MATHEMATICAL SPECTRUM

A MAGAZINE FOR STUDENTS AND TEACHERS OF MATHEMATICS AT SCHOOLS, COLLEGES AND UNIVERSITIES

Volume 18 1985/86 Number 3

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 18 of *Mathematical Spectrum* consists of three issues, of which this is the third. The first was published in September 1985 and the second in January 1986.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

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The Editor, Mathematical Spectrum,
Hicks Building, The University, Sheffield S3 7RH

Beauty from Chaos

K. DEVLIN, *University of Lancaster*

The author took his B.Sc. at King's College, London, in 1968 and his Ph.D. at the University of Bristol in 1972. He has held positions at several British and overseas universities and is currently Reader in Mathematics at the University of Lancaster. His main mathematical interest is set theory and he has written over thirty research papers and five textbooks, including *Sets, Functions and Logic* (Chapman and Hall) for beginning university students. He writes a regular mathematical column in the *Guardian* and has contributed several articles to *Mathematical Spectrum*.

It was Bertrand Russell who wrote, in his 1918 book *Mysticism and Logic*, that: 'Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture.'

Until recently the abundance of beauty in mathematics could only be appreciated by the professional mathematician, used to dealing with the highly abstract domains in which Russell's beauty could be found. But with the coming of the modern computer, large areas of abstract, pure mathematics have become accessible to the man in the street. No better example could be found than the recent work on *Chaos*, where the use of sophisticated computer graphics has demonstrated to the layman and expert alike that beneath some simple-looking mathematical processes there lies a fascinating world where so-called 'chaotic behaviour' produces strangely beautiful patterns, patterns which when analysed more closely turn out to contain regions of chaotic behaviour which produce strangely beautiful patterns, patterns which ... and so on *ad infinitum*.

The basic ideas behind all of the pictures accompanying this article are very simple. First of all you take some simple function of one variable (which may be real or complex), say $y = f(x)$. Then you take some starting value x_0 and iterate the application of your function to this initial value, obtaining a sequence of numbers satisfying the equation

$$x_{n+1} = f(x_n).$$

You then examine the behaviour of the sequence $x_0, x_1, x_2, \ldots, x_n, \ldots$. Various things may occur. The sequence may eventually become constant or oscillate between two or more values. Or there may be one or more numbers around which the values of the sequence seem to cluster. Or there may be no kind of ordered behaviour at all.

This article was abridged from a series which first appeared in the *Guardian* during 1985.

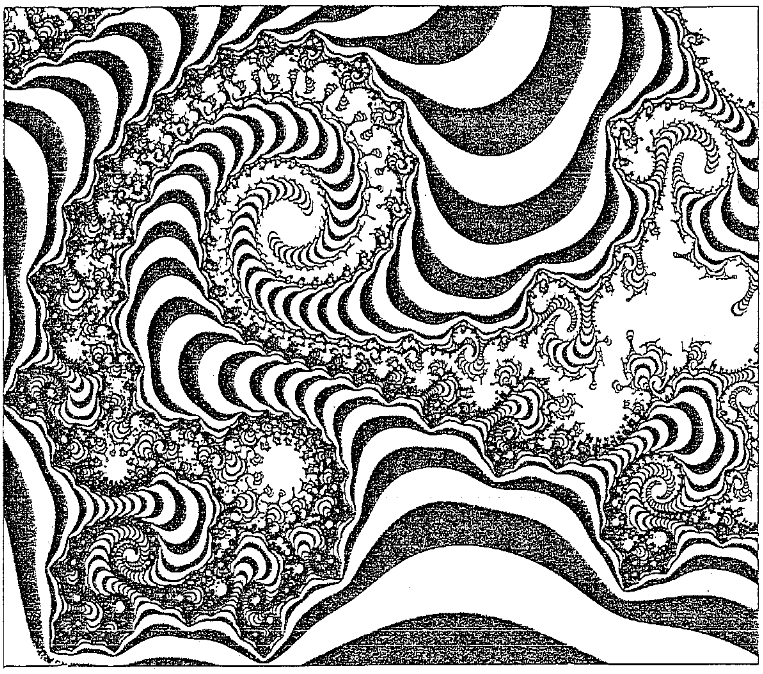


Figure 1. Chaotic art

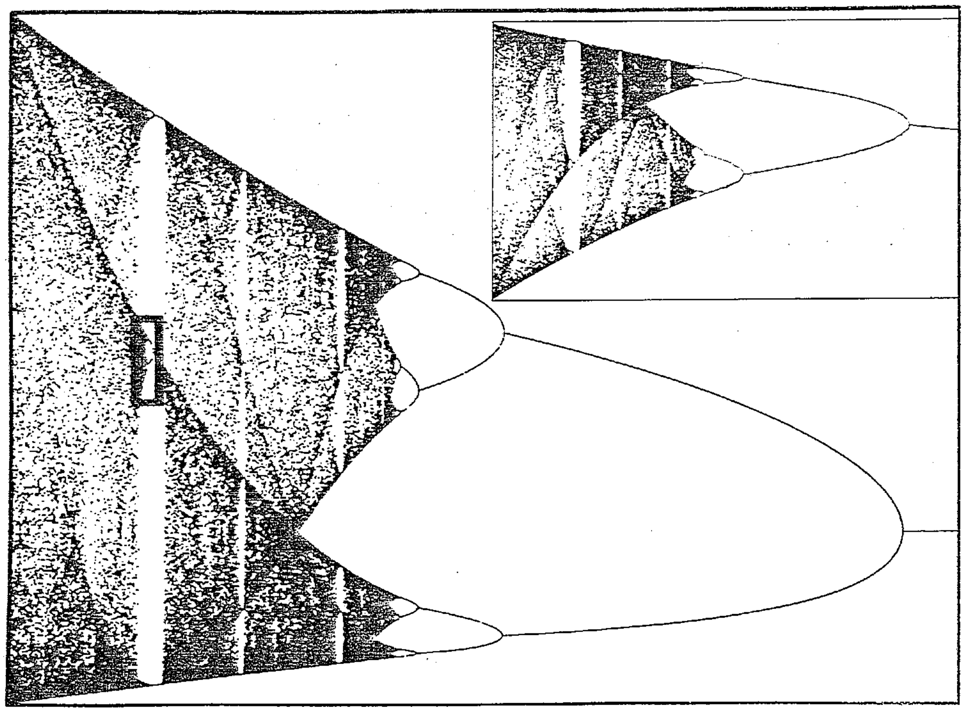


Figure 2. Iteration of $Ax(1-x)$ with a close-up of the boxed region

For example, suppose you take the function $f(x) = x^2$. If your starting value x_0 is between -1 and $+1$ the values of your sequence 'cluster' around 0 (the origin). If you start with $x_0 = 1$ the sequence becomes constant with value 1. If your starting value is greater than 1 or less than -1 the sequence simply grows without limit. To use terminology associated with the study of such iterative processes, the number 0 is an *attractor* for the function

$f(x) = x^2$ in the range between -1 and $+1$.

Though it illustrates the different types of behaviour of an iterative process, the above example is too simple to be of much interest. For what first seems to have been observed in the early 1960s by P. J. Myberg was that, in functions where there is a numerical parameter, the actual value of that parameter plays a critical role in the behaviour of the associated iterative sequence (whereas the initial 'seed' value x_0 does not, apart from being constrained to lie in some given region such as between -1 and $+1$). This is amply illustrated by the parabola function

$$f(x) = Ax(1-x),$$

where A is some fixed value between 1 and 4. The behaviour of the iteration sequence you get from this function is independent of the initial seed value when this is chosen to lie between 0 and 1, but highly dependent on the actual choice of the constant parameter A. For a value of A chosen less than 3, the iteration sequence rapidly converges to a single cluster point. For A between 3 and around 3.57 there occurs a succession of bifurcations. Just above $A = 3$ the iteration sequence settles down to an oscillation between two cluster points. Increase A a little and you find four cluster points. And so on and so on. Above $A = 3.57$ there is chaotic behaviour where the iteration sequence jumps all over the place. Then, around $A = 3.82$ you suddenly find three cluster points. Then chaos takes over once more as A gets larger. This is all illustrated in figure 2, where the iteration sequence is plotted against the vertical axis for values of A increasing along the horizontal axis. (For each value of A the sequence is generated for 5000 steps and the next 120 values are plotted. This provides a reasonable picture of the 'cluster' behaviour of the sequence.)

Even more fascinating is what you discover if you take a close look at the region around the central cluster point at $A = 3.82$. Blow this up (an easy task on a computer) and you find you have a 'carbon copy' of the entire picture of which it is a part. Within this 'baby' you will see a point where chaos gives way to three cluster points. If you examine the region around the middle of one of these you find .... Curious, huh?

Using a microcomputer it is quite straightforward to investigate not only the process discussed above but whatever other functions you want to try. (You will need to experiment in each case to see how far to iterate before you plot values, how many values to plot and what are the 'interesting' values for the parameter, but otherwise it is an easy programming task.)

Things become even more interesting when you allow functions from complex numbers to complex numbers, but you then need to find a different way of graphing the results. Since it requires two dimensions to plot a complex number, you need to use colours or shading to show the effect of the changing parameter. The classic example in this case is the function

$$f(z) = z^2 + c,$$

where the variable z now ranges over the complex numbers and c is a complex constant. Each of figures 1, 3 and 4 is produced using this function, figures 1 and 3 being generated using a large computer at the University of Bremen, figure 4 coming from a BBC Microcomputer. The shading attached to each point c (in the complex plane) depends upon the behaviour of the iteration sequence obtained from that value, and the picture can be thought of as a 'contour map' which represents the cluster behaviour of the corresponding iteration sequence. Rather than give a detailed explanation of what is going on, I shall instead provide a step-by-step guide as to how you can implement the process on a microcomputer and leave you to investigate it for yourself.

