

Mathematicorum

Crux

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CRUX MATHEMATICORUM

Volume 16 #9

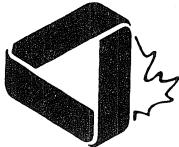
November / novembre

1990

CONTENTS / TABLE DES MATIÈRES

<i>The Olympiad Corner: No. 119</i>	<i>R.E. Woodrow</i>	257
<i>Book Reviews</i>		265
<i>Problems: 1567, 1581-1590</i>		239
<i>Solutions: 1352, 1386, 1458, 1459, 1463-1468</i>		268

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Société mathématique du Canada

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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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Solutions: 1352, 1386, 1458, 1459, 1463–1468		268

THE OLYMPIAD CORNER
No. 119
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first set of problems this month are from the 1990 Asian Pacific Mathematical Olympiad. In this regional correspondence contest selected students from participating countries write the paper at a local site. The participating countries were Australia, Canada, Hong Kong, Korea, Mexico, New Zealand, the Philippines, Singapore, and Thailand. Each country may submit ten students' results on the basis of which awards are made. This year 80 students participated. Each problem was marked out of a possible 7, with 35 being a perfect score. The mean score was 5.9. Six gold, ten silver and twenty-six bronze awards were made. Honourable mention was given to any student whose efforts were deemed worthy by the jury. Thanks go to Professor Bruce Shawyer of Memorial University of Newfoundland for forwarding copies of the contest and results.

1990 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

March 1990

Time allowed: 4 hours

1. In ΔABC , let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle. For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

2. Let a_1, a_2, \dots, a_n be positive real numbers, and let S_k be the sum of products of a_1, a_2, \dots, a_n taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n, \quad \text{for } k = 1, 2, \dots, n-1.$$

3. Consider all the triangles ABC which have a fixed base AB and whose altitude from C is a constant h . For which of these triangles is the product of its altitudes a maximum?

4. A set of 1990 persons is divided into non-intersecting subsets in such a way that

- (a) no one in a subset knows all the others in the subset;

(b) among any three persons in a subset, there are always at least two who do not know each other; and

(c) for any two persons in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

(i) Prove that within each subset, every person has the same number of acquaintances.

(ii) Determine the maximum possible number of subsets.

Note: It is understood that if a person A knows person B , then person B will know person A ; an acquaintance is someone who is known. Every person is assumed to know one's self.

5. Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

*

The second set of Olympiad problems we give this year come from Dr. Matt Lehtinen, Helsinki, Finland. The set consists of the problems from the Fourth Nordic Mathematical Olympiad held April 5, 1990. The contest had students participating from Sweden, Denmark, Norway, Iceland and Finland. The top mark attained was 15 of a possible 20 marks.

FOURTH NORDIC MATHEMATICAL OLYMPIAD

April 5, 1990

Time allowed: 4 hours

1. Let m , n and p be positive odd integers. Show that the number

$$\sum_{k=1}^{\left(\frac{n-1}{2}\right)^p} k^m$$

is divisible by n .

2. Let a_1, a_2, \dots, a_n be real numbers. Show that

$$\sqrt[3]{a_1^3 + a_2^3 + \dots + a_n^3} \leq \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}.$$

When does equality hold?

3. Let ABC be a triangle and P a point inside ABC . Let a line l passing through P but not through A intersect AB and AC (or their extensions over B and C) at Q and R . Find l such that the perimeter of the triangle AQR is least possible.

4. With any positive integer, three operations f , g and h are allowed:

$f(n) = 10n$, $g(n) = 10n + 4$ and $h(2n) = n$, i.e. one may write a zero or a four as the last digit, and if the number is even, one may divide it by two. Prove: starting from 4, every positive integer can be constructed by performing a finite number of the operations f , g and h in some order.

*

To complete the problem sets given this month we include the 1990 Kürschák Competition. The contest was obtained for us from Prof. J. Pataki, Budapest, Hungary by Andy Liu.

1990 KÜRSCHÁK COMPETITION

1. Given are intersecting straight lines e and f and circle C which does not intersect the two lines. For an arbitrary chord AB of C parallel to f , lines AB and e meet at E . Construct AB if AB/EA is maximum.

2. For a positive integer n , denote the sum of n 's decimal digits by $S(n)$. Find those positive integers m for which $S(mk) = S(m)$ for all positive integers k not exceeding m .

3. We are wandering on the co-ordinate plane according to the following rules. From a given point $P(x,y)$, we are allowed to step to one of the following four points: $P_U(x, y + 2x)$, $P_D(x, y - 2x)$, $P_L(x - 2y, y)$, and $P_R(x + 2y, y)$. There is only one further restriction: if we moved from P to Q , then we cannot go back to P at once. Prove that if we start from the point $(1, \sqrt{2})$ then we are not able to return in any number of steps.

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Now we turn to solutions sent in in response to our request for solutions from "the archives". This issue we look at solutions to problems from the 1983 numbers of *Crux*.

1. [1983: 107] *Bulgarian Winter Competition.*

Prove that a regular hexagon with edge of length 2 can be covered by six circular disks of unit radius but cannot be covered by five such disks.

Solution by Richard K. Guy, The University of Calgary.

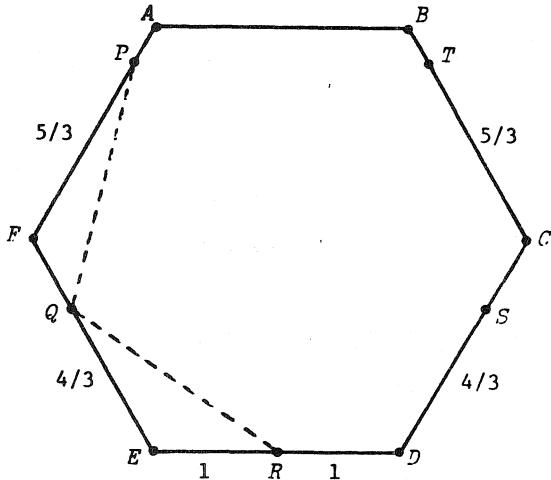
If the hexagon is $ABCDEF$ and O its centre, then the disks on OA, OB, \dots, OF as diameters cover the hexagon.

If five disks covered the hexagon, two vertices must be covered by the same disk, by the pigeon-hole principle. Without loss we may assume these to be A and B . Note that AB must be a diameter of the disk covering them. Consider the points P, Q, R, S, T as shown in the figure, where $FP = CT = 5/3$, $FQ = CS = 2/3$ and R is the midpoint of DE . No two of these can be covered by any one of the remaining 4 disks, giving a contradiction. [For example,

$$PQ^2 = (5/3)^2 + (2/3)^2 - 2(5/3)(2/3)(-1/2) = 39/9 > 2^2,$$

$$QR^2 = (4/3)^2 + 1^2 - 2(4/3)(1)(-1/2) = 37/9 > 2^2.]$$

*



2. [1983: 108] *British Mathematical Olympiad.*

The Fibonacci sequence $\{f_n\}$ is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (n > 2).$$

Prove that there are unique integers a, b, m such that $0 < a < m$, $0 < b < m$, and $f_n - anb^n$ is divisible by m for all positive integers n .

Solution by George Evangelopoulos, law student, Athens, Greece.

We first prove that a triple (a, b, m) of integers satisfies the condition

$$m \mid f_n - anb^n \quad \text{for all integers } n > 0 \quad (1)$$

if and only if

$$ab \equiv 1 \pmod{m} \quad (2)$$

and

$$nab^n + (n-1)ab^{n-1} \equiv (n+1)ab^{n+1} \pmod{m} \quad \text{for all integers } n > 0. \quad (3)$$

Suppose that a, b, m satisfy (1). Then (2) follows from (1) with $n = 1$.

Condition (3) for $n = 1$ has the form

$$ab \equiv 2ab^2 \pmod{m} \quad (4)$$

and thus follows from (1) with $n = 2$ and from (2). Finally, for the other values of $n > 1$, condition (3) holds since from (1)

$$f_n \equiv nab^n \pmod{m},$$

$$f_{n-1} \equiv (n-1)ab^{n-1} \pmod{m},$$

$$f_{n+1} \equiv (n+1)ab^{n+1} \pmod{m},$$

and

$$f_n + f_{n-1} = f_{n+1}. \quad (5)$$

Now suppose a, b, m satisfy (2) and (3). We prove (1) by mathematical induction on n . For $n = 1$ and $n = 2$ (1) follows from (2) and (4) (which is (3) when $n = 1$) since $f_1 = f_2 = 1$. Suppose that for some $n > 1$ we have $f_n - nab^n$ and $f_{n-1} - (n-1)ab^{n-1}$ both divisible by m . Then from (3) and (5) we see that

$$f_{n+1} - (n+1)ab^{n+1} \equiv (f_n - nab^n) + (f_{n-1} - (n-1)ab^{n-1}) \pmod{m}$$

and so is also divisible by m . Thus (1) follows.

So we suppose $0 < a, b < m$ and that (2) and (3) hold. Now observe that, from (2), $\gcd(a, m) = \gcd(b, m) = 1$. Therefore from (3) it follows that each number of the form

$$(n+1)b^2 - nb - (n-1) = (n+1)(b^2 - b - 1) + (b+2)$$

is divisible by m for all positive integers n . This happens if and only if each of the two numbers $b^2 - b - 1$ and $b + 2$ is divisible by m . Consequently, we find that

$$(b^2 - b - 1) - (b+2)(b-3) = 5$$

must be divisible by m . Since $m > 1$, we must have $m = 5$. Now from

$$b+2 \equiv 0 \pmod{5}, \quad 1 \leq b \leq 5$$

and

$$ab \equiv 1 \pmod{5}, \quad 1 \leq a \leq 5$$

we have that $b = 3$ and $a = 2$. To complete the proof, it suffices to remark that the numbers $a = 2, b = 3, m = 5$ satisfy (2), and that $b + 2$ and $b^2 - b - 1$ are divisible by m .

[Editor's note. R.K. Guy of the University of Calgary also sent in a solution which was less elementary as it appealed to Lucas-Lehmer theory.]

5. [1983: 109] *British Mathematical Olympiad.*

If 10 points are within a circle of diameter 5 inches, prove that the distance between some 2 of the points is less than 2 inches.

