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Editor: Léo Sauvé, Algonquin College, 140 Main Street, Ottawa, Ontario, Canada K1S 1C2.

Managing Editor: Dr. Kenneth S. Williams, Canadian Mathematical Society, 577 King Edward Avenue, Ottawa, Ontario, Canada K1N 6N5.

Assistant to the Editors: Mrs. Siu Kuen Lam.

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CONTENTS

THE OLYMPIAD CORNER: 71

M.S. KLAMKIN

This month's problem sets consist of the first and second rounds of the 1985 Dutch Mathematical Olympiad. I am grateful to Jan van de Craats for these problems. The answers (only) of the first round problems will be given here next month. For the second round problems, I solicit, as usual, "nice" solutions from all readers.

1985 DUTCH MATHEMATICAL OLYMPIAD - First Round

March 22, 1985. Time: 3 hours

Give answers only! Each A problem is worth 2 points, each B problem 3 points, and each C problem 4 points.

A1. The pages of a book are numbered from 1 to 1985. How many 1's occur in these numbers?

A2. Determine a number N that has the following properties:

(a) N is the square of a natural number;

(b) N has four digits, each of which is less than 7;

(c) adding 3 to each digit of N gives another square.

A3. Determine the *number* of real solutions of the equation

$$\sin \pi x = \frac{x}{1985}.$$

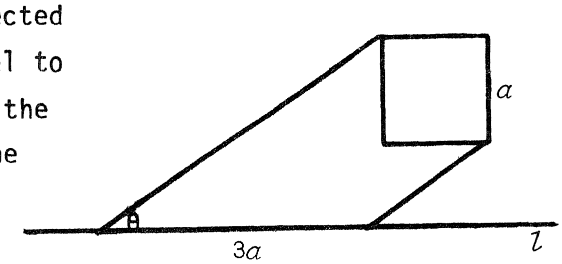
A4. N lots were sold in a lottery (N being a multiple of 10), and every red lot won a prize. Four out of the first one hundred lots sold were red, and of the remaining lots sold two out of every ten were red. At most 15% of the lots sold won a prize.

What is the largest value of N for which the above can be true?

A5. Determine $a + b + c + d$, given that

$$\begin{aligned} 6a + 2b &= 3840 \\ 6c + 3d &= 4410 \\ a + 3b + 2d &= 3080. \end{aligned}$$

A6. A square with edge length a is projected by parallel rays on a line l parallel to one of its sides (see figure). The length of the projection is $3a$. Determine the tangent of the angle θ between l and the projecting rays.



B1. Anita computes the sum of the squares of the first N positive integers, and Bernadette does the same with the squares of the following N integers. The difference of the two sums is 28224. Calculate N.

B2. On one of the sides of an angle A of 60° a point P is given such that $AP = 10$. On the other side of the angle a point Q is chosen in such a way that $AP^2 + AQ^2 + PQ^2$ is a minimum. Compute AQ.

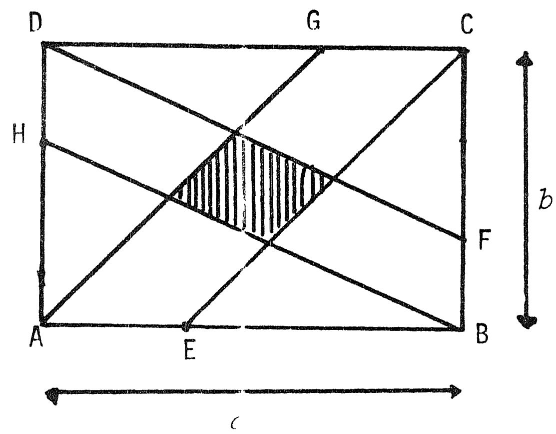
B3. A rectangular block with edge lengths a,b,c has a volume numerically equal to its total surface area. Furthermore, it is given that a,b,c are integers with $a < b < c$. Determine all possible triples (a,b,c) satisfying these conditions.

B4. a, b, and c are positive integers of 2, 3, and 5 digits, respectively, all digits being less than 9. The digits of c are distinct. Furthermore, $a \cdot b = c$, and adding 1 to each digit does not affect the truth of this equation. Determine a, b, and c.

C1. On the sides AB, BC, CD, DA of rectangle ABCD with sides of lengths a and b, points E, F, G, H are chosen such that

$$AE = \tfrac{1}{2}EB,\ BF = \tfrac{1}{2}FC,\ CG = \tfrac{1}{2}GD,\ DH = \tfrac{1}{2}HA,$$

as shown in the figure. Determine the area of the quadrangle bounded by the lines AG, BH, CE, and DF.



C2. The terms a_n of a sequence of positive integers satisfy

$$a_{n+3} = a_{n+2}(a_{n+1} + a_n), \qquad n = 1,2,3,\ldots .$$

Compute a_7 if it is given that $a_6 = 144$.

