Crux Mathematicorum

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 20 #5

May / mai
1994

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Canadian Mathematical Society

Société mathématique du Canada

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Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

SUBSCRIPTION INFORMATION

Crux Mathematicorum is published monthly (except July and August). The subscription rates for ten issues are detailed on the inside back cover. Information on Crux Mathematicorum back issues is also provided on the inside back cover.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum.

ACKNOWLEDGEMENTS

The support of the Department of Mathematics & Statistics of the University of Calgary and of the Department of Mathematics of the University of Ottawa is gratefully acknowledged.

REMERCIEMENTS

Nous rendons hommage à l'appui du département de mathématique et statistique de l'Université de Calgary et du département de mathématiques de l'Université d'Ottawa.

Printed at / imprimé à: Synergy Print and Copy, Ottawa.

ISSN 0705-0348

Publication Mail Registration No. 5432

A FIVE–CIRCLE PROBLEM

Hiroshi Okumura

The recent book [1] introduced the Japanese mathematics of the 18th and 19th centuries to the West. This mathematics deals with such unhackneyed and challenging problems as that book's Example 1.4 [1, pages 7 and 84]:

PROBLEM: *In the circle C of radius r let AB be a chord whose midpoint is M. The circle C_0 of radius r_0 ($r_0 < r/2$) touches AB at M and also touches C internally. Let P be any point on AB distinct from A, B and M; a circle C_2 of radius r_0 (equal to the radius of C_0) touches AB at P on the other side of AB. Distinct circles C_1 and C_3 of radii r_1 and r_3 touch AB, and each touches C internally and C_2 externally. Show that $r = r_1 + r_2 + r_3$.*

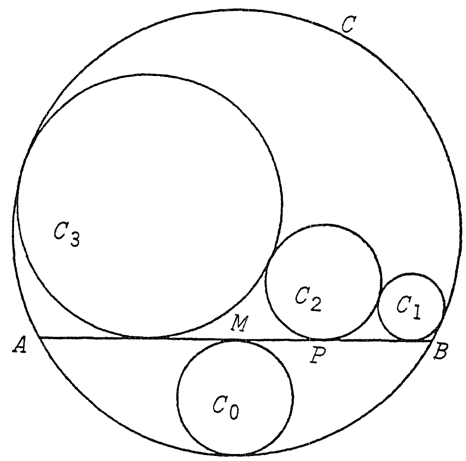

Figure 1

The purpose of this note is to generalize the problem and display periodic patterns of circles based on it. We will see that neither the restriction $r_0 < r/2$ nor $P \neq A, B, M$ are needed for the desired relation among the radii in the problem. We need the following easy lemma.

LEMMA: (i) Common tangents of externally touching circles whose radii are a and b have length $2\sqrt{ab}$.

(ii) Let C be a circle of radius r with a chord AB, and C_0 a circle of radius r_0 touching AB at the midpoint M of AB and touching C internally. Then $AM = 2\sqrt{(r - r_0)r_0}$.

With the aid of the lemma, we get a generalization of the problem as follows.

THEOREM 1: Let t be a secant of a circle C of radius r; let C_1, C_2, C_3 be circles on one side of t and tangent to it, with C_1 and C_3 internally tangent to C, while C_2 is externally tangent to C_1 and C_3. Let C_0 be the circle internally tangent to C on the other side of t, and tangent to t at the midpoint of the segment cut from it by C. If C_i has radius r_i and $r_0 = r_2$, then

$$r = r_1 + r_2 + r_3. \tag{1}$$

Our proof will be an adaptation to our more general setting of the proof found in [1, p. 84], which the authors reproduced from an 1864 Japanese book.

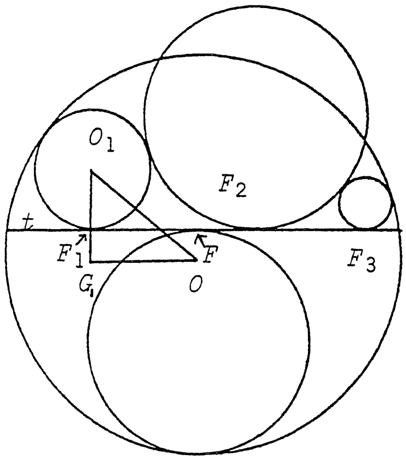

Figure 2

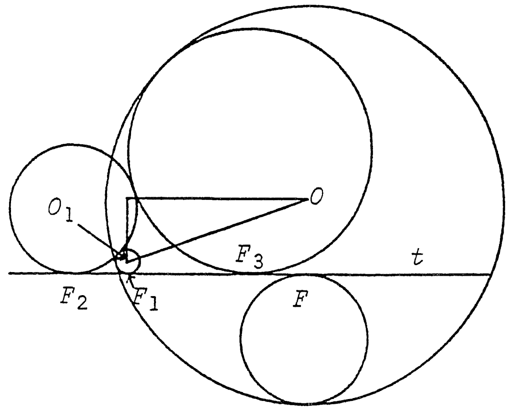

Figure 3

Proof: Let O, O_i be the centers of the circles C and C_i and F, F_i the feet of the perpendiculars from O, O_i to t. From the right triangle OO_1G_1 formed by the lines OO_1, O_1F_1 and the line parallel to t through O, we get

$$F_1F = G_1O = \sqrt{(r-r_1)^2 - (r-2r_2-r_1)^2} = 2\sqrt{(r-r_2-r_1)r_2}.$$

Similarly we obtain $F_3F = 2\sqrt{(r-r_2-r_3)r_2}$. Also we have $F_1F_2 = 2\sqrt{r_1r_2}$ and $F_2F_3 = 2\sqrt{r_2r_3}$ by the lemma.

There are two cases to be considered according as F_2 lies inside C or not (see Figures 2 and 3). We claim that in the former case, F lies between F_1 and F_3. For otherwise, we may assume that F_1, F_2, F_3, F lie in this order. Then we have $F_1F_3 = F_1F_2 + F_2F_3 = F_1F - F_3F$. Substituting, we get

$$2\sqrt{r_1r_2} + 2\sqrt{r_2r_3} = 2\sqrt{(r-r_2-r_1)r_2} - 2\sqrt{(r-r_2-r_3)r_2}$$

or

$$\sqrt{r_1} - \sqrt{r-r_2-r_1} = -\sqrt{r_3} - \sqrt{r-r_2-r_3}.$$

Squaring this, we obtain $-\sqrt{r_1}\sqrt{r-r_2-r_1} = \sqrt{r_3}\sqrt{r-r_2-r_3}$, a contradiction. Therefore F lies between F_1 and F_3 as claimed. In the other case we may assume that F_1 lies between F_2 and F_3, which implies that F_3 must lie between F and F_1: For since $r_2 + r_3 \leq r$ we get $r_2r_3 \leq r_2(r - r_2)$, or $F_2F_3 = 2\sqrt{r_2r_3} \leq 2\sqrt{(r-r_2)r_2}$. Therefore F_2F_3 is less than or equal to one-half of the chord by the lemma; consequently $F_2F_3 \leq F_2F$.

Therefore we have $F_1F_3 = F_1F_2 + F_2F_3 = F_1F + FF_3$ in the former case, and $F_1F_3 = -F_1F_2 + F_2F_3 = F_1F - FF_3$ in the latter. Thus we get

$$\pm 2\sqrt{r_1r_2} + 2\sqrt{r_2r_3} = 2\sqrt{(r-r_2-r_1)r_2} \pm 2\sqrt{(r-r_2-r_3)r_2}.$$

Therefore we have

$$(\pm\sqrt{r_1} - \sqrt{r - r_2 - r_1})^2 = (\pm\sqrt{r - r_2 - r_3} - \sqrt{r_3})^2.$$

Expanding the equation and rearranging we get

$$(r_1 - r_3)(r - r_1 - r_2 - r_3) = 0.$$

Therefore, when $r_1 \neq r_3$ we have (1). Suppose that $r_1 = r_3$. This happens only in the former case. Then we obtain $FF_3 = F_2F_3$ or $2\sqrt{(r - r_2 - r_3)r_2} = 2\sqrt{r_2 r_3}$. Hence we get $r = r_2 + 2r_3$ and the proof is now complete. □

If C_2 touches C externally in the theorem, then C_1 and C_3 coincide. Therefore we get:

COROLLARY: Let t be a secant of a circle C of radius r, and C_0 a circle touching C internally and touching t at the midpoint of the segment cut from it by C. On the other side, circles C_1 and C_2 touch t and touch C internally and externally respectively at the same point. If the radius of C_i is r_i and $r_0 = r_2$, we have

$$r = 2r_1 + r_2.$$

The following theorem shows that the external tangents of C_0, C_1 and C_0, C_3 are equal in Theorem 1. A related result can be seen in [2].

THEOREM 2: Let C and C_0 be circles with C_0 inside and tangent to C, and let the chord k of C be tangent at its midpoint to C_0. Then

(i) If the circle C_1 is tangent to k on the other side from C_0 and touches C internally, then an external common tangent of C_0 and C_1 is half as long as k; and

(ii) if C_1 touches the extension of k on the same side as C_0 and touches C externally, then an internal common tangent of C_0 and C_1 is half as long as k.

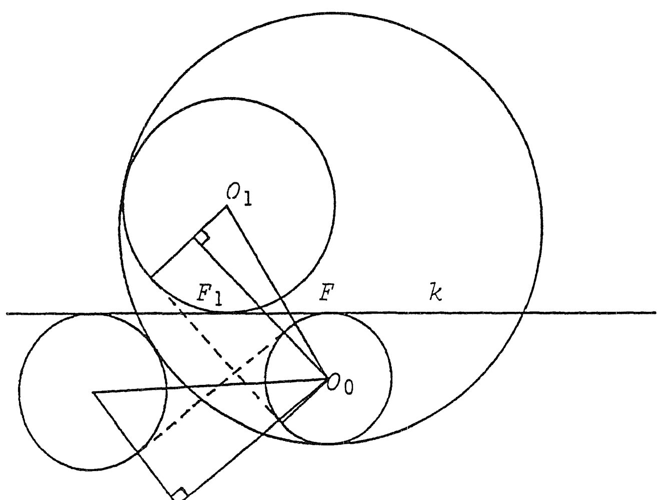

Figure 4

Proof: Let C_1 be an arbitrary circle touching t on the other side and C internally, and F, F_i, O_i, r, r_i as in the proof of Theorem 1 (see Figure 4). Then we have $(FF_1)^2 = 4(r - r_0 - r_1)r_0$ as we saw in the proof of Theorem 1, and $(O_0O_1)^2 = (FF_1)^2 + (r_0 + r_1)^2$ by the Pythagorean theorem. Therefore the square of the external tangents of C_0 and C_1 is equal to

$$(O_0O_1)^2 - (r_0 - r_1)^2 = 4(r - r_0 - r_1)r_0 + (r_0 + r_1)^2 - (r_0 - r_1)^2 = 4(r - r_0)r_0.$$

Hence the tangents are equal to $2\sqrt{(r - r_0)r_0}$, one-half of k. (ii) is proved similarly. □

Returning to Theorem 1, if C_3 degenerates to a point circle in that theorem, C_2 touches t at an intersection of C and t. Since $r = r_0 + r_1$ in this situation, the centers of C, C_0 and C_1 are collinear. This suggests a converse: Let C be a circle and k its chord, C_0 and C_1 circles touching C internally and also touching k at the midpoint on opposite sides. Then the radius of the circle touching k at an end of k on the other side from C_0 and touching C_1 externally is equal to the radius of C_0. But we can prove a more general result (see Figure 5):

THEOREM 3: Let C_0 and C_1 be externally (resp. internally) touching circles, and C the circle touching the two at points different from the point of tangency of C_0 and C_1 such that the three centers are collinear. For any chord k of C perpendicular to the line through the three centers, there is a circle of radius equal to the radius of C_0 touching k at an end of k and touching C_1 externally (resp. internally).

