MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 Mars 2009**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M369. *Proposé par l'Équipe de Mayhem.*

Soit $A(0,0)$, $B(6,0)$, $C(6,4)$ et $D(0,4)$ les sommets d'un rectangle. Par le point $P(4,3)$, on trace d'une part une droite horizontale coupant BC en M et AD en N et d'autre part une droite verticale coupant AB en Q et CD en R. Montrer que AP, DM et BR passent toutes par le même point.

M370. *Proposé par l'Équipe de Mayhem.*

(a) Montrer que $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ pour tous les angles A et B.

(b) Montrer que $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$ pour tous les angles C et D.

(c) Trouver la valeur exacte de $\cos 20° + \cos 60° + \cos 100° + \cos 140°$, sans l'aide d'une calculatrice.

M371. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Un segment AB de longueur 3 contient un point C tel que $AC = 2$. On construit d'un même côté de AB deux triangles équilatéraux ACF et CBE. Déterminer l'aire du triangle AKE si K est le point milieu de FC.

M372. *Proposé par l'Équipe de Mayhem.*

Soit x un nombre réel satisfaisant $x^3 = x + 1$. Trouver des entiers a, b et c de sorte que $x^7 = ax^2 + bx + c$.

M373. *Proposé par Kunal Singh, étudiant, Kendriya Vidyalaya School, Shillong, Inde.*

Les côtés d'un triangle sont mesurés par trois nombres entiers consécutifs et le plus grand angle est le double du plus petit. Déterminer la longueur des côtés du triangle.

M374. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit p un nombre premier fixé, avec $p \geq 3$. Trouver le nombre de solutions de $x^3 + y^3 = x^2y + xy^2 + p^{2009}$, où x et y sont des entiers.

M375. *Proposé par Neculai Stanciu, École Technique Supérieure de Saint Mucenic Sava, Berca, Roumanie.*

Déterminer toutes les solutions réelles du système d'équations

$$\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4 \, ; \qquad x^2 + y^2 + z^2 = 9 \, ; \qquad xyz = \frac{9}{2} \, .$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

M369. *Proposed by the Mayhem Staff.*

A rectangle has vertices $A(0,0)$, $B(6,0)$, $C(6,4)$, and $D(0,4)$. A horizontal line is drawn through $P(4,3)$, meeting BC at M and AD at N. A vertical line is drawn through P, meeting AB at Q and CD at R. Prove that AP, DM, and BR all pass through the same point.

M370. *Proposed by the Mayhem Staff.*

(a) Prove that $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$ for all angles A and B.

(b) Prove that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$ for all angles C and D.

(c) Determine the exact value of $\cos 20° + \cos 60° + \cos 100° + \cos 140°$, without using a calculator.

M371. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Suppose that the line segment AB has length 3 and C is on AB with $AC = 2$. Equilateral triangles ACF and CBE are constructed on the same side of AB. If K is the midpoint of FC, determine the area of $\triangle AKE$.

M372. *Proposed by the Mayhem Staff.*

A real number x satisfies $x^3 = x + 1$. Determine integers a, b, and c so that $x^7 = ax^2 + bx + c$.

M373. *Proposed by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.*

The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

M374. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that p is a fixed prime number with $p \geq 3$. Determine the number of solutions to $x^3 + y^3 = x^2y + xy^2 + p^{2009}$, where x and y are integers.

M375. *Proposed by Neculai Stanciu, Saint Mucenic Sava Technological High School, Berca, Romania.*

Determine all real solutions to the system of equations

$$\frac{1}{x^2} + \frac{4}{y^2} + \frac{9}{z^2} = 4; \quad x^2 + y^2 + z^2 = 9; \quad xyz = \frac{9}{2}.$$

Mayhem Solutions

M332. *Proposed by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.*

A closed right circular cylinder has an integer radius and an integer height. The numerical value of the volume is four times the numerical value of its total surface area (including its top and bottom). Determine the smallest possible volume for the cylinder.

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Let r and h be the radius and the height of the closed right circular cylinder. The volume of such a cylinder is $V = \pi r^2 h$ and the surface area is $A = 2\pi r^2 + 2\pi rh$.

From the hypotheses, $\pi r^2 h = 4(2\pi r^2 + 2\pi rh)$, or $rh = 8r + 8h$, or $rh - 8r - 8h + 64 = 64$, or $(r-8)(h-8) = 64$. Note that $r - 8 > -8$ and $h - 8 > -8$. This gives us the following possibilities:

$r-8$	1	2	4	8	16	32	64
$h-8$	64	32	16	84	4	2	1
r	9	10	12	16	24	40	72
h	72	40	24	16	12	10	9
V	5832π	4000π	3456π	4096π	6912π	16000π	46656π

Thus, the smallest possible volume for the cylinder is 3456π.

Also solved by DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comăneşti, Romania. There was 1 incomplete solution submitted.

M333. *Proposed by the Mayhem Staff.*

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove **1**, **3**, or **6** toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from **36** to **40** inclusive does Brenda have a winning strategy?

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.

We can build a table of winning and losing positions for Anne. Her winning positions start with **1**, **3**, or **6**, since she can immediately win by removing all of the toothpicks.

Starting with **2** toothpicks, Anne must remove **1** toothpick, leaving Brenda with **1**, and so Brenda wins. Starting with **4** toothpicks, Anne must remove **1** or **3** toothpicks, leaving Brenda with **3** or **1** (respectively), and so Brenda wins by removing all of the toothpicks.

Starting with **5** toothpicks, Anne can remove **3** toothpicks, thus leaving Brenda with **2** toothpicks. Since **2** is a losing position for whoever goes first, then Brenda loses, so Anne wins.

So far, **1**, **3**, **5**, and **6** are winning positions for Anne, while **2** and **4** are losing positions for Anne.

Starting with a pile of size n, Anne must reduce the pile to one of size $n-1$, $n-3$, or $n-6$ and pass to Brenda. If the person who goes first has a winning strategy starting with a pile of each of these sizes, then Anne will lose. In other words, if Anne has a winning strategy starting with piles of size $n-1$, $n-3$, and $n-6$, then Anne will lose starting with a pile of size n, as Brenda can implement Anne's strategy for the smaller pile and win, no matter what Anne does. If one or more of these pile sizes are such that the first person does not have a winning strategy, then Anne should reduce to this size, thus preventing Brenda from being able to win, and so Anne herself will win.

We can examine the cases from $n = 7$ to $n = 40$, obtaining the following lists:

Winning positions for Anne: **1**, **3**, **5**, **6**, **7**, **8**, **10**, **12**, **14**, **15**, **16**, **17**, **19**, **21**, **23**, **24**, **25**, **26**, **28**, **30**, **32**, **33**, **34**, **35**, **37**, **39**.

Losing positions for Anne: **2**, **4**, **9**, **11**, **13**, **18**, **20**, **22**, **27**, **29**, **31**, **36**, **38**, **40**.

Therefore, Brenda wins for $n = 36, 38, 40$.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB.
See the Problem of the Month column in [2007 : 15–17] for a similar problem with a more detailed explanation.

M334. *Proposed by the Mayhem Staff.*

(a) Determine all integers x for which $\dfrac{x-3}{3x-2}$ is an integer.

(b) Determine all integers y for which $\dfrac{3y^3+3}{3y^2+y-2}$ is an integer.

I. *Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.*

Let A be an integer such that $A = \dfrac{x-3}{3x-2}$. Then $3A$ is an integer and
$$3A = \frac{3x-9}{3x-2} = \frac{3x-2-7}{3x-2} = 1 - \frac{7}{3x-2}.$$

Thus, $\dfrac{7}{3x-2}$ is an integer; that is, $3x-2$ is a divisor of 7, so $3x-2$ is one of ± 1, ± 7. Since x is an integer, $x = 1$ or $x = 3$. This answers part (a).

Now let B be an integer such that
$$
\begin{aligned}
B &= \frac{3y^3+3}{3y^2+y-2} = y - \frac{y^2-2y-3}{3y^2+y-2} \\
&= y - \frac{(y-3)(y+1)}{(y+1)(3y-2)} = y - \frac{y-3}{3y-2}.
\end{aligned}
$$

Since y is an integer, $\dfrac{y-3}{3y-2}$ is an integer. From the solution to part (a), $y = 1$ or $y = 3$, which answers part (b).

II. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

We show that the only integer solutions to part (a) are $x = 1$ and $x = 3$.

Let $f(x) = \dfrac{x-3}{3x-2}$. Then $f(0) = \dfrac{3}{2}$, $f(1) = -2$, $f(2) = -\dfrac{1}{4}$, and $f(3) = 0$. Of these, only $f(1)$ and $f(3)$ are integers.

If $x > 3$, then $f(x)$ is not an integer, since $3x-2 > x-3 > 0$ for $x > 3$ and so $0 < \dfrac{x-3}{3x-2} < 1$.

If $x \leq -1$, let $x = -s$ where $s \geq 1$. Then $f(x) = f(-s) = \dfrac{s+3}{3s+2}$. Since $3s+2 > s+3 > 0$ for $s \geq 1$, $f(-s)$ is not an integer by a similar argument so, $f(x)$ is not an integer.

Therefore, $f(x)$ is an integer for integer values of x if and only if $x = 1$ or $x = 3$.

Also solved by RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; and TITU ZVONARU, Comăneşti, Romania. There was one incorrect and one incomplete solution submitted.

M335. *Proposed by the Mayhem staff.*

In a sequence of four numbers, the second number is twice the first number. Also, the sum of the first and fourth numbers is **9**, the sum of the second and third is **7**, and the sum of the squares of the four numbers is **78**. Determine all such sequences.

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA.

Let a, b, c, and d represent the first, second, third and fourth number, respectively. We can now write the given information as $b = 2a$, $a + d = 9$, $b + c = 7$ and $a^2 + b^2 + c^2 + d^2 = 78$.

The first three equations allow us to rewrite b, c, and d in terms of a, obtaining $b = 2a$, $c = 7 - b = 7 - 2a$, and $d = 9 - a$.

Therefore,

$$
\begin{aligned}
a^2 + (2a)^2 + (7 - 2a)^2 + (9 - a)^2 &= 78\,, \\
a^2 + 4a^2 + 49 - 28a + 4a^2 + 81 - 18a + a^2 - 78 &= 0\,, \\
5a^2 - 23a + 26 &= 0\,, \\
(5a - 13)(a - 2) &= 0\,,
\end{aligned}
$$

hence $a = \dfrac{13}{5}$ or $a = 2$.

Therefore, the sequences are $a = \dfrac{13}{5}$, $b = \dfrac{26}{5}$, $c = \dfrac{9}{5}$, $d = \dfrac{32}{5}$ and $a = 2$, $b = 4$, $c = 3$, $d = 7$. Both sequences satisfy the given requirements.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; BILLY SUANDITO, Palembang, Indonesia; and TITU ZVONARU, Comăneşti, Romania. There was one incorrect and one incomplete solution submitted.

M336. *Proposed by the Mayhem Staff.*

A lattice point is a point (x, y) in the coordinate plane with each of x and y an integer. Suppose that n is a positive integer. Determine the number of lattice points inside the region $|x| + |y| \le n$.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina, modified by the editor.

We can rewrite the given inequality as the equations $|x| + |y| = 0$ and $|x| + |y| = k$ for $1 \le k \le n$, where $x, y \in \mathbb{Z}$.

The equation $|x| + |y| = 0$ has one integer solution only, namely $(x, y) = (0, 0)$.

Consider next $|x| + |y| = k$, for an integer k with $1 \leq k \leq n$. We can remove the absolute values by splitting into four cases:

Case 1. The integers x and y satisfy $x + y = k$, where $x \geq 0$ and $y \geq 0$. This has solutions $(k, 0)$, $(k - 1, 1)$, ..., $(1, k - 1)$, $(0, k)$, for a total of $k + 1$ solutions.

Case 2. The integers x and y satisfy $x - y = k$, where $x \geq 0$ and $y < 0$. This has solutions $(k - 1, -1)$, $(k - 2, -2)$, ..., $(1, -(k - 1))$, $(0, -k)$, for a total of k solutions.

Case 3. The integers x and y satisfy $-x + y = k$, where $x < 0$ and $y \geq 0$. This case is the same as Case 3, but with the roles of x and y switched, so there are a total of k solutions here as well.

Case 4. The integers x and y satisfy $-x - y = k$, where $x < 0$ and $y < 0$. This has solutions $\big(-1, -(k - 1)\big)$, $\big(-2, -(k - 2)\big)$, ..., $\big(-(k - 2), -2\big)$, $\big(-(k - 1), -1\big)$, for a total of $k - 1$ solutions.

Thus, for each k with $1 \leq k \leq n$, the equation $|x| + |y| = k$ has $(k + 1) + k + k + (k - 1) = 4k$ solutions.

Therefore, the number of lattice points inside the region is

$$1 + \sum_{k=1}^{n} 4k \;=\; 1 + 4\sum_{k=1}^{n} k \;=\; 1 + 4 \cdot (1 + 2 + \cdots + n)$$
$$=\; 1 + 4 \cdot \frac{n(n + 1)}{2} \;=\; 2n^2 + 2n + 1 \,.$$

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain. There were one incorrect and two incomplete solutions submitted.

M337. *Proposed by the Mayhem Staff.*

On sides AB and CD of rectangle $ABCD$ with $AD < AB$, points F and E are chosen so that $AFCE$ is a rhombus.

(a) If $AB = 16$ and $BC = 12$, determine EF.

(b) If $AB = x$ and $BC = y$, determine EF in terms of x and y.

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

We present the solution to (b), which is a general version of the specific case in (a).

Suppose that $AF = FC = CE = EA = m$. Let O be the point of intersection of diagonals AC and EF of rhombus $AFCE$. Note that AC and EF are perpendicular and bisect each other at O.

456

By the Pythagorean Theorem,

$$
\begin{aligned}
CF^2 - FB^2 &= CB^2 , \\
m^2 - (x - m)^2 &= y^2 , \\
m^2 - x^2 + 2mx - m^2 &= y^2 , \\
2mx &= x^2 + y^2 , \\
m &= \frac{x^2 + y^2}{2x} .
\end{aligned}
$$

Now, $AF = m = \frac{x^2 + y^2}{2x}$ and $AC = \sqrt{AB^2 + BC^2} = \sqrt{x^2 + y^2}$. Also, $OA = \frac{1}{2}AC$. Thus, by the Pythagorean Theorem again,

$$
\begin{aligned}
OF^2 &= AF^2 - OA^2 \\
&= \left(\frac{x^2 + y^2}{2x}\right)^2 - \left(\frac{\sqrt{x^2 + y^2}}{2}\right)^2 \\
&= \left(\frac{x^4 + y^4 + 2x^2y^2}{4x^2}\right) - \left(\frac{x^2 + y^2}{4}\right) \\
&= \frac{x^4 + y^4 + 2x^2y^2 - x^4 - x^2y^2}{4x^2} \\
&= \frac{y^4 + x^2y^2}{4x^2} .
\end{aligned}
$$

Therefore,

$$
OF = \sqrt{\frac{y^4 + x^2y^2}{4x^2}} = \frac{\sqrt{y^2(y^2 + x^2)}}{2x} = \frac{y\sqrt{x^2 + y^2}}{2x} ,
$$

and

$$
EF = 2\,OF = \frac{2y\sqrt{x^2 + y^2}}{2x} = \frac{y\sqrt{x^2 + y^2}}{x} .
$$

In part (a), this yields

$$
EF = \frac{12\sqrt{16^2 + 12^2}}{16} = \frac{12(20)}{16} = 15 .
$$

Also solved by *EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB (part (a) only); RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; BILLY SUANDITO, Palembang, Indonesia; LUYAN ZHONG-QIAO, Columbia International College, Hamilton, ON; and TITU ZVONARU, Comăneşti, Romania.*

Problem of the Month

Ian VanderBurgh

Here is a problem that might seem to be not very interesting initially, but turns out to have a whole lot of unexpected solutions.

Problem (2005 Canadian Open Mathematics Challenge) In the grid shown, each row has a value assigned to it and each column has a value assigned to it. The number in each cell is the sum of its row and column values. For example, the "8" is the sum of the value as-

3	0	5	6	−2
−2	−5	0	1	y
5	2	x	8	0
0	−3	2	3	−5
−4	−7	−2	−1	−9

signed to the **3**rd row and the value assigned to the **4**th column. Determine the values of x and y.

It is tempting first of all to give labels to the values that are assigned to the rows and columns in order to be able to dive into some algebra. Let's label the values assigned to the five columns A, B, C, D, E and the values assigned to the five rows a, b, c, d, e.

Each entry in the table gives us an equation involving two of these variables. For example, the −3 in row 4, column 2 gives us $d + B = -3$, and the −9 in row 5, column 5, gives us $e + E = -9$. We could proceed and write down **25** equations, one for each entry in the table. These equations would include **12** variables – the **10** that label the rows and columns together with x and y. We could then spend pages and pages wading through algebra trying to come up with the answers. At this point, we would hope that there has to be a better way. Maybe we should have looked before we leapt!

Here are three neat ways to approach this. (As a point of interest, I was recently talking about this problem with a friend while driving and so neither of us really wanted to do any algebra, and so were forced to come up with better ways to do it.)

Solution 1. If we choose five entries from the table which include one from each row and one from each column, then the sum of these entries is constant no matter how we choose the entries, as it is always equal to

$$A + B + C + D + E + a + b + c + d + e.$$

Can you see why? Here are three ways in which this can be done (looking at the underlined numbers in the two grids below and the grid on the following page):

$\underline{3}$	0	5	6	−2
−2	−$\underline{5}$	0	1	y
5	2	x	$\underline{8}$	0
0	−3	$\underline{2}$	3	−5
−4	−7	−2	−1	−$\underline{9}$

3	0	5	6	−$\underline{2}$
−2	−5	0	$\underline{1}$	y
5	2	$\underline{x}$	8	0
0	−$\underline{3}$	2	3	−5
−$\underline{4}$	−7	−2	−1	−9

Therefore,

$$3 + (-5) + 2 + 8 + (-9)$$
$$= \ (-4) + (-3) + x + 1 + (-2)$$
$$= \ 3 + y + 2 + (-2) + 3 \, ,$$

$\underline{3}$	0	5	6	-2
-2	-5	0	1	$\underline{y}$
5	$\underline{2}$	x	8	0
0	-3	2	$\underline{3}$	-5
-4	-7	$\underline{-2}$	-1	-9

or $-1 = x - 8 = y + 6$. Thus, $x = 7$ and $y = -7$.

Solution 2. Consider the first two entries in row **1**. From the labels above, we have $3 = A + a$ and $0 = B + a$. Subtracting these, we obtain the equation $3 = 3 - 0 = (A + a) - (B + a) = A - B$.

