SKOLIAD No. 113

Robert Bilinski

Please send your solutions to the problems in this edition by *April 1, 2009*. A copy of **MATHEMATICAL MAYHEM Vol. 7** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

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Our contest this month is the National Bank of New Zealand Competition 2005. Our thanks go to Warren Palmer, Otago University, Otago, New Zealand for providing us with this contest and for permission to publish it. We also thank Rolland Gaudet, Collège universitaire de Saint-Boniface, Winnipeg, MB, for translation of this contest.

National Bank of New Zealand
Junior Mathematics Competition 2005
(Years 9 and above) 1 hour allowed

1 (For year 9, Form 3 only). **Note**: In this question the word "digit" means a positive single-digit whole number, that is, a member of the set $\{1, \ldots, 9\}$.

On a long journey by car, Michael was starting to get bored. To keep him amused, his mother asked him some arithmetic questions. The first question she asked was "Can you think of five different digits which add to a multiple of **5**?" Michael answered straight away "That's easy Mum. **1**, **2**, **3**, **4**, and **5** work because $1 + 2 + 3 + 4 + 5 = 15$, and **15** is a multiple **5** because $15 = 5 \times 3$."

Now answer the other questions which Michael's mother asked.

(a) Write down a set of five different digits which which add to **35**.

(b) Write down a set of **three** different digits which add to a multiple of **5**, but which don't include **5** itself or **1**.

(c) How many different sets of **four** different digits are there which add to a multiple of **5**, but which don't include **5** itself or **1**? (Note that writing the same set of numbers in a different order doesn't count here.)

(d) Is it possible to write down a set of **six** different digits which add to a multiple of **5**, but which don't include **5** itself? If it is possible, write down such a set. If it is not possible, **explain** briefly why it cannot be done.

(e) Is it possible to write down a set of **seven** different digits which add to a multiple of **5**, but which don't include **5** itself? If it is possible, write down such a set. If it is not possible, **explain** briefly why it cannot be done.

2. Braille is a code which lets blind people read and write. It was invented by a blind Frenchman, Louis Braille, in 1829. Braille is based on a pattern of dots embossed on a **3** by **2** rectangle. It is read with the fingers moving across the top of the dots.

Altogether there are **63** possible ways to emboss one to six dots on a **3** by **2** rectangle. (We will not count zero dots in this question.)

Figure 1 shows the pattern for the letter h. Note that if we reflect this pattern (down the middle of the rectangle) the result is the Braille letter j, shown in Figure 2.
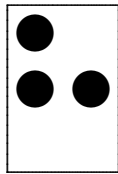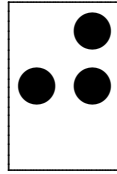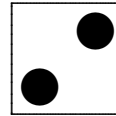

Fig. 1:
the letter h

Fig. 2:
the letter j

Fig. 3: a simplified
version of Braille.

(a) How many different patterns are possible using just one dot?

(b) There are **15** different ways to emboss two dots on a **3** by **2** rectangle. How many ways are there to emboss four dots on a **3** by **2** rectangle? Briefly **explain** your answer.

(c) Including the two patterns shown in Figures 1 and 2, how many possible patterns are there using three dots?

(d) A simplified version of Braille has been proposed. In this version the dots will be embossed on a **2** by **2** rectangle. An example is shown in Figure 3. How many possible patterns would there be in this simplified version (assuming that we will not count zero dots)?

(e) Write down how many possible patterns there will be if we were to develop a more complicated version of Braille using a **4** by **2** rectangle. (Again assume that we would not count zero dots.)

3. Around the year 2000 BC, the Babylonians used a number system based on the number **60**. For example, where we would write **0.25** (meaning "one quarter" or 1/4), they would write something like || 15 | (meaning "fifteen sixtieths" or **15/60**, which does simplify to become 1/4 in our number system).

The table below shows some numbers and their reciprocals written according to the Babylonian system. (A special feature of a number and its

reciprocal is that when you multiply them together, the result is always **1**.) In the table below, each column represents a place value of **1/60** of the previous column. For example, **7 | 30** means **7/60 + 30/3600**.

Number	Reciprocal	Number	Reciprocal
2	|| 30 |	6	|| — |
3	|| 20 |	7	|| — | — | — | — | ...
4	|| — |	8	|| 7 | 30 |
5	|| — |	9	|| — | — |

(a) Write down in the correct order the three missing numbers which should go beside **4**, **5**, and **6**.

(b) If we added our number **2.5** (meaning "two and a half") into the table above, what number would we write beside it to show the Babylonian version of its reciprocal?

(c) The reciprocal of the number **8** is shown in the table as **7 | 30 |**. Using the same notation, what is the reciprocal of the number **9**?

(d) Write down the first two numbers which should go beside **7** in the table above.

(e) Write down the Babylonian version for the reciprocal of our number **192**.

4. When we finally landed on Mars, we discovered that Martians love to play a game called HitBall. In this game two teams of players try to hit a ball between poles placed at each end of a field. The team that scores the most points within one Martian hour is the winner.

There are three ways to score points. An **Inner** scores **7** points, an **Outer** scores **4** points, while a **Wide** scores **2** points.

The first match report sent back to Earth was not very clear because of static, so not all the details are certain. However, we did hear that the Red Team won. They scored "something–seven" points altogether (only the last number could be clearly heard). We also learned that they had **16** successful scoring shots.

(a) Write down in a list (from smallest to largest) the possible numbers of *Inners* which the Red Team could have scored according to this first match report.

(b) A later report added the information that the Red Team scored the same number of *Inners* and *Outers*. Using this extra information, write down how many points the Red Team scored altogether.

(c) Explain why your answer to (b) is the only possible solution. As part of your explanation, make sure you include how many *Inners*, *Outers*, and *Wides* the Red Team scored.

5. During 2004, a Dunedin newspaper held a competition to find a new flag design for the province of Otago. Wendy entered the competition. Her entry was based on the design shown below (Figure 4). Her flag featured a gold cross with a blue background. She also placed a circle into her design. The top and bottom of the circle just touch the corners of the top and bottom triangles, as shown in Figure 4.
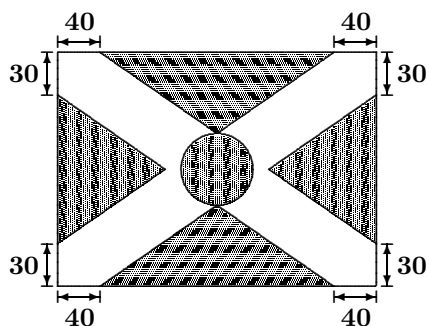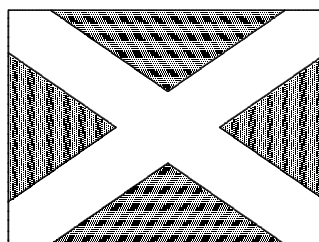

Fig. 4 (not to scale)

Fig. 5 (not to scale)

(a) Wendy designed her flag to be **240** cm long and **150** cm high. If the edges of the cross are **40** cm and **30** cm away from each of the corners, as shown in Figure 4, what is the radius of the centre circle?

(b) Wendy decided to remove the circle from her design (see Figure 5). With the circle removed, what is the total area of the cross?

National Bank of New Zealand
Concours mathématique 2005
Niveau junior
(secondaire 3 et plus) 1 heure au total

1. Note : Dans cette question, le mot "chiffre" représente un entier positif à une position décimale, donc un entier venant de l'ensemble $\{1, 2, \ldots, 9\}$.

Lors d'un long voyage en voiture, Michel s'ennuyait. Comme divertissement, sa mère lui a posé quelques questions d'arithmétique.

La première question : "Donner 5 chiffres différents dont la somme est multiple de **5**."

Michel répond : "Facile, maman ! **1, 2, 3, 4, 5**. Car $1+2+3+4+5 = 15$, qui est un multiple de **5** car $15 = 3 \times 5$."

Répondre aux autres questions de la maman de Michel.

(a) Donner **5** chiffres différents dont la somme est **35**.

(b) Donner trois chiffres différents, incluant ni **1** ni **5**, dont la somme est multiple de **5**.

(c) Combien d'ensembles de quatre chiffres différents y a-t-il, incluant ni **1** ni **5**, dont la somme est un multiple de **5** ?

(d) Est-il possible de donner un ensemble de six chiffres différents, n'incluant pas **5**, dont la somme est un multiple de **5** ? Si possible, fournir un exemple. Sinon, expliquer pourquoi ce n'est pas possible.

(e) Est-il possible de donner un ensemble de sept chiffres différents, n'incluant pas **5**, dont la somme est un multiple de **5** ? Si possible, fournir un exemple. Sinon, expliquer pourquoi ce n'est pas possible.

2. Le braille est un système d'écriture tactile à l'usage de personnes aveugles ou avec de sérieuses déficiences visuelles ; il porte le nom de son inventeur, le Français Louis Braille, en 1829. En braille standard, un caractère est représenté par la combinaison de **1** à **6** points en relief, disposés sur un rectangle **3** de haut par **2** de large.

Au total il y a **63** façons possibles de placer en relief de **1** à **6** points dans ce rectangle.

La Figure 1 donne le caractère braille pour la lettre h. Si on fait la réflexion de ce caractère par rapport à une droite verticale au centre du rectangle, on obtient la Figure 2, donnant le caractère braille de la lettre j.
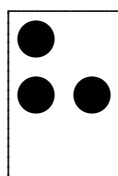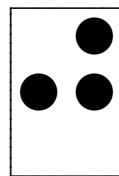

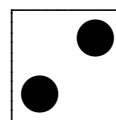Fig. 1 :
la lettre h

Fig. 2 :
la lettre j

Fig. 3 : une version
simplifiée de braille

(a) Combien de caractères braille sont possibles, chacun utilisant un seul point ?

(b) Il y a **15** façons différentes de choisir **2** points dans le rectangle braille. Combien de façons y a-t-il alors pour choisir **4** points dans ce même rectangle ? Expliquer brièvement.

(c) Incluant les caractères braille dans les Figures 1 et 2, combien de caractères sont représentables avec **3** points ?

(d) Un système braille modifié a été proposé. Dans cette version, on utilise plutôt un rectangle de taille **2** par **2**. Un exemple se trouve dans la Figure 3. Combien de caractères sont alors possibles, supposant toujours au moins un point ?

(e) Combien de caractères seraient possibles dans une version de braille sur un rectangle de taille **4** par **2**, supposant toujours au moins un point ?

3. En Babylonie aux environs de l'an 2000 av. J.-C., la population utilisait un système de nombres en base **60**. Par exemple, pour écrire notre **0.25** (en système décimal, donc 1/4), ils auraient écrit quelque chose comme ‖ 15 ǀ (voulant dire "quinze soixantièmes" ou **15/60**, qui réduit à 1/4).

Le tableau ci-dessous donne certains nombres et leurs réciproques, écrits dans le système babylonien. (Rappel : le produit d'un nombre et sa réciproque donne toujours **1**.) Chaque colonne représente **1/60** de la colonne antérieure. Par exemple, **7 | 30** représente **7/60 + 30/3600**.

Nombre	|| Réciproque	Nombre	|| Réciproque
2	|| 30 |	6	|| — |
3	|| 20 |	7	|| — | — | — | — | ...
4	|| — |	8	|| 7 | 30 |
5	|| — |	9	|| — | — |

(a) Donner, en ordre, les nombres devant apparaître à droite de **4**, **5** et **6**.

(b) Si on ajoutait notre nombre décimal **2.5** (donc deux et demi) au tableau, quel nombre se retrouverait comme version babylonienne de sa réciproque ?

(c) La réciproque du nombre **8** est donnée dans le tableau comme || **7 | 30** |. Dans la même notation, quelle est la réciproque du nombre **9** ?

(d) Donner les deux premiers nombres devant aller à droite de **7** au tableau.

(e) Donner la version babylonienne de la réciproque du nombre décimal **192**.

4. En atterrissant sur Mars, on a découvert le jeu préféré des martiens : Ballon-frappe. Dans ce jeu, deux équipes tentent tour à tour de frapper un ballon entre deux poteaux placés à chaque bout du terrain de jeu. L'équipe ayant compté le plus grand nombre de points dans une heure martienne gagne.

Il y a trois façons de compter des points : *I*, comptant **7** points, *O* comptant **4** points et *W* comptant **2** points.

Les résultats du premier match nous ont été communiqués, mais des erreurs de transmission font que certains détails ne sont pas connus. Voici ce qu'on sait. L'équipe Rouge a gagné le match avec un score total de "quelque chose, sept" points, le premier chiffre n'ayant pas été entendu clairement. Nous avons aussi appris que l'équipe Rouge a compté lors de **16** lancers précisément.

(a) Donner une liste de toutes les possibilités de lancer I que l'équipe Rouge aurait pu compter dans ce match, listés en ordre allant du plus petit au plus grand.

(b) Un deuxième communiqué a ajouté cette information supplémentaire : le nombre de lancers I égale le nombre de lancers O. l'aide de cette nouvelle information, fournir le nombre de points comptés au total par l'équipe Rouge.

(c) Expliquer pourquoi la réponse en (b) est unique. Fournir le nombre exact de I, O et W comptés par l'équipe Rouge.

5. En 2004, un journal de Dunedin a tenu un concours visant à proposer un nouveau drapeau pour la province d'Otago. Wendy a participé à ce concours. Sa soumission était basée sur le dessin (voir figure 4). Son drapeau comportait une croix de couleur or, sur un fond bleu. De plus, elle a ajouté un cercle au dessin, dont le haut et le bas touchent tout juste les coins des triangles situés au haut et au bas du drapeau, tel qu'illustré dans la figure 4.
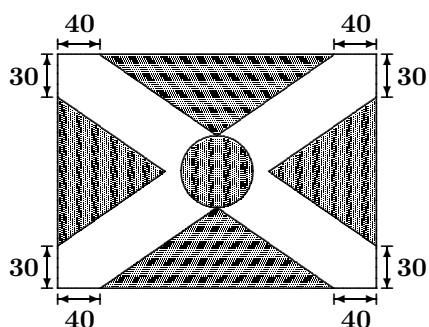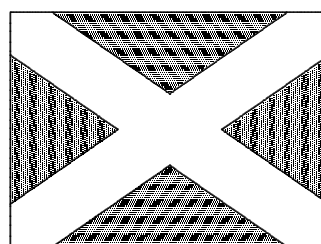

Fig. 4 (pas à l'échelle)

Fig. 5 (pas à l'échelle)

(a) Wendy a pris pour acquis un drapeau de **240** cm de large et de **150** cm de haut. Si les bords de la croix se trouvent à **40** cm et **30** cm des coins, tel qu'illustré à la figure 4, quel est le rayon du cercle?

(b) Enfin, Wendy a décidé d'enlever le cercle de son dessin (voir figure 5). Avec le cercle enlevé, quelle est la surface totale de la croix?

We now give solutions to the Montmorency Contest 2005–06, Sec V, given in $[2008 : 2$–$4]$.

1. Evaluate the following, giving the answer as a fraction:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{2005^2}\right).$$

Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.
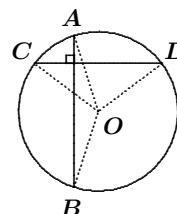
We have

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{2005^2}\right)$$
$$= \left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 + \frac{1}{2005}\right)\left(1 - \frac{1}{2005}\right)$$
$$= \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{2005}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{2005}\right)$$
$$= \left(\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{2005}{2004} \cdot \frac{2006}{2005}\right)\left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{2003}{2004} \cdot \frac{2004}{2005}\right)$$
$$= \frac{2006}{2} \cdot \frac{1}{2005} = \frac{1003}{2005}.$$

Natalia Desy also noted that the formula $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+2}{2n+2}$ *holds for each integer* $n \geq 2$.

2. In a circle having centre O, the chords AB and CD are perpendicular to each other and neither chord passes through the centre. Show that

$$\angle AOD + \angle BOC = 180°.$$

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

We have that $\angle AOC + \angle BOD = 2\angle AEC$, where E is the intersection of AB and CD. Thus, $\angle AOC + \angle BOD = 180°$, from which it follows that $\angle AOD + \angle BOC = 180°$.

No other solutions were submitted.

3. Show that for real x and y where $x \neq 0$ and $y \neq 0$, there is no solution for the equation

$$(x+y)^4 = x^4 + y^4.$$

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON, modified by the editor.

We have

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^4 + y^4.$$

Thus, $2x^2 + 3yx + 2y^2 = 0$ (by simplifying and dividing by $2xy \neq 0$). Then

$$x = \frac{-3y \pm \sqrt{(3y)^2 - 4 \cdot 2 \cdot 2y^2}}{2 \cdot 2} = \frac{-3y \pm |y|\sqrt{-7}}{4}.$$

This means that if y is a real number, then x is not a real number. Therefore, x and y cannot both be non-zero real numbers.

There was one incorrect solution submitted.

4. (a) How many positions on average can a knight reach in one move on a 8×8 chessboard?

(b) What result do we get for an $n \times n$ chessboard?

The movement of a knight has the shape of an "L": a shift of two squares horizontally followed by one vertically, or vice versa.