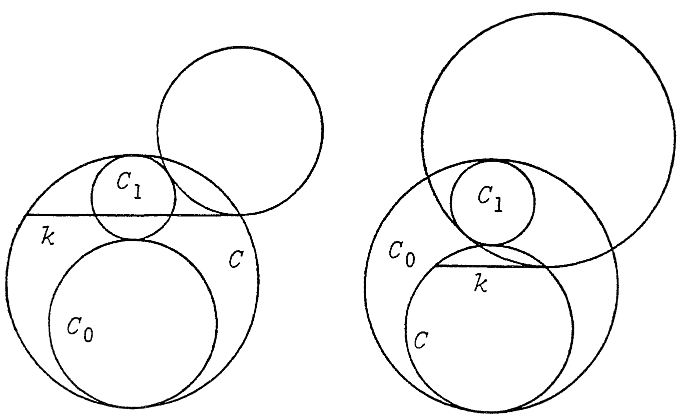

Figure 5

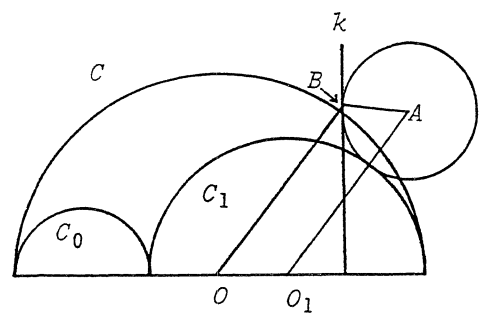

Figure 6

Proof: Let us first assume that C_0 and C_1 touch externally; let r_0 and r_1 be their radii. O and O_1 are the centers of C and C_1 respectively (see Figure 6). Let us draw a chord k of C perpendicular to the line through the three centers and a new circle of radius r_0 and center A touching C_1 externally. If the circle touches k at B and the circle is drawn such that $\overrightarrow{O_1O}$ and $\overrightarrow{AB}$ have the same orientation, then O_1O and AB are equal and parallel, since $O_1O = r_0$. Hence OO_1AB is a parallelogram and we get $OB = r_0 + r_1$. Therefore B lies on C. Thus the theorem follows from the uniqueness of the figure. The internal case is proved similarly. □

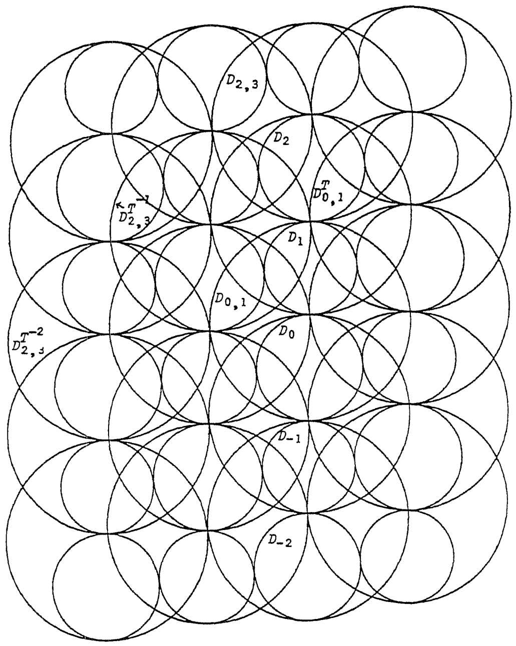

Figure 7

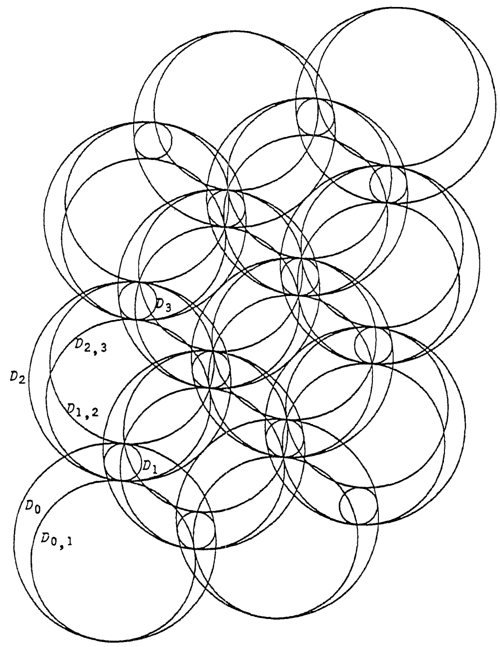

Figure 8

We now extend the three-circle pattern of Theorem 3 to the entire plane (as in Figures 7 and 8). Let $\cdots, D_{-2}, D_{-1}, D_0, D_1, D_2, \cdots$ be distinct circles such that all the centers lie on a line, and D_i and D_{i+1} touch externally (resp. internally) and the radii of D_{2i} and D_{2i+1} are equal to r_0 and r_1, the radii of D_0 and D_1 respectively. For each pair of D_i and D_{i+1} let us draw another circle $D_{i,i+1}$ touching to the two at points different from the point of tangency of D_i and D_{i+1} such that the three centers are collinear. By Theorem 3, there is a translation T such that D_0^T touches D_1 externally (resp. internally) and intersects $D_{0,1}$ at a point where the tangent of D_0^T is perpendicular to the line through the centers of D_0 and D_1. If we draw the images of the whole figure by the translations $T, T^2, \cdots$, and $T^{-1}, T^{-2}, \cdots$, we get Figure 7 (resp. Figure 8).

Let us observe that T equals the product of two translations T_x and T_y, where T_x has the direction perpendicular to the line joining the centers of D_0 and D_1, and T_y has the direction parallel to the line (see Figure 9). If the lengths of T_x and T_y are d_x and d_y, then we have $d_x = \sqrt{r^2 - (d_y - r)^2} = \sqrt{d_y(2r - d_y)}$ by the Pythagorean theorem, where r is the radius of $D_{0,1}$. With this fact it is easily seen that our pattern has the following properties.

1. $\cdots, D_{2,3}^{T^{-2}}, D_{0,1}, D_{-2,-1}^{T_2}, D_{-4,-3}^{T^4}, \cdots, D_{-2n,-2n+1}^{T^{2n}}, \cdots$ are tangent successively and the centers are collinear,

2. $\cdots, D_1^{T^{-1}}, D_0, D_{-1}^{T}, D_{-2}^{T^2}, \cdots$ are tangent successively and the centers are collinear,

3. $D_{-1,0}^{T}$ and $D_{1,2}^{T^{-1}}$ touch at a point where $D_{-1,0}$ and $D_{1,2}$ also touch.

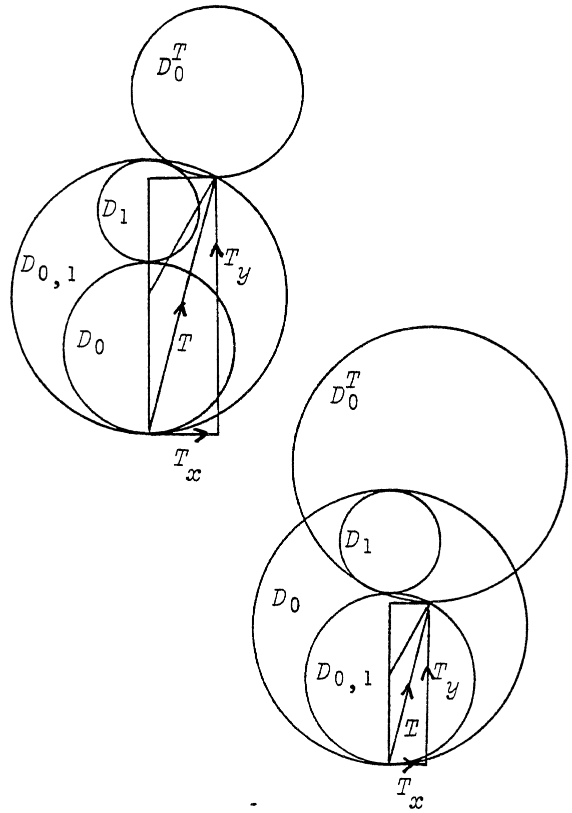

Figure 9

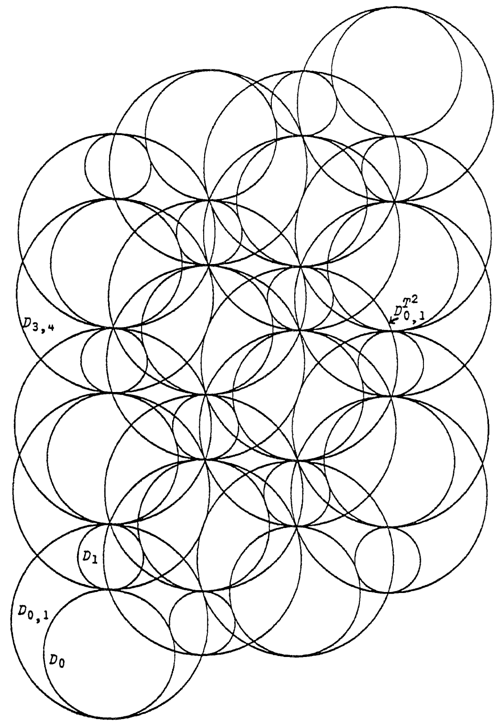

Figure 10

In Figure 7, if the line joining the centers of $D^T_{-1,0}$ and D_1 is perpendicular to the line joining the centers of D_0 and D_1, then $D^T_{-1,0}$ passes through the point where D_1 and D_2 touch. This will happen if and only if $d_y = r_1 + r = r_0 + 2r_1$. Moreover we can adjust the ratio of r_0 and r_1 so that $D_{3,4}$ and $D^{T^2}_{0,1}$ touch externally in this situation. Since $D_{3,4}$ can be obtained by translating $D_{0,1}$ through the distance $2r - 2r_1 = 4r_0 + 2r_1$ along the line through the centers of D_0 and D_1, and $d_x^2 = d_y(2r - d_y) = (r_0 + 2r_1)r_0$, the square of the distance between the centers of $D_{3,4}$ and $D^{T^2}_{0,1}$ is equal to $(2d_x)^2 + (2d_y - 4r_0 - 2r_1)^2 = 4(2r_0^2 + r_1^2)$. We also get

$$4(2r_0^2 + r_1^2) - (2r)^2 = 4(2r_0^2 + r_1^2) - (2r_0 + 2r_1)^2 = 4r_0(r_0 - 2r_1).$$

Therefore $D_{3,4}$ and $D^{T^2}_{0,1}$ touch externally if and only if $r_0 : r_1 = 2 : 1$ (see Figure 10).