Notice that whenever we take entries in columns **1** and **2** from the same row, their difference will always equal $A - B$, which is equal to **3**. Similarly, since the difference between the **0** and the **5** in the first row is **5**, then every entry in column **3** will be **5** greater than the entry in column **2** from the same row. In row **3**, we see that $x = 2 + 5 = 7$.

Also, since the difference between the **6** and the -2 in the first row is **8**, then every entry in column **5** is **8** less than the entry in column **4** from the same row. In row **2**, we see that $y = 1 - 8 = -7$. Thus, $x = 7$ and $y = -7$.

Solution 3. Consider the sub-grid

0	1
x	8

.

Since the **0** is in row **2** and column **3**, then $0 = b + C$. Similarly, $1 = b + D$, $8 = c + D$, and $x = c + C$.

But then $0 + 8 = (b + C) + (c + D) = (c + C) + (b + D) = x + 1$, or $x = 7$.

In a similar way, by looking at the sub-grid

1	y
8	0

we can show that $1 + 0 = y + 8$, or $y = -7$. Thus, $x = 7$ and $y = -7$.

So there are three different but neat solutions to the problem. One footnote to the final solution is that in fact, in any sub-grid of the form

p	q
r	s

, we must have $p + s = q + r$. Can you see why?

Another interesting point about this problem is that it might be easier for those who know less! If we replaced the x and the y with "?" and gave it to someone who didn't know a lot of algebra, they might find the answers faster than those of us who go immediately to algebra. Sometimes, the extra machinery that we have can get in the way.

As 2008 draws to a close, the Mayhem Editor has three enormous sets of thanks to offer. First, to the Mayhem Staff, especially to Monika Khbeis and Eric Robert, for all of their help over the past year. Second, to the Editor-in-Chief of *CRUX with MAYHEM*, Václav Linek, for all of his help and encouragement over the past year (as well as for his sharp eyes!). Third, to the Mayhem readership for their support. Please keep those problems and solutions coming!

THE OLYMPIAD CORNER

No. 274

R.E. Woodrow

With the Winter break coming up, I have decided to focus this issue mainly on providing problems for your puzzling pleasure, and to give some time for the mails to deliver the solutions to problems from 2008 numbers of the *Corner* to restore the readers' solutions file, which is particularly thin for the February 2008 number, as you will see later in the column.

To start you off we have the problems proposed but not used at the 47th IMO in Slovenia 2006. My thanks go to Robert Morewood, Canadian Team Leader at the IMO for collecting them for our use.

47th INTERNATIONAL MATHEMATICAL OLYMPIAD SLOVENIA 2006
Problems Proposed But Not Used

Contributing Countries. Argentina, Australia, Brazil, Bulgaria, Canada, Colombia, Czech Republic, Estonia, Finland, France, Georgia, Greece, Hong Kong, India, Indonesia, Iran, Ireland, Italy, Japan, Republic of Korea, Luxembourg, Netherlands, Poland, Peru, Romania, Russia, Serbia and Montenegro, Singapore, Slovakia, South Africa, Sweden, Taiwan, Ukraine, United Kingdom, United States of America, Venezuela.

Problem Selection Committee. Andrej Bauer, Robert Geretschläger, Géza Kós, Marcin Kuczma, Sventoslav Savchev.

Algebra

A1. Given an arbitrary real number a_0, define a sequence of real numbers $a_0, a_1, a_2, \ldots$ by the recursion

$$a_{i+1} = \lfloor a_i \rfloor \cdot \{a_i\}, \qquad i \geq 0,$$

where $\lfloor a_i \rfloor$ is the greatest integer not exceeding a_i, and $\{a_i\} = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for sufficiently large i.

A2. Let $a_0 = -1$ and define the sequence of real numbers $a_0, a_1, a_2, \ldots$ by the recursion

$$\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0$$

for $n \geq 1$. Show that $a_n > 0$ for $n \geq 1$.

A3. Let $c_0 = 1$, $c_1 = 0$ and define the sequence c_0, c_1, c_2, $\ldots$ by the recursion $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Let S be the set of ordered pairs (x, y) such that

$$x = \sum_{j \in J} c_j \quad \text{and} \quad y = \sum_{j \in J} c_{j-1}$$

for some finite set J of positive integers. Prove that there exist real numbers α, β, m, and M with the property that an ordered pair of non-negative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$. (By convention an empty sum is 0.)

A4. Let a_1, a_2, $\ldots$, a_n be positive real numbers. Prove that

$$\sum_{i<j} \frac{a_i a_j}{a_i + a_j} \leq \frac{n}{2(a_1 + a_2 + \cdots + a_n)} \sum_{i<j} a_i a_j \,.$$

A5. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3 \,.$$

Combinatorics

C1. There are $n \geq 2$ lamps L_1, L_2, $\ldots$, L_n arranged in a row. Each of them is either *on* or *off*. Initially the lamp L_1 is on and all of the other lamps are off. Each second the state of each lamp changes as follows: if the lamp L_i and its neighbours (L_1 and L_n each have one neighbor, any other lamp has two neighbours) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Prove that there are

(a) infinitely many n for which all of the lamps will eventually be off,

(b) infinitely many n for which the lamps will never be all off.

C2. Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside of P. Prove that for every real number x

$$\sum_P x^{a(P)} (1-x)^{b(P)} = 1 \,,$$

where the sum is taken over all convex polygons with vertices in S. (A line segment, a point, and the empty set are convex polygons of 2, 1, and 0 vertices, respectively.)

C3. A cake is shaped as an $n \times n$ square with n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that the arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a sequence of *swaps*, where a *swap* consists of selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and then moving these two strawberries to the other two corners of that rectangle.

C4. An (n,k)-*tournament* is a competition with n players held in k rounds such that

(a) Each player plays in each round, and every two players meet at most once.

(b) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.

Determine all pairs (n,k) for which there exists an (n,k)-tournament.

C5. A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A *diamond* is a unit rhombus with angles of $60°$ and $120°$. Prove that a holey triangle T can be tiled with diamonds if and only if for each $k = 1, 2, \ldots, n$ every upward equilateral triangle of side length k in T contains at most k holes.

C6. Let $\mathcal{P}$ be a convex polyhedron with no parallel edges and no edge parallel to a face other than the two faces it borders. A pair of points on $\mathcal{P}$ are *antipodal* if there exist two parallel planes each containing one of the points and such that $\mathcal{P}$ lies between them. Let A be the number of antipodal pairs of vertices and let B be the number of antipodal pairs of mid-points of edges. Express $A - B$ in terms of the numbers of vertices, edges, and faces of $\mathcal{P}$.

Geometry

G1. Let $ABCD$ be a trapezoid with $AB\|CD$ and $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, such that $AK : KB = DL : LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \quad \text{and} \quad \angle CQD = \angle ABC.$$

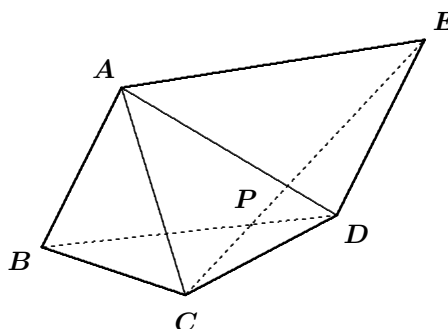Prove that the points P, Q, B, and C are concyclic.
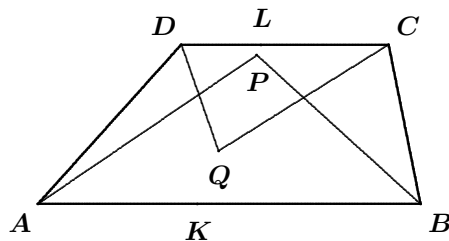
G2. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE;$$
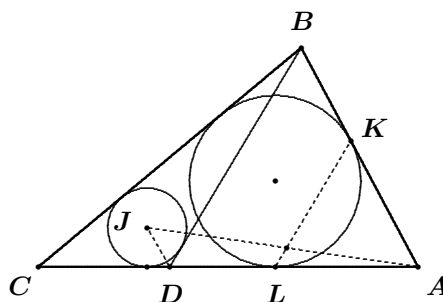$$\angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE intersect at P. Prove that the line AP bisects the side CD.

G3. A point D is chosen on the side AC of a triangle ABC with

$$\angle ACB < \angle BAC < 90°$$

in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triangle BCD. Prove that the line KL intersects the line segment AJ at its mid-point.

G4. In triangle ABC, let J be the centre of the excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.
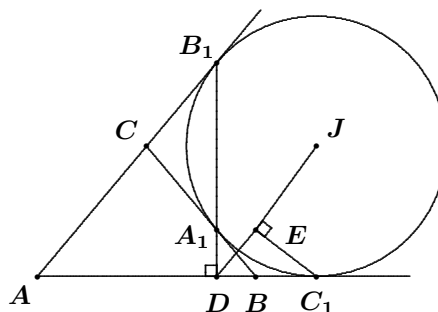
G5. Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internaly tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, and O_1 are on the same side of t. Prove that the lines AO_1, BO_2, EF, and t are concurrent.
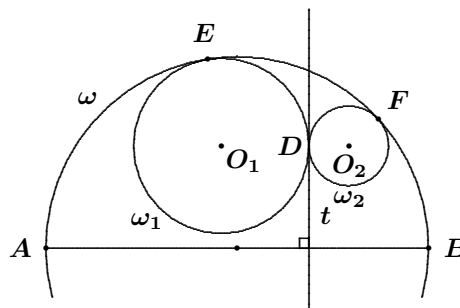
G6. In a triangle ABC, let M_a, M_b, M_c be the respective mid-points of the sides BC, CA, AB and let T_a, T_b, T_c be the mid-points of the arcs BC, CA, AB of the circumcircle of ABC not containing A, B, C, respectively. For each $i \in \{a,b,c\}$, let ω_i be the circle with diameter M_iT_i. Let p_i be the common external tangent to ω_j, ω_k such that $\{i,j,k\} = \{a,b,c\}$ and such that ω_i lies on one side of p_i while ω_j, ω_k lie on the other side. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.
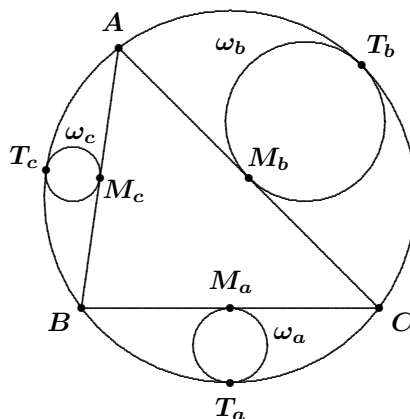
G7. Let $ABCD$ be a convex quadrilateral. A circle passing through A and D and a circle passing through B and C are externally tangent at the point P in the interior of the quadrilateral. Prove that if $\angle PAB + \angle PDC \leq 90°$ and $\angle PBA + \angle PCD \leq 90°$, then $AB + CD \geq BC + AD$.
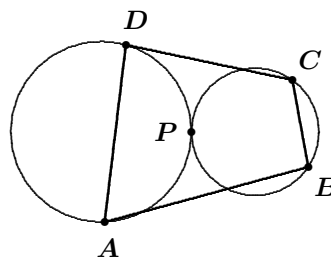
G8. Points A_1, B_1, C_1 are on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively (that is, $A_2 \neq A$, $B_2 \neq B$, $C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the mid-points of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.
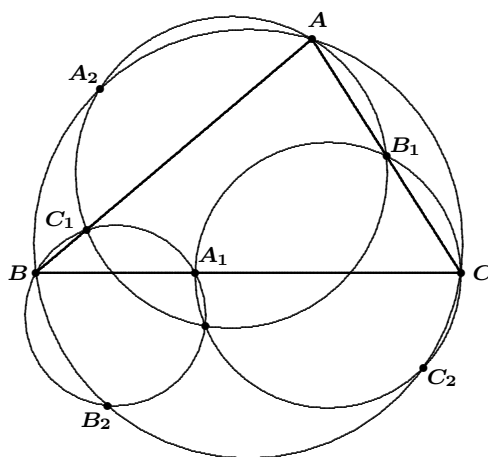
Number Theory

N1. Given $x \in (0, 1)$ let $y \in (0, 1)$ be the number whose n^{th} digit after the decimal point is the $(2^n)^{\text{th}}$ digit after the decimal point of x. Prove that if x is a rational number, then y is a rational number.

N2. For each positive integer n let

$$f(n) \ = \ \frac{1}{n} \left(\left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor \right) \,,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(a) Prove that $f(n+1) > f(n)$ for infinitely many n.

(b) Prove that $f(n+1) < f(n)$ for infinitely many n.

N3. Find all solutions in integers of the equation

$$\frac{x^7 - 1}{x - 1} \ = \ y^5 - 1 \,.$$

N4. Let a and b be relatively prime integers with $1 < b < a$. Define the *weight* of an integer c, denoted by $w(c)$, to be the minimum possible value of $|x| + |y|$ taken over all pairs of integers x and y such that

$$ax + by \ = \ c \,.$$

An integer c is called a *local champion* if $w(c) \geq \max\{w(c \pm a), w(c \pm b)\}$. Find all local champions and determine their number.

N5. Prove that for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.

A second problem set is the 2005/06 Swedish Mathematical Contest. My thanks go to Robert Morewood, Canadian Team Leader at the IMO, for collecting them for our use.

SWEDISH MATHEMATICAL CONTEST 2005/2006
Final Round
November 19, 2005 (Time: 5 hours)

1. Find all solutions in integers x and y of the equation

$$(x + y^2)(x^2 + y) \ = \ (x + y)^3 \,.$$

2. A queue in front of a counter consists of **12** persons. The counter is then closed because of a technical problem and the **12** people are redirected to another one. In how many different ways can the new queue be formed if each person maintains the same position as before, or is one step closer to the front, or is one step farther from the front?

3. In the triangle ABC the angle bisector from A intersects the side BC in the point D and the angle bisector from C intersects the side AB in the point E. The angle at B is greater than $60°$. Prove that $AE + CD < AC$.

4. The polynomial $f(x)$ is of degree four. The zeroes of f are real and form an arithmetic progression, that is, the zeroes are a, $a+d$, $a+2d$, and $a+3d$ where a and d are real numbers. Prove that the three zeroes of $f'(x)$ also form an arithmetic progression.

5. Each square on a 2005×2005 chessboard is painted either black or white. This is done in such a way that each 2×2 "sub-chessboard" contains an odd number of black squares. Show that the number of black squares among the four corner squares is even. In how many different ways can the chessboard be painted so that the above condition is satisfied?

6. All the edges of a regular tetrahedron are of length 1. The tetrahedron is projected orthogonally into a plane. Determine the largest possible area and the least possible area of the image.

———————～∿∿∿～————————

Next we look at an alternate solution to problem 2 of the 17$^{\text{th}}$ Irish Mathematical Olympiad discussed at [2007 : 151; 2008 : 88-89].

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72°$.

Alternate Solution by Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

Let $D' \neq B$ be the second point of intersection of BD with the circle T. Let $\angle ABD = \angle DBC = \beta$. Since $AB = AC$, we have $\angle ACB = 2\beta$. Also, BC is tangent to the circle and AB is a chord, hence $\angle D'AB = \beta$. Let $\angle ABO = \gamma$.
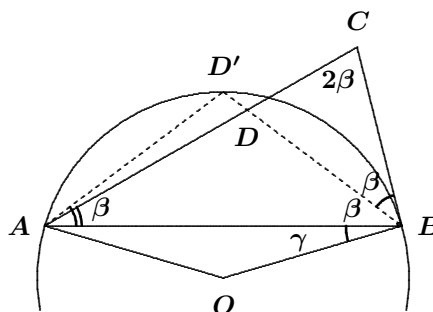
Since BC is tangent to T, we have $2\beta + \gamma = 90°$. Adding the angles in the isosceles triangle ABC yields $4\beta + \angle CAB = 180°$. From these two equations it follows that $\angle CAB = 2\gamma$. Since D is inside T we have

$$\beta = \angle D'AB > \angle CAB = 2\gamma,$$

and therefore $\frac{\beta}{2} > \gamma$. This last inequality together with $2\beta + \gamma = 90°$ yields $2\beta + \frac{\beta}{2} > 90°$, or $\frac{5\beta}{2} > 90°$. Hence, $\beta > 36°$ and $\angle ABC = 2\beta > 72°$.

Next we look at a comment on the solution to problem 6 of the 2007 Italian Olympiad [2007: 149–150; 208: 84–85].

6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'}, \quad y = \frac{BP}{PB'}, \quad z = \frac{CP}{PC'}.$$

Prove that $xyz = x + y + z + 2$.

Comment by J. Chris Fisher, University of Regina, Regina, SK.

The result is a theorem of Euler featured in the **Crux Mathematicorum** article "Euler's Triangle Theorem" by G.C. Shephard in [1999 : 148–153]. Euler's proof is very neat, even nicer than any of the solutions found in **Crux Mathematicorum**. It can be located in Edward Sandifer's book, *How Euler Did It*, and on his webpage: `http://www.maa.org/news/howeulerdidit.html` click on "19th Century Triangle Geometry".

Now we turn to solutions from our readers to problems given in the February 2008 number of the *Corner*. First a solution to a problem of the 11th Form, Final Round, XXXI Russian Mathematical Olympiad given in the *Corner* at [2008: 20–21].

5. *(N. Agakhanov)* Does there exist a bounded function $f : \mathbb{R} \to \mathbb{R}$ with $f(1) > 0$ such that

$$f^2(x + y) \geq f^2(x) + 2f(xy) + f^2(y)$$

for all $x, y \in \mathbb{R}$?

Solved by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that f is such a function. Let $a_0 = 1$ and $a_n = a_{n-1} + \frac{1}{a_{n-1}}$ for $n \geq 1$. Then

$$f^2(a_1) \geq f^2(a_0) + 2f(1) + f^2\left(\frac{1}{a_0}\right) \geq 2f(1).$$

As an induction step, assume that $f^2(a_n) \geq 2nf(1)$ for some $n \geq 1$. Then

$$
\begin{aligned}
f^2(a_{n+1}) &= f^2\left(a_n + \frac{1}{a_n}\right) \\
&\geq f^2(a_n) + 2f(1) + f^2\left(\frac{1}{a_n}\right) \\
&\geq f^2(a_n) + 2f(1) \geq 2(n+1)f(1) \,,
\end{aligned}
$$

completing the induction. Hence $f^2(a_n) \geq 2nf(1)$ for all $n \geq 1$, contradicting the facts that $f(1) > 0$ and f is bounded.

And to complete our files for the *Corner*, we look at a problem of the Taiwan Mathematical Olympiad, Selected Problems 2005, given in [2008: 21–22].