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

We answer part (b) first. Let the bottom left square be $x_{1,1}$, and let $x_{a,b}$ be the square in the ath row above the bottom and the bth column from the left.

From any corner square $x_{1,1}$, $x_{1,n}$, $x_{n,1}$, or $x_{n,n}$ there are **2** knight moves, and a total of **4** such squares.

From any square $x_{1,c}$, $x_{c,1}$, $x_{n,c}$, or $x_{c,n}$, where $3 \le c \le n-2$, there are **4** knight moves, and a total of $4(n-4)$ such squares.

From any square $x_{1,c}$, $x_{c,1}$, $x_{n,c}$, or $x_{c,n}$, where $c = 2$ or $c = n-1$, there are **3** knight moves, and a total of **8** such squares.

From any square $x_{2,2}$, $x_{2,n-1}$, $x_{n-1,2}$, or $x_{n-1,n-1}$, there are **4** knight moves, and a total of **4** such squares.

From any square $x_{2,c}$, $x_{c,2}$, $x_{n-1,c}$, or $x_{c,n-1}$, where $3 \le c \le n-2$, there are **6** knight moves, and a total of $4(n-4)$ such squares.

From any square $x_{d,c}$, where $3 \le c$, $d \le n-2$, there are **8** knight moves, and a total of $(n-4)^2$ such squares.

Thus, the average number of squares a knight can reach on an $n \times n$ chessboard is

$$\frac{\left(8 + 16(n-4) + 24 + 16 + 24(n-4) + 8(n-4)^2\right)}{n^2}$$
$$= \frac{\left(8n^2 - 24n + 16\right)}{n^2}.$$

When $n = 8$ the answer to part (a) is $\dfrac{\left(8n^2 - 24n + 16\right)}{n^2} = \dfrac{21}{4}$.

No other solutions were submitted.

Alex Song's derivation implicitly assumes that $n \ge 4$, but the formula he obtained is also valid for $n = 1$, $n = 2$, and $n = 3$.

5. Show that by placing **5** points inside a right-triangle with sides **6** cm, **8** cm, and **10** cm, at least two of them have a distance smaller than **5** cm.

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

We partition the triangle into four parts. Let $A = (6, 0)$, $B = (0, 0)$, and $C = (0, 8)$ be the vertices of the triangle. To save on writing, let $s = \dfrac{5}{\sqrt{2}} - 0.001$. One part is the square with vertices $(0, 0)$, $(s, 0)$, $(0, s)$, and (s, s) intersected with the triangle. The second part is this square moved s units to the right and also bounded by the triangle. The third part is the first square moved s units up and bounded by the triangle. The last part is the first square moved $2s$ units up and bounded by the triangle. [*Ed.*: One must check that the four squares completely cover the triangle, for example, the point (s, s) is above the line AC.]

By the Pigeonhole Principle, there are **2** points that lie in the same part. Since each part is inside a square of side s, the maximum distance between these two points is less than **5**.

No other solutions were submitted.

6. Three runners, Yves, Jules, and Bertrand meet one day at a circular track. Knowing that they start running at the same moment, that Bertrand, the fastest of the three, does a full revolution of the track in **2** minutes and that Yves does it in **8** minutes, how long does it take Jules to do a full revolution, if we further note that all three runners meet at the same place on the track before Yves, the slowest of the three runners, has done a full revolution?

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON, expanded by the editor.

Bertrand's speed is $\frac{1}{2}$ a lap (revolution) per minute, and Yves' speed is $\frac{1}{8}$ of a lap per minute. If t is the time in minutes when Bertrand and Yves first meet, then

$$\frac{t}{2} - 1 = \frac{t}{8},$$

because Bertrand runs one lap before they first meet. Solving for t gives $t = \frac{8}{3}$ minutes and this first meeting occurs at a point $\frac{1}{3}$ of the way along the track. After first meeting, Bertrand and Yves meet again in $t = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$ minutes at a point $\frac{2}{3}$ of the way along the track. The next time they meet Yves will have done a full revolution, so we only need to think about these two meeting points. Jules is slower than Bertrand, so he cannot catch Yves the first time that Bertrand does. Thus, Jules meets them both at the second meeting point. When Jules does meet them there, he cannot have done less than a lap (otherwise he would be just as slow as Yves) and he cannot have done two laps (otherwise he would be just as fast as Bertrand). Therefore, he runs **1** complete lap before the second meeting point and his speed is $\left(1 + \frac{2}{3}\right) / \frac{16}{3} = \frac{5}{16}$ laps per minute and it takes Jules $\frac{16}{5}$ minutes to run one lap.

No other solutions were submitted.

7. How many numbers are there smaller than **10,000** that contain the digit **7** at least once?

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON.

Write a whole number less than **10,000** in the form $abcd$, where a, b, c, and d are digits. We consider the numbers that do not have a **7** as one of the digits. For these there are **9** choices for each of a, b, c, and d, resulting in $9^4 = 6561$ numbers. Therefore, the total count of numbers that have at least one **7** is $10{,}000 - 6561 = 3439$.

No other solutions were submitted.

8. We build a cone from a circular piece of cardboard having a radius of **10** cm by cutting out a sector from it. Determine the volume of the cone we obtain as a function of the angle, in radians, at the center of the sector.

Recall that Volume $= \frac{1}{3}\pi r^2 h$, where r is the radius of the base of the cone, h is the altitude of the cone, and 2π radians $= 360°$.

Solution by Alex Song, Elizabeth Ziegler Public School, Waterloo, ON, expanded by the editor.

Let θ be the angle of the sector that is cut out from the cone. The circumference of the piece of cardboard is $2\pi \cdot 10 = 20\pi$, and the arc along the sector has length $\frac{\theta}{2\pi} \cdot 20\pi = 10\pi\theta$. If r is the radius of the base of the cone made from the sector, then $2\pi r = 10\pi\theta$, hence $r = 5\theta$. Following from the tip of the cone to the centre of the base to the edge of the base along a radius, we have a right triangle of height h, base 5θ, and hypotenuse 10. By the Pythagorean Theorem, $h^2 + (5\theta)^2 = 10^2$, hence $h = \sqrt{100 - 25\theta^2}$ and the volume of the resulting cone is $\frac{1}{3}\pi(5\theta)^2\sqrt{100 - 25\theta^2} = \frac{25\pi}{3}\theta^2\sqrt{100 - 25\theta^2}$.

$\big[$*Ed.*: In the question it appears the sector is to be discarded and the material remaining forms the cone. However, for $\theta > \pi$, the material remaining is less than what is discarded.$\big]$

No other solutions were submitted.

That brings us to the end of another issue. This month's winner of a past volume of Mayhem is Alex Song. Congratulations, Alex! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by **15 January 2009**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M363. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that A is a six-digit positive integer and B is the positive integer formed by writing the digits of A in reverse order. Prove that $A - B$ is a multiple of **9**.

M364. *Proposed by the Mayhem Staff.*

A semi-circle of radius **2** is drawn with diameter AB. The square $PQRS$ is drawn with P and Q on the semi-circle and R and S on AB. Is the area of the square less than or greater than one-half of the area of the semi-circle?

M365. *Proposed by Alexander Gurevich, student, University of Waterloo, Waterloo, ON.*

Let D be the family of lines of the form $y = nx + n^2$, with $n \geq 2$ a positive integer. Let H be the family of lines of the form $y = m$, where $m \geq 2$ is a positive integer. Prove that a line from H has a prime y-intercept if and only if this line does not intersect any line from D at a point with an x-coordinate that is a non-negative integer.

M366. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The roots of the equation $x^3 + bx^2 + cx + d = 0$ are p, q, and r. Find a quadratic equation with roots $(p^2 + q^2 + r^2)$ and $(p + q + r)$.

M367. *Proposed by George Tsapakidis, Agrinio, Greece.*

For the positive real numbers a, b, and c we have $a + b + c = 6$. Determine the maximum possible value of $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

M368. *Proposed by J. Walter Lynch, Athens, GA, USA.*

An infinite series of positive rational numbers $a_1 + a_2 + a_3 + \cdots$ is the fastest converging infinite series with a sum of 1, $a_1 = \frac{1}{2}$, and each a_i having numerator 1. (By "fastest converging", we mean that each term a_n is successively chosen to make the sum $a_1 + a_2 + \cdots + a_n$ as close to 1 as possible.) Determine a_5 and describe a recursive procedure for finding a_n.

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M363. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Soit A un entier positif de six chiffres et B l'entier positif formé des chiffres de A écrits dans l'ordre inverse. Montrer que $A - B$ est un multiple de 9.

M364. *Proposé par l'Équipe de Mayhem.*

Dans un demi-cercle de rayon 2 et de diamètre AB, on dessine un carré $PQRS$ avec P et Q sur le demi-cercle, et R et S sur AB. L'aire du carré est-elle plus petite ou plus grande que la moitié de celle du demi-cercle?

M365. *Proposé par Alexander Gurevich, étudiant, Université de Waterloo, Waterloo, ON.*

Soit D la famille des droites de la forme $y = nx + n^2$, avec $n \geq 2$, un entier positif. Soit H la famille des droites de la forme $y = m$, avec $m \geq 2$, un entier positif. Montrer que l'ordonnée du point d'intersection d'une droite de H avec l'axe des y est un nombre premier si et seulement si cette droite ne coupe aucune droite de D en un point d'abscisse entière non négative.

M366. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit p, q et r les racines de l'équation $x^3 + bx^2 + cx + d = 0$. Trouver une équation quadratique dont les racines sont $(p^2 + q^2 + r^2)$ et $(p + q + r)$.

M367. *Proposé par George Tsapakidis, Agrinio, Grèce.*

Soit a, b et c trois nombres réels positifs tels que $a + b + c = 6$. Déterminer la valeur maximale possible de $a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

M368. *Proposé par J. Walter Lynch, Athens, GA, É-U.*

Une série infinie de nombres rationnels positifs $a_1 + a_2 + a_3 + \cdots$ est la série la plus rapidement convergente de somme 1, avec $a_1 = \frac{1}{2}$ et chaque a_i de numérateur 1. (Par "la plus rapidement convergente", on entend que chaque terme a_n est tour à tour choisi de manière à ce que la somme $a_1 + a_2 + \cdots + a_n$ soit aussi proche de 1 que possible.) Déterminer a_5 et décrire une procédure récursive pour trouver a_n.

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Mayhem Solutions

The Editor would like to thank Emily Saltstone, Gravenhurst High School, Gravenhurst, ON, for her help in preparing this month's solutions.

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M325. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Let a, b, and c be non-zero digits. A student takes the fraction $\frac{ab}{ca}$, where ab and ca represent the two-digit integers $10a + b$ and $10c + a$, and applies a (false) cancellation law, cancelling the a from the numerator with the a from the denominator. For example, if $a = 6$, $b = 5$, and $c = 2$, the student would obtain $65/26 = 5/2$ (by 'cancelling' the 6s!).

Determine all triples (a, b, c) for which this student actually obtains the correct answer.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain, modified by the editor.

If $\dfrac{10a + b}{10c + a} = \dfrac{b}{c}$, then $c(10a + b) = b(10c + a)$ or $10c(a - b) = b(a - c)$ so $2 \cdot 5 \cdot c(a - b) = b(a - c)$. Therefore, 5 must be a divisor of the right hand side.

Since 5 is a prime number, then 5 must be a divisor of either b or $a - c$. Since a, b, and c are integers such that $1 \le a, b, c \le 9$, then either $a - c = 0$ (which would imply that $a - b = 0$), $b = 5$, or $a - c = 5$.

Case I: We have $a - c = 0$ and $a - b = 0$.
Here $a = b = c$, and we get nine triples (a, a, a), where $a \in \{1, 2, \ldots, 9\}$.

Case II: We have $b = 5$.

If $b = 5$, then the equation becomes $2c(a - 5) = a - c$. Solving for c we obtain, $c = \dfrac{a}{2a - 9}$. We have $a \geq 5$, otherwise $c < 0$, a contradiction. The value $a = 5$ yields $(5, 5, 5)$, which we found in Case I. Of the remaining possibilities, $a = 6$ and $a = 9$ yield two new triples, $(6, 5, 2)$ and $(9, 5, 1)$, respectively.

Case III: We have $a - c = 5$.

If $a - c = 5$, then $2c(a - b) = b$. Substituting $a = c + 5$, we then obtain $2c(c + 5 - b) = b$, whence, $b = \dfrac{2c(c + 5)}{2c + 1} = (c + 5) - \dfrac{c + 5}{2c + 1}$. Now, if $c > 4$, then $2c + 1 > c + 5$, and consequently $0 < \dfrac{c + 5}{2c + 1} < 1$, contradicting the fact that b is an integer. Therefore, $c \leq 4$, and of these possibilities $c = 1$ and $c = 4$ yield the last two triples, $(6, 4, 1)$ and $(9, 8, 4)$.

The thirteen triples (a, b, c) are $(6, 5, 2)$, $(9, 5, 1)$, $(6, 4, 1)$, $(9, 8, 4)$, and all triples (a, a, a), where $a \in \{1, 2, \ldots, 9\}$.

Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; MIGUEL MARAÑÓN GRANDES, student, Universidad de La Rioja, Logroño, La Rioja, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. There were 3 incorrect solutions submitted.

M326. *Proposed by the Mayhem Staff.*

The notation $\underline{a}89\underline{b}$ means the four-digit (base 10) integer whose thousands digit is a, whose hundreds digit is 8, whose tens digit is 9, and whose units digit is b. Determine all pairs of non-zero digits a and b such that $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Since $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$, then the following equations are true:

$$\begin{aligned}
1000a + 890 + b - 5904 &= 1000b + 980 + a\,, \\
1000a + b &= 1000b + a + 5994\,, \\
999a &= 999b + 5994\,, \\
a &= b + 6\,.
\end{aligned}$$

Since both a and b represent non-zero digits, they must be whole numbers between 1 and 9 inclusive. This gives the pairs $(a, b) = (7, 1), (8, 2), (9, 3)$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; RICARD PEIRO, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON; and TITU ZVONARU, Comănești, Romania. There was 1 incorrect solution submitted.

M327.   *Proposed by Lino Demasi, student, University of Waterloo, Waterloo, ON.*

Kaitlyn bought a new eraser. Her new eraser is in the shape of a rectangular prism. She calculates the lengths of the diagonals of the faces to be 10, $\sqrt{61}$, and $\sqrt{89}$. What is the volume of Kaitlyn's eraser?

Solution by Mrinal Singh, student, Kendriya Vidyalaya School, Shillong, India.

Let the three edge lengths of the prism be l, b, and h. Using the Pythagorean Theorem and the given lengths of the face diagonals, we obtain $l^2 + b^2 = 100$ and $b^2 + h^2 = 61$ and $h^2 + l^2 = 89$.

Adding these equations, we obtain $2l^2 + 2b^2 + 2h^2 = 250$, so that $l^2 + b^2 + h^2 = 125$. Therefore, $l^2 = 125 - (b^2 + h^2) = 125 - 61 = 64$, hence $l = 8$ since $l > 0$.

Since $l^2 + b^2 = 100$ and $l = 8$, then $b^2 = 100 - 64 = 36$, hence $b = 6$ since $b > 0$.

Lastly, since $b^2 + h^2 = 61$ and $b = 6$, then $h^2 = 61 - 36 = 25$, hence $h = 5$ since $h > 0$.

The volume of the rectangular prism equals $lbh = 8(6)(5) = 240$.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; JACLYN CHANG, student, Western Canada High School, Calgary, AB; PETER CHIEN, student, Central Elgin Collegiate, St. Thomas, ON; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; JOCHEM VAN GAALEN, grade 9 student, Medway High School, Arva, ON; VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON; and TITU ZVONARU, Comăneşti, Romania. There was 1 incomplete solution submitted.

M328.   *Proposed by Hugo Cuéllar, Columbia Aprendiendo, Zipaquirá, Coloumbia.*

Prove that, if from any positive integer we subtract the sum of each of its digits raised to any odd power (not necessarily the same), then the result is always a multiple of 3.

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan, modified by the editor.

Let $\sum_{k=0}^{n-1} d_k 10^k$ be a positive integer, where $0 \le d_k < 10$ is an integer for each integer $0 \le k < n$ and $d_{n-1} \ne 0$. Let e_k be a positive odd integer for each integer $0 \le k < n$. We need to prove that

$$\sum_{k=0}^{n-1} d_k 10^k \; - \; \sum_{k=0}^{n-1} d_k^{e_k} \; \equiv \; 0 \pmod{3} \, , \tag{1}$$

which is equivalent to proving that

$$\sum_{k=0}^{n-1} d_k 10^k \;\equiv\; \sum_{k=0}^{n-1} d_k^{e_k} \pmod 3 . \qquad (2)$$

To prove (2), it suffices to prove that $d_k 10^k \equiv d_k^{e_k} \pmod 3$ for each k.

Since $10^k \equiv 1 \pmod 3$ for any non-negative integer k, we have $d_k 10^k \equiv d_k \pmod 3$, so it suffices to prove that $d_k \equiv d_k^{e_k} \pmod 3$ for each integer $0 \le k < n$.

This follows from the lemma below.

