Mathematical Spectrum

2003/2004 Volume 36 Number 3

- Generating a conic
- River lengths
- We need some new functions
- A complex series of voyages of discovery

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Why mathematics?

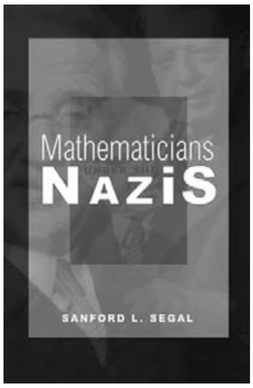
That is a question many young people ask as they decide which subjects to study at school or at university, or contemplate a career. The answers given will be as varied as are people themselves: enjoyment, ‘it’s my best subject’, it’s good for a career, and so on. But is there a higher reason inherent in the subject itself? Does it make you a ‘better person’?

It can be argued that studying any subject has a similar effect, whether in the arts or sciences. Education is in some way a civilising influence. You are perhaps less likely to be attacked in the street by a university graduate than by someone who dropped out of school at 16, although the reasons for this may be found further back. And the graduate may have more sophisticated ways of doing you ill!

But is there something special about the study of mathematics? After all, it is *the* subject which, in essence, is based on strict logical thought. The mathematical method is to start with a set of axioms, that one hopes are consistent, and make deductions from them, the theorems, using rules of logic. Can this method be applied in other areas? We are told that employers like to take on mathematics graduates, not just for their mathematical knowledge but also because of the processes of logical thought that they have acquired and which they can then apply to the diverse problems that confront humankind. There is surely some truth in this. But how much? And is it the whole truth?

I was pulled up sharply by a book that came for review, *Mathematicians Under the Nazis* by Sanford Segal. This is not a book for the general reader; it is too encyclopædic. It is a comprehensive study of the subject for the serious student of mathematical history. Professional university mathematicians will recognize many household names that are now part of the nomenclature of mathematics. But it is fascinating to see how mathematicians in Germany in the 1930s reacted to Nazism. Did their logical training help them to assess the situation they faced and to react to it? It would be hard to argue that from this study. Reactions were as varied as the people themselves. They were probably no worse than students of other disciplines. But were they any better? Probably not. There seems to be something deeper that is involved. I must leave you to decide what that might be. Perhaps we also need to ask another question: how would I have reacted?

Mathematicians Under the Nazis. By Stanford L. Segal. Princeton University Press, 2003. Pp. xxiv+530. Hardback £55.00 (ISBN 0-691-00451-X).



Large powers

Problem. How to evaluate r^x $(r, x \in \mathbb{R})$ where x is so large that it would cause an overflow error in a calculator. First consider

$$r^x = 10^y .$$

Now, taking logs to base 10, we have $x \log r = y$. Therefore,

$$r^x = 10^{x \log r} .$$

Now, let $f = \mathrm{frac}(x \log r)$ and $i = \mathrm{int}(x \log r)$. Then,

$$r^x = 10^f \cdot 10^i .$$

For example, evaluate $z = 5.35^{200}$. Here,

$$\begin{aligned} f &= \mathrm{frac}(200 \log 5.35) = 0.6707564 , \\ i &= \mathrm{int}(200 \log 5.35) = 145 , \\ 10^f &= 4.68550494 . \end{aligned}$$

Therefore, $z \approx 4.6855 \times 10^{145}$. Alternatively,

$$\begin{aligned} 5.35^{100} &= 6.84507486 \times 10^{72} , \\ 5.35^{200} &= 6.84507486^2 \times (10^{72})^2 \\ &= 46.85504984 \times 10^{144} . \end{aligned}$$

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Generating a Conic

GUIDO LASTERS and HUGO STAELENS

We start with a conic, say an ellipse, and a point P on the ellipse. Draw the tangent to the ellipse at P and the diameter OP, where O is the centre of the ellipse. Then every chord A_0A_1 parallel to the tangent at P is bisected by OP; see figure 1.

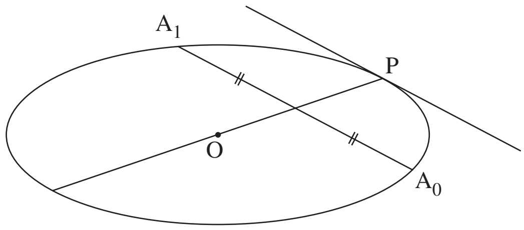

Figure 1.

The same is true of a hyperbola and also of a parabola, where now O has moved to infinity and the diameter OP becomes the straight line through P parallel to the axis of symmetry of the parabola; see figure 2

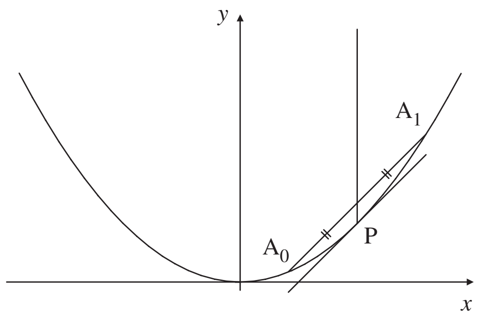

Figure 2.

To prove this property for a parabola, let the equation of the parabola be $y = ax^2$ and let P be the point (k, ak^2). Then the chord has slope $2ak$ and so has equation of the form

$$y = 2akx + c\,.$$

This meets the parabola in points A_0 and A_1 whose x-coordinates x_1, x_2 are solutions of the equation

$$2akx + c = ax^2$$

or

$$ax^2 - 2akx - c = 0\,,$$

so the sum of the roots is

$$x_1 + x_2 = 2k\,.$$

The midpoint of A_0A_1 thus has x-coordinate

$$\frac{x_1 + x_2}{2} = k$$

and so lies on the line $x = k$, as claimed. We leave readers to prove similar properties for an ellipse and a hyperbola — see problem 36.5 in the previous issue.

We can use this property to generate a sequence of points on a conic. We first designate two distinct directions by means of two straight lines m, n. Now designate a point O and distinct straight lines r and s through O, with r not parallel to m and s not parallel to n. Start with a point A_0 not on r and reflect it in r in a direction parallel to m to give the point A_1. Thus, A_0A_1 is parallel to m and r bisects A_0A_1. Now reflect A_1 in s in a direction parallel to n to give the point A_2. Now reflect A_2 in r in a direction parallel to m to give the point A_3. And so on. The sequence of points $A_0, A_1, A_2, A_3, \ldots$ will lie on a conic; see figure 3.

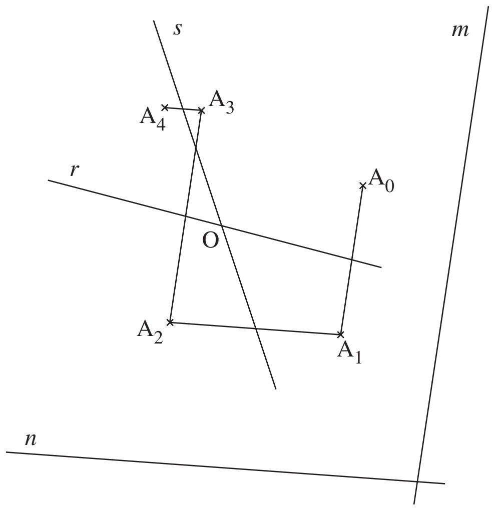

Figure 3.

When r and s are parallel, the conic generated in this way will be a parabola. We can determine its equation relative to suitable axes as follows. Choose axes so that the y-axis is parallel to r and s and lies midway between them, so that r and s will have equations

$$x = k \quad \text{and} \quad x = -k \quad \text{for some } k \neq 0\,.$$

The x-axis is chosen to be any line perpendicular to these. Denote by m, n the slopes of the two designated directions for the reflections, and let A_0 be the point (x_0, y_0). Let the parabola generated by our construction have equation

$$y = ax^2 + bx + c\,.$$

The straight line A_0A_1 has slope m and is parallel to the tangent to the parabola at the point with x-coordinate k, so that

$$m = 2ak + b\,.$$

The straight line A_1A_2 has slope n and is parallel to the tangent to the parabola at the point with x-coordinate $-k$, so that

$$n = -2ak + b\,.$$

Hence,

$$m - n = 4ak\,, \qquad m + n = 2b\,,$$

so that

$$a = \frac{m-n}{4k}\,, \qquad b = \frac{m+n}{2}\,.$$

The point A_0 lies on the parabola, so that

$$y_0 = ax_0^2 + by_0 + c\,,$$

and so

$$c = y_0 - ax_0^2 - bx_0\,.$$

Hence, the parabola has equation

$$y = ax^2 + bx + c$$

or

$$y = a(x^2 - x_0^2) + b(x - x_0) + y_0$$

or

$$y = \frac{m-n}{4k}(x^2 - x_0^2) + \frac{m+n}{2}(x - x_0) + y_0\,.$$

For example, when $k = 1$, $m = 2$, $n = -2$ and A_0 is the point $(0, 0)$, we obtain the equation $y = x^2$.

Similar calculations can be made for the ellipse and the hyperbola.

Guido Lasters *(Tienen, Belgium) teaches in Leuven and* ***Hugo Staelens*** *(Eeklo, Belgium) is retired.*

A Simple Proof of Euler's Inequality in Space

ZHANG YUN

Let R be the radius of the circumscribed sphere of a tetrahedron and let r be the radius of the inscribed sphere of the tetrahedron. Then Euler's famous inequality in space states that

$$R \geq 3r\,. \tag{1}$$

We give here a simple proof of this inequality.

Let O be the circumcentre of the tetrahedron $A_1A_2A_3A_4$. Let s_k $(k = 1, 2, 3, 4)$ denote the area of the face opposite the vertex A_k, let h_k denote the distance from A_k to its opposite face, and let d_k denote the distance from the point O to the face opposite A_k. Then

$$OA_k + d_k \geq h_k,$$

and so

$$R + d_k \geq h_k\,.$$

Thus,

$$s_kR + s_kd_k \geq s_kh_k\,.$$

Adding the four inequalities, we obtain that

$$\begin{aligned} R(s_1 + s_2 + s_3 + s_4) &+ s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4 \\ &\geq s_1h_1 + s_2h_2 + s_3h_3 + s_4h_4\,. \end{aligned}$$

Let V denote the volume of the tetrahedron $A_1A_2A_3A_4$. Then

$$V = \tfrac{1}{3}s_kh_k = \tfrac{1}{3}(s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4)\,,$$

so

$$R(s_1 + s_2 + s_3 + s_4) + 3V \geq 4 \times 3V\,,$$

from which it follows that

$$R(s_1 + s_2 + s_3 + s_4) \geq 9V\,.$$

Since

$$V = \frac{r}{3}(s_1 + s_2 + s_3 + s_4)\,,$$

this gives

$$R(s_1 + s_2 + s_3 + s_4) \geq 9 \times \frac{r}{3}(s_1 + s_2 + s_3 + s_4)\,.$$

Thus,

$$R \geq 3r\,,$$

so the inequality (1) is proved.

In two dimensions rather than three, if R is now the radius of the circumscribed circle of a triangle and r the radius of the inscribed circle, then, by a similar argument, $R \geq 2r$.

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Distribution of River Lengths and the Total Length of Rivers

A. TAN and W. SHENG

1. Introduction

Rivers are among mankind's most vital assets. The earliest civilizations were located near rivers. Today, the greatest cities, as well as the densest population centres, are situated in the major river basins. In high-school geography classes, students learn about the longest rivers in the world. In this article we examine the relationship between the ranks and the lengths of rivers.

Lists of the longest rivers are found in atlases and handbooks. They differ slightly from one another and are invariably incomplete. One of the longest lists (containing 75 longest rivers) is found in reference 1 and is reproduced here in table 1. The ranks and lengths of the rivers (in kilometres) and the continents in which they run are also given.

