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Mathematicorum

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SOME BOOLEAN INEQUALITIES FROM A TRIANGLE

J.L. BRENNER

1. *Introduction.* It is convenient when some mathematical theorems can be established without any argument. The theorems of this article are nontrivial and of that nature. Some of them are even new, and sharper than similar previously known results. The theorems are all derived from the obvious assertion:

1.1. *If two events A and B are mutually exclusive, then $\Pr(A) + \Pr(B) \leq 1$, with equality if and only if $A \vee B$ is exhaustive.*

2. *The inequalities.*

2.01. *THEOREM (Turner and Conway [2]). If $m > 1$ and $n > 1$ are integers, and if $p, q > 0$ with $p+q \leq 1$, then*

$$(1 - p^m)^n + (1 - q^n)^m > 1.$$

To see that this is obvious, draw a triangle DEF with n lines from D to the opposite side and m lines from E to the opposite side. (Figure 2.02 illustrates the case $n = 3$, $m = 2$.) Draw a small circle centered at each of the mn intersection points and then erase one of the crossing line segments inside each circle. If this erasing is done at random, with probability p that the surviving segment points towards D (and probability $q = 1-p$ that it points towards E), it is clear that:

2.03. For each of the n lines from D, the probability that it is solid (unbroken) is p^m .

2.04. The probability that none of the n lines from D is solid is $(1 - p^m)^n$.

2.05. The probability that at least one of the n lines from D is solid is $1 - (1 - p^m)^n$.

Note that 2.03-2.05 are immediate consequences of the probability theorems on independent and on complementary events. Proceeding likewise from point E, we obtain:

2.06. The probability that at least one of the m lines from E is solid is $1 - (1 - q^n)^m$.

Now the events A and B whose probabilities are given by 2.05 and 2.06 are

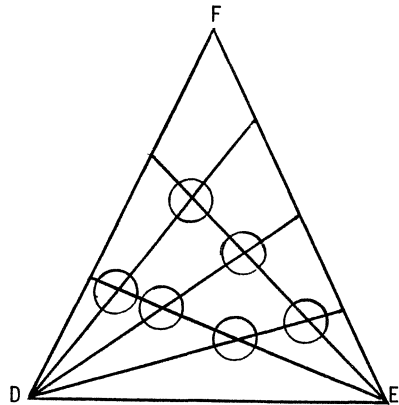


Figure 2.02

mutually exclusive (if one line from D is solid, then *all* lines from E are broken) and the union of these events is not exhaustive; hence, from 1.1,

$$\{1 - (1 - p^m)^n\} + \{1 - (1 - q^n)^m\} < 1,$$

and 2.01 follows, at least when $p+q = 1$. If $p+q < 1$, let $p+q' = 1$. The inequality in 2.01 then holds with q' in place of q , and it therefore holds *a fortiori* with q since $q' > q$. \square

At the moment, 2.01 is proved only for integral $m, n > 1$. For nonintegral m, n , a proof by calculus can be given. If $0 < m, n < 1$, the inequality is reversed. See [1].

Using Figure 2.07, the following can be proved:

2.08. *THEOREM. If $p, q, r > 0$ and $p+q+r \leq 1$, then*

$$(2.09) \quad 1 + p^3 < (1 - q^3)^3 + (1 - r^3)^3.$$

The collinearities and concurrences shown in the figure follow from Ceva's Theorem. Actually, exact concurrence is not needed in the argument. The (Boolean) reasoning will be very similar to that already used.

Suppose that in each small circle two of the three crossing segments are randomly erased, with probabilities q and r that the surviving segment points towards D and E, respectively (and probability $p = 1-q-r$ that it points towards F).

To prove (2.09), note that the following three events are mutually exclusive and that their union is not exhaustive:

2.10. At least one of the three lines interior to angle D is solid.

2.11. At least one of the three lines interior to angle E is solid.

2.12. The median line through F is solid.

The respective probabilities are:

$$\text{Pr}(2.10) = 1 - (1 - q^3)^3,$$

$$\text{Pr}(2.11) = 1 - (1 - r^3)^3,$$

$$\text{Pr}(2.12) = p^3.$$

The sum of these probabilities must be less than 1. This shows that (2.09) holds when $p+q+r = 1$, and, by the same argument used before, it holds *a fortiori* when $p+q+r < 1$. \square

Inequality (2.09) and its proof are easily generalized. For any integer $m > 1$, from a figure similar to 2.07 but with m, m , and $2m-1$ lines interior to angles D, E, and F, respectively, we obtain

$$(2.13) \quad 1 + p^m < (1 - q^m)^m + (1 - r^m)^m.$$

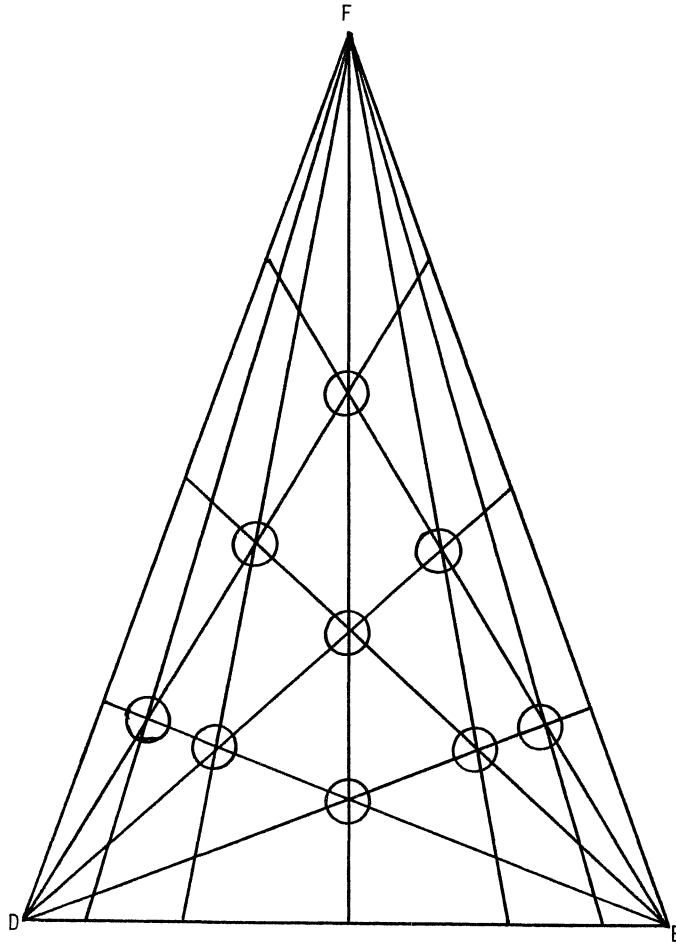


Figure 2.07

While (2.13) remains valid for nonintegral $m > 1$, the proof for such values of m requires more than a picture.

3. *Homogeneous inequalities.* If in (2.09) we set

$$p = \frac{a}{a+b+c}, \quad q = \frac{b}{a+b+c}, \quad r = \frac{c}{a+b+c},$$

where $a, b, c > 0$, we easily obtain the following equivalent homogeneous result:

$$(3.1) \quad (a+b+c)^9 + a^3(a+b+c)^6 < \{(a+b+c)^3 - b^3\}^3 + \{(a+b+c)^3 - c^3\}^3;$$

and the homogeneous equivalent of (2.13) is, for any integer $m > 1$,

$$(3.2) \quad (a+b+c)^{m^2} + a^m(a+b+c)^{m^2-m} < \{(a+b+c)^m - b^m\}^m + \{(a+b+c)^m - c^m\}^m.$$

4. *Further inequalities.* Let k quantities p_1, p_2, \dots, p_k be given such that $p_i > 0$ for all i and $\sum p_i \leq 1$. Then for any real $m > 1$ we have

$$(4.1) \quad k - 1 < \sum_{i=1}^k (1 - p_i^m)^m.$$

For $k = 2$, this is proved for integral m in the present article. In a sequel to appear later in this journal, special cases of (4.1) are proved using experimental design patterns. Other inequalities proved in the sequel use regular or semiregular polygons, and multidimensional arrays. Some of these inequalities are sharper than anything in [1]. See, for example, Crux 843 in this issue.

5. *Acknowledgment.* I thank Henry Fettis for explaining Ceva's Theorem to me.

