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Téléc : 613–733–8994
email : abonnements@smc.math.ca

EDITORIAL

Shawn Godin

Welcome to issue 3 of Volume 38 or to Issues 3 and 4 if you are reading the print edition. We hope you like the new format.

We have a new feature in this issue. Crux readers will recognize the name Michel Bataille. Michel is a regular problem proposer (see, for example, problem 3722 later in this issue), and problem solver (see, for example, problem 3623 later in this issue) in Crux and he was featured in the contributor profiles [2007 : 1]. Michel has agreed to write a regular column for Crux on advanced problem solving techniques. The title of the column is *Focus On...* and will appear every second issue. In the inaugural article, Michel shows how to use ideas of periodicity to solve problems involving the floor function. Future columns will feature topics such as linear recurrence relations and polynomial identities, seeing the geometry behind algebraic problems and using Lagrange multipliers to solve inequalities. We hope you enjoy this new feature and we are looking forward to future columns.

We also have an article this issue by an author who will be familiar to many of our readers. Ross Honsberger has written a number of books on problem solving that are enjoyed by the problem solving community. Many of his books feature interesting problems and their solutions, and many of those problems have come from the pages of Crux. Professor Honsberger has sent us a couple of articles, the first of which appears this month. Enjoy!

As mentioned in previous editorials, we are working on some new features for upcoming issues. Hopefully, you will get a glimpse at what we are working on in issues 5 and 6. As always, your feedback is welcome. Changes are done in an attempt to improve the journal; so we need to know if we are hitting the mark.

Shawn Godin

In Memoriam

Juan-Bosco Romero Márquez, 1945 – 2013

We have just learned that Juan-Bosco Romero Márquez passed away on January 19, 2013 at his home in Avila, Spain. Juan-Bosco has been a valued contributor to ***Crux Mathematicorum*** for almost 25 years. His first contribution was problem 1435 [1989 : 110; 1990 : 185-186], which attracted 26 solvers:

Find all pairs of integers x, y such that

$$(xy-1)^2 = (x+1)^2 + (y+1)^2.$$

He will be sorely missed. Problem 3721, proposed in this issue by his colleague Francisco Javier García Capitán, is dedicated to his memory.

SKOLIAD No. 139

Lily Yen and Mogens Hansen

Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched in 2013.*

In this issue we present the solutions to the 21st Transylvanian Hungarian Mathematical Competition, 2011, given in Skoliad 133 at [2011:194–195].

1. Prove that if a, b, c, and d are real numbers, then

$$a+b+c+d-a^2-b^2-c^2-d^2 \leq 1\,.$$

Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Note that $0 \leq (a-\frac{1}{2})^2 = a^2 - a + \frac{1}{4}$, so $a - a^2 \leq \frac{1}{4}$. Similarly, $b - b^2 \leq \frac{1}{4}$, $c - c^2 \leq \frac{1}{4}$, and $d - d^2 \leq \frac{1}{4}$. Therefore

$$(a-a^2)+(b-b^2)+(c-c^2)+(d-d^2) \leq \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4},$$

so $a+b+c+d-a^2-b^2-c^2-d^2 \leq 1$, as required.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

2. Compare the following two numbers,

$$A = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{\text{2011 copies of 2}} \qquad \text{and} \quad B = \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{\text{2010 copies of 3}}\;;$$

which is larger, A or B? (Note that a^{b^c} equals $a^{(b^c)}$, not $(a^b)^c$.)

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

For any positive integer n, define A_n and B_n as follows:

$$A_n = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n\text{ copies of 2}} \qquad \text{and} \quad B_n = \underbrace{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}_{n\text{ copies of 3}}\,.$$

Note that $A_3 = 2^{2^2} = 2^4 = 16$, and $B_2 = 3^3 = 27$, so $A_3 < B_2$.

Suppose that $A_n < B_{n-1}$ for some positive integer n. Then $2^{A_n} < 3^{B_{n-1}}$, but $2^{A_n} = A_{n+1}$ and $3^{B_{n-1}} = B_n$, so $A_{n+1} < B_n$.

It follows by induction from the previous two paragraphs that $A_{n+1} < B_n$ whenever $n \geq 2$. In particular, $A = A_{2011} < B_{2010} = B$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

3. Find all natural number solutions to each equation:

a. $20x^2 + 11y^2 = 2011$.

b. $20x^2 - 11y^2 = 2011$.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

Part a: If $20x^2 + 11y^2 = 2011$, then $20x^2 \leq 2011$. Since x is a natural number, $1 \leq x \leq 10$. Trying all ten possible values for x yields only one solution: $x = 10$ and $y = 1$.

Part b: If $20x^2 - 11y^2 = 2011$, then $20x^2 = 2011 + 11y^2$, so $2011 + 11y^2$ is divisible by 20. In particular, the ones digit of $2011 + 11y^2$ must be 0. The ones digit of a sum or a product depends only on the ones digits of the numbers being added or multiplied. Therefore, since the ones digit of $2011 + 11y^2$ is 0, the ones digit of $11y^2$ is 9, so the ones digit of y^2 is 9, so the ones digit of y is 3 or 7.

If the ones digit of y is 3, then $y = 10n + 3$ for some integer n. Then

$$\begin{aligned}20x^2 &= 2011 + 11y^2 = 2011 + 11(10n + 3)^2 \\ &= 2011 + 1100n^2 + 660n + 99 = 2110 + 1100n^2 + 660n,\end{aligned}$$

so

$$2x^2 = 211 + 110n^2 + 66n,$$

thus

$$211 = 2x^2 - 110n^2 - 66n = 2(x^2 - 55n^2 - 33).$$

This is not possible, since 211 is odd, so the ones digit of y cannot be 3.

Similarly, if the ones digit of y is 7, then $y = 10n + 7$ for some integer n. Then

$$\begin{aligned}20x^2 &= 2011 + 11y^2 = 2011 + 11(10n + 7)^2 \\ &= 2011 + 1100n^2 + 1540n + 539 = 2550 + 1100n^2 + 1540n,\end{aligned}$$

so

$$2x^2 = 255 + 110n^2 + 154n,$$

thus

$$255 = 2x^2 - 110n^2 - 154n = 2(x^2 - 55n^2 - 77).$$

This is not possible, since 255 is odd, so the ones digit of y cannot be 7 either. Therefore no natural numbers x and y satisfy the equation.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

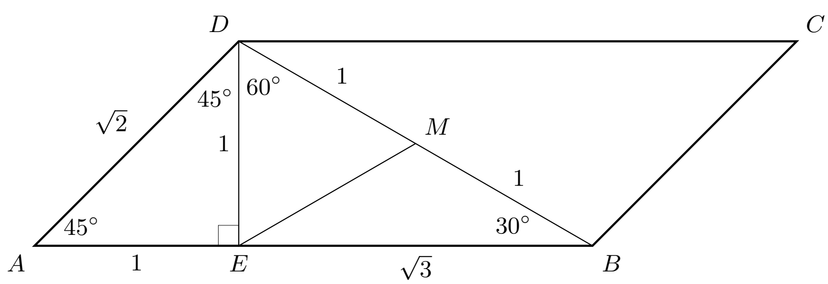
Note that we could have arrived at the same result a little quicker. If $20x^2 - 11y^2 = 2011$, *then* $20x^2 - 2000 = 11y^2 + 11$, *so* $20(x^2 - 100) = 11(y^2 + 1)$. *The left-hand side of this equation is clearly an even number, so* $y^2 + 1$ *must be even, so* y^2 *must be odd, so* y *is odd. Therefore* $y = 2k + 1$ *for some integer* k, *so* $y^2 + 1 = (2k + 1)^2 + 1 = 4k^2 + 4k + 2$, *so* $11(y^2 + 1) = 44k^2 + 44k + 22$. *If you divide this by* 4, *you get* $11k^2 + 11k + 5$ *and remainder* 2. *If you divide* $20(x^2 - 100)$ *by* 4, *the remainder is* 0, *therefore* $20(x^2 - 100)$ *cannot equal* $11(y^2 + 1)$, *so the equation has no (integer) solution.*

4. In the parallelogram $ABCD$, $\angle BAD = 45^\circ$ and $\angle ABD = 30^\circ$. Show that the distance from B to the diagonal AC is $\frac{1}{2}|AD|$.

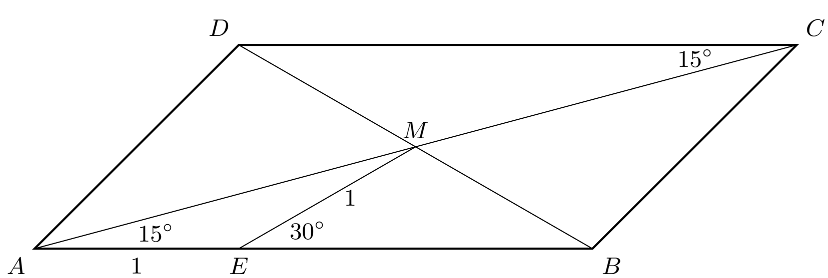
Solution by the editors.

Recall that the 45°–45°–90° triangle has sides in ratio $1 : 1 : \sqrt{2}$, and that the 30°–60°–90° triangle has sides in ratio $1 : \sqrt{3} : 2$. Moreover, recall that the diagonals of a parallelogram bisect each other.

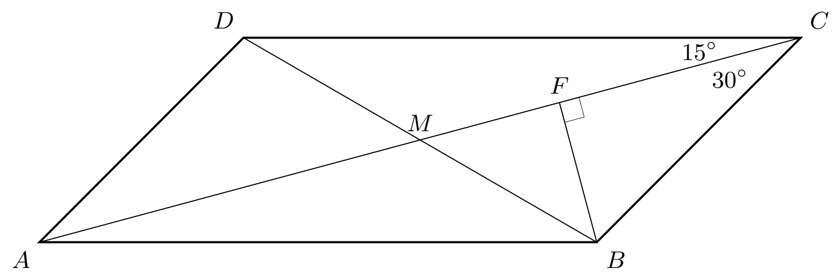
Let E be the closest point on AB to D, and let M be the midpoint of diagonal BD. Assume, without loss of generality, that $|AD| = \sqrt{2}$. Then $\triangle ADE$ is a 45°–45°–90° triangle, so $|AE| = |DE| = 1$. Moreover, $\triangle BDE$ is a 30°–60°–90° triangle, so $|EB| = \sqrt{3}$, and $|BD| = 2$. Since M is the midpoint of BD, $|BM| = |MD| = 1$. This accounts for all the labels in the figure below.



Since $|ED| = |MD|$ and $\angle EDM = 60^\circ$, $\triangle EDM$ is equilateral, and $|EM| = 1$. Therefore $\triangle BME$ is isosceles, and $\angle BEM = 30^\circ$. Thus $\angle AEM = 150^\circ$, and since $\triangle AEM$ is isosceles, $\angle EAC = 15^\circ$. It follows that $\angle ACD = 15^\circ$ since $AB \parallel CD$.



Finally, let F be the point on the diagonal AC closest to B. Since $\angle BCD = \angle BAD = 45^\circ$, $\angle ACB = 30^\circ$, so $\triangle CBF$ is a 30°–60°–90° triangle. Therefore $|BF| = \frac{1}{2}|BC|$.



Since $|BC| = |AD|$, it follows that the distance, $|BF|$, from B to AC is half of AD as required.

A solution by RICHARD I. HESS, Rancho Palos Verdes, CA, USA arrived after the due date.

5. What is the next year with four Friday the 13ths?

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

The table below lists the weekday of the 13th of each month in 2011 and 2012.

