Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

Volume 41 2008/2009 Number 3

- Relative Arithmetic
- Lottery Combinatorics
- Nested Ellipses

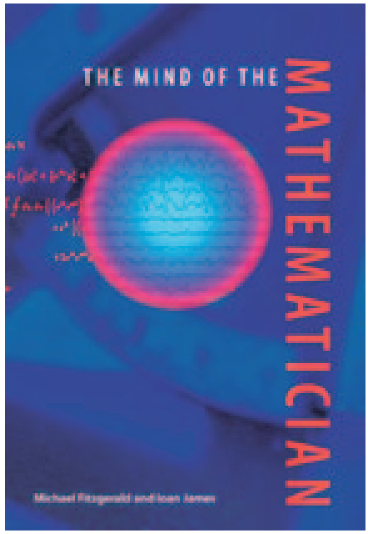
Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

The Mind of the Mathematician



I do not quite know what to make of an interesting book that has come our way (see reference 1). It is written jointly by a psychologist and a mathematician and asks the question 'what makes mathematicians tick?'. There is no doubt that the world at large thinks they (or we?) are an odd lot, living in their own private world, using their own language, asking questions, and giving answers no one understands. Of course, this ignores the mathematical techniques being applied throughout the scientific, industrial, biological, economic, and communications worlds. The language of mathematics may be a closed book to the public at large, but everyone uses their gadgets, handles their finances, relies on their drugs, and their security, in all of which mathematicians play a vital part.

It is revealing to begin to see one's own discipline through the eyes of someone of another discipline, in this case the world of the mathematician seen through the eyes of a psychologist. Are there behavioral patterns common to mathematicians? Does the private thought-world that the mathematician inhabits provide evidence of personality disorders such as Asperger syndrome and autism? I confess I had to look these terms up on the internet, and I am still rather vague about them. What comes first, the chicken or the egg? Does the study of mathematics create the personality traits, or may those with these traits be attracted to mathematics? Having read this book, I still do not know the answer.

The second part of the book, more than half, is given over to potted biographies of twenty of the greatest mathematicians of the last three centuries, analysing their lives and personalities to see if there is any common denominator. The authors do indeed detect evidence of Asperger syndrome in the compulsive private world they inhabited and the difficulty that some had in communicating with others. This can surely be exaggerated. What came over to me was the variety of personalities described, not their uniformity. And what about all those not selected? Were (and are) they the life and soul of the party? Perhaps statistical techniques need to be applied – mathematics again!

It may be that geniuses in all fields, whether mathematics or music, art or literature, or any sphere of activity, pay a high personal price, for which us lesser mortals can only be grateful to them. So do not be put off. Mathematics of itself will not do you any harm and may give you a lifetime of pleasure, despite the times of frustration when you seem to be banging your head against the wall. And you may in some small way be contributing to the sum of human happiness.

Reference

1 Michael Fitzgerald and Ioan James, *The Mind of the Mathematician* (The Johns Hopkins University Press, Baltimore, MD, 2007).

How Many Digits Make a Fibonacci Number?

M. A. NYBLOM

The sequence of Fibonacci numbers $\{F_n\}$, which are defined via the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = 1$, $F_2 = 1$, is arguably the most well-known and studied of all the integer sequences. In recent times, the long outstanding problem of determining all perfect powers in the Fibonacci sequence has finally been settled. In particular, by using some of the deep ideas contained in Wiles's now famous proof of Fermat's last theorem, Bugeaud *et al.* (see theorem 1 in reference 1) have shown that the only perfect powers in the sequence of Fibonacci numbers are $F_1 = 1$, $F_2 = 1$, $F_6 = 2^3$, and $F_{12} = 12^2$. By applying this important result together with some well-known properties of the Fibonacci sequence, we can now show, for integer bases a other than $a = 2$ and $a = 12$, that the number of digits in F_n for $n > 2$ is given by

$$N_a(n) = \begin{cases} \lfloor n \log_a(\Phi) + (1 - \log_a(\sqrt{5}))\rfloor & \text{if } F_n \neq a \text{ or } F_n = a \text{ with } \Phi^n/\sqrt{5} > a, \\ \lceil n \log_a(\Phi) + (1 - \log_a(\sqrt{5}))\rceil & \text{if } F_n = a \text{ with } \Phi^n/\sqrt{5} < a, \end{cases} \tag{1}$$

where $\Phi = (1+\sqrt{5})/2$ is the golden ratio, $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ denotes the integer-part function or floor function, and its companion function, $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$, is known as the ceiling function. To begin, let $a > 1$ be a fixed integer such that $a \neq 2, 12$ and consider the following two cases.

Case 1 ($F_n \neq a$.) Suppose for $n > 2$ that F_n has m digits when written in base a, that is $N_a(n) = m$. By assumption and theorem 1 in reference 1, $a^{m-1} < F_n < a^m$, from which, upon taking logarithms to base a, we find $m-1 < \log_a(F_m) < m$, and so $m = \lfloor \log_a(F_n) \rfloor + 1$. To obtain the required expression for $N_a(n)$, we start with the well-known Binet formula for the nth Fibonacci number, namely

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{2}$$

Now, setting $\Phi = (1+\sqrt{5})/2$ and noting that $|(1-\sqrt{5})/2| < 1$, observe that (2) implies $|\Phi^n/\sqrt{5} - F_n| < 1/\sqrt{5} < \frac{1}{2}$, from which it is easily deduced that

$$0 < \frac{\Phi^n}{\sqrt{5}} + \frac{1}{2} - F_n < 1.$$

Thus, for each $n > 2$, we can find a $\theta_n \in (0, 1)$ such that $F_n = \Phi^n/\sqrt{5} + \frac{1}{2} - \theta_n$, and so $m = \lfloor \log_a(\Phi^n/\sqrt{5} + \frac{1}{2} - \theta_n) \rfloor + 1$. As $F_n > 1$ and $a \neq 2, 12$, we are guaranteed (again by theorem 1 in reference 1) that $F_n \neq a^s$ for any $s \in \mathbb{N}$; consequently, F_n must differ from a positive integer power of a by at least one. Thus, as F_n is an integer, both F_n and $F_n + \eta$, where $|\eta| < 1$, must be bounded between the same consecutive powers of a, that is $F_n, F_n + \eta \in (a^{m-1}, a^m)$. Hence, $\lfloor \log_a(F_n) \rfloor = \lfloor \log_a(F_n + \eta) \rfloor$. So, by choosing $\eta = \theta_n - \frac{1}{2}$, we see that

$$\begin{aligned} N_a(n) &= \lfloor \log_a(F_n) \rfloor + 1 \\ &= \left\lfloor \log_a\left(F_n + \theta_n - \tfrac{1}{2}\right) \right\rfloor + 1 \\ &= \left\lfloor \log_a\left(\frac{\Phi^n}{\sqrt{5}}\right) \right\rfloor + 1 \\ &= \lfloor n \log_a(\Phi) + 1 - \log_a \sqrt{5} \rfloor, \end{aligned}$$

since $|\theta_n - \frac{1}{2}| < \frac{1}{2} < 1$.

Case 2 ($F_n = a$.) In this instance, suppose for some $n > 2$ that we have $F_n = a = (10)_a$. Then, clearly, $N_a(n) = 2$. Recalling again the fact that F_n is the nearest integer to $\Phi^n/\sqrt{5}$, observe that if $\Phi^n/\sqrt{5} > a$ then $\Phi^n/\sqrt{5} \in (a, a^2)$, and so

$$\begin{aligned} \lfloor n \log_a(\Phi) + (1 - \log_a(\sqrt{5})) \rfloor &= \left\lfloor \log_a\left(\frac{\Phi^n}{\sqrt{5}}\right) \right\rfloor + 1 \\ &= 1 + 1 \\ &= N_a(n). \end{aligned}$$

Alternatively, if $\Phi^n/\sqrt{5} < a$ then $\Phi^n/\sqrt{5} \in (1, a)$, and so

$$\begin{aligned} \lfloor n \log_a(\Phi) + (1 - \log_a(\sqrt{5})) \rfloor &= \left\lfloor \log_a\left(\frac{\Phi^n}{\sqrt{5}}\right) \right\rfloor + 1 \\ &= 0 + 1 \\ &= N_a(n) - 1. \end{aligned}$$

The desired expression for $N_a(n)$ now readily follows after recalling that $\lfloor x \rfloor + 1 = \lceil x \rceil$ for any $x \notin \mathbb{Z}$.

Example 1 To illustrate the use of (1) let us first consider the decimal case, namely base 10. In this instance, a quick examination of the first few terms of the Fibonacci sequence reveals that $F_n \neq 10$. Thus,

$$N_{10}(n) = \lfloor n \log_{10}(\Phi) + (1 - \log_{10}(\sqrt{5})) \rfloor. \tag{3}$$

Now, $F_{37} = 24\,157\,817$ and so $N_{10}(37) = 8$. By (3) we see that

$$N_{10}(37) = \lfloor 8.383\,057\,7 \rfloor = 8.$$

Alternatively, suppose that we wanted to know the number of digits in, say, F_{2006}, without directly calculating F_{2006}. Then, again by (3) we find that $N_{10}(2006) = \lfloor 419.879\,72 \rfloor = 419$. In the case of base 3 we have to be a little careful as $F_4 = 3$. However, since $\Phi^4/\sqrt{5} > 3$, we deduce from (1) that $N_3(n) = \lfloor n \log_3(\Phi) + (1 - \log_3(\sqrt{5})) \rfloor$, from which all number-of-digit calculations in this base may be made for F_n when $n > 2$.

Despite our reliance in the above argument on the rather deep result in reference 1, it is still possible in the base 10 case to prove (1) without reference to this result. Indeed, all that is necessary is to establish that there is no Fibonacci number F_n which is a positive integer power of 10. To conclude, we demonstrate using an elementary argument the nonexistence of positive integers n and m to the Diophantine equation $\mathrm{F}_n = 10^m$.

Recall that a positive integer a divides another positive integer b if and only if $(a, b) = a$, where (a, b) denotes the greatest common divisor of a and b. Furthermore, recall from theorem II in reference 2 the standard divisibility property of the Fibonacci numbers, namely $(\mathrm{F}_n, \mathrm{F}_m) = \mathrm{F}_{(n,m)}$. Now, as $\mathrm{F}_3 = 2$ and $\mathrm{F}_5 = 5$, we deduce from the above that 2 and 5 divide F_n if and only if $(3, n) = 3$ and $(5, n) = 5$, respectively, that is, 3 divides n and 5 divides n. Consequently, 10 divides F_n if and only if n is a multiple of 15. Thus, the only Fibonacci numbers F_n which may be a positive power of 10 are those in which $n = 15s$, for some $s \in \mathbb{N}$. If we assume that $\mathrm{F}_{15s} = 10^m$ for some integers $m, s \geq 1$ then, noting that $\mathrm{F}_{15} = 610$, observe that

$$(610, 10^m) = (\mathrm{F}_{15}, \mathrm{F}_{15s}) = \mathrm{F}_{(15,15s)} = \mathrm{F}_{15} = 610,$$

and so 610 must divide 10^m, which is clearly impossible. Hence, there are no Fibonacci numbers which are a positive power of 10.

Acknowledgement

I would like to thank Dr Dan Kildea for his original question which helped inspire this article.

References

1 Yann Bugeaud, Maurice Mignotte and Samir Siksek, Classical and modular approaches to exponential Diophantine equations, I, Fibonacci and Lucas perfect powers, *Ann. Math.* **163** (2006), pp. 969–1018.
2 V. E. Hoggatt, Jr, *Fibonacci and Lucas Numbers* (Houghton Mifflin, Boston, MA, 1969).

Michael Nyblom *is a lecturer in the Department of Mathematics at RMIT University, Melbourne. His research interests include Analysis, Combinatorics, and Number Theory.*

Birthdays and magic squares

If you write your birthday in the form $a/b/c$ (e.g. $25/5/21$), then it can be a row of a magic square with rational entries as follows.

a	b	c
$\frac{-2a+b+4c}{3}$	$\frac{a+b+c}{3}$	$\frac{4a+b-2c}{3}$
$\frac{2a+2b-c}{3}$	$\frac{2a-b+2c}{3}$	$\frac{-a+2b+2c}{3}$

The row-sums, column-sums, and two diagonal-sums are all equal to $a + b + c$.

Midsomer Norton, Bath, UK **Bob Bertuello**

On k-oblong Numbers

KONSTANTINE ZELATOR

1. Introduction

In reference 1 p. 105, the author defines an *oblong* number as a positive integer of the form $n(n+1)$, where n is a positive integer. Thus, the nth oblong number is $2t_n$, where t_n is the nth triangular number.

In this article, we generalize the concept of an oblong number in a natural way.

Definition 1 Let k be a fixed positive integer. A natural number is said to be a k-oblong number, denoted by $0_{k,n}$, if it is of the form $n(n+k)$, for some $n \in \mathbb{Z}^+$. We write $0_{k,n} = n(n+k)$.

Thus, a 1-oblong number is simply an oblong number.

In Section 2 we state the parametric formulas that generate the entire family of Pythagorean triples, a well-known fact in number theory. We make use of these formulas in Section 4. In Section 3 we present a simple precise condition (necessary and sufficient) that a given integer must satisfy in order to be k-oblong. This is easily done in theorem 1. Section 4 is really the main focus of this work. We give a complete description/characterization of those integers which are both square numbers and k-oblong numbers in the case where k is an odd integer and $k \geq 3$. This is done in theorem 2, part (i). Using part (i), one easily establishes part (ii) of the same theorem which says that, when k is a (odd) prime power, p^ℓ, there are exactly ℓ k-oblong numbers which are also square numbers.

An immediate corollary of theorem 2 is that, when k is odd, there are finitely many k-oblong numbers which are also square. Even though we do not treat the case when k is even, the same method can be used to describe the family of numbers which are both square and k-oblong in that case. Again, as in the case of k being odd, that family is finite. (Incidentally, there exist no numbers which are both k-oblong and square when $k = 1$ or 2. This is easily shown – see Section 4.) Section 4 ends with theorem 3, in which a formula for N, the number of integers which are both k-oblong and square numbers, is presented in the case where k is an odd square free integer with at least $\ell \geq 2$ prime factors. The formula for N depends only on ℓ.

In the remark at the end of Section 4 we show that, for a given k there are only finitely many numbers which are both k-oblong and square numbers.

