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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

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# CRUX MATHEMATICORUM

Vol. 11, No. 6

June 1985 ISSN 0705 - 0348

Sponsored by
Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
Publié par le Collège Algonquin, Ottawa

The assistance of the publisher and the support of the Canadian Mathematical Society, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $25 in Canada, $28 (or US$21) elsewhere. Back issues: each $2.50 in Canada, $3.00 (or US$2.50) elsewhere. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-10, $19 in Canada, $20 (or US$15) elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

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## CONTENTS

The Olympiad Corner: 66 . . . . . . . . . . . . . . . . . . . . . . . . M.S. Klamkin 168

Problems - Problèmes: 1045, 1051-1060 . . . . . . . . . . . . . . . . . . . . . . . . 187

Solutions: 478, 859, 931-938 . . . . . . . . . . . . . . . . . . . . . . . . 189

*Nombres* (for the) *Premiers* . . . . . . . . . . . . . . . . . . . . . . . . 200

Mathematical Clerihews . . . . . . . . . . . . . . . . . . . . . . . . Alan Wayne 200

# THE OLYMPIAD CORNER: 66

M.S. KLAMKIN

I begin this month with three new sets of problems: the 1984 Dutch Olympiad, the 1985 Australian Mathematical Olympiad, and the 1985 Bulgarian Mathematical Olympiad. I am grateful to Jan van de Craats, Jim Williams, and Jordan B. Tabov, respectively, for these problems. As usual, I solicit from all readers elegant solutions to all of these problems, with, if possible, extensions or generalizations. Readers submitting solutions should clearly identify the problems by giving their numbers as well as the year and page number of the issue where they appear.

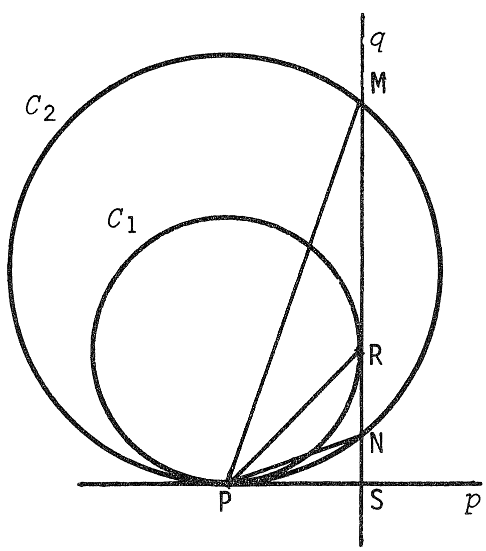
## 1984 DUTCH OLYMPIAD

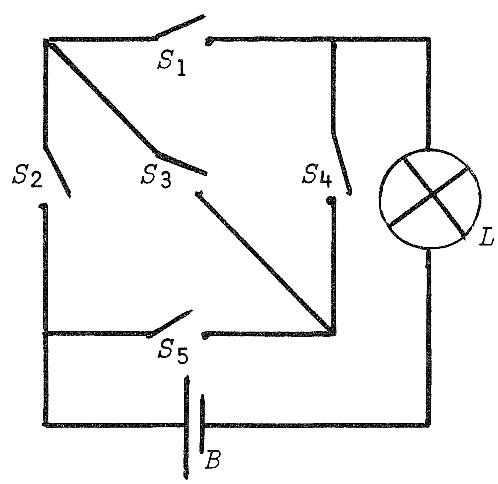
September 14, 1984 - Time: 3 hours

1. Two circles $C_1$ and $C_2$ with radii $r_1$ and $r_2$, respectively, are tangent to the line $p$ at point P. All other points of $C_1$ are inside $C_2$. Line $q$ is perpendicular to $p$ at point S, is tangent to $C_1$ at point R, and intersects $C_2$ at points M and N, with N between R and S, as shown in the figure.

(a) Prove that PR bisects $\angle$MPN.

(b) Compute the ratio $r_1:r_2$ if, moreover, it is given that PN bisects $\angle$RPS.




2. In the given diagram, $B$ is a battery, $L$ is a lamp, and $S_1,S_2,\ldots,S_5$ are switches. The probability that switch $S_3$ is on is 2/3, and it is 1/2 for the other four switches. These probabilities are independent. Compute the probability that the lamp is on.

3. For $n = 1,2,3,\ldots$, let $a_n = 1\cdot4\cdot7\cdot\ldots\cdot(3n-2)/2\cdot5\cdot8\cdot\ldots\cdot(3n-1)$. Prove that, for all $n$,

$$(3n+1)^{-1/2} \le a_n \le (3n+1)^{-1/3}.$$

4. By inserting parentheses in the expression 1:2:3, we get two different numerical values, (1:2):3 = 1/6 and 1:(2:3) = 3/2. Now parentheses are inserted in the expression

$$1:2:3:4:5:6:7:8.$$

(a) What are the maximum and minimum numerical values that can be obtained in this way?

(b) How many different numerical values can be obtained in this way?

*

## THE 1985 AUSTRALIAN MATHEMATICAL OLYMPIAD

Paper I: March 12, 1985 - Time: 4 hours

1. Let $f(n)$ be the sum of the first $n$ terms of the sequence

$$0,1,1,2,2,3,3,4,4,5,5,6,6,\ldots .$$

(a) Give a formula for $f(n)$.

(b) Prove that $f(s+t) - f(s-t) = st$, where $s$ and $t$ are positive integers and $s > t$.

2. If $x,y,z$ are real numbers such that

$$x + y + z = 5 \quad \text{and} \quad yz + zx + xy = 3,$$

prove that $-1 \le z \le 13/3$.

3. Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

Paper II: March 13, 1985 - Time: 4 hours

4. ABC is a triangle whose angles are smaller than 120°. Equilateral triangles AFB, BDC, and CEA are constructed on the sides of and exterior to triangle ABC.

(a) Prove that the lines AD, BE, and CF pass through one point S.

(b) Prove that SD + SE + SF = 2(SA + SB + SC).

5. Find all positive integers $n$ such that

$$n = d_6^2 + d_7^2 - 1,$$

where $1 = d_1 < d_2 < \ldots < d_k = n$ are all positive divisors of the number $n$.

6. Find all polynomials $f(x)$ with real coefficients such that

$$f(x)\cdot f(x+1) = f(x^2+x+1).$$

*

34TH BULGARIAN MATHEMATICAL OLYMPIAD (3rd Stage)

13 and 14 April, 1985

1. If $k$ and $n$ are positive integers, prove that

$$(n^4-1)(n^3-n^2+n-1)^k + (n+1)n^{4k-1}$$

is divisible by $n^5+1$.

2. Determine the values of the parameter $a$ for which the equation

$$\ln 2x \ln 3x = a$$

has two distinct solutions, and find the product of these two solutions.

3. The center of the inscribed sphere and the midpoints of the edges AB and CD of a given tetrahedron ABCD lie on a straight line. Prove that the center of the circumscribed sphere of this tetrahedron lies on the same line.

4. Let $a_n$ and $b_n$ be positive integers satisfying the relation

$$a_n + b_n\sqrt{2} = (2 + \sqrt{2})^n, \quad n = 1,2,3,\ldots .$$

Prove that $\lim_{n\to\infty} (a_n/b_n)$ exists, and find this limit.

5. A triangle ABC of area $S$ is inscribed in a circle $K$ of radius 1. The orthogonal projections of the incenter I of ABC on the sides BC, CA, and AB are $A_1$, $B_1$, and $C_1$, respectively, and $S_1$ is the area of triangle $A_1B_1C_1$. If the line AI meets $K$ in $A_2$, prove that

$$4S_1 = AI\cdot A_2B\cdot S.$$

6. Five given points in the plane have the following property: of any four of them, three are the vertices of an equilateral triangle.

(a) Prove that four of the five points are the vertices of a rhombus with an angle equal to 60°.

(b) Find the number of equilateral triangles having their vertices among the given five points.

*

I now present solutions to some problems published in earlier columns.

4. [1981: 43] *From the* 1980 *regional competition in Mersch, Luxembourg.*

Two circles touch (externally or internally) at the point P. A line touching one of the circles at A cuts the other circle at B and C. Prove that the line PA is one of the bisectors of angle BPC.

*Solution by Andy Liu, University of Alberta.*

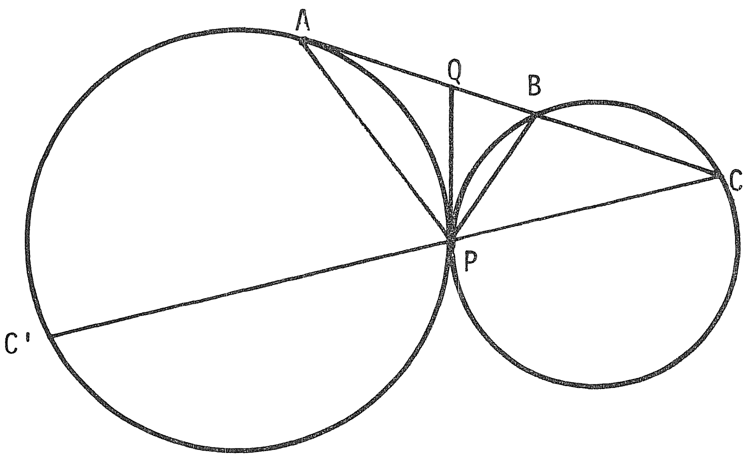



Figure 1

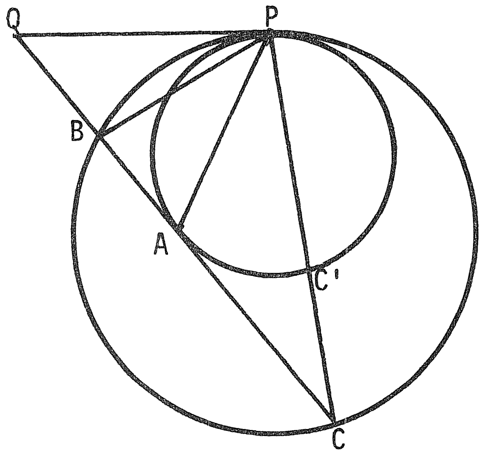



Figure 2

Let the chord CP of one circle, produced if necessary, meet the other circle again in C', and let the common tangent at P meet line AB in Q. Then QA = QP and $\angle QPB = \angle PCB$.

If, as in Figure 1, the circles are externally tangent, then

$$\angle C'PA = \angle PAQ + \angle PCB = \angle APQ + \angle QPB = \angle APB,$$

and PA is the external bisector of $\angle BPC$. If, as in Figure 2, the circles are internally tangent, then

$$\angle C'PA = \angle PAQ - \angle PCB = \angle APQ - \angle QPB = \angle APB,$$

and PA is the internal bisector of $\angle BPC$.