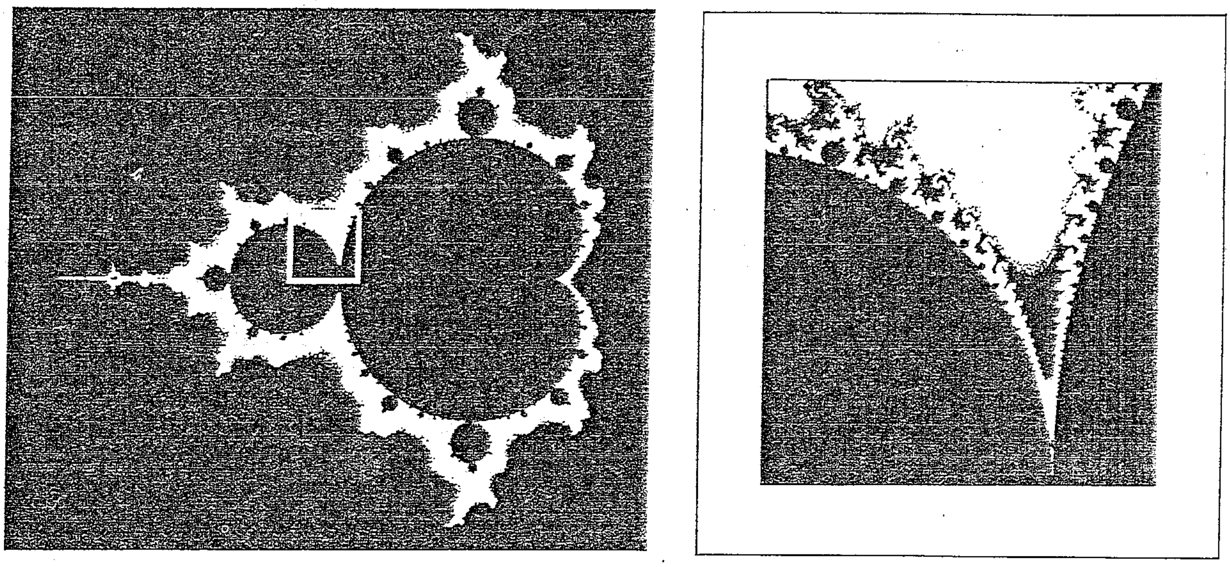


Figure 3. The Mandelbrot set with a close-up of 'Sea-Horse Valley'

To start with, suppose your computer display screen has a graphics resolution of A times B points, and let $K+1$ be the number of distinct colours which can be displayed. (For good results a value of K of around 200 is required. If your machine does not provide this kind of resolution you should take $K = 200$ in what follows and assign colours to various bands within this region. Or you can get nice results using cycled colours. Experiment!)

STEP 0: Set $H = 3/(A-1)$, $V = 3/(B-1)$.

Then for all points (P, Q) of the monitor (P from 0 to $A-1$, Q from 0 to $B-1$) go through the following routine.

STEP 1: Let $M = -2.25 + P^*H$; $N = -1.5 + Q^*V$; $I = 0$; $X = 0$; $Y = 0$.

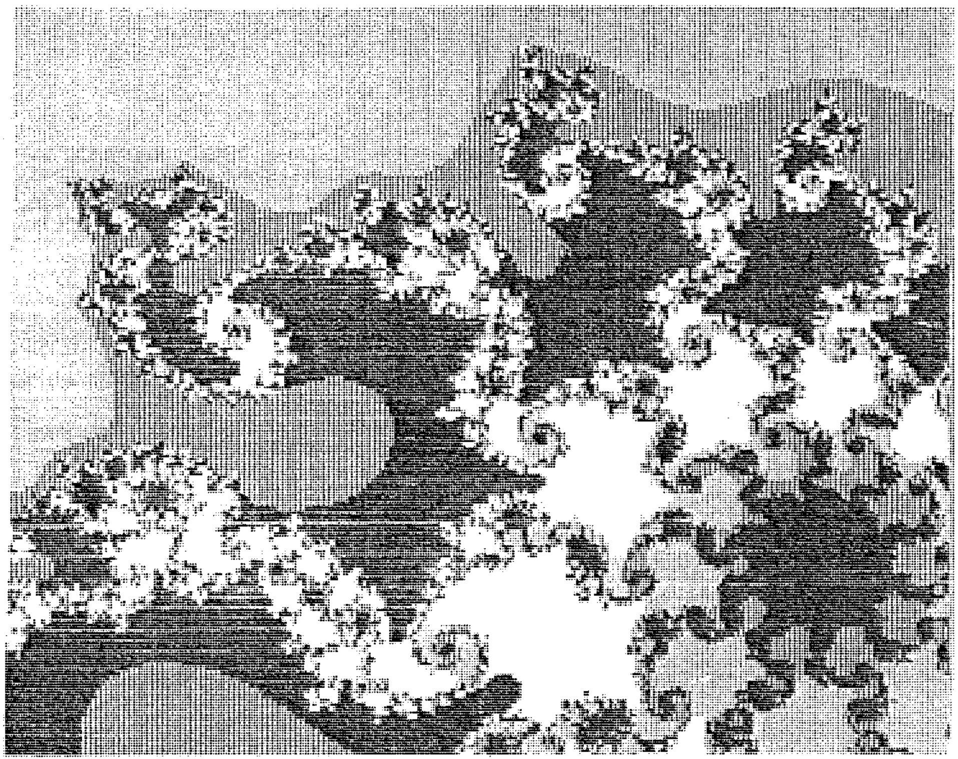


Figure 4. Results obtained with a BBC Micro

STEP 2: Let $R = X^2 + Y^2$; $T = X^2 - Y^2 + M$; $Y = 2*X*Y + N$; $X = T$; $I = I + 1$.

STEP 3: If $R > 100$ assign colour I to point (P, Q) and go to STEP 1; if $I = K$ assign colour 0 (which should be black) to point (P, Q) and go to STEP 1; if $R \leqslant 100$ and $I < K$ then repeat STEP 2.

If you follow the above procedure you will obtain a picture (figure 3) in which the so-called *Mandelbrot set* is shown (in black) in the region -2.25 to $+0.75$ along the real axis and -1.5 to $+1.5$ along the imaginary axis. If you want to obtain a blow-up of part of the set in the region X_{min} to X_{max} and Y_{min} to Y_{max} then in STEP 0 you must take $H = (X_{\text{max}} - X_{\text{min}})/(A-1)$ and $V = (Y_{\text{max}} - Y_{\text{min}})/(B-1)$ and in STEP 1 replace the figures -2.25 and -1.5 by your values for X_{min} and Y_{min}, respectively.

From now on it is up to you. Figure 4 shows what can be obtained using a standard BBC Microcomputer. The other illustrations were all taken from the book *Frontiers of Chaos* published by Mapart of Bremen, an excellent book containing dozens of high-resolution colour illustrations of chaotic art produced as above.

Independence and the Length of a Run of Wins

D. O. FORFAR AND T. W. KEOGH, *Scottish Widows' Fund and Life Assurance Society*

The company for which the authors work is one of the largest and best-known Scottish life assurance companies, located in Edinburgh. D. O. Forfar is Joint Actuary, and T. W. Keogh is an actuarial student. They both graduated in Mathematics from Cambridge University, D. O. Forfar from Trinity College and T. W. Keogh from Pembroke College.

1. The problem stated

Problem 17.8 (in Volume 17 Number 3 of *Mathematical Spectrum*) was stated as follows:

'In a series of independent games between two players, each player has probability $\frac{1}{2}$ of winning each game. The series terminates as soon as either player has won three consecutive games. Find the probability that the series terminates just after the nth game.'

Perhaps more interesting than the question asked is the problem of determining the average length of such a series and the standard deviation of the length. The problem can be generalised to consider the case where the series is won by the first person to win m consecutive games.

We consider first the original problem where three consecutive wins are needed to terminate the series. Call the two players A and B. Any series can be written as a sequence of A's and B's according to who is the winner of each game. Let $u(n)$ be the number of different series which terminate on the nth game. Since there are 2^n distinct sequences of A's and B's, where the length of the sequence is n, the probability that the series terminates on the nth game is $u(n)/2^n$. We find, by enumerating the possible sequences, that $u(1) = 0$, $u(2) = 0$, $u(3) = 2$, $u(4) = 2$, $u(5) = 4$, $u(6) = 6$, $u(7) = 10$ and, in general, it appears that $u(n) = u(n-1)+u(u-2)$ for $n \geqslant 4$.

To prove this, consider the sequence of A's and B's for any series which ends on the $(n-2)$th game. We see that we can form an n-game sequence by prefixing the $(n-2)$-game sequence by AA or BB according to whether the first game of the $(n-2)$-game sequence was B or A, respectively. For any series which ends on the $(n-1)$th game, we can form an n-game sequence by prefixing the $(n-1)$-game sequence by A or B according to whether the first game was B or A, respectively. This covers all the n-game sequences since such a sequence must start $AAB\ldots$, $BBA\ldots$, $AB\ldots$ or $BA\ldots$, and hence

$$u(n) = u(n-1)+u(n-2).$$

2. The problem solved

We now consider the series where m consecutive wins are needed to terminate the series. We wish to derive the average length of such a series and the standard deviation of the length. We again define $u(n)$ to be the number of different series which terminate on the nth game. We define $P_n^{(m)}$ to be the probability that the series ends on the nth game. As before, $P_n^{(m)} = u(n)/2^n$.

Theorem 1. *The following relations hold:*

$$u(i) = \begin{cases} 0 & (1 \leqslant i \leqslant m-1), \\ 2 & (i = m), \\ 2^{i-m} & (m+1 \leqslant i \leqslant 2m-1), \\ u(u-1)+u(i-2)+\ldots+u(i-m+1) & (i \geqslant 2m). \end{cases}$$

Proof. It is clear that the series cannot end before the mth game: accordingly $u(i) = 0$ $(1 \leqslant i \leqslant m-1)$. If the series ends on the mth game then either all the games are won by A (a sequence which we shall write as A^m) or all the games are won by B (a sequence which we shall write as B^m). As these are the only two possibilities, $u(m) = 2$. It is clear that, if the series ends on the ith game, where $m+1 \leqslant i \leqslant 2m-1$, then the last $m+1$ terms of the sequences must be AB^m or BA^m. The first $i-m-1$ terms can be any sequence of A's and B's and there are 2^{i-m-1} such sequences. Hence $u(i) = 2\times 2^{i-m-1} = 2^{i-m}$.

