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E U R E K A

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CONTENTS

A HISTORY OF COMPLEX NUMBERS

CLAYTON W. DODGE, University of Maine at Orono

Throughout the history of mathematics it seems that the rule has been to use each new entity for a few hundred years before deciding just what it is you are using. The calculus, for example, invented toward the end of the seventeenth century, was not fully understood until the nineteenth century. Hazy thinking about its ill-defined concepts produced many absurdities such as the statement by Johann Bernoulli (1667-1748) that "a quantity that is either increased or decreased by an infinitesimal quantity is neither increased nor decreased."

In an attempt to justify using negative, irrational, and imaginary numbers, just over 100 years ago the English mathematician George Peacock (1791-1858) stated his *principle of permanence of forms* that "equal general expressions of arithmetic are to remain equal when the letters no longer denote simple quantities, and hence also when the operation is changed." He was trying to state that the same rules applied to negatives, irrationals, etc. as apply to positive rational numbers. For example, since $a+b = b+a$ whenever a and b are positive rationals, then $a+b = b+a$ should hold for all numbers.

Such fuzzy thinking never really contributed to the advancement of mathematics.

The earliest glimmerings of the existence of imaginary numbers occurred some 1100 years ago, when the Hindu Mahavira made the very intelligent statement that "a negative number has no square root." Furthermore, he wisely made no attempt to work with these nonexistent square roots of negative numbers. Even 1000 years later, in 1847, Augustin Cauchy (1789-1857), the eminent Frenchman, made the same statement, discarding $\sqrt{-1}$ because he felt it had no meaning.

Many students today feel that $\sqrt{-1}$ falls into the same nonexistent category as do the expressions $0/0$, $5/0$, $\infty - \infty$, 0^0, and ∞^0. Often they have been told little more than that, since some quadratic polynomials fail to have two real linear factors and since the quadratic formula gives rise to expressions involving square roots of negative numbers, then we use these square roots in just the same way *as if they were numbers*.

This is the type of argument all too prevalent up to the middle of the nineteenth century: the formula gives us a certain result we have never seen before and is even absurd, but since a *formula* gave it to us, why then it just *must* be correct. In the seventeenth and eighteenth centuries many mathematicians, looking at the divergent infinite series

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots,$$

accepted that $S = 1$ or $S = 0$ according to how the terms were grouped, for we have

$$1 = 1 + (-1 + 1) + (-1 + 1) + \ldots$$

and

$$0 = (1 - 1) + (1 - 1) + (1 - 1) + \ldots .$$

They concluded, by logic that would cause Aristotle to roll over in his grave, that since the answers 0 and 1 were "equally likely," then clearly S should equal 1/2! Oddly enough, this answer, too, can be obtained algebraically. We have

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - 1 + \ldots \\ &= 1 - (1 - 1 + 1 - 1 + 1 - \ldots) \\ &= 1 - S, \end{aligned}$$

so $2S = 1$ and $S = 1/2$. Their crowning achievement was to use $S = 1/2$ in the second grouping above to get

$$\frac{1}{2} = (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + 0 + 0 + \ldots,$$

a truly remarkable result. Folding their hands, they then piously stated that this equation illustrated the fact that out of nothing God created the universe! To which we reply: "Well, half of it, anyway."

The early thinking about complex numbers was not unlike the muddy ideas expressed in other areas of mathematics, although occasionally it was dotted with a bit of insight.

The Hindu Bhaskara Acharya (1114-1185?) recognized that a square root of a positive number can be either positive or negative, but a negative number has no square root, for the "negative number is no square."

Girolamo Cardano (or Cardan) (1501-1576) observed the need for negative and imaginary numbers as roots of equations. He called these numbers false, fictitious, and ingenious but useless, but he still used them. Among other results, he proved that imaginary zeros of polynomials with real coefficients come in pairs and that

$$(5 + \sqrt{-15})(5 - \sqrt{-15}) = 40.$$

Raphael Bombelli (1526-1573) improved algebraic notation, wrote roots of cubic equations as sums of imaginary numbers, and gave rules for handling imaginaries.

Albert Girard (1590?-1633?) conjectured that each polynomial of degree n has exactly n linear factors, the missing real factors to be just made up by imaginary factors. He felt this to be true since some polynomials of low degree n do indeed have n linear factors. No attempt at a proof was made.

More wisely, René Descartes (1596-1650) stated that a polynomial equation of degree n has "no more than n roots." It was Descartes who applied those unfortunate words "real" and "imaginary" to numbers. At that time imaginaries were considered to be "uninterpretable and even self-contradictory." That they were used with ever-increasing faith is one of the great mysteries of human nature.

In 1673 John Wallis (1616-1703) just missed discovering the usual geometric interpretation of complex numbers. He and others used $\surd{-1}$ to denote the imaginary unit. Later writers used $\surd(-1)$ and $\sqrt{-1}$. It was Euler who introduced the symbol i and Gauss who popularized it.

Abraham De Moivre (1667-1754) introduced complex numbers into trigonometry, discovering his famous theorem, which states

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

for n a positive integer.

Leonhard Euler (1707-1783), a firm believer in manipulation of formulas, was the first to state De Moivre's theorem in its present form, and he extended it to all rational values of n. It was in 1777, the year of the birth of Gauss, that he suggested i for $\sqrt{-1}$ and called $a^2 + b^2$ the *norm* of $a + bi$. Euler was a rather ordinary person. For a mathematician.

In order to give some sort of meaning to complex numbers, Caspar Wessel (1745-1818), a Norwegian surveyor for the Danish Academy of Sciences, discovered their geometric interpretation in the plane. His very complete paper, in which he interpreted addition as translation and multiplication as rotation, appeared in 1799, two years after his discovery. Unfortunately, it lay unnoticed by the mathematical world for nearly 100 years. The fame due him was greatly belated, and very little glory has come to him, since his name is not attached to any phase of complex numbers.

As occurs so often in the history of science, another person, the bookkeeper Jean Robert Argand (1768-1822) of Geneva, discovered the same interpretation in 1806. His paper, although not as clear as Wessel's, was widely read. Hence the plane of complex numbers is called an Argand diagram.

Now if $z = a + bi$ and $w = c + di$, then $z + w = a + c + (b + d)i$. Interpreting z and w as translations of the points of the plane, then $z + w$ is the translation formed by first performing the translation z and then performing the translation w. Recalling the forerunner of De Moivre's theorem, that

$$(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \cos(\alpha+\beta) + i \sin(\alpha+\beta),$$

we see that the product of two complex numbers

$$z = r(\cos \alpha + i \sin \alpha) \quad \text{and} \quad w = s(\cos \beta + i \sin \beta),$$

considered as stretches of ratios r and s and rotations through angles α and β, represents a stretch of ratio rs and a rotation through the angle $\alpha+\beta$.

Simply replacing coordinates (a,b) of points in the plane by complex numbers $a+bi$ provides a nice picture, aids the imagination, and has most worthwhile applications in geometry, trigonometry, and electrical circuitry, but in no way does it prove that complex numbers exist in algebra. There was still no justification for using imaginary numbers as roots of equations. But the nineteenth century marked the beginning of efforts to solve the problems in the foundations of mathematics. So the time was ripe for someone to find the true meaning of complex numbers in mathematics, and again two men, independently of each other, provided identical answers.

In 1825 Carl Friedrich Gauss (1777-1855), the greatest mathematician of modern times, stated that "the true metaphysics of $\sqrt{-1}$ is illusive." Thus he recognized the problem at hand, a giant step forward! Since his doctoral dissertation was a proof of the fundamental theorem of algebra, that each polynomial with complex coefficients has at least one complex zero, he was quite well versed in imaginary numbers.

In 1831 he found that "true metaphysics of $\sqrt{-1}$." Keeping in mind the algebra of complex numbers $a+bi$, he defined an algebra of ordered pairs (a,b) of real numbers a and b wherein equality, addition, and multiplication are given by

$$(a,b) = (c,d) \text{ if and only if } a = c \text{ and } b = d\,,$$

$$(a,b)+(c,d) = (a+c,\ b+d),$$

and

$$(a,b)(c,d) = (ac-bd,\ ad+bc).$$

Thus the ordered pair (a,b) is used in place of the form $a+bi$.

