Mathematical Spectrum

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- **Ancient Chinese Mathematicians**
- **The Great Pyramid at Giza**
- **The Pythagorean magic forest**
- **The harmonic series**

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

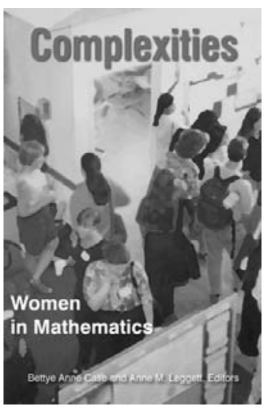
Complexities

A (male) colleague of mine has an imposing poster on the wall of his office of a mathematician whose work has been fundamental for him in his professional life. It is of Emmy Noether (1882–1935), one of the foremost algebraists of the early part of the twentieth century. Her name has become an adjective in the language of algebra with the naming of Noetherian rings. There is an Emmy Noether lecture given by a prominent woman mathematician at the International Congress of Mathematicians, held every four years. Yet she could not be given a paid post at The University of Göttingen. Members of the Senate argued 'what will our soldiers think when they return to the University to find that they are expected to learn at the feet of a woman'. Professor David Hilbert replied 'the Senate is not a bathhouse'. But the Senate would not give way. Lectures were announced under the name of Professor Hilbert but delivered by Fräulein Noether. She was forbidden to teach when the Nazis came to power and left for the United States of America in 1933, where she taught at Bryn Mawr, a then unknown ladies' college, where she gathered the 'Noether girls' around her to replace the 'Noether boys' of Göttingen.

Emmy Noether was not the first woman mathematician. Sophie Germain (1776–1831) experienced the opposition of her family when she tried to study mathematics. They took away her fire, her light, and her clothes to force her from her books. Sophie waited until everyone else slept, wrapping herself in her covers to study by the light of candles. She sent submissions to the great Gauss under the pseudonym of Monsieur Le Blanc. Imagine Gauss' surprise when he discovered the identity of his correspondent!

Ada Lovelace (1815–1852) worked with Charles Babbage on digital computers a century before their invention. She is described as the first hacker! Sonia Kovalevsky, born in 1850 in Russia, read her re-acquired algebra book secretly at night after her father had confiscated it.

In the past, mathematics was regarded as unfitting for women. The University of Cambridge did not grant degrees to women until 1948. Princeton did not admit women to its graduate programme in mathematics until 1968. Thankfully, things have changed since those days. But by how much? It is said that girls now outshine boys in school and, according to figures in this book, 43% of students on undergraduate degree courses in mathematics in the US are



women. The percentage of women on US graduate programmes ('postgraduate' in the UK) drops to 30%. But what about university teachers? Women are very much in the minority there, as any undergraduate will testify. This book asks why this is so. This book is compiled by members of the *Association for Women in Mathematics* (AWM) in the US. Its many contributors are largely, but not wholly, AWM members; one male contributor was spotted! The special problems women face as mathematicians are tackled head on, and helpful advice is given, much of it also relevant to male students. Inevitably, most of the contributions are in a US context, although there is an international survey. Curiously, warmer countries have a significantly greater proportion of mathematicians who are women than colder ones. Discuss! Hence, perhaps, the rather enigmatic title.

As well as biographies past and present, this book contains articles by women doing mathematics in industry and commerce as opposed to in academic life. Curiously, again, they seem to fare much better here. For many readers, these will be the most interesting contributions in the book. Working mathematicians describe their work in government and administration, computer science, aerospace, the oil industry, publishing, national security, biomedical research, and communications.

There is much in this volume for all students of mathematics, but obviously it is written to encourage women to take up, and to stick at, a career in mathematics. About the time this book arrived at the *Mathematical Spectrum* office, an announcement came of the 2005 'Women in Mathematics' day, organized by the *London Mathematical Society*.

We have come a long way since the days of pioneering women like Sophie Germain and Emmy Noether. Not that they saw themselves as pioneers. They merely wanted to work in the subject that they loved.

As a postscript, one of the contributors writes of the birth of her love of mathematics in her second year at school. She noticed that the numbers in the 9-times table, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, all had digits which added up to 9, and wondered why. Now there's a thought!

Reference

1 B. A. Case and A. M. Leggett (eds), *Complexities: Women in Mathematics* (Princeton University Press, 2005).

What is the range of the function

$$f(x) = \frac{2x}{1+x^2}\ ?$$

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Makki Abad Avenue, Sirjan, Iran

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Fundamental Achievements of Ancient Chinese Mathematicians

MICHAL KŘÍŽEK, LIPING LIU and ALENA ŠOLCOVÁ

According to the old Chinese philosophy, all phenomena have arisen from an ancient chaos that has split into two contraries, *Yin* and *Yang*.

1. Introduction

The Chinese civilization is the oldest living civilization (see reference 1). The fact that the old Chinese were good mathematicians is proved by their many original astronomical calculations (e.g. predictions of solar eclipses), the establishment of their lunisolar calendar (see reference 2), and by many other fundamental discoveries that are contained in hundreds of publications about ancient Chinese mathematics. This article surveys the most important of them.

The first calculation instruments in ancient China were various ropes with knots and bars of bamboo, wood, iron, ivory, or nephrite. They were used for storing numerical information and for simple calculation. Later, they were replaced by written symbols.

By the 14th century BC, the Chinese had developed their own decimal system of numbers, whose largest number was 10 000. This is proved by the excavation of thousands of bones and tortoise shells (e.g. in Anyang, Henan Province) dating from the 13th and 14th centuries BC, on which were written astronomical data such as the number of days and months, the origin and decay of a supernova, data about the number of people, animals, etc. (see reference 3). Inscriptions on magic cubes and ceramic or bronze objects of the 14th to the 11th century BC have also been discovered. This is, in fact, the oldest positional number system.

The writing of numbers developed and changed shape differently in the various regions of China. A large number of tables of ancient Chinese digits is contained in reference 3 (pp. 179–190). It is disputed whether zero was discovered in China or if its introduction was stimulated by Indian influences. According to certain sources (see, e.g. reference 4), zero was indicated by an empty space on calculating boards in use by the 4th century BC. However, the small circle did not appear as a symbol for zero until 1247, in ‘Nine Books about Mathematics’ (see references 5 and 6). This tract summarizes results of Chinese mathematicians from the 1st millennium BC. The titular nine books about mathematics have been often rewritten and extended. According to references 3 and 6, their author was probably Zhang Cang, who died in 152 BC. The first note about a negative value dates from the Han Dynasty (206 BC–200 AD) (see references 3 and 4).

2. Discovery of the binary system?

In one of the oldest Chinese books, *Yi Jing* (Book of Changes), from the 8th century BC, there is a picture (the so-called hexagram) containing 8×8 squares (see reference 7). Each square contains six horizontal lines (see figure 1). The broken line represents the ancient Chinese principle of *Yin* while the unbroken line represents the opposing principle of *Yang*. Gottfried Wilhelm Leibniz (1646–1716) connected this hexagram with the discovery of the binary system (see references 3 and 8). If we let zero represent the broken line and one the full line, then the symbols in the squares from left to right in the top row would be the numbers zero to seven. In particular, the first number, in the top-left square, is *zero*, even though this symbol probably was not used for arithmetical operations. The last number, in the bottom-right square, corresponds to 63, which is written 111111_2 in the binary system. The binary system has had to wait until the 20th century (almost 28 centuries later!), and the advent of digital computing, to find widespread application. Thus, compact disks, fax, e-mail, mobile phones, digital photography, and in fact the whole Internet are based on the principles of Yin (0) and Yang (1). The symbols Yin and Yang can be seen today in the South Korean flag (see figure 2).

3. The abacus

Some sinologists think (see references 3 and 6) that perhaps by around the 2nd century AD (but surely by the 6th century AD) calculating bars were being used. They were constructed on a special board similar to a chessboard. This was the origin of the Chinese abacus, which evolved to the point at which it enabled quite complicated calculations to be performed. (The origin of the word *abacus* comes from Egypt. Ancient Egyptians called it *a, ba, ga,* which was later changed to *abax* ($\alpha\beta\alpha\xi$) by the ancient Greeks.) Originally, the abacus was a wooden, metallic, or stone board with vertical or horizontal sections or small holes into which small bars, stones, or balls were placed. Current calculators are, in fact, based on a similar principle. The fact that a stone is or is not in some hole is equivalent to the fact that zero or one is in a memory cell.

In its present form (see figure 3), the abacus is reminiscent of a child's counting-frame. This ingenious remote ancestor of computers is still used in China today. Clever arithmeticians can not only add, subtract, multiply, and divide on an abacus (see reference 9), but also calculate square and cube roots. In the top-left panel of figure 3, an abacus is shown in its basic position. Each of its wooden (or metallic) bars contains seven rings. The five lower rings symbolize the

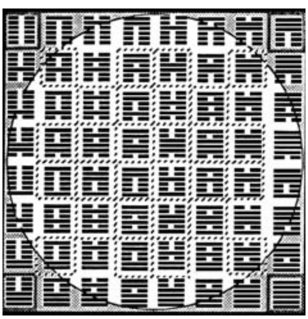

Figure 1

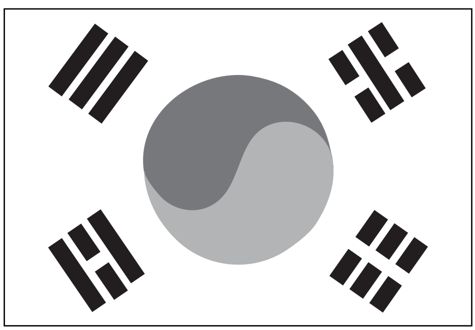

Figure 2

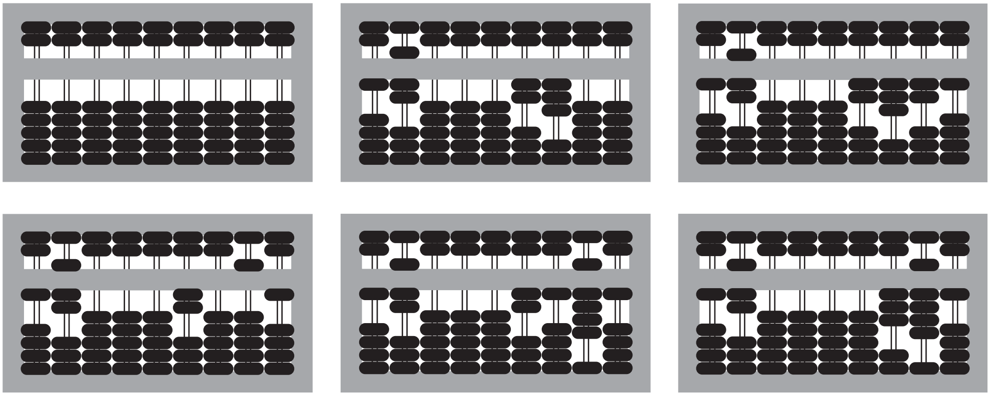

Figure 3

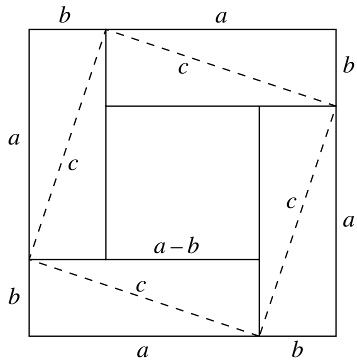

Figure 4

fingers on one hand, while the two upper rings symbolize two hands. The bars correspond to decimal places; there are nine bars in figure 3, but there can be any number. The figure shows an abacus being used to multiply 17 by 23. If the abacus is sufficiently large, we can multiply much larger numbers in a similar manner.

4. Algebra and geometry

In the 1st century BC, Chinese mathematicians knew algorithms for determining square and cube roots. From the beginning of the Christian era, they could solve systems of linear algebraic equations with two or three unknowns. The coefficients of each equation were written in a table from top to bottom. In this way, each column corresponded to one equation. For this ‘matrix’ description, the Chinese used the method *fangcheng* (see reference 3), which is equivalent to Gaussian elimination, i.e. the resulting table had a triangular shape.

In the 3rd century AD, Chinese mathematicians were able to compute the roots of a quadratic equation and prove Pythagoras’ theorem in the following way. From figure 4 we see that the

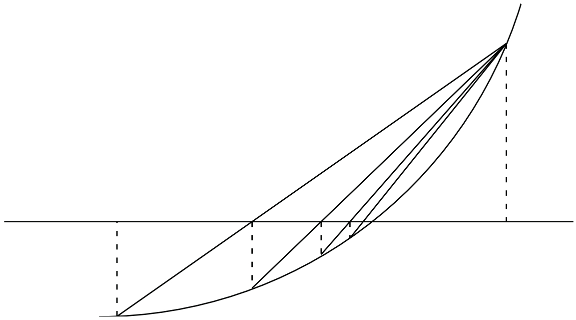

Figure 5

areas of squares with sides of lengths $a + b$ and c satisfy

$$(a + b)^2 = c^2 + 2ab,$$
$$2ab + (a - b)^2 = c^2.$$

Each of these equations yields $a^2 + b^2 = c^2$. These visual proofs of Pythagoras' theorem are due to Liu Hui (see reference 3 (p. 298)) and can be found in many modern-day high-school algebra texts. Liu Hui lived in the 3rd century AD and was unanimously considered to be the most eminent mathematician of ancient China. He wrote a textbook, *Haido Suanjin*, containing computational prescriptions. He also worked on calculating the number π, and found the approximation $157/50 = 3.14$. Zu Chongzhi (429–500) determined lower and upper bounds of π (see reference 10):

$$3.141\,592\,6 < \pi < 3.141\,592\,7\,.$$

Another source (reference 3) introduces a similar value, $355/113 = 3.141\,592\,9\ldots$; see also reference 6. (Recall that Archimedes (287–212 BC) used a circle with the regular inscribed and circumscribed 96-gon to determine an approximate value of π to three decimal places. The Dutch mathematician Ludolf van Ceulen (1540–1600) used a regular polygon with 1 073 741 284 vertices to establish π to 35 decimal places.) Moreover, Zu Chongzhi's son, Zu Xuan, derived a formula for calculating the volume of a ball. Before this it was known that the area of a circle is equal to one-half of the circumference multiplied by one-half of the diameter.

