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MAYHEM SOLUTIONS

Mathematical Mayhem *is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time* **Mathematical Mayhem** *will be discontinued in* **Crux**. *New* **Mayhem** *problems will appear when the journal is relaunched in 2013.*

M488. *Proposed by the Mayhem Staff.*

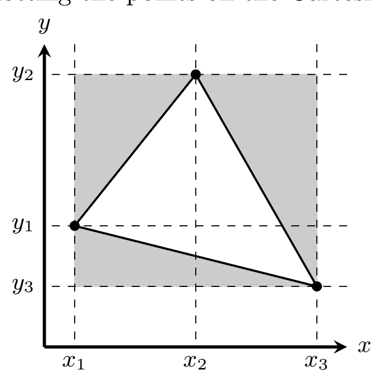
A triangle has vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3).

(a) If $x_1 < x_2 < x_3$ and $y_3 < y_1 < y_2$, determine the area of the triangle.

(b) Show that, if the conditions on x_1, x_2, x_3, y_1, y_2, and y_3 are dropped, the expression from (a) gives either the area or -1 times the area.

Solution by Cássio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil.

(a) Plotting the points on the Cartesian plane, we have:



The easiest way to calculate the area A of the triangle is to consider the rectangle that surrounds the points and disregard three right triangles (see the diagram):

$$\begin{aligned}A &= (x_3 - x_1)(y_2 - y_3) \\ &\qquad - \frac{(x_3 - x_1)(y_1 - y_3)}{2} - \frac{(x_2 - x_1)(y_2 - y_1)}{2} - \frac{(x_3 - x_2)(y_2 - y_3)}{2} \\ &= \frac{1}{2}(x_2(y_1 - y_3) - x_1(y_2 - y_3) + x_3(y_2 - y_1)).\end{aligned}$$

(b) Considering the vertices as points in $\mathbb{R}^3$, that is as $P_1(x_1, y_1, 0)$, $P_2(x_2, y_2, 0)$, and $P_3(x_3, y_3, 0)$, then it is known that the area of the triangle with sides defined by vectors $\overrightarrow{P_1P_2}$, $\overrightarrow{P_1P_3}$, and $\overrightarrow{P_2P_3}$ is given by

$$A = \frac{1}{2}|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}| = \frac{1}{2}|x_2(y_1 - y_3) - x_1(y_2 - y_3) + x_3(y_2 - y_1)|$$

and hence is equal to the expression from (a) or its negative.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; AARON PERKINS and ADRIENNA BINGHAM, students, Angelo State University, San Angelo, TX, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

All other solvers used vectors for both parts. In part (b), if any of the x or y coordinates of two points are equal, the statement is trivially true. As such, we could assume, without loss of generality, that $x_1 < x_2 < x_3$. By looking at the six configurations of the y coordinates, four can be found to be equivalent to part (a) after some reflections, hence the statement is true. The other two cases are equivalent to each other by reflections, and can be shown to be equivalent to the result in (a) or its negative. Readers may enjoy looking at these cases and showing the result holds.

M489. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Prove that if m and n are relatively prime positive integers such that

$$m\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2010}\right) = n,$$

then 2011 divides n.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Observe that

$$\begin{aligned}\sum_{k=1}^{2010} \frac{1}{k} &= \left(1 + \frac{1}{2010}\right) + \left(\frac{1}{2} + \frac{1}{2009}\right) + \cdots + \left(\frac{1}{1005} + \frac{1}{1006}\right) \\ &= \frac{2011}{1(2010)} + \frac{2011}{2(2009)} + \cdots + \frac{2011}{1005(1006)} \\ &= 2011 \sum_{k=1}^{1005} \frac{1}{k(2011-k)}.\end{aligned}$$

Summing the last group of fractions one obtains

$$\sum_{k=1}^{1005} \frac{1}{k(2011-k)} = \frac{S}{2010!}$$

for some positive integer S. Therefore,

$$m\sum_{k=1}^{2010} \frac{1}{k} = 2011m \sum_{k=1}^{1005} \frac{1}{k(2011-k)} = \frac{2011mS}{2010!} = n$$

implies $2011(mS) = (2010!)n$. Thus, 2011 divides the integer $(2010!)n$. Since 2011 is a prime and $\gcd(2011, 2010!) = 1$, it follows that 2011 divides n as was to be shown.

Also solved by ARKADY ALT, San Jose, CA, USA; FELIX BOOS, University of Kaiserslautern, Kaiserslautern, Germany; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ HERNÁNDEZ SANTIAGO, Mexico; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; CODREANU IOAN VIOREL, Secondary School student, Satulung, Maramureş, Romania; and the proposer. One incorrect solution was received.

Hernández points out that Wolstenholme's theorem gives the stronger result, $2011^2 \mid n$. *Cano points out that Shklarsky et al. (The USSR Olympiad Problem Book, Dover Publications, 1994) contains a problem asking for the proof of Wolstenholme's theorem.*

The Missouri State University Problem Solving Group generalized the problem. Letting m, n *and* q *be positive integers with* $q > 2$, $Q = \{i \mid 0 < i < q \text{ and } \gcd(i, q) = 1\}$ *and* $S = \sum_{i \in Q} \frac{1}{i}$, *then if* $mS = n$, *then* q *divides* n.

M490. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

For any positive integer n, let $S(n)$ denote the sum of the digits of n (in base 10). Given a positive integer m, prove that there exists a positive integer n such that $m = \dfrac{S(n^2)}{S(n)}$.

Solution by Geneviève Lalonde, Massey, ON.

For a given m, let

$$n = \sum_{i=1}^{m} 10^{2^i - 1},$$

then we will show that n satisfies the condition in the problem.

Clearly, as n is a sum of different powers of 10, the decimal representation will contain m ones and the rest of the digits will be zero, thus $S(n) = m$. Also

$$n^2 = \sum_{i=1}^{m} 10^{2^{i+1}-2} + 2 \sum_{1 \le j < k \le m} 10^{2^j + 2^k - 2},$$

which we claim has a decimal representation that is made up of only of twos (from the second sum only), ones (from the first sum only) and zeros. This must be true, since if a term from the first sum involves the same power of 10 as a term from the second sum, we would have to have $2^{i+1} - 2 = 2^j + 2^k - 2 \Rightarrow 2^{i+1} = 2^j + 2^k \Rightarrow 2^{i-j+1} = 2^{k-j} + 1$, which means that, since $j < k$, the right hand side is odd while the left hand side is even, a contradiction.

Thus we must have

$$\begin{aligned} S(n^2) &= 1 \times (\text{ \# of terms in first sum}) + 2 \times (\text{ \# of terms in second sum}) \\ &= 1 \times m + 2 \times \binom{m}{2} = m^2 \end{aligned}$$

hence $\dfrac{S(n^2)}{S(n)} = m$, as desired.

Four incorrect solutions were received. All incorrect solutions forgot to take into account that, if not careful, you will have to deal with "carrys" at some point. That is, the numbers 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111 and 111111111 give solutions to the problem for $m = 1, 2, \dots, 9$, respectively; e.g., the largest in the list gives us $S(111111111) = 9$, and $111111111^2 = 12345678987654321$, so $\frac{S(111111111^2)}{S(111111111)} = \frac{81}{9} = 9$. Unfortunately, the next number in this family has $1111111111^2 = 1234567900987654321$, so $S(1111111111) = 10$ but $S(1111111111^2) = 82$.

The solution given is not unique. If $\alpha_1, \alpha_2, \dots, \alpha_m$ are distinct positive integers such that the $\binom{m}{2} + m$ numbers $\{\alpha_i + \alpha_j \mid 1 \le i \le j \le m\}$ are all distinct, then

$$n = \sum_{i=1}^{m} 10^{\alpha_i}$$

also satisfies the conditions of the problem.

M491. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let a, b, and c be given constants, not necessarily distinct. Solve the equation below:

$$\frac{(x-a)^2}{(x-a)^2-(b-c)^2} + \frac{(x-b)^2}{(x-b)^2-(c-a)^2} + \frac{(x-c)^2}{(x-c)^2-(a-b)^2} = 1\,.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

We consider three cases:

Case 1: If $a = b = c = \alpha$, then the left hand side of the equation becomes:

$$\frac{(x-\alpha)^2}{(x-\alpha)^2-(\alpha-\alpha)^2} + \frac{(x-\alpha)^2}{(x-\alpha)^2-(\alpha-\alpha)^2} + \frac{(x-\alpha)^2}{(x-\alpha)^2-(\alpha-\alpha)^2} = 3\frac{(x-\alpha)^2}{(x-\alpha)^2} = 3,$$

so the equation has no solutions.

Case 2: If exactly two of a, b, c are equal, there is only one solution. Without loss of generality, assume $a = b = \alpha \neq c$, then the equation becomes:

$$\frac{(x-\alpha)^2}{(x-\alpha)^2-(\alpha-c)^2} + \frac{(x-\alpha)^2}{(x-\alpha)^2-(c-\alpha)^2} + \frac{(x-c)^2}{(x-c)^2-(\alpha-\alpha)^2} = 1\,.$$

Since $(\alpha - c)^2 = (c - \alpha)^2$, this simplifies to

$$\begin{aligned} \frac{2(x-\alpha)^2}{(x-\alpha)^2-(\alpha-c)^2} + 1 &= 1 \\ \frac{2(x-\alpha)^2}{(x-\alpha)^2-(\alpha-c)^2} &= 0 \Rightarrow x = \alpha\,. \end{aligned}$$

Similarly, if $b = c = \beta \neq a$ then the only solution is $x = \beta$ and if $c = a = \gamma \neq b$ the only solution is $x = \gamma$.

Case 3: If none of a, b, c are equal, there are three solutions. If we factor the denominators, the original equation becomes:

$$\frac{(x-a)^2}{(x-a-b+c)(x-a+b-c)}+\frac{(x-b)^2}{(x-b-c+a)(x-b+c-a)}$$
$$+\frac{(x-c)^2}{(x-c-a+b)(x-c+a-b)}=1\,.$$

The equation is defined for $x\neq a+b-c$, $x\neq b+c-a$, and $x\neq c+a-b$. Setting $u=x-a$, $v=x-b$ and $w=x-c$, the equation becomes

$$\frac{u^2}{u^2-(v-w)^2}+\frac{v^2}{v^2-(w-u)^2}+\frac{w^2}{w^2-(u-v)^2}=1$$
$$\frac{u^2}{(u+v-w)(u-v+w)}+\frac{v^2}{(v+w-u)(v-w+u)}+\frac{w^2}{(w+u-v)(w-u+v)}=1$$
$$\frac{u^2(v+w-u)+v^2(w+u-v)+w^2(u+v-w)}{(u+v-w)(v+w-u)(w+u-v)}=1.$$

Writing $1=\frac{(u+v-w)(v+w-u)(w+u-v)}{(u+v-w)(v+w-u)(w+u-v)}$, expanding the numerators of both sides, bringing all terms together on one side and simplifying we get

$$\frac{-2uvw}{(u+v-w)(v+w-u)(w+u-v)}=0$$

whence $uvw=0$ yielding $x=a$ or $x=b$ or $x=c$.

Also solved by ARKADY ALT, San Jose, CA, USA; FLORENCIO CANO VARGAS, Inca, Spain; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; AARON PERKINS and ADRIENNA BINGHAM, students, Angelo State University, San Angelo, TX, USA; ÁNGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CHRISTOPHER ZHANG, student, Hillsborough High School, Hillsborough, NJ, USA; and the proposer. One incorrect solution was received.

