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CONTENTS

DIOPHANTINE GEOMETRY

A. DAY BRADLEY

The determination of all triangles with integral sides and an angle equal to 60° or 120° requires the solution of Diophantine equations. When some of the numerical solutions are tabulated, certain relationships between the solutions become apparent. These relationships are low-grade but authentic theorems which will, we believe, provide interesting exercises in discovery and proof for average high school students, since only the most rudimentary algebra is required for their proof.

If angle $C = 60^\circ$ in triangle ABC, then, in the usual notation,

$$c^2 = a^2 - ab + b^2; \tag{1}$$

and $C = 120^\circ$ gives

$$c^2 = a^2 + ab + b^2. \tag{2}$$

These two equations have been solved by J. Neuberg and G.B. Mathews [1,2]; C.E. Hilger [3] has applied the solutions to the determination of sets of right triangles with rational sides and equal areas; and Artemas Martin [4] has also discussed equation (1). The solutions given for (1) are

$$a = p^2 - q^2, \quad b = p^2 - 2pq, \quad c = p^2 - pq + q^2, \quad p > 2q \tag{3}$$

and

$$a = p^2 - q^2, \quad b = 2pq - q^2, \quad c = p^2 - pq + q^2, \quad p > q/2. \tag{4}$$

These solutions are easy to verify but not easy to develop *ab ovo*.

Table 1 contains solutions for some relatively prime pairs (p,q) from $(2,1)$ to $(8,3)$. For some values of (p,q), both (3) and (4) apply; in others only (4). When both (3) and (4) apply, the value from (3) is tabulated first.

Inspection of Table 1 leads to three theorems, all quite easy to prove:

1. When (4) is applied to $(3,1)$ and $(3,2)$, the same triangle results; and the same is true for $(4,1)$ and $(4,3)$. The generalization is *that* (4) *applied to* (p,q) *and* $(p,p-q)$ *produces the same triangle*. For if we replace (p,q) by $(p,p-q)$ in (4) we get

$$a = p^2 - (p-q)^2 = 2pq - q^2,$$

$$b = 2p(p-q) - (p-q)^2 = p^2 - q^2,$$

$$c = p^2 - p(p-q) + (p-q)^2 = p^2 - pq + q^2.$$

(p,q)	a	b	c
(2,1)	3	3	3
(3,1)	8	3	7
	8	5	7
(3,2)	5	8	7
(4,1)	15	8	13
	15	7	13
(4,3)	7	15	13
(5,1)	24	15	21
	24	9	21
(5,2)	21	5	19
	21	16	19
(5,3)	16	21	19
(6,1)	35	24	31
	35	11	31
(7,1)	48	35	43
	48	13	43
(7,2)	45	21	39
	45	24	39
(7,3)	40	7	37
	40	33	37
(8,1)	63	48	57
	63	15	57
(8,3)	55	16	49
	55	39	49

Table 1
Triangles with C = 60°

(p,q)	a	b	c
(2,1)	3	5	7
(3,1)	8	7	13
(3,2)	5	16	19
(4,1)	15	9	21
(4,3)	7	33	37
(5,1)	24	11	31
(5,2)	21	24	39
(5,3)	16	39	49
(5,4)	9	56	61
(6,1)	35	13	43
(6,5)	11	85	91
(7,1)	48	15	57
(7,2)	45	32	67
(7,3)	40	51	79
(7,4)	33	72	93
(7,5)	24	95	109
(7,6)	13	120	127
(8,1)	63	17	73
(8,3)	55	57	97
(8,5)	39	105	129
(8,7)	15	161	169

Table 2
Triangles with C = 120°

2. While the numbers in the pairs (5,1), (7,2), and (8,1) are relatively prime, the corresponding triplets (a,b,c) have the common factor 3. The generalization, noted by Artemas Martin, is *that if* $p+q$ *is a multiple of* 3, *then* a, b, *and* c *are all multiples of* 3. This is easily seen to be true by replacing (p,q) by $(p,3k-p)$ in (3) or (4).

3. Comparing the triplets for (3,1) and (5,1), and those for (4,1) and (7,2),

leads to the easily proved generalization *that each triplet for* $(2p-1,p-2)$ *is a triplet for* $(p,1)$ *multiplied by* 3.

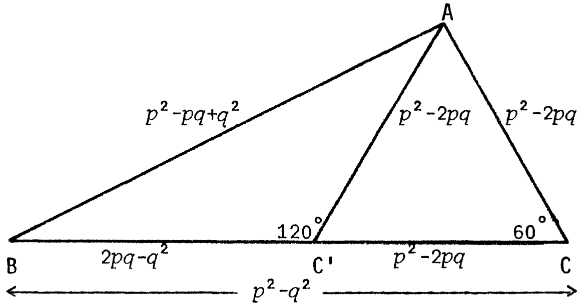

Figure 1

Figure 1 shows the triangle ABC whose sides are given by (3). It is apparent from the figure that one solution of (2) is

$$a = 2pq - q^2, \quad b = p^2 - 2pq, \quad c = p^2 - pq + q^2, \quad p > 2q. \tag{5}$$

Since the values (5) all appear in (3) and (4), several $120°$ triangles can be found from Table 1. For example, the triangles

(5,3,7), (7,8,13), (9,15,21), (16,5,19), ...

corresponding to the pairs (3,1), (4,1), (5,1), (5,2), ...

(p,q) in Table 1	a	b	c	(p,q) in Table 2
(3,1)	3	5	7	(2,1)
(4,1)	8	7	13	(3,1)
(5,1)	15	9	21	(4,1)
(6,1)	24	11	31	(5,1)
(5,2)	5	16	19	(3,2)
(7,2)	21	24	39	(5,2)
(7,3)	7	33	37	(4,3)
(8,3)	16	39	49	(5,3)

Table 3
$120°$ triangles in both Tables 1 and 2

A second solution for the 120° triangle was obtained by Neuberg and Mathews:

$$a = p^2 - q^2, \quad b = 2pq + q^2, \quad c = p^2 + pq + q^2, \quad p > q. \tag{6}$$

In Table 2 are given values of a, b, c for all relatively prime pairs (p,q) from (2,1) to (8,7). The table shows that the triplets for (4,1), (5,2), and (7,4) are three times those for (2,1), (3,1), and (5,1), respectively. Here the easily verified generalization is *that the triplet for* $(p+2,p-1)$ *is three times the triplet for* $(p,1)$.

A comparison of Tables 1 and 2 shows that the 120° triangles of Table 1 are repeated in Table 2. These triangles are shown in Table 3.

In order to find the relationship between Tables 1 and 2, it may be necessary to extend the tables. Further examples will confirm the easily verified relationship that (p,q) in Table 1 and $(p-q,q)$ in Table 2 yield the same triangle when $p > 2q$.

Two additional solutions for the 60° triangle can be found from (6), as shown in Figures 2 and 3.

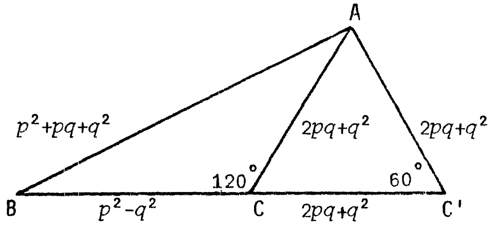

Figure 2

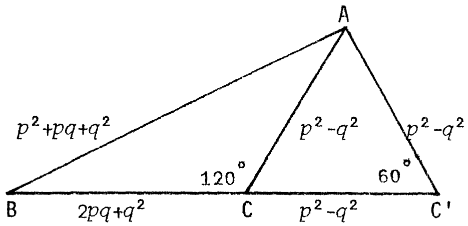

Figure 3

These solutions are

$$a = p^2 + 2pq, \quad b = 2pq + q^2, \quad c = p^2 + pq + q^2, \tag{7}$$

and

$$a = p^2 + 2pq, \quad b = p^2 - q^2, \quad c = p^2 + pq + q^2. \tag{8}$$

Formulas (7) and (8) are related to (4) and (3) as follows: *(p,q) in (7) yields the same triangle as $(p+q,q)$ in (4), and (p,q) in (8) yields the same triangle as $(p+q,q)$ in (3)*

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1. *Mathematical Questions and their Solutions from the Educational Times*, 46 (1887) 97.