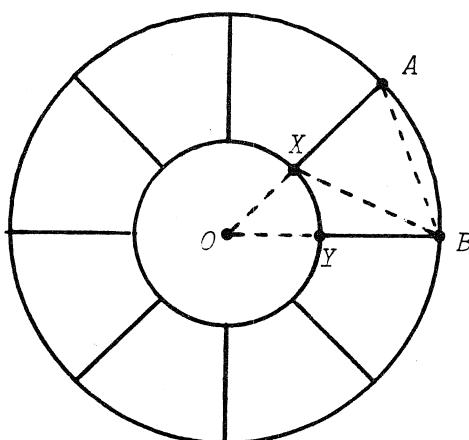
Solution by Richard K. Guy and Jonathan Schaer, The University of Calgary.

Partition the circle into 9 parts, a concentric circle, whose radius is to be chosen, and parts of eight sectors of angle $\pi/4$. By the pigeonhole principle, two of the 10 points are in the same part, and these points are less than d apart, where we choose

$d = \text{diameter of inner circle} = BX = AY$

$$(> AB = 5\sqrt{2} - \sqrt{2}/2 \approx 1.9134).$$

From the triangle BOX , the cosine law gives



$$d^2 = (d/2)^2 + (5/2)^2 - 2(d/2)(5/2)\cos(\pi/4) ,$$

$$3d^2 + 5d\sqrt{2} - 25 = 0 ,$$

$$(6d + 5\sqrt{2})^2 = 350 ,$$

and so

$$d = \frac{5}{6}(\sqrt{14} - \sqrt{2}) < \frac{5}{6}\left(\frac{15}{4} - \frac{7}{5}\right) = \frac{47}{24} < 2 .$$

The better result $d < 5(\sqrt{14} - \sqrt{2})/6 \approx 1.939536521$ can also be given, the strict inequality justified by taking the part $ABXY$, say, as including all of its boundary except the closed segment AX .

[*Editor's note.* George Evangelopoulos, law student, Athens, Greece, submitted a similar solution. In his the inner circle was taken to be of radius 2.]

*

F2414. [1983: 237] *From Középiskolai Matematikai Lapok.*

We have two regular octagons $N_1 = A_1B_1 \cdots H_1$ and $N_2 = A_2B_2 \cdots H_2$. The sides A_1B_1 and A_2B_2 lie on the same line, and also the sides D_1E_1 and D_2E_2 lie on the same line. Furthermore, G_2 coincides with C_1 , and $A_2B_2 < A_1B_1$. The regular octagon N_{i+1} is in the same relation with N_i as N_2 is with N_1 ($i = 2, 3, \dots$). Assuming that all octagons N_i have been constructed, show that the sum $R_1 + R_2 + \dots$ of the radii of their circumcircles converges to the length of the segment A_1M , where $M = A_1C_1 \cap D_1O_2$ and O_2 is the center of N_2 .

Solution by Richard K. Guy, The University of Calgary.

Let O_1 be the centre of N_1 , $X = A_1B_1 \cap E_1D_1$, and K be the midpoint of B_1D_1 . Then

$$2R_1 + 2R_2 + \dots = G_1C_1 + G_2C_2 + \dots = G_1X ,$$

so

$$R_1 + R_2 + \dots = G_1X/2 .$$

Now

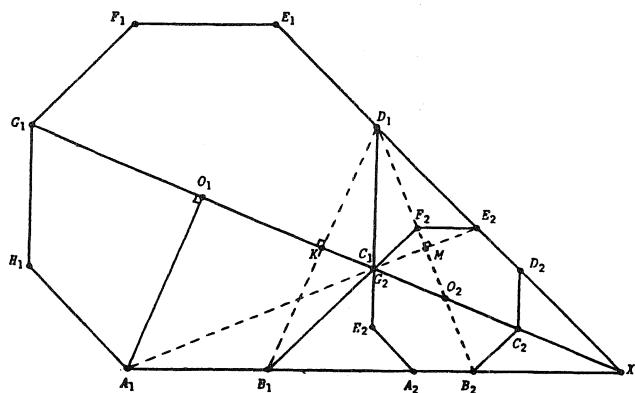
$$\frac{R_1}{O_1X} = \frac{O_1A_1}{O_1X} = \tan \frac{\pi}{8} = \frac{1}{1 + \sqrt{2}} ,$$

so

$$O_1X = R_1(1 + \sqrt{2}) , \quad G_1X = R_1(2 + \sqrt{2}) .$$

By reflection in $D_1G_2H_2$,

$$A_1M = XK = R_1(1 + \sqrt{2} - 1/\sqrt{2}) = \frac{R_1(2 + \sqrt{2})}{2} = \frac{GX_1}{2} = R_1 + R_2 + \dots .$$



M796. [1983: 269] *Problems from KVANT.*

Find $\angle APB$ if P is a point inside a square $ABCD$ such that

$$PA : PB : PC = 1 : 2 : 3 .$$

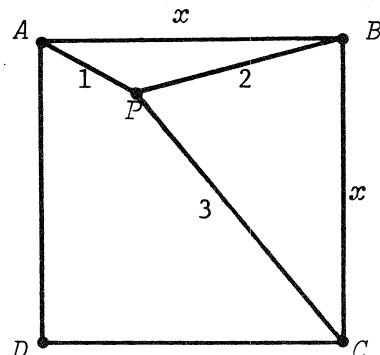
Solution by Richard K. Guy, The University of Calgary.

Put $PA = 1$, $PB = 2$, $PC = 3$, and let x be the side of the square. With $\alpha = \angle ABP$ and $\beta = \angle CBP$ we have $\cos^2 \alpha + \cos^2 \beta = 1$; thus

$$\left(\frac{x^2 + 2^2 - 1^2}{2 \cdot 2 \cdot x} \right)^2 + \left(\frac{x^2 + 2^2 - 3^2}{2 \cdot 2 \cdot x} \right)^2 = 1 ,$$

$$(x^2 + 3)^2 + (x^2 - 5)^2 = 16x^2 ,$$

$$x^4 - 10x^2 = -17 ,$$



and so $x^2 = 5 \pm 2\sqrt{2}$. The lower sign corresponds to P being outside the square. Choosing the upper sign, we find

$$\angle APB = \arccos \frac{1^2 + 2^2 - x^2}{2 \cdot 1 \cdot 2} = \arccos \frac{-2\sqrt{2}}{4} = \frac{3\pi}{4} .$$

If P' is the corresponding point outside the square, $\angle AP'B = \pi/4$.

Note. The Apollonius circles for points in ratio 1:2 from A and B and ratio 2:3 from B and C , and the circle through the concyclic points $APBP'$, form a coaxal system whose radical axis cuts the sides of the square in rational points.

*

Now we move to solutions to three of the problems from the *Arany Daniel Competition 1987, Junior Level* (age 15), which were given in the January 1989 number of the *Corner* ([1989: 5]).

1. The real numbers x, y, z satisfy the following equation:

$$\frac{y^2 + z^2 - x^2}{2yz} + \frac{z^2 + x^2 - y^2}{2zx} + \frac{x^2 + y^2 - z^2}{2xy} = 1 .$$

Prove that two of the three fractions have the value 1.

Solutions by Mathew Englander, Toronto, Ontario; George Evangelopoulos, law student, Athens, Greece; M. Selby, University of Windsor, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Clearly $xyz \neq 0$, so the given equation is equivalent to

$$x(y^2 + z^2 - x^2) + y(z^2 + x^2 - y^2) + z(x^2 + y^2 - z^2) - 2xyz = 0 ,$$

$$x(y^2 + z^2 - 2yz) + z(y^2 - z^2) - y(y^2 - z^2) + x^2(y + z - x) = 0 ,$$

$$x(y - z)^2 - (y - z)^2(y + z) + x^2(y + z - x) = 0 ,$$

$$x^2(y + z - x) - (y - z)^2(y + z - x) = 0 ,$$

and finally

$$(x - y + z)(x + y - z)(y + z - x) = 0.$$

If, say, $x + y - z = 0$ then

$$\frac{y^2 + z^2 - x^2}{2yz} - 1 = \frac{(y - z)^2 - x^2}{2yz} = \frac{(y - z - x)(y - z + x)}{2yz} = 0$$

and

$$\frac{z^2 + x^2 - y^2}{2zx} - 1 = \frac{(z - x - y)(z - x + y)}{2zx} = 0$$

implying the result.

2. The median lines of a convex quadrilateral divide it into four smaller ones. Prove that the sum of the areas of two of the quadrilaterals with no common side equals the sum of the areas of the other pair of such quadrilaterals.

Solution by Bot and Kőszegi, Grade 11, Halifax West High School.

Let $|\mathcal{F}|$ denote the area of figure \mathcal{F} .

Let E, F, G, H be the midpoints of AB, BC, CD, DA , respectively, and let HF and GE intersect at O . Now $|AEO| = |BEO|$, because the altitudes of these triangles are the same and the bases are equal, since E is the midpoint of AB . In the same way $|BFO| = |CFO|$, $|CGO| = |DGO|$ and $|DHO| = |AHO|$. By adding these equations,

$$|AEO| + |AHO| + |CFO| + |CGO| = |DGO| + |DHO| + |BEO| + |BFO|.$$

So

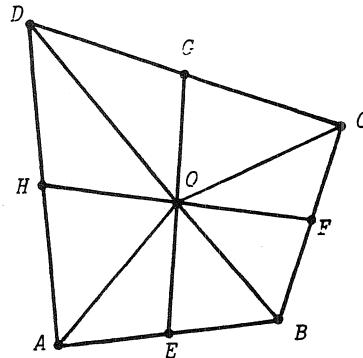
$$|AEOH| + |CGOF| = |DHOG| + |OEBF|,$$

as required.

3. Choose n points on a circle and label them with the numbers $1, 2, \dots, n$. Say that two non-neighbouring points A and B are *connectable* if the points on at least one of the two arcs containing A and B are all labelled with numbers that are less than those for A and B . Prove that the number of connectable pairs of points is $n - 3$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $f(n)$ denote the number of connectable pairs when n points on a circle are labelled. We show that $f(n) = n - 3$ for $n \geq 4$ regardless of the actual labelling of



the points. (Clearly $f(n) = 0$ for $n < 4$, since in these cases there are no non-neighbouring points.) For $n = 4$, the two points which are neighbours of 1 clearly form a connectable pair. Since 1 cannot form a connectable pair with any other point, $f(4) = 1$ follows. Suppose then that $f(n) = n - 3$ for some $n \geq 4$. Consider an arbitrary labelling of $n + 1$ points on a circle. Let s and t denote the two points which are neighbours of (the point labelled) 1. Then clearly s and t are a connectable pair. If we discard the point 1 and subtract one from the remaining n points, then we have a labelling of n points on a circle. Since 1 cannot form a connectable pair with any point, the property of being connectable remains unchanged except for the points s and t , which are now neighbours in the new configuration. It follows that $f(n + 1) = 1 + f(n) = 1 + n - 3 = n - 2$, completing the induction.