Acknowledgement: The author would express his thanks to the referees for their helpful comments and for providing the neat drawings of Figures 7, 8 and 10.

References:

[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Center, Winnipeg, Canada, (1989).

[2] Solution of the Problem 1155, *Mathematics Magazine* **57** (1984), p. 47.

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THE OLYMPIAD CORNER

No. 155

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

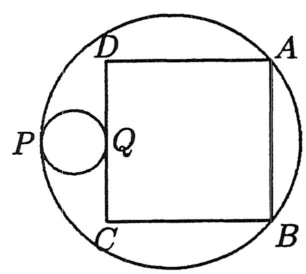
The "pre-Olympiad" set of problems for this month is the 1994 AIME, which is a regular feature of this column. The American Invitational Mathematics Examination was written Thursday, March 31, 1994, and its problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America, and they may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

12th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

March 31, 1994

1. The increasing sequence 3, 15, 24, 48,... consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994$^{\text{th}}$ term of the sequence is divided by 1000?

2. A circle with diameter $\overline{PQ}$ of length 10 is internally tangent at P to a circle of radius 20. Square $ABCD$ is constructed with A and B on the larger circle, $\overline{CD}$ tangent at Q to the smaller circle, and the smaller circle outside $ABCD$. The length of $\overline{AB}$ can be written in the form $m + \sqrt{n}$, where m and n are integers. Find $m + n$.



3. The function f has the property that, for each real number x,

$$f(x) + f(x-1) = x^2.$$

If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

4. Find the positive integer n for which

$$\lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor = 1994.$$

(For real x, $\lfloor x \rfloor$ is the greatest integer $\leq x$.)

5. Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + p(3) + \cdots + p(999).$$

What is the largest prime factor of S?

6. The graphs of the equations

$$y = k, y = \sqrt{3}\,x + 2k, y = -\sqrt{3}\,x + 2k,$$

are drawn in the coordinate plane for $k = -10, -9, -8, \ldots, 9, 10$. These 63 lines cut part of the plane into equilateral triangles of side $2/\sqrt{3}$. How many such triangles are formed?

7. For certain ordered pairs (a, b) of real numbers, the system of equations

$$\begin{aligned} ax + by &= 1 \\ x^2 + y^2 &= 50 \end{aligned}$$

has at least one solution, and each solution is an ordered pair (x, y) of integers. How many such ordered pairs (a, b) are there?

8. The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

9. A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. The probability that the bag will be emptied is p/q, where p and q are relatively prime positive integers. Find $p + q$.

10. In triangle ABC, angle C is a right angle and the altitude from C meets $\overline{AB}$ at D. The lengths of the sides of $\triangle ABC$ are integers, $BD = 29^3$, and $\cos B = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

11. Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to be stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$ or $10''$ or $19''$ to the total height of the tower. How many different tower heights can be achieved using all 94 of the bricks?

12. A fenced, rectangular field measures 24 meters by 52 meters. An agricultural researcher has 1994 meters of fence that can be used for internal fencing to partition the field into congruent, square test plots. The entire field must be partitioned, and the sides of the squares must be parallel to the edges of the field. What is the largest number of square test plots into which the field can be partitioned using all or some of the 1994 meters of fence?

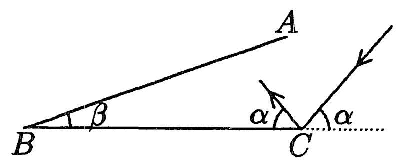
13. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots r_1, $\overline{r_1}$, r_2, $\overline{r_2}$, r_3, $\overline{r_3}$, r_4, $\overline{r_4}$, r_5, $\overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}\,.$$

14. A beam of light strikes $\overline{BC}$ at point C with angle of incidence $\alpha = 19.94°$ and reflects with an equal angle of reflection as shown. The light beam continues its path, reflecting off line segments $\overline{AB}$ and $\overline{BC}$ according to the rule: *angle of incidence equals angle of reflection.* Given that $\beta = \alpha/10 = 1.994°$ and $AB = BC$, determine the number of times the light beam will bounce off the two line segments. Include the first reflection at C in your count.



15. Given a point P on a triangular piece of paper ABC, consider the creases that are formed in the paper when A, B, and C are folded onto P. Let us call P a *fold point* of $\triangle ABC$ if these creases, which number three unless P is one of the vertices, do not intersect. Suppose that $AB = 36$, $AC = 72$, and $\angle B = 90°$. Then the area of the set of all fold points of $\triangle ABC$ can be written in the form $q\pi - r\sqrt{s}$, where q, r, and s are positive integers and s is not divisible by the square of any prime. What is $q + r + s$?

* * *

As an olympiad this month we give the problems of the final round of the 43rd Mathematical Olympiad (1991–92) in Poland. Many thanks to Marcin E. Kuczma, Warszawa, Poland for sending them to me.

43rd MATHEMATICAL OLYMPIAD (1991–92) IN POLAND

Final Round

1. Segments AC and BD intersect in point P so that $PA = PD$, $PB = PC$. Let O be the circumcentre of triangle PAB. Prove that lines OP and CD are perpendicular.

2. Determine all functions f defined on the set of positive rational numbers, taking values in the same set, which satisfy for every positive rational number x the conditions

$$f(x+1) = f(x) + 1 \quad \text{and} \quad f(x^3) = (f(x))^3.$$

3. Prove that the inequality

$$\sum_{n=1}^{r}\left(\sum_{m=1}^{r}\frac{a_m a_n}{m+n}\right) \geq 0$$

holds for any real numbers $a_1, a_2, \ldots, a_r$. Find conditions for equality.

4. Define the sequence of functions $f_0, f_1, f_2, \ldots$ by

$$f_0(x) = 8 \quad \text{for all } x \in \mathbb{R},$$

$$f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)} \quad \text{for } n = 0, 1, 2, \ldots \text{ and for all } x \in \mathbb{R}.$$

For every positive integer n, solve the equation $f_n(x) = 2x$.

5. The regular $2n$-gon $A_1A_2\ldots A_{2n}$ is the base of a regular pyramid with vertex S. A sphere passing through S cuts the lateral edges SA_i in the respective points B_i $(i = 1, 2, \ldots, 2n)$. Show that

$$\sum_{i=1}^{n} SB_{2i-1} = \sum_{i=1}^{n} SB_{2i}.$$

6. Prove that, for every natural k, the number $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.

* * *

We next turn to solutions from the readers to problems given in the March 1993 number of the Corner, where we gave the 14*th Austrian–Polish Mathematics Competition*, [1993: 66–67].

1. Show that there are infinitely many integers $m \geq 2$ such that the equality $\binom{m}{2} = 3\binom{n}{4}$ holds for some integer $n = n(m) \geq 4$. Give the general form of all such m.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; Seung-Jin Bang, Albany, California; by Tim Cross, Wolverley High School, Kidderminster, U.K.; by Matti Lehtinen, Helsinki, Finland; by Panos E. Tsaoussoglou, Athens, Greece; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wang's solution with historical comments.

All such m are characterized by $m = \binom{n-1}{2}$ where $n \geq 4$, i.e. $n(m) = \binom{n-1}{2}$. To see this note that

$$\begin{aligned}
\binom{m}{2} = 3\binom{n}{4} &\iff 4m(m-1) = n(n-1)(n-2)(n-3) \\
&\iff (2m-1)^2 = (n^2-3n)(n^2-3n+2)+1 = (n^2-3n+1)^2 \\
&\iff 2m-1 = n^2-3n+1 \\
&\iff m = \frac{n^2-3n+2}{2} = \binom{n-1}{2}.
\end{aligned}$$

Remarks.

(1). The tricky idea used in the proof above (namely that the product of any 4 consecutive integers plus one is always a perfect square) is not new. It has been used in, for example, Quickie Q765 in *Mathematics Magazine*, Vol. 63, No. 3, 1990, p. 190 (Solution, p. 198). Incidentally, there was an inadvertent inaccuracy in the statement of Q765 which should have been "Show that for any integer $a \neq 0, 2, -2$, the Diophantine equation $(a^2-1)(b^2-1) = c^2-1$ has at least two distinct solutions (b, c) in which $b \neq \pm 1$, and $c \neq \pm 1$". (To avoid trivial solutions.)

(2). The identity $\binom{\binom{n-1}{2}}{2} = 3\binom{n}{4}$ is also known and can be found as Problem #195 on page 6 of Book 3 of "1001 Problems in High School Mathematics" (collected and edited by E. Barbeau, M. Klamkin, and W. Moser, and published by the Canadian Mathematical Society). A direct combinatorial proof can be found on pp. 46–47. It is rather obvious that the present problem was derived from this identity.

2. Determine all triples of real numbers (x, y, z) satisfying the system of equations

$$(x^2 - 6x + 13)y = 20, \qquad (y^2 - 6y + 13)z = 20, \qquad (z^2 - 6z + 13)x = 20.$$

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Tim Cross, Wolverley High School, Kidderminster, U.K.; by Beatriz Margolis, Paris, France; by David Vaughan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Wildhagen's solution.

We rewrite the system as

$$[(x-3)^2 + 4] \cdot y = 20 \tag{1}$$
$$[(y-3)^2 + 4] \cdot z = 20 \tag{2}$$
$$[(z-3)^2 + 4] \cdot x = 20 \tag{3}$$

Because $(t-3)^2 + 4 \geq 4$, (1), (2), (3) imply that $0 < x, y, z \leq 5$. Thus $(x-3)^2 < 9$, $(y-3)^2 < 9$, and $(z-3)^2 < 9$, and $x, y, z > 20/13$. Therefore $(x-3)^2 \leq 4$, $(y-3)^2 \leq 4$ and $(z-3)^2 \leq 4$. This implies that $x, y, z \geq 2$.

But if, for example, $x = 2$, then $y = 4$ by (1), and $z = 4$ (by (2)). In this case (3) does not hold. Therefore $2 < x, y, z \leq 5$.

We shall prove that $x = y = z = 4$. Suppose, for example, that $x \neq 4$. We distinguish two cases.

Case I. $|x - 3| < 1$. Now (1) implies $y > 4$, so (by (2)) $z < 4$, and hence by (3), $x > 4$. This is a contradiction.

