MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

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Mathematical Spectrum Awards for Volume 20

Prizes have been awarded to the following student readers for contributions published in Volume 20:

Miklós Bóna for his article 'Colouring Space' (pages 71–75);
Gregory Economides for a problem submitted and solutions to problems;
Nicholas Shea and Amites Sarkar for solutions to problems.

Readers are reminded that awards of up to £30 are available for articles and of up to £15 for letters, solutions to problems, or other items. To qualify, contributors must be students at school, college or university.

The Life and Work of H. E. Dudeney

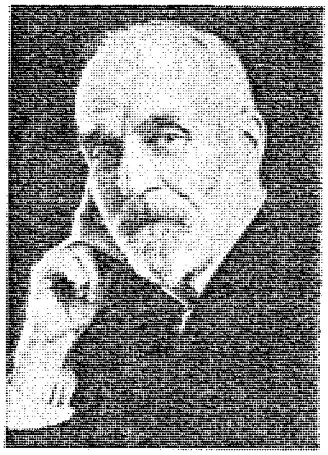
ANGELA NEWING

The author is a medical physicist who works in hospitals in Gloucestershire. One of her hobbies is recreational mathematics, and she composes puzzles for various journals and newspapers in her spare time.

In 1982, at a Symposium of the American Mathematical Society on the theory of networks, Stefan Burr, one of the leading workers in this field of mathematics, was asked to list the important events which he considered to have been turning points in the development of network or graph theory. Naturally, he began with Euler's solution of the recreational problem of the Königsberg Bridges in 1736 (see reference 1) which was the foundation of topology, and then he traced a history which had as many points of reference in recreational mathematics as it did in serious mathematics. The greatest mathematicians in history never hesitated to pursue problems which might be considered 'frivolous' and we are lucky that this was so, because, in this way, they often discovered new branches of mathematics which proved of tremendous value to mankind. Another example is probability theory which developed entirely from gambling and card games. A number of French mathematicians of the sixteenth century who were gamblers developed game strategies, which they then analysed mathematically, and this produced the foundations for modern probability theory.

These are not the only benefits of recreational mathematics. Because this kind of mathematics is fun, it introduces large numbers of people to mathematics who might otherwise not develop any interest in the subject. Modern mathematics textbooks are much easier to digest because they use lighthearted puzzles to illustrate serious mathematics.

Returning to the list of important milestones in network theory, Burr made much of Mei-ko Kwan, whom he said had discovered, in 1962, a general solution to the problem of finding the shortest route which covers every



H. E. Dudeney 1857–1930

path in a given network at least once (see reference 2). In Mei-ko's honour, this type of problem is now known as the 'Chinese Postman's Problem' after the puzzle he used to illustrate his solution (figure 1). This represents a postman's delivery area consisting of nine groups of houses in squares surrounded by streets as shown. The postman enters the territory from the post office at corner *A* and must walk along all the roads of his area before returning to *A*. You might like to try to solve the puzzle by finding the shortest route before reading the answer.

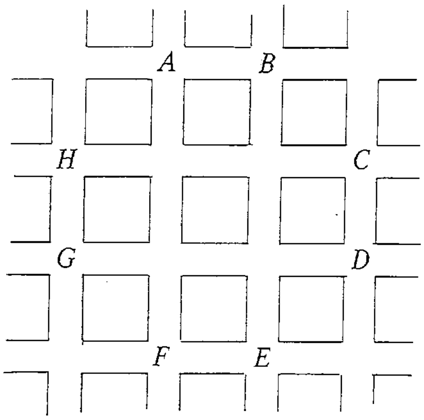


Figure 1

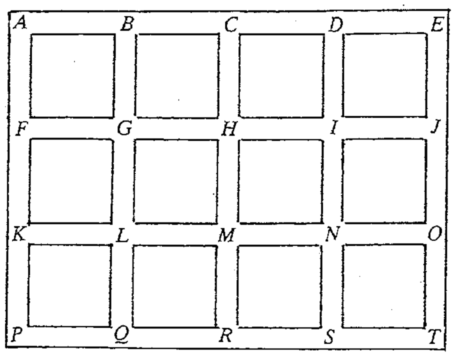


Figure 2

The area can be considered to be a network with closed boundaries outside the nine blocks. It can then be seen to have eight odd nodes or vertices (points where three edges meet). The roads which must be travelled twice are those between the odd nodes—*AB*, *CD*, *EF* and *GH*. There are several correct solutions in which all the other roads are covered only once, but, if the distance between every pair of adjacent points is designated as one unit, the shortest path will always be 28 units long.

Anyone reading H. E. Dudeney's *Amusements in Mathematics*, published first in 1917 (reference 3) will find a similar diagram in puzzle number 247 (see figure 2). His diagram represents the galleries in a coal mine; an inspector must examine all 31 sections AB, BC, etc. The reader is asked to determine the length of the shortest route and, in this case, the route is to start at the mine shaft at point A. This is an even node, which makes the problem slightly different. Dudeney's general solution predates Mei-ko Kwan's by about 50 years.

Reading Dudeney's puzzles, one comes across countless instances like this where later workers have rediscovered some principle which he propounded at the beginning of this century. One is also struck by the fact that most modern puzzle books contain many puzzles which originated with him.

So, who was Henry Ernest Dudeney? He was born at Mayfield, a small village in Kent in southern England, in April 1857. He received the standard elementary education of the mid-nineteenth century, which was not very much, but it was obvious from a very early age that he was bright and intelligent. He learned to play chess and beat everyone he played, he was very musical and became an accomplished pianist and organist, and he developed a great interest in English literature and in mathematics. He managed to teach himself from books, and thus he steadily improved his education.

In an interview he gave towards the end of his life, he recorded how he developed an interest in conjuring as a small child after he went to watch an 'illusionist' perform in the local church hall. He rapidly acquired the skill to perform conjuring tricks with which to entertain his family and friends, and from these he progressed to mathematical puzzles. When he was eight or nine years old he began composing puzzles for his brothers and sisters to solve. Someone suggested that he should send them up to a boys' weekly paper. This he did and, to his great delight, they were published and he was paid five shillings (25p) each for them. His puzzle career had started.

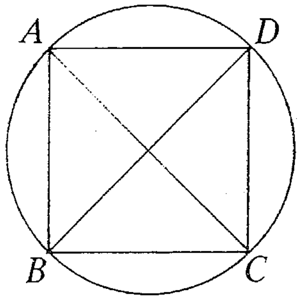


Figure 3

One of his earliest network puzzles is shown in figure 3. The solver is required to draw the figure with one stroke of the pencil, never lifting the pencil from the paper or going over a line twice (i.e. finding an 'Eulerian path'). This network has four odd nodes and one even node, so everyone familiar with Euler's solution of the Königsberg Bridges problem which produced the general rules for Eulerian paths will know that such a path is impossible because there are more than two odd nodes. Dudeney's solution involves a trick which eliminates two of the odd nodes. The paper must first be folded across the middle so that AB and CD are together. The pencil is then inserted in the fold and the two straight lines AB and DC are drawn with one stroke. The pencil is kept in contact with point C while the paper is unfolded, and then the solution is simple. This puzzle was composed when Dudeney was nine, and shows a remarkable grasp of network theory and also of the philosophy of puzzle setting. He had obviously read some of Euler's works by this time and he soon became interested in the writings of Cardano, Fermat, Newton and many others. All the time his fertile mind was devising puzzles based on ideas put forward by earlier mathematicians.

In an article in Volume 18 of *Spectrum*, Keith Devlin put forward an admittedly inefficient method of generating all the prime numbers (reference 4). Mathematicians through the ages have been looking for an efficient formula to express all primes. Some have found expressions which give primes for a particular range of numbers. One such is x^2+x+41 (another discovery of Euler's in 1772 (reference 5)) which gives primes for integral values of x from 0 to 39. This had largely been forgotten or overlooked until Dudeney based a puzzle on it (see 'The Banker's Puzzle', number 134, in *Amusements in Mathematics*, reference 3). This rekindled the interest of mathematicians in searching for formulae rich in primes.

Dudeney became a clerk in the Civil Service when he was a young teenager. He continued to read about mathematics and its history in his spare time and began to submit puzzles for publication. It was not long before he became established as a mathematical wizard, and he was often called 'The Puzzle King' by the popular press.

In the early years of this century, Dudeney's output of puzzles in large numbers of papers and journals was prodigious. Luckily for us, he also began collecting the best of his puzzles for publication in book form, and his first collection, entitled *The Canterbury Puzzles*, appeared in 1907. This was followed in 1917 by *Amusements in Mathematics* which immediately became so popular that reprints were necessary in both 1919 and 1920 and about every second year thereafter until 1951. (Both these books are now available as Dover reprints.)

Hilbert's theorem states that 'If two polygons are equivalent (i.e. have the same area) then it is possible to transform either of them into the other by cutting into a finite number of polygons and re-arranging the pieces'. Recreational mathematicians have had great fun with geometrical dissection problems for centuries, and Dudeney was undoubtedly the master of minimum-dissection puzzles (i.e. converting one shape into another with as few cuts as possible). His conversion of a square into a pentagon in six pieces was a popular classic (figure 4), especially since it had been thought for many years that the minimum dissection involved seven pieces. His most famous dissection problem was to cut an equilateral triangle into four pieces which could be re-assembled to make a square. Dudeney's solution, shown in figure 5, was remarkable in that, if the pieces were hinged at three vertices as shown, they formed a chain which could be folded one way to make the square or the other way to make the triangle. He made a model of these four pieces in mahogany with brass hinges and demonstrated it at a meeting of the Royal Society in May 1905.

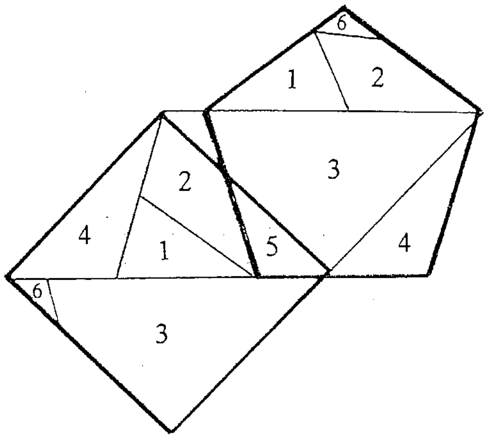


Figure 4

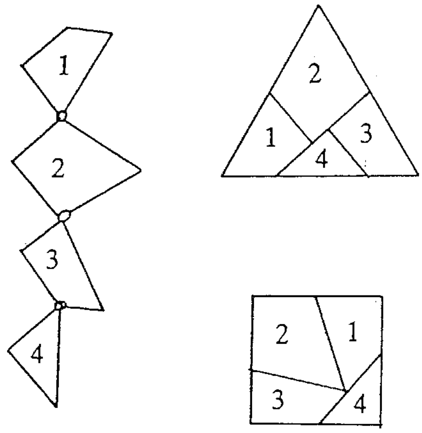


Figure 5

Dudeney showed an interest in Kirkman's Schoolgirl's Problem (reference 6) and published solutions for the formation of triples among groups of n things for several values of n up to 111. He also tackled permutation problems of the maximum number of different arrangements of n people in a ring so that no one had the same two neighbours more than once. The total number of ways in which n people can be arranged in a ring is $\frac{1}{2}(n-1)!$. The number of these arrangements in which everyone has a different pair of neighbours on every occasion cannot exceed $\frac{1}{2}(n-1)(n-2)$ which is the number of ways in which any one person can sit between every possible pair of others. In 1905, Dudeney suggested that $\frac{1}{2}(n-1)(n-2)$ was the number of arrangments for every n, and he set a problem in the *Daily Mail* asking for all the possible arrangments (10) of six people around a table. In the *Canterbury Puzzles* he asked for the 15 possible arrangements of seven people. By this time (1907) a general solution had been discovered for cases in which n was even. Dudeney said that he had found a subtle method for solving all cases, but unfortunately he did not publish it then or later. In 1917, in *Amusements in Mathematics*, he wrote:

> 'A solution is possible for any numbers of persons... but, as I know that a good many mathematicians are still considering the case of $n = 13$, I will not at this stage rob them of the pleasure of solving it by showing the answer.'

