# Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

# MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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# Pierre-Simon de Laplace: 1749–1827

ROGER COOK

## Introduction

Laplace is frequently termed the Isaac Newton of France. Isaac Newton was a great seventeenth century mathematician whose laws of motion determined the mechanics of a particle. In the eighteenth century it was one of Laplace's main ambitions to apply the Newtonian laws to the entire solar system. One major difference between seventeenth and eighteenth century mathematics is that the emphasis in mechanics had moved from geometry to analysis.

His name is also associated with other major mathematical achievements: Laplace's equation is of fundamental importance in fluid mechanics and electromagnetism, Laplace transforms have many applications including the solution of differential equations (see references 4 and 6), and he was one of the first mathematicians to study probability.

He was born at Beaumont-en-Auge, near Trouville in Normandy, on 23 or 28 March 1749 (authors, even different editions of the same encyclopedia (reference 3), differ on the exact date). His parents were peasant farmers and it was unusual at that time for the children of the poor to receive more than the most rudimentary education. However he showed such ability at the village school that wealthy neighbours sponsored him to continue his education at the military academy in Beaumont-en-Auge. By the time he left, at the age of eighteen, he was teaching the mathematics courses at the school.

To develop his mathematical talents fully he then went to Paris. Influential neighbours wrote him letters of introduction, which was the custom at that time. On arriving in Paris, Laplace called on Jean d'Alembert (1717–83) with these introductions, hoping to meet him.

D'Alembert was the permanent secretary of the Academy of Sciences in Paris. One of the leading French mathematicians, he was the first to give a complete solution to the problem of the precession of the equinoxes, made important contributions in the field of partial differential equations and was the originator of the 'ratio test' for the convergence of series. He was the illegitimate son of the Chevalier Destouches and was abandoned by his mother, the Marquise de Tencin (the sister of a cardinal), on the steps of a church. His father arranged for him to be brought up in fairly humble origins by step-parents, but did ensure that he received a good education. He was unimpressed by letters of introduction from 'influential' people and did not receive Laplace.

Fortunately Laplace recognised the problem, possibly because of his own humble origins, and realized that d'Alembert was more likely to be impressed by mathematics than by letters of introduction, and sent him a letter on the principles of mechanics. D'Alembert replied

> Sir, you see that I paid little enough attention to your recommendations: you do not need any. You have introduced yourself better. That is enough for me; my support is your due.

True to his word, D'Alembert used his influence to secure Laplace a post as a professor at the Ecole Militaire, Paris.

## Celestial mechanics

Laplace now began his systematic attempt to describe the mechanics of the Solar System in terms of Newtonian mechanics. There were anomalies in the orbits of the planets Jupiter and Saturn that were not well understood, despite the efforts of Newton, Euler and Lagrange. This raised the question of whether the Solar System was stable. Can we be certain that one day the Earth will not fall into the Sun, or that the Moon will not fall on the Earth?

The problems of trying to predict the motions of the planets when taking into account all their gravitational interactions is still difficult, even with modern computing power.

In 1773, at the age of 24, Laplace proved that the average distances of the planets from the Sun are essentially constant, with minor periodic variations. However his analysis is valid only for an idealized model of the Solar System! The analysis of the motions of the planets became the centre of his life's work. Between 1784 and 1787 Laplace made remarkable discoveries. In particular he showed that the anomalies in the orbits of Jupiter and Saturn arise because their mean distances from the Sun are very nearly in a rational ratio—approximately $\frac{6}{11}$, providing an early demonstration of the connection between dynamical systems and Diophantine approximation.

By this stage the general workings of the Solar System were understood and Laplace's career entered a second stage, where he sought to present the results obtained by three generations of mathematicians and astronomers from a single point of view. This culminated in his major work *Celestial Mechanics* which was published in 5 volumes. The first two volumes appeared in 1799, two more in 1802 and 1805 with the final volume not appearing until 1825. It was intended to be a complete theoretical solution to the mechanical problem presented by the Solar System. A more popular account *Exposition of the System of the World*, leaving out most of the mathematics, appeared in 1796. Indeed, some authors feel that his frequent use of the phrase *Il est aisé à voir* indicate that much of the mathematics had also been left out of *Celestial Mechanics*. J. B. Biot (1774–1862), who assisted Laplace with the proof reading of his work, commented that on occasion he had seen the author himself struggle for over an hour to reconstruct a link in his chain of reasoning. Laplace also put forward the *nebular hypothesis*, that the Solar System condensed out of a cloud of gas and dust surrounding the Sun, apparently unaware that this idea had been suggested by Kant in 1755.

Although the central problem of *Celestial Mechanics*, trying to establish that the Solar System is some sort of perpetual motion machine, is now a mathematical backwater, the methods developed still remain central. Perhaps the most fundamental is the concept of *potential*, which Laplace appropriated from Lagrange. Potential remains an essential idea in fluid mechanics, heat, gravitation and electromagnetism; the equation which it satisfies is called *Laplace's equation*. The third volume contains a paper presented to the French Academy in 1784 which introduces spherical harmonics; this was developed from an idea of Legendre who treated the two-dimensional case. Rouse Ball (reference 7, p. 413) comments 'Legendre had good reason to complain of the way in which he was treated in this matter.'

The article in *Encyclopedia Britannica* (reference 5) quotes from the *Journal des Savants* (1850):

> In the delicate task of apportioning his own large share of merit, he certainly does not err on the side of modesty; but it would perhaps be as difficult to produce an instance of injustice, as of generosity in his estimate of others. Far more serious blame attaches to his all but total suppression in the body of the work – and the fault pervades the whole of his writings – of the names of his predecessors and contemporaries. Theorems and formulae are appropriated wholesale without acknowledgement.

Bell (reference 1) is even more scathing in his comments:

> Laplace stole outrageously wherever he could lay his hands on anything of his contemporaries and predecessors which he could use. From Lagrange he lifted the fundamental concept of the potential, from Legendre he took whatever he needed in the way of analysis; and finally, in his masterpiece *Celestial Mechanics* he deliberately omits references to the work of others incorporated in his own, with the intention of leaving posterity to infer that he alone created the mathematical theory of the heavens.

Then Bell goes on to comment that Laplace's own work is superior to that of many others whom he ignored.

## Probability theory

Laplace became interested in the theory of probability, which had been studied previously by B. Pascal (1623–62) and Pierre de Fermat (1601–65), and put the field on a sounder mathematical basis. His interest arose from the applications in physics and astronomy, rather than the use in games of chance which had been the motivation for earlier work. In 1812 he published *The Analytic Theory of Probability* which is the first recognizably modern treatment of probability. In it Laplace gave an account of the method of least squares, which still forms the basis for error analysis. The method had first been suggested empirically by Gauss and Legendre, but with little theoretical justification. Laplace gives a theoretical basis for the method but it is sketchy and, in places, wrong. He also published a method of generating functions, which provided the foundations for his theory of probability, and the first part of *Analytic Theory* is an exposition of this principle. He also revived the work of the Comte de Buffon (1707–1788), on the needle problem, and the Rev. Thomas Bayes (d. 1761) on probability.

## Turbulent times

Laplace's lifespan, 1749–1827, was a troubled time in French history including both the French Revolution (1789–95) and the Napoleonic Wars (1805–1815). In 1785, at the age of 36, Laplace was elected to full membership of the French Academy of Sciences. In the same year, as a professor at the Ecole Militaire in Paris, he was the mathematics examiner for the sixteen-year-old Napoleon Bonaparte.

During the eighteenth century, prior to the revolution, the French university system had become something of a mathematical backwater, most of the mathematical education being based on military or technical colleges. It was a symptom of a system where major posts were reserved for members of the nobility with a title. In the army it was not possible for a commoner to rise above the rank of captain, 'The competent were not noble and the noble were not competent' (see reference 2, chapter 22).

In 1789 the States General assembled at Versailles on 25 May; it was the first time the assembly had been called since 1614. They formed themselves into a National Assembly. The state was virtually bankrupt and there was civil unrest after a poor harvest and a hard winter. King Louis XVI resisted the demands of the Deputies from the States General and on 14 July crowds stormed the Bastille. Over the next two years various constitutional schemes were tried but the death of Mirabeau, a leader of the assembly, in 1791 ended hopes for a constitutional monarchy. In 1792 the French were invaded in a war with Austria and the Prussians, who were attempting to restore the monarchy. The king was brought to trial, charged with being responsible for these acts, and beheaded on 21 January 1793. Revolts started throughout France and there was a reign of terror until 1795. Regimes formed, split up and then guillotined their enemies. It is estimated that some 17,000 people were executed in these two years. In 1795 a Directory was formed to run the government and Napoleon Bonaparte established his reputation as a commander of the French armies. On 9 November 1799 the Directory fell and Napoleon was given supreme power as First Consul. In 1804 he was created Emperor.

The fall of the Bastille in 1789 heralded more than political changes. In 1790 Talleyrand proposed that the system of weights and measures be revised, and it took until 1799 for the work of the committee to be completed. Laplace was amongst the many French mathematicians and scientists who served on this committee. During the Revolution the educational system was overhauled, with the foundation of the Ecole Polytechnique and Ecole Normale in 1794. Laplace was just one of the many eminent mathematical staff which also included Lagrange, Legendre and Monge. He was one of the first members of the Bureau des Longitudes, and soon afterwards was made its president. Not all the French scientists came through the revolution unscathed. The chemist Antoine Lavoisier, with whom Laplace had worked, was guillotined in 1794. His friend Concordet died in prison, probably committing suicide in preference to the guillotine.

Laplace became an adroit politician, switching his allegiance from one regime to the next, apparently without any political beliefs of his own. 'He aspired to the role of a politician, and has left a memorable example of genius degraded to servility for the sake of a riband and a title' (reference 5). When Napoleon came to power he appointed Laplace as Minister of the Interior, but removed him from office after just six weeks with the comment 'Laplace sought subtleties everywhere, had only doubtful ideas, and finally carried the spirit of the infinitely small into administration.' Since Laplace's replacement was a relative, Lucien Bonaparte, this may not have been a totally honest appraisal. In any case, Napoleon wished to retain the support of Laplace so appointed him to the Senate in 1799. The third volume of *Celestial Mechanics* (published in 1802) contains a dedication to Napoleon. In 1803 Laplace was appointed Chancellor of the Senate, and at the same time named as grand officer of the Legion of Honour. He was made a count on the creation of the Napoleonic Empire.

After Napoleon's defeat in Russia in 1812 it was clear that the empire was crumbling and Laplace switched his allegiance to the Bourbons, voting for the deposition of Napoleon. The dedication to Napoleon was struck out of copies of *Celestial Mechanics* sold after the restoration of the monarchy. Louis XVIII rewarded him, making him a peer in 1815 and a marquis in 1817. Elected to the French Academy in 1816 he was made its president the next year. Laplace died in Paris on 5 March 1827.

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7. W. W. Rouse Ball, *A Short Account of the History of Mathematics* (Dover, London, 1960).


**Roger Cook** *is a professor in the Pure Mathematics Department at the University of Sheffield, where he has been since 1974. During the 25 years at Sheffield he has been able to make study visits to mathematical departments in India and Canada. His student days were spent at University College London and he also spent three years as a post-doctoral fellow at University College Cardiff. His mathematical interests include number theory and combinatorics. His main hobby is photographing ice hockey games.*

We gratefully acknowledge permission granted by Dover Publications Inc., for the reproduction of the illustration of Pierre-Simon de Laplace on p. 49, taken from *A Concise History of Mathematics,* by Dirk Struik (Dover, New York, 1967).

What is the probability that two brothers were both born on a Tuesday given that at least one of them was born on a Tuesday?

(Taken from the *Canberra Times*)

# The Golden Ratio by Origami

P. GLAISTER

You can make more than aeroplanes with a piece of paper.

Of the many well-known mathematical constants my favourite is the golden ratio. For those unfamiliar with this number, a rectangular sheet of paper whose sides are in the golden ratio has the property that, if a square is removed whose side is that of the smaller of the original two sides, then the remaining rectangle is similar to the original one. This means that in figure 1

$$\frac{1}{x} = \frac{x-1}{x},$$

so that $x$ satisfies the quadratic equation

$$x^2 - x - 1 = 0,$$

with positive root $x = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$ as the golden ratio.

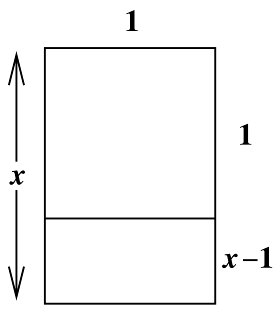


**Figure 1.**

The Greeks regarded the rectangle with sides in the ratio $x : 1$ (the golden rectangle) as particularly pleasing to the eye, and it often appears in classical and classical revival architecture as the shape of a window. The golden ratio also appears in other geometrical contexts. For example, the internal and external constructions of the regular pentagon of unit side shown in figure 2 also leads to the golden ratio as indicated. We leave readers to check this. A property of the golden ratio is that its reciprocal is one less than itself, i.e. $1/x = x - 1$.