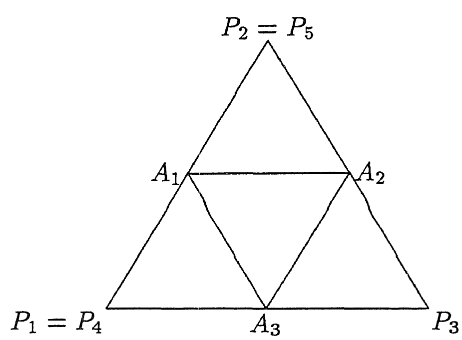
Case II. $|x - 3| > 1$. As $2 < x$ this gives $x > 4$. But now (1) gives $y < 4$, and then (2) gives $z > 4$, so by (3) $x < 4$, again a contradiction. Therefore $|x - 3| = 1$ and $x = 4$. Similarly $y = z = 4$.

3. Given are points A_1, A_2 in the plane. Determine all possible positions of a point A_3 with the following property: there exist an integer $n \geq 3$ and n points $P_1, P_2, \ldots, P_n$ such that the segments $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n, P_nP_1$ have equal length and their midpoints are $A_1, A_2, A_3, A_1, A_2, A_3, \ldots$, in this order.

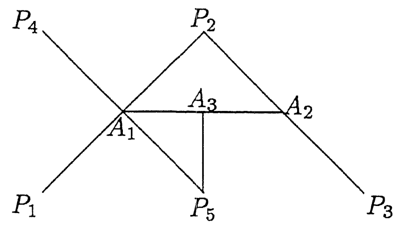
Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; and by Matti Lehtinen, Helsinki, Finland. We give Ardila's answer.

First suppose that for some A_3 we have a solution with $n \geq 5$. Let $2d$ be the common length of $P_1P_2, P_2P_3, \ldots, P_nP_1$. Then $d = A_1P_2 = P_2A_2 = A_1P_5 = P_5A_2$, and this implies P_5 is either coincident with P_2 or the reflection of P_2 through the line A_1A_2.

Case i. $P_5 = P_2$. Then $P_1 = P_4$ is the reflection of $P_2 = P_5$ through A_1 and P_3 is the reflection of P_2 through A_2. As $d = P_1A_1 = A_1P_2 = P_2A_2 = A_2P_3 = P_3A_3 = A_3P_1$ the triangle $A_1A_2A_3$ is equilateral. This yields two positions for A_3: those in which $A_1A_2A_3$ is equilateral. Here, essentially $n = 3$ and P_1 is the reflection of A_2 in the line A_1A_3.



Case ii. P_5 is the reflection of P_2 in the line A_1A_2. Now P_3 is the reflection of P_2 through A_2, P_4 is the reflection of P_5 through A_1, and A_3 is the midpoint of P_3P_4, which is the midpoint of A_1A_2 because $P_4A_1P_3A_2$ is a parallelogram. Since $A_1P_5 = P_5A_2$, $\angle A_1A_3P_5 = 90°$, so $\angle P_4A_1A_3 = \angle A_1A_3P_5 + \angle A_1P_5A_3 \geq 90°$, whence $\angle P_4A_1A_3 > \angle P_4A_3A_1$, implying $P_4A_3 > P_4A_1$, a contradiction.

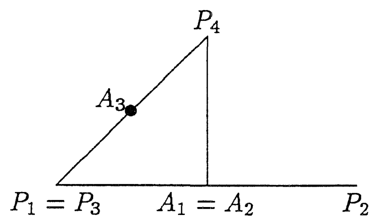


The case $n = 3$ is the same as $n \geq 5$, $P_5 = P_2$.

Now suppose $n = 4$. Now A_1 is the midpoint of P_1P_2, A_2 is the midpoint of P_2P_3, A_3 is the midpoint of P_3P_4, and A_1 is the midpoint of P_4P_1.

The first and last imply that $P_2 = P_4$ and the middle two statements give $A_2 = A_3$, which is a solution letting P_1 be the reflection of the midpoint of A_1A_2 in A_1. Therefore if $A_1 \neq A_2$, A_3 has three positions: the two for which $A_1A_2A_3$ is equilateral and $A_3 = A_2$.

Now consider $A_1 = A_2$. Clearly $P_3 = P_1$, so $n > 3$ unless $P_1 = A_1 = A_2$ whence $A_3 = A_1 = A_2$. P_4, the reflection of P_3 in A_3 has to be such that $P_4A_1 = P_3A_1 = (P_3P_4)/2$. By the triangle inequality, necessarily $P_4 = P_2$ and $A_3 = A_1 = A_2$. Therefore in this case the only solution is $A_3 = A_1 = A_2$ taking $P_1 = A_1$ and any n.



5. Show that the inequality

$$x^2 + y^2 + z^2 + xy + yz + zx \geq 2(\sqrt{x} + \sqrt{y} + \sqrt{z})$$

holds for all positive numbers x, y, z with $xyz = 1$.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; by Seung-Jin Bang, Albany, California; by Tim Cross, Wolverley High School, Kidderminster, U.K.; by Henry J. Ricardo, Tappan, New York; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We use the solutions of Ardila and Wildhagen.

Since $xy = 1/z$, $yz = 1/x$ and $zx = 1/y$, it suffices to show that $t^2 + 1/t \geq 2\sqrt{t}$, for all $t > 0$. This follows from

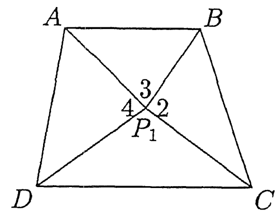
$$t^2 + \frac{1}{t} - 2\sqrt{t} = \left(t - \frac{1}{\sqrt{t}}\right)^2 \geq 0.$$

Notice that this inequality is strict unless $t = 1/\sqrt{t}$, or $t = 1$, so the original inequality is strict unless $x = y = z = 1$.

6. Inside a convex quadrilateral $ABCD$ there is a point P such that the triangles PAB, PBC, PCD, PDA have equal areas. Prove that the area of $ABCD$ is bisected by one of the diagonals.

Solutions by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Tsaoussoglou.

Now $\hat{P}_1 + \hat{P}_2 + \hat{P}_3 + \hat{P}_4 = 360°$. Denoting the area of XYZ by $|XYZ|$, we have $|PAB| = |PBC| = |PCD| = |PDA|$, by assumption. So



$$\overline{PA}\ \overline{PB} \sin \hat{P}_3 = \overline{PB}\ \overline{PC} \sin \hat{P}_2$$
$$= \overline{PC}\ \overline{PD} \sin \hat{P}_1 = \overline{PD}\ \overline{PA} \sin \hat{P}_4 \qquad (1)$$

Thus $\overline{PA}\ \overline{PB}\ \overline{PC}\ \overline{PD} \sin \hat{P}_3 \sin \hat{P}_1 = \overline{PA}\ \overline{PD}\ \overline{PB}\ \overline{PC} \sin \hat{P}_2 \sin \hat{P}_4$. From this, $2 \sin \hat{P}_1 \sin \hat{P}_3 = 2 \sin \hat{P}_2 \sin \hat{P}_4$. Then

$$\cos(\hat{P}_1 - \hat{P}_3) - \cos(\hat{P}_1 + \hat{P}_3) = \cos(\hat{P}_2 - \hat{P}_4) - \cos(\hat{P}_2 + \hat{P}_4).$$

But $\cos(\hat{P}_1 + \hat{P}_3) = \cos(\hat{P}_2 + \hat{P}_4)$, therefore

$$\hat{P}_1 - \hat{P}_3 = \hat{P}_2 - \hat{P}_4 \quad \text{and} \quad \hat{P}_1 + \hat{P}_4 = \hat{P}_2 + \hat{P}_3 = 180°$$

or

$$\hat{P}_1 - \hat{P}_3 = -(\hat{P}_2 - \hat{P}_4) \quad \text{and} \quad \hat{P}_1 + \hat{P}_2 = \hat{P}_3 + \hat{P}_4 = 180°.$$

Thus AC or BD is a diagonal which bisects $ABCD$.

7. For a given integer $n \geq 1$ determine the maximum value of the function

$$f(x) = \frac{x + x^2 + \cdots + x^{2n-1}}{(1 + x^n)^2}$$

over $x \in (0, \infty)$ and find all $x > 0$ for which the maximum is attained.

Solutions by Seung-Jin Bang, Albany, California; and by Siming Zhan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We use the solution by Zhan and Wang.

We claim that the maximum value is $(2n-1)/4$ attained uniquely at $x = 1$. Since $f(x) = f(1/x)$ it suffices to show that $f(x) < f(1)$ for all $x \in (0,1)$. Note that $f(x) = (x - x^{2n})/((1-x)(1+x^n)^2)$ for $x \neq 1$. Differentiating and simplifying we find that $f'(x) = g(x)/((1-x)^2(1+x^n)^3)$ where

$$\begin{aligned} g(x) &= (1-x)(1+x^n)(1 - 2nx^{2n-1}) - (x - x^{2n})[2nx^{n-1}(1-x) - (1+x^n)] \\ &= (1+x^n)[(1-x) + (x - x^{2n})] - 2nx^n(1-x)[x^{n-1}(1+x^n) + (1 - x^{2n-1})] \\ &= (1+x^n)(1-x^{2n}) - 2nx^n(1-x)(1+x^{n-1}). \end{aligned}$$

Clearly $f'(x) > 0 \Leftrightarrow h(x) > 0$ where

$$h(x) = \frac{g(x)}{1-x} = (1+x^n)(1 + x + x^2 + \cdots + x^{2n-1}) - 2nx^n(1 + x^{n-1}).$$

By the Arithmetic–Geometric–Mean–Inequality, we have

$$1 + x + x^2 + \cdots + x^{2n-1} > 2n(1 \cdot x \cdot x^2 \cdots x^{2n-1})^{1/2n} = 2nx^{(2n-1)/2}.$$

Thus

$$\begin{aligned} h(x) &> 2nx^{(2n-1)/2}(1+x^n) - 2nx^n(1+x^{n-1}) \\ &= 2nx^{(2n-1)/2}[(1+x^n) - \sqrt{x}(1+x^{n-1})] \\ &= 2nx^{(2n-1)/2}(1-\sqrt{x})(1-x^{(2n-1)/2}) > 0. \end{aligned}$$

Therefore f is strictly increasing on $(0,1)$ and the proof is complete.

That completes the solutions we have on file for the contest. Here is a challenge to readers to solve problems 4, 8 and 9 and complete the list.

* * *

This makes a good place to end this number of the *Corner*. Please send me your Olympiads, pre-Olympiads, and especially your nice solutions!

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BOOK REVIEWS

Edited by ANDY LIU, University of Alberta.

