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- **Françoise Viète and the quest for π**
- **Constructing a triangle**
- **Bucky Ball models for the continents**

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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François Viète and the Quest for π

G. C. BUSH

Viète's infinite product for $2/\pi$ is obtained by considering the areas of polygons inscribed in a circle and with ever more sides. The proof of a companion formula for $\pi/4$ employs polygons circumscribed to a circle.

Ever since the time of Archimedes, mathematicians — both professional and amateur — have been fascinated by the task of finding better and better rational approximations to π, which is equivalent to finding more and more digits in the decimal representation of π. François Viète (1540–1603) was one of the most successful amateurs to tackle the problem. He was educated as a lawyer and worked first as a private tutor and then as a civil servant to the French parliament. During the war with Spain he was a decoder. His interest in cosmology and astronomy led him to mathematics, especially trigonometry and algebra. Although he had limited time to spend on his scientific hobbies, he was able to work with such a high degree of concentration that he became what Cajori calls the most eminent French mathematician of the 16th century (reference 1).

In about 240 BC, Archimedes obtained the first truly scientific approximation to π. He began by inscribing a regular hexagon inside a circle and calculating the perimeter of the hexagon. He found a simple way to calculate the perimeter when the number of edges of a regular polygon was doubled. By repeatedly doubling the number of edges he was able to derive a lower bound for the value of π. Proceeding similarly with circumscribed polygons, Archimedes also found an upper bound. Archimedes did not have decimal notation available to him, but nevertheless he was able to find fractions as upper and lower bounds that were equivalent to giving the value of π correct to two decimal places. For many centuries Archimedes' method remained the standard for approximating π. In 1579 Viète used this method to find the value of π to 9-digit accuracy. He must have been a very patient man! He began with a hexagon and doubled the number of edges 16 times!

If Viète had been content with this approximation, he may not have been much more than a footnote in the history of π. However, instead of spending more energy on trying to find a few more digits in the value for π, he concentrated on the method itself and arrived at the first algebraic expression for π. We say 'algebraic' with an apology. All the symbols are algebraic, but there is an infinite number of them. We know now that any 'formula' for π must involve an infinite process. What is new in Viète's discovery is that geometry has faded into the background in the proof and the result is expressed in algebraic terms.

Viète's proof employed some ideas from classical geometry that are no longer part of the usual tool kit of a mathematician. Instead, we shall use the proof suggested by Boyer and Merzbach (reference 2) which is based on trigonometry. We can be confident that Viète would have approved of the method since he contributed to the discovery of formulae for the trigonometric functions of multiple and fractional angles.

The basic idea is still that used by Archimedes, except that now areas rather than perimeters are considered. We start with an inscribed square and calculate the area of inscribed regular polygons of 4, 8, 16, … edges. For convenience we shall use a circle of radius 1. Then this sequence of areas should converge to π. From figure 1 it is a simple matter to show that the area of the square is 2.

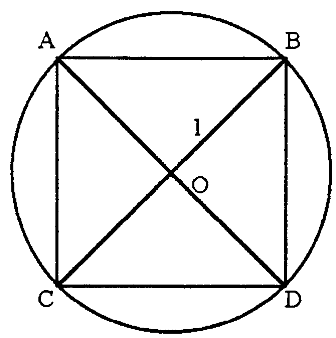

Figure 1.

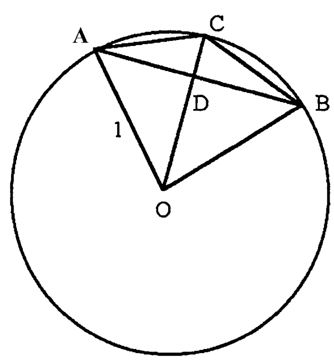

Figure 2.

In figure 2, AB represents one edge of a regular polygon with n edges. AC and CB are the edges that replace it when we double the number of edges. Angle AOB is $(2\pi)/n$, so angle AOD is π/n. Since the radius OA is 1, OD $= \cos(\pi/n)$ and DA $= \sin(\pi/n)$. Then the area of triangle OAB is $\cos(\pi/n)\ \sin(\pi/n)$, and the area a_n of the n-sided polygon is $n\cos(\pi/n)\ \sin(\pi/n)$. Also, $a_{2n} = 2n\cos(\pi/2n)\sin(\pi/2n) = n\sin(\pi/n)$. The sequence $a_4, a_8, \ldots, a_n, a_{2n}, \ldots$ should converge to π.

Viète's formula is usually expressed by using the reciprocals of this sequence.

Since

$$\frac{a_n}{a_{2n}} = \frac{n\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{n\sin\left(\frac{\pi}{n}\right)} = \cos\left(\frac{\pi}{n}\right)$$

i.e.

$$\frac{1}{a_{2n}} = \frac{1}{a_n}\cos\left(\frac{\pi}{n}\right),$$

we are dealing with the sequence

$$\begin{aligned}\frac{1}{a_4} &= \tfrac{1}{2}\,,\\ \frac{1}{a_8} &= \tfrac{1}{2}\cos\left(\frac{\pi}{4}\right)\,,\\ \frac{1}{a_{16}} &= \tfrac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\,,\\ &\cdots.\end{aligned}$$

This is where the half-angle formulae come to our rescue. Since

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}}$$

provided $\alpha/2$ is in the first quadrant,

$$\frac{1}{a_{32}} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}.$$

The pattern is becoming clear, so we have Viète's formula in the form in which it usually appears today:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots. \tag{1}$$

Calculating rational approximations to π from this formula would be quite a chore, especially without calculators or computers. However, its importance is not as a computational tool, but as a theoretical breakthrough.

Evidently, with enough patience, we could calculate more and more accurate rational approximations to π. It is also clear that all of these values will be too small, since all the polygons are inscribed in the circle. The idea of obtaining upper bounds for π by using circumscribed polygons goes back to before Archimedes. Viète appears not to have explored the possibility of combining that idea with his more algebraic methods. However, with our trigonometric methods it is not difficult to find an upper bound companion for Viète's formula.

We again begin with a circle of radius 1, but now we circumscribe regular polygons around it. Starting with a square we repeatedly double the number of edges. Obviously, a square circumscribed about a circle of radius 1 has edges of length 2 and hence has area 4. We need a simple method for carrying out the doubling process. In figure 3, AB represents one edge of a circumscribed polygon with n edges. Angle AOD is π/n, so AD $= \tan(\pi/n)$, since the radius is 1. The area of triangle AOB is $\frac{1}{2}$AB $\cdot$ OD $=$ AD $\cdot$ OD $= \tan(\pi/n)$. If A_n represents the area of the circumscribed regular polygon with n edges, then $A_n = n\tan(\pi/n)$. The sequence $A_4, A_8, A_{16}, \ldots$ converges to π. This time it is easier to deal with the sequence directly, rather than with its reciprocals. We see that

$$\begin{aligned}\frac{A_{2n}}{A_n} &= \frac{2n\tan\left(\frac{\pi}{2n}\right)}{n\tan\left(\frac{\pi}{n}\right)}\\ &= 2\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\cdot\frac{\sin\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)} = \frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}\,,\end{aligned}$$

i.e.

$$A_{2n} = A_n\frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}\,.$$

Expressions for $\cos(\pi/4), \cos(\pi/8), \cos(\pi/16), \ldots$ are available from formula (1) for $2/\pi$. Therefore, after some additional algebra,

$$\begin{aligned}A_4 &= 4,\\ A_8 &= A_4\cdot\frac{2\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)} = A_4\frac{2\sqrt{2}}{2+\sqrt{2}}\,,\\ A_{16} &= A_8\cdot\frac{2\cos\left(\frac{\pi}{8}\right)}{1+\cos\left(\frac{\pi}{8}\right)} = A_8\frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\,,\\ A_{32} &= A_{16}\cdot\frac{2\cos\left(\frac{\pi}{16}\right)}{1+\cos\left(\frac{\pi}{16}\right)} = A_{16}\frac{2\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\,,\\ &\cdots\end{aligned}$$

A clear, although slightly complicated, pattern is emerging to give us a companion to Viète's formula:

$$\frac{\pi}{4} = \frac{2\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\cdot\frac{2\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots. \tag{2}$$

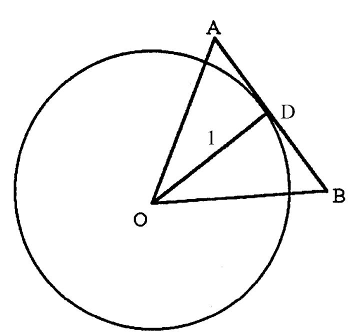

Figure 3.

Viète's formula gives a sequence of approximations converging to π from below; the companion formula (2) gives a sequence converging to π from above. The two formulae together give us a measure of the accuracy of the current approximation. Here are the first few approximations:

n	a_n	A_n
4	2	4
8	2.828	3.314
16	3.061	3.183
32	3.121	3.152

The amount of calculation at each step is considerable, and the convergence is too slow to give much joy to a person who is trying to obtain a very accurate approximation to π. The importance of Viète's formula was that it gave a new direction to the study of π, the direction of infinite algebraic rather than geometric processes. The interest of the companion formula is that it fills rather belatedly a gap in the history of π.

It is of interest to note that (2) can actually be obtained from (1) — see the problem in the Problems and Solutions Section. However, if (2) is derived from (1) in this way, the geometrical interpretation of (2) is lost.

References

1. Florian Cajori, *The History of Mathematics*, 2nd edn. (Macmillan, New York, 1919).
2. Carl B. Boyer and Uta C. Merzbach, *A History of Mathematics*, 2nd edn. (Wiley, New York, 1989).

__G. C. Bush__ is a retired associate professor who taught mathematics and computing in several universities and colleges in Canada and the Middle East. His most recent regular appointment was at Middle East Technical University in Ankara, Turkey. He is a Canadian and now lives in a suburb of Halifax, Nova Scotia. After his retirement he taught on a casual basis at Dalhousie University. He is a co-author of Basic Concepts of Mathematics *and has written several research papers. He frequently serves as a lay preacher in his own and in other churches. He is married and has two grown sons and three grandchildren.*

Circles in the Right Triangle

JENS CARTENSEN

The right triangle seems to many mathematicians to be a thoroughly well-known (or even dull) object in plane geometry. It is amazing, however, how many little known facts about the right triangle lie around waiting to be picked up by the passing mathematician. Below, I shall do some of the 'picking up'.

We shall study a special configuration connected with the right triangle, namely the incircles of the two right triangles in which the altitude on the hypotenuse divides the original triangle. In this simple geometrical picture lie hidden a series of impressive, beautiful and quite unknown gems that deserve a wider appreciation.

For the sake of simplicity we denote by PQ the segment PQ itself, as well as the length | PQ | of the segment. In the same manner A is the point A as well as the measure in degrees of angle A of $\triangle$ABC.

Theorem 1. *Let* D *be the foot of the altitude h from the right angle C. If* r, r_1 *and* r_2 *are the radii of the incircles of the triangles* ABC, ACD *and* BCD *respectively, then*

$$h = r + r_1 + r_2 \,.$$

Proof. The situation is shown in figure 1. The triangles $\triangle$ABC and $\triangle$ACD are similar with the linear ratio c/b. This is also the ratio between the radii of the respective incircles, so that

$$\frac{r}{r_1} = \frac{c}{a} \qquad \text{or} \qquad b = \frac{cr_1}{r}\,. \tag{1}$$

In the same way $\triangle$ABC and $\triangle$BCD are similar with the linear ratio c/a so that

$$\frac{r}{r_2} = \frac{c}{a} \qquad \text{or} \qquad a = \frac{cr_2}{r}\,. \tag{2}$$

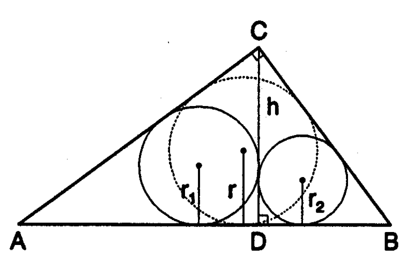

Figure 1.

If s denotes the semiperimeter of the triangle and T its area we have from (1) and (2):

$$2T = c \cdot h = 2r \cdot s = r(a+b+c) = r\Big(\frac{cr_2}{r} + \frac{cr_1}{r} + c\Big)$$
$$= cr_2 + cr_1 + cr,$$

and it follows that

$$h = r + r_1 + r_2\,.$$

Theorem 2. *The radii r, r_1 and r_2 of the three incircles satisfy*

$$r^2 = r_1^2 + r_2^2\,.$$

Proof. From theorem 1 we have

$$r_1 = \frac{rb}{c}\,, \qquad r_2 = \frac{ra}{c}$$

so that

$$r_1^2 + r_2^2 = \frac{r^2(a^2+b^2)}{c^2} = \frac{r^2c^2}{c^2} = r^2.$$

We shall prove a beautiful and little known result about the line connecting the centres R and S of the two smaller inscribed circles.

Theorem 3. *The line* RS *connecting the centres* R *and* S *of the incircles of* △ACD *and* △BCD *is perpendicular to the angle bisector* CG *of the right angle C.*

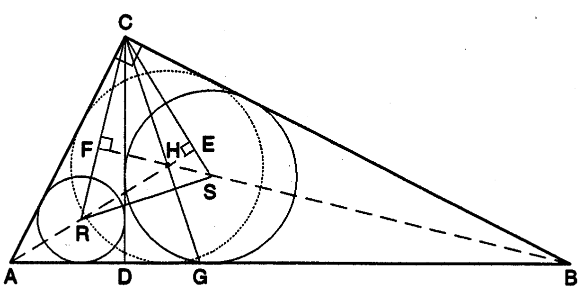

Figure 2.