2. Pythagorean triples

A triple (a, b, c) is called Pythagorean if a, b and c are positive integers such that $a^2 + b^2 = c^2$. In effect, a, b and c are the integer side lengths of a right triangle having c as its hypotenuse length. Then,

$$a = \delta(M^2 - N^2), \qquad b = \delta(2MN), \qquad c = \delta(M^2 + N^2), \tag{1}$$

where M, N and δ are positive integers such that $(M, N) = 1, M > N$, and $M + N \equiv 1(\text{mod}2)$.

Parametric formulas (1) are very well known, and they describe the entire family of Pythagorean triples (up to symmetry; obviously a and b can be switched). We refer the reader to references 2 or 3.

3. When is a given number k-oblong?

Theorem 1 *A given natural number m is k-oblong if and only if it is of the form $m = (N^2 - k^2)/4$ for some positive integer N such that $N > k$ and $N \equiv k(\text{mod}2)$. Then, $m = 0_{k,n}$, the choice of n being $n = (-k + N)/2$.*

Proof Suppose first that $m = (N^2 - k^2)/4$ for some positive integer $N > k$ with $N \equiv k(\text{mod}2)$. We calculate the oblong number $0_{k,n}$: $0_{k,n} = n(n+k)$, with $n = (-k + N)/2$:

$$0_{k,n} = \left(\frac{-k+N}{2}\right)\left(\frac{-k+N}{2} + k\right) = \left(\frac{-k+N}{2}\right)\left(\frac{k+N}{2}\right) = \frac{N^2 - k^2}{4} = m.$$

Now the other direction. Assume that m is a k-oblong number. Then we must have $n(n + k) = m$ for some positive integer n. We obtain,

$$n^2 + kn - m = 0.$$

Hence,

$$n = \frac{-k + \sqrt{D}}{2},$$

where $D = k^2 + 4m$. Then $\sqrt{D} = 2n + k$, which is a positive integer; we may set $2n + k = N$ for some $N \in \mathbb{Z}^+$. Therefore, $D = N^2$, which in turn implies $N^2 = k^2 + 4m$; $m = (N^2 - k^2)/4$. Since $N^2 - k^2 = 4m \equiv 0(\text{mod}4)$, it follows that $N \equiv k(\text{mod}2)$; and $N > k$ since m is a positive integer. Lastly, $n = (-k + \sqrt{D})/2 = (-k + N)/2$. The proof is complete.

4. Square numbers which are also k-oblong numbers

Firstly we point out there exist no 1-oblong (or oblong) or 2-oblong numbers which are also square numbers. This can be shown easily. In the case of oblong numbers: if the square number r^2 (r is a natural number) were also oblong, then $n(n + 1) = r^2$ for some $n \in \mathbb{Z}^+$. The last statement is equivalent to $(2n + 1)^2 = (2r)^2 + 1$, which is impossible, since no two nonzero integer squares can differ by 1. A similar argument holds for 2-oblong numbers.

We can now state and prove the main result of this article.

Theorem 2 *(i) Let k be an odd positive integer, $k \geq 3$, of the form $k = \delta\delta_1\delta_2$, where $\delta, \delta_1, \delta_2$ are positive integers such that $(\delta_1, \delta_2) = 1$ and $\delta_2 > \delta_1$. Then the square number $\delta^2(\delta_2^2 - \delta_1^2)^2/16$ is a k-oblong number,*

$$0_{k,n} = \frac{\delta^2(\delta_2^2 - \delta_1^2)^2}{16} \quad \textit{with} \quad n = \frac{-2k + \delta(\delta_1^2 + \delta_2^2)}{4}.$$

Conversely, if the square number r^2 is k-oblong, then it must be of the above form. That is, $r = \delta(\delta_2^2 - \delta_1^2)/4$ for some $\delta, \delta_1, \delta_2 \in \mathbb{Z}^+$ such that $\delta_2 > \delta_1$ $(\delta_1, \delta_2) = 1$ and $k = \delta\delta_1\delta_2$.

(ii) If $k = p^{\ell}$, where p is an odd prime, $\ell \in \mathbb{Z}^+$, then there are exactly ℓ square numbers which are also k-oblong. These are the numbers $0_{k,n} = p^{2(\ell-i)}(p^{2i}-1)^2/16$, where $n = p^{\ell-i}(p^i-1)^2/4$ for $i = 1, 2, \ldots, \ell$. In particular, when $\ell = 1$ (so that $k = p =$ odd prime), there is exactly one square number which is k-oblong, namely the number $(p^2-1)^2/16$.

Proof (i) Suppose first that k and n are as given. Note that, since $k \equiv \delta \equiv \delta_1 \equiv \delta_2 \equiv 1(\mathrm{mod}2)$, we must have $\pm 2k \equiv 2 \equiv \delta(\delta_1^2 + \delta_2^2)(\mathrm{mod}4)$. Then

$$\begin{aligned} 0_{k,n} &= n(n+k) \\ &= \left[\frac{-2k + \delta(\delta_1^2 + \delta_2^2)}{4}\right]\left[\frac{2k + \delta(\delta_1^2 + \delta_2^2)}{4}\right] \\ &= \frac{\delta^2(\delta_1^2 + \delta_2^2)^2 - 4k^2}{16} \\ &= \frac{\delta^2(\delta_1^2 + \delta_2^2)^2 - 4\delta^2\delta_1^2\delta_2^2}{16} \\ &= \left[\frac{\delta(\delta_2^2 - \delta_1^2)}{4}\right]^2, \end{aligned}$$

a square number.

Now let us prove the converse. Suppose that the square number r^2 is also k-oblong. We must have $0_{k,n} = n(n+k) = r^2$ for some $n \in \mathbb{Z}^+$. Hence,

$$n^2 + kn - r^2 = 0,$$

and so

$$n = \frac{-k + \sqrt{D}}{2} \quad \text{with} \quad D = k^2 + 4r^2. \tag{2}$$

Hence, $D = R^2$, where $R = 2n + k$ and

$$k^2 + (2r)^2 = R^2, \tag{3}$$

so that $(k, 2r, R)$ is a Pythagorean triple. By formulas (1) we must have $R = \delta(M^2 + N^2)$ and either $k = \delta(M^2 - N^2)$ and $2r = \delta(2MN)$; or vice versa. Since k is odd, the latter possibility is ruled out. Therefore,

$$k = \delta(M^2 - N^2), \qquad 2r = \delta(2MN), \qquad R = \delta(M^2 + N^2),$$
$$\text{for some positive integers } \delta, M, N \text{ such that } M > N, (M, N) = 1, M + N \equiv 1(\mathrm{mod}2). \tag{4}$$

Let $\delta_2 = M + N$ and $\delta_1 = M - N$; then $M = (\delta_1 + \delta_2)/2$, $N = (\delta_2 - \delta_1)/2$. Also, δ_1 and δ_2 are both odd integers, with $\delta_2 > \delta_1$ and $(\delta_1, \delta_2) = 1$. (This last condition easily follows from the conditions on M and N found in (4).)

From (4), $k = \delta(M^2 - N^2) = \delta(M-N)(M+N) = \delta\delta_1\delta_2$. Also,

$$R = \delta\left[\left(\frac{\delta_2 + \delta_1}{2}\right)^2 + \left(\frac{\delta_2 - \delta_1}{2}\right)^2\right] = \frac{\delta(\delta_1^2 + \delta_2^2)}{2}, \tag{5}$$

and

$$2r = \delta(2MN) = 2\delta\left(\frac{\delta_2 + \delta_1}{2}\right)\left(\frac{\delta_2 - \delta_1}{2}\right),$$

whence

$$r = \frac{\delta(\delta_2^2 - \delta_1^2)}{4}.$$

Combining (2), (3), and (5) yields

$$n = \frac{-k + \sqrt{D}}{2} = \frac{-k + R}{2} = \frac{-k + (\delta(\delta_1^2 + \delta_2^2))/2}{2} = \frac{-2k + \delta(\delta_1^2 + \delta_2^2)}{4}.$$

The proof is complete.

(ii) When $k = p^\ell$, then $p^\ell = \delta\delta_1\delta_2$, with $(\delta_1, \delta_2) = 1$, $\delta_1 < \delta_2$ (from part (i)), which implies $\delta_1 = 1$, $\delta_2 = p^i$, and $\delta = p^{\ell-i}$, for some positive integer $i \in \{1, 2, \ldots, \ell\}$. By part (i) it follows that there are exactly ℓ square numbers which are also k-oblong, these being the numbers $0_{k,n} = p^{2(\ell-i)}\left(p^{2i} - 1\right)^2/16$ for $i = 1, 2, \ldots, \ell$, where

$$\begin{aligned} n &= \frac{-2k + \delta(\delta_1^2 + \delta_2^2)}{4} \\ &= \frac{-2p^\ell + p^{\ell-i}(p^{2i} + 1)}{4} \\ &= \frac{p^{\ell-i}(p^i - 1)^2}{4}. \end{aligned}$$

Corollary 1 *If k is an odd natural number, then there are finitely many square numbers which are also k-oblong.*

The next result, theorem 3, simply offers a counting formula for the number of square numbers which are also k-oblong when k is an odd square free integer.

Theorem 3 *Let $k = p_1 \cdot p_2 \cdot \cdots \cdot p_\ell$; $p_1, p_2, p_3, \ldots, p_\ell$ are distinct odd primes. There are precisely $N = (3^\ell - 1)/2$ k-oblong numbers which are also square.*

Proof Choose $\delta = 1$. There are 2^ℓ ways of partitioning the set $\{p_1, p_2, \ldots, p_\ell\}$ into two subsets, so there are $2^{\ell-1}$ choices for δ_1, δ_2 (since $\delta_2 > \delta_1$, so each choice of δ_1, δ_2 comes from two partitions). Next choose $\delta = p_i$, giving $\binom{\ell}{1}$ such choices for δ. Then, for each such choice of δ, there are $2^{\ell-2}$ choices for δ_1, δ_2. After that, choose $\delta = p_i p_j$, giving $\binom{\ell}{2}$ such choices for δ; and, hence, for each such δ, $2^{\ell-3}$ choices for δ_1, δ_2. Continuing this way, we find that

$$\begin{aligned} N &= \binom{\ell}{0}2^{\ell-1} + \binom{\ell}{1}2^{\ell-2} + \binom{\ell}{2}2^{\ell-3} + \ldots + \binom{\ell}{\ell-1}2^0 \\ &= \tfrac{1}{2}\left[(1+2)^\ell - 1\right] \\ &= \frac{3^\ell - 1}{2}. \end{aligned}$$

Note

For $\ell = 1$, we have $k = p$ and $N = (3-1)/2 = 1$, as already remarked in theorem 2(ii). When $\ell = 2$, we have $k = p_1 p_2$ and $N = (9-1)/2 = 4$; for $\ell = 3$, we have $k = p_1 p_2 p_3$ and $N = (27-1)/2 = 13$.

Remark Unlike the case of k-oblong numbers, of which only finitely many are also square numbers, there are infinitely many triangular numbers which are also square. This can be seen quite readily:

$$t_n = r^2 \quad \Longleftrightarrow \quad \frac{n(n+1)}{2} = r^2 \quad \Longleftrightarrow \quad n^2 + n - 2r^2 = 0.$$

The last quadratic equation will have a positive integer solution if and only if the discriminant D is a perfect square: $D = K^2$; $1+8r^2 = K^2$; $K^2 - 2(2r)^2 = 1$; so that $(K, 2r)$ is a solution to the well-known Pell equation $x^2 - 2y^2 = 1$, which has infinitely many solutions, all generated from the initial solution $(x_0, y_0) = (3, 2)$ (all such solutions require that y is even and x is odd, as a congruence modulo 4 easily shows. For further details, see reference 3. Also above, n is given by $n = (-1 + K)/2$.

References

1 D. M. Burton, *The History of Mathematics, An Introduction*, 6th edn. (McGraw-Hill, 2007).

2 L. E. Dickson, *History of the Theory of Numbers*, Vol. II (AMS Chelsea Publishing, Providence, Rhode Island, 1992), p. 803.

3 W. Sierpinski, *Elementary Theory of Numbers* (Warsaw, Poland, 1964), p. 480.

Konstantine Zelator *is a native of Athens, Greece. He recieved a diploma in civil engineering from the National Technical University of Athens in 1978. In 1979, he went to the US as a graduate student. He has taught at various universities in the US. His main mathematical interests are elementary number theory and diophantine problems in geometry; also the history of number theory. Currently, he is a visiting Assistant Professor at the University of Toledo in Ohio.*

$$\begin{aligned}
9 \times 9 + 7 &= 88,\\
98 \times 9 + 6 &= 888,\\
987 \times 9 + 5 &= 8888,\\
9876 \times 9 + 4 &= 88\,888,\\
98\,765 \times 9 + 3 &= 888\,888,\\
987\,654 \times 9 + 2 &= 8\,888\,888,\\
9\,876\,543 \times 9 + 1 &= 88\,888\,888,\\
98\,765\,432 \times 9 + 0 &= 888\,888\,888,\\
987\,654\,321 \times 9 + (-1) &= 8\,888\,888\,888.
\end{aligned}$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Relative Arithmetic

PAUL FJELSTAD, GUIDO LASTERS and DAVID SHARPE

In a previous article (see reference 1), it was shown how the usual rule for addition of real numbers has to be modified in Einstein's relativity theory to accommodate the fact that speeds greater than c, the speed of light, are impossible. Thus, for example, we cannot add $\frac{2}{3}c$ and $\frac{3}{4}c$ to give $\frac{17}{12}c$. For real numbers in the interval $(-c, c)$, i.e. lying strictly between $-c$ and c (allowing for negative velocities), the new rule for addition is

$$a +_c b = \frac{a+b}{1+ab/c^2},$$

where the addition and multiplication on the right-hand side are the usual addition and multiplication of real numbers. We refer to this as *relative addition*. Now,

$$\tfrac{2}{3}c +_c \tfrac{3}{4}c = \frac{\frac{2}{3}c + \frac{3}{4}c}{1+\frac{1}{2}} = \tfrac{17}{18}c.$$

In fact, if $-c < a, b < c$, then $-c < a +_c b < c$, so that this defines a binary operation on the interval $(-c, c)$. We also showed or pointed out in reference 1 that

$$a_1 +_c (a_2 +_c a_3) = (a_1 +_c a_2) +_c a_3 \quad \text{for all } a_1, a_2, a_3 \in (-c, c),$$

i.e. $+_c$ is *associative*, that

$$a +_c b = b +_c a \quad \text{for all } a, b \in (-c, c),$$

i.e. $+_c$ is *commutative*, that

$$a +_c 0 = a \quad \text{for all } a \in (-c, c),$$

i.e. 0 is the *zero element*, and that

$$a +_c (-a) = 0 \quad \text{for all } a \in (-c, c),$$

i.e. $-a$ is the *negative* of a for this binary operation. This is all summarized by saying that $((-c, c), +_c)$ is an *abelian group*.

