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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practice or teach mathematics. Its purpose is primarily educational but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the appropriate member of the Editorial Board as detailed on the inside back cover.

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RENSEIGNEMENTS GÉNÉRAUX

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire. Bien que principalement de nature éducative, elle sert aussi à ceux qui la lisent pour des raisons professionnelles, culturelles ou récréative.

Les propositions de problèmes, solutions et courts articles à publier doivent être envoyés au membre approprié du conseil de rédaction tel qu'indiqué sur la couverture arrière.

RENSEIGNEMENTS SUR L'ABONNEMENT

Crux Mathematicorum est publié mensuellement (sauf juillet et août). Les tarifs d'abonnement pour dix numéros figurent sur la couverture arrière. On peut également y retrouver de plus amples renseignements sur les volumes antérieurs de Crux Mathematicorum.

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THE OLYMPIAD CORNER

No. 156

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

This month we begin with the 1994 Canadian Mathematical Olympiad which we reproduce with the permission of the Canadian Mathematical Olympiad Committee of the Canadian Mathematical Society. My thanks to Edward Wang, its chairperson, for sending me the contest, and for agreeing to supply the "official" solutions which we will give in the September number of the Corner.

1994 CANADIAN MATHEMATICAL OLYMPIAD

1. Evaluate the sum $\sum_{n=1}^{1994}(-1)^n\frac{n^2+n+1}{n!}$.

2. Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$ for some positive integer m.

$$\text{(i.e. } (\sqrt{2}-1)^2 = 3-2\sqrt{2} = \sqrt{9}-\sqrt{8}.$$

3. Twenty-five men sit around a circular table. Every hour there is a vote, and each must respond *yes* or *no*. Each man behaves as follows: on the nth vote, if his response is the same as the response of at least one of the two people he sits between, then he will respond the same way on the $(n+1)$th vote as on the nth vote; but if his response is different from that of both his neighbours on the nth vote, then his response on the $(n+1)$th vote will be different from his response on the nth vote. Prove that, however everybody responded on the first vote, there will be a time after which nobody's response will ever change.

4. Let AB be a diameter of a circle Ω and P be any point *not* on the line through A and B. Suppose the line through P and A cuts Ω again in U, and the line through P and B cuts Ω again in V. (Note that in case of tangency U may coincide with A or V may coincide with B. Also, if P is on Ω then $P=U=V$.) Suppose that $|PU| = s|PA|$ and $|PV| = t|PB|$ for some nonnegative real numbers s and t. Determine the cosine of the angle APB in terms of s and t.

5. Let ABC be an acute angled triangle. Let AD be the altitude on BC, and let H be any interior point on AD. Lines BH and CH, when extended, intersect AC and AB at E and R, respectively. Prove that $\angle EDH = \angle FDH$.

* * *

The next set of problems are from the twenty-third annual United States of America Mathematical Olympiad written April 28, 1994. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C., Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68588–0322. As always, we welcome your original "nice" solutions and generalizations. Thanks go to Cecil Rousseau, Memphis State University, for supplying a TEXfile of the contest.

23rd UNITED STATES OF AMERICA MATHEMATICAL OLYMPIAD

April 28, 1994

Time Limit: 3.5 hours

1. Let $k_1 < k_2 < k_3 < \cdots$ be positive integers, no two consecutive, and let $s_m = k_1 + k_2 + \cdots + k_m$ for $m = 1, 2, 3, \ldots$. Prove that, for each positive integer n, the interval $[s_n, s_{n+1})$ contains at least one perfect square.

2. The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, ..., red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, ..., red, yellow, blue?

3. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, and CF, are concurrent. Let P be the intersection of AD and CE. Prove that $CP/PE = (AC/CE)^2$.

4. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers satisfying $\sum_{j=1}^{n} a_j \geq \sqrt{n}$ for all $n \geq 1$. Prove that, for all $n \geq 1$,

$$\sum_{j=1}^{n} a_j^2 > \frac{1}{4}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

5. Let $|U|$, $\sigma(U)$, and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!(n-k)!}$. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.

* * *

As promised last issue where we gave the problems of the A.I.M.E. for 1994, we next give the numerical solutions. The problems and their official solutions are copyrighted by the Committee of the American Mathematics Competitions of the Mathematical Association of America, and may not be reproduced without permission. Detailed solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A. 68488–0322.

1. 063	**2.** 312	**3.** 561	**4.** 312	**5.** 103
6. 660	**7.** 072	**8.** 315	**9.** 394	**10.** 450
11. 465	**12.** 702	**13.** 850	**14.** 071	**15.** 597

* * *

We next turn to solutions from the readers to problems given in the April 1993 number of the Corner. First we give solutions to the four problems of the *XXV Soviet Mathematical Olympiad, 9th form* [1993: 100].

1. Find all integer solutions of the system

$$xz - 2yt = 3, \qquad xt + yz = 1.$$

(Ju. Nesterenko)

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; Tim Cross, Wolverley High School, Kidderminster, U.K.; by George Evagelopoulos, Athens, Greece; by Norvald Midttun, Royal Norwegian Naval Academy; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give the solution by Evagelopoulos.

From the given system by squaring and adding we obtain

$$(xz - 2yt)^2 + 2(xt + yz)^2 = 11$$

or

$$(x^2 + 2y^2)(z^2 + 2t^2) = 11.$$

Since x, y, z, t are integers we have either $x^2 + 2y^2 = 1$ (which means $x^2 = 1$, $y = 0$) or $z^2 + 2t^2 = 1$ (which means $z^2 = 1$, $t = 0$). Substitution into the original equations gives the following four solutions (x, y, z, t) of the system

$$(1,0,3,1),\ (-1,0,-3,-1),\ (3,1,1,0),\ \text{and}\ (-3,-1,-1,0).$$

2. On the blackboard are written n numbers. One may erase any pair of them, say a and b, and write down the number $(a + b)/4$ to replace them. After this procedure is repeated $n - 1$ times, only one number remains on the blackboard. Prove that given that the n numbers at the beginning are all 1, the last number will not be less than $1/n$. (B. Berlov)

Solutions by Christopher J. Bradley, Clifton College, Bristol, U.K.; and by George Evagelopoulos, Athens, Greece.

It is obvious that $1/a + 1/b \geq 4/(a+b)$ (for $a > 0$, $b > 0$). Therefore, the sum S of the reciprocals of the numbers that are written on the blackboard is not increased.

In the beginning we had $S = n$, and as a result, at the end of the procedure, we are going to have $S \leq n$, from which we conclude that the last number is $1/S$ and $1/S \geq 1/n$, which is the solution.

3. There are four straight lines, each two of them intersecting, and no three of them having a common point. Each line is divided into four pieces with two finite length segments among them. The total number of segments is eight. Is it possible that the lengths of these segments are equal to:

(a) $1, 2, 3, 4, 5, 6, 7, 8$?

(b) pairwise different natural numbers? (A. Berzinsh)

Solution by George Evagelopoulos, Athens, Greece.

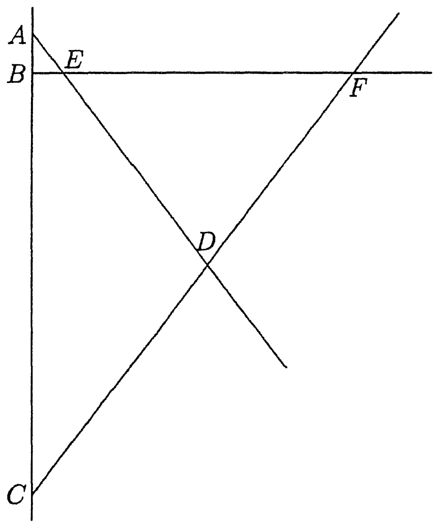


Figure 1

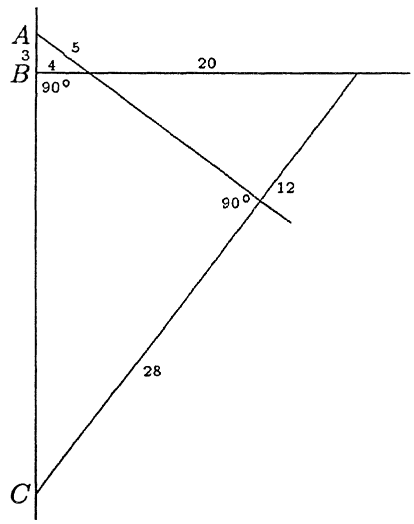


Figure 2

(a) It is not possible, because the segment which has length 1 can be neither side of the triangle ABE, nor of the triangle EFD (see Figure 1). So either $BC = 1$ or $CD = 1$. We may suppose that $BC = 1$. But then $BF = CF$ since the sides of ΔBCF are of integral length. Thus $\cos F = 1 - (1/(2(BF)^2))$.

Applying the law of cosines to the triangle EFD we get $ED^2 = EF^2 + FD^2 - 2EF \cdot ED + (EF \cdot ED)/BF^2$. In the above mentioned sum only the last term is a fraction, which is not possible.

(b) It is possible. See Figure 2.

4. A ticket for a lottery is a card which has 50 empty cells in a line. Each participant writes down in the cells the numbers $1, 2, \ldots, 50$ without repetitions. The organizer of the lottery has his own card with the numbers written on it according to the rule. A ticket wins if at least one number in it coincides with the number in the corresponding cell of the organizer's card. What is the least possible number of cards which the participant must fill to guarantee a win? (A. Berzinsh)

Solutions by George Evagelopoulos, Athens, Greece; and by John Morvay, Springfield, Missouri. We give the solution of Evagelopoulos.

Answer: 26 cards!

