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CONTENTS

THE OLYMPIAD CORNER: 69

M.S. KLAMKIN

I start out with two new problem sets, for which I solicit elegant solutions from all readers: selected problems from the Bulgarian Winter Competition which took place in Varna on January 14, 1985, and selected problems proposed by students at the 1985 U.S.A.M.O. training session. I am grateful to Jordan B. Tabov and Gregg Patruno, respectively, for these problems.

1985 BULGARIAN WINTER COMPETITION

1. Determine each term of the sequence $\{a_1, a_2, a_3, \ldots\}$ of positive numbers, given that $a_4 = 4$, $a_5 = 5$, and

$$\frac{1}{a_1a_2a_3} + \frac{1}{a_2a_3a_4} + \ldots + \frac{1}{a_na_{n+1}a_{n+2}} = \frac{(n+3)a_n}{4a_{n+1}a_{n+2}}$$

holds for $n = 1,2,3,\ldots$. (Grade 9)

2. S_1 and S_2 are two disjoint nonempty finite sets of points in a plane. If every segment whose endpoints are elements of one these two sets contains a point of the other set, prove that all the points of $S_1 \cup S_2$ are collinear. (Grade 9)

3. In a given tetrahedron ABCD, M is the centroid of face ABC, and A_1, B_1, C_1 are the feet of the perpendiculars from A,B,C, respectively, to line DM.

(a) Prove that there exists a triangle EFH whose sides HE, EF, and FH have lengths equal to those of the segments AA_1, BB_1, and CC_1, respectively.

(b) Find the volume of tetrahedron ABCD if the area of triangle EFH is S and the length of segment DM is d. (Grade 10)

4. Determine the number of real solutions of the equation

$$x^2 + 2x\sin x - 3\cos x = 0. \qquad \text{(Grade 11)}$$

5. A plane intersects a given regular square pyramid in a regular pentagon. Determine the angle between a lateral edge and the base of the pyramid. (Grade 11)

*

STUDENT PROPOSALS FROM THE 1985 U.S.A.M.O. TRAINING SESSION

1. *Proposed by Waldemar Horwat, Hoffman Estates, Illinois.*

N being the set of positive integers, determine all functions $f: N \to N$ such that

$$f(f(n)) = n + k,$$

where k is a given odd positive integer.

2. *Proposed by Bjorn Poonen, Winchester, Massachusetts.*

Is there a function f such that, for all real numbers x,

$$f(f(x)) = \sin 3x?$$

3. *Proposed by Bjorn Poonen, Winchester, Massachusetts.*

For a given positive integer n, prove that there are infinitely many positive integers m for which

$$\frac{(m!)^{n-1}}{(n!)^{m-1}}$$

is an integer.

4. *Proposed by Robert Adams, Annandale, Virginia.*

Starting with a given triangle ABC inscribed in a circle, show how to construct a directly homothetic triangle A'B'C' with A',B' on the circle and C' on segment AB.

5. *Proposed by Peter Yu, Richmond, California.*

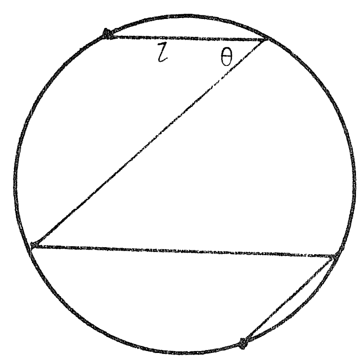
A circle is tangent to the sides AB, BC, CD, and DA of a quadrilateral at points P, Q, R, and S, respectively. If AP = 1, BQ = 2, CR = 2, and DS = 4, determine the area of ABCD.

6. *Proposed by Patrick Brown, Fairfax, Virginia.*

In a given quadrilateral ABCD, line AC is a nonperpendicular bisector of BD. If the bisectors of angles B and D intersect on AC, prove that they meet at right angles.

7. *Proposed by Zinkoo Han, Brooklyn, N.Y.*

A zigzag path across a unit circle consists of n chords lying in two directions. If the initial and terminal points of the path are ends of a diameter and the first chord has length l, find, in terms of l and n, the angle θ between the first and second chords. (The figure illustrates the case $n = 4$.)



$n = 4$

8.* *Proposed by Randall Rose, Plainview, N.Y.*

The Heronian mean $(a + \sqrt{ab} + b)/3$ of two distinct positive numbers is certainly less than the arithmetic mean and greater than the geometric mean. Determine the largest $r \geq 0$ and the smallest $s \leq 1$ for which

$$\left(\frac{a^r + b^r}{2}\right)^{1/r} \le \frac{a + \sqrt{ab} + b}{3} \le \left(\frac{a^s + b^s}{2}\right)^{1/s}$$

are valid inequalities.

9.* *Proposed by Joseph Keane, Pittsburgh, Pennsylvania.*

For $0 < x < 1$, it is conjectured that

$$(1 + x)^{1+x} + (1 + x)^{1-x} - (1 - x)^{1+x} - (1 - x)^{1-x} > 4x.$$

Prove or disprove.

10.* *Proposed by David Grabiner, Claremont, California.*

A sequence of real polynomials $\{P_1, P_2, \ldots, P_n\}$ has the property that for any permutation π of $(1,2,\ldots,n)$ there is a real number x such that

$$P_{\pi(1)}(x) < P_{\pi(2)}(x) < \ldots < P_{\pi(n)}(x).$$

Is it true that $\max_i$ {degree of P_i} $\ge n-1$? If not, find a sharp bound.

*

I now give solutions to some problems published in previous columns, starting with the official solutions to the problems of the 1985 Canadian Mathematics Olympiad.

1. [1985: 134] *From the 1985 Canadian Mathematics Olympiad.*

The lengths of the sides of a triangle are 6, 8, and 10 units. Prove that there is exactly one straight line which simultaneously bisects the area and perimeter of the triangle.

Official solution.

Let ABC be the given triangle, with side lengths BC = 6, CA = 8, and AB = 10. There are three cases to consider.

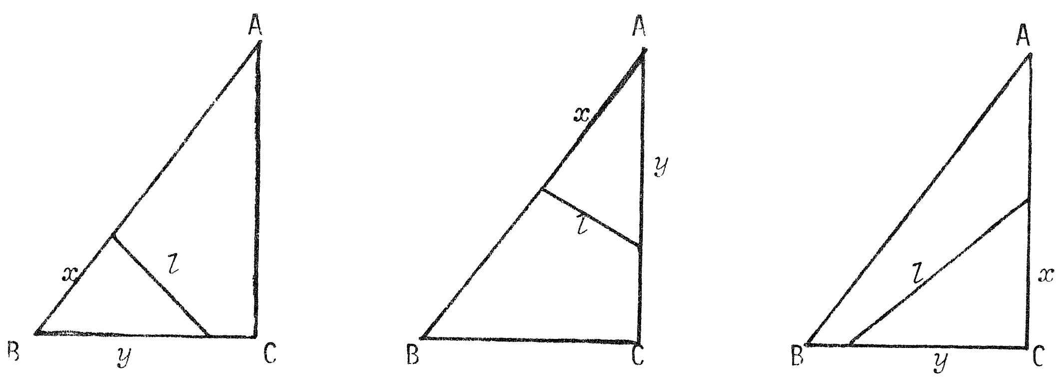


Figure 1 Figure 2 Figure 3

Case 1. Suppose a line l cuts AB and BC, as shown in Figure 1. Then l bisects the area and perimeter of ABC if and only if there are positive numbers x and y, with $x < 10$ and $y < 6$, such that $x+y = 12$ and $xy = 30$. The only possibilities are $\{x,y\} = \{6\pm\sqrt{6}\}$, the roots of $t^2-12t+30 = 0$. Since $6-\sqrt{6} < 6 < 6+\sqrt{6} < 10$, the unique solution for this case is obtained by taking $x = 6+\sqrt{6}$ and $y = 6-\sqrt{6}$.

Case 2. Suppose a line l cuts CA and AB, as shown in Figure 2. Then l bisects the area and perimeter of ABC if and only if there are positive numbers x and y such that $x+y = 12$ and $xy = 40$. Since the roots of $t^2-12t+40 = 0$ are imaginary, there is no solution in this case.

Case 3. Suppose a line l cuts BC and CA, as shown in Figure 3. Then l bisects the area and perimeter of ABC if and only if there are positive numbers x and y, both less than 8, such that $x+y = 12$ and $xy = 24$. Since one root of $t^2-12t+24 = 0$ is $6+2\sqrt{3} > 8$, there is no solution in this case either.

*

2. [1985: 134] *From the 1985 Canadian Mathematics Olympiad.*

Prove that there does not exist an integer which is doubled when the initial digit is transferred to the end.

Official solution.

Suppose, on the contrary, that there is an integer with the stated property, say

$$d_n\cdot 10^n + d_{n-1}\cdot 10^{n-1} + \ldots + d_1\cdot 10 + d_0,$$

where the d_i are digits and $d_n \neq 0$, and let

$$s = d_{n-1}\cdot 10^{n-1} + \ldots + d_1\cdot 10 + d_0.$$

Then

$$2(d_n\cdot 10^n + s) = 10s + d_n, \tag{1}$$

from which $d_n(2\cdot 10^n - 1) = 8s$. Hence $8|d_n$, so that $d_n = 8$. But then

$$2(d_n\cdot 10^n + s) > 10^{n+1} > 10s + d_n,$$

contradicting (1).

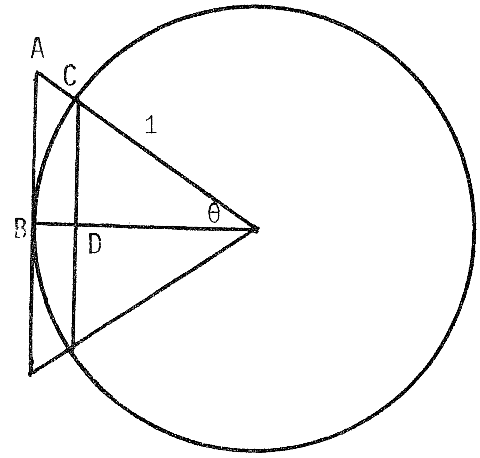
*

3. [1985: 134] *From the 1985 Canadian Mathematics Olympiad.*

Suppose that a given circle has circumference c. Let x denote the length of the perimeter of a regular polygon of 1985 sides circumscribed about the given circle. Let y denote the perimeter of a regular polygon of 1985 sides inscribed in the given circle. Prove that $x + y \geq 2c$.

