

Crux

Published by the Canadian Mathematical Society.



<http://crux.math.ca/>

The Back Files

The CMS is pleased to offer free access to its back file of all issues of *Crux* as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No. 2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol. 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

Mathematicorum

ISSN 0705 - 0348

CRUX MATHEMATICORUM

Vol. 10, No. 6

June-July 1984

Sponsored by
 Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
 Publié par le Collège Algonquin, Ottawa

The assistance of the publisher and the support of the Canadian Mathematical Society, the Carleton University Department of Mathematics and Statistics, the University of Ottawa Department of Mathematics, and the endorsement of the Ottawa Valley Education Liaison Council are gratefully acknowledged.

CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is \$22 in Canada, \$24.50 (or US\$20) elsewhere. Back issues: each \$2.25 in Canada, \$2.50 (or US\$2) elsewhere. Bound volumes with index: Vols. 1&2 (combined) and each of Vols. 3-9, \$17 in Canada, \$18.40 (or US\$15) elsewhere. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the managing editor.

All communications about the content (articles, problems, solutions, etc.) should be sent to the editor. All changes of address and inquiries about subscriptions and back issues should be sent to the managing editor.

Editor: Léo Sauvé, Algonquin College, 140 Main Street, Ottawa, Ontario, Canada K1S 1C2.

Managing Editor: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ontario, Canada K1S 0C5.

Typist-compositor: Kuen Huynh.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

*

*

*

CONTENTS

Geometrical Attitudes	Dan Pedoe	170
A Pair of Prime Magic Cubes	Allan Wm. Johnson Jr.	179
Mathematical Swifties	M.S. Klamkin	179, 193
The Olympiad Corner: 56	M.S. Klamkin	180
Problems - Problèmes		194
Mathematical Clerihews	Alan Wayne	196
Solutions		197
The Puzzle Corner	Alan Wayne	204

GEOMETRICAL ATTITUDES

DAN PEDOE

"I see somebody now!" she exclaimed at last. "But he's coming very slowly—and what curious attitudes he goes into!" (For the Messenger kept skipping up and down, and wriggling like an eel, as he came along, with his great hands spread out like fans on each side.)

"Not at all," said the King. "He's an Anglo-Saxon Messenger—and those are Anglo-Saxon attitudes."

LEWIS CARROLL, *Through the Looking-Glass and What Alice Found There*.

In this article I propose to comment on the geometrical attitudes gone into by the authors of three recent books. They are: *The Mathematical Experience*, by Philip J. Davis and Reuben Hersh [1]; *Sacred Geometry, Philosophy and Practice*, by Robert Lawlor [2]; and *Geometry, A High School Course*, by Serge Lang and Gene Murrow [3]. I have already discussed some aspects of the first book in an article in *Queen's Quarterly* which I called, more simply, "Mathematical Experience" [4], and I have also reviewed *Sacred Geometry* in the same journal [5].

I first came across the third book when I noticed a blurb advertising it which, to use an Anglo-Saxon metaphor, made my hackles rise. This said: "Experienced teachers will notice at once the omissions of items traditionally included in the high school geometry course, such as common tangents, power of a point, and several others, which we regard as having little significance."

The inane condemnation at the end of this declaration prompted me to write to the very capable editor of Springer-Verlag, the publishers, and I expressed my incredulity at the point of view expressed, which recalls the Dieudonné attitude towards large parts of Euclidean geometry. I was assured that the book does not resemble a Dieudonné text, and the editor kindly sent me a copy to review.

Any book by Dieudonné, despite his strictures on conventional geometry, and an occasional wild remark which he cannot substantiate (see [8]), is bound to be an excellent book. The Lang-Murrow book is not. The cover sets the tone of the book. It is derived from the celebrated da Vinci drawing of *Homo sapiens*, modestly endowed, as in all Greek statues, this being a matter of good taste, with his arms and legs spread out and reaching to the circumference of a surrounding circle. This drawing is accurately reproduced, uncensored, in my *Geometry and the Visual Arts* [6]. It is derived from Vitruvius's ideas on human proportions. But on the cover of the Lang-Murrow book *Homo* is completely emasculated, and no longer *sapiens*, perhaps to ensure sales in California, and the result is both deplorable and unappetising. Since, as I shall indicate, the cover does represent the contents of

the book, honesty, if not cheerfulness, has broken through.

But first I wish to consider the Davis-Hersh book. *The Mathematical Experience* consists of a set of rather slight and sometimes trivial essays on various aspects of mathematics, written separately by the two authors over the past ten years, and mixed together in a coffee-table format with apparently no real oversight by either author. As a result, their attitude is sometimes schizophrenic, at least towards geometry. One author declares: "In mathematics many areas show signs of internal exhaustion—for example, the elementary geometry of the circle and triangle, or the classical theory of functions of a complex variable. While one can call on the former to provide five-finger exercises for beginners and the latter for applications to other areas, it seems unlikely that either will ever again produce anything that is both new and startling within its bounded confines."

We can leave the statement on complex variables to others, but the many skilled geometers who read this journal will agree that Euclidean geometry hardly provides five-finger exercises for beginners, and in a fundamental sense is real mathematics. And the pages of *Cruix* amply demonstrate that new and startling theorems and constructions do appear in the geometry of the triangle and the circle.

The other author, however, is enthusiastic about geometry, deplores the omission of geometry from the curriculum, and says: "It is a reasonable conclusion that a mathematical culture that specifically downgrades the spatial, visual, kinesthetic, and nonverbal aspects of thought does not fully use all the capacities of the brain."

These are fine words. But the extensive bibliography (drawn up by neither author?) contains not one title of the many excellent books on geometry which have appeared over the past fifty years.

I sent a copy of my *Queen's Quarterly* article, which included a review of *The Mathematical Experience*, to each author, and one responded, unabashed, and thought it was "a nice review", said that he loved geometry, that not teaching it was a crime, and that he was writing a book on mathematics and art, with the help of all modern appliances, a computer artist, and so on, and were there any books I could recommend? I modestly suggested my own *Geometry and the Visual Arts* [6], which has been translated into Russian, has gone into two editions in Spanish, and still sells after 8 years in the original Penguin edition, besides awaiting translation into Japanese. He had never heard of it, and blamed my publishers. His publishers have evidently done a much better public relations job with *The Mathematical Experience* than mine with my book, and a short review has appeared in *The New Yorker*, there has been a book award, and more books of the same kind, by the same authors,

have appeared and will undoubtedly continue to appear.

I now turn to the Lang-Murrow book. It is not possible to guess which of the authors is responsible for categorising some aspects of geometry as of "little significance". The senior author, Serge Lang, is described in the blurb as a "prominent" mathematician. This is, I feel, an unfortunate description. He has written many books on various branches of mathematics, some of which are thought to be good. His joint effort with Murrow has been "taught successfully...for several years", and we are told that criticisms and suggestions are welcomed. Let us first look at the concepts which are omitted, and are regarded as of "little significance". These include common tangents and power of a point.

Common tangents to circles must involve two or more circles. Some years ago Dieudonné complained that he studied systems of circles as a student, but has never come across them in mathematics. This may explain the Lang-Murrow disdain for common tangents and the power of a point, for the Dieudonné influence has been very strong, and there has been much aping of the maestro's geometrical attitudes. The Lang-Murrow book, however, hardly touches one circle, let alone linear systems of circles, and the discussion of analytical geometry is very mickey mouse, so that the power of a point could hardly have been discussed profitably, even if the concept had not been relegated to limbo. Readers of *Cruix* will have come across a short note entitled "The Power of the Power Concept" [7], written years before the Lang-Murrow book appeared. The power concept can only be of "little significance" to those who know nothing about circle geometry. It is a sound principle to keep silent about anything of which one is ignorant. And this observer has remarked a diagram and a little problem on a common tangent to two circles...on page 303, which must have slipped in when both authors were looking the other way!

From of "little significance" to of "*fundamental* importance" (their emphasis) ... The special concepts of *fundamental* importance include a discussion of the changes in area and volume under "dilation", proofs of the standard volume formulas, vectors, the dot product and its connection with perpendicularity, and transformations.

If a change in area is to be considered, the idea of area has to be discussed. This is done by adding little squares on squared paper and handwaving, the accurate concept being relegated to the calculus. The word "limit" is never mentioned, and the exercises are pure mickey mouse. Volume is treated in a similar fashion. Vectors are indeed introduced: "to translate physical and geometrical concepts into purely algebraic notions, which can be handled efficiently." Unfortunately, the notion of a parallelogram is introduced before vectors are treated, and the student

is asked to prove: "If one pair of opposite sides of a quadrilateral are equal in length and parallel, then the quadrilateral is a parallelogram", a howler which has been avoided in high school texts for many years. The confusion is compounded by the diagram on page 56, where a parallelogram is defined, the diagram being incorrectly lettered.

"The dot product, which is never mentioned at high school level, deserves being included at the earliest possible stage. It provides a beautiful and basic relation between algebra and geometry, in that it can be used to interpret perpendicularity extremely efficiently in terms of coordinates." The term "efficiently" appears often, but the authors have obviously not efficiently surveyed the many books on high school geometry which have appeared in the past, since they have imagined concepts appearing which are absent, and ignore some which do appear. But their statement reads well, and it is sad to note how they wriggle, in a very Anglo-Saxon manner, avoiding the trigonometrical interpretation of the dot product, for they know that trigonometry has been a no-no in high schools since the new math burst on the American scene.