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1. J.L. Brenner, "Some Inequalities from Switching Theory", *J. Combinatorial Theory*, 7 (1969) 197-205.
 2. J.C. Turner and V. Conway, Problem 68-1, *SIAM Review*, 11 (1969) 402-406.
- 10 Phillips Rd., Palo Alto, California 94303.

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MATHEMATICAL CLERICHEWS

Menaechmus of Cyzicus,
A mean old busy cuss,
Gave Alexander directions,
Invented conic sections.

Colin Maclaurin
Could have put more in
His series, so splendid,
Which Taylor extended.

Willebrord Snell
Knew circles quite well.
He studied old tomes
And loxodromes.

Leonhard Euler?
The Germans say "Oiler";
The British say "Yuler",
And that is peculiar.

Girolamo Cardano,
A brilliant, crooked man, OH!
Once stole from (inter alia)
Ferrari and Tartaglia.

Sir Christopher Wren,
Most constructive of men—
Savilian professor,
A Halley successor.

Edward John Routh
From the North, not the South,
Senior Wrangler was reckoned.
James Clerk Maxwell was SECond.

Eratosthenes,
("Beta", if you please)
First found the girth
Of planet Earth.

ALAN WAYNE
Holiday, Florida

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NOTEWORTHY THIRD-ORDER TRIANGULAR ARRAYS

CHARLES W. TRIGG

1. Introduction.

An n th-order triangular array consists of $n(n+1)/2$ elements distributed in n rows in which the k th row contains k elements. Imbedded in a third-order triangular array are three second-order triangular arrays each with a vertex in common with the third-order array. Our current concern is with the sums of the elements in the second-order arrays imbedded in the third-order arrays that have the digits 1, 2, 3, 4, 5, and 6 as elements.

We define *complementary numbers* as two numbers with a sum of 7. *Complementary arrays* are arrays in which corresponding numbers are complementary.

2. Second-order sums in arithmetic progression.

Distribute 1, 3, 5 in that order in some sense (say counterclockwise) on the vertices of a triangle or on its midpoints, and then the complementary set 6, 4, 2 in that order in the same sense on the other three points. Remarkably, regardless of the starting points of the distributions, the sums of the elements of the second-order arrays are in arithmetic progression. The six possible distributions (not counting rotations or reflections) follow, together with their corresponding A.P.'s. Complementary arrays are adjacent to each other.

1	6	1	6	1	6
2 4	5 3	6 2	1 5	4 6	3 1
3 6 5	4 1 2	3 4 5	4 3 2	3 2 5	4 5 2
7, 11, 15	6, 10, 14	9, 11, 13	8, 10, 12	9, 11, 13	8, 10, 12

Again, distribute the complementary sets 1, 2, 3 and 6, 5, 4 in the same sense on the vertices and midpoints of a triangle. As before, in each of the six possible arrays that follow, the sums of the elements of the second-order arrays are in A.P.

1	6	1	6	1	6
4 5	3 2	5 6	2 1	6 4	1 3
2 6 3	5 1 4	2 4 3	5 3 4	2 5 3	5 2 4
10, 12, 14	7, 9, 11	11, 12, 13	8, 9, 10	11, 12, 13	8, 9, 10

In a cyclic arrangement of the first six nonzero digits, the digit sets 2, 3, 4 and 5, 6, 1 may be considered to be sets of consecutive digits, as may their complementary sets 5, 4, 3 and 2, 1, 6. Two of the six third-order arrays where

the first pair of sets appears in *opposite senses* are shown below together with their complementary arrays. Their second-order sums are in A.P.

2	5	6	1
5 6	2 1	2 4	5 3
3 1 4	4 6 3	5 3 1	2 4 6
9, 11, 13	8, 10, 12	8, 10, 12	9, 11, 13

Two of the third-order arrays where the first pair of sets appears in the *same sense* have their second-order sums in A.P. They are shown below together with their complementary arrays.

2	5	1	6
5 1	2 6	2 4	5 3
3 6 4	4 1 3	5 3 6	2 4 1
8, 11, 14	7, 10, 13	7, 10, 13	8, 11, 14

The only third-order triangular arrays (not counting rotations and reflections) of the first six nonzero digits that have second-order sums in A.P. are the twenty recorded above. They constitute $20/(6!/3 \cdot 2) = 1/6$ of all third-order arrays.

3. *Equal second-order sums.*

The sum of the first six digits is 21. In the sum of the sums of the three second-order arrays, the digits on the medial triangle are counted twice. It follows that, if the three second-order sums are to be equal, then the sum of the digits on the medial triangle must be a multiple of 3. This condition is met by either of the complementary sets 1, 3, 5 and 6, 4, 2; and by either of the complementary sets 1, 2, 3 and 6, 5, 4. In each case, two of the six distributions of the complementary sets in *opposite senses* produce arrays with equal second-order sums, as shown below.

1	6	1	6
6 4	1 3	6 5	1 2
3 2 5	4 5 2	2 4 3	5 3 4
11, 11, 11	10, 10, 10	12, 12, 12	9, 9, 9

If these are viewed as A.P.'s with a common difference of zero, then the number of triangular arrays with second-order sums in A.P. is increased to twenty-four, $1/5$ of all third-order arrays.

POSTSCRIPT TO "A 1983 SALMAGUNDI"

CHARLES W. TRIGG

When proofreading "A 1983 Salmagundi" [1983: 8-13], Trigg saw the editor's solicitation of aid for Trigg in representing 19 and 59. Intrigued, Trigg, who believes in self-help, promptly sent to the editor the trig

$$19 = -1 + 9 + 8 + 3 \quad \text{and} \quad 59 = -1 + (\sqrt{9}!)(8 + !3).$$

Postal service being what it is and shouldn't be, Trigg's reply to the editor's SOS did not arrive in time for incorporation in the Salmagundi.

The vanishing breed of good Samaritans is not quite extinct. Four of them rallied 'round in response to the editorial plea. The first to answer the clarion call was R.S. Johnson of Montréal, who offered

$$19 = \sqrt{(1 + .\dot{9})^8} + 3 = \sqrt{(1 - \sqrt{9})^8} + 3.$$

Next to answer one of the editor's calls was Friend H. Kierstead, Jr., of Cuyahoga Falls, with

$$19 = 1 + \sqrt{9} \cdot 8 - 3!.$$

Then Bob Prielipp reported from Oshkosh that, by gosh,

$$19 = (1 + \sqrt{9})! - 8 + 3 \quad \text{and} \quad 59 = -(1 + \sqrt{9})! + 83.$$

Last came Basil Rennie, from down under in Australia, who came up (or down) with the trig sum for 19 given at the beginning of this note and with

$$59 = 1 \cdot \sqrt{9} + \binom{8}{3}.$$

While languishing on the road from Jerusalem to Jericho, Trigg also discovered that

$$\begin{aligned} 19 &= (-1 + \sqrt{9}) \cdot 8 + 3 = (1 + .\dot{9}) \cdot 8 + 3 \\ &= -1 - \sqrt{9}! + [8/.3] = -1 + \sqrt{9}! + 8 + 3! \\ &= 19 + 8[![!(!3)]] \end{aligned}$$

and

$$59 = (1 + \dots + 9) + 8 + 3!.$$

All in all, a soupçon of variety to spice the salmagundi to the editor's taste. Finally, the same Basil Rennie was able to exorcize the distasteful sub-

factorial and excise it from its various prime repositories, thus:

$$\begin{array}{ll}
 11 = 1^9 \cdot (8 + 3) & 41 = -1 + (\sqrt{9}!) \cdot 8 - 3! \\
 17 = 1 \cdot (-9 + [8/.3]) & 47 = -1 \cdot 9 + \binom{8}{3} \\
 23 = -1 + 9 \cdot 8/3 & 53 = -1 \cdot \sqrt{9} + \binom{8}{3} \\
 29 = 1 \cdot \sqrt{9} + [8/.3] & 71 = -1 + \sqrt{9} \cdot 8 \cdot 3 \\
 31 = -1 + [98/3] & 83 = 1^9 \cdot 83 \\
 37 = -19 + \binom{8}{3} & 97 = -1 + 98 - [.3]
 \end{array}$$

According to the Chinese, 1983 is the Year of the Pig. While he stands back and watches it continue to unfold itself, a convalescent Trigg will prepare a Farrago for 1984, the Year of the Rat or, well, Orwell.

