Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 18 #2

February / février

1992

CONTENTS / TABLE DES MATIÈRES

Canadian Mathematical Society

Société mathématique du Canada

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1991 subscription rate for ten issues is \$ 20.00 for members of the Canadian Mathematical Society and \$40.00, for non-members. Back issues: \$4.00 each. Bound Volumes with index: volumes 1 & 2 (combined) and each of 3, 7, 8 & 9: \$10.00 (Volumes 4, 5, 6 & 10 are out-of-print). All prices are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

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577 King Edward
Ottawa, Ontario, Canada K1N 6N5

ACKNOWLEDGEMENTS

The support of the Department of Mathematics and Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

Published by the Canadian Mathematical Society
Printed at Ottawa Laser Copy

ISSN 0705-0348 Publications Mail Registration Number 5432

THE OLYMPIAD CORNER

No. 132

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we give a selection of problems used to select and train I.M.O. teams in three countries. The first problems were some used in the competition to determine the Irish I.M.O. team for 1990. Thanks go to Andy Liu, University of Alberta, for forwarding these problems to me.

SELECTION QUESTIONS FOR THE 1990 IRISH I.M.O. TEAM

1. Find all pairs of integers (x, y) such that $y^3 - x^3 = 91$.

2. Observe that, when the first digit of $x = 714285$ is moved to the end, we get $y = 142857$ and $y = x/5$. Find the smallest positive integer u such that if v is obtained from u by moving the first digit of u to the end, then $v = u/2$.

3. Let $2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, \cdots$ be the sequence of non-squares (i.e., the sequence obtained from the natural numbers by deleting $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, etc.). Prove that the nth term of the sequence is

$$\left[n + \frac{1}{2} + \sqrt{n}\right].$$

(Note that, for x a real number, $[x]$ denotes the greatest integer z with $z \le x$. Thus, for example, $[16/7] = 2$.)

4. Let $n \ge 3$ be a natural number. Prove that

$$\frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3} < \frac{1}{12}.$$

5. Let t be a real number and let

$$a_n = 2\cos\left(\frac{t}{2^n}\right) - 1$$

$(n = 1, 2, 3, \ldots)$. Let b_1 be the product $a_1 \ldots a_n$. Find a formula for b_n which does not involve a product of n terms and deduce that

$$\lim_{n\to\infty} b_n = \frac{2\cos t + 1}{3}.$$

*

The next set of three problems are from the First Test of the Third Chinese National Mathematics Training Camp Examination of April 1988. Thanks go to Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario, who translated and sent them to me.

1988 CHINESE OLYMPIAD TRAINING CAMP

Test I

1. Prove that

$$\frac{xyz(x+y+z+\sqrt{x^2+y^2+z^2})}{(x^2+y^2+z^2)(xy+yz+zx)} \leq \frac{3+\sqrt{3}}{9}$$

for all positive real numbers x, y and z with equality holding if and only if $x=y=z$.

2. Determine the smallest value of the natural number $n>3$ with the property that whenever the set $S_n=\{3,4,\ldots,n\}$ is partitioned into the union of two subsets, at least one of the subsets contains three numbers a, b and c (not necessarily distinct) such that $ab=c$. [Compare with Problem 11 by Morocco, [1988: 226] — E.T.H.W.].

3. A pharmacist has a number of ingredients some of which are "strong". Using these ingredients he is to make 68 different medicines such that each medicine contains 5 different ingredients, at least one of which is "strong" and furthermore, for any 3 ingredients chosen, there is exactly one medicine containing them. Prove or disprove that one of the 68 medicines must contain at least 4 "strong" ingredients.

*

The last test questions for this number are the two problems given the last day of the training camp of the then Soviet Union held in January 1991 in preparation for the contest in Sweden. These were forwarded from Zun Shan to Andy Liu who translated them from Chinese for use in the Corner.

TRAINING TEST OF THE 1991 U.S.S.R. I.M.O. TEAM

Last Day — 4 Hours

1. Let $a_1=1$ and

$$a_{n+1}=\frac{a_n}{2}+\frac{1}{4a_n}, \qquad n\geq 1.$$

Prove that $\sqrt{\frac{2}{2a_n^2-1}}$ is a positive integer for $n>1$.

2. Let n be a positive integer and S_n be the set of all permutations of $\{1,2,\ldots,n\}$. For $\sigma\in S_n$ let $f(\sigma)=\sum_{i=1}^n |i-\sigma(i)|$. Prove that

$$\frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma)=\frac{n^2-1}{3}\ .$$

* * *

Now we turn to answers for Klamkin's Five Quickies given last issue. Thanks to Murray S. Klamkin, University of Alberta, for the problems and their solutions.

FIVE KLAMKIN QUICKIES

1. Determine the extreme values of $r_1/h_1 + r_2/h_2 + r_3/h_3 + r_4/h_4$ where h_1, h_2, h_3, h_4 are the four altitudes of a given tetrahedron T and r_1, r_2, r_3, r_4 are the corresponding signed perpendicular distances from any point in the space of T to the faces.

Solution. If the face areas and volume of the tetrahedron are F_1, F_2, F_3, F_4, and V respectively, then

$$r_1F_1 + r_2F_2 + r_3F_3 + r_4F_4 = 3V,$$

and $h_1F_1 = h_2F_2 = h_3F_3 = h_4F_4 = 3V$. Now eliminating the F_i's, we get

$$r_1/h_1 + r_2/h_2 + r_3/h_3 + r_4/h_4 = 1 \quad \text{(a constant).}$$

2. Determine the minimum value of the product

$$P = (1 + x_1 + y_1)(1 + x_2 + y_2)\dots(1 + x_n + y_n)$$

where $x_i, y_i \geq 0$, and $x_1x_2\dots x_n = y_1y_2\dots y_n = a^n$.

Solution. More generally, consider

$$P = (1 + x_1 + y_1 + \cdots + w_1)(1 + x_2 + y_2 + \cdots + w_2)\dots(1 + x_n + y_n + \cdots + w_n)$$

where $x_1x_2\dots x_n = \xi^n$, $y_1y_2\dots y_n = \eta^n, \dots, w_1w_2\dots w_n = \omega^n$, and $x_i, y_i, \dots, w_i \geq 0$. Then by Hölder's inequality,

$$P^{1/n} \geq \left\{1 + \prod x_i^{1/n} + \prod y_i^{1/n} + \dots + \prod w_i^{1/n}\right\}$$

or

$$P \geq (1 + \xi + \eta + \dots + \omega)^n.$$

In this case $\xi = \eta = a$, so

$$P \geq (1 + 2a)^n.$$

3. Prove that if $F(x,y,z)$ is a concave function of x, y, z, then $\{F(x,y,z)\}^{-2}$ is a convex function of x, y, z.

Solution. More generally $G(F)$ is a convex function where G is a convex decreasing function. By convexity of G,

$$\lambda G\{F(x_1,y_1,z_1)\} + (1-\lambda)G\{F(x_2,y_2,z_2)\} \geq G\{\lambda F(x_1,y_1,z_1) + (1-\lambda)F(x_2,y_2,z_2)\}.$$

By concavity of F,

$$\lambda F(x_1,y_1,z_1) + (1-\lambda)F(x_2,y_2,z_2) \leq F([\lambda x_1 + (1-\lambda)x_2], [\lambda y_1 + (1-\lambda)y_2], [\lambda z_1 + (1-\lambda)z_2]).$$

Finally, since G is decreasing,

$$\lambda G\{F(x_1,y_1,z_1)\} + (1-\lambda)G\{F(x_2,y_2,z_2)\} \geq$$
$$G\{F([\lambda x_1 + (1-\lambda)x_2], [\lambda y_1 + (1-\lambda)y_2], [\lambda z_1 + (1-\lambda)z_2])\}.$$

More generally and more precisely, we have the following known result: if $F(X)$ is a concave function of $X = (x_1, x_2, \ldots, x_n)$ and $G(y)$ is a convex decreasing function of y where y is a real variable and the domain of G contains the range of F, then $G\{F(X)\}$ is a convex function of X.

4. If a, b, c are sides of a given triangle of perimeter p, determine the maximum values of

(i) $(a-b)^2 + (b-c)^2 + (c-a)^2$,
(ii) $|a-b| + |b-c| + |c-a|$,
(iii) $|a-b||b-c| + |b-c||c-a| + |c-a||a-b|$.

Solution. (i) $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(\sum a^2 - \sum bc) \le kp^2$.
Let $c = 0$, so that $k \ge 1/2$. We now show that $k = 1/2$ suffices. Here,

$$2(\sum a^2 - \sum bc) \le \frac{1}{2}\,(a+b+c)^2$$

reduces to

$$2bc + 2ca + 2ab - a^2 - b^2 - c^2 \ge 0.$$

The LHS is 16 times the square of the area of a triangle of sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ or

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

There is equality iff the triangle is degenerate with one side 0.

(ii) $|a-b| + |b-c| + |c-a| \le kp$.
Letting $c = 0$, $k \ge 1$. To show that $k = 1$ suffices, assume that $a \ge b \ge c$, so that

$$|a-b| + |b-c| + |c-a| = 2a - 2c \le a + b + c$$

and there is equality iff $c = 0$.

(iii) $|a-b|\,|b-c| + |b-c|\,|c-a| + |c-a|\,|a-b| \le kp^2$.
Letting $c = 0$, $k \ge 1/4$. To show that $k = 1/4$ suffices, let $a = y+z$, $b = z+x$, $c = x+y$ where $z \ge y \ge x \ge 0$. Our inequality then becomes

$$|x-y|\,|z-y| + |y-z|\,|z-x| + |z-x|\,|x-y| \le (x+y+z)^2$$

or

$$x^2 - y^2 + z^2 + yz - 3zx + xy \le x^2 + y^2 + z^2 + 2yz + 2zx + 2xy$$

or

$$2y^2 + 5zx + 1xy + 1yz \ge 0.$$

There is equality iff $x = y = 0$ or equivalently, $a = b$, and $c = 0$.

5. If A, B, C are three dihedral angles of a trihedral angle, show that $\sin A$, $\sin B$, $\sin C$ satisfy the triangle inequality.