	J	F	M	A	M	J	J	A	S	O	N	D
2011:	Th	Su	Su	W	F	M	W	Sa	Tu	Th	Su	Tu
2012:	F	M	Tu	F	Su	W	F	M	Th	Sa	Tu	Th

The year 2012 has three Fridays but not more than three of any weekday. Any leap year will then have three of some weekday (indeed, three of the weekday that January the 13th happens to be), but no more than three. Similarly, 2011 has three Sundays but not more than three of any weekday, so no non-leap year will have more than three of any weekday.

Thus no year will ever have the 13th of the month land on the same weekday more than three times. In particular, no year will ever have four Friday the 13ths.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

This issue's prize of one copy of ***Crux Mathematicorum*** for the best solutions goes to Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

THE CONTEST CORNER

No. 3

Shawn Godin

The Contest Corner *is a new feature of* ***Crux Mathematicorum****. It will be filling the gap left by the movement of* Mathematical Mayhem *and* Skoliad *to a new on-line journal in 2013. The column can be thought of as a hybrid of* Skoliad, The Olympiad Corner *and the old* Academy Corner *from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers' solutions will begin to appear in the next volume.*

Solutions can be sent to:

Shawn Godin
Cairine Wilson S.S.
975 Orleans Blvd.
Orleans, ON, CANADA
K1C 2Z5

or by email to

`crux-contest@cms.math.ca`.

The solutions to the problems are due to the editor by **1 September 2013**.

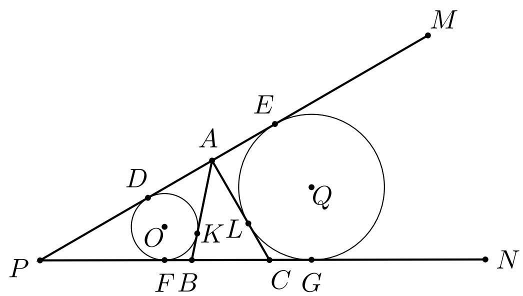
Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface, Winnipeg, MB for translating the problems from English into French.

CC11. Ten boxes are arranged in a circle. Each box initially contains a positive number of golf balls. A move consists of taking all of the golf balls from one of the boxes and placing them into the boxes that follow it in a counterclockwise direction, putting one ball into each box. Prove that if the next move always starts with the box where the last ball of the previous move was placed, then after some number of moves, we get back to the initial distribution of golf balls in the boxes.

CC12. Prove that $\sum \frac{1}{i_1 i_2 \cdots i_k} = 2001$, where the summation taken is over all non-empty subsets $\{i_1, i_2, \cdots, i_k\}$ of the set $\{1, 2, \cdots, 2001\}$.

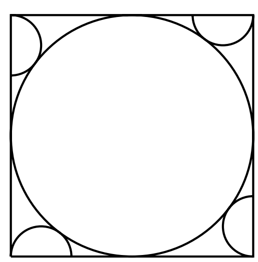
CC13. Triangle ABC has its base on line segment PN and vertex A on line PM. Circles with centres O and Q, having radii r_1 and r_2, respectively, are tangent to both PM and PN, and to the triangle ABC externally at K and L(as shown in the diagram).



(a) Prove that the line through K and L cuts the perimeter of triangle ABC into two equal pieces.
(b) Let T be the point of contact of BC with the circle inscribed in triangle ABC. Prove that $(TC)(r_1) + (TB)(r_2)$ is equal to the area of triangle ABC.

CC14. Evaluate $\int_0^\pi \ln(\sin x)dx$.

CC15. A circle is inscribed in a square. Four semicircles with their flat sides along the edge of the square and tangent to the circle are inscribed in each of the four spaces between the square and circle. What is the ratio of the area of the circle to the total area of the four semicircles?

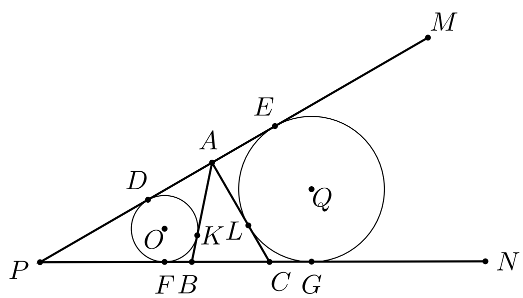


. . . .

CC11. Dix boîtes sont disposées en forme de cercle. Au départ, chaque boîte contient un nombre positif de balles de golf. Un jeu consiste à retirer toutes les balles de golf d'une boîte, pour ensuite les placer une par une dans chacune des boîtes qui suivent en ordre anti-horaire. Démontrer que si on décide de toujours choisir comme prochaine boîte de départ la dernière à recevoir une balle au jeu qui précède, alors, après un certain nombre de jeux, on reviendra à la distribution initiale de balles.

CC12. Démontrer que $\sum \frac{1}{i_1 i_2 \cdots i_k} = 2001$, où la somme est effectuée sur tous les sous-ensembles non vides $\{i_1, i_2, \cdots, i_k\}$ de l'ensemble $\{1, 2, \cdots, 2001\}$.

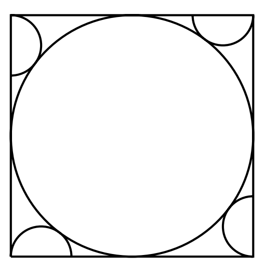
CC13. Le triangle ABC a sa base sur le segment PN et son sommet A sur la ligne PM. Un cercle est tracé avec centre O et rayon r_1, de façon à être extérieurement tangent au triangle ABC au point K, tangent à PM au point D, puis tangent à PN au point F. Similairement, un cercle est tracé avec centre Q et rayon r_2, de façon à être extérieurement tangent au triangle ABC au point L, tangent à PM au point E, puis tangent à PN au point G.



(a) Démontrer que la ligne passant par K et L coupe le périmètre du triangle ABC en deux parties égales.
(b) Soit T le point de contact de BC avec le cercle inscrit du triangle ABC. Démontrer que $(TC)(r_1) + (TB)(r_2)$ est égal à la surface du triangle ABC.

CC14. Évaluer l'intégrale $\int_0^{\pi} \ln(\sin x)dx$.

CC15. Un cercle est inscrit dans un carré. Quatre demi cercles sont ensuite inscrits entre cercle et le carré, de façon à être tangents au cercle, puis avec leurs diamètres situés sur les côtés du carré et aboutissant aux coins. Déterminer le ratio de la surface du cercle par rapport à la surface totale des quatre demi cercles.



THE OLYMPIAD CORNER

No. 301

Nicolae Strungaru

The solutions to the problems are due to the editor by **1 September 2013**.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC71. Define a_n a sequence of positive integers by $a_1 = 1$ and a_{n+1} being the smallest integer so that

$$\operatorname{lcm}(a_1, .., a_{n+1}) > \operatorname{lcm}(a_1, .., a_n)\,.$$

Find the set $\{a_n | n \in \mathbb{Z}\}$.

OC72. Prove that there are infinitely many positive integers so that the arithmetic and geometric mean of their divisors are integers.

OC73. Find all non-decreasing sequences $a_1, a_2, a_3, ...$ of natural numbers such that for each $i, j \in \mathbb{N}$, $i + j$ and $a_i + a_j$ have the same number of divisors. (a non-decreasing sequence is a sequence such that for all $i < j$, we have $a_i \leq a_j$.)

OC74. Let H be the orthocenter of an acute triangle ABC with circumcircle Γ. Let P be a point on the arc BC (not containing A) of Γ, and let M be a point on the arc CA (not containing B) of Γ such that H lies on the segment PM. Let K be another point on Γ such that KM is parallel to the Simson line of P with respect to triangle ABC. Let Q be another point on Γ such that $PQ \parallel BC$. Segments BC and KQ intersect at a point J. Prove that $\triangle KJM$ is an isosceles triangle.

OC75. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0$ be two polynomials with integral coefficients so that $a_n - b_n$ is a prime, $a_n b_0 - a_0 b_n \neq 0$, and $a_{n-1} = b_{n-1}$. Suppose that there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that $r \in \mathbb{Z}$.

OC71. On définit une suite a_n d'entiers positifs avec $a_1 = 1$ et a_{n+1} comme étant le plus petit entier tel que

$$\text{ppcm}(a_1, .., a_{n+1}) > \text{ppcm}(a_1, .., a_n)\,.$$

Trouver l'ensemble $\{a_n | n \in \mathbb{Z}\}$.

OC72. Montrer qu'il y a une infinité d'entiers positifs tels que les moyennes arithmétique et géométrique de leurs diviseurs sont des entiers.

OC73. Trouver toutes les suites non décroissantes $a_1, a_2, a_3, ...$ de nombres naturels telles pour chaque $i, j \in \mathbb{N}$, $i+j$ et $a_i + a_j$ ont le même nombre de diviseurs. (une suite non décroissante est une suite telle que $a_i \leq a_j$ dès que $i < j$.)

OC74. Soit H l'orthocentre d'un triangle acutangle ABC avec Γ comme cercle circonscrit. Soit P un point sur l'arc BC (ne contenant pas A) de Γ, et soit M un point sur l'arc CA (ne contenant pas B) de Γ de sorte que H soit sur le segment PM. Soit K un autre point sur Γ de telle sorte que KM soit parallèle à la droite de Simson de P par rapport au triangle ABC. Soit Q un autre point sur Γ tel que $PQ \parallel BC$. Soit J le point d'intersection des segments BC et KQ. Montrer que $\triangle KJM$ est un triangle isocèle.

OC75. Soit $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ et $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ deux polynômes à coefficients entiers tels que $a_n - b_n$ est un nombre premier, $a_n b_0 - a_0 b_n \neq 0$, et $a_{n-1} = b_{n-1}$. Supposons qu'il existe un nombre rationnel r tel que $P(r) = Q(r) = 0$. Montrer que $r \in \mathbb{Z}$.

OLYMPIAD SOLUTIONS

OC11. For non-empty subsets $A, B \subseteq \mathbb{Z}$ define $A + B$ and $A - B$ by

$$A + B = \{a + b \mid a \in A, b \in B\}\,,\ A - B = \{a - b \mid a \in A, b \in B\}\,.$$

In the sequel we work with non-empty finite subsets of $\mathbb{Z}$.

Prove that we can cover B by at most $\dfrac{|A+B|}{|A|}$ translates of $A - A$, i.e. there exists $X \subseteq \mathbb{Z}$ with $|X| \leq \dfrac{|A+B|}{|A|}$ such that

$$B \subseteq \bigcup_{x \in X} (x + (A - A)) = X + A - A\,.$$

(*Originally question #1 from the 1st selection test, 60th National Mathematical Olympiad Selection Tests for the Balkan and IMO, by Imre Rusza, Hungary.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

For a subset $X \subset B$, consider the family $\mathcal{F}_X := \{A + x | x \in X\}$. Since B is finite, we can find a subset $X \subset B$ so that the elements of $\mathcal{F}_X$ are pairwise disjoint, and X is maximal with this property, that is:

(1) For all $x, y \in X$ with $x \neq y$ we have $(x + A) \cap (y + A) = \emptyset$,

(2) For each $b \in B$ there exists some $x \in X$ so that $(x + A) \cap (b + A) \neq \emptyset$.

We claim that this X has the required properties.
First, by (1) we have

$$|X| \cdot |A| = \sum_{x \in X} |x + A| = |A + X| \leq |A + B| \ .$$

Thus

$$|X| \leq \frac{|A + B|}{|A|} \,.$$

Now, let $b \in B$. Then, by the second condition there exists some $x \in A$ such that

$$(b + A) \cap (x + A) \neq \emptyset \,.$$

Let $z \in (b + A) \cap (x + A)$. Then, there exists $a, a' \in A$ so that

$$z = b + a = x + a' \,.$$

Thus

$$b = x + a' - a \in X + (A - A) \,,$$

which completes the proof.

