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Patterns and Primes in Bernoulli's Triangle

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The author is Reader in Mathematical Biology at Cambridge and a Fellow of Gonville and Caius College. His historical book *Pascal's Arithmetical Triangle* appeared in 1987. He recently published a generalization of the Venn diagram to an arbitrary number of sets (*New Scientist*, 7 January 1989).

In *Mathematical Spectrum* Volume 20 (reference 1) I christened the number triangle of cumulative binomial coefficients after James Bernoulli, who gave it in his *Ars conjectandi* of 1713. It is found from Pascal's triangle (figure 1) by simply summing along the rising diagonals (figure 2). The cumulative binomial coefficients are defined as

$$^{n}B_{r} = \sum_{i=0}^{r} {}^{n}C_{i}.$$

They bear the same relation to the cumulative binomial distribution as do binomial coefficients to the binomial distribution.

1	1	1	1	1	1	1	·
1	2	3	4	5	6	·	
1	3	6	10	15	·		
1	4	10	20	·			
1	5	15	·				
1	6	·					
1	·						
·							

Figure 1. Pascal's triangle of binomial coefficients

As with Pascal's triangle, each number in Bernoulli's triangle is the sum of the adjacent numbers above and to the left of it, but in the case of Bernoulli's triangle the first row consists of the powers of 2, since binomial coefficients for index n sum to 2^n. Furthermore, the falling diagonal starting with the 1 at the beginning of the second row also consists of powers of 2 (for even exponents only), corresponding to exactly half of each n-odd binomial distribution.

1	2	4	8	16	32	64	·
1	3	7	15	31	63	·	
1	4	11	26	57	·		
1	5	16	42	·			
1	6	22	·				
1	7	·					
1	·						
·							

Figure 2. Bernoulli's triangle of cumulative binomial coefficients

Leibniz proved that if n is prime it divides all the binomial coefficients ${}^nC_r$ in the corresponding 'rising diagonal' of Pascal's triangle except ${}^nC_0$ and ${}^nC_n$ (which are both 1). Given that the coefficients in the next diagonal are formed by the addition rule, it is then obvious that all bar the end two on each side will also be divisible by n, and so on, divisibility by n spreading through successive diagonals in an ever-narrowing band. Indeed, any run of coefficients in a single diagonal which all contain the same factor will generate a subtended triangle of numbers divisible by that factor. Furthermore, generalizations of Leibniz's result, notably that, if n is a power of a prime, the prime divides all the ${}^nC_r$ except ${}^nC_0$ and ${}^nC_n$, ensure larger and larger triangles of divisibility.

In consequence, Pascal's triangle exhibits beautiful patterns when rendered modulo a prime, and these patterns are easy to generate since the addition rule of construction continues to operate in the modulo arithmetic. The zeros then correspond to the original numbers which are divisible by the prime in question. Figure 3 shows the pattern of divisibility by 5, black representing divisibility. Other primes produce similar patterns, whilst composite numbers produce less-regular ones. These patterns are well known (see, for example, reference 2), but we can now see that they are *fractal*, that is, self-similar, if we use that word not only to describe self-similarity under repeated magnification but also self-similarity under repeated reduction. For the quarter-plane which they fill has much the same appearance however far away you stand to look at it.

In a like manner we can generate the patterns corresponding to particular remainders. For example figure 4 depicts the incidence of 3s when the triangle is represented modulo 7.

Bernoulli's triangle similarly generates attractive patterns of divisibility. Figure 5 shows divisibility by 5, and figure 6 divisibility by 11. The basic

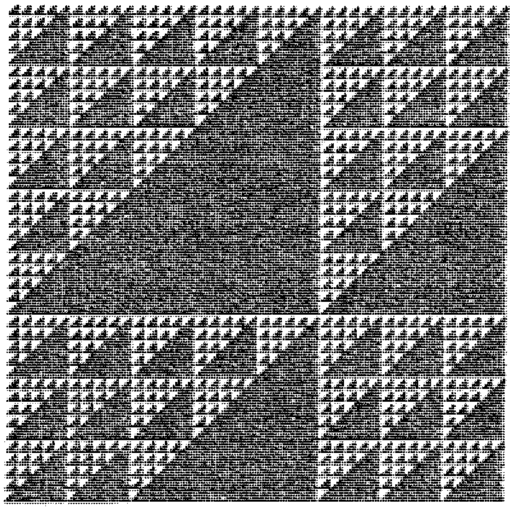

Figure 3. Pascal's triangle showing the pattern of divisibility by 5. In this and the subsequent figures the grid of points is 200 by 200

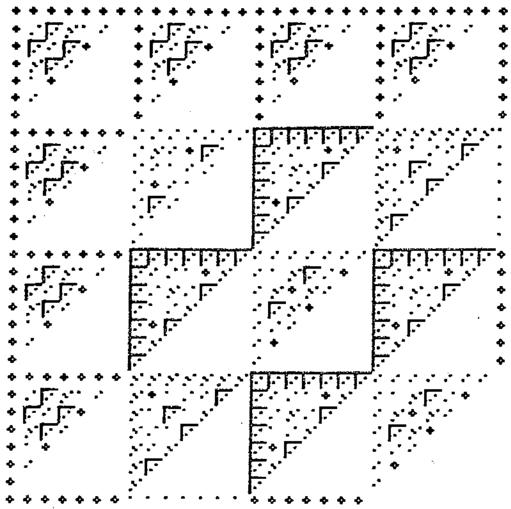

Figure 4. Pascal's triangle modulo 7 showing the pattern of 3s.

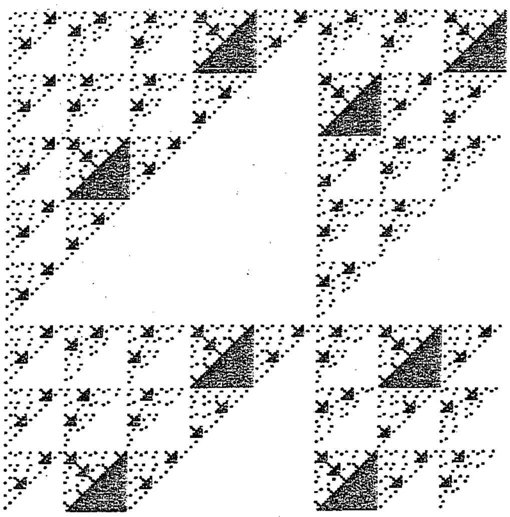

Figure 5. Bernoulli's triangle showing the pattern of divisibility by 5

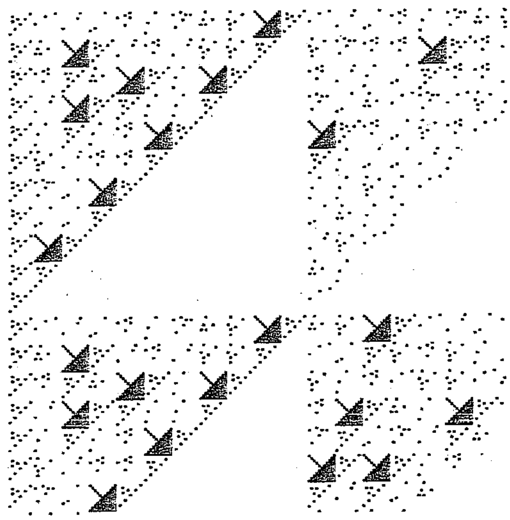

Figure 6. Bernouli's triangle showing the pattern of divisibility by 11

patterns, with their interesting repeat structures, are easily explained on the basis of Leibniz's prime divisor theorem and its generalizations. Thus it is obvious that since all the coefficients (save the terminal 1s) in the rising diagonal of Pascal's triangle for n are divisible by n when n is prime, the corresponding coefficients of Bernoulli's triangle are all 1 when expressed modulo n, and therefore none is divisible by n (neither are the end terms, 1 and 2^n, except trivially the latter for $n = 2$). As a result of all but the last of these coefficients of this rising diagonal being 1 (mod n), and bearing in mind the addition rule of construction, throughout the triangle subtended from them each number will be a power of 2 when expressed modulo n, and therefore not itself divisible by n.

It is similarly clear that the rising diagonal in Bernoulli's triangle corresponding to $n-1$ (n prime) will, when expressed modulo n, read $(1, 0, 1, \ldots, 0, 1)$, which explains the rising diagonals of alternate spaces and dots. As with Pascal's triangle, any continuous line of zeros in a rising diagonal in the modular representation subtends a triangle full of zeros in that representation, and thus creates an arrow-head of dots in the pattern.

A particular consequence of the rising diagonal for n (n prime) in Bernoulli's triangle being all 1s when expressed modulo n is that the penultimate term 2^n-1 (mod n) is 1, and therefore 2^n (mod n) is 2 and we have the congruence $2^{n-1} \equiv 1 \pmod{n}$. This is a special case of Fermat's theorem $\alpha^{n-1} \equiv 1 \pmod{n}$, where n is a prime which is not a divisor of α, a theorem which is itself easily proved by reference to the divisibility of binomial coefficients.

We have already noted that the falling diagonal starting with the 1 in the second row consists entirely of powers of 2. In a divisibility pattern for any number other than a power of 2 it therefore consists only of spaces. Moreover, in any rising diagonal two numbers an equal distance on either side of this line will sum to a power of 2, so that if one of them is divisible by a particular number (other than a power of 2) the other cannot be. Every dot in a divisibility pattern is therefore paired with a space an equal distance on the other side of the falling diagonal, which proves that in every rising diagonal fewer than half the numbers can be divisible by the same divisor (except 2).

In Pascal's triangle all the numbers are composite except those in the first two rows and first two columns, as is easily proved. By contrast, in Bernoulli's triangle primes occur away from the first two rows and columns (for example, 11 in position 3, 3). However, it appears by inspection that there are no primes in the even-numbered columns (other than in column 2), and a computer search revealed none in the even columns 4–64 up to row 31 inclusive. Continued searching finally encountered the prime

$$^{62}B_{13} = \sum_{i=1}^{13} {}^{62}C_i = 11\,108\,452\,651\,921,$$

and this is the smallest prime in any even-numbered column of Bernoulli's triangle (other than column 2).

In the course of the search for prime divisors the following large numbers in Bernoulli's triangle were proved composite:

$$
\begin{aligned}
{}^{54}B_{17} &= 81\,725\,384\,344\,741\\
&= 39\,733\times 2056\,864\,177,\\
{}^{54}B_{21} &= 1204\,030\,473\,149\,641\\
&= 49\,391\times 91\,019\times 267\,829,\\
{}^{64}B_{19} &= 14\,414\,187\,542\,581\,429\\
&= 1335\,053\times 10\,796\,715\,593,\\
{}^{64}B_{31} &= 8307\,059\,966\,383\,480\,541\\
&= 185\,849\times 404\,009\times 110\,635\,901.
\end{aligned}
$$

There is no simple polynomial expression for the cumulative binomial coefficient ${}^nB_r$, but it may be shown that when r is odd it takes the form

$$\frac{(n+1)(\text{a polynomial in } n \text{ of order } r-1 \text{ with integral coefficients})}{r!}.$$

For with r odd the binomial coefficients which make up ${}^nB_r$ can be paired:

$$
\begin{aligned}
&({}^nC_0+{}^nC_1)+({}^nC_2+{}^nC_3)+({}^nC_4+{}^nC_5)+\ldots+({}^nC_{r-1}+{}^nC_r)\\
&\quad= {}^{n+1}C_1+{}^{n+1}C_3+{}^{n+1}C_5+\ldots+{}^{n+1}C_r\\
&\quad= (n+1)\left(1+\sum_{i=3,5,\ldots}^{r}\frac{n(n-1)(n-2)\ldots(n-i+2)}{i!}\right)
\end{aligned}
$$

which is clearly of the stated form.

