SKOLIAD No. 80

Shawn Godin

Please send your solutions to the problems in this edition by *1 April, 2005*. A copy of **MATHEMATICAL MAYHEM Vol. 6** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (∗) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

This month's questions are taken from the Nova Scotia Math League championship held in Halifax on May 15, 2004. Thanks to Richard Hoshino of Dalhousie University and John Grant McLoughlin of the University of New Brunswick for sending these questions.

Nova Scotia Math League Game 4: Group Questions

You will solve the following ten problems together as a team. At the end of 40 minutes, we will collect your answer sheet. Your team will receive 3 points for each correct answer.

1. A *lattice point* is a point (x, y), where the coordinates are both integers. For example, $(3, -4)$ and $(5, 0)$ are lattice points, but $(2, 4.58)$ is not.

Determine the number of lattice points on the circumference of the circle $x^2 + y^2 = 25$.

2. Let $S = \dfrac{2^2 - 1}{2^2} \times \dfrac{3^2 - 1}{3^2} \times \dfrac{4^2 - 1}{4^2} \times \cdots \times \dfrac{2004^2 - 1}{2004^2}$. Express S as a fraction, reduced to lowest terms.

3. A *palindrome* is a number that reads the same forwards and backwards, such as **8338** and **50705**.

Let A and B be four-digit palindromes, and let C be a five-digit palindrome. If $A + B = C$, determine all possible values of C.

4. The circle with equation $x^2 + y^2 = 1$ intersects the line $y = 7x + 5$ at two distinct points, A and B. Let O be the centre of the circle. Find the measure of $\angle AOB$.

5. Determine all integers x such that $(x^2 - 3x + 1)^{x+1} = 1$.

6. Let $f(a, b)$ denote the sum of the integers between a and b, inclusive. For example, $f(1, 5) = 1 + 2 + 3 + 4 + 5 = 15$ and $f(3, 6) = 3 + 4 + 5 + 6 = 18$.

Determine the value of $f(133333, 533333)$.

7. A hexagon and an equilateral triangle have equal perimeters. If the area of the hexagon is $6\sqrt{3}$ square units, what is the area of the triangle?

8. Determine all values of x for which

$$(1999x - 99)^3 = (1234x - 56)^3 + (765x - 43)^3.$$

9. Find all solutions (x, y) in real numbers to

$$\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$$
$$x^2 y + xy^2 = 30.$$

10. Find the number of real solutions to the equation $\sin(x) = \dfrac{x}{315}$.

Next we give the official solutions to the 2003 Maritime Mathematics Competition [2003 : 352–354].

2003 Maritime Mathematics Competition
Concours de Mathématiques des Maritimes 2003

1. When a father distributes a number of candies among his children, each child receives 15 candies and there is one left over. If, however, two friends join the group and the candies are redistributed, then each child receives 11 candies and there are three left over. What is the total number of candies?

Solution: Let x be the number of candies and let y be the number of children (not including the two friends). Then we have

$$x = 15y + 1 \qquad \text{and} \qquad x = 11(y + 2) + 3.$$

Thus, $15y + 1 = 11(y + 2) + 3$ which yields $y = 6$. Now $x = 15(6) + 1 = 91$. Hence, the total number of candies is 91.

2. Pour chaque entier strictement positif n, posons

$$f(n) = (4(1)^2 - 1) \times (4(2)^2 - 1) \times \cdots \times (4n^2 - 1).$$

Par example, $f(1) = 3$ et $f(2) = 3 \times 15 = 45$.

   Trouver toutes les valeurs de n pour lesquelles $f(n)$ est un carré parfait.

*Solution :* Puisque $4k^2 - 1 = (2k)^2 - 1 = (2k - 1)(2k + 1)$, on a

$$\begin{aligned}
f(n) &= (2(1) - 1)(2(1) + 1) \times (2(2) - 1)(2(2) + 1) \times \cdots \\
&\qquad \times (2n - 1)(2n + 1) \\
&= (1)(3) \times (3)(5) \times (5)(7) \times \cdots \\
&\qquad \times (2n - 3)(2n - 1) \times (2n - 1)(2n + 1) \\
&= 3^2 5^2 \cdots (2n - 1)^2 (2n + 1) \\
&= \left((3)(5) \cdots (2n - 1) \right)^2 (2n + 1).
\end{aligned}$$

Pour que $f(n)$ soit un carré parfait, il faut et il suffit que $2n+1$ soit un carré parfait.

Si le nombre impair $2n + 1$ est un carré parfait, il est le carré d'un nombre impair, disons $2a+1$. Dans ce cas, $2n+1 = (2a+1)^2 = 4a^2+4a+1$, donc $n = 2a(a+1)$. Réciproquement, si $n = 2a(a+1)$ pour un certain entier positif a, alors $2n+1 = (2a+1)^2$ est un carré parfait. Les valeurs de n pour lesquelles $f(n)$ est un carré parfait sont donc données par $n = 2a(a+1)$, $a = 1, 2, 3, \ldots$.

3. A 10-metre ladder rests against a vertical wall. The mid-point of the ladder is twice as far from the ground as it is from the wall. What height on the wall does the ladder reach?

Solution: Let the points at which the ladder touches the wall and the ground be A and B, respectively, and let C be the point at which the wall meets the ground. Let P be the mid-point of the ladder, and let point D be on the wall so that PD is parallel to the ground. Finally, let E be the point on the ground such that PE is parallel to the wall.
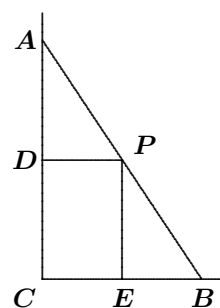
Let x be the length of DP. Then the length of EP is $2x$. Since the ladder is 10 metres long, we have $|AP| = |BP| = 5$.

Now $\angle DAP = \angle EPB$ and $\angle DPA = \angle EBP$, whence, $\triangle DAP$ is congruent to $\triangle EPB$. Thus, $|DA| = |EP| = 2x$ and $|EB| = |DP| = x$. Applying the Pythagorean Theorem to $\triangle DAP$, we obtain $(2x)^2 + x^2 = 5^2$, which yields $x = \sqrt{5}$. Since $|DA| = 2x$ and $|CD| = |EP| = 2x$, we get

$$|AC| = 4x = 4\sqrt{5}.$$

Therefore, the ladder reaches a height of $4\sqrt{5}$ metres on the wall.

4. Trouver un nombre à six chiffres dont le premier chiffre est 1 et qui devient trois fois plus grand si le premier chiffre est déplacé à l'autre bout pour devenir le chiffre des unités.

Solution : Soit $1abcde$ le nombre en question. On a $1abcde = 10^5 + x$ où $x = abcde$. Lorsque le premier chiffre est déplacé à l'autre bout, on obtient $abcde1 = 10x + 1$. Ce nouveau nombre est trois fois plus grand que le premier, ce qui donne

$$\begin{aligned} 10x + 1 &= 3(10^5 + x), \\ 7x &= 3 \times 10^5 - 1, \\ x &= 42857. \end{aligned}$$

Le nombre cherché est donc 142857.

5. Evaluate

$$\sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}}.$$

Solution: Let $x = \sqrt[3]{5 + 2\sqrt{13}}$ and $y = \sqrt[3]{5 - 2\sqrt{13}}$, and let $s = x + y$. Thus, s is the required sum. We have

$$s^3 = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xys.$$

Now $x^3 = 5 + 2\sqrt{13}$ and $y^3 = 5 - 2\sqrt{13}$, and

$$xy = \sqrt[3]{(5 + 2\sqrt{13})(5 - 2\sqrt{13})} = \sqrt[3]{25 - 4(13)} = \sqrt[3]{-27} = -3.$$

Hence,

$$s^3 = 5 + 2\sqrt{13} + 5 - 2\sqrt{13} + 3(-3)s,$$

which simplifies to $s^3 + 9s - 10 = 0$. By inspection, $s = 1$ is a root of this equation. Thus, $s - 1$ is a factor of $s^3 + 9s - 10$. By long division, we obtain $s^2 + s + 10$ as the other factor. Therefore,

$$(s - 1)(s^2 + s + 10) = 0.$$

Now $s^2 + s + 10 = 0$ has no real roots. Hence, the only solution to the above equation is $s = 1$. Therefore,

$$\sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}} = 1.$$

6. Trouver toutes les paires d'entiers positifs (x, y) telles que

$$x^2 - 11y! = 2003.$$

(Par définition, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, etc.)

Solution : Si x est pair, disons $x = 2k$, alors $x^2 = (2k)^2 = 4k^2$ est un multiple de quatre. D'autre part, si x est impair, disons $x = 2k + 1$, alors $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$ est un multiple de quatre plus un. On conclut que pour un certain entier m, $x^2 = 4m$ ou $x^2 = 4m + 1$.

De plus, lorsque $y \geq 4$, $y!$ contient le facteur 4. Le membre de gauche de l'équation, c'est à dire $x^2 - 11y!$, est donc soit un multiple de quatre, soit un multiple de quatre plus un. Par contre, le membre de droite, c'est à dire 2003, est un multiple de quatre plus trois. L'équation n'a donc pas de solutions quand $y \geq 4$.

On vérifie un par un les cas $y = 1$, 2, 3.

Si $y = 1$, on a $x^2 - 11(1!) = 2003$, donc $x^2 = 2014$. On vérifie aisément que 2014 n'est pas un carré parfait. Il n'a donc pas de solutions lorsque $y = 1$.

Si $y = 2$, on a $x^2 - 11(2!) = 2003$, donc $x^2 = 2025$. Cette fois on trouve $x = 45$.

Enfin, si $y = 3$, on a $x^2 - 11(3!) = 2003$, donc $x^2 = 2069$. On vérifie que 2069 n'est pas un carré parfait. Il n'a donc pas de solutions lorsque $y = 3$.

L'équation possède donc la solution unique $(x, y) = (45, 2)$.

Next we give the official solutions to the 2003 W.J. Blundon Mathematics Contest [2003 : 417–418].

The Twentieth W.J. Blundon
Mathematics Contest

1. Solve: $\log_2(9 - 2^x) = 3 - x$.

Solution: Using the definition of the logarithm function, we have

$$\begin{aligned}
\log_2(9 - 2^x) &= 3 - x\,, \\
9 - 2^x &= 2^{3-x}\,, \\
9 - 2^x &= 8 \cdot 2^{-x}\,, \\
9 \cdot 2^x - 2^{2x} &= 8\,, \\
(2^x)^2 - 9 \cdot 2^x + 8 &= 0\,, \\
(2^x - 1)(2^x - 8) &= 0\,, \\
2^x = 1 \quad &\text{or} \quad 2^x = 8\,, \\
x = 0 \quad &\text{or} \quad x = 3\,.
\end{aligned}$$

Checking these in the original equation, we find that they are both solutions.

2. Show that $(\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}}$ is rational, and find its value.

Solution: Let $x = (\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}}$. Then

$$\begin{aligned}
x^3 &= (\sqrt{5} + 2) - 3(\sqrt{5} + 2)^{\frac{2}{3}}(\sqrt{5} - 2)^{\frac{1}{3}} \\
&\quad + 3(\sqrt{5} + 2)^{\frac{1}{3}}(\sqrt{5} - 2)^{\frac{2}{3}} - (\sqrt{5} - 2) \\
&= 4 - 3(\sqrt{5} + 2)^{\frac{1}{3}}(\sqrt{5} - 2)^{\frac{1}{3}}\left((\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}}\right) \\
&= 4 - 3(1)(x)\,.
\end{aligned}$$

Hence, $(\sqrt{5} + 2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}}$ is a zero of

$$P(x) = x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$$

which has $x = 1$ as its only real zero. Therefore, $(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5} - 2)^{\frac{1}{3}} = 1$.

[*Ed.* This problem is similar to problem 5 on the 2003 Maritime Mathematics Competition, for which a solution was given on the preceding page.]

3. If $a^3 + b^3 = 4$ and $ab = \frac{2}{3}$, where a and b are real, find $a + b$.

Solution: Note first that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b)\,.$$

Let $x = a + b$. Then $x^3 = 4 + 3\left(\frac{2}{3}\right)x$, which yields

$$\begin{aligned}
x^3 - 2x - 4 &= 0\,, \\
(x - 2)(x^2 + 2x + 2) &= 0\,.
\end{aligned}$$

Clearly, $x = 2$ is the only real root. Hence, $a + b = 2$.

4. Find x, y, and z such that when any one of them is added to the product of the other two, the result is 2.

Solution: By symmetry, we must have $x = y = z$. Then each equation becomes $x + x^2 = 2$; that is, $x + x^2 - 2 = 0$, or $(x - 1)(x + 2) = 0$. Thus, $x = 1$ or $x = -2$. Finally, $x = y = z = 1$ or $x = y = z = -2$.

5. If a, b, and c are the three zeros of $P(x) = x^3 - x^2 + x - 2$, find $a + b + c$ and $a^2 + b^2 + c^2$.

Solution: If a, b, and c are the roots of the given cubic, then

$$\begin{aligned}
x^3 - x^2 + x - 2 &= (x - a)(x - b)(x - c) \\
&= x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc.
\end{aligned}$$

Equating coefficients of the x^2-terms, we see that $a + b + c = 1$. Squaring this, we get $a^2 + b^2 + c^2 + 2(ab + ac + bc) = 1$. But equating coefficients of the x-terms gives $ab + ac + bc = 1$. Hence,

$$\begin{aligned}
a^2 + b^2 + c^2 + 2(1) &= 1, \\
a^2 + b^2 + c^2 &= -1.
\end{aligned}$$

6. If $\sin x + \cos x = \sqrt{\dfrac{2 + \sqrt{3}}{2}}$, with $0 < x < \frac{\pi}{2}$, find x.

Solution: Squaring the given equation, we get

$$\begin{aligned}
\sin^2 x + 2 \sin x \cos x + \cos^2 x &= \frac{2 + \sqrt{3}}{2}, \\
1 + 2 \sin x \cos x &= 1 + \frac{\sqrt{3}}{2}, \\
\sin 2x &= \frac{\sqrt{3}}{2},
\end{aligned}$$

where $0 < 2x < \pi$. Therefore, $2x = \pi/3$ or $2x = 2\pi/3$. Then $x = \pi/6$ or $x = \pi/3$.

7. Prove that two consecutive odd positive integers cannot have a common factor other than 1.

Solution: Two consecutive odd positive integers can be written as $2k - 1$ and $2k + 1$ for some positive integer k. Suppose p is a common factor of $2k - 1$ and $2k + 1$. Then

$$2k - 1 = mp \quad \text{and} \quad 2k + 1 = np,$$

for some positive integers m and n. Subtracting the first equation from the second gives $2 = (n - m)p$. Thus, $p = 1$ or $p = 2$. But $p \neq 2$, since $2k - 1$ and $2k + 1$ are odd. Therefore, $p = 1$.

8. Triangle ABC has vertices $A(3,1)$, $B(5,7)$, and $C(1,y)$. Find all y so that angle C is a right angle.

Solution:

$$BC \perp AC,$$
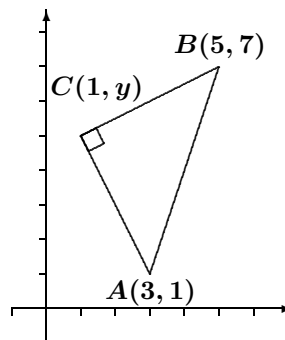$$m_{BC} = \frac{-1}{m_{AC}},$$
$$\frac{7-y}{4} = \frac{-2}{1-y},$$
$$7 - 8y + y^2 = -8,$$
$$y^2 - 8y + 15 = 0,$$
$$(y-5)(y-3) = 0,$$
$$y = 5 \quad \text{or} \quad y = 3.$$

9. In the diagram to the right, $PQ = 8$, $TS = 12$, and $QS = 20$. Find QR so that $\angle PRT$ is a right angle.

Solution: Let $QR = x$. Then

$$(PT)^2 = (PR)^2 + (RT)^2$$
$$= (64 + x^2) + (144 + (20-x)^2)$$
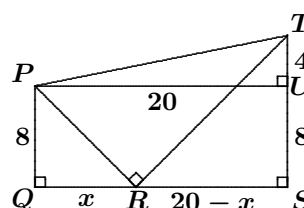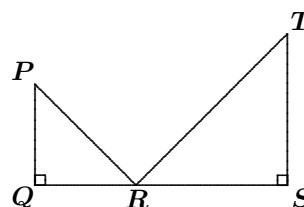$$= 2x^2 - 40x + 608.$$

Also, $(PT)^2 = 20^2 + 4^2 = 416$. Hence,

$$2x^2 - 40x + 608 = 416,$$
$$x^2 - 20x + 96 = 0,$$
$$(x-8)(x-12) = 0,$$
$$x = 8 \quad \text{or} \quad x = 12.$$

10. A square of side 2 is placed with one side on a tangent to a circle of radius 5 so that the square lies outside the circle, and one vertex of the square lies on the circle. A line is drawn from the centre of the circle through the vertex of the square that is not on the tangent and not on the circle. This line cuts the tangent at a point T. If the tangent meets the circle at S, find the length of the line segment TS.
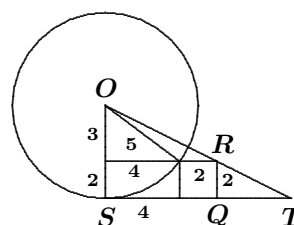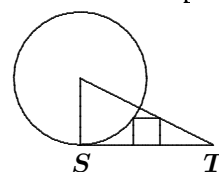
Solution: $\triangle TQR$ and $\triangle TSO$ are similar. Thus,

$$\frac{TQ}{2} = \frac{TQ+6}{5},$$
$$5TQ = 2TQ + 12,$$
$$TQ = 4.$$

Hence, $TS = 6 + 4 = 10$.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Larry Rice (University of Waterloo), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

*Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier avril 2005**. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M157. *Proposé par Neven Jurič, Zagreb, Croatie.*

On sait que les formules $a = x^2 - y^2$, $b = 2xy$, $c = x^2 + y^2$ sont utiles pour trouver des solutions entières de l'équation $a^2 + b^2 = c^2$. Y a-t-il des formules semblables pour trouver des solutions entières de l'équation $a^2 + ab + b^2 = c^2$?

