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Editor: Léo Sauvé, Architecture Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Ollen G. Boshko.

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CONTENTS

AREA CHARACTERIZATIONS OF CURVES: I

ALFRED AEPPLI

In Crux 374 [1979: 140], parabolas are characterized by an area condition. Analogously, various area conditions for plane curves can be considered, formulated with respect to a distinguished point O. This leads to the exponential spiral and, of course, to characterizations of circles with center O.

We use polar coordinates (r,ϕ) in the plane. Let $r=r(\phi)$ be a curve Γ of class C^1, $r>0$ (see Figure 1), and let

$$A_{\psi}^{\psi+\alpha}=\tfrac{1}{2}\int_{\psi}^{\psi+\alpha} r^2(\phi)\,d\phi$$

(the surface area of the shaded region OPQ in Figure 2). Then we have

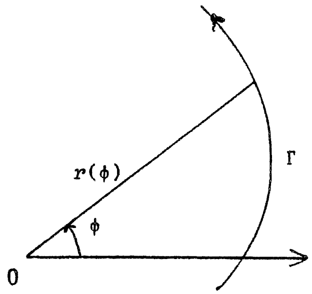

Figure 1

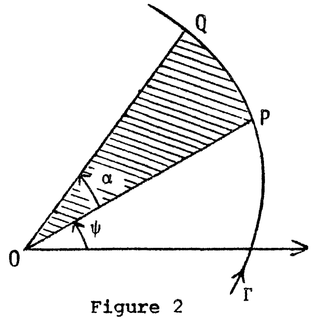

Figure 2

PROPOSITION 1. *If* $A_{\psi}^{\psi+\alpha}=A_{\psi+\alpha}^{\psi+2\alpha}$ *for all* ψ,α, *then* Γ *is a circle with center* O, *i.e.,* $r=$ *constant.*

Proof. Differentiation of

$$\int_{\psi}^{\psi+\alpha} r^2\,d\phi=\int_{\psi+\alpha}^{\psi+2\alpha} r^2\,d\phi$$

with respect to α gives

$$r^2(\psi+\alpha)=2r^2(\psi+2\alpha)-r^2(\psi+\alpha);$$

hence $r^2(\psi+\alpha)=r^2(\psi+2\alpha)$ for all ψ,α and $r^2=$ constant. □

Observe that only continuity of $r=r(\phi)$ is needed in this proof, that is, Γ has to be of Class C^0; and that $A_0^{\alpha}=A_{\alpha}^{2\alpha}$ (for all α) forces Γ to be a circle, that is, the hypothesis in Proposition 1 for some fixed ψ is sufficient.

PROPOSITION 2. *If* $A_{\psi}^{\psi+\alpha}=f(\alpha)A_{\psi+\alpha}^{\psi+2\alpha}$ *for all* ψ,α *then* Γ *is an exponential spiral:*

$$r(\phi)=ke^{c\phi}\quad \textit{for some constants } k>0,\ c\in R$$

and

$$f(\alpha)=e^{-2c\alpha}.$$

Notice that Proposition 1 is contained in Proposition 2 as a special case ($c = 0$). Another consequence of Proposition 2 is of interest: *if* $A_{\psi}^{\psi+\alpha} = \kappa A_{\psi+\alpha}^{\psi+2\alpha}$ *for a constant* κ *and for all* ψ,α, *then* $\kappa = 1$ *and* Γ *is a circle.*

Proof of Proposition 2. Here $f(\alpha) = A_{\psi}^{\psi+\alpha}/A_{\psi+\alpha}^{\psi+2\alpha}$ is independent of ψ; for example,

$$f(\alpha) = A_0^{\alpha}/A_{\alpha}^{2\alpha} = \int_0^{\alpha} r^2\, d\phi \Big/ \int_{\alpha}^{2\alpha} r^2\, d\phi \,.$$

Let $\Phi = \Phi(\phi)$ be a primitive of r^2 (so that $\Phi' = r^2 > 0$). Then the equation

$$A_{\psi}^{\psi+\alpha}/A_{\psi+\alpha}^{\psi+2\alpha} = A_0^{\alpha}/A_{\alpha}^{2\alpha} \qquad \text{(for all } \psi,\alpha)$$

is equivalent to

$$\frac{\Phi(\psi+\alpha) - \Phi(\psi)}{\Phi(\psi+2\alpha) - \Phi(\psi+\alpha)} = \frac{\Phi(\alpha) - \Phi(0)}{\Phi(2\alpha) - \Phi(\alpha)} \qquad \text{(for all } \psi,\alpha),$$

that is,

$$\{\Phi(\alpha) - \Phi(2\alpha)\}\Phi(\psi) + \{\Phi(2\alpha) - \Phi(0)\}\Phi(\psi+\alpha) + \{\Phi(0) - \Phi(\alpha)\}\Phi(\psi+2\alpha) = 0. \qquad (1)$$

This says that, for fixed $\alpha \neq 0$, the three functions $\Phi(\psi)$, $\Phi(\psi+\alpha)$, $\Phi(\psi+2\alpha)$ are linearly dependent over R and that, since $\Phi' > 0$ everywhere, the coefficients in (1) are all nonzero. Thus

$$\left\{\begin{array}{l}\text{each of the functions } \Phi(\psi),\ \Phi(\psi+\alpha),\ \Phi(\psi+2\alpha)\\ \text{is a linear combination of the other two}\\ \text{(over } R\text{, for fixed } \alpha).\end{array}\right. \qquad (1')$$

The Wronskian $W(\Phi(\psi), \Phi(\psi+\alpha), \Phi(\psi+2\alpha))$ is identically zero as a function of ψ (again for fixed α); thus there exists a nontrivial triple of functions $a_0(\psi)$, $a_1(\psi)$, $a_2(\psi)$ such that

$$\left\{\begin{array}{llll} a_0(\psi)\Phi(\psi) & + a_1(\psi)\Phi'(\psi) & + a_2(\psi)\Phi''(\psi) & = 0\\ a_0(\psi)\Phi(\psi+\alpha) & + a_1(\psi)\Phi'(\psi+\alpha) & + a_2(\psi)\Phi''(\psi+\alpha) & = 0\\ a_0(\psi)\Phi(\psi+2\alpha) & + a_1(\psi)\Phi'(\psi+2\alpha) & + a_2(\psi)\Phi''(\psi+2\alpha) & = 0\end{array}\right. \qquad (2)$$

and it follows from (1') that

every pair of equations in (2) implies the third. (2')

If now $\psi = \psi_0$ is kept fixed, the described process for

$$\alpha,\ 2\alpha,\ 4\alpha,\ \ldots,\ 2^m\alpha,\ \ldots,\ 2^{-1}\alpha,\ 2^{-2}\alpha,\ \ldots,\ 2^{-m}\alpha,\ \ldots$$

yields, with the help of (2'),

$$a_0\Phi(\phi_n) + a_1\Phi'(\phi_n) + a_2\Phi''(\phi_n) = 0 \qquad (3)$$

for constants a_0, a_1, a_2 and $\phi_n = \psi_0 + 2^n\alpha$, $n \in Z$ (Z the integers). Take any ϕ_n as a new ψ and any $\phi_{n'} - \phi_n$ as a new α ($n, n' \in Z$, $n \neq n'$), and repeat the process. Do this again and again. The result is (3) for an everywhere dense set of ϕ_n's, which implies

$$a_0\Phi(\phi) + a_1\Phi'(\phi) + a_2\Phi''(\phi) = 0 \tag{4}$$

for all $\phi \in R$.

The solution must be of the form $\Phi(\phi) = c_1 \exp(c_2\phi) + c_3$ for constants c_1, c_2, c_3, except in the case $\Phi'' \equiv 0$, which means $\Phi'(\phi) = r^2 = \text{constant}$ (circle). Hence $r(\phi) = ke^{c\phi}$ for $k > 0$ and $c \neq 0$ or $r = \text{constant}$. Now $f(\alpha) = e^{-2c\alpha}$ follows from $A_{\psi}^{\psi+\alpha} = f(\alpha)\, A_{\psi+\alpha}^{\psi+2\alpha}$. □

A few remarks about Proposition 2 and its proof may now be useful:

(a) The class C^1 assumption for Γ (continuous differentiability) is essential. It is used for the establishment of differential equation (4), and it prevents the mixing of solutions of (4).

(b) As mentioned, $\Phi'' \equiv 0$ is the case of the circle. It appears also as a "limit" of spirals if $c \to 0$.

(c) A slight generalization of Proposition 2 is immediate:

If $A_{\psi}^{\psi+\alpha} = f(\alpha)\, A_{\psi+\alpha}^{\psi+v\alpha}$ for all ψ, α and a fixed $v > 1$, then Γ is an exponential spiral $r(\phi) = ke^{c\phi}$ ($k>0$, $c \in R$) and

$$f(\alpha) = e^{-2c\alpha} \cdot \frac{e^{2c\alpha} - 1}{e^{2c(v-1)\alpha} - 1} .$$

Moreover, if $A_{\psi}^{\psi+\alpha} = \kappa A_{\psi+\alpha}^{\psi+v\alpha}$ for a constant κ, for all ψ, α and a fixed $v > 1$, then $\kappa = 1/(v-1)$ and Γ is a circle.

Consider again a C^1 curve $r = r(\phi) > 0$, i.e.,

$\vec{r} = \vec{r}(\phi) = (x(\phi), y(\phi)) = (r(\phi)\cos\phi,\ r(\phi)\sin\phi)$.