Solution. Let a, b, c be the face angles of the trihedral angle opposite to A, B, C respectively. Since

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

by the Law of Sines for spherical triangles, it suffices to show that $\sin b + \sin c > \sin a$, or

$$2\sin\frac{1}{2}(b+c)\cos\frac{1}{2}(b-c) > 2\sin\frac{1}{2}a\cos\frac{1}{2}a,$$

for any labelling of the angles. We now use the following properties of a, b, c:

(i) they satisfy the triangle inequality, (ii) $0 < a+b+c < 2\pi$.

Hence, $\cos\frac{1}{2}(b-c) > \cos\frac{1}{2}a$. To complete the proof, we show that

$$\sin\frac{1}{2}(b+c) > \sin\frac{1}{2}a.$$

This follows immediately if $b+c \le \pi$; if $b+c > \pi$, then

$$\sin\frac{1}{2}(b+c) = \sin\left\{\pi - \frac{1}{2}(b+c)\right\} > \sin\frac{1}{2}a \quad \left(\text{since } \pi - \frac{b+c}{2} > \frac{a}{2}\right).$$

Comment: More generally, if $a_1, a_2, \ldots, a_n$ are the sides of a spherical n-gon (convex), it then follows by induction over n that

$$\sin a_1 + \sin a_2 + \cdots + \sin a_n > 2\sin a_i, \quad i = 1, 2, \ldots, n.$$

It also follows by induction that

$$|\sin a_1| + |\sin a_2| + \cdots + |\sin a_n| > |\sin(a_1 + a_2 + \cdots + a_n)|$$

for any angles $a_1, a_2, \ldots, a_n$.

* * *

Now we turn to solutions to problems from the October 1990 number of *Crux*, and the 25th Spanish Mathematics Olympiad, First Round, 1988.

1. [1990: 225] *25th Spanish Mathematics Olympiad, 1988.*

Let n be an even number which is divisible by a prime bigger than $\sqrt{n}$. Show that n and n^3 cannot be expressed in the form $1 + (2l+1)(2l+3)$, i.e., as one more than the product of two consecutive odd numbers, but that n^2 and n^4 can be so expressed.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; O. Johnson, student, King Edward School, Birmingham, England; Stewart Metchette, Culver City, California; Bob Prielipp, University of Wisconsin–Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. [We give Wang's solution.]

Call a natural number 'expressible' if it is of the form $1 + (2l+1)(2l+3)$ for some l. If n is even, then $n^2 = 1 + (n-1)(n+1) = 1 + (2l+1)(2l+3)$ where $l = (n-2)/2$, showing that n^2 is expressible. Since n^2 is even when n is even, n^4 is also expressible. On the other hand suppose n were expressible. Then $n = 4l^2 + 8l + 4 = 4(l+1)^2$ showing that the largest prime divisor of n can not exceed $\max\{l+1, 2\}$ which is a contradiction since $\sqrt{n} = 2(l+1)$. Now suppose n^3 were expressible. Then $n^3 = (2(l+1))^2$, implying that n^3 is a perfect 6th power. Hence the largest prime divisor of n^3 (and thus the largest prime divisor of n) can not exceed $(n^3)^{1/6}$ or $\sqrt{n}$, which is again a contradiction.

[*Editor's Note.* Both Johnson and Metchette point out the generalization to other powers of n.]

3. [1990: 225] *25th Spanish Mathematics Olympiad, 1988.*

The natural numbers $1, 2, \ldots, n^2$ are arranged to form an $n \times n$ matrix

$$A = \begin{bmatrix} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ \vdots & \vdots & & \vdots \\ \cdot & & & n^2 \end{bmatrix} .$$

A sequence $a_1, a_2, \ldots$ of elements of A is chosen as follows. The first element a_1 is chosen at random and the row and column containing it are deleted. As long as elements remain, the next element is chosen at random from among the elements that remain, and its row and column are deleted. The process continues until only *one* element is left. Calculate the sum of this last number and all the numbers previously chosen. Show that this sum is independent of the choices made.

Solutions by O. Johnson, student, King Edward School, Birmingham, England; and by Edward T.H. Wang, Wilfrid Laurier University, and Wan-Di Wei, University of Waterloo, Waterloo, Ontario.

The method of selection ensures that we select exactly one element from each row and exactly one element from each column. Since the (i,j) entry of the matrix is $(i-1)n + j$ for $i, j = 1, 2, \ldots, n$, the sum S of the n numbers must be $\sum_{i=1}^{n}(i-1)n + a_i$ where $a_1, a_2, \ldots, a_n$ is a permutation of $1, 2, \ldots, n$. Thus

$$S = n\sum_{i=1}^{n}(i-1) + \sum_{j=1}^{n} j = \frac{n(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2} .$$

5. [1990: 226] *25th Spanish Mathematics Olympiad, 1988.*

Let $ABCD$ be a square and let E be a point inside the square such that ΔECD is isosceles with $\angle C = \angle D = 15°$. What can one say about ΔABE?

Solution by Seung-Jin Bang, Seoul, Republic of Korea.

We may assume that $ABCD$ is a unit square. Let F and G be the points of intersection with AB and CD respectively, of the line through E perpendicular to those sides. The length of the segment EG is $\frac{1}{2}\tan 15° = (2-\sqrt{3})/2$. It follows that the length of the segment EF is $\sqrt{3}/2$. It turns out that $\tan \angle EAF = \sqrt{3}$, that is $\angle EAF = \angle EBF = 60°$. Thus ΔABE is equilateral.

7. [1990: 226] *25th Spanish Mathematics Olympiad, 1988.*

Find the maximum value of the function

$$f(x) = \prod_{k=0}^{7} |x-k|$$

for x in the closed interval $[3,4]$.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Edward T.H. Wang, Wilfrid Laurier University, and Wan-Di Wei, University of Waterloo, Waterloo, Ontario. [We give the solution of Wang and Wei.]

Clearly $f(x) = x(x-1)(x-2)(x-3)(4-x)(5-x)(6-x)(7-x)$. By the arithmetic-geometric-mean inequality we have

$$x(7-x) \le \left(\frac{7}{2}\right)^2, \quad (x-1)(6-x) \le \left(\frac{5}{2}\right)^2,$$

$$(x-2)(5-x) \le \left(\frac{3}{2}\right)^2, \quad (x-3)(4-x) \le \left(\frac{1}{2}\right)^2,$$

with equality holding in any (and hence all) of these inequalities if and only if $x = 7/2$. Thus the maximum value of $f(x)$ is $3^2 \cdot 5^2 \cdot 7^2 \cdot 2^{-8}$ or $11025/256$ attained uniquely when $x = 7/2$.

In exactly the same way one can show that if a is a natural number then for x in $[a, a+1]$ the maximum value of $f(x) = \prod_{k=0}^{2a+1} |x-k|$ is $((2a+1)!!/2^{a+1})^2$ attained uniquely at $x = (2a+1)/2$.

8. [1990: 226] *25th Spanish Mathematics Olympiad, 1988.*

Let m be odd. Show that for each integer $n > 2$, the sum of the mth powers of the numbers less than n that are relatively prime to n is a multiple of n.

Solutions by Bob Prielipp, University of Wisconsin-Oshkosh; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $S = \{k : 1 \le k \le n \text{ such that } (k,n) = 1\}$. Then clearly $k \in S$ iff $n-k \in S$. Furthermore, for $k \in S$, $k \ne n-k$. Otherwise $n = 2k$ would imply that $(n,k) = k$, a contradiction since $n > 2$ implies $k = 1$. Hence the elements of S can be grouped into disjoint pairs $\{k, n-k\}$. The result now follows from the fact that $a+b$ is a factor of $a^m + b^m$ when m is odd.

We have no solutions on file for the other even numbered exercises from the competition. Here is an opportunity, and a challenge to provide nice solutions!

*

The remaining solutions that we give this month are to problems from the November 1990 number of the *Corner*. We first give solutions to some of the problems from the 1990 Asian Pacific Mathematical Olympiad. Here we give only solutions that differ from the official solutions published in the solutions manual for that contest.

2. [1990: 257] *1990 Asian Pacific Mathematical Olympiad.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n, \qquad \text{for } k = 1, 2, \ldots, n-1.$$

Solution by George Evagelopoulos, Athens, Greece.

There are $\binom{n}{k}$ products of the a_i taken k at a time. Any given a_i will appear in exactly $\binom{n-1}{k-1}$ of these products, since once a_i is chosen there are $\binom{n-1}{k-1}$ ways of choosing the other factors of the product. Now the arithmetic and geometric mean of the $\binom{n}{k}$ products are related by

$$\frac{S_k}{\binom{n}{k}} \geq \left[\prod_{i=1}^{n} a_i^{\binom{n-1}{k-1}}\right]^{1/\binom{n}{k}}$$

by the arithmetic-geometric means inequality. Since $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$,

$$S_k \geq \binom{n}{k}\left(\prod_{i=1}^{n} a_i\right)^{k/n}.$$

Hence

$$S_k S_{n-k} \geq \binom{n}{k}\left(\prod_{i=1}^{n} a_i\right)^{k/n}\binom{n}{n-k}\left(\prod_{i=1}^{n} a_i\right)^{\frac{n-k}{n}} = \binom{n}{k}^2\left(\prod_{i=1}^{n} a_i\right),$$

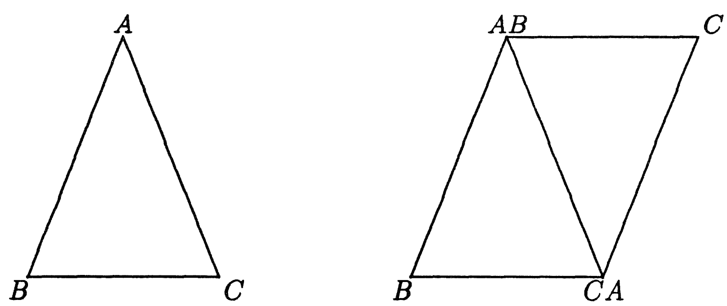
as desired.