M158. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

(a) En n'utilisant que des nombres naturels, trouver les longueurs des côtés des différents triangles rectangles apparaissant dans la figure.
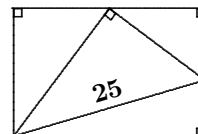
(b) Trouver une expression à valeurs entières pour remplacer le nombre 25 de telle sorte qu'on puisse engendrer une infinité de rectangles dans lesquels est inscrit un triangle rectangle, tous à côtés entiers.

M159. *Proposé par l'Équipe de Mayhem.*

Est-il possible d'arranger les nombres 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 le long d'un cercle de façon que la somme de deux voisins quelconques soit un nombre premier ?

M160. *Proposé par l'Équipe de Mayhem.*

Comme stratège, Napoléon était fier de son armée qu'il voulait très disciplinée, entraînée et mobile. Dans une de ses batailles contre les Alliés, il décida de varier l'arrangement de ses fantassins. Au début, ceux-ci furent groupés en 540 rangées égales ; puis il les groupa en 105 rangées égales, pour finalement les grouper en 216 rangées égales. Le nombre de soldats était le plus petit nombre rendant possibles de tels arrangements.

Combien y en avait-il ?

M161. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Dans un triangle isocèle ABC de base $BC = 1$, la première de deux droites parallèles à la base coupe les côtés égaux AB et AC aux points P et Q, la seconde aux points R et S. Sachant que la distance de PQ à BC est 1 et que le rapport PQ/RS est égal au nombre d'or $\frac{1}{2}(1+\sqrt{5})$, trouver l'aire du trapèze $PQSR$.

M162. *Proposé par Neven Jurič, Zagreb, Croatie.*

On veut construire pas à pas une armature cubique avec des tiges de même longueur, disons 1 mètre, qui peuvent s'emboîter aux extrémités l'une de l'autre pour former une structure cubique rigide. Il en faut 12 pour un premier cube dont l'arête mesure 1 m. A condition d'enlever les tiges qui font double usage, huit de ces cubes peuvent s'adapter afin d'obtenir une deuxième armature dont l'arête mesure cette fois 2 m. Il en faut 27 pour une troisième armature de 3 m d'arête, etc.

Quelle est la plus grande armature cubique qu'on puisse construire si la longueur totale des tiges disponibles suffirait pour relier la terre à la lune, soit 384 000 km ?

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M157. *Proposed by Neven Jurič, Zagreb, Croatia.*

The formulas $a = x^2 - y^2$, $b = 2xy$, $c = x^2 + y^2$ are known to be useful for producing integer solutions of the equation $a^2 + b^2 = c^2$. Are there similar formulas for integer solutions of the equation $a^2 + ab + b^2 = c^2$?

M158. *Proposed by K.R.S. Sastry, Bangalore, India.*

(a) Determine the lengths of the sides of the various right triangles in the figure, given that the lengths are integers.
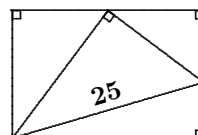
(b) Find an integer-valued expression in place of the number 25 so that an infinite number of different rectangles may be generated, each with a right triangle inscribed as shown, such that the lengths of the sides of the various right triangles are integers.

M159. *Proposed by the Mayhem Staff.*

Is it possible to arrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in a circle so that the sum of any two neighbours is a prime number?

M160. *Proposed by the Mayhem Staff.*

The strategist Napoleon was proud of his army's discipline, training, and mobility. In a battle against the European allies he varied the arrangement of his soldiers. Initially, the soldiers were arranged in 540 equal rows; in the second arrangement, there were 105 equal rows; and in the third case, 216 equal rows. The number of soldiers was the smallest number that enabled all such arrangements.

How many soldiers were there?

M161. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Triangle ABC is isosceles, with $AB = AC$ and $BC = 1$. On the sides AB and AC are points P and Q, respectively, such that $PQ \parallel BC$ and the distance from PQ to BC is 1. On the segments AP and AQ are points R and S, respectively, such that $RS \parallel PQ$.

If $PQ/RS = (1 + \sqrt{5})/2$ (the golden ratio), find the area of the trapezoid $PQSR$.

M162. *Proposed by Neven Jurič, Zagreb, Croatia.*

Rods 1 m in length are used to build a rigid cubic framework. Twelve rods are needed to build a cube of side 1 m. By fitting together 8 of these unit cubes, a cubic framework can be constructed that has side 2 m. By using 27 of the unit cubes, a cubic framework can be constructed that has side 3 m.

Each time the unit cubes are combined, there are duplicate rods along the edges where the cubes fit together. The duplicates may be removed (leaving one rod where there were previously two) and re-used elsewhere.

How large a cubic framework can be created in the manner described above if the total length of the available rods would connect the earth to the moon, assuming a distance of 384 000 km between them?

Mayhem Solutions

M83. *Proposed by the Mayhem Staff.*

Five balls numbered 1 to 5 are put into a box. A ball is drawn at random, its number recorded, and the ball returned to the box. This process is repeated until five numbers have been recorded. If the sum of the recorded numbers is 15, what is the probability that the number 3 was drawn each time?

Solution by Geneviève Lalonde, Massey, ON.

The table below shows the number of ways of drawing five balls with a sum of 15. The total number of ways is 381. Therefore, the probability that every draw was a 3 is $\frac{1}{381}$.

Balls	Number of Ways	Balls	Number of Ways
5, 5, 3, 1, 1	$\frac{5!}{2!2!} = 30$	5, 3, 3, 2, 2	$\frac{5!}{2!2!} = 30$
5, 5, 2, 2, 1	$\frac{5!}{2!2!} = 30$	4, 4, 4, 2, 1	$\frac{5!}{3!} = 20$
5, 4, 4, 1, 1	$\frac{5!}{2!2!} = 30$	4, 4, 3, 3, 1	$\frac{5!}{2!2!} = 30$
5, 4, 3, 2, 1	$5! = 120$	4, 4, 3, 2, 2	$\frac{5!}{2!2!} = 30$
5, 4, 2, 2, 2	$\frac{5!}{3!} = 20$	4, 3, 3, 3, 2	$\frac{5!}{3!} = 20$
5, 3, 3, 3, 1	$\frac{5!}{3!} = 20$	3, 3, 3, 3, 3	$\frac{5!}{5!} = 1$

One incorrect solution was received.

M84. *Proposed by the Mayhem Staff.*

Consider the two functions $f(x) = x^2 - 2ax + 1$ and $g(x) = 2b(a - x)$, where $a, b, x \in \mathbb{R}$. We will consider each pair of constants a and b as a point (a, b) in the ab–plane. Let A be the set of points (a, b) for which the graphs of $y = f(x)$ and $y = g(x)$ do *not* intersect. Find the area of A.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

The graphs of $y = f(x)$ and $y = g(x)$ do not intersect when the equation $x^2 - 2ax + 1 = 2b(a - x)$ has no solution. This happens if and only if $b^2 + a^2 < 1$. Hence, A is the open disk of radius 1 centred at the origin. Thus, the area of A is π.

Also solved by Robert Bilinski, Outremont, QC.

M85. *Proposed by the Mayhem Staff.*

Find a triangle whose integer sides are in arithmetic progression with a common difference of 2 and which has an area of 336.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let a, $a+2$, and $a+4$ be the sides of the triangle. By Heron's Formula, we have

$$336 = \sqrt{\tfrac{1}{16}(a+6)(a+2)(a-2)(3a+6)}\,.$$

Hence, $(a+6)(a+2)(a-2)(3a+6) = 16 \cdot 336^2 = 1806336$, which has the unique positive integer solution $a = 26$. Then the triangle sides are 26, 28, and 30.

Also solved by Robert Bilinski, Outremont, QC.

M86. *Proposé par l'équipe de Mayhem.*

Soit deux nombres entiers positifs a et b. Parmi les nombres

$$a,\ 2a,\ 3a,\ \ldots,\ (b-1)a,\ ba$$

combien y en a-t-il qui soient divisibles par b?

Solution par Robert Bilinski, Outremont, QC.

Si $(a,b) = 1$, alors il y a seulement ba qui est divisible par b. Il y en a donc 1.

Si $(a,b) = b$, alors tous les nombres sont divisibles par b. Il y en a donc b.

Si $(a,b) = k$, alors $a = ks$ et $b = kt$ avec $(s,t) = 1$. La question devient : "Parmi les nombres s, $2s$, $\ldots$, $(kt-1)s$, kts, combien sont divisibles par t?" Dans la suite de nombre on trouvera ts, $2ts$, $\ldots$, $(k-1)ts$, kts, donc k nombres sont divisibles par t. Autrement dit, il y aura toujours (a,b) nombres divisibles par b dans la liste.

En outre résolu par Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine.

M87. *Proposed by the Mayhem Staff.*

Seven people, A, B, C, D, E, F, G, are on one side of a river. To get across the river they have a rowboat, but it can only fit two people at a time. The times that would be required for the people to row across individually are 1, 2, 3, 5, 10, 15, and 20 minutes, respectively. However, when two people are in the boat, the time it takes them to row across is the same as the time necessary for the slower of the two to row across individually. Assuming that no one can cross without the boat, what is the minimum time for all seven people to get across the river?

Solution by Geneviève Lalonde, Massey, ON.

The minimum time is 46 minutes and can be accomplished as shown in the table below:

Cross	Time	Return	Time
A, B	2	A	1
F, G	20	B	2
A, B	2	A	1
D, E	10	B	2
A, C	3	A	1
A, B	2		

One incorrect solution was received.

M88. *Proposed by the Mayhem Staff.*

A set S consists of six numbers. When we take all possible subsets of S containing 5 elements, the sums of the elements of these subsets are 87, 92, 98, 99, 104, and 110, respectively. Determine the six numbers in S.

Solution by Allen O'Hara, grade 10 student, Oakridge Secondary School, London, ON.

Let $S = \{a, b, c, d, e, f\}$. From the information provided, we can create the following set of equations:

$$a + b + c + d + e = 87, \tag{1}$$
$$a + b + c + d + f = 92, \tag{2}$$
$$a + b + c + e + f = 98, \tag{3}$$
$$a + b + d + e + f = 99, \tag{4}$$
$$a + c + d + e + f = 104, \tag{5}$$
$$b + c + d + e + f = 110. \tag{6}$$

By subtracting each equation from the last, we can relate each variable to a:

$$f = a + 23, \quad e = a + 18, \quad d = a + 12, \quad c = a + 11, \quad b = a + 6.$$

Now we can rewrite one of the equations (1)–(6) to determine a, and then obtain the rest of the numbers. For example, we can rewrite equation (6) as

$$(a + 6) + (a + 11) + (a + 12) + (a + 18) + (a + 23) = 110,$$

which can be simplified to:

$$5a + 70 = 110,$$

from which we get $a = 8$. Then $b = 14$, $c = 19$, $d = 20$, $e = 26$, and $f = 31$. Thus, $S = \{8, 14, 19, 20, 26, 31\}$.

Also solved by Robert Bilinski, Outremont, QC; Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC; and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M89. *Proposé par l'Équipe de Mayhem.*

Trouver tous les entiers positifs x pour lesquels $x(x + 60)$ est un carré parfait.

I. *Solution par Robert Bilinski, Outremont, QC.*

Posons $x = n - 30$, alors chercher $x > 0$ tel que $x(x + 60)$ soit un carré parfait revient à chercher $n > 30$ tel que $(n - 30)(n + 30) = n^2 - 900$ soit un carré parfait. Nommons k^2 ce carré parfait que l'on recherche. Ainsi, $n^2 - 900 = k^2$ ou bien $n^2 - k^2 = (n - k)(n + k) = 900 = 2^2 3^2 5^2$.

Nous devons maintenant séparer 900 en deux facteurs a et b tels que $n - k = a$ et $n + k = b$. On aura ainsi que $n = \frac{a + b}{2}$ et $k = \frac{b - a}{2}$. Puisque nous voulons que n et k soient des entiers, il faut que a et b soient de même parité. Puisque un des deux facteurs de 900 doit être pair, les deux le seront. Le problème revient donc à trouver deux facteurs r et s de 225 qui donneront $n = r + s > 30$ et $k = r - s > 0$. Or les paires différentes de facteurs de 225 sont $(1, 225)$, $(3, 75)$, $(5, 45)$, $(9, 25)$ et $(15, 15)$. Elles donnent $n \in \{226, 78, 50, 34\}$ ou bien $x \in \{196, 48, 20, 4\}$.

II. *Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

Let k be a positive integer. The equation $x^2 + 60x - k^2 = 0$ has positive integer solutions if and only if $\sqrt{30^2 + k^2}$ is an integer. Then we must find all Pythagorean triples of the form $(30, k, \sqrt{30^2 + k^2})$. This is equivalent to finding all primitive Pythagorean triples (a, b, c) where $a \mid 30$ or $b \mid 30$. These triples are $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, and $(15, 112, 113)$. Hence, $x = 20$, $x = 48$, $x = 4$, and $x = 196$, respectively.

Also solved by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

M90. *Proposed by the Mayhem Staff.*

Determine the largest positive integer n for which 2002^n is a factor of $2002!$. What happens if 2002 is replaced by 2003 or 2004?

Solution by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

Let $F = 2002!$. Since $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, in $2002!$ there will be $\lfloor 2002/13 \rfloor = 154$ multiples of 13 and $\lfloor 2002/13^2 \rfloor = 11$ multiples of 13^2 and $\lfloor 2002/13^3 \rfloor = 0$ multiples of 13^3. There will be more multiples of 11, 7, and 2. Therefore, the largest integer n such that 2002^n divides $2002!$ is $n = 154 + 11 = 165$. Thus, $n = 165$ for 2002.

Since 2003 is a prime number, in $2003!$ there will be only one multiple of 2003. Thus, only 2003^1 will divide $2003!$. Thus, $n = 1$ for 2003.

Since $2004 = 2^2 \cdot 3 \cdot 167$, in $2004!$ there are $\lfloor 2004/167 \rfloor = 12$ multiples of 167, and since $167^2 > 2004$, there are 0 multiples of 167^2. Therefore, the largest integer n such that 2004^n divides $2004!$ is $n = 12$.

Also solved by Robert Bilinski, Outremont, QC; and Allen O'Hara, grade 10 student, Oakridge Secondary School, London, ON.

M91. *Proposed by Robert Morewood, Burnaby South Secondary School, Burnaby, BC.*

Let k be a four-digit integer. Determine all possible values of k for which k^{2003} ends in the four digits 2003. What happens if 2003 is replaced by 2002 or 2004?

Solution by Geneviève Lalonde, Massey, ON.

Looking at the problem modulo 10, we see that $k^{2003} \equiv 3 \pmod{10}$. Thus, $k \equiv 7 \pmod{10}$.

We next notice that $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, and $7^4 \equiv 1 \pmod{100}$. Let $k \equiv 10a + 7 \pmod{100}$, where a is a single non-negative digit. Then, using the Binomial Theorem, we get

$$\begin{aligned} k^{2003} &\equiv (10a + 7)^{2003} \equiv 7^{2003} + 2003 \cdot 10a \cdot 7^{2002} \\ &\equiv 43 + 3 \cdot 10a \cdot 49 \equiv 43 + 70a \pmod{100} \, . \end{aligned}$$

Then, since k^{2003} ends in the digits 2003, we have

$$\begin{aligned} 43 + 70a &\equiv 3 \pmod{100} \, , \\ 70a &\equiv 60 \pmod{100} \, , \\ 7a &\equiv 6 \pmod{10} \, , \\ a &\equiv 8 \pmod{10} \, . \end{aligned}$$

Hence, $k \equiv 87 \pmod{100}$. With a little help from Maple, we find that $k \equiv 587 \pmod{1000}$ and $k \equiv 587 \pmod{10000}$. Thus, the only "four-digit" number is $k = 0587$.

When 2003 is replaced with 2002 or 2004, there is no number that satisfies the congruence modulo 10. Thus, there are no solutions.

M92. *Proposed by the Mayhem Staff.*

A 3×3 magic square consists of nine distinct values, such that each of the rows, columns, and diagonals have a constant sum. Below is an example of a 3×3 magic square.

Suppose that a 3×3 magic square has a constant sum of T. Let the middle entry of this square be E. Prove that $T = 3E$.

2	9	4
7	5	3
6	1	8

Solution by Rusi Kolev, grade 10 student, Burnaby South Secondary School, Burnaby, BC.

Let the magic square be

a_1	a_2	a_3
a_4	a_5	a_6
a_7	a_8	a_9

Then $a_5 = E$. Since the sum of the elements horizontally, vertically, or diagonally is equal to T, then

$$
\begin{aligned}
3T &= (a_1 + a_5 + a_9) + (a_2 + a_5 + a_8) + (a_3 + a_5 + a_7) \\
&= (a_1 + a_2 + a_3) + 3a_5 + (a_9 + a_8 + a_7) \\
&= T + 3E + T = 3E + 2T,
\end{aligned}
$$

and hence, $T = 3E$.

Also solved by Robert Bilinski, Outremont, QC; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

M93. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

In triangle ABC, suppose that $\tan A$, $\tan B$, $\tan C$ are in harmonic progression. Show that a^2, b^2, c^2 form an arithmetic progression.

[*Note: x, y, z are in harmonic progression if $\dfrac{1}{x}$, $\dfrac{1}{y}$, $\dfrac{1}{z}$ form an arithmetic progression.*]

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Let ABC be an acute-angled triangle. Let AH_a, BH_b, and CH_c be the altitudes corresponding to sides a, b, and c, respectively.

