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* * *

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This issue of Crux Mathematicorum *is dedicated to its founding editor, Léo Sauvé.*

What follows are letters of tribute received in this office in response to a suggestion made in the March issue. They surely speak eloquently of the esteem in which Léo is held by the readership of Crux. *Other readers submitted problems dedicated to Léo; some of them appear as this issue's problem set.*

* * *

We are three Dutch mathematicians, united by a longstanding friendship, and all three addicted to geometry. We have also in common the warmest feelings of admiration and thankfulness towards the founder of *Crux*, Léo Sauvé, who by his skill, his stimulating energy and his organizing talent has raised the journal to international level and has earned the appreciation of the entire mathematical community.

It is a pleasure for us to express these thanks, and also to propose a problem [#1161] dedicated to our friend.

O. Bottema	J.T. Groenman	D.J. Smeenk
Delft	Arnhem	Zaltbommel

* * *

In July 1985 I wrote the following "Letter to the Editor" to *Crux*, but Léo, being not only an excellent editor but also a decent and modest fellow, could not be persuaded to publish it; not even with a disclaimer, e.g. "The opinions expressed in this letter are not necessarily those of the editorial board". While I profoundly regret that he had to stop his work due to ill health, this now gives me a chance to express my admiration and my sincere thanks for many pleasant and interesting hours which I -- and undoubtedly many others around the world -- owe Léo Sauvé.

Letter to the Editor: (dated July 17, 1985)

It has come to my attention that the Editor of Crux Mathematicorum *is perpetrating a fraud on the readers and publishers of the magazine.*

You will find that my name is occasionally given as the author of a solution. I happen to know, however, from my own notes, that the published solutions have three authors. Two of them are never revealed by *the Editor. Their names are Léo Sauvé and Edith Orr. While I admit to being part-author of such solutions, several of them were published in a vastly improved form. I read them and think: "Gee, I wish I had said that!"*

Not only an occasional comma was changed -- no, the whole presentation, and often the English (thank you, Edith), is much better than it had been when I wrote it. This goes beyond the simple duties of routine editing, and it should be properly revealed and acknowledged. I imagine, indeed I hope, that others may have had similar experiences. (After all, who wants to be the only one whose solutions need rewriting?)

To conclude: Crux Mathematicorum *wouldn't be what it is without its readers and contributors. But it wouldn't be anything without Léo and Edith!*

In a certain sense we can still say *Crux* "wouldn't be anything without Léo and Edith", to which I might add the name of our dearly beloved friend Fred Maskell. If they had not started *Eureka* so many years ago, if a world-wide circle of subscribers had not been built up diligently over the years, today *Crux* would not exist, wouldn't be anything. In this sense, Léo's work did not come to a stop when he left the editorial chair. As long as *Crux* lives on, it will owe something to Léo Sauvé.

Dr. Hayo Ahlburg
Benidorm (Alicante), Spain

* * *

I am very sorry indeed to hear about the illness of Léo Sauvé. I give him my very best wishes. I have only known him over the last few months, but he was most kind and helpful regarding a problem or two and stimulated my interest in recreational mathematics which I would not otherwise have discovered. Many thanks to him.

John Flatman
Timmins, Ontario

* * *

Here is a letter I mailed to Léo Sauvé some time ago. If you can use any of it at all, fine. It has already served its purpose.

Dear Professor Sauve:

This long, rambling letter is of no immediate significance, and should be read at your leisure. It has been in my head for a couple of *years -- like good port, aged in oak. Some time ago I promised it in a letter to Professor Maskell whom I knew only through correspondence, but liked immensely. The material may not be worth your time, and it would not hurt me to have you pass it through a secretary first, to mark appropriate parts for you. And remember, fan mail requires no response.*

The first knowledge I had of your journal was about three years ago through a comment either in the Mathematical Gazette *(London) or in Martin Gardner's column. I immediately subscribed, and after a few issues, obtained the bound volumes of the entire run. (So that you know whence this comes, I am a retired radiologist.)*

The first year or two of your journal were particularly interesting, not only because of the problems -- and I am going over the volumes systematically, learning much -- but because of the between-the-lines messages. A small but stalwart group was represented, and there was no pretentiousness: this was mathematics for its own sake. In comparison, the message I received from the early issues of another problems journal was of bombast. It expressed the ideas of principally one man who was not nearly so appreciative of his helpful friends, and who used many pages of each issue advertising his own previous publications. It was a revealing contrast.

Your journal, to an outsider, has been eminently successful. Like a parent you have reared your offspring to maturity where it has become desirable to another: isn't this the aim of all parents! It will have a future of more continuous security.

Before his early death, Fred Maskell sent along to me several notes, in response to my minor financial contributions, and he suggested that they were welcome. His brief words indicated a warm, thoughtful, responsive personality. I obtained the bound volumes through him, of course. Presenting the issues in that form was a great idea. These are respectable volumes, and are permanent library material. No matter what happens with the new publisher, the first eleven volumes will be a satisfying monument to the editor.

After my tour as editor of a radiology journal, I was asked to contribute some material. I was happy to do so, but advised that since my retirement I no longer had an automatic and autonomic response to deadlines. I hope you continue to contribute.

I find that this word processor makes evident what I had hoped would remain latent: my garrulity. Enough!

So, good luck in your next venture, and thanks for Crux Mathematicorum!

E. Frederick Lang, M.D.
Grosse Pointe, Michigan

* * *

Two short items dedicated to Leo Sauve:

1. A double dactyl in honor of the assistant to the Editor. (See [1978: 10] and [1982: 323].)

Eeether Orr, Eyether Orr,*
Edith Orr, Poetess,
Whispering gently to
Léo Sauvé,

Checked that the Editor's
Writing was accurate,
Unreprehensible,
Not too risqué.

*These are Edith's twin sisters.

2. An anagram (quasi-acceptable French, but grammar and meaning often get lost in anagrams).

LÉO SAUVÉ A SOULEVÉ SEL AVOUÉ .

(Literal translation: Léo Sauvé has raised avowed wit. Not-so-literal translation: Léo Sauvé has raised wit to new heights.)

Leroy F. Meyers
The Ohio State University
Columbus, Ohio.

* * *

Whom do I admire and love most: Léo Sauvé, the mathematician, or Léo Sauvé, the human being?

I think of him as a mathematical artist, creator and conductor of *Crux Mathematicorum*, a man of inexpressible dedication to our discipline, of indescribable industriousness and devotion to his work -- all of it overglazed by his incredible modesty. Immeasurably do I value all my personal contacts with him, those delightful letters -- always sagacious, always warm, always witty.

And then -- oh what happiness! -- I received his photograph. It was brief ecstacy: I lost it in a freak theft!

I don't consider myself a mathematician. I am a totally addicted mathematical amateur. And I am 77 years old. Léo Sauvé has given me countless hours of happiness. That profoundly treasured photograph I now carry in my memories and in my heart.

I admire and love Léo Sauvé, the mathematician, as deeply as I feel that way toward him as a human being.

Herta T. Freitag
Roanoke, Virginia

* * *

Léo first wrote to me on March 3rd, 1976, to ask whether I could help with a problem submitted to *Eureka*. Léo sent a copy of *Eureka*, which he had been editing for the past year. I passed then as an "expert" on circles, on the basis of my little book "Circles", one of the first books published by the Pergamon Press. (It is now published, with solutions, as "Circles Viewed Mathematically" by the Dover Press.) I was impressed by *Eureka*, and sent Léo the names of some possible subscribers, as I continued to do for the next ten years, besides problems and articles. One of the readers I suggested was Dan Sokolowsky, and Léo and I first met at Dan's house in Yellow Springs. Léo had described himself before our meeting in rather self-deprecatory terms, mentioning his rotundity and hearing aids. Talking to him however was a joy and his wit in writing is matched by his wit in conversation. The one tax on his invariably genial attitude he would not tolerate was my French accent, which he classed immediately as abominable, and I immediately desisted, his English being impeccable.

Those of us who have witnessed the remarkable development of *Eureka* during the past ten years, even with the change of title to the rather too solid *Crux Mathematicorum* in March 1978, know that Léo is one of the finest editors of recent times. I met him on two further occasions at his home in Ottawa, together with his wife and family, and there were plans for Léo and his wife Carmen to visit my home here in Minneapolis, as an adjunct to a visit to a 90-year-old mother, but this did not happen.

Crux is not now being reprinted, but those of us lucky enough to possess the 10 volumes which have been reprinted in a reduced size (and perhaps can still be purchased) have a treasury of problems, solutions and articles. Many of the solutions have comments by Léo which produce both a chuckle at his dry wit and a gasp at his erudition, in other fields as well as mathematical ones. He has often expressed his pride in the developing readership of *Crux*. May it continue to grow! My latest effort has resulted in the Chengdu (mainland China) branch of the Academia Sinica coming in out of the cold.