It follows that, since the new polynomial has integer coefficients, ${}^nB_r$ (r odd > 1) is composite for all $n+1 > r!$. [*Proof:* ${}^nB_r$ is of the form XY/Z, where $X = n+1$, Y is an integer, $Z = r!$ and $X > Z$. Divide out the highest common factor (if any) of X and Z to obtain xY/z, say, where x and z are therefore coprime. x is then a factor unless $Y = z$, but then ${}^nB_r = x \leqslant X = n+1$, which is false for odd $r > 1$.] Thus there are no primes in any even-numbered column of Bernoulli's triangle, other than column 2, beyond a certain sufficiently-large row number.

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Two Binomial Identities

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Roger Cook is a reader in pure mathematics at the University of Sheffield and an editor of *Mathematical Spectrum*. He was a student at University College London from 1965–71 and then spent three years at University College Cardiff before moving to Sheffield in 1974. His main research interests are in number theory and combinatorics. Outside mathematics his recreation includes photography, travel, and browsing in second-hand bookshops or flea-markets.

Some recent work on signal processing by a colleague, Shaun Quegan, led to the following unusual identities:

$$\binom{2}{1}.1.1^2+\binom{2}{1}.(-1).2^2+\binom{2}{2}.(-1).3^2+\binom{2}{2}.1.4^2 = 1$$

and

$$\binom{4}{1}.1.1^4+\binom{4}{1}.(-1).2^4+\binom{4}{2}.(-1).3^4+\binom{4}{2}.1.4^4$$
$$+\binom{4}{3}.1.5^4+\binom{4}{3}.(-1).6^4+\binom{4}{4}.(-1).7^4+\binom{4}{4}.1.8^4 = 1.$$

The left-hand sides of the above equations started out as the cases $m = 1$ and $m = 2$ of the sum

$$S = \sum_{j=1}^{2m} \binom{2m}{2m-j}(-1)^j[(2j)^{2m}-(2j-1)^{2m}]$$

and, after writing out the cases $m = 1,2$ in the form above, it is easy to see that

$$S = \sum_{r=1}^{4m} \epsilon_r B_r r^{2m},$$

where

$$\epsilon_r = \begin{cases} +1 & \text{when } r \equiv 0,1 \pmod 4, \\ -1 & \text{when } r \equiv 2,3 \pmod 4, \end{cases}$$

B_r is the binomial coefficient $\binom{2m}{[\frac{1}{2}(r+1)]}$ and $[\frac{1}{2}(r+1)]$ denotes the integer part of $\frac{1}{2}(r+1)$. You might like to check on a calculator that when $m = 3$ we still have $S = 1$. In fact another colleague, Peter Harley, checked the identity for me on a computer using a multiple-precision package. We have $S = 1$ for $m = 1,2,\ldots,20$.

It then seemed safe to conjecture that in fact $S = 1$ but to prove it needed a bit of calculus. The sum S is a particular example of sums

$$\sum_{r=1}^{N} c_r r^p$$

and we can set up a generating function

$$f(x) = \sum_{r=1}^{N} c_r e^{rx}$$

and differentiate p times to give

$$f^{(p)}(x) = \sum_{r=1}^{N} c_r r^p e^{rx}.$$

So, putting $x = 0$, we have

$$f^{(p)}(0) = \sum_{r=1}^{N} c_r r^p.$$

For our problem, when $m = 1$ we get

$$\begin{aligned} f(x) &= 2e^x - 2e^{2x} - e^{3x} + e^{4x} \\ &= (1-e^{-x})(1-e^{2x})^2 - 1 + e^{-x} \end{aligned}$$

and when $m = 2$ it is easy to verify that we get

$$(1-e^{-x})(1-e^{2x})^4 - 1 + e^{-x}.$$

In general we get

$$f(x) = (1-e^{-x})(1-e^{2x})^{2m} - 1 + e^{-x}.$$

If we write

$$(1-e^{-x})(1-e^{2x})^{2m} = u(x)\,v(x),$$

where $u(x) = (1-e^{-x})$ and $v(x) = (1-e^{2x})^{2m}$, we have to differentiate the product $2m$ times. The substitution $y = 1-e^{2x}$ makes $v(x) = y^{2m}$, and also $y = 0$ when $x = 0$. Therefore, if we differentiate the product $u(x)\,v(x)$ $2m$ times and substitute $x = 0$, the only non-zero term comes from when we differentiate $v(x)$ $2m$ times, i.e.

$$\left.\frac{d^{2m}}{dx^{2m}} u(x)\,v(x)\right|_{x=0} = u(0) \left.\frac{d^{2m}}{dx^{2m}} v(x)\right|_{x=0}.$$

But $u(0) = 0$ and so, for $m \geqslant 1$,

$$\left.\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}f(x)\right|_{x=0} = \left.\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}\mathrm{e}^{-x}\right|_{x=0} = 1,$$

giving us the required identity.

Now the title of this article refers to two binomial identities. The other one crops up when we look at sums with $2m+1$ terms:

$$\sum_{j=0}^{2m}\binom{2m+1}{2m-j}(-1)^j[(2j+1)^{2m+1}-(2j)^{2m+1}].$$

The cases $m = 0$ and $m = 1$ lead to

$$-0+\binom{1}{0}.1.1 = 1$$

and

$$\binom{3}{2}.(-1).0^3+\binom{3}{2}.1.1^3+\binom{3}{1}.1.2^3+\binom{3}{1}.(-1).3^3$$
$$+\binom{3}{0}.(-1).4^3+\binom{3}{0}.1.5^3 = 7 = 2^3-1.$$

You might like to check that when $m = 2$ the sum is 31, or 2^5-1. Generally the sum is

$$\sum_{r=1}^{4m+1}\binom{2m+1}{1+[\frac{1}{2}r]}\epsilon_r r^{2m+1},$$

where now

$$\epsilon_r = \begin{cases} +1 & \text{if } r \equiv 1,2 \pmod 4, \\ -1 & \text{if } r \equiv 0,3 \pmod 4. \end{cases}$$

The generating function is now

$$f(x) = (\mathrm{e}^{-2x}-\mathrm{e}^{-x})(1-\mathrm{e}^{2x})^{2m+1}+2m+1+\mathrm{e}^{-x}-\mathrm{e}^{-2x}.$$

In much the same way as before we find that the product term has zero derivative at $x = 0$ and so

$$\left.\frac{\mathrm{d}^{2m+1}}{\mathrm{d}x^{2m+1}}f(x)\right|_{x=0} = \left.\frac{\mathrm{d}^{2m+1}}{\mathrm{d}x^{2m+1}}(\mathrm{e}^{-x}-\mathrm{e}^{-2x})\right|_{x=0}$$
$$= 2^{2m+1}-1,$$

giving our second identity

$$\sum_{r=1}^{4m+1}\binom{2m+1}{1+[\frac{1}{2}r]}\epsilon_r r^{2m+1} = 2^{2m+1}-1.$$

Visualization of the *Gleichniszahlen-Reihe*, an Unusual Number Sequence

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Consider the number-theory sequence $u_{r,n}$, where r is the row number, and n the column number:

$$
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
2 & 1 & & & & & \\
1 & 2 & 1 & 1 & & & \\
1 & 1 & 1 & 2 & 2 & 1 & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

The reader probably cannot guess the numerical entries for the next row. However, the answer is actually simple, when viewed in hindsight. To appreciate the answer, it helps to speak the entries in each row out loud. Note that row 2 has two 1s, thereby giving the sequence 2 1 for the third row. Row 3 has one 2 and one 1. Row 4 has one 1, one 2, and two 1s. From this, an entire sequence $u_{r,n}$ can be generated. This interesting sequence was described in a German article (reference 1), where the author called it *Die Gleichniszahlen-Reihe*, which translated means 'the likeness sequence'. The sequence grows rather rapidly. For example, row 15 is:

1321132132211331121321133112111312211213211312111322

22112311311222113111231133211121321132221131211321 1

Row 27 contains 2012 entries. A casual inspection of the sequence indicates a predominance of 1s, with 2s and 3s less common. For rows between 6 and 27, there are about 50% 1s, 30% 2s and 20% 3s. As proved previously (see reference 1), the largest number u can contain is 3. Readers are invited to prove this in the Problems section (p. 138), and also to prove the result that 333 can never occur.

In this article, we are particularly interested in the distribution of 1s, 2s and 3s. While one can simply compute the percentage of occurrence of each digit for a given row, this does not tell us anything about any interesting clusters or peculiar areas of concentration of one digit over another. A technique which has proved useful in overcoming this drawback involves the transformation of the digit strings into characteristic two-dimensional patterns. A single digit is inspected and assigned a direction of movement on a plane. To visualize this (and other) ternary sequences, use a three-way vectorgram where the occurrence of a 1 directs the trace one unit at 0°, a 2 causes a walk at 120°, and a 3 a walk of 240°. Each of these angles is with respect to the x-axis. Figures 1–3 show patterns for row 15 con-

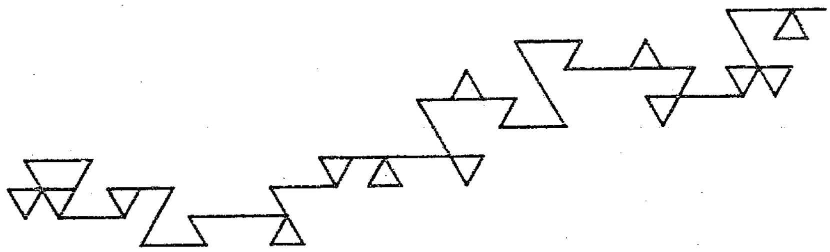

Figure 1. Vectorgram for u when r = 15

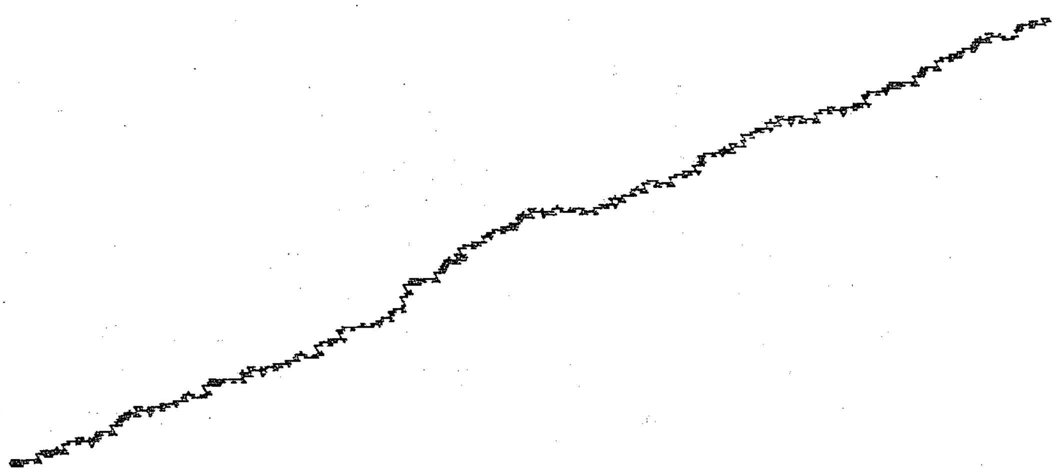

Figure 2. Vectorgram for u when r = 25

taining 102 digits, row 25 containing 1182 digits, and row 33 containing somewhat over 10 000 digits. Different scales were used to fit the graphs on a page. Notice that if the string were to contain only 1s, the walk would be only to the right. As can be seen from the figures, u is far from random. The upward diagonal trend in figures 2 and 3 indicates a mixture