Table 1. Ranks and lengths of the longest rivers

Rank	River	Continent	Length (km)	Rank	River	Continent	Length (km)
1	Nile	Africa	6671	39	Ganges	Asia	2494
2	Amazon	South America	6300	40	Ural	Asia	2428
3	Chang Jiang	Asia	6276	41	Japura	South America	2414
4	Mississippi–Missouri	North America	6019	42	Arkansas	North America	2334
5	Ob–Irtysh	Asia	5411	43	Colorado	North America	2334
6	Yenisey–Angara	Asia	4989	44	Negro	South America	2253
7	Huang He	Asia	4630	45	Dnieper	Europe	2202
8	Amur–Argun	Asia	4416	46	Orange	Africa	2173
9	Lena	Asia	4400	47	Irrawaddy	Asia	2132
10	Congo	Africa	4374	48	Brazos	North America	2107
11	Mackenzie	North America	4241	49	Ohio–Allegheny	North America	2102
12	Mekong	Asia	4200	50	Kama	Europe	2031
13	Missouri–Red Rock	North America	4125	51	Don	Europe	1967
14	Niger	Africa	4101	52	Red	North America	1966
15	Plata–Parana	South America	3943	53	Columbia	North America	1953
16	Mississippi	North America	3778	54	Saskatchewan	North America	1939
17	Murray–Darling	Australia	3718	55	Peace–Finlay	North America	1923
18	Volga	Europe	3531	56	Tigris	Asia	1901
19	Madeira	South America	3240	57	Darling	Australia	1867
20	Purus	South America	3211	58	Angara	Asia	1827
21	Yukon	North America	3185	59	Sungari	Asia	1819
22	St Lawrence	North America	3058	60	Pechora	Europe	1809
23	Rio Grande	North America	3034	61	Snake	North America	1670
24	Syrdarya–Naryn	Asia	2992	62	Churchill	North America	1609
25	Sao Francisco	South America	2914	63	Pilcomayo	South America	1609
26	Indus	Asia	2897	64	Uruguay	South America	1600
27	Danube	Europe	2857	65	Platte–North Platte	North America	1593
28	Salween	Asia	2849	66	Ohio	North America	1578
29	Brahmaputra	Asia	2736	67	Magdalena	South America	1538
30	Euphrates	Asia	2736	68	Pecos	North America	1490
31	Tocantins	South America	2699	69	Oka	Europe	1477
32	Xi	Asia	2601	70	Canadian	North America	1458
33	Amudarya	Asia	2601	71	Colorado	North America	1439
34	Nelson–Saskatchewan	North America	2575	72	Dniester	Europe	1410
35	Orinoco	South America	2575	73	Fraser	North America	1369
36	Zambezi	Africa	2575	74	Rhine	Europe	1319
37	Paraguay	South America	2549	75	Northern Dvina	Europe	1302
38	Kolyma	Asia	2514				

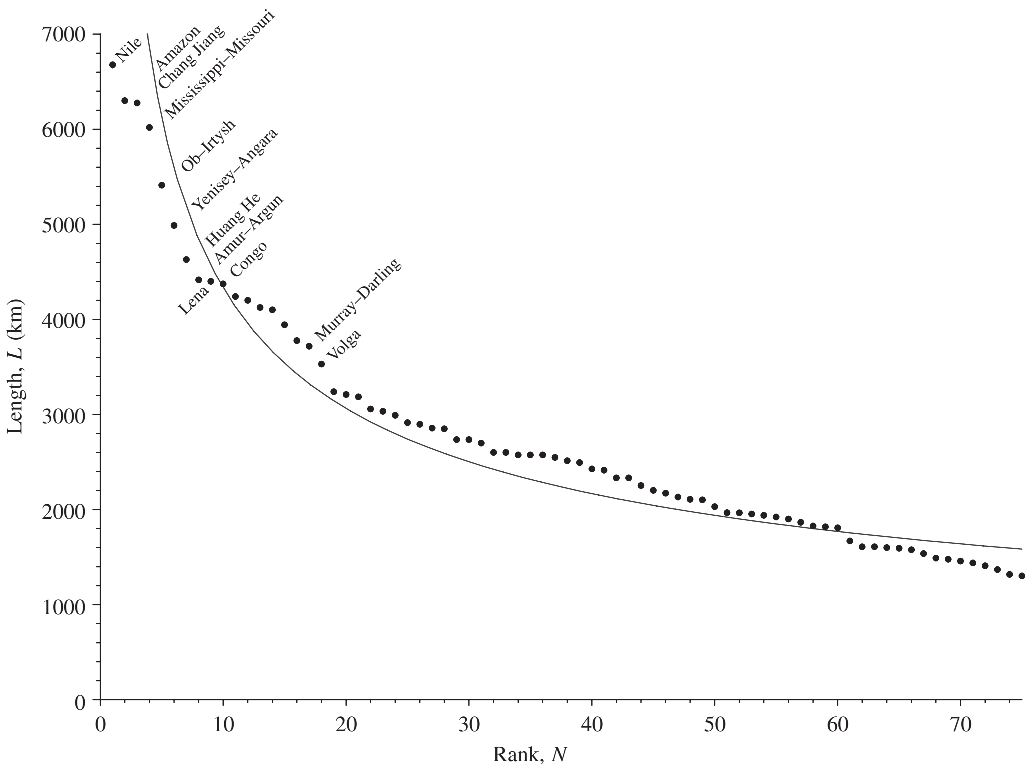

Figure 1. The power function fitted to ranked river lengths. The length $L = C/N^{\beta}$ of the river is plotted against its rank N using the estimates $\hat{C} = 13\,680.68$ and $\hat{\beta} = 0.499\,365\,3$.

2. Length as a power function of rank

A visual examination of the data indicates that the latter are likely to follow a power function ion law, where the length of a river L as a function of its rank N is given by

$$L = CN^{\gamma-1} \tag{1}$$

(see reference 2, pp. 161–164). Here γ is the shape parameter ($1 > \gamma > 0$) and $N \in \{1, 2, \ldots, n\}$, the number of data points ($n = 75$ in our case). It is convenient to rewrite (1) as

$$L = \frac{C}{N^{\beta}}, \tag{2}$$

with $\beta = 1 - \gamma < 1$. Taking the logarithm of both sides of (2), we get

$$\log C - \beta \log N = \log L. \tag{3}$$

Multiplying both sides of (3) by N then gives

$$N \log C - \beta N \log N = N \log L. \tag{4}$$

The normal equations are obtained by summing (3) and (4) over the n data points N:

$$n \log C - \beta \sum_{N=1}^{n} \log N = \sum_{N=1}^{n} \log L \tag{5}$$

and

$$\log C \sum_{N=1}^{n} N - \beta \sum_{N=1}^{n} N \log N = \sum_{N=1}^{n} N \log L. \tag{6}$$

For a least-squares fit of the data points, the constants C and β are obtained by elimination from (5) and (6):

$$\beta = \frac{\sum_{N=1}^{n} N \sum_{N=1}^{n} \log L - n \sum_{N=1}^{n} N \log L}{n \sum_{N=1}^{n} N \log N - \sum_{N=1}^{n} N \sum_{N=1}^{n} \log N} \tag{7}$$

and

$$C = \exp\left[\frac{\beta \sum_{N=1}^{n} \log N + \sum_{N=1}^{n} \log L}{n}\right]. \tag{8}$$

Applying (7) and (8) to the data from table 1 gives the estimates $\hat{C} = 13\,680.68$ and $\hat{\beta} = 0.499\,365\,3$. Figure 1 shows the power function fitted to river lengths thus obtained. It can be observed that the function overestimates the river lengths for $N = 1, \ldots, 9$, underestimates them for $N = 11, \ldots, 50$ and again overestimates them for $N > 60$ near the tail end of the curve. A plausible reason for the overestimation of the first nine lengths is that the lengths of the longest rivers are limited by the sizes of the continents in which they run. The areas of the continents (S) are taken from reference 3 and entered in table 2. Also shown in the table are $\sqrt{S}$ (which is a measure of the linear dimensions of the continents) and the lengths of the longest river on each continent. The fact that the two are comparable indicates that the lengths of the longest rivers are indeed restricted by the sizes of the continents.

Table 2. The areas of the continents compared with the lengths of their longest rivers and how many of the longest rivers are in each continent.

Continent	Area, S (km^2)	$\sqrt{S}$ (km)	Length of longest river (km)	Number of the longest 10 rivers in continent	Number of the longest 75 rivers in continent
Asia	43 608 000	6604	6276	6	21
Africa	30 335 000	5508	6671	2	5
North America	25 349 000	5035	6019	1	24
South America	17 611 000	4197	6300	1	13
Europe	10 498 000	3240	3531	0	10
Australasia	8 923 000	2987	3718	0	2

Table 2 shows that of the 10 longest rivers in the world, 6 (or 60%) are in Asia (the largest continent), 2 are in Africa (the second largest continent) with one each in North and South America (the third and fourth largest continents). The longest rivers of the smallest continents, Australasia and Europe, rank 17th and 18th respectively. Interesting too is the fact that, of the 75 longest rivers, 5 are in Africa and only 2 are in Australasia. Australia is by far the driest continent, and northern Africa has the largest desert in the world.

The total length of the world's 75 longest rivers according to the power function can be approximated by integrating (2) from 1 to n, instead of summing from 1 to n:

$$L_n \approx \int_1^n \frac{C}{N^\beta}\,\mathrm{d}N = \frac{C}{1-\beta}[n^{1-\beta} - 1]\,. \tag{9}$$

Upon substituting our data, we obtain $L_{75} \approx 209\,979$ km. This compares very favourably with the actual value of 209 957 km obtained by summing the lengths in table 1.

3. Length as an exponential function of rank

The river length L may also be expressed as an exponential function of the rank N:

$$L = A\mathrm{e}^{-\alpha N}\,, \tag{10}$$

where $N \in \{1, 2, \ldots\}$.

The two constants A and α can be estimated as follows: take logarithms of both sides of (10) and sum over the n (= 75) data points; then obtain A and α from the normal equations by elimination:

$$\log A = \frac{\sum_{N=1}^n N \sum_{N=1}^n N \log L - \sum_{N=1}^n \log L \sum_{N=1}^n N^2}{\left(\sum_{N=1}^n N\right)^2 - n\sum_{N=1}^n N^2}$$

and

$$\alpha = \frac{n\sum_{N=1}^n N\log L - \sum_{N=1}^n \log L \sum_{N=1}^n N}{(\sum_{N=1}^n N)^2 - n\sum_{N=1}^n N^2}\,.$$

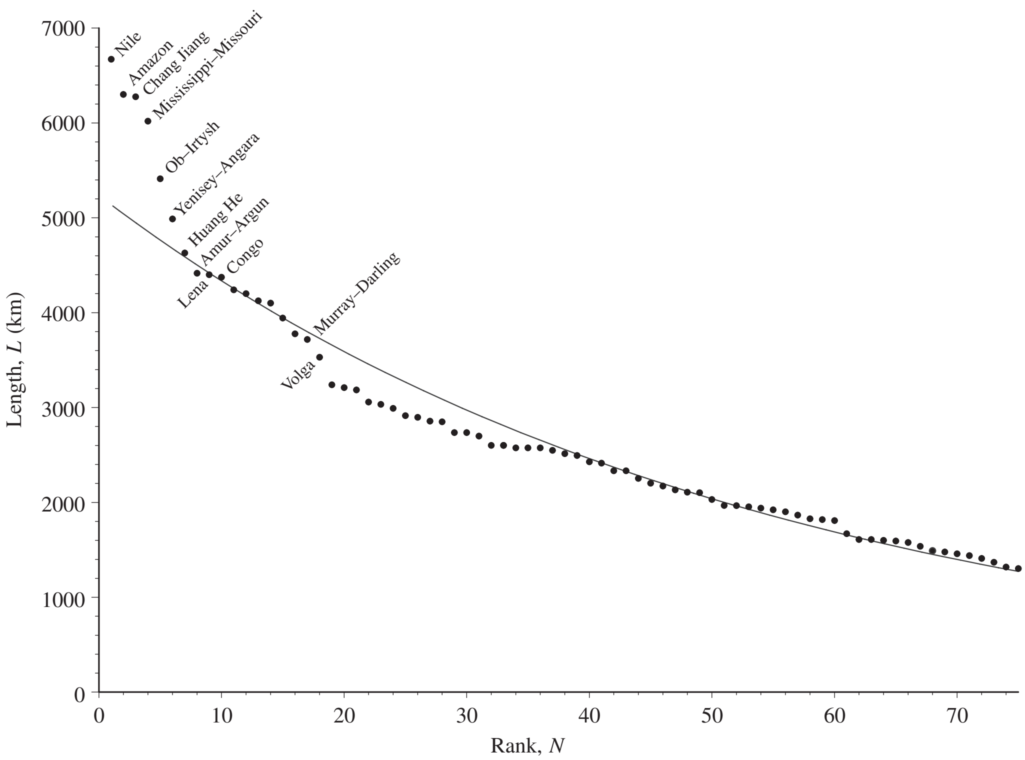

Figure 2. The exponential function fitted to ranked river lengths. The length $L = A\mathrm{e}^{\alpha N}$ of the river is plotted against its rank N using the estimates $\hat{A} = 5235.338$ and $\hat{\alpha} = 0.018\,854\,7$.

Substituting the data from table 1 gives the estimates $\hat{A} = 5235.338$ and $\hat{\alpha} = 0.018\,854\,7$. Figure 2 shows the exponential function fitted to river lengths. This underestimates the river lengths for $N = 1, \ldots, 6$ and overestimates them for $N = 18, \ldots, 36$.

For the exponential function, the total length of the 75 longest rivers can be approximated by the integral

$$L_n \approx \int_1^n A\mathrm{e}^{-\alpha N}\mathrm{d}N = \frac{A}{\alpha}[\mathrm{e}^{-\alpha} - \mathrm{e}^{-n\alpha}] .$$

Upon substitution, we get $L_{75} = 204\,951$ km, which is not too distant from the actual value of 209 957 km. The exponential function furnishes a rough finite value for the total length of all rivers, which is approximated by the integral

$$L_\infty \approx \int_1^\infty A\mathrm{e}^{-\alpha N}\mathrm{d}N = \frac{A}{\alpha}\mathrm{e}^{-\alpha} .$$

If the exponential assumption holds, the total length of all rivers is $L_\infty \approx 272\,440$ km. If, on the other hand, the power function assumption is correct, $L_n \to \infty$ and $n \to \infty$, as can be seen from the approximation (9).

4. Summary

In this article, river length as a function of rank is investigated using the power function and the familiar exponential function. The former gives better agreement for lower values of rank, whereas the latter gives better agreement for larger rank. The exponential function furnishes a finite approximation for the total length of all rivers, while the power function gives an infinite total length.

References

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3. *The New York Times Atlas of the World* (John Bartholomew and Times Books, New York, 1986).

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W. Sheng *recently obtained his Master's degree in physics from Alabama A & M University.*

We Need Some New Functions

MARGEN ÇUKO and PAUL BELCHER

We know that

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x ,$$

but what is

$$\int \frac{1}{\sqrt{1-x^4}}\,\mathrm{d}x$$

equal to? We cannot solve this integral with elementary functions, so let us try and generalize circular functions.

