girls, and the number of such pairs is $\frac{1}{2}n(n - 1)$. (Recall the rule for finding the number of combinations of n things taken r at a time, usually written ^nC_r or $\binom{n}{r}$. In this case, $r = 2$.) Hence $3t = \frac{1}{2}n(n-1)$, or

$$t = n(n-1)/6.$$

Writing $n = 6k + l$ as above, we see that (since t is an integer) 6 must be a divisor of

$$n(n-1) = (6k+l)(6k+l-1) = 36k^2 + 6(2l-1)k + l(l-1).$$

The first two terms are clearly divisible by 6, and so our condition becomes that $l(l-1)$ should be divisible by 6. Of the three possible values $l = 1, 3, 5$, only the first two are satisfactory. So we have strengthened the necessary condition for the existence of an S_n, and we now require

$$n = 6k+1 \quad \text{or} \quad n = 6k+3, \qquad \text{where } k \text{ is an integer.}$$

It should be emphasised that we have only proved that this is a necessary condition; we have not shown that a system S_n exists when n has the required form. Indeed, it is rather surprising that the converse result is true. Kirkman himself had established this fact some years before posing the problem of the 15 schoolgirls, and it was the general question which prompted his famous problem (see reference 2, or, for a modern treatment, pages 236–241 of reference 3). Thus we can say that a system S_n exists *if and only if* n is a number of the form $6k + 1$ or $6k + 3$.

Returning to the fifteen girls, we see that the existence of S_n, in the case $n = 15$, is only the first part of the problem. We are asked to arrange the 35 triples of the system S_{15} into seven sets of five, so that each set of five triples is the list for one day's walk and must contain each girl's name just once. For a general system S_n there are $n(n-1)/6$ triples, and these must be arranged into a certain number of 'days', each 'day' listing every girl once. How many days are needed? Well, we have already calculated that one girl (we called her Ann) walks $\frac{1}{2}(n-1)$ times in a system S_n, and so this must be the number of days in general. The general formulation of Kirkman's problem can now be stated in two parts:

(1) find a system S_n;
(2) partition the triples of S_n into $\frac{1}{2}(n-1)$ 'days' so that each girl is listed once every day.

The second condition imposes an additional numerical constraint on n, as we might expect. Since the list for each day contains n girls arranged in triples, n must be divisible by three. This means that n cannot be of the form $6k + 1$, and so the only possible values of n have the form $6k + 3$. At this point, the reader should try to write down solutions when $n = 3$ (trivial) and $n = 9$ (less trivial, but not hard).

Once again, it must be stressed that we have shown only that the condition $n = 6k + 3$ is necessary for a solution to the general form of the problem, not that it is sufficient. In the second half of the nineteenth century many articles were written concerning this question, and solutions were constructed for many values of n. However, a general solution for all n (of the form $6k + 3$) was not given until as recently as 1968, by two American mathematicians, D. J. Ray-Chaudhuri and R. M. Wilson. The solution is quite complicated, and the interested reader is referred to the original article (reference 4).

Sylvester's problem

In one of the first articles on Kirkman's problem a fascinating extension of it was mentioned. The article (reference 5) was written by Arthur Cayley, but the extended problem was suggested by his friend J. J. Sylvester. Sylvester had noticed that there are 455 $\left(=\binom{15}{3}\right)$ ways of forming a triple of the 15 schoolgirls, and this number is 13 times the number of triples (35) occurring in a solution to Kirkman's problem. So the following question arises. Can the 455 different triples be partitioned into 13 different systems S_{15}, each one of them satisfying the conditions of Kirkman's problem? We may think of each S_{15} as a 'week', so that the 13 'weeks' give us a schedule for one quarter of a year; no triple is to be repeated, and each 'week' and each 'day' is to satisfy the original conditions. (The fact that there are seven days in a week and thirteen weeks in a quarter is a lucky accident!)

In the general case of n girls, the total number of different triples is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

Now a system S_n contains $n(n-1)/6$ triples, so that $n-2$ disjoint S_n's will exhaust all the triples. To generalise Sylvester's problem, we have to arrange the set of all triples into $n-2$ disjoint S_n's, each one of them partitioned into $\frac{1}{2}(n-1)$ 'days' as before. In the special case $n = 15$, Kirkman mistakenly claimed to have a solution. However, the problem is more difficult than he thought, and it was not until recently that a solution was found, by the expertise of Dr R. H. F. Denniston of Leicester University and the aid of a computer (reference 6).

It is still an open problem as to which values of n admit a solution to the general form of Sylvester's problem. We have the condition that n must be a number of the form $6k + 3$, already necessary for Kirkman's problem, but it is not known whether or not this is sufficient. Solutions have been constructed for some small values of n, including of course $n = 15$, and for n equal to a power of 3 (reference 7); but no general construction is known.

The solution for $n = 9$ is rather neat. Suppose the nine girls are denoted by $A, B, C, \ldots, I$, and consider the seven squares:

ABC	ABD	DEG	GHA	GAB	ADE	DGH
DEF	EFG	HIA	BCD	CDE	FGH	IAB
GHI	HIC	BCF	EFI	FHI	IBC	CEF

Each square gives rise to an S_9 in the following way: take each row, each column, and each of the six triples involved in calculating the 'determinant' of the square. For example, the first square gives the triples

ABC	DEF	GHI	(rows)			
ADG	BEH	CFI	(columns)			
AEI	AFH	BFG	BDI	CDH	CEG	(determinant).

These twelve triples form one system S_9, and the seven S_9's so constructed give a solution to Sylvester's problem in the case $n = 9$.

Problems of this kind have returned to popularity in recent years. An important stimulus came from the theory of experimental design, where it is required to test and compare results in a systematic, but unbiased, way. More recently, it has become clear that modern algebra (especially the theory of groups) is a very useful tool for the construction of such designs. Nevertheless, there is still the opportunity for a bright young student to contribute to the subject—perhaps by solving Sylvester's problem in its full generality.

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An Approach to Independent Sets

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1. Introduction to the problem

When teaching a course which introduces probability, one is eventually faced with the task of understanding and defining conditional probability. No matter how well-orchestrated a lecture may be, some student will inevitably ask for one more example and explanation of independent sets than has been prepared. With blissful naiveté, determined to keep the example simple, we offer some finite probability space to the class—say

$$X = \{1, 2, 3, 4, 5\}$$

with uniform probability. We then, quite randomly, choose two sets which we subsequently find are dependent. We choose two more, hoping this time for independence in order to complete the example. After choosing several pairs of dependent sets, one might start to wonder at one's luck. The fact is that this particular space has no non-empty independent proper subsets. It was this observation which caused us to ask just how probable it is, given a finite space with uniform probability, that one will randomly choose two independent sets? In other words, how many independent pairs of non-empty proper subsets are there?

2. Counting independence

Let $X = \{1, 2, 3, \ldots, n\}$ and let p be the uniform probability on X. Sets A and B are *independent* if $p(A)p(B) = p(A \cap B)$. Since $\varnothing$, X are independent of every set A, we say (A, B) is a trivially independent pair if A or B is either empty or the whole space. We are interested in counting how many non-trivial independent ordered pairs (A, B) exist within the total $(2^n - 2)^2$ non-trivial ordered pairs.

We first obtain a characterization of independence based on the number n. For notational convenience, $n(A)$ will denote the number of elements in A and $g(a, b)$ will be the greatest common divisor of a and b.

Theorem 1. A and B are independent if and only if $n(A \cap B) = n(A)n(B)/n$.

Proof. The following are equivalent:

(i) A and B are independent.