Exploring Mathematics With Your Computer, by Arthur Engel. MAA New Mathematical Library series Vol. 35, 1993. Paperback, 301+ix pages, ISBN 0-88385-639-5. *Reviewed by Graham Denham, student, University of British Columbia.*

Over the last ten years, affordable personal computers have appeared that rival or exceed the processing power of the large timesharing systems of previous decades. This has made significant computing resources conveniently available to many mathematicians and students. Computation has had an impact on mathematics in one way by facilitating exhaustive searches that would be impossible by hand; we recall Appel and Haken's proof of the Four–Colour Theorem as an example. At the same time, software such as Maple has automated symbol manipulation.

Arthur Engel's book, however, brings to light another, more subtle use in mathematics for a computer: as an experimental and educational tool to gain insight into a problem, make observations, and formulate conjectures. Sixty-five independent topics are considered. A chapter on number theory introduces algorithms for finding primes, determining the greatest common divisor, and other basics. A sample investigation in this chapter is the empirical observation that the density of relatively prime pairs of integers approaches the value $\pi^2/6$, then the proof that this is indeed the case. Similarly, the chapter on probability uses simulations to suggest results. Another chapter introduces some combinatorial algorithms: it shows how to find solutions to the familiar Eight Queens Problem, and analyzes games such as Nim.

Each section concludes with a substantial set of problems. Some are basic exercises, while others are very challenging. Engel has been involved with the German national mathematics competition and the German Olympiad team for many years; accordingly, some of the problems included are from the Olympiads. No background in mathematics beyond the Olympiad level is assumed.

The programs given in the text are also supplied on a 3.5" diskette included with the book. In order to run the programs, one needs a Pascal compiler. Turbo Pascal is suggested for MS-DOS systems, and Think Pascal for the Macintosh; however, instructions are included for converting them for use with other compilers. Only very basic experience with programming should be necessary to do the exercises, and the appendix summarizing Pascal syntax would be helpful for readers who are not familiar with programming in Pascal. The programs in the text tend to be short and elegant, and enough background is included to make them intelligible to readers of almost any degree of computer experience. One unfortunate aspect of the choice of programming language is that the numerical precision in Pascal is fixed, which introduces some machine-dependent distractions in calculations which are sensitive to precision.

The book is not intended to be a comprehensive introduction to its topics in mathematics, programming, or algorithms; other references are provided for each. It offers, rather, a well-chosen tour through a variety of appealing areas of elementary mathematics and the study of algorithms which, one would hope, inspires the reader with a sense of the title notion: exploring mathematics with a computer.

* * *

The Art of Problem Solving, Volume 1, by S. Lehoczky and R. Rusczyk. Published by Greater Testing Concepts, P. O. Box A-D, Stanford, CA 94309, 1993. Paperback, 360+ pages, solution manual 176+ pages, without ISBN number, US \$25 without solution manual or \$32 with. *Reviewed by Andy Liu.*

This first volume of a projected two-volume set consists of 29 chapters, 11 on algebra, 12 on geometry, five covering miscellaneous topics, and the last chapter with additional problems. There are 259 examples and 324 exercises interspersed with text, and 588 problems at the end of the chapters. The companion manual contains solutions to all the exercises and problems.

This well-written book is aimed at high school students who are interested in problem solving but perhaps inexperienced in mathematics competitions. The concepts under consideration are quite basic, and they are very well illustrated and reinforced by the examples and exercises. More advanced materials are deferred to the second volume. The book opens with familiar topics in elementary algebra to put the readers at ease. Emphasis soon shifts to elementary geometry, with which North American students are in general uncomfortable.

Many of the problems are taken from actual competitions. They are on the whole short and straight-forward, so that the readers can expect to enjoy a reasonable level of achievement. This gives them the technical background as well as some confidence for tackling harder problems later. Nevertheless, successful solvers do have to exhibit some ingenuity and exercise some caution.

The book is written primarily from the students' perspective. Its style is that of an informal discourse, discussing general methods rather than dishing out facts in isolated capsules. The authors, in graduate schools at present, draw on their own experiences in attempting as well as constructing competition papers. They have carefully identified places where the students tend to gloss over, get frustrated or slip up.

This book is highly recommended, as independent reading material or as a valuable supplement to standard textbooks. The authors graciously permit the reproduction of up to three chapters per year for use in mathematics clubs. One of these should be Chapter 28, even if for no other reason than its lucid explanation of some mathematical terms in common use.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **December 1, 1994**, *although solutions received after that date will also be considered until the time when a solution is published.*

1941. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Suppose that the area of the (nonconvex) pentagon $ABOCD$ is equal to the area of triangle OBC. Let P and Q be the points on BC such that $OP\|AB$ and $OQ\|DC$. Prove that

$$[OAB]+[OCD]=2\,[OPQ],$$

where $[XYZ]$ denotes the area of triangle XYZ.

1942. *Proposed by Paul Bracken, University of Waterloo.*

Prove that, for any $a\geq 1$,

$$\left(\sum_{k=0}^{\infty}\frac{1}{(a+k)^2}\right)^2 > 2\sum_{k=0}^{\infty}\frac{1}{(a+k)^3}\,.$$

1943. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In triangle ABC, the median AD is the geometric mean of AB and AC. Prove that

$$1+\cos A=\sqrt{2}\,|\cos B-\cos C|.$$

1944. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton.*
Find the smallest positive integer n so that

$$(n+1)^{2000} > (2n+1)^{1999}.$$

1945. *Proposed by Murray S. Klamkin, University of Alberta.*
Let $A_1A_2\ldots A_n$ be a convex n-gon.

(a) Prove that

$$A_1A_2 + A_2A_3 + \cdots + A_nA_1 \leq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

(b)* Prove or disprove that

$$2\cos\left(\frac{\pi}{n}\right)(A_1A_2 + A_2A_3 + \cdots + A_nA_1) \geq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

1946. *Proposed by N. Kildonan, Winnipeg, Manitoba.*

In a television commercial some months ago, a pizza restaurant announced a special sale on two pizzas, in which each pizza could independently contain up to five of the toppings the restaurant had available (no topping at all is also an option). In the commercial, a small boy declared that there were a total of 1048576 different possibilities for the two pizzas one could order. How many toppings are available at the restaurant?

1947. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has incenter I and centroid G. The line IG intersects BC, CA, AB in K, L, M respectively. The line through K parallel to CA intersects the internal bisector of $\angle BAC$ in P. The line through L parallel to AB intersects the internal bisector of $\angle CBA$ in Q. The line through M parallel to BC intersects the internal bisector of $\angle ACB$ in R. Show that BP, CQ and AR are parallel.

1948. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Are there any nonconstant differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(f(x))) = f(x) \geq 0$$

for all $x \in \mathbb{R}$?

1949. *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

Let D, E, F be points on the sides BC, CA, AB respectively of triangle ABC, and let R be the circumradius of ABC. Prove that

$$\left(\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}\right)(DE + EF + FD) \geq \frac{AB + BC + CA}{R}.$$

1950. *Proposed by Svetlozar Doitchev, Stara Zagora, Bulgaria.*

The equation

$$2 \cdot 3^x + 1 = 7 \cdot 5^y,$$

where x and y are nonnegative integers, has $x = 1$, $y = 0$ as one solution. Find all other solutions.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1849. [1993: 141] *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*

Let three points P, Q, R be on the sides BC, CA, AB, respectively, of a triangle ABC, such that they cut the perimeter of ΔABC into three equal parts; i.e. $QA + AR = RB + BP = PC + CQ$.

(a) Prove that

$$RP \cdot PQ + PQ \cdot QR + QR \cdot RP \geq \frac{1}{12}(a+b+c)^2.$$

(b)* Prove or disprove that the circumradius of ΔPQR is at least half the circumradius of ΔABC.

Solution to part (a) *by the proposers.*

Let

$$\lambda = \frac{b+c-a}{6}, \qquad \mu = \frac{c+a-b}{6}, \qquad \nu = \frac{a+b-c}{6},$$

and

$$\sigma_1 = \lambda + \mu + \nu = \frac{a+b+c}{6}, \quad \sigma_2 = \mu\nu + \nu\lambda + \lambda\mu, \quad \sigma_3 = \lambda\mu\nu.$$

By the cosine law,

$$\begin{aligned}(QR)^2 &= (AQ)^2 + (AR)^2 - 2AR \cdot AQ \cos A \\ &= (AQ+AR)^2\left(\frac{1-\cos A}{2}\right) + (AQ-AR)^2\left(\frac{1+\cos A}{2}\right) \\ &= \frac{4\mu\nu(\lambda+\mu+\nu)^2}{(\lambda+\mu)(\lambda+\nu)} + \frac{\lambda(\lambda+\mu+\nu)}{(\lambda+\mu)(\lambda+\nu)}(AQ-AR)^2,\end{aligned}$$

and similarly for $(RP)^2$ and $(PQ)^2$, so

$$\prod(\mu+\nu) \cdot \sum(QR)^2 = 4\left(\sum\lambda\right)^2 \cdot \sum\mu\nu(\mu+\nu) + \left(\sum\lambda\right) \cdot \sum\lambda(\mu+\nu)(AQ-AR)^2, \quad (1)$$

where the product and summations are cyclic. By the identities $\sum\lambda(\mu+\nu) = 2\sum\mu\nu$ and

$$(p+q+r)(px^2+qy^2+rz^2) = (px+qy+rz)^2 + qr(y-z)^2 + rp(z-x)^2 + pq(x-y)^2,$$

multiplying (1) by $2\sum \mu\nu$ gives

$$
\begin{aligned}
2\sum \mu\nu \cdot \prod(\mu+\nu)\cdot \sum(QR)^2 \\
&= 8\left(\sum \lambda\right)^2 \cdot \sum \mu\nu \cdot \sum \mu\nu(\mu+\nu) + \sum \lambda \cdot \sum \lambda(\mu+\nu)\cdot \sum \lambda(\mu+\nu)(AQ-AR)^2 \\
&= 8\left(\sum \lambda\right)^2 \cdot \sum \mu\nu \cdot \sum \mu\nu(\mu+\nu) \\
&\qquad + \sum \lambda \cdot \sum \mu\nu(\lambda+\mu)(\lambda+\nu)[(BR-BP)-(CP-CQ)]^2 \\
&\qquad\qquad + \sum \lambda \cdot \left(\sum \lambda(\mu+\nu)(AQ-AR)\right)^2 . \qquad (2)
\end{aligned}
$$

But

$$(BR-BP)-(CP-CQ) = BR+CQ+BC-2BC = \frac{2}{3}(a+b+c)-2a = 4\lambda-2\mu-2\nu,$$

etc. Thus, since

$$(\lambda+\mu)(\mu+\nu)(\nu+\lambda) = (\lambda+\mu+\nu)(\mu\nu+\nu\lambda+\lambda\mu)-\lambda\mu\nu = \sigma_1\sigma_2-\sigma_3, \qquad (3)$$

we have from (2) [and a considerable amount of calculation —*Ed.*] that

$$
\begin{aligned}
2\sigma_2(\sigma_1\sigma_2-\sigma_3)\sum(QR)^2 \\
&= 12\sigma_1^3\sigma_2^2 + 16\sigma_1^4\sigma_3 + 108\sigma_1\sigma_3^2 - 120\sigma_1^2\sigma_2\sigma_3 + \sigma_1\left[\sum \lambda(\mu+\nu)(AQ-AR)\right]^2 .
\end{aligned}
$$