Official solution.

Suppose the circle has unit radius, as shown in the figure, so that $c = 2\pi$. Then $x = 2\cdot 1985\text{AB}$, $y = 2\cdot 1985\text{CD}$, and it suffices to show that



$$\text{AB} + \text{CD} \ge \frac{2\pi}{1985},$$

or

$$\tan\theta + \sin\theta \ge 2\theta.$$

Now $\theta/2 = \pi/2\cdot 1985 < \pi/4$, so $\theta/2 < \tan(\theta/2) < 1$ and, with $t = \tan(\theta/2)$,

$$\sin\theta + \tan\theta = \frac{2t}{1+t^2} + \frac{2t}{1-t^2} = \frac{4t}{1-t^4} \ge 4t \ge 2\theta.$$

*

4. [1985: 134] *From the 1985 Canadian Mathematics Olympiad.*

Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k.

Official solution.

For any positive integer n, the exponent of the highest power of 2 dividing $n!$ is

$$[n/2] + [n/2^2] + [n/2^3] + \ldots,$$

where the brackets denote the greatest integer function.

Suppose $n = 2^{k-1}$ for some positive integer k. Then the exponent of the highest power of 2 dividing $n!$ is

$$2^{k-2} + 2^{k-3} + \ldots + 2 + 1 = 2^{k-1} - 1 = n - 1,$$

and so $2^{n-1}|n!$.

Conversely, suppose $2^{n-1}|n!$, and let k be such that $2^{k-1} \le n < 2^k$, so that

$$n = 2^{k-1} + \delta_{k-2}\cdot 2^{k-2} + \delta_{k-3}\cdot 2^{k-3} + \ldots + \delta_1\cdot 2 + \delta_0,$$

where each $\delta_j \in \{0,1\}$. Then

$$[n/2^j] = 2^{k-1-j} + \delta_{k-2}\cdot 2^{k-2-j} + \delta_{k-3}\cdot 2^{k-3-j} + \ldots + \delta_{j+1}\cdot 2 + \delta_j$$

for $j = 1,2,\ldots,k-1$, and $[n/2^j] = 0$ for $j \ge k$. Thus the exponent of the highest power of 2 dividing $n!$ is

$$\sigma = \sum_{j=1}^{\infty}[n/2^j] = (2^{k-1}-1) + \delta_{k-2}(2^{k-2}-1) + \delta_{k-3}(2^{k-3}-1) + \ldots + \delta_2(2^2-1) + \delta_1(2-1)$$

$$= n - 1 - (\delta_{k-2} + \delta_{k-3} + \ldots + \delta_1 + \delta_0)$$
$$\le n - 1.$$

But $\sigma \ge n-1$ from the hypothesis. Hence $\sigma = n-1$, all $\delta_j = 0$, and $n = 2^{k-1}$.

*

5. [1985: 134] *From the 1985 Canadian Mathematics Olympiad.*

Let $1 < x_1 < 2$ and, for $n = 1,2,3,\ldots$, define

$$x_{n+1} = 1 + x_n - \frac{1}{2}x_n^2.$$

Prove that $|x_n - \sqrt{2}| < 2^{-n}$ for $n \ge 3$.

Official solution.

We have $x_{n+1} = f(x_n)$, where

$$f(x) = 1 + x - \frac{1}{2}x^2 = \sqrt{2} + \frac{1}{2}(x - \sqrt{2})(2 - \sqrt{2} - x).$$

Now $f(x)$ decreases from 3/2 to 1 as x increases over the interval [1,2]. Hence

$$1 < x_1 < 2 \implies 1 < x_2 < \frac{3}{2} \implies \frac{11}{8} < x_3 < \frac{3}{2}$$
$$\implies \frac{11-8\sqrt{2}}{8} < x_3 - \sqrt{2} < \frac{12-8\sqrt{2}}{8}$$
$$\implies -\frac{1}{8} < \frac{11-8\sqrt{2}}{8} < x_3 - \sqrt{2} < \frac{12-8\sqrt{2}}{8} < \frac{1}{8},$$

and so $|x_3 - \sqrt{2}| < 2^{-3}$.

We now proceed by induction. Suppose $|x_n - \sqrt{2}| < 2^{-n}$ for some $n \ge 3$. Then

$$\begin{aligned}|x_{n+1} - \sqrt{2}| &= \frac{1}{2}|x_n-\sqrt{2}|\cdot|2-2\sqrt{2}+\sqrt{2}-x_n| \\ &\le \frac{1}{2}|x_n-\sqrt{2}|(|2-2\sqrt{2}| + |x_n-\sqrt{2}|) \\ &\le \frac{1}{2}|x_n-\sqrt{2}|(2\sqrt{2}-2+2^{-3}) \\ &< \frac{1}{2}|x_n-\sqrt{2}| \qquad (\text{since } 2\sqrt{2}-2+2^{-3} < 1) \\ &< 2^{-(n+1)},\end{aligned}$$

and the induction is complete.

*

4. [1981: 73; 1985: 173] *From the 1979 Moscow Olympiad.*

Karen and Billy play the following game on an infinite checkerboard. They take turns placing markers on the corners of the squares of the board. Karen plays first. After each player's turn (starting with Karen's second turn), the markers placed on the board must lie at the vertices of a convex polygon. The loser is the first player who cannot make such a move. For which player is there a winning strategy?

II. *Solution by Leroy F. Meyers, The Ohio State University.*

Andy Liu's partial solution uses symmetry with respect to a line. By using symmetry with respect to a point, it can be shown that Billy has a winning strategy, provided it is assumed that after n moves ($n \geq 3$) the resulting convex n-gon must always be strictly convex.

Let Karen place her first marker at A. Then Billy places his first marker at B, one of the four corners nearest A. Let Karen place her second marker at C, which cannot be on the line AB. Then Billy places his second marker at D so that ABCD is a parallelogram. Thereafter, all markers must be placed between the lines AB and CD. Whenever Karen places a marker at a corner, say K, Billy places a marker at the point symmetric to K with respect to the centre of the parallelogram. Since at most two markers can go on a line, and there are finitely many lines of corners parallel to AB and CD and between them, there are only finitely many moves that can be made. Since Billy can counter every move of Karen, the game must end with Karen paralyzed, i.e., unable to move. □

Similar games can be played on other boards. For example, the same strategy will work for Billy on a half-plane as well as on an infinite strip, provided that B is chosen so that AB is parallel to the edge(s). For a finite $m \times n$ board, Billy can play symmetrically with respect to the centre of the board, *unless* both m and n are odd and Karen occupies the centre with her first marker, in which case I do not know the outcome. I also do not know the outcome for a quadrant.

III. *Further remark by Andy Liu, University of Alberta.*

It is not difficult to show that Billy can win on all $3 \times n$ and $5 \times n$ boards. This is clear if n is even. Suppose n is odd. Then, as mentioned above, Karen must start at the centre A of the board. Billy plays at B adjacent to A with AB parallel to the longer side of the board. When Karen places her second marker, say at C, Billy plays at D, where ABCD or ABDC is a parallelogram. Play ends here for a $3 \times n$ board. For a $5 \times n$ board, there may be one other grid line between and parallel to AB and CD, on which exactly two other markers can be placed. Hence Billy wins again.

*

6. [1981: 73; 1985: 174] *From the 1979 Moscow Olympiad.*

A scientific conference is attended by k chemists and alchemists, of whom the chemists are in the majority. When asked a question, a chemist will always tell the truth, while an alchemist may tell the truth or lie. A visiting mathematician has the task of finding out which of the k members of the conference are chemists and which are alchemists. He must do this by choosing a member of the conference and asking him: "Which is So-and-So, a chemist or an alchemist?" In particular, he can ask a member: "Which are you, a chemist or an alchemist?" Show that the mathematician can accomplish his investigation by asking

(a) $4k$ questions;

(b) $2k - 2$ questions;

(c) $2k - 3$ questions.

(d) [After the Olympiad, it was announced that the minimum number of questions is no greater that $\lceil (3/2)k \rceil - 1$. Prove it.]

II. *Comment by Fred Galvin, University of Kansas.*

A complete solution, including the lower bound as well, and a history of the problem can be found in P.M. Blecher, "On a logical problem", *Discrete Math.*, 43 (1983) 107-110.

*

6. [1985: 2] *From the 1984 Leningrad Olympiad (proposed by S.E. Rushkin).*

Two students take turns in writing their choice of one of the signs +, -, or × between each pair of consecutive integers in the series

$$1 \quad 2 \quad 3 \quad \ldots \quad 100.$$

Prove that there is a strategy such that the student who wrote the first (and hence the last) sign can ensure that, after the ninety-nine signs have been filled in, the arithmetical result is (a) odd, (b) even.

Solution by K.S. Murray, Brooklyn, N.Y.

The parity of the final result is unaffected if only + and × signs are used. We show, more generally, that the first player can win if the sequence of the first 100 consecutive natural numbers is replaced by an even number of arbitrary alternating odd (O) and even (E) integers:

$$O \quad E \quad O \quad E \quad \ldots \quad O \quad E.$$

(b) We show that the first player can win by using only × signs. Note that when a × sign is placed between two of the numbers, the resulting $O \times E$ or $E \times O$ can be replaced by a new even number E. Let the first move be a × between the first two numbers, resulting in a new sequence

$$E \quad O \quad E \quad O \quad \ldots \quad O \quad E.$$

Thereafter, for every move of the second player (denote by $*$), say

$$\ldots \quad E \quad O * E \quad \ldots \qquad \text{or} \qquad \ldots \quad E * O \quad E \quad \ldots,$$

the first player places a $\times$ on the other side of the O, thus

$$\ldots \quad E \times O * E \quad \ldots \qquad \text{or} \qquad \ldots \quad E * O \times E \quad \ldots,$$

resulting in $\ldots E * E \ldots$. Eventually, we end up with all even numbers. This process can easily be formalized by induction.

