Mathematical Spectrum

1995/6 Volume 28 Number 2

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Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year, consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems and to mathematics in the classroom, as well as a computer column.

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G. H. Hardy (1877–1947)

H. BURKILL

A biography of one of the greatest British mathematicians, including some of his mathematical achievements.

For most of the first half of this century British first-year mathematics students regarded the name G. H. Hardy as synonymous with the mixture of excitement and trepidation they associated with the transition from school to university mathematics. At the beginning of this period Hardy's *A Course of Pure Mathematics* was the only textbook that developed elementary analysis rigorously from clearly stated principles. As time went on other books covering similar ground appeared, but Hardy's, going through numerous editions, never lost its freshness and continued as a favourite text until the development of mathematics brought about major changes of syllabus. Hardy himself had, in fact, been the chief instigator of the revolution in university teaching which, in the early 1900s, created the need for new textbooks in English. His passion for mathematics and brilliance in research, coupled with the force of his personality gave him unmatched influence in British mathematics. Educational debates gained immeasurably from the challenge to received wisdom posed by his radical ideas; and one cannot help wondering how he would have viewed today's problems in schools and universities.

Both Hardy's parents were teachers, born into homes too poor to afford the university education that their able offspring merited. His mother Sophia became senior mistress at Lincoln Training College before marrying, in 1875, Isaac Hardy who had moved from a grammar school in Lincolnshire to Cranleigh School in Surrey to teach geography and drawing. After his marriage Isaac Hardy also became bursar of the school and a housemaster, and he remained at Cranleigh for the rest of his life.

Godfrey Harold Hardy was born on 7 February 1877 and his sister two years later. The two children throve on their intellectual upbringing, with Harold exhibiting particular precocity. At the age of two he could write down numbers in the million range, and in church he amused himself by factorizing the hymn numbers. His life-long interest in history also showed early: he embarked on an illustrated History of England, but his wealth of knowledge was his undoing and he only reached the Anglo-Saxon period. Musing on his career towards the end of his life he thought that journalism would have been a reasonable second choice profession for him; indeed at the age of eight he produced a miniature newspaper containing a speech by Gladstone, a leading article, advertisements, and a detailed report of a cricket match. This last was symptomatic of an interest in all ball games that amounted to devotion in the case of cricket. In his adult life cricket came second only to mathematics; it gave him endless aesthetic and intellectual pleasure and in dark days it provided sure diversion and comfort. Cricket even permeated into his mathematics: in a paper published in a Scandinavian journal and intended for an international readership he wrote 'The problem is most easily grasped when stated in the language of cricket .... Suppose that a batsman plays, in a given season, a given 'stock' of innings ...'. It is not surprising that cricket coloured his conversation. For instance his highest praise was 'in the Hobbs class' until in the 1930s he was forced to create the even rarer distinction of a 'Bradman class'. As a boy Hardy accidentally struck his sister with a cricket bat and blinded her in one eye. This tragedy affected neither their mutual affection nor his enthusiasm as a player. Hardy inherited his father's artistic talent, but none of the musical gifts with which both his parents were endowed.

It was natural for Hardy to start his school career at the Cranleigh preparatory school and to proceed from there to the senior school. Incomparably abler than his fellow pupils, he could hardly fail to collect all available prizes. Unfortunately his acute shyness and self-consciousness made the inevitable public presentation an agonizing ordeal. To some extent these traits were with him all his life. For example, regarding himself as ugly, though others judged his face to be exceptionally fine, he could not bear mirrors and covered up any provided in hotel rooms. At the age of 12 Hardy reached the top form of Cranleigh School. This not only testified to his brilliance, but also reflected the very modest standards at a school which had been founded in the 1860s to cater for the sons of the prosperous but ill-educated local farmers and tradesmen. Nevertheless, a Cambridge don (college fellow) was retained by the school to report on the boys' performance and he was so impressed by Hardy's that he sent his work to Winchester College, which promptly awarded him a scholarship. However, Hardy was forced to mark time for a year as he was deemed to be too young for immediate entry.

One might have thought that Hardy, relishing the superb teaching, particularly in mathematics for which he was in a class of one, would have enjoyed his time at Winchester. In fact, he hated the spartan conditions and, after leaving, he never returned. He also bore the school one life-long grudge: he was never coached in cricket and therefore did not realise his potential as a player. It probably did not occur to anyone that this outstandingly intellectual boy had any interest in games. When he was 15 Hardy decided on his aim in life. He had come across an unctuous novelette in which the pious hero is elected to a fellowship at Trinity College, Cambridge, while his dissolute friend, miraculously saved from damnation, becomes a missionary. What fascinated the young Hardy was the description of the fellows drinking port and eating walnuts in the Senior Combination Room; it was this that he determined to experience, and mathematics, his best subject, was to be the instrument. So Hardy entered for the Trinity scholarship examination and was, of course, successful.

Initially Hardy's experience of Trinity was unhappy. Both he and his sister had become atheists, so when he found that weekly attendance in Chapel was expected, he pleaded conscientious scruples. He was, in response, made to inform his devout parents who were naturally pained. From then on his atheism was militant and vociferous. However, what preyed more persistently on Hardy's mind was the mathematics teaching and examination system then in force in Cambridge. The Mathematical Tripos (degree examination) had two parts, taken after two and four years, respectively. The syllabus for Part I consisted mainly of highly manipulative analysis, geometry and applied mathematics. Undergraduates who seemed likely to do well were sent to specialist coaches for intensive training in the high-speed solution of examination questions. The examinees were not only placed in classes, as they are now, but the order of merit in each class was also made public. Anyone high in the list of Wranglers (those in the First Class) was well placed for most careers, and the prestige attached to a Senior Wrangler (the highest placed candidate) was immense. The pressures on able undergraduates were therefore intense, with the whole learning process subordinated to the perfection of examination technique. Hardy, as an obvious candidate for high honours in the Tripos, was placed in the care of one of the famous coaches, but he found the regime so stultifying that he contemplated abandoning mathematics for history. Fortunately he agreed instead to exchange his coach for A. E. H. Love who, though an applied mathematician, sympathised with the Continental revolution which had banished vague, intuitive reasoning in favour of rigorous argument. Moreover Love advised Hardy to read Camille Jordan's great *Cours d'Analyse*, a monument of meticulous logic which had at that time no parallel in the English language. In Hardy's own words: 'I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant'. It was from that time on that Hardy felt a genuine passion for mathematics. In 1898 he took Part I of the Tripos and was placed fourth. Two years later came the rather more respectable Part II in which he came first, and immediately afterwards he was elected to a Trinity Fellowship.

Hardy's first paper was published in 1899, before he had taken Part II of the Tripos. It deals with the evaluation of the integrals of a substantial class; an example of the many attractive particular cases is

$$\int_0^\infty \log\left(1+\frac{\cos\alpha}{\cosh x}\right) dx = \tfrac{1}{8}(\pi^2-4\alpha^2) \quad (|\alpha| \leqslant \pi).$$

This area of analysis was, in fact, one of Hardy's abiding interests and a series entitled *Notes on the Integral Calculus* ran to 69 papers. In 1905 Hardy published his first book, a short tract in which he systematised the integration of various types of functions, a task which, curiously, had not been attempted before.

Of course Hardy was not simply interested in evaluating integrals. For instance he worked extensively on integral transforms, i.e. functions defined as integrals of the form

$$g(y) = \int_a^b f(x)\phi(x,y)\, dx, \tag{$*$}$$

where a and b are fixed, possibly infinite, ϕ is a given function and f is an arbitrary function belonging to a certain class. The function g is then said to be an *integral transform* of f and the problem is to investigate the properties of all the functions g. In particular, can f always be expressed in terms of g by means of a formula similar to (*)? An example of Hardy's results is the striking theorem that, if f belongs to a (specified) large class of functions and

$$\pi g(y) = \int_{-\infty}^\infty \frac{f(x)}{x-y}\, dx,$$

then

$$\pi f(x) = \int_{-\infty}^\infty \frac{g(y)}{x-y}\, dy.$$

In 1908 Hardy made a very brief, but to biologists highly important, excursion into the new science of genetics. Mendel had demonstrated that, when an organism reproduces from a male and a female parent, many characteristics are determined by (in modern terminology) genes which take two alternative forms, say A and a. If both parents carry the same gene A, a situation denoted by (AA), then all offspring have gene A; and, equally, (aa) parents' offspring carry gene a. In the case of (Aa) parents, half the transmitted genes are A and half a. In a letter to the American journal *Science* Hardy showed that the proportions of (AA), (Aa) and (aa) matings remained constant from generation to generation provided only that the sexes are equally distributed among the A and a varieties and that matings in the population are random. For suppose that, in a particular generation, the proportions of the three types of matings are

$$p:2q:r.$$

Then the proportions of the A and a genes in the next generation are $(p+q):(q+r)$ and so the proportions of (AA), (Aa) and (aa) matings are

$$(p+q)^2{:}2(p+q)(q+r){:}(q+r)^2,$$

say $p_1{:}2q_1{:}r_1$. It is easily seen that $p_1{:}2q_1{:}r_1$ equals $p{:}2q{:}r$ if and only if $q^2 = pr$. Since evidently $q_1^2 = p_1r_1$, it follows that the relevant proportions remain constant in succeeding generations. This observation was so significant because, according to Mendel's second law, one of the A, a genes, say the first, is *dominant* (and the other *recessive*) in the sense that, while (AA) and (aa) parentages lead to the A and a characteristics, respectively, mixed, i.e. (Aa) parentage, produces only the A characteristic. Intuition had led some biologists to the erroneous belief that Mendel's second law entailed the eventual swamping of the recessive characteristic by the dominant one. (Traditionalist mathematicians might have taken note that intuition, when not confirmed by rigorous proof, can lead one astray.) Examples of dominant genes are those for brown (as against blue) eyes, and brachydactyly, the possession of stubby fingers, which is rare—and is not getting any more common. Hardy's result became immediately known as his law, but the nomenclature was amended when it emerged that Weinberg, a German physician, had come to the same conclusion at about the same time and it is now known as the Hardy–Weinberg law.

Figure 1. Hardy in about 1900. (Reproduced by permission of the Master and Fellows of Trinity College, Cambridge.)

Until the last few years of his life Hardy worked at a tremendous pace. He was the sole or joint author of well over 300 papers and eleven books; during 1905 alone appeared 16 papers by him, several of them very substantial. His second—and most influential—book, *A Course of Pure Mathematics*, published in 1908 (and referred to above), was part of his campaign to reform the Mathematical Tripos, and through it university mathematics as a whole. Its style has, with reason, been compared to that of a missionary preaching to cannibals. Hardy was not the only reformer, but he was certainly the most ardent and radical one and he did not, in fact, entirely get his way. The numerical ordering of candidates ceased in 1910 and this was the key to changing the whole character of the examination, but Hardy wanted to go much further, abolishing degree classes altogether. His ideal was an easy pass/fail examination which simply ensured that students had a reasonable knowledge of the contents of their courses; and in his 1926 presidential address to the Mathematical Association he made a cogent case for such a system. It is interesting to note that, in this address, he referred sardonically to the then modern educationalists who advocated the abolition of lectures and examinations altogether.

Since Hardy published so much, it might be thought that he devoted most of his waking hours to mathematics, but such a conclusion would be quite wrong, for he held that four hours' research was the daily limit for a mathematician. His mode of life has been described by C. P. Snow, the chemist and novelist who was a close friend of Hardy's for a good many years. Over breakfast he read *The Times*, starting with the cricket scores—the English ones in the summer and the Australian ones in the winter. (J. M. Keynes, the economist, once told him that, if he paid the same attention to the stock exchange reports, he would be a rich man.) Then, until lunch, he did mathematics. The afternoon was spent on his favourite games. He might play some real tennis (a complicated indoor game, the ancestor of lawn tennis) or practise at the nets, but above all he liked watching cricket, if possible in the company of a friend with whom to discuss the game or anything else under the sun. Though he enjoyed conversation over dinner, the port and walnuts in the Combination Room, the prospect of which had brought him to Trinity, soon palled. Some evenings may well have been devoted to literature; Hardy was certainly an exceptionally well-read man.

In 1910, when Hardy was 33, there came public recognition of his achievements with his election to Fellowship of the Royal Society. The same year saw the return, after three years in Manchester, of J. E. Littlewood (1885–1977), a young fellow of Trinity who had already acquired an enviable reputation as a formidable mathematician of the modern school. (An article on Littlewood appeared in *Mathematical Spectrum* Volume 11, pp. 1–10, a year after his death.) Then, in 1911, began the famous collaboration between Hardy and Littlewood. Temperamentally they were very different: Hardy's sensitivity and fervour contrasted with Littlewood's relaxed imperturbability. The disparity showed up in their student attitudes to the Tripos; both disapproved, but Hardy rebelled while Littlewood judged it simplest to put up with the system and went on to be bracketed Senior Wrangler. However, they had very similar mathematical tastes, as is indicated by their independent papers on a problem in the theory of infinite series.

Figure 2. Littlewood in 1924. (Reproduced from *The Pólya Picture Album: Encounters of a Mathematician*, by permission of Birkhäuser.)