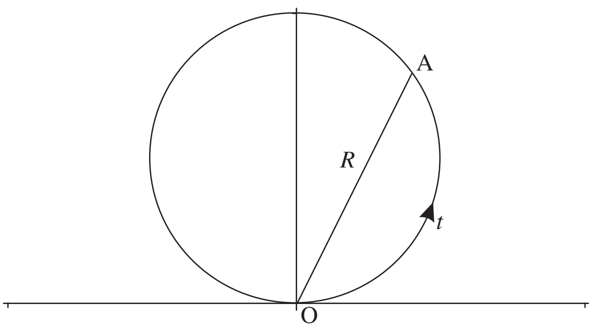

Figure 1.

1. The circular function

Consider the circle given in polar co-ordinates by the equation $r = \sin\theta$ (where r is allowed to be negative). This is the circle with centre $(0, \frac{1}{2})$ and radius $\frac{1}{2}$. The length t of the arc OA of the circle in figure 1 given by the arc-length formula $\int \sqrt{1 + r^2(\mathrm{d}\theta/\mathrm{d}r)^2}\,\mathrm{d}r$ is

$$t = \int_0^R \sqrt{1 + \frac{r^2}{\cos^2\theta}}\,\mathrm{d}r = \int_0^R \frac{1}{\sqrt{1-r^2}}\,\mathrm{d}r$$
$$= [\arcsin r]_0^R = \arcsin R ,$$

where R is the length of the chord OA. Since $t = \arcsin R$, $R = \sin t$. The formula $t = \arcsin R$ is originally defined for $R \in [0, 1]$ and $t \in [0, \pi/2]$ and then extended to $R \in [-1, 1]$ and $t \in [-\pi/2, \pi/2]$ by taking $\arcsin R$ as an odd function and using $\arcsin(-R) = -\arcsin R$. We consider the arc length to be positive when proceeding anticlockwise from O and negative when proceeding clockwise from O.

The domain of the function sin can then be extended to $[-\pi, \pi]$ using the symmetry rules

$$\sin\left(\frac{\pi}{2} + t\right) = \sin\left(\frac{\pi}{2} - t\right)$$

and

$$\sin\left(-\frac{\pi}{2}-t\right)=\sin\left(-\frac{\pi}{2}+t\right)\quad \text{for } t\in\left[0,\frac{\pi}{2}\right]$$

and then further extended to $\mathbb{R}$ using a periodicity of 2π. Basic graphs of $\sin t$ and $\arcsin R$ are given in figure 2.

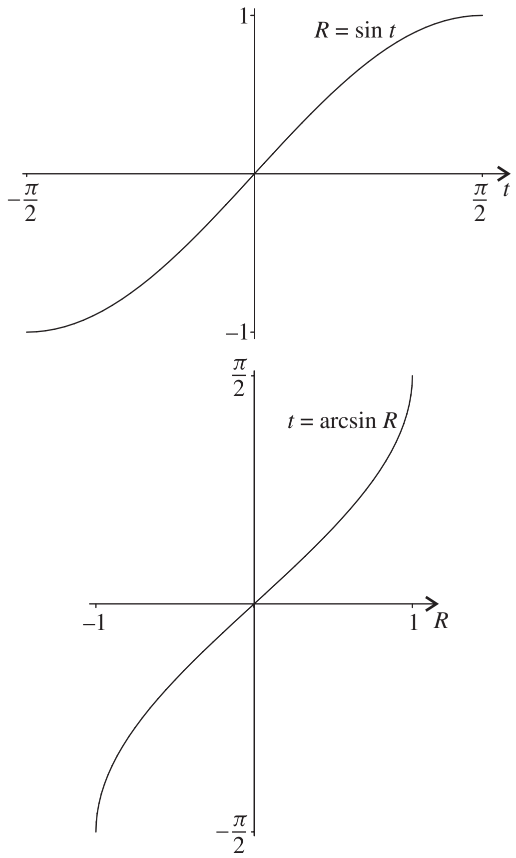

Figure 2.

2. The lemniscate function

Now consider in an analogous way the lemniscate curve $r^2 = \sin 2\theta$. This curve is undefined for $\theta \in (\pi/2, \pi)$ and $\theta \in (3\pi/2, 2\pi)$, whereas for $\theta \in [0, \pi/2]$ and $\theta \in [\pi, 3\pi/2]$ there are two values for r, a positive and a negative one. The length t of the arc OA of the lemniscate curve in figure 3 is given by

$$t=\int_0^R \sqrt{1+r^2\frac{r^2}{\cos^2 2\theta}}\,\mathrm{d}r=\int_0^R \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r\,.$$

If we define a new function arcsinl by

$$\operatorname{arcsinl} R=\int_0^R \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r\,,$$

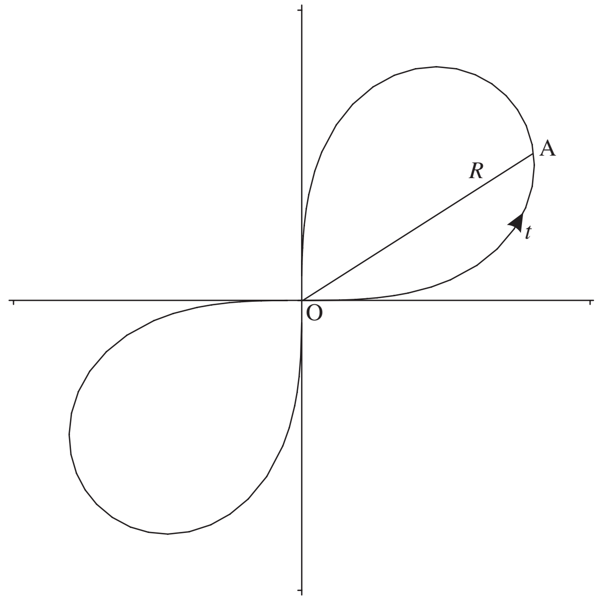

Figure 3.

then $t = \operatorname{arcsinl} R$ for $R \in [-1, 1]$ and $t \in [-w/2, w/2]$, where w is the length of the lemniscate leaf in the first quadrant. Again, initially this definition holds for $R \in [0, 1]$ and $t \in [0, w/2]$ and can then be extended to the larger domain and range by making it an odd function. Denoting the inverse function of arcsinl by sinl, for the sine lemniscate we have $R = \operatorname{sinl} t$. Now we consider the period of sinl. For the usual, circular, sine the period is 2π where π is the circumference of the circle in figure 1 that is traced out twice as θ goes from 0 to 2π. As w is the arc length of one of the lemniscate leaves, we have

$$\tfrac{1}{2}w=\int_0^1 \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r$$

(which is approximately 1.311) and the length of the whole lemniscate will be $4w$ as θ goes from 0 to 2π (since the shape in figure 3 is traced out twice). The function sinl can then have its domain extended by the symmetry rules

$$\operatorname{sinl}\left(\frac{w}{2}+t\right)=\operatorname{sinl}\left(\frac{w}{2}-t\right)$$

and

$$\operatorname{sinl}\left(-\frac{w}{2}-t\right)=\operatorname{sinl}\left(-\frac{w}{2}+t\right)\quad \text{for } t\in\left[0,\frac{w}{2}\right]$$

and then extended again to make it periodic with period $2w$.

In the circle, R is negative when $-\pi/2 < t < 0$; in the lemniscate, R is negative when $-w/2 < t < 0$. Using this symmetry we have the following equalities:

$$\begin{aligned}\operatorname{sinl}(w-t)&=\operatorname{sinl}(t)\,,\\ \operatorname{sinl}(w+t)&=-\operatorname{sinl}(t)\,,\\ \operatorname{sinl}(2w-t)&=\operatorname{sinl}(-t)=-\operatorname{sinl}(t)\,.\end{aligned}$$

Since $\cos t = \sin(\pi/2 - t)$, by analogy we will define the cosine lemniscate function cosl by

$$\operatorname{cosl}(t)=\operatorname{sinl}\left(\frac{w}{2}-t\right).$$

This gives

$$\begin{aligned}\operatorname{cosl}(w+t) &= -\operatorname{cosl}(t)\,,\\ \operatorname{cosl}(w-t) &= -\operatorname{cosl}(t)\,,\\ \operatorname{cosl}(2w-t) &= \operatorname{cosl}(-t) = \operatorname{cosl}(t)\,.\end{aligned}$$

If we set $u = \operatorname{cosl}(s)$ for some s, then $u = \operatorname{sinl}(w/2 - s)$. Using the notation $s = \operatorname{arccosl}(u)$, as $w/2 - s = \operatorname{arcsinl} u$, we then have $\operatorname{arccosl}(u) = w/2 - \operatorname{arcsinl}(u)$. So

$$\begin{aligned}\operatorname{arccosl}(u) &= \int_0^1 \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r - \int_0^u \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r\\ &= \int_u^1 \frac{1}{\sqrt{1-r^4}}\,\mathrm{d}r\,. \qquad (1)\end{aligned}$$

Now we attempt to find an analogue of the equality $\sin^2 t + \cos^2 t = 1$. Let us apply the substitution given by $r^2 + y^2 + r^2y^2 = 1$ to the integral $\int_0^R \left(1/\sqrt{1-r^4}\right)\mathrm{d}r$ (which, by definition, is $\operatorname{arcsinl} R = t$), to give

$$\int_1^{\sqrt{(1-R^2)/(1+R^2)}} \frac{1}{\sqrt{1-\left((1-y^2)/(1+y^2)\right)^2}} \times \frac{-\left(y + y\left((1-y^2)/(1+y^2)\right)\right)}{\sqrt{(1-y^2)/(1+y^2)}\,(1+y^2)}\,\mathrm{d}y\,.$$

This simplifies to

$$\int_{\sqrt{(1-R^2)/(1+R^2)}}^1 \frac{1}{\sqrt{1-y^4}}\,\mathrm{d}y\,,$$

which, from (1), is equal to

$$\operatorname{arccosl}\left(\sqrt{\frac{1-R^2}{1+R^2}}\right).$$

So

$$t = \operatorname{arcsinl}(R) = \operatorname{arccosl}\sqrt{\frac{1-R^2}{1+R^2}}$$

and, hence,

$$R = \operatorname{sinl} t \quad \text{and} \quad \sqrt{\frac{1-R^2}{1+R^2}} = \operatorname{cosl} t\,,$$

giving the identity

$$\operatorname{cosl}^2(t) + \operatorname{sinl}^2(t) + \operatorname{cosl}^2(t)\operatorname{sinl}^2(t) = 1\,. \qquad (2)$$

Since $\operatorname{arcsinl} x = \int_0^x \left(1/\sqrt{1-r^4}\right)\mathrm{d}r$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsinl} x = \frac{1}{\sqrt{1-x^4}}\,.$$

Also, as $\operatorname{arccosl} x = -\int_1^x \left(1/\sqrt{1-r^4}\right)\mathrm{d}r$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccosl} x = -\frac{1}{\sqrt{1-x^4}}\,.$$

Now let $y = \operatorname{sinl} x$. Then $x = \operatorname{arcsinl} y$ and differentiating with respect to y gives

$$1 = \frac{1}{\sqrt{1-y^4}}\,\frac{\mathrm{d}y}{\mathrm{d}x}\,.$$

Hence $\mathrm{d}y/\mathrm{d}x = \sqrt{1-y^4}$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sinl} x &= \sqrt{1-\operatorname{sinl}^4 x} = \frac{2\operatorname{cosl} x}{1+\operatorname{cosl}^2 x}\\ &= \operatorname{cosl} x\,(1+\operatorname{sinl}^2 x)\,.\end{aligned}$$

Similarly, giving the correct sign,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosl} x &= -\sqrt{1-\operatorname{cosl}^4 x} = \frac{-2\operatorname{sinl} x}{1+\operatorname{sinl}^2 x}\\ &= -\operatorname{sinl} x(1+\operatorname{cosl}^2 x)\,.\end{aligned}$$

Now $\operatorname{sinl} x$ and $\operatorname{cosl} x$ are periodic functions, with period $2w$, from $\mathbb{R}$ to $[-1, 1]$. So, for $\operatorname{arcsinl} t$ and $\operatorname{arccosl} t$ to be functions, we need to choose principal values for them. For arcsinl these are between $-w/2$ and $w/2$ and for arccosl these are between 0 and w. So

$$\begin{aligned}\operatorname{arcsinl}\colon [-1,1] &\to \left[-\frac{w}{2}, \frac{w}{2}\right],\\ \operatorname{arccosl}\colon [-1,1] &\to [0, w]\,.\end{aligned}$$

The graphs of $R = \operatorname{sinl} t$ for $t \in [0, w/2]$ and $t = \operatorname{arcsinl} R$ for $R \in [0, 1]$ are given in figure 4.

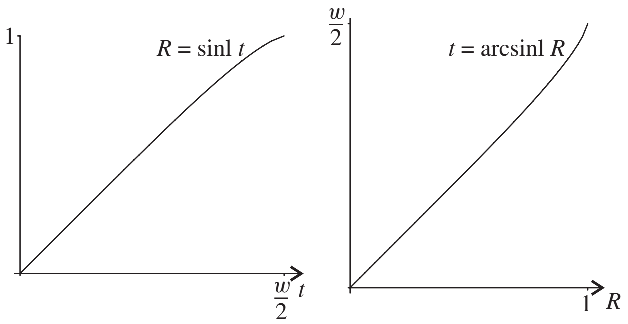

Figure 4.

3. The trefoil function

It now seems natural to look at the 'rose with three petals' given by $r^3 = \sin 3\theta$ (where again we will allow r to be negative).

