# SKOLIAD No. 129

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **June 1, 2011**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest for this month is the City Competition of the Croatian Mathematical Society, 2010, secondary level, grade 1. Our thanks go to Željko Hanjš, University of Zagreb, Croatia, for providing us with this contest and for permission to publish it.

## Compétition 2010 de la Société mathématique croate
### Niveau secondaire, première année

**1**. Soit $n$ un entier positif et $a$ un nombre réel non nul. Simplifier la fraction

$$\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5}\,.$$

**2**. Trouver un entier positif qui, multiplié par $9$ donne un entier compris entre $1100$ et $1200$, et lorsque multiplié par $13$ donne un entier compris entre $1500$ et $1600$.

**3**. Dans le plan, on donne trois cercles de rayon $2$, de sorte que le centre de chacun d'eux se trouve à l'intersection des deux autres. Trouver l'aire de l'intersection des trois disques limités par ces cercles.

**4**. On considère l'entier $n$. Soit $m$ l'entier obtenu à partir de $n$ en y biffant le chiffre des unités. Si $n - m = 2010$, trouver $n$.

**5**. Un sac contient un grand nombre de balles rouges, blanches et bleues. Chaque enfant d'un groupe donné sort du sac au hasard trois balles. Quel est le nombre minimal d'enfants dans ce groupe permettant que deux d'entre eux aient la même combinaison de balles, c.-à-d. le même nombre de balles de chaque couleur ?

**6**. Si $a^2 + 2b^2 = 3c^2$, montrer que

$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c}$$

est un entier positif.

**7**. Un triangle rectangle $ABC$, d'angle droit en $B$ et dont les côtés de l'angle droit mesurent $15$ et $20$, est congruent à un triangle $BDE$ avec l'angle droit en $D$. Le point $C$ est situé strictement à l'intérieur du segment $\overline{BD}$, et les points $A$ et $E$ sont situés du même côté de la droite $BD$.

(a) Trouver la distance entre les points $A$ et $E$.

(b) Trouver l'aire de l'intersection des triangles $ABC$ and $BDE$.

**8**. Soit $p$ et $q$ deux nombres premiers impairs distincts. Montrer que l'entier $(pq+1)^4 - 1$ possède au moins quatre diviseurs premiers différents.

## City Competition of the Croatian Mathematical Society, 2010
## Secondary level, Grade 1

**1**. Let $n$ be a positive integer and $a$ a non-zero real number. Reduce the fraction

$$\frac{a^{3n+1} - a^4}{a^{2n+3} + a^{n+4} + a^5}.$$

**2**. Find a positive integer which when multiplied by $9$ gives an integer between $1100$ and $1200$, and when multiplied by $13$ gives an integer between $1500$ and $1600$.

**3**. Three circles, each with radius $2$, are given in the plane such that the centre of each lies on the intersection of the other two. Determine the area of the intersection of the three disks bounded by those circles.

**4**. Consider the integer $n$. Let $m$ be the integer obtained from $n$ by removing its ones digit. If $n - m = 2010$, find $n$.

**5**. A bag contains a sufficient number of red, white, and blue balls. Each child in a given group takes three balls at random from the bag. What is the smallest number of children in the group that ensures that two of them have taken the same combination of balls, that is, the same number of balls of each colour?

**6**. If $a^2 + 2b^2 = 3c^2$, prove that

$$\left(\frac{a+b}{b+c} + \frac{b-c}{b-a}\right) \cdot \frac{a+2b+3c}{a+c}$$

is a positive integer.

**7**. A right triangle, $\triangle ABC$, with legs of lengths $15$ and $20$ and the right angle at vertex $B$ is congruent to a triangle, $\triangle BDE$, with the right angle at vertex $D$. The point $C$ lies strictly inside the segment $\overline{BD}$, and the points $A$ and $E$ are on the same side of the straight line $BD$.

(a) Find the distance between points $A$ and $E$.

(b) Find the area of the intersection of $\triangle ABC$ and $\triangle BDE$.

**8**. Let $p$ and $q$ be different odd prime numbers. Prove that the integer $(pq+1)^4 - 1$ has at least four different prime divisors.

Next we give the solutions to the City Competition of the Croatian Mathematical Society, 2009, Secondary Level, Grade 1, given in Skoliad 123 at [2010 : 67–68].

**1**. Reduce the fraction

$$\frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a+1)(a+2)}\,.$$

*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

First, factor the numerator:

$$\begin{aligned} a^4 - 2a^3 - 2a^2 + 2a + 1 &= (a^4 - 2a^2 + 1) - 2a^3 + 2a \\ &= (a^2-1)^2 - 2a(a^2-1) = (a^2-1)(a^2-2a-1) \\ &= (a+1)(a-1)(a^2-2a-1)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{a^4 - 2a^3 - 2a^2 + 2a + 1}{(a+1)(a+2)} &= \frac{(a+1)(a-1)(a^2-2a-1)}{(a+1)(a+2)} \\ &= \frac{(a-1)(a^2-2a-1)}{a+2}\,. \end{aligned}$$

*Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.*

*Note that the denominator is already factored as $(a+1)(a+2)$. Therefore, the only candidates for reducing are $a+1$ and $a+2$. If you make $a=-2$ in the numerator, you get $21$, so the expression cannot be reduced by $a+2$. If you make $a=-1$ in the numerator, you get $0$, so reducing by $a+1$ is possible. You can now obtain the answer by polynomial division.*

**2**. If you write the digit $3$ on the left side of a two-digit number, you obtain, of course, a three-digit number. If twice the three-digit number equals $27$ times the two-digit number, what is the original two-digit number?

*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

Let $x$ be the original two-digit number. When the digit $3$ is inserted in front of $x$, the resulting three-digit number is $300+x$. The given relationship between the two numbers is then that $2(300+x) = 27x$. Solving this equation yields that $x = 24$.

*Also solved by ELLEN CHEN, student, Burnaby North Secondary School, Burnaby, BC; LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; and GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.*

**3**. Find the largest integer $n$ such that $3\left(n-\frac{5}{3}\right)-2(4n+1)>6n+5$.

*Solution by Ellen Chen, student, Burnaby North Secondary School, Burnaby, BC.*

If $3(n-\frac{5}{3})-2(4n+1)>6n+5$, then $3n-5-8n-2>6n+5$, so $-5n-7>6n+5$, so $-12>11n$. Thus $n<-\frac{12}{11}\approx-1.09$, so the largest integer value for $n$ is $-2$.

*Also solved by MATTHEW NG, student, St. Francis Xavier Secondary School, Mississauga, ON.*

**4**. Find the number of divisors of $288$.

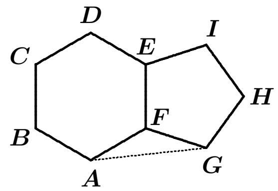
*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

The prime factorisation of $288$ is $2^5\cdot3^2$. Therefore, any divisor of $288$ has the form $2^a\cdot3^b$, where $a$ and $b$ are integers such that $0\le a\le5$ and $0\le b\le2$. You have $6$ choices for $a$ and $3$ choices for $b$, for a total of $6\cdot3=18$ choices. These are $1$, $2$, $3$, $4$, $6$, $8$, $9$, $12$, $16$, $18$, $24$, $32$, $36$, $48$, $72$, $96$, $144$, and $288$.

*Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC.*

*Our solver's method for counting divisors is much easier than listing divisors systematically. If you were not familiar with it, read the solution again.*

**5**. In the figure, $ABCDEF$ is a regular hexagon while $EFGHI$ is a regular pentagon. Determine the angle $\angle GAF$.




*Solution by Natalia Desy, student, SMA Xaverius 1, Palembang, Indonesia.*

The angle sum of an $n$-gon is $180(n-2)$, so the angle sum of a hexagon is $720^\circ$ and the angle sum of a pentagon is $540$. Since the polygons in the problem are regular, $\angle AFE=120^\circ$ and $\angle GFE=108^\circ$. Therefore, $\angle AFG=360^\circ-120^\circ-108^\circ=132^\circ$. Since $FG=EF=AF$, $\triangle AFG$ is isosceles, so

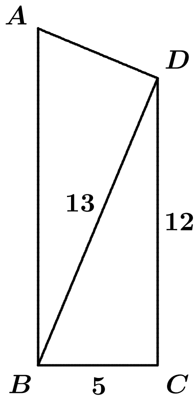
$$\angle GAF=\frac{180^\circ-132^\circ}{2}=24^\circ.$$

*Also solved by ELLEN CHEN, student, Burnaby North Secondary School, Burnaby, BC; and MATTHEW NG, student, St. Francis Xavier Secondary School, Mississauga, ON.*

**6**. In a trapezoid $ABCD$, the angle at $B$ is a right angle, and the diagonal $BD$ is perpendicular to the leg $AD$. The length of the leg $BC$ is $5$, and the length of the diagonal $BD$ is $13$. Find the area of the trapezoid $ABCD$.

*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

For diagonal $BD$ to be perpendicular to $AD$, the parallel sides of the trapezoid must be $AB$ and $CD$, as in the figure. Thus, $\angle ABC = \angle BCD = \angle ADB = 90^\circ$. It now follows from the Pythagorean Theorem that $CD = \sqrt{13^2 - 5^2} = 12$. Moreover, $\angle ABD = \angle BDC$, so $\triangle ABD$ is similar to $\triangle BDC$. Therefore, $\frac{AB}{BD} = \frac{BD}{DC}$, so $\frac{AB}{13} = \frac{13}{12}$, so $AB = \frac{169}{12}$.




The area of trapezoid $ABCD$ is thus

$$\frac{AB + CD}{2} \cdot BC = \frac{\frac{169}{12} + 12}{2} \cdot 5 = \frac{1565}{24}.$$

*Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.*

**7**. At Tihana's birthday party, the first guest arrived the first time the bell rang. Each time the bell rang thereafter the number of guests arriving was two more than the number that had arrived the previous time the bell rang. If the bell rang $n$ times, how many guests attended the party?

*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

From the pattern in the table below it is easy to see that $2n - 1$ guests arrived when the bell rang the $n^{\text{th}}$ time:

| | | | | | | |
|---|---|---|---|---|---|---|
| Time the bell rang | $1^{\text{st}}$ | $2^{\text{nd}}$ | $3^{\text{rd}}$ | $4^{\text{th}}$ | $\ldots$ | $n^{\text{th}}$ |
| Guests arriving | $1$ | $3$ | $5$ | $7$ | $\ldots$ | $2n-1$ |

The total number of guests is then the sum of the numbers in the second row in the table, $1 + 3 + 5 + 7 + \cdots + (2n - 1)$. But this is an arithmetic sum with first term $1$, last term $2n - 1$, and $n$ terms. Therefore, the sum is $\frac{1 + (2n-1)}{2} \cdot n = \frac{2n}{2} \cdot n = n^2$.

*If you are not familiar with our solver's formula for the sum of an arithmetic sequence, you can use Gauss' trick:*

$$\begin{array}{ccccccccccc} 1 & + & 3 & + & \cdots & + & (2n-3) & + & (2n-1) & = & S \\ (2n-1) & + & (2n-3) & + & \cdots & + & 3 & + & 1 & = & S \end{array}$$

*so that*

$$\underbrace{2n + 2n + \cdots + 2n + 2n}_{n \text{ copies}} = 2S$$

*Thus,* $2n^2 = 2S$ *and* $S = n^2$.

**8**. Determine all positive integers $n$ such that $n^2 - 440$ is the square of an integer.

*Solution by Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.*

If $n^2 - 440 = k^2$, where $k$ is a positive integer, then

$$440 \;=\; n^2 - k^2 \;=\; (n+k)(n-k)\,.$$

Therefore, $n+k$ and $n-k$ must both be (positive, integer) divisors of $440$. Since $440 = 2^3 \cdot 5 \cdot 11$, the only divisors are $1$, $2$, $4$, $5$, $8$, $10$, $11$, $20$, $22$, $40$, $44$, $55$, $88$, $110$, $220$, and $440$. [*Ed.: To find the divisors, see the solution to Problem 4 above.*] To reduce the number of cases to check, note that $n+k$ is larger than $n-k$ and that they have the same parity (that is, they are either both even or both odd). That leaves just four cases:

If $n+k = 220$ and $n-k = 2$, then $n = 111$ and $k = 109$.
If $n+k = 110$ and $n-k = 4$, then $n = 57$ and $k = 53$.
If $n+k = 44$ and $n-k = 10$, then $n = 27$ and $k = 17$.
If $n+k = 22$ and $n-k = 20$, then $n = 21$ and $k = 1$.
Thus, the only possible values for $n$ are $21$, $27$, $57$, and $111$.

*Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia.*

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Matthew Ng, student, St. Francis Xavier Secondary School, Mississauga, ON.

We hope that our readers will enjoy the featured contest and that they will share their joy by submitting one or more solutions for publication.

# NOTICE TO CRUX READERS

The CMS is in the process of appointing a new Editor-in-Chief for 2011 as well as finding a number of section editors. The situation is causing severe production problems with the journal and has caused delays in 2010 and is expected to cause delays in the delivery of issues in 2011.

The CMS apologizes for this disruption and delay in service.

Johan Rudnick,
Managing Editor and CMS Executive Director.

# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff member is Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON).

## Mayhem Problems

*Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le* **1 avril 2011**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.*

*La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.*

**M463**. *Proposé par l'Équipe de Mayhem.*

Dans un carré $ABCD$ de côté $2\sqrt{2}$ on dessine un cercle de centre $A$ et de rayon $1$. On dessine un second cercle de centre $C$ de sorte qu'il touche juste le premier au point $P$ sur $AC$. Déterminer l'aire totale des régions à l'intérieur du carré mais à l'extérieur des deux cercles.

**M464**. *Proposé par l'Équipe de Mayhem.*

Soit $\lfloor x \rfloor$ le plus entier n'excédant pas $x$. Par exemple, $\lfloor 3.1 \rfloor = 3$ et $\lfloor -1.4 \rfloor = -2$. Trouver tous les nombres réels $x$ tels que $\lfloor \sqrt{x^2+1} - 1 \rfloor = 2$.

**M465**. *Proposé par Antonio Ledesma López, Institut d'Education Secondaire No. 1, Requena-Valence, Espagne.*

L'entier $20114022$ est divisible par $2011$. Trouver s'il existe un entier positif divisible par $2011$ et dont la somme des chiffres donne $2011$.

**M466**. *Proposé par Pedro Henrique O. Pantoja, étudiant, UFRN, Brésil.*

Trouver toutes les paires $(m, n)$ d'entiers positifs tels que $2^m - 2 = n!$.

**M467**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Trouver tous les nombres réels $x$ pour lesquels

$$(x-2010)^3 + (2x-2010)^3 + (4020-3x)^3 \;=\; 0\,.$$

**M468**. *Proposé par Gheorghe Ghiţă, Collège National "M. Eminescu", Buzău, Roumanie.*

Trouver toutes les paires $(p, q)$ de nombres premiers telles que

$$p+q, \;\; p+q^2, \;\; p+q^3, \;\; p+q^4,$$

soient premiers.

**M469**. *Proposé par Antonio Ledesma López, Institut d'Education Secondaire No. 1, Requena-Valence, Espagne.*

Montrer que pour tous les nombres réels $x$, on a

$$\left(2^{\sin x}+2^{\cos x}\right)^2 \;\geq\; 2^{2-\sqrt{2}}\,.$$

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**M463**. *Proposed by the Mayhem Staff.*

The square $ABCD$ has side length $2\sqrt{2}$. A circle with centre $A$ and radius $1$ is drawn. A second circle with centre $C$ is drawn so that it just touches the first circle at point $P$ on $AC$. Determine the total area of the regions inside the square but outside the two circles.

**M464**. *Proposed by the Mayhem Staff.*

Let $\lfloor x\rfloor$ be the greatest integer not exceeding $x$. For example, $\lfloor 3.1\rfloor = 3$ and $\lfloor -1.4\rfloor = -2$. Find all real numbers $x$ for which $\lfloor\sqrt{x^2+1}-1\rfloor = 2$.

**M465**. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

The integer $20114022$ is divisible by $2011$. Determine if there exists a positive integer that is divisible by $2011$ and whose digits add to $2011$.

**M466**. *Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.*

Determine all pairs $(m, n)$ of positive integers such that $2^m - 2 = n!$.

**M467**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all real numbers $x$ for which

$$(x-2010)^3+(2x-2010)^3+(4020-3x)^3 = 0\,.$$

**M468**. *Proposed by Gheorghe Ghiţă, M. Eminescu National College, Buzău, Romania.*

Determine all pairs $(p, q)$ of prime numbers for which each of $p + q$, $p + q^2$, $p + q^3$, and $p + q^4$ is a prime number.

**M469**. *Proposed by Antonio Ledesma López, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain.*

Prove that, for all real numbers $x$, we have $\left(2^{\sin x} + 2^{\cos x}\right)^2 \geq 2^{2-\sqrt{2}}$.

# Mayhem Solutions

We acknowledge a correct solution to problem M413 by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain, and a correct solution to problem M419 by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Our apologies for these oversights.

**M426**. *Proposed by the Mayhem Staff.*

Determine the number of positive integers less than or equal to $1000000$ that are divisible by all of the integers $2$, $3$, $4$, $5$, $6$, $7$, $8$, $9$, and $10$.

*Solution by Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia.*

A positive integer is divisible by all of the integers from $2$ to $10$ if it is divisible by the least common multiple (lcm) of these numbers.

We can write this list of integers in terms of their prime factorizations as $2$, $3$, $2^2$, $5$, $2 \times 3$, $7$, $2^3$, $3^2$, $2 \times 5$. Therefore, $\mathrm{lcm}(2, 3, 4, 5, 6, 7, 8, 9, 10) = 2^3 \times 3^2 \times 5 \times 7 = 2520$.

Now the largest integer less than or equal to $1\,000\,000$ that is divisible by $2520$ is $2520 \times 396$. This is because the quotient when $1\,000\,000$ is divided by $2520$ is $396$ and the remainder is $2080$.

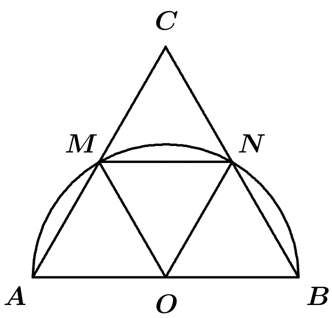
Thus, there are $396$ positive integers less than or equal to $1\,000\,000$ that are divisible by all of the integers from $2$ to $10$. (These $396$ integers are the multiples of $2520$ from $2520 \times 1$ to $2520 \times 396$.)

*Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; AFIFFAH NUUR MILA HUSNIANA, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; JOHN WYNN, student, Auburn University, Montgomery, AL, USA; and INGESTI BILKIS ZULPATINA, student, SMPN 8, Yogyakarta, Indonesia. One incorrect solution was submitted.*

**M427**. *Proposed by the Mayhem Staff.*

A semicircle has diameter $AB$. Equilateral triangle $ABC$ is drawn on the same side of $AB$ as the semicircle. Determine the area of the region that lies inside the triangle and outside the semicircle.

*Solution by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania, modified by the editor.*




Suppose that $r$ is the radius of the semicircle. Let $O$ be the centre of the semicircle and points $M$ and $N$ where the semicircle intersects $AC$ and $BC$, respectively. Join $OM$, $ON$, and $MN$.

Note that $OA = OM = ON = OB$, since each is a radius. Since $\triangle ABC$ is equilateral, $\angle ABC = \angle ACB = \angle BAC = 60^\circ$.

Since $OA = OM$, then $\triangle OMA$ is isosceles and $\angle AMO = \angle MAO = 60^\circ$. This tells us in fact that $\triangle OMA$ is equilateral, because its third angle also equals $60^\circ$. Similarly, $\triangle ONB$ is equilateral.

Now $\angle MON = 180^\circ - \angle MOA - \angle NOB = 180^\circ - 60^\circ - 60^\circ = 60^\circ$. Since $OM = ON$, then in fact $\triangle OMN$ is also equilateral since the remaining two angles are equal and add to $120^\circ$.

Note that $\angle CMN = 180^\circ - \angle AMO - \angle OMN = 180^\circ - 60^\circ - 60^\circ = 60^\circ$. Similarly, $\angle CNM = 60^\circ$, so $\triangle CMN$ is also equilateral.

Since each of $\triangle OMA$, $\triangle ONB$, $\triangle OMN$, and $\triangle CMN$ is equilateral, and each shares a side with one of the others, then these four equilateral triangles all have the same side length and so are all congruent.

The area inside $\triangle ABC$ but outside the semicircle is equal to the area of rhombus $MONC$ minus the area of sector $MON$.

Now rhombus $MONC$ is made up of the two congruent equilateral triangles $MON$ and $CMN$. Each is an equilateral triangle with side length $r$ (the radius of the semicircle), and so each has area $\frac{\sqrt{3}}{4}r^2$. (We could calculate this by constructing an altitude in one of these triangles.) Therefore, the area of rhombus $MONC$ is $2 \cdot \frac{\sqrt{3}}{4}r^2 = \frac{\sqrt{3}}{2}r^2$.

Sector $MON$ has angle $60^\circ$, and so has area $\frac{60^\circ}{360^\circ} \cdot \pi r^2 = \frac{1}{6}\pi r^2$.

Therefore, the area of the region is $\frac{\sqrt{3}}{2}r^2 - \frac{1}{6}\pi r^2$.

*Also solved by NATALIA DESY, student, SMA Xaverius 1, Palembang, Indonesia; GEOFFREY A. KANDALL, Hamden, CT, USA; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. One incorrect solution was submitted.*

**M428**. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all integers $x$ for which

$$(4-x)^{4-x} + (3-x)^{3-x} + 20 \;=\; 4^x + 3^x\,.$$

*Solution by John Wynn, student, Auburn University, Montgomery, AL, USA, modified by the editor.*

We will examine three cases to show that there is only one integer $x$ that satisfies the equation.

**Case 1:** $x \geq 3$. We note first in this case that if $x = 3$ or if $x = 4$, the left side will include a term of the form $0^0$. We could sensibly adopt the convention that this is undefined, that it equals $0$, or that it equals $1$. Using any of these conventions, we first show that neither $x = 3$ nor $x = 4$ is a solution.

Substituting $x = 3$, we see that the left side equals $1^1 + 0^0 + 20$, which is either undefined or equal to $21$ or $22$. When $x = 3$, the right side equals $4^3 + 3^3$, which equals $91$. Therefore, the equation is not satisfied, no matter which convention we adopt.

Substituting $x = 4$, we see that the left side equals $0^0 + (-1)^{-1} + 20$, which is either undefined or equal to $19$ or $20$. When $x = 4$, the right side equals $4^4 + 3^4$, which equals $337$. Therefore, the equation is not satisfied, no matter which convention we adopt.

When $x \geq 5$, we have that $4^x + 3^x \geq 4^5 + 3^5 = 1267$.

Also note that when $x \geq 5$, we have $4 - x \leq -1$ and $3 - x \leq -2$ and so $|4 - x| \geq 1$ and $|3 - x| \geq 2$. Therefore, $|4 - x|^{x-4} \geq 1$ and $|3-x|^{x-3} \geq 2^2 = 4$. Thus, $(4-x)^{4-x} = \dfrac{1}{(4-x)^{x-4}} \leq \dfrac{1}{|4-x|^{x-4}} \leq 1$ and $(3-x)^{3-x} = \dfrac{1}{(3-x)^{x-3}} \leq \dfrac{1}{|3-x|^{x-3}} < 1$. Therefore, when $x \geq 5$, the right side is at least $1267$ and the left side is at most $22$, so no such value of $x$ satisfies the equation.

**Case 2:** $x \leq 1$. When $x \leq 1$, we have $4^x + 3^x \leq 4^1 + 3^1 = 7$. Also, when $x \leq 1$, we have that $4 - x \geq 3$ and $3 - x \geq 2$, so $(4-x)^{4-x} \geq 3^3 = 27$ and $(3-x)^{3-x} \geq 2^2 = 4$. Therefore, the left side is at least $27 + 4 + 20 = 51$ and the right side is at most $7$. Thus, there are no solutions in this case.

**Case 3:** $x = 2$. Here, the left side equals $2^2 + 1^1 + 20 = 25$ and the right side equals $4^2 + 3^2 = 25$, so $x = 2$ is a solution.

In summary, we see that $x = 2$ is the the only integer solution.

*Also solved by HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EVEREST SHI, student, Burnaby North Secondary School, Burnaby, BC; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. Three incomplete solutions were submitted.*

**M429**. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Determine all triples $(a, b, c)$ of positive integers with $a^{(b^c)} = \left(a^b\right)^c$.

*Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.*

The equation $a^{(b^c)} = (a^b)^c$ is equivalent to the equation $a^{(b^c)} = a^{bc}$. We examine a number of different cases.

**Case 1:** $a = 1$. Then the equation is true regardless of the values of $b$ and $c$. Therefore, $(1, b, c)$ is a solution for all positive integers $b$ and $c$.

**Case 2:** $a > 1$. In this case, $a^{(b^c)} = a^{bc}$ is equivalent to $b^c = bc$, which is equivalent to $b^{c-1} = c$ since $b > 0$. We consider subcases where $c = 1$, $c = 2$, and $c > 2$.

**Subcase 2(a):** $a > 1$ and $c = 1$. If $c = 1$, then we have $b^0 = 1$, which is true for all positive integers $b$. Therefore, $(a, b, 1)$ is a solution for all positive integers $a > 1$ and all positive integers $b$.

**Subcase 2(b):** $a > 1$ and $c = 2$. If $c = 2$, then the equation $b^{c-1} = c$ becomes $b = 2$. Therefore, $(a, 2, 2)$ is a solution for all positive integers $a > 1$.

**Subcase 2(c):** $a > 1$ and $c > 2$. If $c > 2$, then $b$ cannot equal $1$, so $b \geq 2$. Using the fact that $2^{c-1} > c$ for $c \geq 3$ (proved at the end of this solution), we see that $b^{c-1} \geq 2^{c-1} > c$, so $b^{c-1} = c$ has no solutions in this case.

In conclusion, the solutions are all triples $(a, b, c)$ of positive integers with (i) $a = 1$, or (ii) $a > 1$ and $c = 1$, or (iii) $a > 1$ and $b = c = 2$.

To finish, we must show that $2^{c-1} > c$ for all positive integers $c \geq 3$. We prove this by mathematical induction on $c$.

If $c = 3$, the inequality becomes $4 = 2^2 > 3$, which is true.

Suppose that $2^{c-1} > c$ for $c = k$ for some positive integer $k \geq 3$.

Consider $c = k + 1$. Since $2^{k-1} > k$ by the induction hypothesis, then $2^k = 2 \cdot 2^{k-1} > 2k$. Since $k \geq 3$, then $2k > k + 1$, so $2^k > k + 1$, or $2^{(k+1)-1} > k + 1$, as required. This completes the proof by induction.

*Also solved by RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain. Seven incorrect solutions were submitted.*

**M430**. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Let $p_n$ be the $n^{\text{th}}$ prime number. Prove that $p_n > 3n$ for all $n \geq 12$.

*Solution by Bruno Salgueiro Fanego, Viveiro, Spain.*

We prove the result by induction on $n$. First, we note that if $n = 12$, then $p_n = p_{12} = 37$ and $3n = 36$, so $p_n > 3n$ when $n = 12$.

Next, we assume that $p_k > 3k$ for some positive integer $k \geq 12$. We will prove that $p_{k+1} > 3(k+1)$.

Note that the first prime larger than $p_k$ is $p_{k+1}$ so $p_{k+1} \geq p_k + 1$. Since $p_k$ is an odd prime (the only even prime is $2$), then $p_k + 1$ is even and so cannot be prime. Thus, $p_{k+1} \geq p_k + 2$.

Also, note that since $p_k > 3k$ and $p_k$ is an integer, then $p_k \geq 3k + 1$.

Altogether, we obtain $p_{k+1} \geq p_k + 2 \geq 3k+1+2 = 3k+3 = 3(k+1)$. But $3(k+1)$ cannot be a prime number since it is divisible by $3$ and it is at least $39$, and $p_{k+1}$ is a prime number, so $p_{k+1} > 3(k+1)$, as required.

Therefore, by induction, $p_n > 3n$ for all positive integers $n \geq 12$.

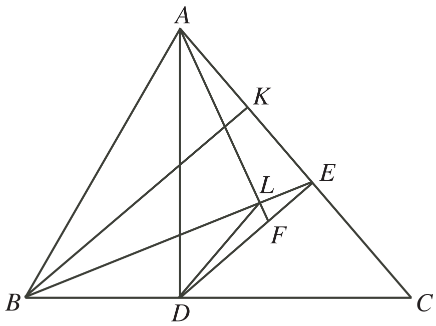
*Also solved by SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; JOSÉ HERNÁNDEZ SANTIAGO, student, Universidad Tecnológica de la Mixteca, Oaxaca, Mexico; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; and RAFAEL MARTINEZ CALAFAT, I.E.S. La Plana, Castellon, Spain. One incomplete solution was submitted.*

## M431. *Proposed by Shailesh Shirali, Rishi Valley School, India.*

In acute triangle $ABC$, the foot of the perpendicular from $A$ to $BC$ is $D$, and the foot of the perpendicular from $D$ to $AC$ is $E$. Point $F$ is located on line segment $DE$ such that $\dfrac{DF}{FE} = \dfrac{\cot C}{\cot B}$. Prove that $AF$ and $BE$ are perpendicular.

*Solution by the proposer, modified by the editor.*

Let $L$ be the point where lines $AF$ and $BE$ intersect each other, and let $K$ be the foot of the perpendicular from $B$ to $AC$. Draw $BK$ and $DL$.




Now $\triangle BKC$ is similar to $\triangle DEC$ since each is right-angled and the triangles share the angle at $C$. Therefore, $\dfrac{BC}{DC} = \dfrac{KC}{EC}$, and so we have $\dfrac{DC+DB}{DC} = \dfrac{EC+EK}{EC}$, or $1 + \dfrac{DB}{DC} = 1 + \dfrac{EK}{EC}$, or $\dfrac{DB}{DC} = \dfrac{EK}{EC}$, or $\dfrac{DC}{DB} = \dfrac{EC}{EK}$.

Since $AD$, $BC$ are perpendicular, $\cot B = \dfrac{DB}{AD}$ and $\cot C = \dfrac{DC}{AD}$. Therefore, $\dfrac{\cot C}{\cot B} = \dfrac{DC}{DB}$, and we then have $\dfrac{EC}{EK} = \dfrac{\cot C}{\cot B} = \dfrac{DF}{FE}$.

Note that $\angle DAE = \angle CBK = 90^\circ - \angle ACB$. Thus, $\triangle AED$ and $\triangle BKC$ are similar since each has a right angle and a second equal angle. Therefore, in these similar triangles, points $F$ and $E$ divide the corresponding sides $ED$ and $KC$ in the same ratio. Also, from the similarity of these two triangles, we have $\dfrac{ED}{EA} = \dfrac{KC}{KB}$.

We will show that this implies that $\angle EAF = \angle KBE$. This will mean that $\angle DAF = \angle CBE$ since $\angle CBK = \angle DAE$. This in turn will tell us that $\angle DAL = \angle DBL$. From this, we can conclude that points $A$, $B$, $D$, and $L$ form a cyclic quadrilateral. Hence, $\angle ALB = \angle ADB = 90^\circ$, and so $AF$ and $BE$ are perpendicular, as required.

It remains to show that $\angle EAF = \angle KBE$. Note that both angles are acute. Also, $\dfrac{FE}{ED} = \dfrac{FE}{FE+DF} = \dfrac{1}{1+\frac{DF}{FE}} = \dfrac{1}{1+\frac{EC}{EK}} = \dfrac{EK}{KC}$. Therefore,

$$\begin{aligned} \tan(\angle EAF) &= \frac{FE}{EA} = \frac{ED \cdot \frac{EK}{KC}}{EA} = \frac{ED}{EA}\cdot\frac{EK}{KC} \\ &= \frac{KC}{KB}\cdot\frac{EK}{KC} = \frac{EK}{KB} = \tan(\angle KBE)\,. \end{aligned}$$

Since acute angles with equal tangents are equal, then $\angle EAF = \angle KBE$, as required, thus completing the proof.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEOFFREY A. KANDALL, Hamden, CT, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.*

# Problem of the Month

## Ian VanderBurgh

This month, we investigate numbers expressed in bases other than $10$.

**Problem** (1986 Canadian Invitational Mathematics Challenge) Find a base $7$ three-digit number which has its digits reversed when expressed in base $9$.