In the resulting algebra one easily proves [6, pp. 387-395] that

$$(a,b)+(0,0) = (a,b), \quad (a,b)(1,0) = (a,b),$$

$$(a,b)+(-a,-b) = (0,0),$$

and, if $(a,b) \neq (0,0)$, then

$$(a,b)(a/(a^2+b^2),\ -b/(a^2+b^2)) = (1,0).$$

Thus $(0,0)$ and $(1,0)$ are the additive and multiplicative identities 0 and 1, and the last two equations give expressions for the additive and multiplicative inverses. One also shows that both addition and multiplication are both commutative and associative, and that multiplication distributes over addition. Since this set of ordered pairs is closed under both operations, we have a field. Here then we have an interesting field of ordered pairs of real numbers, an algebra that assumes no strange square roots or mysterious new symbols.

Next Gauss showed that those ordered pairs of the form $(a,0)$ behave like real numbers a in the sense that

$$(a,0) = (b,0) \text{ if and only if } a=b,$$

$$(a,0)+(b,0) = (a+b,\ 0),$$

and

$$(a,0)(b,0) = (ab,0).$$

That is, ordered pairs $(a,0)$ and $(b,0)$ are equal when and only when the corresponding real numbers a and b are equal, and the sum and product of the ordered pairs correspond to the sum and product of the corresponding real numbers. We say that these two sets are *ring isomorphic*, and we agree to write $a = (a,0)$. That is, since the pairs $(a,0)$ behave like real numbers a, we write them as real numbers, we use $(a,0)$ and a interchangeably.

Now for convenience we define the symbol i to stand for the ordered pair $(0,1)$; $i = (0,1)$. Then we have

$$a+bi = (a,0)+(b,0)(0,1) = (a,0)+(0,b) = (a,b),$$

so we may now write the ordered pair (a,b) in the more familiar $a+bi$ form. Finally we prove that

$$i^2 = (0,1)(0,1) = (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0) = (-1,0) = -1.$$

Now we have shown that our ordered pairs of real numbers are actually the familiar complex numbers. The mathematician is now satisfied of their existence. This is the "true metaphysics of $\sqrt{-1}$" that Gauss discovered. His development has found a square root for -1 without assuming such a root. Logically, the uncertainty of the existence of $\sqrt{-1}$ has been removed; $\sqrt{-1}$ now has a firm, logical foundation in this algebra of ordered pairs.

Gauss did not make his work known, and quite independently, the Irishman Sir William Rowan Hamilton (1805-1865) performed this same feat in 1835 and published his work. Later it was found that Gauss had made the discovery earlier, so the complex plane is sometimes called the Gauss plane. In 1827 Hamilton received the singular honor of being elected unanimously to the vacant Chair of Astronomy at Trinity College, Dublin, while he was still an undergraduate! His main accomplishment in mathematics was his discovery in 1843 of quaternions, an algebra of ordered quadruples which broke the commutative law of multiplication, freeing algebra from its traditional shackles, just as Gauss, Bolyai, and Lobachevsky freed geometry about 1829 with their discovery of non-Euclidean geometry.

It would seem that the ordered pair notation (a,b) is only a thin disguise for $a+bi$, but the algebraic theory of the former is rigorous, eliminating the assumption of that mysterious entity i, which stands for the assumed $\sqrt{-1}$. It is not that we object to having $\sqrt{-1}$; it is just that we should not boldly assume it without first proving that its assumption does not lead to contradictions. Even as late as 1873 the Larousse dictionary stated that imaginary numbers are "impossible" and that algebra had found only two impossible entities: "the negative and the imaginary."

Not satisfied with merely proving the fundamental theorem of algebra and only demonstrating that complex numbers do exist, Gauss also developed the interesting number theory of *Gaussian integers*, complex numbers $a+bi$ where a and b are integers.

Just as the rational integers 0, ±1, ±2, ±3, ... form a subset of the rational or real numbers and have their own arithmetic or number theory, so also do the Gaussian integers have their arithmetic within the complex numbers. There are many parallels between the two systems, some of which are most clever.

The division algorithm, for example, states that if a and b are rational integers with $b>0$, then there are unique integers q and r (the quotient and remainder when a is divided by b) such that

$$a = qb+r \quad \text{and} \quad 0 \le r < b.$$

Because the rational numbers form an ordered field, it makes sense to write $0 \le r < b$. When one tries to state a division algorithm for the Gaussian integers, the problem arises that they cannot be ordered, since the complex numbers do not form an ordered field; so one cannot write a "less than" relation between complex numbers. (Would you say $2+3i < 3+2i$? Or is $3+2i < 2+3i$?)

The genius of Gauss shines through his solution to this problem. For a complex number $a+bi$ he used its norm $a^2+b^2 = |a+bi|^2$. He then proved the following division algorithm for the Gaussian integers: If w and z are Gaussian integers with $z \neq 0$, then there are Gaussian integers q and r such that

$$w = qz+r \quad \text{and} \quad 0 \le |r|^2 < |z|^2.$$

Furthermore, q and r are unique in the sense that, if q_1 and r_1 satisfy the division algorithm for given w and z, and if q_2 and r_2 also satisfy it, then

$$|q_1|^2 = |q_2|^2 \quad \text{and} \quad |r_1|^2 = |r_2|^2.$$

The theory of Gaussian integers paved the way for modern number theory, the higher arithmetic.

To illustrate the division algorithm, let us find the quotient q and remainder r when $8+13i$ is divided by $3+2i$. First perform the division in the usual way:

$$\frac{8+13i}{3+2i} = \frac{8+13i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{50+23i}{13} = \frac{50}{13} + \frac{23}{13}i .$$

The desired quotient q is the nearest Gaussian integer to the obtained quotient; that is, $q = 4+2i$ because 4 is the closest integer to $50/13$ and 2 is the closest integer to $23/13$. Now the remainder r is probably easiest to calculate from the division algorithm equation $r = w - qz$. Thus we have

$$r = 8+13i - (4+2i)(3+2i) = 8+13i - (8+14i) = -i.$$

So we have $q = 4+2i$ and $r = -i$. The reader may enjoy trying this method on his own numbers and showing that the norm of r satisfies the stated inequality.

The typical high school student of today rarely attains even as much understanding of complex numbers as did the early mathematicians. Students still feel that "i is the square root of a number you cannot take the square root of," a self-contradictory statement. Many high school mathematics classes teach that $a+bi$ is nothing more than a nonexistent symbol invented by mathematicians just so that each polynomial equation of degree 2 will have two not necessarily distinct roots. Occasionally complex numbers are pictured as plane numbers to contrast them with real numbers as linear numbers, but virtually never are applications carried further. It is no wonder that imaginary numbers are greeted with skepticism and suspicion.

It is to be hoped that more high school mathematics teachers will learn the applications of complex numbers to geometry, trigonometry, and physical forces so that their students will gain confidence in and understanding of these numbers. Plane analytic geometry is greatly simplified by using complex numbers $a+bi$ instead of coordinates (a,b) for points. Perhaps this application provides the best appreciation of this number system at the high school level.

By treating addition of complex numbers as vector addition and as composition of translations, and then multiplication as a combination stretch-rotation, one finds that that mysterious equation $i^2 = -1$ merely states the very obvious fact that two $90°$ rotations are equivalent to one $180°$ rotation.

Gauss suggested the words *horizontal* and *vertical* to replace "real" and "imaginary," so that students would not be wrongly influenced by misleading terminology. His suggestion, unfortunately, was not adopted.

My own offering recognizes that the symbol i and the abbreviation Im are so firmly entrenched that replacement would be next to impossible. Hence, let us coin a new word to accept these notations. I offer *immy*. "Immy" is short, has no current meaning to prejudice the student, and allows the current notations i and Im to remain. One may well question whether one could succeed in ousting the term "imaginary" when Gauss failed. But it may be that the adoption of a new term is the only way that general acceptance of complex numbers will ever occur. It seems worth a try. Your comments and suggestions are invited.

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* * *

THE IMAGINARY TWINS

The mathematicians and engineers,
On using the square root of minus one,
Don't see eye to eye, they see i to j.
As twins, i and j have an equal right
    To consideration
    From a grateful nation.

Now Clayton the Great, with his magic wand,
Has one yclept *immy*, and all applaud.
The engineers now should take up the cry:
A name for our own or else none for both.
    If they call i *immy*,
    We should call j *jimmy*.

Yes, this is the year when it should be done,
In Canada first, while it still is one,
And then to the world let the names be flung.
And let it be known they're in honour of
    Our good neighbour *Jimmy*
    And his daughter *Amy*.