Lemma If d and e are integers and $e > 0$ is odd, then $d \equiv d^e \pmod 3$.

Proof: Without loss of generality, we may assume that $d = 0$, 1, or 2. If $d = 0$ or $d = 1$, then $d = d^e$, so the result follows. If $d = 2$, then we have $d^e \equiv 2^e \equiv (-1)^e \equiv -1 \equiv 2 \pmod 3$, since e is odd. ∎

This completes our solution.

Also solved by LAURIN M. COLEMAN, student, Auburn University Montgomery, Montgomery, Alabama, USA; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICARD PEIRÓ, IÉS "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON; and TITU ZVONARU, Comăneşti, Romania. There were 3 incomplete solutions submitted.

M329. *Proposed by the Mayhem Staff.*

Determine the value of

$$\cos^2 1° + \cos^2 2° + \cos^2 3° + \cdots + \cos^2 89° + \cos^2 90° .$$

Solution by Kunal Singh, student, Kendriya Vidyalaya School, Shillong, India.

We have

$$\cos^2 1° \;+\; \cos^2 2° \;+\; \cos^2 3° \;+\; \cdots \;+\; \cos^2 89° \;+\; \cos^2 90°$$

$$= \;(\cos^2 1° + \cos^2 89°) \;+\; (\cos^2 2° + \cos^2 88°) \;+\; (\cos^2 3° + \cos^2 87°)$$
$$\quad +\; \cdots \;+\; (\cos^2 44° + \cos^2 46°) \;+\; \cos^2 45° + \cos^2 90°$$

$$= \;(\cos^2 1° + \cos^2(90° - 1°)) \;+\; (\cos^2 2° + \cos^2(90° - 2°))$$
$$\quad +\; \cdots \;+\; (\cos^2 44° + \cos^2(90° - 44°)) \;+\; \left(\frac{1}{\sqrt 2}\right)^2 + 0$$

$$= \;(\cos^2 1° + \sin^2 1°) \;+\; (\cos^2 2° + \sin^2 2°) \;+\; (\cos^2 3° + \sin^2 3°)$$
$$\quad +\; \cdots \;+\; (\cos^2 44° + \sin^2 44°) \;+\; \frac{1}{2}$$

$$= \;\underbrace{1 + 1 + \cdots + 1}_{44 \text{ times}} \;+\; \frac{1}{2} \;=\; 44 + \frac{1}{2} \;=\; \frac{89}{2} .$$

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; JACLYN CHANG, student, Western Canada High School, Calgary, AB; DENISE CORNWELL, student, Angelo State University, San Angelo, TX, USA; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON.

Hernández Santiago and Zvonaru both noted that M329 is the same as M170 from 2004.

M330. *Proposed by the Mayhem Staff.*

If n is a positive integer, the nth triangular number is defined as $T_n = 1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$. Determine all pairs of triangular numbers whose difference is 2008.

Solution by Titu Zvonaru, Comǎneşti, Romania.

We must solve the equation $\frac{1}{2}n(n+1) - \frac{1}{2}m(m+1) = 2008$, which is equivalent to $n^2 - m^2 + n - m = 2 \cdot 2008$, or $(n-m)(n+m+1) = 2^4 \cdot 251$.

Note that $n + m + 1 > n - m$. Also, $(n-m)$ and $(n+m+1)$ have different parity, because their sum is $2n + 1$, which is odd.

Therefore, $n - m = 1$ and $n + m + 1 = 4016$, or $n - m = 16$ and $n + m + 1 = 251$. (Since 251 is a prime number, these are the only two ways to factor $2^4 \cdot 251$ as the product of one even and one odd positive integer.)

In the first case, adding the two equations gives $2n + 1 = 4017$ whence $n = 2008$ and the equation $n - m = 1$ gives $m = 2007$. Repeating this in the second case, we obtain $n = 133$ and $m = 117$.

The two solutions are the pairs (T_{2008}, T_{2007}) and (T_{133}, T_{117}).

Also solved by MIHÁLY BENCZE, Brasov, Romania; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. There were 3 incomplete solutions and 1 incorrect solution submitted.

M331. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, side AB is parallel to DC, and diagonals AC and BD intersect at P.

(a) If the area of $\triangle APB$ is 4 and the area of $\triangle DPC$ is 9,

    (i) prove that $AP : PC = 2 : 3$,

    (ii) explain why the ratio of the area of $\triangle BPC$ to the area of $\triangle BPA$ equals $3 : 2$, and

    (iii) determine the area of trapezoid $ABCD$.

(b) If the area of $\triangle APB$ is x and the area of $\triangle DPC$ is y, determine the area of trapezoid $ABCD$ in terms of x and y.

Solution by Denise Cornwell, student, Angelo State University, San Angelo, TX, USA, modified slightly by the editor.

Suppose that $AP = q$, $CP = r$, $BP = s$, and $DP = t$. We use the notation $[\triangle XYZ]$ to denote the area of $\triangle XYZ$.

(a) Since $AB \parallel CD$ and diagonals AC and BD intersect at P, then $\angle APB = \angle CPD$, $\angle ABP = \angle CDP$, and $\angle BAP = \angle DCP$, so $\triangle APB \sim \triangle CPD$, whence $\frac{q}{r} = \frac{s}{t}$.

  (i) Now, $[\triangle APB] = \frac{1}{2}qs \sin \angle APB$ and $[\triangle CPD] = \frac{1}{2}rt \sin \angle CPD$. Since $\angle APB = \angle CPD$, the ratio of the areas of these triangles is $\frac{4}{9} = \frac{qs}{rt}$. Using the fact that $\frac{q}{r} = \frac{s}{t}$ gives $\frac{4}{9} = \frac{q^2}{r^2}$, or $\frac{2}{3} = \frac{q}{r}$. Therefore, $AP : PC = 2 : 3$.

  (ii) Now, $[\triangle CPB] = \frac{1}{2}rs \sin \angle CPB$ and $[\triangle APB] = \frac{1}{2}qs \sin \angle APB$. Also, $\angle CPB = \pi - \angle APB$, and since $\sin \theta = \sin(\pi - \theta)$ we then have $\sin \angle CPB = \sin \angle APB$.

  Hence, the ratio of the area of the triangles is

  $$\frac{[\triangle CPB]}{[\triangle APB]} = \frac{\frac{1}{2}rs \sin \angle CPB}{\frac{1}{2}qs \sin \angle APB} = \frac{r}{q} = \frac{3}{2}.$$

  (iii) Now we can find the area of the trapezoid by finding the areas of $\triangle CPB$ and $\triangle DPA$.

  $$[\triangle CPB] = \frac{r}{q}[\triangle APB] = \frac{3}{2}(4) = 6.$$

  By a similar argument, we can show that $[\triangle DPA] = 6$. That is, we can switch the pairs of labels A and B, C and D, q and s, and r and t, and then carry out same arguments that we did before. Altogether, $[ABCD] = 4 + 6 + 6 + 9 = 25$.

(b) If $[\triangle APB] = x$ and $[\triangle DPC] = y$, then using the same steps as above we see that $\frac{x}{y} = \frac{q^2}{r^2}$, hence $\frac{q}{r} = \frac{\sqrt{x}}{\sqrt{y}}$. As above,

  $$[\triangle BPC] = \frac{r}{q}[\triangle APB] = \frac{\sqrt{y}}{\sqrt{x}}(x) = \sqrt{xy};$$

  $$[\triangle DPA] = \frac{r}{q}[\triangle APB] = \frac{\sqrt{y}}{\sqrt{x}}(x) = \sqrt{xy}.$$

Thus, the area of the trapezoid $ABCD$ is

$$[ABCD] \; = \; x + \sqrt{xy} + \sqrt{xy} + y \; = \; x + 2\sqrt{xy} + y \; = \; (\sqrt{x} + \sqrt{y})^2 \,.$$

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRO, IÉS "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KUNAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; VINCENT ZHOU, student, Dr. Norman Bethune Collegiate Institute, Agincourt, ON; and TITU ZVONARU, Comănești, Romania. There were 2 incomplete solutions and 1 incorrect solution submitted.

Problem of the Month
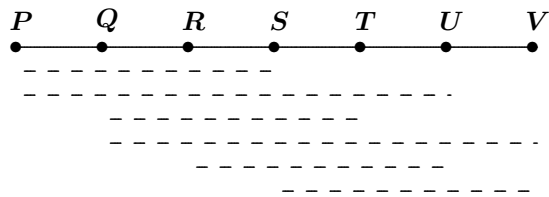
Ian VanderBurgh

It's time for a road trip!

Problem (1997 UK Junior Mathematical Challenge) Seven towns P, Q, R, S, T, U, V lie (in that order) along a road. The table on the right is meant to give all the distances between pairs of towns, in km; for example, the distance from P to S is **23** km. Unfortunately, fifteen of the distances are missing. How many of the missing distances can be calculated from the given information?

P						
?	Q					
?	?	R				
23	?	?	S			
?	**30**	?	?	T		
58	?	**40**	?	?	U	
?	**68**	?	**53**	?	?	V

(A) 0   (B) 1   (C) 6   (D) 12   (E) 15

You have probably seen this type of chart before on the back of a roadmap. We should make sure first that we know how to interpret the chart. From the chart, the distance from Q to T is 30 km, but we don't know the distance from Q to U.

My next inclination is to draw a schematic diagram.

I say the diagram is "schematic" since it is not to scale. Next, we should indicate the information we are given on the diagram. I'm going to do this not by marking the actual lengths, but by indicating under the diagram the lengths that we know using dashed line segments.

Now we should try to find some new lengths. Can you see how to use the given information to find the distance from P to V? See if you can figure this out!

From the information, the distance from P to S is 23 km. We'll write $PS = 23$ as shorthand. Also, $SV = 53$. Therefore, $PV = PS + SV = 23 + 53 = 76$. Can you see what this is saying physically? It says that the distance along the line from P to V equals the distance from P to S plus the distance from S to V. If you're not convinced, try going for a drive! (A question to think about: Why is it important that the points lie along a line?)

So we've got at least 1 of the 15 missing pieces of information sorted out. We can continue by trial and error to find more pieces of information. But is there a strategy that we can use to do this systematically?

Another question: Are some segments more important than others? Put another way, are there segments whose lengths give us the most information about other lengths? The answer is yes – it's the "basic" segments: PQ, QR, RS, ST, TU, and UV which are crucial. All of the other segments are made up from these, so let's try to figure out the lengths of these basic segments. At the very least, this gives us a way to organize our work.

Solution We have the partial chart above and we have already figured out PV. Is there a basic segment we can deduce in one step? Yes, we can figure out UV. Since $PV = 76$ and $PU = 58$, then $UV = PV - UV = 18$.

Let's continue to work from right to left. We don't have enough information yet to determine TU in one step, but since we know UV, then if we knew TV, we could find TU. But $TV = QV - QT = 68 - 30 = 38$. Thus, $TU = TV - UV = 38 - 18 = 20$.

Next, we determine ST. We have $SV = 53$ and $TV = 38$. Therefore, $ST = 53 - 38 = 15$. Also, $SU = ST + TU = 15 + 20 = 35$.

Now we figure out RS. We know that $RU = 40$ and $SU = 35$, so $RS = RU - SU = 5$. Thus, $RT = RS + ST = 5 + 15 = 20$ and $RV = RS + SV = 5 + 53 = 58$.

We're getting closer: QR is next. We know $QT = 30$ and $RT = 20$, so $QR = QT - RT = 10$. Thus, $QS = QR + RS = 10 + 5 = 15$ and $QU = QR + RU = 10 + 40 = 50$.

Lastly, we deduce PQ. We know that $PV = 76$ and $QV = 68$. Thus, $PQ = PV - QV = 8$. Also, $PR = PQ + QR = 8 + 10 = 18$ and $PT = PQ + QT = 8 + 30 = 38$.

Our chart is now complete and we determined all 15 pieces of missing information!

I always find these kinds of problems surprising. It's amazing how much information we can get from so little. But wait – is it so surprising? While there are 21 distances in total that we need, how many of these are truly "independent"? Actually, only 6 distances are (essentially, these 6 basic segments that we looked at earlier). Do you think that it's a coincidence that we were given 6 pieces of information to begin? Do you think that we could do this if we were given only 5 pieces of information to start?

P						
8	Q					
18	10	R				
23	15	5	S			
38	30	20	15	T		
58	50	40	35	20	U	
76	68	58	53	38	18	V

As a postscript, I would suggest that you not approach your local Department of Transportation to see if they want to save on printing costs by removing most of the distances on their local chart...
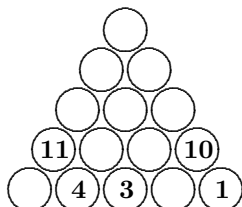
Adding Up

Bruce Shawyer

A daily exercise is offered in the British Newspaper *The Times* and is there called *Add Up*. The stated rules are:

The number in each circle is the sum of the two below it. Work out the top number. Try it in your head, if you can.

A typical example is

With a little bit of work, we can see that the number that must be in the top position is **78**.

There are two ways (at least) to solve this problem. There is the obvious straight forward way of filling in every number on the collection. In the given example, we see the missing numbers in the bottom row:

then fill in the second row, and each subsequent row, to get

However, those with a little mathematical imagination can work out an interesting algebraic formula for the top number, based on the bottom row.

Suppose that the bottom row is

$$a, \quad b, \quad c, \quad d, \quad e.$$
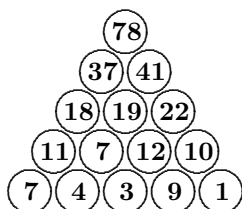
Then the next row is

$$a+b, \quad b+c, \quad c+d, \quad d+e.$$

The following row is

$$(a+b)+(b+c), \quad (b+c)+(c+d), \quad (c+d)+(d+e);$$

that is,

$$a+2b+c, \quad b+2c+d, \quad c+2d+e.$$

The next row is

$$(a+2b+c)+(b+2c+d), \quad (b+2c+d)+(c+2d+e);$$

that is,

$$a+3b+3c+d, \quad b+3c+3d+e.$$

Has the pattern become clear? The top number is

$$a+4b+6c+4d+e.$$

The binomial coefficients are the key to the solution. And, we can see, that if we take a bottom row of $n+1$ numbers $a_0, a_1, \ldots, a_n$, we end up with a top number of $\sum\limits_{k=0}^{n} a_k \binom{n}{k}$.

I will conclude with three challenges to the reader. Consider an *Add Up* diagram with $n+1$ circles in the bottom row.

1. Take the bottom row to consist entirely of 1's. Where in the diagrams do powers of 2 arise?

2. Take the bottom row to be the Fibonacci numbers $\{1, 1, 2, 3, 5, 8, \ldots\}$. What is the top number?

3. The top number is always $\sum\limits_{k=0}^{n} a_k \binom{n}{k}$. Can you prove that?

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THE OLYMPIAD CORNER
No. 273

R.E. Woodrow

We start this number of the *Corner* with the problems of both rounds of the 2005/6 British Mathematical Olympiad. Thanks go to Robert Morewood, 2006 Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

2005/6 BRITISH MATHEMATICAL OLYMPIAD
Round 1

1. Let n be an integer greater than **6**. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by **720**. Is the converse true?

2. Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

  (i) In how many ways can he split them into two teams of six?

  (ii) In how many ways can he split them into three teams of four?

3. In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

4. The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length **1**.

A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find, with proof, the greatest number of cells which can be visited.

5. Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.

6. Let T be a set of **2005** coplanar points with no three collinear. Show that, for any of the **2005** points, the number of triangles it lies strictly within, whose vertices are points in T, is even.

Round 2

1. Find the minimum possible value of $x^2 + y^2$ given that x and y are real numbers satisfying

$$xy(x^2 - y^2) \ = \ x^2 + y^2 \quad \text{and} \quad x \neq 0 \ .$$

2. Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy

$$x^2 - y^2 \ = \ 2^k$$

for some non-negative integer k.

3. Let ABC be a triangle with $AC > AB$. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that $BX = CA$ and $CY = BA$. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle BPC + \angle BAC = 180°$.

4. An exam consisting of six questions is sat by 2006 children. Each question is marked either right or wrong. Any three children have right answers to at least five of the six questions between them. Let N be the total number of right answers achieved by all the children (that is, the total number of questions solved by child $1 +$ the total solved by child $2 + \cdots +$ the total solved by child 2006). Find the least possible value of N.

Next we give the problems of the Bulgarian National Olympiad, National Round, May 20–21, 2006. Thanks again go to Robert Morewood, 2006 Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for us.

BULGARIAN NATIONAL OLYMPIAD
National Round
May 20–21, 2006

1. (Aleksandar Ivanov) Consider the set $A = \{1, 2, 3, \ldots, 2^n\}$, $n \geq 2$. Find the number of subsets B of A, such that if the sum of two elements of A is a power of 2 then exactly one of them belongs to B.

2. (Oleg Mushkarov, Nikolai Nikolov) Let $\mathbb{R}^+$ be the set of all positive real numbers and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that

$$f(x + y) - f(x - y) \ = \ 4\sqrt{f(x)f(y)} \ .$$

for all $x > y > 0$.

(a) Prove that $f(2x) = 4f(x)$ for all $x \in \mathbb{R}^+$.