A series $\sum_{n=0}^{\infty} a_n$ is defined to be convergent if the sequence of partial sums $s_n = a_0 + \cdots + a_n$ converges. But now suppose that the sequence of arithmetic means

$$s_n^{(1)} = \frac{s_0+s_1+\cdots+s_n}{n+1}$$

tends to a limit. Then $\sum a_n$ is said to be *summable* by the $(C,1)$ method (where C stands for Cesàro, an Italian mathematician). It is easy to prove that, if $s_n \to s$ as $n \to \infty$, then $s_n^{(1)} \to s$ also, so that a convergent series is also $(C,1)$ summable. On the other hand, $\sum a_n$ may be $(C,1)$ summable without being convergent, as when $a_0 = 1$ and $a_n = 2(-1)^n$ for $n \geqslant 1$, so that $s_n = (-1)^n$. Thus $(C,1)$ summability is a generalization of convergence. Moreover, the process of taking arithmetic means may be repeated so as to yield the higher means $s_n^{(2)}$, $s_n^{(3)}$, etc. and the associated summability methods $(C,2)$, $(C,3)$, etc. Evidently convergence implies (C,k) summability, but Hardy asked the opposite question: Given a (C,k) summable series $\sum a_n$, what additional condition will ensure that $\sum a_n$ converges? The beautifully simple (but difficult to prove) answer he gave in a 1910 paper was that na_n should be bounded, i.e. that, for some constant K, $|na_n| \leqslant K$ for all n. At Hardy's suggestion Littlewood subsequently proved that the corresponding result holds even when (C,k) summability is replaced by a yet more powerful method (named after the Norwegian Abel). The argument has always been regarded as a remarkable tour de force.

The Hardy–Littlewood collaboration lasted for 35 years, almost until Hardy's death. Producing nearly 100 papers of the highest quality it soared far above any other collaboration in the history of mathematics. It covered many areas, but most notably the theory of numbers. A celebrated example is the Hardy–Littlewood assault on Goldbach's problem on which no impression had previously been made. In 1742 C. Goldbach (in a letter to Euler) conjectured that (i) every even integer greater than 2 is the sum of two primes and (ii) every odd integer greater than 5 is the sum of three primes. (Of course (ii) follows from (i).) Hardy and Littlewood established the existence of the desired representation for every sufficiently large odd integer. Although their proof required an intrusive hypothesis, this was eventually eliminated.

Hardy and Littlewood could not resist another famous unsolved problem. When $z = x+\mathrm{i}y$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges if $x > 1$ since

$$|n^z| = |n^{x+\mathrm{i}y}| = |n^x||n^{\mathrm{i}y}| = n^x|\mathrm{e}^{\mathrm{i}y\ln n}| = n^x$$

and $\sum 1/n^x$ converges for $x > 1$. It may now be proved that there is a differentiable complex function $\zeta(z)$ defined in the whole complex plane, except at the point 1, which coincides with $\sum_{n=1}^{\infty} 1/n^z$ when $x > 1$. The function $\zeta(z)$ is called the *Riemann zeta function* after B. Riemann (1826–1866) who showed that the location of the complex zeros of $\zeta(z)$ (i.e. points z such that $\zeta(z) = 0$) had important number-theoretic consequences. Riemann conjectured that all these zeros were on the line $x = \frac{1}{2}$. This is the famous *Riemann hypothesis* which mathematicians have been trying to prove ever since, but so far without success. Hardy was the first (in 1914) to make an impression on the problem by proving that there were infinitely many zeros on the line $x = \frac{1}{2}$; while he and Littlewood obtained (in 1921) the much improved result that, for some positive constant A, $\zeta(\frac{1}{2}+\mathrm{i}y)$ has at least AY zeros in each interval $-Y \leqslant y \leqslant Y$.

Other areas in which Hardy and Littlewood worked extensively include series of many kinds, especially trigonometric series (which are of the form $\sum (a_n \cos nx + b_n \sin nx)$), and inequalities. Together with G. Pólya (1887–1985) they wrote in 1934 the first—and still the best—book on the latter subject.

'The one romantic incident in his life', according to Hardy himself, began one morning in January 1913 when he received a fat envelope posted in India. It contained nine pages full of strange looking analytical and number-theoretic formulae, all without proof, of which a striking example was

$$\cfrac{1}{1+\cfrac{\mathrm{e}^{-2\pi}}{1+\cfrac{\mathrm{e}^{-4\pi}}{1+\ddots}}} = \left(\sqrt{\tfrac{1}{2}(5+\sqrt{5})} - \tfrac{1}{2}(\sqrt{5}+1)\right)\mathrm{e}^{2\pi/5}.$$

In a covering letter the writer, Srinivasa Ramanujan, introduced himself as a clerk in the Port Trust of Madras, without university mathematical training, who asked for Hardy's opinion of his work and who would generally welcome his advice. Though irritated by this unsolicited call on his time, Hardy was intrigued; and that evening he and Littlewood, after careful study of the manuscript, decided that they were dealing with a genius. It turned out that Ramanujan, born in 1887, came from a town about 200 miles south of Madras. After performing outstandingly at school he was given a scholarship to the local Government College, but he failed the broad-based first-year course because by then he was interested in nothing but mathematics. Attendance at a college in Madras produced the same result, and for the next five years he stayed at home simply doing his own highly individual mathematics. Having seen only a few elderly textbooks he was wholly untouched by modern developments. Marriage eventually obliged him to earn his living and, through the good offices of influential amateur mathematicians, he obtained the clerkship which afforded him plenty of time for pursuing his research. The head of the Port Trust and other British officials wished to help Ramanujan, but as they were unable to evaluate his work, one of them sent a sample to his former professor in London who, though out of his depth, was moderately encouraging. Thereupon Ramanujan was persuaded to write himself to more eminent Cambridge mathematicians. It is not clear why he first approached H. F. Baker, primarily a geometer, who returned his letter without comment. E. W. Hobson, as an analyst evidently more appropriate, was equally unhelpful. So Hardy, in every way the ideal person to turn to, was only his third choice.

Hardy quickly wrote back to Ramanujan, very appreciatively, but naturally anxious to see proofs. Although new results kept coming, proofs, when eventually sent, were unacceptable by modern standards. Arranging a visit by Ramanujan to Cambridge proved less straightforward than expected, but at last, in April 1914, he arrived, supported by a substantial scholarship from the University of Madras. Hardy could now examine Ramanujan's notebooks which contained over 3000 results. Some of these were wrong, others were already known in the West, but a third or more were startlingly new. He also began the delicate task of conveying to Ramanujan the notion of proof and teaching him some of the mathematics he had missed in his isolation. In the process, he said, he learned more from Ramanujan than Ramanujan from him.

During the next four years Ramanujan wrote 20 outstandingly original papers, four in collaboration with Hardy. Of the latter the foremost is that on the number $p(n)$ of partitions of the positive integer n, i.e. the number of ways in which n can be written as the sum of positive integers. Thus

$$4 = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1,$$

so that $p(4) = 5$. Evidently $p(n)$ increases very rapidly with n. In fact, Hardy and Ramanujan showed that $p(n)$ is of the order $e^{A\sqrt{n}}$, where A is a positive constant, and they devised an uncannily exact method for evaluating $p(n)$ for large n. In 1918 Ramanujan was elected to Fellowships first of the Royal Society and then of Trinity College; but sadly by then he had contracted tuberculosis, moving from one nursing home to another in an ineffectual search for a cure. It was during his stay in a London clinic that there occurred the famous interchange with Hardy that so aptly illustrated Littlewood's remark that every positive integer was a personal friend of Ramanujan's. Hardy had taken a taxi from the station and, on arrival, mentioned to Ramanujan that he thought its number, 1729, was rather a dull one. But Ramanujan protested: 'No, Hardy, no! It is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' In the spring of 1919 Ramanujan returned to India, but after a last flowering of his genius he died a year later, at the age of 32.

Figure 3. Ramanujan in 1919—his passport photograph. (Reproduced by permission of the Master and Fellows of Trinity College, Cambridge.)

The war years 1914–1918 were unhappy for Hardy. Having volunteered for military service, though rejected as unfit, he could not be called a pacifist; but he hated the war and, ever the champion of the underdog, he fiercely defended those who were persecuted for their anti-war views or activities. The most prominent of these was Bertrand Russell who was fined for publishing a pamphlet deemed 'prejudicial to recruitment and discipline' and who, as a result, was deprived of his Trinity Fellowship. For a sensitive man like Hardy the frequent quarrels and permanently strained relations with many fellows must have been hard to bear. Fortunately there was the solace of mathematics, pursued on his own and jointly with Ramanujan or with 2nd Lieutenant Littlewood of the Royal Artillery who was seconded to the Ballistics Office in London.

In 1919 Hardy was offered a professorship at Oxford and, with memories of the recent Trinity feuds still fresh, he gladly accepted. The move marked the beginning of the happiest period of his life. He became a fellow of New College, a smaller, more intimate society than Trinity that treasured his unconventional personality. The college even pandered to his refusal to enter religious buildings (which, incidentally, deprived him of much aesthetic pleasure) by inserting a clause in its by-laws enabling a fellow, exceptionally, to cast his vote in the election of the Warden without crossing the chapel threshold. But, above all, Hardy was—and felt—at the top of his mathematical powers. Much of his most memorable research was done during this period and his work with Littlewood was now at its most intense, with 56 joint papers appearing in the 12 years 1920–1931. Littlewood was still in Cambridge, but physical separation was immaterial since they normally communicated in writing even when living in the same college. With time they evolved four principles to govern their collaboration. The most significant, which would have ruined other relationships, was that, when one received a letter from the other, he was under no obligation to read it, let alone answer it. A mutual friend, the Danish mathematician Harald Bohr, reported that once, when Hardy was staying with him, Littlewood sent a succession of bulky envelopes. However, Hardy simply threw them in a corner with the remark 'I suppose I shall want to read them one day'. Hardy and Littlewood also developed a routine for writing up their papers. Littlewood produced the penultimate, skeletal version, while Hardy provided the flesh and connective tissue since, according to Littlewood, 'he took a sensual pleasure in calligraphy and it would have been a deprivation if he didn't make the final copy of a joint paper'.

Littlewood was, in 1928, elected to the newly established Rouse-Ball chair in Cambridge. The older Sadleirian chair, the most prestigious in the country, fell vacant in 1931 with the death of Hobson. It was naturally offered to Hardy who accepted in spite of his happy life in Oxford. A factor in his decision may have been thoughts of old age, for wealthy Trinity was by then the only college that was willing to accommodate and look after its retired fellows. Actually, the Cambridge thirties, though not as much as the Oxford twenties, turned out to be a good time for Hardy, his Indian summer as C. P. Snow called it. He had a large circle of friends and he was still very active mathematically. Jointly with Littlewood he conducted an exhilarating weekly seminar, living proof of the strength and vitality of the world-famous school of analysis that they had created by their teaching and example. Hardy may well have occasionally reflected on the contrast with the mathematical backwater that he found when he arrived in Cambridge not much more than thirty years before.

Hardy's world began to crumble when, early in 1939, he had a heart attack. Although he made a good immediate recovery, from then on his health deteriorated and he was never again able to indulge in the physical activities, such as cricket and real tennis, which throughout his life had meant so much to him. Then, later that year, war broke out again. Hardy hated the evil Nazi regime and had helped many of its victims, but he could not reconcile himself to what he saw as the lunacy of war. C. P. Snow recalls a sometimes angry discussion in which Hardy appeared to recognize intellectually the case for war, though he was unable to accept it emotionally. Yet another cause of unhappiness was the consciousness of his waning mathematical powers. His research output certainly declined after his heart attack, but he wrote two superb textbooks, one published posthumously, for which the mathematical community continues to be grateful. There is a moving indication of his state of mind in his little book *A Mathematician's Apology* written in 1940. In it he speaks of a mathematician's distress when he finds that he has lost the power or desire to create, but goes on: 'It is a pity, but in that case he does not matter a great deal anyhow, and it would be silly to bother about him'. By the early summer of 1947 Hardy felt so desperate that he tried to kill himself. However, he took such a large overdose of barbiturates that he succeeded only in making himself violently sick. Thereafter he resigned himself to a sad, slow death. In November the Royal Society informed him that it had awarded him its highest honour, the Copley Medal. His reaction was sardonic amusement: 'Now I know that I must be pretty near the end. When people hurry up to give you honorific things there is exactly one conclusion to be drawn'. He died on 1 December, the day he was to have been presented with the Copley Medal.

Figure 4. Hardy at the 1941 Oxford v. Cambridge rugby match. (Reproduced by permission of the Master and Fellows of Trinity College, Cambridge.)

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Hardy wrote 11 books. The following were mentioned in the text.

1. *The Integration of Functions of a Single Variable*, Cambridge Tracts in Mathematics and Mathematical Physics, 2 (Cambridge University Press, 1905; 2nd edition 1916).
2. *A Course of Pure Mathematics* (Cambridge University Press, 1908; 10th edition 1952, last reprinted 1975).
3. (with J. E. Littlewood and G. Pólya) *Inequalities* (Cambridge University Press, 1935; 2nd edition 1952).
4. *A Mathematician's Apology* (Cambridge University Press, 1940; with a foreword by C. P. Snow 1967, last reprinted 1992).
5. (with W. W. Rogosinski) *Fourier Series*, Cambridge Tracts in Mathematics and Mathematical Physics, 38 (Cambridge University Press, 1944; 3rd edition 1956).
6. *Divergent Series* (Clarendon Press, Oxford, 1949).

The following biography of Ramanujan also provides much detailed information about Hardy.