2. Leonard Eugene Dickson, *History of the Theory of Numbers*, Volume II, Chelsea, New York, 1952, pp. 405-406.

3. *Mathematical Questions and their Solutions from the Educational Times*, 72 (1900) 30.

4. Artemas Martin, Problem 195, *American Mathematical Monthly*, 21 (1914) 98.

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* * *

OVERCOMING CALCULATOR INDECISION

CLAYTON W. DODGE

One of life's pleasures is to find ways to force machines to do tasks that are normally beyond their capability. It is in this spirit that I offer a program designed to force a programmable calculator to make a decision when no decision routines are built in.

My particular calculator is a Sharp PC-1002, having 8 memories, each capable of holding either data or 8 program steps. No branching or decision capabilities are available. Although the program is written for this unit, having algebraic logic, it is readily adapted to other machines. Its main purpose is to demonstrate that simple decisions can be forced by proper programming, thereby encouraging others to overcome calculator indecision with their own programs.

Step	Key
01	÷
02	2
03	=
04	Sto
05	0
06	+
07	1
08	Exp
09	1
10	0
11	-
12	1

We shall make the calculator apply the Collatz function of Problem 133 [1976: 67, 144, 221]: *if the integer given is even, divide it by 2; if odd, multiply it by 3 and add 1.* So we want the calculator to discern whether a given integer is even or odd. We force this decision by dividing the integer by 2 and examining the remainder. The remainder itself will be used to adjust the operations as we desire. The program steps are listed along with an explanation of the purposes.

Steps 01 to 05 divide the given number n, already on display, by 2 and store the result $n/2$ in memory number 0.

Steps 06 to 15 add and then subtract 10^{10} to the quotient $n/2$. The exponent should be chosen by trial so that your cal-

Step	Key
13	Exp
14	1
15	0
16	-
17	Rcl
18	0
19	×
20	2
21	=
22	+/-
23	Sto
24	1
25	×
26	5
27	+
28	1
29	×
30	Rcl
31	0
32	+
33	Rcl
34	1
35	=
36	Sto
37	0
38	1
39	M+
40	2
41	Rcl
42	1
43	M+
44	3
45	Rcl
46	0
47	End

culator capacity is overflowed just enough to truncate or round the sum to an integer, thus eliminating the .5 from the quotient $n/2$ if the given number n was odd.

Steps 16 to 24 find the difference between the truncated quotient and the original quotient (i.e. either 0 or -.5), multiply it by 2, change its sign (you may not need this step if your calculator rounds internally), and store it in memory 1. Thus memory 1 now contains 0 if n was even or 1 if n was odd.

We want to display $n/2$, the contents of memory 0, if memory 1 contains 0. If memory 1 contains 1, we need to double $n/2$ to get n back again, then multiply n by 3 and add 1. Steps 25 to 35 accomplish this task. Letting x denote the contents of memory 1, we multiply $5x+1$ by $n/2$ and add x to get our function value. Thus we have

$$(5x+1)\frac{n}{2} + x = \frac{n}{2} \qquad \text{when} \quad x = 0$$

and

$$(5x+1)\frac{n}{2} + x = 3n+1 \quad \text{when} \quad n = 1.$$

The desired program terminates here.

If you have a few more program steps available, you can ask the calculator to count the number of times the program is applied. Given enough memory and program space, you may also want the number of times the $3n+1$ value was found. Steps 36 and 37 store the answer for later retrieval, steps 38 to 40 add 1 to memory 2, thus counting the number of times the program is run. Steps 41 to 44 recall the remainder 0 or 1 and add it to memory 3, thus counting the number of times we multiplied by 3 and added 1. Steps 45 to 47 return the answer to the display and stop the program. Of course, memories 2 and 3 should be cleared each time a new number is processed.

Mathematics Department, University of Maine, Orono, Maine 04469.

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TWO PYRAMID PROBLEMS

The following two problems, each of which arises from ancient reports, make good pedagogical fodder.

Problem 1.

There are two versions of how Thales calculated the height of an Egyptian pyramid by shadows. The earlier account, given by Hieronymus, a pupil of Aristotle, says that Thales noted the length of the shadow of the pyramid at the moment when his shadow was the same length as himself. The later version, given by Plutarch, says that he set up a stick and then made use of similar triangles. Both versions fail to mention the difficulty, in either case, of obtaining the length of the shadow of the pyramid, that is, the distance from the apex of the shadow to the center of the base of the pyramid. Devise a method, based on similar triangles and independent of latitude and time of year or day, for determining the height of a pyramid *from two shadow observations.*

Problem 2.

One of the puzzling features of the Great Pyramid of Egypt is the noted fact that the ratio of twice a side of the square base to the pyramid's height yields a surprisingly accurate approximation of the number π, and it has been conjectured that the Egyptians purposely incorporated this ratio in the construction of the pyramid. Herodotus, on the other hand, has stated that the pyramid was built so that the area of each lateral face would equal the area of a square with side equal to the pyramid's height. Show that, if Herodotus is right, the concerned ratio is automatically a remarkable approximation of π.

HOWARD EVES,
University of Maine.

Solutions to these problems appear on page 76 in this issue.

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THE OLYMPIAD CORNER: 13

MURRAY S. KLAMKIN

We have one new Practice Set this month, No. 11, and the solutions to Practice Set 10. For any Practice Set, readers who find what they consider to be better solutions than those published here, or nice generalizations, are invited to submit them to me for possible publication in a later column.

PRACTICE SET 11

11-1. If z_1, z_2, z_3, z_4 are complex numbers of unit modulus, prove that
$|z_1-z_2|^2|z_3-z_4|^2+|z_1+z_4|^2|z_3-z_2|^2 = |z_1(z_2-z_3)+z_3(z_2-z_1)+z_4(z_1-z_3)|^2$.

11-2. Sum the series

$$S=\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots+\frac{a(a+1)\ldots(a+m-1)}{b(b+1)\ldots(b+m-1)}.$$

11-3. If RS denotes any diameter of a given elliptical cross section of a right circular cone whose vertex is P, prove that PR + PS is constant.

*

SOLUTIONS TO PRACTICE SET 10

10-1. If a,b,c,d are positive integers, show that

$$30\,|\,(a^{4b+d}-a^{4c+d}).$$

(Here $m|n$ means that m divides or is a factor of n.)

Solution.

Consider the numbers

$$a^d(a^{4b}-1) \qquad \text{and} \qquad a^d(a^{4c}-1),$$

whose difference is $D=a^{4b+d}-a^{4c+d}$. Each is even and so is their difference D; and each is divisible by 5 since

$$(a^b)^4-1\equiv(a^c)^4-1\equiv 0 \pmod 5$$

by Fermat's Theorem, and so is their difference D. This takes care of divisibility by 10. If a is a multiple of 3, then clearly so is D; and if $a=3m\pm1$, then

$$D=a^d\{(3m\pm1)^{4b}-(3m\pm1)^{4c}\},$$

which is again a multiple of 3. Thus divisibility by 30 is assured.

10-2. Determine the area of the region bounded by the closed curve whose points (x,y) in rectangular coordinates satisfy

$$\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2}=2ky, \tag{1}$$

where the constants satisfy $k\geq a>0$.

Solution.

We will need the following formula from the geometry of the triangle:

$$abc = 4R\Delta, \tag{2}$$

where a, b, c are the sides, R the circumradius, and Δ the area of a triangle.