(a) The first move is

$$O + E \quad O \quad E \quad \ldots \quad O \quad E.$$

Proceeding thereafter as in (b), the first student can ensure that the term to the right of the + comes out even.

*

7. [1985: 3] *From the 1984 Leningrad Olympiad (proposed by A.S. Merkuriev).*

Prove that the integer a can be represented in the form x^2+2y^2, where x and y are integers, if $3a$ can be represented in the same form.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

By hypothesis $3a = x_0^2+2y_0^2$ for some integers x_0 and y_0. Thus $3 \mid (x_0^2-y_0^2)$, so $3 \mid (x_0-y_0)$ or $3 \mid (x_0+y_0)$. If $3 \mid (x_0-y_0)$, then $x_0 = y_0+3k$ for some integer k. Hence

$$3a = (y_0+3k)^2 + 2y_0^2 = 3\{(y_0+k)^2 + 2k^2\},$$

making $a = (y_0+k)^2 + 2k^2$. If $3 \mid (x_0+y_0)$, then $x_0 = -y_0+3j$ for some integer j. Hence

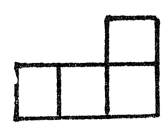
$$3a = (-y_0+3j)^2 + 2y_0^2 = 3\{(y_0-j)^2 + 2j^2\},$$

making $a = (y_0-j)^2 + 2j^2$.

*

8. [1985: 3] *From the 1984 Leningrad Olympiad (proposed by I.V. Itenberg).*

Prove that at least 8 different colours are needed to paint the (infinite) squared plane so that the 4 squares of each L-shaped tetromino (see figure) have 4 different colours.



Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

There are eight possible orientations in the plane for the L-shaped tetromino. Six of these appear in Figure 1, and it is clear that seven different colours (denoted by the numbers 1 to 7) are needed to colour it. All eight orientations appear in Figure 2, and it is clear that seven colours still suffice, provided the colour 1 is used for square x. Seven colours still suffice in Figure 3, provided colour 5 is used for square y; and seven colours still suffice in Figure 4, provided colour 2

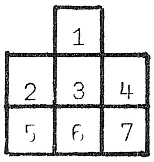


Figure 1

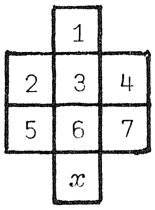


Figure 2

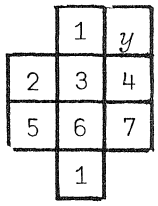


Figure 3

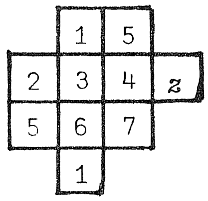


Figure 4

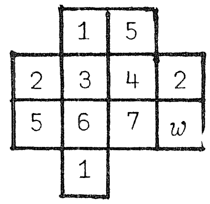


Figure 5

is used for square z. But in Figure 5 a new eighth colour is needed for square w.

To show that eight colours suffice, we can use the module shown in Figure 6, repeat it periodically right and left and up and down, to colour the entire plane.

1	4	5	8
3	6	7	2
5	8	1	4
7	2	3	6

Figure 6

*

9. [1985: 3] *From the 1984 Tournament of Towns Olympiad (proposed by A.B. Pechkovski).*

In a ballroom dance class 17 boys and 17 girls are lined up in parallel rows so that 17 couples are formed. It so happens that the difference in height between the boy and the girl in each couple is not more than 10 cm. Prove that if the boys and the girls were placed in each line in order of decreasing height, then the difference in height in each of the newly formed couples would still be at most 10 cm.

Solution by K. Seymour, Toronto, Ontario.

Let the boys' and the girls' heights in cm be given by the monotonic decreasing sequences $\{b_k\}$ and $\{g_k\}$, respectively, so that

$$b_1 \geq b_2 \geq \ldots \geq b_{17} \quad \text{and} \quad g_1 \geq g_2 \geq \ldots \geq g_{17}.$$

Then the final pairing is

$$\begin{matrix} b_1 & b_2 & \cdots & b_{17} \\ g_1 & g_2 & \cdots & g_{17}. \end{matrix}$$

Suppose that in this final pairing there is an i such that $b_i - g_i > 10$. Then, in the original pairing

$$\begin{matrix} b_1 & b_2 & \cdots & b_{17} \\ g_{\pi(1)} & g_{\pi(2)} & \cdots & g_{\pi(17)}, \end{matrix}$$

where π is a permutation of $\{1,2,\ldots,17\}$, we must have $g_{\pi(i)} > g_i$. Consequently, there is a $j < i$ such that $g_{\pi(j)} < g_i$; so

$$b_j - g_{\pi(j)} \geq b_i - g_i > 10,$$

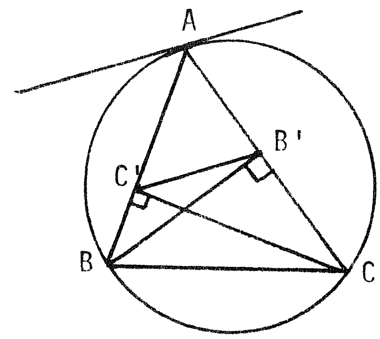
and we have a contradiction. Similarly, the assumption that $g_i - b_i > 10$ for some i in the final pairing leads to a contradiction.

*

10. [1985: 3] *From the 1984 Tournament of Towns Olympiad (proposed by I.F. Sharygin).*

Six altitudes are constructed from the three vertices of the base of a tetrahedron to the opposite sides of the three lateral faces. Prove that all three straight lines joining two base points of the altitudes in each lateral face are parallel to a certain plane.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.



It is a well-known result that, in the figure, B'C' is parallel to the tangent line at A. It then follows for the given problem that the plane tangent to the circumsphere of the given tetrahedron at the vertex opposite the base is parallel to the three lines formed by the base points of the altitudes.

(*Note by M.S.K.* The following theorems are relevant:

The tangential and the orthic triangles of a given triangle are homothetic. [1]

An antiparallel section of a tetrahedron relative to a given face is parallel to the tangent plane to the circumsphere of the tetrahedron at the vertex opposite the face considered. [2])

REFERENCES

1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 98, Theorem 191.

2. __________, *Modern Pure Solid Geometry* (Second Edition), Chelsea, New York, 1964, p. 285, Theorem 759 (p. 247, Theorem 795 in the original 1935 Macmillan edition).

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11. [1985: 3] *From the 1984 Tournament of Towns Olympiad (proposed by A.V. Andjans).*

The two pairs of consecutive natural numbers (8,9) and (288,289) have the following property: in each pair, each number contains each of its prime factors to a power not less than 2.

(a) Find one more pair of consecutive numbers with that property.

(b) Prove that there are infinitely many such pairs.

Solution by Curtis Cooper, Central Missouri State University.

(a) Another such pair is $(675, 676)$.

(b) Since $4x(x+1) + 1 = (2x+1)^2$, it follows that if $(x, x+1)$ is a pair with the desired property, then so is $(4x(x+1), (2x+1)^2)$. Hence there are infinitely many such pairs.

*

12. [1985: 3] *From the 1984 Tournament of Towns Olympiad (proposed by V.G. Il'ichev).*

An infinite (in both directions) sequence of rooms is situated on one side of an infinite hallway. The rooms are numbered by consecutive integers and each contains a grand piano. A finite number of pianists live in these rooms. (There may be more than one of them in some of the rooms.) Every day some two pianists living in adjacent rooms (the kth and $(k+1)$st) decide that they interfere with each other's practice, and they move to the $(k-1)$st and $(k+2)$nd rooms, respectively. Prove that these moves will cease after a finite number of days.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

Without loss of generality, we may make the assumptions more specific in the following way: There are n pianists $P_1, P_2, \ldots, P_n$ inhabiting a set of rooms numbered consecutively, say from left to right, by the integers (positive, negative, and zero). The pianists are ordered from left to right according to their indices, that is, P_1 is leftmost, then P_2, etc., and this ordering is maintained inside a room if it has several inhabitants. Each day, as long as there are inhabitants in adjacent rooms, a *basic operation* takes place in which, for some pair of adjacent rooms, k and $k+1$, the leftmost inhabitant of room k moves to room $k-1$ and the rightmost inhabitant of room $k+1$ moves to room $k+2$. Thus a basic operation preserves the ordering of the pianists.

If P_i and P_{i+1} have more than two rooms between them, then a basic operation affecting either one of them can only bring them closer to each other. If at a certain stage (maybe from the beginning) they have at most two rooms between them (they may even be in the same room), then from then on they will never have more than two rooms between them. It follows that if at the start the settlement has a *breadth* (the number of rooms from that of P_1 to that of P_n inclusively) of r rooms, then its breadth will never exceed $r+3n$ rooms.

Now assume that each pianist has a unit mass and consider the moment of inertia of all the masses with respect to the room numbered 0. Since the settlement has bounded breadth, the moment of inertia of all the masses with respect to room 0 must be bounded. However, note that each basic operation adds 4 to this moment of inertia:

$$(k-1)^2 + (k+2)^2 - (k+1)^2 - k^2 = 4.$$

Therefore only finitely many basic operations are possible.

*

13. [1985: 3] *From the 1984 Tournament of Towns Olympiad (proposed by A.V. Zelevinsky).*

Let $p(n)$ be the number of partitions of the natural number n into natural summands. The *diversity* of a partition is by definition the number of different summands in it. Denote by $q(n)$ the sum of the diversities of all the $p(n)$ partitions of n.

(For example, $p(4) = 5$, the five distinct partitions of 4 being

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1;$$

and $q(4) = 1+2+1+2+1 = 7$.)