*

3. [1981: 45] *From the* 1978 *Rumanian Mathematical Olympiad.*

Let $m \geq 1$ and $n \geq 1$ be integers such that $\sqrt{7} - (m/n) > 0$. Prove that $\sqrt{7} - (m/n) > 1/(mn)$.

*Solution by M.S.K., Andy Liu, and A. Meir, University of Alberta.*

The desired result is certainly true if $m = 1$, for $\sqrt{7} > 2 \geq 2/n$ for all $n \geq 1$. For $m > 1$, the result will follow from the double implication

$$\sqrt{7} - \frac{m}{n} > 0 \quad \Longrightarrow \quad 7n^2 - m^2 \geq 3 \quad \Longrightarrow \quad \sqrt{7} - \frac{m}{n} > \frac{1}{mn}, \qquad (1)$$

the first of which was given as Rider (a) in Problem 11 [1984: 193].

The inequality on the left in (1) implies that

$$7n^2 - m^2 = 8n^2 - (m^2+n^2) = k$$

for some positive integer $k$. Running through all $m,n \equiv 0,1,2,3 \pmod 4$, we find that

$$k \equiv 0,\ 3,\ 4,\ 6, \text{ or } 7 \pmod 8.$$

Hence $k \geq 3$, and the first implication in (1) is established.

As for the second implication in (1), its conclusion is easily found to be equivalent to

$$7n^2 - m^2 > 2 + \frac{1}{m^2},$$

and this is certainly true since the left side is at least 3 and the right side is less than 3.

*

4. [1981: 46] *From the* 1978 *Rumanian Mathematical Olympiad.*

There are $n$ participants in a chess tournament. Each person plays exactly one game with each of the $n - 1$ others, and each person plays at most one game per day. What is the minimum number of days required to finish the tournament?

*Solution by Andy Liu, University of Alberta.*

*Case 1:* $n$ odd. On the first day, at least one player has the bye. This player must still play each of the other $n$-1 players on different days. Hence at least $n$ days are necessary. We show that $n$ days are sufficient. Label the players from 0 to $n$-1, and have player $x$ take on player $y$ on the $r$th day if and only if $x+y \equiv 2r \pmod n$. It is easy to verify that player $x$ has a unique and distinct opponent each day except on the $x$th day when he has the bye.

*Case 2:* $n$ even. Here $n$-1 days are necessary and sufficient. For the proof, leave out one player and use the ($n$-1)-day schedule of Case 1. Each day the extra player can take on the player who would have had the bye.

*

1. [1981: 72] *From the* 1979 *Moscow Olympiad.*

A point A is chosen in a plane.

(a) Is it possible to draw (i) 5 circles, (ii) 4 circles, none of which cover point A, and such that any ray with endpoint A intersects at least two of the circles?

(b) As a variant, is it possible to draw (i) 7 circles, (ii) 6 circles, none of which cover point A, and such that any ray with endpoint A intersects at least three of the circles?

*Solution by Andy Liu, University of Alberta.*

(a) It is possible with 5 circles but not with 4. To show the former, draw five rays from A such that the angle between each pair of adjacent rays is 72°. For each ray, it is possible to describe a circle with centre on the ray, not covering A but intersecting the two adjacent rays. It is easy to see that these 5 circles have the desired property. To show the latter, let B,C,D,E be the centres of any 4 circles not covering A, and draw the rays AB,AC,AD,AE. We assume that these rays are all distinct (the desired result will be even more obvious if two or more of them coincide) and that they are labelled in such a way that the above listing is in cyclic order with $\angle BAC + \angle CAD \geq 180°$. Then the ray with endpoint A which bisects reflex $\angle BAD$ intersects at most only the circle with centre on AC.

(b) It is possible with 7 circles but not with 6. To show the former, draw seven rays from A such that the angle between each pair of adjacent rays is $(360/7)°$. For each ray, it is possible to describe a circle with centre on the ray, not covering A but intersecting the two adjacent rays. It is easy to see that these 7 circles have the desired property. To show the latter, let B,C,D,E,F,G be the centres of any 6 circles not covering A, and draw the rays AB,AC,AD,AE,AF,AG. We assume that these rays are all distinct (the desired result will be even more obvious if two or more of them coincide) and that they are labelled in such a way that the above listing is in cyclic order with $\angle BAC + \angle CAD + \angle DAE \geq 180°$. Then the ray with endpoint A which bisects reflex $\angle BAE$ intersects at most only the circles with centres on AC and AD. □

A generalization is obvious from the proofs of parts (a) and (b).

*

4. [1981: 73] *From the* 1979 *Moscow Olympiad.*

Karen and Billy play the following game on an infinite checkerboard. They take turns placing markers on the corners of the squares of the board. Karen plays first. After each player's turn (starting with Karen's second turn), the markers placed on the board must lie at the vertices of a convex polygon. The loser is the first player who cannot make such a move. For which player is there a winning strategy?

*Partial solution by Andy Liu, University of Alberta.*

Karen does not have a winning strategy. Let Billy place his first marker on one of the four nearest corners adjacent to Karen's first marker. Now let $\ell$ be the perpendicular bisector of the segment joining the two occupied corners. Note that there are no corners on $\ell$. Thereafter, Billy places his marker symmetrically about $\ell$ to the one Karen has just placed. This ensures that Karen cannot win, though she can make the game last indefinitely if Billy sticks to the above strategy.

Still to be resolved is the question whether there is a winning strategy for Billy.

*

6. [1981: 73] *From the* 1979 *Moscow Olympiad.*

A scientific conference is attended by $k$ chemists and alchemists, of whom the chemists are in the majority. When asked a question, a chemist will always tell the truth, while an alchemist may tell the truth or lie. A visiting mathematician has the task of finding out which of the $k$ members of the conference are chemists and which are alchemists. He must do this by choosing a member of the conference and asking him: "Which is So-and-So, a chemist or an alchemist?" In particular, he can ask a member: "Which are you, a chemist or an alchemist?" Show that the mathematician can accomplish his investigation by asking

(a) $4k$ questions;

(b) $2k - 2$ questions;

(c) $2k - 3$ questions.

(d) [After the Olympiad, it was announced that the minimum number of questions is no greater than $[(3/2)k] - 1$. Prove it.]

*Solution by M. Katchalski and Andy Liu, University of Alberta.*

Since (a), (b), and (c) are all weaker results, it would suffice to prove (d), and we will do so. However, because (d) is probably too difficult for an Olympiad problem, we first prove (c).

(c) We prove by induction that $2k - 3$ questions are sufficient for $k \geq 2$. This is certainly true for $k = 2$. For $k = 3$, let the three persons be $x$, $y$, and $z$. We (the visiting mathematician) ask $x$ if $y$ is a chemist. If $x$ answers Yes, then $y$ is indeed a chemist, as otherwise both $x$ and $y$ would be alchemists, violating the majority condition. If $x$ answers No, then $z$ is a chemist, as otherwise $z$ and either $x$ or $y$ would be alchemists. Having determined a chemist with only one question, we can use the other two to ascertain the true nature of the remaining two persons.

Let $k = 2t$ or $2t + 1$ for $t \geq 2$. Label the persons $a_i$ for $1 \leq i \leq t$ and $b_i$ for $1 \leq i \leq k-t$. For $1 \leq i \leq t$, we ask $a_i$ if $b_i$ is a chemist. Let $h = 0$ if all answer Yes; otherwise, by relabelling if necessary, we assume that the answer is No for $1 \leq i \leq h$ and Yes for $h+1 \leq i \leq t$. This uses up $t$ questions.

For $1 \leq i \leq h$, each pair $\{a_i, b_i\}$ contains at least one alchemist. For $h+1 \leq i \leq t$, if $b_i$ is an alchemist then so is $a_i$. Since the chemists are in the majority overall, they are still in the majority in the set $\{b_i\colon h+1 \leq i \leq k-t\}$. By induction, $2(k-t-h) - 3$ questions are sufficient to determine the true nature of the persons in this set.

We can afford one question for each of the remaining $t+h$ persons. The total number of questions is

$$t + 2(k-t-h) - 3 + t + h = 2k - 3 - h \leq 2k - 3,$$

completing the argument.

(d) We first describe the algorithm, for which a flow chart is easily drawn.

(1) Let $S$ be the set of all chemists and alchemists, and label them from 1 to $k$. The sets $A$, $B$, and $C$ are initially empty. YOU are the visiting mathematician. Continue.

(2) If $|S| \le 2$, then by the majority condition on $S$ everyone in $S$ is a chemist. Proceed to (6). If $|S| > 2$, let $x$ be the person with the lowest label in $S$ and continue.

(3) Let $y$ be the person with the second lowest label in $S$. Ask $y$ if $x$ is a chemist. If the answer is Yes, proceed to (5). Otherwise continue.

(4) If $C = \emptyset$, delete $x$ and $y$ from $S$ and put the pair $(x,y)$ in $A$. Return to (2). If $C \ne \emptyset$, let $z$ be the person with the highest label in $C$. Delete $y$ from $S$, $z$ from $C$, and put the pair $(y,z)$ in $B$. Return to (3).

(5) Delete $y$ from $S$ and put him in $C$. If $|S| - |C| > 2$, return to (3). If $|S| - |C| \le 2$, then by the majority condition on $S \cup C$, $x$ must be a chemist. Ask $x$ about everyone but himself in $S \cup C$ and continue.

(6) Ask a known chemist from (2) or (5) about everyone in $A$. For any pair $(y,z)$ in $B$, there is some $x$ on whose profession (i.e., chemist or alchemist) $y$ and $z$ disagree. Moreover, $x$ is not in $B$ so that his profession is known by now. Thus the profession of one of $y$ and $z$ is known from his statement about $x$. We ask the known chemist about the other person in the pair.

Observe that a pair $(x,y)$ is put in $A$ because $y$ claims that $x$ is an alchemist. A pair $(y,z)$ is put in $B$ because $y$ claims that some $x$ is an alchemist while $z$ claims that the same $x$ is a chemist. Hence no pair in $A$ or $B$ consists of two chemists. The deletion of these pairs does not affect the majority condition on $S \cup C$ which is used in (2) with $C = \emptyset$ and in (5). The algorithm must terminate since $|S|$ decreases steadily.

Let there be, at the end, $a$ pairs in $A$, $b$ pairs in $B$, $c$ persons in $C$, and $s$ persons in $S$. If the algorithm terminates via (2) then $c = 0$ and $s \ge 1$. The number of questions asked in (3) is $a+2b$ while that in (6) is $2a+b$. Since $2a+2b+s = k$, we have

$$(a + 2b) + (2a + b) \le k + \lceil k/2 \rceil - 1 = [(3/2)k] - 1.$$

If the algorithm terminates via (5), then $s-c \ge 1$ so that $c \le \lceil (s+c)/2 \rceil$. The number of questions asked in (3) is $a+2b+c$, that in (5) is $c+s-1$, while that in (6) is $2a+b$. Since $2a+2b+c+s = k$, we have

$$(a+2b+c) + (c+s-1) + (2a+b) \le k + \lceil k/2 \rceil - 1 = [(3/2)k] - 1.$$

*

11. [1981: 74] (Corrected) *From the 1979 Moscow Olympiad.*

The *area of the union* of a set of circles is 1. Show that a subset of these circles may be chosen such that no two of the chosen circles intersect, and such that the sum of the areas of the chosen circles is *no less than* 1/9.