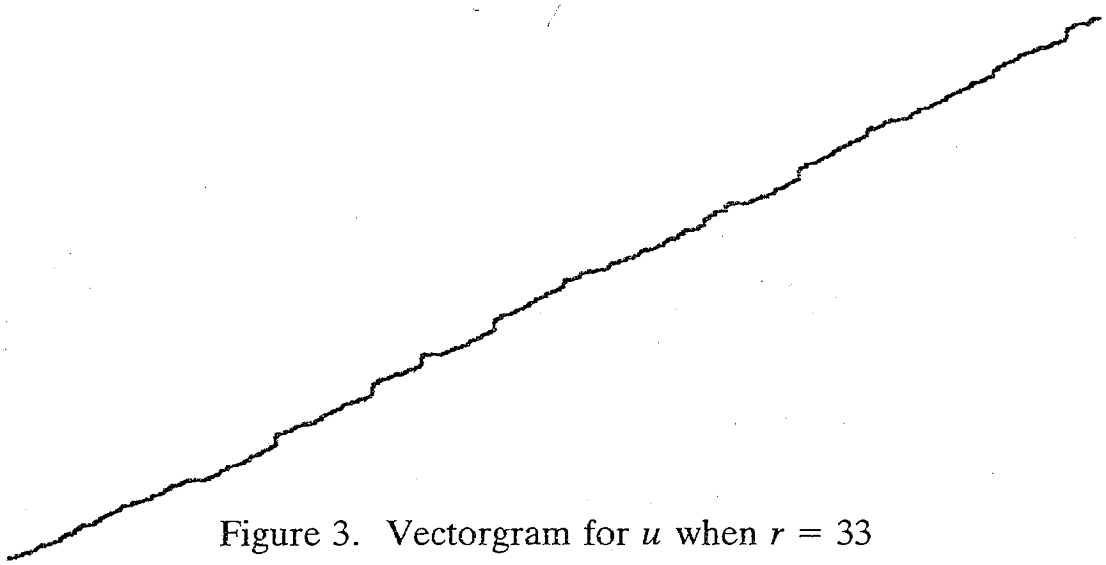

Figure 3. Vectorgram for u when $r = 33$

of predominantly 1s, some 2s and relatively few 3s. The fact that the trends are fairly linear suggests that the ratios are relatively constant throughout the row. Figures 2 and 3 show occurrence of sudden upward bumps which eventually return to the diagonal base line. These bumps indicate a temporary change in the trend to more 2s.

The reader can understand the resulting patterns by considering the directions travelled by various combinations of entries in the sequence. For example, the sequence 1–1–1 is totally x-directed. 1–2–3, 1–3–2, and various cyclic permutations return to the original point:

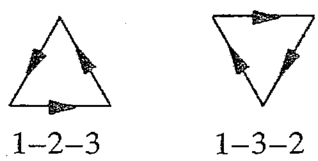

Figure 4

For future experiments, readers may wish to compute the slope of the mean-line of the vectorgram as a function of the row number, or make a plot of the slope of the mean-line versus the number of entries in a row. It appears, from just a few sample points, that the slope of the mean-line increases as a function of row number.

Hopefully readers will uncover or solve additional mysteries with this interesting sequence. Readers interested in the use of eight-way vectorgrams in the characterization of genetic sequences should see reference 2.

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Summing Powers of Integers

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1. Introduction

Most mathematics students will be familiar with the formulae for the sums of the first n integers, these integers squared and these integers cubed. The three results can be written:

$$\sum_{s=1}^{n} s^1 = \tfrac{1}{2}n^2+\tfrac{1}{2}n, \tag{1}$$

$$\sum_{s=1}^{n} s^2 = \tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n, \tag{2}$$

$$\sum_{s=1}^{n} s^3 = \tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2, \tag{3}$$

where we observe a striking feature. Denoting $\sum_{s=1}^{n} s^k$ by $S(k,n)$, for $k \geqslant 0$, we see that

$$S(2,n) = 2\int_0^n S(1,x)\ \mathrm{d}x + \tfrac{1}{6}n, \qquad S(3,n) = 3\int_0^n S(2,x)\ \mathrm{d}x;$$

and it is natural to ask whether this pattern generalises for $S(k,n)$, $k \geqslant 2$.

To investigate this, we might first look at $S(4,n)$, which can be calculated by writing

$$(s+1)^5-s^4 = 5s^4+10s^3+10s^2+5s+1$$

and then summing these equations for s from 1 to n to give

$$(n+1)^5-1 = 5S(4,n)+10S(3,n)+10S(2,n)+5S(1,n)+S(0,n).$$

This gives

$$\begin{aligned} S(4,n) &= \tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n \qquad (4) \\ &= 4\int_0^n S(3,x)\ \mathrm{d}x - \tfrac{1}{30}n; \end{aligned}$$

and, if we then go to $k = 5$, we get in a similar manner that

$$S(5,n) = \tfrac{1}{6}n^6+\tfrac{1}{2}n^5+\tfrac{5}{12}n^4-\tfrac{1}{12}n^2 \tag{5}$$

$$= 5 \int_0^n S(4,x)\ \mathrm{d}x.$$

2. The general recursive power summation formula

All this suggests a general result

$$S(k,n) = k \int_0^n S(k-1,x)\ \mathrm{d}x + A_k n, \tag{6}$$

where A_k is a particular constant (dependent on k), which can be found, at the end, by putting $n = 1$ in (6). Then, assuming that (6) in fact holds for every positive integer k, we can of course deduce all the 'sums of powers' polynomials sequentially. We now justify this assumption.

Initially, in order that (6) should be generally meaningful, we establish that $S(k,n)$ is always a polynomial in n (of degree $k+1$). There is a unique representation for s^k in the form

$$s^k = c_0 + c_1 s + c_2 s(s+1) + \ldots + c_k s(s+1)\ldots(s+k-1),$$

with $c_k \neq 0$. (In fact, $c_k = 1$.) But, using the fact that

$$s(s+1)\ldots(s+i-1) = \frac{s(s+1)\ldots(s+i-1)(s+i) - (s-1)s(s+1)\ldots(s+i-1)}{i+1},$$

we get, on summing for s from 1 to n and then cancelling successive pairs of equal-magnitude negative and positive terms, that

$$\sum_{s=1}^{n} s(s+1)\ldots(s+i-1) = \frac{n(n+1)\ldots(n+i-1)(n+i) - 0}{i+1}.$$

Thus

$$\sum_{s=1}^{n} s^k = c_0 n + c_1 \frac{n(n+1)}{2} + c_2 \frac{n(n+1)(n+2)}{3} + \ldots + c_k \frac{n(n+1)(n+2)\ldots(n+k)}{k+1},$$

which is a polynomial in n of degree $k+1$.

Now, we establish (6).

Proof. Evidently, for any positive integer n,

$$S(k,n) - S(k,n-1) = n^k$$

and so

$$S(k,x) - S(k,x-1) = x^k \tag{7}$$

for all x, since two polynomials which are equal for infinitely many values must be identically equal.

Then, if we differentiate (7) with respect to x, we get

$$S'(k,x)-S'(k,x-1) = kx^{k-1}$$

and, together with (7), this gives

$$S'(k,x)-kS(k-1,x) = S'(k,x-1)-kS(k-1,x-1). \tag{8}$$

Now, whenever the polynomial on the left of (8) has a zero at $x = x_0$ say, then, from (8), it clearly also has another zero at $x = x_0-1$, and so at x_0-2, $x_0-3,\dots$. Thus, if the latter polynomial has one zero, then it has infinitely many zeros. The only conclusion possible is that it is a constant, i.e.

$$S'(k,x) = kS(k-1,x)+A_k, \tag{9}$$

which yields (6) on integrating with respect to x between the limits of $x = 0$ and $x = n$, since $S(k,0) = 0$.

3. Other curiosities

The right-hand side of (5) actually factorises to give

$$S(5,n) = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1).$$

We also know, from (3), that

$$S(3,n) = \tfrac{1}{4}n^2(n+1)^2; \tag{10}$$

and indeed we have, for all odd $k > 1$, that $n^2(n+1)^2$ divides $S(k,n)$. Similarly, for all positive even k, $n(n+1)(2n+1)$ is a divisor of $S(k,n)$. We just prove the first of these results.

Proof. Consider the identity

$$r^k\{(r+1)^k-(r-1)^k\} = r^k(r+1)^k-(r-1)^kr^k \quad (r = 1,\dots,n).$$

Then, adding for $r = 1,\dots,n$, we get

$$\sum_{r=1}^{n} r^k\left\{\sum_{i=0}^{[\frac{1}{2}(k-1)]}\binom{k}{2i+1}r^{k-2i-1}\right\} = \tfrac{1}{2}n^k(n+1)^k,$$

where $[x]$ denotes the 'integer part of x'. Hence

$$\sum_{i=0}^{[\frac{1}{2}(k-1)]}\binom{k}{2i+1}S(2k-2i-1,n) = \tfrac{1}{2}n^k(n+1)^k;$$

and thus

$$S(2k-1,n) = \frac{1}{k}\left\{\frac{n^k(n+1)^k}{2} - \sum_{i=1}^{[\frac{1}{2}(k-1)]}\binom{k}{2i+1}S(2k-2i-1,n)\right\}, \tag{11}$$

where the first argument of $S(\cdot, n)$ on the right decreases in $2s$ from $2k-3$ to $2k-2[\frac{1}{2}(k-1)]-1$ ($= k$ if k is odd and $k+1$ if k is even).

Finally, assume that $n^2(n+1)^2$ divides $S(2r-1, n)$ for $r = 2, \ldots, k-1$, where $k \geqslant 3$. Then, from (11), $n^2(n+1)^2$ divides $S(2k-1, n)$. But the division holds for $r = 2$ (see (10)) and hence the result is proved by induction.

Corollary. It follows then that $A_k = 0$ for all odd $k > 1$, for otherwise the highest power of n which could be a factor would be the first power. (It is clear from (6) that $S(k, n)$ is divisible by n, so that the term

$$k \int_0^n S(k-1, x)\, dx$$

in (6) cannot have a term of degree 1 in n.)

4. Explicit form of summation formula

For all non-negative integers k, if

$$\sum_{s=1}^{n} s^k = b_{1,k} n^{k+1} + b_{2,k} n^k + \ldots + b_{k+1,k} n = \sum_{r=1}^{k+1} b_{r,k} n^{k+2-r},$$

then, from (6),

$$\sum_{s=1}^{n} s^{k+1} = \sum_{r=1}^{k+1} \left(\frac{k+1}{k+3-r}\right) b_{r,k} n^{k+3-r} + \left\{1 - \sum_{r=1}^{k+1} \left(\frac{k+1}{k+3-r}\right) b_{r,k}\right\} n.$$

Also, since $A_k = 0$ for all odd $k > 1$, then evidently $\sum_{s=1}^{n} s^k$ has the form

$$\sum_{s=1}^{n} s^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \sum_{r=3}^{[\frac{1}{2}k]+2} b_{2r-3,k} n^{k+5-2r} \quad (k = 1, 2, \ldots);$$

and, for $3 \leqslant r \leqslant [\frac{1}{2}k]+2$, $b_{2r-3,k}$ is given by the recurrence relation

$$b_{2r-3,k} = \left(\frac{k}{k+5-2r}\right) b_{2r-3,k-1}. \tag{12}$$

From repeated application of (12), we obtain

$$\sum_{s=1}^{n} s^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \sum_{r=3}^{[\frac{1}{2}k]+2} k(k-1)\ldots(k-2r+6) c_r n^{k+5-2r} \quad (k = 1, 2, \ldots),$$

where the constants c_r are independent of k.

