Ceva meets Pythagoras!

K.R.S. Sastry
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I had assumed that nobody could beat Ptolemy in providing a shortest proof of the Pythagorean theorem. To a rectangle you simply apply his "In a convex cyclic quadrilateral, the product of the diagonals equals the sum of the products of the opposite sides" theorem. In fact we can do better. With the assumed knowledge of ratios of line segments, the area measure of a polygonal region, the definitions of trigonometrical ratios and the theorems of Ceva and Menelaus, I am going to describe a construction to you about erecting parallelograms on the sides of a triangle. That will enable us to discover a new necessary and sufficient condition for the concurrency of three cevians of a triangle. Then, by treating the altitudes of a right-angled triangle as the degenerate case of three concurrent cevians, we deduce the Pythagorean theorem. Intrigued?

PRELIMINARIES.

Suppose D, E, F are points respectively on the sides BC, CA, AB of triangle ABC as in Figure 1. Let $BD : DC = 1 : \lambda$, and $CE : EA = 1 : \mu$, but $AF : FB = \nu : 1$. We let a, b, c denote the side lengths BC, CA, AB respectively. You will immediately tell me that

$$\left.\begin{aligned} BD &= \tfrac{a}{1+\lambda}, \quad DC = \tfrac{a\lambda}{1+\lambda}; \\ CE &= \tfrac{b}{1+\mu}, \quad EA = \tfrac{b\mu}{1+\mu}; \\ AF &= \tfrac{c\nu}{1+\nu}, \quad FB = \tfrac{c}{1+\nu}. \end{aligned}\right\} \tag{1}$$

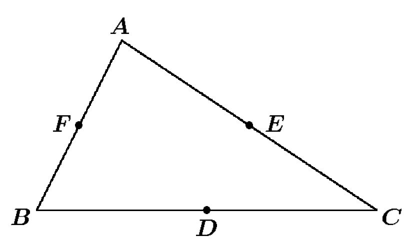


Figure 1

We both know from Ceva that the cevians AD, BE, CF concur if and only if

$$\nu = \lambda\mu. \tag{2}$$

In the title of our discussion, Menelaus did not meet Pythagoras, but he has his role to play. So let us recall that the points D, E, F will be collinear if and only if $\nu = \Leftrightarrow\lambda\mu$.

CONSTRUCTION ($\star$).

In addition to the points D, E, F mentioned above, let us construct parallelograms outwardly on the sides BC, CA, AB having widths w_1, w_2, w_3 respectively. These parallelograms, in general, are not similar but mutually equiangular containing an angle α, $0° < \alpha < 180°$. Furthermore, we split each parallelogram into two by drawing parallels through the points D, E, F (see Figure 2). We use the notation $M[DC, w_1]$ for the area measure of the parallelogram having DC and w_1 as side lengths.

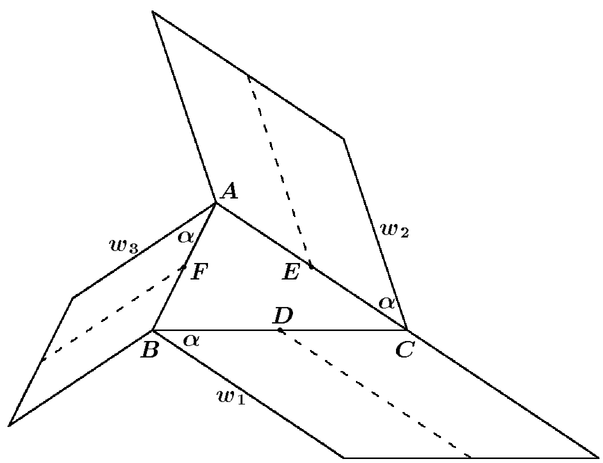


Figure 2

It is now a simple matter for either of us to compute the area measures of the three pairs of parallelograms, each pair sharing exactly one vertex of triangle ABC:

$$\left.\begin{array}{ll} M[DC, w_1] = \frac{a\lambda w_1 \sin\alpha}{1+\lambda}, & M[CE, w_2] = \frac{b w_2 \sin\alpha}{1+\mu}; \\ M[EA, w_2] = \frac{b\mu w_2 \sin\alpha}{1+\mu}, & M[AF, w_3] = \frac{c\nu w_3 \sin\alpha}{1+\nu}; \\ M[FB, w_3] = \frac{c w_3 \sin\alpha}{1+\nu}, & M[BD, w_1] = \frac{a w_1 \sin\alpha}{1+\lambda}. \end{array}\right\} \quad (3)$$

WHEN DO CEVIANS AD, BE, CF CONCUR?

I will provide an answer to that question by relating the area measures of the parallelograms listed in (3).

Theorem 1. The cevians AD, BE, CF concur if and only if there exist mutually equiangular parallelograms of appropriate widths w_1, w_2, w_3, depending on the values a, b, c, λ, μ, so that the construction ($\star$) yields parallelogram pairs of equal area measure in (3).

Proof: Allow me to assume that AD, BE, CF are concurrent cevians. This leaves me with the task of providing consistent expressions for w_1, w_2, w_3 in terms of a, b, c, λ, μ.

From my assumption and (2), $\nu = \lambda\mu$ follows. So I will substitute $\nu = \lambda\mu$ and equate parallelogram area measures of each pair in (3). Observe that the factor $\sin\alpha$ disappears from our equations:

$$\frac{a\lambda w_1}{1+\lambda} = \frac{bw_2}{1+\mu}; \quad \frac{b\mu w_2}{1+\mu} = \frac{c\lambda\mu w_3}{1+\lambda\mu};$$

$$\frac{cw_3}{1+\lambda\mu} = \frac{aw_1}{1+\lambda}; \quad \lambda\mu \neq 0, \quad \lambda, \mu \neq \Leftrightarrow 1.$$

Indeed we have the consistent equations

$$\frac{a\lambda w_1}{1+\lambda} = \frac{bw_2}{1+\mu} = \frac{c\lambda w_3}{1+\lambda\mu} = k \quad \text{say.}$$

These yield, for an appropriate constant k,

$$w_1 = \frac{k(1+\lambda)}{a\lambda}, \quad w_2 = \frac{k(1+\mu)}{b}, \quad w_3 = \frac{k(1+\lambda\mu)}{c\lambda}. \tag{4}$$

True, we get an infinity of dissimilar parallelograms of varying widths as we vary the value of k.

To establish the converse, presently I am obliged to assume that there are parallelograms of appropriate widths w_1, w_2, w_3 so that the construction ($\star$) leads to parallelogram pairs of equal area measure in (3). You will therefore allow me to assume the equations

$$\frac{a\lambda w_1}{1+\lambda} = \frac{bw_2}{1+\mu}; \quad \frac{b\mu w_2}{1+\mu} = \frac{c\nu w_3}{1+\nu}; \quad \frac{cw_3}{1+\nu} = \frac{aw_1}{1+\lambda}.$$

But wait! The above equations imply that

$$\frac{a\lambda\mu w_1}{1+\lambda} = \frac{b\mu w_2}{1+\mu} = \frac{c\nu w_3}{1+\nu} = \frac{a\nu w_1}{1+\lambda}.$$

Hence $\nu = \lambda\mu$ and we have established the concurrency of the cevians AD, BE, CF by virtue of (2).

You will agree with me that, although the proof thus far has proceeded as if λ, μ, ν are all positive, the same goes through should two of them be negative. So, to complete the proof of Theorem 1, I must now answer the question: What if λ (or μ) is 0?

As you can see from Figure 3, when $\lambda = 0$, both points D and F coincide with the vertex B. We therefore determine w_1, w_2, w_3 from the equations

$$aw_1 = \frac{bw_2}{1+\mu}; \quad \frac{b\mu w_2}{1+\mu} = cw_3.$$

This yields
$$w_1 = \frac{k}{a\mu}, \quad w_2 = \frac{k(1+\mu)}{b\mu}, \quad w_3 = \frac{k}{c}$$
for some appropriate constant k. And this springs on us the following delightful surprise!
$$M[AB, w_3] + M[BC, w_1] = M[CA, w_2]. \tag{5}$$

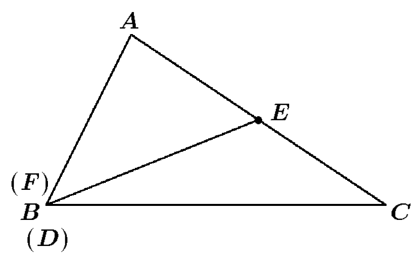


Figure 3

As this equation (5) is an important result, rivalling Pythagoras, let us give it the status of a theorem.

Theorem 2. Let E be a point on side AC of triangle ABC distinct from the vertex A or C. Suppose $CE : EA = 1 : \mu$. Then there exist mutually equiangular parallelograms of appropriate widths w_1, w_2, w_3, depending on the values a, b, c, μ, so that the sum of the area measures of the parallelograms on the sides AB, BC equals the area measure of the parallelogram on side AC.

THE PYTHAGOREAN THEOREM.

I know your impatience is growing at an alarming rate to meet and greet Pythagoras. But I am honour bound to show you, in some specific instance, how I actually compute the w's. For this purpose let us assume that AD, BE, CF are the altitudes of triangle ABC. Then you know $BD = c\cos B$, $DC = b\cos C, \cdots$, $BD : DC = (c\cos B) : (b\cos C) = 1 : \lambda, \cdots$. Hence
$$\lambda = \frac{b\cos C}{c\cos B}, \quad \mu = \frac{c\cos A}{a\cos C}.$$
I am very glad to learn that on using (4) you already deduced the nice conclusion
$$w_1 = ka, \quad w_2 = kb, \quad w_3 = kc. \tag{6}$$

For the proof of the Pythagorean theorem, look at Figure 4. If you have the triangle right-angled at B, the altitudes concur at the vertex B. So from Theorem 2, (5) and (6) we have
$$M[AB, kc] + M[BC, ka] = M[CA, kb].$$

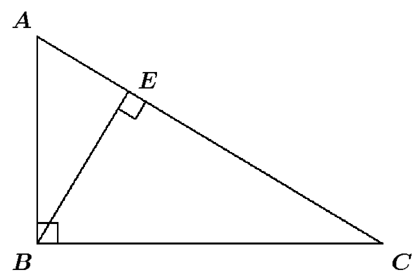


Figure 4

That is

$$kc^2 \sin\alpha + ka^2 \sin\alpha = kb^2 \sin\alpha,$$

which is more than Pythagoras is said to have said. Specifically, $k = 1$, $\alpha = 90°$ yields the Pythagorean theorem. You may wish to establish the more general result and its converse: In triangle ABC the cosine law follows by treating its altitudes as three concurrent cevians.

CONCLUSION.

For the sake of completeness, I will state Theorem 3. It can be proved in the manner of Theorem 1.

Theorem 3. The points D, E, F on the sides of a triangle ABC are collinear if and only if there exist mutually equiangular parallelograms of appropriate widths w_1, w_2, w_3, depending on the values a, b, c, λ, μ, so that the construction ($\star$) yields parallelogram pairs of equal area measure in (3).

ACKNOWLEDGEMENT.

The author thanks the referee for his suggestions.

THE SKOLIAD CORNER

No. 15

R.E. Woodrow

First this issue we pause to correct an error which crept into the solution given for problem 10 of the Eleventh W.J. Blundon Contest, February 23, 1994. Thanks go to Bob Prielipp for noting that we used 6 in place of 5 in the last line on page 155, thus producing a wrong answer! Here is his correction.

10. [1996: 102, 1996: 155] *The Eleventh W.J. Blundon Contest.*

Two numbers are such that the sum of their cubes is 5 and the sum of their squares is 3. Find the sum of the two numbers.

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.