*Solution by Andy Liu, University of Alberta.*

The two italicized phrases in the proposal correct errors in the original formulation of the problem, which may have been due to a faulty translation from the Russian. Problem 13 [1981: 237] is similar but more difficult.

For each circle $C$, let $C'$ be the circle with the same centre but twice the radius. We choose the subset inductively. Let $C_1$ be the largest circle. After $C_1,C_2,\ldots,C_{k-1}$ have been chosen, let $C_k$ be the largest circle with centre outside $C_1' \cup C_2' \cup \ldots \cup C_{k-1}'$. The procedure terminates when no further choices are possible.

We first show that the chosen circles are pairwise disjoint. Let $C_i$ and $C_j$ be two of the chosen circles, with $i < j$. The centre of $C_j$ is outside $C_i'$ and the radius of $C_j$ is less than that of $C_i$. Hence $C_i \cap C_j = \emptyset$.

We now show that the total area of the chosen cirlces is at least 1/9. For each chosen circle $C_i$, let $C_i''$ be the circle with the same centre but three times the radius. Let $C$ be any circle not chosen. Then the centre of $C$ is inside $C_i'$ for some chosen circle $C_i$. Moreover, the radius of $C_i$ is greater than that of $C$. Hence $C$ is contained in $C_i''$. Thus, with the bars denoting area, we have

$$|C_1'' \cup C_2'' \cup \ldots \cup C_k''| > 1$$

and hence

$$|C_1 \cup C_2 \cup \ldots \cup C_k| > \frac{1}{9}.$$

*

1. [1981: 74] *From a selection test for the Rumanian team for the 1978 International Mathematical Olympiad.*

Consider the set $X = \{1,2,3,4,5,6,7,8,9\}$. Show that, for any partition of $X$ into two subsets, one of these contains three numbers such that the sum of two of them equals twice the third.

*Solution by Andy Liu, University of Alberta.*

Our proof is indirect. Suppose, on the contrary, that there are disjoint nonempty subsets $A$ and $B$, with $X = A \cup B$, such that neither $A$ nor $B$ contains numbers $x,y,z$ satisfying $x + z = 2y$. Then clearly $\{1,2,3\}$ is not a subset of $A$ or $B$. We may assume that $1 \in A$.

*Case 1:* Suppose $2 \in A$. Then $3 \in B$. If $4 \in A$, then $6 \in B$ (otherwise $2+6 = 2\cdot 4$ in $A$). For similar reasons, $9 \in A$ and $5 \in B$. Now either $1+7 = 2\cdot 4$ in $A$ or else

$5+7 = 2\cdot6$ in $B$. Hence $4 \in B$. Then $5 \in A$, $8 \in B$, $9 \in B$, and $6 \in A$. Now either $5+7 = 2\cdot6$ in $A$ or $7+9 = 2\cdot8$ in $B$, contradicting our assumption.

*Case 2:* Suppose $2 \in B$. If $3 \in B$, then $4 \in A$, $7 \in B$, $5 \in A$, and $9 \in B$. Now either $4+6 = 2\cdot5$ in $A$ or $3+9 = 2\cdot6$ in $B$. Therefore $3 \in A$. Then $5 \in B$ and $8 \in A$. If $4 \in A$, then $7 \in B$ and either $4+8 = 2\cdot6$ in $A$ or $5+7 = 2\cdot6$ in $B$. Hence $4 \in B$. Now $6 \in A$ and $7 \in B$. Thus either $3+9 = 2\cdot6$ in $A$ or $5+9 = 2\cdot7$ in $B$, and we have the required contradiction. □

Van der Waerden's Theorem states that, for all $k,t > 1$, there exists a least positive integer $w = w(k,t)$ such that, if the set $\{1,2,3,\ldots,w(k,t)\}$ is partitioned in any way into $k$ classes, then at least one class contains numbers in an arithmetic progression of length $t+1$. The above problem asserts that $w(2,2) \le 9$. In fact, it can be shown that $w(2,2) = 9$.

*

3. [1981: 74] *From a selection test for the Rumanian team for the 1978 International Mathematical Olympiad.*

Let $P(x,y)$ be a polynomial in $x,y$ of degree at most 2. Let A, B, C, A', B', C' be six distinct points in the $xy$-plane such that A,B,C are not collinear, A' lies on BC, B' on CA, and C' on AB. Prove that if $P$ vanishes at these six points, then $P \equiv 0$.

*Solution by M.S.K.*

Let $L_1(x,y) = 0$ be the line through B,C,A' and $L_2(x,y) = 0$ the line through C,A,B'. Since a nondegenerate conic intersects a line in at most two points, the conic $P(x,y) = 0$ must be a degenerate one for which

$$P \equiv kL_1L_2,$$

where $k$ is a constant. Now $P$ vanishes at C', but $L_1$ and $L_2$ do not. Hence $k = 0$ and $P \equiv 0$.

*

4. [1981: 74] *From a selection test for the Rumanian team for the 1978 International Mathematical Olympiad.*

Let ABCD be a convex quadrilateral and O the intersection of the diagonals AC and BD. Show that if the triangles OAB, OBC, OCD, and ODA all have the same perimeter, then ABCD is a rhombus. Does this assertion remain true if O is another interior point?

*Solution by M.S.K. and Andy Liu, University of Alberta.*

Let $\partial$ denote perimeter. We have

$$\partial OAB = \partial OBC \implies OA + AB = BC + OC \tag{1}$$

and

$$\partial OCD = \partial ODA \implies OC + CD = DA + OA. \tag{2}$$

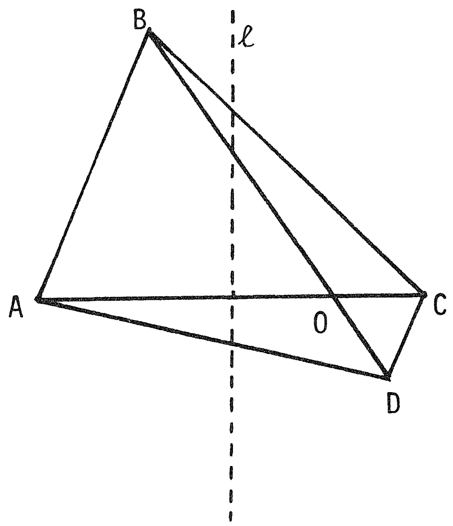



Figure 1

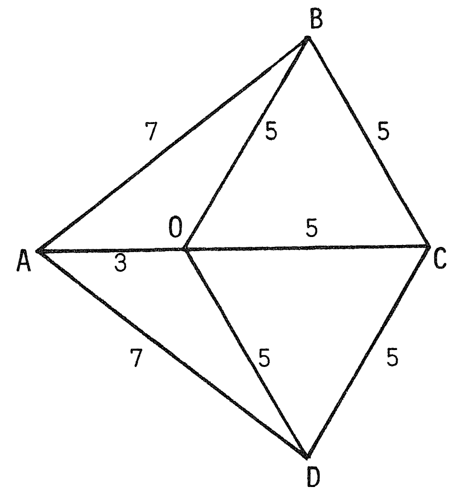



Figure 2

Suppose AB $\neq$ BC. We may assume that the vertices have been labelled so that AB < BC, as shown in Figure 1, and then OC < OA follows from (1). Thus B lies to the left and D to the right of the perpendicular bisector $\ell$ of AC, and so CD < DA. Now OC + CD < OA + DA, contradicting (2). Hence AB = BC, and then adding (1) and (2) shows that CD = DA. Similarly,

$$\partial OBC = \partial OCD \quad \text{and} \quad \partial ODA = \partial OAB \implies BC = CD.$$

Hence AB = BC = CD = DA, and ABCD is a rhombus.

Figure 2 shows that the given assertion does not remain true if O is another interior point.

*

B-5. [1981: 115] *From a recent mathematical competition in Bulgaria.*

$N$ points are given in general position in space (i.e., no four in a plane). There are $\binom{N}{4}$ possible tetrahedra with vertices only at the given points. Prove that, if a plane does not contain any of the $N$ given points, then it can intersect the $\binom{N}{4}$ tetrahedra in at most $N^2(N-2)^2/64$ plane quadrangular cross sections.

*Solution by Andy Liu, University of Alberta.*

A plane will cut a tetrahedron in a quadrangular cross section if and only if two vertices of the tetrahedron lie on each side of the plane. Let $k$ of the $N$ points lie on one side of the given plane and the rest on the other side. Then the number of plane quadrangular cross sections is

$$\binom{k}{2}\binom{N-k}{2} = \frac{1}{4}\{k(N-k)\}\{(k-1)(N-k-1)\} \le \frac{1}{4}\left(\frac{N}{2}\right)^2\left(\frac{N}{2}-1\right)^2 = \frac{N^2(N-2)^2}{64}.$$

Note that Problem 4 [1981: 269] is identical.

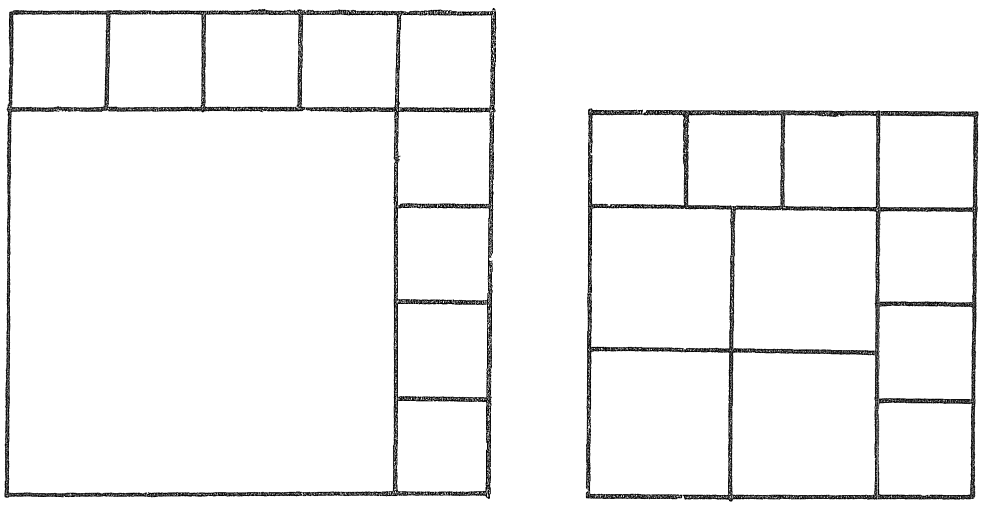
*

7, [1981: 236] *Proposed by France (but unused) at the 1981 I.M.O.*

Determine the smallest natural number $n$ having the property that, for every integer $p$, $p \geq n$, it is possible to subdivide (partition) a given square into $p$ squares (not necessarily equal).