It would interest readers of *Crux* to see a list of all the subscribers, which now, I believe, covers the world.

Dan Pedoe
Minneapolis, Minnesota

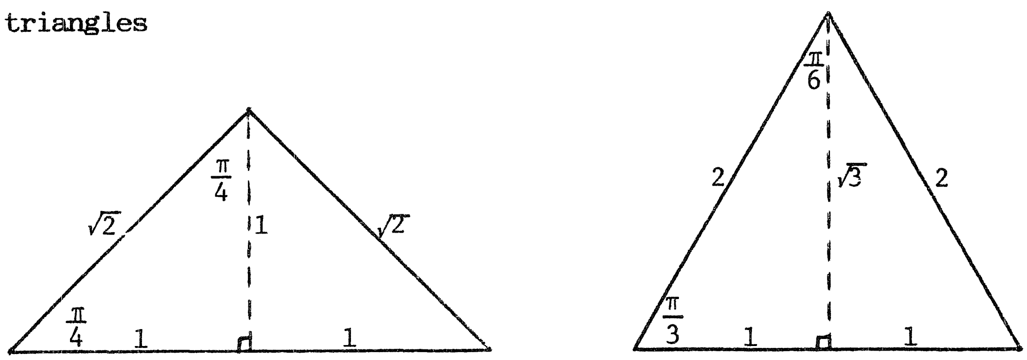
* * *

A LITTLE TRIGONOMETRY

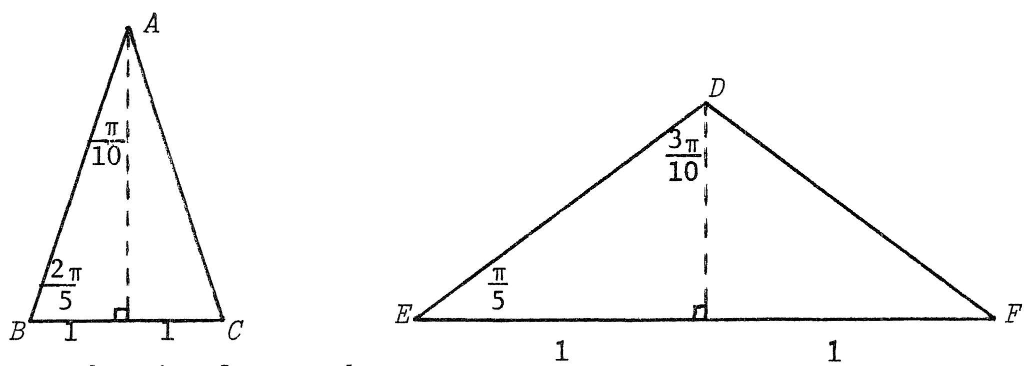
Kenneth S. Williams
Carleton University

To Léo Sauvé with thanks for all his years of service as Editor of Crux.

The values of $\sin\theta$ and $\cos\theta$ for $\theta = \pi/6,\ \pi/4,\ \pi/3$ follow at once from the triangles



In this note we show how the values of $\sin\theta$ and $\cos\theta$ for $\theta = \pi/10,\ \pi/5,\ 3\pi/10,\ 2\pi/5$ can be obtained analogously from the triangles



Let $a = AB$. By the sine law, we have

$$a = \frac{2\sin 2\pi/5}{\sin \pi/5} = 4\cos\pi/5\ ,$$

and, by the cosine law, we have

$$\cos \pi/5 = \frac{a^2 + a^2 - 2^2}{2a^2} = \frac{a^2 - 2}{a^2} .$$

Thus a must satisfy $a^3 - 4a^2 + 8 = 0$, and so

$$a = 2,\ 1 + \sqrt{5} \text{ or } 1 - \sqrt{5} .$$

Clearly $a \neq 2$ as ΔABC is not equilateral and $a \neq 1 - \sqrt{5}$ as a is positive. Hence we have $a = 1 + \sqrt{5}$, and the perpendicular AM from A to BC by Pythagoras's theorem is of length $\sqrt{5 + 2\sqrt{5}}$. The values

$$\sin \pi/10 = \cos 2\pi/5 = \frac{\sqrt{5} - 1}{4} ,$$

$$\sin 2\pi/5 = \cos \pi/10 = \frac{\sqrt{10 + 2\sqrt{5}}}{4} ,$$

then follow from ΔABM. Similarly, using ΔDEF, we obtain the values

$$\sin \pi/5 = \cos 3\pi/10 = \frac{\sqrt{10 - 2\sqrt{5}}}{4} ,$$

$$\sin 3\pi/10 = \cos \pi/5 = \frac{\sqrt{5} + 1}{4} .$$

* * *

THE OLYMPIAD CORNER: 77

M.S. KLAMKIN

All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

I first give a report on the 27th IMO. I am grateful to Cecil Rousseau, the U.S.A. team leader, for supplying the information.

The Twenty-Seventh International Mathematical Olympiad (IMO) was held this year in Warsaw, Poland from July 4 to July 15. Teams from 37 countries took part in the competition, one less than the record number of 38 countries of last year. The maximum team size for each country was 6 students, the same as for the last three years.

The 1987 and 1988 IMO's are to be held in Cuba and Australia, respectively.

The six problems of the competition were assigned equal weights of 7 points each (the same as in the last five IMO's) for a maximum possible individual score of 42 and a maximum possible team score of 252. For comparison purposes, see the last five IMO reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249] and [1985: 202].

First, Second, and Third Prizes were awarded to students with scores in the respective intervals 34-42, 26-33, and 17-25. Congratulations to the following 18 First-Prize winners:

Name	Country	Score
Kos Geza	Hungary	42
Vladimir Roganov	U.S.S.R.	42
Stanislav Smirnov	U.S.S.R.	42
Fang Weimin	China	41
Jurg Jahnel	East Germany	41
Joseph Keane	U.S.A.	41
Marius Dabija	Rumania	40
Zhang Hao	China	39
Bruno Savalle	France	38
Ralph C. Teixera	Brazil	37
Li Ping Li	China	37
Martin A. Harterich	West Germany	36
David Grabiner	U.S.A.	36
Jeremy A. Kahn	U.S.A.	35
Iliya T. Kraichev	Bulgaria	34
Nicolae I. Beli	Rumania	34
Ha Anh Vu	Vietnam	34
Wieland E. Fischer	West Germany	34

For many years now, I and others have been highly critical of the number and the method of awarding prizes (see [1985: 205]). This year I find that the awarding of prizes was particularly lopsided. There were

18 First Prizes for scores in the interval 34-42,

41 Second Prizes for scores in the interval 26-33,

48 Third Prizes for scores in the interval 17-25,

giving a total of 107 prizes among 204 students. Last year, there were

14 First Prizes for scores in the interval 34-42,

36 Second Prizes for scores in the interval 22-33,

52 Third Prizes for scores in the interval 15-21,

for a total of 102 prizes among 209 students.

Usually, one tries to award the prizes in the ratio 1:2:3. To me it would have been much more preferable and more meaningful if the prizes were awarded as

13 First Prizes for scores in 36-42,

26 Second Prizes for scores in 30-35,

40 Third Prizes for scores in 21-29,

for a total of 79 prizes.

If the IMO leaders are so concerned about their students getting prizes, then why don't they go all out and give every student participating one of the three prizes. This would at least reduce the considerable time spent in bickering about the awarding of prizes.

Previously, I had also been highly critical of the method of awarding Special Prizes. Two years ago this was rectified by forming a special committee of some of the jury members to make recommendations for special prizes. In view of this, there should also be a committee of some of the senior jury members to decide on the awarding of 1st, 2nd and 3rd prizes and their basic philosophy should be decided before the papers are graded.