It was not unusual at this time for there to be interest in finding the areas of plane regions and the volumes of spatial bodies (cones, pyramids, etc.). Volumes of bodies with a more complicated shape were calculated by decomposing them into simpler parts.

In the 7th century AD the method *regula falsi* (method of secants) was developed to solve algebraic equations of third order. In this method, each real root is searched iteratively using a sequence of secants (see figure 5). In 1248 Li Zhi introduced negative powers of unknowns, and in 1303 Zhu Shijie eliminated unknowns in a system of four or more algebraic equations. More information about ancient Chinese geometry and algebra can be found in reference 11.

5. Chinese remainder theorem

In his arithmetic book, Sunzi Suanjing introduced the first example of a 'Chinese remainder theorem' problem, one of the most important problems in number theory. The exact origin of his book is not known, but it was written at some time during the period 280–473; see reference 3 (p. 310). (In reference 12 (p. 77), Burton dates the Chinese remainder theorem to the 1st century AD (Sun-Tsu).) This example can be stated as follows. We have an unknown number, x, of objects. By counting them in threes, two objects will remain; by counting them in fives, three objects will remain; and by counting them in sevens, two objects will remain. What is x?

Using modern Gaussian notation, we can write this ancient example as the following system of simultaneous congruences:

$$x \equiv 2 \bmod 3\,, \qquad x \equiv 3 \bmod 5\,, \qquad x \equiv 2 \bmod 7\,,$$

where $x \equiv z \bmod m$ means that m divides $x - z$ without any remainder.

In his book from 1275, Yang Hui describes five more examples similar to this. In modern notation, the Chinese remainder theorem is usually formulated as follows (see references 3 and 13).

Theorem 1 *Let $m_1, m_2, \ldots, m_k$ be mutually coprime natural numbers. Then, for the system of simultaneous congruences*

$$x \equiv z_1 \bmod m_1\,, \qquad x \equiv z_2 \bmod m_2\,, \qquad \cdots \qquad x \equiv z_k \bmod m_k\,,$$

where z_i are integers, there exists one and only one solution, x, modulo M, where

$$M = m_1 m_2 \cdots m_k\,.$$

Another way of interpreting the Chinese remainder theorem is to say that every natural number not greater than $M = m_1 m_2 \cdots m_k$ can be uniquely characterized by the k remainders $(z_1, z_2, \ldots, z_k)$ after division by $(m_1, m_2, \ldots, m_k)$. This is called a *sino-representation*. For instance, if $k = 2$, $m_1 = 3$, and $m_2 = 5$, then we obtain table 1.

We see that the interior of this table contains the numbers zero to 14 each exactly once. Notice that these numbers grow by one in the direction of the diagonals. Table 2, depicting the traditional Chinese calendar (based on the artificial 60-year cycle), has an analogous structure. The rows of table 2 correspond to 12 earthly branches and its columns to ten celestial stems. Each row is devoted to an animal and each double column is associated with one of five elements supposed, in Chinese mythology, to form our universe (cf. reference 14).

The Chinese remainder theorem has many practical applications (see reference 15) in the digital computation of convolutions, Fourier transforms, and quadratic congruences, etc.

6. The Horner scheme and Pascal's triangle

In around the year 1050, Jia Xian introduced a method for calculating the values of a polynomial that in Western literature is known as *Horner's method*. (The British mathematician William George Horner (1786–1837) later rediscovered this method – see *Phil. Trans. R. Soc. London A* **1,** 308–335 (1819). The same scheme was also published by Paolo Ruffini, in 1804.) The value of the function

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

at the point x_0 can be successively calculated as follows: $b_0 = a_0$, $b_1 = a_1 + b_0 x_0$, $b_2 = a_2 + b_1 x_0, \ldots,\ f(x_0) = b_n = a_n + b_{n-1} x_0$. Note that in each step of this method, the

Table 1

z_1	z_2				
	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

Table 2

	Chia	I	Ping	Ting	Wu	Chi	Keng	Hsin	Jen	Kuei	
Tzu	1		13		25		37		49		Mouse
Ch'ou		2		14		26		38		50	Cow
Yin	51		3		15		27		39		Tiger
Mao		52		4		16		28		40	Rabbit
Ch'en	41		53		5		17		29		Dragon
Szu		42		54		6		18		30	Snake
Wu	31		43		55		7		19		Horse
Wei		32		44		56		8		20	Sheep
Shen	21		33		45		57		9		Ape
Yu		22		34		46		58		10	Cock
Hsu	11		23		35		47		59		Dog
Hai		12		24		36		48		60	Pig
	Wood		Fire		Ground		Metal		Water		

result of the previous step is used and only one multiplication and one addition is performed. We therefore need not compute any powers, and the number of arithmetic operations is thus reduced. The Horner scheme can also be written as

$$f(x) = (\cdots((a_0x + a_1)x + a_2)x + \cdots + a_{n-1})x + a_n\,.$$

In the 11th and 12th centuries the Chinese began to employ an analogue of Pascal's triangle of binomial coefficients for higher-order equations. In current mathematical Chinese literature this triangle is called Yang Hui, after the mathematician who published it in 1261 up to the sixth order. One of later forms of this triangle, from 1353, is shown in figure 6. The diagonal connections indicate that Chinese mathematicians already knew the formula

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}, \qquad 0 \le k < n\,,$$

even though they did not write it this way.

7. A primality test

Chinese mathematicians (probably from the Qing period) used the following 'equivalence' for primality testing. (The word 'equivalence' is in quotation marks because it in fact holds in only

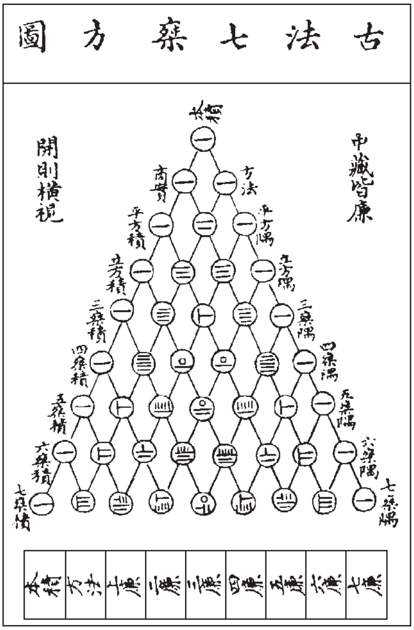

Figure 6

one direction.) The ancient Chinese test says (see reference 15 and reference 16 (pp. 103–105) for interesting discussions concerning this problem)

$$p \text{ is prime } `\Longleftrightarrow' \; p \mid (2^p - 2)\,,$$

i.e. p divides $2^p - 2$. If p is prime then, by the Little Fermat Theorem, $p \mid (n^p - n)$ for any natural number n (see references 13 and 15), and, in particular, for $n = 2$. Therefore, the left-hand side of the 'equivalence' implies the right-hand side.

On the other hand, if p is not a prime less than 341 then we can verify that p does not divide $2^p - 2$. To the authors' knowledge, the ancient Chinese never tried the case of the composite number $p = 341 = 11 \times 31$. It is not hard to check that $2^{10} \equiv 1 \bmod 341$ and, thus, that $2^{340} \equiv 1 \bmod 341$. By multiplying the last congruence by two, we find that 341 divides the number $2^{341} - 2$. Consequently, the implication does not hold from right to left. This could not have been determined by the ancient Chinese mathematicians using abacuses, since the number $2^{341} - 2$ has over 100 digits and the rules for calculation with congruences were not known at that time. Moreover, it is very probable that they never defined the term *prime number*.

There are two other composite three-digit numbers, $p = 561$ and $p = 645$, that divide $2^p - 2$. For the remaining integers less than 1000, the 'equivalence' *is* satisfied. A composite number for which $p \mid (2^p - 2)$ is called a *pseudoprime* (with respect to the base 2). The ancient Chinese test thus gave us a motive to introduce a new mathematical notion – the pseudoprime. Note that pseudoprimes are distributed on the real axis very sparsely, but it has been proved that their number is infinite (see reference 13 (p. 132)).

8. Concluding remarks

The influence on ancient Chinese mathematicians by mathematicians from Japan, Korea, Mongolia, Tibet, India, Vietnam and Islamic countries is described in detail in reference 3. From the travels of missionaries we have evidence of contact between Chinese and European mathematicians from the end of the 16th century (see reference 2).

In 1723 the mathematical encyclopedia *Shuli Jingyun* was published. It contained almost all accessible mathematical knowledge known at that time in China. Ancient Chinese mathematical works did not have the 'definition–theorem–proof' structure. They also did not develop mathematical theory on the basis of axioms, as was done in ancient Greece. Ancient Chinese mathematics had more of a counting character. New ideas were introduced in the form of solved examples and proposed advances (algorithms), etc. To this end, the Chinese used various empirical and heuristic methods of comparison and analogy. Exposition was usually brief, and the results obtained were introduced without proof (which was characteristic not only of Chinese mathematics). The results were transmitted from one generation to the next in spoken or written form; the first Chinese mathematical journal (*Suanxue Bao*) did not appear until 1899.

In this article we have shown that ancient Chinese mathematicians made many fundamental discoveries. Some of these results were rediscovered in Europe only many centuries later, and practical applications not found until the 20th century.

Acknowledgements

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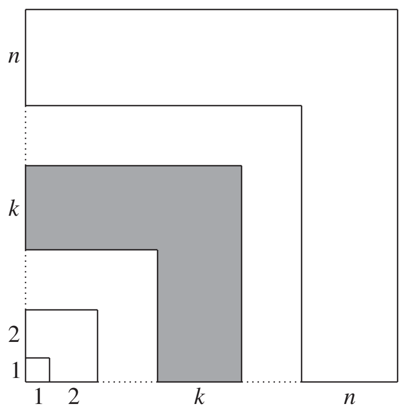
Why is $1^3 + 2^3 + n^3 = (1 + 2 + \cdots + n)^2$?



The area of the shaded region is

$$\begin{aligned} 2k(1 + 2 + \cdots + (k-1)) + k^2 &= 2k \times \tfrac{1}{2}(k-1)k + k^2 \\ &= k^3, \end{aligned}$$

so the area of the whole square is

$$1^3 + 2^3 + \cdots + n^3;$$

it is also equal to

$$(1 + 2 + \cdots + n)^2.$$

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A Spiral of Triangles Related to the Great Pyramid

MATTHEW OSTER and THOMAS J. OSLER

1. Introduction

In this article, we study an interesting spiral built from a single remarkable triangle. The method we use to generate the spiral is shown in figure 1. We start by drawing any right-angled triangle *OAB*, with O at the origin of the x–y plane and *OA* placed along the x-axis. The side lengths are a, b, and c as shown. From point B we construct a line of length b perpendicular to *OB* terminating at C. *OBC* is now our second triangle. We continue in this way to construct the other triangles shown. Our spiral is the rectilinear arc *ABCDE*.

We can start with any right-angled triangle *OAB*, and a rectilinear spiral with interesting mathematical properties will be generated. However, in this article we will concentrate our attention on a very special triangle whose sides are related to the golden ratio, $\phi = \frac{1}{2}(1+\sqrt{5})$, see reference 1. We select the following lengths for the sides of our first triangle: $a = \sqrt{\phi}$, $b = 1$, and $c = \phi$, see figure 2. The reader will see at once that these values satisfy the Pythagorean theorem $c^2 = a^2 + b^2$, so we obtain the well-known equation $\phi^2 = \phi + 1$, see reference 1. Figure 2 also shows the dimensions of the Great Pyramid at Giza in Egypt. Our triangle is nearly similar to the triangle shown in the pyramid since $186.4/115.2 = 1.618\,055\,555\ldots$ while $\phi = 1.618\,033\,989\ldots$. No one knows if this is a happy accident, or if the architect of the pyramid had this mathematical fact in mind. For this reason we will call the triangle with sides $a = \sqrt{\phi}$, $b = 1$, and $c = \phi$ the *pyramid triangle*. In figure 3, we see this new spiral starting with our pyramid triangle. In Section 2 we examine some properties of these triangles and the rectilinear spiral. Later, we investigate a smooth spiral that contains the points $A, B, C, \ldots$. Finally, we evaluate the length of these spirals and the areas generated by the radius vector as it traces the spirals.

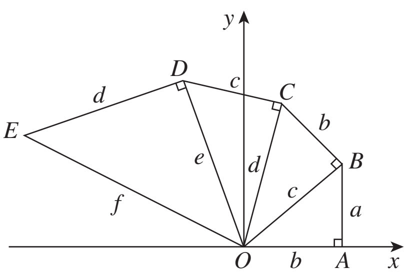

Figure 1 Spiral of triangles.

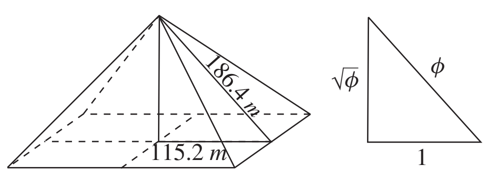

Figure 2 The Great Pyramid at Giza and a related triangle.