C3. A carpenter saws from a block a polyhedron with 30 vertices and 18 faces. The faces are 5 quadrangles, 6 pentagons, and 7 hexagons. How many interior diagonals does it have? (An interior diagonal connects two vertices not in the same face.)

1985 DUTCH MATHEMATICAL OLYMPIAD - Second Round

September 13, 1985. Time: 3 hours

1. For a certain real number p, the equation

$$x^3 + px^2 + 3x - 10 = 0$$

has three real roots a,b,c satisfying $c - b = b - a > 0$. Determine p, a, b, and c.

2. Among the numbers $11n + 10^{10}$, $1 \le n \le 10^{10}$, how many are squares?

3. In a factory, square tables of size 40×40 cm^2 are tiled with four tiles of size 20×20 cm^2. All tiles are the same, and decorated in the same way with an asymmetric pattern such as the letter J. How many different types of tables can be produced in this way?

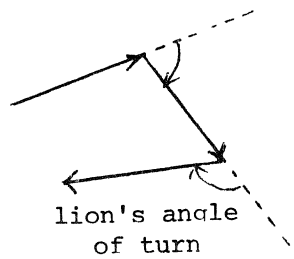
4. A convex hexagon ABCDEF is such that each of the diagonals AD, BE, and CF divides the hexagon into two parts of equal area. Prove that AD, BE, and CF are concurrent.

*

I now present solutions to some problems published earlier in this column.

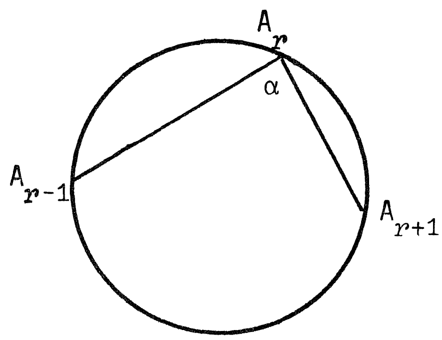
10. [1981: 16] *From a Preparation Problem Set for the 36th Moscow Olympiad (1973).*

A lion runs about the circular arena (radius 10 metres) of a circus tent. Moving along a broken line, he runs a total of 30 km. Show that the sum of the angles through which he turns (see figure) is not less than 2998 radians.



Solution by M.S.K.

To minimize the sum of the angles turned by the lion, each segment of his broken line path, except for the last one, must be as long as possible, i.e., each must be a chord of the circular arena. Also, the segments, except possibly for the last one, must be of equal length. To see this, consider two consecutive segments of the path starting at point A_{r-1} and ending at point A_{r+1} (see figure). Since the angle α is constant for all points A_r in the open arc $\overset{\frown}{A_{r-1}A_{r+1}}$, A_r must be such that $\overline{A_{r-1}A_r} + \overline{A_rA_{r+1}}$ is a maximum. This implies that $\overline{A_{r-1}A_r} = \overline{A_rA_{r+1}}$.



Let the path consist of $n+1$ segments where all segments, except possibly the last one, are equal and each subtends an angle 2θ at the center of the arena. Since the length of each segment is $2R\sin\theta$, where R is the radius, the path length L satisfies

$$L = 2Rn\sin\theta + k, \qquad \text{where } 0 < k \le 2R\sin\theta.$$

The lion turns through an angle 2θ for each pair of consecutive segments. Thus the sum of the angles through which the lion turns is $S(\theta) = 2n\theta$. Eliminating n, we obtain

$$S(\theta) = \frac{\theta(L - k)}{R\sin\theta} > \frac{L - 2R\sin\theta}{R} > \frac{L}{R} - 2,$$

since $\sin\theta \le 1$ and $\theta/\sin\theta > 1$. For $R = 10$ and $L = 30000$, we get

$$S(\theta) > 3000 - 2 = 2998 \text{ radians.}$$

*

11. [1981: 16] *From a Preparation Problem Set for the 36th Moscow Olympiad (1973).*

Show that in any convex equilateral (but not necessarily regular) pentagon one may place an equilateral triangle so that one of its sides coincides with a side of the pentagon and the entire triangle lies within the pentagon.

Solution by M.S.K.

Our proof is indirect. Suppose it cannot be done. If the consecutive vertices of the pentagon are A,B,C,D,E, then in each pair of angles A and B, B and C, C and D, D and E, and E and A, at least one angle must be less than 60°. This implies that at least three of the five angles are each less than 60°. This gives a contradiction, since in any convex pentagon the sum of the angles is 540° whereas our assumption yields

$$A + B + C + D + E < 60^\circ + 60^\circ + 60^\circ + 180^\circ + 180^\circ = 540^\circ.$$

*

6. [1981: 42] *From the 1980 Austrian-Polish Competition.*

Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \le 1$ for all positive integers k and m, prove that, for all positive integers p and q,

$$|(a_p/p) - (a_q/q)| < (1/p) + (1/q). \tag{1}$$

Solution by M.S.K.