As the IMO competition is an individual event, the results are announced officially only for individual team members. However, team standings are usually compiled unofficially by adding up the scores of the individual team members. The team results are given in the following table (where the team size is given if less than 6). Congratulations to the tying winning teams, the U.S.A. and the U.S.S.R.:

Rank	Country	Score (max 252)	Prizes 1st	2nd	3rd	Total Prizes	
1,2	U.S.A.	203	3	3	-	6	
1,2	U.S.S.R.	203	2	4	-	6	
3	West Germany	196	2	4	-	6	
4	China	177	3	1	1	5	
5	East Germany	172	1	3	2	6	
6	Rumania	171	2	2	1	5	
7	Bulgaria	161	1	3	2	6	
8	Hungary	151	1	2	2	5	
9	Czechoslovakia	149	-	3	3	6	
10	Vietnam	146	1	2	2	5	
11	Great Britain	141	-	2	3	5	
12	France	131	1	1	2	4	
13	Austria	127	-	2	2	4	
14	Israel	119	-	2	2	4	
15	Australia	117	-	-	5	5	
16	Canada	112	-	2	1	3	
17	Poland	93	-	-	3	3	
18	Morocco	90	-	1	2	3	
19	Tunisia	85	-	-	1	1	
20	Yugoslavia	84	-	-	2	2	
21	Algeria	80	-	-	2	2	
22	Belgium	79	-	1	2	3	
23	Spain	78	-	1	2	3	Team of 4
24	Brazil	69	1	-	-	1	
25	Norway	68	-	1	-	1	
26	Greece	63	-	-	2	2	
27	Finland	60	-	-	1	1	

28	Colombia	58	-	-	-	-	
29	Sweden	57	-	-	-	-	
30	Turkey	55	-	-	-	-	
31	Mongolia	54	-	-	-	-	
32	Cyprus	53	-	1	-	1	
33	Cuba	51	-	-	-	-	
34	Italy	49	-	-	2	2	
35	Kuwait	48	-	-	-	-	
36	Iceland	37	-	-	-	-	Team of 4
37	Luxemburg	22	-	-	-	-	Team of 2

The members, scores, prizes, and leaders of the Canadian and U.S.A. teams are as follows:

Canadian team:

Ravi D. Vakil	32	(2nd prize)
Steven Siu	29	(2nd prize)
Giuseppe Russo	18	
Bryan Feir	14	
Rocky Lee	14	
Alex Romosan	5	

Leaders: Ron Scoins, University of Waterloo
Ron Dunkley, University of Waterloo

U.S.A. team:

Joseph Keane	41	(1st prize)
David Grabiner	36	(1st prize)
Jeremy Kahn	35	(1st prize)
John A. Overdeck	32	(2nd prize)
Darien G. Lefkowitz	30	(2nd prize)
William P. Cross	29	(2nd prize)

Leaders: Cecil Rousseau, Memphis State University
Gregg Patruno, Columbia University

It is of interest to note that (i) Joseph Keane (U.S.A.) won the only special prize, for his solution of problem #3, (ii) Terence Tao (Australia) at 10 years of age was the youngest student in the competition and won a third prize with a score of 19, (iii) China, which only entered the competition for the first time last year with two students, did very well this time, coming in 4th with a score of 177.

The problems of this year's competition are given below. Solutions to these problems, along with those of the 1986 U.S.A. Mathematical Olympiad, will appear in a booklet, Olympiads for 1986, obtainable later this year for a small charge from

Dr. W.E. Mientka, Executive Director
M.A.A. Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska 68588.

*

THE 27th INTERNATIONAL MATHEMATICAL OLYMPIAD

WARSAW, POLAND

FIRST DAY

July 9, 1986

Time allowed: 4.5 hours

1. Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a,b in the set $\{2,5,13,d\}$ such that $ab - 1$ is not a perfect square.

2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a sequence of points P_1, P_2, $P_3, \ldots$ such that P_{k+1} is the image of P_k under rotation with center A_{k+1} through angle 120° clockwise (for $k = 0,1,2,\ldots$). Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all the five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure *necessarily* comes to an end after a finite number of steps.

SECOND DAY

July 10, 1986

Time allowed: 4.5 hours

4. Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:

(i) $f[xf(y)]\cdot f(y) = f(x + y)$ for all $x,y \geq 0$,

(ii) $f(2) = 0$,

(iii) $f(x) \neq 0$ for $0 \leq x < 2$.

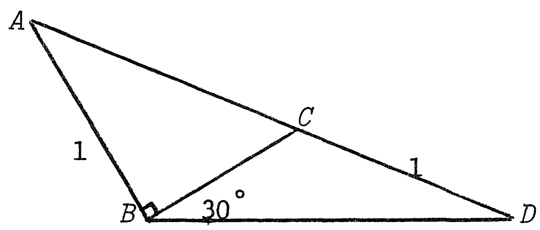
6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the

points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white points and red points on L is not greater than 1? Justify your answer.

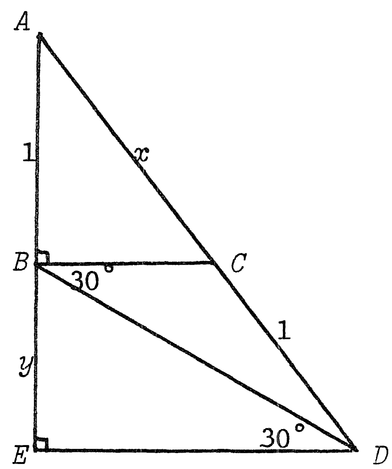
* * *

I now give solutions to the problems of the 1986 Canadian Mathematics Olympiad [1986: 131]. They are all due to H.L. Abbott and Andy Liu, University of Alberta.

1.



In the diagram AB and CD are of length 1 while angles ABC and CBD are 90° and 30° respectively. Find AC.



Solution.

Draw $DE \| BC$ with E on AB extended. Let $AC = x$ and $BE = y$. Since $\Delta ABC \sim \Delta AED$, $x/1 = 1/y$ or $y = 1/x$. Also, since $\angle BDE = \angle DBC = 30^\circ$, we have $DE = y \cot 30^\circ = \sqrt{3}/x$. Then by the Pythagorean theorem,

$$(1 + 1/x)^2 + (\sqrt{3}/x)^2 = (1 + x)^2.$$

This simplifies to

$$(x + 2)(x^3 - 2) = 0.$$

Clearly, since $x \neq -2$, $x = \sqrt[3]{2}$. [See H. Eves, *A Survey of Geometry*, Allyn & Bacon, Boston, 1972, p.183, Problem 10(a).]

*

2. A Mathlon is a competition in which there are M athletic events. Such a competition was held in which only A, B and C participated. In each event p_1 points were awarded for first place, p_2 for second and p_3 for third where $p_1 > p_2 > p_3 > 0$ and p_1, p_2, p_3 are integers. The final score for A was 22, for B was 9 and for C was also 9. B won the 100 metres. What is the value of M and who was second in the high jump?

Solution.

We have $M(p_1 + p_2 + p_3) = 22 + 9 + 9 = 40$ and $p_1 + p_2 + p_3 \geq 1 + 2 + 3 = 6$, and hence $M < 7$. Now we also know that $M > 1$. The only factors of 40 in this range are 2, 4 and 5.

If $M = 2$, then $9 \geq p_1 + p_3$ by B's score. Hence $p_1 \leq 8$ which means A could not have scored 22 points.

If $M = 4$, then $9 \geq p_1 + 3p_3$ by B's score. Since $p_3 \geq 1$, $p_1 \leq 6$. In fact we must have $p_1 = 6$ in order for A's score to reach 22 points. It follows from $p_1 + p_2 + p_3 = 10$ that $p_2 = 3$. Now A's score is at most $3p_1 + p_2 = 21$, a contradiction.

Thus $M = 5$ and $p_1 + p_2 + p_3 = 8$. If $p_3 \geq 2$, then $p_1 + p_2 + p_3 \geq 4 + 3 + 2 = 9$. Hence $p_3 = 1$. We must have $p_1 \geq 5$ in order that A could score 22 points. If $p_1 \geq 6$, then $p_2 \leq 1 = p_3$. Hence $p_1 = 5$ and $p_2 = 2$. Now A's score must consist of $4p_1 + p_2 = 22$, B's score $p_1 + 4p_3 = 9$ and C's score $4p_2 + p_3 = 9$. It follows that A was second in the 100 metres, and C was second in everything else including the high jump.

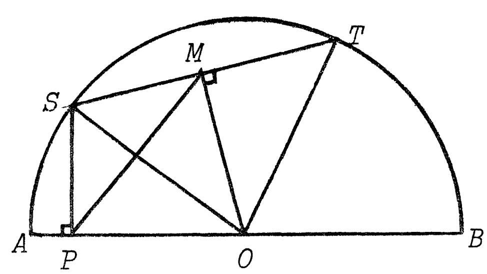
Editorial note: This problem has appeared previously as #8, p.139 in M. Gardner, *New Mathematical Diversions*, Simon & Schuster, N.Y., 1966. The same problem with the numbers 22, 9, 9 changed to 20, 10, 9 appears as #6 in E. Barbeau, M. Klamkin, W. Moser, *1001 Problems in High School Mathematics, Book I*, Canadian Mathematical Society, 1976 and as #1 in the Sixteenth IMO, 1974 [S.L. Greitzer, *International Mathematical Olympiads 1959-1977*, MAA, Washington, D.C. 1978, pp.16, 159].