Since $\tan A$, $\tan B$, $\tan C$ are in harmonic progression, there exists $d \in \mathbb{R}$ such that:

$$
\frac{1}{\tan A} = \frac{1}{\tan B} - d, \tag{1}
$$

$$
\frac{1}{\tan C} = \frac{1}{\tan B} + d. \tag{2}
$$

In addition, we have the following relations:

$$
\frac{1}{\tan A} = \frac{AH_b}{BH_b} = \frac{AH_c}{CH_c}, \tag{3}
$$

$$
\frac{1}{\tan B} = \frac{BH_c}{CH_c} = \frac{BH_a}{AH_a}, \tag{4}
$$

$$
\frac{1}{\tan C} = \frac{CH_a}{AH_a} = \frac{CH_b}{BH_b}. \tag{5}
$$

From (3) and (4), (4) and (5), (3) and (5), respectively, we get:

$$\frac{1}{\tan A} + \frac{1}{\tan B} = \frac{AH_c + BH_c}{CH_c} = \frac{c}{CH_c}, \tag{6}$$

$$\frac{1}{\tan B} + \frac{1}{\tan C} = \frac{BH_a + CH_a}{AH_a} = \frac{a}{AH_a}, \tag{7}$$

$$\frac{1}{\tan A} + \frac{1}{\tan C} = \frac{AH_b + CH_b}{BH_b} = \frac{b}{BH_b}. \tag{8}$$

From (1) and (6), (2) and (7), and (1), (2) and (8), respectively, we obtain

$$\frac{2}{\tan B} = \frac{c}{CH_c} + d, \tag{9}$$

$$\frac{2}{\tan B} = \frac{a}{AH_a} - d, \tag{10}$$

$$\frac{2}{\tan B} = \frac{b}{BH_b}. \tag{11}$$

Let $k = \dfrac{2}{\tan B}$. From equations (9), (10) and (11), it follows that:

$$c^2 = (k - d)^2 \cdot (CH_c)^2, \tag{12}$$

$$a^2 = (k + d)^2 \cdot (AH_a)^2, \tag{13}$$

$$b^2 = k^2 \cdot (BH_b)^2. \tag{14}$$

Let Δ be the area of $\triangle ABC$. Since

$$\Delta = \frac{c \cdot CH_c}{2} = \frac{a \cdot AH_a}{2} = \frac{b \cdot BH_b}{2},$$

we have:

$$(CH_c)^2 = \frac{4\Delta^2}{c^2}, \tag{15}$$

$$(AH_a)^2 = \frac{4\Delta^2}{a^2}, \tag{16}$$

$$(BH_b)^2 = \frac{4\Delta^2}{b^2}. \tag{17}$$

Combining (12) with (15), (13) with (16), and (14) with (17), we obtain:

$$c^2 = 2\Delta(k - d) = 2k\Delta - 2d\Delta,$$
$$b^2 = 2\Delta k,$$
$$a^2 = 2\Delta(k + d) = 2k\Delta + 2d\Delta.$$

These equations show that c^2, b^2, and a^2 are in arithmetic progression with common difference $2d\Delta$.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Here is a problem where it is possible to determine the answer without discovering some of the really interesting things that are happening.

Problem (2002 UK Senior Mathematics Challenge)

A square $XABD$ of side length 1 is drawn inside a circle with diameter XY of length 2. The point A lies on the circumference of the circle. Another square $YCBE$ is drawn. What is the ratio of the area of square $XABD$ to the area of square $YCBE$?

With this type of problem (a geometrical one with lengths given), my first instinct sometimes is to use coordinates, because quite often the answer can be hit upon quickly in this way. This is not particularly elegant, but it works!
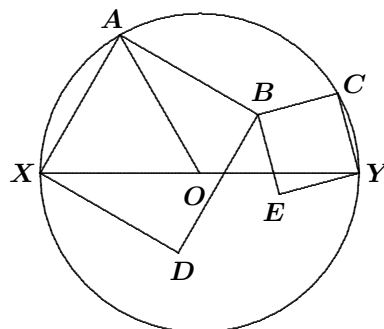
Solution 1

Put the circle in the coordinate plane with its centre, O, at $(0,0)$, X at $(-1, 0)$, and Y at $(1, 0)$.

If we join O to A (which is on the circle), then $OA = OX = 1$, since they are radii, and $AX = 1$, because the square has side length 1. Therefore, $\triangle XAO$ is equilateral; thus, $\angle OXA = 60°$.

With this piece of information and our knowledge of 30–60–90 triangles, we can determine that the coordinates of A are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. In other words, to get from X to A we go $\frac{1}{2}$ unit to the right and $\frac{\sqrt{3}}{2}$ units up.

Since XA and AB are perpendicular, then to go from A to B, we must go $\frac{1}{2}$ unit down and $\frac{\sqrt{3}}{2}$ unit to the right. (We can justify this using similar triangles or vectors or perpendicular slopes.) Therefore, the coordinates of B are $\left(\frac{-1+\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2}\right)$. Now, knowing the coordinates of B and Y, we can determine the length of the diagonal BY of square $YCBE$:

$$
\begin{aligned}
BY &= \sqrt{\left(\frac{-1+\sqrt{3}}{2} - 1\right)^2 + \left(\frac{\sqrt{3}-1}{2} - 0\right)^2} \\
&= \sqrt{\left(\frac{\sqrt{3}-3}{2}\right)^2 + \left(\frac{\sqrt{3}-1}{2}\right)^2} = \sqrt{\frac{12 - 6\sqrt{3}}{4} + \frac{4 - 2\sqrt{3}}{4}} \\
&= \sqrt{4 - 2\sqrt{3}} = \sqrt{3} - 1
\end{aligned}
$$

But $BY = \sqrt{2}\,YC$, and the area of square $YCBE$ is YC^2. Hence, the area of the square is

$$\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 \;=\; \frac{4-2\sqrt{3}}{2} \;=\; 2-\sqrt{3}\,.$$

Thus, the answer to the question is $1 : (2-\sqrt{3})$.

That's a good start—we've at least got an answer to the problem. But, is there a more elegant way? Yes!

Solution 2

Since XY is a diameter, it subtends a right angle anywhere on the circle. Since $\angle XAB = 90°$ then if we extend AB to meet the circle, it must meet the circle at the opposite end of the diameter through X, namely Y. In other words, ABY is a straight line!

Let r be the radius of the circle (we know here that $r = 1$), let p be the side length of square $XABD$ (we know here that $p = 1$), and let q be the side length of square $YCBE$. Since ABY is a straight line with $AB = p$ and $BY = \sqrt{2}q$, then, using the Pythagorean Theorem in triangle XAY, we get
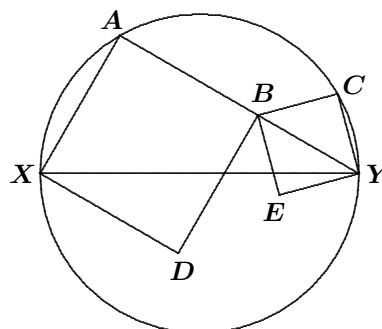
$$
\begin{aligned}
p^2 + (p + \sqrt{2}q)^2 &= (2r)^2\,, \\
p^2 + p^2 + 2\sqrt{2}pq + 2q^2 &= 4r^2\,, \\
\hline
p^2 + \sqrt{2}pq + q^2 &= 2r^2\,.
\end{aligned}
$$

We can solve this for q, using $r = 1$ and $p = 1$, and find again that the desired ratio is $1 : (2-\sqrt{3})$.

Well, that's a bit better. It is more elegant, and more general too, since we didn't use the fact that the radius and the side length of $XABD$ are equal. But wait—there's more! We're still missing something pretty neat.

We know that ABY is a straight line. Rotate this line by 45° counterclockwise about B. Since $\angle ABX = 45°$, the new line passes through X and B. Since $\angle CBY = 45°$, then the new line also passes through C. In other words, XBC is a straight line.

What does this mean? Well, since $\angle XCY = 90°$, the point C must lie on the circle as well! And this works in general—if we draw the square $XABD$ with any (reasonable) side length and A on the circumference of the circle, and if we use B and Y as the diagonal of a new square $YCBE$, then C will always lie on the circumference. Neat! And that's not something we figured out from our original solution.

Pólya's Paragon

What's the difference? (Part 3)

Shawn Godin

In the last two columns ([2004 : 77–78; 2004 : 200–201]) we analyzed sequences by examining the sequences of differences, second differences, and so on. Now we wish to apply our ideas to sequences that are not generated by polynomials.

Consider the sequence $\{a_n\}$ where $a_n = 2^n$ (Question #1 from March's homework). We find that, for this particular sequence, each sequence of differences is equal to the original sequence.

t_n	$_1d_n$	$_2d_n$
1		
	1	
2		1
	2	
4		2
	4	
8		4
	8	
16		

We can extend the sequences easily. We have $_1d_4 = t_4 = 16$; then, since $_1d_4 = t_5 - t_4$, we see that $t_5 = {_1d_4} + t_4 = 16 + 16 = 32$. Now that we know a new term, we can repeat the process to find as many terms as we like. The same process can be used on the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, ... (Question #4 from March's homework).

In general, if any of the sequences of differences can be extended, then we can extend the original sequence. But in some cases the table of differences doesn't yield any patterns. What do we do then?

Consider the sequence $\{b_n\}$ where $b_n = 5^n$ (Question #2 from March's homework). Constructing the table of differences, we get

t_n	$_1d_n$	$_2d_n$	$_3d_n$	$_4d_n$
1				
	4			
5		16		
	20		64	
25		80		256
	100		320	
125		400		
	500			
625				

If we don't see any patterns here, we can consider the *difference table of depth* 2. To construct this, we consider the sequence $t_0, {_1d_0}, {_2d_0}, {_3d_0}, \ldots$, which in our case is 1, 4, 16, 64, 256, ... If nothing appears at this level, we can construct the difference table of depth 3 from the difference table of depth 2 using the same method. If you continue the process you will be led to the powers of 2.

Now we can combine these methods with our inversion formula from the May issue (corrected to remove a typo):

$$t_{n+k} = \sum_{i=0}^{n} \binom{n}{i} {}_i d_k .$$

Example: 0, 2, 9, 31, 97, 291, ...
Constructing the difference table, we get

t_n	${}_1 d_n$	${}_2 d_n$	${}_3 d_n$	${}_4 d_n$	${}_5 d_n$
0					
	2				
2		5			
	7		10		
9		15		19	
	22		29		36
31		44		55	
	66		84		
97		128			
	194				
291					

This isn't much help. Let's construct the difference table of depth 2. Starting with the sequence $\{s_n\} = \{0, 2, 5, 10, 19, 36, \ldots\}$, we get

s_n	${}_1 d_n'$	${}_2 d_n'$	${}_3 d_n'$
0			
	2		
2		1	
	3		1
5		2	
	5		2
10		4	
	9		4
19		8	
	17		
36			

We see that ${}_i d_n = 2^n$ for $i \geq 2$. Using our inversion formula with $k = 0$, we get

$$s_n = \sum_{i=0}^{n} \binom{n}{i} {}_i d_0 = 0 + 2n + \sum_{i=2}^{n} \binom{n}{i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} + n - 1 = 2^n + n - 1 .$$

Hence, applying the inversion formula to the original sequence, we obtain

$$t_n = \sum_{i=0}^{n} \binom{n}{i} \left(2^i + i - 1 \right) .$$

You should now have a variety of tools for attacking sequences. Until next time, happy problem solving!

THE OLYMPIAD CORNER
No. 240

R.E. Woodrow

Our first set of problems for this issue comes from Argentina. My thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use, and to Alberto Nettel and Luz Palacios for translating them into English.

XVII ARGENTINIAN MATHEMATICAL OLYMPIAD
National Competition
Level Three

1. The natural numbers are written in succession, forming a sequence of digits. Determine how many numerical characters are in the natural number that contributes to this sequence the digit at position 10^{2000}.

Note: The natural number that contributes to the sequence the digit at position 10 has 2 numerical characters because it is 10; the natural number that contributes to the sequence the digit at position 10^2 has 2 numerical characters because it is 55.

2. Given a triangle ABC with side AB greater than BC, let M be the midpoint of AC, and let L be the point at which the bisector of $\angle B$ cuts side AC. A straight line is drawn through M parallel to AB, cutting the bisector BL at D, and another straight line is drawn through L parallel to BC, cutting the median BM at E. Show that ED is perpendicular to BL.

3. A board has 32 rows and 10 columns. Paul writes in each square of his board either 1 or -1. Matthew, with Paul's board in sight, chooses one or more columns from his own board and, in these columns, negates all of Paul's numbers (putting 1 where Paul has -1, and -1 where Paul has 1). In the rest of the columns he puts the same numbers as Paul's.

Matthew wins if he can get each row of his board to be different from each row of Paul's board. Otherwise (that is, if any row of Matthew's board is the same as any row of Paul's board), Paul wins.

If they both play to perfection, determine which of them has victory guaranteed.

4. Determine how many pairs of natural numbers (a, b) there are such that 4620 is a multiple of a, 4620 is a multiple of b, and b is a multiple of a.

5. A computer program generates a sequence of numbers by the following rule. The first number is written by Camilo. From that point on, the program does integer division of the last generated number by 18, obtaining an integer quotient and remainder. The sum of this integer quotient and this remainder becomes the next generated number. For example, if Camilo's number is 5291, the computer does $5291 = 293 * 18 + 17$, and generates the number $310 = 293 + 17$. The next generated number will be 21, given that $310 = 17 * 18 + 4$ and $17 + 4 = 21$, etc.

For every initial number that Camilo chooses, there comes a point at which the computer generates the same number over and over. Determine which number this will be, if Camilo's initial number is 2^{110}.

6. An equilateral triangle with area equal to 9 is made out of paper. It is folded in two along a straight line passing through the centre of the triangle and not passing through any of the vertices of the triangle. The result is a four-sided polygon formed by the overlap of the two pieces (created by the fold) and three triangles with no overlap. Determine the minimum possible area of the four-sided polygon created by the overlap.

As the second problem set for this number, we give the 12th Form of the XXI Albanian Mathematical Olympiad for High Schools, Third Round. Thanks again go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them.

XXI ALBANIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOLS
Third Round
March 20, 2000

1. (a) Prove the inequality

$$\frac{(1+x_1)(1+x_2)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} \leq 2^{n-1}, \quad \forall x_1, x_2, \ldots, x_n \in [1, +\infty).$$

(b) When does the equality hold?

2. Consider the sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for all $n > 1$. Find

(a) $\lim\limits_{n\to\infty} x_n$;

(b) $\lim\limits_{n\to\infty} 4^n(2 - x_n)$.

3. Prove that, if $0 < a < b < \dfrac{\pi}{2}$, then

(a) $\dfrac{a}{b} < \dfrac{\sin a}{\sin b}$;

(b) $\dfrac{\sin a}{\sin b} < \dfrac{\pi}{2}\dfrac{a}{b}$.

4. (a) Prove that for any convex n–gon, where $n > 4$, the arithmetic mean of the lengths of its sides is no greater than the arithmetic mean of the lengths of its diagonals.

    (b) Does equality hold for any case?

5. Let a, b, c be the sides of a triangle, and let α, β, γ be the angles opposite the sides a, b, c, respectively.

(a) Prove that $\gamma = 2\alpha$ if and only if $c^2 = a(a + b)$.

(b) Find all triangles such that a, b, c are natural numbers, b is a prime, and $\gamma = 2\alpha$.

MATHEMATICS COMPETITIONS IN FINLAND, 2000–2001
High School Mathematics Contest, Senior Division, 1st Round
October 10, 2000 — Time: 100 minutes

1. Four unit discs are packed together as shown in the figures. The discs are enclosed by a curve of minimal length. In which one of the configurations is the enclosing curve shorter?

2. Determine real x and y such that $5x^2 - 5y^2 - 24xy + 11y + 3x = 0$.

3. Determine all positive integers m and n such that

$$m^2 - n^2 = 270.$$

4. A number of cross-shaped pieces, as shown, are placed on an 8×8 chessboard in such a way that the squares of the pieces and the squares of the chessboard are aligned and the pieces do not overlap each other. We say that the board has been filled if no more pieces can be placed on the board satisfying the conditions above. Determine the smallest possible number of pieces with which the board can be filled.
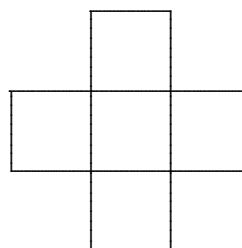
MATHEMATICS COMPETITIONS IN FINLAND, 2000–2001
High School Mathematics Contest, Senior Division, Final Round
February 2, 2001 — Time: 180 minutes

1. Let ABC be a right triangle with hypotenuse AB and altitude CF, where F lies on AB. The circle through F centred at B and another circle of the same radius centred at A intersect on the side CB. Determine $FB : BC$.

2. Two non-intersecting curves have equations $y = ax^2 + bx + c$ and $y = dx^2 + ex + f$, where $ad < 0$. Prove that there exists a straight line having no points in common with the two curves.

3. The positive integers a, b, and c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

4. In the weekly State Lottery, a sequence of seven numbers is picked at random. Each number may be any of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Determine the probability that the sequence is composed of only five different numbers.

5. Determine $n \in \mathbb{N}$ such that $n^2 + 2$ divides $2 + 2001n$.

Now we turn to the solutions we have received to problems of the St. Petersburg Contests 1965–1984 given in May, 2002 [2002: 201–203].

1. There are n glasses each big enough to hold all the water. Initially, all glasses contain the same amount of water. It is allowed to pour from any glass to any other glass as much water as in the second glass. For which values of n is it possible to collect all water into one glass?

Solution by Pierre Bornsztein, Maisons-Laffitte, France, adapted by the editor.

The integers n for which it is possible to put all the water into one glass are those such that $n = 2^k$ for some non-negative integer k.

Let G_1, G_2, ..., G_n denote the glasses. We will denote by (i, j) the operation which consists of pouring water from G_i to G_j.

If $n = 1$, then the glass G_1 contains all the water. If $n = 2$, then the operation $(1, 2)$ puts all the water into G_2. If $n = 2^k$ where $k > 1$, then the sequence $(1, 2)$, $(3, 4)$, ..., $(2^k - 1, 2^k)$ gives a configuration with 2^{k-1} glasses containing equal amounts of water. Repeating this process, we reach a configuration where all the water is in a single glass, which proves that $n = 2^k$ is a possible value for n.

Now consider any $n \geq 2$ for which there is a finite sequence of operations by which all the water can be put into one glass, say G_n. Consider any such sequence, and let N be the number of operations in the sequence. Let x be the initial amount of water in each glass. Thus, the total amount of water is nx. Without any loss of generality, we may assume that the last operation in the sequence is $(n-1, n)$. Just before this operation, both G_n and G_{n-1} contained an amount of water equal to $nx/2$, and every other glass was empty.

We proceed by descending induction. Let i be an integer such that $0 < i \leq N$. Suppose that after i operations, for each $j \in \{1, 2, \ldots, n\}$, the amount of water contained in G_j is of the form $p_j nx / 2^{a_j}$, where a_j and p_j are non-negative integers.

Let (ℓ, m) be the ith operation. For each glass G_j other than G_ℓ and G_m, the amount of water in G_j is the same before the ith operation as after it. By our induction hypothesis, the amount of water in G_m after the ith operation is $p_m nx / 2^{a_m}$; therefore, the amount before the ith operation was $p_m nx / 2^{a_m + 1}$, which is of the desired form. The amount of water in G_ℓ after the ith operation is $p_\ell nx / 2^{a_\ell}$; therefore, the amount before this operation was $\left(\frac{p_\ell}{2^{a_\ell}} + \frac{p_m}{2^{a_m + 1}} \right) nx$, which can be expressed in the desired form.