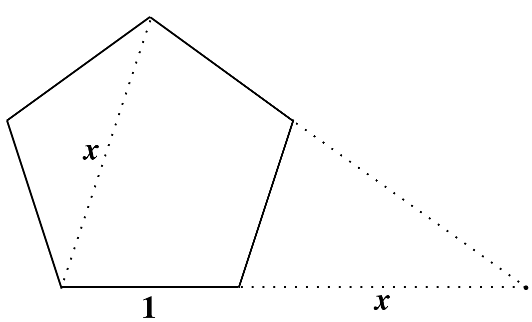


**Figure 2.**

When I was at school we frequently used foolscap paper instead of A4, and a sheet of foolscap could be used to demonstrate this with the ratio of its sides $13''/8'' = 1.625$ approximating $x$ to within less than $\frac{1}{2}\%$. Unfortunately, foolscap is no longer available. My preferred demonstration which gives the golden ratio exactly is the closest I have got to origami, and is as follows.

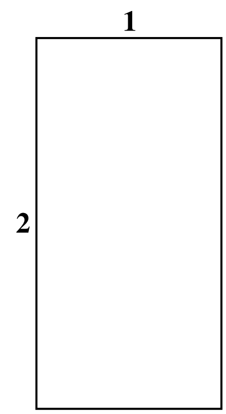


**Figure 3.**

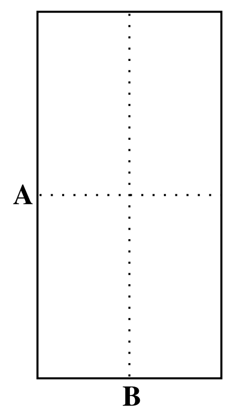


**Figure 4.**

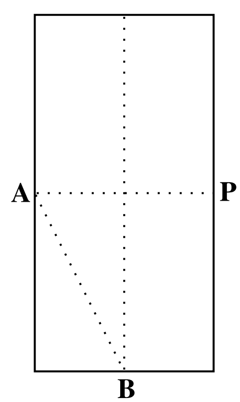


**Figure 5.**

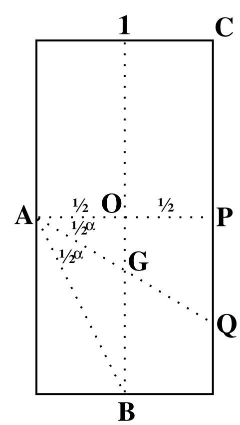


**Figure 6.**

Take a rectangular piece of paper as shown in figure 3 and fold in half both ways as shown in figure 4, and then make a crease along the diagonal AB of one of the smaller rectangles as shown in figure 5. Now fold the diagonal AB to the bisector AP, and then open out to reveal the fold lines shown in figure 6. In figure 6 AQ is the angle bisector of $\widehat{PAB}$ so that $\widehat{PAQ} = \frac{1}{2}\alpha$, where $\widehat{PAB} = \alpha$. Now, by the double angle formula

$$\tan\alpha = \frac{2\tan\frac{1}{2}\alpha}{1-\tan^2\frac{1}{2}\alpha},$$

and since $\tan\alpha = \mathrm{OB}/\mathrm{AO} = 2$, we have

$$\tan^2 \tfrac{1}{2}\alpha + \tan \tfrac{1}{2}\alpha - 1 = 0.$$

The positive root of this equation is

$$\tan \tfrac{1}{2}\alpha = \tfrac{1}{2}(\sqrt{5} - 1) = 1/x\,,$$

and hence the length CQ is

$$\begin{aligned} \mathrm{CQ} &= \mathrm{CP} + \mathrm{PQ} = 1 + \mathrm{AP}\tan \tfrac{1}{2}\alpha \\ &= 1 + 1/x = x = \tfrac{1}{2}(\sqrt{5} - 1)\,, \end{aligned}$$

i.e. precisely the golden ratio.

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# A Property of the Catenary

## FRANK CHORLTON

Readers are challenged to devise a more comprehensive solution to the problem investigated in this article.

It is well known that a uniform heavy chain suspended from its ends assumes the shape of an arc of the catenary, the curve given by

$$y = c\cosh(x/c) \quad (-\infty < x < \infty), \tag{1}$$

where $c$ is a positive constant, if the coordinate system is suitably chosen. An intriguing property of (1) relates the length of an arc of (1) with the area between the arc and the $x$-axis. For let $L(x_1, x_2)$ be the length of any arc $P_1(x_1, y_1)P_2(x_2, y_2)$ with $x_1 < x_2$ and let $A(x_1, x_2)$ be the area below $P_1P_2$. (See figure 1.)

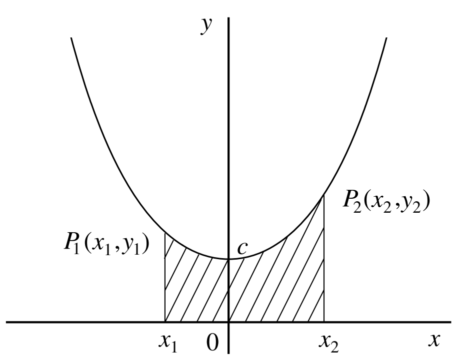


**Figure 1.**

Then

$$\begin{aligned} L(x_1, x_2) &= \int_{x_1}^{x_2} [1 + (y'(x))^2]^{1/2}\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} [1 + \sinh^2(x/c)]^{1/2}\mathrm{d}x \\ &= \int_{x_1}^{x_2} \cosh(x/c)\,\mathrm{d}x, \end{aligned}$$

while

$$A(x_1, x_2) = \int_{x_1}^{x_2} y(x)\,\mathrm{d}x = c\int_{x_1}^{x_2} \cosh(x/c)\,\mathrm{d}x,$$

so that

$$A(x_1, x_2) = cL(x_1, x_2). \tag{2}$$

The question now arises whether the relation (2) for all $[x_1, x_2]$ in an interval implies, with or without the imposition of certain restrictions, that the curve is an arc of a catenary. In other words, to what extent, if any, is there a converse of the result just proved?

So let $C$, given by $y = y(x)$, be a differentiable curve in the upper half of the $(x, y)$-plane, and suppose that there are $a$, $b$ such that (2) holds whenever $a < x_1 < x_2 < b$. This means that

$$A(x_1, x) = cL(x_1, x)$$

if $x_1 \leqslant x \leqslant x_2$ and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}A(x_1, x) = c\frac{\mathrm{d}}{\mathrm{d}x}L(x_1, x),$$

i.e.

$$y(x) = c[1 + (y'(x))^2]^{1/2},$$

which implies that

$$c^2(y'(x))^2 = y^2(x) - c^2. \tag{3}$$

We now consider three cases.

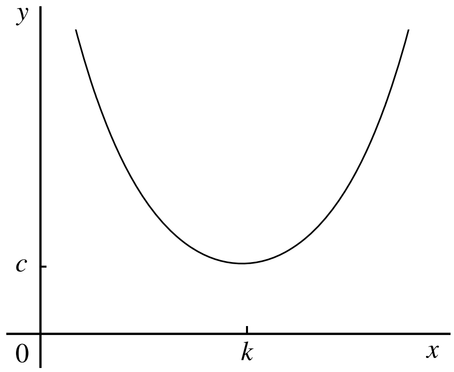


**Figure 2.**

(i) Assume that $y'(x) > 0$ for $a < x < b$. Then, by (3),

$$c\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y^2 - c^2}.$$

Hence

$$\frac{c}{\sqrt{y^2 - c^2}}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and, by integration with respect to $x$,

$$\int \frac{c}{\sqrt{y^2 - c^2}}\,\mathrm{d}y = x - k,$$

where $k$ is a constant. The substitution $y = c\cosh t$ gives $t = (x-k)/c$, so that $\cosh t = \cosh((x-k)/c)$, i.e. $y = c\cosh((x-k)/c)$, for $x_1 \leqslant x \leqslant x_2$. If the intervals $[x_1, x_2]$, $[x_3, x_4]$ in $(a, b)$ overlap, then the constant of integration is the same for both. It follows by continuity that

$$y = c\cosh\frac{x-k}{c} \quad \text{for } a \leqslant x \leqslant b\,. \tag{4}$$

This is, of course, an arc of a catenary with vertex at $(k, c)$ (see figure 2).

(ii) It is proved similarly that (4) holds if $y'(x) < 0$ for $a < x < b$.

(iii) If $y'(x) = 0$ for $a < x < b$, then (3) shows at once that $y(x) = c$ for $a \leqslant x \leqslant b$.

The previous three basic curves can be joined to produce other curves satisfying (2). For instance, the complete catenary arises from (2) together with $y'(x) > 0$ for $x > k$ and $y'(x) < 0$ for $x < k$.

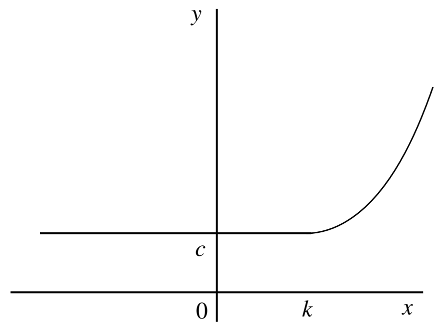


**Figure 3.**

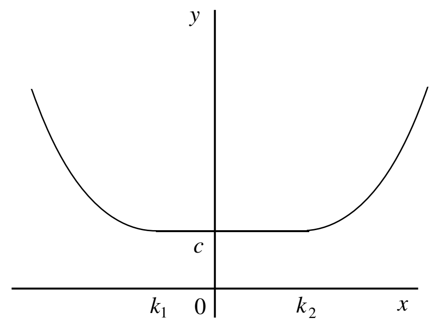


**Figure 4.**

Combinations involving (iii) are shown in figures 3 and 4.

However, are these really the only smooth curves with the $A = cL$ property?

**Frank Chorlton** *was senior lecturer in mathematics at Aston University and is now in retirement. He continues his pursuit of mathematics but is also fond of music, especially that of Bach.*

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Without using a calculator, can you say if

$$\sqrt[43]{\sqrt[7]{\sqrt[3]{\frac{1}{\sqrt{x}}}}} \equiv \frac{1}{\sqrt{x}}\cdot\sqrt[3]{x}\cdot\sqrt[7]{x}\cdot\sqrt[43]{x}\ ?$$

PETER DERLIEN
(University of Sheffield)

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Moira's wedding anniversary is on the same date as Mary's. They were married in different years on a Saturday in August. Each has been married less than 28 years.

List the possibilities for the number of years between the two weddings.

JOHN MACNEILL
(University of Warwick)

# Some Trigonometric Inequalities

MASAKAZU NIHEI

The notion of convexity is introduced and used to prove very simply several trigonometric inequalities which, without its aid, would present real difficulties.

The simplest inequalities considered by us are of the form

$$\sqrt{\tan\alpha\tan\beta} \leqslant \tan\left(\tfrac{1}{2}(\alpha+\beta)\right) \leqslant \tfrac{1}{2}(\tan\alpha+\tan\beta) \quad (0<\alpha,\beta<\tfrac{1}{4}\pi) \quad (1)$$

and we first prove them by use of trigonometry alone.

Since all the terms are positive, to prove the left inequality it is sufficient to show that

$$X = \tan^2\left(\tfrac{1}{2}(\alpha+\beta)\right) - \tan\alpha\tan\beta \geqslant 0.$$

But

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta},$$

so that

$$X = \frac{1-\cos(\alpha+\beta)}{1+\cos(\alpha+\beta)} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}.$$

After some manipulation we see that

$$X = \frac{\cos(\alpha+\beta)(1-\cos(\alpha-\beta))}{\cos\alpha\cos\beta(1+\cos(\alpha+\beta))},$$

which implies that $X \geqslant 0$.

To prove the right inequality, or that

$$Y = \tfrac{1}{2}(\tan\alpha+\tan\beta) - \tan\left(\tfrac{1}{2}(\alpha+\beta)\right) \geqslant 0,$$

we use the formula

$$\tan\theta = \frac{\sin 2\theta}{(1+\cos 2\theta)}.$$

This leads to

$$Y = \frac{\sin(\alpha+\beta)(1-\cos(\alpha-\beta))}{2\cos\alpha\cos\beta(1+\cos(\alpha+\beta))},$$

and so $Y \geqslant 0$.

One of our aims is to obtain a generalization of (1) in which $\alpha$, $\beta$ are replaced by an arbitrary number $n$ of variables and corresponding changes are made to the terms in the chain. (See inequality A(i) below.) The case $n=2$ was seen to be amenable to elementary manipulation, though not trivially so. However, larger values of $n$ require more sophisticated means and we shall use so-called convex functions.

**Definition.** *If the real function $f$ on the interval $I$ is such that*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2} \quad (2)$$

*for all a, b in I, then f is said to be* convex *in I.*

The geometrical interpretation of convexity is evident from figure 1.

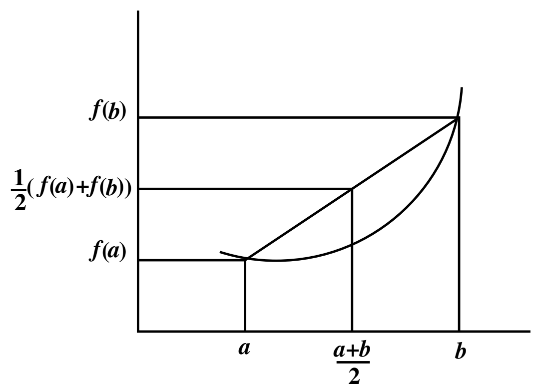


**Figure 1.**

If, in (2), the inequality sign is reversed, $f$ is called *concave*. The term is introduced for the sake of completeness; it will not subsequently feature in this article.

It is remarkable that a convex function, i.e. one that satisfies (2) for all $a$, $b$, automatically satisfies an apparently much more general inequality.

