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Mathematical Model of a Kidney Machine

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1. Introduction

One of the basic problems of applied mathematics is to model a physical system mathematically so that it sheds useful light on the mechanisms working within the system. The model may merely confirm that the basic laws assumed do in fact hold or, at a later stage of development, it may predict new results or new applications.

A typical example familiar in Newtonian mechanics is to model the earth and the sun as point particles in order to obtain the earth's orbit. The basic mechanism of the model relies on Newton's laws of motion and the assumption of the inverse square law of attraction. The equations of motion can then be written down and solved to predict the orbit of the earth with remarkable accuracy. It should however be noted that such a model of the earth is totally inadequate to describe, for instance, the earth's weather. Since the time of Newton, constructing a mathematical model to describe a system quantitatively has been progressively more used in chemistry and the technologies and assists greatly in the understanding and designing of processes or equipment in use in these areas. Recently, subjects such as biology and economics have attempted to use a similar methodology. There have been some outstanding successes, but the method proves to be much more difficult to apply since there is rarely a single mechanism that dominates the system under study. In consequence, there has been much less acceptance of the mathematical modelling approach in these subjects. Some idea of the type of work that can be done at a fairly simple level in biology may be found in Rosen (1967) and Maynard Smith (1971).

2. The dialyser

The present study concerns a bioengineering problem of designing a dialyser and an attempt at constructing a simple model of such a system.

The kidney is the organ of the body that filters out waste material, such as urea, creatinine, excess salts, etc., from the blood. Waste products in the blood pass through the porous walls of the active units in the kidney to the inside of the units. When the kidney fails to function properly waste products build up in the blood to toxic levels and the person becomes ill and eventually dies. To avoid a build-up the removal process is performed artificially through a dialyser or kidney machine. These machines have been in regular use over the past 15–20 years and a great amount of effort has gone into their design, largely to improve their efficiency and to bring down their cost. It is clearly not possible here to consider the clinical limitations and actual practice of dialysis. It is however possible to study the fundamental mechanism that operates in a dialyser by constructing a simple mathematical model.

Basically, the blood which is taken from the body and the cleaning fluid, called the dialysate, are placed in adjacent compartments separated by a membrane. The membrane is usually made out of some sort of reconstituted cellulose such as cellophane. The membrane contains minute pores which are large enough to allow the passage of the relatively small molecules of the waste products such as urea, but too small to allow the flow of blood cells, plasma protein and the like. The flow through the membrane is then determined by the differences in concentration on either side of the membrane, with a diffusion from high to low concentrations. Thus, when the dialysate is made up, normal amounts of such materials as glucose and necessary salts are included, so that equal concentrations of them are maintained on either side of the membrane and they are not removed from the blood stream. On the other hand, for those materials which must be removed from the blood, low or zero concentrations are maintained in the dialysate to ensure maximum removal rates. It is possible, therefore, by careful analysis, to make up the dialysate to suit the needs of each patient.

To improve the efficiency of the diffusion across the membrane, the blood and dialysate are made to flow in opposite directions on either side of the membrane, with the net effect of increasing the effective area for diffusion. The removal rate clearly depends on four major parameters: the flow rates of the blood and dialysate, the size of the dialyser and the permeability of the membrane. The permeability is not easy to vary except by changing to a different material or method of construction of the membrane. In fact a great deal of work has been done to improve the effective area available for diffusion in membranes, but this is not for discussion here. The size of the dialyser is not easy to alter except in the very early design stage, but the two flow rates are easily adjusted and it is on these that the present article will concentrate.

3. Setting up the mathematical model

The model is basically one of diffusion across a membrane and was studied in its present form by Kaplan *et al.* (1968). The configuration of the dialyser is illustrated in Figure 1 and, if we are to proceed, we must make various assumptions about the system. Firstly, it is assumed that all properties depend only on x, the distance along the dialyser. This implies that each property is uniform in height

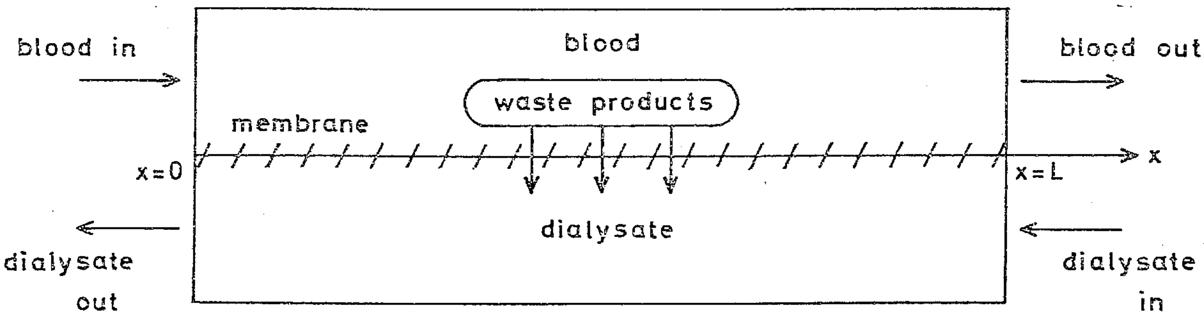

Figure 1. Schematic diagram of a dialyser.

and width, so that in the direction perpendicular to the x-direction a unit cross-section is being considered. Secondly, it is assumed that pressure effects are negligible, so that no fluid transfer across the membrane occurs, only diffusion due to concentration gradients. Thirdly, all properties are assumed to be independent of time. A patient is normally connected to a dialyser for several hours while the throughput of the machine is of the order of a minute. Thus, provided the length of time under consideration is not too long compared with the throughput time, values of the concentrations may be reasonably considered to take steady values independent of time. This assumption may be called a quasi-static assumption. Finally, a crucial quantitative assumption must be made about the permeability of the membrane. At a particular element along the dialyser, x to $x + \delta x$, the concentrations of the blood and dialysate are taken to be $u(x)$ and $v(x)$ respectively (see Figure 2).

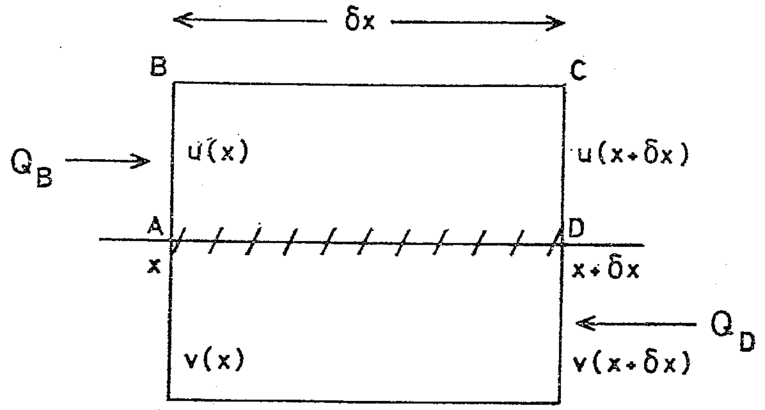

Figure 2. Element of the dialyser.

Since the material diffuses from high to low concentrations, the assumption made is that the amount of material passing through the membrane is proportional to the concentration difference—this is just a form of Fick's law.

Mass transferred through a section of membrane of unit width and length δx from the blood to the dialysate in time δt

$$= K[u(x) - v(x)]\,\delta x\,\delta t. \qquad (1)$$

The proportionality constant K is considered to be independent of x in this calculation and it gives a measure of the mass of material passing through unit area per unit time per unit concentration difference.

The equations can now be completed from the flow rates Q_B, Q_D (ml/min) of the blood and dialysate respectively.

Mass entering section AB of unit cross-section in time $\delta t = Q_B u(x)\,\delta t$.

Mass leaving section CD of unit cross-section in time $\delta t = Q_B u(x + \delta x)\,\delta t$.

The top of the dialyser BC is impermeable. Since the variables are all assumed to have reached their steady values, the net increase of mass in the element is zero and hence

$$0 = Q_B u(x)\,\delta t - Q_B u(x + \delta x)\,\delta t - K[u(x) - v(x)]\,\delta x\,\delta t.$$

Rearranging, we have

$$Q_B\left(\frac{u(x+\delta x)-u(x)}{\delta x}\right)=K[v(x)-u(x)];$$

and taking the limit as $\delta x \to 0$ we obtain

$$Q_B\frac{du}{dx}=K(v-u). \tag{2}$$

A similar calculation for the dialysate gives

$$-Q_D\frac{dv}{dx}=K(u-v). \tag{3}$$

The equations (2) and (3) provide a mathematical description of the dialyser, and a solution of these equations for u and v will give informative results.

4. Solving the equations

Dividing equation (2) by Q_B and equation (3) by Q_D and adding we get the differential equation

$$\frac{d}{dx}(u-v)=-\left(\frac{K}{Q_B}-\frac{K}{Q_D}\right)(u-v)$$

with the simple solution

$$u-v=A\exp(-\alpha x),$$

where A is an arbitrary constant and

$$\alpha=\frac{K}{Q_B}-\frac{K}{Q_D}. \tag{4}$$

Equation (2) can now be re-written

$$\frac{du}{dx}=-\frac{KA\exp(-\alpha x)}{Q_B},$$

which integrates to

$$u=C+\frac{KA}{\alpha Q_B}\exp(-\alpha x); \tag{5}$$

and similarly

$$v=C+\frac{KA}{\alpha Q_D}\exp(-\alpha x), \tag{6}$$

where C is a second arbitrary constant.