EDITH ORR

* * *

LETTERS TO THE EDITOR

Dear Editor:

I have had a lot of fun reading EUREKA.

I agree that the problem in [1976: 216] is very elegant.

For Problem 133 [1976: 67, 144, 221] I once offered a prize, Erdös style; see *Vinculum 8* (Melbourne, Australia, 1971), pp. 4-7. Did Collatz (or someone at Syracuse) propose this problem before 1971? I have also seen it ascribed to Kakutani; I wrote to ask him about it, but he never replied.

For Problem 134 [1976: 68, 151, 173, 222; 1977: 12], the relevance of the regular 18-gon was clarified by G. Bol, Beantwoording van prijsvraag No. 17, *Nieuw Archief voor Wiskunde* (2) 18 (1936), pp. 14-66. That 'prize question' was the complete discussion of concurrent diagonals of regular polygons. Opposite p. 66 he gave one-eighteenth of the complete figure for the 18-gon. See also p. 126 of my *Regular Complex Polytopes* (Cambridge University Press, 1974), where a drawing of a certain complex polyhedron happens to embody an 18-gon with many (but not all) diagonals.

For Problem 163, Comment II [1976: 229], I believe the 'unpaired parentheses' could well be omitted. All that is needed is a convention to the effect that the solidus covers everything that follows:

$$b_0 + a_1/b_1 + a_2/b_2 + \dots .$$

This convention was used (without any apology) by Heilbron in his chapter in the book *Number Theory and Analysis* (Papers in honor of Edmund Landau, Plenum, New York, 1969, pp. 87-96); see also Ball and Coxeter, *Mathematical Recreations and Essays* (12th edition) University of Toronto Press, 1974). For a simple sum of fractions one would write instead

$$b_0 + (a_1/b_1) + (a_2/b_2) + \dots .$$

With best wishes,

H.S.M. COXETER,
University of Toronto.

Dear Editor:

In EUREKA [1976: 208], you asked me to check your report [1975: 5] that the millionth decimal place in the expansion of π is 5. Since I now have a programmable calculator (see [1976: 227]), I let it do the work of computing successive partial sums of Gregory's series

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}.$$

According to Philip Calabrese, "A note on alternating series," *American Mathematical Monthly*, Vol. 69 (1962), pp. 215-217 (reprinted on pp. 352-353 of *Selected Papers on Calculus*, edited by Apostol *et al.*, Mathematical Association of America, 1969), in order to calculate π correctly to a million decimal places, that is, to an accuracy of $\frac{1}{2}\cdot 10^{-1000000}$, $2\cdot 10^{1000000}$ terms are sufficient. I found that my calculator had added 1250 terms in 15 minutes. At this rate, there would be 5000 terms in 1 hour, 120000 terms in 1 day, and fewer than $5\cdot 10^7$ terms in 1 year. Hence the calculation would take more than $4\cdot 10^{999992}$ years. I don't have time to wait! Furthermore, since my calculator displays only 10 digits, I'd need 100000 calculators to display a million digits. I can't afford that many!

But — did I say "a million digits" or "a million decimal places"? There's a difference! A million decimal places for π means a million and one digits. I turned to my infallible source on such matters, Martin Gardner's articles in the *Scientific American*. I found the following:

> [September, 1975, p. 176; reprinted on p. 181 of Gardner's *The Incredible Dr. Matrix*, Scribner's, New York, 1976.] Dr. Matrix reminds his readers that on the basis of the third book of the Old Testament, 14th chapter, 16th verse, he had predicted in 1966 (see my *New Mathematical Diversions*, page 100) that the millionth digit of π would prove to be a 5. This was verified last year, when π was computed in Paris to a million decimal digits. (The millionth *decimal* digit, excluding the initial 3, is 1.)
>
> [November 1976, p. 133.] "I remember," I said [to Dr. Matrix], "that back in 1966, before π was calculated in France to a million places, you predicted that the millionth digit, counting the first 3 as a digit, would be 5."
>
> "And I was exactly right, was I not?"

Steve Conrad's swami friend (see [1976: 117]), his palm having first been suitably crossed with silver, promptly went into a trance and within ten minutes was able to confirm the above findings of Dr. Matrix.

LEROY F. MEYERS,
The Ohio State University.

Editor's comment.

If Professor Meyers does not wish to render this little service to the editor, he has only to say so and there will be an end to it, instead of using the slowness of convergence of the Gregory series as an excuse to get various swamis and quacks to do a questionable verification for him.

Euler would not have shirked the task. As reported on page 39 of *Squaring the Circle and Other Monographs*, by Hobson *et al.* (Chelsea, 1953), in 1755 Euler used the series

$$\tan^{-1} t = \frac{t}{1+t^2}\left\{1 + \frac{2}{3}\frac{t^2}{1+t^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{t^2}{1+t^2}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{t^2}{1+t^2}\right)^3 + \ldots\right\},$$

applied it to the formula

$$\pi = 20\tan^{-1}\frac{1}{7} + 8\tan^{-1}\frac{3}{79},$$

and calculated π to 20 places *in one hour*, without a programmable calculator, electric lighting, or ball-point pen. At that rate he could have calculated π to one million places in 50,000 hours or a mere 5.7 years, say 6 years with coffee breaks.

E.P.B. Umbugio (see [1976: 162]), who many believe is a far greater numerologist than Dr. Matrix, recalls hearing the latter say, at a seminar in advanced numerology at the Grand Academy of Lagado, in Laputa: "This will give you an idea of what is in my mind," as he erased the blackboard. So much for Dr. Matrix.

As for Conrad's swami, he wouldn't know a π if one hit him in the face.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without his permission.

For the problems given below, solutions, if available, will appear in EUREKA *Vol. 3, No. 5, to be published around May 15, 1977. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than May 1, 1977.*

211. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve the cryptarithm $\text{FGB}^2 = \text{MASKEL}$. Since there are nine letters involved, naturally we seek a solution where FGB is divisible by 9.

212. *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

Find four consecutive integers which are divisible by 5, 7, 9, 11 respectively.

213. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

(a) Prove that (in the usual notation) the sides of a triangle are in arithmetic progression if and only if $s^2 = 18Rr - 9r^2$.

(b) Find the corresponding result for geometric progression.

214. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Prove that if the sequence $\{a_i\}$ is an arithmetic progression, then

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{a_k} + \sqrt{a_{k+1}}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}} .$$

215. *Proposed by David L. Silverman, West Los Angeles, California.*

Convert the expression given below from mathematics to English, thereby obtaining the perfect scansion and rhyme scheme of a limerick:

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 5(11) = 9^2 + 0.$$

216. *Proposed by L.F. Meyers, The Ohio State University.*

For which positive integers n is it true that

$$\sum_{k=1}^{(n-1)^2} [\sqrt[3]{kn}] = \frac{(n-1)(3n^2 - 7n + 6)}{4}\,?$$

The brackets, as usual, denote the greatest integer function.

217. *Proposed by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

Find all integer solutions of $n^2(n-1)^2 = 4(m^2-1)$.

218. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Everyone knows that the altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. The median to the hypotenuse also has this property. Does any other segment from vertex to hypotenuse have the property?

219. *Proposed by R. Robinson Rowe, Sacramento, California.*

Find the least integer N which satisfies

$$N = a^{a+2b} = b^{b+2a}, \quad a \neq b.$$

220. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

C is a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. A chord through B meets CD at T and arc AD at U. Prove that there is a circle tangent to CD at T and to arc AD at U.

* * *

A Double Dactyl

YOO-HOO DAVE!

Higgledy-piggledy
David L. Silverman[1]
Juggles adroitly with
Figures and words.

Having a mind that is
Alphanumerical,
When will he write us a
Sonnet on surds?

EDITH ORR

* * *

[1] David L. Silverman is one of America's foremost logomaniac-cum-mathemaniacs. In addition to being a EUREKA subscriber and contributor (see Problem 215 in this issue), he is the editor of the *Problems and Conjectures* and *Computer Challenge Corner* sections of the *Journal of Recreational Mathematics*, and he is an erstwhile *Kickshaws* Editor of *Word Ways*, the Journal of Recreational Linguistics.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

130. [1976: 42, 128] *Proposé par Jacques Marion, Université d'Ottawa.*

Soit A l'anneau $\{z : r \le |z| \le R\}$. Montrer que la fonction $f(z) = \frac{1}{z}$ n'est pas limite uniforme de polynômes sur A.

