Mathematical Spectrum

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- **Generalizations of geometric and arithmetic progressions**
- **Number patterns in different scales**
- **Columns of Fibonacci or Lucas numbers**
- **Odd abundant numbers**

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

More than rabbits

Over the years, two subjects have stood out above all others in popularity amongst contributors to *Mathematical Spectrum*, Fibonacci numbers and Fermat's last theorem. Not surprisingly, readers have been more successful with the former than the latter, and it is to these numbers that we turn in this column.

Fibonacci numbers originated as an exercise in *Liber Abaci* (literally *The Book of the Abacus*) by Leonardo of Pisa, more commonly known as Fibonacci (son of Bonacci), which was first published in 1202. Therein is a problem about rabbits which has become famous:

> Suppose that there are two newborn rabbits, one male and one female. Find the number of rabbits produced in a year if
>
> 1. each pair takes one month to become mature;
> 2. each pair produces a mixed pair every month, from the second month on;
> 3. no rabbits die during the course of the year.

So in month 1 there is one newborn pair, in month 2 they have grown into an adult pair, in month 3 they have produced a newborn pair, giving two pairs, in month 4 the adult pair produces another pair, giving three pairs, in month 5 there are two adult pairs producing two new pairs, and so on. So the sequence of pairs goes

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,$$

so, after a year, there are 144 pairs. In fact, the rule is

$$\mathrm{F}_1 = 1,\ \mathrm{F}_2 = 1 \quad \text{and} \quad \mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2} \quad \text{for } n > 2.$$

This is the famous Fibonacci sequence.

I have always been uneasy about the reproductive peculiarities of these rabbits, so I was relieved to see alternative ways of introducing Fibonacci numbers, and much else besides, in a book by Thomas Koshy (see reference 1). Here is an example.

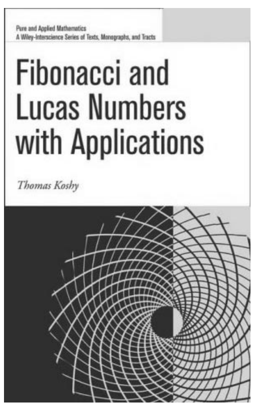

Table 1

n	P_n	Independent sets	u_n
1	a	$\varnothing, \{a\}$	2
2	a b	$\varnothing, \{a\}, \{b\}$	3
3	a b c	$\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}$	5
4	a b c d	$\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}$	8

Consider the *path graph* P_n. This consists of n vertices joined together in a line (a path). We ask how many subsets of vertices P_n possesses such that no two vertices in the subset are joined directly together, and denote this number by u_n. We call such subsets 'independent' sets of vertices of P_n. For trivial reasons, the empty set (written $\varnothing$) is independent, and so is each single-vertex subset. For results for P_n, $1 \leq n \leq 4$, see Table 1.

Table 1 shows the same pattern as the rabbits from month 3! Thus $u_1 = 2$, $u_2 = 3$. Now suppose that $n > 2$, and consider an independent set of vertices of P_n. If the last vertex a_n does not lie in this set, it is an independent set of vertices of P_{n-1}, so there are u_{n-1} such independent sets. If a_n does lie in this set, then a_{n-1} cannot and the independent set is made up of an independent set of vertices of P_{n-2} with a_n added. There are thus u_{n-2} such sets. Hence

$$u_n = u_{n-1} + u_{n-2}\,.$$

Thus we have the Fibonacci sequence, although two places on, i.e. $u_n = F_{n+2}$.

You may, like me, feel happier with this than with the rabbits. In case you think that this is the end of the story, you might like to delve into Koshy's comprehensive book, all 652 pages of it. Even *Mathematical Spectrum* gets a couple of mentions! But be warned, the author evidently subscribes to the philosophy that it is good to keep readers on their toes by making as many mistakes as possible. He may be right!

Coincidentally, on the day I wrote this column, I received an email from Abbas Amini who had noticed that, at least for the first few primes p with the exception of $p = 5$,

$$F_p \equiv \pm 1 \bmod p\,.$$

Thus $F_2 = 1 \equiv 1 \bmod 2$, $F_3 = 2 \equiv -1 \bmod 3$, ($F_5 = 5 \equiv 0 \bmod 5$ is the exception), $F_7 = 13 \equiv -1 \bmod 7$, and so on. He is right. If you want to try a proof, start with Binet's formula

$$F_p = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^p - \left(\frac{1-\sqrt{5}}{2}\right)^p\right].$$

Expand this by the binomial theorem and use Fermat's little theorem to give that, when $p \neq 2$, $2^{p-1} \equiv 1 \bmod p$ and, when $p \neq 5$, $5^{(p-1)/2} \equiv \pm 1 \bmod p$. Happy proving!

Finally for something completely different. In case you think that there is nothing new to be found in mathematics, how about this? There are three primes in arithmetic progression:

$3, 5, 7$. Also, there are five: $5, 11, 17, 23, 29$. The longest sequence of primes in arithmetic progression known to date has 23 terms, with up to 12 digits in them. Ben Green (of Cambridge University) and Terry Tao have just proved that, for every $n \geq 3$, there are n primes in arithmetic progression. So there are a thousand, a million, any number. How about that! It is so new the proof hasn't been published yet. You read it first in *Mathematical Spectrum*!

Reference

1 T. Koshy, *Fibonacci and Lucas Numbers with Applications* (John Wiley, New York, 2001).

Mathematical Spectrum Awards for Volume 36

Prizes have been awarded to the following student readers for contributions in Volume 36:

Margen Çuko for the article 'We Need Some New Functions' (with Paul Belcher);
Will Donovan for various contributions;
Paul Jefferys for various contributions.

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Equal sums of squares

Write some digits in a circle, e.g.

$$\begin{array}{ccccc} & 7 & & 2 & \\ 5 & & & & 4 \\ & & 1 & & \end{array}$$

The sum of the squares of the two-digit numbers read clockwise is equal to the sum going anticlockwise; in this case

$$24^2 + 41^2 + 15^2 + 57^2 + 72^2 = 27^2 + 75^2 + 51^2 + 14^2 + 42^2.$$

Can you prove this generally?

Shyam Lal College (E)
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Generalizations of Geometric and Arithmetic Progressions

PHILIP MAYNARD

In this article we consider the following generalizations of arithmetic and geometric progressions. Take $a, x, y \in \mathbb{R}$ and consider a sequence whose first term is a and whose nth term is x times the previous term plus y. Specifically, consider $\{D_n\} = a, ax + y, ax^2 + y(x+1), \ldots$. It is not hard to see that $D_n = ax^{n-1} + y(x^{n-1} - 1)/(x-1)$. In Lemma 3, below, we shall evaluate the sum $E_n = \sum_{i=1}^{n} D_i$. Note that if $x = 1$, then the sequence is just an arithmetic sequence and, if $y = 0$, then we have a geometric sequence.

First we consider a sequence with nth term $T_n = (a + (n-1)b)x^{n-1}$. That is, the sequence $a, (a+b)x, (a+2b)x^2, \ldots$. Essentially this is a polynomial whose coefficients form an arithmetic progression. We shall need to evaluate the sum $S_n = \sum_{i=1}^{n} T_i$.

Lemma 1 *Let $a, b, x \in \mathbb{R}$, $x \neq 1$ and $n \in \mathbb{N}$. Then we obtain*

$$S_n = \frac{a(x^n - 1)}{x - 1} + \frac{bx}{(x-1)^2}((n-1)x^n - nx^{n-1} + 1).$$

Proof We have

$$S_n = a(1 + x + x^2 + \cdots + x^{n-1}) + bx(1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2}).$$

The first part is just a geometric series. To evaluate S_n we only need to consider $f_n = 1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2}$. Now

$$\begin{aligned} f_n = 1 &+ x + x^2 + \cdots + x^{n-2} \\ &+ x + x^2 + \cdots + x^{n-2} \\ &+ x^2 + x^3 + \cdots + x^{n-2} \\ &\vdots \\ &+ x^{n-3} + x^{n-2} \\ &+ x^{n-2}. \end{aligned}$$

Hence

$$f_n = \frac{x^{n-1} - 1}{x - 1} + x\frac{x^{n-2} - 1}{x - 1} + x^2\frac{x^{n-3} - 1}{x - 1} + \cdots + x^{n-3}\frac{x^2 - 1}{x - 1} + x^{n-2}\frac{x - 1}{x - 1}.$$

Thus

$$\begin{aligned} f_n &= \frac{1}{x-1}((n-1)x^{n-1} - 1 - x - x^2 - \cdots - x^{n-2}) \\ &= \frac{1}{x-1}\left((n-1)x^{n-1} - \frac{x^{n-1} - 1}{x - 1}\right) \\ &= \frac{1}{(x-1)^2}((n-1)x^n - nx^{n-1} + 1), \end{aligned}$$

which completes the proof.

We now work with the sum $L_n = n + (n-1)x + (n-2)x^2 + \cdots + 2x^{n-2} + x^{n-1}$, which we shall need later.

Lemma 2 *Let $x \in \mathbb{R}$, $x \neq 1$ and $n \in \mathbb{N}$. Then we obtain*

$$L_n = \sum_{i=1}^{n}(n+1-i)x^{i-1} = \frac{x^{n+1} - x(n+1) + n}{(x-1)^2}.$$

Proof Let $L_n = \sum_{i=1}^n T_i = S_n$, where $T_i = (a + (i-1)b)x^{i-1}$ with $a = n$ and $b = -1$. Hence, from Lemma 1 we have

$$L_n = \frac{n(x^n - 1)}{x-1} - \frac{x}{(x-1)^2}((n-1)x^n - nx^{n-1} + 1).$$

After a little algebra this simplifies to the expression in the statement of Lemma 2.

We can now evaluate the sum $E_n = \sum_{i=1}^n D_i$.

Lemma 3 *Let $a, b, x \in \mathbb{R}$, $x \neq 1$ and $n \in \mathbb{N}$. Then we obtain*

$$E_n = a\frac{x^n - 1}{x-1} + y\frac{x^n - xn + n - 1}{(x-1)^2}.$$

Proof We have

$$\begin{aligned} E_n &= a(x^{n-1} + x^{n-2} + \cdots + x + 1) \\ &\quad + y((n-1) + (n-2)x + \cdots + 3x^{n-4} + 2x^{n-3} + x^{n-2}). \end{aligned}$$

So, in fact, $E_n = a(x^n - 1)/(x-1) + yL_{n-1}$. The result now follows from Lemma 2.

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Solutions of a Diophantine equation

The equation

$$2x^2 - y^4 - z^4 = 1$$

has an infinite family of solutions in positive integers, namely

$$(x, y, z) = (n^2 + n + 1, n + 1, n).$$

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Number Patterns in Different Scales

ISAO ASIBA and MASAKAZU NIHEI

It is well known that $123\,456\,789 \times 9 + 10 = 1\,111\,111\,111$. In this article we will show that this arrangement holds true for any scale. More arrangements of digits in a general scale are given.

It is well known that, in the decimal scale,

$$123\,456\,789 \times 9 + 10 = 1\,111\,111\,111. \tag{1}$$

Is this true in scale N ($N \geq 2$)? Are there other similar patterns? We first prove the following result.

Lemma 1 *Let n be an integer, $n \geq 2$. Then we obtain*

$$(n-1) + (n-2)n + (n-3)n^2 + \cdots + 2n^{n-3} + n^{n-2} = \frac{n^n - n^2 + n - 1}{(n-1)^2}.$$

Proof Set

$$S = \sum_{k=1}^{n-1} (n-k)n^{k-1}.$$

Then

$$nS = \sum_{k=1}^{n-1} (n-k)n^k$$

and

$$nS - S = \frac{n(n^{n-1} - 1)}{n-1} - (n-1).$$

Divide both sides by $n-1$ to complete the proof.

Now, we will show that (1) holds true in scale N ($N \geq 2$).

Theorem 1 *In scale N ($N \geq 2$), arrange the digits $1, \ldots, N-1$ in increasing order, multiply by $N-1$ and add N. The resulting number is $11\ldots1$ with N digits.*

For example, in scale 8,

$$1\,234\,567 \times 7 + 8 = 11\,111\,111;$$

in scale 7,

$$123\,456 \times 6 + 7 = 1\,111\,111.$$

Proof of Theorem 1 In scale N, we are considering the number

$$N^{N-2} + 2N^{N-3} + \cdots + (N-3)N^2 + (N-2)N + (N-1).$$

From Lemma 1 it follows that

$$\begin{aligned}
&(N^{N-2} + 2N^{N-3} + \cdots + (N-3)N^2 + (N-2)N + (N-1))(N-1) + N \\
&\quad = \frac{(N^N - N^2 + N - 1)(N-1)}{(N-1)^2} + N \\
&\quad = \frac{N^N - 1}{N-1} \\
&\quad = N^{N-1} + N^{N-2} + \cdots + N + 1
\end{aligned}$$

as required.

In the decimal scale, we have

$$12\,345\,679 \times 9 = 111\,111\,111.$$

We now consider this in any scale in the following theorem.

Theorem 2 *In scale N ($N \geq 3$), arrange the digits $1, \ldots, N-3, N-1$ in increasing order and multiply by $N-1$. The resulting number is $11 \ldots 1$ with $N-1$ digits.*

For example, in scale 7,

$$12\,346 \times 6 = 111\,111.$$

Proof of Theorem 2 We have

$$\begin{aligned}
&(N^{N-3} + 2N^{N-4} + \cdots + (N-3)N + (N-1))(N-1) \\
&\quad = \frac{(N^{N-2} + 2N^{N-3} + \cdots + (N-3)N^2 + (N-1)N)(N-1)}{N} \\
&\quad = \frac{1}{N}(N^{N-2} + 2N^{N-3} + \cdots + (N-3)N^2 + (N-2)N + (N-1) - (N-2)N \\
&\qquad\quad - (N-1) + (N-1)N)(N-1) \\
&\quad = ((N^N - N^2 + N - 1)(N-1)^2 + 1)\frac{N-1}{N} \\
&\quad = \frac{N^N - N^2 + N - 1}{N(N-1)} + \frac{N-1}{N} \\
&\quad = \frac{N^{N-1} - 1}{N-1} \\
&\quad = N^{N-2} + N^{N-3} + \cdots + N + 1
\end{aligned}$$

as required.

In the decimal scale, we have

$$123\,456\,789 \times 8 = 987\,654\,312.$$

Adding 9 to this result, we get 987 654 321 on the right-hand side. We consider the corresponding result in scale N ($N \geq 3$) in the following theorem.