**Theorem 1.** *A function convex in an interval I is such that*

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leqslant \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \quad (3)$$

*for all points $x_1,\ldots,x_n$ in $I$.*

*Proof.* Suppose that $f$ is convex on $I$.

(i) We first show by induction that (3) holds when $n=2^p$ $(p=1,2,\ldots)$.

By the definition of convexity, (3) holds for $n=2^1$. Next suppose that (3) holds for $n=2^p$. Then, when $n=2^{p+1}$,

$$f\left(\frac{x_1+\cdots+x_{2^{p+1}}}{2^{p+1}}\right) = f\left(\frac{1}{2}\left(\frac{x_1+\cdots+x_{2^p}}{2^p} + \frac{x_{2^p+1}+\cdots+x_{2^{p+1}}}{2^p}\right)\right)$$

$$\leqslant \tfrac{1}{2}\left(f\left(\frac{x_1+\cdots+x_{2^p}}{2^p}\right) + f\left(\frac{x_{2^p+1}+\cdots+x_{2^{p+1}}}{2^p}\right)\right)$$
$$\leqslant \tfrac{1}{2}\left(\frac{1}{2^p}\left(f(x_1)+\cdots+f(x_{2^p}\right) + \frac{1}{2^p}\left(f(x_{2^p+1})+\cdots+f(x_{2^{p+1}})\right)\right)$$
$$= \frac{1}{2^{p+1}}\left(f(x_1)+\cdots+f(x_{2^{p+1}}\right).$$

Thus (3) holds for $n = 2^{p+1}$ and so for all $n$ which are integral powers of 2.

(ii) Now let $n$ be any integer $> 2$ and, given $x_1, \dots, x_n$ in $I$, put $a = (1/n)(x_1 + \cdots + x_n)$. If $m = 2^n$, then $m > n$ and, taking the point $a$ $m-n$ times, we have

$$x_1 + \cdots + x_n + a + \cdots + a = na + (m-n)a = ma.$$

Hence, by (i),

$$f(a) = f\left(\frac{x_1+x_n+a+\cdots+a}{m}\right)$$
$$\leqslant \frac{1}{m}\left(f(x_1)+\cdots+f(x_n)+f(a)+\cdots+f(a)\right)$$
$$= \frac{1}{m}(f(x_1)+\cdots+f(x_n)) + \frac{m-n}{m}f(a)$$

and so

$$nf(a) \leqslant f(x_1)+\cdots+f(x_n),$$

which is (3). The theorem is therefore proved.

Convexity turns out to be such a useful concept because there is a particularly simple criterion for it. To obtain it we employ the Mean Value Theorem of elementary analysis: *If the function g on the closed interval* $[a, b]$ *is continuous at a, b and differentiable in the open interval (a,b), then there exists a point c in* $(a, b)$ *such that*

$$g(b) - g(a) = (b-a)g'(c).$$

This means that there is at least one point on the graph of $g$ at which the tangent is parallel to the chord subtending the graph. Figure 2 makes the Mean Value Theorem intuitively obvious and we do not now supply a rigorous proof. Such a proof can be found in any first course on analysis.

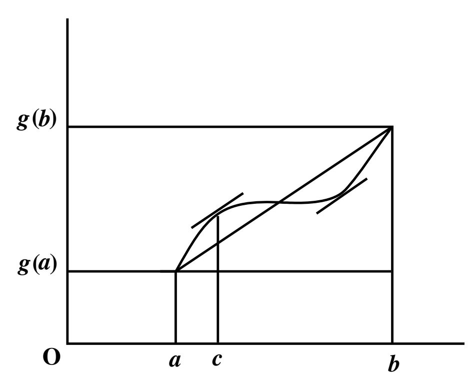

**Figure 2.**

**Theorem 2.** *If the real function f on the interval I is twice differentiable and* $f''(x) > 0$ *for all x in I, then f is convex in I.*

*Proof.* Take any points $a$, $b$ in $I$ with $a < b$. Then, by the Mean Value Theorem, there exists $c_1$ such that

$$a < c_1 < \tfrac{1}{2}(a+b)$$

and

$$f\left(\tfrac{1}{2}(a+b)\right) - f(a) = \tfrac{1}{2}(b-a)f'(c_1). \tag{4}$$

Also there exists $c_2$ such that $\frac{1}{2}(a+b) < c_2 < b$ and

$$f(b) - f\left(\tfrac{1}{2}(a+b)\right) = \tfrac{1}{2}(b-a)f'(c_2). \tag{5}$$

Moreover there exists $d$ such that $c_1 < d < c_2$ and

$$f'(c_2) - f'(c_1) = (c_2 - c_1)f''(d). \tag{6}$$

Subtracting (5) from (4) and using (6) we now have

$$2f\left(\tfrac{1}{2}(a+b)\right) = f(a) + f(b) - \tfrac{1}{2}(b-a)(c_2-c_1)f''(d)$$
$$\leqslant f(a) + f(b),$$

since $b-a, c_2-c_1, f''(d) \geqslant 0$. Therefore $f$ is convex in $I$.

**Corollary 1.** *Suppose that the function f is positive and twice differentiable on the interval I. If also*

$$f(x)f''(x) \leqslant (f'(x))^2 \tag{7}$$

*in I, then, for any points* $x_1, \dots, x_n$ *in I,*

$$\left(\prod_{i=1}^{n} f(x_i)\right)^{1/n} \leqslant f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

*Proof.* We define the function $F$ on $I$ by

$$F(x) = -\log f(x)$$

for all $x$ in $I$. Then $F'(x) = -f'(x)/f(x)$ and, by (7),

$$F''(x) = \frac{(f'(x))^2 - f(x)f''(x)}{(f(x))^2} \geqslant 0.$$

Hence $F$ is convex in $I$ and so, by theorem 1, for $x_1, \dots, x_n$ in $I$,

$$F\left(\frac{x_1+\cdots+x_n}{n}\right) \leqslant \frac{1}{n}\sum_{i=1}^{n} F(x_1).$$

Thus

$$-\log f\left(\frac{x_1+\cdots+x_n}{n}\right) \leqslant -\frac{1}{n}\sum_{i=1}^{n}\log f(x_i)$$

or

$$f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \geqslant \left(\prod_{i=1}^{n}f(x_i)\right)^{1/n}.$$

**Corollary 2.** *The geometric-arithmetic mean inequality: if $x_1, x_2, \ldots, x_n$ are non-negative, then*

$$(x_1x_2\ldots x_n)^{1/n} \leqslant \frac{1}{n}(x_1+x_2+\ldots+x_n).$$

*Proof.* We need only take $f(x) = x$ for $x \geqslant 0$ in corollary 1.

A very thorough account of convex functions is given in reference 1. However, we have now developed the subject sufficiently to prove with ease six chains of inequalities, the first of which generalizes (1).

**Inequalities A**

$$\text{(i)}\quad \left(\prod_{i=1}^{n}\tan x_i\right)^{1/n} \leqslant \tan\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \leqslant \frac{1}{n}\sum_{i=1}^{n}\tan x_i \quad \left(0 < x_1, \ldots, x_n < \tfrac{1}{4}\pi\right).$$

$$\text{(ii)}\quad \left(\prod_{i=1}^{n}\sin x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n}\sin x_i \leqslant \sin\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \quad (0 < x_1, \ldots, x_n < \pi).$$

$$\text{(iii)}\quad \left(\prod_{i=1}^{n}\cos x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n}\cos x_i \leqslant \cos\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \quad \left(0 < x_1, \ldots, x_n < \tfrac{1}{2}\pi\right).$$

*Proof.* (i) If $f(x) = \tan x$, it is easily shown that (7) holds as $\tan^2 x \leqslant 1$ on $(0, \pi/4)$. Hence, by corollary 1, the left inequality holds.

Since $f''(x) > 0$ on $(0, \pi/4)$, $f$ is convex on $(0, \pi/4)$. The right inequality now follows from theorem 1.

(ii) The left inequality is an immediate consequence of the geometric-arithmetic mean inequality (corollary 2).

By theorem 2, the function $-\sin x$ is convex in $(0, \pi)$ and so

$$-\sin\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \leqslant \frac{1}{n}\sum_{i=1}^{n}(-\sin x_i),$$

which is the right inequality.

(iii) This is proved in the same way as (ii).

We note that, in all three chains of inequalities, the geometric mean is the least term.

**Inequalities B**

$$\text{(iv)}\quad \cot\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \leqslant \left(\prod_{i=1}^{n}\cot x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n}\cot x_i \quad \left(0 < x_1, \ldots, x_n < \tfrac{1}{4}\pi\right).$$

$$\text{(v)}\quad \operatorname{cosec}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \leqslant \left(\prod_{i=1}^{n}\operatorname{cosec} x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n}\operatorname{cosec} x_i \quad (0 < x_1, \ldots, x_n < \pi).$$

$$\text{(vi)}\quad \sec\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \leqslant \left(\prod_{i=1}^{n}\sec x_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n}\sec x_i \quad \left(0 < x_1, \ldots, x_n < \tfrac{1}{2}\pi\right).$$

*Proof.* All three chains of inequalities are of the same type. The right inequalities are simply instances of the geometric-arithmetic mean inequality. The left inequality of (iv) is obtained by taking the reciprocal of the left inequality in (i). However, to prove the left inequality in (v) we begin with the extreme terms in (ii) related by

$$\left(\prod_{i=1}^{n}\sin x_i\right)^{1/n} \leqslant \sin\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right).$$

and then take the reciprocal of each side. The left inequality of (vi) is similarly obtained from (iii).

**Reference**

1. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edn (CUP, Cambridge, UK, 1988).

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# A Story from Flocklore

K. R. S. SASTRY

A version of this story may be familiar to you: someone died and left 17 sheep saying that half of them should go to the eldest son, one third to the middle one and one ninth to the youngest. Obviously they could not divide the flock that way so they sought the help of a saint. The saint borrowed a sheep from someone else, added it to the flock of 17, divided the new flock so that the eldest got 9, the middle one 6, the last one 2 and returned the one left out to the lender of the sheep.

You will soon realise that this puzzle has a lot more to offer than the obvious amusement.

## Old story...new problem

Let us consider this puzzle in a more general setting. First, we observe that the fractions are unit fractions with denominators 2, 3, 9 respectively. Next, the flock number $17 = \text{lcm}[2, 3, 9] - 1$. So we consider the unit fractions with denominators $U, V, V^2$. Further, write $\gcd(U, V) = d$. Then there exist relatively prime natural numbers $u, v$ so that $U = du$, $V = dv$. Now

$$\begin{aligned}\text{lcm}[U, V, V^2] &= \text{lcm}[du, dv, d^2v^2] \\ &= \text{lcm}[du, d^2v^2] \\ &= d^2uv^2)/\Delta,\end{aligned}$$

where $\Delta = \gcd(u, d)$. This gives us the flock number $(d^2uv^2)/\Delta - 1$. Preserving the flavour of the old story we may, therefore, state the general problem as follows:

> A man with three children died leaving a flock of $(d^2uv^2)/\Delta - 1$ sheep saying that the children should receive respectively $1/du$, $1/dv$, $1/d^2v^2$ parts of it. Determine the possible values for the flock number.

We know there is at least one solution. Are there others or is it a unique solution?

## The solution of the general problem

Let us invite the saint to perform his trick solution. He adds a borrowed sheep to the flock so that the new flock number is $(d^2uv^2)/\Delta$. The first child receives

$$\left(\frac{d^2uv^2}{\Delta}\right)\left(\frac{1}{du}\right) = \frac{dv^2}{\Delta}.$$

Likewise the second receives $(duv)/\Delta$ and the last $u/\Delta$ sheep. Finally, only one, the borrowed one, remains. Therefore we have to solve the equation

$$\frac{d^2uv^2}{\Delta} - \frac{dv^2}{\Delta} - \frac{duv}{\Delta} - \frac{u}{\Delta} = 1$$

or

$$d^2uv^2 - dv^2 - duv - u = \Delta. \tag{1}$$

We write (1) as a quadratic in $v$:

$$d(du-1)v^2 - duv - (u+\Delta) = 0,$$

and observe that the last term is negative, implying that the product of the two roots for $v$ is negative. Hence one root is positive and the other negative. Our interest lies in the *positive integral* root $v$. To obtain an upper bound on the values of $v$ we recast equation (1) as

$$u(d^2v^2 - dv - 1) = dv^2 + \Delta. \tag{2}$$

The equation (2) shows that $d^2v^2 - dv - 1$ must be an integral divisor of $dv^2 + \Delta$. A necessary condition for this divisibility is that $d^2v^2 - dv - 1 \leqslant dv^2 + \Delta$ or

$$d(d-1)v^2 - dv - (\Delta + 1) \leqslant 0.$$

When the focus is on the values of $d$ then there are two distinct cases to consider: $d > 1$ and $d = 1$.

Suppose first that $d > 1$. Then the above inequality yields

$$\begin{aligned}1 \leqslant v &\leqslant \frac{1}{2d(d-1)}\left[d + \sqrt{\left(d^2 + 4(\Delta+1)d(d-1)\right)}\right] \\ &< \frac{1}{2d(d-1)}\left[d + \sqrt{\left(d^2 + 4(\Delta+1)d^2\right)}\right].\end{aligned}$$

We recall that $\Delta = \gcd(u, d)$ so $\Delta < d$. Therefore the preceding inequalities simplify, in terms of $d$, to

$$1 \leqslant v < \frac{1 + \sqrt{4d+5}}{2(d-1)}.$$

From $1 < (1 + \sqrt{4d+5})/2(d-1)$ we have $d = 2$ or 3. These values imply that either

$$d = 2, \quad v = 1, 2 \quad \text{or } d = 3, \quad v = 1. \tag{3}$$

We need to consider the case of $d = 1$ separately. When $d = 1$, $\Delta = \gcd(u, d) = \gcd(u, 1) = 1$ and equation (2) can be written as

$$u(v^2 - v - 1) = v^2 + 1 = (v^2 - v - 1) + (v + 2).$$

This shows that $v^2 - v - 1$ is a divisor of $v + 2$ and a like analysis now yields

$$d = 1, \quad \Delta = 1; \quad v = 2, 3. \tag{4}$$

First we use (3) and (4) in (2) to determine the possible values of $u$ which in turn enables us to determine the flock numbers $f = (d^2uv^2)/\Delta - 1$.