We could have done this more simply had we spotted the close connection between the formula for relative addition and a well-known formula. We recall the function 'tanh' defined by

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

whose graph is shown in figure 1, so that $-1 < \tanh x < 1$ for all real numbers x. It is a routine matter to check that

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

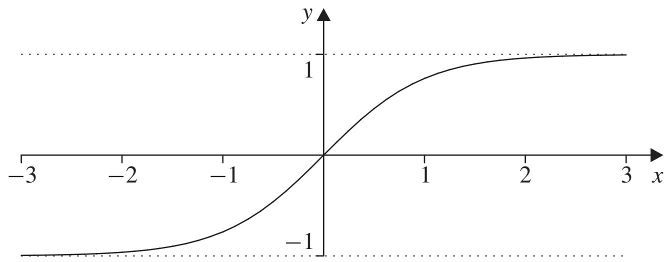


Figure 1 The graph of $y = \tanh x$.

for real numbers x and y, which has a striking resemblance to the formula for relative addition. In fact, if we define the function

$$f\colon \mathbb{R} \to (-c, c)$$

by

$$f(x) = c \tanh \frac{x}{c},$$

we obtain a one–one or bijective mapping from $\mathbb{R}$ to $(-c, c)$; the inverse mapping

$$f^{-1}\colon (-c, c) \to \mathbb{R}$$

is given by

$$f^{-1}(a) = c \tanh^{-1} \frac{a}{c},$$

for $a \in (-c, c)$. Now, for $x, y \in \mathbb{R}$,

$$\begin{aligned} f(x+y) &= c \tanh\left(\frac{x+y}{c}\right) \\ &= c\frac{\tanh(x/c) + \tanh(y/c)}{1 + \tanh(x/c)\tanh(y/c)} \\ &= \frac{c\tanh(x/c) + c\tanh(y/c)}{1 + (c\tanh(x/c))(c\tanh(y/c))/c^2} \\ &= f(x) +_c f(y), \end{aligned}$$

so the usual addition of real numbers corresponds to the relative addition in $(-c, c)$ under the bijective mapping f. We say that f is an *isomorphism* between the two groups; one is a copy of the other.

We could more easily have used f to *define* the relative addition of a and b in $(-c, c)$ by transferring a and b back into $\mathbb{R}$ under f^{-1}, adding them in $\mathbb{R}$, and then transferring them back into $(-c, c)$ under f. Specifically, we could *define*

$$\begin{aligned} a +_c b &= f(f^{-1}(a) + f^{-1}(b)) \\ &= c \tanh \frac{1}{c}\left(c \tanh^{-1} \frac{a}{c} + c \tanh^{-1} \frac{b}{c}\right) \\ &= c\frac{a/c + b/c}{1 + (a/c)(b/c)} \\ &= \frac{a+b}{1 + ab/c^2}, \end{aligned}$$

as before, and this well automatically transfer the abelian group properties of $(\mathbb{R}, +)$ to $((-c, c), +_c)$; we did not need to check them at all!

A word about the mapping f. It would have worked with the slightly simpler formula $f(x) = c \tanh x$. The reason for the formula $f(x) = c \tanh(x/c)$ becomes apparent if we keep x fixed and let $c \to \infty$, as in the classical as opposed to the relativistic situation. We can use l'Hôpital's rule to give, for fixed x,

$$\begin{aligned}\lim_{c\to\infty} c \tanh \frac{x}{c} &= \lim_{c'\to 0} \frac{\tanh c'x}{c'} \\ &= \lim_{c'\to 0} \frac{x \operatorname{sech}^2 c'x}{1} \\ &= x,\end{aligned}$$

so that, as $c \to \infty$, the mapping f tends to the identity mapping and relative addition tends to the usual addition in $\mathbb{R}$.

But there is also multiplication on $\mathbb{R}$, which can likewise be transferred to $(-c, c)$ by f. Specifically, for $a, b \in (-c, c)$, we define *relative multiplication* by

$$\begin{aligned}a \cdot_c b &= f(f^{-1}(a) f^{-1}(b)) \\ &= c \tanh\left(\frac{(c \tanh^{-1}(a/c))(c \tanh^{-1}(b/c))}{c}\right) \\ &= c \tanh\left(c \tanh^{-1} \frac{a}{c} \tanh^{-1} \frac{b}{c}\right).\end{aligned}$$

Now all the properties of addition *and* multiplication in $\mathbb{R}$ can be transferred to $(-c, c)$. The multiplication is commutative and associative, is distributive over addition, there is an identity element

$$1_c = f(1) = c \tanh \frac{1}{c},$$

and every nonzero element has an inverse, the inverse of $a \neq 0$ being

$$f\left(\frac{1}{f^{-1}(a)}\right) = c \tanh\left(\frac{1}{c^2 \tanh^{-1}(a/c)}\right).$$

This is all summarized by saying that $((-c, c), +_c, \cdot_c)$ is a *field* which is isomorphic to the field of real numbers under the isomorphism f. We note that

$$\begin{aligned}\lim_{c\to\infty} 1_c &= \lim_{c\to\infty} c \tanh \frac{1}{c} \\ &= \lim_{c\to\infty} \frac{\tanh(1/c)}{1/c} \\ &= \lim_{c\to\infty} \frac{-(1/c^2) \operatorname{sech}^2(1/c)}{-1/c^2} \\ &= 1,\end{aligned}$$

as expected.

It must be admitted that the definition of relative multiplication in $(-c, c)$, namely

$$a \cdot_c b = c \tanh\left(c \tanh^{-1} \frac{a}{c} \tanh^{-1} \frac{b}{c} \right),$$

does not have the transparency of the relative addition. A little manipulation gives

$$a \cdot_c b = c \tanh\left(\frac{c}{4} \ln \frac{c+a}{c-a} \ln \frac{c+b}{c-b} \right),$$

which is even less transparent. For example,

$$\tfrac{2}{3}c \cdot_c \tfrac{3}{4}c = c \tanh\left(\frac{c}{4} \ln 5 \ln 7 \right).$$

But you cannot have everything!

Reference

1 Guido Lasters and David Sharpe, From squares to circles by courtesy of Einstein, *Math. Spectrum* **38** (2005/2006), pp. 51–55.

Before retiring, ***Paul Fjelstad*** *was a tutor at St. Olaf College in Northfield, MN, USA, which has an emphasis on creative, independent study.*

Guido Lasters *teaches at Redingenhof School, Leuven, Belgium; he is interested in basic mathematics in a broad context.*

David Sharpe*, although retired, still teaches at the University of Sheffield, tutors for the Open University, and is Editor of Mathematical Spectrum.*

1089

$$\begin{array}{r} 832 \\ -238 \\ \hline 594 \\ +495 \\ \hline 1\,089 \end{array} \qquad \begin{array}{r} 726 \\ -627 \\ \hline 099 \\ +990 \\ \hline 1\,089 \end{array}$$

$$1089 \times 9 = 9801,$$

$$\begin{array}{lll} 1089 \times 2 = 2178, & 1089 \times 3 = 3267, & 1089 \times 4 = 4356, \\ 1089 \times 8 = 8712, & 1089 \times 7 = 7623, & 1089 \times 6 = 6534. \end{array}$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Lottery Combinatorics

IAN MCPHERSON and DEREK HODSON

The chance of landing the National Lottery jackpot (or a share of it) by correctly picking six numbers from the set 1 to 49 is about one in 14 million (13 983 816 to be exact). The probability of such a win would be considered negligible by any sensible mathematician, but nevertheless, in the draws on the 6th, 9th, 13th, and 16th of December 2006, a total of nine people drew six correct numbers. To illustrate the organisers' judicious arrangement of the odds we can note that were the Lotto task to pick seven numbers from 50, the chances of winning a jackpot deteriorate to one in 99.8844 million, while picking five numbers from 48 improves the odds to one in 1.712 304 million.

The glitzy televised draw utilises a fairly primitive system of tumbling coloured balls, amounting to, basically, a reassuringly transparent and fair raffle. The draw is claimed to be perfectly random and many checks are in place to ensure that this is so. However, in connection with a related matter, I have frequently been mildly surprised by the regular appearance of 'runs', e.g. two or more adjacent integers. The following draws were recorded towards the end of the year 2006:

22nd November : **14**, **15**, 24, 31, 45, 49.

18th November : 1, **19**, **20**, 32, 42, 46.

11th November : **23**, **24**, 28, **33**, **34**, 41.

28th October : **4**, **5**, **6**, 12, 20, 29.

Are these kind of runs remarkable, or can we expect them to occur fairly frequently? If so, is it a good policy to arrange for, say, a run of two when choosing numbers for your occasional dabble? What *is* the probability that when r numbers are picked randomly from the integers 1 to n, two integers will be adjacent as in the first two examples above? Extending the idea further, what is the probability of a treble run, as in the October example, or of a quadruple, a quintuple, a sextuple, two distinct doubles (as on 11th November), or no runs at all? Regarding runs, there are, in fact, eleven possible outcomes when drawing six numbers from 49, as summarised later in table 3. The total number of possible draws is the number of different ways of picking r items from n with no replacement, i.e.

$$ {}^nC_r = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{1 \times 2 \times \cdots \times r} = \frac{n!}{r!\,(n-r)!}, \qquad (1) $$

the result for $n = 49$, $r = 6$ being given above. For the general case, the derivation of the frequency of each of the possible outcomes (and, hence, its theoretical probability) forms an interesting combinatorial exercise.

The various types of combination listed in table 3 appear to vary widely, e.g. no runs, one run of two + four singles, two runs of two + two singles, a treble + double + single, etc., but rather surprisingly a unified approach can be applied to all cases, resulting in a general formula which can be applied to any draw of r integers from n. It will be clear that depending on the values of these parameters the number of possible outcomes may be more or less than the eleven for the lottery.

Derivation

Consider a combination of r numbers drawn from n:

$$a_1, a_2, a_3, \ldots, a_r.$$

We may define a 'group' from these numbers to be either a single (nonconsecutive) number or a subset of two or more consecutive numbers. Let the number of groups within the combination be g; let the number of different types of groups (i.e. singles, runs of two, runs of three, etc.) be d and the number of groups of type i be g, where $i = 1, 2, \ldots, d$. Note that $g_1 + g_2 + \cdots + g_d = g$.

For each configuration of g groups with d types present there are P possible permutations where:

$$P = \frac{g!}{g_1!\, g_2! \ldots g_d!}. \tag{2}$$

For example, table 1 shows the possible configurations for $r = 4$.

The number of combinations for a particular configuration (ignoring permutations for the moment) will be shown to be:

$$C = {}^{n-r+1}C_g.$$

To see this, first consider a combination of r nonconsecutive numbers (i.e. no runs) from the set 1 to n:

$$a_1, a_2, a_3, \ldots, a_r.$$

This combination may be regarded as being composed of r groups of a single type (thus, $g = r$ and $d = 1$). Because the numbers are nonconsecutive, they can be compressed to an equivalent set of r numbers:

$$a_1, a_2 - 1, a_3 - 2, \ldots, a_r - (r - 1),$$

selected from a reduced set of $n - (r - 1)$ numbers. This gives the possible number of combinations in this case as

$$C = {}^{n-r+1}C_r.$$

Table 1

Configuration	No. of types of groups (d)	Nos. of each type (g_i)	Total no. of groups (g)	Permutations: S - Single D - Run of two T - Run of three Q - Run of four
Four singles	1	$g_1 = 4$	4	1 [SSSS]
Two singles, one run of two	2	$g_1 = 2$, $g_2 = 1$	3	3 [SSD, SDS, DSS]
One single, one run of three	2	$g_1 = 1$, $g_2 = 1$	2	2 [ST, TS]
Two runs of two	1	$g_1 = 2$	2	1 [DD]
One run of four	1	$g_1 = 1$	1	1 [Q]

Next consider the case of a combination of r from n where there are exactly two consecutive numbers a_i and a_{i+1}:

$$a_1, a_2, a_3, \ldots, a_i, a_{i+1}, \ldots, a_r.$$

Now we have $(r-1)$ groups. The number of combinations of such a set of r numbers from n is equivalent to selecting $(r-1)$ nonconsecutive numbers from $(n-1)$, or selecting an equivalent compressed set of $(r-1)$ from a reduced set of size $(n-1)-(r-2) = n-r+1$. Thus,

$$C = {}^{n-r+1}C_{r-1}.$$

Continuing in this manner it may be noted that for a particular value of r, as the incidence of consecutive numbers increases the number of groups g per combination decreases accordingly. In general, if there are k instances of 'consecutiveness', the number of groups will be $g = r-k$. In this case, the number of combinations of such a set of r numbers from n is equivalent to selecting $r-k$ nonconsecutive numbers from $n-k$ or selecting an equivalent compressed set of $r-k$ from a reduced set of size

$$(n-k)-(r-1-k) = n-r+1,$$

giving

$$C = {}^{n-r+1}C_{r-k} = {}^{n-r+1}C_g.$$

Each of these combinations will have a particular configuration of groups and so must be multiplied by the permutation formula (1) to give the total number of selections, N, for that configuration, where N is given by:

$$N = \frac{g!}{g_1!\,g_2!\ldots g_d!} \times {}^{n-r+1}C_g. \tag{3}$$

Extending the previous example, take the draw of $r = 4$ to be from the set $n = 1$ to 7. The number of combinations in each category is given by (2) with $n-r+1 = 4$. All possible combinations are shown in table 2. As a check of the total given in table 2, using (1),

$${}^{7}C_4 = 35.$$

For the National Lottery, the parameters are $n = 49$, $r = 6$, giving the combinatorial frequencies for all eleven categories as shown in table 3, below.

From these figures, the mild surprise on observing a run of two was obviously unjustified, since it occurs on average 38.8% of the time and is in fact the second most likely event, after the 'all unattached', which should occur on average in 50.5% of the draws. The single run of three, as seen on 28th October 2006, is much less likely, appearing on 3.9% occasions. A run of six is the most unlikely result, with a probability of 0.000 003 1. Going by these calculations and assuming two draws per week we should expect to see the 'all unattached' event about once per week (52.5 times per year) and one double between 40 and 41 times per year. A run of six, turning up only once every 3056 years, should cause some excitement among statisticians, since its appearance would be almost as infrequent as the Hale-Bop comet (about every 4000 years). The run of five, on the other hand, should turn up on average once every 71 years (as would the combined runs of four and two), i.e. marginally more frequent than Halley's comet, which comes round once every 76 years.