Let τ_1, τ_2 be the tangents at any two points $P_1, P_2 \in \Gamma$, and let $Q = \tau_1 \cap \tau_2$, as shown in Figure 3. Finally, let A_{P_1Q} and A_{QP_2} be the surface areas of triangles P_1OQ and QOP_2, respectively. Then we have

PROPOSITION 3. If Γ is a C^1 curve $\vec{r} = \vec{r}(\phi)$ with $\vec{r}' \neq \vec{0}$ everywhere, and if $A_{P_1Q} = A_{QP_2}$ for all $P_1, P_2 \in \Gamma$ (whenever the construction is possible), then Γ is given, up to a rotation about 0, by

$$r^2 = r^2(\phi) = r_0^2/(1 + cr_0^2 \sin^2\phi) \tag{5}$$

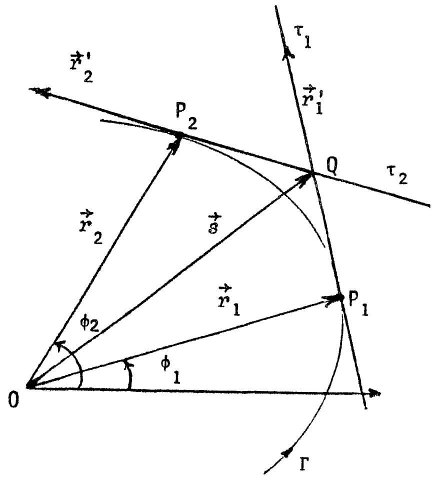

Figure 3

for some constants $r_0 > 0$ *and* c.

Before proving this proposition, we discuss (5) briefly.

(a) If $c = 0$, then Γ is a circle with center 0.

(b) If $c \neq 0$ and $1 + cr_0^2 > 0$, then Γ is a simple closed curve with 0 in its interior. If $c > 0$, then

$$r(0) = r(\pi) \geq r(\phi) \geq r(\pi/2) = r(3\pi/2);$$

and if $c < 0$, then

$$r(0) = r(\pi) \leq r(\phi) \leq r(\pi/2) = r(3\pi/2).$$

(c) If $c \neq 0$ and $1 + cr_0^2 < 0$, then Γ is not compact and $r = r(\phi)$ is defined for ϕ with $\sin^2\phi < 1/|c|r_0^2$. If $r(\phi)$ is defined for

$$-\pi/2 \leq \phi_- < \phi < \phi_+ \leq \pi/2,$$

where $\sin\phi_- = -1/\sqrt{|c|}r_0$ and $\sin\phi_+ = 1/\sqrt{|c|}r_0$, then $r(\phi)\uparrow\infty$ for $\phi\downarrow\phi_-$ and $r(\phi)\uparrow\infty$ for $\phi\uparrow\phi_+$.

Corollary. If, under the hypotheses of Proposition 3, Γ *is assumed to be a closed curve such that* $r(\alpha) \geq r(\phi) \geq r(\beta)$ *for all* ϕ. *for fixed* α, β *with* $|\alpha - \beta| \neq \pi/2, 3\pi/2$ *(mod* 2π*), then* Γ *is a circle with center* 0.

Proof of Proposition 3. As shown in Figure 3, let

$$\overrightarrow{OP}_1 = \vec{r}_1 = \vec{r}(\phi_1) \quad \text{and} \quad \overrightarrow{OP}_2 = \vec{r}_2 = \vec{r}(\phi_2);$$

then

$$\overrightarrow{OQ} = \vec{s} = \vec{s}(\phi_1, \phi_2) = \vec{r}_1 + t_1\vec{r}_1{}' = \vec{r}_2 + t_2\vec{r}_2{}'$$

for

$$t_1(\vec{r}_1{}' \times \vec{r}_2{}') = (\vec{r}_2 - \vec{r}_1) \times \vec{r}_2{}' \tag{6'}$$

and

$$t_2(\vec{r}_1{}' \times \vec{r}_2{}') = (\vec{r}_2 - \vec{r}_1) \times \vec{r}_1{}'. \tag{6''}$$

The condition $A_{P_1Q} = A_{QP_2}$ means that $\vec{r}_1 \times \vec{s} = \vec{s} \times \vec{r}_2$, that is,

$$t_1(\vec{r}_1 \times \vec{r}_1{}') = t_2(\vec{r}_2{}' \times \vec{r}_2). \tag{7}$$

Now

$$\vec{r} = \vec{r}(\phi) = r(\phi)(\cos\phi, \sin\phi)$$

and

$$\vec{r}\,' = \vec{r}\,'(\phi) = r'(\phi)(\cos\phi, \sin\phi) + r(\phi)(-\sin\phi, \cos\phi);$$

hence

$$\vec{r}\times\vec{r}\,' = r^2\vec{k}, \tag{8}$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$ are the basic vectors (1,0,0), (0,1,0), (0,0,1) in R^3.

Note that (7) and (8) imply

$$t_1r_1^2 + t_2r_2^2 = 0. \tag{9}$$

Furthermore,

$$\vec{r}_1\times\vec{r}_2\,' = r_1(r_2\cos(\phi_2-\phi_1) + r_2'\sin(\phi_2-\phi_1))\vec{k}$$

and

$$\vec{r}_2\times\vec{r}_1\,' = r_2(r_1\cos(\phi_2-\phi_1) - r_1'\sin(\phi_2-\phi_1))\vec{k};$$

hence, with (6),

$$t_1(\vec{r}_1\,'\times\vec{r}_2\,') = \{r_2^2 - r_1(r_2\cos(\phi_2-\phi_1) + r_2'\sin(\phi_2-\phi_1))\}\vec{k},$$

$$t_2(\vec{r}_1\,'\times\vec{r}_2\,') = \{-r_1^2 + r_2(r_1\cos(\phi_2-\phi_1) - r_1'\sin(\phi_2-\phi_1))\}\vec{k},$$

and, with (9),

$$r_1^3(r_2\cos(\phi_2-\phi_1) + r_2'\sin(\phi_2-\phi_1)) - r_2^3(r_1\cos(\phi_2-\phi_1) - r_1'\sin(\phi_2-\phi_1)) = 0. \tag{10}$$

For $\phi_1 = 0$, $r_1 = r_0$ and $\phi_2 = \phi$, $r_2 = r$, we get from (10)

$$(r_0^3\sin\phi)r' = (r_0\cos\phi - r_0'\sin\phi)r^3 - (r_0^3\cos\phi)r,$$

that is, the following differential equation for $1/r^2$:

$$-\tfrac{1}{2}r_0^3\sin\phi\,(1/r^2)' = (r_0\cos\phi - r_0'\sin\phi) - (r_0^3\cos\phi)(1/r^2). \tag{11}$$

The general solution of (11) is

$$1/r^2 = (1/r_0^2) + c\sin^2\phi - (r_0'/r_0^3)\sin 2\phi \tag{12}$$

for any $c\in R$. Note that (12) solves (10) as well.

Since $r'(\psi) = 0$ for $\tan 2\psi = r_0'/cr_0^3$, we can achieve $r_0' = 0$ by a rotation, and (5) is established. □

Two final remarks about Proposition 3:

(a) The Corollary follows directly from (10). For a closed Γ, let

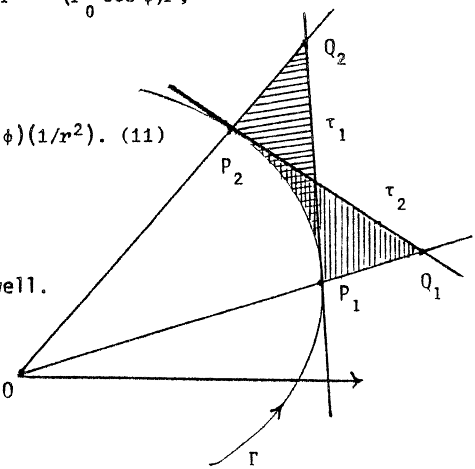

Figure 4

$$r_1 = \max_\phi r(\phi) \geq r(\phi) \geq r_2 = \min_\phi r(\phi).$$

Then $r_1' = r_2' = 0$, and if $\cos(\phi_2 - \phi_1) \neq 0$ then (10) implies $r_1 = r_2$, that is, $r \equiv$ constant.

(b) The condition that triangles OP_1Q_2 and OQ_1P_2 in Figure 4 have equal areas, and hence that curvilinear triangles $P_1P_2Q_1$ and $P_1P_2Q_2$ (shaded in the figure) have equal areas, leads to the same curves (5).

School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church Street S.E., Minneapolis, Minnesota 55455.

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ANOTHER "PROOF" THAT 0 = 1

JAN VAN DE CRAATS

In the Bulgarian magazine *Mathematica* (1980 (4), p. 44), Problem 42 reads: *Solve the equation*

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 7.$$

The published solution runs as follows: Squaring both sides, we get $x + 7 = 49$, so $x = 42$.

A suspicious reader, however, might be tempted to try the same trick once again: with

$$\sqrt{x + \sqrt{x + \ldots}} = 49 - x,$$

squaring both sides yields $x + 7 = (49-x)^2$, from which $x = 42$ or 57. Why should 42 be a better solution than 57? Moreover, arguing in the same vein,

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 0$$

without any doubt yields $x = 0$, so

$$\sqrt{0 + \sqrt{0 + \sqrt{0 + \ldots}}} = 0,$$

but

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}} = 1$$

leads to $x + 1 = 1$ and $x = 0$. Thus

$$0 = \sqrt{0 + \sqrt{0 + \sqrt{0 + \ldots}}} = 1.$$

To clarify this embarrassing situation, it is necessary to state clearly what is meant by the expression

$$\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}\,, \qquad x \geq 0. \tag{1}$$

It seems appropriate to consider the sequence

$$s_1(x) = \sqrt{x}, \qquad s_{n+1}(x) = \sqrt{x + s_n(x)}, \qquad x \geq 0, \quad n = 1,2,3,\ldots .$$

If the sequence $\{s_n(x)\}$ converges to a limit $s(x)$ as $n \to \infty$, it makes sense to define the expression (1) as $s(x)$.