II. *Solution by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If we consider contour integrals round a curve like

$$z = \tfrac{1}{2}(R+r)\exp i\theta,\quad 0 \le \theta \le 2\pi,$$

the integral for any polynomial is zero, while that for $\frac{1}{z}$ is $2\pi i$. Since for any uniformly (or even boundedly) convergent sequence of polynomials, we have

$$\oint \lim = \lim \oint = 0,$$

the theorem follows.

* * *

134. [1976: 68, 151, 173, 222; 1977: 12] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

ABC is an isosceles triangle with $\angle$ABC = $\angle$ACB = 80°. P is the point on AB such that $\angle$PCB = 70°. Q is the point on AC such that $\angle$QBC = 60°. Find $\angle$PQA.

(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

Editor's comment.

Harry Schor, to whom is due solution VI (see [1976: 222; 1977: 12]), wrote in to offer a further simplification of his proof. He says that the congruence of Δs ACS, ACP (see figure in [1976: 222]) is not needed. One need merely note that ΔARS $\cong$ ΔCRP (*ASA*), from which we get RP = RS = RQ, and the required result follows.

Schor, now retired, is a former chairman of the mathematics department at Abraham Lincoln H.S. in Brooklyn, N.Y. He also taught at Brooklyn Polytechnic Institute and at Brooklyn College. Now living at Greenbrae, California, he occupies his leisure hours by teaching part-time at the College of Marin and by writing a series of textbooks for the Globe Book Co. in New York (plug!). Eight volumes (for the slow learner in junior high school) have already been published.

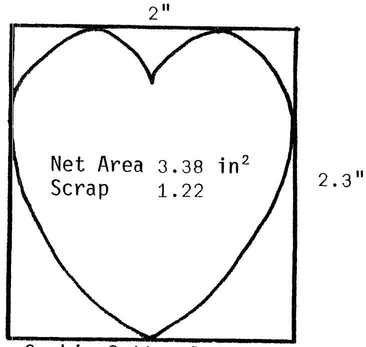
* * *

135. [1976: 68, 153, 223] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

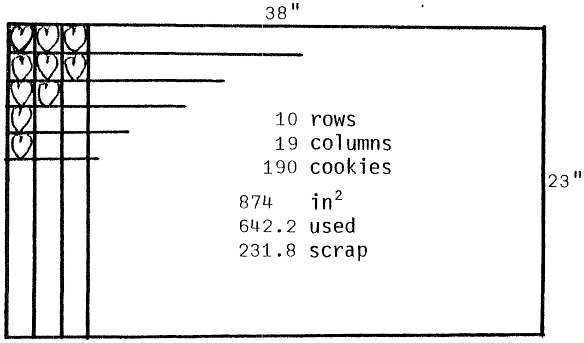
How many 3×5 rectangular pieces of cardboard can be cut from a 17×22 rectangular piece so that the amount of waste is a minimum?

V. *Comment by R. Robinson Rowe, Sacramento, California.*

I'm afraid I couldn't wait for the next editor to send in my solution to the problem, mentioned by the present editor in [1976: 224], about how many cookies can be cut from a 23 by 38 layer of dough with a heart-shaped cookie cutter. The dough was all ready, so I went right ahead.



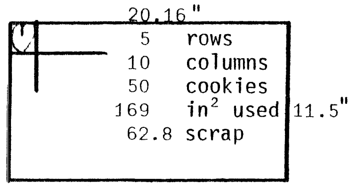
Cookie Cutter Pattern



First Layer

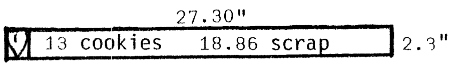
The standard heart-shaped cookie and cookie cutter established by the Society of Culinary Engineers is inscribed in a 2×2.3 in. rectangle, as diagrammed at the right. As shown in the next diagram, 190 cookies can be cut from the specified layer of dough.

My mother had one of these cutters; 70 years ago I watched as she cut and removed the cookies to a baking tin, then collated the scrap, kneaded it into a glob, and rolled the scrap into another layer. As shown in the third diagram, this would yield 50 more cookies.



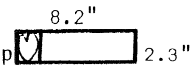
Second Layer

Sequentially collating, kneading and rolling will yield 13, then 4, then 1, making a total of 258 cookies.

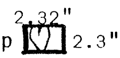


Third Layer

Inevitably, as I watched, there would be a final scrap too small for another heart-shaped cookie. This she would knead, press, pat and sculp into a boy-shaped cookie — for me. There was no waste.

4 cookies 5.34 scrap


4th Layer

1 cookie 1.96 scrap


1 boy-shaped cookie


* * *

140. [1976: 68; 1977: 13] *Proposed by Dan Pedoe, University of Minnesota.*

THE VENESS PROBLEM. A paper cone is cut along a generator and unfolded into a plane sheet of paper. What curves in the plane do the originally plane sections of the cone become? (This problem is due to J.H. Veness.)

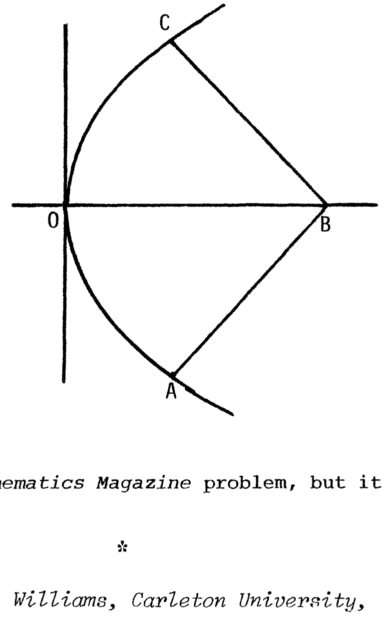
Editor's comment.

Murray S. Klamkin, University of Alberta, informs me that this problem was first proposed in the January 1951 issue of *Mathematics Magazine* by Leo Moser, who was then at the Texas Technological College, and that a solution by Professor Klamkin's Sophomore Calculus Class at the Polytechnic Institute of Brooklyn was published in the same *Magazine*, Vol. 25 (1952), pp. 282-283.

As proposed by Moser, the problem read as follows:

A right circular cone is cut by a plane. The intersection of course is a conic. Find the equation of the curve that this conic goes into if the cone is unrolled on to a plane. In particular, if the cone is a cylinder and the plane cuts the axis of the cylinder at 45°, then the ellipse formed will unroll into a sine curve.

Queried as to when and where J.H. Veness had independently proposed the problem, the proposer replied:



On January 16, 1975, John Veness wrote to me: "Can you tell me something of this? Draw the three normals from $(x_1,0)$, $x_1 > 2a$ (to the parabola $y^2 = 4ax$) and cut out the parabolic "sector" OABC (see figure), and form a "cone" by gluing BC to BA. Does it become an oblique cone with a circular base? Conversely, open an oblique cone, etc."

The idea arose from the rather odd fact that a sector of a circle can be folded so as to become a circular cone, much used with litmus papers, old-fashioned sugar bags, etc. and this aroused Veness' curiosity.

People I consulted had ideas which I thought were wrong, such as the arc length of the parabola, in this instance, being unchanged when it becomes a curve on the cone. I am sure that Veness had never seen the *Mathematics Magazine* problem, but it makes an interesting bit of history.

* * *

149. [1976: 94, 184] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Find the last two digits of 3^{1000}.

III. *Solution and comment by David R. Stone, Georgia Southern College, Statesboro, Georgia.*

Euler's Theorem says that, if a and m are relatively prime, then $a^{\phi(m)} \equiv 1 \pmod{m}$. Since

$$\phi(100) = 100(1-\frac{1}{2})(1-\frac{1}{5}) = 40$$

and 3 and 100 are relatively prime, we have $3^{40} \equiv 1 \pmod{100}$. Hence

$$3^{1000} = (3^{40})^{25} \equiv 1^{25} = 1 \pmod{100},$$

so the last two digits are 01.

I think this is simpler and quicker than the solutions which used a binomial expansion; in particular, it avoids the search for the right binomial to expand. It also avoids direct calculation (e.g. $3^{20} \equiv 1 \pmod{100}$). And it does not depend upon any special properties of 3, since we can also calculate, for example, the last two digits of 7^{1000} or 729^{1000} in this way.

Editor's comment.

Yeah, but try to use this method to find the last *four* digits of 3^{1000}, as our second solver did [1976: 184] with the binomial theorem.