Table 2

Configuration	No. of types of groups (d)	Nos. of each type (g_i)	Incidence of consecutive nos. (k)	Total no. of groups (g)	N	Possible combinations
Four singles	1	$g_i = 4$	0	4	1	1357
Two singles, one run of two	2	$g_1 = 2$, $g_2 = 1$	1	3	12	1356 1346 1246 1367 1347 1247 1467 1457 1257 2467 2457 2357
One single, one run of three	2	$g_1 = 1$, $g_2 = 1$	2	2	12	1345 1235 1456 1236 1567 1237 2456 2346 2567 2347 3567 3457
Two runs of two	1	$g_1 = 2$	2	2	6	1245 1256 1267 2356 2367 3467
One run of four	1	$g_1 = 1$	3	1	4	1234 2345 3456 4567
				Total	35	

How do the above predictions match up to the actual figures? We cannot test for the rarer combinations since not enough time has yet elapsed, but we can count the occurrences of the more frequent events over a reasonably representative period. The Lotto results (Wednesday and

Table 3 National Lottery predicted frequencies and probabilities.

Configuration	No. of types of groups (d)	Nos. of each type (g_i)	Total no. of groups (g)	N	Probability (N/13 983 816)
Six singles	1	$g_1 = 6$	6	7 059 052	0.504 801 6
Four singles + one run of two	2	$g_1 = 4$, $g_2 = 1$	5	5 430 040	0.388 308 9
Three singles + one run of three	2	$g_1 = 3$, $g_2 = 1$	4	543 004	0.038 830 9
Two singles + one run of four	2	$g_1 = 2$, $g_2 = 1$	3	39 732	0.002 841 3
Single + run of five	2	$g_1 = 1$, $g_2 = 1$	2	1892	0.000 135 3
One run of six	1	$g_1 = 1$	1	44	0.000 003 1
Two runs of two + two singles	2	$g_1 = 2$, $g_2 = 2$	4	814 506	0.058 246 3
Three runs of two	1	$g_1 = 3$	3	13 244	0.000 947 1
Run of three + run of two + single	3	$g_1 = 1$, $g_2 = 1$, $g_3 = 1$	3	79 464	0.005 682 6
Run of four + run of two	2	$g_1 = 1$, $g_2 = 1$	2	1892	0.000 135 3
Two runs of three	1	$g_1 = 2$	2	946	0.000 068
Check totals				13 983 816	1.000 000 4

Table 4 A survey of all 1028 Lotto draws from 19th November 1994 to 29th October 2005.

Type of combination	Frequency of that combination	*Actual* relative frequency	*Predicted* relative frequency	Predicted average period between events
Six singles	555	0.539 883 3	0.504 801 6	∼ 1 week
Four singles + one run of two	384	0.373 540 9	0.388 308 9	∼ 9 days
Three singles + one run of three	34	0.033 073 9	0.038 830 9	∼ 13 weeks
Two singles + one run of four	0	0	0.002 841 3	∼ 3 years
Single + run of five	0	0	0.000 135 3	∼ 71 years
One run of six	0	0	0.000 003 1	∼ 3056 years
Two runs of two + two singles	50	0.048 638 1	0.058 246 3	∼ 9 weeks
Three runs of two	0	0	0.000 947 1	∼ 10 years
Run of three + run of two + single	5	0.004 863 8	0.005 682 6	∼ 2 years
Run of four + run of two	0	0	0.000 135 3	∼ 71 years
Two runs of three	0	0	0.000 006 8	142 years
Check totals	1028	1	1.000 000 4	

Saturday, all machines) from 19th November 1994 to 29th October 2005 have been surveyed, making a total of 1028 draws, a reasonable sample on which to test the statistics. The results are shown above in table 4, which lists the actual frequency of each category of combination, the actual relative frequency, the predicted relative frequency, and also in each case the predicted average period between events, i.e. between occurrences of that combination.

The survey figures compare reasonably well with the theory, supporting the presumption of randomness. To answer the question 'is it a good idea to have a run of two on your ticket?', it is not a bad idea since it occurs 38.8% of the time, but the better option would obviously (?) be to have your numbers 'all unattached'. Even more obviously it would be a bad idea to have a run of six.

Ian McPherson *is a former senior lecturer in physics at Glasgow Caledonian University and is now an associate lecturer with the Open University, Faculty of Maths & Computing.*

Derek Hodson *is a lecturer in mathematics at Glasgow Caledonian University. His mathematical interests include applications within both the physical and bio sciences. Outside of mathematics, he enjoys reading contemporary literature and listening to folk and classical music.*

$$
\begin{aligned}
1^3 &= 1 = 1,\\
2^3 &= 2 + 6 = 3 + 5 = 3 + 5,\\
3^3 &= 3 + 9 + 15 = 6 + 9 + 12 = 7 + 9 + 11,\\
4^3 &= 4 + 12 + 20 + 28 = 10 + 14 + 18 + 22 = 13 + 15 + 17 + 19,\\
5^3 &= 5 + 15 + 25 + 35 + 45 = 15 + 20 + 25 + 30 + 35\\
&= 21 + 23 + 25 + 27 + 29,
\end{aligned}
$$

and so on.

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

100

$$
\begin{aligned}
100 &= 1^3 + 2^3 + 3^3 + 4^3\\
&= 111 - 11\\
&= 3 \times 33 + (3 \div 3)\\
&= 5 \times (5 + 5 + 5 + 5)\\
&= 1 + 2 + 3 + 4 + 5 + 6 + 7 + (8 \times 9)\\
&= 12 + 20 + 4 + 64
\end{aligned}
$$

and $12 + 4 = 20 - 4 = 4 \times 4 = 64 \div 4$.

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

On the Natural Exponential Function

ROBERT P. GOVE and JAN RYCHTÁŘ

1. Introduction

The purpose of this article is to provide a short, self-contained exposition on the natural exponential function e^x starting from an accessible definition to the derivation of all of its properties. The natural exponential function is perhaps the most important function in mathematics. Its applications range from mathematics, statistics, and economics to physics and other natural sciences. Possibly because of its wide use and importance, there is a variety of available definitions and approaches to the function. We will discuss the most common definitions, show their strengths and weaknesses and at the end we will select one unified approach that we think is the most appropriate for mathematicians as well as those who only need to learn the basics about e^x.

Wikipedia, see reference 9, lists the following five definitions at the time of writing:

(D1) $e^x := \lim_{n\to\infty}(1 + x/n)^n$.

(D2) e^x is the inverse of $\ln x := \int_1^x (1/s)\, ds$.

(D3) $e^x := \sum_{n=0}^{\infty} x^n/n!$.

(D4) e^x is the only continuous function $f(x)$ satisfying $f(a+b) = f(a)f(b)$ for all $a, b \in \mathbb{R}$ and $f(1) = \sum_{n=0}^{\infty} 1/n!$.

(D5) e^x is the solution of the differential equation $y' = y$ satisfying $y(0) = 1$.

Every single one of the above definitions constitutes one possible approach to the function. From a mathematical point of view, all definitions define the same function and, hence, it does not really matter which one we choose. However, choosing the 'right' approach can significantly affect and potentially improve our understanding of the function and mathematics in general.

We have to say right away that there is, very likely, no best approach. As we will discuss in the next section, each definition has its strengths and weaknesses. What seems to be right for one person may sound completely outrageous for another. Ultimately, the best approach will vary based on the circumstances. The current article argues that if somebody does not know much mathematics, yet for various reasons needs to learn and understand a great deal about e^x because of its applications in economics and the natural sciences, then definition (D5) together with the mathematical methods implicitly contained in this definition are the best way to start.

2. Strengths and weaknesses of different definitions

Definition (D1) contains the most primitive terms. Theoretically, we can understand this definition with minimal mathematical knowledge and background. Another advantage is that this definition of e^x can be naturally motivated by the example of compounding interests. The drawback of this definition is that the derivation of other properties from it is technical; see reference 4, pp. 51 and 133.

In calculus classes, the exponential function is usually defined by (D2); see, e.g. references 1 (p. 428), 7 (p. 425), and 8 (p. 331). Compared to the definition (D1), this approach requires nontrivial knowledge and understanding of definite integrals. Most importantly, there does not seem to be a natural motivation for this definition. Probably the only reason why it is so widely spread is that it fits the current curriculum with the least resistance.

Definition (D3) and the approach to functions through series is universal. One can define other functions in the same way (for example, $\sin x := \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$, or $\cos x := \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$). Moreover, the formula can also be used to define exponentials of matrices (and linear operators in general); see reference 3. These are the reasons why this definition is often used in real and functional analysis; see reference 6, p. 178. However, the formula requires a good knowledge of series and, thus, this definition is not suitable for relative beginners in mathematics. Also, despite its universality, the formula by itself does not provide sufficient motivation.

The property $e^{a+b} = e^a e^b$, which is the core of definition (D4), is the reason why e^x is an important function. Definition (D4) is the most abstract yet the most beautiful definition of e^x. We will see below how this property demonstrates itself by the lack of memory in many natural processes. More importantly, this definition is elegant and its beauty (the mere fact that it is equivalent to all other definitions is beautiful) constitutes the essence of mathematics.

If the authors were not primarily interested in a simple use of e^x in biology and other natural sciences through calculus, but instead aimed for deeper applications in probability and mathematical modelling, they would argue that (D4) is the best way to approach e^x. However, for the rest of the article we will advocate why one should choose (D5) for the definition of the exponential function.

3. e^x as a unique solution of $y' = y,\ y(0) = 1$

Let us start from the beginning.

Definition 1 The natural exponential function e^x is defined to be the only function $y = y(x)$ that satisfies the following two conditions:

(E1) $y'(x) = y(x)$ for all $x \in (-\infty, \infty)$, and

(E2) $y(0) = 1$.

The definition does not really say that such a function exists and is unique. Similar problems arise with other definitions of e^x, yet they can be fixed in a relatively elementary way. Here one needs an advanced tool, namely Picard's theorem (see reference 2, p. 110), to guarantee the existence and uniqueness. A much weaker version of the theorem which is enough for our purposes is stated below.

Theorem 1 *(Picard.) For any numbers k and c_0 there exists a unique function $y(x)$ satisfying*

- $y'(x) = ky(x)$ *for all* $x \in (-\infty, \infty)$, *and*
- $y(0) = c_0$.

We can also argue that, on top of Picard's theorem, we also use differential equations implicitly contained in the definition. Still, this definition is understandable to anybody with the knowledge of a derivative; and there are significant advantages of this definition: (1) a natural motivation and (2) an easy way to derive other properties of e^x.

4. Motivational examples

Let us consider a savings account such that the interest is added to the principal at every moment, and from the moment the interest is added, the account also accrues interest on that interest. Thus, if the interest is 100% per a certain unit of time and $\mathrm{d}x$ denotes a very small portion of time, we have

$$y(x+\mathrm{d}x) \approx y(x) + y(x)\,\mathrm{d}x \qquad \text{for all } x,$$

where $y(x)$ denotes the account balance at time x. This means that

$$\frac{y(x+\mathrm{d}x) - y(x)}{\mathrm{d}x} \approx y(x)$$

and, consequently, it yields our equation $y'(x) = y(x)$. We note that the above derivation is not rigorous but motivational only.

In general, if $y(x)$ represents the amount of a certain quantity at time x, the property $y' = y$ means that the rate of growth is proportional to its size. The savings account example illustrates the common knowledge that money (either in the form of savings or debt) behaves this way. Someone with a background in the natural sciences also thinks of uninhibited growth and/or radioactive decay.

One important feature of a savings account is the lack of the memory within the system. A dollar does not remember whether it was added as a principal or interest and when it happened; it still yields the same amount of interest as it would have if it had been in the principal from the beginning. Also, the future balance of the account depends only on the current balance.

The lack of memory is responsible for the property

$$e^{a+b} = e^a e^b.$$

Indeed, for simplicity assume that e^a is an integer. If we put \$1 into a savings account, in time $x = a$ the balance will be e^a. If we let the account grow for some additional time b and mentally track every single one of the e^a dollars, the new balance will be $e^a e^b$. However, we just had \$1 in that account for the total time $x = a + b$. Thus, the balance is e^{a+b} and, consequently, $e^{a+b} = e^a e^b$. The property of being memory-free is illustrated by the fact that one can 'forget' the past by 'restarting' the clock at time $x = a$.

The example of uninhibited growth of bacteria (see, e.g. reference 1, p. 605) is possibly the best motivation for a person with knowledge of biology. We just have to be careful, because, as always with mathematics, once it starts to touch real life, many of the idealistic assumptions of the mathematical model can easily be violated. To illustrate this point, we need to realize that considering only one bacterium at the beginning is not appropriate, since the system would behave in a discrete way. Starting with N bacteria, for very large N, and considering any collection of N bacteria to be 1 (colony) helps to take care of the discreteness (at least for the initial period of time), especially if we assume that the bacteria do not all split at the same time, but rather their splitting time is uniformly distributed. On the other hand, this idealization is not completely without memory, since once a bacterium splits, the two new bacteria are like identical twins with the same inner clocks; they split together at the same time (after which there would be four identical bacteria, etc.), i.e. the bacteria sort of remember the common ancestors.

5. Properties of e^x

Fact 1 $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

Proof Denote $y(x) = \lim_{n\to\infty}(1 + x/n)^n$. Then we have

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) \\ &= \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{n\to\infty} n\left(1+\frac{x}{n}\right)^{n-1}\frac{1}{n} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\left(1+\frac{x}{n}\right)^{-1} \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-1} \\ &= y(x). \end{aligned}$$

The second equality (where we interchanged the limit and differentiation) follows by reference 6, Theorem 7.17. Thus, we have $y'(x) = y(x)$. Moreover,

$$y(0) = \lim_{n\to\infty}\left(1+\frac{0}{n}\right)^n = 1.$$

Hence, by the uniqueness of the exponential function, $y(x) = e^x$ which is exactly what we wanted to prove.

Fact 2 e^x *is the inverse of* $\ln x := \int_1^x (1/s)\, ds$.