1. A $\triangle ABC$ is given with side lengths a, b, and c. A point P lies inside $\triangle ABC$, and the distances from P to the three sides are p, q, and r, respectively. Prove that

$$
R \leq \frac{a^2 + b^2 + c^2}{18\sqrt[3]{pqr}} \,,
$$

where R is the circumradius of $\triangle ABC$. When does equality hold?

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and George Tsapakidis, Agrinio, Greece. We give Bataille's write-up.

Let F denote the area of $\triangle ABC$. We have the well-known relation $2F = \dfrac{abc}{2R}$, but also from the definition of p, q, and r we have the equation $2F = ap + bq + cr$. Thus, the proposed inequality is equivalent to

$$
\frac{abc}{2(ap + bq + cr)} \leq \frac{a^2 + b^2 + c^2}{18\sqrt[3]{pqr}}
$$

or

$$
(a^2 + b^2 + c^2)(ap + bq + cr) \geq 9abc\sqrt[3]{pqr} \,. \tag{1}
$$

By the AM–GM Inequality,

$$
a^2 + b^2 + c^2 \geq 3\sqrt[3]{a^2b^2c^2} \quad \text{and} \quad ap + bq + cr \geq 3\sqrt[3]{abcpqr} \,,
$$

and the inequality (1) now follows from

$$
(a^2 + b^2 + c^2)(ap + bq + cr) \geq 9\sqrt[3]{a^2b^2c^2} \cdot \sqrt[3]{abc} \cdot \sqrt[3]{pqr} \,.
$$

That completes the *Corner* for this number, and this Volume. As Joanne Canape, who has been translating my scribbles into beautiful LaTeX has decided that twenty-plus years is enough, I want to thank her too for all the help over the time we've worked together.

BOOK REVIEW

John Grant McLoughlin

The Symmetries of Things
By John H. Conway, Heidi Burgiel, and Chaim Goodman–Strauss, published
by AK Peters, Wellesley, MA, 2008
ISBN 978-1-56881-220-5, hardcover, 423+xxv pages, US$75.00
*Reviewed by* **J. Chris Fisher**, *University of Regina, Regina, SK*

The authors set themselves the ambitious goal of producing a book that
appeals to everybody. As far as I can tell from a single reading, they have
succeeded admirably. The first thing anybody would notice about the book
is that it is filled with beautiful and fascinating photographs and computer
drawings. No special knowledge is required for admiring beauty; this book
would be as much at home on the living room coffee table as on the office
shelf. Of course, it is primarily a mathematics book.

The contents have been organized into three parts. The first of them
describes and enumerates the symmetries found in repeating patterns on sur-
faces; it is written at a level suitable for *Crux with Mayhem* readers. This part
might well serve as a textbook for a geometry course directed at university
students specializing in mathematics, education, physical science, or com-
puter science. What makes the authors' approach both novel and elementary
is the introduction of what they call the *orbifold signature notation*. Groups
are not needed here; the concept can be easily described and quickly mas-
tered. Here is the idea. A point in a pattern where two mirrors of symmetry
meet at an angle of $\frac{\pi}{m}$ is called *kaleidoscopic* and is denoted by $*m$; points
having rotational symmetry of order m (but no kaleidoscopic symmetry) are
called *gyrational* and are denoted by m (with no asterisk). If a region has
an oppositely oriented image in the pattern that is not explained by mirrors,
then these two regions must be related by a glide reflection, which here is
called a *miracle* (short for "mirrorless crossing", they say), denoted by $\times$.

To identify the signature of any repeating
plane pattern one writes down the symbols
starting from the middle and working out-
ward. First locate mirror lines and each
kind of kaleidoscopic point, if any (where two
points are of the same kind if they are related
by a symmetry of the pattern); list them after
the asterisk in decreasing order. Next locate
any gyrational points and order them before
any asterisk. Then look for miracles. Typical
signatures are $*632$ for a pattern whose sym-
metry is explained by three kinds of mirrors
that meet in pairs at angles of $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$;
632 (having no asterisk) for a pattern with

$*632$
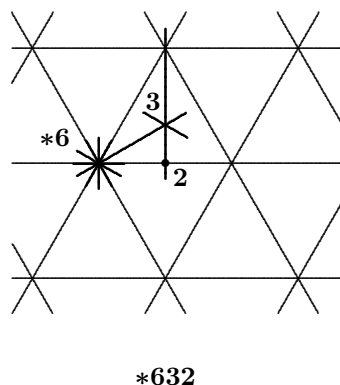
6–fold, 3–fold, and 2–fold gyrational points but no reflections or miracles; **2∗22** for a pattern with two kinds of kaleidoscopic points where a pair of mirrors intersect at right angles, and one point where there is a half-turn symmetry but no mirror.
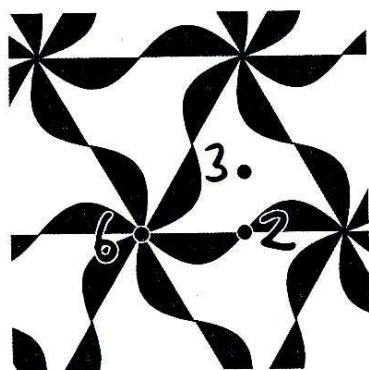
632
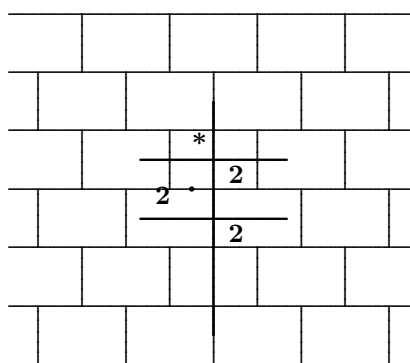
2∗22

Unlike most other notation systems that have been devised for describing plane symmetry, these orbifold signatures can also be used to describe frieze patterns and spherical patterns. (We learn in Part III that they work equally well for describing hyperbolic patterns.) But how does one know that the resulting lists of **17** signatures for plane patterns, **7** signatures for frieze patterns, and **14** signatures for spherical patterns are correct and complete? There is a "Magic Theorem" that assigns a cost to every symbol in the signature in such a way that plane patterns and frieze patterns cost exactly $2 while spherical patterns cost a bit less. That theorem tells us immediately which signatures are feasible. To establish the Magic Theorem, a pattern on the surface is associated with a folded surface they call an *orbifold*. The orbifold is obtained by identifying points related by a symmetry of the pattern (whereby points of an *orbit* are *folded* atop of one another so that a single representative point of every orbit lives on the orbifold). This sounds a bit scary, but the authors manage to explain the details in a gentle way using suitable pictures and simple examples. They then state the Classification Theorem for Surfaces and provide Conway's elementary and intuitive Zip proof. They also prove that these surfaces can be distinguished using Euler's formula (involving the numbers of vertices, edges, and faces of a suitable map on the surface), which they also prove. Since the orbifolds are easily classified using Euler's formula, the corresponding patterns are thereby classified.

Remarkably, all the proofs should satisfy the professional mathematician even though they are directed at an elementary audience. The authors achieve this feat by repeatedly reducing technical difficulties down to problems that are postponed to the following chapter. This way they present one concept at a time, as compared to the typical textbook's initial barrage of poorly motivated definitions and lemmas. Their proofs are every bit as brilliant as their notation. The illustrations are not just beautiful, but they

have been carefully chosen to clarify the exposition. I really appreciated the authors' decision to repeat pictures that they require for illustrating new ideas — instead of making the reader turn back to a picture on an earlier page, they reproduced a smaller version of it whenever needed. The authors clearly have fun coining whimsical new words; their terminology will not appeal to everybody, but the informal nature of their discussions makes for enjoyable reading. I rather liked the word *miracle* in place of the standard, but awkward and misleading term *glide reflection*; however I saw little need for *gyrational* in place of *rotational* or *wandering* in place of *translation*. We will have to wait to see which words catch on.

What I have described so far is the content of the 116 pages of the first nine chapters. Originally, according to the preface, this was all that the authors had intended to write. But they decided it was worthwhile to extend the signature to colour symmetry, and the book grew from there. For a careful reading of Part II the reader needs some group theory and a bit of mathematical maturity. The authors' main goal for this part is to present their analysis and notation for colour symmetry. They enumerate the p–fold colour types for plane, spherical, and frieze patterns (for all primes p). The complete classifications appear in a book for the first time. Along the way the authors show how their orbifold notation corresponds to previous classification systems, which gives them the opportunity to discuss the shortcomings of those systems. Also in this part, they enumerate the isohedral tilings of the sphere and plane, and they extend to $n = 2009$ the Besche-Eick-O'Brien table of the number of abstract groups of each order n.

The informative lists of Part II can probably be understood by readers who might not take an interest in the accompanying technical arguments. Similar comments apply to Part III, which the authors expect to be completely understood only by a few professional mathematicians. Still, as they point out, much of Part III can profitably be explored by other readers, while many more will enjoy inspecting the pretty pictures. Here, among other things, the authors discuss hyperbolic groups and Archimedean polyhedra and tilings; they list the **219** crystallographic space groups (and explain why chemists distinguish **230** groups), and they provide a complete list for the first time in print of the 4–dimensional Archimedean polytopes. Apparently they could have kept writing, but they decided to leave something for the rest of us to do. Their final words are, "A universe awaits — Go forth!"

Old Idaho Usual Here

Robert Israel, Stephen Morris, and Stan Wagon

N. Kildonan [1] raised the following problem. Take an arbitrary word using at most 10 distinct letters, such as $DONALD\ COXETER$. Can one subsititute distinct digits for the 10 letters so as to make the resulting base 10 number divisible by d? The answer depends on d. If d has 100 digits then the answer is clearly NO. If $d = 100$ then the answer is again NO, since ER cannot be 00. If d is 2, the answer is clearly YES: just let the units digit be even; $d = 5$ or $d = 10$ are just as easy.

Kildonan proved that divisibility by $d = 3$ can always be achieved and R. Israel and R. I. Hess extended this to $d = 9$; the case of $d = 7$ was left unresolved. In this note we settle all cases. The reader interested in an immediate challenge should try to prove that divisibility by $d = 45$ is always possible. This appears to be the hardest case.

To phrase things precisely, a *word* is a string made from 10 or fewer distinct letters; for each word and each possible substitution of distinct digits for the letters, there is an associated *value*: the base 10 number one gets after making the substitution. If all substitutions yield a value for the word w that is not divisible by d, then w is called a *blocker* for d. If any word ending (on the right) with w fails to be divisible by d, then w is called a *strong blocker* for d. An integer d is called *attainable* if the value of every sufficiently long word can be made divisible by d by some substitution of distinct digits for letters. Thus d is not attainable if there exist arbitrarily long blockers. The use of arbitrarily long strings is important because, for example, AB is a blocker for 101, but only because it is too short. An integer d is *strongly attainable* if the value of every word can be made divisible by d by an appropriate substitution.

In this paper we will find all attainable integers; moreover, they are all strongly attainable. Note that any divisor of an attainable number is attainable.

Some cases, such as $d = 2$, $d = 5$, or $d = 10$ are extremely easy to attain, and it is just about as easy to attain $d = 4$ or $d = 8$. It takes a little work to show that $d = 3$ and $d = 9$ are attainable (proofs given below). Our main theorem resolves the attainability status of all integers.

Theorem 1 An integer is attainable if and only if it divides one of the integers 18, 24, 45, 50, 60, or 80.

To start, we discuss the cases of $d = 3$ and $d = 9$ for completeness and to introduce the ideas needed later. We use the well known fact that when d is 3 or 9, then d divides a number if and only if d divides the sum of its digits.

The number $d = 3$ is attainable (Kildonan [1]). Given a word, let the 10 letters be grouped as A_i, B_i, and C_i, where each A_i has a multiplicity (perhaps 0) that is divisible by 3, each B_i has a multiplicity of the form $3k+1$, and each C_i has a multiplicity of the form $3k+2$. Look for one, two, or three pairs among the B_i and replace them with digits 1 and 2, and 4 and 5 for the second pair, and 7 and 8 for the third pair. Then look for pairs of the C_i and replace them with digits in any of the still-available pairs among $(1, 2)$, $(4, 5)$, and $(7, 8)$. These substitutions take care of $B_i \cup C_i$ except possibly four letters (since we used three pairs) and we can substitute 0, 3, 6, and 9 for them. The letters A_i can be assigned the remaining digits in any order. Thus, the final number has a digit sum divisible by $d = 3$.

The number $d = 9$ is attainable (Solution II by Israel and Hess [1]). Suppose a word has length n. Suppose some letter occurs k times, where $n - k$ is not divisible by 3. Assign 9 to this letter and assign 0 to 8 arbitrarily to the other letters. Let the value of the resulting number be v (mod 9). Now replace each digit from 0 to 8 by the next higher digit, wrapping back to 0 in the case of 8. This adds $n - k$ to the value modulo 9. But $n - k$ is relatively prime to 9, so we can do this $-v/(n-k)$ times, where the division uses the inverse of $n - k$ modulo 9, in order to obtain the value 0 modulo 9.

The other case is that every letter has a multiplicity $k \equiv n$ (mod 3). If in fact every multiplicity is congruent to n (mod 9), then any assignment will yield a value congruent to $n(0 + 1 + \cdots + 9) = 45n \equiv 0$ (mod 9). Otherwise there is a multiplicity $k \equiv n$ (mod 3) but $k \not\equiv n$ (mod 9), and then we proceed as in the first half of the proof: assign 9 to this letter, 0 to 8 to the other letters, and then cyclically permute the values 0 to 8. Each permutation adds $n - k$ modulo 9 and this will eventually transform the value v, which is divisible by 3, to a value divisible by 9, because 3 divides $n - k$ but 9 does not.

Now to the proof of Theorem 1, which follows from these four lemmas.

Lemma 1 Any integer divisible by a prime greater than 5 is not attainable.

Lemma 2 The largest attainable powers of 2, 3, and 5 are 16, 9, and 25, respectively.

Lemma 3 The numbers 36, 48, 75, 90, 100, and 120 are not attainable.

Lemma 4 The numbers 18, 24, 45, 50, 60, and 80 are attainable.

The ordering of these lemmas indicates how Theorem 1 was found. First the cases of $d = 7$ and $d = 11$ were settled and that led to the general result of Lemma 1. It followed that the only candidates for attainability had the form $2^a 3^b 5^c$. Once the powers of 2, 3, and 5 were resolved (Lemma 2), the candidate list was reduced to the 45 divisors of $3600 = 16 \cdot 9 \cdot 25$. Resolving the situation for those divisors, with some computer help, led to Lemmas 3 and 4. Finally, the computer searches were eliminated and the whole thing was redone by hand. Theorem 1 follows from the lemmas because Lemmas 3 and 4 settle the status of all 45 divisors of 3600.

A key idea is that the ten digits sum to 45. So we begin with Lemma 3, which shows how unattainability is proved. We use the fact that an integer is congruent modulo 9 (hence modulo 3) to the sum of its digits. Let A, B, C, D, E, F, G, H, J, K be the ten letters and let w^g be the concatenation of g copies of word w. The table at right lists the blockers needed for Lemmas 2 and 3; most were found by a computer search.

d	blocker
27	AAB
32	$ABBAB$
36	$(ABCDEFGHJ)^5 J^4 K$
48	$(ABCDEFGH)^2 KKJK$
75	$AABA$
90	$A^6(BCDEFGHJ)^7 JK$
100	AB
120	$ABCDEFGHJJJK$
125	BBA

We show that the words in the table are blockers. The easiest case is $d = 100$, since the value of any word ending in AB is not divisible by 100.

Case 1. The number $d = 36$ can be blocked. We have

$$(ABCDEFGHJ)^5 J^4 K \equiv K + 4J + 5(45 - K) \equiv 4J - 4K \pmod 9 \,.$$

The only way $4(J-K)$ is divisible by 9 is if JK is either 90 or 09, and neither is divisible by 4. Extension on the left by A^{9i} preserves the value modulo 36, because 111111111 is divisible by 9.

Case 2. The number $d = 48$ can be blocked. The rightmost 4 digits of the word $(ABCDEFGH)^2 KKJK$ must be one of 0080, 2272, 4464, 6656, or 8848, as these are the only words of the form $KKJK$ that are divisible by 16. However, now the value of the word modulo 3 is one of the entries below, where we work with vectors and ignore K which occurs three times:

$$2\big((45, 45, 45, 45, 45) - (8, 7, 6, 5, 4) - (0, 2, 4, 6, 8)\big) + (8, 7, 6, 5, 4)$$
$$= (82, 79, 76, 73, 70) \,,$$

and no entry is divisible by 3. Left extension by A^{3i} preserves the value modulo 48.

Case 3. The number $d = 75$ can be blocked. Here we have the congruence $AABA \equiv 51A + 10B \pmod{75}$. Multiplying by 53 transforms the congruence to $3A + 5B \equiv 0 \pmod{75}$. However, $3 \le 3A + 5B \le 69$, so the congruence is never satisfied. Left extension by A^{3i} preserves the value modulo 75.

Case 4. The number $d = 90$ can be blocked. We have

$$A^6(BCDEFGHJ)^7 JK \equiv 6A + 7(45 - A) + J \pmod 9 \,,$$

because K must be 0. The expression simplifies to $J - A$ modulo 9, which cannot be divisible by 9 because 0 is already assigned to K. Left extension by A^{9i} preserves the value modulo 9.

Case 5. The number $d = 120$ can be blocked. We have

$$ABCDEFGHJJJK \equiv 45 + 2J \equiv 2J \pmod 3 \ .$$

However, JJK must be either 440 or 880 to obtain divisibility by 40, therefore, $2J$ is either 8 or 16, and so is not divisible by 3. Left extension by A^{3i} preserves the value modulo 120.

Case 6. The number $d = 32$ can be blocked. Any word ending in $ABBAB$ has a value satisfying $10010A + 1101B \equiv 26A + 13B \pmod{32}$. If this is congruent to 0 modulo 32, then we may cancel 13, leaving $2A + B$. However, this sum is between 1 and $18 + 8 = 26$, so it is not divisible by 32.

Case 7. The number $d = 125$ can be blocked. A number is divisible by 125 if and only if it ends in $125, 250, 375, 500, 625, 750, 875$, or 000. Thus, BBA is a strong blocker for 125.

Case 8. The number $d = 27$ can be blocked. The value of AAB satisfies the congruence $110A + B \equiv 2A + B \pmod{27}$. However, $1 \le 2A + B \le 26$, which is not divisible by 27. This shows nonattainability, because we can add the prefix A^{27i}, which leaves the value modulo 27 unchanged.