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That's all the space we have this month. Send me your nice solutions and competition problem sets!

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BOOK REVIEW

Recent Advances in Geometric Inequalities, by D.S. Mitrinović, J.E. Pečarić, and V. Volenec. Kluwer Academic Press, Dordrecht, 1989. ISBN 90-277-2565-9. Hardcover, xix + 710pp. US \$149. Reviewed by Murray S. Klamkin, University of Alberta.

This is a must book for aficionados of triangle inequalities and the like, of which *Crux* has quite a number. This new volume (which we abbreviate *AGI*) will replace *GI* (*Geometric Inequalities* by O. Bottema et al) as the bible on inequalities of the triangle for a long time to come. *AGI* gives us a comprehensive treatment of a subject which is not so easily treated. It contains several thousand inequalities, not only for elements of figures in the plane (as in *GI*), but also for elements of spherical triangles, tetrahedra, polyhedra, simplices, polytopes, and spheres. Also cited are some 750 authors from journals around the world, and many of them are cited a number of times. Even though this reviewer is quite knowledgeable in this field, very many of the references and inequalities are new to him since they were published in languages other than English and in journals not accessible to him. The authors, and their colleagues who assisted in this compendium, have done a yeoman

service. Like *GI*, *AGI* will be a catalyst to provoke and inspire further research in the field.

Readers submitting notes, problems, and/or solutions in triangle inequalities to *Crux* should be checking through this book to see whether the results are known or dual to known inequalities.

A more extensive review of this book by this reviewer will be appearing in *Mathematical Reviews*.

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PROBLEMS

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1991, although solutions received after that date will also be considered until the time when a solution is published.

1567. (Corrected) *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let

$$f(x_1, x_2, \dots, x_n) = \frac{x_1\sqrt{x_1 + \dots + x_n}}{(x_1 + \dots + x_{n-1})^2 + x_n}.$$

Prove that $f(x_1, x_2, \dots, x_n) \leq \sqrt{2}$ under the condition that $x_1 + \dots + x_n \geq 2$ and all $x_i \geq 0$.

1581* *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*

If T_1 and T_2 are two triangles with equal circumradii, it is easy to show that if the angles of T_2 majorize the angles of T_1 , then the area and perimeter of T_2 is not greater than the area and perimeter, respectively, of T_1 . (One uses the concavity of $\sin x$ and $\log \sin x$ in $(0, \pi)$.) If T_1 and T_2 are two tetrahedra with equal circumradii, and the solid angles of T_2 majorize the solid angles of T_1 , is it true that the volume, the surface area, and the total edge length of T_2 are not larger than the corresponding quantities for T_1 ?

1582. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A nonzero polynomial $P(x) = a_0 + a_1x + \cdots + a_nx^n$ is called n -*palindromic* if $a_j = a_{n-j}$ for $j = 0, 1, \dots, n$. (Note that a_n may equal 0, so that $P(x)$ need not have degree n .) It is plain that if P and Q are n -palindromic then $F = P/Q$ satisfies

$$(*) \quad F(1/x) = F(x) \text{ for all } x \text{ such that } F(x), F(1/x) \text{ are defined.}$$

Prove or disprove that, conversely, every rational function $F \neq 0$ satisfying $(*)$ is the quotient of two n -palindromic polynomials for some n .

1583. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with circumcenter O . Let P, Q be points on the sides AB, AC respectively, such that

$$\overline{BP} : \overline{PQ} : \overline{QC} = \overline{AC} : \overline{CB} : \overline{BA} .$$

Prove that A, P, Q and O are concyclic.

1584. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove that for $\lambda > 1$

$$\left(\frac{\ln \lambda}{\lambda - 1}\right)^3 < \frac{2}{\lambda(\lambda + 1)} .$$

1585. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

We are given a triangle $A_1A_2A_3$ and a real number $r > 0$. Inside the triangle, inscribe a rectangle R_1 whose height is r times its base, with its base lying on side A_2A_3 . Let B_1 be the midpoint of the base of R_1 and let C_1 be the center of R_1 . In a similar manner, locate points B_2, C_2 and B_3, C_3 , using rectangles R_2 and R_3 .

- (a) Prove that lines A_iB_i , $i = 1, 2, 3$, concur.
- (b) Prove that lines A_iC_i , $i = 1, 2, 3$, concur.

1586. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let ABC be a triangle with angles $A \geq B \geq C$ and sides $a \geq b \geq c$, and let $A'B'C'$ be a triangle with sides

$$a' = a + \lambda, \quad b' = b + \lambda, \quad c' = c + \lambda$$

where λ is a positive constant. Prove that $A - C \geq A' - C'$ (i.e., $\Delta A'B'C'$ is in a sense "more equilateral" than ΔABC).

1587*. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

When all the diagonals of a certain convex n -gon are drawn, it is found that no three of them are concurrent at an interior point and that they divide the interior of the n -gon into a square number of regions. Find the possible values of n .

1588. *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Show that

$$\sin B \sin C \leq 1 - \frac{a^2}{(b+c)^2},$$

where a, b, c are the sides of the triangle ABC .

1589. *Proposed by Mihaly Bencze, Brasov, Romania.*

Prove that, for any natural number n ,

$$\sqrt[n]{n!} + \sqrt[n+2]{(n+2)!} < 2 \cdot \sqrt[n+1]{(n+1)!}.$$

1590. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Four familiar circles in the plane of a scalene triangle are the incircle, circumcircle, nine-point circle, and the Spieker circle. Let I, O, F, S be their respective centers. Prove that the lines IO and FS are parallel.

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S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1352. [1988: 174; 1989: 219] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Determine lower and upper bounds for

$$S_r = \cos^r A + \cos^r B + \cos^r C$$

where A, B, C are the angles of a non-obtuse triangle, and r is a positive real number, $r \neq 1, 2$. (The cases $r = 1$ and 2 are known; see items 2.16 and 2.21 of Bottema et al, *Geometric Inequalities*.)

III. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

R.I. Hess [1989: 221–223] has asserted that, for each r , the lower and upper bounds for S_r always occur among the values arising from the "triangles"

$(A, B, C) = (\pi/2, \pi/2, 0)$, $(\pi/2, \pi/4, \pi/4)$, and $(\pi/3, \pi/3, \pi/3)$ (up to permutations); the precise bounds are listed in (6) on [1989: 222]. The assertion was verified when $r \leq 1$ or $r \geq 2$ (see also the proposer's solution [1989: 220]). Analysis by Hess, pursued further by the editor, resulted in the following conjecture [1989: 224] which, if true, would establish Hess's assertion for $1 < r < 2$ as well:

Let $1 < r < 2$, $r \neq 4/3$, and

$$f_r(x) = 2x^r + (1 - 2x^2)^r$$

and let x_0 be the unique solution (not equal to $1/2$) of

$$(g(x))^{r-1} = 2x, \quad 0 < x < 1/\sqrt{2},$$

where

$$g(x) = \frac{x}{1 - 2x^2}.$$

Show that

$$1 < f_r(x_0) < 2 \cdot 2^{-r/2} \quad \text{for } 1 < r < 2 \quad (1)$$

and

$$f_r(x_0) < 3 \cdot 2^{-r} \quad \text{for } 1 < r < 4/3. \quad (2)$$

Here we give a proof of this conjecture.

Let us first remark that the arguments given in [1989: 222–224] imply a good deal of this conjecture quite automatically. Since

$$f_r(x) = 2r[x^{r-1} - 2x(1 - 2x^2)^{r-1}],$$

we note that

$$\text{sign } f_r(x) = \text{sign}(g(x) - \varphi_r(x)) \quad (3)$$

where

$$\varphi_r(x) = (2x)^{1/(r-1)}, \quad 0 < x < 1/\sqrt{2}.$$

By definition, x_0 is the unique root of $g(x) = \varphi_r(x)$ in $(0, 1/2) \cup (1/2, 1/\sqrt{2})$. From

$$g'(x) = \frac{1 + 2x^2}{(1 - 2x^2)^2} \quad \text{and} \quad \varphi_r'(x) = \frac{2}{r-1}(2x)^{(2-r)/(r-1)}$$

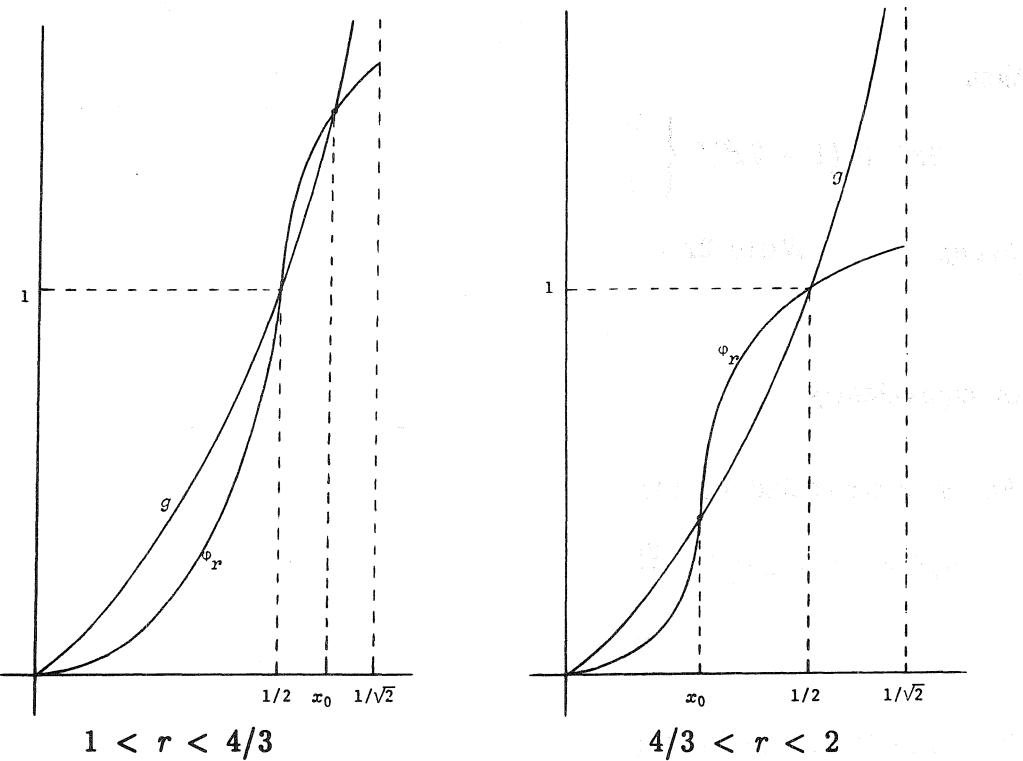
we have

$$\begin{aligned} g(0) &= \varphi_r(0) = 0, & g'(0) &= 1, & \varphi_r'(0) &= 0, \\ g(1/2) &= \varphi_r(1/2) = 1, & g'(1/2) &= 6, & \varphi_r'(1/2) &= \frac{2}{r-1}, \end{aligned}$$

and

$$\lim_{x \rightarrow 1/\sqrt{2}} g(x) = +\infty, \quad \varphi_r(1/\sqrt{2}) < \infty.$$

In order to make the following pictures more legible, the convexity of φ_r has been violated.