7. R. Kanigel, *The Man Who Knew Infinity* (Scribners, London, 1991). (Reviewed in *Mathematical Spectrum*, Vol. 25 (1992/93), p. 25.) □

Before his retirement ***Harry Burkill*** *was a Reader in Pure Mathematics at the University of Sheffield; he edited Mathematical Spectrum from 1974 to 1979. His principal mathematical interest is analysis.*

The Undulation of the Monks

CLIFF PICKOVER

Such as say that things infinite are past God's knowledge may just as well leap headlong into this pit of impiety, and say that God knows not all numbers .... What madman would say so? .... What are we mean wretches that dare presume to limit His knowledge.

St Augustine

The number 69 696 is a remarkable number and certainly my favourite of all the integers. Aside from being almost exactly equal to the average velocity in miles per hour of the Earth in orbit, it is also the surface temperature, in degrees Fahrenheit, of some of the hottest stars. More important are its fascinating mathematical properties.

It is written that a Tibetan monk once presented this number to a student and said: 'What do you find significant about 69 696?'

The student thought for a few seconds, and replied, 'That is too easy, Master. It is the largest undulating square known to humanity.'

The teacher pondered this answer, and himself started to undulate in a mixture of excitement and perhaps even terror.

To understand the monk's passionate response, we must digress to some simple mathematics. *Undulating numbers* are of the form

$$abababab\ldots.$$

For example, 171 717 and 28 282 are undulating numbers. When I first conceived the idea of undulating squares a few years ago, it was not known if any such numbers existed. It turns out that $69\,696 = 264^2$ is indeed the largest undulating square known to humanity, and most mathematicians believe we will never find a larger one.

Dr Noam D. Elkies from the Harvard Mathematics Department wrote to me about the probabilities of finding undulating squares. The chance that a 'random' number around x is a perfect square is about $1/\sqrt{x}$. More generally, the probability is $x^{-1+1/d}$ for a perfect dth power. Since there are (for any k) only 81 k-digit undulants, one would expect to find very few undulants that are also perfect powers, and none that are very large. Dr Elkies believes that listing all cases may be impossible using present-day methods for treating exponential Diophantine equations.

Bob Murphy used the software Maple V to search for undulating squares, and he discovered some computational tricks for speeding the search. For example, he began by examining the last 4 digits of perfect squares (i.e. he computed squares modulo 10 000). Interestingly, he found that the only possible digit endings for squares which undulate are: 0404, 1616, 2121, 2929, 3636, 6161, 6464, 6969, 8484 and 9696. By examining squares modulo 100 000, then modulo 1000 000, then modulo 10 000 000, etc., he found that no perfect square ends in 40 404, 6161 616, 63 636, 464 646 464 or 969 696, thereby allowing him to speed further the search process. Searching all possible endings, he asserts that, if there is an undulating square, it must have more than 1000 digits.

Dr Helmut Richter from Germany is the world's most famous undulation hunter, and he has indicated to me that it is not necessary to restrict the 'modular searches' to powers of 10, and arbitrary primes work very well. He has searched for undulating squares with a million digits or fewer, using a Control Data Cyber 2000. No undulating squares greater than 69 696 have been found.

Other undulants

Randy Tobias of the SAS Institute in North Carolina notes that there are larger undulating squares in other number bases. For example, $292^2 = 85\,264 = 41\,414$ in base 12. And 121 is an undulating square in any base. (121 in base n is $(n+1)^2$.)

Interestingly, we find that there are very few undulating powers of any kind in base 10. For example, a 3-digit undulating cube is $7^3 = 343$. However, Randy Tobias conducted a search for other undulating powers and found that 343 is the *only* undulant he could find that is formed by raising a number to a power $p \geqslant 3$. He has checked this for $3 \leqslant p \leqslant 31$ and for for undulants less than 10^{100}. Undulating powers are indeed rare!

Undulating prime numbers, on the other hand, are more common. For example, Randy has discovered the following huge undulating prime:

$$7+720\times\frac{100^{49}-1}{99}$$
$$= 727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$
$$727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$
$$727\,272\,727\,272\,727\,272\,727\,272\,727\,272\,727$$

(It has 99 digits.) To find this monstrosity, he also used the software program called Maple. The program scanned numbers using two lines:

$$(a*10+b)*(10**(2*(k+1))-1)/99$$
$$a+10*(a*10+b)*(10**(2*(k+1))-1)/99$$

for $0 \leqslant k \leqslant 50$, $1 \leqslant a \leqslant n-1$ and $0 \leqslant b \leqslant n-1$. The Maple **isprime()** function was used to check if a number is prime. Maple makes it possible to work with very large integers.

There are many other undulating primes with many digits. However, there is no undulating prime with an even number of digits, because $abab\,ab\ldots ab = ab\times 101\,01\ldots 01$. I would be interested in hearing from readers who have searched for undulating primes with larger periods of undulation, such as found in the prime number 59 959 959 959 (which does not finish its last cycle of undulation).

Binary undulants beyond imagination

Finally, *binary undulants* are powers of 2 that alternate the adjacent digits 1 and 0 somewhere in their decimal expansion. For example, the 'highest quality' binary undulant I have found is 2^{949}. It has the undulating binary sequence 101 010 in it, which I have placed in parentheses in the following:

$$2^{949} = 4758\,454\,107\,128\,905\,800\,953\,799\,994\,079$$
$$681\,792\,420\,032\,645\,310\,062\,268\,978\,469$$
$$949\,81(1\,010\,10)2\,913\,293\,995\,344\,538\,606$$
$$387\,700\,321\,887\,355\,916\,128\,617\,513\,761$$
$$454\,672\,785\,743\,698\,264\,930\,657\,859\,527$$
$$662\,802\,505\,506\,689\,431\,871\,596\,616\,596$$
$$511\,469\,752\,757\,984\,765\,426\,503\,524\,599$$
$$059\,416\,795\,862\,009\,216\,282\,102\,716$$
$$609\,115\,705\,865\,638\,544\,337\,453\,260$$
$$521\,036\,049\,116\,206\,989\,312.$$

Here 949 is called an undulation seed of order 6, since it gives rise to a 6-digit undulation pattern of adjacent 1's and 0's. When I challenged mathematicians and programmers around the world to produce a higher-order binary undulant, many took up the challenge. The highest quality binary undulant so far known to humanity was discovered by Arlin Anderson of Alabama. He was the first to find that 2^{1802} contains an 8-digit binary undulation. After much hard work he also found that $2^{7694\,891}$ starts with the digits 101 010 101 73..., and a week later he discovered that $2^{1748\,219}$ gives rise to a 10-digit undulant! Since Arlin only checked the last 240 digits of each number, he feels it is almost certain that there is a bigger binary undulation somewhere in the first million powers of 2. Considering that $2^{1\,000\,000}$ contains around 300 000 digits, the chance of finding a 10 101 010 101 or 01 010 101 010 is large. (Arlin uses a custom C-program for large integer computation. The program runs on an Intergraph 6040 Unix workstation and on a 486 PC. Searching 240 digits in 2 million powers of 2 required 15 hours.)

How do binary undulants vary with the base b? For example, for the case of $b = 2$, all powers of 2 are binary undulants.

Further Reading

1. C. Pickover, Is there a double smoothly undulating integer? In *Computers, Pattern, Chaos and Beauty* (St. Martin's Press, New York, 1990).
2. C. Pickover, *Keys to Infinity* (Wiley, New York). □

Clifford Pickover *is a research staff member at the IBM Thomas J. Watson Research Center in Yorktown Heights, New York. He is the author of numerous popular books on mathematics, art and science, including 'Chaos in Wonderland: Visual Adventures in a Fractal World' and 'Mazes for the Mind: Computers and the Unexpected'. He in an associate editor for 'Computers and Graphics' and 'Computers in Physics', and an editorial-board member for 'Speculations in Science and Technology', 'Leonardo' and 'YLEM'.*

The Roseberry Conjecture

FILIP SAJDAK

The Roseberry conjecture states that all natural numbers which are not multiples of 5 have multiples containing only the digits 6 and 7. For example, $23 \times 29 = 667$. This conjecture was proved by several people in 1990. (See the references.) Here we shall prove certain results in the same vein and propose a conjecture.

Theorem 1. Every natural number n has a multiple containing only the digits 0 and 1.

In this case there is no restriction on n.

Proof. We divide successive powers of 10 by n and consider their remainders $r_1, r_2, r_3, \ldots$. Thus

$$10^i \equiv r_i \pmod{n},$$

where $0 \leqslant r_i \leqslant n-1$, for $i = 0, 1, 2, 3, \ldots$.

Eventually we must obtain two remainders the same, so there exist $a, k \in \mathbb{N}$ such that

$$10^a \equiv 10^{a+k} \pmod{n}.$$

Then

$$10^a \equiv 10^{a+k} \equiv 10^{a+2k} \equiv \ldots \pmod{n}.$$

Put

$$Y = 10^a + 10^{a+k} + 10^{a+2k} + \cdots + 10^{a+(n-1)k}.$$

Then $Y \equiv n \times 10^a \equiv 0 \pmod{n}$ and Y is a multiple of n consisting only of the digits 0 and 1.

The proof shows how to construct such a multiple of a given n. If we replace the powers of 10 in the proof by $m\times$(powers of 10), where $1 \leqslant m \leqslant 9$, we have another theorem.

Theorem 2. Let m be an integer between 1 and 9. Then every natural number has a multiple containing only the digits 0 and m.

For an integer $b > 1$ we can use $m\times$(powers of b) and so obtain our final theorem.

Theorem 3. Let m be an integer between 1 and $b-1$. Then every natural number has a multiple containing only the digits 0 and m in base b.

Returning to base 10, as long as we put no restriction on the natural numbers considered, one of the two digits must be zero; multiples of 10 always involve the digit 0, for example. But, as in the original Roseberry conjecture, if we exclude multiples of 5, then we can get away from zero. We propose the following conjecture.

Conjecture. Let a and b be co-prime natural numbers between 1 and 9 with one even and the other odd. Then every natural number which is not a multiple of 5 has a multiple containing only the digits a and b.

In a base other than 10, the corresponding conjecture would also have to exclude some numbers.

References

1. G. Berzsenyi, The Roseberry conjecture, *Quantum* (May 1990).
2. G. Berzsenyi, At sixes and sevens, *Quantum* (November 1990).
3. The Roseberry conjecture, *Mathematical Digest* **83** (April 1991).

□

Filip Sajdak *is 20 years old. He was born in Slovakia and moved to New Zealand in 1992. He is a second-year mathematics student at the University of Auckland. His main interest is the theory of prime numbers.*

The function f is defined in the set of natural numbers by

$$f(n) = \begin{cases} \frac{1}{2}n & (n \text{ even}), \\ 3n+1 & (n \text{ odd}). \end{cases}$$

For a given n, form the sequence n, $f(n)$, $f(f(n))$, $f(f(f(n)))$, etc. With $n = 9$, for example, we obtain the sequence 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Surprisingly, all natural numbers n seem to iterate to 1.

Can anyone prove this, or provide a counterexample?

Junji Inaba
(Student, William Hulme's
Grammar School, Manchester)

The Probability of Consecutive Numbers in Lotto

F. T. HOWARD

Readers may have been intrigued by the sequences of winning numbers in the lotteries of various countries. This article investigates.

In Texas, the game of Lotto is played in the following way. Six different numbers between 1 and 50 are drawn at random, and you win the game if you correctly guess the six numbers (in any order). If we let $C(n,k)$ denote the number of ways of selecting k objects from n distinguishable objects, with no repetitions, i.e. $C(n,k) = n(n-1)\cdots(n-k+1)/k!$, then the probability of winning at Lotto is

$$\frac{1}{C(50,6)} = \frac{1}{15\,890\,700} = 0.000\,000\,0629\ldots.$$

A friend of mine figured out the above probability, and he observed that many of the drawings included a pair of consecutive numbers. He sent me the following question:

> What is the probability that a Lotto selection contains at least one pair of consecutive numbers?

Let us first introduce some notation. Let $F(n,k)$ denote the number of selections of k numbers from $1,2,\ldots,n$ with no consecutive numbers selected, and let $P(n,k)$ be the probability that at least one pair of consecutive numbers is selected. Then the answer to my friend's question is

$$P(50,6) = 1 - \frac{F(50,6)}{C(50,6)} = 1 - \frac{F(50,6)}{15\,890\,700}.$$

Thus the problem is to evaluate $F(50,6)$, or, more generally, $F(n,k)$. In this article we present four different approaches to the problem, and a generalization.

