

Mathematicorum

Crux

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CRUX

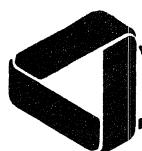
Mathematicorum

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CONTENTS

The Olympiad Corner: No. 103	R.E. Woodrow	65
Problems: 1421-1430		72
Solutions: 189, 1312-1317		75
J.T. Groenman		96



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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

G.W. Sands
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4

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Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

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THE OLYMPIAD CORNER
No. 103
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first set of problems this issue comes from Dimitris Vathis of Chalcis, Greece. It is the 1987 Annual High School Competition of the Greek Mathematical Society. The problems and solutions have appeared in the Greek Mathematical Society publication *Euclid B'*, December 1987, pp. 79-80.

1987 ANNUAL GREEK HIGH SCHOOL COMPETITION

1. Let M be the centroid of a planar quadrilateral $ABCD$. Suppose that K and L are points on the plane of $ABCD$ such that $\overrightarrow{AK} = \overrightarrow{MB}$ and $\overrightarrow{KL} = \overrightarrow{MC}$. Prove that M is the midpoint of the segment LD .
2. Given the mappings $g: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $g(g(x)) = x$ for every real x , and α a real number having $|\alpha| \neq 1$, prove that there is a unique mapping $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\alpha f(x) + f(g(x)) = h(x)$$

for every x in \mathbb{R} .

3. Let A be an $n \times n$ matrix such that

$$A^2 - 3A + 2I = 0,$$

where I is the identity matrix and 0 the zero matrix. Prove that

$$A^{2k} - (2^k + 1)A^k + 2^k \cdot I = 0$$

for every natural number $k \geq 1$.

4. Suppose that for each natural number $n \geq 2$ there is given a real number a_n such that

$$\frac{[a_n]\{a_n\}}{a_n} = n$$

where $[a_n]$ is the greatest integer not exceeding a_n and $\{a_n\} = a_n - [a_n]$. Prove that $|a_n - a_{n-1}| < \frac{1}{1987}$

for every $n > 45$.

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Next we include problems proposed by the Royal Spanish Mathematical Society for the 1987 Spanish Olympiad. These were sent in by Professor Francisco Bellot of Valladolid, Spain.

24TH SPANISH OLYMPIAD - First Round

November 28-29, 1987

1. Express 1987 as a sum of squares of distinct prime numbers in all possible ways. (Proceed rationally.)
2. Show that for each polynomial $p(x)$, there is a number k such that one of the polynomials $p(x) + k$ and $xp(x) + k$ has no real roots, while the other takes the value zero for at least one value of x .

3. Let C be the set of natural numbers

$$C = \{1, 5, 9, 13, 17, 21, \dots\}.$$

Say that a number is "prime for C " if it cannot be written as a product of smaller numbers from C .

- (a) Show that 4389 is a member of C which can be represented in at least two distinct ways as a product of two numbers "prime for C ".
- (b) Find another member of C with the same property.

4. The ancient Egyptians approximated the area of a circle of diameter d in the following manner. At each vertex of a square of side length d , an isosceles right triangle with equal sides $d/3$ is cut off (with the right angle coinciding with the vertex). This gives an octagon which is not regular. The area of this octagon was used as an approximation to the area of the circle. Find the ratio of the area of the circle to the octagon.

Generalize this process for the sphere of diameter d . Given a cube of edge d , in the three faces adjacent to one vertex the octagons described in the last paragraph are drawn. Consider three right prisms with bases these octagons and height d , contained in the cube. The body given by the intersection of these three prisms has a volume which gives an approximation to the volume of the sphere. Find the ratio of the volume of the sphere to the volume of this body.

5. Given the function f defined by

$$f(x) = \sqrt{4 + \sqrt{16x^2 - 8x^3 + x^4}}.$$

- (a) Draw the graph of the curve $y = f(x)$.
(b) Find, without the use of integral calculus, the area of the region bounded by the straight lines $x = 0$, $x = 6$, $y = 0$ and by the curve $y = f(x)$. Note: all the square roots are non-negative.

6. A regular hexagon of area a is decomposed into six parts by drawing lines from an interior point P , perpendicular to the sides of the hexagon. The six parts are labelled anticlockwise, and the areas are a_1, a_2, a_3, a_4, a_5 and a_6 , respectively. For what points P is

$$a_1 + a_3 + a_5 = a_2 + a_4 + a_6?$$

Give bounds for $a_1 + a_3 + a_5$.

7. Let $I_n = (n\pi - \pi/2, n\pi + \pi/2)$, and let f be the function defined by

$$f(x) = \tan x - x.$$

- (a) Show that the equation $f(x) = 0$ has only one root in each interval I_n , $n = 1, 2, 3, \dots$.
(b) If c_n is the root of $f(x) = 0$ in I_n find $\lim_{n \rightarrow \infty} (c_n - n\pi)$.

8. Let E be the set of points (x, y, z) in space which satisfy

$$x + y \leq 1,$$

$$2x + z = 3.$$

Find the points (x, y, z) of E for which $|x| + |y| + |z|$ is least. Find this minimum value and give a geometric interpretation of the situation.

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The last problems this month are those for the 1987 Spanish Mathematical Olympiad proposed by the Valladolid jury. Problems 3, 5, and 6 were the same as above.

24TH SPANISH MATHEMATICAL OLYMPIAD

Valladolid – First Round

November 28–29, 1987

1. Fifteen problems, numbered 1 through 15, are posed on a certain examination. No student answers two consecutive problems correctly.

If 1600 candidates sit the test, must at least two of them answer each question in the same way?

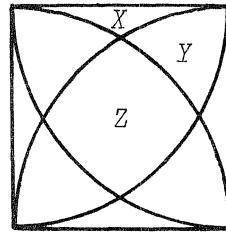
2. Let f be a continuous function on \mathbb{R} such that

(i) $f(n) = 0$ for every integer n , and

(ii) if $f(a) = 0$ and $f(b) = 0$ then $f\left(\frac{a+b}{2}\right) = 0$, with $a \neq b$.

Show that $f(x) = 0$ for all real x .

4. Given a square of side a , as in the figure, find the areas of each of the regions X , Y , Z . (The curves are arcs of circles of radius a .)



7. Solve the following system of equations in the set of complex numbers:

$$|z_1| = |z_2| = |z_3| = 1,$$

$$z_1 + z_2 + z_3 = 1,$$

$$z_1 z_2 z_3 = 1.$$

8. In the triangle ABC , let P be a point of BC , between B and C . Let m , n , t be the lengths of BP , PC and AP respectively. If a , b , c are the sides of ABC , show that

$$b^2m + c^2n = t^2a + mna.$$

As an application, find the lengths of the median and the internal bisector of the triangle with respect to vertex A .

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We return now to the I.M.O. problems proposed but not used at the 28th I.M.O. in Havana.

X1. [1987: 249]

1987

It is given that $x = -2272$, $y = 10^3 + 10^2c + 10b + a$ and $z = 1$ satisfy the equation

$$ax + by + cz = 1$$

where a , b , c are positive integers with $a < b < c$. Find y .

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The given equation can be rewritten as

$$b(1000 + 100c + 10b + a) + c - 2272a - 1 = 0. \quad (*)$$

Let $b = a + u$, $c = a + u + v$ where $u \geq 1$, $v \geq 1$ are integers. Substitution into (*) and simplification gives $f(a) = 0$ where

$$f(a) = 111a^2 + (221u + 100v - 1271)a + (110u^2 + 100uv + 1001u + v - 1).$$

If $u \geq 2$ then $f(a) \geq 111a^2 - 729a + 2642 = g(a)$. Since the discriminant of $g(a)$ is negative, $g(a) > 0$, giving $f(a) > 0$, a contradiction. Hence $u = 1$, and

$$f(a) = 111a^2 + (100v - 1050)a + (101v + 1110).$$

If $v \geq 3$ then

$$f(a) \geq 111a^2 - 750a + 1413 = h(a).$$

Since the discriminant of $h(a)$ is negative, $h(a) > 0$, again impossible. Thus $v = 1$ or 2. If $v = 2$, then $f(a) = 111a^2 - 850a + 1312$, and it is easy to check that $f(a) = 0$ has no integer solutions.

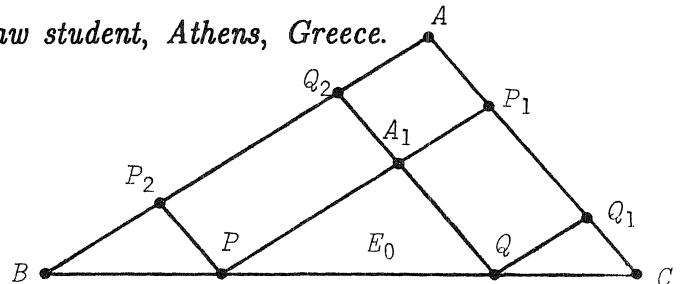
Thus $v = 1$ and $f(a) = 111a^2 - 950a + 1211 = (a - 7)(111a - 173)$. Therefore $a = 7$ and it follows that $b = 8$, $c = 9$ and $y = 1987$.

X2. [1987: 249]

Let PQ be a line segment of fixed length but variable position on the side BC of a triangle ABC , with the order $BPQC$, and let the lines through P , Q parallel to the lateral sides meet AC , AB at P_1 , Q_1 and P_2 , Q_2 , respectively. Prove that the sum of the areas of the trapezoids PQQ_1P_1 and PQQ_2P_2 is independent of the position of PQ on BC .

Solution by George Evangelopoulos, Law student, Athens, Greece.