(b) Find all such functions.

3. (Aleksandar Ivanov, Emil Kolev) Consider the infinite sequence of digits obtained by writing all the positive integers one after another in increasing order. Find the least positive integer k such that among the first k digits of the above sequence every two non-zero digits appear a different number of times.

4. (Aleksandar Ivanov) Let p be a prime number such that p^2 is a divisor of $2^{p-1} - 1$. Prove that for any positive integer n, the integer $(p-1)(p! + 2^n)$ has at least three distinct prime divisors.

5. (Emil Kolev) Let $\triangle ABC$ be such that $\angle BAC = 30°$ and $\angle ABC = 45°$. Consider all pairs of points X and Y such that X is on the ray $\overrightarrow{AC}$, Y is on the ray $\overrightarrow{BC}$, and $OX = BY$, where O is the circumcenter of $\triangle ABC$. Prove that the perpendicular bisectors of the segments XY pass through a fixed point.

6. (Nikolai Nikolov and Slavomir Dinev) Given a point O in the plane, find all sets S in the same plane, containing at least two points, such that for any point $A \in S$, $A \neq O$, the circle with diameter OA is contained in S.

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Next we give the problems used in the Selection of the Indian Team 2006. Thanks again go to Robert Morewood, 2006 Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

INDIAN MATHEMATICAL OLYMPIAD 2006
Team Selection Problems

1. Let n be a positive integer divisible by 4. Find the number of permutations σ of $(1, 2, 3, \ldots, n)$ which satisfy the condition $\sigma(j) + \sigma^{-1}(j) = n+1$ for all $j \in \{1, 2, 3, \ldots, n\}$.

2. (Short list, IMO 2005) Let $ABCD$ be a parallelogram. A line l through the point A intersects the rays BC and DC at X and Y respectively. Let K and L be the centres of the excircles of triangles ABX and ADY, touching the sides BX and DY respectively. Prove that the size of $\angle KCL$ does not depend on the choice of the line l.

3. (Short list, IMO 2005) There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. In each step (if possible) we pick a marker with the white side up that is not an outermost marker, remove it, and turn over the closest marker to the left and the closest marker to the right of it. Prove that one can reach a terminal state of exactly two markers if and only if $(n - 1)$ is not divisible by 3.

4. Let ABC be a triangle and let P be a point in the plane of ABC that is inside the region of the angle BAC but outside triangle ABC.

(a) Prove that any two of the following statements imply the third:

(i) The circumcentre of triangle PBC lies on the ray $\overrightarrow{PA}$.

(ii) The circumcentre of triangle CPA lies on the ray $\overrightarrow{PB}$.

(iii) The circumcentre of triangle APB lies on the ray $\overrightarrow{PC}$.

(b) Prove that if the conditions in (a) hold, then the circumcentres of triangles BPC, CPA, and APB lie on the circumcircle of triangle ABC.

5. Let p be a prime number and let X be a finite set containing at least p elements. A collection of pairwise mutually disjoint p-element subsets of X is called a *p-family*. (In particular, the empty collection is a p-family.) Let A (respectively, B) denote the number of p-families having an even (respectively, odd) number of p-element subsets of X. Prove that A and B differ by a multiple of p.

6. Let ABC be an equilateral triangle, and let D, E, and F be points on BC, CA, and AB respectively. Let $\angle BAD = \alpha$, $\angle CBE = \beta$, and $\angle ACF = \gamma$. Prove that if $\alpha + \beta + \gamma \geq 120°$, then the union of the triangular regions BAD, CBE, and ACF covers the triangle ABC.

7. Let ABC be a triangle with inradius r, circumradius R, and with sides $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{R}{2r} \geq \left(\frac{64a^2b^2c^2}{\left(4a^2 - (b-c)^2\right)\left(4b^2 - (c-a)^2\right)\left(4c^2 - (a-b)^2\right)} \right)^2 .$$

8. The positive divisors $d_1, d_2, \ldots, d_l$ of a positive integer n are ordered

$$1 = d_1 < d_2 < \cdots < d_l = m .$$

Suppose it is known that $d_7^2 + d_{15}^2 = d_{16}^2$. Find all possible values of d_{17}.

9. Let $A_1, A_2, \ldots, A_n$ be arithmetic progressions of integers, each of k terms, such that any two of these arithmetic progressions have at least two common elements. Suppose b of these arithmetic progressions have common difference d_1 and the remaining arithmetic progressions have common difference d_2, where $0 < b < n$. Prove that

$$b \leq 2 \left(k - \frac{d_2}{\gcd(d_1, d_2)} \right) - 1 .$$

10. Find all triples (a, b, c) such that a, b, and c are integers in the set $\{2000, 2001, \ldots, 3000\}$ satisfying $a^2 + b^2 = c^2$ and $\gcd(a, b, c) = 1$.

11. Let u_{jk} be a real number for each $j = 1, 2, 3$ and each $k = 1, 2$; and let N be an integer such that

$$\max_{1 \leq k \leq 2} \sum_{j=1}^{3} |u_{jk}| \leq N.$$

Let M and l be positive integers such that $l^2 < (M+1)^3$. Prove that there exist integers ξ_1, ξ_2, and ξ_3, not all zero, such that

$$\max_{1 \leq j \leq 3} \xi_j \leq M \quad \text{and} \quad \left| \sum_{j=1}^{3} u_{jk}\xi_k \right| \leq \frac{MN}{l}, \quad \text{for } k = 1, 2.$$

12. Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set S such that $|A_j| = 8$ for each j. For a subset B of S, let $F(B) = \{j : 1 \leq j \leq n \text{ and } A_j \subset B\}$. Suppose for each subset B of S, at least one of the following conditions holds:

(a) $|B| > 25$,

(b) $F(B) = \emptyset$,

(c) $\bigcap_{j \in F(B)} A_j \neq \emptyset$.

Prove that $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$.

Another set of problems for your pleasure are those of the South African Mathematical Olympiad, Third Round 2004, Senior Division. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for our use.

THE SOUTH AFRICAN MATHEMATICAL OLYMPIAD 2004
Third Round
Senior Division (Grades 10-12)

1. Let $a = 1111 \cdots 1111$ and $b = 1111 \cdots 1111$, where a has forty ones and b has twelve ones. Determine the greatest common divisor of a and b.

2. Fifty points are chosen inside a convex polygon having eighty sides such that no three of the fifty points lie on the same straight line. The polygon is cut into triangles such that the vertices of the triangles are just the fifty points and the eighty vertices of the polygon. How many triangles are there?

3. Find all real numbers x such that $x\lceil x\lceil x\lceil x\rceil\rceil\rceil = 88$. The notation $\lceil x\rceil$ means: "the least integer which is not less than x".

4. Let ABC be an isosceles triangle with $CA = CB$ and $\angle C > 60°$. Let A_1 and B_1 be two points on AB such that $\angle A_1 C B_1 = \angle BAC$. A circle externally tangent to the circumcircle of triangle $A_1 B_1 C$ is tangent also to the rays CA and CB at points A_2 and B_2 respectively. Prove $A_2 B_2 = 2AB$.

5. Let $n \geq 2$ be an integer. Find the number of integers x with $0 \leq x < n$ and such that x^2 leaves a remainder of 1 when divided by n.

6. Let a_1, a_2, and a_3 be distinct positive integers such that

$$
\begin{aligned}
a_1 &\quad\text{is a divisor of}\quad a_2 + a_3 + a_2 a_3\,, \\
a_2 &\quad\text{is a divisor of}\quad a_3 + a_1 + a_3 a_1\,, \quad\text{and} \\
a_3 &\quad\text{is a divisor of}\quad a_1 + a_2 + a_1 a_2\,.
\end{aligned}
$$

Prove that a_1, a_2, and a_3 cannot all be prime.

As a final set of problems for this number, we give the problems of the two days of the 2006 Vietnamese Olympiad. Thanks go to Robert Morewood, Canadian Team Leader to the 47th IMO in Slovenia, for collecting them for us.

2006 VIETNAMESE MATHEMATICAL OLYMPIAD
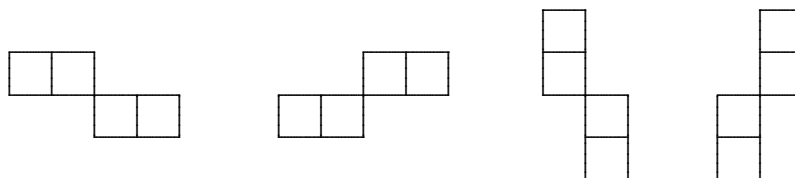## Days 1 and 2
February 23–24, 2006

1. Find all real solutions of the system of equations

$$
\begin{aligned}
\sqrt{x^2 - 2x + 6} \cdot \log_3(6 - y) &= x\,, \\
\sqrt{y^2 - 2y + 6} \cdot \log_3(6 - z) &= y\,, \\
\sqrt{z^2 - 2z + 6} \cdot \log_3(6 - x) &= z\,.
\end{aligned}
$$

2. Let $ABCD$ be a given convex quadrilateral. A point M moves on the line AB but does not coincide with A or B. Let N be the second point of intersection (distinct from M) of the circles (MAC) and (MBD), where (XYZ) denotes the circle passing through the points X, Y, and Z. Prove that

(a) N moves on a fixed circle,

(b) The line MN passes through a fixed point.

3. A rectangular $m \times n$ board is given, where m and n are integers each greater than 3. At each step, one puts 4 marbles into 4 cells of the board (one marble per cell) so that these four cells form one of the following shapes:

After a finite number of steps is it possible for each cell to contain exactly the same (positive) number of marbles if

(a) $m = 2004$ and $n = 2006$?

(b) $m = 2005$ and $n = 2006$?

(At each step any of the four chosen cells may or may not have marbles in them.)

4. Consider the function

$$f(x) \;=\; -x + \sqrt{(x+a)(x+b)}$$

where a and b are distinct positive real numbers. Prove that for every real number s in the interval $(0, 1)$, there exists a unique positive real number α such that

$$f(\alpha) \;=\; \left(\frac{a^s + b^s}{2} \right)^{1/s}.$$

5. Find all polynomials $P(x)$ with real coefficients satisfying

$$P(x^2) + x\big(3P(x) + P(-x)\big) \;=\; P(x)^2 + 2x^2 \,,$$

for all real numbers x.

6. A set of integers T is called *sum-free* if for every two (not necessarily distinct) elements u and v in T, their sum $u + v$ does not belong to T. Prove that

(a) a sum-free subset of $S = \{1, 2, \ldots, 2006\}$ has at most **1003** elements,

(b) any set S consisting of **2006** positive integers has a sum-free subset consisting of **669** elements.

Now we turn to the file of readers' solutions to problems given in the December 2007 number of the *Corner* beginning with a problem of the 2004 Chinese Mathematical Olympiad, given at [2007 : 470].

5. Given any positive integer $n \geq 2$, let a_i ($i = 1, 2, \ldots, n$) be positive integers satisfying $a_1 < a_2 < \cdots < a_n$ and $\sum_{i=1}^{n} \frac{1}{a_i} \leq 1$. Prove that for any real number x,

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2} \right)^2 \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2} .$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

We assume first that $x^2 \geq a_1(a_1 - 1)$. Then x is non-zero and from $(a_i - |x|)^2 \geq 0$, we have $a_i^2 + x^2 \geq 2a_i|x|$ for each i. Thus, $\frac{1}{a_i^2 + x^2} \leq \frac{1}{2a_i|x|}$, and we have

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{1}{2a_i|x|} \right)^2 = \frac{1}{4x^2} \left(\sum_{k=1}^{n} \frac{1}{a_i} \right)^2$$

$$\leq \frac{1}{4x^2} \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2} .$$

Now we suppose that $x^2 < a_1(a_1 - 1)$. Let $\overrightarrow{u} = \left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \ldots, \frac{1}{\sqrt{a_n}} \right)$ and $\overrightarrow{v} = \left(\frac{\sqrt{a_1}}{a_1^2 + x^2}, \frac{\sqrt{a_2}}{a_2^2 + x^2}, \ldots, \frac{\sqrt{a_n}}{a_n^2 + x^2} \right)$. We apply the Cauchy-Schwarz Inequality to $\overrightarrow{u}$ and $\overrightarrow{v}$ to obtain

$$\left(\sum_{i=1}^{n} \frac{1}{a_i^2 + x^2} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{1}{a_i} \right) \left(\sum_{i=1}^{n} \frac{a_i}{(a_i^2 + x^2)^2} \right) \leq \sum_{i=1}^{n} \frac{a_i}{(a_i^2 + x^2)^2} .$$

For each i we have $\frac{a_i}{2} \geq \frac{1}{16}$ and $(a_i^2 + x^2)^2 \geq (a_i^2 + x^2 + \frac{1}{4})^2 - a_i^2$. Furthermore, from $a_1 < a_2 < \cdots < a_n$, it follows that $a_{i+1} \geq a_i + 1$ and $a_{i+1} - \frac{1}{2} \geq a_i + \frac{1}{2}$. Therefore,

$$\frac{a_i}{(a_i^2 + x^2)^2} \leq \frac{a_i}{(a_i^2 + x^2 + \frac{1}{4})^2 - a_i^2}$$

$$= \frac{1}{2} \frac{2a_i}{\left((a_i - \frac{1}{2})^2 + x^2 \right) \left((a_i + \frac{1}{2})^2 + x^2 \right)}$$

$$= \frac{1}{2} \left(\frac{1}{(a_i - \frac{1}{2})^2 + x^2} - \frac{1}{(a_i + \frac{1}{2})^2 + x^2} \right)$$

$$\leq \frac{1}{2} \left(\frac{1}{(a_i - \frac{1}{2})^2 + x^2} - \frac{1}{(a_{i+1} - \frac{1}{2})^2 + x^2} \right) .$$

Finally,

$$\sum_{i=1}^{n} \frac{a_i}{\left(a_i^2 + x^2\right)^2} \leq \frac{1}{2} \sum_{i=1}^{n} \left(\frac{1}{\left(a_i - \frac{1}{2}\right)^2 + x^2} - \frac{1}{\left(a_{i+1} - \frac{1}{2}\right)^2 + x^2} \right)$$

$$\leq \frac{1}{2} \cdot \frac{1}{\left(a_1 - \frac{1}{2}\right)^2 + x^2} \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1 - 1) + x^2} \,,$$

and the proof is complete.

Next we look at solutions to problems of the Singapore Mathematical Olympiad 2004, Open Section, Special Round, given at $\left[2007 : 470\right]$.

3. Let AD be the common chord of two circles Γ_1 and Γ_2. A line through D intersects Γ_1 at B and Γ_2 at C. Let E be a point on the segment AD different from A and D. The line CE intersects Γ_1 at P and Q. The line BE intersects Γ_2 at M and N.

  (i) Prove that P, Q, M, and N lie on the circumference of a circle Γ_3.

  (ii) If the centre of Γ_3 is O, prove that OD is perpendicular to BC.

Solution by Michel Bataille, Rouen, France.

  (i) We have $EP \cdot EQ = EA \cdot ED$, since the chords PQ and AD of Γ_1 intersect at E. Similarly, we obtain $EM \cdot EN = EA \cdot ED$. It follows that $EP \cdot EQ = EM \cdot EN$ and so P, Q, M, and N are concyclic.

  (ii) Since MN and CD intersect at B, we have $BD \cdot BC = BM \cdot BN$. But $BM \cdot BN$ is the power of B with respect to Γ_3, so $BM \cdot BN = BO^2 - \rho^2$ where ρ is the radius of Γ_3. Similarly,
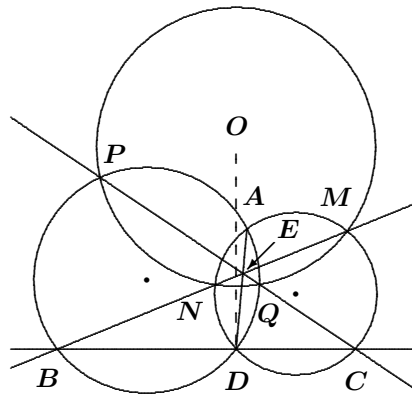
$$CD \cdot CB = CP \cdot CQ = CO^2 - \rho^2 \,,$$

and therefore

$$\begin{aligned} BO^2 - CO^2 &= BD \cdot BC - CD \cdot CB = BC(BD - DC) \\ &= (BD + DC)(BD - DC) = BD^2 - DC^2 \,. \end{aligned}$$

From $BO^2 - CO^2 = BD^2 - DC^2$, it follows that OD is perpendicular to BC.

4. If $0 < x_1, x_2, \ldots, x_n \leq 1$, where $n \geq 1$, show that

$$\frac{x_1}{1 + (n-1)x_1} + \frac{x_2}{1 + (n-1)x_2} + \cdots + \frac{x_n}{1 + (n-1)x_n} \leq 1 \,.$$

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We give the write-up of Maragoudakis.

It is enough to prove that $\dfrac{x}{1+(n-1)x} \leq \dfrac{1}{n}$ for every $x \in (0,1]$. However, this is equivalent to $nx \leq 1 + (n-1)x$, which is equivalent to $x \leq 1$.