To guarantee a win, we can write the following numbers on the 26 cards:

$$\begin{array}{rrrcrrrcr}
1, & 2, & 3, & \ldots, & 25, & 26, & 27, & \ldots, & 50\\
2, & 3, & 4, & \ldots, & 26, & 1, & 27, & \ldots, & 50\\
 & & & & \ldots & & & & \\
25, & 26, & 1, & \ldots, & 23, & 24, & 27, & \ldots, & 50\\
26, & 1, & 2, & \ldots, & 24, & 25, & 27, & \ldots, & 50
\end{array}$$

On the organizer's card at least one of the numbers $1, 2, \ldots, 26$ must lie in one of the first 26 cells, since there are only 24 remaining cells. This number guarantees a win on one of the 26 cards (on which $1, 2, 3, \ldots, 25, 26$ have been permuted cyclically).

Now we are going to prove that no choice of 25 cards guarantees a win. For this reason, we place the 25 cards and the organizer's card as shown in the figure.

a_1	a_2	a_3	$\ldots$	a_{50}
b_1	b_2	b_3	$\ldots$	b_{50}
		$\ldots$		
c_1	c_2	c_3	$\ldots$	c_{50}

organizer's card

We will complete the organizer's card in such a way that not even one number of those that are written on it coincides with the numbers that are written above it. It is obvious that the number 1 can be written on the organizer's card according to this rule.

Let numbers $1, 2, 3, \ldots, a-1$ be already written. We next examine the number a. If it can be written in one of the remaining cells of the organizer's card, this will be done. If this is not possible, namely in the cells that are allowable for a (of which there are at least 25), the numbers $x_1, x_2, \ldots, x_{25}, \ldots, x_k$ have already been written we choose any one of the free cells. In the column above this cell no more than 25 numbers (including a) appear. That means that there are at most 24 distinct numbers different from a appearing.

We choose x_i which does not belong to this column, write it in the free cell, and in the cell for x_i we write the number a. Continuing this procedure we complete the card so that no one of the 25 cards will win!

Comment by David Vaughan and George Evagelopoulos, Athens, Greece.

The number of cards which do not have any match with the organizer's card is D_{50} the derangement number of 50 objects. D_n is given by any one of the following three well-known formulae:

$$D_n = nD_{n-1} + (-1)^n, \quad n = 2, 3, 4, \ldots, D_1 = 0, \tag{1}$$

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right), \tag{2}$$

$$D_n = \|n!e^{-1}\|, \tag{3}$$

the closest integer to $n!e^{-1}$. In a matter of seconds, using MAPLE we have $D_{50} =$ 11 188 719 610 782 480 504 630 258 070 757 734 324 011 354 208 865 721 592 720 336 801, a 65-digit "monster."

* * *

Next I want to mention the efforts of students in Shawn Godin's Grade 9 Mathematics class at St. Joseph Scollard Hall S.S. in North Bay, Ontario. The students "took" the 2nd U.K. Schools Mathematical Challenge which we gave in the February number. Nine of them wrote out explanations for their answers and sent them in to me. Well done: Jodi Badowski, Sara DiSalle, Phillip Gamchine, Lisa Lamothe, Dave Mathias, Susan Miller, Natasha Palumbo, and two anonymous students whose papers were unsigned.

* * *

Now we turn to solutions to problems of the *22nd Austrian Mathematical Olympiad 2nd Round* [1993: 101]. (Next month we will give the solutions received for the final round.)

1. Let a, b be rational numbers such that $\sqrt[3]{a} + \sqrt[3]{b}$ is a rational number $c \neq 0$. Show that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ themselves are rational numbers.

Solutions by Seung-Jin Bang, Albany, California; by Joel Brenner, Palo Alto, California; by Geoffrey A. Kandall, Hamden, Connecticut; by Waldemar Pompe, student, University of Warsaw, Poland; by Bob Prielipp, University of Wisconsin–Oshkosh; and by Chris Wildhagen, Rotterdam, The Netherlands. We give Kandall's solution which was similar to several others.

We have $c^3 = a+b+3\sqrt[3]{a}\,\sqrt[3]{b}(\sqrt[3]{a}+\sqrt[3]{b})$, hence we have $\sqrt[3]{a}\,\sqrt[3]{b} = (c^3-a-b)/3c \in \mathbb{Q}$. Let $k = (\sqrt[3]{a})^2 + \sqrt[3]{a}\,\sqrt[3]{b} + (\sqrt[3]{b})^2$. Note that $k = c^2 - \sqrt[3]{a}\,\sqrt[3]{b} \in \mathbb{Q}$. We have $a - b = (\sqrt[3]{a} - \sqrt[3]{b})k$. If $k \neq 0$ then $\sqrt[3]{a} - \sqrt[3]{b} = (a-b)/k \in \mathbb{Q}$. If $k = 0$ then $a = b$. So, in either case $\sqrt[3]{a} - \sqrt[3]{b} \in \mathbb{Q}$.

Finally $\sqrt[3]{a} = \frac{1}{2}((\sqrt[3]{a} + \sqrt[3]{b}) + (\sqrt[3]{a} - \sqrt[3]{b})) \in \mathbb{Q}$, and $\sqrt[3]{b} = c - \sqrt[3]{a} \in \mathbb{Q}$.

2. Determine all real solutions of the equation

$$\frac{1}{x} + \frac{1}{x+2} - \frac{1}{x+4} - \frac{1}{x+6} - \frac{1}{x+8} - \frac{1}{x+10} + \frac{1}{x+12} + \frac{1}{x+14} = 0.$$

Solutions by Seung-Jin Bang, Albany, California; by Joel Brenner, Palo Alto, California; by Tim Cross, Wolverley High School, Kidderminster, U.K.; by Geoffrey Kandall, Hamden, Connecticut; by Norvald Midttun, Royal Norwegian Naval College; by Waldemar Pompe, student, University of Warsaw, Poland; by Panos E. Tsaoussoglou, Athens, Greece; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We give the solution of Midttun, which was very similar to several of the others.

Putting $x = y - 7$, we get the equation

$$\frac{1}{y-7}+\frac{1}{y-5}-\frac{1}{y-3}-\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{y+3}+\frac{1}{y+5}+\frac{1}{y+7}=0.$$

Combining terms in pairs we get

$$\frac{2y}{y^2-49}+\frac{2y}{y^2-25}=\frac{2y}{y^2-9}+\frac{2y}{y^2-1}\,.$$

This gives a solution of $y = 0$ and $x = -7$, otherwise we obtain

$$\frac{1}{y^2-49}+\frac{1}{y^2-25}=\frac{1}{y^2-9}+\frac{1}{y^2-1}$$

or

$$\frac{48}{(y^2-49)(y^2-25)}=-\frac{16}{(y^2-9)(y^2-1)}\,.$$

Writing $y^2 = z$, we get

$$3(z-9)(z-1)=-(z-49)(z-25)$$

or $z^2 - 38z + 181 = 0$. This yields $z = 19 \pm 6\sqrt{5}$. This finally gives us

$$x_1=-7,\quad x_{2,3,4,5}=\pm\sqrt{19\pm 6\sqrt{5}}-7.$$

3. Determine the number of all square numbers contained in the sequence $\{a_0, a_1, a_2, \ldots\}$ where $a_0 = 91$ and $a_{n+1} = 10a_n + (-1)^n$, $n \geq 0$.

Solutions by Seung-Jin Bang, Albany, California; by Waldemar Pompe, student, University of Warsaw, Poland; by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario; and by Chris Wildhagen, Rotterdam, The Netherlands. We use Wildhagen's solution.

Clearly a_0 and a_1 are not (perfect) squares. It is a straightforward induction to show that for all $n \geq 1$, $a_{2n} \equiv 5 \bmod 8$ and $a_{2n+1} \equiv 3 \bmod 8$. Since any odd square is congruent to 1 mod 8, the sequence contains no squares.

[*Editor's note.* Pompe points out that there is a slight difference betwen the version of this problem given in *Crux* and the one from a German copy of the Austrian Olympiad he received from a friend. The original version gave $a_3 = 91$ instead of $a_0 = 91$. The answer is similar, but a_0, a_1 and a_2 turn out to be square numbers.

4. Let A, B be two points on a circle k of radius r such that $\overline{AB} = c$.

(i) Give a construction of all triangles ABC having k as their circumcircle and such that one of the medians s_a or s_b (through vertices A or B, resp.) is of given length d.

(ii) How have r, c and d to be chosen such that ΔABC is uniquely determined?

Solutions by Waldemar Pompe, student, University of Warsaw, Poland; and by D.J. Smeenk, Zaltbommel, The Netherlands. We use Pompe's solution, which is similar to Smeenk's.

(i) We will show how to construct such a triangle with median $s_a = d$. First draw a circle ℓ of radius $(1/2)r$ internally tangent to circle k at B. Next draw a circle m with centre A and radius d. Let P and Q be the points of intersection of m with ℓ (see Figure 1). Lines BP and BQ intersect the circle k again at C_1 and C_2. Triangles ABC_1 and ABC_2 each satisfy the given conditions. To see this it is enough to prove that AP is a median of ABC_1, i.e. $BP = PC_1$. But this is obvious: the circle ℓ is homothetic to k with scale $1/2$ and centre B. The same construction works for the median s_b.

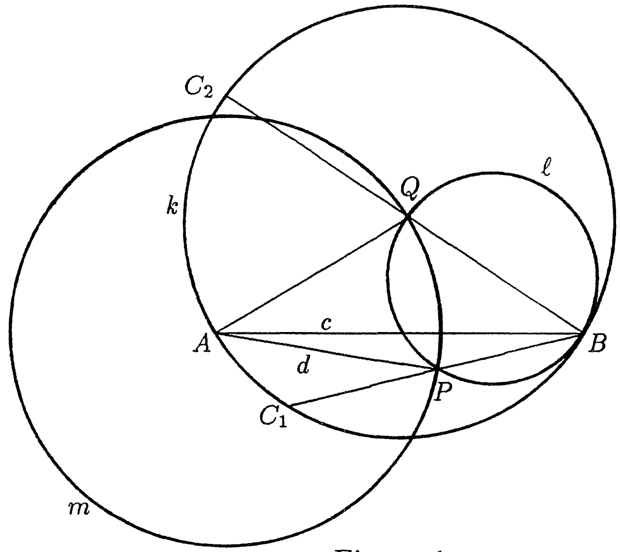


Figure 1

(ii) The triangle ABC is uniquely determined if the circles ℓ and m are tangent to each other (in this case we assume that if we constructed two congruent triangles, we have these triangles uniquely determined; if we assumed that the triangle ABC was uniquely determined iff C is uniquely determined, we would get that there were no such d, c, r!).