Let A_1 be the intersection of PP_1 and QQ_2 . Denote by E_0 the area of triangle A_1PQ . Adopting the notation $[R]$ for the area of a region R , we have



$$\begin{aligned} [PQQ_1P_1] + [PQQ_2P_2] &= 2E_0 + [A_1QQ_1P_1] + [A_1PP_2Q_2] \\ &= 2E_0 + 2[A_1A_1Q] + 2[A_1A_1P] \\ &= 2[APQ] = uh_a = \text{constant}, \end{aligned}$$

where $u = |PQ|$ and h_a is the length of the altitude from A to BC .

[Editor's note: The problem was also solved by J.T. Groenman, Arnhem, The Netherlands.]

X3. [1987: 249]

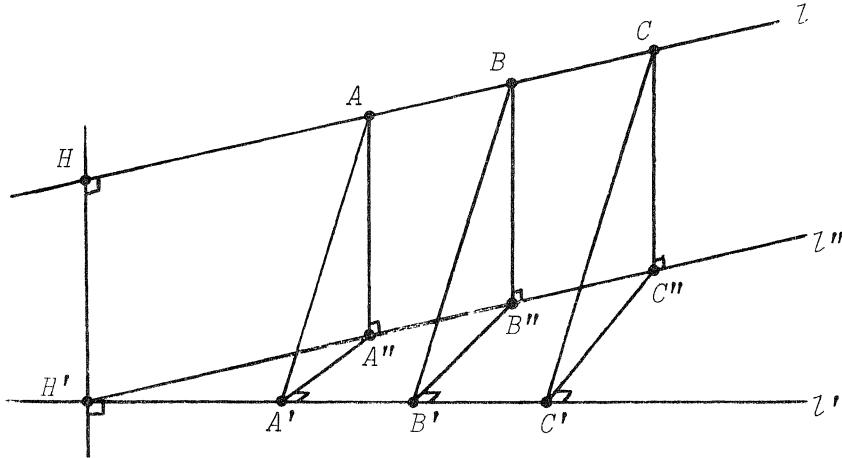
Let l , l' be two lines in 3-space, and let A , B , C be three points taken on l with B as midpoint of the segment AC . If a , b , c are the distances of

A, B, C from l' , respectively, show that

$$b \leq \sqrt{\frac{a^2 + c^2}{2}}$$

with equality holding if l, l' are parallel.

Solution by George Evangelopoulos, Law student, Athens, Greece.



Let a common perpendicular line of l, l' meet them at H, H' respectively and let $|HH'| = d$. Draw the line l'' through H' parallel to l . Drop segments AA'', BB'', CC'' to l'' , parallel to HH' , with

$$|AA''| = |BB''| = |CC''| = |HH'| = d.$$

Now drop the perpendicular segments $A''A', B''B', C''C'$ to l' . Then from the Three-Perpendicular Theorem AA', BB' and CC' will be perpendicular to l' so that we have

$$a = |AA'|, \quad b = |BB'|, \quad c = |CC'|.$$

Denoting $a' = |A''A'|, b' = |B''B'|, c' = |C''C'|$ we have

$$\frac{a^2 + c^2}{2} = \frac{(d^2 + a'^2) + (d^2 + c'^2)}{2} = d^2 + \frac{a'^2 + c'^2}{2}.$$

Thus

$$\frac{a^2 + c^2}{2} \geq d^2 + \left(\frac{a' + c'}{2}\right)^2$$

with equality holding just in case $a' = c'$ ($= 0$), i.e. just in case l and l' are parallel. Now

$$\frac{a^2 + c^2}{2} \geq d^2 + b'^2 = b^2$$

since

$$b' = \frac{a' + c'}{2},$$

giving

$$b \leq \sqrt{\frac{a^2 + c^2}{2}}.$$

[Editor's note: A similar solution was submitted by Zun Shan and Edward T.H. Wang of Wilfrid Laurier University, Waterloo, while an algebraic solution using vectors was sent in by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.]

X4. [1987: 249]

Compute $\sum_{k=0}^{2n} (-1)^k a_k$ where the a_k 's are the coefficients in the expansion

$$(1 - \sqrt{2}x + x^2)^n = \sum_{k=0}^{2n} a_k x^k.$$

Remark by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There must be an error in the statement as the trivial solution is to set $x = -1$ giving

$$\sum_{k=0}^{2n} (-1)^k a_k = (2 + \sqrt{2})^n.$$

Proposed correction and solution by George Evangelopoulos, Law student, Athens, Greece.

Perhaps the problem should read: calculate

$$\sum_{k=0}^{2n} (-1)^k a_k^2.$$

In that case set

$$p(x) = (1 - \sqrt{2}x + x^2)^n = \sum_{k=0}^{2n} a_k x^k.$$

Then

$$p(-x) = \sum_{k=0}^{2n} (-1)^k a_k x^k.$$

Moreover since $p(x) = x^{2n} p(1/x)$ we have $a_k = a_{2n-k}$ for all k . Forming the product

$$p(-x)p(x) = \sum_{j=0}^{4n} c_j x^j,$$

we get

$$c_{2n} = \sum_{k=0}^{2n} (-1)^k a_k a_{2n-k} = \sum_{k=0}^{2n} (-1)^k a_k^2.$$

But

$$p(-x)p(x) = (1 + x^4)^n = \sum_{j=0}^{4n} \frac{n!}{(n-j)! j!} x^{4j}.$$

Thus

$$\sum_{k=0}^{2n} (-1)^k a_k^2 = 0$$

if n is odd, while

$$\sum_{k=0}^{2n} (-1)^k a_k^2 = \frac{n!}{[(n/2)!]^2}$$

if n is even.

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Having finished the I.M.O. problems from the October 1987 number of *Crux*, next issue we will move to those posed in November 1987. Remember to send in your solutions to problems as well as national and regional Olympiad contests.

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P R O B L E M S

Problem proposals and solutions should be sent to the editor, whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1989, although solutions received after that date will also be considered until the time when a solution is published.

1421. Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.

ABC is a triangle with sides a, b, c . The escribed circle to the side a has centre I_a and touches a, b, c (produced) at D, E, F respectively. M is the midpoint of BC .

- (a) Show that the lines I_aD, EF and AM have a common point S_a .
- (b) In the same way we have points S_b and S_c . Prove that

$$\frac{\text{area}(\Delta S_a S_b S_c)}{\text{area}(\Delta ABC)} > \frac{3}{2}.$$

1422. Proposed by George Tsintsifas, Thessaloniki, Greece.

Let $A_1A_2A_3$ be a triangle and M an interior point; $\lambda_1, \lambda_2, \lambda_3$ the barycentric coordinates of M ; and r_1, r_2, r_3 its distances from the sides A_2A_3, A_3A_1, A_1A_2 respectively. Set $A_iM = R_i, i = 1, 2, 3$. Prove that

$$\sum_{i=1}^3 \lambda_i R_i \geq 2 \left[\lambda_1 \cdot \frac{r_2 r_3}{r_1} + \lambda_2 \cdot \frac{r_3 r_1}{r_2} + \lambda_3 \cdot \frac{r_1 r_2}{r_3} \right].$$

1423* Proposed by Murray S. Klamkin, University of Alberta.

Given positive integers k, m, n , find a polynomial $p(x)$ with real coefficients such that

$$(x-1)^n \mid (p(x))^m - x^k.$$

What is the least possible degree of p (in terms of k, m, n)?

1424. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Show that

$$\sum a \tan A \geq 10R - 2r$$

for any acute triangle ABC , where a, b, c are its sides, R its circumradius, and r its inradius, and the sum is cyclic.

1425. Proposed by Jordi Dou, Barcelona, Spain.

Let D be the midpoint of side BC of the equilateral triangle ABC and ω a circle through D tangent to AB , cutting AC in points B_1 and B_2 . Prove that the two circles, distinct from ω , which pass through D and are tangent to AB , and which respectively pass through B_1 and B_2 , have a point in common on AC .

1426. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Prove that if n is a positive integer then

$$512 \mid 3^{2n} - 32n^2 + 24n - 1.$$

1427. *Proposed by David Singmaster, Polytechnic of the South Bank.*

In *Numerous Numerals* (NCTM, 1975), James Henle defines the "fracimal" $.a.b.c\dots$ to be the number

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} + \dots,$$

where a, b, c, \dots is a finite sequence of integers, each greater than 1. Which rational numbers can be represented as fracimals?

1428^{*}. *Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.*

$A_1A_2A_3$ is a triangle with sides a_1, a_2, a_3 , and P is an interior point with distances R_i and r_i ($i = 1, 2, 3$) to the vertices and sides, respectively, of the triangle. Prove that

$$\left(\sum a_1 R_1 \right) \left(\sum r_1 \right) \geq 6 \sum a_1 r_2 r_3$$

where the sums are cyclic.

1429^{*}. *Proposed by D.S. Mitrinovic, University of Belgrade, and J.E. Pecaric, University of Zagreb.*

(a) Show that

$$\sup \sum \frac{x_1^2}{x_1^2 + x_2 x_3} = n - 1,$$

where x_1, x_2, \dots, x_n are n positive real numbers ($n \geq 3$), and the sum is cyclic.

(b) More generally, what is

$$\sup \sum \frac{x_1^{r+s}}{x_1^{r+s} + x_2^r x_3^s},$$

for natural numbers r and s ?

1430. *Proposed by Mihaly Bencze, Brasov, Romania.*

AD, BE, CF are (not necessarily concurrent) Cevians in triangle ABC , intersecting the circumcircle of ΔABC in the points P, Q, R . Prove that

$$\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq 9.$$

When does equality hold?