(i) If $v = 1$ then from (3) $d = 2$ or 3 and the equation (2) becomes

$$u(d^2 - d - 1) = d + \Delta.$$

(a) When $d = 2$ then the above equation gives $u = \Delta + 2$. Also, $\Delta = \gcd(u, d) = \gcd(u, 2) = 1$ or 2. If $\Delta = 1$ then $u = 3$ and $f = 11$ with flock parts containing 2, 6, 3 sheep respectively. If $\Delta = 2$ then $u = 4$ and $f = 7$ with flock parts containing 1, 4, 2 sheep respectively.

(b) When $d = 3$ then the above equation gives $5u = \Delta + 3$. Also $\Delta = \gcd(u, 3) = 1$ or 3 and there is no integer solution for $u$.

(ii) If $v = 2$ then from (4) $d = \Delta = 1$ and from (3) $d = 2$, $\Delta = \gcd(u, 2) = 1$ or 2. The equation (2) now assumes the form

$$u(4d^2 - 2d - 1) = 4d + \Delta.$$

(a) When $d = \Delta = 1$ then the above equation gives $u = 5$ and $f = 19$ with flock parts containing 4, 10, 5 sheep respectively.

(b) When $d = 2$ then the above equation gives $11u = 8 + \Delta$ and $u$ is not an integer for $\Delta = 1$ or 2.

(iii) If $v = 3$ then from (4) $d = \Delta = 1$. Now equation (2) becomes $5u = 10$ to give $u = 2$, $f = 17$ and the flock parts contain 9, 6, 2 sheep respectively.

Hence we conclude that, to retain the flavour of the old story, there are just four flock numbers. For later use let us summarize the findings so far:

7, shared out 1, 4, 2 sheep respectively $(\alpha)$

11, shared out 2, 6, 3 sheep respectively $(\beta)$

17, shared out 9, 6, 2 sheep respectively $(\gamma)$

19, shared out 4, 10, 5 sheep respectively. $(\delta)$

The solution of the general problem is also intriguing: all the flock numbers 7, 11, 17, 19 are primes!

The alert reader may wonder why we took the flock size as $(d^2uv^2)/\Delta - 1$ when, for any $\lambda \in \mathbb{N}$, a flock size $\lambda((d^2uv^2)/\Delta) - 1$ could be divided up this way. However, in this case equation (1) would show that $\lambda$ must be a divisor of 1, Hence we must have $\lambda = 1$.

## Why borrow just one sheep?

A *Mathematical Spectrum* editor asked a significant question: what would be your flock size if the saint were to borrow $b$ sheep, $b \geqslant 1$, in order to employ his trick? This leads to further generalization of the preceding problem. A quick way to see that there is a flock number $f = (d^2uv^2)/\Delta - b$ for each value of $b = 1, 2, 3, \ldots$ is to let $d = 1$ (so $\Delta = 1$), $u = b + 4$, $v = 2$. This gives the flock number $f = 3b + 16$ in which the flock parts contain 4, $2b + 8$, $b + 4$ sheep respectively. The special value $b = k^2 - 2k - 2 = (k-1)^2 - 3$ and $d = \Delta = 1$, $u = 2$, $v = k$ yields another flock number $f = k^2 + 2K + 2$ in which the flock parts contain $k^2$, $2k$, 2 sheep respectively for $k = 3, 4, 5, \ldots$ to ensure that $b > 0$. However, there will be other solutions so let us now concentrate on the resolution of the challenge: to determine all flock numbers $f$ for a given value of $b$, $b \in \mathbb{N}$. We do this by assuming the flock number more generally as

$$f = \lambda\left(\frac{d^2uv^2}{\Delta}\right) - b \tag{5}$$

for some $\lambda \in \mathbb{N}$. As usual, the saint adds $b$ sheep to the existing flock $f$ before dividing the new flock into $1/du$, $1/dv$, $1/d^2v^2$ parts of it. This gives us another analogue of equation (1) to solve, from which we see that $\lambda$ must be a divisor of $b$. Thus $b = \lambda c$ for some natural number $c$. Naturally $c$ is a divisor of $b$ too and we have

$$f = \lambda\left(\frac{d^2uv^2}{\Delta} - c\right) = \lambda f_1,$$

where

$$f_1 = \frac{d^2uv^2}{\Delta} - c\,. \tag{6}$$

Here is a key observation:

*If $f_1 = (d^2uv^2)/\Delta - c$ is the flock size when the saint has to borrow $c$ sheep then (6) says that $f = \lambda f_1$ would be a flock size when the saint has to borrow $b$ sheep.*

The above observation provides a clue to determine completely the values of $f$ for a given value of $b$. Once we have found out all values of $f_1$ in (6) for each divisor $c$ of $b$, we can generate all flock numbers $f$ in (5).

Let us now see how to solve the equation (6) for $f_1$.

## The solution of $f_1 = (d^2uv^2)/\Delta - c$

In determining $f_1$ we recall that the saint adds $c$ sheep to $f_1$ and divides the new flock into $1/du$, $1/dv$, $1/d^2v^2$ parts. This produces an equation analogous to (2):

$$u(d^2v^2 - dv - 1) = dv^2 + c\Delta. \tag{7}$$

We now obtain the bounds on the values of $v$ and $d$ as we did earlier in (3) and (4). Here too we consider the cases of $d = 1$ and $d > 1$.

When $d = 1$ we have $\Delta = 1$ and the equation (7) reduces to

$$u(v^2 - v - 1) = v^2 + c = (v^2 - v - 1) + (v + c + 1)$$

so that $v^2 - v - 1$ divides $v + c + 1$. In particular

$$v^2 - v - 1 \leqslant v + c + 1.$$

This yields

$$2 \leqslant v \leqslant 1+\sqrt{c+3}; \quad d=\Delta=1. \tag{8}$$

Now suppose that $d > 1$. Then from (7) we see that $d^2v^2 - dv - 1 \leqslant dv^2 + c\Delta$. This yields

$$1 \leqslant v < \frac{1+\sqrt{4cd+5}}{2(d-1)} \tag{9}$$

since $\Delta \leqslant d$. For there to be any solution $v$ we must have

$$1 < \frac{1+\sqrt{4cd+5}}{2(d-1)},$$

or

$$d^2-(c+3)d+1<0,$$

$$1 < d < \frac{c+3+\sqrt{(c+3)^2-4}}{2}.$$

Hence

$$1 < d < c+3. \tag{10}$$

Then $4cd+5 \leqslant 4c^2+12c+5 < 4(c+3/2)^2$ so from (9) we have

$$1 \leqslant v < \frac{1+\sqrt{4cd+5}}{2(d-1)} < \frac{c+2}{d-1}. \tag{11}$$

Thus there are only finitely many possibilities for $d$ and $v$. Then $\Delta \leqslant d$ and (7) show that there are only a finite number of flock numbers $f$.

## The solution of $f = \lambda\left(d^2uv^2/\Delta\right) - b$

Our discussion thus far brings us to the following algorithm to obtain the complete solution of the flock numbers $f$.

**Stage 1.** At the first stage we take $\lambda = b$. This gives us

$$f = b\left(\frac{d^2uv^2}{\Delta} - 1\right).$$

So $c = 1$,

$$f_1 = \frac{d^2uv^2}{\Delta} - 1$$

and $f_1$ has the solutions given by $(\alpha)$, $(\beta)$, $(\gamma)$, $(\delta)$. Then as we saw in (6) $f = bf_1$ has the solutions for $f$:

| | |
|---|---|
| $7b$ with flock parts of $b, 4b, 2b$ sheep | (A) |
| $11b$ with flock parts of $2b, 6b, 3b$ sheep | (B) |
| $17b$ with flock parts of $9b, 6b, 2b$ sheep | (C) |
| $19b$ with flock parts of $4b, 10b, 5b$ sheep. | (D) |

**Stage 2.** At the second stage we take $\lambda = b/c$, $c$ being a divisor of $b : 1 < c \leqslant b$ and solve for the solutions $f_1$ using the relations in (7), (8), (10) and (11).

**Stage 3.** At this final stage we take the totality of all the solutions obtained at the second stage together with (A), (B), (C), (D). This produces the complete solution of the flock numbers $f$.

## A numerical illustration

Here is a numerical illustration to enable the reader to obtain a further understanding of the underlying process.

**Problem.** Suppose the saint has to borrow 4 sheep to employ his trick. Determine completely the possible flock numbers $f$.

**Solution.** Here $b = 4$ so $c \in \{1, 2, 4\}$ and

$$f = \lambda\left(\frac{d^2uv^2}{\Delta}\right) - 4, \quad \lambda = \frac{4}{c}.$$

**(I) $c = 1$**. Therefore $\lambda = 4$, $f = 4((d^2uv^2/\Delta) - 1)$ and hence $f_1 = (d^2uv^2/\Delta) - 1$. This corresponds to stage 1 noted earlier, so $f = 4f_1$ has the solutions (A), (B), (C), (D) for $f$ when $b = 4$ :

| | |
|---|---|
| 28 shared out 4, 16, 8 sheep respectively | (A′) |
| 44 shared out 8, 24, 12 sheep respectively | (B′) |
| 68 shared out 36, 24, 8 sheep respectively | (C′) |
| 76 shared out 16, 40, 20 sheep respectively. | (D′) |

**(II) $c = 2$**. Now $\lambda = 2$, $f = 2((d^2uv^2/\Delta) - 2)$ and hence $f_1 = (d^2uv^2/\Delta) - 2$.

If $d = 1$ then $\Delta = 1$ and from (8), $2 \leqslant v \leqslant 1+\sqrt{5}$ so $v = 2$ or 3. There is no acceptable solution in this case because if $v = 2$ then (7) becomes $u = 6$, contradicting $\gcd(u, v) = 1$. If $v = 3$ then (7) becomes $5u = 11$ and there is no integer solution for $u$.

If $d > 1$ then from (10), $1 < d < 5$ and for each value of $d$ we find $v : 1 \leqslant v < 4/(d-1)$, see (11).

When $d = 2$ we have $v = 1$, 2, or 3 and $\Delta = \gcd(u, 2) = 1$ or 2. If $d = 2$ and $\Delta = 1$ then $u$ must be odd. In this case (7) reduces to $u(4v^2 - 2v - 1) = 2v^2 + 2$ which is impossible as the left-hand side is odd and the right-hand side is even. If $d = 2$ and $\Delta = 2$ then $u$ must be even and therefore $v = 1$ or 3 because $\gcd(u, v) = 1$. Now (7) reduces to

$$u(4v^2 - 2v - 1) = 2v^2 + 4.$$

When $v = 1$, we have a new solution $u = 6$, $f_1 = 10$ and $f = 2f_1 = 20$ with flock parts containing 2, 12 and 6 sheep respectively. When $v = 3$ there are no integer solutions.

When $d = 3$ we have $v = 1$ and $\Delta = \gcd(u, d) = 1$ or 3. Presently (7) reduces to $5u = 3 + 2\Delta$, producing the integer solution only when $\Delta = 1$ and then $u = 1$. This gives $f_1 = 7$, $f = 2$ and $f_1 = 14$ with the flock parts containing 6, 6, 2 sheep respectively.

When $d = 4$ we have $v = 1$, and $\Delta = 1$, 2 or 4. These reduce (7) to $11u = 4+2\Delta$ and produce no integer solutions for $u$.

**(III) $c = 4$**. This gives $\lambda = 1$ and $f = f_1 = (d^2uv^2/\Delta) - 4$.

When $d = 1$ we have $\Delta = 1$ and from (8)

$$2 \leqslant v \leqslant 1 + \sqrt{7},$$

so $v = 2$ or 3. Then (7) reduces to

$$u(v^2 - v - 1) = v^2 + 4$$

and there are no integer solutions with $\gcd(u, v) = 1$.

When $d > 1$, from (10) we have $1 < d < 7$ and also $1 \leqslant v < 6/(d - 1)$ for each value of $d$.

When $d = 2$ we have $1 \leqslant v \leqslant 5$, $\Delta = \gcd(u, d) = 1$ or 2 and (7) reduces to

$$u(4v^2 - 2v - 1) = 2v^2 + 4\Delta.$$

This shows that $u$ must be even and since $\gcd(u, v) = 1$, $v$ must be odd. Further $\gcd(u, d) = 2$ so $\Delta = 2$. The only new integer solution we obtain is $u = 10$, $v = 1$ giving a flock size $f = 16$ with flock parts of 1, 10 and 5 sheep respectively.

When $d = 3$ we have $v = 1$ or 2 and $\Delta$ is 1 or 3. This reduces (7) to

$$u(9v^2 - 3v - 1) = 3v^2 + 4\Delta.$$

When $v = 1$ we obtain the solution $u = 3$ with $f = 5$ and the flock parts containing 1, 3 and 1 sheep respectively. When $v = 2$ there are no integer solutions.