Let the two numbers be denoted by x and y. Then $x^3 + y^3 = 5$ and $x^2 + y^2 = 3$. Since $x^2 + y^2 = 3$, $(x^2 + y^2)(x + y) = 3(x + y)$. Thus $(x^3 + y^3) + xy(x+y) = 3(x+y)$. Because $x^3+y^3 = 5$, $xy(x+y) = 3(x+y) \Leftrightarrow 5$. Hence $5 = x^3 + y^3 = (x+y)^3 \Leftrightarrow 3xy(x+y) = (x+y)^3 \Leftrightarrow 9(x+y) + 15$, so

$$(x+y)^3 \Leftrightarrow 9(x+y) + 10 = 0.$$

Let $x + y = s$. Then

$$\begin{gathered} s^3 \Leftrightarrow 9s + 10 = 0 \\ (s \Leftrightarrow 2)(s^2 + 2s \Leftrightarrow 5) = 0 \\ (s \Leftrightarrow 2)(s^2 + 2s + 1 \Leftrightarrow 6) = 0 \\ s = 2 \qquad \text{or} \qquad s + 1 = \pm\sqrt{6}. \end{gathered}$$

Thus $s = 2$ or $s = \Leftrightarrow 1 \pm \sqrt{6}$.

In this issue, we give the problems of the 14th annual American Invitational Mathematics Examination written March 28, 1996. These problems are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, USA 68588–0322. As always we welcome your original "nice" solutions and generalizations which differ from the published official solutions.

14th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

Thursday, March 28, 1996

1. In a *magic square*, the sum of the three entries in any row, column, or diagonal is the same value. The figure shows four of the entries of a magic square. Find x.

x	19	96
1		

2. For each real number x, let $\lfloor x \rfloor$ denote the greatest integer that does not exceed x. For how many positive integers n is it true that $n < 1000$ and that $\lfloor \log_2 n \rfloor$ is a positive even integer?

3. Find the smallest positive integer n for which the expansion of $(xy \Leftrightarrow 3x + 7y \Leftrightarrow 21)^n$, after like terms have been collected, has at least 1996 terms.

4. A wooden cube, whose edges are one centimetre long, rests on a horizontal surface. Illuminated by a point source of light that is x centimetres directly above an upper vertex, the cube casts a shadow on the horizontal surface. The area of the shadow, which does not include the area beneath the cube, is 48 square centimetres. Find the greatest integer that does not exceed $1000x$.

5. Suppose that the roots of $x^3 + 3x^2 + 4x \Leftrightarrow 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$. Find t.

6. In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays. (There are no ties.) Let m/n be the probability that the tournament will produce neither an undefeated team nor a winless team, where m and n are relatively prime positive integers. Find $m + n$.

7. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

8. The *harmonic mean* of two positive numbers is the reciprocal of the arithmetic mean of their reciprocals. For how many ordered pairs of positive integers (x, y) with $x < y$ is the harmonic mean of x and y equal to 6^{20}?

9. A bored student walks down a hall that contains a row of closed lockers, numbered 1 to 1024. He opens the locker numbered 1, and then alternates between skipping and opening each closed locker thereafter. When he reaches the end of the hall, the student turns around and starts back. He opens the first closed locker he encounters, and then alternates between skipping and opening each closed locker thereafter. The student continues wandering back and forth in this manner until every locker is open. What is the number of the last locker he opens?

10. Find the smallest positive integer solution to

$$\tan 19x^\circ = \frac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ \Leftrightarrow \sin 96^\circ}.$$

11. Let P be the product of those roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \leq \theta < 360$. Find θ.

12. For each permutation $a_1, a_2, a_3, \ldots, a_{10}$ of the integers 1, 2, 3, $\ldots$, 10, form the sum

$$|a_1 \Leftrightarrow a_2| + |a_3 \Leftrightarrow a_4| + |a_5 \Leftrightarrow a_6| + |a_7 \Leftrightarrow a_8| + |a_9 \Leftrightarrow a_{10}|.$$

The average value of all such sums can be written in the form p/q, where p and q are relatively prime positive integers. Find $p + q$.

13. In triangle ABC, $AB = \sqrt{30}$, $AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$ and $\angle ADB$ is a right angle. The ratio

$$\frac{\text{Area } (\triangle ADB)}{\text{Area } (\triangle ABC)}$$

can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

14. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?

15. In parallelogram $ABCD$, let O be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Angles CAB and DBC are each twice as large as angle DBA, and angle ACB is r times as large as angle AOB. Find the greatest integer that does not exceed $1000r$.

In the last number, we gave the problems of the European "Kangaroo" Mathematical Challenge written 23 March 1995. Here are solutions.

1.	B	**2**.	E	**3**.	C	**4**.	E	**5**.	C
6.	C	**7**.	C	**8**.	C	**9**.	B	**10**.	D
11.	B	**12**.	27	**13**.	C	**14**.	C	**15**.	D
16.	E	**17**.	A	**18**.	E	**19**.	E	**20**.	C
21.	D	**22**.	D	**23**.	C	**24**.	E	**25**.	E

That completes the Skoliad Corner for this issue. Thanks go to readers who have sent in contest materials and solutions. Please send me more!

Congratulations!

Professor Andy Liu

We are delighted to congratulate Andy Liu, Department of Mathematical Sciences, University of Alberta, Edmonton, on being awarded, this year, the prestigious **David Hilbert International Award** at the World Federation of National Mathematics Competitions meeting at ICME-8, in Seville.

Andy has a long and distinguished record in Mathematics. He has coached both the American and Canadian teams for the IMO, and also participated in the training of teams from Hong Kong and China. He is a well respected problemist. His mathematical interests span discrete mathematics, hypergraph theory, combinatorial geometry, foundations of mathematics, mathematical education and recreational mathematics. He says that the common characteristics of the research problems in which he works are that they are easy to understand, but not so easy to solve.

Andy has long been associated with ***CRUX***, and is at present the Editor of the *Book Reviews* section.

Past recipients of the award, who are associated with ***CRUX***, include Ed. Barbeau, Murray Klamkin and Marcin Kuczma.

THE OLYMPIAD CORNER

No. 175

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin with the 1996 Canadian Mathematical Olympiad which we reproduce with the permission of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. My thanks go to Daryl Tingley, Chair of the Committee, for forwarding the questions and promising to provide "official" and interesting contestant solutions.

1996 CANADIAN MATHEMATICAL OLYMPIAD

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}.$$

2. Find all real solutions to the following system of equations. Carefully justify your answer.

$$\begin{cases} \dfrac{4x^2}{1+4x^2} & = y, \\[2ex] \dfrac{4y^2}{1+4y^2} & = z, \\[2ex] \dfrac{4z^2}{1+4z^2} & = x. \end{cases}$$

3. We denote an arbitrary permutation of the integers $1, \ldots, n$ by $a_1, \ldots, a_n$. Let $f(n)$ be the number of these permutations such that

(i) $a_1 = 1$;

(ii) $|a_i - a_{i+1}| \le 2$, $i = 1, \ldots, n-1$.

Determine whether $f(1996)$ is divisible by 3.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

5. Let $r_1, r_2, \ldots, r_m$ be a given set of m positive rational numbers such that $\sum_{k=1}^{m} r_k = 1$. Define the function f by $f(n) = n \Leftrightarrow \sum_{k=1}^{m}[r_k n]$ for each positive integer n. Determine the minimum and maximum values of $f(n)$. Here $[x]$ denotes the greatest integer less than or equal to x.

The next set of problems are from the twenty-fifth annual United States of America Mathematical Olympiad written May 2, 1996. These problems are copyrighted by the Committee on the American Mathematical Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, N.E., USA 68588–0322. As always we welcome your original, "nice" solutions and generalizations which differ from the published official solutions.

25th UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

Part I 9 a.m. - 12 noon

May 2, 1996

1. Prove that the average of the numbers $n \sin n^\circ$ $(n = 2, 4, 6, \ldots, 180)$ is $\cot 1^\circ$.

2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S. Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A. Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

3. Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $2/3$ the area of triangle ABC.

Part II 1 p.m. - 4 p.m.

May 2, 1996

4. An n-term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a *binary sequence of length* n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10°$, $\angle PBA = 20°$, $\angle PCA = 30°$, and $\angle PAC = 40°$. Prove that triangle ABC is isosceles.

6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

As a third Olympiad for this issue we give the problems of the Italian Mathematical Olympiad written May 6, 1994. My thanks go to Richard Nowakowski who collected this (and many others) when he was Canadian Team Leader to the International Mathematical Olympiad in Hong Kong.

ITALIAN MATHEMATICAL OLYMPIAD
May 6, 1994

1. Show that there exists an integer N such that for all $n \geq N$ a square can be represented as a union of n pairwise non-overlapping squares.

2. Find all integer solutions of the equation

$$y^2 = x^3 + 16.$$

3. A journalist wants to report on the island of scoundrels and knights, where all inhabitants are either scoundrels (and they always lie) or knights (and they always tell the truth). The journalist interviews each inhabitant exactly once, and he gets the following answers:

A_1: on this island there is at least one scoundrel;
A_2: on this island there are at least two scoundrels;
...
A_{n-1}: on this island there are at least $n \Leftrightarrow 1$ scoundrels;
A_n: on this island everybody is a scoundrel.

Can the journalist decide whether there are more scoundrels or more knights?

4. Let r be a line in the plane and let ABC be a triangle contained in one of the halfplanes determined by r. Let A', B', C' be the points symmetric to A, B, C with respect to r; draw the line through A' parallel to BC, the line through B' parallel to AC and the line through C' parallel to AB. Show that these three lines have a common point.

5. Let OP be a diagonal of the unit cube. Find the minimum and the maximum value of the area of the intersection of the cube with a plane through OP.

6. On a 10×10 chessboard the squares are labelled with the numbers $1, 2, \ldots, 100$ in the following way: the first row contains the numbers $1, 2, \ldots, 10$ in increasing order from left to right, the second row contains the numbers $11, 12, \ldots 20$ in increasing order from left to right, ... , the last row contains the numbers $91, 92, \dots, 100$ in increasing order from left to right. Now change the signs of fifty numbers in such a way that each row and each column contains five positive and five negative numbers. Show that after this change the sum of all numbers on the chessboard is zero.

Before turning to reader's solutions to problems from the 1995 numbers of Crux, I want to give some reactions to material from the Corner. First, comments on the 1995 Canadian Mathematical Olympiad [1995: 190; 1995: 223–226] by Murray S. Klamkin, The University of Alberta.

2. Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Comment. This same exact problem appeared in the 1974 U.S.A. Mathematical Olympiad [1]. In the solution given, it was also noted that since $\ln x^x$ is convex for $x > 0$, it follows more generally and immediately by Jensens's inequality that

$$a_1^{a_1} a_2^{a_2} \cdots a_n^{a_n} \geq (a_1 a_2 \cdots a_n)^{(a_1+a_2+\cdots+a_n)/n}.$$

3. Let n be a fixed positive integer. Show that for any nonnegative integer k, the Diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

Comment. The following problem, which implies the solution, appeared in the 1985 U.S.A. Mathematical Olympiad [1]:

Determine whether or not there are any positive integral solutions of the simultaneous equations

$$x_1^2 + x_2^2 + \cdots + x_{1985}^2 = y^3,$$

$$x_1^3 + x_2^3 + \cdots + x_{1985}^3 = z^2,$$

with distinct integers $x_1, x_2, \dots, x_{1985}$.