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

- 189.** [1976: 194; 1977: 74, 193, 252; 1978: 37] *Proposed by Kenneth S. Williams, Carleton University.*

If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

[Editor's note: In a recent problem set he was using to train potential Canadian I.M.O. team members, Andy Liu, University of Alberta, had included a slightly simplified version of *Crux* 189. One solution he received to this problem seemed to him to merit publication in *Crux*. So it does, and here it is! The editor has added the minor "bridge" needed to link *Crux* 189 with the version given to the students. Missing is the comment (noted by Bankoff and Klamkin in their published solutions [1977: 74]) that it suffices to prove the result when the ellipse is a circle.]

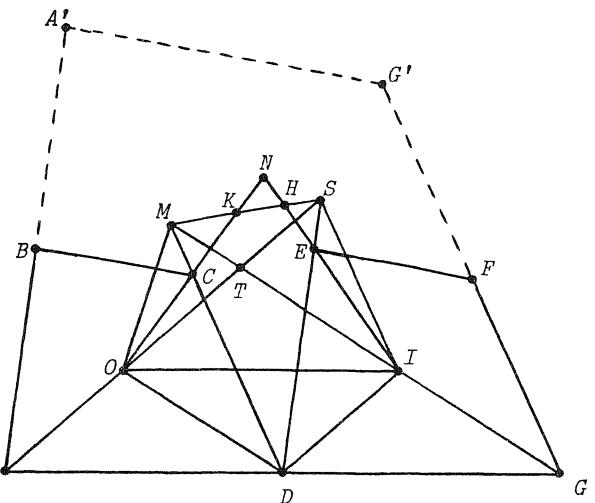
V. *Solution by Martin Bissell and Amy Wong (student), A.N. Myer Secondary School, Niagara Falls, Ontario.*

Let D be the midpoint of AG , and let $ABCD$ and $DEFG$ be congruent quadrilaterals, on the same side of AG , and having inscribed circles with centres O and I respectively. Extend AB to A' and GF to G' so that $AB = BA'$ and $GF = FG'$, and let AO and GI meet at T . Consider $AA'G'G$ to be the quadrilateral in the problem. Then, since $AA'G'G$ is similar to and twice the linear size of $ABCD$ and $DEFG$, C and E will be the midpoints of its diagonals and T will be the centre of its inscribed circle. Thus we must show that C , T and E are collinear.

Put

$$\begin{aligned}\alpha &= \angle OAD = \angle OAB, & \beta &= \angle OBA = \angle OBC, \\ \gamma &= \angle OCB = \angle OCD, & \delta &= \angle ODC = \angle ODA.\end{aligned}$$

Then



$$2\alpha + 2\beta + 2\gamma + 2\delta = 360^\circ,$$

so

$$\alpha + \beta + \gamma + \delta = 180^\circ. \quad (1)$$

Let DC meet GI at M , OC meet IE at N , and AO meet DE at S . Since $AOS \parallel DI$, $OI \parallel ADG$, and I is the incentre of $DEFG$,

$$\angle OSD = \angle SDI = \angle IDG = \angle OID = \alpha,$$

so the points $OSID$ are concyclic. Similarly, $\angle DMI = \angle DOI = \delta$ and $DOMI$ are concyclic. Thus $DOMSI$ are concyclic.

Let MS meet ON and IN at K and H respectively. Then

$$\angle MCK = \angle OCD = \gamma$$

and

$$\angle CMK = \angle DMI + \angle IMS = \angle DOI + \angle IDS = \delta + \alpha,$$

so from $\triangle MKC$ and (1) we get

$$\angle NKH = \angle MKC = \beta.$$

Similarly, $\angle NHK = \gamma$, so from $\triangle NKH$ and (1) we have

$$\angle ONI = \angle KNH = \alpha + \delta.$$

But now

$$\angle OMI = \angle OMD + \angle DMI = \angle OID + \angle DOI = \alpha + \delta = \angle ONI,$$

so $OMNI$ are concyclic. Since $DOMSI$ are concyclic, $ONIMDS$ form a cyclic hexagon. From Pascal's Theorem, C , T and E are collinear.

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1312. [1988: 44] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Find all 27 solutions of the system of equations

$$y = 4x^3 - 3x$$

$$z = 4y^3 - 3y$$

$$x = 4z^3 - 3z.$$

Solution by Colin Springer, student, University of Waterloo.

First note that $|x| \leq 1$, for if $|x| > 1$ then $y = x^3 + 3(x^3 - x)$ implies $|y| > |x|$, and similarly $|z| > |y|$ and $|x| > |z|$, a contradiction.

Thus we may let $x = \cos \theta$, $0 \leq \theta \leq \pi$. Then

$$y = 4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta$$

and thus

$$z = \cos 9\theta, x = \cos 27\theta.$$

Therefore θ is a solution to

$$\cos \theta - \cos 27\theta = 0$$

or

$$2 \sin 13\theta \sin 14\theta = 0.$$

Now this has 27 solutions for θ in $[0, \pi]$, namely

$$\theta = k\pi/13, k = 0, 1, 2, \dots, 13$$

and

$$\theta = k\pi/14, k = 1, 2, \dots, 13,$$

each of which produces a distinct triple

$$(x, y, z) = (\cos \theta, \cos 3\theta, \cos 9\theta)$$

satisfying the given equations.

Also solved by SEUNG-JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Penn State University at Harrisburg; M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; M.A. SELBY, University of Windsor; C. WILDHAGEN, Breda, The Netherlands; JURGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer. One incorrect solution was received.

Lau noted that x satisfies a polynomial equation of degree 27, so there can be at most 27 distinct solutions. Janous generalized the problem to n variables, remarking that a similar question appeared (in Bulgarian) as problem 81, Chapter 2 of Davidov et al, Mathematical Competitions, Sofia, 1977. Kierstead sends his compliments to the proposer for a nice problem!

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1313. [1988: 45] *Proposed by Wendel Semenko, Snowflake, Manitoba.*

Show that any triangular piece of paper of area 1 can be folded once so that when placed on a table it will cover an area of less than $\frac{\sqrt{5} - 1}{2}$.

I. *Solution by C. Wildhagen, Breda, The Netherlands.*

Let ABC be the given triangle with sides $a \leq b \leq c$. Suppose the internal bisector of $\angle A$ meets BC at the point D . We fold $\triangle ABC$ along AD . The resulting triangle covers an area equal to

$$\text{area}(\Delta ABD) = \frac{c}{b+c}.$$

Similarly, if we fold ΔABC along the internal bisector of $\angle C$, the area covered by the resulting figure is $b/(a+b)$. Thus we have only to show:

$$\min \left\{ \frac{b}{a+b}, \frac{c}{b+c} \right\} < \frac{\sqrt{5}-1}{2},$$

or equivalently

$$\min \left\{ \frac{b}{a}, \frac{c}{b} \right\} < \left(\frac{2}{\sqrt{5}-1} - 1 \right)^{-1} = \frac{\sqrt{5}+1}{2} = \tau. \quad (1)$$

If $b/a < \tau$, then we are done. Therefore assume that $b/a \geq \tau$. It follows that

$$\frac{c}{b} < \frac{a+b}{b} = 1 + \frac{a}{b} \leq 1 + \frac{1}{\tau} = \tau.$$

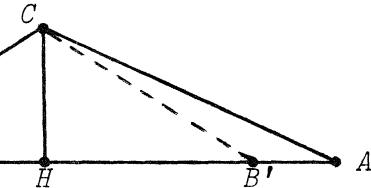
Thus (1) holds, as required.

II. Solution by Jürgen Wolff, Steinheim, Federal Republic of Germany.

Let a, b, c be the sides of the triangle and $a \leq b \leq c$. We distinguish between two cases:

$$(i) \quad b \leq \frac{\sqrt{5}-1}{2}c.$$

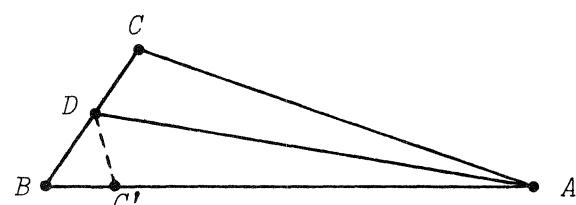
Let CH be the altitude on side c . We fold along CH and have only to consider ΔAHC . Because $\overline{AB} \cdot \overline{HC}/2 = 1$, we have



$$\text{area}(\Delta AHC) = \frac{HC \cdot HA}{2} < \frac{HC \cdot AC}{2} \leq \frac{HC \cdot \sqrt{5}-1}{2} \cdot AB = \frac{\sqrt{5}-1}{2}.$$

$$(ii) \quad b > \frac{\sqrt{5}-1}{2}c.$$

Let AD be the bisector of $\angle A$. We fold along AD , and because C' then lies on AB we have only to consider ΔABD . We have



$$\text{area}(\Delta ABD) = \frac{\text{area}(\Delta ABD)}{\text{area}(\Delta ABC)} = \frac{BD}{BC}.$$

Using the well-known relation $CD/BD = b/c$ we get

$$\frac{BC}{BD} = \frac{BD}{BD} + \frac{CD}{BD} = 1 + \frac{b}{c} > 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2}$$

and so

$$\text{area}(\Delta ABD) = \frac{BD}{BC} < \frac{1}{(1+\sqrt{5})/2} = \frac{\sqrt{5}-1}{2}.$$

[Read on! - Ed.]