Solution 1

A bit string, which is a sequence of 0s and 1s, represents a selection of integers in the following way: a 1 in the mth position means the number m is selected, while a 0 in the mth position means the number m is not selected. Clearly every bit string of length n with exactly k 1s and no consecutive 1s represents a selection of k numbers from $1,2,\ldots,n$ with no consecutive numbers selected. We can easily count the number of such bit strings. First write down the $n-k$ 0s : 0 0 0 ... 0. We can put the k 1s in the spaces between the 0s (including before the first 0 and after the last 0). There are $n-k+1$ spaces and we choose k of them for 1s. Thus there are $C(n-k+1,k)$ ways to make the selections; i.e.

$$F(n,k) = C(n-k+1,k).$$

Therefore the answer to the original question is

$$P(50,6) = 1 - \frac{C(45,6)}{C(50,6)} = 0.487\ldots.$$

Solution 2

We can easily set up a one-to-one correspondence between all selections of k numbers from $1,2,\ldots,n-k+1$ and all selections of k numbers from $1,2,\ldots,n$ with no consecutive numbers selected. Let $a_0,a_1,\ldots,a_{k-1}$ be any selection of k numbers from $1,2,\ldots,n-k+1$ with $a_0 < a_1 < \cdots < a_{k-1}$. Let this selection correspond to $a_0, a_1+1, a_2+2,\ldots,a_{k-1}+k-1$, which contains no consecutive numbers and has $a_{k-1}+k-1 \leqslant n$. Conversely, let $b_0 < b_1 < \cdots < b_{k-1}$ be any selection of k numbers from $1,2,\ldots,n$ with no consecutive numbers, and let it correspond to $b_0, b_1-1, b_2-2,\ldots,b_{k-1}-k+1$. Hence

$$F(n,k) = C(n-k+1,k).$$

Solution 3

Consider placing k indistinguishable objects $x,x,\ldots,x$ into $n+2$ cells (so there are $n+1$ divisions between the cells). The cell dividers can be represented by the $n+1$ symbols $|,|,|,\ldots,|$, and the $n+2$ cells can be represented by the spaces between the dividers, including the space before the first divider and after the last one. For example, if we have two x's and five cells, the symbol $x|||x|$ represents x in the first and fourth cells, with the second, third, and fifth cells empty. Now consider all permutations of the k indistinguishable objects $x||,x||,\ldots,x||$ and the remaining $n-2k+1$ cell dividers $|,|,\ldots,|$. Every permutation represents placing k objects into $n+2$ cells with the last two cells empty, and no two consecutive cells containing objects. For example, if we place all the objects $x||$ at the beginning, then $x||x||\ldots x||||\ldots||$ represents objects in cells $1,3,5,\ldots,2k-1$. Thus every such permutation represents a selection of k numbers from $1,2,\ldots,n$ with no consecutive numbers picked. It is well known that the number of ways of permuting $u+v$ objects with u of one kind and v of another kind is $(u+v)!/u!\,v!$. Thus we have

$$F(n,k) = \frac{(n-k+1)!}{k!\,(n-2k+1)!} = C(n-k+1,k).$$

Solution 4

We can use a combinatorial argument similar to the one used to build Pascal's triangle. Suppose we select k numbers from $1,2,\ldots,n$ with no consecutive numbers selected. If the number n is not picked, then we select the k numbers from $1,2,\ldots,n-1$, and there are $F(n-1,k)$ ways to do this. If the number n is picked, then we select the remaining $k-1$ numbers from $1,2,\ldots,n-2$, and there are $F(n-2,k-1)$ ways to make the selection. Hence we have the recurrence formula

$$F(n,k) = F(n-1,k)+F(n-2,k-1),$$

for $1 \leqslant k < n$, with $F(n,0) = 1$ for $n \geqslant 0$, $F(1,1) = 1$ and $F(n,n) = 0$ when $n > 1$. Using this recurrence, we can construct the triangle

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 0 && \\ & 1 && 3 && 1 && 0 & \\ 1 && 4 && 3 && 0 && 0 \end{array}$$

and so on, where the elements in row $n+1$ are $F(n,0), F(n,1),\ldots, F(n,n)$. With the recurrence we can also use mathematical induction on n to prove $F(n,k) = C(n-k+1,k)$. The equation is true for $n = 1$, $n = 2$ and all k; assume it is true for $n = 1,2,\ldots,m-1$ and all k. Then

$$\begin{aligned} F(m,k) &= F(m-1,k)+F(m-2,k-1) \\ &= C(m-k,k)+C(m-k,k-1) \\ &= C(m-k+1,k). \end{aligned}$$

A generalization

It is interesting that we can easily generalize this problem. Let $F_j(n,k)$ denote the number of selections of k integers from $1,2,\ldots,n$ such that any two integers that are selected differ by at least $j+1$. Thus $F_1(n,k) = F(n,k)$. All of the above arguments, with the exception of Solution 1, easily generalize to give the formula

$$F_j(n,k) = C(n-jk+j,k).$$

Solutions 2 and 4 are especially easy to generalize if we already know the answer. We show next how the answer can be derived by using the approach of Solution 3.

Consider placing k indistinguishable objects $x,x,\ldots,x$ into $n+j+1$ cells (so there are $n+j$ cell dividers). We permute the k indistinguishable objects $x||\ldots|,x||\ldots|,\ldots,x||\ldots|$ (where each x is followed by $j+1$ cell dividers) and the remaining $n+j-k(j+1)$ cell dividers. Each such permutation represents placing k indistinguishable objects into $n+j+1$ cells, with the last $j+1$ cells empty and at least j empty cells separating any two occupied cells. Thus every such permutation represents a selection of k numbers $d_1,d_2,\ldots,d_k$ from $1,2,\ldots,n$ such that $d_{m+1}-d_m > j$ for $m = 1,\ldots,k-1$. Since we are permuting k objects of one kind and $n+j-k(j+1)$ objects of another kind the number of such permutations is

$$F_j(n,k) = \frac{[k+n+j-k(j+1)]!}{k!\,[n+j-k(j+1)]!} = C(n-jk+j,k).$$

We leave it to the reader to generalize the other proofs.

To illustrate the generalization, we return to Texas Lotto. The probability that in the drawing there is at least one pair of numbers differing by less than 3 is

$$1-\frac{F_2(50,6)}{C(50,6)} = 1-\frac{C(40,6)}{C(50,6)} = 0.758\ldots.$$

The probability that in the drawing there is at least one pair of numbers differing by less than 4 is

$$1-\frac{F_3(50,6)}{C(50,6)} = 1-\frac{C(35,6)}{C(50,6)} = 0.897\ldots,$$

and so on.

Since this article was written, another article on the same topic has been published (see the reference below). There is almost no overlap between that article and this one, however. Dr Berman's article contains some good references to related work.

Reference

David M. Berman, Lottery drawings often have consecutive numbers, *The College Mathematics Journal* **25** (1994), 45–47.

□

Fred Howard *teaches mathematics at Wake Forest University in Winston-Salem, North Carolina. His research interests include number theory, combinatorics, and special functions, and he is on the board of directors of the Fibonacci Association. He has never played Lotto, and has no advice for anyone who does.*

Mathematical Spectrum Awards for Volume 27

Prizes have been awarded to the following student readers for contributions published in Volume 27:

- **Chris Holt** for his article 'The domino problem' (page 62);
- **David Pirnes**, **Rosemary Sexton** and **Sam and Jim Yu** for other contributions.

The Editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems, and other items.

A Greek Dialogue

COLIN R. FLETCHER

Bob and Alf are enjoying the delights of a summer's afternoon in an urban park. The birds are singing and people are playing tennis. Bob has just completed a year studying mathematics at one of Britain's newer universities. His brain is being honed to razor sharpness, but the danger of the class peaking too soon has been foreseen, and the appropriate dampening has been built into the course. Alf is resting at the moment, being between jobs. He is not afraid to put his brain to work when he feels so inclined, but sadly this has been a rare event—especially during his school days.

Bob (languidly) It's great, isn't it?

Alf What's that, Bob?

Bob Tennis. Watching other people playing tennis, I mean. That one over there reminds me of Arantxa Sanchez Vicario.

Alf It makes me feel tired just to look at them.

Bob (philosophically) You know there is no finer way of spending an afternoon than sitting in the sun and ...

Alf (playfully) Give over. You'd much prefer to close your eyes and contemplate all things non-energetic.

Bob Well, that's what I was going to say. I'd have made a good ancient Greek; oodles of sun, nothing to do all day except drink ouzo and lie back and think of arithmetica.

Alf Arithmetic?

Bob (scathingly) No, no. *Arithmetica*. The theory of numbers. Properties of the integers. All that stuff. There's some lovely little bits which even you could understand if you put your mind to it. I bet you haven't even heard of perfect numbers.

Alf (in a most superior fashion) Well that's just where you're wrong, clever clogs. We had a lecture on perfect numbers at the extra-mural class.

Bob (wide-eyed) Oh, I don't believe it.

Alf (continuing in a confident vein) Well it's perfectly true. (Sorry about that dreadful pun.) Now let me try to remember what it was all about. Yes, 28 is perfect. I made a mental note of that at the time because I catch the 28 bus to the reservoir when I go fishing. And 28 is $2^2 \times 7$. Ah yes, it all comes back to me now. The formula for perfect numbers is $2^{n-1}(2^n-1)$. I'm not much good at formulae normally, but that one is easy to remember because you just repeat 'two to the n minus one'. Of course, 2^n-1 has to be prime. What escapes me for the moment is what a perfect number actually is.

Bob Oh well, I can help you there. We had a course of lectures on number theory last term from Willy Whitehead. You've got to laugh. His head was upside down with all his hair at the bottom rather than the top. He had a long straggly black beard with a completely bald top. We called him Rasputin for a bit, but then we realised that his name was quite appropriate in more ways than one.

Alf Who's Rasputin?

Bob (not really knowing) Oh, never mind that. Anyway, Willy seemed to know more or less what he was talking about for most of the time. According to him a perfect number is one that equals the sum of its divisors. Six is the first one, 1 + 2 + 3.

Alf Yes, I had forgotten that. Oh, of course. That's why God created the world in six days, to make it perfect.

Bob If you believe that, you'll believe anything. But with your formula we can calculate the perfect numbers. Let's see: for $n = 2$ we get 6, for $n = 3$ we get 28; $n = 4$ is no good because 2^4-1 is not prime.

Alf That's obvious. The number 4 isn't prime so 2^4-1 can't be prime. Fermat knew that in the seventeenth century.

Bob Who?

Alf Pierre de Fermat. He was some French whiz kid of the time. You must have heard of him. He had a last theorem named after him.

Bob (hesitatingly) His name does ring a bell; Willy probably mentioned him. To tell you the truth I got a bit lost. The mathematics was so complicated and it all seemed so unnecessary. I mean if you can solve a problem by going outside the set of integers, why not do so?

Alf (doubtfully) Well, I think the idea is just to see what can be done in the set of integers. It's all to do with beauty and the quest for knowledge. [A sudden thought occurs.] The Fundamental Theorem of Arithmetic would look a bit sick if you could use fractions. Hey, so too would perfect numbers.

Bob I'm not complaining about perfect numbers. They're easy. Now where did we get to? $n = 4$ was no good, $n = 5$ gives 496, $n = 6$ is no good, according to your French friend, and $n = 7$ gives 8128. You know these ancient Greeks were really superb. They have this daft definition which involves division and addition. By rights nothing should have come of it at all because those operations just don't mix, do they? But Euclid gets this marvellous theorem out of it giving a formula for the things. It's quite amazing. And that's not all. They then notice two things. First that these perfect numbers end alternately in 6 and 8. And second that each successive perfect number has an extra digit.

Alf Yes. Our man pointed that out too. And according to him, the Greeks went on to claim that these were general properties of the sequence of perfect numbers. I told him he was a fool. Well no, I didn't quite tell him that. I simply said I found it hard to believe that the Greeks would talk such obvious rhubarb.

Bob (quizzically) What do you mean? They both seemed reasonable guesses to me.

Alf You didn't learn much on that number theory course of yours, did you? It's painfully obvious that the perfect numbers can't increase by one digit each time. Whoops. Correction. It's painfully obvious that the Euclidean perfects can't increase by one digit each time. You do understand the subtle distinction?

Bob (faking exasperation) Yes, yes. There are no others but it hasn't been proved yet.

Alf If you like. Anyway, the point is that, for $2^{n-1}(2^n-1)$ to be perfect, 2^n-1 has to be prime, so n must be odd, forgetting about $n = 2$. Agreed?

Bob You said n had to be prime. Fermat's theorem.

Alf (scornfully) Oh dear. We have to add logic to the list of your deficiencies now. Of course n has to be prime, you clown. What I am saying is, it has to be odd. Do you disagree?

Bob (doubtfully) Well, no. But why not go the whole hog?

Alf Because it would make the argument more awkward. Look, if n has to be odd then the ratio of two consecutive perfect numbers is at least

$$\frac{2^{n+1}(2^{n+2}-1)}{2^{n-1}(2^n-1)},$$

which is more than

$$\frac{4(2^{n+2}-1)}{2^n},$$

which is the same as

$$4\left(2^2-\frac{1}{2^n}\right),$$

which is the same as

$$16-\frac{4}{2^n},$$

which is more than 15.

Bob (feeling bemused) I don't see where all this is leading.

Alf I think we need paper and pencil. 28 is perfect so the next one is at least 28 times 15, which is 420.

Bob But we know it is 496.

Alf OK, OK. The third is 496 so the next one is at least 496 times 15, which is 7440.

Bob But …

Alf (exasperated) Yes, yes, we know it's 8128. Right, so the next one is at least 8128 times 15, which is 121920. QED. There can't be a perfect number with five digits.

Bob I never thought of that.

Alf (sarcastically) I was under the impression that the whole point of a university education was to train a person to think.

Bob (even more sarcastically) Now it is you who is showing a certain deficiency. Why don't we concentrate on the tennis?

Alf There you go again. A man in his prime …. Perhaps I should rephrase that. Shall we continue?

Bob I thought we had finished.

Alf No, there's more to come. You were saying a few moments ago that the ancient Greeks were marvellous mathematicians.

Bob Correct.

Alf And you were quite happy to believe their leap of faith, that perfect numbers alternately end in 6 and 8, and that each perfect number has one more digit than the previous one?

Bob To be honest, I hadn't thought about it. I just accepted what Barbarous Willy had said. I can see now that the extra digit claim was not the work of an astute mind. I suppose the 6, 8 sequence is equally ridiculous.

Alf Well, you tell me. You're the maths student, I'm only attending a course on European Ideas Through the Ages for bored housewives and the unemployed.