Let's review (or learn!) about numbers in different bases. Since the problem talks about three-digit numbers, we'll focus on three-digit numbers. All of what we look at can be extended to numbers with more digits.

When we write the three-digit integer two hundred seventy-three as $273$, we normally mean that this is the base $10$ representation of this integer. Writing $273$ is a way of representing the integer equal to $2\times 10^2+7\times 10+3$. We could write this as $(273)_{10}$ to emphasize that we are thinking of a base $10$ number.

Let's look at base $7$. Any digit in base $7$ must be less than $7$, so the possible digits are $0, 1, 2, 3, 4, 5$, and $6$. The notation $(326)_7$ is an example of a three-digit integer in base $7$. (The subscript of $7$ indicates the base.) This is the base $7$ representation of the integer equal to $3\times 7^2+2\times 7+6$, which equals one hundred sixty-seven. In other words, $(326)_7 = (167)_{10}$.

Let's look at a general base $b$, where $b$ is an integer with $b > 1$. In base $b$, the possible digits are from $0$ to $b-1$, inclusive. An example of a three-digit integer would be $(pqr)_b$, which is the base $b$ representation of the integer equal to $p \times b^2 + q \times b + r$.

We now know enough about numbers in different bases to work out a solution to the problem.

**Solution** We want to find a three-digit base $7$ number $(pqr)_7$ so that when it is converted to base $9$, its representation is $(rqp)_9$. In other words, we want to find a base $7$ number $(pqr)_7$ so that $(pqr)_7 = (rqp)_9$.

Now,

$$\begin{aligned} (pqr)_7 &= p \times 7^2 + q \times 7 + r = 49p + 7q + r\,, \text{ and} \\ (rqp)_9 &= r \times 9^2 + q \times 9 + p = 81r + 9q + p\,. \end{aligned}$$

Therefore, we want $49p + 7q + r = 81r + 9q + p$, or $48p = 80r + 2q$, or $24p = 40r + q$.

We have thus transformed the initial problem into the problem of finding positive integers $p$, $q$, and $r$ with $24p = 40r + q$ and with the added condition that each of $p$, $q$, and $r$ is no more than $6$, since each must be a valid digit in base $7$. Fiddling a bit, you might find the solution $(p, q, r) = (5, 0, 3)$.

In other words, $(503)_7 = (305)_9$, so $(503)_7$ is a base $7$ three-digit number with the required property. ∎

We should probably check our answer by converting both numbers to base $10$. (It's always a good idea to check your answer whenever possible.) Converting each to base $10$, we obtain $(503)_7 = 5 \times 7^2 + 0 \times 7 + 3 = 248$ and $(305)_9 = 3 \times 9^2 + 0 \times 9 + 5 = 248$, so our answer does indeed work.

While the question didn't ask us to do so, let's see if we can determine whether or not there are more solutions.

Let's go back to the last equation $24p = 40r + q$ and rewrite it as $q = 24p - 40r$. We notice that right side can be factored as $8(3p - 5r)$, which is a multiple of $8$. Since $q = 24p - 40r$, then $q$ must also be a multiple of $8$. Since $q$ is a digit, then $q$ must equal $0$ or $8$.

But wait! Not only is $q$ a digit, but it is actually a digit in base $7$ (as well as in base $9$) so it can be no larger than $6$. This tells us that $q$ must be $0$.

Since $q = 0$, the equation $24p = 40r + q$ becomes $24p = 40r$ or $3p = 5r$. The right side is a multiple of $5$, so the left side must also be a multiple of $5$. For $3p$ to be a multiple of $5$, the integer $p$ must be a multiple of $5$. Since $p$ is between $0$ and $6$ inclusive, then $p$ can equal $0$ or $5$. If $p = 0$, then $3p = 5r$ gives $r = 0$; if $p = 5$, then $r = 3$.

Therefore, the possible solutions are $(p, q, r) = (0, 0, 0)$ or $(5, 0, 3)$. The first triple is a solution to the equation $q = 24p - 40r$, but is *not* a solution to the problem, since $(000)_7$ is not a three-digit number in base $7$ as its leading digit is $0$.

Numbers in different bases are fun things to play with, but can appear at first glance not to be terribly useful. This is far from the case — just ask someone interested in computers about binary and hexadecimal representations of integers and they will tell you how useful this theory actually is.

# THE OLYMPIAD CORNER

## No. 290

R.E. Woodrow

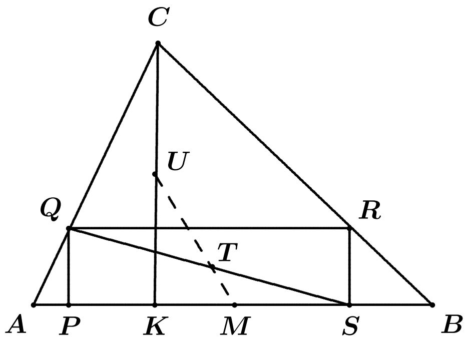
In this last *Corner* of volume 36, we begin reducing the backlog of readers' solutions to make way for a renewed column, with new features, and a new editorial team for 2011. I shall continue to support the *Corner* and the team as we seek to introduce new features. Henceforth no new problem sets will be given. We turn to the balance of solutions from our readers and to the 11th Mathematical Olympiad of Bosnia and Herzegovina at [2009 : 438–439].

**2**. Triangle $ABC$ is given. Determine the set of the centres of all rectangles inscribed in the triangle $ABC$ so that one side of the rectangle lies on the side $AB$ of the triangle $ABC$.

*Solved by Michel Bataille, Rouen, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.*

Let $a = BC$, $b = CA$, $c = AB$ and $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$. We suppose that $\alpha$ and $\beta$ are not obtuse (otherwise the required set is empty).

Let $K$ be the foot of the altitude from $C$, and let $U$, $M$ be the midpoints of $CK$, $AB$, respectively. We show that the required locus is the segment $UM$ excluding its endpoints.




We remark that an inscribed rectangle $PQRS$ with $P$, $S$ on the side $AB$ is entirely determined by the choice of $Q$ on the side $AC$ (with $Q \neq A$, $C$). Let $Q = tC + (1-t)A$, where $t \in (0,1)$. Then we have $R = tC + (1-t)B$ and $S = tK + (1-t)B$. Moreover, since $cK = (a\cos\beta)A + (b\cos\alpha)B$, we have $cS = (ta\cos\beta)A + (tb\cos\alpha + c(1-t))B$.

The centre of $PQRS$ is the midpoint $T$ of $QS$, hence,

$$\begin{aligned} 2cT = c(Q+S) &= (c(1-t) + ta\cos\beta)A + (c(1-t) + tb\cos\alpha)B + ctC \\ &= t((a\cos\beta)A + (b\cos\alpha)B + cC) + (1-t)c(A+B) \\ &= t(cK + cC) + 2(1-t)cM \;=\; 2ctU + 2(1-t)cM\,, \end{aligned}$$

so that $T = tU + (1-t)M$.

It follows that $T$ traces the line segment $UM$ (except for the two endpoints $U$ and $M$) as $t$ varies in $(0,1)$.

**4**. For any two positive integers $a$ and $d$ prove that the infinite arithmetic progression

$$a,\ a+d,\ a+2d,\ \ldots,\ a+nd,\ \ldots$$

contains an infinite geometric progression of the form

$$b,\ bq,\ bq^2,\ \ldots,\ bq^n,\ \ldots\,,$$

where $b$ and $q$ are also positive integers.

*Solved by Mohammed Aassila, Strasbourg, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.*

We take $b = a$ and $q = d + 1$, so that the geometric progression is

$$a,\ a(d+1),\ a(d+1)^2,\ \ldots\ ,a(d+1)^n,\ \ldots\ .$$

It remains to prove that $a(d+1)^n$ is of the form $a + md$; indeed,

$$\begin{aligned} a(d+1)^n &= a\left[1 + \binom{n}{1}d + \binom{n}{2}d^2 + \cdots + \binom{n}{n}d^n\right] \\ &= a + d\left[a\binom{n}{1} + ad\binom{n}{2} + \cdots + ad^{n-1}\binom{n}{n}\right] \;=\; a + dm\,. \end{aligned}$$

**5**. The acute triangle $ABC$ is inscribed in a circle with centre $O$. Let $P$ be a point on the arc $\widehat{AB}$, where $C \not\in \widehat{AB}$. The perpendicular from the point $P$ to the line $BO$ cuts the side $AB$ at point $S$ and the side $BC$ at point $T$. The perpendicular from the point $P$ to the line $AO$ cuts the side $AB$ at point $Q$ and the side $AC$ at point $R$. Prove that:

(a) The triangle $PQS$ is isosceles.

(b) $PQ^2 = QR \cdot ST$.

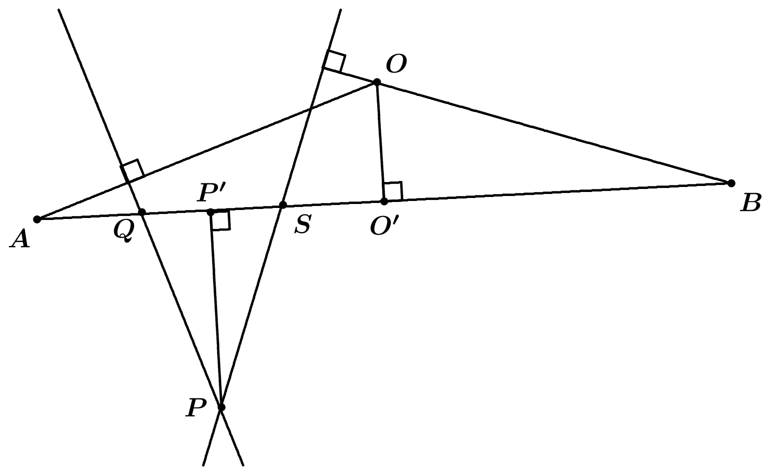
*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Bataille.*

(a) In this question, $P$ does not need to be on the circumcircle of $\triangle ABC$. Actually, we prove the following:

> Let $OAB$ be an isosceles triangle with $OA = OB$ and let perpendiculars to $OA$ and $OB$ meet at $P$ and intersect $AB$ at $Q$ and $S$, respectively (see the figure on the next page). Then $PQ = PS$.

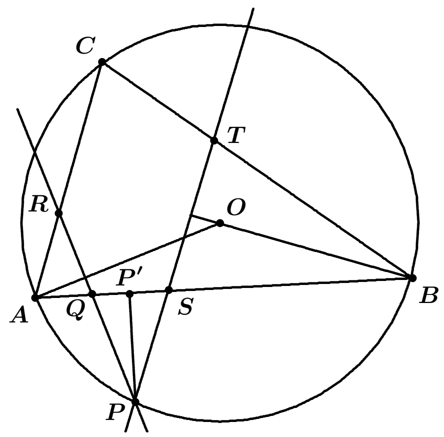
Let $O'$, $P'$ be the orthogonal projections of $O$, $P$ onto $AB$ and let $\mathcal{R}_{OO'}$ and $\mathcal{R}_{PP'}$ denote the reflections in $OO'$ and $PP'$, respectively. Since $PP'$ is parallel to $OO'$, the mapping $\mathcal{R}_{PP'} \circ \mathcal{R}_{OO'}$ is the translation $\mathcal{T}$ with vector

$2\overrightarrow{O'P'}$ and $\mathcal{R}_{PP'}(PQ) = \mathcal{T}\,(\mathcal{R}_{OO'}(PQ))$. Since $PQ$ is perpendicular to $OA$, the line $\mathcal{R}_{OO'}(PQ)$ is perpendicular to $OB = \mathcal{R}_{OO'}(OA)$.




Hence, $\mathcal{R}_{PP'}(PQ)$, which is parallel to $\mathcal{R}_{OO'}(PQ)$, is perpendicular to $OB$ as well and so $\mathcal{R}_{PP'}(PQ) = PS$. Thus the altitude $PP'$ in $\triangle QPS$ also bisects the angle $\angle QPS$ and $\triangle QPS$ is isosceles.

(b) As before, let $P'$ be the orthogonal projection of $P$ onto $AB$. We have $\angle BAO = \angle QPP'$ (acute angle with perpendicular sides); since $\triangle QPS$ and $\triangle OAB$ are isosceles, it follows that




$$\begin{aligned}\angle PQS &= \frac{1}{2}\left(\pi - 2\angle QPP'\right)\\ &= \frac{1}{2}\left(\pi - 2\angle BAO\right)\\ &= \frac{1}{2}\angle AOB = C\,.\end{aligned}$$

Thus, $\angle AQR = C$, and so $\triangle ARQ \sim \triangle ABC$. Similarly, $\triangle TBS \sim \triangle ABC$ and therefore $\triangle TBS \sim \triangle ARQ$. We deduce that $\dfrac{QA}{QR} = \dfrac{ST}{BS}$.

Now, $\angle APB = \pi - C$ and $\angle QPS = \pi - 2C$, so $\angle APQ + \angle BPS = C$. Also, $\angle APQ + \angle PAQ = \pi - \angle AQP = \angle PQS = C$, hence $\angle BPS = \angle PAQ$. As a result, $\triangle AQP \sim \triangle PSB$, and we deduce that $\dfrac{QP}{QA} = \dfrac{BS}{PS}$.

We now have $\dfrac{QP}{QR} = \dfrac{QA}{QR}\cdot\dfrac{QP}{QA} = \dfrac{ST}{BS}\cdot\dfrac{BS}{PS} = \dfrac{ST}{PS}$, and the result follows, since $PS = PQ$.

**6**. Let $a_1, a_2, \ldots, a_n$ be real constants and for each real number $x$ let

$$f(x) = \cos(a_1 + x) + \frac{\cos(a_2 + x)}{2} + \frac{\cos(a_3 + x)}{2^2} + \cdots + \frac{\cos(a_n + x)}{2^{n-1}}.$$

If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2 = m\pi$, where $m$ is an integer.

*Solved by Mohammed Aassila, Strasbourg, France; and Michel Bataille, Rouen, France. We give the solution of Aassila.*

For each $k$ let $z_k = 2^{1-k}(\cos a_k + i \sin a_k)$, and let $z = \cos x + i \sin x$. We have $z_k z = 2^{1-k}(\cos(a_k + x) + \sin(a_k + x))$, and so

$$f(x) = \mathfrak{Re}(z_1 z + z_2 z + \cdots + z_n z) = \mathfrak{Re}(z(z_1 + z_2 + \cdots + z_n)). \qquad (1)$$

Note that $z_1 + z_2 + \cdots + z_n \neq 0$, since otherwise $|z_1| = |z_2 + \cdots + z_n| \leq |z_2| + \cdots + |z_n|$ would imply that $1 \leq 2^{-1} + 2^{-2} + \cdots + 2^{1-n} = 1 - 2^{1-n}$, a contradiction. Hence, $0 \neq z_1 + \cdots + z_n = c = r(\cos\varphi + i \sin\varphi)$. By (1) we have

$$f(x) = \mathfrak{Re}(cz) = r\cos(x + \varphi), \qquad (r \neq 0).$$

If $f(x_1) = f(x_2) = 0$, then $\cos(x_1 + \varphi) = \cos(x_2 + \varphi) = 0$, and hence $x_2 + \varphi - (x_1 + \varphi) = x_2 - x_1 = m\pi$, where $m$ is an integer.

Next we turn to solutions of the Vietnamese Mathematical Olympiad 2006–2007 given at [2009 : 439–440].

**1**. Solve the system of equations

$$\begin{aligned} 1 - \frac{12}{y + 3x} &= \frac{2}{\sqrt{x}}, \\ 1 + \frac{12}{y + 3x} &= \frac{6}{\sqrt{y}}. \end{aligned}$$

*Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Wang's write-up.*

We assume the problem asks for real solutions and we show that the only solution is $(x, y) = (4 + 2\sqrt{3}, 12 + 6\sqrt{3})$.

Adding and subtracting the given equations we obtain

$$\frac{1}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 1, \qquad (1)$$

$$-\frac{1}{\sqrt{x}} + \frac{3}{\sqrt{y}} = \frac{12}{y + 3x}. \qquad (2)$$

From (1) and (2) we obtain (respectively)

$$3\sqrt{x}+\sqrt{y} = \sqrt{xy}\,, \qquad (3)$$

$$3\sqrt{x}-\sqrt{y} = \frac{12\sqrt{xy}}{y+3x}\,. \qquad (4)$$

Multiplying (3) and (4) yields $9x-y=\dfrac{12xy}{y+3x}$, hence

$$\begin{aligned}(9x-y)(3x+y) &= 12xy\,;\\ 27x^2-6xy-y^2 &= 0\,;\\ (9x+y)(3x-y) &= 0\,.\end{aligned}$$

Since $9x+y>0$, we have $y=3x$. Substituting this into the first given equation then yields $1-\dfrac{2}{x}=\dfrac{2}{\sqrt{x}}$, or $x-2\sqrt{x}-2=0$.

Solving we obtain $\sqrt{x}=1\pm\sqrt{3}$. Since $\sqrt{x}>0$, we have $\sqrt{x}=1+\sqrt{3}$ from which $x=4+2\sqrt{3}$ and $y=12+6\sqrt{3}$ follow.

**3**. Triangle $ABC$ has two fixed vertices, $B$ and $C$, while the third vertex $A$ is allowed to vary. Let $H$ and $G$ be the orthocentre and the centroid of $ABC$, respectively. Find the locus of $A$ such that the midpoint $K$ of the segment $HG$ lies on the line $BC$.

*Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's solution.*

Let $O$ be the midpoint of $BC$. We choose a system of coordinates in which the points are $B(-b,0)$, $C(b,0)$, $A(m,n)$. Then $G$ has coordinates $G\left(\dfrac{-b+b+m}{3},\dfrac{0+0+n}{3}\right)=G\left(\dfrac{m}{3},\dfrac{n}{3}\right)$. The slope of the line $AB$ is $\dfrac{n}{m+b}$, and the altitude from $C$ is $y=-\dfrac{m+b}{n}(x-b)$. Since the altitude from $A$ is $x=m$, the orthocentre is then $H\left(m,-\dfrac{m+b}{n}(m-b)\right)$. The point $K$ lies on the line $BC$ if and only if the $y$-coordinate is $0$, that is

$$\begin{aligned}-\frac{m+b}{n}(m-b)+\frac{n}{3}=0 &\iff b^2-m^2+\frac{n^2}{3}=0\\ &\iff \frac{m^2}{b^2}-\frac{n^2}{3b^2}=1\,,\end{aligned}$$

hence the locus of $A$ is the hyperbola $\dfrac{x^2}{b^2}-\dfrac{y^2}{3b^2}=1$, without the points $B(-b,0)$ and $C(b,0)$.

**5**. Let $b$ be a positive real number. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y) = f(x)\cdot 3^{b^y+f(y)-1} + b^x\left(3^{b^y+f(y)-1}-b^y\right)$$

for all real numbers $x$ and $y$.

*Solution by Michel Bataille, Rouen, France.*

We show that the only solutions are the functions $f_1 : t \mapsto -b^t$ and $f_2 : t \mapsto 1 - b^t$.

It is easily checked that $f_1$, $f_2$ are indeed solutions. Conversely, let $f$ be any function satisfying

$$f(x+y) = f(x) \cdot 3^{b^y + f(y) - 1} + b^x(3^{b^y + f(y) - 1} - b^y) \tag{1}$$

for all $x$, $y$ and let $g$ be defined by

$$g(t) = f(t) + b^t \qquad (t \in \mathbb{R})\,.$$

From (1), $g$ is a solution to the functional equation

$$3g(x+y) = g(x)3^{g(y)}\,. \tag{2}$$

In particular, we have $g(x)(3 - 3^{g(0)}) = 0$ for all $x$. If $g(x) = 0$ for all $x$, then $f = f_1$. Otherwise, $g(0) = 1$ and therefore $3g(y) = 3^{g(y)}$ for all $y$. It follows that $g(y) > 0$ and $g(y)\ln 3 - \ln(g(y)) = \ln 3$ for all $y$. A quick study of the function $\phi$ defined by $\phi(t) = t\ln 3 - \ln t$ shows that the equation $\phi(t) = \ln 3$ has two positive solutions, namely $1$ and some real number $\alpha$ with $\alpha \in (0, \frac{1}{\ln 3})$. Note that $\alpha \neq 1$. Thus, $g(y) = \alpha$ or $g(y) = 1$ for all $y$. Now, we observe that (2) and $3g(y) = 3^{g(y)}$ imply $g(x+y) = g(x) \cdot g(y)$. If we had $g(x_0) = \alpha$ for some $x_0$, then we would have $g(2x_0) = \alpha^2$, a contradiction since $\alpha^2 \notin \{\alpha,\, 1\}$. We conclude that we must have $g(x) = 1$ for all $x$, and so $f = f_2$.

**7**. Let $a > 2$ be a real number and

$$f_n(x) \;=\; a^{10}x^{n+10} + x^n + x^{n-1} + \cdots + x + 1$$

for each positive integer $n$. Prove that for each $n$ the equation $f_n(x) = a$ has exactly one real root $x_n \in (0, \infty)$, and that the sequence $\{x_n\}_{n=1}^{\infty}$ has a finite limit as $n$ approaches infinity.

*Solution by Michel Bataille, Rouen, France.*

The function $f_n$ is continuous and strictly increasing on $[0, \infty)$ with $f_n(0) = 1$ and $\lim_{x\to\infty} f_n(x) = \infty$, hence is a bijection from $[0, \infty)$ onto $[1, \infty)$. Since $a \in (1, \infty)$, the equation $f_n(x) = a$ has exactly one real root in $(0, \infty)$, namely $x_n = f_n^{-1}(a)$.

To prove that the sequence $\{x_n\}_{n=1}^{\infty}$ has a finite limit as $n$ approaches infinity, we show that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above.

To this aim, we observe that for positive $x$, the inequality $x > x_n$ is equivalent to $f_n(x) > a$. Since we obviously have $f_n(a) > a$, we obtain that $a > x_n$ for each positive integer $n$, hence $\{x_n\}_{n=1}^{\infty}$ is bounded above. Next, we consider $f_n(x_{n+1})$. We have

$$f_n(x_{n+1}) = a^{10}x_{n+1}^{n+10} + x_{n+1}^n + x_{n+1}^{n-1} + \cdots + x_{n+1} + 1$$

with $a^{10}x_{n+1}^{n+11} + x_{n+1}^{n+1} + x_{n+1}^{n} + \cdots x_{n+1} + 1 = f_{n+1}(x_{n+1}) = a$.

It follows that $x_{n+1}f_n(x_{n+1}) = a - 1$, so that

$$x_{n+1}(f_n(x_{n+1}) - a) \;=\; a - 1 - ax_{n+1}\,.$$

We will prove that $x_{n+1} < 1 - \dfrac{1}{a}$, from which we deduce first $f_n(x_{n+1}) > a$ and then $x_{n+1} > x_n$, so that $\{x_n\}_{n=1}^{\infty}$ is indeed increasing.

Now, $x_{n+1} < 1 - \dfrac{1}{a}$ will follow from $f_{n+1}\left(1 - \dfrac{1}{a}\right) > a$ and ultimately, we are reduced to proving the latter. We calculate

$$\begin{aligned} f_{n+1}\left(1-\frac{1}{a}\right) &= a^{10}\left(1-\frac{1}{a}\right)^{n+11} + \frac{1-\left(1-\frac{1}{a}\right)^{n+2}}{1-\left(1-\frac{1}{a}\right)} \\ &= a^{10}\left(1-\frac{1}{a}\right)^{n+11} + a - a\left(1-\frac{1}{a}\right)^{n+2} \end{aligned}$$

so $f_{n+1}\left(1 - \dfrac{1}{a}\right) - a$ has the same sign as $(1-b)^9 - b^9$, where $b = \dfrac{1}{a} < \dfrac{1}{2}$. But the function $\psi(u) = (1-u)^9 - u^9$ decreases from $1$ to $0$ when $u$ varies from $0$ to $\dfrac{1}{2}$, hence $(1-b)^9 - b^9 > 0$ and $f_{n+1}\left(1 - \dfrac{1}{a}\right) > a$ follows.

Next we turn to solutions from readers to problems of the December 2009 number of the *Corner*. We first look at the Austrian Mathematical Olympiad 2007, National Competition Final Round, Part 1 at [2009 : 497].

**1**. We are given a $2007 \times 2007$ grid. An odd integer is written in each of its cells. Let $Z_i$ be the sum of the numbers in the $i^{\text{th}}$ row and $S_j$ the sum of the numbers in the $j^{\text{th}}$ column for $1 \le i,\, j \le 2007$. Furthermore, let $A = \prod\limits_{i=1}^{2007} Z_i$ and $B = \prod\limits_{j=1}^{2007} S_j$. Show that $A + B$ cannot be equal to zero.

*Solution by Matti Lehtinen, National Defence College, Helsinki, Finland, modified by the editor.*

Look at the grid modulo 4. Assume $a_i$ of the entries in row $i$ are $1$, and $b_i$ are $-1$ modulo 4. Also, let $c_j$ of the entries in column $j$ be $1$ and $d_j$ be $-1$ modulo 4.

Then $Z_i \equiv a_i - b_i = a_i - (2007 - a_i) \equiv 1 + 2a_i$. Note that we have $(1 + 2x)(1 + 2y) \equiv 1 + 2(x + y)$ for integers $x$ and $y$, so it follows that $A = \prod Z_i \equiv \prod(1 + 2a_i) \equiv 1 + 2\sum a_i$.

By similar calculations, $B \equiv 1 + 2\sum c_i$.

However, $\sum a_i = \sum c_i$, since each counts the total number of entries in the grid that are $1$ modulo 4. Then, $A + B \equiv 2 + 4\sum a_i \equiv 2$, so $A + B$ cannot be equal to zero.

**2**. Determine the largest possible value of $C(n)$ for all positive integers $n$, such that

$$(n+1)\sum_{j=1}^{n} a_j^2 - \left(\sum_{j=1}^{n} a_j\right)^2 \;\geq\; C(n)\,,$$

holds for all $n$-tuples $(a_1, a_2, \ldots, a_n)$ of pairwise distinct integers.

*Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.*

It helps to recall an elementary fact from probability or statistics. Set $\overline{a} = \frac{1}{n}\sum\limits_{j=1}^{n} a_j$. Then $\sum\limits_{j=1}^{n}(a_j-\overline{a})^2 = \sum\limits_{j=1}^{n}(a_j^2 - 2a_j\overline{a} + \overline{a}^2) = \sum\limits_{j=1}^{n} a_j^2 - n\overline{a}^2$. In our notation, the expression to be minimized is $n\sum\limits_{j=1}^{n}(a_j-\overline{a})^2 + \sum\limits_{j=1}^{n} a_j^2$. The first sum is invariant to changing the origin and is clearly minimized when the distinct integers are consecutive. It is trivial that the latter sum, for distinct integers, is minimized for even $n = 2m$ when $\{a_1,\, a_2,\, \ldots,\, a_{2m}\}$ is either $\{-m+1,\, -m+2, \ldots, m-1,\, m\}$ or $\{-m,\, -m+1, \ldots, m-2,\, m-1\}$ and for odd $n = 2m+1$ when the set is $\{-m,\, -m+1,\, \ldots,\, m-1,\, m\}$. Recalling the formula for the sum of squares of consecutive integers, we can now do the computations with the original expression of the problem. They yield $C(2m) = \frac{1}{3}(4m^4+2m^3-m^2+m)$ and $C(2m+1) = \frac{2}{3}m(m+1)^2(2m+1)$, or $C(n) = \frac{1}{12}n(n+2)(n^2-n+1)$ and $C(n) = 112n(n-1)(n+1)^2$ for even and odd $n$, respectively.

**3**. Let $M(n) = \{-1,\, -2,\, \ldots,\, -n\}$. For each nonempty subset of $M(n)$ we form the product of the elements. What is the sum of all such products?

*Solved by Michel Bataille, Rouen, France; Matti Lehtinen, National Defence College, Helsinki, Finland; Stan Wagon, Macalester College, St. Paul, MN, USA; and Titu Zvonaru, Comăneşti, Romania. We give Wagon's solution.*

For each set not containing $-1$, its product adds to the product of the set with $-1$ adjoined to yield $0$. This leaves only the set $\{-1\}$ to make a nonzero contribution, so the sum is $-1$.

**4**. Let $n > 4$ be an integer. The $n$-gon $A_0A_1 \ldots A_{n-1}A_n$ (with $A_n = A_0$), is inscribed in a circle, is convex, and is such that the lengths of the sides are $A_{i-1}A_i = i$ for $1 \leq i \leq n$. Let $\phi_i$ be the angle between the line $A_iA_{i+1}$ and the tangent to the circumcircle of the $n$-gon at $A_i$. (Note that the angle between any two lines is at most $90^\circ$.) Determine the value of $\Phi = \sum\limits_{i=0}^{n-1}\phi_i$.

*Solution by Matti Lehtinen, National Defence College, Helsinki, Finland.*

Let $O$ be the centre of the circumscribed circle of polygon $A_0A_1 \ldots A_n$. If $O$ is inside the polygon, $\phi_i = \frac{1}{2}\angle A_iOA_{i+1}$, by the well-known property of the angle between a chord and tangent. So in this case the sum of the $\phi_i$'s equals half the sum of the central angles, that is, $180°$. To show that $O$ is indeed inside the polygon, assume the contrary. Then all points $A_1$, $A_2$, ..., $A_{n-2}$ lie on the shorter arc $A_{n-1}A_n$, and the length of the polygon $A_0A_1 \ldots A_{n-1}$ is less than the length of the arc, which in turn is less than $\frac{1}{2}\pi \cdot A_{n-1}A_n = \frac{1}{2}n\pi$. But the length of the broken line is $1+2+\cdots+n-1 = \frac{1}{2}(n-1)n$. Since $n \geq 5$, $n-1 \geq 4 > \pi$, and we have a contradiction.

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Next we turn to the file for the Austrian Mathematical Olympiad 2007 National Competition Final Round, Part 2 given at [2009 : 498].

**2**. Determine all sextuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of nonnegative integers satisfying the following system of equations:

$$\begin{aligned} x_1x_2(1-x_3) &= x_4x_5\,, & x_4x_5(1-x_6) &= x_1x_2\,,\\ x_2x_3(1-x_4) &= x_5x_6\,, & x_5x_6(1-x_1) &= x_2x_3\,,\\ x_3x_4(1-x_5) &= x_6x_1\,, & x_6x_1(1-x_2) &= x_3x_4\,. \end{aligned}$$

*Solved by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give a solution that combines ideas from both submissions.*

First note that by the cyclic symmetry, when a solution $(a, b, c, d, e, f)$ is obtained the six cyclic permutations are also solutions. Adding the six equations yields

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2 = 0\,,$$

or equivalently

$$x_1x_2(x_3+x_6) + x_3x_4(x_2+x_5) + x_5x_6(x_1+x_4) = 0\,. \tag{1}$$

Each $x_i$ is nonnegative, so at least one factor in each summand must be zero.

Next note that if $x_3 + x_6 = 0$, then $x_3 = 0$ and $x_6 = 0$, satisfying (1). From the original equations we obtain $x_1x_2 = x_4x_5$. If this product is zero we obtain solutions $(0, a, 0, 0, b, 0)$, $(0, a, 0, b, 0, 0)$, and their cyclic variants.

So suppose $x_1x_2 = x_4x_5 \neq 0$, and set $d = \gcd(x_1, x_4)$. Then $x_1 = dr$, $x_4 = ds$ with $r$ and $s$ coprime. Thus, $rx_2 = sx_5$ with $r$ and $s$ coprime, so $s$ divides $x_5$ and we have $x_5 = ra$ and $x_2 = sa$. We then obtain sextuples of the form $(dr, sa, 0, ds, ra, 0)$ with $(r, s)$ coprime, and we note that the cyclic shifts of these arise similarly from the cases $x_2 + x_5 = 0$ and $x_1 + x_4 = 0$.

So we suppose now that $x_3 + x_6 \neq 0$, $x_1 + x_5 \neq 0$, and $x_1 + x_4 \neq 0$. By (1) we have $x_1x_2 = x_3x_4 = x_5x_6 = 0$.

Suppose first $x_1 = 0$. Then $x_4 \neq 0$, so $x_3 = 0$ and so $x_6 \neq 0$ giving $x_5 = 0$. It is easy to check that all sextuples of the form $(0, a, 0, b, 0, c)$ satisfy the equations.