Clearly, for $x = 0$ we have $s_n(x) = 0$ for all n, so $s(0) = 0$. For $x > 0$, it is well known that $\lim_{n\to\infty} s_n(x)$ exists and is equal to the positive root of the equation $t = \sqrt{x+t}$, that is,

$$s(x) = \tfrac{1}{2}(1 + \sqrt{1 + 4x}), \qquad x > 0.$$

It is noteworthy that

$$0 = s(0) \neq \lim_{x \downarrow 0} s(x) = 1$$

and that the sequence of functions $\{s_n(x)\}$ converges nonuniformly to its limit $s(x)$ over any interval that includes $x = 0$. Since, for $x > 0$, $s(x)$ is continuous, greater than 1, monotonically increasing, and unbounded, the equation $s(x) = c$ has a unique positive solution for any $c > 1$. This solution is $x = c^2 - c$. In particular, $c = 7$ yields $x = 42$, which justifies the "official" solution of the Bulgarian problem. However, the equation $s(x) = 1$ has no solution, and this invalidates our "proof" that $0 = 1$.

Department of Mathematics, University of Leiden, Wassenaarseweg 80, Postbus 9512, 2300 R A Leiden, The Netherlands.

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NOTES ON NOTATION: I

LEROY F. MEYERS

All of us have noticed errors in notation made by students in mathematics classes, for example the use of

$$(a)b + c \qquad \text{instead of} \qquad a(b + c).$$

(Professional mathematicians also make notational errors, but these tend to be less glaring. Sometimes the errors are deliberate, such as the "abus de notation" made famous by the brethren Bourbaki.)

It is commonly believed that mathematical notation, at least for those parts of mathematics having a long tradition, is fixed. (Alternate notations for the same concepts may be used, but that is a different matter.) This is not true, and in these notes I shall discuss several situations where there is no commonly accepted agreement on notation.

I often tell my students to avoid using the slant (slash, solidus, shilling sign) to indicate division, and to use the horizontal line instead. (Mathematical printers prefer the slant.) The reason for my preference is that I often see students write $1/2x$ or ${}^{1}/_{2}x$ when they mean $\frac{1}{2}x$ (better $\frac{x}{2}$) and then read it, for use in the next step, as $\frac{1}{2x}$. Of course, errors can be made with the horizontal line, such as writing $\frac{1}{2\,x}$ for $\frac{1}{2}x$ an then reading it as if it were $\frac{1}{2x}$ or $\frac{1}{2^x}$, but such errors are infrequent. The use of $(1/2)x$ or $1/(2x)$, as appropriate, would specify what is meant, but this is often awkward-looking; even worse is the printers' preference: $2^{-1}x$ or $(2x)^{-1}$.

How do *you* interpret a/bc? Is it always $a/(bc)$ or always $(a/b)c$? If sometimes one and sometimes the other, does it make a difference whether b (or another letter) is replaced by a specific numeral, so that $1/2x$ means $(1/2)x$ but $1/kx$ means $1/(kx)$? What if an explicit multiplication sign is used, as in $a/b \cdot c$ or $a/b \times c$? What about $a \div bc$, $a \div b \cdot c$, $a \div b \times c$? What about $a/b/c$ and other, more complicated, expressions?

There is a standard way of interpreting expressions involving only + and -, such as $a - b + c$ and $a - b - c$; association is always to the left: $(a - b) + c$, $(a - b) - c$. Since multiplication and division are in many ways analogous to addition and subtraction, the convention might be made that multiplication and division are also to associate to the left, so that $a \div b \times c$ and $a \div b \div c$ are always to mean $(a \div b) \times c$ and $(a \div b) \div c$, respectively. In fact, this is the interpretation for hand-held calculators which use the "algebraic" system of button-pushing and for many computer languages. In these cases, however, multiplication is indicated explicitly.

What do *you* suggest?

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210.

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THE OLYMPIAD CORNER: 22

MURRAY S. KLAMKIN

Since there was no International Mathematical Olympiad in 1980 (the first break in this annual competition after 20 consecutive years), several regional European Olympiads were held. I give below the problems set at three of these regional competitions. (The problems in the first of the three were translated from the German by Andy Liu, University of Alberta.) I shall also give, later on in this column, some problems from the 1978 Romanian Mathematical Olympiad. For all of these problems, I solicit solutions from all readers of this journal (particularly, but not exclusively, from secondary school students, who should give the name of their school and their grade). I shall, from time to time, publish selected elegant solutions from those I receive.

I. *Oesterreichisch-Polnischer Mathematik Weltbewerb.*

1st day July 3, 1980 $4\frac{1}{2}$ hours

1. Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

2. Let $\{x_n\}$ be a sequence of natural numbers such that
(a) $1 = x_1 < x_2 < x_3 < \ldots$; (b) $x_{2n+1} \le 2n$ for all n.
Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

3. Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540°.

2nd day July 4, 1980 $4\frac{1}{2}$ hours

4. Prove that $\Sigma\{1/(i_1 i_2 \ldots i_k)\} = n$, where the summation is taken over all non-empty subsets $\{i_1, i_2, \ldots, i_k\}$ of $\{1,2,\ldots,n\}$.

5. Let $A_1A_2A_3$ be a triangle and, for $1 \le i \le 3$, let B_i be an interior point of the edge opposite A_i. Prove that the perpendicular bisectors of A_iB_i for $1 \le i \le 3$ are not concurrent.

6. Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \le 1$ for all positive integers k and m, prove that, for all positive integers p and q,

$$|(a_p/p) - (a_q/q)| < (1/p) + (1/q).$$

3rd day July 5, 1980 4 hours

Team Competition

7. Find the greatest natural number n such that there exist natural numbers $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_{n-1}$ with $a_1 < a_2 < \ldots < a_{n-1}$ satisfying the following system of equations:

$$\begin{cases} x_1 x_2 \ldots x_n = 1980, \\ x_i + (1980/x_i) = a_i, \qquad i = 1,2,\ldots,n-1. \end{cases}$$

8. Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

9. Let AB be a diameter of a circle; let t_1 and t_2 be the tangents at A and B, respectively; let C be any point other than A on t_1; and let D_1D_2, E_1E_2 be arcs on the circle determined by two lines through C. Prove that the lines AD_1 and AD_2 determine a segment on t_2 equal in length to that of the segment on t_2 determined

by AE_1 and AE_2.

II. *Competition in Mersch, Luxembourg (Belgium, Great Britain, Luxembourg, The Netherlands, and Yugoslavia).*

1st day July 10, 1980 4 hours

1. Find all functions $f:Q \to Q$ (where Q is the set of all rational numbers) satisfying the following two conditions:

(a) $f(1) = 2$;

(b) $f(xy) = f(x)f(y) - f(x+y) + 1$ for all $x, y \in Q$.

2. Let A,B,C be three collinear points with B between A and C. On the same side of AC are drawn the three semicircles on AB, BC, and AC as diameters. The common tangent at B to the first two semicircles meets the third at E. Let U and V be the points of contact of the other common tangent of the first two semicircles. Calculate the ratio

$$\frac{\text{area of triangle EUV}}{\text{area of triangle EAC}}$$

as a function of $r_1 = \frac{1}{2}AB$ and $r_2 = \frac{1}{2}BC$.

3. Let p be a prime number and n a positive integer. Prove that the following statements (a) and (b) are equivalent:

(a) None of the binomial coefficients $\binom{n}{k}$ for $k = 0,1,\ldots,n$ is divisible by p.

(b) n can be represented in the form $n = p^s q - 1$, where s and q are integers, $s \geq 0$, $0 < q < p$.

2nd day July 11, 1980 4 hours

4. Two circles touch (externally or internally) at the point P. A line touching one of the circles at A cuts the other circle at B and C. Prove that the line PA is one of the bisectors of angle BPC.

5. Ten gamblers started playing each with the same amount of money. Each in turn threw five dice. At each stage the gambler who had thrown paid to each of his nine opponents $1/n$ times the amount which that opponent owned at that moment, where n is the total shown by the dice. They threw and paid one after the other. At the tenth throw the dice showed a total of 12, and after payment it turned out that every gambler had the same sum as he had at the beginning. Determine if possible the totals shown by the dice at each of the other throws.

6. Determine all pairs (x,y) of integers satisfying the equation

$$x^3 + x^2y + xy^2 + y^3 = 8(x^2 + xy + y^2 + 1).$$

III. *Competition in Mariehamn, Finland (Finland, Great Britain, Hungary, and Sweden).*

1st day July 1, 1980 4 hours

1. In triangle ABC the perpendicular bisectors of AB and AC cut BC, produced if necessary, at X and Y, respectively.

(a) Prove that a sufficient condition for $BC = XY$ is $\tan B \tan C = 3$.

(b) Prove that this condition is not necessary and find necessary and sufficient conditions for $BC = XY$.

2. The sequence $a_0, a_1, \ldots, a_n$ is defined by

$$a_0 = \tfrac{1}{2}, \qquad a_{k+1} = a_k + (1/n)a_k^2, \qquad k = 0,1,\ldots,n-1.$$

Prove that $1 - (1/n) < a_n < 1$.