Next we prove Lemma 1. Our first proof of this was a little complicated (see the Proposition that follows), but when we focused on words involving two letters only we discovered Theorem 2, which yields Lemma 1 in all cases except $d = 7$. Recall Euler's theorem, that $a^{\phi(d)} \equiv 1 \pmod d$ when $\gcd(a, d) = 1$. It follows that if d is coprime to 10, then there is a smallest positive integer, denoted by $\mathrm{ord}_d(10)$, such that $10^{\mathrm{ord}_d(10)} \equiv 1 \pmod d$.

Theorem 2 Let d be coprime to 10 and greater than 10 with $e = \mathrm{ord}_d(10)$. Then $w = A^{ke-1}B$ is a blocker for d for any positive integer k.

Proof: Assume first that $k = 1$ so that w is just $A^{e-1}B$. If 3 does not divide d then the value of w satisfies the congruence

$$B + A \sum_{i=1}^{e-1} 10^i \ = \ B - A + A\frac{10^e - 1}{9} \equiv B - A \pmod d \ .$$

Since $d > 10$, d cannot divide $B - A$. Now suppose that 3 divides d and $d > 81$. Suppose the value of w, in the formula just given, is a multiple of d. Then multiplying by 9 yields $9(B - A) + A(10^e - 1) = 9Kd$, and hence d divides $9(B - A)$. However, $A \neq B$ and $-81 \le 9(B - A) \le 81$, so $d > 81$ cannot divide $9(B - A)$, a contradiction.

There remain the cases where 3 divides d and $11 \le d \le 81$, namely $d \in \{21, 27, 33, 39, 51, 57, 63, 69, 81\}$. Suppose that d is one of these but $d \neq 21, 27, 81$; then $\mathrm{ord}_d(10) = \mathrm{ord}_{3d}(10)$. This means that from $B - A + A(10^e - 1)/9 = Kd$, we have $9(B - A) + A(10^e - 1) = 3K(3d)$, whence $3d$ divides $9(B - A)$. Thus, d divides $3(B - A)$, which means that $d \le 27$, a contradiction.

For $d = 21$ the value of w modulo 21 is $B - A$, which is not divisible by 21. For $d = 27$ the value of w modulo 27 is $2A + B$ and $1 \leq 2A + B \leq 26$, so the value is not divisible by d. For $d = 81$ the value of w modulo 81 is $8A + B$ and $1 \leq 8A + B \leq 80$, so the value is not divisible by d.

The extension to the case of general k is straightforward. ∎

The preceding result blocks all primes greater than 10. We need to deal also with $d = 7$. One can give an alternate construction in the general case that includes $d = 7$, and we give the following without proof.

Proposition Suppose that d is coprime to 10 and d does not divide 9. Let

$$w = KJK^eHK^eGK^eFK^eEK^eDK^eCK^eBK^eAK^e,$$

where $e = \mathrm{ord}_d(10) - 1$. Then after any substitution the value of w is congruent to $\dfrac{9(d+1)}{2}$ (mod d), and so is not divisible by d. ∎

For $d = 7$ the word w of the Proposition has length 55. A different approach led to the much shorter example *OLD IDAHO USUAL HERE*, its value is always $3 \cdot 45$ (mod 7). This 17-character word is thus a blocker for $d = 7$, and it can be made arbitrarily long by prepending E^6.

On to Lemma 2. The positive results for $d = 16$ and $d = 25$ are not difficult, but they are omitted as they follow from the cases of $d = 80$ and $d = 50$ (proved below); the case of $d = 9$ was discussed earlier, as were the negative results for $d = 32$, 27, and 125. It remains only to prove Lemma 4.

Case 1. The number $d = 50$ is strongly attainable. Just use 00 or 50 for the rightmost two digits.

Case 2. The number $d = 80$ is strongly attainable. Assign 0 to the rightmost letter; 8 to the next new letter that occurs reading from the right, 4 to the next one, and 2 to the next one after that. The value is then divisible by 16 and also by 5; divisibility by 80 is only affected by the four rightmost digits.

Case 3. The number $d = 18$ is strongly attainable. Given a word, find an assignment that makes it divisible by 9. If the rightmost digit is even, we are done. Otherwise, replace this digit y with $9 - y$. This preserves divisibility by 9 and makes the rightmost digit even.

Case 4. The number $d = 60$ is strongly attainable. The word ends in either AA or BA. In either case, assign 0 to A and 6 to B. Let the eight remaining letters be grouped as A_i, B_i, and C_i as in the proof for $d = 3$. Use the pairs $(1, 2)$, $(4, 5)$, and $(7, 8)$ on whatever pairs of letters can be found within B_i or within C_i. This leaves at most two single letters in the B and C groups. Use 3 and 9 for the two singletons, and any remaining digits for the A group. The final value is then divisible by 3, 4, and 5.

Case 5. The number $d = 24$ is strongly attainable. The idea is to modify the proof for $d = 3$ so as to guarantee divisibility by 8. Recalling the proof that $d = 3$ is strongly attainable, call two letters *matched* if they are replaced in

that proof by **1** and **2**, or by **4** and **5**, or by **7** and **8**. If the word ends in AAA just make sure A is either **0** or **8**. The remaining cases are that the word ends in one of the patterns ABC, ABB, BBA, or ABA.

If the ending is ABC with A and C matched, then use **152** or **192**, depending on whether B is part of a matched pair or not. If A and C are unmatched use **320** or **360** according as B is part of a matched pair or not.

If the ending is ABB with A and B matched, then use **488**, since the matching can use **4** and **8** as well as **1** and **2**. If both A and B are unmatched, then use **600**. If A is matched and B is not, use **800**. If B is matched and A is not, use **088**.

If the ending is BBA, then proceed as if the ending was ABB, but use instead **448**, **336**, **008**, and **880** for the four subcases.

If the ending is ABA, then proceed similarly, using **848**, **696**, **808**, and **080** for the four subcases.

Case 6. The number $d = 45$ is strongly attainable. Let $m(X)$ denote the multiplicity of the letter X in the given word reduced modulo 9; let $\hat{X}$ denote the digit assigned to X. Let A be the rightmost letter and assign **0** to it, thus ensuring divisibility by **5**.

Assume first that the multiplicities of at least eight of the nine remaining digits are all mutually congruent modulo **3**, and assign **9** to the other letter. Let the m-values of the eight letters be $3a_i + c$, where each a_i is a non-negative integer and $c \in \{0, 1, 2\}$. Let L_i be the digits assigned to these eight letters. Since $\sum L_i = 36$, which 9 divides, the value modulo 9 of the word is $3 \sum a_i L_i$. So we want $\sum a_i L_i$ to be divisible by **3**. Assign the pairs $(1, 2)$, $(4, 5)$, and $(7, 8)$ to pairs of letters with equal a_i. Assign **3** and **6** to the remaining two. The total is then divisible by **9** and therefore by **45**.

In the other case we can choose a letter, K say, with $m(K)$ not congruent modulo **3** to the length of the word; therefore S, the sum of the multiplicities of the nine letters other than K, is not divisible by **3**. Let $\hat{K} = 9$.

Case 6a. There is a letter, say B, with $m(B) = m(A)$. Then let $\hat{B} = 1$ and assign the remaining digits arbitrarily.

Case 6b. There is no letter as in Case 6a. Then we can find two letters among B, C, D, E, F, G, H, J, say C and D, with $m(C) \not\equiv m(D) \pmod{3}$. Consider B; we know $m(B) \neq m(A)$. Set $\hat{B}$ to be the non-negative residue modulo 9 of $m(D) - m(C)$ and note that $\hat{B} - \hat{A} \equiv m(D) - m(C) \pmod{9}$ which is not divisible by **3**. Assign unused digits to the letters C and D so that $\hat{C} - \hat{D} \equiv m(B) - m(A) \pmod{9}$; there are enough digits left for this to be possible. Assign the remaining digits arbitrarily.

Now we can treat both cases to get the result. The assignment produces some total value, reduced modulo 9 to v. If $v \neq 0$ then replace each digit between 0 and 8 by the next higher digit, wrapping back to 0 in the case of **8**. This adds S to the value modulo 9 and does not alter $\hat{B} - \hat{A}$ or $\hat{C} - \hat{D}$ modulo 9. But S is relatively prime to **9**, so we can do this $-v/S$ times, where the division uses the inverse modulo 9 of S, in order to achieve divisibility by **9**.

If $\hat{A}$ is **0** or **5**, then we are done.

If $\hat{A}$ is 3 or 6, then switch digits of A and B, where we know that $\hat{B}$ is not divisible by 3; this is because the value modulo 3 of $\hat{B} - \hat{A}$, which starts out nonzero, does not change in the translational step. If we are in Case 6b, then also switch $\hat{C}$ and $\hat{D}$; the net change is

$$(\hat{B} - \hat{A})(m(A) - m(B)) + (\hat{C} - \hat{D})(m(D) - m(C))$$
$$\equiv \ (\hat{B} - \hat{A})(\hat{D} - \hat{C}) + (\hat{C} - \hat{D})(\hat{B} - \hat{A}) \ \equiv \ 0 \ (\text{mod } 9) \ ,$$

so divisibility by 9 is preserved.

As $\hat{A}$ is now not divisible by 3, we can multiply each digit less than 9 by $5/\hat{A} \ (\text{mod } 9)$. This preserves divisibility by 9 and makes $\hat{A} = 5$. The total is now divisible by 45. The proof that $d = 45$ is strongly attainable is complete, as is the proof Lemma 4, and thus the proof of Theorem 1 is complete. ∎

There are several variations to this problem that one might consider, such as using bases other than 10. Another variant is to restrict the alphabet to the two letters A and B. We use the terms 2-*attainable* and 2-*blocker* in this context. Using techniques similar to those presented, we obtained the following result.

Theorem 3 A number is 2-attainable if and only if it divides one of 24, 50, 60, 70, 80, or 90.

The negative part of the proof required finding a 2-blocker for each $d \in \{28, 36, 48, 120, 175\}$. The reader might enjoy finding them; they are all short, of length at most 7.

Acknowledgment

References

[1] N. Kildonan, Problem 1859 (*Solution*), **Crux Mathematicorum**, 20:6, June 1994, pp. 168–170.

Robert Israel
University of British Columbia
Vancouver, BC, Canada
israel@math.ubc.ca

Stephen Morris
Newbury, Berkshire, England
stephenmor@gmail.com

Stan Wagon
Macalester College
St. Paul, MN, USA
wagon@macalester.edu

A Limit of an Improper Integral Depending on One Parameter

Iesus C. Diniz

In this article we will calculate the following limit of an improper integral that depends on one parameter $\lambda \in \mathbb{R}^+$

$$\lim_{\lambda \to 0+} \int_0^\infty \exp(-\lambda \mathcal{K}(x+l)^n) \lambda \mathcal{K} n x^{n-1} \, dx \, ,$$

where $\mathcal{K}$ and l are positive real numbers and n is a positive integer.

This is an interesting example where one cannot interchange the order of the limit and the integral. Through a nice application of elementary tools such as change of variables in integrals and the Binomial Theorem, one is able to obtain this limit.

This problem arose in the calculation of a lower bound for a probability of theoretical interest in the study of multidimensional Poisson point processes, namely

$$\lim_{\lambda \to 0+} \int_0^\infty \exp(-\lambda v_n(1)[(r+l)^n - r^n]) \lambda v_n(1) n r^{n-1} \exp(-\lambda v_n(1) r^n) \, dr \, ,$$

where $v_n(1)$ is the volume of the n–dimensional unit ball, λ is the Poisson intensity, l is the distance between two distinguished points in $\mathbb{R}^n$, and r is the distance from the first point to the closest occurrence in the Poisson point process. We shall show that this limit is equal to 1.

Proposition For all positive real numbers $\mathcal{K}$ and l, and for each positive integer n we have

$$\lim_{\lambda \to 0+} \int_0^\infty \exp(-\lambda \mathcal{K}[(x+l)^n]) \lambda \mathcal{K} n x^{n-1} \, dx \; = \; 1 \, .$$

Proof: The case $n = 1$ is a straightforward calculation:

$$\lim_{\lambda \to 0+} \int_0^\infty \exp(-\lambda \mathcal{K}(x+l)) \lambda \mathcal{K} \, dx$$

$$= \lim_{\lambda \to 0+} \left(-\exp(-\lambda \mathcal{K}(x+l)) \Big|_0^\infty \right)$$

$$= \lim_{\lambda \to 0+} \exp(-\lambda \mathcal{K} l) \; = \; 1 \, .$$

Henceforth, we take $n > 1$.

Making first the change of variable

$$v \,=\, \mathcal{K}x^n\,, \qquad dv \,=\, \mathcal{K}nx^{n-1}\,dx\,,$$

in the integral we obtain

$$\int_0^\infty \lambda \exp\left[-\lambda\left(v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\right)^n\right]\,dv\,.$$

Changing variables again, this time according to

$$u^{\frac{1}{n}} \,=\, v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\,, \qquad dv \,=\, \left(\frac{u}{v}\right)^{\left(\frac{1}{n}-1\right)}\,du\,,$$

we obtain

$$
\begin{aligned}
&\lim_{\lambda\to 0^+} \int_0^\infty \lambda \exp\left[-\lambda\left(v^{\frac{1}{n}} + (l^n\mathcal{K})^{\frac{1}{n}}\right)^n\right]\,dv \\
=\ &\lim_{\lambda\to 0^+} \int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)\left(\frac{u}{v}\right)^{\left(\frac{1}{n}-1\right)}\,du \\
=\ &\lim_{\lambda\to 0^+} \int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)\left[1 - \left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{1}{n}}\right]^{n-1}\,du\,,
\end{aligned}
$$

which by the Binomial Theorem becomes

$$
\begin{aligned}
&\lim_{\lambda\to 0^+} \int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right) \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{k}{n}}\,du \\
=\ &1 + \lim_{\lambda\to 0^+} \sum_{k=1}^{n-1} \binom{n-1}{k} \int_{l^n\mathcal{K}}^\infty \lambda \exp\left(-\lambda u\right)(-1)^k \left(\frac{l^n\mathcal{K}}{u}\right)^{\frac{k}{n}}\,du\,.
\end{aligned}
$$

After yet another change of variable,

$$w \,=\, u^{\frac{1}{n}}\,, \qquad dw \,=\, \frac{1}{n}u^{\left(\frac{1}{n}-1\right)}\,du\,,$$

the preceding expression becomes

$$1 + \lim_{\lambda\to 0^+} \sum_{k=1}^{n-1} \binom{n-1}{k} n \left(-l\mathcal{K}^{\frac{1}{n}}\right)^k \lambda \int_{l\mathcal{K}^{\frac{1}{n}}}^\infty \exp(-\lambda w^n)w^{n-1-k}\,dw\,.$$

It remains to show that

$$\lim_{\lambda\to 0^+} \lambda \int_{l\mathcal{K}^{\frac{1}{n}}}^\infty \exp\left(-\lambda w^n\right) w^{n-1-k}\,dw \tag{1}$$

is zero for each $k \in \{1,\,2,\,\ldots,\,n-1\}$.

To do this, we make a final change of variables

$$z \;=\; \lambda^{\frac{1}{n}} w\,, \qquad dz \;=\; \lambda^{\frac{1}{n}}\, dw\,,$$

which yields

$$\lambda \int_{l\mathcal{K}^{\frac{1}{n}}}^{\infty} \exp\left(-\lambda w^n\right) w^{n-1-k}\, dw$$

$$=\; \lambda^{\frac{k}{n}} \int_{(\lambda l^n \mathcal{K})^{\frac{1}{n}}}^{\infty} \exp\!\left(-z^n\right) z^{n-1-k}\, dz$$

$$<\; \lambda^{\frac{k}{n}} \int_{0}^{\infty} \exp\!\left(-z^n\right) z^{n-1-k}\, dz\,. \qquad (2)$$

It now suffices for us to show that the last integral in (2) is finite for each $k \in \{1, 2, \ldots, n-1\}$.

Setting

$$C \;=\; \int_{0}^{1} \exp\left(-z^n\right) z^{n-1-k}\, dz$$

and noting that if $z \geq 1$, then $z^{n-1-k} \leq z^{n-1}$, we finally obtain

$$\int_{0}^{\infty} \exp\!\left(-z^n\right) z^{n-1-k}\, dz \;\leq\; C + \int_{1}^{\infty} \exp\!\left(-z^n\right) z^{n-1}\, dz$$

$$=\; C + \frac{1}{ne}\,,$$

which is a finite number. This completes the proof. ∎

Acknowledgments

Iesus C Diniz
Institute of Mathematics and Statistics
University of São Paulo
São Paulo, Brazil
iesus@usp.br

Sliding Down Inclines with Fixed Descent Time: a Converse to Galileo's Law of Chords

Jeff Babb

Suppose that a vector is anchored at the origin and lies along the positive x-axis. Consider rotating the vector counter clockwise about the origin through an angle A, with $0 < A < \pi$. Consider a particle of mass m which is initially at rest and then slides, under gravity and without friction, from a starting point on the inclined vector down towards the origin. Let D_A denote the distance of the starting point from the origin. For each value of A, suppose that D_A is chosen to ensure that the particle requires exactly T seconds to reach the origin. Determine the curve characterized by the starting points of the particles.

For the particle under consideration, let $d_A(t)$ be the distance travelled along the vector at time t, $v_A(t) = d'_A(t)$ be the velocity along the vector at time t, and let $a_A(t) = v'_A(t)$ be the acceleration along the vector at time t.

Note that by definition, $D_A = d_A(T)$.

If $a_A(t)$ is some constant K, then

$$d_A(t) \; = \; \frac{K}{2}t^2 \,. \tag{1}$$

This may be confirmed by integrating $a_A(t)$ twice with respect to time and applying the initial conditions $d_A(0) = 0$ and $v_A(0) = 0$ to obtain zero for both constants of integration.

Since the particle is sliding down a frictionless incline in the Earth's gravitational field, the component of acceleration along the incline is $K = g \sin A$, where g is the acceleration due to gravity at the Earth's surface.

If the descent time is fixed at T seconds, then

$$D_A \; = \; d_A(T) \; = \; \frac{g}{2}T^2 \sin A \,. \tag{2}$$

A point (r, θ) in polar coordinates may be expressed in Cartesian coordinates as (x, y), where $x = r \cos \theta$ and $y = r \sin \theta$. Consider the following equation, which is expressed in polar coordinates as

$$r \; = \; 2c \sin \theta \,, \tag{3}$$

where $r > 0$ and $0 < \theta < \pi$. Multiplying both sides of equation (3) by r yields

$$r^2 \; = \; 2cr \sin \theta \tag{4}$$

By setting $x = r\cos\theta$ and $y = r\sin\theta$, equation (4) may be re-expressed in Cartesian coordinates as

$$x^2 + y^2 \;=\; 2cy\,.$$

Subtracting $2cy$ from each side and completing the square on $y^2 - 2cy$ yields

$$x^2 + (y - c)^2 \;=\; c^2 \tag{5}$$

which is the equation of a circle of radius c centred at the point $(0, c)$.