Thus, immediately after the $(i-1)^{\text{th}}$ operation in the sequence, for each $j \in \{1, 2, \ldots, n\}$, the amount of water contained in G_j is of the form $p_j nx / 2^{a_j}$, where a_j and p_j are non-negative integers. This ends the induction step.

It follows that the initial amount of water in G_n is $x = pnx / 2^a$, where a and p are non-negative integers. Thus, $n = 2^a / p$, from which we deduce that p is a power of 2 and so is n.

2. The point C is on the segment AB. A straight line through C intersects the circle with diameter AB at E and F, the circle with diameter AC again at M, and the circle with diameter BC again at N. Prove that $MF = EN$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; Toshio Seimiya, Kawasaki, Japan; and Bob Serkey, Leonia, NJ, USA. We give the write-up of Amengual Covas.

In the figure, the line AM is extended to meet the circle with diameter AB at M'. Since AC and AB are diameters, $\angle AMC$ and $\angle AM'B$ are right angles. The two chords EF and BM', both perpendicular to AM', are parallel to each other. Hence, $EFM'B$ is an isosceles trapezoid.


Since BC is a diameter, $\angle BNC = 90°$. The right triangles MFM' and NEB are congruent, and hence, $MF = EN$.
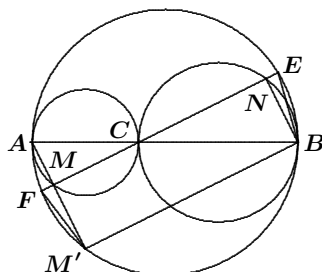
4. The sides of a heptagon $A_1A_2A_3A_4A_5A_6A_7$ have equal length. From a point O inside, perpendiculars are dropped to the sides A_1A_2, A_2A_3, ..., A_7A_1, meeting them, and not their extensions, at H_1, H_2, ..., H_7, respectively. Prove that

$$A_1H_1 + A_2H_2 + \cdots + A_7H_7 \;=\; H_1A_2 + H_2A_3 + \cdots + H_7A_1 \,.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's write-up.
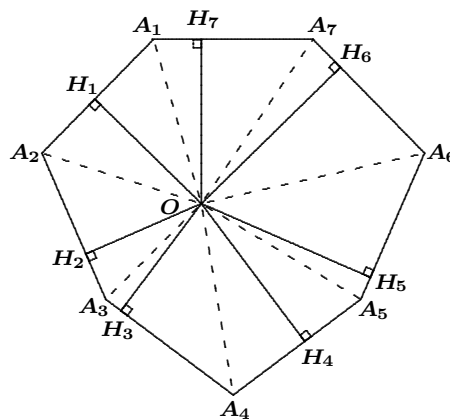
We let a denote the length of the sides of the heptagon, and we put $A_8 = A_1$. For $i = 1$, 2, ..., 7, since $OH_i \perp A_iA_{i+1}$, we have

$$
\begin{aligned}
OA_i^2 - OA_{i+1}^2 &= A_iH_i^2 - H_iA_{i+1}^2 \\
&= (A_iH_i + H_iA_{i+1})(A_iH_i - H_iA_{i+1}) \\
&= a(A_iH_i - H_iA_{i+1})\,.
\end{aligned}
$$

Thus,

$$\sum_{i=1}^{7}(OA_i^2 - OA_{i+1}^2) \;=\; a\sum_{i=1}^{7}(A_iH_i - H_iA_{i+1})\,.$$

Since the sum on the left side is equal to zero, the sum on the right side is also zero. Consequently,

$$\sum_{i=1}^{7} A_iH_i \;=\; \sum_{i=1}^{7} H_iA_{i+1}\,,$$

which is the desired result.

5. There are $2N$ people at a party. Each knows at least N others. Prove that one can always choose four people and place them at a round table so that each person knows both neighbours.

Solution by Pierre Bornsztein, Maisons–Laffitte, France.

We assume that the relation "A knows B" is symmetric. We also assume that $N \geq 2$, since there is no set of four distinct persons if $N = 0$ or $N = 1$.

First consider the case $N = 2$. Let A, B, C, D be the people. If they all know one another, then each person at the table will know his neighbours, no matter what seating arrangement is used. Otherwise, we may suppose that A does not know C. Then A and C must each know B and D, and the seating arrangement A–B–C–D is a solution.

Now suppose $N \geq 3$. Consider the simple graph whose vertices are the people, two of which are joined by an edge if and only if they know each other. The problem is to prove that this graph contains a quadrilateral (that is, a 4–circuit). According to [1] (exercise 361, p. 69), every simple graph with $2N$ vertices and at least $\dfrac{N}{2}(1 + \sqrt{8N - 3})$ edges contains a quadrilateral.

Let A be any one of the vertices, and let $d(A)$ be its degree in the graph. From the given information, we have $d(A) \geq N$. Since any edge AB is counted in both $d(A)$ and $d(B)$, we deduce that the total number of edges is at least N^2. It is easy to verify that $N^2 \geq \dfrac{N}{2}(1 + \sqrt{8N - 3})$ for $N \geq 3$. The conclusion follows.

Reference

[1] L. Lovász, *Combinatorial Problems and Exercises*, North–Holland, 1979.

6. Prove that any non–negative even integer can be uniquely represented as $(x + y)^2 + 3x + y$ where x and y are non–negative integers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bornsztein's solution, adapted by the editor.

Let n be a non–negative even integer, and let x and y be non–negative integers such that
$$(x + y)^2 + 3x + y = n$$
It is easy to see that $n = 0$ if and only if $x = y = 0$. From now on, we assume that $n \geq 2$.

Let a be the positive integer such that $a^2 \leq n < (a + 1)^2$. Note that $a^2 + a - 2$ and $a^2 + a$ are two consecutive even integers. Consequently, the following two cases exhaust all possibilities:

Case 1. $a^2 + a \leq n < (a + 1)^2$.

If $x + y \geq a + 1$, then
$$n = (x + y)^2 + 3x + y \geq (a + 1)^2 > n,$$
a contradiction. If $x + y \leq a - 1$, then
$$\begin{aligned} n &= (x + y)^2 + 3x + y \leq (a - 1)^2 + 3(a - 1) \\ &= a^2 + a - 2 < a^2 + a \leq n, \end{aligned}$$

a contradiction. Therefore, $x + y = a$. Then

$$n = (x + y)^2 + 3x + y = a^2 + a + 2x.$$

Hence, $x = \frac{1}{2}(n - a^2 - a)$ and $y = a - x = \frac{1}{2}(a^2 + 3a - n)$.

Now let x and y be defined by these expressions. Since n is even and $a^2 \equiv a \pmod 2$, both x and y are integers. Since $a \geq 1$, we have

$$a^2 + 3a \geq a^2 + 2a + 1 = (a + 1)^2 > n,$$

from which we deduce that $y > 0$. Since $n \geq a^2 + a$ (for this case), we also have $x > 0$. Thus, the pair (x, y) gives a unique representation of n in the desired form.

Case 2. $a^2 \leq n \leq a^2 + a - 2$.

If $x + y \leq a - 2$, then

$$\begin{aligned} n = (x + y)^2 + 3x + y &\leq (a - 2)^2 + 3(a - 2) \\ &= a^2 - a - 2 < a^2 \leq n, \end{aligned}$$

a contradiction. If $x + y \geq a$, then

$$n = (x + y)^2 + 3x + y \geq a^2 + a + 2x \geq a^2 + a > n,$$

a contradiction. Therefore, $x + y = a - 1$. Then

$$n = (x + y)^2 + 3x + y = (a - 1)^2 + a - 1 + 2x = a^2 - a + 2x.$$

Hence, we have $x = \frac{1}{2}(n - a^2 + a)$ and $y = a - 1 - x = \frac{1}{2}(a^2 + a - n - 2)$.

Now let x and y be defined by these expressions. As in Case 1, both x and y are integers. Since $a^2 \leq n$ and $a > 0$, we have $x > 0$. Since $n \leq a^2 + a - 2$ (for this case), we also have $y > 0$. Thus, the pair (x, y) gives a unique representation of n in the desired form.

7. In triangle ABC, the sides satisfy $AB + AC = 2BC$. Prove that the bisector of $\angle A$ is perpendicular to the line segment joining the incentre and circumcentre of ABC.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's argument.

We assume that $AB \neq AC$. Let I and O be the incentre and circumcentre of $\triangle ABC$, respectively, and let D be the intersection of AI with BC. Since BI and CI bisect $\angle ABD$ and $\angle ACD$, respectively, we have

$$\begin{aligned} \frac{AI}{ID} &= \frac{AB}{BD} = \frac{AC}{CD} \\ &= \frac{AB + AC}{BD + CD} = \frac{AB + AC}{BC} = 2. \end{aligned}$$
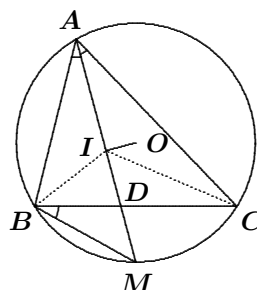
Hence, $AB = 2BD$ and $AI = 2ID$. Consequently,

$$AD = 3ID. \tag{1}$$

Let M be the second intersection of AD with the circumcircle of $\triangle ABC$. Then

$$\angle MBD = \angle MBC = \angle MAC = \angle MAB.$$

Since $\angle BMD = \angle BMA$, we get $\triangle MBD \sim \triangle MAB$. It follows that

$$\frac{DM}{BM} = \frac{BM}{AM} = \frac{BD}{AB} = \frac{1}{2}.$$

Thus,

$$\frac{DM}{AM} = \frac{DM}{BM} \cdot \frac{BM}{AM} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Hence, $AM = 4DM$, from which we obtain

$$AD = 3DM. \tag{2}$$

From (1) and (2) we get $ID = DM$. Thus, $AI = 2ID = ID + DM = IM$. Therefore, $OI \perp AM$. This implies that $AI \perp OI$.

[*Ed.* See the solution to Problem 2870 later in this issue, especially the editorial comments following the solution.]

9. Four pedestrians were moving at uniform velocities along four straight roads in general positions. Two of them met each other as well as the other two. Prove that the other two also met.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give Bradley's write-up.
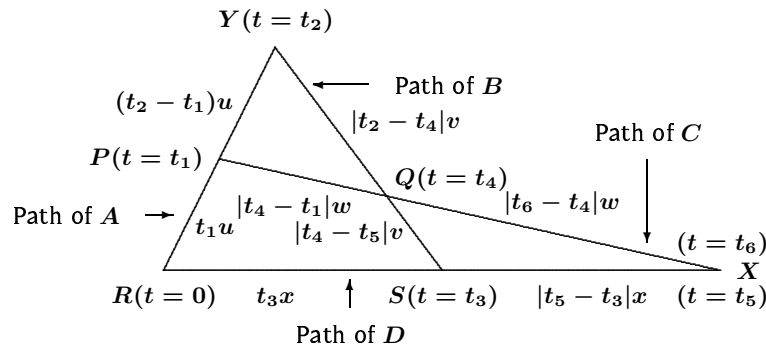
Denote the pedestrians by A, B, C, D. Without loss of generality, we may assume that the first pair to meet was A and D, and that they met at time 0. Next suppose that A and C met at t_1 and that A and B met at t_2, where $t_2 > t_1 > 0$. Suppose further that B and D met at t_3 and that B and C met at t_4. We want to show that C and D also met.

The roads along which C and D were travelling meet at a point X. Suppose that D reached X at time t_5 and C reached X at time t_6. Distances

are as shown in the figure, assuming that t_4 is between t_2 and t_3. We have also denoted by u, v, w, and x the respective speeds of A, B, C, and D.

Using Menelaus' Theorem on $\triangle YRS$ with transversal PQX, we have

$$\frac{RX}{XS} \cdot \frac{SQ}{QY} \cdot \frac{QP}{PR} = -1 = \frac{t_5}{t_3 - t_5} \cdot \frac{t_4 - t_3}{t_2 - t_4} \cdot \frac{t_1 - t_2}{-t_1}. \tag{1}$$

Using Menelaus' Theorem on $\triangle PRX$ with transversal YQS, we have

$$\frac{PY}{YR} \cdot \frac{RS}{SX} \cdot \frac{XQ}{QP} = -1 = \frac{t_2 - t_1}{-t_2} \cdot \frac{t_3}{t_5 - t_3} \cdot \frac{t_4 - t_6}{t_1 - t_4}. \tag{2}$$

Equation (1) gives $t_1 t_2 t_3 + t_1 t_3 t_5 + t_2 t_4 t_5 = t_1 t_2 t_5 + t_1 t_3 t_4 + t_2 t_4 t_5$, and equation (2) gives $t_1 t_2 t_3 + t_1 t_3 t_6 + t_2 t_4 t_5 = t_1 t_2 t_5 + t_1 t_3 t_4 + t_2 t_3 t_6$. Subtracting yields $(t_1 t_3 - t_2 t_3)(t_5 - t_6) = 0$. But $t_1 \neq t_2$; whence, $t_6 = t_5$, implying that C and D meet.

[*Ed.* Here is another approach. We regard the time t as a third dimension in the problem, along with the two spatial dimensions x and y. As the pedestrians move in the xy–plane, they follow paths in xyt–space. Since each pedestrian moves at a uniform velocity along a straight line in the xy–plane, the corresponding paths of the pedestrians in xyt–space are straight lines. Two pedestrians meet at some time if and only if their paths in xyt–space intersect.

Let the paths of the pedestrians in xyt–space be the lines L_1, L_2, L_3, and L_4. According to the given information, each pair of these lines intersects, with one possible exception—say L_3 and L_4. The three lines L_1, L_2, and L_3 must be coplanar, since each pair among them has an intersection. Similarly, L_1, L_2, and L_4 are coplanar. Therefore, all four lines lie in the same plane. Now we note that L_3 and L_4 cannot be parallel, because this would imply that pedestrians 3 and 4 were walking on parallel roads (at the same speed), which is not allowed from the problem statement. (One way to see this is to consider the projections of L_3 and L_4 onto the three coordinate planes in xyt–space.) Therefore, L_3 and L_4 intersect. Thus, pedestrians 3 and 4 meet.]

10. At King Arthur's Court, $2n$ knights gathered at the Round Table. Each has at most $n - 1$ enemies among the others. Prove that Merlin the wizard can devise a seating arrangement such that no knight will be next to any of his enemies.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We assume that the relation "A is not an enemy of B" is symmetric. Consider the simple graph whose vertices are the knights, two of which are joined by an edge if and only if they are not enemies of each other. The problem is then to prove that this graph contains a Hamiltonian circuit.

Let K_1, K_2, $\ldots$, K_{2n} be the knights, and, for $i = 1$, 2, $\ldots$, $2n$, let d_i be the degree of K_i. From the statement of the problem, we have $d_i \geq n$ for each i. Thus, $\min_i \{d_i\} \geq n$.

But, according to [1] (exercise 21-a, p. 67), every simple graph with k vertices satisfying $\min_i \{d_i\} \geq \frac{k}{2}$ (this is known as Dirac's condition) has a Hamiltonian circuit. The conclusion follows.

Reference

[1] L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, 1979.

11. Construct a set of circles with non-zero radii such that exactly one of them passes through each point of three-dimensional space.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem, with its solution, appears as Exercise 12G in P. Halmos, *Problems for Mathematicians Young and Old*, MAA, 1991.

12. Does there exist a positive integer n such that

$$27^n + 84^n + 110^n + 133^n = 144^n ?$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; D.J. Smeenk, Zaltbommel, the Netherlands. We give Bornsztein's solution.

Let

$$f(x) = \left(\frac{27}{144}\right)^x + \left(\frac{84}{144}\right)^x + \left(\frac{110}{144}\right)^x + \left(\frac{133}{144}\right)^x$$

defined on $(0, \infty)$. It is easy to see that the function f is decreasing. Then the equation $f(x) = 1$ has at most one solution in positive real numbers. Since $f(5) = 1$, it follows that $n = 5$ is the only solution of the problem.

14. Prove that

$$\sum_{i,j=1}^{\infty} \frac{a_i a_j}{i+j} \leq \pi \sum_{k=1}^{\infty} a_k^2 .$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

This is a special case ($p = 2$) of Theorem 315 in [1], which states: Let $p > 1$ and $p' = \dfrac{p}{p-1}$. If $\sum\limits_{n=1}^{\infty} a_n^p \leq A$ and $\sum\limits_{n=1}^{\infty} b_n^{p'} \leq B$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} A^{\frac{1}{p}} B^{\frac{1}{p'}} .$$

Moreover, the constant $\dfrac{\pi}{\sin(\pi/p)}$ is the best possible.

Reference

[1] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Mathematical Library, 1952, p. 226–228.

16. Decompose $235^2 + 972^2$ into two factors.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Christopher J. Bradley, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the write-up of Amengual Covas.

Since

$$235^2 + 972^2 = 55225 + 944784 = 1000009 = 1000^2 + 3^2 ,$$

we apply the identity

$$(ab + cd)^2 + (ad - bc)^2 \ = \ (a^2 + c^2)(b^2 + d^2) \,,$$

with $a = 17$, $b = 58$, $c = 2$, $d = 7$, to obtain

$$1000^2 + 3^2 \ = \ (17^2 + 2^2)(58^2 + 7^2) \ = \ 293 \cdot 3413 \,;$$

whence,

$$235^2 + 972^2 \ = \ 293 \cdot 3413 \,.$$

17. Students in a school go for ice cream in groups of at least two. No two students will go together more than once. After $k > 1$ groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most k.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First note that the published statement of this problem was obviously wrong, since it asked for proof that "the number of students in the school is *at least* k". We prove the corrected version as stated above.

The given conditions have the following consequences:

(a) Any two groups have at most one student in common. (Otherwise, two students would be together in two different groups, contrary to the hypothesis.)

(b) No single group contains all the students. (Otherwise, we could not have another group of at least two students, as required, which would contradict (a).)

(c) Each student belongs to at least two groups. (If some student were in only one group, then either that group would contain all the students, contradicting (b), or else the student would not be together with every other student exactly once, contradicting the hypothesis.)

Let S be the set of all the students, and let $n = |S|$. If $x \in S$, we denote by $d(x)$ the number of groups containing x.

