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Volume 11 of *Mathematical Spectrum* will consist of three issues of which this is the second. The first issue was published in September 1978, and the third will appear in May 1979.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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Hicks Building, The University, Sheffield S3 7RH.

Mathematical Spectrum Awards for Volume 10

In 1977 the editors of *Mathematical Spectrum* instituted two annual prizes for contributors who are still at school or are undergraduates in colleges or universities. One prize, to the value of £20, is for an article published in the magazine; another of £10 is for a letter or the solution of a problem. Volume 10 did not contain any articles by authors eligible for the £20 prize. However, the editors have decided to make two awards for letters: a prize of £10 to K. H. Yim for his letter 'Latin Squares and Magic Squares' (Volume 10, Number 2, pp. 61–63), and a prize of £5 to J. R. Ramsden for his letter 'The General Quartic' (Volume 10, Number 1, pp. 25–26).

The 20th International Mathematical Olympiad

JOHN HERSEE
School Mathematics Project

Romania, where the first International Olympiad was held in 1959, was again the host country in 1978. Seventeen countries participated, each sending a team of eight pre-university students.† The team from Great Britain included three members of the 1977 team and was placed third. Romania was first and the U.S.A. second.

Each team is accompanied by a leader and deputy leader. The leaders form the international jury which selects the questions for the competition and agrees on the final results. Participating countries are invited to submit questions for the competition and from these the jury selects three for each of the two four-hour papers. Of course, the questions have to be carefully checked for correctness, and their wording (in English) agreed before being translated into different languages for the competitors.

The two papers are taken on consecutive days and the scripts of each team are first assessed by the team leader and deputy. In addition, for each question the host country provides a panel of two or three coordinators. These coordinators review the assessments of the team leaders to ensure a uniform standard for all teams. It will be appreciated that, while mathematical symbols are generally unambiguous and widely understood, students write in their own language, so the coordination process takes some time. The methods used and the details of the working are carefully studied by the coordinators, for a correct answer is by no means all that is required. The solution must be carefully argued, with no lapses or omissions in the reasoning; any such flaws are penalised. In addition, elegance, originality, or a generalisation of the problem set can earn a special prize.

† Cuba sent only four competitors.

This year only one competitor, an American, scored full marks. Richard Borcherds, of the British team, lost one point for the omission of part of a proof. However, he produced an extremely neat solution to Question 6 and for this was awarded a special prize. First, second and third prizes are awarded to all students who achieve appropriate standards; the British team received one first, two second and two third prizes this year.

After the papers have been taken there is a good deal of time during which the teams are entertained. Our hosts provided a programme of visits to museums and places of interest, and a stay of three days on the Black Sea coast. In the meantime, the work of marking the scripts and agreeing on the results was going ahead at Busteni in the Carpathians where the team leaders, deputies and coordinators stayed. For the last two days, everyone met again in Bucharest. There was time for more sightseeing, shopping and two specially arranged concerts, before the presentation of prizes and the final dinner brought our very enjoyable visit to a close.

The results of the different teams reveal interesting patterns of success in the various questions, reflecting differences in mathematics syllabuses and team selection and training methods. Almost all countries begin their selection process with a national mathematical competition, followed by further tests, but some follow up with an intensive training period for the team.

In Great Britain selection of the team begins with the National Mathematics Contest (NMC) which takes place in March. Any school pupil may enter for this competition which is a 'multiple choice' type of paper.† Those who score high marks in the NMC are invited to enter for the British Mathematical Olympiad (BMO); the best entrants here take the Further International Selection Test (FIST) on which the team is selected. A problem-solving correspondence 'course' is the only form of training given to the team, but it is important to emphasize that participation in this course is not restricted to members of the team. (A stamped, addressed envelope sent to Dr D. Monk, University Department of Mathematics, James Clerk Maxwell Building, Mayfield Road, Edinburgh, will bring details of how to take part.) The limited training that the British Team receives makes our results over the past years all the more pleasing. We shall hope to do well in 1979 when the competition is held in London for the first time.

Questions set in the 1978 Olympiad

1. m and n are natural numbers with $n > m \geq 1$. In their decimal representations, the last 3 digits of 1978^m are equal, respectively, to the last 3 digits of 1978^n. Find m and n such that $m + n$ has its least value.

2. P is a given point inside a given sphere and A, B, C are any three points on the sphere such that PA, PB and PC are mutually perpendicular. Let Q be the vertex diagonally opposite to P in the parallelepiped determined by PA, PB and PC. Find the locus of Q.

† Details of how to enter can be obtained from the Mathematical Association, 259 London Road, Leicester.

3. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), \ldots, f(n), \ldots\}$, $\{g(1), g(2), \ldots, g(n), \ldots\}$, where $f(1) < f(2) < \cdots < f(n) < \cdots$, $g(1) < g(2) < \cdots < g(n) < \cdots$, and $g(n) = f(f(n)) + 1$ for all $n \geq 1$. Determine $f(240)$.

4. In the triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of the triangle ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of the segment PQ is the centre of the incircle of the triangle ABC.

5. Let $\{a_k\}$ $(k = 1, 2, 3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

6. An international society draws its members from six different countries. The list of members contains 1978 names, numbered 1, 2, ..., 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

DETAILED RESULTS

	Question							Prizes			
	1	2	3	4	5	6	Total	1st	2nd	3rd	Special
Romania	46	27	40	40	48	36	237	2	3	2	
U.S.A.	44	36	47	39	48	11	225	1	4	1	
Great Britain	43	20	45	36	45	12	201	1	2	2	1
Vietnam	45	37	15	40	48	15	200		2	6	
Czechoslovakia	41	37	24	40	45	8	196		2	3	
West Germany	34	29	32	36	45	8	184	1	1	2	1
Bulgaria	44	27	18	39	48	6	182		1	3	
France	41	26	30	34	41	7	179		4	2	
Austria	38	23	24	35	43	11	174		3	2	
Yugoslavia	43	14	23	40	40	11	171		1	2	
Holland	33	12	26	31	43	12	157		2		2
Poland	36	18	26	26	48	2	156			2	
Finland	31	14	25	20	28	0	118			1	1
Sweden	26	3	13	32	41	2	117				
Turkey	11	3	9	18	19	6	66				
Mongolia	16	7	3	15	17	3	61				
[Cuba	15	13	5	5	20	10	68			1	]
Maximum	48	56	64	40	48	64	320				

Vieta's Iterative Process for π, with Errors

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The Frenchman François Viète (1540–1603), better known today by the latinized form of his name—Franciscus Vieta—was one of the leading algebraists of the sixteenth century.

Born at Fontenay-le-Comte, Poitou, in 1540, Vieta studied the law, and on completion of his studies he became an advocate in his native town. A well-organised legal career led to Vieta's subsequently acquiring the post of royal privy councillor.

Like so many of his time, Vieta 'dabbled' with mathematics as an amateur. His first major achievement of a mathematical nature was the discovery of the key to a cipher being used by the Spanish. Such a discovery was invaluable to the French, and certainly did much to accelerate Vieta's standing in his society. Phillip II of Spain was convinced of the safety of his cipher, so convinced that when it was broken he accused the French of using sorcery. This accusation was made in the form of an official complaint to the Pope. The paths of politics, religion and science have crossed many times!

Had Vieta achieved no more he would have had some say in political, if not mathematical history. As it happened, however, the advocate turned more towards mathematics later in his life. A very brief account of some of Vieta's contribution to mathematics can be found in reference 1.

At an Open University summer school a few years ago, I gave a lecture on the history of π, including in it an account of a geometrical construction used by Vieta to determine π. The essential idea of Vieta's method is that a regular n-gon (a polygon with n sides) is inscribed in a circle of unit radius. As $n \to \infty$ the perimeter of the n-gon tends to the perimeter 2π of the circle.

Letting X_n be the length of a side of the n-gon, we can find a relationship between X_n and X_{2n} as follows.

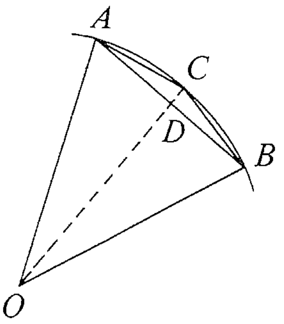

Figure 1

In Figure 1, $OA = OB = OC = 1$, $AB = X_n$, $AC = CB = X_{2n}$. Then, since

$$AD\,.\,DB = (1 + OD)\,.\,DC = (2 - DC)\,.\,DC$$

and

$$DC^2 = X_{2n}^2 - \tfrac{1}{4}X_n^2$$

it can be readily established that

$$X_{2n} = \sqrt{2 - \sqrt{4 - X_n^2}}.$$

The above relationship gives an iterative process for determining successive values of X_n, and as $n \to \infty$ so $nX_n \to 2\pi$. Thus, for sufficiently large values of N, the difference between $N \,.\, X_N/2$ and π will be very small.

Such then was Vieta's constructional method for determining π. It must have been time-consuming calculating the successive values of X_n.

At the summer school I made a quick suggestion that the aforesaid construction, and algebraic result, could be used as the heart of a brief computer program to evaluate π. Some of the people at the summer school responded to my suggestion. One of the programs offered by a student, and later repeated by a colleague of mine, was the BASIC program set out in Figure 2.

```
10  LET N=6
20  LET X=1
30  LET C=0
40  LET X=SQR(2-SQR(4-X*X))
50  LET N=2*N
60  LET C=C+1
70  LET P=N*X/2
80  PRINT C,N,X,P
90  IF C<30 THEN 40
100  END
```

Figure 2

In theory this algorithm should give a very accurate value of π. In practice the summer school student, and my colleague, obtained the result shown in Table 1.

TABLE 1

C	N	X	P
1	12	.517638	3.10583
2	24	.261052	3.13263
3	48	.130806	3.13933
4	96	.065437	3.14097
5	192	3.27221E−02	3.14132
6	384	1.63556E−02	3.14028
7	768	8.17051E−03	3.13748
8	1536	4.08525E−03	3.13748
9	3072	2.01324E−03	3.09233
10	6144	9.76562E−04	3
11	12288	4.88281E−04	3
12	24576	0	0
13	49152.	0	0
14	98304.	0	0
15	196608.	0	0
16	393216.	0	0

and so on to $C=30$.

The initial choices of $n = 6$ and $X = 1$ were both fair and fairly obvious. Certainly this initial choice does not account for the 'error in π'. Indeed, the new initial conditions $n = 4$ and $X = 1.4142$ gave, for the same program, the results in Table 2.

TABLE 2

C	N	X	P
6	256	2.45407E−02	3.14121
7	512	1.22655E−02	3.13996
8	1024	6.11815E−03	3.13249
9	2048	3.04932E−03	3.1225
10	4096	1.46484E−03	3
11	8192	6.90534E−04	2.82843
12	16384	0	0
13	32768.	0	0

and so on as before.