Our proof is by repeated induction. The case $p = q$ is trivial. We first establish the following stronger result for the case $q = 1$:

$$|(a_p/p) - (a_1/1)| \le 1 - \frac{1}{p}. \tag{2}$$

That this holds for $p = 2$ follows from the hypothesis, for

$$|a_2 - a_1 - a_1| \le 1 \implies |(a_2/2) - (a_1/1)| \le 1 - \frac{1}{2}.$$

Suppose (2) holds for some $p \ge 2$. Then

$$|a_{p+1} - a_p - a_1| \le 1 \qquad \text{and} \qquad |a_p - pa_1| \le p - 1$$

together imply that

$$|a_{p+1} - a_1 - pa_1| \le p, \qquad \text{or} \qquad |(a_{p+1}/(p+1)) - (a_1/1)| \le 1 - \frac{1}{p+1}.$$

Thus (2) holds for all p, and (1) holds *a fortiori* for all pairs $(p,1)$. Note that (1) also holds for all pairs $(2p,p)$, since

$$|a_{2p} - a_p - a_p| \le 1 \quad \Rightarrow \quad |(a_{2p}/2p) - (a_p/p)| \le \frac{1}{2p} < \frac{1}{2p} + \frac{1}{p}.$$

We now assume that (1) is valid for some pair p,q with $p \ne q$. Then

$$|a_{p+q} - a_p - a_q| \le 1 \qquad \text{and} \qquad |a_p - p(a_q/q)| < 1 + \frac{p}{q}$$

together imply that

$$|a_{p+q} - a_q - p(a_q/q)| < 1 + \frac{p+q}{q}, \qquad \text{or} \qquad |(a_{p+q}/(p+q)) - (a_q/q)| < \frac{1}{p+q} + \frac{1}{q}.$$

Thus (1) will also be valid for the pair $(q,p+q)$ and symmetrically for the pair $(p,p+q)$. Using this last result together with the valid pairs $(p,1)$, $(p,2p)$ for all p, it follows inductively that (1) is valid for all p,q.

*

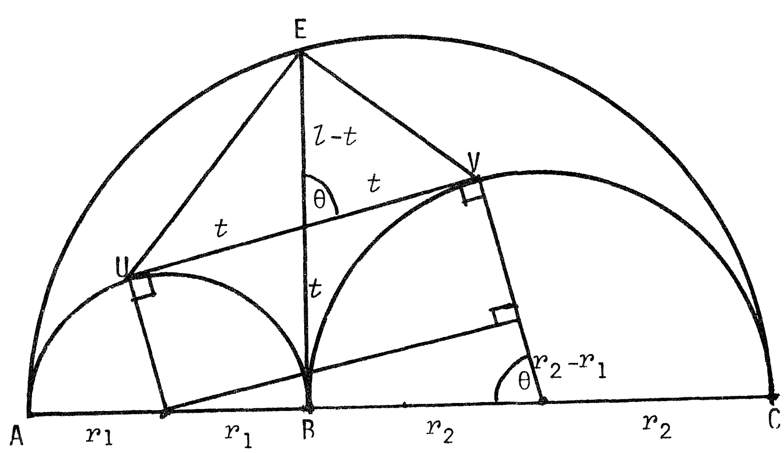
2. [1981: 43] *From the 1980 Competition in Mersch, Luxembourg.*

Let A,B,C be three collinear points with B between A and C. On the same side of AC are drawn the three semicircles on AB, BC, and AC as diameter. The common tangent at B to the first two semicircles meets the third at E. Let U and V be the points of contact of the other common tangent of the first two semicircles. Calculate the ratio

$$\frac{\text{area of triangle EUV}}{\text{area of triangle EAC}}$$

as a function of $r_1 = \frac{1}{2}\text{AB}$ and $r_2 = \frac{1}{2}\text{BC}$.

Solution by M.S.K.



Referring to the firgure, where BE = l, we have $l^2 = 4r_1r_2$ and

$$(2t)^2 = (r_1+r_2)^2 - (r_2-r_1)^2 = 4r_1r_2 = l^2.$$

With the brackets denoting area, we therefore have

$$[\text{EUV}] = t(l-t)\sin\theta = \frac{l^2\sin\theta}{4},$$

$$[\text{FAC}] = l(r_1+r_2),$$

so

$$\frac{[\text{EUV}]}{[\text{EAC}]} = \frac{l\sin\theta}{4(r_1+r_2)} = \frac{l\cdot 2t}{4(r_1+r_2)^2} = \frac{r_1r_2}{(r_1+r_2)^2}.$$

*

5, [1981: 44] *From the 1980 Competition in Mariehamn, Finland.*

A horizontal line (i.e., parallel to the x-axis) is called *triangular* if it intersects the curve with equation

$$y = x^4 + px^3 + qx^2 + rx + s$$

in four distinct points A, B, C, and D (in that order from left to right) in such a way that AB, AC, and AD could be the lengths of the sides of a triangle. Prove that either all or none of the horizontal lines which cut the curve in four distinct points are triangular.