OC12. Let k be a positive integer greater than 1. Prove that for every non-negative integer m there exist k positive integers $n_1, n_2, \ldots, n_k$, such that

$$n_1^2 + n_2^2 + \cdots + n_k^2 = 5^{m+k} \,.$$

(*Originally question #2 from the 53rd National Mathematical Olympiad in Slovenia, 2nd Selection Exam for the IMO 2009.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania. We give the solution of Geupel.

We prove the statement by induction on k.
P(2): If $m = 2q$ then we have

$$(3 \cdot 5^q)^2 + (4 \cdot 5^q)^2 = 5^{m+2} \,.$$

If $m = 2q - 1$ then we have

$$(5^q)^2 + (2 \cdot 5^q)^2 = 5^{m+2} \,.$$

P(3): If $m = 2q$ then we have

$$(3 \cdot 5^q)^2 + (4 \cdot 5^q)^2 + (10 \cdot 5^q)^2 = 5^{m+3} \,.$$

If $m = 2q - 1$ then we have

$$(12 \cdot 5^{q-1})^2 + (15 \cdot 5^{q-1})^2 + (16 \cdot 5^{q-1})^2 = 5^{m+3} \,.$$

Now we show that $P(k) \Rightarrow P(k+2)$:

By $P(2)$, we can find n_1, n_2 so that

$$n_1^2 + n_2^2 = 5^{m+k+1}\,.$$

Also, by $P(k)$ we can find $n_3, .., n_{k+2}$ so that

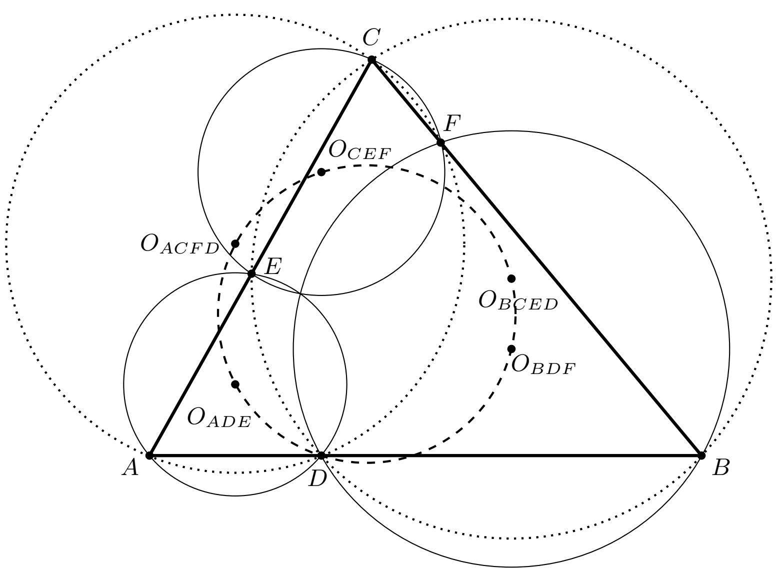
$$n_3^2 + ... + n_{k+2}^2 = 5^{m+k+1}\,.$$

Then

$$n_1^2 + n_2^2 + (2n_3)^2 + ... + (2n_{k+2})^2 = 5^{m+k+1} + 4 \cdot 5^{m+k+1} = 5^{m+k+2}\,.$$

OC13. Let ABC be an acute triangle and let D be a point on the side AB. The circumcircle of the triangle BCD intersects the side AC at E. The circumcircle of the triangle ADC intersects the side BC at F. Let O be the circumcentre of triangle CEF. Prove that the points D and O and the circumcentres of the triangles ADE, ADC, DBF and DBC are concyclic and the line OD is perpendicular to AB. (*Originally question #2 from the 53rd National Mathematical Olympiad in Slovenia, 1st Selection Exam for the IMO 2009.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania . We give the solution of Geupel .



Let us denote by $\alpha = \angle BAC, \beta = \angle CBA, \gamma = \angle ACB$. Let O_{ADE}, O_{BDF}, O_{ACFD}, and O_{BCED} denote the circumcircles of $\triangle ADE$, $\triangle BDF$, quadrilateral $ACFD$, and quadrilateral $BCED$, respectively. Let $O_{CEF} := O$.

The points O_{ACFD} and O_{CEF} are on the perpendicular bisector of CF, thus $O_{ACFD}O_{CEF} \perp CF$. Similarly, $O_{BCED}O_{CEF} \perp CE$.

Since $\angle ECF = \gamma$, we have

$$\angle O_{ACFD} O_{CEF} O_{BCED} \in \{\gamma,\ 180^\circ - \gamma\}. \tag{1}$$

We have $O_{ACFD}O_{ADE} \perp AD$ and $O_{BCED}O_{ADE} \perp DE$. Since B, C, D, and E are concyclic, we have $\angle ADE = 180^\circ - \angle BDE = BCE = \gamma$. Thus,

$$\angle O_{ACFD} O_{ADE} O_{BCED} \in \{\gamma,\ 180^\circ - \gamma\}. \tag{2}$$

Similarly,

$$\angle O_{ACFD} O_{BDF} O_{BCED} \in \{\gamma,\ 180^\circ - \gamma\}. \tag{3}$$

Combining (1), (2), and (3), we get that the points O_{ADE}, O_{BDF}, O_{CEF}, O_{ACFD}, and O_{BCED} are on a common circle Γ. We now prove that $D \in \Gamma$.

Since $O_{ACFD}O_{ADE} \perp AD$ and $O_{BCED}O_{BDF} \perp BD$, we have

$$O_{ACFD}O_{ADE} \parallel O_{BCED}O_{BDF}\,.$$

Thus, the chords $O_{ACFD}O_{BCED}$ and $O_{ADE}O_{BDF}$ of the circle Γ are congruent.

Moreover, in $\triangle ADE$, we have $\angle ADO_{ADE} = 90^\circ - \angle AED = 90^\circ - \beta$. Similarly, $\angle BDO_{BDF} = 90^\circ - \alpha$. Thus,

$$\angle O_{ADE} D O_{BDF} = 180^\circ - \angle ADO_{ADE} - \angle BDO_{BDF} = \gamma\,,$$

so that $D \in \Gamma$.

We now show that $\angle ADO_{CEF} = 90^\circ$.

In $\triangle CEF$, we have $\angle EO_{CEF}F = 2\gamma$. Since

$$\angle EDF = 180^\circ - \angle ADE - \angle BDF = 180^\circ - 2\gamma\,.$$

the points D, E, F, O_{CEF} are concyclic.

By $EO_{CEF} = FO_{CEF}$, we deduce the corresponding arcs are equal, thus

$$\angle EDO_{CEF} = \angle EDF/2 = 90^\circ - \gamma\,.$$

Consequently,

$$\angle ADO_{CEF} = \angle ADE + \angle EDO_{CEF} = \gamma + (90^\circ - \gamma) = 90^\circ\,.$$

OC14. Let a_n, b_n, $n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1$, $b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6\,,$$
$$b_{n+1} = 4a_n + 7b_n + 3\,.$$

Prove that a_n^2 is the difference of two consecutive cubes. (*Originally question #2 from the Singapore Mathematical Olympiad 2010, Open Section, Round 2.*)

Solved by Arkady Alt, San Jose, CA, USA ; Michel Bataille, Rouen, France ; Felix Boos, University of Kaiserslautern, Kaiserslautern, Germany ; Oliver

Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Alex Song, Phillips Exeter Academy, NH, USA and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; Titu Zvonaru, Cománeşti, Romania. We give the solution of Song and Wang.

We prove by induction that

$$a_n^2 = (b_n+1)^3 - b_n^3 = 3b_n^2 + 3b_n + 1\,.$$

The initial step is obvious. We now proceed to proving the inductive step.

$$\begin{aligned}
&a_{n+1} - (3b_{n+1}^2 + 3b_{n+1} + 1)\\
&\quad = (7a_n + 12b_n + 6)^2 - 3(4a_n + 7b_n + 3)^2 - 3(4a_n + 7b_n + 3) - 1\\
&\quad = (49a_n^2 + 144b_n^2 + 168a_nb_n + 84a_n + 144b_n + 36)\\
&\qquad\quad - 3(16a_n^2 + 49b_n^2 + 56a_nb_n + 24a_n + 42b_n + 9) - 3(4a_n + 7b_n + 3) - 1\\
&\quad = a_n^2 - 3b_n^2 - 3b_n - 1 = 0
\end{aligned}$$

which completes the proof.

OC15. A ruler of length ℓ has $k \geq 2$ marks at positions a_i units from one of the ends with $0 < a_1 < \cdots < a_k < \ell$. The ruler is called a *Golomb ruler* if the lengths that can be measured using the marks on the ruler are consecutive integers starting with 1, and each such length be measurable between a unique pair of marks on the ruler. Find all Golomb rulers. (*Originally question #4 from the 60th National Mathematical Olympiad Selection Tests for the Balkan and IMO, 2nd selection test, by Barbu Berceanu.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

For simplicity let us identify a Golomb ruler G with the set $\{a_1, \ldots, a_k\}$ of positions of its marks. Since G can measure the lengths $1, \ldots, \binom{k}{2}$ by definition, we have $a_k = a_1 + \binom{k}{2}$.

Further, we have either $a_2 = a_1 + 1$ or $a_k = a_{k-1} + 1$, since the length $\binom{k}{2} - 1$ can be measured.

Let us call the Golomb ruler $G' = \{a'_1, \ldots, a'_k\}$ a *shift* of G if $a'_i = a_i + d$ for $1 \leq i \leq k$ and a constant d.

Also let us call G' the *reflection* of G if $a'_i = a_k + 1 - a_{k+1-i}$ for $1 \leq i \leq k$.

We will say that G and G' are *equivalent* if G' is a shift of G or of the reflection of G. Equivalence, in fact, is an equivalence relation on the set of all Golomb rulers.

For each equivalence class there exists a unique member such that $a_1 = 1$ and $a_2 = 2$. We will call this Golomb ruler canonical, and we will determine it.

We first claim that there exists no Golomb ruler with $k > 4$.

Let's assume by contradiction that there exists a Golomb ruler with $k > 4$. Then there exists a canonical Golomb ruler G equivalent to it. Let's look at G.

For simplicity, let's denote $n := \binom{k}{2}$. We know that $n \geq 10$ and $1,\ 2,\ n+1 \in G$.

We know that $n-2$ can be measured with G. The only ways we can get $n-2$ is $(n-1)-1$ or $n-2$ or $(n+1)-3$. It is not possible to have $3 \in G$ or $n \in G$, because in this case we would get two ways of measuring 1. Thus, we must obtain measure $n-2$ between 1 and $n-1$, and hence

$$\{1, 2, n-1, n+1\} \subset G\,.$$

Now, $n-4$ can also be measured with G. Thus, one of the pairs $(1, n-3), (2, n-2), (3, n-1), (4, n)$ or $(5, n+1)$ must appear in G. Again, by unique measurability, we cannot have $3, 4, n-2, n-3$ in G [Note that $4 < n-2$]. Thus $5 \in G$.

So far we have

$$\{1, 2, 5, n-1, n+1\} \subset G\,.$$

Case 1: $k = 5$. Then $n = 10$ and our G is $G = \{1, 2, 5, 9, 11\}$. But this is not a Golomb ruler, since $9 - 5 = 5 - 1$.

Case 2: $k \geq 6$. Since the length $n-5$ can be measured using G, at least one of the pairs $\{1,\ n-4\}$, $\{2,\ n-3\}$, $\{3,\ n-2\}$, $\{4,\ n-1\}$, $\{5,\ n\}$, and $\{6,\ n+1\}$ must be contained in G. Moreover $6 < n-4$.