Trigonometry was another school discipline which Dieudonné banished, and his disciples tread ardently in the maestro's footsteps. But Dieudonné was amusing when he said [9] that books on trigonometry filled with formulas are written for astronomers, surveyors, and for writers of books on trigonometry, and schoolchildren should never be deliberately trained for any of these professions. The Lang-Murrow book is not even amusing.

When one submits a manuscript to a publisher, it often comes back with a review by an anonymous referee which is a discourse on how the referee would have written the book. If the publisher wants the book, he will tell you to ignore the referee's comments, unless you can find something useful therein. I have no wish to rewrite the Lang-Murrow book. I merely report some of the nonsensical things it contains, and feel that the more intelligent the student trying to obtain some benefit from reading it, the more confused he will be, and the more frustrated at the lack of any worthwhile exercises he can try his teeth on. The book is a very poor preparation for any college course on geometry, but perhaps it was never intended to lead on to such a course.

It is to be hoped that school libraries do not always throw out old books when new ones appear, and still retain some of the excellent texts on geometry, not necessarily designed only for high school use, published years ago. I was told by a distinguished older mathematician that the worst harm his high school did to him was to give him the impression, when he left to go to university, that he knew all

there was to be known of mathematics. His first year at university was traumatic. A good school library would have saved him from a profound humiliation.

The third book I shall review, *Sacred Geometry, Philosophy and Practice*, is also a glossy coffee-table product, filled with excellent pictures, and the most remarkable interpretation of them. There are 202 illustrations and diagrams, 36 in two colours. But the many statements on geometry will hardly appeal to readers of *Cruix*. The large number of similar books published nowadays show that such interpretations do appeal to and comfort multitudes.

In striking contrast to the Dieudonné "Away with the triangle!", we read: "The formation of any volume structurally requires triangulation, hence the Trinity is the creative basis of all form." This is certainly a sound beginning, and with enough imagination we could reconstruct the universe, and more easily this book, from this statement. We find that: "Geometry as a contemplative practice is personified by an elegant and refined woman, for geometry functions as an intuitive, synthesising, creative yet exact activity of mind associated with the feminine principle. But when these geometric laws come to be applied to the technology of daily life, they are represented by the rational masculine principle; contemplative geometry is transformed into practical geometry."

Females are evidently not given a hard row to hoe, but interpretations of icons differ. The well-known *Melencolia I* by Dürer, not reproduced in this book, but given in mine [6], shows a very large, frustrated and unrefined lady contemplating an unsolved geometrical problem, with symbols of technology all around her, the male technologists, the practical men, if there were any, absent, perhaps enjoying a coffee break.

"Arithmetic is also personified as a woman, but not as grand and noble in attire as Geometry, perhaps symbolically indicating that Geometry was considered a higher order of knowledge. On her thighs (symbolising the generative function) are two geometric progressions. The first series 1, 2, 4, 8 goes down her left thigh, associating the even numbers with the feminine, passive side of the body. The second series 1, 3, 9, 27 goes down her right thigh, associating the odd numbers with the masculine, active side, an association which goes back to the Pythagoreans, who called the odd numbers male and the even female..."

Perhaps I should have issued the usual media warning: "Some readers may object to some of the following utterances!" To turn to more abstract matters, the use of the term "geometric progression" is correct here, although we note that the second geometric progression does not contain all the odd numbers. But later on Lawlor insists on calling the series of *equal* numbers

$$a \cdot b, \quad 2a : 2b, \quad 3a : 3b, \quad 4a : 4b, \quad 5a : 5b, \quad \dots$$

for various values of $a:b$, geometric progressions, with varying interpretations of each resulting series of equal numbers.

When $a:b = 1:2$, we read: "In this geometric demonstration of the relationship between proportion and progression we are reminded of the alchemical axiom that everything in creation is formed from a fixed immutable component (proportion) as well as a volatile, mutable component (progression). The relationship between the fixed and the volatile (between proportion and progression) is a key to sacred geometry."

When $a:b$ is the golden ratio, we see that it is: "...the only continuous proportion that yields a progression in which the terms representing the external universe are an exact, continuous proportional reflection of the internal progression --the creative dream of God. The root 2 progression on the contrary is strictly a procreative power, functioning generatively only on the external plane."

There is much more of this nature, and things become even more involved: "When one becomes two, we have automatically the potential of endless multiplicity through progression, as we have seen. Thus the extreme, essential polarity of the universe, Unity and Multiplicity, is perfectly represented and observable in the simple drawing of a square and its diagonal." The author then follows Theon, so he says, and builds squares assuming that the diagonal of a square is equal to its side!

It is clear that I have been on the wrong track since my retirement. I should have become a guru, peddling Sacred Geometry. I have an uneasy feeling that my attempts over the past 20 years since I came to America to restore geometry to the curriculum, banished by the "new math", may be held partially responsible for the mishmash of ideas which proliferates in some quarters. But, in my defence, I think it is true that Ouspensky [6, p.268] *et al.* preceded me here.

When the irreplaceable Martin Gardner retired, his "Mathematical Games" column in *Scientific American* was replaced by a column (since discontinued) entitled "Metamagical Themas" (an anagram of Gardner's title) and written by Douglas R. Hofstadter. While I still have the floor, I would like to discuss one of Hofstadter's columns, a long enthusiastic article which appeared in the April 1982 *Scientific American*. In the development of the early Christian church, enthusiasm was considered a sin, leading to error and heresy. Hofstadter's article is a hymn of praise to the Chopin *Études*, involves music, geometry of a sort, and, on the whole, demonstrates a deplorable and limiting attitude towards music.

Hofstadter's thesis, that some Chopin Studies show geometrical form, is not to be questioned, but rather obvious. The simplest piano studies, scales, go up and

down the piano, and in musical notation the tops of the notes lie on straight lines. Five-finger exercises in musical notation also exhibit geometrical form. Not all the Studies, however, exhibit geometrical form. The lovely Opus 10, No. 3, does not. Those that do hardly need a computer scientist, brought in by Hofstadter, to show their geometry.

But even a computer scientist could hardly find geometric forms in the Chopin mazurkas and polonaises, and Hofstadter's computer scientist makes no attempt to do so. Yet Hofstadter gushes: "Chopin's music is universal...the mazurkas and polonaises speak to a common set of emotions in everyone. But what are these emotions? How are they so deeply evoked by mere pattern? What is the secret magic of Chopin? I know of no more burning question!"

Mere pattern! In the thirties, another age of desperation, some bright young people asserted that if the score of a musical piece suggested a landscape, the music itself was bound to be good. This notion has recently been revived. Be this as it may, Hofstadter has broadcast a clarion call for us to forget economic depressions, wars in various parts of the world, death squads in El Salvador, murder in Lebanon, and so on, and to address ourselves to answer the most burning question or questions: What are the emotions aroused in everyone (*everyone*, note!) by the mazurkas and polonaises, and how are they evoked by mere (*mere!!!*) pattern?

Geometry, alas, has gone to Hofstadter's head. It may be the result of looking overmuch at computer screens.

A personal note.

Critics are rarely popular. I know that many of my colleagues resent my criticisms, and I must warn any young knights in shining armour who are eager to smite the enemies of geometry that sanctions may be invoked against them which may well affect their careers. Of course, if I relate my experiences I shall probably be accused of paranoia, but my statements can be substantiated.

I grew up in the late period of the amazing development of Italian algebraic geometry. It may be remembered that Mussolini invaded Ethiopia in the 1930's, and that this aroused resentment in the rest of the world, but little action, which confirmed the Italian fascists in their sense of playing a great rôle in world history. I was at the Institute of Advanced Study, in Princeton, from 1935 to 1936, and there was a young Italian geometer there. In a very *de haut en bas* manner, he announced one day that he could prove the four colour theorem. His lordly presentation was found to be fundamentally defective at practically the first step. Politics can affect mathematicians. Being able to get away with political wrongdoing can lead some mathematicians to feel that proofs hardly matter. The ruling geometer

in Italy at this time was Francesco Severi. His early papers were magnificent, but his later ones diminished the stature of Italian geometry by their intellectual arrogance and incorrectness, and effectively brought Italian algebraic geometry to a standstill.

I had one of his casual papers to review, full of so-called theorems which Severi hardly bothered to enunciate, let alone prove, and I noticed that his system of references was rather curious. He gave exact references to his own papers, rarely mentioned the theorems he used if proved by others, and, if he had to mention them, gave no references at all. I pointed this out.

The sequel was amusing. In a later paper Severi referred to "un jeune inconnu" (it was written in French), gave exact references to my papers, but with a blank _____ where my name should have been printed. He also wrote to *Mathematical Reviews* and complained, and no less a mathematician than Claude Chevalley came to my defence. At a subsequent International Congress, Severi asked Beniamino Segre to introduce us, and we conversed, he in Italian French, while I used my best Churchillian French. I remarked that I was hardly "jeune", to which he responded (and I deserved this thrust): "Mais vous êtes bien jeune dans les sciences." This made me laugh, which surprised him, and when subsequently my wife and I passed through Rome, he invited us to dinner, but for various reasons we were unable to accept. His record in protecting geometers from persecution during the Mussolini era was not a good one, and I first met Segre in London, banished by the fascist regime. At the Congress I met young Italian geometers who thanked me for my reviews, saying that at the time they appeared nobody in Italy dared to disagree with Severi, and they knew he was harming geometry.