*Solution by Andy Liu, University of Alberta.*

It is easy to see that a square cannot be partitioned into 5 squares because each corner of the given square must belong to a different square in the partition. We show that the smallest value of $n$ is 6. Let $s$ be the side of the given square. If $p \geq 4$ is even, cut out a square of side $(1-\frac{2}{p})s$ at a corner of the given square. The remaining gnomon can now be partitioned into $p$-1 squares each of side $\frac{2s}{p}$. If $p \geq 7$ is odd, cut out a square $A$ of side $(1-\frac{2}{p-3})s$ at a corner of the given square, and then partition $A$ into four equal squares. The remaining gnomon can now be partitioned into $p$-4 squares each of side $\frac{2s}{p-3}$. We illustrate the cases $p = 10$ and $p = 11$.



*

3, [1981: 269] *From the 1981 British Mathematical Olympiad.*

Given that $a,b,c$ are positive numbers, prove that

$$\text{(i) } a^3 + b^3 + c^3 \geq b^2c + c^2a + a^2b;$$

$$\text{(ii) } abc \geq (b+c-a)(c+a-b)(a+b-c).$$

*Comment by M.S.K.*

These are known elementary inequalities. Inequality (i) is proved and generalized in Problem 6-3 [1979: 198], where it is shown, in particular, that

$$x_1^n + x_2^n + \ldots + x_n^n \geq x_1^{n-k}x_2^k + x_2^{n-k}x_3^k + \ldots + x_n^{n-k}x_1^k,$$

where $x_i \geq 0$, $n \geq k \geq 0$, with equality if and only if all the $x_i$ are equal. Inequality (ii) is proved in several ways and generalized in Problem 2 [1984: 46].

*

4. [1981: 269] *From the 1981 British Mathematical Olympiad.*

*Comment by M.S.K.* This is identical to Problem B-5 [1981: 115], which is proved earlier in this column (page 178).

*

3. [1982: 269] *From a 1962 Peking Mathematics Contest.*

A car can carry fuel which will last a distance of $a$. A distance $d > a$ is to be covered with no refueling station in between. However, the car may go back and forth transporting and depositing fuel en route. What is the most economical scheme to get to the destination if $d = 4a/3$? What if $d = 23a/15$?

*Solution by Andy Liu, University of Alberta.*

Let S be the starting point of the journey and D the destination, so that SD = $d$; and let the fuel capacity of the car be $a$ units, so that one unit of fuel is consumed per unit of distance.

(1) *If $d = 4a/3$, then $2a$ units of fuel are sufficient.* Let X be the point between S and D such that SX = $a/3$. The car fills up at S, drives to X, deposits $a/3$ units of fuel, returns to S where it fills up again, returns to X where it picks up the fuel previously deposited, and drives on to D.

(2) *If $d = 4a/3$, then $2a$ units of fuel are necessary.* Let $b$ be the minimum number of units of fuel required, so that $b \leq 2a$ by (1). Since clearly $b > a$, there is a point X between S and D such that the distance $y$ = SX is covered three times while the distance $x$ = XD is covered just once. We have $x \leq a$ and $3y+x = b$. Now

$$3(x+y) = (3y+x) + 2x \leq b + 2a \leq 4a,$$

so $x+y \leq 4a/3$. Since $x+y = 4a/3$, we must have $b = 2a$.

(3) *If $d = 23a/15$, then $3a$ units of fuel are sufficient.* Let Y be a point between S and D such that SY = $a/5$. The car fills up at S, drives to Y where it deposits $3a/5$ units of fuel, returns to S where it fills up again, returns to Y where it deposits $3a/5$ units of fuel, returns to S where it fills up again, and returns for the last time to Y. At this stage, $2a$ units of fuel are available at Y, and the distance remaining to be covered is YD = $4a/3$. The rest of the journey can therefore be done as in part (1).

(4) *If $d = 23a/15$, then $3a$ units of fuel are necessary.* Let $b$ be the minimum number of units of fuel needed, so that $b \leq 3a$ by part (3). Since clearly $b > 2a$, there exist points Y and X between S and D such that the distance $z$ = SY is covered five times, the distance $y$ = YX is covered three times, and the distance $x$ = XD is

covered just once. We have $x \le a$ and $5z + 3y + x = b$. Moreover, $x + y \le 4a/3$ by part (2). Now

$$5(x+y+z) = (5z+3y+x) + 2(x+y) + 2x \le b + 2\cdot\frac{4a}{3} + 2a \le \frac{23a}{3},$$

so $x+y+z \le 23a/15$. Since $x+y+z = 23a/15$, we must have $b = 3a$.

*

4. [1982: 269] *From a 1962 Peking Mathematics Contest.*

A group of children forms a circle and each child starts with an even number of pieces of candy. Each child then gives half of what he has to his right-hand neighbour. After the transaction, if a child has an odd number of pieces, he or she will receive an extra piece from an external source. Show that, after a finite number of such steps, each child will have the same number of pieces of candy.

*Solution by Andy Liu, University of Alberta.*

Suppose that, at the start of a round of transfers, $2m$ and $2n$ are respectively the greatest and least numbers of pieces of candy that any child has. We claim that at the end of the round

(a) no child has more than $2m$ pieces;

(b) no child has fewer than $2n$ pieces;

(c) a child who started with more than $2n$ pieces still has more than $2n$ pieces;

(d) at least one child who started with $2n$ pieces has more than $2n$ pieces if $m > n$.

It follows from these claims that after a finite number of rounds the greatest and least values will coincide. It remains to justify the claims.

(a) Suppose a child started a round with $2h$ pieces and his left-hand neighbour with $2k$ pieces. Then after the round this child has $h+k$ pieces if $h+k$ is even and $h+k+1$ pieces if $h+k$ is odd. Since $h \le m$, $k \le m$, and $2m$ is even, this child cannot have more than $2m$ pieces.

(b) As in (a), we have $h \ge n$ and $k \ge n$, so $h+k+1 > h+k \ge 2n$.

(c) As in (a), we have $h > n$ and $k \ge n$, so $h+k+1 > h+k > 2n$.

(d) If $m > n$, at least one child who started with $2n$ pieces has a left-hand neighbour who started with $2k$ pieces for some $k > n$. Then $n+k+1 > n+k > 2n$.

*

1. [1983: 137, 239] *From the 1982 Netherlands Olympiad.*

Which of $(17091982!)^2$ and $17091982^{17091982}$ is greater?

II. *Comment by M.S.K.*

More generally, we have

$$(n!)^2 > n^n, \quad n > 2. \tag{1}$$

This inequality goes back to A. Cauchy, *Exercices d'analyse*, Vol. 4, Paris, 1847, p. 106. We have already given one simple proof of (1) in [1983: 239], which depends upon

$$1 \le j \le n \quad \Longrightarrow \quad (j-1)(n-j) \ge 0 \quad \Longrightarrow \quad j(n+1-j) \ge n,$$

with equality just when $j = 1$ and $j = n$. We now give an alternate proof by induction.

It is clear that (1) holds for $n = 3$. Assume that it holds for some integer $n = k \ge 3$. Then it suffices to establish the second inequality in

$$\{(k+1)!\}^2 = (k+1)^2(k!)^2 > (k+1)^2k^k > (k+1)^{k+1},$$

or equivalently, to establish that

$$(1 + \frac{1}{k})^k < 1 + k \tag{2}$$

holds for all integers $k \ge 3$. We show that (2) holds, in fact, for all integers $k \ge 2$.

By the binomial theorem,

$$\begin{aligned}(1 + \frac{1}{k})^k &= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{k}) + \dots + \frac{1}{k!}(1 - \frac{1}{k})(1 - \frac{2}{k})\dots(1 - \frac{k-1}{k})\\ &< 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{k!}\\ &< 1 + (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots )\\ &= 1 + 2\\ &\le 1 + k.\end{aligned}$$

As a rider, we ask for a proof that (2) holds for all real $k > 1$.

*

3. [1983: 137] *From the 1982 Netherlands Olympiad.*

Five marbles are distributed independently and at random among seven urns. What is the expected number of urns receiving exactly one marble?

*Solution by Andy Liu, University of Alberta.*

We calculate the frequency of each distribution in the chart on the following page.

The expected number of urns receiving exactly one marble is

$$\frac{210 + 3150 + 2100\cdot 2 + 8400\cdot 3 + 2520\cdot 5}{16807} = \frac{6480}{2401} \approx 2.7.$$

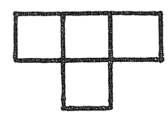
| Distribution | Formation | Arrangement | Frequency |
|---|---|---|---|
| 5 | 1 | 7 | 7 |
| 4 + 1 | 5 | 7•6 | 210 |
| 3 + 2 | 10 | 7•6 | 420 |
| 3 + 1 + 1 | 10 | 7•6•5 | 2100 |
| 2 + 2 + 1 | 10•3/2 | 7•6•5 | 3150 |
| 2 + 1 + 1 + 1 | 10 | 7•6•5•4 | 8400 |
| 1 + 1 + 1 + 1 + 1 | 1 | 7•6•5•4•3 | 2520 |

Total = 16807

*

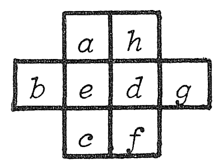
4. [1983: 303; 1985: 110] *From the 1980 Leningrad High School Olympiad, Third Round.*

Is it possible to arrange the natural numbers from 1 to 64 on an $8\times 8$ checkerboard in such a way that the sum of the numbers in any figure of the form shown on the right is divisible by 5? (The figure can be placed on the board with any orientation.)



II. *Comment by M.S.K.*

In the previous solution [1985: 110], it was shown that the arrangement is not possible even if we use 64 different natural numbers from 1 to 78. With a little more effort, the number 78 can be increased to $5\cdot 64 - 1 = 319$.




It was shown that we must have $a \equiv c$ and $b \equiv d$ (all congruences are modulo 5) in the cross *abcde* of the figure. But then we must also have $2a+b+e \equiv 2b+a+e$ ($\equiv 0$) from tetrominoes *abce* and *abde*, so $a \equiv b \equiv c \equiv d$. Similarly, $e \equiv f \equiv g \equiv h$ from the cross *defgh*. Now $3a+e \equiv 3e+a \equiv 0$ from tetrominoes *abce* and *defh*, so $a \equiv e$. Finally, $4a \equiv 0$, so $a \equiv 0$. If the arrangement is possible, the board must therefore be filled with 64 different multiples of 5. But there are only 63 multiples of 5 from 1 to 319.

