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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publications should be sent to the Editors-in-Chief:

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IT'S ELEMENTARY (COMBINATORICS) II

William Moser

[*Editor's note.* This is the second of a series of three articles by Professor Moser. Some reference is made below to equations (1) to (5) appearing in the first article; for these see the previous issue [1992: 257].]

3. Circular displays In some restricted-choice problems the restrictions are best described with $1, 2, 3, \ldots, n$ displayed in a circle (clockwise), which for convenience in typesetting we draw in ellipse-like shape. For example, $\{1, 2, 3, \ldots, 10\}$ displayed in a circle looks like

$$\begin{array}{cccccc} & 10 & 1 & 2 & 3 & \\ 9 & & & & & 4 \\ & 8 & 7 & 6 & 5 & \end{array} \qquad \text{or} \qquad \begin{array}{cccccc} & 8 & 9 & 10 & 1 & \\ 7 & & & & & 2 \\ & 6 & 5 & 4 & 3 & \end{array}$$

and eight more such circular displays obtained by rotation.

A k-choice (1) can be represented by a circular display of k 1's and $n-k$ 0's with one of the n entries capped. For example, $\{2, 3, 5, 9, 10\} \subseteq \{1, 2, 3, \ldots, 11\}$ is described by

$$\begin{array}{ccccccc} & \hat{0} & 1 & 1 & 0 & 1 & \\ 0 & & & & & & 0 \\ & 1 & 1 & 0 & 0 & & \end{array} \qquad \text{or} \qquad \begin{array}{ccccccc} & 0 & \hat{0} & 1 & 1 & 0 & \\ 1 & & & & & & 1 \\ & 1 & 0 & 0 & 0 & & \end{array}$$

(and nine more such circular displays obtained by rotation).

Problem 6.

(a) Let $(n, k \,|\, 1)$ denote the number of k-choices (1) which satisfy: no two chosen integers are adjacent in the circular display

$$\begin{array}{ccccccccc} \ldots & n-2 & n-1 & n & 1 & 2 & 3 & \ldots \\ \vdots & & & & & & & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{6}$$

i.e., $x_{i+1} - x_i - 1 \geq 1$ for $i = 1, 2, 3, \ldots, k-1$, and $x_1 + n - x_k - 1 \geq 1$. Show that

$$(n, k \,|\, 1) = \frac{n}{n-k}\binom{n-k}{k}, \qquad 1 \leq k < n.$$

Solution. We construct the corresponding circular displays of k 1's and $n-k$ 0's, with one entry capped and every 1 followed clockwise by a 0, as follows. Place k 1's in a circle and color any one of the boxes *blue*, so that the boxes are distinguishable: B_1 is the *blue* box and $B_2, B_3, \ldots, B_k$ follow clockwise:

$$\begin{array}{ccccc} \ldots\ 1 \underbrace{\qquad}_{B_k} 1 \overbrace{\underbrace{\qquad}_{B_1}}^{blue} 1 \underbrace{\qquad}_{B_2} 1 \ \ldots\ \ldots \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ \ldots \end{array}$$

Distribute $n-k$ 0's into the k boxes, no box empty (by (4) this can be done in $\binom{n-k-1}{k-1}$ ways), then cap one of the n entries (this can be done in n ways). Now we have $n\binom{n-k-1}{k-1}$ circular displays of k 1's and $n-k$ 0's with one of the n entries capped and one of the boxes colored. Erase the color and note that the displays fall into groups of k each which are congruent by rotation. Take one member of each group and we have

$$\frac{n}{k}\binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k} \quad (k<n)$$

displays, precisely those we want.

(b) More general is the count, for $w \geq 0$, of the number $(n, k \mid w)$ of k-choices (1) satisfying: in the display (6) every chosen integer is followed clockwise by at least w non-chosen integers.

By Lemma 2 of Part I, there are

$$\frac{n}{k}\binom{(n-k-kw)+k-1}{k-1}$$

circular displays of k 1's and $n-k$ 0's with one entry capped and every 1 followed by at least w 0's. Hence

$$(n, k \mid w) = \frac{n}{n-kw}\binom{n-kw}{k}.$$

(c) Still more general is the count, for given non-negative integers $a_1, a_2, \ldots, a_k$, of the number of k-choices (1) satisfying: in the display (6) x_i is followed clockwise by at least a_i non-chosen integers, $i = 1, 2, \ldots, k$. (This problem was posed by Lass (1971).)

We count these k-choices in two subsets, those for which $x_k + a_k > n$ and those for which $x_k + a_k \leq n$.

For the first subset the corresponding circular displays are constructed as follows. Start with k 1's in a circle, label the boxes $B_1, B_2, \ldots, B_k$ in clockwise order, put a_1 0's into B_1, a_2 0's into B_2, ..., a_k 0's into B_k and cap one of these a_k 0's; this can be done in a_k ways. At this point we delete the labels on the boxes, since the capped $\hat{0}$ identifies the box B_k. The displays look like this:

$$\cdots 1 \underbrace{0 \cdots 0\,\hat{0}\,0 \cdots 0}_{a_k \text{ 0's}} \underbrace{\qquad} 1 \underbrace{0 \cdots 0}_{a_1 \text{ 0's}} \underbrace{\qquad} 1 \cdots$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\cdots 1 \underbrace{\qquad} \underbrace{0 \cdots 0}_{a_{i+1} \text{ 0's}} 1 \underbrace{\qquad} \underbrace{0 \cdots 0}_{a_i \text{ 0's}} 1 \cdots$$

Now distribute $n-k-(a_1+a_2+\cdots+a_k)$ 0's into the k indicated boxes; this can be done in

$$\binom{(n-k-a)+k-1}{k-1} = \binom{n-a-1}{k-1}, \quad a = a_1 + a_2 + \cdots + a_k,$$

ways. We have constructed

$$a_k\binom{n-a-1}{k-1}$$

displays, precisely those which correspond to the k-choices in the first subset.

For the second subset, the corresponding circular displays are constructed as follows. We put one symbol y and k 1's in a circle. Place a_1 0's next to (in clockwise fashion) the first 1 following the y, place a_2 0's following the next 1, a_3 next to the third 1, and so on, till we have placed a_k 0's next to the kth 1. The displays look like this:

$$\cdots\ 1\underbrace{0\cdots0}_{a_k\ 0's}\underbrace{\qquad}\ y\ \underbrace{\qquad}\ 1\underbrace{0\cdots0}_{a_1\ 0's}\underbrace{\qquad}$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

Distribute $n-k-a$ 0's into the $k+1$ indicated boxes; this can be done in

$$\binom{(n-k-a)+(k+1)-1}{(k+1)-1}=\binom{n-a}{k}, \qquad a=a_1+a_2+\cdots+a_k,$$

ways. Now put a hat on the entry which immediately follows the y and delete the y. We have constructed precisely those displays which correspond to the k-choices in the second subset. Hence the number of desired k-choices is

$$a_k\binom{n-a-1}{k-1}+\binom{n-a}{k}=\frac{ka_k+n-a}{n-a}\binom{n-a}{k}, \qquad a=a_1+a_2+\cdots+a_k.$$

(d) Suppose a year has n days, and day 1 follows day n. If k people are chosen at random, what is the probability that no two of them have birthdays in any interval of $w+1$ consecutive days, i.e., each birthday is followed by w days when there are no birthdays?

Solution. Obviously, the answer is

$$\frac{k!}{n^k}\cdot\frac{n}{n-kw}\binom{n-kw}{k}.$$

Of course the probability that some pair have birthdays in an interval of $w+1$ consecutive days is

$$P(n,k:w)=1-\frac{k!}{n^k}\cdot\frac{n}{n-kw}\binom{n-kw}{k}.$$

Now, for given n and w let $f(n:w)=\min\{k\mid P(n,k:w)\geq 1/2\}$, i.e., if $f(n:w)$ people are chosen at random then it is more likely than not that some pair have birthdays in an interval of $w+1$ consecutive days. (With fewer than $f(n:w)$ people it is more likely that no interval of $w+1$ consecutive days contains two birthdays.) The ordinary birthday surprise is that $f(365,0)=23$, i.e., with 23 people in a room, there will be a birthday coincidence with probability more than .5. It is also a surprise that $f(365,1)=14$, i.e., with 14 people there will be a birthday coincidence or adjacency with probability more than .5. Some values of $f(365, w)$ are given below.

w	0	1	2	3	4	5	6	7	8	9
$f(365, w)$	23	14	11	9	8	8	7	7	6	6

More values of $P(n, k : w)$ are given by Abramson and Moser (1970).

A linear arrangement of n 0's and 1's is also called a *binary n-bit string.* The string may also be displayed in a circle with one entry capped, and then we call it a *circular* binary n-bit string. In this case the 0's occur in "blocks", as do the 1's, and all blocks have positive length.

$$\cdots 1 \underbrace{0\,0\,0\,0\,0\,0\,0\,0}_{\text{block of 0's}} 1 \cdots \qquad \cdots 0 \underbrace{1\,1\,1\,1\,1\,1\,1\,1}_{\text{block of 1's}} 0 \cdots$$

Let $s(n \,|\, r)$ denote the number of circular binary n-bit strings which have exactly r blocks. Of course $s(n \,|\, 1) = 2$ (the string of n 1's and the string of n 0's). If not all the n bits are alike, the blocks of 0's alternate with the blocks of 1's. Hence $s(n \,|\, 2r - 1) = 0$ if $r = 1, 2, 3, \ldots$.

Problem 7.