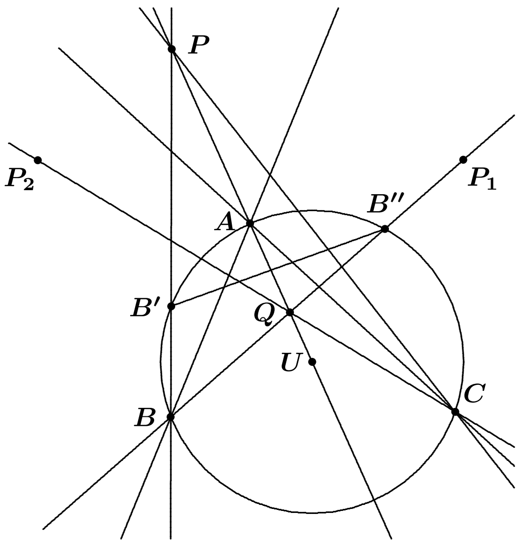
Similarly, taking $x_2 = 0$ yields $x_5 \neq 0$, $x_6 = 0$, $x_3 \neq 0$, and $x_4 = 0$. This gives solutions of the form $(a, 0, b, 0, c, 0)$, a cyclic shift of the previous solution.

Thus, the solutions are the sextuples $(0, a, 0, b, 0, c)$, $(0, a, 0, 0, b, 0)$, $(0, a, 0, b, 0, 0)$, and $(dr, sa, 0, ds, ra, 0)$ with $r$ and $s$ coprime, and all cyclic shifts of these four basic types.

**6**. We are given a triangle $ABC$ with circumcentre $U$. A point $P$ is chosen on the extension of $UA$ beyond $A$. Let $g$ denote the line symmetric to $PB$ with respect to $BA$ and $h$ the line symmetric to $PC$ with respect to $AC$. Let the lines $g$ and $h$ intersect at the point $Q$.

*Solutions by Michel Bataille, Rouen, France.*

*First solution*: Let $B'$ be the second point of intersection of the line $PB$ and the circumcircle $\Gamma$ of $\triangle ABC$ and let $B''$ be its reflection in the line $AP$ (see the figure). Note that $B''$ is on $\Gamma$.




Since $UB = UB'$, we have $\angle UBB' = \angle BB'U$, hence $\angle UB''P = \angle UB'P = 180^\circ - \angle BB'U = 180^\circ - \angle UBB' = 180^\circ - \angle UBP$. It follows that $B''$ is on the circle $(BUP)$.

Now, consider the inversion in the circle $\Gamma$ and let $P'$ be the inverse of $P$. The inverse of the circle $(BUP)$ is the line $BB''$, so that $P'$ is on this line, which is the symmetric of $BP$ in $BA$ ($A$ being the midpoint of the arc $B'B''$ of $\Gamma$, $BA$ bisects $\angle PBB''$). Similarly, $P'$ is on the symmetric of $CP$ in $CA$, and so $P' = Q$. Thus, as $P$ varies on $UA$ beyond $A$, $Q$ traverses the line segment $UA$, the extremities $U$, $A$ being excluded.

*Second solution*: We shall use complex numbers. Without loss of generality, we suppose that the affixes of $U$ and $A$ are $0$ and $1$, and that the circumcircle $\Gamma$ of $\triangle ABC$ is the unit circle. For a point $M \neq U$, $A$, we denote by $m$ the affix of $M$. The symmetric $M'$ of $M$ in $BA$ has an affix of the form $m' = \alpha\overline{m} + \beta$ for some complex numbers $\alpha$, $\beta$ independent of $m$. Writing $M' = B$ when $M = B$ and $M' = A$ when $M = A$, we obtain $\alpha = -b$ and $\beta = 1 + b$ (using $\overline{b} = \frac{1}{b}$). Thus, the affix of the symmetric $P_1$ of $P$ is $p_1 = -b\overline{p} + 1 + b = -bp + 1 + b$.

Similarly, the affix of the symmetric $P_2$ of $P$ in $CA$ is $p_2 = -cp + 1 + c$.

Now, the lines $BP_1$ and $CP_2$ have respective equations

$$z(1 - p\overline{b}) - \overline{z}(1 - pb) = b - \overline{b}\,, \qquad z(1 - p\overline{c}) - \overline{z}(1 - pc) = c - \overline{c}\,,$$

so that the affix of their point of intersection $Q$ is given by the relation

$$q[(1 - p\overline{b})(1 - pc) - (1 - p\overline{c})(1 - pb)] = (b - \overline{b})(1 - pc) - (c - \overline{c})(1 - pb)\,.$$

An easy calculation yields $q = \dfrac{1}{p}$, hence $q$ is a real number in $(0, 1)$ when $p$ varies in $(1, \infty)$, meaning that the required locus of $Q$ is the line segment $UA$, the extremities $U$, $A$ being excluded.

Next we move to solutions to some problems of the XXI Olimpiadi Italiane della Matematica given at [2009 : 499].

**2**. Polynomials with integer coefficients, $p(x)$ and $q(x)$, are *similar* if they have the same degree and the same coefficients (possibly in different order).

(a) If $p(x)$ and $q(x)$ are similar, prove that $p(2007) - q(2007)$ is even.

(b) Is there an integer $k > 2$ such that $p(2007) - q(2007)$ is divisible by $k$ whenever $p(x)$ and $q(x)$ are similar?

*Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA.*

(a) Let $n \geq 0$ be the degree of $p(x)$ and $q(x)$. Then,

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\,, \\ q(x) &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0\,, \end{aligned} \tag{1}$$

where the coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$; $b_0, b_1, \ldots, b_{n-1}, b_n$ are integers. Since $p(x)$ and $q(x)$ are similar, the sequence $b_0, b_1, \ldots, b_{n-1}, b_n$ is a *permutation* of the sequence $a_0, a_1, \ldots, a_{n-1}, a_n$. Hence,

$$a_0 + a_1 + \cdots + a_n = b_0 + b_1 + \cdots + b_n\,,$$

and thus,

$$a_0 + a_1 + \cdots + a_n \equiv b_0 + b_1 + \cdots + b_n \pmod{2}\,. \tag{2}$$

Let $r$ be any odd integer. Then $r \equiv 1 \pmod{2}$, and so $r^k \equiv 1 \pmod{2}$ for any nonnegative integer $k$. Thus,

$$\begin{aligned} p(r) &= a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0 \\ &\equiv a_n + a_{n-1} + \cdots + a_0 \pmod{2}\,. \end{aligned}$$

Likewise, $q(r) \equiv b_n + b_{n-1} + \cdots + b_0 \pmod 2$. Therefore, by (2), we have

$$\begin{aligned} p(r) - q(r) &\equiv (a_n + a_{n-1} + \cdots + a_0) - (b_n + b_{n-1} + \cdots + b_0) \\ &\equiv 0 \pmod 2\ , \end{aligned}$$

so that $p(r) - q(r)$ is even.

The case $r = 2007$ is obviously a particular one.

(b) Let $r$ be a positive integer, $r \geq 3$. We will prove that $p(r) - q(r)$ is divisible by $r - 1 \geq 2$.

Keep the notation for $p$, $q$ as in (1). As before, $b_0$, $b_1$, $\ldots$, $b_n$ is a permutation of $a_0$, $a_1$, $\ldots$, $a_n$, so for each $i$ with $0 \leq i \leq n$, there is a unique $j$ with $0 \leq j \leq n$ such that $a_i = b_j$.

If $i \geq j$, then

$$a_i r^i - b_j r^j = a_i r^i - a_i r^j = a_i r^j \cdot (r^{i-j} - 1)\,.$$

If also $i = j$, then $r^{i-j} - 1 = 1 - 1 = 0$, which is divisible by $r - 1$.

Otherwise $i > j$, and then $r^{i-j} - 1 = (r-1) \cdot (r^{(i-j)-1} + \cdots + r + 1)$, so that $r^{i-j} - 1$ is again divisible by the integer $r - 1 \geq 2$.

Likewise, when $i < j$, the same argument shows that $a_i r^i - b_j r^j$ is divisible by $r - 1$.

It is now clear that we can write the difference $p(r) - q(r)$ as a sum of $(n + 1)$ differences, each divisible by $r - 1$.

This proves that $p(r) - q(r)$ is divisible by $r - 1$.

In particular, $p(2007) - q(2007)$ is divisible by $2007 - 1 = 2006$.

**3**. Triangle $ABC$ has centroid $G$, $D \neq A$ is a point on the line $AG$ such that $AG = GD$, and $E \neq B$ is a point on the line $GB$ such that $GB = GE$. The midpoint of $AB$ is $M$. Prove that the quadrilateral $BMCD$ can be inscribed in a circle if and only if $BA = BE$.

*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Amengual Covas.*

Because each median is trisected by the centroid, $D$ and $E$ are the symmetrics of $G$ with respect to the midpoints of sides $BC$ and $CA$, respectively.

Hence, segments $BC$ and $GD$ bisect each other, and also segments $CA$ and $GE$ bisect each other, so that quadrilaterals $BGCD$ and $CGAE$ are parallelograms.

Thus, $AE \parallel GC$ and $GC \parallel BD$, implying that $AE \parallel BD$. Consequently,

$$\angle BEA = \angle EBD = \angle DCM\,. \tag{1}$$

Therefore,

$$\begin{aligned} BMCD \text{ is cyclic} &\iff \angle MBD + \angle DCM = 180^\circ \\ &\iff (\angle ABE + \angle EBD) + \angle DCM = 180^\circ \\ &\iff \angle ABE + 2\,\angle BEA = 180^\circ \qquad \text{[by (1)]} \\ &\iff \angle ABE + 2\,\angle BEA = \angle ABE + \angle BEA + \angle EAB \\ &\iff \angle BEA = \angle EAB \\ &\iff BE = BA\,, \end{aligned}$$

as desired.

**6**. For each integer $n \geq 2$, find

(a) the greatest real number $c_n$ such that

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} \geq c_n$$

for any positive real $n$-tuple $(a_1, a_2, \ldots, a_n)$ with $a_1 a_2 \cdots a_n = 1$;

(b) the greatest real number $d_n$ such that

$$\frac{1}{1+2a_1} + \frac{1}{1+2a_2} + \cdots + \frac{1}{1+2a_n} \geq d_n$$

for any positive real $n$-tuple $(a_1, a_2, \ldots, a_n)$ with $a_1 a_2 \cdots a_n = 1$.

*Solution by Titu Zvonaru, Comăneşti, Romania.*

(b) Let $t$ be a positive real number, set $a_1 = a_2 = \cdots = a_{n-1} = t$, and set $a_n = \frac{1}{t^{n-1}}$. The inequality becomes

$$\frac{n-1}{t+1} + \frac{t^{n-1}}{t^{n-1}+1} \geq c_n\,.$$

The left side goes to $1$ in the limit as $t \to \infty$, hence $c_n \leq 1$. We will show that in fact $c_n = 1$ is the answer.

Without loss of generality we suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then $a_1 a_2 \leq 1$, and therefore

$$\begin{aligned} \frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n} &> \frac{1}{1+a_1} + \frac{1}{1+a_2} \\ &\geq \frac{1}{1+a_1} + \frac{1}{1+\frac{1}{a_1}} = \frac{1}{1+a_1} + \frac{a_1}{1+a_1} = 1\,. \end{aligned}$$

(b) If $n = 2$, then we have

$$\frac{1}{1+2a_1} + \frac{1}{1+2a_2} \geq d_2 \iff \frac{1}{1+2a_1} + \frac{a_1}{a_1+2} \geq d_2\,.$$

Taking $a_1 = 1$ yields $d_2 \le \frac{2}{3}$. It remains to prove that $\frac{1}{a+2a_1} + \frac{a_1}{a_1+2} \ge \frac{2}{3}$, which is equivalent to

$$\begin{aligned} 3a_1 + 6 + 3a_1 + 6a_1^2 - 2a_1 - 4a_1^2 - 4 - 8a_1 \ge 0\,, \\ 2(a_1-1)^2 \ge 0\,, \end{aligned}$$

and we are done.

If $n \ge 3$, then as above we set $a_1 = a_2 = \cdots = a_{n-1} = x$ and $a_n = \frac{1}{x^{n-1}}$. The inequality becomes

$$\frac{n-1}{1+2x} + \frac{x^{n-1}}{x^{n-1}+2} \ge d_n\,,$$

and letting $x \to \infty$ we find that $d_n \le 1$.

It suffices to prove that

$$\frac{1}{1+2a_1} + \frac{1}{1+2a_2} + \cdots + \frac{1}{1+2a_n} \ge 1.$$

We assume (without loss of generality) that $a_1 \le a_2 \le \cdots \le a_n$; then $a_1a_2a_3 \le 1$, and therefore there exists a positive number $k$ such that $k \le 1$ and $a_1a_2a_3 = k^3$. Now, set

$$a_1 = \frac{knp}{m^2}\,, \quad a_2 = \frac{kpm}{n^2}\,, \quad a_3 = \frac{kmn}{p^2}\,,$$

and applying the Cauchy–Schwarz Inequality, we obtain

$$\begin{aligned} &\frac{1}{1+2a_1} + \frac{1}{1+2a_2} + \frac{1}{1+2a_3} \\ &\qquad = \frac{m^2}{m^2+2knp} + \frac{n^2}{n^2+2kpm} + \frac{p^2}{p^2+2kmn} \\ &\qquad \ge \frac{m^2}{m^2+2np} + \frac{n^2}{n^2+2pm} + \frac{p^2}{p^2+2mn} \\ &\qquad \ge \frac{(m+n+p)^2}{m^2+2mp+n^2+2pm+p^2+2mn} = 1\,. \end{aligned}$$

Therefore,

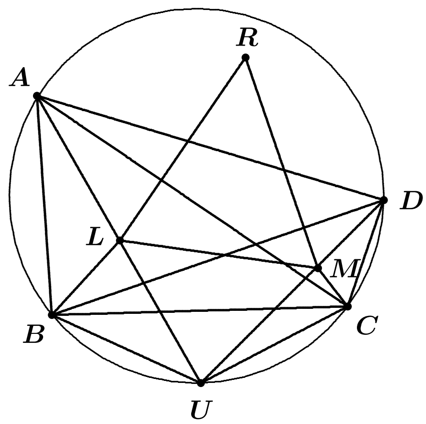
$$d_n = \begin{cases} \frac{2}{3}\,, & \text{if } n = 2 \\ 1\,, & \text{if } n \ge 3\,. \end{cases}$$

Next we turn to solutions from our readers to problems of the 56th Czech and Slovak Mathematical Olympiad Final Round, given at [2009 : 500].

**2**. In a cyclic quadrangle $ABCD$ let $L$ and $M$ be the incentres of triangles $BCA$ and $BCD$, respectively. Let $R$ be the intersection of the perpendiculars from the points $L$ and $M$ onto the lines $AC$ and $BD$, respectively. Show that the triangle $LMR$ is isosceles.

*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and Michel Bataille, Rouen, France. We give Bataille's version.*

We assume that $ABCD$ is convex so that $A$ and $D$ are on the same side of $BC$. Let $\Gamma$ be the circumcircle of $\triangle ABC$ and let $U$ be the midpoint of its arc $BC$ not containing $A$. Note that $AL$ and $DM$ intersect at $U$. Lastly, let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$.




Since $\angle UBC$ and $\angle UAC$ subtend the same arc of $\Gamma$, we have $\angle UBC = \angle UAC = \dfrac{A}{2}$, and so $\angle UBL = \dfrac{A+B}{2}$.

Since we also have $\angle BUL = \angle BUA = \angle BCA = C$, it follows that $\angle BLU = 180^\circ - C - \dfrac{A+B}{2} = \dfrac{A+B}{2} = \angle UBL$ so that $\triangle BUL$ is isosceles. As well, $\triangle MUC$ is isosceles, hence $UB = UC = UL = UM$ and $\angle ULM = \angle UML$.

In addition, we have

$$\angle DMR = 90^\circ - \angle BDU = 90^\circ - \angle BAU = 90^\circ - \frac{A}{2} = \angle ALR\,.$$

Thus,

$$\begin{aligned}\angle RLM &= 180^\circ - \angle ALR - \angle ULM \\ &= 180^\circ - \angle DMR - \angle UML = \angle RML\,,\end{aligned}$$

and the result follows.

**3**. Denote by $\mathbb{N}$ the set of all positive integers and consider all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any $x, y \in \mathbb{N}$,

$$f\big(xf(y)\big) \;=\; yf(x)\,.$$

Find the least possible value of $f(2007)$.

*Solved by Michel Bataille, Rouen, France; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the argument of Wang and Zhao.*

We prove that $f(2007) \geq 18$.

First we show that $f \circ f = i_d$, the identity function.

Setting $x = y = 1$ in the given equation, we have $f(f(1)) = f(1)$. Hence, $f(1) = f(1 \cdot f(1)) = f(1 \cdot f(f(1))) = f(1) \cdot f(1)$, yielding $f(1) = 1$.

Setting $x = 1$ in the given equation then yields $f(f(y)) = yf(1) = y$ for all $y \in \mathbb{N}$. Thus, $f \circ f = i_d$, as claimed. In particular, $f$ is both 1-1 and onto, and $f(x) = y$ implies that $f(y) = x$ since $f$ is its own inverse.

Next we show that $f$ is completely multiplicative, that is, $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. Since $f$ is onto, $\exists d \in \mathbb{N}$ such that $f(d) = b$. Then $d = f(b)$ and $f(ab) = f(af(d)) = df(a) = f(b)f(a)$.

Now we show that $f(p)$ is a prime if $p$ is a prime. Suppose $f(p) = mn$, where $1 < m \leq n$. Then $p = f(f(p)) = f(mn) = f(m)f(n)$, so either $f(m) = 1$ or $f(n) = 1$. Since $f$ is 1-1 and $f(1) = 1$, we have $m = 1$ or $n = 1$, a contradiction.

Note that $f(2007) = f(3^2 \cdot 223) = f(3)^2 \cdot f(223)$ and $f(3)$, $f(223)$ are primes, since $3$ and $223$ are primes. We cannot have $f(3) = 2$ and $f(223) = 3$, for then $f(2) = 3$, contradicting the fact that $f$ is 1-1. Thus, if $f(3) = 2$, then $f(2007) \geq 2^2 \cdot 5 = 20$.

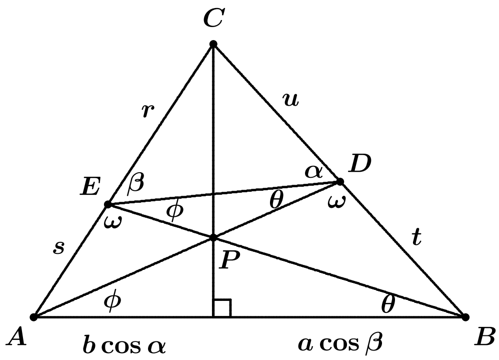
If $f(3) \geq 3$, then $f(223) \geq 2$ and $f(2007) \geq 3^2 \cdot 2 = 18$. The value $f(2007) = 18$ can be achieved by taking $f(2) = 223$, $f(3) = 3$, $f(223) = 2$, having $f$ match up all the remaining primes in pairs, then extending $f$ over the natural numbers. Our proof is complete.

**5**. Triangle $ABC$ is acute with $|AC| \neq |BC|$. The points $D$ and $E$ lie on the interiors of the sides $BC$ and $AC$ (respectively) such that $ABDE$ is a cyclic quadrangle, and the diagonals $AD$ and $BE$ intersect at $P$. If the lines $CP$ and $AB$ are perpendicular, show that $P$ is the orthocentre of triangle $ABC$.

*Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We give the write-up by Kandall.*

Let the angles of $\triangle ABC$ be $\alpha$, $\beta$, $\gamma$ ($\alpha \neq \beta$); $a = BC$, $b = AC$, $r = CE$, $s = EA$, $t = BD$, $u = DC$.

Since $ABDE$ is cyclic,




$$\begin{aligned}
\angle CDE &= 180^\circ - \angle EDB = \alpha\,, \\
\angle CED &= 180^\circ - \angle DEA = \beta\,, \\
\angle EDA &= \angle EBA = \theta\,, \\
\angle DEB &= \angle DAB = \varphi\,, \\
\angle AED &= \angle ADB = \omega\,.
\end{aligned}$$

By the Sine Law, $\dfrac{r}{u} = \dfrac{\sin\alpha}{\sin\beta} = \dfrac{a}{b}$, so $rb = au$; also $\dfrac{s}{\sin\theta} = \dfrac{|AB|}{\sin\omega} = \dfrac{t}{\sin\varphi}$; hence, $s = \dfrac{|AB|\sin\theta}{\sin\omega}$, $t = \dfrac{|AB|\sin\varphi}{\sin\omega}$.

By Ceva's theorem, $\frac{r}{s}\cdot\frac{b\cos\alpha}{a\cos\beta}\cdot\frac{t}{u} = 1$, that is, $\frac{t}{s} = \frac{au\cos\beta}{rb\cos\alpha}$, or $\frac{\sin\varphi}{\sin\theta} = \frac{\cos\beta}{\cos\alpha}$. Thus, $\sin\varphi\cos\alpha = \sin\theta\cos\beta$. Consequently, $\sin(\varphi+\alpha) + \sin(\varphi-\alpha) = \sin(\theta+\beta) + \sin(\theta-\beta)$. But $\alpha+\theta = \beta+\varphi = 180^\circ - \omega$, so $\varphi - \alpha = \theta - \beta$. Thus, $\sin(\varphi+\alpha) = \sin(\theta+\beta)$.

If $\varphi+\alpha = \theta+\beta$, then $(\varphi+\alpha)-(\varphi-\alpha) = (\theta+\beta)-(\theta-\beta)$, hence $\alpha=\beta$; contradiction. Therefore, we must have $(\varphi+\alpha)+(\theta+\beta) = 180^\circ$, that is $(\alpha+\theta)+(\beta+\varphi) = 180^\circ$. It follows that $\alpha+\theta = \beta+\varphi = 90^\circ$, hence $AD$ and $BE$ are altitudes of $\triangle ABC$ and $P$ is the orthocentre.

**6**. Find all ordered triples $(x,y,z)$ of mutually distinct real numbers which satisfy the set equation

$$\{x,y,z\} \;=\; \left\{\frac{x-y}{y-z},\ \frac{y-z}{z-x},\ \frac{z-x}{x-y}\right\}\,.$$

*Solution by Titu Zvonaru, Comăneşti, Romania.*

Since $\frac{x-y}{y-z}\cdot\frac{y-z}{z-x}\cdot\frac{z-x}{x-y} = 1$, we have $xyz = 1$. Thus, there are nonzero real numbers $a$, $b$, $c$ such that $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, and $ab \neq c^2$, $bc \neq a^2$, $ca \neq b^2$.

The set equation then becomes

$$\left\{\frac{a}{b},\frac{b}{c},\frac{c}{a}\right\} = \left\{\frac{a(ac-b^2)}{b(ab-c^2)},\frac{b(ab-c^2)}{c(bc-a^2)},\frac{c(bc-a^2)}{a(ac-b^2)}\right\}\,,$$

which resolves into one of six systems of three equations. The first of these is

$$\left\{\begin{array}{l}\dfrac{a}{b} = \dfrac{a(ac-b^2)}{b(ab-c^2)}\,,\\[2mm] \dfrac{b}{c} = \dfrac{b(ab-c^2)}{c(bc-a^2)}\,,\\[2mm] \dfrac{c}{a} = \dfrac{c(bc-a^2)}{a(ac-b^2)}\,,\end{array}\right. \qquad\Longleftrightarrow\qquad \left\{\begin{array}{l}(b-c)(a+b+c) = 0\,,\\ (c-a)(a+b+c) = 0\,,\\ (a-b)(a+b+c) = 0\,.\end{array}\right. \qquad (1)$$

If $b=c$ and $c=a$, then $ac=b^2$, so it follows that $a+b+c=0$ and we obtain the solution $(-\alpha-\beta,\alpha,\beta)$, with $\alpha\beta\neq 0$.

The next system is

$$\left\{\begin{array}{l}\dfrac{a}{b} = \dfrac{a(ac-b^2)}{b(ab-c^2)}\,,\\[2mm] \dfrac{b}{c} = \dfrac{c(bc-a^2)}{a(ac-b^2)}\,,\\[2mm] \dfrac{c}{a} = \dfrac{b(ab-c^2)}{c(bc-a^2)}\,.\end{array}\right. \qquad\Longleftrightarrow\qquad \left\{\begin{array}{r}(b-c)(a+b+c) = 0\,,\\ ab(ac-b^2) = c^2(bc-a^2)\,,\\ ab(ab-c^2) = c^2(bc-a^2)\,.\end{array}\right. \qquad (2)$$

Subtracting the last two equations, we obtain

$$ab(ac-b^2-ab+c^2) = 0 \Longleftrightarrow (b-c)(a+b+c) = 0\,.$$

It remains to solve the system

$$\begin{cases} (b-c)(a+b+c)=0 \\ ab(ab-c^2)=c^2(bc-a^2) \end{cases}.$$

If $c=-a-b$, then we have $ab(ab-c^2)=c^2(bc-a^2)$, which is equivalent to $ab(ab-a^2-b^2-2ab)=(a+b)^2(-ab-b^2-a^2)$. However, $a^2+b^2+ab>0$, and it follows that the last equation has no solution.

If $b=c$, then $ab(ab-c^2)=c^2(bc-a^2)$, or $ab(ab-b^2)=b^2(b^2-a^2)$, or $(a-b)(2a+b)=0$; and since $a\neq b$, we obtain the solution $(a,b,c)=(-\frac{\alpha}{2},\alpha,\alpha)$, with $\alpha\neq 0$.

Similarly, the system (3) below has solution $(a,b,c)=(\alpha,-\frac{\alpha}{2},\alpha)$, $\alpha\neq 0$, and the system (4) below has solution $(a,b,c)=(\alpha,\alpha,-\frac{\alpha}{2})$, $\alpha\neq 0$.

$$\frac{a}{b}=\frac{c(bc-a^2)}{a(ac-b^2)},\qquad \frac{b}{c}=\frac{b(ab-c^2)}{c(bc-a^2)},\qquad \frac{c}{a}=\frac{a(ac-b^2)}{b(ab-c^2)}; \tag{3}$$

$$\frac{a}{b}=\frac{b(ab-c^2)}{c(bc-a^2)},\qquad \frac{b}{c}=\frac{a(ac-b^2)}{b(ab-c^2)},\qquad \frac{c}{a}=\frac{c(bc-a^2)}{a(ac-b^2)}. \tag{4}$$

We will show that the next system has no solution

$$\begin{cases} \dfrac{a}{b}=\dfrac{b(ab-c^2)}{c(bc-a^2)} \\ \dfrac{b}{c}=\dfrac{c(bc-a^2)}{a(ac-b^2)} \\ \dfrac{c}{a}=\dfrac{a(ac-b^2)}{b(ab-c^2)} \end{cases} \iff \begin{cases} ac(bc-a^2)=b^2(ab-c^2) \\ c^2(bc-a^2)=ab(ac-b^2) \\ bc(ab-c^2)=a^2(ac-b^2) \end{cases} \tag{5}$$

Adding the first two equations, we have $a(a+c)(bc-a^2)=b(ab^2-bc^2+a^2c-ab^2)$, or $(a+c)(bc-a^2)=b(a^2-bc)$, which implies $a+b+c=0$ (because $bc\neq a^2$). With $c=-a-b$, the third equation is equivalent to

$$-b(a+b)(ab-a^2-b^2-2ab)=a^2(-a^2-ab-b^2)\iff a^2+ab+b^2=0,$$

and we do not obtain a solution.

Similarly, the system

$$\frac{a}{b}=\frac{c(bc-a^2)}{a(ac-b^2)},\qquad \frac{b}{c}=\frac{a(ac-b^2)}{b(ab-c^2)},\qquad \frac{c}{a}=\frac{b(ab-c^2)}{a(ac-b^2)}; \tag{6}$$

After transforming back to $x$, $y$, and $z$, we have that the set $\{x,\,y,\,z\}$ can be $\left\{-\dfrac{\alpha+\beta}{\alpha},\,\dfrac{\alpha}{\beta},\,-\dfrac{\beta}{\alpha+\beta}\right\}$ with $\alpha\beta\neq 0$, or $\{x,\,y,\,z\}=\{-\frac{1}{2},\,1,\,-2\}$.

To complete the files for 2009, we give some solutions from our readers to the selected problems of the 2007 Taiwanese Mathematical Olympiad [2009 : 501].

**1**. Prove the following statements:

(a) If $0 < a,\, b \leq 1$, then
$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} \leq \frac{2}{\sqrt{1+ab}}\,;$$

(b) If $ab \geq 3$, then
$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} \geq \frac{2}{\sqrt{1+ab}}\,.$$

*Solution by Titu Zvonaru, Comǎneşti, Romania.*

(a) The inequality is true for $a,\, b > 0$ with $ab \leq 1$. By squaring we obtain
$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{2}{\sqrt{(1+a^2)(1+b^2)}} \leq \frac{4}{1+ab}\,.$$
By the Cauchy–Schwarz Inequality, $(1+a^2)(1+b^2) \geq (1+ab)^2$, thus
$$\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{2}{1+ab} \leq \frac{4}{1+ab}$$
$$\Leftrightarrow 2 + 2a^2 + 2b^2 + 2a^2b^2 - 1 - ab - b^2 - ab^3 - 1 - ab - a^2 - a^3b \geq 0$$
$$\Leftrightarrow a^2 + b^2 - 2ab - ab(a^2+b^2-2ab) \geq 0$$
$$\Leftrightarrow (a-b)^2(1-ab) \geq 0\,,$$
and the last inequality is true.

Equality holds if and only if $(a-b)^2(1-ab) = 0$ and $(1+a^2)(1+b^2) = (1+ab)^2$, that is $a = b$.

(b) If $a = b$, the equality occurs. Suppose $a \neq b$. After squaring, we have
$$\frac{1}{1+a^2} + \frac{1}{1+b^2} - \frac{2}{1+ab} + \frac{2}{\sqrt{(1+a^2)(1+b^2)}} - \frac{2}{1+ab} \geq 0$$
$$\Leftrightarrow \frac{(a-b)^2(ab-1)}{(1+a^2)(1+b^2)} + 2\cdot\frac{1+ab-\sqrt{(1+a^2)(1+b^2)}}{\sqrt{(1+a^2)(1+b^2)}} \geq 0$$
$$\Leftrightarrow \frac{(a-b)^2(ab-1)}{\sqrt{(1+a^2)(1+b^2)}} + 2\cdot\frac{(1+ab)^2-(1+a^2)(1+b^2)}{1+ab+\sqrt{(1+a^2)(1+b^2)}} \geq 0$$
$$\Leftrightarrow \frac{(a-b)^2(ab-1)}{\sqrt{(1+a^2)(1+b^2)}} + 2\cdot\frac{-(a-b)^2}{1+ab+\sqrt{(1+a^2)(1+b^2)}} \geq 0$$
$$\Leftrightarrow (ab-1)(ab+1) + (ab-1)\sqrt{(1+a^2)(1+b^2)}$$
$$- 2\sqrt{(1+a^2)(1+b^2)} \geq 0$$
$$\Leftrightarrow (ab-1)(ab+1) + (ab-3)\sqrt{(1+a^2)(1+b^2)} \geq 0\,,$$
and the last inequality is true.

**2**. Find all positive integers $a$, $b$, $c$, and $d$ such that

$$2^a \;=\; 3^b 5^c + 7^d \,.$$

*Solution by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA, modified by the editor.*

We will prove that there is a unique solution to the equation

$$2^a = 3^b \cdot 5^c + 7^d \tag{1}$$

namely $a = 6$, $b = 1$, $c = 1$, and $d = 2$.

We will make use of the following lemma.

**Lemma 1**. The only solution in positive integers $x$ and $y$, to the diophantine equation.

$$2^x - 1 = 7^y \tag{2}$$

is $x = 3$ and $y = 1$.

*Proof*: Modulo $3$ the equation becomes $(-1)^x + 2 \equiv 1 \pmod 3$, from which we see that $x$ must be odd.

Clearly $x = 1$ is not a solution and $(x, y) = (3, 1)$ is a solution with $x = 3$, so henceforth we assume that $x$ is odd, $x \geq 5$, and $y$ a positive integer.

Modulo $7$ the equation becomes $2^x \equiv 1 \pmod 7$, from which we see that $x$ is divisible by $3$, since modulo $7$ the powers $2^0$, $2^1$, $2^2$, $2^3$, $2^4$, $2^5$, $\dots$ repeat in a cycle of three: $0$, $1$, $4$, $0$, $1$, $4$, $\dots$.

Thus, $x = 6k + 3$, for some positive integer $k$.