Before World War II there were also mathematicians in America who ruled the roost, and some even boasted openly that all university jobs went through their hands. I met one at a Congress in Scotland. He told me of his power, and suggested I should let him know if I wanted a job in America. I preferred not to take up his offer. I have met an American mathematician during my visits and subsequent stay in America who impressed me and others as being mathematically broadminded, cultured, humane, not self-seeking, understanding, approachable, and tolerant. This was Oswald Veblen. He was a fine geometer and human being.

I should mention Dieudonné's reaction to my criticism of his attitude towards geometry. First I should explain, and this can be easily verified, that between 1940 and 1960, say, only a very small number of books appeared which concentrated on the foundations of algebraic geometry, taking over from the Italian school. A pioneering effort was the one by van der Waerden, there was one by André Weil, and

there was the three-volume work *Methods of Algebraic Geometry* by Hodge and myself. These volumes, begun in 1942, the dark days of the war, appeared in 1947, 1952, and 1954. Perhaps I should mention that they were translated into Russian, a sign of international recognition.

Yet, in a long article on the history of algebraic geometry written by Dieudonné in 1972, there is no reference to the Hodge-Pedoe books. Van der Waerden and André Weil are quoted, of course, and since Hodge was a world class mathematician, and had to be mentioned, he is brought in with a reference to harmonic integrals. Algebraic geometers younger than myself, and active, also find this omission amusing.

Finally, one more anecdote. Some years ago I received a warm invitation from Donald Coxeter to give two lectures at the University of Toronto, one to be a general, cultural lecture, and the second a more technical one. These talks were subsequently written up in "Thinking Geometrically" [8]. I did remark, in my second lecture, that Dieudonné's brushing aside of projective geometry in favour of the calculus was sometimes inadvisable. After my lecture, a member of the large audience arose and asserted, and I had not denied this, that Dieudonné was a fine mathematician. There was a chorus of fervent "Hear! Hear!". My critic then walked to the blackboard, scribbled something on it, and said proudly: "*This* is geometry!" Whatever it was, it was incomprehensible, but the hostility was clear, and I found that there were other consequences of my rational but mild criticism of Dieudonné, the general attitude being: "Who are *you* to criticize?" But for my imprudence, I might now be living in Toronto instead of Minneapolis!

REFERENCES

1. Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, Birkhauser, Boston, 1981.
2. Robert Lawlor, *Sacred Geometry, Philosophy and Practice*, Oxford University Press, Toronto, 1982.
3. Serge Lang and Gene Murrow, *Geometry, A High School Course*, Springer-Verlag, New York, 1983.
4. Dan Pedoe, "Mathematical Experience", *Queen's Quarterly*, Kingston, Ontario, Summer 1983, pp. 449-456.
5. _____, Review, *ibid.*, Winter 1983, pp. 1176-1179.
6. _____, *Geometry and the Visual Arts*, Dover Publications, New York, 1983.
7. _____, "The Power of the Power Concept", *Cruce Mathematicorum*, 4 (1978) 62-63.
8. _____, "Thinking Geometrically", *American Mathematical Monthly*, 77 (1970) 711-721.
9. J. Dieudonné, *Linear Algebra and Geometry*, Hermann, Paris, 1964.
1956 East River Terrace, Minneapolis, Minnesota 55414.

A PAIR OF PRIME MAGIC CUBES

ALLAN WM. JOHNSON JR.

Here are 128 different primes assembled into eight magic squares:

2531	6947	7309	1693
1783	7393	7817	1487
7963	2767	503	7247
6203	1373	2851	8053

Top layer

4447	8783	1117	4133
3119	2131	6449	6781
8123	5107	4793	457
2791	2459	6121	7109

409	5387	7517	5167
5791	8147	3803	739
6863	4093	2437	5087
5417	853	4723	7487

6053	2011	6569	3847
4567	5849	4051	4013
2671	5393	3187	7229
5189	5227	4673	3391

8737	1993	1277	6473
6389	1187	3037	7867
1931	7547	6709	2293
1423	7753	7457	1847

2417	7369	2843	5851
6073	2621	5647	4139
6397	3389	6823	1871
3593	5101	3167	6619

6803	4153	2377	5147
4517	1753	3823	8387
1723	4073	8831	3853
5437	8501	3449	1093

Bottom layer

5563	317	7951	4649
4721	7879	2333	3547
1289	4591	3677	8923
6907	5693	4519	1361

The four magic squares on the left are the layers of a magic cube that is pandiagonal. Each of the four magic squares on the right is pandiagonal, and these magic squares are also the layers of a magic cube. The eight magic squares and the two magic cubes all have the same magic sum 18480.

524 S. Court House Road, Apt. 301, Arlington, Virginia 22204.

*

*

*

MATHEMATICAL SWIFTY

"The angles of that triangle are all less than 90° ", Tom observed acutely.

M.S. KLAMKIN

THE OLYMPIAD COPNER: 56

M.S. KLAMKIN

I start off this month with the problems of the 13th U.S.A. Mathematical Olympiad, which took place on May 1, 1984. The problems were set by the Examination Subcommittee of the USAMO, consisting of M.S. Klamkin, University of Alberta (Chairman); J.D.E. Konhauser, Macalester College; Andy Liu, University of Alberta; and C.C. Rousseau, Memphis State University. Solutions to these problems, as well as those of the 1984 International Mathematical Olympiad, can be obtained later this year by writing to Dr. W.E. Mientka, University of Nebraska, 917 Oldfather Hall, Lincoln, Nebraska 68588.

13th U.S.A. MATHEMATICAL OLYMPIAD

May 1, 1984 - Time: $3\frac{1}{2}$ hours

1. The product of two of the four roots of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Determine the value of k .

2. The geometric mean of any set of m nonnegative numbers is the m th root of their product.

(a) For which positive integers n is there a finite set S_n of n distinct positive integers such that the geometric mean of any subset of S_n is an integer?

(b) Is there an infinite set S of distinct positive integers such that the geometric mean of any finite subset of S is an integer?

3. P, A, B, C , and D are five distinct points in space such that

$$\angle APB = \angle BPC = \angle CPD = \angle DPA = \theta,$$

where θ is a given acute angle. Determine the greatest and least values of

$$\angle APC + \angle BPD.$$

4. A difficult mathematical competition consisted of a Part I and a Part II with a combined total of 28 problems. Each contestant solved exactly 7 problems altogether. For each pair of problems, there were exactly two contestants who solved both of them. Prove that there was a contestant who in Part I solved either no problem or at least 4 problems.

5. $P(x)$ is a polynomial of degree $3n$ such that

$$P(0) = P(3) = \dots = P(3n) = 2,$$

$$P(1) = P(4) = \dots = P(3n-2) = 1,$$

$$P(2) = P(5) = \dots = P(3n-1) = 0,$$

and $P(3n+1) = 730$. Determine n .

*

Now I give, through the courtesy of G.J. Butler, the problems of the 16th Canadian Mathematics Olympiad, which took place on May 2, 1984. I shall publish in a subsequent column the official solutions to these problems.

16th CANADIAN MATHEMATICS OLYMPIAD

May 2, 1984 - Time: 3 hours

1. Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.
2. Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes

place:

Alice: Are you going to cover your keys?

Bob: I would like to, but there are only seven colours and I have eight keys.

Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob: You must be careful what you mean by "next to" or "three keys over" since you can turn the key ring over and the keys are arranged in a circle.

Alice: Even so, you don't need eight colours.

The problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered?

3. An integer is *digitally divisible* if (a) none of its digits is zero; (b) it is divisible by the sum of its digits (e.g., 322 is digitally divisible). Show that there are infinitely many digitally divisible integers.
4. An acute-angled triangle has area 1. Show that there is a point inside the triangle whose distances from each of the vertices is at least $(16/27)^{\frac{1}{3}}$.
5. Given any seven real numbers, prove that there are two of them, say x and y , such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

*

Next I give, through the courtesy of Bernhard Leeb, the problems set at Round 1 of the 1984 West German Olympiad. For these problems I solicit elegant solutions from all readers.

1984 WEST GERMAN OLYMPIAD

Round 1

1. Given are $2n$ x 's in a row. Two players alternately change an x into one of the digits 1, 2, 3, 4, 5, and 6. The second player wins if and only if the resulting $2n$ -digit number (in base ten) is divisible by 9. For which values of n is there a winning strategy for the second player?
2. Determine the sum of the squares of all line segments joining two vertices of a regular n -gon with circumradius 1.
3. Prove that the product of the positive integers a and b is even if and only if there exist positive integers c and d such that
$$a^2 + b^2 + c^2 = d^2.$$
4. Inside a square of side 12 is a set of line segments such that, from each point of the square, the distance to the nearest line segment is at most 1. Prove that the total length of the line segments exceeds 70.

*

I gave last month the problems set at the 1984 AIME [1984: 142-144]. The students who took this examination had obtained a score of at least 95 (out of 150) on the AHSME. To simplify the grading process, they were required to give only the answers, which are all integers from 0 to 999. The papers were then machine-graded and, in addition to the scores, various kinds of statistical information were thus obtained. The top students (approximately 90) then went on to write the USAMO. Six students from this final group will be selected at a June training session to represent the U.S.A. in the 1984 IMO, to be held this year in Prague, Czechoslovakia from June 29 to July 9. Similarly, six students will be selected on the basis of their performance in the Canadian Mathematics Olympiad to represent Canada in the IMO.