The length t of the arc OA in figure 5 is given by

$$\int_0^R \sqrt{1 + r^2\frac{r^4}{\cos^2 3\theta}}\,\mathrm{d}r = \int_0^R \frac{1}{\sqrt{1-r^6}}\,\mathrm{d}r\,.$$

We will call this expression $\arcsin_3 R$. Denoting the inverse of this function by $\sin_3$, we have $R = \sin_3 t$. Again we have $R \in [-1, 1]$ after we have extended from $R \in [0, 1]$

for an odd function and negative and positive arc lengths for t. Let w_3 be the length of the leaf of the trefoil in the first quadrant. Then

$$\tfrac{1}{2}w_3 = \int_0^1 \frac{1}{\sqrt{1-r^6}}\,\mathrm{d}r$$

(which is approximately 1.214) and the length of the whole curve will be $6w_3$ as θ goes from 0 to 2π.

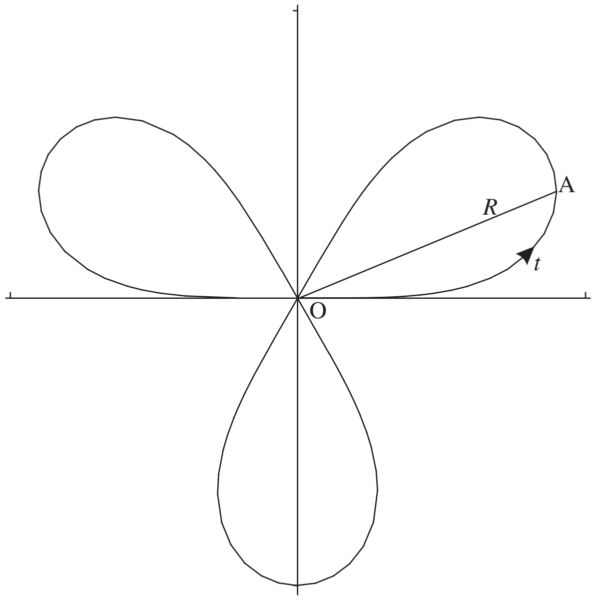

Figure 5.

The function $\sin_3$ can then have its domain extended by the symmetry rules

$$\sin_3\left(\frac{w_3}{2}+t\right) = \sin_3\left(\frac{w_3}{2}-t\right)$$

and

$$\sin_3\left(-\frac{w_3}{2}-t\right) = \sin_3\left(-\frac{w_3}{2}+t\right) \quad \text{for } t \in \left[0, \frac{w_3}{2}\right]$$

and then extended again to make it periodic with period $2w_3$. With this and symmetry, we have

$$\begin{aligned}
\sin_3(w_3-t) &= \sin_3 t\,,\\
\sin_3(w_3+t) &= -\sin_3(t)\,,\\
\sin_3(2w_3-t) &= \sin_3(-t) = -\sin_3(t)\,.
\end{aligned}$$

We now define $\cos_3(t) = \sin_3(w_3/2-t)$, giving

$$\begin{aligned}
\cos_3(w_3+t) &= -\cos_3(t)\\
\cos_3(w_3-t) &= -\cos_3(t)\,,\\
\cos_3(2w_3-t) &= \cos_3(-t) = \cos_3(t)\,.
\end{aligned}$$

Since $\arcsin_3 x = \int_0^x \left(1/\sqrt{1-r^6}\right)\mathrm{d}r$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin_3 x = \frac{1}{\sqrt{1-x^6}}\,.$$

If we set $u = \cos_3(s)$ for some s, then $u = \sin_3(w_3/2-s)$. Hence $w_3/2 - s = \arcsin_3 u$ and $\arccos_3(u) = w_3/2 - \arcsin_3(u)$. So

$$\begin{aligned}
\arccos_3 u &= \int_0^1 \frac{1}{\sqrt{1-r^6}}\,\mathrm{d}r - \int_0^u \frac{1}{\sqrt{1-r^6}}\,\mathrm{d}r\\
&= \int_u^1 \frac{1}{\sqrt{1-r^6}}\,\mathrm{d}r\,. \qquad (3)
\end{aligned}$$

Then $\arccos_3 u = -\int_1^u \left(1/\sqrt{1-r^6}\right)\mathrm{d}r$ gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos_3 x = \frac{-1}{\sqrt{1-x^6}}\,.$$

Now $\sin_3 x$ and $\cos_3 x$ are periodic functions, with period $2w_3$, from $\mathbb{R}$ to $[-1,1]$. So, for $\arcsin_3 t$ and $\arccos_3 t$ to be functions, we need to choose principal values for them. For $\arcsin_3$ these are between $-w_3/2$ and $w_3/2$; for $\arccos_3$ they are between 0 and w_3. So

$$\begin{aligned}
\arcsin_3 &: [-1,1] \to \left[-\frac{w_3}{2}, \frac{w_3}{2}\right],\\
\arccos_3 &: [-1,1] \to [0, w_3]\,.
\end{aligned}$$

To find the connection between $\sin_3 t$ and $\cos_3 t$, we apply the substitution $r^2 + y^2 + 2r^2y^2 = 1$ to the integral $\int_0^R \left(1/\sqrt{1-r^6}\right)\mathrm{d}r$ (which is $\arcsin_3 R = t$), to give

$$\int_1^{\sqrt{(1-R^2)/(1+2R^2)}} \frac{1}{\sqrt{1-\left((1-y^2)/(1+2y^2)\right)^3}} \times \frac{-\left(y(1+2y^2)+2y(1-y^2)\right)}{\sqrt{(1-y^2)/(1+2y^2)}\,(1+2y^2)^2}\,\mathrm{d}y\,.$$

This simplifies to

$$\int_{\sqrt{(1-R^2)/(1+2R^2)}}^1 \frac{1}{\sqrt{1-y^6}}\,\mathrm{d}y\,,$$

which, using (3), is equal to $\arccos_3 \sqrt{(1-R^2)/(1+2R^2)}$. So

$$t = \arcsin_3 R = \arccos_3 \sqrt{\frac{1-R^2}{1+2R^2}}\,.$$

Now, $R = \sin_3 t$ and $\sqrt{(1-R^2)/(1+2R^2)} = \cos_3 t$, giving

$$\cos_3^2 t + \sin_3^2 t + 2\sin_3^2 t \cos_3^2 t = 1\,. \qquad (4)$$

Let $y = \sin_3 x$. Then $x = \arcsin_3 y$ and differentiating with respect to x gives

$$1 = \frac{1}{\sqrt{1-y^6}}\,\frac{\mathrm{d}y}{\mathrm{d}x}\,.$$

Hence, $\mathrm{d}y/\mathrm{d}x = \sqrt{1-y^6}$, that is,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin_3 x &= \sqrt{1-\sin_3^6 x}\\ &= \cos_3 x\sqrt{(1+2\sin_3^2 x)(1+\sin_3^2 x+\sin_3^4 x)}\\ &= \frac{3\cos_3 x\sqrt{1+\cos_2^3 x+\cos_3^4 x}}{\sqrt{(1+2\cos_3^2 x)^3}}\,,\end{aligned}$$

giving the correct sign.

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cos_3 x &= -\sqrt{1-\cos_3^6 x}\\ &= -\sin_3 x\sqrt{(1+2\cos_3^2 x)(1+\cos_3^2 x+\cos_3^4 x)}\\ &= \frac{-3\sin_3 x\sqrt{1+\sin_3^2 x+\sin_3^4 x}}{\sqrt{(1+2\sin_3^2 x)^3}}\,.\end{aligned}$$

4. The general case

To generalise, we will look at the 'rose with n petals', where $n \in \mathbb{N}^+$, given in polar co-ordinates by the equation $r^n = \sin n\theta$ (again allowing r to be negative).

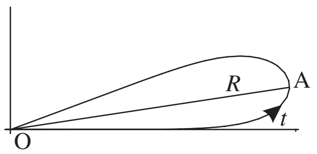

Figure 6.

Figure 6 shows one of the petals in this general case. The length t of the arc OA is given by the arc-length formula as $\int\sqrt{1+r^2(\mathrm{d}\theta/\mathrm{d}r)^2}\,\mathrm{d}r$. So

$$t = \int_0^R \sqrt{1+r^2\frac{r^{2n-2}}{\cos^2 n\theta}}\,\mathrm{d}r = \int_0^R \frac{1}{\sqrt{1-r^{2n}}}\,\mathrm{d}r\,.$$

Let us call the expression on the right-hand side $\arcsin_n R$. We will denote the inverse function of $\arcsin_n$ by $\sin_n$, so $R = \sin_n t$. Again after extending for an odd function we have $R \in [-1, 1]$ and negative and positive arc lengths for t. We now consider the period of the function $\sin_n$. Let w_n be the arc length of one petal. Then

$$\tfrac{1}{2}w_n = \int_0^1 \frac{1}{\sqrt{1-r^{2n}}}\,\mathrm{d}r\,.$$

The length of the whole curve as θ goes from 0 to 2π will be $2nw_n$ since each petal is traced out twice (allowing for positive and negative values of r when taking the root for n even). The function $\sin_n$ can then have its domain extended by the symmetry rules

$$\sin_n\left(\frac{w_n}{2}+t\right) = \sin_n\left(\frac{w_n}{2}-t\right)$$

and

$$\sin_n\left(-\frac{w_n}{2}-t\right) = \sin_n\left(-\frac{w_n}{2}+t\right) \quad \text{for } t \in \left[0, \frac{w_n}{2}\right]$$

and then extended again to make it periodic with period $2w_n$. Using this and symmetry, we have the following equalities:

$$\begin{aligned}\sin_n(w-t) &= \sin_n(t)\,,\\ \sin_n(w+t) &= -\sin_n(t)\,,\\ \sin_n(2w-t) &= \sin_n(-t) = -\sin_n(t)\,.\end{aligned}$$

We now define $\cos_n(t) = \sin_n(w_n/2 - t)$, giving

$$\begin{aligned}\cos_n(w_n+t) &= -\cos_n(t)\,,\\ \cos(w_n-t) &= -\cos_n(t)\,,\\ \cos_n(2w_n-t) &= \cos_n(-t) = \cos_n(t)\,.\end{aligned}$$

Since $\arcsin_n x = \int_0^x \left(1/\sqrt{1-r^{2n}}\right)\mathrm{d}r$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin_n x = \frac{1}{\sqrt{1-x^{2n}}}\,.$$

If $u = \cos_n(s)$, then $u = \sin_n(w_n/2 - s)$. So $w_n/2 - s = \arcsin_n u$, giving $\arccos_n u = w_n/2 - \arcsin_n u$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos_n x = -\frac{1}{\sqrt{1-x^{2n}}}$$

and

$$\begin{aligned}\arccos_n u &= \int_0^1 \frac{1}{\sqrt{1-r^{2n}}}\,\mathrm{d}r - \int_0^u \frac{1}{\sqrt{1-r^{2n}}}\,\mathrm{d}r\\ &= \int_u^1 \frac{1}{\sqrt{1-r^{2n}}}\,\mathrm{d}r\,.\end{aligned}$$

Now $\sin_n x$ and $\cos_n x$ are periodic functions, with period $2w_n$, from $\mathbb{R}$ to $[-1, 1]$. So, for $\arcsin_n t$ and $\arccos_n t$ to be functions we need to choose principal values for them. For $\arcsin_n t$ these are between $-w_n/2$ and $w_n/2$ and for $\arccos_n t$ they are between 0 and w_n. So

$$\begin{aligned}\arcsin_n &: [-1,1] \to \left[-\frac{w_n}{2}, \frac{w_n}{2}\right],\\ \arccos_n &: [-1,1] \to [0, w_n]\,.\end{aligned}$$

The graphs of $\sin_n x$ and $\cos_n x$ are similar to the graphs of $\sin x$ and $\cos x$, but as n increases the curves tend to the section of the straight line $y = x$ from $(0, 0)$ to $(1, 1)$.

It would be natural to conjecture that $\cos_n^2 t + \sin_n^2 t + (n-1)\sin_n^2 t\cos_n^2 t = 1$, but a numerical counter-example with $n = 4$ shows that this is not the case. For example, $\arcsin_4(0.5) = 0.50011$, so $\sin_4(0.50011) = 0.5$, $w_4/2 = 1.16359$, $\cos_4(0.50011) = \sin_4(0.66348) = 0.66210$, and $(0.66210)^2 + (0.5)^2 + 3(0.66210)^2(0.5)^2 = 1.017$, not 1. We do have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin_n x &= \sqrt{1-\sin_n^{2n} x}\,,\\ \frac{\mathrm{d}}{\mathrm{d}x}\cos_n x &= -\sqrt{1-\cos_n^{2n} x}\end{aligned}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin_n x = -n\sin_n^{2n-1} x\,,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos_n x = -n\cos_n^{2n-1} x\,.$$

This idea could be continued for other equations in polar form. For example, we could even apply it to the equation $r = \mathrm{sinl}\,\theta$.

These techniques give integrals that cannot be solved using elementary functions.

*This is an amended version of the extended essay that **Margen Çuko**, an 18-year-old student from Albania, produced for the International Baccalaureate. **Paul Belcher** was his supervisor at Atlantic College.*

More on Sums of Consecutive Numbers

PHILIP MAYNARD and YINGHUI ZHOU

We look at the number of different ways in which an integer can be written as the sum of consecutive integers.