Equation (2) now becomes $2^{6k+3} - 1 = 7^y$, or $(2^{2k+1})^3 - 1 = 7^y$, and the lefthand side can be factored as a difference of cubes

$$(2^{2k+1} - 1) \cdot [(2^{2k+1})^2 + 2^{2k+1} + 1] = 7^y \,. \tag{3}$$

Since $k \geq 1$, both factors on the lefthand side of (3) are positive integers *greater than* $1$. Therefore, since $7$ is a prime; equation (3) implies that

$$\begin{aligned} 2^{2k+1} - 1 &= 7^{t_1} \,, \\ (2^{2k+1})^2 + 2^{2k+1} + 1 &= 7^{t_2} \,, \end{aligned} \tag{4}$$

where $t_1$, $t_2$ are positive integers such that $t_1 + t_2 = y$.

Substituting for $2^{2k+1} = 7^{t_1} + 1$ in the second equation in (4) yields, $(7^{t_1} + 1)^2 + 7^{t_1} + 1 + 1 = 7^{t_2}$, or $7^{2t_1} + 3 \cdot 7^{t_1} + 3 = 7^{t_2}$, an impossibility as this last equation implies (in view of $t_1 \geq 1$, $t_2 \geq 1$) that $7$ divides $3$. ∎

Back to equation (1). Since $a$, $b$, $c$, $d$ are positive integers, the righthand side of (1) must be at least $3 \cdot 5 + 7 = 22$. Thus, by inspection, we see that there are no solutions with $a \leq 5$. For $a = 6$, we have the solution $a = 6$, $b = 1$, $c = 1$, $d = 2$.

Now suppose that $a \geq 7$. First observe that $a$ must be even. Indeed, modulo $3$ the equation (1) becomes $(-1)^a \equiv 1 \pmod 3$, and the claim is established. Thus,

$$a \equiv 0 \pmod 2 \quad \text{and} \quad a \geq 8\,. \tag{5}$$

The next claim is that $b$ and $c$ have the same parity, that is, either both $b$ and $c$ are odd, or both $b$ and $c$ are even. To show this, we assume that $b$ and $c$ are of opposite parity and arrive at a contradiction.

Since $a \geq 8$, $2^a \equiv 0 \pmod 8$, and equation (1) modulo 8 becomes

$$0 \equiv 3^b \cdot 5^c + 7^d \pmod 8\ . \tag{6}$$

If $b$ is odd and $c$ even, then $3^b \equiv 3 \pmod 8$, while $5^c \equiv 1 \pmod 8$. By (6) we obtain $0 \equiv 3 \cdot 1 + 7^d \pmod 8$, or $7^d \equiv -3 \equiv 5 \pmod 8$, which is impossible since $7^d \equiv 7$ or $1 \pmod 8$, for $d$ odd or even, respectively.

If $b$ is even and $c$ odd, then $3^b \equiv 1 \pmod 8$, while $5^c \equiv 5 \pmod 8$; and by (6) we obtain $0 \equiv 1 \cdot 5 + 7^d \pmod 8$, or $7^d \equiv -5 \equiv 3 \pmod 8$, again an impossibility.

We have proved that both $b$ and $c$ are odd, or both $b$ and $c$ are even.

If $b$ and $c$ are odd, then $3^b \cdot 5^c \equiv 3 \cdot 5 \equiv 15 \equiv 7 \pmod 8$, and so by (1) we have $7^d \equiv -7 \equiv 1 \pmod 8$ and $d$ is even.

Otherwise, if $b$ and $c$ are even, then modulo 8 the equation (1) yields $7^d \equiv -1 \equiv 7 \pmod 8$, and then $d$ is odd.

These two cases are dealt with below.

**Case A.** $b \equiv c \equiv 1 \pmod 2$ and $d \equiv 0 \pmod 2$.

Recall from (5) that $a$ is also even. Put $a = 2m$, $d = 2l$, where $m$, $l$ are positive integers with $m \geq 4$ (since by (5), $a \geq 8$). From (1) we obtain $2^{2m} - 7^{2l} = 3^b \cdot 5^c$; or equivalently

$$(2^m - 7^l) \cdot (2^m + 7^l) = 3^b \cdot 5^c \tag{7}$$

By inspection, the two odd factors on the lefthand side of equation (7) are relatively prime; since any common prime divisor would have to divide their sum $2 \cdot 2^m = 2^{m+1}$; so such a prime would have to equal $2$, not possible since these two factors are odd. Now, since the two factors are relatively prime positive integers; and $3$ and $5$ are primes and $2^m - 7^l$ is *the smaller of the two factors*; then, according to equation (7), there are precisely two possibilities:

*Possibility 1*: $2^m - 7^l = 1$ and $2^m + 7^l = 3^b \cdot 5^c$; or

*Possibility 2*: $2^m - 7^l = 3^b$ and $2^m + 7^l = 5^c$.

Possibility 1 is ruled out at once by Lemma 1, since $m \geq 4$.

Possibility 2 is ruled out modulo 8. Indeed, since $m \geq 4$; $2^m \equiv 0 \pmod 8$. And since $b$ is odd, $3^b \equiv 3 \pmod 8$. Thus, by the first equation,

$$0 - 7^l \equiv 3 \pmod 8 \iff 7^l \equiv -3 \equiv 5 \pmod 8\ ,$$

which is a contradiction since $7^l \equiv 7$ or $1 \pmod 8$, as $l$ is odd or even, respectively. The second equation in Possibility 2 yields a similar contradiction since $c$ is odd.

Finally, we consider

**Case B**. $b \equiv c \equiv 0 \pmod 2$ and $d \equiv 1 \pmod 2$.

We put $a = 2\alpha$, $b = 2\beta$, $c = 2\gamma$ where $\alpha$, $\beta$, $\gamma$ are positive integers with $\alpha \geq 4$ (since $a \geq 8$ by (5)). From equation (1) we get

$$2^{2\alpha} - 3^{2\beta} \cdot 5^{2\gamma} = 7^d \iff (2^\alpha - 3^\beta 5^\gamma)(2^\alpha + 3^\beta 5^\gamma) = 7^d\,. \tag{8}$$

Again, by inspection, we see that the two odd positive integers on the left-hand side of (8) are relatively prime; and since $7$ is a prime and $2^\alpha - 3^\beta 5^\gamma$ is the smaller of the two factors; (8) implies

$$\begin{aligned} 2^\alpha - 3^\beta 5^\gamma &= 1\,, \\ 2^\alpha + 3^\beta 5^\gamma &= 7^d\,. \end{aligned} \tag{9}$$

By adding the equations in (9) we obtain

$$2^{\alpha+1} - 1 = 7^d\,, \tag{10}$$

which is impossible by Lemma 1, since $\alpha + 1 \geq 5$.

Therefore, the equation (1) has the *unique* solution in positive integers $a = 6$, $b = 1$, $c = 1$, $d = 2$.

**4**. Let $ABCD$ be a convex quadrilateral. Prove or disprove that there exists a point $E$ in the plane of $ABCD$ such that $\triangle ABE$ is similar to $\triangle CDE$.

*Solved by Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's generalization.*

We generalize as follows: if $A$, $B$, $C$, $D$ are points in the Euclidean plane such that $A \neq B$, $C \neq D$ and $\overrightarrow{AB} \neq \overrightarrow{CD}$, then there exists a point $E$ such that $\triangle ABE$ and $\triangle CDE$ are similar.

The result is obvious if $AB$ and $CD$ are parallel, since then the point of intersection of $AC$ and $BD$ is a suitable point $E$.

In the general case when $AB$ and $CD$ are not parallel, we obtain the result with the help of complex numbers. We denote the complex affix of any point $M$ by $m$ and let $\omega = \dfrac{c-d}{a-b}$. Note that $\omega \neq 1$. Let $E$ be the point whose affix is $e = \dfrac{\alpha}{1-\omega}$ where $\alpha = c - a\omega$. Then, $c = a\omega + \alpha$ and $d = c - a\omega + b\omega = b\omega + \alpha$. Also, $e = e\omega + \alpha$, and therefore

$$\frac{d-e}{c-e} = \frac{b\omega + \alpha - e\omega - \alpha}{a\omega + \alpha - e\omega - \alpha} = \frac{b-e}{a-e}.$$

Thus,

$$\frac{DE}{CE} = \frac{BE}{AE} \quad \text{and} \quad \angle CED = \angle AEB\,,$$

so that $\triangle ABE$ is similar to $\triangle CDE$.

**5**. Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that for all real numbers $x$ and $y$,

$$f(x)f\big(yf(x) - 1\big) \;=\; x^2 f(y) - f(x)\,.$$

*Solution by Michel Bataille, Rouen, France.*

The zero function $\tilde{0} : x \mapsto 0$ and the identity function $\mathrm{id}_{\mathbb{R}} : x \mapsto x$ are obviously solutions. We show that there are no other solutions.

For convenience, denote the given equation by $(E)$, and let $f$ be any solution. Taking $x = y = 0$ in $(E)$ yields $f(0) \cdot (f(-1) + 1) = 0$, so that $f(0) = 0$ if $f(-1) \neq -1$. In this case, we also have $f(x) \cdot f(-1) = -f(x)$ (by taking $y = 0$ in $(E)$), hence $f(x) = 0$ for all $x$. Thus $f = \tilde{0}$.

Now we suppose that $f(-1) = -1$. Taking $y = 0, x = 1$ in $(E)$ yields $f(0) = 0$. Also, taking $y = -1$ in $(E)$ shows that $f(x) = 0$ implies $x = 0$. In particular, $f(1)$ is a nonzero real number which satisfies $f(1)f(f(1) - 1) = f(1) - f(1) = 0$ (taking $x = y = 1$ in $(E)$). Thus, $f(f(1) - 1) = 0$ and $f(1) = 1$. Taking $x = 1$ and $x = -1$ in $(E)$, we obtain

$$f(y-1) = f(y) - 1 \qquad \text{and} \qquad -f(-y-1) = f(y) + 1\,.$$

It is easy to deduce that $f$ is an odd function, and since $f(yf(x) - 1) = f(yf(x)) - 1$, it then follows from $(E)$ that

$$f(x)f(yf(x)) = x^2 f(y) \tag{$E'$}$$

for all real numbers $x$ and $y$.

With $y = x$, $(E')$ gives $f(xf(x)) = x^2$ (for $x \neq 0$, but this is also valid for $x = 0$). Replacing $x$ by $xf(x)$ in $(E')$ yields $f(yx^2) = (f(x))^2 f(y)$ from which we deduce $f(x^2) = (f(x))^2$ and so $f(yx^2) = f(y)f(x^2)$. Since $f$ is odd, an easy consequence is

$$f(uv) = f(u)f(v)$$

for all real numbers $u$ and $v$. Also, for $v \neq 0$,

$$\begin{aligned} f(u+v) &= f\left(v\left(\frac{u}{v} + 1\right)\right) = f(v)f\left(\frac{u}{v} + 1\right) = f(v)\left(f\left(\frac{u}{v}\right) + 1\right) \\ &= f(v)f\left(\frac{u}{v}\right) + f(v) = f(u) + f(v) \end{aligned}$$

hence $f(u+v) = f(u) + f(v)$ for all $u$, $v$.

Thus, $f$ satisfies the conditions $f(1) = 1$, $f(u+v) = f(u)+f(v)$, $f(uv) = f(u)f(v)$ for all real numbers $u$, $v$. As is well-known, this implies that $f = \mathrm{id}_{\mathbb{R}}$.

We finish with a single solution to a problem of the Youth Mathematical Olympiad of the Asociación Venezolana de Competencias Matemáticas, 2006, given at [2009 : 380].

**4**. Joseph, Dario, and Henry prepared some labels. On each label they wrote one of the numbers $2$, $3$, $4$, $5$, $6$, $7$, or $8$. David joined them and stuck one label on the forehead of each friend. Joseph, Dario, and Henry could not see the numbers on their own foreheads, they only saw the numbers of the other two. David said, "You do not have distinct numbers on your foreheads, and the product of the three numbers is a perfect square." Each friend then tried to deduce what number he had on his forehead. Could anyone discover it?

*Solution by Titu Zvonaru, Comănești, Romania.*

Since the three numbers are not distinct, then their product is $a^2b$, where $a$, $b \in \{2, 3, 4, 5, 6, 7, 8\}$. The product $a^2b$ is a perfect square if and only if $b$ is a perfect square, that is $b = 4$.

If one friend sees the label "$a, a$", then the label 4 is on his forehead.

If one friend sees the label "$a, 4$", then the label $a$ is on his forehead.

It follows that each of the three friends can discover what label is on his own forehead.

Since we are introducing changes in editorship of the *Corner* next issue it is appropriate to wrap up this number (and this volume of ***CRUX with MAYHEM***) with thanks to all those who have contributed problems and solutions in 2010:

| | |
|---|---|
| Mohammed Aassila | David E. Manes |
| Arkady Alt | John Grant McLoughlin |
| Miguel Amengual Covas | Henry Ricardo |
| George Apostolopoulos | Leda Sanchez |
| Matthew Babbitt | Bill Sands |
| Michel Bataille | D.J. Smeenk |
| Chip Curtis | Alex Song |
| José Luis Díaz-Barrero | Jan Verster |
| J. Chris Fisher | Stan Wagon |
| Oliver Geupel | Edward T.H. Wang |
| Joe Howard | Dexter Wei |
| Geoffrey A. Kandall | Kaiming Zhao |
| Matti Lehtinen | Konstantine Zelator |
| Hugo Luyo | Titu Zvonaru |

Also, I cannot stress how vital the support of Joanne Canape has been to bringing together the numbers of the *Corner* over these many years.

# BOOK REVIEWS

## Amar Sodhi

*The Calculus Collection, A Resource for AP and Beyond*
Edited by Caren L. Diefenderfer and Roger B. Nelsen
Published by The Mathematical Association of America, 2010
ISBN: 978-0-88385-761-8, hardcover, 507+xx pages, US$74.95
*Reviewed by* **Amar Sodhi***, Sir Wilfred Grenfell College, Corner Brook, NL*

Rushing to the dentist's office you hurriedly grab the latest *College Mathematics Journal*. After all, you want some light reading while waiting for your appointment. While absorbed by an article on the advantages of implicit differentiation you are summoned for dental cleaning. Naturally, calculus is in both the hygienist's and your thoughts as you are having your teeth scraped, and later that afternoon you rush to the library as a germ of an idea takes hold. Sure enough, during the last twenty years, the three main journals of the Mathematics Association of America (MAA) contain enough stimulating papers in calculus to fill a book.

*The Calculus Collection* is a worthy enough title for a volume containing select articles on limits, the derivative, integrals, polynomial approximations, and series, which have been written to inform or amuse anyone with an interest in calculus. Such a volume may provide a battery of ideas to allow an instructor to invigorate a "text-book" calculus course or to demonstrate to the keen student that there is some beauty to behold in an area of mathematics which is invariably taught for its usefulness in science, social science, and engineering. Yes, the MAA has done a service in publishing this book which features a smorgasbord of refereed papers dating from 1991, arranged neatly by topic and judiciously edited by Diefenderfer and Nelsen.

The reason that Diefenderfer and Nelsen only go back to 1991 is quite simple; their book can be viewed as a sequel to *A Century of Calculus* which contains papers which appeared in MAA journals between 1884 and 1991. However, the motivation the editors use to justify publishing *The Calculus Collection* is to provide resource material for advanced placement (AP) calculus. The subtitle of the book, *A Resource for AP and Beyond*, confirms that the book is targeted solely for high school teachers or students who are involved in an AP calculus course. This is unfortunate since this resource book neither contains a wealth of challenge problems (complete with solutions) nor abounds with articles specially written with the AP programme in mind. Rather, it is a collection of papers (91 from *The College Mathematics Journal*, 17 from *Mathematics Magazine*, 12 from the *American Mathematical Monthly* and three from the two other MAA periodicals, *FOCUS* and *Horizons*) which were presumably written to share ideas with the professional mathematical community in general. However, suggestions on how each article can be used in an introductory calculus course is given in an appendix.

Also, in the preface, the editors make it clear that this book is an instructor's (as opposed to a student's) resource manual.

Anybody who enjoys the calculus based articles that can be found in the *College Mathematics Journal* will find a lot to like in this book, and this certainly includes students and instructors of AP calculus. However, from a marketing stance, this "rose" would smell sweeter if it had another subtitle. Might I suggest "The Bedside Calculus Companion"?

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*An Episodic History of Mathematics: Mathematical Culture Through Problem Solving*
by Steven G. Krantz
Mathematical Association of America, 2010
ISBN: 978-0-88385-766-3, hardcover, 381+xii pages, US$67.95
*Reviewed by* ***Ed Barbeau****, University of Toronto, Toronto, ON*

The title of this book is unfortunate, as it does not reflect what is between the covers. To be sure, there is history here as well as many problems and explorations. However, these are not woven into the mathematical material; rather the volume is an exposition of mathematical topics presented in a modern style with the history and problems playing an ancillary role.

Each chapter includes one or more essays on an historical figure or school, followed by accounts of related areas of mathematics. These are punctuated by invitations to the reader to explore an example or extension. At the conclusion of the chapter is a set of exercises, some designated as projects; these come without hints, solutions or commentaries.

There is good coverage of many seminal areas of mathematics: limits, conics, the development of algebra and solution of equations, Cartesian coordinates, differential and integral calculus, complex numbers and the fundamental theorem of algebra, number theory, the Fermat conjecture, the real continuum, the pigeonhole theorem, Ramsey theory, the hyperbolic disc, cardinality, the beginnings of topology, modern abstract algebra, methods of proof, and cryptology.

The historical essays contain a great deal of interesting lore and detail, although the historical judgments are not always completely reliable. For example, Euclid's proof of the infinitude of primes is presented as a proof by contradiction, whereas he really proved that no matter what finite set of primes we have, we can always find one more. Perhaps to the modern reader this is essentially the same, but to the historian, the shading is important. For a more authentic engaging of history, one might turn to the venerable text by Howard Eves, *Introduction to the History of Mathematics* ($6^{\text{th}}$ edition, 1990, Saunders) or Otto Toeplitz, *The Calculus: a Genetic Approach* (1963, 2007). The level of the mathematical presentation is too sophisticated for most secondary students; this book is best recommended for those pre-college students who are advanced or particularly keen and persistent, and for college students in their first two years, where some of the chapters will supplement material in their regular courses.

*Methods for Euclidean Geometry*
by Owen Byer, Felix Lazebnik, and Deirdre L. Smeltzer
Mathematical Association of America, 2010
ISBN: 9-780-88385-763-2, hardcover, 461 + xvi pages, US$69.95
*Reviewed by* ***J. Chris Fisher****, University of Regina, Regina, SK*

This excellent book is quite different from other geometry texts. Its goal is to review and deepen a reader's understanding of Euclidean plane geometry by emphasizing techniques developed after Euclid. The authors focus on the relationship between geometry and mainstream mathematics, reminding us that in previous centuries all mathematicians did geometry. To achieve their goal they feature an ample collection of problems that range from routine to challenging; nearly half the book's 461 pages are devoted to problem statements, hints, and solutions. Although ***CRUX with Mayhem*** was not a source, many of the problems would be attractive to readers of this journal. Some problems appear more than once throughout the book so that the reader can try a variety of methods and compare the merits of each approach. An appendix provides a complete solution to most of the problems, but each chapter concludes with fifteen or so supplemental problems that are not accompanied by solutions.

The first two chapters provide a perfunctory history of geometry (six pages) and discussion of axioms (13 pages). That is followed by four chapters (about 100 pages) that review plane geometry using methods that would be familiar to Euclid. The topics covered are triangles, quadrilaterals, other polygons, circles, length, area, and loci. These chapters review those theorems that students should have seen in high school and complement that material with other basic theorems (such as the theorems of Ceva, Menelaus, and Ptolemy) that they will need when solving problems. The authors provide the simple proofs of many of these results; more importantly, they carefully state the results and, where appropriate, their converses. Examples: they list six necessary and sufficient properties for a quadrilateral to be a parallelogram, and eight properties establishing that a triangle is isosceles. It is crucial that readers be provided with an explicit list of results that they can use to back claims they make in their own proofs. In every chapter the authors provide a few examples of how the basic theorems can be used in problem solving. Although the topics go somewhat beyond what is taught in typical high schools, the authors stop short of introducing $19^{\text{th}}$ century triangle geometry (such as the nine-point circle, which the authors refer to as "baroque problems," a description that made this reviewer choke slightly). Nevertheless, there are fresh and interesting items in most chapters; in the locus chapter, for example, the authors describe how Newton corrected Galileo's claim that the trajectory of a projectile is a parabola. (The trajectory is elliptical unless the earth happens to be flat.) In the area chapter there is a proof that any two plane polygons having the same area are equidecomposable, that is, they can be partitioned by straight lines into an equal finite number of pieces such that corresponding pieces are congruent.

Chapters 7 through 13 form the core of the text. Each of these chapters introduces a postEuclidean technique for solving geometry problems: trigonometry, coordinates (considered central to the authors' approach; there is a separate chapter that uses coordinates to study conics), complex numbers, vectors, affine transformations, and inversions. These topics clearly support the authors' desire that the reader learn mathematics, with geometry providing the content. A $14^{th}$ chapter discusses the use of computer software to supplement the use of coordinates for solving geometry problems. The authors had originally intended for a CD to accompany the text; instead, the reader can download a Maple worksheet that demonstrates how to use Maple to solve some of the problems from earlier chapters; without access to Maple, however, the computer-aided solutions can only be read, not implemented.

The book is published by the Mathematical Association of America as part of its Classroom Resource Materials series, "intended to provide supplemental classroom material for students—laboratory exercises ... [and] textbooks with unusual approaches for presenting mathematical ideas." The authors have used their book for university courses taken by second (and third) year math majors, as well as for courses aimed principally at education majors who plan to teach in high schools. A typical course briefly covered Chapters 3 to 7 (geometry and trigonometry review), then concentrated on material from Chapters 8 (coordinates), 9 (conics), 10 (complex numbers), and 12 (affine transformations). The text has also been used for more demanding courses that include Chapter 13 (inversions). That looks like too much material to fit into any course I would teach; one could make a variety of courses out of any pair of the chapters 7 through 13 after a brief review of the highlights of high-school geometry. Beyond their geometry courses, the authors have used individual chapters to supplement courses they taught in calculus, linear algebra, and abstract algebra. They further suggest that their book could serve as the basis of a capstone course in mathematics, or as a resource for a problem-solving group, or perhaps as a text for the bright high-school student who wants to learn the material on his own. Unfortunately, the book has one glaring fault: its price. With a list price of US$70, the book costs double what I would ask a student to pay for a course that might use perhaps a hundred of its pages. The Math Association of America seems to have gone from a policy of publishing inexpensive, accessible books for students, to using their publishing wing to support other worthy association activities. Whether or not such a policy might be wise, I hesitate to recommend this book as a textbook—there are other available texts that might not be as carefully written nor contain such a rich assortment of material, but they are adequate and cost much less.

# A Solution to Gibson's and Rodgers' Problem in 3 Dimensions

Nguyen Minh Ha

## 1 Introduction

Peter M. Gibson and Michael H. Rodgers [1] posed problem 844 in ***CRUX Mathematicorum*** on iterated triangles inscribed in a circle and a higher dimensional analogue. The first part of their problem is as follows:

> (a) A triangle $A_0B_0C_0$ with centroid $G_0$ is inscribed in a circle $\Gamma$ with centre $O$. The lines $A_0G_0$, $B_0G_0$, $C_0G_0$ meet $\Gamma$ again in $A_1$, $B_1$, $C_1$, respectively, and $G_1$ is the centroid of triangle $A_1B_1C_1$. A triangle $A_2B_2C_2$ with centroid $G_2$ is obtained in the same way from $A_1B_1C_1$, and the procedure is repeated indefinitely, producing triangles with centroids $G_3$, $G_4$, $\ldots$. If $g_n = OG_n$, prove that the sequence $\{g_0, g_1, g_2, \ldots\}$ is decreasing and converges to $0$.

This part was solved by R.B. Killgrove and Dan Sokolowsky [3].

The second part of problem 844 was to determine if a similar result holds for a tetrahedron inscribed in a sphere, or, more generally, for an $n$-simplex inscribed in an $n$-sphere. This latter problem is hitherto unsolved. Here we give a positive answer and a proof in the 3-dimensional case.

## 2 Notation and Preliminary Results

Throughout we will assume that all tetrahedra are nondegenerate or we shall prove that the tetrahedra which arise are nondegenerate.

For convenience we adopt certain notations. Let $S_A$, $S_B$, $S_C$, $S_D$ be the areas of the faces opposite the vertices $A$, $B$, $C$, $D$ of tetrahedron $ABCD$, let $(XYZ)$ be the plane through the three points $X$, $Y$, $Z$, and let $V(WXYZ)$ be the volume of tetrahedron $WXYZ$. For certain special sums, the following notation will be used:

$$\begin{aligned}\sum S_A^2\overrightarrow{LA} &= S_A^2\overrightarrow{LA} + S_B^2\overrightarrow{LB} + S_C^2\overrightarrow{LC} + S_D^2\overrightarrow{LD}\,,\\ \sum AB^2 &= AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2\,.\end{aligned}$$

A dot "$\cdot$" will denote either multiplication of two numbers or the dot product of two vectors, depending on the context.

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We now make some definitions. Let $ABCD$ be a tetrahedron. A plane through the edge $AB$ and the midpoint of the edge $CD$ is called the *median plane* through the edge $AB$ of the tetrahedron. A bisecting plane of the dihedral angle at the edge $AB$ of the tetrahedron is called the *bisector plane* through the edge $AB$ of the tetrahedron. The plane that is the reflection of the median plane through edge $AB$ in the bisector plane through the edge $AB$ is called the *symmedian plane* through the edge $AB$ of the tetrahedron.

Each tetrahedron has six edges and thus has six median planes, six bisector planes, and six symmedian planes.

It is known that the six median planes intersect in a common point which is the centroid of the tetrahedron, and the six bisector planes intersect in a common point which is the centre of the inscribed sphere. The six symmedian planes also intersect in a common point and we shall call this point the *Lemoine point* of the tetrahedron (we will prove this later).

Our main theorem has two parts, the second part being the positive answer to the problem posed by Peter M. Gibson and Michael H. Rodgers in three dimensions.

**Theorem** Let $A_0B_0C_0D_0$ be a tetrahedron with volume $V_0$ and centroid $G_0$ inscribed in a sphere $\Gamma$ with centre $O$. The lines $A_0G_0$, $B_0G_0$, $C_0G_0$, $D_0G_0$ intersect $\Gamma$ again in $A_1$, $B_1$, $C_1$, $D_1$, respectively, and $V_1$ and $G_1$ are the volume and the centroid of tetrahedron $A_1B_1C_1D_1$, respectively. A tetrahedron $A_2B_2C_2D_2$ with volume $V_2$ and centroid $G_2$ is obtained in a similar way from $A_1B_1C_1D_1$, and the procedure is repeated indefinitely, producing tetrahedra with volumes $V_3$, $V_4$, $\ldots$ and centroids $G_3$, $G_4$, $\ldots$. Then,

(1) The sequence $\{V_n\}$ is nondecreasing, and

(2) The sequence $\{OG_n\}$ is nonincreasing and converges to zero.

In order to prove Theorem 1 we need several lemmas.

**Lemma 1** If $M$ is inside tetrahedron $ABCD$, then $\sum V(MBCD)\overrightarrow{MA} = \overrightarrow{0}$.

*Proof*: Choose points $A'$, $B'$, $C'$, $D'$ on the rays $MA$, $MB$, $MC$, $MD$, respectively, so that $M$ is the centroid of tetrahedron $A'B'C'D'$. Note that the volume of each tetrahedron $MB'C'D'$, $MC'D'A'$, $MD'A'B'$, $MA'B'C'$ is one-fourth the volume of tetrahedron $A'B'C'D'$. We have

$$\begin{aligned}
&\sum V(MBCD)\overrightarrow{MA} \\
&\quad= \frac{1}{4}V(A'B'C'D')\sum\frac{V(MBCD)}{V(MB'C'D')}\overrightarrow{MA} \\
&\quad= \frac{1}{4}V(A'B'C'D')\sum\frac{MB\cdot MC\cdot MD}{MB'\cdot MC'\cdot MD'}\overrightarrow{MA} \\
&\quad= \frac{1}{4}V(A'B'C'D')\frac{MA\cdot MB\cdot MC\cdot MD}{MA'\cdot MB'\cdot MC'\cdot MD'}\sum\frac{MA'}{MA}\overrightarrow{MA} \\
&\quad= \frac{1}{4}V(A'B'C'D')\frac{MA\cdot MB\cdot MC\cdot MD}{MA'\cdot MB'\cdot MC'\cdot MD'}\sum\overrightarrow{MA'} = \overrightarrow{0}. \qquad \blacksquare
\end{aligned}$$

**Lemma 2** Tetrahedron $ABCD$ is inscribed in sphere $(O)$. Let $M$ be a point in the interior of the tetrahedron. Let the lines $MA$, $MB$, $MC$, $MD$ meet $(O)$ again at $A'$, $B'$, $C'$, $D'$. Then

$$\frac{V(ABCD)}{V(A'B'C'D')} = \frac{MA \cdot MB \cdot MC \cdot MD}{MA' \cdot MB' \cdot MC' \cdot MD'} .$$

*Proof*: By Lemma 1, we have $\sum V(MBCD)\overrightarrow{MA} = \overrightarrow{0}$. Thus,

$$\begin{aligned}\sum V(MBCD)\frac{MA}{MA'}\overrightarrow{MA'} &= -\sum V(MBCD)\left(-\frac{MA}{MA'}\right)\overrightarrow{MA'} \\ &= -\sum V(MBCD)\overrightarrow{MA} = \overrightarrow{0} .\end{aligned}$$

Since the numbers $V(MBCD)\frac{MA}{MA'}$, $V(MCDA)\frac{MB}{MB'}$, $V(MDAB)\frac{MC}{MC'}$, and $V(MABC)\frac{MD}{MD'}$ are positive, $M$ is inside the tetrahedron $A'B'C'B'$, and hence $V(A'B'C'D') = \sum V(MB'C'D')$.

Note that $MA \cdot MA' = MB \cdot MB' = MC \cdot MC' = MD \cdot MD' = R^2 - OM^2$, where $R$ is the radius of $(O)$. Thus,

$$\begin{aligned} &V(A'B'C'D') \\ &\quad = \sum \frac{V(MB'C'D')}{V(MBCD)} V(MBCD) \\ &\quad = \sum \frac{MB' \cdot MC' \cdot MD'}{MB \cdot MC \cdot MD} V(MBCD) \\ &\quad = \frac{MA' \cdot MB' \cdot MC' \cdot MD'}{MA \cdot MB \cdot MC \cdot MD} \cdot \frac{1}{MA \cdot MA'} \sum V(MBCD)MA^2 \\ &\quad = \frac{MA' \cdot MB' \cdot MC' \cdot MD'}{MA \cdot MB \cdot MC \cdot MD} \cdot \frac{1}{R^2 - OM^2} \sum V(MBCD)MA^2 . \qquad (1)\end{aligned}$$

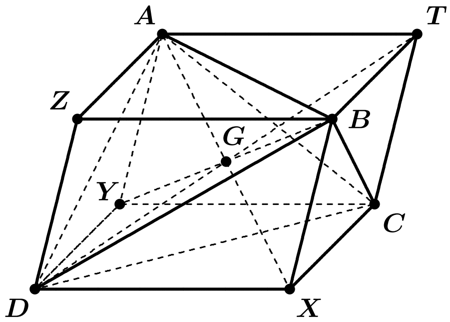
However, we also have

$$\begin{aligned} &V(ABCD)R^2 \\ &\quad = \sum V(MBCD)OA^2 = \sum V(MBCD)\left|\overrightarrow{OM} + \overrightarrow{MA}\right|^2 \\ &\quad = \left(\sum V(MBCD)\right)OM^2 + 2\overrightarrow{OM} \cdot \left(\sum V(MBCD)\overrightarrow{MA}\right) \\ &\qquad + \sum V(MBCD)MA^2 \\ &\quad = V(ABCD)OM^2 + 2\overrightarrow{OM} \cdot \overrightarrow{0} + \sum V(MBCD)MA^2 \\ &\quad = V(ABCD)OM^2 + \sum V(MBCD)MA^2 . \qquad (2)\end{aligned}$$

It follows that $\sum V(MBCD)MA^2 = V(ABCD)(R^2 - OM^2)$.
The lemma now follows from the above identities (1) and (2). ■

**Lemma 3** The opposing edges (three pairs altogether) of a tetrahedron are of equal length if and only if its centroid coincides with the centre of its circumscribed sphere.