It was shown that there are infinitely many solutions. The solution given easily extends to take care of the case if z^2 is replaced by z^{3k+2}. For a single equation, we have more generally that

$$x_1^{a_1} + x_2^{a_2} + \cdots x_n^{a_n} = y^a$$

has infinitely many positive solutions if a is relatively prime to all the a_i's. We just let

$$x_i = u_i\{\sum u_i^{a_i}\}^{kp/a_i} \text{ where } p = \prod a_i \text{ so that } \sum x_i^{a_i} = \{\sum u_i^{a_i}\}^{kp+1}.$$

Then we can take $y = \{\sum u_i^{a_i}\}^l$. Finally, since p and a are relatively prime, intgers k and l exist such that $kp + 1 = al$.

Much more general results are to appear in the proceedings of the Belgian Royal Academy.

In view of the above comments on previously published problems, the Canadian Olympiad Examination Committee should be more careful about the problems they set in their competitions.

Reference

[1] M.S. Klamkin, *U.S.A. Mathematical Olympiads 1972–1986*, Mathematical Association of America, Washington, D.C., 1988, pp. 81, 33–34.

Next we give a comment by Murray S. Klamkin of the University of Alberta about one of the solutions published in the Corner.

2. [1994: 39; 1995: 193–194] 1992 *Czechoslovak Mathematical Olympiad, Final Round.*

Let a, b, c, d, e, f be lengths of edges of a given tetrahedron and S its surface area. Prove that

$$S \leq (\sqrt{3}/6)(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).$$

Comment by Murray S. Klamkin, University of Alberta. It is just as easy as the previous solution to obtain the stronger inequality

$$S \leq (\sqrt{3}/4)\{(def)^{2/3} + (abd)^{2/3} + (bce)^{2/3} + (caf)^{2/3}\}.$$

This follows from the known inequality

$$F \leq (\sqrt{3}/4)(uvw)^{2/3}$$

for the area F of a triangle of sides u, v, w, and which is equivalent to the equilateral triangle being the triangle of maximum area which can be inscribed in a given circle. The rest follows from the AM–GM inequality

$$(uvw)^{2/3} \leq 3(u^2 + v^2 + w^2).$$

The next two solutions are in response to challenges to the readership "to fill the gaps" in solutions to problem sets discussed in 1995 and 1996 numbers of the Corner. My thanks to members of the Singapore team to the 35th IMO.

4. [1994: 64–65; 1995: 274] *9th Balkan Mathematical Olympiad (Romania).*

For every integer $n > 3$ find the minimum positive integer $f(n)$ such that every subset of the set $A = \{1, 2, 3, \ldots, n\}$ which contains $f(n)$ elements contains elements $x, y, z \in A$ which are pairwise relatively prime.

Solution by the joint efforts of Siu Taur Pang and Hoe Teck Wee, silver medallists on the Singapore team to the 35th IMO.

Let $T_n = \{t \mid t \leq n, 2 \mid t \text{ or } 3 \mid t\}$. Therefore, among any three elements of T_n, two have a common factor (either 2 or 3). Hence, $f(n) \geq |T_n| + 1$.

For $4 \leq n \leq 24 < 5^2$, there are $|T_n| \Leftrightarrow 2$ composite numbers in the set $A_n = \{1, 2, 3, \ldots, n\}$ since every composite number in A_n is a multiple of 2 or 3. Hence, there exist three irreducible elements (either 1 or a prime) in any $(|T_n| + 1)$-element subset of A_n, which are in pairs relatively prime.

Therefore, we have, for $4 \leq n \leq 24$, $f(n) = |T_n| + 1$. By the Principle of Inclusion and Exclusion,

$$|T_n| = \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] \Leftrightarrow \left[\frac{n}{6}\right],$$

$$f(n) = \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] \Leftrightarrow \left[\frac{n}{6}\right] + 1. \qquad (*)$$

Assume that $(*)$ also holds for some integer $k > 3$. Consider any $(|T_{k+6}|+1)$-element subset B of A_{k+6}. If at least $f(k)$ elements of B lie in A_k, then there exist three elements in B which are pairwise relatively prime. Otherwise, since $|T_{k+6}| = T_k + 4$, at least five elements of B are from the five-element set $C = \{k+1, k+2, \ldots, k+6\}$. Note that the difference between any two elements of C is at most 5, so the greatest common divisor between any two elements of C is $1, 2, 3, 4$ or 5. If any three elements of $B \cap C$ are odd, then there are three elements in B which are pairwise relatively prime, namely these three odd elements since the difference between any two of these three elements is either 2 or 4 which have no odd factor larger than 1. Otherwise, there are two odd numbers and three even numbers in $B \cap C$. The difference between any two of these even numbers is either 2 or 4, so at most one of these three numbers is divisible by 3, and at most one is divisible by 5. Hence, there exists an even number in $B \cap C$ which is neither divisible by 3 nor 5. Hence, this even number and the two odd numbers are pairwise relatively prime. Thus, $f(k+6) = |T_{k+6}| + 1$, so $(*)$ holds for $k + 6$.

Therefore, we conclude by the Principle of Mathematical Induction that $(*)$ holds for all integers $n > 3$.

5. [1994: 129–130; 1996; 24–27] *Final Round of the 43rd Mathematical Olympiad in Poland.*

The regular $2n$-gon $A_1, A_2, \ldots, A_{2n}$ is the base of a regular pyramid with vertex S. A sphere passing through S cuts the lateral edges SA_i in the respective points B_i $(i = 1, 2, \ldots, 2n)$. Show that

$$\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}.$$

Solution by Hoe Teck Wee, silver medallist on the Singapore team at the 35th IMO.

It can be proved easily, using vectors that for any regular polygon $A_1A_2\ldots A_{2n}$ with centre C (where $n > 1$ is a positive integer) and any point P in space, that

$$\sum_{j=1}^{n} PA_{2j-1}^2 = \sum_{j=1}^{n} PA_{2j}^2,$$

by writing $PA_i \cdot PA_i = CP \cdot CP + CA_i \cdot CA_i \Leftrightarrow 2CA_i \cdot CP$ $(i = 1, 2, \ldots, 2n)$, and using the identity

$$\sum_{j=1}^{n} CA_{2j-1} = \sum_{j=1}^{n} CA_{2j}.$$

Let O denote the center of the sphere passing through S and R denote its radius. Note that we may extend each of the lateral edges so that $OA_i > R$ for $i = 1, 2, \ldots 2n$. Therefore, by considering the power of A_i, we have

$$OA_i^2 \Leftrightarrow R^2 = SA_i \cdot A_iB_i = SA_i \cdot (SA_i \Leftrightarrow SB_i) = SA_i^2 \Leftrightarrow SA_i \cdot SB_i.$$

$$\sum_{j=1}^{n} (SA_{2j-1}^2 \Leftrightarrow SA_{2j-1} \cdot SB_{2j-1}) = \sum_{j=1}^{n} (SA_{2j}^2 \Leftrightarrow SA_{2j} \cdot SB_{2j}).$$

Taking $P = S$, we have

$$\sum_{j=1}^{n} SA_{2j-1}^2 = \sum_{j=1}^{n} SA_{2j}^2 \Rightarrow \sum_{j=1}^{n} SB_{2j-1} = \sum_{j=1}^{n} SB_{2j}.$$

We now turn to reader's solutions to problems from the 1995 numbers of the corner. We begin with the First Stage Exam of the 10th Iranian Mathematical Olympiad [1995: 8–9].

1. Find all integer solutions of

$$\frac{1}{m}+\frac{1}{n}-\frac{1}{mn^2}=\frac{3}{4}.$$

Solutions by Mansur Boase, student, St. Paul's School, London, England; Christopher J. Bradley, Clifton College, Bristol, UK; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Stewart Metchette, Gardena, California, USA; Michael Selby, University of Windsor, Windsor, Ontario; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Hsia's write-up, although most solutions were similar.

$\frac{1}{m}+\frac{1}{n}-\frac{1}{mn^2}=\frac{3}{4}$. Note $m,n\neq 0$. Then

$$\frac{n^2+mn-1}{mn^2}=\frac{3}{4}$$

giving

$$m=\frac{4(n^2-1)}{n(3n-4)}=\frac{4(n+1)(n-1)}{n(3n-4)}.$$

Now $(n+1,n)=1=(n-1,n)$, i.e. n is relatively prime to both $n+1$ and $n-1$. Since m is an integer, $n\mid 4(n+1)(n-1)$, giving $n\mid 4$. Thus $n=\pm1,\pm2,\pm4$.

For

$$n=\pm1,\quad m=0 \text{ which is impossible}$$

$$n=-2,\quad m=\frac{4(-1)(-3)}{(-2)(-10)},\ \text{not an integer}$$

$$n=2,\quad m=3$$

$$n=-4,\quad m=\frac{4(-3)(-5)}{(-4)(-16)},\ \text{not an integer}$$

$$n=4,\quad m=\frac{4(5)(3)}{4\cdot 8},\ \text{not an integer.}$$

Thus the only solution is $(n,m)=(2,3)$.

2. Let X be a set with n elements. Show that the number of pairs (A, B) such that A, B are subsets of X, A is a subset of B, and $A \neq B$ is equal to:

$$3^n \Leftrightarrow 2^n.$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mansur Boase, student, St. Paul's School, London, England; Christopher J. Bradley, Clifton College, Bristol, UK; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Joseph Ling, University of Calgary, Calgary, Alberta; Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; Michael Selby, University of Windsor, Windsor, Ontario; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Boase's argument.

The number of subsets of an n element set is 2^n. Thus the number of pairs (A, B) with $A = B$ is 2^n. As a result we need to show that the number of pairs (A, B) with A a subset of B, (possibly equal to B), is 3^n. We do so by induction on n.

If $n = 1$, let $X = \{a_1\}$. The pairs (A, B) are $(\emptyset, \emptyset)$, $(\emptyset, X)$, (X, X).

Assume, by induction, that for k elements the number of pairs is 3^k. Let $X = \{a_1, \ldots, a_k, a_{k+1}\}$. By hypothesis there are 3^k pairs (A, B) with $A \subset B \subset \{a_1, \ldots, a_k\}$. Each of these can be extended in three ways to yield a pair (A', B') with $A' \subset B' \subset \{a_1, \ldots, a_k, a_{k+1}\}$ as follows:

(i) $A' = A$, $B' = B$

(ii) $A' = A \cup \{a_k\}$, $B' = B \cup \{a_k\}$

(iii) $A' = A$, $B' = B \cup \{a_k\}$.

Obviously no pairs are repeated, so the number of pairs (A', B') is $3 \cdot 3^k = 3^{k+1}$. This completes the induction, as required.

4. Let a, b, c, be rational and one of the roots of $ax^3 + bx + c = 0$ be equal to the product of the other two roots. Prove that this root is rational.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Mansur Boase, St. Paul's School, London, England; Christopher J. Bradley, Clifton College, Bristol, UK; Paul Colucci, student, University of Illinois; Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Michael Selby, University of Windsor, Windsor, Ontario; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Amengual Covas's solution (which was similar to the others).

Let r_1, r_2, r_3, be the roots of the given cubic equation and also let $r_1 = r_2 r_3$. Then

$$r_1 + r_2 + r_3 = 0, \tag{1}$$

$$r_1(r_2 + r_3) + r_1 = r_1r_2 + r_1r_3 + r_2r_3 = \frac{b}{a}, \tag{2}$$

$$r_1r_2r_3 = r_1^2 = \Leftrightarrow\frac{c}{a}. \tag{3}$$

(Note that $a \neq 0$ for the existence of three solutions.) From (1) $r_2+r_3 = \Leftrightarrow r_1$ and substituting for $r_2 + r_3$ in (2) we find that

$$\Leftrightarrow r_1^2 + r_1 = \frac{b}{a},$$

or equivalently

$$r_1 = \frac{b}{a} + r_1^2.$$

Finally, substituting $\frac{-c}{a}$ for r_1^2 from (3), we obtain $r_1 = \frac{b-c}{a}$, which is rational.