Solution by M.S.K. and A. Meir, University of Alberta.

In what follows, it is to be understood that the lines we consider are only the horizontal lines, of equation $y = k$, which intersect the given curve in four distinct points. For a line $y = k$ to be triangular, it is necessary and sufficient that AB + AC > AD, or AB > CD. Clearly, AB - CD $\equiv f(k)$ is a continuous function of k. Suppose a k_1 exists such that $f(k_1) > 0$. We show by an indirect proof that all other lines are then triangular. Assume, on the contrary, that there is a k_2 such that $f(k_2) \le 0$. Then, by continuity, there is a k_3 such that $f(k_3) = 0$. Now consider the equation of the given curve referred to a translated coordinate system (X,Y) whose origin is the midpoint of the segment AD corresponding to $k = k_3$. This equation must then have the form

$$Y = (X-a)(X-b)(X+b)(X+a), \qquad \text{or} \qquad Y = (X^2 - a^2)(X^2 - b^2).$$

Since this equation is symmetric with respect to the Y-axis, we must have AB = CD for all lines. This gives the contradiction. Thus AB > CD for all lines.

Similarly, if there is one line such that AB < CD (or AB = CD), then this property will hold for all lines.

*

3. [1981: 46] *From the Final Round of the 1978 Rumanian Mathematical Olympiad (for 11th class).*

Two persons, A and B, take turns in assigning real values to the empty cells of a 3×3 array until, after nine plays, a complete 3×3 matrix results. Show that, whether or not he plays first, A can choose his entries in such a way that the resulting matrix is singular.

Solution by M.S.K.

We show that A can get a singular matrix if he plays first. All his moves will consist of zero entries and his first move is in the center cell. B then has only two essentially different first moves (to a corner cell or to a middle boundary cell), and his subsequent moves are all forced by his attempt to avoid a singular matrix. With the subscripts indicating the order of the moves, we have the following matrix determinants:

$$\begin{vmatrix} B_1 & B_3 & \\ 0_2 & 0_1 & B_2 \\ 0_4 & 0_3 & \end{vmatrix} \qquad \text{or} \qquad \begin{vmatrix} B_3 & B_1 & \\ 0_2 & 0_1 & B_2 \\ 0_3 & 0_4 & \end{vmatrix}.$$

We show that A can also get a singular matrix if he plays second. We can assume that B's first entry is in the top left cell, since we can always get any entry into that position by interchanging rows and columns, and these operations can at most change the sign of the determinant. If B's first move is a zero, then A can force a singular matrix as follows:

$$\begin{vmatrix} 0 & 0_3 & \\ 0_1 & 0_2 & B_3 \\ B_2 & & \end{vmatrix}.$$

If $B_1 \neq 0$ and A's first move is a zero in the center cell, then there are four essentially different moves for B_2, and in each case A can force a singular matrix. Thus

$$\begin{vmatrix} B_1 & B_3 & \\ B_2 & 0_1 & 0_4 \\ B_4 & 0_2 & 0_3 \end{vmatrix}, \quad \begin{vmatrix} B_1 & 0_4 & 0_3 \\ B_3 & 0_1 & 0_2 \\ B_2 & & B_4 \end{vmatrix}, \quad \begin{vmatrix} B_1 & B_3 & \\ 0_3 & 0_1 & B_4 \\ 0_4 & 0_2 & B_2 \end{vmatrix}, \quad \begin{vmatrix} B_1 & 0_4 & 0_3 \\ B_3 & 0_1 & 0_2 \\ & B_2 & B_4 \end{vmatrix}.$$

*

2. [1981: 74] *From a Rumanian Selection Test for the 1978 I.M.O.*

Let $k, l \geq 1$ be fixed natural numbers. Show that if $(11m-1,\ k) = (11m-1,\ l)$ for any natural number m (where (x,y) means the greatest common divisor of x and y), then there is an integer n such that $k = 11^n \cdot l$.

Solution by M.S.K.