The early examples (1) to (5) would then indicate that

$$\sum_{s=1}^{n} s^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \frac{kn^{k-1}}{12} - \frac{k(k-1)(k-2)n^{k-3}}{720} + \ldots,$$

where we always truncate the right-hand side after the n^2 term when k is odd and $k > 1$, and after the term in n otherwise; which, on rewriting, suggests the convenient form

$$\sum_{s=1}^{n} s^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \sum_{r=0}^{[\frac{1}{2}k]-1} (-1)^r \frac{k(k-1)\dots(k-2r)\,n^{k-2r-1}}{d_r} \quad (k = 1,2,\dots), \tag{13}$$

where the d_r are particular constants (to be determined).

The successive d_r can then be found quite easily sequentially, at least up to $r = 12$ (given an eight-figure calculator and leaving large factorials unexpanded), which are sufficient for determining all the sums of powers up to $k = 27$. (Higher d_r can be obtained using a computer and multi-precision arithmetic.)

In every case, d_i can be found in terms of the earlier $d_0,\dots,d_{i-1}$ by putting $n = 1$ in (13), for $k = 2(i+1)$, and then rearranging the result to give

$$d_i = (-1)^i \Bigg/ \left(\frac{2i+1}{2(2i+3)!} - \sum_{r=0}^{i-1} \frac{(-1)^r}{(2i+1-2r)!\,d_r} \right). \tag{14}$$

For instance, putting $i = 1$ in (14) (and noting that $d_0 = 12$), we get

$$d_1 = (-1)^1 \Bigg/ \left(\frac{3}{2\times5!} - \frac{1}{3!\,12} \right) = 720.$$

5. An example

Putting $k = 9$, we get

$$\sum_{s=1}^{n} s^9 = \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2, \tag{15}$$

having first obtained (in addition to d_0 and d_1, given above) $d_2 = 30\,240$ and $d_3 = 1209\,600$ sequentially from (14); and where we note, as a quick check, that the coefficients on the right-hand side sum to unity.

This particular example was chosen because the result is frequently given incorrectly in tables of formulae. Of course, if you already have the formula for $\sum_{s=1}^{n} s^8$, then it may well be considerably easier to get (15) by using (6) (with $A_9 = 0$).

Even given that we only have (5), successive application of (6), with $A_k = 0$ for $k = 7$ and 9, yields first

$$S(6) = 6\left(\tfrac{1}{42}n^7 + \tfrac{1}{12}n^6 + \tfrac{1}{12}n^5 - \tfrac{1}{36}n^3\right) + A_6 n = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + A_6 n;$$

and, putting $n = 1$, we get

$$1 = \tfrac{1}{7}+\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{6}+A_6,$$

and so $A_6 = \frac{1}{6}-\frac{1}{7} = \frac{1}{42}$.

Next,

$$\begin{aligned} S(7) &= 7\left(\tfrac{1}{56}n^8+\tfrac{1}{14}n^7+\tfrac{1}{12}n^6-\tfrac{1}{24}n^4+\tfrac{1}{84}n^2\right) \\ &= \tfrac{1}{8}n^8+\tfrac{1}{2}n^7+\tfrac{7}{12}n^6-\tfrac{7}{24}n^4+\tfrac{1}{12}n^2; \end{aligned}$$

and, similarly, we get

$$S(8) = \tfrac{1}{9}n^9+\tfrac{1}{2}n^8+\tfrac{2}{3}n^7-\tfrac{7}{15}n^5+\tfrac{2}{9}n^3-\tfrac{1}{30}n$$

and then $S(9)$ as before, in (15).

Powerless Polynomials

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At the time of writing this article, the author had just completed his Master of Science Education degree at Mysore.

In a previous article (*Mathematical Spectrum* Volume 23 Number 3 pages 91–92) we saw that, given an integer $K > 1$, there is an arithmetic progression which does not contain any powers (excluding, of course, first powers) up to and including Kth powers. We remarked that this enables us to write down a polynomial of arbitrary degree with non-negative integer coefficients with a similar property. We can do better than this, however.

Let p be any prime number. Then, for any integer m, p and $pm+1$ are coprime. If $p(pm+1)$ were an rth power, where $r > 1$, so would be p, which of course is not so. Hence the linear polynomial $p(px+1)$ takes no powers when evaluated at positive integers. To obtain such a polynomial of arbitrary degree n, choose a prime number p such that $p \geqslant n+1$. Then, for a similar reason, the polynomial

$$p(px+1)(px+2)\dots(px+n)$$

has this property.

[Editor: The polynomial $100x+15$ has a similar property—see the letter from Douglas Quadling in this issue.]

Probability and Epidemiology

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1. Introduction

Epidemiology is that branch of medical science which investigates the causes, patterns and control of disease in human populations. The word 'epidemic' was originally used as a name for outbreaks of contagious diseases among humans and is derived from the Greek word '*epi-demos*' meaning literally 'upon-people'. Epidemiologists frequently work with rates, proportions, and other quantitative measures of occurrence, prevalence, and causes of disease where the concepts and principles of elementary probability are readily applicable. This article describes some common problems in epidemiological studies where results of elementary probability are readily applied.

2. Measures of association of disease and risk factors: use of conditional probability

Establishing a relationship between various diseases and causes of death is important in epidemiology. The early development of epidemiology was concerned with infection rates, and problems connected with the spread of epidemics. The application of epidemiological methods resulted in discoveries of many important relationships, such as that between smoking and lung cancer, between air pollution levels and respiratory diseases, and between fluorides in water and the decline of dental caries among children.

To illustrate the use of epidemiological methods, suppose we wish to study the relationship between smoking and lung cancer. Consider table 1, which gives the distribution of lung cancer in a population of smokers and non-smokers.

Table 1. Distribution of lung-cancer cases in a population of smokers and non-smokers

	Smokers	Non-smokers	Total
With lung cancer	A	B	$A+B$
Without lung cancer	C	D	$C+D$
Total	$A+C$	$B+D$	$A+B+C+D$

An important quantity in epidemiological studies is *relative risk*. Note that the probability that an individual has lung cancer given that he or she is a smoker is

$$P_1 = P(\text{cancer}|\text{smoker}) = \frac{A}{A+C}.$$

Similarly, the probability that an individual has lung cancer given that he or she is a non-smoker is

$$P_2 = P(\text{cancer}|\text{non-smoker}) = \frac{B}{B+D}.$$

The relative risk (RR) of getting lung cancer for a smoker compared to a non-smoker is defined as the ratio of the two probabilities P_1 and P_2, i.e.,

$$RR = \frac{P(\text{cancer}|\text{smoker})}{P(\text{cancer}|\text{non-smoker})} = \frac{A/(A+C)}{B/(B+D)} = \frac{A(B+D)}{B(A+C)}.$$

Another important quantity commonly used to measure the association risk of a disease is the *odds ratio*. An odds ratio is based on the same concept as the odds of an event. Rather than saying that the probability of an event is 0.3, we may say that the odds for the occurrence of the event are 3 to 7. Note that the odds for getting lung cancer for a smoker are $A/(A+C)$ to $C/(A+C)$, or A to C. Similarly, the odds for getting lung cancer for a non-smoker are $B/(B+D)$ to $D/(B+D)$, or B to D. The odds ratio (OR) is defined as the ratio of the two odds, i.e.,

$$\begin{aligned} OR &= \frac{P(\text{cancer}|\text{smoker}) \div P(\text{no cancer}|\text{smoker})}{P(\text{cancer}|\text{non-smoker}) \div P(\text{no cancer}|\text{non-smoker})} \\ &= \frac{[A/(A+C)] \div [C/(A+C)]}{[B/(B+D)] \div [D/(B+D)]} = \frac{AD}{BC}. \end{aligned}$$

The odds ratio is also known as the *cross-product ratio* since it is the ratio of the products AD and BC of entries from diagonally opposite cells.

Numerical example. Consider table 2, which classifies the frequency of smoking and lung cancer in a population of 100 000 people.

Table 2. Distribution of lung cancer in a population of smokers and non-smokers

	Smokers	Non-smokers	Total
With lung cancer	120	10	130
Without lung cancer	19 880	79 990	99 870
Total	20 000	80 000	100 000

The relative risk of lung cancer for smokers compared to non-smokers is

$$RR = \frac{120/20\,000}{10/80\,000} = 48.0.$$

Thus a smoker has a risk of lung cancer 48 times greater than a non-smoker. The odds ratio of lung cancer for smokers compared to non-smokers is

$$OR = \frac{(120/20\,000) \div (19\,880/20\,000)}{(10/80\,000) \div (79\,990/80\,000)} = \frac{120 \times 79\,990}{10 \times 19\,880} = 48.3.$$

The risk ratio and odds ratio need not be in such close agreement. However, for rare ailments, when the probability that the exposed group will develop the disease is very small, the odds ratio closely approximates the risk ratio.

3. Revelance and incidence: probability as relative frequency

In epidemiology the terms *prevalence* and *incidence* are used to denote probabilities in a special context. The prevalence of a disease is the probability of currently having the disease regardless of the length of time one may have had it. Let D be the event that a randomly chosen individual in a certain population has the disease. Then, using the relative-frequency definition of probability, the prevalence is calculated as

$$P(D) = \frac{\text{number of individuals with the disease}}{\text{total number of individuals in the population}}.$$

The incidence of a disease is the probability of developing a new case of it during some specified time period among all individuals who did not have the disease at the beginning of the time interval. Let I be the event that a randomly chosen individual in a certain population develops a new case of the disease. Then, using the relative-frequency definition of probability, the incidence is calculated as

$$P(I) = \frac{\text{number of new cases of the disease}}{\begin{array}{c}\text{total number of individuals at the beginning of the period}\\ \text{who are free from disease}\end{array}}.$$

Numerical example. Suppose that in a community of 5000 persons, 10 cases of hepatitis were found in a given month. Then the prevalence of the hepatitis is $P(D) = 10/5000 = 0.002$. Thus the probability that a randomly chosen person in the population has hepatitis in any given month is 0.002.

Now, suppose that the number of new cases of hepatitis in the same community during that month was only 2, with 8 cases carried over from the previous month. Then the incidence of hepatitis is $P(I) = 2/4992 = 0.0004$. Thus the probability that a randomly chosen person develops a new case of hepatitis in any given month is 0.0004.

4. Screening tests: an application of Bayes' rule

In many epidemiological studies a common diagnostic procedure is to administer a screening test for the presence or absence of a disease. Unfortunately, many screening tests are not definitive. A *false negative* is defined as an individual who tests as negative but who is actually positive. A *false positive* is defined as an individual who tests as positive but who is actually negative. The *sensitivity* of a test is the probability that the test is positive given that the individual has the disease. The *specificity* of a test is the probability that the test is negative given that the individual does not have the disease.

Epidemiologists are often interested in measuring the *predictive accuracy* of a test. The *predictive accuracy positive* (PA^+) of a screening test is the probability that an individual has the disease given that the test is positive. Similarly, the *predictive accuracy negative* (PA^-) is the probability that the individual does not have the disease given that the test is negative. The higher the predictive accuracy, the more valuable is the test. Ideally, we should like a test such that both PA^+ and PA^- are 1. Then we should be able to diagnose the disease accurately for each patient.

