

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol. 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

Mathematicorum

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 4, No. 10

December 1978

Sponsored by

Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton

A Chapter of the Ontario Association for Mathematics Education

Publié par le Collège Algonquin

(32 issues of this journal, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name EUREKA.)

CRUX MATHEMATICORUM is published monthly (except July and August). The yearly subscription rate is \$10.00. Back issues: \$1.00 each. Bound volumes: Vol. 1-2 (combined), \$10.00; Vol. 3-4, \$10.00 each. Cheques or money orders, payable in Canadian or U.S. funds to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content of the magazine (articles, problems, solutions, permission to reprint, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.

Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.

Typist-compositor: Nancy Makila.

*

*

*

CONTENTS

A Pep Talk on Problem-Solving for Post-Calculus Students . . . Paul J. Campbell	272
The Prime Life of our Years. Robert S. Johnson	278
Algorithms and Pocket Calculators: Square Roots: IV. . . . Clayton W. Dodge	279
Problems - Problèmes.	282
Solutions.	285

For Vol. 4 (1978), the support of Algonquin College, the Samuel Beatty Fund, and Carleton University is gratefully acknowledged,

A PEP TALK ON PROBLEM - SOLVING FOR POST - CALCULUS STUDENTS

PAUL J. CAMPBELL

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery...

GEORGE PÓLYA

EAST IS EAST AND WEST IS WEST, BUT THE EARTH IS ROUND

In mathematics there are two basic types of problems: "problems to find" and "problems to prove". It is the latter type that presents a major hurdle to the post-calculus math student. Although you have had a major experience with proofs in high-school geometry and have encountered increasing emphasis in "problems to find" on the derivation of the solution, the changeover in advanced mathematics to a predominant emphasis on proofs may represent a difficult adjustment for you.

Fortunately, the habits of mind and techniques of solution are similar for both kinds of problem-solving, the main difference being in the emphasis in proof problems on explanation and justification of steps.

YOU AND NOW

At this point, after calculus and a certain exposure to proof mathematics, you may justifiably feel unsure of yourself. How do you start to solve a problem? How do you justify an answer? How do you explain on paper the reasoning that led you to your solution? How do you check that reasoning to be sure it is sound?

Well, it's too early for despair! In fact, your uncertainty is a good sign: you are inquiring about mathematical concepts and the framework of mathematics, things you may have taken for granted and never questioned before—trying to probe the WHY as well as the HOW. In fact, if I had to summarize the difference between calculus and subsequent analysis courses in just a few words, I would say that the latter expand on the HOW of calculus to investigate also the WHY.

Nothing is more interesting for us humans than human activity. The most characteris-

tically human activity is to solve problems, thinking for a purpose, devising means to some end.

GEORGE PÓLYA

YOU CANNOT BE REPLACED BY A COMPUTER

There is no one specific method you can learn that will give you the power to solve every problem—not even every problem within a very restricted part of, say, analysis. That is, there is no set procedure you can go through that will guarantee you will arrive at a solution for every problem.

In mathematical terminology, there is no algorithm for solving problems that I—or anyone—can teach you. Those special classes of problems for whose solutions there exist algorithms, such as long division or finding the derivative of a polynomial, we soon cease to regard as problems in the proper sense and relegate instead to the category of "exercises".

You acquire the ability to solve problems through experience and practice. Still, there are distinguishable patterns of thought in successful problem-solving, and from these patterns it is possible to distill some hints that will be helpful to you in your work.

...three, two, one, BLASTOFF!

HOW TO START: PREPARATION

Just as it is most important in "problems to find" to understand what quantity is sought, it is essential in "problems to prove" to understand the proposition that is to be proved. In particular, each of the terms used in its formulation must make sense to you—not only that, but you must have in the forefront of your mind the precise meaning of each. Even when you think you understand the proposition, it is often useful to begin by rechecking the details of the definitions of the terms used, since the key to the proof may hinge on one such detail.

In fact, I find it most useful to recopy the problem onto a piece of paper to which I then add definitions of the terms, so that all of the relevant information is displayed at once in front of me. The important thing is to reformulate the proposition in the language you find most familiar, and then to carefully distinguish between what is given and what must be proved.

CHECKING IT OUT: EXPERIMENTATION

After you are sure you are in command of the meaning of the proposition, it is

appropriate to ask yourself whether you believe it is true. Surprisingly, skepticism may be the more fruitful attitude, rather than blind faith. For one thing, the exercise or theorem in your textbook may actually be wrong (few textbooks are entirely error-free); for another, it won't be long before you start making conjectures on your own. What is more important, a healthy skepticism will lead you to TRY EXAMPLES in an effort to prove the proposition false.

Now, you may be asking, why should I start by trying to prove it's false when I was asked to prove it's true? Well, aside from the above-mentioned possibility that it *may* be false, successive trials with weirder and weirder examples may do you a double service. First, it may reveal to you why the proposition is true by demonstrating how its various hypotheses come into play—and that is *exactly* what you need to know to solve the problem. Second, it will increase your conviction that the proposition really *is* true, thereby increasing your curiosity and fortifying your motivation. After all, it's pretty hard to construct a proof of something you don't believe in or know much about!

And, in problems where it is helpful, don't forget to DRAW A PICTURE!

GETTING A HANDLE ON IT: INVESTIGATION

Although it is possible in principle to solve the problem by having recourse directly to the definitions, it is usually more efficient to make a survey of known results that may have a bearing on your problem.

Is this problem similar to one you solved previously? Are any of the conditions in the hypothesis present in any of the theorems in your collection? Do any of these theorems have the same conclusion as the proposition to be proved?

What you are trying to do is to establish a connection between the problem's hypotheses and its conclusion. An examination of other things you know may turn up some links that can be fashioned into a chain of reasoning.

Problems
worthy of attack
prove their worth
by hitting back.

PIET HEIN¹, *Grooks* 1.

EYEBALL TO EYEBALL WITH THE DRAGON: FRUSTRATION

After you have worked on the problem for a while without "getting" it, frustration

¹Reprinted by permission of the author, Piet Hein, and PaperJacks Ltd., Toronto, Ontario, The Canadian Paperback Publishers, publishers of *Grooks* 1, 2, 3, 4, 5. PaperJacks Ltd. is a subsidiary of General Publishing Co. Limited.

is bound to set in. When it does, you have reached the crucial point in solving the problem: what you do now makes all the difference in the world. There are a variety of pits you may have fallen into along the way, but conscious attention now to how the problem is affecting you can be the helicopter to lift you up and out. It is especially important to pay attention to these pitfalls at this time, or the rest of the time you spend on the problem may sap your morale and be useless too.

It may be that you fail to understand the problem completely. The reason may be psychological: other things on your mind, a distracting environment, hunger, or other blocks to concentration. Or the reason may be technical: you may not have read thoroughly enough the section of the text from which the problem derives or, having read it, you are as yet still unfamiliar with the relevant definitions and concepts; you may not have tried enough examples to be able to abstract a common feature. It may help now to consult other textbooks in order to read different accounts or see different examples about the same material.

It may be that you have made a minor error in arithmetic or algebra (absolutely the most frequent source of frustration in calculus!), miscopied the problem, misread or misinterpreted the problem, or made a basic error in reasoning (e.g. reversing the hypothesis and conclusion in a theorem). Here is where it pays to be working with a partner, who can check your work for these kinds of mistakes.

Most commonly, you may be *stuck in a loop*. You may be persisting in looking at the problem from a single point of view, though others are possible; or you may be stubbornly sticking to one strategy for solving the problem, without looking into others. Our natural tendency is to run with the first potential method of solution that comes to mind; we do so because it often works. When it does not, though, it's time to reexamine the problem for other strategies. It still may turn out to be the best plan to stick to your original idea, but you will then be able to do so with greater confidence and determination (which is what you need at this point, badly).

DON'T QUIT YET: INCUBATION

If you have given the problem a good try and reached the limit of your frustration, it's time to let your subconscious work on the problem while you do something else (go on to the next problem, go for a walk, clean up your room, etc.). If you haven't really come to grips with the problem, this incubation period won't do you any good; but if you have, it is distinctly possible that the solution or a good

idea will pop into your mind while you're doing something else. You can't count on this, of course, but it's nice when it happens. Normally, what will happen instead is that you will be able to redevote your energies to the problem at another time with a fresh perspective; you may see things that escaped your attention earlier, and you won't be laboring under the totality of built-up frustration that forced you to put the problem aside earlier.

The incubation period is a good time to compare notes with someone else in the class, see your teacher for a hint or two, or return to the sources of your inspiration.

LIGHTNING STRIKES: ILLUMINATION

What makes it all worth-while! It's the "mathematical high" that will keep you going, past future frustration.

