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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

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A New Look at Archimedes

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If we reckon the achievement of a mathematician by the *percentage* increase in the total quantity of interesting mathematics which resulted from his work there is no doubt that Archimedes' achievement was the greatest. Euclid wrote a lot of mathematics, but only a part of the total content of the *Elements* consisted of his own original work. Newton, Leibnitz and Gauss came too late to make a percentage contribution comparable with that of Archimedes. All this is beyond dispute. Yet... because today's mathematics is so much more sophisticated than that of the third century B.C. it is no longer obvious that Archimedes' achievement is as significant as earlier commentators supposed. Perhaps he was able to get as far as he did because the problems he faced were very easy. So the judgement of earlier commentators of Archimedes' greatness need not necessarily remain valid. We might argue, with the advantages of hindsight, that his work looked important only when it was measured by elementary and even rustic yardsticks.

Turning to the popular 'image' of Archimedes we see of course the great eccentric of antiquity: the Dr Who-like figure who disregarded the conventions of dress when he ran naked through the streets of Syracuse; who was a total master of mechanical wizardry; and later, when he repulsed the Roman fleet at Syracuse, was quite absent-mindedly oblivious of his own personal safety when finally challenged by the invading Roman soldiery. Was Archimedes, then, a great mathematician, or merely a great 'character'?

Professor Scott quotes with approval the judgement of earlier writers that Archimedes was the 'greatest intellect of antiquity'. Professor Bell describes him as the most 'modern' of the ancient mathematicians because his style of thinking was free-ranging, not confined within narrow limits. Since the computer revolution and the consequent dawning of the immense potential applicability of mathematics, we may even wonder whether Archimedes was an 'applicable' mathematician in the modern sense.[1] Was he, perhaps, an even more outstanding figure than the histories of mathematics give him credit for?

Can the life of Archimedes, the great technologist-mathematician of Syracuse, serve today as a valid model of the mathematical way of life?

We begin with a short survey of Archimedes' known work. After that we consider the style of his mathematics and some of its implications today.

[1] I.e. whether he thought of mathematics as a modelling kit for simulating situations which involve a multiplicity of predictable possibilities. The computer has opened up this point of view by enabling a mass of hitherto inaccessible problems to be tackled in which there would be too much multiplicity to handle comfortably by paper-and-pencil methods.

Archimedes' works

In Sir Thomas Heath's edition of Archimedes' Works (1897) we find translations of the following books:

On the Sphere and the Cylinder I & II. This is considered by many to be Archimedes' finest book.

(One of Archimedes' discoveries in this book was that each section of the circumscribing cylinder of a sphere has the same area as the corresponding section of the sphere.[2] See Figure 1.)

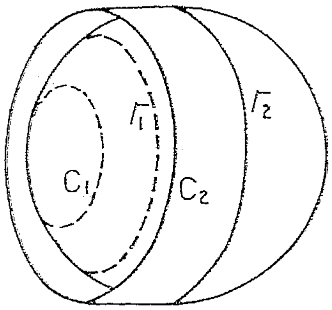

Figure 1

The area of the cylinder equals the area on the sphere between the dotted lines. Circles C_2, Γ_2, have the same radius as the sphere. Circles C_1, C_2, are concentric: so are Γ_1, Γ_2.

Problem 1. Can you prove this by modern methods?

Can you also show, by similar reasoning, that the centroid of a semi-circular arc is $2r/\pi$ units along its axis of symmetry from the centre of the circle?

Measurement of a Circle. This contains Archimedes' classic evaluation of bounds for π.

On Conoids and Spheroids. In this book Archimedes establishes the volumes of revolution of conics such as the parabola and the hyperbola.

On Spirals. This consists of a geometrical investigation in terms of tangents, areas enclosed, etc. of the 'Archimedean Spiral'. (Archimedes defines his *spiral* as follows:

> 'If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral ($\varepsilon\lambda\iota\xi$) in the plane.')

Archimedes proves a sequence of twenty-eight theorems about the spiral, including the rather striking result that the area bounded by the first turn of the spiral and the initial line is equal to one third of the area of the 'first circle' (see Figure 3).

[2] If one considers the total area of the curved surface of the circumscribing cylinder one has an area of $2\pi r \times 2r$ or $4\pi r^2$ as expected.