Theorem 3 *In scale N ($N \geq 3$), consider a number with digits $1, 2, \ldots, N-1$, multiply it by $N-2$ and add $N-1$. Then the resulting number has digits $N-1, N-2, \ldots, 1$.*

For example, in scale 8,

$$1\,234\,567 \times 6 + 7 = 7\,654\,321.$$

Proof of Theorem 3 In scale N, we have

$$123\ldots(N-1) + (N-1)(N-2)\ldots321 + 1 = 1\ldots1, \tag{2}$$

where the number on the right-hand side has N digits. By Theorem 1,

$$(123\ldots(N-1))(N-1) + N = 1\ldots1, \tag{3}$$

where the number on the right-hand side has N digits. From (2) and (3), we obtain

$$(N-1)(N-2)\ldots321 = (123\ldots(N-1))(N-1) + N - (123\ldots(N-1)) - 1.$$

Hence

$$(N-1)(N-2)\ldots321 = (123\ldots(N-1))(N-2) + (N-1),$$

as required.

In the decimal scale, we have

$$987\,654\,321 \times 9 - 1 = 8\,888\,888\,888.$$

We consider the corresponding result in scale N ($N \geq 3$) in the following theorem.

Theorem 4 *In scale N ($N \geq 3$), consider a number with digits $N-1, \ldots, 1$, multiply it by $N-1$ and subtract 1. Then the resulting number has N digits which are all $N-2$.*

For example, in scale 8,

$$7\,654\,321 \times 7 - 1 = 66\,666\,666;$$

in scale 7,

$$654\,321 \times 6 - 1 = 5\,555\,555.$$

The proof of Theorem 4 is similar to that of Theorem 3.

We leave readers to investigate whether there are other similar results.

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Some Properties of a Double Circle Quadrilateral

ZHANG YUN

The article reference 1 introduced some inequalities connecting the lengths of the sides, the area, the radius of the circumcircle and the radius of the inscribed circle of a triangle ABC. In this article, I will discuss some properties of a double circle quadrilateral. A quadrilateral ABCD is called a *double circle quadrilateral* if it has a circumcircle and an inscribed circle.

We denote the lengths of the sides AB, BC, CD, DA by a, b, c, d respectively, and we put $s = \frac{1}{2}(a + b + c + d)$. Also, the radius of the circumcircle is R, the radius of the inscribed circle is r, and the area of ABCD is Δ, see figure 1.

Theorem 1

$$(abcd)^{1/2} = \Delta. \tag{1}$$

Proof Let $\mathrm{T_A}, \mathrm{T_B}, \mathrm{T_C}, \mathrm{T_D}$ be tangent points of AB, BC, CD, DA respectively for the inscribed circle, and put $\mathrm{AT_D} = \mathrm{AT_A} = x, \mathrm{BT_A} = \mathrm{BT_B} = y, \mathrm{CT_B} = \mathrm{CT_C} = z, \mathrm{DT_C} = \mathrm{DT_D} = u$. Then we obtain

$$x + y = a, \qquad y + z = b, \qquad z + u = c, \qquad u + x = d,$$

and so

$$a + c = b + d.$$

Now, using Brahmagupta's formula (see reference 2) for the area of a cyclic quadrilateral,

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

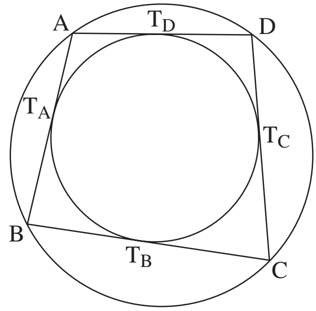

Figure 1 Quadrilateral ABCD.

we have

$$\begin{aligned}\Delta &= \left\{\left[\tfrac{1}{2}(a+b+c+d)-a\right]\left[\tfrac{1}{2}(a+b+c+d)-b\right]\right.\\ &\quad \left.\times\left[\tfrac{1}{2}(a+b+c+d)-c\right]\left[\tfrac{1}{2}(a+b+c+d)-d\right]\right\}^{1/2}\\ &= \left\{\left[\tfrac{1}{2}(a+c+a+c)-a\right]\left[\tfrac{1}{2}(b+d+b+d)-b\right]\right.\\ &\quad \left.\times\left[\tfrac{1}{2}(a+c+a+c)-c\right]\left[\tfrac{1}{2}(b+d+b+d)-d\right]\right\}^{1/2}\\ &= (abcd)^{1/2}.\end{aligned}$$

Corollary 1

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \geq 4s^3r^3.$$

Proof By the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 &\geq 4(a^6b^6c^6d^6)^{1/4}\\ &= 4(abcd)^{3/2},\end{aligned}$$

and so, by Theorem 1,

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \geq 4\Delta^3.$$

Since

$$\Delta = \tfrac{1}{2}r(a+b+c+d) = rs, \tag{2}$$

we have

$$a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \geq 4s^3r^3.$$

Theorem 2

$$s \geq 4r. \tag{3}$$

Proof By (1) and (2), we have

$$\Delta^2 = r^2s^2 = abcd.$$

Using the arithmetic-geometric mean inequality, we obtain

$$r^2s^2 = abcd \leq \left(\frac{a+b+c+d}{4}\right)^4 = \left(\frac{s}{2}\right)^4,$$

and so

$$s \geq 4r.$$

Theorem 3

$$R \geq \sqrt{2}r.$$

Proof The cosine formula applied to the triangles ABC and ACD gives

$$\begin{aligned}\mathrm{AC}^2 &= \mathrm{AB}^2 + \mathrm{BC}^2 - 2\times\mathrm{AB}\times\mathrm{BC}\times\cos\mathrm{B}\\ &= \mathrm{CD}^2 + \mathrm{DA}^2 - 2\times\mathrm{CD}\times\mathrm{DA}\times\cos\mathrm{D}.\end{aligned}$$

Hence

$$a^2+b^2-2ab\cos \mathrm{B} = c^2+d^2-2cd\cos(180-\mathrm{B})$$
$$= c^2+d^2+2cd\cos \mathrm{B},$$

so

$$\cos \mathrm{B} = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

Hence

$$\mathrm{AC}^2 = a^2+b^2-2ab\frac{a^2+b^2-c^2-d^2}{2(ab+cd)} = \frac{(ac+bd)(ad+bc)}{ab+cd},$$

so

$$\mathrm{AC} = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

Also $\Delta = \frac{1}{2}ab\sin \mathrm{B} + \frac{1}{2}cd\sin \mathrm{B}$, so that

$$\sin \mathrm{B} = \frac{2\Delta}{ab+cd}.$$

The sine formula applied to the triangle ABC is

$$\frac{\mathrm{AC}}{\sin \mathrm{B}} = 2R,$$

so

$$\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} = 2R\frac{2\Delta}{ab+cd}.$$

Hence

$$\begin{aligned} R &= \frac{1}{4\Delta}\sqrt{(ab+cd)(ac+bd)(ad+bc)} \\ &= \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}} \\ &\geq \frac{1}{4}\sqrt{\frac{2\sqrt{abcd}\ 2\sqrt{abcd}\ 2\sqrt{abcd}}{abcd}} \\ &= \frac{2\sqrt{2}}{4}\sqrt[4]{abcd} \\ &= \frac{\sqrt{2}}{2}\sqrt{\Delta} \\ &= \frac{\sqrt{2}}{2}\sqrt{rs}. \end{aligned}$$

Therefore, by (3),

$$R \geq \frac{\sqrt{2}}{2}\sqrt{r\times 4r} = \sqrt{2}r.$$

Combining Theorem 2 and the inequality $R \geq \sqrt{(rs/2)}$ from the proof of Theorem 3, we have

$$4r \leq s \leq \frac{2R^2}{r},$$

so that

$$4r^2 \leq \Delta \leq 2R^2.$$

We note that these inequalities are all equalities for a square.

References

1 Z. Yun, An introduction to geometric inequalities, *Math. Spectrum* **32** (1999/2000), pp. 35–37.

2 L. S. Hong, *Elementary Mathematics Review and Studies* (People Education Press, Beijing, China, 1958).

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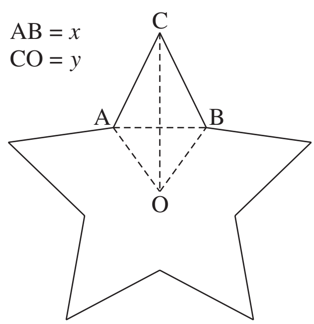
The area of a star

The area of the regular five-pointed star with centre O shown is

$$\tfrac{5}{2}xy.$$

For a regular star with n outer vertices, the area is

$$\tfrac{1}{2}nxy.$$



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Periodic Recurrence Relations of the Type $x, y, y^k/x, \ldots$

JONNY GRIFFITHS

We are all likely to remember the first time we were struck by the way that some mathematical activity 'came back to where it started', sometimes quite unexpectedly. For example, maybe we observed that if you start with 3, and take it away from 7, you get 4. Now take 4 away from 7 and you get back to 3. Does this always work? It is hard to imagine a stronger motivation for embarking on algebra than the following example:

$$x,\ a-x,\ a-(a-x)=x.$$

You can, if you wish, regard this as a recurrence relation (RR) of period 2, where $u_n = a - u_{n-1}$, and where u_0 can be anything.

Can we find any other RRs that behave in the same way? Perhaps the simplest example is $x, a/x, x, \ldots$, which is of period 2. If you wonder what happens if we try to make this more general, and also if we can get away from period 2, we could try

$$x,\ \frac{ax+b}{cx+d}, \ldots.$$

For period 2, doing the algebra implies that the RR must be of the form $(ax+b)/(cx-a)$ to be nontrivial, which includes both of the above RRs as special cases.

What about period 3? The algebra gets a little more complicated, but eventually produces:

$$x,\ \frac{ax-(a^2+ab+b^2)}{x+b},\ \frac{bx+(a^2+ab+b^2)}{-x+a},\ x, \ldots$$

as the nontrivial solution. This idea can be continued for higher periods, although the algebra gets pretty dense!

What happens if you extend this idea to RRs that include more than just the previous term? In other words, instead of looking at $x, f(x), f(f(x)), \ldots,$ we look at $x, y, f(x, y),$ $f(y, f(x, y)), \ldots.$ Can this sequence be periodic?

One of the happiest memories of my years at school is of being shown the following:

$$x,\ y,\ \frac{y+1}{x},\ \frac{1+x+y}{xy},\ \frac{x+1}{y},\ x,\ y, \ldots.$$

The simplicity of the function, and the way in which the algebra organised itself to return to its starting point, struck me as marvellous. I suppose that ever since I have been on the lookout for RRs that return to their starting points. So it wasn't surprising that the other day I found myself looking at

$$x,\ y,\ \frac{1}{xy},\ x,\ y, \ldots,$$

Table 1

n	u_n	v_n
0	1	0
1	0	1
2	j	k
3	jk	$k^2 + j$
4	$j^2 + jk^2$	$k^3 + 2jk$
5	$2j^2k + jk^3$	$k^4 + 3jk^2 + j^2$
6	$j^3 + 3j^2k^2 + jk^4$	$k^5 + 4jk^3 + 3j^2k$
7	$3j^3k + 4j^2k^3 + jk^5$	$k^6 + 5jk^4 + 6j^2k^2 + j^3$

which is of period 3, and

$$x,\ y,\ \frac{y}{x},\ \frac{1}{x},\ \frac{1}{y},\ \frac{x}{y},\ x,\ y, \ldots,$$

which is of period 6.

Could I get a period 4? A period 5? Suddenly, it seemed a fruitful idea to look at $x, y, x^j y^k, \ldots,$ and to see what it would yield. (Actually, looking at $x, y, jx + ky, \ldots$ gives virtually the same mathematics.) In effect, I was producing two more RRs, u_n and v_n, where the nth term in the original RR would be $x^{u_n} y^{v_n}$. The sequence for u_n and v_n is given in table 1. If we take $j = k = -1$, then $u_3 = 1, v_3 = 0, u_4 = 0$ and $v_4 = 1$, then we get our $1/xy$ RR of period 3.

Is period 4 possible? This would mean that $j^2 + jk^2 = 1$ and $k^3 + 2jk = 0$, which leads to the following possible values for (j, k):

$$(1, 0),\ (-1, 0),\ (i, 1 - i),\ (i, i - 1),\ (-i, 1 + i),\ (-i, -i - 1).$$

So, if we are restricting ourselves to real solutions, then it seems that $x, y, x^{-1}, y^{-1}, x, y, \ldots$ is the only possibility for period 4 with this kind of RR. (Note that $x, y, (x+a)(y+a)/(-y), \ldots$ is periodic of period 4.)

How about period 5? This would mean that:

$$2j^2k + jk^3 = 1 \quad \text{and} \quad k^4 + 3jk^2 + j^2 = 0.$$

These give, after lots of substitution, the equation $k^{10} + 11k^5 - 1 = 0$, which is a quadratic in k^5. Solving this gives $k^5 = (-11 \pm 5\sqrt{5})/2$. What do you get if you take the fifth root here? To my delight, you get the following:

$$k = \Phi - 1 \quad \text{or} \quad k = -\Phi,$$

where Φ is the golden ratio $(1+\sqrt{5})/2$ and $j = -1$. This suggests that the RR $x, y, 1/(xy^{\Phi})$ may be of period 5. This holds true, and it is a very enjoyable exercise to check this through, simply using $\Phi^2 = \Phi + 1$. Maybe this is a candidate for the most beautiful RR ever?

It seems that j is very often equal to -1 with the mathematics generated here. Let's simplify things by restricting ourselves to $j = -1$ for the moment. So we are now just looking at RRs of the form:

$$x,\ y,\ \frac{y^k}{x}, \ldots.$$

Table 2

n	u_n	v_n
0	1	0
1	0	1
2	-1	k
3	$-k$	k^2-1
4	$1-k^2$	k^3-2k
5	$2k-k^3$	k^4-3k^2+1
6	$3k^2-1-k^4$	k^5-4k^3+3k
7	$4k^3-3k-k^5$	$k^6-5k^4+6k^2-1$

Table 1 now simplifies to table 2. So we can see that u_n and v_n are almost the same sequence, just differing by a minus sign and one term, i.e.

$$u_n = -v_{n-1}.$$

A graphical calculator and a short program become very useful here to check what happens. Enter, for example, $x = 2$ and $y = 3$, and then a value for k, and you can see what happens as the RR progresses. You can see that if $-2 < k < 2$ then the terms remain bounded, whereas if k is outside this range then the terms tend to infinity. It appears that our only chance of periodic behaviour lies in the range $-2 < k < 2$. Why should this be? (Curiously, the condition for a point to be in the Mandelbrot set is very similar.)