Application of Bayes' rule. Unfortunately, the predictive accuracy of a screening test often cannot be determined directly. However, if we know the prevalence rate of disease in a population, we can use the sensitivity and specificity of the test—which are quantities that physicians can estimate—to compute the predictive accuracy, using a well-known result in probability, *Bayes' rule.*

To illustrate the computation, let us define the following events:

T: the screening test is positive;
D: the individual has the disease;
$\bar{T}$: the screening test is negative;
$\bar{D}$: the individual does not have the disease.

Now, by the earlier discussion,

$$\text{sensitivity} = P(T\,|\,D), \qquad PA^+ = P(D\,|\,T),$$

$$\text{specificity} = P(\bar{T}\,|\,\bar{D}), \qquad PA^- = P(\bar{D}\,|\,\bar{T}).$$

Let $P(D)$ denote the prevalence rate of the disease in the general population, i.e. the probability that a randomly chosen individual in the population has the disease. Now the predictive accuracy of the test can be determined from Bayes' formula:

$$P(D\,|\,T) = \frac{P(T\,|\,D)\,P(D)}{P(T\,|\,D)\,P(D) + P(T\,|\,\bar{D})\,P(\bar{D})},$$

i.e.,

$$PA^{+} = \frac{\text{sensitivity} \times \text{prevalence}}{\text{sensitivity} \times \text{prevalence} + (1 - \text{specificity})(1 - \text{prevalence})} \qquad (1)$$

and

$$P(\bar{D}\,|\,\bar{T}) = \frac{P(\bar{T}\,|\,\bar{D})\,P(\bar{D})}{P(\bar{T}\,|\,\bar{D})\,P(\bar{D}) + P(\bar{T}\,|\,D)\,P(D)},$$

i.e.,

$$PA^{-} = \frac{\text{specificity} \times (1 - \text{prevalence})}{\text{specificity} \times (1 - \text{prevalence}) + (1 - \text{sensitivity}) \times \text{prevalence}}. \qquad (2)$$

Numerical example. Suppose that 85 per cent of patients with hypertension (high blood pressure) and 20 per cent with normotension (normal blood pressure) are diagnosed as hypertensive by an automated blood-pressure machine. To calculate the predictive accuracy of the machine, assume that 25 per cent of the adult population is suffering from hypertension.

Now we have

$$\text{sensitivity} = 0.85, \quad \text{specificity} = 1 - 0.20 = 0.80, \quad \text{prevalence} = 0.25.$$

Thus, by the use of the formulas (1) and (2),

$$PA^{+} = \frac{0.85 \times 0.25}{0.85 \times 0.25 + (1 - 0.80)(1 - 0.25)} = 0.59$$

and

$$PA^{-} = \frac{0.80 \times (1 - 0.25)}{0.80(1 - 0.25) + (1 - 0.85) \times 0.25} = 0.94.$$

Hence a negative result from the machine is very predictive since we are 94 per cent sure that such a person is normotensive. However, a positive result is not very predictive since we are only 59 per cent sure that such a person is hypertensive.

5. Conclusion

The purpose of this article has been to illustrate some simple applications of elementary probability to epidemiology, a branch of medical science. Medicine is an inexact science, which is described as being 'probabilistic' as opposed to 'deterministic'. Probability is widely used in clinical diagnosis, where probabilistic statements are the terms in which a competent prognosis is given. Many interesting and useful applications of probability to medical science have found widespread usage in medical practice.

The 'Inverse' Differential Equation

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The author wrote this article whilst he was in the sixth form.

Last year I stumbled across a difficult, yet simple, problem whilst I was doing my homework. I accidentally dropped the minus sign from the symbol for the inverse of a function, and so ended up calculating a derivative instead of the inverse. This made me wonder about the possibility of making the same mistake and obtaining the correct solution. In other words, are there any solutions to

$$f'(x) = f^{-1}(x)\,?$$

At first, I thought such a function did not exist, but using a simple algebraic approach, I discovered one solution. Suppose

$$f(x) = Ax^n \quad \text{and} \quad f'(x) = f^{-1}(x). \tag{1}$$

Then

$$f'(x) = Anx^{n-1} \quad \text{and} \quad f^{-1}(x) = \left(\frac{x}{A}\right)^{1/n}.$$

Equating the two, we have

$$\left(\frac{x}{A}\right)^{1/n} = Anx^{n-1}.$$

Therefore,

$$1 = A^{n+1}n^n x^{n(n-1)-1}.$$

Comparing coefficients, we have

$$0 = n^2 - n - 1 \tag{2}$$

and

$$1 = A^{n+1}n^n. \tag{3}$$

By (2),

$$n = \frac{1 \pm \sqrt{5}}{2}$$

and, by (3),

$$A = n^{-n/(n+1)}.$$

It seems from this analysis that two solutions have been found, but on closer inspection it can be seen that if we take the root of (2) with the minus sign, then $n < 0$, $n+1 > 0$ and

$$A = n^{-n/(n+1)} = (-\text{ve number})^{(+\text{ve irrational number})}$$

i.e. A is not a real number, and so this is not a solution in functions of a real variable. If we take the positive solution, such a situation does not arise and thus this solution is valid.

Thus one solution to equation (1) is

$$\left(\frac{\sqrt{5}+1}{2}\right)^{-(\sqrt{5}-1)/2} x^{(\sqrt{5}+1)/2} \quad (\text{defined for } x \geqslant 0).$$

I do not know whether this is the only solution. I am currently working on this problem, and would like to know if any other readers of *Mathematical Spectrum* have any success with it.

Another problem which the simple algebraic method allows me to solve is to find one solution (in some cases two) to the equation

$$f^{-1}(x) = f^{(m)}(x).$$

Let us suppose that $f(x) = Ax^n$. Then

$$f^{-1}(x) = \left(\frac{x}{A}\right)^{1/n}$$

and

$$f^{(m)}(x) = An(n-1)\dots(n-m+1)x^{n-m}.$$

Equating, we have

$$\left(\frac{x}{A}\right)^{1/n} = An(n-1)\dots(n-m+1)x^{n-m}.$$

Therefore,

$$1 = A^{n+1}n^n(n-1)^n\dots(n-m+1)^n x^{n(n-m)-1}.$$

Comparing coefficients, we have

$$0 = n^2 - nm - 1, \tag{4}$$

$$1 = A^{n+1}n^n(n-1)^n\dots(n-m+1)^n. \tag{5}$$

By (4),

$$n = \frac{m \pm \sqrt{m^2+4}}{2}$$

and, by (5),

$$A = [n(n-1)\dots(n-m+1)]^{-n/(n+1)}.$$

If we consider the case when m is odd, and take the 'negative root' as a solution for n, then

$$A = [n(n-1)\dots(n-m+1)]^{(\text{+ve irrational number})},$$

yet $n(n-1)\dots(n-m+1)$ is negative since there is an odd number of terms in this product. Therefore, this solution again defines A to be a complex number and the value of n with the minus sign is an invalid solution. No such situation occurs when the positive sign is taken in the value of n. Therefore, when m is odd only one solution is found by this method:

$$f(x) = [n(n-1)\dots(n-m+1)]^{-n/(n+1)}x^n, \quad \text{where } n = \tfrac{1}{2}(m+\sqrt{m^2+4}).$$

If we consider the case when m is even, then the product $n(n-1)\dots(n-m+1)$ is positive for positive and negative n. Therefore, when m is even two solutions are found by this method:

$$f(x) = [n(n-1)\dots(n-m+1)]^{-n/(n+1)}x^n, \quad \text{where } n = \tfrac{1}{2}(m\pm\sqrt{m^2+4}).$$

Note that, when $m = 0$, the solutions obtained for $f(x)$ are $f(x) = \pm x$ and $f(x) = A/n$. These are the most obvious examples of functions that are equal to their inverses. The general solution is

$$F(x,y) = 0, \quad \text{where } F(x,y) = F(y,x).$$

Even though this solution is not an explicit $f(x)$, it is quite easy to show that the above function satisfies the equation. The inverse of a function can be thought of as the reflection of the function in the line $y = x$. When a function is reflected in that line, the result is that x maps to y and y maps to x. Therefore, the inverse of the implicit function is

$$F(y,x) = 0 = F(x,y)$$

which is the required result.

This idea of looking at the inverse as a reflection in the line $y = x$ opens a new line of attack on the differential equations. Certain relations between $f'(x)$ and $[f^{-1}(x)]'$ may allow possible solutions to be deduced.

So, it has been shown that the original equation does, at least, have one solution, but are there others? The fact that $m = 0$ has a solution may lead us to assume that for all m a solution exists.

Just as an aside, is it not strange that, for the solution to the original equation, the value of n was the golden ratio $\frac{1}{2}(1+\sqrt{5})$? This is the number to which the ratio of successive terms in the Fibonacci sequence tends. Perhaps there is a fundamental reason as to why the golden ratio should appear here.

There is another allied equation to which the algebraic method can be applied:

$$f^{-1}(x) = [f^{(m)}(x)]^p.$$

Assuming that

$$f(x) = Ax^n,$$

we have

$$n = \frac{mp \pm \sqrt{p^2m^2+4p}}{2p}$$

and

$$A = [n(n-1)\dots(n-m+1)]^{-np/(np+1)}.$$

Again, whether or not both solutions for n are valid depends upon whether the values of A are real. This, of course, depends on the values of n and p. Are there other solutions to this equation?

This article arose from a simple mistake, and another simple mistake that can arise when we first encounter the idea of inverse functions is the misinterpretation of $f^{-1}(x)$ to be $1/f(x)$. Two new equations now arise due to this:

$$\frac{1}{f(x)} = f'(x) \tag{6}$$

and

$$\frac{1}{f(x)} = f^{-1}(x). \tag{7}$$

If we take (6) first, then

$$1 = f(x)f'(x).$$

Integrating with respect to x, we have

$$[f(x)]^2 = 2x+c,$$

where c is a constant. Since $[f(x)]^2 \geqslant 0$ for all x, then $2x+c \geqslant 0$. Therefore, $x \geqslant -\frac{1}{2}c$ and so, for $x \geqslant -\frac{1}{2}c$,

$$f(x) = \pm\sqrt{2x+c}.$$

Looking at (7), we can use the algebraic method again. Let us suppose $f(x) = Ax^n$ and $1/f(x) = f^{-1}(x)$. Then

$$\frac{1}{Ax^n} = \left(\frac{x}{A}\right)^{1/n}$$

and so

$$1 = \frac{x}{A} A^n x^{n^2}.$$

Re-arranging, we have

$$1 = A^{n-1} x^{n^2+1}.$$

Comparing coefficients, we have

$$0 = n^2+1 \tag{8}$$

and

$$1 = A^{n-1} \tag{9}$$

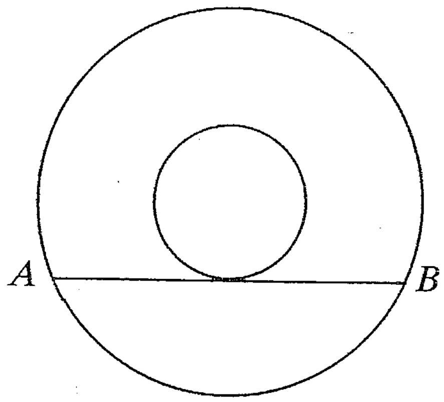
From (8), $n = \pm i$. As a consequence, we cannot find a real solution to the equation by this method. Are there any real solutions?