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THE PUZZLE CORNER

Puzzle No. 37: Alphametic

MODEST mod EST is equal to MOD;
 Solutions thereto once known only to God...
 Find their total number by counting my toes,
 Alphametically true: a rose is a rose.
 Just a dozen short of being too gross
 To have published in a family journal...
 A single solution has this riddle infernal!

With MODEST mod EST still equal to MOD,
 Solutions thereto are now one less than ODD.
 That would make it even for this to be true,
 So two more solutions I want from you.

Now when the above you have put to the test,
 MOD is still equal to MODEST mod EST.
 The total solutions, allow me the pun,
 Must herewith now equal exactly ONE.

HANS HAVERMANN, Weston, Ontario

Answer to Puzzle No. 35 [1983: 105]: Excerpt, except; except, expect.

Answer to Puzzle No. 36 [1983: 105]: Disfigure (D is figure).

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MATHEMATICAL VENERY: I

FRIEND H. KIERSTEAD, JR.

This note was inspired by a note of Charles W. Trigg, also entitled "Mathematical VenerY", which appeared recently in the *Journal of Recreational Mathematics* (Vol. 15 (1982-83), No. 3, pp. 173-174). One meaning of the word "venerY" is "the practice or sport of hunting". This was the original inspiration for the *venereal game*, which consists of assigning "nouns of multitude" to groups, principally of the animal kingdom. This game has been played extensively, and familiar examples abound: a covey of quail, a gam of whales, a pride of lions, etc., etc.

What Trigg has done is to apply the venereal game to mathematics and education, with forty well-chosen examples. Inspired by him, we extend his list as follows:

a trigg of numerologists	a nelson of supercomputers
a dearth of odd perfect numbers	a frustration of Rubik's cubes
a triangle of Pythagorean numbers	a papyrus of Egyptian fractions
a toadstool of Farey series	a golden of Fibonacci numbers
a reversal of palindromes	a johnson of magic squares
a hardy of number theorists	a loyd of puzzlers

and, of course,

a sauv  of editors.

2504 Fifteenth Street, Cuyahoga Falls, Ohio 44223.

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MATHEMATICAL VENERY: II

EDITH ORR

This note was inspired by the above inspired. It is less generally known that the venereal game has been extended to the *other* meaning of the word "venerY": the gratification of sexual desire. We illustrate with a few examples, only five or six of which are already known, the rest having been created especially for this occasion.

We start, innocently enough, with

a peck of kisses

and then, acceleration being rapid when going down the primrose path, continue with

a demimonde of courtesans	an anthology of pros
a wishful of wantons	an essay of trollops
a pander of bawds	a cavalcade of whores
a jam of tarts	a canker of harlots
a flourish of strumpets	a pride of loins.

And when the bloom is off the rose, we have usually

a scruple of magdalenes.

For relevance in a mathematical journal, we stammeringly add

a dedekind of cocottes

and then dismiss the whole lot with

an empty set of virgins.

2206-E Halifax Drive, Ottawa, Ontario K1G 2W6.

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THE OLYMPIAD CORNER: 45

M.S. KLAMKIN

I present three new problem sets this month, the first two through the courtesy of Jan van de Craats of Leiden University. First come the problems set at the 1982 Netherlands Olympiad, for which I solicit elegant solutions. Next come the problems of the recent 1983 Netherlands Invitational Mathematics Examination, in which only answers are required, and which leads to qualification for the Netherlands Olympiad. (Next month I will give the problems set at the new American Invitational Mathematics Examination (AIME), which is of the same type.) The third consists of the problems set at the Fifteenth Canadian Mathematics Olympiad (1983), for which the official solutions will be published here next month. Finally, I give solutions to last month's Practice Set 18 [1983: 107].

*

1982 NETHERLANDS OLYMPIAD

1. Which of $(17091982!)^2$ and $17091982^{17091982}$ is greater?
2. M is the midpoint of AB and P is an arbitrary point on AC. Using only a pencil and a straightedge, construct a point Q on BC such that P and Q are at equal distance from CM. Justify your construction.
3. Five marbles are distributed independently and at random among seven urns. What is the expected number of urns receiving exactly one marble?

4. Determine the greatest common divisor of $n^2 + 2$ and $n^3 + 1$, where $n = 9^{753}$.

*

1983 NETHERLANDS INVITATIONAL MATHEMATICS EXAMINATION

(The marks are 2,3,4, respectively, for the A,B,C problems. Problems A4, B2, C1, and C3 have been changed slightly. The answers, which are all integers, are given at the end of this column.)

- A1. A regular n -gon is inscribed in a circle and another regular n -gon is circumscribed about the same circle. The ratio of the areas of the two regular n -gons is 3:4. Determine n .
- A2. A die rests on the top of a table. Two persons, sitting at opposite sides of the table, can each see three faces of the die. The total number of spots on the three faces one of them can see is 9, and the total for the other is 15. How many spots are on the top face of the die?
- A3. A rectangle is inscribed in a given triangle with two vertices of the rectangle lying on one side of the triangle. If the maximum area of such a rectangle is 12, what is the area of the given triangle?
- A4. What is the smallest amount, in cents, that cannot be made up with at most ten of the coins of denominations 1¢, 5¢, 10¢, 25¢, 50¢, and 100¢?
- A5. AB is a diameter of a circle with centre M. C is a point on the circle distinct from A and from B. The perpendicular from C to AB meets AB at D, and the perpendicular from M to BC meets BC at E. If $DB = 3ME$, calculate angle ABC in degrees.
- A6. The same remainder is obtained when each of 51760, 51982, and 52241 is divided by an integer $n > 1$. Determine n .
- B1. The volume of a regular octahedron is n times that of a regular tetrahedron with the same edge length. Determine n .
- B2. Let x and y be two two-digit natural numbers with $x < y$. The product xy is a four-digit number beginning with 2. If this 2 is deleted, the resulting number is equal to $x+y$. Now $x = 30$ and $y = 70$ yield $xy = 2100$ and $x+y = 100$. There is another pair (x,y) with this property. Determine y in this pair.
- B3. Matches of unit length are used to form various patterns. A triangular pattern consists of an equilateral triangle of side n subdivided into n^2 unit equilateral triangles. For $n = 2$, 9 matches are required; for $n = 4$,

30 matches are required. A square pattern consists of a square of side n subdivided into n^2 unit squares. For $n = 2$, 12 matches are required; for $n = 3$, 24 matches are required.

What is the minimum number of matches which may be used to form a triangular pattern as well as a square pattern? The two patterns are formed one after the other, and all matches must be used in each formation.

B4. In base ten representation, the natural number n consists of 100 nines. How many nines are in the base ten representation of n^3 ?

C1. Consider all integral multiples of 101 which consist of six digits. From each such multiple, create a new number by moving the leftmost digit to the extreme right. This new number is divided by 101 and a remainder r is obtained. Find the sum of all possible values of r ($0 \leq r < 101$).

C2. How many noncongruent triangles can be obtained by joining pairwise three vertices of a regular dodecagon?

C3. The total age of five children is 40. Each of them is older than 1 and younger than 25. The product of the ages of the boys is 13 times the product of the ages of the girls. Next year, the boys' product will again be an integral multiple of the girls' product. Determine the total age of the boys.

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FIFTEENTH CANADIAN MATHEMATICS OLYMPIAD

May 4, 1983 — Time: 3 hours

ANSWER ALL FIVE QUESTIONS — ALL QUESTIONS ARE OF EQUAL VALUE

1. Find all positive integers w, x, y, z which satisfy $w! = x! + y! + z!$.
2. For each real number r , let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r 2^r x + 2^r y)$. Let F be the family of all such transformations, i.e., $F = \{T_r; r \text{ is a real number}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F .
3. The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?
4. Prove that for every prime number p there are infinitely many positive integers n such that p divides $2^n - n$.
5. The geometric mean (G.M.) of k positive numbers a_1, a_2, \dots, a_k is defined to be the (positive) k th root of their product. For example, the G.M. of 3, 4, 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all nonempty subsets of S .

*

PRACTICE SET 18

18-1, [1983: 107] If z_0 is any zero of the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \quad a_0 > 0,$$

where $a_0 \geq a_1 \geq \dots \geq a_n \geq 0$, prove that $|z_0| \leq 1$.

Solution.

