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CONTENTS

FIVE- AND SEVEN-DIGIT PALINDROMIC PRIMES

JACQUES SAUVÉ

Problem 490, which is about palindromic primes, is discussed on pages 288-290 in this issue. As material supplementary to this discussion, I have, at the editor's request, prepared a list of the 93 five-digit and 668 seven-digit palindromic primes. The calculations were done on a PDP-11/45 at the University of Waterloo, and the computer time required was slightly more than one minute.

FIVE-DIGIT PALINDROMIC PRIMES

10301	10501	10601	11311	11411	12421	12721	12821	13331	13831
13931	14341	14741	15451	15551	16061	16361	16561	16661	17471
17971	18181	18481	19391	19891	19991	30103	30203	30403	30703
30803	31013	31513	32323	32423	33533	34543	34843	35053	35153
35353	35753	36263	36563	37273	37573	38083	38183	38783	39293
70207	70507	70607	71317	71917	72227	72727	73037	73237	73637
74047	74747	75557	76367	76667	77377	77477	77977	78487	78787
78887	79397	79697	79997	90709	91019	93139	93239	93739	94049
94349	94649	94849	94949	95959	96269	96469	96769	97379	97579
97879	98389	98689							

SEVEN-DIGIT PALINDROMIC PRIMES

1003001	1008001	1022201	1028201	1035301	1043401	1055501	1062601
1065601	1074701	1082801	1085801	1092901	1093901	1114111	1117111
1120211	1123211	1126211	1129211	1134311	1145411	1150511	1153511
1160611	1163611	1175711	1177711	1178711	1180811	1183811	1186811
1190911	1193911	1196911	1201021	1208021	1212121	1215121	1218121
1221221	1235321	1242421	1243421	1245421	1250521	1253521	1257521
1262621	1268621	1273721	1276721	1278721	1280821	1281821	1286821
1287821	1300031	1303031	1311131	1317131	1327231	1328231	1333331
1335331	1338331	1343431	1360631	1362631	1363631	1371731	1374731
1390931	1407041	1409041	1411141	1412141	1422241	1437341	1444441
1447441	1452541	1456541	1461641	1463641	1464641	1469641	1486841
1489841	1490941	1496941	1508051	1513151	1520251	1532351	1535351
1542451	1548451	1550551	1551551	1556551	1557551	1565651	1572751
1579751	1580851	1583851	1589851	1594951	1597951	1598951	1600061

1609061	1611161	1616161	1628261	1630361	1633361	1640461	1643461
1646461	1654561	1657561	1658561	1660661	1670761	1684861	1685861
1688861	1695961	1703071	1707071	1712171	1714171	1730371	1734371
1737371	1748471	1755571	1761671	1764671	1777771	1793971	1802081
1805081	1820281	1823281	1824281	1826281	1829281	1831381	1832381
1842481	1851581	1853581	1856581	1865681	1876781	1878781	1879781
1880881	1881881	1883881	1884881	1895981	1903091	1908091	1909091
1917191	1924291	1930391	1936391	1941491	1951591	1952591	1957591
1958591	1963691	1968691	1969691	1970791	1976791	1981891	1982891
1984891	1987891	1988891	1993991	1995991	1998991	3001003	3002003
3007003	3016103	3026203	3064603	3065603	3072703	3073703	3075703
3083803	3089803	3091903	3095903	3103013	3106013	3127213	3135313
3140413	3155513	3158513	3160613	3166613	3181813	3187813	3193913
3196913	3198913	3211123	3212123	3218123	3222223	3223223	3228223
3233323	3236323	3241423	3245423	3252523	3256523	3258523	3260623
3267623	3272723	3283823	3285823	3286823	3288823	3291923	3293923
3304033	3305033	3307033	3310133	3315133	3319133	3321233	3329233
3331333	3337333	3343433	3353533	3362633	3364633	3365633	3368633
3380833	3391933	3392933	3400043	3411143	3417143	3424243	3425243
3427243	3439343	3441443	3443443	3444443	3447443	3449443	3452543
3460643	3466643	3470743	3479743	3485843	3487843	3503053	3515153
3517153	3528253	3541453	3553553	3558553	3563653	3569653	3586853
3589853	3590953	3591953	3594953	3601063	3607063	3618163	3621263
3627263	3635363	3643463	3646463	3670763	3673763	3680863	3689863
3698963	3708073	3709073	3716173	3717173	3721273	3722273	3728273
3732373	3743473	3746473	3762673	3763673	3765673	3768673	3769673
3773773	3774773	3781873	3784873	3792973	3793973	3799973	3804083
3806083	3812183	3814183	3826283	3829283	3836383	3842483	3853583
3858583	3863683	3864683	3867683	3869683	3871783	3878783	3893983
3899983	3913193	3916193	3918193	3924293	3927293	3931393	3938393
3942493	3946493	3948493	3964693	3970793	3983893	3991993	3994993
3997993	3998993	7014107	7035307	7036307	7041407	7046407	7057507
7065607	7069607	7073707	7079707	7082807	7084807	7087807	7093907
7096907	7100017	7114117	7115117	7118117	7129217	7134317	7136317
7141417	7145417	7155517	7156517	7158517	7159517	7177717	7190917
7194917	7215127	7226227	7246427	7249427	7250527	7256527	7257527
7261627	7267627	7276727	7278727	7291927	7300037	7302037	7310137

7314137	7324237	7327237	7347437	7352537	7354537	7362637	7365637
7381837	7388837	7392937	7401047	7403047	7409047	7415147	7434347
7436347	7439347	7452547	7461647	7466647	7472747	7475747	7485847
7486847	7489847	7493947	7507057	7508057	7518157	7519157	7521257
7527257	7540457	7562657	7564657	7576757	7586857	7592957	7594957
7600067	7611167	7619167	7622267	7630367	7632367	7644467	7654567
7662667	7665667	7666667	7668667	7669667	7674767	7681867	7690967
7693967	7696967	7715177	7718177	7722277	7729277	7733377	7742477
7747477	7750577	7758577	7764677	7772777	7774777	7778777	7782877
7783877	7791977	7794977	7807087	7819187	7820287	7821287	7831387
7832387	7838387	7843487	7850587	7856587	7865687	7867687	7868687
7873787	7884887	7891987	7897987	7913197	7916197	7930397	7933397
7935397	7938397	7941497	7943497	7949497	7957597	7958597	7960697
7977797	7984897	7985897	7987897	7996997	9002009	9015109	9024209
9037309	9042409	9043409	9045409	9046409	9049409	9067609	9073709
9076709	9078709	9091909	9095909	9103019	9109019	9110119	9127219
9128219	9136319	9149419	9169619	9173719	9174719	9179719	9185819
9196919	9199919	9200029	9209029	9212129	9217129	9222229	9223229
9230329	9231329	9255529	9269629	9271729	9277729	9280829	9286829
9289829	9318139	9320239	9324239	9329239	9332339	9338339	9351539
9357539	9375739	9384839	9397939	9400049	9414149	9419149	9433349
9439349	9440449	9446449	9451549	9470749	9477749	9492949	9493949
9495949	9504059	9514159	9526259	9529259	9547459	9556559	9558559
9561659	9577759	9583859	9585859	9586859	9601069	9602069	9604069
9610169	9620269	9624269	9626269	9632369	9634369	9645469	9650569
9657569	9670769	9686869	9700079	9709079	9711179	9714179	9724279
9727279	9732379	9733379	9743479	9749479	9752579	9754579	9758579
9762679	9770779	9776779	9779779	9781879	9782879	9787879	9788879
9795979	9801089	9807089	9809089	9817189	9818189	9820289	9822289
9836389	9837389	9845489	9852589	9871789	9888889	9889889	9896989
9902099	9907099	9908099	9916199	9918199	9919199	9921299	9923299
9926299	9927299	9931399	9932399	9935399	9938399	9957599	9965699
9978799	9980899	9981899	9989899				

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TWO SLICING PROBLEMS

Two planar areas that can be placed so that they cut off equal chords on each member of a family of parallel lines, or two solids that can be placed so that they intercept equiareal sections on each member of a family of parallel planes, are said to be *Cavalieri congruent*. Two figures that are Cavalieri congruent have, of course, equal areas (in the one case) or equal volumes (in the other case). The following two problems point out a curiosity related to Cavalieri congruence.

Problem 1.

Show that there cannot exist a polygon to which a given circle is Cavalieri congruent.

Problem 2.

On the other hand, show that there exists a polyhedron (actually a tetrahedron) to which a given sphere is Cavalieri congruent.

HOWARD EVES,
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Solutions to these problems appear on page 270 *in this issue.*

* * *

ON MATHEMATICAL BEHAVIOUR

The logician Raymond Smullyan gives the following two examples of mathematical behaviour ([2], see also [1]):

Example 1.

1. To get hot water given an unlighted stove, matches, cold water, and an empty pot: fill pot, light stove, put pot on stove, and wait.