(a) Show that

$$s(n \,|\, 2r) = 2\binom{n}{2r}, \quad 1 \le r < n.$$

Solution. We construct the circular strings by starting with $2r$ strokes in a circle, creating $2r$ indistinguishable boxes, and color one of the boxes blue. The boxes are now distinguishable: B_1 is the blue box and $B_2, B_3, B_4, \ldots, B_{2r}$ follow clockwise. Distribute n like objects (say n x's) into the $2r$ boxes, no box empty. This can be done in $\binom{n-1}{2r-1}$ ways. Replace all the x's in boxes $B_1, B_3, B_5, \ldots, B_{2r-1}$ by 1's, replace all the x's in boxes $B_2, B_4, \ldots, B_{2r}$ by 0's, and cap one of the n $0-1$ entries. This can be done in n ways. Now erase the color and the strokes. The displays fall into sets of r each, congruent by rotation, and we have

$$s(n \,|\, 2r) = \frac{n}{r}\binom{n-1}{2r-1} = 2\binom{n}{2r}, \qquad 1 \le r.$$

(b) Let $s(n \,|\, r, w)$ denote the number of circular binary n-bit strings with exactly r blocks, all of length $\ge w$. (Of course $s(n \,|\, r, 1) = s(n \,|\, r)$ and $s(n \,|\, 2r - 1, w) = 0,\ r \ge 2$.) Show that

$$s(n \,|\, 2r, w) = \frac{2n}{n - 2r(w-1)}\binom{n - 2r(w-1)}{2r}, \qquad 1 \le r,\ 1 \le w.$$

Solution. In the solution of (a), replace *no box empty* by *at least w in each box*; following the instructions now leads to

$$\begin{aligned} s(n \,|\, 2r, w) &= \frac{n}{r}\binom{n - 2r(w-1) - 1}{2r - 1} \\ &= \frac{2n}{n - 2r(w-1)}\binom{n - 2r(w-1)}{2r} = 2(n, 2r \,|\, w - 1), \end{aligned}$$

where $r \geq 1$, $w \geq 1$, by Problem 6(b). When $w = 2$, we have

$$s(n \mid 2r, 2) = \frac{2n}{n-2r}\binom{n-2r}{2r} = 2(n, 2r \mid 1), \qquad r \geq 1,$$

so the number $s_2(n)$ of circular binary n-bit strings with all blocks of length ≥ 2, or equivalently, with no substrings 101 nor 010 is

$$\begin{aligned} s_2(n) &= \sum_{r=1} s(n \mid r, 2) = s(n \mid 1, 2) + \sum_{r=1} s(n \mid 2r, 2) \\ &= 2 + \sum_{r=1} \frac{2n}{n-2r}\binom{n-2r}{2r} = \sum_{r=0} \frac{2n}{n-2r}\binom{n-2r}{2r}. \end{aligned}$$

Values of $s_2(n)$ for small n are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$s_2(n)$	2	2	2	6	12	20	30	46	74	122	200	324	522	842

in agreement with Agur, Fraenkl and Klein (1988) who derived these numbers by a different method.

Let $s(n, k \mid r, w)$ denote the number of circular binary n-bit strings with exactly k bits 1 ($n-k$ bits 0) and exactly r blocks, all of length $\geq w$, ($w \geq 1$).

(c) Show that for $1 \leq k < n$

$$\begin{aligned} s(n, k \mid 2r, w) &= \frac{nr}{(k-r(w-1))(n-k-r(w-1))}\binom{k-r(w-1)}{r}\binom{n-k-r(w-1)}{r} \\ &= \frac{nr}{k(n-k)}(k, r \mid w-1)(n-k, r \mid w-1). \end{aligned}$$

Solution. Of course $r \geq 1$. We construct the displays by starting with $2r$ strokes in a circle, with one of the $2r$ boxes they determine colored blue. The boxes are now distinguishable: B_1 is the blue box and $B_2, B_3, \ldots, B_{2r}$ follow clockwise. Distribute k 1's into the odd-numbered boxes with at least w 1's in each of these boxes. This can be done in $\binom{k-r(w-1)-1}{r-1}$ ways. Distribute $n-k$ 0's into the even numbered boxes with at least w 0's in each of these boxes. This can be done in $\binom{n-k-r(w-1)-1}{r-1}$ ways. Cap one of the n entries $0-1$; this can be done in n ways. Erase the color and the displays fall into sets of r each which are congruent by rotation. Delete the strokes, and we have

$$\begin{aligned} &\frac{n}{r}\binom{k-r(w-1)-1}{r-1}\binom{n-k-r(w-1)-1}{r-1} \\ &\qquad = \frac{nr}{(k-r(w-1))(n-k-r(w-1))}\binom{k-r(w-1)}{r}\binom{n-k-r(w-1)}{r} \end{aligned}$$

displays, precisely those we want.

When $w = 1$ we have

$$s(n, k \,|\, 2r, 1) = \frac{nr}{k(n-k)}\binom{k}{r}\binom{n-k}{r}, \qquad 1 \le k < n, \quad r \ge 1,$$

and hence

$$\binom{n}{k} = \sum_{r=1} s(n, k \,|\, 2r, 1) = \sum_{r=1} \frac{nr}{k(n-k)}\binom{k}{r}\binom{n-k}{r}, \qquad 1 \le k < n,$$

a known identity (Gould 1959, identity 3.30).

Since

$$s(n \,|\, 2r, w) = \sum_{k=1}^{n-1} s(n, k \,|\, 2r, w), \ (r \ge 1)$$

we have the identity

$$2(n, 2r \,|\, w-1) = \sum_{k=1}^{n-1} \frac{nr}{k(n-k)}(k, r \,|\, w-1)(n-k, r \,|\, w-1),$$

i.e.,

$$\frac{2n}{n-2r(w-1)}\binom{n-2r(w-1)}{2r}$$
$$= \sum_{k=1}^{n-1} \frac{nr}{(k-r(w-1))(n-k-r(w-1))}\binom{k-r(w-1)}{r}\binom{n-k-r(w-1)}{r},$$

a possibly new identity. When $w = 1$ this identity reduces to

$$\binom{n-1}{2r-1} = \sum_{k=r}\binom{k-1}{r-1}\binom{n-k-1}{r-1},$$

which is Gould's (1959) identity 3.3 with $r = s$.

(d) For $1 \le r < n$, what is the number of circular binary n-bit strings with precisely r blocks of 0's (alternating, of course, with r blocks of 1's), and no occurrence of 000 nor 111?

Solution. The condition "no occurrence of 000 or 111" is equivalent to "every block (blocks of 0's and blocks of 1's) is of length one or two". Hence there are exactly $4r - n$ blocks of length 1 and $n - 2r$ blocks of length 2. Thus, to construct the desired displays, place $2r$ strokes in a circle; color one of the boxes blue (the boxes are now distinguishable — B_1 is the blue box and $B_2, B_3, \ldots, B_{2r}$ follow clockwise). Now distribute n like symbols, say x's, into the boxes as follows. Choose $n - 2r$ of the $2r$ boxes (this can be done in $\binom{2r}{n-2r}$ ways), put a pair $x\,x$ into these chosen boxes and a single x into the other $4r - n$ boxes. Cap one of the n x's (this can be done in n ways). Replace all the x's in the box containing the capped x by 0 or 1 (2 ways to do this) and replace each of the other x's by 0 or 1 in the obvious fashion (so that in each box all the symbols are 1 or all the symbols are 0, and the blocks of 1's alternate with the blocks of 0's). Delete the $2r$ strokes and we have at this stage $2n\binom{2r}{n-2r}$ displays of n 1's and 0's in a circle, one of these n symbols capped and one of the blocks colored blue. Erase the color and the displays fall into sets of $2r$ each which are congruent by rotation. Choose one display from each set and we have

$$\frac{n}{r}\binom{2r}{n-2r}$$

displays corresponding to the strings we seek.

It follows that the number $g(n)$ of circular binary n-bit strings with no occurrence of 0 0 0 nor 1 1 1 is

$$g(n) = \sum_{r=1} \frac{n}{r}\binom{2r}{n-2r}.$$

Values of $g(n)$ for small n are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$g(n)$	0	2	6	6	10	20	28	46	78	122	198	324	520	842

in agreement with Agur, Fraenkl and Klein (1988), who computed the numbers by a different method.

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THE OLYMPIAD CORNER

No. 140

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The problem set I have selected this month is from a country which has not frequently been featured in the Corner. Many thanks go to Georg Gunther, leader of the Canadian team of the I.M.O. at Sigtuna, Sweden, and again this year in Moscow, for collecting the contest and forwarding it to me.

VIETNAMESE NATIONAL OLYMPIAD IN MATHEMATICS
FOR SECONDARY SCHOOLS

Hanoi, February, 1991

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

holds for arbitrary $x, y, z \in \mathbb{R}$.

2. Let k be an odd integer ($k > 1$). For every positive integer n, denote by $f(n)$ the greatest non-negative integer such that $(k^n - 1) \mid 2^{f(n)}$. Find a formula for $f(n)$ in terms of k and n.

3. Let a right trihedral angle $Oxyz$ and 3 fixed points A, B, C on Ox, Oy, Oz, respectively be given. A variable sphere (E) always meets A, B, C and intersects Ox, Oy, Oz also at A', B', C', respectively. Let M and M' be the centres of triangles $A'BC$ and $AB'C'$. Find the locus of midpoint S of MM'.

4. 1991 students stand in a circle facing the centre to play the following game. Each student calls one number in clockwise order beginning with a student A. The called numbers are $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$ and so on. The students calling numbers 2 and 3 must leave the game immediately. The last student remaining in the game will be rewarded. If a student wants to get a prize which clockwise position will he choose starting from A?