Thus

$$\frac{1}{2}\sigma_2(\sigma_1\sigma_2-\sigma_3)[(QR)^2+(RP)^2+(PQ)^2] \ge 3\sigma_1^3\sigma_2^2 + 4\sigma_1^4\sigma_3 + 27\sigma_1\sigma_3^2 - 30\sigma_1^2\sigma_2\sigma_3. \qquad (4)$$

Next, by the Heron formula

$$\text{area}(ABC) = 9\sqrt{\sigma_1\sigma_3}\ ,$$

and

$$
\begin{aligned}
\text{area}(PQR) &= \text{area}(ABC) - \sum \frac{1}{2}AQ\cdot AR\sin A \\
&= \text{area}(ABC) - \sum[(AQ+AR)^2-(AQ-AR)^2]\frac{\text{area}(ABC)}{4bc} \\
&= 9\sqrt{\sigma_1\sigma_3} - \sigma_1^2\sqrt{\sigma_1\sigma_3}\sum\frac{1}{(\lambda+\mu)(\lambda+\nu)} + \frac{1}{4}\sqrt{\sigma_1\sigma_3}\sum\frac{(AQ-AR)^2}{(\lambda+\mu)(\lambda+\nu)} .
\end{aligned}
$$

Hence, by (3) and $\sum(\mu+\nu) = 2\sigma_1$,

$$(\sigma_1\sigma_2-\sigma_3)\ \text{area}(PQR) = [9(\sigma_1\sigma_2-\sigma_3)-2\sigma_1^3]\sqrt{\sigma_1\sigma_3} + \frac{1}{4}\sqrt{\sigma_1\sigma_3}\sum(\mu+\nu)(AQ-AR)^2.$$

Now we have

$$2\sigma_1(\sigma_1\sigma_2-\sigma_3)\ \text{area}(PQR) = 4\sigma_1^2\sigma_2\sqrt{\sigma_1\sigma_3} + \frac{1}{4}\sqrt{\sigma_1\sigma_3}\left(\sum(\mu+\nu)(AQ-AR)\right)^2 .$$

[More calculation needed here. Intrepid readers wishing to verify it may find the equation

$$(BR - BP) - (AQ - AR) = 2(\lambda + \mu - 2\nu), \text{ etc.}$$

useful.—*Ed.*] Thus

$$(\sigma_1\sigma_2 - \sigma_3)\ \mathrm{area}(PQR) \geq 2\sigma_1\sigma_2\sqrt{\sigma_1\sigma_3}\ . \tag{5}$$

By the Finsler–Hadwiger inequality

$$\sum RP \cdot PQ \geq 2\sqrt{3}\ \mathrm{area}(PQR) + \frac{1}{2}\sum (QR)^2,$$

[see p. 179 of Mitrinović, Pečarić and Volenec, *Recent Advances in Geometric Inequalities*] and using (4) and (5) we have

$$\begin{aligned}
&\sigma_2(\sigma_1\sigma_2 - \sigma_3)\left[\sum RP \cdot PQ - \frac{1}{12}\left(\sum a\right)^2\right] \\
&\quad \geq 4\sigma_1\sigma_2^2\sqrt{3\sigma_1\sigma_3} + 3\sigma_1^3\sigma_2^2 + 4\sigma_1^4\sigma_3 + 27\sigma_1\sigma_3^2 - 30\sigma_1^2\sigma_2\sigma_3 - 3\sigma_1^2\sigma_2(\sigma_1\sigma_2 - \sigma_3) \\
&\quad = 4\sigma_1\sigma_2^2\sqrt{3\sigma_1\sigma_3} + 4\sigma_1^4\sigma_3 + 27\sigma_1\sigma_3^2 - 27\sigma_1^2\sigma_2\sigma_3 \\
&\quad = 4\sigma_1\sigma_2(\sigma_2 - \sqrt{3\sigma_1\sigma_3})\sqrt{3\sigma_1\sigma_3} + \sigma_1\sigma_3(4\sigma_1^3 + 27\sigma_3 - 15\sigma_1\sigma_2) \\
&\quad = \frac{2\sigma_1\sigma_2\sqrt{3\sigma_1\sigma_3}}{\sigma_2 + \sqrt{3\sigma_1\sigma_3}}\sum \lambda^2(\mu - \nu)^2 + \sigma_1\sigma_3\sum \lambda(\mu - \nu)^2 \\
&\qquad\qquad + 2\sigma_3\sum(\mu + \nu - \lambda)^2(\mu - \nu)^2 + 4\sigma_3\sum \mu\nu(\mu - \nu)^2 \\
&\quad \geq 0.
\end{aligned}$$

Therefore [since $\sigma_1\sigma_2 - \sigma_3 > 0$ by (3)] the desired inequality follows.

No other responses to this problem were received. Part (b) *should be ignored; it is a misinterpretation by the editor of part of the proposers' original problem.*

* * * * *

1851. [1993: 169] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum_{i=1}^n x_i^2 = 1$. Prove that

$$\frac{2\sqrt{n} - 1}{5\sqrt{n} - 1} \leq \frac{1}{n}\sum_{i=1}^n \frac{x_i + 2}{x_i + 5} \leq \frac{2\sqrt{n} + 1}{5\sqrt{n} + 1}\ .$$

I. *Solution by Nasha Komanda, Central Michigan University, Mt. Pleasant.*

Put

$$u_i = \frac{x_i + 2}{x_i + 5} \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad u = \frac{1}{n}\sum_{i=1}^n u_i.$$

Then

$$x_i = \frac{5u_i - 2}{1 - u_i}\ .$$

Consider the function

$$f(t) = \left(\frac{5t-2}{1-t}\right)^2 .$$

Since

$$f'(t) = \frac{6(5t-2)}{(1-t)^3} \qquad \text{and} \qquad f''(t) = \frac{6(10t-1)}{(1-t)^4} ,$$

the function f is convex on the interval $(1/10, 1)$. The equality $\sum_{i=1}^n x_i^2 = 1$ implies $-1 \le x_i \le 1$, which in turn implies $1/4 \le u_i \le 1/2$, so all u_i and u belong to the interval $(1/10, 1)$. Using convexity of f, we obtain the inequality

$$\frac{1}{n}\sum_{i=1}^n f(u_i) \ge f\left(\frac{1}{n}\sum_{i=1}^n u_i\right) = f(u) = \left(\frac{5u-2}{1-u}\right)^2 .$$

Since $\sum_{i=1}^n f(u_i) = \sum_{i=1}^n x_i^2 = 1$, we get the inequality

$$-\frac{1}{\sqrt{n}} \le \frac{5u-2}{1-u} \le \frac{1}{\sqrt{n}} ,$$

which implies

$$\frac{2\sqrt{n}-1}{5\sqrt{n}-1} \le u \le \frac{2\sqrt{n}+1}{5\sqrt{n}+1} .$$

II. *Partial solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*

By the arithmetic mean–harmonic mean and Cauchy inequalities,

$$\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i+5} \ge \frac{n}{\sum_{i=1}^n (x_i+5)} = \frac{n}{5n+\sum_{i=1}^n x_i} \ge \frac{n}{5n+\sqrt{n\sum_{i=1}^n x_i^2}} = \frac{\sqrt{n}}{5\sqrt{n}+1} .$$

Thus

$$\frac{1}{n}\sum_{i=1}^n \frac{x_i+2}{x_i+5} = \frac{1}{n}\sum_{i=1}^n \left(1+\frac{-3}{x_i+5}\right) = 1+\frac{1}{n}\sum_{i=1}^n \frac{-3}{x_i+5} \le 1-\frac{3\sqrt{n}}{5\sqrt{n}+1} = \frac{2\sqrt{n}+1}{5\sqrt{n}+1} .$$

Editor's note. This is one of three noncalculus proofs of the right-hand inequality which were sent in. *No* noncalculus proofs of the left-hand inequality were offered, however; a bit strange, especially considering Komanda's nice handling of both inequalities simultaneously.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; A. MCD. MERCER, University of Guelph, Guelph, Ontario; VEDULA N. MURTY, Dover, Pennsylvania; FRANCISCO LUIS R. PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, *University of Warsaw, Poland; HENRY J. RICARDO, Medgar Evers College, Brooklyn, N.Y.; P. TSAOUSSOGLOU, Athens, Greece; and the proposer. One incorrect solution was sent in.*

According to the proposer, the problem arose from a "statistical headache" of one of his colleagues.

Kuczma actually proved a stronger result: for x_i as in the problem, and for any a and b satisfying $0 \le a \le b$ and $b \ge 3$,

$$\frac{a\sqrt{n}-1}{b\sqrt{n}-1} \le \frac{1}{n}\sum_{i=1}^{n}\frac{a+x_i}{b+x_i} \le \frac{a\sqrt{n}+1}{b\sqrt{n}+1} \tag{1}$$

(for the right-hand inequality only $b > 1$ is needed). Komanda's solution can be generalized to yield exactly this result (it makes a nice exercise!), although Kuczma's and Komanda's solutions are not the same. (The proposer's original problem was phrased somewhat differently, and was also more general, although analysis shows that his result is not as general as Kuczma's.)

Recently Kuczma in fact has found a **counterexample** *to the left-hand inequality in (1) for some b's smaller than 3. In particular the inequality fails in the case $n = 2$, $a = 1$, $b = 2$, $x_1 = -3/5$, $x_2 = -4/5$. According to Kuczma's computer calculations, (1) appears valid whenever $b > 2.1213$ approximately, and can fail if b is smaller than this number.*

* * * * *

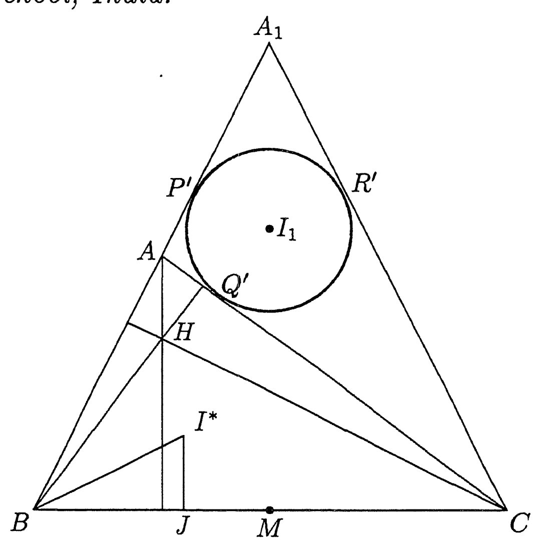
1852. [1993: 169] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an acute triangle with $AB < AC$ and with orthocenter H. Let I^* be the incenter of ΔHBC. The line through I^* perpendicular to BC meets AB and AC at P and Q respectively. Prove that the perimeter of ΔAPQ is equal to $AC - AB$.