*

1. [1984: 282] *From the 1984 Annual Greek High School Competition.*

(a) Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon having its opposite sides parallel. Prove that triangles $A_1A_3A_5$ and $A_2A_4A_6$ have equal areas.

(b) Consider a convex octagon in which all the angles are equal and the length of each side is a rational number. Prove that its opposite sides are equal and parallel.

I. *Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.*

(a) Let $\vec{A}_i$, $i = 1,2,\ldots,6$, denote vectors from some common origin to the respective vertices of the hexagon. Since the opposite sides are parallel, we have

$$(\vec{A}_1 - \vec{A}_2) \times (\vec{A}_4 - \vec{A}_5) = \vec{0},$$
$$(\vec{A}_3 - \vec{A}_2) \times (\vec{A}_5 - \vec{A}_6) = \vec{0},$$
$$(\vec{A}_3 - \vec{A}_4) \times (\vec{A}_6 - \vec{A}_1) = \vec{0}.$$

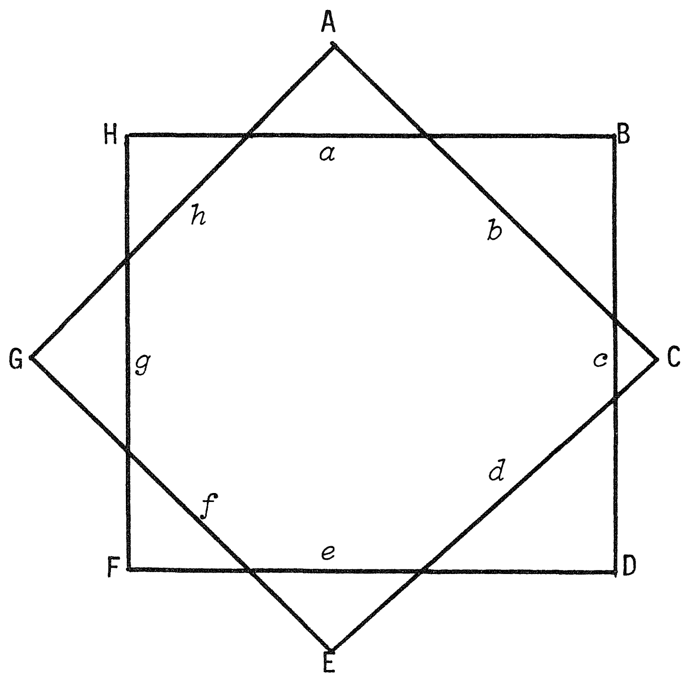
Expanding out and adding these three equations yield an equation equivalent to

$$\vec{A}_1 \times \vec{A}_3 + \vec{A}_3 \times \vec{A}_5 + \vec{A}_5 \times \vec{A}_1 = \vec{A}_2 \times \vec{A}_4 + \vec{A}_4 \times \vec{A}_6 + \vec{A}_6 \times \vec{A}_2.$$

Now, with square brackets denoting area, we have

$$[A_1A_3A_5] = \frac{1}{2}|(\vec{A}_1-\vec{A}_3) \times (\vec{A}_1-\vec{A}_5)| = \frac{1}{2}|\vec{A}_1 \times \vec{A}_3 + \vec{A}_3 \times \vec{A}_5 + \vec{A}_5 \times \vec{A}_1|$$
$$= \frac{1}{2}|\vec{A}_2 \times \vec{A}_4 + \vec{A}_4 \times \vec{A}_6 + \vec{A}_6 \times \vec{A}_2| = \frac{1}{2}|(\vec{A}_2-\vec{A}_4) \times (\vec{A}_2-\vec{A}_6)| = [A_2A_4A_6].$$

(b) Let the side lengths of the octagon be $a,b,c,d,e,f,g,h$ in consecutive order. The sum of its interior angles is 1080°, so each angle is 135°. Hence producing each side in both directions yields two overlapping rectangles ACEG and BDFH, as shown in the figure, and so all pairs of opposite sides of the octagon are parallel. From HB = FD, we get




$$a + \frac{b+h}{\sqrt{2}} = e + \frac{d+f}{\sqrt{2}},$$

from which $(a-e)\sqrt{2} = d+f-b-h$. Since all side lengths are rational, we must have $a = e$, and similarly $b = f$, $c = g$, and $d = h$.

II. *Comment by M.S.K.*

Part (a) is not new. For other proofs of it and references to its earlier occurrences, see my article in this journal [1981: 102-105].

*

2, [1984: 282] *From the 1984 Annual Greek High School Scompetition.*

An equilateral triangle ABC and an interior point P are given such that PA = 5, PB = 4, and PC = 3. Find the length of a side of triangle ABC.

*Comment by M.S.K.*

More generally, let PA = $x$, PB = $y$, and PC = $z$, where P is not necessarily an interior point of triangle ABC. In this more general form the problem is well known. A solution appears in S.I. Jones, *Mathematical Nuts*, S.I. Jones Co., Nashville, Tennessee, 1932, pp. 294-296. Both the general case and the special case $(x,y,z) = (5,4,3)$ appeared later in this journal in Problem 39 [1975: 64-66]. And Bob Prielipp, University of Wisconsin-Oshkosh, notes that the special case appeared still later as Problem 3682 in *School Science and Mathematics*, 78 (1978) 174-176. The results are as follows:

Given three positive numbers $x,y,z$, there exists an equilateral triangle ABC and a point P such that PA = $x$, PB = $y$, and PC = $z$ if and only if

$$K \equiv 2(y^2z^2 + z^2x^2 + x^2y^2) - x^4 - y^4 - z^4 \geq 0,$$

that is, if and only if $x,y,z$ are themselves the side lengths of a triangle (of area $\sqrt{K}/4$). This condition being satisfied, the side length $\ell$ of triangle ABC is given by

$$2\ell^2 = x^2 + y^2 + z^2 \pm \sqrt{3K},$$

where the plus sign or the minus sign is used according as P is an interior point or an exterior point of triangle ABC. For the special case of the proposal, we find that

$$\ell = \sqrt{25 + 12\sqrt{3}} \approx 6.7664.$$

One way of obtaining these results that was not used in the references given above is to equate to zero the volume of a tetrahedron with edge lengths $\ell,\ell,\ell,x,y,z$.

For a similar problem about a square, see Problem M796 [1984: 153-154].

*

3. [1984: 282] *From the 1984 Annual Greek High School Competition.*

In a given triangle ABC, $\angle A = 5\pi/8$, $\angle B = \pi/8$, and $\angle C = \pi/4$. Prove that its angle bisector CZ, the median BE, and the altitude AD are concurrent.

*Solution by René Schipperus, student, Western Canada High School, Calgary, Alberta.*

Let $a,b,c$ be the side lengths in the usual order. By Ceva's theorem, we must show that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AZ}{ZB} = 1.$$

Now

$$\frac{BD}{DC} = \frac{c\cos(\pi/8)}{b\cos(\pi/4)}, \quad \frac{CE}{EA} = 1, \quad \text{and} \quad \frac{AZ}{ZB} = \frac{b}{a}.$$

Hence

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AZ}{ZB} = \frac{c}{a}\cdot\frac{\cos(\pi/8)}{\cos(\pi/4)} = \frac{\sin(\pi/4)}{\sin(5\pi/8)}\cdot\frac{\cos(\pi/8)}{\cos(\pi/4)} = \frac{\cos(\pi/8)}{\sin(5\pi/8)} = 1.$$

*

4. [1984: 282] *From the 1984 Annual Greek High School Competition.*

(a) Find the real roots of the equation

$$(x^2-x-2)^4 + (2x+1)^4 = (x^2+x-1)^4.$$

(b) Find the roots of the equation

$$x^3 - 2ax^2 + (a^2+1)x - 2a + 2 = 0$$

for all real $a$.

(c) Find the range of the function

$$f(x) = \frac{\sqrt{x^2+1} + x - 1}{\sqrt{x^2+1} + x + 1}, \qquad -\infty < x < \infty,$$

and show that $f$ is an odd function.

*Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.*

(a) If we set $A = x^2-x-2 = (x+1)(x-2)$ and $B = 2x+1$, then the given equation becomes

$$A^4 + B^4 = (A + B)^4. \tag{1}$$

It is clear that $A = 0$ and $B = 0$ both satisfy (1), and these yield the real roots -1, 2, and -1/2.

[*Addendum by M.S.K.* To show that these are *all* the real roots, suppose $AB \neq 0$. Then (1) is equivalent to $2y^2 + 3y + 2 = 0$, where $y = A/B$, an equation with two imaginary roots. So there are no more real roots for (1), since it is clear that $y$ is real whenever $x$ is real.]

(b) Since $x = 0$ is a root when $a = 1$, one root of the given equation is $x = a-1$; then synthetic division shows that the remaining two roots are those of $x^2 - (a+1)x + 2 = 0$, that is,

$$x = \frac{a+1 \pm \sqrt{a^2+2a-7}}{2}.$$

(c) For $x \neq 0$, we rationalize the denominator and obtain

$$f(x) = \frac{\sqrt{x^2+1} - 1}{x}, \tag{2}$$

so $f(-x) = -f(x)$. Since also $f(0) = 0$, it follows that $f$ is an odd function.

For $0 < x < \infty$, we can set $x = \tan\theta$, where $0 < \theta < \pi/2$, and then (2) becomes

$$f(x) = g(\theta) = \frac{\sec\theta - 1}{\tan\theta} = \frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2}.$$

Thus $f$ is strictly increasing on $(0,\infty)$ and

$$\lim_{x\to\infty} f(x) = \lim_{\theta\to\pi/2} g(\theta) = 1.$$

Since $f$ is odd and continuous, its range is the interval $(-1,1)$.

*Editor's note.* All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

# PROBLEMS--PROBLÈMES

*Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.*

*To facilitate their consideration, your solutions, typewritten or neatly hand-written on signed, separate sheets, should preferably be mailed to the editor before February 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.*

1051. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $a,b,c$ be the side lengths of a triangle of area $K$, and let $u,v,w$ be positive real numbers. Prove that

$$\frac{ua^4}{v+w} + \frac{vb^4}{w+u} + \frac{wc^4}{u+v} \ge 8K^2.$$

When does equality occur? Some interesting triangle inequalities may result if we assign specific values to $u,v,w$. Find a few.

1052*. *From a Trinity College, Cambridge, examination paper dated June 5, 1901.*

Prove that

$$\frac{1}{1^2\cdot 3^3\cdot 5^2} - \frac{1}{3^2\cdot 5^3\cdot 7^2} + \frac{1}{5^2\cdot 7^3\cdot 9^2} - \dots = \frac{1}{9} - \frac{\pi}{2^6} - \frac{\pi^3}{2^9}.$$

1053. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Exhibit a bijection between the points in the plane and the lines in the plane.