Let k have the form

$$k = 11^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$

where the p_i are distinct primes other than 11, and α and the α_i are nonnegative integers. It follows that a number m exists such that $k/11^{\alpha}$ divides $11m-1$. Consequently, l must have the form

$$l = 11^{\beta} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},$$

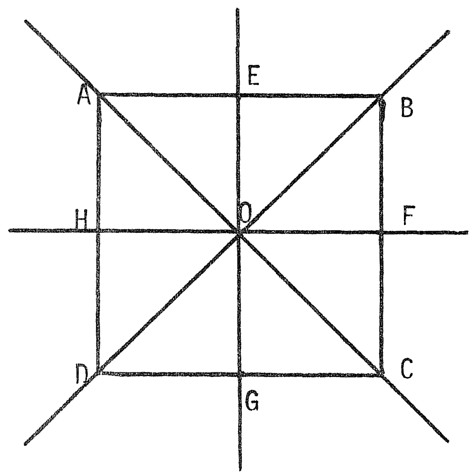
giving the desired result.

*

5. [1981: 74] *From the First Selection Test for the Rumanian team for the 1978 I.M.O.*

Show that there is no square whose vertices are located on four concentric circles whose radii are in arithmetic progression.

Solution by M.S.K.



It is tacit that the four circles are distinct. Our proof is indirect, so we assume that such a square exists and obtain a contradiction. Let ABCD be the square and consider its four axes of symmetry shown in the figure.

The center P of the circles cannot lie on any axis of symmetry, for then the circles would not be distinct. We can assume without loss of generality that P lies in the interior of angle AOE. Then PA is the smallest radius and PC is the largest radius. Also,

$$PA + PC = PB + PD = k, \text{ say.} \qquad (1)$$

Now consider the locus of a point such that the sum of its distances from A and C is k. This is an ellipse e with foci A and C. Similarly, the locus of a point such that the sum of its distances from B and D is k is a congruent ellipse e' with foci B and D. Consequently, P must be a point of intersection of e and e'. This gives the contradiction since by symmetry e and e' can only intersect on the axes EG and HF. □

In a similar vein, we show that there is no square whose vertices are located on four distinct concentric circles whose radii are in geometric progression. In view of the hypotheses, the equivalent of (1) is

$$\frac{PD}{PA} = \frac{PC}{PB} = k, \text{ say.}$$

Now the locus of a point such that the ratio of its distance from D to its distance from A is k is an Apollonian circle c with center on AD. Similarly, the locus of a point such that the ratio of its distance from C to its distance from B is k is a congruent circle c', and c and c' are symmetric with respect to axis EG. Consequently, as before, P must be a point of intersection of c and c', which can only be on EG.

*

4. [1981: 77] *From the Third Selection Test for the Rumanian team for the 1978 I.M.O.*

For a natural number $n \geq 1$, solve the equation

$$\sin x \sin 2x \ldots \sin nx + \cos x \cos 2x \ldots \cos nx = 1.$$

Solution by M.S.K.

For simplicity we restrict our solutions to $0 \leq x < 360^\circ$.

For $n = 1$,

$$\sin x + \cos x = \sqrt{2}\sin(x+45^\circ) = 1 \quad \Longrightarrow \quad x+45^\circ = 45^\circ \text{ or } 135^\circ$$
$$\Longrightarrow \quad x = 0 \text{ or } 90^\circ.$$

For $n = 2$,

$$\sin x \sin 2x + \cos x \cos 2x = \cos(2x-x) = 1 \quad \Longrightarrow \quad x = 0.$$

For $n > 2$, by Hölder's inequality,

$$\begin{aligned}1 &= (\sin^2 x + \cos^2 x)^{1/n}(\sin^2 2x + \cos^2 2x)^{1/n}\ldots(\sin^2 nx + \cos^2 nx)^{1/n}\\ &\geq (\sin^n x + \cos^n x)^{1/n}(\sin^n 2x + \cos^n 2x)^{1/n}\ldots(\sin^n nx + \cos^n nx)^{1/n}\\ &\geq \sin x \sin 2x\ldots\sin nx + \cos x\cos 2x\ldots\cos nx\\ &= 1.\end{aligned}$$

Consequently, equality holds throughout and the only solution is $x = 0$.

Editor's note. All communications about this column should be sent directly to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permisssion.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before July 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1080*. [1985: 250] (Corrected) *Proposed by D.S. Mitrinović, University of Belgrade, Yugoslavia.*

Determine the maximum value of

$$f(a,b,c) = \left|\frac{b-c}{b+c} + \frac{c-a}{c+a} + \frac{a-b}{a+b}\right|,$$

where a,b,c are the side lengths of a nondegenerate triangle.

1101. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

This problem is the dual of Crux 941 [1985: 227]. Independently solve each of the following alphametics in base ten:

6•FLOCK = GEESE,

7•FLOCK = GEESE,

8•FLOCK = GEESE.

1102. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be an n-simplex in n-dimensional Euclidean space. Let M be an interior point of σ_n whose barycentric coordinates are $(\lambda_0,\lambda_1,\ldots,\lambda_n)$ and, for $i = 0,1,\ldots,n$, let p_i be its distance from the $(n-1)$-face

$$\sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n.$$

Prove that $\lambda_0p_0 + \lambda_1p_1 + \ldots + \lambda_np_n \ge r$, where r is the inradius of σ_n.