5. Let a triangle ABC with centre G be inscribed in a circle of radius R. Medians from vertices A, B, C meet the circle at D, E, F, respectively. Prove the inequalities

$$\frac{3}{R} \leq \frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \leq \sqrt{3}\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right).$$

6. Let x, y, z be positive real numbers with $x \geq y \geq z$. Prove

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

7. In a plane consider a set consisting of n different points ($n \geq 3$) which satisfy the following three conditions:

(1) the distance between any two points of S is not more than 1 (in a certain length unit);

(2) for each point A of S there are exactly two "adjacent" points to it, i.e. there are two points A' and A'' of S with the same unit distance $AA' = AA'' = 1$;

(3) for two arbitrary points A and B of S, if A', A'' and B', B'' are adjacent points to A and B, respectively, then $\angle A'AA'' = \angle B'BB''$.

Does there exist such a set S with $n = 1991$ and with $n = 2000$? Why?

8. Let a sequence of positive real numbers $a_1, a_2, \ldots, a_n$ ($a_n \neq a_1$, $n > 2$) be given and suppose that this sequence is not increasing (i.e. $a_k \geq a_{k+1}$ for $k = 1, 2, \ldots, n-1$) or is not decreasing (i.e. $a_k \leq a_{k+1}$ for $k = 1, 2, \ldots, n-1$). Suppose, moreover, that x, y are positive real numbers such that

$$\frac{x}{y} \geq \frac{a_1 - a_2}{a_1 - a_n} \, .$$

Prove

$$\frac{a_1}{a_2x + a_3y} + \frac{a_2}{a_3x + a_4y} + \cdots + \frac{a_{n-1}}{a_nx + a_1y} + \frac{a_n}{a_1x + a_2y} \geq \frac{n}{x+y} \, .$$

9. Given is a sequence of positive real numbers $x_1, x_2, \ldots, x_n, \ldots$ defined by the formula: $x_1 = 1$, $x_2 = 9$, $x_3 = 9$, $x_4 = 1$,

$$x_{n+4} = \sqrt[4]{x_n x_{n+1} x_{n+2} x_{n+3}} \qquad \text{if } n \geq 1.$$

Prove that this sequence is convergent and find its limit.

10. Consider all tetrahedra T satisfying the following two conditions:

(1) each side is not greater than 1 unit in length;

(2) every face is a right triangle.

Write: $S(T) = S_1^2 + S_2^2 + S_3^2 + S_4^2$, where S_j ($j = 1, 2, 3, 4$) are the areas of the four faces of T. Determine the maximum value of $S(T)$.

11. Let n be a natural number and let

$$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}, \tag{1}$$

for $n > 1$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $a_1, a_2, \ldots, a_k$ are positive integers. Put

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ 1 + a_1p_1 + a_2p_2 + \cdots + a_kp_k & \text{if } n \text{ is of form (1)} \end{cases}$$

and

$$f_1(n) = f(n), \; f_2(n) = f(f(n)), \; \ldots, \; f_s(n) = f(f_{s-1}(n)), \; \ldots \, .$$

Prove that for every given natural number β, there is a natural number s_0 such that $f_s(\beta) + f_{s-1}(\beta)$ does not depend on β for each integer $s > s_0$.

12. Let a set X consisting of $2n$ different real numbers ($n \geq 3$) be given. Suppose that a set K of pairs (x, y), where $x, y \in X$ and $x \neq y$, is such that

(1) if $(x, y) \in K$ then $(y, x) \notin K$;

(2) each $x \in X$ appears in at most 19 pairs of K.

Prove that the set X can be divided into 5 non-empty subsets $X_1, \ldots, X_5$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$, and the number of all pairs $(x, y) \in K$ with x, y belonging to the same X_i ($i = 1, \ldots, 5$) is at most $3n$.

* * *

We now return to problems, comments, and solutions sent in over the summer in response to earlier numbers of the Corner this year. Last month we began the solutions sent in by Leroy F. Meyers to problems of the Celebration of Chinese New Year contests given in [1991: 1–2] and discussed in [1992: 100–103]. We resume with his corrections and solutions to the *1981 Celebration of Chinese New Year Contest* [1991: 2]. This will leave only problem 5 unsolved from this set. Any takers for this challenge?

1. What is the coefficient of x^2 when

$$(\ldots(((x-2)^2-2)^2-2)^2-\cdots-2)^2$$

is expanded and like terms are combined?

Correction by Leroy F. Meyers, The Ohio State University.

The last line on [1992: 101] should end with $2 \cdot 4^{2n-2} + 4P''_{n-1}(0)$. The second line on the next page should be $P''_n(0) = -\frac{1}{6}4^n + \frac{1}{6}4^{2n}$, and the end of the next line should be "is thus $-\frac{1}{12}4^n + \frac{1}{12}4^{2n}$."

2. Prove that $1980^{1981^{1982}} + 1982^{1981^{1980}}$ is divisible by 1981^{1981}.

Correction by Leroy F. Meyers, The Ohio State University.

There is a misprint in the first display. Replace $(-1)^r$ by $(-1)^{a^{a+1}-r}$.

4. The base of a tetrahedron is a triangle with side lengths 8, 5 and 5. The dihedral angle between each lateral face and the base is 45°. Determine the volume of the tetrahedron.

Solution by Leroy F. Meyers, The Ohio State University.

The altitude of the tetrahedron to the given triangular face is equal to the inradius of the triangle, since the 45° dihedral angles make that altitude equal to each of the three distances from the foot of the altitude to the sides of the given triangle. Then from the usual formulas for the inradius r and the area K of a triangle in terms of its side lengths a, b, c and semiperimeter $s = (a+b+c)/2$ we obtain the volume of the pyramid as

$$V = \frac{1}{3}rK = \frac{1}{3}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\sqrt{s(s-a)(s-b)(s-c)} = \frac{(s-a)(s-b)(s-c)}{3}\ .$$

With the given data $a = 8$, $b = c = 5$, we have $s = 9$ and

$$V = \frac{1 \cdot 4 \cdot 4}{3} = \frac{16}{3}\ .$$

Thanks again to Leroy F. Meyers for solutions to the first three problems from the *Composition de Mathématiques, Session de 1988* [1991: 33–34]. It should be noted that simple calculators *are allowed* for this contest.

Problem I. Let N, p, n be non-zero whole numbers. Consider a rectangular matrix T having n lines numbered 1 through n, and p columns numbered 1 through p. For $1 \le i \le n$ and $1 \le k \le p$, the entry in row i and column k is an integer a_{ik} satisfying $1 \le a_{ik} \le N$. Let E_i be the set of integers appearing in row i.

Answer Question 1 or 2.

Question 1.

In this question two further conditions are imposed on T:

i. for $1 \le i \le n$, E_i has exactly p elements;

ii. for different values of i and j, the sets E_i and E_j are different.

Let m be the smallest value of N for which, given values for n and p, one can form a matrix T having the preceding properties.

(a) Calculate m for $n = p + 1$.

(b) Calculate m for $n = 10^{30}$ and $p = 1988$.

(c) Determine the limit of m^p/n where p is fixed and n tends to infinity.

Question 2.

In this question we replace the two extra conditions of Question 1 by the two following conditions:

i. $p = n$;

ii. for every ordered pair of positive integers (i, k) with $i + k \le n$, the integer a_{ik} does not belong to the set E_{i+k}.

(a) Show that for distinct i and j the sets E_i and E_j are different.

(b) Show that if n is at least 2^q, where q is a positive integer, then $N \ge q + 1$.

(c) Suppose that $n = 2^r - 1$, where r is a fixed positive integer. Show that $N \ge r$.

Solution by Leroy F. Meyers, The Ohio State University.

The matrix T is a red herring, as can be seen below.

Question 1. The number m is the smallest positive integer N for which the number of p-subsets of an N-element set is at least n.

(a) Obviously, $m = p + 1$ for $n = p + 1$, since $\binom{p+1}{p} = p + 1 > \binom{N}{p}$ for $N < p + 1$.

(b) The smallest integer N such that $\binom{N}{1988} = \binom{N}{N-1988} \ge 10^{30}$ is found by easy use of a calculator to be $1988 + 12 = 2000$.

(c) For fixed p, we have

$$\binom{m}{p} = \frac{m(m-1)\cdots(m-p+1)}{p!} \ge n > \frac{(m-1)(m-2)\cdots(m-p)}{p!} = \binom{m-1}{p},$$

so that

$$\frac{m^p}{n} \ge \frac{m(m-1)\cdots(m-p+1)}{n} \ge p! > \frac{(m-1)(m-2)\cdots(m-p)}{n} \ge \frac{(m-p)^p}{n},$$

which shows that m^p/n tends to $p!$ as n tends to infinity.

Question 2.

(a) We may assume that $i < j$. Then $(a_{i,j-i})$ belongs to E_i but not to E_j, since $E_j = E_{i+(j-i)}$.

(b) If $N \leq q$, then there are at most $2^q - 1$ distinct subsets of $\{1, 2, \ldots, N\}$.

(c) If $N < r$, then the number of nonempty subsets of $\{1, 2, \ldots, N\}$ is less than $2^r - 1$, which is the number n of distinct (by (a)) sets.

Exercise II. Determine, for n a positive integer, the sign of $n^6 + 5n^5 \sin n + 1$. For which positive integers n is it true that

$$\frac{n^2 + 5n \cos n + 1}{n^6 + 5n^5 \sin n + 1} \geq 10^{-4}?$$

Solutions by Leroy F. Meyers, The Ohio State University, and by Pavlos Maragoudakis, student, Athens, Greece. We give Meyers' solution, which uses a hand calculator.