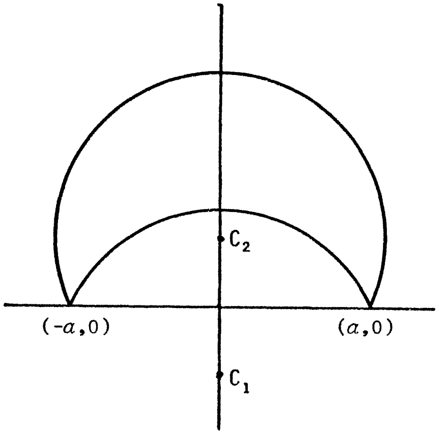
It is clear from (1) that the curve lies entirely in the upper half-plane $y \geq 0$ and that it goes through the points $(a,0)$ and $(-a,0)$. Consider the variable triangle T whose vertices are $(a,0)$, $(-a,0)$, and a point (x,y) in the upper half-plane. Applying (2) to T, we get

$$2a \cdot \sqrt{(x-a)^2 + y^2} \cdot \sqrt{(x+a)^2 + y^2} = 4Ray,$$

where R is the circumradius of triangle T. It now follows from (1) that (x,y) lies on the curve if and only if $R = k$. Thus the curve consists of the intersections with the upper half-plane of the two circles with radius k and centers

$C_1 = (0,-\sqrt{k^2-a^2})$ and $C_2 = (0,\sqrt{k^2-a^2})$.

It is the crescent-shaped curve shown in the figure.



(Now that we know what the curve looks like, a purely algebraic solution suggests itself. We can square equation (1) and show that the resulting equation is equivalent to

$$\{x^2 + (y + \sqrt{k^2 - a^2})^2 - k^2\}\{x^2 + (y - \sqrt{k^2 - a^2})^2 - k^2\} = 0, \qquad y \geq 0.$$

But it is not so easy to discover how to manipulate (1) algebraically when we don't know what we are looking for.)

The required area of the region bounded by the crescent-shaped curve is the difference of the areas of two circular segments. A straightforward calculation shows that the area of the smaller segment is

$$S_1 = k^2 \sin^{-1}\frac{a}{k} - a\sqrt{k^2 - a^2},$$

while that of the larger is $S_2 = \pi k^2 - S_1$. Thus the required area is

$$S_2 - S_1 = \pi k^2 + 2a\sqrt{k^2 - a^2} - 2k^2 \sin^{-1}\frac{a}{k}.$$

10-3. For $a \geq b \geq c \geq 0$, establish the inequality

$$b^m c + c^m a + a^m b \geq bc^m + ca^m + ab^m \tag{1}$$

(a) when m is a positive integer;
(b) find a proof valid for all real $m \geq 1$.

Solution.

It can be verified by inspection that equality holds in (1) when $m = 1$ and also when $a = b$ or $b = c$, so we assume $m > 1$ and $a > b > c \geq 0$. If we let $b = xa$ and $c = ya$, so that $1 > x > y \geq 0$, then (1) is equivalent to

$$1 + xy\left(\frac{x^{m-1} - y^{m-1}}{x - y}\right) \geq \frac{x^m - y^m}{x - y} ,$$

which is itself equivalent to the more compact

$$\frac{1 - x^m}{1 - x} \geq \frac{x^m - y^m}{x - y} . \tag{2}$$

(a) If m is an integer, then (2) is equivalent to

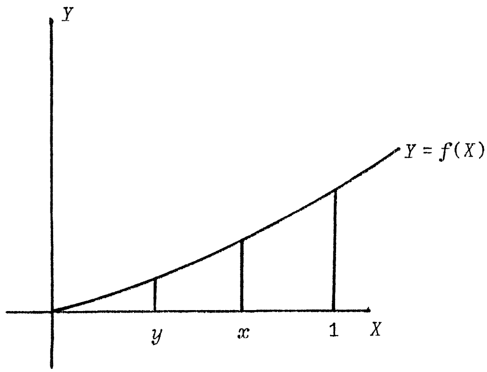
$$1 + x + x^2 + \ldots + x^{m-1} \geq x^{m-1} + x^{m-2}y + \ldots + y^{m-1},$$

which is true since $x^r \geq x^{m-1-r}y^r$ for $r = 0,1,\ldots,m-1$.

(b) To show that (2) holds for arbitrary real $m > 1$, consider the monotonically increasing curve

$$Y = f(X) \equiv mX^{m-1}$$

shown in the figure. The average value of the function in the interval $[x,1]$ is clearly greater than its average value in the interval $[y,x]$. Analytically, we have



$$\int_x^1 f(X)\,dX \geq (1-x)f(x)$$

and

$$(x-y)f(x) \geq \int_y^x f(X)\,dX.$$

Finally,

$$\frac{\int_x^1 mX^{m-1}dX}{1 - x} \geq \frac{\int_y^x mX^{m-1}dX}{x - y} ,$$

that is,

$$\frac{1-x^m}{1-x} \geq \frac{x^m-y^m}{x-y},$$

as required.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

SOLUTIONS TO "TWO PYRAMID PROBLEMS" (see page 72)

Solution to Problem 1.

This problem has become known as *the Thales puzzle*. One may solve it as follows: Drive a vertical pole into the sand near the pyramid. Hammer stakes A and A' into the ground marking the positions of the vertex of the shadow of the pyramid and the top of the shadow of the pole. Wait a couple of hours and hammer stakes B and B' into the ground marking the new positions of the vertex of the shadow of the pyramid and the top of the shadow of the pole. Then

height of pyramid : height of pole = AB : A'B'.

Solution to Problem 2.

Represent the side and altitude of the pyramid by s and h respectively. Then the area of a lateral face is given by

$$(s/4)\sqrt{4h^2+s^2}\ .$$

Now, according to Herodotus, the pyramid was constructed so that $4h^2 = s\sqrt{4h^2+s^2}$ or

$$4 = (s/h)\sqrt{4+(s/h)^2}\ .$$

Solving for s/h we find $s/h \approx 1.57$, whence $2s/h \approx 3.14$.

HOWARD EVES

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MAMA-THEMATICS

Mrs. Gibbs to son Josiah: "And I thought you wanted those arrows for archery practice."

Mrs. Newton to son Isaac: "You'll never accomplish anything idling away your day under an apple tree."

Mrs. Abel to son Niels: "You must learn to group your things neatly."

HOWARD EVES

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1980, although solutions received after that date will also be considered until the time when a solution is published.

521. *Proposed by Sidney Kravitz, Dover, New Jersey.*

No visas are needed for this alphametical Cook's tour:

$$\begin{array}{r} \text{SPAIN} \\ \underline{\text{SWEDEN}} \\ \text{POLAND} \end{array}.$$

522. *Proposed by William A. McWorter, Jr. and Leroy F. Meyers, The Ohio State University.*

A recent visitor to our department challenged us to prove the following result, which is not new. We pass on the challenge.

Prove that

$$\binom{a}{b} \equiv \binom{a_1}{b_1}\binom{a_2}{b_2}\cdots\binom{a_m}{b_m} \pmod{p},$$

where p is a prime and the nonnegative integers a and b have base-p representations $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_m$, respectively, with initial zeros permitted.

The standard conventions for binomial coefficients, namely

$$\binom{r}{0} = \binom{r}{r} = 1 \quad \text{if} \quad r \geq 0 \qquad \text{and} \qquad \binom{r}{s} = 0 \quad \text{if} \quad r < s,$$

are assumed.

523. *Proposed by James Gary Propp, student, Harvard College, Cambridge, Massachusetts.*

Find all zero-free decimal numerals N such that both N and N^2 are palindromes.

524. *Proposed by Dan Pedoe, University of Minnesota.*

Disproving a "theorem" can be as difficult as proving a theorem. One of my students is very keen to extend the Pascal Theorem to two equal circles, and has come up with the following "theorem", which you are asked to disprove:

γ *and* δ *are two equal circles.* A,B,C *are distinct points on* γ, *and* A',B',C' *are distinct points on* δ, *with* AA',BB',CC' *concurrent at a point* V. *Show that the three intersections*

$$BC' \cap B'C, \qquad CA' \cap C'A, \qquad AB' \cap A'B$$

are collinear, so that the Pascal Theorem holds for the hexagon AB'CA'BC'.