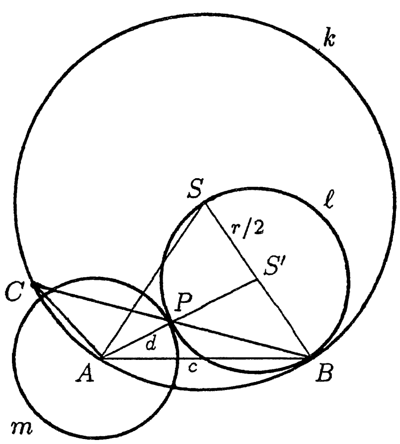


Figure 2

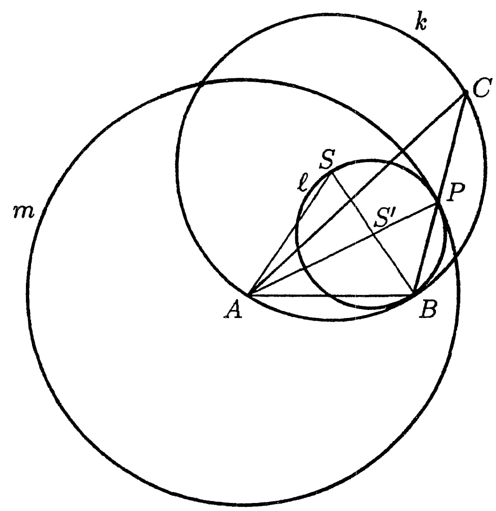


Figure 3

If m is tangent to ℓ externally then $AS' = d + \frac{1}{2}r$ (see Figure 2) is a median of ABS to BS (where S and S' are the centres of k and ℓ, respectively). Therefore $AS' = \frac{1}{2}\sqrt{2AS^2 + 2AB^2 - BS^2}$ (using a well-known formula for the length of the median of a triangle with given sides). This gives $d + \frac{1}{2}r = \frac{1}{2}\sqrt{2r^2 + 2c^2 - r^2}$ or after reduction

$$2d^2 + 2dr = c^2 \qquad \text{for } c < 2r. \tag{4}$$

If m is tangent to ℓ internally (see Figure 3), then similarly we get

$$2d^2 - 2dr = c^2 \qquad \text{for } c < 2r. \tag{5}$$

Therefore if either (1) or (2) holds, then the constructed triangle will be uniquely determined.

* * *

That completes this number. Send me your contests and nice solutions.

* * * * *

BOOK REVIEW

Edited by ANDY LIU, University of Alberta.

Proofs Without Words — Exercises in Visual Thinking, by Roger B. Nelson. Published by the Mathematical Association of America, Washington, D.C., 1993. vi+152 pages, paperbound, ISBN 0-88385-700-6. List US$27.50, MAA member $22.00. *Reviewed by Murray S. Klamkin, University of Alberta.*

"Proofs without words" (PWW's) have been appearing in the MAA journals *Mathematics Magazine* and *The College Mathematics Journal* since 1975. My original impression of a PWW was that it was (almost) a self contained proof given by a diagram. Rufus Isaacs, the first contributor of a PWW to *Mathematics Magazine*, stated in a Letter to the Editor in January 1976 that "All I intended was to stress the rare and secluded pleasure of grasping a mathematical truth from visual evidence alone." The editors in a note after this letter agreed with this, and remarked that they were looking for interesting visual material to illustrate the pages of the *Magazine* and to use as end-of-article fillers. The author in the preface notes that Martin Gardner, in his Mathematical Games column in the October 1973 *Scientific American*, discussed PWW's as "look-see diagrams" and pointed out that "in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance". Since many of the PWW's published subsequently did not really meet the above description, the author wisely notes in the preface that PWW's are not really proofs, but in general are pictures or diagrams that help an observer see why a particular statement may be true and also to see how one might begin to go about proving it is true. So the emphasis is on providing visual clues to the observer to stimulate thought.

There are 144 of these “proofs” arranged by topic into six chapters: Geometry and Algebra; Trigonometry, Calculus and Analytic Geometry; Inequalities; Integer Sums; Sequences and Series; and Miscellaneous. Most of these PWW's have appeared in the MAA journals. The remaining ones are modern renditions of PWW's from ancient China, classical Greece, and India of the twelfth century. At the end of the book are citations for the sources of the PWW's.

I find that some of these PWW's lead to easy proofs and others do not, but they do stimulate mathematical thought. I now give some examples of the PWW's, with some comments.

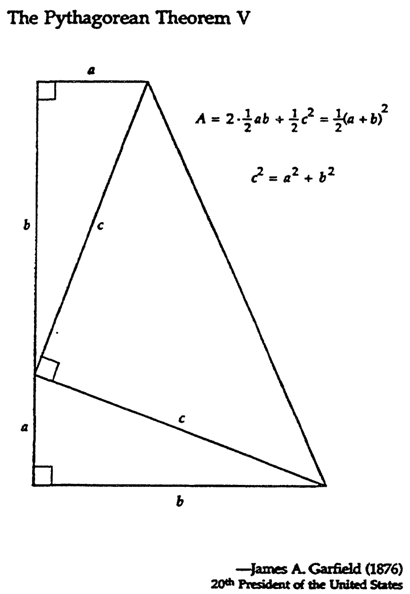


Figure 1

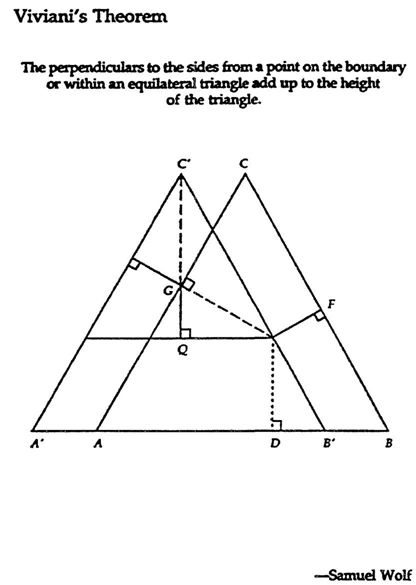
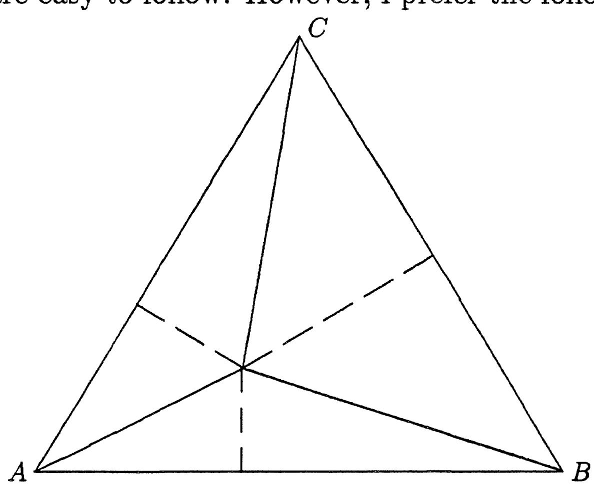


Figure 2

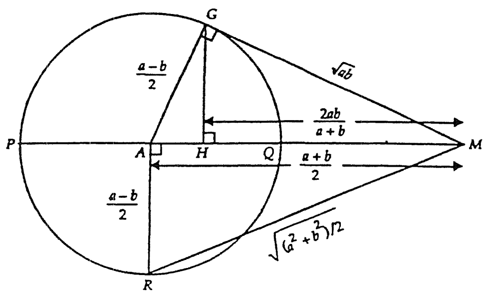
Both of these are easy to follow. However, I prefer the following figure for Viviani's Theorem:



$$\text{area}(\Delta ABC) = \tfrac{1}{2}(h_1a + h_2a + h_3a) = \tfrac{1}{2}ha.$$

This gives a somewhat simpler and more direct proof which extends to the corresponding theorem for regular simplexes.

The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality I



$PM = a,\ QM = b,\ a > b > 0$

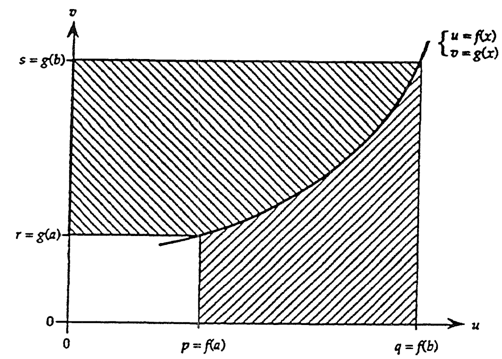
$HM < GM < AM < RM$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{(a^2+b^2)/2}$$

—RBN

Figure 3

Integration by Parts



Area [cross-hatched] + Area [hatched] $= qs - pr$

$$\int_r^s u\,dv + \int_p^q v\,du = uv\Big|_{(p,r)}^{(q,s)}$$

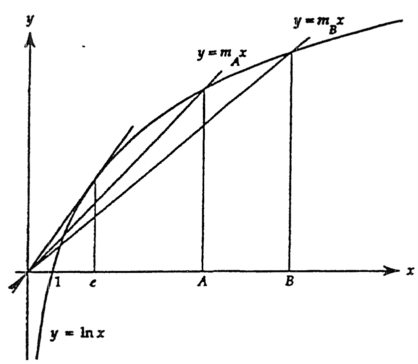
$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

—Richard Courant

Figure 4

These two are easy to follow due to the good figures, labelling, and the equations.