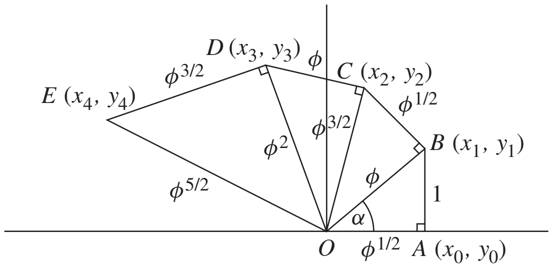

Figure 3 Spiral of pyramid triangles.

2. Similar triangles

In figure 3, we denote the coordinates of the points $A, B, C, \ldots$, respectively, by (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots$. We also denote the angle AOB by α. (Note that $\alpha = 0.666\,239\ldots$ radians $= 38.172\,707\ldots$ degrees.) It is easy to see that all the triangles in the spiral are similar and that the sides of successive triangles differ by a factor of $\sqrt{\phi}$. Thus, the angles of each triangle at the origin are equal to α. The polar coordinates of the vertex (x_n, y_n) are

$$\theta = n\alpha \quad \text{and} \quad r = \phi^{(n+1)/2}. \tag{1}$$

It follows that the rectangular coordinates of the same vertex are

$$x_n = \phi^{(n+1)/2} \cos n\alpha \quad \text{and} \quad y_n = \phi^{(n+1)/2} \sin n\alpha. \tag{2}$$

In the following theorem, we show that the only way to have all the triangles in the spiral similar is to start with our pyramid triangle, or one similar to it.

Theorem 1 *If the triangles that generate the spiral shown in figure 1 are all similar, then the starting triangle has sides $a = 1$, $b = \sqrt{\phi}$, and $c = \phi$, or is a triangle similar to it.*

Proof If the triangles are similar then, from figure 1, we see by examining the first two triangles that

$$\frac{a}{b} = \frac{b}{c} = \frac{b}{\sqrt{a^2 + b^2}}.$$

Thus $a\sqrt{a^2 + b^2} = b^2$, and by squaring we get $a^4 + a^2 b^2 = b^4$, which we rewrite as

$$\left(\frac{b^2}{a^2}\right)^2 = \frac{b^2}{a^2} + 1.$$

This last equation is quadratic in b^2/a^2 and has the solution

$$\frac{b^2}{a^2} = \frac{1 + \sqrt{5}}{2}.$$

We thus have $b = \sqrt{\phi}\, a$ and it follows that $c = \phi a$. Thus the theorem is proved.

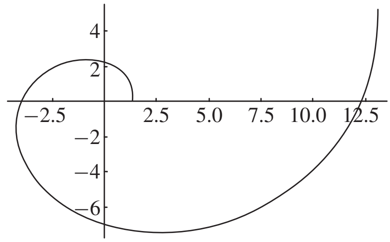

Figure 4 The smooth spiral for $0 < t < 10$.

3. The smooth spiral

In the parametric equations (1) and (2) with the discrete parameter $n = 0, 1, 2, \ldots$, we can easily replace n by a continuous parameter t to obtain a smooth spiral that passes through all the points $A, B, C, \ldots$ of our previous rectilinear spiral. The polar equations now become

$$\theta = \alpha t \quad \text{and} \quad r = \phi^{(t+1)/2}\,, \tag{3}$$

and the Cartesian equations become

$$x(t) = \phi^{(t+1)/2} \cos \alpha t \quad \text{and} \quad y(t) = \phi^{(t+1)/2} \sin \alpha t\,. \tag{4}$$

It will be convenient to rewrite these equations in terms of the exponential function. Let

$$\beta = \frac{\log \phi}{2}\,. \tag{5}$$

(Note that $\beta = 0.240\,605\,912\ldots$.) Then $\phi^{t/2} = \mathrm{e}^{\beta t}$ and the parametric equations in polar form become

$$\theta = \alpha t \quad \text{and} \quad r = \sqrt{\phi}\,\mathrm{e}^{\beta t}\,, \tag{6}$$

and the Cartesian equations become

$$x(t) = \sqrt{\phi}\,\mathrm{e}^{\beta t} \cos \alpha t \quad \text{and} \quad y(t) = \sqrt{\phi}\,\mathrm{e}^{\beta t} \sin \alpha t\,. \tag{7}$$

In figure 4 we present a plot generated by MATHEMATICA® of our smooth spiral.

4. Arc length of the spirals

We begin by finding the length of a segment of our rectilinear spiral.

Theorem 2 *The length of the rectilinear spiral described by (1) and (2) from the point* (x_0, y_0) *to the point* (x_n, y_n) *is*

$$L(n) = \frac{\phi^{n/2} - 1}{\phi^{1/2} - 1}\,. \tag{8}$$

Proof From figure 3, we see that the desired arc length is the sum

$$L(n) = 1 + (\phi^{1/2}) + (\phi^{1/2})^2 + (\phi^{1/2})^3 + \cdots + (\phi^{1/2})^{n-1}\,.$$

This is a geometric series, and it is well known that $1+x+x^2+\cdots+x^{n-1} = (x^n - 1)/(x - 1)$, when $n \neq 1$; therefore we obtain (8) and the theorem is proved.

It is also easy to find the corresponding length of our smooth spiral.

Theorem 3 *The length of the segment of the smooth spiral described by (6) and (7) from the point* $(x(0), y(0))$ *to the point* $(x(t), y(t))$ *is*

$$S(t) = \frac{\sqrt{\phi}}{\log\phi}\sqrt{4\alpha^2 + (\log\phi)^2}\,(\phi^{t/2} - 1)\,. \tag{9}$$

Proof We know that the arc length can be calculated in polar coordinates by

$$S(t) = \int_0^t \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}\,\mathrm{d}t\,. \tag{10}$$

From (6) we see that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \beta\sqrt{\phi}\,\mathrm{e}^{\beta t} \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \alpha\,.$$

Substituting these derivatives into (10) and simplifying, we obtain

$$S(t) = \sqrt{\phi}\sqrt{\alpha^2 + \beta^2}\int_0^t \mathrm{e}^{\beta t}\,\mathrm{d}t = \frac{\sqrt{\phi}\sqrt{\alpha^2 + \beta^2}}{\beta}\,(\mathrm{e}^{\beta t} - 1)\,.$$

Since $\mathrm{e}^{\beta t} = \phi^{t/2}$, and since $\beta = \log\phi/2$, this last expression can be written as (9), and our theorem is proved.

5. Area under the spirals

We begin by finding the area of the first n triangles that generate our rectilinear spiral.

Theorem 4 *The area of the first* n *triangles that generate our rectilinear spiral given by (1) and (2) is*

$$A(n) = \frac{\sqrt{\phi}\,(\phi^n - 1)}{2(\phi - 1)}\,. \tag{11}$$

Proof From figure 3 it is easy to add the areas of the first n triangles. We obtain

$$\frac{\sqrt{\phi}}{2} + \frac{\sqrt{\phi}}{2}\phi + \frac{\sqrt{\phi}}{2}\phi^2 + \frac{\sqrt{\phi}}{2}\phi^3 + \cdots + \frac{\sqrt{\phi}}{2}\phi^{n-1} = \frac{\sqrt{\phi}\,(\phi^n - 1)}{2(\phi - 1)}\,,$$

and the theorem is proved.

We end by finding the area generated by the radius vector as it traces the smooth spiral.

Theorem 5 *The area generated by the radius vector as it traces the smooth spiral given by (6) while the parameter ranges from 0 to* t *is*

$$a(t) = \frac{\alpha\phi}{2\log\phi}(\phi^t - 1)\,. \tag{12}$$

Proof The area is given by the integral

$$a(t) = \tfrac{1}{2}\int_0^t r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\mathrm{d}t\,.$$

Using (6) this integral, after a little computation, becomes

$$a(t) = \frac{\alpha\phi}{2\log\phi}(\mathrm{e}^{2\beta t} - 1)\,.$$

Since $\mathrm{e}^{\beta t} = \phi^{t/2}$, we obtain (12) at once and the theorem is proved.

6. Final remarks

We can investigate the spirals with other initial triangles. Starting with the lengths $a = 1$, $b = 1$, and $c = \sqrt{2}$, the Fibonacci numbers appear in the spiral. If we start with the triangle $a = 1$, $b = \sqrt{3}$, and $c = 2$, our spiral involves the Lucas numbers. Many other spirals of interest are found in reference 2.

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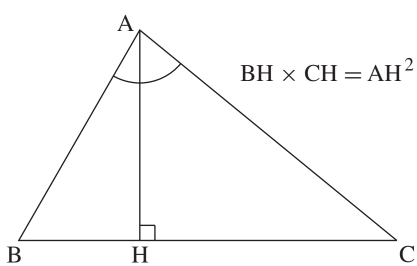
1 R. L. Graham, D. L. Knuth and Oren Patashnik, *Concrete Mathematics: A Foundation for Computer Science* (Addison-Wesley, Reading, MA, 1988).

2 P. J. Davis, *Spirals: From Theodorus to Chaos* (A. K. Peters, Wellesley, MA, 1993).

Matthew Oster *is a mathematics undergraduate at Rowan University. He plans to attend graduate school in the Fall of 2006. He tutors students in pre-algebra and algebra who are preparing to enter college. Besides mathematics, music is also his passion. He has competed in several guitar competitions, played in a few rock bands, and recorded three full length CDs.*

Tom Osler *is professor of mathematics at Rowan University. He is the author of 79 mathematical papers. In addition to teaching university mathematics for the past 44 years, Tom has a passion for long distance running. He has been competing for the past 51 consecutive years. Included in his over 1850 races are wins in three national championships in the late 1960s at distances from 25 kilometres to 50 miles. He is the author of two books on running.*

What is the angle BAC?



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Abbas Roohol Amini

An Asymptotic Approach to Constructing the Hyperbola

DAVID GROCHOWSKI and THOMAS J. OSLER

There are two standard methods that are often used in defining the hyperbola.

(A) The hyperbola is the locus of all points P whose distance from a fixed point F (the focus) divided by its distance to a fixed line D (the directrix) is a constant e (the eccentricity) which is greater than one, i.e.

$$\frac{PF}{PD} = e > 1.$$

(B) The hyperbola is the locus of all points P in which the difference of the distances from two fixed points (the foci F and F') is a constant, i.e.

$$PF - PF' = c.$$

These definitions are related to the *conic sections*, see reference 1.

In this short article, we present another method of defining the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

by visualizing simple constructions based on the asymptote

$$y = \frac{b}{a}x. \tag{2}$$

In figure 1, we see the asymptote (2) drawn as the line $\overline{OA}$. An auxiliary line $\overline{O'A'}$ is constructed parallel to line $\overline{OA}$ with distance a separating them. Now consider any point P on the asymptote. From P draw a line $\overline{PH}$ parallel to the x-axis and a line $\overline{PP'}$ perpendicular to $\overline{OA}$ and meeting $\overline{O'A'}$ at P'. Using O as centre and the length OP' as radius, construct an arc meeting line $\overline{PH}$ at the point Q. We will show that this point Q is on our hyperbola (1). By letting P move from O to infinity on the asymptote $\overline{OA}$, the corresponding points Q constructed as described will trace out the branch of our hyperbola (1) in the first quadrant. It is also evident from our construction that, for every point Q on the hyperbola in the first quadrant, there is a corresponding point P on the asymptote.

We will now demonstrate that the point Q is on the hyperbola (1). Let the coordinates of point P be (x_P, y_P), and let the coordinates of point Q be (x_Q, y_Q). We now have

$$y_P = \frac{b}{a}x_P \quad \text{and} \quad y_Q = y_P. \tag{3}$$

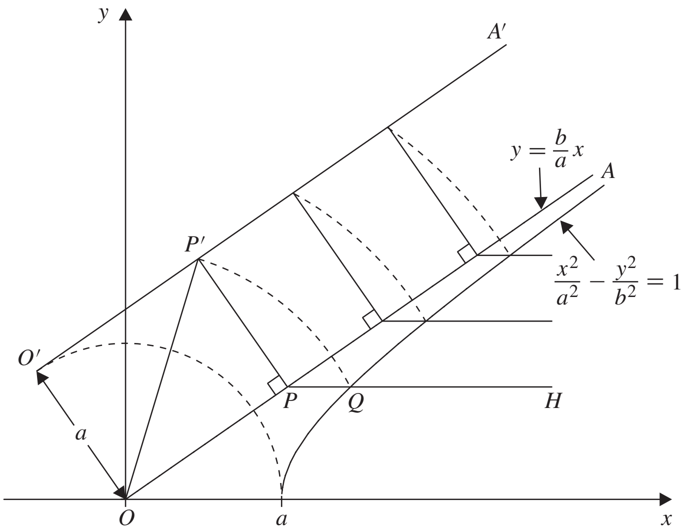

Figure 1

Note that Q is constructed so that $|\overline{OQ}| = |\overline{OP'}|$. But

$$|\overline{OP'}|^2 = a^2 + |\overline{OP}|^2 = a^2 + x_P^2 + y_P^2 \quad \text{and} \quad |\overline{OQ}|^2 = x_Q^2 + y_Q^2.$$

Hence, $a^2 + x_P^2 + y_P^2 = x_Q^2 + y_Q^2$. Since $y_Q = y_p$ we have

$$a^2 + x_P^2 = x_Q^2. \tag{4}$$

Thus, using (3) and (4), we obtain

$$\frac{x_Q^2}{a^2} - \frac{y_Q^2}{b^2} = \frac{a^2 + x_P^2}{a^2} - \frac{y_P^2}{b^2} = \frac{a^2 + x_P^2}{a^2} - \frac{b^2}{a^2}\frac{x_P^2}{b^2} = 1.$$

Thus we have shown that the point Q is on the hyperbola (1). It is easy to construct the remaining branches of the hyperbola in the other three quadrants by reflections in the x-axis and the y-axis.