To complete the solution it is necessary to satisfy the boundary conditions. At entry ($x=0$) the blood has concentration u_0, while the dialysate on entry ($x=L$) has concentration v_1. The value of v_1 is small and will be taken to be zero in the present calculation. Thus

$$u(0)=u_0 \quad \text{and} \quad v(L)=0;$$

and hence

$$u_0=C+A(K/\alpha Q_B),$$
$$0=C+A[K\exp(-\alpha L)/\alpha Q_D].$$

Solving for C and A and substituting into (5) and (6) gives, after a little algebra,

$$u = u_0 \left(\frac{[\exp(-\alpha L)/Q_D] - [\exp(-\alpha x)/Q_B]}{[\exp(-\alpha L)/Q_D] - (1/Q_B)} \right) \tag{7}$$

and

$$v = \frac{u_0}{Q_D} \left(\frac{\exp(-\alpha L) - \exp(-\alpha x)}{[\exp(-\alpha L)/Q_D] - (1/Q_B)} \right). \tag{8}$$

5. Interpretation of the solution

The most important factor that is required by the designer of a dialyser is the amount of waste material removed in unit time. From the definition (1):

Mass leaving blood to dialysate through section of unit width and total length L in unit time

$$= \int_0^L K(u - v)\, dx$$

$$= -Q_B \int_0^L \frac{du}{dx}\, dx \quad \text{(from equation (2))}$$

$$= Q_B[u(0) - u(L)].$$

In the usual jargon of dialysis, this expression is closely related to the clearance Cl, where

$$\text{Cl} = Q_B[u(0) - u(L)]/u(0).$$

Using (7) with $x = L$ and a little straightforward algebra, we have

$$\text{Cl} = Q_B \left(\frac{1 - \exp(-\alpha L)}{1 - (Q_B/Q_D)\exp(-\alpha L)} \right). \tag{9}$$

From the definition of α in (4),

$$\alpha L = \frac{KL}{Q_B}\left(1 - \frac{Q_B}{Q_D}\right).$$

It is now clear that the key parameters in this work are Q_B/Q_D, the ratio of the flow rates, and KL/Q_B, the ratio of the flow across the membrane to the blood flow rate. In normal operating conditions Q_B varies from 100–300 ml/min, Q_D from 200–600 ml/min and KL/Q_B from about 1 to 3.

Consider now the special case of keeping K, Q_B and Q_D constant and varying L. Since $\alpha > 0$, Cl $\to Q_B$, as $L \to \infty$, while as $L \to 0$, Cl $\to 0$. The graph of Cl against L is a steadily rising curve; thus the model indicates that to get maximum clearance the dialyser should be as long as practicable. This result just confirms a fact which appears intuitively obvious. A second case for consideration is the situation when K, L and Q_D are kept constant and Q_B is varied. Some results are available for this case from an experimental small-scale model due to Frost (1974). The best fit curves from the experiments and the curves of Cl against Q_B calculated from (9) with $KL/Q_D = 0{\cdot}5$ and $0{\cdot}3$ are shown in Figure 3. There is seen to be a very good qualitative agreement between the theoretical and experimental curves.

A final special case is made by fixing K, L and Q_B and allowing Q_D to vary. For a given dialyser, Q_D is the easiest parameter to vary, and hence finding an optimum rate for Q_D is the most natural problem associated with a dialyser. A little analysis on equation (9) shows that $\mathrm{Cl} \to 0$ as $Q_D \to 0$. This result may be a little surprising,

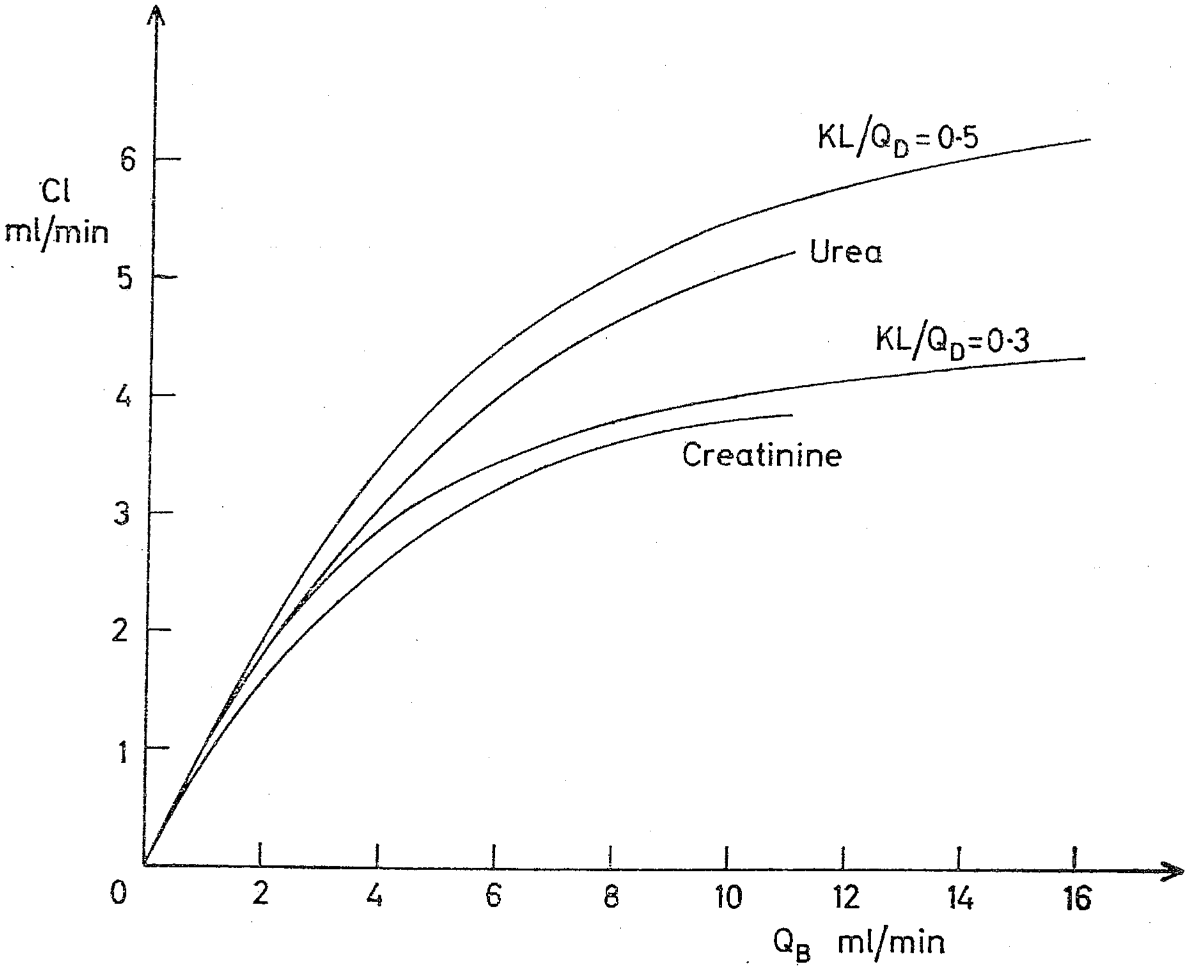

Figure 3. Clearance (ml/min) against blood flow (ml/min) at constant dialysate flow (19·5 ml/min). Results calculated from (9), labelled $KL/Q_D = 0{\cdot}3$ and $0{\cdot}5$, and experimental results taken from small-scale test cell of Frost (1974).

but it is a consequence of the quasi-static assumption. When $Q_D = 0$ the dialysate is stationary and, since steady conditions have been reached, the concentration on either side of the membrane will be the same and hence $v = u = u_0$ for $x > 0$. Thus, near this limit the quasi-static assumption is inappropriate and variations with time are essential to achieve sensible results. On the other hand, as $Q_D \to \infty$,

$$\mathrm{Cl} \to Q_B[1 - \exp(-KL/Q_B)]$$

and it can be shown that Cl increases steadily with Q_D to the above maximum value. Again the model indicates that the flow rate of the dialysate should be as high as possible. The limiting factor is usually the amount of expensive dialysate that can be handled easily (usually 150–200 litres). As speeds rise the flow becomes turbulent and many of the approximations used in the present calculation fail to be valid, so that at the highest values of Q_D the results deduced from (9) are not to be trusted. A comparison can be made with experimental results presented by Frost (1974) for Cl against Q_D. The graph of (9) fits the scatter of experimental points quite satisfactorily.

6. Conclusions

The deductions from the simple model postulated accord reasonably with what is intuitively expected and qualitatively with published experimental results. The model has isolated the key parameters of the problem and it has indicated some of the limiting features of the assumptions made. Certainly there is sufficient evidence that the model is worth pursuing and this is about as much as could be expected from such a simple one. A more realistic model, with the consequent more difficult mathematics, might be expected to predict with some accuracy optimum parameters for the dialyser within the clinical limitations of the apparatus. Such a model would need to include such factors as variations of K with x, the depths of the blood and dialysate channels and pressure differences across the membrane which is required for essential water removal and which leads to x-dependent flow rates.

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Éamon de Valéra—A Mathematical Portrait

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Readers of this article may be surprised to find associated with mathematics the name of the late President of Eire, who died on 29 August 1975, at the age of 92. However, there is such an association and it endured throughout most of his life. I had an indication of this on my first visit to Ireland, in 1945, when I met Dr A. J. McConnell, who was later to become Provost of Trinity College, Dublin. He introduced me to the Dublin Institute for Advanced Studies, where I met the first Director, Erwin Schrödinger, whose name is so intimately associated with the wave-mechanical treatment of quantum theory. I asked how it came about that such a small country as Ireland could give support, at the front-line level, to research into a fundamental part of mathematical physics, which was a source of great activity in most of the leading nations. Dr McConnell explained that Mr de Valéra, the Prime Minister of Ireland at the time, was very interested in mathematics and was so well-disposed towards the subject that he had persuaded the Government in Dublin to set up the Institute.

Many years passed and then, in 1973, during a week's residence in Pembroke College, Cambridge, where I participated in a symposium to celebrate the 70th birthday of the late Professor W. V. D. Hodge, my research supervisor at Cambridge, I heard a surprising statement. It was to the effect that E. T. Whittaker (of 'Whittaker and Watson' fame) had once remarked many years ago that de Valéra was at one time an applicant for the Chair of Mathematics at an Irish University College, but that he had been unsuccessful. 'How different', the remark went, 'might have been the history of Ireland if de Valéra had become a Professor of Mathematics'.

With these experiences before me, I set out to try to find the background to de Valéra's interest and activity in mathematics, and the following is an account of what I have been able to find.

Éamon de Valéra was born in America in 1882 but was sent to Ireland as an infant on the death of his father. At the age of 14 he went to the Christian Brothers School in Charleville, Co. Limerick, despite an unfortunate interview with the Head Brother, during which he was unable to give the factors of $a^3 - b^3$. In 1898 he entered Blackrock College, Dublin, and quickly became fascinated by mathematics, which was very well taught at the school. He went to the University College at Blackrock in 1900 and, according to a timetable he wrote on the back of a sports programme, he studied Euclid from 6.00 to 7.30 a.m., Algebra from 8.00 to 10.00 a.m., Euclid from 10.30 to 11.30 a.m., Conics from 12 noon to 2.30 p.m., Natural Philosophy from 5.00 to 7.00 p.m. and Trigonometry and Theory of Equations from 7.00 to 10.30 p.m. He now realised that mathematics was his strongest subject.