M492. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Prove that

$$\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1]=2011!\left(6^{2010}-1\right)\ .$$

Solution by Christopher Zhang, student, Hillsborough High School, Hillsborough, NJ, USA.

Rearranging the sum we get

$$\begin{aligned}\sum_{k=0}^{2009}(k+1)![6^k(6k+11)-k-1] &= \sum_{k=0}^{2009}(k+1)![6^k(6k+12)-6^k-(k+2)+1]\\ &= \sum_{k=0}^{2009}(k+1)![(k+2)(6^{k+1}-1)-(6^k-1)]\\ &= \sum_{k=0}^{2009}(k+2)!(6^{k+1}-1)-\sum_{k=0}^{2009}(k+1)!(6^k-1)\\ &= \sum_{k=1}^{2010}(k+1)!(6^k-1)-\sum_{k=0}^{2009}(k+1)!(6^k-1)\\ &= (2010+1)!(6^{2010}-1)-(0+1)!(6^0-1)\\ &= 2011!(6^{2010}-1)\end{aligned}$$

as desired.

Also solved by ARKADY ALT, San Jose, CA, USA; FELIX BOOS, University of Kaiserslautern, Kaiserslautern, Germany; FLORENCIO CANO VARGAS, Inca, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; AARON PERKINS and ADRIENNA BINGHAM, students, Angelo State University, San Angelo, TX, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; CODREANU IOAN VIOREL, Secondary School student, Satulung, Maramureş, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

Viorel and Wang generalized the problem and showed that for any natural number n, and any constant a we have

$$\sum_{k=0}^{n}(k+1)![a^k(ak+2a-1)-k-1]=(n+2)!\left(a^{n+1}-1\right).$$

The problem is the special case when $n=2009$ and $a=6$.

M493. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Find all positive integers x that satisfy the equation

$$\frac{x+\left[\sqrt{x}+\sqrt{x+1}\right]}{\left[\sqrt{4x+1}+4022\right]}+\frac{x}{\left[\sqrt{4x+2}\right]+4022}=1\,,$$

where $[x]$ is the integer part of x.

Solution by Florencio Cano Vargas, Inca, Spain.

We first note that $\left[\sqrt{4x+1}+4022\right]=\left[\sqrt{4x+1}\right]+4022$.

Now, we are going to show that

$$\left[\sqrt{4x+1}\right]=\left[\sqrt{4x+2}\right]. \qquad (1)$$

Any perfect square is congruent to 1 (if the number is odd) or to 0 (if the number is even) modulo 4. This means that $4x+1$ can be a perfect square, whereas $4x+2$ can not. No matter if $4x+1$ is a perfect square or not, since x is a positive integer, there exits an integer $k \geq 3$ such that

$$k^2 \leq 4x+1 < (k+1)^2,$$

but $4x+1$ and $4x+2$ are consecutive integers and $4x+2$ cannot be a perfect square so we must have

$$k^2 \leq 4x+1 < 4x+2 < (k+1)^2.$$

Taking square roots yields

$$k \leq \sqrt{4x+1} < \sqrt{4x+2} < k+1$$

and thus

$$[\sqrt{4x+1}] = [\sqrt{4x+2}] = k,$$

establishing (1).

Now we are going to show that

$$\sqrt{4x+1} < \sqrt{x} + \sqrt{x+1} < \sqrt{4x+2}. \tag{2}$$

Note that, since x is a positive integer

$$\begin{aligned}
\sqrt{4x+1} < \sqrt{x} + \sqrt{x+1} < \sqrt{4x+2} \qquad & \\
&\Leftrightarrow 4x+1 < 2x+1+2\sqrt{x(x+1)} < 4x+2 \\
&\Leftrightarrow x < \sqrt{x(x+1)} < x + \frac{1}{2} \\
&\Leftrightarrow x^2 < x^2 + x < x^2 + x + \frac{1}{4} \\
&\Leftrightarrow 0 < x < x + \frac{1}{4},
\end{aligned}$$

and since the last statement is always true, (2) is established.

Then, there will exist a positive integer $k \geq 3$ such that

$$k \leq \sqrt{4x+1} < \sqrt{x} + \sqrt{x+1} < \sqrt{4x+2} < k+1$$

and we have

$$[\sqrt{4x+1}] = [\sqrt{4x+2}] = [\sqrt{x} + \sqrt{x+1}] = k.$$

Thus, we can rewrite the original problem as

$$\frac{x+k}{k+4022} + \frac{x}{k+4022} = 1$$

which simplifies to

$$2x + k = k + 4022$$

which yields the unique solution to the problem $x = 2011$.

Also solved by ARKADY ALT, San Jose, CA, USA; FELIX BOOS, University of Kaiserslautern, Kaiserslautern, Germany; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; CODREANU IOAN VIOREL, Secondary School student, Satulung, Maramureş, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; CHRISTOPHER ZHANG, student, Hillsborough High School, Hillsborough, NJ, USA; and the proposer.

M494. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let z be a complex number such that $|z| = 2$. Find the minimum value of $\left|z - \dfrac{1}{z}\right|$.

Solution by Christopher Zhang, student, Hillsborough High School, Hillsborough, NJ, USA.

We claim that the minimum value of $\left|z - \frac{1}{z}\right|$ is $\frac{3}{2}$.

Representing the problem on the complex plane, let O represent the origin, A the point with coordinates of z and B the point with coordinates of $\frac{1}{z}$. We know that the length of AB is $\left|z - \frac{1}{z}\right|$ and the lengths of OA and OB are 2 and $\frac{1}{2}$, respectively. Hence, by the triangle inequality, $AB = \left|z - \frac{1}{z}\right| \geq OA - OB \geq \frac{3}{2}$. Equality may occur when $z = \pm 2$. This proves the claim.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FLORENCIO CANO VARGAS, Inca, Spain; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; JOSÉ HERNÁNDEZ SANTIAGO, Mexico; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ÁNGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; CODREANU IOAN VIOREL, Secondary School student, Satulung, Maramureş, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and the proposer.

THE CONTEST CORNER

No. 4

Shawn Godin

The Contest Corner *est une nouvelle rubrique offerte pas* ***Crux Mathematicorum****, comblant ainsi le vide suite à la mutation en 2013 de* Mathematical Mayhem *et* Skoliad *vers une nouvelle revue en ligne. Il s'agira d'un amalgame de* Skoliad, The Olympiad Corner *et l'ancien* Academy Corner *d'il y a plusieurs années. Les problèmes en vedette seront tirés de concours destinés aux écoles secondaires et au premier cycle universitaire ; les lecteurs seront invités à soumettre leurs solutions ; ces solutions commenceront à paraître au prochain numéro.*

Les solutions peuvent être envoyées à : Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5 *ou par couriel à* `crux-contest@cms.math.ca`.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC16. Dans un carré magique, les nombres dans chaque rangée, les nombres dans chaque colonne et les nombres dans chaque diagonale donnent des sommes égales. Étant donné le carré magique ci-joint avec $a, b, c, x, y, z > 0$, déterminer le produit xyz en termes de a, b et c.

$\log a$	$\log b$	$\log x$
p	$\log y$	$\log c$
$\log z$	q	r

CC17. Une droite de pente m rencontre la parabole $y = x^2$ en A et B. Si la longueur du segment AB est ℓ quelle est l'équation de la droite en termes de ℓ et m ?

CC18. L'extrémité à gauche d'un élastique de longueur e mèters est attachée à un mur et un enfant malin tient l'extrémité droite. Une fourmi minuscule se trouve à l'extrémité gauche de l'élastique au temps $t = 0$, au moment où elle commence à marcher vers l'extrémité droite, en même temps que l'enfant commence à étirer l'élastique. La fourmi, de plus en plus fatiguée, marche à une vitesse de $1/(ln(t + e))$ centimètres à la seconde, tandis que l'enfant étire l'élastique à une vitesse constante de un mètre à la seconde. L'élastique est infiniment étirable ; la fourmi et l'enfant sont immortels. Calculer le temps en secondes pour que la fourmi atteigne l'extrémité droite de l'élastique, si ce temps existe.

CC19. Évaluer

$$\frac{1}{3+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2013}}}}}}+\frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{\cdots+\frac{1}{2013}}}}}}.$$

CC20. Le Club mathématique organise une “soirée dansante (M, N)”. Toute danse est soit lente, soit rapide. Le disk-jockey a reçu les instructions suivantes : la M^{e} danse après une danse rapide doit obligatoirement être une danse lente, tandis que la N^{e} danse après une danse lente doit obligatoirement être une danse rapide. Pour certaines valeurs de M et N ceci veut dire que la soirée dansante doit terminer tôt, tandis que pour d'autres valeurs de M et N la soirée dansante peut en principe ne jamais avoir de fin. Pour quelles paires ordonnées (M, N) n'y a-t-il aucune borne supérieure pour le nombre de danses ?

. . . .

CC16. In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. Given the magic square shown with $a, b, c, x, y, z > 0$, determine the product xyz in terms of a, b and c.

$\log a$	$\log b$	$\log x$
p	$\log y$	$\log c$
$\log z$	q	r

CC17. A line with slope m meets the parabola $y = x^2$ at A and B. If the length of segment AB is ℓ what is the equation of that line in terms of ℓ and m?

CC18. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(ln(t + e))$ centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band.

CC19. Evaluate

$$\frac{1}{3+\frac{1}{3+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+\frac{1}{2013}}}}}}+\frac{1}{1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{\cdots+\frac{1}{2013}}}}}}.$$

CC20. When the Math Club advertises an “(M, N) sock hop”, this means that the DJ has been instructed that the M^{th} dance after a fast dance must be a slow dance, while the N^{th} dance after a slow dance must be a fast dance. (All dances are slow or fast; the DJ avoids the embarrassing ones where nobody is quite sure what to do.) For some values of M and N this means that the dancing must end early and everybody can start in on the pizza; for other values the dancing can in principle go on forever. For which ordered pairs (M, N) is there no upper bound to the number of dances?

THE OLYMPIAD CORNER

No. 302

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC76. Pour tout entier positif n, soit a_n la plus grande puissance de 2 qui apparaît comme un facteur de $5^n - 3^n$. De plus, soit b_n la plus grande puissance de 2 qui divise n. Montrer que, pour tout n,

$$a_n \leq b_n + 3.$$

OC77. Trouver toutes les fonctions $f : (0,\infty) \to (0,\infty)$ telles que, pour tous les $x, y \in (0,\infty)$, on a

$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy}\,.$$

OC78. Soit $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ la suite des entiers positifs commençant par 1 et suivi par tous les entiers dont la somme des chiffres est divisible par 5. Montrer que pour tout n, on a

$$a_n \leq 5n\,.$$

OC79. Soit D un point différent des sommets sur le côté BC d'un ΔABC. Soit respectivement I, I_1 et I_2 les centres des cercles inscrits de $\Delta ABC, \Delta ABD$ et ΔADC. Soit E le second point d'intersection des cercles circonscrits de ΔAI_1I et ΔADI_2, et soit F le second point d'intersection des cercles circonscrits de ΔAII_2 et ΔAI_1D. Si $AI_1 = AI_2$, montrer que

$$\frac{EI}{FI}\cdot\frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}\,.$$

OC80. Soit G un graphe simple avec $3n^2$ sommets ($n \geq 2$), tel que le degré de chaque sommet de G ne soit pas plus grand que $4n$, qu'il existe au moins un sommet de degré un, et qu'entre deux sommets quelconques, il y ait un chemin de longueur ≤ 3. Monter que le nombre minimal de sommets que G puisse avoir est égal à $\dfrac{7n^2 - 3n}{2}$.