From the uniqueness of x_0 we get the following conclusions.

(i) If $1 < r < 4/3$, then $\varphi_r(1/2) > 6$, so $x_0 > 1/2$ and we have

$$\begin{aligned}\varphi_r &< g \text{ in } (0, 1/2) \cup (x_0, 1/\sqrt{2}), \\ \varphi_r &> g \text{ in } (1/2, x_0).\end{aligned}$$

(ii) If $4/3 < r < 2$, then $\varphi_r(1/2) < 6$, so $x_0 < 1/2$ and we have

$$\begin{aligned}\varphi_r &< g \text{ in } (0, x_0) \cup (1/2, 1/\sqrt{2}), \\ \varphi_r &> g \text{ in } (x_0, 1/2).\end{aligned}$$

In either case, the points $1/2$ and x_0 partition $(0, 1/\sqrt{2})$ into three subintervals, inequality $\varphi_r > g$ holding in the middle subinterval and the opposite inequality in the other two. This in view of (3) means that f_r increases in the two outer subintervals and decreases in between. Accordingly, we get

$$\begin{aligned}f_r(x_0) &< \min\{f_r(1/2), f_r(1/\sqrt{2})\} = \min\{3 \cdot 2^{-r}, 2 \cdot 2^{-r/2}\}, \quad 1 < r < 4/3, \\ f_r(x_0) &> \max\{f_r(0), f_r(1/2)\} = \max\{1, 3 \cdot 2^{-r}\}, \quad 4/3 < r < 2.\end{aligned}$$

These relations establish the upper bounds of (1) and (2) for $1 < r < 4/3$ and the lower bound of (1) for $4/3 < r < 2$. What remains of (1) and (2) to be justified is exactly the contents of the following claim.

Claim. If the two numbers $r \in (1, 4/3) \cup (4/3, 2)$ and $x \in (0, 1/2) \cup (1/2, 1/\sqrt{2})$ are connected by the equation

$$\left(\frac{x}{1 - 2x^2}\right)^{r-1} = 2x, \tag{4}$$

then

$$2x^r + (1 - 2x^2)^r \begin{cases} > 1 & \text{for } 1 < r < 4/3, \text{ i.e. } 1/2 < x < 1/\sqrt{2}, \\ < 2 \cdot 2^{-r/2} & \text{for } 4/3 < r < 2, \text{ i.e. } 0 < x < 1/2. \end{cases} \quad (5)$$

Proof. Write $2x = \sqrt{1-t}$, $-1 < t < 1$, $t \neq 0$. Then (4) becomes

$$\left(\frac{\sqrt{1-t}}{1+t} \right)^{r-1} = \sqrt{1-t} \quad (6)$$

or equivalently

$$(1-t)^{r/2} = (1-t)(1+t)^{r-1}. \quad (7)$$

For x , r connected by (4), the quantity in (5) equals, by (7),

$$\begin{aligned} 2x^r + (1-2x^2)^r &= \frac{2(1-t)^{r/2}}{2^r} + \left(\frac{1+t}{2} \right)^r \\ &= \frac{2(1-t)(1+t)^{r-1}}{2^r} + \frac{(1+t)^r}{2^r} = \frac{(1+t)^{r-1}(3-t)}{2^r}. \end{aligned}$$

The asserted inequalities (5) then take the form

$$(1+t)^{r-1}(3-t) \begin{cases} > 2^r & \text{for } 1 < r < 4/3, \text{ i.e. } -1 < t < 0, \\ < 2 \cdot 2^{r/2} & \text{for } 4/3 < r < 2, \text{ i.e. } 0 < t < 1. \end{cases} \quad (8)$$

On the other hand, we get from (6) that

$$r = \frac{2 \ln(1+t) - 2 \ln(1-t)}{2 \ln(1+t) - \ln(1-t)}.$$

This allows us to express everything in terms of a single variable t . Write

$$u = u(t) = \ln(1+t), \quad v = v(t) = \ln(1-t), \quad w = w(t) = \ln(3-t).$$

By taking logarithms, (8) becomes

$$\frac{uv}{v-2u} + w \begin{cases} > \frac{(2u-2v)}{2u-v} \ln 2 & \text{for } -1 < t < 0, \\ < \frac{(3u-2v)}{2u-v} \ln 2 & \text{for } 0 < t < 1. \end{cases}$$

For $|t| < 1$ we have that $\text{sign}(2u-v) = \text{sign}(t)$. So after small manipulations we obtain the following inequalities to prove:

$$2uw - uv - vw < \begin{cases} (2u-2v)\ln 2 & \text{for } -1 < t < 0, \\ (3u-2v)\ln 2 & \text{for } 0 < t < 1. \end{cases} \quad (9)$$

We shall employ the following quadratic estimates of the logarithmic function:

Lemma. For $0 < y \leq c < 1$,

$$y - \frac{1}{2}y^2 < \ln(1+y) < y - \left(\frac{3}{6} - \frac{2c}{6} \right)y^2,$$

$$y + \frac{1}{2}y^2 < -\ln(1-y) < y + \left(\frac{1}{2} - \frac{2c}{2} \right)y^2.$$

Proof of lemma. The left hand inequalities require no comment. For the right hand inequalities,

$$\ln(1 + y) < y - \frac{y^2}{2} + \frac{y^3}{3} \leq y - \frac{y^2}{2} + \frac{c}{3}y^2 = y - \left(\frac{3-2c}{6}\right)y^2$$

and

$$\begin{aligned} -\ln(1-y) &= y + \frac{y^2}{2} + \frac{y^3}{3} + \dots < y + \frac{y^2}{2}(1+y+y^2+\dots) \\ &= y + \frac{y^2}{2}\left(\frac{1}{1-y}\right) \leq y + \frac{y^2}{2}\left(\frac{1}{1-c}\right). \end{aligned}$$

□

We pass to the proof of (9).

Case (i): $0 < t \leq 1/7$.

By the lemma (with $c = 1/7$),

$$t - \frac{t^2}{2} < u < t - \frac{19}{42}t^2, \quad t + \frac{t^2}{2} < -v < t + \frac{7}{12}t^2.$$

Hence

$$3u - 2v > 5t - \frac{t^2}{2}, \quad 2u - v < 3t - \frac{9}{28}t^2,$$

and

$$-uv < t^2 + \frac{11}{84}t^3 - \frac{19}{72}t^4 < t^2 + t^3.$$

Since $2u - v > 0$ and $w < \ln 3$, the above inequalities imply

$$LHS(9) = (2u - v)w - uv < (3t - \frac{9}{28}t^2)\ln 3 + t^2 + t^3,$$

$$RHS(9) = (3u - 2v)\ln 2 > (5t - \frac{t^2}{2})\ln 2,$$

and thus

$$\begin{aligned} RHS(9) - LHS(9) &> t[5\ln 2 - 3\ln 3 - (1 - \frac{9}{28}\ln 3 + \frac{1}{2}\ln 2)t - t^2] \\ &> t\left(\frac{1}{6} - t - t^2\right) \geq t\left(\frac{1}{6} - \frac{1}{7} - \frac{1}{49}\right) > 0. \end{aligned}$$

Case (ii): $1/7 < t < 1$.

Using the substitution $t = 1 - s$, $0 < s < 6/7$, we have

$$u = \ln(2-s) = \ln 2 + \ln(1-s/2) > 0,$$

$$v = \ln s < 0,$$

$$w = \ln(2+s) = \ln 2 + \ln(1+s/2) > 0.$$

By the lemma (with $y = s/2$ and $c = 3/7$),

$$u < \ln 2 - \frac{s}{2} - \frac{s^2}{8}, \quad w < \ln 2 + \frac{s}{2} - \frac{5}{56}s^2. \quad (10)$$

Rewrite claim (9) as

$$(2w - 3\ln 2)u < (u + w - 2\ln 2)v. \quad (11)$$

In view of (10),

$$u + w - 2 \ln 2 < -\frac{3}{14}s^2, \quad (12)$$

$$2w - 3 \ln 2 < -\ln 2 + s - \frac{5}{28}s^2.$$

Observe that the last expression grows with s . Thus, if $0 < s \leq 4/5$ then

$$2w - 3 \ln 2 < -\ln 2 + \frac{4}{5} - \frac{5}{28}\left(\frac{4}{5}\right)^2 < 0,$$

proving (11), since $LHS(11) < 0 < RHS(11)$ (the last inequality follows from (12)). Assume therefore that $4/5 < s < 6/7$; then

$$2w - 3 \ln 2 < -\ln 2 + \frac{6}{7} - \frac{5}{28}\left(\frac{6}{7}\right)^2 < \frac{1}{25}.$$

By (10),

$$u < \ln 2 - \frac{1}{2} \cdot \frac{4}{5} - \frac{1}{8}\left(\frac{4}{5}\right)^2 < \frac{1}{4}$$

and hence $LHS(11) < 1/100$. On the other hand, from (12) we get

$$RHS(11) > \frac{3}{14}s^2 |\ln s| > \frac{3}{14}\left(\frac{4}{5}\right)^2 \ln \frac{7}{6} > \frac{1}{100}.$$

Thus (11) results, and claim (9) is proved for $0 < t < 1$.