When $d = 4$, 5, or 6 we have $v = 1$ and (7) reduces to

$$u(d^2 - d - 1) = d + 4\Delta \leqslant 5d$$

and it is easy to check that there are no integer solutions.

Hence we conclude that there are precisely eight flock sizes if the saint were to borrow 4 sheep to solve the distribution problem:

$$f = 5,\ 14,\ 16,\ 20,\ 28,\ 44,\ 68 \text{ and } 76.$$

## Conclusion

Another generalization of the original problem is presented in reference 1. Here is yet another for you to try. This may shock those who watch the human population growth!

An old man with $(n + 1)$ children dies leaving a flock of $uv^n - b$ sheep saying that the children should receive respectively $1/u$, $1/v$, $1/v^2, \ldots, 1/v^n$ parts of it; $n = 3, 4, 5, \ldots$; $b \geqslant 1$ and $\gcd(u, v) = 1$. Determine the flock size.


**Reference**

1. K. R. S. Sastry, Problem 2226, *Crux Mathematicorum with Mathematical Mayhem,* **23** (1997) p. 166, and **24** (1998) pp. 186–189.



**K. R. S. Sastry** *studied mathematics at the University of Mysore, India. He then taught mathematics in India and Ethiopia. At present he devotes his time to contributing articles and problem proposals to mathematics journals.*


### Braintwister

**8. Cutting corners**

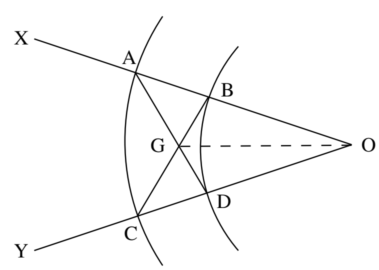
A farmer has a very large rectangular field surrounded by a fence. There is a post near a corner of the field: it is 5 metres from one side of the field and 12 metres from an adjacent side. He wants to build a straight fence through the post which then creates a triangular area fenced off in the corner. He wishes to choose the direction of the fence so that the area fenced off in this way is as small as possible.

**How long will the fence be, and what will the area of the triangle be?**

(The solution will be published next time.)

VICTOR BRYANT

**Bisecting an angle**




To bisect an angle XOY, draw concentric arcs, centre O and let G be the point of intersection of AD and BC as shown. Then OG bisects ∠XOY.

SEYAMACK JAFARI
(Ahwaz, Iran)

# Mathematics in the Classroom

I consider below the presentation of the well-known hyperbolic functions in front of a specific audience. I am referring to the first-year students in the engineering section at the Hellenic Naval Academy. These students are rather overqualified according to the Greek educational system that requires hard entrance exams to universities for all high school graduates. Nevertheless mathematics remains a tough subject for them, not to mention the fact that they have to place emphasis on subjects such as navigation, and in this way mathematics is considered as some kind of forced labour. As I write the definition relations

$$\begin{aligned}\sinh x &= \tfrac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}),\\ \cosh x &= \tfrac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}),\end{aligned}$$

on the blackboard, I can sense a wave of embarrassment spread through the class. It is as if they are asking me, through the look on their faces, why they need all this stuff. Although I told them at the very beginning that I need these functions as a preliminary step to develop further techniques of integration, this explanation does not seem to satisfy them. At this point I resort to the comment given by Professor D. Berkey in his wonderful book *Calculus* (see reference 1):

> Roughly speaking, $\cosh x$ represents the average of exponential growth and exponential decay, while $\sinh x$ represents half the difference between these two phenomena.

Now, looking at their faces I can see a mild smile of satisfaction. Terms like exponential growth and exponential decay refer to applications, and applications always seem to stimulate their interest.

**Similarities and differences**

Similarities and differences fascinate students, especially in the first two years at university level, because they strengthen their notion of the continuation and evolution of science from an early stage to a more complicated one. This also applies to my students, who are interested in seeing whether the classical trigonometric identities remain invariant under the hyperbolic notation. A natural question occurs to them: does the classical trigonometric identity hold in this case, i.e.

$$\sinh^2 x + \cosh^2 x = 1?$$

A lot of books contain the corresponding equality as an exercise or part of their theory. But instead of giving them something specific to prove, by asking them, for example, to prove the relation

$$\cosh^2 x - \sinh^2 x = 1,$$

I urge them to find the relation that holds without assumptions. In this way, I make them understand that mathematicians do not wake up with a formula ready in their minds, but that they try to find out how some concepts are associated by using the theory discovered so far. Starting from $\cosh^2 x$ and applying the definition gives

$$\cosh^2 x = \tfrac{1}{4}(\mathrm{e}^{x} + \mathrm{e}^{-x})^2. \tag{1}$$

The basic algebraic identity $(a+b)^2$ is applied to get

$$\cosh^2 x = \tfrac{1}{4}(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}). \tag{2}$$

Now we apply the trick of adding and subtracting the same number, something they use a lot while in high school and I can see they enjoy it. Thus (2) becomes

$$\begin{aligned}\cosh^2 x &= \tfrac{1}{4}(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x} - 4 + 4)\\ &= \tfrac{1}{4}\left[(\mathrm{e}^{x} - \mathrm{e}^{-x})^2 + 4\right]\\ &= \sinh^2 x + 1,\end{aligned} \tag{3}$$

and introducing $\sinh^2 x$ on the left-hand side of (3), the required equality appears. All steps from (1) to (3) are discussed or written by them. In this way, by taking the initiative in proving formulas on their own, even the simple ones, they start feeling that they are achieving something in the mathematics sessions, and the legend of the forced labour seems to disappear.

**Kyriakos I. Petakos**
Hellenic Naval Academy and
Technological Educational Institute of Athens

**Reference**

1. D. Berkey, *Calculus* (Saunders College Publishing, 1983), p. 492.

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Two players A and B play a game on a $(2n-1)\times(2n-1)$ chessboard, $n>0$. Player A starts in the bottom right-hand corner and on each move takes a step left or up. Player B starts in the bottom left-hand corner and on each move takes a step right or up. If A and B move alternately, starting with A, and B cannot see A's moves, how many times must B ask A for his position to ensure B meets A at some point?

MANSUR BOASE
(53 Hamilton Place, London)

# Computer Column

## Ray tracing

Many readers will have used graphics programs to create two-dimensional (2D) images. The creation of realistic high-quality three-dimensional (3D) images using a computer is much more difficult. One possible approach is to use the method of *ray tracing*. The method is so called because the computer renders a scene by calculating the paths that light rays would follow if that scene were real.

In the real world, rays from a light source travel in all directions. Some rays will reflect off objects and enter your eye (or perhaps a camera) and form an image; but most rays will not contribute to the image. They will be absorbed by your skin, perhaps, or they might pass through a window and out into space. If the ray-tracing software were to calculate the paths of all the light rays that do not form an image, it would be impractical to try to render even a simple scene. In order to render scenes in a reasonable length of time, ray-tracing software cannot duplicate the way that light propagates in the real world. A ray-tracing program has to work backwards. It starts with a 'camera', and traces the rays backwards out into the scene; in this way, only those rays contributing to the final image are examined. When one of these 'viewing rays' hits an object, the software calculates the colour of the surface at that point. It does this by sending a 'shadow ray' back to the light source, to determine whether or not the surface point is in shadow. If the surface is transparent or reflective or has a special texture, further rays are created to determine the surface colour at that point.

Even this simplified ray-tracing strategy requires millions of calculations in order to render a scene. Fortunately, modern personal computers are now powerful enough to render photo-realistic images using the ray-tracing technique. One of the most popular ray-tracing packages is POV-Ray™, which is *freely* available for most major operating systems (`http://www.povray.org/`). In the rest of this article, I will briefly describe how to use POV-Ray to render a simple 3D scene.

You describe a scene in POV-Ray using a 'scene description language': simply type various commands into a plain text file, and then use POV-Ray to read the file and render the image you have described. The scene file consists of three types of elements. First, you must include a camera in the scene and also describe its location and the direction in which it is pointing. POV-Ray uses this information effectively to take a snapshot of the scene. Second, you must include at least one light source, and give its position. A light source can have any colour, and it may be a point source, a spotlight source or something more complicated. Third, your scene must include at least one object. This is what the light source illuminates, and what the camera records.

Before we consider a typical scene file, we should mention the 3D coordinate system used by POV-Ray. The positive $x$-axis points to the right, the positive $y$-axis points up and the positive $z$-axis points *into* the screen (see figure 1).

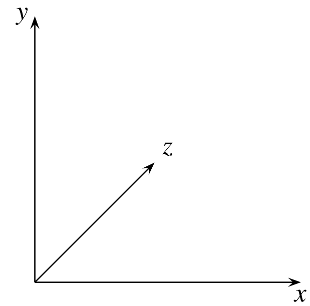


**Figure 1.** A 'left-handed' 3D coordinate system.

We can now begin to describe a scene. We do this by using a simple text editor to write a file — called `test.pov` say — containing commands in the scene description language. As an example, consider the program listing below. The first two lines are 'standard includes'; they enable us to use standard colours and textures. The next statement gives the location of the camera (point (5, 5, −5) in our coordinate system) and the direction in which the camera looks. We then define a white point source of light at location (1, 2, −3), and then set the 'sky' to be white using the background statement. The remaining elements in any scene are the objects at which we want to look.

```
#include "colors.inc"
#include "textures.inc"
camera{
       location  <5.0, 5.0, -5.0>
       look_at   <0.0, 0.0,  0.0> >
}

light_source{ <1, 2, -3> color White }

background { color red 1 green 1 blue 1 }

union{
       sphere{ <0, 0.5, 0> ,  2 }
       sphere{ <2, 1.5, 0.5> ,  1.5 }

       pigment {color White filter 1}
       finish{
              ambient 0
              diffuse 0
              reflection .25
              specular 1
              roughness .001
       }
       interior{ior 1.33}
}

plane{
       <0, 1, 0>,
       -3
       pigment {checker Green Blue}
}
```

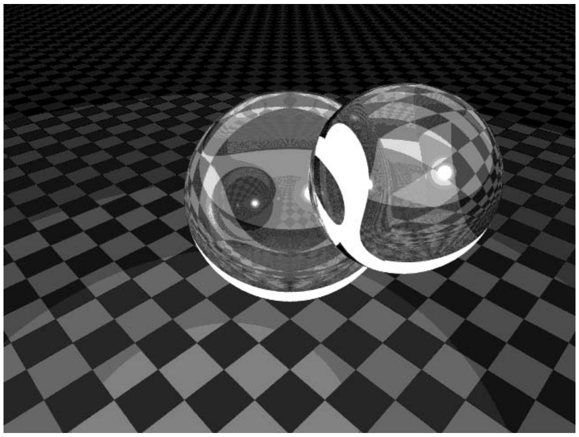

**Figure 2.**

To define a sphere, which is perhaps the simplest object, we need only define its radius and the location of its centre. In this example we define two spheres, one with radius 2 and one with radius 1.5. The *union* statement lets us add these two shapes together, and treat them as one object. We can then transform the object as a whole, and apply a single texture to the entire union. In this case we use the *pigment* statement to make this union of two spheres white, and then we apply a particular *finish* that corresponds to making this object out of glass. The index of refraction of the combined object is 1.33. Finally, we define a plane by giving the surface normal (in this case, vector $\langle 0, 1, 0\rangle$), and the distance the plane is displaced along the normal from the origin (in this case, by 3 units in the negative direction). The *pigment* statement says that the plane is covered by green and blue squares, in a chessboard pattern.

So, when illuminated by a point source of light, what does a union of glass spheres suspended above an infinite chessboard look like? We simply run POV-Ray to find out! The result is shown (in black and white) in figure 2.

You can use POV-Ray to create very much more complicated images than this. You might want to generate objects with a mathematical theme: a three-dimensional Sierpinski gasket made out of marble, for example, or a helix constructed of stone. You will find that you can create many strange and beautiful images using ray-tracing software.

**Stephen Webb**

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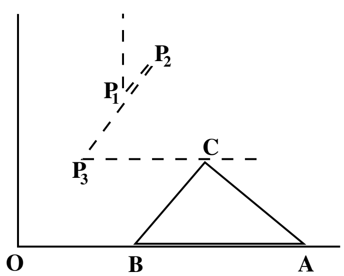
**Solution to Braintwister 7**
(Table tour)

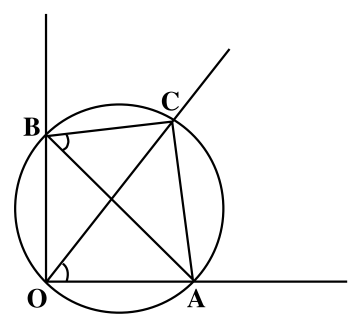
*Answer:* The triangle has sides 20 cm, 21 cm and 29 cm.

*Solution:*
Assume that the triangle is ABC with sides $a$, $b$, $c$ (with $a$ opposite A etc.) and with $a < b < c$. Then, in particular, $b = a + 1$ but we shall not use that until later.




When travelling along one side of the table (with the side AB of the triangle flush with that side) the vertex C travels $c$ centimetres less far than A. However, at each corner C's route is $P_1P_2P_3$, as illustrated on the left above, where (rather surprisingly) $P_1$, $P_2$, and $P_3$ are in a straight line with the corner.