WRITING IT UP: EXPLANATION

The actual writing up of the proof requires in itself some care and patience. You have to remember that the reader of the proof hasn't necessarily gone through all of the same mind trips with this problem that you have. Consequently, you're going to have to be extra careful about telling the reader what you're doing and why you're doing it. The most common error students make who are new to proof-writing is not to write down enough of what they are thinking. It is not enough to just write down a sequence of equations or statements; you should also supply reasons and a commentary linking each step to the next. You might also provide a short introduction describing how the proof will proceed (e.g. by mathematical induction on n , by recourse to the Heine-Borel Theorem, by contradiction, etc.).

Be sure to write in complete sentences and organize your work into paragraphs. Display long equations or expressions on lines by themselves. Supply a figure where appropriate. Be extra careful to explain any notation you introduce or any quantities that do not appear in the statement of the problem.

FIND A CRITIC: CONFIRMATION

By all means, get a fellow student to read your proof before submitting it. The expression of your ideas will profit from his or her comments. If you have to explain anything orally in addition to what you have written, it might be advisable to revise your proof to incorporate your oral remarks. The secret to an effective writing style is practice and feedback.

LOOKING BACK: RECONSIDERATION

Above all, you should read your own proof. Are you convinced? Did you use all the facts that you know? Did you use the entire hypothesis? It's often possible to detect a false proof because it "proves too much": it fails to use an essential fact in the hypothesis. Of course, it's possible that the hypothesis was stronger than necessary. The way to check which is the case, if any, is to examine the proposition you did prove. Ask yourself, is it reasonable?

The opposite error is also common—assuming facts not granted in the hypotheses. Again, you must look and see what you have used.

The positive side of reviewing your work is that maybe you will come upon a better (shorter, more elegant, simpler) proof, or even a better result. You may be inspired to investigate other propositions of a similar nature that occur to you as plausible conjectures.

What you have been obliged to discover by yourself leaves a path in your mind which you can use again when the need arises.

LICHTENBERG

IN CONCLUSION: EDIFICATION

Relax and enjoy yourself; or, at least, congratulate yourself before going on to the next problem.

BIBLIOGRAPHY

1. James L. Adams, *Conceptual Blockbusting: A Guide to Better Ideas*. W.H. Freeman Co., San Francisco, 1974.
2. Thomas Butts, *Problem Solving in Mathematics: Elementary Number Theory and Arithmetic*. Scott, Foresman, Glenview, Illinois, 1973.
3. George Pólya, *How to Solve It: A New Aspect of Mathematical Method*, 2nd edition. Doubleday, Garden City, New York, 1957.
4. Wayne A. Wickelgren, *How to Solve Problems: Elements of a Theory of Problems and Problem Solving*. W.H. Freeman Co., San Francisco, 1974.

Mathematics Department, Beloit College, Beloit, Wisconsin 53511.

*

*

*

To all our readers

A MERRY CHRISTMAS AND A HAPPY NEW PROBLEM-SOLVING YEAR.

Nancy Makila, Fred Maskell, and Léo Sauv .

THE PRIME LIFE OF OUR YEARS

ROBERT S. JOHNSON

The number of prime years in the twentieth century is 13, which is itself a prime number. These years were and will be

1901, 1907, 1913, 1931, 1933, 1949, 1951, 1973, 1979, 1987, 1993, 1997, 1999.

At this stage, we are not too concerned about past history, but this year, 1978 (composite), and next year, 1979 (prime), are of some interest to those of us who like to examine numbers with a microscope.

Peering closely at 1979 first, we find that by using the available digits there are 31 (a prime) ways of making numbers of two, three, and four digits. Eliminating 13 (a prime) composites leaves us with 18 prime numbers, about 60% of the total stock, which is rather remarkable. These primes are

17, 19, 71, 79, 97, 179, 197, 199, 719, 919,

971, 991, 997, 1979, 1997, 7919, 9719, 9791.

It will be noted that there are several instances where the reverse of a prime is also a prime, the most interesting case being 1979 - 9791, where the entire four-digit number is reversed. Also notable is 1979 - 7919, in which adjoining prime pairs are interchanged.

Looking at 1978 similarly, we have a composite year which yields even more primes. This time there are 60 ways of making two-, three-, and four-digit numbers from the available digits, but only $1/3$ (a prime reciprocal) of these yield prime numbers. The 20 primes (two more than for 1979) are

17, 19, 71, 79, 89, 97, 179, 197, 719, 971, 1789,

1879, 1987, 8179, 8719, 8971, 9187, 9781, 9817, 9871.

Comparing these results with those for 1979, we see that changing 9 to 8 exactly halves the number of three-digit primes and exactly doubles the number of four-digit primes. In only two instances do we have adjoining prime pairs: in 8971 and 1789, the latter being the year of the French Revolution. Fortunately time is running out: the Québec Revolution will not occur in the 1978 permutation of these digits, so we are safe at least until 1987 (a prime year for revolution!). The prime 1987 is one of the rare instances in which the digits are decreasing and cyclically consecutive (omitting zero when it's in the way). Only six such primes

are known. They are

19, 43, 109, 1987, 10987, 76543.

As a parting shot before I go, add '78 and '79 and you'll get 157, another prime!

As a Parthian shot after I've gone, adjoin '78 and '79 and you'll get 7879, another prime!

You'll never catch me now.

433 Laird Boulevard, Montréal, Québec H3R 1Y3.

;

*

*

ALGORITHMS AND POCKET CALCULATORS:

SQUARE ROOTS: IV

CLAYTON W. DODGE

In spite of its easy application and its actual use in some computers, Newton's method¹ for finding square roots apparently has not found use in pocket calculator programs. One manufacturer definitely does not use it and, from external appearances, others probably follow suit.

Hewlett-Packard², as reported in the *Hewlett-Packard Journal* [1], uses the long division method for finding square roots. Since algorithms involving many multiplications and divisions are too slow, only additions and subtractions should be used whenever possible. Newton's method does not satisfy this requirement, so Hewlett-Packard turned its attention to a modified version of the long division method, which we now develop by finding $\sqrt{54756}$.

Writing 54756 as 5.4756×10^4 , with the exponent on 10 even, we need examine only the mantissa 5.4756, which lies between 1 and 100. Subtract successive squares of integers from the mantissa until a negative difference appears and then restore the immediately preceding value. Thus we have

$$5.4756 - 1^2 = 4.4756 \geq 0,$$

$$5.4756 - 2^2 = 1.4756 \geq 0,$$

$$5.4756 - 3^2 = -3.5244 < 0.$$

¹See earlier articles in this series: [1978: 96-99, 154-157, 217-222]. (Editor)

²In response to the author's letter of inquiry, Jerry D. Fisner, National Advertising Manager for Hewlett-Packard, very kindly sent copies of the *Hewlett-Packard Journal* containing articles on calculator accuracy and algorithms. The material that follows is taken from one of these articles through the courtesy of the Hewlett-Packard Company.

So 2 is the first digit of the square root, and we retain

$$5.4756 - 2^2 = 1.4756.$$

To avoid multiplication in calculating the squares, we recall that

$$n^2 = 1 + 3 + 5 + \dots + (2n - 1). \quad (1)$$

Hence, only additions need be performed if we subtract successive terms of the series for n^2 . We thus calculate

$$5.4756 - 1 = 4.4756 \geq 0,$$

$$4.4756 - 3 = 1.4756 \geq 0,$$

$$1.4756 - 5 = -3.5244 < 0.$$

The third iteration produces a negative result, so the second iteration is the one retained, and the correct first digit of the square root is 2.