He certainly did know the solution because he later published answers for $n = 27$ and $n = 33$, and it is a great pity that he did not set his method down on paper because no one since has published a solution for odd values of n.

Dudeney became steadily more famous as it was recognised that his position in the world of mathematics was unique. Not only had he generated a tremendous interest in mathematics among members of the public by his popular puzzles in the press, but he had also given professional mathematicians much to think about by his rediscovery of long-forgotten topics.

Another good example concerns a little-known problem, originally posed by Pappus in Alexandria in the 3rd century, which Dudeney printed as puzzle 199 in *Amusements in Mathematics*. A rectangular piece of board has a triangular piece clipped from it as shown in figure 6, in such a way that it balances with CD horizontal when suspended by a thread from point A. It is required to determine the position of A. At first sight this seems a very simple problem of finding AB so that the area of $ABCE$ equals the area of AED. But this is not the case and further study shows that, to balance, the moments of the weights of both sides about A must be equal. One therefore has to find the centres of gravity of the two sides. For AED this is the point where the medians meet, so the answer is that $AB{:}ED = 1{:}\sqrt{3}$. Dudeney gives an interesting method of construction of the correct figure.

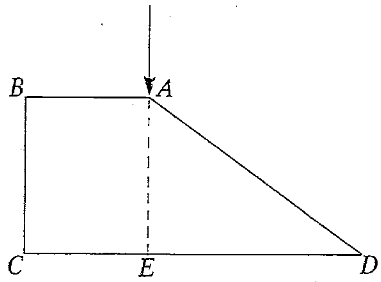


Figure 6

By his puzzle columns he rekindled interest in magic squares—a very ancient area of mathematics—and in 1924 he was invited to write a section on this topic for the new edition of *Encyclopaedia Britannica*.

He became the first mathematician to work with digital roots of numbers (successively adding the digits until a single digit remains). This technique gave a new method of classifying various types of numbers such as squares, triangular numbers, etc.

Dudeney also dabbled in word puzzles, inventing several new forms of word puzzle as well as giving new twists to long-established types. He published a collection of his favourite word puzzles in 1925 called *The World's Best Word Puzzles*, while in 1926 another mathematical collection was published entitled *Modern Puzzles*. This book included the first modern adaption of what has become known as 'The Explorer's Problem':

> 'Nine travellers, each possessing a motor car, meet on the eastern edge of a desert. They wish to explore the interior, always going due west. Each car can travel 40 miles on the contents of the engine tank, which holds a gallon of petrol, and each can carry 9 extra gallon tins of

petrol and no more. Unopened tins can alone be transferred from car to car. What is the greatest distance to which they can penetrate the desert without making any depots of petrol for the return journey?'

This puzzle seems to be based upon an ancient Italian problem from the middle ages dealing with the transport of 90 apples across 30 miles with one apple eaten each mile. There is no record of it appearing in any form in the intervening years. It could be that Dudeney re-invented it.

During his later years, he continued to play chess. When quite an old man, he was able to use his mathematical skill to good effect on an uneven croquet lawn near his home. He was always able to beat visitors by sending his ball in what was apparently the wrong direction: nevertheless, it would go round the hazards and through the hoop.

For the whole of his adult life Dudeney had been a smoker. As he grew older he suffered from bronchitis and sometimes pleurisy each winter, and had to take to his room for long periods. He died from pleurisy two weeks after his 73rd birthday in April 1930.

After his death, his wife persuaded a friend who was a mathematician to help her arrange another collection of puzzles for publication. The result of their efforts, *Puzzles and Curious Problems*, appeared in 1931, and yet another selection, *A Puzzle Mine* came by the same route a couple of years later. Thirty-five years after this, in 1967, Martin Gardner—of *Scientific American* fame—obtained permission from Dudeney's daughter to bring together the best puzzles from the last three volumes into one book, *536 Puzzles and Curious Problems*. This book is still in print (reference 7).

Martin Gardner wrote that

'(He) was England's greatest inventor of puzzles; indeed he may well have been the greatest puzzlist who ever lived. Today there is scarcely a single puzzle book that does not contain (often without credit) dozens of brilliant mathematical problems that had their origins in Dudeney's fertile imagination' (reference 8).

Post Script

The solution to 'Exploring the Desert' is that one car can travel a total distance of 360 miles before turning for home. For the complete solution of this puzzle and a slightly more interesting variation of it, 'Exploring Mount Neverest', see puzzles 49 and 50 in *Modern Puzzles* or 76 and 77 in *536 Puzzles and Curious Problems*.

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Note. I am at present writing a more extended biography of Dudeney. If any reader has any further information about the man and his work, I should be grateful if he or she would get in touch with me at The Rectory, Brimpsfield, Gloucester GL4 8LD.

An Algebra of Points

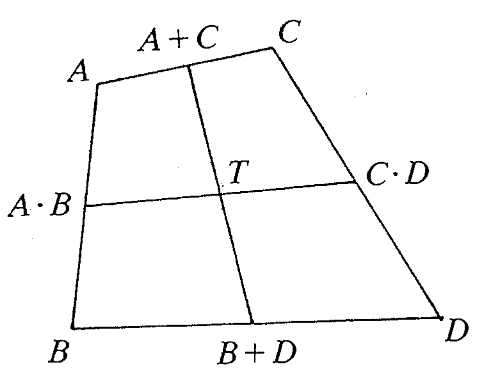
GUIDO LASTERS, *Tienen, Belgium*

The author teaches mathematics at the school of Sint-Truiden, Belgium. He studied at the University of Brussels, where he wrote a thesis applying group theory to non-rigid molecules. His main interest in mathematics is the application of algebra in geometry. His previous note on this subject in *Mathematical Spectrum* was in Volume 18 Number 2.

By considering a fixed triangle KLM in the plane (the triangle may be degenerate), we can define a binary operation on the points of the plane, denoted (say) by $+$, as follows. Given points X and Y, we define $X+Y$ to be that point such that the three points X, Y and $X+Y$ form a triangle similar to KLM, where $X, Y, X+Y$ and K, L, M are in the same order, clockwise or anticlockwise. If we start with a different fixed triangle, we can define another binary operation on the points of the plane, denoted this time by $\cdot$ (say). We shall prove the law

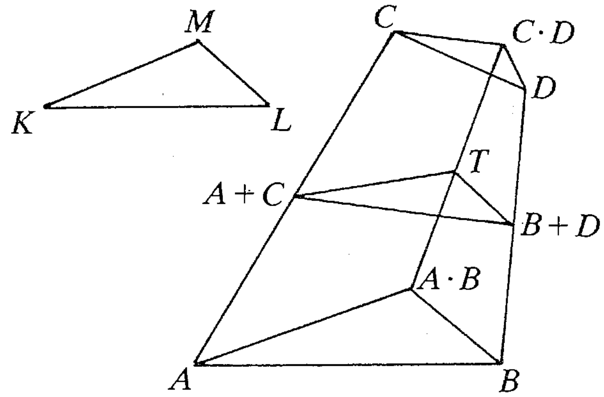
$$(A+C)\cdot(B+D) = (A\cdot B)+(C\cdot D).$$

Before doing this, we give two illustrations of this law out of many possible ones. If $X+Y$ and $X\cdot Y$ are both defined as the midpoint of the line segment XY, we see that the point T in figure 1 is the midpoint of the line segment joining the midpoints of AC and BD and is also the midpoint of the line segment joining the midpoints of AB and CD. In figure 2, $+$ is defined as in figure 1 and $\cdot$ is defined using the fixed triangle KLM.



$T = (A+C)\cdot(B+D) = (A\cdot B)+(C\cdot D)$

Figure 1



$T = (A+C)\cdot(B+D) = (A\cdot B)+(C\cdot D)$

Figure 2

We prove our law using complex numbers, denoting each point by a complex number relative to some axes. Given complex numbers a and b with $a \neq b$, any complex number c can be written as

$$c = (1-k)a+kb$$

for some complex number k: just take

$$k = \frac{c-a}{b-a}.$$

For example, the midpoint of the line segment joining a and b is given by $k = \frac{1}{2}$. Moreover, the triangles given by a, b and $(1-k)a+kb$ and c, d and $(1-k)c+kd$ (with $c \neq d$) are similar, with the points in the same order, clockwise or anticlockwise. For the lengths of the sides of the first triangle are in the proportions

$$|b-a| : |(1-k)a+kb-b| : |a-(1-k)a-kb| = 1 : |k-1| : |k|,$$

and these are independent of a and b. Moreover,

$$(1-k)a+kb = a+k(b-a),$$

and the argument of k determines the ordering of the vertices, clockwise or anticlockwise. But now the law is immediate. For, if we denote the points A, B, C and D by the complex numbers a, b, c and d, respectively, and denote by k and l the complex numbers associated with the binary operations $+$ and $\cdot$, then

$$\begin{aligned}(A+C)\cdot(B+D) &= (1-l)[(1-k)a+kc]+l[(1-k)b+kd] \\ &= (1-k)[(1-l)a+lb]+k[(1-l)c+ld] \\ &= (A\cdot B)+(C\cdot D),\end{aligned}$$

as required.

An Iterative Method of Construction

PETER O'GRADY, *Warwick School*

The author was a scholar at Christ's College, Cambridge. Since 1975 he has taught at Warwick School, where he is learning how little he knew about elementary mathematics when he was at school!

Can you construct an angle of 10°, using only a straight edge and compasses? Nearly!

Draw a straight line and use compasses to mark off 9 equal distances from point A on the line to point B. Construct an equilateral triangle ABC of side 9 units. Construct the bisector of angle B. Now draw an arc of radius 7 units, centre A, meeting angle B's bisector at D. Join AD (see figure 1). You should find that angle BAD is 10°—according to your protractor.

Now produce BD to meet the arc again at E. Join AE, bisect the angle at E and produce AC to meet this bisector at F. Your diagram (see figure 2) should now include every angle up to 180° which is a multiple of 10°!

Unfortunately, the angle BAD is not 10°. You should be able to show that it is about 10.005 200 884°—not bad, but you can do better by bisecting the angle AFE. In fact, if $\angle BAD = (10+\epsilon)°$, then $\angle AFE = (20+\frac{1}{2}\epsilon)°$, and bisecting $\angle AFE$ gives an angle of $(10+\frac{1}{4}\epsilon)°$.

This suggests an iterative method. Given $\angle XPY$ equal to $(10+\delta)°$, choose any point Q on PX and construct the equilateral triangle PQR. Construct QS, the bisector of $\angle PQR$, meeting PY at T. Mark U on QS, where $PU = PT$, and bisect $\angle PUS$. This bisector meets PR produced at V. Finally, bisect $\angle PVU$ to obtain an angle of $(10+\frac{1}{4}\delta)$ (see figure 3).