We can easily get a simple RR for v_n, that is

$$v_n = kv_{n-1} - v_{n-2}.$$

We can solve this RR as follows:

$$\begin{aligned}
&v_0 = 0,\\
&v_1 = 1,\\
&v_n = ({}^{n-1}C_0)k^{(n-1)} - ({}^{n-2}C_1)k^{(n-3)} + \cdots + (-1)^r({}^{n-1-r}C_r)k^{(n-2r-1)} + \cdots,
\end{aligned}$$

which ends with the term in k^1 or k^0. Now, checking this formula (even if the check is rather rough at the edges), we obtain:

$$\begin{aligned}
kv_{n-1} - v_{n-2} &= \sum(-1)^r k^{(n-2r-1)}[({}^{n-2-r}C_r) + ({}^{n-2-r}C_{r-1})]\\
&= \sum(-1)^r k^{(n-2r-1)}[({}^{n-1-r}C_r)]\\
&= v_n.
\end{aligned}$$

So, for $n = 2t$,

$$\begin{aligned}
v_{2t} = {}&({}^{2t-1}C_0)k^{(2t-1)} - ({}^{2t-2}C_1)k^{(2t-3)} + \cdots + (-1)^r({}^{2t-1-r}C_r)k^{(2t-2r-1)}\\
&+ \cdots + (-1)^{t-1}({}^{t}C_{t-1})k^1.
\end{aligned}$$

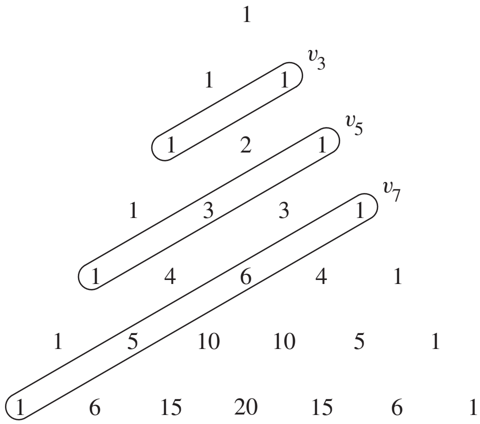

Figure 1

Also, for $n = 2t + 1$,

$$v_{2t+1} = ({}^{2t}C_0)k^{(2t)} - ({}^{2t-1}C_1)k^{(2t-2)} + \cdots + (-1)^r({}^{2t-r}C_r)k^{(2t-2r)} \\ + \cdots + (-1)^{t-1}({}^{t-1}C_{t-1})k^0.$$

For a periodic RR, $v_n = 0$ for some $n > 0$. So we end up trying to solve the following types of equations:

$$({}^8C_0)k^8 - ({}^7C_1)k^6 + ({}^6C_2)k^4 - ({}^5C_3)k^2 + ({}^4C_4) = 0,$$

or

$$({}^7C_0)k^7 - ({}^6C_1)k^5 + ({}^5C_2)k^3 - ({}^4C_3)k = 0.$$

The coefficients here lie in diagonal lines within Pascal's triangle, see figure 1.

Does this kind of equation have real solutions for k between -2 and 2? The possibly surprising news is that these equations seem remarkably keen to give such values. For example, $v_{12} = 0$ gives the following equation:

$$k^{11} - 10k^9 + 36k^7 - 56k^5 + 35k^3 - 6k = 0.$$

This has eleven real roots ($0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{2+\sqrt{3}}, \pm\sqrt{2-\sqrt{3}}$), which are all between -2 and 2. Using a graphical calculator, we can see what period each of these values of k gives: $-1, 0, 1, \pm\sqrt{2}, \pm\sqrt{3}$ and $\pm\sqrt{2 \pm \sqrt{3}}$ give periods 3, 4, 6, 8, 12 and 24 respectively. Carrying this out for all small values of n we obtain the values in table 3.

Firstly this suggests that, whenever $n \geq 3$, it is possible to find a value of k that gives period n, which is very pleasing! So that if we seek a value of k so that $x, y, y^k/x, \ldots$ is of period 2 049, then it seems that such a k exists.

It also seems that if $p = 2n+1$ is prime then there are n values of k that give $x, y, y^k/x$ as periodic of period p, and a further n that give period $2p$. Table 4 shows how many different values of k there are for each period.

It seems that if $x, y, y^k/x$ is periodic, then $x, y, y^{-k}/x$ is too, yet not always with the same period (there may be a halving or a doubling). So $x, y, y^{\sqrt{2}}/x$ gives period 8, and so does $x, y, y^{-\sqrt{2}}/x$. However, $x, y, y^{\Phi}/x, \ldots$ is of period 10, while $x, y, y^{-\Phi}/x, \ldots$ is of period 5.

Table 3

Equation	Number of solutions	Periods
$v_2 = 0$	1	4
$v_3 = 0$	2	3, 6
$v_4 = 0$	3	4, 8, 8
$v_5 = 0$	4	5, 5, 10, 10
$v_6 = 0$	5	3, 4, 6, 12, 12
$v_7 = 0$	6	7, 7, 7, 14, 14, 14
$v_8 = 0$	7	4, 8, 8, 16, 16, 16, 16
$v_9 = 0$	8	3, 6, 9, 9, 9, 18, 18, 18
$v_{10} = 0$	9	4, 5, 5, 10, 10, 20, 20, 20, 20
$v_{11} = 0$	10	11, 11, 11, 11, 11, 22, 22, 22, 22, 22

Table 4

Period	Number of solutions for k	Solutions for k
3	1	-1
4	1	0
5	2	-1.618, 0.618
6	1	1
7	3	-1.802, -0.445, 1.247
8	2	-1.414, 1.414
9	3	-1.877, 0.357, 1.520
10	2	-0.618, 1.618
11	5	-1.919, -1.310, -0.285, 0.831, 1.683
12	2	-1.732, 1.732
13	6	-1.942, -1.497, -0.709, 0.241, 1.136, 1.770
14	3	-1.247, 0.445, 1.802
15	4	-1.956, -0.209, 1.338, 1.827
16	4	-1.848, -0.765, 0.765, 1.848
17	8	***
18	3	***, 1.877
19	9	***
20	4	***, 1.902

Also, it appears that the top solution for k is steadily increasing. This gives us the conjecture that there is a sequence $k_3, k_4, k_5, \ldots,$ with $k_n < k_{n+1} < 2$ for all n, such that $x, y, y^{k_n}/x$ is periodic of period n, with $k_3 = -1, k_4 = 0, k_5 = \Phi - 1, k_6 = 1, k_8 = \sqrt{2}, k_{10} = \Phi$, and $k_{12} = \sqrt{3}$.

So it seems that finding a value of k that gives $v_n = 0$ means that we must have found something periodic. Our experimenting suggests that if $v_n = 0$ then u_n must be either 1 or -1, and that v_{n+1} is always 1 or -1. The 'pressure', if I may call it that, for a RR to go on to y once it has returned to x is enormous, it appears. Why should this be?

A little thought suggests that, if k is a root of $v_n = 0$, then k will be a root of $v_{2n} = 0$, indeed of $v_{an} = 0$. So if, as our experimenting suggests, v_m always has $m - 1$ distinct roots, then we have that, if $n|m$ then $v_n|v_m$.

Now take v_4 and v_8, is it immediately obvious that $k^3 - 2k$ divides $k^7 - 6k^5 + 10k^3 - 4k$ exactly? These polynomials clearly have some interesting properties. The quotient here turns out to be $k^4 - 4k^2 + 2$, what is the significance of this?

One last thought: a brief proof that if $|k| \geq 2$, then the v_n diverge. We have $v_0 = 0$, $v_1 = 1$, and we have that $v_n = kv_{n-1} - v_{n-2}$. Suppose that $|k| \geq 2$, then certainly $|v_1| \geq |v_0| + 1$. Now suppose that $|v_n| \geq |v_{n-1}| + 1$, then we obtain

$$\begin{aligned}
|v_{n+1}| &= |kv_n - v_{n-1}| \\
&\geq ||k||v_n| - |v_{n-1}|| \quad \text{(by the backwards triangle inequality)} \\
&= |k||v_n| - |v_{n-1}| \\
&= |v_n| + (|k| - 1)|v_n| - |v_{n-1}| \\
&\geq |v_n| + (|k| - 1)(|v_{n-1}| + 1) - |v_{n-1}| \\
&= |v_n| + (|k| - 2)|v_{n-1}| + |k| - 1 \\
&\geq |v_n| + 1.
\end{aligned}$$

So, by induction, $|v_n| \geq |v_{n-1}| + 1$ for all n. Hence $|v_n| \geq n$ for all $n \geq 0$, and the v_n diverge.

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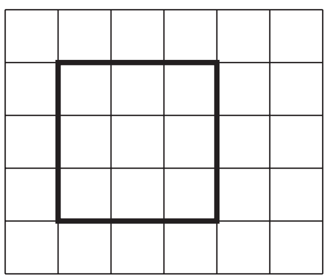
How many squares?

In *Mathematical Spectrum*, Volume 35, Number 1, we asked 'how many rectangles are there in an $m \times n$ array of equal squares?' One of our readers, Bablu Chandra Dey of Kolkata, India, asks 'how many squares are there in an $n \times n$ or, more generally, an $m \times n$ array of equal squares?' (The case $m = 5, n = 6$ is illustrated.)



Columns of Fibonacci or Lucas Numbers

HOMER W. AUSTIN

Introduction

The Fibonacci sequence is defined as a sequence of positive integers such that $F_1 = 1, F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for $n \geq 2$. A Fibonacci-type sequence, called the Lucas sequence, is defined as $L_1 = 1, L_2 = 3$ and $L_{n+1} = L_n + L_{n-1}$, for $n \geq 2$.

There are many known connections between the two sequences. The purpose of this article is to show some patterns that become apparent when the sequences are displayed in columns. The number of columns chosen for displaying a sequence might facilitate the discovery of new patterns. Also, these two sequences are rich in exercises for mathematical induction, congruences and recursion.

Columns of Numbers

When displaying the Fibonacci or Lucas sequence in columns, patterns appear which might not be recognized if viewing the sequence in other ways. For example, if the Fibonacci sequence is printed in five columns, we obtain

$$\begin{array}{rrrrr} 1 & 1 & 2 & 3 & 5 \\ 8 & 13 & 21 & 34 & 55 \\ 89 & 144 & 233 & 377 & 610 \\ 987 & 1597 & 2584 & 4181 & 6765\ldots. \end{array}$$

Looking at the third and fourth rows, it becomes clear that each entry is 11 times the number above it plus the number two rows above it. For example, $89 = 11\times 8 + 1$, i.e.

$$F_{n+10} = 11F_{n+5} + F_n, \quad \text{for all } n \geq 1. \tag{1}$$

This result is easily proved using mathematical induction. It also follows that the digit for units is preserved in each of the five columns. That is, the unit digit of a number is the sum of the unit digits of the two numbers above it. Since $11 \equiv 1 \mod 10$, the result follows directly from (1); see references 1 and 2.

If the Fibonacci sequence is printed in four columns, we obtain

$$\begin{array}{rrrr} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \\ 233 & 377 & 610 & 987\ldots. \end{array}$$

In this case, looking at the third and fourth rows, each entry is 7 times the number one row above it minus the number two rows above it. For example, $34 = 7\times 5 - 1$, i.e.

$$F_{n+8} = 7F_{n+4} - F_n, \quad \text{for all } n \geq 1.$$

This result is also easily proved by mathematical induction. Here, however, the unit digit is not preserved.

The following question naturally arises: 'is there a generalization as to what pattern exists if any number of columns is used to list the Fibonacci numbers?' The answer lies in a result due to Ruggles (see reference 3).

Connection Between the Lucas Sequence and Fibonacci Sequence

Ruggles (see reference 3) obtained general results by looking at Fibonacci-type sequences, and proving connections between them. The following theorem on columns of Fibonacci numbers follows from his work.

Theorem 1 *Let $m \geq 1$ be the number of columns in which the Fibonacci numbers are listed. Then we obtain*

$$\mathrm{F}_{n+2m} = \mathrm{L}_m \mathrm{F}_{n+m} + (-1)^{m+1}\mathrm{F}_n\,, \qquad n \geq 1\,. \tag{2}$$

This theorem shows that any given column in an array of m columns of the Fibonacci sequence can be written if the first two numbers in the column are known. In the statement of Theorem 1, m is a fixed arbitrary positive integer chosen as the number of columns desired for the listing. Theorem 1 also presents an interesting connection between the Fibonacci and Lucas sequences. Ruggles (see reference 3) obtained this result as a special case of a more general result; however, Theorem 1 is readily established by proving four lemmas. All four lemmas are proved using mathematical induction.

Before stating and proving the lemmas, the following result is needed. If $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then

$$\mathrm{L}_n \mathrm{F}_n = (\alpha^n + \beta^n)\frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}} = \mathrm{F}_{2n}, \qquad \text{for } n \geq 1\,. \tag{3}$$

This result is found in many elementary number theory books and can be established using mathematical induction. Information on this theorem and other related results are clearly stated in reference 4, p. 277.

Lemma 1 *For all $n \geq 1$, we obtain* $\mathrm{F}_{4n+1} = \mathrm{L}_{2n}\mathrm{F}_{2n+1} - 1$.

Proof For $n = 1$, we have $\mathrm{F}_5 = \mathrm{L}_2\mathrm{F}_3 - 1$, it follows that $5 = 3\times 2 - 1$, so the statement is verified for $n = 1$. Now, assume that the statement holds for all positive integers n such that $1 \leq n \leq k$. In particular, when $n = k$, $\mathrm{F}_{4k+1} = \mathrm{L}_{2k}\mathrm{F}_{2k+1} - 1$. It follows that

$$\begin{aligned}
\mathrm{L}_{2k+2}\mathrm{F}_{2k+3} - 1 &= \mathrm{L}_{2k+2}(\mathrm{F}_{2k+2} + \mathrm{F}_{2k+1}) - 1 \\
&= \mathrm{L}_{2k+2}\mathrm{F}_{2k+2} + \mathrm{L}_{2k+2}\mathrm{F}_{2k+1} - 1 \\
&= \mathrm{F}_{4k+4} + (\mathrm{L}_{2k} + \mathrm{L}_{2k+1})\mathrm{F}_{2k+1} - 1 \quad \text{(from (3))} \\
&= \mathrm{F}_{4k+4} + \mathrm{L}_{2k}\mathrm{F}_{2k+1} + \mathrm{L}_{2k+1}\mathrm{F}_{2k+1} - 1 \\
&= \mathrm{F}_{4k+4} + \mathrm{F}_{4k+2} + \mathrm{L}_{2k}\mathrm{F}_{2k+1} - 1 \quad \text{(from (3))} \\
&= \mathrm{F}_{4k+4} + \mathrm{F}_{4k+2} + \mathrm{F}_{4k+1} \quad \text{(by inductive assumption)} \\
&= \mathrm{F}_{4k+4} + \mathrm{F}_{4k+3} \\
&= \mathrm{F}_{4k+5}\,,
\end{aligned}$$

so the statement holds for $n = k + 1$, thus the statement holds for all $n \geq 1$.