. . . .

OC76. For any positive integer n, let a_n be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let b_n be the exponent of the largest power of 2 which divides n. Show that

$$a_n \leq b_n + 3$$

for all n.

OC77. Find all functions $f : (0, \infty) \to (0, \infty)$ so that for all $x, y \in (0, \infty)$ we have

$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy}.$$

OC78. Let $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ be the sequence of positive integers starting with 1, followed by all integers with the sum of the digits divisible by 5. Prove that for all n we have

$$a_n \leq 5n.$$

OC79. Let D be a point different from the vertices on the side BC of a ΔABC. Let I, I_1 and I_2 be the incenters of the $\Delta ABC, \Delta ABD$ respectively ΔADC. Let E be the second intersection point of the circumcircles of the ΔAI_1I and ΔADI_2, and let F be the second intersection point of the circumcircles of the triangles ΔAII_2 and ΔAI_1D. If $AI_1 = AI_2$, prove that

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}.$$

OC80. Let G be a simple graph with $3n^2$ vertices ($n \geq 2$), such that the degree of each vertex of G is not greater than $4n$, there exists at least a vertex of degree one, and between any two vertices, there is a path of length ≤ 3. Prove that the minimum number of edges that G might have is equal to $\dfrac{7n^2 - 3n}{2}$.

OLYMPIAD SOLUTIONS

OC16. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \ldots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2 .$$

(*Originally question # 3 from 2010 Singapore Mathematical Olympiad, Senior Section, Round 2.*)

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Felix Boos, University of Kaiserslautern, Kaiserslautern, Germany; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru .

Let $s_k = a_1 + a_2 + \cdots + a_k$.
We first prove by induction that

$$a_n a_{n-1} \geq 2s_{n-1} .$$

The case $n = 2$ follows immediately from $a_2 \geq a_1 + 1 \geq 2$, while the inductive step is the following:

$$\begin{aligned} a_{n+1}a_n &\geq (a_{n-1} + 2)a_n = a_{n-1}a_n + 2a_n \\ &\geq 2s_{n-1} + 2a_n = 2s_n \end{aligned}$$

Also, let's observe that equality can be obtained only if $a_{n+1} = a_{n-1} + 2$ for all n, that is if $a_n = n$.
Now, we prove the problem by induction.
For $n = 1$ the problem is equivalent to

$$a_1^3 \geq a_1^2 \Leftrightarrow a_1 \geq 1 .$$

We now prove the inductive step:

$$\begin{aligned} a_1^3 + a_2^3 + \cdots + a_{n+1}^3 &\geq s_n^2 + a_{n+1}^3 \geq s_n^2 + a_{n+1}^2(a_n + 1) \\ &= s_n^2 + a_{n+1}^2 a_n + a_{n+1}^2 \geq s_n^2 + 2s_n a_{n+1} + a_{n+1}^2 = s_{n+1}^2 \end{aligned}$$

Equality only holds if $a_n = n$ for all n.

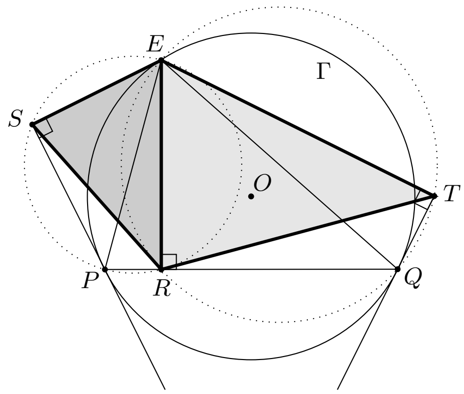
OC17. Prove that the vertices of a convex pentagon $ABCDE$ are concyclic if and only if the following holds

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC) .$$

(*Originally question # 6 from the 60th national mathematical Olympiad Selection Tests for the Balkan and IMO, 2nd selection test.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Lemma 1: *Let E, P, Q be distinct points on a circle Γ. Let R, S, T be the feet of the perpendiculars from E onto the line PQ and onto the tangents to Γ in P and Q, respectively. Then, $ER^2 = ES \cdot ET$.*



Proof: Since $\angle ESP + \angle ERP = 180°$ the quadrilateral ESPR is cyclic. Thus

$$\angle ESR = \angle EPQ\,;\ \angle SRE = \angle SPE\,.$$

Since PS is tangent to Γ, we get

$$\angle SPE = \frac{\widehat{PE}}{2} = \angle EQP$$

Thus

$$\angle ESR = \angle EPQ\,;\ \angle SRE = \angle EQP\,.$$

The same argument now works in $ETQR$: $\angle ETQ + \angle ERQ = 180°$, hence the quadrilateral ETQR is cyclic. Thus

$$\angle ETR = \angle EQP\,;\ \angle ERT = \angle EQT = \angle EPQ\,.$$

Thus, $\Delta SRE \sim \Delta RTE$ which implies our claim.

Corollary 2: *If $ABCDE$ is a convex cyclic pentagon, then the following holds:*

$$d(E,AB)\cdot d(E,CD) = d(E,AC)\cdot d(E,BD) = d(E,AD)\cdot d(E,BC). \qquad (1)$$

Proof: Let A_0, B_0, C_0, D_0 be the feet of the perpendiculars from E onto the tangents to the the circumcircle of the pentagon in A, B, C, D, respectively. By Lemma 1, we have $d(E,AB)^2 = EA_0 \cdot EB_0$, $d(E,CD) = EC_0 \cdot ED_0$ and similar relations for the remaining distances. Hence, each expression in the equation (1) equals to $EA_0 \cdot EB_0 \cdot EC_0 \cdot ED_0$.

Lemma 3: *Let $ABCDE$ be a convex pentagon such that the equation (1) holds. Then, the quadrilateral $ABCD$ is cyclic.*

Proof: Let $m = d(E, AB) \cdot d(E, CD)$, $\alpha = \angle AEB$, $\beta = \angle BEC$, $\gamma = \angle CED$. Then, by the relation (1), we have

$$\begin{aligned}
&\frac{m}{AE \cdot BE \cdot CE \cdot DE} \cdot (AC \cdot BD - AB \cdot CD - AD \cdot BC) \\
&= \frac{d(E, AC) \cdot AC}{AE \cdot CE} \cdot \frac{d(E, BD) \cdot BD}{BE \cdot DE} - \frac{d(E, AB) \cdot AB}{AE \cdot BE} \cdot \frac{d(E, CD) \cdot CD}{CE \cdot DE} \\
&\quad - \frac{d(E, AD) \cdot AD}{AE \cdot DE} \cdot \frac{d(E, BC) \cdot BC}{BE \cdot CE} \\
&= \sin(\alpha + \beta)\sin(\beta + \gamma) - \sin\alpha\sin\gamma - \sin(\alpha + \beta + \gamma)\sin\beta \\
&= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\beta\cos\gamma + \cos\beta\sin\gamma) - \sin\alpha\sin\gamma \\
&\quad - (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma \\
&\qquad\qquad - \sin\alpha\sin\beta\sin\gamma) \cdot \sin\beta \\
&= 0,
\end{aligned}$$

that is, $AC \cdot BD - AB \cdot CD - AD \cdot BC = 0$. By the converse of Ptolemy's Theorem, the points A, B, C, D are concylic.

Corollary 4: *Let $ABCDE$ be a convex pentagon such that the equation (1) holds. Then, the pentagon is cyclic.*

Proof: By Lemma 3, A, B, C, D lie on a common circle Γ. It is enough to show that E also lies on Γ. Let F be the foot of the perpendicular from E onto the line AB. Let Γ be the unit circle in the complex plane and let a, b, c, d, e, f be the coordinates of A, B, C, D, E, F. We have

$$\begin{aligned}
d(E, AB)^2 = (f - e)(\overline{f} - \overline{e}) &= \frac{1}{4}(a + b - e - ab\overline{e})\left(\frac{1}{a} + \frac{1}{b} - \overline{e} - \frac{e}{ab}\right) \\
&= \frac{1}{4ab}(a + b - e - ab\overline{e})^2
\end{aligned}$$

and similar relations for the remaining distances. Direct computation yields

$$16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2 = \left[e^2 - \sigma e - \tau\overline{e} + (ab + cd)e\overline{e} + (a + b)(c + d)\right]^2,$$

where $\sigma = a + b + c + d$ and $\tau = abc + abd + acd + bcd$, and two analogous identities.

Let z denote the principal value of the square root of the complex number $16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2$. By (1), there are numbers $\vartheta_1, \vartheta_2, \vartheta_3 \in \{-1,\ 1\}$ such that

$$\begin{aligned}
z &= \vartheta_1\left[e^2 - \sigma e - \tau\overline{e} + (ab + cd)e\overline{e} + (a + b)(c + d)\right] \\
&= \vartheta_2\left[e^2 - \sigma e - \tau\overline{e} + (ac + bd)e\overline{e} + (a + c)(b + d)\right] \\
&= \vartheta_3\left[e^2 - \sigma e - \tau\overline{e} + (ad + bc)e\overline{e} + (a + d)(b + c)\right]
\end{aligned}$$

By the Pigeonhole Principle, two of the numbers ϑ_1, ϑ_2, ϑ_3 must be equal. If, for example, $\vartheta_1 = \vartheta_2$, then we obtain

$$\begin{aligned}
0 &= \vartheta_1\left[(ab + cd)e\overline{e} + (a + b)(c + d) - (ac + bd)e\overline{e} - (a + c)(b + d)\right] \\
&= \vartheta_1(e\overline{e} - 1)(a - d)(b - c),
\end{aligned}$$

whence $e\bar{e} = 1$, that is $E \in \Gamma$. The other cases $\vartheta_2 = \vartheta_3$ and $\vartheta_3 = \vartheta_1$ are analogous.

From Corollary 2 and Corollary 4, the equivalence stated in the problem follows.

OC18. Suppose $a_1, a_2, \ldots, a_n$ are n non-zero complex numbers, not necessarily distinct, and k, l are distinct positive integers such that $a_1^k, a_2^k, \ldots, a_n^k$ and $a_1^l, a_2^l, \ldots, a_n^l$ are two identical collections of numbers. Prove that each a_j, $1 \le j \le n$, is a root of unity.
(*Originally question # 2 from the problems used in selection of the Indian team for IMO-2009.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let $1 \le j \le n$ be arbitrary. By hypothesis, we can construct inductively a sequence $i_1 = j, i_2, \ldots, i_k, \ldots$ so that for all $m \ge 1$ we have

$$a_{i_m}^k = a_{i_{m+1}}^l .$$

Then, by induction, one can easily prove that

$$a_{i_1}^{k^m} = a_{i_m}^{l^m} .$$

Since $1 \le i_m \le n$ for all m, there exists some $q < r$ such that $i_q = i_r$. Then

$$a_{i_1}^{k^r} = a_{i_r}^{l^r} = a_{i_q}^{l^r} = \left(a_{i_q}^{l^q}\right)^{l^{r-q}} = \left(a_{i_1}^{k^q}\right)^{l^{r-q}} .$$

This implies that

$$a_j^{k^r} = a_j^{k^q l^{r-q}} \Rightarrow a_j^{k^q l^{r-q} - k^r} = 1 .$$

Let's observe that $k^q l^{r-q} - k^r \ne 0$, since otherwise we would have $k^{r-q} = l^{r-q}$, and hence $k = l$. Thus, we proved that there exists some integer $p \ne 0$ so that

$$a_j^p = 1 .$$

This shows that a_j is a root of unity.