III. *Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.*

Take a triangle of area k , and fold it about a segment (called a *crease*) with endpoints on two of the sides. The ratio of the area of the folded triangle to that of the unfolded will be denoted ρ . The problem is then to show that

$$\sup \inf \rho \leq \frac{\sqrt{5} - 1}{2},$$

where the supremum is taken over all triangles and the infimum is taken over all creases of a triangle. We show in fact that

$$\sup \inf \rho = 2 - \sqrt{2};$$

that is, every triangle of area 1 can be folded to cover an area of less than $2 - \sqrt{2}$, and this is best possible!

It will prove convenient to define

$$\varphi = \frac{\rho}{1 - \rho}, \quad \text{i.e. } \rho = \frac{\varphi}{1 + \varphi}.$$

Then ρ is an increasing function of φ , so seeking $\sup \inf \rho$ is equivalent to seeking $\sup \inf \varphi$. Also, we shall use ω to denote the area of the overlap when the triangle is folded. Then it is easy to see that

$$\rho = \frac{k - \omega}{k} \quad \text{and so } \varphi = \frac{k - \omega}{\omega}.$$

Let PQR be a triangle and suppose without loss of generality that the crease has its endpoints on the edges PR and QR . We consider the situation where the direction of the crease is given; let l be the line through R in this direction. Let U and V be on l such that $UP = RP$ and $RQ = VQ$ (and U, V are distinct from R if possible).

At first we limit ourselves to the case that U, R and V are distinct, with R between U and V (Figure 1). Through U and V we draw parallels to PR and QR , respectively, intersecting

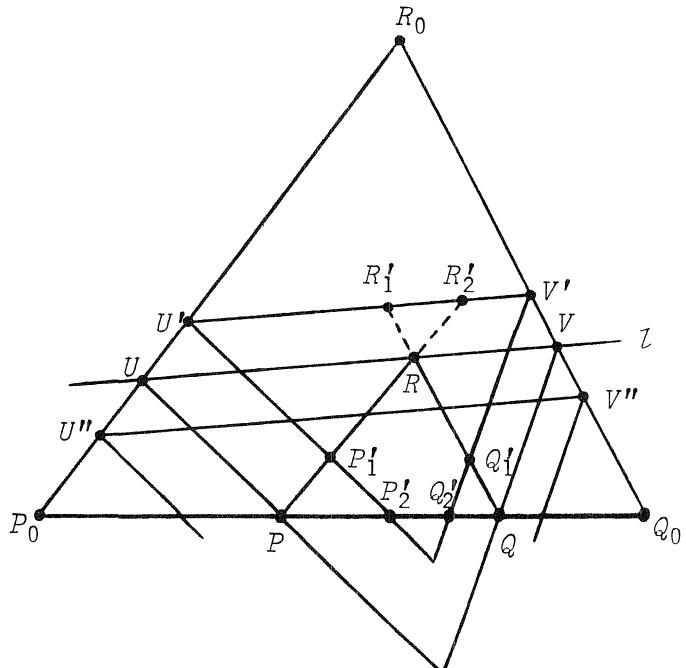


Figure 1

each other in R_0 , and PQ in P_0Q_0 .

Claim: UV is the optimal crease for $\Delta P_0Q_0R_0$ and the given crease direction.

This will enable us to find $\inf \rho$ for $\Delta P_0Q_0R_0$ and the given direction, and since only relative area magnitudes are involved this will also be $\inf \rho$ for ΔPQR and the given direction.

The proof of the claim is simple. Since $\angle R_0UR = \angle URP = \angle RUP$ and $\angle R_0VR = \angle RVQ$, $UPQV$ is precisely the region of overlap for the crease UV . Now consider the competing crease $U'V'$. Using $[F]$ for the area of the figure F , we have

$$[U'P'_2Q'_2V'] = [U'P'_1R'_2] + [RP'_1P'_2Q'_2Q'_1] + [RQ'_1V'R'_2].$$

But by translation, $[U'P'_1R'_2] = [UPR]$, $[RP'_1P'_2Q'_2Q'_1] < [RPQ]$, and $[RQ'_1V'R'_2] < [RQV]$, so

$$[U'P'_2Q'_2V'] < [UPQV].$$

Since the ratio ρ is minimized when the area of overlap is maximized, UV is a better crease than $U'V'$. The possibility $U''V''$ is treated analogously.

Let θ denote the oriented measure of the angle from PQ to the crease URV (positive if counterclockwise, negative otherwise). By adjusting the unit of length we can take the side lengths of ΔPQR to be $\sin P$, $\sin Q$, $\sin R$. Since

$$\angle PUP_0 = 180 - 2(P - \theta),$$

from ΔUP_0P we get

$$P_0P = \frac{\sin Q \sin(2P - 2\theta)}{\sin P},$$

and similarly

$$QQ_0 = \frac{\sin P \sin(2Q + 2\theta)}{\sin Q}.$$

But

$$\begin{aligned} \omega &= [UPQV] = [UPR] + [RPQ] + [RQV] \\ &= [R_0PR] + [RPQ] + [RQR_0] = [R_0PQ], \end{aligned}$$

so

$$\begin{aligned} \varphi &= \frac{k - \omega}{\omega} = \frac{P_0P + Q_0Q}{PQ} \\ &= \frac{\sin Q \sin(2P - 2\theta)}{\sin P \sin R} + \frac{\sin P \sin(2Q + 2\theta)}{\sin Q \sin R}. \end{aligned} \tag{1}$$

Now considering φ as a function of θ ,

$$\varphi'' = -4\varphi < 0,$$

so as θ ranges over any interval, φ will be minimal at one of the interval endpoints. We are interested in the interval of θ values for which the situation is as in Figure

1. If we let θ increase, this situation will eventually change for one of two reasons: either l passes through P_0 , and then $\theta = P/2$; or $l \perp Q_0R_0$, and then $\theta = 90 - Q$. So the right endpoint of our interval is

$$\min\{P/2, 90 - Q\}.$$

Similarly its left endpoint is

$$\max\{-Q/2, -(90 - P)\}.$$

Note that

$$P/2 > 90 - Q \Leftrightarrow P + Q > 180 - Q = P + R \Leftrightarrow Q > R$$

and similarly

$$-Q/2 > -(90 - P) \Leftrightarrow R > P,$$

so we have four cases to examine.

Case (i): $P, Q \leq R$.

The relevant θ -interval is then $[-Q/2, P/2]$. By (1), $\theta = -Q/2$ implies

$$\begin{aligned} \varphi &= \frac{\sin Q \sin(2P + Q)}{\sin P \sin R} + \frac{\sin P}{\sin R} \\ &= \frac{2 \sin Q \sin(2P + Q) + 2 \sin^2 P}{2 \sin P \sin R} \\ &= \frac{\cos 2P - \cos(2P + 2Q) + 1 - \cos 2P}{2 \sin P \sin R} \\ &= \frac{2 \sin^2(P + Q)}{2 \sin P \sin R} = \frac{\sin R}{\sin P}, \end{aligned} \quad (2)$$

and similarly $\theta = P/2$ implies

$$\varphi = \frac{\sin R}{\sin Q}.$$

So for $-Q/2 \leq \theta \leq P/2$,

$$\inf \varphi = \min \left\{ \frac{\sin R}{\sin P}, \frac{\sin R}{\sin Q} \right\}. \quad (3)$$

If we choose a crease such that $\theta > P/2$

then, referring to Figure 2,

$$[USV] = [UVR] < [PV_1R] = [PS_1V_1],$$

so the bisector PV_1 is a better crease than UV . (The case when $\theta > P/2$ and S is outside ΔPQR is left to the reader.) A similar result holds if we choose a crease such that $\theta < -Q/2$. So we conclude that (3) holds in Case (i) for all creases with endpoints on PR and QR .

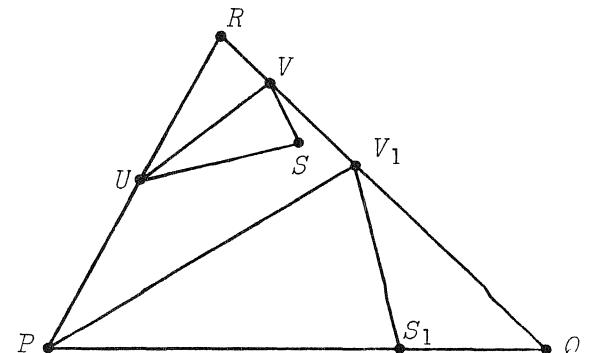


Figure 2

Case (ii): $R \leq P, Q$.

Our interval is then $[-(90 - P), 90 - Q]$. By (1), $\theta = -(90 - P)$ implies

$$\begin{aligned}\varphi &= \frac{\sin P \sin(2Q + 2P - 180)}{\sin Q \sin R} = \frac{\sin P \sin 2R}{\sin Q \sin R} \\ &= \frac{2 \sin P \cos R}{\sin Q},\end{aligned}$$

and similarly $\theta = 90 - Q$ implies

$$\varphi = \frac{2 \sin Q \cos R}{\sin P}. \quad (4)$$

So for $-(90 - P) \leq \theta \leq 90 - Q$,

$$\inf \varphi = \min \left\{ \frac{2 \sin P \cos R}{\sin Q}, \frac{2 \sin Q \cos R}{\sin P} \right\}. \quad (5)$$

To see what happens if θ increases beyond $90 - Q$, we draw Figure 3, which is similar to Figure 1. Here, UV makes angle θ with PQ , where $V = R$ and $UP = PR$, and P_0R_0 is parallel to PR and passes through U . We claim that UV is the optimal crease for ΔP_0QR_0 having the crease direction represented by UV . The proof is very much like the case illustrated by Figure 1, only simpler. The region of overlap for the crease UV is $UPQR$. To show that $U'V'$ is less than optimal, we observe that $U'P'R'V'$ can be translated into UPR and that $P'Q'R \subset PQR$. Similarly, $U''V''$ is not optimal.