525. *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Eliminate α, β, and γ from

$$\cos\alpha + \cos\beta + \cos\gamma = a$$

$$\sin\alpha + \sin\beta + \sin\gamma = b$$

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = c$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = d\,.$$

526*. *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

The following are examples of *chains* of lengths 4 and 5, respectively:

25, 225, 1225, 81225

25, 625, 5625, 75625, 275625.

In each chain, each link is a perfect square, and each link (after the first) is obtained by prefixing a single digit to its predecessor.

Are there chains of length n for $n = 6,7,8,\ldots$?

527. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

(a) You stand at a corner in a large city of congruent square blocks and intend to take a walk. You flip a coin — tails, you go left; heads, you go right — and you repeat the procedure at each corner you reach. What is the probability that you will end up at your starting point after walking n blocks?

(b)* Same question, except that you flip the coin twice: TT, you go left; HH, you go right; otherwise, you go straight ahead.

528. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $k \geq 2$ be a fixed integer. Prove that $\log k$ is the sum of the infinite series

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{k-1} - \frac{k-1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{2k-1} - \frac{k-1}{2k} + \frac{1}{2k+1} + \ldots + \frac{1}{3k-1} - \frac{k-1}{3k} + \ldots .$$

529. *Proposed by J.T. Groenman, Groningen, The Netherlands.*

The sides of a triangle ABC satisfy $a \leq b \leq c$. With the usual notation r, R, and r_c for the in-, circum-, and ex-radii, prove that

$$\operatorname{sgn}(2r + 2R - a - b) = \operatorname{sgn}(2r_c - 2R - a - b) = \operatorname{sgn}(C - 90^\circ).$$

530.* *Proposed by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

Let $A = (a_n)$ be a sequence of positive integers such that a_0 is any positive integer and, for $n \geq 0$, a_{n+1} is the sum of the cubes of the decimal digits of a_n. Prove or disprove that A converges to 153 if and only if 3 is a proper divisor of a_0.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

438. [1979: 109] *Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Eliminate x, y, z from the following three equations:

$$a_i x^2 + b_i y^2 + c_i z^2 + 2f_i yz + 2g_i zx + 2h_i xy = 0, \qquad i = 1,2,3.$$

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let S_i, $i = 1,2,3$, denote the left members of the given equations. Each equation $S_i = 0$ is linear in the 6 homogeneous products of degree 2 in the 3 variables x,y,z:

$$x^2, \quad y^2, \quad z^2, \quad yz, \quad zx, \quad xy. \tag{1}$$

It is well-known that, in a system of n homogeneous linear equations in n unknowns, the eliminant (that is, the necessary and sufficient condition for the system to have a nontrivial solution) is $\Delta = 0$, where Δ is the determinant of the coefficient matrix. But we cannot apply this theorem immediately since we would need 6 equations and have only 3. If we multiply each equation $S_i = 0$ successively by x, y, and z,

$$
\begin{vmatrix}
a_1 & 2h_1 & 2g_1 & b_1 & 2f_1 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_2 & 2h_2 & 2g_2 & b_2 & 2f_2 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_3 & 2h_3 & 2g_3 & b_3 & 2f_3 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & 2h_1 & 2g_1 & 0 & 0 & b_1 & 2f_1 & c_1 & 0 & 0 \\
0 & 0 & 0 & a_2 & 0 & 0 & 2h_2 & 2g_2 & 0 & 0 & b_2 & 2f_2 & c_2 & 0 & 0 \\
0 & 0 & 0 & a_3 & 0 & 0 & 2h_3 & 2g_3 & 0 & 0 & b_3 & 2f_3 & c_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 2h_1 & 2g_1 & 0 & 0 & b_1 & 2f_1 & c_1 \\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 2h_2 & 2g_2 & 0 & 0 & b_2 & 2f_2 & c_2 \\
0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 2h_3 & 2g_3 & 0 & 0 & b_3 & 2f_3 & c_3 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 & 2h_1 & 2g_1 & 0 & 0 & b_1 & 2f_1 & c_1 & 0 \\
0 & 0 & 0 & 0 & a_2 & 0 & 0 & 2h_2 & 2g_2 & 0 & 0 & b_2 & 2f_2 & c_2 & 0 \\
0 & 0 & 0 & 0 & a_3 & 0 & 0 & 2h_3 & 2g_3 & 0 & 0 & b_3 & 2f_3 & c_3 & 0 \\
0 & 0 & a_1 & 0 & 2h_1 & 2g_1 & 0 & b_1 & 2f_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_2 & 0 & 2h_2 & 2g_2 & 0 & b_2 & 2f_2 & c_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_3 & 0 & 2h_3 & 2g_3 & 0 & b_3 & 2f_3 & c_3 & 0 & 0 & 0 & 0 & 0
\end{vmatrix} = 0
$$

we obtain 9 equations linear in the 10 homogeneous products of degree 3 in x,y,z:

$$x^3, \quad x^2y, \quad x^2z, \quad xy^2, \quad xyz, \quad xz^2, \quad y^3, \quad y^2z, \quad yz^2, \quad z^3. \qquad (2)$$

So we are still short one equation. But if, instead, we multiply each equation $S_i = 0$ successively by any 5 of the 6 homogeneous products (1), we obtain 15 equations linear in the 15 homogeneous products of degree 4 in x,y,z:

$$x^4,\ x^3y,\ x^3z,\ x^2y^2,\ x^2yz,\ x^2z^2,\ xy^3,\ xy^2z,\ xyz^2,\ xz^3,\ y^4,\ y^3z,\ y^2z^2,\ yz^3,\ z^4. \quad (3)$$

(In general, the number of homogeneous products of degree m in n variables is

$${}_nH_m = \frac{(m+n-1)!}{m!(n-1)!} .$$

In (1), (2), and (3), we have $n=3$ and $m=2,3,4$, respectively.) Since we now have 15 homogeneous equations in 15 unknowns, the eliminant can be found by equating to zero the determinant of the coefficient matrix of the system.

Each given equation $S_i = 0$ represents, in homogeneous coordinates $(x,y,z)\neq(0,0,0)$, a conic in the complex projective plane. Hence the required eliminant gives a necessary and sufficient condition for three conics to have a point (x,y,z) in common. As far as we know, this eliminant is not given explicitly anywhere in the literature. Since a determinant of order 15 can be routinely evaluated by today's computers, it may be useful for algebraic geometers to have a record of it, so we give it on the preceding page. It was obtained by multiplying each $S_i = 0$ by x^2, then by y^2, z^2, yz, zx, with the coefficients given in the order (3).

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; and KENNETH S. WILLIAMS, Carleton University, Ottawa. A comment was received from O. BOTTEMA, Delft, The Netherlands.

* * *

439. [1979: 109] *Proposed by Ram Rekha Tiwari, The Belsund Sugar Co., P.O. Riga, Bihar, India.*

The palindromic number 252 has the property that it becomes a perfect square when multiplied (or divided) by 7. Are there other such *even* palindromic numbers?

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

The answer is, indeed there are, but apparently not very many. Essentially, the proposer is asking for numbers of the form $28x^2$ which are palindromic. This seemed like a good problem to put to a computer. The program takes only 16 statements, and I was able to find all solutions with fewer than 15 digits in about 15 minutes. They are:

$$252 = 7\cdot 6^2$$

$$2148412 = 7\cdot 554^2$$

$$8812188 = 7\cdot 1122^2$$

$$86197279168 = 7\cdot 110968^2.$$

Observe that all these solutions have an odd number of digits. We are left with the following open questions:

(a) Are there infinitely many such numbers?