⁓⁓⁓⁓⁓

Next we turn to solutions to problems of the 18$^{\text{th}}$ Nordic Mathematical Contest 2004 given at [2007: 471].

2. Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, \ldots$, be the sequence of Fibonacci numbers. Show that there exists a strictly increasing infinite arithmetic sequence of integers which has no numbers in common with the Fibonacci sequence.

Solution by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.

We consider the Fibonacci sequence modulo **8**. The first few terms (**mod 8**) are:

$$\underbrace{1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0}, 1, 1, 2, 3, 5, \ldots,$$

so we see that the two-term subsequence "1, 1" occurs again, hence the Fibonacci sequence is cyclic modulo **8**.

We conclude that the Fibonacci sequence does not contain any terms congruent to **4** or **6** (**mod 8**).

Thus, the arithmetic sequence A, with $A_1 = 4$ (or **6**) and common difference $A_{n+1} - A_n = 8$, has no numbers in common with the Fibonacci sequence.

4. Let a, b, c, and R be the side lengths and the circumradius of a triangle. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Pavlos Maragoudakis, Pireas, Greece; and D.J. Smeenk, Zaltbommel, the Netherlands. We first give the write-up of Amengual Covas.

Let s and r be the semiperimeter and the inradius of the given triangle. We have

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a+b+c}{abc} = \frac{2s}{4Rrs} = \frac{1}{2Rr} \geq \frac{1}{R^2},$$

418

since $R \geq 2r$ (Euler's Inequality).

Equality occurs only if and only if the triangle is equilateral.

We next give the solution by Malikić.

Let s, r, and P be the semiperimeter, the inradius, and the area of the triangle. It is well known that

$$\begin{aligned} \frac{1}{R^2} &= \left(\frac{4P}{abc}\right)^2 = \frac{16P^2}{(abc)^2} \\ &= \frac{16(s(s-a)(s-b)(s-c))}{(abc)^2} \\ &= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{(abc)^2}, \end{aligned}$$

so it is enough to prove

$$\begin{aligned} \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} &= \frac{a+b+c}{abc} \\ &\geq \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{(abc)^2}, \end{aligned}$$

which is equivalent to

$$(a+b-c)(c+a-b)(b+c-a) \leq abc.$$

There are several ways to prove this last inequality.

One may, for instance, expand the left-hand side and reduce the inequality to Schur's Inequality. We give another proof, where we use the fact that $a+b-c > 0$, $b+c-a > 0$, $c+a-b > 0$, and $\sqrt{ab} \leq \frac{a+b}{2}$:

$$\begin{aligned} &(a+b-c)(b+c-a)(a+c-b) \\ &= \prod_{\text{cyclic}} \sqrt{(a+b-c)(b+c-a)} \\ &= \prod_{\text{cyclic}} \frac{(a+b-c)+(b+c-a)}{2} = \prod_{\text{cyclic}} b = abc. \end{aligned}$$

Equality holds if and only if $a = b = c$, in which case the triangle is equilateral.

Next we look at readers' solutions to problems proposed, but not used, at the 2004 IMO in Athens given at [2007: 471–475].

A3. Let a, b, $c > 0$ and $ab + bc + ca = 1$. Prove the inequality

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

From the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$ and the inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$, we get

$$(x + y + z)^2 \geq 3(xy + yz + zx).$$

Setting $x = ab$, $y = bc$ and $c = ca$ into the preceding inequality yields

$$1 = (ab + bc + ca)^2 \geq 3(ab^2c + bc^2a + ca^2b) = 3abc(a + b + c)$$

from which we have

$$3(a + b + c) \leq \frac{1}{abc}. \tag{1}$$

On the other hand, applying the AM–GM Inequality, we have

$$1 = ab + bc + ca \geq 3\sqrt[3]{(abc)^2},$$

hence,

$$(abc)^2 \leq \frac{1}{27}. \tag{2}$$

Using power mean inequalities, and taking into account that $ab + bc + ca = 1$, (1), and (2), we obtain

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 6(a + b + c)}$$

$$= \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{abc} + 6(a + b + c)} \leq \frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{3}{abc}} = 3\sqrt[3]{\frac{1}{abc}} \leq \frac{1}{abc}.$$

Equality holds when $a = b = c = \frac{1}{3}$.

N2. The function ψ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$\psi(n) = \sum_{k=1}^{n}(k, n), \qquad \text{for } n \in \mathbb{N},$$

where (k, n) denotes the greatest common divisor of k and n.

(a) Prove that $\psi(mn) = \psi(m)\psi(n)$ whenever m, $n \in \mathbb{N}$ are relatively prime.

(b) Prove that, for each $a \in \mathbb{N}$, the equation $\psi(x) = ax$ has a solution.

(c) Find all $a \in \mathbb{N}$ such that the equation $\psi(x) = ax$ has a unique solution.

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

First we prove a lemma.

Lemma 1. $\psi(n) = \sum_{d|n} \phi(\frac{n}{d})d$ for $n \in \mathbb{N}$, where the sum is over the positive divisors d of n and ϕ is the Euler Totient Function (that is, $\phi(m)$ is the number of positive integers relatively prime to m and not exceeding m).

Proof. Since each term of $\psi(n) = \sum_{k=1}^{n} (k, n)$ is a positive divisor of n,

$$\sum_{k=1}^{n}(k, n) = \sum_{d|n} |\{1 \le k \le n : (k, n) = d\}| \cdot d = \sum_{d|n} \phi\left(\frac{n}{d}\right) d$$

and the lemma is proved. ∎

Let $m, n \in \mathbb{N}$ be relatively prime, then by the Lemma,

$$
\begin{aligned}
\psi(mn) &= \sum_{d|mn} \phi\left(\frac{mn}{d}\right)d = \sum_{e|m}\sum_{f|n} \phi\left(\frac{mn}{ef}\right)de \\
&= \sum_{e|m}\sum_{f|n} \phi\left(\frac{m}{e}\right)e \cdot \phi\left(\frac{n}{f}\right)f \\
&= \sum_{e|m} \phi\left(\frac{m}{e}\right)e \sum_{f|n} \phi\left(\frac{n}{f}\right)f \\
&= \psi(m)\psi(n)\,.
\end{aligned}
$$

This proves the statement in part (a).

Lemma 2. Let p be a prime number and $\alpha \ge 0$ be an integer. Then

$$\psi(p^\alpha) = (\alpha + 1)p^\alpha - \alpha p^{\alpha-1} = \left(\frac{p-1}{p}\alpha + 1\right)p^\alpha$$

Proof. By Lemma 1, we have

$$
\begin{aligned}
\psi(p^\alpha) &= \sum_{d|p^\alpha} \phi\left(\frac{p^\alpha}{d}\right)d = \sum_{k=0}^{\alpha} \phi(p^{\alpha-k})p^k \\
&= \sum_{k=0}^{\alpha-1} \phi(p^{\alpha-k})p^k + p^\alpha = \sum_{k=0}^{\alpha-1} (p^{\alpha-k} - p^{\alpha-k-1})p^k + p^\alpha \\
&= \sum_{k=0}^{\alpha-1} (p^\alpha - p^{\alpha-1}) + p^\alpha = \alpha(p^\alpha - p^{\alpha-1}) + p^\alpha \\
&= (\alpha + 1)p^\alpha - \alpha p^{\alpha-1} = \left(\frac{p-1}{p}\alpha + 1\right)p^\alpha\,.
\end{aligned}
$$

and the proof is complete. ∎

Corollary. Let $n \in \mathbb{N}$ be odd, then $\psi(n)$ is odd.

Proof. In view of part (a), it suffices to prove that if p is an odd prime number and $\alpha \geq 0$ is an integer, then $\psi(p^\alpha)$ is odd. Clearly $\psi(p^0) = 1$ is odd. If $\alpha \geq 1$, then by Lemma 2, $\psi(p^\alpha) = (\alpha + 1)p^\alpha - \alpha p^{\alpha-1}$ is the difference of two integers of opposite parity, and therefore it is odd. ∎

For any integer $\alpha \geq 0$, we have by Lemma 2

$$\psi(2^\alpha) \; = \; \left(\frac{\alpha}{2} + 1 \right) 2^\alpha \,. \tag{1}$$

For each $a \in \mathbb{N}$ take $\alpha = 2(a-1)$ and $x = 2^\alpha$, then the above equation yields $\psi(x) = ax$. This proves the statement in part (b).

To settle part (c), we will prove that the only integers $a \in \mathbb{N}$ such that $\psi(x) = ax$ has a unique solution are the non-negative powers of 2. For any integer $\alpha \geq 0$, we have by Lemma 2

$$\psi(3^\alpha) \; = \; \left(\frac{2\alpha}{3} + 1 \right) 3^\alpha \,. \tag{2}$$

Let $a \in \mathbb{N}$ be of the form $2^m n$, where $m \geq 0$ is an integer and $n > 1$ is an odd number. By equation (1), the number $x = 2^{2(2^m-1)}$ is a solution to $\psi(x) = 2^m x$ and by equation (2), the number $x = 3^{\frac{3}{2}(n-1)}$ is a solution to $\psi(x) = nx$. Since $2^{2(2^m-1)}$ and $3^{\frac{3}{2}(n-1)}$ are relatively prime, it follows from part (a) that $x = 2^{2(2^m-1)} 3^{\frac{3}{2}(n-1)}$ is a solution to $\psi(x) = ax$. However, by equation (1), the number $x = 2^{2(a-1)}$ is also a solution to $\psi(x) = ax$ that is different from the solution just obtained.

Finally, let $a = 2^k$, where $k \geq 0$ is an integer, and let $x = 2^m n$ be a solution to $\psi(x) = ax$, where $m \geq 0$ is an integer and $n \in \mathbb{N}$ is odd. By part (a) and the equation in (1), we have $\psi(x) = (\frac{m}{2}+1)2^m \psi(n) = 2^{k+m}n$, hence

$$(m + 2)\psi(n) \; = \; 2^{k+1}n \,. \tag{3}$$

Now 2^{k+1} divides the left-hand side of (3), so by the Corollary, it divides $m + 2$ and $m + 2 \geq 2^{k+1}$. If $n > 1$, then $\psi(n) \geq n + 1 > 1$, which implies that the left-hand side of (3) is greater than the right-hand side of (3), a contradiction! Hence, $n = 1$ and by (3) we have $m = 2(2^k - 1)$.

That completes the material for this number of the *Corner*. There are lots of opportunities to send in your nice solutions and generalizations to problems.

BOOK REVIEWS

John Grant McLoughlin

Julian Havil, a master at Winchester College in England, is a type of mathematics teacher that is all too rare, a well-read connoisseur of his subject who is eager to explore its byways and share his discoveries with his pupils and the public at large. (Another such was F.J. Budden, whose fine book *Fascination of Groups*, published by Cambridge, dates back to 1972.) This is Havil's third book, after *Nonplussed! Mathematical Proof of Implausible Ideas* (2007) and *Gamma: Exploring Euler's Constant* (2003), both published by Princeton. There is a lot to savour, and Havil has both the understanding and technical proficiency to convey it to his readers.

This book cannot be read casually. It is especially suitable for mathematics undergraduates, but only for the most capable secondary students and teachers. Professional mathematicians will be familiar with many of the topics, but will undoubtedly find something they have not yet encountered, particularly the puzzles that have recently come onstream.

While most of the topics date from the last century, he does reach back in time for some of the discussion. The infinitely long "trumpet" of Torricelli, with its infinite volume and finite surface area provides the occasion for a discussion of the divergence of the harmonic series and the convergence and evaluation by Euler of the sum of the series of square reciprocals. The study of the length of a run of heads in a sequence of coin tosses includes an account of the work of DeMoivre, an early investigator of the problem. A discussion of the frequency of poker hands is extended to include the effect of wild cards.

There are three enjoyable chapters for those with an affinity for numbers, especially large ones. A treatment of the occurrence of sequences of digits in powers of **2** leads smoothly into Benford's Law, to wit that in a "natural" set of numerical data, the numbers start with the smaller digits more frequently than the larger. The chapter titled Goodstein Sequences packs a surprise. We start the sequence with a number written in the *complete base* **2** *representation*; we start with the standard base **2** representation and (recursively) write all exponents that occur in base **2**. Thus,

$$2136 \;=\; 2^{2^{2+1}+2+1} \;+\; 2^{2^2+2} \;+\; 2^{2^2} \;+\; 2^{2+1}\,.$$

The Goodstein sequence starting with n is defined initially for $r = 2$ by $G_2(n) = n$ (written in complete base **2**). For $r \geq 3$, $G_r(n)$ is obtained by taking $G_{r-1}(n)$, a number written in complete base $r - 1$, changing all the

occurrences of $r - 1$ to r, subtracting 1 and then adjusting the result to get a complete base r representation. Thus,

$$G_3(2136) \ = \ 3^{3^{3+1}+3+1} + 3^{3^3+3} + 3^{3^3} + 2 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 2.$$

With this continual bumping up of the base, one might expect the terms of the sequence to keep increasing rapidly, which indeed they do to begin with. But eventually, they crash to 0 in finite time. This result turns out to be unprovable in ordinary arithmetic, and, indeed, Goodstein established it in 1944 by appealing to the well-ordering of the transfinite ordinals.

Geometry has a small but interesting niche. A brief look at Euclid's axioms leads to a discussion of Cantor cardinality and situations where the whole need not be greater than the part, such as the equipollence of the closed unit interval and n—dimensional Euclidean space. A chapter treats in detail the solution of the Kakeya problem where a needle can be reversed in direction within an arbitrarily small area. The particularly fine last chapter conveys the essence of the argument behind the Banach-Tarski Paradox.

Other paradoxes make an appearance. Within a discussion of complex numbers, we learn about the reality of i^i and determine logarithms of negative numbers. Simpson's paradox is analyzed, although the insight offered by the observation that a mediant of two positive rationals (obtained by adding the numerators and the denominators) depends on their particular representations and can lie anywhere in the interval between them, would have helped. Probably less familiar to the reader would be Braess' paradox that the construction of an additional road in a town may actually worsen the congestion.

Problems of recent notoriety are treated in detail: car and goats (Monty Hall); placement of coloured hats; interactions between two individuals to identify a number pair; and the chance of getting an elevator in your direction. These raise delicate issues in logic, probability and coding theory that are competently handled. The reader will want to check out two card tricks that exemplify the Kruskal and Gilbreath principles, the latter based on the structure preserved in a deck of cards by an imperfect riffle shuffle.

The author analyzes the phenomena in some depth without being tedious. The book is lightened by anecdotes and historical asides; it is well referenced, so that the reader can go to the literature. An interesting feature of the book is the set of illustrations that accompany each of the eighteen chapter headings. There are seventeen different symmetry groups for wallpaper patterns; the basic pattern is presented with Chapter 1 and each of the remaining chapters gives a fundamental region for one of these groups.

While the author dedicates the book to Martin Gardner, the treatment is more overtly mathematical than what Gardner would provide. A better comparator is Sherman Stein's *How the Other Half Thinks: Adventures in Mathematical Reasoning* (McGraw-Hill, 2001) or Ross Honsberger's essays. In any case, this book makes a pleasant and absorbing read.

Twin Problems on Non-Periodic Functions

Eugen J. Ionascu

1   Introduction

Two very similar problems were proposed in the *American Mathematical Monthly* by P.P. Dalyay.

Problem 11111. ([3]) Let f and g be nonconstant, continuous periodic functions mapping $\mathbb{R}$ into $\mathbb{R}$. Is it possible that the function h on $\mathbb{R}$ given by $h(x) = f\big(xg(x)\big)$ is periodic?

Problem 11174. ([4]) Let f and g be nonconstant, continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ and satisfying the following conditions:

1. f is periodic.

2. There is a sequence $\langle x_n \rangle_{n \geq 1}$ such that $\lim\limits_{n \to \infty} x_n = \infty$ and $\lim\limits_{n \to \infty} \left| \dfrac{g(x_n)}{x_n} \right| = \infty$.

3. $f \circ g$ is not constant on $\mathbb{R}$.

Determine whether $h = f \circ g$ can be periodic.

The solutions to Problem 11111 and Problem 11174 appeared in [5] and [8]. In this note we are going to consider a new question which is similar to but more general than each of these problems. The proofs here are based on a particular case of the well-known Stolz-Cesàro Lemma and on the fact that a continuous periodic function on $\mathbb{R}$ is uniformly continuous. The use of the latter idea is not new as it was used in the published solutions of these problems. On the other hand, the use of the Stolz-Cesàro Lemma is a good example of where an old tool from analysis appears unexpectedly (see [1], [6], [7], and [10]). L'Hospital's rule, which is well known to calculus students, is its "differentiable" counterpart.

For each version of L'Hospital's rule, there is an analogous version of the Stolz-Cesàro Lemma. For example, one version of L'Hospital's Rule states that if f and g are two differentiable functions on (a, ∞) such that $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) = \infty$ and $\lim\limits_{x \to \infty} \dfrac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = L$.