Prove that, for all natural numbers n,

(a) $q(n) = 1 + p(1) + p(2) + p(3) + \ldots + p(n-1)$,

(b) $q(n) \le \sqrt{2n}\cdot p(n)$.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

(a) Our solution is by induction on n. It is obvious that $q(1) = 1$ and $q(2) = 2 = 1 + p(1)$. The induction proof will be complete when we have shown that $q(n+1) = q(n) + p(n)$.

Let S be the set of ordered pairs (x,y), where x is a partition of n and y is a partition of $n+1$ obtained by adding an extra term 1 to x or by increasing by 1 one of the summands of x. It is clear that an x with diversity k appears $k+1$ times in S while a y with diversity l appears l times in S. It follows that

$$q(n) + p(n) = |S| = q(n+1).$$

(b) The diversity of n is greatest when the terms of the partition form an arithmetic progression with common difference 1 and each type of term occurs just once, except possibly one them. Thus

$$1 + 2 + \ldots + t \le n, \qquad \text{that is,} \qquad t^2 + t \le 2n.$$

Thus $t \le \sqrt{2n}$ and $q(n) < \sqrt{2n}\cdot p(n)$.

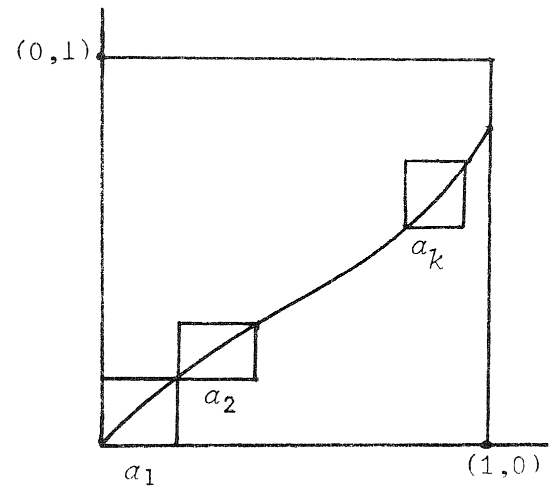
*

14. [1985: 4] *From the 1984 Tournament of Towns Olympiad (proposed by A.V. Andjans).*

Prove that, for any natural number n, the graph of any increasing function $f: [0,1] \to [0,1]$ can be covered by n rectangles each of area $1/n^2$ whose sides are parallel to the coordinate axes.

Solution by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.

We can assume without loss of generality that $f(0) = 0$. We first choose an $a_1 \in (0,1]$ such that the rectangle with sides parallel to the coordinate axes and with opposite vertices $(0,0)$ and $(a_1,f(a_1))$ has area $1/n^2$ (see figure). Then we choose an a_2 with $a_1 < a_1+a_2 \le 1$ such that a similarly oriented rectangle with opposite vertices $(a_1,f(a_1))$ and $(a_1+a_2,f(a_1+a_2))$ has area $1/n^2$, and so on as far as possible. We now ask: Can this process be repeated until we have n rectangles with the point



$$(a_1+a_2+\ldots+a_n,\ f(a_1+a_2+\ldots+a_n))$$

belonging to the unit square? If so, this would imply that

$$a_1 + a_2 + \ldots + a_n \le 1 \qquad \text{and} \qquad \frac{1}{n^2}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \le 1.$$

Since by the Cauchy inequality

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \ge n^2,$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$, this could only occur for the function $f(x) = x$. Otherwise the process can be repeated at most k times, where $k < n$. Then the part of the graph remaining uncovered can be covered by a rectangle of area less than $1/n^2$, so n rectangles of area $1/n^2$ would certainly suffice to cover the entire graph.

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15. [1985: 4] *From the 1984 Moscow Olympiad.*

Let $X = (x_1,x_2,\ldots,x_n)$ be a sequence of n nonnegative numbers whose sum is 1.

(a) Prove that $x_1x_2 + x_2x_3 + \ldots + x_nx_1 \le 1/4$.

(b) Prove that there exists a permutation $Y = (y_1,y_2,\ldots y_n)$ of X such that

$$y_1y_2 + y_2y_3 + \ldots + y_ny_1 \le 1/n.$$

I. *Solution to part (a) by M.S.K.*

The condition $n \geq 4$ was left out of the proposal. Note that for $n = 3$ we can have $x_1 = x_2 = x_3 = 1/3$, giving

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{1}{3} > \frac{1}{4}.$$

We will prove by induction on n that

$$(x_1 + x_2 + \ldots + x_n)^2 \geq 4(x_1x_2 + x_2x_3 + \ldots + x_nx_1), \qquad (1)$$

where $x_i \geq 0$ and $n \geq 4$, and the desired result will follow from $\Sigma x_i = 1$. For $n = 4$, the inequality is equivalent to $(x_1-x_2+x_3-x_4)^2 \geq 0$ with equality if and only if $x_1+x_3 = x_2+x_4$. We assume that (1) holds for some $n = k \geq 4$. Our induction proof will be complete if we can show that this assumption implies that

$$(x_1 + x_2 + \ldots + x_k + x_{k+1})^2 \geq 4(x_1x_2 + x_2x_3 + \ldots + x_kx_{k+1} + x_{k+1}x_1). \qquad (2)$$

Since the two sides of this last inequality are cyclic, we may assume without loss of generality that $x_{k+1} \leq x_i$ for $i = 1,2,\ldots,k$. Then, by the induction hypothesis,

$$(x_1 + x_2 + \ldots + x_{k-1} + (x_k+x_{k+1}))^2$$
$$\geq 4(x_1x_2 + x_2x_3 + \ldots + x_{k-1}(x_k+x_{k+1}) + (x_k+x_{k+1})x_1).$$

Since the right member of this inequality equals the right member of (2) plus

$$4(x_{k-1}x_{k+1} + x_k(x_1-x_{k+1})) \geq 0,$$

then (2) holds and it follows that (1) is valid for $n = k+1$, and thus for all $n \geq 4$.

There is equality for $n \geq 5$ if and only if all the x_i's are zero except possibly for three cyclically consecutive ones, say x_{i-1}, x_i, x_{i+1}, which must then satisfy $x_i = x_{i-1} + x_{i+1}$. □

For related results for both parts (a) and (b), see Problem 1059, "Cyclic Extrema", *Mathematics Magazine*, 53 (1980) 115-116; and Problem 516, *Pi Mu Epsilon Journal*, 7 (1983) 551.

II. *Solution to part (b) by Aage Bondesen, Royal Danish School of Educational Studies, Copenhagen, Denmark.*

For every permutation $Y = (y_1,y_2,\ldots,y_n)$ of X, let

$$S_Y = y_1y_2 + y_2y_3 + \ldots + y_ny_1.$$

Also, let $S = \Sigma S_Y$, where the sum is over all the $n!$ permutations of X. Now, for $i,j \in \{1,2,\ldots,n\}$ with $i \neq j$, the number of permutations of X having x_i immediately before x_j (cyclically) is $n(n-2)!$. Therefore

$$S = n(n-2)!\sum_{\substack{i,j=1\\ i\neq j}}^{n} x_i x_j = n(n-2)!(1 - \sum_{k=1}^{n} x_k^2) \le n(n-2)!(1 - \frac{1}{n}\{\sum_{k=1}^{n} x_k\}^2),$$

or

$$S \le n(n-2)!(1 - \frac{1}{n}) = (n-1)!.$$

Hence the mean value $\bar{S}_Y$ of S_Y satisfies

$$\bar{S}_Y = \frac{S}{n!} \le \frac{1}{n},$$

and so there must be a permutation Y of X such that $S_Y \le 1/n$.

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16. [1985: 4] *From the 1984 Moscow Olympiad.*

Prove that the area of the triangle which is the intersection of a cube of edge 1 with an arbitrary plane tangent to the sphere inscribed in the cube does not exceed 1/2.

Solution by M.S.K.

We introduce a rectangular coordinate system such that four of the vertices of the cube are at (0,0,0), (1,0,0), (0,1,0), (0,0,1), and its center is at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Let the equation of the tangent plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \qquad \text{where} \qquad 0 \le a,b,c \le 1.$$

Since this plane must lie at distance 1/2 from the center of the insphere, we have

$$\frac{\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{2},$$

from which

$$(bc + ca + ab - 2abc)^2 = b^2c^2 + c^2a^2 + a^2b^2. \tag{1}$$

The tangent plane cuts off a triangle of area A, say, and a rectangular tetrahedron of volume V, say, from the cube. Then

$$V = \frac{abc}{6} = \frac{A}{3}\cdot(\text{distance of origin to tangent plane})$$

$$= \frac{A}{3}\cdot\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}},$$

and so

$$A = \frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}.$$

To show that $A \leq 1/2$ is equivalent to showing that

$$L \equiv bc + ca + ab - 2abc \leq 1 \quad \text{for} \quad 0 \leq a,b,c \leq 1.$$

Since L is linear in each of a,b,c, it takes on its maximum at the endpoints, i.e., at $a,b,c = 0$ or 1. Thus L is a maximum for $(a,b,c) = (1,1,1)$, or else for $(1,1,0)$ and permutations thereof. However, only $(1,1,0)$ satisfies (1), and it yields a right triangle with unit legs. But we cannot achieve this result, since this plane corresponds to a face of the cube and does not cut off a triangle. However, we can get the area of the triangle cut off from the cube arbitrarily close to $1/2$ by letting $a = b = 1-\varepsilon$ and $c = \varepsilon$, where ε is an arbitrarily small positive number. □

As an extension of the problem, we find the minimum area cut off. This is more bothersome. We will show that, as expected, it occurs for the symmetric configuration for which

$$a = b = c = \frac{3-\sqrt{3}}{2}, \quad \text{which gives} \quad A = \frac{3\sqrt{3}}{4}(2 - \sqrt{3}) \approx 0.348.$$

Our problem is to minimize

$$bc + ca + ab - 2abc$$

subject to $0 < a,b,c \leq 1$ and to the constraint (1), which is equivalent to

$$a + b + c - 2(bc+ca+ab) + 2abc = 0.$$

Using Lagrangian multipliers, we find that the critical points must satisfy

$$\left(\frac{1}{b} - \frac{1}{c}\right)\left(1 - \frac{1}{2a}\right) = 0, \quad \left(\frac{1}{c} - \frac{1}{a}\right)\left(1 - \frac{1}{2b}\right) = 0, \quad \left(\frac{1}{a} - \frac{1}{b}\right)\left(1 - \frac{1}{2c}\right) = 0.$$

Therefore all the critical points are given by $a = b = c = (3-\sqrt{3})/2$, and by $(a,b,c) = (\frac{1}{2},\frac{1}{2},1)$ and permutations thereof. Finally, the symmetric configuration corresponding to $a = b = c$ turns out to produce the least area.