1054. *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

A paper square ABCD is divided into three strips of equal area by the parallel lines PQ and RS, as shown in Figure 1.

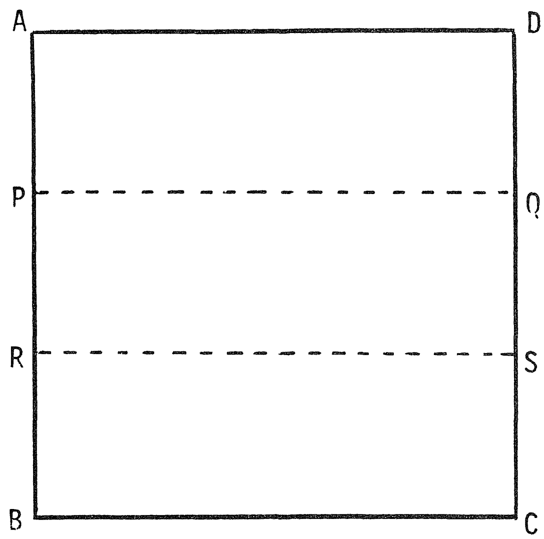


Figure 1

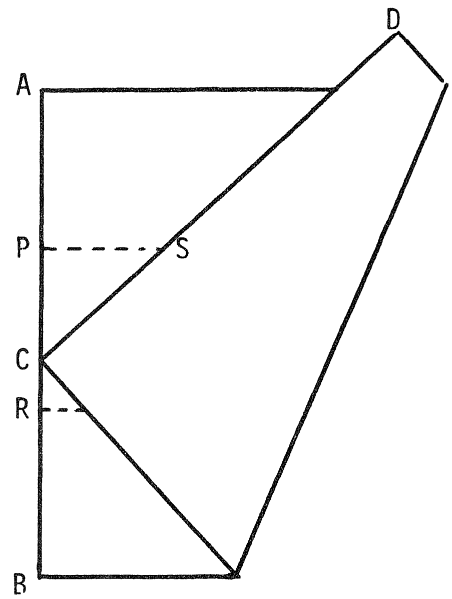


Figure 2

The square is then folded so that C falls on AB and S falls on PQ, as shown in Figure 2. Determine the ratio AC/CB in Figure 2.

1055. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that every fourth-order magic square can be written in the form

| | | | |
|---|---|---|---|
| $F+y$ | $G+x+t+u$ | $G-x-t-u$ | $F-y$ |
| $G-w-t$ | $F+z+v$ | $F-z+t$ | $G+w-v$ |
| $F+x+t$ | $G+y-t$ | $G-y-v$ | $F-x+v$ |
| $G+z$ | $F-w-u-v$ | $F+w+u+v$ | $G-z$ |

where $w = x+y+z$.

*Note.* Deleting the variables $t$, $u$, and $v$ from this square leaves an algebraic form that represents every *pandiagonal* fourth-order magic square (see Problem 605 [1982: 22-23]).

1056. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let ABC be a triangle with sides $a,b,c$ in the usual order. Side BC is divided into $n$ ($\geq 2$) equal parts by the point $P_i$, $i = 1,2,\ldots,n-1$. Let

$$S_n = \frac{1}{n-1}\sum_{i=1}^{n-1}(AP_i)^2$$

be the average value of the squared lengths of the segments $AP_i$. Show that the sequence $\{S_n\}_{n=2}^{\infty}$ is monotonically increasing and evaluate $\lim\limits_{n\to\infty} S_n$ in terms of $a,b,c$.

1057. *Proposed by Jordi Dou, Barcelona, Spain.*

Let $\Omega$ be a semicircle of unit radius, with diameter $AA_0$. Consider a sequence of circles $\gamma_i$, all interior to $\Omega$, such that $\gamma_1$ is tangent to $\Omega$ and to $AA_0$, $\gamma_2$ is tangent to $\Omega$ and to the chord $AA_1$ tangent to $\gamma_1$, $\gamma_3$ is tangent to $\Omega$ and to the chord $AA_2$ tangent to $\gamma_2$, etc. Prove that $r_1+r_2+r_3+\ldots < 1$, where $r_i$ is the radius of $\gamma_i$.

1058. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

Two points X and Y are chosen at random, independently and uniformly with respect to length, on the edges of a unit cube. Determine the probability that

$$1 < XY < \sqrt{2}.$$

1059. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

In his book, *The Modern Geometry of the Triangle* (London, 1913), W. Gallatly denotes by J the circumcentre of triangle $I_1I_2I_3$, whose vertices are the excentres of the reference triangle ABC. On pages 1 and 21 are figures in which J *appears* to be collinear with the incentre and circumcentre of triangle ABC. Are these points *really* collinear?

1060. *Proposed by M.S. Klamkin, University of Alberta.*

If ABC is an obtuse triangle, prove that

$$\sin^2 A\tan A + \sin^2 B\tan B + \sin^2 C\tan C < 6\sin A\sin B\sin C.$$

1045. [1985: 147] *Correction.* The inequality of part (a) should read

$$\frac{x^2}{vw} + \frac{y^2}{wu} + \frac{z^2}{uv} \ge 12.$$

* * *

## S O L U T I O N S

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

478. [1979: 229; 1980: 219] *Proposed by Murray S. Klamkin, University of Alberta.*

Consider the following theorem:

*If the circumcircles of the four faces of a tetrahedron are mutually congruent, then the circumcentre* O *of the tetrahedron and its incentre* I *coincide.*

An editor's comment following Crux 330 [1978: 264] claims that the proof of this theorem is "easy". Prove it.

II. *Comment by the proposer and Andy Liu, University of Alberta.*

The given solution I [1980: 219] appears to be incomplete. This solution states, in part: "In the *proof* of Art. 304 [in the given Altshiller-Court reference [1]], it is shown that if the circumcircles of the four faces are equal, then the tetrahedron is isosceles." Altshiller-Court's proof is based on his statement: "Now equal chords subtend equal angles in equal circles." This statement is questionable, because equal chords also subtend supplementary angles in equal circles. So, to complete the proof, all the supplementary angle cases must be ruled out. So far we do not see any easy way of doing this.

*Editor's comment.*

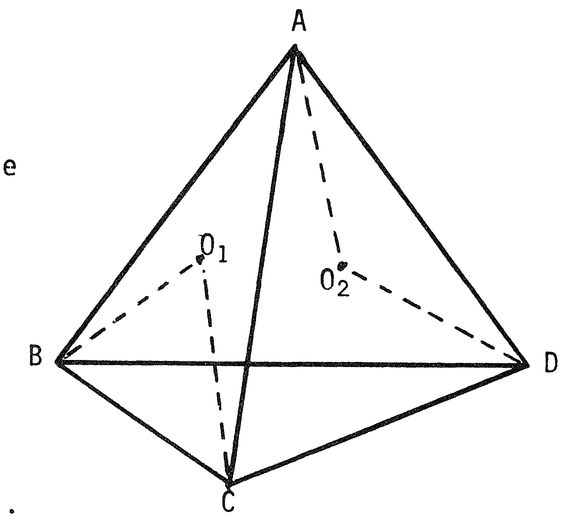
We give a proof of the theorem of Art. 304 which avoids Altshiller-Court's questionable statement. However, this does not resolve the difficulty raised in the above comment II.

304. *THEOREM.* If the circumcentre and the incentre of a tetrahedron coincide, the tetrahedron is isosceles.

*Proof.* If the centres of the inscribed and circumscribed spheres coincide, then

(a) the four faces are equidistant from the centre of the circumscribed sphere, and hence the circumcircles of the four faces are congruent;

(b) the point of contact of the inscribed sphere with each face lies at the circumcentre of that face.




Let $O_1$ and $O_2$ be the circumcentres of the faces ABC and ACD, respectively, of a tetrahedron ABCD satisfying the hypothesis (see figure). It follows from (a) that

$$O_1B = O_1C = O_2A = O_2D;$$

and it follows from (b) and the theorem of Bang (1897), which is stated and proved in [2], that

$$\angle BO_1C = \angle AO_2D.$$

Hence BC = AD. Similarly, CA = BD and AB = CD, so tetrahedron ABCD is isosceles.

REFERENCES

1. Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea, New York, 1964, p. 107.

2. B.H. Brown, "Club Topics", *American Mathematical Monthly*, 33 (1926) 224-225. (Reference supplied by Leon Bankoff.)

* * *

859. [1983: 179; 1984: 307] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let ABC be an acute-angled triangle of type II, that is (see [1982: 64]), such that $A \le B \le \pi/3 \le C$, with circumradius $R$ and inradius $r$. It is known [1982: 66] that for such a triangle $x \ge \frac{1}{4}$, where $x = r/R$. Prove the stronger inequality

$$x \ge \frac{\sqrt{3}-1}{2}.$$

II. *Generalization by M.S. Klamkin, University of Alberta.*

More generally, let ABC by *any* triangle with $A \le B \le C$. If $r$ and $R$ are its inradius and circumradius, we set

$$x = \frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Now let PQR be any triangle such that $P \ge C \ge Q \ge B$; then $A \ge R$. The vector $(P,Q,R)$ majorizes the vector $(C,B,A)$, in the sense that

$$P \ge Q \ge R, \quad C \ge B \ge A, \quad P \ge C, \quad P+Q \ge C+B, \quad P+Q+R = C+B+A.$$

It now follows from the Majorization Inequality [1] that

$$f(A) + f(B) + f(C) \ge f(P) + f(Q) + f(R)$$

for any continuous concave function $f$. Since $\ln\sin(x/2)$ is concave in $(0,\pi)$, we have

$$\ln\sin\frac{A}{2} + \ln\sin\frac{B}{2} + \ln\sin\frac{C}{2} \ge \ln\sin\frac{P}{2} + \ln\sin\frac{Q}{2} + \ln\sin\frac{R}{2},$$

and so

$$x \ge 4\sin\frac{P}{2}\sin\frac{Q}{2}\sin\frac{R}{2}.$$

In particular, if $P = \pi/2$ and $Q = \pi/3$, as in our problem, we get

$$x \ge 4\sin\frac{\pi}{4}\sin\frac{\pi}{6}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}.$$

REFERENCE

1. Edwin F. Beckenbach and Richard Bellman, *Inequalities*, Springer-Verlag, Heidelberg, 1965. p. 30.

* * *

931. [1984: 113] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Replace the letters with digits to obtain a decimal addition:

$$\begin{array}{r} \text{PASTE} \\ \underline{\text{STAPLE}} \\ \text{FASTEN} \end{array}.$$

Then PASTE, STAPLE, or FASTEN somehow your solution to a postcard and send it along to the editor.