Proof. We draw the lines CR, CS and RS as shown in figure 2. We then have

$$\angle\text{RCA} = \tfrac{1}{2}\angle\text{DCA} = \tfrac{1}{2}(90^\circ - A) = 45^\circ - \tfrac{1}{2}A$$

and

$$\angle\text{BCR} = 90^\circ - \angle\text{RCA} = 90^\circ - (45^\circ - \tfrac{1}{2}A) = 45^\circ + \tfrac{1}{2}A.$$

Let BS and CR intersect in F and AR and CS in E. Then we have in △FBC:

$$\angle\text{CFB} = 180^\circ - (\angle\text{BCR} + \tfrac{1}{2}B)$$
$$= 180^\circ - (45^\circ + \tfrac{1}{2}A + \tfrac{1}{2}B) = 90^\circ.$$

This means that the angle bisector FS (or BF) of angle B is an altitude on the side CR in △CRS. In the same manner we find that the angle bisector ER (or AE) of angle A is an altitude on the side CS in △CRS.

Now let H be the intersection of FS and ER, i.e. H is the intersection of the altitudes in △CRS. The third altitude from C in this triangle will pass through H as well, so that CH is perpendicular to RS. But since H is the intersection of the angle bisectors in △ABC, CH is also an angle bisector in △ABC. The point H is of course the incentre of △ABC.

We now consider the lines connecting the centres of the two smaller circles with the right angle C of △ABC.

Theorem 4. *Let* R *and* S *be centres of the incircles of* △ACD *and* △BCD *respectively.* CR *and* CS *intersect the hypotenuse* AB *in* P *and* Q. *Then* AC = AQ *and* BC = BP.

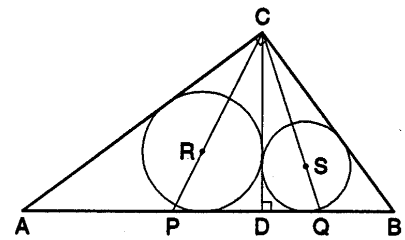

Figure 3.

Proof. We have that (figure 3)

$$\angle\text{BCQ} = \tfrac{1}{2}\angle\text{BCD} = \tfrac{1}{2}A,$$

so that

$$\angle\text{CQA} = 180^\circ - \angle\text{CQB}$$
$$= 180^\circ - (180^\circ - B - \angle\text{BCQ}) = B + \tfrac{1}{2}A.$$

Further we have

$$\angle\text{ACQ} = \angle\text{ACD} + \angle\text{DCQ} = B + \tfrac{1}{2}\angle\text{BCD} = B + \tfrac{1}{2}A.$$

But this means that △ACQ is isosceles and AC = AQ. In the same way we see that △BCP is isosceles, so BC = BP.

The points P and Q as intersection points with the hypotenuse have another interesting property.

Theorem 5. *The incentre* I *in* △ABC *is also the centre of the circumcircle of* △CPQ.

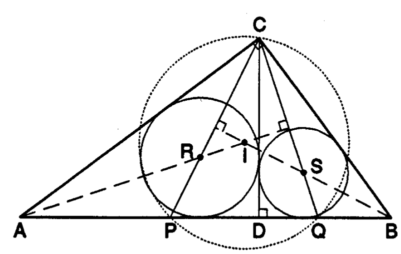

Figure 4.

Proof. Consider figure 4. Since $\triangle$ACQ is isosceles according to theorem 4, the angle bisector of A through R and I is perpendicular to CQ in the midpoint of CQ, i.e. AI is the perpendicular bisector of CQ. Since $\triangle$BCP is isosceles as well, the angle bisector of B through S and I is perpendicular to CP in the midpoint of CP. Thus the perpendicular bisectors of the sides CP and CQ of $\triangle$CPQ intersect in the point I.

We consider the centres R and S of the two small incircles. We have the following theorem.

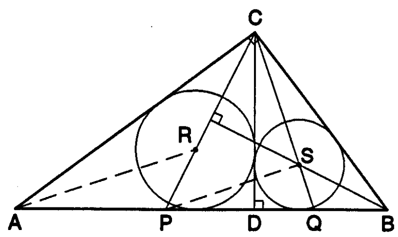

Figure 5.

Theorem 6. PS *is parallel to* AR *and* QR *to* BS *(figure 5).*

Proof. According to theorem 4, $\triangle$BCP is isosceles and BC = BP. But then

$$\angle \mathrm{CPB} = \angle \mathrm{BCP} = \angle \mathrm{BCD} + \angle \mathrm{DCP} = A + \tfrac{1}{2}B.$$

In $\triangle$BCP the line BS is angle bisector of the vertex angle, so BS is also the perpendicular bisector of the base CP. Thus SP = SC, so $\triangle$PSC is isosceles. In this triangle we have

$$\angle \mathrm{SPC} = \angle \mathrm{SCP} = \angle \mathrm{SCD} + \angle \mathrm{DCP} = \tfrac{1}{2}A + \tfrac{1}{2}B.$$

But then

$$\begin{aligned}\angle \mathrm{SPQ} &= \angle \mathrm{CPB} - \angle \mathrm{SPC} = A + \tfrac{1}{2}B - (\tfrac{1}{2}A + \tfrac{1}{2}B)\\ &= \tfrac{1}{2}A = \angle \mathrm{RAP}.\end{aligned}$$

From this we see that AR and PS are parallel. In the same manner QR and BS are parallel.

The centres R, S and I have a further property.

Theorem 7. *The triangles* $\triangle$AIB*,* $\triangle$BSC *and* $\triangle$ARC *are similar.*

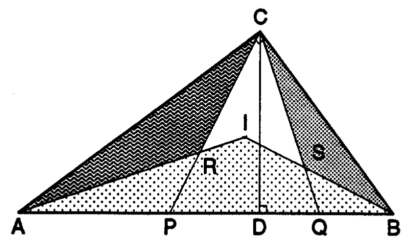

Figure 6.

Proof. We refer to figure 6. In $\triangle IAB$ we have

$$\angle \mathrm{IAB} = \tfrac{1}{2}A \qquad \text{and} \qquad \angle \mathrm{IBA} = \tfrac{1}{2}B.$$

In $\triangle BSC$ we see that

$$\angle \mathrm{SBC} = \tfrac{1}{2}B \qquad \text{and} \qquad \angle \mathrm{SCB} = \tfrac{1}{2}\angle \mathrm{DCB} = \tfrac{1}{2}A.$$

Finally, in $\triangle ARC$

$$\angle \mathrm{CAR} = \tfrac{1}{2}A \qquad \text{and} \qquad \angle \mathrm{ACR} = \tfrac{1}{2}\angle \mathrm{ACD} = \tfrac{1}{2}B.$$

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Solution to Braintwister 10

(Big fleas have little fleas . . .)

Answer: The big flea has walked 3 metres.
Solution: You can use quite tricky calculus to show that the big flea walks $\frac{3}{8}$ of the original distance apart, but you can deduce some crucial facts without calculus.

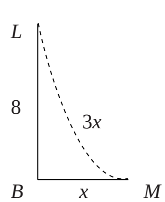

Figure 1.

If x is the distance walked by the big flea (and $3x$ the curved distance walked by the little flea) as shown, then clearly the length of the curved route is less than the distance from L to M via B;

$$\text{i.e.} \quad 3x < 8 + x \quad \text{or} \quad x < 4.$$

Also, the curved route from L to M is clearly longer than the straight-line route from L to M;

$$\text{i.e.} \quad 3x > \sqrt{8^2 + x^2} \quad \text{giving} \quad 8x^2 > 64 \quad \text{or} \quad x > 2\sqrt{2}.$$

Hence simple inequalities show that x lies strictly between $2\sqrt{2}$ and 4. But we are given the extra crucial fact that x is a whole number. Hence $x = 3$.

VICTOR BRYANT

Bucky Ball and Soccer Ball Models for the Earth's Continents

A. TAN

1. The bucky ball and soccer ball models

In 1985, new forms of carbon having cage-like structures were discovered at Rice University in Texas, which subsequently garnered a Nobel prize for the discoverers. The most common form consisted of 60 carbon atoms arranged in a regular fashion on a sphere. The structure resembled the geodesic domes of the architect Buckminster Fuller and was named 'Buckminster fullerene'. That name was quickly shortened to 'bucky ball'. If the carbon atoms are removed from the bucky ball and their bonds are represented by straight lines, a structure resembling that of the football or 'soccer ball' emerges (figure 1).

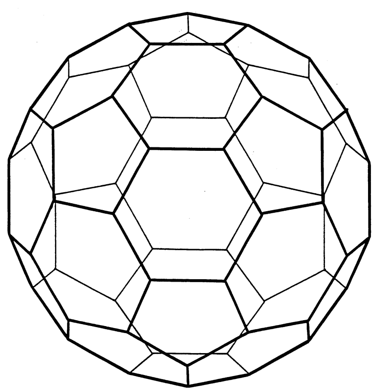

Figure 1. The bucky ball model of carbon-60.

The surface of the bucky ball (or the soccer ball for that matter) consists of 12 regular pentagons and 20 regular hexagons of equal sides. Each pentagon is entirely bounded by five hexagons whereas each hexagon is bounded by a ring consisting of alternate pentagons and hexagons. Two hexagons and a pentagon meet at each vertex.

In this article, the Earth, its continents and oceans, are described in the bucky ball and the soccer ball models. Interestingly, in both models, the areas of the Arctic Ocean and Antarctica are nearly equal to that of a pentagon, whereas the areas of the other continents are not far from those of a pentagon or a hexagon or a combination of both. Interestingly still, the locations of the Arctic Ocean and the continents either coincide with their respective polygons or are not far away from them.

It is commonly stated that the structure of the bucky ball is *identical* to that of the soccer ball. However, that statement is not entirely accurate. Since the carbon–carbon bond, like any other bond, is necessarily rectilinear, this means that the bucky ball is made up of flat pentagons and hexagons. The pentagons and hexagons of the soccer ball, on the other hand, are necessarily curved or 'spherical'. Evidently, the soccer ball model is more appropriate for the Earth than the bucky ball model. However, the bucky ball model utilises plane trigonometry, which is far more straightforward than the soccer ball model, which uses spherical trigonometry. It will be instructive to use both models and find out how the results compare.

2. Areas of pentagons and hexagons in the bucky ball model

The area of an n-sided regular polygon in plane trigonometry is quite easy to obtain. Divide the polygon into n equal isosceles triangles from the centre; calculate the area of each triangle and multiply by n. The areas of the regular pentagon P and the regular hexagon H of the same side d thus obtained are as follows:

$$P = \tfrac{5}{4}\cot\left(\frac{\pi}{5}\right)d^2 \approx 1.720d^2, \tag{1}$$

and

$$H = \tfrac{3}{2}\cot\left(\frac{\pi}{6}\right)d^2 \approx 2.598d^2. \tag{2}$$

The total surface area of the bucky ball is then

$$A = 12P + 20H \approx 72.607d^2. \tag{3}$$

The percentage areas p and h with respect to the total surface area of a pentagon and a hexagon, respectively, are obtained from (1), (2) and (3): $p \approx 2.370$ and $h \approx 3.578$. Thus, for instance, the percentage area of a pentagon and a hexagon is $p + h \approx 5.948$ and that of a pentagon and two hexagons is $p + 2h \approx 9.526$.

3. Areas of pentagons and hexagons in the soccer ball model

Next we shall calculate the areas of the spherical pentagons and hexagons in the soccer ball model. Since it is easier to use right spherical triangles than oblique spherical triangles in spherical trigonometric calculations (see reference 1), we divide each spherical pentagon into 10 right-angled spherical triangles whose vertices meet at the centre of the pentagon, the angle subtended by each being $\pi/5$ (36°) (see figure 2). The hexagon is likewise divided into 12 right-angled spherical triangles, which subtend angles of $\pi/6$ (30°) at the centre (see figure 3).

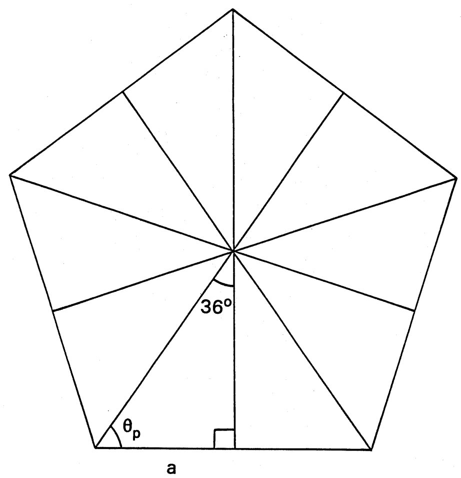

Figure 2. The spherical pentagon divided into 10 right-angled spherical triangles.

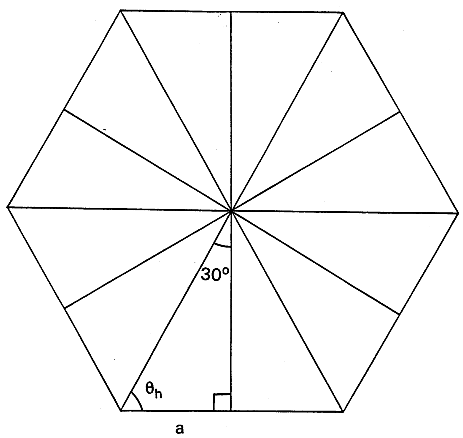

Figure 3. The spherical hexagon divided into 12 right-angled spherical triangles.