The answers to the 15 problems of the 1984 AIME are as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
93	592	1--	649	512	24	997	160	20	119	106	401	15	38	36

*

Finally, I present solutions to various problems proposed earlier.

Q, [1981: 73; 1984: 79] *From the 1979 Moscow Olympiad (for Grade 9).*

Does there exist an infinite sequence $\{a_1, a_2, a_3, \dots\}$ of natural numbers such that no element of the sequence is the sum of any number of other elements and such that, for all n , (a) $a_n \leq n^{10}$; (b) $a_n \leq n\sqrt{n}$?

II. *Solution by H.L. Abbott, University of Alberta.*

The solution given in [1984: 79] is incorrect. The given forbidden sum property does not imply that the subsequence $\{a_1, a_2, \dots, a_n\}$ has $2^n - 1$ distinct non-empty subsets with *distinct* sums. In fact, the answer to part (a) is affirmative.

Proof. Define a sequence of arithmetic progressions A_0, A_1, A_2, \dots recursively as follows: $A_0 = \{1\}$ and, for $k = 1, 2, 3, \dots$, A_k is an arithmetic progression such that

(i) its common difference exceeds the sum of all the elements of

$$A_0 \cup A_1 \cup \dots \cup A_{k-1}$$

and

(ii) the last term is at most twice the first term.

Then, if $A = \bigcup_{k=0}^{\infty} A_k$, the sequence A has the property that no term is the sum of any number of other terms.

It remains to show that it is possible to construct a (monotonically increasing) sequence whose n th term a_n satisfies $a_n \leq n^{10}$ for all n . The following example actually satisfies $a_n \leq n^8$.

For $k = 1, 2, 3, \dots$, let

$$A_k = \{2^{2^{2k-1}} + \lambda \cdot 2^{2^{2k-2}} : \lambda = 0, 1, 2, \dots, 2^{2^{2k-2}} - 1\}.$$

It is easy to show that the A_k satisfy (i) and (ii). Specifically, we have

$$A_0 = \{1\}, \quad A_1 = \{4, 6\}, \quad A_2 = \{256, 272, 288, \dots, 496\}.$$

One may verify directly that, if $a_n \in A_0, A_1, A_2$, then $a_n \leq n^8$. Suppose, then, that $a_n \in A_k$ for some $k \geq 3$. It suffices to consider the case when a_n is the first term of A_k . Then

$$\begin{aligned} n &= 1 + |A_0 \cup A_1 \cup \dots \cup A_{k-1}| \\ &= 1 + 1 + 2^{2^0} + 2^{2^2} + \dots + 2^{2^{2k-4}}. \end{aligned}$$

Thus

$$a_n = 2^{2^{2k-1}} = \left\{ 2^{2^{2k-4}} \right\}^8 \leq n^8.$$

By making the A_k somewhat more complicated, one may show that $a_n \leq cn^6$ is

possible. However, it does not seem to be easy to find the smallest exponent that works. (See [1].)

(b) The answer here is no. The inequality $a_n \leq n\sqrt{n}$ implies that

$$a_1 = 1, \quad a_2 \leq 2, \quad a_3 \leq 5, \quad a_4 \leq 8, \quad \text{and} \quad a_5 \leq 11.$$

It is now easy to verify that no subsequence $\{a_1, a_2, a_3, a_4, a_5\}$ with the above bounds has the forbidden sum property.

REFERENCE

1. P.L. Abbott, "Extremal problems on non-averaging and non-dividing sets", *Pacific Journal of Mathematics*, 91 (1980) 1-12.

*

P-2, [1991: 114] *From a recent Bulgarian mathematical competition.*

Let n be a positive integer, and let x and a be real numbers such that $0 < a < 1$ and $a^{n+1} \leq x \leq 1$. Prove that

$$\prod_{k=1}^n \left| \frac{x - a^k}{x + a^k} \right| \leq \prod_{k=1}^n \frac{1 - a^k}{1 + a^k}.$$

Solution by M.S.K.

For $k = 1, 2, \dots, n$, the function

$$y = \left| \frac{x - a^k}{x + a^k} \right| = \begin{cases} \frac{2a^k}{x+a^k} - 1, & x \in I_1 \equiv [a^{n+1}, a^k], \\ 1 - \frac{2a^k}{x+a^k}, & x \in I_2 \equiv [a^k, 1], \end{cases}$$

is a decreasing function of x on I_1 and increasing on I_2 . Thus y takes on its maximum value at one or both endpoints of $[a^{n+1}, 1]$. Since the product of the n function values at each endpoint equals the right member of the proposed inequality, as seen from

$$\prod_{j=1}^n \frac{a^j - a^{n+1}}{a^j + a^{n+1}} = \prod_{j=1}^n \frac{1 - a^{n+1-j}}{1 + a^{n+1-j}} = \prod_{k=1}^n \frac{1 - a^k}{1 + a^k},$$

it follows that the proposed inequality holds for all $x \in [a^{n+1}, 1]$.

*

F, 2411, [1983: 237] *From Középiskolai Matematikai Lapok (March 1983).*

Show that there is no party of 10 members in which the members have 9, 9, 9, 8, 8, 8, 7, 6, 4, 4 acquaintances, respectively, among themselves. (Acquaintances are supposed to be mutual.)

I. *Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.*

Suppose there is such a party. Leaving out the three who know everyone else, the remaining seven members form a reduced party, say $\{A, B, C, D, E, F, G\}$, with 5, 5, 5, 4, 3, 1, 1 acquaintances (not respectively). Suppose A has 5 acquaintances, say $\{B, C, D, E, F\}$. One of these, say B, has only 1 acquaintance (A) and one, say C, has 5 acquaintances. Then the acquaintances of C must be $\{A, D, E, F, G\}$. Since each of D, E, F knows A and C, then G must be the remaining member with only 1 acquaintance (C). Finally, one of D, E, F must have 5 acquaintances, but this is impossible since none of these can know either B or G.

II. *Comment by M.S.K.*

This problem is generalized by the following two known results [1]:

1.5.6. A sequence $D = (d_1, d_2, \dots, d_n)$ is *graphic* if there is a simple graph (no loops or multiple edges) with degree sequence D. Show that

(a) the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic;

(b) if D is graphic and $d_1 \geq d_2 \geq \dots \geq d_n$, then $d_1 + d_2 + \dots + d_n$ is even and, for $k = 1, 2, \dots, n$,

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(k, d_i).$$

(Erdős and Gallai (1960) have shown that this necessary condition is also sufficient for D to be graphic.)

1.5.7. Let $D = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers, and denote by D' the sequence

$$(d_2-1, d_3-1, \dots, d_{d_1+1}-1, d_{d_1+2}, \dots, d_n).$$

(a)* Show that D is graphic if and only if D' is graphic.

(b) Using (a), describe an algorithm for constructing a simple graph with degree sequence D , if such a graph exists. (V. Havel, S. Hakimi.)

Now back to our problem. Using 1.5.6 (b) on the sequence $(5, 5, 5, 4, 3, 1, 1)$, we find that, for $k = 3$,

$$5 + 5 + 5 > 3 \cdot 2 + (3+3+1+1) = 14,$$

which provides another proof for our problem.

REFERENCE

1. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976, p. 11.

F. 2413, [1983: 237] *From Középiskolai Matematikai Lapok (March 1983).*

Given is a square ABCD. Find the locus of the points P for which

$$PA + PC = \sqrt{2} \max \{PB, PD\}.$$

I. *Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.*

Let the points involved have coordinates

$$A(0,a), \quad B(a,0), \quad C(0,-a), \quad D(-a,0), \quad \text{and} \quad P(x,y)$$

in a rectangular coordinate system. Then

$$\max \{PB, PD\} = \begin{cases} PB & \text{if and only if } x \leq 0, \\ PD & \text{if and only if } x \geq 0. \end{cases}$$

The given equation now becomes

$$\sqrt{x^2 + (y-a)^2} + \sqrt{x^2 + (y+a)^2} = \begin{cases} \sqrt{2} \cdot \sqrt{(x-a)^2 + y^2}, & x \leq 0, \\ \sqrt{2} \cdot \sqrt{(x+a)^2 + y^2}, & x \geq 0. \end{cases} \quad (1)$$

Upon squaring, transposing, and squaring again, each of equations (1) and (2) is found to imply

$$(x^2 + y^2 - a^2)^2 = 0.$$

We conclude that, if P satisfies the given equation, then it must lie on the circumcircle of the square.

II. *Comment by M.S.K.*

To show that the locus is the circumcircle, we must show in addition that, conversely, every point P(x,y) of the circumcircle satisfies the given equation. This follows by applying Ptolemy's theorem to APCB (for $x \leq 0$) and APCD (for $x \geq 0$).

*

M, 799, [1983: 270] *Proposed by S.S. Vallander (in Kvant, April 1983).*

(a) Find a solution of the equation

$$3^{x+1} + 100 = 7^{x-1},$$

and prove that it is unique.

(b) Find two solutions of the equation

$$3^x + 3^{x^2} = 2^x + 4^{x^2},$$

and prove that there are no other solutions.