At the age of 21 (in 1903), he took up his first full-time position as a teacher at Rockwell College near Cashel, but at the same time he continued his University studies. He graduated in 1904 and in 1905 returned to Dublin, where he continued as a teacher. However, he also continued to study mathematics and commenced postgraduate study under Professor A. W. Conway. De Valéra was introduced to the subject of quaternions by Conway and from then onwards he had a special interest in the work of the Irish mathematician, Hamilton.

At this point, it may be of interest to note that in 1906 E. T. Whittaker was appointed Professor of Astronomy in the University of Dublin (that is, Trinity College) with the title of Royal Astronomer of Ireland. The observatory at Dunsink (5 miles from Dublin) had very poor equipment, as there was no fund for the purchase of instruments. As a result, it became tacitly understood that the professor would strengthen the University School of Mathematical Physics, and Whittaker gave courses of advanced lectures in the field (particularly in spectroscopy, astrophysics and electro-optics). De Valéra was one of his pupils from 1906 to 1908. He and Whittaker became firm friends and often discussed mathematics until, in 1912, Whittaker left Ireland on appointment as Professor of Mathematics in the University of Edinburgh.

During the period of his postgraduate study, de Valéra gave lectures in mathematics and mathematical physics for the Royal University at various Colleges, but this work ceased on the establishment of the National University in 1909. He continued as a teacher and in April, 1912, was an applicant for the Chair of Mathematics at University College, Galway. However, he withdrew and a year later was a candidate for the Chair of Mathematical Physics at University College, Cork. He was unsuccessful on this occasion but in October, 1912, became a lecturer at St Patrick's College, Maynooth.

De Valéra was, as is well known, on various occasions confined to prison. Even here he continued to read mathematics. In one prison, for instance, he was allowed to study a book by Poincaré; and in another prison he had all his mathematical notebooks sent to him, together with books by Goursat, Bromwich and Darboux. He had an abiding interest in the subject, which always remained a passion with him and he turned to it at every opportunity. He continued to subscribe to a number of mathematical periodicals and the numerous notes in the margins show that he read them. He corresponded with eminent mathematicians and, even after the almost complete loss of his eyesight in 1952, he continued to take pleasure in having mathematical books and journals read to him. Mathematics was always his relaxation. He was widely read and had a keen interest in postgraduate research. This motivated him to establish the Institute for Advanced Studies, especially the Schools of Theoretical Physics and Cosmic Ray Physics.

Heronian Triangles

K. R. S. SASTRY

1. The Heronian problem

Let a, b, c be the lengths of the sides of a triangle and put $s = \frac{1}{2}(a + b + c)$. Then Heron's[1] formula for the area Δ of the triangle is

$$\Delta = \sqrt{(s(s-a)(s-b)(s-c))}.$$

When a, b, c are positive integers, Δ is sometimes also a positive integer. For instance the triangle whose sides are 11, 13 and 20 units has an area of 66 square units. A triangle is called *Heronian* if its sides a, b, c and area Δ are all positive

[1]Heron (or Hero) of Alexandria lived in the first century A.D. The famous formula which bears his name appeared in one of his books, but was probably known to Archimedes, 300 years earlier.

integers. The *Heronian problem* is to find a set of expressions for the sides which will yield all Heronian triangles. As far as I know, a complete solution of the Heronian problem does not exist as yet.

2. Right-angled Heronian triangles

Let x, y be distinct positive real numbers and assume that $x > y$. A triangle with sides $x^2 - y^2$, $2xy$, $x^2 + y^2$ is clearly right-angled; its hypotenuse is the side of length $x^2 + y^2$. Conversely, suppose that a, b, c are the sides of a right-angled triangle and that c is the hypotenuse. There are, then, real numbers x, y such that

$$a = x^2 - y^2, \quad b = 2xy, \quad c = x^2 + y^2;$$

for they are given by the equations

$$x^2 = \tfrac{1}{2}(a + c), \quad y^2 = \tfrac{1}{2}(c - a).$$

Thus all right-angled triangles are generated by the 'Pythagorean triples'

$$x^2 - y^2, \quad 2xy, \quad x^2 + y^2. \tag{1}$$

The area of a triangle given by (1) is $xy(x^2 - y^2)$. So if x, y are integers, such a triangle is Heronian. On the other hand, not all right-angled Heronian triangles can be obtained from (1) by assigning suitable integral values to x and y; for instance

$$9, \quad 12, \quad 15$$

is such a triangle, since it is easily seen (by experiment) that 15 is not the sum of two perfect squares. However we shall prove that *all* right-angled Heronian triangles are of the form

$$(m^2 - n^2)q, \quad 2mnq, \quad (m^2 + n^2)q, \tag{2}$$

where m, n, q are positive integers.

First we show that any right-angled triangle with integral sides is automatically Heronian. Let a, b, c be integers and suppose that $c^2 = a^2 + b^2$. Then at least one of a, b is even. For if a and b are both odd, so that

$$a = 2h + 1, \quad b = 2k + 1,$$

where h, k are integers, then

$$c^2 = (2h + 1)^2 + (2k + 1)^2 = 4(h^2 + k^2 + h + k) + 2,$$

and so c^2 is even, but not divisible by 4; and this is impossible since c is an integer. Hence the area $\tfrac{1}{2}ab$ is an integer, i.e., the triangle is Heronian.

Suppose now that the triangle with hypotenuse c and neighbouring sides a, b is Heronian. We may take b to be even. The identity $c^2 = a^2 + b^2$ then shows that a and c are either both even or both odd so that, in either case, $a + c$ and $c - a$ are even. Thus

$$x^2 = \tfrac{1}{2}(a + c) \quad \text{and} \quad y^2 = \tfrac{1}{2}(c - a)$$

are both integers; and, since $b = 2xy$ is even, xy is also an integer.

As x^2 is an integer, x is of the form $m\sqrt{q}$, where m, q are integers and q is the product of distinct prime factors; and as xy is an integer, y must therefore be of the form $n\sqrt{q}$, where n is an integer. Hence we have

$$a = (m^2 - n^2)q, \quad b = 2mnq, \quad c = (m^2 + n^2)q.$$

It is clear that any triangle with sides given by (2), where m, n, q are integers, is Heronian. Hence (2) generates all right-angled Heronian triangles when m, n, q run through all positive integral values (with $m > n$).

Of course any given right-angled Heronian triangle may have several representations in the form (2). Thus the triangle with sides 36, 48, 60 is given by $m = 2$, $n = 1$, $q = 12$ or by $m = 4$, $n = 2$, $q = 3$.

3. Other Heronian triangles

We have seen that the Heronian problem has a solution when attention is restricted to right-angled triangles. In the general case I know of only one set of expressions which yields infinitely many Heronian triangles. This is derived below.

Consider, in the first place, two triangles with sides a, b, c and ka, kb, kc respectively. Then Heron's formula shows that, if the first triangle has area Δ, the second will have area $k^2\Delta$. Hence, if a, b, c, Δ are rational numbers, but not all integers, then, for a suitable k, the triangle with sides ka, kb, kc is Heronian.

Now let t, u, v, w be positive integers such that

$$t^2 = u^2 + v^2 + w^2. \tag{3}$$

When t is even, t^2 is of the form $4m$ (where m is an integer) and when t is odd t^2 is of the form $4m + 1$. An argument of the kind used in Section 2 then shows that at least two of u, v, w are even.

The numbers

$$a = \tfrac{1}{2}(t^2 - u^2), \quad b = \tfrac{1}{2}(t^2 - v^2), \quad c = \tfrac{1}{2}(t^2 - w^2)$$

are integers or half integers, and

$$a + b + c = t^2, \quad b + c - a = u^2, \quad c + a - b = v^2, \quad a + b - c = w^2.$$

Since $b + c - a$, $c + a - b$, $a + b - c$ are all positive, it follows that a, b and c are the sides of a triangle. (For if B and C are points a distance a apart, the circles with centres B, C and radii c, b, respectively, intersect at points not lying on the straight line through B and C.) The area

$$\Delta = \tfrac{1}{4}\sqrt{((a + b + c)(b + c - a)(c + a - b)(a + b - c))} = \tfrac{1}{4}tuvw$$

of the triangle is an integer since certainly two of u, v, w are even. Therefore integral solutions of (3) lead to Heronian triangles.

One set of solutions of (3) is obtained from Lagrange's identity

$$(p^2 + q^2 + r^2 + s^2)^2 = (p^2 + q^2 - r^2 - s^2)^2 + (2pr + 2qs)^2 + (2ps - 2qr)^2. \tag{4}$$

If p, q, r and s are integers such that none of the terms in (4) is 0, and if

$$\left.\begin{aligned} a &= \tfrac{1}{2}(t^2 - u^2) = 2(p^2 + q^2)(r^2 + s^2), \\ b &= \tfrac{1}{2}(t^2 - v^2) = \tfrac{1}{2}[(p + r)^2 + (q + s)^2][(p - r)^2 + (q - s)^2], \\ c &= \tfrac{1}{2}(t^2 - w^2) = \tfrac{1}{2}[(p + s)^2 + (q - r)^2][(p - s)^2 + (q + r)^2], \end{aligned}\right\} \quad (5)$$

then the triangle with sides $2a$, $2b$, $2c$ is certainly Heronian and the triangle with sides a, b, c may be so. More generally, the triangle with sides ka, kb, kc is Heronian when k is an even integer, and may also be Heronian for odd k. It seems unlikely that all Heronian triangles are obtainable from (5) in this way. However I have been unable to demonstrate this by exhibiting a Heronian triangle whose sides are not multiples of the numbers a, b, c generated by (5).

A Statistical Problem in Criminology

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1. The prediction of behaviour

Can statistical techniques be used to predict human behaviour? Some people may be dismayed at the thought, and think that in some way this reduces people to automata and undermines their free will. Yet we often behave on the implicit assumption that 'so-and-so will *probably* do such-and-such', and might be willing to specify the odds in favour of this event. If the odds are high, say 9:1, we might say there is a probability of nine-tenths of his doing so; alternatively we might say, more specifically, 'I predict so-and-so will do such-and-such, but I realise there is a small probability (one-tenth) that I will be mistaken'. There are not many situations where we can be certain how a person will behave, and make a definite prediction. However, in some circumstances we may be able to estimate the probability that a person will behave in a certain way. The calculation of the estimate may be complicated and its interpretation and use may even be controversial. The statistician's job is to provide an acceptable method of calculation and to clarify the interpretation. In this article I am going to describe how the problem arose in a piece of criminological research.