I. *Solution by Shailesh Shirali, Rishi Valley School, India.*

Extend BA to A_1 such that $BA_1 = CA_1$. Let (I_1, r_1) be the incircle of ΔA_1AC and let the points of tangency of this circle with sides AA_1, AC, A_1C of the triangle be P', Q', R' respectively. Let M be the midpoint of BC and let J be the foot of the perpendicular from I^* to BC.

We claim: *the line JI^* is tangent to the circle (I_1, r_1).* The proof uses known trigonometric formulas [e.g.,



$$r_1 = AI_1 \sin\left(90 - \frac{A}{2}\right) = AI_1 \cos\frac{A}{2},$$

$$r_1 = I_1C \sin\frac{B-C}{2}$$

and

$$br_1 = 2\,\text{area}(AI_1C) = AI_1 \cdot I_1C \cdot \sin \angle AI_1C,$$

where $\angle AI_1C = 180^\circ - B$. —*Ed.*] to give

$$r_1 = \frac{b\cos(A/2)\sin[(B-C)/2]}{\sin B} = 2R\cos\frac{A}{2}\sin\frac{B-C}{2},$$

where R is the circumradius of ΔABC; and

$$\begin{aligned} JM &= \frac{HC - HB}{2} = R(\cos C - \cos B) = 2R\sin\frac{B+C}{2}\sin\frac{B-C}{2} \\ &= 2R\cos\frac{A}{2}\sin\frac{B-C}{2} \end{aligned}$$

[e.g., see pages 163 and 184 of Johnson's *Advanced Euclidean Geometry*], and so $r_1 = JM$, which implies that JI^* is tangent to the circle (I_1, r_1).

This shows that the line PQ of the problem is tangent to (I_1, r_1) and so (I_1, r_1) is an excircle to ΔAPQ. Therefore

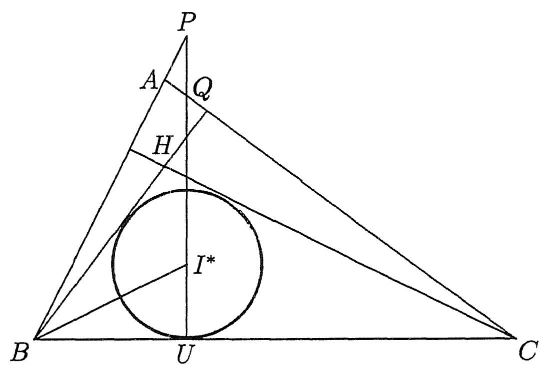
$$\begin{aligned} \text{perimeter } (\Delta APQ) &= 2 \cdot \text{ length of tangent from } A \text{ to } (I_1, r_1) \\ &= AP' + AQ' = AC - AB, \end{aligned}$$

since $BP' = CR' = CQ'$. The result follows.

Indeed, we have a corollary: the tangent to (I_1, r_1) that is parallel to PQ, the point of tangency being diametrically opposite to that of PQ with the circle, meets BC at the point where the excircle to ΔHBC opposite vertex H meets side BC. (This is true by symmetry.)

II. *Solution by P. Penning, Delft, The Netherlands.*

Denote the inradius of ΔHBC by r^* and the intersection of PI^* with BC by U. Then



$$\angle BI^*U = 45^\circ + \frac{C}{2}$$

and

$$BU = r^*\tan\left(45 + \frac{C}{2}\right) = \frac{r^*(1+\sin C)}{\cos C}.$$

With $\angle QCU = C$ and $\angle QUC = 90^\circ$,

$$QU + QC = CU\tan C + CU\cos C = \frac{CU(1+\sin C)}{\cos C}.$$

So

$$\frac{BU}{r^*} = \frac{QU+QC}{CU}$$

or $QU+QC = BU \cdot CU/r^*$. For $PU+PB$ one must find the same expression, because of the symmetry in the expression. So $QU+QC = PU+PB$, and $QC-PB = PU-QU = PQ$. Add $AQ+AP$ to both sides and the result to be proven follows immediately.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA A. LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; L.J. HUT, Groningen, The Netherlands; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

* * * * *

1853. [1993: 169] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers which satisfies the condition

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every $n \geq 1$. Prove that either the sequence converges or $\lim_{n\to\infty} b_n = \infty$.

Solution by Kee-Wai Lau, Hong Kong.

If $\liminf_{n\to\infty} b_n = \infty$ then clearly $\lim_{n\to\infty} b_n = \infty$. So we assume that $\liminf_{n\to\infty} b_n = a < \infty$ and proceed to show that $\lim_{n\to\infty} b_n = a$. By definition of a we see that for any $\varepsilon > 0$ there exists $m \geq 1$ such that $b_m < a + \varepsilon/5$ and $b_n > a - \varepsilon/5$ for all $n \geq m$. Now suppose that there exists $t > m$ such that $b_t \geq a + \varepsilon$. Then

$$b_{t+1} \geq \frac{1}{3}b_t + \frac{2}{3}b_{t-1} \geq \frac{1}{3}(a+\varepsilon) + \frac{2}{3}\left(a - \frac{\varepsilon}{5}\right) = a + \frac{\varepsilon}{5}\ .$$

It follows that $b_n \geq a+\varepsilon/5$ for all $n > t$ [since $b_{t+2} \geq (b_{t+1}+2b_t)/3$ and b_{t+1} and b_t are both $\geq a+\varepsilon/5$, so is b_{t+2}, and so on — *Ed.*]. This contradicts the fact that $\liminf_{n\to\infty} b_n = a$. Thus $b_n < a+\varepsilon$ and hence $|b_n - a| < \varepsilon$ for all $n > m$. Since $\varepsilon > 0$ is arbitrary this shows that $\lim_{n\to\infty} b_n = a$ and completes the solution of the problem.

Also solved by H.L. ABBOTT, University of Alberta; ED BARBEAU, University of Toronto; C. J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; MARCIN E. KUCZMA, Warszawa, Poland; VEDULA N. MURTY, Dover, Pennsylvania; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Israel notes that, more generally, the result is true if the inequality for the b's is replaced by

$$b_n \geq \sum_{k=1}^{m} t_k b_{n-k} \quad \text{for } n > m,$$

where all $t_k > 0$ and $\sum_{k=1}^{m} t_k = 1$. (The given problem is the case $m = 2$, $t_1 = 1/3$, $t_2 = 2/3$.) This can be proved as above, as readers may like to check.

* * * * *

1854. [1993: 169] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

In any convex pentagon prove that the sum of the squares of the diagonals is less than three times the sum of the squares of the sides.

Solution by Edward T. H. Wang, Wilfrid Laurier University, and Siming Zhan, University of Waterloo, Waterloo, Ontario.

We first establish a lemma from which the desired result follows.

Lemma. *For any convex quadrilateral Ω the sum of the squares of the two diagonals is less than or equal to the sum of the squares of the four sides, with equality if and only if Ω is a parallelogram.*

[*Editor's comment by Chris Fisher.* The solvers are too modest: their proof holds for any four points (even repeated) labeled arbitrarily A, B, C, D; they show that

$$(AB)^2 + (BC)^2 + (CD)^2 + (DA)^2 = (AC)^2 + (BD)^2 + 4(MN)^2$$

where M and N are the midpoints of the "diagonals" AC and BD. Other proofs may be found in Nathan Altshiller Court, *College Geometry* (#246, p. 126) and Roger A. Johnson, *Advanced Euclidean Geometry* §97, pp. 68–69.]

Proof. Since

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}, \quad \overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}, \quad \text{and} \quad \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$$

we have

$$\begin{aligned}
&(\overrightarrow{AB})^2 + (\overrightarrow{BC})^2 + (\overrightarrow{CD})^2 + (\overrightarrow{AD})^2 - (\overrightarrow{AC})^2 - (\overrightarrow{BD})^2 \\
&\quad = (\overrightarrow{AB})^2 + (\overrightarrow{AC} - \overrightarrow{AB})^2 + (\overrightarrow{AD} - \overrightarrow{AC})^2 + (\overrightarrow{AD})^2 - (\overrightarrow{AC})^2 - (\overrightarrow{AD} - \overrightarrow{AB})^2 \\
&\quad = (\overrightarrow{AB})^2 + (\overrightarrow{AC})^2 + (\overrightarrow{AD})^2 - 2\,\overrightarrow{AB}\cdot\overrightarrow{AC} + 2\,\overrightarrow{AB}\cdot\overrightarrow{AD} - 2\,\overrightarrow{AD}\cdot\overrightarrow{AC} \\
&\quad = (\overrightarrow{AB} + \overrightarrow{AD} - \overrightarrow{AC})^2 \geq 0,
\end{aligned}$$

with equality if and only if $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$. Since $\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$, we conclude that equality holds if and only if $\overrightarrow{AD} = \overrightarrow{BC}$; i.e., if and only if Ω is a parallelogram. □

To prove the statement of the problem, let $ABCDE$ be a convex pentagon. For $i = 1, 2, 3, 4, 5$ we denote by s_i and l_i the side and diagonal lengths, respectively, starting from A. Then by the lemma we get

$$l_i^2 + l_{i+1}^2 \leq s_i^2 + s_{i+1}^2 + s_{i+2}^2 + l_{i+3}^2 \tag{1}$$

$i = 1, 2, 3, 4, 5$ ($s_6 = s_1$, $l_6 = l_1$, etc.). Adding, $\sum_{i=1}^5 l_i^2 \leq 3\sum_{i=1}^5 s_i^2$ follows. To complete the proof, it suffices to show that equality cannot hold in all five inequalities in (1). In view of our lemma, this means that at least one of the five quadrilaterals formed by the vertices of the pentagon is not a parallelogram. It is clear from the convexity assumption that, for example, $\overrightarrow{BC}$ and $\overrightarrow{CD}$ cannot both be parallel to $\overrightarrow{AE}$.