In the program offered, a counter C was used to pull out of the loop. Another standard way of pulling out of a loop is based on asking if two successive values of whatever is being computed are more or less equal. If we denote the computed approximation for π corresponding to n as P_n, then we pull out of the loop when $|P_{2n} - P_n| < \varepsilon$, for some arbitrarily chosen ε (small). Often as well, we are not interested in all the approximations to our result. That is, in the given program we do not really require all the values given in the P column. Using such a procedure for terminating the operation and asking only for a final result could, for the given problem, quite easily offer the result 3.13748 as a 'best' approximation for π. It can be seen that to obtain this result we have passed through the more accurate value of 3.14132. We have, in fact, overshot the 'correct' result. For the problem given here it does not really matter that we have overshot the correct answer, nor does the error matter. We are not solving a particularly important problem and, after all, we know the correct result in advance. Needless to say, though, the computer is used to solve many significant problems to which we do not know the answer. Similar errors can arise.

The relationship:

$$X_{2n} = \sqrt{2 - \sqrt{(4 - X_n^2)}}$$

between X_n and X_{2n} has been written in a very neat and tidy form. There has always been a tendency to tidy up algebraic forms in mathematics. In fact, in my day we spent quite some time at school simplifying such algebraic expressions. But the relationship between X_n and X_{2n} can also be written as

$$X_{2n} = \sqrt{\tfrac{1}{4}X_n^2 + (1 - \sqrt{1 - \tfrac{1}{4}X_n^2})^2}.$$

This form could hardly be regarded as neat and tidy like the previous one. With the initial values $n = 6$ and $X = 1$ the program is now that of Figure 3.

```
10 LET N=6
20 LET X=1
30 LET C=0
40 LET Y=1-SQR(1-X*X/4)
50 LET X=SQR(X*X/4+Y*Y)
60 LET N=2*N
70 LET C=C+1
80 LET P=N*X/2
90 PRINT C,N,X,P
100 IF C<30 THEN 40
110 END
```

Figure 3

This gave the results shown in Table 3.

TABLE 3

C	N	X	P
1	12	.517638	3.10583
2	24	.261052	3.13263
3	48	.130806	3.13935
4	96	6.54382E−02	3.14103
5	192	3.27235E−02	3.14145
6	384	1.63623E−02	3.14156
7	768	8.18121E−03	3.14158
8	1536	4.09061E−03	3.14159
9	3072	2.04531E−03	3.14159
10	6144	1.02265E−03	3.14159
11	12288	5.11327E−04	3.14159
		and so on until	
30	6.44245E+09	9.75279E−10	3.14159

In a way it is strange that the two equivalent forms

$$\sqrt{2-\sqrt{4-X_n^2}} \qquad \text{and} \qquad \sqrt{\tfrac{1}{4}X_n^2+(1-\sqrt{1-\tfrac{1}{4}X_n^2})^2}$$

should produce different results. Let us now examine, briefly, why this is so. Indeed we are attempting to answer the question 'why is the more clumsily written form the better of the two?'

First of all, every time a machine makes a calculation there is the possibility of some form of rounding error. For an iterative process, like the one described here, each time we pass through a loop we can experience an accumulation of such errors. It is quite possible for this accumulation of errors to be sufficient to offset any gain in accuracy obtained by passing through the loop some more times.

The process shown here certainly converges to π. Indeed if P_n and P_{2n} are (in theory) two successive approximations for π then

$$|P_{2n}-\pi| < |P_n-\pi| \qquad \text{for all } n.$$

But each P_n is calculated by nX_n, where n is large and X_n small. So a small error in X_n can bring a rather significant error in P_n.

Let us assume that the theoretically calculated values of P_n are, necessarily, subject to an error e_n, so that we are offered the result $P_n + e_n$. Then, although

$$|P_{2n} - \pi| < |P_n - \pi| \qquad \text{for all } n,$$

it is quite possible to have

$$|P_{2n} + e_{2n} - \pi| > |P_n + e_n - \pi|.$$

In other words, each time we pass through a loop we gain something due to the convergence of the iterative process; yet on the other hand we can lose something due to the accumulation of computed errors. It brings us back to the question of 'swings and roundabouts'.

Let us look a little more closely at the two processes

$$X_{2n} = \sqrt{2 - \sqrt{4 - X_n^2}}$$

and

$$X_{2n} = \sqrt{\tfrac{1}{4}X_n^2 + (1 - \sqrt{1 - \tfrac{1}{4}X_n^2})^2}.$$

For the first process, given input data X_n, the computer, or a human being, will calculate

X_n^2	which is going to become small,
$4 - X_n^2$	which will be more or less 4,
$\sqrt{4 - X_n^2}$	which will be more or less 2,
$2 - \sqrt{4 - X_n^2}$	which is virtually zero.

In the second case, at the time when $2 - \sqrt{4 - X_n^2}$ is virtually zero, in the second loop, then $1 - \sqrt{1 - \frac{1}{4}X_n^2}$ is also virtually zero, i.e. $X_{2n} \simeq \frac{1}{2}X_n$. Consequently, as we double n we halve X_n and leave the product nX_n invariant. Since $X_{2n} \simeq \frac{1}{2}X_n$ the computed relative error in X_{2n} will be virtually the same as that in X_n. This process will thus, having reached an optimum accuracy of π, tend to stabilise on that value. But this will not be so for the first process. As $2 - \sqrt{4 - X_n^2}$ approaches zero the relative error in X_n^2, and consequently P_{2n} can become greater than the relative error in X_n (and P_n). This, then, is the key to the discrepancy in the two results.

The two subjects, error analysis and convergence of iterative procedures, are delicate matters, and indeed topics for contemporary research. There are things we can do, a change of programming language maybe; but as far as I know there is no simple answer to the problems. Certainly we have to be alert to the situation. When all is said and done we have to rely to a large extent on our experience and feel for the circumstances. Development of such intuition allows us to view the two forms of X_{2n}

$$\sqrt{2 - \sqrt{4 - X_n^2}} \qquad \text{and} \qquad \sqrt{\tfrac{1}{4}X_n^2 + (1 - \sqrt{1 - \tfrac{1}{4}X_n^2})^2}$$

and make a (mental) judgement along the following lines:

> We are dealing with some very small numbers which will be multiplied by some fairly large ones. This can lead to substantial errors. The second form is

explicitly dominated by $\frac{1}{2}X_n$. This places a bound on the relative error for our result. Such a bound is not offered, not explicitly offered, by the first form. Thus, if one of these two forms is to give us some odd results it is more likely to be the first. So I'll choose the second.

Modern computers can work to a great degree of accuracy and at a shuddering speed. They have the power to solve more problems in a few seconds than a man could in a life-time. Yet, unless handled with a degree of sensitivity, the same technology can make an equivalent number of errors. Life seems to be full of similar dichotomies.

I should like to thank the referee for the following comment: There is a useful rule that to reduce rounding errors in a program avoid

(a) taking the difference of two nearly equal numbers
(b) dividing by a small number.

Reference

1. J. F. Scott, *A History of Mathematics* (Taylor and Francis, London, 1958).

A Biologist's Magic Urn

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Students of biology and genetics will know that several interesting combinatorial and probabilistic processes arise in the formulation of problems in their fields. We describe here one such process, presented for convenience in the abstract form of coloured balls in an urn. In some brief concluding remarks, we mention the significance of our conclusions to certain biological problems.

Imagine an urn containing a fixed number M of balls. Each is of a single colour, but several balls may have the same colour. Thus if $M = 100$ the urn might contain, for example 59 red, 27 blue, 12 yellow, 1 green and 1 white balls at a given time. At the discrete time points $t = 1, 2, 3, \ldots$, let us perform the following operation.

First, a ball is taken at random from the urn and replaced after its colour is noted. A further ball is now drawn at random from the urn; this may possibly be that drawn initially. This second ball is now repainted: with probability $1 - u$ it is given the same colour as the first one drawn, and with probability u an entirely new colour not currently or previously represented in the urn. It is then replaced in the urn. This sequence of operations is repeated indefinitely in what we might call the *colour replacement process*. In biological applications u is usually rather small, say of order 10^{-5}.

The actual colours present at any one time are of minor interest to us: indeed it is clear that any one colour, once introduced, will persist during the colour replacement process for a random period of time and then be eliminated, never to reappear. On the other hand the *configurations* of the numbers of balls of different colours are of considerable interest, and a given configuration can occur at any time point. In the above example, the observed configuration of $M = 100$ balls partitioned into $K = 5$ colours, can be written $\{59, 27, 12, 1, 1\}$, but since the actual colours are unimportant we could write it equally well as $\{12, 1, 27, 1, 59\}$ or indeed in several other ways.

Once the colour replacement process has gone on for a sufficiently long time, a stationary distribution of configurations will emerge; our main aim is to examine some of the properties of this distribution. First, the number of possible configurations is $p(M)$, the number of partitions of M into positive integers. Exact and asymptotic values for $p(M)$ may be found in Abramowitz and Stegun (1964) (reference 1): thus, for example, $p(100) = 190, 569, 292$. The configuration given above is just one of these. If we write an arbitrary configuration as $\{M_1, M_2, \ldots, M_K\}$, where $\sum_{i=1}^{K} M_i = M$, it turns out that the stationary configuration probability distribution, established by Trajstman (1974) (reference 4), is

$$\Pr\{M_1, \ldots, M_K\} = \binom{v + M - 1}{M}^{-1} v^K (M_1 M_2 \ldots M_K)^{-1} \prod_{j=1}^{M} (\alpha_j!)^{-1}. \quad (1)$$

In this formula $v = Mu/(1-u)$, K is the (random) number of colours in the configuration in question and α_j is the number of times the number j (of balls of a particular colour) appears in the configuration; thus $\alpha_1 = 2$, $\alpha_{12} = \alpha_{27} = \alpha_{59} = 1$, with all other α_j's $= 0$ in the above configuration. We now list some interesting properties of the stationary distribution (1).

Property 1. Suppose a sample of m balls is taken at random (without replacement) from the urn once stationarity has been reached. Such a sample will yield a configuration $\{m_1, m_2, \ldots, m_k\}$ where $\sum_{i=1}^{k} m_i = m$ and k is the (random) number of colours observed in the sample. Then the probability of this sample is

$$\Pr\{m_1, m_2, \ldots, m_k\} = \binom{v + m - 1}{m}^{-1} v^k (m_1 m_2 \ldots m_k)^{-1} \prod_{j=1}^{m} (\alpha_j!)^{-1}, \quad (2)$$

where α_j is now the number of times the number j appears in the *sample* configuration. Hence the sampling distribution has a form identical with the distribution for the entire urn, thus obeying the important statistical requirement of partition consistency.

