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# EUREKA

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CONTENTS

# THE THEOREMS OF CEVA AND MENELAUS

DAN PEDOE, University of Minnesota

There is a story of the late Emil Artin at a conference held in India, where some elementary geometry of the triangle was discussed, bursting out impatiently: "What *are* these theorems of Ceva and Menelaus that everyone is talking about?" Once upon a time every schoolboy knew that the Ceva theorem refers to a point and a triangle (Figure 1) and the Menelaus theorem to a line and a triangle (Figure 1). Let the triangle be ABC, the point O, and let the intersections of AO, BO and CO with BC, CA and AB respectively be A', B' and C'. Let the line intersect the sides BC, CA and AB in L, M and N respectively. Then the Ceva Theorem says that:

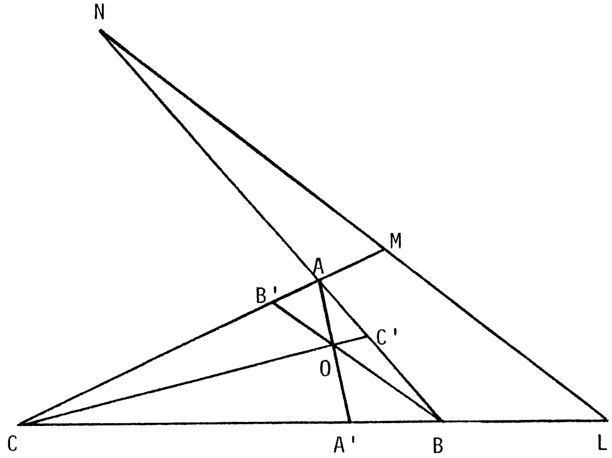


Figure 1

$$\frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} \cdot \frac{\overline{AC'}}{\overline{C'B}} = 1,$$

and the Menelaus Theorem that:

$$\frac{\overline{BL}}{\overline{LC}} \cdot \frac{\overline{CM}}{\overline{MA}} \cdot \frac{\overline{AN}}{\overline{NB}} = -1.$$

In each case we are dealing with the product of three position-ratios. If X, Y, Z are three collinear points, and we agree to write the position-ratio $\overline{XY}/\overline{YZ}$ of Y with respect to X and Z as $(XZ)_Y$, the Ceva Theorem is

$$(BC)_{A'}(CA)_{B'}(AB)_{C'} = 1,$$

and the Menelaus Theorem is

$$(BC)_L(CA)_M(AB)_N = -1.$$

It is not difficult to prove these theorems (see [2], pp. 28, 31, 249-250) and there are various methods which are possible. In a course I have been giving recently, I showed that *assuming* the Menelaus Theorem, that of Ceva can be obtained, using Menelaus twice, and a little arithmetic. I set my students the converse problem: *assuming* Ceva, prove Menelaus. They were unable to do this. When I tried, I also failed!

In such a predicament, one calls on friends, and I asked Dan Sokolowsky, well known to readers of this magazine, for help. He succeeded where I had failed, using Ceva seven times. Knowing that it was possible made all the difference, and I now present a symmetrical method of obtaining the one theorem from the other in three applications. Dan Sokolowsky tells me that Clayton W. Dodge [1] does it in six applications.

[If we begin with the Ceva Theorem, and call the points where B'C', C'A' and A'B' meet the sides BC, CA and AB of the triangle L, M and N, we know that the points L, M and N are collinear. This follows, say, from the fact that the triangles ABC and A'B'C' are in perspective from O, and the Desargues Theorem (see

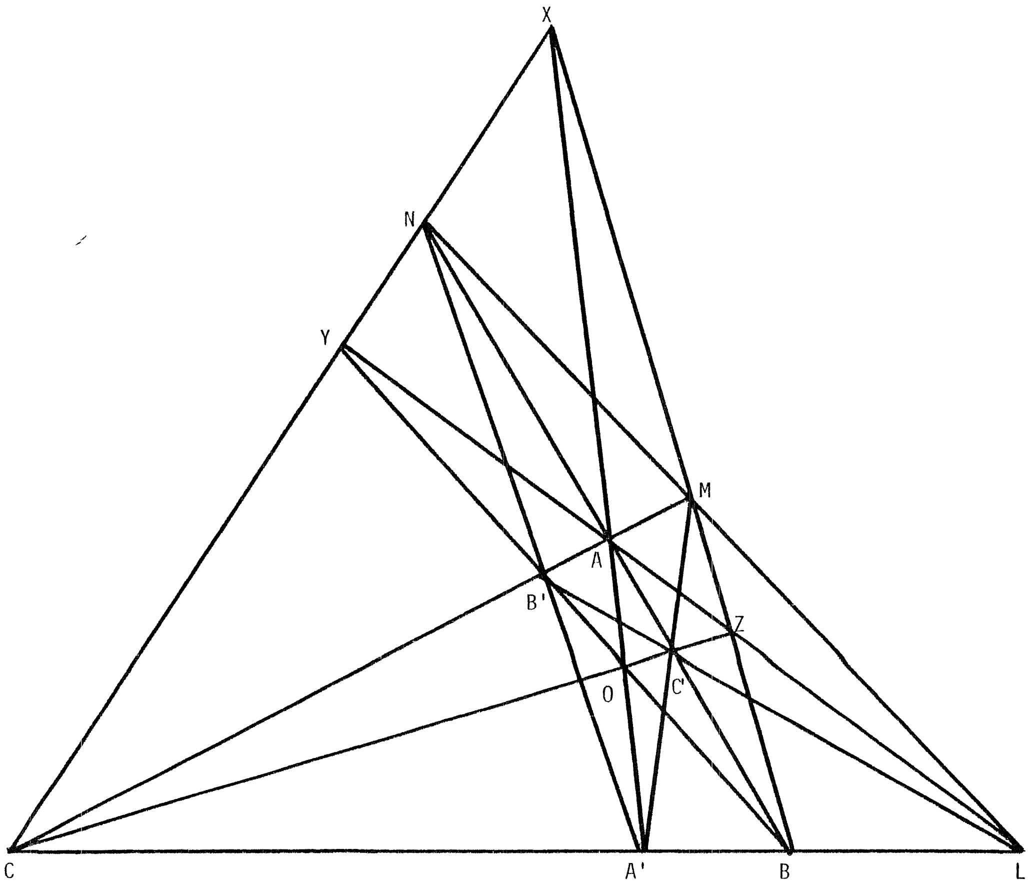


Figure 2

[2], p. 251). The point O therefore defines the line LMN.

If we start with the line LMN, a dual procedure must produce the point O, and it was by the use of this procedure that I obtained the complete Figure 2.

Assuming Ceva three times, for the triangle ABC and the points X, Y and Z in turn, we have:

$$(BC)_{A'}(CA)_M(AB)_N = 1,$$

$$(CA)_{B'}(AB)_N(BC)_L = 1,$$

$$(AB)_{C'}(BC)_L(CA)_M = 1.$$

Multiply these equations together, and we have

$$[(BC)_{A'}(CA)_{B'}(AB)_{C'}][(BC)_L(CA)_M(AB)_N]^2 = 1,$$

and since $(BC)_{A'}(CA)_{B'}(AB)_{C'} = 1$, we find that

$$(BC)_L(CA)_M(AB)_N = \pm 1.$$

It is interesting to note that a square root had to be taken in Dan Sokolowsky's proof. We can decide that the above product must equal -1 in Euclidean geometry, invoking the Pasch axiom, or any other suitable consideration.

To obtain Ceva from Menelaus, we again use triangle ABC, and the following sets of collinear points in succession: LB'C', MC'A' and then NA'B', and we have:

$$(BC)_L(CA)_{B'}(AB)_{C'} = -1,$$

$$(CA)_M(AB)_{C'}(BC)_{A'} = -1,$$

$$(AB)_N(BC)_{A'}(CA)_{B'} = -1,$$

and multiplication gives:

$$[\text{Menelaus}][\text{Ceva}]^2 = -1,$$

so that

$$(BC)_{A'}(CA)_{B'}(AB)_{C'} = \pm 1,$$

and we choose the plus sign in this case.

I cannot remember using duality in triangle geometry before. One lives and learns!

REFERENCES

1. C.W. Dodge, *Euclidean Geometry and Transformations*, Addison Wesley, 1972, pp. 22-23.

2. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, 1970.

# TEN ELEMENTS ON A PENTAGRAM

CHARLES W. TRIGG, Professor Emeritus, Los Angeles City College

When the sides of a regular pentagon are extended until they intersect, a pentagram is formed. This five-pointed star consists of five line segments that together form a closed path. Every line has exactly one point in common with every other line, so the ten intersection points lie by fours on the five lines.

An element pattern may be established by assigning an element to each of the intersection points. The pattern consists of five interwoven quartets of elements. Two patterns will be considered identical if one goes into the other by rotation or reflection. Essentially, a particular element can appear only in one of two positions — a point of the star or a vertex of the pentagon. For any quartet, if one element is positioned, the other members can be permuted into 3! orders. Thus any quartet can appear on its line in exactly 2(3!) or 12 orders (not counting reflections). It follows from the tightly interwoven relationship of the quartets that every basic pattern on the pentagram can appear in 12 different guises, all having the same five quartets. The quartets may appear in different positions on the pentagrams and their elements may be in different orders.