5. [1990: 258] *1990 Asian Pacific Mathematical Olympiad.*

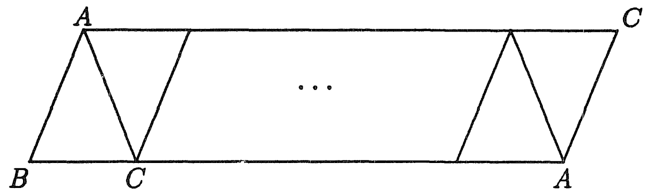
Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

Solution by George Evagelopoulos, Athens, Greece.

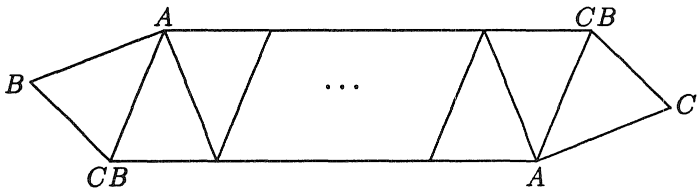
Consider an isosceles triangle S with $\angle B = \angle C$ and $\angle A < 60°$. One can form a parallelogram T by juxtaposing AC and BA as shown.



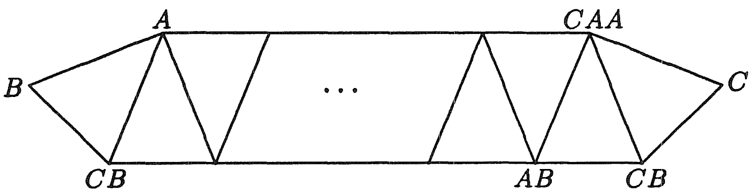
We can align n copies of T to form a parallelogram using $2n$ triangles congruent to S:



A hexagon can now be formed in two ways: (i) add one copy of S at each end, identifying AC and AB in each case;



or (ii) add one copy of S at the left as in (i) and two copies of S at the right, as shown.



In both cases, $\angle A < 60°$ ensures that all six angles of the hexagon are less than $180°$ so each hexagon is convex. Using either (i) or (ii) one can build convex hexagons that can be decomposed into exactly n congruent triangles for $n = 4, 5, 6, \ldots$.

[*Editor's remark:* the official solution obtains *right* triangles.]

*

The last two solutions are to the first two problems of the Fourth Nordic Mathematical Olympiad held April 5, 1990.

1. [1990: 258] *Fourth Nordic Mathematical Olympiad.*

Let m, n and p be positive odd integers. Show that the number

$$\sum_{k=1}^{(n-1)^p} k^m$$

is divisible by n.

Solutions by Seung-Jin Bang, Seoul, Republic of Korea; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since $x + y$ is a factor of $x^m + y^m$ for all odd positive integers m, we have for each $k = 1, 2, \ldots, (n-1)^p/2$ that $k^m + [(n-1)^p - (k-1)]^m$ is divisible by $k + [(n-1)^p - (k-1)] = (n-1)^p + 1$. Now since $(n-1)^p + 1 = n^p - \binom{p}{1}n^{p-1} + \cdots + \binom{p}{p-1}n - 1 + 1 \equiv 0 \mod n$, we conclude that

$$\sum_{k=1}^{(n-1)^p} k^m = \sum_{k=1}^{(n-1)^p/2} (k^m + [(n-1)^p - (k-1)]^m) \equiv 0 \mod n.$$

2. [1990: 258] *Fourth Nordic Mathematical Olympiad.*
Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\sqrt[3]{a_1^3 + a_2^3 + \cdots + a_n^3} \le \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

When does equality hold?

Solution(s) by David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Since the LHS of the given inequality does not decrease if any a_i is replaced by $|a_i|$ we may assume, without loss of generality, that $a_i \ge 0$ for all $i = 1, 2, \ldots, n$.

We give two solutions, the first of which establishes a more general result.

Solution I. The result is a special case of the following:

Theorem. If $a_i \ge 0$ for $i = 1, 2, \ldots, n$ and $0 < p \le q$ then

$$(a_1^q + a_2^q + \cdots + a_n^q)^{1/q} \le (a_1^p + a_2^p + \cdots + a_n^p)^{1/p}$$

with equality holding if and only if either $p = q$ or at most one $a_i \ne 0$.

Proof. The claimed inequality is equivalent to

$$a_1^q + \cdots + a_n^q \le (a_1^p + a_2^p + \cdots + a_n^p)^{q/p} \tag{1}$$

which we shall prove by induction on n. The case $n = 1$ is trivial. For $n = 2$, we need to show that

$$a_1^q + a_2^q \le (a_1^p + a_2^p)^{q/p}. \tag{2}$$

If $a_1 = 0$, there is nothing to prove. Assume $a_1 \ne 0$. Then (2) is equivalent to

$$1 + \left(\frac{a_2}{a_1}\right)^q \le \left(1 + \left(\frac{a_2}{a_1}\right)^p\right)^{q/p}.$$

Consider $f(x) = (1 + x^p)^{q/p} - (1 + x^q)$ for $x \ge 0$. Then

$$f'(x) = \frac{q}{p}(1 + x^p)^{(q/p)-1} \cdot px^{p-1} - qx^{q-1} = qx^{p-1}((1 + x^p)^{(q/p)-1} - x^{q-p}) > 0$$

since $(1 + x^p)^{(q/p)-1} > (x^p)^{(q/p)-1} = x^{q-p}$. Since $f(0) = 0$, we conclude that $f(x) > 0$ and thus (2) follows. It is easily checked that (2) is strict if both $a_1, a_2 > 0$.

Now suppose that (1) holds for some $n \ge 2$. Then using the induction hypothesis and (2) we have, with $y = (a_1^p + \cdots + a_n^p)^{1/p}$,

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_{n+1}^2 &= (a_1^2 + \cdots + a_n^2) + a_{n+1}^2 \le (a_1^p + \cdots + a_n^p)^{q/p} + a_{n+1}^q \\ &= y^q + a_{n+1}^q \le (y^p + a_{n+1}^p)^{q/p} = (a_1^p + \cdots + a_n^p + a_{n+1}^p)^{q/p}, \end{aligned}$$

completing the induction. The last inequality is easily seen to be strict when at least two a_i's are nonzero by rearranging the terms, if necessary, so that a_{n+1} is one of these nonzero terms. For the proposed problem, this means that equality holds iff either $p = q$ or at most one $a_i \neq 0$. $\square$

[*Editor's Note.* This result is known as "Jensen's Theorem". See eq. 1.4.1, p. 4 of Hardy, Littlewood, and Pólya, *Inequalities.*]

Solution II. We use the Cauchy-Schwarz Inequality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

Setting $b_i = a_i^2$, then we have

$$\begin{aligned}(a_1^3 + a_2^3 + \cdots + a_n^3)^2 &\leq (a_1^2 + a_2^2 + \cdots + a_n^2)(a_1^4 + a_2^4 + \cdots + a_n^4)\\ &\leq (a_1^2 + a_2^2 + \cdots + a_n^2)^3 \qquad (3)\end{aligned}$$

since clearly $a_1^4 + a_2^4 + \cdots + a_n^4 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)^2$. Since equality holds in (3) if and only if either $n = 1$ or $a_i \neq 0$ for at most one i, we see that, for the proposed problem, equality holds iff either $a_i = 0$ for all i or $n = 1$ and $a_1 > 0$.

* * *

That's the Corner for this month. Contest season is upon us. Send me your regional contests and your nice solutions.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **September 1, 1992,** *although solutions received after that date will also be considered until the time when a solution is published.*

1711*. *Combination of independent proposals by Herta T. Freitag, Roanoke, Virginia, and by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $0 < r < 1$, let $x_1, x_2, \ldots, x_k$ be fixed positive numbers, and define

$$x_{n+k+1} = \sum_{i=1}^{k} x_{n+i}^r$$

for $n \geq 0$. Show that the sequence $\{x_n\}$ converges and determine its limit.

1712. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the minimum value of

$$\frac{16\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)+1}{\tan(A/2)\tan(B/2)\tan(C/2)}$$

where A, B, C are the angles of a triangle.

1713. *Proposed by Jeremy Bern, student, Ithaca H.S., Ithaca, N.Y.*
For a fixed positive integer n, let K be the area of the region

$$\left\{z : \sum_{k=1}^{n}\left|\frac{1}{z-k}\right| \geq 1\right\}$$

in the complex plane. Prove that $K \geq \pi(11n^2+1)/12$.

1714. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Let P and Q be two points lying in the interior of $\angle BAC$ of ΔABC, such that the line PQ is the perpendicular bisector of BC, and such that $\angle ABP + \angle ACQ = 180°$. Prove that $\angle BAP = \angle CAQ$.

1715. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*
Evaluate the sum

$$\sum_{k=0}^{n-2}\frac{1}{k!}\left(\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^{n-k}\frac{1}{(n-k)!}\right)$$

for $n \geq 2$.

1716. *Proposed by Jordi Dou, Barcelona, Spain.*
Equilateral triangles $A'BC, B'CA, C'AB$ are erected outward on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the other intersections of Ω with the lines $A'A, B'B, C'C$, respectively. Prove that $AA'' + BB'' + CC'' = AA'$. [It is known that AA', BB' and CC' are concurrent; e.g., see [1991: 308].]

1717. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
For each positive integer n, let $f(n)$ denote the number of ordered pairs (x, y) of nonnegative integers such that $n = x^2 - y^2$. For example, $f(9) = 2$ as $9 = 3^2 - 0^2 = 5^2 - 4^2$ are the only representations. Find a formula for $f(n)$.

1718. *Proposed by Juan Bosco Romero Márquez, Universidad de Valladolid, Spain.*
Let $\mathcal{C}$ be a central conic with foci F_1 and F_2, and let X and Y be the points where the tangent to $\mathcal{C}$ at a point P on $\mathcal{C}$ meets the axes (extended) of $\mathcal{C}$. Prove that

$$PX \cdot PY = PF_1 \cdot PF_2.$$

1719*. *Proposed by E. Lindros, Le Colisée, Québec.*

Let k and l be two positive real numbers less than 1, and consider a square of side 1 whose sides are horizontal and vertical. A horizontal line of length k is drawn at random inside the square (the line cannot stick outside the square). Independently, a vertical line of length l is drawn at random inside the square. What is the probability that the two lines intersect?