1. Introduction

It was shown in references 1 and 2 that a number n can be expressed as the sum of two or more consecutive numbers if and only if n is not a power of 2. We call such numbers *cs-numbers* and an expression for n as a sum of consecutive numbers a *representation*. Some numbers allow more representations than others; for example, 6 can be written in just one way, $6 = 1+2+3$, but $9 = 2+3+4$ and $9 = 4+5$. In this article we determine the number of different representations that a cs-number can have. However, we begin by considering this problem in a different form. A positive integer $n \in \mathbb{N}$ is called a *ces-number* if it can be written as the sum of two or more consecutive even numbers. For example, the smallest ces-number is $6 = 2 + 4$. In the next section we shall consider properties of such numbers.

2. Sums of consecutive even numbers

We begin by stating a simple necessary and sufficient condition for a number to be a ces-number.

Lemma 1. *An integer $n \in \mathbb{N}$ is a ces-number if and only if $n = (i+1)(2t+i)$ for some $t, i \in \mathbb{N}$.*

Proof. An integer is a ces-number if and only if it can be expressed as $n = 2t + (2t+2) + (2t+4) + \cdots + (2t+2i)$ for some $i, t \in \mathbb{N}$. Now, we can write the right-hand side of this expression as $2((i+1)t + 1 + 2 + \cdots + i)$. Applying the well-known formula $\sum_{j=1}^{i} j = i(i+1)/2$, we have $n = (i+1)(2t+i)$.

Let $f(n)$ be the number of different ways in which n can be expressed as the sum of two or more consecutive even numbers. So $f(n) \neq 0$ if and only if n is a ces-number. Clearly, $f(n) = 0$ if n is odd. More precisely, we have the following lemma.

Lemma 2. *Let $n \in \mathbb{N}$. Then $f(n) = 0$ if and only if n is odd or $n = 2^r$ for some $r \in \mathbb{N}$.*

Proof. We first show that, if n is odd or $n = 2^r$, then $f(n) = 0$. Equivalently, we show that, if $f(n) \neq 0$, then $n \neq 2^r$ and n is even. From lemma 1, we know that $f(n) \neq 0$ if and only if $n = (i+1)(2t+i)$ for some $i, t \in \mathbb{N}$. If i is odd, then $i+1$ is even and $2t+i$ is odd, so n is the product of an odd number (> 1) and an even number. The same is true if i is even. Thus, since n is a multiple of an even number, it too must be even. But, since n is also a multiple of an odd number, it cannot be of the form $n = 2^r$ for some $r \in \mathbb{N}$.

Next we show that, if $f(n) = 0$, then n is odd or $n = 2^r$. In fact, we show that, if $n = ab$ where $a > 1$ is odd and b is even, then $f(n) \neq 0$ (which is clearly equivalent). There are two cases to consider. Firstly, assume that $a > b$. Then take $i = b - 1$ and $t = (a - b + 1)/2$. It is easy to see that i and t are positive integers and that $n = (i+1)(2t+i)$, so that $f(n) \neq 0$ by lemma 1. Similarly, if $a < b$, then take $i = a - 1$ and $t = (b - a + 1)/2$. Again i and t are positive integers and $n = (i+1)(2t+i)$, so that $f(n) \neq 0$.

For any positive integer $n \in \mathbb{N}$, we define $S(n)$ to be the number of different ways that n can be expressed as the difference of two squares, $n = x^2 - y^2$, with both x and y odd integers. Since $x^2 - y^2 = (x-y)(x+y)$, we will be able to use the above result along with the following lemma to obtain a formula for $f(n)$.

Lemma 3. *Let $n \in \mathbb{N}$ be a ces-number. Then $f(n) = S(4n) - 1$.*

Proof. We first show that $f(n) \leq S(4n) - 1$. Take any representation of n as a sum of consecutive even numbers, say $n = 2t + (2t+2) + (2t+4) + \cdots + (2t+2i)$, where $i, t \in \mathbb{N}$. Then, by lemma 1, $n = (i+1)(2t+i)$, so $i^2 + i(2t+1) + 2t - n = 0$. Solving this quadratic equation in i we find that

$$i = -\frac{2t+1}{2} \pm \sqrt{t^2 - t + \tfrac{1}{4} + n}\,.$$

Now, since i is a positive integer, it follows that $t^2 - t + \frac{1}{4} + n = x^2/4$ for some $x \in \mathbb{N}$ with x odd. It then follows that $(t - \frac{1}{2})^2 = x^2/4 - n$. Again, since t is an integer, it must be the case that $x^2/4 - n = y^2/4$ for some $y \in \mathbb{N}$ with y odd (namely $y = 2t - 1$), and so

$$4n = x^2 - y^2\,.$$

Since x and y are uniquely determined by i and t, we can conclude that $f(n) \le S(4n) - 1$. The term -1 is due to the fact that there are no x and y corresponding to $4n = (n+1)^2 - (n-1)^2$: this would imply that $x = 2t+1$ and $y = 2t - 1$; since $4n = x^2 - y^2$, we would obtain that $n = 2t$ and this implies that $i = 0$, a contradiction.

We now show that $f(n) \ge S(4n) - 1$. Suppose that $4n = x^2 - y^2$ with x, y odd and $(x-y)/2 > 1$, i.e. any of the $S(4n)$ ways of writing $4n = x^2 - y^2$ except $4n = (n+1)^2 - (n-1)^2$. Note that $m^2 = 1+3+5+\cdots+(2m-1)$ for any $m \in \mathbb{N}$, so

$$\begin{aligned} x^2 - y^2 &= 1 + 3 + 5 + \cdots + (2y-1) + (2y+1) + \cdots \\ &\qquad + (2x-1) - (1 + 3 + 5 + \cdots + (2y-1)) \\ &= (2y+1) + (2y+3) + \cdots \\ &\qquad + (2x-3) + (2x-1)\,. \end{aligned}$$

Taking these terms in consecutive pairs (which is possible since x and y are odd) and dividing by 4, we obtain

$$n = (y+1) + (y+3) + \cdots + (x-1)\,.$$

Thus, from the expression $4n = x^2 - y^2$, we have obtained a representation of n as a sum of consecutive even numbers since x and y are odd. We can also see that, for different x, y these representations are non-trivial and different since they have length $(x-y)/2 > 1$.

It now remains to determine $S(4n)$ for any $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. Then the ways of writing m as $x^2 - y^2$, where $x, y \in \mathbb{N}$, are in one-to-one correspondence with the ways of factoring m as $m = ab$ with a and b either both even or both odd since, if $m = x^2 - y^2$, then $m = (x-y)(x+y)$, and, if $m = ab$ (with a and b both odd or both even), then

$$m = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2.$$

Theorem 1. *Let $n \in \mathbb{N}$ be a ces-number and suppose that $n = 2^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ for some odd primes $p_2, \ldots, p_r$ and some integers $a_i \ge 1$. Then*

$$f(n) = (a_2+1)(a_3+1)\cdots(a_r+1) - 1.$$

Proof. From the correspondence given above (and lemma 3), $f(n) + 1$ is the number of ways of writing $4n$ as the product ab of two even numbers, where one factor is 2 times an odd number. In other words, $f(n)+1$ is the number of ways of writing n as a product cd, where c is odd and d is even. The result then follows easily.

For example, the result tells us that $f(12) = f(2^2 3) = 1$; the unique expression of 12 as the sum of consecutive even numbers is $12 = 2 + 4 + 6$. Note that it follows that, if n is even, then $f(n) = f(2^r n)$ for every $r \in \mathbb{N}$.

Corollary 1. *Let $n \in \mathbb{N}$ be a ces-number. Then $f(n) = 1$ if and only if $n = 2^r p$ for some odd prime p and some $r \in \mathbb{N}$.*

Proof. This follows directly from theorem 1.

In a similar way, we can note that $f(n) = 2$ if and only if $n = 2^r p^2$ for some odd prime p and $r \in \mathbb{N}$.

3. The number of sums of consecutive numbers

Recall that a positive integer n is a cs-number if it can be expressed as the sum of two or more consecutive numbers. Let $F(n)$ be the number of ways of representing n in this way. It is easy to see that $F(n) = f(2n)$. Hence, we have the following result, the first part of which was shown in references 1 and 2.

Corollary 2. *Let $n \in \mathbb{N}$. Then n is a cs-number if and only if it is not a power of 2. Furthermore, if $n = 2^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ for some odd primes $p_2, \ldots, p_r$ with a_1 possibly zero, then*

$$F(n) = (a_2+1)(a_3+1)\cdots(a_r+1) - 1.$$

Proof. The first part follows from the observation that $F(n) = f(2n)$ and lemma 2. The second part follows again since $F(n) = f(2n)$ and from theorem 1.

References

1. R. Cook and D. Sharpe, Sums of arithmetic progressions, *Math. Spectrum* **27** (1994/95), pp. 30–32.
2. R. Cook and D. Sharpe, Sums of arithmetic progressions, *Fibonacci Quart.* **33** (1995), pp. 218–221.

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From Gauss to Pythagoras — A Complex Series of Voyages of Discovery

P. GLAISTER

Complex numbers are a powerful tool used in many areas of mathematics, science and engineering, and Carl Friedrich Gauss, often referred to as the 'Prince of Mathematicians', was one of the first to give a coherent account of these. One popular application of complex numbers is in finding sums of finite and infinite series involving sines and cosines, and often based on geometric series. For example, starting with the series

$$C = \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots\,, \tag{1}$$

defining the corresponding series with sines

$$S = \frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^3}\sin 3\theta + \cdots \tag{2}$$

and then combining (1) and (2) using the imaginary unit $\mathrm{i} = \sqrt{-1}$, we have

$$\begin{aligned} C + \mathrm{i}S &= \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta) + \frac{1}{2^2}(\cos 2\theta + \mathrm{i}\sin 2\theta) \\ &\quad + \frac{1}{2^3}(\cos 3\theta + \mathrm{i}\sin 3\theta) + \cdots \\ &= \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta) + \frac{1}{2^2}(\cos\theta + \mathrm{i}\sin\theta)^2 \\ &\quad + \frac{1}{2^3}(\cos\theta + \mathrm{i}\sin\theta)^3 + \cdots \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{2^2}(\mathrm{e}^{\mathrm{i}\theta})^2 + \frac{1}{2^3}(\mathrm{e}^{\mathrm{i}\theta})^3 + \cdots\,, \end{aligned} \tag{3}$$

where we have used De Moivre's theorem in the form

$$\cos n\theta + \mathrm{i}\sin n\theta = \mathrm{e}^{\mathrm{i}n\theta} = (\mathrm{e}^{\mathrm{i}\theta})^n = (\cos\theta + \mathrm{i}\sin\theta)^n \tag{4}$$

for integral n. Clearly, the right-hand side of (3) is a geometric series with first term and common ratio $\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$, and hence

$$C + \mathrm{i}S = \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}\,, \tag{5}$$

where we have used the formula

$$\alpha + \alpha r + \alpha r^2 + \cdots = \frac{\alpha}{1-r} \tag{6}$$

for the sum of a geometric series, which is convergent when $|r| < 1$, noting that $|\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}| = \frac{1}{2}|\mathrm{e}^{\mathrm{i}\theta}| = \frac{1}{2} < 1$. To determine C from (5), we require the real part of the right-hand side. Multiplying numerator and denominator by the complex conjugate

$$\overline{1 - \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\theta}} = 1 - \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\theta},$$

we have

$$\begin{aligned} C + \mathrm{i}S &= \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}\,\frac{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}} \\ &= \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{4}}{1 + \frac{1}{4} - \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})} \\ &= \frac{\frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta) - \frac{1}{4}}{\frac{5}{4} - \cos\theta} \\ &= \frac{2\cos\theta - 1 + 2\mathrm{i}\sin\theta}{5 - 4\cos\theta}\,. \end{aligned} \tag{7}$$

Taking the real part of (7), we have

$$\begin{aligned} C &= \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots \\ &= \frac{2\cos\theta - 1}{5 - 4\cos\theta}\,, \end{aligned} \tag{8}$$

and taking the imaginary part we obtain the sum of the sine series (2).

On close inspection of (8), we notice that, if $\theta = \pi/3$, then $\cos\theta = \frac{1}{2}$ and the right-hand side vanishes, i.e.

$$\frac{1}{2}\cos\frac{\pi}{3} + \frac{1}{2^2}\cos\frac{2\pi}{3} + \frac{1}{2^3}\cos\frac{3\pi}{3} + \cdots = 0\,, \tag{9}$$

which I found surprising, and felt needed further explanation. Since $\cos\pi/3 = \frac{1}{2}$, $\cos 2\pi/3 = -\frac{1}{2}$, $\cos 3\pi/3 = -1$, $\cos 4\pi/3 = -\frac{1}{2}$, $\cos 5\pi/3 = \frac{1}{2}$ and $\cos 6\pi/3 = 1$, with all further terms just repeats of these through periodicity, the left-hand side of (9) is

$$\begin{aligned} &\frac{1}{2}\times\frac{1}{2} + \frac{1}{2^2}\times-\frac{1}{2} + \frac{1}{2^3}\times-1 + \frac{1}{2^4}\times-\frac{1}{2} + \frac{1}{2^5}\times\frac{1}{2} \\ &\quad + \frac{1}{2^6}\times 1 + \frac{1}{2^7}\times\frac{1}{2} + \frac{1}{2^8}\times-\frac{1}{2} + \frac{1}{2^9}\times-1 \\ &\quad + \frac{1}{2^{10}}\times-\frac{1}{2} + \frac{1}{2^{11}}\times\frac{1}{2} + \frac{1}{2^{12}}\times 1 + \cdots \\ &= \frac{1}{2}\times\Big(\frac{1}{2} + \frac{1}{2^7} + \cdots\Big) - \frac{1}{2}\times\Big(\frac{1}{2^2} + \frac{1}{2^8} + \cdots\Big) \\ &\quad - 1\times\Big(\frac{1}{2^3} + \frac{1}{2^9} + \cdots\Big) - \frac{1}{2}\times\Big(\frac{1}{2^4} + \frac{1}{2^{10}} + \cdots\Big) \\ &\quad + \frac{1}{2}\times\Big(\frac{1}{2^5} + \frac{1}{2^{11}} + \cdots\Big) + 1\times\Big(\frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\Big) + \cdots \\ &= \Big(\frac{1}{2}\times 1 - \frac{1}{2}\times\frac{1}{2} - 1\times\frac{1}{2^2} - \frac{1}{2}\times\frac{1}{2^3} \\ &\quad + \frac{1}{2}\times\frac{1}{2^4} + 1\times\frac{1}{2^5}\Big)\Big(\frac{1}{2} + \frac{1}{2^7} + \cdots\Big) \end{aligned}$$

$$= \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{4} - \frac{1}{16} + \frac{1}{32} + \frac{1}{32}\right) \times \frac{1/2}{1 - 1/2^6}$$
$$= 0 \times \frac{32}{63} = 0\,,$$

confirming the result in (9).