(b) Are there any such numbers with an even number of digits? (These would all be divisible by 11^2.)

On the other hand, it is easy to show that there are infinitely many *odd* palindromic numbers which become perfect squares when multiplied (or divided) by 7. For example, there is the infinite set

$$343\cdot(10^n+1)^2, \qquad n = 2,3,4,\ldots \tag{1}$$

There are also solutions not of the form (1). For example, apart from the obvious

7 and 343, we have:

$$763101367 = 7 \cdot 10441^2$$

$$782222287 = 7 \cdot 10571^2$$

$$38786068783 = 7 \cdot 74437^2$$

$$72783738727 = 7 \cdot 101969^2$$

$$3853563653583 = 7 \cdot 741963^2.$$

All of these solutions also have an odd number of digits, so open question (b) can be asked of these numbers as well.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; JOHN OMAN and BOB PRIELIPP, The University of Wisconsin-Oshkosh; BASIL C. RENNIE, James Cook University of North Queensland, Australia; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; and CHARLES W. TRIGG, San Diego, California.

Editor's comment.

That's a pretty thin crop of solutions for 15 minutes of computer time. Charlotte Armstrong once wrote [1]: "He was a young mathematics instructor ...; he was accustomed to using his mind and not his fingers." This editor, who is now getting to be pretty long in the tooth, and in whom the ratio of agility of mind to fingers is therefore reversed, feels he could have obtained those four solutions in half the time on the ideal instrument for finding palindromes: an ambidextrous abacus.

Prielipp mentioned the following related problem proposed by Gustavus J. Simmons in *The Fibonacci Quarterly: Let S be the set of n such that n^3 is a palindrome but n is not. Is S empty, finite, or infinite?* The entire "solution" given in [2] consisted of the following comment by the proposer: "Since 2201^3 is the palindrome 10662526601, S is not empty. This is all that is known about the set S."

Now where's that abacus ... .

REFERENCES

1. Charlotte Armstrong, in "The Witch's House," *The Charlotte Armstrong Treasury*, Coward, McCann & Geohegan, Inc., 1972, p. 15.

2. Problem B-183, *The Fibonacci Quarterly*, December 1970, p. 551.

* * *

440. [1979: 110] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

My favourite proof of the well-known result

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

uses the identity

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}$$

and the inequality

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x, \qquad 0 < x < \frac{\pi}{2}$$

to obtain

$$\frac{\pi^2}{(2n+1)^2} \cdot \frac{n(2n-1)}{3} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{(2n+1)^2} \left\{ n + \frac{n(2n-1)}{3} \right\},$$

from which the desired result follows upon letting $n \to +\infty$.

Can any reader find a new elementary proof simpler than the above? (Many references to this problem are given by E.L. Stark in *Mathematics Magazine*, 47 (1974) 197-202.)

Comments were received from RICHARD A. GIBBS, University of New Mexico, Albuquerque, N.M.; and from the proposer.

Editor's comment.

The proposer's intent was to obtain, for the well-known result $\zeta(2) = \pi^2/6$, a new proof that is at least different from and no less elementary than the one given in the proposal. "Elementary" means here "not requiring concepts from advanced real or complex analysis." Whether or not such a proof would be "simpler" could then be left to individual judgment.

There is an abundant literature on the subject, in addition to the reference given in the proposal. We mention only a few references ([1], [2], [7]), in most of which many additional references can be found. The proof given in our proposal is due to Papadimitriou [5], but it had earlier been discovered independently by Holme [4]. This proof is certainly elementary, since it uses from real analysis only the most rudimentary notion of convergence of a sequence. No new proofs were submitted and, of the existing proofs from the literature mentioned in the comments received, all, in the editor's opinion, were less elementary than Papadimitriou's. Proofs that antedate Papadimitriou's and compare with it in elementariness (although they are perhaps marginally less "simple") can be found in [3], [6], and [8].

For a nonelementary approach which features the immediacy of the result, it is hard to beat putting $x = \pi$ in the Fourier expansions

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\right)$$

or

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right).$$

REFERENCES

1. Tom M. Apostol, "Another Elementary Proof of Euler's Formula for $\zeta(2n)$," *American Mathematical Monthly*, 80 (April 1973) 425-431.

2. Bruce C. Berndt, "Elementary Evaluation of $\zeta(2n)$," *Mathematics Magazine*, 48 (May 1975) 148-154. (Reference supplied by the proposer.)

3. F. Gerrish, *Pure Mathematics*, Volume II, Cambridge University Press, 1960, pp. 543-544.

4. Finn Holme,"En enkel beregning av $\sum_{k=1}^{\infty}(1/k^2)$," *Nordisk Mat. Tidskr.*, 18 (1970) 91-92.

5. Ioannis Papadimitriou,"A Simple Proof of the Formula $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$," *American Mathematical Monthly*, 80 (April 1973) 424-425.

6. D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Company, San Francisco, 1962, Problem 233, pp. 53, 339-341.

7. E.L. Stark, "Another Way to Sum $\Sigma(1/k^2)$", *The Mathematical Gazette*, 63 (March 1979) 47.

8. A.M. Yaglom and I.M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Volume II, Holden-Day, 1967, Problem 145*a*, pp. 24, 131.

* * *

441. [1979: 131] *Proposed by Sunder Lal, Institute of Advanced Studies, Meerut University, Meerut, India (part (a)); and the editor (part (b)).*

(a) Solve the decimal alphametic

```
 ASHA
  GOT
  THE
-----
MEDAL .
```

(b) Who was ASHA and what did he or she do to deserve THE MEDAL?

I. *Solution and comment by J.A.H. Hunter, Toronto, Ontario.*

(a) There are 5 possible values for MEDAL, stemming from

```
 9639     9389     9329     9389     9329
  725      725      846      427      548
  530      580      620      780      820
-----,   -----,   -----,   -----,   -----.
10894    10694    10795    10596    10697
```

In each case, the values corresponding to S and G are interchangeable, making 10 solutions in all.

(b) ASHA BHOSLE is the famous Indian chanteuse, and may well be the "Asha" who got the medal.

Comment. This was a bad problem, in that no unique value for MEDAL was pinpointed. In fact, no very satisfactory way to pinpoint a necessarily unique MEDAL — of course with a "play on words" — can be found here. However, it could be acceptable to stipulate: "The MEDAL might be prime judging by its appearance, but appearances can be deceptive!"

II. *Comment by J.A. McCallum, Medicine Hat, Alberta.*

This is terrible. No medal for CRUX on this one!

III. *"Solution" to part (b) by Charles W. Trigg, San Diego, California.*

A conclave of acronymists, unable to agree among themselves, offer their several identifications:

A Suave Hungarian Argufier, who won THE MEDAL in a debating contest.

A Swift Hurdling African, Olympic Medalist in the steeplechase.

A Skinny Hungry Arkansan, winner of a pie-eating contest.

A Spear-Hurling Abyssinian, winner of the javelin throw.

Etc.

Also solved by GÖRAN ÅBERG, Själevad, Sweden (4 solutions); LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas (2 solutions); ALLAN WM. JOHNSON JR., Washington, D.C. (10 solutions); EDGAR LACHANCE, Ottawa, Ontario (2 solutions); J.A. McCALLUM, Medicine Hat, Alberta (10 solutions); CHARLES W. TRIGG, San Diego, California (6 solutions); KENNETH M. WILKE, Topeka, Kansas (10 solutions); and the proposer (1 solution).

Editor's comment.

Okay, okay, okay, so Asha got 5 reversible medals.

This is what happens when a proposer submits an alphametic, includes a single answer and no solution, and the editor, poor sap, who couldn't solve one of those things if his life depended on it, is taken in.

Hunter's identification of Asha is plausible and may well be correct. But it is strange that none of our readers from India (including the proposer) saw fit to provide us with an authoritative identification.