Property 2. Suppose a sample of m balls has been taken as above, revealing k colours and the configuration $\{m_1, m_2, \ldots, m_k\}$. A further $(m + 1)$th ball is now taken from the urn. What is the probability that it will be of a colour different from the k colours noted for the first m balls? Now the probability of drawing the configuration $\{m_1, m_2, \ldots, m_k, 1\}$ with $m + 1$ balls can be found by replacing m by $m + 1$ and $\{m_1, \ldots, m_k\}$ by $\{m_1, \ldots, m_k, 1\}$ in (2). The probability that a singleton is drawn last is $(\alpha_1 + 1)/(m + 1)$, since $\alpha_1 + 1$ is the number of singletons in the configuration $\{m_1, m_2, \ldots, m_k, 1\}$. Thus the probability of drawing this configuration with a singleton as the last ball is

$$\binom{v + m}{m + 1}^{-1} v^{k+1}(m_1 \ldots m_k)^{-1}(\alpha_1 + 1)(m + 1)^{-1}\{(\alpha_1 + 1)!\}^{-1} \prod_{j=2}^{m} (\alpha_j!)^{-1}$$

$$= \binom{v + m}{m + 1}^{-1} v^{k+1}(m_1 \ldots m_k)^{-1} \prod_{j=1}^{m} (\alpha_j!)^{-1}. \quad (3)$$

The probability, then, that a new colour is drawn in the $(m + 1)$th draw, given the configuration $\{m_1, \ldots, m_k\}$ at the first draw, is the ratio of this to (2). This reduces to

$$\Pr\{\text{new colour in } (m + 1)\text{th draw}\} = v/(v + m). \quad (4)$$

Note the surprising fact that this probability is entirely independent of what was observed in the first m draws, and in particular of the number of colours observed in these! This property in fact characterizes the distribution (1), at least under rather mild restrictions; for any other distribution, the probability of observing a new colour in the $(m + 1)$th draw would depend, as intuition perhaps suggests, on what had been observed in the first m draws.

Property 3. Let us next suppose that a single ball is taken at random from the urn and its colour noted. All balls of this colour are then removed from the urn. If $n (1 \leqslant n \leqslant M)$ balls are so removed, we seek the probability distribution of the colour configuration of the remaining $M - n$ balls. It is not hard to see, from (1), that this distribution is precisely the same as that which would arise if the urn had only contained $N - n$ balls throughout the entire process! This suggests a form of non-interference between the colours which is similar to that of Property 2 and which again characterizes the distribution (1). We refer to this property in a biological context later.

Property 4. Suppose $v = Mu/(1 - u)$ is unknown, and we wish to estimate it, given a sample observation $\{m_1, m_2, \ldots, m_k\}$. Then v must be estimated using k only; to use the numbers $m_1, m_2, \ldots, m_k$ in the estimation procedure merely introduces irrelevant information which reduces its efficiency. This fact can be established from (2) by showing that the conditional distribution of $m_1, \ldots, m_k$, given k, is independent of v; thus, given k, the numbers $m_1, \ldots, m_k$ can give no further information about v.

Several of the properties mentioned above are counter-intuitive. This is perhaps even more so for properties associated with the *history* of the colour replacement process. Several examples are given by Kelly (1976) (reference 3); we shall note only one here.

Property 5. At any time one of the colours in the urn will be the oldest, so that it will have been introduced before any of the other colours present. The age of the oldest colour at time t is a random variable, and we would expect its probability distribution to depend on its current frequency: the more frequent are the balls of a particular colour C, the older we might perhaps expect C to be. In fact this is not so; the age of the oldest colour is completely independent of its current frequency. Thus if we know C to be the oldest colour, then the probability that its age is j does not depend on the frequency of the balls of this colour.

As was noted earlier, the urn model is an abstract form of various population models in biology. In a genetical context, the balls in the urn are the genes in a population and the ball colour is the gene type. Alternatively the balls in the urn could represent animals in a given area, and their colour denote the species of the animal. In the latter context, Property 3 shows, provided the replacement procedure described is biologically realistic, that if all the animals of a given species are withdrawn from the area, the probability configuration of the numbers of animals in the remaining species is the same as if the species withdrawn had never existed. This characterizes the concept of non-interaction between species, and may also be connected with Property 2. In the genetical context, u is a mutation rate and Property 4 shows that this must be estimated using only the numbers of different types of genes rather than (as often in past biological practice) their frequencies. The remaining properties, as well as many others, also have biological relevance; the interested reader may pursue these further in the references below.

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, 1964).
2. N. L. Johnson and S. Kotz, *Urn Models and their Applications* (Wiley-Interscience, New York, 1977).
3. F. Kelly, On stochastic population models in genetics, *J. Appl. Prob.* **13** (1976), 127–131.
4. A. C. Trajstman, On a conjecture of G. A. Watterson, *Adv. Appl. Prob.* **6** (1974), 489–493.

A Bayesian Look at the Jury System

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While searching for examples to illustrate the application of Bayes' Theorem for a class in statistics at our college, I recalled seeing an example related to trial by jury in some forgotten text. The general theme involved analysing the situation from the posterior probability viewpoint; that is, if a defendant is found guilty by the jury what chance is there that the defendant is actually innocent? Conviction of innocent persons is obviously a very serious error and one expects that our justice system reduces the chance of such events to very small (ideally zero) values.

A related (but perhaps not so serious) problem is the release of guilty persons by the system and again one expects that such occurrences will be rare (and in the eyes of some) preferably not occur at all.

In attempting to analyse both these questions several assumptions about actual probabilities of certain events is inevitable and lead to the present discussion. Let us define the events involved as follows:

G^+ = event the defendant is in fact guilty,
G^- = event the defendant is in fact not guilty,
J^+ = event the jury finds the defendant guilty,
J^- = event the jury finds the defendant not guilty.

Then $(J^+|G^+)$ is the event the jury finds a guilty person guilty, $(J^-|G^+)$ is the event the jury finds a guilty person not guilty and so forth.

For discussion purposes, consider the evaluation of $P(G^+|J^-)$; that is, the probability the defendant actually is guilty given that the jury finds him not guilty.

From Bayes' Theorem we have:

$$P(G^+|J^-) = \frac{P(J^-|G^+).P(G^+)}{P(J^-|G^+).P(G^+) + P(J^-|G^-).P(G^-)}.$$

Now $P(J^-|G^+)$ measures the extent to which the jury acquits a guilty person and one expects that its value be small (nearly zero). In fact, of course, the actual value is not known but values between zero and 0·10 might be reasonable. Similarly $P(J^-|G^-)$ measures the extent to which the jury frees an innocent person and one would expect its value to be high (nearly unity). Again, the actual value is not known but values between 0·9 and 0·99 might be reasonable. Finally $P(G^+)$ measures the extent to which the defendant is actually guilty and really corresponds to the efficiency of police detection methods since they presumably only charge the person(s) involved if their investigation leads to that conclusion. Hopefully the question of police corruption and ineptitude is greatly exaggerated and if a person is brought to trial it is in the belief that such a person is in fact guilty. Values for $P(G^+)$ between 0·8 and 0·99 might be reasonable. I will return to this question in the next part of the article for further development.

To illustrate, let us take the following values:

$$P(J^-|G^+) = 0{\cdot}05,\ P(J^-|G^-) = 0{\cdot}99,\ P(G^+) = 0{\cdot}90.$$

Then

$$P(G^+|J^-) = \frac{0{\cdot}05 \times 0{\cdot}90}{(0{\cdot}05 \times 0{\cdot}90) + (0{\cdot}99 \times 0{\cdot}10)} \qquad \text{since } P(G^-) = 1 - P(G^+)$$

$$= \frac{0{\cdot}045}{0{\cdot}045 + 0{\cdot}099} = \frac{0{\cdot}045}{0{\cdot}144} = 0{\cdot}3125.$$

That is, in about 31 % of cases where the jury acquits the defendant, the person was in fact guilty. No wonder it is difficult (and rightly so it would seem) for the average person to believe that a person acquitted by the system is truly innocent. Or conversely, that an acquitted person finds it difficult to shake off the taint of being brought to trial.

For the same assumed values consider now the value of $P(G^-|J^+)$. Again we have by Bayes' Theorem:

$$P(G^-|J^+) = \frac{P(J^+|G^-) \,.\, P(G^-)}{P(J^+|G^-) \,.\, P(G^-) + P(J^+|G^+) \,.\, P(G^+)};$$

since $P(J^+|G^-) = 1 - P(J^-|G^+)$ and $P(J^+|G^+) = 1 - P(J^-|G^+)$ we can write this as

$$P(G^-|J^+) = \frac{0{\cdot}01 \times 0{\cdot}10}{(0{\cdot}01 \times 0{\cdot}10) + (0{\cdot}95 \times 0{\cdot}90)}$$

$$= \frac{0{\cdot}001}{0{\cdot}001 + 0{\cdot}855}$$

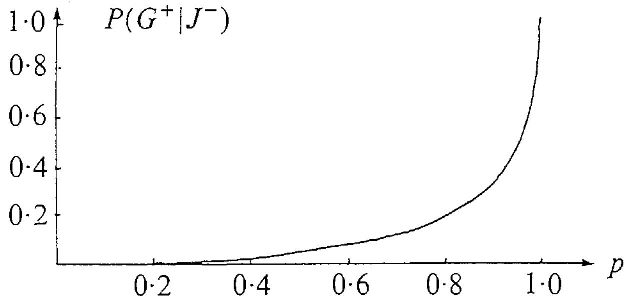
$$= 0{\cdot}0012.$$

Here we see the virtues of the system illustrated since this result indicates that only about 1 in 1000 defendants are incorrectly convicted by the jury. Although even one is undesirable (as may have happened in the Christie–Evans case) it reflects the axiom of 'innocent until proved guilty' in a very material way.

Now the above calculations only intend to serve as illustrations of what might possibly be the case in practice and the second part of this article looks at implications for varying values of the assumed values, in particular, that for $P(G^+)$. If $P(G^+) = p$, say, then for the first case we have:

$$P(G^+|J^-) = \frac{0{\cdot}05p}{0{\cdot}05p + 0{\cdot}99\,(1-p)} = \frac{5p}{5p + 99 - 99p} = \frac{5p}{99 - 94p}.$$

It might be interesting to ask how varying values of p affect the value of $P(G^+|J^-)$ and the graph in Figure 1 indicates the variation. This result reflects the somewhat paradoxical result in that as police efficiency increases ($p \to 1$) the probability that guilty persons are acquitted increases towards certainty! It would appear at first sight that the police ought to do very little work since the more efficient they become the more likely it is that guilty persons will be acquitted!



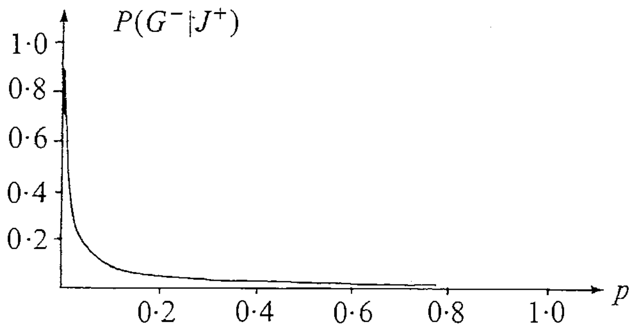
p	0	0·2	0·4	0·6	0·8	0·9	0·95	0·99	1·0
$P(G^+\|J^-)$	0	0·012	0·033	0·070	0·17	0·31	0·49	0·83	1·0

Figure 1. Graph and table for $P(G^+|J^-)$ versus p.