But for reasons of unique measurability, the numbers $n-4$, $n-3$, $n-2$, 4, n, and 6 are not elements of G. This is a contradiction, as desired.

We know now that $k \leq 4$.

If $k = 2$, the only canonical Golomb ruler is $\{1, 2\}$.

If $k = 3$, then $\binom{3}{2} + 1 = 4$, thus the only canonical Golomb ruler is $\{1, 2, 4\}$.

If $k = 4$, then $\binom{4}{2} + 1 = 7$, thus any canonical Golomb ruler must contain $\{1, 2, 7\}$ and one more number. Let that number be x. The only ways to measure a segment of length 4 is if $x - 1 = 4$ or $x - 2 = 4$ or $7 - x = 4$, and it is easy to see that only $x = 5$ creates a Golomb ruler.

Thus, the only canonical Golomb Rulers are

$$\{1, 2\}\,;\ \{1, 2, 4\}\,;\ \{1, 2, 5, 7\}\,.$$

This means that all the Golomb Rulers are the following

$$\{\{d, d+1\}\,;\ \{d, d+1, d+3\}\,;\ \{d, d+2, d+3\}\,;\ \{d, d+1, d+4, d+6\}\,;$$
$$\{d, d+2, d+5, d+6\} | d > 0\}\ .$$

BOOK REVIEWS

Amar Sodhi

Expeditions in Mathematics
Edited by Tatiana Shubin, David F. Hayes, and Gerald L. Alexanderson
MAA Spectrum Series. The Mathematical Association of America 2011
ISBN: 978-0-88385-571-3, 312 + xiv pp., hardcover, US$60.95
Reviewed by ***S. Swaminathan****, Dalhousie University, Halifax, N. S.*

Almost every professor who teaches the first year mathematics class notices, and gets frustrated with, the gap that exists between the school and university curriculum. There is serious and hard mathematics lurking behind the innocent high school courses. This was noticed even at the beginning of the 20th century when Felix Klein published his famous book, *Elementary Mathematics from an Advanced Standpoint*. The Bay Area Mathematical Adventures (BAMA) program, that was begun in the Fall of 1998, aims at a remedy to this situation by sponsoring lectures targeted at bright middle school or high school students and their teachers. First class mathematicians are invited to the campuses of San Jose State University and Santa Clara University to speak to students on a broad range of topics of current interest in mathematics. The book under review is the second volume published by MAA containing written versions of some of the best talks given at these lectures.

There are five chapters titled General; Number Theory; Geometry & Topology; Combinatorics & Graph Theory; and Applied Mathematics. Under these headings, the topics include progress toward proving the twin primes conjecture; surprising mathematical paradoxes; facts about unnatural sequences of integers; applications of topology to questions in chemistry; ways of deciding when a tangle of string is actually a knot; how the medieval ranking of angels was related to the location of the planets, and by whom; the volume of a tetrahedron formed by a space rhombus; how the heavenly bodies seem to behave when viewed from the tropics and from the Southern Hemisphere; the latest techniques in cryptography; and determining preferences in voting.

Here are a few highlights to illustrate the wide scope of the lectures:

John Stillwell on 'Yearning for the Impossible': "*In the past, mathematicians have been faced with seeming impossibilities, yet mathematics grew by embracing them. I want to talk about few such events, where the struggle with an apparent impossibility became a turning point in the development of mathematics. The impossible is sometimes a valuable new idea.*"

Tom Davis on 'The Mathematics of Sudoku': "*The article begins by examining some logical and mathematical approaches to solving Sudoku puzzles beginning with the most obvious and continuing on to more and more sophisticated techniques. Later a few of the more mathematical aspects are discussed.*"

John H. Conway and Tim Hsu: 'Some Very Interesting Sequences': "*We hope that anyone reading this is a Nerd. Nerds have been interested in sequences of numbers throughout the ages... We will be discussing some of our favorite sequences and their remarkable properties, let's face it — only a real Nerd is likely to be interested.*"

D.A.Goldston: 'Are There Infinitely Many Twin Primes?': An exhaustive survey of this open problem': "*You are welcome to prove this conjecture and become famous, but be warned that a great deal of effort has already been expended on this problem... Therefore put in some effort into understanding what has been learned about primes in the last 200 years.*"

J.B.Conrey: 'The Riemann Hypothesis': This article describes one of the most fundamental unsolved problems in mathematics.

Erica Flapen: 'A Topological Approach to Molecular Chirality': "*Topology is the study of deformations of geometric figures. Chemistry is the study of molecular structures. At first glance these fields seem to have nothing in common. But let's take a closer look to see how these fields come together in the study of molecular symmetries.*"

Helmer Aslaksen: 'Heavenly Mathematics: Observing the Sun and the Moon from Different Parts of the World': Section headings are: Introduction, Where does the Sun Rise?; Which way does the Sun move in the course of the day; What time does the Sun rise?; What does the orbit of the Moon look like?; What does a waxing crescent look like?

Steven G. Krantz: 'Zero Knowledge Proofs': "*In today's complex world, and with the advent of high-speed digital computers, there are new demands on the technology of cryptography. The present brief article will discuss some of these considerations.*"

Helen Moore: 'Mathematical Recipe': "*If you dream of discovering a cure for cancer, you would contribute to this cause by majoring math in college (while taking plenty of biology), going to graduate school, and becoming an applied mathematician working in mathematical biology. In this chapter, you will see examples for HIV and leukemia in which mathematics played a key role in answering important questions.*"

Francis Edward Su: 'The Agreeable Society Theorem': "*In this article, we shall examine preferences in the context of voting, and the use of mathematics to make some predictions about how people vote.*"

Thus this book will enrich the reader's knowledge and appreciation of the role of mathematics in various spheres of thought.

FOCUS ON ...

No. 1

Michel Bataille

Integer Part and Periodicity

Introduction

A T-periodic function which vanishes over a period, i.e. on an interval $[a, a+T)$, is the zero function. This obvious property can, rather unexpectedly, lead to elegant proofs for some identities involving the integer part function. For a simple and classical example, consider the general formula

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = \lfloor nx \rfloor \qquad (1)$$

where n is any positive integer and x any real number. Various proofs of (1) are possible, but my favorite one is as follows:

Fix n and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor - \lfloor nx \rfloor$. Recalling that $\lfloor x + k \rfloor = \lfloor x \rfloor + k$ whenever k is an integer, an easy calculation yields $f\left(x + \frac{1}{n}\right) = f(x)$, showing that f is a $\frac{1}{n}$-periodic function. In addition, if $0 \le x < \frac{1}{n}$, then $\lfloor nx \rfloor$ and all terms $\left\lfloor x + \frac{i}{n} \right\rfloor$ in the sum vanish, hence $f(x) = 0$. Thus, the periodic function f is zero over a period, hence is the zero function. $\square$

Other examples will show that this connection between the floor function and periodicity is worth being kept in mind.

Another simple example

Let us start with another familiar result: $\frac{x}{n}$ and $\frac{\lfloor x \rfloor}{n}$ have the same integer part (as before x is any real number and n any positive integer). Inspired by the example above, we obtain the following short and elegant proof:

For $x \in \mathbb{R}$, let $g(x) = \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor$. Then, $g(x+n) = g(x)$ for all real numbers x and for $x \in [0, n)$, both $\left\lfloor \frac{x}{n} \right\rfloor$ and $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor$ vanish (the latter since $\lfloor x \rfloor \in \{0, 1, \ldots, n-1\}$), hence $g(x) = 0$. $\square$

Generalizing one mathematical challenge out of 500

One of the problems of [1] requires a proof of the equality $\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor$ for positive integers n. Actually, the result remains valid if one allows n to be any real number. To see this, consider the function h defined for all real numbers x by the formula

$$h(x) = \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+2}{6} \right\rfloor + \left\lfloor \frac{x+4}{6} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor - \left\lfloor \frac{x+3}{6} \right\rfloor .$$

It is straightforward to check the periodicity of h: $h(x+6) = h(x)$ for all x.

Now, suppose that $-3 \leq x < -2$. Then, examining each term in succession, we obtain $h(x) = (-1) + (-1) + 0 - (-2) - 0 = 0$. It is elementary to verify that, in a similar way, $h(x) = 0$ when $-2 \leq x < 0$, $0 \leq x < 2$ and $2 \leq x < 3$. As a result, $h(x) = 0$ for $x \in [-3, 3)$ and we are done.

A more involved example

In conclusion and as a last example, we offer the following generalization of (1) proposed by Mihály Bencze in problem 1785 of the Mathematics Magazine in 2007:

$$\sum_{j=1}^{n} \left\lfloor x + \frac{j-1}{n} \right\rfloor^k = n\lfloor x\rfloor^k + \left((\lfloor x\rfloor + 1)^k - \lfloor x\rfloor^k\right) \lfloor n\{x\}\rfloor . \tag{1}$$

Here k is a positive integer and $\{x\}$ denotes the fractional part of x, i.e. $\{x\} = x - \lfloor x\rfloor$.

The interesting featured solution by M. Gertz and D. Jones (see [2]) does not use periodicity, but a solution in the vein of the above examples can also be given:

As before, let $f(x) = \ell(x) - r(x)$ where $\ell(x)$ and $r(x)$ are the left-hand side and the right-hand side of (2). For $x \in [0, \frac{1}{n})$, we easily obtain $\ell(x) = r(x) = 0$, hence $f(x) = 0$. To prove the equality $f\left(x + \frac{1}{n}\right) = f(x)$, we first observe that $\ell(x + \frac{1}{n}) = \ell(x) - \lfloor x\rfloor^k + (\lfloor x\rfloor + 1)^k$. As for $r(x + \frac{1}{n})$, we distinguish two cases.

Case 1: If $x + \frac{1}{n} < \lfloor x\rfloor + 1$, then $\lfloor x + \frac{1}{n}\rfloor = \lfloor x\rfloor$ so that $\lfloor n\{x + \frac{1}{n}\}\rfloor = \lfloor nx\rfloor + 1 - n\lfloor x\rfloor = \lfloor n\{x\}\rfloor + 1$. It readily follows that $r\left(x + \frac{1}{n}\right) = r(x) + (\lfloor x\rfloor + 1)^k - \lfloor x\rfloor^k$.

Case 2: If $x + \frac{1}{n} \geq \lfloor x\rfloor + 1$, then $\lfloor x + \frac{1}{n}\rfloor = \lfloor x\rfloor + 1$. We now obtain $\lfloor n\{x + \frac{1}{n}\}\rfloor = \lfloor n\{x\} - (n-1)\rfloor = 0$ (the latter because $\lfloor x\rfloor + 1 \leq x + \frac{1}{n} < \lfloor x\rfloor + 1 + \frac{1}{n}$ so that $n\lfloor x\rfloor \leq nx - (n-1) < n\lfloor x\rfloor + 1$). It follows that $r(x + \frac{1}{n}) = n(\lfloor x\rfloor + 1)^k$. Since $\lfloor n\{x\}\rfloor = n-1$, we have $r(x) = \lfloor x\rfloor^k + (n-1)(\lfloor x\rfloor + 1)^k$ and again $r(x + \frac{1}{n}) - r(x) = (\lfloor x\rfloor + 1)^k - \lfloor x\rfloor^k$.

In both cases $f(x + \frac{1}{n}) = f(x)$. $\square$

Other examples would be interesting and are welcome!

References

[1] E.J. Barbeau, M.S. Klamkin, W.O.J. Moser, *Five Hundred Mathematical Challenges*, MAA, 1995, p. 4.

[2] *Math. Magazine*, Vol. 81, No 5, December 2008, p. 379-80.