2. The failure of borstal boys

When boys are released from borstal it is usually the case that just over half are convicted of another offence within eighteen months of release. If these are described as 'failures' and the rest as 'successes' it is found that the failures tend to differ in several ways from the successes. For example, they tend to have had a first conviction at an earlier age, to have more previous convictions and to have been younger when they were actually sentenced to borstal. They were more likely to have committed their latest offence alone rather than in the company of others. The failures are more likely to have spent some time in an institution of some sort (e.g. a children's home or approved school) than the successes. In the area of their work the successes have, on the average, held down a job longer, are more likely to have had a definite trade and so on.

If these statements are turned round, we can see that, for example, if a boy has lived in an institution he is more likely to be a failure than if he has not. This is also the case if he committed his offence alone.

The following tables show the proportion of failures for a large group of boys, classified according to different factors (see reference 1).

TABLE 1. Percent failure by age

Age	No. of boys	% failure
16	19	74
17	126	67
18	209	52
19	194	52
20	103	50
All	651	55

TABLE 2. Percent failure by number of associates

No. of associates	No. of boys	% failure
0	250	64
1	187	56
2	123	46
3	53	43
4	24	38
5+	14	29
All	651	55

TABLE 3. Percent failure by confinement in an institution

	No. of boys	% failure
Been in institution	376	61
Not been in institution	275	47
All	651	55

TABLE 4. Percent failure by trade

	No. of boys	% failure
Had trade	121	40
Had no trade	530	58
All	651	55

Table 1 shows that the failure rate for boys aged 16 at sentence was 0·74, and this decreased to 0·50 for boys aged 20. The most dramatic factor is the number of associates (Table 2); those who committed their offence alone had a failure rate of 0·64, and this decreased to 0·29 for those with five or more associates. (Note, however, that the number in the latter group was only 14, so the estimate is not very accurate.)

3. Estimation of probabilities

Now, it was found that the effect of these factors was fairly constant over a period of several years, and the overall failure rate was also fairly constant. It is reasonable, therefore, to use these failure rates as the basis for estimating the probability that a given boy, not in the sample, will be a failure. If the boy was aged 16, we could estimate his chance of failure as 0·74, but suppose also that he had had no associates, had been in an institution and had no definite trade: how can we estimate his chance of failure? The total number of boys with any particular combination of factors is likely to be very small (there are $5 \times 6 \times 2 \times 2 = 120$ possibilities, and only 651 boys) and cannot give an accurate estimate. How can the results from all the factors be combined to give a satisfactory estimate of the probability?

One possible solution is to divide all the factors into two groups, and score *one* if the rate is higher than average, and *zero* if lower. The score for a boy is then the total number of adverse factors affecting him. The following hypothetical table shows what may result.

TABLE 5. Percent failure by number of adverse factors

Score	Number	% failure
0	60	30
1	130	49
2	351	58
3	90	65
4	20	70
	651	55

This method gives valid estimates for a boy falling in any group. However, it may not be the best method. For one thing it does not use all the information about the age or the number of associates, for another it gives equal weight to the different factors, although some of these may be more important than others. Thirdly, it ignores possible inter-relationships between the factors.

If we look at the problem in a general way, we can see that what we would like is to estimate p, the probability of failure, from the values of the factors, which can here be called x_1, x_2, x_3, x_4; the first two x's take the values in Tables 1 and 2, and the latter two take values 0 or 1 as before. We want to find a function of (x_1, x_2, x_3, x_4) such that $p = f(x_1, x_2, x_3, x_4)$ estimates the probability of failure; we require $0 \leqslant p \leqslant 1$. The simplest solution is a linear function, for example:

$$p = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4.$$

The problem then, is to estimate the β's from the observations, which will, in turn, enable us to estimate p. This may be satisfactory over part of the range of p, but is in danger of giving values of p that are negative or exceed 1. A more satisfactory model would be to take:

$$p = \exp\left(\sum_{i=0}^{4} \gamma_i x_i\right) \Big/ \left[1 + \exp\left(\sum_{i=0}^{4} \gamma_i x_i\right)\right];$$

that is,

$$\log [p/(1 - p)] = \sum_{i=0}^{4} \gamma_i x_i, \qquad (x_0 = 1)$$

where we now need to estimate the γ's. If p is near to a half, a good approximation is obtained from $\log [p/(1 - p)] \simeq 4(p - \frac{1}{2})$, and the β's and γ's bear a simple relationship to each other. In our particular study we found the linear approximation was fairly accurate if $0{\cdot}2 < p < 0{\cdot}8$.

Methods of estimating the γ coefficients from the data are available (see reference 2) and we can examine how well our model fits the observations. By including extra terms to measure interactions we can test whether these give us a better 'fit' for the model, and terms that do not increase the 'fit' can be excluded. A model obtained in this way combines all the information into one equation, so that the estimate of the probability of failure will be more accurate than when calculated individually from small groups.

4. Some practical applications

There are several reasons for estimating the probability of failure, which can be regarded as a measure of risk for each boy. In this study of borstal boys (reference 1), we were concerned to measure the effect on the failure rate of changing the way the institution was run. However, an increase in the failure rate might have been due to a worse type of boy being sent to the borstal; it would only be meaningful to compare groups of boys who were 'equal risks' to start with—that is, boys with approximately the same p values.

The following table shows them grouped according to their p values, which were estimated using two extra variables in addition to those listed in Tables 1–4.

TABLE 6. Estimated probability of failure related to failure rate for two regimes

	Old regime		New regime*	
Estimated probability of failure	No. of boys	Observed % failure	No. of boys	Observed % failure
<0·25	40	23	30	20
0·25–0·45	139	34	42	36
0·45–0·55	125	51	45	49
0·55–0·75	266	64	85	60
>0·75	81	84	102	80
All	651	55	304	58

* The original figures have been altered slightly to illustrate the statistical point more clearly.

Comparing each line of the table, we can see no consistent change, and the differences were not in fact significant; so it was concluded that, despite the hopes of the staff, the new regime was in fact no more successful than the old. It could very well have happened, however, that even with the overall increase the decrease in the failure rate for some groups in the table, or for all groups combined, was significant.

It should be noted that estimates of the probability of failure are based *only* on the factors in the equation. If other information is available, this may alter the probability and should be incorporated into the estimate. It is also important to realise that the estimate can only be used for boys from other borstals, and at other times, if the same relationships have been shown to hold.

These so-called prediction techniques have been used for many years in the U.S. in relation to the selection of prisoners released on parole; this may affect the time of release and the amount of supervision needed. Many other factors besides those mentioned here are included, several of them relating to the home background. The method is also now used in this country to help assess whether prisoners should be given parole.

The mathematical problems relating to this type of statistical analysis have been clarified only in the last two years; other methods have been put forward and are discussed in reference 3.

References

1. A. E. Bottoms and F. H. McClintock, *Criminals Coming of Age* (Heinemann, London, 1973).
2. D. R. Cox, Some procedures connected with the logistic qualitative response curve. In *Research Papers in Statistics*, ed. F. N. David (Wiley, New York, 1966).
3. M. A. Walker, The development of a new instrument of prediction in penal treatment. Institute of Criminology, Cambridge (mimeo) (1973).

Letters to the Editor

Dear Editor,

Formal and informal proofs

I distinguish three different sorts of proof.

1. *Formal proof.* This is the kind of proof employed in formal logic. There is a finite set of rules which lays down conditions under which one statement leads to another; and each step in a proof must be justified by one of these rules or else by appeal to a previously proved result. To be effective this method of proof requires a language and symbolism of its own, and even then it is long and cumbersome. Therefore, although it is unassailable, it is not used in 'ordinary' mathematics.

2. *Mathematician's proof.* This type of proof is used all the time by mathematicians. It is written in ordinary language instead of the formal one used in formal proofs. The purpose of this kind of proof is to indicate to the mathematician reading it how he could (if he really wanted to) construct a formal proof. In practice one studies a mathematician's proof until one is convinced that one *could* construct a formal proof. This conviction is psychological and, when it has been attained, it might be said that the proof has been 'understood'.

A (supposed) mathematician's proof may fail in its purpose. There are two possible reasons for this: either the indications given do not lead to a formal proof, or the person reading them has not been able to follow those indications. In the first case we say that the proof is incorrect, and in the second case that it is difficult.

It is clear that a mathematician's proof aimed at beginners should contain fuller indications than one intended for a mature mathematician who can easily fill in the details for himself. A mathematician's proof should therefore contain sufficient details to be readily followed by a mathematician with the mathematical maturity of one who might reasonably be expected to read the text, and yet it should not contain so many details as to become tedious.

3. *Teacher's proof.* A perusal of texts used at schools and some of the texts used at universities shows that the 'proofs' used are not always mathematician's proofs as described above. The 'proofs' used in such texts are tissues of plausible arguments whose purpose is to convince the reader of the truth of what is advanced (rather than of the possibility of constructing a formal proof). These 'proofs' often fail in their purpose. The schoolboy says that he has 'understood' the 'proof' if he is convinced. Many schoolboys are credulous. So the schoolboy and the mathematician do not always mean the same thing when they say that they have understood a text; and it sometimes happens that the schoolboy 'understands' a text that the mathematician knows he has not understood.

The mathematics teacher has to advance specious arguments—it's what the examiner wants. 95% of the students are only too eager to take the path of least resistance and accept lock, stock and barrel anything their teacher says. Their object is to satisfy the examiner. But is it right to accustom students to accept specious arguments? Is it not dreadful intellectual arrogance to present to students arguments that one doesn't believe to be correct?

This letter will have served its purpose if it gives rise to a storm of protest. Storms clear the air and that cannot be a bad thing.

Some mathematicians may not agree with my notion of a mathematician's proof. We are all entitled to our own opinions. Also some teachers may not agree with what I have said about their proofs. Can we hope for some interesting comments from them?

Yours sincerely,

JOHN STRANGE

(25 Offington Avenue, Broadwater, Sussex)

* The writer of the above letter is a practising school teacher. The Editor would be glad to receive correspondence on the important matter that he has raised. Views expressed in this or any subsequent letter on the subject do not, of course, necessarily express editorial opinion.