*

3. A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

Solution.

Let O be the centre of the semicircle. Then $\angle SOT$ is constant, and so is $\angle SOM$. Since $\angle SMO = \angle SPO = 90^\circ$, S, P, O and M are concyclic. Hence $\angle SPM = \angle SOM$, and is constant too.



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4. For positive integers n and k, define $F(n,k) = \sum_{r=1}^{n} r^{2k-1}$. Prove that $F(n,1)$ divides $F(n,k)$.

Solution.

We use the fact that $a^{2k-1} + b^{2k-1}$ is divisible by $a + b$.

Case (i): $n = 2t$. We have $F(2t,1) = \sum_{r=1}^{2t} r = t(2t + 1)$. Now

$$F(2t,k) = \sum_{r=1}^{t} (r^{2k-1} + (2t + 1 - r)^{2k-1})$$

and so $F(2t,k)$ is divisible by $2t + 1$. Also,

$$F(2t,k) = \sum_{r=1}^{t-1} (r^{2k-1} + (2t - r)^{2k-1}) + t^{2k-1} + (2t)^{2k-1}$$

and so $F(2t,k)$ is divisible by t. Since $(t,2t + 1) = 1$, $F(2t,1)$ divides $F(2t,k)$.

Case (ii): $n = 2t + 1$. We have $F(2t + 1,1) = (t + 1)(2t + 1)$. Now

$$F(2t + 1,k) = \sum_{r=1}^{t} (r^{2k-1} + (2t + 2 - r)^{2k-1}) + (t + 1)^{2k-1}$$

is divisible by $t + 1$. Also,

$$F(2t + 1,k) = \sum_{r=1}^{t} (r^{2k-1} + (2t + 1 - r)^{2k-1}) + (2t + 1)^{2k-1}$$

is divisible by $2t + 1$. Since $(t + 1,2t + 1) = 1$, $F(2t + 1,1)$ divides $F(2t + 1,k)$.

*

5. Let u_1, u_2, u_3,... be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$. Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

Solution.

For $n = 1,2,3,\ldots$, let $v_n \equiv u_n \pmod{1986}$ with $0 \leq v_n < 1986$. A pair of consecutive terms (v_n,v_{n+1}) can be any one of the pairs $(0,0)$, $(0,1)$,..., $(1985,1985)$. Since there are only finitely many such pairs and the sequence is infinite, there exist positive integers m and k such that $v_m = v_{m+k}$ and

$v_{m+1} = v_{m+k+1}$. Since $u_{n+2} = u_{n+1}^2 - u_n$, it is easy to see that $v_{m+2} = v_{m+k+2}$, $v_{m+3} = v_{m+k+3}, \ldots$. On the other hand, we also have $u_n = u_{n+1}^2 - u_{n+2}$. Hence, $v_{m-1} = v_{m+k-1}$, $v_{m-2} = v_{m+k-2}, \ldots$. It follows that the sequence $\{v_n\}$ is purely periodic. Since $u_3 = 45^2 - 39 = 1986$, $v_3 = 0$. Hence $u_n = 0$, that is 1986 divides u_n, for infinitely many n.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before April 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

<u>1137</u>*. [1986: 79] (Revised) *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s}$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

The following problems were all submitted especially for this issue and are all dedicated to Léo Sauvé. More problems were received for this purpose than can appear at this time; they will be included in future issues with the notation "dedicated to Léo Sauvé".

<u>1161</u>. *Proposed by O. Bottema, Delft, The Netherlands; J.T. Groenman, Arnhem, The Netherlands; and D.J. Smeenk, Zaltbommel, The Netherlands.*

The dihedral angle on the edge A_iA_j of a given tetrahedron $A_1A_2A_3A_4$

is denoted by α_{ij} $(= \alpha_{ji})$, for $1 \le i,\ j \le 4,\ i \ne j$. Determine a necessary and sufficient condition on the α_{ij} for the midpoints of the four altitudes of the tetrahedron to be coplanar.

1162. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $G = \{A_1, A_2, \ldots, A_{n+1}\}$ be a point set of diameter D (that is, $\max A_iA_j = D$) in E^n. Prove that G can be contained in a slab of width d, where

$$d \le \begin{cases} 2D/\sqrt{2n+2} & \text{for } n \text{ odd} \\[2ex] D\cdot\sqrt{\dfrac{2(n+1)}{n(n+2)}} & \text{for } n \text{ even.} \end{cases}$$

(A *slab* is a closed connected region in E^n bounded by two parallel hyperplanes. Its *width* is the distance between these hyperplanes.)

1163. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Aichi, Japan.*

Draw two parallel tangent lines to an ellipse with semiaxes a and b, and draw a circle tangent to the ellipse (externally) and to both lines. Prove that the center of the circle is at distance $a + b$ from the center of the ellipse.

1164. *Proposed by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.*

In ΔABC, $\angle C = 2\angle B$, and a point P in the interior of ΔABC satisfies $AP = AC$ and $PB = PC$. Show that AP trisects $\angle A$.

1165*. *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

For fixed $n \ge 5$, consider an n-gon P imbedded in a unit cube.

(i) Determine the maximum perimeter of P if n is odd.

(ii) Determine the maximum perimeter of P if it is convex (which implies it is planar).

(iii) Determine the maximum volume of the convex hull of P if also $n < 8$.

1166. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*

Let A and B be positive integers such that the arithmetic progression $\{An + B\colon n = 0,1,2,\ldots\}$ contains at least one square. If M^2 $(M > 0)$ is the smallest such square, prove that $M < A + \sqrt{B}$.

1167. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

Determine the greatest real number r such that for every acute triangle ABC of area 1 there exists a point whose pedal triangle with respect to ABC is right-angled and of area r.

1168. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

Let $S = \sum_{i=1}^{k} F_{(2i-1)n}$ where n is odd and F_m denotes a Fibonacci number. Determine a Lucas number L_a such that L_aS is a Fibonacci number.

1169. *Proposed by Andy Liu, University of Alberta, Edmonton, Alberta; and Steve Newman, University of Michigan, Ann Arbor, Michigan.*

[To Léo Sauvé who, like J.R.R. Tolkien, created a fantastic world.]

(i) *The Fellowship of the Ring.* Fellows of a society wear rings formed of 8 beads, with two of each of 4 colours, such that no two adjacent beads are of the same colour. No two members wear indistinguishable rings. What is the maximum number of fellows of this society?

(ii) *The Two Towers.* On two of three pegs are two towers, each of 8 discs of increasing size from top to bottom. The towers are identical except that their bottom discs are of different colours. The task is to disrupt and reform the towers so that the two largest discs trade places. This is to be accomplished by moving one disc at a time from peg to peg, never placing a disc on top of a smaller one. Each peg is long enough to accommodate all 16 discs. What is the minimum number of moves required?

(iii) *The Return of the King.* The King is wandering around his kingdom, which is an ordinary 8 by 8 chessboard. When he is at the north-east corner, he receives an urgent summons to return to his summer palace at the south-west corner. He travels from cell to cell but only due south, west, or south-west. Along how many different paths can the return be accomplished?

1170. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

In the plane of triangle ABC, let P and Q be points having trilinears $\alpha_1 : \beta_1 : \gamma_1$ and $\alpha_2 : \beta_2 : \gamma_2$, respectively, where at least one of the products $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ is nonzero. Give a Euclidean construction for the point $P*Q$ having trilinears $\alpha_1\alpha_2 : \beta_1\beta_2 : \gamma_1\gamma_2$. (A point has trilinears $\alpha : \beta : \gamma$

if its signed distances to sides *BC*, *CA*, *AB* are respectively proportional to the numbers α, β, γ.)

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

870*. [1983: 209] *Proposed by Sidney Kravitz, Dover, New Jersey.*

Of all the simple closed curves which are inscribed in a unit square (touching all four sides), find the one which has the minimum ratio of perimeter to enclosed area.

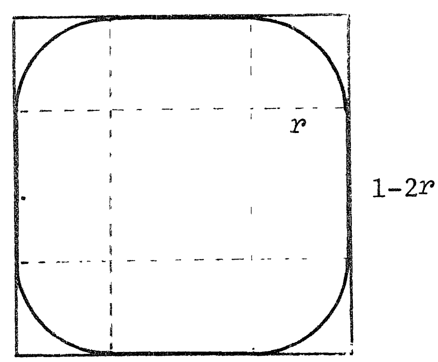
Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

An intuitive feeling for the solution can be obtained by considering the problem as a two-dimensional soap bubble problem. Air, bounded by a soap film, is admitted into a square chamber through a pinhole in the middle of the square. For a given area of air, the bounding curve of the soap bubble will have minimum perimeter. If the area A is at most $\pi/4$, then, by the isoperimetric theorem, the curve will be a circle of radius $\sqrt{A/\pi}$, perimeter $P = 2\sqrt{\pi A}$, and

$$P/A = 2\sqrt{\pi/A}\ ,$$

which obviously decreases as A increases, reaching its smallest value 4 when $A = \pi/4$. The bounding curve is then the circle inscribed in the square.