Let a be a student. By (c) above, there are two distinct groups G and G' such that $a \in G$ and $a \in G'$. Since $|G| \geq 2$ and $|G'| \geq 2$, there exist students $b \in G$ and $c \in G'$ such that $b \neq a$ and $c \neq a$. Moreover, by (a), we have $b \notin G'$ and $c \notin G$. There is a group G'' which contains b and c (since b and c must be together once). Using (a), we have $a \notin G''$. It follows that $d(a) < k$.

We have

$$\sum_{\substack{(x,G) \\ x \in G}} 1 \ = \ \sum_{x} \left(\sum_{\substack{G \\ x \in G}} 1 \right) \ = \ \sum_{x} d(x) \,.$$

On the other hand,

$$\sum_{\substack{(x,G) \\ x \in G}} 1 \;=\; \sum_{G} \left(\sum_{\substack{x \\ x \in G}} 1 \right) \;=\; \sum_{G} |G| \,.$$

Therefore,

$$\sum_{x} d(x) \;=\; \sum_{G} |G| \,. \tag{1}$$

Let x be a student, and let G be a group not containing x. For each student $a \in G$, there exists a group G_a which contains a and x. Note that $G \neq G_a$, since $x \notin G$. Moreover, if $b \in G$ and $b \neq a$, the groups G_a and G_b are distinct (since G is the unique group containing both a and b). It follows that

$$d(x) \geq |G| \,. \tag{2}$$

Now suppose, for a contradiction, that $n > k$. Then, from (2), for each pair (x, G) such that $x \notin G$, we have

$$0 \;<\; k - d(x) \;\leq\; k - |G| \;<\; n - |G| \,.$$

Then

$$\frac{|G|}{n - |G|} \;<\; \frac{d(x)}{k - d(x)} \,. \tag{3}$$

Summing all the inequalities (3) obtained for all the pairs (x, G) such that $x \notin G$, we get

$$\sum_{\substack{(x,G) \\ x \notin G}} \frac{|G|}{n - |G|} \;<\; \sum_{\substack{(x,G) \\ x \notin G}} \frac{d(x)}{k - d(x)} \,. \tag{4}$$

Let G be a group. The number of students not belonging to G is $n - |G|$. Therefore,

$$\sum_{\substack{(x,G) \\ x \notin G}} \frac{|G|}{n - |G|} \;=\; \sum_{G} \left(\sum_{\substack{x \\ x \notin G}} \frac{|G|}{n - |G|} \right)$$

$$=\; \sum_{G} (n - |G|) \frac{|G|}{n - |G|} \;=\; \sum_{G} |G| \,.$$

Also,

$$\sum_{\substack{(x,G) \\ x \notin G}} \frac{d(x)}{k - d(x)} \;=\; \sum_{x} \left(\sum_{\substack{G \\ x \notin G}} \frac{d(x)}{k - d(x)} \right)$$

$$=\; \sum_{x} (k - d(x)) \frac{d(x)}{k - d(x)} \;=\; \sum_{x} d(x) \,.$$

It follows that (4) may be rewritten as

$$\sum_G |G| < \sum_x d(x) \, ,$$

which contradicts (1). Then $n \le k$, as claimed.

Remark. The inequality is the best possible, in the following sense. Let $S = \{x_1, x_2, \ldots, x_n\}$. If $G_1 = \{x_2, \ldots x_n\}$, and $G_i = \{x_1, x_i\}$ for $i = 2, \ldots, n$, then the conditions of the problem are satisfied and $k = n$.

18. We choose 2^{p-1} subsets from a set with p elements such that any three have a common element. Prove that they all have a common element.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Lemma. Let $n \ge 1$ be an integer, and let S be a set with n elements. Let A_1, $A_2, \ldots, A_k$ be distinct subsets of S such that, for all i and j, the intersection $A_i \cap A_j$ is not empty. Then $k \le 2^{n-1}$.

Proof of lemma. For $i = 1, \ldots, k$, let $B_i = S \backslash A_i$. Then $B_1, B_2, \ldots, B_k$ are distinct subsets of S. If $B_i = A_j$ for some i and j, then we find that $A_i \cap A_j = A_i \cap (S \backslash A_i) = \emptyset$, which contradicts the hypothesis. It follows that $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ are distinct subsets of S. Thus, $2k \le 2^n$; that is, $k \le 2^{n-1}$. ∎

We now prove the desired result by induction for $p \ge 3$. (If $p = 1$ or $p = 2$, then the condition that any three subsets have a common element is vacuously true, since we can choose only $2^{p-1} < 3$ subsets. The result is false for these cases.)

If $p = 3$, let $S = \{x_1, x_2, x_3\}$. We choose four sets A_1, A_2, A_3, A_4 among $\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}$, such that the intersection of any three of them is non-empty. Clearly, none of the four can be $\emptyset$. Also, the sets $\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}$ cannot all be chosen, because their intersection is empty. It follows that one of the 1–element sets must be chosen, say $\{x_1\}$. Then each of the chosen sets must contain x_1. Thus, the conclusion holds for $p = 3$.

Let $p \ge 4$ be a fixed integer. Suppose that the result holds for a set with $p - 1$ elements. Let $S = \{x_1, x_2, \ldots, x_p\}$, and let $A_1, A_2, \ldots, A_{2^{p-1}}$ be distinct subsets of S such that

$$A_i \cap A_j \cap A_k \ne 0, \quad \text{for all } i < j < k.$$

Let n_p be the number of sets A_i that contain x_p. If $n_p < 2^{p-2}$, then the number of sets A_i not containing x_p is $2^{p-1} - n_p > 2^{p-1} - 2^{p-2} = 2^{p-2}$. These sets form a collection of more than 2^{p-2} distinct subsets of $\{x_1, x_2, \ldots, x_{p-1}\}$ such that $A_i \cap A_j \ne \emptyset$ for all i and j, contradicting the lemma. Thus $n_p \ge 2^{p-2}$.

With no loss of generality, we may now suppose that x_p belongs to $A_1, A_2, \ldots, A_{n_p}$.

Case 1. $n_p > 2^{p-2}$.

Let $A_i' = A_i \setminus \{x_p\}$ for $i = 1, 2, \ldots, n_p$. Then $A_1', A_2', \ldots, A_{n_p}'$ form a collection of more than 2^{p-2} distinct subsets of $\{x_1, x_2, \ldots, x_{p-1}\}$. Using the lemma, we deduce that there exist r and s such that $A_r' \cap A_s' = \emptyset$. It follows that $A_r \cap A_s = \{x_p\}$. For each $k \notin \{r, s\}$, since $A_r \cap A_s \cap A_k \neq \emptyset$, we have $A_r \cap A_s \cap A_k = \{x_p\}$, from which we deduce that x_p is a common element of all the A_i's.

Case 2. $n_p = 2^{p-2}$.

Let k be such that $x_p \notin A_k$ and define $B_k = A_k \cup \{x_p\}$. Then $B_k \neq A_k$. Suppose that $B_k \neq A_j$ for all j. Then the sets $A_1, A_2, \ldots, A_{2^{p-1}}, B_k$ form a collection of $2^{p-1} + 1$ distinct subsets of S, and the intersection of any pair of distinct sets in this collection is non-empty, contradicting the lemma.

Therefore, for each k such that $x_p \notin A_k$, there exists j such that $A_k \cup \{x_p\} = A_j$. Since the number of A_i's which do not contain x_p is the same as the number of A_i's which contain x_p, we deduce that the collection of the A_i's is

$$A_1, A_2, \ldots, A_{n_p}, A_1 \cup \{x_p\}, A_2 \cup \{x_p\}, \ldots, A_{n_p} \cup \{x_p\},$$

where $x_p \notin A_i$ for $i = 1, 2, \ldots, n_p$.

Now $A_1, A_2, \ldots, A_{n_p}$ are 2^{p-2} distinct subsets of $\{x_1, x_2, \ldots, x_{p-1}\}$ such that, for all $i < j < k$, we have $A_i \cap A_j \cap A_k \neq \emptyset$. By the induction hypothesis, they share a common element, say a. Clearly, $a \in A_i \cup \{x_p\}$ for each i, and therefore, a is a common element of all the A_i's. This ends the induction step and the proof.

19. Let a, b and c be real numbers with sum 0. Prove that

$$\frac{a^7 + b^7 + c^7}{7} = \left(\frac{a^5 + b^5 + c^5}{5}\right)\left(\frac{a^2 + b^2 + c^2}{2}\right).$$

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; Pierre Bornsztein, Maisons-Laffitte, France; and Vedula N. Murty, Dover, PA, USA. We give Bradley's solution.

Let a, b, c be the roots of $x^3 + qx - r = 0$, and let $S_k = a^k + b^k + c^k$. Then $S_1 = a + b + c = 0$ and

$$\begin{aligned}
S_2 &= a^2 + b^2 + c^2 \\
&= (a + b + c)^2 - 2(ab + bc + ca) = -2q
\end{aligned}$$

Now we have

$$\begin{aligned}
S_3 + qS_1 - 3r &= 0 &\implies& & S_3 &= 3r \\
S_4 + qS_2 - rS_1 &= 0 &\implies& & S_4 &= 2q^2 \\
S_5 + qS_3 - rS_2 &= 0 &\implies& & S_5 &= -3qr - 2qr = -5qr \\
S_7 + qS_5 - rS_4 &= 0 &\implies& & S_7 &= 5q^2r + 2q^2r = 7q^2r.
\end{aligned}$$

Then $\dfrac{S_7}{7} = q^2r = \dfrac{S_5}{5} \cdot \dfrac{S_2}{2}$.

21. Segments AC and BD intersect at point E. Points K and M, on segments AB and CD, respectively, are such that the segment KM passes through E. Prove that $KM \leq \max\{AC, BD\}$.

Solved by Christopher J. Bradley, Bristol, UK; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

The following lemma is well known, and its proof will not be given.

Lemma. If P is a point on the side BC of triangle ABC, then $AP \leq \max\{AB, AC\}$.

Case 1. $AB \parallel CD$. (See the diagram on the left below.)
Let X, Y be points on CD such that $KX \parallel AC$ and $KY \parallel BD$. Since $AKXC$ and $BKYD$ are both parallelograms, we have
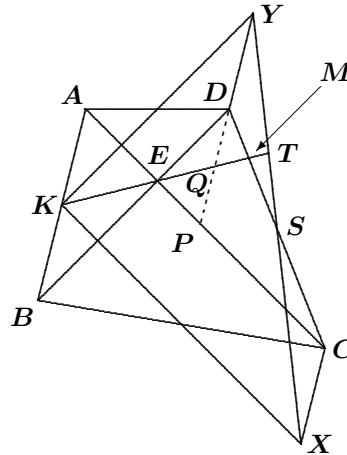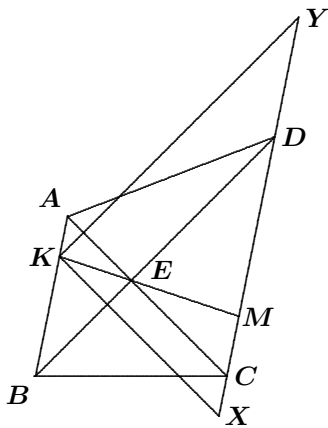
$$KX = AC \quad \text{and} \quad KY = BD.$$

Since M is a point on the segment XY, we have, by the lemma,

$$KM \leq \max\{KX, KY\}.$$

Therefore,

$$KM \leq \max\{AC, BD\}.$$

Case 2. $AB \nparallel CD$. (See the diagram on the right above.)
In this case, when we consider quadrilateral $ABCD$, we have either $\angle A + \angle D > 180°$ or $\angle B + \angle C > 180°$. We may assume without loss of generality that $\angle A + \angle D > 180°$. The line through D parallel to AB meets EC and EM at P and Q, respectively. Then P and Q are points on the segments EC and EM, respectively. By Menelaus' Theorem for $\triangle DPC$, we have

$$\frac{PQ}{QD} \cdot \frac{DM}{MC} \cdot \frac{CE}{EP} = 1.$$

Therefore,

$$\frac{PQ}{QD} \cdot \frac{DM}{MC} = \frac{EP}{CE} < 1 \,;$$

that is, $\frac{CM}{MD} > \frac{PQ}{QD}$. Since $DP \parallel AB$, we get $\frac{PQ}{QD} = \frac{AK}{KB}$. Thus,

$$\frac{CM}{MD} > \frac{AK}{KB} \,. \tag{1}$$

Let X be a point such that $CX \parallel AK$ and $KX \parallel AC$. Let Y be a point such that $DY \parallel BK$ and $KY \parallel BD$. Let S be the intersection of XY with CD.

Then $CX = AK$, $KX = AC$, $YD = KB$, and $KY = BD$. Since $CX \parallel AB \parallel YD$, we have

$$\frac{CS}{SD} = \frac{CX}{YD} = \frac{AK}{KB} \,.$$

Consequently, using (1), we have $\frac{CS}{SD} < \frac{CM}{MD}$. Hence, M is a point on the segment DS.

Let T be the intersection of KM with SY. Then M is a point on the segment KT. Thus, $KM \leq KT$. Since T is a point on the segment XY, we have, by the lemma,

$$KT \leq \max\{KX, KY\} = \max\{AC, BD\} \,.$$

Therefore, $KM \leq \max\{AC, BD\}$.

22. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} (a+k)^{k-1} (b+n-k)^{n-k-1} = (a+b+n)^{n-1} \left(\frac{1}{a} + \frac{1}{b} \right) \,.$$

Solution by Michel Bataille, Rouen, France.

Let S denote the left side of the above equation, and let $N = n - 1$. Using $\binom{n}{k} = \binom{N}{k} + \binom{N}{k-1}$, we write $S = S_1 + S_2$, where

$$S_1 = \sum_{k=0}^{N} \binom{N}{k} (a+k)^{k-1} (b+N+1-k)^{N-k}$$

$$\text{and} \quad S_2 = \sum_{k=1}^{N+1} \binom{N}{k-1} (a+k)^{k-1} (b+N+1-k)^{N-k} \,.$$

Working with S_1 first, we note that, for each $k = 0, 1, \ldots, N$,

$$\binom{N}{k}(a+k)^{k-1}(b+N+1-k)^{N-k}$$

$$= \binom{N}{k}(a+k)^{k-1}(a+b+N+1-(a+k))^{N-k}$$

$$= \binom{N}{k}(a+k)^{k-1}\sum_{j=0}^{N-k}\binom{N-k}{j}(a+b+N+1)^j(-(a+k))^{N-k-j}$$

$$= \sum_{j=0}^{N-k}\binom{N}{k}\binom{N-k}{j}(-1)^{N-k-j}(a+b+N+1)^j(a+k)^{N-1-j}$$

$$= \sum_{j=0}^{N-k}\binom{N}{j}\binom{N-j}{k}(-1)^{N-k-j}(a+b+N+1)^j(a+k)^{N-1-j}$$

Summing over k, and interchanging the order of summation, we get

$$S_1 = \sum_{j=0}^{N}\binom{N}{j}(a+b+N+1)^j\sum_{k=0}^{N-j}\binom{N-j}{k}(-1)^{N-k-j}(a+k)^{N-1-j}.$$

Now we make use of the following general result: for every polynomial P with degree $< m$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}P(k) = 0.$$

(The sum on the left side is $\Delta^m P(0)$, where Δ is the difference operator defined by $\Delta P(x) = P(x+1) - P(x)$.) It follows that the inner sum in the above expression for S_1 is 0 except when $j = N$. Thus,

$$S_1 = \binom{N}{N}(a+b+N+1)^N\binom{0}{0}(-1)^0(a+0)^{-1} = \frac{1}{a}(a+b+n)^{n-1}.$$

For S_2, we have

$$S_2 = \sum_{k=1}^{N+1}\binom{N}{N-k+1}(a+k)^{k-1}(b+N+1-k)^{N-k}$$

$$= \sum_{j=0}^{N}\binom{N}{j}(a+N+1-j)^{N-j}(b+j)^{j-1}$$

Applying the same argument as for S_1, we get $S_2 = \frac{1}{b}(a+b+n)^{n-1}$. The desired result follows.

Comment by Pierre Bornsztein, Maisons-Laffitte, France.

This problem is proposed as exercise 44-b (with solution) in L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, 1979.

23. The plane is divided into regions by n lines in general positions. Prove that at least $n - 2$ of the regions are triangles.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First note that, if the line ℓ meets the interior of the non-degenerate triangle ABC, then ℓ must cross two sides of the triangle, say $[AB]$ at M and $[AC]$ at N. It follows that AMN is a non-degenerate triangle. Thus, if a line meets the interior of a triangle, it divides the triangle into two regions, at least one of which is a triangle.

Let $n \geq 3$ be an integer. Let the plane be divided into regions by n lines $\ell_1, \ell_2, \ldots, \ell_n$ in general position. Without loss of generality, we may suppose that, for $i = 1, 2, \ldots, n - 1$, the line ℓ_i meets ℓ_n at M_i, such that the points $M_1, M_2, \ldots, M_{n-1}$ are pairwise distinct (since the lines are in general position) and in this order on ℓ_n.

For $i \leq n - 2$, the lines ℓ_i, ℓ_{i+1}, and ℓ_n form a triangle T_i, where M_i and M_{i+1} are two vertices of T_i. Moreover, from the order of the M_i's on the line ℓ_n, any two of the triangles $T_1, T_2, \ldots, T_{n-2}$ have no interior point in common. Thus, we have exactly $n - 2$ triangles.

Let $k \leq n - 2$ be fixed. If none of the lines ℓ_i meets the interior of T_k, then T_k is one of the regions. Otherwise, a line ℓ_i meets the interior of T_k. Let i_1 be the least integer such that ℓ_{i_1} meets the interior of T_k. Then, from the initial remark above, the line ℓ_{i_1} divides T_k into two regions, at least one of which is a triangle, say T_{k_1}. Note that none of the lines ℓ_i meets the interior of T_{k_1} for $i \leq i_1$.

If, for $i > i_1$, none of the lines ℓ_i meets the interior of T_{k_1}, then T_{k_1} is one of the regions. In the other case, substituting T_{k_1} for T_k, we follow the same reasoning as above. Since, at each step, the number of lines remaining to be considered is decreasing, this process will eventually stop, giving us a region which is a triangle.

With this process, each of the T_i's leads to a triangular region. Since any two of the regions T_i have no interior point in common, the triangular regions are distinct, and we are done.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations. Over the next several issues, we will be making an effort to shorten the time between the appearance of a problem set and publication of the solutions.