1720. *Proposed by P. Penning, Delft, The Netherlands.*

The osculating circle at point P (not a vertex) of a conic intersects the conic in one other point Q. Find a simple construction for Q, given the conic itself, its axes and the tangent at P.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1508. [1990: 20; 1991: 89] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $a \le b < c$ be the lengths of the sides of a right triangle. Find the largest constant K such that

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \ge Kabc$$

holds for all right triangles and determine when equality holds. It is known that the inequality holds when $K = 6$ (problem 351 of the *College Math. Journal*; solution on p. 259 of Volume 20, 1989).

II. *Generalization by Murray S. Klamkin, University of Alberta.*

Instead of right triangles, we consider all triangles of fixed largest angle $C = 2\theta$ and determine the maximum K. Here $30^\circ \le \theta \le 90^\circ$, and without loss of generality we can take $c = 1$. Then

$$\max K = \min\left(\frac{a^2(b+1)+b^2(1+a)+a+b}{ab}\right) = \min\left(a+b+\frac{1}{a}+\frac{1}{b}+\frac{a}{b}+\frac{b}{a}\right)$$

subject to

$$1 = a^2+b^2-2ab\cos 2\theta. \tag{1}$$

We will show that the minimum is taken on when $a = b$. Clearly $a/b + b/a$ is minimized when $a = b$. Since

$$a+b+\frac{1}{a}+\frac{1}{b} = (a+b)\left(1+\frac{1}{ab}\right),$$

it now suffices to show that

$$P \equiv (a+b)^2\left(1+\frac{1}{ab}\right)^2$$

takes on its minimum for $a = b$. Letting

$$x = ab = \frac{ab}{c^2} = \frac{\sin A \sin B}{\sin^2 C} = \frac{\cos(B-A) - \cos(B+A)}{2\sin^2 C},$$

we find that

$$0 \le x \le \frac{1+\cos C}{2\sin^2 C} = \frac{1}{2(1-\cos 2\theta)},$$

where the upper bound occurs for $A = B$, i.e., for $a = b$. Also,

$$\frac{1}{4} \le \frac{1}{2(1-\cos 2\theta)} \le 1.$$

It follows from (1) that

$$(a+b)^2 = 1 + 2ab(1+\cos 2\theta) = 1 + 4x\cos^2\theta.$$

Hence

$$P = (1 + 4x\cos^2\theta)\left(1 + \frac{1}{x}\right)^2 = 1 + \frac{2}{x} + \frac{1}{x^2} + 4\left(x + \frac{1}{x}\right)\cos^2\theta + 8\cos^2\theta.$$

Since $x \le 1$, P is minimized when x is a maximum, that is, when $a = b$.

Thus from (1),

$$1 = 2a^2 - 2a^2\cos 2\theta = 4a^2\sin^2\theta$$

or $a = 1/(2\sin\theta)$, so

$$\max K = 2a + \frac{1}{2a} + 2 = 2 + 4\sin\theta + \frac{1}{\sin\theta} = \left(2\sqrt{\sin\theta} - \frac{1}{\sqrt{\sin\theta}}\right)^2 + 6.$$

This means that $\max K$ is an increasing function of θ in the interval $30° \le \theta \le 90°$. Thus from $1/2 \le \sin\theta \le 1$,

$$\min_{30° \le \theta \le 90°} \max K = 6 \quad \text{and} \quad \max_{30° \le \theta \le 90°} \max K = 7.$$

Thus for all angles C between 60° and 180° there is only a spread of 1 in $\max K$.

*   *   *   *   *

1531. [1990: 108; 1991: 156] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that

$$\frac{v+w}{u} \cdot \frac{bc}{s-a} + \frac{w+u}{v} \cdot \frac{ca}{s-b} + \frac{u+v}{w} \cdot \frac{ab}{s-c} \ge 4(a+b+c),$$

where a, b, c, s are the sides and semiperimeter of a triangle, and u, v, w are positive real numbers. (Compare with *Crux* 1212 [1988: 115].)

IV. *Comment by Murray S. Klamkin, University of Alberta.*

At the end of his solution [1991: 160], Walther Janous leaves open the problem of proving or disproving that

$$\sqrt{a(s-a)}+\sqrt{b(s-b)}+\sqrt{c(s-c)} \geq 2s\sqrt{r/R}\,. \tag{1}$$

We prove a generalization of (1) using results in my solution [1991: 157]. We have that for $p \geq 1$,

$$[3^{p-1}(a^p+b^p+c^p)]^{1/p} \geq a+b+c \geq \sqrt{12F\sqrt{3}}\,, \tag{2}$$

where F is the area of the triangle. Now if a, b, c are sides of a triangle, then so are $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$, $\sqrt{c(s-c)}$, and the area of this triangle is $F/2$ [see VII.5.2, page 113 of D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*]. Hence a dual inequality to (2) is

$$[3^{p-1}([a(s-a)]^{p/2}+[b(s-b)]^{p/2}+[c(s-c)]^{p/2})]^{1/p} \geq \sqrt{6F\sqrt{3}}\,. \tag{3}$$

Now

$$\sqrt{6F\sqrt{3}} \geq 2s\sqrt{r/R}\,,$$

as this is equivalent (via $F=rs$) to $3\sqrt{3}\,R \geq 2s$, a known inequality corresponding to the fact that the maximum perimeter triangle inscribed in a given circle is the equilateral one. Consequently (3) becomes

$$[a(s-a)]^{p/2}+[b(s-b)]^{p/2}+[c(s-c)]^{p/2} \geq 3\left(\frac{2s}{3}\sqrt{\frac{r}{R}}\right)^p\,. \tag{4}$$

Inequality (1) is just the special case of (4) when $p=1$.

*   *   *   *   *

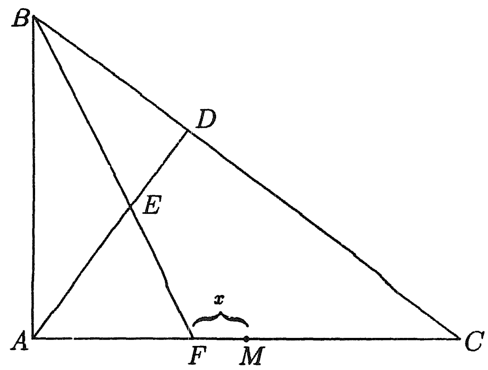
1601. [1991: 13] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a right-angled triangle with the right angle at A. Let D be the foot of the perpendicular from A to BC, and let E and F be the intersections of the bisector of $\angle B$ with AD and AC respectively. Prove that $\overline{DC} > 2\overline{EF}$.

I. *Solution by the Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen.*

From the figure (where M is the midpoint of AC),

$$\angle FEA = \angle DEB = \angle AFB$$



and thus

$$\frac{EF/2}{AF} = \frac{AF}{BF}\,,$$

i.e.

$$2EF = \frac{4(AF)^2}{BF}\,.$$

Also

$$\frac{DC}{AC} = \frac{AC}{BC}, \quad \text{so} \quad DC = \frac{(AC)^2}{BC}.$$

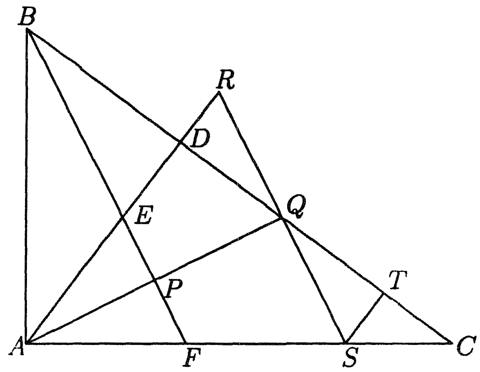
Hence (since BF is the bisector of $\angle B$)

$$\frac{DC}{2EF} = \frac{(AC)^2 \cdot BF}{BC \cdot 4(AF)^2} > \frac{(AC)^2}{4(AF)^2} \cdot \frac{BA}{BC} = \frac{(AC)^2}{4(AF)^2} \cdot \frac{AF}{FC}$$
$$= \frac{(AM)^2}{AF \cdot FC} = \frac{(AM)^2}{(AM-x)(AM+x)} = \frac{(AM)^2}{(AM)^2 - x^2} > 1,$$

so $DC > 2EF$.

II. *Solution by Charles H. Jepsen, Grinnell College, Grinnell, Iowa.*

In the figure, $APQ \perp BF$, $RQS \| BF$ and $ST \perp BC$. Since BF bisects $\angle B$, $AP = PQ$, thus $RS = 2EF$. We finish the proof by showing $DC > RS$. From



area $(ADC) >$ area $(ADTS) =$ area (ARS)

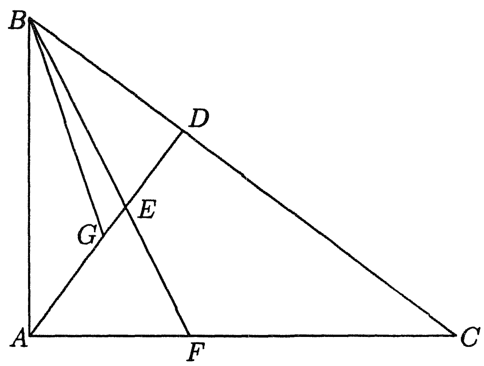
we have

$$\frac{1}{2}\, DC \cdot AD > \frac{1}{2}\, RS \cdot AQ > \frac{1}{2}\, RS \cdot AD,$$

and so $DC > RS$.