To confirm that C's route is straight, look at the right-hand figure. The two right-angles ensure that OABC is cyclic and then angles subtended in a circle show that the angle COA always equals the angle B of the triangle. You can soon calculate that $P_1$ is distance $b$ from O, that $P_2$ is $c$ from O, and that $P_3$ is $a$ from O. Hence at each corner C travels $(c - b) + (c - a)$.




Therefore in one tour of the perimeter the distance travelled by A exceeds that of C by $4c - 4(2c - a - b)$, and we are told that this is 48. Hence we get three equations:

$$a + b - c = 12, \quad b = a + 1, \quad a^2 + b^2 = c^2.$$

These give that $c = 2a - 11$ and substituting for $b$ and $c$ in the third equation soon gives $a^2 - 23a + 60 = 0$ with $a = 20$ ($b = 21$, $c = 29$) as the only viable solution.

VICTOR BRYANT

# Letters to the Editor

Dear Editor,

*Unit fractions*

Mr Turner (his letter in Volume 31, Number 2) is being a wee bit unfair to Ahmes, on two counts. First, Ahmes (described elsewhere as a scribe rather than a priest) was copying a work written two centuries earlier, and thus recording the collective wisdom of earlier Egyptian mathematicians rather than his own work. Secondly, the expression of $2/(2n+1)$ as a sum of unit fractions is fundamental to Egyptian arithmetic since their whole system was based on (a) doubling, halving and adding — i.e. they were effectively using a binary based system — (b) with the exception of 2/3, all fractions had to be in terms of unit fractions; and (c) for some odd reason, when they expressed a fraction in terms of unit fractions these all had to be different (otherwise, of course, $2/(2n+1)$ is easily written as $1/(2n+1) + 1/(2n+1)$!).

Stuart Hollingdale (*Makers of Mathematics*, Penguin, 1988) says that the Rhind papyrus gives the method of calculaton, as well as the final result, for all $n$ up to 50, so to that extent we *do* know how the unit fractions were derived, although I do not have access to this information.

Hollingdale gives a worked example for 2/7, but the algorithm used is not clearly spelt out. However, the earlier comment that the Egyptian arithmetic is binary based suggests such an algorithm.

As an example, suppose we wish to express 2/29 — the example quoted by Mr Turner — as a sum of unit fractions. First turn 29 into binary, i.e. 11101, and then divide through by the highest power of 2 in the expression — in this case 16 — to give $29/16 = 1.1101$. Then add to both sides powers of $\frac{1}{2}$ to make the right-hand side equal to 2 — in this case 1/8 (0.001) and 1/16 (0.0001).

Then we have

$$\frac{29}{16} + \frac{1}{8} + \frac{1}{16} = 2$$

or, dividing through by 29,

$$\frac{1}{16} + \frac{1}{232} + \frac{1}{464} = \frac{2}{29}.$$

Sadly, however, this, although correct, does not match the equally correct expression quoted by Mr Turner.

So — can anyone explain how the specific Rhind papyrus figures were arrived at? Clearly there is not a unique representation of $2/(2n+1)$ as a sum of unit fractions, but the derivation above would seem to be the simplest procedure to use.

Yours sincerely,

ALAN. D. COX
(Pen-y-Maes, Ostrey Hill,
St Clears, Carms, SA33 4AJ, UK.)

Dear Editor,

*On a problem of K. R. S. Sastry*

In reference 1, K. R. S. Sastry noticed that by adjoining the consecutive integers 183 and 184 one obtains a perfect square:

$$183\,184 = 428^2\,. \tag{1}$$

He asks us to find another consecutive pair with such a property. Instead of just finding some more examples, it would have been interesting to characterize all pairs of consecutive integers with the above property.

Let $n \geqslant 1$ be a positive integer. Denote by $\mathcal{P}(n)$ the set of all integers $a$ such that both $a$ and $a+1$ have exactly $n$ digits and such that by adjoining $a$ and $a+1$ one obtains a perfect square. With our notation, characterizing all pairs of consecutive integers with the above property reduces to describing $\mathcal{P}(n)$ for various values of $n$. This was already done by Alan D. Cox in reference 2 for $n \leqslant 6$.

We will first give a result which gives some information on the size of $\mathcal{P}(n)$. Let $N(n)$ be the number of elements of $\mathcal{P}(n)$. For every positive integer $k > 1$, let $\omega(k)$ be the number of distinct prime divisors of $k$. We have the following theorem.

**Theorem 1.** *Let* $m = 10^n + 1$. *Then*

$$2^{\omega(m)-1} - 1 \leqslant N(n) \leqslant 2(2^{\omega(m)-1} - 1). \tag{2}$$

From theorem 1, it can be shown that $\mathcal{P}(n)$ is non-empty if and only if $10^n + 1$ is not a prime. In particular, $\mathcal{P}(1)$ and $\mathcal{P}(2)$ are empty. For example,

$$\begin{aligned} 1001 &= 7 \times 11 \times 13, \\ 10\,001 &= 73 \times 137, \\ 100\,001 &= 11 \times 9091, \\ 1000\,001 &= 101 \times 9091, \end{aligned}$$

so $\omega(3) = 3$, $\omega(4) = 2$, $\omega(5) = 2$, $\omega(6) = 2$ and $3 \leqslant N(3) \leqslant 6$, $1 \leqslant N(4) \leqslant 2$, $1 \leqslant N(5) \leqslant 2$, $1 \leqslant N(6) \leqslant 2$. In fact, from reference 2, $N(3) = 4$, $N(4) = 1$, $N(5) = 2$, $N(6) = 2$.

Assume now that $n$ is such that $10^n + 1$ is not a prime. One can describe $\mathcal{P}(n)$ as follows.

**Theorem 2.** *Let* $m = 10^n + 1$. *Then the following statements hold.*

1. *Let* $d_1$ *and* $d_2$ *be two proper divisors of* $m = 10^n + 1$ *such that* $\gcd(d_1, d_2) = 1$ *and* $d_1 d_2 = m$. *Then, the system*

$$\begin{cases} d_1 a_1 - d_2 a_2 = \pm 2, \\ 10^{n-1} \leqslant a_1 a_2 < 10^n - 1 \end{cases} \tag{3}$$

*has at least one and at most two positive solutions* $(a_1, a_2)$. *If one denotes* $a = a_1 a_2$, *then* $a \in \mathcal{P}(n)$.

2. *Let* $\{d_1, d_2\} \neq \{d_1' d_2'\}$ *be two distinct pairs of divisors of* $m$ *with the above property. Let* $(a_1, a_2)$ *and* $(a_1', a_2')$ *be solutions of systems* (3) *for* $\{d_1, d_2\}$ *and* $\{d_1', d_2'\}$, *respectively. Then* $a_1 a_2 \neq a_1' a_2'$.

3. *If* $a \in \mathcal{P}(n)$, *then there exist two divisors* $\{d_1, d_2\}$ *of* $m$ *with the above property and two positive integers* $a_1$ *and* $a_2$ *satisfying* (3) *such that* $a = a_1 a_2$.

The proof of theorem 2 is very technical and is beyond the scope of this article.

Notice that theorem 2 completely describes $\mathcal{P}(n)$ . Theorem 1 follows easily from theorem 2 by a counting argument. Moreover, using theorem 2 one can implement a fast algorithm for finding $\mathcal{P}(n)$ for any given value of $n$. In fact, solving system (3) for given divisors $d_1$ and $d_2$ of $m$ reduces to Euclid's algorithm for finding the highest common divisor. We implemented this algorithm using *Mathematica* and found $\mathcal{P}(n)$ for $n \leqslant 10$. The table lists all solutions $(a_1, a_2)$ of system (3) for all choices of divisors $\{d_1, d_2\}$ of $m$ such that $d_1 < d_2$, $\gcd(d_1, d_2) = 1$ and $d_1 d_2 = m$. We used $x$ to denote the number whose square is obtained by adjoining $a$ and $a+1$. The values of $a$ marked with a '⋆' are the ones that do not satisfy the inequality of (3) and therefore do not lead to elements $a$ of $\mathcal{P}(n)$. The values $n = 1$ and $n = 2$ do not appear in the table because $\mathcal{P}(1) = \mathcal{P}(2) = \emptyset$. K. R. S. Sastry's example $a = 183$ is obtained for $n = 3$, $d_1 = 7$, $d_2 = 143$, $a_1 = 61$ and $a_2 = 3$.

A close inspection of the table reveals the fact that $a = 66 \times 8$, $a = 6666 \times 68$ and $a = 666\,666 \times 668$ are elements of $\mathcal{P}(3)$, $\mathcal{P}(6)$ and $\mathcal{P}(9)$ respectively. One might wonder if this is coincidental or if these elements are just part of a general pattern. In fact, one can easily show that these elements are members of the following parametric family of elements of $\mathcal{P}(n)$ when $n$ is a multiple of 3.

**Proposition.** *Let* $k \geqslant 1$ *be a positive integer. The number*

$$a = \underbrace{66\ldots6}_{2k \text{ times}} \times \underbrace{66\ldots6}_{k-1 \text{ times}} 8$$

*is an element of* $\mathcal{P}(3k)$.

*Proof.* Clearly

$$a > 6 \times 10^{2k-1} \times 6 \times 10^{k-1} = 3.6 \times 10^{3k-1},$$

and

$$a > 7 \times 10^{2k-1} \times 7 \times 10^{k-1} = 4.9 \times 10^{3k-1}.$$

Hence, both $a$ and $a+1$ have exactly $n = 3k$ digits. Write

$$\begin{aligned} a &= \tfrac{6}{9} \cdot (10^{2k} - 1) \cdot \left(\tfrac{6}{9} \cdot (10^{k-1} - 1) \times 10 + 8\right) \\ &= \tfrac{4}{9} \cdot (10^{2k} - 1) \times (10^k + 2). \end{aligned}$$

Then the number obtained by adjoining $a$ and $a+1$ is

$$\begin{aligned} 10^{3k} a + a + 1 &= (10^{3k} + 1)a + 1 \\ &= \tfrac{4}{9} \times (10^{2k} - 1) \times (10^k + 2) \cdot (10^{3k} + 1) + 1. \end{aligned}$$

| $n$ | $d_1$ | $d_2$ | $a_1$ | $a_2$ | $a$ | $x$ | $a_1$ | $a_2$ | $a$ | $x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 143 | 82 | 4 | 328 | 573 | 61 | 3 | 183 | 428 |
| 3 | 11 | 91 | 25 | 3 | ⋆ 75 | | 66 | 8 | 528 | 727 |
| 3 | 13 | 77 | 12 | 2 | ⋆ 24 | | 65 | 11 | 715 | 846 |
| 4 | 73 | 137 | 107 | 57 | 6099 | 7810 | 30 | 16 | ⋆ 480 | |
| 5 | 11 | 9091 | 3306 | 4 | 13 224 | 36 365 | 5785 | 7 | 40 495 | 63 636 |
| 6 | 101 | 9901 | 3235 | 33 | 106 755 | 326 734 | 6666 | 68 | 453 288 | 673 267 |
| 7 | 11 | 909 091 | 495 868 | 6 | 2975 208 | 5454 547 | 413 223 | 5 | 2066 115 | 4545 454 |
| 8 | 17 | 5882 353 | 3114 187 | 9 | 28 027 683 | 52 941 178 | 2768 166 | 8 | 22 145 328 | 47 058 823 |
| 9 | 7 | 142 857 143 | 122 448 980 | 6 | 734 693 880 | 857 142 859 | 20 408 163 | 1 | ⋆20 408 163 | |
| 9 | 11 | 90 909 091 | 8264 463 | 1 | ⋆ 8264 463 | | 82 644 628 | 10 | 826 446 280 | 909 090 909 |
| 9 | 13 | 76 923 077 | 29 585 799 | 5 | 147 928 995 | 384 615 386 | 47 337 278 | 8 | 378 698 224 | 615 384 615 |
| 9 | 19 | 52 631 579 | 49 861 496 | 18 | 897 506 928 | 947 368 423 | 2770 083 | 1 | ⋆ 2770 083 | |
| 9 | 77 | 12 987 013 | 12 312 363 | 73 | 898 802 499 | 948 051 950 | 674 650 | 4 | ⋆ 2698 600 | |
| 9 | 91 | 10 989 011 | 2656 684 | 22 | ⋆58 447 048 | | 8332 327 | 69 | 574 930 563 | 758 241 758 |
| 9 | 133 | 7518 797 | 6048 957 | 107 | 647 238 399 | 804 511 280 | 1469 840 | 26 | ⋆38 215 840 | |
| 9 | 143 | 6993 007 | 3325 346 | 68 | 226 123 528 | 475 524 477 | 3667 661 | 75 | 275 074 575 | 524 475 524 |
| 9 | 209 | 4784 689 | 183 146 | 8 | ⋆ 1465 168 | | 4601 543 | 201 | 924 910 143 | 961 722 488 |
| 9 | 247 | 4048 583 | 1344 064 | 82 | 110 213 248 | 331 983 807 | 2704 519 | 165 | 446 245 635 | 668 016 194 |
| 9 | 1001 | 999 001 | 332 335 | 333 | 110 667 555 | 332 667 334 | 666 666 | 668 | 445 332 888 | 667 332 667 |
| 9 | 1463 | 683 527 | 612 044 | 1310 | 801 777 640 | 895 420 371 | 71 483 | 153 | ⋆10 936 889 | |
| 9 | 1729 | 578 369 | 109 385 | 327 | ⋆35 768 895 | | 468 984 | 1402 | 657 515 568 | 810 873 337 |
| 9 | 2717 | 368 053 | 155 647 | 1149 | 178 838 403 | 422 892 898 | 212 406 | 1568 | 333 052 608 | 577 107 103 |
| 9 | 19 019 | 52 579 | 14 724 | 5326 | ⋆78 420 024 | | 37 855 | 13 693 | 518 348 515 | 719 964 246 |
| 10 | 101 | 99 009 901 | 39 211 842 | 40 | 1568 473 680 | 3960 396 041 | 59 798 059 | 61 | 3647 681 599 | 6039 603 960 |

Let $X = 10^k$. The above expression becomes

$$10^{3k}a + a + 1 = \tfrac{4}{9} \times (X^2 - 1) \times (X + 2) \times (X^3 + 1) + 1 .$$

From the identity

$$\tfrac{4}{9} \times (X^2 - 1) \times (X + 2) \times (X^3 + 1) + 1 = \left(\frac{1 - 2X + 2X^2 + 2X^3}{3}\right)^2 ,$$

it follows that $10^{3k}a + a + 1 = x^2$, where

$$x = \left(\frac{1 - 2 \cdot 10^k + 2 \cdot 10^{2k} + 2 \cdot 10^{3k}}{3}\right) .$$

Notice that $x$ is a positive integer.