We now consider the series which ends on the ith game, where $i \geqslant 2m$. From any series which ends on the $(i-1)$th game, we can form an i-game sequence by prefixing the $(i-1)$-game sequence by A or B according to whether the first game was B or A, respectively. From any series which ends on the $(i-2)$th game, we can form an i-game sequence by prefixing the $(i-2)$-game sequence by AA or BB according to whether the first game of the sequence was B or A, respectively. This argument can be repeated until one considers finally a series which ends on the $(i-m+1)$th game. The $(i-m+1)$-game sequence can be prefixed by A^{m-1} or B^{m-1} according to whether the first game was B or A, respectively. Since any i-game sequence must start with AB, BA, A^2B, B^2A, $A^{m-1}B$ or $B^{m-1}A$, we have

$$u(i) = u(i-1)+\ldots+u(i-m+1) \quad (i \geqslant 2m).$$

We now define the following series:

$$S_0 = \sum_{n=1}^{\infty} \frac{u(n)}{2^n} = \sum_{n=1}^{\infty} P_{n^{(m)}},$$

$$S_1 = \sum_{n=1}^{\infty} \frac{nu(n)}{2^n} = \sum_{n=1}^{\infty} nP_n^{(m)},$$

$$S_2 = \sum_{n=1}^{\infty} \frac{n^2u(n)}{2^n} = \sum_{n=1}^{\infty} n^2P_n^{(m)}.$$

We require to show that $S_0 = 1$, i.e. that $P_n^{(m)}$ is an 'honest' distribution for the length of the series. The average length of the series is S_1 and the standard deviation is $(S_2 - S_1^2)^{\frac{1}{2}}$. ($S_1$ and S_2 represent the first and second moments of the distribution function.)

Theorem 2. $S_0 = 1$.

Proof. We consider the expression

$$S_0 - \frac{S_0}{2} - \frac{S_0}{2^2} - \ldots - \frac{S_0}{2^{m-1}}.$$

If we expand S_0 by using the formula

$$S_0 = \sum_{n=1}^{\infty} u(n)/2^n,$$

we can sweep up terms to the same power of $\frac{1}{2}$. Let the coefficient of $(\frac{1}{2})^j$ be $a(j)$. We see that, since $u(j) = 0$ $(1 \leqslant i \leqslant m-1)$, we have $a(j) = 0$ for $1 \leqslant j \leqslant m-1$. We see that $a(m) = u(m) = 2$, and that

$$a(j) = \begin{cases} u(j) - u(j-1) - \ldots - u(m) & (m+1 \leqslant j \leqslant 2m-1), \\ u(j) - u(j-1) - \ldots - u(j-m+1) & (j \geqslant 2m). \end{cases}$$

Using the recurrence relation proved in theorem 1, we see that $a(j) = 0$ for $j \geqslant 2m$. Using the formula $u(j) = 2^{j-m}$ for $m+1 \leqslant j \leqslant 2m-1$, we have

$$\begin{aligned} a(j) &= 2^{j-m} - 2^{j-1-m} - \ldots - 2 - u(m) \quad (m+1 \leqslant j \leqslant 2m-1) \\ &= 2^{j-m} - (2^{j-m} - 2) - 2, \\ &= 0. \end{aligned}$$

Hence

$$a(j) = \begin{cases} u(m) = 2 & (j = m), \\ 0 & (j \neq m). \end{cases}$$

The original expression thus equals

$$\frac{u(m)}{2^m} = \frac{1}{2^{m-1}}.$$

It follows that

$$S_0 - S_0\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{m-1}}\right) = \frac{1}{2^{m-1}}.$$

We know that

$$T_0 = x+x^2+\ldots+x^{m-1} = \frac{x(1-x^{m-1})}{1-x},$$

and we can evaluate T_0 at $x = \frac{1}{2}$. Hence

$$S_0 - S_0\left(1-\frac{1}{2^{m-1}}\right) = \frac{1}{2^{m-1}},$$

so that $S_0 = 1$.

Theorem 3. $S_1 = 2^m - 1$.

Proof. We consider the expression

$$S_1 - \frac{S_1+S_0}{2} - \frac{S_1+2S_0}{2^2} - \frac{S_1+(m-1)S_0}{2^{m-1}}.$$

If the coefficient of $(\frac{1}{2})^j$ in this expression is $b(j)$, it is easily shown that $b(j) = ja(j)$, where $a(j)$ is the coefficient from the expression in theorem 2. Hence

$$b(j) = \begin{cases} ma(m) = 2m & (j = m), \\ 0 & (j \neq m). \end{cases}$$

Hence the expression has the value $m/2^{m-1}$.

By using the fact that, if $T_1 = x+2x^2+\ldots+(m-1)x^{m-1}$ then $(1-x)T_1 = T_0-(m-1)x^m$, we can evaluate T_1 with $x = \frac{1}{2}$. We can then show that $S_1 = 2^m - 1$.

Theorem 4. $S_2 = 2^{2m+1} - 2^m(2m+1) - 1$.

Proof. We form the expression

$$S_2 - \frac{S_2+2S_1+S_0}{2} - \ldots - \frac{S_2+2(m-1)S_1+(m-1)^2S_0}{2^{m-1}}.$$

If this expression is equal to

$$\sum_{j=1}^{\infty} \frac{c(j)}{2^j},$$

it is easily shown that $c(j) = j^2a(j)$, where $a(j)$ is the coefficient from theorem 2. Hence

$$c(j) = \begin{cases} 2m^2 & (j = m), \\ 0 & (j \neq m). \end{cases}$$

Hence the above expression is equal to $m^2/2^{m-1}$.

If

$$T_2 = x + 2^2x^2 + 3^2x^3 + \ldots + (m-1)^2x^{m-1},$$

then

$$(1-x)T_2 = 2T_1 - T_0 - (m-1)^2x^m$$

and we can evaluate T_2 with $x = \frac{1}{2}$. We can then show that

$$S_2 = 2^{2m+1} - 2^m(2m+1) - 1.$$

The average length of the series is thus $S_1 = 2^m - 1$ and the standard deviation of the length is $(S_2 - S_1^2)^{1/2} = [2^{2m} - 2^m(2m-1) - 2]^{1/2}$. For large m the average length of the series and the standard deviation are of the same order, approximately 2^m.

These results were tested by computer simulation. The computer could generate many series of games and calculate their average length and standard deviation. The results are given in table 1.

Table 1. Average length and standard deviation of 1000 simulated series of games

	Theory		Simulation	
	Average length	S. D.	Average length	S. D.
$m = 2$	3	1.4	2.97	1.37
$m = 3$	7	4.7	6.98	4.67
$m = 10$	1023	1014	1025	962

3. The problem illustrated

We can make use of these results in considering the results of the university boat races between Oxford and Cambridge. The 131 boat races held so far have resulted in 68 wins for Cambridge, 62 for Oxford with one dead heat. The actual results were as follows (C is a Cambridge win, O an Oxford win and D a dead heat):

O C C C C O C C C O O O C O C O C O O O O O O O O O C C C C C O C

D O C O O O O C O C C C C O O O O O O O O O C C O C C C O C C C O

O O O O C C C C O C C C C C C C C C C C C C O O C O C C C C C O C

O C C C C O O C C O C O O O C C C C C C O C O O O O O O O O O O.

Hence, on the hypothesis that the races are independent trials (we shall show later that this is not the case) and that each university has an equal chance of winning, (and ignoring the dead heat) the number of wins for Cambridge has a binomial distribution of $(130, \frac{1}{2})$ i.e. with mean 65.5 and standard deviation 5.7. Assuming the binomial distribution is approximately normal because of the large number of races, we can see that the 68/62 split is not unexpected.

However, we see that there have been two runs of 9 Oxford wins (1861–69 and 1890–98) and one run of 10 wins (1976–85) and one run of 13 Cambridge wins (1924–36). The results which we have previously obtained show that, under the assumptions made, the series which terminates after the 9th consecutive win has an average length of 511 games (S.D. 503). The average length of a series which terminates after the 10th consecutive win has an average length of 1023 games (S.D. 1014) and the series which terminates after the 13th consecutive win has an average length of 8191 games (S.D. 8179).

These results suggest that, whereas the number of wins for one side is not statistically different from that for the other, the wins are not independent but tend to be 'clumped' together; i.e. there are more 'runs' of wins than one would expect by chance alone. One can test for 'clumpiness' by looking at the number of times there is a change of winner. If the results were independent and it was equally probable that either university won, then the number of changes of winner would, ignoring the dead heat, be binomially distributed as $(129, \frac{1}{2})$ with mean 64.5 and standard deviation 5.7. The actual number of changes is 46, which is significantly less than expected, being more than two standard deviations below the mean.

There is a further test for clumpiness called the Stevens test (which finds practical application among actuaries as a test of the graduation of mortality tables). To apply this test we need to know the number of 'runs' or groups of Cambridge wins, which is 23. The test states that the number of runs for the side which has won n_1 times is distributed with mean $n_1(n_1+1)/(n_1+n_2)$ and standard deviation $n_1 n_2/(n_1+n_2)^{3/2}$, where n_1 and n_2 are the number of wins for the two sides.

Putting $n_1 = 68$ and $n_2 = 62$ (ignoring the dead heat) we have mean $= 36$ and S.D. $= 2.8$. Since 23 is more than two standard deviations from the mean, we conclude again that there is strong evidence of 'clumpiness'. This should perhaps come as no surprise—strong crew members are often in the winning boat for three years running and good coaches (like Dan Topolsky) may take charge of a crew for several years.

A more sophisticated proof of the main results in this article is available from the authors on request.

Arithmetic in the Countryside —The Four Seasons

K. AUSTIN, *University of Sheffield*

Keith Austin was born on a smallholding in the village of Cuddington, on the Cheshire plain, with its patchwork of green fields and hedges. He now teaches pure mathematics at the University of Sheffield and lives within sight of the Peak District National Park, with its towering hills, rugged moors and quiet dales.