Trigg's contributions to the CRUX Problem Section have been many and valuable. But this time his tongue in cheek solution (unless that's his bubble gum sticking out) shows that he did not deign to treat this problem with the seriousness it deserved. But we're still glad to have him aboard, even with Abyssinians.

So long, Trigg. Abyssinia.

* * *

442. [1979: 131] *Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Prove that the equation of *any* quartic (a plane curve of order 4) may, in an infinity of ways, be thrown into the form

$$aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV = 0,$$

where $U = 0$, $V = 0$, $W = 0$ represent three conics.

Solution by Gali Salvatore, Perkins, Québec.

Let $Q(x,y,z) = 0$ be a given quartic, in homogeneous coordinates $(x,y,z) \neq (0,0,0)$. The number of homogeneous products of degree m in n variables is

$${}_nH_m = \frac{(m+n-1)!}{m!(n-1)!}.$$

For our quartic, we have $m = 4$ and $n = 3$, so Q contains ${}_3H_4 = 15$ terms in x^4, x^3y, ..., z^4, and the curve $Q = 0$ is uniquely determined by 14 distinct points on it. We will show that there are infinitely many triples (U,V,W), where $U = 0$, $V = 0$, $W = 0$ are conics, and, for each such triple, constants a, b, c, $2f$, $2g$, $2h$ such that the quartic curve

$$aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV = 0 \tag{1}$$

coincides with the given curve $Q = 0$ for 14 distinct points, and thus our theorem will be established.

Let $(P_1,P_2,\ldots,P_{15})$ be an ordered set of 15 distinct points on the given curve $Q = 0$. There are infinitely many such ordered sets, and with each of them we can associate a triple (U,V,W), where $U = 0$ is the unique conic on the 5 points $P_1,\ldots,P_5$; $V = 0$ is the unique conic on the 5 points $P_6,\ldots,P_{10}$; and $W = 0$ is the unique conic on the 5 points $P_{11},\ldots,P_{15}$. Suppose such a triple (U,V,W) is substituted in (1); then we have a quartic with 6 undetermined coefficients a, b, c, $2f$, $2g$, $2h$. We will select values for these coefficients such that the 14 points $P_1,\ldots,P_{14}$ will lie on (1). If $P_i = (x_i,y_i,z_i)$, we will use U_i, V_i, W_i to denote $U(x_i,y_i,z_i)$, etc.

The 5 points $(0,V_i,W_i)$, $i = 1,\ldots,5$ (which may or may not be distinct), lie on at least one conic. Let one of the equations of such a conic be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0. \tag{2}$$

For the coefficients b, c, $2f$ in (1), we select the coefficients of y^2, z^2, yz, respectively, in (2). As a result, we have

$$U_i = 0 \quad \text{and} \quad bV_i^2 + cW_i^2 + 2fV_iW_i = 0, \qquad i = 1,\ldots,5,$$

and the points $P_1,\ldots,P_5$ lie on (1) regardless of the subsequent choice of values for a, $2g$, $2h$.

The 5 points $(U_i,0,W_i)$, $i = 6,\ldots,10$, lie on at least one conic whose equations are of the form (2). One of the equations for such a conic has as its coefficient of z^2 the value c previously chosen. We select the coefficients of x^2 and zx in that equation and assign these values to a and $2g$, respectively, in (1). As a result, we have

$$V_i = 0 \quad \text{and} \quad aU_i^2 + cW_i^2 + 2gW_iU_i = 0, \qquad i = 6,\ldots,10,$$

and the points $P_6,\ldots,P_{10}$ also lie on (1) regardless of the subsequent choice of $2h$.

The 4 points $(U_i,V_i,0)$, $i = 11,\ldots,14$, lie on infinitely many conics whose equations are of the form (2). Let $S_1 = 0$ and $S_2 = 0$ be two such distinct conics, where

$$S_1 = a_1x^2 + b_1y^2 + \ldots + 2h_1xy \quad \text{and} \quad S_2 = a_2x^2 + b_2y^2 + \ldots + 2h_2xy.$$

All conics in the two-parameter family $\lambda S_1 + \mu S_2 = 0$ also pass through the same 4 points. If we assign to λ and μ values which satisfy the linear system

$$\lambda a_1 + \mu a_2 = a$$

$$\lambda b_1 + \mu b_2 = b,$$

where a and b are the values previously assigned in (1), the resulting conic from the family will have the equation

$$ax^2 + by^2 + \ldots + (2\lambda h_1 + 2\mu h_2)xy = 0. \tag{3}$$

The value we assign to our last undetermined coefficient $2h$ in (1) is the coefficient of xy in (3). As a result, we have

$$W_i = 0 \quad \text{and} \quad aU_i^2 + bV_i^2 + 2hU_iV_i = 0, \qquad i = 11,\ldots,14,$$

and the points $P_{11},\ldots,P_{14}$ also lie on (1), which completes the proof.

One other (doubtful) solution was received.

Editor's comment.

The proposer mentioned that the statement in our problem is given without proof in George Salmon's *A Treatise on the Higher Plane Curves*, G.E. Stechert and Co., New York, 1934, page 242.

The other solution received, which the editor (perhaps in his ignorance) has labeled "doubtful", is conceptually much simpler than the one given above. We paraphrase it as follows:

"We set

$$aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV = Q,$$

where $Q = 0$ is the given quartic and equate the coefficients of corresponding terms to find the unknown coefficients on the left. Of the 6 coefficients a, b, c, $2f$, $2g$, $2h$, only 5 need be determined, since any one of them can be taken to be 1 without loss of generality. Similarly, for each conic $U = 0$, $V = 0$, $W = 0$, only 5 coefficients need be determined. So there are in all 20 unknown coefficients to be determined. Since Q contains 15 terms, comparing coefficients yields 15 equations in 20 unknowns. For such a system, there are infinitely many solutions for the 20 unknowns."

The statement made in the last sentence would certainly be true if the 15 equations were linear in the 20 unknowns, but they are not. The argument may still be valid, but it seemed to the editor that some additional explanation was needed at this point. So this proof is, at the very least, incomplete. If some reader can clinch the argument, this could provide a proof much simpler than the one we have featured.

* * *

443, [1979: 132] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

(a) Here are seven consecutive squares for each of which its decimal digits sum to a square:

81, 100, 121, 144, 169, 196, 225.

Find another set of seven consecutive squares with the same property.

(b)* Does there exist a set of more than seven consecutive squares with the same property?

Partial solution by the proposer.

For any positive integer n, the seven consecutive squares

$$(10^n + i)^2, \quad i = -1, 0, 1, 2, 3, 4, 5, \tag{1}$$

have decimal digit sums (*DDS*) of $9n$, 1, 4, 9, 16, 16, 9, respectively. These results are obvious, except possibly when $i = -1$ where we have

$$DDS\,(10^n - 1)^2 = DDS\,\{10^n(10^n - 2) + 1\} = 9(n-1) + 8 + 1 = 9n.$$

Hence (1) gives seven consecutive squares each with a square *DDS* if and only if $n = k^2$ for some positive integer k. The seven squares in the proposal are obtained by setting $n = 1$. For $n = 4$, we get the set

$$9999^2 \quad 10000^2 \quad 10001^2 \quad 10002^2 \quad 10003^2 \quad 10004^2 \quad 10005^2, \tag{2}$$

which provides an answer to part (a) of the problem. Other answers can be obtained by setting $n = 3^2, 4^2, 5^2, \ldots$.

The set (1) cannot be extended to eight consecutive squares with square *DDS* since

$$DDS\,(10^{k^2} - 2)^2 = DDS\,\{10^{k^2}(10^{k^2} - 4) + 4\} = 9(k^2 - 1) + 6 + 4 = 9k^2 + 1,$$

which is never a square, and

$$DDS\,(10^{k^2} + 6)^2 = DDS\,(10^{2k^2} + 12 \cdot 10^{k^2} + 36) = 13,$$

which is not a square.