I. *Solution to part (a) by Daniel Ropp, student, Stillman Valley High School, Illinois.*

It is easy to verify that $x = 4$ is a solution. We show that it is the only (real) solution. This follows from

$$\begin{aligned} 7^{x-1} - 3^{x+1} &= 7^{x-4}(7^3 \cdot 3^5) + 3^5(7^{x-4} - 3^{x-4}) \\ &= 100 \cdot 7^{x-4} + 3^{x+1} \left\{ \left(\frac{7}{3}\right)^{x-4} - 1 \right\}, \end{aligned}$$

which is greater or less than 100 according as x is greater or less than 4.

II. *Solution to part (b) by M.S.K.*

We will use the following two lemmas:

LEMMA 1. If $a > b \geq 0$, then $a^t - b^t$ is increasing for $t \geq 0$.

Proof. $a^t - b^t = a^t(1 - (b/a)^t)$, a product of two increasing functions.

LEMMA 2. If the function F is strictly convex and $a+b = c+d$, where $a \geq b$ and $a > \max\{c, d\}$, then

$$F(a) + F(b) > F(c) + F(d).$$

(The inequality is reversed if F is strictly concave.)

Proof. This follows immediately from the Majorization Inequality [1].

By inspection, $x = 0$ and $x = 1$ are solutions. To show that these are the only two (real) solutions, it suffices to establish that

$$4^{x^2} - 3^{x^2} > 3^x - 2^x \quad \text{for } x < 0, \quad (1)$$

$$4^{x^2} - 3^{x^2} < 3^x - 2^x \quad \text{for } 0 < x < 1, \quad (2)$$

$$4^{x^2} - 3^{x^2} > 3^x - 2^x \quad \text{for } x > 1. \quad (3)$$

Proof of (1). The left side is positive and the right side negative.

Proof of (2). By Lemma 1, it suffices to establish the stronger inequality

$$4^x - 3^x < 3^x - 2^x,$$

and this follows from Lemma 2 since t^x is concave for $0 < x < 1$.

Proof of (3). By Lemma 1 again, it suffices to establish the stronger inequality

$$4^{x^2} - 3^{x^2} > 3^{x^2} - 2^{x^2},$$

and this follows from Lemma 2 since t^{x^2} is convex for $x > 1$. \square

I would welcome an elegant but less sophisticated solution for part (b). As a rider, prove or disprove that

$$y = 4^{x^2} - 3^{x^2} - 3^x + 2^x$$

is a convex function of x for all real x .

REFERENCE

1. Edwin F. Beckenbach and Richard Bellman, *Inequalities*, Springer-Verlag, New York, 1965, p. 30.

*

1. [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 8).*

An arithmetic progression contains an odd number of terms. The sum of the terms in the even-numbered positions equals the sum of the terms in the odd-numbered positions. Find the sum of all the terms in the progression.

Solution by P. Findlay, Toronto.

Let a , d , and $2n+1$ be the first term, common difference, and number of terms of the arithmetic progression. The sum of the n terms in the even-numbered positions is

$$S_0 \equiv \frac{n}{2}\{2(a+d) + (n-1) \cdot 2d\} = n(a+nd)$$

and the sum of the $n+1$ terms in the odd-numbered positions is

$$S_1 \equiv \frac{n+1}{2}(2a + n \cdot 2d) = (n+1)(a+nd).$$

Since these two sums are equal, we must have $a+nd = 0$, and then the sum of all the terms is

$$S_0 + S_1 = (2n+1)(a+nd) = 0.$$

*

2. [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grades 8, 9, 10).*

A trapezoid has perpendicular diagonals and altitude 4. Find the area of the trapezoid if one diagonal has length 5.

Solution by P. Findlay, Toronto.

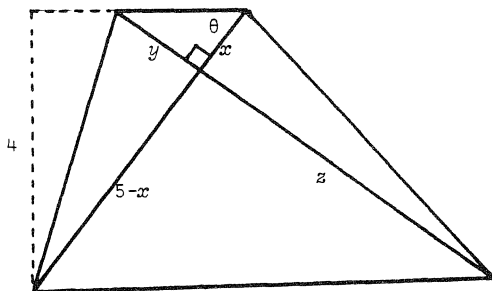
As seen from the figure, the area of the trapezoid is $5(y+z)/2$. By similar triangles,

$$\frac{y}{z} = \frac{x}{5-x}, \text{ or } \frac{y}{y+z} = \frac{x}{5}.$$

Thus the area is

$$\frac{5}{2} \cdot 5 \cdot \frac{y}{x} = \frac{25}{2} \tan \theta = \frac{50}{3},$$

since $\tan \theta = 4/3$.



*

3, [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grades 8, 9).*

Find all natural numbers n with unit digit 4 which are such that the sum of the squares of the digits in n is not less than n .

Solution by Mark Kantrowitz, Maimonides School, Brookline, Massachusetts.

Let $n = 4 + 10a_1 + 10^2a_2 + 10^3a_3 + \dots$, where each digit $a_i \geq 0$. Then the inequality

$$4^2 + a_1^2 + a_2^2 + a_3^2 + \dots \geq 4 + 10a_1 + 10^2a_2 + 10^3a_3 + \dots$$

is equivalent to

$$(a_1 - 5)^2 - 13 \geq a_2(10^2 - a_2) + a_3(10^3 - a_3) + \dots \quad (1)$$

It is evident that (1) cannot hold if any of a_2, a_3, \dots is nonzero; and if they are all zero, then (1) holds if and only if $a_1 = 0, 1$, or 9 . So the only solutions are $n = 4, 14$, and 94 .

*

4, [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 8).*

Find all primes p such that $2p^4 - p^2 + 16$ is a perfect square.

Solution by Mark Kantrowitz, student, Maimonides School, Brookline, Massachusetts.

It follows from

$$(3n)^2 \equiv 0, \quad (3n+1)^2 \equiv 1, \quad (3n+2)^2 \equiv 1 \pmod{3}$$

that the square of every integer is congruent to 0 or 1 modulo 3. Hence, for all integers p ,

$$2p^4 - p^2 + 16 \equiv 2p^2 - p^2 + 16 \equiv p^2 + 1 \pmod{3}.$$

Now let p be any prime. If $p \equiv 1$ or $2 \pmod{3}$, then $p^2 + 1 \equiv 2 \pmod{3}$ and the given expression cannot be a square. So any solution must satisfy $p \equiv 0 \pmod{3}$. The only such prime is $p = 3$, and for this value we have

$$2p^4 - p^2 + 16 = 169 = 13^2.$$

*

5, [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grades 9, 10).*

The function f satisfies $f(0) = 1$ and, for any natural number n ,

$$1 + f(0) + f(1) + f(2) + \dots + f(n-1) = f(n).$$

Find $f^2(0) + f^2(1) + \dots + f^2(n)$.

Solution by Gali Salvatore, Perkins, Québec.

A simple induction proof shows that $f(n) = 2^n$. Hence

$$\sum_{k=0}^n f^2(k) = \sum_{k=0}^n 4^k = \frac{4^{n+1} - 1}{3}.$$

*

6, [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 9).*

Construct the common tangents to the parabolas

$$y = -x^2 + 2x \quad \text{and} \quad y = x^2 + 2.5.$$

Show that the points of tangency are the vertices of a parallelogram.

Solution by Paul Findlay, Toronto.

The line $y = mx + b$ is tangent to the parabola $y = rx^2 + 2sx + t$ if and only if the equation

$$rx^2 + 2sx + t = mx + b$$

has a double root, and the vanishing of its discriminant gives the necessary and sufficient condition

$$(2s - m)^2 = 4r(t - b).$$

The line $y = mx + b$ is therefore a common tangent to the two given parabolas if and only if

$$(2 - m)^2 = 4b \quad \text{and} \quad m^2 = 4(2.5 - b),$$

from which we get $(m, b) = (-1, \frac{9}{4})$ or $(3, \frac{1}{4})$.

The points of tangency turn out to be

$$(-\frac{1}{2}, -\frac{5}{4}), \quad (-\frac{1}{2}, \frac{11}{4}), \quad (\frac{3}{2}, \frac{3}{4}), \quad (\frac{3}{2}, \frac{19}{4}),$$

and it is clear that these are the vertices of a parallelogram.

*

7, [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 10).*

Show that $\sqrt[3]{2 + \frac{10}{3\sqrt{3}}} + \sqrt[3]{2 - \frac{10}{3\sqrt{3}}} = 2$.

Solution by Gali Salvatore, Perkins, Québec.

Let $x = a+b$ denote the left member of the given equation, where a and b are the two cube roots. From the identity

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

and the fact that $ab = 2/3$, we obtain $x^3 - 2x - 4 = 0$. By Descartes' rule of signs, this equation has just one positive root, which is found by inspection to be $x = 2$.

*

2. [1983: 302] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 10).*

The point O is the midpoint of diagonal AC_1 of the cube $ABCD A_1 B_1 C_1 D_1$, and M is the midpoint of segment OC_1 . Through M we draw all possible segments limited by the surface of the cube which are bisected by the point M. What set is formed on the surface of the cube by the endpoints of these segments?

Solution by Daniel Ropp, student, Stillman Valley High School, Illinois.