* * *

173. [1976: 171] *Proposed by Dan Eustice, The Ohio State University.*

For each choice of n points on the unit circle ($n \geq 2$), there exists a point on the unit circle such that the product of the distances to the chosen points is greater than or equal to 2. Moreover, the product is 2 if and only if the n points are the vertices of a regular polygon.

Editor's comment.

In submitting this problem and Problem 165 [1976: 135, 230], the proposer wrote to the editor: "The two results as stated are not new with me; I found them in a file of problems I'd collected. Perhaps I found them in the *American Mathematical Monthly* several (> 8) years ago since, as you stated in [1976: 96], *everything* has appeared before in the *Monthly*."

The editor's rash statement about the *Monthly* will not be disproved in this instance, since Steven R. Conrad discovered that Problems 165 and the present one form parts (1) and (2) of Problem E 1023 in [1], proposed by D.J. Newman and H.S. Shapiro.

In [1], the problem is stated as follows:

THEOREM 1. Let $z_1,\ldots,z_n$ be n distinct points on the unit circle, $n \geq 2$. If the z_i are not the vertices of a regular polygon, then there exists a point z on the unit circle such that the product of the distances from z to the z_i is greater than 2.

This is equivalent to

THEOREM 2. Let $z_1,\ldots,z_n$ *be* n *distinct points on the unit circle,* $n \geq 2$. *If, for every* z *on the unit circle, the product of the distances from* z *to the* z_i *is* ≤ 2, *then the* z_i *are the vertices of a regular polygon.*

But neither of these theorems seems to correspond exactly to the problem as proposed here.

REFERENCE

1. Solution to Problem E 1023 , *The American Mathematical Monthly*, 60 (1953) 119-121.

* * *

174. [1976: 171] *Proposed by Leroy F. Meyers, The Ohio State University.*

The function whose value at each rational number is 1 and at each irrational number is 0 is known to be discontinuous on the entire real line R. Describe a function which is defined on R and is continuous and differentiable at each point in a set E (specified below), but is discontinuous at each point not in E.

(a) $E = \{0\}$;

(b) E is a finite set;

(c) E is denumerable.

Solution by the proposer.

Let f be the everywhere discontinuous function described in the proposal, let g be a nonnegative everywhere differentiable function, and let $F = f\cdot g$. We will use the following

LEMMA. The function F *is differentiable on the set of zeros of* g, *and is continuous nowhere else.*

Proof. If $g(a) = 0$, then g has a local minimum at a, and so $g'(a) = 0$. Then

$$F'(a) = \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} = \lim_{x\to a} \frac{f(x)g(x) - 0}{x - a} = \lim_{x\to a} f(x)\cdot\frac{g(x) - g(a)}{x - a} = 0\ ,$$

since f is bounded and $\lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a} = g'(a) = 0$.

If $g(a) \neq 0$, then there is an open interval I containing a such that $g(x) > 0$ for all $x \in I$. If F is continuous at a, then $f = F/g$ is also continuous at a, contrary to the hypothesis.

Now, from the lemma, if $g(x)$ is

(i) x^2,

(ii) $(x-x_1)^2\cdot\ldots\cdot(x-x_n)^2$ for selected points $x_1,\ldots,x_n$,

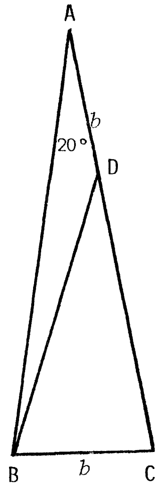
or (iii) $\sin^2 x$,

then F satisfies (a), (b), or (c), respectively.

* * *

175. [1976: 171] *Proposed by Andrejs Dunkels, University of Luleå, Sweden.*

Consider the isosceles triangle ABC in the figure, which has a vertical angle of 20°. On AC, one of the equal sides, a point D is marked off so that $|AD| = |BC| = b$. Find the measure of $\angle ABD$.



I. *Solution without words by the proposer, which shows that the required angle is* 10° *(see Figure 1).*

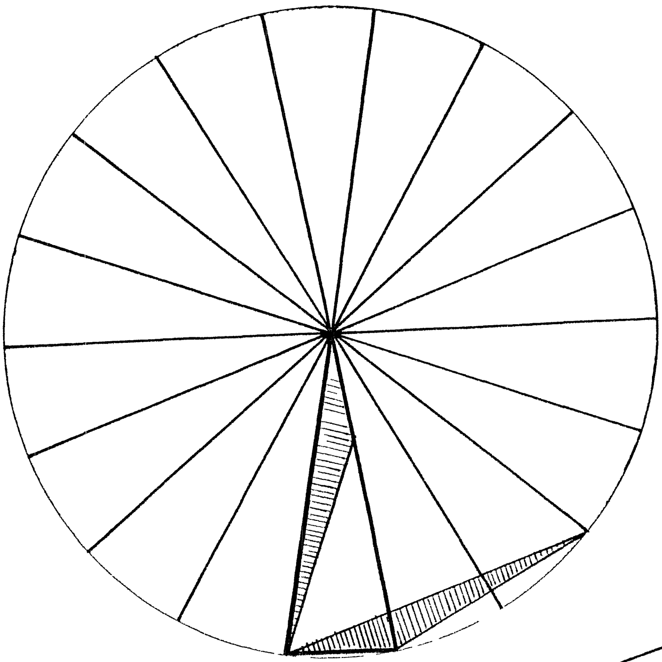


Figure 1

II. *Solution by Doug Dillon, Brockville, Ont.*

If ΔABE is equilateral (see Figure 2), then $\Delta EAD \cong \Delta ABC$ (*SAS*) and ΔEBD is isosceles. Since $\angle BED = 60° - 20° = 40°$, it follows that $\angle EBD = 70°$ and $\angle ABD = 10°$.

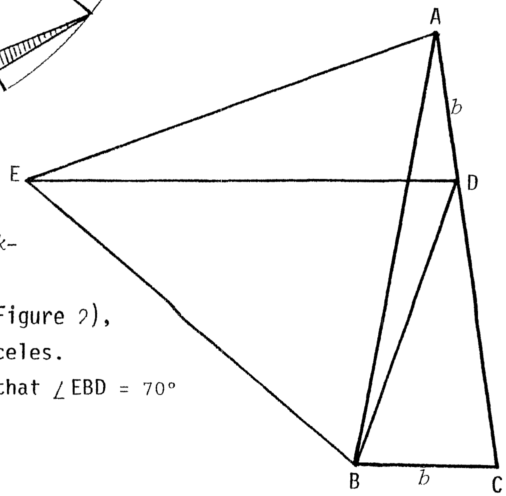


Figure 2

III. *Comment by the proposer.*

An unexpected by-product of this fascinating triangle is shown in Figure 3. By symmetry, ADEB is an isosceles trapezoid; hence $\angle$CDE = 20° and DE = EC = b.

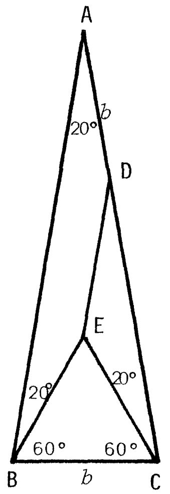


Figure 3

Also solved by LEON BANKOFF, *Los Angeles, California;* WALTER BLUGER, *Department of National Health and Welfare, Ottawa (partial solution);* ANDRÉ BOURBEAU, *École Secondaire Garneau, Vanier, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* R. ROBINSON ROWE, *Sacramento, California;* DAN SOKOLOWSKY, *Antioch College, Yellow Springs, Ohio (two solutions);* CHARLES W. TRIGG, *San Diego, California;* KENNETH M. WILKE, *Topeka, Kansas;* KENNETH S. WILLIAMS, *Carleton University, Ottawa; and the proposer (second solution).*

Editor's comment.

The other solvers, all but two of whom submitted trigonometric solutions of varying degrees of complexity, will surely applaud at the limpid beauty of the two solutions given above. The proposer wrote that he first heard of this problem some 8 years ago when he was teaching at the Kenya Science Teacher's College in Nairobi, but was unable to trace it further.

* * *

177. [1976: 171] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

P is a point on the diameter AB of a circle whose centre is C. On AP, BP as diameters, circles are drawn. Q is the centre of a circle which touches these three circles. What is the locus of Q as P varies?

Solution by Gali Salvatore, Ottawa, Ont.

It is clear that for each point P on AB (see figure) there are two tritangent circles (Q) which are symmetrical with respect to AB; hence the required locus consists of two portions symmetrical with respect to AB. We will find the upper half only; the rest can be obtained by reflection in AB.