In Figure 1, the nodes of the pentagram have been identified by Roman numerals, thus fixing them in position. The placement of the numerals is such that when the 12 guises of a basic element pattern (with elements $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, and $j$) are recorded in compact tabular form, the invariance of composition of some of the quartets is quite evident. The five quartets are $(a, b, c, d)$, $(d, e, f, g)$, $(g, h, b, i)$, $(i, c, e, j)$, and $(j, f, h, a)$.

Figure 2 shows a second guise of the element pattern of Figure 1 with element $d$ still on a star point. Figure 3 shows another guise (the seventh in the table) wherein element $d$ is on a vertex of the pentagon.

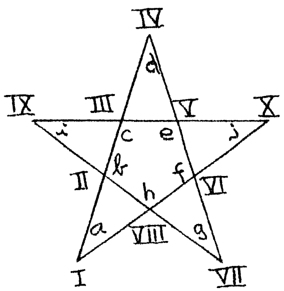


Figure 1

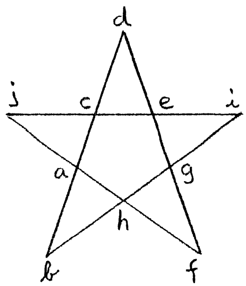


Figure 2

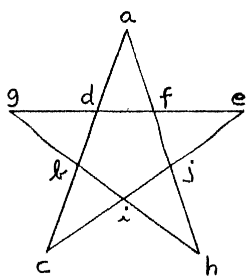


Figure 3

| I | II | III | IV | V | VI | VII | VIII | IX | X |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ | $j$ |
| $b$ | $a$ | $c$ | $d$ | $e$ | $g$ | $f$ | $h$ | $j$ | $i$ |
| $a$ | $c$ | $b$ | $d$ | $g$ | $f$ | $e$ | $j$ | $i$ | $h$ |
| $c$ | $a$ | $b$ | $d$ | $g$ | $e$ | $f$ | $j$ | $h$ | $i$ |
| $c$ | $b$ | $a$ | $d$ | $f$ | $e$ | $g$ | $i$ | $h$ | $j$ |
| $b$ | $c$ | $a$ | $d$ | $f$ | $g$ | $e$ | $i$ | $j$ | $h$ |
| $c$ | $b$ | $d$ | $a$ | $f$ | $j$ | $h$ | $i$ | $g$ | $e$ |
| $b$ | $c$ | $d$ | $a$ | $f$ | $h$ | $j$ | $i$ | $e$ | $g$ |
| $c$ | $a$ | $d$ | $b$ | $g$ | $i$ | $h$ | $j$ | $f$ | $e$ |
| $a$ | $c$ | $d$ | $b$ | $g$ | $h$ | $i$ | $j$ | $e$ | $f$ |
| $b$ | $a$ | $d$ | $c$ | $e$ | $i$ | $j$ | $h$ | $f$ | $g$ |
| $a$ | $b$ | $d$ | $c$ | $e$ | $j$ | $i$ | $h$ | $g$ | $f$ |

Table I

The 12 guises on a pentagram of a basic element pattern

Thus every magic pentagram can be converted into 11 other magic pentagrams.

The line sums of the Roman numerals are X, XXII, XXVI, XXVII, and XXV. On this basis the pentagram is antimagic.

* * *

## THE 1977 CARLETON UNIVERSITY MATHEMATICS COMPETITION

A number of Entrance Awards to Carleton University for the study of Mathematics will again be offered by the Department of Mathematics, based upon its Fifth Annual Mathematics Competition. The number of awards offered will depend upon funds available, but the minimum value of each award will be at least $500. These awards are contingent upon the student registering in the first year of a mathematics program at Carleton University.

The competition, which consists of two parts, is open to all Ontario high school students. Part A, which consists of a series of problems, is circulated to Ontario high schools around December 1, and solutions must be returned by the candidate before February 10, 1977 to: Mathematics Competition (Attention: Dr. K.S. Williams), Department of Mathematics, Carleton University, Ottawa, Ontario K1S 5B6.

Successful candidates will be invited to write Part B at Carleton University in March 1977. All entrants are eligible for book prizes, but only those accepted for admission to Carleton University will be considered for an Entrance Award.

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## POLITICAL GEOMETRY

From *The New York Times*, December 7, 1975, Section 4, p. 3:

> . . . As Secretary of State Henry A. Kissinger said, China and the United States "can pursue parallel policies where their interests converge" but still politely disagree in other areas.

(Thx to L.F. Meyers)

*An important new book*

REVIEW

*Geometry and the Liberal Arts*, by Dan Pedoe. A Peregrine Book published in 1976 by Penguin Books Ltd, to be distributed in Canada by Penguin Books Canada Ltd, 41 Steelcase Road West, Markham, Ontario L3R 1B4. 296 pages (Canadian price not yet available).

This fascinating and very informative account, written by one of the world's leading geometers, should help to rekindle interest in geometry in the heart of mankind.

Born in London in 1910, Professor Pedoe attended Cambridge University (1930-35), spent some time at the Institute for Advanced Study (1935-36), and was instructor, then reader in various universities in Britain. Later he became the first professor of mathematics at the University of Khartoum, Sudan (1952-59), then professor at the University of Singapore (1959-62). Since then he has been in the United States, first at Purdue University (1962-64), and then at the University of Minnesota where he has been since 1964. In addition to his well-known books *Methods of Algebraic Geometry* (written in collaboration with Sir William Hodge), *The Gentle Art of Mathematics*, *A Geometric Introduction to Linear Algebra*, *Circles*, and *A Course of Geometry for Colleges and Universities*, he made three films, still worth seeing, with the Minnesota College Geometry Project: *Orthogonal Projection*, *Central Similarities*, and *Inversion*. He has twin sons, both cardiologists in London. His hobbies are music, ceramic sculpture, painting in oils, and cooking for discerning guests (make a reservation!).

Now that we have met the author, let us take a look at *Geometry and the Liberal Arts*. The book fills the readers with excitement in revealing the fascination of the great practitioners of geometry, like Vitruvius, Leonardo and Dürer. It gives readers not only a diversion into the byways of history but also several glimpses of geometry's lasting appeal to famous artists, scientists, and philosophers. With his enormous scholarly experience and unusual love for the subject, the author guides the reader through the discoveries stimulated by the theory of perspective and discusses form in architecture, projective geometry, and mathematical curves as well as various aspects of space. All these different areas contain a generous selection of topics, which are threaded together very nicely by explanatory comments when necessary. Exercises at the end of each chapter are also given for those readers who may wish to experience the aesthetic appeal of geometry by carrying out some simple constructions and thereby enter into the more exciting domain of mathematical proofs. Very little previous acquaintance with algebra and geometry is assumed on the whole — except for some of the later sections — making this book worthwhile and easily accessible to students majoring in the liberal arts, as well as to the general public.

Appearing as a handy paperback with an attractive cover showing a detail of "Armadio Aperto," a marquetry panel by Giovanni da Verona, the book contains twenty-four plates of immense historical value. These plates include the famous illustration of Vitruvius' concepts of human proportions, war machines, water clock, Archimedean screw, church designs by Leonardo da Vinci, the Golden ratio as used by Sir Theodore Cook in the analysis of a Botticelli Venus and other delightful items. There are nine very interesting chapters.

Chapter 1 is a beautiful introduction to those aspects of geometry that are reflected in the work of Vitruvius. Vitruvius' design of a Roman theatre, the Archimedean screw, and war machines (siege tower) are some of the remarkable things described here that reflect the existence, even then, of a high-class technology.

Chapter 2 reveals the interest and excitement of practical geometry, as revealed in the work of Albrecht Dürer. The theory of perspective which stimulated the development of geometry in the time of Dürer is discussed. The ideas of orthogonals, vanishing line and point, equidistant tranversals, Dürer's approach to conic sections and his trailblazing use of planar nets for three-dimensional models, all make stimulating reading.

Chapter 3, on Leonardo da Vinci, is an extensive study of the third, after Vitruvius and Dürer, of the three great historical characters and their attitudes towards geometry. Leonardo's investigations into different kinds of shadow and his research on the proportions and movement of the human figure are all very appealing.

Chapter 4 discusses form in architecture and other important theories which have been fashionable from time to time. Dürer's extensive research on human proportions are once again spotlighted. His observation of the following relation:

$$(\text{neck to hip})(\text{knee to ankle}) = (\text{hip to knee})^2$$

is indeed amusing.

Chapter 5 is on Euclid's *Optics* and Chapter 6 on his great *Elements of Geometry*, which is still an inspiration to modern geometry, and which once shared the honour with the Bible of being the world's greatest best seller.

Chapter 7 introduces both Cartesian and Projective Geometry and shows how some of the great theorems in projective geometry can be used in perspective drawings.

Chapter 8 discusses some lovely curves from conic sections to equiangular spirals. The reader is shown how to produce these curves in a variety of ways. The focal properties of conic sections are also given for those who possess some mathematical vigor.

The final chapter deals with "space" from a number of different aspects, with adventures in two, three and four dimensions, concluding with interesting descriptions from Abbott's *Flatland, A Romance of Many Dimensions.*

The bibliography at the end will prove an excellent supplementary reading list for those who wish to pursue the subject to greater depths.

To conclude, the book is a simple and easily understandable account of geomtry filled with wit and humour! I urge all students and teachers of geometry to get a copy of this unique and delightful text and derive enjoyment from it.