3. Consider the equation

$$x^n + 1 = y^{n+1},$$

where $n \geq 2$ is a natural number. Prove that no positive integer solutions (x,y) exist for which x and $n+1$ have no common factor.

2nd day July 2, 1980 4 hours

4. A convex polygon with $2n$ sides is inscribed in a circle and $n-1$ of its n pairs of opposite sides are parallel. For which values of n is it true that the remaining pair of opposite sides must be parallel?

(In the polygon $A_1A_2\ldots A_{2n}$, A_1A_2 and $A_{n+1}A_{n+2}$, for example, are a pair of opposite sides.)

5. A horizontal line (i.e., parallel to the x-axis) is called *triangular* if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in four distinct points A, B, C, and D (in that order from left to right) in such a way that AB, AC, and AD could be the lengths of the sides of a triangle. Prove that either all or none of the horizontal lines which cut the curve in four distinct points are triangular.

6. Find, with proof, the digit immediately to the left and the digit immediately to the right of the decimal point in the decimal expansion of the number

$$(\sqrt{2} + \sqrt{3})^{1980}.$$

*

In 1978, Romania placed first in the unofficial team standings in the International Mathematical Olympiad. Consequently, it should be of interest to see the kinds of problems used by Romanians in their own competitions. Through the courtesy of the Romanian Ministry of Education, I have received a booklet containing, in both Romanian and English, the problems given at the final round of the 1978 national Romanian Mathematical Olympiad (for their 9th, 10th, 11th, and 12th classes), which took place at Galati in April 1978, as well as the problems given in four selection tests held in April and June 1978 to select the Romanian team members for the I.M.O. There are 43 problems in all. I give for now only the 16 problems from their national Olympiads; the 27 selection test problems will appear subsequently. Solutions, as usual, are solicited from all readers.

Final Round - 9th class

1. Determine the range of the function f defined for all real x by

$$f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x+1}.$$

2. Let ABCD be an arbitrary convex quadrilateral and M a point on the diagonal AC. The parallel to AB [resp. DC] through M intersects BC in P [AD in Q].

(a) Prove that

$$MP^2 + MQ^2 \geq \frac{AB^2 \cdot DC^2}{AB^2 + DC^2}.$$

When does equality hold?

(b) Determine the locus of the midpoint of PQ as M ranges over the segment AC.

3. Let $m \geq 1$ and $n \geq 1$ be integers such that $\sqrt{7} - (m/n) > 0$. Prove that $\sqrt{7} - (m/n) > 1/(mn)$.

4. Find all real triples (x,y,z) such that

$$\sqrt{x} + \sqrt{y-1} + \sqrt{z-2} = \tfrac{1}{2}(x+y+z).$$

Final Round - 10th class

1. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of real numbers and $\phi\colon A \to A$ a bijective map. Suppose that $a_1 < a_2 < \ldots < a_n$ and that

$$a_1 + \phi(a_1) < a_2 + \phi(a_2) < \ldots < a_n + \phi(a_n).$$

Show that ϕ coincides with the identical map of the set A. Is this result still true if A is replaced by the set of all integers?

2. Find an integer $k \geq 1$ such that the expression

$$\sin kx \cdot \sin^k x + \cos kx \cdot \cos^k x - \cos^k 2x$$

does not depend on x.

3. Given is a convex polyhedron with $n \geq 5$ (blank) faces and exactly three edges emanating from each vertex. Two persons play the following game: each player in turn signs his name on one of the (remaining) blank faces. To win, a player must sign his name on three faces with a common vertex. Show that there is a winning strategy for the first player.

4. There are n participants in a chess tournament. Each person plays exactly one game with each of the $n-1$ others, and each person plays at most one game per day. What is the minimum number of days required to finish the tournament?

Final Round - 11th class

1. Let $f\colon R \to R$ be a real function defined by $f(x) = 0$ if x is irrational and $f(p/q) = 1/q^3$ if p and q are integers with $q > 0$ and p/q irreducible. Show that f has a derivative at each irrational point $x_0 = \sqrt{k}$, where k is a natural number that is not a perfect square.

2. An infinite sequence $c_1, c_2, c_3, \ldots$ of natural numbers is defined as follows: for each $n \geq 1$, c_n is equal to the smallest integral power of 4 that is greater than or equal to n. Let

$$b_n = c_1 + c_2 + \ldots + c_n - \frac{1}{5}.$$

(a) Express b_n/n^2 in terms of c_n/n.

(b) Prove that, for any real d in the interval [4/5, 5/4], there is a sequence of natural numbers $n_k \to \infty$ such that $\lim_{k\to\infty} b_{n_k}/(n_k)^2 = d$.

3. Two persons, A and B, take turns in assigning real values to the empty cells of a 3×3 array until, after nine plays, a complete 3×3 matrix results. Show that, whether or not he plays first, A can choose his entries in such a way that the resulting matrix is singular.

4. Let $f\colon R \to R$ be the function defined by

$$f(x) = x|x-a_1| + |x-a_2| + \ldots + |x-a_n|,$$

where $a_1, a_2, \ldots, a_n$ are fixed real numbers. Find a condition for f to be everywhere differentiable.

Final Round - 12th class

1. Let $f\colon R \to R$ be a continuous function and let

$$g(x) = f(x)\int_0^x f(t)\,dt.$$

Prove that, if g is decreasing, then $f \equiv 0$ on R.

2. Let P and Q be two polynomials (neither identically zero) with complex coefficients. Show that P and Q have the same roots (with the same multiplicities) if and only if the function f: $C \to R$ defined by $f(z) = |P(z)| - |Q(z)|$ has a constant sign for all $z \in C$ if it is not identically zero.

3. Let F be the set of all continuous functions f: $[0,1] \to [0,2]$ such that

$$\int_0^1 f(x)\,dx = 1.$$

(a) As f ranges over F, find all the possible values of

$$\int_0^1 x\,f(x)\,dx.$$

(b) Show that, if both f and f^2 belong to F, then f is a constant function.

4. Let $P(z)$ be a polynomial with complex coefficients. Prove that

$$(P \circ P \circ \ldots \circ P)(z) - z$$

is divisible by $P(z) - z$, where the composition $\circ$ is taken any finite number of times.

*

2nd NOTICE TO CANADIAN STUDENTS

It now appears likely that there will be financial support for travel to send an 8-student Canadian team to participate in the International Mathematical Olympiad to be held in the U.S.A. on July 8-20, 1981. The team will be selected on the basis of performance in the 1981 Canadian Mathematical Olympiad. To be eligible for the I.M.O., a student must not have reached the age of 20 by July 8, 1981 and also must not be regularly enrolled in a college or university. It is also likely that there will be financial support for the team to participate in honoring ceremonies and in a training session to be held at the University of Saskatchewan from June 24 to July 7, 1981. If enough financial support is available, four more students who have done well in the Canadian Mathematical Olympiad and who will not graduate until at least the following year will also be invited to the training session.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

605. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Can every fourth-order pandiagonal magic square (magic also along the broken diagonals) be written in the form

$F+y$	$G+x$	$G-x$	$F-y$
$G-w$	$F+z$	$F-z$	$G+w$
$F+x$	$G+y$	$G-y$	$F-x$
$G+z$	$F-w$	$F+w$	$G-z$

where $w=x+y+z$?

606.* *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in Euclidean space R^n and let $\sigma_n' = A_0'A_1'\ldots A_n'$ be an n-simplex similar to and inscribed in σ_n, and labeled in such a way that

$$A_i' \in \sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n, \qquad i=0,1,\ldots,n.$$

Prove that the ratio of similarity

$$\lambda \equiv A_i'A_j'/A_iA_j \geq 1/n.$$

[If no proof of the general case is forthcoming, the editor hopes to receive a proof at least for the special case $n=2$.]

607.* *Proposed by Gali Salvatore, Perkins, Québec.*

Let S be the set of all positive integers that are, in base ten, palindromes with fewer than 10 digits. S contains an abundance of multiples of 3, of 9, and of 27, but it contains only *one* multiple of 81. It is the largest number in the set:

$$999999999 = 81 \cdot 12345679.$$

What is there in the structure of palindromes that explains this strange behaviour?

608. *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

ABC is a triangle with sides of lengths a, b, c and semiperimeter s. Prove that

$$\cos^4 \tfrac{1}{2}A + \cos^4 \tfrac{1}{2}B + \cos^4 \tfrac{1}{2}C \le \tfrac{1}{2}\frac{s^3}{abc},$$

with equality if and only if the triangle is equilateral.

609.* *Proposed by Jack Garfunkel, Flushing, N.Y.*

$A_1B_1C_1D_1$ is a convex quadrilateral inscribed in a circle and M_1, N_1, P_1, Q_1 are the midpoints of sides B_1C_1, C_1D_1, D_1A_1, A_1B_1, respectively. The chords A_1M_1, B_1N_1, C_1P_1, D_1Q_1 meet the circle again in A_2, B_2, C_2, D_2, respectively. Quadrilateral $A_3B_3C_3D_3$ is formed from $A_2B_2C_2D_2$ as the latter was formed from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely. Prove that quadrilateral $A_nB_nC_nD_n$ "tends to" a square as $n \to \infty$.

What happens if $A_1B_1C_1D_1$ is not convex?

610. *Proposed by Hayo Ahlburg, Benidorm, Alicante, Spain.*

If m is odd and $(m,5) = 1$, prove that $m^{21} - m \equiv 0 \pmod{13200}$.

611. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

This decimal cryptarithmic addition is dedicated to the memory of the late Sidney Penner, formerly Problem Editor of the *New York State Mathematics Teachers' Journal*:

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 SIDNEY
 EDITOR
 SIDNEY
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 PENNER.
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Editor Sidney Penner was unique, and so is EDITOR SIDNEY PENNER.

* * *

MAMA-THEMATICS

"My son Ernst has such great ideals," bragged Frau Kummer.

ALAN WAYNE

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

492, [1979: 291; 1980: 291] *Proposed by Dan Pedoe, University of Minnesota.*

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}$AB are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{2}$AB?

II. *Comment by the proposer.*

Students were asked to find the third vertex C of an equilateral triangle ABC, given the segment AB, a rusty compass of unspecified radius r, and a ruler. This is simple. Kevin Panzer, however, sent in the 5-circle diagram with no comment, no proof, no writing of any sort to accompany his figure. I had never encountered this construction, and felt it was worth experimenting to see whether it was correct, with, of course, $r \geq \frac{1}{2}$AB.

I was able to confirm the validity of the construction analytically, and sent it to Dr. E.T. Whiteside, the editor of Newton's Mathematical Papers, with whom I was corresponding at the time. He wrote that he had never come across it, and sent me a fairly long Euclidean proof, using angles. I then saw a very short proof, which is as follows (see figures in [1980: 293]):

The circle with centre X passes through A, Z, and C. Triangle AZX is equilateral. The angle subtended by AZ at X is twice the angle subtended at C, and therefore angle ACZ is 30°. Since Z is on the perpendicular bisector of AB, this proves that triangle ABC is equilateral.

The fact that students may discover, even inadvertently, new constructions in geometry seemed to stimulate my student audiences on my recent visit to India. I have suggested the 5-circle diagram as a logo for the Institute of Technology at the University of Minnesota.

An additional reference to the literature should be: Arthur Hallerberg, "The Geometry of the Fixed Compass," *Mathematics Teacher,* 52 (April 1959) 230-244.

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493, [1979: 291; 1980: 294] *Proposed by R.C. Lyness, Suffolk, England.*

(a) A, B, C are the angles of a triangle. Prove that there are positive x,y,z, each less than $\frac{1}{2}$, simultaneously satisfying

$$y^2\cot\tfrac{1}{2}B + 2yz + z^2\cot\tfrac{1}{2}C = \sin A,$$

$$z^2\cot\tfrac{1}{2}C + 2zx + x^2\cot\tfrac{1}{2}A = \sin B,$$

$$x^2\cot\tfrac{1}{2}A + 2xy + y^2\cot\tfrac{1}{2}B = \sin C\ .$$

(b) In fact, $\frac{1}{2}$ may be replaced by a smaller $k > 0.4$. What is the least value of k?

II. *Comment by M.S. Klamkin, University of Alberta.*

Here we tie up the first "loose end" mentioned in the editor's comment on this problem [1980: 296], that is, we prove that, when A and r_1 are fixed, $a = BC$ is least when $B = C$.

We start with the result of Lob and Richmond referred to in the editor's comment, modified only by the explicit appearance of r as a factor (Lob and Richmond had assumed a unit inradius):

$$r_1 = \frac{r(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{2(1+\tan\frac{1}{4}A)}.$$

Since

$$r = \frac{\Delta}{s} = \frac{2R\ \sin A\ \sin B\ \sin C}{\sin A + \sin B + \sin C}$$

and $a = 2R\sin A$, where A is fixed, we equivalently have to minimize R in

$$r_1 = \frac{R\sin A\sin B\sin C(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{(\sin A+\sin B+\sin C)(1+\tan\frac{1}{4}A)}.$$

But since r_1 also is fixed, we need in fact only minimize

$$F \equiv \frac{\sin A+\sin B+\sin C}{\sin B\sin C(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}. \tag{1}$$

Using $\sin A+\sin B+\sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$ and the familiar expansions for $\sin 2x$ and $\sin(x+y)$, it is easy to show that (1) is equivalent to

$$\cos\tfrac{1}{2}A/F = 8\sin\tfrac{1}{4}B\sin\tfrac{1}{4}C\sin\tfrac{1}{4}(\pi+B)\sin\tfrac{1}{4}(\pi+C),$$

the right side of which must be maximized for minimum F. This is accomplished by noting that, since $\log\sin x$ is concave for $0 < x < \pi$, we have $\sin u\sin v \le \sin^2\frac{1}{2}(u+v)$ with equality if and only if $u = v$. Hence each of

$$\sin\tfrac{1}{4}B\sin\tfrac{1}{4}C \qquad \text{and} \qquad \sin\tfrac{1}{4}(\pi+B)\ \sin\tfrac{1}{4}(\pi+C)$$

takes on its maximum value when $B = C$.

* * *

511. [1980: 43] *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the following alphametic, which was inspired by the editor's comment following the solution of Crux 251 [1978: 43]:

$$\begin{array}{r} MY \\ FAIR \\ \underline{LADY} \\ ELIZA \end{array}.$$

You will know there is only one solution when she raises her little pinkie: in a LADY the digits are always rising.

I. *Solution by Charles W. Trigg, San Diego, California.*

Immediately, $E = 1$ and $F = 9$. The first three columns (from the right) give

$$2Y + R = A + 10k, \tag{1}$$

$$M + I + D + k = Z + 10m, \tag{2}$$

$$2A + m = I + 10. \tag{3}$$

If $A = 4$, then $m = 2$, $I = 0$, and (2) cannot hold. As a result of the nature of the LADY's digits, we must have $A = 5$ or 6 and the possible partial solutions

$$(A, D, Y) = (5, 6, 7), (5, 6, 8), (5, 7, 8), \text{ and } (6, 7, 8).$$

The first three of these, it is seen from (1), lead to the duplications $R = 1$, $R = 9$, and $R = 9$, respectively. So $(A, D, Y) = (6, 7, 8)$, $R = 0$ and $k = 1$ from (1), $m = 1$ from (2), $I = 3$ from (3), $M = 4$ and $Z = 5$ from (2), leaving $L = 2$. Thus the unique solution is

$$\begin{array}{r} 48 \\ 9630 \\ \underline{2678} \\ 12356 \end{array},$$

so it appears that the "pinkie" raised by (or from) the orderly ELIZA was her 4-finger.

II. *Comment by J.A. McCallum, Medicine Hat, Alberta.*

I had sent this very same alphametic in May 1975 to J.A.H. Hunter, who published it a few months later in one of his syndicated columns. But we used a different stipulation to make the answer unique: that ELIZA was prime since Eliza Dolittle is certainly a prime role. Where would Julie Andrews ever have been without it?

Also solved by J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; G.D. KAYE, Ottawa, Ontario; EDGAR LACHANCE, Ottawa, Ontario; POONAM and RAMA KRISHNA MANDAN, Bombay, India (independently); J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; CHRIS NYBERG, East York, Ontario; MARK PRYSANT, student, Princeton University; HYMAN ROSEN, student, The Cooper Union, New York, N.Y.; DONVAL R. SIMPSON, Fairbanks, Alaska; ROBERT TRANQUILLE, Collège de

Maisonneuve, Montréal, Québec; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

McCallum's ELIZA was a prim unbending prime, and ours is deliciously composite and frangible. So, the name may be the same but the same LADY she ain't.

* * *

512. [1980: 43] *Proposed by Chris Nyberg, East York, Ontario.*

Let m and n be repdigits consisting of the *same* nonzero digit and consider the continued fraction equations

$$\sqrt{m} = n + \cfrac{2n}{n + \cfrac{n}{n + \cfrac{n}{n + \dots}}}, \qquad \sqrt{77} = 7 + \cfrac{7+7}{7 + \cfrac{7}{7 + \cfrac{7}{7 + \dots}}}.$$

The striking second equation, which is true in base ten, shows that in this base the first equation has at least the solution $m = 77$, $n = 7$. Show that, in every base $B > 3$, the first equation has a unique solution.

Solution by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.

The positive integer pair (m,n) is a solution of the given equation if and only if there is a positive number A such that

$$\sqrt{m} = n + \frac{2n}{A} \qquad (1)$$

and

$$A = n + \frac{n}{A}. \qquad (2)$$

Furthermore, (2) shows that we must have

$$A = \frac{n + \sqrt{n^2 + 4n}}{2},$$

and it is easy to verify that, with this value of A, (1) is equivalent to

$$m = n(n + 4). \qquad (3)$$

Thus the solutions of the given equation consist of all the positive integer pairs (m,n) which satisfy (3).

If $B > 3$, the pair (m,n) defined by

$$m = (B-3)B + (B-3), \qquad n = B - 3$$

(suggested by the solution $(m,n) = (77, 7)$ in base ten) satisfies (3) and hence is a solution. We show that it is the only solution in which m and n are both repdigits in base B with the same repeated nonzero digit.