5. Find all primes p such that $(2^{p-1} \Leftrightarrow 1)/p$ is a square.

Solutions by Christopher J. Bradley, Clifton College, Bristol, UK; Stewart Metchette, Gardena, California, USA; Michael Selby, University of Windsor, Windsor, Ontario; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bradley's solution.

Lemma 1. The only solution in natural numbers of $2^n = x^2 \Leftrightarrow 1$ is $n = 3$, $x = 3$.

Proof. $2^n = x^2 \Leftrightarrow 1 = (x \Leftrightarrow 1)(x + 1)$. The factors of 2^n that differ by 2 are 2^2 and 2^1 with $n = 3$ and no others.

Lemma 2. The only solutions in natural numbers of $2^n = x^2 + 1$ are $x = 0$, $n = 0$, $n = 1$, $x = 1$.

Proof. The cases mentioned apart, $n \geq 2$ and the lefthand side is $0 \bmod 4$. Now $x^2 \equiv 0, 1 \bmod 4$, so $x^2 + 1 \equiv 1, 2 \bmod 4$. Contradiction establishes the result.

Let $p = 2$. Then $(2^{p-1} \Leftrightarrow 1)/2 = \frac{1}{2}$, so p must be odd. Put $p = 2k+1$. Then $\frac{2^{p-1}-1}{p} = \frac{2^{2k}-1}{2k+1}$ which is an integer by Fermat's Little Theorem. Now $\frac{2^{2k}-1}{2k+1} = \frac{(2^k+1)(2^k-1)}{2k+1}$, and since $2k + 1$ is prime it must be the case that $2k+1 \mid 2^k + 1$ or $2k+1 \mid 2^k \Leftrightarrow 1$. Indeed $2^k + 1$ and $2^k \Leftrightarrow 1$ are coprime, so the only ways in which $\frac{2^{p-1}-1}{p}$ can be a perfect square is for *either* $\frac{2^k+1}{2k+1}$ and $2^k \Leftrightarrow 1$ to be perfect squares *or* $2^k + 1$ and $\frac{2^k-1}{2k+1}$ to be both perfect squares.

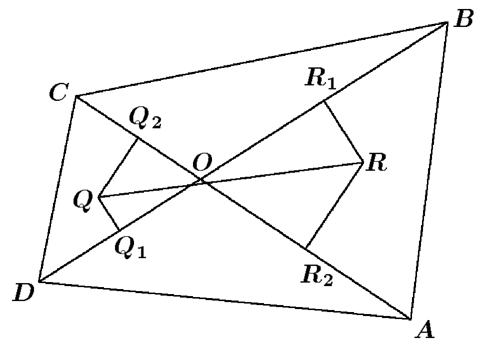
Case 1. $2^k \Leftrightarrow 1$ a perfect square. By Lemma 2 we have $k = 0$ or $k = 1$. Then $\frac{2^k+1}{2k+1} = 2$ $(k = 0)$ or 1 $(k = 1)$. So $k = 1$, $p = 3$ gives a solution.

Case 2. $2^k + 1$ a perfect square. By Lemma 1 we have $k = 3$. Then $\frac{2^k-1}{2k+1} = \frac{7}{7} = 1$, so $p = 7$ gives a solution, and there can be no others.

6. Let O be the intersection of diagonals of the convex quadrilateral $ABCD$. If P and Q are the centres of the circumcircles of AOB and COD show that

$$PQ \geq \frac{AB+CD}{4}.$$

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.



Let P_1, Q_1, P_2, Q_2 be the midpoints of OB, OD, OA and OC respectively.

Then

$$PQ \geq P_1Q_1 \Rightarrow PQ \geq \tfrac{1}{2}BD,$$

$$PQ \geq P_2Q_2 \Rightarrow PQ \geq \tfrac{1}{2}AC.$$

From this, we get

$$\begin{aligned} PQ &\geq \frac{1}{4}(BD+AC) \\ &\geq \frac{1}{4}(OB+OD+OA+OC) \\ &\geq \frac{1}{4}[(OB+OA)+(OC+OD)]. \end{aligned}$$

Now $OB+OA > AB$ and $OC+OD > CD$. Thus $PQ \geq \frac{1}{4}(AB+CD)$.

That completes the Corner for this issue. Please send me your nice solutions as well as Olympiad contests for use in the future.

THE ACADEMY CORNER

No. 4

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In the February 1996 issue, we gave the first set of problems in the Academy Corner. Here we present solutions to the first three questions, as sent in by Šefket Arslanagić, Berlin, Germany.

Memorial University Undergraduate Mathematics Competition 1995

1. Find all integer solutions of the equation $x^4 = y^2 + 71$.

   Solution. From the equation $x^4 = y^2 + 71$, we get

$$(x^2 \Leftrightarrow y)(x^2 + y) \quad = \quad 71 \cdot 1.$$

Since 71 is prime, we have

$$\mathrm{I}\left\{\begin{array}{l} x^2 \Leftrightarrow y = 1 \\ x^2 + y = 71 \end{array}\right. \quad \text{or} \quad \mathrm{II}\left\{\begin{array}{l} x^2 \Leftrightarrow y = 71 \\ x^2 + y = 1, \end{array}\right.$$

so that

$$2x^2 = 72, \qquad \text{or} \qquad 2x^2 = 72,$$

giving

$$x_{1,2} = \pm 6, \quad y_{1,2} = 35, \quad x_{3,4} = \pm 6 \quad y_{3,4} = \Leftrightarrow 35.$$

That is,

$$(x, y) \in \{(6, 35),\ (\Leftrightarrow 6, 35),\ (6, \Leftrightarrow 35),\ (\Leftrightarrow 6, \Leftrightarrow 35)\}.$$

The other possibility is $(x^2 \Leftrightarrow y)(x^2 + y) = (\Leftrightarrow 71) \cdot (\Leftrightarrow 1)$, which gives: $x = \pm 6i \notin \mathbb{Z}$.

Thus we have found all solutions.

2. (a) Show that $x^2 + y^2 \geq 2xy$ for all real numbers x, y.

   (b) Show that $a^2 + b^2 + c^2 \geq ab + bc + ca$ for all real numbers a, b, c.

Solution.

(a) $x^2 + y^2 \geq 2xy \Leftrightarrow (x \Leftrightarrow y)^2 \geq 0$ for all $x, y \in \mathbb{R}$.

(b) From (a), we get

$$a^2 + b^2 \geq 2ab; \quad a^2 + c^2 \geq 2ac; \quad b^2 + c^2 \geq 2bc,$$

that is

$$2(a^2 + b^2 + c^2) \quad \geq \quad 2(ab + ac + bc)$$

or

$$a^2 + b^2 + c^2 \quad \geq \quad ab + ac + bc, \quad \text{for all } a, b \in \mathbb{R}.$$

Remark: Let a, b, $c \in \mathbb{C}$. We observe that the equality

$$x^3 \Leftrightarrow 1 = 0, \text{ that is, } (x \Leftrightarrow 1)(x^2 + x + 1) = 0$$

has roots $x_1 = 1$, $x_{2,3} = \frac{1}{2}(\Leftrightarrow 1 \pm i\sqrt{3})$, so that $x_1 \neq x_2 \neq x_3$. Let $a = x_1$, $b = x_2$, $c = x_3$, giving $a + b + c = 0$, $ab + ac + bc = \frac{1}{2}(\Leftrightarrow 1 + i\sqrt{3}) + \frac{1}{2}(\Leftrightarrow 1 \Leftrightarrow i\sqrt{3}) + \frac{1}{4}(1 + 3) = 0$ and $a^2 + b^2 + c^2 = (a + b + c)^2 \Leftrightarrow 2(ab + ac + bc) = 0$.

(i) Further, in (b) the equality holds because

$$a^2+b^2+c^2 = ab+ac+bc \Leftrightarrow \frac{1}{2}[(a \Leftrightarrow b)^2+(a \Leftrightarrow c)^2+(b \Leftrightarrow c)^2] = 0$$

if and only if $a = b = c \in \mathbb{R}$.

(ii) Also, (b) holds as well, since $a^2 + b^2 + c^2 = ab + ac + bc$ for $a, b, c \in \mathbb{C}$ and $a \neq b \neq c$.

3. Find the sum of the series

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{99}{100!}.$$

Solution. We have

$$\frac{1}{k!} \Leftrightarrow \frac{1}{(k+1)!} = \frac{1}{k!} \Leftrightarrow \frac{1}{(k+1)} = \frac{k}{(k+1)!}.$$

Thus

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{n}{(n+1)!} = 1 \Leftrightarrow \frac{1}{(n+1)!}.$$

For $n = 99$, we get

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots + \frac{99}{100!} = 1 \Leftrightarrow \frac{1}{100!}.$$

BOOK REVIEWS

Edited by ANDY LIU

She Does Math! real-life problems from women on the job, edited by Marla Parker.

Published by the Mathematical Association of America, 1995, paperbound, 272+ pages, ISBN# 0-88385-702-2, list price US$24.00.

Reviewed by **Katherine Heinrich**, *Simon Fraser University*.

Thirty eight women working in a variety of careers (engineering, mathematics, computing, biology, management) describe their lives, their feelings about mathematics and the work they currently do. In all cases, competence, confidence and skill in mathematics are essential to their work. Shelley J. Smith avoided math and chose archaeology because it required no math. When she reached graduate school she knew it was needed (to compute shapes of pottery from broken pieces and to do statistical analysis). She overcame her fear and discovered not only that she could do math but that she also enjoyed it.

These are "regular" women who, on discovering what they wanted to do, pursued it. They met the usual challenges: of changing goals (Donna McConnaha Sheehy started in commercial art but after a work-study experience in the Forest Service, moved to civil engineering); of having children (Beth MacConnell, with a new baby, still had to rise at 4 a.m. to track grizzly bears); and of taking care of parents (Nancy G. Roman retired from her job as astronomer at NASA to take care of her mother) to mention a few.

Their stories are written for young women of all ages: telling them it is possible to be successful and to have interesting and challenging jobs; and that the first step is to develop math skills and to have confidence in one's abilities.

With each story the author presents problems based on either her work or her other interests. All are mathematical. The problems range from the very simple (formatted stock prices) to the far too obscure and difficult (determining prism diopters, determining a star's perigalactic distance). Many were confusing with important aspects undefined – they could be answered only after one first studied the solution. Many were simply substitutions into an unexplained formula - these same problems could often be solved by careful thought without the formula.

Some required formulas and ideas from physics which were presented only in the solutions, and then often without explanation. Some were simple arithmetic and others required calculus. On the other hand, there were some well-written problems that encouraged the reader to explore and search for understanding: such problems were regrettably in the minority.

Perhaps the greatest difficulty I had was in trying to understand who the book was written for. I fear that in trying to reach all young women, regardless of math background or feelings about math, it will fail them all. I hope that any future edition will begin by targeting the audience and then ensuring that the problems and their solutions are written specifically for them.

Mathematical Literacy

Here are the answers to the questions posed in the February 1996 issue.

1. Who thought that the binary system would convince the Emperor of China to abandon Buddhism in favour of Christianity?

   Leibnitz

2. Who asked which king for one grain of wheat for the first square of a chess board, two grains for the second square, four grains for the third square, and so on?

   Grand Vizier Sissa Ben Dahir asked King Shirhâm

3. In which well-known painting, by whom, does a Magic Square appear?

   ***Melancholia* by Dürer**

4. Where was bread cut into "Cones, Cylinders, Parallelograms, and several other Mathematical Figures"?

   ***Laputa* — (Gulliver's Travels)**

5. Which mathematician said: "Philosophers count about two-hundred and eighty eight views of the sovereign good"?

   Pascal

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that* **1 March 1997**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX, *preferably in* LaTeX2e). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2151. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$\triangle ABC$ is a triangle with $\angle B = 2\angle C$. Let H be the foot of the perpendicular from A to BC, and let D be the point on the side BC where the excircle touches BC. Prove that $AC = 2(HD)$.