In Memoriam:
Murray S. Klamkin, 1921–2004

Andy Liu

Murray Seymour Klamkin was born on March 5, 1921 in Brooklyn, New York. He had a most productive and fulfilling life, divided between industry and academia. Although he was highly successful in everything he attempted, he will probably be remembered most for his involvement in mathematics problem-solving and competitions.

He authored or edited four problem books and left his mark in every major journal which had a problem section. He was the editor of the problem section of SIAM Review for a long time, and was well known to readers of this journal as the inaugural editor of our Olympiad Corner.

Murray was heavily involved in the USA Mathematical Olympiad and the USA National Team in the International Mathematical Olympiad, serving as deputy leader from 1975 to 1979, and as leader from 1981 to 1984. After moving to Canada, he made significant contributions to the Canadian Mathematical Olympiad as well as provincial mathematics competitions.

He was arguably the world's best-known mathematics problem-solver, his immense talent shining through his incisive insight, his clinical efficiency, and his phenomenal memory. His research interest ranged over the whole spectrum of mathematics, from Applied Mathematics to Spherical Geometry. However, we may single out his expertise in inequalities in general and in the Triangle Inequality in particular.

Murray enjoyed remarkably good health during his long life. On August 6, 2004, a combination of intestinal tumour and heart failure took him away from us. His passing marks the end of an era in the world of mathematics competitions and problem-solving. He will be deeply missed.

Ed. We plan to print a Murray Klamkin commemorative issue in 2005. If you would like to have something included in this issue, please ensure that we receive it by **March 31, 2005**. Also, be sure to identify your submission as being for that commemorative issue.

BOOK REVIEW
John Grant McLoughlin

Puzzlers' Tribute: A Feast for the Mind
Edited by David Wolfe and Tom Rodgers, published by A.K. Peters, 2002
ISBN 1-56881-121-7, hardcover, 420 pages, US$35.00.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredricton, NB.*

Gardner, Singmaster, Golomb, Conway, Kim, Smullyan, Penrose, Pickover, Berlekamp, . . . Wow! This is a special collection of problems and puzzles. This book ought to find itself in the libraries of recreational mathematical enthusiasts. The book stems from periodical Gathering(s) for Gardner (G4G1 to G4G4) at which mathematicians, magicians, and puzzlists meet to honour Martin Gardner. *Puzzlers' Tribute* is dedicated to the memory of Mel Stover, Harry Eng, and David Klarner (three participants at the gatherings). Appropriately, the initial section features eight articles that make up "The Toast Tributes". *Crux Mathematicorum* readers will certainly recognize the names of co-authors of "Three Problems", a piece by Andy Liu and Bill Sands. The note on the authors states, "Bill Sands likes winters on the Canadian prairie, and was an editor of *Crux Mathematicorum* for ten years, but is otherwise relatively sane."

Indeed the collection of over sixty articles is too rich to review in a page or two, let alone a paragraph. This recreational math enthusiast highly recommends the book. The editors, David Wolfe and Tom Rodgers, extend an invitation to http://www.g4g4.com, where one will find articles, problems, materials, and links pertinent to the participants and flavour of the gathering.

In Code: A Mathematical Journey
By Sarah Flannery with David Flannery, published by Algonquin Books of Chapel Hill, 2001
ISBN 1-56512-377-8, paperbound, 360 pages, US$13.95.
Reviewed by **John Grant McLoughlin**, *University of New Brunswick, Fredricton, NB.*

This popular general-interest book is (auto)biographical in that it follows the mathematical journey of Sarah Flannery from childhood mathematical puzzles in the kitchen to elementary number theory and, subsequently, cryptography. One appeal of the book is that it is easy to read regardless of mathematical background. Only Chapters 6 and 7 and the appendices require any mathematical sophistication. The story surrounds the mathematics, though it is the remarkable mathematical achievements of the young author that created the interest in the story, as told by herself and her father, a professor of mathematics in Cork.

Some Reducibility Criteria for
$AX^4 + BX^2 + C$

Natalio H. Guersenzvaig

Let Z be an arbitrary unique factorization domain. We established the following results in [1].

Theorem 1 Let $f(X)$ be any non-zero polynomial in $Z[X]$. The following statements are equivalent:

 (i) $f(X^2)$ is reducible in $Z[X]$;

 (ii) $f(X)$ is reducible in $Z[X]$, or there exist polynomials $G(X)$ and $H(X)$ in $Z[X]$ and a unit u of Z (that is, an invertible element of $Z \setminus \{0\}$) such that
$$uf(X) = G^2(X) - XH^2(X).$$

Corollary 1 Let $f(X)$ be any polynomial of $Z[X]$ which is irreducible in $Z[X]$. Assume that $f(X)$ has leading coefficient A and constant term C. In addition, suppose that either uA is not a square in Z for each unit u of Z or that AC is not a square in Z. Then $f(X^2)$ is irreducible in $Z[X]$.

For domains of characteristic 2 (that is, for domains in which $-1 = 1$ (see [3, p. 193])), we have a more precise result.

Corollary 2 Suppose that Z has characteristic 2. Let $f(X)$ be any polynomial of $Z[X]$ which is irreducible in $Z[X]$. Let A be the leading coefficient of $f(X)$. Then the following statements are equivalent.

 (i) $f(X^2)$ is reducible in $Z[X]$.

 (ii) There exists a unit u of Z such that every coefficient of $uf(X)$ is a square in Z.

Proof. The equivalence of (i) and (ii) follows at once from Theorem 1 and $(a_1 + \cdots + a_n)^2 = a_1^2 + \cdots + a_n^2$ for any $a_1, \ldots, a_n \in Z$. $\square$

There are more precise results for biquadratic polynomials in $Z[X]$; that is, for $f(X^2)$, where $f(X)$ is a quadratic polynomial of $Z[X]$ (for biquadratic polynomials over a field, see [5, pp. 137–139]). In order to give elementary proofs, we recall that a non-zero polynomial of $Z[X]$ is called *primitive* if the greatest common divisor of its coefficients is a unit of Z. A quadratic polynomial $f(X) = AX^2 + BX + C \in Z[X]$ is irreducible in $Z[X]$ if and only if $f(X)$ is a primitive polynomial whose discriminant, $B^2 - 4AC$, is not a square in Z.

We have the following characterizations of the biquadratic polynomials of $Z[X]$ which are reducible in $Z[X]$.

Theorem 2 Let $f(X) = AX^2 + BX + C$ be any primitive quadratic polynomial of $Z[X]$. Let $\tilde{f}(X) = X^2 + 2ABX + A^2(B^2 - 4AC)$. The following three statements are equivalent:

(i) $f(X^2)$ is reducible in $Z[X]$;

(ii) $f(X)$ is reducible in $Z[X]$, or there exists $S \in Z$ with $AC = S^2$ such that $A(2S - B)$ is a square in Z;

(iii) $f(X)$ is reducible in $Z[X]$, or some square element of Z is a root of $\tilde{f}(X)$.

Proof. Certainly the three statements are all true if $f(X)$ is reducible in $Z[X]$. Thus, we may assume that $f(X)$ is irreducible in $Z[X]$.

We first suppose that (i) is true. It follows from Theorem 1 that there exist polynomials $G(X)$ and $H(X) \in Z[X]$ and a unit u of Z such that

$$u(AX^2 + BX + C) = G^2(X) - XH^2(X).$$

Thus, $G(X)$ is linear and $H(X)$ is constant, say

$$G(X) = \alpha X + \gamma \quad \text{and} \quad H(X) = \beta.$$

Then $uA = \alpha^2$, $uC = \gamma^2$, and $2\alpha\gamma - uB = \beta^2$. Hence, $AC = (u^{-1}\alpha\gamma)^2$ and $u(2u^{-1}\alpha\gamma - B) = \beta^2$. Then (ii) is true with $S = u^{-1}\alpha\gamma$, because

$$A(2S - B) = (u^{-1})^2(uA)(u(2S - B)) = (u^{-1}\alpha\beta)^2.$$

Now suppose that (ii) is true; that is, there exist S and T in Z such that $AC = S^2$ and $A(2S - B) = T^2$. Then (iii) is true with $\tilde{f}(T^2) = 0$, because

$$\begin{aligned}
\tilde{f}(X^2) &= (X^2 + AB)^2 - 4A^2S^2 \\
&= (X^2 - 2AS + AB)(X^2 + 2AS + AB) \\
&= (X^2 - T^2)(X^2 + 2AS + AB).
\end{aligned}$$

Finally, suppose that (iii) is true. It is clear that the discriminant of $\tilde{f}(X)$ is equal to $16A^3C$, which is a square in Z because $\tilde{f}(X)$ has a root in Z. Then there exist S and T in Z such that $AC = S^2$ and $\tilde{f}(T^2) = 0$. Hence,

$$0 = (T^2 + AB)^2 - 4A^2S^2 = (T^2 - A(2S - B))(T^2 - A(2(-S) - B)).$$

Therefore, since $AC = S^2 = (-S)^2$, we may assume that $AC = S^2$ and $A(2S - B) = T^2$. It follows from this that

$$\begin{aligned}
Af(X^2) &= A^2X^4 + ABX^2 + AC = (AX^2 + S)^2 - T^2X^2 \\
&= (AX^2 - TX + S)(AX^2 + TX + S) \\
&= d^2(A'X^2 - T'X + S')(A'X^2 + T'X + S'), \qquad (1)
\end{aligned}$$

where $d = \gcd(A, T, S)$ and $A' = A/d$, $T' = T/d$, $S' = S/d$. Hence, $A'X^2 - T'X + S'$ and $A'X^2 + T'X + S'$ are primitive polynomials of $Z[X]$. Conversely, a well-known result of Gauss (see [3, p. 317]) establishes that the product of primitive polynomials is also a primitive polynomial. Then, since $f(X^2)$ is primitive, from (1), we get $A = d^2$, from which (i) follows. □

Remark. The polynomial $f(X^2)$ has exactly four roots in Z (counting multiplicities) if there exist S and T in Z such that $S^2 = AC$, $\tilde{f}(T^2) = 0$, and $T^2 - 4AS$ is a square in Z; otherwise, it has no roots. Of special importance is the case when $T^2 - 4AS$ is not a square in Z, since $f(X^2)$ then has two irreducible quadratic factors, but no roots.

Now we prove, without using complex numbers, that any "generalized" biquadratic polynomial of $\mathbb{R}[X]$ is reducible in $\mathbb{R}[X]$.

Corollary 3 Let m be any positive integer, and let

$$F(X) = AX^{4m} + BX^{2m} + C$$

be a polynomial in $\mathbb{R}[X]$ with $A \neq 0$. Then $F(X)$ is reducible in $\mathbb{R}[X]$.

Proof. It is clear, via the change of variable $Y = X^m$, that it suffices to prove the case $m = 1$. We may also assume $A > 0$. Now suppose that $F(X)$ is irreducible in $\mathbb{R}[X]$. It follows from (ii) of Theorem 2 that $B^2 - 4AC < 0$ and that, for any real number S, we have $AC < S^2$ or $A(2S - B) < 0$. Since this is clearly equivalent to the contradiction $B^2 < 4AC$ and $C < 0$, the proof is complete. □

As an application of Theorem 2, we consider (for any a, $b \in Z$) the polynomial
$$F_{a,b}(X) = X^4 - 2(a + b)X^2 + (a - b)^2 .$$

We first determine necessary and sufficient conditions on a and b so that $F_{a,b}(X)$ is reducible in $Z[X]$ (generalizing this way problem A3 of the 2001 Putnam Competition [2, pp.829–831]).

If Z has characteristic 2, it is clear that $F_{a,b}(X) = (X^2 + a - b)^2$. Next suppose that the characteristic of Z is different from 2. From Theorem 2, it follows at once that $F_{a,b}(X)$ is reducible in $Z[X]$ if and only if at least one of the elements $4((a + b)^2 - (a - b)^2) = 16ab$, $2(a - b) + 2(a + b) = 4a$, and $-2(a - b) + 2(a + b) = 4b$ is a square in Z; that is, if and only if at least one of the elements a, b, and ab is a square in Z.

We note the following consequences:

1. The polynomial $F_{a,b}(X)$ is reducible in $\mathbb{Z}_p[X]$ for any prime p, because, by the law of quadratic reciprocity, the product of non-squares in $\mathbb{Z}_p$ is always a square in $\mathbb{Z}_p$. (Thus, it is easy to find irreducible polynomials of $\mathbb{Z}[X]$ which are reducible in $\mathbb{Z}_p[X]$ for any prime integer p; the case $a = 5$, $b = 41$ is considered in [4, pp. 133–134, 330]).

2. $X^4 - 2(Y+W)X^2 + (Y-W)^2$ is irreducible in $\mathbb{Q}[X, Y, W]$.

3. Let α and β denote arbitrary square roots of a and b, respectively, in some extension field of the field of quotients of Z, say Q_Z (see [3, pp. 210, 344]). Since $F_{a,b}(\alpha + \beta) = 0$, we have also given an elementary proof that $F_{a,b}(X)$ is the minimal polynomial of $\alpha + \beta$ over Q_Z if and only if none of the elements a, b, or ab is a square in Z (which constitutes, for specific values of a, $b \in \mathbb{Z}$, a very common exercise in any introductory course on extension fields).

With more generality, by taking $A = 1$ we get from Theorem 2 the following characterization of the elements of some algebraic closure of Q_Z (see [4, pp. 351, 359–360]) whose minimal polynomial over Q_Z is a biquadratic polynomial of $Z[X]$.

Corollary 4 Suppose that Z does not have characteristic 2. Let $\overline{Q_Z}$ be any algebraic closure of Q_Z, and let ξ be any element of $\overline{Q_Z}$. Let $F(X)$ be the minimal polynomial of ξ over Q_Z. The following statements are equivalent:

(i) $F(X) = X^4 + BX^2 + C$ is a biquadratic polynomial of $Z[X]$.

(ii) We can write ξ in some of the following four ways:

$$\pm\sqrt{\frac{B \pm \sqrt{B^2 - 4C}}{2}}\,.$$

where B and C are elements of Z such that $B^2 - 4C$ is not a square in Z, and $B \pm 2\sqrt{C}$ are not squares in Z if C is a square in Z.

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PROBLEMS

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Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

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2963. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle acutangle. Soit respectivement r et R les rayons des cercles inscrit et circonscrit, s le semi-périmètre, c'est-à-dire $s = \frac{1}{2}(a+b+c)$. Soit m_a la longueur de la médiane de A à BC, et soit w_a la longueur de la bissectrice intérieure de l'angle A, mesurée de A au côté BC. Les longueurs m_b, m_c, w_b et w_c sont définies de manière analogue. Montrer que :

(a) $\dfrac{3s^2 - r^2 - 4Rr}{8sRr} \leq \displaystyle\sum_{\text{cyclique}} \dfrac{m_a}{aw_a} \leq \dfrac{s^2 - r^2 - 4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \leq \displaystyle\sum_{\text{cyclique}} \dfrac{m_a^2}{b^2 + c^2} \leq \dfrac{4R + r}{4R}$.

2964. *Proposé par Joe Howard, Portales, NM, USA.*
(Inspiré du Problème 80.D, Math. Gazette 80 (489) (1996) p. 606.)

Soit $x \in \left(0, \frac{\pi}{2}\right)$. Montrer que :

(a) $\left[\dfrac{2 + \cos x}{3}\right]\left[\dfrac{2(1 - \cos x)}{x^2}\right] > \dfrac{1 + \cos x}{2}$;

(b) $\dfrac{2 + \cos x}{3} < \sqrt{\dfrac{1 + \cos x}{2}} < \dfrac{2(1 - \cos x)}{x^2}$.

2965. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Soit $ABCD$ un parallélogramme. A l'aide d'une règle sans graduation, trouver un point M sur AB tel que $AM = \frac{1}{5}AB$.

2966. *Proposé par Mikhael Kotchetov, Université Memorial de Terre-Neuve, St. Jean, NL.*

On considère deux cercles disjoints et non-congruents Γ_1 et Γ_2, de centres respectifs O_1 et O_2. Soit Q le point d'intersection des deux tangentes communes, t_1 et t_2, qui ne coupent pas le segment O_1O_2. Une tangente commune, t_c, qui coupe le segment O_1O_2 intersecte les tangentes t_1 et t_2 en E_1 et E_2, respectivement.

Soit P le point milieu du segment O_1O_2. Montrer que P, Q, E_1 et E_2 sont sur un même cercle.

2967. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a_1, a_2, ..., a_n des nombres réels positifs et

$$E_n \;=\; \sum_{i=1}^{n}\left(\sum_{j=0}^{n-1} a_i^{j}\right)^{-1}.$$

Si $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$, montrer que $E_n \;\geq\; n\left(\sum_{j=0}^{n-1} r^{j}\right)^{-1}$ pour :

(a) $n = 2$,   (b) $n = 3$,   (c)$\star$ $n \geq 4$.

2968. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a_1, a_2, ..., a_n des nombres réels positifs et

$$E_n \;=\; \frac{1 + a_1 a_2}{1 + a_1} + \frac{1 + a_2 a_3}{1 + a_2} + \cdots + \frac{1 + a_n a_1}{1 + a_n}.$$

Soit $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$.

(a) Montrer que $E_n \;\geq\; \dfrac{n(1 + r^2)}{1 + r}$ pour $n = 3$ et $n = 4$.

(b)$\star$ Montrer que ou réfuter $E_n \;\geq\; \dfrac{5(1 + r^2)}{1 + r}$.

2969. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b, c, d et r des nombres réels positifs tels que $r = \sqrt[4]{abcd} \geq 1$. Montrer que

$$\frac{1}{(1 + a)^2} + \frac{1}{(1 + b)^2} + \frac{1}{(1 + c)^2} + \frac{1}{(1 + d)^2} \;\geq\; \frac{4}{(1 + r)^2}.$$

2970. *Proposé par Titu Zvonaru, Bucarest, Roumanie.*

Si m et n sont des entiers positifs tels que $m \geq n$, et si a, b, $c > 0$, montrer que

$$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \;\geq\; \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}.$$

2971. *Proposé par Michel Bataille, Rouen, France.*

Pour a, b, $c \in (0, 1)$, trouver la plus petite borne supérieure et la plus grande borne inférieure de $a + b + c + abc$ en tenant compte de la contrainte $ab + bc + ca = 1$.