*Proof*: Let tetrahedron $ABCD$ have centroid $G$ and let $O$ be the centre of its circumscribed sphere.




Let $(\alpha)$, $(\alpha')$ be two parallel planes that contain $AB$, $CD$, respectively; let $(\beta)$, $(\beta')$ be two parallel planes that contain $AC$, $DB$, respectively; and let $(\gamma)$, $(\gamma')$ be two parallel planes that contain $AD$, $BC$, respectively. The pairs of planes $(\alpha)$, $(\alpha')$; $(\beta)$, $(\beta')$; and $(\gamma)$, $(\gamma')$ define a parallelepiped, which we denote by $ATBZ.YCXD$ (see the figure at right).

It is evident that $CD = TZ$, $DB = YT$, $BC = ZY$ and $G$ is the common midpoint of the diagonals of the parallelepiped $ATBZ.YCXD$. Hence, the following conditions are equivalent.

(a) $AB = CD$, $AC = DB$, $AD = BC$.

(b) $AB = TZ$, $AC = YT$, $AD = ZY$.

(c) $ATBZ$, $AYCT$, $AZDY$ are rectangles.

(d) $ATBZ.YCXD$ is a rectangular parallelepiped.

(e) $AX = BY = CZ = DT$.

(f) $GA = GB = GC = GD$.

(g) $G$ coincides with $O$. ∎

A tetrahedron is said to be *quasiregular* if it satisfies one of the two equivalent conditions stated in Lemma 3.

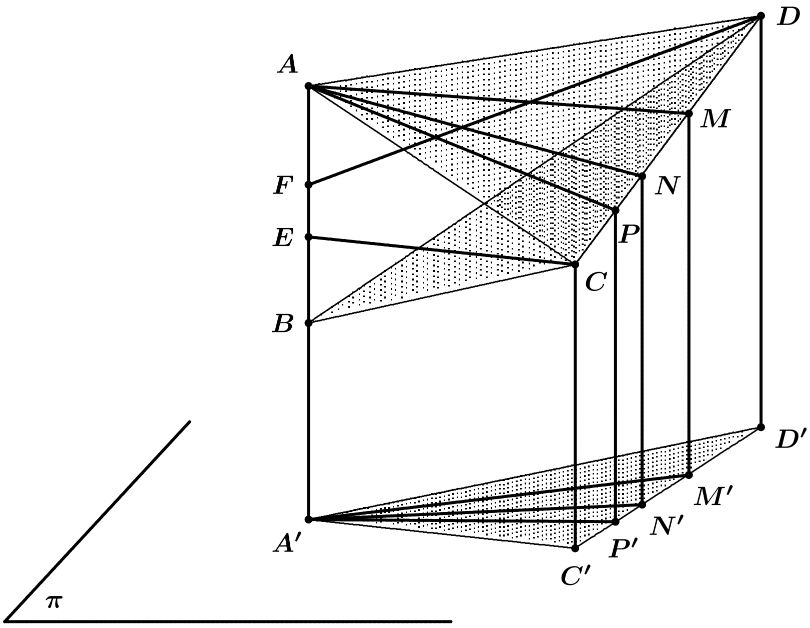
**Lemma 4** Six symmedian planes of tetrahedron $ABCD$ intersect at one common point $L$ defined by

$$\sum S_A^2 \overrightarrow{LA} = \overrightarrow{0}\,.$$

We note that this point is uniquely defined by the above equality and is referred to as the Lemoine point (as aforementioned).

More generally, for each quadruple of positive real numbers $(\alpha, \beta, \gamma, \delta)$ there exists a unique point $P$ in the interior of the tetrahedron such that $\sum \alpha\overrightarrow{PA} = \overrightarrow{0}$, and conversely for each point $P$ in the interior of the tetrahedron $ABCD$ there is a unique quadruple of positive real numbers $(\alpha, \beta, \gamma, \delta)$ such that $\alpha + \beta + \gamma + \delta = 1$ and $\sum \alpha\overrightarrow{PA} = \overrightarrow{0}$.

*Proof of Lemma 4*: Let the median plane, the bisector plane, and the symmedian plane through the edge $AB$ of the tetrahedron meet the edge $CD$ at $M$, $N$, and $P$, respectively.




Let $(\pi)$ be a plane perpendicular to the line $AB$. Let $A'$ be the orthogonal projection of $A$, $B$ onto the plane $(\pi)$, and let $C'$, $D'$, $M'$, $N'$, $P'$ be the orthogonal projections of $C$, $D$, $M$, $N$, $P$ onto the plane $(\pi)$, respectively.

It is evident that in triangle $A'C'D'$ the segments $A'M'$, $A'N'$, $A'P'$ are, respectively, the median, the bisector, and the symmedian from the vertex $A'$. By the symmedian property [2], we have

$$\frac{P'C'}{P'D'} = \left(\frac{A'C'}{A'D'}\right)^2 .$$

From this, and the fact that $CC'$, $DD'$, $PP'$ are parallel, we have

$$\frac{PC}{PD} = \left(\frac{A'C'}{A'D'}\right)^2 .$$

Suppose that $E$, $F$ are respectively the orthogonal projections of $C$, $D$ on $AB$. It is easily seen that $A'C' = EC$, $A'D' = FD$.

Thus,

$$\frac{PC}{PD} = \left(\frac{EC}{FD}\right)^2 = \frac{\left(\frac{1}{2}AB \cdot EC\right)^2}{\left(\frac{1}{2}AB \cdot FD\right)^2} = \frac{S_D^2}{S_C^2} .$$

This implies that $S_C^2\overrightarrow{PC} + S_D^2\overrightarrow{PD} = \overrightarrow{0}$. On the other hand, since $\sum S_A^2\overrightarrow{LA} = \overrightarrow{0}$, we have

$$S_A^2\overrightarrow{LA} + S_B^2\overrightarrow{LB} + S_C^2\left(\overrightarrow{LP} + \overrightarrow{PC}\right) + S_D^2\left(\overrightarrow{LP} + \overrightarrow{PD}\right) = \overrightarrow{0}\,.$$

This means that

$$S_A^2\overrightarrow{LA} + S_B^2\overrightarrow{LB} + (S_C^2 + S_D^2)\overrightarrow{LP} + (S_C^2\overrightarrow{PC} + S_D^2\overrightarrow{PD}) = \overrightarrow{0}\,.$$

Consequently,

$$S_A^2\overrightarrow{LA} + S_B^2\overrightarrow{LB} + (S_C^2 + S_D^2)\overrightarrow{LP} = \overrightarrow{0}\,,$$

so that $L$ lies in $(ABP)$, the symmedian plane through the edge $AB$ of the tetrahedron $ABCD$.

Therefore, $L$ lies in all six symmedian planes of tetrahedron $ABCD$. ■

**Lemma 5** If $M$ is in the interior of tetrahedron $ABCD$ and $H$, $K$, $I$, $J$ are the orthogonal projections of $M$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$, respectively, then

$$\sum \frac{S_A}{MH}\overrightarrow{MH} = \overrightarrow{0}\,.$$

*Proof*: Let $S(UVW)$ denote the area of triangle $UVW$. Let the inscribed sphere of tetrahedron $ABCD$ touch the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$ at $X$, $Y$, $Z$, $T$, respectively. Let $P$, $r$ be the centre and radius of the inscribed sphere, respectively.

From the planar analogue of Lemma 1 (see also [4]),

$$S(XCD)\overrightarrow{XB} + S(XDB)\overrightarrow{XC} + S(XBC)\overrightarrow{XD} = \overrightarrow{0}\,,$$

so it follows that

$$\begin{aligned} S(XCD)(\overrightarrow{XP} + \overrightarrow{PB}) + S(XDB)(\overrightarrow{XP} + \overrightarrow{PC}) & \\ + S(XBC)(\overrightarrow{XP} + \overrightarrow{PD}) &= \overrightarrow{0}\,. \end{aligned}$$

Hence, $S_A\overrightarrow{PX} = S(XCD)\overrightarrow{PB} + S(XDB)\overrightarrow{PC} + S(XBC)\overrightarrow{PD}$, and also

$$\begin{aligned} S_B\overrightarrow{PY} &= S(YDA)\overrightarrow{PC} + S(YAC)\overrightarrow{PD} + S(YCD)\overrightarrow{PA}\,, \\ S_C\overrightarrow{PZ} &= S(ZAB)\overrightarrow{PD} + S(ZBD)\overrightarrow{PA} + S(ZDA)\overrightarrow{PB}\,, \\ S_D\overrightarrow{PT} &= S(TBC)\overrightarrow{PA} + S(TCA)\overrightarrow{PB} + S(TAB)\overrightarrow{PC}\,. \end{aligned}$$

Moreover, we note that

$$\begin{aligned} &S(ZAB) = S(TAB)\,, \quad S(XCD) = S(YCD)\,, \quad S(YAC) = S(TAC)\,, \\ &S(ZBD) = S(XDB)\,, \quad S(ZDA) = S(YDA)\,, \quad S(TBC) = S(XBC)\,; \end{aligned}$$

so, by using Lemma 1, we have

$$\begin{aligned}
\sum \frac{S_A}{MH}\overrightarrow{MH} &= \frac{1}{r}\sum S_A \frac{PX}{MH}\overrightarrow{MH} = \frac{1}{r}\sum S_A \overrightarrow{PX} \\
&= \frac{1}{r}\sum \left(S(XCD)\overrightarrow{PB} + S(XDB)\overrightarrow{PC} + S(XBC)\overrightarrow{PD}\right) \\
&= \frac{1}{r}\sum\left(S(YCD) + S(ZDB) + S(TBC)\right)\overrightarrow{PA} \\
&= \frac{1}{r}\sum \left(S(XCD) + S(XDB) + S(XBC)\right)\overrightarrow{PA} \\
&= \frac{1}{r}\sum S_A\overrightarrow{PA} = \frac{3}{r^2}\sum \frac{1}{3}S_A \cdot PX \cdot \overrightarrow{PA} \\
&= \frac{3}{r^2}\sum V(PBCD)\overrightarrow{PA} = \overrightarrow{0}\,. \qquad \blacksquare
\end{aligned}$$

The planar analogue of the next lemma can be found in [4].

**Lemma 6** Suppose that any three of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$ are not coplanar, that $x$, $y$, $z$, $t$, $x'$, $y'$, $z'$, $t'$ are nonzero, and that the equations $x\overrightarrow{a}+y\overrightarrow{b}+z\overrightarrow{c}+t\overrightarrow{d} = \overrightarrow{0}$ and $x'\overrightarrow{a} + y'\overrightarrow{b} + z'\overrightarrow{c} + t'\overrightarrow{d} = \overrightarrow{0}$ hold. Then

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{t}{t'}\,.$$

*Proof*: By isolating $\overrightarrow{d}$ we have

$$\frac{x}{t}\overrightarrow{a} + \frac{y}{t}\overrightarrow{b} + \frac{z}{t}\overrightarrow{c} = -\overrightarrow{d} = \frac{x'}{t'}\overrightarrow{a} + \frac{y'}{t'}\overrightarrow{b} + \frac{z'}{t'}\overrightarrow{c}\,.$$

Since $\overrightarrow{a}$ $\overrightarrow{b}$, $\overrightarrow{c}$ are not coplanar, it follows that

$$\frac{x}{t} = \frac{x'}{t'}\,,\quad \frac{y}{t} = \frac{y'}{t'}\,,\quad \frac{z}{t} = \frac{z'}{t'}\,,$$

which implies that $\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{t}{t'}$. $\blacksquare$

**Lemma 7** Let $M$ be any point in the interior of tetrahedron $ABCD$. Let $H$, $K$, $I$, $J$ be the orthogonal projections of the point $M$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$. Then $M$ is the centroid of tetrahedron $HKIJ$ if and only if $M$ is the Lemoine point of the tetrahedron $ABCD$.

*Proof*: We shall show the equivalence of the following statements.

(a) The point $M$ is the centroid of tetrahedron $HKIJ$.

(b) The equation $\overrightarrow{MH} + \overrightarrow{MK} + \overrightarrow{MI} + \overrightarrow{MJ} = \overrightarrow{0}$ holds.

(c) The equation $\frac{S_A}{MH} = \frac{S_B}{MK} = \frac{S_C}{MI} = \frac{S_D}{MJ}$ holds.

(d) The equation $\dfrac{S_A^2}{\frac{1}{3}MH\cdot S_A} = \dfrac{S_B^2}{\frac{1}{3}MK\cdot S_B} = \dfrac{S_C^2}{\frac{1}{3}MI\cdot S_C} = \dfrac{S_D^2}{\frac{1}{3}MJ\cdot S_D}$ holds.

(e) The equation $\dfrac{S_A^2}{V(MBCD)} = \dfrac{S_B^2}{V(MCDA)} = \dfrac{S_C^2}{V(MDAB)} = \dfrac{S_D^2}{V(MABC)}$ holds.

(f) The equation $\sum S_A^2\overrightarrow{MA} = \overrightarrow{0}$ holds.

(g) The point $M$ is the Lemoine point of the tetrahedron $ABCD$.

Parts (a) and (b) are equivalent by properties of the centroid. Lemma 5 and Lemma 6 imply the equivalence of (b) and (c). Clearly, (c), (d), and (e) are equivalent. Lemma 1 and Lemma 6 imply that (e) and (f) are equivalent, while Lemma 4 implies that (f) and (g) are equivalent. ■

**Lemma 8** Let $ABCD$ be a tetrahedron and $X$, $Y$, $Z$, $T$ points on the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$, respectively. The sum $\sum XY^2$ is minimized if and only if $X$, $Y$, $Z$, $T$ are the orthogonal projections of the Lemoine point of $ABCD$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$.

*Proof*: Let $M$ be the centroid of tetrahedron $XYZT$ and $H$, $K$, $I$, $J$ the orthogonal projections of $M$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$. We have

$$\begin{aligned}
\sum XY^2 &= \sum \left|\overrightarrow{MX} - \overrightarrow{MY}\right|^2 = 3\sum MX^2 - 2\sum \overrightarrow{MX}\cdot\overrightarrow{MY} \\
&= 4\sum MX^2 - \left|\sum \overrightarrow{MX}\right|^2 = 4\sum MX^2 \geq 4\sum MH^2 \\
&= \frac{4}{\sum S_A^2}\left(\sum MH^2\right)\left(\sum S_A^2\right) \geq \frac{4}{\sum S_A^2}\left(\sum S_A MH\right)^2 \\
&= \frac{4}{\sum S_A^2}\left(\sum 3V(MBCD)\right)^2 \geq \frac{36}{\sum S_A^2}V^2(ABCD)\,.
\end{aligned}$$

Therefore, $\sum XY^2 \geq \dfrac{36}{\sum S_A^2}V^2(ABCD)$, with equality if and only if the following three conditions are satisfied:

(a) The points $X$, $Y$, $Z$, $T$ are the orthogonal projections of $M$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$.

(b) The equation $\dfrac{MH}{S_A} = \dfrac{MK}{S_B} = \dfrac{MI}{S_C} = \dfrac{MJ}{S_D}$ holds.

(c) The point $M$ is in the interior of the tetrahedron $ABCD$.

By Lemma 5 and Lemma 7, these conditions are satisfied if and only if $X$, $Y$, $Z$, $T$ are the orthogonal projections of the Lemoine point of tetrahedron $ABCD$ onto the planes $(BCD)$, $(CDA)$, $(DAB)$, $(ABC)$. ■

## 3 Proof of Theorem 1

Let $R$ be the radius of the circumsphere, $\Gamma$, of tetrahedron $A_0B_0C_0D_0$.

*Proof of part* (1): Note that

$$\begin{aligned} G_0A_0 \cdot G_0A_1 = G_0B_0 \cdot G_0B_1 &= G_0C_0 \cdot G_0C_1 \\ &= G_0D_0 \cdot G_0D_1 = R^2 - OG_0^2\,, \end{aligned}$$

$$\sum G_0A_0^2 = \left(\sum OA_0^2\right) - 4OG_0^2 = 4\left(R^2 - OG_0^2\right) \tag{3}$$

Using (3), Lemma 2, and the AM-GM Inequality, we have

$$\begin{aligned} \frac{V_0}{V_1} &= \frac{G_0A_0 \cdot G_0B_0 \cdot G_0C_0 \cdot G_0D_0}{G_0A_1 \cdot G_0B_1 \cdot G_0C_1 \cdot G_0D_1} \le \left(\frac{1}{4}\sum \frac{G_0A_0}{G_0A_1}\right)^4 \\ &= \left(\frac{1}{4}\sum \frac{G_0A_0^2}{G_0A_0 \cdot G_0A_1}\right)^4 = \frac{1}{\left(4(R^2 - OG_0^2)\right)^4}\left(\sum G_0A_0^2\right)^4 \\ &= \frac{1}{\left(4(R^2 - OG_0^2)\right)^4}\left(\sum \left(OA_0^2 - 4OG_0^2\right)\right)^4 \\ &= \frac{1}{\left(4(R^2 - OG_0^2)\right)^4}\left(4(R^2 - OG_0^2)\right)^4 = 1\,. \end{aligned}$$

Thus, $V_0 \le V_1$.

We remark that by (3) and Lemma 3, the following are equivalent.

(a) The volumes of successive tetrahedra are equal, that is, $V_0 = V_1$.

(b) The equation $\dfrac{G_0A_0}{G_0A_1} = \dfrac{G_0B_0}{G_0B_1} = \dfrac{G_0C_0}{G_0C_1} = \dfrac{G_0D_0}{G_0D_1}$ holds.

(c) The equation $\dfrac{G_0A_0^2}{G_0A_0 \cdot G_0A_1} = \dfrac{G_0B_0^2}{G_0B_0 \cdot G_0B_1} = \dfrac{G_0C_0^2}{G_0C_0 \cdot G_0C_1}$
$= \dfrac{G_0D_0^2}{G_0D_0 \cdot G_0D_1}$ holds.

(d) The equation $G_0A_0 = G_0B_0 = G_0C_0 = G_0D_0$ holds.

(e) The centroid $G_0$ coincides with $O$.

(f) The tetrahedron $A_0B_0C_0D_0$ is quasiregular.

Repeating this procedure, we have $V_0 \le V_1 \le V_2 \le \cdots$, and $\{V_n\}$ is a nondecreasing sequence.

*Proof of part* (2): Let $(\alpha)$, $(\beta)$, $(\gamma)$, and $(\delta)$ be the planes through the points $A_0$, $B_0$, $C_0$, $D_0$ respectively and perpendicular to $A_0G_0$, $B_0G_0$, $C_0G_0$, and $D_0G_0$ in that order.

Let $A_0' = (\beta) \cap (\gamma) \cap (\delta)$, $B_0' = (\gamma) \cap (\delta) \cap (\alpha)$, $C_0' = (\delta) \cap (\alpha) \cap (\beta)$, and $D_0' = (\alpha) \cap (\beta) \cap (\gamma)$.

Since $G_0$ is the centroid of tetrahedron $A_0B_0C_0D_0$, by Lemma 7 $G_0$ is the Lemoine point of the tetrahedron $A_0'B_0'C_0'D_0'$.

Let $A_1'$, $B_1'$, $C_1'$, $D_1'$ be the reflections of $A_1$, $B_1$, $C_1$, $D_1$ in $O$. Then $A_1'$, $B_1'$, $C_1'$, $D_1'$ are on the planes $(\alpha)$, $(\beta)$, $(\gamma)$, $(\delta)$, respectively.

By Lemma 8, $\sum(A_1'B_1')^2 \geq \sum(A_0B_0)^2$. Since $\sum(A_1'B_1')^2 = \sum(A_1B_1)^2$, we obtain

$$\sum(A_1B_1)^2 \geq \sum(A_0B_0)^2 .$$

Furthermore, we have

$$\begin{aligned}\sum(A_0B_0)^2 &= \sum\left|\overrightarrow{OA_0} - \overrightarrow{OB_0}\right|^2 = 12R^2 - 2\sum \overrightarrow{OA_0}\cdot\overrightarrow{OB_0}\\ &= 16R^2 - \left|\sum\overrightarrow{OA_0}\right|^2 = 16R^2 - \left|4\overrightarrow{OG_0}\right|^2 = 16(R^2 - OG_0^2)\,,\end{aligned}$$

and $\sum(A_1B_1)^2 = 16(R^2 - OG_0^2)$ is deduced similarly.

Therefore, $OG_0 \geq OG_1$, and by Lemma 7 equality holds if and only if $A_1'$, $B_1'$, $C_1'$, $D_1'$ respectively coincide with $A_0$, $B_0$, $C_0$, $D_0$. In other words, $G_0$ coincides with $O$. By Lemma 3 this occurs if and only if $A_0B_0C_0D_0$ is a quasiregular tetrahedron.

We now know that $\{OG_n\}$ is a nonincreasing sequence bounded below by $0$, so the following limit exists:

$$\lim_{n\to\infty} OG_n\,. \tag{4}$$

Let $\overline{\Gamma}$ be the closed ball with boundary $\Gamma$. Since $\overline{\Gamma}$ is closed and bounded, by the Bolzano–Weierstrass Theorem there is an increasing sequence of positive integers $\{n_k\}$ such that each of the sequences $\{A_{n_k}\}$, $\{B_{n_k}\}$, $\{C_{n_k}\}$, $\{D_{n_k}\}$, $\{G_{n_k}\}$, $\{A_{n_k+1}\}$, $\{B_{n_k+1}\}$, $\{C_{n_k+1}\}$, $\{D_{n_k+1}\}$, $\{G_{n_k+1}\}$ is convergent in $\overline{\Gamma}$. Let the respective limits of these sequences be $A_0^*$, $B_0^*$, $C_0^*$, $D_0^*$, $G_0^*$, $A_1^*$, $B_1^*$, $C_1^*$, $D_1^*$, $G_1^*$; that is, $A_{n_k} \to A_0^*$, $B_{n_k} \to B_0^*$, and so forth.

It is evident that

$$OG_0^* = \lim_{k\to\infty} OG_{n_k}\,, \qquad OG_1^* = \lim_{k\to\infty} OG_{n_k+1}\,. \tag{5}$$

Since $\lim\limits_{n\to\infty} OG_n$ exists, it follows from (5) that

$$OG_0^* \;=\; OG_1^*\,. \tag{6}$$

Let $V_n$ be the volume of $A_nB_nC_nD_n$. The sequence $\{V_n\}$ is nondecreasing by part (1), and is bounded above by the volume of $\Gamma$ and bounded below by $V_0 > 0$. Therefore, $\lim\limits_{n\to\infty} V_n$ exists and is positive, and it follows that $\lim\limits_{n_k\to\infty} V_{n_k} = \lim\limits_{n_k\to\infty} V_{n_k+1} > 0$.

If either tetrahedron $A_0^*B_0^*C_0^*D_0^*$ or $A_1^*B_1^*C_1^*D_1^*$ were degenerate, then we would have $\lim\limits_{n_k\to\infty} V_{n_k} = 0$ or $\lim\limits_{n_k\to\infty} V_{n_k+1} = 0$, a contradiction.

Thus, $A_0^*B_0^*C_0^*D_0^*$ and $A_1^*B_1^*C_1^*D_1^*$ are nondegenerate tetrahedra.

On the other hand, $\mathbf{\Gamma}$ is closed and bounded, so $\mathbf{\Gamma}$ contains $A_0^*$, $B_0^*$, $C_0^*$, $D_0^*$, $A_1^*$, $B_1^*$, $C_1^*$, $D_1^*$. Since $G_{n_k}$ and $G_{n_k+1}$ are the respective centroids of the tetrahedra $A_{n_k}B_{n_k}C_{n_k}D_{n_k}$ and $A_{n_k+1}B_{n_k+1}C_{n_k+1}D_{n_k+1}$ for all $n_k$, we have that $G_0^*$ and $G_1^*$ are the respective centroids of tetrahedra $A_0^*B_0^*C_0^*D_0^*$ and $A_1^*B_1^*C_1^*D_1^*$. Since $A_{n_k+1}$, $B_{n_k+1}$, $C_{n_k+1}$, $D_{n_k+1}$ are the respective intersections of the lines $A_{n_k}G_{n_k}$, $B_{n_k}G_{n_k}$, $C_{n_k}G_{n_k}$, $D_{n_k}G_{n_k}$ with $\mathbf{\Gamma}$, it then follows that $A_1^*$, $B_1^*$, $C_1^*$, $D_1^*$ are the respective intersections of the lines $A_0^*G_0^*$, $B_0^*G_0^*$, $C_0^*G_0^*$, $D_0^*G_0^*$ with $\mathbf{\Gamma}$.

By the above remarks, the tetrahedra $A_0^*B_0^*C_0^*D_0^*$ and $A_1^*B_1^*C_1^*D_1^*$ are related to one another in the same way that the tetrahedra $A_0B_0C_0D_0$ and $A_1B_1C_1D_1$ are related to one another.

By the same reasoning as in the first part of the proof, $OG_0^* \geq OG_1^*$, with equality only when $A_0^*B_0^*C_0^*D_0^*$ is a quasiregular tetrahedron. However, we showed in (6) that equality does indeed hold. This implies that $G_0^*$ coincides with the circumcentre $O$ of the sphere. Then $OG_0^* = 0$, so that $\lim\limits_{n\to\infty} OG_n = \lim\limits_{n\to\infty} OG_{n_k} = OG_0^* = 0$. ∎

## 4 Acknowledgments

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# Inequalities Involving Reciprocals of Triangle Areas

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In this paper we begin with the study of a new inequality about the reciprocals of triangle areas in an arbitrary quadrilateral. Using a familiar fact as a lemma we prove this inequality and find the conditions of equality. We also prove a similar inequality for triangles and generalize it to arbitrary polygons. We also describe a situation in which the lemma does not work. At the end of the paper we propose a problem for further investigation.

**Problem 1**. Let $ABCD$ be a convex quadrilateral and $K$, $L$, $M$, and $N$ be arbitrary points on corresponding sides $AB$, $BC$, $CD$, and $DA$ (see Figure 1). Let $KM$ and $LN$ intersect at the point $O$. Denote the areas of triangles $ANK$, $BKL$, $CLM$, and $DMN$ by $S_1$, $S_2$, $S_3$, and $S_4$; and denote the areas of triangles $ONK$, $OKL$, $OLM$, and $OMN$ by $T_1$, $T_2$, $T_3$, and $T_4$, respectively. Prove that

$$\frac{1}{S_1}+\frac{1}{S_2}+\frac{1}{S_3}+\frac{1}{S_4}\geq\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}+\frac{1}{T_4}\,.$$

We need the following lemma, which is a generalization of a fact given in [1]. In what follows square brackets around a figure denote the area of the figure.

**Lemma 1**. Let $ABCD$ be a given convex quadrilateral, and let a line through the intersection point $O$ of diagonals $AC$ and $BD$ intersect the sides $AD$ and $BC$ at the points $K$ and $L$. Then the sum $\frac{1}{[AOK]}+\frac{1}{[BOL]}$ is minimal if and only if $KL \parallel AB$.

*Proof.* Suppose $KL$ is not parallel to $AB$, and let the line through $O$ and parallel to $AB$ intersect $AD$ and $BC$ at the points $K_1$ and $L_1$, respectively (see Figure 2). We must prove that

$$\frac{1}{[AOK]}+\frac{1}{[BOL]}\geq\frac{1}{[AOK_1]}+\frac{1}{[BOL_1]}\,.$$

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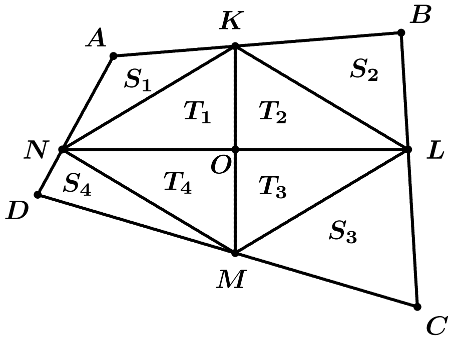


Figure 1

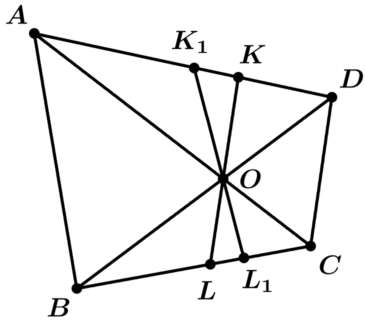


Figure 2

Without loss of generality we suppose that $K_1$ is closer to $A$ than $K$. Then we can write the last inequality as

$$\frac{1}{[BOL]} - \frac{1}{[BOL_1]} \geq \frac{1}{[AOK_1]} - \frac{1}{[AOK]},$$

$$\frac{[OLL_1]}{[BOL][BOL_1]} \geq \frac{[OKK_1]}{[AOK][AOK_1]},$$

$$\frac{OL \cdot OL_1 \sin \angle LOL_1}{2[BOL][BOL_1]} \geq \frac{OK \cdot OK_1 \sin \angle KOK_1}{2[AOK][AOK_1]}.$$

Since $K_1L_1 \parallel AB$, then $\frac{OL_1}{[BOL_1]} = \frac{OK_1}{[AOK_1]}$; hence $\frac{OL}{[BOL]} \geq \frac{OK}{[AOK]}$, which holds since $L$ is closer to line $AB$ than $K$. Lemma 1 is proved. ■

*Solution of Problem 1.* Take $A'$, $B'$ on the rays $NA$, $LB$ so that $A'B'$ passes through $K$ and so that $A'B' \parallel LN$ (see Figure 3). By Lemma 1,

$$\frac{1}{S_1} + \frac{1}{S_2} \geq \frac{1}{[A'NK]} + \frac{1}{[B'KL]}. \quad (1)$$

Take $D'$, $C'$ on the rays $ND$, $LC$ so that $D'C'$ passes through $M$ and $D'C' \parallel LN$. By Lemma 1,

$$\frac{1}{S_3} + \frac{1}{S_4} \geq \frac{1}{[C'LM]} + \frac{1}{[D'MN]}. \quad (2)$$

Now, we apply Lemma 1 to the quadrilateral $A'B'C'D'$. Take $A''$, $D''$ on the rays $KA'$, $MD'$ so that $A''D''$ passes through $N$ and so that $A''D'' \parallel KM$.

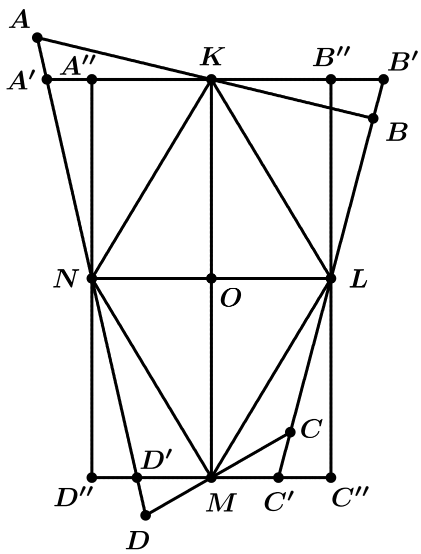


Figure 3

Similarly, take the points $B''$ and $C''$ on the corresponding rays $KB'$ and $MC'$ such that $B''C''$ passes through $L$ and $B''C'' \parallel KM$. By Lemma 1,

$$\frac{1}{[A'NK]} + \frac{1}{[D'MN]} \geq \frac{1}{[A''NK]} + \frac{1}{[D''MN]}\,, \tag{3}$$

$$\frac{1}{[B'KL]} + \frac{1}{[C'LM]} \geq \frac{1}{[B''KL]} + \frac{1}{[C''LM]}\,. \tag{4}$$

The quadrilaterals $A''NOK$, $B''KOL$, $C''LOM$, and $D''MON$ are parallelograms, so $T_1 = [A''NK]$, $T_2 = [B''KL]$, $T_3 = [C''LM]$, and $T_4 = [D''MN]$. By (1)-(4), we obtain the inequality in Problem 1, with equality if and only if $AD \parallel BC \parallel KM$ and $AB \parallel CD \parallel LN$. ∎

**Problem 2**. (Janous' inequality [5]) Let $K$, $L$, $M$ be points on the sides $BC$, $CA$, $AB$ of triangle $ABC$ (see Figure 4). Denote the areas of triangles $KLM$, $ALM$, $BKM$, $CKL$ by $S_0$, $S_1$, $S_2$, $S_3$, respectively. Prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \geq \frac{3}{S_0}\,.$$

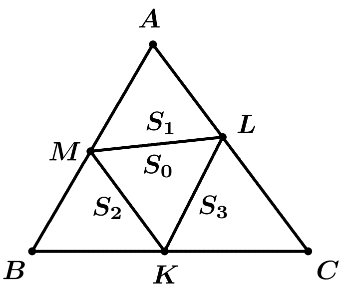


Figure 4

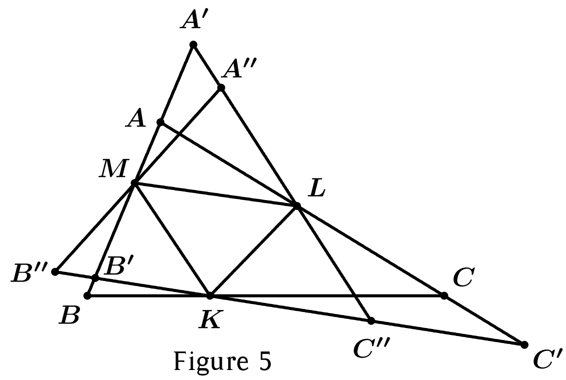


Figure 5

*Solution*. The proof in this case is similar, so we only indicate the main steps.