In our discussion, the x-axis was the axis of the hyperbola and the origin was the centre of the hyperbola. Consider now the general case where we are given any two intersecting lines as asymptotes, the axis which is a line that bisects one of the angles of intersection, and the focus, which is a point on the axis. How do we use our method to construct the corresponding hyperbola? Let c be the distance from the focus to the centre and let α be the angle formed by the axis and an asymptote. Then we have $a = c\cos\alpha$. We can now construct a line, corresponding to $O'A'$ in figure 1, that is parallel to an asymptote and a distance a from it. Given any point P on the asymptote, the corresponding point Q on the hyperbola will be on a line PH that is parallel to the axis. The reader should have no trouble describing the remaining steps.

Reference

1 D. A. Brannan, M. F. Esplen and J. J. Gray, *Geometry* (Cambridge University Press, 1999, p. 5).

David Grochowski *is currently attending Rowan University, where he majors in both Computer Science and Mathematics, with a minor in Physics. Ideally, Dave would like to attend graduate school and receive his PhD in Computer Science. His main research interests concern algorithms, operating systems, and computer theory. He also enjoys the company of his dog, Pepper, critically listening to angst-filled punk rock, and eating at diners late at night, when he really should be doing his course work.*

Tom Osler *is professor of mathematics at Rowan University. He is the author of 79 mathematical papers. In addition to teaching university mathematics for the past 44 years, Tom has a passion for long-distance running. He has been competing for the past 51 consecutive years. Included in his over 1850 races are wins in three national championships in the late 1960s at distances from 25 kilometres to 50 miles. He is the author of two books on running.*

What is the remainder when 249^{249} is divided by 1000?

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What is

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \, ?$$

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The Primitive Pythagorean Magic Forest

WENJIN HUANG and PAUL BELCHER

A primitive Pythagorean triple x, y, z is a solution, in positive integers, of the equation

$$x^2 + y^2 = z^2, \tag{1}$$

where $(x, y, z) = 1$, i.e. x, y, and z have a highest common factor of 1. Thus, x and y cannot both be even (x and y both odd would mean that $z^2 \equiv 2 \bmod 4$ which is impossible); so we conclude that x and y are of opposite parity and that z is odd. Without loss of generality, we will take x as odd and y as even.

A magic forest exists whose borders extend to the north and to the east of an origin. A tree can only grow in a certain position if the two perpendicular distances to the borders and the distance to the origin are all positive integers that do not have a common factor. Thus a tree grows at point (x, y) if and only if x and y are the two legs of a primitive Pythagorean triple. Again, without loss of generality, we will only let the tree grow if x is odd and y is even. We wish to count the trees in our forest.

The approach from x

Let $x = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where p_i is an odd prime and $a_i \in \mathbb{N}^*$ (the set of natural numbers excluding zero), since it is impossible to have $x = 1$. Rewriting (1), we have

$$x^2 = z^2 - y^2 = (z + y)(z - y) = p_1^{2a_1} p_2^{2a_2} \cdots p_n^{2a_n}.$$

Since $(y, z) = 1$ and x is odd, it follows that $(z + y, z - y) = 1$. This implies that $p_1^{2a_1} p_2^{2a_2} \cdots p_n^{2a_n}$ has to be divided into two products such that they have no common factor and the larger of the two is $z + y$ and the smaller is $z - y$. This can be done in exactly $2^n/2$ ways. Thus we have proved the following result.

Result 1 *There are 2^{n-1} sets of positive integer solutions for a primitive Pythagorean triple if $x = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where p_i is an odd prime and $a_i \in \mathbb{N}^*$. This tells us how many trees there are in a column of the magic forest.*

The above analysis also gives us a method of constructing all the primitive Pythagorean triples for a particular value of x. For example, if $x = 105 = 3 \times 5 \times 7$ then $z + y$ and $z - y$ could, respectively, take the values 15^2, 21^2, 35^2, 105^2 and 7^2, 5^2, 3^2, 1 giving z and y, respectively, as 137, 233, 617, 5513 and 88, 208, 608, 5512, creating the following four primitive Pythagorean triples:

$$105^2 + 88^2 = 137^2, \qquad 105^2 + 208^2 = 233^2,$$
$$105^2 + 608^2 = 617^2, \qquad 105^2 + 5512^2 = 5513^2.$$

The approach from y

It is well established that primitive Pythagorean triples have the format

$$x = r^2 - s^2, \qquad y = 2rs, \qquad z = r^2 + s^2,$$

where $r, s \in \mathbb{N}^*$, $(r, s) = 1$, $r > s$, and r and s have opposite parity. Moreover, for each pair (x, y) there is only one pair (r, s), and for each pair (r, s) there is only one pair (x, y). It follows that y is a multiple of 4. Let $y = 2^k p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$, where $k > 1$, p_i is an odd prime, and $a_i \in \mathbb{N}^*$. Then $rs = 2^{k-1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$ and, since $(r, s) = 1$, $r > s$, we have 2^{n-1} possibilities for r and s. Thus we have proved the following result.

Result 2 *There are 2^{n-1} sets of positive integer solutions for a primitive Pythagorean triple if $y = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1 = 2$, $a_1 \geq 2$, p_i is an odd prime for $i \geq 2$, and $a_i \in \mathbb{N}^*$. This tells us how many trees there are in a row of the magic forest.*

The above analysis also gives us a method of constructing all the primitive Pythagorean triples for a particular value of y. For example, if $y = 60$ then $rs = 2 \times 3 \times 5$, so r and s, respectively, could take the values 2×3, 2×5, 3×5, $2 \times 3 \times 5$ and 5, 3, 2, 1 creating the following four primitive Pythagorean triples:

$$11^2 + 60^2 = 61^2, \qquad 91^2 + 60^2 = 109^2,$$
$$221^2 + 60^2 = 229^2, \qquad 899^2 + 60^2 = 901^2.$$

The approach from z

From page 219 of reference 1, we have ‘every prime of the form $4k + 1$ can be expressed as the sum of two squares, $r^2 + s^2$’.

From page 13 of reference 1, we have ‘if natural numbers r and s have no common factors, then any odd prime divisor of $r^2 + s^2$ is of the form $4k + 1$’.

We now have to venture outside the set $\mathbb{N}$ and school mathematics. We will work in $\mathbb{Z}[\mathrm{i}] = \{x + \mathrm{i}y \mid x, y \in \mathbb{Z}\}$. In $\mathbb{Z}[\mathrm{i}]$, the units (divisors of 1) are ± 1 and $\pm\mathrm{i}$; for example, 5 is not a prime since $5 = (2 + \mathrm{i})(2 - \mathrm{i})$. It is well established that factorization into primes in $\mathbb{Z}[\mathrm{i}]$ is unique, apart from multiplicative factors of units and order. Let $z = r^2 + s^2$, where $r, s \in \mathbb{N}^*$, $(r, s) = 1$, $r > s$, and r and s have opposite parity. Then z is odd and, from page 13 of reference 1, we obtain $z = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where p_j is a prime of the form $4k + 1$ and $a_j \in \mathbb{N}^*$. As p_j is a prime of the form $4k + 1$ then, using page 219 of reference 1, it can be written as the sum of two squares, i.e. $p_j = c_j^2 + d_j^2$, where c_j and d_j are positive integers. So in $\mathbb{Z}[\mathrm{i}]$, p_j factorizes as $p_j = (c_j + \mathrm{i}d_j)(c_j - \mathrm{i}d_j)$, where both these factors are irreducible as they have prime norm of p_j. So

$$(r + \mathrm{i}s)(r - \mathrm{i}s) = \prod_{j=1}^{n} (c_j + \mathrm{i}d_j)^{a_j} (c_j - \mathrm{i}d_j)^{a_j}.$$

We cannot allocate both $c_j + \mathrm{i}d_j$ and $c_j - \mathrm{i}d_j$ to $(r + \mathrm{i}s)$ (or to $(r - \mathrm{i}s)$), as this would imply that $p_j | r + \mathrm{i}s$ in $\mathbb{Z}[\mathrm{i}]$ giving $r + \mathrm{i}s = p_j(t + \mathrm{i}u)$ for $t, u \in \mathbb{Z}$ and, thus, $p_j | r$ and $p_j | s$ in $\mathbb{Z}$, which contradicts $(r, s) = 1$. So, using the unique factorization in $\mathbb{Z}[\mathrm{i}]$, we must have

$$r + \mathrm{i}s = u \prod_{j=1}^{n} (c_j \pm \mathrm{i}d_j)^{a_j} \quad \text{and} \quad r - \mathrm{i}s = u^{-1} \prod_{j=1}^{n} (c_j \mp \mathrm{i}d_j)^{a_j},$$

for some unit u, or vice versa. Let us call the first expression A and denote it by $u(c+\mathrm{i}d)$ and the second expression B and denote it by $u^{-1}(c-\mathrm{i}d)$. We have to ensure that $r>s>0$. There are eight possibilities to consider:

(i) $c>0,\ d>0,\ c>d$, where we have to chose A with $u=1$ for $r+\mathrm{i}s$,

(ii) $c>0,\ d>0,\ c<d$, where we have to chose B with $u=-\mathrm{i}$ for $r+\mathrm{i}s$,

(iii) $c>0,\ d<0,\ c>-d$, where we have to chose B with $u=1$ for $r+\mathrm{i}s$,

(iv) $c>0,\ d<0,\ c<-d$, where we have to chose A with $u=\mathrm{i}$ for $r+\mathrm{i}s$,

(v) $c<0,\ d>0,\ -c>d$, where we have to chose B with $u=-1$ for $r+\mathrm{i}s$,

(vi) $c<0,\ d>0,\ -c<d$, where we have to chose A with $u=-\mathrm{i}$ for $r+\mathrm{i}s$,

(vii) $c<0,\ d<0,\ -c>-d$, where we have to chose A with $u=-1$ for $r+\mathrm{i}s$,

(viii) $c<0,\ d<0,\ -c<-d$, where we have to chose B with $u=\mathrm{i}$ for $r+\mathrm{i}s$.

There are two possibilities for each choice of sign, giving 2^n possibilities in all, but, due to the fact that in each case we must make a choice between A and B, we have 2^{n-1} possibilities for r and s, all of which are different due to the unique factorization. Thus we have proved the following result.

Result 3 *If $z=p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}$, where p_j is a prime of the form $4k+1$ and $a_j\in\mathbb{N}^*$, then there are exactly 2^{n-1} sets of positive integer solutions for a primitive Pythagorean triple, with this value of z as the hypotenuse. If z is not of this form, then there are no primitive Pythagorean triples, with this value of z as the hypotenuse. This tells us how many trees there are on a quarter circle of radius z, with centre at the origin, of the magic forest.*

We can use the above analysis in $\mathbb{Z}[\mathrm{i}]$ to construct the 2^{n-1} primitive Pythagorean triples for a particular value of z. We will illustrate this with $z=1105=5\times 13\times 17=(2+\mathrm{i})(2-\mathrm{i})(3+2\mathrm{i})(3-2\mathrm{i})(1+4\mathrm{i})(1-4\mathrm{i})$. This generates four possibilities for $r+\mathrm{i}s$ ($24+23\mathrm{i}$, $32+9\mathrm{i}$, $31+12\mathrm{i}$, $33+4\mathrm{i}$), creating the following four primitive Pythagorean triples:

$$47^2+1104^2=1105^2, \qquad 943^2+576^2=1105^2,$$
$$817^2+744^2=1105^2, \qquad 1073^2+264^2=1105^2.$$

Results 1, 2, and 3 can be put together to give the following final conclusion.

Result 4 *If $v=p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}$, where p_i is a prime and $a_i\in\mathbb{N}^*$ is a possible leg or hypotenuse of a primitive Pythagorean triple, then there are, in total, 2^{n-1} sets of positive integer solutions for a primitive Pythagorean triple, with this value of v as the leg or the hypotenuse.*

Reference

1 G. H. Hardy and E. M. Wright, *An Introduction to The Theory of Numbers*, 5th edition (Oxford University Press, 1979).

This investigation is a condensed and extended version of the Mathematics Extended Essay that ***Wenjin Huang****, a 19-year-old Atlantic College student from China, produced as part of her assessment for the International Baccalaureate Diploma.*

Paul Belcher *was her Extended Essay tutor. The authors wish to thank the referee for his most helpful comments that were incorporated to shorten the article.*

What is the area of the region enclosed by the graphs

$$y = x + 1 \qquad \text{and} \qquad y = |x| + |x - 1|\ ?$$

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Factorize

$$n^4 + 5n^2 + 9 \quad \text{and} \quad n^4 + n^2 + 25.$$

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Abbas Roohol Amini

Intersections of sets of integers

Can you find a sequence $A_1, A_2, A_3, \ldots$ of sets of positive integers such that the intersection of any finite number of them is not empty, but their intersection is?

Tehran
Iran

H. A. Shah Ali

Partial Sums of the Harmonic Series Close to an Integer

ALAN HUNTER

It is well known that the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

diverges, and so a partial sum of this series that is greater than any given integer can be found. The sum of the first n terms can be expressed as

$$\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + \varepsilon_n,$$

where γ is Euler's constant $(0.577\ldots)$, and it is well known that ε_n, the error term, decreases to zero as n increases. Define $f(n)$ to be the smallest number of terms needed to be added to equal or go above an integer n, which means that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{f(n)} \geq n$$

and

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{f(n)-1} < n.$$

The first few values of $f(n)$ for positive n are given in table 1. This is sequence A004080 in 'The On-Line Encyclopedia of Integer Sequences' (see http://www.research.att.com/∼njas/sequences/).