2972. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

(a) Montrer que si $0 \le \lambda \le 4$, alors, pour tous les nombres réels positifs x, y, z, t,

$$(t^2 + 1)(x^3 + y^3 + z^3) + 3(1 - t^2)xyz$$
$$\ge \quad (1 + \lambda t)(x^2 y + y^2 z + z^2 x) + (1 - \lambda t)(xy^2 + yz^2 + zx^2) \, .$$

(b) Si $t = \frac{1}{4}$ et $\lambda = 4$, l'inégalité ci-dessus devient

$$17(x^3 + y^3 + z^3) + 45xyz \quad \ge \quad 32(x^2 y + y^2 z + z^2 x) \, .$$

Trouver toutes les valeurs positives de δ telles que l'inégalité

$$x^3 + y^3 + z^3 + 3\delta xyz \quad \ge \quad (1 + \delta)(x^2 y + y^2 z + z^2 x)$$

soit valable pour tous x, y, z qui sont : (i) des nombres réels positifs ; (ii) des longueurs des côtés d'un triangle.

2973. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne (consacré à Toshio Seimiya).*

Soit ABC un triangle rectangle non-isocèle d'angle droit en A et tel que $AC > AB$. Soit D le pied de la hauteur abaissée de A sur le côté BC. Soit G le point d'intersection de la droite AD et de la parallèle à AB passant par C. Soit E et F des points tels que $ACGE$ et $BFGE$ forment respectivement des rectangles. Soit H l'intersection de AG et BF. Soit finalement O_1 et O_2 les intersections respectives des diagonales des quadrilatères $CDHF$ et $BDGE$.

Montrer que les triangles ABC, DFE et DO_1O_2 sont semblables.
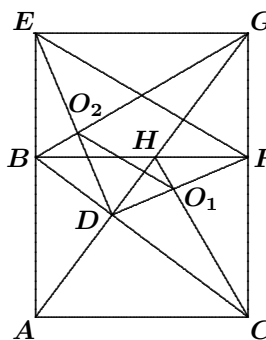
2974. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

On donne un triangle ABC et soit P un point quelconque sur la droite BC. Soit A_1 l'intersection de AP (prolongée peut-être) avec la droite par B parallèle à AC, et soit A_2 l'intersection de AP (prolongée peut-être) avec la droite par C parallèle à AB.

Montrer que l'aire du triangle ABC est la moyenne géométrique des aires des triangles A_1BC et A_2BC.

2975. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

On donne un quadrilatère inscriptible convexe dont les longueurs des côtés sont m, n, p et q, pris dans cet ordre. Si les diagonales mesurent d et d', montrer que $\sqrt{mp + nq} \leq \frac{1}{2}(d + d')$

. . . .

2963. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a + b + c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\dfrac{3s^2 - r^2 - 4Rr}{8sRr} \leq \displaystyle\sum_{\text{cyclic}} \dfrac{m_a}{aw_a} \leq \dfrac{s^2 - r^2 - 4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \leq \displaystyle\sum_{\text{cyclic}} \dfrac{m_a^2}{b^2 + c^2} \leq \dfrac{4R + r}{4R}$.

2964. *Proposed by Joe Howard, Portales, NM, USA.*
(Inspired by Problem 80.D, Math. Gazette 80 (489) (1996) p. 606.)

Let $x \in \left(0, \frac{\pi}{2}\right)$. Show that:

(a) $\left[\dfrac{2 + \cos x}{3}\right] \left[\dfrac{2(1 - \cos x)}{x^2}\right] > \dfrac{1 + \cos x}{2}$;

(b) $\dfrac{2 + \cos x}{3} < \sqrt{\dfrac{1 + \cos x}{2}} < \dfrac{2(1 - \cos x)}{x^2}$.

2965. *Proposed by Titu Zvonaru, Bucharest, Romania.*

Let $ABCD$ be a parallelogram. Using only an unmarked straightedge, find a point M on AB such that $AM = \frac{1}{5}AB$.

2966. *Proposed by Mikhael Kotchetov, Memorial University of Newfoundland, St. John's, NL.*

Consider non-intersecting and non-congruent circles Γ_1 and Γ_2 with centres O_1 and O_2, respectively. Let Q be the point of intersection of the two common tangents, t_1 and t_2, which do not intersect the line segment O_1O_2. A common tangent, t_c, which intersects the segment O_1O_2 meets the tangents t_1 and t_2 at E_1 and E_2, respectively.

Let P be the mid-point of the line segment O_1O_2. Prove that P, Q, E_1, and E_2 are concyclic.

2967. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, ..., a_n be positive real numbers, and let

$$E_n = \sum_{i=1}^{n} \left(\sum_{j=0}^{n-1} a_i^j \right)^{-1}.$$

If $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$, prove that $E_n \geq n \left(\sum_{j=0}^{n-1} r^j \right)^{-1}$ for:

(a) $n = 2$,   (b) $n = 3$,   (c)★ $n \geq 4$.

2968. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, ..., a_n be positive real numbers, and let

$$E_n = \frac{1 + a_1 a_2}{1 + a_1} + \frac{1 + a_2 a_3}{1 + a_2} + \cdots + \frac{1 + a_n a_1}{1 + a_n}.$$

Let $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$,.

(a) Prove that $E_n \geq \dfrac{n(1 + r^2)}{1 + r}$ for $n = 3$ and $n = 4$.

(b)★ Prove or disprove that $E_n \geq \dfrac{n(1 + r^2)}{1 + r}$ for $n = 5$.

2969. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d, and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{4}{(1+r)^2}.$$

2970. *Proposed by Titu Zvonaru, Bucharest, Romania.*

If m and n are positive integers such that $m \geq n$, and if a, b, $c > 0$, prove that

$$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \geq \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}.$$

2971. *Proposed by Michel Bataille, Rouen, France.*

For a, b, $c \in (0, 1)$, find the least upper bound and the greatest lower bound of $a + b + c + abc$, subject to the constraint $ab + bc + ca = 1$.

2972. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

(a) Prove that if $0 \le \lambda \le 4$, then, for all positive real numbers x, y, z, t,

$$(t^2 + 1)(x^3 + y^3 + z^3) + 3(1 - t^2)xyz$$
$$\ge \ (1 + \lambda t)(x^2 y + y^2 z + z^2 x) + (1 - \lambda t)(xy^2 + yz^2 + zx^2) \,.$$

(b) For $t = \frac{1}{4}$ and $\lambda = 4$, the above inequality becomes

$$17(x^3 + y^3 + z^3) + 45xyz \ \ge \ 32(x^2 y + y^2 z + z^2 x) \,.$$

Find all positive values of δ such that the inequality

$$x^3 + y^3 + z^3 + 3\delta xyz \ \ge \ (1 + \delta)(x^2 y + y^2 z + z^2 x)$$

holds for all x, y, z which are: (i) positive real numbers; (ii) side lengths of a triangle.

2973. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain (dedicated to Toshio Seimiya).*

Let ABC be a non-isosceles right triangle with right angle at A and $AC > AB$. Let D be the foot of the altitude from A to the side BC. Let G be the point of intersection of the line AD (extended) with the line through C which is parallel to AB. Let E be the point such that $ACGE$ is a rectangle, and let F be the point such that $BFGE$ is a rectangle. Let H be the intersection of AG and BF. Let O_1 be the intersection of the diagonals of the quadrilateral $CDHF$, and let O_2 be the intersection of the diagonals of the quadrilateral $BDGE$.
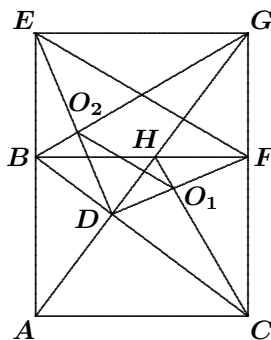
Prove that the triangles ABC, DFE, and $DO_1 O_2$ are similar.

2974. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be any point on the line BC in $\triangle ABC$. Let A_1 be the intersection of AP (possibly extended) with the line through B which is parallel to AC, and let A_2 be the intersection of AP (possibly extended) with the line through C which is parallel to AB.

Prove that the area of $\triangle ABC$ is the geometric mean of the areas of $\triangle A_1 BC$ and $\triangle A_2 BC$.

2975. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given an inscribed convex quadrilateral with sides of length m, n, p, q, taken in order around the quadrilateral, and diagonals of length d and d', prove that $\sqrt{mp + nq} \le \frac{1}{2}(d + d')$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have discovered that problem 2962 is identical to problem 2860, whose solution has already been printed [2003 : 315–316]. Consequently, problem 2860 is closed, and no solutions for it will be accepted. We apologize for this oversight.

2864. [2003 : 397] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

If a, b, c are the sides of an acute angled triangle, prove that

$$\sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2}\sqrt{a^2 - b^2 + c^2} \leq ab + bc + ca.$$

I. *Solution by Christopher J. Bradley, Bristol, UK.*

Let $x^2 = b^2 + c^2 - a^2$, $y^2 = c^2 + a^2 - b^2$, and $z^2 = a^2 + b^2 - c^2$, where x, y, $z > 0$. Then $a^2 = \frac{1}{2}(y^2 + z^2)$, $b^2 = \frac{1}{2}(z^2 + x^2)$, and $c^2 = \frac{1}{2}(x^2 + y^2)$. The given inequality becomes

$$2(xy + yz + zx) \leq \sum_{\text{cyclic}} \sqrt{y^2 + z^2}\sqrt{z^2 + x^2},$$

$$\text{or} \quad \sum_{\text{cyclic}}(yz + zx) \leq \sum_{\text{cyclic}} \sqrt{y^2 + z^2}\sqrt{z^2 + x^2}, \quad\quad (1)$$

which is true, since $(y^2 + z^2)(z^2 + x^2) - (yz + zx)^2 = (z^2 - xy)^2 \geq 0$.

Equality holds in (1) if and only if $z^2 = xy$, $x^2 = yz$, and $y^2 = zx$. These conditions are equivalent to $x = y = z$. Hence, equality holds in the proposed inequality if and only if $x = y = z$; that is, if and only if $a = b = c$.

II. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

By Cauchy's Inequality, for any x, y, z, $w \geq 0$, we have

$$\sqrt{xy} + \sqrt{zw} \leq \sqrt{(x + z)(y + w)}.$$

Hence,

$$\sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2}\sqrt{a^2 - b^2 + c^2}$$

$$= \frac{1}{2} \sum_{\text{cyclic}} \left(\sqrt{a^2 + b^2 - c^2}\sqrt{a^2 - b^2 + c^2} \right.$$

$$\left. + \sqrt{c^2 + a^2 - b^2}\sqrt{c^2 - a^2 + b^2} \right)$$

$$\leq \frac{1}{2} \sum_{\text{cyclic}} \sqrt{(2a^2)(2c^2)} = \sum_{\text{cyclic}} ac.$$

III. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

By the convexity of $\sqrt{x}$, we have

$$\frac{1}{2}\left(\sqrt{a^2 + b^2 - c^2} + \sqrt{a^2 - b^2 + c^2}\right)$$
$$\leq \sqrt{\frac{(a^2 + b^2 - c^2) + (a^2 - b^2 + c^2)}{2}} = a.$$

Hence, $\sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2} \leq a + b + c$, which becomes the desired inequality upon squaring.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; BABIS STERGIU, Chalkida, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Bucharest, Romania; and the propser.

About half of the submitted solutions used the same standard substitutions given in Solution I above, together with the AM-GM Inequality and/or Cauchy's Inequality.

Alt remarked that if we multiply the given inequality by 2 and add $a^2 + b^2 + c^2$ to both sides, then we obtain

$$\sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2} \leq a + b + c.$$

If we now let K and s denote the area and the semiperimeter of the triangle, then using

$$a^2 + b^2 - c^2 = 2ab\cos C = 4\left(\tfrac{1}{2}ab\sin C\right)\cot C = 4K\cot C$$

and two similar identities, the inequality above can be rewritten as

$$\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C} \leq \frac{s}{K}.$$

Janous obtained the following inequality which is stronger than the proposed one:

$$\sum_{\text{cyclic}} a(b + c)\cos A \leq ab + bc + ca.$$

Using substitutions similar to those in Solution I above, followed by Cauchy's Inequality and the Power Mean Inequality, Klamkin established the more general result that for all $n \geq 2$,

$$\sum_{\text{cyclic}} \sqrt[n]{a^n + b^n - c^n} \, \sqrt[n]{a^n - b^n + c^n} \leq ab + bc + ca,$$

provided the quantities under the radicals are all non-negative.

2865. [2003 : 398] *Proposed by George Baloglou, SUNY Oswego, Oswego, NY, USA.*

Suppose that D, E, F are the points at which the concurrent lines AD, BE, CF meet the sides of a given triangle ABC. Let p_1 and p_2 be the perimeters and δ_1 and δ_2 the areas of $\triangle ABC$ and $\triangle DEF$, respectively. Prove that

(a) $2p_2 \leq p_1$ if AD, BE, and CF are angle bisectors;

(b) $2p_2 \leq p_1$ if AD, BE, and CF are altitudes;

(c) $3p_2 \leq 2p_1$ for all D, E, F if and only if $\triangle ABC$ is equilateral;

(d) $4\delta_2 \leq \delta_1$ for all D, E, F and arbitrary $\triangle ABC$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) This is CRUX problem **2502** [2000 : 45; 2001 : 53–54].

(b) The wording of the problem suggests that $\triangle ABC$ is acute. (The result does not hold for non-acute triangles.) This problem is known as Fagnano's Problem [1]: of all the triangles incscribed in $\triangle ABC$, the orthic triangle attains the minimum perimeter. In particular, the orthic triangle has perimeter no greater than $\frac{1}{2}p_1$.

(c) If $\triangle ABC$ is not equilateral, and BC is the longest side, then we can choose D to be the mid-point of BC. Let $E \to C$ and $F \to B$ to get $p_2 \to 2BC > \frac{2}{3}p_1$.

Conversely, suppose $\triangle ABC$ is equilateral with $AB = 1$. Let $u = AF$, $v = BD$, and $w = CE$. Then $0 < u, v, w < 1$ and

$$p_2 \; = \; f(u, v, w) \; = \; \sum_{\text{cyclic}} \sqrt{u^2 + (1 - w)^2 - u(1 - w)}\,.$$

Then

$$\frac{\partial f}{\partial u} \; = \; \frac{2u - (1 - w)}{2\sqrt{u^2 + (1 - w)^2 - u(1 - w)}} + \frac{v - 2(1 - u)}{2\sqrt{v^2 + (1 - u)^2 - v(1 - u)}}\,.$$

Setting $\frac{\partial f}{\partial u} = 0$, we get

$$[2u - (1 - w)]^2 \cdot [v^2 + (1 - u)^2 - v(1 - u)]$$
$$= \; [v - 2(1 - u)]^2 \cdot [u^2 + (1 - w)^2 - u(1 - w)]\,,$$

which reduces to

$$u(u + w - 1)v^2 \; = \; (1 - u)(1 - u - v)(1 - w)^2\,. \tag{1}$$

This can further be manipulated into

$$uv(1 - u)(1 - w)(1 - v - w) \; = \; 0\,, \tag{2}$$

by using Ceva's relation

$$uvw \;=\; (1-u)(1-v)(1-w)\,. \tag{3}$$

[*Ed.* See the note following the list of solvers below.] Hence, $1-v-w=0$; that is, $v = 1-w$ and $w = 1-v$. Thus, $u = \frac{1}{2}$. By symmetry, we conclude $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ is the only critical point of f. Note that $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{3}{2}$. Also, as the concurrent point approaches a side of $\triangle ABC$, the function f clearly approaches its maximum value of 2. Hence, $\frac{1}{2}p_1 \leq p_2 \leq \frac{2}{3}p_1$ for non-degenerate $\triangle DEF$.

(d) This part is also well known [2].

References

[1] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965, pp. 359–361.

[2] M.S. Klamkin, *International Mathematical Olympics 1978–1985*, MAA, 1999, p. 13 and pp. 86–88 (problem G.I./4 and its solutions).

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain parts (a), (b), (d); CHRISTOPHER J. BRADLEY, Bristol, UK parts (b), (d); CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VEDULA N. MURTY, Dover, PA, USA parts (b), (d); PANOS E. TSAOUSSOGLOU, Athens, Greece parts (b), (d); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

For the convenience of the reader, we give one way of obtaining equation (2) from (1) using (3):

$$uv^2(1-u-w) + (1-u)(1-u-v)(1-w)^2 \;=\; 0\,,$$

$$uv^2(1-u-w+uw) - u^2v^2w$$
$$+\,(1-u)(1-u-v+uv)(1-w)^2 - uv(1-u)(1-w)^2 \;=\; 0\,,$$

$$uv^2(1-u)(1-w) - u^2v^2w$$
$$+\,(1-u)^2(1-v)(1-w)^2 - uv(1-u)(1-w)^2 \;=\; 0\,,$$

$$uv^2(1-u)(1-w) - uv(1-u)(1-v)(1-w)$$
$$+\,uvw(1-u)(1-w) - uv(1-u)(1-w)^2 \;=\; 0\,,$$

$$uv(1-u)(1-w)\big[v-(1-v)+w-(1-w)\big] \;=\; 0\,,$$

$$2uv(1-u)(1-w)(v+w-1) \;=\; 0\,.$$

2866. [2003 : 398] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

For two given circles Γ_1 and Γ_2, the lines l and m are external common tangents. The line l touches Γ_1 and Γ_2 at A and B, respectively, and the line m touches Γ_1 and Γ_2 at C and D, respectively. Suppose that M is the mid-point of the segment AB, and that P and Q are the second intersections of MC and MD with Γ_1 and Γ_2, respectively.

Prove that A, B, P, and Q are concyclic.

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since $MA^2 = MB^2$, we see that M is on the radical axis of Γ_1 and Γ_2. It follows that $MP \cdot MC = MQ \cdot MD$, so that $CDQP$ is cyclic; hence, $\angle MPQ = \angle CDQ$. Let O_1 and O_2 denote the centres of Γ_1 and Γ_2, respectively. Then through some angle chasing, we have

$$\begin{aligned}
\angle APQ + \angle QBA &= \angle APM + \angle MPQ + \angle QBA \\
&= 180^\circ - \angle CPA + \angle CDQ + \angle QBA \\
&= \tfrac{1}{2}\angle AO_1C + \angle DBQ + \angle QBA \\
&= \angle AO_1O_2 + \angle DBA \\
&= \angle AO_1O_2 + \angle O_1O_2B = 180^\circ,
\end{aligned}$$

and the result follows.