If $A > \pi/4$, however, a circle of radius $\sqrt{A/\pi}$ will no longer fit into the unit square. It seems reasonable, and is in fact true, that when $\pi/4 < A < 1$, the curve of minimum perimeter will consist of four quadrants of circles, say of radius r, near the corners of the square, and four straight segments of length $1 - 2r$ connecting successive quadrants. Hence the ratio of perimeter to area is given by



$$f(r) = \frac{2\pi r + 4(1 - 2r)}{1 - 4r^2 + \pi r^2} = \frac{4 - 2cr}{1 - cr^2} \qquad \text{for} \quad 0 \le r \le \frac{1}{2}\ ,$$

where $c = 4 - \pi$. The endpoints of this interval correspond to the inscribed circle and the entire square. The curve giving the minimum of f will then be the answer to the problem. We work with the reciprocal of this function,

$$g(r) = \frac{1 - cr^2}{4 - 2cr} = \frac{r}{2} + \frac{1 - 2r}{2(2 - cr)} .$$

Then

$$g'(r) = \frac{1}{2} + \frac{2(2 - cr)(-2) - (1 - 2r)(-2c)}{4(2 - cr)^2}$$
$$= \frac{1}{2} + \frac{c - 4}{2(2 - cr)^2}$$
$$= \frac{1}{2} - \frac{\pi}{2(2 - cr)^2} ,$$

so g' decreases, and equals 0 when

$$(2 - cr)^2 = \pi$$
$$r = \frac{2 - \sqrt{\pi}}{c} = \frac{2 - \sqrt{\pi}}{4 - \pi} = \frac{1}{2 + \sqrt{\pi}} .$$

Hence the maximum of g, i.e. the minimum of f, occurs at this value of r, and then, since $cr = 2 - \sqrt{\pi}$,

$$f(r) = \frac{2\sqrt{\pi}}{1 - \dfrac{2 - \sqrt{\pi}}{2 + \sqrt{\pi}}} = 2 + \sqrt{\pi} \approx 3.77245.$$

Also solved by JORDI DOU, Barcelona, Spain; G.P. HENDERSON, Campbellcroft, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; JORDAN B. TABOV, Sofia, Bulgaria; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

Most solvers, Meyers included, justified the shape of the solution. I have suppressed these details in the published proof, since it turns out that the problem is a known one; it occurs as problem 47, page 29 (solution pp.168-173) in L.A. Graham, *Ingenious Mathematics Problems and Methods*, Dover, N.Y., 1959. Murray Klamkin points out that a history of this and related problems is given in David Singmaster and D.J. Souppouris, A constrained isoperimetric problem, *Math. Proc. Camb. Phil. Soc.* 83 (1978) 73-82.

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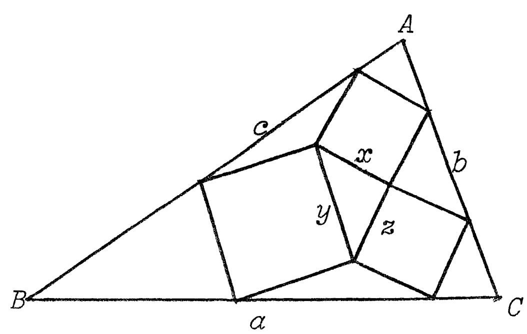
<u>1013</u>. [1985: 50; 1986: 119] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai City, Aichi, Japan.*

This problem is about "Malfatti" squares, named by analogy with Malfatti circles. The concept is illustrated in the adjoining figure.



(a) Given a triangle ABC, show how to construct its three Malfatti squares.

(b) *The Malfatti squares problem.* Given the sides a, b, c of a triangle, calculate the sides x, y, z of its Malfatti squares.

(c) *The reverse Malfatti squares problem.* Given the sides x, y, z of the Malfatti squares of a triangle, calculate the sides a, b, c of the triangle.

Further comment by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.

I would like to make some points in connection with my solution of this problem, which I should have made earlier.

In that solution the locus described on p.120 should have been described as: the locus relative to $\angle DEF$, ray EU, *and ray* ED. The point P of the subsequent discussion is then correctly described as the intersection of the loci L_1 and L_2, where L_1 is the locus relative to $\angle BAC$, rays AG and AB, and L_2 is the locus relative to $\angle ABC$, rays BG and BA.

It should be noted (which I failed to point out earlier) that the locus relative to $\angle DEF$, rays EU and ED degenerates to the single point E when $EU \perp ED$ (as is also true in that case for the locus relative to $\angle DEF$, rays EU and EF). In application to the construction problem under consideration this corresponds to the case where for some median of ΔABC, say m_c, we have $m_c \perp AC$ (or $m_c \perp BC$). It is easy to see that under these circumstances angle C is necessarily obtuse. Hence we can be assured that if angles A and B are acute (which we can assume without loss of generality to be the case since every triangle has at least 2 acute angles) that the relevant loci L_1 and L_2 described above are non-degenerate, and intersect uniquely at a point P for reasons as earlier given.

This is all that is needed to guarantee completion of the construction, since P in turn determines unique squares QPX_1X_2, PRY_1Y_2, hence unique points Q and R, by the construction prescribed in the solution. The rest of the argument is the same as before.

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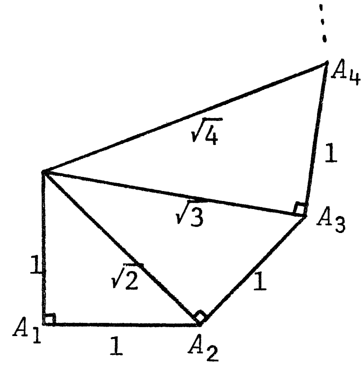
<u>1014</u>. [1985: 50; 1986: 125] *Proposed by Shmuel Avital, Technion-Israel Institute of Technology, Haifa, Israel.*



The points $A_1, A_2, A_3, \ldots$ are chosen, by the familiar construction illustrated in the figure, in such a way that $OA_n = \sqrt{n}$, $n = 1,2,3,\ldots$.

(a) What is the nature of the smooth spiral that passes through $A_1, A_2, A_3, \ldots$?

(b) Find, in terms of n, an explicit formula for the measure of the rotation that ray OA_1 must undergo to bring it into coincidence with ray OA_n.

Partial solution to (a) by David Singmaster, Polytechnic of the South Bank, London, England.

I start with the initial radius of 1 along the x-axis. The nth triangle then has sides $1, \sqrt{n}, \sqrt{n+1}$ and so the angle at the origin is given by

$$\theta_n = \tan^{-1} 1/\sqrt{n} \approx 1/\sqrt{n}.$$

Observe that $\Sigma\theta_n \longrightarrow \infty$, so that our "Pythagorean spiral" actually encircles the origin infinitely often.

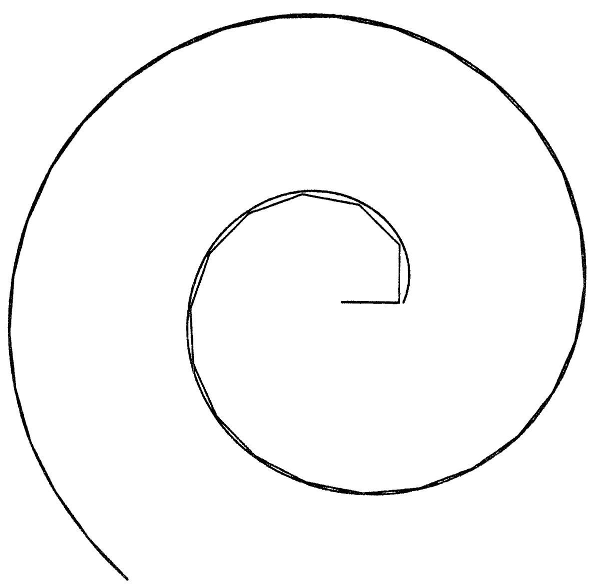
Now consider the movement from the point at radius $\sqrt{n}$ to the point at radius $\sqrt{n+1}$. The change in angle $d\theta$ is just θ_n. The change in radius, dr, is

$$\sqrt{n+1} - \sqrt{n} \approx \frac{1}{2\sqrt{n}} \approx \frac{1}{2}\,d\theta\ .$$

So we have $\frac{dr}{d\theta} \longrightarrow \frac{1}{2}$ and the asymptotic curve is $r = \frac{1}{2}\theta + r_0$ which is an Archimedean spiral. I have computed that

$$1.0781 < r_0 < 1.0814$$

with the upper bound being more accurate. The following diagram shows both the polygonal curve (up to $n = 39$) and its spiral asymptote.