Approaching the second case in the same way we find:

$$P(G^-|J^+) = \frac{0{\cdot}01(1-p)}{0{\cdot}01(1-p)+0{\cdot}95p} = \frac{1-p}{1-p+95p} = \frac{1-p}{1+94p}.$$

The graph showing this relationship is given in Figure 2.



p	0	0·01	0·02	0·05	0·10	0·3	0·5	0·8	1·0
$P(G^-\|J^+)$	1·0	0·51	0·34	0·17	0·087	0·024	0·01	0·003	0

Figure 2. Graph and table for $P(G^-|J^+)$ versus p

This result is more appealing in the sense that as the police work more efficiently the probability of an innocent person being convicted approaches zero.

Interested readers might consider generalising the situation further by not assuming actual values for $P(J^-|G^+)$, etc., as I have done in this article and see what apparent implications there are for the events of interest as these other unknown probabilities vary. I invite readers troubled by the paradoxical result found above to discover for themselves a meaningful explanation by considering actual values as p approaches unity and interpret them accordingly.

In conclusion, it would be interesting to hear from readers who have thought along the above lines and may have interesting comment, actual data or related problems.

References

1. J. L. Hodges, Jr. and E. L. Lehmann, *Basic Concepts of Probability and Statistics*, (Holden-Day, San Francisco, 1970).
2. F. Mosteller, R. E. K. Rourke and G. B. Thomas, Jr., *Probability with Statistical Applications* (Addison-Wesley, Reading, Mass., 1973).

Board Polynomials and some of their Applications

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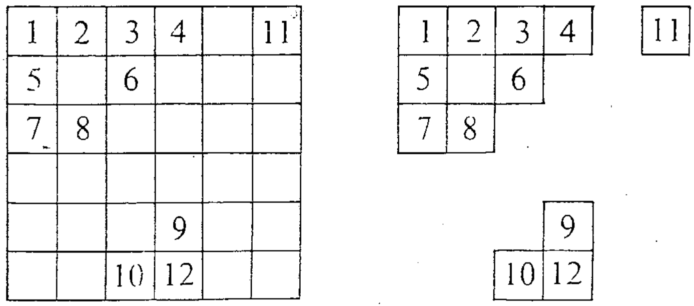
1. Definition of board polynomials

A rook is a chess piece that is restricted to move along horizontal or vertical lines and can take other pieces situated in the same row or column as itself. We shall be interested in finding the number of ways of placing n rooks on arbitrarily shaped boards in such a way that no rook can be taken by any other rook. Consider for example Figure 1(a), where the board consists of the numbered squares. A rook placed on square 3 can take any other rook placed on the squares 1, 2, 4, 6, 10 or 11. Subsequently we shall, for simplicity, often omit the background boards in our sketches. Thus the board of Figure 1(a) would appear as in Figure 1(b).



(a) (b)

Figure 1

This leads to the concept of a rook polynomial $R(x, C)$ for an arbitrarily shaped board C. The polynomial is defined by

$$R(x, C) = r_0(C) + r_1(C)x + r_2(C)x^2 + \cdots + r_m(C)x^m,$$

where $r_k(C)$ is the number of ways of placing k non-taking rooks on C, and m is the largest number of non-taking rooks that can be placed on the board. Note that $r_k(C)$ is zero for $k > m$ and, by convention, $r_0(C)$ is defined to be 1. Thus we may write

$$R(x, C) = \sum_{k=0}^{\infty} r_k(C)x^k.$$

In chess the bishop moves and takes only along diagonal lines and the queen moves and takes along horizontal, vertical and diagonal lines; and we can define the

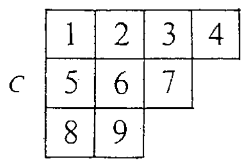


Figure 2

bishop polynomial $B(x, C)$ and the queen polynomial $Q(x, C)$ in the same way as the rook polynomial. For example, when C is the board in Figure 2,

$$\begin{aligned}R(x, C) &= 1 + 9x + 15x^2 + 9x^3,\\ B(x, C) &= 1 + 9x + 26x^2 + 26x^3 + 8x^4,\\ Q(x, C) &= 1 + 9x + 9x^2.\end{aligned}$$

It is natural to ask if the queen polynomial is related to the rook and bishop polynomials, since the queen combines the roles of the bishop and the rook. We shall look at this relationship in Section 2. In later sections we shall see how these polynomials can be applied to some permutation problems, and we shall discuss the failure of certain cellular structures.

For complicated boards the calculation of $R(x, C)$ is simplified by the use of the following two theorems.

Theorem 1. Given a board C, choose a square s of C. Let D_R denote the board obtained by deleting from C every square in the same row or column as s, including s itself. Let E denote the board obtained from C by deleting only the chosen square s. Then

$$R(x, C) = xR(x, D_R) + R(x, E).$$

Proof. When k non-taking rooks are placed on C, then the square s is either used or not used. If the square s is not used then the k rooks can be placed on C in $r_k(E)$ ways. If the square s is used, then there are $r_{k-1}(D_R)$ ways of placing the other $k - 1$ rooks on C. Thus

$$r_k(C) = r_k(E) + r_{k-1}(D_R) \qquad (k > 0),$$

so that

$$\begin{aligned}R(x, C) &= 1 + \sum_{k=1}^{\infty} r_k(C)x^k\\ &= 1 + \sum_{k=1}^{\infty} r_k(E)x^k + \sum_{k=1}^{\infty} r_{k-1}(D_R)x^k\\ &= R(x, E) + xR(x, D_R).\end{aligned}$$

If a board C is made up of two sub-boards F_R and G_R such that no row or column in the background board of C intersects both F_R and G_R, then F_R and G_R are said to be *non-interfering*. Figure 3 illustrates such a board C; here F_R consists of the squares 1, 2, 3, 4 and G_R consists of the squares 5, 6, 7, 8, 9.

Theorem 2. If C is composed of two non-interfering boards F_R and G_R, then

$$R(x, C) = R(x, F_R)\, R(x, G_R).$$

Proof. We can place k rooks on C by placing t rooks on F_R in $r_t(F_R)$ ways and placing $k - t$ rooks on G_R in $r_{k-t}(G_R)$ ways, for any t with $0 \leq t \leq k$. Thus, summing over all possible values of t we have

$$r_k(C) = r_0(F_R)r_k(G_R) + r_1(F_R)r_{k-1}(G_R) + \cdots + r_k(F_R)r_0(G_R).$$

The expression on the right is the coefficient of x^k in the product $R(x, F_R)R(x, G_R)$ and so the result follows.

We can establish similar theorems for bishop and queen polvnomials. For example, Theorem 1 becomes

$$B(x, C) = xB(x, D_B) + B(x, E),$$

where D_B is now the board obtained from C by deleting the chosen square s together with all squares on the two diagonals through s, and R is defined, as before, by deleting s from C. In the same way,

$$Q(x, C) = xQ(x, D_Q) + Q(x, E),$$

where

$$D_Q = D_R \cap D_B,$$

the intersection of D_R and D_B, consists of all the squares that are common to D_R and D_B.

For the results analogous to Theorem 2 we have

$$B(x, C) = B(x, F_B)B(x, G_B)$$

and

$$Q(x, C) = Q(x, F_Q)Q(x, G_Q),$$

where in the case of bishop polynomials two boards F_B and G_B are non-interfering if F_B and G_B have no diagonal in common. For queen polynomials two boards F_Q and G_Q are non-interfering if F_Q and G_Q have no rows, columns or diagonals in common. For the bishop polynomial of the ordinary 8×8 square chess board, the boards made up of the black squares and the white squares, respectively, are non-interfering.

2. Relationships between the board polynomials

We define the board C' as that obtained from the board C by first rotating C clockwise through 45° in the plane and then rotating each square through 45°. Thus, when C is the board in Figure 2, C' is the board of numbered squares shown in Figure 4.

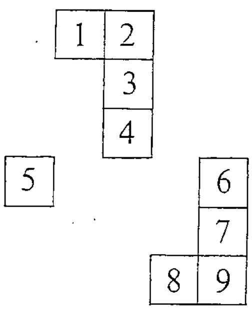


Figure 3

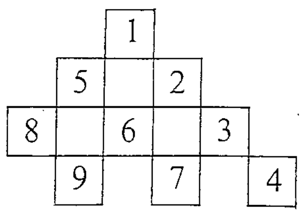


Figure 4

The rook polynomial for this board is given by

$$R(x, C') = 1 + 9x + 26x^2 + 26x^3 + 8x^4,$$

which is identical with the bishop polynomial of the original board C. This procedure can always be adopted for finding $B(x, C)$, since the rows and columns of C' are simply the diagonals of C.

We note that re-ordering the rows (or columns) of a board leaves the rook polynomial unchanged. This is so since two rooks can take on the re-ordered board if and only if they can take on the original board. Thus we can construct a simpler board $\overline{C}$ whose rook polynomial is the same as that of C'. Figure 5 demonstrates clearly that $\overline{C}$ is made up of two non-interfering sub-boards F and G.

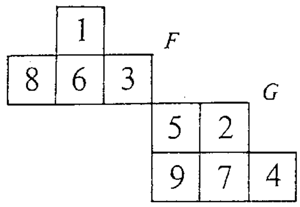


Figure 5

Thus

$$\begin{aligned} B(x, C) &= R(x, \overline{C}) \\ &= R(x, F)R(x, G) \\ &= (1 + 4x + 2x^2)(1 + 5x + 4x^2) \\ &= 1 + 9x + 26x^2 + 26x^3 + 8x^4. \end{aligned}$$

We now establish a relationship between the queen, rook and bishop polynomials of a particular board.

Consider first two entirely separate boards of numbered squares, as for example in Figure 6, where some of the numbers are common to both boards. We shall find the number of ways of placing k non-taking rooks on both boards such that if a particular rook is placed on the square numbered n on C then it must also be placed on square n on D. Thus we are enumerating the number of ways of placing k non-taking rooks on C and on D. We shall denote this number by $r_k(C \wedge D)$ and we can define the rook polynomial of the composite board C and D, written $C \wedge D$, by

$$R(x, C \wedge D) = r_0(C \wedge D) + r_1(C \wedge D)x + \cdots + r_m(C \wedge D)x^m.$$

Thus for the boards C and D shown in Figure 6, we have

$$R(x, C \wedge D) = 1 + 5x + 3x^2.$$

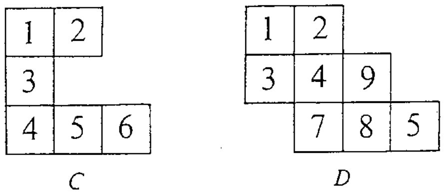


Figure 6

Note that we cannot place any rooks on squares 6, 7, 8 or 9 since these are not common to both boards.