In this article, we will find a recurrence-type relation which will give an approximation of $f(n)$ from the previous term $f(n-1)$.

Table 1

n	$f(n)$	n	$f(n)$
1	1	9	4 550
2	4	10	12 367
3	11	11	33 617
4	31	12	91 380
5	83	13	248 397
6	227	14	675 214
7	616	15	1 835 421
8	1 674	16	4 989 191

Proposition 1 *We have*

$$\frac{f(n+1)}{f(n)} \to \mathrm{e} \quad \text{as } n \to \infty.$$

Proof Let

$$d_{f(n)} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{f(n)} - n,$$

so that $d_{f(n)}$ can be thought of as how much you 'overshoot' the integer n. Then we obtain $d_{f(n)} \geq 0$ and

$$d_{f(n)} - \frac{1}{f(n)} < 0.$$

Thus,

$$0 \leq d_{f(n)} < \frac{1}{f(n)}.$$

Because $1/f(n)$ clearly tends to zero as n tends to infinity, $d_{f(n)}$ thus tends to zero as well by the sandwich principle. Also,

$$d_{f(n)} = \log f(n) + \gamma + \varepsilon_{f(n)} - n,$$

so that

$$\begin{aligned}
\frac{f(n+1)}{f(n)} &= \exp\left(\log \frac{f(n+1)}{f(n)}\right) \\
&= \exp(\log f(n+1) - \log f(n)) \\
&= \exp(n + 1 - \gamma - \varepsilon_{f(n+1)} + d_{f(n+1)} - (n - \gamma - \varepsilon_{f(n)} + d_{f(n)})) \\
&= \exp(1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} - d_{f(n)} + d_{f(n+1)}).
\end{aligned}$$

Since $1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} - d_{f(n)} + d_{f(n+1)} \to 1$ as $n \to \infty$, the result follows.

From this, we can see that, for large n, $f(n+1) \approx \mathrm{e} f(n)$. In fact, we can get a stronger result than this. But first we need the following lemma.

Lemma 1 *We have*

$$\frac{1}{f(n)} - \varepsilon_{f(n)} > \frac{1}{f(n+1)} - \varepsilon_{f(n+1)}, \quad \text{for all } n \in \mathbb{N}.$$

Proof For all n, we obtain

$$\mathrm{e}^{1/n} > 1 + \frac{1}{n}.$$

Taking logarithms of both sides, we get

$$\frac{1}{n} > \log\left(1 + \frac{1}{n}\right) = \log\left(\frac{n+1}{n}\right) = \log(n+1) - \log n,$$

so

$$\log n > \log(n+1) - \frac{1}{n}.$$

By adding $\gamma - \sum_{i=1}^{n-1}(1/i)$ to both sides, we get

$$\log n + \gamma - \sum_{i=1}^{n-1}\frac{1}{i} > \log(n+1) + \gamma - \sum_{i=1}^{n}\frac{1}{i}.$$

This can be rewritten as

$$\frac{1}{n} - \left(\sum_{i=1}^{n}\frac{1}{i} - \log n - \gamma\right) > \frac{1}{n+1} - \left(\sum_{i=1}^{n+1}\frac{1}{i} - \log(n+1) - \gamma\right),$$

in other words

$$\frac{1}{n} - \varepsilon_n > \frac{1}{n+1} - \varepsilon_{n+1}.$$

This creates a strictly monotonically decreasing sequence $\{1/n - \varepsilon_n\}$, so the result follows.

This leads on to our main proposition.

Proposition 2 *We have*

$$f(n)\exp\left(1 - \frac{1}{f(n)}\right) < f(n+1) < f(n)\exp\left(1 + \frac{1}{f(n)}\right).$$

Proof For the lower bound, we know from the proof of Proposition 1 that

$$\begin{aligned}\frac{f(n+1)}{f(n)} &= \exp(1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} - d_{f(n)} + d_{f(n+1)}) \\ &> \exp\left(1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} - \frac{1}{f(n)} + 0\right) \\ &\qquad\qquad \text{(because } d_{f(n)} < 1/f(n) \text{ and } d_{f(n+1)} \geq 0) \\ &> \exp\left(1 - \frac{1}{f(n)}\right) \qquad \text{(since } \varepsilon_{f(n)} > \varepsilon_{f(n+1)}).\end{aligned}$$

For the upper bound, we can see that

$$\begin{aligned}\frac{f(n+1)}{f(n)} &= \exp(1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} - d_{f(n)} + d_{f(n+1)}) \\ &< \exp\left(1 - \varepsilon_{f(n+1)} + \varepsilon_{f(n)} + 0 + \frac{1}{f(n+1)}\right).\end{aligned}$$

From Lemma 1, we get

$$\varepsilon_{f(n)} - \varepsilon_{f(n+1)} < \frac{1}{f(n)} - \frac{1}{f(n+1)},$$

so the above value is less than

$$\exp\left(1 + \frac{1}{f(n)} - \frac{1}{f(n+1)} + \frac{1}{f(n+1)}\right) = \exp\left(1 + \frac{1}{f(n)}\right).$$

Combining these two results gives the required result.

Table 2

n	$f(n-1)\exp\left(1-\dfrac{1}{f(n-1)}\right)$	$f(n)$	$f(n-1)\exp\left(1+\dfrac{1}{f(n-1)}\right)$
17	13 562 024.51	13 562 027	13 562 029.95
18	36 865 408.83	36 865 412	36 865 414.26
19	100 210 576.81	100 210 581	100 210 582.25
20	272 400 598.63	272 400 600	272 400 604.06
21	740 461 598.32	740 461 601	740 461 603.75
22	2 012 783 311.95	2 012 783 315	2 012 783 317.38

In table 2, we provide some values to demonstrate this fact.

It can be shown (by expressing the exponentials as a Taylor series) that the difference between the upper and lower bounds tends to 2e as n tends to infinity, and so Proposition 2 can give the value of $f(n)$ to within about 5, but never exactly.

__Alan Hunter__ is a mathematics student at the University of Glasgow. This is his first published article. He found this result whilst preparing a talk on the harmonic series.

Making magic squares

Start with, say, the following four magic squares:

	8	1	6	5	0	4	6	7	2	8	1	6
	3	5	7	2	3	4	1	5	9	3	5	7
	4	9	2	2	6	1	8	3	4	4	9	2
magic sum:		15			9			15			15	

We can combine these to give the magic square

$$\begin{array}{ccc} 8568 & 1071 & 6426 \\ 3213 & 5355 & 7497 \\ 4284 & 9639 & 2142 \end{array}$$

with magic sum $15 \times 10^3 + 9 \times 10^2 + 15 \times 10 + 15 = 16\,065$.

Shyam Lal College (E)
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Vinod Tyagi

An Approach to Solving Equations with Multiple Modulus Signs

STUART SIMONS

Consider the equation $f(x) = 0$ where $f(x)$ involves one or more sets of modulus signs. The usual technique for tackling this is an algebraic approach (based on the result $|g(x)| = g(x)$ if $g \geq 0$ and $|g(x)| = -g(x)$ if $g \leq 0$), whereby we remove the modulus signs and thus reduce our equation to a form which can be readily solved. However, if there are n sets of modulus signs this approach can yield up to 2^n equations, each subject to n inequalities. Each of these equations must then be solved and only those solutions which satisfy the relevant inequalities then contribute to the final solution set. For n much above unity it is clear that this approach can be very time consuming; the purpose of this article is to suggest an alternative graphical treatment which offers more insight into the solutions, as well as being significantly more efficient in certain cases, both in deciding the number of solutions and also in determining them.

Our approach is best explained by tackling a particular example; we consider

$$|||x-2|-3|-1| = b, \tag{1}$$

where b is a specified nonnegative constant. For given b we are interested firstly in deciding how many solutions there are and secondly in determining them.

We begin by briefly reviewing the algebraic approach. Here we remove, in turn, the three sets of modulus signs, leading to tables 1–3.

In tables 1 and 2, we have given the possible algebraic forms that $|x-2|$ and $||x-2|-3|$ can take when the modulus signs are removed, together with the relevant inequalities. In table 3, we have given the same data for $|||x-2|-3|-1|$, together with the equivalent single or double

Table 1 Possibilities for $|x-2|$.

algebraic forms	conditions on x
$x-2$	$x \geq 2$
$2-x$	$x \leq 2$

Table 2 Possibilities for $||x-2|-3|$.

algebraic forms	conditions on x
$x-5$	$x \geq 2,\ x \geq 5$
$5-x$	$x \geq 2,\ x \leq 5$
$-1-x$	$x \leq 2,\ x \leq -1$
$1+x$	$x \leq 2,\ x \geq -1$

Table 3 Possibilities for $|||x-2|-3|-1|$.

algebraic forms	conditions on x		
$x-6$	$x \geq 2,\ x \geq 5,\ x \geq 6$	$\Longrightarrow$	$x \geq 6$
$6-x$	$x \geq 2,\ x \geq 5,\ x \leq 6$	$\Longrightarrow$	$5 \leq x \leq 6$
$x-4$	$x \geq 2,\ x \leq 5,\ x \geq 4$	$\Longrightarrow$	$4 \leq x \leq 5$
$4-x$	$x \geq 2,\ x \leq 5,\ x \leq 4$	$\Longrightarrow$	$2 \leq x \leq 4$
$x+2$	$x \leq 2,\ x \leq -1,\ x \geq -2$	$\Longrightarrow$	$-2 \leq x \leq -1$
$-2-x$	$x \leq 2,\ x \leq -1,\ x \leq -2$	$\Longrightarrow$	$x \leq -2$
x	$x \leq 2,\ x \geq -1,\ x \geq 0$	$\Longrightarrow$	$0 \leq x \leq 2$
$-x$	$x \leq 2,\ x \geq -1,\ x \leq 0$	$\Longrightarrow$	$-1 \leq x \leq 0$

inequality which defines when that particular algebraic form is valid. In order to solve (1) we now equate b to, in turn, each of the eight forms for $|||x-2|-3|-1|$ shown in table 3, and hence obtain $x = pb + q$ where, for each of the eight solutions, $p = +1$ or -1 and q takes a value from the set $\{0, \pm 2, \pm 4, \pm 6\}$. We now require that each of these solutions should satisfy the accompanying inequality and this imposes constraints on the value of b for which each solution is valid. Thus, for example, from the second entry in table 3 we obtain $6 - x = b$, giving $x = 6 - b$ subject to the inequality $5 \leq 6 - b \leq 6$, which shows that the solution holds when $0 \leq b \leq 1$. The final stage of the calculation is then to divide the interval $[0, \infty)$ in b into a set of sub-intervals within each of which the solutions for x remain unchanged. This will then give us the number of solutions for each such sub-interval, together with the detailed algebraic form of these solutions. We shall not follow through the details of this programme,

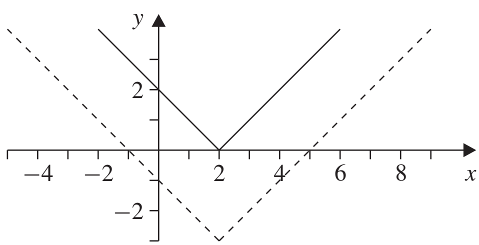

Figure 1

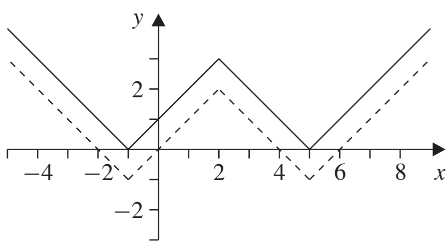

Figure 2

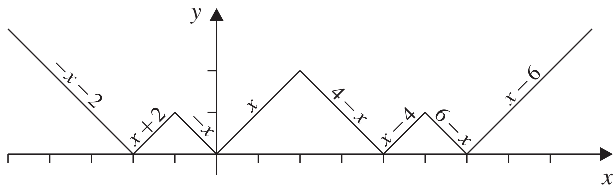

Figure 3

but we note that altogether the implementation of this algebraic approach is rather lengthy and time-consuming, and we therefore now switch to developing a significantly quicker graphical approach to the solution of (1).

Our graphical approach is based on sketching the graph of $y = |||x-2|-3|-1|$. We begin by considering the graph of $y|x-2|$, shown as the solid line in figure 1. The dashed line in figure 1 gives the graph of $y|x-2|-3$ and, reflecting in the x-axis those portions of the graph which lie below the x-axis, this yields the graph of $y = ||x-2|-3|$, which is shown as the solid line in figure 2. Repeating this two-step procedure firstly gives the graph of $y = ||x-2|-3|-1$ (the dashed line in figure 2), followed by the graph of $y = |||x-2|-3|-1|$ (figure 3). The line segments which together form figure 3 each have a linear equation of the form $y = f(x)$, and the relevant function for each segment (which is easily obtained from where it intersects the x-axis and whether its gradient is $+1$ or -1) is shown against that segment. Now, the solutions of (1) correspond to the points of intersection of the graph $y = |||x-2|-3|-1|$ with the straight line $y = b$ and thus we readily obtain, for nonnegative values of b, the number of solutions, see table 4. Within each of the intervals for b shown in table 4, it is very simple to calculate explicit solutions for x in terms of b. Thus, for example, in the interval $1 < b < 2$, the four solutions arise from the intersection of $y = b$ with the four lines $y = -2-x$, $y = x$, $y = 4-x$, and $y = x-6$, and the solutions of (1) in this interval are therefore $x = -2-b$, $x = b$, $x = 4-b$, and $x = 6+b$.