In an unsuccessful attempt to discover a set of more than seven consecutive squares with square *DDS*, I examined by computer the first million squares to identify each square with a square *DDS*, the squares thus identified being monitored to detect all sets of six or more consecutive squares with the desired property. The enumeration shows that among the million squares $1^2, 2^2, \ldots, (10^6)^2$, there is no set of more than seven consecutive squares with square *DDS*, exactly two sets of seven (the one given in the proposal and (2)), and exactly four sets of six not obtainable from (1). The latter are

$$11117^2 \quad 11118^2 \quad 11119^2 \quad 11120^2 \quad 11121^2 \quad 11122^2,$$

$$41267^2 \quad 41268^2 \quad 41269^2 \quad 41270^2 \quad 41271^2 \quad 41272^2,$$

$$185798^2 \quad 185799^2 \quad 185800^2 \quad 185801^2 \quad 185802^2 \quad 185803^2,$$

and

$$246008^2 \quad 246009^2 \quad 246010^2 \quad 246011^2 \quad 246012^2 \quad 246013^2.$$

It turns out, unexpectedly, that these four sets all have the same sequence of *DDS*:

$$49 \quad 36 \quad 25 \quad 25 \quad 36 \quad 49.$$

This fact may turn out to be of significance in the search, among the Himalayas of numbers, for the elusive set of eight consecutive squares with square *DDS*.

Also solved (part (a) only) by FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; CHARLES W. TRIGG, San Diego, California; and KENNETH M. WILKE, Topeka, Kansas. One incorrect solution was received.

Editor's comment.

The incorrect solution received was apparently based on a misreading of the problem.

One solver gave as one of his answers to part (a) the seven consecutive integers

$$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3,$$

whose squares have square *DDS*. But the squares of these numbers don't form seven consecutive squares, so this does not constitute an answer to the problem as posed.

Part (b) of the problem remains open.

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444. [1979: 132] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

A circle is inscribed in a square ABCD. A second circle on diameter BE touches the first circle, as shown in Figure 1. Show that AB = 4BE.

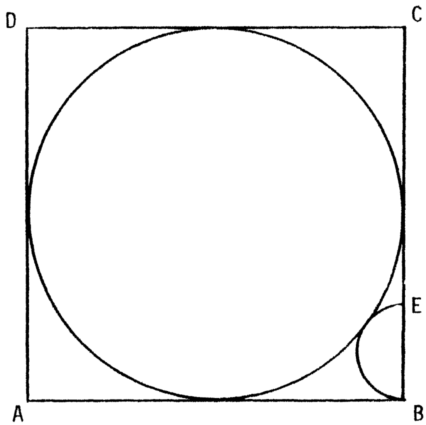

Figure 1

Solution by Leon Bankoff, Los Angeles, California.

The square in Figure 1 is a window that shows only a part of the scenery and hides the most interesting portion. Step outside with me to see the complete picture (Figure 2).

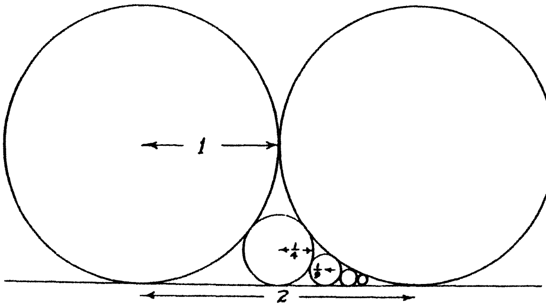

Figure 2

We will need the following problem proposed in 1940 by C.W. Trigg [1]:

If a and b are the radii of two spheres, tangent to each other and to a plane, show that the radius x of the largest sphere which can pass between them is given by the formula

$$x^{-1/2} = a^{-1/2} + b^{-1/2}. \tag{1}$$

Let C_1 stand for either of the large equal circles in Figure 2, and let C_2, C_3, ... be the remaining circles in decreasing order of size. If r_n is the radius of C_n, then $r_1 = 1$ and it follows from (1) by induction that $r_n = 1/n^2$ for $n = 1,2,3,\ldots$. In particular, our proposed problem follows from the fact that $r_2 = 1/4$.

Still more useful information is contained in Figure 2. The segment of the common tangent included between its contacts with two consecutive circles, C_{n-1} and C_n, is

$$\sqrt{\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)^2 - \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right)^2} = \frac{2}{n(n-1)} = \binom{n}{2}^{-1},$$

and it follows from Figure 2 that

$$\sum_{n=2}^{\infty}\binom{n}{2}^{-1} = 2.$$

This is a theorem I had stated and proved in 1956 [2]. Its proof provides a fruitful interplay between algebra and geometry.

Also solved by MICHAEL ABRAMSON, 14, student, Benjamin N. Cardozo H.S., Bayside, N.Y.; W.J. BLUNDON, Memorial University of Newfoundland; RICHARD BURNS, East Longmeadow H.S., East Longmeadow, Massachusetts; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia; HOWARD EVES, University of Maine; J.T. GROENMAN, Groningen, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, New Jersey; HYMAN ROSEN, Cooper Union, Brooklyn, N.Y.; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DONALD P. SKOW, Griffin & Brand, Inc., McAllen, Texas; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia (two solutions); CHARLES W. TRIGG, San Diego, California (three solutions); FERRELL WHEELER, student, Forest Park H.S., Beaumont, Texas; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; JOHN A. WINTERINK, Albuquerque Technical Vocational Institute; and the proposer.

Editor's comment.

There were several easy ways, both synthetic and analytic, of proving the theorem as stated. The most popular among solvers was the following:

$$(R - r)^2 + R^2 = (R + r)^2 \Longrightarrow R = 4r,$$

where R and r are the radii of the two circles in Figure 1, and AB = 4BE follows immediately.

This problem was, in fact, one of the easy ones we put in from time to time to encourage more readers to cease being passive spectators and jump into the action. The experiment was a modest success, for several names never or seldom seen before appear in the list of solvers. *It is hoped* that these readers will continue to contribute *hopefully*, because the quality of this journal depends to a very large extent upon the editor having a wide range of solutions to choose from.

REFERENCES

1. Problem E432, *American Mathematical Monthly*, 48 (1941) 267-268. Proposed by C.W. Trigg, 47 (1940) 487.

2. Problem 92, *Pi Mu Epsilon Journal*, 2 (Spring 1958) 378-379. Proposed by Leon Bankoff in the Fall 1956 issue.

* * *

445. [1979: 132] *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

Consider a family of parabolas escribed to a given triangle. To each parabola corresponds a focus F and a point S of intersection of the lines joining the vertices of the triangle to the points of contact with the opposite sides. Prove that all lines FS are concurrent.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let ABC be the given triangle, with side lengths a, b, c. We use areal coordinates, with ABC as the triangle of reference. We assume for now that $b \neq c \neq a \neq b$. For a justification of the unexplained steps in this demonstration, see [1].