1103. *Proposed by Roger Izard, Dallas, Texas.*

Three concurrent cevians through the vertices A,B,C of a triangle meet the lines BC,CA,AB in D,E,F, respectively, and the internal bisector of angle A meets BC in V. If A,F,D,V,E are all concyclic, prove that AD ⊥ BC.

1104*. *Proposed by D.S. Mitrinović, University of Belgrade, Yugoslavia.*

As a by-product, Giuseppina Masotti Biggiogero (*Ist. Lombardo, Sci. Lett. Rend. Cl. Sci. Math. Nat.*, (3) 14 (83) (1950) 735-752) obtained the following result about a square matrix (a_{ij}) of order $m+1$:

If

$$a_{ij} = \begin{cases} c, & \text{if } i = j, \\ m-j+1, & \text{if } i = j+1, \\ -j+1, & \text{if } i = j-1, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\det (a_{ij}) = \begin{cases} c(c^2+2^2)(c^2+4^2)\ldots(c^2+m^2), & \text{if } m \text{ is even,} \\ (c^2+1^2)(c^2+3^2)\ldots(c^2+m^2), & \text{if } m \text{ is odd.} \end{cases}$$

Prove this assertion.

1105*. *Proposed by Lászlő Csirmaz, Mathematical Institute of the Hungarian Academy of Sciences.*

The Extraterrestials' Secret Invisible Submarine (ESIS) carries a deadly weapon through the Grid Ocean $(Z\times Z)$. You, as the Captain of the Moast Advanced Space Command, are the only person in the universe who can save (wo)mankind by destroying the ESIS. Your Secret Agent reported that the command computer of the enemy's submarine was programmed to travel at a uniform speed along a straight line so that the ESIS must be in a grid point at every full hour, and only then is its energy shield weak enough to blow it up. You have infinite supplies, but only countably many rockets. The rockets can reach any grid point within an hour. Your equipment, however, allows one launching per hour. Well, Captain, could you save me?

1106. *Proposed by Jack Garfunkel, Flushing, N.Y.*

The directly similar triangles ABC and DEC are both right-angled at C. Prove that

(a) AD $\perp$ BE;

(b) AD/BE equals the ratio of similitude of the two triangles.

1107. *Proposed by Jordi Dou, Barcelona, Spain.*

An equilateral skew pentagon has angles A,B,C,D all equal to 90°. Give a Euclidean construction for, and calculate the measure of, angle E, given that it is not 60°. (See Problem 1 [1985: 240].)

1108. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Solve the equation $4\sec x + 3\csc x = 10$ for $0 < x < 360^\circ$.

1109. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is a triangle with orthocentre H. A rectangular hyperbola with centre H intersects line BC in A_1 and A_2, line CA in B_1 and B_2, and line AB in C_1 and C_2. Prove that the points P,Q,R, the midpoints of A_1A_2, B_1B_2, C_1C_2, respectively, are collinear.

1110*. *Proposed by M.S. Klamkin, University of Alberta.*

How many different polynomials $P(x_1, x_2, \ldots, x_m)$ of degree n are there for which the coefficients of all the terms are 0's or 1's and

$$P(x_1, x_2, \ldots, x_m) = 1 \quad \text{whenever} \quad x_1 + x_2 + \ldots + x_m = 1?$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

979. [1984: 263] *Proposed by R.B. Killgrove, Alhambra, California.*

(a) Find a ring of smallest possible order which has nonzero products but in which all squares are zero.

(b) Characterize all rings R with the property that, for all $x,y \in R$, $x \neq y$ implies $xy = 0$.

Solution by the proposer.

(a) We will show that no ring of order less than 8 has the desired property, and then find a ring of order 8 that has.

Let $(R,+,\circ)$ be a ring in which the Abelian group $(R,+)$ is cyclic, with generator 1, say. If all squares are zero, then in particular $1\circ 1 = 0$ and every product is of the type

$$(1+1+\ldots+1)\circ(1+1+\ldots+1) = 1\circ 1 + 1\circ 1 + \ldots + 1\circ 1 = 0.$$

So we may restrict our attention to rings $(R,+,\circ)$ in which the Abelian group $(R,+)$ is noncyclic. All Abelian groups of orders 2, 3, 5, 6, and 7 are cyclic, but there is one noncyclic Abelian group of order 4 and there are two of order 8.

For the ring $(R,+,\circ)$ of order 4 with a noncyclic additive group, we may without loss of generality assume that $(R,+)$ and $(R,\circ)$ are given by Tables 1 and 2, respectively, the second of which is dictated by the left distributive law. Suppose all squares are zero. Then $a = d = a+b+c+d = 0$, so $b+c = 0$. Then $b = c$ since $1 = -1$, and the multiplication is commutative. At this point, all products are zero except possibly for $1\circ 2$, $1\circ 3$, and $2\circ 3$, all of which are equal to b.