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17. [1985: 4] *From the 1984 All-Union Olympiad (proposed by A.A. Fomin).*

(a) The product of n integers is equal to n, and their sum is 0. Prove that n is divisible by 4.

(b) Prove that, for every natural number n divisible by 4, there exist n integers with product n and sum 0.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

(a) Let $x_1,x_2,\ldots,x_n$ be the given integers, and let P be their product and S their sum.

Suppose n is odd. Then all the x_i are odd since $P = n$. Hence S must be odd, it being the sum of an odd number of odd integers. But this contradicts the hypothesis that $S = 0$. Now suppose n is of the form $4k+2$. Since $P = n$, exactly one of the x_i is even and none of them is divisible by 4. We assume without loss of generality that x_1 is the even factor of P. Then $x_1/2$ is odd, and so is

$$S = \frac{x_1}{2} + \frac{x_1}{2} + x_2 + \ldots + x_n,$$

since it is the sum of an odd numbers of odd integers. Again, this contradicts the fact that $S = 0$. Therefore n is divisible by 4.

(b) Suppose n is divisible by 4. If $n = 4(2k-1)$, then for the following n integers

$$-\frac{n}{2}, \quad 2, \quad (\frac{n}{2}-2)\ 1\text{'s}, \quad (\frac{n}{4})\ 1\text{'s}, \quad (\frac{n}{4})\ (-1)\text{'s},$$

we have $P = n$ and $S = 0$. If $n = 4(2k)$, then for the following n integers

$$\frac{n}{2}, \quad 2, \quad (\frac{n}{2}+2)\ (-1)\text{'s}, \quad (\frac{n}{4}-2)\ 1\text{'s}, \quad (\frac{n}{4}-2)\ (-1)\text{'s},$$

we have $P = n$ and $S = 0$.

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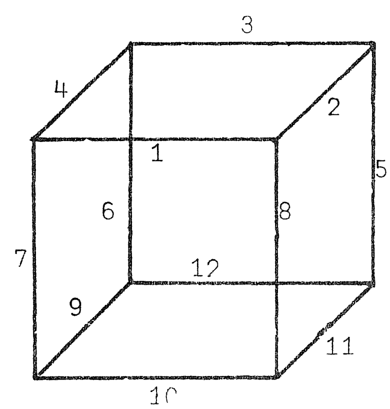
18. [1985: 4] *From the 1984 All-Union Olympiad (proposed by N. Agakharov).*

Two boys paint the twelve edges of a white cube in two colours: red and green. The first chooses any three edges and paints them red; the second chooses some other three edges and paints them green; then, once more, the first paints three edges red, and, finally, the second paints the last three edges green. Can the first boy make all four edges of a face red?

Solution by Jordi Dou, Barcelona, Spain.

Let the edges of the cube be numbered as in the figure. If the second boy chooses well, then the first boy cannot make all four edges of a face red, because he cannot prevent the second boy from choosing one of the triads



(1,5,9), (2,6,10), (3,7,11), or (4,8,12).

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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A MESSAGE FROM THE MANAGING EDITOR

Members of the Canadian Mathematical Society can initiate or renew a subscription to *Crux Mathematicorum* as part of the annual renewal of their membership to the C.M.S. They will benefit from a special subscription price of \$22.50 (\$25 for nonmembers). Communications from all subscribers intended for the editor and for the managing editor should be sent to their respective addresses as given on the front page of this issue.

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1081. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

For a given integer $b > 1$, evalutate

$$\int_0^\infty \left\lfloor \log_b \left\lfloor \frac{\lceil x \rceil}{x} \right\rfloor \right\rfloor dx.$$

(The *floor* of x, denoted by $\lfloor x \rfloor$, is the largest integer $\le x$; and the *ceiling* of x, denoted by $\lceil x \rceil$, is the smallest integer $\ge x$.)

1082. *Proposed by O. Bottema, Delft, The Netherlands.*

The midpoints of the edges A_3A_4, A_4A_1, A_1A_2 of a tetrahedron $A_1A_2A_3A_4$ are B_2, B_3, B_4, respectively; l_1 is the line through A_1 parallel to A_2A_3; and l_2, l_3, l_4 are the lines A_2B_2, A_3B_3, A_4B_4, respectively.

(a) Show that the four lines l_i have two conjugate imaginary transversals t and t'.

(b) If S_i is the intersection of t and l_i, and S_i' that of t' and l_i ($i = 1,2,3,4$), show that S_i and S_i' are equianharmonic quadruples of points.

1083.* *Proposed by Jack Garfunkel, Flushing, N.Y.*

Consider the double inequality

$$\frac{2}{\sqrt{3}}\Sigma\sin A \le \Sigma\cos\frac{B-C}{2} \le \frac{2}{\sqrt{3}}\Sigma\cos\frac{A}{2},$$

where the sums are cyclic over the angles A,B,C of a triangle. The left inequality has already been established in this journal (Problem 613 [1982: 55, 67, 138]). Prove or disprove the right inequality.

1084. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that every symmetrical fourth-order magic square can be written as

$F+w+y$	$F-w+z$	$F-w-z$	$F+w-y$
$F-x-y$	$F+x-z$	$F+x+z$	$F-x+y$
$F+x-y$	$F-x-z$	$F-x+z$	$F+x+y$
$F-w+y$	$F+w+z$	$F+w-z$	$F-w-y$

1085. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $\sigma_n = A_0A_1\ldots A_n$ be a regular n-simplex in R^n, and let π_i be the hyperplane containing the face $\sigma_{n-1} = A_0A_1\ldots A_{i-1}A_{i+1}\ldots A_n$. If $B_i \in \pi_i$ for $i = 0,1,\ldots,n$, show that

$$\sum_{0\le i<j\le n} |\vec{B_iB_j}| \ge \frac{n+1}{2}\cdot e,$$

where e is the edge length of σ_n.

1086. *Proposed by M.S. Klamkin, University of Alberta.*

The medians of an n-dimensional simplex $A_0A_1\ldots A_n$ in R^n intersect at the centroid G and are extended to meet the circumsphere again in the points $B_0,B_1,\ldots,B_n$, respectively.

(a) Prove that

$$A_0G + A_1G + \ldots + A_nG \le B_0G + B_1G + \ldots + B_nG.$$

(b)* Determine all other points P such that

$$A_0P + A_1P + \ldots + A_nP \le B_0P + B_1P + \ldots + B_nP.$$

1087. *Proposed by Robert Downes, student, Moravian College, Bethlehem, Pennsylvania.*

Let a,b,c,d be four positive numbers.

(a) There exists a regular tetrahedron ABCD and a point P in space such that PA = a, PB = b, PC = c, and PD = d if and only if a,b,c,d satisfy what condition?

(b) This condition being satisfied, calculate the edge length of the regular tetrahedron ABCD.

(For the corresponding problem in a plane, see Problem 39 [1975: 64; 1976: 7].)

1088*. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*

If R,r,s are the circumradius, inradius, and semiperimeter, respectively, of a triangle with largest angle A, prove or disprove that

$$s \gtreqless 2R + r \quad \text{according as} \quad A \lesseqgtr 90^\circ.$$

1089. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find the range of the function f: $R \to R$ defined by

$$f(\theta) = \sum_{k=1}^{\infty} 3^{-k} \cos k\theta, \qquad \theta \in R.$$

1090. *Proposed by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.*

Let Γ be a circle with center O, and A a fixed point distinct from O in the plane of Γ. If P is a variable point on Γ and AP meets Γ again in Q, find the locus of the circumcenter of triangle POQ as P ranges over Γ.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

682. [1981: 274; 1982: 287; 1983: 23] A comment was received from BENO ARBEL, Tel Aviv University, Israel.

* * *

962. [1984: 216] *Proposed by S.C. Chan, Singapore.*

Two gamblers play a game with 5 coins, taking alternate turns. At each turn the player calls either "Heads" or "Tails" and then throws the 5 coins. If fewer than 3 coins fall as he has called, he pays one dollar into the pool; if 3 or 4 coins fall as he has called, he does not pay anything into the pool; if all 5 coins fall as he has called, he receives the contents of the pool.

Determine the probabilities

(a) that the pool is paid out after exactly n turns;

(b) that after n turns since the pool was last paid out if contains r dollars;

(c) that when the pool is paid out it contains r dollars.

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

At each throw the probability of less than three correct calls is 1/2, the probability of three or four is 15/32, and the probability of five is 1/32.

The state of a game can most easily be represented by a unit lattice on which the abscissa is the number of throws, n, since the last payout, and the ordinate is the current size, r, of the pool. Thus the lattice point (n,r) represents the state of a game when n throws have been completed and there are r dollars in the pool. Let the probability of arriving at the state (n,r) be $P(n,r)$. The state (n,r) can be attained from state $(n-1,r-1)$ with probability 1/2 and from state $(n-1,r)$ with probability 15/32.