(ii) $p(A)p(B) = p(A \cap B)$.

(iii) $\dfrac{n(A)}{n} \cdot \dfrac{n(B)}{n} = \dfrac{n(A \cap B)}{n}$.

(iv) $\dfrac{n(A) \cdot n(B)}{n} = n(A \cap B)$.

Corollary. If A and B are independent then n divides $n(A)n(B)$.

As a result of Theorem 1, we are able to characterize those uniform spaces which have no non-trivial independent pairs.

Theorem 2. (X, p) contains no non-trivial independent pairs if and only if n is prime.

Proof. If (A, B) is a non-trivial pair then $1 \leq n(A) \leq n - 1$ and $1 \leq n(B) \leq n - 1$. If n divides $n(A) \cdot n(B)$ then $n > g(n(A), n) > 1$ and $n > g(n(B), n) > 1$. Thus n is not prime.

If n is not prime then $n = ab$, where $1 < a, b < n$. Since $a(1 - (b/n)) < n(1 - (b/n))$, we have $a + b - 1 < n$. If $A = \{1, 2, \ldots, a\}$ and $B = \{a, a + 1, \ldots, a + b - 1\}$ then (A, B) is a non-trivial pair of independent sets of (X, p).

The example in the introduction was of this form where $n = 5$. One can see that the choice of A and B, even when n divides $n(A)n(B)$, can only be made with care if one requires independence. For example, if $n = 6$ and $A = \{1, 2, 3\}$ with $B = \{1, 2\}$, then $n = 6$ divides $n(A)n(B) = 2 \cdot 3 = 6$, but A and B are not independent.

Theorem 3. If n divides ab, where $1 < a, b < n$, then there exist non-trivial independent sets A and B such that $n(A) = a, n(B) = b$, and $n(A \cap B) = ab/n$.

Proof. Observing that $a < a + b - (ab/n) < n$, we define $A = \{1, 2, \ldots, a\}$ and $B = \{a - (ab/n) + 1, a - (ab/n) + 2, \ldots, a - (ab/n) + b\}$.

We now know that every ordered pair (a, b), for which n divides ab, engenders independent pairs. How many such pairs are there?

Theorem 4. Let $0 < a < n$ and $0 < b < n$ be given such that n divides ab. There are

$$\frac{n!}{\left(a - \dfrac{ab}{n}\right)!\left(b - \dfrac{ab}{n}\right)!\left(\dfrac{ab}{n}\right)!\left(n - a - b + \dfrac{ab}{n}\right)!}$$

ordered pairs (A, B) of independent sets with $n(A) = a$ and $n(B) = b$.

Proof. This is a counting problem. We just count how many ways there are of arranging n objects in four boxes where there must be $a - (ab/n)$ in the first box (the set $A \sim (A \cap B)$), $b - (ab/n)$ in the second box (the set $B \sim (A \cap B)$), ab/n in the third box (the set $A \cap B$), and. finally, $n - (a + b - (ab/n))$ in the fourth box (the set $X \sim (A \cup B)$).

To count the number of non-trivial independent ordered pairs, we count how many sets can be paired with sets with a given number of elements and then add over all possible sizes for non-trivial subsets.

Theorem 5. There exist

$$\sum_{a=1}^{n-1}\sum_{i=1}^{d(a)-1} \frac{n!}{\left(a - i\dfrac{a}{d(a)}\right)!\left(i\dfrac{n}{d(a)} - i\dfrac{a}{d(a)}\right)!\left(i\dfrac{a}{d(a)}\right)!\left(n - a - i\dfrac{n}{d(a)} + i\dfrac{a}{d(a)}\right)!}$$

non-trivial independent ordered pairs (A, B). Here $d(a) = g(a, n)$ and the inner sum is taken as zero if $d(a) = 1$.

Proof. Let $0 < a < n$ be given. If $ab = kn$ and if $d(a) = g(a, n)$, then $g(a, n/d(a)) = 1$. Then also $g(a/d(a), n/d(a)) = 1$. Therefore $(a/d(a))b = k(n/d(a))$. So $n/d(a)$ divides b which implies that $b = i(n/d(a))$ for i between 1 and $d(a) - 1$ (if

$0 < b < n$). Thus, for Theorem 4, the number of non-trivial ordered pairs (A, B) where $n(A) = a$ is

$$\sum_{i=1}^{d(a)-1} \frac{n!}{\left(a - i\frac{a}{d(a)}\right)!\left(i\frac{n}{d(a)} - i\frac{a}{d(a)}\right)!\left(i\frac{a}{d(a)}\right)!\left(n - a - i\frac{n}{d(a)} + i\frac{a}{d(a)}\right)!}.$$

We conclude the proof by adding over all numbers a, such that $1 \leq a < n$.

Now that we know the number of non-trivial ordered pairs of independent events we denote as $I(n)$, we calculate the probability of choosing such a pair from all ordered pairs of events as $I(n)/(2^n - 2)^2$.

3. Some examples and observations

It is perhaps more reasonable to ask for the probability that, given the event A, we draw a set B such that (A, B) is an independent pair of non-trivial sets. Since the set A is given, we know $n(A)$. There are

$$\binom{n}{a} = \frac{n!}{a!(n-a)!}$$

sets A available, with $n(A) = a$. Theorem 1 indicates that two sets with the same number of elements have the same number of non-trivial sets of which they are independent. That number is given by

$$\frac{1}{\binom{n}{a}} \sum_{i=1}^{d(a)-1} \frac{n!}{\left(a - i\frac{a}{d(a)}\right)!\left(i\frac{n}{d(a)} - i\frac{a}{d(a)}\right)!\left(i\frac{a}{d(a)}\right)!\left(n - a - i\frac{n}{d(a)} + i\frac{a}{d(a)}\right)!},$$

which we shall call $N(A)$. The probability of drawing a non-trivial independent pair with the set A as the first element is $N(A)/(2^n - 2)$.

The case where $n = 2m$ and $a = m \geq 1$ is interesting. Here $d(a) = g(m, 2m) = m$. Then

$$\sum_{i=1}^{m} \frac{(2m)!}{\{(m-i)!\}^2\{i!\}^2}$$

is the number of non-trivial independent pairs where the first set has m elements.

For a fixed A, with $n(A) = m$,

$$N(A) = \frac{1}{\binom{2m}{m}} \sum_{i=1}^{m} \frac{(2m)!}{\{(m-i)!\}^2\{i!\}^2} = \sum_{i=1}^{m} \binom{m}{i}^2 = \binom{2m}{m} - 2.$$

More generally, if $a = g(a, n)$, then the number of non-trivial ordered pairs (A, B) for a fixed A, with $n(A) = a$, is

$$N(A) = \frac{1}{\binom{ak}{a}} \sum_{i=1}^{a-1} \frac{(ak)!}{(a-i)!\,(ki-i)!\,i!\,(ak-a-ik+i)!} = \sum_{i=1}^{a-1} \binom{a}{i}\binom{(k-1)a}{(k-i)i}.$$

4. The denumerable case

In Section 2, we found that some finite uniform spaces have no non-trivial independent events. We now consider the countably infinite probability space (X, p), where $X = \{1, 2, 3, 4, \ldots\}$. Note, however, that we can no longer conceive of a uniform probability. Furthermore, we can no longer effectively change X; if we want to change the nature of the space, we must change the distribution. We therefore consider conditions on p which guarantee independent events.

We first obtain a lemma which applies to any infinite series and which yields a sufficient condition for the existence of non-trivial independent events.