II. *Solution by Daniel Reisz, Vincelles, France.*

[*Editor's comment.* The following proof uses very much the same argument as in Solution I, except that here the angles are directed. It is interesting to compare the use of undirected and directed angles. For the undirected angles $\angle XYZ$ and $\angle XY'Z$ inscribed in a circle, $\angle XYZ = \angle XY'Z$ or $\angle XYZ = 180^\circ - \angle XY'Z$ according as Y and Y' lie on the same arc or on the opposite arcs determined by X and Z. For the directed angles from YX to YZ and from $Y'X$ to $Y'Z$ we have $(YX, YZ) = (Y'X, Y'Z)$—in Reisz's notation—for any four points X, Y, Y', Z on a circle.]

Remarques.

1. $MP \cdot MC = MA^2 = MB^2 = MQ \cdot MD$, donc les quatre points C, P, Q, D sont cocycliques.

2. Pour des raisons de symétrie (la droite des centres O_1O_2 est médiatrice commune) les cordes AC et BD sont parallèles.

3. On notera (d, d') la mesure modulo π de l'angle des droites d et d'.

   En vertu des différantes cocyclicités on peut écrire:

$$\begin{aligned}
(PA, PQ) &= (PA, PC) + (PC, PQ) \\
&= (CA, CD) + (DC, DQ) \\
&= (CA, DQ) = (DB, DQ) = (BA, BQ),
\end{aligned}$$

d'où la cocyclicité des points A, B, P, Q.

III. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let i be the inversion with respect to the circle Γ with centre M and radius $r = MA = MB$. Then $i(A) = A$ and $i(B) = B$. By the Secant–Tangent Theorem, we deduce that $r^2 = MP \cdot MC$ and $r^2 = MQ \cdot MD$.

Therefore, $i(P) = C$ and $i(Q) = D$. But A, B, C, and D are concyclic, since $ABCD$ is an isosceles trapezoid with $AC \parallel BD$ and $AB = CD$. This circle cannot pass through the mid-point M of AB; it is therefore inverted by i into a circle through A, B, P, and Q, as required.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; (Bellot also submitted a second solution); CHRISTOPHER J. BRADLEY, Bristol, UK; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, student, New York University, NY, USA; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2867. [2003 : 399] *Proposed by Proposed by Antreas P. Hatzipolakis and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given two points B and C, find the locus of the point A such that the centre of the nine-point circle of $\triangle ABC$ lies on the line BC.

I. *Solution by Joel Schlosberg, student, New York University, NY, USA.*

Introduce a coordinate system with $B = \left(-\frac{1}{2}a, 0\right)$ and $C = \left(\frac{1}{2}a, 0\right)$, where a is the length of segment BC, and let A have coordinates (X, Y). If D, E, and F are the mid-points of sides BC, AC, and AB, respectively, then $D = (0, 0)$, $E = \left(\frac{1}{4}a + \frac{1}{2}X, \frac{1}{2}Y\right)$, and $F = \left(-\frac{1}{4}a + \frac{1}{2}X, \frac{1}{2}Y\right)$. The nine-point circle is the circumcircle of $\triangle DEF$; thus, its centre N must be on the perpendicular bisector of EF, which is the line $x = \frac{1}{2}X$. If N is also on the line BC (which has equation $y = 0$), then N must have coordinates $\left(\frac{1}{2}X, 0\right)$. Then, since $ND = NE$ are radii of the nine-point circle, we have

$$ND^2 = \left(\tfrac{1}{2}X\right)^2 = NE^2 = \left(\tfrac{1}{4}a\right)^2 + \left(\tfrac{1}{2}Y\right)^2.$$

Hence, X and Y satisfy

$$x^2 - y^2 = \tfrac{1}{4}a^2.$$

Conversely, for any position of vertex A whose coordinates satisfy this equation [*Ed.*: except $A = B$ or $A = C$, where the triangle is degenerate], $N = \left(\frac{1}{2}X, 0\right)$ is the centre of the nine-point circle, since it is equidistant from D, E, and F. Therefore, the locus of the vertex A is given by the equation above, which describes a rectangular hyperbola with major axis BC [*Ed.*: with the points B and C removed].

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

By the solution and comment to problem 2765 [2003 : 349–351], the nine-point centre lies on BC if and only if $|B - C| = \pi/2$. Consider $C = B + \pi/2$. Suppose that $B = (-1, 0)$, $C = (1, 0)$, and $A = (x, y)$, with $x > 1$. Let $P = (x, 0)$ and $Q = (2x - 1, 0)$. Then $B = \angle CAP = \angle PAQ$; thus [since $A + 2B = \pi/2$], $AB \perp AQ$. Hence, $\dfrac{y}{x+1} = \dfrac{x-1}{y}$; that is,

$$x^2 - y^2 \;=\; 1.$$

We get the other branch ($x < -1$) of the hyperbola when $B = C + \pi/2$. In conclusion, the locus of A is the rectangular hyperbola centred at the midpoint of BC, with the vertices B, C excluded.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Bucharest, Romania; and the proposers.

Bradley remarked that as a consequence of this problem, we obtain the hyperbolic analogue of the familiar theorem about angles inscribed in semicircles: When BC is the diameter of a semicircle containing A, then $B + C = \pi/2$; when BC is the major axis of a rectangular hyperbola containing A, then $|B - C| = \pi/2$.

2868. [2003 : 399] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, we have $c^4 = a^4 + b^4$.

(a) Show that $\triangle ABC$ is acute angled.

(b) Determine the range of $\angle ACB$.

(c)$\star$ How can we generalize to $c^n = a^n + b^n$?

Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.

Suppose that $c^n = a^n + b^n$ with $n > 2$ (n real). We claim that $\triangle ABC$ is acute-angled and that

$$\cos^{-1}\left(\frac{2 - 2^{\frac{2}{n}}}{2}\right) \;\leq\; C \;<\; \frac{\pi}{2}.$$

Lemma. If $n > m > 0$, then $(1 + \lambda^n)^{\frac{1}{n}} < (1 + \lambda^m)^{\frac{1}{m}}$ for all $\lambda > 0$.

Proof. Set $F(\lambda, n) = (1 + \lambda^n)^{\frac{1}{n}}$. Then

$$\frac{\partial F}{\partial n} \;=\; \frac{\lambda^n \ln(\lambda^n) - (1 + \lambda^n)\ln(1 + \lambda^n)}{n^2(1 + \lambda^n)^{1 - \frac{1}{n}}}.$$

Now set $f(t) = t \ln t$. Then $f(t)$ is decreasing on $(0, e^{-1})$ and increasing on (e^{-1}, ∞), with a minumum at $t = e^{-1}$. If $t \geq e^{-1}$, then $f(t) < f(1 + t)$, and if $0 < t < e^{-1}$, we have $f(t) < 0$ and $f(1 + t) > 0$, implying that, again, $f(t) < f(1 + t)$. Hence, $\dfrac{\partial F}{\partial n} < 0$. Thus, F is decreasing as a function of n. The statement in the lemma now follows. ∎

From $c^n = a^n + b^n$, we see that C is the largest angle. Set $\lambda = b/a$. Then $c = (a^n + b^n)^{\frac{1}{n}} = a\,(1 + \lambda^n)^{\frac{1}{n}} < a(1 + \lambda^2)^{\frac{1}{2}}$, by the lemma. Hence, $c^2 < a^2(1 + \lambda^2) = a^2 + b^2$. Therefore, $C < 90°$, and thus, $\triangle ABC$ is acute-angled.

By the Law of Cosines, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + b^2 - (a^n + b^n)^{\frac{2}{n}}}{2ab}$$
$$= \frac{1 + \lambda^2 - (1 + \lambda^n)^{\frac{2}{n}}}{2\lambda},$$

Define

$$g(\lambda, n) = \frac{1 + \lambda^2 - (1 + \lambda^n)^{\frac{2}{n}}}{2\lambda}.$$

Applying the lemma (with $m = 2$), we see that $g(\lambda, n) > 0$ for all $\lambda > 0$ and for all $n > 2$. We claim that $\lim\limits_{n \to \infty} \big(g(\lambda, n) - g(1, n)\big) \leq 0$ for all $\lambda > 0$.

First note that $g(1, n) = \dfrac{2 - 2^{\frac{2}{n}}}{2}$ for all n. Hence, $\lim\limits_{n \to \infty} g(1, n) = \dfrac{1}{2}$.

Case 1. $0 < \lambda < 1$.

Then $\lim\limits_{n \to \infty} (1 + \lambda^n)^{\frac{1}{n}} = 1$; hence, $\lim\limits_{n \to \infty} g(\lambda, n) = \dfrac{(1 + \lambda^2) - 1}{2\lambda} = \dfrac{\lambda}{2}$. Thus,

$$\lim_{n \to \infty} (g(\lambda, n) - g(1, n)) = \frac{\lambda - 1}{2} < 0.$$

Case 2. $\lambda > 1$.

Then $\lim\limits_{n \to \infty} (1 + \lambda^n)^{\frac{1}{n}} = \lambda$; hence, $\lim\limits_{n \to \infty} g(\lambda, n) = \dfrac{(1 + \lambda^2) - \lambda^2}{2\lambda} = \dfrac{1}{2\lambda}$. Thus,

$$\lim_{n \to \infty} (g(\lambda, n) - g(1, n)) = \frac{1 - \lambda}{2\lambda} < 0.$$

Case 3. $\lambda = 1$.

Then $\lim\limits_{n \to \infty} (g(\lambda, n) - g(1, n)) = 0$.

This completes the proof of our claim above.

Now we will show that, for each fixed n, the function $g(\lambda, n)$ has its maximum when $\lambda = 1$. For $n \geq m$, we have

$$(g(\lambda, n) - g(1, n)) - (g(\lambda, m) - g(1, m))$$
$$= \frac{1}{2\lambda}\left(1 + \lambda^m)^{\frac{1}{m}} - (1 + \lambda^n)^{\frac{1}{n}}\right) \geq 0,$$

using the lemma. Then

$$g(\lambda, m) - g(1, m) \ \leq \ \lim_{n \to \infty} \left(g(\lambda, n) - g(1, n) \right) \ \leq \ 0 \,,$$

and thus, $g(\lambda, m) \leq g(1, m)$, for all m.

Finally, note that $\lim_{\lambda \to \infty} g(\lambda, n) = 0$. It follows that the range of $g(\lambda, n)$, as a function of λ, is $(0, g(1, n)] = \left(0, \dfrac{2 - 2^{\frac{2}{n}}}{2} \right]$. Thus,

$$\cos^{-1} \left(\frac{2 - 2^{\frac{2}{n}}}{2} \right) \ \leq \ C \ < \ \frac{\pi}{2} \,.$$

Also solved by MICHEL BATAILLE, Rouen, France (parts (a) and (b)); CHRISTOPHER J. BRADLEY, Bristol, UK (parts (a) and (b) and a partial solution to part (c)); CON AMORE PROBLEM GROUP, The Danish University of Education Copenhagen, Denmark; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA (parts (a) and (b)); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA (parts (a) and (b)); ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA (parts (a) and (b)); G. TSINTSIFAS, Thessaloniki, Greece (parts (a) and (b) and a partial solution to part (c)); KENNETH M. WILKE, Topeka, KS, USA (parts (a) and (b)); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2869. [2003 : 399] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given rectangle $ABCD$ with area S, let E and F be points on sides AB and AD, respectively, such that $[CEF] = \frac{1}{3}S$, where $[PQR]$ denotes the area of $\triangle PQR$.

Prove that $\angle ECF \leq \frac{\pi}{6}$.

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $a = AB$, $b = BC$, $\alpha = \angle BCE$, $\theta = \angle ECF$ and $\beta = \angle FCD$.
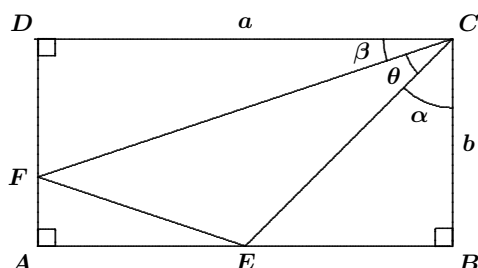
Then $\cos(\alpha + \beta) = \cos(\frac{\pi}{2} - \theta) = \sin\theta$ and we have

$$
\begin{aligned}
ab = S &= 3[CEF] = 3\left(\tfrac{1}{2} \cdot CE \cdot CF \cdot \sin\theta\right) \\
&= \frac{3}{2} \cdot \frac{b}{\cos\alpha} \cdot \frac{a}{\cos\beta} \cdot \sin\theta \\
&= \frac{3ba\sin\theta}{\cos(\alpha+\beta) + \cos(\alpha-\beta)} = \frac{3ba\sin\theta}{\sin\theta + \cos(\alpha-\beta)} \, .
\end{aligned}
$$

Hence, $\sin\theta = \frac{1}{2}\cos(\alpha - \beta) \leq \frac{1}{2}$. Therefore, $\theta \leq \frac{\pi}{6}$. Equality holds when $\alpha = \beta = \frac{\pi}{6}$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GUSTAVO KRIMKER, Universidad CAECE, Buenos Aires, Argentina; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Woo commented that the problem, as stated, does not exclude cases where E and F lie on the lines AB and AD (as opposed to the segments AB and AD). He then proceeded to investigate the possible cases. All other solvers (including the proposer) implicitly assumed that E and F lie on the segments AB and AD, respectively.

2870. [2003 : 399] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Given triangle ABC with incentre I, circumcentre O, and centroid G, suppose that $\angle AIO = 90°$. Prove that $IG \| BC$.

Solution by the proposer.

We denote the circumcircle of $\triangle ABC$ by Γ. Let E be the second intersection of AI with Γ, and let D be the intersection of AE and BC. Let AG meet BC at M. Since G is the centroid and AM is a median, $AG : GM = 2 : 1$.

We have $\angle ABI = \angle DBI$ and

$$
\begin{aligned}
\angle BAE &= \angle BAI = \angle IAC \\
&= \angle EAC = \angle EBC \\
&= \angle EBD .
\end{aligned}
$$

Then $\angle BIE = \angle BAI + \angle ABI = \angle EBD + \angle DBI = \angle EBI$, and hence $BE = EI$. Since $\angle BAE = \angle EBD$, it follows that $\triangle ABE$ is similar to $\triangle BDE$, and therefore $AB : BD = AE : BE = AE : EI$.
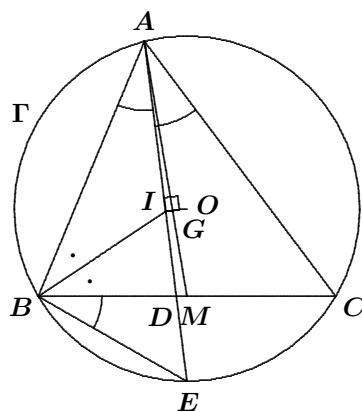
We are given that $\angle AIO = 90°$; that is, $OI \perp AE$. This implies that $AI = IE$, and hence $AE : EI = 2 : 1$. Then $AB : BD = 2 : 1$. Since $\angle ABI = \angle IBD$, we get $AI : ID = AB : BD = 2 : 1$. Thus, $AI : ID = AG : GM$, which implies that $IG \parallel DM$. Thus, $IG \parallel BC$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania.

Several solvers observed that the converse is also true: if $IG \parallel BC$, then $\angle AIO = 90°$. To prove this, just reverse the steps in the last paragraph of the featured solution. Furthermore, $IG \parallel BC$ if and only $2a = b + c$, where a, b, and c are the lengths of the sides. This was a Klamkin Quickie from 1996. (See [1996 : 61], or the reprinted version in [2001 : 79]. For these references, we thank Specht.)

Amengual Covas claims that, for triangles where $2a = b + c$, other relationships are given in [1978 : 78–79] and the following references:

[1] F. G.-M., Exercices de géométrie, 7th edition, Paris, pp. 464–465.
[2] F. G.-M., Exercices de trigonométrie, Paris, pp. 413–414, 481.
[3] I. Shariguin, Problemas de geometría Planimetría, editorial Mir, 1989, pp. 96–97.

2871. [2003 : 400] *Proposed by Mihály Bencze, Brasov, Romania.*

In $\triangle ABC$, denote the sides by a, b, c, the symmedians by s_a, s_b, s_c, and the circumradius by R. Prove that
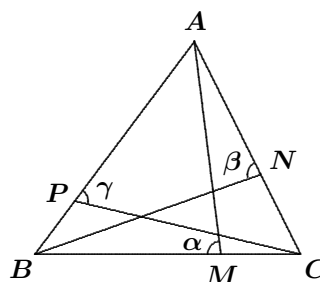
$$\frac{bc}{s_a} + \frac{ca}{s_b} + \frac{ab}{s_c} \leq 6R.$$

Solution by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.

We prove a more general inequality.

Theorem. Let ABC be a triangle and let M, N, P, be arbitrary points on the line segments BC, CA, BA, respectively. Then

$$\frac{bc}{AM} + \frac{ca}{BN} + \frac{ab}{CP} \leq 6R.$$

Proof: Let $\alpha = \angle AMB$, $\beta = \angle BNA$, $\gamma = \angle APC$, and let Δ be the area of $\triangle ABC$. We have

$$\frac{1}{2}a \cdot AM \sin \alpha = \Delta = \frac{abc}{4R},$$

Hence, $\dfrac{bc}{AM} = 2R\sin\alpha$. Similarly, $\dfrac{ca}{BN} = 2R\sin\beta$ and $\dfrac{ab}{CP} = 2R\sin\gamma$. Thus,

$$\frac{bc}{AM} + \frac{ca}{BN} + \frac{ab}{CP} \;=\; 2R(\sin\alpha + \sin\beta + \sin\gamma) \;\leq\; 6R\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece (two solutions); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

All other solvers proved the original and some noted that it was not necessary to take symmedians. Bataille noted that the proof actually gives a stronger result:

$$\max\left\{\frac{bc}{s_a}, \frac{ca}{s_b}, \frac{ab}{s_c}\right\} \;\leq\; 2R\,.$$

Indeed, it is clear from the above proof that

$$\max\left\{\frac{bc}{AM}, \frac{ca}{BN}, \frac{ab}{CP}\right\} \;\leq\; 2R\,,$$

where AM, BN, CP are any three cevians of $\triangle ABC$.

Crux Mathematicorum
with Mathematical Mayhem

Crux Mathematicorum

Mathematical Mayhem