Let

$$f(x) = n^6 + 5n^5 \sin n + 1 = n^5(n + 5 \sin n) + 1.$$

If $n \geq 5$, then $f(n) \geq n^5(n-5) + 1 > 0$. If $1 \leq n \leq 3$, then $\sin n > 0$, so that $f(n) \geq n^5(n-0) + 1 > 0$. However, more delicate calculations are needed for $n = 4$, since $\sin 4 \approx -\sin 49.2°$. If we use the bounds $45° < 4 - \pi < 54°$, noting that $\sin 45° = \sqrt{2}/2 \approx 0.707$ and $\sin 54° = (1+\sqrt{5})/4$, we find that $476.7 > f(4) > -45.2$, which doesn't determine the sign of $f(4)$. Hence we are forced to use electronic calculators or computers or (perish the thought!) trigonometric and logarithmic tables to find that $f(4) \approx 222.2$, so that $f(n) > 0$ for all positive integers n.

The second part of the problem begs for a calculator. The following table gives the first few values (rounded) of $g(n) = (n^2 + 5n \cos n + 1)/(n^6 + 5n^5 \sin n + 1)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$g(n)$	.757	.004	−.005	.018	.051	.002	.004	.00014	.00006	.00008	.00013	.00008

It appears that $g(n) \geq 10^{-4}$ just when n is 1, 2, 4, 5, 6, 7, 8, or 11. This is in fact true. First we note that if $n \geq 5$, then $n^2 + 5n \cos n + 1 \geq n(n-5) + 1 > 0$. If $n \geq 13$, then

$$g(n) = \frac{n(n + 5\cos n) + 1}{n^5(n + 5 \sin n) + 1} \leq \frac{n(n+5)+1}{n^5(n-5)+1} \leq \frac{235n^2/169}{8n^6/13} = \frac{235}{104n^4} \leq .00008 < 10^{-4},$$

since for $n \geq 13$ we have

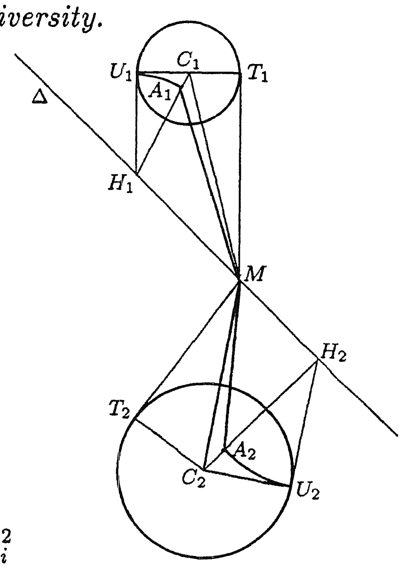
$$n(n+5) + 1 = n^2\left(1 + \frac{5}{n} + \frac{1}{n^2}\right) \leq \frac{235}{169}n^2$$

and

$$n^5(n-5) + 1 > n^6\left(1 - \frac{5}{n}\right) \geq \frac{8}{13}n^6.$$

Exercise III. Consider two spheres Σ_1 and Σ_2 and a straight line Δ which does not meet them. For $i = 1$ and $i = 2$, let C_i be the centre of Σ_i, H_i the orthogonal projection of C_i on Δ, r_i the radius of Σ_i, and let d_i be the distance of C_i to Δ. Let M be a point on Δ, and for $i = 1$ and $i = 2$, let T_i be the point of contact with Σ_i of a plane tangent to Σ_i and passing through M; set $\delta_i(M) = MT_i$. Situate M on Δ so that $\delta_1(M) + \delta_2(M)$ is minimized.

Solution by Leroy F. Meyers, The Ohio State University.



The points H_2 and M and the distances d_2 and $\delta_2(M)$ are not changed if $\sum_2$ is rotated about Δ as axis so that C_2 is in the plane containing Δ and C_1 and is on the opposite side of Δ from C_1. Neither $\delta_1(M)$ nor $\delta_2(M)$ is changed if we assume that T_1 and T_2 lie in this plane. Hence we now have a plane problem, namely that of finding M to minimize the sum of the lengths of the tangents from M to the circles Γ_i with radii r_i and respective centres C_i distance d_i from Δ. Let U_i be a point of contact of Γ_i with a tangent line from H_i. Then

$$\begin{aligned}(\delta_i(M))^2 &= MT_i^2 = MC_i^2 - r_i^2 = MH_i^2 + H_iC_i^2 - r_i^2 \\ &= MH_i^2 + H_iU_i^2 + r_i^2 - r_i^2 = MH_i^2 + H_iA_i^2 = MA_i^2,\end{aligned}$$

where A_i is the point on the segment H_iC_i whose distance from H_i is $H_iA_i = \sqrt{d_i^2 - r_i^2}$. Since A_1 and A_2 are uniquely determined by $\sum_1$, $\sum_2$, and Δ, the problem reduces to that of finding the point M on Δ which minimizes $MA_1 + MA_2$. This point M is well known to be the point of intersection of the segment A_1A_2 with Δ.

*

We received solutions from Michael Selby, University of Windsor, to two problems from the April number, #7, 8 [1991: 102] for which solutions have been recently discussed [1992: 203]. For the May number John Morvay, Springfield, Missouri, sent in a solution to 1 [1991: 130] and he and Michael Selby sent solutions to 2 [1991: 130]. In both cases solutions were discussed in the interim [1992: 227]. A solution to 3 [1991: 131] was sent in by Murray S. Klamkin, University of Alberta, but it was misfiled because of an error in the page reference. Over the summer one arrived from Michael Selby as well. The solutions appeared in [1992: 230].

*

The next solution we give this number responds to my challenge of the last number, but its author must be a mind reader since the solution arrived in the summer and the October issue of *Crux* has not yet (at this writing) reached the subscribers.

6. [1991: 129] *40th Mathematical Olympiad in Poland.*
Prove that the inequality

$$\left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)^{1/2} \geq \left(\frac{abc+abd+acd+bcd}{4}\right)^{1/3}$$

holds for any positive numbers a, b, c, d.

Solution by Michael Selby, University of Windsor.
Let

$$A = \frac{a+b+c+d}{4}, \quad B = \frac{ab+ac+ad+bc+bd+cd}{6}, \quad C = \frac{abc+abd+acd+bcd}{4},$$

and $D = abcd$. Define

$$p(x) = (x+a)(x+b)(x+c)(x+d) = x^4 + 4Ax^3 + 6Bx^2 + 4Cx + D.$$

Now $p(x) = 9$ has four real nonzero roots, $-a$, $-b$, $-c$, and $-d$. Since $p(x)$ has all real roots $p'(x)$ and $p''(x)$ has real roots, that is $12x^2 + 24Ax + 12B = 0$ has only real roots. From this

$$4A^2 \geq 4B, \quad \text{or} \quad A^2 \geq B. \tag{1}$$

Since $p'(x)$ has only nonzero real roots, where $p'(x) = 4x^3 + 12Ax^2 + 12Bx + 4C$, we have $C \neq 0$, and $p'(0) \neq 0$. Let $x = 1/y$. Then $4/y^3 + 12A/y^2 + 12B/y + 4C = 0$ has only real roots. Simplifying, we obtain $4Cy^3 + 12By^2 + 12Ay + 4 = 0$ has only real roots. Differentiating, $12Cy^2 + 24By + 12A = 0$ has only real roots. From this $4B^2 \geq 4AC$, and $B^2 \geq AC$. From (1), since $B \geq 0$, $A \geq B^{1/2}$, giving $B^2 \geq AC \geq B^{1/2}C$ or $B^{3/2} \geq C$.

Therefore $(B^{3/2})^{1/3} \geq C^{1/3}$ or $B^{1/2} \geq C^{1/3}$. By a similar technique, one can prove the following. *Let $a_1, a_2, \ldots, a_n$ be positive numbers, and let*

$$A_k = \left(\sum_{i_1<i_2<\cdots<i_k} a_{i_1}\, a_{i_2}\, \ldots\, a_{i_k}\right) \Big/ \binom{n}{k}.$$

Then if $k_1 \geq k_2 \geq 1$, $A_{k_2}^{1/k_2} \geq A_{k_1}^{1/k_1}$. For more details see the *Theory of Equations* by J.V. Uspensky, pp. 115–116.

[*Editor's Comment.* Murray S. Klamkin, University of Alberta, comments that these inequalities are just the MacLaurin inequalities, and gives the reference: G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, London, 1934, p. 52.]

*

Next we have two generalizations to problems for which the solutions were discussed in the October number.

1. [1991: 163; 1992: 231] *15th All Union Mathematical Olympiad, Tenth Grade.*
Find natural numbers $a_1 < a_2 < \ldots < a_{2n+1}$ which form an arithmetic sequence such that the product of all terms is the square of a natural number.

Generalization and solution by Murray S. Klamkin, University of Alberta.

More generally, we can make the product be the mth power of a natural number whenever m is relatively prime to $2n+1$. Let $a_1, a_2, \ldots, a_{2n+1}$ be any consecutive terms of an increasing arithmetic progression of natural numbers. Then so are $ka_1, ka_2, \ldots, ka_{2n+1}$. The product of the latter terms is $k^{2n+1}(a_1 \ldots a_{2n+1}) \equiv k^{2n+1}P$. Now let $k = P^r$, so that $k^{2n+1}P = P^{(2n+1)r+1}$. Since $2n+1$ is relatively prime to m, there always exist natural numbers r and s satisfying

$$(2n+1)r + 1 = ms.$$

In particular, if $m = 2$, r can be any odd natural number. If $m = 3$, we can choose $r = 2$ if $2n+1 \equiv 1 \bmod 3$ and $r = 1$ if $2n+1 \equiv 2 \bmod 3$.