Take the points $B'$ and $C'$ on the corresponding rays $MB$ and $LC$ so that $B'C'$ passes through $K$ and also so that $B'C' \parallel ML$ (see Figure 5).

Similarly, take $A'$, $C''$ on the rays $MA$, $KC'$ so that $A'C''$ passes through $L$ and $A'C'' \parallel MK$, and finally take $A''$, $B''$ on the rays $LA'$, $KB'$ so that $A''B''$ passes through $M$ and $A''B'' \parallel KL$. As in the proof of Lemma 1, we compare successive pairs of reciprocal areas to obtain

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \geq \frac{1}{S_1} + \frac{1}{[B'MK]} + \frac{1}{[C'LK]} \geq$$

$$\frac{1}{[A'ML]} + \frac{1}{[B'MK]} + \frac{1}{[C''LK]} \geq \frac{1}{[A''ML]} + \frac{1}{[B''MK]} + \frac{1}{[C''LK]} = \frac{3}{S_0}\,.$$

The last equality follows from the fact that $KL$, $LM$, $MK$ are midlines of triangle $A''B''C''$, so $[A''ML] = [B''MK] = [C''LK] = S_0$. ∎

The following general problem can be solved in a similar manner, and is left to the reader.

**Problem 3**. Let $A_0A_1\dots A_{n-1}$ be an arbitrary convex polygon and $B_i$ be an arbitrary point on the side $A_iA_{i+1}$ for $i = 0, 1, \dots, n-1$ (all indices are taken modulo $n$). Let the diagonals $B_{i-2}B_i$ and $B_{i-1}B_{i+1}$ intersect at $C_i$ for $i = 0, 1, \dots, n-1$. Prove that

$$\sum_{i=0}^{n-1} [A_iB_iB_{i-1}]^{-1} \geq \sum_{i=0}^{n-1} [C_iB_iB_{i-1}]^{-1} .$$

After these successful applications of Lemma 1 we must note that blind use of Lemma 1 may lead in some cases to contradictory results. Consider the following problem.

**Problem 4**. Let $M$ be an arbitrary point inside triangle $ABC$ (see Figure 6). Let $A_1$ and $A_2$, $B_1$ and $B_2$, $C_1$ and $C_2$ be arbitrary points on the corresponding sides $BC$, $CA$ and $AB$ such that the lines $A_1B_2$, $B_1C_2$, $C_1A_2$ intersect at $M$. Denote the areas of triangles $MA_1A_2$, $MB_1B_2$, $MC_1C_2$, and $ABC$ by $S_1$, $S_2$, $S_3$, and $S$, respectively. Find all possible values of the parameter $\lambda$ for which the following inequality holds:

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \geq \frac{\lambda}{S} .$$

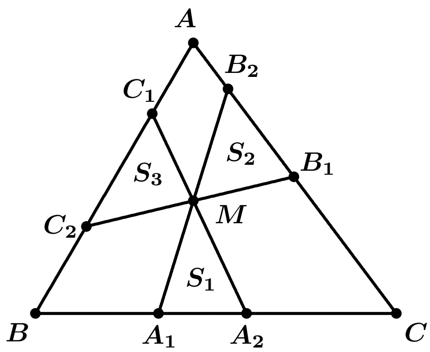


Figure 6

**Remark 1**. By Lemma 1, if the quadrilateral $AC_1MB_2$ and the triangle $ABC$ are fixed, then the sum $\frac{1}{S_2} + \frac{1}{S_3}$ is minimal if $B_1C_2 \parallel B_2C_1$. Similarly, the sum $\frac{1}{S_1} + \frac{1}{S_3}$ is minimal if $A_2C_1 \parallel A_1C_2$, and the sum $\frac{1}{S_1} + \frac{1}{S_2}$ is minimal if $A_1B_2 \parallel A_2B_1$.

**Remark 2**. It was proved in [2], Problem 1, (see also [3]) that it is possible to construct the lines $A_1B_2$, $B_1C_2$, $C_1A_2$ so that they meet at $M$ and so that $A_1B_2 \parallel A_2B_1$, $B_1C_2 \parallel B_2C_1$, $C_1A_2 \parallel C_2A_1$.

**Remark 3**. It was proved in [3] (it follows also from the results of [2]) that if $A_1B_2 \parallel A_2B_1$, $B_1C_2 \parallel B_2C_1$, $C_1A_2 \parallel C_2A_1$, then

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \geq \frac{27}{S} .$$

Can we deduce from these remarks that the last inequality is always true? It is surprising to find that the answer to Problem 4 is not $\lambda \leq 27$

as we expected, but $\lambda \leq 18$. Indeed, it was proved in [7] (see also [4], pages 184, 200) that the inequality in Problem 4 holds true when $\lambda = 18$ and equality occurs when $A_1B_2$, $B_1C_2$, $C_1A_2$ are the medians of triangle $ABC$. Therefore, additional constructions in Problem 1 and Problem 2 are necessary parts of the solutions. In conclusion we propose a new problem for independent study.

**Problem 5**. Let $ABCD$ be a convex quadrilateral whose diagonals $AC$ and $BD$ intersect at the point $O$. Construct the line $EF$ passing through $O$, where the points $E$ and $F$ are on the corresponding sides $AD$ and $BC$, such that the sum

$$\frac{1}{[AOE]} + \frac{1}{[BOF]} + \frac{1}{[COF]} + \frac{1}{[DOE]}$$

is minimized. Is it possible that the constructed line $EF$ passes through the intersection point of the lines $AB$ and $CD$?


### Acknowledgements

I must note that the present paper was inspired by the problems, papers, and books of I.F. Sharygin [6]. I also thank the reviewer for his helpful comments.


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# A Generalization of Mayhem Problem M396 Involving Pythagorean Triangles

Konstantine Zelator

The motivation behind this work is Mayhem problem M396 in the May 2009 issue of ***CRUX with MAYHEM*** [1]. Let us restate the problem.

**M396**. The rectangle $ABCD$ has side lengths $AB = 8$ and $BC = 6$. Circles with centres $O_1$ and $O_2$ are inscribed in triangles $ABD$ and $BCD$. Determine the distance between $O_1$ and $O_2$.

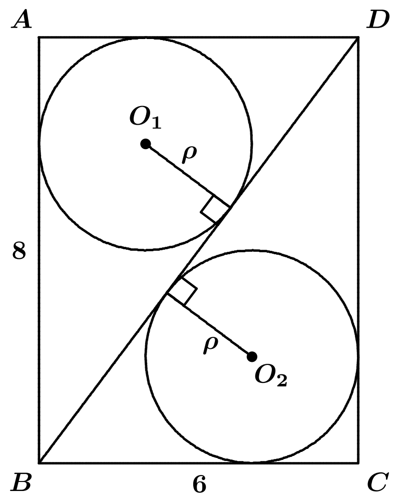


Figure 1

As we shall see, the distance $O_1O_2$ is $2\sqrt{5}$. The points $O_1$ and $O_2$ are the incentres of the congruent right triangles $ABD$ and $BCD$, which are in fact Pythagorean triangles with a common hypotenuse $BD$ of length 10. Note that the quadrilateral $BO_1DO_2$ is, in fact, a parallelogram with the diagonals $O_1O_2$ and $BD$ intersecting at their common midpoint. Now, picture the general case in which the rectangle $ABCD$ is formed by glueing together two congruent Pythagorean triangles $ABD$ and $BCD$. It turns out that the distance between the two incentres is always an irrational number (a quadratic irrational). Also, of the four side lengths $O_1D = BO_2$ and $BO_1 = DO_2$, two (equal) ones are always irrational. The other two (equal) ones can be, in fact, integers. We give precise conditions as to when this occurs; otherwise, they are also irrational.

In the general case, we will denote the incentres by $I_1$ and $I_2$ instead of $O_1$ and $O_2$. Also, for reasons of convenience, relabel the rectangle $ABCD$ as $BCAD$, as in Figure 2 on the next page. In Figure 2, $BI_1AI_2$ is a parallelogram and $\rho$ stands for the inradii of the two congruent right triangles $BCA$ and $ADB$.

As usual we set $BC = a$, $CA = b$, $AB = c$, and we also introduce $y = BT_2 = BT_3$, $x = AT_3 = AT_1$, and $z = CT_2 = CT_1 = \rho$; where $T_1$, $T_2$, and $T_3$ are the three points of tangency of the incircle of triangle $BCA$ with the sides $CA$, $CB$, and $BA$, respectively.

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Our main result is

**Theorem** With the above notation,

(a) The side length $\ell_2 = AI_1 = BI_2$ is always an irrational number.

(b) The side length $\ell_1 = AI_2 = BI_1$ is an integer precisely when either $m = k_1^2 - k_2^2$ and $n = 2k_1k_2$; or $m = 2k_1k_2$ and $n = k_1^2 - k_2^2$; where $k_1$ and $k_2$ are relatively prime positive integers of opposite parity and with $k_1 > k_2$; and such that $m > n$.

(c) The length of the diagonal $I_1I_2$ is always an irrational number.

A triple $(a, b, c)$ of positive integers $a$, $b$, and $c$, with $c$ being the hypotenuse length, is said to be a *Pythagorean triple* precisely when $a^2 + b^2 = c^2$. The parametric formulas we will use are well known, and they generate the entire family of Pythagorean triples (or triangles corresponding to these triples).

The interested reader can find a wealth of historical information in L.E. Dickson's monumental book, *History of the Theory of Numbers, Vol. II* [2], as well as in W. Sierpinski's book, *Elementary Theory of Numbers* [4]. For a more textbook type approach, see Rosen [3]; and for a derivation of formulas (1), refer to Sierpinski [4] or Rosen [3].

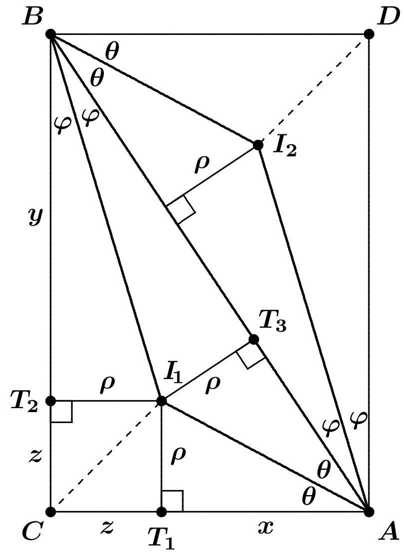


Figure 2

**Lemma 1** Let $a$, $b$, $c$ be positive integers. A triple $(a, b, c)$ is Pythagorean, with $c$ being the hypotenuse length, if and only if ($a$ and $b$ may be switched),

$$a = d(m^2 - n^2)\,, \quad b = d(2mn)\,, \quad c = d(m^2 + n^2)\,, \tag{1}$$

for some positive integers $m$, $n$, $d$ such that $m > n$, $\gcd(m, n) = 1$, and $m + n \equiv 1 \pmod 2$. If $d = 1$ the Pythagorean triple is called *primitive*. ∎

In Figure 3, a triangle $ABC$ is shown with side lengths $AB = c$, $BC = a$, $CA = b$ and with incentre $I$. Also, $T_1$, $T_2$, and $T_3$ are the three points of tangency of the incircle of $ABC$ with the sides $AC$, $CB$, and $BA$, respectively; and $\rho$ is the radius of the incircle. We put $x = AT_1 = AT_3$, $y = BT_2 = BT_3$, and $z = CT_2 = CT_1$.

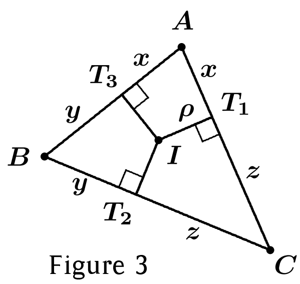


Figure 3

Clearly, we have $x+y=c$, $y+z=a$, $z+x=b$; from which we obtain $x=s-a$, $y=s-b$, $z=s-c$, where $s=\dfrac{a+b+c}{2}$ is the semiperimeter.

Applying these formulas and (1) to the Pythagorean triangle $BCA$ in Figure 2, we obtain by straightforward calculations that

$$z=\rho=dn(m-n)\,,\quad y=dm(m-n)\,,\quad x=dn(m+n)\,. \tag{2}$$

We will also need the well-known Parallelogram Law:

**Lemma 2** Let $ABCD$ be a parallelogram with diagonal lengths $d_1=BD$, $d_2=AC$ and side lengths $\ell_1=AB=DC$, $\ell_2=BC=AD$. Then

$$2(\ell_1^2+\ell_2^2)=d_1^2+d_2^2\,.$$

Now we can compute the side lengths, as well as the two diagonal lengths, of the parallelogram $BI_1AI_2$ in Figure 2, in terms of the integers $m$, $n$, and $d$ in formulas (1). These are the side lengths $BI_1=\ell_1=AI_2$ and $AI_1=\ell_2=BI_2$, and the diagonal lengths $AB=c=d(m^2+n^2)$ and $I_1I_2$.

To compute $\ell_1=BI_1=AI_2$, examine the right triangle $I_1BT_2$. We have $(I_1B)^2=(BT_2)^2+(I_1T_2)^2$, or $\ell_1^2=y^2+\rho^2$, so by (2) we obtain $\ell_1^2=d^2(m-n)^2\left[n^2+m^2\right]$, and therefore

$$\ell_1=BI_1=AI_2=d(m-n)\sqrt{n^2+m^2}\,. \tag{3}$$

To compute $\ell_2=AI_1=BI_2$, examine the right triangle $AI_1T_1$. We have $\ell_2^2=x^2+\rho^2$, so by (2) we obtain $\ell_2^2=d^2n^2\left[(m+n)^2+(m-n)^2\right]$, or $\ell_2^2=2d^2n^2(m^2+n^2)$. Therefore,

$$\ell_2=dn\sqrt{2(m^2+n^2)}\,. \tag{4}$$

To compute the diagonal length $I_1I_2$, we apply Lemma 2 to the parallelogram $BI_1AI_2$. We have $2(\ell_1^2+\ell_2^2)=c^2+(I_1I_2)^2$, and by formulas (1), (3), and (4) we obtain

$$2\left[d^2(m^2+n^2)(m-n)^2+2d^2n^2(m^2+n^2)\right]=d^2\left(m^2+n^2\right)^2+(I_1I_2)^2\,.$$

Solving for $(I_1I_2)^2$ yields

$$(I_1I_2)^2=d^2\left[2(m^2+n^2)(m-n)^2+4n^2(m^2+n^2)-(m^2+n^2)^2\right]\,,$$

and after some algebra we arrive at $(I_1I_2)^2=d^2\cdot\left[(m-n)^4+4n^4\right]$, or

$$I_1I_2=d\sqrt{(m-n)^4+4n^4}\,. \tag{5}$$

Note that in the case of Mayhem problem M396, we have $d=2\,m=2$, $n=1$, $a=6$, $b=8$, and $c=10$ in (1). Thus, by (5) we see that $I_1I_2$ (or $O_1O_2$ in the notation of that problem) is $2\sqrt{5}$.

Since in (1) one of the integers $m$, $n$ is even and the other odd, the integer $2(m^2+n^2)$ is twice an odd integer and thus, it cannot be a perfect or integer square. Therefore, (4) shows that $\ell_2$ is always an irrational number, establishing part (a) of our main theorem.

On the other hand, we see from (3) that $\ell_1$ is an irrational number when $m^2+n^2$ is not a square; and when $m^2+n^2$ is a square, only then will the side length $\ell_1$ be an integer. Since $m$ and $n$ are relatively prime (and of different parity), it follows that $m^2+n^2$ is a square if and only if the numbers $m$ and $n$ are the leg lengths of a primitive Pythagorean triple. Now part (b) of our main theorem follows from Lemma 1.

Finally, part (c) of our main theorem follows from equation (5) and Lemma 3 below, which we prove for the sake of completeness. We remark that Lemma 3 is also given as Exercise 6 in Section 13.2 of Rosen's book [3].

**Lemma 3** The diophantine equation

$$x^4+4y^4=z^2 \tag{6}$$

has no solution in positive integers $x$, $y$, $z$.

*Proof*: The proof rests on the fact that the system of equations

$$\begin{aligned} x^2-y^2 &= z^2\,, \\ x^2+y^2 &= w^2\,, \end{aligned}$$

has no solution in positive integers $x$, $y$, $z$, $w$. This result has been attributed to P. Fermát, and a proof can be found in Sierpinski's book [4] (pp. 38-42), which uses the method of infinite descent introduced by Fermat.

Now suppose to the contrary that $x$, $y$, $z$ are positive integers satisfying, (6). Let $\delta$ be the greatest common divisor of $x$ and $y$. Then $x=\delta x_1$ and $y=\delta y_1$, where $x_1$ and $y_1$ are relatively prime positive integers. We thus obtain $\delta^4(x_1^4+4y_1^4)=z^2$. Since $\delta^4 \mid z^2$, it follows that $\delta^2$ must be a divisor of $z$. Let $z=\delta^2 z_1$, for some positive integer $z_1$. Accordingly, we obtain

$$x_1^4+4y_1^4=z_1^2\,. \tag{7}$$

Since $x_1$ and $y_1$ are relatively prime, one is odd and the other even; or both are odd. The latter case is eliminated by an argument modulo $8$ shows. Recall that the square of an odd integer is congruent to $1$ modulo $8$. If $x_1$ and $y_1$ were both odd, then $z_1$ would also be odd by (7). But then,

$$x_1^4+4y_1^4 \equiv 1+4\cdot 1 \equiv 5 \pmod 8\ , \quad \text{while } z_1^2 \equiv 1 \pmod 8\ .$$

Therefore, $x_1$ is odd and $y_1$ is even, or vice-versa. However, the case where $y_1$ is odd and $x_1$ is even can be reduced to the case where $x_1$ is odd and $y_1$ is even. Indeed, if $x_1$ is even and $y_1$ odd, then $x_1=2x_2$ for some positive integer $x_2$, and so by (7) we have $4(4x_2^4+y_1^4)=z_1^2$. Obviously, $2 \mid z_1$, so $z_1=2z_2$. Therefore, $4x_2^4+y_1^4=z_2^2$, which is an equation like (7) with $y_1$

odd (and $x_2$ even, by the modulo $8$ argument above). It is now clear that we only have to treat the case where $x_1$ is odd and $y_1$ is even in (7).

We write (7) in the form

$$(x_1^2)^2 + (2y_1^2)^2 = z_1^2\,, \tag{8}$$

and observe that $x_1^2$ and $2y_1^2$ are relatively prime integers, since $x_1$ is odd and relatively prime to $y_1$. Thus, $(x_1^2,\ 2y_1^2,\ z_1)$ is a primitive Pythagorean triple, and by (1) we must have

$$x_1^2 = r^2 - s^2\,, \qquad 2y_1^2 = 2rs\,, \qquad z_1 = r^2 + s^2 \tag{9}$$

for coprime positive integers $r$, $s$ with $r > s$ and $r + s \equiv 1 \pmod 2$. Then

$$(r-s)(r+s) = x_1^2 \quad \text{and} \quad y_1^2 = rs\,. \tag{10}$$

Note that since $r$ and $s$ are relatively prime and of opposite parity, the odd integers $r - s$ and $r + s$ must also be relatively prime. Consequently, it follows from the equations in (10) that each of the four positive integers $r - s$, $r + s$, $r$, $s$ is a perfect square. In particular,

$$\begin{aligned} r_1^2 - s_1^2 &= u_1^2\,, \\ r_1 + s_1^2 &= u_2^2\,, \end{aligned}$$

for positive integers $r_1$, $s_1$, $u_1$, $u_2$; contradicting the fact (which we stated at the outset) that such a system has no solution in positive integers. ■

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# The CRUX Open: Unsolved Problems in CRUX through Vol. 36

J. Chris Fisher

**133** [1976:67, 144-150, 221]. The "Collatz $3x + 1$ problem."
See also problem **1370** [1988:203; 1989:281-3; 1990:317; 1993:304].
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**266** [1977:190; 1978:75-76].
**283** [1977:250; 1978:115, 195-6].
**339** [1978:102, 292].
**342** [1978:133, 297; 1980:319-22].
**355(c)(d)** [1978:160; 1979:78-80, 168-171].
**410** [1979:17, 296-9].
**434(b)** [1979:108; 1980:59].
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**1580** [1990:240; 1991:308].
**1615(b)** [1991:44; 1992:82-83].
**1666(b)** [1991:207; 1992:191-2].
**1737(b)** [1992:110; 1993:90-91].
**1754** [1992:175; 1993:151-2; 1994:196-9; 1995:236-8].
**1872** [1993:234; 1994:201].
**1965** [1994:194; 1995:207].
**2014(b)** [1995:52; 1996:45-47].
**2025(b)** [1995:90; 1996:125-126].
**2049** [1995:158; 1996:183-4].
**2173** [1996:275; 1997:169, 440-2].
**2210** [1997:47; 1998:128].
**2247(b)** [1997:244; 1998:366-71].
**2285** [1997:432; 1998:522-5].
**2286(b)** [1997:432].
**2294(b)** [1997:502; 1998:530-2].
**2386(b)** [1998:426; 1999:516].

**2399** [1998:506; 1999:531].
**2405(b)** [1999:48, 238; 2000:53].
**2474** [1999:368; 2000:443]. **2476** [1999:429; 2000:445].
**2520(2)** [2000:115; 2001:218-9].
**2535(2)** [2000:179; 2001:279]. Part (2) is related to a well-known open problem.
**2558** [2000:304; 2001:466]. **2561** [2000:305; 2001:408].
**2593** [2000:498; 2001:556]. **2597** [2000:499; 2001:559].
**2616** [2001:137; 2002:252]. **2623** [2001:138; 2002:252].
**2669** [2001:404; 2002:252]. Klamkin offered a $50 prize, but no valid solution was submitted.
**2673(c)(d)** [2001:405; 2002:468-471].
**2686** [2001:461; 2002:543-4].
**2724** [2002:174; 2003:180-1; 2004:44-46].
**2725(2)** [2002:175; 2003:181-2].
**2735** [2002:179; 2003:190]. **2832** [2003:176; 2004:184].
**2838** [2003:238; 2004:192]. **2847** [2003:242; 2004:255].
**2904** [2004:39, 42; 2005:52]. The known proof is not elementary; an elementary proof is sought.
**2949** [2004:231, 234; 2005:255]. **2950** [2004:231, 234; 2005:256].
**2967(c)** [2004:368, 371; 2005:407-8].
**2968(b)** [2004:368, 371; 2005:409-10].
**2977(b)** [2004:429, 432; 2005:468-70].
**3105(b)** [2006:45, 48; 2007:52-4].
**3132(d)** [2006:172, 174; 2007:185-6].
**3145** [2006:239, 241; 2007:248].
**3169** [2006:395, 397; 2007:375-6].
**3195(b)** [2006:515, 518; 2007:496-9].
**3225(b)** [2007:112, 115, 297; 2008:124-127].
**3231** [2007:170, 173; 2008:183].
**3268(b)** [2007:367, 369; 2008:374-5].
**3277** [2007:428, 430; 2008:436].
**3325(b)** [2008:104, 106; 2009:127-128].
**3326(b)** [2008:170, 173; 2009:177-8].
**3327(b)** [2008:170, 173; 2009:179].
**3474** [2009:397, 399; 2010:416]. **3492** [2009:515, 518; 2010:503].

**M118** [2003:427; 2004:470].
**Totten–M7(c)** [2009:271-272, 274; 2010:141].

# PROBLEMS

*Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le* **1er avril 2010**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

*Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.*

*La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.*

**3574**. Correction. *Proposé par Michel Bataille, Rouen, France.*

Soit $x$, $y$ et $z$ trois nombres réels tels que $x + y + z = 0$. Montrer que

$$\sum_{\text{cyclique}} \cosh x \;\leq\; \sum_{\text{cyclique}} \cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1 + 2\prod_{\text{cyclique}} \cosh x\,.$$

**3588**. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit $ABC$ un triangle rectangle d'hypoténuse $c = AB$. Soit $w_a$ et $w_b$ les longueurs respectives des bissectrices issues de $A$ et $B$. Montrer que

$$w_a + w_b \;\leq\; 2c\sqrt{2-\sqrt{2}}\,.$$

**3589**. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Trouver tous les entiers $n > 6$ pour lesquels il existe un $n$-gone convexe avec un point intérieur $P$ tel que $PA_i = A_iA_{i+1}$ pour chaque $i$, les indices étant pris modulo $n$.

**3590**. *Proposé par G.W. Indika Amarasinghe, Université de Kelaniya, Kelaniya, Sri Lanka.*

Soit $ABPC$ un quadrilatère tel que $BC$ coupe en deux le segment $AP$ et que $AP$ soit la bissectrice de l'angle $BAC$. Soit $a = BC$, $b = AC$, $c = AB$, $p = BP$ et $q = PC$. Montrer que

$$\frac{p^2}{c} + \frac{q^2}{b} \;=\; b + c\,.$$

**3591**. *Proposé par Michel Bataille, Rouen, France.*

Soit $\mathcal{E}$ une ellipse de centre $O$. En exactement quatre points $P$ de $\mathcal{E}$, la tangente à $\mathcal{E}$ forme un angle de $45^\circ$ avec $OP$. Quelle est l'excentricité de $\mathcal{E}$ ?

**3592★**. *Proposé par Faruk Zejnulahi et Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

Soit $a$, $b$ et $c$ des nombres réels non négatifs tels que $a+b+c=3$. Démontrer si oui ou non les inégalités ci-dessous sont valides.

$$\frac{19}{20} \leq \frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2} \leq \frac{27}{20}\,.$$

**3593**. *Proposé par Daryl Tingley, Université du Nouveau-Brunswick, Fredericton, NB.*

Montrer que pour tous les entiers non négatifs $n$, le chiffre distinct de zéro le plus à droite dans l'écriture de $(4\cdot 5^n)!$ est 4. De plus, montrer que si $n \geq k \geq 0$, alors la chaîne de $k+1$ chiffres consécutifs, avec ce chiffre 4 à droite, est indépendant de $n$.

**3594**. *Proposé par Michel Bataille, Rouen, France.*

Soit $x$, $y$ et $z$ trois inconnues et $A = (y-z)(y+x)(x+z)$, $B=(z-x)(z+y)(y+x)$, $C=(x-y)(x+z)(z+y)$. Trouver tous les polynômes $P$, $Q$, $R \in \mathbb{C}[x,y,z]$ tels que

$$\frac{x^2P+y^2Q+z^2R}{xP+yQ+zR}=\frac{x^2A+y^2B+z^2C}{xA+yB+zC}\,.$$

**3595**. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Soit $a$, $b$ et $n$ entiers positifs tels que $a<b$ et $n<a+b$, et tels que

exactement $\frac{1}{n}$ des entiers $a^2$, $a^2+1$, $a^2+2$, $\dots$, $b^2$ sont des carrés. (1)

Répondre aux deux questions suivantes :

(a) Montrer qu'aussi, exactement $\frac{1}{n}$ des entiers consécutifs $(n-a)^2$, $(n-a)^2+1$, $(n-a)^2+2$, $\dots$, $b^2$ sont des carrés.

(b) D'une part exactement $\frac{1}{n}$ des entiers $1$, $2$, $\dots$, $n^2$ sont des carrés, et d'autre part exactement $\frac{1}{n}$ des entiers $(n-1)^2=n^2-2n+1$, $n^2-2n+2$, $\dots$, $n^2$ sont des carrés. Ainsi, pour tout entier $n\geq 3$, les valeurs $a=1$, $b=n$ et $a=n-1$, $b=n$ satisfont toujours (1). Pour quels entiers $n\geq 3$ ces valeurs sont-elles les seules solutions de (1) ?

**3596**. *Proposé par Paolo Perfetti, Département de Mathématiques, Université de Rome, "Tor Vergata", Rome, Italie.*

Soit $x$, $y$ et $z$ trois nombres réels positifs. Montrer que

$$\sum_{\text{cyclique}}\frac{x(y+z)}{(x+2y+2z)^2} \leq \sum_{\text{cyclique}}\frac{(x+y)(x+y+2z)}{(3x+3y+4z)^2}\,.$$

**3597**. *Proposé par Johan Gunardi, étudiant, SMPK 4 BPK PENABUR, Jakarta, Indonésie.*

Cent étudiants passent un examen consistant en $50$ questions "vrai" ou "faux". Montrer qu'il existe trois étudiants dont les réponses coïncident pour au moins $13$ questions.

**3598**. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Le quadrilatère $ABCD$ possède à la fois un cercle circonscrit et un cercle inscrit, celui-ci de centre $I$.

Poser $a = AB$, $b = BC$, $c = CD$ et $d = DA$. Montrer que

$$\frac{IB^2}{ab} + \frac{IC^2}{bc} + \frac{ID^2}{cd} + \frac{IA^2}{da} \;=\; 2\,.$$

**3599** ★. *Proposé par Cristinel Mortici, Valahia Université de Târgovişte, Roumanie.*

Soit $m$ et $n$ deux entiers positifs tels que $2^m - 3^n \geq n$. Montrer que

$$2^m - 3^n \geq m\,.$$

**3600**. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $k \geq 1$ un entier. Montrer que

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} \frac{1}{(n_1+n_2+\cdots+n_k)!} \;=\; (-1)^{k-1}\left(e\sum_{j=0}^{k-1}\frac{(-1)^j}{j!} \;-\; 1\right)\,.$$

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**3574**. Correction. *Proposed by Michel Bataille, Rouen, France.*

Let $x$, $y$, and $z$ be real numbers such that $x + y + z = 0$. Prove that

$$\sum_{\text{cyclic}} \cosh x \;\leq\; \sum_{\text{cyclic}} \cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1 + 2\prod_{\text{cyclic}} \cosh x\,.$$

**3588**. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let $ABC$ be a right-angled triangle with hypotenuse $c = AB$. Let $w_a$ and $w_b$ be the lengths of the angle bisectors from $A$ and $B$, respectively. Prove that

$$w_a + w_b \;\leq\; 2c\sqrt{2-\sqrt{2}}\,.$$

**3589**. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Find all integers $n > 6$ for which there exists a convex $n$-gon with an interior point $P$ such that $PA_i = A_iA_{i+1}$ for each $i$, where indices are taken modulo $n$.

**3590**. *Proposed by G.W. Indika Amarasinghe, University of Kelaniya, Kelaniya, Sri Lanka.*

Let $ABPC$ be a quadrilateral such that $BC$ bisects the segment $AP$ and $AP$ bisects $\angle BAC$. Let $a = BC$, $b = AC$, $c = AB$, $p = BP$, and $q = PC$. Prove that
$$\frac{p^2}{c} + \frac{q^2}{b} \;=\; b + c\,.$$

**3591**. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{E}$ be an ellipse with centre $O$. At exactly four points $P$ of $\mathcal{E}$, the tangent to $\mathcal{E}$ makes a $45^\circ$ angle with $OP$. What is the eccentricity of $\mathcal{E}$?

**3592★**. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let $a$, $b$, and $c$ be nonnegative real numbers such that $a + b + c = 3$. Prove or disprove that
$$\frac{19}{20} \;\leq\; \frac{1}{1+a+b^2} + \frac{1}{1+b+c^2} + \frac{1}{1+c+a^2} \;\leq\; \frac{27}{20}\,.$$

**3593**. *Proposed by Daryl Tingley, University of New Brunswick, Fredericton, NB.*

Show that for all nonnegative integers $n$ the rightmost nonzero digit of $(4 \cdot 5^n)!$ is 4. Furthermore, show that if $n \geq k \geq 0$, then the string of $k+1$ consecutive digits with this digit 4 at the right is independent of $n$.