OC19. There were 64 contestants at a chess tournament. Every pair played a game that ended either with one of them winning or in a draw. If a game ended in a draw, then each of the remaining 62 contestants won against at least one of these two contestants. There were at least two games ending in a draw at the tournament. Show that we can line up all the contestants so that each of them has won against the one standing right behind him.
(*Originally question # 3 from 53rd national mathematical Olympiad in Slovenia, 3rd selection Exam for the IMO 2009.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

We prove by induction that the claim holds for any number $n \ge 4$ of contestants.

Let us write $A > B$, $A < B$, or $A \sim B$ if A won against B, A lost against B, or A tied with B, respectively. Let $\mathcal{P}$ denote the property that, if a game ended in a draw, then each of the remaining $n-2$ players won against at least one of these two contestants. Observe that no player tied with more than one other player. For, if A tied with B and C, where $B \neq C$, then by hypothesis $\mathcal{P}$, we have $B > C$ and $C > B$, a contradiction.

We are now ready to prove our initial claim by mathematical induction.

In the base case there are four players A, B, C, and D. Since no contestant had more than one tie, there is no loss of generality in assuming $A \sim B$ and $C \sim D$. By symmetry we may further assume $A > C$. Condition $\mathcal{P}$ now yields $C > B$ and $B > D$. Then,

$$A > C > B > D$$

is a chain with the desired property, which completes the base case.

For the induction step, assume that the claim holds for each tournament of $n = k \geq 4$ contestants. Consider a tournament of $k+1$ players $A_1, \ldots, A_{k+1}$. If each contestant had a game that ended in a draw, then the total number of draws is

$$\left\lceil \frac{k+1}{2} \right\rceil \geq 3,$$

and there are at least two draws in the subtournament of the players $A_1, \ldots, A_k$. Otherwise there is a player, say A_{k+1}, without any draws. In each case there are two draws in the subtournament of the contestants $A_1, \ldots, A_k$. By the induction hypothesis, these k contestants can be arranged in a descending chain, say, $A_1 > \cdots > A_k$. By the property $\mathcal{P}$, the contestant A_{k+1} won against another player. Let i be the least index such that $A_{k+1} > A_i$.

If $i = 1$, then

$$A_{k+1} > A_1 > A_2 > \cdots > A_k$$

is a descending chain of length $k+1$ and we are done.

It remains to consider $i > 1$. If $A_{i-1} \sim A_{k+1}$ then A_i lost against both A_{i-1} and A_{k+1}, which contradicts $\mathcal{P}$. Thus, $A_{i-1} > A_{k+1}$. We obtain the chain

$$A_1 > \cdots > A_{i-1} > A_{k+1} > A_i > \cdots > A_k.$$

This completes the induction.

OC20. Given an integer $n \geq 2$, determine the maximum value the sum $x_1 + x_2 + \cdots + x_n$ may achieve, as the x_i run through the positive integers subject to $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.
(*Originally question #10 from 60th national mathematical Olympiad selection tests for the Balkan and IMO, 4th selection test.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Zvonaru.

We denote
$$s_n = x_1 + x_2 + \cdots + x_n \,.$$
We claim that the maximum value s_n can achieve is $2n$.

If $x_1 = 0$ then $x_2 + \cdots + x_n = 0$, thus $s_n = 0$.

Now, suppose $x_1 \geq 1$. If $x_n = 1$, then $x_1 = ... = x_n = 1$ and hence $n = x_1 + \cdots + x_n = x_1 \cdots x_n = 1$, a contradiction to $n \geq 2$. Hence, for the rest of the proof we can assume
$$1 \leq x_1 \leq x_2 \leq .. \leq x_n \text{ and } x_n \geq 2 \,.$$
For $n = 2$, the equality $x_1 + x_2 = x_1 x_2$ is equivalent to $(x_1 - 1)(x_2 - 1) = 1$, hence $x_1 = x_2 = 2$ and $s_2 = 4$.

Suppose now that $n \geq 3$. It is easy to see that the equality
$$x_1 + \cdots + x_n = x_1 \cdots x_n$$
can be rewritten as
$$(x_1 - 1)(x_2 - 1) + (x_1 x_2 - 1)(x_3 - 1) + \cdots + (x_1 x_2 \cdots x_{n-1} - 1)(x_n - 1) = n - 1 \,.$$
Thus
$$(x_1 x_2 \cdots x_{n-1} - 1)(x_n - 1) \leq n - 1 \,. \tag{1}$$
Since $(x_1 x_2 \cdots x_{n-1} - 1) x_n = x_1 + \cdots + x_{n-1} \neq 0$,
$$x_n - 1 \leq n - 1 \Rightarrow x_n \leq n \,.$$
Now, substituting $x_1 \cdots x_{n-1} = \dfrac{s_n}{x_n}$ into (1), we get
$$\begin{aligned} \left(\frac{s_n}{x_n} - 1\right)(x_n - 1) \leq n - 1 &\Leftrightarrow s_n \frac{x_n - 1}{x_n} \leq n - 1 + x_n - 1 \\ &\Leftrightarrow s_n \leq \frac{x_n^2 + (n-2)x_n}{x_n - 1} \,. \end{aligned}$$
Let's observe that
$$\begin{aligned} \frac{x_n^2 + (n-2)x_n}{x_n - 1} \leq 2n &\Leftrightarrow x_n^2 - (n+2)x_n + 2n \leq 0 \\ &\Leftrightarrow (x_n - 2)(x_n - n) \leq 0 \,, \end{aligned}$$
which is true since $2 \leq x_n \leq n$. This shows that $s_n \leq 2n$.

Moreover, when $x_n = n$, $x_{n-1} = 2$, $x_{n-2} = \cdots = x_1 = 1$, we have $s_n = 2n$, which shows that $2n$ is the maximum value the sum can achieve.

BOOK REVIEWS

Amar Sodhi

Rediscovering Mathematics: You Do the Math by Shai Simonson
Classroom Resource Materials, Mathematical Association of America, 2011
ISBN 978-0-88385-770-0, Hardcover, 207 + xxxi pages, US$64.95
ISBN 978-0-88385-912-4, e-book, 207 + xxxi pages, US$39
ISBN 978-0-88385-912-4, Softcover, 207 + xxxi pages, US$42
Reviewed by ***Edward Barbeau****, University of Toronto, Toronto, ON*

How can we encourage students to get beyond a mindless approach to mathematics and become active learners who will strive for insight, understanding and creativity? This book, whose author teaches computer science at Stonehill College in Easton, MA, addresses this question explicitly. He directs his book to teachers who he hopes will be able to "reshape the popular perception of mathematics – one child at a time" as well as anyone "looking for a guide to revisit and reconsider mathematics". The background required is arithmetic, basic algebra and some geometry; the reader is assumed to have little experience of mathematics beyond the traditional school classroom.

He begins with advice: don't miss the big picture; don't be passive; slow down; own the mathematics. This is illustrated by an imaginary (and somewhat artificial) exchange between a professional and reader of a mathematical passage. Then the mathematics, a mixed bag of topics, is developed through a sequence of questions, exercises and problems designed to encourage participation. Most of the material will be familiar to readers of this journal: repeating and terminating decimals, averages, mental computation, checking, pythagorean triples, fractions, convergent and divergent series, numerical patterns, rates, variation, percentages, algorithms, Pythagoras' theorem, quadratic equations, probability, three-dimensional solids and area. There are a few snippets of greater interest, such as near misses to Fermat's theorem (e.g. $13^5 + 16^5 = 17^5 + 12$), caroms, the RSA encryption method and solids whose faces are all pentagons and hexagons. Although the treatment is salted with some anecdotes and points of information, it mainly consists of a graded sequence of exercises, each immediately followed by the solution. Although the reader is strongly encouraged to stop and consider the question before looking at the solution, I wonder whether this will in fact occur.

The difficulty with this approach is that a book is a one-way communication from author to reader. Any learning situation requires the teacher and learner to find common ground from which to proceed; each comes with his own worldview. The danger is that, without an opportunity for negotiation, the teacher can proceed to develop a topic without being aware that the reader has hit a stumbling block because of a fundamental difference in outlook. In the present book, I can see this occurring in the discussions on averages and probabilistic expectation. On the other hand, a two-jug liquid pouring problem makes a nice entrée to the Euclidean algorithm. A highly disciplined reader who is willing to struggle with difficulties will gain from this book, but much of the book reads like a regular textbook.

It might have been a better strategy to defer discussion and solutions to the end of the chapter. Sometimes we come to understand gradually or haphazardly, by going ahead, getting stuck and backtracking. Thus, the reader should be discouraged from looking for outside help too soon. Where feasible, students should be helped to construct their own examples. Perhaps the presentation can be punctuated by blanks for the student to fill in or the occasional "why?" inserted into an argument. One author who has done this with greater success is A.D. Gardiner [1, 2].

This book would be suitable for secondary teachers looking for advice and material to encourage more independence among their students, as well as college teachers who have to teach an appreciation course to a general audience.

References

[1] A.D. Gardiner, *Discovering Mathematics: The Art of Investigation*, Oxford, 1987.

[2] A.D. Gardiner, *Mathematical Puzzling*, Oxford,1987.

What's Luck Got to Do with It? The History, Mathematics, and Psychology of the Gambler's Illusion by Joseph Mazur
Princeton University Press, 2011
ISBN: 978-1-40083-445-7, hardcover, 277 + xvii pages, US$29.95
Reviewed by ***Dave Ehrens****, Macalester College, St. Paul, MN, USA*

In addressing the question of why people gamble, or more specifically why people believe they have to win, Joseph Mazur delves into a great deal of history, with enough mathematics and psychology, to make us think that the problem of gambling won't be solved any time soon. The colorful stories from history, mixed with formulas and theories, give us quite a picture of one of our favorite addictions.

Part 1: The History

The book starts with historical and anthropological evidence of games of chance, including a number of histories of words we now use. I was familiar with "knucklebones", but "your lot in life" has a game of chance in its background as well. If the courtier loudly says, "The king has left the building", but you see the king at the next table, then it means that it's legal to gamble. The pendulum swings of the attitudes towards gambling in law through the centuries are displayed in a nice way, full of anecdotes from many time periods. A culture may have laws against gambling because of societal pressure or because the royalty want to keep it to the upper class.

In modern eras, gambling was logically related to the invention of insurance for the large shipping firms. Betting on the markets and betting on dice were shown to be similar to the addicted gambling man. Now, much like a good documentary, the author selects details to show the reader from a vast array of possibilities. He concentrates on certain colorful times and places, showing the similar effects of addiction to gambling.

In this chapter he makes quite clear the fallacy in his subtitle. If a person is successful at a game or investment, he may believe it is because he is good at it, or that the luck he has experienced is part of him instead of random. Positive reinforcement through arbitrary effect may make someone feel invincible. This leads to tremendous losses at the tables and in the markets.

Part 2: The Mathematics

The second part of the book starts with some classics of probability classes, with sampling and spaces and fractions, but all the math is mixed in with anecdotes about teachers and gamblers, making it much more palatable for the lay reader than a typical probability text. A small scene from *Casablanca* is well placed before the discussion switches from coin flipping to the more complex roulette.