Thus, as depicted in the figure at right, the locus of points defined by equation (2) is a circle resting upon the origin $(0, 0)$, centred at $\left(0, \frac{g}{4}T^2\right)$ on the vertical axis, and of radius $\frac{g}{4}T^2$.

On an historical note, the motivation for this problem arose while considering the Law of Chords, which was stated and proven by Galileo Galilei in his 1638 masterpiece *Dialogues Concerning Two New Sciences*. Galileo considered rates of descent along a vertical circle and, with his Proposition VI, established the Law of Chords (see [1], p. 212):
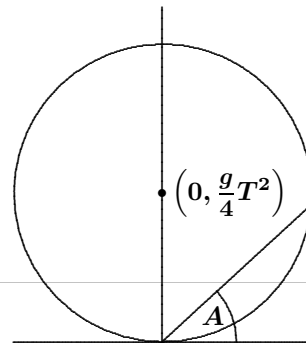
> If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference, the times of descent along these chords are each equal to the other.

Galileo's proof of the Law of Chords is presented via a series of geometric propositions, which require familiarity with many of Euclid's theorems.

The question the author wished to address was whether the vertical circle is the only curve with the property that descent time to the lowest point on the curve is constant for all chords. This paper demonstrates that the vertical circle, or one of its component arcs intersecting at the lowest point of the circle, is indeed the only such curve.

References

[1] Galilei, Galileo. 1638/1952 *Dialogues Concerning Two New Sciences*, translated by H. Crew and A. de Salvio, Encyclopaedia Britannica, Chicago, 1952.

Jeff Babb
Department of Mathematics and Statistics
University of Winnipeg
Winnipeg, MB, R3B 2E9
Canada
j.babb@uwinnipeg.ca

PROBLEMS

*Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er juin 2009**. Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB et Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

3371. Correction. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit ABC un triangle de côtés respectifs a, b et c, et soit M un de ses points intérieur. Les droites AM, BM et CM coupent respectivement les côtés opposés aux points A_1, B_1 et C_1. Les droites passant par M et perpendiculaires aux côtés coupent respectivement BC, CA et AB en A_2, B_2, et C_2. Soit p_1, p_2 et p_3 les distances respectives de M aux côtés BC, CA et AB. Montrer que

$$\frac{[A_2B_2C_2]}{[A_1B_1C_1]} = \frac{(ap_1 + bp_2)(bp_2 + cp_3)(cp_3 + ap_1)}{2a^2b^2c^2}\left(\frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3}\right),$$

où $[KLM]$ désigne l'aire du triangle KLM.

3389. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Pour $a \in \mathbb{R}$, la suite (x_n) est définie par $x_0 = a$ et $x_{n+1} = 4x_n - x_n^2$ pour tout $n \geq 0$. Montrer qu'il existe un nombre infini de valeurs $a \in \mathbb{R}$ telles que la suite (x_n) est périodique.

3390. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Montrer que si A, B, C et D sont les solutions de

$$X^2 = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix},$$

alors $A^{2007} + B^{2007} + C^{2007} + D^{2007} = O$, où O est la matrice nulle de taille 2×2.

3391. *Proposé par Michel Bataille, Rouen, France.*

Soit $ABCD$ un quadrilatère convexe tel que AC et BD se coupent à angle droit en P, et soit respectivement I, J, K et L les milieux de AB, BC, CD et DA. Montrer que les cercles (PIJ), (PJK), (PKL) et (PLI) sont congruents si et seulement si $ABCD$ est cyclique.

3392. *Proposé par Michel Bataille, Rouen, France.*

Soit A, B, C, D et E concycliques avec respectivement V et W sur les droites AB et AD. Montrer que si la droite CE, la parallèle à CB par V et la parallèle à CD par W sont concourantes, alors les triangles EVB et EWD sont semblables. La réciproque est-elle vraie ?

3393. *Proposé par Dragoljub Milošević et G. Milanovac, Serbie.*

Soit le triangle ABC, où $a = BC$, $b = AC$, $c = AB$ et où s est le demi périmètre. Montrer que

$$\frac{y+z}{x} \cdot \frac{A}{a(s-a)} + \frac{z+x}{y} \cdot \frac{B}{b(s-b)} + \frac{x+y}{z} \cdot \frac{C}{c(c-a)} \geq \frac{9}{s^2},$$

où les angles A, B et C sont mesurés en radians et x, y et z sont des nombres réels positifs quelconques.

3394. *Proposé par Dragoljub Milošević et G. Milanovac, Serbie.*

Soit $ABCD$ un tétraèdre ; désignons par h_A et m_A les longueurs respectives de la hauteur et de la médiane issues du sommet A sur la face opposée BCD. Si V est le volume du tétraèdre, montrer que

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{128}{\sqrt{3}} V.$$

3395. *Proposé par Taichi Maekawa, Takatsuki, Préfecture d'Ōsaka, Japon.*

Soit le triangle ABC ; dénotons par H son orthocentre et par R le rayon de son cercle circonscrit. Montrer que $4R^3 - (l^2 + m^2 + n^2)R - lmn = 0$, où $AH = l$, $BH = m$ et $CH = n$.

3396. *Proposé par Neven Jurič, Zagreb, Croatie.*

Soit n un entier positif et, pour i, j et k dans $\{1, 2, \ldots, n\}$, posons

$$\begin{aligned}
a_{ijk} = {} & 1 + \mathrm{mod}(k - i + j - 1, n) + n\,\mathrm{mod}(i - j + k - 1, n) \\
& + n^2\,\mathrm{mod}(i + j + k - 2, n),
\end{aligned}$$

où $\mathrm{mod}(a, n)$ est le résidu de a modulo n, choisi parmi 0, 1, $\ldots$, $n - 1$. Pour quels n le cube portant les valeurs a_{ijk} est-il un cube magique ? (Ici, le mot "magique" se réfère au fait que la somme des a_{ijk} est constante si deux des indices restent fixes et l'autre varie, et de plus que les sommes des diagonales principales du cube sont aussi égales à cette même constante.)

3397. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Évaluer
$$\lim_{n\to\infty}\frac{1}{n^2}\int_0^n\frac{\sqrt{n^2-x^2}}{2+x^{-x}}\,dx\,.$$

3398. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit l'équation
$$\left\lfloor\frac{n}{10}\right\rfloor+\left(n-10\left\lfloor\frac{n}{10}\right\rfloor\right)\cdot10^{\lfloor\log_{10}n\rfloor}=\frac{2n}{3}\,,$$

(a) montrer que $n=5294117647058823$ est une solution de cette équation,

(b) ★ déterminer toute autre solution entière et positive de cette équation.

3399. *Proposed by Vincenţiu Rădulescu, Université de Craiova, Craiova, Roumanie.*

Montrer qu'il n'existe aucune fonction positive et deux fois différentiable $f:[0,\infty)\to\mathbb{R}$ telle que $f(x)f''(x)+1\le0$ pour tout $x\ge0$.

3400. *Proposé par Yakub N. Aliyev, Université d'Etat de Bakou, Bakou, Azerbaïdjan.*

Pour des entiers positifs m et k, posons $(m)_k=m(1+10+10^2+\cdots+10^{k-1})$; par exemple, $(1)_2=11$ et $(3)_4=3333$. Déterminer tous les nombres réels α tels que
$$\left\lfloor10^n\sqrt{(1)_{2n}+\alpha}\right\rfloor=(3)_{2n}-\left\lfloor\frac{5-9\alpha}{6}\right\rfloor$$

est valide pour tout entier positif n, où $\lfloor x\rfloor$ dénote le plus grand entier ne dépassant pas x.

·.·

3371. Correction. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with a, b, and c the lengths of the sides opposite the vertices A, B, and C, respectively, and let M be an interior point of $\triangle ABC$. The lines AM, BM, and CM intersect the opposite sides at the points A_1, B_1, and C_1, respectively. Lines through M perpendicular to the sides of $\triangle ABC$ intersect BC, CA, and AB at A_2, B_2, and C_2, respectively. Let p_1, p_2, and p_3 be the distances from M to the sides BC, CA, and AB, respectively. Prove that
$$\frac{[A_2B_2C_2]}{[A_1B_1C_1]}=\frac{(ap_1+bp_2)(bp_2+cp_3)(cp_3+ap_1)}{2a^2b^2c^2}\left(\frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3}\right)\,,$$

where $[KLM]$ denotes the area of triangle KLM.

3389. *Proposed by Mihály Bencze, Brasov, Romania.*

For $a \in \mathbb{R}$ define a sequence (x_n) by $x_0 = a$ and $x_{n+1} = 4x_n - x_n^2$ for all $n \geq 0$. Prove that there exist infinitely many $a \in \mathbb{R}$ such that the sequence (x_n) is periodic.

3390. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that if A, B, C, and D are the solutions of

$$X^2 = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix},$$

then $A^{2007} + B^{2007} + C^{2007} + D^{2007} = O$, where O is the 2×2 zero matrix.

3391. *Proposed by Michel Bataille, Rouen, France.*

Let $ABCD$ be a convex quadrilateral such that AC and BD intersect in right angles at P, and let I, J, K, and L be the mid-points of AB, BC, CD, and DA, respectively. Show that the circles (PIJ), (PJK), (PKL), and (PLI) are congruent if and only if $ABCD$ is cyclic.

3392. *Proposed by Michel Bataille, Rouen, France.*

Let A, B, C, D, and E be concyclic with V and W on the lines AB and AD, respectively. Show that if the line CE, the parallel to CB through V, and the parallel to CD through W are concurrent, then triangles EVB and EWD are similar. Does the converse hold?

3393. *Proposed by Dragoljub Milošević and G. Milanovac, Serbia.*

Let ABC be a triangle with $a = BC$, $b = AC$, $c = AB$, and semiperimeter s. Prove that

$$\frac{y+z}{x} \cdot \frac{A}{a(s-a)} + \frac{z+x}{y} \cdot \frac{B}{b(s-b)} + \frac{x+y}{z} \cdot \frac{C}{c(c-a)} \geq \frac{9}{s^2},$$

where the angles A, B, and C are measured in radians and x, y, and z are any positive real numbers.

3394. *Proposed by Dragoljub Milošević and G. Milanovac, Serbia.*

Let $ABCD$ be a tetrahedron with h_A and m_A the lengths of the altitude and the median from vertex A to the opposite face BCD, respectively. If V is the volume of the tetrahedron, prove that

$$(h_A + h_B + h_C + h_D)(m_A^2 + m_B^2 + m_C^2 + m_D^2) \geq \frac{128}{\sqrt{3}} V.$$

3395. *Proposed by Taichi Maekawa, Takatsuki City, Osaka, Japan.*

Let triangle ABC have orthocentre H and circumradius R. Prove that $4R^3 - (l^2 + m^2 + n^2)R - lmn = 0$, where $AH = l$, $BH = m$, and $CH = n$.

3396. *Proposed by Neven Jurič, Zagreb, Croatia.*

Let n be a positive integer, and for i, j, and k in $\{1, 2, \ldots, n\}$ let

$$a_{ijk} = 1 + \mathrm{mod}(k-i+j-1,n) + n\,\mathrm{mod}(i-j+k-1,n) + n^2\mathrm{mod}(i+j+k-2,n),$$

where $\mathrm{mod}(a,n)$ is the residue of a modulo n in the range $0, 1, \ldots, n-1$. For which n is the cube with entries a_{ijk} a magic cube? (Here "magic" means that the sum of a_{ijk} is constant if two indices are fixed and the third index varies, and also the sums along the great diagonals of the cube are equal to this constant.)

3397. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Evaluate

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^n \frac{\sqrt{n^2-x^2}}{2+x^{-x}}\,dx.$$

3398. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Given the equation

$$\left\lfloor \frac{n}{10} \right\rfloor + \left(n - 10\left\lfloor \frac{n}{10}\right\rfloor\right)\cdot 10^{\lfloor \log_{10} n\rfloor} = \frac{2n}{3},$$

(a) show that $n = 5294117647058823$ is a solution,

(b) ★ find all other positive integer solutions of the equation.

3399. *Proposed by Vincenţiu Rădulescu, University of Craiova, Craiova, Romania.*

Prove that there does not exist a positive, twice differentiable function $f : [0,\infty) \to \mathbb{R}$ such that $f(x)f''(x) + 1 \leq 0$ for all $x \geq 0$.

3400. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

For positive integers m and k let $(m)_k = m(1+10+10^2+\cdots+10^{k-1})$, for example, $(1)_2 = 11$ and $(3)_4 = 3333$. Find all real numbers α such that

$$\left\lfloor 10^n\sqrt{(1)_{2n}+\alpha}\right\rfloor = (3)_{2n} - \left\lfloor \frac{5-9\alpha}{6}\right\rfloor$$

holds for each positive integer n, where $\lfloor x\rfloor$ is the greatest integer not exceeding x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

———————～⌒～⌒～———————

3229. [2007 : 170, 172; 2007 : 179–181] *Proposed by Mihály Bencze, Brasov, Romania.*

(a) Let x and y be positive real numbers, and let n be a positive integer. Prove that

$$(x+y)^n \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} \geq n+1+2 \sum_{i=1}^{n} \sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \geq (n+1)^2.$$

(b)★ Let $x_1, x_2, \ldots, x_k$ be positive real numbers, and let n be a positive integer. Determine the minimum value of

$$(x_1 + x_2 + \cdots + x_k)^n \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \ldots, i_k \geq 0}} \frac{i_1! i_2! \cdots i_k!}{n! x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}}.$$

Solution to (b) by Sergey Sadov, Memorial University of Newfoundland, St. John's, NL.

We may assume that $x_1 + x_2 + \cdots + x_k = 1$ without loss of generality. The minimum occurs when $x_1 = x_2 = \cdots = x_k = \frac{1}{k}$. We will obtain this as a consequence of a more general proposition. For convenience, let $X = (x_1, x_2, \ldots, x_k)$ be a vector in $\mathbb{R}_+^k$ (that is, $x_i > 0$ for all i) and let $Q = (q_1, q_2, \ldots, q_k)$ be a multi-index with non-negative integer entries q_i. Let

$$|Q| = \sum_{i=1}^{n} q_i;$$

$$|X| = \sum_{i=1}^{n} x_i;$$

$$Q! = q_1! q_2! \cdots q_k!;$$

$$X^Q = x_1^{q_1} x_2^{q_2} \cdots x_k^{q_k}.$$

Finally, let $\Delta = \{X \in \mathbb{R}_+^k : |X| = 1\}$ be the positive unit simplex in $\mathbb{R}^k$ of dimension $k-1$.

Theorem 1 With the above notation, let

$$P(X) \;=\; \sum_{|Q|=n} \frac{c_Q}{X^Q}$$

be a homogeneous rational function of degree $-n$ in the variables x_i. Suppose that $c_Q \geq 0$ for each Q and that $P(X)$ is a symmetric function, that is, interchanging any x_i and x_j does not change the value of $P(X)$. Then the minimum value of $P(X)$ over the simplex Δ exists and is attained when $x_1 = x_2 = \cdots = x_k = \dfrac{1}{k}$.

Proof. If $P(X)$ is identically zero, then there is nothing to prove, so we assume at least one coefficient c_Q is not zero. Note that $P(x)$ has a minimum value, since $P(X)$ is continuous on Δ and tends to infinity as any of the x_i approaches 0. Suppose that the minimum occurs at a point with at least two unequal coordinates. Without loss of generality (due to the symmetry of $P(X)$) we may assume that $x_1 > x_2$. We will show that by slightly decreasing x_1 and slightly increasing x_2 (while keeping all other variables and the sum $x_1 + x_2$ unchanged) the value of $P(X)$ will become smaller, contrary to our assumption. Fixing $x_3, x_4, \ldots, x_n$ causes $P(X)$ to become a symmetric function of two variables

$$F(u,v) \;=\; P(u,v,x_3,\ldots,x_n) \;=\; \sum_{j+s\leq n} \frac{A_{j,s}}{u^j v^s}\,,$$

where the coefficients $A_{j,s}$ depend on $x_3, x_4, \ldots, x_n$. Note that each $A_{j,s}$ is non-negative and at least one of these is positive, and that $A_{j,s} = A_{s,j}$. Thus, $F(u,v)$ is a linear combination with non-negative coefficients, not all zero, of functions of the form

$$F_{j,s}(u,v) \;=\; \frac{1}{u^j v^s} \;+\; \frac{1}{u^s v^j}\,,$$

where j and s are non-negative integers with $j+s \leq n$. Now let $u(t) = x_1 - t$ and $v(t) = x_2 + t$, so that $\dfrac{du}{dt} = -1$ and $\dfrac{dv}{dt} = 1$. It suffices to prove that the "time derivative" $\dfrac{d}{dt} F_{j,s}\big(u(t),v(t)\big)$ is negative at $t = 0$. We have

$$\frac{d\,F_{j,s}\big(u(t),v(t)\big)}{dt} \;=\; \frac{\partial F_{j,s}(u,v)}{\partial v} \;-\; \frac{\partial F_{j,s}(u,v)}{\partial u}$$

$$=\; \left(\frac{j}{u} - \frac{s}{v}\right)\frac{1}{u^j v^s} \;+\; \left(\frac{s}{u} - \frac{j}{v}\right)\frac{1}{u^s v^j}\,.$$

Differentiating once again, we obtain

$$\frac{d^2 F_{j,s}\big(u(t),v(t)\big)}{dt^2} \;=$$

$$\left(\frac{j}{u^2} + \frac{s}{v^2} + \left(\frac{j}{u} - \frac{s}{v}\right)^2\right)\frac{1}{u^j v^s} \;+\; \left(\frac{j}{v^2} + \frac{s}{u^2} + \left(\frac{j}{v} - \frac{s}{u}\right)^2\right)\frac{1}{u^s v^j}\,.$$

Hence, $\dfrac{d^2 F}{dt^2} > 0$. Since $\dfrac{dF}{dt} = 0$ when $u = v$ (this occurs when $t = \dfrac{u-v}{2}$), it follows that $\dfrac{dF}{dt} < 0$ when $t = 0$. We have obtained a contradiction by assuming $x_1 > x_2$ at a point achieving the minimum. This proves that the minimum occurs when all the x_i are equal. ∎

It follows that the minimum sought in part (b) is

$$k^n \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \ldots, i_k \geq 0}} \frac{i_1! i_2! \ldots i_k!}{n!} \,.$$

No other solutions to part (b) were received.
Regarding part (a) Sadov remarks that

$$\min_{x+y=1} \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} = 2^n \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} = n + 1 + 2 \sum_{j=0}^{n-1} \sum_{i=0}^{n-j} \frac{\binom{n}{j}}{\binom{n}{i+j}} \,,$$

where the first equality follows Theorem 1 and the last expression is the minimum obtained by Bataille [2007 : 179–181]. *He notes that the first equality yields a minimum of at least* 2^{n+1}, *considerably improving the lower bound of* $(n+1)^2$ *if* $n > 4$. *He observes that if* $b_i = \binom{n}{i} x^{n-i} y^i$, *then by the AM–HM Inequality*

$$\left(\frac{1}{n+1} \sum \frac{1}{b_i} \right)^{-1} \leq \frac{1}{n+1} \sum b_i = \frac{(x+y)^n}{n+1} \,,$$

hence $(x+y)^2 \sum b_i^{-1} \geq (n+1)^2$, *which yields a quick proof of part (a).*

Sadov comments that the function $P(X)$ *in Theorem 1 is Schur-convex, referring to* [1] *for the definition of this term and applications. He indicates that the AM-GM Inequality and the AM–HM Inequality can be obtained by taking* $P(X) = (x_1 x_2 \cdots x_n)^{-1}$ *and* $P(X) = \sum x^i$ *in Theorem 1, respectively.*

He mentions that particular cases (and other theorems of a more general nature) of Theorem 1 can be found in [2], *Chapter 3, Section G, Examples G.1.k and G.1.m; though he believes that Theorem 1 is present somewhere in the existing literature.*

Finally, he refers to [3], *Section 2.18, for a treatment of (the related) Muirhead's Inequality.*

References

[1] M.L. Clevenson and W. Watkins, Majorization and the Birthday Inequality, *Math. Magazine*, vol. 64, No. 3 (1991), pp. 183–188.