Since the areas of spherical triangles are entirely determined by their angles, the task at hand is to find the angles θ_p and θ_h of figures 2 and 3. For a right spherical triangle having angles *A, B* and $\pi/2$, one of several trigonometric relations is given by (see reference 1)

$$\cos A = \sin B \cos a. \tag{4}$$

Applying (4) to the pentagonal and hexagonal triangles, we get

$$\cos a = \frac{\cos(\pi/5)}{\sin\theta_p} = \frac{\cos(\pi/6)}{\sin\theta_h}. \tag{5}$$

Further, at each vertex, two hexagons meet one pentagon, which imposes the condition

$$2\theta_p + 4\theta_h = 2\pi. \tag{6}$$

Substituting θ_p from (6) into (5) and solving for θ_h, we get

$$\theta_h = \cos^{-1}\left[\frac{\cos(\pi/5)}{\cos(\pi/6)}\right] \approx 1.0848 \text{ rad} \approx 62.155°, \tag{7}$$

whence, from (6)

$$\theta_p = \pi - 2\theta_h \approx 0.971985 \text{ rad} \approx 55.691°. \tag{8}$$

Now, the area of a right spherical triangle having angles *A*, *B* and $\pi/2$ is given by (see reference 1)

$$\Delta = \left(A + B - \frac{\pi}{2}\right)R^2, \tag{9}$$

where *R* is the radius of the sphere. Thus, the areas of the pentagonal and hexagonal spherical triangles are, respectively,

$$\Delta_p = \left(\theta_p - \frac{3\pi}{10}\right)R^2, \tag{10}$$

and

$$\Delta_h = \left(\theta_h - \frac{\pi}{3}\right)R^2. \tag{11}$$

Here *R* is the radius of the Earth. The total areas of the spherical pentagons and hexagons are 10 and 12 times Δ_p and Δ_h, respectively. Using the earlier notations in the bucky ball model, we have: $P \approx 0.295R^2$ and $H \approx 0.451R^2$; whence $A = 12.566R^2 \approx 4\pi R^2$; $p \approx 2.348$ and $h \approx 3.591$; $p + h \approx 5.939$ and $p + 2h \approx 9.530$. These figures from the more accurate soccer ball model compare fairly well with those obtained in the cruder bucky ball model.

4. The continental areas as pentagons and hexagons

We can now examine the areas of the continents and their relationship, if any, to the polygon areas. The areas of the continents and the Arctic Ocean are taken from *The New York Times Atlas of the World* (reference 2) and entered in table 1. Also shown in the table are the respective areas as percentages of the Earth's surface area and as fractions of appropriate polygon areas. It is interesting to note that the continental areas are all quite close to units of pentagons or hexagons or combinations of the two. The Arctic Ocean and Antarctica are slightly larger than a pentagon whereas South America is within 4 percent of a hexagon. Australasia (including New Zealand and New Guinea) is only about 3/4 of a pentagon while North America is 38 percent larger than a hexagon. If one subtracts Greenland and the Canadian and Caribbean islands, or Central America, North America better approximates a hexagon. Africa is almost exactly equal to a pentagon plus a hexagon (within one-tenth of a percent), while Eurasia is marginally larger than a pentagon plus two hexagons. In summary, the pentagon represents Australasia and Antarctica, whereas the hexagon represents North and South America; Africa is represented by a pentagon plus a hexagon while Eurasia is represented by a pentagon and two hexagons. The total land masses of the world closely approximate (to within 7 percent) four pentagons and five hexagons.

Table 1. Continental areas as percentages of Earth's surface area.

Continent/Ocean	Area (km^2)	Percentage	Fraction of polygon
Arctic Ocean	14 056 000	2.755	1.173 p
Antarctica	13 340 000	2.615	1.114 p
Australasia	8 923 000	1.749	0.745 p
South America	17 611 000	3.452	0.961 h
North America	25 349 000	4.969	1.384 h
Africa	30 335 000	5.945	1.001 $(p + h)$
Eurasia	54 106 000	10.605	1.113 $(p + 2h)$
Total Land Surface	148 941 000	29.200	1.067 $(4p + 5h)$
Total Surface Area	510 073 000	100.000	1.000 $(12p + 2h)$

Next we shall examine if the continents are located near their respective polygons. For this purpose, the locations of the polygons on the globe are first determined. The coordinates of the vertices of the polygons are calculated using spherical trigonometry. The procedure is straightforward but painstaking and will not be elaborated here. As stated earlier, the bucky ball and the soccer ball surfaces consist of 12 pentagons and 20 hexagons of the same side. Each pentagon is completely surrounded by five hexagons so that no two pentagons are adjacent to one another. In the bucky ball model, the north pole and the south pole are centred on two anti-podal pentagons, each of which is surrounded by five hexagons. The remaining 10 hexagons straddle the equator and the balance of 10 pentagons are distributed around 30° circles of latitude in the two hemispheres.

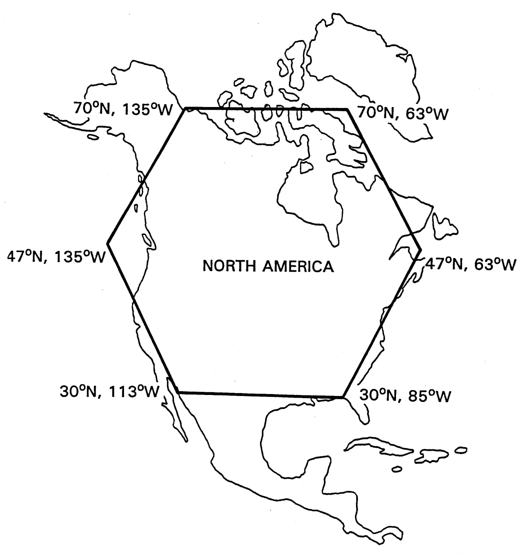

Figure 4. Hexagonal representation of North America.

Naturally, the Arctic Ocean and Antarctica occupy the polar pentagons. We start with the Arctic Ocean pentagon which is surrounded by five hexagons — one represents North America; two represent Eurasia; and the remaining two are oceanic hexagons in the north Atlantic and north Pacific Oceans. Figures 4–9 depict all the continental land masses together with their representative polygons, the coordinates of whose vertices are clearly marked. North America straddles over a northern hemispheric hexagon adjacent to the north polar pentagon (figure 4) whereas South America is situated on an equatorial hexagon which is largely in the southern hemisphere (figure 5). The North and South American hexagons are separated by an oceanic pentagon in the western Atlantic Ocean. The Antarctic pentagon (figure 6) is anti-podal to the Arctic Ocean pentagon. An oceanic hexagon separates it from the South American hexagon.

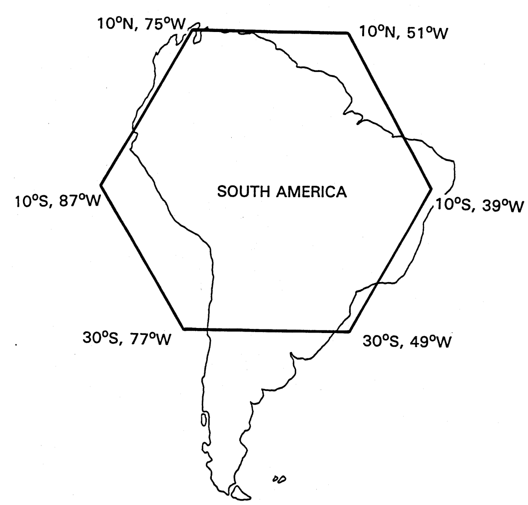

Figure 5. Hexagonal representation of South America.

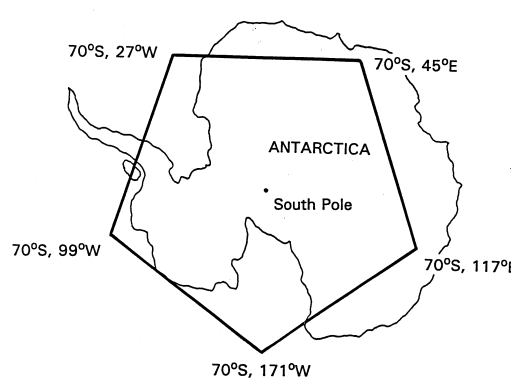

Figure 6. Pentagonal representation of Antarctica.

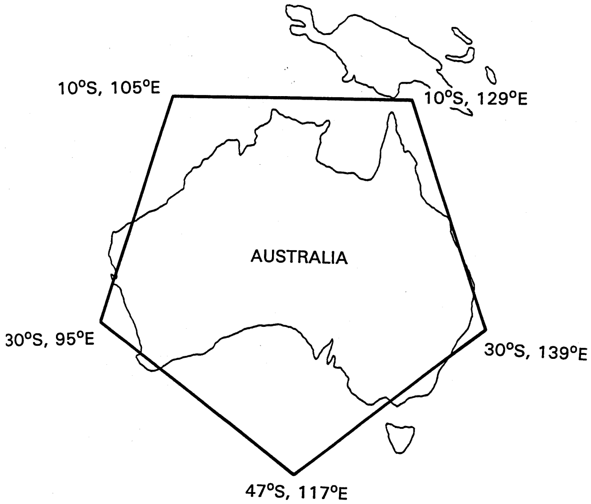

Figure 7. Pentagonal representation of Australasia. Australia has been moved 18° westward in longitude.

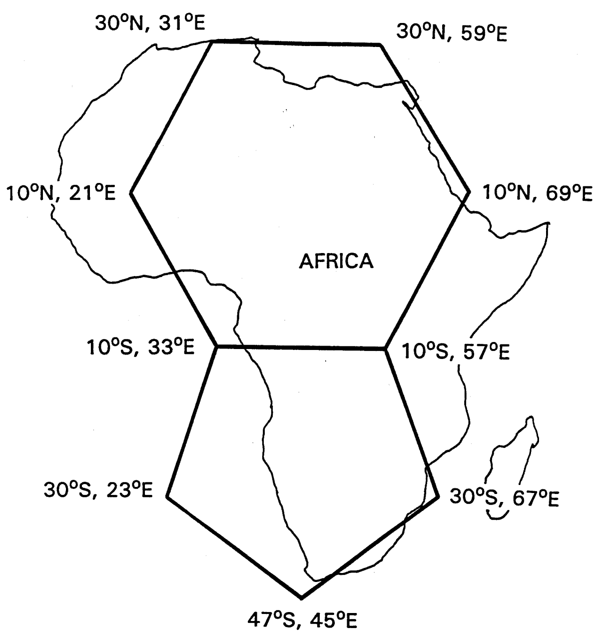

Figure 8. Polygonal representation of Africa. Africa has been moved 25° eastward in longitude and 6° southward in latitude.

Figure 7 shows the Australasian pentagon. Australia fits into the pentagon after it is moved 18° westward longitudinally (no latitudinal movement is necessary). New Zealand lies outside the range of the figure. Figure 8 shows the hexagon–pentagon pair representing Africa. In this case, Africa was moved 25° eastward longitudinally and 6° southward in latitude. An adjacent pentagon–hexagon pair required lesser movement, but the hexagon–pentagon pair of figure 8 gives a more appropriate geometrical representation. For, geomorphologically, Africa consists of two parts: a larger low plateau region of the north and a smaller high plateau region of the south. Here, the hexagon represents northern Africa while the pentagon represents southern Africa.

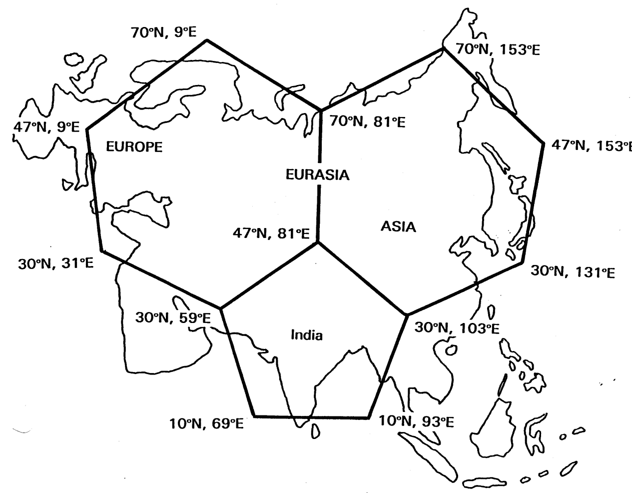

Figure 9. Polygonal representation of Eurasia.

Lastly, figure 9 shows the polygonal representation of Eurasia. Europe and Asia, strictly speaking, are not separate continents, but parts of the giant Eurasian continent. Its representation requires three polygons — (i) the western hexagon consisting of Europe and western Asia; (ii) the eastern hexagon consisting of eastern Asia; and (iii) the southern pentagon representing India. The latter, even though relatively small in size, is appropriately called a subcontinent because it was geologically a part of the southern supercontinent Gondwanaland (a term derived from a region of southern India) which collided with the northern landmass Laurasia to produce the lofty Himalayas and the massive Tibetan plateau.

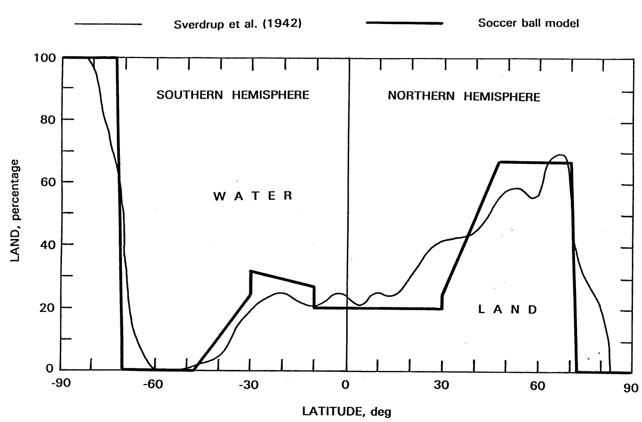

Figure 10. Land–water distribution of the Earth.