Now shift the decimal point to the right one group of (two) digits, so our partial root now is $a = 20$ and the remainder is 147.56. As in (1), we have

$$(2a + b)b = (2a + 1) + (2a + 3) + \dots + (2a + (2b - 1)), \quad (2)$$

so we subtract successive terms of this series from the remainder, obtaining

$$147.56 - 41 = 106.56 \geq 0,$$

$$106.56 - 43 = 63.56 \geq 0,$$

$$63.56 - 45 = 18.56 \geq 0,$$

$$18.56 - 47 = -28.44 < 0.$$

Hence 3 is the next digit and 18.56 is the new remainder. Again shifting the decimal point one group to the right, we obtain the divisor 230 and remainder 1856. We repeat the algorithm, getting

$$1856 - 461 = 1395 \geq 0,$$

$$1395 - 463 = 932 \geq 0,$$

$$932 - 465 = 467 \geq 0,$$

$$467 - 467 = 0 \geq 0,$$

$$0 - 469 = -469 < 0.$$

So the third and last digit is 4, and the required square root is

$$2.34 \times 10^2 = 234.$$

One clever modification simplifies the procedure greatly: *multiply everything by 5*. Then (1) and (2) become

$$5n^2 = 5 + 15 + 25 + \dots + (10n - 5)$$

and

$$5(2a + b)b = (10a + 5) + (10a + 15) + \dots + (10a + (10b - 5)).$$

The modified algorithm thus starts with $5 \times 5,4756 = 27,3780$, and we have

$$27,3780 - 5 = 22,3780 \geq 0,$$

$$22,3780 - 15 = 7,3780 \geq 0,$$

$$7,3780 - 25 = -17,6220 < 0.$$

The next-to-last remainder is restored and the first digit of the root is the 2 in the 25 just subtracted. For successive stages we shift the decimal point in the remainder two places to the right, and to get the required $10a + 5$ for the next series of subtractions we insert a zero before the terminal 5 in the last minuend 25, obtaining 205. Then we calculate

$$737,80 - 205 = 532,80 \geq 0,$$

$$532,80 - 215 = 317,80 \geq 0,$$

$$317,80 - 225 = 92,80 \geq 0,$$

$$92,80 - 235 = -142,20 < 0.$$

The 23 from the last minuend gives the first two digits of the root, and the next-to-last difference 92,80 is restored. Another shift prepares us for the next iteration. We get

$$9280 - 2305 = 6975 \geq 0,$$

$$6975 - 2315 = 4660 \geq 0,$$

$$4660 - 2325 = 2335 \geq 0,$$

$$2335 - 2335 = 0 \geq 0,$$

$$0 - 2345 = -2345 < 0.$$

Now the digits of the root 202 are all but the final 2 in the last minuend and the remainder, if needed, is the next-to-last difference, in this case 0.

The Hewlett-Packard calculators use this modification of the long division algorithm to calculate a 12-digit mantissa, round back to 10 digits, and then affix the proper power of 10 exponent, as discussed earlier. So a process that, at first glance, seems more tedious and time-consuming can actually be faster for a machine. In this case we require one division (to divide the exponent by 2), one multiplication (to multiply the mantissa by 5), and then only additions, subtractions, and shifts, a very simple process for a calculator.

I suspect that Newton's method is not used for calculating square roots in any

of the current pocket calculators. By timing the square root process in the small sample of calculators available to me, I have observed less time required for

$$\sqrt{1.0000002} = 1.0000001 \quad \text{than for} \quad \sqrt{99.999999} = 9.9999999,$$

quite consistent with the Hewlett-Packard long division process and wholly inconsistent with Newton's divide-and-average method which should take the same amount of time for either root.

REFERENCE

1. W.E. Egbert, Personal Calculator Algorithms I: Square Roots. *Hewlett-Packard Journal*, 28 (May 1977) 22-24.

Mathematics Department, University of Maine, Orono, Maine 04469.

*

*

*

PROBLEMS - - PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk () after a number indicates a problem submitted without a solution.*

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before March 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

391. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Here is a word definition that is also an alphametic:

A
SUN
DRIED
GRAPE
RAISIN

Solve this decimal addition, bearing in mind that of course the digits of UP go that way.

392* *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Find all natural numbers n for which $n^8 - n^2$ is not divisible by 504.

393, Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.

If $f_n(a_i) = (a_i - a_1) \dots (a_i - a_{i-1})(a_i - a_{i+1}) \dots (a_i - a_n)$, prove that, for

$k = 0, 1, \dots, n-2$,

$$\sum_{i=1}^n \frac{a_i^k}{f_n(a_i)} = 0.$$

(This is given without proof in H.F. Baker's *An Introduction to Plane Geometry*, Chelsea, Bronx, N.Y., 1971, p. 340.)

394, Proposed by Harry D. Ruderman, Hunter College Campus School, New York.

A wine glass has the shape of an isosceles trapezoid rotated about its axis of symmetry. If R , r , and h are the measures of the larger radius, smaller radius, and altitude of the trapezoid, find $r : R : h$ for the most economical dimensions.

395* Proposed by Kenneth S. Williams, Carleton University, Ottawa.

In Crux 247 [1977: 131; 1978: 23, 37] the following inequality is proved:

$$\frac{1}{2n^2} \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{a_n} \leq A - G \leq \frac{1}{2n^2} \frac{\sum_{1 \leq i < j \leq n} (a_i - a_j)^2}{a_1},$$

where A (resp. G) is the arithmetic (resp. geometric) mean of a_1, \dots, a_n . This is a refinement of the familiar inequality $A \geq G$. If H denotes the harmonic mean of a_1, \dots, a_n , that is,

$$\frac{1}{H} = \frac{1}{n} \left(\frac{1}{a_1} + \dots + \frac{1}{a_n} \right),$$

find the corresponding refinement of the familiar inequality $G \geq H$.

396, Proposed by Viktors Linis, University of Ottawa.

Given is the following polynomial with some undetermined coefficients denoted by stars:

$$x^{10} + *x^9 + *x^8 + \dots + *x^2 + *x + 1.$$

Two players, in turn, replace one star by a real number until all stars are replaced. The first player wins if all zeros of the polynomial are imaginary, the second if at least one zero is real. Is there a winning strategy for the second player?

397, Proposed by Jack Garfunkel, Forest Hills H.S., Flushing, N.Y.

Given is $\triangle ABC$ with incenter I . Lines AI , BI , CI are drawn to meet the

incircle (I) for the first time in D, E, F respectively. Prove that

$$(AD + BE + CF)\sqrt{3}$$

is not less than the perimeter of the triangle of maximum perimeter that can be inscribed in circle (I).

398. *Proposed by Murray S. Klamkin, University of Alberta.*

Find the roots of the $n \times n$ determinantal equation

$$\left| \frac{1}{x\delta_{rs} + k_r} \right| = 0,$$

where δ_{rs} is the Kronecker delta (= 1 or 0 according as $r=s$ or $r \neq s$).

399*. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

A prime magic square of order 3 is a square array of 9 distinct primes in which the three rows, three columns, and two main diagonals all add up to the same *magic constant*. What prime magic square of order 3 has the smallest magic constant

- (a) when the 9 primes are in arithmetic progression;
- (b) when they are not.

400. *Proposed by Andrejs Dunkels, University of Luleå, Sweden.*

In the false bottom of a chest which had belonged to the notorious pirate Capt. Kidd was found a piece of parchment with instructions for finding a treasure buried on a certain island. The essence of the directions was as follows.

"Start from the gallows and walk to the white rock, counting your paces. At the rock turn left through a right angle and walk the same number of paces. Mark the spot with your knife. Return to the gallows. Count your paces to the black rock, turn right through a right angle and walk the same distance. The treasure is midway between you and the knife."

However, when the searchers got to the island they found the rocks but no trace of the gallows remained. After some thinking they managed to find the treasure anyway. How?

(This problem must be very old. I heard about it in my first term of studies at Uppsala.)

*

*

*

Several readers include with their solutions photocopies of relevant references. This is extremely helpful to the editor, who does not have immediate access to a good mathematical library. Others are urged to do likewise when convenient.

*

*

*

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

332, [1978: 100, 266] Late solution by DONALD P. SKOW, McAllen H.S., McAllen, Texas.

334, [1978: 101] Proposed by Philip D. Straffin, Beloit College, Beloit, Wisconsin.

Let A, B, C be three fixed noncollinear points in the plane, and let X_0 be the centroid of $\triangle ABC$. Call a point P in the plane *accessible* from X_0 if there is a sequence of points $X_0, X_1, \dots, X_n = P$ such that X_{i+1} is closer than X_i to at least two of the points A, B, C ($i = 0, 1, \dots, n-1$). Characterize the set of points in the plane which are accessible from X_0 .

(One application is to the effect of agenda control on committee decisions. In this interpretation A, B, C are the preferred points of the three committee members in a two-dimensional policy space, and the X_i are proposals to be voted on by majority rule.)

Solution by the proposer.

All points in the plane are accessible from X_0 . This can be proved by a variety of means, including a non-constructive topological approach which shows that the set of accessible points is both open and closed in the plane. Here is a constructive geometric proof.

Let r_a, r_b, r_c denote respectively reflections in the lines BC, CA, AB. The points X_0 and $r_a X_0$ are equidistant from B as well as from C. Since $r_a X_0$ is not on the line BC, there are points arbitrarily close to $r_a X_0$ which are closer than X_0 to both B and C, and hence are accessible from X_0 in one step. Similarly, applying r_b and then r_c shows that there are points arbitrarily close to $r_c r_b r_a X_0$ which are accessible from X_0 in three steps. But it is a standard fact in transformational geometry (proved in [1], for example) that $r_c r_b r_a$ is a glide reflection with nonzero displacement. (A *glide reflection* is the "sum", in either order, of a reflection in a line l and a displacement parallel to l .) Hence iterating $r_c r_b r_a$ shows that there are points Q arbitrarily far from X_0 which are accessible from X_0 in $3k$ steps.