Keith believes that once a person has understood a piece of mathematics he comes under its influence and is changed into a Mr Hyde. This prevents him from making real contact with others who wish to learn that piece of mathematics. The present article was prompted in part by his hope that in the very act of writing the mathematics within a context which has a strong emotional atmosphere he will be distracted from the mathematical influence and revert to Dr Jekyll.

We folk who live in the heart of the English countryside have become attuned over the years to the ways of nature. Our lives are in harmony, not only with the furry and feathered ones and the flowers and trees, but also with the natural wonders of arithmetic. While those boys and girls who win their scholarships study algebra and geometry at the grammar school in the nearby market town, those who remain at the village school learn more of the ways of our natural friends $1, 2, 3, 4, \ldots$ and addition, subtraction, multiplication and division in the arithmetic lessons.

Just as the birds and animals change their habits as the seasons pass and the plants their form, so with arithmetic each topic falls to its appropriate time of year.

Autumn

As the leaves turn to gold and the rich harvest is gathered in, as our feathered summer visitors wing their way into the sun and the shortening days are clothed in mist, so our thoughts turn to those branches and twigs of arithmetic which are, in the natural way of things, associated with autumn.

Now autumn is a time when nature paints her most glorious pictures and we can but look on in wonder. So it is with arithmetic; we shall watch nature weave her way with the numbers and marvel at the intricacies of her patterns.

Let us think of two numbers; 18 and 28 come to my mind. Now let us see how they naturally intertwine with one another. Consider the multiples of 18,

$$18, 36, 54, 72, 90, 108, \ldots,$$

and divide each by 28, noting just the remainder,

$$18, 8, 26, 16, 6, 24, 14, 4, 22, 12, 2, 20, 10, 0, 18, 8, 26, 16, 6,$$

and so on.

Doesn't that bring a lump to your throat as you see the order of the numbers, so natural and yet so mysterious. Note how the numbers return to 18 and then begin to cycle for ever. One is forced to ask oneself why, and yet at the back of one's mind is the feeling that it could happen in no other way, as if it had been destined from the beginning of time.

Now let us look, as if through a microscope in the mind's eye, at the intricate detail—the atoms, nay, electrons which make up these numbers. The numbers are not a random selection, but each bears a mark—a mark which guarantees its right to be present. What is this mark? Why, each number is divisible by 2—the smallest number, apart from 0, which is there. Thus as the mighty oak grows from the tiny acorn, so all these numbers are the multiples of 2.

Why, we ask ourselves, is 2 the key that links 18 and 28 so that it is as if their very destiny is dependent upon it? No sooner is the question past our lips than the answer floods our mind—so natural, just as a mother hen feeds her chicks—2 is the highest common factor of 18 and 28, a fact so fitting and right that were it not so we would bury our arithmetic in the depths of the ocean and begin again to construct a new edifice of numbers.

But now the leaves have fallen from the trees and lie carpeting the ground against the chill times that are ahead. Many furry ones have entered the land of slumber and the fields are bare and quiet. All nature waits for the changing of the times as autumn gives way to winter.

Winter

It is Christmas Eve. The moonlight reflects on the blanket of crisp white snow which covers the village and the sound of carols floats across the still night air from the church where the villagers are gathered. Can this really be part of the hard winter season? The calendar says 'yes', but in our hearts we know Christmas is a time out of time, beyond the natural order of days, a corner of winter which takes us, however briefly, out of our ordinary lives to things above.

Now January is upon us, when man and nature seem at odds and battle for the very soil of the earth. Throughout autumn the poet within us ruled our heads and hearts as we looked in wonder upon the sequence of numbers that grew naturally before our eyes without help from man or beast. But now, with sterner times upon us, the ploughman, the practical man inside each of us, takes the stage with a question. In these days when nature is not the friend we knew in better times, can we discover the secrets of the autumnal sequence in a way which has man's hand on the reins, rather than by acting simply as a passive observer of the natural order of things?

The human spirit rises to such a challenge and breaks its chains with the ancient process which has been handed down from generation to generation.

Divide 28 by 18 to obtain a remainder of 10.

Divide 18 by 10 to obtain a remainder of 8.

Divide 10 by 8 to obtain a remainder of 2.

Divide 8 by 2 to obtain a remainder of 0.

At this point we retrace our steps.

$$8 = 4\times2+0;$$

$$10 = 1\times8+2, \qquad \text{so } 2 = 10-1\times8;$$

$$18 = 1\times10+8, \qquad \text{so } 2 = 10-1\times(18-1\times10) = 2\times10-1\times18;$$

$$28 = 1\times18+10, \qquad \text{so } 2 = 2\times(28-1\times18)-1\times18 = 2\times28-3\times18;$$

$$2 = 2\times28-3\times18+\tfrac{1}{2}\times28\times18-\tfrac{1}{2}\times18\times28 = 11\times18-7\times28.$$

If we now look back to the autumnal sequence, we see indeed that the 11th multiple of 18 gives a remainder of 2 when divided by 28.

We continue with the other multiples of 2, for example,

$$\begin{aligned}6 &= 33\times18-21\times28-28\times18+18\times28\\ &= 5\times18-3\times28\end{aligned}$$

and the 5th multiple of 18 did indeed give a remainder of 6 when divided by 28.

Thus we have the message of winter, that man is charged not simply to receive his destiny from nature but to strive and play his part in bringing it about.

Spring

Drip, drip, drip, ... goes the last of the melting snow as it clears from the slowly opening leaves on the trees. The little lambs are gambolling in the fields and the whole village is alive with an air of anticipation. It is a time to stand and admire as all around us everything springs into life.

See how the first shy shoot peeps from the soil; it grows and then divides into two, each part grows and then divides again, and so on, time after time, until a majestic ash tree stands before us. So it is with numbers at this season of spring. Let us choose a number, not too large yet not too small, say 11 088.

Now 11 088 is our first shoot. Watch as it divides into 924×12, and then into $132\times7\times12$, and then into $6\times22\times7\times12$, and finally into $2\times3\times2\times11\times7\times2\times3\times2$. We can go no further for we have reached all prime numbers, just like the flowers that appear at the ends of the stems.

Now let us imagine the plant 11 088 has closed up overnight and we watch it open again the next day. As the sun's rays touch it, so we see unfolding 9×1232 and then $9\times154\times8$, $9\times11\times14\times8$, $3\times3\times11\times2\times7\times2\times2\times2$. The same primes as before!

Is this pure chance or will we always get the same primes however it unfolds? Surely we always get the same, for are not those primes hiding in the tightly clenched 11 088? For instance, we know 7 divides 11 088 so, if this now unfolds into two numbers, can we be sure one of them contains 7? Let us try it. $11\,088 = 66\times168$. Now 7 does not divide 66. Must it therefore divide 168?

As the plant grows in the spring from the seed sown in the autumn and winter, so our answer comes from the arithmetic of those past two seasons.

Since 7 is prime, it only has factors 1 and 7 and since 7 is not a factor of 66, the highest common factor of 7 and 66 is 1. Thus we find

$$1 = 19\times7-2\times66.$$

If we now multiply by 168 we have

$$168 = 19\times7\times168-2\times11\,088.$$

As 7 divides the number on the right it must also divide 168.

Thus we were right in our belief that the primes are always the same however 11 088 unfolds. This truth climaxes the arithmetic of spring in just the same way that the village life in spring is brought to its crowning achievement by the Easter services in the church.

Summer

Children running and laughing through the tall green grass and playing by a quiet pool. A cricket match on the village green and the church fête on the vicarage lawn. There is so much activity for country folk in the summer. And so it is with the arithmetic of those beautiful sunny days; it is a lively, bonny arithmetic, full of eventful happenings.

We begin by picking some numbers. Choose any you like but they should have no common factors. How about 5, 7, 8 and 9? Next we need a large sheet of paper and we write down the following pattern, based on our numbers.

0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 ...

0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 ...

0 1 2 3 4 5 6 7 0 1 2 3 4 5 6 7 0 1 2 3 4 5 6 7 0 1 2 3 4 5 6 7 ...

0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 5 6 7 8 0 1 2 3 4 ...

Make sure your sheet is very, very big for you will have $5\times7\times8\times9 = 2520$ vertical columns. You need not go any further for the pattern will repeat. Now look at those 2520 columns and see if you can find two the same.

You didn't find any, did you? That was because they are all different. You are surprised! You think that with all that activity I cannot be certain two the same will not occur. But I can, and I will tell you why.

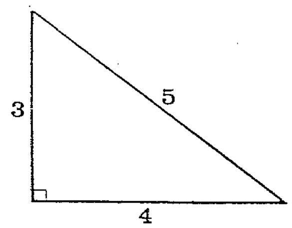
If two columns are the same then, by the first row, they must be a multiple of 5 apart. Similarly they must be a multiple of 7, 8 and 9 apart. But, as these four numbers do not have any common factors, we can say the two columns must be separated by a number which is a multiple of $5\times7\times8\times9$.

If you think very carefully about that last sentence you will see its truth is due to the arithmetic of spring.

Now take any number less than 2520, say 1939, and divide it by 5, 7, 8 and 9. We get the remainders 4, 0, 3 and 4, which is the 1940th column of the pattern. Thus each of the numbers less that 2520 has a unique array of remainders that belongs to that number alone.

And you know, all we who are God's creatures are unique; whether we are boy or girl, man or woman, or even the smallest harvest mouse, we are all different ... and all rather special.

The Area of a Pythagorean Triangle



The area of a $3, 4, 5$ right-angled triangle is 6, correct? Did you know that, although the area of a right-angled triangle with integer sides is always an integer, it is never a perfect square?

Ruth Lawrence,
St Hugh's College,
Oxford.

A Question on Invariant Lines

C. NOON AND I. M. RICHARDS, *Penwith Sixth Form College*

Chris Noon was a student at Penwith Sixth Form College, Penzance, when this article was written: he went to Corpus Christi College, Oxford, in September 1985. Ian Richards teaches at Penzance, having previously studied at Exeter University, where he received a Ph.D. in 1983. Both have lived at Pendeen, perhaps the remotest outpost of mathematics in England, being only five miles from Land's End.