V.P. MADAN
Red Deer College and
University of Western Ontario

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# PROBLEMS--PROBLÈMES

*Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without solution.*

*Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without his permission.*

*For the problems given below, solutions, if available, will appear in* EUREKA *Vol. 3, No. 4, to be published around April 15, 1977. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than April 1, 1977.*

**201.** *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve the cryptarithm $\text{LEO}^2 = \text{SAUVE}$.

**202.** *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Prove that any real number can be approximated within any $\epsilon > 0$ as the difference of the square roots of two natural numbers.

**203.** *Proposed by Charles W. Trigg, San Diego, California.*

On page 157 of his *Amusements in Mathematics*, Dover (1958), H.E. Dudeney makes the statement: "If you add together the digits of any number, and then as often as necessary, add the digits of the result, you must ultimately get a number composed of one figure. This last number I call the *digital root.*"

Prove or disprove: The digital root of every even perfect number greater than 6 is 1.

204. *Proposed by R. Robinson Rowe, Sacramento, California.*

A common $8 \times 10$-inch plate of coordinate paper is $W = 80$ spaces wide by $L = 100$ spaces long with $8000$ small squares.

(a) Including larger ones, how many squares are there?

(b) How many oblongs (nonsquare rectangles) are there?

205. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Find the least common multiple of the numbers

$$(29!)(37!) \quad \text{and} \quad (23!)(41!).$$

206. *Proposed by Dan Pedoe, University of Minnesota.*

A circle intersects the sides BC, CA and AB of a triangle ABC in the pairs of points X, X', Y, Y' and Z, Z' respectively. If the perpendiculars at X, Y and Z to the respective sides BC, CA and AB are concurrent at a point P, prove that the respective perpendiculars at X', Y' and Z' to the sides BC, CA and AB are concurrent at a point P'.

207. *Proposed by Ross Honsberger, University of Waterloo.*

Prove that $\dfrac{2r+5}{r+2}$ is always a better approximation to $\sqrt{5}$ than $r$.

208. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $a$ and $b$ be real numbers such that $a \geq b \geq 0$. Determine a matrix $X$ such that

$$X^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

209. *Proposed by L.F. Meyers, The Ohio State University.*

Suppose that the sequence $(a_n)_{n=1}^{\infty}$ of nonnegative real numbers converges to $0$. Show that there exists a sequence $(e_n)_{n=1}^{\infty}$ each of whose terms is $1$ or $-1$ such that $\sum_{n=1}^{\infty} e_n a_n$ converges.

210. *Proposed by Murray S. Klamkin, University of Alberta.*

P, Q, R denote points on the sides BC, CA and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{\mathrm{BP}}{\mathrm{BC}} = \frac{\mathrm{CQ}}{\mathrm{CA}} = \frac{\mathrm{AR}}{\mathrm{AB}} = k \quad (\neq 0,\ 1/2,\ 1),$$

then PQR (in some order) is similar to ABC.

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# S O L U T I O N S

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

**132.** [1976: 67, 142, 172] *Proposed by the editor.*

If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ for $\theta = \alpha, \beta, \gamma$, prove that the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of eccentric angles $\alpha, \beta, \gamma$ are concurrent if and only if

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

III. *Comment by B.C. Rennie, James Cook University of North Queensland, Australia.*

Let

$$D = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix}$$

and

$$S = \sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta).$$

These are connected by the equation

$$D = 4S \sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta).$$

This equation is true without any restriction on the values of the three angles, for in fact it can be regarded as an identity between two polynomials in complex exponentials such as $\exp\frac{1}{2}i\alpha$. It is a difficult equation to prove, but C. Smith [2] gives a neat proof due to Prof. Anglin, and Durell and Robson give another ingenious method, quoted in [1].

The simplification of mathematical expressions was an important topic when I was a schoolboy, but it tends to be neglected these days. Any mathematician looking at the determinant $D$ should see firstly that $D$ vanishes when two of the angles are equal, secondly that a factor $\sin\frac{1}{2}(\beta-\gamma)$ may be taken out, leaving a trigonometric polynomial, and thirdly that two similar factors can be taken out because of symmetry. After that, a bit of hard work should reveal the expression $S$.

The condition for concurrence of the three normals is $D = 0$, and the factorization of $D$ distinguishes two cases: the trivial concurrence when two of the points coincide corresponds to the vanishing of one of the factors such as $\sin\frac{1}{2}(\beta-\gamma)$; and the non-trivial concurrence, when the three points are distinct, corresponds to the vanishing of $S$.

Now how can a result on concurrent normals help us to find a centre of curvature? At any point $\alpha$ (that is, $x = a\cos\alpha$ and $y = b\sin\alpha$), the centre of curvature is the limit of the intersection of the normals at $\alpha$ and $\beta$, as $\beta$ tends to $\alpha$. First suppose that the point $\alpha$ is not at one end of a principal axis, that is, $\alpha$ is not a multiple of a right angle. Keeping $\alpha$ fixed and letting $\beta$ tend to $\alpha$, the angle $\gamma$ satisfies $S = 0$ and is a differentiable function of $\beta$ for all $\beta$ sufficiently near $\alpha$. The centre of curvature at $\alpha$ is the limit of the intersection of the normals at $\alpha$ and $\beta$, or at $\alpha$ and $\gamma$, and is therefore the intersection of the normals at $\alpha$ and $\gamma_0 = \lim\gamma$. This $\gamma_0$ is a solution of the equation $S(\alpha,\alpha,\gamma) = 0$ or $\sin(\alpha+\gamma) = -\frac{1}{2}\sin 2\alpha$, and $\gamma_0$ cannot equal either $\alpha$ or $\alpha+\pi$, and so the two normals intersect properly. If $\alpha$ is at the end of one of the principal axes, the last condition fails.

REFERENCES

1. Editor's comment, EUREKA, Vol. 2, p. 143.

2. C. Smith, *Conic Sections*, Third Edition, Macmillan, London, 1910, pp. 171-172.

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134. [1976: 68, 151, 173, 222] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

ABC is an isosceles triangle with $\angle ABC = \angle ACB = 80°$. P is the point on AB such that $\angle PCB = 70°$. Q is the point on AC such that $\angle QBC = 60°$. Find $\angle PQA$.

(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

*Editor's comment.*

Solution VI [1976: 222] was incorrectly credited to Don Baker. He recently wrote to say that he had received the solution from Harry Schor, Brooklyn, N.Y., sent it to Steven R. Conrad, who then forwarded it to the editor. Somewhere in the shuffle, Schor's name unaccountably dropped out of sight. I am happy to restore to Schor credit for the best yet solution to this problem. Conrad wrote to say that the problem was proposed by Schor in [4]. The related problem with angles 50°-60° instead of 70°-60° (see [1976: 152]) was mentioned by E.M. Langley in [1], reappeared in [2], and again in [3]. Mathematical archeologists please note.

REFERENCES

1. *Mathematical Gazette*, Vol. 11 (1922-1923), p. 173.

2. *School Science and Mathematics*, April 1939, p. 379.

3. *The American Mathematical Monthly*, 57 (1950) 260, and 58 (1951) 38.

4. *The New York State Mathematics Teachers' Journal*, 1974, pp. 173-174, Problem 25.

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140. [1976: 68] *Proposed by Dan Pedoe, University of Minnesota.*

*THE VENESS PROBLEM.* A paper cone is cut along a generator and unfolded into a plane sheet of paper. What curves in the plane do the originally plane sections of the cone become? (This problem is due to J.H. Veness.)

*Solution by Viktors Linis, University of Ottawa.*

In spherical coordinates $(\rho,\theta,\phi)$ the equation of the conic can be obtained by the intersection of the double cone

$$\{\phi = \alpha = \text{const.} \cup \phi = \pi - \alpha,\ 0 < \alpha < \pi/2\} \tag{1}$$

and a plane

$$\rho(a\cos\theta\sin\phi + b\cos\theta\sin\phi + c\cos\phi) = d. \tag{2}$$

Using (1) and solving for $\rho$ we have for $c \neq 0$

$$\rho = \frac{A}{B\cos(\theta-\theta_0)\pm 1} \tag{3}$$

where $A = \frac{d}{c}\sec\alpha$, $B = \frac{\sqrt{a^2+b^2}}{c}\tan\alpha$ and $\theta_0$ is determined by

$$\sin\theta_0 = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\theta_0 = \frac{a}{\sqrt{a^2+b^2}}, \quad 0 \le \theta_0 < 2\pi.$$

For $d \neq 0$ equation (3) represents a nondegenerate conic (ellipse, parabola, hyperbola) depending on $|B| < 1$, $= 1$, $> 1$. For $d = 0$ the conic is degenerate (a point or a pair of lines). If $c = 0$ and $d \neq 0$ we have a hyperbola which lies in a plane parallel to the axis of the cone. (Note that $\rho \ge 0$, $0 \le \theta \le 2\pi$.) Unfolding the cone by cutting it along a generator ($\theta = \text{const.}$) and developing the surface on a plane, the mapping can be described in polar coordinates $(r,\phi)$ by

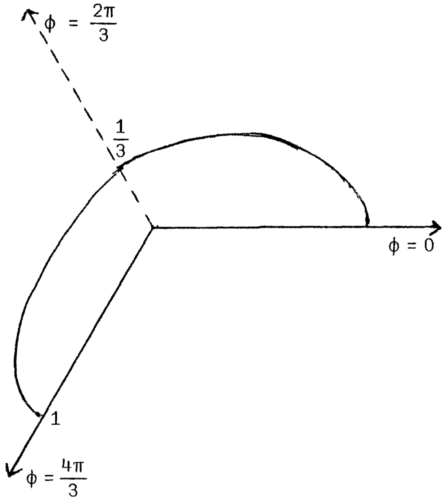
$$r = \rho, \quad \phi = \theta\sin\alpha = k\theta \quad (0 < k < 1). \tag{4}$$