Lemma 2 *Let k be a fixed arbitrary positive integer. Then*

$$F_{n+4k} = L_{2k}F_{n+2k} - F_n\,, \qquad n \geq 1\,.$$

Proof For $n = 1$, it follows that $F_{4k+1} = L_{2k}F_{2k+1} - 1$, by Lemma 1. Assume that $F_{n+4k} = L_{2k}F_{n+2k} - F_n$ for all $1 \leq n \leq j$. Then we obtain

$$\begin{aligned}
L_{2k}F_{j+1+2k} - F_{j+1} &= L_{2k}F_{j+1+2k} - F_j - F_{j-1} \\
&= L_{2k}F_{j+2k} + L_{2k}F_{j-1+2k} - F_j - F_{j-1} \\
&= (L_{2k}F_{j+2k} - F_j) + (L_{2k}F_{j-1+2k} - F_{j-1}) \\
&= F_{j+4k} + F_{j-1+4k} \quad \text{(by inductive assumption)} \\
&= F_{j+1+4k}\,,
\end{aligned}$$

so the statement holds for $n = j + 1$. Thus, the statement holds for all $n \geq 1$.

Lemma 3 *We have*

$$L_{2n+1}F_{2n} + 1 = F_{4n+1}\,, \qquad n \geq 1\,. \tag{4}$$

Proof For $n = 1$, $L_3F_2 + 1 = 4 \times 1 + 1 = 5 = F_5$, therefore (4) is verified for $n = 1$. Now assume that (4) holds when $n = k$. Then we obtain

$$\begin{aligned}
L_{2(k+1)+1}F_{2(k+1)} + 1 &= L_{2k+3}F_{2k+2} + 1 \\
&= (L_{2k+1} + L_{2k+2})F_{2k+2} + 1 \\
&= L_{2k+1}F_{2k+2} + L_{2k+2}F_{2k+2} + 1 \\
&= L_{2k+1}(F_{2k} + F_{2k+1}) + F_{4k+4} + 1 \\
&= L_{2k+1}F_{2k} + L_{2k+1}F_{2k+1} + F_{4k+4} + 1 \\
&= L_{2k+1}F_{2k} + F_{4k+2} + F_{4k+4} + 1 \\
&= (L_{2k+1}F_{2k} + 1) + F_{4k+2} + F_{4k+4} \\
&= F_{4k+1} + F_{4k+2} + F_{4k+4} \quad \text{(by inductive assumption)} \\
&= F_{4k+3} + F_{4k+4} \\
&= F_{4k+5}\,,
\end{aligned}$$

so the statement holds for $n = k + 1$. Thus, the statement holds for all $n \geq 1$.

Lemma 4 *Let k be a fixed arbitrary positive integer, then*

$$F_{n+4k+2} = L_{2k+1}F_{n+2k+1} + F_n\,, \qquad n \geq 1\,. \tag{5}$$

Proof For $n = 1$, we obtain

$$\begin{aligned}
F_{4k+3} &= F_{4k+2} + F_{4k+1} \\
&= F_{4k+2} + L_{2k+1}F_{2k} + 1 \quad \text{(by Lemma 3)} \\
&= L_{2k+1}F_{2k+1} + L_{2k+1}F_{2k} + 1 \\
&= L_{2k+1}(F_{2k+1} + F_{2k}) + 1 \\
&= L_{2k+1}F_{2k+2} + 1\,,
\end{aligned}$$

so (5) is verified for $n = 1$. Now, assume that (5) holds for all $1 \leq n \leq j$. Then we obtain

$$\begin{aligned} L_{2k+1}F_{(j+1)+2k+1} + F_{j+1} &= L_{2k+1}F_{j+2k+1} + L_{2k+1}F_{(j-1)+2k+1} + F_{j+1} \\ &= L_{2k+1}F_{j+2k+1} + F_j + L_{2k+1}F_{(j-1)+2k+1} + F_{j-1} \\ &= F_{j+4k+2} + F_{(j-1)+4k+2} \\ &= F_{(j+1)+4k+2}\,, \end{aligned}$$

so the statement holds for $n = j + 1$. Thus, the statement holds for all $n \geq 1$.

The proof of Theorem 1 follows directly from these four lemmas.

Proof of Theorem 1 Let m be the number of columns of Fibonacci numbers.

Case 1 (m even). We have $m = 2k$ for some positive integer k. The proof follows from Lemma 2.

Case 2 (m odd). We have $m = 1$ or $m = 2k + 1$, where k is some positive integer. If $m = 1$, then the theorem is trivially true (since $L_1 = 1$). If $m = 2k + 1$, then the proof follows from Lemma 4.

Note that a similar theorem exists for columns of Lucas numbers by substituting all Fibonacci numbers in (2) with the corresponding Lucas numbers.

Discussion of Theorem 1 and Some Derived Results

Theorem 1 guarantees that the unit digit will be preserved in the m columns if $L_m \equiv 1 \mod 10$. Of course, $L_5 = 11$ and $L_{13} = 521$ are both congruent to $1 \mod 10$; however, there are other Lucas numbers that are congruent to $1 \mod 10$. It is easy to prove, using mathematical induction, that

$$L_m \equiv L_{m+12} \mod 10, \qquad m \geq 1\,. \tag{6}$$

So, from (6), it can be inferred that, if $L_m \equiv 1 \mod 10$, then $L_{m+12} \equiv 1 \mod 10$. Also, the only two of the first twelve Lucas numbers that are congruent to $1 \mod 10$ are $L_1 = 1$ and $L_5 = 11$. Thus, we obtain the following theorem.

Theorem 2 *We have* $L_m \equiv 1 \mod 10$ *if and only if* $m = 12k + 1$ *or* $m = 12k + 5$*, where* k *is a positive integer.*

Pythagorean Triples

A triple of positive integers (x, y, z) is said to be a *Pythagorean triple* if $x^2 + y^2 = z^2$. A display of Fibonacci numbers in columns facilitates finding Pythagorean triples. Pythagorean triples can be constructed by factoring even squares. It is an easy exercise in algebra to verify the following lemma (see references 5 and 6).

Lemma 5 *For positive integers* r, b *and* w, *the triple* $(r + w, b + w, r + b + w)$ *is a Pythagorean triple if and only if*

$$w^2 = 2rb\,.$$

Furthermore, the triple in Lemma 5 is *primitive* if and only if r and b are relatively prime. A Pythagorean triple (x, y, z) is primitive if x, y and z do not share a common factor greater than 1. A display of Fibonacci numbers in columns aids in finding a suitable value for w to generate Pythagorean triples.

Consider the Fibonacci numbers F_{6n}, where n is a positive integer. If the Fibonacci numbers are written in $m = 3$ columns, then from (2) it can be deduced that

$$F_{6n} = F_{3(2n)} = 4F_{3(2n-1)} + F_{3(2n-2)}, \qquad n \geq 2.$$

So, $F_{6n} - F_{6n-6} = 4F_{6n-3}$. Squaring both sides of this equation yields $(F_{6n} - F_{6n-6})^2 = 2(8(F_{6n-3})(F_{6n-3}))$. Setting $r = 8(F_{6n-3})$, $b = F_{6n-3}$ and $w = F_{6n} - F_{6n-6}$, from Lemma 5, we have that the triple

$$\big(8(F_{6n-3}) + F_{6n} - F_{6n-6}\,,\ F_{6n-3} + F_{6n} - F_{6n-6}\,,\ 9F_{6n-3} + F_{6n} - F_{6n-6}\big)$$

is a Pythagorean triple for all $n \geq 2$. In this case, the triples will not be primitive because r and b are not relatively prime. Many other representations for triples can be generated in a similar manner.

Digits of Units Preserved in Thirteen Columns

Displaying Fibonacci and Lucas numbers in columns provides a visual mechanism for discovering hidden properties that the two sequences possess. For example, consider the following display of Fibonacci numbers in thirteen columns:

1	1	2	3	5	8	13	21	34	55	89	144	233
377	610	...										
196 418	317 811	...										

By (2),

$$F_{n+26} = L_{13}F_{n+13} + F_n \equiv F_{n+13} + F_n \mod 10,$$

because $L_{13} = 521 \equiv 1 \mod 10$. So the unit's digits in each column satisfy the same recurrence relation as the Fibonacci sequence $\bmod 10$. In the second column, they begin with $F_2 = 1 \equiv 1 \mod 10$ and $F_{15} = 610 \equiv 0 \mod 10$, so they form the sequence $1, 0, 1, 1, 2, 3, 5, \ldots$ which, from the third term on, is just the sequence of unit digits of the Fibonacci sequence.

Concluding Remarks

In closing, we note that the process of displaying Fibonacci and Lucas numbers in columns and looking for connections holds promise for uncovering other interesting connections. More hidden secrets of these sequences could possibly be divulged, secrets that seem to be shrouded in mystery. For example, who would have suspected that the Fibonacci sequence preserves its unit digits in the second column of a thirteen column array of itself!

Acknowledgment This article is dedicated to James Madison University Professor Charles W. Ziegenfus, who introduced me to the Fibonacci sequence almost forty years ago, and to the memory of Herta Freitag, who always inspired me with her knowledge and love of mathematics.

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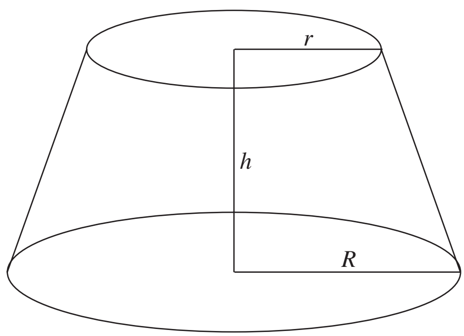
Volume of a frustrum

The volume of the frustrum of the right circular cone shown is

$$\tfrac{1}{3}\pi(R^2 + Rr + r^2)h.$$

Can you prove it?



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A. Hoger Kurdy

Odd Abundant Numbers

JAY L. SCHIFFMAN

The natural numbers can be divided into three types, the *abundant*, *deficient*, and *perfect* numbers. If $\sigma(n)$ denotes the sum of all the positive divisors of n (including 1 and n), then n is classified as *deficient* if $\sigma(n) < 2n$, *perfect* if $\sigma(n) = 2n$, and *abundant* if $\sigma(n) > 2n$. For example, 5 is deficient ($\sigma(5) = 1 + 5 = 6 < 10 = 2 \times 5$), 28 is perfect ($\sigma(28) = 1+2+4+7+14+28 = 56 = 2 \times 28$), and 12 is abundant ($\sigma(12) = 1+2+3+4+6+12 = 28 > 24 = 2 \times 12$). Empirical evidence might erroneously lead us to believe that all the positive odd integers are deficient, particularly if the search did not exceed 500. Indeed, odd abundant numbers exist with 945 as the first member. The purpose of this article is to explore the sequence $u_n = 945 + 630n$ for the initial 52 values of n ($0 \le n \le 51$). What transpires is rather curious and remarkable; our formula always generates an odd abundant number. In a manner similar to E. B. Escott's prime generating formula, $f(n) = n^2 - 79n + 1601$, which produces prime outputs for the initial 80 whole numbers (Escott's formula fails when $n = 80$ which yields the composite output $1681 = 41^2$), our formula fails when $n = 52$. Such investigations are nonetheless useful for two reasons, firstly, never abandon a problem after only the observation of a few cases and, secondly, the formulation of conjectures is an essential ingredient in mathematical discovery. We initiate our discussion with the following results, whose proofs are available in most elementary number theory references, including reference 1.

1. Every prime number as well as every power of a prime is deficient.

2. Every divisor of a perfect number or a deficient number is deficient.

3. Every multiple of a perfect number or an abundant number is abundant.

4. Using the formula for the sum of a finite geometric progression, we have $\sigma(p^n) = (p^{n+1} - 1)/(p - 1)$, where p is any prime.

5. The function σ is a multiplicative number–theoretic function in the sense that

$$\sigma(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})\sigma(p_3^{\alpha_3}) \cdots \sigma(p_r^{\alpha_r})$$

   for distinct primes $p_1, \dots, p_r$ and distinct whole numbers $\alpha_1, \dots, \alpha_r$.

We are now in a position to prove that 945 is abundant. To see this, observe $\sigma(945) = \sigma(3^3 \times 5 \times 7) = \sigma(3^3)\,\sigma(5)\,\sigma(7) = ((3^{3+1} - 1)/(3 - 1))(5 + 1)(7 + 1) = 40 \times 6 \times 8 = 1920 > 1890 = 2 \times 945$.

The following theorem is an immediate consequence of the third result above.

Theorem 1 *There exist infinitely many odd abundant numbers.*

Table 1 shows all the odd abundant numbers less than 50 000. Outcomes of the form $945 + 630n$ are denoted by an asterisk in the first column. Not all odd abundant numbers are of this form.