4. [1991: 163; 1992: 232] *15th All Union Mathematical Olympiad, Tenth Grade.*

Let $a \geq 0$, $b \geq 0$, $c \geq 0$, and $a+b+c \leq 3$. Prove

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \leq \frac{3}{2} \leq \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

Generalization by Murray S. Klamkin, University of Alberta.

Let $a_1, a_2, \ldots, a_n \geq 0$; $a_1 + a_2 + \cdots + a_n \leq nA$; $m, \lambda > 0$; $F(x)$ an increasing function of x; and $G(x)$ a decreasing function of x. Then

$$\sum_{i=1}^{n} F\left(\frac{A^m a_i^m}{A^{2m} + a_i^{2m}}\right) \leq nF\left(\frac{1}{2}\right) \tag{1}$$

and with equality if all the a_i's equal A. Also

$$\sum_{i=1}^{n} G(\lambda + a_i) \geq nG(\lambda + A) \tag{2}$$

and with equality if all the a_i's equal A.

Proof: Note that (1) follows immediately from

$$A^m a_i^m \leq \frac{1}{2}(A^{2m} + a_i^{2m})$$

(with equality if and only if $a_i = A$).

For (2), we have by Jensen's inequality that

$$\sum_{i=1}^{n} G(\lambda + a_i) \geq nG(\lambda + \sum_{i=1}^{n} a_i/n) \geq nG(\lambda + A).$$

The proposed inequalities correspond to the special cases when $n = 3$, $A = n = \lambda = 1$, $F(x) = x$, and $G(x) = 1/x$.

* * *

That pretty much brings us up to date with correspondence from the summer, but it uses all the space available this month. We'll start the new year with solutions to problems proposed but not used at the 31st IMO in China. Send me your contests and nice solutions.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **July 1, 1993**, *although solutions received after that date will also be considered until the time when a solution is published.*

1791. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Γ is an ellipse with foci F and F'. Let P, Q be points on Γ, and let A be the intersection of PQ with the minor axis of Γ. Prove that

$$\left|\frac{PF-PF'}{QF-QF'}\right| = \frac{AP}{AQ} .$$

1792. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x, y \geq 0$ such that $x + y = 1$, and let $\lambda > 0$. Determine the best lower and upper bounds (in terms of λ) for

$$(\lambda+1)(x^\lambda + y^\lambda) - \lambda(x^{\lambda+1} + y^{\lambda+1}).$$

1793. *Proposed by Murray S. Klamkin, University of Alberta.*

Prove that in any n-dimensional simplex there is at least one vertex such that the n edges emanating from that vertex are possible sides of an n-gon.

1794. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Pairs of numbers from the set $\{7, 8, \ldots, n\}$ are adjoined to each of the 20 different (unordered) triples of numbers from the set $\{1, 2, \ldots, 6\}$, to obtain twenty 5-element sets $A_1, A_2, \ldots, A_{20}$. Suppose that $|A_i \cap A_j| \leq 2$ for all $i \neq j$. What is the smallest n possible?

1795. *Proposed by Hayo Ahlburg, Benidorm, Spain.*

A triangle has the angles $A < B < C$. Angle D is then defined by $\tan A + \tan B + \tan C = \tan D$. Find a triangle for which A, B, C, D are in arithmetic progression.

1796. *Proposed by Ji Chen, Ningbo University, China.*

If A, B, C are the angles of a triangle, prove that

$$\sum \sin B \sin C \leq 3 \sum \sin(B/2) \sin(C/2),$$

where the sums are cyclic.

1797. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Solve the equation $2^x - 5 = 11^y$ in positive integers.

1798. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Show that there is a three-term arithmetic progression a_1, a_2, a_3 of positive integers so that $a_1a_2a_3 = x^{1992} - y^{1992}$ for distinct positive integers x, y.

1799. *Proposed by Shiko Iwata, Gifu, Japan.*

Let P be on the circumcircle of triangle ABC. D and E are the feet of the perpendiculars from P to BC and CA, respectively. L and M are the midpoints of AD and BE, respectively. Show that $DE \perp LM$.

1800. *Proposed by Weixuan Li, University of Ottawa, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Call a permutation π on $\{1, 2, \ldots, n\}$ an *equidistance permutation* if there is a constant $c \neq 0$ such that $|\pi(i) - i| = c$ for all $i \in \{1, 2, \ldots, n\}$. Find the number of equidistance permutations for $n = 1800$.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1588. [1990: 268; 1991: 315] *Proposed by D.M. Milošević, Pranjani, Yugoslavia.*

Show that

$$\sin B \sin C \leq 1 - \frac{a^2}{(b+c)^2} ,$$

where a, b, c are the sides of the triangle ABC.

II. *Solution by Jun-hua Huang, student, The 4th Middle School of Nanxian, Hunan, China.*

The result may be sharpened to

$$\sin B \sin C = \cos^2\left(\frac{B-C}{2}\right)\left(1 - \frac{a^2}{(b+c)^2}\right) . \tag{1}$$

Let AE be the internal bisector of angle A, and put $\theta = \angle AEC$. We know

$$AE = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{1}{b+c}\sqrt{bc[(b+c)^2 - a^2]} = \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)} .$$

Then, since

$$\frac{AE}{\sin B} = \frac{c}{\sin \angle AEB} = \frac{c}{\sin\theta} \quad \text{and} \quad \frac{AE}{\sin C} = \frac{b}{\sin\theta} ,$$

we have

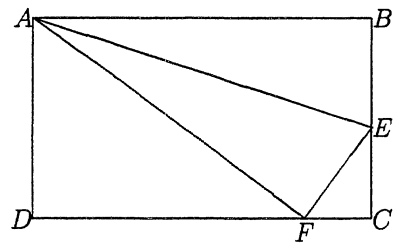
$$1 - \frac{a^2}{(b+c)^2} = \frac{(AE)^2}{bc} = \frac{\sin B \sin C}{\sin^2\theta} = \frac{\sin B \sin C}{\sin^2(B + A/2)} = \frac{\sin B \sin C}{\cos^2[(B-C)/2]} ,$$

which is (1).

* * * * *

1696. [1991: 302] *Proposed by Ed Barbeau, University of Toronto.*

An $8\frac{1}{2}$ by 11 sheet of paper is folded along a line AE through the corner A so that the adjacent corner B on the longer side lands on the opposite longer side CD at F. Determine, with a minimum of measurement or computation, whether triangle AEF covers more than half the quadrilateral $AECD$.



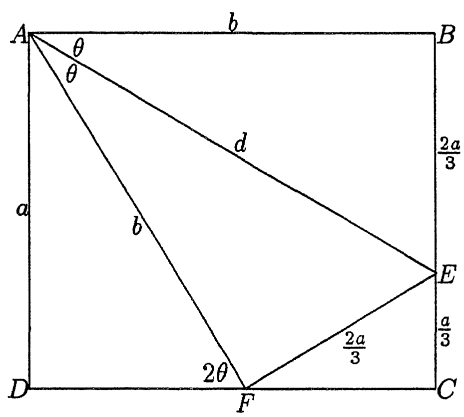
Solution by Richard I. Hess, Rancho Palos Verdes, California.

Note that

$$[AEF] < \frac{1}{2}[AECD] \iff [ABE] < \frac{1}{3}[ABCD] \iff \overline{BE} < \frac{2}{3}\overline{BC},$$

where $[X]$ is the area of figure X. An easy measurement verifies that $\overline{BE} < \frac{2}{3}\overline{BC}$ if one actually folds an $8\frac{1}{2} \times 11$ sheet of paper. Thus $[AEF] < \frac{1}{2}[AECD]$.

Without the ability to make any measurement, consider the rectangle at right in which



$$d^2 = b^2 + \frac{4}{9}a^2 \qquad (1)$$

and

$$\frac{a}{b} = \sin 2\theta = 2\sin\theta\cos\theta = 2 \cdot \frac{2a}{3d} \cdot \frac{b}{d} = \frac{4ab}{3d^2},$$

hence $d^2 = 4b^2/3$ and (from (1))

$$\frac{b^2}{a^2} = \frac{4}{3}.$$

Thus for $b^2/a^2 \le 4/3$ we have $[AEF] \ge [AECD]/2$. But for $a = 17/2$ and $b = 11$,

$$\frac{b^2}{a^2} = \frac{484}{289} > \frac{4}{3},$$

so $[AEF] < \frac{1}{2}[AECD]$.

Also solved by H.L. ABBOTT, University of Alberta; GENE ARNOLD, Ferris State University, Big Rapids, Michigan; CHARLES ASHBACHER, Cedar Rapids, Iowa; SAM BAETHGE, Science Academy, Austin, Texas; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; ANDY LIU, University of

Alberta; P. PENNING, Delft, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; KENNETH M. WILKE, Topeka, Kansas; JOSÉ YUSTY PITA, Madrid, Spain; and the proposer.

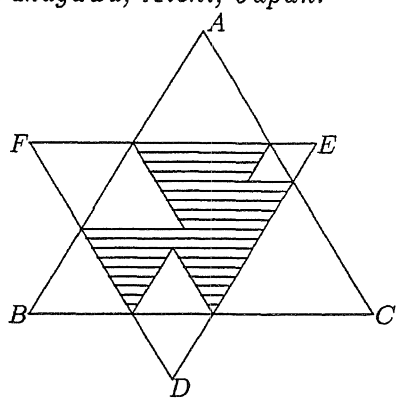
Several solvers gave similar answers, some noting that the condition $\overline{BE}/\overline{BC} < 2/3$ *can also be tested by folding. Some solvers pointed out that (as given above) the rectangle such that triangle AEF covers exactly half of AECD has dimensions* 2 *by* $\sqrt{3}$, *and in this case the point F will be the midpoint of DC (this observation yields another easy "solution by folding").*

The problem was first printed (by the proposer) in the Spring 1991 issue of the University of Toronto Alumni Magazine, *and there elicited another twenty or so responses.*

*   *   *   *   *

1698. [1991: 302] *Proposed by Hidetosi Fukagawa, Aichi, Japan.*

ABC is an equilateral triangle of area 1. DEF is an equilateral triangle of variable size, placed so that the two triangles overlap, with $DE \parallel AB$, $EF \parallel BC$, $FD \parallel CA$, and D, E, F not in ΔABC, as shown. The corners of ΔDEF sticking outside ΔABC are then folded over. Find the maximum possible area of the uncovered (shaded) part of ΔDEF.