$A^B > B^A$ for $e \le A < B$



$$e \le A < B \Rightarrow m_A > m_B$$

$$\Rightarrow \frac{\ln A}{A} > \frac{\ln B}{B}$$

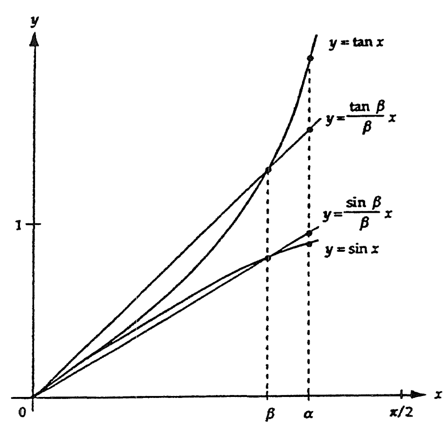
$$\Rightarrow A^B > B^A$$

—Charles D. Gallant

Figure 5

Aristarchus' Inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \Rightarrow \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$



$$\sin\alpha < \frac{\sin\beta}{\beta}\alpha;\quad \frac{\tan\beta}{\beta}\alpha < \tan\alpha$$

$$\therefore \frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}$$

—RBN

Figure 6

On letting $A = e$ and $B = \pi$ in Figure 5, we get $e^\pi > \pi^e$ which is given in the PWW on the adjoining page and which is obtained from the graph of $y = (\ln x)/x$.

More generally it follows from Figure 5 in the same way that if $y = F(x)$ is an increasing concave function of x and e_1 is the abscissa of the point of contact of a tangent line from $(0,0)$ to the curve, then $F(A)/A > F(B)/B$ for $e_1 \leq A < B$. Also, if we draw secants from the point $(h\ln(e/h), 0)$ for $h \geq e$ to the ln curve we would obtain $A^B/B^A > (B/A)^{h\ln(h/e)}$ for $h \leq A < B$.

What is being proved in Figure 6 is that $(\tan x)/x$ is an increasing function and $(\sin x)/x$ is a decreasing function in $(0, \pi/2)$, which incidentally follows easily via derivatives knowing that $\tan x \geq x \geq \sin x$. In the figure it is assumed that $\tan x$ is convex, that $\sin x$ is concave, that the line $y = (x\tan\beta)/\beta$ lies above $y = \tan x$, and that the line $y = (x\sin\beta)/\beta$ lies below $y = \sin x$, in an interval including the origin. The latter two conditions also require that $\tan x > x > \sin x$.

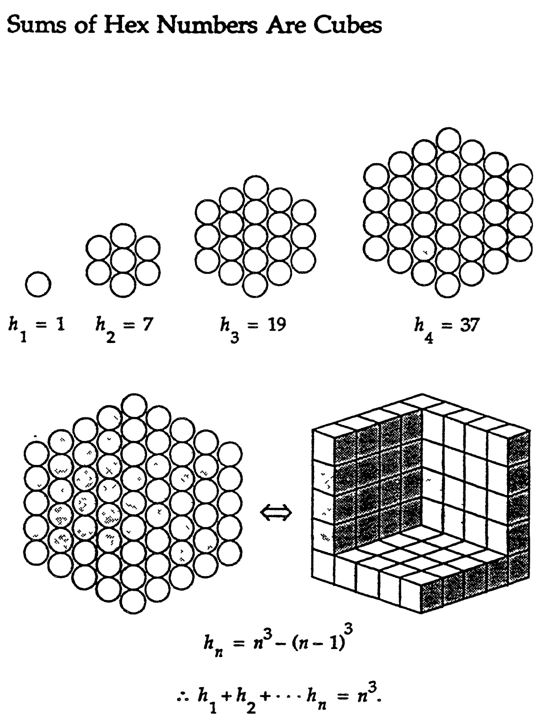


Figure 7

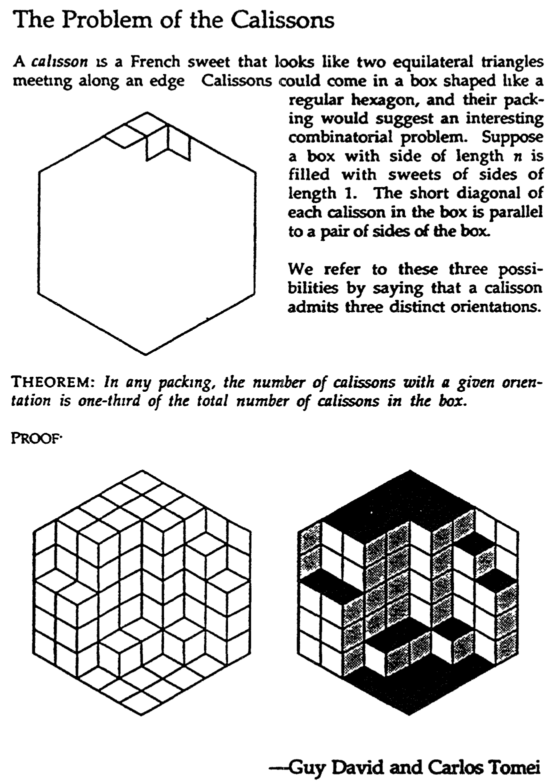


Figure 8

In both of these problems you can imagine you are counting unit cubes. It is to be noted that the Calisson problem was taken from a paper by David and Tome (*Amer. Math. Monthly* 96 (1989) 429–430). In that paper, the authors note that it is not obvious that filling a hexagon corresponds to some arrangement of cubes within the walls and the floor of a cube. They give a sketch of a proof for this which takes $1\frac{1}{2}$ pages. In a subsequent Letter to the Editor (*A.M.M.* 97 (1990) 131), Fred Galvin gives a nice short proof. The following generalization appeared as a Tournament of the Towns Problem:

> "A regular hexagon is cut up into n parallelograms of equal area. Prove that n is divisible by 3."

A solution by Andy Liu will appear in Book 3 of a collection of these problems.

I am all for giving a picture of any mathematical result if possible (provided it is not complicated), since "a picture is worth a thousand words". In my view many of the PWW's here would be enhanced pedagogically if they included at least a few words. The title of these "proofs" would then have to be changed to the not so catchy phrase "proofs without or with few words". My colleague Andy Liu feels that one should not push too far just to get PWW's, lest they turn into "proofs with words missing".

Coincidentally, while preparing this review, I received a PWW of Menelaus' Theorem via Centroids to referee. The one diagram had weights at the intersections of a line with the sides of a triangle plus an equivalent set of weights at the vertices. Although I know Menelaus' Theorem and geometric applications quite well, having all these weights on one figure without any explanation was rather confusing. My recommendation was not to publish it as is, but to publish it as a nice short note on applications of the centroid method if the equivalent set of weights was deleted from the figure and then placed on a second figure and some short explanation was included.

Despite my few reservations, this book is a nice collection of results which certainly will enhance one's geometric sense and stimulate mathematical thought.

* * * * *

PROBLEMS

Problem proposals and solutions should be sent to B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **January 1, 1995**, *although solutions received after that date will also be considered until the time when a solution is published.*

1951. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a cyclic quadrilateral, and P is the intersection of diagonals AC and BD. A line ℓ through P meets AB and CD at E and F respectively. Let O_1 and O_2 be the circumcenters of ΔPAB and ΔPCD, and let Q be the point on O_1O_2 such that $PQ \perp \ell$. Prove that $EP : PF = O_1Q : QO_2$.

1952. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

The convex cyclic quadrilateral $ABCD$ is such that each of its diagonals bisects one angle and trisects the opposite angle. Determine the angles of $ABCD$.

1953. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine a necessary and sufficient condition on real constants $r_1, r_2, \ldots, r_n$ such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq (r_1x_1 + r_2x_2 + \cdots + r_nx_n)^2$$

holds for all real $x_1, x_2, \ldots, x_n$.

1954. *Proposed by Vedula N. Murty, Maharanipeta, India.*

Let ABC be a triangle with $\angle A < \pi/2$ and $\angle B \leq \angle C$. The tangents to the circumcircle of ABC at B and C meet at D. Put $\theta = \angle OAD$, where O is the circumcentre. Prove that

$$2\tan\theta = \cot B - \cot C.$$

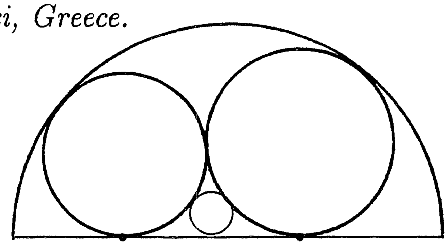
1955. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*

Find all integer solutions of the system of equations

$$x^2 + 9y^2 + 9z^2 + 4u^2 = 1981, \qquad x + y + z + u = 54.$$

1956. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In a semicircle of radius 4 there are three tangent circles as in the figure. Prove that the radius of the smallest circle is at most $\sqrt{2} - 1$.



1957. *Proposed by William Soleau, New York.*

A 9 by 9 board is filled with 81 counters, each being green on one side and yellow on the other. Initially, all have their green sides up, except the 31 marked with circles in the diagram. In one move, we can flip over a block of adjacent counters, vertically or horizontally only, provided that at least one of the counters at the ends of the block is on the edge of the board. Determine a shortest sequence of moves which allows us to flip all counters to their green sides.

			O			O		
			O		O	O		
			O		O	O		
		O	O		O	O		
		O		O			O	O
O	O			O				
	O	O			O			
	O	O			O			
O			O	O		O	O	O

1958. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Find the tetrahedron of maximum volume given that the sum of the lengths of some four edges is 1.

1959. *Proposed by John Selfridge, Northern Illinois University, DeKalb.*

Show that there is a (not too large) integer N so that, for every integer $n \geq N$, you can form a square by multiplying together distinct integers between n^2 and $(n+1)^2$. For instance, the product $27 \cdot 28 \cdot 30 \cdot 32 \cdot 35 = 5040^2$ shows that you can do it for $n = 5$. But you can't do it for $n = 6$, so N has to be at least 7.