We assume that $|z_0| > 1$ and find a contradiction. Consider the function

$$(1-z)f(z) = -a_0 z^{n+1} + (a_0 - a_1)z^n + \dots + (a_{n-1} - a_n)z + a_n.$$

For all z we have

$$\begin{aligned} |(1-z)f(z)| &\geq |a_0 z^{n+1}| - |(a_0 - a_1)z^n + \dots + (a_{n-1} - a_n)z + a_n| \\ &\geq a_0 |z|^{n+1} - \{(a_0 - a_1)|z|^n + \dots + (a_{n-1} - a_n)|z| + a_n\}. \end{aligned}$$

In particular, for $z = z_0$ we have

$$\begin{aligned} 0 &= |(1-z_0)f(z_0)| \geq a_0 |z_0|^{n+1} - \{(a_0 - a_1)|z_0|^n + \dots + (a_{n-1} - a_n)|z_0| + a_n\} \\ &\geq a_0 |z_0|^{n+1} - |z_0|^n \{(a_0 - a_1) + \dots + (a_{n-1} - a_n) + a_n\} \\ &= a_0 (|z_0|^{n+1} - |z_0|^n) \\ &> 0, \end{aligned}$$

and we have the desired contradiction.

18-2, [1983: 107] A, B, C, D, E are five coplanar points such that no two of the lines joining these points in pairs are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to the lines which join the remaining points in pairs. Determine the maximum number of points of intersection of these perpendiculars.

Solution.

This problem appears in N.Y. Vilenkin, *Combinatorics*, Academic Press, New York, 1971, with the following solution:

There are $\binom{5}{2} = 10$ lines joining the points A, B, C, D, E in pairs. There are 4 lines passing through each point. It follows that we can drop 6 perpendiculars from each point. Consider any 2 points, say, B and C. The perpendiculars dropped from B to the lines passing through C intersect all perpendiculars from C. There

are 3 lines issuing from C which do not pass through B. Hence we can drop perpendiculars from B to these 3 lines. These perpendiculars intersect the perpendiculars from C in $3 \cdot 6 = 18$ points. Each of the perpendiculars from B to the remaining 3 lines (not passing through C) intersects only 5 of the perpendiculars from C; this is so because each of the perpendiculars from B in question is parallel to a perpendicular from C. As a result, we get additional 15 points. Hence the perpendiculars issuing from 2 points intersect in $18 + 15 = 33$ points. Since there are 10 pairs of points, there are $33 \cdot 10 = 330$ points of intersection. However, some of these points coincide. Specifically, the points A,B,C,D,E determine $\binom{5}{3} = 10$ triangles. The altitudes of such a triangle belong to our set of perpendiculars, but they intersect in 1 rather than in 3 points. This means a loss of 2 points per triangle, and so a loss of $10 \cdot 2 = 20$ points. It follows that our perpendiculars intersect in 310 points.

18-3. [1983: 107] Show how to construct a triangle having its vertices on three given skew lines so that the centroid of the triangle coincides with a given point.

Solution.

This problem appears without solution in Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea, New York, 1964, page 21, Exercise 8.

Let (a,b,c) be the given point and l_1, l_2, l_3 the three given skew lines (assumed to be *pairwise* skew). For $i = 1, 2, 3$, let (a_i, b_i, c_i) be a point and (h_i, k_i, l_i) a nonzero vector on l_i . The vectorial equation of l_i is then

$$P_i \equiv (x_i, y_i, z_i) = t_i(h_i, k_i, l_i) + (a_i, b_i, c_i), \quad i = 1, 2, 3 \quad (1)$$

where t_i ranges over the reals. The centroid of triangle $P_1P_2P_3$ will coincide with (a,b,c) if and only if

$$\begin{cases} h_1t_1 + h_2t_2 + h_3t_3 = 3a - a_1 - a_2 - a_3, \\ k_1t_1 + k_2t_2 + k_3t_3 = 3b - b_1 - b_2 - b_3, \\ l_1t_1 + l_2t_2 + l_3t_3 = 3c - c_1 - c_2 - c_3. \end{cases} \quad (2)$$

It follows from our assumption that the coefficient matrix, M , of (2) is of rank 2 or 3. If it is of rank 3, then $\det M \neq 0$, the system (2) has a unique solution (t_1, t_2, t_3) , and triangle $P_1P_2P_3$ is uniquely determined and constructible. If M is of rank 2, then there are infinitely many solution triangles, provided the augmented matrix of (2) is also of rank 2. We can then take, for example, an arbitrary point P_1 on l_1 , then, t_1 being known, solve (2) for t_2 and t_3 , and finally obtain P_2 and P_3 from (1). \square

It would be interesting to have a purely synthetic construction for this problem.

The two-dimensional version, where the three given lines are coplanar (no two parallel), is considerably easier and there are infinitely many solution triangles for any given centroid in the plane of the three lines.

ANSWERS FOR THE 1983 NETHERLANDS INVITATIONAL MATHEMATICS EXAMINATION

A1	A2	A3	A4	A5	A6	B1	B2	B3	B4	C1	C2	C3
6	5	24	444	30	37	4	88	84	199	90	12	30

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

841. *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

The adjoined alphametic was inspired by Gertrude Stein's put-down of Oakland, California [1983: 55]. It is in the lowest possible base, and the punctuation is of literary, rather than arithmetic, significance.

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      O D D:
    T H E R E'S
      N O
      T H E R E
    T H E R E
  O A K L A N D
  
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842. *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

Find (a) necessary and (b)* sufficient conditions on a, b, c for the system

$$x + \frac{1}{x} = a, \quad y + \frac{1}{y} = b, \quad xy + \frac{1}{xy} = c$$

to be consistent, that is, to have at least one common solution (x, y) . (These necessary and sufficient conditions constitute the *eliminant* of the system.)

843, Proposed by J.L. Brenner, Palo Alto, California.

For integers $m > 1$ and $n > 2$, and real numbers $p, q > 0$ such that $p + q = 1$, prove that

$$(1-p^m)^n + np^m(1-p^m)^{n-1} + (1-q^n - npq^{n-1})^m > 1.$$

844*, Proposed by Peter M. Gibson, University of Alabama in Huntsville, and Michael H. Rodgers, student at the same university.

(a) A triangle $A_0B_0C_0$ with centroid G_0 is inscribed in a circle Γ with center O . The lines A_0G_0, B_0G_0, C_0G_0 meet Γ again in A_1, B_1, C_1 , respectively, and G_1 is the centroid of triangle $A_1B_1C_1$. A triangle $A_2B_2C_2$ with centroid G_2 is obtained in the same way from $A_1B_1C_1$, and the procedure is repeated indefinitely, producing triangles with centroids G_3, G_4, \dots .

If $g_n = OG_n$, prove that the sequence $\{g_0, g_1, g_2, \dots\}$ is decreasing and converges to zero.

(b) Prove or disprove that a result similar to (a) holds for a tetrahedron inscribed in a sphere, or, more generally, for an n -simplex inscribed in an n -sphere.

845*, Proposed by B.M. Saler, Agincourt, Ontario.

Let r_1, r_2, r_3 be the focal radii (all from the same focus F) of the points P_1, P_2, P_3 , respectively, on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. A circle with centre F and radius $r = \sqrt[3]{r_1r_2r_3}$ intersects the focal radii r_1, r_2, r_3 in P'_1, P'_2, P'_3 , respectively. Find the ratio of the areas of triangles $P_1P_2P_3$ and $P'_1P'_2P'_3$.

(This is Theorema Elegantissimum from *Acta Eruditorum*, A.D. 1771, page 131, by an unknown author.)

846, Proposed by Jack Garfunkel, Flushing, N.Y. and George Tsintsifas, Thessaloniki, Greece.

Given is a triangle ABC with sides a, b, c and medians m_a, m_b, m_c in the usual order, circumradius R , and inradius r . Prove that

$$(a) \quad \frac{m_a m_b m_c}{m_a^2 + m_b^2 + m_c^2} \geq r;$$

$$(b) \quad 12Rm_a m_b m_c \geq a(b+c)m_a^2 + b(c+a)m_b^2 + c(a+b)m_c^2;$$

$$(c) \quad 4R(am_a + bm_b + cm_c) \geq bc(b+c) + ca(c+a) + ab(a+b);$$

$$(d) \quad 2R\left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}\right) \geq \frac{m_a}{m_b m_c} + \frac{m_b}{m_c m_a} + \frac{m_c}{m_a m_b}.$$

847, Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Prove that

$$\sum_{j=0}^n \frac{\binom{n}{2j-1}}{5^j} = \frac{1}{2}(0.4)^n F_n,$$

where F_n is the n th Fibonacci number. (Here we make the usual assumption that $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.)