Now

$$\omega = [UPQR] = [PQR_0],$$

so that

$$\varphi = \frac{k - \omega}{\omega} = \frac{P_0P}{PQ}.$$

We can again take $PQ = \sin R$ and (from ΔP_0UP)

$$\frac{P_0P}{\sin(180 - (2P - 2\theta))} = \frac{UP}{\sin P} = \frac{PR}{\sin P} = \frac{\sin Q}{\sin P},$$

so

$$\varphi = \frac{\sin Q \sin(2P - 2\theta)}{\sin P \sin R}. \quad (6)$$

As with (1) we see that, for any θ -interval, φ is minimal at one of the endpoints.

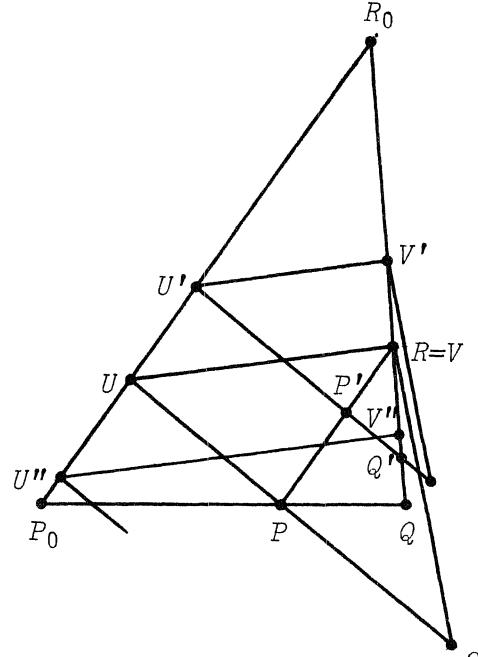


Figure 3

The interval of θ values for which the situation is as in Figure 3 is $[90 - Q, P/2]$. By (6), $\theta = 90 - Q$ implies

$$\varphi = \frac{\sin Q \sin(2P + 2Q - 180)}{\sin P \sin R} = \frac{2 \sin Q \cos R}{\sin P},$$

while $\theta = P/2$ implies

$$\varphi = \frac{\sin Q}{\sin R}.$$

So, for $\theta \in [90 - Q, P/2]$,

$$\inf \varphi = \min \left\{ \frac{2 \sin Q \cos R}{\sin P}, \frac{\sin Q}{\sin R} \right\}. \quad (7)$$

As in the preceding case, if θ increases above $P/2$ the creases we get are not as good as the optimal one for $\theta = P/2$.

Finally, for $\theta \in [-Q/2, -(90 - P)]$ we get, similar to (7),

$$\inf \varphi = \min \left\{ \frac{2 \sin P \cos R}{\sin Q}, \frac{\sin P}{\sin R} \right\}. \quad (8)$$

As before, letting θ drop below $-Q/2$ yields nothing of interest. We can now sum up this case, using (5), (7) and (8): if $R \leq P, Q$ then over all creases with endpoints on PR and QR ,

$$\inf \varphi = \min \left\{ \frac{2 \sin P \cos R}{\sin Q}, \frac{2 \sin Q \cos R}{\sin P}, \frac{\sin P}{\sin R}, \frac{\sin Q}{\sin R} \right\}. \quad (9)$$

Case (iii): $P \leq R \leq Q$.

Here the θ -interval for which Figure 1 is relevant is $[-Q/2, 90 - Q]$. By (1), (2) and (4), if $\theta \in [-Q/2, 90 - Q]$,

$$\inf \varphi = \min \left\{ \frac{2 \sin Q \cos R}{\sin P}, \frac{\sin R}{\sin P} \right\}. \quad (10)$$

As earlier, values of θ below $-Q/2$ or above $P/2$ yield nothing. From (7) and (10), we get: if $P \leq R \leq Q$ then over all creases with endpoints on PR and QR ,

$$\inf \varphi = \min \left\{ \frac{2 \sin Q \cos R}{\sin P}, \frac{\sin R}{\sin P}, \frac{\sin Q}{\sin R} \right\}. \quad (11)$$

Case (iv): $Q \leq R \leq P$.

Reasoning as in (iii) we get: if $Q \leq R \leq P$ then over all creases with endpoints on PR and QR ,

$$\inf \varphi = \min \left\{ \frac{2 \sin P \cos R}{\sin Q}, \frac{\sin R}{\sin Q}, \frac{\sin P}{\sin R} \right\}. \quad (12)$$

We now consider a triangle ABC with edges a, b, c , and with $A \leq B \leq C$. With $P = A, Q = B, R = C$, from (3) we see that for creases connecting AC with BC ,

$$\inf \varphi = \min \left\{ \frac{c}{a}, \frac{c}{b} \right\} = \frac{c}{b}.$$

With $P = B$, $Q = C$, $R = A$, from (9) we see that for creases connecting AB with AC ,

$$\begin{aligned}\inf \varphi &= \min \left\{ \frac{2b \cos A}{c}, \frac{2c \cos A}{b}, \frac{b}{a}, \frac{c}{a} \right\} \\ &= \min \left\{ \frac{2b \cos A}{c}, \frac{b}{a} \right\}.\end{aligned}$$

With $P = A$, $Q = C$, $R = B$, from (11) we see that for creases connecting AB with BC ,

$$\inf \varphi = \min \left\{ \frac{2c \cos B}{a}, \frac{b}{a}, \frac{c}{b} \right\}.$$

All this may be summed up as follows: for ΔABC ,

$$\inf \varphi = \min \left\{ \frac{2b \cos A}{c}, \frac{2c \cos B}{a}, \frac{b}{a}, \frac{c}{b} \right\}.$$

Now

$$\frac{2b \cos A}{c} \cdot \frac{c}{b} = 2 \cos A < 2,$$

so

$$\frac{2b \cos A}{c} < \sqrt{2} \quad \text{or} \quad \frac{c}{b} < \sqrt{2}.$$

Since this is true however ΔABC is chosen,

$$\sup \inf \varphi \leq \sqrt{2}.$$

On the other hand, consider ΔABC where $a = \sqrt{2} - 1$, $b = 1$, and $c = \sqrt{2} - \epsilon$, where ϵ is a very small positive number. Here

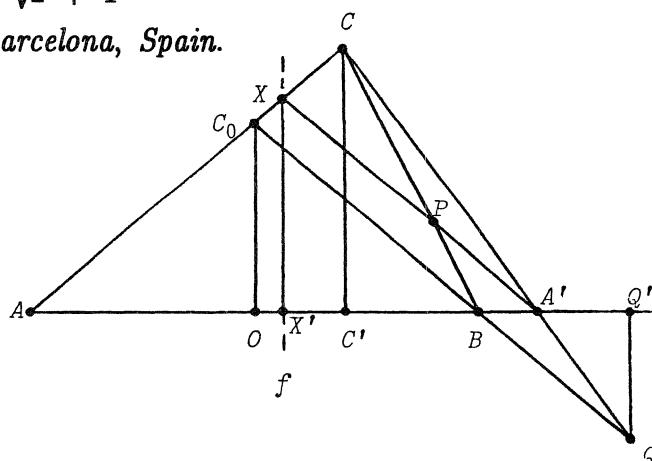
$$\frac{2b \cos A}{c} \approx \sqrt{2}, \quad \frac{2c \cos B}{a} \approx 4 + 2\sqrt{2}, \quad \frac{b}{a} = \sqrt{2} + 1, \quad \frac{c}{b} \approx \sqrt{2},$$

so $\inf \varphi$ (for this triangle and for all creases) is $\approx \sqrt{2}$, the approximation being better the smaller ϵ is. Thus we conclude $\sup \inf \varphi = \sqrt{2}$, hence

$$\sup \inf \rho = \frac{2}{\sqrt{2} + 1} = 2 - \sqrt{2}.$$

IV. Solution by Jordi Dou, Barcelona, Spain.

Let ΔABC be a triangle of area $[ABC] = 1$, and with sides $a \leq b \leq c$. Let C' be the foot of the altitude CC' , O the midpoint of AB , C_0 on AC such that $C_0O \perp AB$. Put $CC' = h$, $C_0O = h_0$, $AC' = kc$ (where $1/2 \leq k < 1$).