(b) It is easily seen that the state (n,r) can be reached along any one path with probability $(1/2)^r(15/32)^{n-r}$, and there are exactly $\binom{n}{r}$ distinct allowable paths from $(0,0)$ to (n,r). Thus

$$P(n,r) = \binom{n}{r}\left(\frac{1}{2}\right)^r\left(\frac{15}{32}\right)^{n-r}.$$

(a) The probability of a payout after n throws with r dollars in the pool is $(1/32)P(n-1,r)$, and thus the probability of a payout after n throws is

$$\begin{aligned} P_n &= \frac{1}{32}\sum_{r=0}^{n-1} P(n-1,r) = \frac{1}{32}\sum_{r=0}^{n-1}\binom{n-1}{r}\left(\frac{1}{2}\right)^r\left(\frac{15}{32}\right)^{n-1-r} \\ &= \frac{1}{32}\left(\frac{1}{2} + \frac{15}{32}\right)^{n-1} = \frac{1}{32}\left(\frac{31}{32}\right)^{n-1} \\ &= \frac{1}{31}\left(\frac{31}{32}\right)^n. \end{aligned}$$

Note that

$$\sum_{n=1}^{\infty} P_n = \frac{1}{31}\sum_{n=1}^{\infty}\left(\frac{31}{32}\right)^n = 1,$$

as one would expect.

(c) The probability of a payout with r dollars in the pool is

$$\begin{aligned} P_r &= \frac{1}{32}\sum_{n=r}^{\infty} P(n,r) = \frac{1}{32}\sum_{n=r}^{\infty}\binom{n}{r}\left(\frac{1}{2}\right)^r\left(\frac{15}{32}\right)^{n-r} = \frac{1}{32}\left(\frac{1}{2}\right)^r\left(1 - \frac{15}{32}\right)^{-r-1} \\ &= \frac{1}{17}\left(\frac{16}{17}\right)^r. \end{aligned}$$

Note that in the above derivation we have used the identity [1]

$$\sum_{n=m}^{\infty}\binom{n}{m}x^{n-m} = (1 - x)^{-m-1}, \qquad |x| < 1.$$

Note also that

$$\sum_{r=0}^{\infty} P_r = \frac{1}{17}\sum_{r=0}^{\infty}\left(\frac{16}{17}\right)^r = 1,$$

as one would expect.

Also solved by the proposer.

Editor's comment.

The proposer wrote that part (b) seemed to cause difficulty to many people he has asked, and indeed his own answer to part (b) differs from the one found above. This probably results from a different interpretation of the problem. As we interpret part (b), the answer should reduce to 1/2 when $n = r = 1$. This agrees with the value found above but not with the proposer's.

REFERENCE

1. Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964 (or Dover, N.Y., 1965), p.822.

* * *

963. [1984: 216] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find consecutive squares that can be split into two sets with equal sums.

Solution by Susie Lanier and David R. Stone, Georgia Southern College, Statesboro, Georgia.

Whenever we find it convenient, we will denote $a^2+b^2+\ldots+t^2$ by $[a,b,\ldots,t]$. The equation $[a,a+1] = [a+2]$ has only the (nonnegative integer) solution $a = 3$, so

$$3^2 + 4^2 = 5^2$$

is the only solution for three consecutive squares. There are no solutions for four consecutive squares because neither of the equations $[a,a+1,a+2] = [a+3]$ or $[a,a+3] = [a+1,a+2]$ has a nonnegative integer solution. Similarly, we find that for five consecutive squares the solutions are

$$2^2 + 4^2 + 5^2 = 3^2 + 6^2,$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2;$$

that there are no solutions for six consecutive squares; that the solutions for seven consecutive squares are

$$1^2 + 2^2 + 4^2 + 7^2 = 3^2 + 5^2 + 6^2,$$
$$5^2 + 7^2 + 8^2 + 10^2 = 6^2 + 9^2 + 11^2,$$
$$13^2 + 14^2 + 16^2 + 17^2 = 15^2 + 18^2 + 19^2,$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2;$$

and that *some* of the solutions for eight consecutive squares are

$$0^2 + 3^2 + 5^2 + 6^2 = 1^2 + 2^2 + 4^2 + 7^2,$$
$$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2,$$
$$35^2 + 38^2 + 40^2 + 41^2 = 36^2 + 37^2 + 39^2 + 42^2,$$
$$150^2 + 153^2 + 155^2 + 156^2 = 151^2 + 152^2 + 154^2 + 157^2.$$

Now for some more general results. The result mentioned above for six consecutive squares is a special case of the following theorem.

THEOREM 1. For all $k \geq 1$, $4k+2$ consecutive squares cannot be split into two subsets having equal sums.

Proof. Suppose, on the contrary, that there are $n = 4k+2$ consecutive squares that can be split into two subsets having equal sums. Then there is a permutation $(i_0, i_1, \ldots, i_{n-1})$ of $(0,1,\ldots,n-1)$ and an integer $m < n$ such that, for some $a \geq 0$,

$$[a+i_0,\ a+i_1,\ \ldots,\ a+i_{m-1}] = [a+i_m,\ a+i_{m+1},\ \ldots,\ a+i_{n-1}].$$

This equation can be rewritten as

$$\sum_{u=0}^{n-1} e_u(a + i_u)^2 = 0,$$

where each e_u is 1 or -1, or as

$$\sum_{u=0}^{n-1} e_u a^2 + 2a\sum_{u=0}^{n-1} e_u i_u + \sum_{u=0}^{n-1} e_u i_u^2 = 0. \tag{1}$$

Now (all congruences are modulo 2) each $e_u \equiv 1$, so

$$\sum_{u=0}^{n-1} e_u a^2 \equiv na^2 = (4k+2)a^2 \equiv 0$$

and

$$\sum_{u=0}^{n-1} e_u i_u^2 \equiv (0^2+1^2) + (2^2+3^2) + \ldots + \{(n-2)^2+(n-1)^2\} \equiv \frac{n}{2} = 2k+1 \equiv 1.$$

Now (1) gives $0 + 0 + 1 \equiv 0$, and we have a contradiction. □

We now consider *balanced solutions* for sets of n consecutive squares, by which we mean a splitting of the set into two *halves* having equal sums. In view of Theorem 1 and the statement made earlier about four consecutive squares, a balanced solution may exist only if $n = 4k$, where $k \geq 2$.

THEOREM 2. For each $k \geq 2$, *every* set of $4k$ consecutive squares has at least one balanced solution.

Proof. Any set of 8 or 12 consecutive squares has a balanced solution, for it is easily verified that, for any $a \geq 0$,

$$[a,\ a+3,\ a+5,\ a+6] = [a+1,\ a+2,\ a+4,\ a+7]$$

and

$$[a,\ a+2,\ a+6,\ a+7,\ a+8,\ a+10] = [a+1,\ a+3,\ a+4,\ a+5,\ a+9,\ a+11].$$

The desired result now follows by decomposing the $4k$ consecutive squares into subsets of 8 or 12 consecutive squares and then combining the balanced solutions of these subsets. Note that the splitting process can usually be done in several ways, corresponding to the ways in which $4k$ can be split into 8's and 12's; and there may also be other ways not using the 8 or 12 patterns. □

We now consider *m-unbalanced solutions* for sets of n consecutive squares, by which we mean splitting the n squares into two subsets whose cardinalities differ by m, and having equal sums.

1-unbalanced solutions. Here a set of $n = 2k+1$ consecutive squares is to be split into two subsets of $k+1$ and k squares having equal sums. Let a^2 be the smallest square in the set. We may assume that $a > 0$, since if $a = 0$ we can drop a^2 and have left $2k$ consecutive squares with either no solution or infinitely many balanced solutions. Let

$$L = [a,\ a+1,\ \dots,\ a+k] \quad \text{and} \quad U = [a+k+1,\ a+k+2,\ \dots,\ a+2k].$$

LEMMA. We have

$$L \gtreqless U \quad \text{according as} \quad a \gtreqless k(2k+1).$$

Proof. With the familiar $f(t) = 1^2+2^2+\dots+t^2 = t(t+1)(2t+1)/6$, we have

$$L = f(a+k) - f(a-1) \quad \text{and} \quad U = f(a+2k) - f(a+k),$$

from which

$$\begin{aligned} L - U &= 2f(a+k) - f(a-1) - f(a+2k) \\ &= \text{(much tedious algebra)} \\ &= (a+k)\{a - k(2k+1)\}, \end{aligned}$$

and the desired result follows. □

THEOREM 3. For any given $k \ge 1$, there is at least one and at most finitely many 1-unbalanced solutions for sets of $n = 2k+1$ strictly positive consecutive squares. Moreover, the largest 1-unbalanced solution is the one where $a = k(2k+1)$.

Proof. Let

$$L' = [a+i_0,\ a+i_1,\ \dots,\ a+i_k] \quad \text{and} \quad U' = [a+i_{k+1},\ a+i_{k+2},\ \dots,\ a+i_{2k}],$$

where $(i_0, i_1, \dots, i_{2k})$ is a permutation of $(0,1,\dots,2k)$. It is a consequence of the lemma that, whenever $a > k(2k+1)$,

$$L' \ge L > U \ge U'.$$

This shows that a 1-unbalanced solution $L' = U'$ is possible only if $a \le k(2k+1)$, that is, in at most finitely many cases. Moreover, for the largest value $a = k(2k+1)$, we obtain a 1-unbalanced solution by taking the identity permutation $i_u = u$ for $u = 0,1,\dots,2k$, since then we have $L' = L = U = U'$. □

2-unbalanced solutions. Here a set of $n = 4k$ consecutive squares, with $k \ge 2$, is to be split into two subsets of $2k+1$ and $2k-1$ squares having equal sums. In this case we may allow the possibility that the smallest square in the set is $a^2 = 0$.

THEOREM 4. For any given $k \geq 2$, there is at least one and at most finitely many 2-unbalanced solutions for sets of $n = 4k$ consecutive squares.

Proof. At least one 2-unbalanced solution is obtained from a balanced solution of the set $\{0^2, 1^2, 2^2, \ldots, (4k-1)^2\}$ by transposing the 0^2 to the other side of the equation. To show that at most finitely many 2-unbalanced solutions exist, we proceed as in the proofs of the lemma and Theorem 3, and we find that a 2-unbalanced solution exists only if the smallest square a^2 satisfies

$$a \leq k(2k - 2 + \sqrt{4k^2-2}). \quad \square$$

Corollary. A set of $4k$ consecutive squares, where $k \geq 2$, has only balanced solutions if the smallest square a^2 satisfies

$$a > k(2k - 2 + \sqrt{4k^2-2}). \quad \square$$

3-unbalanced solutions. Here a set of $n = 2k+1$ consecutive squares is to be split into two subsets of $k+2$ and $k-1$ squares having equal sums. As noted at the beginning of the solution, there are only 1-unbalanced solutions if $k = 1$, 2, or 3, so we assume that $k \geq 4$.