5. Latitudinal distribution of land in the soccer ball model

Finally, it is instructive to calculate the latitudinal distribution of land (and water) in the soccer ball model and see how it compares with the actual distribution. The procedure consists of summing the percentage of land over longitude for each fixed latitude at regular intervals. The result is shown in figure 10. According to the soccer ball model, the north polar region (90°N–73°N) is entirely water (the Arctic Ocean) whereas the south polar region (90°S–73°S) is entirely land (Antarctica). The equatorial region (10°S–30°N) is only 20 percent land. The greatest concentration of land (apart from the south polar region) is in the upper mid-latitude region in the northern hemisphere. Overall, water dominates over land on the surface of the Earth. Also shown in figure 10 is the actual land–water distribution according to reference 3. On the whole, the agreement between the model and observation is quite favourable.

6. Conclusion

In summary, the landmasses of the world curiously approximate groups of polygons in the bucky ball and soccer ball models of the Earth. Even their locations, by and large, coincide with their respective polygons. Only Australia and Africa required significant translation to match their respective polygons. Are these interesting facts merely coincidental or are there fundamental reasons behind them? These are a few questions which should be looked into. One thing is certain, though. The pentagonal and hexagonal areas seem to represent stable land masses in the theory of continental drift.

Acknowledgement

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Constructing a Triangle

GUIDO LASTERS and DAVID SHARPE

1. The problem stated

We are given a circle and three non-parallel straight lines l_1, l_2, l_3. The aim is to construct a triangle which has the given circle as its circumcircle and whose sides are parallel to l_1, l_2, l_3. (See figure 1.)

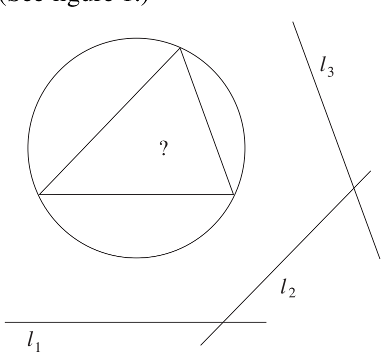

Figure 1.

The crucial word here is 'construct'. We use this in the sense that the ancient Greek mathematicians used it, namely construct using only a straight edge (a ruler) and a compass (and of course a pencil!). To avoid any misunderstanding, a compass is a mathematical compass not a magnetic compass! So we can use a compass to draw a circle with a given centre and we can set the radius of the circle by two given points; and we can draw a straight line passing through given points. But no measuring device is allowed, so we may not use the calibration of the ruler, nor may we measure angles.

2. The first ruler and compass construction

If we are given a straight line l and a point P not on l, a well-known construction enables us to draw the perpendicular to l through P.

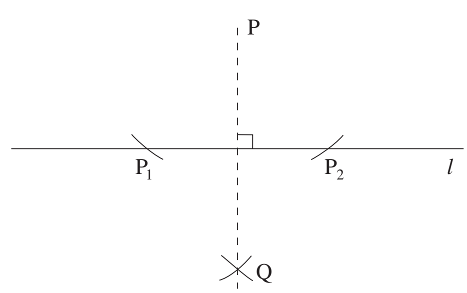

Figure 2.

We use a compass to mark off an arc with centre P cutting l at P_1 and P_2. Then, taking P_1 and P_2 as the centres, we mark off arcs of the same radii to intersect at Q (see figure 2). A simple argument using congruent triangles will show that PQ is perpendicular to l.

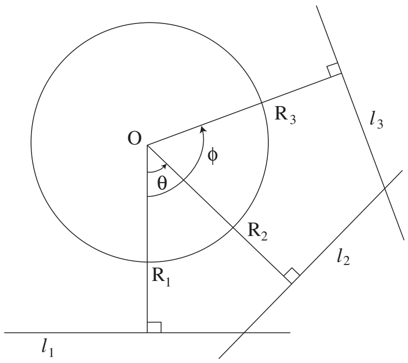

Figure 3.

We can use this construction to draw the perpendiculars through the centre of the circle to the three lines l_1, l_2 and l_3. Denote by R_1, R_2 and R_3 the points where these perpendiculars cut the circle. There are two possibilities for each of these points; either possibility will do. (See figure 3.)

3. The second ruler and compass construction

We are given a line l and a point S on l. We are also given two lines m_1, m_2 (meeting at K, say) at an angle θ to each other. The problem is to construct a line through S at an angle θ to l (see figure 4).

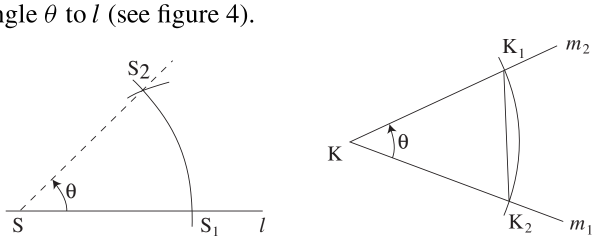

Figure 4.

Use the compass to mark an arc, centre K, cutting m_1, m_2 at K_1, K_2. With the same radius, mark an arc centre S to cut l at S_1. Now draw an arc centre S_1 radius K_1K_2. Call the point where these two arcs intersect S_2. Construct the line SS_2. Then triangles SS_1S_2 and KK_1K_2 are congruent, so angle S_1SS_2 is θ. In a similar way, we can also construct the angle θ in the opposite sense (i.e. a clockwise rotation of θ from l in figure 4).

4. The solution of the problem

We return to figure 3. Denote angle R_1OR_2 by θ and angle R_1OR_3 by ϕ. We can use the construction in Section 3 to rotate OR_1 by angles $\phi - \theta, \theta - \phi$ and $\theta + \phi$ from OR_1 (a negative rotation being clockwise) to OT_1, OT_2, OT_3 respectively (see figure 5). Now

$$\frac{(\phi-\theta)+(\theta-\phi)}{2} = 0,$$

$$\frac{(\theta-\phi)+(\theta+\phi)}{2} = \theta,$$

$$\frac{(\theta+\phi)+(\phi-\theta)}{2} = \phi,$$

so OR_1 bisects angle T_1OT_2, OR_2 bisects angle T_2OT_3 and OR_3 bisects angle T_3OT_1. A simple argument using congruent triangles will now give that T_1T_2 is perpendicular to OR_1 and so is parallel to l_1, that T_2T_3 is perpendicular to OR_2 and so is parallel to l_2 and T_3T_1 is perpendicular to OR_3 and so is parallel to l_3. Thus triangle $T_1T_2T_3$ meets the requirements and solves the problem.

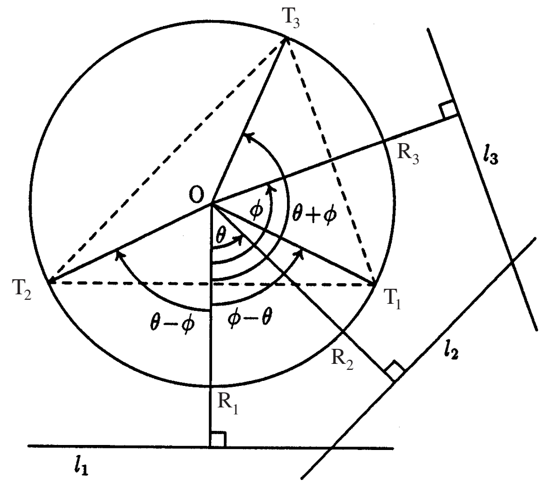

Figure 5.

5. The problem extended

But why stick to triangles? What about a quadrilateral, or a pentagon or, in general, an n-sided polygon?

Denote n given lines by $l_1, \ldots, l_n$, denote by $R_1, \ldots, R_n$ points where the perpendiculars to these lines from the centre O of the circle meet the circle, and denote by $T_1, \ldots, T_n$ the vertices of the proposed n-sided polygon. We can use the above construction to determine points $R_1, \ldots, R_n$. Using some line as starting point (it does not matter which), denote by $a_1, \ldots, a_n$ the angles that the lines $OR_1, \ldots, OR_n$ make with this direction. Denote by $b_1, \ldots, b_n$ the angles that the sought-after lines $OT_1, \ldots, OT_n$ make with the base direction. Then, to make T_1T_2 perpendicular to OR_1 and so parallel to l_1, to make T_2T_3 perpendicular to OR_2 (and so parallel to l_2) etc., and finally T_nT_1 perpendicular to OR_n, we need

$$\frac{b_1+b_2}{2} = a_1\,, \quad \frac{b_2+b_3}{2} = a_2\,, \ldots, \quad \frac{b_n+b_1}{2} = a_n\,. \quad (*)$$

The question is: given $a_1, \ldots, a_n$, can $b_1, \ldots, b_n$ be found to satisfy the equations $(*)$?

At this point an intriguing thing happens. The answer depends on whether n is odd or even. Consider first the case when n is odd, and write

$$\begin{array}{rcl} b_1 &=& a_1 - a_2 + a_3 - \cdots - a_{n-1} + a_n\,, \\ b_2 &=& a_2 - a_3 + a_4 - \cdots - a_n + a_1\,, \\ & \cdots & \\ b_n &=& a_n - a_1 + a_2 - \cdots - a_{n-2} + a_{n-1}\,. \end{array}$$

Then

$$\frac{b_1+b_2}{2} = a_1\,, \quad \frac{b_2+b_3}{2} = a_2\,, \;\ldots\,, \quad \frac{b_n+b_1}{2} = a_n$$

and the construction can be carried out.

Now suppose that n is even. If, for given $a_1, \cdots, a_n$, the equations $(*)$ hold, then

$$a_1 - a_2 + a_3 - \cdots + a_{n-1} - a_n = \frac{b_1+b_2}{2} - \frac{b_2+b_3}{2} + \frac{b_3+b_4}{2} - \cdots + \frac{b_{n-1}+b_n}{2} - \frac{b_n+b_1}{2} = 0.$$

Thus there has to be a special relationship satisfied by $a_1, \ldots, a_n$, so the initial lines would need to be specially positioned.

Hence the construction can be carried for any given lines $l_1, l_2 \ldots, l_n$ if and only if n is odd. Thus triangles, pentagons, heptagons etc. can be constructed, but quadrilaterals, hexagons, octagons etc. cannot.

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Urquhart's Theorem

K. R. S. SASTRY

This is an account of a remarkably simple looking theorem and the long quest for a comparably simple proof.

The discovery of Urquhart's theorem

The year is 1964, and the occasion is a symposium on the theory of relativity at the meeting of the Australian Mathematical Society in Adelaide, when M. L. Urquhart announces the following theorem.

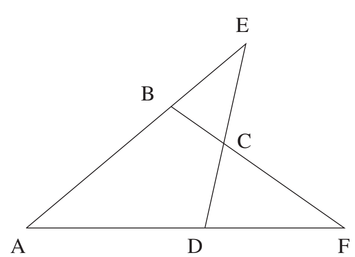

Figure 1. Urquhart's theorem.

Theorem. *In figure 1, if*

$$\mathrm{AB} + \mathrm{BC} = \mathrm{AD} + \mathrm{DC}, \ \textit{then}\ \mathrm{AE} + \mathrm{EC} = \mathrm{AF} + \mathrm{FC}. \quad (*)$$

This theorem has an interesting history; we now tell you its story.

Urquhart discovered this elegant theorem as a consequence of his work on the theory of relativity, and he considered it to be the most elementary theorem of Euclidean geometry. According to reference 1 he justified this opinion by pointing out that the formulation of the theorem requires only the concepts of straight line and distance. Szekeres, in reference 2, provides the context in which the result occurred: Urquhart had discovered a remarkable theorem in relativity and its Euclidean version is the above theorem.

Urquhart had a great love for geometry. He was instrumental in the formation of the Mathematical Association of Tasmania (Australia) whose chosen emblem is figure 1. (See references 3, 4.) His Euclidean proof of the theorem consists of two parts: first show that there is a circle to which the lines AE, AF, CE, CF are tangential, and therefrom deduce the required result. Strangely, as reference 5 says, Urquhart was uninterested in publishing his work and so others rediscovered or refined his proof. Furthermore, it was generally believed that an elementary proof involving lines and distances without the use of the circle with its four tangents might not be possible. However, 33 years later the impossible became possible, as is shown in reference 6.

A reincarnation and more history

A few mathematicians suspected that a theorem as elegant as Urquhart's must be a rediscovery, but the initial search of earlier geometry literature proved futile. However, Barton and Pedoe had other ideas. The statement $(*)$ reminded them of a property of ellipses. An ellipse is a set of points each of which has a constant sum of distances from two fixed points called the foci of the ellipse; see figure 2. If all members of a family of ellipses have the same foci, then the family is called confocal. In the light of this, if you look at figure 1, ($*$) says: consider a family of confocal ellipses with foci at A and C. If, now, the points B, D are on one ellipse of this family, then there is an ellipse of the same family on which E and F necessarily lie.

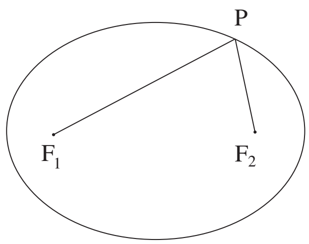

Figure 2. If F_1, F_2 are the foci of the ellipse, $PF_1 + PF_2$ is constant for all points P of the ellipse.

Barton was able to prove the preceding reincarnation of Urquhart's theorem by using the tools of trigonometry and coordinate geometry. (See reference 7.) On the other hand, Pedoe describes his own failure in reference 8. Barton further speculated that the reincarnation may be found in early literature on geometry. Deakin's extensive search could only locate the French geometer Chasles' result in J. L. Coolidge's book on conic sections and quadratic surfaces: if, from two points on a conic, tangents be drawn to a confocal conic, the four straight lines will touch a circle (reference 4). (For the definitions of conics and their foci see any elementary book on coordinate geometry.) Urquhart's theorem may, in fact, be deduced from Chasles', but now Deakin unexpectedly came across a two-page note by A. D. M. — Augustus De Morgan — which put an end to this search, at least for a while (reference 9).