A different parametric family of elements of $\mathcal{P}(n)$ for $n = 3k$ is obtained by considering the complementary solutions of system (3) to the ones pointed out in the above proposition. These complementary solutions are the ones listed at the same rows of the table as the ones previously considered but in the left columns labelled $a_1$ and $a_2$. This solution has the form,

$$a = \underbrace{33\ldots3}_{k-1 \text{ times}}\,2\,\underbrace{33\ldots3}_{k-1 \text{ times}}\,5 \times \underbrace{33\ldots3}_{k \text{ times}}$$

for $n = 3k$ and $k \geqslant 2$. For $k = 1$ this gives $a = 25 \times 3 = 75$, which does not satisfy the inequality of (3).

Finally we propose the following problem.

**Problem.** For $n \geqslant 1$, *find an exact formula for* $N(n)$.

One can notice from the examples that the bounds on $N(n)$ given by (2) of theorem 1 are sharp. This means that both the left and the right bounds can be achieved for suitable values of $n$. For example from the table, one can see that the lower bound is achieved when $n = 4$ and that the upper bound is achieved when $n = 5, 6, 7, 8$, and 10. On the other hand, $N(9) = 20$ is considerably different from both the lower bound 15 and the upper bound 30 given by (2) of theorem 1.

Notice also that solving the above problem is equivalent to characterizing those pairs of divisors $\{d_1, d_2\}$ of $m = 10^n + 1$ satisfying the property from the hypothesis of Theorem 2 such that system (3) has exactly one solution (or equivalently, such that system (3) has exactly two solutions).

Yours sincerely,
FLORIAN LUCA,
(Visiting Asst. Professor of Mathematics,
Syracuse University, New York)

**References**

1. K. R. S. Sastry, Problem, *Mathematical Spectrum* **30** (1997/8), p. 9.
2. A. D. Cox, Letter to the Editor, *Mathematical Spectrum* **30** (1997/8), p. 42

Dear Editor,

*Areas of Polygons*

We have been investigating the areas $A_n$ of $n$-sided regular polygons with fixed perimeter $p$ and their relation to the area $A$ of the circle with perimeter $p$.

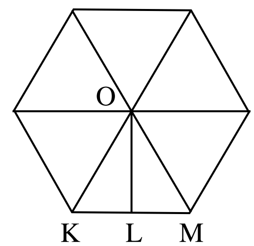



**Figure 1.**

In figure 1 (where $n$ is taken to be 6) KM= $p/n$ and the angle MOL is $\pi/n$. Hence ML/OL= $\tan(\pi/n)$, so that OL= $(p/2n)\cot(\pi/n)$; and the area of the triangle OKM is $(p^2/4n^2)\cot(\pi/n)$, while the area of the polygon is

$$A_n = \frac{p^2}{4n\tan(\pi/n)} .$$

The circle with perimeter $p$ has radius $p/2\pi$ and so its area is

$$A = \frac{p^2}{4\pi} .$$

Thus

$$A_n = A\frac{\pi/n}{\tan(\pi/n)}$$

or, if $\pi/n = x$,

$$A_n = A\frac{x}{\tan x} .$$

Now it is known from geometrical considerations that

$$\sin x < x < \tan x$$

and therefore

$$\cos x < \frac{\sin x}{x} < 1.$$

Since $\cos x \to 1$ as $x \to 0$, it follows that $(\sin x)/x \to 1$ as $x \to 0$, and therefore also that

$$\frac{x}{\tan x} = \frac{x}{\sin x}\cos x \to 1 \quad \text{as } x \to 0 .$$

Hence

$$A_n \to A \quad \text{as } n \to \infty .$$

Finally put

$$f(x) = \frac{x}{\tan x} .$$

Then

$$f'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x} < 0 .$$

So $f(x)$ decreases as $x$ increases, or $f(x)$ increases as $x$ decreases. Hence $A_n$ increases and tends to $A$ as $n$ increases and tends to $\infty$, which implies that $A_n < A$ for all $n$. Thus the area of the circle with perimeter $p$ is greater than the area of the $n$-sided regular polygon with perimeter $p$.

Yours sincerely,
ANDALEEB AHMED AND ALEX COWARD
(Woodhouse Sixth Form College, London)

Dear Editor,

*Problem 30.5*

The solution given by Jeremy Young in Volume 31, p. 21 can be generalized to prove that

$$\begin{aligned} a^{p+2} + b^{p+2} + c^{p+2} + abc(a^{p-1} + b^{p-1} + c^{p-1}) \\ \geqslant 2(a^{(p/2)+1}b^{(p/2)+1} + b^{(p/2)+1}c^{(p/2)+1} \\ + c^{(p/2)+1}a^{(p/2)+1}), \end{aligned}$$

where $a$, $b$, $c$ and $p$ are positive real numbers.

Yours sincerely,
ZHANG YUN, Graduate Student
(Northwest Normal University,
Zan Zhou, China)

Dear Editor,

*Lazy multiplication*

As a break from deeper problems, I offer the following: consider two 2-digit numbers $a$ and $b$ and their 100 complements $\bar{a}$ and $\bar{b}$, i.e.

$$a = 100 - \bar{a} \quad \text{and} \quad b = 100 - \bar{b}\,.$$

Then, the product $ab = (a - \bar{b})100 + \bar{a}\bar{b}$. Note that $(a - \bar{b}) = (b - \bar{a})$. This may easily be proved. For numbers close to 100, this provides a quick answer, for example,

$$\begin{aligned} &88(\text{compl} = 12) \times 97(\text{compl} = 3) \\ &\quad = (88 - 3)100 + 12 \times 3 \\ &\quad = 8536\,. \end{aligned}$$

Provided $\bar{a}\bar{b} < 100$, there is no 'carry' to be considered.

This method may be extended to numbers just over 100. In this case,

$$a = 100 + \bar{a}, \quad b = 100 + \bar{b}$$

and

$$ab = (a + \bar{b})100 + \bar{a}\bar{b}.$$

For example,

$$\begin{aligned} 119 \times 104 &= (119 + 4)100 + 19 \times 4 \\ &= 12\,376. \end{aligned}$$

Larger numbers may be treated in the same way. For example,

$$\begin{aligned} 991 \times 996 &= (991 - 4)1000 + 9 \times 4 \\ &= 987\,036. \end{aligned}$$

(Note that here 36 must fill 3 places; 036 must be used.)

This principle may be further extended to multiples of 100 or 1000, etc. Thus, if

$$a = 100n \pm \bar{a}, \quad b = 100n \pm \bar{b},$$

then

$$ab = (a \pm \bar{b})100n + \bar{a}\bar{b}.$$

For example, if

$$\begin{aligned} a &= 294, \quad \bar{a} = 6, \\ b &= 289, \quad \bar{b} = 11, \end{aligned}$$

then $n = 3$ and

$$\begin{aligned} ab &= 294 \times 289 = 283 \times 300 + 6 \times 11 \\ &= 84\,966. \end{aligned}$$

Yours sincerely,
BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG)

# Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

## Problems

**31.9** A ladder 3 metres long is leaning against a wall, and there is a point on the ladder which is 1 metre from the wall and 1 metre from the ground. How far is the ladder from the wall?

(Submitted by Peter Davidson, Chelmsford)

**31.10** What is the probability that, when arranged to form a non-decreasing sequence, the six numbers chosen in the UK national lottery (distinct numbers between 1 and 49) alternate odd-even or even-odd?

(Submitted by Jeremy Young, Nottingham High School)

**31.11** (i) The sequence $(p_n)$ satisfies the relation

$$p_{n+1} = (p_0 p_1 \dots p_n) + 2,$$

for $n = 0, 1, 2, \dots$, and $p_0 = 3$. Find $p_n$.
(ii) The sequence $(q_n)$ satisfies the relation

$$q_{n+1} = (q_0 q_1 \dots q_n) + 4,$$

for $n = 0, 1, 2, \dots$. What can be said about $q_n$ for $n \geqslant 2$?

(Submitted by J. A. Scott, Chippenham)

**31.12** Prove that, for every positive integer $n$,

$$\sqrt[n]{n} < 1 + \frac{1}{\sqrt{n}}.$$

(Submitted by H. A. Shah Ali, Tehran)

## Solutions to Problems in Volume 31 Number 1

**31.1** Prove that, for positive real numbers $a$, $b$, $c$,

$$\frac{abc(a+b+c+\sqrt{(a^2+b^2+c^2)})}{(a^2+b^2+c^2)(ab+bc+ca)} \leqslant \frac{3+\sqrt{3}}{9}.$$

Can this be generalized?

*Solution* by Andrew Holland, Nottingham High School

$$\begin{aligned}\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} &\leqslant \frac{a+b+c}{3}\\ &\leqslant \sqrt{\frac{a^2+b^2+c^2}{3}}\end{aligned}$$

for all positive real numbers $a$, $b$, $c$ (the harmonic mean $\leqslant$ the arithmetic mean $\leqslant$ the quadratic mean) so

$$\frac{3abc}{ab+bc+ca} \leqslant \sqrt{\frac{a^2+b^2+c^2}{3}},$$

so

$$\frac{abc}{(ab+bc+ca)\sqrt{a^2+b^2+c^2}} \leqslant \frac{\sqrt{3}}{9}.$$

Also

$$a+b+c \leqslant \sqrt{3}\sqrt{a^2+b^2+c^2},$$

so

$$\frac{a+b+c+\sqrt{a^2+b^2+c^2}}{\sqrt{a^2+b^2+c^2}} \leqslant \sqrt{3}+1.$$

Hence

$$\begin{aligned}\frac{abc\left(a+b+c+\sqrt{a^2+b^2+c^2}\right)}{(a^2+b^2+c^2)(ab+bc+ca)} &\leqslant \frac{\sqrt{3}}{9}(\sqrt{3}+1)\\ &\leqslant \frac{3+\sqrt{3}}{9}.\end{aligned}$$

This generalizes to

$$\frac{a_1+a_2+\cdots+a_n+\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}{(a_1^2+a_2^2+\cdots+a_n^2)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)} \leqslant \frac{n+\sqrt{n}}{n^2}$$

for positive real numbers $a_1$, $a_2, \dots,$ $a_n$, with the same proof.

Also solved by Jeremy Young.

**31.2** Are there three rational numbers whose product is 1 and whose sum is zero?

*Solution* by Jeremy Young

Suppose there are, and write them as $a_1/b_1$, $a_2/b_2$, $a_3/b_3$, where $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, $b_3$ are non-zero integers and $\operatorname{hcf}(a_1, b_1) = 1$, $\operatorname{hcf}(a_2, b_2) = 1$, $\operatorname{hcf}(a_3, b_3) = 1$. Because their sum is zero,

$$a_1 b_2 b_3 + a_2 b_3 b_1 + a_3 b_1 b_2 = 0. \tag{1}$$

Because their product is 1,

$$a_1 a_2 a_3 = b_1 b_2 b_3. \tag{2}$$

Suppose that $p$ is a prime number dividing $a_1$. From (2), $p \mid b_1 b_2 b_3$ and $p \nmid b_1$, so either $p \mid b_2$ or $p \mid b_3$. If $p \mid a_2$, then $p \nmid b_2$ and, by (1), $p \mid a_3b_1b_2$, so $p \mid a_3$. So $p \mid a_3$ and $p \mid b_3$, which is impossible. Hence $p \nmid a_2$, and similarly $p \nmid a_3$. Let $p^m$ be the highest power of $p$ to divide $a_1$. Then, by (2), $p^m$ is the highest power of $p$ to divide $b_2b_3$ so we can write $b_2 = p^r u$ and $b_3 = p^s v$ where $r + s = m$ and $p \nmid u$, $p \nmid v$.

Assume first that $r < s$ We have $p^{r+s+m} \mid a_1b_2b_3$ so, by (1), $p^{2m} \mid (a_2b_3b_1 + a_3b_1b_2)$ so $p^{2m} \mid (a_2b_3 + a_3b_1)$, i.e. $p^{2m} \mid (a_2p^sv + a_3p^ru)$. Hence $p^{2m-r} \mid (a_2p^{s-r}v + a_3u)$. But this means that $p \mid a_3u$, which is not true. Hence $r \not< s$, and similarly $s \not< r$. Hence $r = s$. Thus $p^{2r} \mid a_1$, $p^r \mid b_2$, $p^r \mid b_3$ and these are the highest such powers of $p$. Also $p \nmid a_2$, $p \nmid a_3$. Similarly for $a_2$, $a_3$. It follows that we can write

$$\begin{aligned} a_1 &= x^2, \quad a_2 = y^2, \quad a_3 = z^2\,, \\ b_1 &= yz, \quad b_2 = zx, \quad b_3 = xy\,, \end{aligned}$$

for some pairwise coprime integers $x$, $y$, $z$. Now (1) gives

$$xyz(x^3 + y^3 + z^3) = 0,$$

so $x^3 + y^3 + z^3 = 0$. But there are no solutions to this equation by Fermat's Last Theorem for the case of exponent 3.