Question: A transformation of the plane is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find all the invariant lines of the transformation.

This question appeared in an A-level paper of the Oxford Local Examinations of 1979. The solutions are the lines $y = -\frac{1}{2}x$ and $y = x+c$, where c is any real number. The question is not an unusual or difficult one, but it has prompted us to ask, 'Which matrices have invariant lines not passing through the origin and what are those lines?', for here is a matrix with such lines. We restrict ourselves to 2×2 matrices with real entries and, without recourse to any sophistication, show the following to be true.

Statement 1. If M is a matrix with invariant lines not passing though the origin, then 1 is an eigenvalue of M.

Statement 2. If 1 is an eigenvalue of the matrix M, then M has a second eigenvalue k, which may or may not be equal to 1. Every line parallel to an eigenvector for k is invariant under M.

Statement 3. If 1 is an eigenvalue of M, then the only invariant lines of M are those described in statement 2, together with the lines on which the eigenvectors for 1 lie.

Together these three statements provide a complete answer to our inquiry. So now we prove these.

Proof of statement 1. We assume that there is a line L not passing through the origin that is invariant under M. Then there are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that L consists of all points of the form $\boldsymbol{a}+t\boldsymbol{b}$, as t varies over the real numbers. In order that we have a full line, $\boldsymbol{b} \neq \boldsymbol{0}$. In order that the line does not pass through the origin, $\boldsymbol{a}$ is not a multiple of $\boldsymbol{b}$. We apply M to both $\boldsymbol{a}$ and $\boldsymbol{b}$ to obtain points $M\boldsymbol{a}$ and $M\boldsymbol{b}$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, we can write $M\boldsymbol{a}$ and $M\boldsymbol{b}$ as linear combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$:

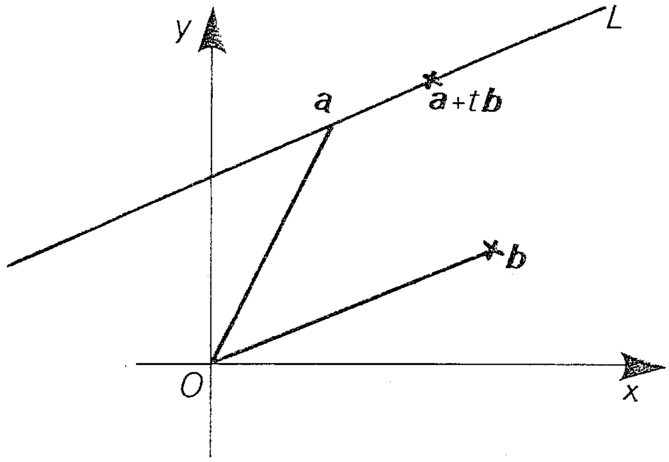


Figure 1. The straight line L through the point $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

$$M\boldsymbol{a} = p\boldsymbol{a}+q\boldsymbol{b}, \tag{1}$$

$$M\boldsymbol{b} = r\boldsymbol{a}+s\boldsymbol{b}, \tag{2}$$

where p, q, r and s are real numbers. But, since $\boldsymbol{a}$ lies on the invariant line L, then $M\boldsymbol{a}$ must lie on it also. Therefore $p = 1$ and so formula (1) can be written again as

$$M\boldsymbol{a} = \boldsymbol{a}+q\boldsymbol{b}.$$

We can write down the image of L under the operation of M:

$$\begin{aligned} M(\boldsymbol{a}+t\boldsymbol{b}) &= M\boldsymbol{a}+tM\boldsymbol{b} \\ &= \boldsymbol{a}+q\boldsymbol{b}+t(r\boldsymbol{a}+s\boldsymbol{b}) \\ &= (1+rt)\boldsymbol{a}+(q+st)\boldsymbol{b}. \end{aligned}$$

Now, since L is invariant, the coefficient of $\boldsymbol{a}$ in $M(\boldsymbol{a}+t\boldsymbol{b})$ must be 1. Hence $1+rt = 1$. From this $rt = 0$. Now t is allowed to vary over the real numbers and so may be taken to be non-zero. We deduce that $r = 0$. Formula (2) now becomes

$$M\boldsymbol{b} = s\boldsymbol{b}. \tag{3}$$

Thus, $\boldsymbol{b}$ is an eigenvector of M with eigenvalue s. If s is equal to 1, then the statement that 1 is an eigenvalue of M is established. Let then $s \neq 1$. Since this is so, the vector $\boldsymbol{a}+[q/(1-s)]\boldsymbol{b}$ is properly defined. This vector is fixed by M, because

$$\begin{aligned} M\left(\boldsymbol{a}+\frac{q}{1-s}\boldsymbol{b}\right) &= M\boldsymbol{a}+M\frac{q}{1-s}\boldsymbol{b} \\ &= \boldsymbol{a}+q\boldsymbol{b}+\frac{sq}{1-s}\boldsymbol{b} \end{aligned}$$

$$= a + \frac{q(1-s)+sq}{1-s} b$$

$$= a + \frac{q}{1-s} b.$$

So then $a + [q/(1-s)]b$ is an eigenvector for M with eigenvalue 1. So, whether $s = 1$ or not, 1 is an eigenvalue of M. Statement 1 is now proved.

Proof of statement 2. Let 1 be an eigenvalue of the matrix M. Then, since M is a 2×2 matrix, its characteristic equation is a quadratic equation. We know that 1 is a root of this equation, and so there must be a second real root k. We consider separately the cases where $k \neq 1$ and $k = 1$.

Case (i) Let $k \neq 1$. The matrix M has eigenvectors e_1 and e_k for 1 and k, respectively. Since $k \neq 1$, these cannot be collinear. We have

$$Mte_1 = te_1, \qquad Mse_k = kse_k,$$

where s and t are any real numbers, from which

$$M(te_1 + se_k) = te_1 + kse_k. \tag{4}$$

If we now fix t and allow s to vary, equation (4) indicates that the line parallel to e_k consisting of all points $te_1 + se_k$ is invariant under M. By a suitable choice of t, any line parallel to e_k can be shown to be invariant in this way. So statement 2 is shown to be true in the case $k \neq 1$.

Case (ii) Let $k = 1$. If M has two non-parallel eigenvectors e_1 and f_1, then any vector can be written as $te_1 + sf_1$, where t and s are real. Applying M to any such vector leaves it unchanged. Thus M leaves every point fixed and must therefore be the identity. We assume then that M does not have two non-parallel eigenvectors with eigenvalue 1, that is, all eigenvectors are parallel. Take e_1 to be an eigenvector. Take a to be a vector not parallel to e_1. Let $v = te_1 + sa$. What happens to v under M?

$$M(te_1 + sa) = Mte_1 + Msa = te_1 + Msa.$$

Since a and e_1 are not parallel, we can express Ma in terms of a and e_1. Let $Ma = pe_1 + qa$, where p and q are real. We apply M to Ma to give

$$M^2a = pe_1 + Mqa.$$

We now subtract

$$Ma = pe_1 + qa \tag{5}$$

to give

$$(M^2 - M)a = q(M - I)a,$$

$$M(M-I)\boldsymbol{a} = q(M-I)\boldsymbol{a}.$$

Thus $(M-I)\boldsymbol{a}$ is an eigenvector of M. Therefore $(M-I)\boldsymbol{a}$ is parallel to $\boldsymbol{e}_1$. Now

$$(M-I)\boldsymbol{a} = p\boldsymbol{e}_1+q\boldsymbol{a}-\boldsymbol{a} = p\boldsymbol{e}_1+(q-1)\boldsymbol{a}.$$

Since $(M-I)\boldsymbol{a}$ is parallel to $\boldsymbol{e}_1$, this means that $q = 1$. Returning to formula (5), we have

$$M\boldsymbol{a} = p\boldsymbol{e}_1+\boldsymbol{a}.$$

Then, for any real s and t, we have

$$\begin{aligned} M(t\boldsymbol{e}_1+s\boldsymbol{a}) &= t\boldsymbol{e}_1+Ms\boldsymbol{a} \\ &= t\boldsymbol{e}_1+sp\boldsymbol{e}_1+s\boldsymbol{a} \\ &= (t+sp)\boldsymbol{e}_1+s\boldsymbol{a}. \end{aligned} \tag{6}$$

Now, if we fix s and allow t to vary, the expression $t\boldsymbol{e}_1+s\boldsymbol{a}$ describes all points on a line parallel to $\boldsymbol{e}_1$. By a suitable choice of s, we can pick out any such line. Formula (6) indicates that any such line is invariant. Summarising case (ii), we have that either $M = I$ or every line parallel to an eigenvector for 1 is invariant under M. In either case it is true that any line parallel to an eigenvector for the eigenvalue 1 is invariant. Therefore statement 2 is proved in the case where $k = 1$ and so the proof is complete.

Proof of statement 3. We suppose that M has 1 as an eigenvalue. Where should we look for invariant lines? Statement 2 indicates that every line parallel to the eigenvectors for the second eigenvalue k is invariant. The proof of statement 1 up to equation (3) establishes that, if there are to be other invariant lines, then they must lie parallel to the eigenvectors for the eigenvalue 1. Now if $k = 1$ these lines are certainly invariant, from statement 2. If $k \neq 1$, it can be seen from formula (4) that the lines parallel to the eigenvectors for the eigenvalue 1 are not invariant apart from the line through the origin. So statement 3 is proved.

Approximating $\sqrt{2}$

$\sqrt{2}$ is an irrational number, so it cannot be written as the ratio of two integers. All right, suppose you have a rational approximation a/b to $\sqrt{2}$ (with a and b both positive numbers). Then $(a+2b)/(a+b)$ will give you a better approximation to $\sqrt{2}$. Can you prove this?

Computer Column

MIKE PIFF

Suppose that messages consisted only of strings of arbitrary length from the alphabet A, B, C, D, E and F, these letters having the observed frequencies

A 0.2, B 0.2, C 0.2, D 0.19, E 0.19, F 0.02.