Now let P be any point in the plane, let

$$m = \max \{d(A, X_0), d(B, X_0), d(C, X_0)\},$$

and let Q be any point accessible from X_0 in $3k$ steps such that

$$d(Q, X_0) > d(P, X_0) + 2m.$$

We claim that P is closer than Q to all of A, B, C. For instance,

$$\begin{aligned} d(A, P) &\leq d(A, X_0) + d(X_0, P) < m + d(Q, X_0) - 2m = d(Q, X_0) - m \\ &\leq d(Q, A) + d(A, X_0) - m \leq d(Q, A), \end{aligned}$$

and similarly $d(B, P) < d(B, Q)$ and $d(C, P) < d(C, Q)$.

This proves that P is accessible from X_0 , for one can go from X_0 to a suitable point Q in $3k$ steps and then to P in one more step.

The implication for decision-making in a three-person committee operating by majority rule in a two-dimensional policy space is that he who controls the agenda has absolute power, at least if members vote in the mechanistic way described in the problem!

Generalizations of this phenomenon in a political context have been studied by Robert McKelvey in [2] and [3].

Also solved by F.G.B. MASKELL, Collège Algonquin, Ottawa.

Editor's comment.

Maskell pointed out that, whatever political advantage there may be in choosing the centroid for the initial point X_0 , the fact is of no geometric import. X_0 can be any interior point of $\triangle ABC$.

For related material by our proposer see [4].

REFERENCES

1. I.M. Yaglom, *Geometric Transformations I*, Random House/Singer, New York, 1962, p. 54.
2. R.D. McKelvey, Intransitivities in multidimensional voting models and some implications for agenda control, *Journal of Economic Theory*, 12 (1976) 472-482.
3. ———, *General conditions for global intransitivities in formal voting models*, paper presented to the Public Choice Society, 1977.
4. Philip D. Straffin Jr., The Power of Voting Blocs: An Example, *Mathematics Magazine*, 50 (January 1977) 22-24.

*

*

*

335, [1978: 101] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Trouver une condition nécessaire et suffisante pour que l'équation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

ait l'une de ses racines égale au carré de l'autre.

Solution by Charles W. Trigg, San Diego, California.

Suppose the roots of the given equation are r and r^2 , so that $r^2 + r = -b/a$ and $r^3 = c/a$. Since

$$(r^2 + r)^3 = r^3(r+1)^3 = r^3\{r^3 + 3(r^2 + r) + 1\},$$

we have the necessary condition

$$\left(-\frac{b}{a}\right)^3 = \frac{c}{a}\left(\frac{c}{a} - \frac{3b}{a} + 1\right), \quad (1)$$

which is equivalent to

$$b^3 + ca(c+a) = 3abc. \quad (2)$$

This condition is also sufficient. For suppose (2) holds and let the roots of the given equation be R and r . Substituting $-b/a = R+r$ and $c/a = Rr$ in the equivalent condition (1) gives

$$(R+r)^3 = Rr\{Rr + 3(R+r) + 1\},$$

which is equivalent to

$$(R^2 - r)(R - r^2) = 0$$

and shows that one root is the square of the other.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; STEVE CURRAN for the Beloit College Solvers, Beloit, Wisconsin; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; J.A.H. HUNTER, Toronto, Ontario; ROBERT S. JOHNSON, Montréal, Québec; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; F.G.B. MASKELL, Collège Algonquin, Ottawa; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, The University of Wisconsin-Oshkosh; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; the late R. ROBINSON ROWE, Sacramento, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposer.

Editor's comment.

Although none of the solutions submitted could truthfully be called incorrect, only five or six were considered to be satisfactory. One, for example, gave as the necessary and sufficient condition "that $c/a = r^3$ and $-b/a = r + r^2$ both hold simultaneously for the same value of r ", which rather seems to beg the question. Others merely spouted truisms as they skated around the question without firmly coming to grips with it. But the most subtle weakness, present in six of the solutions, was to "derive" as the necessary and sufficient condition something like

$$\xi = -\sqrt[3]{-2\tau} - \sqrt[3]{2\sigma}, \quad (3)$$

which is not equivalent to (2) unless τ , ξ , and σ be real, a condition which was

not guaranteed by the proposal. Indeed, (3) is ambiguous when a , b , and c are complex, since it does not specify *which* cube root is to be used.

*

*

*

336, [1978: 101] *Proposed by Viktors Linis, University of Ottawa.*

Prove that if in a convex polyhedron there are four edges at each vertex, then every planar section which does not pass through any vertex is a polygon with an even number of sides.

Solution by F.G.B. Maskell, Algonquin College, Ottawa.

In the given polyhedron, each vertex is of degree 4 (i.e. is incident with 4 edges). A plane which does not pass through any vertex cuts this polyhedron into two convex polyhedra with a common face F which is a convex polygon of, say, m sides, and we must show that m is even.

Let P be one of the two polyhedra resulting from the section. P has m vertices of degree 3 (the vertices of F) and its remaining vertices, say n in number, are all of degree 4. The sum of the degrees of the vertices of P is thus $3m + 4n$. Since each edge of P is incident with two vertices, the number of edges of P , which must be an integer, is

$$\frac{3m + 4n}{2} = \frac{3m}{2} + 2n,$$

and so m must be even.

Also solved by RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; ROBERT S. JOHNSON, Montréal, Québec; and LEROY F. MEYERS, The Ohio State University.

Editor's comment.

The proposer found this problem in the Russian journal *Kvant* (No. 6, 1977, Problem M403). Their solution, if any, was not submitted with the proposal, and I have no idea what it is.

Gibbs used Euler circuits to prove the following more general result:

If a planar section of a convex polyhedron passes through no vertex, then the number of sides of the section will have the same parity as the number of vertices of odd degree of the original polyhedron which lie on either side of the plane.

Since, in the original polyhedron, the number of vertices of odd degree must be even, it is clear that this theorem can be verified from either of the two polyhedra resulting from the section. The verification can be done more easily with the method (and the notation) used in our featured solution, as follows:

Suppose, as before, that F has m sides but that P has n_0 original vertices of even degree and n_1 of odd degree. The number of edges of P is then

$$\frac{3m + 2jn_0 + (2k+1)n_1}{2} = \frac{3m+n_1}{2} + jn_0 + kn_1,$$

so m and n_1 must have the same parity.

*

*

*

337, [1978: 101] Proposed by V.G. Hobbes, Westmount, Québec.

If p and q are primes greater than 3, prove that $p^2 - q^2$ is a multiple of 24.

Solution by Kenneth S. Williams, Carleton University, Ottawa.

More generally, we show that if p and q are both integers of the form $6k \pm 1$ (all primes greater than 3, and many other numbers, are of this form), then $p^2 - q^2$ is a multiple of 24.

Let $p = 6m + a$ and $q = 6n + b$, where $a^2 = b^2 = 1$; then

$$\begin{aligned} p^2 - q^2 &= (6m + a)^2 - (6n + b)^2 \\ &= 36(m^2 - n^2) + 12(am - bn) \\ &= 36(a^2m^2 - b^2n^2) + 12(am - bn) \\ &= 12(am - bn)\{3(am + bn) + 1\}. \end{aligned}$$

Clearly one of $am - bn$ and $3(am + bn) + 1$ is even, so $24 \mid p^2 - q^2$.

Also solved by MYOUNG HEE AN, 1978 graduate, Long Island City H.S., New York; LEON BANKOFF, Los Angeles, California; W.J. BLUNDON, Memorial University of Newfoundland; LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.A.H. HUNTER, Toronto, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; J.D.E. KONHAUSER, Macalester College, Saint Paul, Minnesota; ARTHUR KROEBER, sophomore, St. Ann's School, Brooklyn, N.Y.; B.B. KUNDU, Research Scholar, Indian Institute of Technology, Kharagpur, India; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; LAI LANE LUEY, Willowdale, Ontario; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; F.G.B. MASKELL, Collège Algonquin, Ottawa; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, The University of Wisconsin-Oshkosh; JEREMY PRIMER, student, Columbia H.S., Maplewood, N.J.; HYMAN ROSEN, Yeshiva University H.S., Brooklyn, N.Y.; RICHARD ROSENTHAL, St. Ann's School, Brooklyn, N.Y.; the late R. ROBINSON ROWE, Sacramento, California; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; CHARLES W. TRIGG, San Diego, California; ALAN WAYNE, Pasco-Hernando Community College, New Port Richey, Florida; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Only six of the twenty-nine solvers seemed to be aware that the property extended to *all* numbers of the form $6k \pm 1$, not just to primes greater than 3.

*

*

*

regardless of the size of the circle.) Then ACDB is an inscribed rectangle whose diagonals AD and BC can be drawn with the card (using Crux 257 if necessary) to intersect at the required centre.