[2] A. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, 1979.

[3] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1952.

3289. [2007 : 485, 487] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be a triangle for which there exists a point D in its interior such that $\angle DAB = \angle DCA$ and $\angle DBA = \angle DAC$. Let E and F be points on the lines AB and CA, respectively, such that $AB = BE$ and $CA = AF$. Prove that the points A, E, D, and F are concyclic.

A composite of similar solutions by John G. Heuver, Grande Prairie, AB and George Tsapakidis, Agrinio, Greece.

Triangles ADC and BDA are similar (because their angles are assumed to be equal), whence

$$\frac{CD}{AD} = \frac{CA}{AB} = \frac{2CA}{2AB} = \frac{CF}{AE}.$$

It follows that $\triangle DCF \sim \triangle DAE$ ($\angle DCF = \angle DCA = \angle DAB = \angle DAE$, while the adjacent sides are proportional from the previous step), so that $\angle AFD \ (= \angle CFD) = \angle AED$. Noting that E and F both lie on the same side of AD, we conclude that A, E, F, and D lie on the same circle, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; GEOFFREY A. KANDALL, Hamden, CT, USA; ANDREA MUNARO, student, University of Trento, Trento, Italy; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Bataille comments that D can be constructed as the point inside $\triangle ABC$ where the circle FAE intersects the circle through B, C, and the circumcentre O. It lies on the first circle by the result of this problem. It lies on the second circle because $\angle BOC = \angle BDC = 2\angle BAC$ as follows: on the one hand the angle at A is inscribed in the circle BAC that is centred at O and is therefore $\frac{1}{2}\angle BOC$; on the other hand the similar triangles ADC and BDA fit together at A and at D in such a way that the two exterior angles at D (that form $\angle BDC$) sum to twice the sum of the two interior angles at A (that form $\angle BAC$).

3290. [2007 : 485, 487] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $ABCD$ be a trapezoid with $AD \parallel BC$. Denote the lengths of AD and BC by a and b, respectively. Let M be the mid-point of CD, and let P and Q be the mid-points of AM and BM, respectively. If N is the intersection of DP and CQ, prove that N belongs to the interior of $\triangle ABM$ if and only if $\frac{1}{3} < \frac{a}{b} < 3$.

Solution by Joel Schlosberg, Bayside, NY, USA.

There is a misleading subtlety in the statement of the problem: we shall see that the conclusion fails should $AD = BC$; in other words, our trapezoid $ABCD$ must not be a parallelogram.

Since all of the conditions of the problem are invariant under an affine transformation, we can assume without loss of generality that

$$AB \perp AD \quad \text{and} \quad AB = 4.$$

We therefore introduce a Cartesian coordinate system with the origin at A, and with B on the positive x–axis and D on the positive y–axis; then A, B,

C, and D have coordinates

$$A(0,0)\,, \quad B(4,0)\,, \quad C(4,b)\,, \quad D(0,a)\,,$$

where a and b are positive. It follows that the coordinates of M, P, and Q are

$$M\left(2, \frac{a+b}{2}\right)\,, \quad P\left(1, \frac{a+b}{4}\right)\,, \quad Q\left(3, \frac{a+b}{4}\right)\,,$$

so that the lines DP and CQ satisfy

$$y \;=\; \left(\frac{b-3a}{4}\right)x \;+\; a \quad \text{and} \quad y \;=\; \left(\frac{3b-a}{4}\right)x \;+\; a-2b\,,$$

whence $N = DP \cap CQ$ has coordinates

$$N\left(\frac{4b}{a+b}, \frac{(a-b)^2}{a+b}\right)\,.$$

A point in the plane is on the same side of AB as M if and only if its y–coordinate is positive. The y–coordinate of N satisfies

$$\frac{(a-b)^2}{a+b} \;\geq\; 0\,,$$

with equality if and only if $a = b$. Since $P = DP \cap AM$ has x–coordinate 1, a point on line DP is on the same side of AM as B if and only if its x–coordinate exceeds 1. Lastly, since $Q = CQ \cap BM$ has x–coordinate 3, a point on line CQ is on the same side of BM as A if and only if its x–coordinate is less than 3. Therefore, N is within the interior of $\triangle ABM$ if and only if $a \neq b$ and $1 < \dfrac{4b}{a+b} < 3$. This inequality is equivalent to

$$\frac{1}{3} \;<\; \frac{a}{b} \;<\; 3\,,$$

which is the desired inequality; it is equivalent to N belonging to the interior of $\triangle ABM$ as long as $a \neq b$.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3291. $\lceil 2007:485, 487\rceil$ *Proposed by Virgil Nicula, Bucharest, Romania.*

Let ABC be an isosceles triangle with $AB = AC$. Find all points P such that the sum of the squares of the distances of the points A, B, and C from any line through P is constant.

Solution by Michel Bataille, Rouen, France.

We will introduce a Cartesian coordinate system. Label the points as $A(0, a)$, $B(-b, 0)$, $C(b, 0)$, and $P(u, v)$, where a and b are positive and u and v are variables. The equation of a line ℓ through P is $x \cos\theta + y \sin\theta = p$, where $p = u \cos\theta + v \sin\theta$ and θ is an arbitrary real number. Let $d(Q, \ell)$ denote the distance between a point Q and the line ℓ, and let

$$S = d(A, \ell)^2 + d(B, \ell)^2 + d(C, \ell)^2 .$$

We then have $d(A, \ell)^2 = (a \sin\theta - p)^2$, $d(B, \ell)^2 = (p + b \cos\theta)^2$, and $d(C, \ell)^2 = (p - b \cos\theta)^2$. After a simple calculation, we obtain

$$
\begin{aligned}
S \;=\;& (\cos 2\theta)\left(\frac{3(u^2 - v^2)}{2} + av - \frac{a^2}{2} + b^2\right) \\
&+ (\sin 2\theta)(3uv - au) + \frac{3(u^2 + v^2)}{2} - av + \frac{a^2}{2} + b^2 .
\end{aligned}
$$

The number S is independent of θ if and only if

$$3uv - au = 0 \quad \text{and} \quad 3(u^2 - v^2) + 2av - a^2 + 2b^2 = 0 ,$$

which gives

$$u = 0 \quad \text{and} \quad -3v^2 + 2av - a^2 + 2b^2 = 0 , \tag{1}$$

or

$$v = \frac{a}{3} \quad \text{and} \quad 3\left(u^2 - \frac{a^2}{9}\right) + \frac{2a^2}{3} - a^2 + 2b^2 = 0 . \tag{2}$$

Now, (1) is satisfied if and only if $0 < a \le b\sqrt{3}$ (that is, $\angle B = \angle C \le 60°$) and the coordinates of P are

$$\left(0, \frac{a + \sqrt{2(3b^2 - a^2)}}{3}\right) \quad \text{or} \quad \left(0, \frac{a - \sqrt{2(3b^2 - a^2)}}{3}\right) .$$

Similarly, (2) is satisfied if and only if $a \ge b\sqrt{3}$ (that is, $\angle B = \angle C \ge 60°$) and the coordinates of P are

$$\left(\frac{\sqrt{2(a^2 - 3b^2)}}{3}, \frac{a}{3}\right) \quad \text{or} \quad \left(-\frac{\sqrt{2(a^2 - 3b^2)}}{3}, \frac{a}{3}\right) .$$

In conclusion, if ABC is equilateral, only the centre of ABC is a solution for P. Otherwise, there are two solutions for P:

$$P\left(0, \frac{a \pm \sqrt{2(3b^2 - a^2)}}{3}\right) \qquad (\text{if } \angle A > 60°) ,$$

$$P\left(\pm\frac{\sqrt{2(a^2 - 3b^2)}}{3}, \frac{a}{3}\right) \qquad (\text{if } \angle A < 60°) .$$

In both cases, the two solutions are symmetric about the centroid of the triangle; in the first case, the two points lie on the median through A and in the second case the two points lie on a line parallel to BC and passing through the centroid of $\triangle ABC$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; FRANCISCO JAVIER GARCÍA CAPITÁN, IES Álvarez Cubero, Priego de Córdoba, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA (2 solutions); SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3292. [2007 : 485, 488] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, and d be arbitrary real numbers. Show that

$$11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6$$
$$\geq 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The proof is by contradiction. Assume that

$$11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6$$
$$< 98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d.$$

We consider the quadratic $f(x) = 11x^2 + px + q$ with

$$
\begin{aligned}
p &= 22b - 98c - 38d - 12, \\
q &= 11b^2 + 221c^2 + 131d^2 + 202cd + 48c + 6 - 98bc - 38bd \\
&\quad - 12b - 12d.
\end{aligned}
$$

Then $f(a) < 0$, and the leading coefficient of f is positive, hence f has two distinct real roots; that is, the discriminant of f is positive. By computing the discriminant, we find $p^2 - 44q = -120(c + 6d - 1)^2 \leq 0$, a contradiction.

Also solved by APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, É-U; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3293. $[2007 : 485, 488]$ *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\prod_{k=1}^{n} \frac{\arcsin\left(\dfrac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right)}{\arctan\left(\dfrac{1}{\sqrt{3k+1}}\right)} = 3^n \,.$$

Composite of similar solutions by Michel Bataille, Rouen, France and Douglass L. Grant, Cape Breton University, Sydney, NS, modified by the editor.

For each $k = 1, 2, \ldots, n$ let P_k denote the corresponding factor under the product sign. We prove that in fact, $P_k = 3$ for each k.

Note first that $27k^3 + 54k^2 + 36k + 8 = (3k+2)^3$. For fixed k, let $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3k+1}}\right)$. Then $\tan\theta = \dfrac{1}{\sqrt{3k+1}}$ implies $\sin\theta = \dfrac{1}{\sqrt{3k+2}}$.

Since $\tan^{-1}$ is an increasing function, we have

$$0 \,<\, \theta \,\le\, \tan^{-1}\left(\frac{1}{2}\right) \,<\, \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \,=\, \frac{\pi}{6}\,,$$

hence, $0 < 3\theta < \dfrac{\pi}{2}$.

From

$$\begin{aligned}
\sin(3\theta) \;&=\; 3\sin\theta - 4\sin^3\theta \\
&=\; \frac{3}{\sqrt{3k+2}} - \frac{4}{\sqrt{(3k+2)^3}} \;=\; \frac{9k+2}{\sqrt{(3k+2)^3}}\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sin^{-1}\left(\frac{9k+2}{\sqrt{27k^3+54k^2+36k+8}}\right) \\
&=\; \sin^{-1}\left(\frac{9k+2}{\sqrt{(3k+2)^3}}\right) \;=\; \sin^{-1}(\sin 3\theta) \;=\; 3\theta\,,
\end{aligned}$$

and it follows that $P_k = \dfrac{3\theta}{\theta} = 3$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, and KARL HAVLAK, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3294. ⌈2007 : 486, 488⌉ *Proposed by Mihály Bencze, Brasov, Romania.*

For all positive integers m and n, show that

$$m(m+1)n^2(n+1)^2(2n^2+2n-1) - n(n+1)m^2(m+1)^2(2m^2+2m-1)$$

is divisible by **720**.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

Let $f(k) = k(k+1)\left(2k^2 + 2k - 1\right)$. Write

$$
\begin{aligned}
A\left(m,n\right) &= m\left(m+1\right)n^2\left(n+1\right)^2\left(2n^2+2n-1\right) \\
&\quad - n\left(n+1\right)m^2\left(m+1\right)^2\left(2m^2+2m-1\right) \\
&= mn\left(m+1\right)\left(n+1\right)\left[f\left(n\right)-f\left(m\right)\right].
\end{aligned}
$$

Let $C(m,n) = mn(m+1)(n+1)$ and $D(m,n) = f(n) - f(m)$, so that $A(m,n) = C(m,n)\,D(m,n)$. Since $720 = 2^4 3^2 5$, it suffices to show that $A(m,n)$ is divisible by **16**, **9**, and **5**.

(a) Divisibility by **16**. The residues of $f(n)$ modulo 4 are given in the following table.

n (mod 4)	0	1	2	3
$f(n)$ (mod 4)	0	2	2	0

   (i) If $n \equiv 0$ or $3 \pmod 4$, or $m \equiv 0$ or $3 \pmod 4$, then $C(m,n)$ is divisible by **8** and $D(m,n)$ is divisible by **2**, so that $A(m,n)$ is divisible by **16**.

   (ii) Otherwise, $C(m,n)$ is divisible by **4** and $D(m,n)$ is divisible by **4**, and therefore, $A(m,n)$ is divisible by **16**.

(b) Divisibility by **9**. The residues of $f(n)$ modulo 9 are given in the next table.

n (mod 9)	0	1	2	3	4	5	6	7	8
$f(n)$ (mod 9)	0	6	3	6	6	6	3	6	0

   (i) If $n \equiv 0$ or $8 \pmod 9$, or $m \equiv 0$ or $8 \pmod 9$, then $C(m,n) \equiv 0 \pmod 9$, so that $A(m,n)$ is divisible by **9**.

   (ii) If $n \equiv 2$ or $6 \pmod 9$, or $m \equiv 2$ or $6 \pmod 9$, then $C(m,n)$ and $D(m,n)$ are each divisible by **3**, and therefore, $A(m,n)$ is divisible by **9**.

   (iii) Otherwise, $f(n) \equiv 6 \pmod 9$ and $f(m) \equiv 6 \pmod 9$, so that $D(m,n)$ is divisible by **9**. Thus, $A(m,n)$ is divisible by **9**.

(c) Divisibility by 5. The residues of $f(n)$ modulo 5 are given in the table below.

$n \pmod 9$	0	1	2	3	4
$f(n) \pmod 9$	0	1	1	1	0

    (i) If $n \equiv 0$ or $4 \pmod 5$, or $m \equiv 0$ or $4 \pmod 5$, then $C(m,n)$ is divisible by 5, so that $A(m,n)$ is divisible by 5.

    (ii) Otherwise, $D(m,n)$ is divisible by 5, and therefore, $A(m,n)$ is divisible by 5.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ANDREA MUNARO, student, University of Trento, Trento, Italy; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3295. $[2007 : 486, 488]$ *Proposed by Michel Bataille, Rouen, France.*

Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded function. For $x > 0$, let

$$f(x) = \sup\{u(t) : t > \ln(1/x)\}$$
$$\text{and} \quad g(x) = \sup\{u(t) - xe^{-t} : t \in \mathbb{R}\}.$$

Prove that $\displaystyle\lim_{x \to 0+} f(x) = \lim_{x \to \infty} g(x)$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

The proof consists of showing that both limits are equal to the limit $L = \limsup\limits_{t \to \infty} u(t)$, which exists since the function $u(t)$ is bounded. Let K be such that $|u(t)| < K$ for all $t \in \mathbb{R}$. Given any $\epsilon > 0$ and any $t_0 > 0$, we have

  (i) there exists some $t > t_0$ such that $L - \epsilon < u(t)$, and

  (ii) there exists some $t_1 > t_0$ such that for all $t > t_1$, $u(t) < L + \epsilon$.

Let $\epsilon > 0$ be given. For a fixed $x > 0$ we have $\displaystyle\lim_{t \to \infty} xe^{-t} = 0$, so there exists $t_0 > 0$ such that $xe^{-t} < \epsilon$ for any $t > t_0$. By part (i), there exists $t > t_0$ such that $L - \epsilon < u(t)$, hence

$$L - 2\epsilon < u(t) - xe^{-t} \leq g(x).$$

Thus, $g(x) > L - 2\epsilon$ for each $x > 0$.

On the other hand, there exists $t_1 \in \mathbb{R}$ such that $u(t) < L + \epsilon$ for all $t > t_1$. Since $e^{-t_1} > 0$, let $M > 0$ be such that $K - Me^{-t_1} < L + \epsilon$. We claim that if $x > M$, then $g(x) \leq L + \epsilon$. For this it suffices to show that $u(t) - xe^{-t} < L + \epsilon$ whenever $x > M$ and $t \in \mathbb{R}$.

Indeed, if $x > M$ and $t \leq t_1$, then $u(t) - xe^{-t} < K - Me^{-t_1} < L + \epsilon$, while if $x > M$ and $t > t_1$, then $u(t) - xe^{-t} < u(t) < L + \epsilon$.

Therefore, for $x > M$ we have $L - 2\epsilon < g(x) < L + 2\epsilon$, hence $\lim\limits_{x \to \infty} g(x) = L$.

Finally, writing $S(v) = \sup\{u(t) : t > v\}$ and making two changes of variable in the limit yields

$$\lim_{x \to 0^+} f(x) \;=\; \lim_{x \to 0^+} S(\ln(1/x)) \;=\; \lim_{y \to \infty} S(\ln y) \;=\; \lim_{z \to \infty} S(z) \;=\; L\,.$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3296. $[2007 : 486, 488]$ *Proposed by Michel Bataille, Rouen, France.*

Find the greatest constant K such that

$$\frac{b^2 c^2}{a^2(a-b)(a-c)} + \frac{c^2 a^2}{b^2(b-c)(b-a)} + \frac{a^2 b^2}{c^2(c-a)(c-b)} > K$$

for all distinct positive real numbers a, b, and c.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

We prove that $K = 10$.