Table 1

Odd abundant number n	Prime factorization of n	$\sigma(n)$	$2n$	Odd abundant number n	Prime factorization of n	$\sigma(n)$	$2n$
945*	$3^3 \times 5 \times 7$	1 920	1 890	26 145*	$3^2 \times 5 \times 7 \times 83$	52 416	52 290
1 575*	$3^2 \times 5^2 \times 7$	3 224	3 150	26 565	$3 \times 5 \times 7 \times 11 \times 23$	55 296	53 130
2 205*	$3^2 \times 5 \times 7^2$	4 446	4 410	26 775*	$3^2 \times 5^2 \times 7 \times 17$	58 032	53 550
2 835*	$3^4 \times 5 \times 7$	5 808	5 670	27 405*	$3^3 \times 5 \times 7 \times 29$	57 600	54 810
3 465*	$3^2 \times 5 \times 7 \times 11$	7 488	6 930	28 035*	$3^2 \times 5 \times 7 \times 89$	56 160	56 070
4 095*	$3^2 \times 5 \times 7 \times 13$	8 736	8 190	28 215	$3^3 \times 5 \times 11 \times 19$	57 600	56 430
4 725*	$3^3 \times 5^2 \times 7$	9 920	9 450	28 665*	$3^2 \times 5 \times 7^2 \times 13$	62 244	57 330
5 355*	$3^2 \times 5 \times 7 \times 17$	11 232	10 710	28 875	$3 \times 5^3 \times 7 \times 11$	59 904	57 750
5 775	$3 \times 5^2 \times 7 \times 11$	11 904	11 550	29 295*	$3^3 \times 5 \times 7 \times 31$	61 440	58 590
5 985*	$3^2 \times 5 \times 7 \times 19$	12 480	11 970	29 835	$3^3 \times 5 \times 13 \times 17$	60 480	59 670
6 435	$3^2 \times 5 \times 11 \times 13$	13 104	12 870	29 925*	$3^2 \times 5^2 \times 7 \times 19$	64 480	59 850
6 615*	$3^3 \times 5 \times 7^2$	13 680	13 230	30 555*	$3^2 \times 5 \times 7 \times 97$	61 152	61 110
6 825	$3 \times 5^2 \times 7 \times 13$	13 888	13 650	31 185*	$3^4 \times 5 \times 7 \times 11$	69 696	62 370
7 245*	$3^2 \times 5 \times 7 \times 23$	14 976	14 490	31 395	$3 \times 5 \times 7 \times 13 \times 23$	64 512	62 790
7 425	$3^3 \times 5^2 \times 11$	14 880	14 850	31 815*	$3^2 \times 5 \times 7 \times 101$	63 648	63 630
7 875*	$3^2 \times 5^3 \times 7$	16 224	15 750	32 175	$3^2 \times 5^2 \times 11 \times 13$	67 704	64 350
8 085	$3 \times 5 \times 7^2 \times 11$	16 416	16 170	32 445*	$3^2 \times 5 \times 7 \times 103$	64 896	64 890
8 415	$3^2 \times 5 \times 11 \times 17$	16 848	16 830	33 075*	$3^3 \times 5^2 \times 7^2$	70 680	66 150
8 505*	$3^5 \times 5 \times 7$	17 472	17 010	33 345	$3^3 \times 5 \times 13 \times 19$	67 200	66 690
8 925	$3 \times 5^2 \times 7 \times 17$	17 856	17 850	33 495	$3 \times 5 \times 7 \times 11 \times 29$	69 120	66 990
9 135*	$3^2 \times 5 \times 7 \times 29$	18 720	18 270	33 915	$3 \times 5 \times 7 \times 17 \times 19$	69 120	67 830
9 555	$3 \times 5 \times 7^2 \times 13$	19 152	19 110	34 125	$3 \times 5^3 \times 7 \times 13$	69 888	68 250
9 765*	$3^2 \times 5 \times 7 \times 31$	19 968	19 530	34 155	$3^3 \times 5 \times 11 \times 23$	69 120	68 310
10 395*	$3^3 \times 5 \times 7 \times 11$	23 040	20 790	34 965*	$3^3 \times 5 \times 7 \times 37$	72 960	69 930
11 025*	$3^2 \times 5^2 \times 7^2$	22 971	22 050	35 805	$3 \times 5 \times 7 \times 11 \times 31$	73 728	71 610
11 655*	$3^2 \times 5 \times 7 \times 37$	23 712	23 310	36 225*	$3^2 \times 5^2 \times 7 \times 23$	77 376	72 450
12 285*	$3^3 \times 5 \times 7 \times 13$	26 880	24 570	36 855*	$3^4 \times 5 \times 7 \times 13$	81 312	73 710
12 705	$3 \times 5 \times 7 \times 11^2$	25 536	25 410	37 125	$3^3 \times 5^3 \times 11$	74 880	74 250
12 915*	$3^2 \times 5 \times 7 \times 41$	26 208	25 830	37 485*	$3^2 \times 5 \times 7^2 \times 17$	80 028	74 970
13 545*	$3^2 \times 5 \times 7 \times 43$	27 456	27 090	38 115*	$3^2 \times 5 \times 7 \times 11^2$	82 992	76 230
14 175*	$3^4 \times 5^2 \times 7$	30 008	28 350	38 745*	$3^3 \times 5 \times 7 \times 41$	80 640	77 490
14 805*	$3^2 \times 5 \times 7 \times 47$	29 952	29 610	39 375*	$3^2 \times 5^4 \times 7$	81 224	78 750
15 015	$3 \times 5 \times 7 \times 11 \times 13$	32 256	30 030	39 585	$3 \times 5 \times 7 \times 13 \times 29$	80 640	79 170
15 435*	$3^2 \times 5 \times 7^3$	31 200	30 870	40 425	$3 \times 5^2 \times 7^2 \times 11$	84 816	80 850
16 065*	$3^3 \times 5 \times 7 \times 17$	34 560	32 120	40 635*	$3^3 \times 5 \times 7 \times 43$	84 480	81 270
16 695*	$3^2 \times 5 \times 7 \times 53$	33 696	33 390	41 055	$3 \times 5 \times 7 \times 17 \times 23$	82 944	82 110
17 325*	$3^2 \times 5^2 \times 7 \times 11$	38 688	34 650	41 895*	$3^2 \times 5 \times 7^2 \times 19$	88 920	83 790
17 955*	$3^3 \times 5 \times 7 \times 19$	38 400	35 910	42 075	$3^2 \times 5^2 \times 11 \times 17$	87 048	84 150
18 585*	$3^2 \times 5 \times 7 \times 59$	37 440	37 170	42 315	$3 \times 5 \times 7 \times 13 \times 31$	86 016	84 630
19 215*	$3^2 \times 5 \times 7 \times 61$	38 688	38 430	42 525*	$3^5 \times 5^2 \times 7$	90 272	85 050
19 305	$3^3 \times 5 \times 11 \times 13$	40 320	38 610	42 735	$3 \times 5 \times 7 \times 11 \times 37$	87 552	85 470
19 635	$3 \times 5 \times 7 \times 11 \times 17$	41 472	39 270	43 065	$3^3 \times 5 \times 11 \times 29$	86 400	86 130
19 845*	$3^4 \times 5 \times 7^2$	41 382	39 690	44 415*	$3^3 \times 5 \times 7 \times 47$	92 160	88 830
20 475*	$3^2 \times 5^2 \times 7 \times 13$	45 136	40 950	44 625	$3 \times 5^3 \times 7 \times 17$	89 856	89 250
21 105*	$3^2 \times 5 \times 7 \times 67$	42 432	42 210	45 045*	$3^2 \times 5 \times 7 \times 11 \times 13$	104 832	90 090
21 735*	$3^3 \times 5 \times 7 \times 23$	46 080	43 470	45 675*	$3^2 \times 5^2 \times 7 \times 29$	96 720	91 350
21 945	$3 \times 5 \times 7 \times 11 \times 19$	46 080	43 890	45 885	$3 \times 5 \times 7 \times 19 \times 23$	92 160	91 770
22 275	$3^4 \times 5^2 \times 11$	45 012	44 550	46 035	$3^3 \times 5 \times 11 \times 31$	92 160	92 070
22 365*	$3^2 \times 5 \times 7 \times 71$	44 928	44 730	46 305*	$3^3 \times 5 \times 7^3$	96 000	92 610
22 995*	$3^2 \times 5 \times 7 \times 73$	46 176	45 990	47 025	$3^2 \times 5^2 \times 11 \times 19$	96 720	94 050
23 205	$3 \times 5 \times 7 \times 13 \times 17$	48 384	46 410	47 355	$3 \times 5 \times 7 \times 11 \times 41$	96 768	94 710
23 625*	$3^3 \times 5^3 \times 7$	49 920	47 250	47 775	$3 \times 5^2 \times 7^2 \times 13$	98 952	95 550
24 255*	$3^2 \times 5 \times 7^2 \times 11$	53 352	48 510	48 195*	$3^4 \times 5 \times 7 \times 17$	104 544	96 390
24 885*	$3^2 \times 5 \times 7 \times 79$	49 920	49 770	48 825*	$3^2 \times 5^2 \times 7 \times 31$	103 168	97 650
25 245	$3^3 \times 5 \times 11 \times 17$	51 840	50 490	49 665	$3 \times 5 \times 7 \times 11 \times 43$	101 376	99 330
25 515*	$3^6 \times 5 \times 7$	52 464	51 030	49 725	$3^2 \times 5^2 \times 13 \times 17$	101 556	99 450
25 935	$3 \times 5 \times 7 \times 13 \times 19$	53 760	51 870	49 875	$3 \times 5^3 \times 7 \times 19$	99 840	99 750

The number 33 705 corresponding to the value of $945 + 630n$ for $n = 52$ is missing from Table 1, so this formula fails to always produce an odd abundant number. In fact, $\sigma(33\,705) = \sigma(3^2 \times 5 \times 7 \times 107) = \sigma(3^2)\,\sigma(5)\,\sigma(7)\,\sigma(107) = 13 \times 6 \times 8 \times 108 = 67\,392 < 67\,410 = 2 \times 33\,705$, so 33 705 is deficient.

A second observation from Table 1 is that none of our integers of the form $945 + 630n$ are square-free. (An integer is *square-free* if it is not divisible by the square of any prime.) It is clear that all are divisible by 9.

We might erroneously believe from looking at Table 1 that all odd abundant numbers are divisible by 3 and end in the digit 5. The following counterexamples serve to refute these notions.

Counterexample 1 The odd abundant number 81 081 does not terminate in the digit 5, and is the first such number. In fact, $\sigma(81\,081) = \sigma(3^4 \times 7 \times 11 \times 13) = \sigma(3^4)\,\sigma(7)\,\sigma(11)\,\sigma(13) = 121 \times 8 \times 12 \times 14 = 162\,624 > 162\,162 = 2 \times 81\,081$.

Counterexample 2 The odd abundant number $1\,382\,511\,906\,801\,025 = 5^2 \times 7^2 \times 11^2 \times 13^2 \times 17^2 \times 19^2 \times 23^2$ is not a multiple of 3. It can be verified that $\sigma(1\,382\,511\,906\,801\,025 = 31 \times 57 \times 133 \times 183 \times 307 \times 381 \times 553 = 2\,781\,811\,913\,132\,763 > 2\,765\,023\,813\,602\,050 = 2 \times 1\,382\,511\,906\,801\,025$.

Theorem 2 *There are infinitely many odd abundant numbers terminating in any odd digit.*

Proof Since 81 081 is abundant, any odd integer multiple of this integer will again be an odd abundant number. The first few odd abundant integers terminating in digits other than 5 are 81 081, 153 153, 171 171, 189 189, 207 207, 243 243 and 297 297.

In conclusion, the set of odd abundant numbers lends itself to interesting investigations in number theory. The next step, for example, might be to explore square-free odd abundant numbers that have a maximum of five prime factors. With technology such as modern graphical calculators and MATHEMATICA®, further investigations are possible.

Reference

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+ equals ×

Find all pairs of real numbers whose sum is equal to their product.

Tienen, Belgium **Guido Lasters**

Computer Column

Power computing

Imagine you're in charge of a large research project, with a mountain of data waiting to be analysed. You desperately need all the computing horsepower you can get your hands on, but where should you go to get it? From simulations of protein folding to searches for new subatomic particles, it seems that everyone needs the latest and greatest. The problem isn't just confined to the academic world, in this online, on-demand world, everyone from business executives to gamers is clamouring for more power and speed.

Once upon a time, the answer would have been simple: buy a bigger computer, probably a room-filling giant known as a *mainframe*. This might have given you the power you needed, but it would have been expensive to buy, would have brought everything to a grinding halt if it broke down, and generally been lacking in flexibility. Nowadays, there are several other ways to go, from simple off-the-shelf solutions to bespoke, custom-engineered approaches.

The key to all of these systems is parallelism: where mainframes usually have a few, very powerful, main processors, modern systems have a large number (often thousands) of less powerful chips. One advantage of this is that it often allows manufacturers to use cheap, common chips like those in an ordinary PC; another is that the machine (and often even the program that's currently running) can keep going even if a few chips fail. For these reasons, virtually all of the world's most powerful supercomputers are now built this way. The current world leader, NEC's *Earth Simulator*, uses 5 120 chips, and IBM is currently working on a 65 000-processor monster as part of its Blue Gene project, designed to study protein folding.

Parallel computers need to be used with care, keeping thousands of processors in work isn't always easy. For this reason, they are most useful for handling large amounts of data, where each piece needs to be analysed in much the same way. Fortunately, that's usually what's needed!

The interesting part about all this is that these processors don't all need to be in one box, thanks to high-speed networks and the internet, we can now spread the load around the room, or around the world.

Clusters: the beehive solution

Rather than having one huge monolithic computer, why not have a group (or *cluster*) of small, cheap computers linked tightly together? That way, it can expand to fit your needs, and take advantage of whatever you have available, rather than throwing out old PCs, you can simply add them to the cluster!

Network connections are now fast enough that, as far as most users of a cluster are concerned, it may as well be one large computer. There are still tasks for which having one big box is better (the more different strands of the program need to talk to each other, or get hold of the same data, the more the strain on the connections shows) but clusters can be a cheap way of creating a powerful system. The Beowulf Project, in particular, has demonstrated how almost any collection of spare PCs can be turned into a useful cluster, much to the delight of universities with limited budgets!

Despite their apparent differences though, at heart clusters are much like more integrated parallel computers; to be effective, a cluster still has to be grouped together in one place, and there still has to be a ‘queen bee’ that takes control and ensures that the work is divided up fairly. To break out onto the internet, we need a different approach.

Grids: a more flexible solution

Even if the name seems unfamiliar, your computer may already be part of a grid! If you run SETI@HOME®, FOLDING@HOME® or any other software that puts your spare computer power to a more productive use, your computer is a member of one. Like a cluster, a *grid* is a large group of computers working together. Unlike clusters though, the connections are much looser and there is no central controller.

A cluster can be thought of as being like an office, with workers doing what their boss tells them, grids are more like a collaboration, where each person handles their own schedule and doesn’t take on what they can’t handle. If you want a job done, you send a request out across the grid and wait for a computer that has enough spare capacity to reply and accept it. Similarly, you can retrieve any piece of publicly accessible information held by any computer on the grid without having to know where it is. The key to all this is *middleware* (software that each computer on the grid runs, and which acts as a mediator). In the case of SETI@HOME, this job is pretty straightforward (i.e. sending back results and requesting new data) but can get much more complicated for more powerful grids, where it becomes important to make sure that the computer that takes a job on is the one best able to do it, and not just any one that can cope with it.

The end result is that although a grid still looks, to the user, like one big computer, it behaves very differently. Grids are much more flexible than clusters, even being able to use up spare capacity on computers that are already being used for something else, but jobs have to be divided up into pieces small enough to be dealt with by individual computers. With the gain in flexibility, we seem to have lost the most basic quality of parallel systems, which is the ability to have programs use several processors at once. In practice though, this often doesn’t matter, as with SETI@HOME, many tasks naturally break down into a series of entirely independent jobs.

Probably the world’s largest grid project is the one currently being undertaken by CERN, the European Organization for Nuclear Research. Their latest experiment, the large hadron collider (LHC), will begin operation in 2007 and will generate 12–14 petabytes of data per year, the equivalent of more than 20 million CDs. This is a thousand times as much information as is published in book form anywhere in the world each year, and about 10% of *all* the information we generate in a year! Not surprisingly, analysing it all will be a monumental task, requiring the equivalent of 70 000 of today’s fastest PCs. Already, the initial phase encompasses 30 sites and 4 000 PCs, located across Europe and in the USA, Canada, Russia, India, Japan, and Taiwan. When it is fully operational, more than 6 000 researchers will be working on the data, collaborating in a way that has never been possible before.