2. To get hot water given an unlighted stove, matches, and a pot filled with cold water: pour out water, thereby reducing the problem to the already solved Case 1.

3. To get hot water given a lighted stove and a pot filled with cold water: turn off stove and pour out water, thereby reducing to Case 1; alternatively, turn off stove, thereby reducing to Case 2.

Example 2.

1. To put out the fire given a hydrant, a disconnected hose, and a house on fire: connect hose and squirt house.

2. To put out the fire given a hydrant, a disconnected hose, and a house not on fire: set fire to house, thereby reducing to Case 1.

The behaviour of the mathematician in Example 1 is perfectly logical if he has a robot programmed to solve Case 1 from start to finish (with no variations, robots being what they are). Hence, in Case 2, it makes sense for him to perform the simple task of emptying the pot and then to turn the job over to the robot, rather than to light the stove, put the pot on the stove, and wait. In Case 3,emptying the pot and turning off the stove still beat putting the pot on the stove and waiting for the water to get hot. Alternatively, if the robot is programmed to solve Case 2, it makes sense to just turn off the stove and leave the rest to the robot.

It is in Example 2 that a lesson may be learned. Many mathematicians have the habit of making things out to be more difficult than they really are, creating difficulties where there are none. While the effect may be blazing, it is hardly worth doing.

REFERENCES

1. Martin Gardner, "Mathematical Games," *Scientific American,* May 1977, p. 134.

2. Raymond M. Smullyan, *What is the Name of this Book?*, Prentice-Hall, Englewood Cliffs, New Jersey, 1978, pp. 189-190.

A. LIU, University of Alberta.

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SOLUTIONS TO "TWO SLICING PROBLEMS" (see page 269)

Solution to Problem 1. Parallel chords cut off by a pair of coplanar lines vary linearly in length, whereas parallel chords cut off by a circle do not.

Solution to Problem 2. Let AB and CD be two line segments in space such that: (1) $AB = CD = 2r\sqrt{\pi}$, where r is the radius of the sphere; (2) AB and CD are each perpendicular to the line joining their midpoints, this join having length $2r$; (3) AB is perpendicular to CD. It can easily be shown that the tetrahedron ABCD may serve as the comparison solid. □

The interested reader may care to try to establish the following more difficult curiosity: *Though there exist pairs of tetrahedra of the same volume that are not Cavalieri congruent, any pair of triangles of the same area are Cavalieri congruent.*

HOWARD EVES

* * *

THE OLYMPIAD CORNER: 19

MURRAY S. KLAMKIN

On Mathematical Olympiads.

The International Mathematical Olympiad (IMO) was initiated by Rumania in 1959 [1]. It is for students who have not yet started at university and have not reached their 20th year. So far it has run for 21 consecutive years. The last one was held in England in 1979. Unfortunately, no country was willing to be the host country for 1980 (this involves a lot of work, organization, and expense). It is still uncertain whether or not the IMO will continue to be a yearly event. There are pressures to change it to a biennial event. The 1981 IMO is set to be held in the U.S.A. in July (in Washington, D.C.). More information concerning this will be given in a subsequent issue.

The U.S.S.R. teams have had outstanding results in the IMO's. A measure of their success is the number of individual prizes won as well as the *unofficial* team placement. A good part of the reason for this success is the support given in the U.S.S.R. to mathematics (as well as to other intellectual endeavours and sports) for a long time. This should be contrasted with the relatively meager support the U.S.A. and Canada have provided for their gifted children. By and large, programs for gifted children are still considered here to be elitist, notwithstanding the fact that many of our future intellectual leaders and statesmen should be coming from this pool of gifted children. Unfortunately, many of them either "drop out" or else fail to live up to their potential because they are bored with many of their programs which present little or no challenge. Considering the present competition between the "Communist System" and the "Free Enterprise System", we can ill afford this neglect. That the U.S.S.R. is in dead earnest about education is indicated by recent initiatives taken by them for all secondary school students [2].

H. Freudenthal [3] has given a highly recommended report on mathematical olympiads in many countries. The report also contains many problems from various countries as well as a list of 105 references to papers mostly in English. There is in [3] a particularly interesting section on local and national olympiads in the Soviet Union, and I regret that there is no room here to quote it *in extenso*. It shows to what lengths Soviet authorities are prepared to go to discover and foster the development of mathematical talent.[1] I give below a few excerpts from this section, but

[1] Unfortunately, there is also discrimination in mathematics for certain minority groups. See [1980: 145].

readers are urged to study the full report.

Local olympiads.

The name 'olympiads' for mathematical contests among high school students seems to have been first used in Russia...

The first was held in Leningrad in the spring of 1934. Moscow followed in 1935... The first Moscow Olympiad had 314 participants, in 1964 the number was about 4000.

The Moscow olympiads (and most of the other local olympiads) are run in two rounds, in spring, with a fortnight interval... the 1953 Moscow olympiad was entered by 1350 students, 517 of which were admitted to the second round, in which 262 succeeded, 3 got a first prize, 15 a second prize, 24 a third prize, and 69 got certificates of merit. The prizes consist of small mathematical libraries. Certificates of merit are also awarded to the teachers of winning students.

... The time allowed for solving the problems is 4-5 hours in both the first and second round. The degree of difficulty of the problems has much increased in the course of the years. Their character is much like that of the Eötvös contest problems though in general the appeal to a creative mind is perhaps less strong and they often require more familiarity with intentionally cultivated techniques. They are traditional mathematics, except for the use of some artifices like Dirichlet's drawer principle.

... The Committee read the participant's answers and grade them according to a not too formal system. Such criteria as elegance and originality of the solution play an important role.

... Another new feature are Moscow olympiads for the 4th-6th grades (10 to 13-year-olds), separated according to these grades.

... In the course of the years, in particular after 1945, the olympiads and related activities have spread to other major cities of the Soviet Union, where universities or pedagogical institutes existed...

National olympiads.

In 1960 the Moscow Olympiads Committee took an initiative to organize a geographically broader olympiad, the first all-Russian (actually all-Union) olympiad, in which teams of 13 *oblast'* (provinces) of the R.S.F.S.R. and 9 Union-states participated... This enterprise has annually been repeated, in different places...

These olympiads are played on four geographic levels: 1st round, school olympiads; 2nd, city and 'rayon' olympiads; 3rd, *oblast'*, *krai*, republic (province or state) olympiads; 4th, final round. In this pattern the Moscow olympiad is a third round competition. There are also 'correspondence instruction olympiads' and television olympiads... The level of the television olympiads is subjected to great variations. People who succeeded in the correspondence instruction olympiad are admitted to the 3rd round of the national olympiads. The problems of the final round are differentiated according to the four highest school classes...

The final round of the 1967 U.S.S.R. mathematical olympiad took place in Tbilisi ... It seems that the first round is entered by hundreds of thousands of competitors.

Thus in the Soviet Union. The following "Notice to Canadian Students" reflects the Canadian reality. It would be appreciated if teachers would post a copy of the Notice on their school bulletin boards.

NOTICE TO CANADIAN STUDENTS

So far the Canadian Mathematical Olympiad Committee has not succeeded in obtaining financial support for travel to send an 8-student Canadian team to participate in the July 1981 IMO to be held in the U.S.A. One possible alternative is to participate with that subset of students who would be selected to be team members on the basis of their performance in the 1981 Canadian Mathematical Olympiad and who can finance their own travel to New York City and return from Washington, D.C. Once in the U.S.A., all expenses (room, board, travel) will be provided for by the host country.

It may also be possible for the selected team members to participate in a pre-training session for approximately one month starting around 8 June 1981 at the U.S.A. Military Academy at West Point, N.Y. together with the U.S.A. team. This will probably entail a cost of approximately $5.00/day for room and board.

REFERENCES

1. S.L. Greitzer, *International Mathematical Olympiads 1959-1977*, Mathematical Association of America, Washington, D.C., 1978.

2. I. Wirszup *Preliminary report on the present status of Soviet mathematics and science training at the pre-university level*, Mathematics Department, University of Chicago, Chicago, Illinois 60637.

3. H. Freudenthal (Ed.), "ICMI Report on Mathematical Contests in Secondary Education (Olympiads)I", *Educational Studies in Mathematics*, 2 (1969) 80-114.

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Recently, through the courtesy of Willie Yong, I received a number of U.S.S.R. National Olympiad and Moscow Olympiad preparation problem sets. These were translated by a teacher in New York whom I will acknowledge as soon as I learn his name. Since I believe that these problems are quite good and challenging, I now include the Ninth U.S.S.R. National Olympiad set of 1974 and expect to include other sets in subsequent issues. Since there are quite a few problems in each set, I will only publish selected elegant solutions which are submitted to me. Solutions by secondary school students would be particularly welcome. Solvers are encouraged to reveal the approximate time they spent on each problem.