Finally, we illustrate our technique using the following three equations:

$$|||x-2|-3|-1| = ||x-3|-6|, \tag{2}$$

$$|||x-2|-3|-1| = ||x-3|-6|+4, \tag{3}$$

$$|||x-2|-3|-1| = kx \quad \text{for a specified constant } k. \tag{4}$$

Table 4

values of b	numbers of solutions
$b = 0$	4
$0 < b < 1$	8
$b = 1$	6
$1 < b < 2$	4
$b = 2$	3
$2 < b < \infty$	2

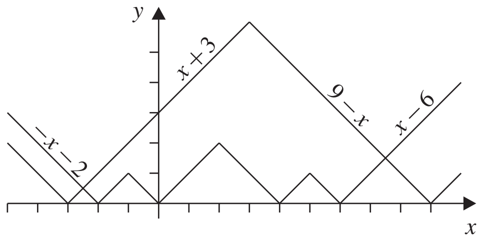

Figure 4

Table 5

values of k	numbers of solutions
$k < -1$	1 (at $x = 0$)
$k = -1$	all x in $[-1, 0]$
$-1 < k < 0$	3
$k = 0$	4
$0 < k < \frac{1}{5}$	5
$k = \frac{1}{5}$	4
$\frac{1}{5} < k < 1$	3
$k = 1$	all x in $[0, 2]$
$1 < k$	1

Following the approach used earlier, we show the graph of $y = ||x-3|-6|$ together with that of $y = |||x-2|-3|-1|$ in figure 4, and it is clear that (2) will have two solutions. One will be at the intersection of the lines $y = 3+x$ and $y = -2-x$, that is at $x = -\frac{5}{2}$, and the other at the intersection of $y = 9-x$ and $y = x-6$, that is at $x = \frac{15}{2}$. Equation (3) is similar to (2) except that the graph of the right-hand side will be that of $y = ||x-3|-6|$ translated four units in the positive y-direction, and it is then readily seen that there will be no intersections with the graph of $y = |||x-2|-3|-1|$, and thus (3) has no solutions. To solve (4), we consider the graph of $y = kx$ superimposed on that of $y = |||x-2|-3|-1|$ (as shown in figure 3). The intersections of the two graphs yield the various solutions of (4) and, for different values of k, the numbers of solutions are shown in table 5. In each case, the detailed solutions are readily obtained by considering the intersections of $y = kx$ with the relevant line segments. Thus, for $-1 < k < 0$, the three solutions will be at $x = 0$ and at the intersections of $y = kx$ with $y = x+2$ and $y = -x-2$, i.e. at $x = 0$, $x = 2/(k-1)$, and $x = -2/(k+1)$. Similar remarks apply to the other intervals in k given in table 5.

It is clear that the graphical approach we have described can be used for a variety of equations involving multiple modulus signs. As such it offers a worthwhile alternative to the algebraic approach, being in many cases both simpler and quicker to implement.

Stuart Simons *was Reader in Applied Mathematics in the University of London before his retirement, with main interests in transport theory and the mathematical theory of aerosols. His current interests include developing novel approaches to relatively elementary problems.*

Generalizations of the Difference of Two Squares

PHILIP MAYNARD

It is well known that, for a fixed $n \in \mathbb{N} = \{1, 2, \ldots\}$, solutions to the equation $n = x^2 - y^2$, for $x, y \in \mathbb{N}$, are in one-to-one correspondence with factorizations of $n = ab$ with a and b, $a \neq b$, both even or both odd. The correspondence is given by

$$n = ab \leftrightarrow n = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2.$$

In this article we consider a more general equation. Specifically, for $n \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\} = \{0, 1, \ldots\}$ we consider

$$nx^i y^j = x^2 - y^2, \tag{1}$$

for $x, y \in \mathbb{N}$. In general, there are no solutions to (1), as our first lemma shows.

Lemma 1 *Let $n, i, j \in \mathbb{N}$. Then the equation $nx^i y^j = x^2 - y^2$ has no solutions for $x, y \in \mathbb{N}$.*

Proof In contradiction to the statement, we assume that for $i, j, n, x, y \in \mathbb{N}$ we have $nx^i y^j = x^2 - y^2$. Let $(x, y) = m$, for some $m \in \mathbb{N}$, denote the highest common factor of x and y. Then $x = m\alpha$ and $y = m\beta$ for some $\alpha, \beta \in \mathbb{N}$ with $(\alpha, \beta) = 1$, and, thus,

$$nm^{i+j-2}\alpha^i \beta^j = \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta), \tag{2}$$

where $i + j \geq 2$. However, since $(\alpha, \alpha + \beta) = (\alpha, \alpha - \beta) = 1$, it follows that if $i \geq 1$ then $\alpha = 1$. In the same way, if $j \geq 1$ then $\beta = 1$ and, consequently, $x = y$, which implies that $n = 0$, contradicting our assumption and proving the lemma.

As explained above, the case in which $i = j = 0$ is well understood. We now concentrate on the cases in which $i = 1$, $j = 0$ and $i = 0$, $j = 1$. First, we note that we need only consider the equation $nx = x^2 - y^2$, since for any $x, y, n \in \mathbb{N}$ we have

$$nx = x^2 - y^2 \Leftrightarrow n(x - n) = y^2 - (x - n)^2. \tag{3}$$

For $n \in \mathbb{N}$ we shall denote the number of solutions to the equation $n = x^2 - y^2$ by $\rho(n)$. As noted above, $\rho(n)$ is just the number of factorizations of $n = ab$ with a and b, $a \neq b$, both even or both odd. An integer is said to be *square-free* if it is the product of distinct prime numbers.

Lemma 2 *Let $n \in \mathbb{N}$. Then the equation $nx = x^2 - y^2$ has*

$$\sum_{\{d,n:\ d \mid n,\ d \text{ is square-free}\}} \rho\left(\frac{n}{d}\right)$$

solutions for $x, y \in \mathbb{N}$.

Proof To a fixed n and any $\alpha, \beta, t \in \mathbb{N}$ for which

$$\frac{n}{t} = \alpha^2 - \beta^2, \tag{4}$$

we can associate a solution to

$$nx = x^2 - y^2 \tag{5}$$

by setting $x = t\alpha^2$ and $y = \alpha\beta t$. To prove the lemma we must prove the following three statements.

Claim 1. For any solutions, x and y, to (5) there exist α, β, t satisfying (4) and for which $x = t\alpha^2$ and $y = \alpha\beta t$.

To show this, assume that $nx = x^2 - y^2$ for $x, y \in \mathbb{N}$. Let $m = (x, y)$ for some $m \in \mathbb{N}$, such that $x = m\alpha$ and $y = m\beta$ for some $\alpha, \beta \in \mathbb{N}$ with $(\alpha, \beta) = 1$. Then $n\alpha = m(\alpha+\beta)(\alpha-\beta)$. It follows that $\alpha \,|\, m$; say $m = \alpha t$ for some $t \in \mathbb{N}$. Therefore, $n = t(\alpha^2 - \beta^2)$, or $n/t = \alpha^2 - \beta^2$. Note that $t \,|\, n$, since $\alpha^2 - \beta^2 \in \mathbb{N}$. The existence of these values of α, β, and t suffices to prove the claim.

Claim 2. Assume that $n/t = \alpha^2 - \beta^2$ and that t is not square-free; say $d^2 \,|\, t$. Let $T = t/d^2$. Then, under the correspondence (4)–(5), the equations $n/t = \alpha^2 - \beta^2$ and $n/T = (d\alpha)^2 - (d\beta)^2$ give rise to the same equation.

This is simple to demonstrate, since by (5) we have $x = t\alpha^2$ and $y = \alpha\beta t$. From the second equation we obtain $x' = T(d\alpha)^2 = x$ and $y' = d\alpha d\beta T = y$.

Claim 3. Different solutions to (4) (with t square-free) give rise to different solutions to (5).

We show this as follows. Let $n/t = \alpha^2 - \beta^2$ and $n/t' = \alpha'^2 - \beta'^2$ hold for $\alpha, \beta, \alpha', \beta' \in \mathbb{N}$ and positive, square-free integers t and t'. Assume that $t \neq t'$ and, to establish a contradiction, assume that $\alpha^2 t = \alpha'^2 t'$. There exists some prime, p, such that $p \,|\, t$ but $p \nmid t'$. Thus, $p \,|\, \alpha'$, meaning that p appears to an even power in both $\alpha'^2 t'$ and, by assumption, $\alpha^2 t$. However, then $p^2 \,|\, t$, a contradiction.

In the case $t = t'$ the claim is clear.

Example 1 Consider the case in which $n = 42$. There are three square-free $d \in \mathbb{N}$ for which $d \,|\, n$ and $\rho(n/d) \neq 0$, namely $d = 2, 6, 14$. In the case $d = 2$ there are two factorizations of n/d, namely $21 = 3 \cdot 7$ and $21 = 21 \cdot 1$, and these yield $21 = 5^2 - 2^2$ and $21 = 11^2 - 10^2$ respectively; for example, $42 = 2(5^2 - 2^2)$. This has the form of (4) with $n = 42$, $t = 2$, $\alpha = 5$, and $\beta = 2$. By the correspondence (5) we must take $x = t\alpha^2$ and $y = \alpha\beta t$, that is, $x = 50$ and $y = 20$. Then $42 \cdot 50 = 50^2 - 20^2$. Likewise, the second factorization gives $42 \cdot 242 = 242^2 - 220^2$.

In the same way, $d = 6$ and $d = 14$ give one solution each, $42 \cdot 96 = 96^2 - 72^2$ and $42 \cdot 56 = 56^2 - 28^2$ respectively. Finally, by using (3) we obtain $42 \cdot 8 = 20^2 - 8^2$, $42 \cdot 200 = 220^2 - 200^2$, $42 \cdot 54 = 72^2 - 54^2$, and $42 \cdot 14 = 28^2 - 14^2$, in the respective cases.

There are two cases left to consider for i and j in (1). These are $i \geq 2$, $j = 0$ and $i = 0$, $j \geq 2$. Consider the former case. Let $(x, y) = m$ for some $m \in \mathbb{N}$. Then $x = m\alpha$ and $y = m\beta$ for some $\alpha, \beta \in \mathbb{N}$ with $(\alpha, \beta) = 1$, and (1) becomes $nm^{i-2}\alpha^i = (\alpha + \beta)(\alpha - \beta)$. This implies that $\alpha = 1$ and, hence, $x \,|\, y$. However, $x > y$, so (1) has no solutions for $i \geq 2$ and $j = 0$. In the second case we only need the following simple lemma.

Lemma 3 *Let $n, j \in \mathbb{N}$ with $j \geq 2$. Then the equation $ny^j = x^2 - y^2$ (for $x, y \in \mathbb{N}$) has a solution if and only if the equation $nm^{j-2} = \alpha^2 - 1$ has a solution for some $\alpha, m \in \mathbb{N}$.*

Proof First assume that, for some $x, y \in \mathbb{N}$, we have $ny^j = x^2 - y^2$. Let $(x, y) = m$ for some $m \in \mathbb{N}$, such that $x = m\alpha$ and $y = m\beta$ for some $\alpha, \beta \in \mathbb{N}$ with $(\alpha, \beta) = 1$. Then $nm^{j-2}\beta^j = \alpha^2 - \beta^2$. Since $(\alpha, \beta) = 1$, it follows that $\beta = 1$ and, thus, $nm^{j-2} = \alpha^2 - 1$. Conversely, if $nm^{j-2} = \alpha^2 - 1$ for some $\alpha, m \in \mathbb{N}$, then set $x = m\alpha$ and $y = m$. It follows that $ny^j = x^2 - y^2$.

For example, the equation $ny^2 = x^2 - y^2$ (i.e. (1) with $j = 2$ and $i = 0$) has a solution if and only if $n = \alpha^2 - 1$ for some $\alpha \in \mathbb{N}$. In contrast to all the previously considered cases, here there is an infinite number of solutions, since $ny^2 = x^2 - y^2$ becomes $\alpha^2 y^2 = x^2$; thus, we take $x = \alpha t$ and $y = t$ for any $t \in \mathbb{N}$.

__Philip Maynard__ received his PhD degree, in the area of combinatorics, at the University of East Anglia. Since then he has continued this research and also enjoys writing mathematical papers with a more recreational content.

Product of two numbers by restrictive addition

Let us find the product of any two randomly chosen numbers, say 27 and 17. The method involves repeatedly taking half of one number until the number 1 is obtained, while repeatedly taking twice the other number, as follows:

27	17
13	34
6	68
3	136
1	272

The first column shows the repeated halving of 27, giving 13, 6, 3, and, finally, 1. The fractional parts (where they occur) are not considered. The second column shows the repeated doubling of 17, giving 34, 68, 136, and 272, where we stop because the first column has reached 1.

From the second column, we add the numbers corresponding to odd numbers in the first column. The odd numbers in column one are 27, 13, 3, and 1, and the corresponding numbers in the second column are 17, 23, 136, and 272.

Adding 17, 34, 136, and 272 gives 459, which is the required product. This method works in general.

Shyam Lal College (E) **Vinod Tyagi**
Delhi University

Mathematics in the Classroom

What's in a constant?