De Morgan (1806–1871) was a prolific mathematician who, in 1828, became the first professor of mathematics at the University of London, soon transformed into University College. In 1865 he was elected first president of the London Mathematical Society which grew into the British national organisation for pure mathematicians.

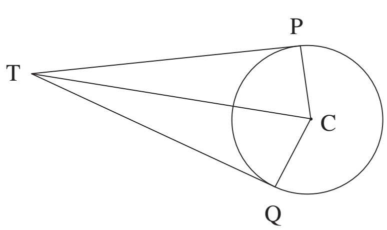

Figure 3. If C is the centre of the circle, $P\hat{C}T = Q\hat{C}T$.

De Morgan anticipated Urquhart's theorem from his work on conics. The reader will have no difficulty in establishing this simple property of the circle. At points P and Q on the circle let the tangents be drawn to meet at the point T. Then TP, TQ subtend equal angles at C (figure 3). However, did you know that a similar result holds more generally for a conic? This was a well-known result in De Morgan's days. In his words:

> The property of the conic sections ... is the following: if the tangents at P and Q meet in the point T and if S be one of the foci, PT and QT subtend equal angles at S except only when P and Q are on different branches of an hyperbola, in which case the angles are supplemental (reference 9).

He further comments that the proof of this simple proposition using the properties of conics is difficult but a simple proof may be based on a theorem that gives the condition for four lines to be tangents to a circle. Unfortunately, this theorem is unavailable in Euclid's *Elements*. In reference 9, De Morgan provides a highly condensed proof of the theorem, thereby anticipating Urquhart by 123 years, and then uses the theorem to establish the equiangular property of the conics mentioned earlier. This very same proof has been rediscovered by everyone who has proved Urquhart's theorem and is available from many sources (see references 2, 3, 9, 10). In reference 6 three distinct new proofs appear. Here we provide yet another.

A new proof of Urquhart's theorem

The present proof rests on the following well-known Euclidean geometry results. Their proofs can be found in college geometry texts, e.g. reference 11.

(i) The circumcentre of a triangle is the meet of the perpendicular bisectors of the sides.

(ii) In triangle ADE the excentre opposite A is the meet of the internal bisector of $D\hat{A}E$ and the external bisectors of the other two angles. There exists a circle centred at the excentre so that the lines AD, AE, DE are tangents to it. Let K and L be the points of tangency of AD, AE. Then $AK = AL =$ semi-perimeter of $\triangle ADE$.

Proof of Theorem ($*$). We assume that $AB + BC = AD + DC$ and deduce that $AE + EC = AF + FC$.

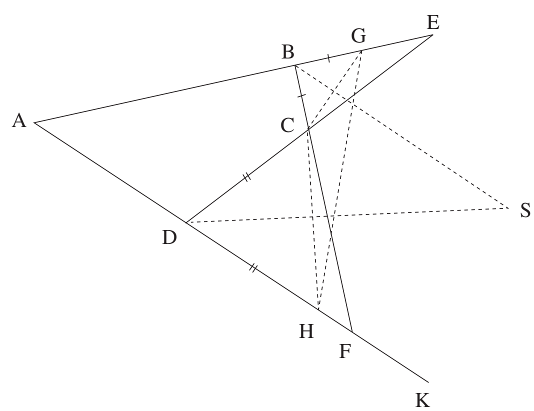

Figure 4. Construction for new proof of Urquhart's theorem.

Let G be the point on BE so that $BG = BC$ and H on DF so that $DH = DC$. We connect CG, CH, GH and draw BS, DS bisecting the angles EBC, FDC displayed in figure 4.

The hypothesis $AB + BC = AD + DC$ implies that $\triangle AGH$ is isosceles. BGC and DCH too are isosceles triangles so BS and DS have the additional property of being the perpendicular bisectors of CG, CH respectively. Then (i) says that S must be the circumcentre of $\triangle$CGH. Hence S must also be on the perpendicular bisector of GH which incidentally is the internal bisector of angle A (because $\triangle$AGH is isosceles).

Now (ii) says that S must be the excentre opposite A of both the triangles ADE and ABF. It is easy to see that these triangles have a common excircle opposite A. Denote by K the tangency point of this excircle with the line ADF. We then have, see (ii),

$$\begin{aligned} \text{AK} &= \text{semi-perimeter of } \triangle\text{ABF} \\ &= \text{semi-perimeter of } \triangle\text{ADE}, \end{aligned}$$

$$\text{i.e.} \quad \tfrac{1}{2}(\text{AB}+\text{BF}+\text{AF}) = \tfrac{1}{2}(\text{AD}+\text{DE}+\text{AE}),$$

$$\text{or} \quad (\text{AB}+\text{BC})+(\text{AF}+\text{CF}) = (\text{AD}+\text{DC})+(\text{AE}+\text{CE}).$$

It is now clear that the use of the hypothesis yields the result as required.

A suitably modified construction may be employed for the proof of another form of Urquhart's theorem.

In figure 1, if

$$\text{AB} - \text{BC} = \text{AD} - \text{DC}, \text{ then AE} - \text{EC} = \text{AF} - \text{FC}. \quad (**)$$

We further comment that the converse statements of $(*)$ and $(**)$ are true and may be proved similarly. The reader's attention is drawn to reference 6 in which a more elaborate form of Urquhart's theorem is stated and proved in five equivalent forms. However, for the present purposes $(*)$ and $(**)$ suffice.

A unified form of Urquhart's theorem

De Morgan's criteria for the four lines AB, BC, CD, AD of figure 1 to be simultaneously tangents to a circle are contained in $(*)$ and $(**)$. If $(*)$ holds then the tangent circle is external to the convex quadrilateral ABCD and if $(**)$ holds then that circle is the incircle of ABCD. In either case we may say that the perimeter of the quadrilateral ABCD is bisected by the diagonal AC. By using the idea of perimeter bisection it is possible to combine $(*)$ and $(**)$ to give a statement of a unified form of Urquhart's theorem. To do so, we introduce the concept of a complete quadrilateral.

Consider a convex quadrilateral ABCD in which no pair of opposite sides are parallel. Let AB, DC meet in the point E and AD, BC in F. Then the resulting configuration of six distinct points A, B, C, D, E, F is called a *complete quadrilateral*. These six points are the vertices of three distinct quadrilaterals: the convex quadrilateral ABCD, the singly concave quadrilateral AECF and the doubly concave quadrilateral BEDF. Observe that AC is the common diagonal of the convex quadrilateral ABCD and the singly concave quadrilateral AECF. With this preparation we may state the unified form of Urquhart's theorem as follows:

In a complete quadrilateral the diagonal common to the convex and the singly concave quadrilaterals bisects the perimeter of one of them if and only if it bisects the perimeter of the other. (See figure 5.)

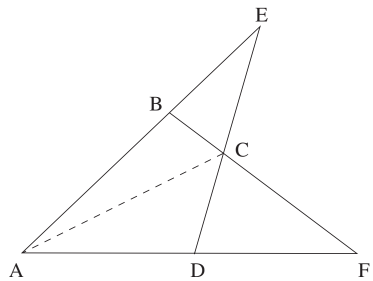

Figure 5. Unified form of Urquhart's theorem: AC bisects the perimeter of ABCD if and only if it bisects the perimeter of AECF.

Acknowledgement

I am grateful to Dr. Michael A. B. Deakin of the Department of Mathematics, Monash University, Australia, who sent me copies of the references listed below with the exception of numbers 6, 7 and 11.

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K. R. S. Sastry *studied mathematics at the University of Mysore, India. He taught mathematics in India and then in Ethiopia. Presently he devotes his time to contributing articles and problem proposals to mathematics journals.*

Mathematics in the Classroom

Towards a greater understanding of mathematics

Unesco has designated the year 2000 as World Mathematical Year with the aim of raising the public's awareness of the enormous range of important practical applications of mathematics in the real world. In response, the DfEE is arranging a series of one-day events taking place throughout the country during the year which are designed to give all an opportunity to do some maths and see what maths can do. (The locations are Plymouth, Exeter and South West in April; Manchester and Liverpool in May; Leicester, Loughborough and Nottingham in July; York in October and finally London in January 2001.)

In addition a campaign sponsored by the Isaac Newton Institute in Cambridge will display posters in the London underground which will try to address the image problem that maths has amongst non-mathematicians. These posters will carry such slogans as *Maths takes off* (about aerodynamics), *Maths makes waves* (about weather forecasting), *Maths is cool* (concerned with iceberg dynamics), *Maths stirs* (about turning water vapour into hurricanes) and the opening poster in the campaign, *Maths counts*. As well as slogans, the posters will feature eye-catching pictures and occasional puzzles.

Maths counts features a picture of a sunflower and asks for the pattern in the following sequence to be identified

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89, \ldots \qquad (*)$$

It then explains that this sequence is attributed to Leonardo Fibonacci who used it as early as the 12th century as a model for the growth of a population of rabbits. Since then it has been found to have applications in an array of natural phenomena which includes the spiral patterns of sunflower seeds and pine cones as well as the angles of divergence of adjacent leaves on a branch.

This reminded me of another problem that was solved by Fibonacci:

Find a square which remains a square if it is decreased by 5 or increased by 5.

Clearly the answer is not an integer, but the solution will be left to the reader with the hint that expressing 10 as the unlikely fraction $(80 \times 18)/12^2$ might help.

Generalising the Fibonacci sequence

As mathematicians we want to be able to derive the nth term, u_n, of the Fibonacci sequence $(*)$ above, in terms of n. Can you prove by induction that this is actually

$$u_n = \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\} \Big/ \sqrt{5}\ ?$$

The poster goes on to introduce the sequence of fractions

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \ldots,$$

which approaches the golden ratio, a special number in maths which is of enormous importance in art and architecture. This can be expressed as $2/(1+\sqrt{5})$ which plays an important part in the formula for u_n given above.

To gain some idea of the importance of this golden ratio, consider the following problem. Consider figure 1. The square is cut into two congruent triangles and two congruent trapeziums. Is it possible to choose x and y so that the square can be transformed into a rectangle as shown?

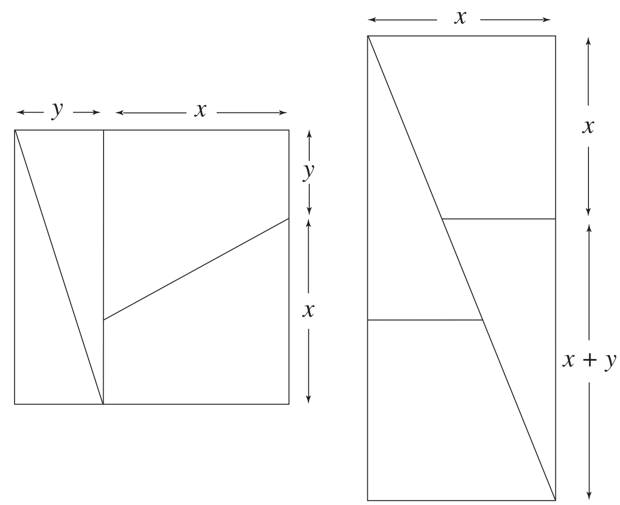

Figure 1.

Can you show that this can be achieved, but only if x and y are in the golden ratio, i.e., if

$$\frac{y}{x} = \frac{2}{1+\sqrt{5}}\ ?$$

The new century promises to bring major discoveries in biotechnology that could change the way in which we live. These will depend on mathematical techniques as well as computer power so never before has it been more important for us all to have a grasp of what mathematics can achieve. Let us hope that this campaign makes some inroads in assisting us all to improve our understanding of what mathematics can do for us.

Carol Nixon

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Letters to the Editor

Dear Editor,

The convergence of the sequence $(1+\frac{1}{n})^n$

Readers of *Mathematical Spectrum* may be interested in a proof of the convergence of the sequence $(1+\frac{1}{n})^n$ which is based on the inequality connecting the arithmetic and geometric means of a set of non-negative numbers. The inequality states that, if $a_1, a_2, \dots, a_n$ are arbitrary non-negative numbers, then

$$(a_1 a_2 \dots a_n)^{1/n} \le (a_1 + a_2 + \cdots + a_n)\frac{1}{n}. \qquad (*)$$

Actually, equality holds in $(*)$ if and only if $a_1 = a_2 = \cdots = a_n$, but this additional clause is not needed in the present context.

We now apply $(*)$ to the $n+1$ numbers

$$1+\frac{1}{n}, 1+\frac{1}{n}, \dots, 1+\frac{1}{n}, 1$$

obtaining

$$\left[\left(1+\frac{1}{n}\right)^n \cdot 1\right]^{1/(n+1)} \le \left[n\left(1+\frac{1}{n}\right)+1\right]\frac{1}{n+1} = 1+\frac{1}{n+1},$$

i.e. $$\left(1+\frac{1}{n}\right)^n \le \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Thus the sequence $\left(1+\frac{1}{n}\right)^n$ increases.

Similarly, considering the means of the $n+1$ numbers

$$\tfrac{1}{2}, \tfrac{1}{2}, 1, \dots, 1$$

we have

$$\left(\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot 1 \dots 1\right)^{1/(n+1)} \le [1+(n-1)]\frac{1}{n+1} = \frac{n}{n+1}$$

and so $$\tfrac{1}{4} \le \left(\frac{n}{n+1}\right)^{n+1},$$

i.e. $$4 \ge \left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n.$$

Hence the sequence $\left(1+\frac{1}{n}\right)^n$ is bounded above by 4.