The mathematical section is not long and, due to the author's manner of presentation, one could easily move past the formulas if the mathematics wasn't of much interest to the reader. Then again, if the formulas were at the right level for the reader, say a clever middle school student or a person well versed in algebra, then the book is deficient in providing enough to learn the math. It does, however, provide enough vocabulary to lead you to an online or library search.

Part 3: The Analysis

A description of the whole book can be found in just Part 3. The complex ideas of mathematics of psychology are introduced, and the book does not intend to draw scientific conclusions. Instead, it illustrates the steps that one might take to reach the conclusions. This is a fine way to introduce people to the discussion, and perhaps convince them of the mistakes people make when they assume too much about their success in gambling.

Mazur explains some of the reasoning that gamblers use, illustrating differences that are not differences in math, but are differences in the mind. Using a wide variety of anecdotes from Dostoyevsky to *Deal or No Deal*, we see a number of important points. Is it easier to gamble with your own money, or the house's? Does it make a difference whether you think of the money as already won?

A weakness in this section might be found in the lack of psychological vocabulary. In section one there are actual historical faces. In section two we have a nice variety of mathematical formulae and definitions. While the third section has vivid descriptions of places and events, you don't feel like the conclusions are grounded in recent science simply because of the lack of use of technical vocabulary. That said, the terms used are easy to digest by the general public, and the points come across clearly.

Copious appendices at the end of the book contain useful information, but may muddy the message. The index of games provided is useful, as is the glossary of gambling terms. The mathematical formulae provided seem to vary dramatically in their detail. Some take a great deal of time on what could be a simple matter, while other explanations seem to make large leaps to conclusions. Having taught many of the listed ideas, I'm probably not the target audience, but the lack of consistent depth was a bit disturbing. The endnotes were useful but divided into "Notes" and "Callouts". I couldn't tell why one piece of information would get one label or the other, but each one was quite useful and entertaining.

I found "What's Luck got to Do with it?" to be entertaining and engaging. I don't think that it will convince people with gambling problems to change their ways, but the bits of history, with gambling in high class and low, may be a satisfying distraction for them.

Unsolved Crux Problems

In Crux, no problem is ever closed . . . some don't even get opened! Below are two more problems from the vaults that have defied solvers to date. Good luck!

909*. [1984 : 20; 1985 : 94–95] *Proposed by Stan Wagon, Smith College, Northampton, Massachusetts, USA.*

For which positive integers n is it true that, whenever an integer's decimal expansion contains only zeros and ones, with exactly n ones, then the integer is not a perfect square?

1077*. [1985 : 249; 1987 : 93] *Proposed by Jack Garfunkel, Flushing, NY, USA.*

For $i = 1, 2, 3$, let C_i be the centre and r_i the radius of the Malfatti circle nearest A_i in triangle $A_1A_2A_3$. Prove that

$$A_1C_1 \cdot A_2C_2 \cdot A_3C_3 \;\geq\; \frac{(r_1 + r_2 + r_3)^3 - 3r_1r_2r_3}{3}\,.$$

When does equality occur?

RECURRING CRUX CONFIGURATIONS 6

J. Chris Fisher

Triangles for Which OI Is Parallel to a Side

Triangle ABC has sides a, b, and c, circumcentre O, circumradius R, incentre I, inradius r, orthocentre H, and centroid G. We will explore how these points and parameters are related when $OI||BC$, beginning with two equivalent properties.

Property 1. *In any triangle that is not equilateral, $OI||BC$ if and only if $AI \perp IH$.*

Crux readers were introduced to this property by Leon Bankoff in Problem 659 [1981 : 204; 1982 : 215-216]. He essentially proved it in [4] in order to deduce

Property 2. *In any triangle that is not equilateral, $OI||BC$ if and only if $\cos B + \cos C = 1$.*

Property 2 is simply the familiar formula $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ applied to the triangle for which O and I are on the same side of BC and the distance OD shown in the accompanying figure, namely the distance from O to BC, equals r (whence $\cos A = \frac{r}{R}$). It is easily seen that any of the three equivalent properties implies that *the triangle must be acute*: B and C must be acute angles for their cosines to sum to 1; from $\cos A = \frac{r}{R}$ and Euler's inequality ($2r < R$), we deduce that $60^\circ < \angle A < 90^\circ$. For his proof of Property 1 Bankoff used $IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$, which is valid for any triangle.

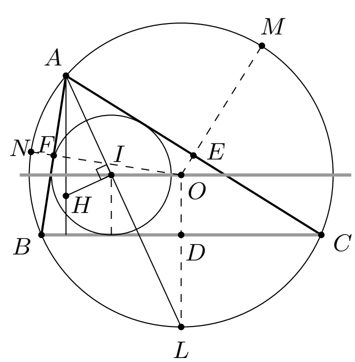


Figure 1: Triangle ABC has $OI\|BC$ and $AI \perp IH$.

Problem 388 [1978: 251; 1979: 201-202] (proposed by W.J. Blundon). A line containing the circumcentre and the incentre of a triangle is parallel to a side of the triangle if and only if

$$s^2 = \frac{(2R-r)^2(R+r)}{R-r}.$$

The proof by Orlando Ramos begins with the observation that a line containing the circumcentre and incentre is parallel to a side if and only if

$$(R\cos A - r)(R\cos B - r)(R\cos C - r) = 0.$$

He reduced this equality to the desired form with the help of formulas involving sums of cosine products that can be found in resources devoted to triangle geometry.

Problem 659 [1981: 204; 1982: 215-216] (proposed by Leon Bankoff). In any triangle in which $OI||BC$,

(a) if the internal bisector of angle A meets the circumcircle in L, $\frac{AI}{AL} = \cos A$;

(b) the circumcircle of triangle AIH has radius r;

(c) $AI \cdot AL = 2Rr = \sqrt{R \cdot AI \cdot BI \cdot CI}$;

(d) $\sin^2\frac{B}{2} + \sin^2\frac{C}{2} = \frac{1}{2}$; $\quad \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{3}{2}$;

(e) $\tan^2\frac{A}{2} = \frac{R-r}{R+r}$.

Kesiraju Satyanarayana observed (in a comment to the following problem) that because the radius of the excircle opposite vertex A, call it r_1, satisfies $r_1 = \tan\frac{A}{2}$, it follows from the equations in Problems 388 and 659(e) that *if the line OI is parallel to a side, then*

$$R = \frac{r + r_1}{2}.$$

The next problem also follows easily from Problem 659.

Problem 660 [1981: 204, 274; 1982: 216-217] (Proposed by Leon Bankoff). As shown in the figure, we denote the midpoints of the arcs BC, CA, AB by L, M, N, and the midpoints of the sides by D, E, F. Then $IO||BC$ if and only if

$$DL + EM + FN = s\tan\frac{A}{2}.$$

Problem 1751 [1992: 175; 1993: 208-209; 1994: 297] (Proposed by Toshio Seimiya). If ABC is an acute triangle for which $AB < AC$ and $IO = \frac{1}{2}(AC - AB)$, then

$$\text{area}(IAO) = \frac{1}{2}[\text{area}(ABO) - \text{area}(CAO)].$$

The solution by Dou and Heuver showed in the first step that the condition $IO = \frac{1}{2}(AC - AB)$ is equivalent to $IO||BC$. (It seems likely that the acuteness of the angles can be proved and could therefore be omitted from the assumptions.)

Problem 1773 [1992: 237; 1993: 148-149] (Proposed by Toshio Seimiya). As in Problem 660 we let L, M, N denote the midpoints of the arcs BC, CA, AB not containing A, B, C, respectively. Let P and Q be the feet of the perpendiculars from L to AB and AC, respectively. If $NP = MQ$ but $AB \neq AC$, then $MQ = R$.

The solution established that $NPLO$ and $OLQM$ are parallelograms. As a byproduct of the proof it was noted that the conditions of this problem ($NP = MQ$ but $AB \neq AC$) are equivalent to requiring $\cos B + \cos C = 1$ in a nonequilateral triangle, and we have yet another property that our class of triangles possesses.

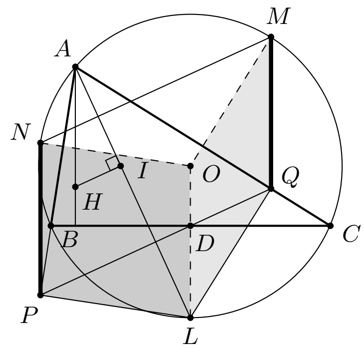


Figure 2: Triangle ABC has $NP = MQ$ but $AB \neq AC$.

Problem 3520 [2010: 108, 110; 2011: 123-124] (proposed by Ricardo Barroso Campos) called for a construction of a triangle ABC for which $IO||BC$, without specifying the given data. Readers provided the unique solutions given

- O, R, and $\angle A$ with $60° < \angle A < 90°$ — use the fact that the midpoint of the arc BC is equidistant from B, C, O, and I;

- O, R, and the distance d_c from O to the side AB — use $d_b + d_c = R$, which is obtained from $\cos B + \cos C = 1$ by multiplying both sides by R;

- O, I, and r — use $R - r$ equals the distance from O to the foot of the perpendicular from I to BC;

- a and an acute $\angle B$ — use $\cos B + \cos C = 1$;

- b and c — apply the cosine law to $\cos B + \cos C = 1$ and solve for a to obtain

$$a = \frac{bc + \sqrt{b^4 - b^2c^2 + c^4}}{b + c}.$$

Shailesh Shirali commented that the preceding equation for a implies that there are no integer-sided triangles having $IO||BC$ (because the expression under the radical sign has an integer square root only if $b = c = 0$).

Concluding Remarks. In the second essay of this series we discussed triangles in which the Nagel line is parallel to BC: $IG||BC$ if and only if $AI \perp IO$; in such triangles the sides are in arithmetic progression whence, equivalently, the sines of its angles are in arithmetic progression. If, however, in a triangle the tangents of the angles are in arithmetic progression, then it is the Euler line that is parallel to a side. More precisely, in a nonequilateral triangle, $OG||BC$ if and only if $\tan B + \tan C = 2\tan A$ which, in turn, is equivalent to $\tan B \tan C = 3$. Apparently this last triangle class has appeared among the problems of **Crux** only as part of Problem 2432 [1999: 173; 2000: 186-187]; its properties can be explored in textbooks such as [2], and in other journals such as [3] and [1]. In [1] the authors show that when segment BC is fixed, the locus of points A for which triangle ABC has $OG||BC$ is an ellipse minus its four vertices; the minor axis of the ellipse is BC while its major axis has length $\sqrt{3}BC$. Finally, in the ninth essay in this series we shall see that $\tan B \tan C = -1$ if and only if $OA||BC$; the locus of points A for which triangle ABC has $OA||BC$ is a rectangular hyperbola (minus its vertices) with major axis BC.

References

[1] Wladimir G. Boskoff and Bogdan D. Suceavă, When is Euler's Line Parallel to a Side of a Triangle? *College Mathematics Journal*, **35**:4 (Sep. 2004) 292-296.

[2] H.S.M. Coxeter, *Introduction to Geometry*, Wiley, 1969, Exercise 1.6.8, pages 18, 422.

[3] Robert Glaskell, Solution to Problem E259 (proposed by Mannis Charos, 1937, p. 104), *American Mathematical Monthly*, **44**:8 (1937) 541.