Consider the triangle ABC folded along the line f perpendicular to AB , and let $f \cap AC = X$, $f \cap AB = X'$. Vertex A falls on point A' of AB , where $AX' = X'A'$. Let $P = XA' \cap CB$, and let $\beta = [XPC] + [A'PB]$ be the nonoverlapped area, $\sigma = [XCPA'X']$ the total folded area. Then $2\sigma = 1 + \beta$, and σ is minimal when β is minimal. The minimum of β occurs when $XP = PA'$, since then the variations of $[XPC]$ and $[A'PB]$ are equal and of opposite sign. In this case, $[XPC] = [PA'C]$ and so

$$\beta = [BCA'] = \frac{BA' \cdot h}{2}.$$

Putting $BA' = mc$, from $ch/2 = [ABC] = 1$ we have $\beta = m$. Let Q be the intersection of C_0B and CA' , and Q' be the foot of the perpendicular from Q to AB . From $C_0B \parallel XA'$ and $XP = PA'$ we have $C_0B = BQ$, $AO = OB = BQ' = c/2$, and $QQ' = C_0O = h_0$. Hence

$$\begin{aligned} \frac{1}{2k} &= \frac{c/2}{kc} = \frac{AO}{AC'} = \frac{C_0O}{CC'} = \frac{QQ'}{CC'} = \frac{A'Q'}{A'C'} = \frac{BQ' - BA'}{AB - AC' + BA'} \\ &= \frac{c/2 - mc}{c - kc + mc} = \frac{1 - 2m}{2 - 2k + 2m} \end{aligned}$$

and so

$$m = \frac{4k - 2}{4k + 2} = \frac{2k - 1}{2k + 1}.$$

Thus the minimum value σ^* of σ is

$$\sigma^* = \frac{1 + \beta}{2} = \frac{1 + m}{2} = \frac{2k}{2k + 1}. \quad (1)$$

Consider now the same triangle ABC folded along the bisector AD of $\angle A$. Then

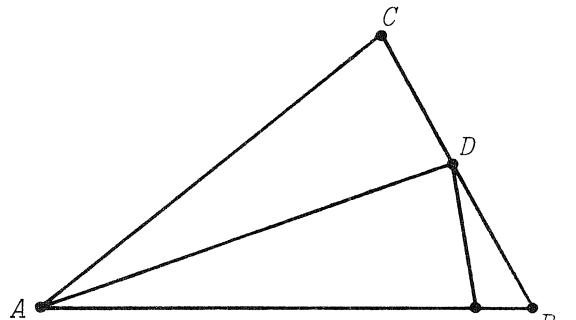
$$\begin{aligned} \sigma &= [ABD] = \frac{c}{c + b} = \frac{c}{c + kc/\cos A} \\ &= \frac{\cos A}{\cos A + k} < \frac{1}{1 + k}. \end{aligned} \quad (2)$$

When $k < \sqrt{2}/2$, we have from (1)

$$\sigma^* = \frac{2k}{2k + 1} < \frac{\sqrt{2}}{\sqrt{2} + 1} = 2 - \sqrt{2},$$

and when $k \geq \sqrt{2}/2$, we have from (2)

$$\sigma < \frac{1}{1 + \sqrt{2}/2} = 2 - \sqrt{2}.$$



Therefore we have demonstrated the proposition, with the upper bound $(\sqrt{5} - 1)/2$ reduced to $2 - \sqrt{2}$.

Moreover, by choosing triangles ABC of area 1 so that $AC' = \frac{\sqrt{2}}{2}AB$ (i.e. $k = \sqrt{2}/2$) and $CC' \rightarrow 0$ (i.e. $\angle A \rightarrow 0$) it follows that this value $2 - \sqrt{2}$ cannot be further reduced, assuming the following assertion: the minimum area covered by any folded triangle is obtained either by folding along one of the angle bisectors (in particular the angle whose edges form a ratio nearest to 1), or by folding along a certain perpendicular to one of the sides (in particular the side whose altitude divides it into a ratio nearest to 1). This assertion seems intuitively plausible, and its demonstration is likely easy but long and detailed.

V. *Editor's comments.*

It was with surprise and delight that the editor received, almost simultaneously, two proofs that the upper bound in this problem could be replaced by $2 - \sqrt{2}$, and that this new bound was best possible (one argument was not quite complete in this respect). The occasion calls for both Bondesen's and Dou's solutions to appear in full; however two nice solutions of the original problem have also been included.

Actually, solution III was the second surprise, courtesy of Bondesen, that the editor found in his mailbox. Earlier Bondesen had sent a proof of the bound $(\sqrt{5} - 1)/2$, much like solution I, but in the form of an entertaining tale of Sherlock Holmes being challenged by Professor Moriarty to solve the problem in question. (Holmes achieves $(\sqrt{5} - 1)/2$, without claiming it to be best possible; it is not revealed whether this sufficed to thwart the evil Professor.) For the editor, this story fulfilled an old fantasy, namely that Holmes (one of his heroes) would just once have taken an interest in mathematics. How would he have done? (As a detective, of course, Holmes has no peers. Léo Sauvé put it best [1976: 130]: there's no police like Holmes!)

Two generalizations suggest themselves.

(i) Is the triangle the "hardest to fold" (in a sense the "least symmetrical") among all convex plane figures? To be precise, is it true that any convex plane figure of area 1 can be folded once so as to cover an area of less than $2 - \sqrt{2}$? On the other hand, T. Bisztriczky, an expert colleague, claims that for any $\epsilon > 0$ there is a *nonconvex* region of the plane of area 1 which, however it is once folded, will always cover an area of at least $1 - \epsilon$. The editor would like to see a proof of this too.

(ii) How about three dimensions? By a *fold* of a tetrahedron we mean the reflection of the portion of the tetrahedron on one side of a cutting plane into the

other side. Into how small a volume can every tetrahedron of volume 1 be folded with one fold?

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD K. GUY, University of Calgary; WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

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1314. [1988: 45] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Solve the determinantal equation

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ x & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = 0$$

for x .

Solution by Robert E. Shafer, Berkeley, California.

We assume $a_i \neq a_j$ if $i \neq j$, since if $a_i = a_j$ then the determinant is zero for all x . Let

$$z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_{n-1} z + \alpha_n$$

be a polynomial for which a_1, a_2, \dots, a_n are roots. Multiply all elements in the first row of the determinant by α_n , in the second row by α_{n-1}, \dots , and in the n th row by α_1 , and add all these rows to the $(n+1)$ st row. We obtain

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \\ x + \sum_{i=1}^n \alpha_i & 0 & 0 & \cdots & 0 \end{vmatrix} = 0,$$

from which we get

$$(-1)^n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} \cdot \left(x + \sum_{i=1}^n \alpha_i \right) = 0 ,$$

or

$$x = - \sum_{i=1}^n \alpha_i .$$

However, since

$$z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n \equiv (z - a_1)(z - a_2) \cdots (z - a_n),$$

we have (putting $z = 1$)

$$x = 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_n).$$

Also solved by SEUNG-JIN BANG, Seoul, Korea; ERIC HOLLEMAN, student, Memorial University of Newfoundland; KEE-WAI LAU, Hong Kong; DANIEL B. SHAPIRO, Ohio State University; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer. Four other readers submitted unsimplified solutions.

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1315. [1988: 45] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with medians AD , BE , CF and median point G . We denote $\Delta AGF = \Delta_1$, $\Delta BGF = \Delta_2$, $\Delta BGD = \Delta_3$, $\Delta CGD = \Delta_4$, $\Delta CGE = \Delta_5$, $\Delta AGE = \Delta_6$, and let R_i and r_i denote the circumradius and inradius of Δ_i ($i = 1, 2, \dots, 6$). Prove that

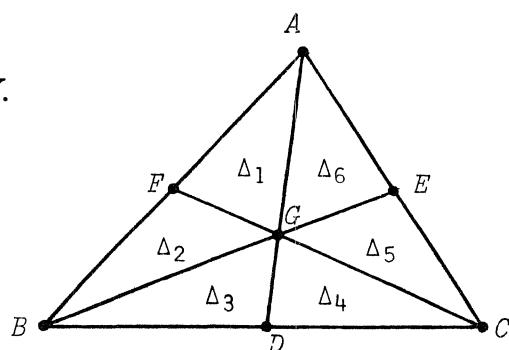
$$(i) \quad R_1 R_3 R_5 = R_2 R_4 R_6 ;$$

$$(ii) \quad \frac{15}{2r} < \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6} < \frac{9}{r} ,$$

where r is the inradius of ΔABC .

Solution by Jack Garfunkel, Flushing, N.Y.

Denote the sides of triangle ABC by a , b , c and the medians to these sides by m_a , m_b , m_c . Without loss of generality let the area of ΔABC equal 1, so that each of the Δ_i has area $1/6$. We have



$$AG = 2m_a/3, BG = 2m_b/3, CG = 2m_c/3, \\ GD = m_a/3, GE = m_b/3, GF = m_c/3.$$

Using $abc/4R$ as the formula for the area of a triangle, we get

$$R_1 = \frac{(2m_a/3)(m_c/3)(c/2)}{2/3} = \frac{cm_am_c}{6}.$$

Similarly,

$$R_2 = \frac{cm_bm_c}{6}, R_3 = \frac{am_am_b}{6}, R_4 = \frac{am_am_c}{6}, \\ R_5 = \frac{bm_bm_c}{6}, R_6 = \frac{bm_am_b}{6}.$$

So

$$R_1R_3R_5 = \frac{abcm_a^2m_b^2m_c^2}{6^3} = R_2R_4R_6.$$

For part (b), we use rs as the formula for the area of a triangle, where r is the inradius and s is the semiperimeter. We get

$$\frac{1}{r_1} = 6\left(\frac{m_a}{3} + \frac{m_c}{6} + \frac{c}{4}\right) = \frac{4m_a + 2m_c + 3c}{2}$$

and similarly

$$\frac{1}{r_2} = \frac{4m_b + 2m_c + 3c}{2}, \frac{1}{r_3} = \frac{4m_b + 2m_a + 3a}{2}, \\ \frac{1}{r_4} = \frac{4m_c + 2m_a + 3a}{2}, \frac{1}{r_5} = \frac{4m_c + 2m_b + 3b}{2}, \\ \frac{1}{r_6} = \frac{4m_a + 2m_b + 3b}{2}.$$

Clearly

$$\frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6} \\ = 3(m_a + m_b + m_c) + \frac{3}{2}(a + b + c).$$

We have to show that

$$\frac{15}{2r} < 3(m_a + m_b + m_c) + \frac{3}{2}(a + b + c) < \frac{9}{r}.$$

Since $rs = 1$, $r = 2/(a + b + c)$ and the above reduces to

$$\frac{3}{4}(a + b + c) < m_a + m_b + m_c < a + b + c$$

which is true; see item 8.1 of O. Bottema et al, *Geometric Inequalities*.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; BOB PRIELIPP, University of Wisconsin-Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student,

University of Waterloo; and the proposer.