Introduce a rectangular coordinate system in space such that the vertices of the cube have the following coordinates:

$A(0,0,0)$, $B(4,0,0)$, $C(4,4,0)$, $D(0,4,0)$, $A_1(0,0,4)$, $B_1(4,0,4)$, $C_1(4,4,4)$, $D_1(0,4,4)$.

The point M then has coordinates $(3,3,3)$, and a point (x,y,z) lies in the surface of the cube if and only if it satisfies one of the relations

$$x = 0, \quad 0 \leq y, z \leq 4; \quad (1)$$

$$x = 4, \quad 0 \leq y, z \leq 4; \quad (2)$$

$$y = 0, \quad 0 \leq x, z \leq 4; \quad (3)$$

$$y = 4, \quad 0 \leq x, z \leq 4; \quad (4)$$

$$z = 0, \quad 0 \leq x, y \leq 4; \quad (5)$$

$$z = 4, \quad 0 \leq x, y \leq 4. \quad (6)$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on different faces of the cube. Then M is the midpoint of segment PQ if and only if

$$x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = 6.$$

It is clear that neither P nor Q can lie in face (1), (3), or (5). Assume that P lies in face (2) and Q in face (4). Then M is the midpoint of PQ if and only if

$$P = (4, 2, z_1) \quad \text{and} \quad Q = (2, 4, 6-z_1), \quad 2 \leq z_1 \leq 4.$$

Similarly, if P lies in face (2) and Q in face (6), or if P lies in face (4) and Q lies in face (6), we find that M is the midpoint of PQ if and only if

$$P = (4, y_1, 2) \quad \text{and} \quad Q = (2, 6-y_1, 4), \quad 2 \leq y_1 \leq 4,$$

and

$$P = (x_1, 4, 2) \quad \text{and} \quad Q = (6-x_1, 2, 4), \quad 2 \leq x_1 \leq 4,$$

respectively.

The required set therefore consists of six line segments. These form a closed path going

from center of face (2) to midpoint of B_1C_1
 to center of face (6)
 to midpoint of C_1D_1
 to center of face (4)
 to midpoint of CC_1
 back to center of face (2).

*

9, [1983: 303] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 9).*

Find the primes p and q if it is known that the equation $x^4 - px^3 + q = 0$ has an integer root.

Solution by Gali Salvatore, Perkins, Québec.

The integer root of $f(x) \equiv x^4 - px^3 + q = 0$ must be one of ± 1 or $\pm q$. But

$$f(-1) = 0 \quad \Rightarrow \quad p + q = -1,$$

$$f(q) = 0 \quad \Rightarrow \quad p - q = \frac{1}{q^2},$$

$$f(-q) = 0 \quad \Rightarrow \quad p + q = -\frac{1}{q^2},$$

none of which is possible for primes p and q . Therefore $f(1) = 0$, which implies that $p - q = 1$. So we must have $p = 3$ and $q = 2$.

*

10, [1983: 303] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 9).*

From a point A outside a circle of radius R two tangents AB and AC are drawn, where B and C are the points of tangency. Let $BC = a$. Show that

$$4R^2 = r^2 + r_0^2 + \frac{1}{2}a^2,$$

where r is the inradius of triangle ABC and r_0 is its circumradius.

Comment by Daniel Ropp, student, Stillman Valley High School, Illinois.

The statement of the problem is incorrect. This can be seen by letting $R = 5$ and $a = 8$. Then $AB = AC = 20/3$, $r = 2$, $r_0 = 25/6$, and

$$r^2 + r_0^2 + \frac{1}{2}a^2 = 4 + \frac{625}{36} + 32 \neq 100 = 4R^2.$$

[Perhaps someone can come up with a correct version of the problem. (M.S.K.)]

*

11. [1983: 303] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 10).*

If m and n are natural numbers such that $m/n < \sqrt{2}$, show that

$$\frac{m}{n} < \sqrt{2} \left(1 - \frac{1}{4n^2}\right).$$

Solution by E.F. Lang, Grosse Pointe, Michigan.

Since $m^2 < 2n^2$, we must also have $m^2 \leq 2n^2 - 1$. Therefore

$$\frac{m}{n} \leq \sqrt{2} \cdot \sqrt{1 - \frac{1}{2n^2}} < \sqrt{2} \left(1 - \frac{1}{4n^2}\right).$$

Riders (by A. Meir and M.S.K.). Show that

$$(a) \quad m/n < \sqrt{7} \quad \Rightarrow \quad 7m^2 - m^2 \geq 3,$$

$$(b) \quad m/n > \sqrt{7} \quad \Rightarrow \quad m - n\sqrt{7} \geq 1/2m.$$

*

12. [1983: 303] *From the Second Round of the 1980 Leningrad High School Olympiad (for Grade 10).*

Prove that the equation

$$\sin(\sin x + x^2 + 1) + (\sin x + x^2 + 1)^2 = x - 1$$

has no real root.

Solution by Daniel Popp, student, Stillman Valley High School, Illinois.

If we set

$$y = \sin x + x^2 + 1, \tag{1}$$

the given equation becomes

$$x = \sin y + y^2 + 1. \tag{2}$$

Since $|\sin \theta| \leq 1$ for all θ , (1) and (2) imply that

$$x^2 \leq y \leq x^2 + 2 \quad \text{and} \quad y^2 \leq x \leq y^2 + 2.$$

Therefore $x^4 \leq y^2 \leq x$ and $y^4 \leq x^2 \leq y$, and these imply that $0 \leq x, y \leq 1$. Since the function (1) is increasing for $x \in [0, 1]$, we have $y \geq \sin 0 + 0^2 + 1 = 1$. But we also have $y \leq 1$. Therefore $y = 1$. However, this does not lead to any real solution.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

*

*

*

MATHEMATICAL SWIFTY

"Pick an epsilon as small as you please," Tom challenged arbitrarily.

M.S. KLAMKIN

P P O B L E M S - - P R O B L È M E S

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before January 1, 1985, although solutions received after that date will also be considered until the time when a solution is published.

951, Proposed by Allan Wm. Johnson Jr., Washington, D.C.

In ancient times, a war galley was armed with a *rostrum*, the Latin word for the metal-shod beam jutting from the bow for use as a ram to sink enemy vessels. The Romans displayed the beaks of ships captured in battle around the platform in the Roman Forum where the populace assembled to hear orations, pleadings, etc. Because such war trophies adorned the area, the Romans called this platform the *Rostra*, which is the Latin plural of "rostrum". Later, because of its connection with the Forum, "rostra" evolved in Latin to denote any of several public speaking platforms in Rome. Hence an English word meaning "pulpit or platform for public speaking" was a Latin word meaning "naval ram".

Solve the following etymological multiplication in hexadecimal numbers

$$\begin{array}{r} \text{NAVAL} \\ \text{RAM} \\ \hline \text{*****} \\ \text{ROME*} \\ \text{ROSTRUM} \end{array}$$

given that this alphametic has digits such that

$$A < N < U < M < O < V < S < E < T < L < R,$$

952, Proposed by Jack Garfunkel, Flushing, N.Y.

Consider the following double inequality, where the sum and product are cyclic over the angles A,B,C of a triangle:

$$\Sigma \sin^2 A \leq 2 + 16 \Pi \sin^2 \frac{A}{2} \leq \frac{9}{4}.$$

The inequality between the first and third members is well known, and that between the second and third members is equivalent to the well-known $\Pi \sin(A/2) \leq 1/8$. Prove the inequality between the first and second members.

953, *Proposed by Herta T. Freitag, Roanoke, Virginia.*

The Lucas sequence $\{L_n\}$ is defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_n + L_{n+1} = L_{n+2}, \quad n = 0, 1, 2, \dots$$

For any positive integer n , let

$$a = L_{n-1}L_{n+2} \quad \text{and} \quad b = 2L_nL_{n+1}$$

be the legs of a right triangle.

(a) Establish the following facts about this triangle.

(i) The hypotenuse, c , is integral (and so the triangle is Pythagorean).

(ii) The sum of the hypotenuse and one of the legs is the square of a Lucas number.

(iii) The semiperimeter s , the inradius r , and the exradius r_c are each the product of two consecutive Lucas numbers.

(iv) The exradii r_a and r_b are each the product of two Lucas numbers whose subindices differ by 2.

(b) Obtain the following limits:

$$(i) \lim_{n \rightarrow \infty} r_c/r_b, \quad (ii) \lim_{n \rightarrow \infty} r_c/r_a, \quad (iii) \lim_{n \rightarrow \infty} r_b/r_a.$$

954, *Proposed by W.J. Blundon, Memorial University of Newfoundland.*

The notation being the usual one, prove that each of the following is a necessary and sufficient condition for a triangle to be acute-angled:

(a) $IH < r\sqrt{2},$

(b) $OH < R,$

(c) $\cos^2 A + \cos^2 B + \cos^2 C < 1,$

(d) $r^2 + r_a^2 + r_b^2 + r_c^2 < 8R^2,$

(e) $m_a^2 + m_b^2 + m_c^2 > 6R^2.$

955, *Proposed by Geng-zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.*

If the real numbers A, B, C, a, b, c satisfy

$$A + a \geq b + c, \quad B + b \geq c + a, \quad C + c \geq a + b,$$

show that

$$Q \equiv Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy \geq 0$$

holds for all real x, y, z such that $x+y+z = 0$.