[4] R. Sivaramakrishnan, Problem 758, *Mathematics Magazine*, **43**:5 (Nov. 1970) 285-286. Solution by Leon Bankoff. (Appeared in Mar. 1970.)

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobret 2013***. Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7, et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8, et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal et Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

3707. *Correction. Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit k un entier positif. Montrer que

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2 - x^2)((2\pi)^2 - x^2)\cdots((k\pi)^2 - x^2)}dx = 0\,.$$

3731. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs tels que $a + b + c = 1$. Montrer que

$$a^{n+1} + b^{n+1} + c^{n+1} \geq (a^2 + b^2 + c^2)^n,$$

pour tous les entiers non négatifs n.

3732. *Proposé par Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan.*

Un cercle de rayon 1 roule sur l'axe des x en direction de la parabole d'équation $y = x^2$. Trouver les coordonnées du point de contact lorsque le cercle touche la parabole.

3733. *Proposé par Angel Dorito, Geld, ON.*

Soit f et g deux polynômes distincts, non constants et de degré au plus 2, de sorte que $f(x) - g(x) = f(g(x)) - g(f(x))$ pour tous les nombres réels x. Montrer qu'exactement un des deux, f ou g, doit être linéaire et trouver toutes les valeurs de sa pente.

3734. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

On donne une paire de triangles ABC et DEF avec un point D' sur BC.

(a) Décrire comment localiser les points E' sur CA et F' sur AB de telle sorte que $\Delta D'E'F'$ soit directement semblable à ΔDEF.

(b) Pour quel point D' sur BC l'aire de $\Delta D'E'F'$ (voir (a)) est-elle minimale ?

3735. *Proposé par Yakub N. Aliyev, Université de Qafqaz, Khyrdalan, Azerbaïdjan.*

Soit A, B, C et D quatre points sur une droite ℓ, pris dans cet ordre, et soit M un point non sur ℓ de sorte que $\angle AMB = \angle CMD$. Montrer que

$$\frac{\sin \angle BMC}{\sin \angle AMD} > \frac{|BC|}{|AD|}.$$

3736. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $f : [0, \infty) \to [0, \infty)$ une fonction continue bornée. Evaluer

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x}\, dx}\,.$$

3737. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

On donne quatre nombres réels a, b, c, d. Trouver le plus grand nombre k tel que l'inégalité suivante soit satisfaite.

$$\frac{1}{a^3+b^3} + \frac{1}{b^3+c^3} + \frac{1}{c^3+d^3} + \frac{1}{a^3+c^3} + \frac{1}{b^3+d^3} + \frac{1}{a^3+d^3} \geq \frac{k}{(a+b+c+d)^3}.$$

3738. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle inscrit dans le cercle Γ et soit respectivement A' et B' les pieds des perpendiculaires abaissées de A et B. Le cercle de diamètre BA' coupe une seconde fois BB' en M et Γ en P. Montrer que A, M et P sont colinéaires.

3739. *Proposé par Cristinel Mortici, Valahia Université de Târgovişte, Roumanie.*

Soit $f, g : \mathbb{R} \to \mathbb{R}$ deux fonctions strictement monotones. Montrer que

(i) Si $f \circ f + g \circ g$ est continue, alors f et g sont continues.

(ii) Si $f \circ g + g \circ f$ est continue, alors f et g sont continues.

3740. *Proposé par Yunus Tuncbilek, Ataturk High School of Science, Istanbul, Turkey.*

Dans un triangle ΔABC, soit respectivement R, r, r_a, r_b et r_c les rayons des cercles circonscrit, inscrit et exinscrits. Trouver le plus grand k satisfaisant

$$r_a^2 + r_b^2 + r_c^2 + (1 + 4k)r^2 \geq (7 + k)R^2.$$

. . . .

3707. *Correction. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let k be a positive integer. Prove that

$$\int_0^\infty \frac{\sin^{2k} x}{(\pi^2 - x^2)((2\pi)^2 - x^2)\cdots((k\pi)^2 - x^2)}dx = 0\,.$$

3731. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$a^{n+1} + b^{n+1} + c^{n+1} \geq (a^2 + b^2 + c^2)^n,$$

for all nonnegative integers n.

3732. *Proposed by Ataev Farrukh Rakhimjanovich, Westminster International University, Tashkent, Uzbekistan.*

A circle of radius 1 is rolling on the x-axis in the first quadrant towards the parabola with equation $y = x^2$. Find the coordinates of the point of contact when the circle hits the parabola.

3733. *Proposed by Angel Dorito, Geld, ON.*

Suppose that f and g are different, nonconstant polynomials of degree at most 2 so that $f(x) - g(x) = f(g(x)) - g(f(x))$ for all real numbers x. Prove that exactly one of the two functions, f or g, must be linear and find all possible values of its slope.

3734. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given a pair of triangles ABC and DEF with a point D' on BC.

(a) Describe how to locate points E' on CA and F' on AB such that $\Delta D'E'F'$ is directly similar to ΔDEF.

(b) For which point D' on BC is the area of $\Delta D'E'F'$ (from part (a)) a minimum?

3735. *Proposed by Yakub N. Aliyev, Qafqaz University, Khyrdalan, Azerbaijan.*

Let A, B, C, and D be points on line ℓ in this order, and let M be a point not on ℓ such that $\angle AMB = \angle CMD$. Prove that

$$\frac{\sin \angle BMC}{\sin \angle AMD} > \frac{|BC|}{|AD|}.$$

3736. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0, \infty) \to [0, \infty)$ be a bounded continuous function. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^\infty \ln^n(1 + e^{nf(x)})e^{-x}\,dx}\,.$$

3737. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Four nonnegative real numbers a, b, c, d are given. Find the greatest number k such that the following inequality is valid.

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+d^3}+\frac{1}{a^3+c^3}+\frac{1}{b^3+d^3}+\frac{1}{a^3+d^3}\geq\frac{k}{(a+b+c+d)^3}.$$

3738. *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in circle Γ and let A' and B' be the feet of the altitudes from A and B, respectively. Let the circle with diameter BA' intersect BB' a second time at M and Γ at P. Prove that A, M, P are collinear.

3739. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*

Let $f, g : \mathbb{R} \to \mathbb{R}$ be strictly monotone functions. Prove that:

(i) If $f \circ f + g \circ g$ is continuous, then f and g are continuous.

(ii) If $f \circ g + g \circ f$ is continuous, then f and g are continuous.

3740. *Proposed by Yunus Tuncbilek, Ataturk High School of Science, Istanbul, Turkey.*

Let R, r, r_a, r_b, r_c represent the circumradius, inradius and exradii, respectively, of ΔABC. Find the largest k that satisfies

$$r_a^2+r_b^2+r_c^2+(1+4k)r^2\geq(7+k)R^2.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3492 $\star$. [2009 : 515, 518; 2010 : 558] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let P be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB, \angle PBC, \angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that P is its centroid.

Solution by Tomasz Cieśla, student, University of Warsaw, Poland.

The claim is not true; we shall see that there are infinitely many counterexamples. Consider points A', B, C on a circle with centre P such that $\angle PA'B = \angle PBA' = \angle PBC = \angle PCB = \arccos\sqrt{\frac{2}{3}}$. For $\alpha \in (0, \pi)$ the rotation about line PB through the angle α maps point A' into a point A (preserving angles $PA'B$ and PBA'). Let D be the reflection of point B in the plane PCA (so that $\angle PCD = \angle PCB$ and $\angle PDA = \angle PBA = \angle PBA'$.) Then the tetrahedron $ABCD$ satisfies

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \arccos\sqrt{\frac{2}{3}},$$

as required. Note that we can choose α so that $ABCD$ is not regular; in fact, there is only one value of α which produces a regular tetrahedron (occurring when $\angle PAC = \arccos\sqrt{\frac{2}{3}}$). Values of α close to that ensure that P lies in the interior of $ABCD$ and, therefore, provide counterexamples to the problem as it was stated.

No other correspondence about this problem has been received.

3631. [2011: 171, 173] *Proposed by Michel Bataille, Rouen, France.*

Let $\{x_n\}$ be the sequence satisfying $x_0 = 1$, $x_1 = 2011$, and $x_{n+2} = 2012x_{n+1} - x_n$ for all nonnegative integers n. Prove that

$$\frac{(2010 + x_n^2 + x_{n+1}^2)(2010 + x_{n+2}^2 + x_{n+3}^2)}{(2010 + x_{n+1}^2)(2010 + x_{n+2}^2)}$$

is independent of n.

Solution by Arkady Alt, San Jose, CA, USA.

More generally, let a be an integer and let $\{x_n\}$ be determined by $x_{-1} = x_0 = 1$, $x_1 = a + 1$ and $x_{n+2} = (a + 2)x_{n+1} - x_n$ for $n \geq 0$. Since

$$\begin{aligned} x_n x_{n+2} - x_{n+1}^2 &= x_n[(a+2)x_{n+1} - x_n] - x_{n+1}^2 = x_{n+1}[(a+2)x_n - x_{n+1}] - x_n^2 \\ &= x_{n-1}x_{n+1} - x_n^2, \end{aligned}$$

it follows that

$$x_n x_{n+2} - x_{n+1}^2 = x_0 x_2 - x_1^2 = (a^2 + 3a + 1) - (a + 1)^2 = a$$

for $n \geq 0$. Therefore, for $n \geq 0$,

$$\begin{aligned}
\frac{(a + x_n^2 + x_{n+1}^2)(a + x_{n+2}^2 + x_{n+3}^2)}{(a + x_{n+1}^2)(a + x_{n+2}^2)} &= \frac{(x_{n-1}x_{n+1} + x_{n+1}^2)(x_{n+1}x_{n+3} + x_{n+3}^2)}{(x_n x_{n+2})(x_{n+1}x_{n+3})} \\
&= \frac{x_{n+1}(x_{n-1} + x_{n+1})x_{n+3}(x_{n+1} + x_{n+3})}{(x_{n+1}x_{n+3})(x_n x_{n+2})} \\
&= \frac{[(a+2)x_n][(a+2)x_{n+2}]}{x_n x_{n+2}} = (a+2)^2.
\end{aligned}$$

Taking $a = 2010$ yields the value 2012^2 for the expression in the problem.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comănești, Romania; and the proposer. Apostolopoulos and the proposer had solutions similar to the one given, while the remaining solvers solved the recursion and used the formula for the general term. One additional person simply gave the answer with no justification.

3632. [2011: 171, 173] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let k be a real number such that $0 \leq k \leq 56$. Prove that the equation below has exactly two real solutions:

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; k(x^2 - 7x) + 720\,.$$

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

We prove that the result holds as long as $k < 945/16 = 59.0625$. The difference $P(x)$ betweeen the two sides of the equation is given by

$$\begin{aligned}
P(x) &= [(x-1)(x-6)][(x-2)(x-5)][(x-3)(x-4)] - k(x^2 - 7x) - 720 \\
&= [(x^2 - 7x) + 6][(x^2 - 7x) + 10][(x^2 - 7x) + 12] - k(x^2 - 7x) - 720 \\
&= (x^2 - 7x)[(x^2 - 7x)^2 + 28(x^2 - 7x) + (252 - k)] \\
&= \frac{1}{16}(x^2 - 7x)\{[(2x-7)^2 - 49]^2 + 112[(2x-7)^2 - 49] + (4032 - 16k)\} \\
&= \frac{1}{16}x(x-7)[(2x-7)^4 + 14(2x-7)^2 + (945 - 16k)].
\end{aligned}$$

When $k < 945/16$, the final factor is positive for all real x, so that the only real roots of $P(x)$ are 0 and 7.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, *USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comăneşti, Romania; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.*

It is easy to determine the situation for other values of k. When $k = 945/16$, then $x = 7/2$ is an additional root of $P(x)$. When $k > 945/16$, then the final factor vanishes for one positive and one negative value of $(2x-7)^2$. The positive value gives rise to two real roots of the final factor. However, these two roots are 0 and 7 when $k = 252$. Thus, $P(x)$ has two roots if and only if $k < 945/16$ or $k = 252$, three roots when $k = 945/16$ and four roots when $k \neq 252$ and $k > 945/16$.