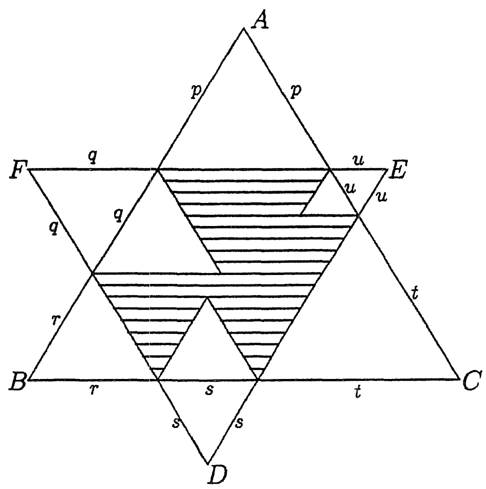
Solution by John G. Heuver, Grande Prairie Composite H.S., Grande Prairie, Alberta.

The triangles that make up the unshaded part of ΔABC are all equilateral with sides p, q, r, s, t, u respectively as shown. The area of this unshaded part is then

$$\tau = (p^2 + q^2 + r^2 + s^2 + t^2 + u^2)\frac{\sqrt{3}}{4} \ ,$$

where

$$2p + q + 2r + s + 2t + u = 3AB = \frac{6}{\sqrt[4]{3}} \ , \qquad (1)$$



since ΔABC has area 1. In order that the shaded area be maximal we want to find the smallest value of

$$p^2 + q^2 + r^2 + s^2 + t^2 + u^2$$

subject to (1). We have by Cauchy's inequality

$$(2^2+1^2+2^2+1^2+2^2+1^2)(p^2+q^2+r^2+s^2+t^2+u^2) \geq (2p+q+2r+s+2t+u)^2$$

with equality if

$$\frac{p}{2} = q = \frac{r}{2} = s = \frac{t}{2} = u = \frac{2}{5\sqrt[4]{3}},$$

which implies that $\tau = 3/5$. Hence the maximum shaded area is 2/5.

Also solved by SAM BAETHGE, Science Academy, Austin, Texas; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; and the proposer.

The problem was quoted from the 1879 Japanese book Kyokusu Taisei.

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1699. [1991: 302] *Proposed by Xue-Zhi Yang and Ji Chen, Ningbo University, China.*

Let $R, r, h_a, h_b, h_c, r_a, r_b, r_c$ be the circumradius, inradius, altitudes, and exradii of a triangle. Prove that

$$\sqrt{\frac{2R}{r}+5} \leq \sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \leq \sqrt{\frac{4R}{r}+1}.$$

Solution by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.

For the left hand side we prove the slightly stronger

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \geq \sqrt{\frac{2R}{r} + 3\sqrt[3]{\frac{4R}{r}} - 1} \geq \sqrt{\frac{2R}{r}+5}\,, \tag{1}$$

the last inequality holding because of $R/r \geq 2$. Since

$$r_a = \frac{F}{s-a} \quad \text{and} \quad h_a = \frac{2F}{a}, \quad \text{etc.},$$

where F is the area and s the semiperimeter of the triangle [e.g. p. 189 of [1]], and also

$$\sum \frac{a}{2(s-a)} = \frac{2R}{r} - 1$$

(see for example "Equalities and Inequalities in the Triangle", Soltan, Meidman, Kishinev, 1982) [or (22), p. 54 of [2]], we have

$$\left(\sum \sqrt{\frac{r_a}{h_a}}\right)^2 = \sum \frac{r_a}{h_a} + 2\sum \sqrt{\frac{r_b r_c}{h_b h_c}}$$

$$= \sum \frac{a}{2(s-a)} + \sum \sqrt{\frac{bc}{(s-b)(s-c)}}$$

$$= \frac{2R}{r} - 1 + \sum \sqrt{\frac{bc}{(s-b)(s-c)}} \,. \tag{2}$$

Using the A.M.–G.M. inequality and

$$F = sr = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}\,,$$

we obtain

$$\sum \sqrt{\frac{bc}{(s-b)(s-c)}} \ge 3\sqrt[3]{\frac{abc}{(s-a)(s-b)(s-c)}} = 3\sqrt[3]{\frac{4RFs}{F^2}} = 3\sqrt[3]{\frac{4R}{r}}\,,$$

which with (2) gives (1), equality holding when the triangle is equilateral.

For the right hand side we apply the Cauchy–Schwarz inequality and the known relations

$$r_a + r_b + r_c = 4R + r, \quad \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

[e.g., p. 189 of [1]]. We have

$$\sum \sqrt{\frac{r_a}{h_a}} \le \sqrt{r_a + r_b + r_c} \cdot \sqrt{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} = \sqrt{4R + r} \cdot \sqrt{\frac{1}{r}} = \sqrt{\frac{4R}{r} + 1}\,.$$

Equality again holds for the equilateral triangle.

References:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

[2] D.S. Mitrinović, J.E. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; JUN-HUA HUANG, student, The 4th Middle School of Nanxian, Hunan, China; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; BOB PRIELIPP, University of Wisconsin–Oshkosh; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposers. The left hand inequality was proved by MURRAY S. KLAMKIN, University of Alberta.

The proposers also ask whether

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \ge 3\left(\frac{R}{2r}\right)^{5/12}; \tag{3}$$

it turned out that almost the identical inequality, with the exponent 5/12 *lowered to* 1/3, *was later sent in, with solution, by Huang! Does someone have a comment on* (3)*?*

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1700. [1991: 302] *Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Suppose a and b are two elements of a group satisfying $ba = ab^2$, $b \neq 1$ and $a^{31} = 1$. Determine the order of b.

Solution by Gene Arnold, Ferris State University, Big Rapids, Michigan.

The order of b is $2^{31} - 1$.

First, it will be shown by induction that

$$b^{2^n} = (a^{-1})^n b a^n, \quad n = 1, 2, 3, \dots . \tag{1}$$

For $n = 1$, (1) is $b^2 = a^{-1}ba$ and follows from $ba = ab^2$. Suppose (1) is true for $n - 1$; then

$$b^{2^n} = b^{2^{n-1}} b^{2^{n-1}} = (a^{-1})^{n-1} b a^{n-1} (a^{-1})^{n-1} b a^{n-1} = (a^{-1})^{n-1} b^2 a^{n-1} = (a^{-1})^n b a^n.$$

Thus (1) holds for all $n \geq 1$.

Using $n = 31$ in (1) yields $b^{2^{31}} = b$ so that $b^{2^{31}-1} = 1$. Since $2^{31} - 1$ is a Mersenne prime, $b \neq 1$ and the order of b must divide $2^{31} - 1$, it follows that b has order $2^{31} - 1$.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; F.J. FLANIGAN, San Jose State University, San Jose, California; RICHARD I. HESS, Rancho Palos Verdes, California; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; CHARLES H. JEPSEN, Grinnell College, Grinnell, Iowa; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta; M. PARMENTER, Memorial University of Newfoundland, St. John's; DAVID G. POOLE, Trent University, Peterborough, Ontario; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; and the proposers.

Bang and Bellot point out the similar problem 4.4.4, *page* 147 *of Loren C. Larson's* Problem Solving Through Problems, *Springer–Verlag,* 1983. *Heuver refers to p.* 68 *of I. Grossman and W. Magnus,* Groups and their Graphs, *M.A.A.,* 1964. *The proposers were motivated by another similar problem in* Mathematical Mayhem, *namely* C23 *on p.* 25 *of Volume* 3 *No.* 4 *(solution on p.* 32 *of Volume* 4 *No.* 1*).*

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1701*. [1992: 12] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

If ABC is a triangle, prove or disprove that

$$R \geq 4 \max \left\{ \frac{h_a \cos A}{1 + 8\cos^2 A}, \frac{h_b \cos B}{1 + 8\cos^2 B}, \frac{h_c \cos C}{1 + 8\cos^2 C} \right\},$$

where h_a, h_b, h_c are the altitudes of the triangle and R is its circumradius.

Solution by Marcin E. Kuczma, Warszawa, Poland.

True. The asserted inequality

$$R \geq \frac{4h_a \cos A}{1 + 8\cos^2 A} \tag{1}$$

will be derived from the following, slightly more general (and more natural, perhaps):

$$R \geq \frac{h_a}{1 + \cos A}\ . \tag{2}$$

To prove (2), note

$$\begin{aligned}\frac{h_a}{R} &= \frac{2\ \text{area}(\Delta ABC)}{Ra} = \frac{4R^2 \sin A \sin B \sin C}{R \cdot 2R \sin A}\\ &= 2\sin B \sin C = \cos(B-C) - \cos(B+C)\\ &\leq 1 - \cos(B+C) = 1 + \cos A.\end{aligned}$$

And since (for $\cos A > 0$)

$$\frac{1 + 8\cos^2 A}{4\cos A} = \frac{1}{2}\left(2\cos A + \frac{1}{2\cos A}\right) + \cos A \geq 1 + \cos A,$$

we have

$$\frac{h_a}{1 + \cos A} \geq \frac{4h_a \cos A}{1 + 8\cos^2 A}$$

(for $\cos A < 0$, this is obvious). Thus (1) follows from (2).