Dear Editor,

Derived sequences

Whilst playing with an electronic calculator (instead of revising for exams!) I came across the following interesting sequences:

$a_n = n^n$	$b_n = a_{n+1}/a_n$	$c_n = b_{n+1} - b_n$
1		
	4	
4		2·75
	6·75	
27		2·731 481
	9·481 481	
256		2·725 550
	12·207 031	
3125		2·722 889
	14·929 920	
46 656		2·721 465
	17·651 385	
823 543		2·720 613
	20·371 998	
16 777 216		2·720 063
	23·092 061	
387 420 489		2·719 688
	25·811 749	
10 000 000 000		

The terms in the third sequence appear to tend to e. Can you prove it? The proof is, I believe, sufficiently testing to warrant inclusion in *Mathematical Spectrum*. It is to be found in the Appendix.

I feel that the problem is much better motivated when set out in this way, rather than in the following boring (and typically mathematical) fashion: 'Prove that

$$\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \to e \quad \text{as} \quad n \to \infty.\text{'} \qquad (*)$$

Moreover, discovering the problem in this way led me to investigate what would happen if the order of operations (*viz.* taking ratios of successive terms, and then differences between successive ratios) was reversed. This did not lead to anything very interesting. But, on taking differences once more, I obtained the next table.

$a_n = n^n$	$b'_n = a_{n+1} - a_n$	$c'_n = b'_{n+1}/b'_n$	$d'_n = c'_{n+1} - c'_n$
1			
	3		
4		7·666 667	
	23		2·288 855
27		9·956 522	
	229		2·571 862
256		12·528 384	
	2 869		2·644 499
3 125		15·172 883	
	43 531		2·673 870
46 656		17·846 753	
	776 887		2·688 631
823 543		20·535 384	
	15 953 673		2·697 089
16 777 216		23·232 473	
	370 643 273		2·702 382
387 420 489		25·934 855	
	9 612 579 511		
10 000 000 000			

Remarkably, the sequence d'_n also appears to tend to e. Here the general term is

$$\frac{(n+2)^{n+2}-(n+1)^{n+1}}{(n+1)^{n+1}-n^n} - \frac{(n+1)^{n+1}-n^n}{n^n-(n-1)^{n-1}}$$

and the rigorous proof that this has the limit e must be pretty messy.

However, I was prompted to investigate the problem further and to look at other sequences (a_n). It is easy to prove the following results.

(i) (a_n) is an arithmetic progression: $c_n \to 0$, $d'_n = 0$ for all n.
(ii) (a_n) is a geometric progression: $c_n = d'_n = 0$ for all n.
(iii) (a_n) is the Fibonacci sequence (i.e. $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$ for $n \geqslant 1$): $c_n = d'_{n+1} \to 0$.
(iv) $a_n = n^k$, where k is constant: $c_n, d'_n \to 0$.
(v) $a_n = n!$: $c_n = 1$ for all n, $d'_n \to 1$.

Given the sequence (a_n), the sequences (c_n) and (d'_n) will be called the α-derived sequence and the β-derived sequence respectively. In each of the examples (i)–(v) both derived sequences converge and the two limits are equal. The first property does not always hold, for the sequence

$$(1, 1, 2, 8, 56, 616, 9\,856, 216\,832, 6\,288\,128, 232\,660\,736, \ldots)$$

is such that $c_n = n$. In this case, however, d'_n also tends to ∞.

Note that the derived sequence limits give a rough measure of the rate of increase of a function. Arithmetic and geometric progressions increase fairly slowly and have derived sequence limits equal to 0; $n!$ increases more rapidly and has derived sequence limit 1, while n^n increases more rapidly still and has derived sequence limit e. It is clear that the sequence constructed above so that $c_n = n$ is about to 'take off' and will soon have terms larger than n^n.

The cases $a_n = (-1)^n$ or $a_n = (-2)^n$ are covered by example (ii) above, so that $c_n = d'_n = 0$ for all n; while if $a_n = (-n)^n$, then $c_n \to -e$ and the figures I have obtained suggest that $d'_n \to -e$ also. This leads to the following conjecture: the derived sequences of (a_n) have limit l if and only if the derived sequences of $(|a_n|)$ have limit $|l|$.

We conclude with two unsolved problems. (1) Do the sequences (c_n) and (d'_n) always both converge or diverge, and if they converge, are the two limits necessarily the same? (2) Show rigorously that, if $a_n = n^n$, then $d'_n \to e$ as $n \to \infty$; and that, if $a_n = (-n)^n$, then $d'_n \to -e$.

Appendix. For the proof of (*) we have to take three results for granted. First, we have the two limiting relations

$$\text{(a)} \left(1+\frac{1}{x}\right)^x \to e \qquad \text{and} \qquad \text{(b)}\ x\log\left(1+\frac{1}{x}\right) \to 1$$

as $x \to \infty$. These are, in fact, equivalent since

$$\log\left(1+\frac{1}{x}\right)^x = x\log\left(1+\frac{1}{x}\right) \qquad \text{and} \qquad \log e = 1.$$

We could prove (b) by expanding $\log(1 + 1/x)$ in powers of $1/x$, which is permissible when $|x| > 1$.

Secondly, we need the Mean Value Theorem which says that, if the function f is differentiable in the interval $a \leqslant x \leqslant b$, then there exists a number p such that $a < p < b$ and

$$f(b) - f(a) = (b - a)f'(p).$$

The geometrical interpretation is simply that the graph of $y = f(x)$ has a point $(p, f(p))$ at which the tangent is parallel to the chord joining the points $(a, f(a))$ and $(b, f(b))$. A sketch makes the theorem extremely plausible.

To prove (*) we first put

$$f(x) = \frac{x^x}{(x-1)^{x-1}} \quad (x > 1),$$

so that

$$\log f(x) = x\log x - (x-1)\log(x-1)$$

and

$$f'(x) = \frac{x^x}{(x-1)^{x-1}}\log\frac{x}{x-1}.$$

The Mean Value Theorem ensures that, given n, there exists a number p_n between n and $n+1$ such that

$$f(n+1) - f(n) = f'(p_n).$$

Thus, if $q_n = p_n - 1$,

$$\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} = \frac{p_n{}^{p_n}}{(p_n-1)^{p_n-1}}\log\frac{p_n}{p_n-1}$$
$$= \left(1+\frac{1}{q_n}\right)^{q_n}\left(1+\frac{1}{q_n}\right)\left[q_n\log\left(1+\frac{1}{q_n}\right)\right].$$

Since $q_n > n + 1$, $q_n \to \infty$ as $n \to \infty$. Hence the first term on the right tends to e, while the second and third terms tend to 1. This proves (*).

Yours sincerely,

M. I. Wenble

(148 Broomspring Lane, Sheffield 10)

Dear Editor,

May I add a postscript to my article (*Mathematical Spectrum*, Vol. 6, No. 1, pp. 2–7) and to that by T. J. Fletcher (*Mathematical Spectrum*, Vol. 7, No. 2, pp. 53–59) on the analysis of a simple 'Blow's game'. The connection between the two analyses is expanded, and another, more general method of calculating the absorption probabilities is demonstrated, giving a fuller analysis of the game to which 'minimax' games theory can then be applied.

The link between the game, which is a discrete process, and the continuous water-flow analogy discussed by Fletcher can be illustrated by a machine which plays the game with sand. There are seven boxes for the sand, representing the capital of one player, and a mechanism which distributes the contents of each box for each go according to the transition probabilities. Before each go, a pound of sand is fed into the starting box (in my example, that representing state 3). Let $\boldsymbol{r}_i$ be the vector of the contents of each box just before the ith go; we see that

$$\boldsymbol{r}_0 = (0, 0, 0, 1, 0, 0, 0), \quad \boldsymbol{r}_1 = \boldsymbol{r}_0 T + \boldsymbol{r}_0,$$
$$\boldsymbol{r}_2 = \boldsymbol{r}_0(T^2 + T + I), \ldots, \boldsymbol{r}_n = \boldsymbol{r}(T^n + \cdots + I),$$

where T is the matrix of transition probabilities.

We cannot then assume that in the steady state

$$\boldsymbol{r} = \boldsymbol{r}_0(I + T + T^2 + \cdots) = \boldsymbol{r}_0(I - T)^{-1},$$

for $I - T$ is singular and some components of $\boldsymbol{r}$ are unbounded (as sand is being continually fed into the machine). The final state equation is

$$\boldsymbol{m} = \lim_{N\to\infty} \boldsymbol{r}_{N+1} - \boldsymbol{r}_N = \lim_{N\to\infty} \boldsymbol{r}_0 T^{N+1},$$

where $\boldsymbol{m}$ is the steady-state probability vector; this is the formula used in my original article.

To obtain Fletcher's formula we observe that

$$T = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ x_1 & & & & y_1 \\ \ldots & & Q & & \ldots \\ x_5 & & & & y_5 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}, \quad T^n = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ x'_1 & & & & y'_1 \\ \ldots & & Q^n & & \ldots \\ x'_5 & & & & y'_5 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

where Q is the reduced matrix. We obtain

$$\boldsymbol{r}_n = (r_0{}^{(n)}, \boldsymbol{x}, r_0{}^{(n)})$$

where

$$\boldsymbol{x} = (p_1(n), p_2(n), \ldots, p_5(n))$$
$$= (p_1(0), p_2(0), \ldots, p_5(0))(I + Q + Q^2 + \cdots + Q^n).$$

Thus, as $n \to \infty$,

$$\boldsymbol{x}(p_1(0), p_2(0), \ldots, p_5(0))(I - Q)^{-1}.$$

To obtain the final probabilities, we can imagine emptying the end boxes (of states 0 and 6) and having a further go. Then

$$\boldsymbol{r} = (0, \boldsymbol{x}, 0)T$$

where r_0 and r_6 will give the probabilities of losing and winning respectively.