Clearly at most 3 bits would be sufficient to store any of these letters and so any message of length n could be stored in $3n$ bits. We could encode a message more efficiently by varying the length of the binary string associated with each letter, for example,

A 0, B 1, C 00, D 01, E 10, F 11,

were it not for the fact that then the sequence 010 could stand for ABA, DA or AE, with even worse ambiguity in longer messages.

In 1952, D. A. Huffman described an algorithm to encode efficiently and unambiguously when the letter frequencies are known. In the frequency list above, we group the two least frequent letters E and F (or D and F), adding their frequencies. Think of this as replacing any E or F by a G, say. The new letters are sorted into descending order of frequency:

EF 0.21, A 0.2, B 0.2, C 0.2, D 0.19

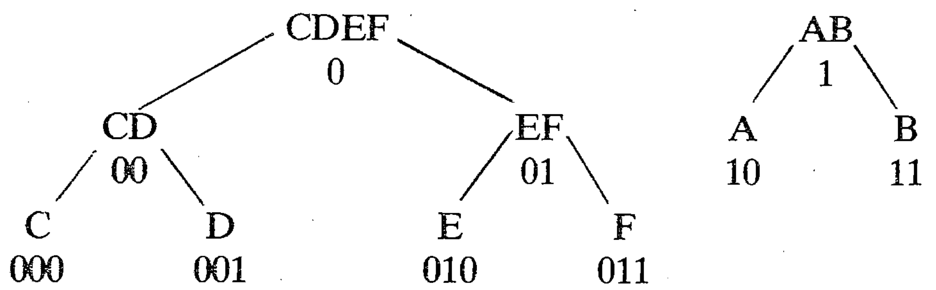
and the process is repeated on the new list:

CD 0.39, EF 0.21, A 0.2, B 0.2,

AB 0.4, CD 0.39, EF 0.21,

CDEF 0.6, AB 0.4.

We now use the following rule for encoding:



Thus the code is

A 10, B 11, C 000, D 001, E 010, F 011,

which uses on average $2.4n$ bits for an n-letter message, a significant saving. Also any valid message can be uniquely decoded; for instance, try

00101001110000010100111011011010010001.

A second example, which would require $2n$ bits naively encoded:

A 0.5, B 0.3, C 0.19, D 0.01,

A 0.5, B 0.3, CD 0.2,

A 0.5, BCD 0.5.

A 0   BCD 1

B 10   CD 11

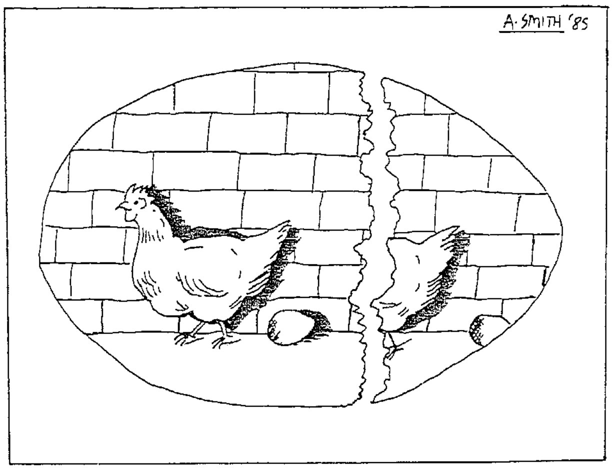
C 110   D 111

giving code

A 0, B 10, C 110, D 111,

which requires about $1.7n$ bits for an n-letter message.

Fowl Mathematics



A hen and a half lay an egg and a half in a day and a half. How many eggs will 12 hens lay in 12 days?

Allan J. Maclean,
Dunfermline.

Letters to the Editor

Dear Editor,

Divisor patterns

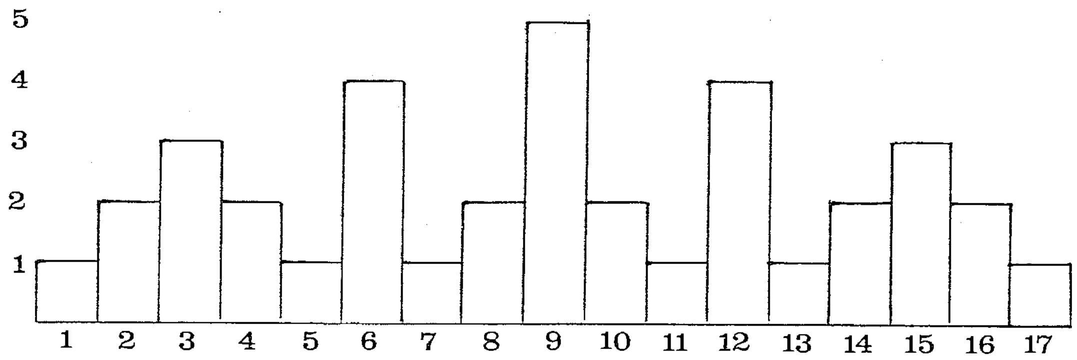
I have developed a simple way of making interesting patterns from the divisors of an integer. The procedure for doing this is as follows:

(i) Select a positive integer and call it x.

(ii) List all the positive divisors of x including 1 but excluding x itself in numerically ascending order, and label them the 1st, 2nd, 3rd, etc.

(iii) Draw a vertical bar chart with $x-1$ vertical bars. To obtain the height of the yth bar from the left, find the highest common factor of y and x. If this is the kth divisor of x then the height of the yth bar is k. The resulting bar chart is a symmetrical pattern.

Example: $x = 18$. The positive divisors of 18 smaller than 18 itself are 1, 2, 3, 6 and 9. The resulting divisor pattern is shown in the figure.



This procedure can be applied to any positive integer, but prime numbers simply give a row of bars all the same height. These bar charts are interesting in that they reflect the divisor properties of the number x. Prime numbers give flat, uninteresting bar charts, but numbers with many factors produce impressive charts with many tall towers and symmetrical patterns. It is stimulating to try to spot similarities in the bar charts of two related numbers, for instance of 60 and 120.

When one draws bar charts for many different numbers, one can begin to classify numbers into 'families', e.g. 4, 6, 10, 14, 22, 26, 34, 38, 46 and 58 all belong to the same family of numbers (all of them being prime numbers multiplied by 2) and all of them produce similar charts.

I have never come across a pattern that displays the divisibility properties of numbers, so I thought that readers might be interested in this idea.

Yours sincerely,

MATTHEW BALL
(Student, Birkenhead School,
55 Shrewsbury Road,
Claughton, Birkenhead, Merseyside)

Dear Editor,

On sums of unlike powers

Having a passion for playing with numbers and finding interesting relations between them, I have searched for solutions of the following equations:

$$x^4+y^4 = z^2+1, \tag{1}$$

$$x^3+y^3 = z^3\pm 1, \tag{2}$$

and the odd-looking equation

$$x^3+y^3 = z^2+\delta \quad (\delta = \pm 1 \text{ or } 0) \tag{3}$$

(where all solutions are in the *positive* integers).

The first two equations are obviously those which come as near as possible to the Fermat equation, $x^n+y^n = z^n$, without actually getting there. (I consider it my goal in life to find a counterexample to Fermat's Last 'Theorem'.) At the time of writing I have the following results.

Equation (1). It is convenient to consider $x \geqslant y$. There are 20 solutions for $x \leqslant 320$, the largest being $253^4+157^4 = 68\,591^2+1$ and $319^4+113^4 = 102\,559^2+1$. Interestingly, three of these solutions have $x = 239$:

$$239^4+104^4 = 58\,136^2+1,$$

$$239^4+143^4 = 60\,671^2+1,$$

$$239^4+208^4 = 71\,656^2+1.$$

Equation (2). For $x \leqslant 300$ there are seven solutions. I give them all:

$$8^3+6^3 = 9^3-1, \qquad 138^3+71^3 = 144^3-1,$$

$$10^3+9^3 = 12^3+1, \qquad 138^3+135^3 = 172^3-1,$$

$$94^3+64^3 = 103^3+1, \qquad 144^3+73^3 = 150^3+1,$$

$$235^3+135^3 = 249^3+1.$$

There seems to be no predominance (certainly at this stage) of + over − or vice versa.

Equation (3). For $x \leqslant 175$ there are 14 'primitive' solutions with $\delta = 0$. (If x, y, z is a solution, then so is k^2x, k^2y, k^3z for every k. Such a solution is termed 'non-primitive'.) There are also 25 solutions for $\delta = -1$. Strangely (or is it?), there are only four solutions for $\delta = +1$:

$$93^3+85^3 = 1191^2+1, \qquad 95^3+51^3 = 995^2+1,$$

$$129^3+76^3 = 1608^2+1, \qquad 157^3+69^3 = 2049^2+1.$$

Can I ask interested readers to check these figures and proceed further? I should be delighted to exchange information (such as mine is).

Note. Would anyone with a spare CRAY or ILLIAC please let me know.

To finish, I give some interesting results obtained during the above search:

$$37^2+11^2 = 23^2+31^2 \quad\text{and}\quad 37^3+11^3 = 228^2;$$
$$13^2+13^2 = 17^2+7^2 \quad\text{and}\quad 13^4+13^4 = 239^2+1^2;$$
$$32^2+17^2 = 23^2+28^2 \quad\text{and}\quad 32^4+17^4 = 1064^2+1^2.$$

Also

$$63^3-63 = 58^3+38^3, \qquad 127^3-127 = 118^3+74^3, \qquad 134^3-134 = 117^3+93^3.$$

Is this accidental?

Yours sincerely,

JOSEPH MCLEAN
(M.Sc. Student, University of Glasgow,
9, Larch Road,
Glasgow, G41 5DA.)

Dear Editor,

Expectations in Family Planning

On pages 16–19 of *Mathematical Spectrum*, Volume 18, D. J. Colwell and J. R. Gillett consider $N(m,n)$, the expected size of family needed to get m boys and n girls when each birth has probability p of being a boy and probability $q = 1-p$ of being a girl. They show that

$$N(m,n) = 1+pN(m-1,n)+qN(m,n-1) \quad \text{for } m,n \geqslant 1$$

and use the negative binomial distribution to get starting values

$$N(m,0) = \frac{m}{p} \quad (m \geqslant 1) \quad\text{and}\quad N(0,n) = \frac{n}{q} \quad (n \geqslant 1)$$

for the application of the above recurrence relation.