NINTH U.S.S.R. NATIONAL OLYMPIAD (1974)

1. Triangle ABC is rotated about the centre of its circumscribed circle by an angle less than 180° to form triangle $A_1B_1C_1$. If $BC \cap B_1C_1 = A_2$, $CA \cap C_1A_1 = B_2$, and $AB \cap A_1B_1 = C_2$, prove that triangles ABC and $A_2B_2C_2$ are similar.

2. Two players play the following game on a triangle ABC of unit area. The first player picks a point X on side BC, then the second player picks a point Y on CA, and finally the first player picks a point Z on AB. The first player wants triangle XYZ to have the largest possible area, while the second player wants it to have the smallest possible area. What is the largest area that the first player can be sure of getting?

3. The vertices of a convex 32-gon lie on the points of a square lattice whose squares have sides of unit length. Find the smallest perimeter such a figure can have.

4. On a 13×13 square piece of graph paper the centres of 53 of the 169 squares are chosen. Show that there will always be 4 of these 53 points which are the vertices of a rectangle whose sides are parallel to those of the paper.

5. Three ants crawl along the sides of a triangle ABC in such a way that the centroid of the triangle they form at any given moment remains fixed. Show that this centroid coincides with the centroid of triangle ABC if one of the ants travels along the entire perimeter of triangle ABC.

6. A certain number of 0's, 1's, and 2's are written on a blackboard. Two unequal digits are erased and the third digit is written in their place (e.g., 2 is written if 0 and 1 are erased). This operation is repeated until no two distinct digits remain on the blackboard. Show that if only one digit remains at the end of the game, then this digit is independent of the order in which the digits were erased.

7. In a convex hexagon $A_1A_2A_3A_4A_5A_6$, let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of diagonals $A_6A_2, A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1$, respectively. Show that if hexagon $B_1B_2B_3B_4B_5B_6$ is convex, then its area is $\frac{1}{4}$ the area of $A_1A_2A_3A_4A_5A_6$.

8. Show that with the digits 1 and 2 one can form 2^{n+1} numbers, each having 2^n digits, and every two of which differ in at least 2^{n-1} places.

9. On a 7×7 square piece of graph paper, the centres of k of the 49 squares

are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest k for which this is possible?

10. A large cube measuring k units on each edge is to be formed of smaller unit cubes, each coloured either black or white. Can this be done so that for any unit cube exactly two of its neighbours have the same colour as the unit cube itself? (Two cubes are called *neighbours* if they share a common face.)

11. A horizontal strip is given in the plane, bounded by straight lines, and n lines are drawn intersecting this strip. Every two of these lines intersect inside the strip and no three of them are concurrent. Consider all paths starting on the lower edge of the strip, passing along segments of the given lines, and ending on the upper edge of the strip, which have the following property: travelling along such a path, we are always going upward, and when we come to the point of intersection of two of the lines we must change over to the other line to continue following the path. Show that, among these paths,

(a) at least $\frac{1}{2}n$ of them have no point in common;

(b) there is some path consisting of at least n segments;

(c) there is some path passing along at most $\frac{1}{2}n+1$ of the lines;

(d) there is some path which passes along each of the n lines.

12. Given is a polynomial $P(x)$ whose coefficients are (i) natural numbers, (ii) integers. Denote by a_n the sum of the digits in the decimal representation of $P(n)$. Show that there is some number which occurs infinitely often in the sequence $a_1, a_2, a_3, \ldots$.

13. In a plane is given a finite set of polygons, every two of which have a common point. Show that there exists a line which intersects all the polygons.

14. Prove that, for positive a,b,c, we have

$$a^3+b^3+c^3+3abc \geq bc(b+c)+ca(c+a)+ab(a+b).$$

15. Quadrilateral ABCD is inscribed in a circle. It is rotated about the centre of the circle through an angle less than 180° to form quadrilateral $A_1B_1C_1D_1$. Show that the points

$$AB \cap A_1B_1, \quad BC \cap B_1C_1, \quad CD \cap C_1D_1, \quad DA \cap D_1A_1$$

are the vertices of a parallelogram.

16. Twenty teams are participating in the competition for the championships both of Europe and the world in a certain sport. Among them, there are k European teams (the results of their competitions for world champion count also towards the European championship). The tournament is conducted in round robin fashion. What is the largest value of k for which it is possible that the team getting the (strictly) largest number of points towards the European championship also gets the (strictly) smallest number of points towards the world championship, if the sport involved is

(a) hockey (0 for a loss, 1 for a tie, 2 for a win);

(b) volleyball (0 for a loss, 1 for a win, no ties).

17. Given real numbers

$$a_1, a_2, \ldots, a_m \quad \text{and} \quad b_1, b_2, \ldots, b_n,$$

and positive numbers

$$p_1, p_2, \ldots, p_m \quad \text{and} \quad q_1, q_2, \ldots, q_n,$$

we form an $m \times n$ array in which the entry in the ith row ($i = 1,2,\ldots,m$) and jth column ($j = 1,2,\ldots,n$) is

$$\frac{a_i + b_j}{p_i + q_j}.$$

Show that in such an array there is some entry which is no less than any other in the same row and no greater than an other in the same column

(a) when $m=2$ and $n=2$,

(b) for arbitrary m and n.

*

The problem about a property of 19^{19} proposed by Dougin Walker [1980: 210] is not new. It has appeared previously in Angela Dunn (Editor), *Mathematical Bafflers*, McGraw-Hill, New York, 1964, pp. 187-188. A new edition was published in 1980 by Dover Publications, New York.

*

SOLUTIONS TO PRACTICE SET 15

15-1. Determine an n-digit number (in base 10) such that the number formed by reversing the digits is nine times the original number. What other multiples besides nine are possible?

Solution.

We use the notation $\overline{ab} = 10a + b$, $\overline{abc} = 100a + 10b + c$, etc., and we exclude (for now) leading and final zeros. We first show that if

$$N \equiv \overline{ab\ldots xy} \cdot k = \overline{yx\ldots ba} \equiv N', \quad k \neq 1,$$

then $k = 4$ or 9.

If $k = 5$, then $a = 1$ for otherwise N would have more digits than N'; and N' is not divisible by 5 when $a = 1$. Similarly, $k \neq 6$ or 8. If $k = 7$, then again $a = 1$. But then we must have $y = 3$ for N' to end in 1, with the resulting contradiction

$$N = \overline{1b\ldots x3} \cdot 7 > \overline{3x\ldots b1} = N'.$$

If $k = 2$, then $a \leq 4$, and $a = 2$ or 4 since N' is even. If $a = 4$, then y, the initial digit of N', must equal 8 or 9, but neither $\overline{4b\ldots x8} \cdot 2$ nor $\overline{4b\ldots x9} \cdot 2$ ends in 4. If $a = 2$, then $y = 4$ or 5, but neither $\overline{2b\ldots x4} \cdot 2$ nor $\overline{2b\ldots x5} \cdot 2$ ends in 2. Finally, if $k = 3$, then $a \leq 3$. If $a = 1$, then $y = 7$ and $N < N'$. If $a = 2$ then $y = 4$, and if $a = 3$ then $y = 1$; and in each case $N > N'$. So $k = 4$ or 9.

We now assume $k = 4$ and find all numbers $\overline{ab\ldots xy}$ such that

$$N \equiv \overline{ab\ldots xy} \cdot 4 = \overline{yx\ldots ba} \equiv N'.$$

For N and N' to have the same number of digits, we must have $a = 1$ or 2; hence $a = 2$ since N' is even. Now y, the initial digit of N', must equal 8 or 9. In fact, since $4y$ ends in 2, we must have $y = 8$ and

$$\overline{2b\ldots x8} \cdot 4 = \overline{8x\ldots b2}.$$

Since $23 \cdot 4 > 90$, we now have $b = 0$, 1, or 2. At the same time, the digit in the tens' place in the product $\overline{x8} \cdot 4$ is odd for any x, so $b = 1$. Knowing the last two digits 12 of the product $\overline{21\ldots x8} \cdot 4$, we conclude that $x = 2$ or 7. Since $21 \cdot 4 > 82$, it follows that $x = 7$ and the required numbers are of the form $21\ldots78$. The smallest satisfactory answer is 2178.

Answers with more than four digits must satisfy

$$\overline{21uv\ldots rs78} \cdot 4 = \overline{87sr\ldots vu12}. \tag{1}$$

If there are k unassigned digits on each side, we have

$$84 \cdot 10^{k+2} + 312 + \overline{uv\ldots rs00} \cdot 4 = 87 \cdot 10^{k+2} + 12 + \overline{sr\ldots vu00},$$

from which

$$M \equiv \overline{uv\ldots rs} \cdot 4 + 3 = \overline{3sr\ldots vu} \equiv M'. \tag{2}$$

We conclude from (2) that $M-3$ starts with 29 or 3 and that $u=7$, 8, or 9. Indeed, since M' is odd, we must have $u=7$ or 9. We consider these two cases separately.