IV. *Solution by the proposer.*

The answer to the question is YES if a VISA card is considered to be a *markable* straightedge. (Its length does not matter by Crux 257.) Mark two points on the card at a distance less than one-fifth the length of the card (possibly much less if the circle is small) and use them to mark off five equally-spaced points on the boundary of the circle. Draw the line determined by the first and fourth points and the line determined by the second and fifth points. The line joining the third point to the intersection of these two lines passes through the center of the circle. Repeat the process at another place on the circle. This determines the center of the circle.

V. *Second solution by Clayton W. Dodge.*

Buy a compass and charge it to your VISA account. With the compass, and the card as a straightedge, the center of the circle is easily found.

If there is a bit of unused credit in your VISA account, but not enough for the price of a new compass, go to a flea market, buy a *rusty compass* (one with a fixed opening), and charge its price of a few cents to your VISA account. It is known [2, p. 174] that all Euclidean constructions can be carried out with a straightedge (your VISA card) and a rusty compass. It does not matter how close to disintegration your rusty compass is, for you will need to use it to draw only one circle and can then discard it [2, p. 175].

VI. *Solution by Jeremy Primer, student, Columbia H.S., Maplewood, N.J.*

[If you are so deep in debt that you cannot even use your VISA card to buy a rusty compass], punch two holes in the card (better wait until it has expired), stick a pin in one hole and a pencil in the other and you have a compass with a fixed opening. You are then able to carry out all Euclidean constructions, for your VISA card serves as both a straightedge and a rusty compass.

Also solved by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; PAUL J. CAMPBELL for the Beloit College Solvers, Beloit, Wisconsin; HIPPOLYTE CHARLES, Waterloo, Québec; ANDREJS DUNKELS, University of Luleå, Sweden; ROBERT S. JOHNSON, Montréal, Québec; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; LEROY F. MEYERS, The Ohio State University; and the late R. ROBINSON ROWE, Sacramento, California.

Editor's comment.

Some solvers claimed they could, in a single operation, use the card to erect a perpendicular to a line at a given point of the line, which is manifestly impossible because of the rounded corners. Possibly they are cash customers who have never seen a credit card. And one used the card to draw tangents to the circle "by eye". If the eye is used, as well as the card, as a quasi-Euclidean instrument, then much more is possible, like trisecting an angle or winking at pretty mathematicians (as a prelude to some possibly non-Euclidean operations).

REFERENCES

1. A.S. Smogorzhevskii, *The Ruler in Geometrical Constructions* (translated from the Russian by Halina Moss). Blaisdell, New York, 1961, pp. 75-77.
2. Howard Eves, *A Survey of Geometry*, Revised Edition. Allyn and Bacon, Boston, 1972.

*

*

*

330* [1978: 102] Proposed by Steven P. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.

Is $\binom{37}{2} = 666$ the only binomial coefficient $\binom{n}{r}$ whose decimal representation consists of a single digit repeated $k(\geq 3)$ times?

Comment by Leroy F. Meyers, The Ohio State University.

The proposer obviously meant to exclude such trivial answers as

$$\binom{77777}{1} = \binom{77777}{77776} = 77777,$$

so the problem should be amended to read:

Are $\binom{37}{2} = \binom{37}{35} = 666$ the only binomial coefficients $\binom{n}{r}$, with $1 < r < n-1$,

whose decimal representation consists of $k(\geq 3)$ identical digits?

I verified (with my calculator, only partially automatedly) that there are no other solutions with up to 16 digits. One reason why more such binomial coefficients are unlikely to exist (subjective probability—mine) is that at least one of the numbers $n, n-1, \dots, n-r+1$ must be divisible by the largest prime divisor of the repunit

$$R_k = \frac{10^k - 1}{9} = 11\dots 1 \quad (k \text{ 1's})$$

when $\binom{n}{r}$ has a decimal representation consisting of k identical digits. If R_k is

prime, or if its largest prime factor is relatively large, then even $\binom{n}{2}$ is likely to be too large to be divisible by R_k (i.e. to contain k identical digits). If all the prime divisors of R_k are relatively small, then $\binom{n}{2}$ or $\binom{n}{3}$ is unlikely to be divisible by all these divisors if it is to be less than 10^k .

Comments were also received from J.D.E. KONHAUSER, Macalester College, St. Paul, Minnesota; JEREMY PRIMER, student, Columbia H.S., Maplewood, N.J.; and the late R. ROBINSON ROWE, Sacramento, California.

Editor's comment.

The other comments received did little more than point out the trivial answers that the wording of the problem unfortunately allowed.

*

*

*

340. [1978: 102] Proposed by Léo Sauvé, Algonquin College (editor).

(This is designed to help those readers who find it hard to shake the habit of sending in only answers to proposed problems.)

Find a problem whose answer is $\frac{22}{7} - \pi$.

I. Problem by Kenneth S. Williams, Carleton University, Ottawa.

Evaluate the integral

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

Solution. We have

$$\begin{aligned} I &= \int_0^1 \left(4 - 4x^2 + 5x^4 - 4x^6 + x^8 - \frac{4}{1+x^2} \right) dx \\ &= 4x - \frac{4x^3}{3} + x^5 - \frac{2x^6}{3} + \frac{x^7}{7} - 4 \tan^{-1} x \Big|_0^1 \\ &= \frac{22}{7} - \pi. \end{aligned}$$

Euler's beta function yields the related integral

$$\int_0^1 x^4(1-x)^4 dx = B(5,5) = \frac{(4!)^2}{9!} = \frac{1}{630},$$

which can be used to estimate $\frac{22}{7} - \pi$. Since clearly

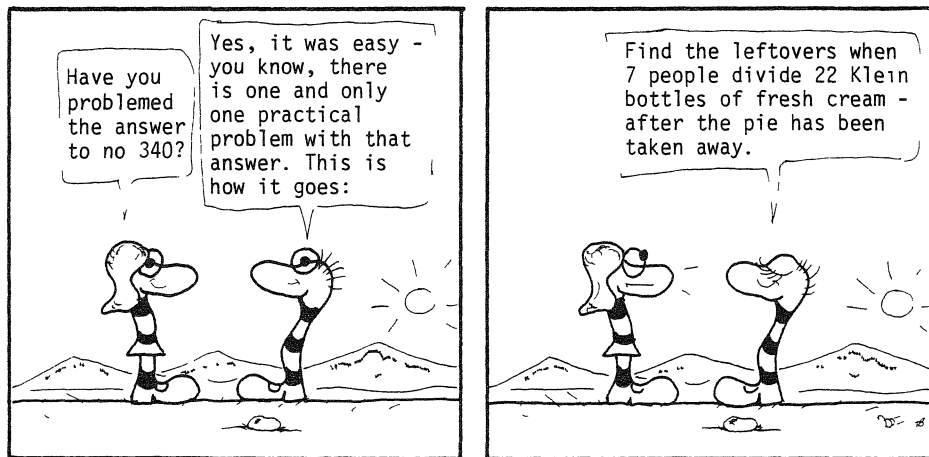
$$\frac{1}{2} \int_0^1 x^4(1-x)^4 dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx < \int_0^1 x^4(1-x)^4 dx,$$

we have

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}.$$

II. Problem by Andrejs Dunkels, University of Luleå, Sweden.

FOOTIES AT NO 340



III. Comment by Peter A. Lindstrom, Genesee Community College, Batavia, N.Y.

This problem (i.e. receiving only answers without solutions) is one that all Problem Editors have to face. The following shows that it is *not* a new problem:

A certain rich Man had 100 Orchards, in each Orchard was 100 Appletrees, under each Appletree was 100 Hogsties, in each Hogstie was 100 Sows, and each Sow had 100 Pigs. Question, How many Sow-Pigs were there among them?

Note, The Answers to this Question won't be accepted without the Solution.

This item appeared, exactly as given above, in Benjamin Franklin's *Poor Richard's Almanack* for 1734. A faithful reproduction can be found in *The Complete Poor Richard Almanacks*, published by Benjamin Franklin, Reproduced in Facsimile with an Introduction by Whitfield J. Bell, Jr., Volume I, 1733-1747, Imprint Society, Barre, Massachusetts, 1970, page 43.

IV. Problem by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.

I am unrepentantly sending in only a problem for the proposed answer, but no solution:

What is the absolute error made in assuming that $\pi = \frac{22}{7}$?

Also problemed by LOUIS H. CAIROLI, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono (two problems); MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus (two problems); ROBERT S. JOHNSON, Montréal, Québec; J.D.E. KONHAUSER, Macalester College, St. Paul, Minnesota; N. KRISHNASWAMY, student, Indian Institute of Technology, Kharagpur, India; ANDY LIU, University of Alberta, Edmonton; BASIL C. RENNIE, James Cook University of North Queensland, Australia; the late R. ROBINSON ROWE, Sacramento, California; KENNETH M. WILKE, Topeka, Kansas (two problems); and the proposer.