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 1

$\circ$	0	1	2	3
0	0	0	0	0
1	0	a	b	$a+b$
2	0	c	d	$c+d$
3	0	$a+c$	$b+d$	$a+b+c+d$

Table 2

Now

$b = 1 \implies b = 1\circ 2 = b\circ 2 = (1\circ 2)\circ 2 = 1\circ(2\circ 2) = 1\circ 0 = 0$ (contradiction),

$b = 2 \implies b = 1\circ 2 = 1\circ b = 1\circ(1\circ 2) = (1\circ 1)\circ 2 = 0\circ 2 = 0$ (contradiction),

$b = 3 \implies b = 1\circ 3 = 1\circ b = 1\circ(1\circ 3) = (1\circ 1)\circ 3 = 0\circ 3 = 0$ (contradiction).

Therefore $b = 0$ and all squares are zero.

We now describe a ring $(R,+,\circ)$ of order 8, with a noncyclic additive group $(R,+)$, which has nonzero products but in which all squares are zero. Its additive group is that of the field GF(8) $= (R,+,\cdot)$ and its multiplication is defined in terms of that of GF(8) by

$$a\circ b = a\cdot b\cdot(a+1)\cdot(b+1)\cdot(a+b)\cdot(a+b+1). \tag{1}$$

Clearly $\circ$ is commutative. We will know that $(R,+,\circ)$ is a ring when we have shown that $\circ$ is associative and that the left distributive law holds. We will prove this later.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Table 3

$\circ$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1
3	0	0	0	0	1	1	1	1
4	0	0	1	1	0	0	1	1
5	0	0	1	1	0	0	1	1
6	0	0	1	1	1	1	0	0
7	0	0	1	1	1	1	0	0

Table 4

The additive structure is given by Table 3. It is clear from (1) that

$$a\circ b = 0 \quad \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \quad \text{or} \quad a = 1 \quad \text{or} \quad b = 1 \quad \text{or} \quad a = b \quad \text{or} \quad a = b+1.$$

This accounts for all the zero products in Table 4. For the remaining products in Table 4, the six factors on the right of (1) must all be distinct and none must equal 0 or 1, that is,

$$a\circ b = 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 \qquad \text{(in some order)}. \tag{2}$$

Now the multiplicative group of nonzero elements of GF(8) is a cyclic group of order 7, so no element other than 1 is its own inverse (relative to $\cdot$). Therefore the six factors on the right of (2) can be partitioned into three pairs of inverses, and we have

$$a \circ b = 1 \cdot 1 \cdot 1 = 1.$$

This accounts for the remaining products in Table 4.

To show that $(R,+,\circ)$ is indeed a ring, we note that, from Table 4,

$$(a \circ b)\circ c = \begin{cases} 0 \circ c = 0 \\ \quad\text{or} \\ 1 \circ c = 0 \end{cases} \qquad \text{and} \qquad a \circ (b \circ c) = \begin{cases} a \circ 0 = 0 \\ \quad\text{or} \\ a \circ 1 = 0 \end{cases}$$

so $\circ$ is associative; and it can be verified that

$$\begin{aligned} a\circ(b+c) &= (a^2+a)\cdot(b^4+c^4+a^2b^2+a^2c^2+a^2b+ab^2+a^2c+ac^2+b^2+c^2+ab+ac) \\ &= a\circ b + a \circ c, \end{aligned}$$

so the left distributive law holds.

(b) Zero rings (i.e., rings in which every product is zero) of all orders clearly have the desired property, as has the ring of order 2 defined by

+	0	1
0	0	1
1	1	0

and

$\circ$	0	1
0	0	0
1	0	1

.

We show that no other ring has the desired property.

Let $(R,+,\circ)$ be a ring in which R contains at least three distinct elements, and such that, for all $x,y \in R$, $x \neq y$ implies $x \circ y = 0$. Let a and b be two distinct nonzero elements of R. Then $a \neq a+b$, so $a\circ(a+b) = a\circ a + a \circ b = 0$. But also $a \neq b$, so $a \circ b = 0$. Hence $a \circ a = 0$, all squares are zero, and $(R,+,\circ)$ is a zero ring.

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984. [1984: 291] *Proposed by J.C. Fisher and H.N. Gupta, University of Regina.*

For which $k \geq 3$ is $\binom{k}{2} - 1$ a prime power p^n?

Solution by Dmitri Thoro, San Jose State University.