848, *Proposed by Charles W. Trigg, San Diego, California.*

Tetrahedral numbers have the form $T(n) = n(n+1)(n+2)/6$ and triangular numbers have the form $n(n+1)/2$. The third tetrahedral number, 10, is also the fourth triangular number. Show that at least one-third of the tetrahedral numbers are also polygonal numbers.

849, *Proposed by J.T. Groerman, Arnhem, The Netherlands.*

The functions defined by

$$f(x) = \frac{Ax^7}{7} + \frac{Bx^5}{5} + \frac{19}{35}x$$

and

$$g(x) = \frac{Bx^7}{7} + \frac{Ax^5}{5} - \frac{1}{35}x,$$

where A and B are primes, have integral values for each integer x . Find the smallest possible values of A and B .

850, *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Let $x = r/R$ and $y = s/R$, where r, R, s are the inradius, circumradius, and semiperimeter, respectively, of a triangle with side lengths a, b, c . Prove that

$$y \geq \sqrt{x}(\sqrt{6} + \sqrt{2-x}),$$

with equality if and only if $a = b = c$.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

689, [1981: 276; 1982: 307] Late solution: LEON BANKOFF, Los Angeles, California.

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702, [1982: 14, 323] (Revised version) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are three distinct points A_1, B_1, C_1 on a circle Γ , and three arbitrary real numbers l, m, n whose sum is 1. Show how to determine a point M such that, if A_1M, B_1M, C_1M meet Γ again in A, B, C , respectively, then

$$[MBC] = l[ABC], \quad [MCA] = m[ABC], \quad [MAB] = n[ABC], \quad (1)$$

where the brackets denote the signed area of a triangle.

II. *Solution by the proposer (revised by the editor).*

Prompted by the editor's comment [1982: 324], we give a more complete solution to the revised version of the problem given above. We will need the following lemma.

LEMMA. *Let the arbitrary (and independently chosen) points M, A, B, C have affixes z, α, β, γ in the complex plane. If $l+m+n = 1$, then (1) holds if and only if*

$$z = l\alpha + m\beta + n\gamma. \quad (2)$$

Proof. If (1) holds, then (l, m, n) are the barycentric coordinates of M with respect to triangle ABC , and (2) follows. Conversely, suppose (2) holds. To show that the geometric significance of l, m, n is that given in (1), we consider complex numbers as vectors and use exterior multiplication [1]. We have

$$\begin{aligned} z \wedge \beta \wedge \gamma &= (l\alpha + m\beta + n\gamma) \wedge \beta \wedge \gamma \\ &= l(\alpha \wedge \beta \wedge \gamma) + m(\beta \wedge \beta \wedge \gamma) + n(\gamma \wedge \beta \wedge \gamma) \\ &= l(\alpha \wedge \beta \wedge \gamma), \end{aligned}$$

from which follows $[MBC] = l[ABC]$. With this and two similar results, we have (1). \square

With the lemma in place, we are ready to tackle our problem. Let $\alpha_1, \beta_1, \gamma_1$, be the affixes of the given points A_1, B_1, C_1 , respectively, and, again, let z, α, β, γ be the affixes of M, A, B, C , where now A, B, C depend upon the choice of M , as described in the proposal. It follows from the lemma that M, A, B, C satisfy (1) if and only if (2) holds. It is clear that (1) never holds if M is on Γ , so we restrict our search to points M not on Γ .

The power of any point M (not on Γ) with respect to Γ is

$$k \equiv (\overline{z-\alpha})(z-\alpha_1) = (\overline{z-\beta})(z-\beta_1) = (\overline{z-\gamma})(z-\gamma_1) \neq 0,$$

and so

$$\frac{lk}{z-\alpha_1} = l\overline{z} - l\overline{\alpha}, \quad \frac{mk}{z-\beta_1} = m\overline{z} - m\overline{\beta}, \quad \frac{nk}{z-\gamma_1} = n\overline{z} - n\overline{\gamma}.$$

Adding corresponding members of these equations, and noting that $l+m+n = 1$, we obtain

$$k\left(\frac{l}{z-\alpha_1} + \frac{m}{z-\beta_1} + \frac{n}{z-\gamma_1}\right) = \bar{z} - (\bar{l}\alpha + \bar{m}\beta + \bar{n}\gamma).$$

Since $k \neq 0$, (2) holds if and only if

$$\frac{l}{z-\alpha_1} + \frac{m}{z-\beta_1} + \frac{n}{z-\gamma_1} = 0. \quad (3)$$

The required points M are those whose affixes are roots of (3). Since (3) is equivalent to the quadratic equation

$$z^2 - \{(m+n)\alpha_1 + (n+l)\beta_1 + (l+m)\gamma_1\}z + (l\beta_1\gamma_1 + m\gamma_1\alpha_1 + n\alpha_1\beta_1) = 0,$$

there are in general two satisfactory points M, and these points are constructible.

A comment was received from DAN PEDOE, University of Minnesota.

Editor's comment.

Pedoe showed that the problem can be generalized still further by letting Γ be an ellipse, since an affine transformation preserves area ratios, and maps a circle onto an ellipse.

REFERENCE

1. D. Pedoe, *A Course of Geometry for Colleges and Universities*, Cambridge University Press, London, 1970, p. 52.

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716, [1982: 48; 1983: 62] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A student has been introduced to common logarithms and is wondering how their values can be calculated. He decides to obtain their binary representations (perhaps to see how a computer would do it). Help him by finding a simple algorithm to generate numbers $b_n \in \{0,1\}$ such that

$$\log_{10} x = \sum_{n=1}^{\infty} b_n \cdot 2^{-n}, \quad 1 \leq x < 10.$$

II. *Comment by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

Since a digital computer already has the capability of performing general arithmetic operations in binary, it is not necessary to calculate the bits of $\log_{10} x$ one at a time.

Most routines to calculate logarithms expect the argument to be in floating-point notation and deliver the result in the same form, although many of the calculations may be carried out in fixed-point. The following method is based on the algorithm used by the CP-V Operating System used on the Xerox Sigma 9 computer, but similar algorithms are used on most medium and large computers.

A floating-point number x may be represented as

$$x = 2^n \cdot f, \quad \frac{1}{2} \leq f < 1,$$

where n is the exponent and f is the mantissa. Then

$$\log_2 x = n + \log_2 f = n - \frac{1}{2} + \log_2 \sqrt{2} f.$$

Now let $y = 1 - \sqrt{2}/(f + \frac{1}{2}\sqrt{2})$; then

$$\log_2 \sqrt{2} f = \log_2 e \cdot \ln \frac{1+y}{1-y} = 2 \log_2 e \cdot y \left(1 + \frac{y^2}{3} + \frac{y^4}{5} + \dots \right).$$

The series in y^2 is approximated by a Chebyshev polynomial in $z = y^2$ over the range $0 \leq z < .02944$. Three terms are used for single precision (24-bit mantissa) and seven terms for double precision (56 bits). Note that the multiplier $2 \log_2 e$ can be absorbed into the Chebyshev coefficients. Finally, the value of $\log_2 x$ is multiplied by $\log_{10} 2$ to convert to the common logarithm (or $\ln 2$ to obtain the natural logarithm). This method thus requires 5 multiplies and 1 divide for single precision, and 9 multiplies and 1 divide for double precision.

For values of x close to unity, the series expansion is used directly, with two terms for single precision and five for double precision.

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723, [1982: 77; 1983: 91] Proposed by George Tsintsifas, Thessaloniki, Greece.

Let G be the centroid of a triangle ABC , and suppose that AG, BG, CG meet the circumcircle of the triangle again in A', B', C' , respectively. Prove that

- (a) $GA' + GB' + GC' \geq AG + BG + CG$;
- (b) $AG/GA' + BG/GB' + CG/GC' = 3$;
- (c) $GA' \cdot GB' \cdot GC' \geq AG \cdot BG \cdot CG$.