I should now like to demonstrate a method for calculating the probabilities of winning from any starting position. We can view the winning and losing states as *absorbing*; once these are entered, it is impossible to move to another state, as the game is finished. We now introduce the *absorption probabilities*

$$\alpha_{iC} = \Pr\{\text{starting in state } i \text{ and finishing in state } C\},$$

where C is an absorbing state. In our example, this is either the winning state or the losing state. Clearly, if we start in state C, then we cannot move from it, and so we must finish in state C, so that

$$\alpha_{CC} = 1. \tag{1}$$

Because they are probabilities, we note that

$$0 \leqslant \alpha_{iC} \leqslant 1. \tag{2}$$

Consider the first step; p_{ij} is the probability of jumping from the initial state i to some other state j. Suppose that after the first step we are in any one of the states j, then

$$\begin{aligned}\alpha_{iC} &= \Pr\{\text{starting in state } i \text{ and finishing in state } C\}\\ &= \sum_j \Pr\{\text{starting in state } i\text{, moving from } i \text{ to } j \text{ in the first step, and finishing in state } C\}\\ &= \sum_j \Pr\{\text{starting in state } i\text{, moving from } i \text{ to } j \text{ in the first step}\}.\\ &\qquad \Pr\{\text{starting in state } j \text{ and finishing in state } C\},\end{aligned}$$

since each step is an *independent* event. Thus

$$\alpha_{iC} = \sum_j p_{ij}\alpha_{jC}.$$

If we arrange the α_{iC} as a column vector α_C, then this can be written as

$$\alpha_C = P\alpha_C.$$

Solving this equation with conditions (1) and (2) gives us the absorption probabilities. When we apply this to the game, as we are seeking the probabilities of A's winning, we put $C = 6$, $\alpha_{66} = 1$, $\alpha_{06} = 0$, $P = T_1T_3$, T_1T_4, T_2T_3, T_2T_4 in turn.

We then solve $(P - I)\alpha_6 = 0$; although $(P - I)$ itself is singular, specifying values of α_{06} and α_{66} gives us 5 equations in 5 unknowns. The probabilities of A's winning are

Initial capital	T_1T_3 A I (2,0) B I (2,0)	T_1T_4 A I (2,0) B II (1,1)
5	0·8	0·826
4	0·727	0·710
3	0·533	0·517
2	0·364	0·349
1	0·273	0·173
	T_2T_3 A II (1,1) B I (2,0)	T_2T_4 A II (1,1) B II (1,1)
5	0·913	0·909
4	0·739	0·727
3	0·567	0·545
2	0·387	0·364
1	0·232	0·182

The results have been set out to assist in the strategy analysis; the connecting line links the results given in my original article.

For an initial capital of 2, 3 or 4 units, the optimum strategy for both players is strategy II (for the reasons given in the original article), and it is a stable solution because there is no advantage in letting the opponent declare his choice first. The analysis is more subtle for an initial capital of 1 or 5 units. As an example we will consider the former case:

		B	
		I	II
A	I	0·27	0·17
	II	0·23	0·18

If B picks strategy I (II), A should pick strategy I (II), so A may gain an advantage by letting B declare first. However, if A picks strategy I (II) B should pick strategy II (II). Thus if both players know the winning probabilities, B will pick strategy II and A will therefore also choose this strategy.

The general 'minimax' method of games analysis will give the same answer. The players examine the possible consequences of each choice of strategy. For example, A notices that if he adopts a strategy I then his minimum probability of winning is 0·17 (if B picks strategy II). They both do this for their possible choices of strategy

A	I	0·17	B	I	0·77
	II	0·18		II	0·83

and then choose the strategy which gives the maximum 'minimum probability of winning':

A	II	0·18	B	II	0·83

We see that even if the players are allowed to change their strategies as the game progresses, they will in fact maintain their initial choice of strategy II.

The method of calculating absorption probabilities is explained in more detail in *An Introduction to Probability Theory and its Applications* Vol. 1 (3rd Edn) by William Feller (John Wiley, New York, 1968).

Yours sincerely,
D. J. Blow
(Corpus Christi College, Cambridge)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

8.7. (Submitted by B. G. Eke, University of Sheffield.) If m, n are odd integers, show that $m^2 - n^2$ is divisible by 8.

8.8. Is it possible for three consecutive binomial coefficients to be (a) in arithmetic progression, (b) in geometric progression?

8.9. The polynomial f has complex coefficients, and all its roots have positive real parts. Show that all the roots of the derivative of f have positive real parts.

Solutions to Problems in Volume 8, Number 1

8.1. Let n be a positive integer. Show that (a) if $2^n - 1$ is prime, then n is prime, (b) if $2^n + 1$ is prime, then n must be a power of 2. Is the converse of (a) true?

Solution by Martin Spencer (University of Leeds)

(a) Suppose that $2^n - 1$ is prime. Then $n > 1$. Write $n = rs$, where r, s are positive integers. Then

$$2^n - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 1),$$

so either $2^r - 1 = 1$ or $2^r - 1 = 2^n - 1$, i.e. either $r = 1$ or $r = n$. Thus n is prime.

(b) Suppose that $2^n + 1$ is prime, and write $n = 2^k l$, where k, l are integers, $k \geqslant 0$, $l \geqslant 1$ with l odd. Then

$$2^n + 1 = (2^{2^k} + 1)(2^{2^k(l-1)} - 2^{2^k(l-2)} + \cdots - 2^{2^k} + 1),$$

so $2^n + 1 = 2^{2^k} + 1$ and $n = 2^k$.

The converse of (a) is false; for example,

$$2^{11} - 1 = 2047 = 23 \times 89.$$

8.2. A projectile is fired upwards from a cliff 45 metres high at an angle of 45° to the horizontal and lands in the sea at a distance 360 metres from the foot of the cliff. The operation is then repeated, but this time a wind of speed 2 metres/sec is blowing on shore. How does this affect the range of the projectile? With the wind blowing, could the range of the projectile be increased by altering the angle of inclination? (You may take the acceleration due to gravity to be 10 metres/sec^2.)

Solution

Suppose that the resolved part of the initial velocity in the horizontal and vertical directions is u metres/sec. With no wind blowing, suppose that the time of flight is t seconds. Then

$$360 = ut$$

and

$$-45 = ut - \tfrac{1}{2}(10)t^2,$$

which give $t = 9$ and $u = 40$. Now suppose that a wind of 2 metres/sec is blowing on shore. The vertical motion is the same as before, so the time of flight is again 9 seconds. But the new range will be

$$(u - 2)9 = 342 \text{ metres}.$$

Hence the range is decreased by 18 metres.

We now fire the projectile at an angle α^0 to the horizontal, still with the wind blowing. The initial velocity is $40\sqrt{2}$ metres/sec. Suppose that the range is r metres and the time of flight t seconds. Then

$$r = (40\sqrt{2}\cos\alpha - 2)t$$

and

$$-45 = 40\sqrt{2}\,(\sin\alpha)t - 5t^2.$$

Thus

$$\frac{dr}{d\alpha} = (40\sqrt{2}\cos\alpha - 2)\frac{dt}{d\alpha} - 40\sqrt{2}\,(\sin\alpha)t \qquad (1)$$

and

$$0 = 40\sqrt{2}\,(\sin\alpha)\frac{dt}{d\alpha} + 40\sqrt{2}\,(\cos\alpha)t - 10t\frac{dt}{d\alpha}. \qquad (2)$$

Suppose that the range is maximum when $\alpha = 45$. Then $(dr/d\alpha)_{\alpha=45} = 0$. Also $(t)_{\alpha=45} = 9$. If we substitute these values in (1), we obtain

$$\left(\frac{dt}{d\alpha}\right)_{\alpha=45} = \frac{360}{38};$$

and from (2) we obtain

$$\left(\frac{dt}{d\alpha}\right)_{\alpha=45} = \frac{360}{50}.$$

This is a contradiction, so the range is not maximum when $\alpha = 45$, and therefore could be increased by altering the angle of inclination.

8.3. The real series $\Sigma\, a_n$, $\Sigma\, b_n$ are such that $\Sigma\, a_n$ is convergent, no a_n is zero and $b_n/a_n \to 1$ as $n \to \infty$. Does the series $\Sigma\, b_n$ have to be convergent?

Solution

The answer is 'no'. For put

$$a_n = \frac{(-1)^{n-1}}{n}, \quad b_n = \frac{(-1)^{n-1}}{n} + \frac{1}{(n+1)\log(n+1)}.$$

Then $\Sigma\, a_n$ is convergent and $b_n/a_n \to 1$ as $n \to \infty$, yet $\Sigma\, 1/(n+1)\log(n+1)$ is divergent, so $\Sigma\, b_n$ is divergent.

Proof of a result of Pappus (see the article 'A New Look at Archimedes' by C. P. Ormell in Volume 8 Number 1, p. 4). The result is as follows:

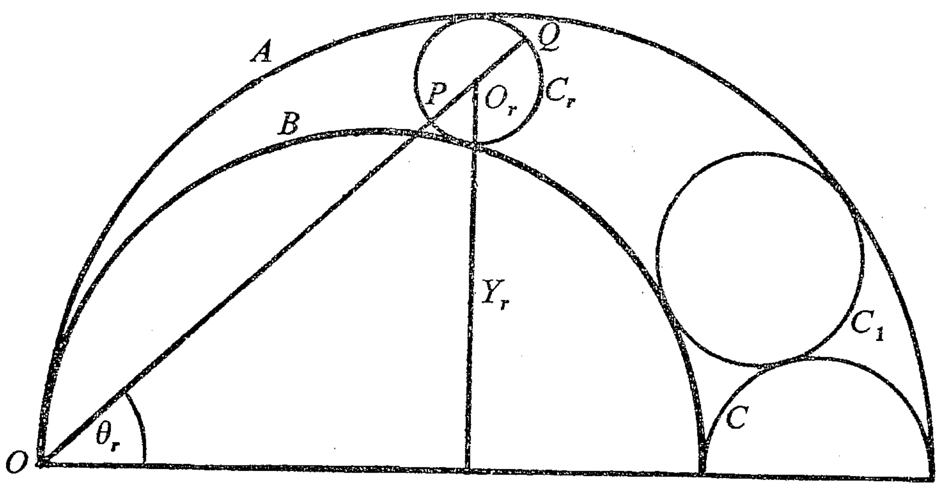

Figure 1

Three semi-circles A, B, C are drawn on the same base-line such that A contains B and C. Each semi-circle touches the other two at the ends of its diameter. A sequence of circles $C_1, C_2, C_3, \ldots$ is constructed such that C_1 touches A, B, C, and C_2 touches A, B, C_1, and C_3 touches A, B, C_2, and so on. The diameter of C_r is denoted by d_r, and the perpendicular distance from the centre O_r of C_r to the common base-line is y_r. Show that $y_r = rd_r$ for all r.