Editor's comment.

The integral in our problem I was located by Rennie in [1], and by Konhauser in [2] as well as in the 29th William Lowell Putnam Mathematical Competition which took place on December 7, 1968.

This astonishing integral shows that if in mathematics there is not an answer to every problem there is at least a problem for every answer. This is in sharp contrast with medical science, for example, where it is not unheard of for someone to find a cure for which there is no known disease.

The following delightful poem¹ is particularly appropriate here:

LAST THINGS FIRST

Solutions to problems
are easy to find:
the problem's a great
contribution.
What is truly an art
is to wring from your mind
a problem to fit
a solution.

PIET HEIN, *Grooks* 3.

REFERENCES

1. J.C. Burkill, *A First Course in Mathematical Analysis*. Cambridge University Press, 1962, p. 148.
2. J.C. Burkill and H.M. Cundy, *Mathematical Scholarship Problems*. Cambridge University Press, 1962, p. 38.

*

*

*

¹Reprinted by permission of the author, Piet Hein, and PaperJacks Ltd., Toronto, Ontario, The Canadian Paperback Publishers, publishers of *Grooks* 1, 2, 3, 4, 5. PaperJacks Ltd. is a subsidiary of General Publishing Co. Limited.

341. [1978: 133] *Proposé par Herman Nyon, Paramaribo, Surinam.*

L'addition décimale suivante est doublement vraie, et SEPT est divisible par 7. Reconstituer ses chiffres.

$$\begin{array}{r} \text{TROIS} \\ \text{TROIS} \\ \text{SEPT} \\ \text{SEPT} \\ \hline \text{VINGT} \end{array}$$

I. *Solution by A.*

...so there are exactly two solutions to the alphametic:

$$23059 + 23059 + 9812 + 9812 = 65742,$$

$$23859 + 23859 + 9012 + 9012 = 65742.$$

But in neither of them is SEPT divisible by 7. So the problem as proposed has no solution.

II. *Solution by B.*

...so there are exactly eight solutions to the alphametic, in only one of which SEPT is divisible by 7:

$$25814 + 25814 + 4732 + 4732 = 61092.$$

III. *Solution by C.*

...so the unique solution with SEPT divisible by 7 is

$$25814 + 25814 + 4732 + 4732 = 61092.$$

IV. *Solution by D.*

...so there are exactly two solutions to the alphametic:

$$42013 + 42013 + 3864 + 3864 = 91754,$$

$$42813 + 42813 + 3064 + 3064 = 91754.$$

But only the first of these has SEPT divisible by 7 and provides the unique answer to our problem.

V. *Solution by E.*

...so the unique solution with SEPT divisible by 7 is

$$42013 + 42013 + 3864 + 3864 = 91754.$$

VI. *Solution by F_1 and F_2 (independently).*

...so the unique solution with SEPT divisible by 7 is

$$23759 + 23759 + 9142 + 9142 = 65802.$$

VII. *Solution by Allan W. Johnson Jr., Washington, D.C.*

A computer enumeration (print-out enclosed) shows that this decimal alphametic has exactly *eighteen* solutions, in exactly three of which SEPT is divisible by 7. These are

$$25814 + 25814 + 4732 + 4732 = 61092,$$

$$42013 + 42013 + 3864 + 3864 = 91754,$$

$$23759 + 23759 + 9142 + 9142 = 65802.$$

Thinking that we might find in English the uniqueness that eludes us in French, I did a computer enumeration (print-out enclosed) of the decimal alphametic

$$\text{THREE} + \text{THREE} + \text{SEVEN} + \text{SEVEN} = \text{TWENTY}.$$

There are six solutions, none of which have SEVEN divisible by 7, but three of which have THREE divisible by 3. These are

$$16755 + 16755 + 35852 + 35852 = 105214,$$

$$16755 + 16755 + 35954 + 35954 = 105418,$$

$$16077 + 16077 + 57879 + 57879 = 147912.$$

All solutions received are accounted for above.

Editor's comment.

No comment.

*

*

*

342* [1978: 133] *Proposed by James Gary Propp, Great Neck, N.Y.*

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets S_1, S_2, \dots, S_n as follows: for each positive integer m , we have $m \in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m \in S_n$ for all sufficiently large m . (If $n = 2$, this is essentially equivalent to Problem 226 [1977: 205].)

Editor's comment.

No solutions were received for this problem, which therefore remains open.

*

*

*

343. [1978: 133] *Proposed by Steven F. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

It was proved in Problem 166 [1976 : 231] that the greatest integer function satisfies the functional equation

$$f(nx) = \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

for all real x and positive integers n . Are there other functions which satisfy this equation? Find as many as possible.

Partial solution by Leroy F. Meyers, The Ohio State University.

The greatest integer function, whose values we denote as usual by

$$f(x) = [x], \quad (1)$$

is replicative¹, as noted in the proposal. We will identify several more replicative functions and investigate some properties common to all such functions.

(a) The function whose values are given by $f(x) = x - \frac{1}{2}$ is replicative, for

$$\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = \sum_{k=0}^{n-1} \left(x - \frac{1}{2} + \frac{k}{n}\right) = n\left(x - \frac{1}{2}\right) + \frac{1}{n} \cdot \frac{(n-1)n}{2} = nx - \frac{1}{2} = f(nx).$$

(b) The functional equation in the proposal is obviously homogeneous linear; hence if f_1 and f_2 are replicative functions, and α and β are constants, then $\alpha f_1 + \beta f_2$ is also replicative. In particular, the functions defined by

$$f_\alpha(x) = \alpha \left(x - \frac{1}{2}\right) \quad (2)$$

for any constant α are all continuous replicative functions [and furthermore *every* continuous replicative function is of the form (2)]².

(c) Suppose g is replicative and define h by $h(x) = g(1-x)$; then h is replicative since

$$\begin{aligned} h(nx) &= g(1-nx) = g\left(n\left(\frac{1}{n} - x\right)\right) = \sum_{k=0}^{n-1} g\left(\frac{1}{n} - x + \frac{k}{n}\right) \\ &= \sum_{j=0}^{n-1} g\left(\frac{1}{n} - x + \frac{n-1-j}{n}\right) = \sum_{j=0}^{n-1} g\left(1 - \left(x + \frac{j}{n}\right)\right) \\ &= \sum_{j=0}^{n-1} h\left(x + \frac{j}{n}\right). \end{aligned}$$

¹Any function which satisfies the functional equation in the proposal for all positive integers n and for any real x is called *replicative* by Knuth [1, p. 42], and it will be convenient to use this term consistently from now on. (Editor)

²Added by the editor. For a proof see [1, p. 478].

It now follows from (b) that the function f defined by

$$f(x) = g(x) + g(1-x)$$

is replicative whenever g is replicative. In particular, for $g(x) = [x]$, we obtain the replicative function defined by

$$f(x) = [x] + [1-x] = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

(d) Let g and h be distinct replicative functions such that $g(x) = h(x)$ for $0 < x < 1$. Such functions exist; for example, the constant-zero function (defined by (2) for $\alpha = 0$), and the functions defined by (1) and (3): all three vanish for $0 < x < 1$. We show that any two such functions can be combined to form new replicative functions (other than mere linear combinations of already known ones, which are replicative by (b)). Let g_- be the restriction of g to $x < 1$, h_+ the restriction of h to $x > 0$, and let $f = g_- \cup h_+$, that is,

$$f(x) = \begin{cases} g(x) & \text{if } x < 1, \\ h(x) & \text{if } x > 0. \end{cases}$$

We show that f is replicative. If $x \leq 0$, then $nx \leq 0 < 1$ and $x + k/n < 1$; and if $x > 0$, then $nx > 0$ and $x + k/n > 0$. The functional equation is thus satisfied by f for all x since it is satisfied by g_- and h_+ .

For example, with the three replicative functions mentioned in the last paragraph, six new replicative functions can be formed, one of them given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ [x] & \text{if } x \geq 0. \end{cases} \quad (4)$$

Another one is given by (compare with (3))

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a positive integer,} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

(e) Any replicative function is completely determined by its values on any open interval containing both 0 and 1, but not by its values on the closed interval $[0,1]$ alone. For suppose the replicative function f is known on the open interval (a,b) , where $a < 0 < 1 < b$. If $x > 0$, then for some positive integer n we have $0 < x/n < b-1$; and then $f(x)$, or $f\left(n \cdot \frac{x}{n}\right)$ is determined by the values of f at

$$\frac{x}{n}, \frac{x}{n} + \frac{1}{n}, \dots, \frac{x}{n} + \frac{n-1}{n},$$

all of which lie in the interval $(0, b)$ since

$$\frac{x}{n} + \frac{n-1}{n} = \frac{x}{n} + 1 - \frac{1}{n} < b - 1 + 1 - \frac{1}{n} < b.$$

Similarly, if $x \leq 0$ then for some positive integer n we have $a < x/n \leq 0$, and $f(x)$ is then determined by its values at places between x/n and $x/n + (n-1)/n$, both of which lie in the interval $(a, 1)$. Observe, however, that the functions defined by (1) and (4) coincide on $[0, 1]$ (on $[0, +\infty)$ in fact) but have different values for all $x < 0$.