Suppose (m,n) is a solution of (3) where, in base $B > 3$, m is a p-digit repdigit and n is a q-digit repdigit, the repeated digit in each case being d; then we have

$$m = \frac{d(B^p - 1)}{B - 1} \quad \text{and} \quad n = \frac{d(B^q - 1)}{B - 1},$$

where $p > q \geq 1$ and $1 \leq d \leq B - 1$. Now

$$\frac{m}{n} = n + 4 = \frac{B^p - 1}{B^q - 1}$$

is an integer, and it is well known that this implies that $q|p$, say $p = rq$, where $r \geq 2$. Thus

$$n = \frac{B^{rq} - 1}{B^q - 1} - 4$$

$$= B^{(r-1)q} + B^{(r-2)q} + \ldots + B^q - 3,$$

and this is clearly not a repdigit in base B unless $q = 1$ and $r = 2$. The unique solution is therefore given by

$$n = B - 3 \quad \text{and} \quad m = n(n+4) = (B-3)B + (B-3).$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; G.D. KAYE, Ottawa, Ontario; MARK PRYSANT, student, Princeton University; HYMAN ROSEN, student, The Cooper Union, New York, N.Y.; DONALD P. SKOW, Griffin & Brand, Inc., McAllen, Texas; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment

The weakest part in some of the other solutions was in proving that m contained two digits and n one digit in base B. Some underachievers simply assumed that fact and then bravely went on to solve the resulting emasculated problem.

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513. [1980: 43] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Define the *density* $d(A)$ of a set A of natural numbers in the usual way,

$$d(A) = \lim_{n\to\infty} \frac{|\{m \in A : m \leq n\}|}{n},$$

provided this limit exists. (Here $|S|$ = number of elements in set S.) Also, associate to each set A of natural numbers the reciprocal series $\Sigma_{a\in A}(1/a)$.

(a) Can a set of density 0 have a divergent reciprocal series?

(b) Can a set of positive density have a convergent reciprocal series?

Solution by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.

Both parts of this problem are known. They appear without proof as an exercise in Niven and Zuckerman [1].

(a) The answer is YES. One obvious example is to take A to be the set of all primes. The fact that $d(A) = 0$ follows immediately from the Prime Number Theorem, and it is well known that the series of prime reciprocals diverges. We give also a less obvious example, which depends upon much weaker results in analysis.

Let $A = \{a_n\}_{n=1}^{\infty}$, where a_n is any integer such that

$$n \ln n < a_n < (n+1) \ln (n+1), \qquad n = 1,2,3,\ldots .$$

Thus

$$a_1 = 1, \qquad \frac{n}{a_n} < \frac{1}{\ln n}, \qquad n = 2,3,4,\ldots$$

and

$$\frac{1}{a_n} > \frac{1}{(n+1) \ln (n+1)}, \qquad n = 1,2,3,\ldots .$$

Now $d(A) = 0$ follows from

$$d(A) = \lim_{n\to\infty} \frac{n}{a_n} \le \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \ge \sum_{n=1}^{\infty} \frac{1}{(n+1) \ln (n+1)} = \sum_{n=2}^{\infty} \frac{1}{n \ln n},$$

and since the series on the right diverges (integral test), so does the one on the left.

(b) To prove that the answer here is NO, we will use the following well-known theorem, which can be found in many calculus texts (e.g., O'Neil [2]):

If $\{b_n\}$ is a decreasing sequence cf positive numbers, and if Σb_n converges, then $\lim_{n\to\infty} nb_n = 0$.

Let $A = \{a_n\}$ be a sequence of natural numbers, which we may assume to be increasing, such that $d(A) > 0$. Then A is an infinite sequence, for otherwise $d(A) = 0$. Suppose furthermore that Σb_n converges, where $b_n = 1/a_n$. Since $\{b_n\}$ is a decreasing sequence, it follows from the above theorem that

$$d(A) = \lim_{n\to\infty} \frac{n}{a_n} = \lim_{n\to\infty} nb_n = 0,$$

a contradiction.

Also solved by ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; MARC SACKUR, Collège Stanislas, Montréal, Québec; and the proposer.

Editor's comment.

Part (b) can also be found, with a hint for a proof, in Klambauer [3].

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1. Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers*, Third Edition, John Wiley & Sons, New York, 1972, p. 249, Problem 2.

2. Peter O'Neil, *Advanced Calculus*, Macmillan, New York, 1975, p. 41, Ex. 9.

3. Gabriel Klambauer, *Mathematical Analysis*, Marcel Dekker, Inc., New York, 1975, p. 42, Ex. 73.

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514. [1980: 43] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

If $\alpha+\beta+\gamma=0$, prove that, for $n=0,1,2,\ldots$,

$$\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3} = \alpha\beta\gamma(\alpha^n+\beta^n+\gamma^n) + \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}).$$

I. *Comment by Bob Prielipp, The University of Wisconsin-Oshkosh.*

This is Problem 261 on page 518 of Hall and Knight's *Higher Algebra* [1]. A solution, which we reproduce faithfully below, is given on page 349 of the same authors' *Solutions* book [2]:

Let α, β, γ denote the roots of the cubic equation

$$x^3+qx+r = 0.$$

Multiply this equation by x^n, substitute in succession α, β, γ for x, and add; then

$$(\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3}) + q(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}) + r(\alpha^n+\beta^n+\gamma^n) = 0;$$

but $q=\beta\gamma+\gamma\alpha+\alpha\beta=\frac{1}{2}\{(\alpha+\beta+\gamma)^2-(\alpha^2+\beta^2+\gamma^2)\} = -\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)$; and $r=-\alpha\beta\gamma$; whence the result at once follows.

II. *Solution by Andy Liu, University of Alberta.*

Since $\alpha+\beta+\gamma=0$, the desired result follows immediately from the identity

$$\begin{aligned}\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3} &- \alpha\beta\gamma(\alpha^n+\beta^n+\gamma^n) - \tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1})\\ &= \tfrac{1}{2}(\alpha+\beta+\gamma)\{\alpha^{n+1}(\alpha-\beta-\gamma)+\beta^{n+1}(\beta-\gamma-\alpha)+\gamma^{n+1}(\gamma-\alpha-\beta)\}.\end{aligned}$$

III. *Solution by Howard Eves, University of Maine.*

Using $\beta+\gamma=-\alpha$, etc., we have

$$\begin{aligned}\text{right member} &= \alpha^{n+1}\{\beta\gamma+\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\} + \text{two similar terms}\\ &= \tfrac{1}{2}\alpha^{n+1}\{\alpha^2+(\beta+\gamma)^2\} + \text{two similar terms}\\ &= \tfrac{1}{2}\alpha^{n+1}(\alpha^2+\alpha^2) + \text{two similar terms}\\ &= \alpha^{n+3}+\beta^{n+3}+\gamma^{n+3}\\ &= \text{left member}.\end{aligned}$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; RICHARD A. GIBBS, University of New Mexico; J.T. GROENMAN, Arnhem, The Netherlands; ALLAN WM. JOHSON JR., Washington, D.C.; G.D. KAYE, Ottawa, Ontario; M.S. KLAMKIN, University of of Alberta; VIKTORS LINIS, University of Ottawa; WILLIAM MOSER, McGill University; V.N. MURTY, Pennsylvania State University, Capitol Campus; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; CHRIS NYBERG, East York, Ontario; MARK PRYSANT, student, Princeton University; SANJIB KUMAR ROY, Indian Institute of Technology, Kharagpur, India; MICHAEL SELBY, University of Windsor; FERRELL WHEELER, student, Forest Park H.S., Beaumont Texas; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University; and the proposer.

Editor's comment.

As Linis pointed out in his solution, the proof does not depend on n, so the identity holds whenever the powers α^k, β^k, γ^k can be defined for $k = n$, $n+1$, $n+3$. Disregarding the trivial case when $\alpha = \beta = \gamma = 0$, if α,β,γ are complex and $\alpha\beta\gamma \neq 0$, the identity holds for all integers n, negative as well as positive. If they are real and not all zero, one of the numbers α,β,γ must be negative; so the exponent n must be of the form p/q, where p and q are relatively prime positive integers with q odd, and p can even be negative if $\alpha\beta\gamma \neq 0$.

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515. [1980: 43] *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

Given is a circle with center O and an inscribed triangle ABC. Diameters AA', BB', CC' are drawn. The tangent at A' meets BC in A", the tangent at B' meets CA in B", and the tangent at C' meets AB in C". Show that the points A", B", C" are collinear.

I. *Solution by G.C. Giri, Midnapore College, West Bengal, India.*

We show that the theorem holds more generally for two triangles ABC and A'B'C' that are inscribed in the same conic and are in perspective from a point O.

Let

$$BC \cdot A'B' = X, \quad CA \cdot B'C' = Y, \quad AB \cdot C'A' = Z,$$

$$BC \cdot C'A' = X', \quad CA \cdot A'B' = Y', \quad AB \cdot B'C' = Z'.$$

If we apply Pascal's Theorem successively to the hexagons AA'B'B'BC and ABB'B'C'C, we find, first that

$$AA' \cdot B'B = O, \quad A'B' \cdot BC = X, \quad B'B' \cdot CA = B''$$

are collinear, and then that

$$AB \cdot B'C' = Z', \quad BB' \cdot C'C = O, \quad B'B' \cdot CA = B''$$

are collinear. Thus O,X,Z',B" are collinear, and similarly O,Y,X',C" and O,Z,Y',A" are collinear. If we now apply Desargues' Two-Triangle Theorem to the triangles ZY'A and X'XO, which are in perspective from A', we find that the three points

$$ZY' \cdot X'X = A'', \quad Y'A \cdot XO = B'', \quad AZ \cdot OX' = C''$$

are collinear.