2152. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $0 \leq x_1 \leq \ldots \leq x_n \leq \frac{\pi}{2}$ be such that $\sum_{k=1}^{n} \sin x_k = 1$.

Consider the set S_n of all sums $x_1 + \ldots + x_n$.

1. Show S_n is an interval.
2. Let l_n be the length of S_n. What is $\lim_{n\to\infty} l_n$?

2153. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Suppose that a, b, $c \in \mathbb{R}$. If, for all $x \in [\Leftrightarrow 1, 1]$, $|ax^2 + bx + c| \leq 1$, prove that

$$|cx^2 + bx + a| \leq 2.$$

2154. *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a convex pentagon, the medians are concurrent. If the concurrence point sections each median in the same ratio, find its numerical value. (A median of a pentagon is the line segment between a vertex and the midpoint of the third side from the vertex.)

2155. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove there is no solution of the equation

$$\frac{1}{x^2} + \frac{1}{y^8} = \frac{1}{z^2}$$

in which y is odd and x, y, z are positive integers with highest common factor 1.

Find a solution in which $y = 15$, and x and z are also positive integers.

2156. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

$ABCD$ is a convex quadrilateral with perpendicular diagonals AC and BD. X and Y are points in the interior of sides BC and AD respectively such that

$$\frac{BX}{CX} = \frac{BD}{AC} = \frac{DY}{AY} .$$

Evaluate

$$\frac{BC \cdot XY}{BX \cdot AC} .$$

2157. *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove that $2^{1997\cdot 1996} \Leftrightarrow 1$ is exactly divisible by 1997^2.

2158. *Proposed by P. Penning, Delft, the Netherlands.*

Find the smallest integer in base eight for which the square root (also in base eight) has 10 immediately following the 'decimal' point.

In base ten, the answer would be 199, with sqrt $(199) = 14.10673\ldots$.

2159. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let the sequence $\{x_n, n \geq 1\}$ be given by

$$x_n = \frac{1}{n}(1 + t + \ldots + t^{n-1})$$

where $t > 0$ is an arbitrary real number.

Show that for all $k, l \geq 1$, there exists an index $m = m(k, l)$ such that $x_k \cdot x_l \leq x_m$.

2160. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$\triangle ABC$ is a triangle with $\angle A < 90^\circ$. Let P be an interior point of ABC such that $\angle BAP = \angle ACP$ and $\angle CAP = \angle ABP$. Let M and N be the incentres of $\triangle ABP$ and $\triangle ACP$ respectively, and let R_1 be the circumradius of $\triangle AMN$. Prove that

$$\frac{1}{R_1} = \frac{1}{AB} + \frac{1}{AC} + \frac{1}{AP}\,.$$

2161. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n \Leftrightarrow 1)(3n \Leftrightarrow 1)}.$$

2162. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, the Cevian lines AD, BE, and CF concur at P. $[XYZ]$ is the area of $\triangle XYZ$. Show that

$$\frac{[DEF]}{2[ABC]} = \frac{PD}{PA} \cdot \frac{PE}{PB} \cdot \frac{PF}{PC}$$

2163. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Prove that if $n, m \in \mathbb{N}$ and $n \geq m^2 \geq 16$, then $2^n \geq n^m$.

Corrections.

2139. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3}R'\sqrt{R'^2 \Leftrightarrow (O'P)^2},$$

where $[XYZ]$ denotes the area of triangle XYZ.

Note: $3\sqrt{3}R'$ instead of $3\sqrt{3R'}$, as was printed in the May 1996 issue.

Also, problem 2149 was ascribed to Juan-Bosco Morero Márquez instead of Juan-Bosco Romero Márquez.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Some solvers are reminded about the instructions given in the **Problems** section above:

> *... please send your proposals and solutions on signed and separate ... sheets of paper.*

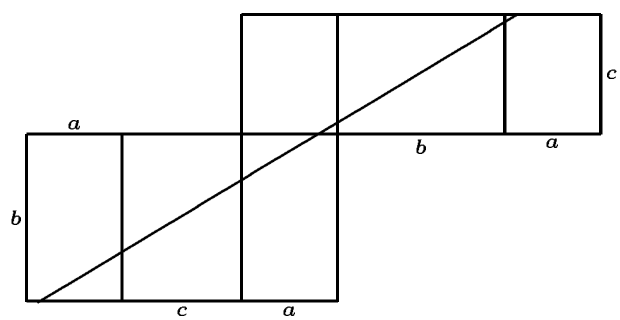
The emphasis here is on "**signed**". The secretarial staff may not notice if a submitted solution is not signed, and when it comes time for an editor to read the file, there is no way of knowing who the anonymous person is!

1823. [1993: 77; 1994: 54] *Proposed by G. P. Henderson, Campbellcroft, Ontario.*

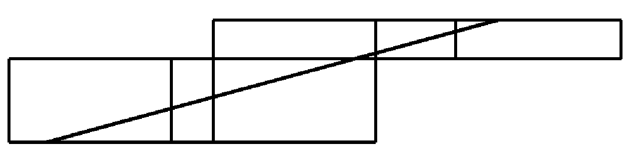
A rectangular box is to be decorated with a ribbon that goes across the faces and makes various angles with the edges. If possible, the points where the ribbon crosses the edges are chosen so that the length of the closed path is a local minimum. This will ensure that the ribbon can be tightened and tied without slipping off. Is there always a minimal path that crosses all six faces just once?

Correction and editor's comment by Bill Sands.

The previous editor's remark on [1994: 55] that Jordi Dou's second example of a minimal path, namely



does not result in any further boxes that can be decorated, is **false**. As Gerd Baron has since pointed out, and Dou and the proposer knew all along, there are boxes that can be decorated in the above way but whose sides a, b, c do not satisfy the triangle inequality. For example, $a = 4$, $b = 2$, $c = 1$ yield



with the ribbon as shown.

The following is derived from Baron's and the proposer's correct solutions for this problem. From the first figure above, one can see that such a ribbon exists in this case if and only if

$$\frac{b}{2a+c} < \frac{b+c}{2a+b+c} < \frac{b}{c}\,,$$

where the middle fraction is the slope of the slanted line. This simplifies to

$$c^2 - b^2 + 2ac > 0 \qquad \text{and} \qquad b^2 - c^2 + 2ab > 0.$$

By permuting a, b, c we obtain two other sufficient conditions for an a by b by c box to allow a ribbon:

$$a^2 - c^2 + 2ab > 0 \qquad \text{and} \qquad c^2 - a^2 + 2bc > 0,$$

and

$$b^2 - a^2 + 2bc > 0 \qquad \text{and} \qquad a^2 - b^2 + 2ac > 0.$$

Moreover, these (and the previous solution) exhaust all possible ways for a ribbon to cross all six faces of the box. If we assume that $a \geq b \geq c$, then an a by b by c box will **not** possess a ribbon satisfying the condition of the problem exactly if the following three inequalities all hold:

$$b^2 - a^2 + 2bc \leq 0, \qquad c^2 - b^2 + 2ac \leq 0, \qquad b + c \leq a.$$

For example, the 4 by 3 by 1 box does not have a ribbon.

So in retrospect, only Baron, Dou and the proposer sent in complete solutions of this problem. The previous editor (that is, me, Bill Sands) thanks Baron and the proposer for calling attention to my too–hasty reading of the solutions the first time around.

2044. [1995: 158] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Suppose that $n \geq m \geq 1$ and $x \geq y \geq 0$ are such that

$$x^{n+1} + y^{n+1} \leq x^m - y^m.$$

Prove that $x^n + y^n \leq 1$.

A typographical error in the LaTeX codes used in typesetting resulted in Toshio Seimiya, Kawasaki, Japan being listed as the solver. The solver was indeed Heinz-Jürgen Seiffert, Berlin, Germany. The editor offers his apologies to Heinz-Jürgen Seiffert for any embarrassment caused by this error.

2048. [1995: 158] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Find the least integer n so that, for every string of length n composed of the letters $a, b, c, d, e, f, g, h, i, j, k$ (repetitions allowed), one can find a non-empty block of (consecutive) letters in which no letter appears an odd number of times.

Solution by Douglas E. Jackson, Eastern New Mexico University, Portales, New Mexico, USA.

Let S be a string of length $2^{11} = 2048$ of the 11 letters, a, b, $\ldots$, k. For $i = 1, ,, \ldots , 2048$, let $n_a(i)$, $n_b(i)$, $\ldots$, $n_k(i)$ be the number of occurrences **mod** 2 of the letters a, b, $\ldots$, k respectively in the first i positions of S. Then, for each position i in S, we have an associated binary string, $n(i) = n_a(i)n_b(i)\ldots n_k(i)$, for length 11. If, for some i, we have $n(i) = 00\ldots0$, then in the first i positions of S, each letter appears an even number of times. If no position is associated with the zero bit string, then, since there are only 2047 non-zero bit strings of length 11, there must exists i and j such that $1 \leq i \leq j \leq 2048$ and $n(i) = n(j)$. [The Pigeon-hole Principle one more time! — Ed.]

Then each letter appears an even number of times in positions $i + 1$ through j of S. Therefore, any string on 11 letters with length at least 2048 will have a block in which no letter appears an odd number of times.

In the following sequence of strings, each string is constructed by taking two copies of the previous string and inserting a new letter between these copies. This construction allows a simple inductive argument, that in these strings, every non-empty block contains some letter an odd number of times. Let B be a block in one of the strings of this sequence, other than the first. If B contains the central (new) letter, then obviously it contains that letter an odd number of times (once). Otherwise, B must be a block of the previous string, and hence is inductively assumed to contain some letter an odd number of times.

$$\begin{array}{l} a \\ a\ b\ a \\ a\ b\ a\ \ c\ \ a\ b\ a \\ a\ b\ a\ c\ a\ b\ a\ \ d\ \ a\ b\ a\ c\ a\ b\ a \\ \quad\ldots \end{array}$$

Clearly string number 11 in this sequence has 11 letters and length 2047. So the minimum value of n which forces the required property is 2048.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; ASHSIH KR. SINGH, student, Kanpur, India; HOE TECK WEE, student, Hwa Chong Junior College, Singapore (all with virtually identical arguments to the one given above); and the proposer.

Wee actually considered the general case when the number of letters is k instead of 11. Using the same argument, he showed that the answer is 2^k.

2052. [1995: 202] *Proposed by K.R.S. Sastry, Dodballapur, India.*

The infinite arithmetic progression $1 + 3 + 5 + 7 + \ldots$ of odd positive integers has the property that all of its partial sums

$$1, \quad 1 + 3, \quad 1 + 3 + 5, \quad 1 + 3 + 5 + 7, \quad \ldots$$

are perfect squares. Are there any other infinite arithmetic progressions, all terms positive integers with no common factor, having this same property?

Solution by Toby Gee, student, the John of Gaunt School, Trowbridge, England Let the n^{th} term of the arithmetic progression be $a + (n \Leftrightarrow 1)d$, and let the sum of the first n terms be

$$S_n = an + \frac{n(n \Leftrightarrow 1)d}{2}.$$

Let p be any prime and consider $n = 2p$. Then, since S_{2p} is a perfect square, we have

$$S_{2p} = 2ap + p(2p \Leftrightarrow 1)d = y^2$$

for some integer y. Thus $p \mid y$. Let $y = px$. Then

$$2a + 2pd \Leftrightarrow d = px^2 \equiv 0 \pmod{p}$$

Therefore, $p \mid (2a \Leftrightarrow d)$. For all sufficiently large primes p we have $p > 2a \Leftrightarrow d$, implying $2a = d$, whence $S_n = an + an(n \Leftrightarrow 1) = an^2$. Since these numbers have no common factor, we must have $a = 1$, so there are no other such progressions.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary; and the proposer. There were two incorrect solutions submitted.