For a general board C we can construct C' as described previously and by re-ordering the rows (or columns) produce a simpler board $\overline{C}$. Two queens are in non-taking positions on C if and only if the squares they stand on are non-taking positions for both rooks and bishops on C. This is equivalent to these squares being non-taking positions for rooks on C and rooks on $\overline{C}$. Thus

$$Q(x,C) = R(x, C \wedge \overline{C}).$$

We illustrate this result by finding the queen polynomial for C_3, the 3×3 square chess board, shown in Figure 7 together with $\overline{C_3}$.

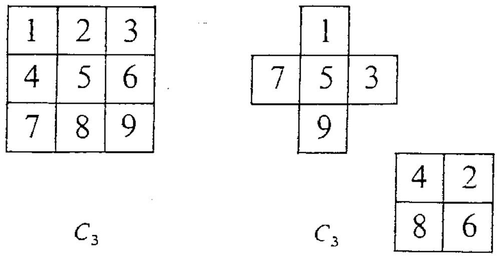


Figure 7

We can establish analogues of Theorems 1 and 2 for finding $R(x, A \wedge B)$ for two boards A and B. Choosing square 5 and using the analogue of Theorem 1 we obtain

$$Q(x, C_3) = R(x, C_3 \wedge \overline{C_3}) = xR(x, H \wedge G) + R(x, M \wedge N)$$

where H, G, M and $N = N_1 \cup N_2$ are shown in Figure 8.

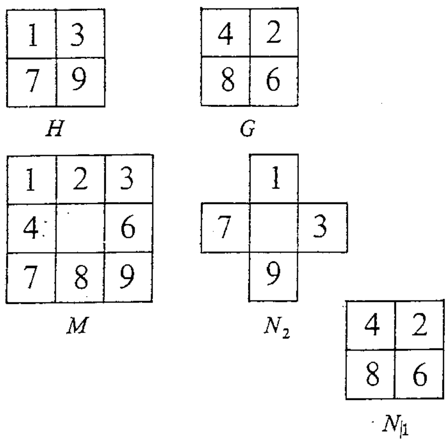


Figure 8

Clearly

$$R(x, H \wedge G) = r_0(H \wedge G) = 1,$$

since H and G have no square in common. We obtain $R(x, M \wedge N)$ from first principles.

There are eight ways of placing one rook on $M \wedge N$ since M and N have eight squares in common. Consider now the number of ways of placing two rooks on N_1. The only possibilities are (2, 8) and (4, 6) but, for each of these, two non-taking rooks cannot be placed on M. Thus these are not non-taking rooks for $M \wedge N$. Similarly, if two non-taking rooks are placed on N_2, the only possibilities are (1, 7), (1, 3), (7, 9) and (3, 9). Rooks in these positions cannot be placed on M. To count the number of ways of placing two non-taking rooks on $M \wedge N$ we consider one rook on N_1 and one on N_2. The possibilities are (1, 2), (1, 4), (1, 8), (1, 6), (3, 4), (3, 2), (3, 8), (3, 6), (7, 2), (7, 4), (7, 8), (7, 6), (9, 4), (9, 2), (9, 8) and (9, 6). Of these only (1, 8), (1, 6), (3, 4), (3, 8), (7, 2), (7, 6), (9, 4) and (9, 2) are valid positions for two non-taking rooks on M. That is, there are eight ways of placing two non-taking rooks on $M \wedge N$. A similar argument shows that there are no ways of placing three non-taking rooks on $M \wedge N$, so that

$$R(x, M \wedge N) = 1 + 8x + 8x^2.$$

Therefore

$$Q(x, C_3) = R(x, C_3 \wedge \overline{C_3}) = 1 + 9x + 8x^2.$$

The corresponding result for C_4, the 4×4 square chess board, is given by

$$Q(x, C_4) = 1 + 16x + 44x^2 + 24x^3 + 2x^4.$$

As a natural extension of these results we can define the king polynomial $K(x, C)$ for a board C. Analogues of Theorems 1 and 2 are, as before, directly applicable. For Theorem 2 the non-interfering boards F_K and G_K are such that no square of F_K is adjacent to a square of G_K.

For example consider the board A of numbered squares shown in Figure 9.

	1	2	3	
A	4			
			5	6

Figure 9

The board F_K consists of the squares 1, 2, 3 and 4 whilst G_K consists of the squares 5 and 6.

$$K(x, F_K) = 1 + 4x + 2x^2$$

and

$$K(x, G_K) = 1 + 2x.$$

Therefore, using the non-interfering property of these boards, we obtain

$$K(x, A) = K(x, F_K)K(x, G_K) = 1 + 6x + 10x^2 + 4x^3.$$

3. Permutations with restrictions

Consider the problem of finding the number of permutations of A, B, C, D in which A's forbidden position is the third, B's is the fourth, C's are the first and third and D's are the second and fourth. This means, for example, that $ABCD$ is not a valid permutation, since D appears in a forbidden position, but $BCDA$ is a possible permutation.

A practical illustration of this is the assignment of four jobs to four people not all of whom are able to do every job.

We can construct a board F (see Figure 10) in which the squares of a particular row correspond to the permitted positions of a particular letter in the permutation. For example, it is seen from the second row of board F that B may appear in any of the first three positions but not in the fourth position. Thus a valid permutation of A, B, C, D is equivalent to putting four non-taking rooks on F, since any one letter must occupy exactly one position (that is every row must contain one rook) and each position is occupied exactly once (every column must contain one rook). The permutation $BCDA$ corresponds to placing rooks on squares 3, 4, 7 and 10 of board F. Thus the total number of permitted permutations of A, B, C and D is given by $r_4(F)$.

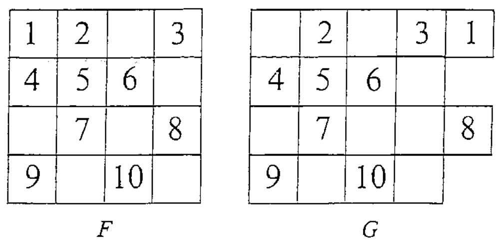


Figure 10

The methods of Section 1 show that

$$R(x, F) = 1 + 10x + 29x^2 + 26x^3 + 5x^4.$$

Thus

$$r_4(F) = 5,$$

and this is the answer to our original problem.

Suppose now that an extra condition is imposed: if A occupies the first position then C cannot occupy the fourth position.

In the practical illustration above, this corresponds to the situation that C cannot have the fourth job when A has the first.

We can construct board G in such a way that whatever is permitted on the board F is also permitted on the board G with the one exception that squares 1 and 8 cannot both be used in the same permutation. That is, A cannot appear first when C appears fourth. The problem is now equivalent to finding $r_4(F \wedge G)$.

Note that some permutations are allowed on board G which are not allowed on board F but these will not be allowed on board $F \wedge G$.

Since

$$R(x, F \wedge G) = 1 + 10x + 28x^2 + 23x^3 + 4x^4,$$

the answer to this more restrictive problem is 4.

4. Application to the failure of cellular structures

In a cellular structure faults may occur which could cause failure under certain conditions. For example consider a metal plate which may be considered as being made up of cells forming a regular 4×4 square board which will fracture if two faults are adjacent. If μ is the expected number of faults on the board, i.e. the number of faulty cells, the Poisson distribution gives the approximate probabilities of a total of r faults as

$$P(r) = e^{-\mu}\frac{\mu^r}{r!} \qquad (r = 0, 1, 2, \ldots).$$

(The approximation will usually be acceptable for small μ.) The probability p of failure is given by

$$p = \sum_{r=2}^{\infty} P(r)p_{r,2},$$

where $p_{r,2}$, the probability that two of the r faults are adjacent, may be found by considering the coefficients of $K(x, C_4)$.

It can be shown that

$$K(x, C_4) = 1 + 16x + 78x^2 + 140x^3 + 79x^4.$$

Thus

$$p_{2,2} = 1 - \frac{78}{\binom{16}{2}},$$

since there are $\binom{16}{2}$ ways of placing two kings on the board C_4 of which 78 consist of non-adjacent pairs of kings. Similarly

$$p_{3,2} = 1 - \frac{140}{\binom{16}{3}},$$

$$p_{4,2} = 1 - \frac{79}{\binom{16}{4}},$$

and

$$p_{r,2} = 1 \qquad \text{for } r \geq 5.$$

Therefore

$$p = e^{-\mu}\frac{\mu^2}{2!}\left(1 - \frac{78}{\binom{16}{2}}\right) + e^{-\mu}\frac{\mu^3}{3!}\left(1 - \frac{140}{\binom{16}{3}}\right) + e^{-\mu}\frac{\mu^4}{4!}\left(1 - \frac{79}{\binom{16}{4}}\right) + \sum_{r=5}^{\infty} e^{-\mu}\frac{\mu^r}{r!},$$

and since $e^{\mu} = \sum_{r=0}^{\infty} \mu^r/r!$,

$$p = 1 - e^{-\mu}\left(1 + \mu + \frac{13}{40}\mu^2 + \frac{1}{24}\mu^3 + \frac{79}{43680}\mu^4\right).$$

The table shows p for various values of μ.

μ	0·5	1	2	3	4	5
p	0·04	0·13	0·37	0·59	0·76	0·86

If the criterion for failure is that two faults occur within two squares of one another we proceed similarly, but we have to replace $K(x, C_4)$ by another polynomial.

The king polynomial itself can be generalised by constructing a new chess piece that can take any other piece that is situated in a $(2n + 1) \times (2n + 1)$ board centred on it. See Figure 11 for the case $n = 2$. This leads to a range-n polynomial since essentially the new chess piece has a 'sphere of influence' of n squares in all directions.

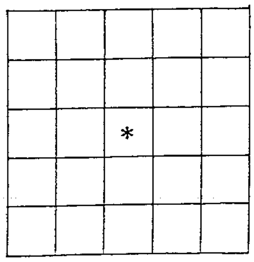


Figure 11. The chesspiece situated in the special cell marked can take any other piece on this sub-board.

Of course the range-1 polynomial is the king polynomial. Denoting the range-2 polynomial of a board by $K_2(x, C)$ we can easily show that

$$K_2(x, C_4) = 1 + 16x + 30x^2 + 12x^3.$$

Returning now to the problem of failure of C_4 which occurs when two faults are within two squares of one another we find that the probability of failure is given by

$$p = e^{-\mu}\frac{\mu^2}{2!}\left(1 - \frac{30}{\binom{16}{2}}\right) + e^{-\mu}\frac{\mu^3}{3!}\left(1 - \frac{12}{\binom{16}{3}}\right) + \sum_{r=4}^{\infty} e^{-\mu}\frac{\mu^r}{r!},$$

i.e.

$$p = 1 - e^{-\mu}\left(1 + \mu + \frac{1}{8}\mu^2 + \frac{1}{280}\mu^3\right)$$

The table below shows p for various values of μ.