Case (iii): $-1/4 \leq t < 0$.

By the lemma (with $y = -t$, $c = 1/4$),

$$-t - \frac{t^2}{2} < v < -t - \frac{5}{12}t^2$$

and

$$-t + \frac{t^2}{2} < -u < -t + \frac{2}{3}t^2.$$

Hence

$$v - u < -2t + \frac{t^2}{4}, \quad v - 2u > -3t + \frac{t^2}{2} > 0,$$

$$-uv < t^2 - \frac{t^3}{4} - \frac{5}{18}t^4 < t^2 - \frac{t^3}{4}.$$

These inequalities, together with $w > 1$, give

$$LHS(9) = -(v - 2u)w - uv < \left(3t - \frac{t^2}{2}\right) + \left(t^2 - \frac{t^3}{4}\right) = 3t + \frac{t^2}{2} - \frac{t^3}{4},$$

whereas

$$RHS(9) = (-2 \ln 2)(v - u) > 4t \ln 2 - \frac{t^2}{2} \ln 2.$$

Thus

$$RHS(9) - LHS(9) > |t| \left[3 - 4 \ln 2 - \frac{|t|}{2}(1 + \ln 2) - \frac{t^2}{4} \right]$$

$$\geq |t| \left[3 - 4 \ln 2 - \frac{1}{8}(1 + \ln 2) - \frac{1}{4^3} \right] > 0.$$

Case (iv): $-1 < t < -1/4$.

This time we substitute $t = s - 1$, $0 < s < 3/4$. So

$$\begin{aligned} u &= \ln s < 0, \\ v &= \ln(2 - s) = \ln 2 + \ln\left(1 - \frac{s}{2}\right) > 0, \\ w &= \ln(4 - s) = 2 \ln 2 + \ln\left(1 - \frac{s}{4}\right) > 0. \end{aligned}$$

Applying the second line of the lemma, putting first $y = s/2$, and then $y = s/4$ and $c = 1/2$, we obtain

$$v < \ln 2 - \frac{s}{2} - \frac{s^2}{8}, \quad w > 2 \ln 2 - \frac{s}{4} - \frac{s^2}{16}. \quad (13)$$

Now we rewrite claim (9) in the form

$$(2 \ln 2 - w)v < -(2w - v - 2 \ln 2)u. \quad (14)$$

It follows from (13) that

$$v < \ln 2 - \frac{s}{2}, \quad 2 \ln 2 - w < \frac{s}{4} + \frac{s^2}{16},$$

whence

$$\begin{aligned} LHS(14) &< \left(\frac{s}{4} + \frac{s^2}{16}\right)\left(\ln 2 - \frac{s}{2}\right) < \frac{s}{4}\left[\ln 2 + \left(\frac{\ln 2}{4} - \frac{1}{2}\right)s\right] \\ &< \frac{s}{4}\ln 2 < \frac{3}{16}\ln 2. \end{aligned}$$

(13) also implies

$$2w - v - 2 \ln 2 > \ln 2.$$

This together with $-u = |\ln s| > \ln(4/3) > 1/4$ gives us

$$RHS(14) > \frac{\ln 2}{4} > LHS(14).$$

Hence (9) is settled also in this case. This ends the proof of the claim.

Is there no nicer proof?!

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1386. [1988: 269; 1990: 21] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2\cdots A_n$ be a polygon inscribed in a circle and containing the centre of the circle. Prove that

$$n - 2 + \frac{4}{\pi} < \sum_{i=1}^n \frac{a_i}{\hat{a}_i} \leq \frac{n^2}{\pi} \sin \frac{\pi}{n},$$

where a_i is the side A_iA_{i+1} and \hat{a}_i is the arc A_iA_{i+1} .

II. *Comments by Murray S. Klamkin, University of Alberta.*

In the following we consider the case when the circumcenter does not lie in the interior of the polygon. For this case we presume that for all but the

largest side, the arcs are the corresponding minor ones, while for the largest side the arc is the major one.

As in the published solution [1990: 22], $(\sin x)/x$ is concave in $[0, \pi/2]$ (note that $x = 0$ is a removable singularity). If $2\varphi_1, 2\varphi_2, \dots, 2\varphi_n$ are the angles subtended by the sides of the polygon at the center, with $2\varphi_1$ the largest, then $\pi \geq \varphi_1 \geq \pi/2$. Note that we are assuming the circumcenter either lies on the largest side ($\varphi_1 = \pi/2$) or in the exterior of the polygon ($\pi/2 < \varphi_1 < \pi$) or else the polygon is degenerate with all sides 0 so that $\varphi_1 = \pi$ and $\varphi_2 = \varphi_3 = \dots = 0$. It then follows that

$$(\pi, 0, \dots, 0) \succ (\varphi_1, \varphi_2, \dots, \varphi_n) \succ \left(\frac{\pi}{2}, \frac{\pi}{2(n-1)}, \dots, \frac{\pi}{2(n-1)} \right).$$

It now follows by the majorization inequality [1] (and $a_i/\hat{a}_i = (\sin \varphi_i)/\varphi_i$) that

$$n-1 \leq \sum_{i=1}^n \frac{a_i}{\hat{a}_i} \leq \frac{2}{\pi} + \frac{2(n-1)^2}{\pi} \sin\left(\frac{\pi}{2(n-1)}\right).$$

The upper bound is achieved when one side is a diameter and the remaining sides are all equal. The lower bound is achieved for a degenerate polygon with all sides 0.

Now we give some extensions of the previous results using the following:

- (A) *If $G(x)$ is convex in $[a, b]$ and $F(x)$ is a nondecreasing convex function over the range of $G(x)$ for x in $[a, b]$, then $F(G(x))$ is convex in $[a, b]$.*

Note that

$$\frac{d^2}{dx^2}(F(G(x))) = F'(G(x))G''(x) + F''(G(x))(G'(x))^2 \geq 0.$$

Hence if $F(x)$ is a nondecreasing convex function over the range of $-(\sin x)/x$ for x in $[0, \pi/2]$, then $F(-(\sin x)/x)$ is convex, and so by the majorization inequality,

$$2F\left(\frac{-2}{\pi}\right) + (n-2)F(-1) \geq \sum F\left(\frac{-\sin \varphi_i}{\varphi_i}\right) \geq nF\left(-\frac{n}{\pi} \sin \frac{\pi}{n}\right)$$

for the case that the circumcenter either lies in the interior or on the boundary of the polygon. For the left hand bound, the polygon is degenerate with two sides equal to a diameter and the rest of the sides zero. For the right hand bound, the polygon is regular. For the case when the center lies either on a side or in the exterior of the polygon,

$$F(0) + (n-1)F(-1) \geq \sum_{i=1}^n F\left(\frac{-\sin \varphi_i}{\varphi_i}\right) \geq F\left(\frac{-2}{\pi}\right) + (n-1)F\left(\frac{-2(n-1)}{\pi} \sin\left(\frac{\pi}{2(n-1)}\right)\right).$$

For the left hand bound, the polygon is degenerate with all sides equal to zero, while for the right hand bound one side is a diameter and the remaining sides are equal. In particular for $F(x) = x$ we recapture the previous inequalities.

For another example, we use $F(x) = -\ln(-x)$ and $G(x) = -(\sin x)/x$ in (A) to conclude that $-\ln((\sin x)/x)$ is convex in $[0, \pi/2]$. Then by the majorization inequality,

$$\left(\frac{n}{\pi} \sin \frac{\pi}{n}\right)^n \geq \prod_{i=1}^n \frac{a_i}{\hat{a}_i} \geq \frac{4}{\pi^2}$$

for the case when the center of the polygon lies in the interior or on a side of the polygon, and

$$\frac{2}{\pi} \left(\frac{2(n-1)}{\pi} \sin \left(\frac{\pi}{2(n-1)} \right) \right)^{n-1} \geq \prod_{i=1}^n \frac{a_i}{\hat{a}_i} \geq 0$$

for the case when the center lies either on a side or in the exterior of the polygon.

Reference:

- [1] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

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1458. [1989: 178] *Proposed by Edward T.H. Wang, Wilfrid Laurier University.*

(a) Characterize all natural numbers n for which there exists a group with exactly two elements of order n .

(b) Characterize all natural numbers n for which there exists a group with exactly three elements of order n .

I. *Solution by David Poole, Trent University, Peterborough, Ontario.*

Let G be a group with an element a of order n .

(a) Suppose G has exactly two elements of order n .

Case (i): $n > 2$. In this case, $a \neq a^{-1}$ and these are the two elements of order n . Now a^m also has order n for all $1 < m < n-1$ such that $(m,n) = 1$. Therefore a necessary condition on n is that $\phi(n) = 2$, where ϕ is the Euler phi-function. Hence $n = 3, 4$ or 6 . The groups \mathbb{Z}_3 , \mathbb{Z}_4 and \mathbb{Z}_6 show that this condition is also sufficient.

Case (ii): $n = 2$. Let $a = a^{-1}$ and $b = b^{-1}$ be the two elements of order 2. If $ab = ba$ then ab also has order 2 and is distinct from both a and b ; if $ab \neq ba$ then bab^{-1} has order 2 and is distinct from both a and b . Thus, in either case, G has at least three elements of order 2, a contradiction.

Therefore there exists a group with exactly two elements of order n if and only if $n = 3, 4$, or 6 .

(b) Suppose G has exactly 3 elements of order n .

Case (i): $n > 2$. Since every element can be paired with its (distinct) inverse and both have the same order, there must be an even number of elements of order n . Hence there are no groups of the type we seek in this case.

Case (ii): $n = 2$. $\mathbb{Z}_2 \times \mathbb{Z}_2$ has three elements of order 2.

Therefore there exists a group with exactly three elements of order n if and only if $n = 2$.

II. *Generalization by Duane M. Broline, Eastern Illinois University, Charleston.*

For $k < 10$, we find all natural numbers n for which there is a group with exactly k elements of order n .

In the following table, the group in the n th row and k th column is a group with exactly k elements of order n . If an entry is blank, no group with the desired properties exists. The cyclic group of order n is represented by \mathbb{Z}_n , the dihedral group with $2n$ elements by D_{2n} , and the quaternion group of order 8 by Q_8 .