*Answer by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

The unique answer is

$$\begin{array}{r} 91328 \\ \underline{321958}. \\ 413286 \end{array}$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; J.A. McCALLUM, Medicine Hat, Alberta; GLEN E. MILLS, Pensacola Junior College, Florida; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

*Editor's comment.*

Only three solvers (including the proposer) sent in detailed solutions, and in all cases these involved an unconscionable amount of brute force which it would be unproductive to reproduce here *in extenso*. The other solvers sent in only the answer. We decided to feature Rabinowitz's because his was the only one which fully met the requirements of the proposal: it was actually pasted, stapled, and fastened (by paper clip) to a postcard.

* * *

932. [1984: 113] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $p \equiv 7 \pmod 8$ be a prime so there is an integer $w$ such that

$$w^2 \equiv 2 \pmod p.$$

Prove that the value of the Legendre symbol $(\frac{2+w}{p})$ is given by

$$\left(\frac{2+w}{p}\right) = \begin{cases} +1, & \text{if } p \equiv 15 \pmod{16}, \\ -1, & \text{if } p \equiv 7 \pmod{16}. \end{cases}$$

*Solution by the proposer.*

The congruence $x^2 \equiv 2 \pmod p$ has exactly two solutions, namely $w$ and $-w$. It does not matter which solution we use in the evaluation of the Legendre symbol $(\frac{2+w}{p})$ as the values of $(\frac{2+w}{p})$ and $(\frac{2-w}{p})$ are the same, since

$$\left(\frac{2+w}{p}\right)\left(\frac{2-w}{p}\right) = \left(\frac{4-w^2}{p}\right) = \left(\frac{4-2}{p}\right) = \left(\frac{2}{p}\right) = 1.$$

Furthermore, as $p \equiv 7 \pmod 8$, there are integers $e$ and $f$ such that

$$p = e^2 - 2f^2, \quad e > 0,\ f > 0,$$

where

$$(e,2) = (f,2) = (e,f) = (e,p) = (f,p) = 1,$$

and we may choose

$$w \equiv \frac{e}{f} \pmod p.$$

With this choice of $w$, we have

$$\left(\frac{2+w}{p}\right) = \left(\frac{2 + e/f}{p}\right) = \left(\frac{f}{p}\right)\left(\frac{e+2f}{p}\right). \tag{1}$$

Next we have, by the law of quadratic reciprocity,

$$\left(\frac{f}{p}\right) = \left(\frac{-p}{f}\right).$$

As $-p \equiv -e^2 \pmod{f}$, we obtain

$$\left(\frac{-p}{f}\right) = \left(\frac{-e^2}{f}\right) = \left(\frac{-1}{f}\right),$$

so that

$$\left(\frac{f}{p}\right) = \left(\frac{-1}{f}\right). \tag{2}$$

Also by the law of quadratic reciprocity,

$$\left(\frac{e+2f}{p}\right) = \left(\frac{-p}{e+2f}\right).$$

As $-p \equiv -2(e+f)^2 \pmod{e+2f}$, we obtain

$$\left(\frac{-p}{e+2f}\right) = \left(\frac{-2}{e+2f}\right) = \left(\frac{-1}{e+2f}\right)\left(\frac{2}{e+2f}\right).$$

Now

$$\left(\frac{-1}{e+2f}\right) = -\left(\frac{-1}{e}\right)$$

and

$$\left(\frac{2}{e+2f}\right) = -\left(\frac{-2}{e}\right)\left(\frac{-1}{f}\right),$$

so

$$\left(\frac{-p}{e+2f}\right) = \left(\frac{2}{e}\right)\left(\frac{-1}{f}\right). \tag{3}$$

Hence from (1), (2), and (3) follows

$$\left(\frac{2+w}{p}\right) = \left(\frac{2}{e}\right) = \begin{cases} +1, & \text{if } p \equiv 15 \pmod{16}, \\ -1, & \text{if } p \equiv 7 \pmod{16}, \end{cases}$$

as $e^2 \equiv p+2 \pmod{16}$. □

This problem is a special case of a much more general theorem which is proved in a paper by the proposer, Kenneth Hardy, and Christian Friesen, to be published in *Acta Arithmetica*.

Also solved by ANON, Erewhon-upon-Spanish River; and J.T. GROENMAN, Arnhem, The Netherlands (partial solution).

*Editor's comment.*

Anon's solution uses Kummer's theory of the factorization of odd rational primes in cyclotomic fields. He also shows that

$$\left(\frac{2+w}{p}\right) = \begin{cases} +1, & \text{if } p \equiv 1 \pmod{16}, \\ -1, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

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933. [1984: 113] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

A truncated triangular pyramid with base ABC and lateral edges $AA_1$, $BB_1$, and $CC_1$ is given. Let P be an arbitrary point in the plane of triangle $A_1B_1C_1$, and let Q be the common point of the planes $\alpha$, $\beta$, and $\gamma$ passing respectively through $B_1C_1$, $C_1A_1$, and $A_1B_1$, and parallel respectively to the planes PBC, PCA, and PAB. Prove that the tetrahedron ABCQ and the pyramid $ABB_1A_1C$ have equal volumes.

*Solution by Jordi Dou, Barcelona, Spain.*

Let $S$ and $S_1$ denote respectively the areas of triangles ABC and $A_1B_1C_1$; $h$ and $h_1$ the altitudes of tetrahedra ABCP and $A_1B_1C_1Q$; $V$ and $V'$ the volumes of pyramid $ABB_1A_1C$ and tetrahedron ABCQ; and, finally, let $T$ be the volume of the given truncated pyramid. We must show that $V = V'$.

Since the tetrahedra ABCP and $A_1B_1C_1Q$ are similar, we have

$$\frac{h}{h_1} = \frac{AB}{A_1B_1} = \frac{\sqrt{S}}{\sqrt{S_1}},$$

and so $h_1 = h\sqrt{S_1}/\sqrt{S}$. Therefore

$$V' = \frac{1}{3}(h+h_1)S = \frac{1}{3}h(S + \sqrt{SS_1}).$$

Now

$$T = \frac{1}{3}h(S + \sqrt{SS_1} + S_1) \implies V' = T - \frac{1}{3}hS_1.$$

Since

$$V = T - (\text{volume of } A_1B_1C_1C) = T - \frac{1}{3}hS_1,$$

we conclude that $V = V'$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

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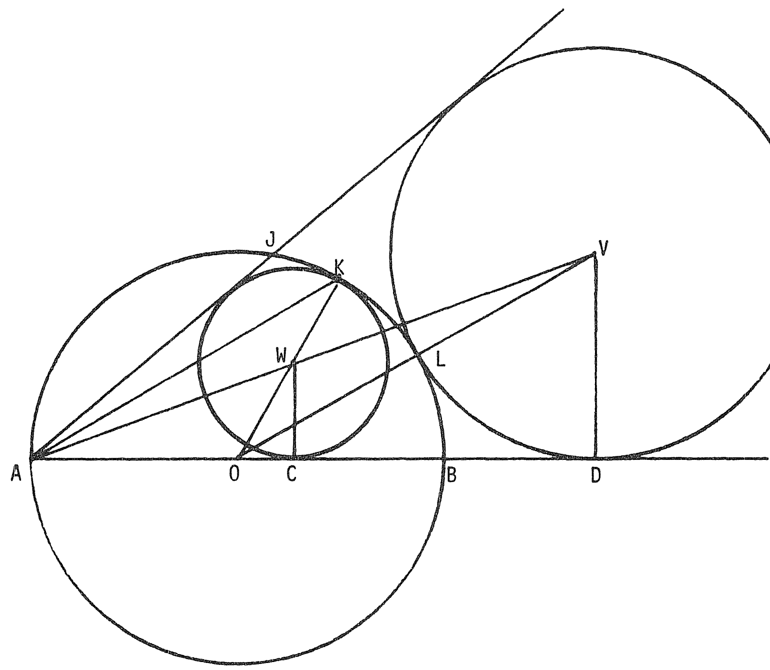
934. [1984: 114] *Proposed by Leon Bankoff, Los Angeles, California.*

As shown in the figure, the diameter AB, a variable chord AJ, and the intercepted minor arc JB of a circle (O) form a mixtilinear triangle whose inscribed circle (W) touches arc JB in K and whose mixtilinear excircle (V) touches arc JB in L. The projections of W and V upon AB are C and D, respectively. As J moves along the circumference of circle (O), the ratio of the arcs KL and LB varies.




(a) When arcs KL and LB are equal, what are their values?

(b) Show that BD is equal to the side of the inscribed square lying in the right angle of triangle ADV.

*Solution by the proposer.*

(a) If $\widehat{KL} = \widehat{LB}$, then $\angle KAB = \angle LOB$. Consequently, AK $\|$ OV and triangles AWK and VWO are similar, so AW/WV = WK/OW. But in the similar triangles ACW and ADV,

$$\frac{AW}{WV} = \frac{CW}{DV-CW} = \frac{WK}{DV-WK}.$$

Hence DV-WK = OW and the radii of circles (O) and (V) are equal. Thus DV/OV = 1/2, and so $\angle VOD = 30°$. Therefore $\widehat{KL} = \widehat{LB} = 30°$.

(b) We have $(OB+BD)^2 + DV^2 = (OL+LV)^2$, and this is equivalent to

$$BD\cdot(BD+2OB) = 2OB\cdot DV,$$

or to BD/AB = DV/AD. If the perpendicular to AD erected at B cuts AV in H, we have BH/AB = DV/AD. Hence BD = BH.

Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; and DAN SOKOLOWSKY, Brooklyn, N.Y.

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935. [1984: 114] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

It is easy to show that (0,0) and (9,9) are the only solutions $(x,y)$ of the equation

$$x^3 + y^3 = 18xy$$

in which $x = y$. Find a solution in integers $(x,y)$ with $x > y$ and show that it is unique.