Returning to figure 2, you can use this procedure with P taken at F, as indicated in figure 4, and carry on indefinitely.

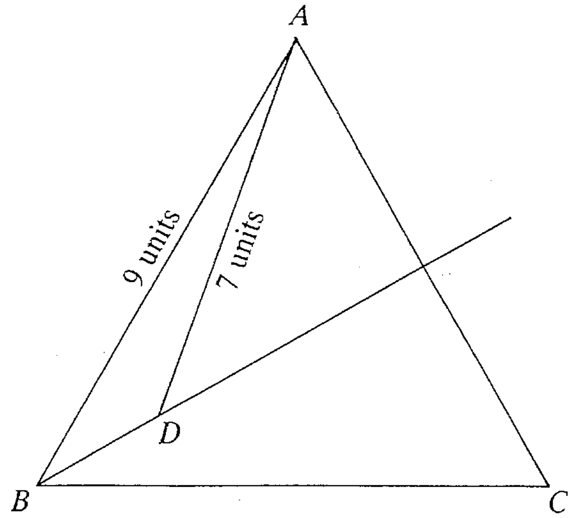


Figure 1

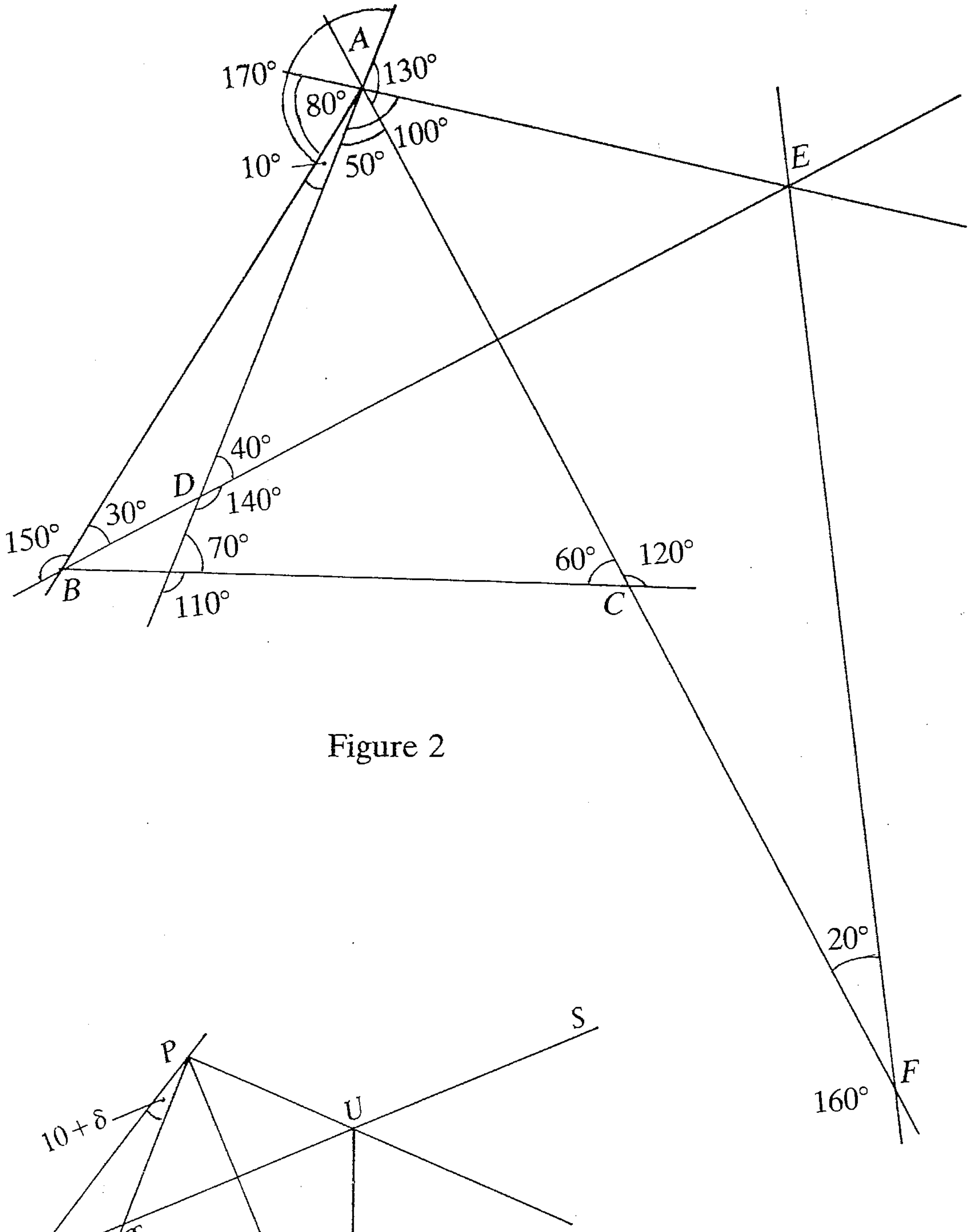


Figure 2

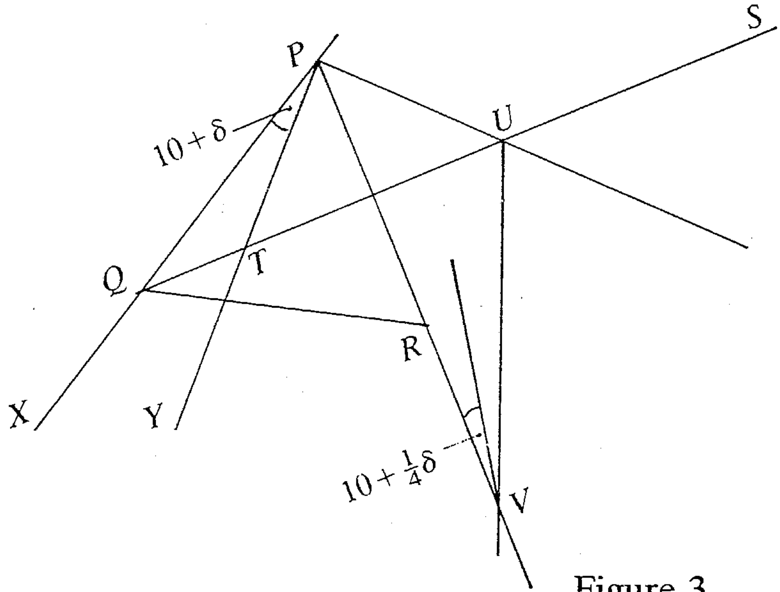


Figure 3

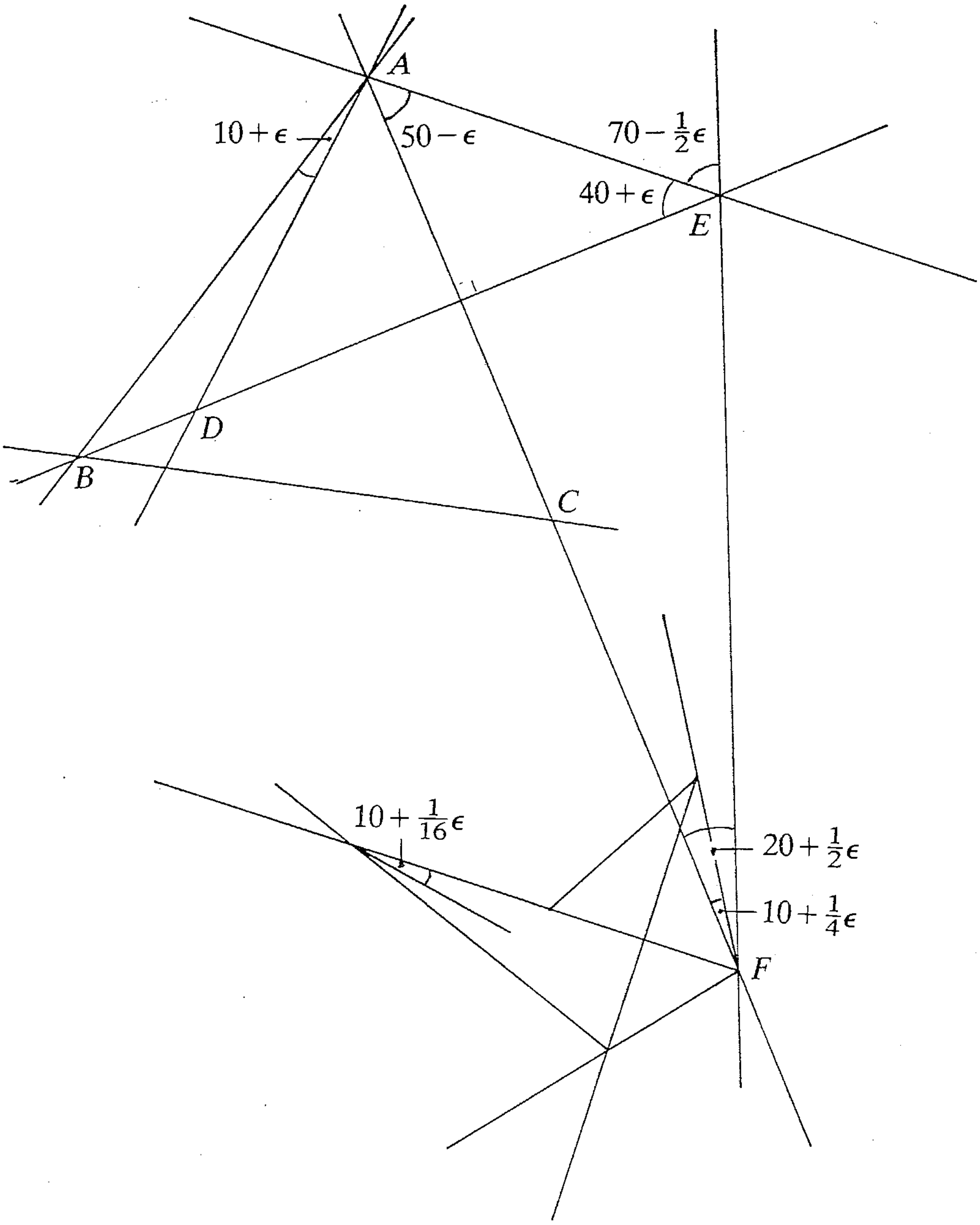


Figure 4

Some remarks

(a) The method is stable: if your pencil slips a little at some stage, you should recover from the mistake.

(b) You could start with a guess for 10° and then follow the iterative procedure.

(c) (See figure 3.) An alternative method would have been to divide $\angle PTS$ into four equal angles of $(10+\frac{1}{4}\delta)°$.

(d) After seven iterations the error should be less than one millionth of a degree, but ...

(e) ... you will need, as always, a very sharp pencil!

Summing the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$

NICHOLAS SHEA, *Gresham's School, Holt, Norfolk*

The author wrote this article whilst studying for his A-levels, with the intention of going on to read mathematics at university.

1. Introduction

Consider the curve $y = 1/x^2$ for $x \geqslant 1$. It is of infinite length, yet the area bounded by the curve, the x-axis and the line $x = 1$ is

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \left[-\frac{1}{x}\right]_1^\infty = 1.$$

Again, consider the curve $y = 1/x$ for $x \geqslant 1$. The area bounded by this curve, the x-axis and the line $x = 1$ is infinite, whereas the volume swept out when this region is revolved round the x-axis is finite; for

$$\text{the area} = \int_1^\infty \frac{1}{x}\,\mathrm{d}x = [\ln x]_1^\infty = \infty, \qquad \text{the volume} = \pi\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = \pi.$$

These facts may initially appear somewhat surprising; they arise from the different dimensions for length, area and volume.