Note also that the asymptotic formula shows that r should increase by π for every complete revolution.

Further comment by the Editor.

Hans Havermann points out that his related problem of whether the lines OA_i, OA_j, $i \neq j$, ever coincide (#789 in *J. Recr. Math.* 11 (1978-79) 301) has in fact been solved in the negative by Duane Allen (*J. Recr. Math.* 13 (1980-81) 300-303).

* * *

1016. [1985: 50; 1986: 128] *Proposed by Andrew P. Guinand, Trent University, Peterborough, Ontario.*

(a) Show that, for the triangle with angles 120°, 30°, 30°, the nine-point centre lies on the circumcircle.

(b) Characterize all the triangles for which the nine-point centre lies on the circumcircle.

Further solution by O. Bottema, Delft, The Netherlands.

The published solutions to (b) [1986: 128] contained a variety of characterizations of such triangles ABC, including

$$\cos A \cos B \cos C = -3/8 \tag{1}$$

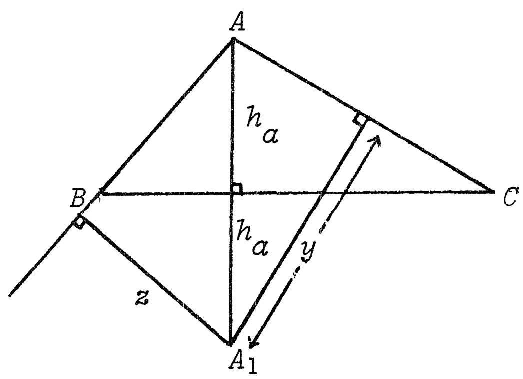
and

$$\cos^2 A + \cos^2 B + \cos^2 C = 7/4 , \tag{2}$$

which are equivalent since for any triangle

$$2 \cos A \cos B \cos C + \cos^2 A + \cos^2 B + \cos^2 C - 1 = 0 \tag{3}$$

holds. We derive here a *geometrical* characterization of the triangles under consideration. Let A_1 be the reflexion of A into BC, B_1 that of B into CA, C_1 that of C into AB. The triangle $A_1B_1C_1$ may be called the *reflexion triangle* of ABC. We ask whether there exist triangles for which the reflexion triangle is degenerate, which means that A_1, B_1, C_1 are collinear.



Let x, y, z be the (signed) distances of A_1 to the sides a, b, c respectively, and h_a the altitude of A. Then

$$x = -h_a$$

$$y = 2h_a \cos C$$

$$z = 2h_a \cos B.$$

Hence the homogeneous triangle coordinates of A_1 are

$$(-1, 2\cos C, 2\cos B)$$

and analogously for B_1 and C_1. The three reflexion points are collinear if and only if

$$\begin{vmatrix} -1 & 2\cos C & 2\cos B \\ 2\cos C & -1 & 2\cos A \\ 2\cos B & 2\cos A & -1 \end{vmatrix} = 0 ,$$

that is,

$$-1 + 4(\cos^2 A + \cos^2 B + \cos^2 C) + 16\cos A\cos B\cos C = 0,$$

which in view of (3) gives us (1) and (2). The conclusion is: *the triangles for which the Feuerbach centre lies on the circumcircle are precisely those for which the reflexion points are collinear.*

We call these triangles ℓ-triangles for short. From (1) it follows that an ℓ-triangle is obtuse. Let C be the obtuse angle and $\cos C = -t$, $0 < t < 1$. Then from (1) and (2),

$$\cos^2 A + \cos^2 B = \frac{7}{4} - t^2$$

and

$$2\cos A\cos B = \frac{3}{4t}$$

and so

$$(\cos A - \cos B)^2 = \frac{7}{4} - t^2 - \frac{3}{4t} = -\frac{1}{4t}(4t^3 - 7t + 3)$$
$$= -\frac{1}{4t}(t-1)(2t+3)(2t-1).$$

Thus we must have $2t - 1 \geq 0$, which implies $\cos C \leq -\frac{1}{2}$. Hence the obtuse angle of an ℓ-triangle satisfies $C \geq 120^\circ$. For the border case $C = 120^\circ$, we have $A = B = 30^\circ$, the solution mentioned by the proposer. In this case A_1 and B_1 coincide.

* * *

<u>1022</u>. [1985: 82] *Proposé par Armel Mercier, Université de Québec à Chicoutimi.*

Soient k un entier positif et i un entier vérifiant $0 \leq i \leq k$. Montrer que

$$\sum_{j=0}^{k} \frac{(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}(q^i - q^j)^k}{q^{jk}} = \prod_{j=1}^{k} (q^j - 1) ,$$

où $\begin{bmatrix} k \\ j \end{bmatrix}$ désigne le coefficient binomial de Gauss défini par $\begin{bmatrix} k \\ 0 \end{bmatrix} = 1$ et

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(q^k - 1)(q^{k-1} - 1) \ldots (q^{k+1-j} - 1)}{(q - 1)(q^2 - 1) \ldots (q^j - 1)} , \; 0 < j \le k ,$$

et q est une variable réelle arbitraire.

Solution par le proposeur.

Nous considerons le fonction

$$\frac{(q^i - 1 - (q - 1)x)^{k+1}}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]}$$

que nous pouvons écrire dans la forme

$$\frac{(q^i - 1 - (q - 1)x)^{k+1}}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]} = (-1)^{k+1}(q - 1)^{k+1} + \frac{h(x)}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]} , \tag{1}$$

où $h(x)$ est un polynôme de degré plus petit ou égal à k. Ainsi

$$\frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1) \right] h(x)}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]} = \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1) \right] (q^i - 1 - (q - 1)x)^{k+1}}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]} + (q - 1)^{k+1} \prod_{\ell=1}^{k} (q^\ell - 1). \tag{2}$$

Nous avons le développement en fractions partielles

$$\frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1) \right] h(x)}{\prod_{j=0}^{k} \left[x - \frac{q^j - 1}{q - 1} \right]} = \sum_{j=0}^{k} \frac{A_j}{x - \frac{q^j - 1}{q - 1}} ,$$

où

$$A_j = \lim_{x \to \frac{q^j - 1}{q - 1}} \left\{ \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] h(x) \left[x - \frac{q^j - 1}{q - 1}\right]}{\prod_{\ell=0}^{k} \left[x - \frac{q^\ell - 1}{q - 1}\right]} \right\} .$$

Employant la règle de L'Hôpital,

$$A_j = \lim_{x \to \frac{q^j - 1}{q - 1}} \left\{ \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] \left\{h'(x) \left[x - \frac{q^j - 1}{q - 1}\right] + h(x)\right\}}{\sum_{\ell=0}^{k} \sum_{\substack{t=0 \\ t \neq \ell}}^{k} \left[x - \frac{q^t - 1}{q - 1}\right]} \right\}$$

$$= \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] h\left[\frac{q^j - 1}{q - 1}\right]}{\prod_{\substack{t=0 \\ t \neq j}}^{k} \left[\frac{q^j - 1}{q - 1} - \frac{q^t - 1}{q - 1}\right]}$$

$$= \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] h\left[\frac{q^j - 1}{q - 1}\right] (q - 1)^k}{\prod_{\substack{t=0 \\ t \neq j}}^{k} (q^j - q^t)}$$

$$= \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] h\left[\frac{q^j - 1}{q - 1}\right] (q - 1)^k}{\prod_{t=0}^{j-1} (q^j - q^t) \cdot \prod_{t=j+1}^{k} (q^t - q^j) \cdot (-1)^{k-j}}$$

$$= \frac{(-1)^j \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right] h\left[\frac{q^j - 1}{q - 1}\right] (q - 1)^k}{\prod_{t=0}^{j-1} q^t \cdot \prod_{t=1}^{j} (q^t - 1) \cdot q^{j(k-j)} \prod_{t=1}^{k-j} (q^t - 1)}$$

$$= \frac{(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} h\left[\frac{q^j - 1}{q - 1}\right](q-1)^k}{q^{j(j-1)/2}\, q^{jk-j^2}}$$

$$= \frac{(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}\, h\left[\frac{q^j - 1}{q - 1}\right](q-1)^k}{q^{jk}} .$$