$n \backslash k$	1	2	3	4	5	6	7	8	9
1	\mathbb{Z}_1								
2	\mathbb{Z}_2		\mathbb{Z}_2^2		D_8		\mathbb{Z}_2^3		D_{16}
3		\mathbb{Z}_3						\mathbb{Z}_3^2	
4		\mathbb{Z}_4		$\mathbb{Z}_2 \times \mathbb{Z}_4$		Q_8		$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	
5				\mathbb{Z}_5					
6		\mathbb{Z}_6				$\mathbb{Z}_2 \times \mathbb{Z}_6$		$\mathbb{Z}_3 \times \mathbb{Z}_6$	
7						\mathbb{Z}_7			
8				\mathbb{Z}_8				$\mathbb{Z}_2 \times \mathbb{Z}_8$	
9						\mathbb{Z}_9			
10				\mathbb{Z}_{10}					
12				\mathbb{Z}_{12}				$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	
14						\mathbb{Z}_{14}			
16								\mathbb{Z}_{16}	
18						\mathbb{Z}_{18}			
20								\mathbb{Z}_{20}	

It is routine to verify that the given groups have the stated properties. The following lemmas establish that no examples of groups exist for the remaining values of (n, k) shown.

Lemma 1. If G is a finite group and n is a natural number, then the number of elements in G of order n is a multiple of $\phi(n)$ (the Euler phi-function).

Proof. We use $o(g)$ to represent the order of an element g . Let

$$A = \{g \in G \mid o(g) = n\}.$$

Define a relation \sim on A by $g \sim h$ if and only if the group $\langle g \rangle$ generated by g is equal to the group generated by h . Clearly this relation is an equivalence relation. Further, if $g \in A$, the equivalence class of g is $\{g^r \mid 1 \leq r < n, (r, n) = 1\}$. Thus each equivalence class has $\phi(n)$ elements, and the cardinality $|A|$ of A is divisible by $\phi(n)$, as claimed. ■

Lemma 2. Let p be a prime and G a finite group having at least one element of order p . If G has exactly k elements of order p , then $k \equiv -1 \pmod{p}$.

Proof. Let $A = \{g \in G \mid o(g) = p\}$ and let t be a fixed element of A . Define a relation \sim on A by $g \sim h$ if and only if $g = t^{-i}ht^i$ for some integer i . Clearly this relation is an equivalence relation.

The equivalence class of a specific element a of A is $\{t^{-i}at^i \mid i \in \mathbb{Z}\}$. Since t has order p , this class has at most p elements. If it has fewer than p elements, then $t^{-r}at^r = a$ for some r , $1 \leq r < p$. Therefore, the group generated by t^r is contained in C_a , the centralizer of a . Because $(r, p) = 1$, $\langle t^r \rangle = \langle t \rangle$ and $t \in C_a$. Thus the equivalence class of an element in A has either p elements or 1 element, and in the latter case the element is in C_t .

Let B be the set of all elements of A whose equivalence class under \sim has size 1. Define a relation \approx on $B \cup \{1\}$ by $g \approx h$ if and only if $g = ht^i$, for some integer i . It is routine to show that this relation is an equivalence relation. Since $B \subset C_t$ and t has order p , the equivalence class of a specific element $g \in B$ is $\{gt^i \mid i = 0, 1, \dots, p-1\}$. If the equivalence class of g under \approx has fewer than p elements, then $g = gt^i$ for some i , $0 < i < p$. Therefore $t^i = 1$, which is impossible. Thus each equivalence class under \approx has p elements.

Hence

$$|A| = |A - B| + |B \cup \{1\}| - 1 \equiv -1 \pmod{p},$$

which completes the proof. ■

Lemma 3. If p is an odd prime and G is a finite group with more than $p - 1$ elements of order $2p$, then G has more than $2(p - 1)$ elements of order $2p$.

Proof. The proof is by contradiction. Suppose the lemma is false and that G is a minimal counterexample. By Lemma 1, the number of elements of order $2p$ is a multiple of $p - 1$. Hence G has exactly $2(p - 1)$ elements of order $2p$.

Let $x \in G$ be an element of order $2p$ and set $H = \langle x \rangle$. Suppose $g^{-1}Hg \neq H$ for some $g \in G$. Since H and thus $g^{-1}Hg$ contains $p - 1$ elements of order $2p$ and none are generators of H , all elements of order $2p$ in G are in $H \cup g^{-1}Hg$. It follows that G contains at most two different conjugates of H and thus $[G:N] = 2$, where N is the normalizer of H . Therefore N is a normal subgroup of G . Hence

$$g^{-1}Hg \subset g^{-1}Ng = N,$$

and N contains more than $p - 1$ elements of order $2p$. But N contains at most $2(p - 1)$ elements of order $2p$ and the minimality of G forces $G = N$, i.e. H is a normal subgroup of G .

Since G has more than $p - 1$ elements of order $2p$, there exists $y \in G \setminus H$ with $o(y) = 2p$. Now $H\langle y \rangle = \{hy^i \mid h \in H, i \in \mathbb{Z}\}$ is a subgroup of G (since H is normal in G) which satisfies the hypothesis of the lemma but not the conclusion. Since G is a minimal counterexample, $H\langle y \rangle = G$. Further, by the argument above, $\langle y \rangle$ is a normal subgroup of G . In particular then, the order of G divides into $|\langle x \rangle| \cdot |\langle y \rangle| = 4p^2$.

Because $\langle x^p \rangle$ is the unique subgroup of order 2 in the normal subgroup $\langle x \rangle$, it follows that $[g^{-1}x^pg = x^p \text{ for all } g \in G \text{ and thus}]$ $\langle x^p \rangle$ is normal in G . Similar arguments show that $\langle x^2 \rangle$, $\langle y^p \rangle$, and $\langle y^2 \rangle$ are all normal subgroups of G . Since each of these subgroups is of prime order, each centralizes all of the others and it follows that G is abelian. [Editor's note. At this point expert colleague H.K. Farahat answered the editor's call for help with the following clarification. The p elements $x, xy^{-1}, y^2xy^{-2}, \dots, y^{p-1}xy^{-(p-1)}$ all have order $2p$, and all lie in $\langle x \rangle$ which contains $p - 1$ elements of order $2p$. Thus two of them must be equal, which implies that $y^i xy^{-i} = x$ for some $1 \leq i \leq p - 1$, that is, x commutes with y^i . But as above we also have $y^p x = xy^p$. It follows that $xy = yx$, and so G is abelian.]

In particular, the Sylow 2-subgroup of G is $\langle x^p \rangle \langle y^p \rangle$ and is isomorphic to either \mathbb{Z}_2 or \mathbb{Z}_2^2 . The Sylow p -subgroup of G is $\langle x^2 \rangle \langle y^2 \rangle$ and is isomorphic to either \mathbb{Z}_p or \mathbb{Z}_p^2 . Thus G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_p$, $\mathbb{Z}_2 \times \mathbb{Z}_p^2$, $\mathbb{Z}_2^2 \times \mathbb{Z}_p$, or $\mathbb{Z}_2^2 \times \mathbb{Z}_p^2$. [Editor's note. One could avoid mention of Sylow subgroups by adding to this list $\mathbb{Z}_4 \times \mathbb{Z}_p$ and $\mathbb{Z}_4 \times \mathbb{Z}_p^2$, which are then ruled out as below.] Since these groups have $p - 1$, $p^2 - 1$, $3(p - 1)$, and $3(p^2 - 1)$ elements of order $2p$, respectively, no counterexample to the lemma exists and the proof is complete. ■

Also solved by KEE-WAI LAU, Hong Kong; PETER ROSS, Santa Clara University, Santa Clara, California; and the proposer. Two incorrect solutions were received.

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1459. [1989: 178] Proposed by Toshio Seimiya, Kawasaki, Japan.

$\triangle ABC$ is an isosceles triangle in which $AB = AC$ and $\angle A < 120^\circ$. Let D be the point on side BC such that $BD = 2DC$, and let E be the point on segment AD such that $\angle BED = 2\angle DEC$. Prove that $\angle BED = \angle BAC$.

Solution by Hans Engelhardt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

The angles of $\triangle ABC$ are denoted α , β , γ . Let $\varepsilon = \angle DEC$, let F be the intersection of AD and the circumcircle of $\triangle ABC$, and let C' be on BF so that EC' bisects $\angle BEF$. By the law of sines,

$$\frac{EB}{BD} = \frac{\sin \angle BDA}{\sin 2\varepsilon} = \frac{\sin \angle BDA}{2\sin \varepsilon \cos \varepsilon}$$

and

$$\frac{EC}{DC} = \frac{\sin \angle CDA}{\sin \varepsilon} = \frac{\sin \angle BDA}{\sin \varepsilon}.$$

Thus, since $BD = 2DC$,

$$EC = EB \cos \varepsilon. \quad (1)$$

Since $\triangle ABC$ is isosceles, $\angle BFA = \angle CFA = \beta = \gamma$. By axial symmetry with AF as axis, $EC' = EC$. Because of (1), $\angle BC'E$ must be 90° . Thus $\varepsilon = 90^\circ - \beta$, so

$$\angle BED = 2\varepsilon = 180^\circ - \beta - \gamma = \alpha = \angle BAC.$$

Also solved by HARRY ALEXIEV, Zlatograd, Bulgaria; BENO ARBEL, Tel Aviv University, Israel; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

As was pointed out by some of the solvers, the condition $\angle A < 120^\circ$ guarantees that point E lies on the segment AD .

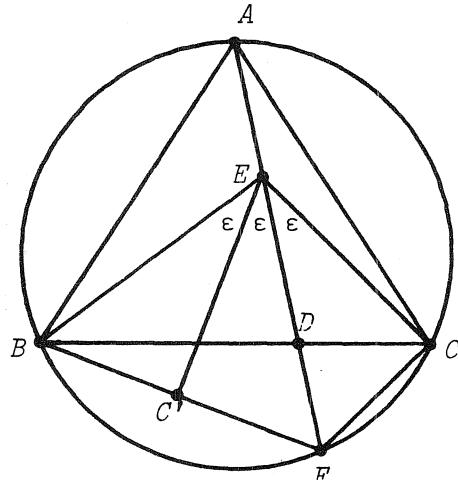
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1463. [1989: 207] Proposed by Stanley Rabinowitz, Westford, Massachusetts.