Proof Let $y(x)$ denote the inverse of $\ln x$. By the fundamental theorem of calculus (see reference 1, p. 403),

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Thus, by the theorem on differentiation of inverse functions (see reference 1, p. 249),

$$y'(x) = \frac{1}{1/y(x)} = y(x).$$

Because $\ln 1 = 0$, we get $y(0) = 1$ and, thus, by the uniqueness of the exponential function, $y(x) = e^x$ which we wanted to prove.

Fact 3 $e^x = \sum_{n=0}^{\infty} x^n/n!$.

Proof Denote $y(x) = \sum_{n=0}^{\infty} x^n/n!$. Then we have

$$y'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} n\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = y(x),$$

where the second equality holds because of the theorem on differentiation of power series; see reference 1, p. 704. Thus, $y'(x) = y(x)$; and since $y(0) = \sum_{n=0}^{\infty} 0^n/n! = 1$, we must have, by the uniqueness of the exponential function, $y(x) = e^x$.

Fact 4 $e^{a+b} = e^a e^b$ *for all* $a, b \in (-\infty, \infty)$.

Proof Fix $a \in (-\infty, \infty)$ and consider the function

$$y(x) = e^{a+x}.$$

Since

$$y'(x) = e^{a+x} = y(x), \qquad y(0) = e^a,$$

the uniqueness property provided by Picard's theorem yields

$$y(x) = e^a e^x,$$

since $e^a e^x$ is another function that shares the above properties. Putting $x = b$ gives the result.

Fact 5 *The natural exponential function* e^x *has the following properties.*

(i) e^x *is continuous and differentiable for all* x.

(ii) $e^x > 0$ *and* $e^{-x} = 1/e^x$ *for all* x.

(iii) e^x *is strictly increasing and concave up.*

(iv) $e^x \geq 1 + x$ *for all* x.

(v) $\lim_{x\to\infty} e^x = \infty$ *and* $\lim_{x\to-\infty} e^x = 0$.

Proof (i) Since e^x solves the equation $y' = y$, it must be differentiable. Moreover, every differentiable function is continuous (see, e.g. reference 1, p. 184).

(ii) If there is x such that $e^x \leq 0$, then, by the intermediate value theorem (see reference 1, p. 149), there must be x' such that $\exp(x') = 0$ (because $\exp(0) = 1 > 0$). It follows that

$$\begin{aligned} 1 &= e^0 \\ &= e^{x'+(-x')} \\ &= e^{x'} e^{-x'} \quad \text{(Fact 4)} \\ &= 0 \cdot e^{-x'} \\ &= 0, \end{aligned} \tag{1}$$

which is a contradiction. Hence, there is no x such that $e^x \leq 0$, i.e. $e^x > 0$ for all x. It follows from (1) that $e^{-x} = 1/e^x$.

(iii) Since, by definition and (ii),

$$(e^x)' = e^x > 0,$$

e^x is strictly increasing. Since, again by the definition and (ii),

$$(e^x)'' = \big((e^x)'\big)' = \big(e^x\big)' = e^x > 0,$$

e^x is concave up.

(iv) The tangent line to $y = e^x$ at $x = 0$ has the equation $y = \exp(0) + \exp'(0)x = 1 + x$. Since, by (iii), e^x is concave up, the graph of e^x must be above the tangent line and the inequality follows.

(v) The first part follows directly from (iv). Indeed,

$$\lim_{x\to\infty} e^x \geq \lim_{x\to\infty} (1 + x) = \infty.$$

Thus, by (ii),

$$\lim_{x\to-\infty} e^x = \lim_{x\to-\infty} \frac{1}{\exp(-x)} = 0,$$

which proves the second part.

6. Conclusion and discussion

We have defined e^x in a relatively elementary way as the unique solution of the initial value problem

$$y'(x) = y(x), \qquad y(0) = 1,$$

and showed how easily all properties of the exponential function follow. In particular, we have demonstrated the power of uniqueness. Indeed, the main idea behind most of the proofs was to guess a function, check its properties, and uniqueness guaranteed the rest.

Let us also mention that our approach is universal in some sense. It is relatively easy to see that the functions $\cos x$ and $\sin x$ can be defined in a very similar manner: as the real part and the imaginary part, respectively, of the unique solution of the initial value problem

$$y'(x) = iy(x), \qquad y(0) = 1,$$

and clearly we do not have to stop here.

There are drawbacks to our approach. The most obvious one is the necessity to use Picard's theorem—a deep statement behind most of the reasoning—which we used without a proof. We argue that this is not really a drawback, rather a standard trend in modern science. We see that trend even in mathematics. Very few people can completely or even partially understand Wiles' proof of Fermat's last theorem; see reference 10. Yet almost everybody understands the statement itself and its use. Moreover, mathematics nowadays is so broad and complex that the acceptance of a nontrivial statement without a proof is inevitable; confront reference 5.

In our opinion the more serious drawback is that with our approach the reader is literally shielded from the exposure to many beautiful mathematical concepts and ideas that would otherwise be needed to understand the exponential function in its full strength. We would be much happier if everybody had to study the exponential function from at least five different points of view, each one corresponding to a different definition and each one requiring a different approach. However, one has to be realistic. For us, as mathematicians at heart, it may be hard to accept the existence of people whose sole purpose is not to study mathematics. Yet, such people exist. And for those people, these few pages may indeed constitute everything they need to know about e^x. Yes, they will not learn much mathematics. This hopefully means they will not be scared too much and possibly even like mathematics by suddenly seeing it as something understandable and not boring. And this is exactly what we wanted to achieve. After all, is it not every mathematician's secret mission to convert as many people to be as devoted to mathematics as oneself?

Acknowledgements

The research was conducted as part of the project 'UBM/RUI: Using Collaborative Undergraduate Research to Train Students in Mathematics and Biology at The University of North Carolina at Greensboro'. The authors would like to thank the NSF for their funding support through the grant 0634182, and anonymous referees for valuable comments which have improved the quality of the paper.

References

1 H. Anton, I. Bivens and S. Davis, *Calculus Early Transcendentals*, 8th edn. (John Wiley, New York, 2005).
2 W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations*, 8th edn. (John Wiley, Hoboken, NJ, 2005).
3 N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory* (Interscience, New York, 1958).
4 R. Larson, R. Hostetler and B. H. Edwards, *Calculus I*, 4th edn. (Houghton Mifflin Company, 2007).
5 M. B. Nathanson, Desperately seeking mathematical truth, *Notices Amer. Math. Soc.* **55** (2008), p. 773.
6 W. Rudin, *Real and Complex Analysis*, 3rd edn. (McGraw-Hill, New York, 1987).
7 J. Steward, *Calculus, Early Transcendetals*, 5th edn. (Brook/Cole, Pacific Grove, 2003).
8 D. Varberg *et al.*, *Calculus*, 8th edn. (Prentice Hall, Upper Saddle River, NJ, 2000).
9 Available at http://en.wikipedia.org/wiki/Characterizations_of_the_exponential_function.
10 A. Wiles, Modular elliptic curves and Fermat's Last Theorem, *Ann. Math.* **141** (1995), pp. 443—551.

Robert Gove *is an undergraduate research assistant of* ***Jan Rychtář****.*

$$
\begin{aligned}
9^2 &= 81,\\
99^2 &= 9801,\\
999^2 &= 998\,001,\\
9999^2 &= 99\,980\,001,\\
99\,999^2 &= 9\,999\,800\,001,
\end{aligned}
$$

and so on.

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Another Geometric Vision of the Hyperbola

THOMAS J. OSLER

There are two standard methods that are often used in defining the hyperbola.

(a) The hyperbola is the locus of all points P whose distance from a fixed point F (the focus) divided by its distance to a fixed line D (the directrix) is a constant e (the eccentricity) greater than one, i.e.

$$\frac{PF}{PD} = e > 1.$$

(b) The hyperbola is the locus of all points P in which the difference of the distances from two fixed points (the foci F and F') is a constant, i.e.

$$PF - PF' = c.$$

These definitions are related to the *conic sections* as described on page 5 of reference 1.

In a recent article (see reference 2) an unusual method of constructing the hyperbola was shown that is based on the asymptotes of the curve. In this short article we present another unusual geometric method of constructing the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

In figure 1 we see a portion of a circle of radius a centered at the origin O of coordinates. The vertical line $x = b$ is also shown. Construct the ray OA making any angle θ with the x-axis and intersecting the line $x = b$ at B and intersecting the circle at T. From the point T,

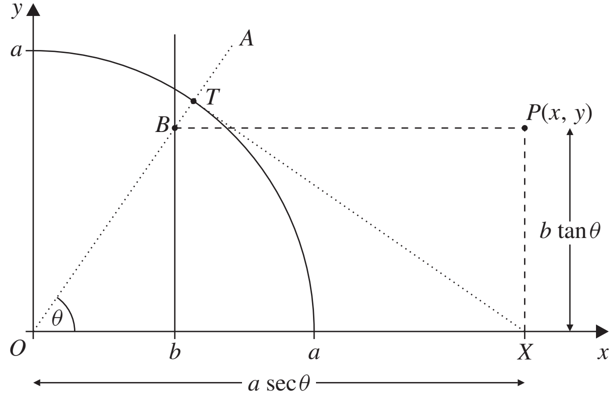


Figure 1

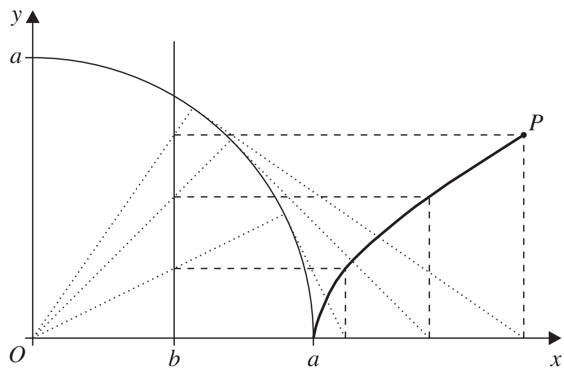


Figure 2

construct a tangent to the circle intersecting the x-axis at the point X. Construct a horizontal line through B and a vertical line through X meeting at P. We will show that the point P is on the hyperbola (1). By allowing θ to vary from 0 to 2π we generate the entire hyperbola described by (1). In figure 2 we see the portion of the hyperbola from the vertex at $(a, 0)$ to the point P being generated by this method.

It is easy to justify this construction. Examining figure 1 we see that the coordinates of the point P are $x = a \sec\theta$ and $y = b \tan\theta$. From the identity

$$\sec^2\theta - \tan^2\theta = 1$$

we see at once that (1) is true.

References

1 D. A. Brannan, M. F. Esplen and J. J. Gray, *Geometry* (Cambridge University Press, 1999).

2 D. Grochowski and T. J. Osler, An asymptotic approach to constructing the hyperbola, *Math. Spectrum* **38** (2005/2006), pp. 113–115.

Tom Osler *is professor of mathematics at Rowan University. In addition to teaching university mathematics for the past 46 years, Tom has a passion for long distance running. He has been competing for the past 53 consecutive years. Included in his over 1950 races are wins in three national championships in the late 1960s at distances from 25 kilometers to 50 miles. He is the author of two running books.*

Olympics and Paralympics

An Olympic swimmer A and a Paralympic swimmer B start at opposite ends of the 50 m Olympic pool. A swims at 2 m/s and B at 1.4 m/s. They start swimming simultaneously, maintaining constant speeds. How long in time will it take for both swimmers to reach their starting points simultaneously?

Midsomer Norton, Bath, UK **Bob Bertuello**

Exploring Ideas for Improving the Convergence Rate in Gauss–Seidel Iteration

ROBERT COMBS, JON KUHL
and JENNIFER SWITKES

Consider the linear system

$$\begin{aligned} 10x_1 - x_2 &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ -2x_2 + 10x_3 &= 6, \end{aligned}$$

from reference 1, which has the exact solution $\boldsymbol{x} = [x_1, x_2, x_3]$, where $x_1 = \frac{473}{475} \approx 0.995\,789$, $x_2 = \frac{91}{95} \approx 0.957\,895$, and $x_3 = \frac{376}{475} \approx 0.791\,579$.

This system initiated a very lively class discussion this past summer, with student co-authors Jon and Robert ultimately contributing the ideas that led to this article.

One method for approximating solutions to our system is Jacobi iteration. In this method we solve the first equation in the system for x_1, the second equation for x_2, and the third equation for x_3, which for our example leads to

$$\begin{aligned} x_1 &= \tfrac{1}{10}x_2 + \tfrac{9}{10}, \\ x_2 &= \tfrac{1}{10}x_1 + \tfrac{1}{5}x_3 + \tfrac{7}{10}, \\ x_3 &= \tfrac{1}{5}x_2 + \tfrac{3}{5}. \end{aligned}$$

Next, we realize that this potentially represents a way to update an initial approximate solution $\boldsymbol{x}^{(0)} = [x_1^{(0)}, x_2^{(0)}, x_3^{(0)}]^\top$ to a new approximate solution $\boldsymbol{x}^{(1)} = [x_1^{(1)}, x_2^{(1)}, x_3^{(1)}]^\top$ by means of the iteration scheme shown below:

$$\begin{aligned} x_1^{(1)} &= \tfrac{1}{10}x_2^{(0)} + \tfrac{9}{10}, \\ x_2^{(1)} &= \tfrac{1}{10}x_1^{(0)} + \tfrac{1}{5}x_3^{(0)} + \tfrac{7}{10}, \\ x_3^{(1)} &= \tfrac{1}{5}x_2^{(0)} + \tfrac{3}{5}. \end{aligned}$$

Now, we generalize in order to update from a kth to a $(k+1)$th approximate solution:

$$x_1^{(k+1)} = \tfrac{1}{10}x_2^{(k)} + \tfrac{9}{10}, \tag{1a}$$

$$x_2^{(k+1)} = \tfrac{1}{10}x_1^{(k)} + \tfrac{1}{5}x_3^{(k)} + \tfrac{7}{10}, \tag{1b}$$

$$x_3^{(k+1)} = \tfrac{1}{5}x_2^{(k)} + \tfrac{3}{5}. \tag{1c}$$

For example, with $\boldsymbol{x}^{(0)} = [0, 0, 0]^\top$, we obtain

$$\boldsymbol{x}^{(1)} = \left[\tfrac{1}{10}(0) + \tfrac{9}{10},\ \tfrac{1}{10}(0) + \tfrac{1}{5}(0) + \tfrac{7}{10},\ \tfrac{1}{5}(0) + \tfrac{3}{5}\right]^\top = [0.9, 0.7, 0.6]^\top,$$

$$\boldsymbol{x}^{(2)} = \left[\tfrac{1}{10}(0.7) + \tfrac{9}{10},\ \tfrac{1}{10}(0.9) + \tfrac{1}{5}(0.6) + \tfrac{7}{10},\ \tfrac{1}{5}(0.7) + \tfrac{3}{5}\right]^\top = [0.97, 0.91, 0.74]^\top.$$

We can continue iterating in this manner. There is theory regarding convergence to the actual solution, and in this example the approximate solutions are indeed converging to the actual solution.