956, *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let

$$S(n,p) = \sum_{k=0}^n \binom{n}{k} k^p,$$

where n is a positive integer and p a nonnegative integer. Prove that

(a) $S(n,2) \cdot S(n,3) - S(n,1) \cdot S(n,2)$ is the square of an integer for every n ;

(b) $S(n,2) \cdot S(n,4) - S(n,1) \cdot S(n,3)$ cannot be the square of an integer for any n .

957, *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let a, b, c be the sides of a triangle with circumradius R and area K .

Prove that

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \geq \frac{2K}{R},$$

with equality if and only if the triangle is equilateral.

958, *Proposed by M.S. Klamkin, University of Alberta.*

If A_1, A_2, A_3 are the angles of a triangle, prove that

$$\tan A_1 + \tan A_2 + \tan A_3 \geq \text{or} \leq 2(\sin 2A_1 + \sin 2A_2 + \sin 2A_3)$$

according as the triangle is acute-angled or obtuse-angled, respectively. When is there equality?

959, *Proposed by Sidney Kravitz, Dover, New Jersey.*

Two houses are located to the north of a straight east-west highway.

House A is at a perpendicular distance a from the road, house B is at a perpendicular distance $b \geq a$ from the road, and the feet of the perpendiculars are one unit apart. Design a road system of minimum total length (as a function of a and b) to connect both houses to the highway.

960, *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

If the altitude of a triangle is also a symmedian, prove that the triangle is either an isosceles triangle or a right triangle.

*

*

*

MATHEMATICAL CLERIHUEWS

John von Neumann
(Not a German)
Could readily explain
Computers and the brain.

Isaac Barrow,
Far from narrow
And very fair,
Gave up his Chair.

ALAN WAYNE, Holiday, Florida

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

554, [1980: 183; 1981: 184] Proposed by G.C. Giri, Midnapore College, West Bengal, India.

A sequence of triangles $\{\Delta_0, \Delta_1, \Delta_2, \dots\}$ is defined as follows: Δ_0 is a given triangle and, for each triangle Δ_n in the sequence, the vertices of Δ_{n+1} are the points of contact of the incircle of Δ_n with its sides. Prove that Δ_n "tends to" an equilateral triangles as $n \rightarrow \infty$.

II. Comment by Geng-zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.

The following computation makes more precise the parenthetical comment, at the end of the published solution, concerning the rapidity of convergence of the areas to 0. We use the notation of that solution, together with the formulas

$$A_{n+1} = 90^\circ - \frac{1}{2}A_n \quad (1)$$

and

$$\frac{K_{n+1}}{K_n} = \frac{2r_n}{s_n} (\cos \frac{1}{2}A_n) (\cos \frac{1}{2}B_n) (\cos \frac{1}{2}C_n) \quad (2)$$

derived therein. The well-known formulas

$$s_n - a_n = r_n \cot \frac{1}{2}A_n \quad \text{and} \quad r_n^2 s_n = (s_n - a_n)(s_n - b_n)(s_n - c_n)$$

yield

$$r_n \cos \frac{1}{2}A_n = (r_n \cot \frac{1}{2}A_n) \sin \frac{1}{2}A_n = (s_n - a_n) \sin \frac{1}{2}A_n$$

and, in combination with (2),

$$\begin{aligned} \frac{K_{n+1}}{K_n} &= \frac{2}{r_n^2 s_n} (r_n \cos \frac{1}{2}A_n) (r_n \cos \frac{1}{2}B_n) (r_n \cos \frac{1}{2}C_n) \\ &= \frac{2}{r_n^2 s_n} (s_n - a_n)(s_n - b_n)(s_n - c_n) (\sin \frac{1}{2}A_n) (\sin \frac{1}{2}B_n) (\sin \frac{1}{2}C_n) \\ &= 2(\sin \frac{1}{2}A_n) (\sin \frac{1}{2}B_n) (\sin \frac{1}{2}C_n). \end{aligned}$$

But from (1) we have

$$\sin \frac{1}{2}A_n = \frac{\sin A_n}{2 \cos \frac{1}{2}A_n} = \frac{\sin A_n}{2 \sin A_{n+1}},$$

so that

$$\frac{K_{n+1}}{K_n} = \frac{1}{4} \cdot \frac{\sin A_n}{\sin A_{n+1}} \cdot \frac{\sin B_n}{\sin B_{n+1}} \cdot \frac{\sin C_n}{\sin C_{n+1}},$$

and an easy induction yields

$$\frac{K_n}{K_0/4^n} = \frac{\sin A_0 \sin B_0 \sin C_0}{\sin A_n \sin B_n \sin C_n}.$$

Since $\lim_{n \uparrow} A_n = 60^\circ$, we have finally

$$\lim_{n \uparrow} \frac{K_n}{K_0/4^n} = \frac{8\sqrt{3}}{9} \sin A_0 \sin B_0 \sin C_0.$$

*

*

*

801. [1983: 21; 1984: 116] *Proposed by Sidney Kravitz, Dover, New Jersey.*

At this time (late January 1983), Canada's peripatetic Prime Minister, the apostle of the North-South dialogue, is catching his breath at home. So let us lose no time in proposing the following alphametic before he takes off again (or bows out):

PIERRE
ELLIOTT.
TRUDEAU

Letter (to the managing editor) from the Right Honourable Pierre Elliott Trudeau, Prime Minister of Canada.

Dear Mr. Maskell:

It was kind of you to send me an advance copy of Crux Mathematicorum, Volume 10, Number 4.

I appreciated learning that I was the subject of Proposal No. 801 submitted by Sydney Kravitz of Dover, New Jersey, and amazed to see that it was another American, Kenneth M. Wilke, of Topeka, Kansas who submitted the only full-blown solution to the problem.

Many thanks for this enjoyable brief glimpse of the mathematical community at play.

With my very best wishes, and

Sincerely,



827. [1983: 79] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For $i = 1, 2, 3$, let A_i be the vertices of a triangle with opposite sides a_i , let B_i be an arbitrary point on a_i , and let M_i be the midpoint of $B_j B_k$. If the lines b_i are perpendicular to a_i through B_i , and if the lines m_i are perpendicular to a_i through M_i , prove that the b_i are concurrent if and only if the m_i are concurrent.

Solution by Howard Eves, University of Maine.

Denote the figure made up of triangle $B_1 B_2 B_3$ and the three lines b_1, b_2, b_3 by B , and that made up of triangle $M_1 M_2 M_3$ and the three lines m_1, m_2, m_3 by M . Clearly, the homothety with center at the common centroid D of the two triangles $B_1 B_2 B_3$ and $M_1 M_2 M_3$, and with ratio $-1/2$, maps B onto M . The desired result follows immediately. Moreover, if C_b and C_m are the respective points of concurrency for triangles $B_1 B_2 B_3$ and $M_1 M_2 M_3$, then C_b, C_m, D are collinear and $C_b D = 2 D C_m$.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D. J. SMEENK, Zaltbommel, The Netherlands; JORDAN B. TABOV, Sofia, Bulgaria; GEORGE TSINTSIFAS, Thessaloniki, Greece; JAN VAN DE CRAATS, University of Leiden, The Netherlands; and the proposer.

Editor's comment.

Tsintsifas noted that this is Problem 1120 in his book *Geometry* (published in Greek), but that he does not now recall his own source for the problem.

*

*

*

828. [1983: 80] *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*

Characterize the pairs of positive integers $(2a, 2b)$, where $a < b$, for which

$$x = \frac{-a^2 + a\sqrt{a^2 - ab + b^2}}{b - 2a + \sqrt{a^2 - ab + b^2}}$$

is an integer. One example is $(2a, 2b) = (24, 45)$, which yields $x = 5$. (Note that $0 < a < b$ implies $x > 0$.)

Joint solution by John Oman and Bob Prielipp, University of Wisconsin-Oshkosh.

The denominator of the given expression and its conjugate never vanish (otherwise $a = 0$ or $a = b$), so we can rationalize the denominator and obtain

$$x = \frac{A + B - \sqrt{A^2 - AB + B^2}}{6}, \quad (1)$$

where $A = 2a$ and $B = 2b$ (and therefore $0 < A < B$). This is the smaller root of the quadratic

$$12x^2 - 4(A+B)x + AB = 0. \quad (2)$$

Suppose the positive integer pair (A, B) gives a positive integer x in (1). We first write (2) in the equivalent form

$$(A - 4x)(B - 4x) = (2x)^2.$$

It is easy to verify that $0 < A < B$ implies that

$$B - 4x > A - 4x > 0.$$

If p is a prime divisor of $2x$, then p^2 divides $A-4x$ or p^2 divides $B-4x$ or p divides both $A-4x$ and $B-4x$. Thus, if g is the *g.c.d.* of $A-4x$ and $B-4x$, there are positive integers m and n , with $m < n$, such that

$$A - 4x = gm^2, \quad B - 4x = gn^2, \quad 2x = gmn,$$

and from this follows

$$A = gm(m+2n), \quad B = gn(n+2m). \quad (3)$$

Conversely, let the pair (A, B) be defined by (3) for some positive integers g, m, n such that $m < n$ and gmn is even. Then we find that

$$\sqrt{A^2 - AB + B^2} = g(m^2 + mn + n^2),$$

and (1) yields the integer $x = gmn/2$.

Therefore the required characterization is (3), where g, m, n are positive integers with $m < n$ and gmn even. The example given in the proposal corresponds to $g = 1, m = 2, n = 5$.