1960. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Two perpendicular lines and a circle C pass through a common point. Three line segments AB, CD, EF, with endpoints on the two perpendicular lines, are tangent to C at their midpoints. Prove that the length of one segment is equal to the sum of the lengths of the other two.

* * * * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1637. [1991: 114; 1992: 125] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

II. *Solution by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

We show that, as conjectured on [1992: 125], the bound may be improved to $9\sqrt{3}/\pi$ for all triangles, with equality holding when ABC is equilateral.

Assume $A \geq B \geq C$ and let

$$M = \frac{\sin B + \sin C}{A} + \frac{\sin C + \sin A}{B} + \frac{\sin A + \sin B}{C}$$

and

$$m = \frac{2\sin[(B+C)/2]}{A} + 4 \cdot \frac{\sin A + \sin[(B+C)/2]}{B+C} \, .$$

We first prove that $M \geq m$. Note

$$\begin{aligned}\frac{\sin C + \sin A}{B} &- 2 \cdot \frac{\sin A + \sin[(B+C)/2]}{B+C} \\ &= -\frac{1}{BC(B+C)}\left[2BC\left(\sin\frac{B+C}{2} - \sin C\right) + C(B-C)(\sin A + \sin C)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\sin A + \sin B}{C} &- 2 \cdot \frac{\sin A + \sin[(B+C)/2]}{B+C} \\ &= \frac{1}{BC(B+C)}\left[2BC\left(\sin B - \sin\frac{B+C}{2}\right) + B(B-C)(\sin A + \sin B)\right].\end{aligned}$$

Thus using

$$\sin B + \sin C = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} \quad \text{etc.}$$

we get

$$M - m = \frac{2}{A}\left(\cos\frac{B-C}{2} - 1\right)\sin\frac{B+C}{2} + \frac{1}{BC(B+C)}\left[2BC\left(\sin B + \sin C\right.\right.$$
$$\left.\left. - 2\sin\frac{B+C}{2}\right) + (B-C)^2\sin A + (B-C)(B\sin B - C\sin C)\right]$$
$$= -\frac{4}{A}\sin^2\frac{B-C}{4}\sin\frac{B+C}{2} + \frac{1}{BC(B+C)}\left[4BC\sin\frac{B+C}{2}\left(\cos\frac{B+C}{2} - 1\right)\right.$$
$$\left. + (B-C)^2(\sin A + \sin B) + C(B-C)(\sin B - \sin C)\right].$$

Note that, from $A \geq B \geq C$ and $A \leq \pi - B$,

$$\sin A \geq \sin B \geq \sin\frac{B+C}{2} \quad \text{and} \quad \frac{B-C}{4} \geq \sin\frac{B-C}{4}.$$

Hence

$$(B-C)^2\sin A \geq 16\sin^2\frac{B-C}{4}\sin\frac{B+C}{2} \geq 8\left(\frac{B+C}{2}\right)^2\sin^2\frac{B-C}{4}\sin\frac{B+C}{2}$$
$$\geq 8BC\sin^2\frac{B-C}{4}\sin\frac{B+C}{2} = 4BC\sin\frac{B+C}{2}\left(1 - \cos\frac{B-C}{2}\right),$$

whence

$$M - m \geq -\frac{4}{A}\sin^2\frac{B-C}{4}\sin\frac{B+C}{2} + \frac{(B-C)^2\sin B}{BC(B+C)}$$
$$\geq \frac{16}{BC(B+C)}\sin\frac{B+C}{2}\sin^2\frac{B-C}{4} - \frac{4}{A}\sin\frac{B+C}{2}\sin^2\frac{B-C}{4}$$
$$\geq 4\sin\frac{B+C}{2}\sin^2\frac{B-C}{4}\left(\frac{16}{(B+C)^3} - \frac{1}{A}\right) \geq 0.$$

Thus $M \geq m$ as claimed.

Now we will prove that

$$m = \frac{2}{A}\sin\frac{B+C}{2} + \frac{4}{B+C}\left(\sin A + \sin\frac{B+C}{2}\right) \geq \frac{9\sqrt{3}}{\pi}, \tag{1}$$

which will establish the result. Set $x = (B+C)/2$, so that $A = \pi - 2x$, and note $0 < x \leq \pi/3$. Then (1) is equivalent to

$$\frac{\sin x}{\pi - 2x} + \frac{\sin 2x + \sin x}{x} \geq \frac{9\sqrt{3}}{2\pi}$$

or

$$\frac{\sin x}{x}\cdot\frac{\pi - x}{\pi - 2x} + \frac{\sin 2x}{x} \geq \frac{9\sqrt{3}}{2\pi}. \tag{2}$$

Case (i): $\pi/4 < x \leq \pi/3$. Then (2) becomes

$$(\pi - x)\sin x + (\pi - 2x)\sin 2x - \frac{9\sqrt{3}}{2\pi}(\pi x - 2x^2) \geq 0.$$

Let

$$f(x) = (\pi - x)\sin x + (\pi - 2x)\sin 2x - \frac{9\sqrt{3}}{2\pi}(\pi x - 2x^2).$$

Then

$$f'(x) = -\sin x + (\pi - x)\cos x - 2\sin 2x + 2(\pi - 2x)\cos 2x - \frac{9\sqrt{3}}{2\pi}(\pi - 4x)$$

and

$$\begin{aligned} f''(x) &= -(\pi - x)\sin x - 2\cos x - 8\cos 2x - 4(\pi - 2x)\sin 2x + \frac{18\sqrt{3}}{\pi} \\ &\geq -\frac{3\pi}{4}\cdot\frac{\sqrt{3}}{2} - \sqrt{2} - 4\left(\pi - \frac{\pi}{2}\right) + \frac{18\sqrt{3}}{\pi} > 0. \end{aligned}$$

So $f'(x) \leq f'(\pi/3) = 0$; thus $f(x) \geq f(\pi/3) = 0$, and (2) follows.

Case (ii): $0 < x \leq \pi/4$. Then

$$\frac{\sin x}{x} \geq \frac{\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}}{\pi},$$

and therefore

$$\frac{\sin x}{x}\cdot\frac{\pi - x}{\pi - 2x} + \frac{\sin 2x}{x} \geq \frac{2\sqrt{2}}{\pi}\left(\frac{\pi - x}{\pi - 2x} + 2\cos x\right).$$

Thus to prove (2) it suffices to show that

$$\frac{\pi - x}{\pi - 2x} + 2\cos x \geq \frac{9\sqrt{6}}{8},$$

or

$$\frac{1}{1 - 2x/\pi} + 4\cos x \geq \frac{9\sqrt{6}}{4} - 1. \tag{3}$$

Note that

$$\frac{1}{1 - 2x/\pi} > 1 + \frac{2}{\pi}x + \frac{4}{\pi^2}x^2,$$

and thus

$$\frac{1}{1 - 2x/\pi} + 4\cos x > 1 + \frac{2}{\pi}x + \frac{4}{\pi^2}x^2 + 4\cos x := F(x).$$

Because

$$F''(x) = \frac{8}{\pi^2} - 4\cos x < \frac{8}{\pi^2} - 2\sqrt{2} < 0$$

for $0 < x \le \pi/4$, F is concave for $0 < x \le \pi/4$. Hence

$$F(x) \ge \min\left\{F(0), F\left(\frac{\pi}{4}\right)\right\} = F\left(\frac{\pi}{4}\right) = \frac{7}{4} + 2\sqrt{2} > \frac{9\sqrt{6}}{4} - 1,$$

and (3) follows, completing the proof.

[*Editor's note.* The conjecture that the bound may be improved to $9\sqrt{3}/\pi$ was made by Richard I. Hess, Rancho Palos Verdes, California, and Murray S. Klamkin, University of Alberta.

Notice that now the sum in this problem has the same lower bound as the cyclic sum $\sum(a/S_a)$ of *Crux* 1651 [1992: 154], also proposed by Tsintsifas! Here a, b, c are the sides of a triangle, and S_a, S_b, S_c are the arc lengths along the incircle, between the points where it touches the sides of the triangle. Are these two sums comparable in general?]

* * * * *

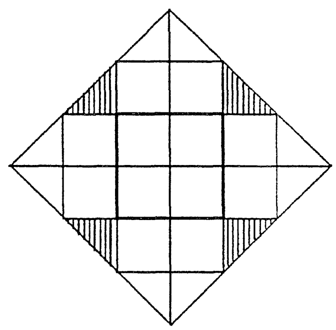
1856. [1993: 170] *Proposed by Jisho Kotani, Akita, Japan.*

Find the rectangular brick of largest volume that can be completely wrapped in a square piece of paper of side 1 (without cutting the paper).

Solution by P. Penning, Delft, The Netherlands.

The largest possible volume is $\sqrt{2}/27$.

Two perpendicular sides of the brick are equal to $\sqrt{2}/3$, and the third side (perpendicular to these two) has half that length. The sides of the brick are parallel to the diagonals of the paper. Let the centre of the paper coincide with the centre of the largest (bottom) face of the brick. The top face is then exactly covered by four triangles of paper. A fraction 1/9 of the paper (shaded in the picture) is not used for coverage.



Also solved by the proposer. Four other readers sent in solutions different from the above and from each other. Three of them may have thought the brick had to be placed parallel to the sides of the paper.

The proposer's solution includes the easy calculus derivation that the above volume is maximal for all bricks whose sides are parallel to the diagonals of the paper, and that the largest wrappable brick with sides parallel to the sides of the paper is smaller, having base $(3+\sqrt{3})/6$ *by* $\sqrt{3}/6$, *height* $(3-\sqrt{3})/6$, *and volume* $\sqrt{3}/36$.

* * * * *

1859. [1993: 170] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Let w be any n-letter "word" ($n \ge 1$) which contains at most 10 different letters, like MATHEMATICORUM or AAAEEIIIAAOOOUUUU. Prove that you can replace the letters of w by decimal digits (different letters replaced by different digits, the first letter of w not replaced by 0) so that the resulting n-digit number is a multiple of 3.