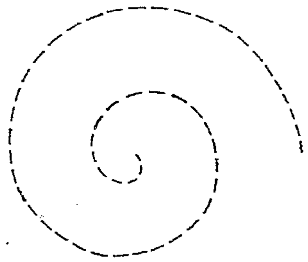

Figure 2

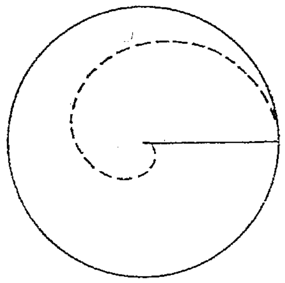

Figure 3

Problem 2. Can you prove this result?

On the Equilibrium of Planes I & II. This is a study of the centres of gravity of laminas of various shapes, including those bounded by parabolas.

The Sand Reckoner. Archimedes defines a series of 'orders' and 'periods' of astronomically large numbers: an ancient version of the Googol.[3] (1 myriad = 10,000, a myriad myriads = 100,000,000. Archimedes calls the numbers up to 10^8 numbers of the *first order*. Numbers between 10^8 and 10^{16} he calls numbers of the *second order*. And so on. . . . Eventually he reaches the myriad-myriadth order. This number is denoted by P.

Numbers from 1 to P are described as numbers of the *first period*. The second period consists of numbers from P to 10^8P, . . . and so on up to the *myriad-myriadth* period. The largest number as defined by Archimedes' scheme is 'a myriad-myriad units of the myriad-myriadth order of the myriad-myriadth period'.

Problem 3. Can you express this number as a power of 10?

The Quadrature of the Parabola. This is an investigation of the area of the segment of a parabola.

On Floating Bodies I & II. This contains Archimedes' pioneering work on hydrostatics, including of course the famous 'Archimedes' Principle'.

Here Archimedes' treatment is more subtle than one might expect. He treats the surface of the water as being a spherical surface (see Figure 4) and establishes, for example, that a solid whose cross-section is a segment of a parabola (Figure 5) will revert to the position in which its axis of symmetry is vertical.

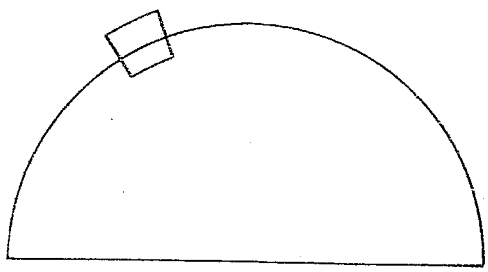

Figure 4

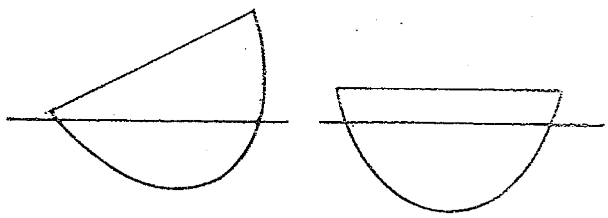

Figure 5

[3] See Kasner and Newman (1949), p. 20. (The 'Googol' is defined as 10^{100}.)

Book of Lemmas. This book contains various geometrical proofs, including the one which is the basis of the following result due to Pappus.

The diagram (Figure 6) shows a semi-circle containing two smaller semi-circles touching each other, and the larger semi-circle, at the end points of their diameters. A series of inscribed circles is constructed with centres $\underset{\cdot}{o}_1, \underset{\cdot}{o}_2, \underset{\cdot}{o}_3 \ldots$ etc.[4]

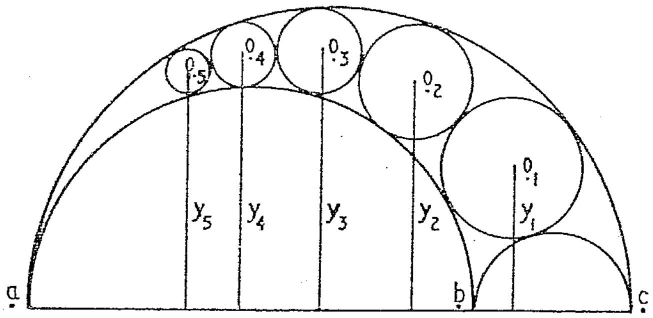

Figure 6

Let the diameter of the circle centre $\underset{\cdot}{o}_1$ be d_1. Let the diameter of the circle centre $\underset{\cdot}{o}_2$ be d_2, and so on. . . . Pappus' result (based on Archimedes' Proposition 6) is that, for $r = 1, 2, \ldots,$

$$y_r = rd_r.$$ [5]

On Mechanical Theorems, Method is not in Sir Thomas's volume. In this book Archimedes describes how he first arrived at many of his results in geometry: by performing 'thought experiments' in mechanics. This book, which had long been thought to be lost, was discovered in Constantinople in 1906 by J. L. Heiberg.