III. *Solution by Giannis G. Kalogerakis, Canea, Crete, Greece.*

We know that $DC/AD = AD/BD$, i.e.,



$$DC = \frac{(AD)^2}{BD}.$$

Also, the triangles ABF and BED are similar, hence $BF/BE = AB/BD$. Thus we find

$$EF = BF - BE = \frac{BE \cdot AB - BE \cdot BD}{BD}.$$

We therefore have to show that

$$\frac{(AD)^2}{BD} > \frac{2BE(AB - BD)}{BD},$$

or

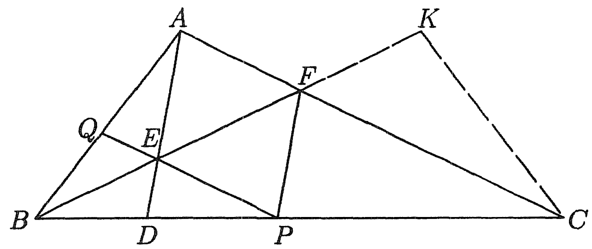
$$(AB)^2 - (BD)^2 > 2BE(AB - BD),$$

which simplifies to $AB + BD > 2BE$. But if BG is the median (G the midpoint of AD), then

$$AB + BD > 2BG > 2BE.$$

IV. *Generalization by Shailesh Shirali, Rishi Valley School, India.*

Actually we need only the fact that $\angle BAC = \angle ADB$; the right angle is not really necessary.



To start with, note that

$$(BF)^2 < BA \cdot BC;$$

for if K is defined to be the point where the ray BF meets the circumcircle of ΔABC, then triangles ABF and KBC are similar, so $AB/BF = KB/BC$ and the stated result follows from $BF < BK$.

Now draw $FP \| AD$ with P on BC; P is symmetric to A with respect to BF. From the similarity of triangles ABF and DBE, we find that ΔAEF is isosceles, so $AE = AF$. Also $AF = FP$ by symmetry, so AE is equal and parallel to FP and therefore $AEPF$ is a rhombus, with $PE \| CA$. Extend PE to meet AB at Q. From the various pairs of similar triangles in the figure we have

$$\frac{EF}{QA} = \frac{BF}{BA}, \quad \frac{EF}{PC} = \frac{BF}{BC}, \quad DP = QA \text{ (by symmetry)},$$

and so

$$(EF)^2 = (BF)^2 \frac{QA \cdot PC}{BA \cdot BC} < QA \cdot PC = DP \cdot PC \le \left(\frac{DP + PC}{2}\right)^2 = \left(\frac{DC}{2}\right)^2,$$

or $EF < (DC)/2$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia (three solutions); SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAG JONSSON, Uppsala, Sweden; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The variety of solutions submitted for this problem was quite astonishing; above are four of the best. Shirali was the only reader to notice the more general result given in solution IV. *Readers may enjoy discovering whether the other given solutions can be adapted to yield the same thing.*

* * * * *

1602. [1991: 14] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Suppose $x_1, x_2, \ldots, x_n \in [0,1]$ and $\sum_{i=1}^n x_i = m + r$ where m is an integer and $r \in [0,1)$. Prove that

$$\sum_{i=1}^n x_i^2 \leq m + r^2.$$

I. *Solution by Richard I. Hess, Rancho Palos Verdes, California.*

Consider maximizing $\sum_{i=1}^n x_i^2$ subject to $\sum_{i=1}^n x_i = m + r$. For $x_i + x_j = c$,

$$x_i^2 + x_j^2 = x_i^2 + (c - x_i)^2 = 2x_i^2 - 2x_i c + c^2$$

is maximized when

$$x_i = \begin{cases} c \text{ or } 0 & \text{if } c < 1, \\ 1 \text{ or } c - 1 & \text{if } c \geq 1 \end{cases}$$

[i.e., at an "endpoint", since the graph of $y = 2x^2 - 2xc + c^2$ is a parabola opening upwards]. This means that for maximum $\sum_{i=1}^n x_i^2$, at most one of the x_i's can be different from 0 or 1, allowing one to maximize $\sum_{i=1}^n x_i^2$ by letting m of the x_i be 1, one x_i be r, and the remaining $n - m - 1$ of the x_i be 0. Thus

$$\sum_{i=1}^n x_i^2 \leq m + r^2,$$

with equality holding in the above case.

II. *Generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We prove

THEOREM. Let q be a positive real number and $f : [0, q] \longrightarrow \mathbf{R}$ be a convex function. Furthermore, let $x_1, \ldots, x_n \in [0, q]$ (where $n \geq 2$) be such that $\sum_{i=1}^n x_i = mq + r$, where m is an integer and $0 \leq r < q$. Then

$$\sum_{i=1}^n f(x_i) \leq (n - m - 1)f(0) + mf(q) + f(r).$$

We first prove the following:

LEMMA. Let $f : [\alpha, \beta] \longrightarrow \mathbf{R}$ be a convex function and $a, b, c, d \in [\alpha, \beta]$ such that $a \leq c, d \leq b$ and $a + b = c + d$. Then

$$f(a) + f(b) \geq f(c) + f(d).$$

Proof. If equality occurs at least once in $a \leq c, d \leq b$, the claimed inequality is clear (being also an equality). Thus let $a < c, d < b$. The convexity of f yields

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(d)}{b - d}.$$

Comparing the first and third terms of this inequality, we get the claimed one. □

Proof of theorem. We proceed by induction. If $n = 2$, then either

(i) $m = 0$, and then $x_1 + x_2 = 0 + r$ where $0 \le x_1, x_2 \le r$,

or

(ii) $m = 1$, and then $x_1 + x_2 = q + r$, where $r \le x_1, x_2 \le q$;

and in both cases we're done by the lemma. [A third case, namely $m = 2$, $x_1 = x_2 = q$, $r = 0$, is easily taken care of. —*Ed.*]

Let us assume the validity of the theorem up to n. Let $\sum_{i=1}^{n+1} x_i = mq + r$ and put $x_{n+1} = x$. We now apply the induction hypothesis to $\sum_{i=1}^{n} x_i = mq + r - x$.

Case (i): $0 \le x \le r$. Then $0 \le r - x < q$, so

$$\begin{aligned}\sum_{i=1}^{n+1} f(x_i) &= \sum_{i=1}^{n} f(x_i) + f(x)\\ &\le (n-m-1)f(0) + mf(q) + f(r-x) + f(x)\\ &\le (n-m)f(0) + mf(q) + f(r)\end{aligned}$$

(where we have used the lemma for $f(r-x) + f(x) \le f(0) + f(r)$).

Case (ii): $r < x \le q$. Then

$$\sum_{i=1}^{n} x_i = (m-1)q + q + r - x, \quad 0 \le q + r - x < q,$$

and thus

$$\begin{aligned}\sum_{i=1}^{n+1} f(x_i) &= \sum_{i=1}^{n} f(x_i) + f(x)\\ &\le (n-m)f(0) + (m-1)f(q) + f(q+r-x) + f(x)\\ &\le (n-m)f(0) + mf(q) + f(r)\end{aligned}$$

(where we have used the lemma for $f(q+r-x) + f(x) \le f(r) + f(q)$). Done! □

Choosing $q = 1$ and $f(x) = x^2$ we get the original problem.

Also solved by MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; BEATRIZ MARGOLIS, Paris, France; JEAN-MARIE MONIER, Lyon, France; CORY C. PYE, student, Memorial University of Newfoundland, St. John's; DAVID C. VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

The method of choice for most solvers was induction on n. However, the proposer's solution was somewhat like Hess's (solution I*). Vaughan gave a generalization weaker than that of Janous. Klamkin proved essentially the same theorem as Janous, but did it by applying the majorization inequality to*

$$(x_1, x_2, \ldots, x_n) \prec (q, q, \ldots, q, r, 0, \ldots, 0)$$

(in which there are m q*'s and* $n - m - 1$ 1*'s).*

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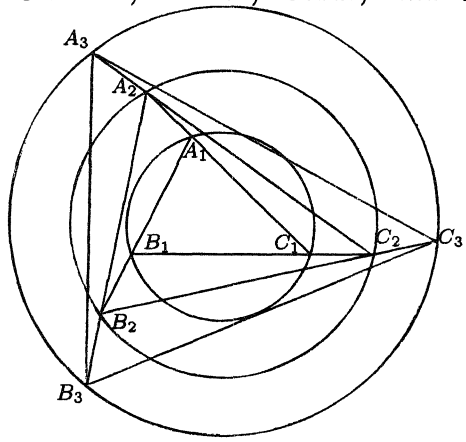
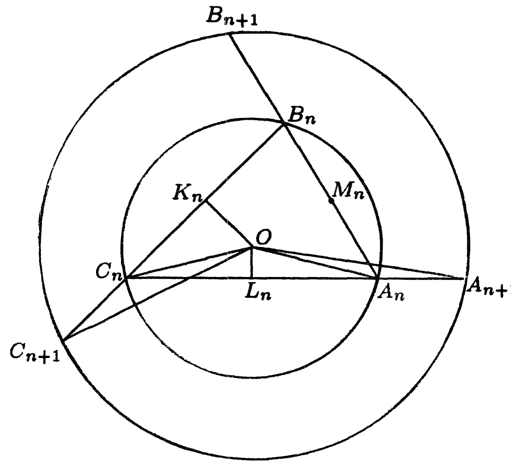
1603. [1991: 14] *Proposed by Clifford Gardner, Austin, Texas, and Jack Garfunkel, Flushing, N.Y.*



Given is a sequence $\Gamma_1, \Gamma_2, \ldots$ of concentric circles of increasing and unbounded radii and a triangle $A_1B_1C_1$ inscribed in Γ_1. Rays $\overrightarrow{A_1B_1}, \overrightarrow{B_1C_1}, \overrightarrow{C_1A_1}$ are extended to intersect Γ_2 at B_2, C_2, A_2, respectively. Similarly, $\Delta A_3B_3C_3$ is formed in Γ_3 from $\Delta A_2B_2C_2$, and so on. Prove that $\Delta A_nB_nC_n$ tends to the equilateral as $n \to \infty$, in the sense that the angles of $\Delta A_nB_nC_n$ all tend to $60°$.

Solution by Marcin E. Kuczma, Warszawa, Poland.



Let triangle $A_nB_nC_n$ have angles A_n, B_n, C_n and let

$$d_n = \max\{|A_n - B_n|, |B_n - C_n|, |C_n - A_n|\}.$$

We must show $\lim_{n\to\infty} d_n = 0$.