(f) Two real numbers x_1 and x_2 are said to be *rationally related* just when there are rational numbers $r \neq 0$ and s such that $x_2 = rx_1 + s$. It is easily seen that being rationally related is an equivalence relation which partitions the real numbers into infinitely many equivalence classes, one of them being the set of all rational numbers. Now the functional equation relates the values of f only at rationally related places, and so the way that f is defined on one equivalence class is independent of the way it is defined on any other equivalence class. This opens the way to define an unlimited number of new replicative functions. However, f cannot be defined arbitrarily on any given equivalence class, since the functional equation restricts it considerably. It suffices if f is defined on a given equivalence class (or union of such) by the restriction to that equivalence class (or union) of a known replicative function. For example, f is replicative if

$$f(x) = \begin{cases} x - \frac{1}{2} & \text{if } x \text{ is rational (or algebraic),} \\ [x] & \text{if } x \text{ is irrational (or transcendental).} \end{cases}$$

A comment was received from JEREMY PRIMER, student, Columbia H.S., Maplewood, N.J.

Editor's comment.

Reference [1] was located by Primer. In it, in addition to several of the functions (1)-(5), Knuth lists the following two interesting replicative functions:

$$f(x) = g(x - [x]), \text{ where } g \text{ is replicative;}$$

$$f(x) = \log |2 \sin \pi x|, \text{ if the value } f(x) = -\infty \text{ is allowed.}$$

REFERENCE

1. Donald E. Knuth, *The Art of Computer Programming*, Second Edition, Volume 1: Fundamental Algorithms, Addison-Wesley, 1973, pp. 42, 477-478.

*

*

*

INDEX TO VOLUME 4, 1978

AUTHORS OF ARTICLES AND NOTES

- AEPPLI, ALFRED. More on the theorems of Ceva and Menelaus, 153.
BOTTEMA, O. A problem of Pieter Nieuwland, 92.
CAMPBELL, PAUL J. A pep talk on problem-solving for post-calculus students, 272.
DODGE, CLAYTON W. Algorithms and pocket calculators: Square roots: I, 96.
_____. Algorithms and pocket calculators: Square roots: II, 154.
_____. Algorithms and pocket calculators: Square roots: III, 217.
_____. Our own Mother Goose, 223.
_____. Algorithms and pocket calculators: Square roots: IV, 279.
DUNKELS, ANDREJS. Footies at the Taylor's, 5.
_____. En effeuillant la marguerite, 90.
_____. A riddle, 216.
EDITOR, THE. Our journal has a new name, 89.
FISCHLER, ROGER. Théories mathématiques de la Grande Pyramide, 122.
GIBBS, RICHARD A. Thinking of the Fundamental Theorem of Arithmetic? Think again!, 158.
HENNEY, ALAN G. and DAGMAR R. Computer oriented solutions, 212.
HUNTER, J.A.H. and SAUVÉ, LÉO. Barbeau consecutive products, 244.
JOHNSON, ROBERT S. Remainders from the library of E.P.B. Umbugio, 188.
_____. The prime life of our years, 278.
KLAMKIN, MURRAY S. A polynomial functional equation, 32.
LIU, A. A direct proof of Pick's Theorem, 242.
MANDAN, SAHIB RAM. A sample of ancient Indian mathematics, 186.
MEYERS, LEROY F. Conics and converses, 2.
NOËL, DR. A.H. A Formula for polytopes?, 223.
NYON, HERMAN. Another ten-digit man, 60.
ORR, EDITH. The Decapus, 240.
PEDOE, DAN. The power of the power concept, 62.
QUADLING, DOUGLAS. Fibonacci on the road, 185.
ROWE, the late R. ROBINSON. High School: letters, logic and sin, 6.
_____. Après la trigonométrie, quoi?, 63.
_____. Harvard was old hat, 129.
_____. Cashing in on mathematics, 189.
SALVATORE, GALI and SHIMA, TIMOTHY. Of coconuts and integrity, 182.
SAUVÉ, LÉO and HUNTER, J.A.H. Barbeau consecutive products, 244.
SHIMA, TIMOTHY and SALVATORE, GALI. Of coconuts and integrity, 182.
SHOOK, T.W., II. An application of the Dirac impulse function to vertebrate anatomy, 188.
TIWARI, RAM REKHA. A panmagic square for 1978, 270.
TRIGG, CHARLES W. Supplementary list of references to the Morley Theorem, 132.

TITLES OF ARTICLES AND NOTES

- ALGORITHMS AND POCKET CALCULATORS: SQUARE ROOTS: I, II, III, IV. Clayton W. Dodge, 96, 154, 217, 279.
ANOTHER TEN-DIGIT MAN. Herman Nyon, 60.

- APPLICATION OF THE DIRAC IMPULSE FUNCTION TO VERTEBRATE ANATOMY. T.W. Shook, II, 188.
- APRÈS LA TRIGONOMÉTRIE, QUOI? The late R. Robinson Rowe, 63.
- BARBEAU CONSECUTIVE PRODUCTS. J.A.H. Hunter and Léo Sauvé, 244.
- CASHING IN ON MATHEMATICS. The late R. Robinson Rowe, 189.
- COMPUTER ORIENTED SOLUTIONS. Alan G. and Dagmar R. Henney, 212.
- CONICS AND CONVERSES. Leroy F. Meyers, 2.
- DECAPUS, THE. Edith Orr, 240.
- DIRECT PROOF OF PICK'S THEOREM, A. A. Liu, 242.
- EN EFFEUILLANT LA MARGUERITE. Andrejs Dunkels, 90.
- FIBONACCI ON THE ROAD. Douglas Quadling, 185.
- FOOTIES AT THE TAYLOR'S. Andrejs Dunkels, 5.
- FORMULA FOR POLYTOPES?, A. Dr. A.H. Noël, 223.
- HARVARD WAS OLD HAT. The late R. Robinson Rowe, 129.
- HIGH SCHOOL: LETTERS, LOGIC AND SIN. The late R. Robinson Rowe, 6.
- HOW'S THAT AGAIN? L.F.M., 249.
- MORE ON THE THEOREMS OF CEVA AND MENELAUS. Alfred Aeppli, 153.
- OF COCONUTS AND INTEGRITY. Timothy Shima and Gali Salvatore, 182.
- OOPS! 131.
- OUR JOURNAL HAS A NEW NAME. The Editor, 89.
- OUR OWN MOTHER GOOSE. Clayton W. Dodge, 223.
- PANMAGIC SQUARE FOR 1978, A. Ram Rekha Tiwari, 270.
- PEP TALK ON PROBLEM-SOLVING FOR POST-CALCULUS STUDENTS. Paul J. Campbell, 272.
- POLYNOMIAL FUNCTIONAL EQUATION, A. Murray S. Klamkin, 32.
- POTPOURRI. 99.
- POWER OF THE POWER CONCEPT. Dan Pedoe, 62.
- PRIME LIFE OF OUR YEARS, THE. Robert S. Johnson, 278.
- PROBLEM OF PIETER NIEUWLAND, A. O. Bottema, 92.
- PUBLISHING NOTE. 34, 226.
- REMAINDERS FROM THE LIBRARY OF E.P.B. UMBUGIO. Robert S. Johnson, 188.
- RIDDLE, A. Andrejs Dunkels, 216.
- R. ROBINSON ROWE, 1896-1978. 152.
- SAMPLE OF ANCIENT INDIAN MATHEMATICS, A. Sahib Ram Mandan, 186.
- SUPPLEMENTARY LIST OF REFERENCES TO THE MORLEY THEOREM. Charles W. Trigg, 132.
- THÉORIES MATHÉMATIQUES DE LA GRANDE PYRAMIDE. Roger Fischler, 122.
- THINKING OF THE FUNDAMENTAL THEOREM OF ARITHMETIC? THINK AGAIN! Richard A. Gibbs, 158.

LETTERS TO THE EDITOR

- | | |
|-----------------------------|-----------------------|
| Leon Bankoff, 8. | Leroy F. Meyers, 270. |
| Dagmar Henney, 8. | Dan Pedoe, 33. |
| Allan Wm. Johnson Jr., 190. | Dan Sokolowsky, 33. |

REVIEW

- Kenneth M. Wilke, 248.

PROBLEMS AND SOLUTIONS

The numbers refer to the pages in which the corresponding name appears with

a problem proposal, a solution, or a comment.