The Gauss–Seidel iteration method attempts to improve upon the convergence rate of the Jacobi method by updating values as soon as possible, as shown below:

$$x_1^{(k+1)} = \tfrac{1}{10}x_2^{(k)} + \tfrac{9}{10}, \tag{2a}$$
$$x_2^{(k+1)} = \tfrac{1}{10}x_1^{(k+1)} + \tfrac{1}{5}x_3^{(k)} + \tfrac{7}{10}, \tag{2b}$$
$$x_3^{(k+1)} = \tfrac{1}{5}x_2^{(k+1)} + \tfrac{3}{5}. \tag{2c}$$

That is, rather than using the old value $x_1^{(k)}$ in the update for $x_2^{(k+1)}$, instead we use the new value $x_1^{(k+1)}$ that has just been computed. Similarly, in the update for $x_3^{(k+1)}$ we use the new value $x_2^{(k+1)}$ that has just been computed. In general, we use new values as soon as they become available. Now, with $\boldsymbol{x}^{(0)} = [0, 0, 0]^\top$, we obtain

$$\begin{aligned}
\boldsymbol{x}^{(1)} &= \left[\tfrac{1}{10}(0) + \tfrac{9}{10}, \tfrac{1}{10}(0.9) + \tfrac{1}{5}(0) + \tfrac{7}{10}, \tfrac{1}{5}(0.79) + \tfrac{3}{5}\right]^\top \\
&= [0.9, 0.79, 0.758]^\top, \\
\boldsymbol{x}^{(2)} &= \left[\tfrac{1}{10}(0.79) + \tfrac{9}{10}, \tfrac{1}{10}(0.979) + \tfrac{1}{5}(0.758) + \tfrac{7}{10}, \tfrac{1}{5}(0.9495) + \tfrac{3}{5}\right]^\top \\
&= [0.979, 0.9495, 0.7899]^\top,
\end{aligned}$$

and so on. Once again, theory guarantees for this example, that the approximate solutions are converging to the actual solution, and our results strongly suggest the potential for Gauss–Seidel iterations to converge much more rapidly than Jacobi iterations. This is often, though not always, true.

There are often convergence rate benefits of rearranging the order of the equations in the original system in order to try to maximize the coefficients on the main diagonal; many numerical analysis texts discuss this (see, for example, reference 1). Such a rearrangement changes (1a)–(1c), which in turn changes (2a)–(2c).

This past summer in a numerical methods course, student co-authors Robert and Jon suggested something slightly different. They suggested initially keeping the order of the equations as written, in order to obtain a Jacobi iteration scheme, but then rearranging the order of the equations in the Jacobi iteration scheme before moving to a Gauss–Seidel iteration. That is, they suggested keeping (1a)–(1c) as written but rearranging the order of these equations. Each of them had an insightful idea that impacted on the rearrangement they chose.

Robert's idea: minimize the number of equations using 'old' values

In the rearrangement shown below, only one equation—the equation for $x_2^{(k+1)}$—involves $x_1^{(k)}$, $x_2^{(k)}$, and $x_3^{(k)}$:

$$\begin{aligned}
x_2^{(k+1)} &= \tfrac{1}{10}x_1^{(k)} + \tfrac{1}{5}x_3^{(k)} + \tfrac{7}{10}, \\
x_1^{(k+1)} &= \tfrac{1}{10}x_2^{(k+1)} + \tfrac{9}{10}, \\
x_3^{(k+1)} &= \tfrac{1}{5}x_2^{(k+1)} + \tfrac{3}{5}.
\end{aligned}$$

Now, with $\boldsymbol{x}^{(0)} = [0, 0, 0]^\top$, we obtain

$$\boldsymbol{x}^{(1)} = [0.97, 0.7, 0.74]^\top, \qquad \boldsymbol{x}^{(2)} = [0.9945, 0.945, 0.789]^\top,$$

etc.

Jon's idea: maximize benefit to a 'critical' variable

In the arrangement shown below, the variable x_2 is critical to both x_1 and x_3. Therefore, Jon chose to rearrange (1a)–(1c) in such a way that he would update to $x_2^{(k+1)}$ using $x_1^{(k+1)}$ and $x_3^{(k+1)}$:

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{10}x_2^{(k)} + \tfrac{9}{10}, \\ x_3^{(k+1)} &= \tfrac{1}{5}x_2^{(k)} + \tfrac{3}{5}, \\ x_2^{(k+1)} &= \tfrac{1}{10}x_1^{(k+1)} + \tfrac{1}{5}x_3^{(k+1)} + \tfrac{7}{10}. \end{aligned}$$

Now, with $\boldsymbol{x}^{(0)} = [0, 0, 0]^\top$, we obtain

$$\boldsymbol{x}^{(1)} = [0.9, 0.91, 0.6]^\top, \qquad \boldsymbol{x}^{(2)} = [0.991, 0.9555, 0.782]^\top,$$

etc.

Tables 1–4 summarize our results using Jacobi iteration, Gauss–Seidel iteration, Robert's idea, and Jon's idea.

Table 1 Jacobi iteration.

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	0	0.900	0.9700	0.991 000	0.994 500 00	0.995 550 000	0.995 725 000	0.995 777 500
$x_2^{(k)}$	0	0.700	0.9100	0.945 000	0.955 500 00	0.957 250 000	0.957 775 000	0.957 862 500
$x_3^{(k)}$	0	0.600	0.7400	0.782 000	0.789 000 00	0.791 100 000	0.791 450 000	0.791 555 000

Table 2 Gauss–Seidel iteration.

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	0	0.900	0.9790	0.994 950	0.995 747 50	0.995 787 375	0.995 789 369	0.995 789 468
$x_2^{(k)}$	0	0.790	0.9495	0.957 475	0.957 873 75	0.957 893 688	0.957 894 684	0.957 894 734
$x_3^{(k)}$	0	0.758	0.7899	0.791 495	0.791 574 75	0.791 578 738	0.791 578 937	0.791 578 947

Table 3 Robert's idea.

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	0	0.970	0.9945	0.995 725	0.995 786 25	0.995 789 313	0.995 789 466	0.995 789 473
$x_2^{(k)}$	0	0.700	0.9450	0.957 250	0.957 862 50	0.957 893 125	0.957 894 656	0.957 894 733
$x_3^{(k)}$	0	0.740	0.7890	0.791 450	0.791 572 50	0.791 578 625	0.791 578 931	0.791 578 947

Table 4 Jon's iteration.

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	0	0.900	0.9910	0.995 550	0.995 777 50	0.995 788 875	0.995 789 444	0.995 789 472
$x_2^{(k)}$	0	0.910	0.9555	0.957 775	0.957 888 75	0.957 894 438	0.957 894 722	0.957 894 736
$x_3^{(k)}$	0	0.600	0.7820	0.791 100	0.791 555 00	0.791 577 75	0.791 578 888	0.791 578 944

Comparison of the rates of convergence

One way to measure the error in an approximate solution to the system is to use the Euclidean norm

$$||\boldsymbol{x} - \boldsymbol{x}^{(k)}|| = \sqrt{(x_1 - x_1^{(k)})^2 + (x_2 - x_2^{(k)})^2 + (x_3 - x_3^{(k)})^2},$$

which corresponds to the distance between points in three-dimensional space. In table 5 we show the error results for Jacobi iteration, Gauss–Seidel iteration, Robert's idea, and Jon's idea.

For this example, at least, both Robert's idea and Jon's idea seem to be good! Interestingly, a look at error ratios,

$$\frac{||\boldsymbol{x} - \boldsymbol{x}^{(k)}||}{||\boldsymbol{x} - \boldsymbol{x}^{(k+1)}||} \quad \text{for large } k,$$

indicates essentially identical rates of convergence for Gauss–Seidel iteration, Robert's idea, and Jon's idea, with Jacobi iteration in a distant fourth place, as shown in table 6.

Table 5 Comparison of the iteration methods: $||\boldsymbol{x} - \boldsymbol{x}^{(k)}||$.

k	1	2	3	4	5	6	7
Jacobi	0.335	0.0750	0.016 800	3.75×10^{-3}	8.38×10^{-4}	1.87×10^{-4}	4.19×10^{-5}
Gauss–Seidel	0.196	0.0188	0.000 942	4.71×10^{-5}	2.36×10^{-6}	1.18×10^{-7}	5.89×10^{-9}
Robert's Idea	0.264	0.0132	0.000 661	3.30×10^{-5}	1.65×10^{-6}	8.26×10^{-8}	4.13×10^{-9}
Jon's Idea	0.219	0.0110	0.000 549	2.74×10^{-5}	1.37×10^{-6}	6.86×10^{-8}	3.43×10^{-9}

Table 6 Comparison of the iteration methods: $||\boldsymbol{x}-\boldsymbol{x}^{(k)}||/||\boldsymbol{x}-\boldsymbol{x}^{(k+1)}||$ for large k.

	Error ratio for large k
Jacobi	4.47
Gauss–Seidel	20
Robert's idea	20
Jon's idea	20

Further investigations

Co-author Jon looked at implementing his idea and Robert's idea on a system from reference 2:

$$9x_1 + x_2 + x_3 = 9, \tag{3a}$$
$$2x_1 + 10x_2 + 3x_3 = 10, \tag{3b}$$
$$3x_1 + 4x_2 + 11x_3 = 11. \tag{3c}$$

Here, the Jacobi iteration scheme is given by

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\big[9 - x_2^{(k)} - x_3^{(k)}\big], \\ x_2^{(k+1)} &= \tfrac{1}{10}\big[10 - 2x_1^{(k)} - 3x_3^{(k)}\big], \\ x_3^{(k+1)} &= \tfrac{1}{11}\big[11 - 3x_1^{(k)} - 4x_2^{(k)}\big]. \end{aligned}$$

Without rearrangement, the Gauss–Seidel scheme here is

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\big[9 - x_2^{(k)} - x_3^{(k)}\big], \\ x_2^{(k+1)} &= \tfrac{1}{10}\big[10 - 2x_1^{(k+1)} - 3x_3^{(k)}\big], \\ x_3^{(k+1)} &= \tfrac{1}{11}\big[11 - 3x_1^{(k+1)} - 4x_2^{(k+1)}\big]. \end{aligned}$$

Since this system does not have any missing terms, Robert's idea in this case does not suggest a particular rearrangement of equations. Jon's idea also needs interpretation here; since his idea involves rearranging in order to maximize the benefit to a 'critical' variable, he chose to rearrange the equations as follows:

$$\begin{aligned} x_3^{(k+1)} &= \tfrac{1}{11}\big[11 - 3x_1^{(k)} - 4x_2^{(k)}\big], \\ x_2^{(k+1)} &= \tfrac{1}{10}\big[10 - 2x_1^{(k)} - 3x_3^{(k+1)}\big], \\ x_1^{(k+1)} &= \tfrac{1}{9}\big[9 - x_2^{(k+1)} - x_3^{(k+1)}\big]. \end{aligned}$$

He chose this rearrangement as a result of viewing x_1 as the most 'critical', owing to its largest multiplier of $\frac{1}{9}$, and, similarly, viewing x_2 as the second most 'critical'. This rearrangement uses two updated values in computing x_1 and one updated value in computing x_2. As before, he used $\boldsymbol{x}^{(0)} = [0, 0, 0]^\top$. In comparing rates of convergence, Jon chose to compute $||\boldsymbol{x}^{(6)} - \boldsymbol{x}^{(5)}||$ and we also computed $||\boldsymbol{x}^{(14)} - \boldsymbol{x}^{(13)}||$; often, when we are solving problems for which

Table 7 Comparison of iteration methods for (3a)–(3c).

	$\|\|\boldsymbol{x}^{(6)} - \boldsymbol{x}^{(5)}\|\|$	$\|\|\boldsymbol{x}^{(14)} - \boldsymbol{x}^{(13)}\|\|$
Jacobi	0.032 644 344 81	0.000 052 126 522 16
Gauss–Seidel	0.000 021 729 58	0.000 000 000 000 20
Jon's idea	0.000 050 502 03	0.000 000 000 000 06

we do not know the actual solution, we measure convergence rates by comparing successive approximations like this. The results are shown in table 7, and in the long run at least they seem to indicate that Jon's idea is good, though the errors are so tiny that we wonder what effect round-off error is having in all of this as well!

Finally, we put forth the following system as an example to investigate which regular Gauss–Seidel iteration works best:

$$\begin{aligned} 4x_1 + 3x_2 &= 24, \\ 3x_1 + 4x_2 - x_3 &= 30, \\ -x_2 + 4x_3 &= -24. \end{aligned}$$

We do not claim that Robert's and Jon's ideas fully generalize. To begin with, as we explored example problems, we realized that their ideas make the most sense in the context of systems with missing terms. Secondly, their ideas do not seem to always produce better results. So, we simply comment that rearranging the order of the equations as you move from a Jacobi to Gauss–Seidel iteration does have an impact on the results and can sometimes improve results, and we encourage the reader to experiment with his or her own ideas in solving linear systems by elegant and interesting iteration methods.

References

1 K. Atkinson and W. Han, *Elementary Numerical Analysis*, 3rd edn. (John Wiley, New Jersey, 2004), page 304.

2 R.L. Burden and J.D. Faires, *Numerical Analysis*, 7th edn. (Brooks/Cole, Pacific Grove, CA, 2001), page 451.

Jon Kuhl *is an applied mathematics major at California State Polytechnic University, Pomona. He enjoys solving problems and finding real-world applications for what he studies, while he strives to understand the confusing and to simplify the complex.*

Robert Combs *is a senior working on a degree in Pure Mathematics at California State Polytechnic University, Pomona; he is interested in puzzles and has a Bachelors of Science in Computer Information Systems. It is his background in programming that brought out his interest in finding a more efficient algorithm for Gauss–Seidel.*

Jennifer Switkes *is an associate professor of mathematics at California State Polytechnic University, Pomona, with the privilege of collaborating with great students like Jon and Robert.*

Nested Ellipses

G. T. VICKERS

1. Two ellipses

Two ellipses are to be fitted into a right-angled corner so that their axes are parallel to one another and also parallel to the respective sides of the corner. This problem was suggested by the requirement to design a decorative bracket. Only one quarter of the larger ellipse would actually be constructed; the smaller ellipse provides strength and, perhaps, elegance. Such a bracket is shown in figure 1. This problem is a generalization of that considered by Burley and Smith (reference 1), but only the 'easy' case in which one curve is a circle was considered in that article.