I then became intrigued as to what might happen for

$$C = a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots\,, \qquad (10)$$

an obvious generalisation of (1), where a is real with $|a| < 1$. We leave readers to show that the corresponding sum is

$$C = \frac{a\cos\theta - a^2}{1 + a^2 - 2a\cos\theta} \qquad (11)$$

and this is zero when $\cos\theta = a$, so, from (10) and (11),

$$\cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \cdots = 0\,,$$

a fascinating result I am sure you agree. For example,

$$\frac{1}{3}\cos\theta + \frac{1}{3^2}\cos 2\theta + \frac{1}{3^3}\cos 3\theta + \cdots = 0$$

when $\cos\theta = \frac{1}{3}$. But there is more …

A more complicated series than (1) is

$$C = 1\times\frac{1}{2}\cos\theta + 2\times\frac{1}{2^2}\cos 2\theta + 3\times\frac{1}{2^3}\cos 3\theta + \cdots\,,$$

where the individual terms have been multiplied by the natural numbers. Repeating the steps leading up to (3) by forming $C + \mathrm{i}S$ using the corresponding sine series S, we obtain

$$C + \mathrm{i}S = 1\times\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta} + 2\times\frac{1}{2^2}(\mathrm{e}^{\mathrm{i}\theta})^2 + 3\times\frac{1}{2^3}(\mathrm{e}^{\mathrm{i}\theta})^3 + \cdots\,, \qquad (12)$$

which is an *arithmetic-geometric* series. The formula for the sum in (12) is readily derived by differentiating (6) with respect to r to give

$$\alpha + 2\alpha r + 3\alpha r^2 + \cdots = \frac{\alpha}{(1-r)^2}\,,$$

which is also convergent when $|r| < 1$, and substituting $\alpha = r = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$, so that

$$C + \mathrm{i}S = \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}{(1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta})^2}\,. \qquad (13)$$

Multiplying the numerator and denominator of (13) by the complex conjugate

$$\overline{(1 - \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\theta})^2} = (1 - \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\theta})^2,$$

we have

$$\begin{aligned} C + \mathrm{i}S &= \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}{(1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta})^2}\,\frac{(1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta})^2}{(1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta})^2} \\ &= \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}(1 + \frac{1}{4}\mathrm{e}^{-2\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})}{\left((1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta})(1 - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta})\right)^2} \\ &= \frac{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{8}\mathrm{e}^{-\mathrm{i}\theta} - \frac{1}{2}}{(\frac{5}{4} - \cos\theta)^2} \\ &= \frac{8\mathrm{e}^{\mathrm{i}\theta} + 2\mathrm{e}^{-\mathrm{i}\theta} - 8}{(5 - 4\cos\theta)^2} \\ &= \frac{10\cos\theta - 8 + 6\mathrm{i}\sin\theta}{(5 - 4\cos\theta)^2}\,, \end{aligned} \qquad (14)$$

and, taking the real part of (14),

$$\begin{aligned} C &= 1\times\frac{1}{2}\cos\theta + 2\times\frac{1}{2^2}\cos 2\theta + 3\times\frac{1}{2^3}\cos 3\theta + \cdots \\ &= \frac{10\cos\theta - 8}{(5 - 4\cos\theta)^2}\,. \end{aligned} \qquad (15)$$

This time, the sum of the series in (15) is zero when $\cos\theta = \frac{4}{5}$, which does not seem particularly special or interesting. After a moment's thought, however, we recall that θ is the smaller acute angle in a $(3, 4, 5)$ Pythagorean triangle. Is this a coincidence?

Quickly turning to another case, say

$$C = 1\times\frac{1}{3}\cos\theta + 2\times\frac{1}{3^2}\cos 2\theta + 3\times\frac{1}{3^3}\cos 3\theta + \cdots$$

(saving the general case to maintain the suspense!), readers may like to show that the algebra now gives

$$\begin{aligned} C &= 1\times\frac{1}{3}\cos\theta + 2\times\frac{1}{3^2}\cos 2\theta + 3\times\frac{1}{3^3}\cos 3\theta + \cdots \\ &= \frac{30\cos\theta - 18}{(10 - 6\cos\theta)^2}\,, \end{aligned}$$

which is zero when $\cos\theta = \frac{3}{5}$, i.e. when θ is now the larger acute angle in a $(3, 4, 5)$ Pythagorean triangle, so nothing really new has been discovered.

Pressing on with the next case, readers will find that

$$\begin{aligned} C &= 1\times\frac{1}{4}\cos\theta + 2\times\frac{1}{4^2}\cos 2\theta + 3\times\frac{1}{4^3}\cos 3\theta + \cdots \\ &= \frac{68\cos\theta - 32}{(17 - 8\cos\theta)^2}\,, \end{aligned}$$

so that the sum is zero when $\cos\theta = \frac{8}{17}$, i.e. when θ is the larger acute angle in an $(8, 15, 17)$ Pythagorean triangle.

For the general case

$$C = 1\times a\cos\theta + 2\times a^2\cos 2\theta + 3\times a^3\cos 3\theta + \cdots\,,$$

the sum can be determined in the same way as (15) by replacing $\frac{1}{2}$ by $a < 1$ in the first line of (14). After some simplification, the corresponding sum is

$$\begin{aligned} C &= 1\times a\cos\theta + 2\times a^2\cos 2\theta + 3\times a^3\cos 3\theta + \cdots \\ &= \frac{(a + a^3)\cos\theta - 2a^2}{(1 - 2a\cos\theta + a^2)^2} \end{aligned} \qquad (16)$$

and this is zero when

$$\cos\theta = \frac{2a^2}{a+a^3} = \frac{2a}{1+a^2}\,.$$

Setting $a = n/m$ for positive integers m, n with $m > n$, we have

$$\cos\theta = \frac{2n/m}{1+n^2/m^2} = \frac{2mn}{m^2+n^2}\,,$$

so that θ is again one of the angles in a Pythagorean triangle, this time with sides m^2-n^2, $2mn$ and m^2+n^2 since $(m^2-n^2)^2+(2mn)^2 = (m^2+n^2)^2$. (Interestingly, the denominator in (16) can only vanish when $\cos\theta = (1+a^2)/2a$, which happens to be the reciprocal of the value above, and of course has no real solutions for θ.)

For example, when $m = 3$ and $n = 2$, $a = n/m = \frac{2}{3}$ and (16) becomes

$$1\times\frac{2}{3}\cos\theta + 2\times\left(\frac{2}{3}\right)^2\cos 2\theta + 3\times\left(\frac{2}{3}\right)^3\cos 3\theta + \cdots = 0$$

when

$$\cos\theta = \frac{2mn}{m^2+n^2} = \frac{12}{13}\,,$$

so that θ is the smaller acute angle in a $(5, 12, 13)$ Pythagorean triangle.

Now I wonder what Pythagoras would have thought of that?

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Mathematics in the Classroom

The binomial and negative binomial distributions

Consider an experiment with two possible outcomes, not necessarily equally likely, that are labelled generically 'success' and 'failure'. In such circumstances, the binomial distribution allows us to calculate the probability of obtaining exactly x successes and y failures in n trials as follows:

$$\Pr(x \text{ successes}) = \binom{n}{x} p^x q^{n-x}\,, \qquad x = 0, 1, 2, \ldots, n\,,$$

where $0 < p < 1$ is the probability of a success and $q = 1-p$ is the probability of a failure. The mean and variance of this distribution are given by $\mu = np$ and $\sigma^2 = npq$ respectively.

To illustrate, let the experiment consist of cutting a pack of playing cards with a success being an ace with probability $p = \frac{1}{13}$ and failure being any other card with probability $q = \frac{12}{13}$. Therefore, the probability of obtaining, say, exactly two aces and four non-aces in six trials is given by

$$\binom{6}{2}\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^4 = 0.0819\,.$$

Suppose now that we wish to calculate the probability that the nth trial produces the xth success. (Note that this is subtly different from n trials producing x successes.)

For the nth trial to produce the xth success, the first $n-1$ trials must have produced exactly $x-1$ successes and therefore $(n-1)-(x-1) = n-x$ failures *and* the nth trial must produce a success. The probability of this sequence of events is given by

$$\begin{aligned}
&\Pr(n\text{th trial gives } x\text{th success})\\
&\quad = \left[\binom{n-1}{x-1} p^{x-1} q^{n-x}\right] p\\
&\quad = \binom{n-1}{x-1} p^x q^{n-x}\,, \qquad n = x, x+1, \ldots\,.
\end{aligned}$$

The above equation represents the negative binomial or Pascal distribution, which appears in the literature in a variety of equivalent guises, viz.

$$\binom{x+y-1}{x-1} p^x q^y \quad \text{or} \quad \binom{x+y-1}{y} p^x q^y \quad \text{or} \quad \binom{n-1}{n-x} p^x q^y\,.$$

The mean and variance of this distribution are given by

$$\mu = \frac{xq}{p} = \frac{x(1-p)}{p}$$

and

$$\sigma^2 = \frac{xq}{p^2} = \frac{x(1-p)}{p^2}$$

respectively.

Returning to our card-cutting experiment for illustration, the probability of the seventh cut producing the third ace is thus

$$\binom{7-1}{3-1} p^3 q^4 = 0.0063\,.$$

The interested reader can easily confirm that any version of the negative binomial above will duplicate this result.

From the foregoing, we can see how the binomial and negative binomial distributions differ from one another. A binomial random variable is a count of the number, x, of successes in a predetermined number, n, of trials, i.e. the number, n, of trials is fixed and the number, x, of successes is random. Conversely, a negative binomial random variable is a count of the number, n, of trials required to obtain a predetermined number, x, of successes, i.e. the number of successes, x, is fixed and the number of trials, n, is random. In this sense, the negative binomial can be considered the opposite, or negative, of the binomial.

Fitting a negative binomial distribution: an example from soccer

During the 2002/03 soccer season, Kilmarnock F.C. scored 47 goals over 38 league games and finished fourth in the Scottish Premier League. The actual distribution of these goals is shown in table 1 from which it can be calculated that the mean number of goals per game was 1.24 with a variance of 1.39.

Table 1. Kilmarnock F.C. goals scored, 2002/03.

Number of goals (y)	Observed frequency of y goals	Theoretical frequency of y goals
0	10	12
1	15	13
2	11	8
3	0	3
4	1	1
5	0	0
6	1	1
Total games	38	38

Since football teams vary their players and tactics and play against opposition of varying calibre on grounds of varying quality, it is unrealistic to assume that the probability of a goal, and hence the expected number of goals for a particular team, will remain constant from one game to another. In such circumstances, as Pollard has suggested (see reference 1), the appropriate statistical distribution to fit to goal-scoring data may be the negative binomial distribution rather than the more popular and easily computed Poisson distribution.

To fit a negative binomial distribution to the above data, we make use of the following:

$$\mu = \frac{x(1-p)}{p} \quad \text{and} \quad \sigma^2 = \frac{x(1-p)}{p^2}$$

from which

$$p = \frac{x}{\mu - x} \quad \text{and} \quad q = \frac{\mu}{\mu + x}$$

and

$$x = \frac{\mu^2}{\sigma^2 - \mu} \quad \text{and} \quad p = \frac{\mu}{\sigma^2}.$$

Thus, by substitution,

$$\binom{x+y-1}{y} p^x q^y = \binom{x+y-1}{y}\Big(\frac{x}{\mu+x}\Big)^x \Big(\frac{\mu}{\mu+x}\Big)^y,$$

where $y = 0, 1, 2, 3, \ldots, x > 0$ and $0 < p < 1$.

As indicated above, $\mu = 1.24$ and $\sigma^2 = 1.39$, which allows us to calculate that

$$x = \frac{1.24^2}{1.39 - 1.24} = 10.25$$

and

$$p = \frac{1.24}{1.39} = 0.89$$

so that $q = 0.11$. Since x is an integer by definition, it should be rounded off to $x = 10$. If we now define y as the number of goals scored per game, then the probability of scoring, say, $y = 4$ goals is

$$\binom{10+4-1}{4}(0.89)^{10}(0.11)^4 = 0.03264.$$

Thus, the theoretical or expected frequency of four goals per game is $38 \times 0.03264 = 1.24$, or 1 rounded to the nearest whole number. The theoretical frequency of $y = 0, 1, 2, \ldots, 6$ goals is shown in table 1.