Let S be the foot of the perpendicular from Q upon AB. Without loss of generality, assume $|CB| = 3$ and choose R on the perpendicular bisector of AB, as shown in the figure, so that $|CR| = 4$. Let $\angle QCS = \alpha$. If the radius of circle (Q) is r, then $|CQ| = 3 - r$ and, by Pappus' Ancient Theorem (see, e.g., [1]-[5]), $|QS| = 2r$, so that

$$(3 - r)\sin\alpha = 2r. \qquad (1)$$

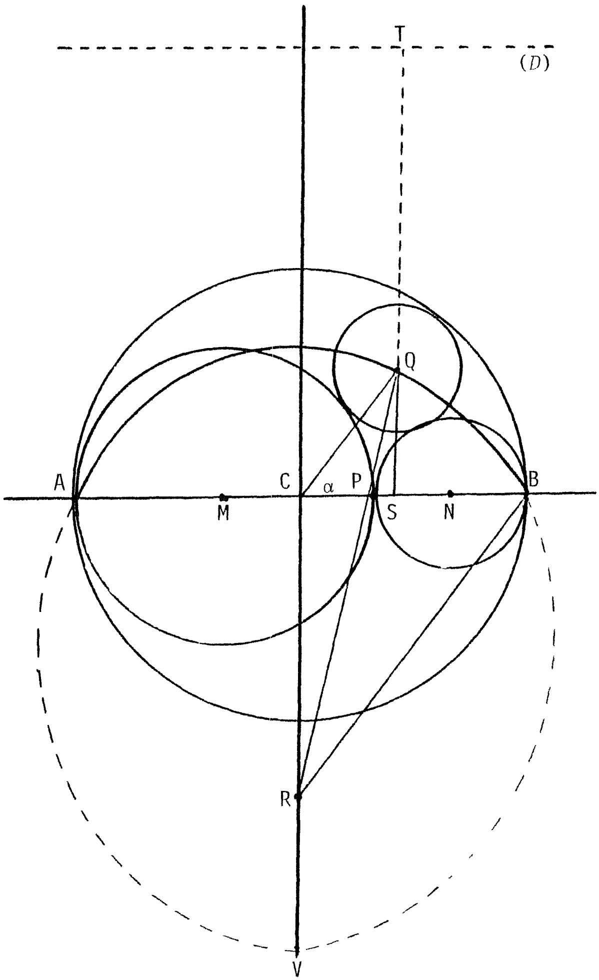
By (1) and the law of cosines applied to ΔCQR, we have

$$\begin{aligned} |QR|^2 &= 16 + (3-r)^2 - 8(3-r)\cos(90° + \alpha) \\ &= 16 + (3-r)^2 + 8(3-r)\sin\alpha \\ &= 16 + (3-r)^2 + 16r \\ &= (5+r)^2. \end{aligned}$$

Thus

$$|QC| + |QR| = (3 - r) + (5 + r) = 8 = |CB| + |BR|,$$

and Q lies on an ellipse with foci C and R, and latus rectum AB. More precisely, Q lies on the elliptical arc subtended by latus rectum AB. We will show that this elliptical arc is the required locus.



If Q is any point on the elliptical arc, there is a unique circle (Q) tangent to circle (C) (see figure); a unique circle (M) tangent internally to circle (C) at A and externally to circle (Q) and meeting AB again in, say, P; and a unique circle (N) on PB as diameter. It suffices to show that circle (Q) is tangent to circle (N).

Let $QS \perp AB$ and $\angle QCS = \alpha$. If the radius of circle (Q) is r, then

$$|CQ| = 3 - r, \quad |QR| = 8 - |CQ| = 5 + r,$$

and

$$\begin{aligned}|QS| &= (3 - r)\sin\alpha \\ &= -(3 - r)\cos(90° + \alpha) \\ &= (3 - r) \cdot \frac{|QR|^2 - |CQ|^2 - |CR|^2}{2|CQ||CR|} \\ &= (3 - r) \cdot \frac{(5 + r)^2 - (3 - r)^2 - 16}{2 \cdot (3 - r) \cdot 4} \\ &= 2r,\end{aligned}$$

and it follows from Pappus' Ancient Theorem that circle (Q) is tangent to circle (N), which completes the proof.

Also solved by LEON BANKOFF, *Los Angeles, California and* ZELDA KATZ, *Beverly Hills, California (jointly)*; DOUG DILLON, *Brockville, Ont.*; CLAYTON W. DODGE, *University of Maine at Orono*; G.D. KAYE, *Department of National Defence, Ottawa*; R. ROBINSON ROWE, *Sacramento, California*; DAN SOKOLOWSKY, *Antioch College, Yellow Springs, Ohio*; *and the proposer*.

Editor's comment.

The first part of Sokolowsky's proof is very neat. With the figure and conventions used in the proof given above, it goes as follows:

Let P be a point on segment AB and let (Q) be the corresponding tritangent circle. Draw line (D) 6 units above AB, as shown in the figure, and let SQ produced meet (D) in T. By Pappus' Ancient Theorem, $|QS| = 2r$. Since $|CQ| = 3 - r$, we have

$$|QT| = |ST| - |QS| = 6 - 2r = 2|CQ|,$$

and so Q lies on an ellipse with focus C, directrix (D), and eccentricity $\frac{1}{2}$. A slight modification of this argument then enables him to show that if P lies on either extension of segment AB, then Q lies on the complementary arc AVB of the ellipse. He then shows, conversely, that if Q is any point except V on the ellipse, then it is the centre of a tritangent circle. Hence if P is allowed to range over the entire *line* AB, the complete locus consists of the ellipse shown in the figure (vertex V excluded), together with the reflection of this curve in line AB.

However, in contrast with his neat proof of the first part, Sokolowsky's purely geometric proof of the converse is quite long and complicated, and it would seem that the method used in our featured solution, with slight modifications, could more easily furnish quick proofs of both the direct and converse parts for the entire locus.

Referring to his difficulties with the proof of the converse, Sokolowsky wrote: "As you see, showing that the points Q were on an ellipse was easy, but to find out whether every point on the ellipse was on the locus was another matter. I think I succeeded but I don't envy the poor [expletive deleted] who tried to do this analytically. Wow!"

Quite so. It is now my duty to report that, apart from Sokolowsky, all the other solvers submitted proofs by analytic geometry of the first part of the theorem only. Not one of them had a word to say about the necessity of proving a converse. I will not speculate as to the reason for this omission. But it would have been entirely fitting if they had given some slight obeisance to the principles by which they live and teach, if only by a brief mention such as: "Conversely, if Q is any point on the elliptical arc, it is clear that . . . ," followed by a bit of arm waving.

So, students, repeat after me: a locus is a set consisting of all those points, *and only those points*, which satisfy. . . .

REFERENCES

1. Carl B. Boyer, *A History of Mathematics*, Wiley, 1968, p. 208.

2. Clayton W. Dodge, *Euclidean Geometry and Transformations*, Addison-Wesley, 1972, p. 195.

3. Howard Eves, *A Survey of Geometry* (Revised Edition), Allyn and Bacon, 1972, p. 133.

4. C. Stanley Ogilvy, *Excursions in Geometry*, Oxford University Press, 1969, p. 55.

5. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970, p. 89.

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178. [1976: 171] *Proposed by Gali Salvatore, Ottawa, Ontario.*

Prove or disprove that the equation $ax^2 + bx + c = 0$ has no rational root if a, b, c are all odd integers.

I. *Comment by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

The problem is not new. It has appeared in *The Mathematics Student Journal*, Nov. 1966, Problem 248; in *Mathematical Challenges II* (a compilation of *MSJ* problems, p. 9; in *The USSR Olympiad Problem Book*, Problem 57; and it is quite similar to Problem 1 of the 1907 Eötvös Competition, as listed in *The Hungarian Problem Book II.*

Probably the best solution appeared in the first reference given above (now called *The Mathematics Student*). As submitted by Malcolm Cross, then a student at Webster Groves H.S. in Webster Groves, Missouri, it reads:

If p/q is an irreducible rational root of the given equation, then p and q must both be odd, since they are factors of the odd numbers c and a, respectively. Substituting $x = p/q$ in the equation gives

$$ap^2 + bpq + cq^2 = 0.$$

This states that the sum of three odd integers is zero, an impossibility. Therefore the given equation has no rational root.