AN, MYOUNG HEE: 70.
ARENDS, PETER: 226.
AVITAL, SHMUEL: 84.
BAKER, DON: 116.
BANKOFF, FRANCINE: 251.
BANKOFF, LEON: 13, 27, 35, 52, 78,
148, 207, 263, 267.
BÉTANCOURT, ROBERT: 39.
BLUNDON, W.J.: 13, 58, 80, 85, 88,
136, 162, 251.
BOTTEMA, O.: 17, 110, 193, 207, 233.
BURGMEIER, JAMES W: 178.
BURNS, RICHARD: 230.
CAMPBELL, PAUL J.: 87, 107.
CARTWRIGHT, D.I.: 38.
CDC-7600: 104.
CHARLES, HIPPOLYTE: 36, 49, 101, 161,
174, 192, 208, 228, 250, 286.
COHEN, CECILE M.: 150.
CONRAD, STEVEN R.: 12, 19, 70, 86,
102, 133, 148, 198, 224, 282, 292,
297.
DAVIS, C.A.: 36, 231.
DHOMBRES, JEAN: 172.
DIETSCHKE, KRISTIN: 230.
DODGE, CLAYTON W.: 27, 28, 37, 43,
54, 73, 109, 114, 142, 167, 238,
290, 291.
DUNKELS, ANDREJS: 284, 294.
DWORSCHAK, H.G.: 205.
ECKER, MICHAEL W.: 35, 172, 209, 226.
EDDY, ROLAND H.: 191.
EDITOR, THE: 29.
FIELD, M.J.: 38.
FIELD, RICHARD S.: 43.
FISHER, DAVID: 49.
FRIETAG, HERTA T.: 228.
GARFUNKEL, JACK: 12, 65, 202, 255,
283.
GIBBS, RICHARD A.: 22, 118, 119.
GIRI, G.C.: 196.
GOLDMAN, AL: 254.
GOODMAN, A.W.: 115, 195.
GREITZER, SAMUEL L.: 116.
HENDERSON, G.P.: 226.
HOBBS, V.G.: 101, 225, 289.
HORNSTEIN, BARRY: 52.
HUNTER, J.A.H.: 100, 264.
JANES, LEIGH: 36, 235.
JOHNSON, ALLAN WM., JR.: 177, 226,
235, 265, 282, 297.
JOHNSON, ROBERT S: 37, 41, 53, 55,
75, 116, 118, 195.
JUST, ERWIN: 114.
KARR, DAVID: 235.
KAUFMAN, IRWIN: 12, 196.
KAY, DAVID C.: 229.
KESSLER, GILBERT W.: 39, 66, 72, 105,
134, 147, 224, 262, 284.
KLAMKIN, MURRAY S.: 13, 50, 67, 81,
87, 112, 115, 134, 135, 161, 170,
193, 225, 255, 263, 284.
KLEIMAN, MARK: 260.
KOST, LARRY L.: 178.
KRAVITZ, SIDNEY: 35, 159, 204, 250.
LINDSTROM, PETER A.: 56, 294.
LINIS, VIKTORS: 11, 17, 57, 101, 116,
133, 177, 178, 192, 283, 288.
LIU, A.: 115, 192, 260.
LUEY, LAI LANE: 115, 192.
MANDAN, SAHIB RAM: 21, 71, 192, 239,
283.
MASKELL, F.G.B.: 40, 66, 110, 117,
161, 164, 170, 260, 288.
McCOLL, BRUCE: 11, 20, 180.
McWORTER, W.A., JR.: 12, 54, 101, 109,
115, 135, 199, 290, 291.
MEYERS, LEROY F.: 11, 19, 22, 40, 50,
56, 57, 100, 110, 111, 117, 134,
139, 160, 176, 209, 225, 252, 266,
267, 292, 298.
NELSON, HARRY L.: 53, 102.
NYON, HERMAN: 11, 18, 133, 174, 296.
PEDOE, DAN: 110.
PENNER, SIDNEY: 107, 114, 159, 225.
PRIELIPP, BOB: 74, 117, 164.
PRIMER, JEREMY: 252, 291.
PROBLEM-SOLVING GROUP AT LARRY'S BAR:
55.
PROPP, JAMES GARY: 36, 133, 160, 230,
297.
RAMOS, ORLANDO: 35, 159, 227.
RENNIE, BASIL C.: 24, 66, 162, 195,
255, 257, 258.
REYNOLDS, LINDSAY: 79.
ROKHSAR, DANIEL: 75.
ROSEN, HYMAN: 176.
ROWE, the late R. ROBINSON: 35, 44,
50, 55, 101, 108, 134, 142, 147,
149, 160, 191, 203, 205, 228,
268, 269.

CHARLES, HIPPOLYTE: 29,30,69,70,109,130,
268.
CLOSS, MICHAEL P.: 110.
CONRAD, STEVEN R.: 10,22,27,42,45,53,54,
65,82,130,142,166,189,202,226.
DePEIZA, RADFORD: 166.
DILLON, DOUG: 28,30,49,53,56,165.
DODGE, CLAYTON W.: 9,28,29,30,42,58,65,
79,105,131,136,140,157,159,164,199,
205,206,228,261,262,263,264,298,299.
DUNKELS, ANDREJS: 49,50,136.
DWORSCHAK, H.G.: 70,112,298.
EUSTICE, DAN: 47,68.
EWEN, IRA: 22.
FIELD, RICHARD S.: 154.
FISHER, DAVID: 154.
FLEGLER, DANIEL: 54,165.
FREITAG, HERTA T.: 172.
GAUSS, CARL FRIEDRICH: 240,253.
GIBBS, RICHARD A.: 251,268.
GOODMAN, A.W.: 250.
GREITZER, SAMUEL L.: 19,53,57,136.
GRIFFITHS, T.J.: 70,173.
HONSBERGER, ROSS: 10,79,144.
HORNSTEIN, BARRY: 155.
JOHNSON, ROBERT S.: 154,251.
JUST, ERWIN: 250.
KARAM, JOHN: 83,195.
KESSLER, GILBERT W.: 43,57,132,172,189,
227,297.
KLAMKIN, MURRAY S.: 10,65,75,112,137,144,
155,160,196,203,226,240,251,258,298.
KLEIMAN, MARK: 69.
KONHAUSER, J.D.E.: 113.
LaMACCHIA, SAM: 197.
LeSAGE, JACK: 199.
LINDSTROM, PETER A.: 155.
LINIS, VIKTORS: 13,22,23,69,104,105,174,
236,238,252,257,258,259,260.
LYNCH, JUDY: 164.
LYNCH, J. WALTER: 144.
MARION, JACQUES: 44.
MASKELL, F.G.B.: 58,172,227,297.
McCOLL, BRUCE: 42,65,130,165,200,266.
McNAMEE, JOHN J.: 130,265.
McWORTER, W.A., Jr.: 112,155,227,250.
MEYERS, LEROY F.: 10,23,25,26,42,48,54,
69,71,72,73,77,81,111,136,137,159,169,
173,174,175,191,198,227,238,251.
NELSON, HARRY L.: 67,106,108,155.
NYON, HERMAN: 265.
ORR, EDITH: 235.
PEDOE, DAN: 10,13,46,75,79,133,143,266.
PENNER, SIDNEY: 227,250.
PRIELIPP, BOB: 262.
RAM MANDAN, SAHIB: 74,112,189,205.
RENNIE, BASIL C.: 11,25,44,105,112,168,
173,174,252,261,297.
RIDGE, H.L.: 139,203.
ROKHSAR, DANIEL: 9,73,137,190.
ROWE, R. ROBINSON: 10,43,45,58,66,71,77,
78,85,140,164,167,173,174,202,205,227,
233,236,251,267.
RUDERMAN, HARRY D.: 297.
SALVATORE, GALI: 50,53,111,134,190,233,
257.
SAUVÉ, JACQUES: 155.
SAUVÉ, LÉO (EDITOR): 11,23,133,134,228,
298.
SILVERMAN: DAVID L.: 42,66,167,168,205.
SOKOLOWSKY, DAN: 26,43,65,131,133,143,
160,175,190,197,204.
STONE, DAVID R.: 30,43,47,76,82,142,157,
172,173,174,198,234,297.
TRIGG, CHARLES W.: 9,16,57,58,66,109,112,
130,138,156,168,200,203,226,230,265,
297.
VANBRUGGHE, BERNARD: 19,70.
VENESS, JOHN: 190.
WAYNE, ALAN: 189,250.
WHEELER, DAVID: 190.
WILKE, KENNETH M.: 66,76,81,131,138,174,
190,231,298.
WILLIAMS, KENNETH S.: 10,12,20,30,44,46,
50,56,74,108,131,132,142,157,191,193,
252.