The Stolz-Cesàro analog of this version of L'Hospital's Rule is: For two sequences $\langle x_n \rangle_{n \geq 1}$ and $\langle y_n \rangle_{n \geq 1}$, if $\lim\limits_{n \to \infty} x_n = \infty$ and $\lim\limits_{n \to \infty} \dfrac{y_{n+1} - y_n}{x_{n+1} - x_n} = L$, then $\lim\limits_{n \to \infty} \dfrac{y_n}{x_n} = L$

Indeed, one may think of the derivative $y'(a)$ as a special quotient $y'(a) = \frac{y(a + \delta) - y(a)}{\delta}$, where δ takes its smallest possible *infinitesimal* value. In the discrete case we write $y_n = y(n)$ for $n \in \mathbb{N}$, and we let δ take its smallest value, namely $\delta = 1$, to obtain the discrete derivative $y'(n) = \frac{y(n + 1) - y(n)}{1} = y_{n+1} - y_n$. Thus, $\lim\limits_{n \to \infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = L$ indicates (in this interpretation) that a ratio of discrete derivatives tends to L.

The following problem may be solved by applying both the Stolz–Cesàro Lemma and L'Hospital's Rule. We include it for the interested reader.

Problem Let $x_0 \in (0, \pi)$ and define the sequence $\langle x_n \rangle_{n \geq 0}$ by the recursion $x_{n+1} = \sin x_n$, $n \geq 0$. Show that $\lim\limits_{n \to \infty} x_n \sqrt{n} = \sqrt{3}$.

Solution The required limit is equivalent to $\lim\limits_{n \to \infty} \frac{1/x_n^2}{n} = \frac{1}{3}$. By the Stolz–Cesàro Lemma, it suffices to show that

$$\lim_{n \to \infty} \frac{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}{(n + 1) - n} = \frac{1}{3},$$

or equivalently

$$\lim_{n \to \infty} \left(\frac{x_n^2 - \sin^2 x_n}{x_n^2 \sin^2 x_n} \right) = \frac{1}{3}.$$

Since $0 < \sin x < x$ whenever $x \in (0, \pi)$, the sequence $\langle x_n \rangle$ is decreasing and bounded below by 0 and so it converges to a limit $\ell \in [0, \pi)$. This limit ℓ satisfies $\sin \ell = \ell$, hence $\ell = 0$.

Thus, it suffices to show that $\lim\limits_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{1}{3}$, which can be done by applying L'Hospital's Rule:

$$
\begin{aligned}
\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} &= \lim_{x \to 0} \frac{(x - \sin x)(x + \sin x)}{x^2 \sin^2 x} \\
&= \lim_{x \to 0} \left(\frac{x - \sin x}{x^3} \right) \lim_{x \to 0} \left(1 + \frac{\sin x}{x} \right) \lim_{x \to 0} \left(\frac{x}{\sin x} \right)^2 \\
&= \left(\lim_{x \to 0} \frac{1 - \cos x}{3x^2} \right) \cdot (1 + 1) \cdot 1^2 \\
&= 2 \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{3}.
\end{aligned}
$$

We will use a variant of the Stolz–Cesàro Lemma to prove our main theorem, where we weaken the conditions in Problem 11174 but obtain the same conclusion.

Theorem 1 Let f and g be nonconstant, continuous functions from $\mathbb{R}$ into $\mathbb{R}$ that satisfy the following conditions:

(i) The function f is periodic.

(ii) There exist sequences $\langle x_n \rangle_{n \geq 1}$ and $\langle y_n \rangle_{n \geq 1}$ such that

$$\inf_n |x_n - y_n| > 0 \qquad \text{and} \qquad \lim_{n \to \infty} \left| \frac{g(x_n) - g(y_n)}{x_n - y_n} \right| = \infty \,.$$

Then the function $h = f \circ g$ is not periodic.

2 Some Facts from Real Analysis

The variant of the Stolz–Cesàro Lemma that we will use is stated next.

Lemma 1 Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences such that $\langle b_n \rangle$ is increasing and $\lim\limits_{n \to \infty} b_n = \infty$. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \infty$, then $\limsup\limits_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \infty$.

For completeness we include a proof of Lemma 1 along classical lines. Assume to the contrary that $\gamma = \limsup\limits_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \gamma + 1 \,,$$

or equivalently

$$a_{n+1} - a_n \leq (b_{n+1} - b_n)(\gamma + 1) \,, \tag{1}$$

for $n \geq n_0$. Adding up the inequalities in (1) for $n = k, \, k + 1, \, \ldots, \, l$, where $n_0 \leq k < l$, we obtain

$$a_{l+1} - a_k \leq (b_{l+1} - b_k)(\gamma + 1) \,.$$

For sufficiently large l the term b_{l+1} is positive and we may divide the last inequality by b_{l+1} and then let $l \to \infty$. Using the hypothesis we then obtain $\infty \leq \gamma + 1$, a contradiction.

We now recall that a (real- or complex-valued) function with domain $\mathcal{D} \subset \mathbb{R}$ is *uniformly continuous* on $\mathcal{D}$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, \, y \in \mathcal{D}$ and $|x - y| < \delta$. The following basic fact about continuous functions on a closed interval is all we need, though it can be generalized considerably (see [9], Theorem 4.19).

Theorem 2 Every continuous, real-valued function whose domain is a closed interval $\mathcal{D} = [a, b]$ is uniformly continuous on $\mathcal{D}$.

As an easy consequence of this theorem, we have

Corollary 1. Every continuous, periodic function f on $\mathbb{R}$ is uniformly continuous.

This can be seen by applying Theorem 2 to the restriction of f to the closed interval $[0, 2T]$, where $T > 0$ is a period of f. That is, given $\epsilon > 0$, by Theorem 2 there is a δ such $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [0, 2T]$ and $|x - y| < \delta$. Thus, if $x, y \in \mathbb{R}$ and $|x - y| < \delta$, then there exist integers n and m such that both $x_1 = x - nT$ and $y_1 = y - mT$ are in $[0, 2T]$ and $|x_1 - y_1| < \delta$, so that $|f(x) - f(y)| = |f(x_1) - f(y_1)| < \epsilon$.

The idea of our proofs is to show that the function h is not uniformly continuous. It then follows from Corollary 1 that h is not periodic.

Let us see how Problem 11111 can be solved using Theorem 1. Since g is not constant there exist a and b such that $g(a) - g(b) \neq 0$. Let $T > 0$ be a period of g, and define the sequences $\langle x_n \rangle_{n \geq 1}$ and $\langle y_n \rangle_{n \geq 1}$ by $x_n = a + nT$ and $y_n = b + nT$. Then $|x_n - y_n| = |a - b| > 0$ and

$$\lim_{n \to \infty} \frac{|x_n g(x_n) - y_n g(y_n)|}{|x_n - y_n|}$$

$$= |a - b|^{-1} \lim_{n \to \infty} \left| a g(a) - b g(b) + nT(g(a) - g(b)) \right| = \infty,$$

which says that f and the function g_1 on $\mathbb{R}$ given by $g_1(x) = xg(x)$ both satisfy the conditions (i) and (ii) in Theorem 1, hence, $h = f \circ g_1$ is not periodic. We have solved Problem 11111.

To show that Problem 11174 can be solved using Theorem 1, we need the weaker version of the Stolz-Cesàro Lemma given in Lemma 1.

Let us assume that f, g, and $\langle x_n \rangle_{n \geq 1}$ satisfy the three conditions in Problem 11174. There is a subsequence $\langle x_{n_k} \rangle_{k \geq 1}$ of $\langle x_n \rangle_{n \geq 1}$, such that $x_{n_{k+1}} - x_{n_k} \geq 1$ for all k, and for which either $\lim_{k \to \infty} \frac{g(x_{n_k})}{x_{n_k}} = \infty$ or $\lim_{k \to \infty} \frac{g(x_{n_k})}{x_{n_k}} = -\infty$. Without loss of generality we may suppose the former, because the latter case follows from the former case by replacing g with $-g$ and f with $f_1(x) = f(-x)$, $x \in \mathbb{R}$. By Lemma 1 we have

$$\limsup_{k \to \infty} \frac{g(x_{n_{k+1}}) - g(x_{n_k})}{x_{n_{k+1}} - x_{n_k}} = \infty,$$

which proves the existence of the two sequences in the hypothesis (ii) of Theorem 1. Hence, Theorem 1 can be applied to f and g and we deduce that $h = f \circ g$ is not periodic. We have solved Problem 11174.

3 Proof of Theorem 1

Let f and g satisfy the hypotheses of Theorem 1. Since g is continuous and satisfies condition (ii), the interval $I_n = g([x_n, y_n])$ (or $I_n = g([y_n, x_n])$)

has length greater than the period T of f for sufficiently large n. Hence, f and $h = f \circ g$ have the same range. Since f is not constant, h is not constant. Therefore, there exist α and β such that $f(g(\alpha)) \neq f(g(\beta))$, and we let $\epsilon_0 = |f(g(\alpha)) - f(g(\beta))| > 0$. As we said in the introduction, the key idea is to prove that h is not uniformly continuous. In fact, we will show that the definition of uniform continuity is not satisfied for this ϵ_0.

We fix $n \in \mathbb{N}$ large enough so that $|g(x_n) - g(y_n)| > 2T$, and we denote by $\sharp(g(\alpha))$ the number of integers k for which $g(\alpha) + kT$ is in I_n. It is then easy to see that

$$\sharp(g(\alpha)) \; > \; \frac{|g(x_n) - g(y_n)|}{T} - 1 \; > \; 1 \,.$$

Similarly, we denote by $\sharp(g(\beta))$ the number of integers k for which $g(\beta) + kT$ is in I_n. Similarly, we have $\sharp(g(\beta)) > 1$.

It is clear that the values $g(\alpha) + kT$, $k \in \mathbb{Z}$, interlace with those of $g(\beta) + kT$, $k \in \mathbb{Z}$. Using again the fact that g is continuous and by repeated application of the Intermediate Value Theorem, we can find two finite sequences $\langle u_k \rangle$ and $\langle v_k \rangle$ in the interval $[x_n, y_n]$ (or $[y_n, x_n]$) both increasing and interlacing and such that $g(u_k) = g(\alpha) + l_k T$ and $g(b_k) = g(\beta) + s_k T$, with l_k, $s_k \in \mathbb{Z}$. The number of intervals of the form $[u_k, v_k)$ (or $[v_k, u_k)$) is at least

$$M \; = \; \min \left\{ 2\big(\sharp((g(\alpha)) - 1\big), 2\big(\sharp((g(\beta)) - 1\big) \right\} \; \geq \; 2 \,.$$

These intervals form a partition of a subinterval of $J_n = [x_n, y_n]$ (or of $J_n = [y_n, x_n]$) of length $|x_n - y_n|$. Then one of these intervals has length at most $\frac{|x_n - y_n|}{M}$. We denote such an interval by $[\zeta_n, \eta_n]$ and notice that

$$|\zeta_n - \eta_n| \; \leq \; \frac{|x_n - y_n|}{M} \; < \; \frac{|x_n - y_n|}{2\left(\dfrac{|g(x_n) - g(y_n)|}{T}\right) - 4}$$

$$= \; \frac{1}{\dfrac{2}{T}\left(\dfrac{|g(x_n) - g(y_n)|}{|x_n - y_n|}\right) - \dfrac{4}{|x_n - y_n|}} \; \to \; 0 \quad \text{as} \quad n \to \infty, \qquad (2)$$

and $|f(g(\zeta_n)) - f(g(\eta_n))| = \epsilon_0$. Given $\delta > 0$, we may choose n so large that $|\zeta_n - \eta_n| < \delta$. This can be done because of (2). For such an n we still have $|h(\zeta_n) - h(\eta_n)| \geq \epsilon_0$, which proves that h is not uniformly continuous.

4 Conclusion

We would like to leave the reader with a natural question: Can Theorem 1 be generalized to almost periodic functions? There are various concepts of almost periodicity, but here we will only give Bohr's definition:

A continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *almost periodic* if for all $\epsilon > 0$, there is an $L > 0$ such that every interval of length L contains an ϵ-period, that is, a number T such that $|F(x+T) - F(x)| < \epsilon$ for all $x \in \mathbb{R}$.

What is interesting and related to the question above is the fact that every almost periodic function is uniformly continuous (see [2]).

Acknowledgment

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PROBLEMS

3376. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The vertices of quadrilateral $ABCD$ lie on a circle. Let K, L, M, and N be the incentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$, respectively. Show that quadrilateral $KLMN$ is a rectangle.

3377. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle B = 2\angle C$. The interior bisector of $\angle BAC$ meets BC at D. Let M and N be the mid-points of AC and BD, respectively. Suppose that A, M, D, and N are concyclic. Prove that $\angle BAC = 72°$.

3378. *Proposed by Mihály Bencze, Brasov, Romania.*

Let x, y, and z be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{xy}{xy + x + y} \leq \sum_{\text{cyclic}} \frac{x}{2x + z}\,.$$

3379. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a_1, a_2, ..., a_n be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{a_i + (n-1)a_{i+1}} \geq 1\,,$$

where the subscripts are taken modulo n.

3380. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, x, y, and z be real numbers. Show that

$$\frac{(a^2 + x^2)(b^2 + y^2)}{(c^2 + z^2)(a - b)^2} + \frac{(b^2 + y^2)(c^2 + z^2)}{(a^2 + x^2)(b - c)^2} + \frac{(c^2 + z^2)(a^2 + x^2)}{(b^2 + y^2)(c - a)^2}$$
$$\geq \frac{a^2 + x^2}{|(a - b)(a - c)|} + \frac{b^2 + y^2}{|(b - a)(b - c)|} + \frac{c^2 + z^2}{|(c - a)(c - b)|}\,.$$

3381★. *Proposed by Shi Changwei, Xi`an City, Shaan Xi Province, China.*

Let n be a positive integer. Prove that

(a) $\left(1 - \dfrac{1}{6}\right)\left(1 - \dfrac{1}{6^2}\right)\cdots\left(1 - \dfrac{1}{6^n}\right) > \dfrac{4}{5}$;

(b) Let $a_n = a_1 q^n$, where $0 < a_1 < 1$ and $0 < q < 1$. Evaluate

$$\lim_{n\to\infty}\prod_{i=1}^{n}(1 - a_i).$$

3382. *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\sin\left(\frac{P_{n+2}}{4P_n P_{n+1}}\right) + \cos\left(\frac{P_{n+2}}{4P_n P_{n+1}}\right) < \frac{3}{2}\sec\left(\frac{3P_n + 2P_{n-1}}{4P_n P_{n+1}}\right),$$

where P_n is the nth Pell number, which is defined by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$.

3383. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with $\angle BAC \neq 90°$, let O be its circumcentre and let M be the mid-point of BC. Let P be a point on the ray MA such that $\angle BPC = 180° - \angle BAC$. Let BP meet AC at E and let CP meet AB and F. If D is the projection of the mid-point of EF onto BC, show that

(a) AD is a symmedian of $\triangle ABC$;

(b) O, P, and the orthocentre of $\triangle EDF$ are collinear.

3384. *Proposed by Michel Bataille, Rouen, France.*

Show that, for any real number x,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n-1} k\cdot\left\lfloor x + \frac{n-k-1}{n}\right\rfloor = \frac{\lfloor x\rfloor + \{x\}^2}{2},$$

where $\lfloor a\rfloor$ is the integer part of the real number a and $\{a\} = a - \lfloor a\rfloor$.

3385. *Proposed by Michel Bataille, Rouen, France.*

Let $p_1, p_2, \ldots, p_6$ be primes with $p_{k+1} = 2p_k + 1$ for $k = 1, 2, \ldots, 5$. Show that

$$\sum_{1\leq i < j\leq 6} p_i p_j$$

is divisible by 15.

3386. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Evaluate the integral

$$\int_0^\infty e^{-x} \left(\int_0^x \frac{e^{-t} - 1}{t} \right) \ln x \, dx \,.$$

3387. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH, USA.*

Let $k > l \geq 0$ be fixed integers. Find

$$\lim_{x \to \infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)} \right) \,,$$

where ζ is the Riemann Zeta function.

3388. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA, in memory of Murray S. Klamkin.*

For all real $x \geq 1$, show that

$$\frac{1}{2}\sqrt{x-1} + \frac{(x-1)^2}{\sqrt{x-1} + \sqrt{x+1}} < \frac{x^2}{\sqrt{x} + \sqrt{x+2}} \,.$$

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3376. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Les sommets d'un quadrilatère $ABCD$ sont situés sur un cercle. Soit respectivement K, L, M et N les centres des cercles inscrits des triangles ABC, BCD, CDA et DAB. Montrer que le quadrilatère $KLMN$ est un rectangle.

3377. *Proposé par Toshio Seimiya, Kawasaki, Japon.*

Soit ABC un triangle avec $\angle B = 2\angle C$. La bissectrice intérieure de l'angle BAC coupe BC en D. Soit M et N les points milieux respectifs de AC et BD. Supposons que A, M, D et N soient cocycliques. Montrer que l'angle $BAC = 72°$.

3378. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit x, y et z trois nombres réels positifs. Montrer que

$$\sum_{\text{cyclique}} \frac{xy}{xy + x + y} \leq \sum_{\text{cyclique}} \frac{x}{2x + z} \,.$$

3379. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $a_1, a_2, \ldots, a_n$ nombres réels positifs. Montrer que

$$\sum_{i=1}^{n} \frac{a_i}{a_i + (n-1)a_{i+1}} \geq 1,$$

où les indices sont calculés modulo n.