We may now apply the familiar χ^2 goodness-of-fit test in the usual way. Pooling the theoretical frequency of 3, 4, 5 and 6 goals to ensure that all such frequencies equal 5 or more, a χ^2 of 3.57 indicates that the negative binomial distribution provides an acceptable fit to this data.

Reference

1. R. Pollard, Goal scoring and the negative binomial distribution, *Math. Gazette* **69** (1985), pp. 45–47.

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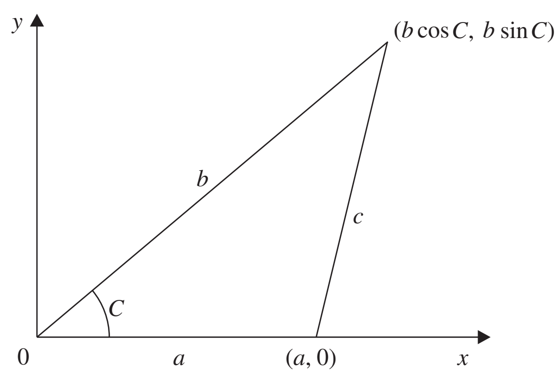
The Computer Column will return in the next issue.

Prove the cosine formula at a glance



Use the diagram to prove the cosine formula

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

GUIDO LASTERS
Tienen, Belgium.

Letters to the Editor

Dear Editor,

Sum = product

I read with interest Seyamack Jafari's 'A polygon spiral' in *Mathematical Spectrum*, Volume 35, Number 3. From figure 1 we have

$$\sin\theta = \frac{1}{x_0}, \qquad \cos\theta = \frac{1}{x_1},$$

$$\sin^2\theta + \cos^2\theta = 1, \qquad \frac{1}{x_0^2} + \frac{1}{x_1^2} = 1,$$

$$a^2 = x_0^2 + x_1^2 = x_0^2 \times x_1^2, \tag{1}$$

$$\sin\phi = \frac{1}{a}, \qquad \cos\phi = \frac{1}{x_2},$$

therefore,

$$\frac{1}{a^2} + \frac{1}{x_2^2} = 1$$

and

$$b^2 = a^2 + x_2^2 = a^2 x_2^2 = x_0^2 x_1^2 x_2^2.$$

Also, by (1),

$$b^2 = a^2 + x_2^2 = x_0^2 + x_1^2 + x_2^2. \tag{2}$$

Continue this way for c, d etc.

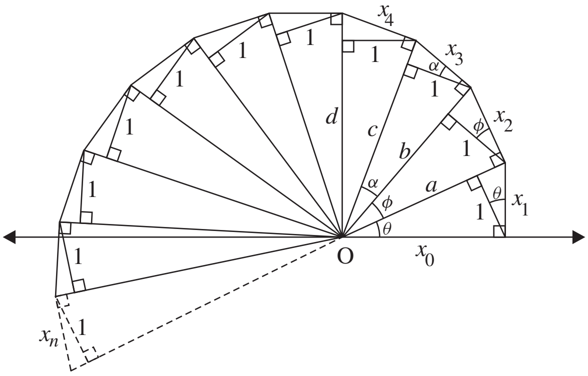


Figure 1.

The question arises: can we find *rational* numbers $x_0^2, x_1^2, x_2^2, \ldots$ that satisfy

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2 = x_0^2 \times x_1^2 \times x_2^2 \times \cdots \times x_n^2.$$

Put $\theta = 30°$. Then

$$x_0 = \frac{1}{\sin 30} = 2, \qquad x_1 = \frac{1}{\cos 30} = \frac{2}{\sqrt{3}}$$

and

$$2^2 + \left(\frac{2}{\sqrt{3}}\right)^2 = 2^2 \times \left(\frac{2}{\sqrt{3}}\right)^2,$$

that is,

$$4 + \frac{4}{3} = 4 \times \frac{4}{3}.$$

Further,

$$\sin^2\phi = \frac{1}{a^2} = \frac{1}{x_0^2 x_1^2} = \frac{3}{16}, \qquad \cos^2\phi = \frac{1}{x_2^2}.$$

Hence,

$$\frac{3}{16} + \frac{1}{x_2^2} = 1,$$

whence, $x_2^2 = \frac{16}{13}$. Then

$$x_0^2 + x_1^2 + x_2^2 = 4 + \frac{4}{3} + \frac{16}{13} = \frac{256}{39}$$

and

$$x_0^2 \times x_1^2 \times x_2^2 = 4 \times \frac{4}{3} \times \frac{16}{13} = \frac{256}{39}.$$

Continuing,

$$\sin\alpha = \frac{1}{b}, \qquad \cos\alpha = \frac{1}{x_3}, \qquad \frac{1}{b^2} + \frac{1}{x_3^2} = 1,$$

$$\begin{aligned} c^2 &= b^2 + x_3^2 = b^2 x_3^2 \\ &= x_0^2 \times x_1^2 \times x_2^2 \times x_3^2 \\ &= x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{(by (2))} \end{aligned}$$

and

$$\sin^2\alpha = \frac{1}{b^2} = \frac{1}{x_0^2 \times x_1^2 \times x_2^2} = \frac{1}{4 \times \frac{4}{3} \times \frac{16}{13}} = \frac{39}{256},$$

$$\cos^2\alpha = \frac{1}{x_3^2}.$$

Hence,

$$\frac{39}{256} + \frac{1}{x_3^2} = 1,$$

whence, $x_3^2 = \frac{256}{217}$. Thus,

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0^2 \times x_1^2 \times x_2^2 \times x_3^2 = \frac{65536}{8463}.$$

It will be found that, for $n > 0$,

$$x_n^2 = \frac{a_n}{b_n}, \quad \text{where } a_n = 2^{2^n} \text{ and } b_n = a_n - \prod_{i=0}^{n-1} b_i,$$

with $b_0 = 1$, and the sum (or product) of the x_i^2 up to and including x_n^2 is given by

$$\frac{c_n}{d_n}, \qquad \text{where } c_n = 2^{2^{n+1}} \text{ and } d_n = \prod_{i=1}^{n} b_i,$$

all of which increase quite rapidly.

Other values for cos and sin will give other solutions.

Yours sincerely,

Bob Bertuello
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG,
UK.)

Dear Editor,

π day, Sunday, March 14, 1:59pm

The number is π, 3.1415926535 . . . . It's the number you get when you divide the circumference of a circle by its diameter, and it can't be expressed as a fraction; it goes on forever without repetition. In this era when maths and mathematicians have become sexy again, come to San Francisco's Exploratorium and gather around the π shrine to perform π-related rites and eat ritual food, pie, in honour of this special number. And come to the Exploratorium and attend the US premier of *Porridge Pulleys and Pi: Two Mathematical Journeys*, directed by George Paul Csicsery, presented in conjunction with the Mathematical Sciences Research Institute, Berkeley. It's all part of π day celebrations, which culminate, appropriately enough, at 1:59pm on March 14. That's the third month, the fourteenth day, at 1:59pm, corresponding to the approximation 3.14159 to π.

In addition to the film and other activities, at 1:59pm celebrants are invited to circumambulate the π shrine approximately 3.14 times. The π shrine is a small brass plate engraved with π to a hundred digits located on the floor on the mezzanine level of the Exploratorium, the prototype for hands-on science museums around the world. Add beads to a ritual π string, where each bead color designates a value for π to over 1600 digits, and growing. Music based on the number π provides the ambience.

The film Porridge *Porridge Pulleys and Pi: Two Mathematical Journeys*, features Vaughan Jones, one of the world's foremost knot theorists, and Hendrik Lenstra, a number theorist, as well as everything from genomics, music, and elliptic curves to Homer and the history of π. Meet Vaughan Jones and other real live mathematicians from MSRI following the screening.

Most important of all, people eat pie. Lemon meringue, apple, peach, whatever. The public is invited to attend π day. It would be nice also to bring along a pie to share.

And as icing on the pie, March 14 is Albert Einstein's birthday.

Yours sincerely,

Raphael Rosen
(The Exploratorium,
3601 Lyon Street,
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USA.)

[Even though the publication of this letter misses π day, we thought readers would be interested to know what they have missed. Maybe next year! — Ed.]

Dear Editor,

'Squarps' and the lead–end digit trade

Consider the process of taking a positive integer whose leading and ending digits are distinct and none of whose digits are zeros, and exchanging the leading and ending digits. (We will only consider numbers without zeros for digits as this will eliminate any ambiguity that arises when zeros are dropped in the switching procedure.) Let's call this process the 'lead–end digit trade'. For example, 12 345 after the lead–end digit trade becomes 52 341.

Now let's look at the number 1156, a perfect square containing no zeros for digits. The number 6151 obtained by exchanging its leading and ending digits turns out to be a prime. Let's refer to perfect squares which give a prime number like this as *squarps* for short. I had the idea for this letter after reading Chapter 80 on 'Parasite Numbers' in reference 1, in particular the last question in the 'Furthering Exploring' section. However, the chapter mentioned deals with a different problem entirely.

If we use a computer to search for squarps we will find a lot of them. I used the software package PARI-GP, which is freely available from the Internet (see reference 2). The first few squarps are 16, 196, 361, 784, 1156, 1444, 3136, 3364, 3481, 3721, 7225, 7396, 11 236, 12 769, 15 376.

Since squarps are relatively abundant, what would a plot of these numbers look like? Figure 1 shows the points (n, n^2) such that n^2 is a squarp for $0 < n < 1000$. This graph seems to suggest that squarps come in 'clumps' with large gaps between them. What would a plot of squarps for $0 < n < 100\,000$ tell us?

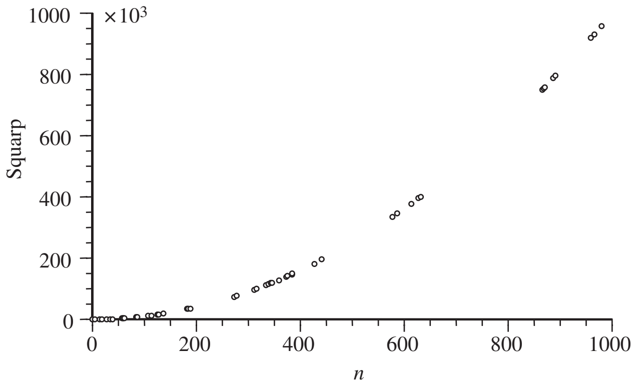

Figure 1. Distribution of squarps.

This suggests some generalizations and some questions.

- Squarts. Find perfect squares that become triangular numbers after performing the lead–end digit trade. The first squart is $5\,239\,521 = 2289^2$ since $1\,239\,525 = 1574\times(1574 + 1)/2$. Three others are 69 172 489, 62 154 977 481 and 88 844 532 624.
- Squarcs. Find perfect squares that become perfect cubes after performing the lead–end digit trade. I haven't found any squarcs.
- Give an efficient method of searching for these numbers via computer. My method isn't very efficient.

References

1. C. Pickover, *Wonders of Numbers* (Oxford University Press, 2001).
2. G. Niklasch, PARI/GP Homepage, http://www.parigp-home.de/.

Yours sincerely,

Jason Earls
(516 W. Oklahoma,
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USA.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

36.9 Let a, b, c, d be positive real numbers such that $a+b \geq c+d$ and $ab \leq cd$. Prove that $a^m + b^m \geq c^m + d^m$ for all integers m.

(Submitted by Hassan Shah Ali, Tehran, Iran)

36.10 When n people each throw a pair of dice, what is the probability that they all get the same total?

(Submitted by Anand Kumar, Patna, India)

36.11 The point P moves in a plane so that the tangents through P to a given ellipse are at right angles. Determine the locus of P. If these tangents touch the ellipse at M, N, find the envelope of MN.

(Submitted by Seyamack Jafari, Khozestan, Iran)

36.12 What is

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ\ ?$$

(Submitted by Zhang Yun, The First Middle School of Jinchang City, China)

Solutions to Problems in Volume 36 Number 1

36.1 Let $A_1A_2 \ldots A_n$ be a regular n-sided polygon. Show that $(PA_1)^2 + (PA_2)^2 + \cdots + (PA_n)^2$ is the same for all points P on the circumscribed circle of the polygon.

Solution by J. Craig, who proposed the problem

Let

$$A_k = \left(r\cos\frac{2\pi(k-1)}{n},\ r\sin\frac{2\pi(k-1)}{n} \right)$$

for $k = 1, \ldots, n$, and

$$P = (r\cos\theta, r\sin\theta).$$

Then

$$\begin{aligned}\sum_{k=1}^{n}(PA_k)^2 &= r^2\sum_{k=1}^{n}\left[\left(\cos\frac{2\pi(k-1)}{n} - \cos\theta\right)^2 + \left(\sin\frac{2\pi(k-1)}{n} - \sin\theta\right)^2\right] \\ &= 2nr^2 - 2r^2\sum_{k=1}^{n}\left(\cos\frac{2\pi(k-1)}{n}\cos\theta + \sin\frac{2\pi(k-1)}{n}\sin\theta\right) \\ &= 2nr^2 - 2r^2\sum_{k=1}^{n}\cos\left(\frac{2\pi(k-1)}{n} - \theta\right) \\ &= 2nr^2 - 2r^2\mathrm{Re}\left\{\sum_{k=1}^{n} e^{(2\pi(k-1)/n-\theta)i}\right\} \\ &= 2nr^2 - 2r^2\mathrm{Re}\left\{e^{-i\theta}\left(\frac{1-e^{2\pi i}}{1-e^{2\pi i/n}}\right)\right\} \\ &= 2nr^2.\end{aligned}$$

J. Craig points out that the same holds true when P lies on any circle concentric to the circumscribed circle of the polygon. If the circle on which P lies has radius R, then

$$\sum_{k=1}^{n}(PA_k)^2 = n(r^2 + R^2).$$

A similar point is made by S. Hayes.