I. *Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Germany.*

First, the number created from w will be a multiple of 3 if and only if the sum of all digits used is a multiple of 3. Also, if any letter occurs three or more times one can remove this letter three times without loss of divisibility by 3. It follows that we can assume the letters of w occur at most two times. Let m be the number of letters which occur exactly once, and n the number of letters which occur exactly twice, so $m + n \leq 10$.

If $m + n \leq 4$ choose the digits 3, 6, 9, 0.

If $m > 1$, replace any two letters which occur exactly once by the digits 1 and 2 (or by 4 and 5 or by 7 and 8). If $n > 1$, replace any two letters which occur exactly twice by the digits 1 and 2 (or by 4 and 5 or by 7 and 8). Repeat this method until there are at most four different letters left. Finally replace the last (at most four) letters by 3, 6, 9, 0. [Note that by this method one can easily avoid replacing the first letter of w by 0.]

II. *Solution by Robert B. Israel, University of British Columbia.*

In fact, it is possible to make the word's value a multiple of 9.

Note first that we can ignore the restriction that the word cannot start with 0, because interchanging 0 with 9 does not change the value mod 9. Moreover, since $10 \equiv 1 \bmod 9$, the word's value is congruent mod 9 to the sum of its digits. I can assume there are exactly 10 different letters (some of which may occur 0 times).

Now suppose there is some letter that occurs k times in w, where $n - k$ is not divisible by 3. Assign 9 to this letter, and assign 0 to 8 arbitrarily to the other letters. Suppose this produces a value v mod 9. Now replace each digit $d \leq 8$ by $d + 1 \bmod 9$ (so 8 gets replaced by 0). This is again a valid assignment of digits to the letters, and the value is congruent to $v + n - k \bmod 9$. Since $n - k$ and 9 are relatively prime, performing this operation the appropriate number of times will produce a value congruent to $0 \bmod 9$.

Now consider the case where every letter occurs a number of times congruent to n mod 3. Thus for any valid assignment of digits to letters, the value will be congruent to $n(0 + 1 + \cdots + 9) \equiv 0 \bmod 3$. If there is a letter that occurs k times where $n - k$ is divisible by 3 but not by 9, then again we can produce a value divisible by 9 by the method of the previous paragraph.

We are left with the case where every letter occurs a number of times congruent to n mod 9. But then for every valid assignment of digits to letters, the value is congruent to $n(0 + 1 + \cdots + 9) \equiv 0 \bmod 9$.

Also solved by F.J. FLANIGAN, San Jose State University, San Jose, California; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; NEVEN JURIĆ, Zagreb, Croatia; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; MARCIN E. KUCZMA, Warszawa, Poland; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.

Godin's solution is the same as Engelhaupt's. Wildhagen gives a generalization involving base b digits.

Hess also shows (by computer) that w can be made divisible by 9. He points out that w can be made divisible by 6 by ensuring that the replacement yielding a number divisible *by 3 also replaces the final letter of w by an even digit (and in fact this is easy to do with Solution* I*).*

It is obvious that there are substitutions making w divisible by 4 *or* 5*, and it is only a little harder to do* 8*. Among single-digit divisors, this leaves only* 7*. Can the word w of the problem always be made divisible by* 7*? (This question was raised by Flanigan and the proposer.)*

For divisors of two or more digits, one would have to rule out the word aaa...a whenever the integer 111...1 *of the same length is prime. Perhaps readers can obtain positive results for some further divisors if all words considered have a large enough number of different letters.*

*   *   *   *   *

1860*. [1993: 170] *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*

Prove or disprove that

$$\sum \frac{\cos[(A-B)/4]}{\cos(A/2)\cos(B/2)} \geq 4,$$

where the sum is cyclic over the angles A, B, C of a triangle.

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.

The inequality is true and follows from

$$\begin{aligned}\sum \frac{\cos[(A-B)/2]}{\cos(A/2)\cos(B/2)} &= \sum\left(1+\tan\frac{A}{2}\tan\frac{B}{2}\right)\\ &= 3+\sum\tan\frac{A}{2}\tan\frac{B}{2}\\ &= 3+1=4.\end{aligned}$$

Here $\sum\tan(A/2)\tan(B/2)=1$ by elementary trigonometry: since $C/2=90^\circ-(A+B)/2$,

$$\begin{aligned}\sum\tan\frac{A}{2}\tan\frac{B}{2} &= \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\\ &= \tan\frac{A}{2}\tan\frac{B}{2}+\frac{\tan(A/2)+\tan(B/2)}{\tan[(A+B)/2]}\\ &= \tan\frac{A}{2}\tan\frac{B}{2}+\left(1-\tan\frac{A}{2}\tan\frac{B}{2}\right)=1.\end{aligned}$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; NASHA KOMANDA, Central Michigan University, Mt. Pleasant; MARCIN E. KUCZMA, Warszawa, Poland; and KEE-WAI LAU, Hong Kong. A partial solution was also sent in.

*   *   *   *   *

1861. [1993: 203] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $f : \mathbf{R}^+ \to \mathbf{R}$ be an increasing and concave function from the positive real numbers to the reals. Prove that if $0 < x \le y \le z$ and n is a positive integer then

$$(z^n - x^n)f(y) \ge (z^n - y^n)f(x) + (y^n - x^n)f(z).$$

Solution by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.

We prove the result more generally, for any real number $n \ge 1$. If two of x, y and z are equal, then the inequality is trivial, so we need only consider $0 < x < y < z$.

By Jensen's inequality we have

$$f\left(\frac{z-y}{z-x}x + \frac{y-x}{z-x}z\right) \ge \frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z)$$

which simplifies to

$$f(y) \ge \frac{(z-y)f(x) + (y-x)f(z)}{z-x},$$

and using $z > y > x$ we obtain

$$(z-x)f(y) \ge (z-y)f(x) + (y-x)f(z), \tag{1}$$

$$(z-y)(f(y)-f(x)) \ge (y-x)(f(z)-f(y)),$$

and finally

$$\frac{f(y)-f(x)}{y-x} \ge \frac{f(z)-f(y)}{z-y}. \tag{2}$$

The inequality in (1) is reversed when f is an increasing and convex function (because Jensen's inequality is reversed). Using this fact with the increasing and convex function $g(r) = r^n$ (with $n \ge 1$) we obtain

$$(z-x)y^n \le (z-y)x^n + (y-x)z^n \tag{3}$$

which reduces to

$$\frac{y-x}{y^n - x^n} \ge \frac{z-y}{z^n - y^n}. \tag{4}$$

Multiplying (2) and (4) we obtain

$$\frac{f(y)-f(x)}{y^n - x^n} \ge \frac{f(z)-f(y)}{z^n - y^n},$$

or

$$(z^n - y^n)(f(y)-f(x)) \ge (y^n - x^n)(f(z)-f(y)),$$

which simplifies to precisely what we wished to prove.

Also solved by SEUNG-JIN BANG, Seoul, Korea; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; CYRUS HSIA, student, Woburn C. I., Toronto, *Ontario; ROBERT B. ISRAEL, University of British Columbia; NEVEN JURIĆ, Zagreb, Croatia; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; DALE VARBERG, Hamline University, St. Paul, Minnesota; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer.*

Some other solvers also note n can be any real number ≥ 1, or in fact that the nth powers in the problem could be replaced by any convex function.

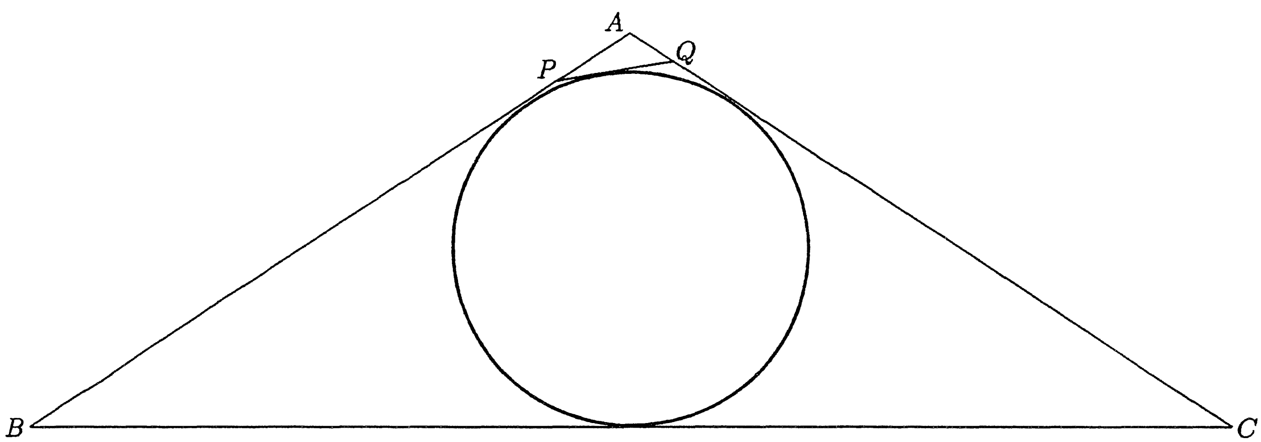
Inequality (3) *has appeared recently in Solution* I *of* Crux 1813 [1994: 22]. *As a couple of solvers point out,* Crux 1868 *is also somewhat related to this problem.*

* * * * *

1862. [1993: 203] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle with $AB = AC$ and $\angle A = 120°$. Let P, Q be points on sides AB, AC respectively so that PQ is tangent to the incircle of ΔABC. Prove that $BP \cdot CQ$ is equal to twice the area of quadrilateral $PBCQ$.

Solution by Geoffrey A. Kandall, Hamden, Connecticut.