II. *Solution by Doug Dillon, Brockville, Ont.; and Samuel L. Greitzer, Rutgers University, New Brunswick, N.J. (independently).*

The roots of the given equation, multiplied by a, are the solutions of the quadratic $y^2 + by + ac = 0$ whose rational roots, if any, are integers. Now, since the sum of the roots is the odd number $-b$, the roots will have opposite parity, so their product must be even. However, this product is the odd number ac, so we have a contradiction.

III. *Solution d'André Bourbeau, École Secondaire Garneau, Vanier, Ont.*

Dans l'équation donnée, posons

$$a = 2m + 1, \quad b = 2n + 1, \quad c = 2p + 1.$$

Si les racines sont rationnelles, le discriminant $b^2 - 4ac$ doit être un carré impair, c'est-à-dire

$$(2n+1)^2 - 4(2m+1)(2p+1) = (2x+1)^2,$$

ce qui entraîne

$$n(n+1) - x(x+1) = (2m+1)(2p+1).$$

Or cette dernière équation est impossible, puisque ses deux membres sont de parités différentes.

IV. *Solution by L.F. Meyers, The Ohio State University.*

If a, b, and c are odd, then

$$b^2 - 4ac \equiv 1 - 4 \not\equiv 1 \pmod 8,$$

so that the discriminant of the equation cannot be a perfect square. Hence the equation has no rational root.

Also solved by DON BAKER, *Presidio Junior H.S., San Francisco, California;* CLAYTON W. DODGE, *University of Maine at Orono;* G.D. KAYE, *Department of National Defence, Ottawa;* DANIEL ROKHSAR, *Susan Wagner H.S., Staten Island, N.Y.;* R. ROBINSON ROWE, *Sacramento, California;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia;* CHARLES W. TRIGG, *San Diego, California;* KENNETH M. WILKE, *Topeka, Kansas (two solutions);* KENNETH S. WILLIAMS, *Carleton University, Ottawa; and the proposer.*

Editor's comment.

It is not surprising that Conrad was able to locate this problem in *The Mathematics Student*, since he is its associate editor. For subscription information about this journal, write to James R. Tewell, Circulation Manager, 1906 Association Drive, Reston, Virginia 22091.

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179. [1976: 171] *Proposed by Steven R. Conrad, Benjamin N. Cardozo High School, Bayside, N.Y.*

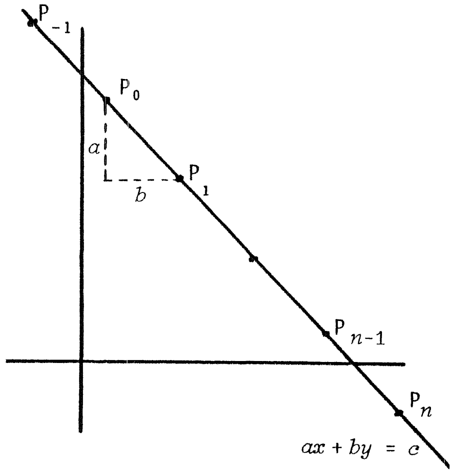
The equation $5x + 7y = c$ has exactly three solutions (x,y) in positive integers. Find the largest possible value of c.

Adapted from the solution submitted by Daniel Flegler, Mathematics Coordinator, Waldwick Public Schools, Waldwick, N.J.

More generally, suppose the equation $ax + by = c$ has exactly n positive integer solutions, where a, b, c, n are positive integers and a, b are relatively prime. Let $P_0 = (x_0, y_0)$ be the positive solution with least abscissa x_0. It is clear from the figure (or it is known from the theory of Diophantine equations) that the n positive integer solutions are

$$P_t = (x_0+bt,\ y_0-at),\quad t = 0,1,\ldots,n-1.$$

If we compare the abscissas of P_{-1}, P_0 and the ordinates of P_{n-1}, P_n, we find



$$x_0 - b \le 0, \quad y_0 - a(n-1) > 0,$$
$$x_0 > 0, \quad y_0 - an \le 0,$$

and so

$$1 \le x_0 \le b, \quad an - a + 1 \le y_0 \le an.$$

Hence we must have

$$c \le a \cdot b + b \cdot an, \quad c \ge a \cdot 1 + b \cdot (an - a + 1), \tag{1}$$

that is,

$$(n-1)ab + a + b \le c \le (n+1)ab. \tag{2}$$

Finally the bounds in (2) are sharp, since (1) shows that equality is attained in (2) when

$$(x_0, y_0) = (1,\ an-a+1) \quad \text{and} \quad (x_0, y_0) = (b, an).$$

In the present problem, where $a = 5$, $b = 7$, $n = 3$, we get from (2): $82 \le c \le 140$.

Also solved by DON BAKER, *Presidio Junior H.S., San Francisco, California;* WALTER BLUGER, *Department of National Defence, Ottawa (answer only);* DOUG DILLON, *Brockville, Ont.;* CLAYTON W. DODGE, *University of Maine at Orono;* LEROY F. MEYERS, *The Ohio State University;* R. ROBINSON ROWE, *Sacramento, California;* DAVID R. STONE, *Georgia Southern College, Statesboro, Georgia (two solutions);* KENNETH M. WILKE, *Topeka, Kansas;* KENNETH S. WILLIAMS, *Carleton University, Ottawa; and the proposer. Two incorrect solutions were received.*

Editor's comment.

A trivial, and obvious, modification of the above proof shows that, if zero solutions are allowed, then we have

$$(n-1)ab \le c \le (n+1)ab - a - b, \tag{3}$$

and this gives $70 \le c \le 128$ when $a = 5$, $b = 7$, $n = 3$.

The proposer pointed out that this problem occurred, exactly as proposed here, in an old New York City Interscholastic Mathematics League contest in which the official answer (which was unappealed!) was 128. He also wrote that a related problem occurs in Hall and Knight [1]. Now it happens that the problem given in [1] is incorrect, in spite of the fact that the book, first published in 1887, went through four editions, many printings with and without corrections, and that [1] is the *forty-fourth* printing of the book. I now give the problem as it appears in [1], so readers can judge for themselves.

Shew that the greatest value of c in order that the equation $ax + by = c$ may *have exactly n solutions in positive integers is* $(n+1)ab-a-b$, *and that the least value of* c *is* $(n-1)ab+a+b$, *zero solutions being excluded.*

It appears that the problemist who made up the NYCIML contest (as well as our own two incorrect solvers, who managed to convince themselves that the answer to our problem was 128) may have relied a bit too heavily on Hall and Knight. The mistake becomes understandable, if not forgivable, when one knows how Hall and Knight went about writing their celebrated book (see p. 58 in this issue).

Meyers wrote that the following problem occurs in Uspensky and Heaslet [3]:

Find the greatest number c *for which the equation* $5x+7y=c$ *has exactly nine solutions in nonnegative integers.*

The answer given is 338, which accords with (3). Meyers also found the upper bound of (3) in [2].

REFERENCES

1. H.S. Hall and S.R. Knight, *Higher Algebra*, 4th edition, Macmillan, London, 1964 reprint, p. 291, Problem 24.

2. William J. LeVeque, *Elementary Theory of Numbers*, Addison-Wesley, 1962, p. 34, Problem 4.

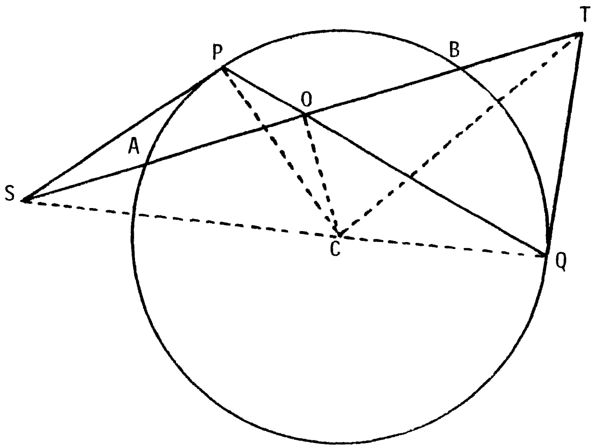
3. J.V. Uspensky and M.H. Heaslet, *Elementary Number Theory*, McGraw-Hill, 1939, p. 66, Problem 17.

*   *   *

180. [1976: 172] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Through O, the midpoint of a chord AB of an ellipse, is drawn any chord POQ. The tangents to the ellipse at P and Q meet AB at S and T, respectively. Prove that AS = BT.