Denote by O the common center of all circles and by R_n the radius of the nth circle, and write $t_n = R_n/R_{n+1}$. Since the radii are unbounded, the infinite product $t_1t_2t_3\ldots$ is equal to 0.

Let K_n, L_n, M_n be the midpoints of B_nC_n, C_nA_n, A_nB_n, respectively. Then

$$OK_n = R_n \cos(\angle K_nOC_n) = R_n \cos A_n$$

and

$$OK_n = R_{n+1} \cos(\angle K_nOC_{n+1}).$$

Hence $\angle K_nOC_{n+1} = f_n(A_n)$, where

$$f_n(x) = \arccos(t_n \cos x), \quad 0 < x < \pi. \tag{1}$$

Likewise, $\angle L_nOA_{n+1} = f_n(B_n)$ and $\angle M_nOB_{n+1} = f_n(C_n)$.

Note the angle equality

$$\angle K_nOC_{n+1} + \angle C_{n+1}OA_{n+1} = \angle K_nOC_n + \angle C_nOL_n + \angle L_nOA_{n+1},$$

i.e.

$$f_n(A_n) + 2B_{n+1} = A_n + B_n + f_n(B_n). \tag{2}$$

(The argument has to be slightly modified if $\Delta A_nB_nC_n$ is obtuse-angled, but the formula (2) is valid in that case, as well; to avoid the need of considering this as a special case we could work with oriented angles.)

Now we rewrite equation (2) together with its two cyclic counterparts in the form

$$2B_{n+1} = A_n + B_n + f_n(B_n) - f_n(A_n),$$
$$2C_{n+1} = B_n + C_n + f_n(C_n) - f_n(B_n),$$
$$2A_{n+1} = C_n + A_n + f_n(A_n) - f_n(C_n).$$

It is easily seen by differentiating (1) that $|f_n'(x)| \le t_n$, thus

$$|f_n(x) - f_n(y)| \le t_n|x - y|$$

for all x, y [by the mean value theorem]; i.e., f_n is contractive with Lipschitz constant t_n. So we get

$$\begin{aligned} 2|B_{n+1} - C_{n+1}| &\le |A_n - C_n| + |f_n(B_n) - f_n(C_n)| + |f_n(A_n) - f_n(B_n)| \\ &\le |A_n - C_n| + t_n|B_n - C_n| + t_n|A_n - B_n| \end{aligned}$$

and similarly

$$2|C_{n+1} - A_{n+1}| \le |B_n - A_n| + t_n|C_n - A_n| + t_n|B_n - C_n|,$$

$$2|A_{n+1} - B_{n+1}| \le |C_n - B_n| + t_n|A_n - B_n| + t_n|C_n - A_n|.$$

Since the configuration is cyclic, we may assume that B_n is the middle angle, i.e., either $A_n \ge B_n \ge C_n$ or $A_n \le B_n \le C_n$ holds, so that $|A_n - B_n| + |B_n - C_n| = |A_n - C_n| = d_n$. Then

$$\begin{aligned} 2|B_{n+1} - C_{n+1}| &\le |A_n - C_n| + t_n|A_n - C_n| = (1 + t_n)d_n, \\ 2|C_{n+1} - A_{n+1}| &\le |B_n - A_n| + t_n|C_n - A_n| + |B_n - C_n| = (1 + t_n)d_n, \\ 2|A_{n+1} - B_{n+1}| &\le |C_n - B_n| + |A_n - B_n| + t_n|C_n - A_n| = (1 + t_n)d_n, \end{aligned}$$

and consequently

$$d_{n+1} \le \frac{1}{2}\,(1 + t_n)d_n. \tag{3}$$

Since the product $\prod_{n=1}^{\infty} t_n$ is zero, so is $\prod_{n=1}^{\infty}(1 + t_n)/2$. [*Editor's note.* Helpful colleague Len Bos supplied the following proof of this. One easily checks that

$$\frac{1 + x}{2} \le x^{1/3} \quad \text{for} \quad \delta \le x \le 1,$$

for some positive constant $\delta < 1$. Then if infinitely many of the t_n's are less than δ,

$$\prod_{n=1}^{\infty} \frac{1 + t_n}{2} \le \prod_{t_n < \delta} \frac{1 + t_n}{2} \le \prod_{t_n < \delta} \frac{1 + \delta}{2} = 0;$$

while if only finitely many of the t_n's are less than δ,

$$\prod_{n=1}^{\infty} \frac{1 + t_n}{2} \le \prod_{t_n \ge \delta} \frac{1 + t_n}{2} \le \prod_{t_n \ge \delta} t_n^{1/3} = \left(\prod_{t_n \ge \delta} t_n\right)^{1/3} = 0;$$

in either case $\prod_{n=1}^{\infty}(1 + t_n)/2 = 0$.]

From (3),

$$d_n \leq d_1 \prod_{k=1}^{n-1} \frac{1+t_k}{2} ,$$

and therefore $\lim_{n\to\infty} d_n = 0$.

Also solved (the same way) by the proposers. One other reader sent in a partial solution in which he showed that d_n (in the above proof) decreases, but didn't show that it went to zero.

* * * * *

1604. [1991: 14] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Ever active Pythagoras recently took a stroll along a street where only Pythagoreans lived. He was happy to notice that the houses on the left side were numbered by squares of consecutive natural numbers while the houses on the right were numbered by fourth powers of consecutive natural numbers, both starting from 1. Each side had the same (reasonably large) number of houses. At some point he noticed a visitor. "It is awesome!" said the visitor on encountering Pythagoras. "Never did I see houses numbered this way." In a short discussion that followed, the visitor heard strange things about numbers. And when it was time to part, Pythagoras asked "How many houses did you see on each side of the street?" and soon realized that counting was an art that the visitor had never mastered. "Giving answers to my questions is not my habit", smilingly Pythagoras continued. "Go to a *Crux* problem solver, give the clue that the sum of the house numbers on one side is a square multiple of the corresponding sum on the other side and seek help."

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.

Let n be the number of houses on either side. The sums of house numbers on the left and right sides are respectively

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} ,$$

so their ratio being a square implies

$$\frac{3n^2+3n-1}{5} = q^2.$$

Thus

$$n = \frac{-3+\sqrt{21+60q^2}}{6} = \frac{-3+\sqrt{21+15r^2}}{6}$$

with $r = 2q$. We want solutions of the Pell equation

$$z^2 - 15r^2 = 21 \tag{1}$$

from which we will then get $n = (z-3)/6$ (when this is an integer). Equation (1) has smallest solutions

$$(z_1, r_1) = (6,1) \quad \text{and} \quad (z_2, r_2) = (9,2), \tag{2}$$

and the Pell equation $u^2-15v^2 = 1$ has smallest solution $(u,v) = (4,1)$. Therefore solutions of (1) are given by

$$\begin{pmatrix} z_{i+2} \\ r_{i+2} \end{pmatrix} = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} z_i \\ r_i \end{pmatrix},$$

or

$$z_{i+2} = 4z_i + 15r_i, \quad r_{i+2} = z_i + 4r_i.$$

Starting with the solutions (2), one gets

z_k	r_k	$n = (z_k-3)/6$
6	1	—
9	2	**1**
39	10	**6**
66	17	—
306	79	—
519	134	**86**
2409	622	**401**
4086	1055	—
18966	4897	—
32169	8306	**5361**
$\vdots$	$\vdots$	$\vdots$

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; C. FESTRAETS-HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SIDNEY KRAVITZ, Dover, New Jersey; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; D.J. SMEENK, Zaltbommel, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Most solvers guessed that either 86 *or* 401 *was the intended answer. Hess chose* 86 *because, as he says, "Pythagoreans are probably a rather selective group".*

Janous wonders for which positive integer values of k and l the quotient

$$\sum_{i=1}^{n} i^k \Big/ \sum_{i=1}^{n} i^l$$

is a polynomial with rational coefficients. The example $(k,l) = (4,2)$ occurs in this problem.

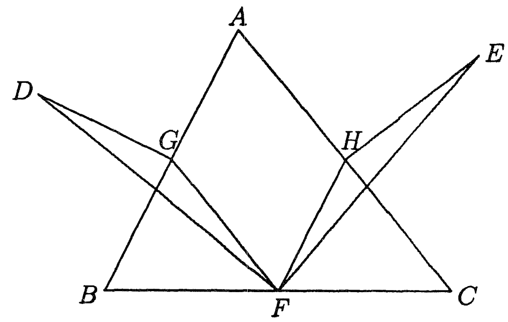
* * * * *

1605. [1991: 14] *Proposed by M.S. Klamkin and Andy Liu, University of Alberta.*

ADB and AEC are isosceles right triangles, right-angled at D and E respectively, described outside ΔABC. F is the midpoint of BC. Prove that DFE is an isosceles right-angled triangle.

Solution by Leon Bankoff, Los Angeles, California.

Let G, H be the midpoints of AB, AC respectively. It is easily seen that $DG = AG = HF$ and that $HE = AH = GF$. Also, $\angle DGF = \angle FHE$ since the extensions of their corresponding sides are mutually perpendicular. Hence triangles DGF and FHE are congruent with $DF = FE$. Furthermore, since corresponding sides of triangles DGF and FHE are mutually perpendicular, DF is perpendicular to FE. Thus DEF is a right-angled isosceles triangle.



Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SAM BAETHGE, Science Academy, Austin, Texas; SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria (three solutions); JASON COLWELL, student, Edmonton; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; HERTA T. FREITAG, Roanoke, Virginia; RICHARD I. HESS, Rancho Palos Verdes, California; JEFF HIGHAM, student, University of Toronto; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CHARLES H. JEPSEN, Grinnell College, Grinnell, Iowa; DAG JONSSON, Uppsala, Sweden; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; JEAN-MARIE MONIER, Lyon, France; P. PENNING, Delft, The Netherlands; JOHN RAUSEN, New York, N.Y. (two solutions); MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Bunuel, Madrid, Spain; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; CHRIS WILDHAGEN, Rotterdam, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposers (three solutions).

Janous, Rausen, and the proposers observed that the result also holds if the triangles ADB *and* AEC *are constructed inwardly on* ΔABC. *Janous and the proposers further show that if* ADB *and* AEC *are similar and isosceles but not right-angled, then they will not be similar to* ΔDFE.