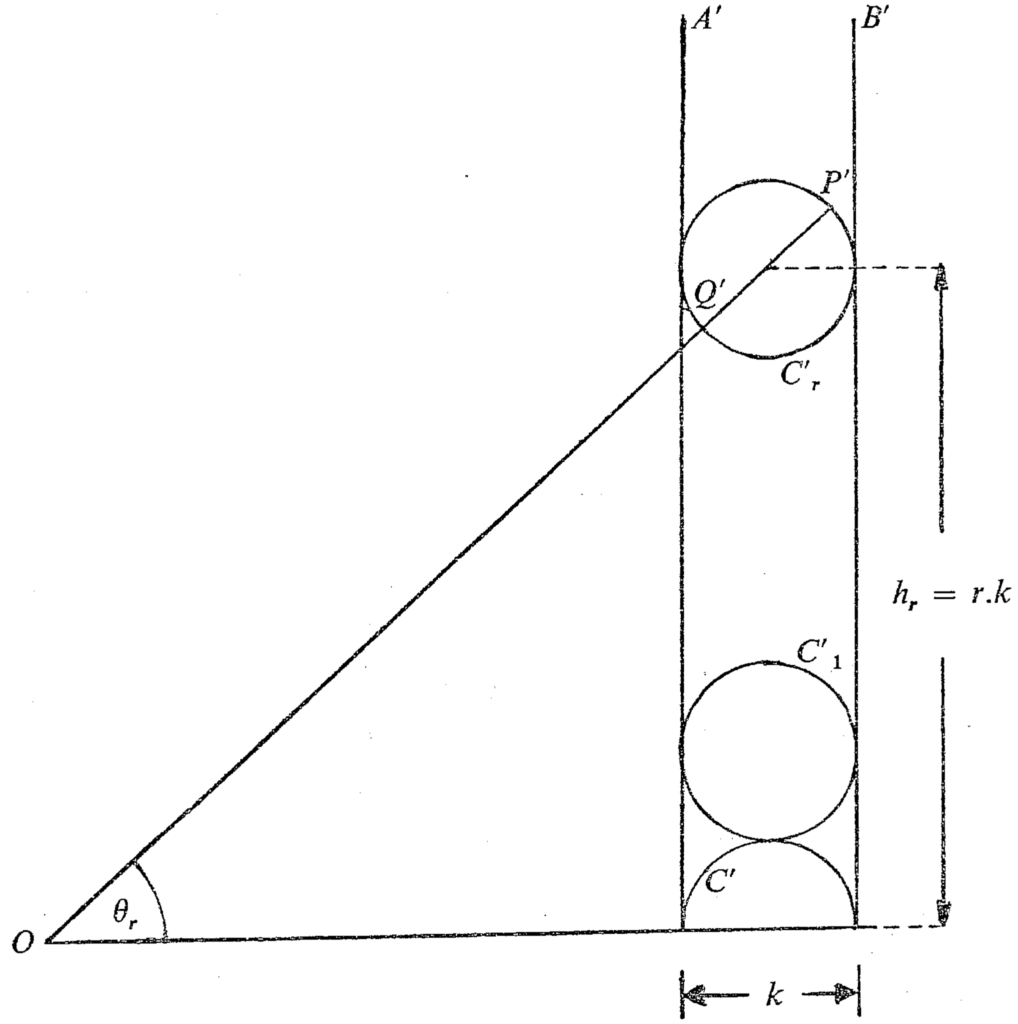

Figure 2

We have received the following proof from two readers, A. R. Pargeter of Tiverton and J. P. Green of Uppingham.

We invert with respect to a circle centre O, radius say s (see Figure 1), and denote the images of points and circles by appending a dash (so that, for example, the circle A inverts to the straight line A'). The inverted system is seen in Figure 2. Now

$$y_r = \tfrac{1}{2}(OP + OQ)\sin\theta_r, \quad d_r = OQ - OP.$$

Since, for example, $OP \,.\, OP' = s^2$, this gives

$$y_r = \frac{s^2}{2}\left\{\frac{1}{OP'} + \frac{1}{OQ'}\right\}\sin\theta_r, \quad d_r = s^2\left\{\frac{1}{OQ'} - \frac{1}{OP'}\right\}.$$

Hence

$$\frac{y_r}{d_r} = \frac{(OP' + OQ')\sin\theta_r}{2(OP' - OQ')} = \frac{h_r}{k} = \frac{rk}{k} = r$$

(see Figure 2). Thus $y_r = rd_r$, as required.

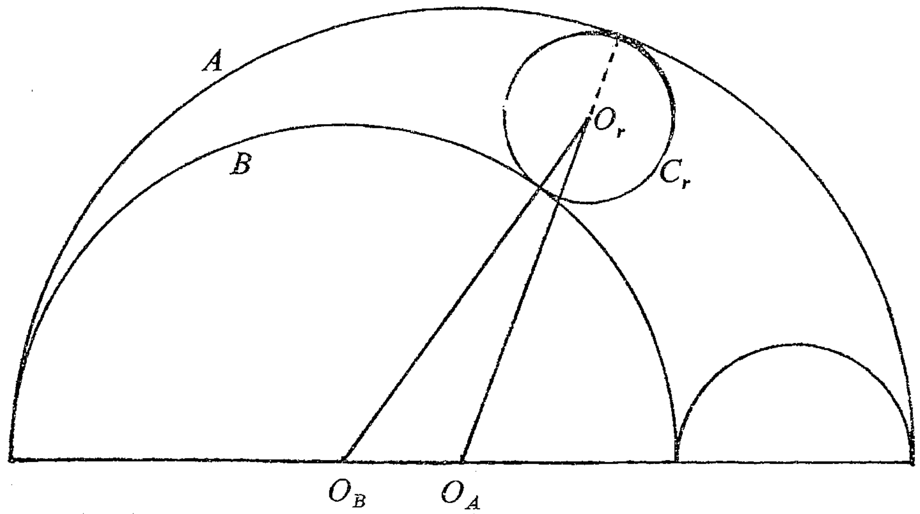

Figure 3

J. P. Green also makes the following observation.
Denote the centres of the circles A, B by O_A, O_B respectively, and let their respective radii be a, b (see Figure 3). Then

$$O_AO_r = a - \frac{d_r}{2} \quad \text{and} \quad O_BO_r = b + \frac{d_r}{2},$$

so that

$$O_AO_r + O_BO_r = a + b$$

and the centres of the circles $C_1, C_2, C_3, \ldots$ lie on an ellipse whose foci are the centres of the circles A, B.

Book Reviews

Mathematics Applicable Series. Schools Council Sixth Form Mathematics Project. Heinemann Educational Books Ltd, London, 1975.
Mathematics Changes Gear. Pp. viii+53. £1·02.
Understanding Indices. Pp. viii+130. £1·80.
Geometry from Coordinates. Pp. viii+85. £1·60.
Logarithmic/Exponential. Pp. viii+169. £1·95.

To the student. Once upon a time, mathematics came in two distinct flavours; 'pure' and 'applied'. These flavours were sampled at school in different lessons, with the use of different books, and they seemed to have little in common. Some students only tried one flavour. But gradually it has been realised that mathematics can be used, or *applied*, in many different situations and that almost all 'pure' mathematics finds application in solving some practical problem. More than this, the 'spur' for much mathematical discovery has often been the need to solve a new practical problem. Underlying both of these points is the idea of *mathematical modelling*: expressing a problem in mathematical terms and symbols, solving the resulting relations between the symbols, and thus, returning to the original situation, proposing a solution to the problem.

These books develop an applicable approach to sixth-form mathematics. Although a lot of pure mathematics is covered in them, everything is set in the context of a practical situation or an application. It is especially pleasing that the authors have succeeded in this approach with both advanced and elementary ideas.

Whatever your mathematical 'diet'—from 'double mathematics' A-level to subsidiary mathematics in support of other subjects—you will find a lot in these books to excite and interest you and much fun in working through them. If you already know the mathematics, you will certainly be surprised by many of the applications of it, and your knowledge will be deepened; if you are learning the mathematics, these books provide a new and stimulating approach. All these books contain background information on the applications, extending the explanation in the main text, and hints and answers to the exercises. They are attractively produced and good value for money.

Make sure that your school has copies.

To the teacher. If you, like me, are looking for new applications of the mathematics you teach; if you are looking for new ways to introduce old topics; if you want ideas which will help you to develop the ideas of 'modelling', then read these books. Teachers' editions are in preparation; in the meantime these first titles show the authors' approach and give promise of a most valuable series. No sixth-form classroom should be without them!

Mathematics Changes Gear

A short book expounding the philosophy of 'mathematics applicable'. It is intended for use either as an introduction to the series (it might be difficult in places for beginners), or for use independently.

Understanding Indices

All the usual rules are here, but treated in the applications context.

Geometry from Coordinates

Mainly on the straight line, but a few other loci appear briefly.

Logarithmic/Exponential

The most advanced book of the four. Much of the work is of second year A-level standard; e^x, ln x, integration, differentiation, standard series are all here and, of course, a host of applications.

Clifton College, Bristol — JOHN HERSEE

Concepts of Modern Mathematics. By IAN STEWART. Penguin Books Ltd, Harmondsworth, 1975. Pp. viii+315. £0·80.

This book is 'a description of the aims, methods, problems and applications of modern mathematics'. It is aimed at the intelligent layman with 'only a smattering of algebra, geometry and trigonometry'. However, the text requires more than 'a little concentration'. In particular the chapters on algebraic topology are very hard indeed. The author proceeds from axiomatic definitions which are concise and clear if one is familiar with this approach to mathematics but could otherwise be hopelessly confusing. However, this book contains a lot of material (especially on topology) which is not available in other books at this level.

For an interested and intelligent sixth-former this could prove a very stimulating book, but I fear that the intelligent layman is likely to find it beyond him. The price is reasonable, and this should provide an additional incentive for any such sixth-former.

Sixth Form, St. Edward's School, Oxford CHRISTOPHER MAYERS

A Foundation Course in Modern Algebra. By D. J. BUONTEMPO. Macmillan Publishers, London, 1975. Pp. viii+248. £2·95.

The aim of this book is to provide a first course in modern algebra for students who have followed a mathematics course at G.C.E. A level.

The basic algebraic ideas of relations and binary operations are introduced in the first two chapters and, by way of application, the integers are constructed from the natural numbers. Groups and subgroups are studied in Chapter 3, where proofs of the theorems of Lagrange and Cayley may be found. I liked the chapter on matrices in which their connection with functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ is explored. There is also a chapter on rings, integral domains and fields, and one on Boolean algebra.