Let $BP = p$, $CQ = q$, $AB = AC = 1$ (so $BC = \sqrt{3}$). Since $PQ + BC = BP + CQ$ for the circumscribed quadrilateral $PBCQ$, we have $PQ = p + q - \sqrt{3}$. By the cosine law applied to ΔAPQ,

$$(p+q-\sqrt{3})^2 = (1-p)^2+(1-q)^2-2(1-p)(1-q)\left(-\frac{1}{2}\right) = (1-p)^2+(1-q)^2+(1-p)(1-q),$$

which simplifies to

$$pq = (2\sqrt{3}-3)(p+q) = \sqrt{3}(2-\sqrt{3})(p+q),$$

i.e.

$$p+q = \frac{2+\sqrt{3}}{\sqrt{3}}\,pq.$$

Now

$$\begin{aligned}\text{area }(PBCQ) &= \text{ area }(ABC) - \text{ area }(APQ)\\ &= \frac{1}{2}\sin 120^\circ - \frac{1}{2}(1-p)(1-q)\sin 120^\circ\\ &= \frac{\sqrt{3}}{4}(p+q-pq) = \frac{\sqrt{3}}{4}\left(\frac{2+\sqrt{3}}{\sqrt{3}}-1\right)pq = \frac{1}{2}pq = \frac{1}{2}BP\cdot CQ,\end{aligned}$$

as required.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; ŠEFKET ARSLANAGIĆ, Nyborg, Denmark; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; RANDY HO, student, University of Arizona; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; FRANCISCO L.R. PIMENTEL, Fortaleza, Brazil; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, The Netherlands; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. Two other readers sent in solutions assuming that PQ and BC were parallel.

Ardila and Bellot mention the related problem 4 of the 1993 Iberoamerican Mathematical Olympiad. In fact, Bellot reveals that our proposer Seimiya also proposed this problem!

Penning notes that the result still holds if, say, P is chosen on the ray BA beyond A, so that PBCQ is a nonconvex, but still not self-intersecting, quadrilateral. The above proof works with only minor changes.

* * * * *

1863. [1993: 203] *Proposed by Murray S. Klamkin, University of Alberta.*

Are there any integer solutions of the equation

$$(x+y+z)^5 = 80xyz(x^2+y^2+z^2)$$

such that none of x, y, z are 0?

Solution by the proposer.

No. The identity

$$(x+y+z)^5 - (-x+y+z)^5 - (x-y+z)^5 - (x+y-z)^5 = 80xyz(x^2+y^2+z^2)$$

follows by expanding out. Hence the given equation is equivalent to

$$(-x+y+z)^5 + (x-y+z)^5 + (x+y-z)^5 = 0. \tag{1}$$

Since Fermat's last "theorem" is known to be valid for exponent 5, there are only the trivial solutions $x = 0$, $y + z = 0$ and symmetrically. [For if $-x + y + z = 0$, (1) becomes $0^5 + (2z)^5 + (2y)^5 = 0$, which implies $z = -y$ and so $x = 0 = y + z$.]

The mysterious procedure by which the proposer conjures up identities such as the one above seems to be unavailable to the rest of us! Just two readers guessed that the answer to the problem was no, and neither had a complete proof.

P. Penning, Delft, The Netherlands, investigated when the number 80 *of the problem can be replaced by a positive integer N so that the resulting equation has an all-nonzero integer solution. Via a computer search, he found only two such values of N: $N = 81$, with solutions $x = y = z$; and $N = 108$, with solutions $4x = 4y = z$. Any others?*

* * * * *

1864. [1993: 203] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Consider the three excircles of a given triangle ABC. Let P be the radius of the circle containing and internally tangent to these three circles. Prove that $P \geq 7r$, where r is the inradius of ΔABC.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

This will be a solution "from the books". First, in [2], pp. 111–114, the following expression for P in terms of the exradii is given:

$$P = \frac{s^4}{4r_1r_2r_3} + \frac{r_1r_2r_3}{4s^2} . \tag{1}$$

Second, in [4], p. 114, ex. 16.29 (or see p. 190 of [3]), the formula $r_1r_2r_3 = s^2r$ is proved, and so (1) becomes

$$P = \frac{s^2 + r^2}{4r} , \tag{2}$$

and the proposed inequality $P \geq 7r$ is equivalent to $s^2 + r^2 \geq 28r^2$ or $s^2 \geq 27r^2$, and this last is known, being item 5.11 from the "Bottema Bible" [1].

Remarks: (1) The circle externally tangent to the three excircles is of course the nine-point circle of ABC, of radius $R/2$, where R is the circumradius of ABC.

(2) In the Belgian journal *Mathesis* (1927) p. 94, the following interesting problem from Victor Thébault is given:

> Let Γ be the circle internally tangent to the excircles of the triangle ABC. Draw the circles ω_a, ω_b, ω_c which are internally tangent to two of the excircles and externally tangent to the third excircle. Let Γ' be the circle tangent (externally or internally) to ω_a, ω_b, ω_c. Then Γ and Γ' are tangent.

Mathesis includes a very quick solution by inversion.

(3) Problem 3508 of the *American Mathematical Monthly*, 1932, pp. 550–552, asks to determine the radius of the circle circumscribing the escribed circles of triangle ABC, with an excellent solution by the famous Belgian problemist René Goormaghtigh and an

also excellent editorial note by Otto Dunkel. In this solution, formula (2) for the desired radius is established by inversion.

References:

[1] O. Bottema et al., *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1968.

[2] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.

[3] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

[4] T. Lalesco, *La Géométrie du Triangle*, J. Gabay, Paris, 1987.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer.

Lau and Pompe gave the same solution as Bellot.

* * * * *

1865. [1993: 203] *Proposed by Christopher J. Bradley, Clifton College, Bristol, U.K.*

Find an integer-sided right-angled triangle with sides x^2-1, y^2-1, z^2-1 where x, y, z are integers.

Solution by David E. Manes, State University of New York, Oneonta.

Right triangles with sides 70224, 82368, 108240 and 99, 168, 195 satisfy the conditions of the problem. Note that if x, y, z are odd, that is, $x = 2a+1$, $y = 2b+1$, and $z = 2c+1$ for integers a, b, c, then the equation $(x^2-1)^2+(y^2-1)^2 = (z^2-1)^2$ can be reduced to

$$\left(\frac{a(a+1)}{2}\right)^2 + \left(\frac{b(b+1)}{2}\right)^2 = \left(\frac{c(c+1)}{2}\right)^2,$$

or $t_a^2 + t_b^2 = t_c^2$ where $t_n = n(n+1)/2$ is the nth triangular number. W. Sierpiński in *Elementary Theory of Numbers*, Harper, New York, 1964, p. 55, points out that the only known solution of this equation is given by $t_{132} = 8778$, $t_{143} = 10296$, $t_{164} = 13530$. As a result, the equation $(x^2-1)^2+(y^2-1)^2 = (z^2-1)^2$ has a solution in odd integers

$$x = \mathbf{265}, \quad y = \mathbf{287}, \quad \text{and} \quad z = \mathbf{329}$$

which yields the first triangle. Sierpiński further points out that

$$x = \mathbf{10}, \quad y = \mathbf{13}, \quad \text{and} \quad z = \mathbf{14}$$

is a solution in which not all the numbers x, y, and z are odd. This solution yields the second triangle. Finally, Sierpiński notes that it is not known whether the equation $(x^2-1)^2+(y^2-1)^2 = (z^2-1)^2$ has infinitely many solutions in positive integers > 1.

Both the above solutions also found by RICHARD I. HESS, Rancho Palos Verdes, California; and BOB PRIELIPP, University of Wisconsin–Oshkosh. One solution (nearly *always the smaller) found by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHARLES ASHBACHER, Cedar Rapids, Iowa; NEVEN JURIĆ, Zagreb, Croatia; PAVLOS B. KONSTADINIDIS, student, University of Arizona, Tucson; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; and the proposer. One incorrect solution was received.*

Prielipp in fact also found the Sierpiński reference.

*   *   *   *   *

1866. [1993: 203] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Perpendicular chords AC, BD of a given circle intersect in point J. Let P, Q, R be the orthogonal projections of J onto segments AB, BC, CD respectively and let N be the midpoint of AD. Prove that N, P, Q, R are concyclic.

Comment by the Editor.

It was pointed out on [1994: 85] that the published solution to *Crux* 1836 also settles *Crux* 1866. So we don't need to give another solution here.

Solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; JORDI DOU, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie Composite H.S., Grande Prairie, Alberta; CYRUS HSIA, student, Woburn C.I., Toronto, Ontario; L.J. HUT, Groningen, The Netherlands; GEOFFREY A. KANDALL, Hamden, Connecticut; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; TOSHIO SEIMIYA, Kawasaki, Japan; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Ardila, Bellot, Dou and Pompe pointed out the connection with Crux 1836.

*   *   *   *   *

1867. [1993: 203] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

(a) Celebrate Canadian Confederation (July 1, 1867) by finding a 6-digit number CANADA (C, A, N, D distinct decimal digits with $C \neq 0$) which is divisible by 1867.

(b) Prove (not using a computer) that there is no CANADA divisible by 1887.

Solution by Pavlos B. Konstadinidis, student, University of Arizona, Tucson.

(a) Since 1867 is prime, I tried to find a CANADA that is a multiple of 3 (since $A+A+A = 3A$) and 10 (making $A = 0$), that is, a multiple of 30. So I tried the 17 6-digit numbers 30×1867, 60×1867, ..., 510×1867. I found that

$$270 \times 1867 = \mathbf{504090}$$

is a solution. [*Editors note.* There are two other solutions,

$$204 \times 1867 = \mathbf{380868} \quad \text{and} \quad 521 \times 1867 = \mathbf{972707}.]$$

(b) Since $1887 = 3 \times 17 \times 37 = 111 \times 17$,

$$\begin{aligned}
1887 \mid \text{CANADA} &\Rightarrow 111 \mid \text{CANADA} = 10101\text{A} + 10\text{D} + 1000\text{N} + 100000\text{C} \\
&\Rightarrow 111 \mid 10\text{D} + 1000\text{N} + 100000\text{C} \quad (\text{because } 10101 = 91 \times 111) \\
&\Rightarrow 111 \mid \text{D} + 100\text{N} + 10000\text{C} \quad (\text{because 10 and 111 are coprime}) \\
&\Rightarrow 111 \mid \text{D} + 100\text{N} + 10\text{C} \quad (\text{because } 10000 = 90 \times 111 + 10) \\
&\Rightarrow 111 \mid \text{NCD} \Rightarrow \text{N} = \text{C} = \text{D},
\end{aligned}$$

which is absurd, because N, C and D are different.