In addition to the books listed above it is believed that Archimedes wrote several further books: on polyhedra, on extending the 'naming of numbers', on levers, on centres of gravity, on optics, on sphere making and on the *Calendar*. Though references are made to these books by various ancient authors, no manuscripts of these works have, as far as we know, survived. Archimedes' work, then, constitutes a considerable whole. The extant works printed by Sir Thomas Heath in his English translation of 1897 amount to 326 pages of quite difficult 'elementary mathematics'.

Two examples of Archimedes' work

We turn now to look at two examples of Archimedes' work in greater detail.

First, Archimedes' investigation of bounds for the value of π. Archimedes' method is to find an estimate for the length of the circumference of a unit circle. He uses half the perimeter of an inscribed regular polygon of 96 sides for the lower bound, and half the perimeter of a circumscribed regular polygon of 96 sides for the upper bound.

[4] The dots underneath these letters indicate that they denote points; this is an experimental notation which will be used in the series *Mathematics Applicable* to be published by Heineman Educational Books in 1975. Capitals are reserved for sets of points, e.g., curves and surfaces.

[5] The editor would be pleased to receive from readers proofs of this rather difficult theorem; the best proof will be published in the next issue but one.

Archimedes' actual result is that

$$\frac{14688}{4673\frac{1}{2}} > \pi > \frac{6336}{2017\frac{1}{4}},$$

but in the interests of a simple, memorable enunciation he settles for the slightly weaker result that

$$3\tfrac{1}{7} > \pi > 3\tfrac{10}{71}.$$

Both Archimedes' inequalities are, in fact, derived from a kind of iteration based on repeated doubling of the number of sides of the polygons.

Figure 7 shows a regular hexagon. Six times the segment marked x_1 will be 3 units, so we begin with $\pi > 3$. (Half the perimeter of a unit circle is π units: half the perimeter of the hexagon is less than π.)

Figure 8 shows a regular polygon of twelve sides (dodecagon). Twelve times the segment marked x_2 will therefore provide a better lower bound for π.

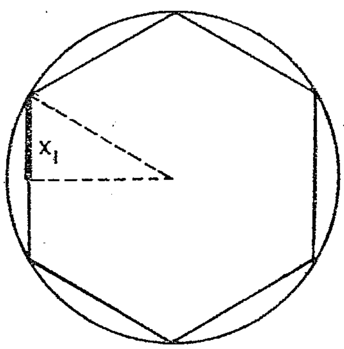

Figure 7

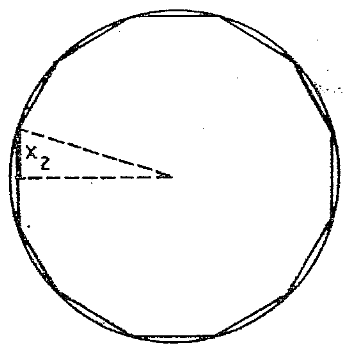

Figure 8

Problem 4. Evaluate $12x_2$ using only elementary geometrical results.

Incidentally, in the final result (Proposition 3) of the *Measurement of a Circle* Archimedes quickly reduces the problem to that of finding what are, in effect, inequalities for $\tan(\pi/48)$ and $\sin(\pi/48)$; he does not consider it necessary to append illustrations of any of the inscribed or circumscribing polygons.

Problem 5. The diagram (Figure 9) shows a circle of radius 1 unit. x_r is half the side of a regular polygon of r sides inscribed in a unit circle. x_{2r} is half the side of a regular polygon of $2r$ sides inscribed in the unit circle. Show that

$$x_{2r} = \left(\frac{1 - (1 - x_r^2)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}.$$

Secondly we consider Archimedes' method for finding the area of a segment of a parabola.

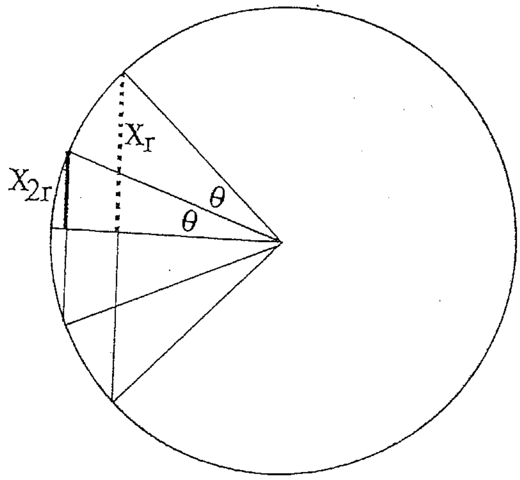

Figure 9

Suppose that Σ is a parabola and ap is part of the locus of the mid-points of chords parallel to a given line L; then Archimedes shows that y^2/x is constant for all points p of the locus.