We have $\binom{k}{2} - 1 = 2^1$ and 5^1 for $k = 3$ and 4, respectively. Let $k \geq 5$. Then either $k = 2t$ or $k-1 = 2t$, where $t \geq 2$. Thus

$$\binom{k}{2} - 1 = (2t \pm 1)(t \mp 1) = p^n.$$

This implies that $2t \pm 1 = p^a$ and $t \mp 1 = p^b$, from which

$$2p^b \pm 3 = p^a, \qquad p^b = 3^1, \qquad \text{and} \qquad p^a = 3^2 \qquad \text{or} \qquad 3^1.$$

Thus $\binom{k}{2} - 1 = 3^3$ or 3^2, and $k = 8$ or 5. All solutions are included in $k = 3,4,5,8$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California (incomplete solution); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARK KANTROWITZ, student, Maimonides School, Brookline, Massachusetts (incomplete solution); FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; EDWIN M. KLEIN, University of Wisconsin-Whitewater; SUSIE LANIER and DAVID R. STONE, Georgia Southern College, Statesboro (jointly); BOB PRIELIPP, University of Wisconsin-Oshkosh; KENNETH M. WILKE, Topeka, Kansas; and the proposers.

* * *

985. [1984: 292] *Proposed by John J. Martinez, Gonzaga High School, Washington, D.C.*

Let ABC be a triangle with sides a,b,c, and let AP,BQ,CR be concurrent cevians terminating in the opposite sides at P,Q,R. We use square brackets to denote the area of a triangle.

(a) If AP,BQ,CR are the internal angle bisectors of the triangle, prove that [PQR] = [BPR] if and only if a,b,c, in some order, are in arithmetic progression.

(b) If AP,BQ,CR are the altitudes of the triangle and p,q,r are the sides of triangle PQR, prove that

$$\frac{[PQR]}{[ABC]} = \frac{2pqr}{abc}.$$

Solution by George Tsintsifas, Thessaloniki, Greece.

(a) With l = BP/PC, m = CQ/QA, n = AR/RB, it is known [1] that

$$\frac{[PQR]}{[ABC]} = \frac{lmn + 1}{(l+1)(m+1)(n+1)}.$$

Since here $l = c/b$, $m = a/c$, $n = b/a$, we obtain

$$\frac{[PQR]}{[ABC]} = \frac{2abc}{(b+c)(c+a)(a+b)}. \qquad (1)$$

Also

$$\frac{[BPR]}{[ABC]} = \frac{BP\cdot BR}{ac} = \frac{ac}{(b+c)(a+b)}. \qquad (2)$$

Hence, from (1) and (2),

$$[PQR] = [BPR] \iff 2b = c+a \iff a,b,c \text{ are in A.P.}$$

(b) Let ρ be the circumradius of triangle ABC (unaccustomed notation to avoid confusion with the point R). The nine point circle of triangle ABC coincides with the circumcircle of triangle PQR, so the radius of the latter is $\rho/2$. (Note that this is true for *any* triangle ABC, even for a right triangle.) Hence

$$\frac{[PQR]}{[ABC]} = \frac{pqr/2\rho}{abc/4\rho} = \frac{2pqr}{abc}.$$

Also solved by H. FUKAGAWA, Yokosuka High School, Aichi, Japan; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, College of William and Mary, Williamsburg; and the proposer.

Editor's comment.

Relative to part (b), Klamkin noted that it would be of interest to determine all points S in the plane such that the three concurrent cevians through S give the same result as the orthocentre.

Some solvers assumed, tacitly or otherwise, that triangle ABC was acute for part (b); and one solver "proved" that part (b) never holds if triangle ABC is obtuse. We give one counterexample to show that this last solver is all wet. If

$$a = 16, \quad b = \sqrt{41}, \quad c = 13, \quad \text{then } A \approx 106^\circ, \quad [ABC] = 40,$$

$$p = -a\cos A = \frac{368}{13\sqrt{41}}, \quad q = b\cos B = \frac{12\sqrt{41}}{13}, \quad r = c\cos C = \frac{52}{\sqrt{41}}, \quad [PQR] = \frac{88320}{6929},$$

and

$$\frac{[PQR]}{[ABC]} = \frac{2208}{6929} = \frac{2pqr}{abc}.$$

REFERENCE

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, Wiley, Toronto, 1969, p. 212, Problem 6 (solution p. 444).

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A MESSAGE FROM THE MANAGING EDITOR

Starting with the February issue, *Crux Mathematicorum* will have a new editor. He is Dr. Bill Sands of the University of Calgary. From now on, all communications concerning the content of *Crux* should be sent to him at the following address:

Dr. Bill Sands
Department of Mathematics
University of Calgary
Calgary, Alberta
Canada T2N 1N4

I would also like to take this opportunity to thank our retiring editor Léo Sauvé, who has dedicated over ten years of his life to editing *Crux*. His untiring efforts have established *Crux* as a world-class problem-solving journal. Unfortunately Léo has spent the last month in hospital. I am sure you all join me in wishing him a speedy recovery.