Remark by Christopher J. Bradley, Clifton College, Bristol, UK.
The correct generalization of the

$$1 \qquad 1+3 \qquad 1+3+5 \qquad 1+3+5+7 \qquad \ldots$$

phenomenon is to be found not in the problem stated, but rather by producing the partial sums of an arithmetic progression such as

$$15 \qquad 15+33 \qquad 15+33+51 \qquad 15+33+51+69 \qquad \ldots$$

($a = 15$, $d = 18$) and then adding 1 providing

$$16 \qquad 49 \qquad 100 \qquad 169 \qquad \ldots$$

and then tacking on 1 at the beginning giving

$$1^2, \quad 4^2, \quad 7^2, \quad 10^2, \quad 13^2, \quad \ldots$$

The integers whose squares appear are always in arithmetic progression themselves.

It always works. For example, if we want to do the reverse for

$$1^2, \quad 5^2, \quad 9^2, \quad 13^2, \quad 17^2, \quad \dots$$

we look at 24, 80, 168, 288, etc. and perceive this to be

$$24, \quad 24+56, \quad 24+56+88, \quad 24+56+88+120, \quad \dots$$

which are the partial sums for $a = 24$, $d = 32$. Proofs covering these remarks are straightforward. In the above, d is twice a perfect square and $a = \frac{1}{2}d + \sqrt{2d}$. It also works when squares other than 1 (or 1^2) are tacked on provided a and d are carefully chosen.

BRADLEY mentions that he has seen this in some magazine before, but does not recall when or where. Perhaps a reader can supply us with the proper reference.

2056. [1995: 203] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find a polynomial of degree five whose roots are the tenth powers of the roots of the polynomial $x^5 - x - 1$.

I *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

If $x^5 - 1 = x$ has roots α_i $(i = 1, 2, \dots, 5)$, then the substitution $y = x^5$ will provide an equation with roots α_i^5. Now, $(x^5 - 1)^5 = x^5$, so the equation required is $(y - 1)^5 = y$; that is, $y^5 - 5y^4 + 10y^3 - 10y^2 + 4y - 1 = 0$. If we now put $u = y^2$, the resulting equation in u will have roots α_i^{10}.

Since $y(y^4 + 10y^2 + 4) = 5y^4 + 10y^2 + 1$, we have, by squaring, that $u(u^2 + 10u + 4)^2 = (5u^2 + 10u + 1)^2$. Multiplying out and collecting like terms, we get $u^5 - 5u^4 + 8y^3 - 30u^2 - 4u - 1 = 0$, and the polynomial on the left of this equation is the answer.

II *Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.*

Let r_i $(1 \le i \le 5)$ be the roots of the given polynomial. Since $x^5 = x + 1$, we have $r_i^{10} = (r_i + 1)^2$. It is easy to construct a polynomial with roots $r_i + 1$; it is

$$f(x) = (x - 1)^5 - (x - 1) - 1 = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x - 1.$$

Now, the polynomial

$$\begin{aligned} -f(x)f(-x) &= (x^5 - 5x^4 + 10x^3 - 10x^2 + 4x - 1) \\ &\quad \times (x^5 + 5x^4 + 10x^3 + 10x^2 + 4x + 1) \\ &= x^{10} - 5x^8 + 8x^6 - 30x^4 - 4x^2 - 1 \end{aligned}$$

is even, and has roots $r_i + 1$, $\Leftrightarrow(r_i + 1)$ $(1 \leq i \leq 5)$. Replacing each x^{2k} term with an x^k term, we get a polynomial $x^5 \Leftrightarrow 5x^4 + 8x^3 \Leftrightarrow 30x^2 \Leftrightarrow 4x \Leftrightarrow 1$, with roots $(r_i + 1)^2$, that is, r_i^{10}.

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RONALD HAYNES, student, Memorial University of Newfoundland, St. John's, Newfoundland; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; MITKO CHRISTOV VINCHEO, Rousse, Bulgaria; and the proposer. One reader submitted a one-line answer only, and another submitted a partially incorrect answer. The solutions given by Hess, Murty and the proposer are very similar to Solution I above.

2058. [1995: 203] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Let a, b, c be integers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3.$$

Prove that abc is the cube of an integer.

Solution by Michael Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland, modified by the editor.

Without loss of generality, we may assume that $\gcd(a, b, c) = 1$, since, if $d \mid a, b, c$ and $a' = \frac{a}{d}$, $b' = \frac{b}{d}$, $c' = \frac{c}{d}$, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$ if and only if $\frac{a'}{b'} + \frac{b'}{c'} + \frac{c'}{a'} = 3$, and abc is a perfect cube if and only if so is $a'b'c'$. Rewrite the given equation as

$$a^2c + b^2a + c^2b \;=\; 3abc. \qquad (1)$$

If $abc = \pm 1$, then we are done. Otherwise, let p be any prime divisor of abc. Then from (1), it is clear that p divides exactly two of a, b and c. By symmetry, we may assume that $p \mid a$, $p \mid b$ and $p \nmid c$.

Suppose that $p^m \parallel a$ (that is, $p^m \mid a$, but $p^{m+1} \nmid a$) and $p^n \parallel b$. Then $p^{m+n} \mid 3abc$. We claim that $n = 2m$.

If $n < 2m$, then $n + 1 < 2m$, and thus $p^{n+1} \mid a^2c$. Since $p^{n+1} \mid b^2a$, but $p^{n+1} \nmid c^2b$, we have $p^{n+1} \nmid a^2c + b^2a + c^2b$. This is a contradiction since $n + 1 \leq m + n$ implies $p^{n+1} \mid 3abc$.

If $n > 2m$, then $n \geq 2m + 1$, and thus $p^{2m+1} \mid c^2b$. Since $p^{2m+1} \mid b^2a$, but $p^{2m+1} \not| \, a^2c$, we have that $p^{2m+1} \not| \, a^2c + b^2a + c^2b$. This is again a contradiction since $2m + 1 < m + n$ implies that $p^{2m+1} \mid 3abc$.

Hence $n = 2m$, which implies that $p^m \parallel abc$. It follows immediately that abc is a perfect cube.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Six solvers submitted incorrect solutions. All of them assumed inadvertently that a, b, c are positive. (One other reader sent in a partial solution based on this assumption.) In this case, the problem becomes trivial since, from the Arithmetic-Geometric Mean inequality, one can deduce immediately that $a = b = c$. Several other incomplete, or partially incomplete, solutions were also received.

Using an argument similar to the one given above, Bosley showed that all solutions are given by $a = b = c$ or $a = kr(r+s)^2$, $b = ksr^2$, $c = \Leftrightarrow k(r+s)s^2$ for some integers k, r, s, provided that $abc \neq 0$. Setting $k = 1$, $r = n$, $s = n+1$ ($n \neq 0$, $\Leftrightarrow 1$) yields the infinite family of solutions (a, b, c), where $a = n(2n+1)^2$, $b = n^2(n+1)$, $c = \Leftrightarrow(n+1)^2(2n+1)$ [together with all integer multiples of them, and triples obtained by cyclically permuting a, b and c. — Ed.].

Setting $k = 1$, $r = n \Leftrightarrow 1$, $s = n + 1$, or $k = 1$, $r = n + 1$, $s = n \Leftrightarrow 1$ ($n \neq 0$, ± 1) yields the infinite families of solutions (a, b, c), where $a = 4n^2(n \Leftrightarrow d)$, $b = (n^2 \Leftrightarrow 1)(n \Leftrightarrow d)$, $c = \Leftrightarrow 2n(n+d)^2$, where $d = \pm 1$. All these families were also found by Penning. Setting $k = 1$, $r = 2$, $s = 3$, one obtains the solution $(50, 12, \Leftrightarrow 45)$, also found by Hurthig. The special case $(4, 1, \Leftrightarrow 2)$, obtained by letting $k = r = s = 1$ was also found by Chronis, Flannigan, Tsoussaglou, and the proposer.

Janous asked whether the "n-analogue" of this problem is true (the case $n = 2$ is trivial); for example, if $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4$, where a, b, c, d are integers, must $abcd$ be a perfect 4th power? The answer is clearly negative, as shown by the example $(a, b, c, d) = (\Leftrightarrow 2, 4, 1, 1)$, found by this editor.

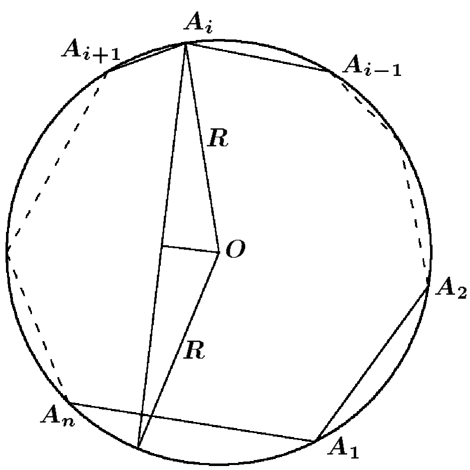
2059. [1995: 203] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Let $A_1A_2\ldots A_n$ be an n-gon with centroid G inscribed in a circle. The lines A_1G, $A_2G, \ldots, A_nG$ intersect the circle again at $B_1, B_2, \ldots, B_n$. Prove that

$$\frac{A_1G}{GB_1}+\frac{A_2G}{GB_2}+\cdots+\frac{A_nG}{GB_n}=n.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let O and R be the centre and the radius of the circumscribed circle $A_1A_2\ldots A_n$.



Since

$$\frac{A_iG}{GB_i}=\frac{A_iG^2}{A_iG\cdot GB_i}=\frac{A_iG^2}{OA_i^2\Leftrightarrow OG^2}=\frac{A_iG^2}{R^2\Leftrightarrow OG^2},$$

for $i=1,\ 2,\ \ldots,\ n$, we get

$$\sum_{i=1}^{n}\frac{A_iG}{GB_i} = \frac{\sum_{i=1}^{n}A_iG^2}{R_2\Leftrightarrow OG^2}. \qquad (1)$$

Since G is the centroid of $A_1A_2\ldots A_n$, we have

$$\sum_{i=1}^{n}\overrightarrow{A_iG} = \vec{0}.$$

Thus we have

$$\sum_{i=1}^{n}\overrightarrow{A_iG}\cdot\overrightarrow{GO} = \left(\sum_{i=1}^{n}\overrightarrow{A_iG}\right)\cdot\overrightarrow{GO} = \vec{0}\cdot\overrightarrow{GO} = 0.$$

Hence we have

$$\begin{aligned}
\sum_{i=1}^{n} \overrightarrow{A_iO}^2 &= \sum_{i=1}^{n} \left(\overrightarrow{A_iG} + \overrightarrow{GO}\right)^2 \\
&= \sum_{i=1}^{n} \overrightarrow{A_iG}^2 + 2\sum_{i=1}^{n} \overrightarrow{A_iG}\cdot\overrightarrow{GO} + \sum_{i=1}^{n} \overrightarrow{GO}^2 \\
&= \sum_{i=1}^{n} \overrightarrow{A_iG}^2 + n\,\overrightarrow{GO}^2, \quad \text{that is} \\
nR^2 &= \sum_{i=1}^{n} A_iG^2 + n\,GO^2.
\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} A_iG^2 = n(R^2 \Leftrightarrow GO^2). \qquad (2)$$

From (1) and (2), we obtain that $\displaystyle\sum_{i=1}^{n} \frac{A_iG}{GB_i} = n$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; P. PENNING, Delft, the Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2060. [1995: 203] *Proposed by Neven Jurić, Zagreb, Croatia.*

Show that for any positive integers m and n, the integer

$$\left\lfloor \left(m + \sqrt{m^2 \Leftrightarrow 1}\right)^n \right\rfloor$$

is odd ($\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

I *Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Let $x_n = \left(m + \sqrt{m^2 \Leftrightarrow 1}\right)^n + \left(m \Leftrightarrow \sqrt{m^2 \Leftrightarrow 1}\right)^n$. Then, using the Binomial Theorem, we get

$$x_n = 2\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} m^{n-2k} \left(m^2 \Leftrightarrow 1\right)^k,$$

and so x_n is an even integer.