Lemma. Let $0 < p_i < 1$, $\sum_{i=1}^{\infty} p_i = b$, and $p_i > p_{i+1}$. For each a, $0 < a \leq b$, there is a subsequence $\{r_j\}$ of $\{p_i\}$ for which $\sum_{j=1}^{\infty} r_j = a$ if and only if $p_i \leq \sum_{k=1}^{\infty} p_{i+k}$ for every i.

Proof. Suppose $p_i > \sum_{k=1}^{\infty} p_{i+k}$. Then an a between $\sum_{k=1}^{\infty} p_{i+k}$ and p_i cannot be the sum of a subsequence of the p_i. (Recall that $p_{i-1} > p_i$ for every i.)

Now suppose $p_i \leq \sum_{k=1}^{\infty} p_{i+k}$ for every i. Let r_1 be the first p_i such that $r_1 < a$. Let r_2 be the first p_i, other than r_1, such that $r_1 + r_2 < a$. Continue in this manner until either $r_1 + r_2 + \cdots + r_n = a$ for some n or until an infinite subsequence is constructed from p_i.

If $\sum_{i=1}^{\infty} r_i = a$, we are done. If not, then there are two cases to consider. Case 1 is where $\sum_{i=1}^{\infty} r_i < a$ *and* after some point $r_i = p_j, r_{i+1} = p_{j+1}, \ldots$. In this case, assuming i and j are minimal, we have $a < r_1 + r_2 + \cdots + r_{i-1} + p_{j-1} \leq r_1 + r_2 + \cdots + r_{i-1} + p_j + \cdots < a$. Thus Case 1 is not possible.

Case 2 is where $\sum_{i=1}^{\infty} r_i < a$ and $r_i = p_j, r_{i+1} = p_k$, with $k - j > 1$, for infinitely many i. In this case, for some positive ε, $\sum_{i=1}^{n} r_i \leq a - \varepsilon$. This implies that $\sum_{i=1}^{n-1} r_i + p_j \leq a - \varepsilon$, whereas $\sum_{i=1}^{n-1} r_i + p_{j-1} > a$, with this occurring for infinitely many j. This states that $p_{j-1} - p_j > \varepsilon$ for some subsequence of the p_i, which contradicts the fact that the limit of the p_i is zero. So Case 2 has also been eliminated. We must conclude that $\sum_{i=1}^{\infty} r_i = a$.

We should point out that any denumerable probability space can be rearranged so that $p_i \geq p_{i+1}$ and that we may choose a subsequence of the p_i which is strictly decreasing. It is the third condition which may fail to hold.

Theorem 6. Let $X = \{1, 2, 3, \ldots\}$ with probabilities p_i such that $p_i > p_{i+1}$ and $p_i \leq \sum_{k=1}^{\infty} p_{i+k}$. Then there are non-trivial independent events in X.

Proof. First observe that $p_1/(p_1 + p_2) < 1$ so that $p_1(1 - p_1 - p_2)/(p_1 + p_2) < (1 - p_1 - p_2) = \sum_{k=3}^{\infty} p_k$. By the lemma, there exists a subsequence p_{i_j} from among $p_3, p_4, p_5, \ldots$ such that

$$\sum_{j=1}^{\infty} p_{i_j} = \frac{p_1(1 - p_1 - p_2)}{p_1 + p_2}.$$

Now let $A = \{1, 2\}$ and $B = \{1, i_1, i_2, \ldots\}$. This pair is a non-trivial independent pair since $p(A \cap B) = p_1 = (p_1 + p_2)[p_1 + p_1(1 - p_1 - p_2)/(p_1 + p_2)] = p(A)p(B)$.

The condition stated in Theorem 6 is sufficient but not necessary, as the next theorem shows.

Theorem 7. If $X = \{1, 2, 3, 4, \ldots\}$ with $p_n = (1 - r)r^{n-1}, 0 < r < 1$, then there exist non-trivial independent events in X.

Proof. Let $A = \{1, 2\}$ and $B = \{1, 3, 5, 7, \ldots\}$.

$$\begin{aligned} p(A)p(B) &= [(1 - r) + (1 - r)r][(1 - r)(1 + r^2 + r^4 + \cdots)] \\ &= (1 - r^2)(1 - r)\frac{1}{1 - r^2} \\ &= 1 - r \\ &= p(A \cap B). \end{aligned}$$

Therefore, A and B are independent.

Notice that $p_n = 2/3^n$ satisfies the hypothesis of Theorem 7 but not of Theorem 6.

5. Conclusions and suggestions

The student by now should realize that the concept of independence is imprecise and frequently non-intuitive, and that one must be careful when drawing inferences based on independence or its offspring, correlation. The subtle nature of the definition is quickly exposed by the inescapable tie to number theory. We are hardly the first to point out this relationship, (we would encourage the interested student to read the excellent little book by Marc Kac (reference 2)), but we feel this approach, if used in the classroom, could give the student an early appreciation of the inseparability of the areas of combinatorics, number theory, and probability (references 1 and 3). As a closing remark we point out that this approach also illustrates that many interesting, fruitful and pleasurable problems can be found within the blend of these areas and we hope the student sees that the ideas presented here can lead one to other questions and investigations.

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A Computer Program To Play Othello

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Introduction

The study of game-playing computer programs has become a firmly established branch of artificial intelligence. Playing games on a computer is very popular and produces a number of benefits. It amuses visitors to the Computer Centre—and the staff—it provides examples for decision-taking programs and it presents stimulating problems to research workers. The problems created by the games are easier to solve than real-life ones as the games are subject to well-defined rules which limit the degree of difficulty. The games have provided an opportunity for researchers to apply the techniques and ideas first suggested by Von Neumann and Morganstern (reference 10) and Turing (reference 9) and also to develop new ones.

There are now a large number of programs in existence which have been written to play various games on a computer. The games range in complexity from Nim and Noughts-and-Crosses, (Tic-Tac-Toe) to Kalah and Draughts (Checkers) and to more advanced games like Chess and Go. Nim and Noughts-and-Crosses have well-known strategies, so these games serve only as tutorial exercises. Kalah or Whari is a game of moderate complexity and there are many examples of very good programs at the Massachusetts Institute of Technology and Stanford University which can beat human players. Bell (reference 1) has written a program to play Kalah on Atlas. This program learnt from its own mistakes so that it did not lose twice in the same way. Samuel (reference 6) has developed a program that uses learning techniques to play Draughts. The current program plays to a very high standard but cannot yet beat the best human players. Chess is an even more complex game and the problems in writing successful programs have exercised the best researchers and some Chess grandmasters. The current programs have a long way to go before they can play human opponents on equal terms. Nevertheless Newborn (reference 4) feels that this is not surprising as the current programs have had a life of only six years, and he thinks it will probably take about fifteen years to develop a really powerful Chess program. So far the programs that play Go have only reached novice standard.

It is interesting to investigate how the methods developed and established by these programs can be applied to other newer games. Othello is a game that has been in existence for some decades but has recently become popular and thus widely available. As Othello is of a similar degree of complexity to Draughts it makes an ideal target for a computer program, and so far there do not appear to be many programs that play Othello.

Othello

Othello is played by two players on an 8×8 board consisting of 64 identical squares. It is marketed by Peter Pan Playthings Ltd, Peterborough, U.K. There is a

British Othello Federation and the game is also popular in Japan and in the U.S.A. Othello is a variation of Reversi which can be played on a larger board.

Othello is played with 64 counters, which are coloured black on one side and white on the other. One of the players plays black (henceforth known as Black) and the other, White, plays white. The purpose of the game, for Black, is to have more black counters on the board than white counters, at the end of the game. This is reached when all the squares are occupied, or when one player loses all his counters or neither player can move.