If $u=9$, we have

$$M \equiv \overline{9v\ldots rs}\cdot 4+3 = \overline{3sr\ldots v9} \equiv M'. \tag{3}$$

Since $4s+3$ ends in 9, we have $s=9$ for the only alternative $s=4$ implies $M>M'$. As in going from (1) to (2), with $s=9$ we find that (3) implies

$$\overline{v\ldots r}\cdot 4+3 = \overline{3r\ldots v}. \tag{4}$$

It follows from (2) and (4) that if $\overline{uv\ldots rs}$ satisfies (2) with $u=s=9$, then dropping the initial and final 9's yields a number $\overline{v\ldots r}$ with the same property as $\overline{uv\ldots rs}$. In particular, $\overline{uv\ldots rs}$ can be any one of the numbers 9, 99, 999, ..., from which we get the numbers

21978, 219978, 2199978, ...,

all of which satisfiy (1).

If $u=7$, then we get from (2)

$$\overline{7v\ldots rs}\cdot 4+3 = \overline{3sr\ldots v7}.$$

An analysis similar to that for the case $u=9$ shows that we must have $s=1$, $v=8$, and $r=2$, so that $\overline{uv\ldots rs}$ is of the form 78...21. This analysis also shows that if the initial 78 and final 21 are dropped, then the remaining integer is one of the answers to our problem, that is, multiplying it by 4 reverses the digits.

To recapitulate, we have shown that, for the case $k=4$, any answer to the problem which differs from all the numbers in the sequence

$$0, \quad 2178, \quad 21978, \quad 219978, \quad 2199978, \quad \ldots \tag{5}$$

has the same combination of digits at the beginning and at the end, that this combination is one of the numbers (5), and that if this combination is dropped from the beginning and the end the resulting number (here we allow leading and final zeros) is also an answer to the problem. Thus all answers to the problem consist of concatenations of one of the types

$$P_1P_2\ldots P_{n-1}P_nP_{n-1}\ldots P_2P_1 \tag{6}$$

or

$$P_1P_2\ldots P_{n-1}P_nP_nP_{n-1}\ldots P_2P_1, \tag{7}$$

where each P_i is one of the numbers (5). Here are some examples:

2197821978,

2199782178219978,

21780219978002199780 2178.

A similar analysis for the case $k = 9$ (which we leave to the reader) shows that all answers are concatenations of one of the types (6) or (7), where each P_i is a number in the sequence

$$0, \quad 1089, \quad 10989, \quad 109989, \quad 1099989, \; \ldots \; . \qquad \square$$

This problem appears in [1], and the above solution was edited from the one given in that excellent reference, which consists of 350 problems (with solutions) from Russian Olympiads and mathematics hobby groups in Moscow.

REFERENCE

1. D.O. Shklyarsky, N.N. Chentsov, and I.M. Yaglom, *Selected Problems and Theorems in Elementary Mathematics*, Mir Publishers, Moscow, 1979, pp. 17, 112-115. (The U.S.A. distributor of Mir Publishers is Imported Publications Inc., 320 West Ohio St., Chicago, Illinois 60610. They have a brochure listing quite a number of translated Russian mathematics problem books. The prices of the books are reasonable.)

15-2. Solve the following system of equations:

$$\begin{array}{l} ax_1 + bx_2 + bx_3 + \ldots + bx_n = c_1, \\ bx_1 + ax_2 + bx_3 + \ldots + bx_n = c_2, \\ \vdots \\ bx_1 + bx_2 + bx_3 + \ldots + ax_n = c_n. \end{array}$$

(In the left member of each equation, all the coefficients except one are b's and the remaining one is a.)

Solution.

The system is trivial if $a = b$, so we assume $a \neq b$. Subtracting the rth equation from the first, we get

$$(a - b)(x_1 - x_r) = c_1 - c_r, \quad r = 2,3,\ldots,n,$$

from which

$$x_r = x_1 - \frac{c_1 - c_r}{a - b}, \quad r = 2,3,\ldots,n. \tag{1}$$

Adding all the equations gives

$$\{a+(n-1)b\}\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} c_k$$

which, by using (1), becomes

$$\{a+(n-1)b\}\left\{nx_1 - \sum_{k=1}^{n}\frac{c_1-c_k}{a-b}\right\} = \sum_{k=1}^{n} c_k. \tag{2}$$

If $a+(n-1)b \neq 0$, then x_1 is uniquely determined by (2) and the remaining x_r by (1). Alternatively, any x_r can be obtained from (2) by replacing x_1 and c_1 by x_r and c_r, respectively. Finally, if $a+(n-1)b = 0$, there is no solution unless $\Sigma c_k = 0$, in which case x_1 is arbitrary and the remaining x_r are then uniquely determined by (1). □

In this problem, the row vectors of the coefficient matrix are cyclic permutations of

$$(a,b,b,\ldots,b) \qquad (n\text{-}1\ b\text{'s}).$$

As a rider, solve the more general system where the row vectors of the coefficient matrix are cyclic permutations of

$$(a,a,\ldots,a,b,b,\ldots,b) \qquad (m\ a\text{'s and } n\text{-}m\ b\text{'s}).$$

15-3. Three circular arcs BC, CA, AB, of fixed total length l, are constructed outwardly on the sides of a given triangle ABC, each passing through two vertices, so that the area they enclose is a maximum (for the given l). Show that the radii of the three arcs are equal. (The problem has been restated for greater clarity.)

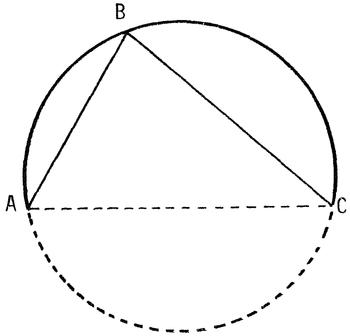
Solution.

It is assumed that the total length l is not less than the perimeter of the triangle but not too large (we shall consider this point subsequently). Although it is intuitively clear that a maximum area does exist for a given l, nevertheless we establish this fact by a continuity argument. Let the lengths of the arcs AB, BC be l_1, l_2, respectively; then the length of arc CA is $l-l_1-l_2$. Since the area of a segment of a circle is a continuous function of the lengths of its bounding arc and bounding chord, the area bounded by the three arcs is a continuous function of l_1 and l_2 over the closed domain

$$l_1 \geq \text{AB}, \quad l_2 \geq \text{BC}, \quad l_1+l_2 \leq l-\text{CA}. \tag{1}$$

Consequently this area takes on its maximum value (also its minimum value) for some values of l_1, l_2 in the domain (1).

Assume that arc CA has length l_3 when the area bounded by the three arcs is a maximum. We will show that the arcs AB and BC, of fixed total length $l - l_3$, have equal radii if the sum of the areas of the circular segments AB and BC is a maximum. Increase or decrease angle B, keeping AB and BC fixed in length, until the length of the portion of the circumcircle through A,B,C (above AC) is also $l - l_3$, as shown in the figure. This can be done if l is not too large (see discussion of this point at the end). It now follows by the Isoperimetric Theorem for the circle (see [1]) that the sum of the areas of the segments AB and BC, of fixed total arc length, is a maximum when they are segments of the same circle. Similarly, the arc CA (in the original configuration) must have the same radius. There also result equal radii if we consider the corresponding problem for an arbitrary convex polygon instead of a triangle.



We now show that the above proof imposes an upper bound on l. For simplicity, we consider the case when $AB = BC$. Letting angle B vary while keeping AB and BC fixed in length, it is not difficult to show that the maximum length of the part of the circumcircle of ABC above AC occurs when angle $B = 0$. Thus our proof requires that

$$\pi AB \geq \text{arc } AB + \text{arc } BC \geq 2AB.$$

Presumably, one could by calculus establish the desired result for larger l. However, if l gets too large we run into a complication. For sufficiently large l, two of the circular arcs become tangent at a vertex. Then, for still larger l, the two corresponding segments have a nonempty intersection whose area is counted twice. This can be corrected for, but it makes finding the solution much more difficult. □

This problem appeared in the Spring 1960 issue of *Pi Mu Epsilon Journal*, where it was proposed by M.S.Klamkin and D.J. Newman, and the proposers' solution, equivalent to the above but more succinct, appeared in the Spring 1963 issue of the same journal. The more general problem of determining a closed curve of given length which passes through a set of given points and encloses a maximum area was treated by Steiner [2]. For the Steiner problem in the case of an equilateral triangle ABC with large enough l it is to be expected that the maximizing configuration will consist of three congruent segments AA', BB', CC' along extensions of the three medians and three congruent circular arcs B'C', C'A', A'B'. If this be true, there still remains the problem of determining the lengths of the arcs and the segments. This is a standard calculus problem but most likely one without an explicit solution. It would be interesting, in the corresponding Steiner problem for an arbitrary triangle, if the three circular arcs of the maximizing configuration still turn out to have the equal radii property.

As a rider, consider the original problem in which "maximum" is replaced by "minimum".