I set my students the following indefinite integral:

$$\int \frac{4\,\mathrm{d}x}{\sqrt{4x - x^2}}.$$

The 'correct solution' was to write it as

$$\int \frac{4\,\mathrm{d}x}{\sqrt{4 - (x-2)^2}},$$

make the substitution $x - 2 = 2\sin\theta$, and obtain the answer

$$4\sin^{-1}\frac{x-2}{2} + c, \quad \text{where } c \text{ is a constant.}$$

One of my students, Lin Ga-Yi, obtained the answer

$$8\sin^{-1}\frac{\sqrt{x}}{2} + c,$$

by making two substitutions, first $x = u^2$ and then $u = 2\sin\theta$. It took me some thought to realize that the two answers are the same, i.e. $4\sin^{-1}((x-2)/2)$ and $8\sin^{-1}(\sqrt{x}/2)$ differ only by a constant. In fact, if we put

$$\alpha = \sin^{-1}\frac{\sqrt{x}}{2} \quad \text{and} \quad \beta = \sin^{-1}\frac{x-2}{2},$$

then

$$\begin{aligned}
\sin(2\alpha - \beta) &= \sin 2\alpha\cos\beta - \cos 2\alpha\sin\beta \\
&= 2\sin\alpha\cos\alpha\cos\beta - (\cos^2\alpha - \sin^2\alpha)\sin\beta \\
&= 2\frac{\sqrt{x}}{2}\frac{\sqrt{4-x}}{2}\frac{\sqrt{4x-x^2}}{2} - \left(\frac{4-x}{4} - \frac{x}{4}\right)\frac{x-2}{2} \\
&= \frac{4x - x^2}{4} - \frac{4-2x}{4}\frac{x-2}{2} \\
&= \frac{4x-x^2}{4} + \frac{x^2 - 4x + 4}{4} \\
&= 1,
\end{aligned}$$

so that $2\alpha - \beta = \frac{1}{2}\pi + 2n\pi$, for some integer n, whence $8\alpha - 4\beta = 2\pi + 8n\pi$.

It's a good job I didn't forget the constant!

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Taipei, Taiwan

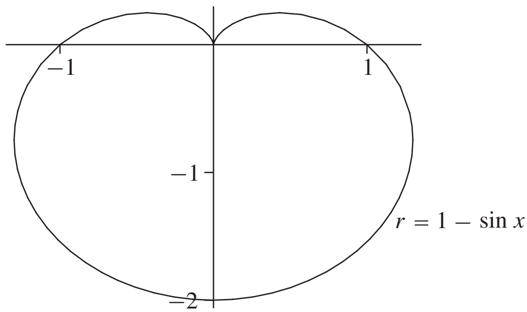
Bor-Yann Chen

Letters to the Editor

Dear Editor,

Mathematics, with love

You asked in your column in Volume 37, Number 3, how one might woo one's beloved with trigonometry. I would start with the graph of $r = 1 - \sin x$. You could run a challenge in *Mathematical Spectrum* to get the best heart-shaped curve, with credit given for simplicity.



A thought occurred to me today while looking at the *Mathematical Spectrum* logo. Could light rays entering from the left be found, obeying Snell's law, so that the four white areas in the diagram are equal?

Yours sincerely,
Jonny Griffiths
(Paston College
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Norfolk NR28 9JL
UK)

Dear Editor,

Extracting cube roots

Joyce Davies' letter *Square roots by subtraction* in Volume 38, Number 1, reminded me of a method for extracting cube roots, which I was very excited to discover as a schoolboy. It is an extension of the square root algorithm given by Joyce, and it may be extended to deal with fourth and higher roots, although the arithmetic gets very heavy.

The theory derives from the identity

$$(10x + a)^3 = 1\,000x^3 + (300x^2 + 30xa)a + a^3.$$

One decimal figure at a time, a, is to ten times the previous estimate x. Essentially the method is that of Horner for solving any polynomial equation. I lay out an example below tabulating the working for convenience (which I confess was computated using a spreadsheet!), see table 1.

Table 1

x	$300x^2$	estimate of a	a	$q = 300x^2$ $+30xa + a^2$	qa	difference	block	carry
							34	34
0	0	3	3	9	27	7	567	7 567
3	2 700	$\frac{7567}{2700}$	2	2 884	5 768	1 799		1 799 000
32	307 200	$\frac{1799000}{307200}$	5	312 025	1 560 125	238 875		238 875 000
325	31 687 500	$\frac{238875000}{31687500}$	7	31 755 799	222 290 593	16 584 407		16 584 407 000
3 257	3 182 414 700	$\frac{16584407000}{3182414700}$	5					

Here are the steps of the algorithm.

1. Divide the digits into blocks of three with one block ending at the decimal point.
2. Find the integer between 1 and 9, inclusive, whose cube is closest to, but not greater than, the first block integer.
3. Place this integer above the first block. It is the first significant figure in the answer.
4. Subtract its cube from the first block of digits and bring down the next block of three digits. We call the number so formed the *carry* and denote it by C.
5. Now we want $qa - (300x^2 + 30xa + a^2)a \leq C$, where x is the integer formed from the digits in the answer so far calculated, and qa is as large as possible. Estimate a by evaluating the integer part of $C/300x^2$. If the calculated value is too large then decrease the estimate of a, say by 1, and try again until you succeed. Place the digit a above the next block. This is the next significant figure in the answer.
6. Subtract qa from the carry C and bring down the next block creating the next carry.
7. Repeat steps 5 and 6 until you have the accuracy you want. Insert the decimal point in the answer above its position in the number being rooted.

The working may be laid out as follows.

$$
\sqrt[3]{34\,567} \;=\; \begin{array}{l}
\quad 3 \quad 2. \quad 5 \quad 7 \quad 5 \\
\hline
\multicolumn{1}{|l}{34\ 567.000\ 000} \\
\multicolumn{1}{|l}{\underline{27}} \\
\multicolumn{1}{|l}{\ \ 7\ 567} \\
\multicolumn{1}{|l}{\ \ \underline{5\ 768}} \\
\multicolumn{1}{|l}{\ \ 1\ 799\ 000} \\
\multicolumn{1}{|l}{\ \ \underline{1\ 560\ 125}} \\
\multicolumn{1}{|l}{\qquad 238\ 875\ 000} \\
\multicolumn{1}{|l}{\qquad \underline{222\ 290\ 593}} \\
\multicolumn{1}{|l}{\qquad\ \ 16\ 584\ 407}
\end{array}
$$

As a sixth former I recall doing all the arithmetic by hand on scraps of paper. I certainly never thought of laying the results out in a neat table as shown above! Also, I am quite sure that I often got the answer wrong. Calculators are a great boon!

So, corrected to 2 decimal places, the cube root of 34 567 is 32.58.

Yours sincerely,

Alastair Summers
(57 Conduit Road
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UK)

Dear Editor,

Multiples of 11

It is well known that a number written in base 10 is a multiple of 11 if and only if the sum of its even-numbered digits differs from the sum of its odd-numbered digits by a multiple of 11. Thus, for example,

$$\begin{aligned} 1\,474 &= 11 \times 134 \quad \text{and} \quad (4+4)-(1+7)=0, \\ 979 &= 11 \times 89 \quad \text{and} \quad (9+9)-7=11, \\ 2\,090 &= 11 \times 190 \quad \text{and} \quad (0+0)-(2+9)=-11. \end{aligned}$$

The reason for this is that

$$a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^r a_r \equiv a_0 - a_1 + a_2 - \cdots + (-1)^r a_r (\bmod 11).$$

The examples given differ in that the first has its even-numbered digit sum equal to its odd-numbered digit sum, whereas the others do not. We ask how many multiples of 11 up to 10 000 (say) have such sums unequal. First consider when the difference is 11. For numbers smaller than 1 000, say cba, we have $c + a - b = 11$, or

$$c + a = b + 11.$$

Thus, $11 \le c + a \le 18$. There are eight possibilities when $c + a = 11$ (c from 2 to 9), seven possibilities when $c + a = 12$ (c from 3 to 9), down to one possibility when $c + a = 18$ ($c = a = 9$), giving $8 + 7 + \cdots + 1 = \Delta_8$, the eighth triangular number, altogether.

For numbers between 1 000 and 1 999, say $1cbd$, we have $(c + a) - (1 + b) = 11$, or

$$(c + a) = b + 12.$$

This time $12 \le c + a \le 18$ and there are $7 + 6 + \cdots + 1 = \Delta_7$, the seventh triangular number. And so on. So the number smaller than 10 000 is

$$\Delta_8 + \Delta_7 + \cdots + \Delta_1,$$

the eighth *tetrahedral number* Tetr_8. Now,

$$\Delta_n = \frac{1}{2} n(n+1) = \binom{n+1}{2},$$

a binomial coefficient, so that

$$\begin{aligned}
\Delta_n &= \binom{n+2}{3} - \binom{n+1}{3}, \\
\Delta_{n-1} &= \binom{n+1}{3} - \binom{n}{3}, \\
&\vdots \\
\Delta_2 &= \binom{4}{3} - \binom{3}{3}, \\
\Delta_1 &= \binom{3}{3}.
\end{aligned}$$

Summing these, we obtain

$$\mathrm{Tetr}_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n = \binom{n+2}{3}.$$

Hence, $\mathrm{Tetr}_8 = \binom{10}{3} = 120$, and this is the required number when the difference is 11. A similar calculation shows that there are also 120 numbers where the difference is -11.

There are $\mathrm{Tetr}_9 = 165$ numbers between 10 000 and 20 000 where the difference is 11 and $\mathrm{Tetr}_{10} = 220$ between 20 000 and 30 000. It would seem that this pattern continues.

The first number where the difference is 22 is 40 909, and 409 090 where the difference is -22. I have not looked into patterns for these or other differences.

Yours sincerely,

S. Hayes
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UK)

$$1^5 + 2^5 + \cdots + 13^5 = 11^2(1^3 + 2^3 + \cdots + 13^3) = 1001^2.$$

Is this a special case of something more general?

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Muneer Jebreel

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

38.9 Let $P(x)$ be a polynomial with integer coefficients and constant term 5928 such that $P(5) = 2006$. Prove that 12, 19, and 26 are not roots of $P(x)$.

(Submitted by Mihály Bencze, Brasov, Romania)

38.10 A right-angled triangle has a point on its hypotenuse at fixed distances a and b from its other two sides. What is its minimum area?

(Submitted by Peter Morris, Salsjöbaden, Sweden)

38.11 For a positive real number x, determine the number of triangular numbers less than or equal to x. (A 'triangular number' is a positive integer of the form $1 + 2 + \cdots + n$, where n is a positive integer.)

(Submitted by Michael Nyblom, RMIT University, Melbourne, Australia)

38.12 Determine all natural numbers n for which there exists a permutation $a_1, \dots, a_n$ of $0, 1, \dots, n-1$ such that the remainders when $a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\cdots+a_n$ are divided by n are all distinct.

(Submitted by H. A. Shah Ali, Tehran, Iran)

Solutions to Problems in Volume 38 Number 1

38.1 Determine all natural numbers n and all permutations $(p_1, \dots, p_n)$ of $(1, \dots, n)$ such that $|k - p_k| = 1$ for $k = 1, \dots, n$.

Solution

The only possibility is that $p_1 = 2$. Now we must have $p_2 = 1$. Then $p_3 = 4$ and $p_4 = 3$, etc. Thus, n is even and there is only one such permutation for each even n.

38.2 The Fibonacci and Lucas sequences are defined by

$$\mathrm{F}_0 = 0,\ \mathrm{F}_1 = 1, \qquad \mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2} \quad \text{for } n \geq 2$$

and

$$\mathrm{L}_0 = 2,\ \mathrm{L}_1 = 1, \qquad \mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2} \quad \text{for } n \geq 2$$

respectively. Prove that

(i) $\mathrm{F}_{3n} + (-1)^n \mathrm{F}_n = \mathrm{L}_n \mathrm{F}_{2n}$,

(ii) $\mathrm{L}_{3n}\mathrm{F}_{3n} + \mathrm{F}_n \mathrm{L}_n = \mathrm{F}_{2n}\mathrm{L}_{2n}^2$,

for all $n \geq 0$.

Solution by Henry Ricardo, Medgar Evers College, New York

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then $\alpha\beta = -1$ and it is well known that

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \qquad L_n = \alpha^n + \beta^n, \qquad \text{for all } n \geq 0.$$

(i)
$$\begin{aligned} F_{3n} + (-1)^n F_n &= \frac{1}{\sqrt{5}}(\alpha^{3n} - \beta^{3n}) + (\alpha\beta)^n \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \\ &= \frac{1}{\sqrt{5}}(\alpha^{3n} + \alpha^{2n}\beta^n - \alpha^n\beta^{2n} - \beta^{3n}) \\ &= (\alpha^n + \beta^n)\frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) \\ &= L_n F_{2n}. \end{aligned}$$

(ii)
$$\begin{aligned} L_{3n}F_{3n} + F_n L_n &= (\alpha^{3n} + \beta^{3n})\frac{1}{\sqrt{5}}(\alpha^{3n} - \beta^{3n}) + \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)(\alpha^n + \beta^n) \\ &= \frac{1}{\sqrt{5}}(\alpha^{6n} - \beta^{6n} + \alpha^{2n} - \beta^{2n}) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n})(\alpha^{4n} + \alpha^{2n}\beta^{2n} + \beta^{4n} + 1) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n})(\alpha^{4n} + 2\alpha^{2n}\beta^{2n} + \beta^{4n}) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n})(\alpha^{2n} + \beta^{2n})^2 \\ &= F_{2n}L_{2n}^2. \end{aligned}$$

38.3 Sum the finite series

$$\sum_{r=0}^{n} \frac{(n-r)!\,(-1)^{n-r}}{(n-r)!\,r!\,r!}.$$

Solution by Henry Ricardo, Medgar Evers College, New York

Since

$$\begin{aligned} (1-x)^n &= \binom{n}{n} - \binom{n}{n-1}x + \binom{n}{n-2}x^2 - \cdots + (-1)^n\binom{n}{0}x^n, \\ \frac{1}{(1-x)^{n+1}} &= 1 + (n+1)x + \frac{(n+1)(n+2)}{2!}x^2 + \frac{(n+1)(n+2)(n+3)}{3!}x^3 + \cdots, \end{aligned}$$

the coefficient of x^n in the expansion of $(1-x)^n/(1-x)^{n+1}$ is

$$(-1)^n\binom{n}{0}\binom{n}{0} + (-1)^{n-1}\binom{n}{1}\binom{n+1}{1} + (-1)^{n-2}\binom{n}{2}\binom{n+2}{2} + \cdots + (-1)^{n-n}\binom{n}{n}\binom{2n}{n},$$

which is the given series. Thus, the given series is the coefficient x^n in the expansion of $(1-x)^{-1}$, i.e. it is 1.