During the description of the play that follows, reference is made to Figure 1, which shows the numbering system used in the computer program. Black is considered as the program and White the opponent. The game commences with four counters on the board, black counters on squares 29 and 36 and white counters on squares 28 and 37. Black moves first. Whenever a player moves he must make a capture. A capture is made by Black when he places his counter on squares 20, 27, 38 or 45. In any one of these four positions Black sandwiches one of the White counters between the black counter just played and one of the black counters already on the board. The white counter which has been sandwiched is captured and turned over—reversed—to become black. Suppose, for example, that Black placed his counter on square 20. Then the white counter on 28 is captured and Black then has four counters on squares 20, 28, 29 and 36 and White one counter on square 37. White now has three possible moves in reply: 35, 19 or 21. In each case one black counter is captured and as a result both Black and White have three counters. The game proceeds in this way, each player moving in turn, when able. If a player cannot move, because he cannot capture, the other player has another turn.

Counters can be captured in any one of eight directions, vertically up or down, horizontally to the left or right, and diagonally in the compass directions north-west, north-east, south-east, south-west. Counters can be recaptured in more than one direction in any move. Also, if more than one white counter is sandwiched between

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Figure 1

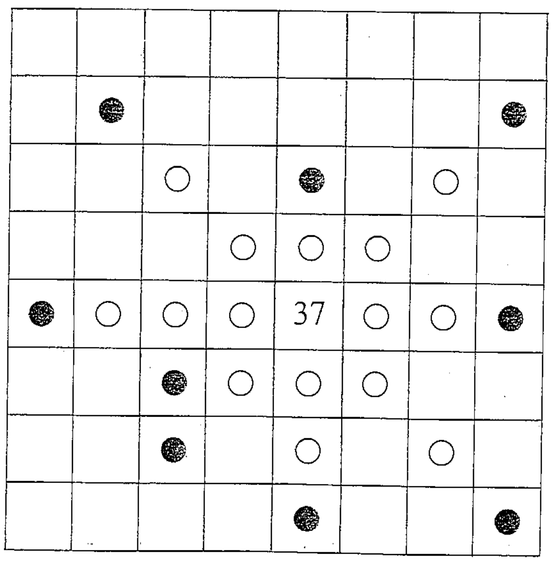

Figure 2

two black counters, all are captured. In Figure 2, if a black counter is placed on square 37, all the white counters shown would be captured. Whereas this is unlikely and would lead to Black winning (White has no counters) it is quite common for Black or White to gain ten or more counters in a move.

The four corner squares 1, 8, 57 and 64 are key squares because once they are gained they can never be recaptured. Possession of the corner square often leads to the capture of the edge squares and major diagonal squares. Although capturing the corners does not always lead to a win, their capture is a worthwhile goal. It is also advantageous for Black to place black counters on the edges as these counters can lead to the possession of rows and columns. Moreover they can only be captured in two of the eight directions. On the other hand occupying the squares diagonally adjacent to the corners 10, 15, 50 and 55 is usually hazardous. It quite frequently enables the opponent to gain the corner. This argument applies to a certain extent to the squares on the second and seventh rows and columns, those neighbouring the edges.

Once the corner is gained the squares around it should be captured and holes should be avoided. In Figure 2, square 37 can be thought of as a hole from White's point of view and makes White vulnerable. Often a black counter that is completely surrounded by White's counters can be subsequently used to capture the white counters. It acts just like an agent who has been successfully planted in alien territory. If Black can force White to make an injudicious move at any stage, it gives Black a tremendous advantage. This situation can be achieved if Black can restrict the number of possible moves that White can make. The fewer moves that a player has to choose from, the less chance he has of making a good move and thus move limitation is a sound strategy.

As players gain experience they learn the positions and combinations of counters that lead to subsequent advantage and those that conceal danger. The game is sufficiently complex to allow many twists and changes of fortune. Part of the fascination of Othello is that the result of the game is often in doubt until the last counter is placed on the board.

Game-playing techniques

The methods of playing games on computers have been well described by Michie (reference 3) and Slagle (reference 8). The class of games to which Othello belongs is called *two-person games with perfect information*. Each player has a full sight of the board and there is no element of chance. Since each game is a sequence of choices, the moves that each player makes are best represented by a tree called the game tree.

The structure of a tree resembles that of a botanical tree, but for convenience it is usually drawn downwards rather than upwards. Common examples of this type of tree are genealogical trees which depict people's ancestry. The nodes, or branch points, in the tree represent the positions in the game before a move is made. Each node stores information about all 64 squares on the board and has a link upwards to the parent (the previous position). The root node is the topmost node and represents the board at the commencement of the game.

If all possible moves were made for a game, a complete game tree would be formed. The terminal nodes would then correspond to the end of the game and the outcome would be known and could be represented as $+1$ for a Black win, -1 for a White win and 0 for a draw. In this way each terminal node would have a value associated with it. The other nodes in the tree would be evaluated by using a backing-up procedure of which the minimax is the most common. In this procedure it is assumed that both Black and White make their best move at all times and follow an optimal strategy.

The minimax procedure will be described by reference to Figure 3. Terminal Nodes *G–Q* have been evaluated in accordance with the above convention and their values are indicated. Nodes *D*, *E* and *F* are called Max nodes because Black is in play and Black moves to the highest-scored node, a maximum. Thus the nodes are given the values 1, 0 and 1 respectively. On the other hand nodes *B* and *C* are called Min nodes because White is in play and White moves to the lowest-scored node, a minimum. Nodes *B* and *C* are given the values 0 and 1. Node *A*, the root node, corresponds to Black moving and is assigned the value 1, which indicates that the game is a potential win for Black. Black's best opening move is therefore to node *C*. Unfortunately the production of a complete game tree is possible only for very simple games and Shannon (reference 7) has calculated for Chess that there are 10^{120} different move sequences and that a computer would take 10^{90} years to find the best opening move!

As it is not generally possible to construct complete game trees, various pruning techniques have to be applied to reduce the size of the tree. One of the most common is to restrict the lookahead to 3 moves—3-ply—and to have a tree of 4 levels. This means that the terminal nodes are not true terminal nodes and they do not correspond to the end of the game. The terminal nodes need to be scored and this is done by using an evaluation function which assesses the strength of the position from Black's point of view. The evaluation function is difficult to construct, it is usually linear, and its components are the factors which are considered significant in