REFERENCES

1. G. Polyá, *Mathematics and Plausible Reasoning, I,* Princeton University Press, Princeton, N. J., 1954, pp. 168-189.

2. J. Steiner, *Gesammelte Werke, II,* pp. 75-91.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1 .

*   *   *

LETTER TO THE EDITOR

Dear Editor:

I was interested to see the discussion by Kenneth S.Williams [1980: 204] of the identity

$$\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1}=1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{1}{2n-1}.$$

It is essentially the same as

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\ldots-\frac{1}{2n}$$

which I asked my analysis class to prove last term. Of course induction was the most popular method, but Miss Judith Lum Wan gave a different solution which seemed to me to be rather elegant. Consider the difference between each side of the equation and the harmonic sum $1 + 1/2 + 1/3 + \ldots + 1/2n$. For the left-hand side it is the harmonic sum $1 + 1/2 + \ldots + 1/n$, and for the right-hand side it is $2/2 + 2/4 + \ldots + 2/2n$ which is the same.

B.C. RENNIE, James Cook
University of North Queensland.

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PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

581. *Proposed by Alan Wayne, Pasco-Hernando Community College, New Port Richey, Florida.*

The Soldier's Farewell

```
My love, since we must  P A R T
               There's  A L O E  in my soul;
Oh, hear the drumbeats  R O L L
                  That  T E L L  how throbs my heart!
```

In the "word square" above, each word represents a four-digit decimal integer which is a perfect square. The letters are one-to-one images of the digits. Restore the digits.

582. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

In how many ways can five distinct digits A, B, C, D, E be formed into four decimal integers AB, CDE, EDC, BA for which the mirror-image multiplication

$$AB \cdot CDE = EDC \cdot BA$$

is true? (For example, the mirror-image multiplication $AB \cdot CD = DC \cdot BA$ is true for $13 \cdot 62 = 26 \cdot 31$.)

583. *Proposed by Charles W. Trigg, San Diego, California.*

A man, being asked the ages of his two sons, replied: "Each of their ages is one more than three times the sum of its digits." How old is each son?

584. *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

If a triangle is isosceles, then its centroid, circumcentre, and the centre of an escribed circle are collinear. Prove the converse.

585. *Proposed by Jack Garfunkel, Flushing, New York.*

Consider the following three inequalities for the angles A, B, C of a triangle:

$$\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2} \ge 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \tag{1}$$

$$\csc\frac{A}{2}\cos\frac{B-C}{2} + \csc\frac{B}{2}\cos\frac{C-A}{2} + \csc\frac{C}{2}\cos\frac{A-B}{2} \ge 6, \tag{2}$$

$$\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2} \ge 6. \tag{3}$$

Inequality (3) is well-known (*American Mathematical Monthly* 66 (1959) 916) and it is trivially implied by (2). Prove (1) and show that (1) implies (2).

586. *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

(a) Given a natural number n, show that the equation

$$9n^3 = 6abn + ab(a+b)$$

has no solution in natural numbers a and b.

(b) Using (a), or otherwise, show that none of the following expressions is a perfect square for any natural number n:

$$36n^3 + 36n^2 + 12n + 1,$$
$$12n^3 + 36n^2 + 36n + 9,$$
$$4n^3 + 36n^2 + 108n + 81.$$

* * *

VERNER E. HOGGATT, Jr.

(In Memoriam)

There was a game one played a few years ago in which one attempted to represent randomly chosen positive integers by arithmetic expressions that involved each of the ten digits 0, 1, ..., 9 once and only once. The game was completely solved when

Verner E. Hoggatt, Jr.

discovered that, for any nonnegative integer n,

$$\log_{(0+1+2+3+4)/5}\{\log_{\sqrt{\sqrt{\ldots\sqrt{(-6+7+8)}}}}{}^{9}\} = n,$$

where there are n square roots in the second logarithmic base. (Notice that the ten digits appear in their natural order.)

HOWARD EVES

Verner E. Hoggatt, Jr., a former student of Howard Eves and the founder of *The Fibonacci Quarterly*, died on 12 August 1980. (Editor)

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

387. [1978: 251; 1979: 201; 1980: 46, 114] A comment was received from BENGT MÅNSSON, Lund, Sweden.

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477. [1979: 229; 1980: 218] Late solution: V.N. MURTY, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.

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483. [1979: 265; 1980: 223] Late comment from SAHIB RAM MANDAN, Bombay, India.

* * *

484. [1979: 265; 1980: 252] *Proposed by Gali Salvatore, Perkins, Québec.*

Let A and B be two independent events in a sample space, and let χ_A, χ_B be their characteristic functions (so that, for example, $\chi_A(x)=1$ or 0 according as $x\in A$ or $x\notin A$). If $F=\chi_A+\chi_B$, show that at least one of the three numbers

$$a = P(F=2), \quad b = P(F=1), \quad c = P(F=0)$$

is not less than $4/9$.

II. *Further comments by M.S. Klamkin, University of Alberta.*

As extensions of this problem, we had in solution I considered the determination of

$$A(m,n) = \min_{0\le P_{ij}\le 1} \max_k P_k$$

and

$$B(m,n) = \max_{0\le P_{ij}\le 1} \min_k P_k,$$

where

$$P_0 + P_1 t + \ldots + P_{mn}t^{mn} \equiv \prod_{i=1}^{m} (P_{i0}+P_{i1}t+\ldots+P_{in}t^n) \qquad (1)$$

and

$$0\le P_{ij}\le 1, \qquad \sum_{j=0}^{n} P_{ij} = 1, \qquad i=1,2,\ldots,m.$$

We had conjectured that

$$A(m,1) = \frac{\binom{m}{[m/2]}}{2^m}, \quad m \text{ odd} \tag{2}$$

and

$$B(m,1) = \frac{1}{2^m}, \quad \text{all } m. \tag{3}$$

First we give a simple proof of (3). Here we have

$$P_0 + P_1 t + \ldots + P_m t^m = (P_{10} + P_{11}t)(P_{20} + P_{21}t)\ldots(P_{m0} + P_{m1}t),$$

from which

$$P_0 = P_{10}P_{20}\ldots P_{m0} \quad \text{and} \quad P_m = (1 - P_{10})(1 - P_{20})\ldots(1 - P_{m0}).$$

Now $x(1-x) \le \frac{1}{4}$ in [0,1], so $P_0 P_m \le \frac{1}{4^m}$; hence

$$\min_k P_k \le \min\{P_0, P_m\} \le \max_{0 \le P_{ij} \le 1} \min\{P_0, P_m\} = \sqrt{\frac{1}{4^m}} = \frac{1}{2^m},$$

and equality holds throughout when $P_{ij} = \frac{1}{2}$ for all i,j, thus establishing (3).

The conjecture for $A(m,1)$ in (2) is valid and its value for all m was established by J.D. Dixon [1] in a somewhat different context when he generalized the following problem of L. Moser and J.R. Pounder [2]:

If $ax^2 + bx + c$ is a polynomial with real coefficients and real roots, then $\max\{a,b,c\} \ge 4(a+b+c)/9$.

Dixon showed more generally that if $a_0 + a_1 x + \ldots + a_m x^m$ is a polynomial of degree m with real coefficients and only real roots, then

$$\max_k a_k \ge \binom{m}{s}\frac{(m-s)^{m-s}(s+1)^s}{(m+1)^m}(a_0 + a_1 + \ldots + a_m), \tag{4}$$

where $s = [m/2]$, and equality is actually attained when all the roots are equal. When m is odd and the sum of the a_k is 1 (as in our problem), (4) reduces to (2). Thus we have, for all m,

$$A(m,1) = \binom{m}{s}\frac{(m-s)^{m-s}(s+1)^s}{(m+1)^m}, \quad s = [m/2]. \tag{5}$$

Seven years later and unaware of the Dixon result, W.O.J. Moser [3] proposed a problem which is a special case of (5):

Let m identical weighted coins, each falling heads with probability x, be tossed, and let $P_k(x)$ be the probability that exactly k of them fall heads. Evaluate

$$F_m = \min_{0 \le x \le 1} \max_{k=0,1,\ldots,m} P_k(x).$$

The published solution by D.Ž. Djoković gives the result

$$F_m = \binom{m}{s}\frac{s^s(m+1-s)^{m-s}}{(m+1)^m}, \qquad s = [m/2]. \tag{6}$$

Although (6) agrees with (5) for even m, it is incorrect for odd m. The error comes from failing to consider one of the possible cases.

We now give conjectured values for $A(2,n)$ and $B(2,n)$ for $n > 1$. Even if it turns out that our values are incorrect, they will at least provide good bounds.