A Typical Problem on an Entrance Exam for the École Polytechnique

Ross Honsberger

Construct a triangle given its area and perimeter and one of its angles.

(a) Let the desired triangle be ABC with area K, perimeter p, and given angle $\angle A$. Let h denote the length of the altitude to base $BC(= a)$. Then

$$K = \frac{1}{2}ah,$$

and if a is known, then h can be calculated.

Now, if the dimensions a, h, and $\angle A$ of ABC are known, then the triangle can easily be constructed as follows.

1. On a segment $BC = a$ as chord, construct the segment of a circle containing the given $\angle A$.
2. At a distance h from BC construct straight line XY parallel to BC to cross the circle at A and possibly at a second point A'.

Then ABC is the desired triangle.

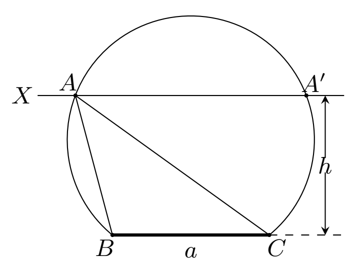


Figure 1

Since reflection in the perpendicular bisector of BC takes ABC to $A'BC$, the triangles are identical, and hence there exists *only one* triangle having the dimensions a, h, and $\angle A$. That is to say, the solution triangle is unique, and clearly, everything hinges on finding a: then h can be found from K, and with the given $\angle A$, ABC can be constructed as described.

We don't have to worry about our triangle having the given perimeter for, being unique, the only perimeter it can possibly have is p. Thus it remains only to determine a, which we can do as follows.

The formula for the area of a triangle gives

$$K = \frac{1}{2}bc \sin A,$$

from which

$$bc = \frac{2K}{\sin A}.$$

Hence bc can be found from the given values of K and $\angle A$.

From the law of cosines we have

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

giving

$$b^2 + c^2 - a^2 = 2bc\cos A,$$

which can be found from the known values of bc and $\angle A$.

Now,

$$p - a = b + c.$$

Squaring gives

$$p^2 - 2ap + a^2 = b^2 + 2bc + c^2.$$

Hence

$$p^2 - 2ap - 2bc = b^2 + c^2 - a^2,$$

which was just determined above, and from which a can be determined from the known values of p and bc, completing the solution.

(b) A Sequel: Much Ado About Nothing

We were always taught that, faced with a given perimeter p, it is generally a good idea to unfold a figure along a segment of length p. Accordingly, let sides AB and AC of a required triangle ABC be folded out along the line of BC to yield a segment PQ of length p (Figure 2).

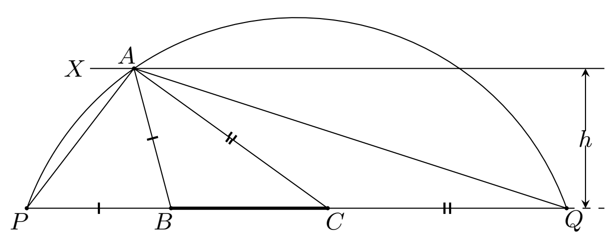


Figure 2

This creates two isosceles outer triangles whose base angles are $\frac{B}{2}$ and $\frac{C}{2}$, respectively (being half their exterior angles B and C). Hence

$$\angle PAQ = \frac{B}{2} + A + \frac{C}{2} = \left(\frac{B}{2} + \frac{A}{2} + \frac{C}{2}\right) + \frac{A}{2} = 90^\circ + \frac{A}{2}.$$

Thus vertex A lies on the arc of the segment of a circle on chord PQ which contains an angle of $90^\circ + \frac{A}{2}$. This arc can be drawn since $\angle A$ is given.

Now, the altitude h to BC can be calculated as described above (first find a, then use $h = \frac{2K}{a}$) and a straight line XY parallel to PQ can be drawn at a distance h from it. Thus vertex A lies on both XY and the arc and is accordingly one of their points of intersection. Now let's begin our construction (Figure 3).

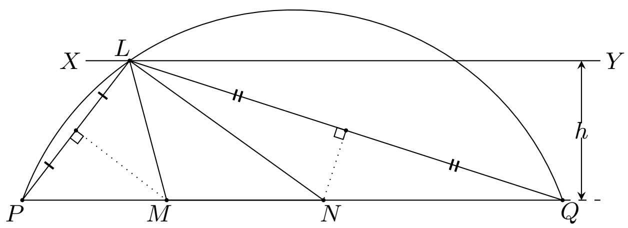


Figure 3

On segment $PQ = p$ as chord, construct the segment of a circle containing an angle of $90^\circ + \frac{A}{2}$. After calculating h, construct XY to meet the arc at L. Then draw the perpendicular bisectors of LP and LQ to get M and N on PQ. Then ΔLMN is a required triangle.

It is customary in the solution of a construction problem to follow the proposed construction with a proof that it accomplishes the desired result. Therefore we are under the obligation to prove that ΔLMN has perimeter p, $\angle L = \angle A$, and has the given area K.

Clearly, the perpendicular bisectors take care of the perimeter. Also, the total angle at L is

$$\frac{M}{2} + L + \frac{N}{2} = 90^\circ + \frac{L}{2} = 90^\circ + \frac{A}{2}$$

(by construction) , implying $\angle L = \angle A$.

But what is the area of ΔLMN? We know its altitude h to MN, but we didn't specifically make $MN = a$ ($= BC$). We could have chosen to draw only the perpendicular bisector of LP to get M and then laid off $MN = a$ (which was found in the calculation of h). This would give ΔLMN the required area K, but then, to show the perimeter equals p we would be required to show that $NQ = LN$, which could be troublesome. Anyway, we don't have to worry about this since we chose to draw the second perpendicular bisector to get N. However, all this rambling doesn't take away our need to show the area of ΔLMN is K.

I stewed about this on and off for some considerable time and finally abandoned the whole approach. It was only after completing the solution in part (a) that it dawned on me that the solution in part (b) *doesn't need any proof at all*: obviously we have made in Figure 3 an exact copy of Figure 2, and therefore ΔLMN, being a copy of ΔABC, enjoys all the required properties.

I can only plead temporary insanity in losing sight of the goal of our construction. I hope I'm alright again now.

Ross Honsberger
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, ON
rosshonsberger@rogers.com

PROBLEMS

Solutions to problems in this issue should arrive no later than ***1 September 2013****.* *An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

Note*: Due to an editorial mix-up, problems 3712 [2012 : 63, 65] and 3719 [2012 : 65,67] were the same problem. In this issue we replace 3719 with a new problem.*

3719. *Replacement. Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $a, b, c > 0$, then

$$\frac{a}{\sqrt{b^2+\frac{1}{4}bc+c^2}}+\frac{b}{\sqrt{c^2+\frac{1}{4}ca+a^2}}+\frac{c}{\sqrt{a^2+\frac{1}{4}ab+b^2}}\geq 2.$$

3721. *Proposed by Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.*

Given the triangle ABC and two isogonal cevians AA', AA'', call B', C' the orthogonal projections of B, C on AA' and B'', C'' the orthogonal projections of B, C on AA''. If $P = B'C'' \cap C'B''$ and $Q = B'B'' \cap C'C''$, show that P lies on line BC and Q lies on the altitude through A. *Dedicated to the memory of Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

3722. *Proposed by Michel Bataille, Rouen, France.*

Prove that

$$\left(\frac{1}{4}-4\cos^2\frac{2\pi}{17}\cos^2\frac{8\pi}{17}\right)\left(\frac{1}{4}-4\cos^2\frac{3\pi}{17}\cos^2\frac{5\pi}{17}\right)+4\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{5\pi}{17}\cos\frac{8\pi}{17}=0.$$

3723. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c be positive real numbers such that $a + b + c = 1$. If n is a positive integer, prove that

$$\frac{(3a)^n}{(b+1)(c+1)}+\frac{(3b)^n}{(c+1)(a+1)}+\frac{(3c)^n}{(a+1)(b+1)}\geq\frac{27}{16}.$$

3724. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one of the legs. One circle of the pair is tangent to the hypotenuse. The other circle is tangent to the other leg. Determine the radii of the circles in both cases.

3725. *Proposed by Cîrnu Mircea, Bucharest, Romania.*

Prove that the sequence of nonzero real numbers, $x_1, x_2, \ldots$, is a geometric progression if and only if it satisfies the recurrence relation

$$nx_1x_n = \sum_{k=1}^{n} x_k x_{n+1-k},\ n = 1, 2, \ldots .$$

3726. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C, s, r, R represent the angles (measured in radians), the semi-perimeter, the inradius and the circumradius of a triangle, respectively. Prove that

$$\left(\frac{A}{B} + \frac{B}{C} + \frac{C}{A}\right)^3 \geq \frac{2s^2}{Rr}.$$

3727. *Proposed by J. Chris Fisher, University of Regina, Regina, SK.*

Let $ABCD$ and $AECF$ be two parallelograms with vertices E and F inside the region bounded by $ABCD$. Prove that line BE bisects segment CF if and only if BF meets AD in a point G that satisfies

$$\frac{DA}{DG} = \frac{BF}{FG}.$$

3728. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Given a continuous function $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ that satisfies

$$\int_0^{\frac{\pi}{2}} \Big([f(x)]^2 - 2f(x)(\sin x - \cos x)\Big)\, dx = 1 - \frac{\pi}{2},$$

show that

$$\int_0^{\frac{\pi}{2}} f(x)dx = 0.$$

3729. *Proposed by Vo Quoc Ba Can, Can Tho University of Medicine and Pharmacy, Can Tho, Vietnam.*

If a, b, c are the side lengths of a triangle, prove that

$$\frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab} \leq \frac{3(a+b+c)}{ab+bc+ca}.$$

3730. *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Points D, E and F are the feet of the perpendiculars from some point P in the plane to the lines BC, CA and AB determined by the sides of an equilateral triangle ABC. Prove that the cevians AD, BE, CF are concurrent (or parallel) if and only if at least one of D, E or F is a midpoint of its side.

. . . .

3719. *Proposé par Pham Van Thuan, Université de Science des Hanoï, Hanoï, Vietnam.*

Montrer que si $a, b, c > 0$, alors

$$\frac{a}{\sqrt{b^2+\frac{1}{4}bc+c^2}}+\frac{b}{\sqrt{c^2+\frac{1}{4}ca+a^2}}+\frac{c}{\sqrt{a^2+\frac{1}{4}ab+b^2}} \geq 2.$$

3721. *Proposé par Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.*

Donné un triangle ABC et deux céviennes isogonales AA', AA'', notons B', C' les projections orthogonales de B', C' sur AA' , et B'', C'' les projections orthogonales de B, C sur AA''. Si $P = B'C'' \cap C'B''$ et $Q = B'B'' \cap C'C''$, alors P est sur la droite BC et Q est sur la hauteur issue de A. *En mémoire de Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

3722. *Proposé par Michel Bataille, Rouen, France.*

Montrer que

$$\left(\frac{1}{4}-4\cos^2\frac{2\pi}{17}\cos^2\frac{8\pi}{17}\right)\left(\frac{1}{4}-4\cos^2\frac{3\pi}{17}\cos^2\frac{5\pi}{17}\right)+4\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{5\pi}{17}\cos\frac{8\pi}{17}=0.$$

3723. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b, c trois nombres réels positifs tels que $a+b+c=1$. Si n est un entier positif, montrer que

$$\frac{(3a)^n}{(b+1)(c+1)}+\frac{(3b)^n}{(c+1)(a+1)}+\frac{(3c)^n}{(a+1)(b+1)} \geq \frac{27}{16}.$$

3724. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

À l'intérieur d'un triangle rectangle de côtés 3, 4, 5, on dessine deux cercles égaux tangents entre eux et à l'un des côtés. Un des cercles est tangent à l'hypothénuse, l'autre est tangent à l'autre côté. Trouver les rayons des cercles dans les deux cas.