It follows that x_1 (Figure 11) is a quarter of $x_2 (x_2 = ap_2)$. Archimedes proceeds to show that the area of the shaded triangle ar_1b is a *quarter* of the area of the triangle ap_2b.

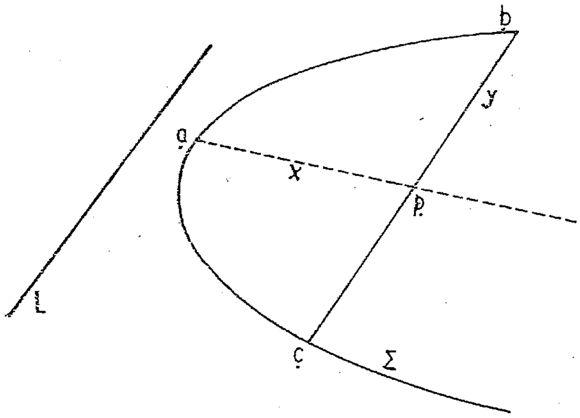

Figure 10
$cb // L$, $cp = pb$.

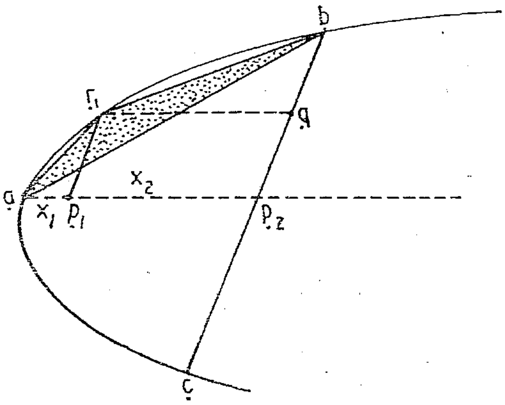

Figure 11
$qb = p_2q$, $r_1q // ap_2$, $r_1p_1 // p_2q$, $x_2 = ap_2$.

Problem 6. Prove that the area of Δar_1b is a quarter of the area of Δap_2b.

This result is the key to Archimedes' method for finding the area of the segment. We may apply the same argument to the second shaded triangle in Figure 12 and show that its area is equal to a quarter of the area of the triangle ap_2c.[6]

[6] So the sum of the areas of the dotted triangles is equal to a quarter of the area of Δabc.

Similarly the sum of the areas of the dark triangles in Figure 13 is equal to a quarter of the area of the upper shaded triangle, ar_1b. And so on. . . . If the process is continued indefinitely we obtain the total area enclosed by the segment, i.e., the sum

$$A + \tfrac{1}{4}A + (\tfrac{1}{4})^2 A + (\tfrac{1}{4})^3 A + \cdots,$$

where A is the area of the original triangle abc.

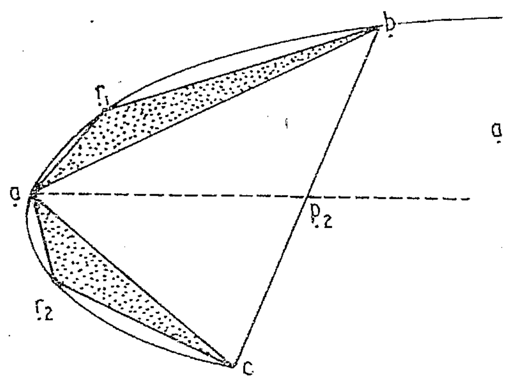

Figure 12

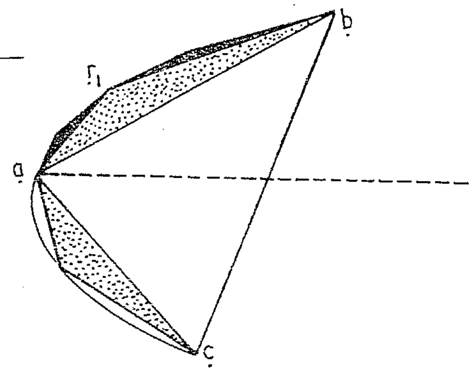

Figure 13

Problem 7. Find this area in terms of A.