For $A(2,n)$, we choose the P_{ij}'s as in solution I [1980: 255]. This gives $P_0 = 0$ and $P_k = 1/2n$ for $k > 0$. Thus our conjecture is

$$A(2,n) = \frac{1}{2n}. \tag{7}$$

We also obtain (7) with the choice

$$P_{1j} = \begin{cases} \dfrac{1}{2n}, & \text{for } j = 0,n, \\ \dfrac{1}{n}, & \text{for } 0 < j < n \end{cases} \quad \text{and} \quad P_{2j} = \begin{cases} \dfrac{1}{2}, & \text{for } j = 0,n, \\ 0, & \text{for } 0 < j < n. \end{cases}$$

For $B(2,n)$, we set $P_{1j} = 1/(n+1)$ for all j and $P_{2j} = \frac{1}{2}$ or 0 as above. This gives $P_n = 1/(n+1)$ and all the remaining $P_k = 1/(2n+2)$. Thus our conjecture is

$$B(2,n) = \frac{1}{2n+2}. \tag{8}$$

Since we must have $P_0 + P_1 + \ldots + P_{2n} = 1$ for any choice of the P_{ij}'s (just set $t = 1$ in (1)), the average value of the P_k's is always $1/(2n+1)$. Hence the exact values of $A(2,n)$ and $B(2,n)$ must satisfy

$$A(2,n) \geq \frac{1}{2n+1} \geq B(2,n).$$

Consequently, even if (7) and (8) are incorrect, they do provide good bounds.

Finally, we bring attention to a paper of MacLeod and Roberts [4] dealing with similar problems but with different norms.

REFERENCES

1. J.D. Dixon, "Polynomials with real roots," *Canadian Mathematical Bulletin*, 5 (1962) 259-263.

2. L. Moser, J.R. Pounder, Problem 53, *ibid.*, 5 (1962) 70.

3. W.O.J. Moser, Problem 154, *ibid.*, 12 (1969) 683-684.

4. R.A. MacLeod, F.D.K. Roberts, "Equalizing the coefficients in a product of polynomials", *ibid.*, 16 (1973) 531-539.

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489. [1979: 266; 1980: 263] Inadvertently omitted from the list of solvers: VIKTORS LINIS, University of Ottawa.

* * *

490. [1979: 266] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Are there infinitely many palindromic primes (e.g., 131, 70207)?

I. *Comment by Friend H. Kierstead, Cuyahoga Falls, Ohio.*

For even n, every n-digit palindrome is divisible by 11 since the alternating sum of its digits, which is 0, is divisible by 11. So 11 is the only palindromic prime with an even number of digits.

We now consider odd n. As in Gabai and Coogan [1], we define

α_n = number of n-digit palindromic primes,

A_n = number of (positive) n-digit palindromes $(= 9\cdot 10^{(n-1)/2})$,

β_n = number of n-digit primes,

B_n = number of (positive) n-digit integers $(= 9\cdot 10^{n-1})$,

and

$$P(n) = \frac{\alpha_n}{A_n}\bigg/\frac{\beta_n}{B_n} = 10^{(n-1)/2}\left(\frac{\alpha_n}{\beta_n}\right).$$

From the exact data given in [1] and updated by Nelson in [2], we have

$$P(1) = \frac{4}{4} = 1,$$

$$P(3) = 10\left(\frac{15}{143}\right) \approx 1.04895,$$

$$P(5) = 10^2\left(\frac{93}{8363}\right) \approx 1.11204,$$

$$P(7) = 10^3\left(\frac{668}{586081}\right) \approx 1.13977,$$

$$P(9) = 10^4\left(\frac{5172}{45086079}\right) \approx 1.14714.$$

Several people (e.g., Beiler [3] and Card [4]) have stated flatly that there are infinitely many palindromic primes but, to the best of our knowldege, no proof of this statement has ever been adduced. The above evidence, however, shows that the statement is probably true. Indeed it suggests that, at least for small n, there are proportionally more primes in the set of n-digit palindromes than there are primes in the set of n-digit integers, and that the proportion increases with n.

II. *Unedited extract from a comment submitted by Professor X at University Y.*

In 1950, Moser [5] remarked that primes (>5) end in 1, 3, 7, or 9, and palindromic primes (other than 11) must have an odd number of digits. He listed 14 three-digit and 88 five-digit palindromic primes, along with the larger

1818181, 7878787, 3535353, and 7272727.

(The last one, as printed, had an additional 27 tacked on, a proofing error.)

Comments were also received from HIPPOLYTE CHARLES, Waterloo, Québec; and CHARLES W. TRIGG, San Diego, California.

Editor's comment.

The least that can be said about Professor X is that he has exhibited stupefying naïveté in parroting Moser's statements without checking at least some of them. Why should Moser *list* only 14 three-digit palindromic primes when there are 15, and only 88 five-digit ones when there are 93? These facts can easily be checked in even a small table of primes. As to Moser's sample of exactly 4 seven-digit palindromic primes, behold:

$$1818181 = 31 \cdot 89 \cdot 659,$$
$$7878787 = 7 \cdot 19 \cdot 59239,$$
$$3535353 = 3^3 \cdot 23 \cdot 5693,$$
$$7272727 = 7^2 \cdot 11 \cdot 103 \cdot 131.$$

Misery loves company. So Professor X will be glad to learn that, fourteen years after Moser, Beiler [3] also gives a sample of exactly 4 seven-digit palindromic primes. They are (you guessed it) Moser's "primes". What is more, Beiler gives them without reference, thereby announcing, at least by implication, that they are his own "discoveries", since he can hardly claim that they are "well-known" primes. It is only fitting that he should now be hoist by his own petard.

One can only conjecture what Moser and the editor of *Scripta Mathematica* were thinking of when they published such baloney. The late Leo Moser was a mathematician and problemist of the first rank. He also had, as those who knew him can testify (this editor was privileged to meet him once), a highly-developed sense of humor and he loved a jape surpassing well. So perhaps ... (we leave the thought unfinished). Wherever he is right now, Leo Moser is probably having a good laugh at the gullibility of mortals.

In his famous Rede Lecture in 1959, the late C.P. Snow delineated and brought the world's consciousness to bear upon the gap between the "two cultures" (science and the humanities). As a result of efforts made since then to bridge the gap (not least by Snow himself), one would expect that by now every cultured person should know enough mathematics to recognize a small multiple of 3 (one of Leo Moser's "primes" was a multiple of 3). Yet in the April 1980 issue of the august *Atlantic*, a magazine that is just dripping with culture and history, Horace Judson (in "The

Rage To Know") claims that 1023 is a prime! The gap is widening, not being bridged.

As far as we know, the 93 five-digit palindromic primes and the 668 seven-digit ones have never been actually listed in the literature. Because of the mis-information floating around as a result of Leo Moser's "jape", and because lists of palindromes are so convenient to read by people who, like Professor X and Beiler, never know whether they are coming or going, it will be useful to record them in this issue. Rather than burying them in the Problem Section, we give them in a separate article (pages 266-268) so that they can later be more easily located in the index to this volume. We leave it to a more voluminous publication to publish the list of the 5172 nine-digit palindromic primes, one of which, a lovely example of the up-hill-and-down-dale type, is

$$345676543.$$

(Readers, remembering that the editor is another Leo, will probably want to check this.)

REFERENCES

1. Hyman Gabai and Daniel Coogan, "On Palindromes and Palindromic Primes", *Mathematics Magazine*, 42 (November 1969) 252-254.

2. Harry L. Nelson, Editor's Comment on Problem 700, *Journal of Recreational Mathematics*, 12 (1979-80) 66-67.

3. Albert H. Beiler, *Recreations in the Theory of Numbers*, Dover, New York, 1964, p. 228.

4. Leslie E. Card, "Patterns in Primes", *Journal of Recreational Mathematics*, 1 (1968) 93-99.

5. Leo Moser, "Palindromic Primes", *Scripta Mathematica*, 16 (March-June 1950) 127-128.

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491. [1979: 291] *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

(Dédié au souvenir de Victor Thébault, jadis inspecteur d'assurances à Le Mans, France.)

Résoudre la cryptarithmie décimale suivante:

$$\text{UN} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} + \text{DEUX} = \text{ONZE}.$$

I. *Solution by Kenneth M. Wilke, Topeka, Kansas.*

Clearly, $D = 1$ and $O \geq 5$. Since X even implies $N = E$, it follows that X is odd and $|E - N| = 5$. The problem is equivalent to

$$\frac{10 \cdot \text{DEUX}}{2} = \text{ONZE} - \text{UN},$$

from which $[DE/2] = 0$, and we have only the possibilities

$$(N,E,O) = (9,4,7),\quad (8,3,6),\quad (7,2,6),\ (3,8,9),\ (2,7,8),\quad \text{or}\quad (0,5,7).$$

If E is even then $N < 5$, and so $(N,E,O) = (3,8,9)$. But then $N = 3$ requires $U = 5$, which leaves $X = 7$ and produces no solution. If E is odd then $N > 5$, and so $(N,E,O) = (8,3,6)$. Now $N = 8$ requires $U = 5$ and $X = 7$, from which DEUX = 1357, UN = 58, and ONZE = 6843. The unique solution is

$$58 + 1357 + 1357 + 1357 + 1357 + 1357 = 6843.$$

II. *Comment by Donval R. Simpson, Fairbanks, Alaska.*

Variants of this problem can be obtained by changing base 10 to some other base $b \geq 7$. An even more fundamental change is to change the language. In Spanish, for example, we have in base 10:

$$\text{UNO} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} + \text{DOS} = \text{ONCE}$$

with (at least) the solution

$$891 + 213 + 213 + 213 + 213 + 213 = 1956.$$

Also solved by J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; DONVAL R. SIMPSON, Fairbanks, Alaska; CHARLES W. TRIGG, San Diego, California; and the proposer.