Any escribed parabola has an equation of the form

$$\lambda x^2 + \mu y^2 + \nu z^2 - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy = 0,$$

where $\lambda + \mu + \nu = 0$, and its focus is

$$F = \left(\frac{a^2}{\lambda}, \frac{b^2}{\mu}, \frac{c^2}{\nu}\right).$$

The points of contact with the sides are

$$\left(0, \frac{1}{\mu}, \frac{1}{\nu}\right), \quad \left(\frac{1}{\lambda}, 0, \frac{1}{\nu}\right), \quad \left(\frac{1}{\lambda}, \frac{1}{\mu}, 0\right);$$

and their joins to the opposite vertices concur at the point

$$S = \left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\nu}\right),$$

since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\mu} & \frac{1}{\nu} \\ \frac{1}{\lambda} & \frac{1}{\mu} & \frac{1}{\nu} \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{\lambda} & 0 & \frac{1}{\nu} \\ \frac{1}{\lambda} & \frac{1}{\mu} & \frac{1}{\nu} \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 0 & 1 \\ \frac{1}{\lambda} & \frac{1}{\mu} & 0 \\ \frac{1}{\lambda} & \frac{1}{\mu} & \frac{1}{\nu} \end{vmatrix} = 0.$$

Since

$$\begin{vmatrix} \frac{1}{b^2-c^2} & \frac{1}{c^2-a^2} & \frac{1}{a^2-b^2} \\ \frac{a^2}{\lambda} & \frac{b^2}{\mu} & \frac{c^2}{\nu} \\ \frac{1}{\lambda} & \frac{1}{\mu} & \frac{1}{\nu} \end{vmatrix} = \frac{1}{\mu\nu} + \frac{1}{\nu\lambda} + \frac{1}{\lambda\mu} = \frac{\lambda+\mu+\nu}{\lambda\mu\nu} = 0,$$

it follows that all lines FS are concurrent in the point

$$K = \left(\frac{1}{b^2-c^2}, \frac{1}{c^2-a^2}, \frac{1}{a^2-b^2}\right).$$

Since areal coordinates are uniquely determined only to within ratios, the same point K is given by

$$K = \left((a^2-b^2)(a^2-c^2),\ (b^2-c^2)(b^2-a^2),\ (c^2-a^2)(c^2-b^2)\right). \tag{1}$$

If triangle ABC is strictly isosceles, then the required point K is still given by (1). If $b = c \neq a$, for example, then all lines FS go through A, and (1) gives

$$K = (1,0,0) = A.$$

However, equilateral triangles must be excluded from the discussion since then F and S coincide and the lines FS do not exist. This explains why (1) is undefined when $a = b = c$.

Also solved by F.G.B. MASKELL, Algonquin College, Ottawa; and the proposer.

REFERENCE

1. E.H. Askwith, *Analytical Geometry of the Conic Sections*, Adam and Black, 1908, Chapter XIV, pp. 273-316.

* * *

446. [1979: 132] *Proposed by the late R. Robinson Rowe, Sacramento, California.*

An errant knight stabled at one corner of an $N \times N$ chessboard is "lost", but happens to be at the diagonally opposite corner. If he moves at random, what is the probable number of moves he will need to get home (a) if $N = 3$ and (b) if $N = 4$?

Solution and comment by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

(a) Number the squares as shown in Figure 1. Then the problem is to determine the expected (or average) number of moves required to get from square 0 to square 4. We will assume that the knight is allowed to return to square 0.

0	3	2
3		1
2	1	4

Figure 1

Let P_i be the expected number of times that the knight lands on a square i. Then the required expected number of moves is

$$E = P_0 + P_1 + P_2 + P_3 + P_4. \qquad (1)$$

It is easy to see from the figure that we must have

$$P_4 = \tfrac{1}{2}P_3, \quad P_3 = \tfrac{1}{2}P_2, \quad P_2 = \tfrac{1}{2}(P_1 + P_3), \quad P_0 = \tfrac{1}{2}P_1, \qquad (2)$$

and that the values obtained from (2) must satisfy

$$P_1 = 1 + P_0 + \tfrac{1}{2}P_2, \qquad (3)$$

since the first move is always to a square 1. Since $P_4 = 1$ (the game is over when the knight lands on square 4), equations (2) yield successively $P_3 = 2$, $P_2 = 4$, $P_1 = 6$, $P_0 = 3$, and these values satisfy (3). Thus, from (1),

$$E = 3 + 6 + 4 + 2 + 1 = 16.$$

(b) The solution for $N = 4$ is similar. The labeling of the squares in Figure 2 is self-explanatory. The equations to be solved are

0	3	2	5
3	4	1	2
2	1	4	3
5	2	3	6

Figure 2

$$\begin{aligned} &P_6 = \tfrac{1}{4}P_1, && P_0 = \tfrac{1}{4}P_1, && P_1 = 1 + \frac{1}{3}P_2 + P_0, \\ &P_2 = \frac{2}{3}P_3 + \tfrac{1}{2}P_1, && P_3 = \tfrac{1}{2}P_4 + \frac{2}{3}P_2, && P_5 = \tfrac{1}{2}P_4, \end{aligned} \qquad (4)$$

and the values obtained from (4) must satisfy

$$P_4 = P_5 + \frac{1}{3}P_3. \qquad (5)$$

Since the game ends when the knight lands on square 6, we have $P_6 = 1$ and then, successively from (4), $P_1 = 4$, $P_0 = 1$, $P_2 = 6$, $P_3 = 6$, $P_4 = 4$, $P_5 = 2$, and these values satisfy (5). The required number of moves is

$$E = \sum_{i=0}^{6} P_i = 24.$$

Comment. When $N > 4$, the computational difficulty of setting up and solving the simultaneous equations takes the problem out of the recreational category. Therefore the problem was simulated by Monte Carlo methods on a computer. The results were:

N	10000 trials	100000 trials	Analytic solution
3	15.99	16.01	16
4	23.67	23.98	24
5	59.20	59.07	59.9395
6	100.09		
7	150.37		
8	214.22		

The analytic result for $N = 5$ was obtained by solving the simultaneous equations by an iterative method on the computer. It is tempting to conjecture, on the basis of the results for $N = 3$ and $N = 4$, that the expected number of moves is always an integer. However, the analytical result for $N = 5$ shows that the conjecture is false.

The situation is different if the game starts and ends at the knight's home corner. The expected number of moves is then $E = 4(N-1)(N-2)$, and thus always an integer. This was shown elsewhere by the proposer [1].

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MILTON P. EISNER, J. Sargeant Reynolds Community College, Richmond, Virginia (part (a) only); and the proposer.

REFERENCE

1. R. Robinson Rowe, "Roundtripping a Chessboard," *Journal of Recreational Mathematics*, 4 (October 1971) 267.

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MAMA-THEMATICS

Mrs. Hamilton to son William R.: "I'm sure you're in the right field, son."

DAVID R. STONE

ADDENDA...

Here is additional information to amplify the editor's footnote in [1980: 3].

First of all, Altshiller-Court refers incorrectly to page 253 of the *Annales de mathématiques*: it should be page 353. The author unidentified by Altshiller-Court is identified on page 230 of the *Annales* simply as "M.", so we are not much further ahead. The problem solved by M. on that page is the following: *Partager un tétraèdre en deux parties équivalentes, par un plan qui coupe deux couples d'arêtes opposées, de manière que l'aire de la section soit un minimum?* The problem had earlier been proposed anonymously, on pages 127-128, so it may have been by one of the editors of the *Annales*, J.D. Gergonne and J.E. Thomas-Lavernède.

On page 353 of the *Annales* begins a "Mémoire sur le tétraèdre" with the coy by-line "Par M.J.L. ..., abonné." One of the items proved in that *Mémoire* is the following: *Le point où se coupent les droites qui joignent les milieux des arêtes opposées d'un tétraèdre est le centre de gravité du volume de ce tétraèdre.* There are several other items more or less distantly related to Theorem 1 of Bottema's article, but nowhere is it stated unequivocally as Bobillier does in reference [6] [1980: 4].

There is one final feature too good to omit. The *Tome Premier* of the *Annales de mathématiques* is given as being published A NISMES, DE L'IMPRIMERIE DE LA VEUVE BELLE.

LEON BANKOFF

...AND CORRIGENDA

The number in the top-right corner of the matrix in Problem 1 of "Two Timely Problems" [1980: 8] appears as 213. It should be 159. Readers will please correct their copy.

HOWARD EVES

The following corrections should be made in my article "Pascal Redivivus: II" [1979: 281-287]:

1. On page 286, lines 8 and 9 from the bottom, the final part of the paragraph, beginning with "and the major axis of the hyperbola ...," should be replaced by "and one of the principal axes of the hyperbola lies in the angle between the asymptotes which contains points of the hyperbola. This is the transverse axis."

2. In the last paragraph on the same page and in the first paragraph on page 287, replace "major axis" wherever it occurs by "transverse axis."

DAN PEDOE