Maybe other readers will let us know about more problems that arise due to misreading or misunderstanding the terminology, or maybe someone could solve

$$f^{-1}(x) = f(x) + f'(x)$$

or

$$f^{-1}(x) = xf'(x) + f''(x).$$



If $AB = 20\,\text{cm}$, what is the area of the region between the concentric circles shown?

III. Infinity and Geometry

JOSEPH ROSENBLATT, *Ohio State University*

This article follows on from two previous ones on infinity by Professor Rosenblatt, in Volume 23 Numbers 2 and 3.

> The ideas that are basic to [my prints] often bear witness to my amazement and wonder at the laws of nature which operate in the world around us. He who wonders discovers that this is in itself a wonder. By keenly confronting the enigmas that surround us, and by considering and analyzing the observations that I had made, I ended up in the domain of mathematics. Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists.
>
> From the introduction to *The Graphic Work of M. C. Escher* by M. C. Escher.

Let us look at an example of a special type of set discovered by Cantor. One description of this set is obtained by describing what to remove from $[0,1]$ to leave behind the set in question. First, one removes the middle third $(\frac{1}{3},\frac{2}{3})$. Then one removes the two middle-thirds of the remaining two closed intervals: $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. So we remove the intervals $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$. Now there remain four disjoint closed intervals $[0,\frac{1}{9}]$, $[\frac{2}{9},\frac{1}{3}]$, $[\frac{2}{3},\frac{7}{9}]$, and $[\frac{8}{9},1]$. We now remove the middle thirds of these four intervals, leaving behind eight pairwise-disjoint closed intervals. And we continue this process indefinitely. The set that is left behind is called Cantor's middle-thirds set. Let us denote it by C. C is not empty since it contains at least the points $0,\frac{1}{3},\frac{2}{3},1$, for example. But what else is in this set? 'Not much,' I hear you say! You could say, 'After all, the first time we remove an interval of length $\frac{1}{3}$. Then we remove two intervals of length $\frac{1}{9}$, then 4 intervals of length $\frac{1}{27}$, etc.. So in total we remove intervals whose total length is given by the series

$$\tfrac{1}{3}+2(\tfrac{1}{9})+4(\tfrac{1}{27})+\ldots = \tfrac{1}{2}(\tfrac{2}{3}+\tfrac{4}{9}+\tfrac{8}{27}+\ldots) = \tfrac{1}{2}(a+a^2+a^3+\ldots),$$

where $a=\frac{2}{3}$. By the formula for the geometric series, this sum is

$$\frac{1}{2}\left(\frac{1}{1-a}-1\right) = \tfrac{1}{2}(3-1) = 1\,!$$

So we have removed almost everything in constructing this set.' But nonetheless, the Cantor set is not countable. Indeed, it has the cardinality of the continuum. The easiest way to see this is to show that C consists of all those real numbers in $[0,1]$ which can be expanded in a base-three decimal expansion that does not contain the digit 1; that is, we can write any number x $(0 \leqslant x \leqslant 1)$ as a series

$$\tfrac{1}{3}a_1+\tfrac{1}{9}a_2+\tfrac{1}{27}a_3+\ldots$$

where $a_1, a_2, a_3, \ldots$ are the digits, 0, 1 and 2. The Cantor set consists of all numbers having some such expansion in which none of the a_i is equal to 1. It is now easy to see that the Cantor set is uncountable by the same argument that showed that the real numbers are not countable (see reference 2). On the other hand, this set, being a set of real numbers, cannot have cardinality larger than the continuum. So C must have cardinality exactly that of the continuum. Thus, this Cantor set offers the paradoxical behaviour that it is large in the sense of cardinality, but small in the sense of length.

The French mathematician Henri Lebesgue, the father of the modern theory of the integral, used Cantor's example to construct an interesting continuous function. There are several ways to define this function, now called the Cantor–Lebesgue function. The easiest is first to define this function f on the intervals which were removed from $[0,1]$ in forming the Cantor set. We make f constant on these intervals, but different constants for different intervals. Let $f(x)$ be $\frac{1}{2}$ for x in the first middle third. Then define $f(x) = \frac{1}{4}$ in the first of the next pair of middle thirds, i.e. for x in $(\frac{1}{9},\frac{2}{9})$, and $f(x) = \frac{3}{4}$ for x in $(\frac{7}{9},\frac{8}{9})$. One continues this pattern of definition to define $f(x)$ for all x in $[0,1]\backslash C$. Then, because these intervals are dense in $[0,1]$, there is a unique extension of f to all of $[0,1]$ which makes f a continuous function. The graph of this function is illustrated in figure 1.

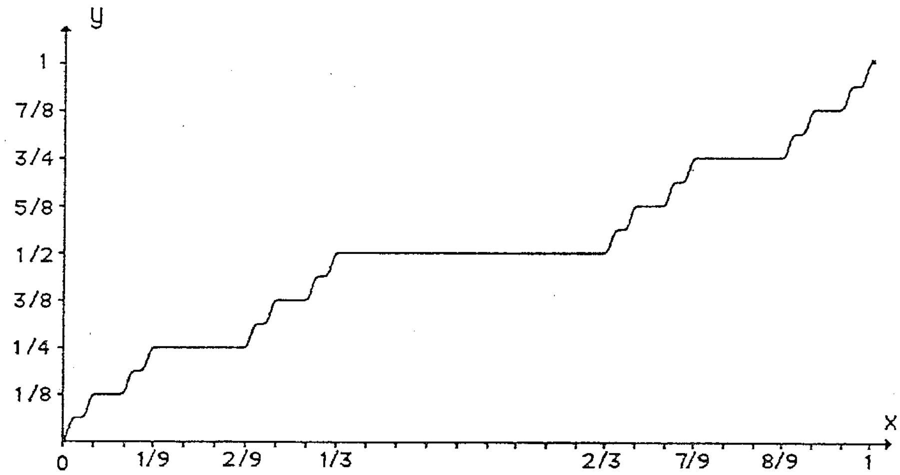


Figure 1. Cantor–Lebesgue function $y = f(x)$

This function has the curious property that it rises continuously from 0 at $x = 0$ to 1 at $x = 1$, and yet its derivative is 0 (the graph is horizontal) everywhere on $[0,1]\backslash C$, which was a set whose total length was 1. Think of this graph as the picture of distance y versus time x for an interstellar rocket ship. The rocket appears never actually to move through space in an appreciable time interval in any visible, continuous fashion. For a while it will be stationary, then suddenly (in less than the blink of the eye), it will be farther along. It is as if, during times x in the Cantor set, the rocket has entered some type of hyperspace in order to move. But yet the motion is continuous. Of course, the point is that the graph is not differentiable everywhere. If some $y = F(x)$ did have a derivative which was zero everywhere (the graph has a horizontal tangent *everywhere*) and $F(0) = 0$, then $F(1) = 0$ too. The points in the Cantor set are what gives the rocket time to move, even though in total these times comprise no time at all. Indeed, if you imagine the Cantor–Lebesgue function as the graph of distance versus time for the motion of a tortoise, you can also get a modern version of Zeno's paradox. Almost all the time the tortoise would not be moving at all, and yet could travel 100 miles in one hour, while Achilles, who was running as quickly as he possibly could, would be left behind!

Another geometrical paradox which is a favourite of mine is called the *painter's paradox*. This paradox concerns a large tube (like an old-fashioned hearing aid) which, while it has finite volume, also has infinite surface area. Specifically, consider the graph of $y = 1/x$ restricted to those x values which are greater than or equal to 1. If we rotate this curve about the x-axis, it sweeps out a surface of an infinitely long tube. See figure 2.

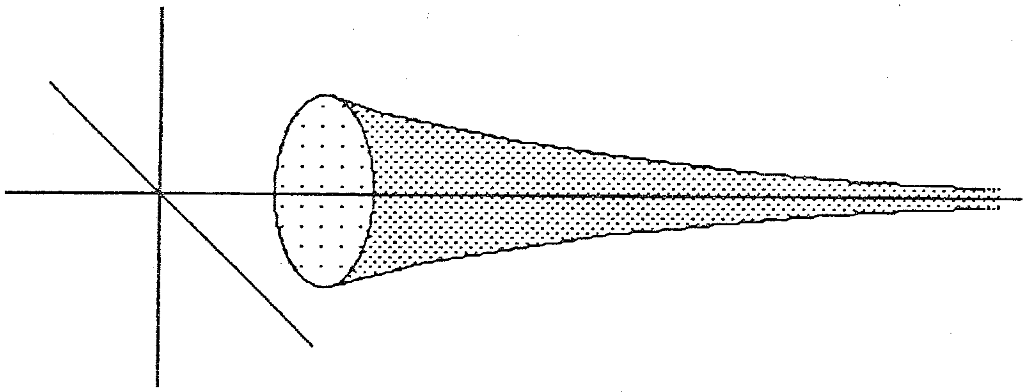


Figure 2. The painter's paradox

This tube contains only a finite amount of volume. However, the surface area is infinite. The painter then finds that if he paints the inside surface of the tube, it will take for ever to finish. However, pouring a finite amount of paint into the tube will coat the interior surface. Is this reasonable? One can perhaps say that this is a mathematical proof of the existence of atoms. The only way to resolve this paradox is if real paint would clog up the end of the tube as it narrows down smaller than the size of the paint molecules. Most people would argue that such a graph does not really exist since it goes on into infinity in space, so there is no paradox at all. However, we can achieve the same result, finite volume together with infinite surface area, in a geometrical shape that is contained in a bounded volume. It would look something like a flower vase made by someone with a great love of curves. For an explicit example, we can take the volume of rotation of the graph $y = \sqrt{x}\sin(1/x) + x$ $(0 \leqslant x \leqslant 1)$ rotated about the x-axis. See figure 3 for an illustration of this solid. The painter's paradox arises because we allow ourselves to carry out geometrical constructions either to the infinitely large or to the infinitely small. Therein lies the secret of these odd creations.

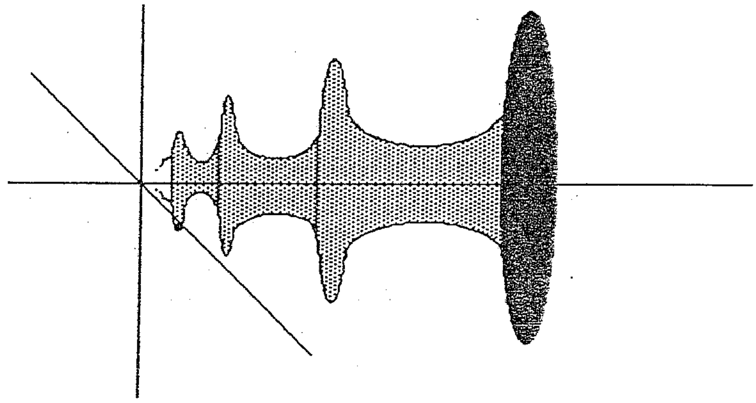


Figure 3. Another painter's paradox

Being able to carry out geometrical constructions to the infinitesimal allowed Giuseppe Peano in the late nineteenth century to build strange space-filling curves. He constructed a continuous curve $P(t) = (x(t), y(t))$ which depends on t as the independent variable such that, as t varies from 0 to 1, the point P passes through every point in the unit square U in the plane; that is, there is a continuous function $f\colon [0, 1] \to U$ which is onto! These curves were considered so remarkable that they are now often called Peano curves in honour of their inventor. See reference 1 for some illustrations. It is not easy to construct such functions by explicit simple formulae. More often they are constructed as limits of simple procedures. Such curves are generally non-differentiable and pass through the same point many times. It is interesting to observe that these curves seem to defy the idea of volume and dimension, just as Cantor thought he had done when he showed that the plane and the line have the same cardinality (see reference 2). What is not the case is that such curves can be 1–1 also. This would defy the basic principle of dimension that there are no 1–1 continuous onto mappings from the unit interval to the unit square.