Since $0 < \left(m \Leftrightarrow \sqrt{m^2 \Leftrightarrow 1}\right)^n = \dfrac{1}{m + \sqrt{m^2 \Leftrightarrow 1}} \leq 1$, we have that

$$x_n - 1 \leq \left(m + \sqrt{m^2 - 1}\right)^n < x_n, \quad \text{and thus that}$$
$$\left\lfloor \left(m + \sqrt{m^2 - 1}\right)^n \right\rfloor = x_n - 1, \quad \text{which is odd.}$$

II *Solution by P. Penning, Delft, the Netherlands.* (*modified slightly by the editor*).

Let $u = m + \sqrt{m^2 - 1}$ and

$$F_n = u^n + u^{-n} = \left(m + \sqrt{m^2 - 1}\right)^n + \left(m - \sqrt{m^2 - 1}\right)^n .$$

Then $F_1 = 2m$. Since

$$2mF_n = F_1 F_n = \left(u + u^{-1}\right)\left(u^n + u^{-n}\right) = F_{n+1} + F_{n-1},$$

we obtain the recurrence relation

$$F_{n+1} \;=\; 2m\, F_n - F_{n-1}.$$

Since $F_2 = 2\left(2m^2 - 1\right)$ is even, we conclude that F_n is even for all n. Since $u^n \geq 1$, we have $0 < u^{-n} \leq 1$, and it follows that $\lfloor u^n \rfloor = \lfloor F_n - u^{-n} \rfloor = F_n - 1$, which is odd.

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; SEUNG-JIN BANG, Seoul, Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE; student, Memorial University, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer.

Most submitted solutions are similar to one of the two solutions above. Bellot Rosado pointed out that the problem is a very old one. He located it as Part (a) of Problem 363, Americal Mathematical Monthly, 1912, p. 51. *He also gave four similar or related problems.*

2061. [1995: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with centroid G, and P is a variable interior point of ABC. Let D, E, F be points on sides BC, CA, AB respectively such that $PD \parallel AG$, $PE \parallel BG$, and $PF \parallel CG$. Prove that $[PAF]+[PBD]+[PCE]$ is constant, where $[XYZ]$ denotes the area of triangle XYZ.

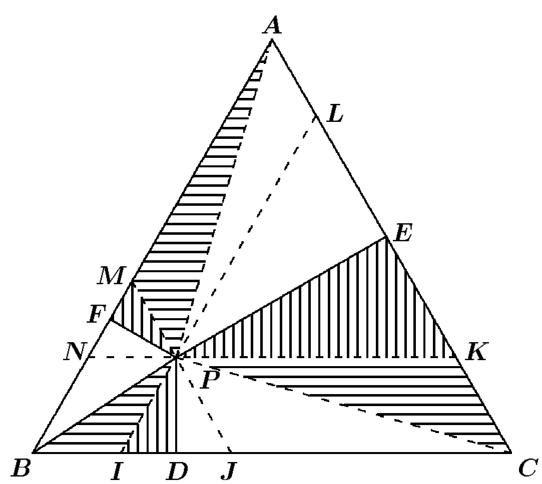
I. *Solution by Jaosia Jaszunska, student, Warsaw, Poland.*

In the figure, the line through P:

parallel to AB intersects the sides BC and CA at I and L respectively;

parallel to BC intersects the sides CA and AB at K and N respectively;

parallel to CA intersects the sides AB and BC at M and J respectively.



Since PD and AG are parallel, the dilatation that takes triangle PIJ to ABC, takes D to the mid-point of BC. Thus D is the mid-point of IJ. Similarly, E and F are the mid-points of KL and MN respectively. It is easily seen that the shaded area is equal to the unshaded area. [The vertically shaded pieces such as PID form half of a triangle, while the horizontally shaded pieces such as PBI form half of a parallelogram.] Thus

$$[PAF]+[PBD]+[PCE]=\tfrac{1}{2}[ABC],$$

so that the sum is constant as desired.

[*Editor's comment*: Several solvers mentioned that since the result belongs to affine geometry, one can assume without loss of generality that the given triangle is equilateral. We have therefore used an equilateral triangle in the figure.]

II. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

The result holds for any point P in the plane – there is no need to restrict P to the interior of the given triangle. In areal coordinates, $\overrightarrow{AG} = (\Leftrightarrow\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, and since $PD \| AG$, the point D must have coordinates $(\lambda \Leftrightarrow \frac{2\kappa}{3}, \mu + \frac{\kappa}{3}, \nu + \frac{\kappa}{3})$, where P is $(\lambda, \mu\nu)$, normalised so that $\lambda + \mu + \nu = 1$, and κ is chosen so that D lies on BC. Thus $\kappa = 3\lambda/2$ and D is $(0, \mu + \frac{\lambda}{2}, \nu + \frac{\lambda}{2})$. From the formula for ratio of areas,

$$\frac{[PBD]}{[ABC]} = \begin{vmatrix} \lambda & 0 & 0 \\ \mu & 1 & \mu + \frac{\lambda}{2} \\ \nu & 0 & \nu + \frac{\lambda}{2} \end{vmatrix} = \frac{1}{2}\lambda^2 + \nu\lambda.$$

Similar formulae hold for the other areas, from which we deduce that

$$\begin{aligned} &[PAF] + [PBD] + [PCE] \\ &\quad = \; [ABC]\left(\tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\mu^2 + \tfrac{1}{2}\nu^2 + \nu\lambda + \lambda\mu + \mu\nu\right) \\ &\quad = \; \tfrac{1}{2}(\lambda + \mu + \nu)^2[ABC] \\ &\quad = \; \tfrac{1}{2}[ABC], \end{aligned}$$

which is constant for a given triangle, as P varies.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer.

2062. [1995: 234] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Find a positive integer n so that both the continued roots

$$\sqrt{1995 + \sqrt{n + \sqrt{1995 + \sqrt{n + \cdots}}}}$$

and

$$\sqrt{n + \sqrt{1995 + \sqrt{n + \sqrt{1995 + \cdots}}}}$$

converge to positive integers.

Solution by Miguel Angel Cabezón Ochoa, Logroño, Spain (loosely translated from the Spanish).

Let $a_0 = 0$, $a_1 = \sqrt{1995}$, and $a_{p+2} = \sqrt{1995 + \sqrt{n + a_p}}$ for $p \geq 0$. Similarly let $b_0 = 0$, $b_1 = \sqrt{n}$, and $b_{p+2} = \sqrt{n + \sqrt{1995 + b_p}}$ for $p \geq 0$.

We will now show by induction that $a_p < K$ for all $p \geq 0$, where $K = \max\{1995, n\}$. This is clearly true for $p = 0$ and $p = 1$. Suppose that $a_p < K$ for all p, $0 \leq p < t$ with $t \geq 2$. Then

$$\begin{aligned} a_t &= \sqrt{1995 + \sqrt{n + a_{t-2}}} < \sqrt{K + \sqrt{K + K}} = \sqrt{K + \sqrt{2K}} \\ &= K\sqrt{\frac{1}{K} + \frac{\sqrt{2K}}{K^2}} = K\sqrt{\frac{1}{K} + \sqrt{\frac{2}{K^3}}} < K \end{aligned}$$

whence $a_p < K$ for all $p \geq 0$.

Next we show that $\{a_p\}$ is an increasing sequence by induction. It is clear that $a_0 < a_1 < a_2$. Suppose that $a_p < a_{p+1}$ for some $p \geq 0$. Then

$$a_{p+2} = \sqrt{1995 + \sqrt{n + a_p}} < \sqrt{1995 + \sqrt{n + a_{p+1}}} = a_{p+3}$$

which establishes that the sequence is increasing.

Similar proofs exist for the sequence $\{b_p\}$. Thus both $\{a_p\}$ and $\{b_p\}$ converge to limits, say x and y, respectively. This leads to the equations $x = \sqrt{1995 + y}$ and $y = \sqrt{n + x}$. Let $y = (p + 44)^2 \Leftrightarrow 1995$; then $x = p + 44$ and $n = y^2 \Leftrightarrow x$. (Editor's comment: this is clearly a solution for all integer values of $p \geq 1$.) The first two values for p give the following solutions: $(p, n, x, y) = (1, 855, 45, 30)$ and $(p, n, x, y) = (2, 14595, 46, 121)$.

Also solved by SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TIM CROSS, King Edward's School, Birmingham, England; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio, USA; JAMSHID KHOLDI, New York, NY, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; CORY PYE, student, Memorial University, St. John's, Newfoundland; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ASHSIH KR. SINGH, student, Kanpur, India; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incomplete and one incorrect solution submitted.

Most solvers found all of the solutions above, but some only found one. Very few solvers directly examined the question of convergence of the continued roots.

2063. [1995: 234] *Proposed by Aram A. Yagubyants, Rostov na Donu, Russia.*

Triangle ABC has a right angle at C.

(a) Prove that the three ellipses having foci at two vertices of the given triangle, while passing through the third, all share a common point.

(b) Prove that the principal vertices of the ellipses of part (a) (that is the points where an ellipse meets the axis through its foci) form two pairs of collinear triples.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

(a) Let P be the fourth vertex of the rectangle $APBC$. Then

$$PA = BC,\ PB = AC,\ PC = AB,$$

whence

$$PA + PB = CB + CA,\ PB + PC = AC + AB,\ PC + PA = BA + BC,$$

which means that each of the three ellipses passes through P.

(b) Let M_1, M_2, M_3 be the centres of the three ellipses, and therefore the mid-points of AB, BC and CA respectively. Denote the principal vertices of the ellipses by X_i and Y_i, with B lying between A and X_1 and between C and X_2, and with C lying between A and X_3. Since

$$M_1X_1 = \frac{AC + BC}{2},\ M_2X_2 = \frac{AB + AC}{2},\ M_3X_3 = \frac{AB + BC}{2},$$

we get with $AB = c$, $BC = a$, $AC = b$ (and recalling that $a^2 + b^2 = c^2$),

$$\begin{aligned}
\frac{AX_1}{BX_1}\cdot\frac{BX_2}{CX_2}\cdot\frac{CX_3}{AX_3} &= \frac{\frac{a+b+c}{2}\,\frac{\Leftrightarrow a+b+c}{2}\,\frac{a\Leftrightarrow b+c}{2}}{\frac{a+b\Leftrightarrow c}{2}\,\frac{a+b+c}{2}\,\frac{a+b+c}{2}} \\
&= \frac{\Leftrightarrow a^2 + 2ab \Leftrightarrow b^2 + c^2}{a^2 + 2ab + b^2 \Leftrightarrow c^2} \\
&= \frac{2ab}{2ab} = 1.
\end{aligned}$$

This means, according to Menelaus' Theorem, that X_1, X_2, X_3 are collinear.