Let L denote the left side of the given inequality. Since L is completely symmetric in a, b, and c, we may assume without loss of generality that $a < b < c$.

Note first that $L = \dfrac{P}{Q}$ where $Q = a^2 b^2 c^2 (b-a)(c-a)(c-b)$ and $P = b^4 c^4 (c-b) - c^4 a^4 (c-a) + a^4 b^4 (b-a)$.

Observing that $P = 0$ when $c = b$ or $b = a$ or $c = a$, we find by straightforward but tedious computations that

$$\begin{aligned}
P &= b^4 c^4(c-b) - a^4\left(c^5 - b^5\right) + a^5\left(c^4 - b^4\right) \\
&= (c-b)\Big(b^4 c^4 - a^4\left(c^4 + c^3 b + c^2 b^2 + cb^3 + b^4\right) \\
&\quad + a^5\left(c^3 + c^2 b + cb^2 + b^3\right)\Big)
\end{aligned}$$

$$= (c-b)\Big(c^4\,(b^4-a^4)\ +\ a^4(a-b)\,(c^3+c^2b+cb^2+b^3)\Big)$$

$$= (c-b)(b-a)\Big(c^4\,(b^3+b^2a+ba^2+a^3)$$
$$\quad -\ a^4\,(c^3+c^2b+cb^2+b^3)\Big)$$

$$= (c-b)(b-a)\Big(b^3\,(c^4-a^4)\ +\ cb^2a\,(c^3-a^2)$$
$$\quad +\ c^2ba^2\,(c^2-a^2)+c^3a^3(c-a)\Big)$$

$$= (c-b)(b-a)(c-a)\Big(b^3\,(c^3+c^2a+ca^2+a^3)$$
$$\quad +\ cb^2a\,(c^2+ca+a^2)\ +\ c^2ba^2(c+a)+c^3a^3\Big).$$

Hence, $\dfrac{P}{Q} = \dfrac{W}{a^2b^2c^2}$, where

$$\begin{aligned}W\ =\ & b^3\,(c^3+c^2a+ca^2+a^3)\ +\ cb^2a\,(c^2+ca+a^2)\\ & +\ c^2ba^2(c+a)+c^3a^3\,.\end{aligned}$$

By writing W as a sum of 10 terms and using the AM–GM Inequality, we readily see that $W \geq 10\,(a^{20}b^{20}c^{20})^{1/10} = 10a^2b^2c^2$, from which it follows that $L \geq 10$. Since a, b, and c are distinct, equality cannot hold. Thus, $L > 10$.

Finally, if we set $a = 1$, $b = 1 + \varepsilon$, and $c = 1 + 2\varepsilon$ and let $\varepsilon \to 0^+$, then the value of L can be made arbitrarily close to 10 from the right. Hence, $K = 10$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposer. There were one incorrect and three incomplete solutions (which only showed that $L > 10$ and then concluded immediately that $K = 10$).

3297. [2007 : 486, 488] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

If A, B, and C are the angles of a triangle, prove that

$$\sin A + \sin B \sin C\ \leq\ \frac{1+\sqrt{5}}{2}\,.$$

When does equality hold?

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

The following are equivalent

$$\sin A \; + \; \sin B \sin C \;\; \leq \;\; \frac{1+\sqrt{5}}{2}\,;$$
$$2\sin A \; + \; \cos(B - C) \; - \; \cos(B + C) \;\; \leq \;\; 1 + \sqrt{5}\,;$$
$$2\sin A \; + \; \cos A \; + \; \cos(B - C) \;\; \leq \;\; 1 + \sqrt{5}\,.$$

Let φ be the first quadrant angle with $\cos\varphi = \dfrac{2}{\sqrt{5}}$ and $\sin\varphi = \dfrac{1}{\sqrt{5}}$ (the angle φ is approximately $26.6°$). The last inequality then becomes

$$\sqrt{5}\sin(A + \varphi) \; + \; \cos(B - C) \;\; \leq \;\; 1 + \sqrt{5}\,.$$

However, $\sin(A + \varphi) \leq 1$ and $\cos(B - C) \leq 1$, so the last inequality is true. Equality holds when $A = \arcsin\dfrac{2}{\sqrt{5}} \approx 63.4°$ and $B = C \approx 58.3°$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

Janous proved a more general result, namely, for $\lambda > 0$

$$\lambda \sin A + \sin B \sin C \;\; \leq \;\; \frac{1 + \sqrt{4\lambda^2 + 1}}{2}\,,$$

with equality if and only if $A = \arccos\left(\frac{1}{\sqrt{4\lambda^2+1}}\right)$ and $B = C$.

3298. [2007 : 486, 489] *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*

Let ABC be a triangle of area $\frac{1}{2}$ in which a is the length of the side opposite vertex A. Prove that

$$a^2 + \csc A \;\; \geq \;\; \sqrt{5}\,.$$

[*Ed.:* The proposer's only proof of this is by computer. He is hoping that some **CRUX with MAYHEM** reader will find a simpler solution.]

Solution by Kee-Wai Lau, Hong Kong, China.

Let $b = AC$, $c = AB$ and let S denote the area of triangle ABC.

Since $S = \frac{1}{2}bc\sin A = \frac{1}{2}$, we obtain $bc = \csc A \geq 1$.

By the Law of Cosines we have (regardless of the sign of $\cos A$) that

$$
\begin{aligned}
a^2 + \csc A \; = \; a^2 + bc \; &= \; b^2 + c^2 - 2bc\cos A + bc \\
&\geq \; b^2 + c^2 - 2bc\sqrt{1 - \sin^2 A} + bc \\
&= \; b^2 + c^2 - 2\sqrt{b^2c^2 - (bc\sin A)^2} + bc \\
&= \; (b^2 + bc + c^2) - 2\sqrt{b^2c^2 - 1} \\
&\geq \; 3bc - 2\sqrt{b^2c^2 - 1}\,.
\end{aligned}
$$

For $x \geq 1$, let $y = 3x - 2\sqrt{x^2 - 1}$. Then y is positive and from $(y - 3x)^2 = 4(x^2 - 1)$ we get $5x^2 - 6xy + y^2 + 4 = 0$. Since x is real, the discriminant of the quadratic polynomial above must be non-negative.

Thus, $(-6y)^2 - 20(y^2 + 4) \geq 0$, or $16y^2 - 80 \geq 0$, from which we obtain $y \geq \sqrt{5}$. The result now follows by setting $x = bc$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece (two solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece (two solutions); OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; SOTIRIS LOURIDAS, Aegaleo, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; KONSTANTINE ZELATOR, University of Toledo, Toledo, OH, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incorrect solution submitted.

From the proof given above, it is easy to deduce that equality holds if and only if $b = c$ and $a = \sqrt[4]{\frac{4}{5}}$, in which case $A = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.19°$. This was pointed out by Barbara, Geupel, Hess, Konečný, Tsapakidis, and Zvonaru.

Both Barbara and Demis generalized to an arbitrary triangle of area $k > 0$, proving that $a^2 + \csc A \geq \sqrt{8k + 1}$ and the lower bound $\sqrt{8k + 1}$ is the best possible.

3299. [2007 : 487, 489] *Proposed by Victor Oxman, Western Galilee College, Israel.*

Given positive real numbers a, b, and w_b, show that

(a) if a triangle ABC exists with $BC = a$, $CA = b$, and the length of the interior bisector of angle B equal to w_b, then it is unique up to isomorphism;

(b) for the existence of such a triangle in (a), it is necessary and sufficient that

$$
b > \frac{2a\,|a - w_b|}{2a - w_b} \; \geq \; 0\,;
$$

(c) if h_a is the length of the altitude to side BC in such a triangle in (a), we have $b > |a - w_b| + \frac{1}{2}h_a$.

502

Solution by Michel Bataille, Rouen, France.

(a) For convenience we write $w = w_b$. Let the interior bisector of $\angle B$ meet AC at W. We assume that $\triangle ABC$ exists with $BC = a, CA = b$, and $BW = w$, and shall produce a (implicitly defined) formula for the third side $c = AB$. We recall that $w = \dfrac{2ac\cos\frac{B}{2}}{a+c}$, so that

$$2a\cos\frac{B}{2} - w = \frac{aw}{c}.$$

By the Law of Cosines we have $WC^2 = a^2 + w^2 - 2aw\cos\dfrac{B}{2}$; using the standard formula $WC = \dfrac{ab}{a+c}$, we therefore have

$$\frac{a^2b^2}{(a+c)^2} = a^2 - w\left(2a\cos\frac{B}{2} - w\right) = a^2 - \frac{aw^2}{c}.$$

It follows that c must be the unique positive solution of $f(x) = a$, where f is the decreasing function on $(0, \infty)$ given by

$$f(x) = \frac{ab^2}{(a+x)^2} + \frac{w^2}{x}.$$

Thus, if a triangle ABC with the given parameters does exist, then its side lengths are uniquely determined, and (a) is proved.

(b) First, it is easy to see that the two given inequalities are equivalent to the conjunction of the following three inequalities

$$w < 2a; \quad (2a+b)w < 2a(a+b); \quad 2a(a-b) < (2a-b)w. \quad (1)$$

Second, the existence of a suitable triangle ABC is equivalent to the fact that the solution c of $f(x) = a$ satisfies $|a - b| < c < a + b$; that is,

$$f(|a-b|) > a > f(a+b). \quad (2)$$

To show that (1) and (2) are equivalent, we first suppose that the inequalities in (1) hold. The inequality $f(a+b) < a$ reduces to the equivalent inequality $(2a+b)w < 2a(a+b)$, which holds by (1). As for $f(|a-b|) > a$, it is equivalent to

$$ab^2\,|a-b| + w^2\left(a + |a-b|\right)^2 > a\,|a-b|\left(a + |a-b|\right)^2.$$

This reduces to $w^2b^2 > 0$ when $a \leq b$, so it holds then; but it also holds if $b < a$ since then it becomes $w(2a - b) > 2a(a - b)$, which holds by (1). We have proved that (1) implies (2).

Conversely, we assume that (2) holds (that is, that a triangle ABC with the given parameters exists). Then $\dfrac{w}{2a} = \dfrac{c}{a+c}\cdot\cos\dfrac{B}{2}$; hence $w < 2a$.

Moreover, from $f(a + b) < a$ we obtain $(2a + b)w < 2a(a + b)$. As for the condition $2a(a - b) < (2a - b)w$, it follows from $f(|a - b|) > a$ if $a > b$, and from $(2a - w)(a - b) < aw$ if $a \leq b$ (because $2a > w$, the left side is negative). The desired equivalence follows.

(c) Note that $b > |a - w| + \frac{1}{2}h_a$ is equivalent to

$$ab \ > \ a\,|a - w| \ + \ \text{Area}(ABC)\,;$$

that is, to $ab > a\,|a - w| + \left(\dfrac{a + c}{2}\right) w \sin \dfrac{B}{2}$. Since $a\,|a - w| < \dfrac{(2a - w)b}{2}$ (from part (b)), the latter will certainly hold if

$$b \ \geq \ (a + c) \sin \frac{B}{2}\,.$$

This inequality is equivalent to

$$\sin B \ \geq \ (\sin A + \sin C) \sin \frac{B}{2}\,,$$

or to

$$2 \cos \frac{B}{2} \ \geq \ 2 \sin \left(\frac{A + C}{2}\right) \cos \left(\frac{A - C}{2}\right)\,,$$

or finally to

$$1 \geq \cos \left(\frac{A - C}{2}\right)\,,$$

which is certainly true. The result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA (part (c) only); and the proposer.

Parts (a) and (b) of our problem appear on page 11 of D.S. Mitrinović et al., Recent Advances in Geometric Inequalities, *Kluwer Academic Publishers, 1989 as the first of 40 existence results from a 1952 paper (in Czech) by G. Petrov.*

In addition to his solution, Oxman also addressed the question of constructibility. Exercise 4 on page 142 of Günter Ewald's Geometry: An Introduction *(Wadsworth Publ., 1971) says that in general a triangle cannot be constructed by ruler and compass given the lengths a, b, and w_b, even when that triangle exists. The author suggests that the proof of his claim can be simplified by taking both the given side lengths equal to 1. The formula $f(x) = 1$ from part (a) of the featured solution (with $a = b = 1$, and w^2 chosen to be rational) is a cubic equation with rational coefficients. One simply has to choose a value of w for which the resulting cubic equation has no rational root. The theory of Euclidean constructions then tells us that the positive root, namely c, cannot be constructed by using ruler and compass.*

3300. $[2007 : 487, 489]$ *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, and c be positive real numbers. For any positive integer n define

$$F_n \ = \ \left(\frac{3(a^n + b^n + c^n)}{a + b + c} - \sum_{\text{cyclic}} \frac{b^n + c^n}{b + c}\right)\,.$$

(a) Prove that $F_n \geq 0$ for $n \leq 5$.

(b)$\star$ Prove or disprove that $F_n \geq 0$ for $n \geq 6$.

504

Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.

Since $F_1 = 0$, we take $n > 1$. We note that $\left(x^{n-1} - y^{n-1}\right)(x - y) \geq 0$ for all positive x and y, with equality if and only if $x = y$. We have

$$
\begin{aligned}
(a + b + c)F_n &= 3\left(a^n + b^n + c^n\right) - (a + b + c)\sum_{\text{cyclic}} \frac{b^n + c^n}{b + c} \\
&= \left(a^n + b^n + c^n\right) - \sum_{\text{cyclic}} \frac{a\left(b^n + c^n\right)}{b + c} \\
&= \sum_{\text{cyclic}} \left[a^n - \frac{a\left(b^n + c^n\right)}{b + c}\right] \\
&= \sum_{\text{cyclic}} \left[\frac{ab\left(a^{n-1} - b^{n-1}\right)}{(b + c)} + \frac{ac\left(a^{n-1} - c^{n-1}\right)}{(b + c)}\right] \\
&= \sum_{\text{cyclic}} \frac{ab\left(a^{n-1} - b^{n-1}\right)(a - b)}{(b + c)(c + a)} \geq 0 \, .
\end{aligned}
$$

Equality holds if and only if $a = b = c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece (part (a) only); STAN WAGON, Macalester College, St. Paul, MN, USA (part (a) only); TITU ZVONARU, Comănești, Romania; and the proposer.

Cîrtoaje mentioned that this problem was posted (together with a solution similar to the one featured above) by Wolfgang Berndt (Spanferkel) on the Mathlinks Forum website http://www.mathlinks.ro/Forum/viewtopic.php?p=607167 in August 2006. Barbara, Cîrtoaje, and Dergiades proved the following generalization: If a_1, a_2, ..., a_m are positive real numbers, $m \geq 2$, and

$$
F_n = \frac{m(a_1^n + a_2^n + \cdots + a_m^n)}{a_1 + a_2 + \cdots + a_m} - \sum_{\text{cyclic}} \frac{a_2^n + \cdots + a_m^n}{a_2 + \cdots + a_m} \, ,
$$

then $F_n \geq 0$ for all $n \geq 1$. Alt ultimately proved that if a, b, c, p, and q are positive real numbers and

$$
F(p, q) = \frac{3(a^p + b^p + c^p)}{a^q + b^q + c^q} - \sum_{\text{cyclic}} \frac{a^p + b^p}{a^q + b^q} \, ,
$$

then $(p - q)F(p, q) \geq 0$.

YEAR END FINALE

This marks my first year as Editor/Co-Editor of *CRUX with MAYHEM*, and my first year end finale. As I write in this space, which Jim Totten wrote in just one year ago, I am reminded of him and the non-permanence of things. Just as the surf swirls sand and pebbles on a beach, so does time alter words and ourselves. Islands disappear and are built up somewhere else. When I remember Jim's incredible enthusiasm and passion for mathematics and his outreach work, I am moved. An ocean is there for us to feel its power and its cool refreshing spray, to play in it and be cleansed by its soothing sound.

Which reminds me that issues of *CRUX with MAYHEM* are now delayed by several weeks! My apologies to all the readers for this considerable delay, and in 2009 we will work hard to close the gap.

Many people have served this past year (I will borrow Jim's habit of capitalizing their names). First, I shall forever be grateful to the late JIM TOTTEN for all the support he gave me during the time I Co-Edited with him (until the end of February, 2008). After Jim's sudden passing on March 9, 2008, three individuals in particular came to my rescue. I thank CHRIS FISHER for his humour and good spirits in those tough early days. I thank BRUCE CROFOOT for his solid support as Mission Control in the CRUX office at Thompson Rivers University and for listening to me when I needed someone to talk to. I thank BRUCE SHAWYER for his continuing help and advice on *CRUX with MAYHEM*, indeed for effectively stepping into the role of Editor-at-Large (which Jim was going to take up after stepping down as Editor-in-Chief). I also thank Bruce S. for his hospitality in Newfoundland and for providing me with a much needed sense of continuity.

I thank the members of the Editorial Board for their hard work and for bringing their unique talents to *CRUX*: JEFF HOOPER, the Associate Editor, for his careful reading and sound judgment; IAN VANDERBURGH, the *MAYHEM* Editor, whose Problem of the Month column is a joy to read and who is ahead of schedule (!); ROBERT BILINSKI, for serving as Skoliad Editor for the past four years; ROBERT WOODROW, for handling the Olympiad Corner on top of his immense administrative duties; JOHN GRANT McLOUGHLIN, for his solid support as Book Reviews Editor and for his fantastic proof reading skills. This is John's last issue as Book Reviews Editor, but he will continue as the Guest Editor for the Jim Totten special issue of May, 2009. I welcome AMAR SODHI to the Board who is taking over as Book Reviews Editor from John. I thank my colleague here in Winnipeg and Articles Editor JAMES CURRIE, for taking on the position and then turning a backlog of articles into a cornucopia. I thank the Problems Editors for their continuing support and hard work: ILIYA BLUSKOV, CHRIS FISHER, MARIA TORRES, and EDWARD WANG. The job of being a Problems Editor is perhaps one of the most difficult on the Board, with tight deadlines and a deluge of problem proposals and solutions in all areas of mathematics coming from all over the world etched, scrawled, or digitally recorded on every conceivable medium known to man. A big thank you to the four of you.

I thank GRAHAM WRIGHT, the Managing Editor, for his solid support and also for providing that vital sense of continuity.

Others who are not on the Editorial Board but whose work is just as crucial are MONIKA KHBEIS and ERIC ROBERT who serve on the *Mayhem* staff; a big thank you Monika and Eric. I thank JEAN-MARC TERRIER for translating the problems that appear in *CRUX with MAYHEM*. The Canadian Mathematical Society recently honoured Jean-Marc at the 2008 Winter Meeting for his many years of service. It is

well deserved! I look forward to working with Jean-Marc in the New Year. As well, I welcome ROLLAND GAUDET of Collège Universitaire Saint-Boniface, Winnipeg. Rolland is also helping with translations, especially in times of crisis! I thank our past *CRUX* editor BILL SANDS for his proof reading and sound advice.