The problem can also be solved by using the identity

$$(4x)^2 + (B-A)^2 = (A+B-8x)^2 - 4\{12x^2 - 4(A+B)x + AB\}.$$

For then (2) is satisfied if and only if $4x$, $B-A$, and $A+B-8x$ (note that each is positive) form a Pythagorean triple. Using the usual characterization of such triples,

$$4x = 2gmn, \quad B - A = g(n^2 - m^2), \quad A+B-8x = g(n^2 + m^2),$$

we obtain again (3). \square

Using (3), it is possible to begin investigating related questions, such as:

- (a) Which values of x can be obtained?
- (b) Which values of A or B appear in at least one pair?
- (c) In how many pairs can a given value of A or B appear?
- (d) Can the "primitive" triples (A, B, x) be characterized?

Finally, minor modifications to the above procedure will characterize the pairs (A, B) for which the larger root of (2),

$$x = \frac{A + B + \sqrt{A^2 - AB + B^2}}{6},$$

is integral.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; and KENNETH M. WILKE, Topeka, Kansas.

*

*

*

829, [1983: 80] *Proposed by O. Bottema, Delft, The Netherlands.*

Let ABC be a triangle with sides a, b, c in the usual order, centroid G, incenter I, and circumcenter O. Prove that

$$[GIO] = - \frac{(a+b+c)(b-c)(c-a)(a-b)}{48[ABC]},$$

where the square brackets denote the signed area of a triangle.

Solution by Leon Bankoff, Los Angeles, California.

To arrive at the desired formula, we must assume that triangle ABC is oriented positively, so that $[ABC] = rs > 0$, where r and s are the inradius and semiperimeter, respectively, of the triangle.

Let H and R be the orthocenter and circumradius, respectively, of triangle ABC. It is very well known (see [1]-[5]) that

$$[HIO] = -2R^2 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}.$$

This expression was converted by Blundon [6] to the more elegant

$$[HIO] = - \frac{(b-c)(c-a)(a-b)}{8r}$$

(a result mentioned earlier in this journal [1976: 216]). Now we have

$$\begin{aligned} [GIO] &= \frac{1}{3}[HIO] = - \frac{(b-c)(c-a)(a-b)}{24r} = - \frac{(a+b+c)(b-c)(c-a)(a-b)}{48rs} \\ &= - \frac{(a+b+c)(b-c)(c-a)(a-b)}{48[ABC]}. \end{aligned}$$

Also solved by HOWARD EVES, University of Maine; J.T. GROENMAN, Arnhem, The Netherlands (two solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANA-RAYANA, Gagan Mahal Colony, Hyderabad, India (two solutions); D.J. SMEENK, Zaltbommel, The Netherlands; JORDAN B. TABOV, Sofia, Bulgaria; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

REFERENCES

1. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 200, Ex. 2. (No solution given.)

2. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1930, p. 18, Ex. 17. (Solution given in [3].)

3. _____, *Key to Advanced Trigonometry*, G. Bell and Sons, London, 1930, pp. 21-22.

4. *Mathematical Questions*, *Educational Times*, Vol. XLIII, 1885, p. 60, Problem 3733. (The problem reappeared in [5] with a different proposer.)

5. *Ibid.*, Vol. XLV, 1886, p. 44, Problem 7978.

6. W.J. Blundon, (proposer of) Aufgabe 541, *Elemente der Mathematik*, 23 (January 1968) 16.

*

*

*

830, [1983: 80] Proposed by M.S. Klamkin, University of Alberta.

Determine all real λ such that

$$|z_1 \bar{z}_1(z_3 - z_2) + z_2 \bar{z}_2(z_1 - z_3) + z_3 \bar{z}_3(z_2 - z_1) - iz_1 \lambda| = |(z_2 - z_3)(z_3 - z_1)(z_1 - z_2)|,$$

where z_1, z_2, z_3 are given distinct complex numbers and $z_1 \neq 0$.

Solution by the proposer (revised by the editor).

Consider the triangle (assumed to be nondegenerate) whose affixes in the complex plane are z_1, z_2, z_3 , oriented in the sense of increasing subindices, and let Δ, R, ω denote its signed area, circumradius, and affix of its circumcenter, respectively. It is known that

$$\omega = \frac{z_1 \bar{z}_1(z_3 - z_2) + z_2 \bar{z}_2(z_1 - z_3) + z_3 \bar{z}_3(z_2 - z_1)}{-4i\Delta}$$

and

$$R = \frac{|(z_2 - z_3)(z_3 - z_1)(z_1 - z_2)|}{4|\Delta|}.$$

The given equation is therefore equivalent to

$$|\omega + \frac{z_1 \lambda}{4\Delta}| = R. \quad (1)$$

It is clear that (1) is satisfied if and only if

$$-\frac{z_1 \lambda}{4\Delta} = z \quad (2)$$

for some point z on the circumcircle, and then that λ is real if and only if $z = z_1$ or $z = z_4$, where z_4 is the point where the line through the origin and z_1 meets the circumcircle again. For $z = z_1$, we obtain from (2)

$$\lambda = \lambda_1 = -4\Delta.$$

To obtain the value of λ when $z = z_4$ in (2), we first note that the power of the origin with respect to the circumcircle is given, in both magnitude and sign, by the real number

$$\bar{z}_1 z_4 = |\omega|^2 - R^2.$$

Hence $z = z_4$ in (2) when

$$\lambda = \lambda_2 = - \frac{4\Delta(|\omega|^2 - R^2)}{|z_1|^2}.$$

Also solved by G.P. HENDERSON, Campbellcroft, Ontario.

Editor's comment.

Henderson found the values of λ_1 and λ_2 explicitly in terms of z_1, z_2, z_3 :

$$\lambda_1 = i \Sigma \bar{z}_1 (z_2 - z_3),$$

$$\lambda_2 = \frac{i}{|z_1|^2} \Sigma \bar{z}_1 \bar{z}_2 z_3 (z_2 - z_1),$$

where the sums are cyclic.

*

*

*

831. [1983: 112] Proposed by J.A. McCallum, Medicine Hat, Alberta.

The adjoined decimal alphametic must represent the occasional feelings of our harried French editor. Fortunately, his surname ensures his salvation, and its primeness ensures his uniqueness.

ASSEZ
ASSEZ
ASSEZ
SAYS
LEO
SAUVE

Solution by Charles W. Trigg, San Diego, California.

Reading from the right, the columns determine the following equations:

$$3Z + S + 0 = E + 10k, \quad k \leq 4, \quad (1)$$

$$4E + Y + k = V + 10m, \quad m \leq 4, \quad (2)$$

$$3S + A + L + m = U + 10n, \quad n \leq 4, \quad (3)$$

$$4S + n = A + 10p, \quad p \leq 4, \quad (4)$$

$$3A + p = S. \quad (5)$$

Eliminating S from (4) and (5), we obtain $11A = 6p - n$, so $A = 1$ or 2 . Since $A = 2 \Rightarrow (p, n) = (4, 2) \Rightarrow S = 10$, we must have $\underline{A = 1}$, $\underline{p = 2}$, $\underline{n = 1}$, and $\underline{S = 5}$.

Since SAUVE is prime, $E = 3, 7$, or 9 . From (2), $m > 0$. Hence, from (2) and (3), $\underline{L = 2}$, $\underline{U = 9}$, $\underline{m = 1}$, and $\underline{E = 3}$. Then, from a table of primes, $\underline{V = 7}$. Equation (2) now requires that $Y + k = 5$, and this is possible only if $\underline{Y = 4}$ and $\underline{k = 1}$. Equation (1) is now equivalent to $3Z + 0 = 8$, and the only digits still available are $0, 6$, and 8 . Hence $\underline{Z = 0}$, $\underline{0 = 8}$, and the unique reconstruction is

15530
15530
15530
5145
238
51973*

Also solved by MEIR FEDER, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDWIN M. KLEIN, University of Wisconsin-Whitewater; JACK LESAGE, Eastview Secondary School, Barrie, Ontario; GLEN E. MILLS, Pensacola Junior College, Florida; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; RAM REKHA TIWARI, Radhaur, Bihar, India; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer. One incorrect solution was received.

Editor's comment.

Tiwari noted that SAUVE is the SEAVth prime. To which we add the corollary that SEAV = YOU•AE. Our wife (this sounds polyandrous), with only a little prodding, stoutly declares that AE stands for Admirable Editor. *Q.E.D.*

By the way, we are rapidly running out of alphametics to publish. Readers who would like to see this feature continued will have to submit some. They should be interesting in the words and have a unique solution.

*

*

*

THE PUZZLE CORNER

Puzzle No. 55: Rebus (11)

I I
S 2

Some rigid motions, if you please,

In three dimensions, may be these.

Puzzle No. 56: Last-letter changes (8 = ONE, TWO; 9 = FIRST, LAST)

"A ONE looks something like a catenary."

This statement might be called a TWO.

"A ONE looks like a FIRST," to the contrary,

Is more than LAST, and isn't true.

Puzzle No. 57: Homonym (9, 8)

Yon ailing mathematician, view:

His kidney's ONE, a fact we rue;

But still his mind is full of TWO.

ALAN WAYNE, Holiday, Florida

Answer to Puzzle No. 53 [1984: 154]: Countersign.

Answer to Puzzle No. 54 [1984: 154]: Summands (Sum M and S).