THEOREM 5. For any given $k \geq 4$, there are at most finitely many 3-unbalanced solutions for sets of $n = 2k+1$ consecutive squares.

Proof. Proceeding as in the proofs of the lemma and Theorem 3, we find that a 3-unbalanced solution exists only if the smallest square a^2 satisfies

$$a \leq \frac{k^2 - 2k - 2 + \sqrt{k^4+2k^3-3k^2-4k-2}}{3}. \quad \square$$

It can be verified from Theorems 3 and 5 that, for odd m, as expected, the upper bounds for the smallest square a^2 in an m-unbalanced solution decrease as m increases, but the algebra involved in finding this upper bound is prohibitive if $m > 3$. But there *are* m-unbalanced solutions for at least some odd $m \geq 3$. Here, for example, is a 5-unbalanced solution:

$$4^2 + 5^2 + 6^2 + 7^2 + 9^2 + 10^2 + 11^2 + 12^2 + 13^2 = 8^2 + 14^2 + 15^2 + 16^2.$$

But in such cases the smallest square must be relatively small. It may then happen that there are several solutions for the same set of $n = 2k+1$ consecutive squares. For example, the 13-element set $\{2^2, 3^2, \ldots, 14^2\}$ has thirteen solutions, eight of which are 1-unbalanced and five 3-unbalanced:

$$2^2 + 3^2 + 4^2 + 7^2 + 8^2 + 13^2 + 14^2 = 5^2 + 6^2 + 9^2 + 10^2 + 11^2 + 12^2,$$
$$2^2 + 3^2 + 6^2 + 8^2 + 9^2 + 12^2 + 13^2 = 4^2 + 5^2 + 7^2 + 10^2 + 11^2 + 14^2,$$
$$4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 11^2 + 14^2 = 2^2 + 3^2 + 9^2 + 10^2 + 12^2 + 13^2,$$
$$2^2 + 4^2 + 5^2 + 7^2 + 10^2 + 12^2 + 13^2 = 3^2 + 6^2 + 8^2 + 9^2 + 11^2 + 14^2,$$

$$2^2 + 4^2 + 5^2 + 8^2 + 9^2 + 11^2 + 14^2 = 3^2 + 6^2 + 7^2 + 10^2 + 12^2 + 13^2,$$
$$2^2 + 5^2 + 7^2 + 8^2 + 10^2 + 11^2 + 12^2 = 3^2 + 4^2 + 6^2 + 9^2 + 13^2 + 14^2,$$
$$3^2 + 4^2 + 6^2 + 9^2 + 10^2 + 11^2 + 12^2 = 2^2 + 5^2 + 7^2 + 8^2 + 13^2 + 14^2,$$
$$3^2 + 4^2 + 5^2 + 6^2 + 9^2 + 12^2 + 14^2 = 2^2 + 7^2 + 8^2 + 10^2 + 11^2 + 13^2,$$
$$2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 10^2 + 11^2 + 14^2 = 7^2 + 8^2 + 9^2 + 12^2 + 13^2,$$
$$2^2 + 3^2 + 4^2 + 5^2 + 7^2 + 8^2 + 12^2 + 14^2 = 6^2 + 9^2 + 10^2 + 11^2 + 13^2,$$
$$2^2 + 3^2 + 4^2 + 7^2 + 8^2 + 10^2 + 11^2 + 12^2 = 5^2 + 6^2 + 9^2 + 13^2 + 14^2,$$
$$2^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 12^2 + 13^2 = 3^2 + 9^2 + 10^2 + 11^2 + 14^2,$$
$$2^2 + 4^2 + 5^2 + 6^2 + 7^2 + 9^2 + 10^2 + 14^2 = 3^2 + 8^2 + 11^2 + 12^2 + 13^2.$$

It will be found that the sum of the squares on both sides of each equation is 507, so 507 can be expressed as the sum of distinct squares in at least 26 different ways.

Also solved by M.S. KLAMKIN, University of Alberta; KENNETH M. WILKE, Topeka, Kansas (two solutions); and the proposer. In addition, one worthless "solution" was received.

Editor's comment.

For special, related, or more general results, Klamkin referred to Dickson [1] and Wilke mentioned Vidger [2].

REFERENCES

1. Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Vol. II, Ch. XXIV.

2. J.S. Vidger, "Consecutive integers having equal sums of squares", *Mathematics Magazine*, 38 (1965) 35-42.

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964*. [1984: 217] *Proposed by R.B. Killgrove, Alhambra, California.*

Let T be the image of the Euler ϕ-function, that is,

$$T = \{\phi(n): n = 1,2,3,\ldots\}.$$

Prove or disprove that T is a *Dirichlet set*, as defined in the proposer's article "Elementary Dirichlet Sets" [1984: 206-209, esp. p. 206 and last paragraph p. 209].

Editor's comment.

No solution was received for this problem, which therefore remains open.

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965. [1984: 217] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2A_3$ be a nondegenerate triangle with sides $A_2A_3 = a_1$, $A_3A_1 = a_2$,

$A_1A_2 = a_3$, and let $PA_i = x_i$ $(i = 1,2,3)$, where P is any point in space. Prove that

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} \geq \sqrt{3}, \tag{1}$$

and determine when equality occurs.

Solution by Wong Ngai-ying, Hong Kong.

Let Q be the orthogonal projection of P onto the plane of the triangle, and let $QA_i = y_i$, $i = 1,2,3$. The desired result (1) will follow from

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} \geq \frac{y_1}{a_1} + \frac{y_2}{a_2} + \frac{y_3}{a_3} \tag{2}$$

and

$$\frac{y_1}{a_1} + \frac{y_2}{a_2} + \frac{y_3}{a_3} \geq \sqrt{3}. \tag{3}$$

Now (2) certainly holds since $x_i \geq y_i$, $i = 1,2,3$. To establish (3), let z, z_1, z_2, z_3 be the affixes of Q, A_1, A_2, A_3, respectively, in the complex plane, and let

$$f(z) = \frac{(z-z_2)(z-z_3)}{(z_1-z_2)(z_1-z_3)} + \frac{(z-z_3)(z-z_1)}{(z_2-z_3)(z_2-z_1)} + \frac{(z-z_1)(z-z_2)}{(z_3-z_1)(z_3-z_2)}. \tag{4}$$

Since the quadratic equation $f(z) = 1$ has the three distinct roots z_1, z_2, z_3, we therefore have $f(z) \equiv 1$. Now the absolute values of the three terms on the right side of (4) are y_2y_3/a_2a_3, y_3y_1/a_3a_1, and y_1y_2/a_1a_2, respectively; hence

$$\frac{y_2y_3}{a_2a_3} + \frac{y_3y_1}{a_3a_1} + \frac{y_1y_2}{a_1a_2} \geq |f(z)| = 1. \tag{5}$$

Now

$$\left(\frac{y_1}{a_1} + \frac{y_2}{a_2} + \frac{y_3}{a_3}\right)^2 \geq 3\left(\frac{y_2y_3}{a_2a_3} + \frac{y_3y_1}{a_3a_1} + \frac{y_1y_3}{a_1a_2}\right), \tag{6}$$

and (3) follows from (5) and (6).

Suppose equality holds in (3). Then it holds in (6), so

$$\frac{y_1}{a_1} = \frac{y_2}{a_2} = \frac{y_3}{a_3};$$

and it holds in (5), so $y_i/a_i = 1/\sqrt{3}$, $i = 1,2,3$, and, in particular, Q is not a vertex of the triangle. Let $\theta_1 = \angle A_2QA_3$, $\theta_2 = \angle A_3QA_1$, and $\theta_3 = \angle A_1QA_2$. We find $\cos\theta_1$ by applying the law of cosines to triangle A_2QA_3, whose sides are $a_2/\sqrt{3}$, $a_3/\sqrt{3}$, and a_1, then do likewise for θ_2 and θ_3, and substitute in the identity

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1 + 2\cos\theta_1\cos\theta_2\cos\theta_3,$$

which is valid because Q is not a vertex, i.e., $\Sigma\theta_i = 0$ or 2π. The result is

$$(a_1^2 + a_2^2 + a_3^2)\{(a_2^2-a_3^2)^2 + (a_3^2-a_1^2)^2 + (a_1^2-a_2^2)^2\} = 0.$$

Therefore the triangle is equilateral and Q is its centre. If equality also holds in (2), then $x_i = y_i$ for $i = 1,2,3$, and P = Q. Hence, if equality holds in (2) and (3), and hence in (1), then the triangle is equilateral and P is its centre. Conversely, if the triangle is equilateral and P is its centre, then equality clearly holds in (1).

Also solved by M.S. KLAMKIN, University of Alberta; and the proposer.

Editor's comment.

The other two solutions received were roughly equivalent to our featured solution. But they both saved space by taking inequality (5) directly from [1], a reference that may not be available to some of our readers.

REFERENCE

1. M.S. Klamkin, "Triangle inequalities from the triangle inequality", *Elemente der Mathematik*, 34 (1979) 49-55, No. (2.1).

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966. [1984: 217] *Proposed by Charles W. Trigg, San Diego, California.*

A topless prism has square vertical faces hinged to the sides of a regular n-gonal base, edge e. The top coinciding vertices of adjacent vertical faces are connected with cords of length e. The vertical faces are permitted to fall outward until stopped by the cords. What angles do the faces then make with the surface upon which the box rests for various values of n?

(The case $n = 4$ is considered in Problem 3898, *School Science and Mathematics*, 83 (February 1983) 177-179.)

Solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.