Bob Right, honour is at stake, so here goes. If n is odd, say 3, then 2^{n-1} is 4. If n is 5 then 2^{n-1} ends in 6. The next ends in 4, and so on.

Alf Come on, Bobby, you can do it.

Bob I'd sooner be doing something else. If 2^{n-1} ends in 4, then 2^n ends in 8 and 2^n-1 ends in 7, so $2^{n-1}(2^n-1)$ ends in 8. If 2^{n-1} ends in 6, then 2^n ends in 2 and 2^n-1 ends in 1, so $2^{n-1}(2^n-1)$ ends in 6. Whoopee. So the Greeks did get this one right. Perfect numbers do end alternately in 8 and 6.

Alf You are forgetting that 2^n-1 has to be prime.

Bob (furious with himself) Oh ****. That's all your fault. You were the one insisting that we thought of n as odd rather than prime. All right, back to basics. The case $n = 7$ gave us 8128. We miss out $n = 9$, and the next perfect number is given by $n = 11$, and this will end in 8, the same as 8128.

Alf Wrong.

Bob What's wrong now?

Alf The number $2^{11}-1 = 2047$ is not prime. The next perfect number comes from $n = 13$ and …

Bob (triumphantly) And this ends in 6, so the sequence is all right after all.

Alf Wrong.

Bob (completely confused) I give up.

Alf It ends in 6, but so does the next perfect number, when n is 17. Neither property holds in general.

Bob (distractedly) So the Greeks messed this one up then? You can't win 'em all I suppose. Still, I am surprised that they were so spectacularly awry, and in such a naive fashion.

Alf Wrong.

Bob We have just proved they were wrong.

Alf Wrong. Use your brain, man. We have proved that these conjectures are wrong, but the Greeks never made them. I checked up on it when our tutor came out with all this rubbish. One textbook writer, a certain Nicomachus, pointed out that the first four perfect numbers do satisfy both properties, but he didn't generalise. Then a later commentator, Iamblichus, offered the suggestion that maybe the next perfect number after 8128 is found amongst the first myriads, the following one amongst the second myriads, and so on. The myriads are the numbers lying between powers of ten thousand, so what Iamblichus is putting forward as a possibility is that

$$10\,000 < \text{5th perfect number} < 10\,000^2,$$
$$10\,000^2 < \text{6th perfect number} < 10\,000^3,$$
$$10\,000^3 < \text{7th perfect number} < 10\,000^4, \quad \ldots.$$

Bob (with a shrug) I don't think much of that as a guess. Why pick on 10 000? These later Greeks weren't much cop.

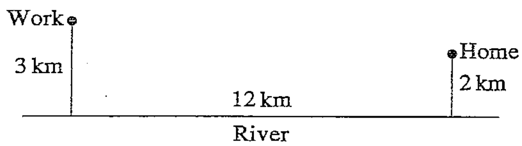
Alf (in exasperation) The country's taxes are certainly being wasted on you. Engage brain before opening mouth. Look, Iamblichus may have been unlucky enough to be born before your educational establishment was founded, but that doesn't necessarily make him a complete moron. His hypothesis, on the face of it, does seem worthless, but as I told my man at night school, maybe one should search behind even a worthless hypothesis just on the off chance that a reason for it can be found. Here is a Greek bearing gifts. Don't you see? The 5th perfect number is 33 million odd, so it does lie between 10 000 and $10\,000^2$. The reason for this bizarre speculation of Iamblichus is that he did actually know the 5th perfect number. What else could it mean? Keep your faith in the Greeks, my friend.

Bob (his eyes straying to the baseline) I am beginning to wish that I had never started this conversation. Anyone for tennis?

□

__Colin Fletcher__ is a historian of mathematics at the University of Wales, Aberystwyth. He is continually amazed by the mathematical exploits of the ancient Greeks.

Show me the way to go home ...



A man wants to go home from work, but he also has to go to the river to draw water to cook his dinner. What is the shortest distance in which he can do this?

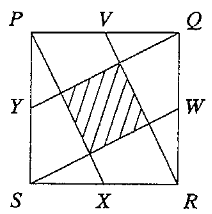
Junji Inaba
(Student, William Hulme's Grammar School, Manchester)

The mathematics of wage bargaining!

When a trades union demanded higher wages of the president of a company, he replied as follows: 'one year is 365 days, but for convenience say 366 days. You work 8 hours a day, which is $\frac{1}{3}$ of a day, so 122 days. In a year there are 52 Saturdays and 52 Sundays, so you only work $122-(2\times52) = 18$ days. But you have 18 days paid holiday in a year, so you are not working at all. Why do I have to pay you any extra wages?

Junji Inaba
(Student, William Hulme's Grammar School, Manchester)

PQRS is a square and *V*, *W*, *X* and *Y* are the midpoints of its sides, as shown. What is the area of the shaded region in terms of the area if *PQRS*?



KAMLESH GAYA
(Student, Mauritius)

The 1996 puzzle

Readers are invited to try our annual puzzle, to express the numbers 1 to 100 in terms of the digits of the year in order, using only the operations of +, −, ×, ÷, √, ! and concatenation (i.e. forming 19 from 1 and 9). For example,

$$1 = 1+\sqrt{9}+\sqrt{9}-6.$$

An Elementary Inequality with Applications

FENG YUEFENG

A new inequality

Let p and q be given positive numbers with $p-1 \geqslant q > 0$ and let a_i and b_i $(1 \leqslant i \leqslant n)$ be positive numbers. Then

$$\frac{1}{n}\sum_{i=1}^{n}\frac{a_i^p}{b_i^q} \geqslant \frac{\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p}{\left(\frac{1}{n}\sum_{i=1}^{n}b_i\right)^q}. \tag{1}$$

If $0 < p < 1$ and $q > 0$, then the inequality (1) holds in reverse.

Proof. Let $f(x) = x^p$. Since $p \geqslant 1+q > 1$, then $f(x)$ is convex for $x > 0$. Define $r = q/(p-1) \leqslant 1$. We have

$$\begin{aligned}\frac{(\sum a_i)^p}{(\sum b_i^r)^p} &= f\left(\frac{\sum a_i}{\sum b_i^r}\right)\\ &= f\left(\frac{b_1^r a_1}{b_1^r\sum b_i^r}+\frac{b_2^r a_2}{b_2^r\sum b_i^r}+\cdots+\frac{b_n^r a_n}{b_n^r\sum b_i^r}\right)\\ &\leqslant \frac{b_1^r}{\sum b_i^r}f\left(\frac{a_1}{b_1^r}\right)+\frac{b_2^r}{\sum b_i^r}f\left(\frac{a_2}{b_2^r}\right)+\cdots+\frac{b_n^r}{\sum b_i^r}f\left(\frac{a_n}{b_n^r}\right)\\ &= \frac{1}{\sum b_i^r}\sum\frac{a_i^p}{b_i^{r(p-1)}}.\end{aligned}$$

Therefore

$$\frac{(\sum a_i)^p}{(\sum b_i^r)^{p-1}} \leqslant \sum\frac{a_i^p}{b_i^{r(p-1)}}.$$

On the other hand, $g(x) = x^r$ $(0 < r \leqslant 1)$ is concave for $x > 0$, so

$$\sum b_i^r = \sum g(b_i) \leqslant ng\left(\frac{1}{n}\sum b_i\right) = n\left(\frac{1}{n}\sum b_i\right)^r.$$

Therefore

$$\begin{aligned}\frac{1}{n}\sum\frac{a_i^p}{b_i^q} &= \frac{1}{n}\sum\frac{a_i^p}{b_i^{r(p-1)}}\\ &\geqslant \frac{1}{n}\,\frac{(\sum a_i)^p}{(\sum b_i^r)^{p-1}} \geqslant \frac{\frac{1}{n}\left(\sum a_i\right)^p}{\left\{n\left(\frac{1}{n}\sum b_i\right)^r\right\}^{p-1}}\\ &= \frac{\frac{1}{n}\left(\sum a_i\right)^p}{n^{p-1}\left(\frac{1}{n}\sum b_i\right)^{r(p-1)}} = \frac{\left(\frac{1}{n}\sum a_i\right)^p}{\left(\frac{1}{n}\sum b_i\right)^q},\end{aligned}$$

which establishes (1).

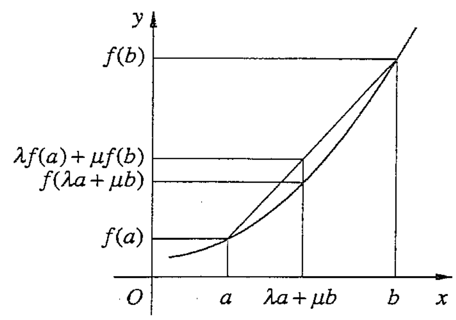
If $0 < p < 1$ and $q > 0$, then $f(x) = x^p$ is concave and we have

---

f is *convex* means

$$f(\lambda a+\mu b) \leqslant \lambda f(a)+\mu f(b),$$

where $\lambda+\mu = 1$.



The inequality extends to n terms:

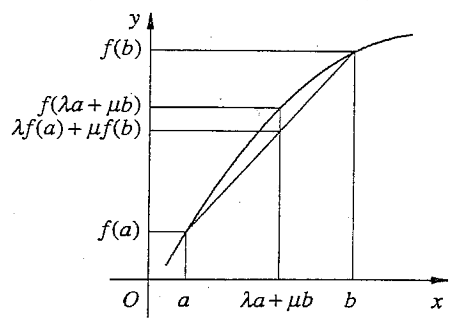
$$f(\lambda_1 a_1+\cdots+\lambda_n a_n) \leqslant \lambda_1 f(a_1)+\cdots+\lambda_n f(a_n),$$

where $\lambda_1+\cdots+\lambda_n = 1$.

f is *concave* means

$$f(\lambda a+\mu b) \geqslant \lambda f(a)+\mu f(b),$$

where $\lambda+\mu = 1$.



The inequality extends to n terms:

$$f(\lambda_1 a_1+\cdots+\lambda_n a_n) \geqslant \lambda_1 f(a_1)+\cdots+\lambda_n f(a_n),$$

where $\lambda_1+\cdots+\lambda_n = 1$.

$$\frac{(\sum a_i)^p}{(\sum b_i^r)^p} = f\left(\frac{\sum a_i}{\sum b_i^r}\right)$$

$$= f\left(\frac{b_1^r a_1}{b_1^r \sum b_i^r} + \frac{b_2^r a_2}{b_2^r \sum b_i^r} + \cdots + \frac{b_n^r a_n}{b_n^r \sum b_i^r}\right)$$

$$\geqslant \frac{b_1^r}{\sum b_i^r} f\left(\frac{a_1}{b_1^r}\right) + \frac{b_2^r}{\sum b_i^r} f\left(\frac{a_2}{b_2^r}\right) + \cdots + \frac{b_n^r}{\sum b_i^r} f\left(\frac{a_n}{b_n^r}\right)$$

$$= \frac{1}{\sum b_i^r} \sum \frac{a_i^p}{b_i^{r(p-1)}}.$$

Therefore

$$\frac{(\sum a_i)^p}{(\sum b_i^r)^{p-1}} \geqslant \sum \frac{a_i^p}{b_i^{r(p-1)}}.$$

But $g(x) = x^r$ $(r = q/(p-1) < 0)$ is convex for $x > 0$, so

$$\sum b_i^r = \sum g(b_i) \geqslant ng\left(\frac{1}{n}\sum b_i\right) = n\left(\frac{1}{n}\sum b_i\right)^r.$$

Therefore

$$\frac{1}{n}\sum \frac{a_i^p}{b_i^q} = \frac{1}{n}\sum \frac{a_i^p}{b_i^{r(p-1)}}$$

$$\leqslant \frac{1}{n}\frac{(\sum a_i)^p}{(\sum b_i^r)^{p-1}} \leqslant \frac{\frac{1}{n}\left(\sum a_i\right)^p}{\left\{n\left(\frac{1}{n}\sum b_i\right)^r\right\}^{p-1}}$$

$$= \frac{\frac{1}{n}\left(\sum a_i\right)^p}{n^{p-1}\left(\frac{1}{n}\sum b_i\right)^{r(p-1)}} = \frac{\left(\frac{1}{n}\sum a_i\right)^p}{\left(\frac{1}{n}\sum b_i\right)^q}.$$

This completes the proof.

An application

Example 1 is a problem from the 20th International Mathematical Olympiad, example 2 is from the 28th Olympiad and example 3 is from the 31st Olympiad.

Example 1. Let a_i $(1 \leqslant i \leqslant n)$ be distinct positive integers. Then

$$\sum_{i=1}^{n} \frac{a_i}{i^2} \geqslant \sum_{i=1}^{n} \frac{1}{i}.$$

Proof. By the inequality (1), we have

$$\sum \frac{a_i}{i^2} = \sum \frac{\frac{1}{i^2}}{\frac{1}{a_i}} \geqslant \frac{\left(\sum \frac{1}{i}\right)^2}{\sum \frac{1}{a_i}}$$

$$\geqslant \frac{\left(\sum \frac{1}{i}\right)^2}{\sum \frac{1}{i}}$$

$$= \sum \frac{1}{i}.$$

The last inequality holds because the a_i are all different.