Aussi, d'après (1) on a

$$h\left[\frac{q^j - 1}{q - 1}\right] = (q^i - 1 - (q^j - 1))^{k+1} = (q^i - q^j)^{k+1}.$$

Ainsi (2) devient

$$\sum_{j=0}^{k} \frac{(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}\,(q^i - q^j)^{k+1}\,(q-1)^k}{q^{jk}\left[x - \frac{q^j - 1}{q - 1}\right]}$$

$$= \frac{(-1)^k \left[\prod_{\ell=1}^{k} (q^\ell - 1)\right](q^i - 1 - (q-1)x)^{k+1}}{\prod_{j=0}^{k}\left[x - \frac{q^j - 1}{q - 1}\right]} + (q-1)^{k+1} \prod_{\ell=1}^{k} (q^\ell - 1),$$

ou

$$\sum_{j=0}^{k} \frac{(-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2}\,(q^i - q^j)^k}{q^{jk}} \cdot \frac{(q^i - q^j)}{(q-1)\left[x - \frac{q^j - 1}{q - 1}\right]}$$

$$= \frac{(-1)\left[\prod_{\ell=1}^{k} (q^\ell - 1)\right]\left[x - \frac{q^i - 1}{q - 1}\right]^k}{\prod_{\substack{j=0 \\ j\neq i}}^{k}\left[x - \frac{q^j - 1}{q - 1}\right]} + \prod_{\ell=1}^{k} (q^\ell - 1) . \tag{3}$$

Maintenant nous laissons tendre x vers $\frac{q^i - 1}{q - 1}$. Puis

$$\lim_{x \to \frac{q^i-1}{q-1}} \left\{ \frac{q^i - q^j}{(q-1)\left[x - \frac{q^j-1}{q-1}\right]} \right\} = \frac{q^i - q^j}{(q-1)\left[\frac{q^i-1}{q-1} - \frac{q^j-1}{q-1}\right]} = 1$$

et (3) devient

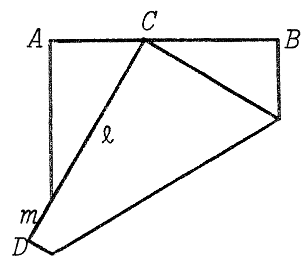
$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j+1)/2} (q^i - q^j)^k = \prod_{\ell=1}^{k} (q^\ell - 1),$$

qui est le resultat.

* * *

1025. [1985: 83; 1986: 156] *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

A paper square $ABCD$ is folded so that vertex C falls on AB and side CD is divided into two segments of lengths ℓ and m, as shown in the figure. Find the minimum value of the ratio ℓ/m.



Comment by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

In solution I [1986: 56], the minimization of $\ell = -x + \frac{2}{2-x}$ was accomplished by calculus. It is to be noted that (even more simply) by the A.M.-G.M. inequality,

$$\ell = -2 + (2 - x) + \frac{2}{2-x} > -2 + 2\sqrt{2}\ ,$$

equality occurring when

$$2 - x = \frac{2}{2-x}\ ,$$

that is, when $x = 2 - \sqrt{2}$.

* * *

1028. [1985: 83] *Proposed by David Singmaster, Polytechnic of the South Bank, London, England.*

Students learning modular arithmetic are pleasantly perplexed by

$$4\cdot 5 \equiv 6 \pmod 7).$$

Solve the following (and possibly other) generalizations:

(a) $a(a + 1) \equiv a + 2 \pmod{a + 3}$.

(b) $a(a + 1) \equiv a + 2 \pmod{m}$, where m is not necessarily $a + 3$.

(c) $a(a + 1) \equiv (a + 2)(a + 3) \pmod{m}$.

I. *Composite of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, and Kenneth M. Wilke, Topeka, Kansas.*

(a) Since $a \equiv -3 \pmod{a + 3}$,

$$a(a + 1) \equiv a + 2 \pmod{a + 3} \iff (-3)(-2) \equiv -1 \pmod{a + 3}$$
$$\iff (a + 3) \mid 7$$

and so the only positive solution for a is $a = 4$, the example given.

(b) The given congruence is equivalent to

$$a^2 \equiv 2 \pmod{m},$$

i.e. 2 is a quadratic residue mod m.

Suppose m is odd. Since for odd primes p, $\left(\frac{2}{p}\right) = 1$ if and only if $p \equiv \pm 1 \pmod 8$, m must be a product of primes of the form $8k \pm 1$. Moreover $\left(\frac{2}{p}\right) = 1$ if and only if $\left(\frac{2}{p^n}\right) = 1$ for any $n \geq 1$, and so by the Chinese Remainder Theorem any number m of this form will have 2 as a quadratic residue.

Suppose m is even. Then it is easy to see that m must be of the form 2ℓ where ℓ is odd. As above, ℓ must be a product of primes of the form $8k \pm 1$, and for any such ℓ there will be a solution of $x^2 \equiv 2 \pmod{\ell}$, say $x = a$. Since $\ell - a$ will then also be a solution, we may assume a is even, in which case $a^2 \equiv 2 \pmod{m}$ will hold.

Thus we have that the congruence in (b) will have a solution if and only if m is a product of primes of the form $8k \pm 1$, or twice such a product.

(c) The given congruence is equivalent to

$$2(2a + 3) \equiv 0 \pmod{m}.$$

Obviously this has a solution (for a) if and only if $m = 2n$, n odd.

II. *Special case of* (c) *by the proposer.*

If we further assume $m = a + 4$, then

$$a(a + 1) \equiv (a + 2)(a + 3) \pmod{m} \iff 4a + 6 \equiv 0 \pmod{m}$$
$$\iff 4m - 10 \equiv 0 \pmod{m}$$
$$\iff m \mid 10$$

and we have two interesting solutions

$$1 \cdot 2 \equiv 3 \cdot 4 \pmod 5 \quad \text{and} \quad 6 \cdot 7 \equiv 8 \cdot 9 \pmod{10}.$$

III. *Generalizations by Richard K. Guy, University of Calgary, Calgary, Alberta.*

The congruence

$$(d - k - \ell)(d - k - \ell + 1) \ldots (d - \ell - 1)$$
$$\equiv (d - \ell)(d - \ell + 1) \ldots (d - 1) \bmod d$$

holds just if

$$d \,\Big|\, \left[\frac{(k + \ell)!}{\ell!} - (-1)^{k+\ell}\ell!\right] \tag{1}$$

where, for positive solutions, $d > k + \ell$. Examples:

$k = 2,\ \ell = 1$ gives the classical $4\cdot5 \equiv 6 \bmod 7$

$k = 1,\ \ell = 2$ gives $2 \equiv 3\cdot4 \bmod 5$

$k = 3,\ \ell = 1$ gives $19\cdot20\cdot21 \equiv 22 \bmod 23$

$k = 3,\ \ell = 3$ gives $13\cdot14\cdot15 \equiv 16\cdot17\cdot18 \bmod 19$

and so on.

With a lot of effort (not worth it!) you can produce an infinity of solutions of the type

$$3\cdot4\cdot5\cdot6 \equiv 7\cdot8 \bmod(9 + 10)$$

$$92\cdot93\cdot94 \equiv 95\cdot96\cdot97 \bmod(98 + 99)$$

but the plus sign introduces a jarring note?

A much nicer generalization is to mod $d(d + 1)$. Now (as well as (1)) we must have

$$(d + 1) \,\Big|\, \left[\frac{(k + \ell + 1)!}{(\ell + 1)!} - (-1)^{k+\ell}(\ell + 1)!\right] .$$

Examples:

$$57\cdot58\cdot59 \equiv 60\cdot61 \bmod 62\cdot63$$

$$15\cdot16\cdot17\cdot18 \equiv 19\cdot20\cdot21\cdot22 \bmod 23\cdot24$$

$$8\cdot9\cdot10 \equiv 11\cdot12\cdot13\cdot14\cdot15\cdot16\cdot17 \bmod 18\cdot19\cdot20$$

and so on.

Finally, we note the remarkable

$$1 \equiv 2\cdot3\cdot4 \equiv 5\cdot6\cdot7\cdot8 \equiv 9\cdot10\cdot11 \equiv -(12\cdot13\cdot14)$$
$$\equiv 15\cdot16\cdot17\cdot18 \equiv 19\cdot20\cdot21\cdot22 \bmod 23.$$

Also solved by Sister DONNA M. GRIBSCHAW, C.D.P., John Carroll University, University Heights, Ohio; J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

* * *

1029. [1985: 83] *Proposed by Farshad Khorrami, student, The Ohio State University.*

Find necessary and sufficient conditions on the complex numbers a and b so that each root of

$$z^2 + az + b = 0$$

has absolute value less than 1.