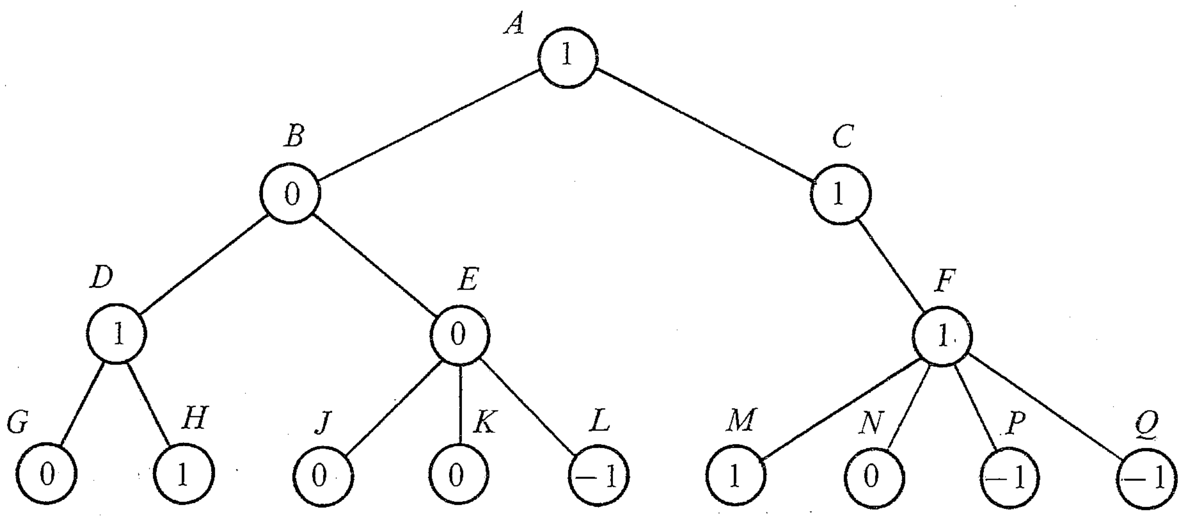

Figure 3

leading to a win. The choice of the parameters and the weights assigned to them have a critical effect on the performance of the program. Samuel (reference 6) solved this problem by making his program play games against itself. Each player then had different parameters and the program used learning techniques to improve its performance. Essentially the program saved the parameters of the winning player and built up a weighted average. The program used the best twenty out of about forty parameters for any run. Samuel's program has been one of the most successful learning programs. The computer's best move is found by applying the minimax algorithm to the pruned game tree.

The game tree is usually constructed by using a depth first-move generation, which means that the first choice at each level is generated until the terminal nodes are reached. When all of the nodes at this level have been generated and evaluated, the appropriate score is backed up to the parent. The next choice at the parent's level is then taken and a new set of nodes at the lower level is generated. By proceeding in this manner until all possible choices have been considered all the required nodes are generated. There are many variations on this procedure and these have been well described by Slagle (reference 8).

Heuristics are also an integral part of game-playing programs. Heuristics, or rules of thumb, are reckoned to be plausible guesses as to the best move at any stage, but they do not guarantee success. Each game has its own special heuristics which are formed from experience gained in studying the game. Chandrasekaran (reference 2) has observed that programs using moves generated solely by game trees and the minimax algorithm play games which are not of sufficiently high standard and that heuristics are needed to improve the quality of play.

The computer program

An experimental program was written to play Othello on a computer. It was written in Fortran IV, which enabled it to be run on the ICL System 4-70, the ICL 2980 and the PDP11-34. The program used a depth first-move generator with a lookahead limited to 2 ply, which meant that a complete game could be played in 3–4 minutes on the ICL System 4-70. The aim of the program was to gain enough information as quickly and efficiently as possible so that an effective evaluation function could be devised. From Samuel's work it was clear that the choice of parameters and their associated weights was crucial. It was also apparent that the best way to gain the information was to allow the program to play itself, with each 'player' having differently weighted parameters.

The evaluation function was made up of ten parameters, the weights assigned to them were read as data, allowing variation. The parameters were:

(i) corner, (ii) edge, (iii) counter,
(iv) good square, (v) bad square, (vi) four square,
(vii) inner zone, (viii) danger zone, (ix) diagonals,
(x) move advantage.

The advantage was calculated by finding how many more black counters occupied the relevant squares than white counters. The good squares were those adjacent to the corner squares on the edges. The bad squares were squares 10, 15, 50 and 55; the four squares were the squares 19, 22, 43 and 46; and the inner zone consisted of the inner parts of the third and fifth rows and columns. The diagonals were the inner parts of the major diagonals. These parameters were chosen to represent position. The move advantage was calculated by multiplying the number of possible replies to any move by the appropriate weight. The integer weights were varied but were normalised so that the sum of the weights for any player was equal to 100. The program was structured and consisted of nine subroutines which

(i) generated the moves,
(ii) checked for legal moves involving captures,
(iii) copied nodes in the tree,
(iv) called the evaluation function,
(v) backed-up the scores,
(vi) carried out a capture,
(vii) displayed the board,
(viii) tested for the end of the game.
(ix) placed a counter in the corner if required.

The maximum number of nodes generated in any run was always less than 36.

Discussion

The Othello program was run several times with a selection of the parameters available. The weights were varied for each run. In the early runs each 'player' had a restricted set of parameters, as this made the effect of each individual parameter easier to detect. From these runs, corner and edge advantages were confirmed. However, piece or counter advantage with a large weight sometimes seemed to be a handicap. It appeared that it was not wise to attempt to eliminate all the opponent's counters, at least in the early stages of the game. Attempts to reward position on the board met with mixed success. Although the good squares required positive weight and the bad squares negative weight, possession of the central squares did not seem to be an overwhelming advantage. Presumably this was because the central positions are very volatile and the counters here are reversed very frequently. Move advantage turned out to be a valuable parameter and it was interesting to see how varying its weight affected the program's performance. Analysis of many of the games showed a certain reluctance on the players' part to move into the corners. This was overcome by a heuristic, which was incorporated by a subroutine, to force Black to play the corners wherever possible. When all these parameters were combined and Black was given the move advantage, it produced one outstanding game which is given in the table below, where Black won by 56 extra pieces. In this game Black played very well, gaining the corners and then consolidating by moving out diagonally in large black triangles.

This game posed the question, 'What are the best weights for the parameters?' Samuel answered this question for Draughts by writing a learning program and allowing the program to choose the best parameters and their weights from a list of about forty parameters. From the runs of Othello it could be seen that the best weights for the parameters could be found empirically—and that a balanced set was best. The need for extra parameters was also felt. With the current ten parameters and the optimum weighting the program is capable of playing a very sound game without making any obvious errors. To raise the standard of the program a longer lookahead is needed as well as some number-of-move-dependent parameters. Just as in Chess there are a definite opening, middle game and end game in Othello. Different parameters are needed for each and the weights of some of the parameters can change during the game. The increase in the number of parameters required is not surprising as Samuel used 40 for his Checkers program and Ostrich, a Chess-playing program, used 17 in its scoring function (Newborn, reference 4).

Extra parameters are needed to locate holes, to allow for vacant squares adjacent to good or bad squares, and squares already gained next to bad squares. Once Black has gained a corner, say square 1, the weight associated with possession of this square must change. It is no longer a square to be captured and its value to Black is demonstrated whenever it is used to capture other squares. Black must now attempt to capture squares 2, 9 and 10, and thus the weights associated with these squares must be altered in Black's favour. The game in Table 1 shows how Black must move out of the corners. Parameters are also needed to take account of rows of like or unlike counters. If there is a white counter on square 25, it is unwise for Black to capture all the squares 26–31, as this leaves Black vulnerable to White placing a counter on square 32 and gaining the whole row. Black can counter the advantage White has gained by placing a white counter on corner square 8, by allowing White to occupy squares 2–6. Black gains square 7 and then captures squares 2–6 by placing a black counter on square 1. This is another example of a hole. Various gambits can be included if deep lookaheads are allowed.

TABLE 1

Move	Black	White	Move	Black	White
1	20	19	17	8	63
2	27	21	18	62	15
3	22	35	19	7	—
4	43	30	20	6	—
5	18	14	21	4	—
6	46	17	22	3	—
7	13	55	23	54	53
8	38	39	24	11	—
9	47	5	25	60	61
10	31	40	26	52	59
11	64	26	27	58	—
12	44	51	28	42	—
13	32	24	29	34	50
14	23	16	30	57	—
15	45	48	31	33	—
16	56	12	32	10	—

Black wins by 56 pieces.