But there are even more bizarre ways in which mathematicians have found that volume in space does not seem to be conserved. The mathematicians Stephen Banach and Alfred Tarski early in this century used some work of Felix Hausdorff on the existence of special rotations in three dimensions to prove an amazing result about the apparent non-preservation of volume in three dimensions. This paradox has been called the *Banach–Tarski paradox*. Simply put, they showed that one can divide the solid ball of radius 1 into nine parts, and then take four of the parts and assemble them by rotating and translating the pieces in space to make another solid ball of radius 1, while at the same time the other five pieces can be assembled again by rotations and translations in space to make a second solid ball of radius 1. Thus, from one object, two objects of the same volume are created in a manner that should never have increased volume in the first place! This strange event was almost without precedent in mathematics. It showed that there really is no reasonable way that we can assign the notion of volume to arbitrary subsets of space and have this notion of volume preserved under motion that does not change relative distances (as rotations and translations do not).

It is interesting to mathematicians especially that the Banach–Tarski paradox requires a fundamental use of the Axiom of Choice. Without this axiom, this paradox does not happen. What is the Axiom of Choice? It says that given any set of non-empty sets, there is another set which contains exactly one element from each of the many sets. That is, if you have a collection of (possibly infinitely many) boxes containing marbles (possibly infinitely many marbles in each box), then you can make a new box with one marble from each of the original boxes. This axiom may seem intuitive to you, but, as Bertrand Russell pointed out, it isn't really anything but an axiom to be assumed or to be discarded. Can you imagine an infinite number of pairs of shoes? It would be easy to choose one shoe from each pair. Just choose the left shoe. But how do we specifically describe choosing one sock from each of infinitely many pairs of socks, socks not really being distinguishable left from right? This is harder and shows that some axiom of sock choice is needed.

One doesn't begin to understand the importance of the Axiom of Choice until one sees some of its uses. The Axiom of Choice is needed to prove that, given any two infinite sets A and B, the cardinality of A is less than or equal to the cardinality of B, or vice versa. Also, the Axiom of Choice is needed to prove that a countable union of countable sets is countable. Furthermore, this axiom is used to show that every vector space has a basis. These different applications of this one axiom show how important it can be. The fact that the Axiom of Choice also leads to strange theorems, like the Banach–Tarski paradox, needs to be seen in context with the axiom's other uses. In any case, with or without the Axiom of Choice, there are many odd and wonderful facts in geometry that we can discover when we allow infinity into our mathematics.

References

1. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (W. H. Freeman, San Francisco, 1977).
2. J. Rosenblatt, Infinity and enumeration, *Mathematical Spectrum* **23**, 44–54.

Computer Column

MIKE PIFF

Graphics and Modula-2

Pascal was ill equipped to handle graphics. Any interface had to be as an extension to the compiler, and hence not portable. Modula-2 solves this problem by allowing extensions by means of separate modules. We shall look at this method by examining the extension mentioned in the last issue. One type of module is a DEFINITION module, and this describes what is available. An example is

```
DEFINITION MODULE Graphics;
PROCEDURE BeginGraph;
PROCEDURE EndGraph;
PROCEDURE PutPixel(column, row, colour:CARDINAL);
END Graphics.
```

This is completely portable between computers and compilers.

The other module is an IMPLEMENTATION module, and will contain the specific code needed to accomplish the extension described in the definition module. This will depend on the particular computer and monitor being used. For instance, on a computer which is permanently in graphics mode, such as an Atari ST, *BeginGraph* and *EndGraph* will merely clear the screen. *PutPixel* would implement a *Line A* trap, either using *inline* code or by means of another provided module.

Here is an example of how the graphics module could be implemented on a CGA or VGA monitor on a PC compatible. The MODULE *System* is assumed to make available the low-level facilities needed to implement these features; otherwise, see if there is an *inline* facility, or a module called *DOS* or *MSDoS*, say.

```
IMPLEMENTATION MODULE Graphics;
FROM System IMPORT Trap, AX, BX, CX,
DX;
FROM InOut IMPORT WriteString,
  WriteLn, WriteCard;
CONST
  TextMode=7; GraphMode=6;(*640 by 200*)
  (* GraphMode=17 or 18 for VGA 640 by 480 *)
  pagefn=5; plotfn=12; GraphInt=16; high=256;
  Page=0;
PROCEDURE SetMode(val:CARDINAL);
BEGIN
  AX:=val; Trap(GraphInt);
END SetMode;
PROCEDURE SetPage(p:CARDINAL);
BEGIN
  AX:=pagefn×high+p; Trap(GraphInt);
END SetPage;
PROCEDURE BeginGraph;
BEGIN
  SetPage(Page); SetMode(GraphMode);
END BeginGraph;
PROCEDURE PutPixel(column, row,
  colour:CARDINAL);
BEGIN
  CX:=column; DX:=row;
  AX:=plotfn×high+colour;
  BX:=high×Page; Trap(GraphInt);
END PutPixel;
PROCEDURE EndGraph;
BEGIN
  SetPage(Page); SetMode(TextMode);
END EndGraph;
END Graphics.
```

Letter to the Editor

Dear Editor

Powerless arithmetic progressions

The question at the end of Mr Prakash's article in Volume 23 Number 3, whether there is an arithmetic progression which contains no powers, is rather easily answered.

The only powers which can end ...5 are those whose bases end ...5. But if a number ends ...5, then its successive powers *either* all end ...25 *or* end alternately ...25 and ...75. So an arithmetic progression such as

$$15, 115, 215, 315, \ldots$$

contains no powers.

Yours sincerely,
DOUGLAS QUADLING
(12 Archway Court,
Barton Road,
Cambridge CB3 9LW)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

23.10. (Submitted by A. W. F. Edwards, Gonville and Caius College, Cambridge—see the article in this issue)

(a) Let $n = p^\lambda$, where p is a prime number and λ is a positive integer. Let r be an integer such that $1 \leqslant r \leqslant n-1$. Show that p divides the binomial coefficient ${}^nC_r$.

(b) Let r and n be integers such that $2 \leqslant r \leqslant n-2$. Show that ${}^nC_r$ is composite.

23.11. (Submitted by Clifford Pickover, IBM Thomas J. Watson Research Center—see the article in this issue)
Prove that no $u_{r,n}$ exceeds 3 and that 333 can never occur.

23.12. (Submitted by M. Movahhedian, Isfahan University of Technology) The point A_1 has coordinates $(3, 2, 1)$. Find points A_2 and A_3 on the planes $y = x$ and $y = 0$, respectively, such that the circumference of triangle $A_1A_2A_3$ is a minimum.

Solutions to Problems in Volume 22 Number 3

23.4. Find all positive integers x such that the decimal representation of $x+2$, when written backwards, equals $\frac{1}{2}x$.

Solution

We shall sketch a solution. Write $x = a_n a_{n-1} \cdots a_0$ in decimal representation, where $a_0, \ldots, a_n$ are integers between 0 and 9 with $a_n > 0$. Then, in decimal representation,

$$x+2 = \begin{cases} a_n a_{n-1} \ldots a_2 a_1 (a_0+2) & \text{if } a_0 \leqslant 7, \\ a_n a_{n-1} \ldots a_2 (a_1+1)(a_0-8) & \text{if } a_0 > 7,\ a_1 < 9, \\ a_n a_{n-1} \ldots (a_2+1) 0 (a_0-8) & \text{if } a_0 > 7,\ a_1 = 9,\ a_2 < 9, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (a_n+1) 0 \ldots 00 (a_0-8) & \text{if } a_0 > 7,\ a_1 = \ldots = a_{n-1} = 9,\ a_n < 9, \\ 100 \ldots 00 (a_0-8) & \text{if } a_0 > 7,\ a_1 = \ldots = a_n = 9. \end{cases}$$

We require

$$\tfrac{1}{2}(a_n a_{n-1} \cdots a_0)$$

$$= \begin{cases} (a_0+2) a_1 a_2 \ldots a_{n-1} a_n & \text{if } a_0 \leqslant 7, \\ (a_0-8)(a_1+1) a_2 \ldots a_{n-1} a_n & \text{if } a_0 > 7,\ a_1 < 9, \\ (a_0-8) 0 (a_2+1) \ldots a_{n-1} a_n & \text{if } a_0 > 7,\ a_1 = 9,\ a_2 < 9, \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (a_0-8) 00 \ldots 0 (a_n+1) & \text{if } a_0 > 7,\ a_1 = \ldots = a_{n-1} = 9,\ a_n < 9, \\ (a_0-8) 000 \ldots 01 & \text{if } a_0 > 7,\ a_1 = \ldots = a_n = 9. \end{cases}$$

In the last case, a_0 has to be 8, whence $\frac{1}{2}(a_n a_{n-1} \cdots a_0) > 1$; so this cannot occur. Consider all other cases. The next step is to show that a_1 has to be odd, for the assumption that a_1 is even leads to impossibilities in both cases a_n is even and a_n is odd. Next it is seen that a_n has to be odd. Thus we have

$$\tfrac{1}{2}(a_0+10) = a_n \text{ or } a_n+1 \tag{1}$$

from the units digit and

$$\tfrac{1}{2}(a_n-1) = a_n+2 \text{ or } a_0-8 \tag{2}$$

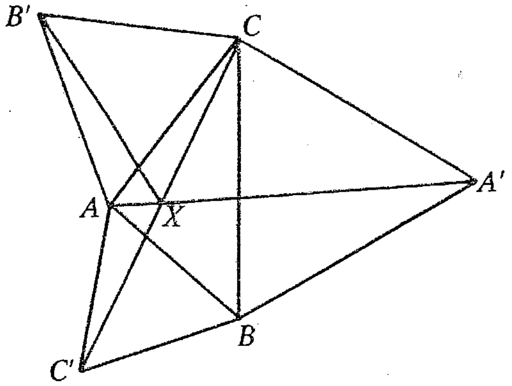
from the 10^n digit; and cases a_n+1 in (1) and a_0+2 in (2) do not occur together. Only cases a_n in (1) and a_0+2 in (2) give possible values, and we deduce from these that $a_n = 5$, $a_0 = 0$. We now have

$$\tfrac{1}{2}(5a_{n-1} \cdots a_1 0) = 2a_1 \ldots a_{n-1} 5$$

in decimal representation. We can eliminate all cases except a_2 odd, a_{n-1} odd and obtain that $a_1 = a_{n-1} = 9$. Exactly the same argument gives that $a_2 = a_{n-2} = 9$. It is now easy to see that we end up with the numbers 50, 590, 5990, ... as the only possibilities. Since these numbers do have the required property, the solution is complete.

23.5. ABC is a triangle and ABC', BCA' and CAB' are equilateral triangles drawn on the sides AB, BC and CA, respectively, of ABC, exterior to ABC. Prove that AA', BB' and CC' are concurrent and of equal length.

Solution



Rotate $\triangle BC'C$ through 60° clockwise about B to map onto $\triangle BAA'$. Thus AA' and CC' have the same length and are at an angle of 60° to each other. Similarly for BB' and AA', showing that AA', BB' and CC' have equal length.