Since a bounded increasing sequence converges, we have therefore shown that the sequence $\left(1+\frac{1}{n}\right)^n$ converges.

The above proof is short and simple, but it needs to be remembered that the proof of $(*)$ on which it is based is by no means trivial.

Yours sincerely,

YANG JIN
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Fuling Teachers' College,
Chongqing 408003
People's Republic of China)

Dear Editor,

The Puzzle King

I was recently given a book by my son who had bought it from an antiquarian bookshop. The book is *The Puzzle King* by John Scott, published by E. J. Labry. The book is not dated, but some detective work dates it at 1898/1899.

In it, problem No. 189 states:

> Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

After solving the problem, I tried to check my answer against the one at the back of the book and was surprised to find the following:

> (189) Imperfect. (Sample of questions we receive daily. Give it to your friends: it will annoy them.)

I hope I do not annoy too many!

Yours sincerely,

BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
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Dear Editor,

Throwing Elliptical Shields on Floorboards

In his article 'Throwing Elliptical Shields on[to] Floorboards' (Volume 32 Number 1, pp. 10–13), P. Glaister was awestruck by the fact that the arc length of the ellipse is the only characteristic of the figure that determines the probability that the ellipse should cross one of the lines. Although this seems surprising at first, it is simply derived by looking at the problem in a more general way. In fact, it applies to all convex figures that fit between the lines in all orientations. This follows by extending an approach that I first learned from Bill Taylor at the University of Canterbury, New Zealand.

Consider the ordinary Buffon needle problem in which a needle of length L is thrown randomly onto a plane with lines ruled a distance W apart. Instead of focussing on the probability of a crossing, we focus on the expected number of crossings (ignoring the case, which has probability 0, that the needle lies along a line). If $W < L$, these are the same. Divide the needle into a large number of small pieces. The number of crossings is the sum of the numbers of crossings in the pieces. Therefore, even though the pieces are attached to each other and are therefore not independent, the expected number of crossings is the sum of the expected numbers for the pieces. If the pieces are of equal length, this shows that the expected number of crossings is proportional to the length of the needle. But we do not need to assume that the needle is straight; the same applies to any shape, even a shoelace (more precisely, a rectifiable curve) or many separate needles. To find the constant of proportionality, consider a circular wire of diameter W, and hence of circumference πW. This always makes two crossings except when, with probability 0, it is tangential to two lines. Thus the expected number for this circle is 2. If the length of the needle is L, therefore, the expected number of crossings is $2L/\pi W$.

Of course, a planar object crosses a line if and only if its boundary crosses; and as the expected number of boundary crossings is independent of the shape, it just depends on the length of the boundary according to the same formula $2L/\pi W$.

If the length of the minor axis of the ellipse is greater than W, obviously the probability of crossing is 1, so, in general, the probability of crossing is expected to be a more complicated problem, as P. Glaister found. When the major axis has length less than W, the probability of crossing must be half the expected number of crossings, i.e. $L/\pi W$. The same applies to every small enough convex figure. If the figure is not convex, however, its boundary can cross the same line more than twice and the relationship is shape dependent. Even so, with a reasonable definition of the number of crossings for a general object with rectifiable boundary, the expected number of crossings is half the expected number of boundary crossings. This is a more useful result than the probability of crossing because the number of crossings contains more information than the binary variable which counts all multiple crossings as 1. Therefore, to estimate π, it is better to find the average number of crossings in a sample of throws rather than the proportion of throws in which there is at least one crossing.

Yours sincerely,

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New Zealand)

Dear Editor,

Unlucky 13 (Volume 32, Number 2, p. 42)

The conjecture is made that, given a large positive integer K then, if positive real numbers $a_1, a_2, \ldots, a_n$ are chosen, where n may be any positive integer, and $a_1 + a_2 + \cdots + a_n = K$, then the choice of $a_1, \ldots, a_n$ and n that maximises the product $a_1 \ldots a_n$ occurs when $a_1 = a_2 = \cdots = a_n \approx e$. This is indeed true, as can be seen.

To prove that, given some particular value for n, the product $a_1 \ldots a_n$ is maximised when $a_1 = a_2 = \cdots = a_n$, we know that $a_1 + a_2 + \cdots + a_n = K$. Then

$$\frac{K}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Using the arithmetic mean–geometric mean inequality, we have

$$\frac{K}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

with equality when $a_1 = a_2 = \cdots = a_n$. Thus

$$\left(\frac{K}{n}\right)^n \geq a_1, a_2, \ldots, a_n$$

and the maximum product for a given n is $\left(\frac{K}{n}\right)^n$ when $a_1 = a_2 = \cdots = a_n$.

To prove that, when K is large, the value of n that maximises $\left(\frac{K}{n}\right)^n$ is $n \approx \frac{K}{\mathrm{e}}$ (since $\frac{K}{\mathrm{e}}$ will not be integers), consider the function $f(x) = \left(\frac{K}{x}\right)^x$, where K is a large positive real number and x is a continuous positive real variable. Then

$$x^x f(x) = K^x. \tag{1}$$

Differentiating (1), we have

$$(1 + \ln x)x^x f(x) + x^x f'(x) = K^x \ln K.$$

Since $x^x f(x) = K^x$, this gives

$$(1 + \ln x)K^x + x^x f'(x) = K^x \ln K,$$

i.e.

$$f'(x) = \frac{K^x}{x^x}[\ln K - \ln x - 1].$$

A turning point will occur when $f'(x) = 0$. Clearly, $K^x/x^x > 0$ for all positive x. Hence $f'(x) = 0$ if and only if

$$\ln K - \ln x - 1 = 0. \tag{2}$$

Equation (2) gives $x = \mathrm{e}^{\ln K - 1} = \frac{K}{\mathrm{e}}$ as the only turning point, and $f\left(\frac{K}{\mathrm{e}}\right) = \left(\frac{K}{(K/\mathrm{e})}\right)^{K/\mathrm{e}} = \mathrm{e}^{K/\mathrm{e}}$. For large K, $\mathrm{e}^{K/\mathrm{e}}$ is very large and $f(1) = K$. And as $x \to \infty$, $f(x) \to 0$ since $\frac{K}{x} \to 0$.

Thus $f(x) = \left(\frac{K}{x}\right)^x$ is maximised for $x = \frac{K}{e}$. If K is large, the integer n maximising $\left(\frac{K}{n}\right)^n$ will be approximately $\frac{K}{\mathrm{e}}$. So $n \approx \frac{K}{\mathrm{e}}$ maximises the product $a_1 a_2 \ldots a_n$ (with $a_1 = \cdots = a_n$ as before). But $a_1 + \cdots + a_n = K$ and $a_1 = \cdots = a_n$, so $na_1 = K$ and $a_1 = \frac{K}{n} \approx \frac{K}{(K/e)} \approx \mathrm{e}$. So, for large K, the product $a_1 a_2 \ldots a_n$ is maximised for $a_1 + a_2 + \cdots + a_n = K$ when $a_1 = a_2 = a_3 = \cdots = a_n \approx \mathrm{e}$.

Yours sincerely,

Peter Allen
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Dear Editor,

Unlucky 13 (Volume 32, Number 2, p. 42)

In his letter, Jim Whiteman moved away from integer factors to reals in order to obtain a higher product. If we move one step further and obtain a non-integer number of factors, we will indeed arrive at the magic value e. Thus, if we assume each factor to be x, the number of factors will be $\frac{13}{x}$ and their product will be $x^{13/x}$.

I have plotted $y = x^{13/x}$ on my Casio fx–7700GE graphic calculator which shows the graph given in figure 1 and, by zooming into the maximum point, I could read a value of 119.3944, which agrees with $\mathrm{e}^{13/\mathrm{e}}$. This is confirmed if we differentiate the function $x^{13/x}$ to find its maximum, which occurs when $\ln x = 1$, i.e. $x = \mathrm{e}$.

N.B. This value of x applies to any starting value and not only to 13.

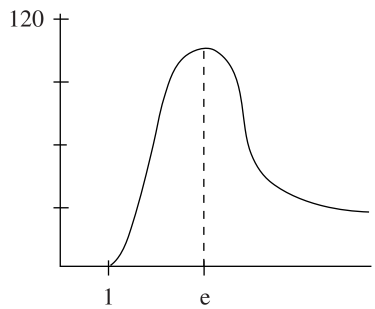

Figure 1.

Yours sincerely,

BOB BERTUELLO,
(12 Pinewood Road,
Midsomer Norton,
Bath, BA3 2RG)

Dear Editor,

Unlucky 13 (Volume 32, Number 2, p. 42)

I write regarding Jim Whiteman's fascinating problem.

Having done some spreadsheet experimenting myself, I came to the conclusion that 2.6 was not a freak result just because it goes exactly into 13. In fact, the best products always come from dividing the number concerned into identical parts. How many parts? Divide by e! There are some examples in table 1.

By this method, although we are not constrained to integer partitions x, we are constrained to an integer number d of them. But as n gets larger, d gets closer to n/e and hence $x \to \mathrm{e}$ and $p \to \mathrm{e}^{n/e}$.

I have come up with two results here:

(i) The highest product is given by dividing the number concerned into equal parts: i.e. d equal parts will always produce a greater product than d unequal parts.

(ii) Each part must be as close to e as possible, given that the number of parts is an integer.

Proof of the first result

First, consider only 2 parts. Let them be either equal, x, or unequal, $(x + a)$ and $(x - a)$. If x and a are real then

$$x^2 > (x + a)(x - a)$$

since $$(x + a)(x - a) = x^2 - a^2.$$

For more than 2 parts, there are many scenarios which can, I think, be broken down into:

(i) $(x+a)(x-a)$ pairs, with or without an extra x or two, for which the proof is as above.

(ii) The number can be partitioned as m lots of $(x + a)$ and an $(x - b)$, where $b = ma$. This follows Jim Whiteman's approach. The product will then be $(x + a)^m(x - b)$, which may be expanded using the binomial theorem to show, eventually, that

$$\begin{aligned}
&(x + a)^m(x - b) \\
&\quad = (x + a)^m(x - ma) \\
&\quad = x^{m+1} + \frac{ma^2}{2!}(-1 - m)x^{m-1} \\
&\qquad + \frac{m(m-1)a^3}{3!}(-2 - 2m)x^{m-2} \\
&\qquad + \frac{m(m-1)(m-2)a^4}{4!}(-3 - 3m)x^{m-3} \\
&\qquad + \cdots - ma^{m+1}.
\end{aligned}$$

The workings-out are left to the interested reader! It can be seen that, after the first term, the others are all negative, given that x, a and n are all positive, thus

$$(x + a)^m(x - b) < x^{m+1}.$$

(iii) A combination of the above, and the above proofs.

Number n	Number of parts $d = n/\mathrm{e}$ (nearest integer)	Each part $x = n/d$	Product $p = x^d$	Best result using integers
5	2	2.5	6.25	$2 \times 3 = 6$
13	5	2.6	118.81 ...	$2^2 \times 3^3 = 108$
22	8	2.75	3270.8 ...	$2^2 \times 3^6 = 2916$
1000	368	2.717391 ...	$5.86\ldots \times 10^{159}$	$2^2 \times 3^{32} \approx 1 \times 10^{159}$

Table 1.

Proof of the second result

Let the number to be partitioned be n. Let each part be x. Let the number of parts, $d = n/x$ (this allows d to be non-integral, which would be our best-case scenario). Then the product,

$$p = x^d = x^{n/x}.$$

Taking logs we have

$$\ln p = \frac{n}{x}\ln x.$$

Differentiating, we have

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{n}{x}\cdot\frac{1}{x} - \frac{n}{x^2}\ln x\\ &= \frac{n}{x^2} - \frac{n}{x^2}\ln x\\ &= \frac{n}{x^2}(1-\ln x),\end{aligned}$$

so

$$\frac{\mathrm{d}p}{\mathrm{d}x} = p\frac{n}{x^2}(1-\ln x) = x^{n/x}\cdot\frac{n}{x^2}(1-\ln x) = 0$$

when $\qquad 1 - \ln x = 0,$

i.e. when $\qquad 1 = \ln x$

i.e. when $\qquad x = \mathrm{e}.$

Many thanks to Jim Whiteman for finding such an interesting problem in a year 7 question.

Yours sincerely,
MERLIN ELLIS
(61 West Grove,
Woodford Green,
Essex IG8 7NR)

Braintwister

11. A Millennium sum

I had 10 cards and on each was a different digit, and I then tore up one of the cards. With two of the remaining cards I formed a two-figure number, with three others I formed a three-figure perfect square, and with the other four I formed a four-figure number. Just one of those three numbers was divisible by the digit which I had torn up, and the sum of the three numbers was 2000.

Which digit had I torn up? (Easier than you might think.)
What were the three numbers which added to 2000?

VICTOR BRYANT

Professor J. M. Gani

On Australia Day, 26 January 2000, the Governor General of Australia awarded Professor Gani Membership of the Order of Australia. The citation referred to Professor Gani's research contributions in probability, statistics and biomathematics, and to his establishment, supported by his late wife Ruth, of the Applied Probability Trust, publisher of *Mathematical Spectrum*.

Everyone associated with *Mathematical Spectrum* will wish to congratulate Professor Gani, founder of this magazine, and Managing Editor since 1968, on the honour that has been bestowed on him.