Just as the inclusion of the move advantage parameter and the corner heuristic gave Black a significant advantage, so the implementation of the above ideas as weighted parameters and heuristics will enable the program to play more skilful players. Analysis of these games will lead to even better programs.

Comments

Move 10: Black prepares for capture of the corner square 64. Move 16: Black prepares for capture of the corner square 8. Move 18: Black controls the right-hand side of the board although White has twelve more pieces. Black begins to move diagonally out from corners 8 and 64. Move 19: a run of five successive Black moves! From now on White either has no moves or only one with no choice! Move 32: Black wins, White has only 1 counter left. The game ends, as neither player can move, although 6 squares are left vacant.

Conclusion

Othello has proved to be a game worthy of detailed study by means of a computer program. Many interesting results have already been found with a small number of parameters, a special heuristic and a 2-ply game tree. The need for further parameters has been demonstrated. It is also apparent that learning techniques could be introduced and these should improve the program's playing performance. Although there are many ideas still to be implemented before the Othello program can seriously challenge human opponents, this experimental program has served as an excellent tool in the design of an even more powerful program to play Othello on a computer.

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Letters to the Editor

Dear Editor,

The Pythagorean theorem in three dimensions

In Volume 11, Number 2, p. 60, A. M. Khidr proved this theorem by means of a three-dimensional geometrical construction. Here is a proof involving only plane geometry and some algebraic manipulation.

As in A. M. Khidr's letter, let OA, OB, OC be three mutually perpendicular line segments and denote the areas of the triangles ABC, OBC, OCA, OAB by e, e_1, e_2, e_3, respectively. We wish to show that

$$e^2 = e_1^2 + e_2^2 + e_3^2. \tag{1}$$

If the lengths of BC, CA, AB are p, q, r, then, by Hero's formula for the area of a triangle,

$$\begin{aligned} 16e^2 &= (p+q+r)(-p+q+r)(p-q+r)(p+q-r) \\ &= -[p^2-(q+r)^2][p^2-(q-r)^2] \\ &= -p^4 + 2(q^2+r^2)p^2 - (q^2-r^2)^2 \\ &= 2(q^2r^2 + r^2p^2 + p^2q^2) - (p^4+q^4+r^4). \end{aligned} \tag{2}$$

Next, let a, b, c be the lengths of OA, OB, OC. Then, by the usual Pythagorean theorem,

$$b^2 + c^2 = p^2, \qquad c^2 + a^2 = q^2, \qquad a^2 + b^2 = r^2;$$

and so also

$$b^2 - c^2 = r^2 - q^2, \qquad c^2 - a^2 = p^2 - r^2, \qquad a^2 - b^2 = q^2 - p^2.$$

Since $e_1 = \frac{1}{2}bc, e_2 = \frac{1}{2}ca, e_3 = \frac{1}{2}ab$, we therefore have

$$\begin{aligned} &16(e_1^2 + e_2^2 + e_3^2) \\ &\quad = 4(b^2c^2 + c^2a^2 + a^2b^2) \\ &\quad = [(b^2+c^2)^2 - (b^2-c^2)^2] + [(c^2+a^2)^2 - (c^2-a^2)^2] + [(a^2+b^2)^2 - (a^2-b^2)^2] \\ &\quad = [p^4 - (r^2-q^2)^2] + [q^4 - (p^2-r^2)^2] + [r^4 - (q^2-p^2)^2] \\ &\quad = 2(q^2r^2 + r^2p^2 + p^2q^2) - (p^4+q^4+r^4). \end{aligned} \tag{3}$$

The identity (1) immediately follows from (2) and (3).

Yours sincerely,

M. T. L. Bizley
(55 High Street, Epsom, Surrey)

Dear Editor,

There is an interesting extension to Problem 11.5 in Volume 11, Number 2 of *Mathematical Spectrum*.

If the notation is extended so that $A \times B \times C$ means the midpoint of the triangle ABC, then Napoleon's Theorem may be stated thus:

$$(A \times B \times (B + A)) + (B \times C \times (C + B)) = (A \times C \times (A + C)).$$

A student solving the original problem should find the above relatively easy to prove.

Perhaps the first task is to give some 'geometric meaning' to the notation, for there can be no doubt that this is a superb example of how notation can obscure the beauty of mathematics.

Yours faithfully,
J. D. WARWICK
(Faringdon School, Faringdon, Oxon)

Dear Editor,

A Bayesian look at the jury system

On page 46 of Volume 11, Number 2, Hille incorrectly interprets the symbol $P(G^+|J^-)$ as 'the probability that guilty persons are acquitted' and as a consequence identifies a paradox. The paradox disappears when the correct interpretation is given to $P(G^+|J^-)$, namely the probability that acquitted persons are guilty. Similarly with the symbol $P(G^-|J^+)$ on page 47.

Such an error could be confusing for some of your readers.

Yours sincerely,
D. CULPIN
(Division of Mathematics and Statistics, CSIRO, Lindfield, NSW, Australia)

(A similar point has been made by some other readers.)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

12.1. If $n_1, \ldots, n_r$ are integers greater than one with sum n, show that

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_r}{2} \leq \binom{n-r+1}{2},$$

with strict inequality when $r > 1$, where () denotes the binomial coefficient.

12.2. For which real numbers x is

$$\{x + \sqrt{(x^2+1)}\}^{1/3} + \{x - \sqrt{(x^2+1)}\}^{1/3}$$

an integer?

12.3. (Submitted by R. C. Lyness). A circle is tangent internally to the circumcircle of triangle ABC and also to the sides AB, AC at P, Q respectively. Prove that the midpoint of segment PQ is the centre of the incircle of triangle ABC.

Solutions to Problems in Volume 11, Number 2

11.4. A man rows with uniform speed v m.p.h. in a straight line against a current of c m.p.h. After 1 hour his hat falls off; after another hour he notices, turns back and catches up with his hat where he first started rowing. Find v/c. If now his hat falls off after 1 mile instead of 1 hour, with all the other statements as previously, determine c and comment on the fact that c is independent of v in this case.

Solution

In the first instance, the man rows $2(v - c)$ miles before he turns round; he then retraces his distance at $(v + c)$ m.p.h., which takes him $2(v - c)/(v + c)$ hours. The hat has covered $v - c$ miles at c m.p.h., taking $(v - c)/c$ hours. Hence

$$\frac{2(v-c)}{v+c} + 1 = \frac{v-c}{c},$$

which simplifies to give $v/c = 3$.

In the second instance, the man rows $1 + v - c$ miles before he turns round; he retraces his steps in $(1 + v - c)/(v + c)$ hours. The hat covers the 1 mile in time $(1/c)$ hours. Hence

$$1 + \frac{1 + v - c}{v + c} = \frac{1}{c},$$

which simplifies to give $c = 1/2$. Thus, if the river is flowing at $(1/2)$ m.p.h., the man will always catch up with his hat where he first started rowing no matter how strong a rower he might be.

11.5. A sum and product are defined on the points of the plane as follows: $A + B$ is the unique point such that A, B and $A + B$ form an equilateral triangle, described in an anticlockwise direction, $A \times B$ is the midpoint of the straight line joining A and B. Show that

$$A \times (B + C) = (B + A) \times (A + C).$$

Solution

In the Argand diagram, suppose that A, B represent the complex numbers a, b. If $A + B$ represents the complex number x, then

$$x - a = (b - a)\, e^{2\pi i/3},$$

from which

$$x = a(1 - e^{2\pi i/3}) + b\, e^{2\pi i/3}.$$

Also, $A \times B$ represents the complex number $(a + b)/2$. It follows that $A \times (B + C)$ represents

$$\tfrac{1}{2}\{a + b(1 - e^{2\pi i/3}) + c\, e^{2\pi i/3}\},$$

whereas $(B + A) \times (A + C)$ represents

$$\tfrac{1}{2}\{b(1 - e^{2\pi i/3}) + a\, e^{2\pi i/3} + a(1 - e^{2\pi i/3}) + c\, e^{2\pi i/3}\},$$

and these are the same.