Both parts also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; RICHARD I. HESS, Rancho Palos Verdes, California; ROBERT B. ISRAEL, University of British Columbia; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; FRANCISCO L.R. PIMENTEL and FERNANDO A. PIMENTEL, Fortaleza, Brazil; and the proposer. Part (a) *only solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHARLES ASHBACHER, Cedar Rapids, Iowa; TIM CROSS, Wolverley High School, Kidderminster, U.K.; and C.S. METCHETTE, Culver City, California.*

Over half the solvers found all three solutions in (a). *Most used a computer, although the proposer for one found them all by hand. Hess, Israel and the Pimentels gave solutions to* (b) *which are similar to Konstadinidis's nice argument.*

Israel notes that you can also make AMERICA *divisible by* 1776*: e.g.,* AMERICA = 2587632. *However, we might also observe that since no* CANADA *is divisible by* 111, CANADA *cannot be divisible by* 1776*!*

* * * * *

1868. [1993: 203] *Proposed by De-jun Zhao, Chengtun High School, Xingchang, China.*

Let $n \geq 3$, $a_1 > a_2 > \cdots > a_n > 0$, and $p > q > 0$. Show that

$$a_1^p a_2^q + a_2^p a_3^q + \cdots + a_{n-1}^p a_n^q + a_n^p a_1^q > a_1^q a_2^p + a_2^q a_3^p + \cdots + a_{n-1}^q a_n^p + a_n^q a_1^p.$$

Solution by Chris Wildhagen, Rotterdam, The Netherlands.

Let

$$L(n) = a_1^p a_2^q + a_2^p a_3^q + \cdots + a_{n-1}^p a_n^q + a_n^p a_1^q, \quad R(n) = a_1^q a_2^p + a_2^q a_3^p + \cdots + a_{n-1}^q a_n^p + a_n^q a_1^p$$

where $n \geq 2$. We shall prove by induction on n that $L(n) > R(n)$ for all $n \geq 3$. Since $L(2) = R(2)$ and

$$L(n+1) = L(n) + a_n^p a_{n+1}^q + a_{n+1}^p a_1^q - a_n^p a_1^q,$$
$$R(n+1) = R(n) + a_n^q a_{n+1}^p + a_{n+1}^q a_1^p - a_n^q a_1^p,$$

it suffices to show that $a_n^p a_{n+1}^q + a_{n+1}^p a_1^q - a_n^p a_1^q > a_n^q a_{n+1}^p + a_{n+1}^q a_1^p - a_n^q a_1^p$ or

$$a_{n+1}^p (a_1^q - a_n^q) + a_1^p (a_n^q - a_{n+1}^q) > a_n^p (a_1^q - a_{n+1}^q). \qquad (1)$$

Put $u = a_1^q$, $v = a_n^q$, $w = a_{n+1}^q$ and $\lambda = p/q$. Then $u > v > w > 0$ and $\lambda > 1$, and (1) becomes $w^\lambda(u-v) + u^\lambda(v-w) > v^\lambda(u-w)$ or

$$\frac{u-v}{u-w}w^\lambda + \frac{v-w}{u-w}u^\lambda > v^\lambda. \tag{2}$$

Since the function $f(t) = t^\lambda$ is convex on the interval $(0, +\infty)$ for $\lambda > 1$, we have [e.g., by Jensen's Inequality] that

$$f(v) = f\left(\frac{u-v}{u-w}w + \frac{v-w}{u-w}u\right) < \frac{u-v}{u-w}f(w) + \frac{v-w}{u-w}f(u)$$

from which (2) follows.

Also solved by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; ROBERT B. ISRAEL, University of British Columbia; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; and the proposer.

Most of the submitted solutions, including the one by the proposer, use induction. Israel obtained a generalization.

Klamkin notes that the inequality of the problem has occurred in Crux *before, as* (3) *in his generalization of* Crux 897 [1986: 141].

As was mentioned before, Crux 1861 *is related to this problem. In fact, inequality* (2) *(or rather its equivalent form in the previous line) also appeared as* (3) *in the solution of* 1861 *(this issue) as well as on* [1994: 22].

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1869. [1993: 203] *Proposed by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

For every positive integer n, let a_n be the biggest odd factor of n. Calculate the sum of the series

$$\frac{a_1}{1^3} + \frac{a_2}{2^3} + \frac{a_3}{3^3} + \cdots .$$

Solution by Cyrus Hsia, student, Woburn C.I., Toronto, Ontario.

The given series can be written as an infinite set of geometric series as follows:

$$\begin{array}{lcl}
\left.\frac{1}{1^3} + \frac{1}{(1\cdot 2)^3} + \frac{1}{(1\cdot 2^2)^3} + \frac{1}{(1\cdot 2^3)^3} + \cdots\right\} & = & \frac{1}{1-1/2^3} = \frac{8}{7} \\
\left.+\frac{3}{3^3} + \frac{3}{(3\cdot 2)^3} + \frac{3}{(3\cdot 2^2)^3} + \frac{3}{(3\cdot 2^3)^3} + \cdots\right\} & = & \frac{1/3^2}{1-1/2^3} = \frac{8}{7}\left(\frac{1}{3^2}\right) \\
\left.+\frac{5}{5^3} + \frac{5}{(5\cdot 2)^3} + \frac{5}{(5\cdot 2^2)^3} + \frac{5}{(5\cdot 2^3)^3} + \cdots\right\} & = & \frac{1/5^2}{1-1/2^3} = \frac{8}{7}\left(\frac{1}{5^2}\right) \\
\left.+\frac{7}{7^3} + \frac{7}{(7\cdot 2)^3} + \frac{7}{(7\cdot 2^2)^3} + \frac{7}{(7\cdot 2^3)^3} + \cdots\right\} & = & \frac{1/7^2}{1-1/2^3} = \frac{8}{7}\left(\frac{1}{7^2}\right) \\
+\cdots & & \vdots
\end{array}$$

All the natural numbers are represented once and only once, because any number $(2a-1)2^l$ is in the ath row and the lth column. Therefore the sum of the series becomes simply

$$\frac{8}{7}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)=\frac{8}{7}\left(\frac{\pi^2}{8}\right)=\frac{\pi^2}{7}\ ,$$

where the known sum

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8} \tag{1}$$

comes from e.g. page 219 of William Dunham, *Journey Through Genius*, Penguin Books, New York, 1990. [*Editor's note.* Some other solvers derived (1) from the (maybe more familiar) sum

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}\ ,$$

via

$$\begin{aligned}\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots &= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right)\\ &= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)\\ &= \frac{3}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{8}\ .\end{aligned}$$]

Also solved by H.L. ABBOTT, University of Alberta; FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia; SEUNG-JIN BANG, Seoul, Republic of Korea; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, U.K.; TIM CROSS, Wolverley High School, Kidderminster, U.K.; JORDI DOU, Barcelona, Spain; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; ESTEBAN INDURAIN, Universidad Pública de Navarra, Pamplona, Spain; ROBERT B. ISRAEL, University of British Columbia; NEVEN JURIĆ, Zagreb, Croatia; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; CHRIS WILDHAGEN, Rotterdam, The Netherlands; and the proposer. One other reader sent in only an approximation.

The proposer found the problem in the Belgian journal Mathesis, 1888, *Problem* 531, *proposed by E. Césaro.*

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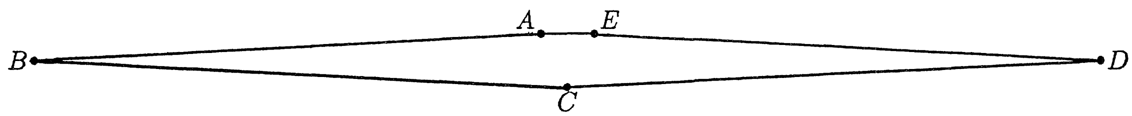
1870*. [1993: 204] *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*
In any convex pentagon $ABCDE$ prove or disprove that

$$AC \cdot BD + BD \cdot CE + CE \cdot DA + DA \cdot EB + EB \cdot AC$$
$$> AB \cdot CD + BC \cdot DE + CD \cdot EA + DE \cdot AB + EA \cdot BC.$$

(*Note*: the first sum involves diagonals, the second sum involves sides.)

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.



The inequality is not true for every convex pentagon. Consider a pentagon for which $AC \to 0$ and $CE \to 0$. The left hand side of the inequality is approaching $DA \cdot EB$ and the right hand side is approaching $AB \cdot CD + BC \cdot DE + DE \cdot AB$ which clearly may be greater than the left hand side.

The same counterexample was found by FEDERICO ARDILA, student, Colegio San Carlos, Bogotá, Colombia.

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LETTER TO THE EDITOR

For several years I have searched for a copy or authorized photocopy of the 1952 edition of the book *Modern College Geometry* by N. A. Court (Barnes and Noble, New York). I would be very grateful if a reader could send me information on some institution (a university library, for example) to which I can address my request for an authorized photocopy (it seems impossible to obtain a copy).

Thank you very much,

Francisco Bellot Rosado
Dos de Mayo, 16-8.° Dcha.
E-47004 Valladolid
Spain.

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Short articles intended for publication should be sent to Dr. Hanson, contest problem sets and solutions to Olympiad Corner problems should be sent to Dr. Woodrow and other problems and solutions to Dr. Sands.

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