Compare the result with similar results obtained by calculus methods for symmetric segments cut off from the curves $y = x^4$, $y = x^{2n}$. Would these results be obtainable by Archimedes' method?

Archimedes obtains what is in fact the sum of the series above (Proposition 24, *The Quadrature of the Parabola*), but he does not jump in straightaway to infer that the area of the segment is kA, where k is the constant concerned. Archimedes uses a much more rigorous argument which says merely that the area of the segment is greater than, or equal to, kA. By a comparable argument he shows that it must also be less than, or equal to, kA. So the conclusion follows that the area of the segment equals kA.

Archimedes' style

The foregoing exposition shows the way in which Archimedes used extreme care in stating his arguments, and was not prepared to assume that an area was simply the sum of an infinite series of terms. Archimedes accepted the standard Greek view that an infinite series was by definition an *incompletable* series[7] and therefore that it could not, by definition, have a sum.

Today we take for granted a more general concept of what constitutes a 'sum'. If a series continues indefinitely one might say that it had no sum; alternatively one can say that its 'sum' (which would otherwise not exist) will be defined as the *limit* of its partial sums S_n, as $n \to \infty$. Professor Scott remarks that Archimedes did not possess the concept of a limit, and that it was this which drove him to use such

[7] Zeno's paradoxes showed the hazards of treating infinite series as completable series.

roundabout (and impressive) rigour. On the other hand, Archimedes' most interesting work nearly all revolves round problems involving limiting processes, and we may hardly deny him the substance of the *idea* of a limiting process. What made the difference was that, to the Greeks, the area of a segment was a 'real' entity, whereas anything which involved recognising the result of an infinite sequence of operations was rather suspect.

The residue of more than two millennia of critical thinking in logic and epistemology has led us finally to reverse these attitudes. We now see that it is the concept of 'area' which must be questioned as the precise idea, and that it is the concept of 'limit' which is the more fundamental and the more firmly defined.[8]

What is remarkable about Archimedes is not that he failed to handle limiting processes with the economy of method to which modern mathematicians are accustomed, but that he tackled problems involving limiting processes in the first place. In a word, he attacked the most difficult problems of his day. Limiting processes are involved explicitly or implicitly in his work on π, on the surface area of the sphere, on spirals, on equilibrium, on the quadrature of the parabola and on floating bodies. His awareness of the direction of future progress in mathematics emerges so strongly that, as Professor Scott remarks, we may think of him less as a mathematician of the Greek school than the first mathematician of the school of Newton and Leibnitz: and he wrote nineteen centuries before Descartes!

One can see in Archimedes' work the pattern of intellectual self-challenge which has become typical of much of the best mathematical research.[9] Again and again Archimedes identifies problems which clearly 'ought' to be within the range of his geometrical dialectic.[10] For example, it ought to be possible to define a process which will divide the surface area of a sphere into two parts in the ratio 1:2 or 1:k.

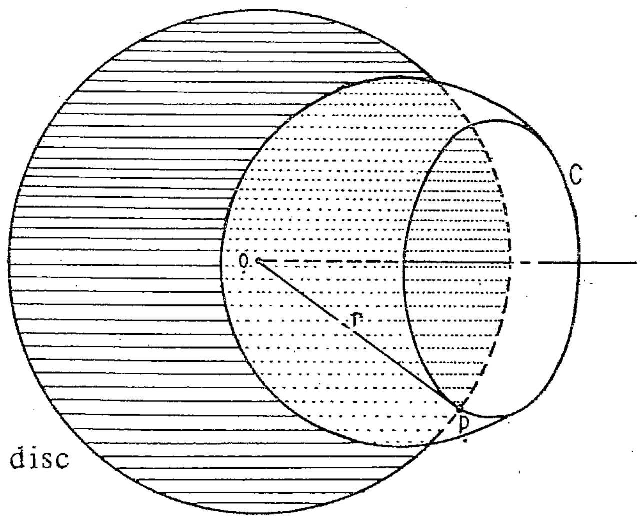

Figure 14

[8] In effect Euclidean geometry has become an application of the mathematics of the real number system.

[9] This may be described as 'trying to fill the gaps in our knowledge'. Not all mathematical research is done with this objective.

[10] I.e., reasoning.

Archimedes tackles this, and discovers the striking result that the area of the curved surface of a segment of a sphere bounded by a circle C is equal to that of a plane disc whose radius is op, the direct distance from o to any point on C. (o is the point of the sphere on the axis of symmetry of the segment. (See Figure 14.))