* * *

492. *Proposed by Dan Pedoe, University of Minnesota.*

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}$AB are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{2}$AB?

Solution by William A. McWorter, Jr., and Leroy F. Meyers, both from The Ohio State University (jointly).

We can draw only circles with fixed radius r, so the notation (P) for a circle with center P and radius r will be unambiguous and convenient.

(a) The only centers for drawing circles which are available at the beginning of the construction are A and B, and so we draw (A) and (B), which must intersect since $r \geq \frac{1}{2}$AB. Let the points of intersection be Z and Z' (where Z' = Z if $r = \frac{1}{2}$AB). If B $\in$ (A), then r = AB, and so Z (or Z') is the third vertex of the equilateral triangle ABZ (or ABZ'), obtained by exactly two uses of the rusty compass. (This is Book I, Proposition 1 of Euclid.)

Otherwise, the required third vertex is not on any of the circles already drawn. The only new points which can be used as centers are Z and Z'. (If $\frac{1}{2}$AB < r < AB, then (A) and (B) intersect AB in two additional points A' and B'. The circles (A') and (B') will then intersect in two points K and K' whose distance from A and B is less than AB, as is easy to check. Hence these four uses of the rusty compass will not suffice.) We now draw (Z), and let (A) $\cap$ (Z) = {X,X'} and (B) $\cap$ (Z) = {Y,Y'}, with the notation chosen so that XY and X'Y' are parallel to AB. (The intersections consist of two points, since {A,B} $\subset$ (Z).) None of the points on any of the three circles drawn so far is the required third vertex, and so we need to draw at least two more circles to determine it. Hence at least five circles are needed. In fact, if (X) $\cap$ (Y) = {Z,C}, then C is the required third vertex, as we will show. (Note that these five circles are just the ones anyone would try first.)

Now YZXC and YZY'B are rhombi, since all sides have length r. Hence CX $\parallel$ YZ $\parallel$ BY', and so CXY'B is a parallelogram and CB = XY'. Using directed arcs on (Z), we have

$$\text{arc } XY' = \text{arc } XA + \text{arc } AY' = \text{arc } Y'B + \text{arc } AY' = \text{arc } AB,$$

since triangles AZX and BZY are equilateral of side r. Hence XY' = AB, and so CB = AB. By the symmetry of the construction, also CA = AB, since C lies on the perpendicular bisector of AB. Hence ABC is equilateral. It may be noted that if (X') $\cap$ (Y') = {Z,C'}, then ABC' is also equilateral.

(b) We lay off points $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ on straight lines at the angle of 60° to AB on the same side of it, beginning at A and B, respectively, so that

$$AA_1 = A_1A_2 = \ldots = A_{k-1}A_k = BB_1 = B_1B_2 = \ldots = B_{k-1}B_k = r.$$

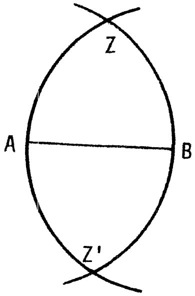
This is done by constructing, in order,

$$X_1 = AB \cap (A), \quad A_1 \in (A) \cap (X_1), \quad X_2 \in (X_1) \cap (A_1), \quad A_2 \in (A_1) \cap (X_2), \quad \text{etc.}$$

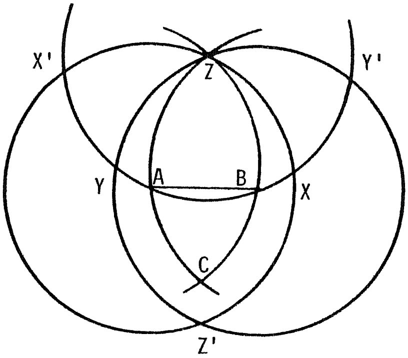
and similarly

$$Y_1 = AB \cap (B), \quad B_1 \in (B) \cap (Y_1), \quad Y_2 \in (Y_1) \cap (B_1), \quad B_2 \in (B_1) \cap (Y_2), \quad \text{etc.}$$

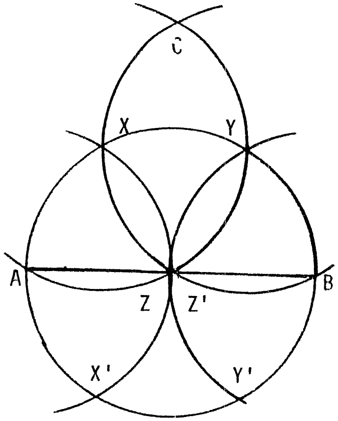
We choose k so that $A_kB_k \leq 2r$, as must happen for some k. We use part (a) to construct an equilateral triangle A_kB_kC on the appropriate side of A_kB_k. Then ABC is equilateral. □



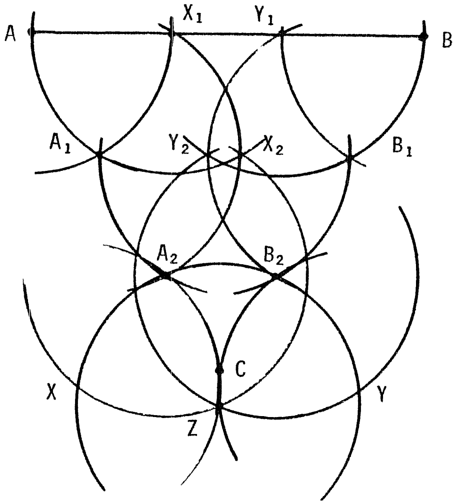
r = AB



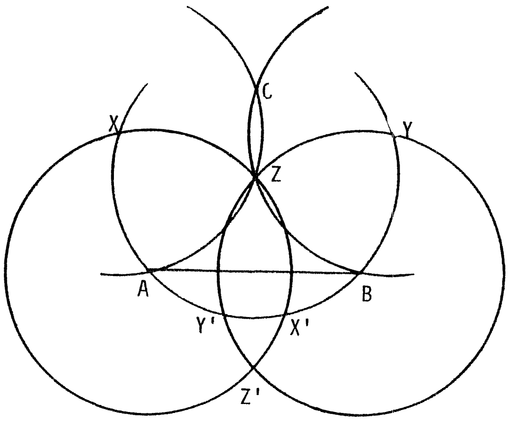
r > AB



$r = \frac{1}{2}$AB



$r < \frac{1}{2}$AB



$\frac{1}{2}$AB < r < AB

We have been unable to construct the third vertex C of an equilateral triangle ABC given only the points A and B (not the entire segment AB) if $r < \frac{1}{2}$AB.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; NGO TAN, student, J.F.Kennedy H.S., Bronx, N.Y.; and HERMAN NYON, Paramaribo, Surinam.

Solutions to part (a) only were submitted by JACK GARFUNKEL, Flushing,N.Y.; G.C. Giri, Midnapore College, West Bengal, India; FRIEND H. KIERSTEAD, Jr., Cuyahoga Falls, Ohio; JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

Editor's comment.

All solutions to part (a) were essentially the same. The proposer noted that this part of the problem and its solution were suggested to him by Kevin Panzer, a student at the University of Minnesota. The problem of solving part (b) when only the points A and B, not the entire segment AB, are given (which may have been what the proposer intended since only the points A and B are needed for part (a)) remains open. For more information about geometry with a rusty compass, see [1] and [2].

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1. Howard Eves, *A Survey of Geometry* (Revised Edition), Allyn and Bacon, Boston, 1972, p. 174.

2. A.N. Kostovskii, *Geometrical Constructions Using Compasses Only*, Blaisdell Publishing Co., New York, 1961, pp. 69-70.

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493. [1979: 291] *Proposed by R.C. Lyness, Suffolk, England.*

(a) A, B, C are the angles of a triangle. Prove that there are positive x,y,z, each less than $\frac{1}{2}$, simultaneously satisfying

$$y^2 \cot\frac{B}{2} + 2yz + z^2\cot\frac{C}{2} = \sin A,$$

$$z^2 \cot\frac{C}{2} + 2zx + x^2\cot\frac{A}{2} = \sin B,$$

$$x^2 \cot\frac{A}{2} + 2xy + y^2\cot\frac{B}{2} = \sin C.$$

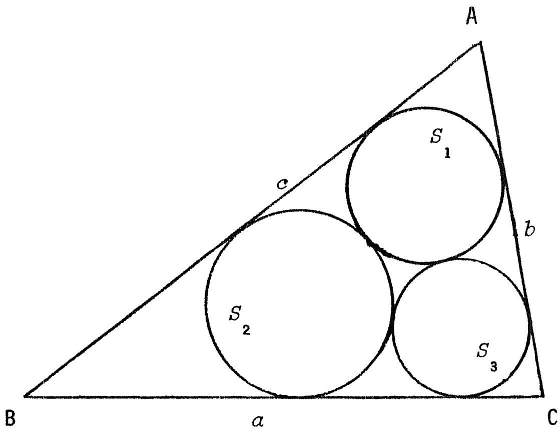
(b) In fact, $\frac{1}{2}$ may be replaced by a smaller $k > 0.4$. What is the least value of k?