My colleagues in the Dept. of Mathematics and Statistics have lent their support. Those who have taken pity on this Editor-in-Chief are ANNA STOKKE, ROSS STOKKE, TERRY VISENTIN, and JEFF BABB. Our secretary, JULIE BEAVER, has also helped out in a pinch as little emergencies have arisen. I thank the previous Dean of Science, GABOR KUNSTATTER, for his understanding and foresight in supporting *CRUX with MAYHEM* at the University of Winnipeg.

The LaTeX expertise of JOANNE CANAPE at the University of Calgary, and TAO GONG and JUNE ALEONG at Wilfrid Laurier University goes a long way to producing high quality copy. Joanne is finishing up her work in this area, so after my one year of service here I thank her for giving twenty! I welcome JILL AINSWORTH on board who is taking over from Joanne to prepare the Olympiads.

Thanks go to the University of Toronto Press and to Thistle Printing, in particular TAMI EHRLICH. The quality finished copy and the purple covers are just lovely.

I thank you MICHAEL DOOB and CRAIG PLATT for technical support, and JUDI BORWEIN for putting *CRUX* on the net.

Someone special who has helped me through my first year with her careful proof reading and knowledge of geometry is CHARLENE PAWLUCK. Thank you for sharing your copy of Euclid with me, and so much more.

Finally, I thank you, the readers, for all that you have done for me and for the journal. *CRUX with MAYHEM* is built from your contributions and all the time, care and creativity that you put into your submissions is reflected in these pages. I wish I could mention all the truly marvellous people that I have gotten to know in the last year, but this margin is too small to hold all the names and praises.

I close with a call for a new Skoliad Editor. Please refer anyone to us you may think is suitable. Skoliad is missing from this issue, but will be back next year; from now on please send all Skoliad materials directly to the Editor-in-Chief (or resend your past submissions if you did not receive a reply from us).

I wish each of you the very best in 2009 in all areas of life,

Václav (Vazz) Linek

INDEX TO VOLUME 34, 2008

Proposers and solvers appearing in the SOLUTIONS section in 2008:

Proposers

Anonymous Proposer 3211
Gregory Akulov 3285
Arkady Alt 3300
Michel Bataille 3217, 3237, 3238, 3251, 3252, 3266, 3267, 3295, 3296
Mihály Bencze 3204, 3205, 3206, 3213, 3214, 3216, 3228, 3229, 3230, 3239, 3240, 3253, 3292, 3293, 3294
K.S. Bhanu 3284
Alper Cay 3258
Vasile Cîrtoaje 3209, 3210, 3274, 3275, Klamkin–05,
M.N. Deshpande 3283, 3284
José Luis Díaz-Barrero 3212, 3247, 3263, 3269, 3281, 3282
Marian Dinca 3205
J. Chris Fisher 3224
Ovidiu Furdui 3218, 3226, 3227, 3261, 3262, 3277, 3288
G.P. Henderson 3201
Ignotus 3231
Neven Jurič 3259, 3276, 3286
Geoffrey A. Kandall 3235
Václav Konečný 3256
Tidor Mitev 3236
Virgil Nicula 3242, 3241, 3260, 3264, 3265, 3270, 3271, 3273, 3278, 3279, 3280, 3287, 3289, 3290, 3291

Victor Oxman 3299
Francisco Palacios Quiñonero 3212
Pantelimon George Popescu 3269, 3281, 3282
Achilleas Pavlos Poryfyriadis 3223
D.E. Prithwijit 3272
Stanley Rabinowitz 3297, 3298
Juan-Bosco Romero Márquez 3221
Bill Sands 3257, 3268
D.J. Smeenk 3202, 3203, 3250
Marian Tetiva 3220, 3246
George Tsapakidis 3225
G. Tsintsifas 3232, 3233, 3234, 3243, 3244, 3254, 3255
Pham Van Thuan 3222
Dan Vetter 3219
Harley Weston 3224
Shaun White 3208, 3215
John Wiest 3268
Peter Y. Woo 3207
Paul Yiu 3245
Titu Zvonaru 3248, 3249

Featured Solvers — Individuals

Mohammed Aassila 3248, 3249
Arkady Alt 3241, 3263, 3276
Šefket Arslanagič 3249, 3253, 3257, 3264
Dionne Bailey 3228, 3237, 3247, 3269, 3271
Roy Barbara 3214, 3232, 3235
Ricardo Barroso Campos 3267
Michel Bataille 3204, 3205, 3209, 3213, 3217, 3222, 3227, 3229, 3232, 3245, 3249, 3256, 3257, 3264, 3269, 3272, 3273, 3276, 3278, 3281, 3283, 3285, 3286, 3288, 3291, 3293, 3299
Brian D. Beasley 3220
Mihály Bencze 3230, 3253
Manuel Benito 3253, 3261, 3262
Elsie Campbell 3228, 3237, 3247, 3269, 3271
Cao Minh Quang 3230, 3253, 3281, 3300
Óscar Ciaurri Ramírez 3253, 3261, 3262
Vasile Cîrtoaje Klamkin–05, 3275
Chip Curtis 3204, 3221, 3226, 3234, 3261, 3274, 3276, 3294
Apostolis K. Demis 3211, 3219, 3224, 3255, 3273
Charles R. Diminnie 3221, 3228, 3237, 3247, 3269, 3271
Emilio Fernández Moral 3253, 3261, 3262
J. Chris Fisher 3238
Francisco Javier García Capitán 3221
Oliver Geupel 3263, 3273, 3282, 3284, 3292
Douglass L. Grant 3293
P.C. Hammer 3254
Karl Havlak 3271
John Hawkins 3240
Richard I. Hess 3221, 3232, 3283, 3286
John G. Heuver 3245, 3289
Joe Howard 3287
Walther Janous 3245
Geoffrey A. Kandall 3221
Paula Koca 3271
Václav Konečný 3202, 3232, 3244
Kee-Wai Lau 3242, 3239, 3246, 3298
Tom Leong 3253, 3260
Nguyen Thanh Liem 3249

Taichi Maekawa 3250, 3279
Thanos Magkos 3276
Salem Maličkić 3214, 3220, 3234, 3245, 3249, 3280
Dragoljub Milošević 3214
Andrea Munaro 3253, 3265, 3280
Vedula N. Murty 3221
José H. Nieto 3201, 3202, 3257, 3259
Michael Parmenter 3252, 3284
Alex Remorov 3249
Henry Ricardo 3204,
Juan-Bosco Romero Márquez 3221
Luz Roncal 3261, 3262
Xavier Ros 3212, 3221, 3236, 3250, 3257
Sergey Sadov 3229
Bill Sands 3268
Joel Schlosberg 3215, 3233, 3290
D.J. Smeenk 3202, 3221, 3235, 3264, 3279, 3297
T. Jefferson Smith 3254
David Stone 3240
Edmund Swylan 3202, 3206, 3207, 3238
Marian Tetiva 3220
Phi Thai Thuan 3249
Daniel Tsai 3251
Panos E. Tsaoussoglou 3249
George Tsapakidis 3289
G. Tsintsifas 3232
Apostolis Vergos 3208
Vo Quoc Ba Can 3210, 3216, 3223
Edward T.H. Wang 3253
Shaun White 3208
Peter Y. Woo 3203, 3205, 3218, 3221, 3225, 3244, 3245, 3295, 3296
Paul Yiu 3245
Roger Zarnowski 3237
Titu Zvonaru 3243, 3270

Featured Solvers — Groups

Missouri State University Problem Solving Group 3259, 3266

Skidmore College Problem Solving Group 3245

Other Solvers — Individuals

Mohammed Aassila 3223, 3237, 3241, 3281, 3283
Gregory Akulov 3285
Arkady Alt 3221, 3222, 3223, 3226, 3227, 3230, 3236, 3246, 3247, 3248, 3249, 3281, 3287, 3300
Miguel Amengual Covas 3203, 3223, 3250
Christos Anastassiades 3250
George Apostolopoulos 3264, 3265, 3270, 3276, 3278, 3284, 3285, 3286, 3287, 3289, 3291, 3293, 3298
Šefket Arslanagić 3203, 3206, 3221, 3222, 3223, 3248, 3249, 3250, 3251, 3256, 3260, 3265, 3270, 3271, 3272, 3276, 3281, 3284, 3285, 3287, 3289, 3293, 3294, 3297, 3298, 3300
Dionne T. Bailey 3222, 3223, 3251, 3253, 3263, 3276, 3285, 3286, 3287, 3293
Roy Barbara 3201, 3202, 3203, 3213, 3221, 3222, 3223, 3233, 3234, 3236, 3237, 3238, 3242, 3241, 3243, 3244, 3247, 3250, 3251, 3252, 3257, 3264, 3265, 3266, 3270, 3271, 3272, 3278, 3285, 3286, 3287, 3290, 3291, 3293, 3294, 3297, 3299, 3300
Ricardo Barroso Campos 3203, 3250, 3279, 3289,
Michel Bataille 3201, 3202, 3206, 3207, 3208, 3211, 3221, 3223, 3228, 3230, 3233, 3234, 3235, 3236, 3237, 3238, 3242, 3237, 3238, 3239, 3243, 3244, 3246, 3247, 3248, 3250, 3251, 3252, 3253, 3255, 3262, 3263, 3265, 3266, 3267, 3270, 3271, 3279, 3280, 3282, 3284, 3287, 3289, 3290, 3292, 3294, 3295, 3296, 3297, 3298
Brian D. Beasley 3294, 3297
Francisco Bellot Rosado 3202, 3203
Mihály Bencze 3204, 3205, 3213, 3214, 3216, 3223, 3228, 3229, 3239, 3247, 3249, 3269, 3292, 3293, 3294
Manuel Benito 3251, 3255, 3257, 3258
K.S. Bhanu 3284
Paul Bracken 3222, 3262, 3286
Katrina Bricker 3221
Whitney Bullock 3257
Sarah Burnham 3271
Elsie M. Campbell 3212, 3222, 3223, 3251, 3253, 3263, 3276, 3285, 3286, 3287, 3293
Cao Minh Quang 3212, 3214, 3221, 3222, 3223, 3235, 3236, 3242, 3246, 3247, 3248, 3249, 3250, 3251, 3260, 3265, 3270, 3276, 3287
Óscar Ciaurri Ramírez 3251, 3255, 3257, 3258
Vasile Cîrtoaje 3209, 3210, 3274, 3300
Chip Curtis 3202, 3203, 3206, 3216, 3217, 3219, 3222, 3223, 3227, 3230, 3235, 3237, 3238, 3239, 3241, 3243, 3244, 3245, 3246, 3247, 3248, 3251, 3252, 3256, 3257, 3259, 3262, 3264, 3265, 3266, 3269, 3270, 3271, 3275, 3278, 3281, 3283, 3284, 3286, 3287, 3289, 3290, 3291, 3292, 3293, 3296, 3297, 3298, 3299, 3300
Paul Deiermann 3284, 3285
Apostolis K. Demis 3202, 3203, 3205, 3207, 3213, 3214, 3221, 3233, 3234, 3235, 3238, 3251, 3256, 3258, 3264, 3265, 3267, 3270, 3289, 3292, 3297, 3298
Hasan Denker 3222, 3223
Nikolaos Dergiades 3300
M.N. Deshpande 3284
José Luis Díaz-Barrero 3212, 3247, 3263, 3269, 3281, 3286, 3287
Charles R. Diminnie 3212, 3222, 3223, 3250, 3251, 3253, 3263, 3276, 3285, 3286, 3287, 3293
Marian Dinca 3205
Oleh Faynshteyn 3293, 3297, 3298
Emilio Fernández Moral 3251, 3255, 3257, 3258
Ovidiu Furdui 3218, 3226, 3227, 3261, 3262, 3288
Ian June L. Garces 3251
Francisco Javier García Capitán 3202, 3290, 3291
Oliver Geupel 3257, 3264, 3266, 3268, 3276, 3278, 3279, 3281, 3283, 3286, 3287, 3289, 3290, 3293, 3294, 3295, 3296, 3297, 3298, 3299, 3300
Karl Havlak 3253, 3276, 3285, 3286, 3287, 3293
John Hawkins 3212, 3237, 3240, 3241, 3257
G.P. Henderson 3201
Richard I. Hess 3203, 3206, 3208, 3219, 3221, 3222, 3227, 3250, 3251, 3253, 3262, 3265, 3270, 3276, 3287, 3292, 3294, 3296, 3297, 3298
John G. Heuver 3203, 3221, 3256
Joe Howard 3204, 3212, 3221, 3222, 3223, 3227, 3235, 3236, 3237, 3246, 3247, 3248, 3251, 3260, 3271, 3276, 3281
Peter Hurthig 3250
Walther Janous 3239, 3241, 3246, 3247, 3248, 3249, 3250, 3251, 3253, 3257, 3260, 3261, 3262, 3263, 3265, 3266, 3267, 3269, 3270, 3271, 3272, 3273, 3274, 3291, 3292, 3294, 3297, 3300
Neven Jurič 3259, 3276, 3286
Natalie Kalmink 3221
Geoffrey A. Kandall 3202, 3235, 3250, 3289
Paula Koca 3253, 3276, 3285, 3286
Václav Konečný 3203, 3221, 3234, 3235, 3243, 3250, 3256, 3258, 3291, 3297, 3298

Kee-Wai Lau 3208, 3222, 3223, 3236, 3237, 3241, 3247, 3249, 3248, 3250, 3253, 3260, 3271, 3285, 3286, 3292, 3297
Tom Leong 3251, 3257
Kathleen E. Lewis 3257
Carl Libis 3276
Sotiris Louridas 3298
Taichi Maekawa 3250
Thanos Magkos 3221, 3260, 3286, 3287, 3297
Salem Malikić 3202, 3203, 3207, 3208, 3211, 3216, 3221, 3222, 3223, 3230, 3233, 3235, 3236, 3237, 3240, 3241, 3243, 3244, 3246, 3247, 3248, 3250, 3251, 3253, 3256, 3257, 3259, 3260, 3266, 3270, 3271, 3272, 3276, 3281, 3284, 3287, 3293, 3297
Dung Nguyen Manh 3281, 3287
Pavlos Maragoudakis 3221, 3251
Petrus Martins 3253
G. Milanovac 3221, 3222, 3223
D.M. Milošević 3222
Dragoljub Milošević 3221, 3222, 3223
Todor Mitev 3236
Andrea Munaro 3235, 3236, 3251, 3256, 3264, 3276, 3278, 3287, 3289, 3294
Vedula N. Murty 3214, 3221, 3222, 3223
N. Nadeau 3262
Virgil Nicula 3241, 3242, 3260, 3264, 3265, 3270, 3271, 3273, 3278, 3279, 3280, 3287, 3289, 3290, 3291
José H. Nieto 3205, 3221, 3237, 3248, 3251, 3252, 3253, 3258
Victor Oxman 3299
Francisco Palacios Quiñonero 3212
Michael Parmenter 3237, 3238, 3250, 3287, 3294
Paolo Perfetti 3281
Pantelimon George Popescu 3269, 3281
Achilleas Pavlos Porfyriadis 3223
D.E. Prithwijit 3272
Stanley Rabinowitz 3297, 3298
Alex Remorov 3226, 3229, 3230, 3234, 3235, 3236, 3237, 3238, 3241, 3243, 3244, 3246, 3247, 3248, 3250
Henry Ricardo 3204, 3212, 3247, 3293
Juan-Bosco Romero Márquez 3221, 3223, 3297
Luz Roncal 3251, 3255, 3257, 3258
Xavier Ros 3227, 3237, 3247, 3248, 3249, 3251, 3253, 3259, 3260, 3262, 3281, 3286, 3287
Deborah Salas-Smith 3257
Bill Sands 3257
Robert P. Sealy 3286
Joel Schlosberg 3202, 3203, 3204, 3205, 3207, 3211, 3218, 3221, 3227, 3235, 3238, 3244, 3250, 3253, 3255, 3265, 3266, 3270, 3271, 3272, 3276, 3278, 3283, 3285, 3287, 3292, 3293, 3297
Bob Serkey 3287
Andrew Siefker 3253
D.J. Smeenk 3203, 3221, 3223, 3233, 3234, 3244, 3250, 3265, 3278, 3287
Digby Smith 3286
David R. Stone 3212, 3237, 3240, 3241, 3257
Edmund Swylan 3203, 3217, 3221, 3234, 3236, 3237, 3243, 3244, 3250, 3257
Alexandros Sygelakis 3209, 3212
Winfer C. Tabares 3251
Ashley Tangeman 3253
Marian Tetiva 3246
Daniel Tsai 3203, 3286
Panos E. Tsaoussoglou 3203, 3222, 3223, 3236, 3248, 3250, 3287, 3297, 3298, 3300
George Tsapakidis 3222, 3223, 3265, 3270, 3276, 3286, 3287, 3291, 3293, 3297
G. Tsintsifas 3233, 3234, 3243, 3244, 3254, 3255
Pham Van Thuan 3222
Phi Thai Thuan 3236, 3248
Apostolis Vergos 3212, 3248
Dan Vetter 3219
Vo Quoc Ba Can 3214, 3222
Stan Wagon 3208, 3222, 3223, 3292, 3300
Peter Y. Woo 3201, 3202, 3207, 3211, 3212, 3213, 3214, 3220, 3222, 3223, 3225, 3233, 3234, 3235, 3236, 3238, 3241, 3243, 3246, 3247, 3248, 3250, 3251, 3252, 3253, 3254, 3255, 3256, 3258, 3259, 3262, 3264, 3265, 3267, 3270, 3271, 3273, 3276, 3278, 3279, 3280, 3281, 3284, 3285, 3287, 3289, 3290, 3291, 3292, 3293, 3294, 3297, 3298, 3299
Bingjie Wu 3257, 3272, 3281, 3287
Konstantine Zelator 3203, 3265, 3266, 3272, 3294, 3297, 3298
Titu Zvonaru 3202, 3203, 3221, 3222, 3223, 3233, 3234, 3235, 3236, 3237, 3238, 3244, 3246, 3247, 3249, 3250, 3263, 3264, 3265, 3266, 3267, 3271, 3272, 3281, 3287, 3289, 3290, 3291, 3293, 3294, 3297, 3298, 3300

Other Solvers — Groups

Ateneo Problem Solving Group 3221, 3251
Missouri State Univeristy Problem Solving Group 3253, 3257, 3272

Skidmore College Problem Solving Group 3221, 3237, 3291, 3294, 3297
University of Regina Math Central Consultants 3211