A curious prime sequence

5	53 + 8 = 61	197 + 26 = 223	503 + 44 = 547	971 + 62 = 1033
5 + 8 = 13	61 + 10 = 71	223 + 28 = 251	547 + 46 = 593	1033 + 64 = 1097
13 + 10 = 23	71 + 12 = 83	251 + 30 = 281	593 + 48 = 641	1097 + 66 = 1163
23 + 8 = 31	83 + 14 = 97	281 + 32 = 313	641 + 50 = 691	1163 + 68 = 1231
31 + 6 = 37	97 + 16 = 113	313 + 34 = 347	691 + 52 = 743	1231 + 70 = 1301
37 + 4 = 41	113 + 18 = 131	347 + 36 = 383	743 + 54 = 797	1301 + 72 = 1373
41 + 2 = 43	131 + 20 = 151	383 + 38 = 421	797 + 56 = 853	1373 + 74 = 1447
43 + 4 = 47	151 + 22 = 173	421 + 40 = 461	853 + 58 = 911	1447 + 76 = 1523
47 + 6 = 53	173 + 24 = 197	461 + 42 = 503	911 + 60 = 971	1523 + 78 = 1601

BABHI CHANDRA DEY
Calcutta

[**Editor:** Can any reader find a longer sequence of primes (p_n) where the differences $\Delta_i = p_{i+1} - p_i$ between successive terms are such that $|\Delta_{i+q} - \Delta_i| = 2$ for each i ?]

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

32.9 Deduce formula (2) from formula (1) on page 50, in the article 'François Viète and the quest for π' by G. C. Bush.

32.10 The three points U, V, W have distinct x-coordinates u, v, w respectively and $k = v - u$. Construct the straight line through U parallel to VW and denote by P the point on this line with x-coordinate $w + k$. Construct the straight line through W parallel to PV and denote by Q the point on this line with x-coordinate $v + k$. Show that the straight line through V parallel to UQ is the tangent at V to the unique curve with equation of the form $y = ax^2 + bx + c$ which passes through U, V, W.

(Submitted by Guido Lasters, Tienen, Belgium)

32.11 Two people arrange to meet at a certain place in town between 3 pm and 4 pm for tea. They decide that each will wait for up to 10 minutes between these times and then leave if the other has not turned up. If all times between 3 pm and 4 pm are equally likely for the arrival of each person, what is the probability that they will meet?

(Submitted by S. L. Platz, Thatcham)

32.12 What are the last two digits of the following numbers, where m, n are integers with $m > 0$?

$$(10n-5)^{2m}, \quad (20n-5)^{2m+1}, \quad (20n-15)^{m} \quad \text{with } m \geq 2.$$

(Submitted by Zhang Yun, First Middle School of Jinchang City, China)

Solutions to Problems in Volume 32 Number 1

32.1 Let $n \geq 1$ be an integer and let $\theta_1, \ldots, \theta_n$ be positive real numbers such that $\theta_1 + \cdots + \theta_n < \frac{1}{2}\pi$. Prove that

$$\frac{(1-\sin\theta_1)(1-\sin\theta_2)\ldots(1-\sin\theta_n)}{(1-\cos\theta_1)(1-\cos\theta_2)\ldots(1-\cos\theta_n)} \geq \frac{1-\sin(\theta_1+\theta_2+\cdots+\theta_n)}{1-\cos(\theta_1+\theta_2+\cdots+\theta_n)}.$$

Solution by Peter Allen (Nottingham High School)

Let

$$f(\theta) = \frac{1-\sin\theta}{1-\cos\theta}.$$

As $\theta \to 0$, $f(\theta) \to \infty$, $f\left(\frac{\pi}{4}\right) = 1$, $f\left(\frac{\pi}{2}\right) = 0$ and

$$\begin{aligned} f'(\theta) &= \frac{(1-\cos\theta)(-\cos\theta) - (1-\sin\theta)\sin\theta}{(1-\cos\theta)^2} \\ &= \frac{1-\cos\theta-\sin\theta}{(1-\cos\theta)^2} \\ &= \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{(1-\cos\theta)^2} < 0 \qquad \text{for } 0 < \theta < \frac{\pi}{2}. \end{aligned}$$

Hence f is a decreasing function in the interval $\left(0, \frac{\pi}{2}\right)$. We may assume that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$, so that only θ_1 may be greater than $\frac{\pi}{4}$.

The result is clear when $n = 1$, so let $k \geq 1$ and assume the result when $n = k$. Suppose that $\theta_1 + \cdots + \theta_{k+1} < \frac{\pi}{2}$. Then $\theta_1 + \cdots + \theta_k < \frac{\pi}{2}$ so, by the inductive hypothesis,

$$\frac{(1-\sin\theta_1)\ldots(1-\sin\theta_k)}{(1-\cos\theta_1)\ldots(1-\cos\theta_k)} \geq \frac{1-\sin(\theta_1+\cdots+\theta_k)}{1-\cos(\theta_1+\cdots+\theta_k)}.$$

Since $k+1 \geq 2$, $\theta_{k+1} < \frac{\pi}{4}$, so

$$\frac{1-\sin\theta_{k+1}}{1-\cos\theta_{k+1}} > 1.$$

Also

$$\frac{1-\sin(\theta_1+\cdots+\theta_k)}{1-\cos(\theta_1+\cdots+\theta_k)} > \frac{1-\sin(\theta_1+\cdots+\theta_{k+1})}{1-\cos(\theta_1+\cdots+\theta_{k+1})},$$

since f is a decreasing function. Hence

$$\begin{aligned} &\frac{(1-\sin\theta_1)\ldots(1-\sin\theta_{k+1})}{(1-\cos\theta_1)\ldots(1-\cos\theta_{k+1})} \\ &\qquad > \frac{(1-\sin\theta_1)\ldots(1-\sin\theta_k)}{(1-\cos\theta_1)\ldots(1-\cos\theta_k)} \\ &\qquad \geq \frac{1-\sin(\theta_1+\cdots+\theta_k)}{1-\cos(\theta_1+\cdots+\theta_k)} \\ &\qquad > \frac{1-\sin(\theta_1+\cdots+\theta_{k+1})}{1-\cos(\theta_1+\cdots+\theta_{k+1})}, \end{aligned}$$

which proves the inductive step.

32.2 Find the sum of the infinite series

$$\frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \cdots + (-1)^{n+1}\frac{n}{(n+1)!} + \cdots.$$

Solution by Chun Chung Tang (Impington Village College, Cambridge)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots,$$

so

$$(x-1)e^x = \Big(x + x^2 + \frac{x^3}{2!} + \cdots + \frac{x^{n+1}}{n!} + \cdots\Big) - \Big(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\Big)$$
$$= -1 + \Big(\frac{2}{2!} - \frac{1}{2!}\Big)x^2 + \cdots + \Big(\frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}\Big)x^{n+1} + \cdots$$
$$= -1 + \frac{x^2}{2!} + \frac{2x^2}{3!} + \cdots + \frac{nx^{n+1}}{(n+1)!} + \cdots .$$

Hence

$$(x-1)e^x + 1 = \frac{x^2}{2!} + \frac{2x^2}{3!} + \cdots + \frac{nx^{n+1}}{(n+1)!} + \cdots .$$

Now put $x = -1$ to give

$$\frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \cdots + (-1)^{n+1}\frac{n}{(n+1)!} + \cdots = 1 - \frac{2}{e}.$$

Also solved by Peter Allen, and Boryann Chen (University of California, Irvine).

32.3 Let $n \geq 1$ be an integer, let $a_1, \ldots, a_n$ be positive real numbers and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers smaller than 1. Prove that

$$\frac{\lambda_1\lambda_2\ldots\lambda_n}{\lambda_2\lambda_3\ldots\lambda_n a_1 + \lambda_1\lambda_3\ldots\lambda_n a_2 + \cdots + \lambda_1\lambda_2\ldots\lambda_{n-1}a_n}$$
$$+ \frac{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_n)}{\big[(1-\lambda_2)(1-\lambda_3)\cdots(1-\lambda_n)a_1 + (1-\lambda_1)(1-\lambda_3)\cdots(1-\lambda_n)a_2 + \cdots + (1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_{n-1})a_n\big]}$$
$$\leq \frac{1}{a_1 + a_2 + \cdots + a_n}.$$

Solution by Zhang Yun, who proposed the problem

The given inequality is equivalent to

$$\frac{1}{\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n}} + \frac{1}{\frac{a_1}{1-\lambda_1} + \cdots + \frac{a_n}{1-\lambda_n}} \leq \frac{1}{a_1 + \cdots + a_n},$$

or to

$$(a_1 + \cdots + a_n)\Big(\frac{a_1}{1-\lambda_1} + \cdots + \frac{a_n}{1-\lambda_n} + \frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n}\Big) \leq \Big(\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n}\Big)\Big(\frac{a_1}{1-\lambda_1} + \cdots + \frac{a_n}{1-\lambda_n}\Big),$$

or to

$$(a_1 + \cdots + a_n)\Big(\frac{a_1}{\lambda_1(1-\lambda_1)} + \cdots + \frac{a_n}{\lambda_n(1-\lambda_n)}\Big) \leq \Big(\frac{a_1}{\lambda_1} + \cdots + \frac{a_n}{\lambda_n}\Big)\Big(\frac{a_1}{1-\lambda_1} + \cdots + \frac{a_n}{1-\lambda_n}\Big).$$

Now,

$$\text{RHS} - \text{LHS} = \sum_{i<j} a_i a_j \Big[\frac{1}{\lambda_i(1-\lambda_j)} + \frac{1}{\lambda_j(1-\lambda_i)} - \frac{1}{\lambda_j(1-\lambda_j)} - \frac{1}{\lambda_i(1-\lambda_i)}\Big]$$
$$= \sum_{i<j} a_i a_j \frac{\big[\lambda_j(1-\lambda_j) + \lambda_i(1-\lambda_i) - \lambda_i(1-\lambda_i) - \lambda_j(1-\lambda_j)\big]}{\lambda_i\lambda_j(1-\lambda_i)(1-\lambda_j)}$$
$$= \sum_{i<j} a_i a_j \frac{\lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j}{\lambda_i\lambda_j(1-\lambda_i)(1-\lambda_j)}$$
$$= \sum_{i<j} a_i a_j \frac{(\lambda_i - \lambda_j)^2}{\lambda_i\lambda_j(1-\lambda_i)(1-\lambda_j)} \geq 0,$$

so the inequality is proved.

32.4 Let $n > 1$ be an integer and let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Prove that

$$(4n-4)\,\alpha_1\alpha_n \leq (\alpha_1 + \cdots + \alpha_n)^2,$$

and determine when equality occurs.

Solution by Chun Chung Tang

First assume that $0 \leq \alpha_1 \leq \cdots \leq \alpha_n$. Now

$$\big(\alpha_n - (n-1)\alpha_1\big)^2 \geq 0,$$

so

$$\big(\alpha_n - (n-1)\alpha_1\big)^2 \geq 4(n-1)\alpha_1\alpha_n$$

so

$$(\alpha_n + \cdots + \alpha_1)^2 \geq \big(\alpha_n + (n-1)\alpha_1\big)^2 \geq 4(n-1)\alpha_1\alpha_n .$$

If equality occurs, then $\alpha_1 + \cdots + \alpha_{n-1} = (n-1)\alpha_1$ and $\alpha_n = (n-1)\alpha_1$, so $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ (say) and $\alpha_n = (n-1)\alpha$, i.e.

$$(\alpha_1, \ldots, \alpha_n) = \big(\alpha, \alpha, \ldots, \alpha, (n-1)\alpha\big),$$

for some α. Conversely, if this holds then there is equality.

Next assume that $\alpha_1 \leq \cdots \leq \alpha_n \leq 0$. Then

$$\big(\alpha_1 - (n-1)\alpha_n\big)^2 \geq 0$$

so

$$\big(\alpha_1 + (n-1)\alpha_n\big)^2 \geq 4(n-1)\alpha_1\alpha_n$$

so

$$\alpha_1 + \cdots + \alpha_n \leq \alpha_1 + (n-1)\alpha_n \leq 0$$

so

$$(\alpha_1 + \cdots + \alpha_n)^2 \geq \big(\alpha_1 + (n-1)\alpha_n\big)^2 \geq 4(n-1)\alpha_1\alpha_n .$$

If equality occurs, then $\alpha_1 + \cdots + \alpha_n = \alpha_1 + (n-1)\alpha_n$ and $\alpha_1 = (n-1)\alpha_n$, so $\alpha_2 = \cdots = \alpha_n = \alpha$ (say) and $\alpha_1 = (n-1)\alpha$, i.e.

$$(\alpha_1, \ldots, \alpha_n) = \big((n-1)\alpha, \alpha, \ldots, \alpha\big)$$

for some α. Conversely, if this holds then there is equality.

If $\alpha_1 < 0$ and $\alpha_n > 0$, then LHS < 0 but RHS ≥ 0 and the result clearly holds; equality cannot occur.

Reviews

Statistics Explained. By JOHN PARRY LEWIS and ALASDAIR TRAILL. Addison-Wesley, Harlow, UK, 1998. Pp. 608. £19.95 (ISBN 0-201-17802-8).

The authors describe this book as having been written for students studying a wide range of subject areas, and who approach their course on statistics with reluctance and much trepidation.

Despite this promising intent, the front cover of the book is rather uninspiring and did not really grab my attention. The contents are easy to follow but there are rather a lot of sections to search through when looking for a topic. On the other hand, the 'how to's' were very helpful and would be useful for exam revision. However, the text was a little too condensed for me, and in each chapter, I found it difficult to establish quickly just where questions and information began and ended.

Perhaps this is a book for the library for use as an occasional reference rather than a study text which is constantly at one's side.

Student, Solihull Sixth Form College JENNY BARNFIELD

Introducing Statistics. By GRAHAM UPTON and IAN COOK. OUP, Oxford, 1999. Pp. 432. Paperback £12.50 (ISBN 0-1991-4561-x).