I. *Solution by Bob Prielipp, University of Wisconsin-Oshkosh.*

We show that the only solution in integers with $x > y$ is $(x,y) = (8,4)$. Thus the complete solution set of the given Diophantine equation is

$$\{(0,0),\ (9,9),\ (8,4),\ (4,8)\}.$$

It is clear that in any solution $x$ and $y$ must have the same parity, so we can set $x = u+v$ and $y = u-v$, where $u$ and $v$ are integers and $v > 0$ (since $x > y$). The given equation is then found to be equivalent to

$$u^2(u-9) + 3v^2(u+3) = 0.$$

It follows that $u \neq 0$ and $(u-9)(u+3) < 0$, so $u \in \{-2,-1,1,2,\ldots,8\}$. We then find that only $u = 6$ yields an integral value of $v$, namely $v = 2$, so the unique solution is

$$(x,y) = (u+v, u-v) = (8,4).$$

II. *Joint solution by J.L. Brenner, Palo Alto, California; and Lorraine L. Foster, California State University, Northridge.*

More generally, we look for the solutions $(x,y,m)$ to the Diophantine equation

$$x^3 + y^3 = mxy, \tag{1}$$

where $x$ and $y$ are integers and $m$ is a nonzero integer. If $(x,y,m)$ is a solution, then so is $(-x,-y,-m)$, so we may assume that $m > 0$. The only solutions with $x = y$ are $(0,0,m)$ for any $m$ and $(m/2,m/2,m)$ when $m$ is even. Equation (1) being symmetric in $x$ and $y$, we may therefore restrict our further search to solutions in which $x > y$. Clearly, any such solutions must satisfy either $x > y > 0$ or $x > 0 > y$; we call these *Type I* and *Type II* solutions respectively. Since, for any integer $c \geq 1$,

$$(x,y,m) \text{ is a solution} \iff (cx,cy,cm) \text{ is a solution},$$

it suffices to find the *primitive solutions*, in which $(x,y,m) = 1$.

*THEOREM 1.* The primitive Type I solutions of (1) are given by

$$(x,\ y,\ m) = (e^2f,\ ef^2,\ e^3+f^3), \tag{2}$$

where $e$ and $f$ are integers such that $e > f > 0$ and $(e,f) = 1$.

*Proof.* It is easy to verify that (2) is in fact a primitive Type I solution of (1). Conversely, let $(x,y,m)$ be such a solution, and let $(x,y) = d$, so that $x = du$ and $y = dv$, where $(u,v) = 1$. Let $e = (d,u)$ and $f = (d,v)$. Then $u = eg$, $v = fh$, and $d = efd_1$, with

$$(eg,fh) = (d_1,f) = (d_1,g) = (d_1,m) = 1.$$

Now from (1), $d(e^3g^3 + f^3h^3) = megfh$, so $d_1(e^3g^3 + f^3h^3) = mgh$. This implies that $d_1 = 1$ (since $(d_1,mgh) = 1$). Hence $g \mid f^3h^3$, so $g = 1$ (since $(g,fh) = 1$). Similarly $h = 1$, and solution (2) results. □

With a slight modification of the above proof, we obtain

*THEOREM 2.* The primitive Type II solutions of (1) are given by

$$(x, y, m) = (k^2r, -kr^2, r^3-k^3),$$

where $k$ and $r$ are integers such that $r > k > 0$ and $(r,k) = 1$.

*Corollary.* Since $(r^2+kr+k^2)|m$, we must have $r < \sqrt{m}$.

*Remarks.*

(a) For any solution $(x,y,m)$ of (1), primitive or not, if $m \neq 2$ then $(x,y) > 1$. For $x|y^3$ and $y|x^3$, so that, if $(x,y) = 1$ then $x = 1$, $y = \pm 1$, and $m = 2$ (since $m = 0$ is excluded).

(b) The proposer's equation has no Type II solution since $m = 18$ has no factors of the form $r^3 - k^3$, where $r > k > 0$. (We try all $r \le 4$ in this case.) Furthermore, $(c,e,f) = (2,2,1)$ is easily seen to be the only solution to $c(e^3 + f^3) = 18$ with $e > f > 0$. It follows that the unique solution requested by the proposer is $(8,4,18)$, a nonprimitive Type I solution.

(c) Unless $m \equiv 0, \pm 1$, or $\pm 2 \pmod 9$, equation (1) has no primitive solution.

(d) The number $m = 91 = 4^3 + 3^3 = 6^3 - 5^3$ is the least $m$ for which (1) has two primitive solutions (one of each type). For even $t > 0$, let $m(t)$ be the common value of

$$(9t^4 - 3t)^3 + (9t^3 - 1)^3 = (9t^4)^3 - 1^3.$$

It is easy to see that $9t^4-3t > 9t^3-1$ and $(9t^4-3t, 9t^3-1) = 1$. Hence there exist infinitely many integers $m(t)$ for which equation (1) has at least two primitive solutions (one of each type).

(e) For $m = 721 = 9^3 - 2^3 = 16^3 - 15^3$, equation (1) has two primitive Type II solutions.

(f) Finally, recalling Ramanujan's remark about Hardy's taxicab number, we can state that the number

$$m = 1729 = 10^3 + 9^3 = 12^3 + 1^3$$

is the least $m$ for which equation (1) has two primitive Type I solutions.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; W.J. BLUNDON, Memorial University of Newfoundland; DALE S. COOPER, Lake Superior High School, Terrace Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; J. WALTER LYNCH, Georgia Southern College, Statesboro; J.A. McCALLUM, Medicine Hat, Alberta; SHIN MOCHIZUKI, student, Phillips Exeter Academy, New Hampshire; RICHARD PARRIS, Phillips Exeter Academy, New Hampshire; D.J. SMEENK, Zaltbommel, The Netherlands; J. SUCK, Essen, West Germany; JORDAN B. TABOV, Sofia, Bulgaria; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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936. [1984: 114] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find all eight-digit palindromes in base ten that are also palindromes in at least one of the bases two, three, ..., nine.

*Solution by the proposer.*

There are exactly fifteen such palindromes. In the (computer-generated) list that follows, the numbers on the left are in base ten and those on the right are in the indicated bases.

$$10088001 = 10040304001_{\text{five}}$$

$$10400401 = 10130303101_{\text{five}}$$

$$13500531 = 110011100000000001110011_{\text{two}}$$

$$15266251 = 1303113031_{\text{six}}$$

$$24466442 = 2232222322_{\text{six}}$$

$$27711772 = 1221010220101221_{\text{three}}$$

$$27711772 = 24043234042_{\text{five}}$$

$$30322303 = 30230303203_{\text{five}}$$

$$47633674 = 44143234144_{\text{five}}$$

$$53822835 = 3031110111303_{\text{four}}$$

$$55366355 = 323151323_{\text{eight}}$$

$$61255216 = 1342442431_{\text{seven}}$$

$$65666656 = 146505641_{\text{nine}}$$

$$65977956 = 113342243311_{\text{five}}$$

$$83155138 = 12210110201101221_{\text{three}}$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California.

*Editor's comment.*

Hess also found the fifteen palindromes in the above list. Unfortunately, he also listed a sixteenth one,

$$87100178 = 12350505321_{\text{six}},$$

which is incorrect. It was a very near miss, though.

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937. [1984: 115] *Proposed by Jordi Dou, Barcelona, Spain.*

ABCD is a trapezoid inscribed in a circle $\phi$, with AB || DC. The midpoint of AB is M, and the line DM meets the circle again in P. A line $l$ through P meets line BC in A', line CA in B', line AB in C', and the circle again in F'.

Prove that (A'B', C'F') is a harmonic range.

*Solution by Dan Pedoe, University of Minnesota.*

The theorem is true if $\phi$ is any conic, as the solution will show.

We note that the cross ratio (AB, M∞) is harmonic, since M divides AB; therefore the pencil D(AB, M∞) is harmonic, which implies that the points (AB, PC) form a harmonic set on the conic. Now let AA' cut the conic again in X. The involution of points on the conic, centre A', maps A,B,P,C into X,C,F',B, respectively and, since cross ratios are preserved under an involution, the set of points (XC, F'B) is harmonic on the conic. This implies that the pencil A(XC, F'B) is harmonic and, taking a section with the line $l$, the range (A'B', F'C') is harmonic, and so is (A'B', C'F').

Also solved by the proposer.

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938. [1984: 115] *Proposed by Charles W. Trigg, San Diego, California.*

Is there an infinity of pronic numbers of the form $a_nb_n$ in the decimal system? (A *pronic number* is the product of two consecutive integers. The symbol $x_n$ indicates $x$ repeated $n$ times. For example, $5_32_3$ = 555222.)

*Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria (revised by the editor).*

The answer is yes. We show that $a_nb_n$ is pronic for all $n$ = 1,2,3,... if and only if $(a,b)$ = (1,2), (4,2), or (9,0), the corresponding consecutive factors being

$$3_n\cdot(3_n+1) = 1_n2_n, \quad 6_n\cdot(6_n+1) = 4_n2_n, \quad 9_n\cdot(9_n+1) = 9_n0_n. \qquad (1)$$

It is easy to verify that each of the relations (1) holds for all $n = 1,2,3,\ldots$ . Conversely, let $x$ be a positive integer such that

$$x(x+1) = a_nb_n \equiv \frac{1}{9}(a\cdot 10^n + b)(10^n - 1)$$

holds for some arbitrary but fixed positive integer $n$. Then, by the quadratic formula,

$$x = \frac{1}{2}\{-1 + \frac{1}{3}\sqrt{4a\cdot 10^{2n} - 4(a-b)\cdot 10^n - (4b-9)}\}.$$

The expression under the square root sign is a quadratic polynomial in $10n$, and it is a perfect square for arbitrary $n$ if and only if its discriminant vanishes, or, equivalently, if and only if

$$(a + b)^2 = 9a.$$

Therefore $a$ must be a perfect square, that is, $a$ = 1, 4, or 9; the corresponding values of $b$ are 2, 2, and 0; and the corresponding values of $x$ are $3_n$, $6_n$, and $9_n$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; STEWART METCHETTE, Culver City, California; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

*Editor's comment.*

For positive integers $x$ and $d$, let the product $x(x+d)$ be called a *d-pronic number*. Klamkin asked for which $d$ are there infinitely many $d$-pronic numbers of the form $a_n b_{n+1}$, or $a_{n+1} b_n$, or, more generally, of the form $a_m b_n$ with $m \neq n$. Janous made the following more modest request: For which $d \geq 2$ are there infinitely many $d$-pronic numbers of the form $a_n b_n$? The proposer showed that $d = 2$ is such a number, for

$$6_n \cdot (6_n + 2) = 4_n 8_n, \quad n = 1,2,3,\ldots .$$

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## *NOMBRES* (FOR THE) *PREMIERS*

René Lévesque, premier of the province of Québec, recently announced his resignation after twenty-five years in politics. His successor as leader of the Parti Québécois (and premier) will be chosen in the fall of 1985.

*Maclean's* magazine (July 1, 1985) writes that "... he [Lévesque] was expelled from Collège Garnier in Quebec City ... for getting only one mark out of 100 in a mathematics exam."

One of the possible candidates to succeed Lévesque as leader of the Parti Québécois and premier of the province is Gilbert Paquette, formerly a mathematics professor at Laval University.

So things may soon be looking up for mathematics in Québec.

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## MATHEMATICAL CLERIHEWS

Emil Artin
Had a part in
Rings as staples
(With George Whaples).

Julius Wilhelm Richard Dedekind
Wrote a piece on "Was die Zahlen sind."
The most unkindest cut that Brutus made
Is not the kind that Dedekind displayed.

ALAN WAYNE
Holiday, Florida