Also solved by ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Most solvers mentioned that equality holds if and only if the triangle is equilateral, which is also the case for Kuczma's stronger inequality.

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1702. [1992: 12] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABDE$ and $BCFG$ are squares described externally upon the sides of an acute triangle ABC with $\overline{AB} < \overline{BC}$. Let M and N be the midpoints of BC and AC, respectively, and let S be the intersection of BN and GM. Suppose that M, C, S, N are concyclic. Prove that $\overline{MD} = \overline{MG}$.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

We denote $\angle BMG = \angle CMS = \psi$, where

$$\sin\psi = 2/\sqrt{5}, \quad \cos\psi = 1/\sqrt{5}. \qquad (1)$$

As $MCNS$ is concyclic,

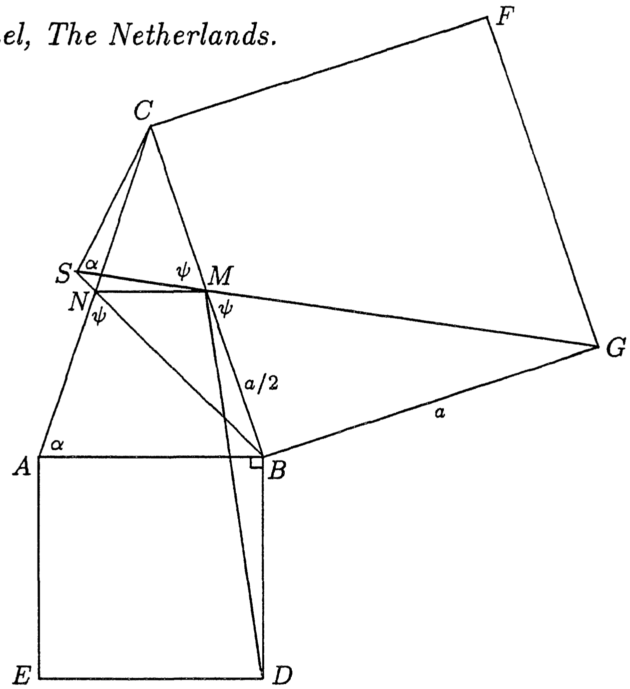
$$\angle ANB = \angle CNS = \angle CMS = \psi \qquad (2)$$

and

$$\angle CSM = \angle CNM = \angle CAB = \alpha. \qquad (3)$$

From (2) and (3), $\Delta BNA \sim \Delta CMS$, so

$$BA : NA = CS : MS,$$



i.e.

$$\frac{2c}{b} = \frac{\sin\psi}{\sin\angle MCS} = \frac{\sin\psi}{\sin(\alpha+\psi)} = \frac{\sin\psi}{\sin\alpha\cos\psi + \cos\alpha\sin\psi}, \qquad (4)$$

where a, b, c are the sides of ΔABC as usual. It follows from (1) and (4) that

$$b = c\sin\alpha + 2c\cos\alpha,$$

so

$$b^2 = bc\sin\alpha + 2bc\cos\alpha = ac\sin\beta + b^2 + c^2 - a^2,$$

or

$$a^2 = ac\sin\beta + c^2. \qquad (5)$$

By the cosine law in ΔDBM,

$$(MD)^2 = \frac{a^2}{4} + c^2 - ac\cos\left(\frac{\pi}{2}+\beta\right) = \frac{a^2}{4} + c^2 + ac\sin\beta. \qquad (6)$$

By (5) and (6),

$$(MD)^2 = \frac{a^2}{4} + a^2 = \frac{5}{4}a^2 = (MG)^2,$$

so $MD = MG$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ILIYA BLUSKOV, Technical University, Gabrovo, Bulgaria; JORDI DOU, Barcelona, Spain; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia; MARCIN E. KUCZMA, Warszawa, Poland; KEE-WAI LAU, Hong Kong; and the proposer.

Jurić and Kuczma point out that one must assume $S \neq N$ (equivalently, MG and AB are not parallel, i.e. $\tan\beta \neq 2$*), else the conclusion of the problem may not hold.*

Dou and Kuczma observe that the conditions $AB < BC$ and ΔABC acute are not necessary. The above proof goes through with no (or almost no) change.

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1703. [1992: 12] *Proposed by Murray S. Klamkin, University of Alberta.*

Determine the maximum and minimum values of

$$x^2 + y^2 + z^2 + \lambda xyz,$$

where $x + y + z = 1$, $x, y, z \geq 0$, and λ is a given constant.

Solution by Marcin E. Kuczma, Warszawa, Poland.

Denote $x^2 + y^2 + z^2 + \lambda xyz$ by $F_\lambda(x,y,z)$. We claim:

$$\min_\Delta F_\lambda = \begin{cases} (\lambda+9)/27 & \text{if } \lambda \leq 9/2, \\ 1/2 & \text{if } \lambda \geq 9/2, \end{cases} \qquad \max_\Delta F_\lambda = \begin{cases} 1 & \text{if } \lambda \leq 18, \\ (\lambda+9)/27 & \text{if } \lambda \geq 18, \end{cases} \tag{1}$$

where $\Delta = \{(x,y,z) :\ x,y,z \geq 0,\ \ x+y+z = 1\}$. (The three numbers: $(\lambda+9)/27$, $1/2$, 1, are the values of F_λ at $(1/3,1/3,1/3)$, $(1/2,1/2,0)$, $(1,0,0)$, respectively.)

The claim can be reduced to the following two inequalities:

$$x^2 + y^2 + z^2 + 18xyz \leq 1 \tag{2}$$

and

$$2x^2 + 2y^2 + 2z^2 + 9xyz \geq 1 \tag{3}$$

for $(x,y,z) \in \Delta$. Indeed: inequality (2) means that $F_{18}(x,y,z) \leq 1$, and hence $F_\lambda(x,y,z) \leq 1$ for any $\lambda \leq 18$ because $F_\lambda(x,y,z)$ is increasing in λ; inequality (3) means that $F_{9/2}(x,y,z) \geq 1/2$, and hence $F_\lambda(x,y,z) \geq 1/2$ for any $\lambda \geq 9/2$, for the same reason. This settles "a half" of each statement in (1). For the "other halves", what remains to show is that

$$F_\lambda(x,y,z) - \frac{\lambda+9}{27} = x^2 + y^2 + z^2 + \Big(xyz - \frac{1}{27}\Big)\lambda - \frac{1}{3} \ \begin{cases} \geq 0 & \text{if } \lambda \leq 9/2, \\ \leq 0 & \text{if } \lambda \geq 18. \end{cases} \tag{4}$$

This expression is *decreasing* in λ (as $xyz \leq ((x+y+z)/3)^3 = 1/27$), and therefore it would be enough to prove the upper line of (4) for $\lambda = 9/2$ and the lower line of (4) for $\lambda = 18$. Well; and these are exactly the inequalities (3) and (2), again.

The proof of (2) is almost automatic:

$$\begin{aligned} 1 - x^2 - y^2 - z^2 &= (x+y+z)^2 - (x^2+y^2+z^2) = 6 \cdot \frac{yz + zx + xy}{3} \\ &\geq 6((yz)(zx)(xy))^{1/3} = 6(xyz)(xyz)^{-1/3} \\ &\geq 6(xyz)(1/27)^{-1/3} = 18xyz. \end{aligned}$$

As for (3): we may assume $x \le y \le z$, so that $x \le 1/3$; hence, in particular, $9x - 4 < 0$. And since $yz \le (y+z)^2/4$, we get

$$\begin{aligned}2x^2 + 2y^2 + 2z^2 + 9xyz - 1 &= 2x^2 + 2(y+z)^2 + (9x-4)yz - 1\\ &\ge 2x^2 + 2(y+z)^2 + (9x-4)((y+z)^2/4) - 1\\ &= 2x^2 + 2(1-x)^2 + (9x-4)((1-x)^2/4) - 1\\ &= (x/4)(1-3x)^2 \ge 0.\end{aligned}$$

So, (2) and (3) are settled and the claim (1) is proved.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, The Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One incorrect solution was received.

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1704. [1992: 13] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Two chords of a circle (neither a diameter) intersect at right angles inside the circle, forming four regions. A circle is inscribed in each region. The radii of the four circles are r, s, t, u in cyclic order. Show that

$$(r - s + t - u)\left(\frac{1}{r} - \frac{1}{s} + \frac{1}{t} - \frac{1}{u}\right) = \frac{(rt - su)^2}{rstu}\,.$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.

Taking a coordinate system with origin at the point of intersection of the two perpendicular chords (which are the axes), the centers of the four circles inscribed in the four regions, of radii r, s, t, u, are respectively

$$O_1(r,r), \quad O_2(-s,s), \quad O_3(-t,-t), \quad O_4(u,-u).$$

Then we have

$$(\overline{O_1O_3})^2 = 2(r+t)^2, \quad (\overline{O_2O_4})^2 = 2(s+u)^2.$$

The lengths of the external common tangents to two of the four circles are the following (notation self-evident):

$$t_{12} = r + s, \quad t_{13}^2 = 2(r+t)^2 - (r-t)^2 = (r+t)^2 + 4rt,$$

$$t_{14} = r + u, \quad t_{24}^2 = (s+u)^2 + 4su, \quad t_{23} = s + t, \quad t_{34} = t + u.$$

The theorem of Casey (see [1], p. 120–121 or [2], p. 121–127) says that if the four circles O_i are tangent to the large circle, then

$$t_{12}t_{34} + t_{14}t_{23} = t_{13}t_{24}.$$

This is (in our case)

$$(r+s)(t+u)+(r+u)(s+t)=\sqrt{(r+t)^2+4rt}\cdot\sqrt{(s+u)^2+4su}$$

or, squaring,

$$[(r+t)^2+4rt][(s+u)^2+4su]-[(r+s)(t+u)+(r+u)(s+t)]^2=0. \tag{1}$$

On the other hand, the proposed equality can be written

$$(r-s+t-u)(-rst+rsu-rtu+stu)-(rt-su)^2=0. \tag{2}$$

The left hand of (1) is equal to 4 times the left hand of (2) (with the help of DERIVE, for instance, this is a matter of seconds). So, equations (1) and (2) are equivalent, and we are done.