The distances along the generators remain invariant; the polar angles $\phi$, $\theta$ are proportional: the circumference corresponding to $0 \le \theta \le 2\pi$ is mapped onto an arc corresponding to $0 \le \phi \le 2\pi\sin\alpha$. Substituting (4) into (3) we have the equation of the *pseudo-conics*

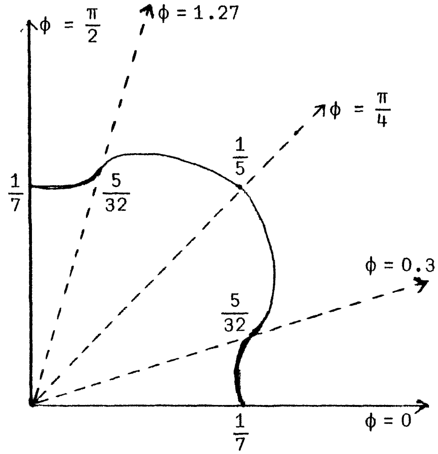
$$r = \frac{A}{B\cos\left(\dfrac{\phi-\phi_0}{k}\right)\pm 1} \quad (\text{where } \phi_0 = k\theta_0), \quad 0 \le \phi \le 2\pi k. \tag{5}$$

These curves fall into three classes as in the ordinary cases, depending on $|B| \lesseqgtr 1$. The curves are located within a circular sector of angle $2\pi k$; in the case of *pseudo-hyperbola* there are two branches, one in each sector. The curves have an axis of symmetry: $\phi = \phi_0 + k\pi$. (continued on p. 16)

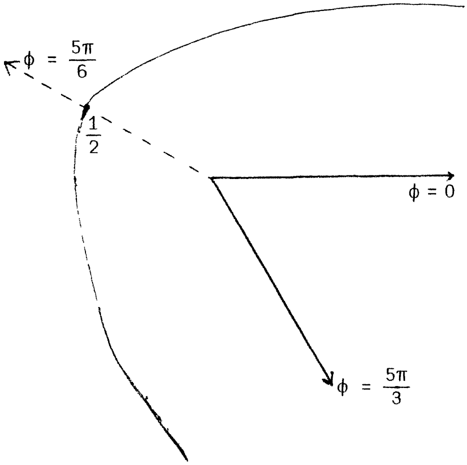
I$(i)$ $k = \frac{2}{3}$ , $B = \frac{1}{2}$ , $r = \dfrac{1}{2 - \cos\frac{3\phi}{2}}$



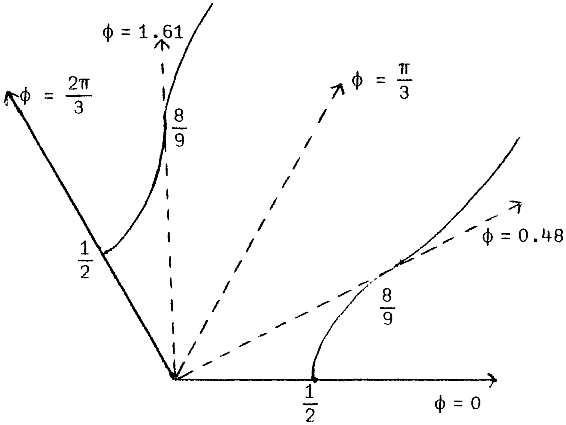

I$(ii)$ $k = \frac{1}{4}$ , $B = \frac{1}{6}$ , $r = \dfrac{1}{6 + \cos 4\phi}$



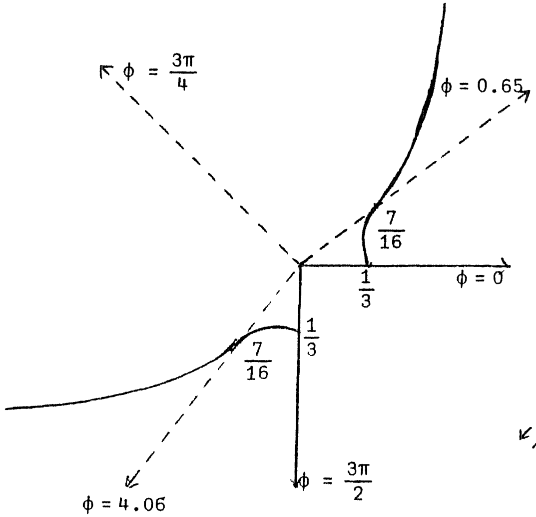

II$(i)$ $k = \frac{5}{6}$ , $B = 1$, $r = \dfrac{1}{1 - \cos\frac{6\phi}{5}}$



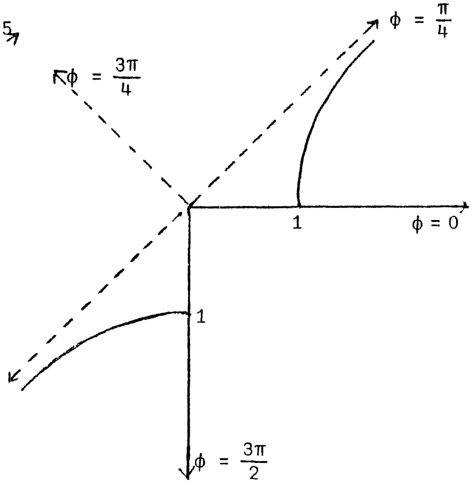

II$(ii)$ $k = \frac{1}{3}$ , $B = 1$, $r = \dfrac{1}{1 + \cos 3\phi}$



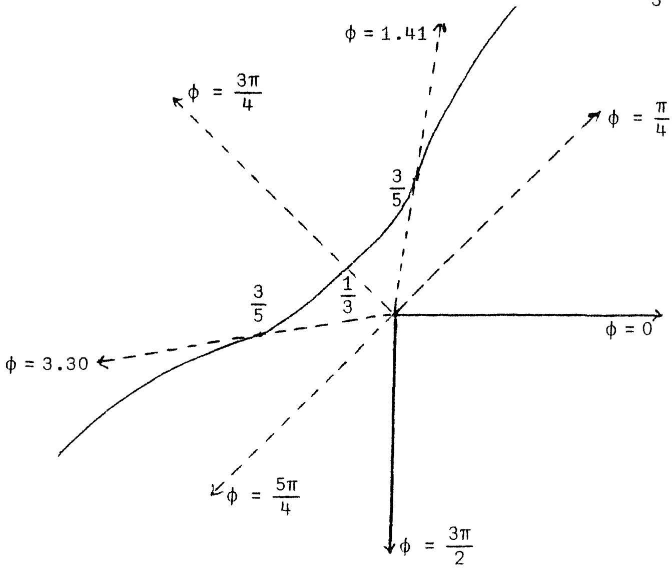
III(*i*) $k = \frac{3}{4}$, $B = 2$, $r = \dfrac{1}{2\cos\frac{4\phi}{3} + 1}$




III(*ii*) $k = \frac{3}{4}$, $B = 2$, $r = \dfrac{1}{2\cos\frac{4\phi}{3} - 1}$




III(*iii*) $k = \frac{3}{4}$, $B = 2$, $r = \dfrac{1}{1 - 2\cos\frac{4\phi}{3}}$



We assume that $\phi_0 = 0$ (which amounts to a rotation of the cone about its axis) and that the cut is along this generator. For normalization we take $A/B = 1$.

If $|B| < 1$ then $r$ is bounded by $\frac{A}{1+B}$ and $\frac{A}{1-B}$ ; the curve has two inflection points (i.e. where $rr'' - 2r'^2 - r^2 = 0$) if $|B| \geq \frac{k^2}{1-k^2}$ , otherwise the curve is convex.

If $|B| = 1$ the curve is unbounded and $r \to \infty$ for only one direction, $\phi = 0$ (or $\phi = k\pi$); the inflection points occur if $k^2 < \frac{1}{2}$ , otherwise the curve is convex.

If $|B| > 1$ the curve is unbounded in two directions determined by $\cos\frac{\phi}{k} = \frac{1}{B}$ . The two separate branches of the curve are given by the $\pm$ choices in (5).

Examples of some typical cases are given in the diagrams on the preceding pages.

*Also solved by* F.G.B. MASKELL, *Algonquin College*.

* * *

**145.** [1976: 94, 181, 224] *Proposed by Walter Bluger, Department of National Health and Welfare, Ottawa, Ont.*

A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 collinear points are equal.

Construct a magic pentagram with the ten smallest possible positive primes.

III. *Comment by Charles W. Trigg, San Diego, California.*

In the magic pentagram 10 distinct integers are arranged in 5 quartets. No quartet has two elements in common with another, and every quartet has an element in common with every other one. The sum of the elements in each quartet is the same, the magic constant.