Also solved by J. CHRIS FISHER, University of Regina; MURRAY S. KLAMKIN, University of Alberta; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; KEE-WAI LAU, Hong Kong; FRANCISCO L. R. PIMENTEL, Fortaleza, Brazil; TOSHIO SEIMIYA, Kawasaki, Japan; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Fisher proves that a stronger result follows for any pentagon, convex or not, from Bachmann's theory of n-gons as expounded in [1] *(see especially pp.* 32–33*). Klamkin independently conjectures such a result for n-gons, and indeed Fisher's argument is valid for all* $n \geq 3$. *In particular: if* A_j, $j = 1, 2, \ldots, n$, *are any* $n \geq 3$ *points in the plane (possibly collinear and not necessarily distinct) then*

$$4\cos^2\left(\frac{\pi}{n}\right) \cdot [(A_1A_2)^2 + (A_2A_3)^2 + \cdots + (A_nA_1)^2] \geq (A_1A_3)^2 + (A_2A_4)^2 + \cdots + (A_nA_2)^2$$

with equality if and only if $A_1A_2 \ldots A_n$ *is the image of a regular (convex) n-gon under a linear transformation. (In the given problem, where* $n = 5$, $2\cos(\pi/5)$ *is the golden section* $\tau = (\sqrt{5}+1)/2$, *and* $\tau^2 < 3$ *as desired.) Pimentel uses inequality* 2 *from* [2, p. 421] *to solve the problem using the golden section, but he does not propose any generalization.*

Klamkin conjectures that the above inequality can be reversed for convex n-gons if the multiplier is replaced by $2/3$: *that is,*

$$\frac{2}{3}[(A_1A_2)^2 + (A_2A_3)^2 + \cdots + (A_nA_1)^2] < (A_1A_3)^2 + (A_2A_4)^2 + \cdots + (A_nA_2)^2. \qquad (2)$$

He notes that the degenerate n-gon with $A_1A_n = 2$ *and the remaining* $n-2$ *vertices coinciding at the midpoint of* A_1A_n *shows that* $2/3$ *cannot be replaced by a larger constant. Can anyone prove or disprove* (2) *for convex* $A_1A_2 \ldots A_n$*?*

For a related problem, also by Klamkin, see Crux 1945, *this issue.*

References:

[1] J. Chris Fisher, D. Ruoff, and J. Shilleto, Perpendicular polygons, *Amer. Math. Monthly*, **92** (1985) 23–37.

[2] D. S. Mitrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities.*

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1855. [1993: 169] *Proposed by Bernardo Recamán, United World College of Southern Africa, Mbabane, Swaziland.*

Twelve friends agree to eat out once a week. Each week they will divide themselves into 3 groups of 4 each, and each of these groups will sit together at a separate table. They have agreed to meet until any two of the friends will have sat at least once at the same table at the same time. What is the minimum number of weeks this requires?

Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

Five weeks is required.

First note that each person sits with 3 people each week. Since there are 11 other people for him to sit with this requires at least 4 weeks.

Second, since there are 12 people altogether there are a total of $\binom{12}{2} = 66$ pairs that must be realized. At each of the 3 tables there are $\binom{4}{2} = 6$ pairs realized (although in weeks after the first some of these pairs might have been already accounted for). So in the first week a total of $3 \times 6 = 18$ of the total 66 pairs have been realized.

Since there are 3 groups of 4, then for any week after the first week at each table there must be at least 2 people that have sat together the first week (by the pigeonhole principle there must be 4 at a table and there are only 3 tables from which to come from so there must be a duplication). This means that the maximum number of new pairs realized after the first week is $6 - 1 = 5$ per table or $3 \times 5 = 15$ per week. Thus it is impossible to realize all possible pairings in the minimum 4 weeks because at most $18 + 15 + 15 + 15 = 63$ pairings would be realized.

Repeating pairs seems to satisfy our task in a few number of weeks. If we broke up all the people into pairs (i.e. 6 pairs) it would yield the following. First week, 18 pairs. Each week the 6 pairs are repeated so 12 new pairs will appear. It would thus seem that in 5 weeks we would have $18 + 12 + 12 + 12 + 12 = 66$ pairs, which is required, and it can be shown that it is satisfied by the following example:

Week #	Table 1	Table 2	Table 3
1	1 2 3 4	5 6 9 10	7 8 11 12
2	1 2 5 6	3 4 7 8	9 10 11 12
3	1 2 7 8	3 4 9 10	5 6 11 12
4	1 2 9 10	3 4 11 12	5 6 7 8
5	1 2 11 12	3 4 5 6	7 8 9 10

Also solved by ED BARBEAU, University of Toronto; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; DOUGLAS E. JACKSON, Eastern New Mexico University, Portales; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, N.Y.; and the proposer. Two other readers sent in incorrect answers due to misunderstanding the problem.

Komanda also mentions that five weeks would even be sufficient for 16 *friends, divided into four groups of four each week, to eat out so that each two friends sit together at least once.*

The problem was in part suggested by the proposer's experience of working in a school which admitted 108 *first-graders each year. They were divided into three classes of* 36. *He wonders: if such a group of* 108 *students is similarly divided each year as they advance from grade to grade, is it possible that after* 12 *years every two students would have been in the same class at least once? If not, how many years are required?*

* * * * *

1857. [1993: 170] *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Prove that, for any positive integer n,

$$1 < \frac{27^n (n!)^3}{(3n+1)!} < \sqrt{2}\,.$$

Solution by F. J. Flanigan, San Jose State University, San Jose, California.

Write, for positive integers n,

$$u_n = \frac{27^n (n!)^3}{(3n+1)!}\,.$$

We prove that (u_n) is a monotonically increasing sequence with $u_1 = 9/8$ and $\lim_{n\to\infty} u_n = 2\pi\sqrt{3}/9 < \sqrt{2}$.

We check that $u_1 = 9/8$ and then observe that

$$u_n = \frac{27n^3}{(3n+1)3n(3n-1)} \cdot \frac{27^{n-1}((n-1)!)^3}{(3(n-1)+1)!} = \frac{1}{1-1/(9n^2)} \cdot u_{n-1},$$

for $n \geq 1$, where $u_0 = 1$. This has two consequences: the factorization

$$u_n = \frac{1}{1-1/9} \cdot \frac{1}{1-1/36} \cdot \dots \cdot \frac{1}{1-1/(9n^2)} = \frac{1}{\prod_{k=1}^{n}(1-1/(9k^2))} \tag{1}$$

and the monotonicity

$$\frac{9}{8} = u_1 < u_2 < \cdots < u_n < \cdots\,.$$

Now to the upper bound. Here the theory of sequences intrudes, reminding us that (u_n), being strictly increasing but (presumably) bounded above (by the $\sqrt{2}$ of the statement), should converge to a limit which will be its *least* upper bound. Thus, rather than deal with $\sqrt{2}$, we elect to seek $\lim_{n\to\infty} u_n$, if at all possible. The factorization (1) reveals the decisive fact: the denominator is the nth partial product, with $z = 1/3$, of the familiar function

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$$

[e.g., see example 2, page 356 of J. E. Marsden, *Basic Complex Analysis*, W. H. Freeman, 1973]. Thus

$$\lim_{n\to\infty} u_n = \frac{\pi/3}{\sin(\pi/3)} = \frac{\pi/3}{\sqrt{3}/2} = \frac{2\pi\sqrt{3}}{9}\,,$$

as claimed.

Also solved by H. L. ABBOTT, University of Alberta; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, *Germany; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; A. MCD. MERCER, University of Guelph, Guelph, Ontario; LEROY F. MEYERS, The Ohio State University; VEDULA N. MURTY, Dover, Pennsylvania; FRANCISCO L. R. PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; HENRY J. RICARDO, Medgar Evers College, Brooklyn, N. Y.; SHAILESH SHIRALI, Rishi Valley School, India; DAVID VAUGHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One reader sent in a proof of the lower bound only.*

Over half the solvers gave the best upper bound of $2\pi\sqrt{3}/9$, *several using the above method. Another popular technique was to use Stirling's formula*

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1,$$

which yields the same result. Other solvers came up with a startling assortment of less sharp upper bounds, all improving on $\sqrt{2}$, *including:*

$$\frac{54}{54-\pi^2}, \quad e^{\pi^2/48}, \quad e^{1/4}, \quad \frac{3\sqrt{3}}{4}, \quad \frac{27}{20} \quad \text{and} \quad \frac{27\sqrt[3]{3}}{28}\ !$$

Janous generalized the given inequality to

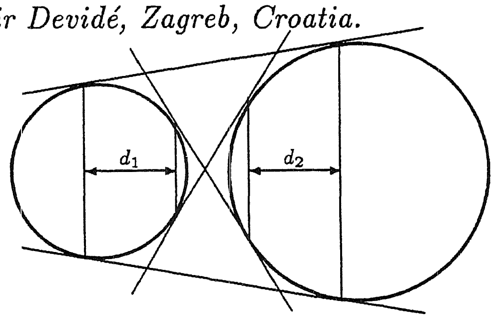
$$\frac{1}{k!} \le \frac{(2k+1)^{(2k+1)n}(n!)^{2k+1}}{[(2k+1)n+k]!} < \left(\frac{2\pi}{2k+1}\right)^k \frac{1}{\sqrt{2k+1}}$$

for $n \ge 0$, *where* k *is a positive integer. (The given inequality is the case* $k = 1$.*) Flanigan in fact also gave the case* $k = 2$.

* * * * *

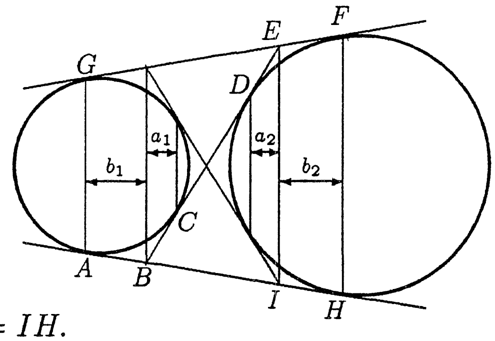
1858. [1993: 170] *Proposed by Vladimir Devidé, Zagreb, Croatia.*

The four common tangents to two circles are drawn, and their points of tangency connected by chords as shown in the diagram. (These chords are parallel, as they are all perpendicular to the line joining the centres of the circles.) Prove that $d_1 = d_2$.



Solution by Waldemar Pompe, student, University of Warsaw, Poland.

See the figure. We have:



$$\begin{aligned} 2BC + CD &= AB + BD = AB + BH \\ &= GE + EF = EC + EF \\ &= 2ED + CD \end{aligned}$$

which gives

$$BC = ED \qquad (1)$$

and then of course

$$AB = EF = IH.$$

From these two equalities follows that $a_1 = a_2$ and $b_1 = b_2$. Therefore $d_1 = a_1 + b_1 = a_2 + b_2 = d_2$.

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; SAM BAETHGE, Science Academy, Austin, Texas; ED BARBEAU, University of Toronto; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA A. LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain (two solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; L.J. HUT, Groningen, The Netherlands; ROBERT B. ISRAEL, University of British Columbia; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia; GIANNIS G. KALOGERAKIS, Canea, Crete, Greece; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University; DAN PEDOE, Minneapolis, Minnesota; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; ALBERT W. WALKER, Toronto, Ontario; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Ardila, Heuver and Meyers all give the same solution as Pompe. Meyers notes in fact that equation (1) *has already appeared in* Crux *as problem* 712 [1983 : 56]*!*

Pompe (in a second solution) and a few other solvers use the fact that the radical axis of the two circles bisects the common tangents of the circles.

The proposer actually published this result (in Croatian) in 1953, ten years after he discovered it while he was a high school student. We have Neven Jurić to thank for suggesting to the proposer that his result would be a good problem for Crux.

Meyers remembers the proposer from a visit he made to Ohio State University in the early 1970's.

Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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