If the semi-axes of the larger ellipse are a and b and those of the smaller ellipse are a' and b', then it is to be expected that there will be just one condition upon the four numbers a, b, a', and b' which will ensure that such a configuration is possible. The solution (that is, finding this relation) will be by coordinate geometry. It is convenient to choose axes centred upon the point of contact, as shown in figure 2. The equation of an ellipse which passes through the origin has axes parallel to the coordinate axes, and has semi-axes of length a and b is

$$\left(\frac{x-ac}{a}\right)^2+\left(\frac{y-bs}{b}\right)^2=1, \tag{1}$$

where $c=\cos\theta$ and $s=\sin\theta$ for some angle θ. Now the slope of the tangent to this ellipse at the origin is $-bc/(as)$. Thus, if there is a second ellipse with parameters a', b', and θ' (and

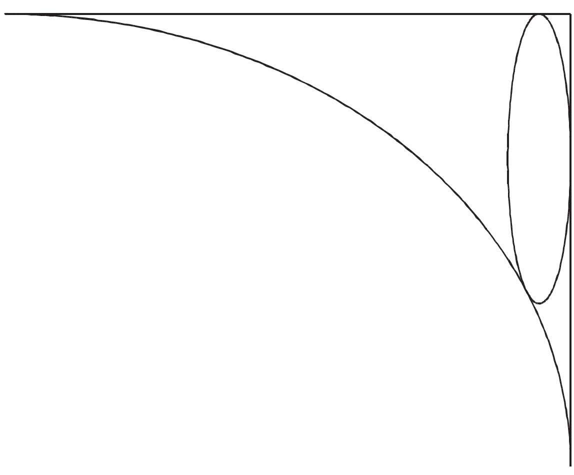


Figure 1 The design of a bracket consisting of a quarter of a large ellipse and a complete small ellipse.

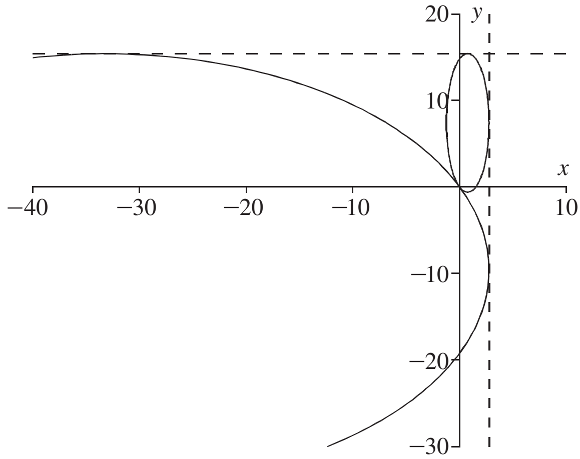


Figure 2 The coordinate system.

$c' = \cos\theta',\ s' = \sin\theta'$), then the condition that they shall touch at the origin is

$$\frac{bc}{as} = \frac{b'c'}{a's'}. \tag{2}$$

Note that the geometry of the situation implies that $c < 0, s < 0$ and $c' > 0, s' > 0$. Hence, $\pi < \theta < 3\pi/2$ and $0 < \theta' < \pi/2$.

The greatest values of x and y on the ellipse given by (1) are $a(1+c)$ and $b(1+s)$. Thus, the condition that the ellipses will fit into the corner gives

$$a(1+c) = a'(1+c'), \tag{3}$$

$$\text{and} \quad b(1+s) = b'(1+s'). \tag{4}$$

There is now the trigonometric problem of eliminating θ and θ' from (2), (3), and (4) in order to obtain the relationship between the lengths of the four axes.

Instead of directly eliminating the angles, it is easier to eliminate the four lengths to obtain

$$\frac{s(1+s)}{c(1+c)} = \frac{s'(1+s')}{c'(1+c')}. \tag{5}$$

With the introduction of

$$T = \tan\left(\frac{\theta}{2}\right) \quad \text{and} \quad T' = \tan\left(\frac{\theta'}{2}\right),$$

and using the standard half-angle formulae,

$$\sin\theta = \frac{2T}{1+T^2} \quad \text{and} \quad \cos\theta = \frac{1-T^2}{1+T^2}$$

(together with their primed friends), (5) becomes

$$\frac{T(T+1)}{1-T} = \frac{T'(T'+1)}{1-T'}.$$

Discarding the unwanted solution $T' = T$, we obtain

$$T' = \frac{T+1}{T-1},$$

and so

$$\tan\theta' = \frac{2T'}{1-T'^2} = \frac{1-T^2}{2T} = \cot\theta.$$

Because of the known ranges in which θ and θ' lie, it follows that

$$\theta + \theta' = \frac{3\pi}{2}. \tag{6}$$

Introduce p and q so that

$$p = \sqrt{\frac{a'}{a}} \quad \text{and} \quad q = \sqrt{\frac{b'}{b}}.$$

Equations (2) and (6) now give

$$\frac{s}{c} = \frac{p}{q} \quad \text{and} \quad \frac{s'}{c'} = \frac{q}{p}.$$

Hence,

$$-s = c' = \frac{p}{\sqrt{p^2+q^2}} \quad \text{and} \quad -c = s' = \frac{q}{\sqrt{p^2+q^2}}.$$

Furthermore, (3) gives

$$p^2 = \frac{a'}{a} = \frac{1+c}{1+c'} = \frac{\sqrt{p^2+q^2}-q}{\sqrt{p^2+q^2}+p}.$$

This can be unwrapped to give

$$pq + 1 = \sqrt{2}(p+q), \tag{7}$$

or, equivalently,

$$(\sqrt{2}-p)(\sqrt{2}-q) = 1 \quad \text{or even} \quad 1 - pq = \sqrt{2(p^2+q^2)}.$$

In terms of the four lengths, this first condition is

$$\sqrt{ab} + \sqrt{a'b'} = \sqrt{2a'b} + \sqrt{2ab'}. \tag{8}$$

So the problem posed has been solved.

1.1. Additional properties

- It is left as an algebraic exercise to show that the coordinates of the corner of the bracket are (X, Y), where

$$X = \frac{aa'}{a - \sqrt{2aa'} + a'} \quad \text{and} \quad Y = \frac{bb'}{b - \sqrt{2bb'} + b'}.$$

  These can be interpreted as giving the coordinates $(-X, -Y)$ of the point of contact with respect to axes centred on the corner.

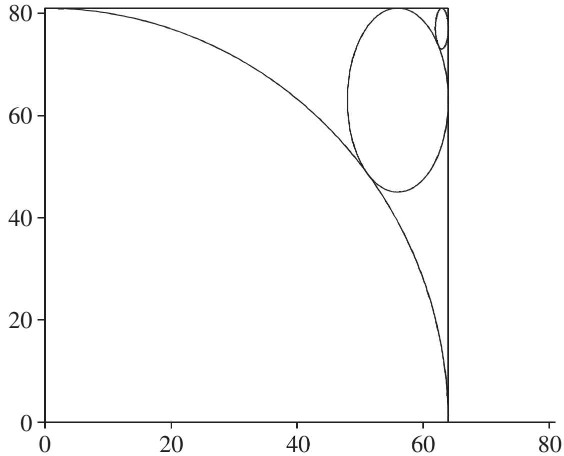


Figure 3 Three nested ellipses. The semi-axes are (64,81), (8,18), and (1,4).

- Suppose that a and b are given and that it is required to find a' and b' so that the ellipses fit into the corner as before but, in addition, the area of the second ellipse will be as large as possible. This is equivalent to maximising $a'b'$ subject to the constraint of (8). You might like to show that this occurs when

$$\frac{a'}{a} = (\sqrt{2} - 1)^2 = \frac{b'}{b}.$$

- It may be noticed in the above algebra that the sign of various square roots was silently chosen. You might like to consider the significance of other choices.

- What is the answer if it is required that one or other of the ellipses is a circle?

- Show that if an ellipse with semi-axes a and b can be fitted between two circles (with all three curves wedged in the corner and the ellipse touching each of the circles), then $ab = Rr$ and $a + b = (\sqrt{R} - \sqrt{r})^2/2$, where R and r are the radii of the circles. Also, there is only a solution if $R/r \geq (\sqrt{2} + 1)^4$.

2. Three ellipses

Suppose that the bracket is sufficiently large that it is desirable to insert a third ellipse, as shown in figure 3. Specifically, if a and b are the semi-axes of the largest ellipse and a' and b' are the semi-axes of the smallest ellipse, when is it possible to insert an intermediate ellipse with semi-axes x and y?

Using (8) for each pair of touching ellipses gives

$$\sqrt{xy} + \sqrt{ab} = \sqrt{2ay} + \sqrt{2xb}$$

and

$$\sqrt{a'b'} + \sqrt{xy} = \sqrt{2xb'} + \sqrt{2a'y}.$$

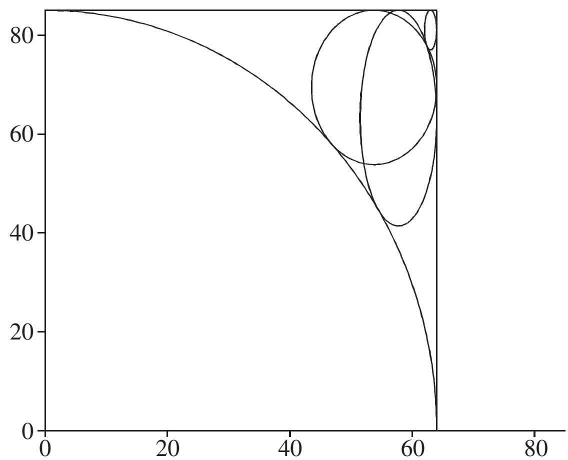


Figure 4 Two possible intermediate ellipses. The semi-axes of the bounding ellipses are (64,85) and (1,4). The other two have semi-axes (6.25,21.80) and (15.59,21.80).

These give

$$\sqrt{y} = \frac{\sqrt{ab} - \sqrt{a'b'} - \sqrt{2x}(\sqrt{b} - \sqrt{b'})}{\sqrt{2a} - \sqrt{2a'}},$$

and so

$$(\sqrt{2b} - \sqrt{2b'})(x + \sqrt{aa'}) = \sqrt{x}(\sqrt{ab} - \sqrt{a'b'} + 2\sqrt{a'b} - 2\sqrt{ab'}),$$

which is just a quadratic for $\sqrt{x}$. With

$$P = \sqrt[4]{\frac{a'}{a}} \quad \text{and} \quad Q = \sqrt[4]{\frac{b'}{b}},$$

the condition for real roots becomes

$$PQ + 1 \geq \sqrt{2}(P + Q), \tag{9}$$

which is reminiscent of (7). If this condition is satisfied then there are two possible values for x and, hence, two different ellipses may be inserted. This is demonstrated in figure 4.

In the marginal case, that is, a, b, a', and b' are such as to give equality in (9), then of course there is only one value for each of x and y, and these are given by

$$x = \sqrt{aa'} \quad \text{and} \quad y = \sqrt{bb'}.$$

Figure 3 in fact is for such a case.

We can now contemplate a large (or even infinite) number of nested ellipses. Suppose that the largest ellipse has semi-axes a_1 and b_1 and that successive ellipses have sizes a_n and b_n. In general, there will be a choice of sizes at each stage as we choose a yet smaller ellipse to fit into the corner. But, if we say that there is only one ellipse that can be inserted between any alternate pair (between the first and third, second and fourth, third and fifth, etc.), then the sizes will satisfy

$$a_n = A\alpha^n \quad \text{and} \quad b_n = B\beta^n,$$

where

$$\sqrt{\alpha\beta} + 1 = \sqrt{2\alpha} + \sqrt{2\beta},$$

which is just (7) in disguise.

If $\alpha = \beta$ then the common value is $(\sqrt{2} - 1)^2$. Curiously, we get back to the case of maximising the area. Also, the centres of all of the ellipses lie on a straight line which passes through the corner and all of the points of contact.

Reference

1 D. M. Burley and R. A. Smith, The railway station problem, *Math. Spectrum* **34** (2001/2002), pp. 58–60.

Glenn Vickers *is a graduate of Sheffield University and then gained a PhD from Queen Mary College, London in Astrophysics. He has now retired from the Applied Mathematics Department in Sheffield where he taught (and has publications in) a wide range of topics including genetics, galactic structure, and evolutionary game theory.*

Powers with the same digits

$$13^2 = 169, \qquad 157^2 = 24\,649, \qquad 913^2 = 833\,569,$$
$$14^2 = 196, \qquad 158^2 = 24\,964, \qquad 914^2 = 835\,396,$$

$$32^2 = 1024, \qquad 32^4 = 1\,048\,576,$$
$$49^2 = 2401, \qquad 49^4 = 5\,764\,801,$$

$$345^3 = 41\,063\,625,$$
$$384^3 = 56\,623\,104,$$
$$405^3 = 66\,430\,125.$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini

Pigeon power

A train is heading towards a marker post along a straight track at constant speed v. When it is distance d from the post, a pigeon flies from the post towards the train at constant speed $w > v$. When it reaches the train, it immediately flies back to the post, then to the train, and so on, at the same speed w. How far has the pigeon flown when the train reaches the post?