Langman [1] states that 72 is the smallest magic constant for a pentagram with distinct prime elements. There are 25 different quartets of distinct prime integers each of which sums to 72. They are:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *3 | 5 | 11 | 53 | 3 | 17 | 23 | 29 | *5 | 11 | 13 | 43 | *7 | 11 | 17 | 37 |
| *3 | 5 | 17 | 47 | *5 | 7 | 13 | 47 | *5 | 11 | 19 | 37 | 7 | 11 | 23 | 31 |
| *3 | 5 | 23 | 41 | *5 | 7 | 17 | 43 | *5 | 13 | 17 | 37 | 7 | 13 | 23 | 29 |
| *3 | 7 | 19 | 43 | *5 | 7 | 19 | 41 | 5 | 13 | 23 | 31 | 7 | 17 | 19 | 29 |
| *3 | 11 | 17 | 41 | *5 | 7 | 23 | 37 | 5 | 17 | 19 | 31 | 11 | 13 | 17 | 31 |
| *3 | 13 | 19 | 37 | 5 | 7 | 29 | 31 | *7 | 11 | 13 | 41 | 11 | 13 | 19 | 29 |
| | | | | | | | | | | | | 13 | 17 | 19 | 23 |

53 occurs in only one quartet so it cannot be an element of the pentagram.

The two quartets that contain 47 also contain 5, so 47 cannot be a pentagram element.

Of the three quartets containing 43 the only pair in which the other elements are distinct is 3, 7, 19, 43 and 5, 11, 13, 43. There are two quartets that contain 3 and have exactly one element in common with each of these quartets, namely:

$$\begin{array}{cccc} \underline{3} & 7 & 19 & \underline{43} \\ \underline{5} & 11 & 13 & \underline{43} \\ \underline{3} & \underline{5} & 23 & 41 \end{array} \quad \text{and} \quad \begin{array}{cccc} \underline{3} & 7 & 19 & \underline{43} \\ 5 & \underline{11} & 13 & \underline{43} \\ \underline{3} & \underline{11} & 17 & 41 \end{array} .$$

In neither case is there another quartet that contains 41 and also exactly one nonduplicated integer in common with each of the first two quartets. Hence 43 cannot be an element, nor can any quartet containing it appear in the pentagram.

The four quartets containing 41 can be paired in just two ways, and each pair can be combined with but one available quartet containing 3. Thus

$$\begin{array}{cccc} \underline{3} & 5 & 23 & \underline{41} \\ 7 & 11 & \underline{13} & \underline{31} \\ \underline{3} & \underline{13} & 19 & 37 \end{array} \quad \text{and} \quad \begin{array}{cccc} \underline{3} & 11 & 17 & \underline{41} \\ 5 & 7 & \underline{19} & \underline{41} \\ \underline{3} & 13 & \underline{19} & 37 \end{array} .$$

No available quartet containing 5 can be combined with either of these trios, so 41 cannot be an element.

Only two pairings can be made from among the five quartets containing 37. Each pair can be combined with only one quartet containing 3. Thus

$$\begin{array}{cccc} \underline{3} & 13 & \underline{19} & \underline{37} \\ \underline{5} & 7 & \underline{23} & \underline{37} \\ \underline{3} & \underline{17} & \underline{23} & 29 \\ \underline{5} & \underline{17} & \underline{19} & 31 \end{array} \quad \text{and} \quad \begin{array}{cccc} \underline{3} & 13 & 19 & \underline{37} \\ 7 & 11 & \underline{17} & \underline{37} \\ \underline{3} & \underline{17} & 23 & 29 \end{array} .$$

Only in the first case is there another quartet with exactly one nonduplicated element in common with each of the quartets in the trio. The common elements are underlined. The four elements not underlined should constitute the fifth quartet of the pentagram, but their sum is 80, not 72. Hence, 37 cannot be an element.

At this stage, fifteen of the 25 quartets (those marked with an asterisk *) have been eliminated. Of the remaining quartets, only one contains a 3, those containing 5 also contain 31, and the last six quartets contain only eight distinct integers among them. Consequently, there is *no* pentagram with distinct prime elements that has a magic sum of 72.

One might be inclined to suspect that Langman, too, had found the margins of his book too small. However, on page 16 of *Play Mathematics*, he starts a list of "Prime Numbers less than 1000" with 1.

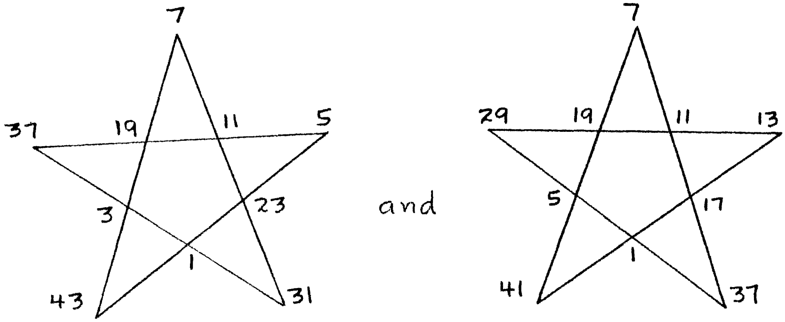
Inclusion of unity among the primes adds for consideration 18 quartets that sum to 72, namely:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 7 | 61 | 1 | 5 | 23 | 43 | 1 | 11 | 13 | 47 | 1 | 13 | 17 | 41 |
| 1 | 3 | 31 | 37 | 1 | 5 | 29 | 37 | 1 | 11 | 17 | 43 | 1 | 17 | 23 | 31 |
| 1 | 5 | 7 | 59 | 1 | 7 | 11 | 53 | 1 | 11 | 19 | 41 | 1 | 19 | 23 | 29 |
| 1 | 5 | 13 | 53 | 1 | 7 | 17 | 47 | 1 | 11 | 23 | 37 | | | | |
| 1 | 5 | 19 | 47 | 1 | 7 | 23 | 41 | 1 | 11 | 29 | 31 | | | | |

Proceeding, as before, to operate on the 43 quartets, two groups of five interrelated quartets are found, namely:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 23 | 43 | | 1 | 13 | 17 | 41 |
| 3 | 7 | 19 | 43 | | 5 | 7 | 19 | 41 |
| 1 | 3 | 31 | 37 | and | 1 | 5 | 29 | 37 . |
| 5 | 11 | 19 | 37 | | 7 | 11 | 17 | 37 |
| 7 | 11 | 23 | 31 | | 11 | 13 | 19 | 29 |

The corresponding pentagrams are:




Other distributions of the quartets are possible, as noted on pages 5 and 6 of this issue.

Still to be established are the minimal magic constants for magic prime pentagrams, with or without inclusion of 1 as a prime.

At least five proofs, essentially different, of the nonexistence of magic pentagrams with integer elements 1 to 10, inclusive, have appeared in the literature: [1], [2], [3], [4], [5].

*Editor's comment.*

The above comment shows that the two readers mentioned in the editor's comment [1976: 225] had between themselves found the only two possible "prime" pentagrams with line sum 72. If they read this before it is too late, perhaps they will now agree to embrace each other, instead of shooting at each other following the editor's heartless suggestion on page 225.

Incidentally, Madachy says in [6] that the magic pentagram with least possible line sum for positive integers is the one with line sum 24 used by the proposer in his solution II in [1976: 181].

REFERENCES

1. Harry Langman, *Play Mathematics*, Hafner, New York, 1962, pp. 80-83.

2. C.W. Trigg, Solution to Problem 113, *Pi Mu Epsilon Journal*, 3 (Fall 1960), 119-120.

3. N.M. Dongre, "More about Magic Star Polygons," *The American Mathematical Monthly*, 78 (November 1971), 1025.

4. Ian Richards, "Impossibility," *Mathematics Magazine*, 48 (November 1975), 249-262 (p. 257).

5. William Pruitt, on p. 262 of reference [4].

6. Joseph S. Madachy, *Mathematics on vacation*, Scribner's, 1966, pp. 98-99.

* * *

**153.** [1976: 110, 196] *Proposé par Bernard Vanbrugghe, Université de Moncton.*

Montrer que les seuls entiers positifs qui vérifient l'équation

$$ab = a + b$$

sont $a = b = 2$.

VI. *Solution by Samuel L. Greitzer, Rutgers University.*

$$ab = a + b \implies (a-1)(b-1) = 1, \quad \text{whence } a = b = 2.$$

*Editor's comment.*

The above wish-I'd-thought-of-it solution arrived too late to be included among those published in [1976: 196], where it would surely have headed the list.

Call a *k-satisfactory sequence* a $k$-tuple $S = (a_1, a_2, \ldots, a_k)$ of natural numbers with $a_1 \le a_2 \le \ldots \le a_k$ such that $\Sigma a_i = \Pi a_i$, and let $v(S)$ denote this common value. In my earlier comment [1976: 197], I stated that there is at least one $k$-satisfactory sequence for every $k \ge 2$. Two readers have written to ask for a proof of this statement. In [1], E.P. Starke proves it by saying that, for any $k \ge 2$, the set consisting of $k$ and 2 and $k-2$ ones is a $k$-satisfactory set. I have also received, from E.P. Starke via Steven R. Conrad, reference [2] which gives upper and lower bounds for $v(S)$.

See also Problem 172 in this issue.

Murray S. Klamkin wrote that since the given relation can be written

$$1 = \frac{1}{a} + \frac{1}{b},$$

another extension of the problem is the optic equation

$$\frac{1}{a} = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n},$$

for which see [3].