Example 2. Let a, b and c be the sides of a triangle and put $a+b+c = 2s$. Then (for every positive integer n)

$$\sum \frac{a^n}{b+c} \geqslant (\tfrac{2}{3})^{n-2} s^{n-1}.$$

$$\left(\sum \frac{a^n}{b+c} \text{ is shorthand for } \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b}.\right)$$

Proof. When $n = 1$, by Cauchy's inequality (namely $\sum x_i^2 \sum y_i^2 \geqslant \left(\sum x_i y_i\right)^2$ for real numbers x_i and y_i $(1 \leqslant i \leqslant n)$),

$$\sum \frac{a}{b+c} = \sum \frac{a+b+c}{b+c} - 3 = (a+b+c)\sum \frac{1}{b+c} - 3$$

$$= \tfrac{1}{2}\sum (b+c) \sum \frac{1}{b+c} - 3$$

$$\geqslant \tfrac{1}{2}\left(\sum 1\right)^2 - 3 = \tfrac{3}{2},$$

which is the required inequality.

When $n > 1$, by the inequality (1) we have

$$\sum \frac{a^n}{b+c} \geqslant 3\times \frac{\left(\frac{1}{3}\sum a\right)^n}{\frac{1}{3}\sum (b+c)} = \frac{9\times(\frac{1}{3}\times 2s)^n}{4s} = (\tfrac{2}{3})^{n-2} s^{n-1}.$$

Example 3. Let a, b, c and d be positive numbers with $ab+bc+cd+da = 1$. Put $s = a+b+c+d$. Then

$$\sum \frac{a^3}{s-a} \geqslant \tfrac{1}{3}.$$

Proof. By the inequality (1), we have

$$\sum \frac{a^3}{s-a} \geqslant 4^{2-3} \frac{\left(\sum a\right)^3}{\sum (s-a)} = \tfrac{1}{12} s^2$$

$$= \tfrac{1}{12}[(a+c)+(b+d)]^2$$

$$\geqslant \tfrac{1}{12}\times 4(a+c)(b+d) = \tfrac{1}{3}.$$

A generalisation of Hölder's inequality

Hölder's inequality. Let a_i and b_i $(1 \leqslant i \leqslant n)$ be positive numbers. Let $p > 1$, $q > 1$ and $pq = p+q$. Then

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \geqslant \sum_{i=1}^{n} a_i b_i.$$

A generalisation. Let a_i and b_i $(1 \leqslant i \leqslant n)$ be positive numbers. Let $p > 1$, $q > 1$ and $pq \geqslant p+q$. Then

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\frac{1}{n}\sum_{i=1}^{n} b_i^q\right)^{1/q} \geqslant \frac{1}{n}\sum_{i=1}^{n} a_i b_i. \qquad (2)$$

If $p > 0$, $q > 0$ and $p < 1$ or $q < 1$, then the inequality (2) holds in reverse.

Proof. Since $pq \geqslant p+q$, then $p \geqslant (p/q)+1$. By inequality (1), we have

$$\frac{1}{n}\sum a_i^p = \frac{1}{n}\sum \frac{(a_ib_i)^p}{(b_i^q)^{p/q}} \geqslant \frac{\left(\frac{1}{n}\sum a_ib_i\right)^p}{\left(\frac{1}{n}\sum b_i^q\right)^{p/q}},$$

so that

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p} \geqslant \frac{\frac{1}{n}\sum a_ib_i}{\left(\frac{1}{n}\sum b_i^q\right)^{1/q}},$$

that is,

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p}\left(\frac{1}{n}\sum b_i^q\right)^{1/q} \geqslant \frac{1}{n}\sum a_ib_i.$$

If $p > 0$, $q > 0$ and $p < 1$ or $q < 1$, we assume $p < 1$. By the reverse of inequality (1), we have

$$\frac{1}{n}\sum a_i^p = \frac{1}{n}\sum \frac{(a_ib_i)^p}{(b_i^q)^{p/q}} \leqslant \frac{\left(\frac{1}{n}\sum a_ib_i\right)^p}{\left(\frac{1}{n}\sum b_i^q\right)^{p/q}},$$

so that

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p} \leqslant \frac{\frac{1}{n}\sum a_ib_i}{\left(\frac{1}{n}\sum b_i^q\right)^{1/q}},$$

that is,

$$\left(\frac{1}{n}\sum a_i^p\right)^{1/p}\left(\frac{1}{n}\sum b_i^q\right)^{1/q} \leqslant \frac{1}{n}\sum a_ib_i.$$

This completes the proof. □

__Feng Yuefeng__ teaches at the attached middle school of Hunan Normal University in Changsha, China. He has won the accolade of being the best young senior teacher in Hunan. He has published four books and around 140 articles. Two of his students have won gold medals in the International Mathematical Olympiad.

Computer Column

Surfing the net

The World Wide Web now holds a vast amount of information on mathematics; but how to find it? A good starting point is *Yahoo*, the catalogue of the Internet. Point your browser to

`http://www.yahoo.com/Science/Mathematics/`

and you will find 25 links to popular mathematical sites, and 20 sub-categories that each contain many more links. The sub-categories cover such topics as Geometry, Numbers and Fractals, as well as information on courses, electronic journals and employment opportunities for mathematicians.

I needed information on a particular mathematician, so I opened the History sub-category, and followed the link to the *MacTutor History of Mathematics* site at

`http://www-groups.dcs.st-and.ac.uk/~history`

This site has a collection of over 550 biographies of mathematicians, and a series of documents on the development of various branches of mathematics. There is also a 'curves index', which contains data (including diagrams and generating equations) on 60 of the most famous curves in mathematics.

I then followed a link to the *Mathematical Quotation Server* at

`http://math.furman.edu/~mwoodard/mquot.html`

which lets you conduct a keyword search through their quotation database. (My search revealed 36 quotations including the keyword 'geometry', but unfortunately not the one I wanted!)

Finally, for light relief, I returned to *Yahoo* and followed a link to

`http://juniper.tc.cornell.edu`
`:8000/spiro/spiro.html`

which is the address of the WWW spirograph. A spirograph is formed by having a circle rotate around the edge of a fixed circle; when a pen is placed at a point along the radius of the rotating circle, an image is formed. The WWW spirograph uses some simple mathematics to generate the image on screen: you supply the radii of the fixed and rotating circles, and how far the 'pen' is placed from the edge of the rotating circle, then press 'Go'. Seconds later you have the spirograph on your monitor.

You can find all the sites mentioned here, and many others, by following links from the *Mathematical Spectrum* home page:

`http//www.shef.ac.uk/~apt/apt4.html`

Stephen Webb □

Mathematics in the Classroom

The aim of this regular feature is to provide a forum in which ideas useful in the classroom can be shared. Readers are invited to write in with any ideas or questions which they would like to be aired.

Shrinking circles

In Mathematical Spectrum, Volume 26, Number 4 Tamara Curnow tackled analytically the following problem. Take a unit circle and inscribe an equilateral triangle; in the triangle inscribe a second circle, then a square, another circle, a regular pentagon, a circle, and so on. The limit of the sequence of circles is clearly a circle, but what is its radius? Other solutions are discussed in a letter from David Singmaster in Volume 27, Number 3. The problem can also be investigated by pupils of KS4 mathematics with the aid of a spreadsheet. Using a spreadsheet in a classroom enables pupils to attack problems that prove too demanding for a calculator and that are too complicated to solve using analytical techniques (see references 1 and 2).

The investigation is first tackled using a calculator so that the pupils can develop an understanding of the mathematics (geometry (reference 3), trigonometry (reference 4) and algebra) required to work through the problem. It also helps pupils realise the value of a spreadsheet (reference 5) and introduces them to the idea of radian measure (reference 6). The investigation I use requires pupils to have a starting radius of 5 cm and they are asked to calculate the 'final' radius to 3 decimal places on the calculator and to 9 decimal places on the spreadsheet. They are then asked to change the initial radius to 10 cm, then to 20 cm and then to any radius of their own choice. This helps them search for a formula to predict the 'final' radius from any given initial radius.

A pupil worksheet (using Excel) may look like the one in table 1, where the polygon reached has 16 385 sides. The value in cell E2 is the 'final' radius divided by the initial radius, a calculation only suggested to the pupils if they cannot discover the formula themselves!

The Excel formulae used in the workspace are given in table 2.

The accuracy of the value in cell E2 can be improved by increasing the number of polygon sides still further. The value I achieved for a starting radius of 1 unit was 0.114 959 357. This figure was reached after completely filling two Excel 4 worksheets.

The convergence is very slow because of the very small change in the cosine of angle A in figure 1 for small angles. The polygon which produced the value 0.114 959 357 had over 33 000 sides.

IT in the classroom is best used to enhance learning and to speed up calculation, as shown in this investigation, or, for example, using a curve-sketching package, to understand the role of the coefficients when studying quadratic curves.

The 'shrinking circles' investigation can be extended by searching for a limiting radius if the polygons circumscribe the circles. Alternatively, Shape, Space and Measures can be taken to Level 9 by considering the limiting radius when cuboids enclose spheres.

Table 1

	A	B	C	D	E
1	Number of Polygon Sides	Triangle angle A in figure1 [radians]	Next Radius	Starting Radius	
2	3	1.047197551	10	20	0.114976667
3	4	0.785398163	7.071067812		
4	5	0.628318531	5.720614028		
5	6	0.523598776	4.954197074		
6	7	0.448798951	4.463577329		
7	8	0.392699082	4.123807736		
16379	16380	0.000191794	2.299533552		
16380	16381	0.000191783	2.29953351		
16381	16382	0.000191771	2.299533467		
16382	16383	0.000191759	2.299533425		
16383	16384	0.000191748	2.299533383		
16384	16385	0.000191736	2.299533341		

Table 2

2	3	=(180/A2)*PI()/180	=D2*COS(B2)	20	=C16384/D2
3	=A2+1	=(180/A3)*PI()/180	=C2*COS(B3)		
4	=A3+1	=(180/A4)*PI()/180	=C3*COS(B4)		
5	=A4+1	=(180/A5)*PI()/180	=C4*COS(B5)		
6	=A5+1	=(180/A6)*PI()/180	=C5*COS(B6)		

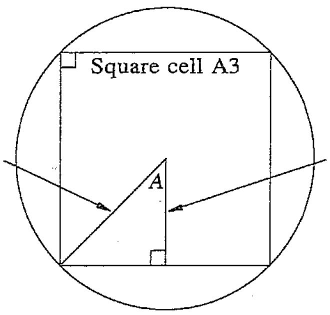


Figure 1. This shows how the problem relates to the worksheet in table 1.

References

1. D. Benjamin, A Spreadsheet estimate of π, *Spreadsheet User* Volume 1 Number 1 (Sheffield Hallam University, May 1994).
2. D. Benjamin, Advanced Level Mathematics and Spreadsheets, *Spreadsheet User* Volume 1 Number 2 (Sheffield Hallam University, November 1994).
3. Calculating angles of polygons, *Shape, Space and Measures*, Level 6.
4. Calculating the 'next radius', *Shape, Space and Measures*, Level 6.
5. Using IT in mathematics (National Curriculum recommendation) and Ma1, Level 9.
6. KS4, Level 9.

David Benjamin □

Letters to the Editor

Dear Editor,

Fibonacci numbers

Whilst reading about the Fibonacci numbers, I did some investigations of my own which threw up the following 'results' that seem to have very few exceptions. I cannot show that these exceptions are the only ones, but my limited numerical results have shown no others.

Let F_n be the nth Fibonacci number.

(a) If $n > 4$, then F_n+1 and F_n-1 are both composite. The exception is F_n-1 for $n = 6$ because $8-1 = 7$, which is prime.

(b) If $n > 3$, then there is a Fibonacci number which is a factor of F_n-1 and there is also a Fibonacci number which is a factor of F_n+1.

(c) If $n > 3$, then, if F_j is the smallest Fibonacci number which is a factor of F_n+1 or F_n-1 (for $2 < j < n$) then F_j is itself prime. The first exception is for $n = 36$, because

$$\begin{aligned} F_{36}-1 &= 14\,930\,352-1 = 14\,930\,351 \\ &= 4181\times 3571 \\ &= F_{19}\times 3571 \\ &= 37\times 113\times 3571. \end{aligned}$$

Yours sincerely,

MARTIN D. SANDFORD
(4600 Steeles Avenue East,
Markham,
Ontario L3R 0L2, Canada)

Dear Editor,

The Smarandache function

This has appeared a number of times in *Mathematical Spectrum*. For a positive integer n, $S(n)$ is defined to be the smallest positive integer m such that $m!$ is divisible by n. The first 17 terms of the sequence $(S(n))$ are

$$1,\ 2,\ 3,\ 4,\ 5,\ 3,\ 7,\ 4,\ 6,\ 5,\ 11,\ 4,\ 13,\ 7,\ 5,\ 6,\ 17.$$

We pose three questions for readers.

(a) How many n are such that

$$S(n)+S(n+1) = S(n+2)+S(n+3)?$$

I have found

$$\begin{aligned} S(6)+S(7) &= S(8)+S(9), \\ S(7)+S(8) &= S(9)+S(10), \\ S(28)+S(29) &= S(30)+S(31). \end{aligned}$$

(b) How many n are such that

$$S(n)-S(n+1) = S(n+2)-S(n+3)?$$

I have found

$$\begin{aligned} S(1)-S(2) &= S(3)-S(4), \\ S(2)-S(3) &= S(4)-S(5), \\ S(49)-S(50) &= S(51)-S(52). \end{aligned}$$

(c) How many n are such that

$$S(n)+S(n+1)+S(n+2) = S(n+3)+S(n+4)+S(n+5)?$$

I have found

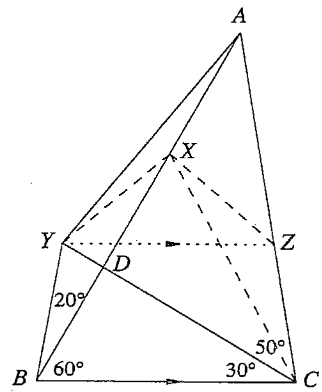
$$S(5)+S(6)+S(7) = S(8)+S(9)+S(10).$$

Yours sincerely,

JORGE CASTILLO
(PO Box 722, Vail,
Arizona 85461, USA)

Dear Editor,

The geometrical problem of Junji Inaba
Volume 28 Number 1 Page 18



ΔCBD is a $(30°, 60°, 90°)$ triangle. Draw ΔCYX congruent to ΔCYB and YZ parallel to BC.