Alternatively it can be shown that, for $m \geqslant 0$ and $n \geqslant 0$,

$$N(m,n) = \frac{m}{p} - mp^m \sum_{r=0}^{n-1} \binom{m+r}{r} q^r + \frac{n}{q} - nq^n \sum_{r=0}^{m-1} \binom{n+r}{r} p^r. \tag{1}$$

This can be obtained from Colwell and Gillett's result

$$N(m,n) = \sum_{u=0}^{\infty} (m+n+u)\left(\frac{(m+n+u-1)!}{(m-1)!\,(n+u)!}p^m q^{n+u} + \frac{(m+n+u-1)!}{(n-1)!\,(m+u)!}p^{m+u}q^n\right)$$

by writing $(m+n+u)\dfrac{(m+n+u-1)!}{(m-1)!\,(n+u)!}$ as $m\dfrac{(m+n+u)!}{m!\,(n+u)!}$ or $m\dbinom{m+n+u}{n+u}$ and then using

$$\begin{aligned} mp^m \sum_{u=0}^{\infty} \binom{m+n+u}{n+u} q^{n+u} &= mp^m \left\{ \sum_{r=0}^{\infty} \binom{m+r}{r} q^r - \sum_{r=0}^{n-1} \binom{m+r}{r} q^r \right\} \\ &= mp^m \left\{ (1-q)^{-m-1} - \sum_{r=0}^{n-1} \binom{m+r}{r} q^r \right\}. \end{aligned}$$

While formula (1) gives an expression for $N(m, n)$ in terms of a finite sum, no simple closed expression for this appears to exist, but in the case $m = n$ an alternative form is available. Thus, from (1),

$$\frac{N(n,n)}{n} = \frac{1}{pq} - p^n \sum_{r=0}^{n-1} \binom{n+r}{r} q^r - q^n \sum_{r=0}^{n-1} \binom{n+r}{r} q^r$$

and so

$$\frac{N(n+1,n+1)}{n+1} - \frac{N(n,n)}{n} = S+S', \tag{2}$$

where it is easily shown that

$$\begin{aligned} S &= (q-1)p^n \sum_{r=0}^{n} \binom{n+1+r}{r} q^r + p^n \sum_{r=0}^{n-1} \binom{n+r}{r} q^r \\ &= p^n \binom{2n+1}{n} q^{n+1} - p^n \binom{2n}{n} q^n, \end{aligned}$$

and similarly that

$$\begin{aligned} S' &= (p-1)q^n \sum_{r=0}^{n} \binom{n+1+r}{r} p^r + q^n \sum_{r=0}^{n-1} \binom{n+r}{r} p^r \\ &= q^n \binom{2n+1}{n} p^{n+1} - q^n \binom{2n}{n} p^n. \end{aligned}$$

Thus (2) simplifies to

$$p^n q^n \left\{ \binom{2n+1}{n} - 2\binom{2n}{n} \right\} = -\frac{1}{2n+1} \binom{2n+1}{n} p^n q^n.$$

Then, since $N(1,1) = \dfrac{1}{pq} - p - q = \dfrac{1}{pq} - 1$ for $n \geqslant 1$, we get

$$\begin{aligned} N(n,n) &= nN(1,1) + n \sum_{r=1}^{n-1} \left\{ \frac{N(r+1,r+1)}{r+1} - \frac{N(r,r)}{r} \right\} \\ &= \frac{n}{pq} - n \sum_{r=1}^{n} \frac{1}{2r-1} \binom{2r-1}{r} (pq)^{r-1}. \end{aligned}$$

For $p = \frac{1}{2}$ this gives $N(1,1) = 3$, $N(2,2) = 5.5$, $N(3,3) = 7.875$ and $N(4,4) = 10.1875$ which can be compared with the values 3.0031, 5.5065, 7.8853 and 10.2015 given by Colwell and Gillett for $p = 0.5138$.

Yours sincerely,

A. V. Boyd
(University of the Witwatersrand,
Johannesburg, South Africa)

Dear Editor,

Family planning—a probabilistic approach

I enjoyed the article by Colwell and Gillett in Volume 18 Number 1 of *Mathematical Spectrum*. I have long been fascinated by the use of difference equations in probability theory and wish to point out that the basic relationship (1) in the article can be obtained more quickly and simply by a conditioning argument.

Let us consider $N(m, n)$, the expected number of births until the couple has at least m boys and n girls, and condition on the first birth, which will be a boy with probability p or a girl with probability q. Since the expected number of births after the first birth will be one smaller than $N(m, n)$, we see that $N(m, n)$ satisfies the equation

$$\begin{aligned} N(m,n) &= p[N(m-1,n)+1]+q[N(m,n-1)+1] \\ &= 1+pN(m-1,n)+qN(m,n-1). \end{aligned}$$

Yours sincerely,

JAMES O. FRIEL
(Department of Mathematics,
California State University,
Fullerton, CA 92634, U.S.A)

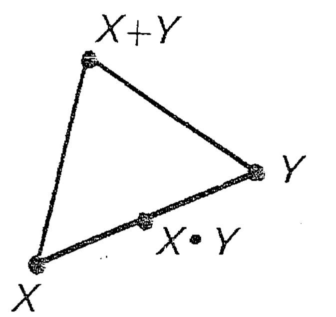
Dear Editor,

The distributive law in geometry

In *Mathematical Spectrum* Volume 18 Number 2, p. 41, Guido Lasters gives some examples of the occurrence of the distributive law in geometry. Some years ago, as a first-year undergraduate, I came up with the following manifestation of the 'undistributive' law!

Let X, Y be points in the plane, let $X \cdot Y$ denote the midpoint of XY, and let X, Y and $X+Y$ form an equilateral triangle in clockwise order. Then

$$X \cdot (Y+Z) = (X+Z) \cdot (Y+X)!$$



Yours sincerely,

MIKE PIFF
University of Sheffield

Dear Editor,

Karmarkar's algorithm

The article on page 14 of Volume 18 Number 1 concerning Karmarkar's new linear-programming algorithm was interesting. On pages 673–674 and 683–689 in Gilbert Strang's *Introduction to Applied Mathematics* (Wellesley–Cambridge Press, Box 157, Wellesley, MA 02181, USA, US\$39.00) one can find an introduction to this algorithm.

A version of Karmarkar's algorithm is produced by Eastern Software Products Inc (PO Box 15328 Alexandria, VA 22309, USA. KLP88 Version 5.02 Karmarkar's LP Algorithm, US\$99.00). They claim that their experience shows that this algorithm runs more slowly than the simplex algorithm.

Yours sincerely,

GEORGE E. MATTHEWS
(Medicine Hat College,
299 College Drive, S.E.,
Medicine Hat, Alberta T1A 3Y6, Canada)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

18.7. (Submitted by Louis Funar, University of Craiova, Romania) Twenty non-overlapping squares lie inside a square of side 1. Show that there are four of these squares the sum of the lengths of whose sides does not exceed $2/\sqrt{5}$.

18.8. Obtain a result connecting the areas of the faces of a right-angled tetrahedron.

18.9. Show that

$$\int_{x_1}^{x_2} e^{2\pi i x}\, dx$$

is zero if and only if $x_2 - x_1$ is an integer.

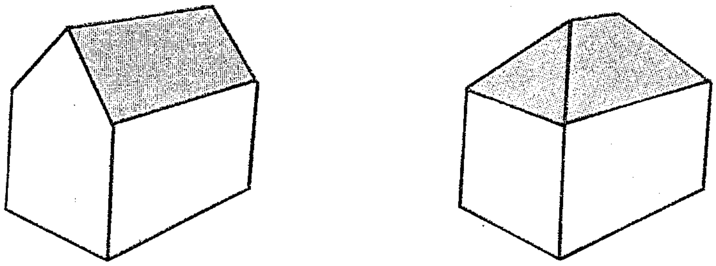
A rectangle is subdivided into smaller rectangles whose sides are parallel to the sides of the large rectangle. By considering the double integral

$$\iint e^{2\pi i(x+y)}\, dx\, dy$$

over the large rectangle, or otherwise, show that, if at least one of the sides of each of the smaller rectangles is an integer, then at least one of the sides of the large rectangle is an integer.

Solutions to Problems in Volume 18 Number 1

18.1. Two houses have the same dimensions for their rectangular bases. One is gabled and the other has a cottage roof, and the slopes of their roofs are the same. Which is more economical on roof-felting?



gabled roof　　　　　　cottage roof

Solution 1 by Adrian Hill (Trinity College, Cambridge)

If the base area is A and the slope of the roofs is θ, then, as in both cases every portion of the base area is projected onto the roof at the same angle, the roof area in both cases is $A\sec\theta$, and they are equally economical.

Solution 2 by Ian Coxon (Tynemouth College)

First consider houses with a square base of side d. The gabled roof has area $2dl$, with an obvious meaning for l. The cottage roof comes to a point and has area $4\times\frac{1}{2}dl = 2dl$, the same as that of the gabled roof. For houses with non-square roofs, the extra roofing inserted is the same for each, and again the roof areas are the same.

Also solved by Christopher Reed (aged 12, The Ipswich School), Dominic Joyce (Queen Elizabeth's Hospital, Bristol), Philip Wadey (University of Exeter), Joseph McLean (University of Glasgow), Kevin Buzzard (The Royal Grammar School, High Wycombe), Ashley Brooks (University of Warwick), Jean Corriveau (Brandon University, Manitoba, Canada), Mark Heron (Tynemouth College), Roger O'Brien (Tynemouth College), Richard Dobbs (Magdalen College, Oxford) and Michael McQuillan (University of Glasgow).

18.2. Which of e^{π} and π^{e} is larger? (Calculators are not allowed!)

Solution 1 by Ashley Brooks (University of Warwick)

For $x > 0$,

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots > 1+x.$$

Put $x = \dfrac{\pi}{e}-1$ to give $e^{(\pi/e)-1} > \dfrac{\pi}{e} \Rightarrow e^{\pi/e} > \pi \Rightarrow e^{\pi} > \pi^{e}$.