REFERENCES

1. E.P. Starke, solution to E 2262, *The American Mathematical Monthly*, 78 (1971) 1021-1022.

2. Carl Hurd, solution to E 2447, *The American Mathematical Monthly*, 82 (1975) 78-80.

3. L.E. Dickson, *History of the Theory of Numbers*, Chelsea, 1952, Vol. II, p. 689.

* * *

**154.** [1976: 110, 159, 197, 225] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

Let $p_n$ denote the $n$th prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds $p_{n+1}$ given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each $r$ $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let $\ell$ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

*Example.* Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| 1 | 3 | 5 | 9 | 7 | 15 | 11 | 27 | 25 | 21 | 13 | 45 |

We observe that $\ell = 17 = p_7$.

III. *Comment by the proposer.*

Let $N$ and $O$ denote the set of positive integers and odd positive integers, respectively. We define a function $f : N \to O$ by

$$f(p_1^{a_1} \ldots p_n^{a_n}) = p_2^{a_1} \ldots p_{n+1}^{a_n}, \qquad f(1) = 1.$$

It follows from the fundamental theorem of arithmetic, the unique factorization theorem, that $f$ is a bijection. For $n = 1,2,3,\ldots$, set

$$L_n = \{f(1),\ f(2),\ \ldots,\ f(p_n - 1)\}$$

and define

$$\ell_{n+1} = \min_{k \in O - L_n} k\ ;$$

then clearly $p_n < \ell_{n+1} \le p_{n+1}$. For example:

$$\begin{array}{lll} L_1 = \{1\}, & O - L_1 = \{3,5,7,\ldots\}, & \ell_2 = 3; \\ L_2 = \{1,3\}, & O - L_2 = \{5,7,9,\ldots\}, & \ell_3 = 5; \\ L_3 = \{1,3,5,9\}, & O - L_3 = \{7,11,13,\ldots\}, & \ell_4 = 7; \\ L_4 = \{1,3,5,7,9,15\}, & O - L_4 = \{11,13,17,19,\ldots\}, & \ell_5 = 11. \end{array}$$

The assertion of the problem is then

*Conjecture* 1. $\ell_{n+1} = p_{n+1}$, $n = 1,2,3,\ldots$ .

We show that Conjecture 1 is equivalent to the following:

*Conjecture* 2. *If* $a_1, \ldots, a_{n-1}$ *are any* $n-1$ $(\ge 1)$ *nonnegative integers such that*

$$p_2^{a_1} \cdots p_n^{a_{n-1}} < p_{n+1},$$

*then*

$$p_1^{a_1} \cdots p_{n-1}^{a_{n-1}} < p_n.$$

(Conjecture 2 is trivially true if all the $a_i$ are zero, so we can ignore this case.)

THEOREM. *Conjecture* 1 $\Longleftrightarrow$ *Conjecture* 2.

*Proof*. (a) Assume Conjecture 1 and let $a_1, \ldots, a_{n-1}$ be any $n-1$ $(\ge 1)$ nonnegative integers such that

$$p_2^{a_1} \cdots p_n^{a_{n-1}} < p_{n+1}.$$

If $p_2^{a_1} \cdots p_n^{a_{n-1}} \le p_n$, then certainly we have (excluding the trivial case where all $a_i = 0$)

$$p_1^{a_1} \cdots p_{n-1}^{a_{n-1}} < p_2^{a_1} \cdots p_n^{a_{n-1}} \le p_n,$$

so we may suppose that

$$p_n < p_2^{a_1} \cdots p_n^{a_{n-1}} < p_{n+1}. \tag{1}$$

Let

$$\lambda = p_1^{a_1} \cdots p_{n-1}^{a_{n-1}},$$

so that

$$f(\lambda) = p_2^{a_1} \cdots p_n^{a_{n-1}}.$$

If $\lambda \ge p_n$, then $f(\lambda) \notin L_n$; hence $f(\lambda) \in O - L_n$, and so $\ell_{n+1} \le f(\lambda)$. But $\ell_{n+1} = p_{n+1}$ by Conjecture 1, so that $p_{n+1} \le f(\lambda)$, contradicting (1). Hence we must have $\lambda < p_n$, and so Conjecture 1 implies Conjecture 2.

(b) Now assume Conjecture 2 and suppose $\ell_{n+1} \ne p_{n+1}$ for some $n \ge 2$, so that $p_n < \ell_{n+1} < p_{n+1}$. If $\ell_{n+1} = p_2^{a_1} \cdots p_n^{a_{n-1}}$, then, by Conjecture 2,

$$p_1^{a_1}\cdots p_{n-1}^{a_{n-1}} < p_n,$$

and so

$$\ell_{n+1} = p_2^{a_1}\cdots p_n^{a_{n-1}} = f(p_1^{a_1}\cdots p_{n-1}^{a_{n-1}}) \in L_n,$$

contradicting $\ell_{n+1} \in O - L_n$. Hence $\ell_{n+1} = p_{n+1}$, and Conjecture 2 implies Conjecture 1.

It is interesting to note that Conjectures 1 and 2 are both false if there exist two pairs of consecutive primes $p_{r-1}$, $p_r$, $p_n$, $p_{n+1}$ such that

$$\sqrt{p_n} < p_{r-1} < p_r < \sqrt{p_{n+1}}\ .$$

For such primes would satisfy $p_r^2 < p_{n+1}$ and $p_{r-1}^2 > p_n$, whereas, under Conjecture 2, $p_r^2 < p_{n+1}$ would imply $p_{r-1}^2 < p_n$. It seems to be a hard problem to determine if such primes exist.

* * *

155. [1976: 110, 198] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y., and Ira Ewen, James Madison H.S., Brooklyn, N.Y.*

A plane is *tessellated* by regular hexagons when the plane is the union of congruent regular hexagonal closed regions which have disjoint interiors. A *lattice point* of this tessellation is any vertex of any of the hexagons.

Prove that no four lattice points of a regular hexagonal tessellation of a plane can be the vertices of a regular 4-gon (square).

This theorem may be called the 4-*gon conclusion.*

(This problem was originally written for the 1976 New York State Math League Meet, held on May 1, 1976.)

*Editor's comment.*

The proposers, digging deeper into their files, discovered that this problem, in slightly different form, had originally been proposed in [1] by Murray S. Klamkin, from whom they had obtained permission to use it in the 1976 New York State Math League Meet. Several solutions were published in [1], including a proof of the following generalization by O. Buggisch, Darmstadt, BRD:

*If all the vertices of a regular n-gon ($n > 2$) are lattice points in a plane tessellated by equilateral triangles, then $n = 3$ or $n = 6$.*

REFERENCE

1. *Elemente der Mathematik*, 30 (1975) 14-15, Aufgabe 709.

* * *

160. [1976: 111, 203] *Proposed by Viktors Linis, University of Ottawa.*

Find the integral part of $\sum_{n=1}^{10^9} n^{-2/3}$ .

This problem is taken from the list submitted for the 1976 Canadian Mathematics Olympiad (but not used on the actual exam).

*Editor's comment.*

It is always a pleasure to be able to trace an interesting problem back to its originator. This problem was submitted to the Canadian Mathematics Olympiad Committee by Murray S. Klamkin, University of Alberta, as a member of the Olympiad Committee. Professor Klamkin is also Problems Editor of *SIAM Review*, as well as a EUREKA subscriber and contributor.

* * *

167. [1976: 136] *Proposed by the editor.*

The first half of the Snellius-Huygens double inequality

$$\frac{1}{3}(2\sin\alpha+\tan\alpha) > \alpha > \frac{3\sin\alpha}{2+\cos\alpha}, \quad 0<\alpha<\frac{\pi}{2},$$

was proved in Problem 115. Prove the second half in a way that could have been understood before the invention of calculus.

I. *Solution by Viktors Linis, University of Ottawa.*

To prove the required inequality by "Huygensian" methods, we can again (see [1976: 112]) use the infinite series for Arc tan $t$. Expressing $\sin\alpha$ and $\cos\alpha$ in terms of $t = \tan\frac{\alpha}{2}$, we obtain after elementary simplifications

$$(3+t^2)\,\text{Arc}\tan t > 3t,$$

from which we get the equivalent inequality

$$\sum_{n=2}^{\infty}(-1)^n\left(\frac{3}{2n+1}-\frac{1}{2n-1}\right)t^{2n-4} = \frac{4}{15}-\frac{8}{35}t^2+\ldots > 0, \quad 0<t<1.$$

This is an alternating monotone decreasing series whose sum lies between $\frac{4}{15}$ and $\frac{4}{15}-\frac{8}{35}=\frac{4}{105}$, which proves the assertion.

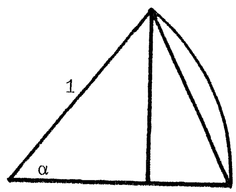
II. *Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.*

The inequality to be proved is equivalent to

$$3(\alpha-\sin\alpha) > \alpha(1-\cos\alpha), \quad 0<\alpha<\frac{\pi}{2}, \tag{1}$$

which is more delicate than that of Problem 115.