3725. *Proposé par Cîrnu Mircea, Bucarest, Roumanie.*

Montrer que la suite de nombres réels non nuls $x_1, x_2, \ldots$ est une progression géométrique si et seulement si elle satisfait la relation de récurrence

$$nx_1x_n = \sum_{k=1}^{n} x_k x_{n+1-k},\ n = 1, 2, \ldots.$$

3726. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Dans un triangle, soit respectivement A, B, C, s, r, R les angles (mesurés en radians), le demi-périmètre, le rayon du cercle inscrit et celui du cercle circonscrit. Montrer que

$$\left(\frac{A}{B} + \frac{B}{C} + \frac{C}{A}\right)^3 \geq \frac{2s^2}{Rr}.$$

3727. *Proposé par J. Chris Fisher, Université de Regina, Regina, SK.*

Soit $ABCD$ et $AECF$ deux parallélogrammes dont les sommets E et F sont situés à l'intérieur de la région bornée par $ABCD$. Montrer que la droite BE coupe le segment CF en son point milieu si et seulement si BF coupe AD en un point G satisfaisant

$$\frac{DA}{DG} = \frac{BF}{FG}.$$

3728. *Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.*

On donne une fonction continue $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ satisfaisant

$$\int_0^{\frac{\pi}{2}} \left([f(x)]^2 - 2f(x)(\sin x - \cos x)\right) dx = 1 - \frac{\pi}{2},$$

montrer que

$$\int_0^{\frac{\pi}{2}} f(x)dx = 0.$$

3729. *Proposé par Vo Quoc Ba Can, Université de Médecine et Pharmacie de Can Tho, Can Tho, Vietnam.*

Si a, b, c sont les longueurs des côtés d'un triangle, montrer que

$$\frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ca} + \frac{a+b}{c^2+ab} \leq \frac{3(a+b+c)}{ab+bc+ca}.$$

3730. *Proposé par Nguyen Thanh Binh, Hanoï, Vietnam.*

Soit D, E et F les pieds des perpendiculaires abaissées d'un point P du plan sur les droites BC, CA et AB déterminées par les côtés d'un triangle équilatéral ABC. Montrer que les céviennes AD, BE, CF sont concourantes (ou parallèles) si et seulement si au moins un des D, E ou F est le point milieu de son côté.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2049⋆. [1995 : 158; 1996 : 183-184] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1 respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1} + \frac{1}{GB_1} + \frac{1}{GC_1} + \frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right).$$

II. *Solution by Tomasz Cieśla, student, University of Warsaw, Poland.*

Murray Klamkin proved a generalization of the right-hand inequality [1996:183-184] for an n-dimensional simplex. He then conjectured that a generalization of the left-hand inequality likewise held: Let G and O be the centroid and circumcentre of an n-dimensional simplex $A_0A_1...A_n$ inscribed in a sphere of radius R. Let each line A_iG meet the sphere again in A_i' $(i = 0, \cdots, n)$. Then

$$\frac{n+1}{R} \leq \sum \frac{1}{A_i'G}.$$

If P and Q are the endpoints of the diameter through G, then $A_iG \cdot A_i'G = PG \cdot QG = (R+OG)(R-OG) = R^2 - OG^2$, whence our inequality is equivalent to $R\sum A_iG \geq (n+1)(R^2 - OG^2)$. Moreover, because $(n+1)(R^2 - OG^2) = \sum A_iG^2$ (a known equality that can be proved, for example, by a vector argument with the origin at O), the problem is reduced to proving that

$$R\sum A_iG \geq \sum A_iG^2.$$

Surprisingly, we can prove it using only the triangle inequality! Looking at triangle A_iOG we can write $|OA_i - A_iG| \leq OG$. After squaring we get $OA_i^2 + A_iG^2 - 2OA_i \cdot A_iG \leq OG^2$, which is equivalent to $2R \cdot A_iG \geq A_iG^2 + R^2 - OG^2$. Summing up $n+1$ such inequalities, we get

$$2R\sum A_iG \geq (n+1)(R^2 - OG^2) + \sum A_iG^2 = 2\sum A_iG^2.$$

Done!

Equality holds if and only if points A_i, O, G are collinear for all i. This happens when $O = G$, or all points A_i lie on the line OG. Clearly, in the latter case, the simplex is degenerate, and every vertex coincides with X or Y, where XY is a diameter of the sphere. In the three-dimensional case, $O = G$ implies that the tetrahedron is isosceles; see, for example, the recent solution to problem 478 [2012 : 68-70].

No other solutions have been received.

3621. [2011 : 113, 115] *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let a, b, and c be nonnegative real numbers with $a+b+c=1$. Prove that

$$\frac{27}{128}[(a-b)^2+(b-c)^2+(c-a)^2]+\frac{4}{1+a}+\frac{4}{1+b}+\frac{4}{1+c} \;\le\; \frac{3}{ab+bc+ca}\,.$$

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, expanded by the editor.

Let $s=ab+bc+ca$ and $t=abc$. Since

$$\begin{aligned}(a-b)^2+(b-c)^2+(c-a)^2 &= 2\sum_{\text{cyclic}} a^2-2\sum_{\text{cyclic}} ab\\ &= 2\left(\left(\sum_{\text{cyclic}} a\right)^2-3\sum_{\text{cyclic}} ab\right)=2(1-3s)\end{aligned}$$

and

$$\begin{aligned}\frac{4}{1+a}+\frac{4}{1+b}+\frac{4}{1+c} &= \frac{4\displaystyle\sum_{\text{cyclic}}(1+b)(1+c)}{(1+a)(1+b)(1+c)}\\ &= \frac{4\left(3+2\displaystyle\sum_{\text{cyclic}} a+\sum_{\text{cyclic}} ab\right)}{1+\displaystyle\sum_{\text{cyclic}} a+\sum_{\text{cyclic}} ab+abc}=\frac{4(5+s)}{2+s+t}\end{aligned}$$

the given inequality is equivalent, in succession, to

$$\begin{aligned}\frac{27(1-3s)}{64}+\frac{4(5+s)}{2+s+t} &\le \frac{3}{s}\\ \frac{27(1-3s)}{64}+\left(\frac{4(5+s)}{2+s+t}-9\right) &\le \frac{3}{s}-9=\frac{3-9s}{s}\\ \frac{2-5s-9t}{2+s+t} &\le \frac{3(1-3s)(64-9s)}{64s}\\ \frac{-1+4s-9t}{2+s+t} &\le 3(1-3s)\left(\frac{64-9s}{64s}-\frac{1}{2+s+t}\right)\\ \frac{-1+4s-9t}{2+s+t} &\le \frac{3(1-3s)\left((64-9s)(2+s+t)-64s\right)}{64s(2+s+t)} \qquad (1)\end{aligned}$$

Now we apply Schur's Inequality which states that

$$\sum_{\text{cyclic}} a^{\lambda}(a-b)(a-c)\ge 0$$

for any $\lambda\ge 0$. Setting $\lambda=1$ we then have

$$\sum_{\text{cyclic}} a^3-\sum_{\text{cyclic}} ab(a+b)+3abc\ge 0. \qquad (2)$$

Note that

$$\sum_{\text{cyclic}} a^3 = \left(\sum_{\text{cyclic}} a\right)^3 - 3\sum_{\text{cyclic}} ab(a+b) - 3abc \tag{3}$$

and

$$\sum_{\text{cyclic}} ab(a+b) = \left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} ab\right) - 3abc. \tag{4}$$

Substituting (3) and (4) into (2) we obtain

$$\left(\sum_{\text{cyclic}} a\right)^3 - 4\left(\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} ab\right) + 9abc \geq 0$$

so $1 + 9t \geq 4s$ or $-1 + 4s - 9t \leq 0$. On the other hand,

$$1 - 3s = \left(\sum_{\text{cyclic}} a\right)^2 - 3\sum_{\text{cyclic}} ab = \sum_{\text{cyclic}} a^2 - \sum_{\text{cyclic}} ab = \frac{1}{2}\sum_{\text{cyclic}} (a-b)^2 \geq 0$$

so $3s \leq 1$ and

$$(64 - 9s)(2 + s + t) \geq (64 - 3)(2) = 122 \geq (122)(3s) > 64s.$$

Hence (1) holds and the proof is complete. If equality holds in (1) then we must have $s = \frac{1}{3}$ and $4s - 9t - 1 = 0$ so $t = \frac{1}{27}$. But $\frac{1}{27} = t = abc \leq \left(\frac{a+b+c}{3}\right)^3 = \frac{1}{27}$ so equality holds which implies $a = b = c = \frac{1}{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy(two proofs); and the proposer.

Lau pointed out that in order for the inequality to make sense, we must stipulate that at most one of a, b, and c is zero. Perfetti proved the stronger result in which $\frac{27}{128}$ is replaced by $\frac{1}{4}$.

3622★. [2011 : 113, 116] *Proposed by George Tsapakidis, Agrinio, Greece.*

Let $ABCD$ be a quadrilateral.

(a) Find sufficient and necessary conditions on the sides and angles of $ABCD$, so that there is an inner point P such that two perpendicular lines through P divide the quadrilateral $ABCD$ into four quadrilaterals of equal area.

(b) Determine P.

The problem remains open. We received comments from Václav Konečný, Big Rapids, MI, USA and from Peter Y. Woo, Biola University, La Mirada, CA, USA. There is a simple topological argument (which applies to any bounded region of the Euclidean plane) proving the existence of two perpendicular lines that divide the area into four regions of equal area, but that argument fails to address our problem's most interesting aspects. Is the choice of perpendicular lines unique for a quadrilateral? *No* for a square, *yes* for any other parallelogram, and still open (as far as we know) for the general quadrilateral. Are the four equal-area regions that result all quadrilaterals? *No* for a nonsquare rhombus (the regions are all triangles), *yes* for other parallelograms, while the answer varies for other quadrilaterals; for example, if the length of one of its sides is close to zero (so that the quadrilateral is nearly a triangle), then one of the four equal-area regions might be a triangle and one a pentagon. As for part (b), even for the simple case of an isosceles trapezoid, the location of the point P in terms of the sides and angles might be too complicated to be of any interest.

3623. [2011 : 114, 116] *Proposed by Michel Bataille, Rouen, France.*

Let z_1, z_2, z_3, z_4 be distinct complex numbers with the same modulus, $\alpha = |(z_3 - z_2)(z_3 - z_4)|$, $\beta = |(z_1 - z_2)(z_1 - z_4)|$ and

$$u(\epsilon) = \frac{\alpha(z_1 - z_4) + \epsilon\beta(z_3 - z_4)}{\alpha(z_1 - z_2) + \epsilon\beta(z_3 - z_2)}.$$

Prove that $u(+1)$ or $u(-1)$ is a real number.

Solution by the proposer.

We take z_1, z_2, z_3, z_4 to represent the points M_1, M_2, M_3, M_4 on a circle Γ centered at the origin of the Argand Plane. Note that α and β are positive real numbers and that the four numbers $\alpha(z_1 - z_4) \pm \beta(z_3 - z_4)$, $\alpha(z_1 - z_2) \pm \beta(z_3 - z_2)$ are nonzero (since the vectors $\overrightarrow{M_1M_4}$ and $\overrightarrow{M_3M_4}$ are linearly independent, as are $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_3M_2}$). We distinguish two cases according as lines M_1M_3 and M_2M_4 are parallel or not.