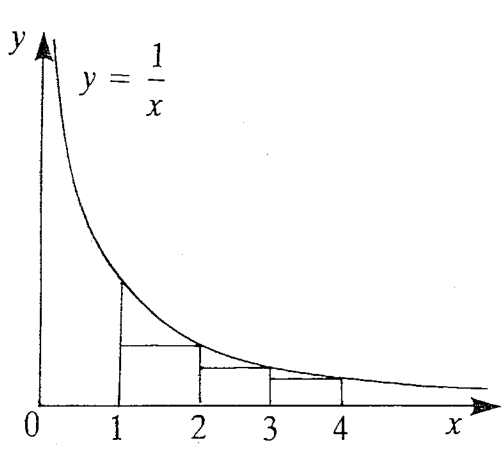


Figure 1

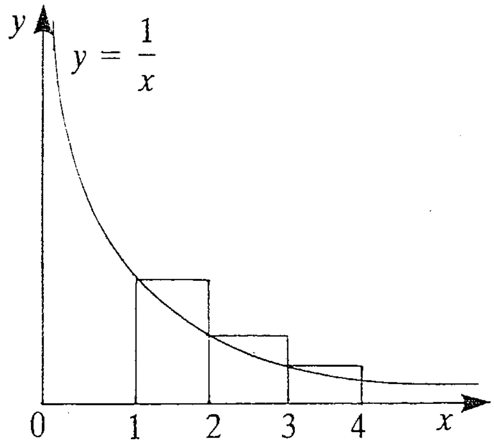


Figure 2

We may confirm the results for the curve $y = 1/x$ if we approximate the area by means of rectangles as in figure 1 and the volume by means of cylinders as in figure 2. We see that

$$\text{the area} > \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots,$$

$$\text{the volume} < \pi\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right).$$

2. Euler's method

Consider a polynomial

$$a_0+a_1x+\ldots+a_nx^n$$

of degree n with $a_0 \neq 0$, and denote its roots by $\alpha_1,\ldots,\alpha_n$. Then the polynomial

$$a_0y^n+a_1y^{n-1}+\ldots+a_n$$

has roots $1/\alpha_1,\ldots,1/\alpha_n$ and the sum of the roots is

$$\frac{1}{\alpha_1}+\ldots+\frac{1}{\alpha_n} = -\frac{a_1}{a_0}.$$

Euler said that, if we denote the roots of the 'infinite polynomial' or power series

$$a_0+a_1x+a_2x^2+\ldots$$

with $a_0 \neq 0$ by $\alpha_1,\alpha_2,\ldots,$ then

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\ldots = -\frac{a_1}{a_0}.$$

It is not clear how this can be justified for power series, however. Now

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots,$$

and this is zero for $x = 0, \pm\pi, \pm 2\pi,\ldots,$ so that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots = 0$$

when $x = \pm\pi, \pm 2\pi,\ldots$. If we put $y = x^2$, we have that

$$1-\frac{y}{4!}+\frac{y^2}{5!}-\ldots = 0$$

when $y = \pi^2, (2\pi)^2,\ldots$. Thus, according to Euler's assertion,

$$\frac{1}{\pi^2}+\frac{1}{(2\pi)^2}+\frac{1}{(3\pi)^2}+\ldots = \frac{1}{6},$$

i.e.

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}.$$

Now the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$ does not converge to any finite limit. This can be seen by taking the first term, then the next two together, then the next four, and so on. Since

$$\tfrac{1}{3}+\tfrac{1}{4} > \tfrac{1}{4}+\tfrac{1}{4} = \tfrac{1}{2}, \qquad \tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8} > \tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8} = \tfrac{1}{2},$$

and so on, we see that

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\ldots+\left(\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n}\right) > \frac{n}{2} \to \infty \quad \text{as } n \to \infty,$$

so the series does not have a finite sum. On the other hand, with the series of squares we have

$$\frac{1}{2^2}+\frac{1}{3^2} < \frac{1}{2^2}+\frac{1}{2^2} = \frac{2}{2^2} = \frac{1}{2},$$

$$\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2} < \frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2} = \frac{4}{4^2} = \frac{1}{4},$$

$$\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\ldots+\frac{1}{15^2} < 8\times\frac{1}{8^2} = \frac{1}{8}$$

and so on. Thus

$$\frac{1}{1^2}+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{4^2}+\ldots+\frac{1}{7^2}\right)+\left(\frac{1}{8^2}+\ldots+\frac{1}{15^2}\right)+\ldots < 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots,$$

and this is a geometric series with common ratio $\tfrac{1}{2}$ and so has sum $1/(1-\tfrac{1}{2}) = 2$. Thus the series of squares is convergent. The question naturally arises: what is its sum?

I shall describe my attempts to obtain this sum. My starting point was an argument given by the Swiss mathematician Leonhard Euler in the eighteenth century. Euler did not explain why this method works, and I cannot see how it can be made rigorous.

3. The expansion for $\arcsin x$

Euler's method suggested to me that it might be worth looking at the series

$$\arcsin x = x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots.$$

(Note that $\arcsin x$ is also denoted by $\sin^{-1} x$.) We have

$$\int_0^1 \frac{\arcsin x}{x}\,dx = \int_0^1 \left(1+\frac{x^2}{3}+\frac{x^4}{5}+\ldots\right) dx$$

$$= \left[x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\ldots\right]_0^1$$

$$= 1+\frac{1}{3^2}+\frac{1}{5^2}+\dots \tag{1}$$

$$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\dots\right)$$

$$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right)-\frac{1}{2^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)$$

$$= \frac{3}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots\right). \tag{2}$$

However, I was unable to evaluate the integral. A similar use of the arc tangent and other expansions also led to integrals which I was unable to evaluate, for example

$$\int_0^1 \frac{\arctan x}{x}\,dx, \qquad \int_0^1 \frac{1-\sqrt{1-x^2}}{x}\,dx, \qquad \int_0^{\frac{1}{2}\pi} (1-\cos\theta)\cot\theta\; d\theta.$$

Perhaps this is not too surprising, as most integrals cannot be solved by elementary means.

4. Logarithmic expansions

I was more successful when I tried logarithmic series and found two arguments for evaluating the sum of the series. First,

$$\ln\frac{1+x}{1-x} = \ln(1+x)-\ln(1-x) = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right).$$

If we replace x by $e^{i\theta}$, we obtain

$$\ln\frac{e^{-\frac{1}{2}i\theta}+e^{\frac{1}{2}i\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}} = 2\left(e^{i\theta}+\frac{e^{3i\theta}}{3}+\frac{e^{5i\theta}}{5}+\dots\right),$$

$$\ln\frac{2\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = 2\left(e^{i\theta}+\frac{e^{3i\theta}}{3}+\frac{e^{5i\theta}}{5}+\dots\right),$$

$$\ln i+\ln\cot\tfrac{1}{2}\theta = 2\left(e^{i\theta}+\frac{e^{3i\theta}}{3}+\frac{e^{5i\theta}}{5}+\dots\right).$$

Writing $\ln i = a+ib$ with a and b real, we have

$$i = e^a e^{ib} = e^a(\cos b+i\sin b).$$

Thus $a = 0$, $b = \frac{1}{2}\pi$, so that $\ln i = \frac{1}{2}i\pi$. Thus, if we equate imaginary parts, we obtain

$$\tfrac{1}{2}\pi = 2\left(\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\dots\right),$$

$$\int_{\frac{1}{2}\pi}^{\pi} \tfrac{1}{2}\pi \; d\theta = 2\int_{\frac{1}{2}\pi}^{\pi}\left(\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \ldots\right) d\theta,$$

$$\tfrac{1}{2}\pi[\theta]_{\frac{1}{2}\pi}^{\pi} = 2\left[-\cos\theta - \frac{\cos 3\theta}{3^2} - \frac{\cos 5\theta}{5^2} - \ldots\right]_{\frac{1}{2}\pi}^{\pi},$$

$$\tfrac{1}{2}\pi(\pi - \tfrac{1}{2}\pi) = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right],$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8},$$

$$\frac{3}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = \frac{\pi^2}{8} \quad \text{[see (1), (2) in section 3]},$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6},$$

which is Euler's answer.

I subsequently modified this method to obtain the result more directly as follows:

$$\ln(1 + e^{i\theta}) = e^{i\theta} - \frac{e^{2i\theta}}{2} + \frac{e^{3i\theta}}{3} - \ldots,$$

$$\ln e^{\frac{1}{2}i\theta}\left(e^{\frac{1}{2}i\theta} + e^{-\frac{1}{2}i\theta}\right) = e^{i\theta} - \frac{e^{2i\theta}}{2} + \frac{e^{3i\theta}}{3} - \ldots,$$

$$\tfrac{1}{2}i\theta + \ln(2\cos\tfrac{1}{2}\theta) = e^{i\theta} - \frac{e^{2i\theta}}{2} + \frac{e^{3i\theta}}{3} - \ldots,$$

so that, taking imaginary parts, we have

$$\tfrac{1}{2}\theta = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \ldots,$$

$$\int_{-\pi}^{\pi} \theta \times \tfrac{1}{2}\theta \; d\theta = \int_{-\pi}^{\pi} \theta(\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \ldots)\, d\theta$$

$$[\tfrac{1}{6}\theta^3]_{-\pi}^{\pi} = \left[\theta\left(-\cos\theta + \frac{1}{2^2}\cos 2\theta - \frac{1}{3^2}\cos 3\theta + \ldots\right)\right]_{-\pi}^{\pi}$$

$$- \int_{-\pi}^{\pi}\left(-\cos\theta + \frac{1}{2^2}\cos 2\theta - \frac{1}{3^2}\cos 3\theta + \ldots\right) d\theta,$$

$$\tfrac{1}{3}\pi^3 = 2\pi\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right),$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}.$$

5. Fourier series

The use of $\cos n\theta$ and $\sin n\theta$ in the above arguments may suggest the use of Fourier series to those familiar with them. These provide a rigorous method of obtaining the sum of our series.

The Fourier series of a function $f(x)$ is given by

$$a_0+(a_1\cos x+a_2\cos 2x+\ldots)+(b_1\sin x+b_2\sin 2x+\ldots),$$

where

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \quad (n\geqslant 1),$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x.$$

We take $f(x)=x^2$. Since $f(x)=f(-x)$, all the b_n are zero, and

$$a_0=\frac{1}{\pi}\int_0^{\pi} x^2\,\mathrm{d}x=\tfrac{1}{3}\pi^2,$$

$$\begin{aligned}a_n&=\frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}2x\frac{\sin nx}{n}\,\mathrm{d}x\\&=-\frac{4}{\pi n}\left[-x\frac{\cos nx}{n}\right]_0^{\pi}-\frac{4}{\pi n}\int_0^{\pi}\frac{\cos nx}{n}\,\mathrm{d}x\\&=\frac{4}{n^2}(-1)^n \quad \text{for } n\geqslant 1.\end{aligned}$$

It is known that the Fourier series of x^2 converges to x^2 in the range $-\pi\leqslant x\leqslant\pi$, so that

$$x^2=\tfrac{1}{3}\pi^2+4\left(-\cos x+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\right).$$

If we now put $x=\pi$, we obtain

$$\pi^2=\tfrac{1}{3}\pi^2+4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right),$$

which gives

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}.$$

This method can be used to find the sum of the series

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots,$$

and the series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots$$

can be shown to converge to $\frac{1}{32}\pi^3$. It is left to the reader to determine the sum of the series

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots .$$

My Logical Lady

KEITH AUSTIN, *University of Sheffield*

Keith Austin is a lecturer in pure mathematics. He believes that, although mathematics is said to be both an art and a science, the current attitude to mathematics lays much more emphasis on the scientific side than on the artistic side. Perhaps the appeal of mathematics could be extended to a wider audience by showing more of its artistic aspects. One possible field for this is philosophy and logic and this is the setting for this present article, which is written as a story rather than as a scientific report.