Analogously, Y_1, Y_2, Y_3 are collinear.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ROBYN M. CARLEY (student) and ANA WITT, Austin Peay State University, Clarksville, Tennessee, USA; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; CAMILLA FOX, Toronto, Ontario; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; WALDEMAR POMPE, student, University of Warsaw, Poland; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. One anonymous solution was received - see note at the start of this section.

Bellot Rosado and López Chamorro, Cabezón Ochoa, Dou, Pompe and Seimiya all sent in solutions that are essentially the same as Perz's, whose solution was chosen by the time-honoured method of selecting the superior paper after having thrown the lot down the stairs. The other solvers used coordinates; teachers may wish to note that this problem provides a non-standard exercise that illustrates effectively several aspects of analytic geometry.

2064. [1995: 234] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Show that

$$3\max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}\geq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

for arbitrary positive real numbers a, b, c.

I. *Solution by Sabin Cautis, student, Earl Haig Secondary School, North York, Ontario.*

Assume without loss of generality that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\,.$$

Using the AM–GM inequality,

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3\sqrt[3]{\frac{abc}{bca}}=3,$$

thus

$$3\max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\quad\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}=3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$

$$\begin{aligned}
&\geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\\
&\geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3\\
&= (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).
\end{aligned}$$

II. *Solution by Chris Wildhagen, Rotterdam, the Netherlands.*

It is evident that

$$x+y\geq 2z \qquad \Longrightarrow \qquad \max\{x,y\}\geq z$$

for all real numbers x, y and z. Hence it is sufficient to show that

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \quad + \quad \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

or

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)\geq 6.$$

This is obviously true, since if $p,q>0$ then

$$\frac{p}{q}+\frac{q}{p}=\frac{p^2+q^2}{pq}\geq 2.$$

III. *Editorial comments.*

Note that Solution II says that "max" in the problem statement can be replaced by "average". This stronger result was noticed by several other solvers as well. Here are some other remarks and generalizations made by readers.

If a,b,c are the sides of a triangle, then the inequality is true if "max" is replaced by "min", that is, the simpler inequality

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{1}$$

holds whenever a,b,c are the sides of a triangle. The proposer included this problem as #2 in his list of six Quickies published in the February 1996 Olympiad Corner (for a solution, see [1996: 59–60]). The proposer also notes that this inequality need not hold otherwise. For example, it fails when $a=8$, $b=3$ and $c=1$.

Bradley proves that in fact (1) *holds whenever* $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ *are the sides of a triangle*. (This happens whenever a,b,c are sides of a triangle and in other cases besides.) He first proves that if x,y,z are the sides of a triangle and l,m,n are any real numbers, then

$$\begin{aligned}
&x^2l^2+y^2m^2+z^2n^2 \Leftrightarrow (y^2+z^2 \Leftrightarrow x^2)mn\\
&\quad \Leftrightarrow (z^2+x^2 \Leftrightarrow y^2)nl \Leftrightarrow (x^2+y^2 \Leftrightarrow z^2)lm \quad \geq \quad 0,
\end{aligned} \tag{2}$$

by completing the square. (If we multiply the left side of (2) by $4x^2$, then it can be rewritten in the form

$$\begin{aligned} &\left(2x^2 l \Leftrightarrow (z^2 + x^2 \Leftrightarrow y^2)n \Leftrightarrow (x^2 + y^2 \Leftrightarrow z^2)m\right)^2 \\ +&(x+y+z)(x+y \Leftrightarrow z)(y+z \Leftrightarrow x)(z+x \Leftrightarrow y)(m \Leftrightarrow n)^2, \end{aligned} \tag{3}$$

with the help of the identity

$$\begin{aligned} &(x+y+z)(x+y \Leftrightarrow z)(y+z \Leftrightarrow x)(z+x \Leftrightarrow y) \\ =\ & 4x^2 z^2 \Leftrightarrow (z^2 + x^2 \Leftrightarrow y^2)^2, \end{aligned} \tag{4}$$

and (3) is clearly ≥ 0 if x, y, z are sides of a triangle.) Bradley then puts $a = x^2 = n$, $b = y^2 = l$, $c = z^2 = m$ in (2); the result is equivalent to (1) and holds if $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are the sides of a triangle. Consultation with helpful expert (and current proposer) Murray Klamkin yielded the following alternate way to derive (2). It is known (for example, see Theorem 3, page 306 of Gantmacher's *Matrix Theory (The Theory of Matrices), Vol. I*, Chelsea, 1959, or any other advanced book on matrix theory) that (2) holds if and only if the matrix

$$\begin{bmatrix} x^2 & \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} & \dfrac{y^2 \Leftrightarrow z^2 \Leftrightarrow x^2}{2} \\ \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} & y^2 & \dfrac{x^2 \Leftrightarrow y^2 \Leftrightarrow z^2}{2} \\ \dfrac{y^2 \Leftrightarrow z^2 \Leftrightarrow x^2}{2} & \dfrac{x^2 \Leftrightarrow y^2 \Leftrightarrow z^2}{2} & z^2 \end{bmatrix}$$

is non-negative definite, which is true if and only if the principal minors are non-negative. Since the minor x^2 is non-negative, and the determinant of the entire matrix is zero (just add all the rows), all we have to prove is that

$$\begin{vmatrix} x^2 & \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} \\ \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} & y^2 \end{vmatrix} \geq 0.$$

But this is true when x, y, z are the sides of a triangle because

$$4\begin{vmatrix} x^2 & \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} \\ \dfrac{z^2 \Leftrightarrow x^2 \Leftrightarrow y^2}{2} & y^2 \end{vmatrix} = 4x^2y^2 \Leftrightarrow (z^2 \Leftrightarrow x^2 \Leftrightarrow y^2)^2$$

which is ≥ 0 by a version of (4).

Janous shows that for any $n \geq 3$ and for all $x_1, \ldots, x_n > 0$,

$$(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \leq \frac{n}{n \Leftrightarrow 1} \sum_{1 \leq k < l \leq n} \left(\frac{x_k}{x_l} + \frac{x_l}{x_k}\right),$$

which generalizes the result of Solution II and can be proved the same way. This can be written as

$$\frac{M_1(x_1,\ldots,x_n)}{M_{-1}(x_1,\ldots,x_n)} \le M_1\left(\left\{\frac{x_k}{x_l} : k \ne l\right\}\right),$$

where M_t denotes the t^{th} *power mean*:

$$M_t(x_1,\ldots,x_n) = \left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t}.$$

He asks for the minimum $t > 0$ such that

$$\frac{M_1(x_1,\ldots,x_n)}{M_{-1}(x_1,\ldots,x_n)} \le M_t\left(\left\{\frac{x_k}{x_l} : k \ne l\right\}\right)$$

for all $x_1,\ldots,x_n > 0$. (His conjecture: $t = 1$.) Janous also writes the inequality of Solution II as

$$M_1(a,b,c)M_1\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right) \le \frac{1}{2}\left[M_1\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right) + M_1\left(\frac{b}{a},\frac{c}{b},\frac{a}{c}\right)\right],$$

and then wonders what the minimum $t > 0$ is so that

$$M_1(a,b,c)M_1\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right) \le M_t\left[M_1\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right), M_1\left(\frac{b}{a},\frac{c}{b},\frac{a}{c}\right)\right].$$

For an integer $n > 3$ and positive real numbers $a_1, a_2, \ldots, a_n$, it is *not always true* that

$$n \cdot \max\left\{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1},\ \frac{a_2}{a_1} + \frac{a_3}{a_2} + \cdots + \frac{a_1}{a_n}\right\}$$
$$\ge (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

A counterexample when $n = 4$ is $(a_1, a_2, a_3, a_4) = (4, 2, 1, 2)$, for which the left side of the inequality is 20 and the right side is $81/4$.

For readers who still need more problems to solve, here are two more, inspired by the above contributions.

(i) Find the smallest $t > 0$ so that (1) holds whenever a^t, b^t, c^t are the sides of a triangle. Could the answer be $t = 1/2$? (Klamkin's quickie says $t \le 1$, and Bradley proved that $t \le 1/2$.)

(ii) Find the smallest $t > 0$ so that

$$\frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le t\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) + (1 \Leftrightarrow t)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

whenever $a, b, c > 0$ satisfy

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

(The original problem says $t \leq 1$, and Solution II shows that $t \leq 1/2$.)

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, New Mexico, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; SAI C. KWOK, San Diego, California, USA; KEE-WAI LAU, Hong Kong; THOMAS LEONG, Staten Island, New York, USA; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ASHSIH KR. SINGH, student, Kanpur, India; PANOS E. TSAOUSSOGLOU, Athens, Greece; JOHANNES WALDMANN, Friedrich-Schiller-Universität, Jena, Germany (two solutions); EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. There was also one anonymous solution.

Many solvers gave solutions similar to I *or* II. *Konečný included a generalization to n numbers which is weaker than Janous's generalization given above.*

2065. [1995: 235] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Find a monic polynomial $f(x)$ of lowest degree and with integer coefficients such that $f(n)$ is divisible by 1995 for all integers n.

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

We have $1995 = 3 \cdot 5 \cdot 7 \cdot 19$. It can be shown that the congruence $f(x) \equiv 0 \pmod{p}$ of degree n with integer coefficients has at most n solutions for prime p (see, for example, Niven, Zuckerman, and Montgomery's *An Introduction to the Theory of Numbers*, 5^{th} edition, p. 93). Hence if $f(n) \equiv 0 \pmod{19}$ for all n and $f(x)$ is monic, $f(x)$ must be of degree 19 or higher. On the other hand,

$$f(x) = x(x+1)(x+2)\cdots(x+18),$$

of degree 19, is easily seen to be divisible by 1995 for all integers x.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; an anonymous solver; and the proposer. There were five incorrect solutions submitted.

Pompe comments that "this problem has been discussed earlier in the literature; see Problem 17 in Donald J. Newmann's A Problem Seminar*".*

2066. [1995: 235] Proposed by John Magill, Brighton, England.

The inhabitants of Rigel III use, in their arithmetic, the same operations of addition, subtraction, multiplication and division, with the same rules of manipulation, as are used by Earth. However, instead of working with base ten, as is common on Earth, the people of Rigel III use a different base, greater than two and less than ten.

```
    BC
AB)CBC
   AB
   BDC
   BDC
  ----
```

This is the solution to one of their long division problems, which I have copied from a school book. I have substituted letters for the notation originally used. Each of the letters represents a different digit, the same digit wherever it appears.

As the answer to this puzzle, substitute the correct numbers for the letters and state the base of the arithmetic of Rigel III.

Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

From the long division, note that $B \times AB = AB$ so B must be 1; $CB \Leftrightarrow AB = BD$ so D must be zero; since there was no "borrowing" from the C in the previous subtraction, $C = A + 1$. From $C \times AB = BDC$ and the first three conditions, we get $(A + 1) \times A = 10$; thus the base must be the product of consecutive numbers. Since the base is greater than two and less than ten, it must be six; thus $A = 2$ and $C = 3$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario;

KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; J.A. MCCALLUM, Medicine Hat, Alberta; JOHN GRANT MCLOUGHLIN, Okanagan University College, Kelowna, British Columbia; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; CHRIS WILDHAGEN, Rotterdam, the Netherlands; SUSAN SCHWARTZ WILDSTROM, Kensington, Maryland, USA; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

Congratulations!

The editors would like to congratulate some regular contributors for making their country's team for the 37th International Mathematical Olympiad, held in July 1996, in Mumbai, India, and on their performances at the IMO. Any student who makes a national team is a champion is his/her own right.

Name	Country	Award (if any)
Carl Bosley	United States	Gold Medal
Sabin Cautis	Canada	Bronze Medal
Adrian Chan	Canada	Bronze Medal
Toby Gee	Great Britain	Silver Medal
Ashish Kr. Singh	India	