Conclusions

Grids have been hailed as the future of high-performance computing, and it is easy to see why. At one end of the scale, SETI@HOME has managed to clock up the equivalent of over two million PC-years of processing time, essentially for free. At the other end, CERN’s grid not only allows the pooling of computing resources from around the world, it will also allow the scientists involved to work as if they are all on one big campus. Who says that computing is just about doing the same old thing, only bigger and better?

Websites

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8 Top 500 Supercomputer Sites: http://www.top500.org/

Peter Mattsson

Letters to the Editor

Dear Editor,

To find the cube root of 63

A young child has been given a building-block puzzle. The puzzle comprises a cube of volume 63 cm^3 and three equally shaped and sized pieces which, when fitted to the cube, one to each of three adjacent faces, makes a cube of volume 64 cm^3. The joiner who made the puzzle knew that those three pieces would only fit if they were of a particular shape and size. Each piece, which we call a 'plinth', has the following measurements:

$$\begin{aligned} \text{base}: &\quad \text{square, each side 4 cm,} \\ \text{top}: &\quad \text{square, each side } \sqrt[3]{63}\text{ cm,} \\ \text{height}: &\quad 4-\sqrt[3]{63}\text{ cm,} \\ \text{volume}: &\quad 0.\dot{3}\text{ cm}^3. \end{aligned}$$

An initial approximation for $\sqrt[3]{63}$ is $x_0 = 4$. For a first approximation x_1, we say that

$$\text{volume of the plinth } = 0.\dot{3} = x_0^2(x_0 - x_1),$$

so

$$x_1 = x_0 - \frac{0.\dot{3}}{x_0^2} = 4 - \frac{0.\dot{3}}{4^2} = 3.979\,1\dot{6}.$$

For a second approximation x_2, we use a plinth with base $x_1 \times x_1$, top $x_2 \times x_2$ and height $x_1 - x_2$. Its volume is

$$0.\dot{3} - \frac{4^3 - x_1^3}{3},$$

so, as an approximation,

$$0.\dot{3} - \frac{4^3 - x_1^3}{3} = x_1^2(x_1 - x_2).$$

Thus,

$$x_2 = x_1 - \frac{0.\dot{3} - (4^3 - x_1^3)/3}{x_1^2} = 3.979\,057\,210\,91.$$

If we continue in this way, we have the iteration given by

$$x_{n+1} = x_n - \frac{0.\dot{3} - (4^3 - x_n^3)/3}{x_n^2},$$

which gives $x_3 = 3.979\,705\,720\,79$. Further iterations do not alter the approximation to $\sqrt[3]{63}$ up to the 10th decimal place.

Yours sincerely,
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Dear Editor,

Equal sums of cubes

I would like to comment on 'Equal sums of cubes' in *Mathematical Spectrum*, Volume 36, Number 3. For the square

$$\begin{matrix} a & d \\ b & c \end{matrix},$$

the given examples illustrate the following general identity:

$$(ka+d)^3+(kd+c)^3+(kc+b)^3+(kb+a)^3 \equiv (ka+b)^3+(kb+c)^3+(kc+d)^3+(kd+a)^3,$$

where a, b, d, c or a, d, b, c are in arithmetical progression, and $k = 10$. So the given rule works in any base. For example, with $k = 100$, we may construct the following square:

$$\begin{matrix} 17 & 61 \\ 39 & 83 \end{matrix},$$

and we find

$$1761^3 + 6183^3 + 8339^3 + 3917^3 = 1739^3 + 3983^3 + 8361^3 + 6117^3 = 881\,817\,163\,000.$$

I tried to generalize the pattern to pentagons (and fourth powers) and failed, but did discover that it works for a triangular array using squares rather than cubes. This can be proved easily as $(ka+c)^2 + (kc+b)^2 + (kb+a)^2 \equiv (ka+b)^2 + (kb+c)^2 + (kc+a)^2$. There is no need for any constraint on a, b, c. Also it can immediately be generalised to any size of polygon, for example for the hexagon

$$\begin{matrix} & 1 & 3 & \\ 6 & & & 4 \\ & 8 & 7 & \end{matrix},$$

we obtain

$$13^2 + 34^2 + 47^2 + 78^2 + 86^2 + 61^2 = 16^2 + 68^2 + 87^2 + 74^2 + 43^2 + 31^2 = 20\,735.$$

This leads naturally to the question 'what happens if we remove the condition that a, b, c, d are terms of an arithmetical progression?'. The following algebra shows that a square works with the looser condition $a + c = b + d$:

$$\begin{aligned}&\{(ka+d)^3 + (kd+c)^3 + (kc+b)^3 + (kb+a)^3\}\\ &\quad - \{(ka+b)^3 + (kb+c)^3 + (kc+d)^3 + (kd+a)^3\}\\ &\qquad = 3k(k-1)\{ad(a-d) + dc(d-c) + cb(c-b) + ba(b-a)\}\\ &\qquad = 3k(k-1)(b-d)(c-a)(a-b+c-d).\end{aligned}$$

Of the cases which make the rule work, $k = 0$ or 1, $b = d$ and $c = a$ are trivially obvious. For example, if $b = d$ or $c = a$ then going round the square in opposite directions gives identical cubes. This leaves $a + c = b + d$.

There are, in fact, essentially 50 different squares of the type given by Bansal and Tyagi which work other than trivial ones, for example

$$\begin{matrix} 2 & 8 \\ 3 & 9 \end{matrix},$$

which yields

$$28^3 + 89^3 + 93^3 + 32^3 = 23^3 + 39^3 + 98^3 + 82^3 = 1\,564\,046.$$

Finally, it is possible to construct non-trivial squares which yield the same pattern with fourth powers. Repeating the above algebra, with index 4 instead of 3, yields

$$4k(k-1)(k+1)(b-d)(c-a)(a^2 - b^2 + c^2 - d^2 + ac - bd).$$

It is now simply a matter of selecting a, b, c, d such that $a^2 + ac + c^2 = b^2 + bd + d^2$. There are essentially only two possibilities if a, b, c, d are integers less than 10 (other than the trivial cases when at least one pair of a, b, c, d are equal), i.e.

$$\begin{matrix} 0 & 5 \\ 3 & 7 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 6 \\ 5 & 9 \end{matrix},$$

yielding

$$5^4 + 57^4 + 73^4 + 30^4 = 3^4 + 37^4 + 75^4 + 50^4$$

and

$$16^4 + 69^4 + 95^4 + 51^4 = 15^4 + 59^4 + 96^4 + 61^4.$$

Readers may like to see if they can generalise these results further.

Yours sincerely,

Alastair Summers
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Dear Editor,

Mathematics and morality

In his comments on *Mathematicians Under the Nazis* by Sanford Segal (*Mathematical Spectrum*, Volume 36, Number 3), David Sharpe asks 'Did the logical training (of mathematicians in Germany in the 1930s) help them to assess the situation they faced and to react to it?' According to the book reviewed the answers were 'no' and 'no'.

I don't find this at all surprising. A mathematical training only really helps you to resolve problems which have sound data and clear guidelines. In real life there is almost always insufficient or unreliable data and we have no time to mess about, any decision we make may well be better than none. The Nazi case is a bit extreme since your life was at risk. A more contemporary question is whether or not Saddam Hussein had weapons of mass destruction in Iraq. There was no way of knowing for sure at the time, so it was a matter of making a shrewd guess, something mathematicians don't get a lot of practice in (except perhaps in the area of fuzzy logic). Incidentally, Anatole Kaletsky, who writes on economics in *The Times*, during the autumn of 2002 assessed the probability of America going to war against Iraq using 'mathematical' arguments, and concluded that it was highly unlikely.

Of course, assessing a situation correctly has nothing to do with having the courage to do something to change it. Nonetheless, since the Enlightenment it has generally been assumed that 'rationalism' has a moral tinge to it, and so, since mathematics is the 'jewel in the crown' of rationalism, maybe there is something 'good' about a mathematical training after all. To me this argument doesn't stand up. In the nineteenth century, Otto von Bismarck, by a deliberate and brilliantly successful policy of bullying, lying and unprovoked war, made Germany the leading power in Europe. What, from a German point of view, was irrational about that? Even Hitler himself, or his government, pursued a well-conceived and very successful Keynesian economic policy; rationally speaking, Hitler only went off the rails when, blinded by emotion, he invaded Russia.

Yours sincerely,

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Dear Editor,

Sums of consecutive numbers

Seeing the article 'Sums of consecutive numbers', by P. Maynard and Y. Zhou (see Volume 36, Number 3, p. 60), it occurred to me that there might be a more direct proof than the one given, and indeed there is. To remind readers, the problem is to find the number of ways of expressing a positive integer n as the sum of consecutive positive integers. I will speak of such a sum as simply a 'sum' in what follows.

I set out to prove that for every *pair* of odd factors p, q of n such that $2^k pq = n$, there are exactly *two* sets of sums, except when $p = q$ when there is one. It follows that the number of sums is equal to the number of odd factors of n. For convenience, I am including the trivial case of a sum consisting of the single element n itself.

The number of elements in a sum, m, must satisfy $m|n$ (i.e. m divides n) if m is odd, or $(m/2)|n$ if m is even, as is clear from noting that

$$n = \frac{m(2a + m - 1)}{2}, \tag{1}$$

where a is the first term of the corresponding sum. In the case where n is even, if we try $m = 2^j p$, where p is an odd factor of n and $1 \leq j \leq k$, then $2a + m - 1$ is odd and 2^k does not divide $m/2$ so 2^k does not divide $n = m(2a + m - 1)/2$, which is a contradiction. So $m = p$, say, or $m = 2^{k+1}p$, where p is an odd factor of n. Note that, for even m, we require only $(m/2)|n$ not $m|n$, so $m = 2^{k+1}p$ is possible.

Conversely let's see when, for any given odd factor p of n, we may construct a sum with p or $2^{k+1}p$ elements. We must have $a > 0$, as we are only counting sums in which all the elements are positive. Let $n = 2^k pq$, where p and q are odd, then

1. if $m = p$, then $2^k q = a + (p-1)/2$ (see (1)), and so $a > 0 \iff 2^k q > (p-1)/2 \iff p < 2^{k+1}q + 1 \iff p < 2^{k+1}q$ since p is odd,
2. if $m = 2^{k+1}p$, then $q = 2a + 2^{k+1}p - 1$, and so $a > 0 \iff q > 2^{k+1}p$.

Note that p and q being odd ensures that a is an integer in both these cases. Now consider $m = q$ and $m = 2^{k+1}q$. In a similar way $m = q$ works if and only if $q < 2^{k+1}p$ and $m = 2^{k+1}q$ works if and only if $p > 2^{k+1}q$.

Combining these results, we have exactly *two* of $m = p, m = 2^{k+1}p, m = q, m = 2^{k+1}q$ working, since exactly two of the restrictions $p < 2^{k+1}q, q > 2^{k+1}p, q < 2^{k+1}p, p > 2^{k+1}q$ must be true, recalling that p and q are odd. The only slight flaw occurs if $p = q$, in which case we note that $p < 2^{k+1}p$ so there is correspondingly exactly one sum with p elements. The proof is complete as we have a 2–2 correspondence between pairs of odd factors of n and sums (and a 1–1 correspondence for p when $n = 2^k p^2$).

If we now exclude the trivial case in which $m = 1$, it follows that the number of sums is given, as stated by Maynard and Zhou, by $f(n) = \prod(\alpha_i + 1) - 1$, where $n = 2^k \prod p_i^{\alpha_i}$, where p_i are the odd prime factors of n. Here I have used the well-known formula for the number of factors of an integer, and applied it to $\prod p_i^{\alpha}$.

I now illustrate how this gives all the sums for $n = 156$, i.e. $2^2 \times 3 \times 13$. For $p = 3$, $q = 13$, we obtain $p < 8q$, so $m = 3$ works. Now $n = m(2a + m - 1)/2$ implies that $a = (8q - (m-1))/2 = 51$. Also $q < 8p$, so $m = 13$ works and $a = (8p - (m-1))/2 = 6$. For $p = 1, q = 39$ we exclude $m = 1$. Now $q > 8p$, so $m = 8p = 8$ works and $a = (q - (m-1))/2 = 16$. The corresponding sums are $51 + 52 + 53, 6 + 7 + 8 + \cdots + 18$ and $16 + 17 + \cdots + 23$.

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

37.5 Prove Brahmagupta's formula for the area Δ of a cyclic quadrilateral with sides a, b, c, d, namely

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where

$$s = \tfrac{1}{2}(a+b+c+d).$$

(See the article by Zhang Yun in this issue, pp. 57–60)

37.6 Let $x_1, \dots, x_n$ be real numbers such that

$$0 < x_1 \le x_2 \le \cdots \le x_n.$$

Prove that

$$\cos^{-1}\frac{x_1}{x_n} \le \cos^{-1}\frac{x_1}{x_2} + \cos^{-1}\frac{x_2}{x_3} + \cdots + \cos^{-1}\frac{x_{n-1}}{x_n}.$$

(Submitted by Mihály Bencze, Săcele-Négyfalu, Romania)

37.7 Let x be a real number. Determine

$$\lim_{n\to\infty}\frac{[x]+[2x]+\cdots+[nx]}{n^2},$$

where $[\cdot]$ denotes the integer-part function.

(Submitted by Anand Kumar, Ramanujan School of Mathematics, Patna, India)

37.8 A set of $n \ge 6$ consecutive integers is partitioned into two subsets A and B, both with at least three elements. Prove that there exist a_1, a_2 in A and b_1, b_2 in B such that $a_1 - a_2 = b_1 - b_2 \ne 0$.

(Submitted by H. A. Shah Ali, Tehran, Iran)

Solutions to Problems in Volume 36 Number 3

36.9 Let a, b, c, d be positive real numbers such that $a+b \geq c+d$ and $ab \leq cd$. Prove that $a^m + b^m \geq c^m + d^m$ for all integers m.

Solution by H. A. Shah Ali, who proposed the problem

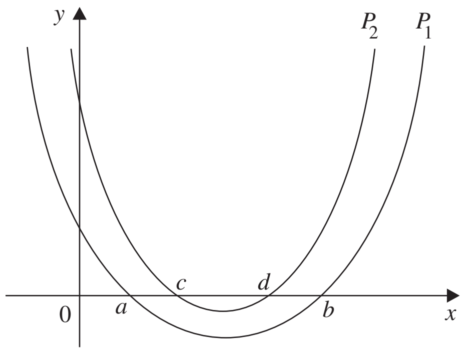
Without loss of generality, we can assume that $a \leq b$ and $c \leq d$. Put

$$P_1(x) = x^2 - (a+b)x + ab,$$
$$P_2(x) = x^2 - (c+d)x + cd,$$

so that a, b are the zeros of P_1 and c, d are the zeros of P_2. For each $x \geq 0$,

$$P_1(x) \leq P_2(x),$$

so the graphs of P_1 and P_2 are related as shown in the figure. Hence $a \leq c \leq d \leq b$.