Problem 8. Can you prove this result using modern calculus methods?

Archimedes freely uses intuitive ideas (mainly mechanical) in his initial investigations of a problem: he then defines the formal results he will try to prove: then, using methods which combine economy, elegance and impressive rigour, he proves the lemmas and other preliminary results which he needs, and which lead finally to the main result of the inquiry.

Archimedes was unquestionably the great mathematician of the ancient world: in insight and intuitive thinking, in the identification of interesting problems, in judgement of impossible problems, in rigour and in his anticipation of the ideas of the calculus, he was without equal. If we accept that the judgement of any mathematician's work must be *in relation* to the state of mathematics of his day, Archimedes was the greatest mathematician.

But was he more than this? He is the only mathematician of the highest rank who was also an equal master of technology. That he invented the water screw, made a working model ('orrery') of the solar system, developed the lever and invented the compound pulley, is well known. It is quite clear that the 'military engines' he devised for the defence of Syracuse *worked*. The achievement implicit in this is all the more astonishing when we set it in perspective against the primitive technology of the day, and the fact that he must have had to work hard to establish credibility even to begin.

Yet Archimedes himself did not rate his achievements in technology very highly: his view of the world was one in which mathematics provided a route to wisdom about the nature of things, and the value of the *truth* thus revealed was far greater than transient advantages in launching ships, raising water or detecting alloyed gold. (Archimedes did not have the advantage of the modern formalist insight that mathematical truths and truths about 'the nature of the world' are quite separate things.)[11] Archimedes' attitude towards mathematics was basically an aristocratic one. As such it captured the imagination of scholars and mathematicians up till quite recent times: Archimedes' mathematical way of life was a marvellous way of life.

But because Archimedes was such an outstandingly creative technologist, his disdain of his own technological achievement has come to assume a special significance which it may not properly deserve. It seems to justify a rejection of the value of practical imagination in mathematics, and hence to support a type of mathematical thinking very different in kind from that practised by Archimedes

[11] More accurately, the two kinds of truth are separable. This does not reduce the great importance of mathematics in constructing explanations of physical phenomena.

himself. Just as a serious intention to form an historical judgement must surely lead us to assess Archimedes' work in *relation* to the baseline of knowledge which existed before he began, so a serious intention to analyse the success of his methods of thought ought to lead us to disregard those of his attitudes which merely reflect the prevailing social structure of his day. Archimedes happened to be born into the upper class of an aristocratic society, and it is hardly surprising that in some respects he mirrored the presuppositions of this society. Yet what was characteristically Archimedean was not his conventional expression of disdain for technology, but a lifelong interest in technological innovation which managed to survive the prevailing conventions of a society in which aristocrats were not expected to involve themselves in such lowly matters. And there is really little doubt—particularly since the discovery of his *Method*—that Archimedes' practical imagination was the motor which drove him to many of his mathematical discoveries. (In his long investigation of the stability of floating bodies we may see direct hints of the thinking underlying the grappling hooks which so dramatically overturned Marcellus' Roman galleys.)

Today the social pattern which produced Archimedes has long since disappeared. The platonism[12] which he took for granted lasted for two millennia but has finally given way to a more existentialist point of view, in which time as well as space is 'real', in which spontaneity and transient effects count and in which the mathematician can no longer easily contract out.[13] Perhaps the most important thing we can learn from the life of Archimedes is that the practical, the imaginative and the mathematical are compatible mental qualities. So, if we subtract from Archimedes' thought the surface details which were coloured by his particular social, philosophical and mathematical presuppositions, what remains is a kind of practical–theoretical dynamism which is still fascinating, and highly relevant, today.

Notes on the problems

Problem 1. This reduces to showing that $2\pi a\delta x$ is equal to $2\pi y\delta s$ where δs is the width of the elemental strip on the sphere which corresponds to a strip of width δx on the cylinder.

Problem 2. This involves evaluating $\int_0^{2\pi} \frac{1}{2}r^2\,d\theta$ when $r = k\theta$.

Problem 3. $P = 10^{800,000,000}$. The largest number defined in the scheme is $P^{100,000,000}$ or $10^{8\times 10^{16}}$.

Problem 4. $12x_2 \approx 3{\cdot}1$.

Problem 5. $x_r = \sin 2\theta = 2\sin\theta\cos\theta$. Express this in terms of x_{2r}.

[12] This was based on the view that only things which are timeless are 'real'.

[13] Archimedes himself of course did not contract out. But the aristocratic conception of mathematics which he accepted implied that the mathematician should contract out.