II. Comment by Leon Bankoff, Los Angeles, California.

Part (a) of this problem is essentially the same as part (1) of Problem E 2505 (proposed by Jack Garfunkel) published in the *American Mathematical Monthly*, 81 (1974) 1111, for which solutions (one by W.J. Blundon, and one by Paul Erdős and M.S. Klamkin) were published in the *Monthly*, 83 (1976) 59-60.

III. Generalization by M.S. Klamkin, University of Alberta.

More generally, we establish the following analogous results for an n -simplex $A_0 A_1 \dots A_n$ with centroid G , circumcenter O , and circumradius R :

- (a') $GA'_0 + GA'_1 + \dots + GA'_n \geq A_0 G + A_1 G + \dots + A_n G$;
- (b') $A_0 G/GA'_0 + A_1 G/GA'_1 + \dots + A_n G/GA'_n = n + 1$;
- (c') $GA'_0 \cdot GA'_1 \cdot \dots \cdot GA'_n \geq A_0 G \cdot A_1 G \cdot \dots \cdot A_n G$.

From the power of a point theorem, we have

$$A_i G \cdot GA_i' = R^2 - OG^2, \quad i = 0, 1, \dots, n; \quad (1)$$

and substituting for GA_i' in (a'), we obtain the equivalent inequality

$$(R^2 - OG^2) \sum \frac{1}{A_i G} \geq \sum A_i G, \quad (2)$$

where the dummy index (here and later) runs from 0 to n . Now

$$A_i G = \frac{n}{n+1} \cdot m_i, \quad (3)$$

where m_i denotes the median of the simplex from vertex A_i ; hence (2) is equivalent to

$$R^2 - OG^2 \geq \left(\frac{n}{n+1}\right)^2 \sum m_i / \sum (1/m_i). \quad (4)$$

Since $\sum m_i / (n+1) \geq (n+1) / \sum (1/m_i)$, the following stronger inequality implies (4), and hence (a'):

$$(a'') \quad R^2 - OG^2 \geq \frac{n^2}{(n+1)^4} \{\sum m_i\}^2.$$

To establish (a''), as well as (b') and (c'), we will need to express $R^2 - OG^2$ in terms of $\sum m_i^2$. With 0 being the origin of all vectors, we have

$$(n+1)\vec{G} = \vec{A}_0 + \vec{A}_1 + \dots + \vec{A}_n$$

and

$$m_i = \frac{n+1}{n} |\vec{G} - \vec{A}_i|;$$

hence

$$n^2 m_i^2 = (n+1)^2 (|\vec{G}|^2 + R^2 - 2\vec{G} \cdot \vec{A}_i)$$

and

$$\begin{aligned} n^2 \sum m_i^2 &= (n+1)^3 (|\vec{G}|^2 + R^2) - 2(n+1)^3 |\vec{G}|^2 \\ &= (n+1)^3 (R^2 - OG^2). \end{aligned} \quad (5)$$

Inequality (a'') is now seen to be equivalent to

$$\frac{\sum m_i^2}{n+1} \geq \left\{ \frac{\sum m_i}{n+1} \right\}^2,$$

which is a special case of the power mean inequality.

For (b') we have, from (1), (3), and (5),

$$\frac{\sum \frac{A_i G_i}{GA_i}}{\sum \frac{A_i G_i^2}{R^2 - OG^2}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{\sum m_i^2}{R^2 - OG^2} = n + 1.$$

Finally, (c') is equivalent to each of the following inequalities,

$$\Pi(A_i G \cdot GA_i) \geq \Pi(A_i G)^2, \quad (R^2 - OG^2)^{n+1} \geq \Pi\left\{\frac{n}{n+1} \cdot m_i\right\}^2,$$

and

$$\frac{\sum m_i^2}{n+1} \geq \Pi m_i^{2/(n+1)},$$

the last of which is a special case of the A.M.-G.M. inequality.

In (a') and (c'), there is equality if and only if all the medians m_i are equal or, equivalently, if and only if O and G coincide. If $n > 2$, this does not imply that the simplex is regular. For $n = 3$, for example, it suffices that the tetrahedron be isosceles (each edge equals its opposite edge).

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733, [1982: 106; 1983: 121] Proposed by Jack Garfunkel, Flushing, N.Y.

A triangle has sides a, b, c , and the medians of this triangle are used as sides of a new triangle. If r_m is the inradius of this new triangle, prove or disprove that

$$r_m \leq \frac{3abc}{4(a^2 + b^2 + c^2)},$$

with equality just when the original triangle is equilateral.

III. Solution by Leon Bankoff, Los Angeles, California.

Let $r, R, s, m_a, m_b, m_c, h_a, h_b, h_c$ be the inradius, circumradius, semiperimeter, medians, and altitudes, respectively, of the given triangle. [As shown in solution I, or otherwise], the proposed inequality is equivalent to

$$2R(m_a + m_b + m_c) \geq a^2 + b^2 + c^2. \quad (1)$$

To establish (1), we will use the known result $m_a/h_a \geq (b^2 + c^2)/2bc$, with equality just when $b = c$, and two similar results [1]. Since $h_a = 2rs/a$, etc., and $rs = abc/4R$, we have

$$m_a \geq \frac{b^2 + c^2}{4R}, \quad m_b \geq \frac{c^2 + a^2}{4R}, \quad m_c \geq \frac{a^2 + b^2}{4R},$$

and (1) follows, with equality if and only if $a = b = c$.

REFERENCE

1. Problem E 2471, proposed by George Tsintsifas, solution by Leon Bankoff and Leonard Goldstone, *American Mathematical Monthly*, 82 (1975) 523-524.

736, [1982: 107] Proposed by George Tsintsifas, Thessaloniki, Greece.

Given is a regular n -gon $V_1V_2\dots V_n$ inscribed in a unit circle. Show how to select, among the n vertices V_i , three vertices A, B, C such that

- (a) the area of triangle ABC is a maximum;
- (b) the perimeter of triangle ABC is a maximum.

Solution by M.S. Klamkin, University of Alberta.

Since there are only finitely many triangles to consider, there exists a triangle of maximum area and also one (not necessarily the same) of maximum perimeter. The *permissible* triangles ABC are those which contain the circumcenter in their interior or on their boundary, since it is clear that for the other triangles neither the area $[ABC]$ nor the perimeter ∂ABC is a maximum.

For any permissible triangle ABC , let the angles subtended by its sides at the center of the circle be, in decreasing order, $\theta_1, \theta_2, \theta_3$. Then we have

$$2[ABC] = \sin \theta_1 + \sin \theta_2 + \sin \theta_3 \quad (1)$$

and

$$\frac{1}{2}\partial ABC = \sin \frac{1}{2}\theta_1 + \sin \frac{1}{2}\theta_2 + \sin \frac{1}{2}\theta_3. \quad (2)$$

We claim that both (1) and (2) are maximized when

$$\begin{cases} \theta_1 = \theta_2 = \theta_3 = \frac{2\pi m}{3m}, & \text{if } n = 3m; \\ \theta_1 = \frac{2\pi(m+1)}{3m+1}, \theta_2 = \theta_3 = \frac{2\pi m}{3m+1}, & \text{if } n = 3m+1; \\ \theta_1 = \theta_2 = \frac{2\pi(m+1)}{3m+2}, \theta_3 = \frac{2\pi m}{3m+2}, & \text{if } n = 3m+2. \end{cases}$$

(An equivalent formulation would be to say that $[ABC]$ and ∂ABC are both maximized when the three central angles are as nearly equal as possible.)

For let ϕ_1, ϕ_2, ϕ_3 , in decreasing order, be the central angles for any permissible triangle ABC . We must have

$$\phi_1 \geq \theta_1, \quad \phi_1 + \phi_2 \geq \theta_1 + \theta_2, \quad \phi_1 + \phi_2 + \phi_3 = \theta_1 + \theta_2 + \theta_3,$$

that is, the vector (ϕ_1, ϕ_2, ϕ_3) majorizes $(\theta_1, \theta_2, \theta_3)$. By the Majorization Inequality [1], we have

$$f(\phi_1) + f(\phi_2) + f(\phi_3) \geq f(\theta_1) + f(\theta_2) + f(\theta_3)$$

for any convex function f , and the desired result follows from the fact that $-\sin x$ and $-\sin \frac{1}{2}x$ are both convex for $0 \leq x \leq \pi$.