This book is designed to meet the needs of students who are taking Statistics as part of a single-subject A-level only and has evolved from *Understanding Statistics* but comes in a shorter version. It takes into account the core of the new syllabuses, so it is up to date for single-subject syllabuses from 1999.

Introducing Statistics provides a sound introduction to the subject. Particularly useful are the large numbers of worked examples and questions (over 800 in all with approximately 300 from past papers). Also very useful for revision are the clear chapter summaries and a complete set of answers.

The topics covered are broken down into digestible segments and are aided by the good layout and clear diagrams. All of this combines to create an excellent book which I recommend to both staff and statistics students alike.

Student, Solihull Sixth Form College JAMES CARPENTER

Understanding Statistics. By GRAHAM UPTON and IAN COOK. OUP, Oxford, 1999. Pp. 750. Paperback £12.50 (ISBN 0-1991-4391-9).

This book is aimed at students undertaking courses in statistics both as an A-level in its own right and as part of an A-level maths course. It is also recommended for first-year undergraduates who need background knowledge in the subject.

This is a lively book which is both interesting and stimulating. I found that it was well organised with all formulae derived simply and in a carefully explained way. If you are still left in doubt about any topic then the 'Notes' give good advice which helps to consolidate your understanding. The chapter summaries really aid revision as does working through the exam questions (there are over 1000 exercises, many from past papers). This book will definitely assist me through my A-level statistics course and will no doubt still be in use when I am analysing data that I will encounter in my degree studies.

Student, Solihull Sixth Form College HOLLY ATKIN

MEI Structured Mathematics: Pure Mathematics 6. By TERRY HEARD and DAVID MARTIN. Hodder & Stoughton, London, 1998. Pp. 285. Paperback £9.99 (ISBN 0-340-688017).

This is the final book in the pure maths strand of the MEI Structured Maths A-level scheme. Accessible primarily to double-subject students, there are five independent chapters covering vectors and matrices, limiting processes, multivariable calculus, differential geometry and abstract algebra. Written in the same style as the previous five texts, the level and nature of the topics covered mean that it would also have relevance to some undergraduate courses.

Solihull Sixth Form College CAROL NIXON

MEI Structured Mathematics: Statistics 5 & 6. By ALEC CRYER, MICHAEL DAVIES, BOB FRANCIS and GERALD GOODALL. Hodder & Stoughton, London, 1998. Pp. 218. Paperback £9.99 (ISBN 0-340-701323).

This book covers the final two modules in the statistics strand of the MEI Structured Maths A-level scheme and continues in the same user-friendly style of the earlier statistics text books written specifically for this course. The first five chapters meet the requirements of Statistics 5 and cover tests for proportions, variances, errors in hypothesis testing, and probability and moment generating functions. The second five chapters cover topics in Statistics 6, addressing maximum likelihood estimators, bivariate distributions, Markov chains, analysis of variance, and regression. These are modules that are only likely to be accessed by further maths students, so few of us will have an opportunity to see this book in action.

Solihull Sixth Form College CAROL NIXON

In Search of SUSY: Supersymmetry and the Theory of Everything. By JOHN GRIBBIN. Penguin Books, London, 1998. Pp. xv+144. Paperback £6.99 (ISBN 0-14-027582-7).

The subject of this book is the on-going search for a 'theory of everything', a single set of equations which explain all particles and forces. As a result, this slim paperback covers not only the supersymmetry (SUSY) of the title, but the whole development of modern physics. Topics include quantum principles, fields, gauge theory, different types of particles, extra dimensions, and superstrings.

Fashionable guides to modern science are often afraid of offering the detail needed for proper understanding, 'explaining' giant leaps only by reference to mysterious equations which we never see. Fortunately, *In Search of SUSY* is more ambitious than most. Every new concept is bravely tackled as clearly and fully as is possible for the general audience, and the book even dares to introduce a little maths! The author genuinely attempts not to over-simplify or conceal the limits of our knowledge. He observes, for example, that the wave–particle duality is not as ridiculous as it may first seem, for both concepts 'are only metaphors for something that we cannot properly comprehend or understand'. Furthermore, the depth of study is achieved without losing our sense of discovery and excitement.

Explaining modern science without advanced mathematics will always be a struggle, yet I think this is among the better attempts. For all those who have a little background knowledge and who are keen to find out more, this serious and intelligent book will be very satisfying.

Student, Nottingham High School JEREMY YOUNG

Combinatorics: A Problem Oriented Approach. By DANIEL A. MARCUS. MAA, Washington, 1998. Pp. x+136. Paperback \$28.00 (ISBN 0-88385-710-3).

This slim book is part of the MAA's 'Classroom Resource Materials' series, rather than the excellent 'New Mathematical Library'. It is distinguished by its large number of problems (over 450), which encourage the reader to play an active role. As it is a textbook, these questions tend to be not very demanding, and stick fairly closely to the material covered. Answers are not always given.

The first part of the book discusses standard results for counting strings, combinations, distributions and partitions of elements. Many of the earlier results are neatly expressed in terms of binomial coefficients. For example, a typical problem is to find the number of distributions of m identical balls into n distinct boxes. The answer is ${}^{m-1}C_{n-1}$ if a ball must go in each box, but ${}^{n+m-1}C_{n-1}$ if empty boxes are allowed. Catalan numbers, multinomial coefficients, derangements and Stirling numbers are also touched upon.

The second half of the book then covers four other counting methods: the inclusion–exclusion principle, recurrence relations, generating functions, and the Pólya–Redfield method. Generating functions have coefficients which give the answers to combinatorial problems. In a simple example we are asked to find the number of three-letter combinations from the set {A,B,C}, with A and B included at most once and twice, respectively. The answer is the number of terms of degree 3 in $(1 + a)(1 + b + b^2)(1 + c + c^2 + c^3)$, or simply the coefficient of x^3 in the function $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)$. The Pólya–Redfield method is used to count distinct ways of labelling or colouring symmetrical figures. In its introductory example, it is used to show that there are six cyclically-distinct four-letter words with letters {A,B}.

With material ranging from basic to quite technical methods, *Combinatorics* should contain something new for most students, and give plenty of opportunity to practise it. However, the material may lack the depth needed to stretch more enthusiastic readers. In summary, this is a concise, well-organised and readable introduction to the principles of combinatorics, is certainly recommended.

Student, Nottingham High School JEREMY YOUNG

The Art and Craft of Problem Solving. By PAUL ZEITZ. John Wiley, New York, 1999. Pp. xvii+334. Hardback £19.99 (ISBN 0-471-13571-2).

The average student can be likened to someone who gets stronger by doing repetitive exercises at the gym. In contrast, according to this book's introduction, the problem solver is an explorer who goes on expeditions: 'The problem solver gets hot, cold, wet, tired, and hungry. The problem solver gets lost, and has to find his or her way. The problem solver gets blisters. The problem solver climbs to the top of mountains, sees hitherto undreamed-of vistas. The problem solver arrives at places of amazing beauty, and experiences ecstasy which is amplified by the effort expended to get there. When the problem solver returns home, he or she is energized by the adventure, and cannot stop telling others about their wonderful experiences.'

One of the pleasures of this book is the enthusiasm of its author, a contestant and now trainer for the US International Maths Olympiad team. Through this text he aims to introduce more bright students to problem solving, both as a way of learning about maths, and as a recreation in itself.

The first half of the book is concerned with the general principles of problem solving. It starts with very basic ideas, such as the importance of drawing diagrams. It then discusses a few classic tricks, these being the use of symmetry, extreme elements, the pigeonhole principle and invariants. Finally, graph theory, complex numbers and generating functions are discussed in sections on 'crossover tactics' which may be used to reinterpret a problem.

The second half of the book introduces relevant theory of algebra, inequalities, combinatorics, number theory and (unusually for a book on problem solving) calculus. The obvious omission is geometry, which, as the author points out, is well covered elsewhere. The text throughout is aimed at beginners, but the large collection of worked examples and questions covers a range of difficulties. For example:

> *Prove that the product of four consecutive natural numbers cannot be the square of an integer.*
>
> *Let S be a region in the plane with area greater than the positive integer n. Show that it is possible to translate S so that it covers at least n+1 lattice points.*
>
> *A rectangle is tiled with smaller rectangles, each with at least one side of integral length. Prove that the tiled rectangle must also have at least one side of integral length.*

The structure of the book will be familiar, with general principles followed by instructive, often well-known examples. Each short section finishes with 10 or 20 problems. A few of these are taken from competitions, but there is a good variety of sources. How much help should be available with exercises is a matter of opinion, but here one-line comments are given to a small selection of the problems. There is apparently an *Instructor's Resource Manual* with more support, and a reference is given to `www.wiley.com/college/zeitz`, which I was unable to find.

With explanations that are always clear and thorough, this book is very easy to read. Example solutions are presented informally and at a gentle pace, with comment on the motivating thought processes and each problem's key step, or 'crux move'. Sample numbers are sometimes used instead of letters, followed by the observation that this generalises, although there are also some formal proofs. A good list of references is given at the end, and often strongly recommended in the text.

This is a book aimed at beginners, but with enough variety to appeal to a wider audience. The opening focus on general approaches and the right mental attitude, while in some ways stating the obvious, is very relevant to the practical experience of problem solving. The specific theory covered is basic but essential, while the large number of problems and worked examples (over 660) is another big advantage. Most impressive of all is how approachable the text is. I hope that the added expense of publishing in hardback does not discourage problem solvers from buying this book.

Student, Nottingham High School JEREMY YOUNG

Euler: The Master of Us All. By WILLIAM DUNHAM. MAA, Washington, 1998. Pp. 192. Paperback \$29.95 (ISBN 0-88385-328-0).

Have you ever stopped and peered over the shoulder of an artist as he paints a landscape? Reading this book is rather like that. The subject is Leonhard Euler, the great 18th century Swiss mathematician. 'Read Euler, read Euler. He is the master of us all', wrote Laplace. Having read this fascinating book, few would wish to contradict Laplace's verdict.

The author gives cameos of Euler's work in eight areas. In number theory we see Euler exploring perfect numbers such as 6, those which are equal to the sum of their proper positive divisors ($6 = 1+2+3$). There is a chapter on Euler's work on logarithms, a 'most natural and fruitful concept', to quote the master. His treatment of infinite series, with his beautiful formula

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$$

is breathtaking. His ignoring of any problems of convergence of infinite series will bring a smile to all students who these days would be pilloried by nitpicking tutors for doing what he did.

How about Euler's 'proof' of the formula

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots=\frac{1}{1-\frac{1}{2}}\times\frac{1}{1-\frac{1}{3}}\times\frac{1}{1-\frac{1}{5}}\times\frac{1}{1-\frac{1}{7}}\times\cdots?$$

What does it mean, since both sides are infinite? Or 'Euler's identity'

$$e^{ix}=\cos x+i\sin x.$$

Put $x=\pi$ and you get the extraordinary formula

$$e^{i\pi}=-1.$$

Euler did not succeed in everything he attempted. His attempt to prove that every polynomial with real coefficients can be factorised into linear and quadratic factors was not successful. He confessed that 'we cannot give any general rules for finding the roots of equations which exceed the fourth degree'. Answers to these questions had to wait until the 19th century mathematicians who stood on Euler's shoulders.

There is no area of mathematics which Euler did not investigate. He proved that the orthocentre, the centroid and the circumcentre of a triangle lie on a straight line, the 'Euler line'. He obtained the formula

$$n!\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^n}{n!}\right]$$

for the number of rearrangements of n objects in which no object stays fixed (connected with e?). He proved that the number of ways of expressing a natural number as a sum of distinct natural numbers (such as $9=4+3+2$) is the same as the number of ways of expressing the same number as a sum of odd numbers (such as $9=5+1+1+1+1$). Readers can watch Euler investigating all of these in this book. Yet this is but a small part of his work. His contributions to mechanics, astronomy, physics are not covered.

The publication of Euler's *Opera Omnia*, his collected works, began in 1911 and is still not complete. Currently it runs to 25 000 pages. Without exaggeration, it was said of Euler that 'he calculated as men breathe and eagles sustain themselves in the wind'. What an example to us all!

Read this book. Something of Euler's enthusiasm, if not his genius, may rub off on you!

The University of Sheffield DAVID SHARPE

Other books received

The Eightfold Way: The Beauty of Klein's Quartic Curve Edited By SILVIO LEVY. CUP, Cambridge, UK, 1999. Pp. 331. Hardback £35.00 (ISBN 0-521-66066-1).

Hodder Mathematics: Higher 1. By CATHERINE BERRY, DIANA COWEY, DAVE FAULKNER, NIGEL GREEN, CHRISTINE WOOD. Hodder & Stoughton, London, 1999. Pp. 269. Paperback £9.99 (ISBN 0-340-705523).

Hodder Mathematics: Higher 2. By CATHERINE BERRY, DIANA COWEY, DAVE FAULKNER, NIGEL GREEN, CHRISTINE WOOD. Hodder & Stoughton, London, 1999. Pp. 246. Paperback £9.99 (ISBN 0-340-711949).

These books are designed to cover all GCSE syllabuses – higher tier.

Introduction to Numerical Ananlysis By ALASTAIR WOOD. Addison-Wesley, Harlow, Essex, 1999. Pp. xii+349. Paperback £19.95 (ISBN 0-201-34291-X).

The Simpler? Polyhedra. By PATRICK TAYLOR. Nattygrafix, Ipswich, 1999. Pp. 79. Softback £6.00 (ISBN 0-9516701-4-X).

Achievements in Mathematics: Revision and Practice. By M. E. WARDLE AND A. LEDSHAM. OUP, Oxford, 1999. Pp. viii+232. Softback £7.50 (ISBN 0-19-914745-0).

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