Prove that if n and r are integers with $n > r$ then



$$\sum_{k=1}^n \cos^2 r \left(\frac{k\pi}{n} \right) = \frac{n}{4r} \binom{2r}{r} .$$

Solution by M. Selby, University of Windsor.

Since

$$\cos \frac{k\pi}{n} = \frac{1}{2}(e^{ik\pi/n} + e^{-ik\pi/n}) ,$$

we have

$$\begin{aligned} \sum_{k=1}^n \cos^2 r \left(\frac{k\pi}{n} \right) &= \frac{1}{4r} \sum_{k=1}^n (e^{ik\pi/n} + e^{-ik\pi/n})^{2r} \\ &= \frac{1}{4r} \sum_{k=1}^n \sum_{s=0}^{2r} \binom{2r}{s} (e^{ik\pi/n})^{2r-s} (e^{-ik\pi/n})^s \\ &= \frac{1}{4r} \sum_{k=1}^n \sum_{s=0}^{2r} \binom{2r}{s} e^{2i\pi k(r-s)/n} \\ &= \frac{1}{4r} \sum_{s=0}^{2r} \binom{2r}{s} \sum_{k=1}^n e^{2i\pi k(r-s)/n} . \end{aligned}$$

Since $r < n$, $|r - s| < n$ and hence $e^{2i\pi(r-s)/n} = 1$ if and only if $r = s$. For $r \neq s$,

$$\sum_{k=1}^n e^{2i\pi k(r-s)/n} = \frac{e^{2\pi i(r-s)/n}(e^{2\pi i(r-s)} - 1)}{e^{2\pi i(r-s)/n} - 1} = 0 ,$$

while for $r = s$

$$\sum_{k=1}^n e^{2i\pi k(r-s)/n} = \sum_{k=1}^n 1 = n .$$

Therefore

$$\sum_{k=1}^n \cos^2 r \left(\frac{k\pi}{n} \right) = \frac{1}{4r} \binom{2r}{r} \cdot n .$$

Also solved (usually the same way) by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; JILL HOUGHTON, Sydney, Australia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, New York; VEDULA N. MURTY, Pennsylvania State University,

Middletown; P. PENNING, Delft, The Netherlands; CHRIS WILDHAGEN, Breda, The Netherlands; KENNETH S. WILLIAMS, Carleton University; and the proposer.

Janous observes that since

$$\int_0^\pi \cos^{2r} x \, dx = \binom{2r}{r} \frac{\pi}{4^r}$$

(cf. p. 421 of Gradshteyn and Ryzhik, Tables of Series, Products and Integrals, vol. I, 1981), this integral equals its rectangular approximation (Riemann sum)

$$\frac{\pi}{n} \sum_{k=1}^n \cos^{2r} \left(\frac{k\pi}{n} \right) .$$

Williams notes that the similar problem

$$\sum_{k=0}^{r-1} (-1)^k \cos^n \left(\frac{k\pi}{r} \right) = \frac{r}{2^{n-1}}$$

is Problem 79 of The Red Book (100 practice problems for undergraduate mathematics competitions), K.S. Williams and K. Hardy, Integer Press, Ottawa, 1988.

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1464. [1989: 207] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC , so that $A' \in BC$, $B' \in CA$, $C' \in AB$.

(a) Prove that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} \quad (1)$$

if and only if the centroids G , G' of the two triangles coincide.

(b) Prove that if (1) holds, and either the circumcenters O , O' or the orthocenters H , H' of the triangles coincide, then ΔABC is equilateral.

(c)* If (1) holds and the incenters I , I' of the triangles coincide, characterize ΔABC .

I. *Solution to (a) and (b) by Murray S. Klamkin, University of Alberta.*

(a) Let \mathbf{A} , \mathbf{B} , \mathbf{C} denote vectors from the centroid of ΔABC to the respective vertices, so that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$. If (1) is true, then

$$\mathbf{A}' = x\mathbf{B} + (1-x)\mathbf{C}, \quad \mathbf{B}' = x\mathbf{C} + (1-x)\mathbf{A}, \quad \mathbf{C}' = x\mathbf{A} + (1-x)\mathbf{B} \quad (2)$$

for some $0 \leq x \leq 1$. It now follows immediately that

$$\mathbf{A}' + \mathbf{B}' + \mathbf{C}' = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

or that $G = G'$.

We now assume $G = G'$. Here

$$\mathbf{A}' = x\mathbf{B} + (1-x)\mathbf{C}, \quad \mathbf{B}' = y\mathbf{C} + (1-y)\mathbf{A}, \quad \mathbf{C}' = z\mathbf{A} + (1-z)\mathbf{B}$$

for $0 \leq x, y, z \leq 1$, and we will show $x = y = z$, which implies (1). Since $\mathbf{A}' + \mathbf{B}' + \mathbf{C}' = \mathbf{0}$,

$$(1 - y + z)\mathbf{A} + (1 - z + x)\mathbf{B} + (1 - x + y)\mathbf{C} = \mathbf{0}.$$

Replacing \mathbf{A} by $-\mathbf{B} - \mathbf{C}$ gives

$$(x + y - 2z)\mathbf{B} + (2y - z - x)\mathbf{C} = \mathbf{0}.$$

Thus $x + y - 2z = 0$ and $2y - z - x = 0$, or $x = y = z$.

(b) Suppose (1) and that $O = O'$. Here we let O be the origin of our vectors so that $|\mathbf{A}| = |\mathbf{B}| = |\mathbf{C}| = R$ and $|\mathbf{A}'| = |\mathbf{B}'| = |\mathbf{C}'|$. Using (2),

$$|\mathbf{A}'|^2 = x^2R^2 + (1 - x)^2R^2 + 2x(1 - x)\mathbf{B} \cdot \mathbf{C}.$$

Since

$$2\mathbf{B} \cdot \mathbf{C} = 2R^2\cos 2A = 2R^2 - 4R^2\sin^2 A = 2R^2 - a^2,$$

we get

$$|\mathbf{A}'|^2 = x^2R^2 + (1 - x)^2R^2 + 2x(1 - x)R^2 - x(1 - x)a^2 = R^2 - x(1 - x)a^2.$$

Similarly

$$|\mathbf{B}'|^2 = R^2 - x(1 - x)b^2 \quad \text{and} \quad |\mathbf{C}'|^2 = R^2 - x(1 - x)c^2.$$

Hence $a = b = c$ (it is tacitly assumed that $x \neq 0$ or 1).

Suppose (1) and that $H = H'$. Here we let H be the origin of our vectors so that

$$\mathbf{A} \cdot (\mathbf{B} - \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} - \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} - \mathbf{B}) = 0$$

or $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = \lambda$. Also $\mathbf{A}' \cdot \mathbf{B}' = \mathbf{B}' \cdot \mathbf{C}' = \mathbf{C}' \cdot \mathbf{A}'$. Then from (2), with $y = 1 - x$,

$$\mathbf{A}' \cdot \mathbf{B}' = (x^2 + xy + y^2)\lambda + xyC^2.$$

Similarly,

$$\mathbf{B}' \cdot \mathbf{C}' = (x^2 + xy + y^2)\lambda + xyA^2 \quad \text{and} \quad \mathbf{C}' \cdot \mathbf{A}' = (x^2 + xy + y^2)\lambda + xyB^2.$$

Then $A^2 = B^2 = C^2$ which easily implies ΔABC is equilateral (again it is tacitly assumed that $x \neq 0$ or 1).

Here are some related open problems. If two distinct triangles, one inscribed in the other, have two of: the same orthocenters; the same circumcenters; the same incenters; must they be equilateral? In three dimensions, can two distinct non-isosceles tetrahedra, one properly inscribed in the other, have the same centroids, the same incenters, and the same circumcenters?

II. Solution to the case $H = H'$ of part (b) by the proposer.

The triangles ABC and $A'B'C'$ have the same centroid and the same orthocenter. Therefore from Euler's theorem (the Euler line) [i.e. O , G , H are collinear with G one-third of the way from O to H] we know that the triangles have the same circumcenter, that is our problem is exactly the case $O = O'$.

Also solved (parts (a) and (b) only) by L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

No correct solutions for part (c) were sent in. Hut gave a solution which appears to have a serious flaw. Janous commented that in trying part (c) he "got lost in awful and disgusting expressions."

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1465. [1989: 207] *Proposed by Murray S. Klamkin, University of Alberta.*

A_1, A_2, \dots, A_n and B_1, B_2, \dots, B_n are two sets of points on an n -dimensional unit sphere of center O such that each of the two sets of vectors A_i and B_i (A_i denoting the vector from O to A_i , etc.) are orthonormal. Show that for any $0 \leq r \leq n$ the two simplexes

$$OA_1 \cdots A_r B_{r+1} \cdots B_n \quad \text{and} \quad OB_1 \cdots B_r A_{r+1} \cdots A_n$$

have equal volumes.

Combined solutions of Graham Denham, student, University of Alberta, and Marcin E. Kuczma, Warszawa, Poland.

Let M be the matrix representing the orthonormal system (B_i) with respect to the basis (A_i) . [So letting B be the matrix with columns B_i and A the matrix with columns A_i , we have $B = AM$.] Let M_1 be the matrix with columns $A_1, \dots, A_r, B_{r+1}, \dots, B_n$, and let M_2 be the matrix with columns $B_1, \dots, B_r, A_{r+1}, \dots, A_n$. Then the volumes of the given simplexes are $|\det M_1|/n!$ and $|\det M_2|/n!$, respectively, so we wish to establish that $|\det M_1| = |\det M_2|$. Since

$$M_1 = A \cdot \left[\begin{array}{c|c} 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{array} \middle| M'' \right] \quad \text{and} \quad M_2 = A \cdot \left[\begin{array}{c|c} M' & 0 \\ 1 & \vdots \\ 0 & \ddots & 1 \end{array} \right],$$

where

$$M = \left[\begin{array}{c|c} M' & M'' \end{array} \right]$$

is the partition of M into $n \times r$ and $n \times (n - r)$ blocks, and $\det A = \pm 1$ (since A is orthonormal), the assertion is that the two $n \times n$ matrices

$$\left[\begin{array}{c|c} 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{array} \middle| M'' \right] \quad \text{and} \quad \left[\begin{array}{c|c} M' & 0 \\ 1 & \vdots \\ 0 & \ddots & 1 \end{array} \right]$$

have equal determinants up to sign. But $M = A^{-1}B$ is orthonormal, so

$$M^T \left[\begin{array}{c|c} M' & 0 \\ \hline 1 & 0 \\ \hline 0 & 1 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & M' \\ \cdot & \cdot & 1 \\ 0 & 1 & \end{array} \right],$$

and we are done from $\det M = \pm 1$.