3380. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, b, c, x, y et z six nombres réels positifs. Montrer que

$$\frac{(a^2 + x^2)(b^2 + y^2)}{(c^2 + z^2)(a - b)^2} + \frac{(b^2 + y^2)(c^2 + z^2)}{(a^2 + x^2)(b - c)^2} + \frac{(c^2 + z^2)(a^2 + x^2)}{(b^2 + y^2)(c - a)^2}$$
$$\geq \frac{a^2 + x^2}{|(a-b)(a-c)|} + \frac{b^2 + y^2}{|(b-a)(b-c)|} + \frac{c^2 + z^2}{|(c-a)(c-b)|}.$$

3381★. *Proposé par Shi Changwei, Xi`an City, Province de Shaan Xi, Chine.*

Soit n un entier positif. Montrer que

(a) $\left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{6^2}\right)\cdots\left(1 - \frac{1}{6^n}\right) > \frac{4}{5}$;

(b) Soit $a_n = a_1 q^n$, où $0 < a_1 < 1$ et $0 < q < 1$. Trouver la valeur de

$$\lim_{n \to \infty} \prod_{i=1}^{n}(1 - a_i).$$

3382. *Proposé par José Luis Díaz-Barrero et Josep Rubió-Massegú, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit n un entier positif. Montrer que

$$\sin\left(\frac{P_{n+2}}{4P_n P_{n+1}}\right) + \cos\left(\frac{P_{n+2}}{4P_n P_{n+1}}\right) < \frac{3}{2}\sec\left(\frac{3P_n + 2P_{n-1}}{4P_n P_{n+1}}\right),$$

où P_n est le n-ième nombre de Pell, défini par $P_0 = 0$, $P_1 = 1$ et $P_n = 2P_{n-1} + P_{n-2}$ pour $n \geq 2$.

3383. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle, d'angle en A différent de $90°$, O le centre de son cercle circonscrit et M le point milieu de BC. Soit P un point sur le rayon MA tel que l'angle en P du triangle BPC soit le supplémentaire de celui en A. Soit E l'intersection de BP avec AC et F celle de CP avec AB. Si D est la projection sur BC du point milieu de EF, montrer que

(a) AD est une symédiane du triangle ABC;

(b) O, P et l'orthocentre du triangle EDF sont colinéaires.

3384. *Proposé par Michel Bataille, Rouen, France.*

Pour un nombre réel quelconque x, montrer que

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k \cdot \left\lfloor x + \frac{n-k-1}{n} \right\rfloor = \frac{\lfloor x \rfloor + \{x\}^2}{2},$$

où $\lfloor a \rfloor$ est la partie entière du nombre réel a et $\{a\} = a - \lfloor a \rfloor$.

3385. *Proposé par Michel Bataille, Rouen, France.*

Soit p_1, p_2, ..., p_6 nombres premiers tels que $p_{k+1} = 2p_k + 1$ avec $k = 1, 2, \ldots, 5$. Montrer que

$$\sum_{1 \le i < j \le 6} p_i p_j$$

est divisible par 15.

3386. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Évaluer l'intégrale

$$\int_0^\infty e^{-x} \left(\int_0^x \frac{e^{-t} - 1}{t} \right) \ln x \, dx.$$

3387. *Proposé par Ovidiu Furdui, Université de Toledo, Toledo, OH, É-U.*

Pour des entiers donnés k et l, avec $k > l \ge 0$, calculer

$$\lim_{x \to \infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)} \right),$$

où ζ désigne la fonction zéta de Riemann.

3388. *Proposé par Paul Bracken, Université du Texas, Edinburg, TX, USA, à la mémoire de Murray S. Klamkin.*

Pour tous les nombres réels $x \ge 1$, montrer que

$$\frac{1}{2}\sqrt{x-1} + \frac{(x-1)^2}{\sqrt{x-1} + \sqrt{x+1}} < \frac{x^2}{\sqrt{x} + \sqrt{x+2}}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3276. [2007 : 428, 430] *Proposed by Neven Jurič, Zagreb, Croatia.*

A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers satisfies the recurrence relation $a_{n+3} = a_{n+1} + a_n$ for $n \geq 0$. Simplify

$$\sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2} \, .$$

Solution submitted independently by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.

Clearly, all the terms of the given sequence are positive. Set

$$A = \sqrt{a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2 - a_{n+2}^2 + a_{n+1}^2 - a_n^2} \, .$$

By the recurrence relation, $a_{n+3} = a_{n+1} + a_n$, $a_{n+4} = a_{n+2} + a_{n+1}$, and $a_{n+5} = a_{n+3} + a_{n+2} = a_{n+2} + a_{n+1} + a_n$, so that

$$a_{n+5}^2 + a_{n+4}^2 + a_{n+3}^2$$
$$= (a_{n+2} + a_{n+1} + a_n)^2 + (a_{n+2} + a_{n+1})^2 + (a_{n+1} + a_n)^2$$
$$= 2a_{n+2}^2 + 3a_{n+1}^2 + 2a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n \, .$$

Thus,

$$A^2 = a_{n+2}^2 + 4a_{n+1}^2 + a_n^2 + 4a_{n+2}a_{n+1} + 2a_{n+2}a_n + 4a_{n+1}a_n$$
$$= (a_{n+2} + 2a_{n+1} + a_n)^2 \, .$$

Therefore,

$$A = a_{n+2} + 2a_{n+1} + a_n$$
$$= (a_{n+2} + a_{n+1}) + (a_{n+1} + a_n)$$
$$= a_{n+4} + a_{n+3}$$
$$= a_{n+6} \, .$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS,

436

Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; CARL LIBIS, University of Rhode Island, Kingston, RI, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ANDREA MUNARO, student, University of Trento, Trento, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3277. [2007 : 428, 430] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

The Lucas numbers L_n satisfy the recurrence relation $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Let k be an even positive integer. Find

$$\lim_{n \to \infty} \left(\left\{ \sqrt[k]{L_n} \right\} - \left\{ \sqrt[k]{L_{n-k}} + \sqrt[k]{L_{n-2k}} \right\} \right),$$

where $\{x\}$ is the fractional part of x (that is, $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the integer part of x).

Comment: All three submissions claimed that the limit is 0, but no satisfactory argument was provided that the limit exists for every positive even integer k. Problem 3277, therefore, remains open.

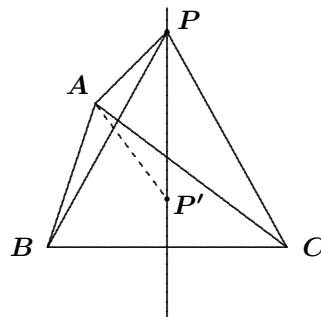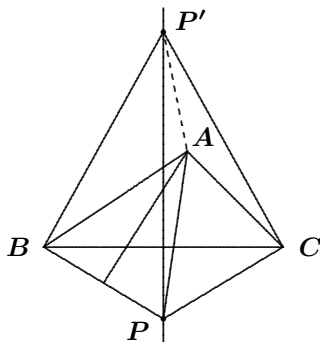**3278**. [2007 : 428, 431] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let P be a point in the plane of $\triangle ABC$ such that $PC = PB$ and $PA = AB$. Let x be the measure of $\angle PBC$. Prove that

$$\sin(B - C) = 2 \sin C \cos(B + 2\varepsilon x),$$

where $\varepsilon = 1$ if the line BC separates the points P and A, and $\varepsilon = -1$ otherwise.

Solution by Michel Bataille, Rouen, France.

Let $AB = c$, $BC = a$, and $CA = b$. First, suppose that BC separates P and A (see the figure on the left).

Then, $\cos x = \dfrac{a/2}{PB}$ and $\cos(x+B) = \cos(\angle PBA) = \dfrac{PB/2}{c}$, so that $2\cos x\cos(x+B) = \dfrac{a}{2c} = \dfrac{\sin A}{2\sin C}$. This yields $\cos(2x+B)+\cos B = \dfrac{\sin A}{2\sin C}$ and, since $\sin A = \sin(B+C) = \sin B\cos C + \sin C\cos B$, we obtain

$$2\sin C\cos(B+2x) \;=\; \sin B\cos C - \sin C\cos B = \sin(B-C)\,,$$

as desired.

If P and A are on the same side of BC, then we have $\cos x = \dfrac{a/2}{PB}$ and $\angle PBA = x - B$ or $\angle PBA = B - x$, depending on the location of point P. [*Ed.*: The cases are (a) P outside $\triangle ABC$ and A inside $\triangle PBC$ (figure on the left, P' replacing P), (b) P outside $\triangle ABC$ and A outside $\triangle PBC$ (figure on the right), and (c) P inside $\triangle ABC$ (figure on the right, P' replacing P).] In any case, $\dfrac{PB/2}{c} = \cos(x-B)$, and, in the same way as above, we obtain $2\cos x\cos(x-B) = \dfrac{a}{2c}$, which leads to $2\sin C\cos(B-2x) = \sin(B-C)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANDREA MUNARO, student, University of Trento, Trento, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3279. [2007 : 428, 431] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let O, I, R, and r be the circumcentre, incentre, circumradius, and inradius of $\triangle ABC$, and let a, b, and c be the lengths of the sides of $\triangle ABC$ opposite the angles A, B, and C, respectively. Let IO meet the lines AB and AC at M and N, respectively. Prove that the points B, C, N, and M are concyclic if and only if $h_a = R + r$ (where h_a is the altitude to the side BC), and, in this case, we also have $\dfrac{1}{MN} = \dfrac{1}{a} + \dfrac{1}{b+c}$.

A composite of similar solutions by Taichi Maekawa, Takatsuki City, Osaka, Japan and D.J. Smeenk, Zaltbommel, the Netherlands.

For an arbitrary triangle ABC let D be the foot of the altitude from A to BC and let P be the foot of the perpendicular from I to AD; thus $PD = r$. Moreover, it is always the case that AI bisects $\angle DAO$; that is, $\angle DAI = \angle IAO$. Because $h_a = AP + PD = AP + r$, we therefore have

$$\begin{aligned} h_a \;=\; R + r \;&\Longleftrightarrow\; AO \;=\; AP \\ &\Longleftrightarrow\; \triangle API \cong \triangle AOI \\ &\Longleftrightarrow\; \angle AOI \;=\; 90^\circ\,. \end{aligned}$$

The first part of the problem therefore reduces to proving that

$$B,\, C,\, N,\text{ and } M \text{ lie on a circle} \;\Longleftrightarrow\; IO \perp AO\,.$$

This is not quite correct, however: the four cyclic points must be distinct to force a nontrivial condition. To that end we will assume the equivalent condition that no two angles of $\triangle ABC$ are equal.

Under this assumption, we have that (because $M \in AB$ and $N \in AC$) B, C, N, and M lie on a circle if and only if $\angle ANM = \angle ABC$, if and only if $\angle ANM$ equals the angle between the chord AC and the tangent to the circumcircle at A (on the side that contains the arc AC opposite B), if and only if MN is parallel to that tangent, if and only if $AO \perp MN$.

Since MN is the same line as IO, the proof of the first part is complete.

For the claim concerning $\frac{1}{MN}$, we assume that B, C, N, and M lie on a circle, in which case $\angle ANM = \angle B$ and $\angle AMN = \angle C$. Let E and F be the feet of the perpendiculars from I to AC and AB, respectively. Then in the right triangles INE and IMF we have

$$NI = \frac{r}{\sin B} \quad \text{and} \quad IM = \frac{r}{\sin C}.$$

Hence, $MN = MI + IN = \dfrac{r}{\sin C} + \dfrac{r}{\sin B}$, and the Law of Sines gives us

$$
\begin{aligned}
MN &= r\left(\frac{2R}{c} + \frac{2R}{b}\right) = 2rR\sin A\left(\frac{b+c}{bc\sin A}\right) \\
&= \frac{ar(b+c)}{2\operatorname{Area}(ABC)} = \frac{ar(b+c)}{r(a+b+c)}.
\end{aligned}
$$

Thus,

$$\frac{1}{MN} = \frac{a+b+c}{a(b+c)} = \frac{1}{a} + \frac{1}{b+c},$$

as claimed.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Both Geupel and the proposer avoid appealing to the Law of Sines to prove the final claim as follows: The triangles ABC and ANM are assumed to be similar. Since the angle bisector AI is common to both triangles, if W is the point where AI meets BC we have

$$\frac{AI}{IW} = \frac{MN}{CB}.$$

*Since $CB = a$ and it is known that $\dfrac{AI}{IW} = \dfrac{b+c}{a+b+c}$ (see, for example, Nathan Altshiller Court, College Geometry, page **75**, Theorem **121**), it follows that*

$$MN = \frac{a(b+c)}{a+b+c},$$

which is the reciprocal of the desired equality.

3280. [2007 : 429, 431] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let O and R be the circumcentre and circumradius, respectively, of $\triangle ABC$. Let E and F be points on AB and AC, respectively, such that O is the mid-point of segment EF. Let A' be the point where the line AO meets the circumcircle Γ of $\triangle ABC$ a second time, and let P be the point on the line EF such that $A'P \perp EF$. Prove that the lines EF, BC, and the tangent line to Γ at A' are concurrent, and that $\angle BPA' = \angle CPA'$.

A composite of solutions by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina and by Andrea Munaro, student, University of Trento, Trento, Italy.

One sees that E is uniquely defined as the point where AB intersects the image of the line AC under the halfturn about O, while F is the intersection of AC with the image of AB under that halfturn. More relevant for us, however, is that because A and A' are interchanged by that halfturn it follows that $A'E \| AF$ and $A'F \| AE$. Because $A'P \perp PF$ (given) and $A'C \perp FC$ (because $A'A$ is a diameter of the circumcircle Γ of $\triangle ABC$), the quadrilateral $A'PFC$ is cyclic, whence the directed angles satisfy $\angle A'PC = \angle A'FC$. The latter angle equals $\angle BAC$ (because their sides are parallel), so that

$$\angle A'PC = \angle A \, .$$

Analogously, $A'BEP$ is cyclic and

$$\angle BPA' = \angle BEA' = \angle A \, .$$

Consequently,

$$\angle BPA' \, = \, \angle A'PC \, = \, \angle A \, ,$$

which is the second claim that we were to prove.

For the concurrency claim, we will prove that EF, BC, and the tangent to Γ at A' all contain the image of P under the inversion defined by Γ. We note that $\angle BOC = 2\angle A$, and we have just seen that $\angle BPC = 2\angle A$; we therefore conclude that the points $B, P, O,$ and C lie on a circle. Inversion in Γ fixes the line EF (because that line passes through the centre O of Γ), it takes line BC to circle BOC (which, we have seen, contains P), and it takes the tangent (to Γ) at A' to the circle on diameter OA' (which contains P because $A'P \perp PO$). We therefore see that the images (under inversion) of the three lines contain P, so that these three lines themselves must concur at the inverse of P, as claimed. [*Editor's comment.* Malikić contributed the nice treatment of angles; it was Munaro's idea to invert the figure.]

Also solved by MICHEL BATAILLE, Rouen, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

3281. $[2007 : 429, 431;$ Correction, 2008: 44, 46$]$ *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let $a_1, a_2, \dots, a_n$ be positive real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k^{\frac{n+1}{2}} \right)^n \leq \prod_{k=1}^{n} \left(\sum_{j=1}^{n} a_j^k \right).$$

I. *Solution by Michel Bataille, Rouen, France.*

Let $A_k = \sum_{j=1}^{n} a_j^k$ and let $P = \prod_{k=1}^{n} A_k$ denote the right side of the given inequality. Then $P^2 = (A_1 A_n)(A_2 A_{n-1}) \cdots (A_n A_1)$.

For each k we have, by the Cauchy-Schwarz Inequality, that

$$A_k A_{n+1-k} = \left(\sum_{j=1}^{n} \left(a_j^{\frac{k}{2}} \right)^2 \right) \left(\sum_{j=1}^{n} \left(a_j^{\frac{n+1-k}{2}} \right)^2 \right)$$

$$\geq \left(\sum_{j=1}^{n} a_j^{\frac{k}{2}} a_j^{\frac{n+1-k}{2}} \right)^2 = \left(\sum_{j=1}^{n} a_j^{\frac{n+1}{2}} \right)^2 .$$

Hence, $P^2 \geq \left(\sum_{j=1}^{n} a_j^{\frac{n+1}{2}} \right)^{2n}$, from which the result follows.

II. *Solution by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Note first that $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$, and let $b_{k\ell} = \sqrt[n]{a_k^\ell}$ for each $1 \leq k, \ell \leq n$. For each k we have $a_k^{\frac{n+1}{2}} = \left(a_k^{1+2+\cdots+n} \right)^{1/n} = b_{k1} b_{k2} \cdots b_{kn}$.