Sherlock Holmes and I had just left a matinee at Covent Garden when we found ourselves caught by a shower of rain. Fortunately we were outside a shop selling umbrellas and so we entered and soon selected a suitable one. The girl at the counter explained that it was £5.19s.11d. plus purchase tax which she found, from a table, to be 18s.0d.† She asked if we minded waiting while she did the addition by hand as her till had broken down. She wrote down the sum and after a while said, '£6.17s.11d.'

'Correct,' said Holmes.

'How do you know?' she replied.

'£6.0s.0d. plus 18s.0d. is £6.18s.0d., and £5.19s.11d. is one penny less than £6.0s.0d. so I took one penny from £6.18s.0d.'

'It sounds very complicated. But if you say so then it must be right.'

'My dear girl, it is not right because I said so; I said so because it is right.'

'Sounds like the same thing to me.'

†Before 1971 British currency was expressed in pounds, shillings and pence. £1 = 20 shillings; 1 shilling = 12 pence.

I thought Holmes was going to hit the girl with his new umbrella and so I pointed to the door.

'Holmes, isn't that Moriarty going by outside?'

'Come on Watson, we mustn't lose him.' Holmes rushed to the door, flung it open and dashed off down the street. I turned to the girl who was clearly shaken by Holmes' manner, and helped her to calm down. Soon Holmes returned.

'I could not see him at all. Are you sure it was Moriarty, Watson? Now, young lady, here is your money.'

While she was getting his change, Holmes remarked, 'Her total disregard for the ways of logic will hold that girl back all her life.'

'Could she learn to be logical?'

'Certainly! I could teach her to think clearly in a matter of months.'

'Here is your change, sir. Could you really teach me how to think like you in such a short time?'

'That sounds a fair challenge, Holmes. Are you going to take on the young lady?'

'Agreed. I shall be pleased to give Elizabeth some lessons in logic.'

'Holmes, how on earth did you learn her name?'

'Elementary, my dear Watson; it is written on her lapel badge.'

Elizabeth paid her first visit to Baker Street a week later.

'Mr Holmes, what is logic?'

'Suppose there is a situation which you cannot see; it may be in another room or in another country or in my imagination. I tell you some facts about the situation and you work out, in some way, other facts about the situation. Your working out is a *logical deduction*.

'Let me give you an example. I have written down a 4×4 array of letters such that each row and each column contains a, b, c and d. Also I will tell you some of the letters, so

$$\begin{array}{cccc} a & \_ & \_ & \_ \\ \_ & b & \_ & \_ \\ \_ & \_ & c & \_ \\ \_ & \_ & \_ & d. \end{array}$$

What can you deduce about the array?'

Elizabeth thought for a while and used pencil and paper.

'Your array is either

$$\begin{array}{cccc} a & d & b & c \\ c & b & d & a \\ d & a & c & b \\ b & c & a & d \end{array} \quad \text{or} \quad \begin{array}{cccc} a & c & d & b \\ d & b & a & c \\ b & d & c & a \\ c & a & b & d \end{array}\text{ .'}$$

'Well done. Now, note the two types of facts involved; *particular* facts like "there is a in the top left position" and *general* facts like "each row and each column contains a, b, c and d". The general fact involves words with a logical significance like "each", and you have to deduce particular facts from the general fact, like "row 1 contains a, b, c and d", etc.'

'So for logic, I need to know the significance of certain words.'

'That's right, Elizabeth. Now, that problem was not too difficult because the fact you deduced was a particular fact. The going is harder if we try to deduce a general fact. Consider this example. I tell you that the rain in Spain is mainly on the plain. Can you deduce that the plain in Spain is mainly in the rain?'

'They sound so alike that I would have thought they meant the same thing.'

'This is a very common fault. You must go from the words, which sound the same, to what they say about the situation. Let us draw a picture (figure 1). You are told that the shaded area is a small fraction of the Rain rectangle. I asked you to deduce that the dotted area is a small fraction of the Plain rectangle.'

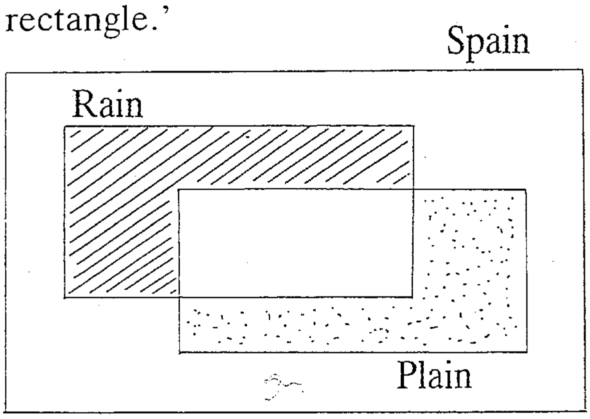


Figure 1

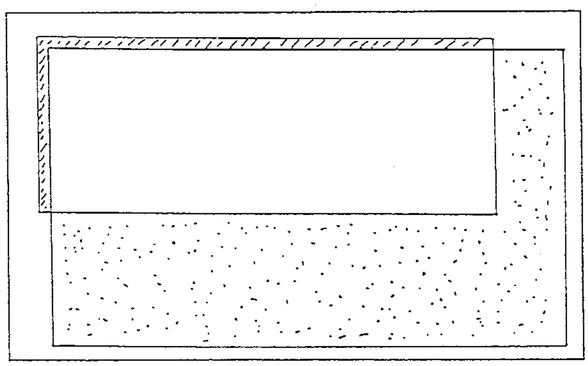


Figure 2

'I don't think it can be deduced, because you might have something like this (figure 2). The shaded area is a small fraction but the dotted is not.'

'She's got it, Watson!'

'Holmes, what about the logic that people such as politicians use in everyday arguments?'

'Really, Watson! They would like to think it was logic, but it simply consists of people stating the facts, or opinions made to sound like facts, which support their side of the argument; there is no deduction involved.'

Thus Elizabeth's lessons in logic went on for a number of months until one day Holmes announced, 'I think you are ready for your test. You will accompany Watson and me to the meeting of the London Mathematical Society which is taking place next week at Burlington House.'

So it was that the three of us found ourselves in the centre of a large set of mathematicians talking excitedly about their latest work.

'Let me claim, although I am not absolutely sure it is true, ...'

'... and that is going to give me ...'

'Let me impose a little more structure on my ...'

'Holmes, how is Elizabeth going to cope?'

'I have told her to look for the logical structure which is independent of the mathematical objects and to make logically correct replies. Listen to the young man talking to her.'

'It turns out that this is equal to that. Now this is not always the case, in fact it is generally not the case .... Sorry, I didn't catch your name.'

'Elizabeth. Do you have a simple counter-example?'

'Yes. It is something that looks like this ... which is what we wanted.'

'Do you think you can strengthen the result in some way?'

'I tried ... but it only worked in the $p = 1$ case.'

'We really want some idea of what is the best-possible result. I feel your third hypothesis is too strong and possibly can be omitted all together.'

'Gosh, do you think so? That would be very exciting if it were true. Look, can I send you a copy of what I have done so far, and a preprint of the other result?'

'That would be lovely.'

Back at Baker Street, Holmes was on top of the world.

'A marvellous performance, Elizabeth. You shocked the lot of them when you said that old conjecture was not statistically likely.'

Just then there was a knock at the door and Inspector Lestrade entered. 'Mr Holmes, we need your help to decode a secret message. The message is in 0's and 1's and this is coded into words. The following words have been decoded:

page – 1010, adjust – 001000, mack – 0000,
happen – 001100, glaze – 10000.

We want to decode "box", "quiver" and "way".'

'Just let me get pencil and paper, Lestrade. I shall want to write out the alphabet and do some counting.'

'I would not bother, Mr Holmes,' said Elizabeth. 'The code is so simple that a three-year-old child could decode the words just by looking at them. The answers you want are

box – 000, quiver – 100000, way – 001.'

Holmes looked astonished and Lestrade began to laugh. 'Thank you, Miss. You will have to watch out, Mr Holmes.'

'Elizabeth has advanced well in her logic,' I remarked.

'Lestrade grinned, 'Your trouble, Mr Holmes, is that you are still at the "elementary" stage.'

1989

Here is our annual puzzle. The aim is to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$, ! and concatenation (i.e. forming 19 from 1 and 9, for example). Thus

$$1 = (1\div 9) + (8\div 9),$$

for example (but $2 = 1^9 - 8 + 9$ is not allowed).

You should find it easier than last year and be able to express all the numbers in this way. How about going as far as 150? One of our readers, Mike Wenble, has written to say that he failed with five numbers up to 150.

Computer Column

MIKE PIFF

Polygons

The first program in the column was written by Paul de Sa of Newcastle Royal Grammar School. The idea is to be found on page 176 of *The Mathematical Experience* by P. J. Davis and R. Hersh, Pelican Books (1986). Draw almost any polygon and replace it by the polygon formed by taking the midpoints of its sides, in sequence. Repeat this process, and the polygons generally converge on an elliptical shape.

```
 10 REM***(c) Paul de Sa 1988
 20 REM Modified Mike Piff 1988
 30 REM for Mathematical Spectrum
 40 MODE0:VDU 23;8202;0;0;0;:Z%=0:N%=30
 50 DIM a%(N%),u%(N%):PROCinit:PROCdraw
 60 REPEAT
 70    PROCpause:PROCdraw
 80    PROCmid:PROCdraw
 90    Z%=Z%+1:PRINT TAB(1,1);Z%
100 UNTIL FALSE
110 DEF PROCpause
120 FOR I%=1 TO 1000:NEXT
130 ENDPROC
140 DEF PROCinit
150 FOR L%=1 TO N%:X%=RND(1000):Y%=RND(1000)
160 a%(L%)=X%:u%(L%)=Y%:NEXT
170 a%(0)=a%(N%):u%(0)=u%(N%)
180 ENDPROC
190 DEF PROCdraw
200 GCOL 3,1
210 MOVE a%(0),u%(0)
220 FOR C%=1 TO N%:DRAW a%(C%),u%(C%):NEXT
230 GCOL0,1
240 ENDPROC
250 DEF PROCmid
260 FOR L%=0 TO N%-1
270 a%(L%)=(a%(L%)+a%(L%+1))/2
280 u%(L%)=(u%(L%)+u%(L%+1))/2
290 NEXT
300 u%(N%)=u%(0):a%(N%)=a%(0)
310 ENDPROC
```