Solution 2 by Dominic Joyce (Queen Elizabeth's Hospital, Bristol)

Let

$$y = \frac{\ln x}{x},$$

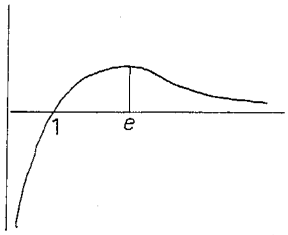
then

$$\frac{dy}{dx} = \frac{1-\ln x}{x^2},$$

which is zero when $x = e$. The graph of $y = \dfrac{\ln x}{x}$ (see figure) has a maximum at

$x = \mathrm{e}$, so that

$$\frac{\ln \mathrm{e}}{\mathrm{e}} > \frac{\ln \pi}{\pi}, \qquad \text{which gives} \quad \pi \ln \mathrm{e} > \mathrm{e} \ln \pi, \quad \text{whence } \mathrm{e}^{\pi} > \pi^{\mathrm{e}}.$$



Also solved by Jeremy Rosten (The Haberdashers' Aske's School, Elstree), Adrian Hill (Trinity College, Cambridge), Joseph McLean (University of Glasgow), Kevin Buzzard (The Royal Grammar School, High Wycombe), Richard Dobbs (Magdalen College, Oxford), Michael McQuillan (University of Glasgow), Lászlo Cseh (Babes-Bolyai University, Romania) and Ruth Lawrence (St. Hugh's College, Oxford).

18.3. Prove that there exist integers a and b such that $|a|, |b| \leqslant 10^7$ and

$$0 < |a + b\sqrt{2}| < 3 \times 10^{-7}.$$

Solution by Jeremy Rosten (The Haberdashers' Aske's School, Elstree)
Let $n = 4 \times 10^6$, and consider the fractional parts of the real numbers

$$0, \quad \sqrt{2}, \quad 2\sqrt{2}, \quad 3\sqrt{}, \quad \ldots, \quad n\sqrt{2},$$

arranged in increasing order. These are all greater than or equal to zero and less than 1, and there are $n+1$ of them, so two of them must differ by less than $1/n$, i.e., there exist integers p and q with $0 \leqslant p < q \leqslant n$, such that

$$|(q\sqrt{2} - [q\sqrt{2}]) - (p\sqrt{2} - [p\sqrt{2}])| < \frac{1}{n},$$

where $[x]$ denotes the largest integer not greater than x. Put $a = [p\sqrt{2}] - [q\sqrt{2}]$ and $b = q - p$. Then a and b are integers and

$$|a + b\sqrt{2}| < \frac{1}{n} = 2.5 \times 10^{-7} < 3 \times 10^{-7}.$$

Now $b \neq 0$, so $|a + b\sqrt{2}| > 0$ because $\sqrt{2}$ is irrational. Also, $|b| \leqslant 4 \times 10^6 < 10^7$ and $|a| \leqslant [4 \times 10^6 \sqrt{2}] < 10^7$.

Jeremy points out that, if we put $n = 7 \times 10^6$, we can improve the result to show that there exist integers a and b such that $|a|, |b| < 10^7$ and

$$0 < |a + b\sqrt{2}| < 1.43 \times 10^{-7}.$$

Also solved by Adrian Hill (Trinity College, Cambridge), Dominic Joyce (Queen Elizabeth's Hospital, Bristol), Kevin Buzzard (The Royal Grammar School, High Wycombe), Michael McQuillan (University of Glasgow) and Ruth Lawrence (St. Hugh's College, Oxford).

That was the year that was (3)

In Volume 18 Number 1 we asked readers to express the numbers 1 to 100 using the digits of the year 1985 in their correct order, using only the operations +, −, ÷, √ and !. Seven of the numbers have eluded readers (and the editor!): Richard Wheatley (St. John's College, Cambridge) provided the following beautiful solution:

$$41 = \sqrt{}(1+\{[(\sqrt{}9)!+8]\times 5!\}).$$

Interesting if illegal suggestions are

$$62 = 1+\frac{8!}{[(\sqrt{}9)!]!}+5, \qquad 63 = 1\times 9\times(8-\sqrt{}\sqrt{}\sqrt{}\ldots 5)$$

from Kevin Buzzard and friends, of the Royal Grammar School, High Wycombe.

Problem 17.5

In Volume 18 Number 1 we gave a solution to Problem 17.5. The original problem was as follows: 'Prove that, when p and q are prime numbers greater than 5, then p^4-q^4 is always divisible by 10'. Several readers pointed out that it is in fact divisible by 80. But, even better, it is always divisible by 240. Readers may like to prove this. Moreover, this cannot be improved, because

$$11^4-7^4 = 240\times 51, \qquad 13^4-11^4 = 240\times 58.$$

Book Reviews

Elements of Relativity Theory. By D. F. Lawden. John Wiley & Sons, Chichester, 1985. Pp. x+108. Paperback £4·95.

This is an excellent little book. It provides a very readable introductory account of relativity at a level which should be accessible to sixth-formers and first-year undergraduates. The background ideas and historical framework are lucidly given, yet there is good coverage of the standard problems and exercises. Both worked examples and problems (with answers but not worked solutions) are included. The level of mathematics required to follow the arguments and do most of the problems is quite low, probably O-level would suffice for most of the book. Although this occasionally leads to some rather involved working it does mean that a wide range of students should find it intelligible. Basic ideas have been deservedly stressed and the author is not afraid to repeat the central parts. However, the reader must be prepared to read the book carefully—this is not a criticism, but rather a warning that relativity does contain some ideas which will appear strange until familiarity is achieved.

The first four chapters (each of about twenty pages) cover special relativity. First the central importance of inertial frames and the propagation of light are fully discussed and then the Lorentz transformation is obtained. The derivation is by the k-calculus. Do not be put off by this word, it is really a neat method which is more attractive than a formal mathematical proof. It also has the merit of greatly aiding one's appreciation of one of the main ideas, namely that all inertial frames are equivalent. This leads on naturally to the transformation of velocities (how different observers see the motion of a particle) and then to relativistic mechanics. There are enough descriptions of experiments to make the account satisfying to physics students without causing too many difficulties for someone whose background may be rather limited.

The last chapter, also of twenty pages, gives a flavour of the ideas that go into general relativity. No attempt is made to enter into the mathematics but a remarkably clear account is given which should be extremely helpful to anyone who wishes to have some understanding of what general relativity is about or to pursue the subject at a deeper level.

This book deserves to have a wide readership. It is well written, covers the necessary topics, gives the reader a 'feel' for the subject and, not least, is interesting.

University of Sheffield GLENN T. VICKERS

Mathematics for the Nonmathematician. By MORRIS KLINE. Dover Publications, Inc, New York, and Constable & Company Ltd, London, 1985. Pp. xiii + 641. Paperback £10·95.

I like most things about this book apart from its title, which I think is misleading. It is certainly a book for those who are interested in mathematics, maybe as non-specialists; but when I was a sixth-former (and future specialist in mathematics) I should like to have had this book available to read. The coverage is very wide: from the basic concepts of arithmetic, through Euclidean and projective geometry, the calculus, the theories of gravitation, the non-Euclidean geometries, some aspects of algebra, to probability and statistics. Morris Kline is a good expositor who is able to put across mathematical ideas without too many technicalities intruding. There is an excellent section on the properties of light and the simple mathematics involved, and another (to me especially exciting) section on the use of isotopes and the startling results which are suggested by simple arithmetic alone. The chapter on projective geometry is beautifully anticipated by a discussion of perspective in drawing and painting. These are but tiny morsels in a feast of mathematics, and I recommend you to buy (or borrow) this book and read it—whether you think you are a mathematician or a non-mathematician. It was first published in 1967 under the title *Mathematics for Liberal Arts* and is an abridgement and revision of an even earlier book *Mathematics: A Cultural Approach.*

University of Sheffield HAZEL PERFECT

Software Review

Micros in Mathematical Education (MIME)—MECHANICS. John Wiley and Sons, Software, 1985. £34·95 per unit.

The MIME project emanates from Loughborough University of Technology under the direction of Professor A. C. Bajpai. The general aim is to use micros to enhance the learning of A-level mathematics. The demonstration pack I received was for the BBC B machine but was not clear whether versions are available for other machines. Currently there are thirteen units available, or soon to be available, on mechanics with a section from four of these on the demonstration disc. Others on analytical, numerical and statistical techniques are promised.

The literature says that the programs are for class-teacher use or for self-learning by the student. I would find them difficult to use in the first way but I can see significant use in the second context. The programs themselves are well written and fully interactive with some nice graphics. As with many such programs it is very easy to go through them without any real understanding of what is going on, so they are not for the unmotivated or unsupervised. The serious student could find them extremely useful if he has missed some particular material, has had difficulty with a topic, has little physical feel for mechanics, needs a different set of examples, or needs a change from traditional teaching techniques.

The quality of the programs is excellent. It is easy to move about the various units from well-constructed menus and it is quite difficult to lose the program under study. For some of the topics, work cards of good quality are given and are helpful. In fact, most of the programs are best studied with a pencil and paper to hand, even when work cards are not provided. Teacher's notes are available but these were not included in the package I received.

Of the four programs provided on the demonstration disc I had only two criticisms. Firstly, the abbreviation psb (press space bar), which is probably well known to everyone except me, was not explained until after it was used for the first time; I had to break into the program to discover its meaning. Secondly, from the section on lunar landing, which might have been called lunar crashing since the module hits the moon with non-trivial velocity, I found I learned very little except perhaps that, under constant gravity, velocity varies linearly and distance quadratically with time.

It is well worth approaching John Wiley, Software, to obtain the demonstration disc to check whether the material suits your needs.

University of Sheffield D. M. BURLEY

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ISSN 0025-5653

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Editor—Mathematical Spectrum,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Published by the Applied Probability Trust
Typeset by the University of Nottingham
Printed in England by Galliard (Printers) Ltd, Great Yarmouth