Therefore, by the generalized Hölder Inequality, we have

$$\sum_{k=1}^{n} a_k^{\frac{n+1}{2}} = \sum_{k=1}^{n} b_{k1} b_{k2} \cdots b_{kn}$$

$$\leq \left(\sum_{k=1}^{n} b_{k1}^n \right)^{1/n} \left(\sum_{k=1}^{n} b_{k2}^n \right)^{1/n} \cdots \left(\sum_{k=1}^{n} b_{kn}^n \right)^{1/n}$$

$$= \left(\sum_{k=1}^{n} a_k \right)^{1/n} \left(\sum_{k=1}^{n} a_k^2 \right)^{1/n} \cdots \left(\sum_{k=1}^{n} a_k^n \right)^{1/n}$$

$$= \prod_{k=1}^{n} \left(\sum_{j=1}^{n} a_j^k \right)^{1/n} ,$$

from which the given inequality follows.

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina, DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; TITU ZVONARU, Comănești, Romania; and the proposers.

Wu proved a generalization: $\left(\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{m} a_{ij} \right)^{m} \leq \prod\limits_{j=1}^{m} \left(\sum\limits_{i=1}^{n} a_{ij}^{m} \right)$ *whenever a_{ij} is positive for $1 \leq i \leq n$ and $1 \leq j \leq m$.*

The proposed inequality is the special case when $m = n$ and $a_{ij} = a_i^{j/n}$.

3283. $[2007 : 429, 432]$ *Proposed by M. N. Deshpande, Nagpur, India.*

Of the $n!$ permutations σ of $(1, 2, \ldots, n)$, for how many is σ^3 the identity permutation?

A composite of solutions by Michel Bataille, Rouen, France and Richard I. Hess, Rancho Palos Verdes, CA, USA.

The permutation σ^3 is the identity if and only if σ is itself the identity or a product of disjoint 3-cycles. We will investigate the case $n = 8$ explicitly and then generalize.

Cycle type	Total number of permutations
$(1)(2)(3)\cdots(8)$	1
$(1,2,3)(4)(5)(6)(7)(8)$	$2\binom{8}{3} = 112$
$(1,2,3)(4,5,6)(7)(8)$	$\frac{1}{2}\left(2\binom{8}{3}\right)\left(2\binom{5}{3}\right) = 1120$

Thus, for $n = 8$ the desired number is $1 + 112 + 1120 = 1233$.

In general, to find the number of permutations of n elements into k 3-cycles, we choose the first 3-cycle in $\binom{n}{3}$ ways, the next in $\binom{n-3}{3}$ ways, and so on until the kth 3-cycle, chosen in $\binom{n-3(k-1)}{3}$ ways. The product of these binomial coefficients is

$$\binom{n}{3}\binom{n-3}{3}\cdots\binom{n-3(k-1)}{3} = \frac{n!}{(n-3k)!(3!)^k}.$$

There are two possible cyclic permutations of each triple (such as $(1, 2, 3)$ and $(1, 3, 2)$ for the triple $\{1, 2, 3\}$), which means that we must multiply the above product by 2^k; moreover, these k 3-cycles can be chosen in $k!$ orders, so that we must divide the product by $k!$. We therefore have

Cycle type	Total number of permutations
no 3-cycle	$\dfrac{2^0}{0!}\dfrac{n!}{(n-0)!(3!)^0} = \dfrac{1}{3^0 0!}\dfrac{n!}{(n-0)!} = 1$
1 3-cycle	$\dfrac{2^1}{1!}\dfrac{n!}{(n-3)!(3!)^1} = \dfrac{1}{3^1 1!}\dfrac{n!}{(n-3)!}$
2 3-cycles	$\dfrac{2^2}{2!}\dfrac{n!}{(n-6)!(3!)^2} = \dfrac{1}{3^2 2!}\dfrac{n!}{(n-6)!}$
$\vdots$	$\vdots$
k 3-cycles	$\dfrac{2^k}{k!}\dfrac{n!}{(n-3k)!(3!)^k} = \dfrac{1}{3^k k!}\dfrac{n!}{(n-3k)!}$

Of course, k can be any number from 0 to $\lfloor \frac{n}{3} \rfloor$. It follows that the total number of permutations for which σ^3 is the identity is

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{n!}{3^k k!(n-3k)!} \ .$$

The first eight values are $1, 1, 3, 9, 21, 81, 351, 1233$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and JOEL SCHLOSBERG, Bayside, NY, USA. There were three incorrect submissions.

If we denote by $P(n)$ the number of permutations for which σ^3 is the identity, then alternative expressions obtained by our correspondents are

$$
\begin{aligned}
P(n) &= \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} \frac{(3k)!}{3^k k!} = 1 + \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \frac{n(n-1)\cdots(n-3k+1)}{3^k k!} \\
&= P(n-1) + (n-1)(n-2)P(n-3) \ .
\end{aligned}
$$

Using an argument analogous to our featured solution, Geupel proved that for any positive integer n and any prime p, the number $P(n, p)$ of permutations σ of $(1, 2, \ldots, n)$ such that σ^p is the identity is

$$P(n, p) = \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \binom{n}{kp} \frac{(kp)!}{p^k k!} \ .$$

For a discussion of the case $p = 2$ he refers to Arthur Engel, Problem-solving Strategies *(Springer, 1998), Chapter 5, Problem 15, pages 101 and 107.*

Our sequence $P(n)$ can be found in the on-line encyclopedia of integer sequences: http://www.research.att.com/~njas/sequences/ *(enter $1, 1, 3, 9, 21, 81, 351$). That web page provides several references, one of which gives (without proof) the formula of Chowla, Herstein, and Scott (from 1952) for the number $P(n, m)$ of permutations for which σ^m is the identity, where m is any given integer: If $d_0 = 1, d_1, d_2, \ldots, d_l = m$ are the divisors of m, then $P(n, m)/(n!)$ is the coefficient of x^n in the Taylor expansion of*

$$e^x e^{\frac{1}{d_1} x^{d_1}} e^{\frac{1}{d_2} x^{d_2}} \cdots e^{\frac{1}{d_l} x^{d_l}} \ .$$

In particular, for our problem $m = 3$ and $P(n, 3) = P(n)$ equals $n!$ times the coefficient of x^n in the Taylor expansion of $e^x e^{\frac{1}{3} x^3}$.
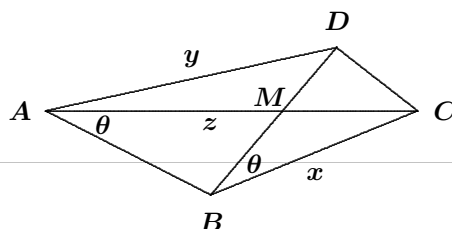
3284. [2007 : 429, 432] *Proposed by K.S. Bhanu and M.N. Deshpande, Institute of Sciences, Nagpur, India.*

Let x, y, and z be positive real numbers which satisfy $x^2 + y^2 = z^2$. Construct a line segment AC with length z. Let B be any point such that $BC = x$ and $90° < \angle ABC < 180°$. Let M be a point on AC such that $\angle MAB = \angle MBC$. Let D be the point on line BM on the opposite side of AC from B such that $AD = y$. Show that $\angle ADM = \angle DCM$.

Solution by Oliver Geupel, Brühl, NRW, Germany; and Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

Since the triangles CAB and CBM are similar, we have successively

$$\frac{MC}{BC} = \frac{CB}{CA},$$

$$\frac{MC}{x} = \frac{x}{z},$$

$$MC = \frac{x^2}{z}.$$

Then we have

$$AM = z - \frac{x^2}{z} = \frac{z^2 - x^2}{z} = \frac{y^2}{z}.$$

Thus, $\dfrac{y}{z} = \dfrac{AM}{y}$, or equivalently $\dfrac{AD}{AC} = \dfrac{AM}{AD}$, which implies that triangles AMD and ADC are similar. Therefore,

$$\angle ADM = \angle ACD = \angle DCM,$$

as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposers.

3285. [2007 : 429, 432] *Proposed by Gregory Akulov, student, University of Regina, Regina, SK.*

Solve the following for x:

$$x \left(\sqrt{3 - 2x + \sqrt{5(1 - x^2)}} + \sqrt{\frac{3}{2}} \right) = \sqrt{\frac{2}{3}}.$$

Solution by Michel Bataille, Rouen, France.

The equation can be rewritten as

$$x\sqrt{6}\left(\sqrt{3 - 2x + \sqrt{5(1-x^2)}}\right) = 2 - 3x\,,$$

so that the given equation is equivalent to $0 < x < \frac{2}{3}$ and

$$6x^2\left(3 - 2x + \sqrt{5\left(1-x^2\right)}\right) = (2-3x)^2\,.$$

Since $(3 - 2x)^2 - (2 - 3x)^2 = 5\left(1 - x^2\right)$, we successively rewrite the above equation as

$$6x^2 = \frac{(2-3x)^2\left(3 - 2x - \sqrt{5\left(1-x^2\right)}\right)}{(3-2x)^2 - 5\left(1-x^2\right)}\,,$$

$$6x^2 + 2x - 3 = -\sqrt{5\left(1-x^2\right)}\,.$$

Setting $\alpha = \arccos\left(\frac{2}{3}\right)$ and $x = \cos\theta$ where $\theta \in (\alpha, \frac{\pi}{2})$, the last equation is equivalent to each of

$$-\sqrt{5}\sin\theta = 6\cos^2\theta + 2\cos\theta - 3\,,$$

$$-\frac{\sqrt{5}}{3}\sin\theta - \frac{2}{3}\cos\theta = 2\cos^2\theta - 1 = \cos(2\theta)\,,$$

$$\cos(\theta - \alpha) = \cos(\pi - 2\theta)\,.$$

Hence, $\theta = \frac{\pi + \alpha}{3}$ and $x = \cos\left(\frac{\pi + \arccos\left(\frac{2}{3}\right)}{3}\right)$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; KEE-WAI LAU, Hong Kong, China; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incomplete and two incorrect solutions submitted.

3286. [2007 : 430,432] *Proposed by Neven Jurič, Zagreb, Croatia.*

Is it possible to find a function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = 1 + x\int_0^1 f(t)\,dt + x^2\int_0^1 \left[f(t)\right]^2 dt\,?$$

All submitted solutions were similar to those of Michel Bataille, Rouen, France and Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let $a = \int_0^1 f(t)\ dt$ and $b = \int_0^1 [f(x)]^2 dt$. Thus, $f(t) = 1 + at + bt^2$ and we have

$$a \;=\; \int_0^1 \left(1 + at + bt^2\right) dt \;=\; \left(t + \frac{at^2}{2} + \frac{bt^3}{3}\right)\bigg|_0^1 \;=\; 1 + \frac{a}{2} + \frac{b}{3}\,,$$

hence $a = 2 + \dfrac{2b}{3}$. Also

$$
\begin{aligned}
b \;&=\; \int_0^1 \left(1 + 2at + \left(2b + a^2\right)t^2 + 2abt^3 + b^2 t^4\right) dt \\
&=\; \left(t + at^2 + \frac{\left(2b + a^2\right)t^3}{3} + \frac{abt^4}{2} + \frac{b^2 t^5}{5}\right)\bigg|_0^1 \\
&=\; 1 + a + \frac{2b + a^2}{3} + \frac{ab}{2} + \frac{b^2}{5} \\
&=\; 1 + 2 + \frac{2b}{3} + \frac{1}{3}\left(2b + 4 + \frac{8b}{3} + \frac{4b^2}{9}\right) + b + \frac{b^2}{3} + \frac{b^2}{5}\,.
\end{aligned}
$$

Hence,

$$\frac{92b^2}{135} + \frac{20b}{9} + \frac{13}{3} \;=\; 0\,,$$

and the two roots of the equation $92b^2 + 300b + 585 = 0$ are

$$b \;=\; \frac{-75 \pm 3i\sqrt{870}}{46}\,.$$

Thus, such a function f does not exist. However, the complex valued function $f(z) = 1 + \left(\dfrac{21 \pm i\sqrt{870}}{23}\right) z + \left(\dfrac{-75 \pm 3i\sqrt{870}}{46}\right) z^2$ does satisfy the equation of the problem.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; DIONNE BAILEY, ELSIE CAMPBELL, CHARLES DIMINNIE, KARL HAVLAK and PAULA KOCA, Angelo State University, San Angelo, TX, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; ROBERT P. SEALY, Mount Allison University, Sackville, NB; DIGBY SMITH, Mount Royal College, Calgary, AB; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3287. $[2007 : 430, 432]$ *Proposed by Virgil Nicula, Bucharest, Romania.*

Let x, y, and z be positive real numbers satisfying

$$xy + yz + zx + xyz \; = \; 4 \, .$$

Prove that

(a) $(x+2)(y+2)+(y+2)(z+2)+(z+2)(x+2) \; = \; (x+2)(y+2)(z+2)$;

(b) there is a triangle whose sides have lengths $(x+2)(y+2)$, $(y+2)(z+2)$, and $(z+2)(x+2)$.

Solution by Joe Howard, Portales, NM, USA.

Let $a = x + 2$, $b = y + 2$, and $c = z + 2$. A simple calculation shows that

$$\begin{aligned}
4 \; &= \; xy + yz + zx + xyz \\
&= \; (a-2)(b-2) \; + \; (b-2)(c-2) \; + \; (c-2)(a-2) \\
&\quad + (a-2)(b-2)(c-2) \\
&= \; abc \; - \; (ab + bc + ca) \; + \; 4 \, .
\end{aligned}$$

Hence, $abc = ab + bc + ca$, and the equation in part (a) holds.

Clearly a, b, and c are positive. By part (a)

$$\begin{aligned}
ab + bc - ca \; &= \; ab + bc + ca - 2ca \\
&= \; abc - 2ca \\
&= \; ac(b - 2) \; = \; acy \; > \; 0 \, .
\end{aligned}$$

Similarly, $bc + ca - ab > 0$ and $ab + ca - bc > 0$, and the result in part (b) follows.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; KARL HAVLAK, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; ANDREA MUNARO, student, University of Trento, Trento, Italy; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; XAVIER ROS, student, Universitat Politècnica de Catalunya, Barcelona, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BINGJIE WU, student, High School Affiliated to Fudan University, Shanghai, China; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3288. [2007 : 430, 432] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let n be a positive integer. Evaluate the sum:

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \frac{2^{n-2i-1}}{n-2i} \ ,$$

where $\lfloor x \rfloor$ is the integer part of x.

Solution by Michel Bataille, Rouen, France.

Let S_n denote the sum to be evaluated. Then

$$S_n \ = \ \frac{1}{2} \int_0^2 \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} x^{n-2i-1} \right) dx \ . \tag{1}$$

From a known result (for example, see the solution to **Crux** problem 3217 in [2008: 112-3]) we have

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} x^i \ = \ F_n(x)$$

for non-negative x, where

$$F_n(x) \ = \ \frac{1}{\sqrt{1+4x}} \left(\left(\frac{1+\sqrt{1+4x}}{2} \right)^n \ - \ \left(\frac{1-\sqrt{1+4x}}{2} \right)^n \right) \ . \tag{2}$$

Calculating $x^{n-1} F_n \left(\frac{1}{x^2} \right)$ for $x > 0$ using (1) and then (2) yields

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} x^{n-2i-1}$$

$$= \ \frac{1}{2^n \sqrt{x^2+4}} \left(\left(x+\sqrt{x^2+4} \right)^n - \left(x-\sqrt{x^2+4} \right)^n \right) \ , \tag{3}$$

a formula that still holds if $x = 0$ [*Ed.*: if $x = 0$, n is odd, and $i = \frac{n-1}{2}$, then the last term of the sum in (3) is 0^0, which we interpret as 1]. It then follows from (1) and (3) that

$$S_n \ = \ \frac{1}{2^{n+1}} \int_0^2 \frac{\left(x+\sqrt{x^2+4} \right)^n - \left(x-\sqrt{x^2+4} \right)^n}{\sqrt{x^2+4}} \ dx \ .$$

Substituting $x = 2\sinh t = e^t - e^{-t}$ and $dx = 2\cosh t\, dt$ in the above integral yields

$$S_n \ = \ \int_0^{\ln(1+\sqrt{2})} \frac{e^{nt} + (-1)^{n-1}e^{-nt}}{2} \ dt \ .$$

Thus, if n is odd,

$$
\begin{aligned}
S_n \;=\; \int_0^{\ln(1+\sqrt{2})} \cosh(nt)\, dt \;&=\; \frac{1}{n} \sinh\left(\ln\left(1+\sqrt{2}\right)^n\right) \\
&=\; \frac{1}{2n}\left(\left(\sqrt{2}+1\right)^n - \left(\sqrt{2}-1\right)^n\right),
\end{aligned}
$$

and if n is even,

$$
\begin{aligned}
S_n \;=\; \int_0^{\ln(1+\sqrt{2})} \sinh(nt)\, dt \;&=\; \frac{1}{n} \cosh\left(\ln\left(1+\sqrt{2}\right)^n\right) - \frac{1}{n}\cosh(0) \\
&=\; \frac{1}{2n}\left(\left(\sqrt{2}+1\right)^n + \left(\sqrt{2}-1\right)^n\right) - \frac{1}{n}.
\end{aligned}
$$

Also solved by the proposer, whose proof used Fibonacci polynomials and similar arguments as given in the featured solution above.

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