In terms of the vertices V_i , maximum area and perimeter will be attained with the selections

$$\begin{aligned} V_1, V_{m+1}, V_{2m+1}, & \text{ if } n = 3m; \\ V_1, V_{m+2}, V_{2m+2}, & \text{ if } n = 3m+1; \\ V_1, V_{m+2}, V_{2m+3}, & \text{ if } n = 3m+2. \quad \square \end{aligned}$$

A similar proof would show that if, for some k such that $3 \leq k \leq n$, k vertices A_j are selected among the n vertices V_i , then the permissible polygon $A_1A_2\dots A_k$ will have maximum area and perimeter when the k central angles are as nearly equal as possible.

Also solved by JORDI DOU, Barcelona, Spain; LEROY F. MEYERS, The Ohio State University; and the proposer.

REFERENCE

1. E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Heidelberg, 1965, p. 30.

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737, [1982: 107] *Proposed by Charles W. Trigg, San Diego, California.*

(a) Distribute the nine decimal nonzero digits so as to form six prime integers. In how many ways can this be done?

(b) Distribute the ten decimal digits so as to form six prime integers. In how many ways can this be done?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Primes can only end in 1,2,3,5,7,9; and since six primes are required, each of these six digits must appear once as a unit digit. Let the required primes be

$$A = \underline{\quad}1, \quad B = 2, \quad C = \underline{\quad}3, \quad D = 5, \quad E = \underline{\quad}7, \quad F = \underline{\quad}9.$$

(a) The only unassigned digits are 4,6,8. A and F must each be two- or three-digit numbers, and C and E must each be one- or two-digit numbers. Now F cannot have three digits, since no combination of two of 4,6,8 makes it a prime. Hence $F = 89$. If A has two digits, then $A = 41$ or 61 . These two possibilities yield one and two solutions, respectively. If A has three digits, then $A = 461$ or 641 , and each possibility yields one solution. The five solutions are tabulated below.

A	B	C	D	E	F
41	2	3	5	67	89
61	2	43	5	7	89
61	2	3	5	47	89
461	2	3	5	7	89
641	2	3	5	7	89

(b) The only unassigned digits are 0,4,6,8. Each of A and F has at least two and at most four digits. If F has four digits, then $F = 6089$, 8069 , or 8609 , and each of these possibilities yields one solution. If F has three digits, then $F = 409$ or 809 . The first possibility yields one solution, and the second yields five solutions (obtainable from those of (a) by replacing 89 by 809). Suppose now that F has two digits, that is, $F = 89$. Then A cannot have four digits because 4061, 4601, 6041, and 6401 are all composite. If A has three digits, then $A = 401$ or 601 , the first of which yields one solution and the second two solutions. If A has two digits, satisfactory values of C and E can be found only for $A = 41$, which produces one solution. There are thus thirteen solutions in all, and these are tabulated below.

A	B	C	D	E	F
41	2	3	5	7	6089
41	2	3	5	7	8069
41	2	3	5	7	8609
61	2	83	5	7	409
41	2	3	5	67	809
61	2	43	5	7	809
61	2	3	5	47	809
461	2	3	5	7	809
641	2	3	5	7	809
401	2	3	5	67	89
601	2	43	5	7	89
601	2	3	5	47	89
41	2	3	5	607	89

Thus (a) has 5 solutions and (b) has 13 solutions, and 5 and 13 are both primes.

Complete solutions were submitted by MILTON P. EISNER, Mount Vernon College, Washington, D.C.; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer. Incomplete solutions were submitted by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; and RAM REKHA TIWARI, Radhaur, Bihar, India.

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738, [1982: 107] Proposed by Stanley Rabinowitz, Digital Equipment Corp.,
Merrimack, New Hampshire.

Find, in terms of p, q, r , a formula for the area of a triangle whose vertices are the roots of

$$x^3 - px^2 + qx - r = 0$$

in the complex plane.

Solution by Viktors Lin's, University of Ottawa.

Under the standard transformation $x = z + p/3$ (a simple translation which leaves the required area unchanged), the given equation becomes

$$z^3 + 3Hz + G = 0, \quad (1)$$

where

$$H = \frac{q}{3} - \frac{p^2}{9} \quad \text{and} \quad G = -r + \frac{pq}{3} - \frac{2p^3}{27}. \quad (2)$$

By Cardan's method, the roots of (1) are found to be

$$z_1 = A + B,$$

$$z_2 = \omega A + \omega^2 B,$$

$$z_3 = \omega^2 A + \omega B,$$

where ω is a primitive cube root of unity,

$$A = \text{any value of } \sqrt[3]{\frac{1}{2}(-G + \sqrt{G^2 + 4H^3})}, \quad (3)$$

and

$$B = -\frac{H}{A}. \quad (4)$$

Since $z_3 = -(z_1 + z_2)$, the real nonnegative area Δ of the triangle with vertices z_1, z_2, z_3 is

$$\Delta = \frac{3}{4} |z_1 \bar{z}_2 - z_2 \bar{z}_1|.$$

A straightforward calculation gives

$$z_1 \bar{z}_2 - z_2 \bar{z}_1 = \omega(\omega-1)(|A|^2 - |B|^2);$$

hence

$$\Delta = \frac{3\sqrt{3}}{4} \left| |A|^2 - |B|^2 \right|.$$

The value of Δ can now be expressed in terms of p, q, r via (2), (3), and (4). The end result is not pretty.

Received were an incomplete solution from DAN PEDOE, University of Minnesota; a comment from O. BOTTEMA, Delft, The Netherlands; and one incorrect solution.

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739, [1982: 107] Proposed by G.C. Giri, Midnapore College, West Bengal, India.

Prove that, if the incentre I of a triangle is equidistant from the circumcentre O and the orthocentre H, then one angle of the triangle is 60° .

Solution by W.J. Blundon, Memorial University of Newfoundland.

Let α, β, γ be the angles of the triangle. We prove the proposed theorem and its converse:

$$OI = IH \iff \text{at least one of } \alpha, \beta, \gamma \text{ is } 60^\circ. \quad (1)$$

With the usual meanings for R, r, s , the well-known relations

$$OI^2 = R^2 - 2Rr \quad \text{and} \quad IH^2 = 4R^2 + 4Rr + 3r^2 - s^2$$

give $OI^2 - IH^2 = s^2 - 3(R+r)^2$, and (1) can be replaced by the equivalent

$$s^2 - 3(R+r)^2 = 0 \iff \text{at least one of } \alpha, \beta, \gamma \text{ is } 60^\circ. \quad (2)$$

We give two proofs of (2).

(a) Equivalence (2) follows immediately from the known result [1978: 59]

$$s^2 - 3(R+r)^2 = R^2(2 \cos \alpha - 1)(2 \cos \beta - 1)(2 \cos \gamma - 1).$$

(b) Alternatively, equivalence (2) results if we put $\theta = 60^\circ$ in the following known theorem [1]: *At least one of the angles of a triangle has measure θ if and only if*

$$s = 2R \sin \theta + r \cot \frac{\theta}{2}.$$

Also solved by LEON BANKOFF, Los Angeles, California; J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India (two solutions); D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer. Comments were received from DAN PEDOE, University of Minnesota; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire.

Editor's comment.

Pedoe and Rabinowitz found the problem (without solution) in Hobson [2], and Klamkin found it (with a hint for a solution) in Durell and Robson [3]. The hint in [3] was: Prove that either $AO = AH$ or A, O, I, H are concyclic. To complete this solution, it will be helpful to look at the solution of Crux 724 [1983: 92].

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1. W.J. Blundon, "Generalization of a Relation Involving Right Triangles", *American Mathematical Monthly*, 74 (1967) 566-568.
2. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 201, Ex. 10.
3. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1959, p. 18, Ex. 19.

740. [1982: 107] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Find all functions $y = y(x)$ which are defined and continuous for all $x > 0$ and satisfy $y + 1/y = x + 1/x$.

Solution by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.