C/o A. A. Khan, Regional Office, Indian Overseas Bank,
Ashok Marg, Lucknow, India

M. A. Khan

Letters to the Editor

Dear Editor,

The sum of the first n squares

Readers may be interested in the following derivation of

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$

from

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1).$$

Put $A = 1 + 2 + \cdots + n,\ S = 1^2 + 2^2 + \cdots + n^2$. Then

$$\begin{aligned} 1 + 2 + 3 + 4 + \cdots + n &= A, \\ 2 + 3 + 4 + \cdots + n &= A - \tfrac{1}{2}(1^2 + 1), \\ 3 + 4 + \cdots + n &= A - \tfrac{1}{2}(2^2 + 2), \\ &\ \vdots \\ n &= A - \tfrac{1}{2}((n-1)^2 + (n-1)). \end{aligned}$$

If we add these equations, we have

$$(1\times 1)+(2\times 2)+(3\times 3)+\cdots+(n\times n) = nA - \tfrac{1}{2}(1^2+2^2+\cdots+(n-1)^2) - \tfrac{1}{2}(1+2+\cdots+(n-1)),$$

or

$$S = n\tfrac{1}{2}n(n+1) - \tfrac{1}{2}(S - n^2) - \tfrac{1}{2}\tfrac{1}{2}(n-1)n,$$

or

$$\tfrac{3}{2}S = \tfrac{1}{4}n(2n^2 + 2n + 2n - n + 1),$$

which gives

$$S = \tfrac{1}{6}n(n+1)(2n+1).$$

Yours sincerely,

Abbas Roohol Amini
(10 Shahid Azam Lane
Makki Ahad Avenue
Sirjan
Iran)

Dear Editor,

Nested Ellipsoids

The article *Nested Ellipses* in this issue considered the problem of packing two ellipses into a right-angled corner so that each ellipse touched both sides of the corner and they also touch each other. This note extends this to ellipsoids (and higher dimensional figures). Not surprisingly, it has not been found possible to express the answer as neatly as in the two-dimensional case, but quite a lot may be discovered.

In the article referred to, axes were chosen with origin at the point of contact of the ellipses. That does not seem to be so useful here, so the origin is taken to be the corner. The equation of an n-dimensional ellipsoid which touches every coordinate plane (and whose axes are parallel to the coordinate axes) is

$$\sum_{i=1}^{n}\left(\frac{x_i}{a_i}-1\right)^2 = 1,$$

where the a_i are the semi-axes. A normal to this figure is

$$\sum_{i=1}^{n}\frac{2}{a_i}\left(\frac{x_i}{a_i}-1\right)\boldsymbol{e}_i.$$

Hence, if we have two touching ellipsoids (with semi-axes a_i and a_i') then

$$\sum_{i=1}^{n}\left(\frac{x_i}{a_i}-1\right)^2 = 1 = \sum_{i=1}^{n}\left(\frac{x_i}{a_i'}-1\right)^2 \quad \text{and} \quad \frac{(x_i-a_i)}{a_i^2}\frac{a_i'^2}{(x_i-a_i')} = \lambda \quad (1 \le i \le n).$$

The elimination of x_i gives

$$\lambda^2\sum_{i=1}^{n}\frac{a_i^2(a_i-a_i')^2}{(a_i'^2-\lambda a_i^2)^2} = 1 = \sum_{i=1}^{n}\frac{a_i'^2(a_i-a_i')^2}{(a_i'^2-\lambda a_i^2)^2}. \tag{1}$$

These two equations determine λ and a relationship between the a_i and the a_i'.

The problem of fitting in the ellipsoid of greatest volume (between the origin and a given ellipsoid) amounts to maximising $\Pi_{i=1}^{n} a_i'$ subject to (1). It is easily seen that this implies that $a_i' = \mu a_i$ $(1 \le i \le n)$. Furthermore, this new condition (that the two ellipsoids are similar) gives

$$\mu = \frac{\sqrt{n}-1}{\sqrt{n}+1}.$$

If $n = 2$ then $\mu = (\sqrt{2}-1)^2$, as given in the article.

Fitting in a hypersphere (that is, $a_i' = b$ for all i), does not seem to lead to a 'nice' result. When $n = 2$ (and $a_1' = a_2' = b$) the article implies that the answer is

$$\sqrt{2b} = \pm\sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_1+a_2} \quad \text{and} \quad \lambda = \pm b/\sqrt{a_1 a_2}$$

(only 8 of the possible 16 sign choices are permitted). But, starting from (1), it is not at all clear how to arrive at this result. So the chances of progress with larger values of n would seem slight. Maybe you can find a better way to travel.

Yours sincerely,

G. T. Vickers
(Department of Applied Mathematics
The University of Sheffield
Sheffield S3 7RH
UK)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

41.9 Determine the area enclosed by the central region of the curve with equation

$$r = \theta + \frac{1}{\theta} \quad (\theta > 0)$$

in polar coordinates.

(Submitted by Jonny Griffiths, Paston College, Norfolk)

41.10 What is the sum of the infinite series

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{f_{2n+1}},$$

where (f_n) denotes the Fibonacci sequence? (Note: $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for all $n \geq 2$.)

(Submitted by Abbas Roohol Amini, Sirjan, Iran)

41.11 Determine all positive integers n for which

$$\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}$$

is an integer.

(Submitted by Hossein Behforooz, Utica College, New York)

41.12 For a triangle ABC, prove that

$$\cos^2 \frac{A}{2} = \frac{a(s-a)}{bc}$$

($s = \frac{1}{2}(a+b+c)$), and that $\cos^2(A/2)$, $\cos^2(B/2)$ and $\cos^2(C/2)$ are the sides of a triangle.

(Submitted by Mihaly Bencze, Brasov, Romania)

Solutions to Problems in Volume 41 Number 1

41.1 Is there a number greater than 1 which leaves the same remainder when it divides into 1716, 2154, and 4271?

Essentially the same solution was received from Gian Paolo Almirante (Milan), Daniel Fretwell (University of Sheffield), Melynda West (Northwest Missouri State University), and Dmitri Skjorshammer (Harvey Mudd College, Claremont, California)

We look for a positive integer $n > 1$ such that

$$1716 = hn + r, \qquad 2154 = kn + r, \qquad 4271 = ln + r$$

for some positive integers $h, k, l,$ and r. Thus, n must divide $2154 - 1716 = 438$ and $4271 - 2154 = 2117$. Now, $438 = 2 \times 3 \times 73$ and $2117 = 29 \times 73$ in prime factors, so the only possibility is $n = 73$. In fact,

$$1716 = 23 \times 73 + 37, \qquad 2154 = 29 \times 73 + 37, \qquad 4271 = 58 \times 73 + 37,$$

and the common remainder is 37.

41.2 (i) Given positive integers $m, n, t,$ show that the equation

$$\frac{1}{x^m} + \frac{1}{y^n} = \frac{1}{z^t}$$

has no solution in positive integers x, y, z.
(ii) Fermat's Last Theorem states that, for a given integer $n \geq 3,$ the equation

$$x^n + y^n = z^n$$

has no solution in positive integers x, y, z. Prove that, when $n \geq 3,$ the equation

$$\frac{1}{x^n} + \frac{1}{y^n} = \frac{1}{z^n}$$

has no solution in positive integers x, y, z.

Solution by Daniel Fretwell, University of Sheffield

(i) Suppose a solution exists. Then

$$\begin{aligned} y^n z^t + x^m z^t &= x^m y^n \\ \text{so} \quad x^m z^t &= y^n (x^m - z^t). \end{aligned}$$

If p is a prime number dividing y, then $p \mid x^m z^t$, so that $p \mid x$ or $p \mid z$. But y, x and y, z are coprime, so no such p exists. Hence, $y = 1$ and

$$\frac{1}{x^m} + 1 = \frac{1}{z^t},$$

which is impossible because the left-hand side is greater than 1 but the right-hand side is less than or equal to 1.

(ii) Suppose such a solution exists. Then

$$(yz)^n + (xz)^n = (xy)^n$$

which cannot happen by Fermat's Last Theorem.

41.3 Let c_1 and c_2 be two given unequal numbers. The function f is differentiable everywhere and satisfies the condition

$$c_1 f(x-y) + c_2 f(x+y) \leq (c_1 + c_2) f(x)$$

for all real numbers x and y. Determine all such functions f.

Solution by Spiros Andriopoulos, who proposed the problem

For real numbers x, h with $h > 0$, putting $y = h$ we get

$$c_1 f(x-h) + c_2 f(x+h) \leq (c_1 + c_2) f(x),$$

which gives

$$c_1 \left(\frac{f(x-h) - f(x)}{-h} \right) \geq c_2 \left(\frac{f(x+h) - f(x)}{h} \right).$$

If we let $h \longrightarrow 0$, this gives

$$c_1 f'(x) \geq c_2 f'(x).$$

Now if we put $y = -h$, we get

$$c_1 f(x+h) + c_2 f(x-h) \leq (c_1 + c_2) f(x),$$

which gives

$$c_1 \left(\frac{f(x+h) - f(x)}{h} \right) \leq c_2 \left(\frac{f(x-h) - f(x)}{-h} \right).$$

If we let $h \longrightarrow 0$, this gives

$$c_1 f'(x) \leq c_2 f'(x).$$

Since $c_1 \neq c_2$, the only possibility is that $f'(x) = 0$ for all x, so f must be a constant function. Every constant function satisfies the condition.

41.4 The regular n-sided polygon $A_1 A_2 \cdots A_n$ is inscribed in the unit circle. What is the product of the distances $A_1 A_n, A_2 A_n, \cdots, A_{n-1} A_n$?

Solution by Theodoros Valahas and Andreas Boukas, who proposed the problem

The points $A_1, \cdots, A_n$ can be denoted by the complex roots of unity, say $z_1, \cdots, z_{n-1}, 1$, respectively, so that

$$z^n - 1 = (z - z_1)(z - z_2) \cdots (z - z_{n-1})(z - 1),$$

whence

$$(z - z_1)(z - z_2) \cdots (z - z_{n-1}) = z^{n-1} + z^{n-2} + \cdots + z + 1.$$

Now

$$
\begin{aligned}
(A_1A_n) \times (A_2A_n) \times \cdots \times (A_{n-1}A_n) &= |(1-z_1)(1-z_2)\cdots(1-z_{n-1})| \\
&= 1+1+\cdots+1 \quad (n \text{ terms}) \\
&= n.
\end{aligned}
$$

Reviews

Modeling for Insight: A Master Class for Business Analysts. By Stephen G. Powell and Robert J. Batt. John Wiley, Chichester, 2008. Paperback, 465 pages, £52.95 (ISBN 0-470-17555-2).

Most business analysts are familiar with using spreadsheets to organize data and build routine models. However, analysts often struggle when faced with examining new and ill-structured problems. *Modeling for Insight* is a one-of-a-kind guide to building effective spreadsheet models and using them to generate insights. With its hands-on approach, this book provides readers with an effective *modeling process* and specific modeling *tools* to become a master modeler.

Statistics and Data with R: An Applied Approach Through Examples. By Yosef Cohen and Jeremiah Y. Cohen. John Wiley, Chichester, 2008. Hardback, 599 pages, £50.00 (ISBN 0-470-75805-2).

R, an Open Source software, has become the de facto statistical computing environment. It has an excellent collection of data manipulation and graphics capabilities. It is extensible and comes with a large number of packages that allow statistical analysis at all levels—from simple to advanced—and in numerous fields including Medicine, Genetics, Biology, Environmental Sciences, Geology, Social Sciences and much more. The software is maintained and developed by academicians and professionals and as such, is continuously evolving and up to date. *Statistics and Data with R* presents an accessible guide to data manipulations, statistical analysis and graphics using R.

Time Series Data Analysis Using EViews. By I. Gusti Ngurah Agung. John Wiley, Singapore, Asia, 2009. Hardcover, 609 pages, £75.00 (ISBN 978-0-470-82367-5).

This book provides a hands-on practical guide to using the most suitable models for analysis of statistical data sets using EViews— an interactive Windows-based computer software program for sophisticated data analysis, regression, and forecasting—to define and test statistical hypotheses. Rich in examples and with an emphasis on how to develop acceptable statistical models. *Time Series Data Analysis Using EViews* is a perfect complement to theoretical books presenting statistical or econometric models for time series data. The procedures introduced are easily extendible to cross-section data sets.

An essential tool for advanced undergraduate and graduate students taking finance or econometrics courses. Statistics, life sciences, and social science students, as well as applied researchers, will also find this book an invaluable source.

Introduction to the Practice of Statistics. By David S. Moore, George P. McCabe and Bruce Craig. W. H. Freeman, New York, 6th edn., 2009. Hardback, 709 pages, £44.99 (ISBN 1-4292-16220-0).

This book was first published almost twenty years ago as an introductory statistics text intended for students from a variety of disciplines. The authors wanted students to see that statistics was not just a series of formula-driven exercises but a powerful tool for understanding our world, and as such was widely used on undergraduate courses. This sixth edition has been revised and updated in line with new technology, but retains its emphasis on working with real data to convey core concepts and practical applications.

Those familiar with earlier versions will recognise the core chapters which look at data handling, and study probability and inference. Four further chapters, moving into logistic regression, non-parametric and bootstrap methods as well as quality control, are available on the book's CD-ROM and website. Presentation is attractively colourful and helpfully laid out with a wealth of pictures and diagrams. Examples are surprisingly up-to-date in their data sources (for example, one relates to the length of audio files on an iPod), and although there has been some criticism about a lack of sufficient worked examples, this is already a weighty tome.

This book has been selling well for some time now and continues to be a valuable resource as an elementary introduction to statistics, particularly for social science students. Tried and tested, it continues to be recommended for reading.

Carol Nixon

Statistics: A Very Short Introduction. By David J. Hand. OUP Inc., New York, 2008. Paperback, 114 pages, £7.99 (ISBN 978-0-19-923356-4).

The VSI pocket-sized series was launched in 1995 as a general introduction series for the intelligent reader and now consists of over 190 titles. These accessible books are designed either to change the way you think about things that interest you, or are intended to be good introductions to subjects you know nothing about. I approached this book from the perspective of the former and was immediately engaged.

Starting from the premise that statistical ideas and methods underlie just about every aspect of modern life, it follows that we need to have some grasp of them if we are to have any understanding of our world. With this objective, the text sets the study of statistics in context, describing its history and giving up-to-the-minute examples of its use and impact in everyday life with hardly a number in sight and only a few diagrams.

The chapter titles look routine enough. They summarize methods of gathering and evaluating data, and explain the role played by probability in statistical methods, proceeding to an exploration of estimation and inference. This all takes place in bite-sized sections made all the more effective by the use of quotations (wonder who did say *being a statistician means never having to say you are certain!*) and illustrations from our very real world which range from spam filtering to the problems behind the Challenger space-shuttle disaster.

Aimed at readers interested in statistics and its applications, and with no prior mathematical knowledge required, this book is a fascinating explanation of how statistics work, and how we can decipher them to unearth what is really going on behind each data set. Unreservedly recommended reading—even small enough to carry around in your pocket for those quiet moments when you are in need of inspiration.

Carol Nixon

Mathematical Spectrum

Volume 41 2008/2009 Number 3

ISSN 0025-5653

http://www.appliedprobability.org
Published by the Applied Probability Trust
Printed by Berforts South East Ltd, Stevenage, Hertfordshire, UK