II. *Solution by Sahib Ram Mandan, Bombay, India.*

This problem is a special case of part (a) of the following

THEOREM. If $T = ABC$ *and* $T' = A'B'C'$ *are two triangles inscribed in a conic* S *and in perspective from a point* O, *then*

(a) *the triangle* t' *formed by the tangents* a', b', c' *to* S *at* A', B', C', *respectively, is in perspective to* T *(and so the three points*

$$A'' = a' \cdot BC, \quad B'' = b' \cdot CA, \quad C'' = c' \cdot AB$$

are collinear in the axis of perspectivity d *of* T *and* t'*);*

(b) *the triangle* t *formed by the tangents* a, b, c *to* S *at* A, B, C, *respectively, is in perspective to* T' *with* d' *(say) as axis of perspectivity;*

(c) *the triangles* t *and* t' *are in perspective with their axis of perspectivity* e *coinciding with that of* T *and* T' *as polar of* O *for* S*;*

(d) *the triangle formed by the three points*

$$D = OA \cdot BC, \quad E = OB \cdot CA, \quad F = OC \cdot AB$$

is in perspective to T *with the polar* f *of* O *for* T *as axis of perspectivity;*

(e) *the lines* d, d', e, f *concur with the polar* f' *of* O *for* T' *at a point* O' *conjugate to* O *for* S.

Proof. Let T be taken as the triangle of reference and $O = (1,1,1)$. Then the equation of S takes the form

$$pyz + qzx + rxy = 0 \qquad\qquad (p+q+r \neq 0)$$

with (x,y,z) as the current homogeneous coordinates, and

$$A' = (-p,\ q+r,\ q+r), \quad B' = (r+p,\ -q,\ r+p), \quad C' = (p+q,\ p+q,\ -r).$$

(a) The equations of BC, a'; CA, b'; and AB, c' are respectively

$$x = 0, \quad (q+r)^2x + pqy + rpz = 0;$$
$$y = 0, \quad pqx + (r+p)^2y + qrz = 0;$$
$$z = 0, \quad rpx + qry + (p+q)^2z = 0;$$

and it is clear that A", B", C" all lie on the line

$$d: \quad px + qy + rz = 0.$$

(b)-(e) We find

$$d': \quad (q+r-p)x + (r+p-q)y + (p+q-r)z = 0,$$
$$e: \quad (q+r)x + (r+p)y + (p+q)z = 0,$$
$$f: \quad x + y + z = 0,$$
$$f': \quad (2q+2r-p)x + (2r+2p-q)y + (2p+2q-r)z = 0,$$

and d, d', e, f, f' concur at

$$0' = (q-r,\ r-p,\ p-q). \quad \square$$

In the special case of our proposed problem, when S is a circle with 0 as centre, e becomes the line at infinity and d, d', f, f' are all parallel to one another.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; HOWARD EVES, University of Maine; J.T.GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and the proposer.

Editor's comment.

Most solvers proved the special case of the proposal in a straightforward manner, using similar triangles and the Theorem of Menelaus.

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516. [1980: 44] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Remove the last digit from a positive decimal integer, multiply the removed digit by 5, and add the product to the remaining digits of the decimal integer, thereby obtaining a new decimal integer. Repeat this process again and again, until a single digit results, as in the example

$$13258 \to 1365 \to 161 \to 21 \to 7.$$

Characterize the decimal integers for which this repetitive process terminates in the digit 7.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

It will be convenient to assume that the process does not terminate when a single-digit number is reached, so that, for example, $21 \to 7 \to 35 \to \ldots$; and we will use the notation

$$(a_1, a_2, \ldots, a_n)$$

to represent the *cycle* $a_1 \to a_2 \to \ldots \to a_n$ if the a_i are all distinct and $a_n \to a_1$.

If $n \to n'$, where $n = 10m + d$, $0 \le d \le 9$, then $n' = m + 5d$. Since

$$m \ge 5 \implies 9m - 36 > 0 \implies 9m - 4d > 0 \implies 10m + d > m + 5d \implies n > n',$$

it follows that $n > n'$ whenever $n \ge 50$ and that, for any n, the *chain* $n \to n' \to n'' \to \ldots$ must eventually reach a number less than 50. The numbers less than 50 are partitioned by the process into three disjoint cycles:

cycle 1 contains only one number:

$$(49);$$

cycle 2 contains the remaining multiples of 7:

$$(7, 35, 28, 42, 14, 21);$$

cycle 3 contains all the remaining numbers:

$$(1, 5, 25, 27, 37, 38, 43, 19, 46, 34, 23, 17, 36, 33,$$
$$18, 41, 9, 45, 29, 47, 39, 48, 44, 24, 22, 12, 11, 6,$$
$$30, 3, 15, 26, 32, 13, 16, 31, 8, 40, 4, 20, 2, 10).$$

Our task is now to characterize the integers which belong to cycle 2 or to a chain ending in cycle 2.

With n and n' as before, we have $5n = 5(10m+d) = (m+5d) + 49m = n' + 49m$; hence

$$7|n \iff 7|n' \quad \text{and} \quad 49|n \iff 49|n'.$$

Thus all multiples of 49 are in cycle 1 or in a chain ending in cycle 1; all other multiples of 7 are in cycle 2 or in a chain ending in cycle 2; and all remaining integers are in cycle 3 or in a chain ending in cycle 3. The required characterization is therefore: all numbers divisible by 7 but not by 49.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; RICHARD A. GIBBS, University of New Mexico; G.C. GIRI, Midnapore College, West Bengal, India; T.J. GRIFFITHS, London, Ontario; G.D. KAYE, Ottawa, Ontario; M.S. KLAMKIN, University of Alberta; ANDY LIU, University of Alberta; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.;

HERMAN NYON, Paramaribo, Surinam; CHRIS NYBERG, East York, Ontario; BOB PRIELIPP, The University of Wisconsin-Oshkosh; MARK PRYSANT, student, Princeton University; SANJIB KUMAR ROY, Indian Institute of Technology, Kharagpur, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; CHARLES W. TRIGG, San Diego, California; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

The proposer noted that this problem exploits divisibility criteria that Dickson [1] credits to Niegemann and Zbikowski.

REFERENCE

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Vol. I, Chelsea, New York, 1966, p. 339.

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517. [1980: 44] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given is a triangle ABC with altitudes h_a, h_b, h_c and medians m_a, m_b, m_c to sides a, b, c, respectively. Prove that

$$h_b/m_c + h_c/m_a + h_a/m_b \le 3,$$

with equality if and only if the triangle is equilateral.

Solution by M.S. Klamkin and A. Meir, University of Alberta (jointly).

The inequality in this problem is equivalent to another one by the same proposer [1], for which solutions were published in [2] and [3]. We establish here the stronger result

$$h_1/m_a + h_2/m_b + h_3/m_c \le 3, \tag{1}$$

where (h_1, h_2, h_3) is any permutation of (h_a, h_b, h_c), with equality if and only if the triangle is equilateral. It is clear from (1) that there is no loss of generality in assuming that $a \le b \le c$, with the consequent

$$h_a \ge h_b \ge h_c \quad \text{and} \quad m_a \ge m_b \ge m_c. \tag{2}$$

With the assumptions (2), it follows by an elementary rearrangement inequality (see [4]) that

$$h_a/m_a + h_b/m_b + h_c/m_c \le h_1/m_a + h_2/m_b + h_3/m_c \le h_a/m_c + h_b/m_b + h_c/m_a. \tag{3}$$

Apart from telling us which of the six permutations (h_1, h_2, h_3) will minimize and maximize the left side of (1), (3) shows that (1) will follow from

$$h_a/m_c + h_b/m_b + h_c/m_a \le 3, \tag{4}$$

which we proceed to establish.

By Cauchy's inequality,

$$(h_a/m_c + h_b/m_b + h_c/m_a)^2 \le (h_a^2 + h_b^2 + h_c^2)(1/m_c^2 + 1/m_b^2 + 1/m_a^2),$$

so we need only show that

$$(\Sigma h_a^2)(\Sigma 1/m_a^2) \le 9, \tag{5}$$

where the sums (and product) here and later are cyclic over a,b,c. Since $h_a = 2K/a$, etc., where K is the area of the triangle, and $4m_a^2 = 2b^2 + 2c^2 - a^2$, etc., (5) can be written in the form

$$16K^2(\Sigma a^{-2})\{\Sigma(2b^2 + 2c^2 - a^2)^{-1}\} \le 9. \tag{6}$$

Using $16K^2 = \Sigma(2b^2c^2 - a^4)$, (6) can be manipulated into the equivalent form

$$F(a^2,b^2,c^2) \equiv a^2b^2c^2\Pi(2b^2 + 2c^2 - a^2) - \{\Sigma(2b^2c^2 - a^4)\}(\Sigma b^2c^2)^2 \ge 0.$$

An easy but tedious calculation shows that $F(x,y,z)$ and $\partial F/\partial z$ both vanish for $x = y$. Thus, by symmetry, we must have $F(x,y,z) = (y-z)^2(z-x)^2(x-y)^2$, and (5) follows from

$$F(a^2,b^2,c^2) = (b^2-c^2)^2(c^2-a^2)^2(a^2-b^2)^2 \ge 0. \tag{7}$$

Thus (4) and hence (1) are established.

Suppose equality holds in (1); then it also holds in (4). Now we get

$$a = b \quad \text{or} \quad b = c \quad \text{or} \quad c = a$$

from (7) and

$$h_a m_c = h_b m_b = h_c m_a$$

from the equality in Cauchy's inequality. Since

$$a = b \Rightarrow h_a = h_b \Rightarrow m_c = m_b \Rightarrow c = b,$$

$$b = c \Rightarrow h_b = h_c \Rightarrow m_b = m_a \Rightarrow b = a,$$

and $c = a \Rightarrow a = b = c$ follows from $a \le b \le c$, we conclude that the triangle is equilateral. The converse is trivial.