I. *Solution by Kee-wai Lau, Hong Kong.*

A necessary and sufficient condition is given by

$$|b|^2 + |b\bar{a} - a| < 1.$$

Let the roots be α and β so that $\alpha + \beta = -a$ and $\alpha\beta = b$. Then the above condition is equivalent to

$$1 - |\alpha\beta|^2 - |\alpha\beta(-\bar{\alpha} - \bar{\beta}) + \alpha + \beta| > 0,$$

or

$$1 - |\alpha\beta|^2 - |\beta(1 - \alpha\bar{\alpha}) + \alpha(1 - \beta\bar{\beta})| > 0,$$

hence, letting

$$f(\alpha,\beta) = 1 - |\alpha\beta|^2 - |\beta(1 - |\alpha|^2) + \alpha(1 - |\beta|^2)|,$$

it suffices to show that $f(\alpha,\beta) > 0$ if and only if $|\alpha| < 1$ and $|\beta| < 1$.

First, if $|\alpha| < 1$ and $|\beta| < 1$, then

$$\begin{aligned}|\beta(1 - |\alpha|^2) + \alpha(1 - |\beta|^2)| &\le |\beta|\cdot(1 - |\alpha|^2) + |\alpha|\cdot(1 - |\beta|^2)\\ &= (1 - |\alpha\beta|)\cdot(|\alpha| + |\beta|),\end{aligned}$$

so

$$\begin{aligned}f(\alpha,\beta) &\ge 1 - |\alpha\beta|^2 - (1 - |\alpha\beta|)\cdot(|\alpha| + |\beta|)\\ &= (1 - |\alpha\beta|)(1 + |\alpha\beta| - |\alpha| - |\beta|)\\ &= (1 - |\alpha\beta|)(1 - |\alpha|)(1 - |\beta|)\\ &> 0,\end{aligned}$$

as claimed.

For the other direction, first note that if $|\alpha\beta| \ge 1$ then $f(\alpha,\beta) \le 0$ trivially. Thus we can assume that $|\alpha\beta| < 1$ and, without loss of generality, $|\alpha| < 1$ and $|\beta| \ge 1$. Then

$$\begin{aligned}|\beta(1 - |\alpha|^2) + \alpha(1 - |\beta|^2)| &\ge |\beta|\cdot|1 - |\alpha|^2| - |\alpha|\cdot|1 - |\beta|^2|\\ &= |\beta|\cdot(1 - |\alpha|^2) - |\alpha|\cdot(|\beta|^2 - 1)\\ &= (1 - |\alpha\beta|)\cdot(|\alpha| + |\beta|),\end{aligned}$$

so

$$\begin{aligned}f(\alpha,\beta) &\le 1 - |\alpha\beta|^2 - (1 - |\alpha\beta|)\cdot(|\alpha| + |\beta|)\\ &= (1 - |\alpha\beta|)(1 - |\alpha|)(1 - |\beta|)\\ &\le 0,\end{aligned}$$

and the solution is complete.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

As $(-a \pm \sqrt{a^2 - 4b})/2$ are the roots in question, the necessary and sufficient condition is

$$|a + \sqrt{a^2 - 4b}| < 2 \quad \text{and} \quad |a - \sqrt{a^2 - 4b}| < 2.$$

Also solved by LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; and M.A. SELBY, University of Windsor, Windsor, Ontario. There was one partial solution received.

* * *

1030. [1985: 83] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given are two obtuse triangles with sides a, b, c and p, q, r, the longest sides of each being c and r, respectively. Prove that

$$ap + bq < cr.$$

I. *Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

More generally, let a_r, b_r, c_r, $r = 1,2,\ldots,n$, be the sides of n obtuse triangles ($n \geq 2$) with $c_r > a_r$, b_r for each r. We will show that

$$c_1c_2 \ldots c_n > a_1a_2 \ldots a_n + b_1b_2 \ldots b_n.$$

Since the triangles are obtuse,

$$c_r^2 > a_r^2 + b_r^2$$

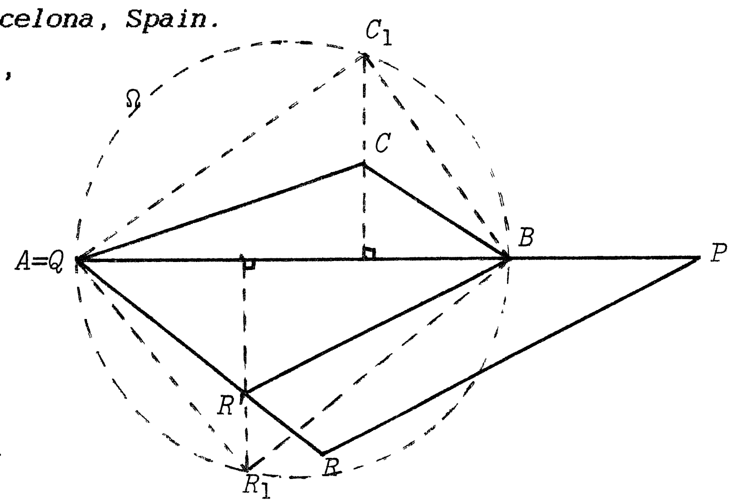
for each r, and so by Hölder's inequality

$$\begin{aligned} c_1^2c_2^2 \ldots c_n^2 &> (a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) \\ &\geq [(a_1a_2 \ldots a_n)^{2/n} + (b_1b_2 \ldots b_n)^{2/n}]^n. \end{aligned}$$

Thus

$$\begin{aligned} c_1c_2 \ldots c_r &> [(a_1a_2 \ldots a_n)^{2/n} + (b_1b_2 \ldots b_n)^{2/n}]^{n/2} \\ &\geq (a_1a_2 \ldots a_n)^{2/n\cdot n/2} + (b_1b_2 \ldots b_n)^{2/n\cdot n/2} \quad \text{since } n \geq 2 \\ &= a_1a_2 \ldots a_n + b_1b_2 \ldots b_n. \end{aligned}$$

II. *Solution by Jordi Dou, Barcelona, Spain.*



Let the triangles be ABC and PQR, oriented as in the figure, with A and Q identified, AB lying along QP (assuming $AB \leq QP$), and C and R on opposite sides of QP. Let R' on QR be such that BR' is parallel to PR. Let Ω be the circle with diameter AB. Since ABC and PQR are obtuse, points C and R' lie inside Ω. Let C_1, R_1 be on Ω, on the same side of AB as C, R' respectively, such that C_1C and R_1R' are perpendicular to AB.

From Ptolemy's theorem applied to the quadrilateral AC_1BR_1,

$$AB^2 = AB\cdot C_1R_1 = AC_1\cdot BR_1 + C_1B\cdot R_1A$$

$$> AC\cdot BR' + CB\cdot R'A.$$

Since triangles PQR and BAR' are similar,

$$\frac{BR'}{RP} = \frac{R'A}{QR} = \frac{AB}{PQ}.$$

Thus

$$AB\cdot PQ > AC\cdot RP + CB\cdot QR,$$

which is the required result.

Also solved by SAM BAETHGE, San Antonio, Texas; LEON BANKOFF, Los Angeles, California; O. BOTTEMA, Delft, The Netherlands; R.H. EDDY, Memorial University of Newfoundland, St. John's, Newfoundland; JACK GARFUNKEL, Flushing, New York; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

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1031. [1985: 121] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Independently solve the alphametics

$$\begin{array}{r} \text{MAN} \\ \text{WINS} \\ \hline \text{MAID} \end{array} \qquad \text{and} \qquad \begin{array}{r} \text{MAN} \\ \text{WEDS} \\ \hline \text{WIFE} \end{array},$$

but FIND A MAN AND A WOMAN common to both alphametics.

Solution.

$$\begin{array}{r} 207 \\ 1876 \\ \hline 2083 \end{array} \qquad \text{and} \qquad \begin{array}{r} 207 \\ 1639 \\ \hline 1846 \end{array}$$

with FIND A MAN AND A WOMAN being

4873 0 207 073 0 15207.

Found by J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Valencia Community College, Orlando, Florida; J. SUCK, Essen, Federal Republic of Germany; and the proposer.

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A QUESTION FROM THE MANAGING EDITOR

A number of readers have expressed interest in having the Canadian Mathematical Society produce bound (indexed) volumes of *Crux* each year as was done for the first ten years of *Crux's* existence. These bound volumes were expensive to produce and lost money. Would you like to see these bound volumes continued, and if so, would you be prepared to pay a much higher price per volume (perhaps \$30 - \$40)? Please let me have your views on this matter.

K.S. Williams