It is apparent from considerations of symmetry that, after the vertical faces have been permitted to fall, the top face of the polyhedron is a regular $2n$-gon with edges of length e. The inradius of a regular n-gon is $r_n = \frac{1}{2}e\cot(\pi/n)$ and that of a regular $2n$-gon is $r_{2n} = \frac{1}{2}e\cot(\pi/2n)$. The angle, θ, that the sloping faces make with the horizontal is given by

$$\cos\theta = \frac{1}{e}(r_{2n} - r_n) = \frac{1}{2}(\cot\frac{\pi}{2n} - \cot\frac{\pi}{n}) = \frac{1}{2\sin(\pi/n)}. \qquad (1)$$

We solve this equation for θ for various values of n. The results are:

$$\begin{array}{ll} n = 2,\ \ \theta = 60^\circ; & n = 5,\ \ \theta \approx 31^\circ 43'; \\ n = 3,\ \ \theta \approx 54^\circ 44'; & n = 6,\ \ \theta = 0^\circ; \\ n = 4,\ \ \theta = 45^\circ; & n > 6,\ \ \theta = 0^\circ. \end{array}$$

Note that when $n = 6$ the squares are horizontal, and when $n > 6$ formula (1) no longer applies and the cords are no longer taut. The regular 2-gon is, of course, degenerate, but the figure can be constructed (although, unlike the others, it is unstable) and (1) gives the correct result.

Also solved by SAM BAETHGE, San Antonio, Texas; and the proposer.

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967. [1984: 217] *Proposed by Jordi Dou, Barcelona, Spain.*

Let ABC be a triangle with sides b = CA and c = AB of fixed length, with $b < c$, and variable angle A. If S is the intersection of the symmedian of vertex A with the circumcircle of the triangle, construct the limit segment of AS as angle A tends to 180°.

Solution by the proposer.

Let A' be the foot of the median through A and M its intersection with the circumcircle. Arcs BM and SC are equal, so MS $\parallel$ BC. The pencil M(BC,A'S) is harmonic, so (BC,AS) is a harmonic range on the circumcircle. Keeping vertices A and B fixed, when $\angle$BAC tends to 180° the point C tends to the point C' on line AB such that AC' = b, and point S tends to the point S' on line AB such that (BC',AS') is a harmonic range. Therefore to obtain S' it suffices to construct the harmonic conjugate of A with respect to BC', and it is clear that AS' = lim AS. Since

$$\frac{2}{AS'} = \frac{1}{AC'} - \frac{1}{AB} = \frac{1}{b} - \frac{1}{c} = \frac{c-b}{bc},$$

we have lim AS = AS' = $2bc/(c-b)$.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands.

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968. [1984: 217] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

For real numbers a,b,c, let $S_n = a^n + b^n + c^n$. If $S_1 \geq 0$, prove that

$$12S_5 + 33S_1S_2^2 + 3S_1^5 + 6S_1^2S_3 \geq 12S_1S_4 + 10S_2S_3 + 20S_1^3S_2.$$

When does equality occur?

Solution by Vedula N. Murty, Pennsylvania State University, Capitol Campus.

We first express S_4 and S_5 in terms of S_1,S_2,S_3. The equation whose roots are a,b,c is

$$z^3 - pz^2 + qz - r = 0, \tag{1}$$

where

$$p = a+b+c = S_1, \quad q = bc+ca+ab = \frac{1}{2}(S_1^2 - S_2), \quad r = abc = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3). \tag{2}$$

If we multiply (1) by z, substitute successively $z = a,b,c$ and then add the results, we get

$$S_4 - pS_3 + qS_2 - rS_1 = 0 \tag{3}$$

and similarly

$$S_5 - pS_4 + qS_3 - rS_2 = 0. \tag{4}$$

Substituting the values (2) in (3) and (4) gives

$$S_4 = \frac{1}{6}(S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2)$$

and

$$S_5 = \frac{1}{6}(S_1^5 - 5S_1^3S_2 + 5S_1^2S_3 + 5S_2S_3).$$

With these values of S_4 and S_5, the proposed inequality turns out to be equivalent to

$$3S_1(S_1^2 - 3S_2)^2 = 3(a+b+c)\{(b-c)^2 + (c-a)^2 + (a-b)^2\}^2 \geq 0,$$

and this is certainly true since $S_1 \geq 0$. Equality holds just when

$$a + b + c = 0 \quad \text{or} \quad a = b = c.$$

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; M.S. KLAMKIN, University of Alberta; R.C. LYNESS, Southwold, England; and the proposer.

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969. [1984: 217] *Proposed by M.S. Klamkin, University of Alberta.*

Find a 3-parameter solution of the Diophantine equation

$$\frac{x}{x^2+w^2} + \frac{y}{y^2+w^2} + \frac{z}{z^2+w^2} = \frac{2w^2}{\sqrt{(x^2+w^2)(y^2+w^2)(z^2+w^2)}}. \tag{1}$$

Solution by the proposer (revised by the editor).

It is clear that (x,y,z,w) is a solution of (1) if and only if (kx,ky,kz,kw) is a solution for any $k > 0$, so it will suffice for our problem to find a 3-parameter family of *primitive* solutions, corresponding to $k = 1$.

We first show that (1) is satisfied by infinitely many solutions of

$$yz + zx + xy = w^2, \tag{2}$$

so it will make sense to look for solutions of (1) among those of (2). It is clear that (1) is satisfied by any solution of (2) in which $w = 0$ and $xyz \neq 0$. For solutions of (2) with $w \neq 0$, we can choose x,y,z such that

$$\frac{x}{w} = \cot A, \quad \frac{y}{w} = \cot B, \quad \frac{z}{w} = \cot C, \tag{3}$$

where A,B,C are the angles of a triangle. This can be done in infinitely many ways for each $w \neq 0$. Substituting (3) into (2) and then into (1) gives

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

and

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$

respectively, both of which are valid triangle identities. Hence every solution (3) satisfies (2) and (1).

To solve (2), we use the method of Desboves and assume that

$$x = a + \lambda p, \quad y = a + \lambda q, \quad z = \lambda r, \quad w = a + \lambda t, \tag{4}$$

where $\lambda \neq 0$, is a solution of (2). Substituting these values into (2) gives

$$\lambda = \frac{a(2t - p - q - 2r)}{qr + rp + pq - t^2}.$$

Since (2) is homogeneous, we can multiply (4) by $qr+rp+pq-t^2$ and obtain

$$\begin{aligned} x &= a(qr+rp+pq-t^2) + ap(2t-p-q-2r), \\ y &= a(qr+rp+pq-t^2) + aq(2t-p-q-2r), \\ z &= ar(2t-p-q-2r), \\ w &= a(qr+rp+pq-t^2) + at(2t-p-q-2r). \end{aligned}$$

With $m = t-p$ and $n = t-q$, we get

$$\begin{aligned} x &= a\{r(m-n) - m^2\}, \\ y &= a\{r(n-m) - n^2\}, \\ z &= ar(m+n-2r), \\ w &= a\{mn - r(m+n)\}. \end{aligned}$$

It will be found that these values satisfy (2) for all a,m,n,r, but they satisfy (1) for all m,n,r if and only if $a < 0$. (To facilitate the latter verification process, we note that

$$\begin{aligned} x^2 + w^2 &= a^2(m^2+n^2)(m^2-2rm+2r^2), \\ y^2 + w^2 &= a^2(m^2+n^2)(n^2-2rn+2r^2), \\ z^2 + w^2 &= a^2(m^2-2rm+2r^2)(n^2-2rn+2r^2).) \end{aligned}$$

Consequently, we take $a = -1$ and obtain the following 3-parameter primitive solution set of (1):

$$\begin{aligned} x &= m^2 - r(m-n), \\ y &= n^2 - r(n-m), \\ z &= r(2r-m-n), \\ w &= r(m+n) - mn. \end{aligned}$$

Editor's comment.

Equation (2) is extensively discussed in L.J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, pp. 291-292.

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970. [1984: 217] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a,b,c and m_a,m_b,m_c denote the side lengths and median lengths of a triangle. Find the set of all real t and, for each such t, the largest positive constant λ_t, such that

$$\frac{m_a m_b m_c}{abc} \geq \lambda_t \cdot \frac{m_a^t + m_b^t + m_c^t}{a + b + c}$$

holds for all triangles.

Solution by Jordan B. Tabov, Sofia, Bulgaria.

We take an arbitrary real t and consider the family F of triangles with sides $a = 1$, $b = 1$, and $c = x$, where $0 < x < 2$. Since for this family

$$\lim_{x\to 2} \frac{m_a m_b m_c}{abc} = \frac{\frac{3}{2}\cdot\frac{3}{2}\cdot 0}{1\cdot 1\cdot 2} = 0$$

and

$$\lim_{x\to 2} \frac{m_a^t + m_b^t + m_c^t}{a + b + c} = \frac{(\frac{3}{2})^t + (\frac{3}{2})^t + \lim_{x\to 2} m_c^t}{1 + 1 + 2} \geq \frac{1}{2}(\frac{3}{2})^t > 0,$$

it follows that there is no positive constant λ_t with the stated property, and so the required set is the empty set. □

If the inequality in the proposal is reversed, then for the same family F we have

$$\lim_{x\to 0} \frac{m_a m_b m_c}{abc} = \infty \quad \text{and} \quad \lim_{x\to 0} \frac{m_a^t + m_b^t + m_c^t}{a + b + c} = (\frac{1}{2})^t + \frac{1}{2},$$

so again there is no positive constant λ_t with the stated property, and the required set is again empty.

Also solved by M.S. KLAMKIN, University of Alberta.

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THE GREEN BOOK

The Green Book: 100 practice problems for undergraduate mathematics competitions, compiled with hints and solutions by Kenneth Hardy and Kenneth S. Williams, Carleton University, 173 pages, soft cover. Obtainable at $15 per copy from Integer Press, P.O. Box 6613, Station J, Ottawa, Ontario, Canada K2A 3Y7.

These 100 problems (several of which were taken from *Crux Mathematicorum*) constitute a useful source of practice material for students who are preparing for the William Lowell Putnam and other undergraduate mathematical competitions.

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