μ	0·05	1	2	3	4	5
p	0·07	0·22	0·52	0·74	0·87	0·94

We can also use the range-2 polynomials to build up a model of the breakdown of long-chain structures. Consider the range-2 polynomial $K_2(n)$ for a chain of n squares in a single row. Using the appropriate analogue of Theorem 1 and choosing one of the end squares as the special square we have

$$K_2(n) = xK_2(n - 3) + K_2(n - 1).$$

Since $K_2(0) = 1$, $K_2(1) = 1 + x$, $K_2(2) = 1 + 2x$, this last result may be used recursively to obtain

$$K_2(3) = 1 + 3x,$$
$$K_2(4) = 1 + 4x + x^2,$$
$$K_2(5) = 1 + 5x + 3x^2,$$
$$K_2(6) = 1 + 6x + 6x^2,$$

and

$$K_2(7) = 1 + 7x + 10x^2 + x^3.$$

Let p denote the probability that a square is faulty. Further, suppose that the chain structure breaks down if two faulty squares are separated by at most one non-faulty square. We can derive an expression for the probability Q that the chain structure will break down.

We have

$$Q = \sum_{s=2}^{n} P(s)\left(1 - \frac{r_2(s)}{\binom{n}{s}}\right),$$

where $K_2(n) = \sum_{s=0}^{\infty} r_2(s)x^s$ and $P(s)$ is the probability that there are s faulty squares in the chain. Also, from the binomial distribution

$$P(s) = \binom{n}{s}p^s(1-p)^{n-s} \qquad (s = 0, 1, \ldots, n).$$

For $n = 10$,

$$K_2(10) = 1 + 10x + 28x^2 + 20x^3 + x^4;$$

and the table below shows Q for different values of p.

p	0·05	0·1	0·2	0·3	0·4
Q	0·04	0·13	0·40	0·66	0·84

A possible realisation would be a biological situation in which faulty cells can be repaired provided no two faults are within two cells of each other.

In both of the applications in this section either the Poisson or binomial distribution could be used depending on the information available.

Letters to the Editor

Dear Editor,

Heronian triangles

You have recently had a number of items on Heronian triangles (Sastry in 8:3, Tagg in 9:2, Pargeter in 9:2 and 10:3, Strange in 10:1, Sands in 10:1). Perhaps the following bibliographical information will be of interest. Much of this comes from L. E. Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, pp. 191–201. Unfortunately he uses the term Heronian for triangles with rational sides and area. I shall refer to these as rational, reserving Heronian for triangles with integral sides and area. It is clear that knowledge of either type of triangle determines the other type.

The general idea of juxtaposing Pythagorean triangles to get a rational triangle seems to be due to G. C. Bachet in his Commentary on Diophantus (1621) (Dickson, pp. 191–192, item 3). Euler (ibid., p. 193, item 7) was the first to give a formula for all rational triangles. Unfortunately, this is contained in posthumous, undated, papers and the proof is lacking. D. N. Lehmer (ibid., p. 199, item 43) appears to be the first to derive Euler's results and to publish a determination of all rational triangles in 1899–1900. Sands' recent letter shows that it is quite easy to derive Euler's results, so there seems little doubt that Euler himself had done so. However, without detailed examination of sources, it is difficult to determine priority, especially since Dickson's notes are often rather condensed. For instance Brahmegupta (7th century) (ibid., p. 191, item 1) gives a formula which is the same as that obtained by a juxtaposition (but it is not clear if he did it this way or treated it as a general process) and which is essentially equivalent to Euler's formula. There are also several papers after Euler and before Lehmer which use juxtaposition and/or Euler's formula, but Dickson's notes are not clear as to whether they show the completeness of the formula. Lehmer's derivation uses the rationality of the sines and cosines of the angles, so is probably similar to Strange's argument. Dickson's notes do show that the 1905 derivation of H. Schubert (ibid., p. 199, item 47) is identical to that of Strange.

While looking up the above items in Dickson, I noted that the Japanese mathematician Nakane Genkai first found all the Heronian triangles with consecutive integer sides in 1722 (ibid., p. 192, item 6a), though Dickson could not tell if the proof was complete. Genkei's results are the same as Strange's.

Some more recent work is found in *Reviews in Number Theory*, W. J. LeVeque, ed., American Mathematical Society, Providence, 1974, Vol. 2, p. 72. For convenience of those who do not have access to this collection, the items are from *Mathematical Reviews:* **22,** No. 4656; **23,** No. A107; **27,** No. 4792; **31,** No. 121. The first (in Italian) gives parametric formulae for some Heronian triangles with sides in arithmetic progression. The second (in Romanian, with Russian and French summaries) asserts that the general solution is not known and gives some partial results. The third is listed only by title and is in a Romanian journal. The fourth (in Russian) gives a formula for all Heronian triangles which is similar to, but more complicated than and not immediately equivalent to, the formulae of Tagg, Strange and Sands.

W. Sierpinski, *Pythagorean Triangles*, Yeshiva University, New York, 1962, has a chapter on Heronian and rational triangles (pp. 59–66). He proves that 'each rational triangle can be obtained by the union of two right-angled triangles with rational sides' and he observes that not every Heronian triangle is obtained as a union of two Pythagorean ones. He also finds the Heronian triangles with consecutive integer sides.

I have just discovered a section in A. P. Domoryad, *Mathematical Games and Pastimes*, Pergamon, Oxford, 1963 (Russian original, 1961), pp. 31–32, on Pythagorean and Heronic

Triples. He says 'it is easy to prove ([21]) that any of the altitudes of a "Heronic triangle" ... gives two right-angled triangles with rational sides ...'. Note ([21]) is explained on p. 253, though it is rather cryptic due to three misprints in one sentence.

Oystein Ore, in his *Invitation to Number Theory*, Random House, New York, 1967, states on pp. 59–60 'we have no general formula giving them all', referring to Heronian triangles. This inspired John R. Carlson's 'Determination of Heronian triangles', *Fibonacci Quarterly* **8** (1970), 499–506 and 551. (See also David Singmaster, 'Some corrections to Carlson's "Determination of Heronian triangles", ibid., **11** (1973) 157–158.)

In addition Carlson makes an observation which is the subject of Mr Pargeter's letter in Volume 10, No. 3; namely that, if the sides of a Heronian triangle have a common factor, then, when the sides are divided by this factor, the resulting triangle is also Heronian. However, comparing Carlson's short proof with Pargeter's longer proof, I found that both are incomplete. Carlson accidentally assumes that $s = (a + b + c)/2$ is an integer, an error which I did not see before, while Pargeter shows that a, b, c cannot all be odd and then asserts that they must all be even. But his proof actually shows that $t = 2s$ cannot be odd, which is what is needed. The following combines features of both proofs.

Theorem. Let k, a, b, c be positive integers. Then a, b, c are the sides of a Heronian triangle if and only if ka, kb, kc form a Heronian triangle.

Proof. The 'only if' is clear. Suppose ka, kb, kc are the sides of a Heronian triangle of area B. Let $t = a + b + c$. If t is even, then the semi-perimeter $s = t/2$ is an integer and the area A of the triangle with sides a, b, c is $A = \{s(s-a)(s-b)(s-c)\}^{1/2}$ which is the square root of an integer. Now B is an integer and $B = k^2A$, so A is rational. But a rational square root of an integer must be an integer, so we are done when t is even.

Pargeter's argument carries through almost verbatim to show that t cannot be odd. For the sake of completeness I reproduce it in condensed form. Suppose t is odd and let $u = t - 2a$, $v = t - 2b$, $w = t - 2c$, so $t = u + v + w$ and u, v, w are all odd. Now $16B^2 = 16k^4A^2 = k^4tuvw$ is an integral square. Considering congruence classes (mod 4) for u, v, w, one sees that $tuvw \equiv 3 \pmod 4$ always holds. But an odd square is always $\equiv 1 \pmod 4$. Thus t cannot be odd and the proof is complete.

Further, some small comments. Strange feels that the formulae of Tagg and Pargeter are incomplete. Both Tagg and Pargeter explicitly observe that their formulae only determine Heronian triangles up to similarity and Tagg even gives an example where a common factor occurs, leading to a reduction. Sands' observation that the legs of a rational Pythagorean triangle can be obtained in either order by re-parametrizing can be used in the integral case to simplify my corrected version of Carlson's Corollary 1, as follows.

A triangle is Heronian if and only if its sides can be represented as

(i) $a(u^2 + v^2)$, $b(r^2 + s^2)$, $a(u^2 - v^2) + b(r^2 - s^2)$, where $auv = brs$,

or

(ii) a reduction by a common factor of a triangle given by (i).

Because of (ii), there is no harm in multiplying (i) through by $uvrs$ and then cancelling $auv = brs$. This gives Euler's formula again. Euler's formula is equivalent to those of Brahmegupta, Schubert, Tagg and Strange as well.

Finally, perhaps readers might be interested in Problem E 2687 from the *American Mathematical Monthly* Vol. **84**, No. 10 (December 1977), p. 820, proposed by Ronald Evans. 'Does there exist a triangle with rational sides whose base equals its altitude?'.

Yours sincerely,

DAVID SINGMASTER
(Polytechnic of the South Bank)

Dear Editor,

The Pythagorean theorem in three dimensions

A parallel result to the Pythagorean theorem is obtained in three-dimensional Euclidean space when right-angled triangles are replaced by right-angular tetrahedrons. We define a right-angular tetrahedron as the tetrahedron in which three of its edges are mutually perpendicular. The face opposite to the vertex at which these three perpendicular edges meet will be called the hypotenuse face.

The result is then, in a right-angular tetrahedron, the square of the area of the hypotenuse face is equal to the sum of the squares of the areas of the other three faces.

To show this, let OA, OB, and OC be three mutually perpendicular line segments with lengths a, b, and c respectively. Draw the line perpendicular to BC from A to meet BC in Q. The plane of OAQ will be perpendicular to the plane of OBQ and OQ will be perpendicular to BC (and Q will lie between B and C). Let e denote the area of the triangle ABC, e_1, e_2, and e_3 denote the areas of the triangles OBC, OCA, and OAB respectively, then

$$\begin{aligned} CQ &= OC \cos OCB \\ &= OC\left(\frac{OC}{BC}\right) \\ &= \frac{c^2}{\sqrt{(b^2+c^2)}}; \end{aligned}$$

and so

$$\begin{aligned} e &= \tfrac{1}{2}BC \,.\, AQ \\ &= \tfrac{1}{2}\sqrt{b^2+c^2}\,.\,\sqrt{a^2+c^2-\frac{c^4}{b^2+c^2}} \\ &= \tfrac{1}{2}\sqrt{(b^2+c^2)\,.\,(c^2+a^2)-c^4} \\ &= \tfrac{1}{2}\sqrt{(bc)^2+(ca)^2+(ab)^2}, \end{aligned}$$

and it follows that

$$\begin{aligned} e^2 &= (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ca)^2 + (\tfrac{1}{2}ab)^2 \\ &= e_1^2 + e_2^2 + e_3^2; \end{aligned}$$

which is the required result.

Yours sincerely,

A. M. KHIDR
(Kuwait University)

Editorial Note. A number of notes on this subject appear in the *Mathematical Gazette* (September 1978).