The defining equation for the required functions is equivalent to

$$(y - x)(y - \frac{1}{x}) = 0,$$

and so $y = x$ or $1/x$ for all $x > 0$. The continuous functions

$$y_1(x) = x \quad \text{and} \quad y_2(x) = \frac{1}{x}$$

are obvious solutions; and since y_1 and y_2 have the same value at $x = 1$, there are exactly two additional continuous solution functions:

$$y_3(x) = \min\{x, \frac{1}{x}\} \quad \text{and} \quad y_4(x) = \max\{x, \frac{1}{x}\}.$$

Also solved by LEROY F. MEYERS, The Ohio State University; STANLEY RABINOWITZ, Digital Equipment Corp; Merrimack, New Hampshire; and the proposer (two solutions). Incomplete solutions were submitted by MILTON P. EISNER, Mount Vernon College, Washington, D.C.; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; W.C. IGIPS, Danbury, Connecticut; J.A. McCALLUM, Medicine Hat, Alberta; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; and KENNETH M. WILKE, Topeka, Kansas.

Editor's comment.

There was an easily avoidable trap concealed in this problem, and all but one of our incomplete solvers promptly fell into it: they found only the obvious solution functions y_1 and y_2 . The remaining incomplete solver did not fall into the trap, but only because he did not address himself squarely to the problem. He found that all solution functions must satisfy

$$y = \frac{1}{2}(x + \frac{1}{x} \pm |x - \frac{1}{x}|),$$

but he did not specify either the number or the nature of the functions continuous for $x > 0$ that satisfy this relation. In fact, continuity was nowhere mentioned in his solution.

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741. [1982: 135] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the doubly true addition

$$7\cdot\text{THREE} + 5\cdot\text{FIVE} + 4\cdot\text{ELEVEN} = \text{NINETY}.$$

Solution by the proposer.

The proposed addition is equivalent to the following system of equations:

$$12E + 4N = Y + 10c_1, \quad (1)$$

$$c_1 + 11E + 5V = T + 10c_2, \quad (2)$$

$$c_2 + 7R + 5I + 4V = E + 10c_3, \quad (3)$$

$$c_3 + 7H + 5F + 4E = N + 10c_4, \quad (4)$$

$$c_4 + 7T + 4L = I + 10c_5, \quad (5)$$

$$c_5 + 4E = N. \quad (6)$$

We have $N \geq 4E$ by (6), so either $E = 1$ with $4 \leq N \leq 9$, or else $E = 2$ with $N = 8$ or 9. We eliminate all but one of the eight possible values of (E, N) with the help of the values of c_5 , c_1 , and Y resulting from (6) and (1).

$$(E, N, c_5, c_1, Y) = (1, 4, 0, 2, 8) \Rightarrow T = 3 \text{ by (2)} \Rightarrow c_5 \geq 2 \text{ by (5)} \\ \Rightarrow \text{contradiction for } c_5.$$

$$(E, N, c_5, c_1, Y) = (1, 5, 1, 3, 2) \Rightarrow T = 4 \text{ or } 9 \text{ by (2)} \Rightarrow c_5 \geq 2 \text{ by (5)} \\ \Rightarrow \text{contradiction for } c_5.$$

$$(E, N, c_5, c_1, Y) = (1, 6, 2, 3, 6) \Rightarrow N = Y.$$

$$(E, N, c_5, c_1, Y) = (1, 7, 3, 4, 0) \Rightarrow T = 5 \text{ by (2) and } c_4 \geq 1 \text{ by (4)} \\ \Rightarrow L = 0 \text{ by (5)} \Rightarrow Y = L.$$

$$(E, N, c_5, c_1, Y) = (2, 8, 0, 5, 6) \Rightarrow T = 7 \text{ by (2)} \Rightarrow c_5 \geq 4 \text{ by (5)} \\ \Rightarrow \text{contradiction for } c_5.$$

$$(E, N, c_5, c_1, Y) = (2, 9, 1, 6, 0) \Rightarrow T = 3 \text{ or } 8 \text{ by (2)} \Rightarrow c_5 \geq 2 \text{ by (5)} \\ \Rightarrow \text{contradiction for } c_5.$$

The remaining possibilities, $(E, N) = (1, 8)$ and $(1, 9)$, are not so easily disposed of. Our systematic approach is to enumerate V in (2) to obtain c_2 and T ; R in (3) using congruences (mod 5) to obtain c_3 and I ; H in (4) using congruences (mod 5) to obtain c_4 and F ; and finally the tenth letter L is defaulted to the last digit available for allocation. In this way [the details are omitted (Editor)], $(E, N) = (1, 9)$ is eliminated, and $(E, N) = (1, 8)$ leads to the unique solution

$$7 \cdot 57911 + 5 \cdot 3261 + 4 \cdot 101618 = 828154. \quad \square$$

The related doubly true addition

$$2 \cdot \text{THREE} + 8 \cdot \text{FIVE} + 4 \cdot \text{ELEVEN} = \text{NINETY}$$

also has a unique solution:

$$2 \cdot 27511 + 8 \cdot 3981 + 4 \cdot 101814 = 494126.$$

Also solved by MEIR FEDER, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; J.A. McCALLUM, Medicine Hat, Alberta; KENNETH M. WILKE, Topeka, Kansas; and ANNELIESE ZIMMERMANN, Bonn, West Germany. One incorrect solution was received.

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742, [1982: 135] *Proposed by Charles W. Trigg, San Diego, California.*

Find three three-digit primes that are composed of the nine nonzero digits and have a sum that is a triangular number.

I. *Solution by the proposer.*

The upper and lower bounds of the sums of three three-digit primes formed from the nine nonzero digits have been established as 2421 [1] and 999 [2]. The three primes are odd, so their sum is odd. The sum of the digits in the primes is 45, so the sum of the primes is a multiple of 9. Within the established range, the only triangular numbers meeting these requirements are 1035, 1431, 1485, and 1953. There are only four possibilities for the units' digits of the three primes:

$$\left\{ \begin{array}{ll} 1, 3, 7 & 11; \\ 1, 3, 9 & 13; \\ 1, 7, 9 & 17; \\ 3, 7, 9 & 19. \end{array} \right. \quad \text{with sum}$$

Thus 1035 and 1485 are eliminated.

If the sum is 1431, the tens' digits, chosen from the six available to sum to 12 or 22, are 2,4,6 (but then the hundreds' digits would sum to more than 14) or 5,8,9 with 2,4,6 as the hundreds' digits. These can be associated with the units' digits 1,3,7 in $(3!)^2$ ways. Of these, only two sets consist of three primes, namely:

$$491 + 683 + 257 = 1431 = 691 + 283 + 457.$$

If the sum is 1953, the tens' digits are 2,5,7 with 4,6,8 as the hundreds' digits; or else 2,4,8 with 5,6,7 as the hundreds' digits. Of the $2 \cdot (3!)^2$ possible arrangements of these digits with the units' digits 1,3,9, only two consist of three primes, namely:

$$421 + 673 + 859 = 1953 = 821 + 479 + 653.$$

II. *Comment by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

The same Charles W. Trigg proposed this same problem in 1969 [3]. The following year [4], a solution involving a computer search revealed that the only solutions are [the four given above]. These agree with a computer search of my own

that I have just conducted. Trigg, however, put us to shame in [5], where he showed how to arrive at these solutions without the use of a computer.

While I was writing the program, I also found the complete set of three three-digit primes (comprising the nine nonzero digits) that sum to a square. They are:

$$39^2 = 1521 = 281 + 593 + 647,$$

$$= 283 + 547 + 691,$$

$$= 293 + 587 + 641,$$

$$33^2 = 1089 = 149 + 257 + 683.$$

Also solved by SAM BAETHGE, Southwest High School, San Antonio, Texas; the COPS of Ottawa; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; MEIR FEDER, Haifa, Israel; ERNEST W. FOX, South Lancaster, Ontario; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; J.A. McCALLUM, Medicine Hat, Alberta; BOB PRIELIPP, University of Wisconsin-Oshkosh; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; and ANNELIESE ZIMMERMANN, Bonn, West Germany.

Editor's comment.

Not all solvers found all four answers. The proposer's solution given here is an improvement on his earlier solution in [5]. But the improvement does not appear to be enough to justify a republication of the problem, which must therefore be due to an oversight of our proposer.

A related problem by our proposer [6] asks for a unique triangular number, consisting of nine distinct digits, which is the concatenation (rather than the sum) of three primes.

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4. Solution to Problem 735, *Mathematics Magazine*, 43 (March 1970).
5. Charles W. Trigg, comment on Problem 735, *Mathematics Magazine*, 43 (November 1970) 289.
6. Solution to Problem 769 (proposed by Charles W. Trigg), *Mathematics Magazine*, 44 (May 1971) 168.

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