38.4 Prove that, for all real numbers x, y, z,

$$\sqrt{x^2+xy+y^2}+\sqrt{x^2+xz+z^2}\geq\sqrt{y^2+yz+z^2}.$$

Solution by Mihály Bencze, Brasov, Romania

If A, B, and C have respective coordinates $(0, y)$, $(x\sqrt{3}/2, -x/2)$, and $(-z\sqrt{3}/2, -z/2)$, then

$$\begin{aligned}
AB &= \sqrt{\frac{3}{4}x^2+\left(y+\frac{x}{2}\right)^2}=\sqrt{x^2+xy+y^2},\\
BC &= \sqrt{\frac{3}{4}(x+z)^2+\frac{1}{4}(x-z)^2}=\sqrt{x^2+xz+z^2},\\
AC &= \sqrt{\frac{3}{4}z^2+\left(y+\frac{z}{2}\right)^2}=\sqrt{y^2+yz+z^2},
\end{aligned}$$

and $AB + BC \geq AC$, by the triangle inequality.

Ramanujan's Continued Fraction

As is recounted in Kanigel's book *The Man Who Knew Infinity* (Abacus, 1991), a Hindu friend of Ramanujan's, Mahalanobis, when he and Ramanujan were both at the University of Cambridge, read out to him a puzzle from *Strand* magazine about an inhabitant of Louvain (which had just been burned by the German army). A Belgian lived in a house on a long street which was numbered 1, 2, 3, ... consecutively along his side of the street. The number of his house had a curious property: the sum of all the house numbers that came before it was the same as the sum of all the house numbers that came after it. The magazine stated that there were more than 50 and fewer than 500 houses on that side of the street. What was the Belgian's house number? Ramanujan thought for a moment and then dictated the first few convergents of a continued fraction which included *all* the solutions to the problem (not just the one falling within the 50–500 range).

What was the house number and how many houses were there on that side of the street?

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Sebastian Hayes

Reviews

Reality Conditions. Short Mathematical Fiction. By Alex Kisman. MAA, Washington, DC, 2005. Paperback, 260 pages, \$37.50 (ISBN 0-88385-552-6).

This unusual book is a collection of sixteen short stories by Alex Kisman, an Associate Professor of Mathematics at the College of Charleston in South Carolina. The singular feature is that each of the stories has a mathematical theme, but (except for one or two minor exceptions) these are handled without using mathematical notation. In his preface, the author explains his motivation as follows. 'There were ideas, stories, and facts that I wished had appeared in some work of fiction but had not. From there, it was not a very big leap to actually writing some stories. So, now you know, each of these stories contains some ideas that I'm trying to get across and is also supposed to be enjoyable to read.'

There is not space here to comment on each story, nor would that be appropriate. I will deal with one or two in an attempt to illustrate the author's approach. In the story I enjoyed the most, called *The Math Code*, the eminent Professor Coburn is kidnapped and a ransom note is received. The kidnapper mistakenly allowed Coburn to include the following two sentences in this note to demonstrate that he does indeed hold Coburn. 'Yesterday in my office, I proved a new theorem. It was that for every pair L and B in Minkowski space, there is an A so that B times L tensor A is a subspace of Hilbert space.' His colleagues of course realise that this makes no mathematical sense, but one of them is able to deduce from it the place where Coburn is being held and the name of the kidnapper. In another story, called *The Object*, an algorithm written by Alice Wu to predict molecular structures, and programmed by Sophia Yakimov, appears to produce a regular polyhedron with 37 vertices and triangular faces. When a three-dimensional model of this impossible structure, 'the object', is constructed, Sophia vanishes when she handles it. The remainder of the story attempts to explain how this could occur via two different universes that look just like flat Euclidean space except near the object. I found this story, like some others, to be rather far-fetched. A commendable feature of this book is the involvement of many female mathematicians. Not surprisingly, for a book written by an American mathematician, most of the mathematical themes are what I (an applied mathematician) would describe as pure mathematics. An exception is provided by one of the shortest stories in this book, called *Maxwell's Equations*. Here, however, I think the author misses the point by over-emphasis of the wave equation which is a consequence of Maxwell's equations, rather than one of the equations themselves. I believe that the correct way to view Maxwell (arguably the greatest Scotsman of all time and arguably the greatest scientist of the 19th century) is as someone whose sole aim was to understand the physics of a problem by describing it accurately using mathematics, i.e. as a classical applied mathematician just like Newton (arguably the greatest Englishman of all time!).

This book contains some factual errors (for example reference to 'Trumpington Road' in Cambridge rather than 'Trumpington Street') and some grammatical errors (for example reference to 'Field's Medal' on page 139 in a story where 'Fields Medal' is used correctly on page 161). Nevertheless, the book is interesting and its originality is very welcome. I would like to think it would stimulate others to develop the author's theme, but I would then hope to see rather better writing and a wider coverage of mathematics.

The University of Sheffield **Philip Chatwin**

Martin Gardner's Mathematical Games. By Martin Gardner. MAA, Washington, DC, 2005. CD-ROM, $55.95 (ISBN 0-88385-545-3).

This is a superb and user-friendly resource; it gives all of Martin Gardner's 'Mathematical Games' columns from *Scientific American*. The CD-ROM is divided into 15 sections; it has a useful search device and an interview with Martin Gardner by Don Albers. Each section is comprised of about 20 to 25 chapters. Accompanying the CD-ROM is a small booklet with two articles about Martin Gardner, by Peter Renz and Don Albers. The CD-ROM is great for dipping into to remind you of articles that you have previously read, for example the famous Game of Life devised by John Conway. If you have not read any of Martin Gardner's *Scientific American* columns or his books, then this CD-ROM should come as a store of much enjoyment for you. One realises again how great Martin is at explaining mathematical concepts and ideas to the general public, whether they have a mathematical background or not. He is perhaps the best writer to demonstrate that items labelled as 'recreational mathematics' can be extremely interesting, thought provoking, relevant to modern life, severely challenging when it comes to proofs, and utilize ideas from many other branches of mathematics as well as leading to developments in the other branches.

At a cost of $55.95, this CD-ROM is a bargain. It is good to have a personal copy, but I would also recommend it for all school, college, and university libraries.

Atlantic College **Paul Belcher**

An Introduction to Statistics Using Resampling Methods and Microsoft Office Excel. By Phillip I. Good. John Wiley, Chichester, 2005. Paperback, 232 pages, £34.50 (ISBN 0-471-73191-9).

Introduction to Statistics Through Resampling Methods and R/S-PLUS. By Phillip I. Good and Clifford E. Lunneborg. John Wiley, Chichester, 2005. Paperback, 230 pages, £34.50 (ISBN 0-471-71575-1).

This book is produced in two versions enabling readers to use their preferred programming language to illustrate new concepts and assist with the completion of exercises. It aims to introduce statistical methodology to a wide audience, simply and intuitively, through resampling from the data at hand. As such, the text can provide a first course in statistics and quantitative reasoning for students of, for example, biology, dentistry, medicine, psychology, sociology, or public health.

The early chapters cover the familiar areas of descriptive statistics, probability, distributions, hypothesis testing, design of experiments, and regression, whilst later chapters focus on developing models, reporting research findings, and the best approach to problem solving. The use of EXCEL® (or R®) is integrated throughout and there are reassuringly helpful images to show what the computer screen should look like after each step. Exercises abound; readers are certainly encouraged and supported to discover solutions on their own.

Use of information and communications technology (ICT) makes handling large data sets relatively simple. Consequently, it makes eminent sense to develop software know-how alongside statistical expertise. These books do just that and add an interesting dimension to the acquisition of skills in applying statistical methods.

Carol Nixon

Mathematics in Service to the Community: Concepts and Models for Service-Learning in the Mathematical Sciences. Edited by Charles R. Hadlock. MAA, Washington, DC, 2005. Paperback, 264 pages, $48.95 (ISBN 0-88385-176-8).

Service-learning for undergraduates is defined in the opening chapter as involving activities which 'take place within a service framework where additional experience with civic engagement or social contribution will be obtained'. The next three chapters describe projects in which this has been achieved in the three areas of mathematical modelling, statistics, and education-oriented mathematics, in the hope that readers will then be able to develop and facilitate similar projects with their students. A fifth chapter details how to turn a rough idea into an action plan and a final chapter gives a 'how-to guide for designing and teaching a service-learning course'.

Because organizations frequently lack the resources to undertake projects that they would like to do, partnerships with academic institutions that can assist them to meet such needs can be welcome. This has led to some interesting collaborations, such as a project with the Baltimore City Fire Department to study staffing methods with the aim of minimizing overtime costs, which provided six undergraduates with stimulating real-world work. Another project, occupying three students, came up with some efficient snowplough routes for Morris, Minnesota.

Various data-interpretation projects are described in the statistics-based chapter. One of these has involved working with the American Red Cross; another with a meals-on-wheels organization. But it is the education-oriented projects that have provided more easily implemented activities such as tutoring of younger students.

Overall, this volume brings together a wealth of experience gained by those teachers who have been keen to enhance the delivery of curricular material and thus contribute to the effectiveness of mathematics learning. Transferrability of practice may not always be possible, but there is little doubt that in reading this book some seeds of ideas will be sown that can only lead to the enrichment of the mathematical experience of undergraduates.

Carol Nixon

Revise for Core Maths 2, 3 and 4. By Greg Attwood, Alistair Macpherson, Bronwen Moran, Joe Petran, Keith Pledger, Geoff Staley and Dave Wilkins. Heinemann, Oxford, 2005. Paperback, 80 pages each, £4.75 each (ISBN 0-435-51123-8, 0-435-51125-4 and 0-435-51124-6).

All three books provide key point summaries, worked examples, revision exercises, exam questions, test-yourself problems, and helpful cross-referencing to the original textbooks for the Edexcel syllabus for three compulsory pure mathematics modules of AS-level and A-level Mathematics. These can only be useful revision material for any student on this course.

Carol Nixon

Understanding Our Quantitative World. By Janet Anderson and Todd Swanson. MAA, Washington, DC, 2005. Hardback, 320 pages, $51.50 (ISBN 0-88385-738-3).

This rather inviting title promises an account that the book does not really live up to. Intended for general education mathematics courses, the text aims to develop the mathematical skills that are 'useful for informed citizens'. However, this book disappoints in the way that it tries to achieve this, offering little that is different from other introductory texts.

Interpretation of graphs, simple functions, and statistical information provides the focus of the chapters, whilst activities and class exercises based on data from newspapers, magazines, and the World Wide Web form the means by which students acquire the skills of interpreting quantitative information encountered in the real world. No answers are provided for the numerical exercises – presumably because they are not felt to be needed if problems are undertaken as class-based activities.

The functions theme includes an investigation of linear, exponential, logarithmic, periodic, power, and multivariable functions. The use of a graphical calculator is assumed; some of the activities require additional programs for graphical calculators.

The statistics theme provides an exploration of various statistical graphs, descriptive statistics, and linear, exponential, and power regression, again with a dependence on graphical calculators. Probability and random samples are also included in the final two chapters.

This book is written in a conversational style, suggesting that it could be used by students working on their own. But the omission of numerical answers or hints to the why?, which?, and how? questions would, I feel, mitigate against this.

It is certainly refreshing to see some real data used in the exercises, and to know that the course has been designed so that students are encouraged to be proactive rather than passive in their learning. But with the dependence of some areas on a specific graphical calculator, this book may well become time-expired when this calculator is superseded by a more sophisticated model.

Carol Nixon

Real Infinite Series. By Daniel D. Bonar and Michael J. Khoury, Jr. MAA, Washington, DC, 2006. Paperback, 272 pages, $49.95 (ISBN 0-88385-745-6).

From the cover: 'This is an introductory treatment of infinite series, bringing the reader from basic definitions and tests to advanced results. An up-to-date presentation is given, making infinite series accessible, interesting, and useful to a wide audience, including students, teachers, and researchers.'

Other books received

A Tour through Mathematical Logic. By Robert S. Wolf. MAA, Washington, DC, 2004. Hardback, 408 pages, $52.95 (ISBN 0-88385-036-2).

The Mathematics Companion: Essential and Advanced Mathematics for Scientists and Engineers. By A. C. Fischer-Cripps. IOP, Bristol, 2005. Paperback, 207 pages, £11.99 (ISBN 0-750-31020-0).

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LONDON MATHEMATICAL SOCIETY

POPULAR LECTURES 2006

The dates for this year's lectures are still to be confirmed but they will take place in London, in July, and in Birmingham, in September.

The 2006 lecturers will be:

Dr Emma McCoy
(Imperial College)
'From magic squares to Sudoku'

Dr John Haigh
(University of Sussex)
'How likely is that?'

Once the details are finalised, the information will be posted on the LMS website (www.lms.ac.uk).

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