Let X be the point of intersection of AA' and CC', and join B' to X. Now $\angle AXC = 120°$ and $\angle AB'C = 60°$ and so $AB'CX$ is a cyclic quadrilateral. Thus $\angle B'XC = \angle B'AC = 60°$ (angles in the same segment). But $B'B$ and $C'C$ are at 60° to each other, so that X must lie on $B'B$, i.e. AA', BB' and CC' are concurrent. (The proof needs a slight amendment if $\angle BAC > 120°$.)

23.6. Does there exist a set S of points in three-dimensional space which has a finite non-empty intersection with every plane?

Solution

Let S be the set of points (x, x^3, x^5), where $x \in \mathbb{R}$, and consider an arbitrary plane $ax+by+cz+d = 0$, where a, b and c are not all zero. Then (x, x^3, x^5) lies in this plane if and only if

$$ax+bx^3+cx^5+d = 0$$

and this equation has at least one solution and at most five solutions.

All the fives

Can you make the following equation correct by inserting one straight line segment?

$$5+5+5+5=555$$

Reviews

Makers of Mathematics. By STUART HOLLINGDALE. Penguin, London, 1989. Pp. xiv + 433. £7.99 (ISBN 0-14-022732-6).

When your teacher introduces you to Blogg's theorem, what is your immediate reaction? Oh no, not another theorem! Or, who was Bloggs? When did he live and where? How was he led to his theorem, and what led from it? It is hardly an exaggeration to say that all mathematics should be learned in a historical context.

A generation ago most students of mathematics were brought up on E. T. Bell's *Men of Mathematics*, and enjoyed his pithy comments and stories about the lives of the mathematically famous, even if they did not always believe what they read. The present volume is more sober, more scholarly, and less comprehensive than Bell, but it should be on the bookshelf of every mathematics student.

The giants of mathematics come over well in this volume. Archimedes provides a landmark in the early part of the journey, and a quarter of the book is devoted to Newton and his immediate circle. But it is in describing the achievements of Einstein that the author's prose comes near to poetry. The author stops short of recent discoveries in such areas as chaos and fractals, but it is perhaps too soon to assess the contribution of individual mathematicians in these fields. All in all, a worthy successor to Bell.

University of Sheffield D. W. SHARPE

Guide to Mathematical Modelling. By DILWYN EDWARDS and MIKE HAMSON. Macmillan Education, Basingstoke, 1989. Pp. x + 277. Paperback £9.95 (ISBN 0-333-45935-0).

'Mathematical modelling' write the authors, 'can be learnt only by doing.' With this in mind, the reader is encouraged to have a go at each of the many examples and exercises in this book, and preferably discuss them with some friends. When looked at cynically, the book could be seen as a collection of isolated techniques from algebra, statistics and calculus, but if you try the questions and read the worked examples in detail, the modelling methodology set up in Chapter 3 becomes increasingly helpful and the authors' approach appears increasingly relevant.

After an introduction about situations in which mathematical modelling is important, a string of diverse and interesting examples is produced. These range from the purely arithmetical 'Home-decorating' one to the one about the evacuation of a building in an emergency, a situation which I thought, at first, would be impossible to model in any way at all. In the next chapter, the book draws a few common themes together, a methodology is discussed, and several terms from modelling are defined. The authors swiftly follow this up with chapters on dimensional analysis, choosing functions to match general patterns of growth, and incorporating data. Chapters 7 and 8 form the 'backbone of the text' and are about using random numbers and differential equations. Later, the authors set up some guidelines for writing reports.

This book is one of a series intended to ease the transition from school mathematics to mathematics in higher education. It is therefore aimed mainly at first-year undergraduates. However, there are few prerequisites and so a sixth-former could probably cope.

Trinity College, Cambridge AMITES SARKAR

Computer-Aided Engineering Mathematics. By M. STEWART TOWNEND AND DAVID C. POUNTNEY. Ellis Horwood, Chichester, 1989. Pp. 160. £14.95 (ISBN 0-7458-0794-1).

The steps involved in solving a problem may include realising that there is a problem, becoming acquainted with what the problem is about, formulating the problem, constructing a mathematical model, checking that the model does its job, analysing the model, interpreting the results of this analysis, using these results, and, finally, observing what happens. The 20 case studies in this book mostly begin with the construction of the model but focus on its analysis. The widely used NAG library of routines is assumed to be available, and listings of programs in FORTRAN 77 which call these routines are given at the end of each chapter. A reader familiar with almost any computer language should be able to understand these programs.

The chapters are arranged by mathematical topic (ordinary but not partial differential equations, numerical integration, equations, systems of linear equations, eigenvectors, curve fitting and optimization) with enough detail to remind the reader about Chebyshev polynomials, Laplace transforms or whatever the technique may be. The authors continually raise the issues which affect the choice of method and the accuracy and reliability of results. There are exercises.

I enjoyed this book. It can be recommended to engineering undergraduates, and also to those responsible for leaving matrices out of the 'common core' for A-level mathematics.

The Royal School, Wolverhampton JOHN MACNEILL

Figuring. By SHAKUNTALA DEVI. Penguin, London, 1990. Pp. 157. £3.99.

This is an interesting book for those who have a limited experience of number patterns, and as a first reader in the subject it should prove valuable. The definitions at the beginning are useful, and the first chapter, exploring the digits and searching for individuality in the numbers, sets the right pace for the rest of the book.

If you think that just reading this book will unlock the mysteries of manipulating numbers quickly and accurately, you will be disappointed. There is, as many of us know, no magic—black or white—in arithmetic. It is hard work, a good memory and an experience of recognizing number patterns quickly that enable us to improve our mental agility.

As I read this book I was constantly thinking how it would be useful in the classroom. I do not think children would readily pick this off the shelf, because they tend to go for a more exciting presentation; it is surprising what colour and a cartoon can do! As a book to read and explore when looking for ideas to help extend a GCSE or A-level investigation, it would undoubtedly be helpful. It is not a book to read from cover to cover, nor for younger children to study without help. Many of the tasks it explains are not new—in terms of the deluge of recreational maths, puzzle and games books now flooding into schools, this represents only a small shower.

I was perturbed to read on page 20 that the number 1 is of course a prime number! Such a basic mistake made me wary of other 'facts' presented in the book. I was also intrigued, and I am sure my students would be too, to know where the 'deceptively straightforward' number 526 315 789 473 684 210 came from in the first place. What method (or magic?) enabled Shakuntala Devi to discover this number and other special numbers like it?

I can recommend this book for the maths shelf. It may encourage readers to have more confidence in their mental abilities and not resort to calculators to perform even simple arithmetic. Most of us, though, will continue to wonder at the spectacular ability of Shakuntala Devi and enjoy watching her television appearances whilst still relying on paper, pencil and calculator.

Honywood School, Coggeshall D. T. KUYPER

A Dictionary of Real Numbers. By JONATHAN BORWEIN AND PETER BORWEIN. Wadsworth and Brooks/Cole, Pacific Grove, California, 1990. Pp. viii + 424. (ISBN 0-534-12840-8).

This book is designed for research scientists who encounter numbers computationally and wish to discover if they have some simple form. It lists more than 100 000 eight-digit numbers in the interval between 0 and 1 that arise as the first eight digits of special values of familiar functions. Using this book you can recognize, for example, that the number 0.933 716 63 may be $2\log_{10}\frac{1}{2}(e+\pi)$. On the other hand, it may not!

University of Sheffield R. J. WEBSTER

The Mathematics of Games. By JOHN D. BEASLEY. Oxford University Press, 1989. Pp. viii+169. Hardback £14.95 (ISBN 0-19-853206-7), paperback £5.96 (ISBN 019-286-107-7).

'If the playing of games is a natural instinct of all humans, the analysis of games is just as natural an instinct of mathematicians. Who should win? What is the best move? What are the odds of a certain chance event? How long is a game likely to take? When we are presented with a puzzle, are there standard techniques that will help us to find a solution? Does a particular puzzle have a solution at all? These are natural questions of mathematical interest, and we shall direct our attention to all of them.' Thus John Beasley, in his introduction (Chapter 1), encapsulates the theme of his book.

Chapters 2 and 3 look at card and dice games respectively, and demonstrate some results which may be surprising. If, when designing a board of snakes and ladders, you want to place a snake so as to minimise a player's chance of climbing a particular ladder, where do you put it? Guess, and then read Chapter 3. Chapter 4 starts the discussion of games which depend both on chance and on skill. Chapter 5 looks at ways of estimating the skill of a player. Chapter 6 looks at the determination of a player's optimal strategy in a game where one player knows something that the other does not. Games of pure skill are considered in Chapters 7–10. Finally, Chapter 11 looks at automatic games, and shows that there is no general procedure for deciding whether an automatic game terminates.

For me the highlights of this book include: The analysis of the queens on a cylinder, Conway's of the solitaire army, and Hutching's of the game known as 'sylver coinage'.

A good A-level in Pure Mathematics is necessary if the reader is to be able to follow the analyses which have been included, and to obtain some of the many results which are given without proof.

Pages 162–166, which are devoted to further reading, will be welcomed by those readers whose appetites have been whetted by particular topics.

'In no sense has this been intended as a book of instruction; its sole object has been to entertain ... .' Thus the author begins the last paragraph of the final chapter. This is a well-written and attractive book, and can be recommended to all enthusiasts for mathematical games.

Medical School, University of Newcastle upon Tyne — GREGORY D. ECONOMIDES

Experiments in Topology. By STEPHEN BARR. Dover, New York, 1989. Republication of the edition published by T. Y. Crowell, New York, 1964. Pp. 210. Paperback £4.20 (ISBN 0-486-25933-1).

The study of topology is one of the cornerstones of modern mathematics. However, the abstract terms and definitions in which it is usually couched can make it very difficult for the beginner, both from the technical point of view and from that of intuition—having a feel for what is actually going on.

Any book which can ameliorate these difficulties is of great value, and this is such a book. The author describes the nature of topology, then goes on to discuss various specific topics such as the structure of Moebius strips, Klein bottles and projective planes, the colouring of maps and the nature of networks. He ends with a discussion of some of the technicalities of sets and continuity which are needed for a more rigorous treatment of the subject.

The book was originally published in 1964 and much work has been done in the past 25 years which would modify some of the statements made. For example, the four-colour conjecture has been proved (although not everyone is convinced!) and much more is known about knots.

However, the book is clear and easy to read and retains a freshness of approach despite its age. It can be thoroughly recommended to anyone of any age with an interest in mathematics, whether technical or recreational.

Oakham School — G. N. THWAITES

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Imperial College London — Friday 28 June 1991

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Four Encounters with Sierpiński's Gasket

Dr Alan Beardon
Wallpaper Patterns in Different Geometries

SHEFFIELD: 7.00 p.m. Ian Stewart; 8.00 p.m. refreshments; 8.30 p.m. Alan Beardon. Lecture Theatre 2, Hicks Building, University of Sheffield, Hounsfield Rd, Sheffield S3 7RH. Admission free without ticket. Enquiries to Dr A. K. Austin, Department of Pure Mathematics, The University, Sheffield S3 7RH.

LONDON: 7.30 p.m. Ian Stewart; 8.30 p.m. refreshments; 9.00 p.m. Alan Beardon. The Great Hall, Sherfield Building, Imperial College, South Kensington, London SW7. Admission free with ticket in advance. Apply by Friday 21 June to Miss S. M. Oakes, London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL, enclosing a stamped addressed envelope.

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