Partial solution by the proposer.

(a) Let S_1, S_2, S_3 be three circles each touching the other two externally and each touching a different pair of (unextended) sides of a triangle ABC, as shown in the figure. If r_1, r_2, r_3 are the radii of these circles, then we have

$$r_2\cot\frac{B}{2} + 2\sqrt{r_2r_3} + r_3\cot\,\frac{C}{2} = a = 2R\sin A,$$

where R is the circumradius of the triangle, and two similar equations. For a given triangle, there is only one ratio $r_1 : r_2 : r_3$. The circles are called *Malfatti cirles* and their unique existence is easily proved by continuity, as we now show. Remove the restriction that S_2 and S_3 touch each each other and start with $r_1 = r$, the inradius of the triangle. Then S_2 and S_3 do not meet. As r_1 is decreased, S_2 and S_3 approach each other and when r_1 is sufficiently small they intersect in



two points. Thus there is one and only one value of r_1 between 0 and r for which all three circles touch one another.

If we set

$$x^2 = r_1/2R, \quad y^2 = r_2/2R, \quad z^2 = r_3/2R,$$

we obtain the system of equations in the proposal. This system therefore has a unique solution in positive reals x,y,z. Since

$$r_i/2R < r/2R \le \tfrac{1}{4}, \qquad i = 1,2,3,$$

we have $0 < x,y,z < \frac{1}{2}$.

(b) For given angle A and Malfatti radius r_1, we will assume that the length of BC $= a$ is least when $r_2 = r_3$, that is, when B = C. This seems obvious but I cannot find a simple proof. Since $2R\sin A$ is least when $r_2 = r_3$, it follows that $x = \sqrt{r_1/2R}$ is greatest when $y = z$. Now the first equation in the proposal becomes

$$2y^2(1 + \cot\frac{B}{2}) = \sin A$$

or, since $A = \pi - 2B$,

$$y^2 = \tfrac{1}{2}\sin 2B/(1 + \cot\frac{B}{2}), \tag{1}$$

and the third becomes

$$x^2\tan B + 2xy + y^2\cot\frac{B}{2} = \sin B\,. \tag{2}$$

If we eliminate y from (1) and (2), e.g., by expressing the trigonometric ratios in terms of $t = \tan(B/2)$, we obtain

$$x = \sqrt{\tfrac{1}{2}(1 - \tan\tfrac{B}{2})} \cdot (-\cos B + \cos\tfrac{B}{2} + \sin\tfrac{B}{2}).$$

Using numerical techniques, we find the approximation

$$x_{\max} \approx 0.409148 \text{ when } B \approx 67^\circ 03'.$$

Editor's comment.

Interested readers are invited to try to tie up the following loose ends in the solution to this problem:

i) Prove that, when A and r_1 are fixed, BC is least when $B = C$.

ii) If $M = x_{\max} \approx 0.409148$ then, for all triangles, $0 < x, y, z \le M$.

iii) It is probably too much to expect that the exact value of M can be found, but it may be possible to characterize geometrically the triangle in which $x = M$.

The following information may be useful:

It was recently proved in this journal [1980: 242] that, when A and the inradius r are fixed, BC is least when $B = C$.

Lob and Richmond (quoted by Goldberg in [1]) have shown that, for any triangle with inradius $r = 1$, the Malfatti radii are

$$r_1 = \frac{(1+v)(1+w)}{2(1+u)}, \quad r_2 = \frac{(1+w)(1+u)}{2(1+v)}, \quad r_3 = \frac{(1+u)(1+v)}{2(1+w)},$$

where $u = \tan(A/4)$, $v = \tan(B/4)$, $w = \tan(C/4)$.

The Malfatti Problem dates from 1803. Solutions and historical information about the problem can be found, e.g., in Coolidge [2], Dörrie [3], and F. G.-M. [4].

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1. Michael Goldberg, "On the Original Malfatti Problem," *Mathematics Magazine*, 40 (November 1967) 241-247.

2. Julian Lowell Coolidge, *A Treatise on the Circle and the Sphere*, Chelsea Pub. Co., Bronx, N.Y., 1971, pp. 174-179.

3. Heinrich Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, New York, 1965, pp. 147-151.

4. F. G.-M., *Exercices de Géométrie*, Quatrième édition, Mame et Fils, Tours, 1907, pp. 710-712.

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494. [1979: 292] *Proposed by Rufus Isaacs, Baltimore, Maryland.*

Let r_j, $j = 1, \ldots, k$, be the roots of a polynomial with integral coefficients and leading coefficient 1.

(a) For p a prime, show that

$$p \mid \sum_j (r_j^p - r_j).$$

(Note that the sum is an integer, since it is a symmetric polynomial of the roots, and hence a polynomial of the coefficients.)

This generalizes Fermat's Little Theorem.

(b) Prove or disprove the corresponding extension of Gauss's generalization of Fermat's Theorem: for any positive integer n,

$$n \mid \sum_j \left(\sum_{d \mid n} r_j^d \, \mu(n/d) \right),$$

where μ is the Möbius function.

Solution of part (a) by the proposer.

In the expansion of $(\sum_j r_j)^p$, each multinomial coefficient is of the form

$$\frac{p!}{i_1! \ldots i_k!}, \qquad \text{with } \sum_{j=1}^{k} i_j = p\,.$$

If some $i_j = p$, then the remaining i_j's are all zero and the corresponding coefficient is 1; in all other cases the coefficient is a multiple of p. Thus

$$\left(\sum_{j=1}^{k} r_j \right)^p = \sum_{j=1}^{k} r_j^p + pQ\,, \tag{1}$$

where Q is a polynomial in the r_j. Since the left member and the sum on the right in (1) are polynomials symmetric in the r_j, so is Q, and thus Q is an integer. Accordingly,

$$\left(\sum_{j=1}^{k} r_j \right)^p \equiv \sum_{j=1}^{k} r_j^p \pmod{p}. \tag{2}$$

By Fermat's Theorem, the left side of (2) is congruent to Σr_j modulo p, and our conclusion follows. □

When the polynomial of the proposal is of degree 1, say $x - a$, our conclusion is the well-known $p \mid a^p - a$.

Editor's comment.

The proposer wrote that he discovered the generalization in part (a) many years ago and took some pride in it until he showed it to the late A.A. Albert, who stared at it intently for ten seconds and then uttered one word: "Trivial!" Our other readers, none of whom submitted a solution, apparently thought otherwise.

Part (b) remains open. The proposer wrote that he verified it for a number of numerical cases.

* * *

REVIEW

Mathematical Solitaires and Games (Volume 1 in the series *Excursions in Recreational Mathematics)*, Benjamin L. Schwartz (Editor). Baywood Publishing Co., 120 Marine St., Farmingdale, N.Y. 11735, 1980, 160 pp., $6.00 (soft cover).

Subscribers and nonsubscribers to the *Journal of Recreational Mathematics* alike should welcome this first volume in the *Excursions in Recreational Mathematics* series. Nonsubscribers have a golden opportunity to sample a choice selection of articles from the *Journal*. These should whet their appetite for Recreational Mathematics. Each section is preceded by a lucid introduction to the subject matter. Subscribers will value the grouping together of articles on the same topic or similar topics for easy reference and comparison. The chronology of a problem and its solution (as far as that may go) may be traced. Moreover, some of the articles are from the earlier volumes of the *Journal* as well as from its predecessor, the *Recreational Mathematics Magazine*, and these would be difficult to find elsewhere.

Mathematical Solitaires and Games consists of 27 articles grouped under the following headings:

Section 1: Solitaire Games with Toys.

Section 2: Competitive Games.

Section 3: Solitaire Games.

Bonus Section: The Four-Color Problem.

The articles are of rather uneven quality, inevitable in a publication of this nature, but collectively the effect is quite inspiring. The publisher, the editor, and the Advisory Committee do maintain very stringent criteria in article selection. The line of division between Sections 1 and 3 may appear a little thin. The inclusion of the Bonus Section is an excellent concept.

A.C.-F. LIU,
University of Alberta.

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MAMA-THEMATICS

Mrs. Hamilton to son William: "And I thought you learned your times table correctly years ago."

Mrs. Occam to son William: "And I thought you were going to grow a beard and had thrown away your shaving equipment."

Mrs. Boole to son George: "I only hope that some day you will learn something about the simple process of thought."

HOWARD EVES

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