(a) If $M_1M_3||M_2M_4$, then M_1, M_2, M_3, M_4 are the vertices of an isosceles trapezoid inscribed in Γ, so that $M_3M_4 = M_1M_2$ and $M_3M_2 = M_1M_4$. Thus,

$$\alpha = |z_3 - z_2||z_3 - z_4| = M_3M_2 \cdot M_3M_4 = M_1M_4 \cdot M_1M_2 = |z_2 - z_1||z_4 - z_1| = \beta,$$

and then

$$u(-1) = \frac{z_1 - z_4 - (z_3 - z_4)}{z_1 - z_2 - (z_3 - z_2)} = \frac{z_1 - z_3}{z_1 - z_3} = 1.$$

(b) Suppose that lines M_1M_3 and M_2M_4 intersect at a point M that is represented by the complex number z. Note that because the z_i are distinct, none of them can equal z; the real numbers λ and μ defined by $z - z_1 = \lambda(z - z_3)$ and $z - z_4 = \mu(z - z_2)$ are therefore nonzero. Because $\lambda \neq 1$, the first equation says that $z = \frac{1}{1-\lambda}(z_1 - \lambda z_3)$; plugging that value of z into the second equation gives us

$$\frac{z_1 - z_4 - \lambda(z_3 - z_4)}{z_1 - z_2 - \lambda(z_3 - z_2)} = \mu.$$

It therefore suffices to prove that $|\lambda| = \frac{\beta}{\alpha}$. Because the M_i are concyclic, we deduce that ΔM_3MM_4 and ΔM_2MM_1 are inversely similar, as are ΔM_1MM_4 and ΔM_2MM_3. Consequently,

$$\frac{M_2M_1}{M_3M_4} = \frac{MM_1}{MM_4} = \frac{MM_2}{MM_3} \quad \text{and} \quad \frac{M_1M_4}{M_2M_3} = \frac{MM_1}{MM_2} = \frac{MM_4}{MM_3}.$$

As a result we have

$$\left(\frac{M_2M_1}{M_3M_4} \cdot \frac{M_1M_4}{M_2M_3}\right)^2 = \frac{MM_1}{MM_4} \cdot \frac{MM_2}{MM_3} \cdot \frac{MM_1}{MM_2} \cdot \frac{MM_4}{MM_3} = \left(\frac{MM_1}{MM_3}\right)^2;$$

that is, $\left(\frac{\beta}{\alpha}\right)^2 = \lambda^2$, and the desired equality $|\lambda| = \frac{\beta}{\alpha}$ follows.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany.

3624. [2011 : 114, 116] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}\right).$$

I. Solution based on the approach of AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $a_0 = 0$, and, for $n \geq 1$, let

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n},$$

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_k = \sum_{k=1}^{n} (a_k - a_{k-1}) a_k,$$

and

$$T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_{k-1} = \sum_{k=1}^{n} (a_k - a_{k-1}) a_{k-1}.$$

Then

$$S_n + T_n = \sum_{k=1}^{n} (a_k - a_{k-1})(a_k + a_{k-1}) = \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2) = a_n^2,$$

and

$$S_n - T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (a_k - a_{k-1}) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Therefore $S_n = \frac{1}{2}\left[a_n^2 + \sum_{k=1}^{n} \frac{1}{k^2}\right]$. The desired sum is equal to

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left[(\log 2)^2 + \frac{\pi^2}{6}\right].$$

II. Solution following approach of Richard I. Hess, Rancho Palos Verdes, CA, USA; Kee-Wai Lau, Hong Kong, China; the Missouri State University Problem Solving Group, Springfield, MO; and the proposer.

For positive integer m, let

$$S_m = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{1\le k\le n\le m} \frac{(-1)^{n-1}}{n}\frac{(-1)^{k-1}}{k}.$$

Interchanging the order of summation and relabeling the indices yields

$$\begin{aligned} S_m &= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{n=k}^{m} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n}^{m} \frac{(-1)^{k-1}}{k} \\ &= \sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n+1}^{m} \frac{(-1)^{k-1}}{k}. \end{aligned}$$

Adding the two expressions for S_m yields that

$$S_m = \frac{1}{2}\left[\sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k}\right].$$

The required sum is

$$\lim_{m\to\infty} S_m = \frac{\pi^2}{12} + \frac{(\log 2)^2}{2}.$$

III. Solution by Oliver Geupel, Brühl, NRW, Germany(abridged).

When $a_n = \sum_{k=1}^{n}(-1)^{k-1}/k$, $b_n = \sum_{k=1}^{n} k^{-2}$ and $c_n = \sum_{k=1}^{n}(-1)^{k-1}a_k/k$, it can be proved by induction that

$$c_n = \frac{1}{2}(a_n^2 + b_n).$$

The required sum is equal to

$$\lim_{n\to\infty} c_n = \frac{1}{2}(\log 2)^2 + \frac{1}{12}\pi^2.$$

IV. Solution based on those of Anastasios Kotronokis, Athens, Greece; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and Albert Stadler, Herrliberg, Switzerland.

Since

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \int_0^1 \sum_{k=1}^{n} (-x)^{k-1} dx = \int_0^1 \frac{1-(-x)^n}{1+x}\, dx,$$

the proposed sum is equal to

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1\frac{1}{1+x}dx-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1\frac{(-x)^n}{1+x}\,dx\\ &= (\log 2)^2-\int_0^1\frac{1}{1+x}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-x)^n}{n}\,dx\\ &= (\log 2)^2-\int_0^1\frac{\log(1-x)}{1+x}\,dx\\ &= (\log 2)^2-\left[\frac{(\log 2)^2}{2}-\frac{\pi^2}{12}\right]=\frac{(\log 2)^2}{2}+\frac{\pi^2}{12}.\end{aligned}$$

No other solutions were received.

Perfetti provided a justification for the interchange of summation and integration in (IV), while Kotronis gave this determination of the final integral:

$$\begin{aligned}-\int_0^1\frac{\log(1-t)}{1+t}\,dt &= \int_0^1\int_{-1}^0\frac{1}{1+t}\cdot\frac{t}{1+yt}\,dydt=\int_{-1}^0\int_0^1\frac{t}{(1+t)(1+yt)}\,dtdy\\ &= \int_{-1}^0\int_0^1\left[\frac{1}{(y-1)(1+t)}-\frac{1}{(y-1)(1+yt)}\right]dtdy\\ &= \int_{-1}^0\left[\frac{\log 2}{y-1}-\frac{\log(1+y)}{y(y-1)}\right]dy=-(\log 2)^2+\int_{-1}^0\left[\frac{\log(1+y)}{y}-\frac{\log(1+y)}{y-1}\right]dy\\ &= -(\log 2)^2-\int_0^1\frac{\log(1-x)}{x}\,dx+\int_0^1\frac{\log(1-x)}{1+x}\,dx,\end{aligned}$$

so that

$$\begin{aligned}\int_0^1\frac{\log(1-t)}{1+t}\,dt &= \frac{(\log 2)^2}{2}+\frac{1}{2}\int_0^1\frac{\log(1-x)}{x}\,dx=\frac{(\log 2)^2}{2}-\frac{1}{2}\int_0^1\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\,dx\\ &= \frac{(\log 2)^2}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\int_0^1\frac{x^{n-1}}{n}\,dx=\frac{(\log 2)^2}{2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{(\log 2)^2}{2}-\frac{\pi^2}{12}.\end{aligned}$$

3625. [2011 : 114, 116] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, and c be positive real numbers. Prove that

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\;\le\;2\sqrt{1+\frac{abc}{(a+b)(b+c)(c+a)}}\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, *NRW, Germany; Titu Zvonaru, Comăneşti, Romania; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer (independently).*

Applying the Cauchy-Schwarz Inequality, we obtain that

$$\begin{aligned}
&\left(\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\right)^2 \\
&\qquad \le (a(b+c)+b(c+a)+c(a+b)) \\
&\qquad\qquad \times\left(\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)}\right) \\
&\qquad = 2(ab+bc+ca)\left(\frac{2(a+b+c)}{(a+b)(b+c)(c+a)}\right) \\
&\qquad = 4\frac{(a+b)(b+c)(c+a)+abc}{(a+b)(b+c)(c+a)} = 4\left(1+\frac{abc}{(a+b)(b+c)(c+a)}\right).
\end{aligned}$$

from which the desired result follows. Equality holds if and only if $a(a+b)(b+c)^2 = b(b+c)(c+a)^2 = c(c+a)(a+b)^2$, which is equivalent to $abc = a^2c+b^2a-c^2b = b^2a+c^2b-a^2c = c^2b+a^2c-b^2a$ or $a=b=c$.

Also solved by MICHEL BATAILLE, Rouen, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and ALBERT STADLER, Herrliberg, Switzerland.

These solvers also used the Cauchy-Schwarz Ineqaulity. The proposer attributed his solution to Zhao Bin, Ningbo, China. Geupel noted that the proposer posed the same problem to the Art of Problem Solving in 2006 (http://www.artofproblemsolving.com/Forum/viewtopic.php?t=78720).

3626. [2011: 170, 172] *Proposed by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece.*

Let x, y, and z be positive real numbers such that $x^2+y^2+z^2=3$. Prove that

$$\frac{1+x^2}{z+2}+\frac{1+y^2}{x+2}+\frac{1+z^2}{y+2} \ \ge\ 2\,.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $f(x,y,z)$ denote the left side of the given inequality. By the AM-GM Inequality we have

$$\frac{1+x^2}{z+2}+\frac{1}{9}(1+x^2)(z+2) \ge \frac{2}{3}(1+x^2)$$

so

$$\frac{1+x^2}{z+2} \ge \frac{4}{9}(1+x^2)-\frac{1}{9}z(1+x^2). \tag{1}$$

Similarly,

$$\frac{1+y^2}{x+2} \ge \frac{4}{9}(1+y^2)-\frac{1}{9}x(1+y^2) \tag{2}$$

and

$$\frac{1+z^2}{y+2} \ge \frac{4}{9}(1+z^2)-\frac{1}{9}y(1+z^2). \tag{3}$$

Summing (1), (2), (3), and using $x^2+y^2+z^2=3$ we have

$$\begin{aligned}
f(x,y,z) &\geq \frac{4}{9}(6)-\frac{1}{9}(x(1+y^2)+y(1+z^2)+z(1+x^2)) \\
&\geq \frac{8}{3}-\frac{1}{9}\left(\frac{1+x^2}{2}(1+y^2)+\frac{1+y^2}{2}(1+z^2)+\frac{1+z^2}{2}(1+x^2)\right) \\
&= \frac{8}{3}-\frac{1}{18}(3+2(x^2+y^2+z^2)+(x^2y^2+y^2z^2+z^2x^2)). \qquad (4)
\end{aligned}$$

Since clearly $x^2y^2+y^2z^2+z^2x^2 \leq x^4+y^4+z^4$ we have

$$3(x^2y^2+y^2z^2+z^2x^2) \leq (x^2+y^2+z^2)^2. \qquad (5)$$

From (4) and (5) we can conclude that

$$\begin{aligned}
f(x,y,z) &\geq \frac{8}{3}-\frac{1}{18}\left(3+2(x^2+y^2+z^2)+\frac{1}{3}(x^2+y^2+z^2)^2\right) \\
&= \frac{8}{3}-\frac{1}{18}(3+6+3)=2.
\end{aligned}$$

It is clear that equality holds if and only if $x=y=z=1$.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy(two proofs); TITU ZVONARU, Cománeşti, Romania; and the proposer.