It is clear from the figure that $\alpha-\sin\alpha$ is twice the area of a segment of central angle $\alpha$ in a circle of radius 1. This segment can be considered as the limit of inscribed polygonal segments. Thus




$$\begin{aligned}\alpha-\sin\alpha &= \lim_{n\to\infty}\left(3^n\sin\frac{\alpha}{3^n}\right) - 3^0\sin\frac{\alpha}{3^0}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(3^k\sin\frac{\alpha}{3^k} - 3^{k-1}\sin\frac{\alpha}{3^{k-1}}\right)\\ &= \sum_{k=1}^{\infty}3^{k-1}\left(3\sin\frac{\alpha}{3^k}-\sin\frac{3\alpha}{3^k}\right).\end{aligned}$$

The trigonometric identity $3\sin\beta - \sin 3\beta = 4\sin^3\beta$ for all $\beta$, together with the inequality $\sin\beta > \beta - \frac{\beta^3}{6}$ for $0 < \beta < \frac{\pi}{2}$, proved in solution IV to Problem 115 [1976: 137-138], now yields

$$\begin{aligned}\alpha - \sin\alpha &= \sum_{k=1}^{\infty} \left(3^{k-1} \cdot 4\sin^3\frac{\alpha}{3^k}\right)\\ &> \sum_{k=1}^{\infty} 4\cdot 3^{k-1}\left(\frac{\alpha}{3^k} - \frac{\alpha^3}{6\cdot 3^{3k}}\right)^3\\ &= \sum_{k=1}^{\infty} \frac{4\alpha^3}{3^{2k+1}}\left(1 - \frac{\alpha^2}{6\cdot 3^{2k}}\right)^3.\end{aligned}$$

Since $3 > 1$ and $-\frac{\alpha^2}{6\cdot 3^{2k}} > -1$, Bernoulli's inequality $(1+h)^t > 1 + ht$ if $h > -1$ and $t > 1$, followed by the formula for summing a geometric series with ratio $\frac{1}{9}$ or $\frac{1}{81}$, now yields

$$\begin{aligned}\alpha - \sin\alpha &> \sum_{k=1}^{\infty} \frac{4\alpha^3}{3^{2k+1}}\left(1 - \frac{\alpha^2}{2\cdot 3^{2k}}\right)\\ &= \sum_{k=1}^{\infty}\left(\frac{4\alpha^3}{3\cdot 9^k} - \frac{4\alpha^5}{6\cdot 81^k}\right)\\ &= \frac{4\alpha^3}{3}\cdot\frac{1}{8} - \frac{4\alpha^5}{6}\cdot\frac{1}{80}\\ &= \frac{\alpha^3}{6} - \frac{\alpha^5}{120}.\end{aligned} \tag{2}$$

Hence, if $0 < \alpha < \frac{\pi}{2}$,

$$\sin\alpha < \alpha - \frac{\alpha^3}{6} + \frac{\alpha^5}{120} \tag{3}$$

and

$$3(\alpha - \sin\alpha) > \frac{\alpha^3}{2} - \frac{\alpha^5}{40}. \tag{4}$$

Now we can use (3) to estimate $1 - \cos\alpha$. In fact,

$$\begin{aligned}1 - \cos\alpha = 2\sin^2\frac{\alpha}{2} &< 2\left(\frac{\alpha}{2} - \frac{(\alpha/2)^3}{6} + \frac{(\alpha/2)^5}{120}\right)^2\\ &= 2\left(\frac{\alpha^2}{4} - \frac{\alpha^4}{48} + \frac{\alpha^6}{48^2} + \frac{\alpha^6}{32\cdot 120} - \frac{\alpha^8}{2^7\cdot 720} + \frac{\alpha^{10}}{2^{10}\cdot 120^2}\right)\\ &< \frac{\alpha^2}{2} - \frac{\alpha^4}{24} + \frac{\alpha^6}{720},\end{aligned}$$

since the sum of the last two terms, $-\frac{\alpha^8}{2^7\cdot 720}\left(1 - \frac{\alpha^2}{160}\right)$, is certainly negative if $0 < \alpha < \frac{\pi}{2}$. Hence

$$\cos\alpha > 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \frac{\alpha^6}{720} \tag{5}$$

and

$$\alpha(1-\cos\alpha) < \frac{\alpha^3}{2} - \frac{\alpha^5}{24} + \frac{\alpha^7}{720}. \qquad (6)$$

In view of (4) and (6), (1) will be established as soon as we know that

$$\frac{-\alpha^5}{40} > \frac{-\alpha^5}{24} + \frac{\alpha^7}{720} \quad \text{or} \quad \alpha^2 < 12.$$

This is certainly true if $0 < \alpha < \frac{\pi}{2}$, so the proof is complete.

Bernoulli's inequality, as used in this proof, can be proved without calculus, especially since the exponent is only 3.

It is interesting to note that inequalities (3) and (5) are obviously related to the Taylor expansions of the sine and cosine functions.

*Also solved by* R. ROBINSON ROWE, *Sacramento,California.*

*Editor's comment.*

Solution I is short and neat and well within the historical guidelines imposed by the proposal, but solution II is a *tour de force* which I felt deserved publication in spite of its length. It seems to be as close as we are likely to get to a purely geometric solution, since it uses a minimum of analysis and that of a most rudimentary kind, and it offers the fascinating spectacle of the Taylor coefficients arising from a purely geometric context.

* * *

170. [1976: 136, 170] (Corrected) *Proposed by Leroy F. Meyers, The Ohio State University.*

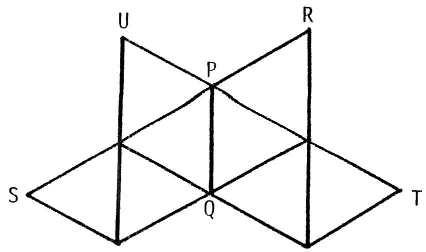
Is it possible to partition the plane into three sets $A$, $B$, and $C$ (so that each point of the plane belongs to exactly one of the sets) in such a way that

(a) a counterclockwise rotation of $120°$ about some point P takes $A$ onto $B$, and

(b) a counterclockwise rotation of $120°$ about some point Q takes $B$ onto $C$?

I. *Solution by B.C. Rennie, James Cook University of North Queensland, Australia.*

The answer is NO. Let $X$ and $Y$ denote the given rotations about P and Q respectively. Consider the points of the lattice shown in the figure, where all the triangles are equilateral.




Q is in $A$ because it is a fixed point of $Y$;

$R = X$Q is in $B$;

$S = Y$R is in $C$, therefore not in $A$;

$T = X$S is not in $B$;

$U = Y$T is not in $C$;

$U = X$R is not in $B$;

$U = X^{-1}$Q is not in $A$.

The last three statements give a contradiction.

II. *Comment by the proposer.*

This is a special case, with a much simpler proof (which I discovered earlier), of Theorem 5 of my paper, "Some possible and some impossible tripartitions of the plane," *Canadian Mathematical Bulletin*, Vol. 11 (1968), pp. 415-421.

*Also solved by the proposer.*

*Editor's comment.*

The above solution assumes $P \neq Q$, which is not guaranteed by the proposal. This is easily remedied by observing that P is in $C$ because it is a fixed point of $X$; hence $P \neq Q$ since Q is in $A$.

Both the proposer and the above solver noted that the theorem is trivially true if *onto* is replaced by *into*, as in the original proposal. One need only let both $A$ and $B$ be empty, and let $C$ be the entire plane.

* * *

171. [1976: 170] *Proposed by Dan Sokolowsky, Yellow Springs, Ohio.*

Let $P_1$ and $P_2$ denote, respectively, the perimeters of triangles ABE and ACD shown in Figure 1. Without using circles, prove that

$$P_1 = P_2 \implies AB + BF = AD + DF.$$

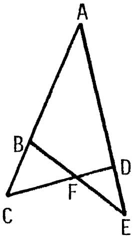



Figure 1

*Solution by the proposer.*

The proof will hinge on the following

*LEMMA*. Let $\Delta$s ARS, ACD have angle A in common and

$$AR = AS = AC + CD + DA = P,$$

as shown in Figure 2. If the bisectors of angles RCD, SDC meet at O, then the perpendicular bisectors of AR and AS meet at O.

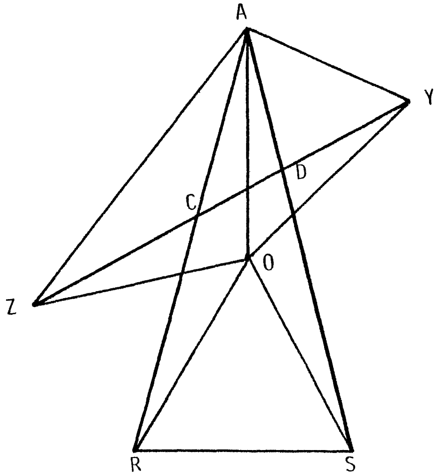



Figure 2

*Proof*. It suffices to show $OA = OR = OS$. Extend CD to Y and Z (see Figure 2), so that $CZ = AC$ and $DY = AD$; then

$$YZ = YD + DC + CZ = P.$$

Since $\Delta$s ACZ, ADY are isosceles, OC and OD are the perpendicular bisectors of AZ and AY, respectively; hence $OA = OY = OZ$. Also, O is equidistant from AC, YZ, and AD; hence OA bisects $\angle CAD$, and $OR = OS$ then follows from the congruency of $\Delta$s OAR, OAS. Now

$$\angle OAC = \angle OAZ - \angle CAZ = \angle OZA - \angle CZA = \angle OZC,$$

so that $\triangle OAR \simeq \triangle OZY$ and $OY = OR$; hence $OA = OR = OS$, which proves the lemma.

Coming back to Figure 1, we can now prove the

*THEOREM.* $P_1 = P_2 = P \implies AB + BF = AD + DF$.