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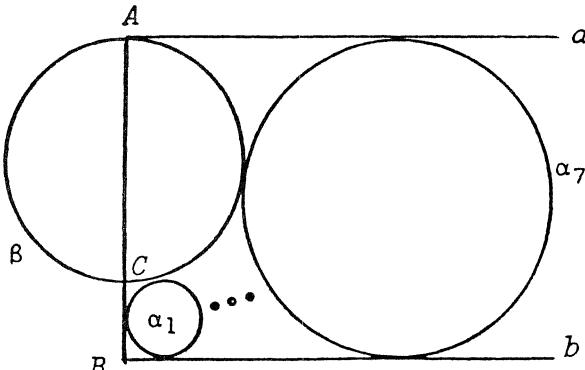
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1316. [1988: 45] *Proposed by Jordi Dou, Barcelona, Spain.*

AB is a segment of unit length

and lines a , b are perpendicular to AB at A and B respectively. C is a point on the segment AB and β is the circle of diameter AC . Suppose that a chain of exactly seven circles $\alpha_1, \dots, \alpha_7$ can be inscribed around β and between a and b as shown (i.e., α_1 is tangent to AB , β , and b ; α_2 to α_1 , β , and b ; ... ; α_7 to α_6 , β , a , and b). Find a simple expression for the distance BC .



Solution by Richard I. Hess, Rancho Palos Verdes, California.

In the figure at right let the inner circle b' have radius R and the outer circle β' have radius $R + 2r$, so fixed that there are $2n + 1$ small circles exactly fitting in the annulus. Letting

$$\theta_n = \frac{\pi}{2n+1},$$

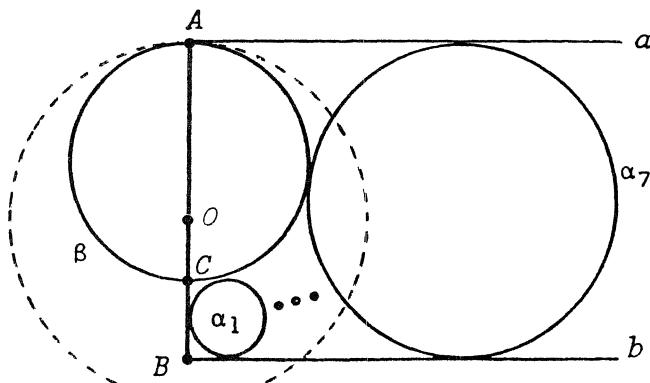
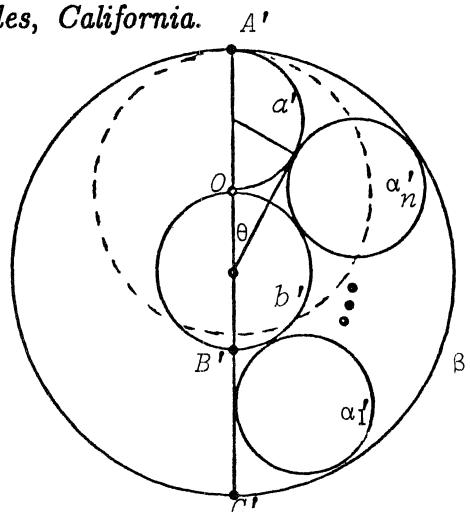
we have

$$\sin \theta_n = \frac{r}{R+r}$$

so

$$R = r(\csc \theta_n - 1).$$

Perform an inversion about the dotted circle (center O , radius $2r$) to produce the second figure at right. (For $n = 7$ this is the figure in the problem.) Circles and straight lines transform into circles and straight lines. Letting O be the origin, we have coordinates



$$B' = (0, 2r(\csc \theta_n - 1)), \text{ so } B = \left(0, \frac{2r}{\csc \theta_n - 1}\right),$$

$$C' = (0, 2r \csc \theta_n), \text{ so } C = (0, 2r \sin \theta_n),$$

$$A' = (0, -2r), \text{ so } A = (0, -2r).$$

Since $AB = 1$,

$$r = \frac{1}{2} \left(1 + \frac{1}{\csc \theta_n - 1} \right)^{-1} = \frac{1 - \sin \theta_n}{2}.$$

Thus

$$\begin{aligned} BC &= 2r \left(\frac{1}{\csc \theta_n - 1} - \sin \theta_n \right) \\ &= (1 - \sin \theta_n) \left(\frac{1}{\csc \theta_n - 1} - \sin \theta_n \right) \\ &= \sin^2 \theta_n = \sin^2 \left(\frac{\pi}{2n+1} \right). \end{aligned}$$

For $n = 7$ we get

$$BC = \sin^2 \left(\frac{\pi}{15} \right).$$

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; HIDETOSI FUKAGAWA, Aichi, Japan; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer. The solutions of Springer and the proposer were also by inversion.

Fukagawa mentions that a similar problem appeared on an 1834 Japanese sangaku and was also recorded by the Japanese mathematician Yoshida in 1842. The general result appears (in Japanese) in Shiko Iwata, Encyclopedia of Geometry (1988).

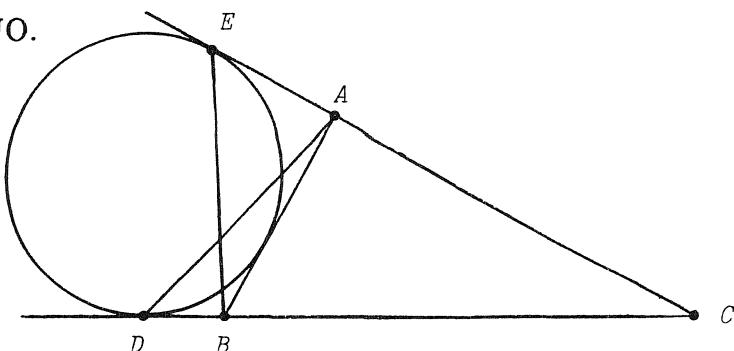
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- 1317.** [1988: 45] Proposed by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen.

Crux 1133 [1987: 225] suggests the following problem. In a triangle ABC the excircle touching side AB touches lines BC and AC at points D and E respectively. If $AD = BE$, must the triangle be isosceles?

- I. Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The answer is NO.



Indeed, considering triangles BAE and ADB , and using $AE = s - b$ and $BD = s - a$ (where s is the semiperimeter), we get

$$BE^2 = c^2 + (s - b)^2 - 2c(s - b)\cos(180 - A),$$

$$AD^2 = c^2 + (s - a)^2 - 2c(s - a)\cos(180 - B),$$

and so, since $AD = BE$,

$$b^2 - 2bs + 2c(s - b)\cos A = a^2 - 2as + 2c(s - a)\cos B. \quad (1)$$

Now

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc} - 1 = \frac{2s(s - a)}{bc} - 1$$

and similarly

$$\cos B = \frac{2s(s - b)}{ac} - 1.$$

Hence (1) becomes

$$b^2 - 2bs + \frac{4s(s - a)(s - b)}{b} + 2bc = a^2 - 2as + \frac{4s(s - a)(s - b)}{a} + 2ac,$$

i.e.

$$(b - a) \left(b + a - 2s - \frac{4s(s - a)(s - b)}{ab} + 2c \right) = 0,$$

i.e.

$$(b - a) \left(c - \frac{4s(s - a)(s - b)}{ab} \right) = 0. \quad (2)$$

If $a \neq b$ then (2) becomes

$$abc = 4s(s - a)(s - b). \quad (3)$$

Put $a = 1$, $b = 2$ in (3). We then get

$$l(c) = 4c = (c + 3)(c + 1)(c - 1) = (c + 3)(c^2 - 1) = r(c). \quad (4)$$

Since $l(1) > r(1)$, $l(3) < r(3)$, and $l(2) \neq r(2)$, there exists $c \in (1, 3)$, $c \neq 2$, satisfying (4).

Remark. Letting R be the circumradius and F the area of ΔABC , equation (3) in fact can also be written

$$4RF = \frac{4F^2}{s - c},$$

i.e.

$$R = \frac{F}{s - c} = r_c, \quad (5)$$

where r_c is the exradius to side c .

II. *Solution by L.J. Hut, Groningen, The Netherlands.*

Since

$$CD = CE = s,$$

we have from triangles ACD and BCE that

$$AD^2 = b^2 + s^2 - 2bs \cos C$$

and

$$BE^2 = a^2 + s^2 - 2as \cos C.$$

If $AD = BE$ then

$$a^2 - b^2 = 2s(a - b)\cos C.$$

If also $a \neq b$ then

$$a + b = 2s \cos C. \quad (6)$$

Putting $2s = a + b + c$, this can be written

$$c \cos C = (a + b)(1 - \cos C) = 2(a + b)\sin^2(C/2). \quad (7)$$

From

$$\frac{a + b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{2 \sin \frac{C}{2} \cos \frac{C}{2}} = \frac{\cos \left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}},$$

(7) becomes

$$\begin{aligned} \cos C &= 2 \sin \left(\frac{C}{2}\right) \cos \left(\frac{A-B}{2}\right) \\ &= 2 \cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right) \\ &= \cos A + \cos B. \end{aligned} \quad (8)$$

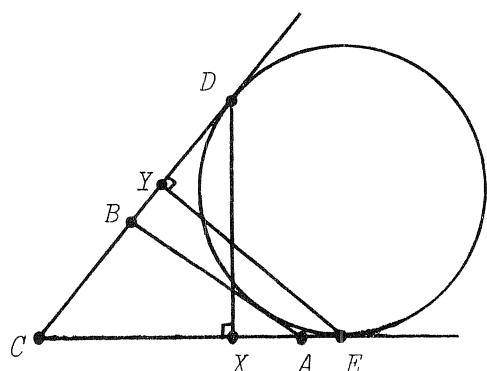
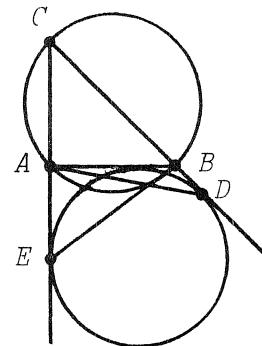
Equation (8) is possible. Take for example $A = 90^\circ$, $B = C = 45^\circ$ (see the above figure).