3627. [2011: 170, 172] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all quadruples a,b,c,d of positive real numbers that are solutions to the system of equations

$$\begin{aligned}
a+b+c+d &= 4\,, \\
\left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd) &= 16\,.
\end{aligned}$$

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

By the Arithmetic-Geometric Means Inequality,

$$abcd \leq \left(\frac{a+b+c+d}{4}\right)^4 = 1,$$

with equality if and only if $a=b=c=d=1$. Also,

$$\begin{aligned}
16 &= \left(\frac{1}{a^{12}}+\frac{1}{b^{12}}+\frac{1}{c^{12}}+\frac{1}{d^{12}}\right)(1+3abcd) \\
&\geq \left(\frac{4}{a^3b^3c^3d^3}\right)(4abcd) = \frac{16}{a^2b^2c^2d^2} \geq 16.
\end{aligned}$$

Since equality must hold throughout, $a = b = c = d = 1$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Cománeşti, Romania; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.

3628. [2011 : 170, 173] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, c and r be the edge-lengths and the inradius of a triangle ABC. Find the minimum value of the expression

$$E = \left(\frac{a^2b^2}{a+b-c} + \frac{b^2c^2}{b+c-a} + \frac{c^2a^2}{c+a-b} \right) r^{-3}.$$

I. Solution by Titu Zvonaru, Cománeşti, Romania.

We shall see that the minimum value of E is $72\sqrt{3}$, its value when ΔABC is equilateral. Let F be the area of the triangle, and $s = \frac{1}{2}(a+b+c)$ be its semiperimeter. It is known that $F = sr$, $s \geq 3r\sqrt{3}$ (item 5.11 in [1]), and $ab + bc + ca \geq 4F\sqrt{3}$ (item 4.5 in [1]). Applying the Cauchy-Schwarz inequality to the vectors $\left(\frac{ab}{\sqrt{a+b-c}}, \frac{bc}{\sqrt{b+c-a}}, \frac{ca}{\sqrt{c+a-b}} \right)$ and $\left(\sqrt{a+b-c}, \sqrt{b+c-a}, \sqrt{c+a-b} \right)$, we have

$$\begin{aligned} E \geq \frac{(ab+bc+ca)^2}{a+b-c+b+c-a+c+a-b} \cdot r^{-3} &= \frac{(ab+bc+ca)^2}{2s} \cdot \frac{1}{r} \cdot \frac{s^2}{F^2} \\ &= \frac{(ab+bc+ca)^2}{2} \cdot \frac{1}{F^2} \cdot \frac{s}{r} \\ &\geq \frac{16F^2 \cdot 3}{2} \cdot \frac{1}{F^2} \cdot 3\sqrt{3} \quad = \quad 72\sqrt{3}. \end{aligned}$$

In an equilateral triangle we have $a = b = c$, and $r = \frac{a}{2\sqrt{3}}$; for these values E attains its lower bound, namely $72\sqrt{3}$.

II. Solution by Kee-Wai Lau, Hong Kong, China.

We use the notation of the previous solution; along with the inequality $s \geq 3\sqrt{3}r$, we use here Euler's inequality $R \geq 2r$ together with $F = \sqrt{s(s-a)(s-b)(s-c)} = rs$ and $abc = 4srR$ (where R is the circumradius). By the AM-GM inequality we have

$$E \geq 3 \left(\frac{a^4b^4c^4}{(a+b-c)(b+c-a)(c+a-b)} \right)^{1/3} r^{-3}.$$

Using our equations for F and for abc, we see that the right-hand side of this inequality equals

$$3\left(\frac{a^4b^4c^4s}{8F^2r^9}\right)^{1/3} \quad = \quad 3\left(\frac{32s^3R^4}{r^7}\right)^{1/3}.$$

By the two inequalities referred to above we obtain

$$3\left(\frac{32s^3R^4}{r^7}\right)^{1/3} \geq 72\sqrt{3}.$$

Thus $E \geq 72\sqrt{3}$. Equality holds when triangle ABC is equilateral; therefore, the minimum value of E is $72\sqrt{3}$.

References

[1] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

Most of the submitted solutions were quite similar to a featured solution.

3629. [2011 : 170, 173] *Proposed by Michel Bataille, Rouen, France.*

Find the greatest positive integer m such that 2^m divides

$$2011^{\left(2013^{2016}-1\right)} - 1.$$

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

For a positive integer x, let $e(x)$ denote the exponent of 2 in the prime factorization of x. Clearly, $e(x_1x_2x_3\cdots x_r) = e(x_1) + e(x_2) + e(x_3) + \cdots + e(x_r)$. We will use the following properties of e to solve the stated problem.

If a and n are odd, then

$$e\left(a^n - 1\right) = e(a-1). \tag{1}$$

If a is odd and n is even, then

$$e\left(a^n + 1\right) = 1. \tag{2}$$

Proof of (1): Since $(a^n - 1) = (a-1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$ and the right factor is an odd sum of odd integers, hence it is odd and the result follows.

Proof of (2): Set $n = 2t$. As a is odd, $a^2 \equiv 1 \pmod 4$, so $a^n + 1 \equiv (a^2)^t + 1 \equiv 2 \pmod 4$ and the result follows.

We will use the above results to show that $m = 9$ for the given problem.

Set $\omega = 2011$, $\theta = 2013$ and $A = \theta^{2016} - 1$. The number from the problem is $N = \omega^A - 1$.

We have $A = (\theta^{32})^{63} - 1$. By (1) we get $e(A) = e\left(\theta^{32} - 1\right)$. Factoring yields

$$\theta^{32} - 1 = \left(\theta^{16} + 1\right)\left(\theta^{8} + 1\right)\left(\theta^{4} + 1\right)\left(\theta^{2} + 1\right)(\theta + 1)(\theta - 1).$$

From (2) we get $e\left(\theta^{16} + 1\right) = e\left(\theta^{8} + 1\right) = e\left(\theta^{4} + 1\right) = e\left(\theta^{2} + 1\right) = 1$. Furthermore, $e(\theta + 1) = e(2 \times 1007) = 1$ and $e(\theta - 1) = e(4 \times 503) = 2$. Hence,

$$e(A) = e\left(\theta^{32} - 1\right) = 1 + 1 + 1 + 1 + 1 + 2 = 7.$$

Therefore, we can set $A = 2^7 B = 128B$ for some odd number B.

Now, $N = \left(\omega^{128}\right)^B - 1$ so, by (1), $e(N) = e\left(\omega^{128}\right)$. Factoring yields

$$\omega^{128} - 1 = \left(\omega^{64} + 1\right)\left(\omega^{32} + 1\right)\left(\omega^{16} + 1\right)\left(\omega^{8} + 1\right)\left(\omega^{4} + 1\right)\left(\omega^{2} + 1\right)(\omega + 1)(\omega - 1).$$

From (2) we get $e\left(\omega^{64} + 1\right) = e\left(\omega^{32} + 1\right) = e\left(\omega^{16} + 1\right) = e\left(\omega^{8} + 1\right) = e\left(\omega^{4} + 1\right) = e\left(\omega^{2} + 1\right) = 1$. Furthermore, $e(\omega+1) = e(4 \times 503) = 2$ and $e(\omega-1) = e(2 \times 1005) = 1$. Finally,

$$m = e(N) = e\left(\omega^{128} - 1\right) = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 1 = 9$$

and we are done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JENNIFER DEMPSEY and MICHAEL MURPHY, St. Bonaventure University, St. Bonaventure, NY, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was received.

3630. [2011 : 170, 173] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$\frac{ab(b+c)}{2+c} + \frac{bc(c+a)}{2+a} + \frac{ca(a+b)}{2+b} \leq 2\,.$$

Solution by Titu Zvonaru, Cománeşti, Romania, modified and expanded by the editor.

Note first that

$$\sum_{\text{cyclic}} ab^2c = abc(a + b + c) = 3abc. \qquad (1)$$

Clearing the denominators and using (1), the given inequality is equivalent, in succession, to:

$$\sum_{\text{cyclic}} (ab^2 + abc)(ab + 2a + 2b + 4) \le (ab + 2a + 2b + 4)(4 + 2c)$$

$$\sum_{\text{cyclic}} (a^2b^3 + a^2b^2c + 2a^2b^2 + 2a^2bc + 2ab^3 + 2ab^2c + 4ab^2 + 4abc)$$
$$\le 4ab + 8a + 8b + 16 + 2abc + 4ac + 4bc + 8c$$

$$\sum_{\text{cyclic}} a^2b^3 + abc \sum_{\text{cyclic}} ab + 2 \sum_{\text{cyclic}} a^2b^2 + 2 \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} 4ab^2 + 24abc$$
$$\le 40 + 4 \sum_{\text{cyclic}} ab + 2abc$$

$$\sum_{\text{cyclic}} a^2b^3 + abc \sum_{\text{cyclic}} ab + 2 \sum_{\text{cyclic}} a^2b^2 + 2 \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} 4ab^2 + 22abc$$
$$\le 40 + 4 \sum_{\text{cyclic}} ab. \tag{2}$$

Since

$$(a + b + c)^3 - 3(ab + bc + ca) = \frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) \ge 0$$

we have $3(ab + bc + ca) \le (a + b + c)^2 = 9$, so

$$\sum_{\text{cyclic}} ab \le 3. \tag{3}$$

Furthermore, in the published solution to Crux problem 3527 [2011 : 177] (proposed by the same proposer of the current problem), George Apostolopoulos proved that if a, b, and c are nonnegative real such that $a + b + c = 3$, then

$$ab^2 + bc^2 + ca^2 + abc \le 4. \tag{4}$$

Using (1) and (4) we obtain successively that

$$(a + b + c)(ab^2 + bc^2 + ca^2 + abc) \le 12$$
$$\sum_{\text{cyclic}} a^2b^2 + \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} abc^2 + abc(a + b + c) \le 12$$
$$\sum_{\text{cyclic}} a^2b^2 + \sum_{\text{cyclic}} ab^3 + 6abc \le 12. \tag{5}$$

From (4) and (5) we see that in order to establish (2), it suffices to show that

$$\sum_{\text{cyclic}} a^2b^3 + abc \sum_{\text{cyclic}} ab + 6abc \le 4 \sum_{\text{cyclic}} ab. \tag{6}$$

[*Ed.: Note that (2) would follow if we added 4(4) + 2(5) to (6).*]

Now (6) is equivalent, in succession, to

$$\sum_{\text{cyclic}}(a^2b^3+ab^2c^2+a^3bc+a^2b^2c)+6abc \le 4\sum_{\text{cyclic}}ab+\sum_{\text{cyclic}}(ab^2c^2+a^3bc)$$

$$\sum_{\text{cyclic}}ab(ab^2+bc^2+ca^2+abc)+6abc \le 4\sum_{\text{cyclic}}ab+abc\sum_{\text{cyclic}}(a^2+ab) \qquad (7)$$

Since $ab^2+bc^2+ca^2+abc \le 4$ by (4), it remains to show that $6abc \le abc\sum_{\text{cyclic}}(a^2+ab)$ or $\sum_{\text{cyclic}}(a^2+ab) \ge 6$ which is true since

$$\sum_{\text{cyclic}}(a^2+ab)=\sum_{\text{cyclic}}a^2+\sum_{\text{cyclic}}ab=(a+b+c)^2-\sum_{\text{cyclic}}ab=9-\sum_{\text{cyclic}}ab\ge 6$$

by (3). This establishes (7) and completes the proof.

If equality holds, then we have $a=b=c$ or $abc=0$ and $ab^2+bc^2+ca^2=4$. Suppose $c=0$, then solving $a+b=3$ and $ab^2=4$ we obtain $a(3-a)^2=4$ or $a^3-6a^2+9a-4=0$ or $(a+2)^2(a-1)=0$ so $a=1$ and $b=2$. Hence, either $a=b=c=1$ or $(a,b,c)=(1,2,0),(0,1,2)$, or $(2,0,1)$. It is readily checked that all these ordered triples yield equality.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. There were also two submitted solutions which either had errors or contained claims without proof.