Also solved by the proposer.

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- 1466.** [1989: 207] *Proposed by J.T. Groenman, Arnhem, The Netherlands, and D.J. Smeenk, Zaltbommel, The Netherlands.*

On the sides of $\Delta A_1A_2A_3$, and outside $\Delta A_1A_2A_3$, we draw similar triangles $A_3A_2B_1$, $A_1A_3B_2$ and $A_2A_1B_3$, with geocentres G_1 , G_2 and G_3 respectively. The geocentres of triangles $A_1B_3B_2$, $A_2B_1B_3$, $A_3B_2B_1$ and $A_1A_2A_3$ are Γ_1 , Γ_2 , Γ_3 and G , respectively. It is known that G is the geocentre of $\Delta B_1B_2B_3$ as well (see *Mathematics Magazine* 50 (1985) 84–89). Show that Γ_1G_1 has midpoint G , length $\frac{2}{3}|A_1B_1|$, and is parallel to A_1B_1 .

Solution by J. Chris Fisher and Dieter Ruoff, University of Regina.

There are so many red herrings floating around that *Crux* may have to clear this solution with the Fisheries Board before sending it out of the country!

The definition of $\Delta B_1B_2B_3$ is just about irrelevant. One needs only that $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$ have the same centre of gravity (or geocentre, if you prefer). We may as well assume that the A_i and B_i are vectors, and that the common centre is the origin: that is,

$$\sum A_i = \sum B_i = 0.$$

Then (taking subscripts modulo 3),

$$G_i = \frac{1}{3}(B_i + A_{i+1} + A_{i+2}), \quad \Gamma_i = \frac{1}{3}(A_i + B_{i+1} + B_{i+2}),$$

and the midpoint of Γ_iG_i is

$$\frac{1}{2}(G_i + \Gamma_i) = \frac{1}{2}(B_i + A_{i+1} + A_{i+2} + A_i + B_{i+1} + B_{i+2}) = 0.$$

In addition,

$$\begin{aligned} \overrightarrow{\Gamma_i G_i} &= G_i - \Gamma_i = \frac{1}{3}(B_i + A_{i+1} + A_{i+2} - A_i - B_{i+1} - B_{i+2}) \\ &= \frac{2}{3}(B_i - A_i) = \frac{2}{3}\overrightarrow{A_i B_i}. \end{aligned}$$

And how do we know that the A_i and B_i of the original proposal have the same centre of gravity? Consider the figure in the Gaussian plane and let the letters for the points also stand for the corresponding complex numbers. Then the similarity of the triangles $A_{i+2}A_{i+1}B_i$ implies that a complex number α exists for which

$B_i - A_{i+2} = \alpha(A_{i+1} - A_{i+2})$, $i = 1, 2, 3$. (*)
 From this one sees at once that $\sum B_i = 0$ if $\sum A_i = 0$.

Apparently there is no need to assume that the points B_i lie outside $\Delta A_1 A_2 A_3$ in order to satisfy equation (*) – the B_i just have to be similarly placed with respect to the sides $A_{i+1} A_{i+2}$ of $\Delta A_1 A_2 A_3$. That includes positions on and inside the triangle. But more importantly, (*) itself does not have to be fulfilled in order that the above proof be valid.

We suppose that a similar exercise could be devised for any pair of n -tuples of points A_i and B_i if one takes care in defining the G_i and Γ_i (so that there is sufficient cancellation).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; STANLEY RABINOWITZ, Westford, Massachusetts; JOHN RAUSEN, New York, N.Y.; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposers.

Pedoe mentions that the result that G is the geocentre of $\Delta B_1 B_2 B_3$ is given as Exercise 8.3 of his Geometry: A Comprehensive Course, Dover, 1988.

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1467. [1989: 207] Proposed by Toshio Seimiya, Kawasaki, Japan.

Find the locus of a point P in the interior of square $A_1 A_2 A_3 A_4$ such that

$$\angle PA_1 A_2 + \angle PA_2 A_3 + \angle PA_3 A_4 + \angle PA_4 A_1 = \pi.$$

I. Solution by Jordi Dou, Barcelona, Spain.

See the notation of the figure.

P' is the point symmetric to P with respect to the perpendicular bisector of $A_1 A_2$. A' is on $A_2 A_3$ so that $QA' = QA_3$. We let $\alpha_i = \angle PA_i A_{i+1}$, $i = 1, 2, 3, 4$ ($A_5 = A_1$) and $F(P) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Then we have

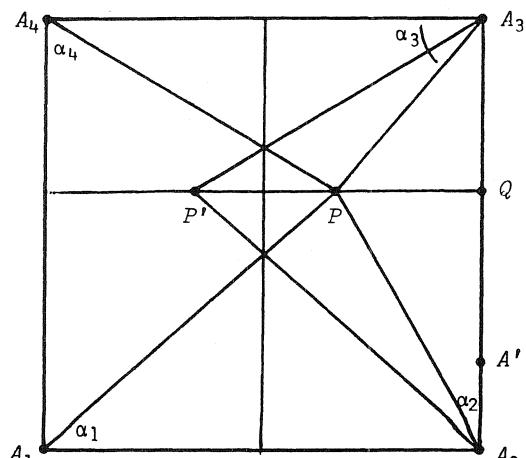
$$\alpha_1 + \alpha_2 = \angle P' A_2 A_1 + \alpha_2 = \frac{\pi}{2} - \angle PA_2 P'$$

and

$$\alpha_3 + \alpha_4 = \alpha_3 + \angle P' A_3 A_2 = \frac{\pi}{2} + \angle PA_3 P'$$

so that

$$F(P) = \pi - \angle PA_2 P' + \angle PA_3 P'.$$



The condition $F(P) = \pi$ of the problem is thus equivalent to

$$\angle PA_2P' = \angle PA_3P' (= \angle PA'P') .$$

Therefore either (i) $P = P'$ or (ii) P, P', A' and A_2 are concyclic. If (i), P lies on the perpendicular bisector of a side. If (ii), we have

$$QP \cdot QP' = QA' \cdot QA_2$$

and also

$$QP + QP' = QA' + QA_2 (= A_1A_2) .$$

It follows that $\{QP, QP'\} = \{QA', QA_2\}$, so $QP = QA'$ and $QP' = QA_2$, therefore P and P' lie on the diagonals A_1A_3 and A_2A_4 . Thus the locus consists of the points of the four axes of symmetry, interior to the square.

II. *Solution by P. Penning, Delft, The Netherlands.*

Let the sum of the four angles be S . By enumerating the vertices the other way round, the angles change into their complements and the new sum is $S' = 2\pi - S$. So if $S = \pi$ then $S' = \pi$. Apparently for any point where the direction of enumeration is irrelevant, the sum is equal to π . In general any point on a mirror-line satisfies this condition. In the case of the square there are four such mirror-lines: the diagonals and the perpendicular bisectors of the sides.

To show that these are the only solutions we argue as follows. The tangent of any angle in the sum is a bilinear function of the coordinates of P . The tangent of the sum of the four angles is of the fourth degree in both numerator and denominator. So the condition that the sum has a specified value leads to an equation of degree four in the coordinates. The four straight lines found for the square form together the complete solution.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. There were three incorrect solutions submitted.

Janous remembered the problem from a Russian or Romanian book, but cannot locate the source. Perhaps a reader can help. He also recalls that a second part to the problem stated: if P lies inside an equilateral triangle $A_1A_2A_3$ such that

$$\angle PA_1A_2 + \angle PA_2A_3 + \angle PA_3A_1 = \frac{\pi}{2} ,$$

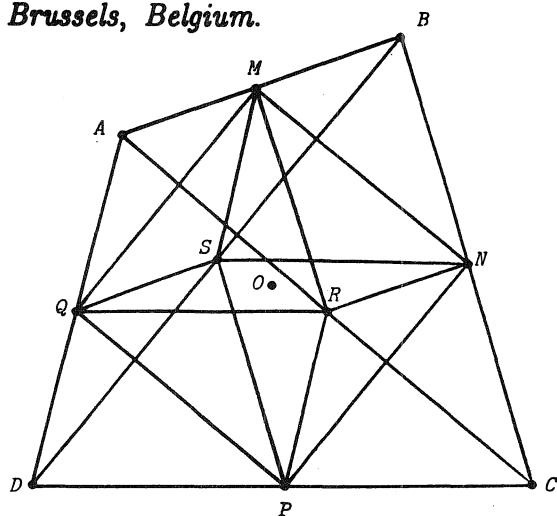
then P must lie on one of the axes of symmetry of the triangle. Might this suggest a generalization to a regular n -gon?

1468. [1989: 208] *Proposed by Jordi Dou, Barcelona, Spain.*

Prove that the midpoints of the sides and diagonals of a quadrilateral lie on a conic.

I. *Solution by C. Festraets-Hamoir, Brussels, Belgium.*

Soit M, N, P, Q les milieux des côtés AB, BC, CD, DA et R, S les milieux des diagonals AC, BD respectivement. Les quadrilatères $MNPQ, MRPS, NRQS$ sont des parallélogrammes de même centre O . Le centre et 3 points déterminent une conique; il y a donc une conique de centre O passant par M, N, R . Par symétrie par rapport à son centre, cette conique passe aussi par P, Q, S .



II. *Solution by Chris Fisher, University of Regina.*

Label the vertices $P_1P_2P_3P_4$ and denote the midpoint of P_iP_j by (ij) . Since the line joining midpoints $(ik), (jk)$ of the sides of $\Delta P_iP_jP_k$ is parallel to the base P_iP_j , the hexagon $(12)(23)(13)(34)(14)(24)$ has opposite sides parallel. Thus by Pascal's Theorem (ignoring possible degenerate cases) the six points lie on a conic.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York, N.Y.; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

Pedoe comments that the result is a known and venerable theorem, and is given as Exercise 80.7, page 355 of his book Geometry: A Comprehensive Course, Dover, 1988.

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