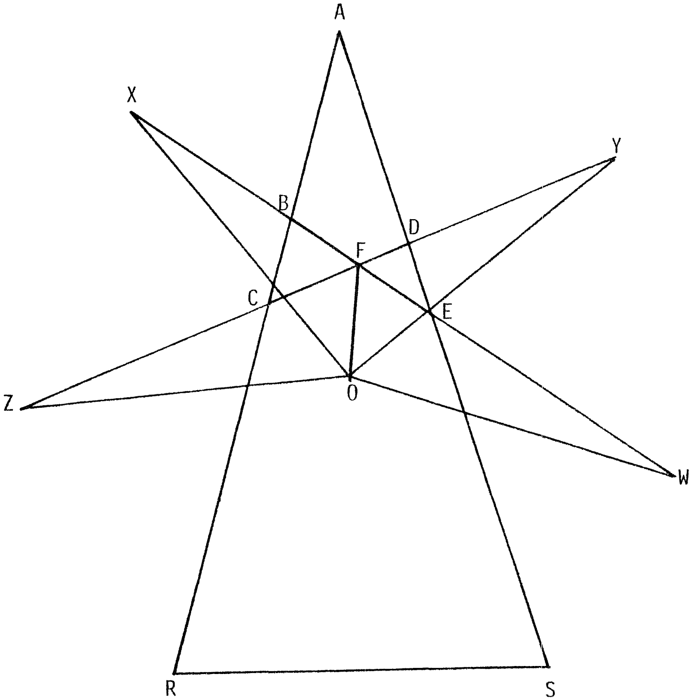



Figure 3

*Proof.* Extend BE to X and W (see Figure 3) so that $BX = AB$, $EW = AE$; extend CD to Y and Z so that $DY = AD$, $CZ = AC$; extend AB to R and AD to S so that $AR = AS = P$. By the lemma, the bisectors of $\angle$s RCD, SDC, RBE, SEB all meet in a point O, and the proof of the lemma shows that

$$OX = OW = OA = OY = OZ.$$

Since $WX = YZ = P$, we have $\triangle OWX \simeq \triangle OYZ$; hence O is equidistant from WX, YZ, and OF bisects $\angle WFZ$. Since $\angle OXF = \angle OYF$, $\triangle OFX \simeq \triangle OFY$ and $FX = FY$, that is, $AB + BF = AD + DF$.

*   *   *

172. [1976: 170] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Find all sets of five positive integers whose sum equals their product. Prove that you have obtained all solutions.

I. *Solution by Doug Dillon, Brockville, Ont.*

Suppose equal the product and sum of the five positive integers $x_i$, $i = 1,2,\ldots,5$, where $x_i \le x_{i+1}$; then

$$x_1x_2x_3x_4x_5 = x_1 + x_2 + x_3 + x_4 + x_5 \le 5x_5;$$

hence $x_1x_2x_3x_4 \le 5$ and we need only consider the following cases:

$$x_1x_2x_3x_4 = 1 \implies x_5 = 4 + x_5,$$

and this gives no solution;

$$x_1x_2x_3x_4 = 2 \implies 2x_5 = 5 + x_5 \implies x_5 = 5,$$

and this gives the solution $(1,1,1,2,5)$;

$$x_1x_2x_3x_4 = 3 \implies 3x_5 = 6 + x_5 \implies x_5 = 3,$$

and this gives the solution $(1,1,1,3,3)$;

$$x_1x_2x_3x_4 = 4 \implies 4x_5 = 6 + x_5 \implies x_5 = 2,$$

which provides solution $(1,1,2,2,2)$, or

$$4x_5 = 7 + x_5,$$

which gives no solution;

$$x_1x_2x_3x_4 = 5 \implies 5x_5 = 8 + x_5 \implies x_5 = 2,$$

giving a solution already obtained.

So the only solutions are $(1,1,1,2,5)$, $(1,1,1,3,3)$, and $(1,1,2,2,2)$.

II. *Solution by Clayton W. Dodge, University of Maine at Orono.*

In Problem E 2262 in *The American Mathematical Monthly*, whose solution appears in Vol. 78 (1971), pp. 1021-1022, a related problem is considered. In my editorial comment to that solution I stated that, using $k$ instead of 5 positive integers in the set, if exactly $n$ of those $k$ integers are greater than 1, then $k \ge 2^n - n$. For if $\Sigma_{i=1}^k a_i = \Pi_{i=1}^k a_i$ under the above conditions, then

$$\Sigma a_i \ge 2n + (k - n) = n + k \quad \text{and} \quad \Pi a_i \ge 2^n,$$

with equality in each case if and only if each of the $n$ integers equals 2. Furthermore, any increase in any one of the $k$ integers increases the product by at least 50% but increases the sum by less than that (when $n > 1$). Hence we get maximal $n$ for given $k$ (or minimal $k$ for given $n$) by setting $n + k = 2^n$. For $n = 3$ we have $k = 2^3 - 3 = 5$, so at most three integers can be nonunit, and when three integers are nonunit, they must be all twos. Hence {2,2,2,1,1} is a solution and the only solution of this form.

When only two integers are nonunit, we have $\{a,b,1,1,1\}$ in which

$$ab = a+b+3, \quad \text{so} \quad a = \frac{b+3}{b-1}.$$

Here $a$ is an integer only for $b = 2$, $b = 3$, and $b = 5$. Then $a = 5$, $a = 3$, and $a = 2$ respectively, giving the new solutions {2,5,1,1,1} and {3,3,1,1,1}. If only one integer $a \neq 1$, then we have $\{a,1,1,1,1\}$, which requires that $a + 4 = a$, an impossibility. Thus we have found all solutions, namely

$$\{2,2,2,1,1\}, \{2,5,1,1,1\}, \text{ and } \{3,3,1,1,1\}.$$

III. *Further comment by Clayton W. Dodge.*

As indicated in my solution II above, I was the referee selected to write up the solutions to elementary problem E 2262 in *The American Mathematical Monthly*. Since that solution was published, I extended the computer program to show there are second solutions for all numbers up to 15,000, except for the values given in the published solution: 2, 3, 4, 6, 24, 114, 174, and 444. Furthermore, in examining the computer printout, I noticed a pattern which was easy to prove in general: *if $k$ is a multiple of* 6 *whose last digit is* 2, *then*

$$\{(k+3)/15,2,2,2,2,1,\ldots,1\} \quad \textit{with } k-5 \textit{ ones}$$

*is a second k-solution (k-satisfactory set).* Thus one need only test multiples of 6 ending in 0 and 4, and no patterns have been observed for these numbers.

I felt the problem interesting enough to publish it as a couple of sections in my mimeographed freshman general mathematics book *Mathematics: Something to Think About.* Unfortunately, publishers have rebelled at the next to last word in the title: students at that level do not especially like to have to *think* about mathematics.

So E 2262 is still not completely solved, but a tiny bit of progress has been made.

See also Problem 153 in this issue.

*Also solved by* ANDRÉ BOURBEAU, *École Secondaire Garneau, Vanier, Ont.;* G.D. KAYE, *Department of National Defence, Ottawa;* F.G.B. MASKELL, *Algonquin College, Ottawa;* L.F. MEYERS, *The Ohio State University;* R. ROBINSON ROWE, *Sacramento, California;* KENNETH M. WILKE, *Topeka, Kansas; and* KENNETH S. WILLIAMS, *Carleton University, Ottawa. The proposer supplied the correct answer, but no proof.*

* * *

176. [1976: 171] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Soit $f: R \to R$ une fonction différentiable paire. Montrer que sa dérivée $f'$ n'est pas paire, à moins que $f$ ne soit une fonction constante.

I. *Essence of the solutions submitted by Doug Dillon, Brockville, Ont.; Clayton W. Dodge, University of Maine at Orono; David R. Stone, Georgia Southern College, Statesboro, Georgia; and Kenneth S. Williams, Carleton University, Ottawa.*

Since $f$ is even and differentiable, we have

$$f(x) = f(-x), \quad f'(x) = -f'(-x),$$

so that $f'$ is odd. If $f'$ is also even, then we must have $f'(x) = 0$ and $f(x) = \text{const.}$, since the zero function is the only one that is both odd and even.

II. *Solution du proposeur.*

Supposons $f'$ paire, de sorte que $f'(-x) = f'(x)$. On a alors

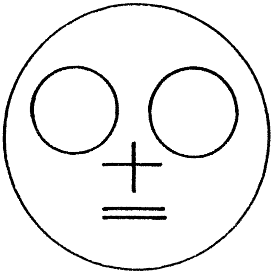
$$\int_{-x}^{0} f'(t)\,dt = \int_{0}^{x} f'(t)\,dt,$$

c'est-à-dire $f(0) - f(-x) = f(x) - f(0)$, d'où $f(x) + f(-x) = 2f(0)$. Puisque $f(-x) = f(x)$, on conclut que $f(x) = f(0) = \text{const.}$

* * *

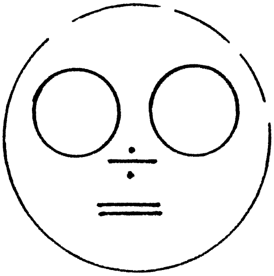
VARIATIONS ON A THEME BY BANKOFF II

The theme by Leon Bankoff



$0 + 0 = 0$

Variation No. 4 by C.W. Trigg



I'm so confused that my head is splitting.

* * *

O GOD! O MONTREAL![1]

From Montreal's *Sunday Express*, January 9, 1977, p. 16.

> Prudential's Leo Carligan received his son's typewritten report card, which read: "English, Fair...French, Good...History, Poor...Mathematics, oGod."

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[1]The title is taken from Samuel Butler's *A Psalm of Montreal*.