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

11.4. (Submitted by B. G. Eke, University of Sheffield.) A man rows with uniform speed v m.p.h. in a straight line against a current of c m.p.h. After 1 hour his hat falls off; after another hour he notices, turns back and catches up with his hat where he first started rowing. Find v/c. If now his hat falls off after 1 mile instead of 1 hour, with all the other statements as previously, determine c and comment on the fact that c is independent of v in this case.

11.5. A sum and product are defined on the points of the plane as follows: $A + B$ is the unique point such that A, B and $A + B$ form an equilateral triangle, described in an anticlockwise direction, $A \times B$ is the midpoint of the straight line joining A and B. Show that

$$A \times (B + C) = (B + A) \times (A + C).$$

11.6. (Submitted by A. K. Austin, University of Sheffield.) A number of cards are dealt into m not necessarily equal piles. They are then collected together and redealt into $m + k$ piles, where $k > 0$. Show that there are at least $k + 1$ cards which are in smaller piles in the second dealing than in the first.

Solutions of Problems in Volume 10, Number 3

For background and terminology to problems 10.7 and 10.8, see Sections 2 and 5 of the article 'A Medley of Squares', Volume 10 Number 3, pp. 72–81.

10.7. Let S be any finite system of similarly oriented squares of equal size in the plane, and denote by $A(S)$ the total area covered by S. Show that it is always possible to find a discrete subsystem T of S such that $A(T) \geq \frac{1}{6}A(S)$.

Solution

Draw coordinate axes in the plane of and parallel to the edges of the squares. Then there exists a line $x = a$ such that a square in S, say Q_1, has one of its edges along this line while no square of S has any points strictly to the right of this line. (If there are several squares with an edge along $x = a$, take any one of them.) We now select Q_1 as a member of T and reject all its associates. The associates of Q_1 occupy an area at most equal to $6A(Q_1)$. Now repeat the same process for the system left when Q_1 and its associates have been removed from S; and carry on in the same way until S is exhausted. The resulting subsystem T is then plainly discrete and satisfies the asserted inequality.

10.8. Accepting as known the existence of a continuous space-filling curve, demonstrate the existence of a continuous curve which passes through every point of the entire plane.

Solution

If a curve joins the points A, B, we shall call them the 'terminal points' of the curve. It is clear that if Γ_1 is a continuous curve joining A and B and Γ_2 a continuous curve joining B and C, then Γ_1 and Γ_2 form together a continuous curve with terminal points A, C.

We take as known the existence of a continuous curve Γ which passes through every point of a given square σ. Denote by A, B the terminal points of σ and by L, M the north-west and south-east vertices of σ. Then the segment LA, the curve Γ and the segment BM constitute together a continuous curve, say Γ', which passes through every point of σ and has L, M as its terminal points.

Let A_1, A_2, A_3, A_4 be the vertices (in order) of the square σ_1 specified by the inequalities $-1 \leq x \leq 1$, $-1 \leq y \leq 1$; let B_1, B_2, B_3, B_4 be the vertices (in order) of the square σ_2: $-2 \leq x, y \leq 2$; let C_1, C_2, C_3, C_4 be the vertices (in order) of the square σ_3: $-3 \leq x, y \leq 3$; and so on. Denote by Γ_1 a continuous curve passing through every point of σ_1 and with terminal points A_1, A_3. Denote by L_1 the segment A_3B_1. Denote by Γ_2 a continuous curve passing through every point of σ_2 and with terminal points B_1, B_3. Denote by L_2 the segment B_3C_1. Denote by Γ_3 a continuous curve passing through every point of σ_3 and with terminal points C_1, C_3; and so on. Then $\Gamma_1, L_1, \Gamma_2, L_2, \Gamma_3, \ldots$ constitute a curve passing (infinitely often) through every point of the entire plane.

10.9. The following algorithm describes a geometrical procedure:

(0) take any triangle ABC;
(1) circumscribe a circle around ABC;
(2) draw tangents l, m, n at A, B, C;
(3) let $A = m \cap n$, $B = n \cap l$, $C = l \cap m$;
(4) go to (1).

Describe the angles of $\triangle ABC$ after reaching (3) for the nth time, and determine under what circumstances the angle at A takes its initial value again.

Now begin with a cyclic quadrilateral $ABCD$ instead of a triangle, and carry out the analogous construction. Show that, if it is possible to pass beyond (1) for the second time, then

$$AB^2 + CD^2 = d^2,$$

where d is the diameter of the circle circumscribing $ABCD$.

Solution

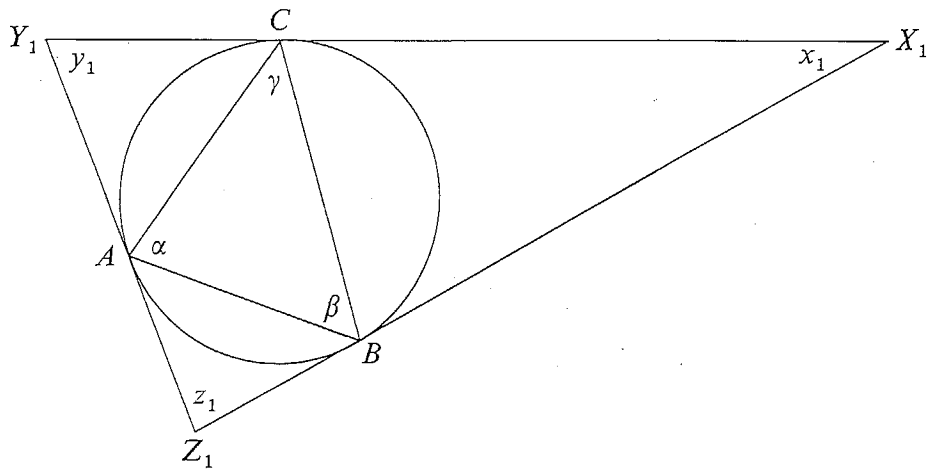


Figure 1

From Figure 1, we have

$$x_1 = \pi - 2\alpha,\ y_1 = \pi - 2\beta,\ z_1 = \pi - 2\gamma.$$

In general, if we denote the angles of the triangle obtained after reaching stage (3) for the rth time by x_r, y_r, z_r, we have

$$x_r = \pi - 2x_{r-1},\ y_r = \pi - 2y_{r-1},\ z_r = \pi - 2z_{r-1}.$$

Thus

$$\begin{aligned}
x_n &= \pi - 2x_{n-1} \\
&= \pi - 2(\pi - 2x_{n-2}) \\
&= (1-2)\pi + 2^2 x_{n-2} \\
&= (1-2)\pi + 2^2(\pi - 2x_{n-3}) \\
&= (1 - 2 + 2^2)\pi - 2^3 x_{n-3} \\
&= \cdots \\
&= (1 - 2 + 2^2 - \cdots + (-1)^{n-1}2^{n-1})\pi + (-1)^n 2^n \alpha \\
&= (1 + (-1)^{n-1}2^n)\frac{\pi}{3} + (-1)^n 2^n \alpha \\
&= \frac{\pi}{3} + (-1)^n 2^n\left(\alpha - \frac{\pi}{3}\right),
\end{aligned}$$

with similar formulae for y_n, z_n. The angle at A takes its initial value again if and only if $x_n = \alpha$ for some $n > 0$, and this happens if and only if $\alpha = \pi/3$.

Suppose we now begin with a cyclic quadrilateral $ABCD$. If we can pass beyond stage (1) for the second time, then $A'B'C'D'$ can be circumscribed in a circle (see Figure 2), and

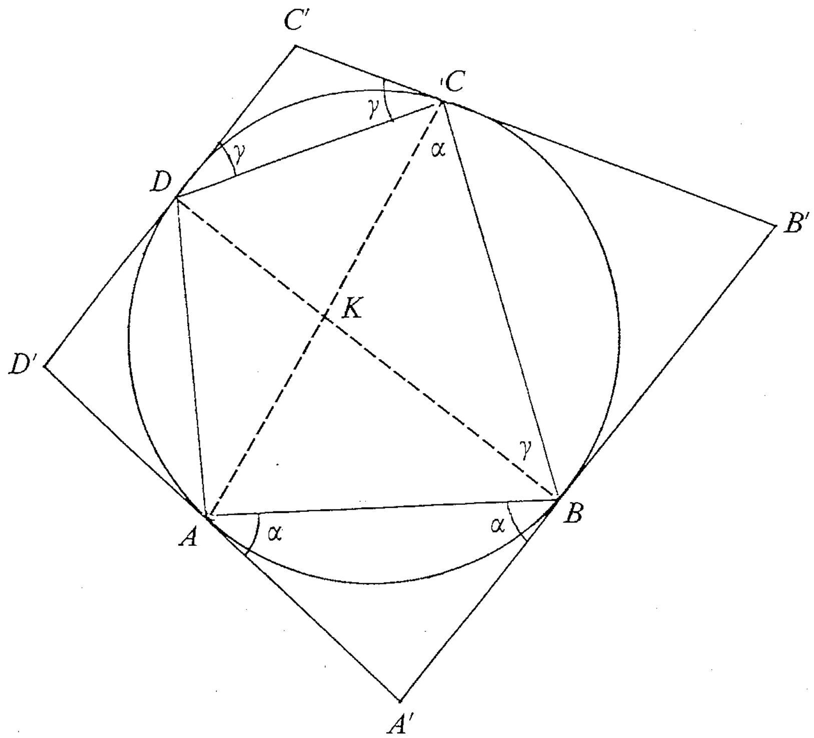


Figure 2

$$\begin{aligned}
&\angle AA'B + \angle CC'D = \pi \\
&\Rightarrow (\pi - 2\alpha) + (\pi - 2\gamma) = \pi \\
&\Rightarrow \alpha + \gamma = \frac{\pi}{2} \\
&\Rightarrow AC \perp BD \\
&\Rightarrow AB^2 + CD^2 = AK^2 + KB^2 + CK^2 + KD^2.
\end{aligned}$$

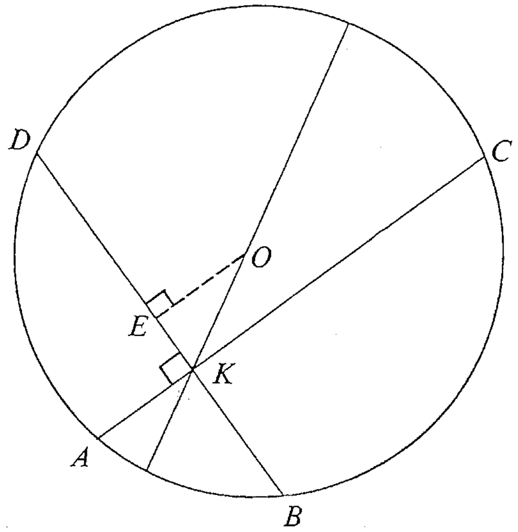


Figure 3

Denote by O, r the centre and radius of the circle, and put $AK = a$, $BK = b$, $CK = c$, $DK = d$, $OK = x$. Then from Figure 3 we see that

$$ac = bd = (r - x)(r + x)$$
$$\Rightarrow r^2 - x^2 = ac = bd.$$

From $\triangle OEK$, we have

$$x^2 = \left(\frac{c - a}{2}\right)^2 + \left(\frac{d - b}{2}\right)^2$$
$$\Rightarrow 4x^2 = a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$$
$$\Rightarrow 4r^2 = a^2 + b^2 + c^2 + d^2,$$

i.e. $AB^2 + CD^2 = d^2$.