From $a+b \geq c+d$ we have

$$b - d \geq c - a \geq 0. \tag{1}$$

Also, for a positive integer m,

$$\sum_{k=1}^{m} b^{m-k} d^{k-1} \geq \sum_{k=1}^{m} c^{m-k} a^{k-1} \geq 0, \tag{2}$$

since $b \geq c$ and $d \geq a$. If we multiply in equalities (1) and (2), we obtain

$$b^m - d^m \geq c^m - a^m,$$

whence

$$a^m + b^m \geq c^m + d^m. \tag{3}$$

From $ab \leq cd$, we have

$$a^{-m} b^{-m} \geq c^{-m} d^{-m}. \tag{4}$$

If we multiply inequalities (3) and (4), we obtain

$$a^{-m} + b^{-m} \geq c^{-m} + d^{-m}.$$

This deals with negative powers; the case $m = 0$ is clear.

36.10 When n people each throw a pair of dice, what is the probability that they all get the same total?

Solution by Anand Kumar, who proposed the problem

The sample space is of size 6^{2n}. The number of events where the total is 2 is 1^n, with total 3 is 2^n, up to total 7 is 6^n, then total 7 is 5^n, total 8 is 4^n, up to total 12 is 1^n. Hence the required probability is

$$\frac{2(1^n + 2^n + 3^n + 4^n + 5^n) + 6^n}{6^{2n}}.$$

36.11 The point P moves in a plane so that the tangents through P to a given ellipse are at right angles. Determine the locus of P. If these tangents touch the ellipse at M, N, find the envelope of MN.

Solution

Choose axes so that the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and let P have coordinates (α, β). If a tangent to the ellipse through P touches the ellipse at $\mathrm{M}(x_1, y_1)$, then it has equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$$

and

$$\frac{\alpha x_1}{a^2} + \frac{\beta y_1}{b^2} = 1, \qquad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

Eliminate y_1 to give

$$x_1^2(a^2\beta^2 + b^2\alpha^2) - 2\alpha a^2 b^2 x_1 + a^4(b^2 - \beta^2) = 0. \tag{5}$$

Similarly

$$y_1^2(a^2\beta^2 + b^2\alpha^2) - 2\beta a^2 b^2 y_1 + b^4(a^2 - \alpha^2) = 0. \tag{6}$$

If the other tangent through P touches the ellipse at $\mathrm{N}(x_2, y_2)$, then x_2, y_2 satisfy the same equations as x_1, y_1. The tangents through P are perpendicular if and only if

$$\frac{y_1 - \beta}{x_1 - \alpha}\,\frac{y_2 - \beta}{x_2 - \alpha} = -1,$$

which gives

$$\alpha^2 + \beta^2 - \alpha(x_1 + x_2) - \beta(y_1 + y_2) + x_1x_2 + y_1y_2 = 0. \tag{7}$$

Equation (5) gives

$$x_1 + x_2 = \frac{2\alpha a^2 b^2}{a^2\beta^2 + b^2\alpha^2}, \qquad x_1x_2 = \frac{a^4(b^2 - \beta^2)}{a^2\beta^2 + b^2\alpha^2},$$

and similarly for y_1, y_2 from (6). If we substitute these equations for x_1, x_2, y_1, y_2 into (7) and simplify, we obtain

$$(\alpha^2 + \beta^2 - a^2 - b^2)(a^2\beta^2 + b^2\alpha^2 - a^2b^2) = 0.$$

But

$$a^2\beta^2 + b^2\alpha^2 = a^2b^2 \iff \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1,$$

i.e. if and only if P lies on the ellipse. Hence

$$\alpha^2 + \beta^2 = a^2 + b^2, \tag{8}$$

so the locus of P is the circle

$$x^2 + y^2 = a^2 + b^2.$$

The equation of MN is

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1, \tag{9}$$

since $(x_1, y_1), (x_2, y_2)$ both satisfy this equation. The ellipse with equation

$$\frac{x^2}{\mathrm{A}^2} + \frac{y^2}{\mathrm{B}^2} = 1$$

has tangent at $(\mathrm{X}_1, \mathrm{Y}_1)$ given by

$$\frac{x\mathrm{X}_1}{\mathrm{A}^2} + \frac{y\mathrm{Y}_1}{\mathrm{B}^2} = 1, \tag{10}$$

where

$$\frac{\mathrm{X}_1^2}{\mathrm{A}^2} + \frac{\mathrm{Y}_1^2}{\mathrm{B}^2} = 1. \tag{11}$$

If we compare (10) with (9) and (11) with (8), then

$$\frac{\mathrm{X}_1}{\mathrm{A}^2} = \frac{\alpha}{a^2}, \qquad \frac{\mathrm{Y}_1}{\mathrm{B}^2} = \frac{\beta}{b^2},$$
$$\frac{\mathrm{X}_1^2}{\mathrm{A}^2} = \frac{\alpha^2}{a^2 + b^2}, \qquad \frac{\mathrm{Y}_1^2}{\mathrm{B}^2} = \frac{\beta^2}{a^2 + b^2}.$$

This gives

$$\mathrm{A} = \frac{\mathrm{X}_1}{A} \div \frac{\mathrm{X}_1}{A^2} = \frac{a^2}{\sqrt{a^2 + b^2}},$$

and

$$X_1 = \frac{\alpha a^2}{a^2 + \beta^2}.$$

Similarly,

$$B = \frac{b^2}{\sqrt{a^2 + b^2}}, \qquad Y_1 = \frac{\beta b^2}{a^2 + b^2}.$$

Thus the ellipse

$$\frac{x^2(a^2 + b^2)}{a^4} + \frac{y^2(a^2 + b^2)}{b^4} = 1$$

has the point $(\alpha a^2/(a^2 + b^2), \beta b^2/(a^2 + b^2))$ on it, and its tangent at this point is the line MN. Hence MN envelopes this ellipse.

36.12 What is

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ\ ?$$

Solution (a) by Jonathan Smith, Gresham's School, Holt

The nth roots of unity are $e^{2\pi i r/n}$, for $r = 0, 1, \ldots, n-1$. Hence the n roots of the equation $z^n = e^{i\alpha n}$ are given by

$$z = e^{i(\alpha + 2\pi r/n)},$$

for $r = 0, 1, \ldots, n-1$. Since the coefficient of z^{n-1} (with $n > 1$) in the equation is 0, the sum of these roots is zero, i.e.

$$\sum_{r=0}^{n-1} e^{i(\alpha + 2\pi r/n)} = 0.$$

If we equate real parts, we obtain

$$\sum_{r=0}^{n-1} \cos\left(\alpha + \frac{2\pi r}{n}\right) = 0$$

in radians, or

$$\sum_{r=0}^{n-1} \cos\left(\alpha + \frac{360r}{n}\right)^\circ = 0$$

in degrees. If we put $n = 5$ and $\alpha = 5$, we obtain the required result.

Solution (b) by Alice Chen, Roedean School

Denote the expression by R. Then

$$\begin{aligned} R \sin 72^\circ &= \sin 72^\circ \cos 5^\circ + \sin 72^\circ \cos 77^\circ + \sin 72^\circ \cos 149^\circ \\ &\quad + \sin 72^\circ \cos 221^\circ + \sin 72^\circ \cos 293^\circ \\ &= \tfrac{1}{2}[(\sin 77^\circ + \sin 67^\circ) + (\sin 149^\circ - \sin 5^\circ) \\ &\qquad + (\sin 221^\circ - \sin 77^\circ) + (\sin 293^\circ - \sin 149^\circ) + (\sin 365^\circ - \sin 221^\circ)] \\ &= 0. \end{aligned}$$

Also solved by Siu Long Au, Diocesan Boys' School, Kowloon, Hong Kong.

Reviews

Mathematical Diamonds. By Ross Honsberger. MAA, Washington, DC, 2003. Paperback, 256 pages, \$35.95 (ISBN 0-88385-332-9).

What would you call a sweet maths problem that combines simplicity of statement with elegance of solution? Ross Honsberger has given such a problem various names down the years: a gem, a plum, a morsel, a chestnut, and now in his latest book, a diamond. These pages contain a terrific collection of problems that are great fun to attempt, and hopefully to solve too, although some of the answers require a flash of something special. Usually stated in a single sentence, they have solutions that can be either brief or quite lengthy, but throughout there is a concern for clarity in exposition that is admirable. I would say that the standard is Further Mathematics level and a little above, by and large, but that is not to say there is nothing here for other A-level students to enjoy too. Let me give you a few tasters:

‘One day a restless lion roamed about in his circular cage of radius ten metres along a polygonal path of total length 30 km. Prove that he must have turned through a total angle of at least 2998 radians.’

‘What is the area of the largest hexagon H that can be inscribed in a unit square S?’

‘Can you find a way of arranging the natural numbers into a sequence a_1, a_2, a_3, a_4, ... such that each natural number appears once and only once, and so that the sum of the first k terms is always divisible by k?’

The author starts by saying ‘God must love mathematicians — he has given them so much to enjoy…’ and indeed, if enjoyment is a yardstick, his book ably provides yet further evidence of God’s existence.

Paston College, Norfolk **Jonny Griffiths**

Mathematical Evolutions. Edited by Abe Shenitzer and John Stillwell. MAA, Washington, DC, 2001. Paperback, 304 pages, \$35.00 (ISBN 0-88385-536-4).

This book gathers together a collection of essays and lecture transcripts on the history and philosophy of maths. It aims to give an overview of what maths is, where it comes from, and where it is going.

The material is made accessible by its division into short, roughly ten-page chunks, which survey a part of mathematics and assess its development and the people that shaped it. Topics covered range from analysis and algebra to the logical foundations of maths and its applications to probability. Many fascinating stories are well told. For instance, it seems remarkable that our concept of a ‘function’ had to be ‘discovered’. But in fact debate lingered on into the 20th century about what exactly a function should be, and many great mathematicians had very different ideas on the subject. Other highlights include a tale of one future mathematician’s arguments with his lecturer on the paradoxes of the infinite and infinitesimals. We get some insight into the personalities behind some dry issues of mathematical history.

This is not a ‘popular science’ book, as it surveys areas of serious mathematical research. Definitions and concepts are explained well throughout. Sixth form students, if they avoid getting bogged down in the details, will see some ‘real maths’ in action. Some parts deal with graduate topics, but these are balanced by broader and more approachable discussions. For the teacher or lecturer, this is a well-written survey, with sharp insights into the status of logic and the way analysis should be taught, amongst other things.

This is a treasure-trove of a book for those interested in the human and historical side of maths.

Student, Queens' College, Cambridge **Will Donovan**

Dissections: Plane & Fancy. By Greg N. Frederickson. Cambridge University Press, 2003. Paperback, xi+310 pages, £16.95 (ISBN 0-521-52582-9).

This book, which was originally published as a hardback in 1997, is concerned with both tessellations and dissections. The latter are basically the division of one or more plane figures in such a way that they can be reassembled in a different form. Many readers will be familiar with simple dissections for the demonstration of the identity

$$a^2 - b^2 = (a - b)(a + b)$$

and Pythagoras theorem. Tessellations are different ways of covering some plane area by the repeated use of one or more standard shapes.

This type of activity is an important part of recreational mathematics, and gives an interesting variation from the more usual attractions of pure and applied maths, statistics and numerics. Its development over many centuries and in many different countries is very well illustrated by the inclusion of short biographies of some of the people responsible. These notes appear regularly throughout the book. Besides mathematicians and engineers, numerous people without an academic background are included. Many of their ideas have been popular as puzzles in newspapers and magazines.

While considerable attention is paid to polygons, stars and crosses, there are also sections on curved figures and three-dimensional dissections. The printing is of a high standard and the numerous diagrams are very clearly presented. Some of these are easy to understand, but many of them need close attention. The notation and various trigonometric statements will intrigue some readers and others may be encouraged to try their hand at model making. Judging by the attractive front cover, some of these models could be used as very elegant Christmas decorations. There are problems to solve throughout the book with illustrated answers given at the back for anyone who becomes frustrated after a reasonable attempt. Besides the normal index there is a second index for the individual dissections. For those readers who become captivated by recreational mathematics, there is an extensive bibliography of nine pages.

This is an interesting book, which should be a good source of project work. It is a valuable addition for any library in school, college or home.

Stourbridge, West Midlands **Robert J. Clarke**

101 Careers in Mathematics. Edited by Andrew Sterrett. MAA, Washington, DC, 2003. Paperback, 360 pages, £28.50 (ISBN 0-88385-728-6).

This book is a collection (139 to be precise) of short career-to-date summaries written by American maths graduates. Of the contributors, 103 have gone on to take further postgraduate qualifications, including 25 who have Masters degrees in maths and 33 who have PhDs. Each summary is printed over a double page but half of the space is taken up with titles and a photograph. The latter is presumably there to reassure readers that maths graduates look normal (they do).

The publication is meant as a resource for high school students who are considering taking a maths degree. By dipping into the pages they can quickly appreciate that maths graduates find employment in a great variety of challenging and rewarding occupations. The message comes over loud and clear that 'the world needs maths graduates'. The subjects have been carefully chosen to cover different ages, careers and companies, and a helpful contents page allows you to pick out those of interest.

Clearly the Mathematical Association of America has a vested interest in attracting students into maths degree courses, so all of the summaries are of successful people with nothing but positive things to say about their choice of maths. Some are clearly keen to blow their own trumpet. This rich diet can become a little sickening after a while.

All of the subjects are American and so are the courses, companies and universities referred to. As a result it is a little difficult to fully understand some of the summaries. Nevertheless, the contributors clearly explain why their mathematical backgrounds are useful in their chosen job, and what maths, if any, they make use of.

This is not a book which you would wish to read from cover to cover, but it would be a useful addition to school libraries. Those unsure about whether a maths degree is the right choice for them would do well to refer to it.

Student, Stamford School **W. C. Chadwick**

777 Mathematical Conversation Starters. By John dePillis. MAA, Washington, DC, 2002. Paperback, 344 pages, $37.95 (ISBN 0-88385-540-2).

This unusual book is conceived by its author as providing an antidote for anyone who regards mathematics as a conversation-stopping disease. It immediately put me in mind of the apocryphal mathematician who ruefully observed, 'When I tell people at parties that I am a mathematician, they tend to shun me, but when I tell them that I work in analysis, I have a much more interesting time!'. DePillis has ranged far and wide in assembling his collection of 777 provocative pieces. In character, these vary from quotations and anecdotes to gobbets of mathematics; in tone, they flit from serious exposition to light-hearted verse and cartoons. The latter (and some of the longer pieces on logic and modern physics) are the author's own interpolations, and the nature of the book (which is arranged alphabetically by themes) makes it one to dip into rather than read from cover to cover. A selection of headings from the book conveys its flavour: abstraction, algebra, axiom of choice, certainty, deduction v. induction, elegance, four-colour problem, frustration and pleasure, Gilbert and Sullivan parodies, infinity, intuition (examples that challenge …), 'mathematics is …', 'mathematicians are …', Newton, probability, progress, QED made easy (fallacies!), symbols.