About half of the solvers identified values of k less than 945/16 as yielding two solutions, but only four picked up $k = 252$. Nine solvers employed the substitution $y = x^2 - 7x$ which led to their analyzing the equation $y(y^2 + 28y + 252 - k) = 0$, while Geupel and Stadler let $u = \frac{x+7}{2}$ and analyzed the equation $(u^2 - 49)(u^4 + 14u^2 + 945 - 16k) = 0$.

3633. [2011 : 171, 173] *Proposed by Razvan Tudoran, Universitatea de Vest din Timisoara, Timisoara, Romania; and Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $g_1(x) = x$ and for natural numbers $n > 1$ define $g_n(x) = x^{g_{n-1}(x)}$. Let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(x) = g_n(x)$, where $n = \left\lfloor \frac{1}{x} \right\rfloor$. For example, $f\left(\frac{1}{3}\right) = \frac{1}{3}^{\frac{1}{3}^{\frac{1}{3}}}$. Here $\lfloor a \rfloor$ denotes the floor of a. Determine $\lim\limits_{x\to 0^+} f(x)$ or prove it does not exist.

[*Ed.: Note when the problem was published, Razvan Turdoran's name was mistakenly omitted from the problem. Our apologies to Razvan.*]

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified slightly by the editor.

The limit does not exist. We consider $\lim\limits_{x\to 0^+} f(x)$ when $x = \dfrac{1}{2n+1}$ and $x = \dfrac{1}{2n}$ separately where $n \in \mathbb{N}$.

In the published solution to problem #922 of the *College Mathematics Journal* (Vol. 42, No. 2, March 2011; pp. 152-155) the following results were obtained:

(a) $$\frac{1}{2n+1} < f\left(\frac{1}{2n+1}\right) < \frac{1}{\ln(2n+1)},$$

(b) $$\left(\frac{1}{2n}\right)^{\frac{1}{\ln(2n)}} < f\left(\frac{1}{2n}\right) < \left(\frac{1}{2n}\right)^{\frac{1}{2n}}.$$

From (a) $\lim\limits_{n\to\infty} f\left(\dfrac{1}{2n+1}\right) = 0$ follows immediately. Hence, if $\lim\limits_{x\to 0^+} f(x)$ exists, then letting $n = \left\lfloor \dfrac{1}{x} \right\rfloor$ we must have $\lim\limits_{n\to\infty} f\left(\dfrac{1}{2n}\right) = 0$ which is impossible in view of (b) since $\lim_{n\to\infty}\left(\frac{1}{2n}\right)^{\frac{1}{2n}} = 1$ [*Ed: This can be shown easily by using the l'Hôpital's rule.*] and $\left(\frac{1}{2n}\right)^{\frac{1}{\ln(2n)}} = e^{-1} > 0$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposers.

The proposers of the current problem are the same as the problem in the College Mathematics Journal. They simply combine the results in (a) and (b) and came to the immediate conclusion about $\lim_{x\to 0^+} f(x)$.

3634. [2011: 171, 174] *Proposed by Michel Bataille, Rouen, France.*

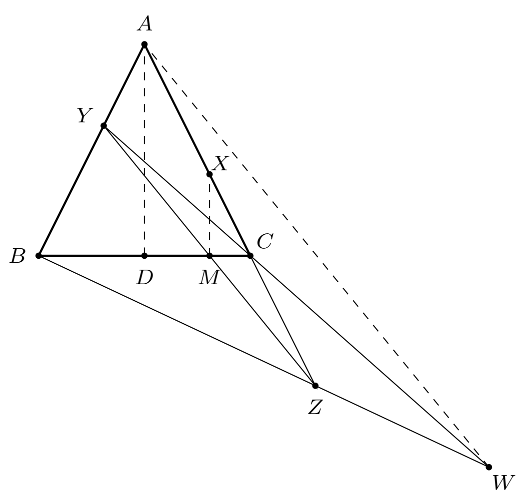
ABC is an isosceles triangle with $AB = AC$. Points X, Y and Z are on rays $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{AC}$ respectively with $AZ > AC$ and $AX = BY = CZ$.

(a) Show that the orthogonal projection of X onto BC is the midpoint of YZ.

(b) If BZ and YC intersect in W, show that the triangles CYA and CWZ have the same area.

Composition of solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Titu Zvonaru, Cománeşti, Romania.

First, we will assume that $X \neq C$, as otherwise the problem becomes trivial. Let D be the midpoint of BC, and let YZ and BC intersect in M.



a) By Menelaus' Theorem applied to $\triangle AYZ$ and the transversal $B-M-C$, we have: $\frac{AB}{BY}\cdot\frac{YM}{MZ}\cdot\frac{ZC}{CA} = 1$ so $\frac{YM}{MZ} = 1$; hence M is the midpoint of YZ. Similarly, Menelaus' Theorem applied to $\triangle ABC$ and the transversal $Z - M - Y$ yields:

$$\begin{aligned}\frac{AY}{YB}\cdot\frac{BM}{MC}\cdot\frac{CZ}{ZA}=1 \;&\Leftrightarrow \frac{BM}{MC}=\frac{ZA}{AY}=\frac{AC+ZC}{AC-ZC}\\ &\Leftrightarrow \frac{BM+MC}{MC}=\frac{2AC}{AC-AX}\\ &\Leftrightarrow \frac{2DC}{MC}=\frac{2AC}{XC}\Leftrightarrow\frac{DC}{MC}=\frac{AC}{XC}.\end{aligned}$$

Hence $XM \parallel AD$, which proves the result that the midpoint M is the orthogonal projection of X onto BC.

b) By Ceva's Theorem applied to $\triangle YBZ$ and the point C, we have:

$$\frac{YA}{AB}\cdot\frac{BW}{WZ}\cdot\frac{ZM}{MY}=1.$$

Since M is the midpoint of YZ, we conclude that $\frac{YA}{AB}=\frac{ZW}{WB}$, which shows that $AW \parallel YZ$. Now, if $[ABC]$ denotes the area of $\triangle ABC$, then we have

$$[CYA]=[AYZ]-[CYZ]=[WYZ]-[CYZ]=[CWZ],$$

as desired.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and the proposer.

3635. [2011 : 172, 174] *Proposed by Mehmet Sahin, Ankora, Turkey.*

Let ABC be an acute-angled triangle with circumradius R, inradius r, semiperimeter s, and with points $A' \in BC, B' \in CA$, and $C' \in AB$ arranged so that

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ.$$

Prove that:

(a) $|BC'||CA'||AB'| = abc$;

(b) $\dfrac{|AA'|}{|BC'|}\dfrac{|BB'|}{|CA'|}\dfrac{|CC'|}{|AB'|}=\tan A\tan B\tan C$;

(c) $\dfrac{Area(A'B'C')}{Area(ABC)}=\dfrac{4R^2}{s^2-(2R+r)^2}-1.$

Solution by Titu Zvonaru, Comăneşti, Romania.

(a) From the right-angled triangle $AC'C$ we have

$$\cos A=\frac{AC}{AC'}=\frac{b}{c+BC'}=\frac{a\cos C+c\cos A}{c+BC'},$$

whence

$$BC' = \frac{a\cos C}{\cos A}.$$

Similarly $CA' = \dfrac{b\cos A}{\cos B}$ and $AB' = \dfrac{c\cos B}{\cos C}$, and the equation in (a) follows immediately.

(b) In right triangles ABA', BCB', and CAC' we have $AA' = c\tan B, BB' = a\tan C$, and $CC' = b\tan A$, respectively. Together with part (a) the product of these line segments produces the equality of part (b).

(c) Let us use square brackets to denote areas. Since $\angle C'BA' = 180^\circ - B$, we have (with the help of part (a))

$$\begin{aligned}[][A'BC'] = \frac{BC' \cdot BA' \cdot \sin\angle C'BA'}{2} &= \frac{a\cos C}{\cos A}\cdot\frac{c}{\cos B}\cdot\frac{\sin B}{2}\\ &= \frac{ac\sin B}{2}\cdot\frac{\cos C}{\cos A\cos B} = [ABC]\frac{\cos C}{\cos A\cos B}.\end{aligned}$$

With the analogous results for triangles $B'CA'$ and $C'AB'$ we have

$$\begin{aligned}[][A'B'C'] &= [ABC] + [A'BC'] + [B'CA'] + [C'AB']\\ &= [ABC]\cdot\left(1 + \frac{\cos C}{\cos A\cos B} + \frac{\cos A}{\cos B\cos C} + \frac{\cos B}{\cos A\cos C}\right)\\ &= [ABC]\cdot\left(1 + \frac{\cos^2 A + \cos^2 B + \cos^2 C}{\cos A\cos B\cos C}\right).\end{aligned}$$

From $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$ the final equation becomes

$$\frac{[A'B'C']}{[ABC]} = \frac{1}{\cos A\cos B\cos C} - 1.$$

The problem has now been reduced to proving that

$$\frac{1}{\cos A\cos B\cos C} = \frac{4R^2}{s^2 - (2R+r)^2}, \tag{1}$$

but a proof of this formula can be found in [2, p. 56, formula 37] and [1, formula 205].

[*Ed.* The references were provided by Arslanagić and by Bellot; instead of supplying a reference, Zvonaru used the sine law to reduce equation (1) to a better-known equation.]

References

[1] Anonymous, *Relations entre les éléments d'un triangle: Recueil de 273 formules relatives au triangle avec leurs démonstrations*, Paris, Librairie Nony & Cie, 1893.

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; and the proposer.

3636. [2011 : 172, 174] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a+b+c+d=2$. Prove that

$$ab(a^2+b^2+c^2)+bc(b^2+c^2+d^2)+cd(c^2+d^2+a^2)+da(d^2+a^2+b^2) \ \leq\ 2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $x=a+c$ and $y=b+d$. Then $x+y=2$, and we have

$$\begin{aligned}\sum_{\text{cyclic}} ab(a^2+b^2+c^2) &\leq (ab+bc+cd+da)(a^2+b^2+c^2+d^2+2ac+2bd)\\ &= (a+c)(b+d)\left((a+c)^2+(b+d)^2\right) = x^3y+xy^3\\ &= \frac{1}{8}\left((x+y)^4-(x-y)^4\right) \leq \frac{1}{8}(x+y)^4 = 2.\end{aligned}$$

The example $(a,b,c,d)=(1,1,0,0)$ shows that the inequality is sharp.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

From the proof featured above it is easy to see that equality holds if and only if $(a,b,c,d)=(1,1,0,0)$ *or* $(0,1,1,0)$ *or* $(0,0,1,1)$ *or* $(1,0,0,1)$*. This was explicitly pointed out by AN-anduud problem solving group and Arslanagić.*

3637. [2011: 172, 174] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let x be a real number with $|x|<1$. Determine

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right)\,.$$

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Observe that

$$\begin{aligned}
S'(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+1+x+x^2+\cdots+x^{n-1}\right)\\
&= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+\frac{1-x^n}{1-x}\right) = \frac{-x}{1-x}\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1}\\
&= \frac{-x}{(1-x)(1+x)^2} = \frac{1}{4}\left[\frac{2}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}\right].
\end{aligned}$$

Noting that $S(0)=0$, we deduce that

$$\begin{aligned}
S(x) &= \frac{1}{4}\left[2-\frac{2}{1+x}-\ln(1+x)+\ln(1-x)\right]\\
&= \frac{1}{4}\left[\frac{2x}{1+x}+\ln\frac{1-x}{1+x}\right] = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}.
\end{aligned}$$

Editor's comment: We can also write

$$S'(x) = \frac{1}{2}\left[\frac{x}{(1+x)^2}-\frac{1}{1-x^2}\right],$$

which leads to $S(x)=\frac{1}{2}[x(1+x)^{-1}-\tanh^{-1}x]$.

II. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

$$\begin{aligned}
S(x) &= -\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\sum_{k=n+1}^{\infty}\frac{x^k}{k}\right)\\
&= -\sum_{n=2}^{\infty}\frac{x^n}{n}\sum_{j=1}^{n-1}(-1)^{j-1}j = -\sum_{n=2}^{\infty}\frac{x^n}{n}(-1)^n\left\lfloor\frac{n}{2}\right\rfloor\\
&= \sum_{n=2}^{\infty}(-1)^{n+1}\left\lfloor\frac{n}{2}\right\rfloor\frac{x^n}{n} = -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\sum_{k=1}^{\infty}\frac{kx^{2k+1}}{2k+1}\\
&= -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\frac{1}{2}\sum_{k=1}^{\infty}x^{2k+1}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k+1}}{2k+1}\\
&= -\frac{1}{2}\frac{x^2(1-x)}{1-x^2}-\frac{1}{4}\ln\frac{1+x}{1-x}+\frac{x}{2}\\
&= \frac{1}{2}\left[\frac{-x^2}{1+x}+x\right]+\frac{1}{4}\ln\frac{1-x}{1+x} = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}.
\end{aligned}$$

III. Solution by Michel Bataille, Rouen, France.

For $|x| < 1$, we use the Taylor representation

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n dt$$

with $f(x) = \ln(1-x)$ to obtain

$$\begin{aligned} \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + \int_0^x \frac{(x-t)^n}{n!} \cdot \frac{-n!}{(1-t)^{n+1}} dt \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \int_0^x \frac{u^n}{1-u} du, \end{aligned}$$

where the two integrals are related by the substitution $(u-1)t = u-x$. Therefore

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} \int_0^x \frac{(-1)^n n u^n}{1-u}\, du = \int_0^x \frac{1}{1-u} \left(\sum_{n=1}^{\infty} (-1)^n n u^n \right) du \\ &= \int_0^x \frac{-u}{(1-u)(1+u)^2}\, du = \frac{1}{4} \int_0^x \left[\frac{2}{(1+u)^2} - \frac{1}{1-u} - \frac{1}{1+u} \right] du \\ &= \frac{x}{2(1+x)} + \frac{1}{4} \ln \frac{1-x}{1+x}. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3638. [2011: 234, 237] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and let points D, E, F lie on lines BC, CA, AB, respectively, such that

$$BD : DC = \lambda : 1-\lambda, \quad CE : EA = \mu : 1-\mu, \quad AF : FB = \nu : 1-\nu.$$

Show that DEF is a pedal triangle with regard to ΔABC if and only if

$$(2\lambda - 1)BC^2 + (2\mu - 1)CA^2 + (2\nu - 1)AB^2 = 0.$$

Solution by the proposer.

Let A', B', and C' be the midpoints of BC, CA, and AB, respectively. Since $\overrightarrow{BD} = \lambda\overrightarrow{BC}$ and $\overrightarrow{CD} = (\lambda - 1)\overrightarrow{BC}$, we have $(2\lambda - 1)\overrightarrow{BC} = 2\overrightarrow{A'D}$. Similarly, $(2\mu - 1)\overrightarrow{CA} = 2\overrightarrow{B'E}$ and $(2\upsilon - 1)\overrightarrow{AB} = 2\overrightarrow{C'F}$ so that the given condition is equivalent to

$$\overrightarrow{A'D} \cdot \overrightarrow{BC} + \overrightarrow{B'E} \cdot \overrightarrow{CA} + \overrightarrow{C'F} \cdot \overrightarrow{AB} = 0. \tag{1}$$

Now, suppose that D, E, F are the orthogonal projections of some point P onto BC, CA, AB, respectively. Then,

$$\overrightarrow{A'D}\cdot\overrightarrow{BC} = \overrightarrow{A'P}\cdot\overrightarrow{BC} = -\frac{1}{2}(\overrightarrow{PB}+\overrightarrow{PC})\cdot(\overrightarrow{PC}-\overrightarrow{PB}) = \frac{1}{2}(PB^2-PC^2)$$

and similarly,

$$\overrightarrow{B'E}\cdot\overrightarrow{CA} = \frac{1}{2}(PC^2-PA^2), \quad \overrightarrow{C'F}\cdot\overrightarrow{AB} = \frac{1}{2}(PA^2-PB^2),$$

hence (1) holds, by addition.

Conversely, suppose that (1) holds and let P be the point of intersection of the perpendiculars to BC at D and to CA at E. Then, $\overrightarrow{A'P}\cdot\overrightarrow{BC} = \overrightarrow{A'D}\cdot\overrightarrow{BC}$ and $\overrightarrow{B'P}\cdot\overrightarrow{CA} = \overrightarrow{B'E}\cdot\overrightarrow{CA}$ and from (1), we obtain

$$\begin{aligned}\overrightarrow{C'F}\cdot\overrightarrow{AB} &= \overrightarrow{PA'}\cdot\overrightarrow{BC}+\overrightarrow{PB'}\cdot\overrightarrow{CA} = \frac{1}{2}(PC^2-PB^2)+\frac{1}{2}(PA^2-PC^2)\\ &= \frac{1}{2}(PA^2-PB^2) = -\overrightarrow{PC'}\cdot\overrightarrow{AB}.\end{aligned}$$

Thus, $\overrightarrow{AB}\cdot\overrightarrow{PF}=0$ and so $PF\perp AB$, as desired.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; and TITU ZVONARU, Cománeşti, Romania.

3639. [2011: 234, 237] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c = 3$. Prove that

$$\frac{a^2b}{a+b+1}+\frac{b^2c}{b+c+1}+\frac{c^2a}{c+a+1} \le 1\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

In his solution to Problem 3549 [2011 : 253], Arkady Alt proved that

$$27(a^2b+b^2c+c^2a+abc)\le 4(a+b+c)^3$$

whenever $a, b, c > 0$. When in addition $a+b+c=3$, it follows that

$$\begin{aligned}&(4-a)(4-b)(4-c)-a(4-a)(4-b)-b(4-b)(4-c)-c(4-c)(4-a)\\ &\qquad = 4(a^2+b^2+c^2)+8(ab+bc+ca)-32-(a^2b+b^2c+c^2a)-abc\\ &\qquad \ge 4(a+b+c)^2-32-\frac{4}{27}(a+b+c)^3=0,\end{aligned}$$

so that

$$a(4-a)(4-b)+b(4-b)(4-c)+c(4-c)(4-a)\le(4-a)(4-b)(4-c).$$

Since

$$\begin{aligned}\frac{a^2b}{a+b+1} &= \frac{1}{4}\left[a^2b + abc\left(\frac{a}{4-c}\right)\right],\\ \frac{b^2c}{b+c+1} &= \frac{1}{4}\left[b^2c + abc\left(\frac{b}{4-a}\right)\right],\\ \frac{c^2a}{c+a+1} &= \frac{1}{4}\left[c^2a + abc\left(\frac{c}{4-b}\right)\right],\end{aligned}$$

the left side of the desired inequality is equal to

$$\begin{aligned}&\frac{1}{4}(a^2b+b^2c+c^2a) + \frac{abc}{4}\left[\frac{a}{4-c}+\frac{b}{4-a}+\frac{c}{4-b}\right]\\ &= 1 - \frac{abc}{4} + \frac{abc}{4(4-a)(4-b)(4-c)}\\ &\qquad\times\left[a(4-a)(4-b)+b(4-b)(4-c)+c(4-c)(4-a)\right]\\ &\leq 1.\end{aligned}$$

Also solved by KEE-WAI LAU, Hong Kong, China; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3640. [2011: 234, 237] *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Consider the function $f(x) = -\sqrt[3]{4x^6+6x^3+3}$.

(a) Find the fixed points of $f(x)$, if any.

(b) Find the periodic points with period 2 of $f(x)$, if any.

(c) Prove that $x=-1$ is the unique real number such that x and $f(x)$ are both integers.

Solution by the proposer.

(a) The equation, $x = f(x)$ for the fixed points is equivalent to

$$0 = 4x^6 + 7x^3 + 3 = (x^3+1)(4x^3+3).$$

Thus, the fixed points are $x=-1$ and $x = -(3/4)^{1/3}$.
(b) Points of period 2 satisfy $x = f(f(x))$, which works out to

$$0 = 64u^4 + 192u^3 + 216u^2 + 109u + 21 = (4u^2+7u+3)(16u^2+20u+7),$$

where $u = x^3$. The roots of the first factor yield the fixed points, while the second factor does not have real roots. Therefore, there are no points of prime period 2.

(c) Let $y = f(x)$, so that

$$-y^3 = 4x^6 + 6x^3 + 3 = (2x^3+1)^2 + (2x^3+1) + 1.$$

It is known that the only solutions of the diophantine equation $-y^3 = z^2 + z + 1$ are $(y, z) = (-1, -1), (-7, -19), (-7, 18)$, but only the first of these leads to the only integer solution $(x, y) = (-1, -1)$ of the equation.

Alternatively, if we set $w = 1 + 2x^3$, $u = 2w^3 - 1$, $v = -2xyw$, we find that

$$\begin{aligned} u^2 - v^3 - 1 &= (4w^6 - 4w^3 + 1) + 8x^3y^3w^3 - 1 = 4w^3(w^3 - 1 + 2x^3y^3) \\ &= 4w^3(8x^9 + 12x^6 + 6x^3 + 2x^3y^3) = 8x^3w^3(4x^6 + 6x^3 + 3 + y^3) = 0. \end{aligned}$$

If x and y are integers, then u and v are integers that satisfy the Catalan equation $u^2 - v^3 = 1$. The only solutions of this equation are $(u, v) = (\pm 1, 0), (\pm 3, 2)$. Only $(u, v) = (-3, 2)$ gives integer values for both x and y, and we find that x and $f(x)$ are integers if and only if $x = -1$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

*The diophantine equation $-y^3 = x^2 + x + 1$ was solved by Nagell in 1921 (*Norsk Mat. Forenings Skrifter, Ser. I, No. 2 and No. 3*), while the equation $u^2 - v^3 = 1$ was solved by Euler in 1738. A recent comprehensive history of the diophantine equation $x^p - y^q = 1$, where p and q are not less than 2, is given in the book,* Catalan's conjecture*, by René Schloof (Springer, 2008).*