Problem 6. First show that Δar_1b is half the area of Δabq. There are triangles on the same base (ab) and their heights are proportional to the two parts of the interval rq.

Problem 7. $\frac{4}{3}A$, $\frac{8}{5}A(4n/(2n+1))A$. No. The mid-point of parallel chords lie on a line only when $n = 1$.

Problem 8. This is done by a method similar to Problem 1.

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An Urn Problem with a Quality-Control Application

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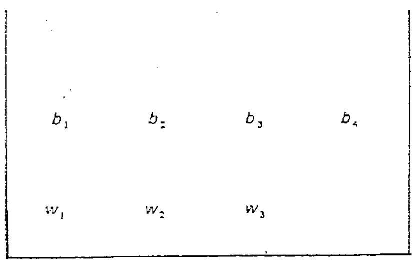
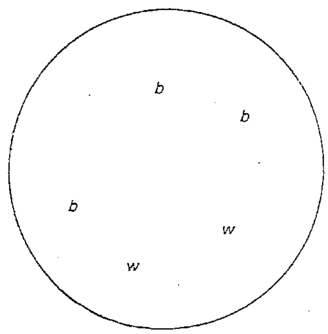
1. The hypergeometric probability law

Suppose that a raffle is being held, that N tickets have been sold and that you have bought B of them. If n prize-winning tickets are subsequently picked out, what are your chances of winning a prize, or, more generally, what are your chances of winning x prizes? This problem, like many others, is equivalent to the following:

An urn contains N balls of which B are black and $N - B$ are white; if n balls are selected randomly from the urn, find the probability that x of them are black. (Such a collection of n balls would be called a random sample from the urn.)



Urn



Sample

A special case such as $N = 7$, $B = 4$, $n = 5$, $x = 3$ points the way to the solution in general. If we label the balls b_1, b_2, b_3, b_4, w_1, w_1, w_3, where b stands for black and w for white, then we see that there are 12 ways in which a sample of 5 balls can be made up so that just 3 are black. The 12 ways are:

$$\begin{aligned}
&b_1b_2b_3 \text{ with } w_1w_2 \text{ or } w_1w_3 \text{ or } w_2w_3,\\
&b_1b_2b_4 \text{ with } w_1w_2 \text{ or } w_1w_3 \text{ or } w_2w_3,\\
&b_1b_3b_4 \text{ with } w_1w_2 \text{ or } w_1w_3 \text{ or } w_2w_3,\\
&b_2b_3b_4 \text{ with } w_1w_2 \text{ or } w_1w_3 \text{ or } w_2w_3.
\end{aligned}$$

Notice that the 3 black balls in the sample can be chosen from the 4 black balls in the urn in $\binom{4}{3} = 4$ ways,[1] and there are $\binom{3}{2} = 3$ ways of choosing the white balls. Hence there are $4 \times 3 = 12$ ways in which 3 black balls and 2 white ones can be selected. Similarly there are $\binom{4}{2} \times \binom{3}{3} = 6$ samples containing just 2 black balls and $\binom{4}{4} \times \binom{3}{1} = 3$ samples containing just 4 black balls.

Since we cannot have fewer than 2 or more than 4 black balls in any sample, this makes $6 + 12 + 3 = 21$ possible samples in all. Therefore in a long series of repeated experiments we would expect to observe samples containing 2, 3 and 4 black balls in the ratio 6:12:3. In other words the probability that any given sample contains 3 black balls is $12/(6 + 12 + 3) = 4/7$.

We could have calculated the total number of different samples directly, since the colour of the balls is irrelevant. We simply wish to choose 5 balls from 7, which we can do in $\binom{7}{5} = 21$ ways. Thus there are 21 different samples which can be chosen.

In the general case the x black balls can be chosen in $\binom{B}{x}$ ways, the $n - x$ white balls in $\binom{N-B}{n-x}$ ways, and there are $\binom{N}{n}$ samples in all, so that

$$\Pr\{x \text{ black balls}\} = p_x = \frac{\binom{B}{x}\binom{N-B}{n-x}}{\binom{N}{n}}. \tag{1}$$

This formula holds for $x = 0, 1, \ldots, n$ (provided that $B \geqslant n$ and $N - B \geqslant n$),[2] and we say that the number of black balls observed in the sample is governed by the *hypergeometric* probability law.

[1] The notation $\binom{n}{r}$ or ${}^nC_r$ (read n choose r) is used to denote the number of different ways in which r items can be chosen from n items, if the order in which they are chosen is not taken into account. It can easily be shown that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $x! = x(x-1)\ldots3.2.1$ and $0! = 1$.