Wilke points out the similar problem Crux 540—*see especially the published solution on* [1981: 127]. *(Interestingly,* Crux 540 *was proposed by our featured solver Leon Bankoff!) Bellot reduced the problem to exercise* 24(c), *pages* 39 *and* 96–97 *of I.M. Yaglom,* Geometric Transformations I, *NML* 8, *M.A.A.*, 1962.

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1606*. [1991: 14] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For integers $n \geq k \geq 1$ and real x, $0 \leq x \leq 1$, prove or disprove that

$$\left(1-\frac{x}{k}\right)^n \geq \sum_{j=0}^{k-1}\left(1-\frac{j}{k}\right)\binom{n}{j}x^j(1-x)^{n-j}.$$

Solution by G.P. Henderson, Campbellcroft, Ontario.

We will prove that the inequality is true. We have

$$\begin{aligned}\left(1-\frac{x}{k}\right)^n &= \left[(1-x)+\left(1-\frac{1}{k}\right)x\right]^n = \sum_{j=0}^{n}\left(1-\frac{1}{k}\right)^j\binom{n}{j}x^j(1-x)^{n-j}\\ &\geq \sum_{j=0}^{k-1}\left(1-\frac{1}{k}\right)^j\binom{n}{j}x^j(1-x)^{n-j}\\ &\geq \sum_{j=0}^{k-1}\left(1-\frac{j}{k}\right)\binom{n}{j}x^j(1-x)^{n-j},\end{aligned}$$

where the last step follows because

$$\left(1-\frac{1}{k}\right)^j \geq 1-\frac{j}{k}, \quad j=0,1,\ldots$$

as is easily proved by induction on j.

Also solved (almost the same way) by SEUNG-JIN BANG, Seoul, Republic of Korea; and MANUEL BENITO, I.B. Sagasta, Logroño, Spain.

The problem came from a statistical question put to the proposer by one of his colleagues.

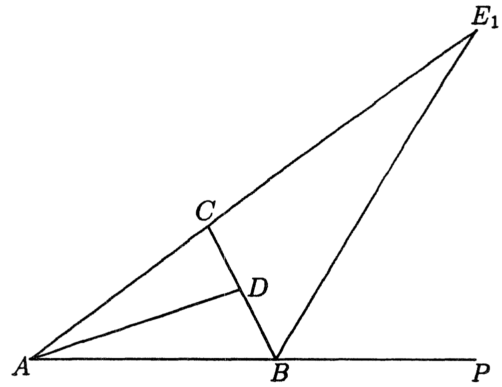
* * * * *

1607. [1991: 15] *Proposed by Peter Hurthig, Columbia College, Burnaby, B.C.*

Find a triangle such that the length of one of its internal angle bisectors (measured from the vertex to the opposite side) equals the length of the external bisector of one of the other angles.

I. *Solution by Andy Liu, University of Alberta.*

We shall find a triangle ABC such that the exterior bisector BE_1 of $\angle ABC$ and the interior bisector AD of $\angle BAC$ are equal in length to each other **and to** AB. Extend AB to P. Let $\angle BAD = x$. Then $\angle CAD = x$, $\angle BE_1C = 2x$ and $\angle E_1BC = \angle E_1BP = 4x$. Also, $\angle ABD = \angle ADB = 90° - x/2$. It follows that



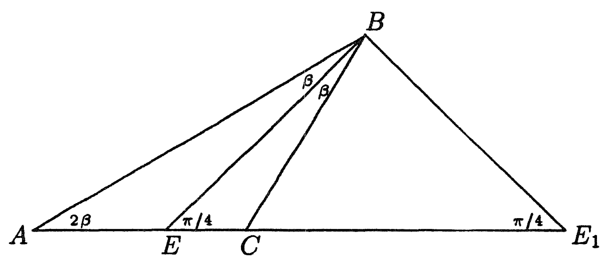
$$180° = \angle E_1BP + \angle E_1BC + \angle ABD = 8x + 90° - x/2.$$

Hence $x = 12°$, and we have

$$\angle BAC = 24°, \quad \angle ABC = 84°, \quad \angle ACB = 72°.$$

II. *Solution by Shailesh Shirali, Rishi Valley School, India.*

Let the lengths of the internal and external angle bisectors associated with an angle X be denoted t_X, T_X, respectively. Then a simple instance where the equality $t_A = T_B$ holds is obtained by placing an additional restriction, namely that angle A equals angle B. Then, with $t_A = t_B = \overline{BE}$, $T_B = \overline{BE_1}$, and $\angle ABE = \beta$ as shown in the diagram, we have $\beta + 2\beta = \pi/4$ and so $\beta = \pi/12$. Thus triangle ABC with angles



$$(A, B, C) = \left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right) = (30°, 30°, 120°)$$

satisfies the relation $t_A = T_B$.

Proceeding more generally, let us note that by elementary trigonometry,

$$t_A^2 = bc\left(1 - \left(\frac{a}{b+c}\right)^2\right), \quad T_B^2 = ca\left(\left(\frac{b}{c-a}\right)^2 - 1\right).$$

Then the relation $t_A = T_B$ implies that

$$\frac{b[(b+c)^2 - a^2]}{(b+c)^2} = b\left(1 - \left(\frac{a}{b+c}\right)^2\right) = a\left(\left(\frac{b}{c-a}\right)^2 - 1\right) = \frac{a[b^2 - (c-a)^2]}{(c-a)^2}.$$

This can be substantially simplified to yield

$$ba^2 - (b^2 + c^2 + 3bc)a + c^2(b+c) = 0.$$

This is a quadratic equation in a (if we fix b and c). Without loss, we may put $b = 1$ and then the above equation yields

$$a = \frac{(c+1)^2 + c - \sqrt{(c^2 + c + 3)^2 - 8}}{2}. \tag{1}$$

If we denote the function on the right side of (1) by $f(c)$, then the following relations are easily verified:

(i) $c > 0$ implies $0 < f(c) < c$ and also $f(c) + 1 > c$;

(ii) $c > \sqrt{3}-1$ implies $f(c)+c > 1$, while the triangle inequality fails for $c \leq \sqrt{3}-1$;

(iii) $(f(c), 1, c)$ is a feasible triple for the sides of a triangle for any $c > \sqrt{3} - 1$.

This procedure obviously gives us infinitely many triangles that have the required property. For instance, $c = 1$ gives the triple

$$(a, b, c) = \left(\frac{5 - \sqrt{17}}{2}, 1, 1\right)$$

and $c = 2$ gives

$$(a, b, c) = \left(\frac{11 - \sqrt{73}}{2}, 1, 2\right).$$

The solution listed at the beginning corresponds to $c = \sqrt{3}$.

III. *Solution by Jordi Dou, Barcelona, Spain.*

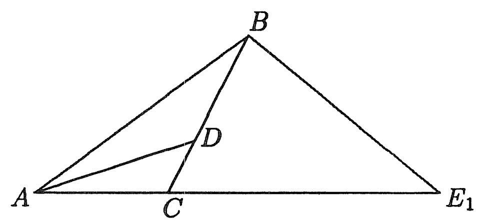
[Dou first gave the 30°, 30°, 120° example in Solution II. — *Ed.*]

We have the following interesting result: *if the interior bisector AD is equal to the exterior bisector BE_1, then $\angle A \leq \varphi$, where φ is approximately* 30.214335°.

To establish this result, consider triangle ABC with $AD = 1$. Then

$$AB = \frac{\sin(C + A/2)}{\sin(C + A)}$$

and so



$$BE_1 = \frac{AB \sin A}{\sin((C - A)/2)} = \frac{\sin A \sin(C + A/2)}{\sin(C + A)\sin((C - A)/2)} = F(A, C). \tag{2}$$

The condition $AD = BE_1$ is $F(A, C) = 1$. For a fixed value of A, the function $F(A, C) = F_A(C)$ is defined for values of C with $A < C < 180° - A$. $F_A(C)$ goes to ∞ as $C \longrightarrow A$ and as $C \longrightarrow 180° - A$. Since $F_A(C) > 0$, $F_A(C)$ has an absolute minimum at $C = \gamma_A$, say. When $F_A(\gamma_A) = 1$, then $A = \varphi$. With formula (2) and a hand calculator, one obtains for φ the value given above. (I don't like it that φ is so close to $\pi/6$ but not exactly!)

We see that if $A < \varphi$, there exist two triangles ABC such that $AD = BE_1 = 1$. If $A = \varphi$, there exists a single triangle ABC such that $AD = BE_1 = 1$. In this case, angle C is $\gamma_A \approx 113.46°$. If $A, B, C > \varphi$ there exists no triangle ABC which satisfies the condition.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer. There were two incorrect solutions sent in.

The solutions were about evenly split between those containing the condition on the sides a,b,c (Solution II*) and those containing the condition on the angles A,B,C (Solution* III*). Most solvers, including the proposer, found the* $30°, 30°, 120°$ *triangle given in Solutions* II *and* III*. The proposer also spotted the* $24°, 84°, 72°$ *triangle given by Liu in Solution* I*. Are there any other solutions with all angles integral?*

Kuczma showed that for every $a \in (2-\sqrt{3},1)$ there is a unique triangle of semiperimeter 1 *(and side opposite A of length a) such that the internal bisector of angle A equals the external bisector of angle B. Festraets-Hamoir noted that the angles of such a triangle ABC satisfy the nice equation*

$$\cos(A/2)\sin(B/2) = \cos^2(C/2).$$

The problem was suggested by the result (due to Bottema) that there are nonisosceles triangles with two external angle bisectors of equal length.

*   *   *   *   *

1608. [1991: 15] *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Suppose n and r are nonnegative integers such that no number of the form $n^2+r-k(k+1)$, $k=1,2,\ldots$, equals -1 or a positive composite number. Show that $4n^2+4r+1$ is 1, 9, or prime.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Assume that $m = 4n^2+4r+1$ is neither 1 nor prime. Then it has an odd prime divisor $p=2k+1$, $p \le \sqrt{m}$ $(k \ge 1)$.