Then ΔXBC is equilateral and Δs XYB and XZC are isosceles and congruent. Thus $XY = XZ$. Also

$$\begin{aligned} \angle XAZ &= 40° \quad \text{(angle sum of } \Delta ADC); \\ \angle AZX &= 40° \quad \text{(sum of opposite angles of } \Delta XZC); \\ \angle XYZ &= \angle XZY \\ &= \angle AZY - \angle AZX \\ &= 80° - 40° = 40°. \end{aligned}$$

Thus the triangles YXZ and AXZ are congruent (AAS). Hence $AZ = YZ$ and ΔZAY is isoceles with $\angle AZY = 80°$. Hence $\angle ZAY = 50°$.

Yours sincerely,

BRIAN STONEBRIDGE
(Department of Computer Science,
Univeristy of Bristol)

Dear Editor,

A large pair of twin primes

I am pleased to report the twin primes

$$6797\,727\times 2^{15\,328}-1, \qquad 6797\,727\times 2^{15\,328}+1$$

each with 4662 digits. They were discovered by my computer (more precisely, a computer on loan to me but paid for by the M500 Society) at about 4 a.m. on Tuesday 18 July 1995, but I had left the machine unattended for some time; it was not until the 19th that I became aware of them.

The computer power was modest by today's standards for this type of work: a 33-MHz 486 microprocessor, later upgraded to 100 MHz. Efficiency was therefore important. I used the best algorithms known to me and programmed everything from scratch in a combination of Yuji Kida's UBASIC and PC assembler language. For instance, we are well out of the range where one can effectively multiply numbers together in the way we were taught at school. Instead, I had to use the more sophisticated Schönhage–Strassen method, based on the fast Fourier transform modulo F_8, the eighth Fermat number.

But the main requirement was a great deal of patience. I decided to search through multiples of $2^{15\,328}$, the upper limit of my computer program. About 229 out of every 230 candidates were eliminated fairly quickly by a combination of the sieve of Eratosthenes together with straightforward trial division by primes up to about a million. That left over 30 000 numbers to be examined using the Fermat test, each taking 8.5 minutes at 33 MHz, reducing to 2 minutes 50 seconds after the 100-MHz upgrade.

An account will appear in M500, the periodical of the M500 Society, the mathematics society of the Open University.

Yours sincerely,
TONY FORBES
(22 St Albans Road,
Kingston upon Thames,
Surrey KT2 5HQ)

Dear Editor,

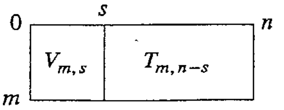
The domino problem

The article by Chris Holt with this title in *Mathematical Spectrum* Volume 27, Number 3, page 62, deals with the problem of enumerating the number of dissections $T_{m,n}$ of a rectangle m units high by n wide into pieces 1×2 or 2×1 termed 'dominoes', for the particular case $m = 3$ and n even. (Of course, $T_{m,n} = 0$ when mn is odd.) A recurrence relation can be expressed for the general case in the following form, previously given by me in *Chessics* (1986) Volume 2, Number 28, page 138:

$$T_{m,n} = \sum_{s=1}^{n} V_{m,s}T_{m,n-s},$$

where $T_{m,0}$ is taken to be 1 (to simplify the expression) and $V_{m,s}$ is the number of $m\times s$ dissections without vertical fault lines (a fault line being a straight line of domino edges passing across the whole rectangle, dividing it into two smaller rectangles).

The argument to prove this formula is illustrated in the figure, $V_{m,s}$ being the number of ways of dissecting the part of the rectangle before the first vertical fault line (or the whole rectangle when there are no fault lines) and $T_{m,n-s}$ the number of ways of dissecting the remainder of the rectangle (becoming $T_{m,0}$ when there are no fault lines).



If all values of $V_{m,s}$ are known for $1 \leqslant s \leqslant n$, then $T_{m,n}$ can be calculated. Fortunately, many of the values of $V_{m,s}$ are zero. By direct construction it is easy to show that the only non-zero values of $V_{m,s}$ up to $m = 4$ are: $V_{1,2} = V_{2,1} = V_{2,2} = 1$, $V_{3,2} = 3$, $V_{3,2h} = 2$ (for $h > 1$), $V_{4,1} = 1$, $V_{4,2} = 4$, $V_{4,2h-1} = 2$ and $V_{4,2h} = 3$, and these values enable one to calculate $T_{m,n}$ for $1 \leqslant m \leqslant 4$, as follows.

When $m = 1$ all values are trivially 1: $T_{1,0} = 1$, $T_{1,2k} = V_{1,2}T_{1,2k-2} = T_{1,2k-2}$ $(k > 0)$.

When $m = 2$ we have

$$T_{2,0} = 1,$$
$$T_{2,1} = V_{2,1}T_{2,0} = 1,$$
$$T_{2,n} = V_{2,1}T_{2,n-1}+V_{2,2}T_{2,n-2},$$

which (since the Vs are 1s) is the Fibonacci relation. (Result by W. L. Patten, *American Mathematical Monthly* 1961.)

When $m = 3$ we find

$$T_{3,0} = 1,$$
$$T_{3,2} = V_{3,2}T_{3,0} = 3,$$
$$T_{3,2h} = V_{3,2}T_{3,2h-2}+V_{3,4}T_{3,2h-4}+V_{3,6}T_{3,2h-6}+\cdots$$
$$= 3T_{3,2h-2}+2T_{3,2h-4}+S$$

and

$$T_{3,2h-2} = V_{3,2}T_{3,2h-4}+V_{3,4}T_{3,2h-6}+\cdots$$
$$= 3T_{3,2h-4}+S$$

(the rest of the sum, S, being the same in each case, since all the Vs beyond $V_{3,2}$ equal 2).

Subraction gives

$$T_{3,2h}-T_{3,2h-2} = 3T_{3,2h-2}-T_{3,2h-4},$$

which simplifies to Holt's recurrence.

When $m = 4$ we similarly find

$$T_{4,n} = T_{4,n-1}+4T_{4,n-2}+2T_{4,n-3}+3T_{4,n-4}+S'$$

and

$$T_{4,n-2} = T_{4,n-3}+4T_{4,n-4}+S'$$

(the terms denoted S' being the same in each case, since the Vs beyond $V_{4,2}$ are alternately 2 and 3). Subtraction gives the recurrence:

$$T_{4,n} = T_{4,n-1}+5T_{4,n-2}+T_{4,n-3}-T_{4,n-4},$$

for which however we require the *four* initial values $T_{4,0} = 1$, $T_{4,1} = 1$, $T_{4,2} = 5$ and $T_{4,3} = 11$. The next five values work out to: $T_{4,4} = 36$, $T_{4,5} = 95$, $T_{4,6} = 281$, $T_{4,7} = 781$ and $T_{4,8} = 2245$. From the square of the last figure we may deduce that the number of ways of 'dominizing' a chessboard is more than 5 million!

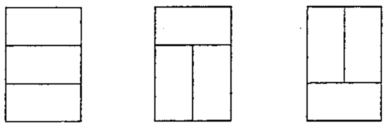
To carry the enumeration to larger m, a recurrence to calculate $V_{m,s}$ would be helpful. Unfortunately, the simple relation conjectured for this purpose in the *Chessics* article cited is erroneous. Perhaps your readers can make more progress with this part of the problem.

Yours sincerely,
GEORGE JELLIS
(63 Eversfield Place,
St Leonards-On-Sea,
East Sussex, TN37 6DB)

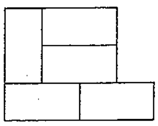
Dear Editor,

The domino problem

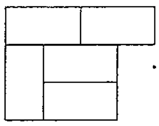
With reference to Chris Holt's article in Volume 27, Number 3, I can answer his question as follows. T_n is made up of T_{n-1} for each array beginning



i.e. a total of $3T_{n-1}$, *and* in addition those arising from

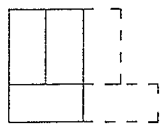


and

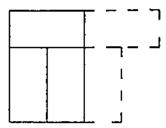


.

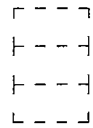
These are equivalent in number to those arising from



and



which are in fact T_{n-1} in number but excluding those of a $3{\times}2(n-1)$ array which have no vertical domino at the start, i.e. starting



which are T_{n-2} in number. Hence there are $T_{n-1}-T_{n-2}$ in addition to the $3T_{n-1}$ referred to above. Therefore

$$T_n = 3T_{n-1}+T_{n-1}-T_{n-2}.$$

Yours sincerely,
Alastair Summers
(Teacher, Stamford School)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

28.5 Prove that there are no natural numbers n which make 3^n+n^3 a cube.
(Submitted by Kenichiro Kashihara, Kanagawa, Japan.)

28.6 A triangle has angles α, β and γ which are whole numbers of degrees, and $\alpha^2+\beta^2=\gamma^2$. Find all possibilities for α, β and γ.
(Submitted by Sam Yu, University of South Dakota.)

28.7 Evaluate

$$\int \sqrt{\sec^2 x+A}\,\mathrm{d}x,$$

where $A \geqslant 0$ is a constant.
(Submitted by Junji Inaba, William Hulme's Grammar School, Manchester.)

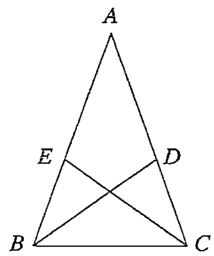
28.8 Prove that

$$\prod_{n=1}^{p-1}(1+pn^{-1}) \equiv 1 \pmod{p},$$

where p is a prime number.
(Submitted by Lee Talbot, De Montfort University.)

Solutions to Problems in Volume 27 Number 3

27.9 In the diagram, BD and CE bisect angles ABC and ACB respectively, and $BD=CE$. Prove that triangle ABC is isosceles.



Solution by Toby Gee, The John of Gaunt School, Trowbridge

We prove that, if $\angle B \neq \angle C$, then $BD \neq CE$.

Lemma 1. If two chords of a circle subtend different acute angles at the circumference, then the smaller chord has the smaller angle.

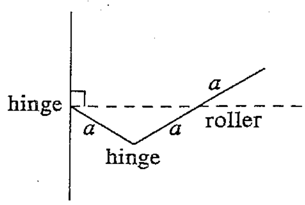
Proof. The shorter chord is further from the centre of the circle than the larger one, so it subtends a smaller angle at the centre and thus at the circumference than the larger one.

Lemma 2. If $\angle C > \angle B$ then $CE < BD$.

Proof. Let D' be the point on BD in from D such that $\angle D'CE = \frac{1}{2}\angle B$. Then E, B, C and D' lie on a circle. But $\angle B < \frac{1}{2}(\angle B+\angle C)$, i.e. $\angle CBE < \angle D'CB < 90°$, so, by Lemma 1, $CE < D'B$. So $CE < BD' < BD$.

Also solved by Can-Anh-Minh, University of California at Berkeley, Sam and Jim Yu, Junji Inaba and Noah Rosenberg, Rice University, Texas.

27.10 A door has two leaves, one being twice the size of the other, joined by a hinge. The smaller leaf is hinged to the wall and the larger has a roller half way along which moves along the dotted line shown. How much floor space is needed, and what is the locus of a point on the door when it is opened?



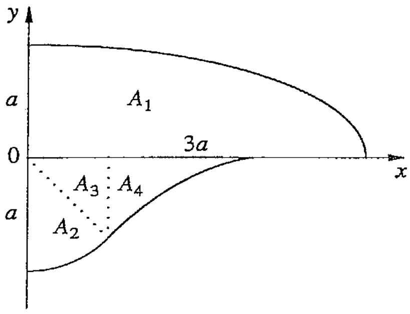
Solution by Toby Gee

We find the locus first. A point on the smaller leaf moves on a quarter-circle with centre at the wall hinge; a point P on the larger leaf distant r from the other hinge has the locus

$$x = (r+a)\cos\theta, \quad y = (r-a)\sin\theta.$$

This is a quarter-ellipse with axis lengths $2(r+a)$ and $2|r-a|$.

The diagram shows the floor space.



The region A_1 is a quarter-ellipse and so has area $\frac{1}{4}\pi a \times 3a = \frac{3}{4}\pi a^2$. A_2 is $\frac{1}{8}$ of a circle with area $\frac{1}{8}\pi a^2$. A_3 is a right-angled isoceles triangle with short side $\frac{1}{2}a\sqrt{2}$ and so has area $\frac{1}{4}a^2$. The equation of the larger door when the angle with the x-axis is θ is

$$y = x\tan\theta - 2a\sin\theta.$$

To find the equation of the envelope of these lines we differentiate with respect to θ to give

$$0 = x\sec^2\theta - 2a\cos\theta,$$

i.e. $x = 2a\cos^3\theta$, whence $y = -2a\sin^3\theta$. Hence the area of A_4 is

$$\int_{\frac{1}{4}\pi}^{0} (-y)\ dx = \int_{0}^{\frac{1}{4}\pi} 2a\sin^3\theta \times 6a\cos^2\theta\sin\theta\ d\theta,$$

which works out to be $\frac{3}{16}\pi a^2 - \frac{1}{4}a^2$. Hence the total area is

$$\tfrac{3}{4}\pi a^2 + \tfrac{1}{8}\pi a^2 + \tfrac{1}{4}a^2 + (\tfrac{3}{16}\pi a^2 - \tfrac{1}{4}a^2) = \tfrac{17}{16}\pi a^2.$$

27.11 Find a formula for the volume of a tetrahedron in terms of the lengths of the three edges which meet at a vertex and the three face angles at that vertex.