For a generalization of this problem, see the letter from J. D. Warwick in this issue.

11.6. A number of cards are dealt into m not necessarily equal piles. They are then collected together and redealt into $m + k$ piles, where $k > 0$. Show that there are at least $k + 1$ cards which are in smaller piles in the second dealing than in the first.

Solution

Suppose not. Then there are at most k cards in smaller piles in the second dealing than in the first, and these are contained in at most k piles. Thus there are at least m piles in the second dealing with the property that every card in these piles is in a pile at least as large as the pile in which it occurs in the first dealing. Let these piles have $q_1, q_2, \ldots, q_m$ cards, with $q_1 \leq q_2 \leq \cdots \leq q_m$, and let the original piles have $p_1, \ldots, p_m$ cards, with $p_1 \leq p_2 \leq \cdots \leq p_m$. Then

$$p_1 + p_2 + \cdots + p_m > q_1 + q_2 + \cdots + q_m.$$

Suppose that $p_1 \leq q_1, \ldots, p_{s-1} \leq q_{s-1}, p_s > q_s$; such an s certainly exists. Then each card in the first s piles in the second dealing must come from the first $s - 1$ piles in the first dealing. Thus

$$\begin{aligned} &p_1 + \cdots + p_{s-1} \geq q_1 + \cdots + q_s \\ &\Rightarrow q_1 + \cdots + q_{s-1} \geq q_1 + \cdots + q_s \\ &\Rightarrow q_s \leq 0, \end{aligned}$$

and this is not so.

Book Reviews

Exploratory Data Analysis. By JOHN W. TUKEY. Addison-Wesley Publishing Company, Inc, London, 1977. Pp. xvi + 688. £14·40.

According to the author, a leading statistician, this book is based on the following notable principle:

It is important to understand what you CAN DO before you learn to measure how WELL you seem to have DONE it.

Most introductory textbooks in statistics pay only lip service to data analysis. In them, histograms and measures of location and dispersion are dispensed with quickly in the first chapter never to reappear in the rest of the book, where the now standard array of test and estimation procedures is developed with varying degrees of competence. This state of affairs is entirely understandable. It is very much more difficult to give an authoritative account of how to analyse data effectively than it is to write about a well-established mathematical theory of statistical inference, and to illustrate that theory with artificial data. What sets the practice of statistics apart from other mathematically oriented disciplines is that every genuine problem, amenable to a statistical treatment, is unique. We may be guided by general principles of analysis and be armed with a box of statistical techniques, but our solution should not be directed by the principles and techniques. We should expect that different people would analyse the problem differently: there is no uniquely correct answer.

It is extremely difficult to write about analysing data effectively and Professor Tukey, realising this only too well, restricts himself to the slightly less difficult task of describing and applying the techniques of exploratory data analysis without becoming too involved with the intricacies of the particular subject area, be it biology, agriculture, economics, psychology, etc. He puts it like this: a Scotland Yard detective would not do particularly well as a Texas ranger trailing cattle thieves, but he would be far from useless because he has certain general understandings of detective work that would help him anywhere; and Tukey describes exploratory data analysis as numerical detective work. This is the '...what you CAN DO...' and mathematical statistics, when appropriate, tells you '...how WELL you seem to have DONE it.'. Tukey has honoured the principle throughout, although there remain doubts in some places about the manner of its presentation.

This book is unique for many reasons: amongst others, standard distributions are only mentioned in passing in the earlier chapters, there are no χ^2-, F- or t-tests; there is virtually no mention of a *mean* or variance, data sets abound, and the mathematical level is below O-level.

The last point is not meant as a criticism. On the contrary, it means that the book is accessible to a wide audience and also serves to underline the point that statistics is not mathematics. In fact, professional statisticians should find this book useful. The reviewer is certainly one who has benefited from studying it and applying the techniques described in exploring complex problems. Anyone with an interest in trying to understand data who has a knowledge of arithmetic, including logarithms, roots and straight-line graphs, could benefit from *participating* in this book.

Before describing the contents of the book, let me air a general criticism. In an attempt presumably to break away from the shackles of the more traditional presentation, the book is strewn with a completely new jargon that requires a ten-page glossary. This is particularly irritating when perfectly acceptable words are dispensed with; for example, *H-spread* is used instead of *interquartile range*. One defence is that most of the material is presented for the first time, but expressions like 3*RSSH*3*RSSH*3 that appear in the chapters on smoothing, although logical in their context, are a little difficult to digest. Another general comment is that some explanations are less than helpful, sometimes to the point of confusing what often transpires to be a simple idea. Also when giving very detailed arithmetic presentations, as Tukey does (including useful and helpful suggestions about the construction and presentation of tables and graphs) inaccuracies and inconsistencies are almost unavoidable and the reader must be constantly on the look-out for possible errors.

A brief description and comments about what constitutes the backbone of the book and the recommended basic course should be sufficient as a review of EDA; namely, Chapters 1 to 7, and 10 and 11.

In Chapter 1, *Scratching Down Numbers*, the *stem-and-leaf* plot is introduced, in my opinion, a superior alternative to the conventional histogram. Amongst others, it has the advantage of both storing and displaying data; that is, it has the visual effectiveness of the histogram but most of the numerical information is retained. An elementary example should suffice to illustrate this simple but novel technique. The marks in a recent undergraduate exam at Durham University were as follows: 81, 55, 57, 76, 71, 88, 67, 83, 41, 55, 62, 101, 85, 72, 86, 59, 75. Stem-and-leaf plots for these data are:

3		3	
4	1	4	1
5	5759	5	5579
6	72	6	27
7	6125	7	1256
8	18356	8	13568
9		9	
10	1	10	1
	(a)		(b)

The 'stem', in this example the tens, are to the left of the vertical line and the 'leaves', the units, are to the right of the stem. Thus the line 5|5759 in (a) refers to the marks 55, 57, 55, 59, as they appear in order. The final version (b), if required, orders the leaves within stems. We thus have a histogram on its side with the depth of each bar formed by the number of leaves alongside the corresponding stem. The effort required is less than that to record the numbers. Medians, quartiles, and other summary information described in Chapter 2, *Schematic Summaries*, can be obtained directly from the stem-and-leaf plot.

The box and whisker plot is introduced in Chapter 2 as an additional way of displaying a single *batch* of data, although its true value is only realised when it is used in Chapter 4, *Effective Comparison*. Possible relationships between locations and spreads of several batches that are to be compared can be investigated by the graphical methods of Chapter 5, *Plots of Relationship*. Knowledge of such a relationship can indicate transformations of the data, such as logarithms or square roots, that make comparisons of batches more meaningful. Transformation or re-expression, first introduced in Chapter 3, *Easy*

Re-expression, is an important idea that permeates the whole book. In most texts it may receive a paragraph of discussion but thereafter is usually forgotten. Another theme that is present throughout may be summed up by the equation

$$DATA = SMOOTH + ROUGH.$$

The *SMOOTH* is an estimate of a particular structure imposed on the data and the *ROUGH* can give useful information regarding the appropriateness of the structure, outlying observations, data transformations, and variation. An example would be fitting a straight line to a scatter of points in the (x, y)-plane. The fitted line is the *SMOOTH* and the deviations constitute the *ROUGH*. Chapters 10 and 11 on *Two-Way Tables* are further illustrations of this theme. Data on suicide rates classified by age and country can often be modelled by assuming that there are fixed age and country effects which are additive on some scale. In these two chapters the methods of fitting and checking an appropriate model—the *SMOOTH*—are discussed in some detail. Finally, Chapter 7, *Smoothing Sequences*, is concerned primarily with smoothing time series data using medians of three and modifications. No specified model is envisaged here, but the principle is the same and the object of the analysis is to give a good visual impression of the underlying trend. Apart from a cluster of four chapters on distributions at the end, which I found very obscure in places, the remainder of the book is concerned with extending the ideas introduced in the basic course.