The second program is one to draw what are called Voronoi polygons. Choose n coloured base points in the plane, and then colour every other point in the rest of the plane in the same colour as the base point nearest to it in distance. The plane will then split into polygons along the boundaries between two equally near points, looking rather like soap bubbles. The program is set up to choose the base points randomly, and to colour them in flashing colours, so they don't disappear completely when the plane is filled in.

```
10 MODE2
20 ON ERROR GOTO 90
30PROCInitialise
40REPEAT
50 PROCGetNextPoint
60 PROCFindColourOfPoint
70 PROCPlotPoint
80UNTIL FNFinished
90PROCFinish
100 END
110DEF PROCInitialise
120 LOCAL i%
130 MaxCentre%=7
140 Colour%=1
150 DIM X%(MaxCentre%),Y%(MaxCentre%)
160SW%=160:SH%=256
170 NrfPixels%=SW%*SH%
180i%=0
190step%=3773
200CLG
210 FOR i%=1 TO MaxCentre%
220  X%(i%)=RND(SW%):Y%(i%)=RND(SH%)
230  GCOL0,MaxCentre%+i%+1:
PLOT69,X%(i%)*8,Y%(i%)*4
240 NEXT
250 VDU23,1,0;0;0;0;
260ENDPROC
270DEF PROCGetNextPoint
280i%=i%+step%
290 IF i%>=NrfPixels% THEN i%=i%-NrfPixels%
300x%=i% DIV SH%:y%=i% MOD SH%
310ENDPROC
320DEF PROCFindColourOfPoint
330 LOCAL c%,bestdist%,dist%
340 bestdist%=FNDist(x%,y%,X%(1),Y%(1))
350 Colour%=1
360 FOR c%=2 TO MaxCentre%
370  dist%=FNDist(x%,y%,X%(c%),Y%(c%))
380  IF dist%<bestdist% THEN Colour%=c%:
bestdist%=dist%
390 NEXT
400 ENDPROC
410 DEF FNDist(x%,y%,a%,b%)=(x%-a%)*(x%-a%)
+(y%-b%)*(y%-b%)
420 DEF PROCPlotPoint
430 IF POINT(x%,y%)=0 THEN GCOL 0,Colour%:
PLOT69,x%*8,y%*4
440 ENDPROC
450 DEF FNFinished=(i%=0)
460 DEF PROCFinish
470 x=GET
480 VDU23,1,1;0;0;0;
490 ENDPROC
```

If you replace `FNDist` by some other function, for example

`|x%-a%| + |y%-b%|,`

then you find the shape of the polygons changes. Try experimenting with other meaningful concepts of distance. For instance, what would happen if the screen were regarded as a projection of the surface of the earth?

A mathematics problem-solving group has been set up for students at Bilkent University in Ankara, Turkey, and they are keen to make contact with other groups or clubs of a similar nature. If you belong to such a group and would like to exchange problems and ideas with the group at Bilkent University, write to

Professor A. Bülent Özgüler,
Department of Electrical & Electronic Engineering,
Bilkent University,
P.O.B. 8, 06572 Maltepe,
Ankara, Turkey

Letters to the Editor

Dear Editor,

Mathematics versus chemistry

In mathematics there is Euler's Rule

$$\text{vertices} + \text{faces} = \text{edges} + 2$$

for a polyhedron or plane graph. In chemistry there is the Phase Rule

$$\text{phases} + \text{degrees of freedom} = \text{components} + 2.$$

I should be grateful if any reader could provide an explanation as to why these are exactly analogous.

Yours sincerely,
F. P. LYNCH
112 Salisbury Road,
Radcliffe,
Manchester M26 0WG

Dear Editor,

Factorisation

It was with great interest that I read Ian Stewart's article in *Mathematical Spectrum*, Volume 20, Number 3 on factorising large integers, as this topic is one of my favourites. As is the way of mathematics, however, some of the information is already (June 1988) out of date and I should like to mention some more recent results.

Thus, for instance, the largest non-prime Mersenne number which has been successfully factorised is $2^{509}-1$, as far as I know, and may be higher. For the record, $2^{509}-1$ factorises in the following manner:

$$12\,619\,129 \times 19\,089\,479\,845\,124\,902\,223 \times 647\,125\,715\,643\,884\,876\,759\,057 \times p104,$$

where $p104$ is a prime with 104 digits which I refrain from presenting in its entirety.

I also draw attention to two articles from Keith Devlin's Micromaths column in the *Guardian*, dating from 21 April and 12 May 1988, regarding the largest numbers factorised by a general factorisation method, these being, respectively, $(5^{160}-1)/(5^{32}-1)$, with 90 digits, and $(6^{131}-1)/(5\times263\times3931\times6551)$, with 92 digits. The first factorisation (which held the record for only a few weeks) was obtained by Robert Silverman in the USA, using a cluster of 24 Sun-3 minicomputers in a parallel network, and the second was obtained in the Netherlands by a team from the Centre for Mathematics and Computer Science in Amsterdam using an NEC SX-2 supercomputer.

I should dearly love to be able to make use of such computing power in my own little factorising survey of integers of the form $nx^n \pm 1$ for $2 \leqslant x \leqslant 10$ and all reasonable n. I have used various programs on various computers, most notably a program called Decomprime for the Apple Macintosh, which uses the elliptic-curve method and is written in Pascal, with assembly-language routines for the arithmetic calculations in order to give a reasonably fast performance. (I hasten to add that, although I have written many factorising programs, I didn't write this one; the author is a Frenchman named Dominic Bernardi.)

Some of the most interesting factorisations I have discovered are:

$$72\times2^{72}-1 = 434\,445\,574\,513\times782\,630\,568\,047,$$

$$86\times3^{86}+1 = 5\times161\,247\,323\times4929\,811\,475\,669\,351\times2331\,489\,844\,383\,610\,103,$$

$$70\times3^{70}-1 = 8741\,418\,487\times20\,044\,902\,695\,132\,106\,492\,648\,467,$$

$$56\times5^{56}-1 = 973\,808\,911\times5689\,806\,403\times10\,374\,144\,317\times1352\,025\,088\,559,$$

$$50\times8^{50}-1 = 7\times5953\times70\,092\,079\times24\,858\,733\,217\times5211\,303\,513\times18\,860\,602\,377\,791.$$

Some of the more interesting primes discovered during my survey are:

$$54\times3^{54}+1 \quad (28 \text{ digits}), \qquad 42\times5^{42}-1 \quad (31 \text{ digits}),$$

$$34\times7^{34}+1 \quad (31 \text{ digits}), \qquad 86\times3^{86}-1 \quad (43 \text{ digits}),$$

$$67\times4^{67}+1 \quad (43 \text{ digits}), \qquad 91\times6^{91}+1 \quad (73 \text{ digits})$$

The 'most wanted' factorisations I should like are of:

$$84\times3^{84}+1, \quad 67\times3^{67}-1, \quad 80\times3^{80}-1, \quad 85\times4^{85}+1, \quad 65\times4^{65}-1,$$

$$53\times6^{53}+1, \quad 54\times6^{54}-1, \quad 41\times8^{41}-1, \quad 50\times9^{50}+1, \quad 52\times9^{52}-1,$$

some of which would take more than a few seconds on a supercomputer.

In closing, I mention that the Mersenne number $2^{110\,503}-1$ has been proved prime by Walter Colquitt, only the third largest known and mysteriously overlooked in previous searches.

Yours sincerely

JOSEPH MCLEAN
9 Larch Road,
Glasgow G41 5DA

Dear Editor,

Pythagorean triangles

Following my letter on approximating $\sqrt{2}$ using Pythagorean triplets in Volume 21 Number 1, another interesting point on this matter has come to light.

As outlined before, to achieve a good approximation to $\sqrt{2}$, it is necessary to have a right-angled triangle with the two non-hypotenuse sides differing by as small an amount as possible, preferably one.

The first Pythagorean triangle with two consecutive integers for its non-hypotenuse sides is clearly the $3, 4, 5$ triangle: to obtain this we let $p = 2$ and $q = 1$ in the formulae

$$x = p^2-q^2, \qquad y = 2pq, \qquad z = p^2+q^2$$

for a primitive Pythagorean triplet which was in the previous letter. (Here, p and q are coprime integers with $p > q > 0$ and one of p and q is even and the other is odd.) More triangles of this form can be obtained by the successive iteration $q_{n+1} = p_n$, $p_{n+1} = 2p_n + q_n$. Thus the next triangle with two consecutive integer sides is obtained when $q = 2$ and $p = 5$, giving a 20, 21, 29 triangle, and the next when $q = 5$ and $p = 12$, giving a 119, 120, 169 triangle.

I think that this iteration provides all such triangles.

Yours sincerely,

PAUL DE SA
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[Paul de Sa's conjecture is true. It is known that these triangles are all given by the formulae

$$x_1 = 3, \qquad y_1 = 4, \qquad z_1 = 5,$$

$$x_{n+1} = 3x_n + 2z_n + 1, \qquad y_{n+1} = x_{n+1} + 1, \qquad z_{n+1} = 4x_n + 3z_n + 2 \quad (n = 2, 3, \ldots);$$

see W. Sierpiński *Elementary Theory of Numbers* (Warsaw, 1964) p. 44. A little effort will show that Paul de Sa's formulae give the same triples.—Editor.]

Dear Editor,

A factorial identity from difference tables

It is well known that difference tables constructed for powers of natural numbers lead eventually to factorial expressions. Thus the second difference for successive squares is 2!, the third difference for successive cubes is 3!, and so on.

$$\begin{array}{ccccccccc} 1^2 & & 2^2 & & 3^2 & & 4^2 & & 5^2 \\ & 3 & & 5 & & 7 & & 9 & \\ & & 2 & & 2 & & 2 & & \end{array} \qquad \begin{array}{ccccccccccc} 1^3 & & 2^3 & & 3^3 & & 4^3 & & 5^3 & & 6^3 \\ & 7 & & 19 & & 37 & & 61 & & 91 & \\ & & 12 & & 18 & & 24 & & 30 & & \\ & & & 6 & & 6 & & 6 & & & \end{array}$$

Obviously, some general expression must account for regularity of this kind and, when I was a sixth-former, many years ago, I found that it could be explained by the identity

$$n! = A^n - \binom{n}{1}(A-1)^n + \binom{n}{2}(A-2)^n - \ldots + (-1)^n(A-n)^n,$$

where A is any number and n is a positive integer. This may or may not be a well-established result, but it is certainly little known.* Treatments of difference tables seem to be more concerned with the constancy of the final difference than with its factorial nature.

I find this identity interesting because it provides a good opportunity for developing a proof by induction, where it is used that the expression is an identity in A holding for all values of A.

We may ask whether the infinite series

$$A^n - n(A-1)^n + \tfrac{1}{2}n(n-1)(A-2)^n - \dots,$$

where n is not now a positive integer, is convergent. Preliminary tests suggest that it diverges, though very slowly, but I invite readers to supply a proof.

Yours sincerely,
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*The identity given in Peter Rowlands' letter occurs in Riordan's *Combinatorial Identities* p. 119.—Editor.

Dear Editor,

Twin primes

In Volume 18 Number 2, p. 33 and in Volume 19 Number 1, pp. 19–20, a formula for prime numbers is discussed. (We note, in passing, that the same formula is discussed in reference 1.) It may be of interest for the reader to know about a similar formula for an interesting subclass of primes, the so-called 'twin primes'. We recall that the prime p is said to be a twin prime if $p+2$ is again a prime. For example, 3, 5, 11, 17, 41, etc., are twin primes. A famous unsolved problem is whether the set of all twin primes is infinite.