John MacNeill, one of our editors, points out that a more general result is true, namely that, in 2- or 3-dimensional space, if $A_1, A_2, \ldots, A_n$ are points equidistant from a point O the sum of whose position vectors relative to O is zero, and if P moves so that its distance from O is constant, then $PA_1^2 + \cdots + PA_n^2$ is constant. To see this, we denote the position vector of A_k relative to O by $\boldsymbol{a}_k$ for $k = 1, \ldots, n$ and the position vector of P by $\boldsymbol{r}$. Put $|\boldsymbol{a}_1| = \cdots = |\boldsymbol{a}_n| = h$ and $|\boldsymbol{r}| = \ell$. Then

$$\begin{aligned}PA_k^2 &= (\boldsymbol{r} - \boldsymbol{a}_k)\cdot(\boldsymbol{r} - \boldsymbol{a}_k) \\ &= |\boldsymbol{r}|^2 - 2\boldsymbol{r}\cdot\boldsymbol{a}_k + |\boldsymbol{a}_k|^2,\end{aligned}$$

so

$$\begin{aligned}\sum_{k=1}^{n}(PA_k)^2 &= n\ell^2 - 2\boldsymbol{r}\cdot(\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n) + nh^2 \\ &= n(h^2 + \ell^2).\end{aligned}$$

36.2 A 3×3 *magic square* is defined as a square array of natural numbers in which the sums of the elements in each row, each column and on each of the two diagonals are the same, s say. Express the element in the centre in terms of s and show how to obtain different 3×3 magic squares with a given central element. Define a 3×3 *multiplication magic square* by replacing addition by multiplication. What are the corresponding results for these?

Solution by Emanuel Emanouilidis, who proposed the problem

Denote the central element by x. If we add the two diagonals and the middle row and column, we see that

$$4s = 3s + 3x\,,$$

so $x = s/3$. All squares of the form

$$\begin{array}{ccc} x+a & x-a-b & x+b \\ x+b-a & x & x+a-b \\ x-b & x+a+b & x-a \end{array}$$

where a, b are natural numbers such that $a+b < x$, are magic squares with central element x.

For a multiplication magic square, if we multiply instead of add, we obtain $x = p^{1/3}$, where p is the product of the entries in a row, column or diagonal. All squares of the form

$$\begin{array}{ccc} xa & \dfrac{x}{ab} & xb \\ \dfrac{xb}{a} & x & \dfrac{xa}{b} \\ \dfrac{x}{b} & xab & \dfrac{x}{a} \end{array}$$

will be multiplication magic squares with central element x, where a, b are positive divisors of x.

36.3 The positive real numbers $a_1, \dots, a_n$ are such that $a_1+a_2+\cdots+a_n = n$. Show that $a_1^{3/2}+a_2^{3/2}+\cdots+a_n^{3/2} \geq n$.

Solution by Milton Chowdhury, who proposed the problem

Since

$$1+\delta \leq \left(1+\frac{\delta}{2}\right)^2,$$

when $\delta > -1$,

$$(1+\delta)^{1/2} \leq 1+\left(\frac{\delta}{2}\right).$$

Hence,

$$(1+\delta)^{3/2} = \frac{(1+\delta)^2}{(1+\delta)^{1/2}} \geq \frac{(1+\delta)^2}{1+\delta/2} \geq 1+\frac{3\delta}{2}\,.$$

Let $\delta = a_i - 1$. Then

$$a_i^{3/2} \geq \frac{3a_i}{2} - \tfrac{1}{2}\,,$$

so

$$\sum_{i=1}^{n} a_i^{3/2} \geq \frac{3}{2}n - \frac{n}{2} = n\,.$$

36.4 Let $s_n = n! - n^n/\mathrm{e}^{n-1}$ for $n = 1, 2, 3, \dots$. Prove that the sequence (s_n) is strictly increasing and unbounded.

Solution by Hassan Shah Ali, who proposed the problem

We have

$$\begin{aligned} s_{n+1} &= (n+1)! - \frac{(n+1)^{n+1}}{\mathrm{e}^n} \\ &= (n+1)\Big[n! - \frac{n^n}{\mathrm{e}^{n-1}} + \frac{n^n}{\mathrm{e}^{n-1}} - \Big(\frac{n+1}{\mathrm{e}}\Big)^n\Big] \\ &= (n+1)\Big[s_n + \Big(\frac{n}{\mathrm{e}}\Big)^n\Big(\mathrm{e} - \Big(1+\frac{1}{n}\Big)^n\Big)\Big], \end{aligned}$$

but

$$\begin{aligned} \Big(1+\frac{1}{n}\Big)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\Big(\frac{1}{n}\Big)^2 + \cdots + \frac{n!}{n!}\Big(\frac{1}{n}\Big)^n \\ &\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < \mathrm{e}\,. \end{aligned}$$

Hence,

$$s_{n+1} > (n+1)s_n\,.$$

Also $s_n \geq 0$ for all n; we can prove this inductively. When $n = 1$, $s_1 = 0$. Suppose that $n! \geq n^n/\mathrm{e}^{n-1}$ for some n. Then

$$\begin{aligned} (n+1)! &\geq (n+1)\frac{n^n}{\mathrm{e}^{n-1}} \\ &= \frac{(n+1)^{n+1}}{\mathrm{e}^n}\,\frac{\mathrm{e}}{(1+1/n)^n} > \frac{(n+1)^{n+1}}{\mathrm{e}^n}\,. \end{aligned}$$

Hence, $s_{n+1} > (n+1)s_n \geq s_n$ for all n. Also, for all $n \geq 2$, $s_{n+1} > (n+1)s_n \geq (n+1)s_2$ and $s_2 > 2s_1 = 0$, so the sequence (s_n) is unbounded.

Equal sums of cubes

$$\begin{array}{cc} 0 & 2 \\ 1 & 3 \end{array}$$

$$\underbrace{2^3 + 23^3 + 31^3 + 10^3}_{\text{pairs read clockwise}} = \underbrace{1^3 + 13^3 + 32^3 + 20^3}_{\text{pairs read anticlockwise}}$$

The same is true for the arrays

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}\quad \begin{array}{cc} 2 & 4 \\ 3 & 5 \end{array}\quad \begin{array}{cc} 3 & 5 \\ 4 & 6 \end{array}\quad \begin{array}{cc} 4 & 6 \\ 5 & 7 \end{array}$$

$$\begin{array}{cc} 5 & 7 \\ 6 & 8 \end{array}\quad \begin{array}{cc} 6 & 8 \\ 7 & 9 \end{array}\quad \begin{array}{cc} 0 & 4 \\ 2 & 6 \end{array}\quad \begin{array}{cc} 1 & 5 \\ 3 & 7 \end{array}\quad \begin{array}{cc} 2 & 6 \\ 4 & 8 \end{array}$$

$$\begin{array}{cc} 3 & 7 \\ 5 & 9 \end{array}\quad \begin{array}{cc} 0 & 6 \\ 3 & 9 \end{array}$$

Seema Bansal and Vinod Tyagi
Shyam Lal College (E)
Delhi University

Reviews

Fun and Fundamentals of Mathematics. By JAYANT V. NARLIKAR AND MANGALA NARLIKAR. Universities Press, Hyderabad, 2002. Pp. 194. Paperback £17.95 (ISBN 81-7371-398-7).

The blurb on the back of this book claims that the book 'introduces fundamental ideas in Mathematics through interesting puzzles', its motivation being that 'many bright students … are bored with routine class work … and will enjoy these puzzles which will sharpen their logical reasoning'. It certainly succeeds, being engaging to the reader, and offering a series of exercises highlighting certain key ideas inherent in many forms of problem solving. Its range of topics is as diverse as it is entertaining, from noughts and crosses to the famous bridges of Königsberg, and from warfare to Gödel's theorem. There are sixteen chapters each dealing with a different and interesting facet of mathematics of interest to this book's target age group of 12–18 year olds.

I would unreservedly recommend this book to anyone at all interested in mathematics, as its refreshing approach to the topic of problem solving is a welcome break from the stodgy fare of the stipulated syllabi in schools. Indeed, in addition to being a useful stimulus to 12–18 year olds, it could also be of interest to many adults, both as an aid to understanding, and as a source of new ideas.

Student, Berkhamsted Collegiate School PAUL JEFFERYS

Mathematical Apocrypha: Stories and Anecdotes of Mathematicians and the Mathematical. By STEVEN G. KRANTZ. MAA, Washington, DC, 2002. Pp. 232. Paperback $32.95 (ISBN 0-88385-539-9).

If you regard mathematicians as a rather eccentric breed, then you will be confirmed in this view by reading this book. It is not a book about mathematics, but about mathematicians, a cornucopia of their idiosyncrasies. There is Dirichlet, who made an exception to his rule of never writing letters when his first child was born: he wired his father-in-law with the message '$2 + 1 = 3$'. Or Dedekind, who read his own obituary in a publication of the German Mathematical Society, giving the day, month and year of his death; he wrote to the editor: 'at least the year is wrong'. Or Gauss, who, deep in mathematical thought, received a report that his wife was dying. 'Tell her to wait a few minutes until I have finished', was his reply. Then there is an article that Littlewood wrote which ended with the statement 'Thus σ should be made as small as possible'. When it appeared in print, the statement was omitted; instead a speck appeared which turned out to be the tiniest σ anyone had ever seen.

This book is full of such stories. The centre of gravity is inevitably American, and reference is made to mathematics which will be beyond the scope of most of our readers. But don't let that put you off. It is a fascinating volume to browse through.

University of Sheffield DAVID SHARPE

Hinged Dissections: Swinging and Twisting. By GREG N. FREDERICKSON. Cambridge University Press, 2002. Pp. 300. Hardback £35.00 (ISBN 0-521-81192-9).

This is a book packed with dissections, in which a geometrical figure is taken apart and rearranged to form a new transformed figure. What makes these dissections particularly special is that the rearrangement is accomplished using hinges; in the simplest case, hinges are placed at the corners of adjacent plane pieces, and rotating the pieces around them accomplishes the transformation. So for instance, a triangle can be cut into four pieces and the pieces hinged in such a way that they can be swung around and re-formed into a square. Seeing such constructions (and far more complex ones) on the page is tremendous visual fun.

The book consists for the most part of a catalogue of 'hinged' dissections, arranged according to the techniques used for discovering them and the figures involved. The techniques are fascinating due to their surprising use of symmetry and tessellation. The sheer weight of examples means that this is a book squarely aimed at the enthusiast; there is an enormous amount of material here, and it requires some passion to wade through it. But there is variety also: different types of hinges are flirted with, and a number of striking three-dimensional dissections are given. There are also exercises, complete with answers. A detective story ('The Curious Case of the Brass Hinges') unfolds in short chapters throughout the book, trying to get to the bottom of an apparent mathematical cover-up instigated by Henry Dudeney, a puzzle-column writer from the turn of the last century.

Frederickson's book is full of gorgeous results, described in lively style; it will reward serious study, and provide a massive store of puzzles.

Queens' College, Cambridge WILL DONOVAN

The A to Z of Mathematics: A Basic Guide. By THOMAS H. SIDEBOTTOM. John Wiley, New York, 2002. Pp. 474. Paperback $54.95 (ISBN 0-471-15045-2).

Instead of taking the textbook approach to teaching maths, this book tries to make the subject more accessible by presenting topics in an encyclopædia format, from 'absolute value' to 'Zeno's paradox'. This means that rather than having to wade through a whole branch of maths to find out about a single area, you can simply look it up. For example, instead of studying statistics until you understand what lies behind combinations and permutations, you can find it under 'C' in *The A to Z of Mathematics*.

The book contains three types of article: explanations (such as 'algebra'), definitions ('proportion'), and other items worth mentioning ('hexomino', or the 'four colour problem'). Rather than just telling the reader about the subjects, it explains each in enough depth to be able to give some understanding of the concepts; and articles often include worked examples to demonstrate. For the more common subjects, such as 'algebra' or 'probability', there are in-depth explanations with plenty of examples showing different variations on the topic, whereas for smaller subjects, such as 'factorial' there is just a simple explanation and example. However, explanations generally go into enough depth to leave the reader with enough knowledge to understand the basics of the subject.

The author states that the book is aimed at 'the millions of people who would love to understand math but are turned away by fear of its complexity', and in general articles start from a basic enough level to avoid scaring off readers. Of course, not all articles can work from an assumption of no background knowledge — many articles state algebraic formulae or use trigonometry, but no subject is used that isn't covered elsewhere in the book, and related areas are usually referenced at the end of an article. Overall, the book lives up to its title, and could be useful to anyone, whether they're trying to pick up the basics or a maths student looking for a quick reference book.

Student, Paston College, Norfolk STEPHEN WALKER

Other books received

Quaternions and Rotation Sequences. By JACK B. KUIPERS. Princeton University Press, 2002. Pp. 371. Paperback £24.95 (ISBN 0-691-10298-8).

Index

Volume **34** (2001/02) Volume **35** (2002/03) Volume **36** (2003/04)

Articles

Mathematical Spectrum
2003/2004 Volume 36 Number 3

ISSN 0025-5653

Published by the Applied Probability Trust
Printed by Pear Tree Press Ltd, Stevenage, Herts, UK