III. *Solution by Colin Springer, student, University of Waterloo.*

Let perpendiculars from D, E meet CE, CD at X, Y , respectively. Then $|DX| = |EY|$, so $|AD| = |BE|$ if and only if $|AX| = |BY|$. Since $|CX| = |CY|$, if $\triangle ABC$ is non-isosceles A and B must be on opposite sides of X and Y , respectively.

Now if $\angle C$ were equal to 60° XY wouldn't touch the excircle, since

$|XY| = |DY| = |XE|$ whereas if XY touched the excircle $|XY| = 2|XE| = 2|DY|$. Thus, choose a value for $\angle C$ less than 60° such that XY doesn't intersect the



excircle. Then $|XC| > |XE|$ and $|YC| > |YD|$. For B close to Y , A is between X and E , so $|XA| > |YB|$. For B close to C , A is close to E , so $|XA| < |YB|$. Therefore for some point B between Y and C , $|XA| = |YB|$. At this point $|AD| = |BE|$, but ΔABC is clearly non-isosceles.

IV. *Solution by T. Seimiya, Kawasaki, Japan.* (Translated by H. Fukagawa.)

We show that

Theorem. Let the excircle of ΔABC touching side AB touch BC at D and AC at E , and suppose $AD = BE$.

(i) If $C \geq 60^\circ$ or $C \leq 2\alpha \approx 42.94^\circ$ where

$$\alpha = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right),$$

then $CA = CB$;

(ii) if $2\alpha < C < 60^\circ$, then CA does not necessarily equal CB .

Proof. (i) Otherwise, we may suppose $CA > CB$. Choose B' on CB extended so that $AC = CB'$. Then $\Delta CEB' \cong \Delta CDA$, so

$$B'E = AD = BE \quad (1)$$

which means that $\Delta BEB'$ is isosceles.

Let K be the excenter with respect to side AB , and let CK meet the circumcircle of ΔABC in point M . Then it is known [e.g. Theorem 292, page 185 of R.A. Johnson, *Advanced Euclidean Geometry*] that

$$AM = MB = MK,$$

and from $\Delta CAM \cong \Delta CB'M$ we have that $AM = MB'$. Thus

$$MB = MB'. \quad (2)$$

From (1) and (2), the line EM passes through the midpoint N of BB' , and $EM \perp CD$. Therefore $EN \parallel KD$. From $\angle EAM = \angle CBM > 90^\circ$ follows $EM > AM = MK$. From

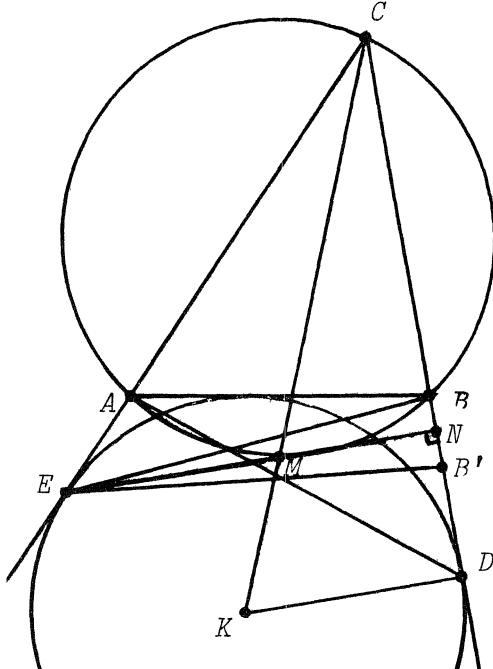
$$\angle EKM = \angle MKD = \angle CMN = \angle EMK$$

we have $EK = EM > MK$, which means

$$C = 90^\circ - \angle CEN = 90^\circ - \angle CEM = \angle KEM < 60^\circ.$$

Letting $\theta = C/2$, from $EM = EK$ and $\angle KEM = C$ follows

$$EM = \frac{KM}{2 \sin \theta}.$$



From

$$\frac{MN}{EM} = \frac{CN}{CE} = \cos C = \cos 2\theta$$

follows

$$MN = EM \cos 2\theta = \frac{\cos 2\theta}{2 \sin \theta} \cdot KM.$$

Since $MN < MB = KM$,

$$\begin{aligned} \cos 2\theta &< 2 \sin \theta, \\ 2 \sin^2 \theta + 2 \sin \theta - 1 &> 0, \end{aligned}$$

and thus

$$\sin \theta > \frac{\sqrt{3} - 1}{2}.$$

Hence $C = 2\theta > 2\alpha$.

(ii) Assuming $2\alpha < C < 60^\circ$, we construct a triangle ABC such that $AD = BE$ but $AC \neq BC$. Let $\angle XCY$ be the given angle C , and let K be any point on its bisector. D and E are the feet of the perpendiculars from K to CY and CX respectively. N is the foot of the perpendicular from E to CY , and EN meets CK in M .

Draw a circle μ with center M and radius MK ; we show that μ meets the segment CD (and so also CE) in two points. For this it is sufficient to show that $MN < MK < MD$. First,

$$\begin{aligned} \angle KEM &= 90^\circ - \angle CEN = C, \\ \angle EMK &= \angle CMN = 90^\circ - C/2, \end{aligned}$$

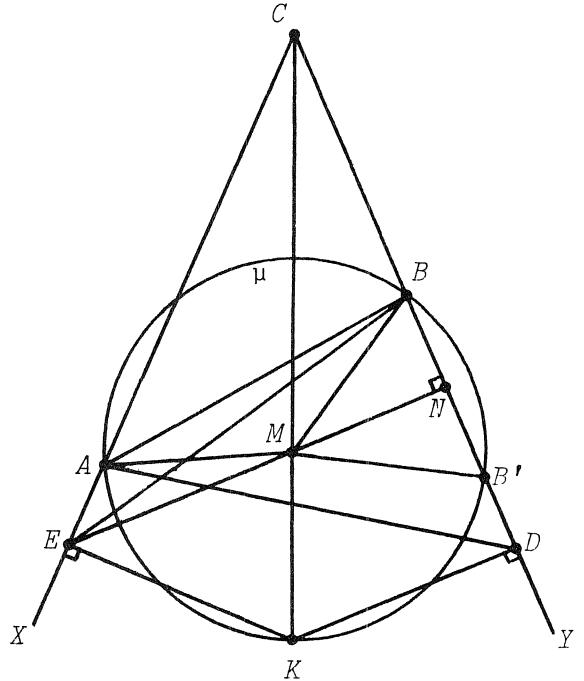
and thus $\angle EMK = \angle EKM$, so $MD = EM > MK$ since $\angle KEM = C < 60^\circ$. Also, since $C > 2\alpha$ we have as above that $\cos 2\theta < 2 \sin \theta$ where $\theta = C/2$. From

$$\frac{MN}{EM} = \frac{CN}{CE} = \cos C = \cos 2\theta$$

we obtain

$$MN = EM \cos 2\theta < 2EM \sin \theta = 2EM \cdot \sin(\angle KEM/2) = MK.$$

Thus we may choose points A on $\mu \cap CX$ and B, B' on $\mu \cap CY$ so that $CA = CB' > CB$. Since $\angle CAM = \angle CB'M = \angle MBB'$, the four points C, A, M, B are concyclic, so M is on the circumcircle of $\triangle ABC$. Also $AM = MB$, that is, M is



the midpoint of the arc AB of the circumcircle. Since $AM = MB = MK$, K must be the excenter of ΔABC relative to side AB . Finally, $AD = EB' = EB$.

[*Editor's note:* One can also derive part (ii) of this theorem from Springer's solution (III).]

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; and the proposer.

The proposer's solution was very much like Springer's. The proposer also answered (in the negative) the analogous problem for exspheres of tetrahedra.

The example of an isosceles right triangle, given in solution II, was also pointed out by Engelhaupt and Hess.

Equations (5) (by Janous) and (8) (by Hut) are especially nice conditions for the existence of a counterexample. Are they equivalent?

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J.T. GROENMAN

With regret, the editor informs the readers of *Crux Mathematicorum* of the passing, in early March, of one of *Crux*'s long-time regular contributors, Dr. J.T. Groenman of Arnhem, The Netherlands. Dr. Groenman's name has been appearing on these pages since 1980, and in virtually every issue the past few years. He was one of the most frequent correspondents with this office. He solved problems in a likeably ingenuous manner, writing in a rather old-fashioned inky script that, although a challenge to eyes spoiled by the word processor, gave his solutions an unusual warmth. The editor shall miss his letters. He was also a prolific problem proposer; in fact there remain in the files of *Crux* well over a year's worth of his proposals still to be used. Dr. Groenman will thus be entertaining *Crux* readers for some time to come.

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