**3594**. *Proposed by Michel Bataille, Rouen, France.*

Let $x$, $y$, $z$ be three indeterminates and $A = (y-z)(y+x)(x+z)$, $B = (z-x)(z+y)(y+x)$, $C = (x-y)(x+z)(z+y)$. Find all polynomials $P$, $Q$, $R \in \mathbb{C}[x,y,z]$ such that
$$\frac{x^2P + y^2Q + z^2R}{xP + yQ + zR} = \frac{x^2A + y^2B + z^2C}{xA + yB + zC}\,.$$

**3595**. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Let $a$, $b$, $n$ be positive integers satisfying $a < b$ and $n < a + b$, and so that

exactly $\frac{1}{n}$ of the integers $a^2$, $a^2+1$, $a^2+2$, $\ldots$, $b^2$ are squares. (1)

Do the following :

(a) Prove that also exactly $\frac{1}{n}$ of the consecutive integers $(n-a)^2$, $(n-a)^2+1$, $(n-a)^2+2$, $\ldots$, $b^2$ are squares.

(b) Exactly $\frac{1}{n}$ of the integers $1, 2, \ldots, n^2$ are squares, and also exactly $\frac{1}{n}$ of the integers $(n-1)^2 = n^2 - 2n + 1, n^2 - 2n + 2, \ldots, n^2$ are squares. Thus, for every integer $n \geq 3$, the values $a = 1$, $b = n$ and $a = n - 1$, $b = n$ always satisfy (1). For which integers $n \geq 3$ are these the only solutions of (1)?

**3596**. *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $x$, $y$ and $z$ be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{x(y+z)}{(x+2y+2z)^2} \leq \sum_{\text{cyclic}} \frac{(x+y)(x+y+2z)}{(3x+3y+4z)^2}.$$

**3597**. *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

One hundred students take an exam consisting of $50$ true or false questions. Prove that there exist three students whose answers coincide for at least $13$ questions.

**3598**. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

The quadrilateral $ABCD$ has both a circumscribed circle and an inscribed circle, the latter with centre $I$. Put $a = AB$, $b = BC$, $c = CD$, and $d = DA$. Prove that

$$\frac{IB^2}{ab} + \frac{IC^2}{bc} + \frac{ID^2}{cd} + \frac{IA^2}{da} = 2.$$

**3599** ★. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*

Let $m$ and $n$ be positive integers such that $2^m - 3^n \geq n$. Prove that

$$2^m - 3^n \geq m.$$

**3600**. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k \geq 1$ be a nonnegative integer. Prove that

$$\sum_{n_1, n_2, \ldots, n_k = 1}^{\infty} \frac{1}{(n_1 + n_2 + \cdots + n_k)!} = (-1)^{k-1} \left( e \sum_{j=0}^{k-1} \frac{(-1)^j}{j!} - 1 \right).$$

# SOLUTIONS

*Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.*

We have received a late batch of correct solutions to problems 3478, 3479, 3480, 3481, 3482, 3483, and 3486 from Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

**3488**. [2009 : 515, 517] *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let $a$, $b$, and $c$ be positive real numbers. Prove that

$$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab} \leq \sqrt{\frac{a^{-1}+b^{-1}+c^{-1}}{a+b+c}}\,.$$

*Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Let $t=(t_1,t_2,\dots,t_n)$ and $s=(s_1,s_2,\dots,s_n)$ be arbitrary $n$-tuples of nonnegative real numbers. We will write $t \succ s$ if

(i) $t_1 \geq \cdots \geq t_n$ and $s_1 \geq \cdots \geq s_n$,

(ii) $\sum\limits_{i=1}^{k} t_i \geq \sum\limits_{i=1}^{k} s_i$ for all $k=1,\,2,\,\dots,\,n$, with equality when $k=n$.

Let $\mathbb{R}_+$ denote the set of positive real numbers, let $P$ be the set of all permutations of $\{1,\,2,\,\dots,\,n\}$, and define $[t]:\mathbb{R}_+^n \to \mathbb{R}$ by

$$[t](x) \;=\; \sum_{\sigma\in P} x_{\sigma(1)}^{t_1}x_{\sigma(2)}^{t_2}\cdots x_{\sigma(n)}^{t_n} \qquad \text{for all } x=(x_1,x_2,\dots,x_n)\,.$$

Muirhead's inequality states that if $t \succ s$, then $[t] \geq [s]$. Here, as usual, $[t] \geq [s]$ means that $[t](x) \geq [s](x)$ for all $x \in \mathbb{R}_+^n$. Now, by squaring and simplifying, the given inequality is equivalent to $A \geq B$, where

$$\begin{aligned} A &= 12[8,5,1]+23[7,4,3]+16[6,6,2]+12[8,4,2]+4[7,6,1] \\ &\qquad +4[9,3,2]+8[7,7,0]\,, \\ B &= 12[7,5,2]+22[6,5,3]+26[6,4,4]+\frac{5}{2}[8,3,3]+\frac{33}{2}[5,5,4]\,. \end{aligned}$$

But this last inequality holds by these applications of Muirhead's inequality:

$$\begin{aligned} [8,5,1] &\geq [7,5,2]\,, \\ [7,4,3] &\geq [6,5,3] \quad\text{and}\quad [7,4,3]\geq[6,4,4]\,, \\ [8,4,2] &\geq [8,3,3] \quad\text{and}\quad [8,4,2]\geq[6,4,4]\,, \\ [6,6,2] &\geq [6,4,4] \quad\text{and}\quad [6,6,2]\geq[5,5,4]\,, \end{aligned}$$

$$\begin{aligned}
[7,6,1] &\geq [5,5,4]\,,\\
[9,3,2] &\geq [5,5,4]\,,\\
[7,7,0] &\geq [5,5,4]\,.
\end{aligned}$$

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incomplete solution was submitted.*

**3489**. [2009 : 515, 517] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $n$ be a nonnegative integer. Prove that

$$\frac{1}{2^{n-1}}\sum_{k=0}^{n}\sqrt{k}\binom{2n}{k} \;\leq\; \sqrt{n\left(2^{2n}+\binom{2n}{n}\right)}\,.$$

*A composite of similar solutions by George Apostolopoulos, Messolonghi, Greece; and Albert Stadler, Herrliberg, Switzerland.*

By using the elementary facts that $\binom{2n}{k}=\binom{2n}{2n-k}$ for $0\leq k\leq 2n$ and $k\binom{2n}{k}=2n\binom{2n-1}{k-1}$ for $1\leq k\leq 2n$, and also the Cauchy–Schwarz Inequality, we have that

$$\begin{aligned}
\left[\sum_{k=0}^{n}\sqrt{k}\binom{2n}{k}\right]^2 &= \left[\sum_{k=0}^{n}\sqrt{\binom{2n}{k}}\sqrt{k}\sqrt{\binom{2n}{k}}\right]^2\\
&\leq \left[\sum_{k=0}^{n}\binom{2n}{k}\right]\left[\sum_{k=0}^{n}k\binom{2n}{k}\right]\\
&= \frac{1}{2}\left[\binom{2n}{n}+\sum_{k=0}^{2n}\binom{2n}{k}\right]\left[2n\sum_{k=1}^{n}\binom{2n-1}{k-1}\right]\\
&= \frac{1}{2}\left[\binom{2n}{n}+2^{2n}\right]\left[2n\sum_{k=0}^{n-1}\binom{2n-1}{k}\right]\\
&= \frac{1}{2}\left[\binom{2n}{n}+2^{2n}\right]\left[2n\cdot\frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\right]\\
&= \frac{1}{2}\left[2^{2n}+\binom{2n}{n}\right]\left(n\cdot 2^{2n-1}\right)\\
&= n\left(2^{2n}+\binom{2n}{n}\right)\cdot 2^{2n-2}\,,
\end{aligned}$$

from which the claimed inequality follows immediately.

*Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE CAMPBELL, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.*

**3490**. [2009 : 515, 518] *Proposed by Michael Rozenberg, Tel-Aviv, Israel.*

Let $a$, $b$, and $c$ be nonnegative real numbers such that $a+b+c=1$. Prove that

(a) $\sqrt{9-32ab}+\sqrt{9-32ac}+\sqrt{9-32bc}\geq 7$;

(b) $\sqrt{1-3ab}+\sqrt{1-3ac}+\sqrt{1-3bc}\geq\sqrt{6}$.

*Solution to part (a) by Oliver Geupel, Brühl, NRW, Germany; solution to part (b) by George Apostolopoulos, Messolonghi, Greece, modified by the editor.*

(a) For nonnegative integers $\ell$, $m$, and $n$, let $[\ell,m,n]=\sum\limits_{\text{symm.}} a^{\ell}b^{m}c^{n}$.

The following inequality is a consequence of Muirhead's Theorem,

$$\begin{aligned}
&27\prod_{\text{cyclic}}\Big(9(a+b+c)^2-32ab\Big)-\Big(11(a+b+c)^2+16(ab+bc+ca)\Big)^3\\
&=9176\,[6,0,0]+34320\,[5,1,0]-36336\,[4,2,0]+50184\,[4,1,1]\\
&\qquad-\;54352\,[3,3,0]+100320\,[3,2,1]-103312\,[2,2,2]\\
&\geq 0\,.
\end{aligned}$$

We put $a+b+c=1$ in the above, and we observe that by the AM-GM Inequality $\sum\limits_{\text{cyclic}}\sqrt{(9-32ab)(9-32bc)}\geq 3\Big(\prod\limits_{\text{cyclic}}(9-32ab)\Big)^{1/3}$. It follows that $\sum\limits_{\text{cyclic}}\sqrt{(9-32ab)(9-32bc)}\geq 11+16(ab+bc+ca)$, and we deduce

$$\left(\sum_{\text{cyclic}}\sqrt{9-32ab}\right)^2\;\geq\;49\,,$$

from which the inequality in (a) follows.

(b) Let $x=3a$, $y=3b$, and $z=3c$. Then $x$, $y$, and $z$ are nonnegative real numbers such that $x+y+z=3$, and we are to show that

$$\sum_{\text{cyclic}}\sqrt{3-xy}\;\geq\;3\sqrt{2}\,. \tag{1}$$

Note first that $\sum_{\text{cyclic}} \sqrt{\frac{(3+z)^2}{8}} = \frac{1}{\sqrt{8}} \sum_{\text{cyclic}} (3+z) = \frac{1}{\sqrt{8}}(9+3) = 3\sqrt{2}$.

Also,

$$\begin{aligned}\frac{(3+z)^2}{8} - \left(3 - \frac{(x+y)^2}{4}\right) &= \frac{(3+z)^2}{8} - 3 + \frac{(3-z)^2}{4} \\ = \frac{1}{8}\left(9 + 6z + z^2 - 24 + 18 - 12z + 2z^2\right) &= \frac{3}{8}(z-1)^2 . \qquad (2)\end{aligned}$$

Hence,

$$\frac{(3+z)^2}{8} \geq 3 - \frac{(x+y)^2}{4} , \qquad (3)$$

and (1) is equivalent to

$$\begin{aligned}&\sum_{\text{cyclic}} \left( \sqrt{3 - xy} - \sqrt{3 - \frac{(x+y)^2}{4}} \right) \\ &\qquad \geq \sum_{\text{cyclic}} \left( \sqrt{\frac{(3+z)^2}{8}} - \sqrt{3 - \frac{(x+y)^2}{4}} \right) . \qquad (4)\end{aligned}$$

Let $H$ and $K$ denote the left and right side of (4), respectively. Then

$$\begin{aligned}H &= \frac{1}{4} \sum_{\text{cyclic}} \frac{(x-y)^2}{\sqrt{3-xy} + \sqrt{3 - \frac{(x+y)^2}{4}}} \\ &\geq \frac{1}{4} \sum_{\text{cyclic}} \frac{(x-y)^2}{\sqrt{3}+\sqrt{3}} = \frac{1}{8\sqrt{3}} \sum_{\text{cyclic}} (x-y)^2 . \qquad (5)\end{aligned}$$

On the other hand, using (2) and (3), we have

$$\begin{aligned}K &= \sum_{\text{cyclic}} \frac{\frac{(3+z)^2}{8} - 3 + \frac{(x+y)^2}{4}}{\sqrt{\frac{(3+z)^2}{8}} + \sqrt{3 - \frac{(x+y)^2}{4}}} = \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{\sqrt{\frac{(3+z)^2}{8}} + \sqrt{3 - \frac{(x+y)^2}{4}}} \\ &\leq \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{2\sqrt{3 - \frac{(x+y)^2}{4}}} \leq \frac{3}{8} \sum_{\text{cyclic}} \frac{(z-1)^2}{2\sqrt{3 - \frac{9}{4}}} = \frac{\sqrt{3}}{8} \sum_{\text{cyclic}} (z-1)^2 . \qquad (6)\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{\text{cyclic}} (x-y)^2 &= 3\left(\sum_{\text{cyclic}} x^2\right) - \left(\sum_{\text{cyclic}} x\right)^2 = 3\left(\sum_{\text{cyclic}} x^2\right) - 9 \\ &= 3\left(\sum_{\text{cyclic}} x^2\right) - 6\left(\sum_{\text{cyclic}} x\right) + 9 = 3\left(\sum_{\text{cyclic}} (z-1)^2\right) . \qquad (7)\end{aligned}$$

From (5), (6), and (7) we get $H \geq K$, establishing (4), and hence (1).

*Part (b) was also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer. Two incomplete solutions were submitted.*

*The case of equality was not requested, though Geupel claimed equality precisely when $a = b = c = 1/3$, but the proposer noted that equality also occurs when $a = b = 1/2$, $c = 0$.*

**3491**. [2009 : 515, 518] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $a_1, a_2, \ldots, a_{n+1}$ be positive real numbers where $a_{n+1} = a_1$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} \ \geq\ \frac{1}{4}\sum_{i=1}^{n} a_i\,.$$

*Solution by George Apostolopoulos, Messolonghi, Greece.*

Let

$$\begin{aligned} A &= \sum_{i=1}^{n} \frac{a_i^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)}\,, \\ B &= \sum_{i=1}^{n} \frac{a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)}\,. \end{aligned}$$

Then

$$A - B = \sum_{i=1}^{n} \frac{a_i^4 - a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} = \sum_{i=1}^{n} a_i - a_{i+1} = 0\,,$$

and hence $A = B$.

We now show that for all positive real numbers $a$ and $b$ we have

$$a^4 + b^4 \ \geq\ \frac{(a+b)^2(a^2+b^2)}{4}\,.$$

Indeed, using the inequality $(x+y)^2 \leq 2(x^2+y^2)$ twice we obtain

$$(a+b)^2(a^2+b^2) \ \leq\ 2(a^2+b^2)^2 \ \leq\ 4(a^4+b^4)\,.$$

Hence,

$$\begin{aligned} 2A \ =\ A + B &= \sum_{i=1}^{n} \frac{a_i^4 + a_{i+1}^4}{(a_i + a_{i+1})(a_i^2 + a_{i+1}^2)} \\ &\geq\ \frac{1}{4}\sum_{i=1}^{n}(a_i + a_{i+1}) \ =\ \frac{1}{2}\sum_{i=1}^{n} a_i\,. \end{aligned}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

*Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.*

**3492★**. [2009 : 515, 518] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $P$ be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB$, $\angle PBC$, $\angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that $P$ is its centroid.

The problem remains open. The only submission, from Peter Y. Woo, Biola University, La Mirada, CA, USA, gave a counterexample where $ABCD$ is a degenerate tetrahedron. In particular, he provided an elegant proof that if $P$ is the centre of a parallelogram $ABCD$ with sides $AD = BC = 3\sqrt{2}$ and $AB = CD = \sqrt{6}$, and diagonals $AC = 2\sqrt{3}$ and $AC = 6$, then

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \arccos\sqrt{\frac{2}{3}}\,.$$

This certainly addresses the question that was asked, and it suggests that there are infinitely many tetrahedra with an interior point $P$ that satisfies the given angle requirement, but it fails to provide an explicit nondegenerate example.

**3494**. [2009 : 516, 518] *Proposed by Michel Bataille, Rouen, France.*

Let $n > 1$ be an integer and for each $k = 1, 2, \ldots, n$ let

$$\sigma(n,k) \;=\; \sum_{1\le i_1<\cdots<i_k\le n} i_1 i_2 \cdots i_k\,.$$

Prove that

$$\sum_{k=1}^{n} \frac{\ln n}{n+1-k}\cdot\sigma(n,k) \;\sim\; (n+1)! \;\sim\; \sum_{k=1}^{n} \frac{n+1-k}{\ln n}\cdot\sigma(n,k)\,,$$

where $f(n) \sim g(n)$ means that $\dfrac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

*Solution by the proposer.*

Let

$$\begin{aligned} P_n(x) &= (x+1)(x+2)\cdots(x+n) \\ &= x^n + \sigma(n,1)x^{n-1} + \cdots + \sigma(n,n-1)x + \sigma(n,n)\,. \end{aligned}$$

If $U_n$ denotes $\sum\limits_{k=1}^{n} \dfrac{\sigma(n,k)}{n+1-k}$, then

$$U_n \;=\; \left(\int_0^1 P_n(x)\,dx\right) - \frac{1}{n+1}\,.$$

Clearly, $\dfrac{P_n'(x)}{P_n(x)} = \dfrac{1}{x+1} + \dfrac{1}{x+2} + \cdots + \dfrac{1}{x+n}$, so that for all $x \in [0,1]$,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} \;\le\; \frac{P_n'(x)}{P_n(x)} \;\le\; 1 + \frac{1}{2} + \cdots + \frac{1}{n}\,. \tag{1}$$

Multiplying by $P_n(x)$ and integrating over $[0,1]$ leads to

$$(H_{n+1} - 1)\left(U_n + \frac{1}{n+1}\right) \;\le\; P_n(1) - P_n(0) \;\le\; H_n\left(U_n + \frac{1}{n+1}\right)\,,$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the $n^{\text{th}}$ harmonic number. Since we also have $P_n(1) - P_n(0) = (n+1)! - n! = \frac{n}{n+1} \cdot (n+1)!$, we obtain

$$\frac{n}{n+1} \cdot \frac{(n+1)!}{H_n} - \frac{1}{n+1} \;\le\; U_n \;\le\; \frac{n}{n+1} \cdot \frac{(n+1)!}{H_{n+1} - 1} - \frac{1}{n+1}$$

for all positive integers $n$. Recalling that $H_n \sim \ln(n)$, the Squeeze Theorem for limits yields $\lim\limits_{n\to\infty} \dfrac{U_n \ln(n)}{(n+1)!} = 1$, that is,

$$\sum_{k=1}^{n} \frac{\ln(n)}{n+1-k} \cdot \sigma(n,k) \sim (n+1)!\,.$$

Let $V_n = \sum\limits_{k=1}^{n} (n+1-k)\sigma(n,k)$. From (1) and $P_n(1) = (n+1)!$, we deduce that

$$(H_{n+1} - 1)(n+1)! \;\le\; P_n'(1) \;\le\; H_n(n+1)!\,.$$

Also, for $n > 1$,

$$\begin{aligned} V_n &= \sum_{k=1}^{n} (n-k)\sigma(n,k) + \sum_{k=1}^{n} \sigma(n,k) \\ &= P_n'(1) - n + (n+1)! - 1 \\ &= P_n'(1) + (n+1)! - (n+1)\,, \end{aligned}$$

so that

$$\begin{aligned} \frac{H_{n+1} - 1}{\ln(n)} + \frac{1}{\ln(n)} - \frac{1}{n!\ln(n)} &\le \frac{V_n}{(n+1)!\ln(n)} \\ &\le \frac{H_n}{\ln(n)} + \frac{1}{\ln(n)} - \frac{1}{n!\ln(n)}\,. \end{aligned}$$

Again, the Squeeze Theorem yields $(n+1)! \sim \sum\limits_{k=1}^{n} \dfrac{n+1-k}{\ln(n)} \cdot \sigma(n,k)$, and the proof is complete.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and ALBERT STADLER, Herrliberg, Switzerland.*

**3495**. [2009 : 516, 518] *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

Let $a$, $b$, $c$ be positive real numbers with $a+b+c=2$. Prove that

$$\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c} \;\leq\; \sum_{\text{cyclic}}\frac{\left(a^2+bc\right)}{b+c} \;\leq\; \frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\,.$$

*A combination of solutions by George Apostolopoulos, Messolonghi, Greece and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified by the editor.*

For vectors $a=(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ with real entries, the notation $a\prec b$ means that $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$ and $a_1+a_2+\cdots+a_j\leq b_1+b_2+\cdots+b_j$ holds for each $j=1,\,2,\,\ldots,\,n-1$.

Since $a+b+c=2$, the inequality on the left is equivalent to

$$\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a}{b+c}\right)\frac{a+b+c}{2}\leq\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}$$

or

$$\sum_{\text{symmetric}}\left(a^4+a^2bc\right)\geq\sum_{\text{symmetric}}\left(a^4+a^2bc\right).$$

Schur's Inequality yields

$$\sum_{\text{symmetric}}\left(a^4+a^2bc\right)\;\geq\;2\sum_{\text{symmetric}}\left(a^3b\right).$$

Now using Muirhead's inequality for $(2,2,0)\prec(3,1,0)$ we obtain

$$\sum_{\text{symmetric}}\left(a^3b\right)\geq\sum_{\text{symmetric}}\left(a^2b^2\right),$$

which proves the inequality on the left.

Now the inequality on the right is equivalent to

$$\sum_{\text{cyclic}}\frac{a^2+bc}{b+c}\leq\left(\frac{1}{2}+\sum_{\text{cyclic}}\frac{a^2}{b^2+c^2}\right)\frac{a+b+c}{2},$$

or

$$\begin{aligned}&\sum_{\text{symmetric}}\left(2a^9b+4a^8bc+7a^7b^2c+a^7b^3+2a^4b^4c^2\right)\\&\qquad\geq\sum_{\text{symmetric}}\left(2a^6b^4+a^5b^5+5a^5b^3c^2+a^4b^3c^3+5a^5b^4c+2a^6b^3c\right).\end{aligned}$$

Using Muirhead's inequality repeatedly we obtain:

$$(6,4,0) \prec (9,1,0) \implies \sum_{\text{symmetric}} 2a^9b \geq \sum_{\text{symmetric}} 2a^6b^4$$

$$(6,2,2) \prec (7,2,1) \implies \sum_{\text{symmetric}} 2a^7b^2c \geq \sum_{\text{symmetric}} 2a^6b^2c^2$$

$$(6,3,1) \prec (8,1,1) \implies \sum_{\text{symmetric}} 2a^8bc \geq \sum_{\text{symmetric}} 2a^6b^3c$$

$$(5,4,1) \prec (8,1,1) \implies \sum_{\text{symmetric}} 2a^8bc \geq \sum_{\text{symmetric}} 2a^5b^4c$$

$$(5,5,0) \prec (7,3,0) \implies \sum_{\text{symmetric}} a^7b^3 \geq \sum_{\text{symmetric}} a^5b^5$$

$$(5,3,2) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^5b^3c^2$$

$$(4,3,3) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^4b^3c^3$$

$$(5,4,1) \prec (7,2,1) \implies \sum_{\text{symmetric}} a^7b^2c \geq \sum_{\text{symmetric}} a^5b^4c$$

Also, by the AM-GM Inequality, we have

$$\sum_{\text{symmetric}} (2a^6b^2c^2 + 2a^4b^4c^2) \geq \sum_{\text{symmetric}} 4a^5b^3c^2 \,.$$

We add all these inequalities, and we are done.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incomplete solution was submitted.*

*Zvonaru observed that this problem appeared in the book* Old And New Inequalities, Vol. 2, *by Vo Quoc Ba Can and Cosmin Pohoata, Gil Publishing House, 2008.*

**3496**. [2009 : 516, 519] *Proposed by Elias C. Buissant des Amorie, Castricum, the Netherlands.*

Prove the following equations:

(a) $\tan 72^\circ = \tan 66^\circ + \tan 36^\circ + \tan 6^\circ$.

(b)★ $\tan 84^\circ = \tan 78^\circ + \tan 72^\circ + \tan 60^\circ$;

[*Ed.: The proposer gave six more relations of the form* $f(\theta) = \sum_{i=1}^{4} \tan k_i\theta = 0$ *for* $k_i \in \mathbb{Z}$ *and* $\theta = 2\pi/n$ *with* $n|360$, *not included here for lack of space.*]

*Composite of solutions by Kee-Wai Lau, Hong Kong, China and D.J. Smeenk, Zaltbommel, the Netherlands.*

With the help of appropriate trigonometric identities, both equations can be reduced to properties of the golden section $\tau = \dfrac{1+\sqrt{5}}{2}$, which is the positive root of the quadratic equation

$$\tau^2 \;=\; \tau + 1\,. \tag{1}$$

Because $\tau$ is the ratio of a diagonal to a side of a regular pentagon, it satisfies

$$\cos 36^\circ = \frac{\tau}{2} \quad \text{and} \quad \cos 72^\circ = \frac{1}{2\tau}\,. \tag{2}$$

(a) The following equations are equivalent.

$$\begin{aligned}
\tan 72^\circ &= \tan 66^\circ + \tan 36^\circ + \tan 6^\circ\,,\\
\tan 72^\circ - \tan 36^\circ &= \tan 66^\circ + \tan 6^\circ\,,\\
\frac{\sin(72^\circ - 36^\circ)}{\cos 72^\circ \cos 36^\circ} &= \frac{\sin(66^\circ + 6^\circ)}{\cos 66^\circ \cos 6^\circ}\,,\\
2\sin 36^\circ \cos 66^\circ \cos 6^\circ &= 2\sin 72^\circ \cos 72^\circ \cos 36^\circ \;=\; \sin 144^\circ \cos 36^\circ\,,\\
2\sin 36^\circ \cos 66^\circ \cos 6^\circ &= \sin 36^\circ \cos 36^\circ\,,\\
2\cos 66^\circ \cos 6^\circ &= \cos 36^\circ\,,\\
\cos 72^\circ + \cos 60^\circ &= \cos 36^\circ\,,\\
\cos 72^\circ - \cos 36^\circ + \frac{1}{2} &= 0\,,
\end{aligned}$$

and the last equality follows immediately from equations (1) and (2).

(b) The following equations are equivalent.

$$\begin{aligned}
\tan 84^\circ &= \tan 78^\circ + \tan 72^\circ + \tan 60^\circ\,,\\
\tan 84^\circ - \tan 60^\circ &= \tan 78^\circ + \tan 72^\circ\,,\\
\frac{\sin(84^\circ - 60^\circ)}{\cos 84^\circ \cos 60^\circ} &= \frac{\sin(78^\circ + 72^\circ)}{\cos 78^\circ \cos 72^\circ} \;=\; \frac{1}{2\cos 78^\circ \cos 72^\circ}\,,\\
\cos 84^\circ &= 4\sin 24^\circ \cos 72^\circ \cos 78^\circ\,,\\
\sin 6^\circ &= 2(\sin 96^\circ - \sin 48^\circ)\cos 78^\circ\,,\\
\sin 6^\circ &= (\sin 174^\circ + \sin 18^\circ) - (\sin 126^\circ - \sin 30^\circ)\,,\\
\sin 6^\circ &= \sin 6^\circ + \cos 72^\circ - \cos 36^\circ + \frac{1}{2}\,,
\end{aligned}$$

and the last equality follows immediately from the equations (1) and (2) just as in part (a).

*Both parts were also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE*

*CAMPBELL, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; JAN VERSTER, Kwantlen University College, BC; and TITU ZVONARU, Comănești, Romania. STAN WAGON, Macalester College, St. Paul, MN, USA gave a computer verification.*

*Part (a) was also solved by ARKADY ALT, San Jose, CA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

*Wagon used* Mathematica *to confirm that there are seven* $4$*-tuples* $(a, b, c, d)$ *of distinct integers between* $0$ *and* $90$ *(other than the pair featured in our problem) that satisfy the relation* $\tan a^\circ = \tan b^\circ + \tan c^\circ + \tan d^\circ$*, namely*

$$(60; 42, 36, 6),\quad (72; 60, 42, 24),\quad (78; 66, 60, 36),\quad (78; 72, 42, 36)$$
$$(60; 50, 20, 10),\quad (70; 60, 40, 10),\quad (80; 70, 60, 50).$$

*The first four are clearly related to the golden section as in our featured pair, while the final three seem to be related to the regular enneagon (or nonagon, if you prefer) as discussed in "Trigonometry and the Nonagon" by Andrew Jobbings (see www.arbelos.co.uk/papers.html). It is amusing to note that the proposer thought that he had found one that fails to fit either of the two patterns, but it turns out that* $\tan 62^\circ$ *differs from* $\tan 48^\circ + \tan 24^\circ + \tan 18^\circ$ *by about* $10^{-5}$*. Wagon further produced a list of* $49$ *such equations allowing repeated angles, and determined that there were no such* $3$*-term equations and no such* $5$*-term equations.*

**3497**. [2009 : 516, 519] *Proposed by Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina.*

Let $P$ be a point in the interior of triangle $ABC$, and let $r$ be the inradius of $ABC$. Prove that $\max\{AP, BP, CP\} \geq 2r$.

**I.** *Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Recall that the convex hull of a triangle $T$ is the union of its interior and boundary. If $C$ is a circle with radius $r$ in the convex hull of a triangle $T_1$ with inradius $r_1$, then $r \leq r_1$. (Here is a proof of this simple fact: Consider the three tangents to $C$ that are parallel to the sides of $T_1$ and separate the centre of $C$ from the corresponding sides; they form a triangle that is similar to $T_1$ for which $C$ is the incircle. Since all points of $C$ are inside or on $T_1$, the ratio of the sides of the new triangle to the sides of $T_1$—which is also the ratio of the inradii—could be at most $1$; that is, $r \leq r_1$.) Let $T = \triangle ABC$ be an arbitrary triangle with incircle $C$ and inradius $r$, and let $P$ be a point in the convex hull of $T$. Without loss of generality, we may assume that max$\{AP$, $BP$, $CP\} = AP$ and show that $AP \geq 2r$. Extend (if necessary) the segments $PB$ to $PB_1$ and $PC$ to $PC_1$ such that $PB_1 = PC_1 = PA$. Then $P$ is the circumcentre of triangle $T_1 = \triangle AB_1C_1$, and $PA$ its circumradius; let $r_1$ denote its inradius. Note that because $P$ is assumed to lie in the convex hull of $T$, $T$ must lie in the convex hull of $T_1$; consequently the incircle of $T$ also lies in that convex hull, so that (from our simple fact)

$$r_1 \geq r\,.$$

By Euler's inequality, $AP \geq 2r_1$, whence $AP \geq 2r$, as desired.

**II.** *Solution by Michel Bataille, Rouen, France.*

Generalization: The following result holds for any point $P$ in the plane of $\triangle ABC$. Let $R$, $O$, $a$, $b$, and $c$ be the circumradius, circumcentre, and sides of $\triangle ABC$, and let $M = \max\{AP, BP, CP\}$; then

(a) *if $\triangle ABC$ is acute, $M \geq R \geq 2r$, with $M = 2r$ if and only if $P = O$ and the triangle is equilateral*;

(b) *if $\triangle ABC$ is not acute, $M \geq \dfrac{\max\{a,b,c\}}{2} \geq 2r$, with $M = 2r$ if and only if $P$ is the midpoint of the longest side.*

Let $A'$, $B'$, and $C'$ be the midpoints of the sides opposite vertices $A$, $B$, and $C$, respectively. For part (a) we fix points $D$, $E$, and $F$ on the perpendicular bisectors of the sides so that the rays $[OD)$, $[OE)$, and $[OF)$ are opposite the rays $[OA')$, $[OB')$, and $[OC')$, respectively. The whole plane is the union of the nonoverlapping angles $\angle EOF$, $\angle FOD$, and $\angle DOE$. Without loss of generality we can assume that $P$ is in or on the sides of angle $\angle EOF$ (bounded by the rays $[OE)$ and $[OF)$) so that $M = PA$. Let $E_0$ on $AB$ and $F_0$ on $AC$ be such that $OE_0 || AC$ and $OF_0 || AB$. Note that because $O$ is in the interior of $\triangle ABC$, $E_0$ and $F_0$ belong to the rays $[AB)$ and $[AC)$, while $OE_0 \perp OE$ and $OF_0 \perp OF$. Since $A$ is in the interior of $\angle E_0OF_0$, the angle $\angle POA$ is obtuse, hence $M = PA \geq OA = R$, with equality exactly when $P = O$. The inequality $R \geq 2r$ is Euler's inequality, with $R = 2r$ exactly when $\triangle ABC$ is equilateral, so the proof of part (a) is complete.