It is a well-known fact, proved by L. Euler, that the series

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \dots = \sum \frac{1}{p},$$

p being a prime, is divergent. A striking result of the Norwegian mathematician Viggo Brun (reference 2) is that the series formed by the reciprocals of the twin primes converges. (The sum of the series of reciprocals of the twin primes is called the 'Brun constant', and it is an unsolved problem to determine the arithmetical nature of this number, i.e., whether it is rational, irrational or a transcendental number.)

We now propose a function whose range of values is exactly the set of twin primes. We use the following result of Clement (reference 3): n is a twin prime if and only if

$$4[(n-1)!+1]+n \equiv 0 \pmod{n[n+2]}.$$

Now put

$$k = mn(n+2) - 4[(n-1)!+1] - n, \qquad p = \tfrac{1}{2}(n-3)[|k^2-1| - (k^2-1)] + 3.$$

Then, as m and n vary, the range of values of p is precisely the set of twin primes.

Of course, this formula is very inefficient, since for 'most' values of m and n, p will be equal to 3, but I believe it to be of some interest. (See reference 4 for other related facts.)

References
1. R. Honsberger, *Mathematical Gems II*, Chapter 4 (Mathematical Association of America, 1976).
2. V. Brun, *Le crible d'Eratosthène et le théorème de Goldbach* (Viden, 1920), Skrifter 1 No. 3.
3. P. A. Clement, Congruences for sets of primes, *Amer. Math. Monthly* **56** (1949), 23–25.
4. Vasile Ion Istrăţescu, *Number Theory: An Introduction*. (Book in preparation. Contains more details about prime-representing functions, twin-prime-representing functions etc., with full references.)

Yours sincerely,
V. I. ISTRĂŢESCU
Kiel,
West Germany

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

21.5. (Submitted by Peter Rowlands, De la Salle Sixth Form College, Salford)
Prove the identity in the letter on pages 63 and 64.

21.6. (Submitted by Stephen Gourley, Mansfield College, Oxford)
Let $0 \leqslant x \leqslant \frac{1}{2}\pi$. Prove that

$$(\pi^2+24)x-4x^3 \leqslant 12\pi \sin x.$$

21.7 (Submitted by Malcolm Smithers, The Open University)
The letters AACEINNOSTW can be rearranged and the resulting sequences of letters are ordered in a dictionary ordering (so that the second sequence is AACEINNOSWT). How far along the sequence do you have to go to reach a famous mathematician?

21.8 (Submitted by Seung-Jin Bang, Seoul, Korea)
Show that the function

$$f(x) = \frac{\displaystyle\int_0^x \sin^q t \, dt}{\displaystyle\int_0^x \sin^p t \, dt},$$

where $p > q$ and $0 < x \leqslant \frac{1}{2}\pi$, is strictly decreasing.

Solutions to Problems in Volume 20 Number 3

20.9 Prove that $15^n - 2^{3n+1} + 1$ is divisible by 98 for all positive integers n.

Two solutions by Amites Sarkar (Winchester College)

1. We use induction on n. The result is true when $n = 1$. Assume that

$$98 \mid (15^k - 2^{3k+1} + 1)$$

for some natural number k. Then

$$\begin{aligned}
& 98 \mid 15(15^k - 2^{3k+1} + 1) \\
\Rightarrow\ & 98 \mid (15^{k+1} - 2^{3(k+1)+1} + 1 - 7 \times 2^{3k+1} + 14) \\
\Rightarrow\ & 98 \mid [15^{k+1} - 2^{3(k+1)+1} + 1 - 14(8^k - 1)] \\
\Rightarrow\ & 98 \mid [15^{k+1} - 2^{3(k+1)+1} + 1 - 14(8-1)(8^{k-1} + 8^{k-2} + \ldots + 1)] \\
\Rightarrow\ & 98 \mid [15^{k+1} - 2^{3(k+1)+1} + 1 - 98(8^{k-1} + 8^{k-2} + \ldots + 1)] \\
\Rightarrow\ & 98 \mid (15^{k+1} - 2^{3(k+1)+1} + 1).
\end{aligned}$$

This completes the inductive step.

2. $15^7 = 170\,859\,375 \equiv 1 \pmod{98}$ so that $15^{7r} \equiv 1 \pmod{98}$. Also

$$2^{21} = 2097\,152 \equiv 50 \pmod{98} \quad \text{and} \quad 50^2 = 2500 \equiv 50 \pmod{98},$$

so that $2^{21r} \equiv 50^r \equiv 50 \pmod{98}$. Thus $2^{21r+1} \equiv 100 \equiv 2 \pmod{98}$. If we divide n by 7, we obtain $n = 7r + k$, where r and k are integers with $0 \leqslant k \leqslant 6$. Thus

$$\begin{aligned}
15^n - 2^{3n+1} + 1 &= 15^{7r+k} - 2^{3(7r+k)+1} + 1 \\
&= 15^{7r}15^k - 2^{21r+1}2^{3k} + 1 \\
&\equiv 1 \times 15^k - 2 \times 2^{3k} + 1 \pmod{98} \\
&\equiv 15^k - 2^{3k+1} + 1 \pmod{98}.
\end{aligned}$$

Therefore we need only check that the required congruence holds for $0 \leqslant n \leqslant 6$. This can easily be done.

Also solved by Peter Denison (Royal Grammar School, High Wycombe), Nicholas Shea (Gresham's School, Holt), Eddie Cheng (Memorial University of Newfoundland), Gregory Economides (Royal Grammar School, Newcastle upon Tyne), Henry Schaefer (Saint Bonaventure University, USA) and Reha Tütüncü (Problem Solving Group, Bilkent University, Ankara, Turkey).

20.10 Let x_1, x_2, x_3 and x_4 be positive real numbers. Show that

$$\frac{x_1+x_3}{x_1+x_2} + \frac{x_2+x_4}{x_2+x_3} + \frac{x_3+x_1}{x_3+x_4} + \frac{x_4+x_2}{x_4+x_1} \geqslant 4,$$

and determine when equality occurs.

Solution by Haluk Yilmaz (Problem Solving Group, Bilkent University)

For any positive real numbers a and b,

$$(a-b)^2 \geqslant 0 \;\Rightarrow\; (a+b)^2 \geqslant 4ab \;\Rightarrow\; \frac{a+b}{ab} \geqslant \frac{4}{a+b} \;\Rightarrow\; \frac{1}{a} + \frac{1}{b} \geqslant \frac{4}{a+b}. \qquad (*)$$

The expression can be written as

$$
\begin{aligned}
(x_1+x_3)\left(\frac{1}{x_1+x_2}+\frac{1}{x_3+x_4}\right)&+(x_2+x_4)\left(\frac{1}{x_2+x_3}+\frac{1}{x_1+x_4}\right)\\
&\geqslant (x_1+x_3)\left(\frac{4}{x_1+x_2+x_3+x_4}\right)+(x_2+x_4)\left(\frac{4}{x_1+x_2+x_3+x_4}\right)\\
&= 4.
\end{aligned}
$$

Equality occurs in (*) if and only if $a = b$, so that equality will occur if and only if $x_1+x_2 = x_3+x_4$ and $x_2+x_3 = x_1+x_4$, and these hold if and only if $x_1 = x_3$ and $x_2 = x_4$.

20.11 Let

$$F(x,y,z) = \frac{1-z}{xz+y}(1-e^{-(xz+y)})+\frac{1-z}{xz+y-x}(e^{-(xz+y)}-e^{-x})+\frac{1}{x}(1-e^{-x}),$$

where $x, y > 0$, $0 < z < 1$ and $xz+y \neq x$. Show that $F(x,y,z)$ is positive.

We received no solutions to this problem, so we give the solution by Seung-Jin Bang, who submitted the problem. (Added in proof: A solution has since been received from Gregory Economides.)

By the mean-value theorem, there exist real numbers θ_1, θ_2 and θ_3 such that

$$F(x,y,z) = (1-z)e^{-\theta_1}-(1-z)e^{-\theta_2}+e^{-\theta_3}$$

and $0 < \theta_1 < xz+y$, $0 < \theta_3 < x$ and θ_2 is a real number between x and $xz+y$. If $xz+y < x$, then $\theta_1 < \theta_2$ and $e^{-\theta_1} > e^{-\theta_2}$, so that $F(x,y,z)$ is positive. If $xz+y > x$, then $0 < \theta_3 < x < \theta_2$. Thus

$$-(1-z)e^{-\theta_2}+e^{-\theta_3} = e^{-\theta_3}[1-(1-z)e^{-(\theta_2-\theta_3)}]$$

is positive and again $F(x,y,z)$ is positive.

20.12 A set of $n > 3$ objects is to be arranged in random order. One object is labelled A, one L and one N; the rest are unlabelled. Assuming that all $n!$ orderings are equally likely, find the probabilities of the events

E_2: at least one of either L or N is in the first or second position, and both L and N precede A;

E_3: either L or N is in the first position, and both L and N precede A.

Repeat the calculations when there are just two labelled objects, one labelled A and one labelled L/N.

Solution by Amites Sarkar

With three objects,

$$
\begin{aligned}
\mathrm{pr}(E_2) = {} & 2[\mathrm{pr}(L \text{ is in 1st position})\mathrm{pr}(N \text{ precedes } A \text{ if } L \text{ is in 1st position})\\
&+\mathrm{pr}(L \text{ is in 2nd position})\mathrm{pr}(N \text{ precedes } A \text{ if } L \text{ is in 2nd position})]\\
&-\mathrm{pr}(L \text{ and } N \text{ occupy 1st and 2nd positions in any order})\\
= {} & 2\left(\frac{1}{n}\times\frac{1}{2}+\frac{1}{n}\times\frac{1}{2}\right)-\frac{2}{n(n-1)} = \frac{2(n-2)}{n(n-1)},
\end{aligned}
$$

$$\begin{aligned}\mathrm{pr}(E_3) &= 2[\mathrm{pr}(L \text{ is in 1st position})\mathrm{pr}(N \text{ precedes } A \text{ if } L \text{ is in 1st position})]\\ &= 2\left(\frac{1}{n}\times\frac{1}{2}\right) = \frac{1}{n}.\end{aligned}$$

With two objects,

$$\begin{aligned}\mathrm{pr}(E_2) &= \mathrm{pr}(L/N \text{ is in 1st position})\mathrm{pr}(L/N \text{ precedes } A \text{ if } L/N \text{ is in 1st position})\\ &\quad + \mathrm{pr}(L/N \text{ is in 2nd position})\mathrm{pr}(L/N \text{ precedes } A \text{ if } L/N \text{ is in 2nd position})\\ &= \frac{1}{n}\times 1+\frac{1}{n}\times\frac{n-2}{n-1}\\ &= \frac{2n-3}{n(n-1)},\end{aligned}$$

$$\begin{aligned}\mathrm{pr}(E_3) &= \mathrm{pr}(L/N \text{ is in 1st position})\mathrm{pr}(L/N \text{ precedes } A \text{ if } L/N \text{ is in 1st position})\\ &= \frac{1}{n}\times 1 = \frac{1}{n}.\end{aligned}$$

Also solved by Nicholas Shea and Gregory Economides.

20.13 Show that the ordinary binomial distribution cannot be used to compute P_0^* as defined on page 80 of Volume 20 Number 3.

As we received no solutions, we give the solution of the proposer of the problem who also wrote the article on which the problem was based. (Added in proof: A solution has since been received from Gregory Economides.)