In summary, I find that Professor Tukey has presented an honest and exciting approach to data analysis—a message can be extracted from data when explored correctly. The author asks the reader to be innovative and sceptical, and if this attitude is also adopted towards the book (in place of an acceptance of it as gospel), there is much to be gained.

Data Analysis and Regression. By FREDERICK MOSTELLER and JOHN W. TUKEY. Addison-Wesley Publishing Company, Inc, London 1977. Pp. xviii + 588. £14·40.

Although it includes some of the material presented in EDA, this book is much more advanced and should follow EDA plus a course in theoretical statistics. In addition, the reader needs to have had considerable experience in the theory and applications of statistics in order to benefit from studying DA & R. It is not, therefore, suitable for consideration at the level normally associated with books reviewed in *Mathematical Spectrum*.

University of Durham A. H. SEHEULT

Mathematical Carnival. By MARTIN GARDNER. Penguin Books Ltd, Harmondsworth, Middlesex, 1978. Pp. x + 274. £0·95.

For those who know a book of this title by this author: yes, it is the same book as that first published in the U.S.A. in 1975 and in 1976 in Britain, now available in the Pelican Series. So you can afford to have your own copy. Secondly, for those familiar with Martin Gardner's books and articles in *Scientific American*, this is the seventh book of collected articles presented together with points arising from the magazine's readers.

For those (and I suppose there may be a few) who are enthusiastic about recreational mathematics but have not yet made acquaintance with these books of Martin Gardner, this one is typical of them, and as such is warmly recommended. You will love it. The author turns his mathematical mind to an amazing variety of topics. Some are closely related to 'conventional' school mathematics, for example, Pascal's triangle, the trisection of an angle and the equation of an ellipse. Others are drawn from higher things such as hypercubes and Cantor's sets (for which all alephs are infinite but some alephs are more infinite than others). Much of the subject matter is unashamedly low-brow: card tricks, coin puzzles, mental arithmetic frauds, persuading hard-boiled eggs in and out of milk bottles and a delicious game called Sprouts which (as its inventor claims) really is consumed avidly by a three-year-old.

Next year in Stockholm, last year in Marienbad, shooting bears at the poles or using Compton's tube in darkest Africa, everywhere Martin Gardner's pellucid analysis and sense of fun cause delight to dawn.

Answers are supplied and the style is always sympathetic to the reader, who is never made to feel inferior, even when out of his mathematical depth. The book is an obvious choice for any secondary school library and many teachers—as well as pupils—will want to have a copy of their own. But finally, a note for those who find mathematical games and puzzles irksome. Not to worry: an interest in these things has always been a sufficient but not a necessary condition for the existence of a mathematical mind.

St. Paul's School, London C. J. J. COLLIER

Proof in Mathematics. By P. R. BAXANDALL, W. S. BROWN, G. St. C. ROSE and F. R. WATSON. Institute of Education, University of Keele, 1978. Pp. v + 130. £0·75 (including postage).

Most of us would agree that proofs are an essential part of mathematics. Proofs dominate university courses, and play a fairly prominent role in sixth-form work. But in spite of all this detailed study, many sixth-formers and even undergraduates remain hazy and uncertain about what a proof actually does, or why it is wanted. This book tries to tackle the idea of proof in general. It contains a rich supply of particular examples of proofs, which are included to illustrate general principles: proof by induction, disproof by counterexample, proof of impossibility, and so on.

The introduction describes the book as a 'background book' for teachers and student teachers which the authors hope will also be accessible to sixth-formers and first-year undergraduates. There is much of value here for all these groups. Moreover, in general the material is lucidly explained. A sixth-former should be able to read most of it without help from a teacher, although it would be good for the teacher to read it as well! The most difficult parts, such as the discussion of the role of axioms and the foundations of mathematics, are sensibly separated from the broader issues in a series of appendices (which occupy just over half the pages). Most of the chapters include exercises; hints and solutions for these are provided. There is also a good bibliography.

Because it aims at being elementary, there are places where the discussion is oversimplified. An example of this is the comparison of Euclid's proof that the base angles of an isosceles triangle are equal with Pappus' version. The statement given here that 'Pappus' proof is simpler, clearer and therefore better than Euclid's . . .' ignores a number of other issues which are important to this comparison. (For a more balanced account see Stewart, *Concepts of Modern Mathematics* (Penguin, 1975) Chapter 2.)

The book is 'home-produced' by offset-litho. This means that the print is small, and that the pages are rather flimsily held together by three staples down the centre. On the other hand, it is sold at an extremely low price. It is to be hoped that this will encourage many students to buy it for themselves; they will certainly profit from reading it.

University of Birmingham WILLIAM WYNNE WILLSON

Mathematics: An Introduction to its Spirit and Use. With introductions by MORRIS KLINE. W. H. Freeman and Company Limited, Reading, 1979. Pp. 249. £8·90 (hard cover); £4·70 (soft cover).

A collection of 40 articles, in this case chosen from *Scientific American* over approximately the last 30 years, stands or falls by the wisdom of the selector. Morris Kline has made the selection and grouped them into six sections each accompanied by his introductory essay. Care is taken to ensure that the technical level of the mathematics required to appreciate the articles is not high; it is not beyond that of a student in the first year of an 'A' Level Science course, or that of an able Arts student. Thus no calculus is required.

The opening section on history covers the lives of some famous mathematicians e.g. Cardano or Pascal, as well as the developments of some branches of mathematics. There is an essay on the Rhind papyrus by James R. Newman, and one on the invention of analytical geometry by Carl Boyer. Unsolved problems and novel ways of tackling problems are considered in the essays on number. Some examples of a not too demanding nature are set and the solutions given at the end of the book. Prime numbers and Cantorian sets are specially discussed. The section on geometry has many articles by Martin Gardner, and this is the part that should appeal most to the problem-loving mathematician. Euclidean and projective geometry as well as topology are considered. Some of the articles on statistics and probability make heavy demands on a sixth former's grasp of ideas. However, Martin Gardner's account of some probability paradoxes is not difficult to grasp and yet very thought-provoking.

There is a section on symbolic logic and computers. The first essay is slightly difficult to follow for a beginner owing to a miscolouring on the left-hand diagram referring to Eulerian circles on p. 172; the blue shading has been misprinted, so that it extends over the whole of a circle rather than part. On the next page the diagrams have been misprinted on the right instead of the left side. Overall, the diagrams are very clear and well-presented, and since they are numerous the value of the book is considerably enhanced. The final section is devoted to the application of mathematics, first to problems in sound and then to modern topics such as the theory of games and operational research.

The appearance, printing and general format of the book are attractive. It seems a valuable addition to a school library.

St. Mary's Sixth Form College, Middlesborough F. A. JACKSON

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Contents

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