Comments identifying our proposer's equivalent inequality in the *Monthly* were received from ROLAND H. EDDY, Memorial University of Newfoundland; BOB PRIELIPP, The University of Wisconsin-Oshkosh; and DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio.

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2. B.J. Venkatachala and C.R. Pranesachar, Solution to Problem E 2715, *American Mathematical Monthly*, 86 (1979) 705-706.

3. C.S. Gardner, Solution to Problem E 2715, *American Mathematical Monthly*, 87 (1980) 304.

4. G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, 1952, p. 261.

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518. [1980: 44] *Proposed by Charles W. Trigg, San Diego, California.*

The sequence of positive integers is partitioned into the groups

$$1,\ (2,3),\ (4,5,6),\ (7,8,9,10),\ (11,12,13,14,15),\ \ldots\ .$$

Find the sum of the integers in the nth group.

Solution de Robert Tranquille, Collège de Maisonneuve, Montréal, Québec.

Le dernier nombre du $m$ième groupe est le $m$ième nombre triangulaire $T_m = \frac{1}{2}m(m+1)$. Le $n$ième groupe est une progression arithmétique de n termes, de différence commune 1, et de premier terme $T_{n-1}+1$. La somme demandée est donc

$$\tfrac{1}{2}n\{2(T_{n-1}+1) + (n-1)\} = \tfrac{1}{2}n(n^2+1).$$

Also solved by JAMES BOWE, Erskine College, Due West, South Carolina; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; ROLAND H. EDDY, Memorial University of Newfoundland; HOWARD EVES, University of Maine; JACK GARFUNKEL, Flushing, N.Y.; RICHARD A. GIBBS, University of New Mexico; G.C. GIRI, Midnapore College, West Bengal, India; T.J. GRIFFITHS, London, Ontario; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; G.D. KAYE, Ottawa, Ontario; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio (two solutions); PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; ANDY LIU, University of Alberta; BENGT MÅNSSON, Lund, Sweden; J.A. McCALLUM, Medicine Hat, Alberta; V.N. MURTY, Pennsylvania State University, Capitol Campus; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; CHRIS NYBERG, East York, Ontario; HERMAN NYON, Paramaribo, Surinam; MICHAEL PARMENTER, Memorial University of Newfoundland; BOB PRIELIPP, The University of Wisconsin-Oshkosh; MARK PRYSANT, student, Princeton University; SANJIB KUMAR ROY, Indian Institute of Technology, Kharagpur, India; MATS RÖYTER, Chalmers University of Technology, Göteborg, Sweden; DONVAL R. SIMPSON, Fairbanks, Alaska; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia (two solutions); FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University; and the proposer (two solutions). A comment was received from M.S. KLAMKIN, University of Alberta.

Inadvertently omitted from the above list: HYMAN ROSEN, student, The Cooper Union, New York, N.Y.

Editor's comment.

For this problem, practically any one of the 39 solutions received could have been featured. We selected Tranquille's solution precisely because the problem was easy, to enable readers to improve at least their French, if not their mathematics.

Wheeler noted that a special case of this problem, asking for the sum of the integers in the 21st group, was on the 1967 Annual H.S. Mathematics Examination (see [1]), and that a similar problem by our own proposer (with a different partitioning scheme) appeared in a recent issue of *The Pentagon* [2]. Digging deeper, the editor found our problem in Briggs and Bryan [3], where it is stated that the problem was given, at some unspecified time in the dim past, in a University of London examination. Mathematical archeologists can take it from there.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Howard Eves noted a remarkable property associated in some way with our problem. If the first n^2 natural numbers are written in order in an $n \times n$ array, increasing from left to right and from top to bottom (as shown in our figure for $n = 5$), then the array is quasi-magic in the sense that the sum of any n numbers, no two of which are in the same row or column, is a constant, the *conjuring number* for the array. (Eves had given another array with the same property in the first of of his "Two Timely Problems" [1980: 7,10].) The conjuring number for the $n \times n$ array, Eves stated without proof, is the answer to the present problem, $\frac{1}{2}n(n^2+1)$. The editor knows that a proof of this property lies buried in the proof of Crux 604 [1981: 19], to which he now refers readers.

REFERENCES

1. Charles T. Salkind and James M. Earl, *The MAA Problem Book III*, Mathematical Association of America, 1973, p. 21.

2. Charles W. Trigg (proposer), Problem 318, *The Pentagon*, Fall 1979, p. 30.

3. William Briggs and G.H. Bryan, *The Tutorial Algebra*, Vol. I (Sixth Edition revised by George Walker), University Tutorial Press, London, 1954, p. 176, Ex. 19.

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519. [1980: 44] *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Let

$$\phi(x) = \frac{\sin \pi x}{x(1-x)}, \qquad 0 < x < 1.$$

Prove that ϕ is increasing for $0 < x < \frac{1}{2}$ and decreasing for $\frac{1}{2} < x < 1$.

Solution by Bengt Månsson, Lund, Sweden.

Since $\phi(1-x) = \phi(x)$ for $0 < x < 1$, the curve is symmetrical with respect to the line $x = \frac{1}{2}$, and it suffices to show that ϕ is increasing in the interval $(0,\frac{1}{2})$ or that

$$\phi'(x) \geq 0, \qquad 0 < x < \tfrac{1}{2}. \tag{1}$$

The restriction $0 < x < \frac{1}{2}$ will be implicit in the analysis that follows.

Since $\cos \pi x / x^2(1-x)^2 > 0$, differentiation shows that

$$\operatorname{sgn} \phi'(x) = \operatorname{sgn} \{\pi x(1-x) - (1-2x)\tan \pi x\} \equiv \operatorname{sgn} f(x);$$

and since $\tan^2 \pi x > 0$, further differentiation shows that

$$\operatorname{sgn} f'(x) = \operatorname{sgn} \{(2x-1)\pi + 2\cot \pi x\} \equiv \operatorname{sgn} g(x).$$

Now $g(\frac{1}{2}) = 0$ and $g'(x) = -2\pi \cot^2 \pi x < 0$ imply that $g(x) > 0$; then $f(0) = 0$ and $f'(x) > 0$ imply that $f(x) > 0$, and (1) follows.

Also solved by VIKTORS LINIS, University of Ottawa; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; MICHAEL SELBY, University of Windsor; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and the proposer. M.S. KLAMKIN, University of Alberta, sent in a comment and, in addition, one very incomplete solution and two incorrect solutions were received.

Editor's comment.

Klamkin revealed that this problem appeared almost simultaneously, with the same proposer, in this journal and in *The MATYC Journal*, 14 (1980) 72-73. This is probably the result of the proposer's left hand not knowing what his right hand was doing. Or else one editor is rifling the files of another editor.

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520. [1980: 44] *Proposed by M.S. Klamkin, University of Alberta.*

If two chords of a conic are mutually bisecting, prove that the conic cannot be a parabola.

I. *Solution by Howard Eves, University of Maine.*

Consider the parabola $y = x^2$ cut by a secant line $y = mx + b$. The x-coordinates x_1, x_2 of the endpoints of the chord formed by the secant line are given by the roots of the quadratic equation $x^2 - mx - b = 0$. Since $(x_1 + x_2)/2 = m/2$, it follows that all chords sharing the same midpoint must have the same slope, and must therefore coincide with one another. Thus a parabola cannot have two distinct mutually bisecting chords.

II. *Solution by Leroy F. Meyers, The Ohio State University.*

Suppose that the nondegenerate and noncoinciding chords AB and CD of a parabola bisect each other at E. Then ACBD is a parallelogram, and the line joining the midpoints of the parallel chords AC and BD must meet the line joining the midpoints of the parallel chords AD and BC at E. But the line joining the midpoints of two noncoinciding but parallel chords of a parabola must be parallel to the unique axis of the parabola. Hence the midpoint lines cannot meet, and we have the desired contradiction.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; G.C. GIRI, Midnapore College, West Bengal, India; T.J. GRIFFITHS, London, Ontario; J.T. GROENMAN, Arnhem, The Netherlands; ANDY LIU, University of Alberta; SAHIB RAM MANDAN, Bombay, India; BENGT MÅNSSON, Lund, Sweden; J.A. McCALLUM, Medicine Hat, Alberta; DAN PEDOE, University of Minnesota; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; and the proposer (two solutions).

Editor's comment.

Several solvers submitted projective proofs, both analytic and synthetic, and some of them were quite short and elegant, but no more so than the "low-energy" proofs given above, which can be understood by any high school student not majoring in basket-weaving.

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INTERNATIONAL CONFERENCE ON TEACHING STATISTICS

Preliminary Announcement

The International Statistical Institute is pleased to announce that the First International Conference on Teaching Statistics will be held in Sheffield, England, from 8-13 August 1982. For a copy of the first announcement write to

The Conference Secretary,
International Conference on the Teaching of Statistics,
Department of Probability and Statistics,
The University,
Sheffield S3 7RH,
England.

The objective of the conference is to improve the quality of statistics teaching on a world-wide basis. Key goals include fostering international cooperation among teachers of statistics and promoting the interchange of ideas about teaching materials, methods and content. Invited speakers of international repute will address the plenary meetings. There will also be many workshops, discussion groups and contributed paper sessions. Teaching from the school to the college level as well as other forms of teaching will be included. There will also be sessions on teaching statistics to government and industrial practitioners.

The Canadian coordinator for this Conference is

Professor Donald G. Watts,
Department of Mathematics and Statistics,
Queen's University,
Kingston, Canada K7L 3N6.