I have two reservations about the book. First, the definition of 'binary operation' is non-standard and leads to needless difficulty. In fact, the definitions on page 152 of left and right distributivity do not make sense unless binary operations are assumed closed. Second, the section on the Boolean algebra of propositions is rather confusing owing to a lack of precision in the definition of 'proposition' which is given as 'a sentence which is a statement that is either true or false but not both'. A definition of 'statement' is not supplied and the role of the variable in a proposition is not explained. I think a sixth-former would find this section very difficult to follow.

Apart from the above points, the book is carefully written. Each topic is well introduced and is seen to build on previous work. There is an ample stock of problems and answers are provided at the back of the book. With one exception, all chapters should prove useful reading for sixth-formers who are preparing for further courses in mathematics as well as for students in Colleges of Education or in their first year at University or Polytechnic.

University of Durham J. BOLTON

A First Course in Abstract Algebra. By P. J. HIGGINS. Van Nostrand Reinhold, Wokingham, 1975. Pp. 158. £4·50 cloth; £2·25 paperback.

This introduction to abstract algebra is intended primarily for use with first-year undergraduates, but there is much substance in the author's suggestion that it might prove useful to teachers, both established and in training. It is also suggested that the book might serve as an introduction to students of the liberal arts. This is probably more in doubt, not so much because of the material but the mode of presentation. The style is essentially formal—of the 'theorem-proof-corollary' type. It is extremely well executed with short interludes of motivation, explanation and discussion of tactics, and this presentation helps to introduce the reader not only to the subject matter but to formal mathematical style. For the committed, the treatment is excellent. For the uncommitted, it is probably not gentle enough and the information content would appear too dense.

The algebraic structures discussed are groups, rings and fields and the main sources of examples and illustrations are the integers and polynomials. The first three chapters lay the foundations for the remainder. After an introductory section,there is an introduction to the basic language and techniques of sets, functions, operations, relations and quotient sets. The third chapter discusses the basic assumptions made about the integers and introduces the notions of congruence and residue classes, notions which are a major theme in subsequent chapters.

The inclusion of these three chapters, and the manner of their presentation, set a tone of attention to detail and concern to carry the reader with the development of the material, which is maintained throughout the book. One frequently hears of abstract algebra as offering the power of generalisation and as an aid to unification, but too often this is only lip-service. In this book, a genuine attempt is made to expose general lines of thought and methods of attack important in abstract algebra, whilst at the same time avoiding the trap of too much 'talking about' and not enough 'doing'. There is a high density of mathematical content with plenty of good exercises and examples. To anybody seeking a serious but accessible introduction to abstract algebra, this book can be highly recommended.

The Open University NORMAN GOWAR

Number Theory. By T. H. JACKSON. Routledge and Kegan Paul, London, 1975. Pp. i+88. £1·50.

Gauss said of the theory of numbers that it possessed a 'magical charm which has made it the favourite science of the greatest mathematicians'. It has been no less the favourite recreation of generations of lesser mortals! Part of the subject's charm lies in the great difficulty which is often experienced in proving simple general theorems suggested by numerical evidence and the wide range of powerful mathematical tools which are frequently required to do so. Indeed the theory of numbers has provided the motivation for research in complex analysis, algebra, geometry and topology and many unsolved problems of the higher arithmetic await further developments in those fields. For all those reasons there are strong arguments for including number theory in all undergraduate courses and in A-level courses.

This little book conforms to the expected high standards of the Library of Mathematics series. After introducing the reader to the basic ideas concerning factorisation and the prime numbers, it deals with congruences (the finite arithmetic of many school courses).

The ideas are developed in Chapter III, where a detailed study is made of quadratic congruences of the type $ax^2 + bx + c \equiv 0 \pmod{p}$. (This means that a, b, c, are integers and x is an integer for which $ax^2 + bx + c$ is divisible by the prime number p.) By using the technique of completing the square, one can reduce that study to a study of congruences of the type $x^2 \equiv d \pmod{p}$, where p does not divide d. This leads to one of the most remarkable and beautiful results in the whole subject. To state it we introduce a symbol (d/p) which equals 1 if there is an x satisfying $x^2 \equiv d \pmod{p}$ and equals -1 if there is no such x. Now let p, q be odd prime numbers. Then the *Law of Quadratic Reciprocity* (first proved by Gauss) asserts that

$$\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{\frac{1}{2}(p-1).\frac{1}{2}(q-1)}.$$

The generalisations of this theorem are still the subject of research.

The final chapter studies the equally attractive and far-reaching question of the representation of numbers as sums of squares. For example: *the only natural numbers which are sums of two integer squares are products of powers of 2, powers of primes of the form* $4k + 1$ *and even powers of primes of the form* $4k + 3$. On the other hand, *every natural number is a sum of four integer squares.*

The whole book offers an attractive first course in number theory and contains nothing that could not be mastered (admittedly with some effort) by an A-level student.

College of St Hild and St Bede, Durham J. V. ARMITAGE

Groups. By JOHN MASON. Transworld Student Library, London, 1975. Pp. 125, £0·85.

This little book is an introduction to group theory intended for private study. It presents the basic ideas of groups, subgroups, cosets, homomorphisms and permutation groups at a leisurely pace and proves three main theorems: Lagrange's theorem on the order of subgroups of a finite group, the first isomorphism theorem, and Cayley's theorem that every group can be realised as a group of permutations. The style is chatty and the text is liberally sprinkled with worked exercises and detailed instructions. The reader needs to be well acquainted with sets and functions, for the two-page summary of set theory is unhelpful. There are several errors that could seriously mislead the unwary.

The book has two unusual features. The main innovation is the use of the so-called 'Cayley cards' (I do not think that Cayley would be grateful for the attribution) which are included as a cardboard fold-out at the back of the volume. These constitute a simple graphical device for representing and composing permutations, and are fun to play with. They give the student direct contact with the group operation in a particular permutation group and are helpful in assimilating the group axioms and some of their elementary consequences. But this initially good idea is very much overworked as a basis for the understanding of later material. It is very noticeable that, in passages where this teaching aid is temporarily forgotten, the quality of the exposition is much improved. (In particular, the account of quotient groups is very clear.)

The second unusual feature is the emphasis placed on the reader's understanding of his own learning process. Many (like the reviewer) will find this irritating and distracting. Apart from the ugliness of the jargon used ('the axioms represent a major concrete/abstract conceptual boundary'), it seems unlikely that students will profit from thinking about learning and about mathematics at the same time. The text would be improved if some of the psychological motivation were replaced by better mathematical motivation, for example some information on what groups are actually used for. However, if you find abstractions difficult but are at home with sets and functions, if you are introspective and if you like a text that tells you exactly what to do at each stage, then this book may suit you well.

King's College London — P. J. HIGGINS

A First Course in Quantum Mechanics. By H. CLARK. Van Nostrand Reinhold Company Ltd, London, 1974. £7·50 cloth; £4·00 paperback.

There are now many undergraduate textbooks on quantum mechanics, covering basically the same material. This particular addition to their number follows essentially traditional lines. Its most distinctive feature is that it does not assume much mathematical knowledge on the part of its readers; for example on page 43 we are told that i is the square root of (-1) and, in Chapter 6, we are given an introduction to the algebra of matrices. The usual chapter on formal classical dynamics is included, although it is not used in the remainder of the book and, to this reviewer, seems to be an unnecessary hurdle for the reader of an introduction to quantum mechanics. It would have been better to have used this space to expand the section on scattering, which is very brief. There is a chapter on group theory which gives a good introduction to the ideas and shows why they are useful.

This book has clearly been written for students—their interests, rather than concern for presenting an elegant description of quantum mechanics, are always paramount. Only the students who read it can pass judgement on how successful it is, but most students (particularly those with weak mathematical background) may find it among the more accessible of the books on quantum mechanics.

It is a neat, compact volume and deserves to be successful.

University of Durham — E. J. SQUIRES

Dynamics. By W. E. WILLIAMS. Van Nostrand Reinhold Company Ltd, London, 1975. Pp. 154. £2·25, paperback; £4·50, cloth.

This book treats in seven chapters particle motion including central orbits; rotating frames; moments of inertia; and rigid body motion including Lagrange's equations. The treatment uses vector methods and is generally clear and concise. Over half the book is devoted to rigid body motion. There are exercises of limited scope at the end of each chapter, but no answers. The index is very brief.

On page 34 the author states that 'to ease the solving of particular problems, coordinate systems other than the Cartesian one have to be used', yet only plane polars are introduced and no mention is made of s, ψ and p, r coordinates. The chapter on central orbits deals almost exclusively with motion under the inverse square law of force, using polar coordinates. The preface mentions that the book is directed primarily at British first-year university mathematics honours degree students, but it seems possible that the book covers more material than is required for the first year, and that the depth of this coverage is not great enough.

Insufficient care has been taken to avoid starting a sentence with a symbol, and there are some consequent ambiguities. For example at the top of page 17 the uninitiated reader (for whom the book is presumably intended) might be led to believe that $V(x_1) = E \,.\, F_x$.

University of Durham D. H. WILSON

Notes on Contributors

D. M. Burley, a graduate of London University, has taught in the University of Glasgow and is currently Senior Lecturer in Applied Mathematics and Computing Science in the University of Sheffield. His article has developed from a course of lectures on mathematical modelling, a topic which he feels is much neglected in school and university curricula.

L. S. Goddard obtained his first degree from the University of Sydney and his doctorate from Cambridge. After lecturing in Scotland he became Professor of Mathematics in the University of Tasmania, but in 1967 returned to Britain to occupy the Chair of Mathematics at the newly founded University of Salford.

His professional interests are now in finite mathematics and in the history of mathematics.

K. R. S. Sastry obtained the degree of M.Sc. from the University of Mysore. Subsequently he lectured in mathematics at a number of colleges in India, and he is now teaching at a secondary school in Addis Ababa. His article on Heronian Triangles is the result of a desire to foster among his pupils an interest in mathematics outside the usual school curriculum.

Monica Walker is a Research Fellow in the Centre for Criminological Studies in the University of Sheffield. She was formerly a Lecturer in Statistics in the University of London, and before coming to Sheffield worked for some time at the Institute of Criminology in Cambridge.

Contents

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