Neither the binomial nor multinomial distribution can be used to compute the probability P_0^*. Let us number the states and territories from 1 to 8 in the order given in tables 1 and 2 of the article. Define random variables X_i and Y_i $(i = 1, 2, \ldots, 8)$ by

$$X_i = \begin{cases} 1 & \text{(if event } E_2 \text{ occurs in state } i\text{)},\\ 0 & \text{(otherwise)},\end{cases} \qquad Y_i = \begin{cases} 1 & \text{(if event } E_3 \text{ occurs in state } i\text{)},\\ 0 & \text{(otherwise)},\end{cases}$$

and set $T = Y_1 + Y_2 + \ldots + Y_8$. Then P_0^* is the probability of the event

$$E_0 = \{T \geqslant 6,\ X_1+Y_1 = 1,\ X_2+Y_2 = 1,\ \ldots,\ X_8+Y_8 = 1\}.$$

Let p_i and q_i denote, respectively, the probability of events E_2 and E_3 in state i $(i = 1, 2, \ldots, 8)$ (see the last two columns of table 1). Then

$$P_0^* = \sum\left(\prod_{i=1}^{8} p_i^{x_i} q_i^{1-x_i}\right),$$

where the summation is over all the x's consistent with E_0.

The only connection with the binomial distribution is that there are two outcomes of interest, E_2 and E_3, in each state ('trial'). Unlike the binomial distribution, these outcomes are not mutually exclusive. More importantly, the probability of either outcome varies from state to state.

Correction. Malcolm Smithers has pointed out that there is an error in the published solution to Problem 20.7 in Volume 21 Number 1. The 19 187th term in $C(19, 11)$ is 1011111100011101000, or 391 400 to base 10.

Reviews

The Recursive Universe. By WILLIAM POUNDSTONE. Oxford Paperbacks, 1987. Pp. 252. £5·95 (ISBN 0-19-285173-x).

This is a beautifully written, well researched and accessible book. The basic idea considered by William Poundstone is that the apparently boundless complexity of the Universe is arbitrated by a few simple laws. To illustrate that great complexity of structure and dynamics *can* arise from the operation of a few elementary rules, the author considers the computer game of 'Life'. The book consists of a series of essays describing, in a clear and stimulating way, carefully chosen problems from physics interspersed with allegorical forays into aspects of the 'Life' game. The real point of the book is to demonstrate how, in our collective experience, complex and fascinating problems in physics have been resolved in terms of elegant and simple *ideas*. In this, the book is a success. But, beyond this, it is a well-worthwhile 'read' for its descriptive essays in physics. So, in order to learn to be excited by the ideas behind thermodynamics, information theory and self-reproducing machines—read this book!

University of Sheffield — DAVID ROSCOE

Taxicab Geometry: An Adventure in Non-Euclidean Geometry. By EUGENE F. KRAUSE. Dover Publications, New York, 1986. Pp. viii+88. £3·15 (ISBN 0-486-25202-7).

This interesting little book, consisting mainly of exercises and questions, is designed to be worked through (I should recommend a quiet weekend and plenty of squared paper!) and the gentle progression, clear diagrams and selected answers are an incentive to give it a try.

The idea is very simple: the standard Euclidean distance function from $A = (a_1, a_2)$ to $B = (b_1, b_2)$, i.e. $\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$, representing the distance as the crow flies, is replaced by the 'taxicab' distance i.e. $|a_1-b_1|+|a_2-b_2|$, representing the distance a taxi would have to drive along roads parallel to the axes. Through the exercises one is introduced to the 'taxi circle' and the 'taxi' equivalents of many familiar shapes.

The author is keen to demonstrate that taxicab geometry is not just a simple example of a non-Euclidean geometry but also that it serves as a sensible model for urban geography. Several of the exercises explore this theme and the reader is frequently called upon to help solve the accommodation problems of Alice and Bruno, an ideal American couple!

The more mathematically minded will enjoy the appendix, where the axioms for Euclidean and taxicab geometry are compared, and also the 'taxi' solution to the three-point Steiner problem, which demonstrates some of the beauty of this alternative geometry.

The book is written in an easily readable style and should be accessible and of interest to members of a sixth form.

Repton School — G. B. ATTWOOD

Introduction to Statistics. A Computer Illustrated Text. By A. W. BOWMAN and D. R. ROBINSON. Adam Hilger, Bristol, 1987. Pp. ix+205. £15 basic pack.

This text contains material covered in most first courses in statistics although probability theory is only briefly discussed, the reader being referred to the companion volume *Introduction to Probability* for a more detailed treatment. The book comprises six chapters covering graphical display of data, probability and sampling, estimation, confidence intervals and statistical testing including some non-parametric tests and linear regression. Three appendices follow which give clear guidance on using the programs, several of which have a facility for reading data from a file on disc, thus enabling students to use their own data. The stated purpose of the accompanying software is to explain and illustrate the ideas introduced in the text, so that the text should be read with the micro to hand. Certainly the book in isolation has limited use, as the programs have been well-integrated into the text.

Without doubt this makes a novel change from the usual introductory textbooks and, with the move towards the greater use of computers in teaching, will be welcomed by many. The software is very easy to use; previous computer experience is almost unnecessary. Like any practical work in this area, these programs provide an invaluable contribution to a student's understanding of the basic concepts of statistics. However, it is probably necessary to revert to more traditional texts to ensure that this solid foundation is firmly built upon. Although its price will probably prohibit one copy per student, every statistics department should certainly invest in this book—its use with students either individually or in groups can only be extremely beneficial.

Solihull Sixth Form College CAROL NIXON

A First Course in Coding Theory. By RAYMOND HILL. Oxford Applied Mathematics and Computing Science Series, Clarendon Press, Oxford, 1986. Pp. xii+251. £30·00 hardback (ISBN 0-19-853804-9), £15·00 paperback (ISBN 0-19-853803-0).

Have you looked at any new books recently? Next time you have one in your hands, look inside the first few pages or on the book cover for its ISBN (International Standard Book Number): this is a 10-digit number like

0-19-853804-9

which contains information about the book (the language in which it is written, the name of the publisher, and so on). These numbers are designed, by means of the last so-called 'check' digit (9 in the above example), so that if, in the reproduction or transmission or copying of the number, a mistake is made in one digit, or two digits are transposed, then the result can be recognized as faulty, and not an allowable ISBN; thus the error can be detected. Transposition of two digits can easily happen in typing. The ISBN is an example of a codeword from an 'error-detecting code'.

Do you remember, perhaps when you were at Junior School or Middle School, or perhaps later, seeing photographs of planets (Mars, Jupiter, Saturn) that had been transmitted back to Earth by spaceships such as Viking 1 and Voyagers 1 and 2? In fact, the photographing of other planets by spaceships was already taking place in the 1960s. In 1965, the spaceship Mariner 4 took 22 complete photographs of Mars. Each picture was broken down into 200×200 picture elements. Each element was assigned a binary 6-tuple representing one of 64 brightness levels from white (000000) to black (111111). These binary 6-tuples were then transmitted back to Earth so that the picture could be reconstructed.

But interference in the transmission of such messages across vast distances of space could have caused received messages to differ from the transmitted ones, resulting in distortion of the picture. To combat this difficulty, each binary 6-tuple was first encoded, before transmission, by the addition of 26 additional bits to turn it into a 32-bit codeword from a certain Reed–Muller code. This is designed so that, even if several (up to 7, in fact) errors occur in the transmission of one of these 32-bit codewords, then the intended message can still be recovered from the received 32-bit string. The Reed–Muller code used is an example of a 7-error correcting code.

The mathematical theory of error-correcting codes uses concepts from various different areas of (mainly pure) mathematics, such as projective planes and design theory from combinatorics, vector spaces, finite fields, rings and groups from algebra and quadratic residues from number theory. It is an exciting subject because the undergraduate can see several topics studied in apparently disconnected contexts being brought together and used in a surprising way. Raymond Hill's book is a good introduction for a student who would like to learn something about error-correcting codes: not much knowledge is assumed on the part of the reader (for example, Chapters 3 and 4 introduce finite fields and vector spaces), some difficult technical proofs are omitted (and the interested reader can chase up the references given to other sources) and many examples and exercises are given to help the reader to consolidate the ideas. In short, Dr Hill's book can be strongly recommended.

University of Sheffield R. Y. SHARP

Mathematical Recreations and Essays. By W. W. ROUSE BALL and H. S. M. COXETER. Thirteenth edition. Dover Publications, New York, 1987. Pp. xvii + 428. £7·60 softback (ISBN 0-486-25357-0).

This is the latest edition of the book you never read whilst at school because it looked old and dusty. Now you're a teacher your students think the same. My school library holds two copies—editions of 1928 and 1920. The book is still mainly written in Ball's own language which is quaint but not unpleasant, but it has a much more appealing presentation. There have been changes—illustrating changes in our perceptions of mathematics and changes in our lives. £.s.d. (remember it?) is out. In the 1920 edition ciphers were confined, with the string, to part 2—only just connected with maths. Yet, in the new edition there is a major post-Ball chapter on the subject of cryptography.

It is always surprising that so many mathematical problems are easy to describe and grasp, yet so difficult to solve. This precious institution is full of them.

Although the book is too difficult for young secondary students to tackle alone, much of the material, when digested and re-presented by teachers, could be used with these children. The book will appeal to older students with mathematical inclination (by definition) but not all chapters will appeal to everyone. I would gladly sacrifice the chapter on calculating prodigies in favour of the long abandoned chapter on string, whereas I find the arithmetical recreations are compulsive reading.

My only reservation is that recent and some not so recent work is neglected. I feel that cryptography should mention Türing, computers, etc.—the most modern reference is 'the most recent (cipher) machines are electrical in operation'. This comment has been out of place for all of the microcomputer decade. Again there is not even a footnote to the four-colour map theorem discussion to indicate developments in recent years.

Should you buy the book? If there is no post-decimalisation copy in your library spend the paltry £7·60. Put the old discarded version on your own bookshelf. Otherwise hope for a further updated centenary edition in four years' time.

Queen Elizabeth's High School, Gainsborough CHRIS DU FEU

Time Travel and other Mathematical Bewilderments. By MARTIN GARDNER. W. H. Freeman, Oxford, 1987. Pp. ix+295. £16·50 hardback (ISBN 0-7167-1924-x), £11·50 paperback (ISBN 0-7167-1925-8).

This is the twelfth collection based on Martin Gardner's columns in *Scientific American*. If you are familiar with any of the others, don't waste your time reading a review—get on to the book. The uninitiated (are there any of you?), please read on. Each chapter is a self-contained unit and has no particular connection with any other. The topics covered are diverse and include, for example, maps, magic squares and music. My favourite parts are the paragraphs on the game hex, the chapter written for the April *Scientific American* issue and the IQ test to find the symbol which is the 'most different'. The book is absorbing—Martin Gardner's writing is as fresh and witty as ever. A-level students approaching the book intelligently will have little difficulty and yet the mathematical concepts are often very deep. Ideas are suggested which require little mathematical knowledge but which will amply reward mathematical investigation. Students will get more of the flavour of abstract mathematics from one chapter of this book than from a month of routine examination mathematics.

Like its companions, this book is neither to be taken lightly, nor to be rushed. To do any part justice needs pencil and paper, thought and much time. To deny it a place on your school library shelves will be to the detriment of your students—mathematicians and other ranks alike.

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