Solution by Doug Dillon, Brockville, Ont.



Since linear transformations preserve proportions in a line, it is sufficient to show that the theorem holds for a circle.

Let the configuration described in the proposal be drawn on a circle (C), as shown in the figure. Since $CO \perp AB$, $CP \perp PS$, and $CQ \perp QT$, it follows that COPS and COTQ are cyclic quadrilaterals. Thus

$$\angle OSC = \angle OPC \quad \text{and} \quad \angle OTC = \angle OQC.$$

Since $\angle OPC = \angle OQC$, it follows that $\angle OSC = \angle OTC$ and $\Delta OSC \simeq \Delta OTC$. Hence

$$AS = OS - OA = OT - OB = BT.$$

Also solved by CLAYTON W. DODGE, *University of Maine at Orono;* DAN PEDOE, *University of Minnesota; and* SAHIB RAM MANDAN, *Indian Institute of Technology, Kharagpur, India.*

Editor's comment.

All other solvers noted that the theorem holds for any conic, not merely for ellipses. Dodge and Ram Mandan recognized it as a special case of the Generalized Butterfly Theorem [2], and Pedoe observed that it could be considered as an exercise on Desargues' Involution Theorem and its dual [1, p. 184]. Pedoe also located the problem as No. 11 in [1, p. 189].

REFERENCES

1. C.V. Durell, *Projective Geometry*, Macmillan (London), 1945.

2. Howard Eves, *A Survey of Geometry*, Revised Edition, Allyn and Bacon, 1972, pp. 255-256.

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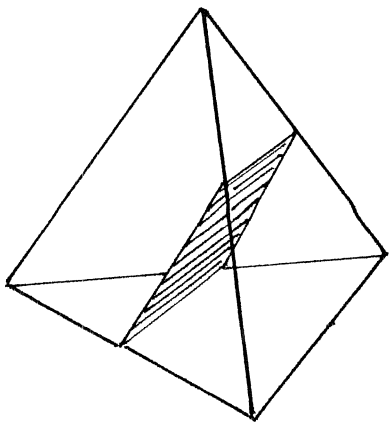
181. [1976: 193] *Proposed by Charles W. Trigg, San Diego, California.*

A polyhedron has one square face, two equilateral triangular faces attached to opposite sides of the square, and two isosceles trapezoidal faces, each with one edge equal to twice a side, e, of the square. What is the volume of this pentahedron in terms of a side of the square?

I. *Essence of the solutions submitted by Leon Bankoff, Los Angeles, California; Samuel L. Greitzer, Rutgers University; Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.; and the proposer.*

Two such pentahedrons, joined at their square faces, form a regular tetrahedron of edge $2e$ (see figure). Hence the volume of one of the pentahedrons is

$$\frac{1}{2}\left[\frac{(2e)^3\sqrt{2}}{12}\right] = \frac{e^3\sqrt{2}}{3}.$$



II. *Comment by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

There was once a puzzle consisting of two plastic solids each of the shape described in this problem. The puzzle consisted of fitting the two pieces together to form a regular tetrahedron.

Also solved by LEON BANKOFF, *(second solution)*; G.D. KAYE, *Department of National Defence, Ottawa*; GILBERT W. KESSLER, *(second solution)*; F.G.B. MASKELL, *Algonquin College, Ottawa*; R. ROBINSON ROWE, *Sacramento, California*; *and the proposer (second solution). Two incorrect solutions were received.*

Editor's comment.

I have on my desk one of those puzzles referred to by Kessler. It was sent to me by Leon Bankoff along with his two solutions.

* * *

182. [1976: 193] *Proposed by Charles W. Trigg, San Diego, California.*

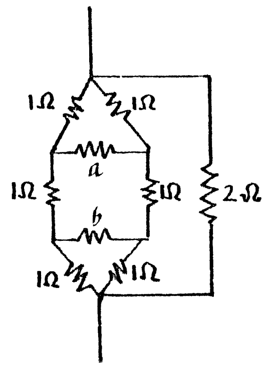
A framework of uniform wire is congruent to the edges of the pentahedron in the previous problem. If the resistance of one side of the square is 1 ohm, what resistance does the framework offer when the longest edge is inserted in a circuit?

Essence of the solutions submitted by Leon Bankoff, Los Angeles, California; Clayton W. Dodge, University of Maine at Orono; F.G.B. Maskell, Algonquin College, Ottawa; R. Robinson Rowe, Sacramento, California; and the proposer.

The diagram illustrates the network of the framework. The resistors labelled a and b can be ignored since there is no voltage drop between the lines they connect. We are dealing here with three lines connected in parallel, two of which offer a resistance of $3\,\Omega$ each and one a resistance of $2\,\Omega$. The effective resistance, R, is calculated by the relation

$$\frac{1}{R} = \frac{1}{3} + \frac{1}{3} + \frac{1}{2},$$

whereupon $R = 6/7\,\Omega$.



* * *

HALL AND KNIGHT

or

$z + b + x = y + b + z$

When he was young his cousins used to say of Mr Knight:
'This boy will write an Algebra — or looks as if he might.'
And sure enough when Mr Knight had grown to be a man,
He purchased pen and paper and an inkpot, and began.

But he very soon discovered that he couldn't write at all,
And his heart was filled with yearnings for a certain Mr Hall;
Till, after thirty years of doubt, he sent his friend a card:
'Have tried to write an Algebra, but find it very hard.'

Now Mr Hall himself had tried to write a book for schools,
But suffered from a handicap: he didn't know the rules.
So when he heard from Mr Knight and understood his gist,
He answered him by telegram: 'Delighted to assist.'

So Mr Hall and Mr Knight they took a house together,
And they worked away at algebra in any kind of weather,
Determined not to give it up until they had evolved
A problem so constructed that it never could be solved.

'How hard it is,' said Mr Knight, 'to hide the fact from youth
That x and y are equal: it is such an obvious truth!'
'It is,' said Mr Hall, 'but if we gave a b to each,
We'd put the problem well beyond our little victim's reach.

'Or are you anxious, Mr Knight, lest any boy should see
The utter superfluity of this repeated b?'
'I scarcely fear it,' he replied, and scratched his grizzled head,
'But perhaps it *would* be safer if to b we added z.'

'A brilliant stroke!' said Hall, and added z to either side;
Then looked at his accomplice with a flush of happy pride.
And Knight, he winked at Hall (a very pardonable lapse).
And they printed off the Algebra and sold it to the chaps.

E.V. RIEU[1]

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[1]From *A Choice of Comic and Curious Verse*, edited by J. M. Cohen, Penguin Books, 1975, p.333.

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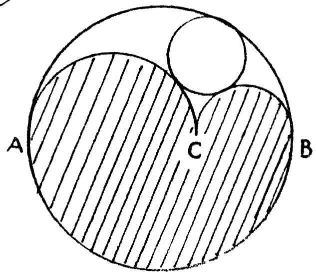
EUREKA ~ We Love You

An Archimedean Valentine Greeting



Choose a point C anywhere on line AB and construct semicircles on the same side of diameters $AC = 2r_1$, $CB = 2r_2$, and $AB = 2r$. Reflect the outer semicircle in AB, as in the figure. We now have a shaded HEART-SHAPED area and an unshaded curvilinear triangle known as THE SHOEMAKER'S KNIFE OF ARCHIMEDES. Let φ denote the radius of the circle inscribed in the KNIFE. then —

$$\frac{\varphi}{r} = \frac{r^2 - r_1^2 - r_2^2}{r^2 + r_1^2 + r_2^2} = \frac{\text{Area of the SHOEMAKER'S KNIFE}}{\text{Area of the HEART}}$$

Leon Bankoff

Editor's comment:

Through its editor, EUREKA gratefully accepts and reciprocates the kind feelings expressed by Dr. Bankoff.

A careful look at the drawing shows that, while Cupid with his bow and eros is attempting to make readers lose their heads by falling in love, he himself is most appropriately in danger of losing his own.

In an accompanying note, Dr. Bankoff wrote: "In connection with the equations shown at the bottom of the VALENTINE, I hereby offer the magnificent prize of a ONE-YEAR SUBSCRIPTION to EUREKA for the most elegant demonstration submitted."

Readers should submit their solutions, before May 1, 1977, directly to Dr. Leon Bankoff, 6360 Wilshire Blvd., Los Angeles, California 90048. The winning solution selected by Dr. Bankoff will be published in a later issue of EUREKA.