Case (i): $p < \sqrt{m}$. Then

$$n^2+r-k(k+1) = \frac{4n^2+4r+1-(4k^2+4k+1)}{4} = \frac{m-p^2}{4}$$

is a positive integer, divisible by p. In view of the assumption it must be equal to p. So

$$n^2+r-k(k+1) = 2k+1,$$

i.e.

$$n^2+r-(k+1)(k+2) = 2k+1-2(k+1) = -1,$$

in contradiction to the other condition of the problem.

Case (ii): $p = \sqrt{m}$. Then

$$4n^2+4r+1 = (2k+1)^2$$

whence

$$n^2+r-k(k-1) = 2k,$$

a contradiction again, unless $k=1$, in which case $4n^2+4r+1=9$, as desired.

Also solved by MARGHERITA BARILE, student, Università degli Studi di Genova, Italy; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; and the proposer. One partial solution was received.

Janous calls the problem "lovely", but notes that a simpler formulation is possible merely by replacing n^2+r by N, say. He then asks for which integers $N \geq 0$ the condition of the problem holds — only finitely many?

The problem is a generalization of a problem in the 1990 Korean Math Olympiad.

*   *   *   *   *

1609. [1991: 15] *Proposed by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.*

P is a point in the interior of a tetrahedron $ABCD$ of volume V, and F_a, F_b, F_c, F_d are the areas of the faces opposite vertices A, B, C, D, respectively. Prove that

$$PA \cdot F_a + PB \cdot F_b + PC \cdot F_c + PD \cdot F_d \geq 9V.$$

Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

Let h_a, h_b, h_c, h_d be the altitudes from vertices A, B, C, D, and let d_a, d_b, d_c, d_d be the distances from P to the faces opposite vertices A, B, C, D, respectively. Then

$$PA \geq h_a - d_a, \quad \text{etc.}$$

Since $h_aF_a = 3V$, etc., we get (with sums cyclic)

$$\sum PA \cdot F_a \geq \sum h_aF_a - \sum d_aF_a = 4 \cdot 3V - 3V = 9V,$$

with equality when P is the intersection of the altitudes.

Also solved by CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer.

Janous notes that an analogous inequality is known for n-dimensional simplices:

$$\sum F_iR_i \geq n^2V,$$

where F_i is the $(n-1)$-dimensional area of the face opposite vertex A_i, and $R_i = A_iP$. See item 2.29, pp. 494–5 of D.S. Mitrinović et al, Recent Advances in Geometric Inequalities.

Klamkin gave a generalization.

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1610. [1991: 15] *Proposed by P. Penning, Delft, The Netherlands.*

Consider the multiplication $d \times dd \times ddd$, where $d < b-1$ is a nonzero digit in base b, and the product (base b) has six digits, all less than $b-1$ as well. Suppose that, when d and the digits of the product are all increased by 1, the multiplication is still true. Find the lowest base b in which this can happen.

Solution by Sam Baethge, Science Academy, Austin, Texas.

We have (in base b)

$$d^3(1)(11)(111) + 111111 = (d+1)^3(1)(11)(111),$$

or

$$1001 = 11(3d^2 + 3d + 1).$$

Then

$$b^3 + 1 = (b+1)(3d^2 + 3d + 1),$$
$$b^2 - b + 1 = 3d^2 + 3d + 1,$$

and so

$$b(b-1) = 3d(d+1).$$

This equation is satisfied by $d = 1$, $b = 3$ which we reject since the product $[1 \cdot 11 \cdot 111 = 1221_3]$ is less than 6 digits. The lowest base which satisfies the problem is $b = 10$, and then $d = 5$, where

$$5(55)(555) = 152625, \quad 6(66)(666) = 263736.$$

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; KEIR COLBO, student, Memorial University of Newfoundland, St. John's; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; STEWART METCHETTE, Culver City, California; SHAILESH SHIRALI, Rishi Valley School, India; and the proposer. Two incorrect solutions were sent in.

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1611. [1991: 43] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with angles A, B, C (measured in radians), sides a, b, c, and semiperimeter s. Prove that

$$(i) \sum \frac{b+c-a}{A} \geq \frac{6s}{\pi}; \qquad (ii) \sum \frac{b+c-a}{aA} \geq \frac{9}{\pi}.$$

Solution by Ian Goldberg, student, University of Toronto Schools.

First, for positive x, y, the function $f(x,y) = x/y$ is linear or concave up in each variable. Therefore, by Jensen's inequality,

$$\frac{x_1/y_1 + x_2/y_2 + x_3/y_3}{3} \geq \frac{(x_1+x_2+x_3)/3}{(y_1+y_2+y_3)/3},$$

or

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3} \geq 3\left(\frac{x_1+x_2+x_3}{y_1+y_2+y_3}\right). \tag{1}$$

Let $y_1 = A$, $y_2 = B$, $y_3 = C$; then this becomes

$$\frac{x_1}{A} + \frac{x_2}{B} + \frac{x_3}{C} \geq \frac{3}{\pi}(x_1+x_2+x_3). \tag{2}$$

Now for part (i), let $x_1 = b+c-a$, $x_2 = c+a-b$, $x_3 = a+b-c$, so that $x_1+x_2+x_3 = 2s$, and the result follows.

For part (ii), let

$$x_1 = \frac{b+c-a}{a}\ , \quad x_2 = \frac{c+a-b}{b}\ , \quad x_3 = \frac{a+b-c}{c}\ ;$$

then by (1)

$$\sum_{i=1}^{3} x_i \geq 3\left(\frac{(b+c-a)+(c+a-b)+(a+b-c)}{a+b+c}\right) = 3,$$

and the result follows by (2).

We can also solve *Crux* 1637 [1991: 114], by the same proposer, the same way. Let $x_1 = \sin B + \sin C$, $x_2 = \sin C + \sin A$, $x_3 = \sin A + \sin B$. For $0 \leq \alpha \leq \pi/2$, $\sin\alpha \geq 2\alpha/\pi$; thus

$$\sum_{i=1}^{3} x_i = 2(\sin A + \sin B + \sin C) \geq 2\cdot\frac{2}{\pi}\ (A+B+C) = 4,$$

and so from (2)

$$\sum \frac{\sin B + \sin C}{A} \geq \frac{3}{\pi}\sum_{i=1}^{3} x_i \geq \frac{12}{\pi}\ ,$$

which is *Crux* 1637.

Also solved by ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; C. FESTRAETS-HAMOIR, Brussels, Belgium; STEPHEN D. HNIDEI, Windsor, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; VEDULA N. MURTY, Penn State Harrisburg; and the proposer.

Janous showed more generally that for $\lambda \geq 1$, $\mu \geq 0$,

$$\sum \frac{(b+c-a)^\lambda}{A^\mu} \geq \frac{3^{\mu-\lambda+1}(2s)^\lambda}{\pi^\mu}$$

and

$$\sum \frac{(b+c-a)^\lambda}{a^\lambda A^\mu} \geq \frac{3^{\mu+1}}{\pi^\mu}\ .$$

These can be proved as above, using $f(x,y) = x^\lambda/y^\mu$. *He also pointed out the related inequality 4.2, page 168 of D.S. Mitrinović et al,* Recent Advances in Geometric Inequalities.

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1612*. [1991: 43] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be positive real numbers. Show that

$$\sum \frac{y^2 - x^2}{z + x} \geq 0,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

Solution by Margherita Barile, student, Università degli Studi di Genova, Italy.

Let
$$S = \frac{y^2 - x^2}{z + x} + \frac{z^2 - y^2}{x + y} + \frac{x^2 - z^2}{y + z} .$$
Since the sum is cyclic over x, y, z we only need to consider the following two cases.

Case 1: $x \le y \le z$. We have to prove that
$$\frac{z^2 - y^2}{x + y} + \frac{y^2 - x^2}{z + x} \ge \frac{z^2 - x^2}{y + z} .$$
But $x + y \le y + z$ and $z + x \le y + z$, so that
$$\frac{z^2 - y^2}{x + y} + \frac{y^2 - x^2}{z + x} \ge \frac{z^2 - y^2 + y^2 - x^2}{y + z} = \frac{z^2 - x^2}{y + z} .$$
Case 2: $x \le z \le y$. The result is equivalent to
$$\frac{y^2 - x^2}{z + x} \ge \frac{y^2 - z^2}{x + y} + \frac{z^2 - x^2}{y + z} ,$$
which is true because $z + x \le x + y$ and $z + x \le y + z$.

Equality holds if and only if $x = y = z$.

Also solved by H.L. ABBOTT, University of Alberta; HAYO AHLBURG, Benidorm, Spain; BENO ARBEL, Tel-Aviv University, Israel; ŠEFKET ARSLANAGIĆ, Trebinje, Yugoslavia; SEUNG-JIN BANG, Seoul, Republic of Korea; MANUEL BENITO, I. B. Sagasta, Logroño, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; DANIEL BROWN, North York, Ontario; IAN GOLDBERG, student, University of Toronto Schools; JEFF HIGHAM, student, University of Toronto; STEVE HNIDEI, Windsor, Ontario; PETER HURTHIG, Columbia College, Burnaby, B.C.; L.J. HUT, Groningen, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; ANDY LIU, University of Alberta; PAVLOS MARAGOUDAKIS, student, University of Athens, Greece; BEATRIZ MARGOLIS, Paris, France; VEDULA N. MURTY, Penn State Harrisburg; M. PARMENTER, Memorial University of Newfoundland, St. John's; TOSHIO SEIMIYA, Kawasaki, Japan; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. There was one anonymous solution sent in. Two other readers submitted incorrect solutions due to considering only one of the above two cases.

The proposer proved more generally that for any $\lambda \ge 0$,
$$\sum \frac{y^\lambda - x^\lambda}{z + x} \ge 0$$
for all positive x, y, z, which can be shown as above. Even more generally, Klamkin replaced y^2 by $F(y)$, etc., where $F(t)$ is a nondecreasing function of t for $t \ge 0$, to obtain
$$\sum \frac{F(y) - F(x)}{z + x} \ge 0$$
for all positive x, y, z.

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