Correction to the solution of Problem 10.3. The problem was as follows. The positive real numbers p, q, r are such that $q \neq r$ and $2p = q + r$. Show that

$$\frac{p^{q+r}}{q^q r^r} < 1.$$

The solution published in Volume 10 Number 3 was valid only for positive integers. To prove the result for all positive real numbers, we resort to the calculus.

If $r = qx$, then

$$\left(\frac{q + r}{2}\right)^{q+r} < q^q r^r \Leftrightarrow \left(\frac{q + qx}{2}\right)^{q+qx} < q^q (qx)^{qx}$$
$$\Leftrightarrow \left(\frac{q + qx}{2}\right)^{1+x} < q(qx)^x$$
$$\Leftrightarrow \left(\frac{1 + x}{2}\right)^{1+x} < x^x$$
$$\Leftrightarrow (1 + x)(\log(1 + x) - \log 2) < x \log x.$$

Put

$$f(x) = (1 + x)(\log(1 + x) - \log 2) - x \log x.$$

Then

$$\begin{aligned} f'(x) &= (1 + x) \cdot \frac{1}{1 + x} + \log(1 + x) - \log 2 - x \cdot \frac{1}{x} - \log x \\ &= \log \frac{1 + x}{2x} \\ &= \log \frac{1}{2}\left(1 + \frac{1}{x}\right). \end{aligned}$$

Thus

$$\begin{aligned} f'(x) &> 0 \qquad \text{for } 0 < x < 1, \\ f'(1) &= 0 \\ f'(x) &< 0 \qquad \text{for } x > 1, \end{aligned}$$

so that f has a strict maximum at $x = 1$. Since $f(1) = 0, f(x) < 0$ for $0 < x < 1$ and for $x > 1$. The result follows from this.

Book Reviews

An Introduction to University Mathematics. By J. L. SMYRL. Hodder and Stoughton, London, 1978. Pp. xi + 740. £8·25 (paperback); £12·50 (hardback).

This book of 740 pages represents the fruits of a great deal of patient and careful work. It consists essentially of two books, the first one (first 315 pages) consisting of algebra, number systems and geometry (with chapter headings including two- and three-dimensional coördinate geometry, vectors, isomorphism, algebraic structure, complex numbers, mappings and matrices, etc.) and the second one of calculus (a fairly comprehensive first course on differentiation and integration with applications, but containing also a chapter on functions of two variables with partial differentiation and double integration).

Although the book includes modern language and notation, it is in many senses a fairly traditional text especially suitable for practical users of mathematics such as engineers. Some mathematicians may complain that the book perpetuates many of the slight defects of traditional technical volumes, such as ignoring special cases in definitions, etc. (e.g. not defining $\mathbf{a} \wedge \mathbf{b}$ when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$), using language not entirely precise (e.g. speaking of *the* angle between lines (and planes) without careful consideration of what this means), implying that results like $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$ can be easily established by the reader, using results on continuity without comment (e.g. that differentiability implies continuity), implying that the justification given for $(d/dy) f^{-1}(y) = 1/f'(x)$ is a 'proof', and so on. In some of the chapters, results are assumed that are often no longer covered at school, for instance, the formula for tan θ with θ an angle between lines of gradients m_1, m_2, the formula for the perpendicular distance from a point to a line, orthogonal circles, the division identity for real polynomials, etc. Since the book arose out of an attempt to revise Margaret M. Gow's book *A Course in Pure Mathematics* and contains many examples from that book, it is not surprising that Dr Smyrl's book evokes in various places in the text and in the problems a strong nostalgic reminder of the past; unfortunately too many students today cannot cope with the technical demands required for some of this work. The treatment of determinants and matrices is fairly unusual for a present-day text, especially the use of determinants before the introduction of matrix arrays.

The book is very free of misprints, has clear diagrams and is well printed. It presents very clearly the power of mathematical modelling and the use of mathematical techniques. There are many exercises which, coupled with the answers provided, should help to develop and strengthen the grasp of the wide range of methods and techniques covered. The book should be very useful for many courses on mathematics and its applications at sixth-form school level and at the level of general first-year and some second-year classes in colleges and universities.

University of Glasgow J. HUNTER

Vectors: Pure and Applied. By DAVID HOLLAND and TERENCE TREEBY. Edward Arnold (Publishers) Ltd, London, 1978. Pp. x + 259. £5·50.

The authors of a textbook on vectors for sixth forms have to face two major questions. As with much else in mathematics, vectors are much easier to use than to define: to what extent should a book at this level attempt to probe theoretical subtleties before exploring the applications? Then there are so many applications, in geometry, linear algebra, mechanics and elsewhere. How much background knowledge can be assumed in a topic book?

Clearly a great deal of thought has been given to these questions in writing *Vectors: Pure and Applied.* The opening words are commendably honest: 'This is a long and apparently formidable chapter . . .'. And so it is. The 'average Advanced Level student who has little experience of vectors', for whom the book is intended, is asked to cope in three pages with the ideas of directed line segment, equivalent line segments, family of equivalent line segments (called a free vector family, $\mathbf{v}$), member of a free vector family (which is thus a directed line segment, but is called a free vector), set of all free vector families, and position vector (with the notation v introduced, and abandoned one page later). Then come vector quantities, bound vectors (which also seem to be directed line segments), and localised line vectors. There are many detailed comments which could be made (e.g. in Worked Example 1.5 the pairs of triangles in the figure should be similar, the sense of the $\mathbf{m}_1$ arrow in (a) is wrong, and so is the final answer), but the main point is that this chapter as a whole is pitched at too sophisticated a level for most sixth formers. The pace of the remaining chapters is generally much gentler, though the section on work and energy in Chapter 14 is difficult, with line integrals slipping in almost unannounced.

The treatment of mechanics starts from scratch and takes six chapters. Most of this is clear and helpful, though the description of the forces acting at a hinge (pp. 116, 120) is dangerously wrong, and the modulus of elasticity is the force required to double the length of a string, not to produce unit extension (p. 131). Although there is a full treatment of polar coordinates, with radial and transverse unit vectors, this does not lead to a study of motion in a circle. Localised line vectors are introduced in Chapter 1 via the example of pulling a sack on level ice (surely not a rigid body, unless frozen solid), but no discussion of the equilibrium of a rigid body follows, and the localised line vectors are never used. So, although the mechanics sections are extensive, they will need supplementing for A level.

The linear algebra in Chapter 2 is carefully done, except for the three sentences about linear dependence preceding Exercise 2e, which are all wrong. Knowledge of determinants and isomorphism is assumed here, and the next chapter uses all the sixth-form techniques of differentiation. The practice of using the same letter for dummy variable and limit in definite integrals leads inevitably to nonsense such as 'assuming that . . . $\mathbf{x} = \mathbf{x}$ and $\mathbf{v} = \mathbf{v}$ when $\mathbf{t} = \mathbf{t}$' (p. 224).

The centroid proof on p. 72 fails if the origin is in the plane of the triangle since $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are then dependent. The brief summary of conics is not likely to be much help to anyone who does not know it already (particularly since circles as defined here are always points), and the use of vectors is here confined almost entirely to writing column vectors instead of coordinates. The definition of a cone on p. 216 assumes that the centre is at the origin, and there is confusion between single and double cones.

The book is attractively produced with a full and useful set of exercises, but sadly the shortcomings of the text make it unsuitable for the average student at whom it is aimed.

City of London School T. J. HEARD

Numerical Analysis—A First Year Course. By R. F. CHURCHHOUSE. Christopher Davies (Publishers) Ltd, Swansea, 1978. Pp. 73. £1·50.

This is a carefully written and very readable introduction to selected topics in elementary numerical analysis. Interpolation (Lagrange form and Newton's divided difference formula), error detection by forward differences, numerical integration (trapezium, mid-point and Simpson's rules), and the solution of non-linear equations (bisection, false position, secant and Newton–Raphson methods) are discussed, always with due attention to the effects of rounding and truncation errors.

As the book is based on a course of only about 18 lectures for first-year university students, its scope is rather limited. In a longer introductory course one would expect to find a discussion of the numerical solution of systems of linear equations, least squares approximation, and perhaps some methods for solving initial value problems for ordinary differential equations. Within his chosen limitations Professor Churchhouse's treatment of the subject is admirable. In just two places, both in the chapter on interpolation, the book lacks the thoroughness evident elsewhere. Firstly, though the Lagrange polynomial and Newton's divided difference formula are established and used, the important fact of uniqueness is neither proved nor stated. Secondly, the discussion of forward differences does not reveal their role in interpolation; an explicit statement of Newton's forward difference formula, as a particular case of the divided difference formula, would be useful.

This book is an excellent introduction to numerical analysis at sixth-form level, and it deserves a place in any school mathematics library. The only prerequisite beyond basic A-level mathematics is Taylor's theorem which is used in the last two chapters. The material covered will not suffice for some first-year courses at universities and polytechnics, but students attending such courses will find the book worth reading, and where its scope is adequate it deserves consideration as a text-book.

University of Durham J. COLEMAN

A Basic Course in Statistics. By G. M. CLARKE and D. COOKE. Edward Arnold (Publishers) Ltd, London, 1978. Pp. 368. £6·50.

Basic courses in statistics are often no more than a collection of recipes with little thought given to the way the material is presented or ordered. This book is different.

The authors have realised that many academic and professional institutions require their students to study basic statistical techniques and that there is an accepted body of topics which ought to be covered. Consequently they have written a textbook which discusses these topics, expressing their belief that statistics is concerned with the collection and analysis of data and that the importance of statistics is reflected by how well this collection and analysis is performed.

The order in which they discuss the various topics is different and interesting. Populations and samples are mentioned very early (Chapter 3) and the first twelve chapters are devoted to work with discrete variables including discussion of estimation, data collecting and significance testing, thus avoiding the use of calculus and the exponential function at an early stage, a point brought out in the preface. The second part of the book deals with continuous random variables, confidence intervals, the use of the Chi-squared distribution, the Poisson distribution and concludes with chapters on correlation and regression. The only chapters which appear to lack body are the chapters on populations and on samples and data collecting.

Throughout the book there are plenty of clear definitions and at all times the authors strive to give lucid explanations relating to the work being discussed. Even if full proofs are not possible at this level, they try to justify any calculations done in order to avoid treating the material superficially. Worked examples and exercises are plentiful and they should prove useful to students, as indeed should the whole book. It is a very welcome addition to the statistical literature available at this level.

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Notes on Contributors

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