References:

[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems (Sangaku)*, Charles Babbage Research Centre, Winnipeg, 1989.

[2] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer.

Janous points out the related problem Crux 1627 [1992: 95], *and the proposer mentions another occurrence of this circle configuration in problem* 1.5.11 *of the Fukagawa–Pedoe book listed above.*

Is there an easy (non-computer) demonstration that (1) *and* (2) *are equivalent?*

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1705. [1992: 13] *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*

Let $n \geq 2$ and $b_0 \in [2, 2n-1]$ be integers, and consider the recurrence

$$b_{i+1}=\begin{cases} 2b_i-1 & \text{if } b_i \leq n,\\ 2b_i-2n & \text{if } b_i > n.\end{cases}$$

Let $p = p(b_0, n)$ be the smallest positive integer such that $b_p = b_0$.

(a) Find $p(2, 2^k)$ and $p(2, 2^k + 1)$ for all $k \geq 1$.

(b) Prove that $p(b_0, n) \mid p(2, n)$.

Solution by Andy Liu, University of Alberta.

Let $m = n - 1$ and $a_i = b_i - 1$. Then $1 \leq a_0 \leq 2m$ and

$$a_{i+1}=\begin{cases} 2a_i & \text{if } a_i \leq m,\\ 2a_i-(2m+1) & \text{if } a_i > m.\end{cases}$$

In other words, we are simply doubling and reducing modulo $2m+1$, if necessary, so that $1 \le a_i \le 2m$ for all i. Now $p = p(a_0, m)$ is the length of the cycle to which a_0 belongs. For instance, if $m = 7$ the cycles are

$$(1,2,4,8), \quad (3,6,12,9), \quad (5,10), \quad \text{and} \quad (7,14,13,11).$$

Thus $p(1,7) = 4$.

(a) Replacing n by $m+1$ and b_i by $a_i + 1$ as above, the problem becomes to find $p(1, 2^k - 1)$ and $p(1, 2^k)$. We claim that

$$p(1, 2^k - 1) = k + 1, \quad p(1, 2^k) = 2(k+1).$$

Clearly $2^{k+1} \equiv 1 \bmod [2(2^k - 1) + 1]$ and $2^t \not\equiv 1 \bmod (2^{k+1} - 1)$ for $1 \le t \le k$. This justifies the first claim. Since

$$2^{2(k+1)} - 1 = (2^{k+1} + 1)(2^{k+1} - 1) \equiv 0 \bmod (2 \cdot 2^k + 1),$$

$2^{2(k+1)} \equiv 1 \bmod (2^{k+1} + 1)$, and hence $p(1, 2^k) \mid 2(k+1)$. Since none of the numbers 2^t, $1 \le t \le k+1$, can be congruent to 1 modulo $2^{k+1} + 1$, we must have $p(1, 2^k) = 2(k+1)$.

(b) It is true that $p(a_0, m) \mid p(1, m)$ for all a_0. Suppose $p(1, m) = t$; then $2^t \equiv 1 \bmod (2m+1)$, so that $2^t a_0 \equiv a_0 \bmod (2m+1)$. It follows that $p(a_0, m) \mid t$.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; NEVEN JURIĆ, Zagreb, Croatia; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. Part (a) *only solved by CHARLES ASHBACHER, Cedar Rapids, Iowa; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

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1706. [1992: 13] *Proposed by Jordi Dou, Barcelona, Spain.*

Given four lines in general position in a plane, and a point P in the plane, a pair of lines through P will usually cut off a segment from each of the four given lines. Construct such a pair of lines so that the midpoints of the four segments are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.

We denote the given lines by a, b, c, d. Let Γ be the parabola which touches the given lines. If P is exterior to Γ we draw the two tangents ℓ_1, ℓ_2 from P to Γ; then ℓ_1, ℓ_2 are the desired lines.

Proof. Let A_i, B_i, C_i and D_i be the points of intersection of a, b, c and d with ℓ_i $(i = 1, 2)$. Because Γ is a parabola that touches a, b, c, d, we have

$$A_1B_1 : B_1C_1 : C_1D_1 = A_2B_2 : B_2C_2 : C_2D_2.$$

Hence, the midpoints of A_1A_2, B_1B_2, C_1C_2, D_1D_2 are collinear (see, for example, H.S.M. Coxeter, *Introduction to Geometry*, exercise 13.6.2, page 216).

Remark. When they exist, the lines ℓ_i can be constructed. Let W be the *Wallace point* of a, b, c, d (the common point of the circumscribed circles of the four triangles formed by the four given lines, as in Theorem 196 in Roger A. Johnson, *Modern Geometry*, p. 139). Then the feet of the perpendiculars from W lie on a line g. If the circle with diameter PW intersects g in two points T_1 and T_2, then the lines called for in the problem are $\ell_i = PT_i$. (Compare the comment following *Crux* 1597 [1992: 26–27].)

Editorial Comment by Chris Fisher. The midpoint line also passes through the midpoints of the segments from P to the points T_1, T_2 where ℓ_1 and ℓ_2 are tangent to the parabola. It follows that the solution lines ℓ_i exist only if P is exterior to the parabola. When there exists a line of midpoints one can define the points T_i on ℓ_i for which the midpoint line bisects the segments PT_i. Let Γ be the parabola tangent to ℓ_i at T_i $(i = 1, 2)$. Seimiya's argument implies that a line is tangent to Γ if and only if the segment cut from it by ℓ_1 and ℓ_2 is bisected by the midpoint line.

Also solved by the proposer.

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YEAR-END WRAPUP

Another year has come and gone, and so the editor again takes the opportunity to pass on a few comments received from readers on some past *Crux* problems, as well as to extend some thank-you's.

1511 [1991: 92]. Murray Klamkin points out a typo in the displayed equation at the end of the editor's comments on [1991: 93]: the π should be x.

1627 [1992: 95]. A late solution was sent in by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain, who also mention that a special case of this problem, namely when one of the two chords is a diameter, was a problem in *Mat. v Škole* in 1974, and appears as item 12, page 440 of Mitrinović, Pečarić, Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.

1669 [1992: 215]. Richard Guy, University of Calgary, observes that the definition of "algebraic integer" given in solution II should have read "a root of a **monic** polynomial with integer coefficients".

1677 [1992: 223]. Richard Guy also points out that the published solutions to this problem both need some further justification. The main result missing from both solutions is a proof that the series indeed converges. This can be done by forming a new series, which will be purely alternating, by adding together each consecutive pair of positive terms of the original series: thus

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots$$

becomes

$$\frac{3}{2} - \frac{2}{3} + \frac{9}{20} - \frac{2}{6} + \frac{15}{56} - \frac{2}{9} + \cdots.$$

This new series then converges by the alternating series test. Moreover, the partial sums of the new series form a subsequence of the partial sums of the original series, and the missing partial sums of the original series are "between" the partial sums of the new series. Therefore the original series will converge to the same limit as the new series.

Now in Solution I, $P(1) = \lim_{x\to 1} P(x)$ follows by Abel's Theorem (e.g., page 160 of Rudin's *Principles of Mathematical Analysis*). And in Solution II we are justified in considering only a subsequence of the partial sums.

Late solutions were received from Hayo Ahlburg, Benidorm, Spain (**1600**); John Oman and Bob Prielipp, University of Wisconsin–Oshkosh (**1627**); and Kenneth M. Wilke, Topeka, Kansas (**1665**).

Many thanks to the following people for their assistance to the editor and other members of the Editorial Board during 1992, in giving advice regarding problems, articles, and solutions: *ED BARBEAU, LEN BOS, ANDREW BREMNER, CHARLES EDMUNDS, DOUG FARENICK, BRUCE GILLIGAN, HARAGAURI GUPTA, WALTER JANOUS, JAMES P. JONES, CLARK KIMBERLING, MURRAY KLAMKIN, JOANNE MCDONALD, RICHARD MCINTOSH, STANLEY RABINOWITZ, JONATHAN SCHAER, JIM TOMKINS, EDWARD T.H. WANG*, and *HARLEY WESTON.*

Not listed are the Editorial Board members themselves, who have (to this editor's gratitude) assumed some of the duties in creating *Crux*, while also contributing their expertise generally. In particular, Andy Liu continues to handle the Book Reviews section and Denis Hanson has taken charge of the Articles section; readers can tell that both sections are healthier for the change! Richard Guy was once again the editor's chief source of information on number theory and trigonometry. Meanwhile Chris Fisher as always is the resident *Crux* expert on classical geometry, and readers would have noticed his apt commentary on several solutions in the past year. Chris, Denis Hanson, and Richard Nowakowski all did write-ups for some problems this year.

Special thanks to *JOANNE LONGWORTH*, whose knowledge of LaTeX, and mathematics, is a priceless asset to the editor. Nearly all of what you read in *Crux* has been keyboarded in by Joanne's magic fingers!

And so it only remains for the editor to wish all *Crux* readers peace, prosperity and good ideas in 1993.

send in problems send in problems send in problems send in problems send in problems

HAPPY NEW YEAR!

send in problems send in problems send in problems send in problems send in problems

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