**31.3** (a) What are the probabilities of correctly picking exactly $r$ numbers in the UK national lottery for values of $r$ from 0 to 6?

(b) What is the probability that the winning numbers will have a common factor larger than 1?

(The winning numbers are an unordered random choice of six distinct numbers from 1 to 49.)

*Solution* by Jeremy Young

(a) The probability of correctly picking exactly $r$ numbers is

$$\binom{6}{r}\binom{43}{6-r}\Big/\binom{49}{6}.$$

(b) The probability that all six numbers have a common prime factor is

$$\binom{\left[\frac{49}{p}\right]}{6}\Big/\binom{49}{6}.$$

The possibilities for $p$ are 2, 3, 5, 7. This counts twice the $\binom{8}{6}$ cases of numbers divisible by 6 since they are divisible by 2 and by 3. Hence the probability is

$$\left(\binom{24}{6}+\binom{16}{6}+\binom{9}{6}+\binom{7}{6}-\binom{8}{6}\right)\Big/\binom{49}{6},$$

which is approximately 0.01.

**31.4** Does the equation

$$x^4 = 2^n + 3^n + 5^n$$

have any solution for positive integers $x$,$n$?

*Solution* by Jeremy Young

For $n$ even,

$$2^n + 3^n + 5^n \equiv (-1)^n + 0^n + (-1)^n \equiv 2 \pmod 3\,,$$

yet $x^4 \equiv 0$ or $1 \pmod 3$. For $n$ odd,

$$2^n + 3^n + 5^n \equiv 2^n + 3^n + (-2)^n \equiv 3^n \pmod 7\,.$$

Now $3^6 \equiv 1 \pmod 7$, so $3^{6k+1} \equiv 3 \pmod 7$, $3^{6k+3} \equiv 6 \pmod 7$, $3^{6k+5} \equiv 5 \pmod 7$. But

$$\begin{aligned} x \equiv 0 \pmod 7 &\Rightarrow x^4 \equiv 0 \pmod 7\,, \\ x \equiv \pm 1 \pmod 7 &\Rightarrow x^4 \equiv 1 \pmod 7\,, \\ x \equiv \pm 2 \pmod 7 &\Rightarrow x^4 \equiv 2 \pmod 7\,, \\ x \equiv \pm 3 \pmod 7 &\Rightarrow x^4 \equiv 4 \pmod 7\,. \end{aligned}$$

So in both cases the equation has no solution.

Also solved by Andrew Holland, George Robinson (Gresham's School, Holt).

# Reviews

**Review of Laboratory Experiences in Group Theory.** By Ellen Maycock Parker. MAA, Washington, 1996. Pp. 112. Paperback $26.50 (ISBN 0-88385-705-7).

This book is a laboratory manual designed to accompany a disc of software, called Exploring Small Groups, which investigates groups of order 16 or less via their Cayley tables. The book comes with the software disc. The aim of the manual is to enable students to explore the early ideas of group theory in the context of specific examples. The book only contains statements of definitions and theorems which it uses. Any course following this book is expected to use another text in parallel for proofs etc.

The book is divided into laboratory sessions. Some of these sessions require a significant amount of preparatory work before the computer is used. Students are encouraged to do certain calculations by hand before relying on the computer.

The program itself consists of setting up a Cayley table, either from scratch or by choosing an in-built one. The notation for the elements of the group can be changed to suit the student at this stage. The software will check the group axioms and generate subgroups, cosets, the centre, commutator subgroup, and factor groups from the table. I found it reasonably straightforward to use, though I suspect some students will have difficulty in finding the menus they need. The first session could be fairly hectic for a demonstrator if the class is of any size!

The book contains a large number of exercises, starting with checking whether Cayley tables form a group and ending with looking for Sylow subgroups. Diligent students will benefit from the large amount of practice they will obtain. They will also build up a considerable knowledge of small groups. Many of the questions they are required to answer are open-ended and require students to make conjectures. How successful this is will depend on the nature of the class and, of course, on the teacher. The book quite rightly suggests that students should collaborate with each other.

My reservations with this approach are more fundamental. The emphasis on Cayley tables will encourage students to regard groups as sets of elements to manipulate, i.e. passive objects, instead of sets of operations, i.e. active objects. The first chapter does indeed introduce symmetries and permutations but these then get forgotten in the morass of Cayley table calculations. No one would normally suggest that the Associative law is checked directly from a Cayley table, even with a computer to do the calculation. Although a well constructed Cayley table can show up a factor group effectively, the lack of reference to the nature of the elements hides the pedagogically useful idea that the equivalence classes may be rotations say, or reflections of a particular type. It would be important that any course using this text made students aware of the many easy examples of infinite groups.

Overall I believe the book succeeds well in its aim, which is to accompany the particular piece of software. I feel however that any user would need to use it with care to prevent students getting a rather narrow and mechanistic view of Group Theory.

Trinity and All Saints University College, Leeds.

CAMILLA JORDAN

**Numbers & Proofs** By R. B. J. T. ALLENBY. Arnold, London, 1997. Pp. x+274. Paperback £14.99 (ISBN 0-340-6765-31).

G. Pólya's famous *How to Solve It* and Allenby's *Numbers & Proofs* have the same aim in helping the reader to become more adept at solving problems. However, where Pólya does this by describing a method to be applied to each problem and takes his specific examples mainly from geometry, Allenby (whose problems mainly concern numbers) tries to give the reader an insight into the thought processes a mathematician might have when confronted with a new question.

First of all comes a chapter on 'The need for proof' in which Allenby discusses what constitutes a proof and why they are necessary to mathematics. Following that, the chapters are set out in an order that starts by exploring what mathematical statements actually mean, translating these into forms that are more conductive to analysis, and finally chapters that describe different mathematical areas and techniques. Special mention must go to Chapter 13: 'Fallacies and paradoxes – and mistakes'. The chapter is especially useful becuase it forces you to examine critically every step in a proof, and in this way to learn a lot more about proofs than just by examining correct arguments.

Each chapter contains a lot of prose, which I found informal yet also immensely relevant. Allenby includes lots of 'Discussions', which are sections intended to give the reader ideas on how it might be best to proceed with proofs. These allow you, in effect, to see how an experienced mathematician might start to tackle a problem and to encounter the various successes and blind alleys to which he is led. In addition to the 'Discussions' are 'Comments' and 'What we have learned' sections that discuss the general principles introduced by the proofs.

In order that the reader can practise constructing proofs of his/her own, there is a satisfying number of exercises spread throughout all of the chapters. In addition, there is an Appendix of hints and solutions to the exercises, in case any defeat you. The final chapter of the book is just a collection of questions, of varying difficulty and topic, which provide further practice, in addition to the earlier problems.

In conclusion, reading this book is valuable instruction for all new undergraduates and others who wish to learn about constructing proofs.

Student, St Olave's Grammar School — ANDREW LOBB

**Geometry Turned On! Dynamic Software in Learning, Teaching and Research.** Edited by JAMES R. KING & DORIS SCHATTSNEIDER. MAA, Washington, 1997. Pp. xiv+206. Paperback \$39.95 (ISBN 0-88385-099-0).

There is much software available that makes computer graphics and animation an instrument for creating and manipulating geometric displays; for example, various *Cabri* versions, *Geometry Inventor, The Geometer's Sketchpad, The Geometric Supposer* and *Supersupposer, Geomview, Mathematical Bouncing Ball Lab* and *Physics Explorer: One Body*. This timely and valuable book addresses issues regarding the relationship of such software to formal mathematics and to mathematical education and describes a number of fascinating instances of geometrical exploration.

There are four parts as follows:

*Personal Reflections on Investigation, Discovery, and Proof* (5 chapters, 46 pages) is of interest from start to finish. I kept having to put the book down—to do some geometry.

*Making Geometry Dynamic in the Classroom* (14 chapters, 122 pages) is more for dipping into; it treats a wide range of topics including triangle areas, loci, transformations and group symmetries.

*Dynamic Visualization in History, Perception, Optics, and Aerodynamics* (4 chapters, 32 pages) was unfortunately made largely unreadable in the review copy by a printing fault.

*The Worlds of Dynamic Geometry; Issues in Design and Use* (3 chapters, 20 pages), deals with non-Euclidean geometry and declarative geometric programming. The *worlds* referred to include Poincaré's Euclidean model of non-Euclidean hyberbolic geometry, in which the student may be surprised by the behaviour of the software 'straight-edge', mystery worlds whose geometric properties the student might discover by exploration and worlds in which a geometric configuration retains its pre-declared properties (such as, H is the orthocentre of triangle ABC) even when its elements are redeployed.

One appendix gives half a dozen lines about each author, including E-mail addresses, and another gives details of software suppliers and their web sites.

The web site at

`http://forum.swarthmore.edu/dynamic/`

is a general source of material for dynamic software for geometry, at which the subdirectory `geometry turned on` holds material relating to this book.

University of Warwick JOHN MACNEILL

**Fermat's Last Theorem: Unlocking the Secret of an Ancient Mathematical Problem.** By AMIR D. ACZEL. Penguin, London, 1998. Paperback £5.99 (ISBN 0-14-026708-5).

This book charts the progress to a proof of a theorem Pierre de Fermat (1601–1665) claimed during his lifetime. Upon picking up this book I expected 140 pages of pure mathematical jargon which I would be unable, nor even wish, to understand. However, I was pleasantly surprised. He has managed in this one volume to combine many mathematical ideas and concepts with plain, readable English, taking the reader on a journey of maths through the ages.

The book consists of a number of sections which all bear some relevance to the Fermat problem. Most of these are about some mathematical concept which, when applied, can be used to prove the Fermat Theorem. The book also manages to provide us with a humorous insight into some of the mathematicians involved in the Fermat problem.

The book is based around a mathematical idea which, however, does not contain much 'maths', but rather lightly touches upon ideas far too complex for a work of fiction.

In this way the book is not only a good read, but one which is lightly educational without being overly boring.

Student, Gresham's School, Holt. BEN MULLEY

**The Contest Problem Book V** Edited by GEORGE BERZSENY AND STEPHEN B. MAURER. MAA, Washington, 1997. Pp. 308. Paperback $27.95 (ISBN 0-88385-640-9).

This is the fifth book of problems and solutions from the American High School Mathematics Examinations (AHSME) covering the six examinations in 1983–1988. It is also the first compilation of the follow-up American Invitational Mathematics (AIME) which began in 1983 as an intermediate step between the AHSME and the USA Mathematics Olympiad. The AIME has a unique answer format – every answer is an integer from 0 to 999.

Most problems are solvable using school arithmetic, algebra, geometry and trigonometry. There are also miscellaneous problems on combinations, elementary number theory and probability, but no calculus. Many of the problems lend themselves to modification, extension and generalization, and would provide simulating material for class investigations.

Perhaps one or two examples would give the best impression. Amongst those which my class (ranging from 12 to 17 years old) and I enjoyed are the following.

1. Exactly three of the angles of a convex polynomial are obtuse. What is the maximum number of sides of the polygon?

2. A cube of cheese $C = \{(x, y, z) \mid 0 \leqslant x, y, z \leqslant 1\}$ is cut along the planes $x = y$, $y = z$, and $z = x$.
   How many pieces are there?

3. Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

This book also contains sections on 'dropped problems'. These are all very instructive problems which, for various reasons, failed to be selected. Amongst them is the 'alternative' quadratic formula $2c/(-b \pm \sqrt{b^2-4ac})$, and its relevance in computing roots of certain quadratics.

There are many admirable features of this book. Often more than one solution is offered. There is an index, enabling the reader to locate problems on chosen topics. There are response frequency tables for the multiple choice AHSME problems. And there is a 15-page guide to the problem literature.

This book is not only a fine collection of entertaining problems, or useful material for the Maths Contest enthusiast. It also provides some elegant and rigorous mathematic arguments accessible to any intelligent school pupil. Much can be learnt from such a book, and every school should have a copy.

Queen Mary's Grammar School STEPHEN ROUT

**Other books received**

**Writing in the Teaching and Learning of Mathematics.** By JOHN MEIER AND THOMAS RISHEL. MAA, Washington, 1998. Pp. 114. Paperback $18.95 (ISBN 0-88385-158-X).

**Non Euclidean Geometry.** By H. M. S. COXETER. MAA, Washington, 1998. Pp. 336. Paperback $30.95 (ISBN 0-88385-522-4).

**MEI Structured Mathematics: Pure Mathematics 6.** By TERRY HEARD AND DAVID MARTIN. Hodder & Stoughton Educational, London, 1998. Pp. 285. Softback $14.99 (ISBN 0 340 68801 7).

**MEI Structured Mathematics: Statistics 5 & 6.** By ALEC CRYER, MICHAEL DAVIES, BOB FRANCIS AND GERALD GOODALL. Hodder & Stoughton Educational, London, 1998. Pp. 218. Softback $15.99 (ISBN 0 340 70132 3).

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