Thus, for example, I was very happy to learn:

- What sentence Bertrand Russell wrote before he wrote '… mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true'.
- What Gauss wrote after 'Mathematics is the queen of the sciences …'.
- Tom Banchoff's anecdote of the herbs' distributor who used a vector space to keep track of orders, with one coordinate for parsley, one for sage, one for rosemary and one for … oregano (because the 4th dimension isn't necessarily thyme!).
- That at least one of $\pi + \mathrm{e}$ and $\pi\mathrm{e}$ must be transcendental (or else both e and π would be algebraic as roots of the quadratic equation $x^2 - (\pi + \mathrm{e})x + \pi\mathrm{e} = 0$).

- ' "Obvious" is the most dangerous word in mathematics' (E. T. Bell).
- Anagrams: A decimal point = I'm a dot in place; eleven plus two = twelve plus one.

Whether the author has succeeded in casting a sufficiently tempting bait to lure non-mathematicians, I hesitate to judge, but there is plenty here for the mathematically inclined reader to savour and enjoy.

Tonbridge School, Kent **Nick Lord**

Environmental Mathematics in the Classroom. Edited by B. A. Fusaro and P. C. Kenschaft. MAA, Washington, DC, 2003. Paperback, 268 pages, $49.95 (ISBN 0-88385-714-6).

Mathematics is an international language with its own special grammar and vocabulary. Sadly, however, most people fail to become even remotely fluent in it and are terrified by this apparently abstract and esoteric subject, which seems to have no relevance to their daily lives. Almost certainly this is a result of uninspiring, unimaginative and purely mechanical teaching in schools.

This little volume addresses this problem and attempts to make a contribution to the popularisation of mathematics by showing how relatively simple procedures can contribute to the analysis and understanding of environmental problems. This is not to suggest that the book is directed solely at environmental scientists. It has much to offer other specialists such as biologists and economists, and would be extremely useful in enhancing the mathematical understanding of senior school students and undergraduates.

The book comprises 14 chapters, each by a different author, and each addressing a particular issue such as population growth, oil spills, air quality, or weather prediction. Of particular interest to the reviewer was the simple model of an epidemic, which is clearly very relevant given popular misconceptions surrounding the spread of AIDS, SARS, influenza etc.

The style is relaxed and very user-friendly, and exercises are offered at the end of each chapter. The editors have deliberately chosen to exclude any reference to calculus; this is perhaps a slight deficiency which they may care to address in any subsequent edition.

Glasgow Caledonian University **John C. B. Cooper**

Teaching First, a Guide for New Mathematicians. By Thomas W. Rishel. MAA, Washington, DC, 2000. Paperback, 150 pages, $19.00 (ISBN 0-88385-165-2).

Way back in the mists of time, further than I care to remember, I was about to start my final year as a research student when my head of department gave me a syllabus and a box of chalk and said 'Go and take this class'. It was an algebra class for non-specialist students new to the university. A more practical applied mathematician gave me two pieces of advice: (1) keep your notes on the desk, (2) propriety forbids me to mention the second! I don't know who was more terrified, the audience or the instructor. After the first lecture, a Miss Lockett (I cannot recall her first name) went in tears to the head of department's office to tell him that she didn't think she was going to be able to do pure mathematics. Fortunately, she was waylaid by the secretary before she could destroy my career before it had begun. She went on to get a high mark in the end of term exam!

Things have changed in universities since those halcyon days. Now there are induction courses and mentors and evaluation sheets. This book, by an experienced teacher, is written for Teaching Assistants in the US. It is very much geared to the US situation, but nevertheless has wisdom for everyone who is about to step onto the first rung of the teaching ladder. In fact, it can be read, with profit, by more experienced teachers. The question is: can you teach teaching?

University of Sheffield **David Sharpe**

A Friendly Mathematics Competition: Thirty-five Years of Teamwork in Indiana. Edited by Rick Gillman. MAA, Washington, DC, 2002. Paperback, 196 pages, $29.95 (ISBN 0-88385-808-8).

This book is essentially an outline of the Indiana College Mathematics Competition. In essence, each college enters a team of three students, who work together on the test, which contains approximately seven questions, for two hours. The contest was formed as a reaction to the Putnam Exam, so its problems are supposed to be somewhat easier, and more representative of the undergraduate curriculum.

The book contains the test papers from 1966 to 2000, along with a solution to each problem and an index of the problems, sorted by topic. The problems themselves are diverse in topic, containing subjects such as number theory, trigonometry, and inequalities. Although topics such as field theory, Riemann sums, and analytic geometry are unfamiliar to most school students, the majority of problems and their solutions can be understood by someone with a good knowledge of A-level mathematics.

A disappointing aspect of the book is the lack of multiple solutions — only one solution is given to each problem, yet others do exist. Hence, I feel that justice is not done to the elegance of some problems. Furthermore, I feel that it would be instructive to the student if different solutions were shown. Nevertheless, the book is a useful resource of problems, and the questions provide a fresh look at some ideas in mathematics. Indeed, some of the questions might be used as stimulus material for the brighter A-level students by their teachers. Thus, whilst this book does not fully do justice to the problems, it would certainly be of use to anyone interested in problem solving.

Student, Berkhamsted Collegiate School **Paul Jefferys**

Teaching Statistics Using Baseball. By James Albert. MAA, Washington, DC, 2003. Paperback, 304 pages, $45.00 (ISBN 0-88385-727-8).

This is an introductory textbook designed for students with little prior knowledge about the discipline of statistics and who the author feels are likely to have anxieties about mathematics and computation. Opening with simple descriptive statistics for a single data set, the book progresses through relationships between variables, probability distributions, statistical inference and Markov chains, each concept illustrated by the use of a set of data that is, unsurprisingly, baseball related.

I heartily endorse the author's view that the student has a much better chance of understanding concepts in probability and statistics if they are described in a familiar context. But, despite reading the excellent chapter at the end of the book on an introduction to baseball, this remains an unfamiliar situation and, for me, the book does not work.

However, I am sure that the book would be of interest to the average baseball enthusiast. Certainly the author is very evidently fascinated with the topic and has experience of successful delivery of course material in this context. But if the name of Rickey Henderson means nothing to you, and stolen bases, slugging percentages, strikeouts and at-bats offer no more meaning than their face value, then this book is probably not for you. I found myself struggling with the concepts of the game, rather than the concepts of the subject.

If the book had been translated to the context of David Beckham's goal scoring rate, both from the spot and as a result of a free kick through the wall, with records of times between these magic moments as well as crowd size when they happened, that would have been very different! Perhaps there is a book waiting to be written here which would do as much for the advancement of statistical learning on this side of the Atlantic as Albert's probably does on the other.

Solihull Sixth Form College **Carol Nixon**

The Oxford Dictionary of Statistical Terms. Edited by Yadolah Dodge. Oxford University Press, 2003. Hardback, 498 pages, £25.00 (ISBN 0-19-850994-4).

This dictionary is the sixth edition of the original 1957 work commissioned by the International Statistical Institute (ISI). The Editorial Board includes the Editor Yadolah Dodge, assisted by Sir David Cox, Daniel Commenges, Anthony Davison, Patty Solomon and Susan Wilson, and a further advisory board of five well-known statisticians. The result of their efforts is a comprehensive statistical dictionary of 438 pages, with a reference list of 60 pages.

Rather than attempt a systematic review of the entries, six definitions from throughout the book are selected for discussion.

Demography: 'A broad social science discipline concerned with the study of human populations.' This brief definition is followed by ten lines giving supplementary details of this area of study.

GLIM: 'Acronym for Generalized Linear Interactive Modelling.... GLIM is widely respected by statisticians but its use has been largely replaced by larger mainstream statistical packages and systems.'

Linkage analysis: 'This uses statistical models to infer whether a disease gene is linked to given genetic markers, namely on the same chromosome.' A further five lines of explanation follow.

Order statistics: 'When a sample of variable values is arrayed in ascending order of magnitude these ordered values are known as order statistics.' Four more lines expand on this definition.

Statistical literacy: 'The ability to understand and critically evaluate statistical results coupled with the ability to appreciate the contributions that statistical thinking can make in public and private professional and personal decisions.'

Weighted regression: 'Any form of regression analysis in which values for different individuals are given different weights....'

These definitions are precise and, as can be seen, are expressed in clear and simple language; they should prove helpful to laymen and students, as well as statisticians. The Editorial Board is to be congratulated on its achievement in providing this sixth edition of the ISI's dictionary. The book deserves to find its way onto the shelves of every university library and of many personal libraries.

Australian National University, Canberra **Joe Gani**

Exploratory Examples for Real Analysis. By Joanne E. Snow and Kirk Weller. MAA, Washington, DC, 2003. Paperback, 100 pages, £30.95 (ISBN 0-88385-734-0).

From the cover: 'This text supplement contains 12 exploratory exercises designed to facilitate students' understanding of the most elemental concepts encountered in a first real analysis course: notions of boundedness, supremum/infimum, sequences, continuity and limits, limit suprema/infima, and pointwise and uniform convergence. In designing the exercises, the authors ask students to formulate definitions, make connections between different concepts, derive conjectures, or complete a sequence of guided tasks designed to facilitate concept acquisition. Each exercise has three basic components: making observations and generating ideas from hands-on work with examples, thinking critically about the examples, and answering additional questions for reflection'.

Mathematical Miniatures. By Svetoslav Savchev and Titu Andreescu. MAA, Washington, DC, 2002. Paperback, 230 pages, $28.50 (ISBN 0-88385-645-X).

The authors met in 1993 at an International Mathematics Olympiad, where they swiftly conceived the idea for this excellent book as a collection of fresh and non-standard problem-solving techniques that would be aesthetically pleasing at the same time. The result, clearly a labour of love, is arranged into groups of five short chapters, each bringing together three or four problems under a common theme. You will find here off-beat circle theorems, unusual inequalities, quirky logic puzzles and much more besides. Often there is some interesting historical footnote to the central problem, and often too there is humour in the way the problem has been posed. The chapters are interspersed with 'Coffee Breaks', that consist of a few simpler stand-alone problems. The questions themselves come from olympiads and journals for the most part, and are excellent preparation for a student soon to enter an olympiad, or for an able mathematician who would like a challenge. The analogy with a rich mathematical tool chest is tempting: if you are looking for a new chisel or rasp, then this book is a good place to look. But the authors prefer 'an anthology of mathematical verse', and there is certainly a strong sense of the poetry of good mathematics here.

Paston College, Norfolk **Jonny Griffiths**

Oval Track and Other Permutation Puzzles. By John O. Kiltinen. MAA, Washington, DC, 2003. Paperback, 304 pages, $42.50 (ISBN 0-88385-725-1).

Many readers will be familiar with the Rubik cube, and its family of other 'permutation puzzles', where the object is to restore harmony to a muddled pattern. Arguably, there are so many possible arrangements of the pieces of these puzzles that most of them have never been seen by a human. However, all of these puzzles have a common theme: it is easy to almost solve them. But then, to fit the final few pieces into position requires you to disturb some others, and the puzzle becomes more difficult than it looks. A combination of human intelligence and patience can solve the Rubik cube, but solve the 'numbered Hungarian rings'? Not a chance.

They can all be solved easily, however, by the systematic application of group theory. That is what this book is about.

It is difficult for an author to convey ideas about permutations in a compact form on paper; they normally resort to many almost identical step-by-step diagrams. Kiltinen neatly sidesteps this by including with his book a CD-ROM with simulations of all the puzzles he discusses. He also develops and uses various easy-to-follow notation.

So this is actually a cross between a software manual, a textbook, and a recreational mathematics book, written in Kiltinen's hands-on style. He imagines you interspersing sessions with the book with time spent solving things for yourself using the CD-ROM. He includes many ideas for further investigation, for those who want more. The necessary group theory is there when you need it, and he demonstrates abstract concepts in a clear way, usually involving duct tape and flexible plastic tubing.

Most of the material here is accessible to A-level students, although it was written for undergraduates or above. Familiarity with basic group theory would be a big help in tackling this book.

Student, Gresham's School **Jonathan Smith**

Proof in Mathematics ('If', 'Then' and 'Perhaps'). A Collection of Material Illustrating the Nature and Variety of the Idea of Proof in Mathematics. Edited by P. R. Baxandall, W. S. Brown, G. St C. Rose and F. R. Watson. Keele Mathematical Education Publications, 2002. 121 pages, £9.99.

This booklet about proof and the language of mathematics is aimed at teachers, sixth-form students and first-year undergraduates. It provides a useful introduction to the idea of proof as well as teaching specific techniques, such as induction. It could provide much needed help to mathematics undergraduates in developing precision of thought and language skills.

The main part of the booklet consists of seven chapters covering reasoning, implication, reading and writing proofs and various strategies of proof. The need for precision and proof is well illustrated, motivated by plausible conjectures that turn out to be false. Helpful analogies are used, including the comparison of induction to a cascade of dominoes. There are examples and exercises in each chapter. Hints and solutions are provided; these are quite useful, although sometimes rather brief. There is some really good practical advice, such as the recommendation to read work aloud. Many nice examples are given and there is useful analysis of mistakes and false 'proofs'. Much of the main text should be accessible to sixth formers and would be useful for students in transition to university and for first-year undergraduates.

There are ten appendices, some containing considerably more advanced material. This includes tastes of non-Euclidean geometry, vector spaces, Cantor's diagonal argument and mention of the four-colour theorem. There is much here to stretch the stronger student and to stimulate enthusiasm for further study. The inclusion of a case study (see appendix 8) is unusual and welcome. It may be very useful for students to see how someone else approached a problem, complete with all the mistakes, false leads and partial results that are part of the mathematical experience.

Any book devoted to proof and the language of mathematics must try to lead by example and to adhere to very high standards of clarity. This booklet is generally nicely written, with occasional lapses (A5.7, for example). The typesetting is rather irritating, however, with subscripts, superscripts and labelling of diagrams unclear in places. There are quite a few typographical errors. For the most part, this is a minor irritation, but in appendix 6 it is a serious impediment, as there almost all subscripts and superscripts have been mistyped.

It is explained in the preface that this is a reissue of a document first produced in 1978 and made available to readers at very little cost. If resources could be found to typeset it to modern publication standards, that would be an excellent thing.

University of Sheffield **Sarah Whitehouse**

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