For part (b) we first suppose that $\angle BAC$, say, is obtuse. Then $O$ is exterior to $\triangle ABC$ with line $BC$ separating $O$ from $A$, and the plane is the union of the three angles $\angle EOF$, $\angle EOA'$, and $\angle FOA'$. If $P$ is in $\angle EOF$ then $M = PA \geq R > \frac{a}{2}$ (much as in part (a)). Otherwise, without loss of generality, we can suppose that $P$ is in $\angle EOA'$, in which case $M = PC \geq A'C = \frac{a}{2}$. To check that the minimum value of $M$, namely $\frac{a}{2}$, occurs when $P = A'$, note that $A$ and $A'$ are on the same side of the perpendicular bisector of the segment $AC$, so that $A'A < A'C$; that is, if $P = A'$, then $M = A'C = A'B = \frac{a}{2}$. If $\angle BAC = 90^\circ$, this argument can easily be adapted to show that $M \geq \dfrac{a}{2} = R$. To complete the proof we show that in the present case we have $\dfrac{a}{2} \geq 2r$. Let $h = AH$ be the altitude from $A$, and let $A_0$ be the point on the ray $[HA)$ such that $\angle BA_0C = 90^\circ$. We want to show that $ah \geq 4rh$; that is, that $a+b+c \geq 4h$ (since $\dfrac{ah}{2} = \dfrac{r(a+b+c)}{2} =$ area$(\triangle ABC)$). But

$$h \leq HA_0 = \sqrt{HB \cdot HC} \leq \frac{HB + HC}{2} = \frac{a}{2},$$

whence $a \geq 2h$; moreover, $b$, $c \geq h$, so that $a + b + c \geq 4h$, as desired.

*Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo,* *Bosnia and Herzegovina; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; VICTOR PAMBUCCIAN, Arizona State University West, Phoenix, AZ, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; GEORGE TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There were two incomplete submissions.*

*Tsintsifas extended the result to $n$-dimensional Euclidean space: For a point $P$ in the interior of the simplex $A_1A_2\dots A_{n+1}$, $\max\{A_1P, A_2P, \dots, A_{n+1}P\} \geq nr$.*

*Janous pointed out that the inequality follows from the more general assertion that $AP + BP + CP \geq 6r$, which is item 12.14 of* O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

**3498**. [2009 : 517, 519] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $F_n$ be the $n^{\text{th}}$ Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For each positive integer $n$, prove that

$$\sqrt{\frac{F_{n+3}}{F_n}} + \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}} > 1 + 2\left(\sqrt{\frac{F_n}{F_{n+3}}} + \sqrt{\frac{F_{n+1}}{F_n + F_{n+2}}}\right).$$

*Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

Let $x = \sqrt{\frac{F_{n+3}}{F_n}}$ and $y = \sqrt{\frac{F_n + F_{n+2}}{F_{n+1}}}$. The claimed inequality is successively equivalent to

$$\begin{aligned} x + y &> 1 + 2\left(\frac{1}{x} + \frac{1}{y}\right), \\ \left(1 - \frac{2}{xy}\right)(x+y) &> 1. \end{aligned}$$

It thus suffices to show that the following two inequalities hold:

$$1 - \frac{2}{xy} \geq \frac{1}{3}, \tag{1}$$

$$x + y > 3. \tag{2}$$

Set $\lambda = \frac{F_{n+1}}{F_n}$. Then

$$\begin{aligned} xy &= \sqrt{\frac{F_{n+3}}{F_n} \cdot \frac{F_{n+2} + F_n}{F_{n+1}}} \\ &= \sqrt{\frac{(2F_{n+1} + F_n)(F_{n+1} + 2F_n)}{F_nF_{n+1}}} \\ &= \sqrt{(2\lambda + 1)\left(1 + \frac{2}{\lambda}\right)}. \end{aligned}$$

Hence, (1) is equivalent to each of

$$\begin{aligned}\sqrt{(2\lambda+1)\left(1+\frac{2}{\lambda}\right)} &\geq 3\,,\\ \frac{2\,(\lambda-1)^2}{\lambda} &\geq 0\,,\end{aligned}$$

and the latter is clearly true.

By the AM–GM Inequality,

$$\begin{aligned}x+y &> 2\cdot\sqrt[4]{\frac{F_{n+3}\,(F_n+F_{n+2})}{F_nF_{n+1}}}\\ &= 2\cdot\sqrt[4]{(2\lambda+1)\left(1+\frac{2}{\lambda}\right)}\,.\end{aligned}$$

For (2), it thus suffices to show that

$$(2\lambda+1)\left(1+\frac{2}{\lambda}\right) > \frac{81}{16}\,,$$

which is equivalent to

$$\frac{32\lambda^2-\lambda+32}{16\lambda} > 0\,,$$

which is clearly true.

*Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. Two incomplete solutions were submitted.*

**3499★**. [2009 : 517, 519] *Proposed by Bernardo Recamán, Instituto Alberto Merani, Bogotá, Colombia.*

A building has $n$ floors numbered $1$ to $n$ and a number of elevators all of which stop at both floors $1$ and $n$, and possibly other floors. For each $n$, find the least number of elevators needed in this building if between any two floors there is at least one elevator that connects them non-stop.

For example, if $n = 6$, nine elevators suffice: $(1,6)$, $(1,5,6)$, $(1,4,6)$, $(1,3,4,6)$, $(1,2,4,5,6)$, $(1,2,5,6)$, $(1,2,6)$, $(1,3,5,6)$, and $(1,2,3,6)$.

*Solution by George Apostolopoulos, Messolonghi, Greece.*

The answer is $\left\lfloor\dfrac{n^2}{4}\right\rfloor$.

To see that at least this many elevators are needed, consider the set

$$P = \left\{(x,y) \in \mathbb{Z}^2 : 1 \le x, y \le n,\ x \le \frac{n}{2},\ y > \frac{n}{2}\right\}.$$

Any elevator can connect at most one pair of floors in the set $P$, and the cardinality of $P$ is $\left\lfloor \frac{n^2}{4} \right\rfloor$, so at least this many elevators are needed.

To show that $\left\lfloor \frac{n^2}{4} \right\rfloor$ elevators suffice, we give a construction in two cases.

**Case 1:** $n = 2k$. Here $k^2$ elevators are needed. Let integers $i$ and $j$ be restricted so that $1 \le i \le k$ and $k+1 \le j \le 2k$, and describe each elevator by the tuple of floors it stops at. The elevators are then

$$\begin{cases} (1, i, j, 2k)\,, & \text{if } i + j = 2k+1\,, \\ (1, 2k+1-j, i, j, 2k)\,, & \text{if } i + j > 2k+1\,, \\ (1, i, j, 2k+1-j, 2k)\,, & \text{if } i + j < 2k+1\,. \end{cases}$$

**Case 2:** $n = 2k+1$. Here $k^2 + k$ elevators are needed. Let integers $i$ and $j$ be restricted so that $1 \le i \le k$ and $k+1 \le j \le 2k+1$, and describe each elevator by the tuple of floors it stops at. The elevators are then

$$\begin{cases} (1, i, j, 2k+1)\,, & \text{if } i + j = 2k+2\,, \\ (1, 2k+2-j, i, j, 2k+1)\,, & \text{if } i + j > 2k+2\,, \\ (1, i, j, 2k+2-j, 2k+1)\,, & \text{if } i + j < 2k+2\,. \end{cases}$$

This completes the proof.

*Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; D.P. MEHENDALE (Dept. of Electronics) and M.R. MODAK, (formerly of Dept. Mathematics), S. P. College, Pune, India; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MORTEN H. NIELSEN, University of Winnipeg, Winnipeg, MB; and PETER Y. WOO, Biola University, La Mirada, CA, USA. Two incomplete solutions were submitted.*

# YEAR END FINALE

As a preliminary we refer readers to the notice on p. 486 of this issue regarding the status of ***CRUX with MAYHEM***.

My term as Editor-in-Chief of ***CRUX with MAYHEM*** officially ended last year, but I have continued a little while longer so that we could bring you this last issue of volume 36. The backlog of articles has now been cleared, and the journal is ready for a new editor and a new team.

After careful consideration of my other duties and personal obligations (and partly also due to the difficulties in finding a new editor), I have decided to step down shortly after the completion of this volume.

I thank the Managing Editor, JOHAN RUDNICK, and the CMS Publications Committee Chair, KEN DAVIDSON, for their service this past year. I wish them both the best of luck and success in nurturing ***CRUX*** and having it grow in the future.

I thank the staff at the CMS head office in Ottawa for administering ***CRUX*** and for their perseverance in the face of having to relocate their offices so many times. In particular I thank DENIS AKOULOV, LAURA ALYEA, DENISE CHARRON, and STEVE LA ROCQUE. Denis has taken over managing subscriptions from Laura, Denise deals with matters related to publishing, and Steve has been speedily putting the issues up on the web. Their work is much appreciated!

I thank JOANNE CANAPE at the University of Calgary, for providing decades of help in preparing the *Olympiad Corner*, and LOUIS MASTORAKOS at Wilfrid Laurier University, for help with preparing the ***CRUX*** solutions. A big thank you goes to MATHIAS PIELAHN, the ***CRUX*** journal assistant, for efficiently processing large chunks of the correspondence we have received over the last two years.

I thank TAMI EHRLICH and the folks at Thistle Printing for adding their magic to the camera-ready PDFs that I send to them. The Pandora font sits bold and heavy, set among acres of white and purple covers, like a pitch-black bull in a field of clover.

I thank past ***CRUX*** editor BRUCE SHAWYER for his kindness and help over the years, and past ***CRUX*** editor BILL SANDS, who is a fountain of knowledge and one of the sharpest proof readers I know. My colleague TERRY VISENTIN has also helped proof the copy.

I thank JEAN-MARC TERRIER for providing French translations, and for his uplifting emails. I also thank ROLLAND GAUDET for providing French translations, and for his very fast turn-around time.

The end of 2010 has seen many board members completing their terms, and some new faces coming on board.

DZUNG MINH HA of Ryerson University completed his term as Problems Editor, and I thank him for his precision and strict moderation of the problems. JONATAN ARONSSON of the University of Manitoba also completed a terms as Problems Editor, and I thank him for bringing his enthusiasm for problem solving to the board. IAN VANDERBURGH stepped down as ***Mayhem*** Editor, and it was a pleasure working with him these past three years. (In that regard, I thank SHAWN GODIN for his help with moderating some Mayhem Problems.) ROBERT WOODROW has completed his term as *Olympiad Corner*, and I thank him for an incredible 30 years of support for ***CRUX*** and the *Corner*. My colleague JAMES CURRIE completed his term as Articles Editor and I thank him for keeping that section organized and ***CRUX*** well stocked with material these last three years.

I welcome CHRIS GRANDISON of Ryerson University and ROB CRAIGEN of the University of Manitoba on board as Problems Editors in 2011.

I thank LILY YEN and MOGENS LEMVIG HANSEN for continuing as ***Skoliad*** Editors and for the marvellous job that they do. I thank JEFF HOOPER, for continuing as Associate Editor, and for his sound advice over my term as Editor-in-Chief. I thank ~~EDWARD WANG for continuing as Problems Editor this past year despite having~~ retired, and for his wealth of experience and his good advice. NICOLAE STRUNGARU has done a great job as Problems Editor since coming on board two years ago, and I admire his skills at solving the problems!

I thank AMAR SODHI for keeping the book reviews section in good order and for his light-hearted sense of humour, which often was the perfect antidote for the stress of editing.

I thank CHRIS FISHER for his prodigious output of high quality contributions to ***CRUX with MAYHEM*** which far exceed his duties as Problems Editor, and for his constant support these past three years.

The Department of Mathematics and Statistics at the University of Winnipeg made it possible to host ***CRUX with MAYHEM*** here in Winnipeg the last three years, and in that regard I thank our Dean of Science, ROD HANLEY, for the continued commitment of the University of Winnipeg.

I thank my wife CHARLENE for her support during this past year, for her very substantial assistance in putting together this last issue, and her great proof reading!

I thank PETER ARPIN for his help with moderating problems during my term.

Most importantly, I sincerely thank all of the readers and wonderful people I have corresponded with these last three years. The constellation of ***CRUX*** is strewn with stars, and I am happy to have seen it on a clear night. My only regret is that I have been late with the issues these past three years, and my only hope is that it was worth the wait.

I wish all of you joy, tranquillity, and the realization of your hopes and aspirations in the New Year.

Václav (Vazz) Linek


# Crux Mathematicorum
## with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek

## Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
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## Mathematical Mayhem

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Former Editors / Anciens Rédacteurs: Philip Jong, Jeff Higham, J.P. Grossman,
Andre Chang, Naoki Sato, Cyrus Hsia, Shawn Godin, Jeff Hooper, Ian VanderBurgh

# INDEX TO VOLUME 36, 2010

## Proposers and solvers appearing in the SOLUTIONS section in 2010:

### Proposers

**Anonymous Proposer** 3525, 3566
**Yakub N. Aliyev** 3505, 3518
**Arkady Alt** 3556, 3570, 3571, 3585
**G.W. Indika Amarasinghe** 3590
**Šefket Arslanagić** 3584, 3592
**Vahagn Aslanyan** 3555, 3562
**Ricardo Barroso Campos** 3520
**Michel Bataille** 3514, 3529, 3532, 3545, 3546, 3553, 3574, 3575, 3591, 3594
**Mihály Bencze** 3534, 3561
**K.S. Bhanu** 3531
**János Bodnár** 3516
**Paul Bracken** 3500
**N. Javier Buitrago Aza** 3552
**Cao Minh Quang** 3526, 3533
**Shai Covo** 3586
**M.N. Deshpande** 3531
**Max Diaz** 3565
**José Luis Díaz-Barrero** 3502, 3515, 3539, 3547, 3572
**A.A. Dzhumadil'daeva** 3573
**Ovidiu Furdui** 3512, 3530, 3550, 3551, 3578, 3580, 3600
**Samuel Gómez Moreno** 3536
**Johan Gunardi** 3558, 3597
**Ignotus** 3587
**Walther Janous** 3535
**Hung Pham Kim** 3508, 3509, 3527, 3549
**Hiroshi Kinoshita** 3528
**Mikhail Kochetov** 3563
**Václav Konečný** 3517, 3589
**Panagiote Ligouras** 3582
**Jian Liu** 3569
**Thanos Magkos** 3559
**Dorin Mărghidanu** 3521, 3522
**Marian Marinescu** 3537
**Dragoljub Milošević** 3588
**Cristinel Mortici** 3599
**Nguyen Duy Khanh** 3519
**Victor Oxman** 3538
**Pedro Henrique O. Pantoja** 3506
**Paolo Perfetti** 3557, 3583, 3596
**Pham Huu Duc** 3507, 3554
**Pham Van Thuan** 3511, 3548, 3560, 3564
**Cosmin Pohoaţă** 3510, 3542
**Pantelimon George Popescu** 3539
**Mariia Rozhkova** 3504
**Josep Rubió-Massegú** 3515
**Sergey Sadov** 3563
**Mehmet Şahin** 3543, 3544, 3576, 3577
**Bill Sands** 3595
**Hassan A. ShahAli** 3501, 3513
**Bruce Shawyer** 3503
**Slavko Simic** 3523
**D.J. Smeenk** 3540, 3541
**Zhi-min Song** 3581
**Albert Stadler** 3567, 3568
**Daryl Tingley** 3593
**Peter Y. Woo** 3579
**Li Yin** 3581
**Katsuhiro Yokota** 3528
**Zhang Yun** 3598
**Faruk Zejnulahi** 3592
**Titu Zvonaru** 3524

### Featured Solvers — Individuals

**Anonymous Solver** 3449
**Arkady Alt** 3421, 3423, 3443, 3450, 3459, 3462, 3470
**Miguel Amengual Covas** 3478
**George Apostolopoulos** 3410, 3453, 3454, 3457, 3460, 3465, 3479, 3487, 3489, 3490(b), 3491, 3495, 3499
**Alberto Arenas Gómez** 3406
**Šefket Arslanagić** 3432, TOTTEN-07
**Roy Barbara** 3416, 3428, 3452, 3497
**Edward J. Barbeau** 3477
**Michel Bataille** 3403, 3406, 3408, 3410, 3414, 3418, 3429, 3446, TOTTEN-05, 3473, 3481, 3482, 3494
**Cao Minh Quang** 3420, 3454
**Chip Curtis** 3409, 3411, TOTTEN-02, TOTTEN-09, 3473, 3498
**Paul Deiermann** 3430
**Charles R. Diminnie** 3431
**Dung Nguyen Manh** 3402, 3412, 3422, 3448, 3450
**J. Chris Fisher** TOTTEN-05
**Ovidiu Furdui** TOTTEN-03
**Francisco Javier García Capitán** 3467
**Oliver Geupel** 3404, 3405, 3413, 3419(a), 3420, 3424, 3427, 3428, 3429, 3433, 3444, TOTTEN-04, TOTTEN-06, TOTTEN-10, TOTTEN-12, 3468, 3481, 3483, 3484, 3490(a)
**John Hawkins** 3440
**Richard I. Hess** 3407, 3458
**John G. Heuver** TOTTEN-01, 3439, 3464
**Joe Howard** 3402, 3453, 3475
**Peter Hurthig** 3445
**Salvatore Ingala** 3477
**Walther Janous** TOTTEN-07
**Václav Konečný** 3434
**Kee-Wai Lau** 3410, 3455, 3469, 3496
**Tom Leong** 3458
**Thanos Magkos** 3448, 3450, 3462, 3466
**Cristinel Mortici** 3472
**Paolo Perfetti** 3486, 3488, 3495
**Joel Schlosberg** 3417, 3430, 3463, 3468, 3469
**Harry Sedinger** 3426
**D.J. Smeenk** 3496
**Albert Stadler** 3410, 3425, 3435, 3438, 3442, TOTTEN-11, 3451, 3458, 3489
**David Stone** 3440
**Edmund Swylan** 3434, TOTTEN-08, 3471
**Panos E. Tsaoussoglou** 3450
**Vo Quoc Ba Can** 3419(a)
**Peter Y. Woo** 3401, 3410, 3456, 3476
**Titu Zvonaru** 3437

### Featured Solvers — Groups

**Missouri State University Problem Solving Group** 3436, 3441
**Hunedoara Problem Solving Group** 3447

### Other Solvers — Individuals

**Anonymous Solver** 3433
**Mohammed Aassila** 3459
**Zafar Ahmed** 3459
**Yakub N. Aliyev** 3424
**Arkady Alt** 3406, 3410, 3415, 3416, 3417, 3420, 3422, 3436, 3444, 3445, 3446, 3447, TOTTEN-04, TOTTEN-10, TOTTEN-11(a), TOTTEN-12, 3451, 3452, 3453, 3454, 3457, 3460, 3461, 3469, 3471, 3473(a), 3478, 3479, 3480, 3483, 3485, 3489, 3491, 3496(a), 3497, 3498
**Miguel Amengual Covas** 3411, 3436, 3439, 3475
**George Apostolopoulos** 3402, 3407, 3411, 3412, 3413, 3415, 3422, 3423, 3426, 3427, 3434, 3435, 3436, 3439, 3440, 3443, 3444, 3445, 3446, 3447, 3450, TOTTEN-04, TOTTEN-05, TOTTEN-08, TOTTEN-11, TOTTEN-12, 3451, 3452, 3454, 3456, 3458, 3459, 3461, 3462, 3463, 3464, 3465, 3467, 3469, 3470, 3472, 3473, 3475, 3476, 3478, 3481, 3485, 3494, 3496, 3497, 3498
**Michele Arnold** 3452
**Šefket Arslanagić** 3402, 3407, 3411, 3412, 3420, 3421, 3422, 3426, 3435, 3436, 3439, 3443, 3444, 3445, 3446, 3450, TOTTEN-08, TOTTEN-11(a), TOTTEN-12, 3453, 3454, 3460, 3461, 3464, 3469, 3473(a), 3478, 3485, 3495, 3497
**Matthew Babbitt** 3402, 3407
**Dionne T. Bailey** 3402, 3452, 3478, 3489, 3496
**Roy Barbara** 3402, 3406, 3407, 3411, 3420, 3422, 3426, 3427, 3430, 3432, 3434, 3435, 3436, 3440, 3441, 3446, 3447, TOTTEN-04, TOTTEN-10, TOTTEN-11, 3468, 3469, 3473(a), 3475, 3477, 3478, 3479, 3496
**Edward J. Barbeau** 3479
**Cătălin Barbu** 3471
**Ricardo Barroso Campos** 3402, 3403, 3426, 3429, 3436, 3439, 3463

**Michel Bataille** 3401, 3402, 3404, 3405, 3407, 3409, 3411, 3415, 3416, 3417, 3420, 3424, 3426, 3427, 3430, 3431, 3432, 3433, 3434, 3435, 3436, 3439, 3440, 3444, 3445, 3447, 3448, 3450, TOTTEN-01, TOTTEN-02, TOTTEN-04, TOTTEN-08, TOTTEN-10, TOTTEN-11(a), 3451, 3452, 3453, 3454, 3455, 3456, 3457, 3458, 3460, 3463, 3464, 3467, 3469, 3470, 3471, 3472, 3475, 3476, 3477, 3478, 3479, 3480, 3483, 3485, 3489, 3496, 3498
**Jesi Bayless** 3452
**Brian D. Beasley** 3416, 3426, 3478, 3479, 3498
**Francisco Bellot Rosado** 3439
**Mihály Bencze** 3402, 3411, 3420, 3446, 3447, TOTTEN-12
**Mihaela Blanariu** 3469, 3470
**Paul Bracken** 3420, 3422, 3433, TOTTEN-04, TOTTEN-07, 3451, 3473(a), 3478, 3485
**Scott Brown** 3452, 3453, 3460
**Elias C. Buissant des Amorie** 3496(a)
**Elsie M. Campbell** 3402, 3452, 3478, 3489, 3496
**Cao Minh Quang** 3402, 3406, 3410, 3412, 3415, 3421, 3422, 3423, 3436, 3443, 3444, 3445, 3478, 3483
**Bao Changjin** 3479
**Chip Curtis** 3402, 3406, 3407, 3408, 3410, 3412, 3415, 3416, 3418, 3420, 3421, 3422, 3426, 3428, 3429, 3431, 3435, 3436, 3438, 3440, 3445, 3446, 3447, 3450, TOTTEN-08, TOTTEN-11(a), 3451, 3452, 3453, 3454, 3455, 3458, 3460, 3461, 3463, 3464, 3475, 3477, 3478, 3479, 3481, 3482, 3485, 3487, 3489, 3490(b), 3491, 3496
**Paul Deiermann** 3416
**Calvin Deng** 3479
**Joseph DeVincentis** 3468
**José Luis Díaz-Barrero** 3406, 3409, 3418, 3427, 3448, 3451, 3465, 3482, 3489, 3498
**Charles R. Diminnie** 3402, 3417, 3420, 3426, 3430, 3431, 3452, 3477, 3478, 3489, 3496, 3498
**Marian Dincă** 3467, 3471
**Dung Nguyen Manh** 3410, 3415, 3421, 3443, 3444, 3445, 3478, 3480
**Keith Ekblaw** 3469
**Aaron Essner** 3478
**Mark Farrenburg** 3478
**Oleh Faynshteyn** TOTTEN-11(a), 3475, 3478, 3481, 3485
**Hidetoshi Fukugawa** 3440
**Ovidiu Furdui** TOTTEN-02, TOTTEN-03, 3465, 3469, 3470
**Francisco Javier García Capitán** 3401, 3410, 3418, TOTTEN-04
**Oliver Geupel** 3401, 3402, 3403, 3406, 3407, 3409, 3410, 3411, 3412, 3414, 3415, 3416, 3417, 3418, 3421, 3422, 3426, 3430, 3431, 3432, 3434, 3435, 3436, 3437, 3439, 3440, 3441, 3442, 3443, 3445, 3446, 3447, 3448, 3450, TOTTEN-01, TOTTEN-02, TOTTEN-03, TOTTEN-05, TOTTEN-08, TOTTEN-09, TOTTEN-11(a), 3451, 3452, 3453, 3454, 3455, 3456, 3457, 3458, 3459, 3460, 3462, 3463, 3464, 3465, 3466, 3467, 3469, 3470, 3471, 3472, 3473, 3475, 3476, 3477, 3478, 3479, 3482, 3485, 3486, 3488, 3489, 3490(b), 3491, 3495, 3496, 3498, 3499
**Douglass L. Grant** 3452
**Miquel Grau-Sánchez** 3406, 3448
**Luke E. Harmon** 3478
**John Hawkins** 3426, 3428, 3436, 3438
**José Hernández Santiago** 3426
**Richard I. Hess** 3402, 3406, 3416, 3420, 3422, 3426, 3435, 3436, 3438, 3440, 3441, 3442, 3452, 3465, 3469, 3470, 3473, 3478
**John G. Heuver** 3402, 3403, 3414, 3436, 3443, 3452, 3463, 3475, 3496
**Richard Hoshino** TOTTEN-08, TOTTEN-09, 3454
**Joe Howard** 3406, 3410, 3425, 3443, 3444, 3445, 3450, TOTTEN-11(a), 3452, 3454, 3473(a), 3478, 3481, 3483, 3485, 3496, 3497
**Peter Hurthig** 3440, 3444
**Salvatore Ingala** 3454
**Bianca-Teodora Iordache** 3480
**Walther Janous** 3402, 3404, 3406, 3407, 3410, 3411, 3412, 3417, 3420, 3422, 3426, 3427, 3430, 3432, 3433, 3435, 3436, 3437, 3440, 3442, 3443, 3444, 3445, 3446, 3447, 3448, 3450, TOTTEN-01, TOTTEN-04, TOTTEN-08, TOTTEN-09, TOTTEN-10, TOTTEN-11(a), TOTTEN-12, 3451, 3452, 3453, 3454, 3455, 3457, 3458, 3460, 3461, 3462, 3463, 3464, 3465, 3467, 3469, 3470, 3471, 3472, 3473(a), 3478, 3479, 3480, 3481, 3482, 3483, 3486, 3488, 3489, 3491, 3495, 3496, 3497, 3498
**Iyoung Michelle Jung** 3442
**Geoffrey A. Kandall** 3475
**Sung Soo Kim** 3442
**Gerhard Kirchner** 3467
**Václav Konečný** 3401, 3402, 3429, 3436, 3440, 3441, 3475, 3476, 3497
**Kee-Wai Lau** 3402, 3406, 3407, 3408, 3411, 3412, 3422, 3426, 3432, 3436, 3443, 3444, 3445, 3447, 3450, 3453, 3454, 3465, 3466, 3472, 3473, 3478, 3481, 3497
**Tuan Le** 3466, 3467
**Tom Leong** TOTTEN-04, TOTTEN-06, TOTTEN-09, 3452, 3457, 3465
**Kathleen E. Lewis** 3407, 3440
**Joshua Long** 3452
**Sotiris Louridas** 3462
**Cezar Lupu** 3415
**Phil McCartney** 3478, 3485
**Thanos Magkos** 3410, 3426, 3436, 3443, 3444, 3445(a), TOTTEN-08, TOTTEN-11(a), 3453, 3454, 3461, 3470, 3471
**Salem Malikić** 3412, 3420, 3421, 3422, 3432, 3436, 3443, 3446, 3450, 3497
**David E. Manes** 3406, 3407, 3412
**Dorin Mărghidanu** 3491
**D.P. Mehendale** 3499
**Georges Melki** 3440
**Dragoljub Milošević** 3435, 3436, 3443, 3445, 3450, 3478, 3485
**M.R. Modak** 3402, 3406, 3407, 3408, 3414, 3415, 3416, 3420, 3464, 3475, 3476, 3477, 3478, 3479, 3483, 3499
**Cristinel Mortici** 3402, 3426, 3435, 3436, 3439, 3440, 3445, 3446, 3447, 3450, 3465, 3467, 3469, 3470, 3471, 3475, 3479, 3480
**Troy Mulholland** 3426
**Morten H. Nielsen** 3499
**José H. Nieto** 3402, 3406, 3407
**Moubinool Omarjee** 3469, 3470
**Victor Pambuccian** 3497
**Pedro Henrique O. Pantoja** 3452, 3467, 3469
**Michael Parmenter** 3426
**Paolo Perfetti** 3354, 3466, 3467, 3469, 3470, 3473, 3478, 3479, 3480, 3489, 3491
**Pham Huu Duc** 3437, 3486, 3488
**Cosmin Pohoaţă** 3495
**Pantelimon George Popescu** 3418, TOTTEN-01
**John Postl** 3452
**Bernardo Recamán** 3468
**Daniel Reisz** 3440
**Juan-Bosco Romero Márquez** 3402, TOTTEN-04, 3469, 3470, 3478
**Xavier Ros** 3465, 3472
**Michael Rozenberg** 3490(b)
**Josep Rubió-Massegú** 3482
**Peter Saltzman** 3468
**Bill Sands** TOTTEN-06
**Joel Schlosberg** 3402, 3406, 3407, 3416, 3420, 3426, 3431, 3433, 3435, 3436, 3439, 3440, 3446, 3447, 3448, TOTTEN-09, 3452, 3464, 3470, 3475, 3478, 3479
**Jonathan Schneider** 3479
**Mosca Sebastiano** 3439
**Bob Serkey** 3402, 3478
**Bruce Shawyer** 3434, 3458
**Slavko Simic** 3408
**Tigran Sloyan** 3401
**D.J. Smeenk** 3403, 3411, 3414, 3439, TOTTEN-10, 3452
**Digby Smith** 3407, 3426
**Albert Stadler** 3401, 3402, 3405, 3406, 3407, 3408, 3411, 3412, 3413, 3415, 3416, 3417, 3418, 3420, 3422, 3423, 3426, 3428, 3430, 3431, 3433, 3434, 3436, 3437, 3440, 3441, 3443, 3444, 3445, 3446, 3447, 3448, 3449, TOTTEN-03, TOTTEN-04, TOTTEN-06, TOTTEN-09, TOTTEN-10, 3452, 3453, 3454, 3457, 3465, 3469, 3470, 3473(a), 3477, 3478, 3479, 3482, 3485, 3486, 3487, 3488, 3494, 3496, 3497, 3498
**David R. Stone** 3426, 3428, 3436, 3438
**Ercole Suppa** 3439
**Edmund Swylan** 3420, 3424, 3426, 3429, 3439, 3440, 3441, 3452, 3458, 3460, 3463, 3475, 3478, 3496, 3497
**Vasile Teodorovici** 3402, 3407, 3452
**Tran Quang Hung** 3460, 3461
**Salvatore Tringali** 3426
**Panos E. Tsaoussoglou** 3440, 3443, 3445, 3452, 3454, 3478, 3481, 3485, 3496(a)
**George Tsintsifas** 3497
**Jan Verster** 3496
**Vo Quoc Ba Can** 3413, 3437
**Stan Wagon** 3443, 3444, 3445(b), 3468, 3487, 3496
**Haohao Wang** 3452, 3478, 3479
**Wei-Dong** 3450
**Luke Westbrook** 3478
**Jerzy Wojdylo** 3452, 3478, 3479
**Peter Y. Woo** 3402, 3403, 3404, 3405, 3407, 3411, 3412, 3414, 3415, 3420, 3421, 3422, 3423, 3439, 3440, 3441, 3445, 3447, 3450, TOTTEN-01, TOTTEN-05, TOTTEN-10, TOTTEN-12, 3452, 3453, 3455, 3459, 3460, 3461, 3463, 3464, 3467, 3469, 3470, 3471, 3475, 3478, 3480, 3481, 3483, 3496(a), 3497, 3499
**Konstantine Zelator** 3402, 3411, 3426, 3436, 3452, 3464, 3475
**Titu Zvonaru** 3402, 3410, 3411, 3414, 3415, 3420, 3422, 3426, 3435, 3436, 3440, 3443, 3444, 3445, 3450, TOTTEN-08, 3459, 3461, 3464, 3471, 3475, 3478, 3479, 3485, 3495, 3496

## Other Solvers — Groups

**Hunedoara Problem Solving Group** 3439, 3440, 3441, 3443, 3444, 3445, 3450
**Missouri State University Problem Solving Group** 3426, 3428, 3431, 3440, 3442, 3499
**Skidmore College Problem Solving Group** 3458, 3478