Solution by Toby Gee

Let the tetrahedron be $OABC$ with $\alpha = \angle BOC$, $\beta = \angle COA$, $\gamma = \angle AOB$, $\boldsymbol{a} = OA$, $\boldsymbol{b} = OB$ and $\boldsymbol{c} = OC$. The volume V is $\frac{1}{3}$(area of the base)×height, and so is $\frac{1}{6}$×the volume of the parallelepiped with sides OA, OB and OC. Hence

$$6V = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ etc. Hence

$$36V^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{a} & \boldsymbol{a}\cdot\boldsymbol{b} & \boldsymbol{a}\cdot\boldsymbol{c} \\ \boldsymbol{b}\cdot\boldsymbol{a} & \boldsymbol{b}\cdot\boldsymbol{b} & \boldsymbol{b}\cdot\boldsymbol{c} \\ \boldsymbol{c}\cdot\boldsymbol{a} & \boldsymbol{c}\cdot\boldsymbol{b} & \boldsymbol{c}\cdot\boldsymbol{c} \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$$

$$= a^2b^2c^2(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma).$$

Hence

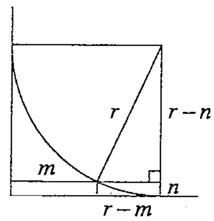
$$V = \tfrac{1}{6}abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}.$$

Also solved by Junji Inaba and Can-Anh-Minh.

27.12 A circular table with radius r is pushed into the corner of a square room. A point P on the edge of the table is m units from one wall and n units from the other, where $m, n < r$. Prove that, if m, n and r are all integers and m and n are coprime, then either m or n is a perfect square and the other is even.

Solution 1 by Toby Gee

From the diagram,



$$(r-m)^2 + (r-n)^2 = r^2.$$

Further, $r-m$, $r-n$ and r have highest common factor 1 because m and n are coprime. Hence, by a standard result on Pythagorean triples, we can write

$$r-m = 2xy, \qquad r-n = x^2-y^2, \qquad r = x^2+y^2$$

for some natural numbers x and y (or the same with m and n interchanged). Then

$$m = (x-y)^2, \qquad n = 2y^2$$

or vice versa.

Solution 2 by Junji Inaba (not using the result about Pythagorean triples)

As in solution 1,

$$(r-m)^2 + (r-n)^2 = r^2,$$

so

$$r^2 - 2rm + m^2 - 2rn + n^2 = 0,$$

so

$$[r-(m+n)]^2 = 2mn,$$

so $2mn$ is a perfect square. Since m and n are coprime, it follows that one of m and n is twice a perfect square and the other is a perfect square.

Also solved by Can-Anh-Minh and Noah Rosenberg.

□

What is the probability that two brothers were born on a Tuesday, given that at least one of them was born on a Tuesday?

Taken from the *Canberra Times*

Reviews

Poetry of the Universe. By ROBERT OSSERMAN. Weidenfeld and Nicholson, 1995. Pp. 172. Hardback £18.95 (ISBN 0-297-81516-4).

Starting in Ancient Greece and finishing at the current edge of cosmological research, the author leads the reader in an exploration of the history of the ideas that have allowed man to embrace the universe and come to terms with its immensity, structure and history.

This book is not a highly mathematical manuscript of equations and formulae, but a well-written flowing account, in a clear and comprehensible manner, of pioneering mathematicians with revolutionary ideas, allowing their meaning, the effect they had on contemporary thinking and the underlying philosophical progression to eclipse the actual mathematics. The reader becomes absorbed in a gloriously evolving story, not a long, complex, developing mathematical theory.

The essence of many diverse theories and concepts has been extracted and cleverly explained using clear analogies and simple diagrams, making the book suitable for the patient lay reader. A list of notes and references supplements the book, so that any person with an interest in cosmology will find the book a stimulating source of information.

I thoroughly enjoyed the book. I found it to have the right length and the right emphasis on each subject that arose. At no time did I feel the author was labouring a point or brushing over a technicality that required a suitable explanation. Even having studied cosmology to a high level, I found the author's analogies enlightening and informative.

It is a brilliant book that does itself justice. It would grace the bookshelf of anybody interested in mathematics.

PIERINO GATTEI

Woolly Thoughts. By PAT ASHFORTH AND STEVE PLUMMER. Souvenir Press, London, 1994. Pp. 128. Paperback £10.99 (ISBN 0-287-63196-9).

This book is written with great enthusiasm which is very catching. I suspect that many readers will find themselves swept along on a wave of enthusiasm and will be inspired to have a go for themselves. The colour pictures certainly encourage confidence. I would, however, volunteer a word of warning. A project such as the knitting of a cardigan will take time for planning, time for knitting and artistic flair in putting colours and shapes together. Before starting make sure that you have enough enthusiasm to see the undertaking through, even when it doesn't seem as easy as it first appeared.

Much of the volume is given over to the practicalities of knitting—how to make stitches and how to choose yarn and needles etc. The sketches illustrating the text are quite delightful and the sections are well cross-referenced. It should be easy to locate the instructions which are needed at any particular stage. It does, however, pass lightly over some of the snags. For example, a very inexperienced knitter would not find it easy to get nice neat edges for the square initially and may have to put in a little practice. Perhaps starting with making something small is the answer. There are illustrations of cushion covers in the book and they could make a suitable project for the beginner.

The mathematics which goes into the actual design is interesting, but it will be far more time-consuming than the book suggests. The reader may need several attempts at the arithmetic and will need practice, concentration and perseverance. There are some very interesting ideas for overall patterns and some readers may find it enjoyable to learn some mathematical ideas in this context rather than just reading from a text book.

University of Sheffield — MARY HART

Bilingual (English/Chinese) Edition of Essentials of Statistical Methods. By T. P. HUTCHINSON, translated into Chinese by JULIAN Z. WANG. Rumsby Scientific Publishing, P.O. Box 76, Rundle Mall, Adelaide 5000, Australia, 1994. Pp. iii+99. Paperback A$10.00 (ISBN 0-646-20458-0).

This is a concise introduction to the elements of statistical methods, consisting of three parts: I. Data description, II. Probability and III. Inference. The author suggests its use as an adjunct to a course of statistics lectures and tutorial classes and hopes the book will prove of use as a 'memory jogger'.

Each page contains a left-hand column in English, together with a right-hand column in Chinese. The book is intended for students whose first language is Chinese attending statistics courses in English-speaking universities, of whom there are many in Australia.

Australian National University — JOE GANI
Canberra, Australia.

All The Math That's Fit To Print—Articles from the Manchester Guardian. By KEITH DEVLIN. The Mathematical Association of America, Washington DC, 1994. Pp. 345. Paperback $29.50 (ISBN 0-88385-515-1).

There cannot be many national daily newspapers that run a regular mathematics column, nor many mathematicians who could sustain such a column to interest the interested layman. The present volume consists of selections from Keith Devlin's *Micromaths* column in the Guardian from 1983 to 1989. It makes fascinating reading. The column appeared fortnightly in the Computing Guardian, so the emphasis is on computing. But there is far more here than just for the computer buff.

Keith Devlin is a mathematician with the rare gift of being able to communicate the subject to the person in the street. No one who has dipped into this volume will again ask 'How do you do research into mathematics?' The topics cover searches for large prime numbers, factorizing large numbers, Fermat's last theorem, the Riemann hypothesis, the four-colour theorem, to name but a few. He keeps returning to topics previously discussed, updating and occasionally correcting the information. His skill is to make mathematics alive and interesting and to involve his readers. You will want to try to write a program to find a number with a fascinating property, or to get a pencil and paper and solve a puzzle.

What this volume tells its readers is that Mathematics is News. Devlin suggests that it should be kept in the smallest room. My only fear is that family harmony may be put under strain as members queue up outside the locked door to take their turn to read all the math that's fit to print.

University of Sheffield DAVID SHARPE

New Mathematical Diversions. By MARTIN GARDNER. The Mathematical Association of America, Washington DC, 1995. Pp. 272. Paperback \$19.95 (ISBN 0-88385-517-8).

This is a revised edition of Martin Gardner's classic *New Mathematical Diversions*. As usual, the book consists of expanded versions of Gardner's *Scientific American* columns; this revised edition also includes twelve pages of 'recent updates', as well as an updated bibliography. Many readers will need no introduction to Gardner's work; those who have not encountered it before should appreciate that whilst this book is written in a simple style that could be understood by the non-specialist, there is much here to interest the mathematicians. As well as new games, humorous stories, and magic tricks, there are also hard (and even unsolved) problems, for example the best packing of spheres in 3-space.

My favourite chapters were 'Group Theory and Braids', which provides a gentle introduction to group theory and permutation groups, 'The Ellipse', which has several intriguing questions (e.g. Prove that no regular polygon with more than 4 sides may have all its vertices upon a non-circular ellipse), and 'The Calculus of Finite Differences', which is a good introduction to this neglected subject. Doubtless other readers would also find many items of interest amongst the twenty chapters.

If you do not already own a copy of this book, buy it; if you do, the revised edition probably is not worth getting, as there is very little new material.

Student at The John of Gaunt School TOBY GEE
Trowbridge

The Guiness Book of Mindbenders. By ROBERT EASTWAY AND DAVID WELLS. Guiness, London, 1995. Paperback £6.99 (ISBN 0-85112-668-5).

This collection of 109 puzzles contains, according to the press release, around 40 classic puzzles 'presented from intriguingly quirky new angles' and about 60 new puzzles. However, very few of the puzzles seem to me to be entirely new; the majority of them are either restatements or variations on the theme of well-known problems. The experienced puzzler is unlikely to find much of interest here, but as a book for a novice it can be recommended; the puzzles are nicely laid out, with clear and concise solutions which appear to be mostly correct (I spotted only one error in a solution).

From a mathematical point of view the book is somewhat disappointing; it includes mathematical puzzles, but it seems afraid to solve them in a mathematical manner, instead relying on 'guesswork'.

In conclusion, this book is probably only suitable for the novice; more seasoned readers would be better off with Wells' own *Penguin Dictionary of Curious and Interesting Puzzles*.

Student at The John of Gaunt School TOBY GEE
Trowbridge

Game Theory and Strategy. By PHILIP D. STRAFFIN. The Mathematical Association of America, Washington DC, 1993. Pp. x+244. Paperback \$27.50 (ISBN 0-88385-637-9).

Before I encountered game theory, 'the mathematical theory of conflict and cooperation', I had frequently heard expressions such as 'pareto optimal', 'non-zero sum payoffs' and 'a prisoner's dilemma situation' in economic analysis and elsewhere, but did not know that they were well-defined mathematical concepts. The fact that the language of game theory is everyday language shows how much it is integrated into our existence, and this is amply illustrated in this book. The author believes that 'the breadth and depth of applications here is greater than in any other elementary treatments of game theory'.

In three sections, the author introduces the reader to the tools of game theory, which are the strategies that can maximise a player's payoffs or negotiate a fair or reasonable settlement. Interspersed with the theory are chapters which show how to wield the most recently acquired tools, by applying them to 'real life' examples such as how Jamaican fishers would rationally place their pots and how the cost of hydroelectric power should have been shared in India. Also, perhaps because it is difficult to quantify the payoffs to games such as battles between members of a species with different temperaments or because experimental data are not necessary or appropriate, reality is modelled by a simplified game and arbitrary payoffs. I found these chapters refreshing breaks from the difficulty of conceptualising the mathematics. There are usually exercises at the end of each chapter, particularly the ones which introduce new strategies. These range from simple practice in the new algorithms to proofs omitted in the chapter and investigation of related topics.

As it says in the 'Note to the Reader', this book is supposed to be accessible to 'a large audience of high-school students and laymen', and this is true: GCSE algebra would suffice. However, the audience that would bother to access it is probably considerably smaller. The narrative is not mechanical—there are some amusing personal touches—but the ideas are not over-explained, and this conciseness can make the book seem text-bookish, as do the numerous, rigorously numbered (necessary) diagrams and lists. So I found it an interesting but somewhat laborious read, and would only suggest buying it if you have a strong desire to know the workings of game theory.

Student at Dame Allan's Girls' School POLLY SHAW
Newcastle upon Tyne □

The Mandelbrot set

The illustrations on pp. 16 and 17 of *Mathematical Spectrum* Volume 18 Number 1 originally appeared as Figures 22, 24, 26 and 40 in *Frontiers of Chaos* by H.-O. Peitgen and P. H. Richter, published in 1985 by MAP-ART (Forschungsgruppe Komplexe Dynamik, Universität Bremen).

Mathematical Spectrum

1995/6 Volume 28 Number 2

ISSN 0025-5653

Published by the Applied Probability Trust
Typeset by The Pi-squared Press, Nottingham, UK
Printed by Galliard (Printers) Ltd, Great Yarmouth, UK