[2] These two conditions arise because we cannot find more black or white balls in the sample than there are in the urn. In general (1) holds for max $(0, n - N + B)$ to min (n, B) inclusive.

2. Approximation to the hypergeometric probabilities

If N is large compared with n, so that the proportion of black balls in the urn remains almost constant throughout the sampling process, then it can be shown that $\binom{B}{x}\binom{N-B}{n-x}\Big/\binom{N}{n}$ may be approximated by $\binom{n}{x}$ $(1-p)^{n-x}p^x$, where $p = B/N$ is the proportion of black balls originally in the urn. This means that p_x depends on B and N only through the ratio B/N.

Thus p_x is approximately equal to the binomial probability, i.e.

$$p_x \doteqdot \binom{n}{x}(1-p)^{n-x}p^x \quad (x = 0, 1, \ldots, n). \tag{2}$$

Notice that by giving x the values $0, 1, \ldots, n$ in turn we obtain the terms in the binomial expansion of $(a+b)^n$ with $a = 1 - p$ and $b = p$.

We have now obtained some degree of simplification in the formula for p_x since

(i) the formula for the binomial probabilities involves 3 variables, the hypergeometric involves 4;
(ii) factorials of large numbers can be difficult to handle accurately without the use of special tables; for instance $100! \doteqdot 9{\cdot}3 \times 10^{157}$.

From (2) we have $p_0 = (1-p)^n$, and, since $1 - p = \exp[\ln(1-p)]$,

$$p_0 = \{\exp[\ln(1-p)]\}^n = \exp[n\ln(1-p)]. \tag{3}$$

Now

$$\ln(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3}\cdots,$$

and if p is small, say less than 0·1, terms like $p^2/2$, $p^3/3$ are much smaller than p, so that

$$\ln(1-p) \doteqdot -p. \tag{4}$$

If you find the above argument unconvincing, you can give p a few values to see just how close the approximation is, for instance

$$\begin{aligned}\ln(1-0{\cdot}1) &= -0{\cdot}1053605,\\ \ln(1-0{\cdot}01) &= -0{\cdot}0100503,\\ \ln(1-0{\cdot}001) &= -0{\cdot}0010005.\end{aligned}$$

Therefore, combining (3) and (4), we have $p_0 \doteqdot \exp(-np)$. If n is greater than about 25, then, by a similar argument we can show that in the case $p \leqslant 0{\cdot}1$,

$$p_x \doteqdot \frac{\exp(-\mu)\mu^x}{x!} \quad \text{where } \mu = np \quad \text{and} \quad x = 0, 1, \ldots, n.$$

This is called Poisson's approximation to the binomial probabilities. Note that there are now only two parameters, $\mu = np$ and x. We may summarise the above approximation as follows:

$$\boxed{\text{Hypergeometric}} \xrightarrow[\text{relative to } n]{N \text{ large}} \boxed{\text{Binomial}} \xrightarrow[p \text{ small}]{n \text{ large}} \boxed{\text{Poisson}}$$

As an example consider an urn with 500 balls of which 50 are black, and suppose we require the probability that 2 black ones are found among 35 selected at random, so that

$$N = 500, \quad B = 50, \quad n = 35, \quad x = 2.$$

The hypergeometric formula gives

$$p_2 = \frac{\binom{50}{2}\binom{450}{33}}{\binom{500}{35}} = \frac{(1{\cdot}225 \times 10^3)(1{\cdot}245 \times 10^{50})}{(8{\cdot}33 \times 10^{53})} = 0{\cdot}183.$$

Now $p = B/N = 0{\cdot}1$, so the binomial formula gives

$$p_2 = \binom{35}{2}(1 - 0{\cdot}1)^{33}(0{\cdot}1)^2 = 0{\cdot}184;$$

and $np = 3{\cdot}5$, so the Poisson approximation is

$$\frac{\exp(-3{\cdot}5)(3{\cdot}5)^2}{2!} = 0{\cdot}185.$$

3. A quality-control technique

A much-used quality-control technique is acceptance sampling. We are presented with a collection of N items (called a batch), and by examining or testing items from the batch we have to decide whether its overall quality is acceptable. Assuming that each item may be described as either 'good' or 'defective', then by quality of the batch we mean the proportion p of defectives; for a batch with B defectives $p = B/N$. A batch will be of acceptable quality if $p \leqslant p^*$ where p^* is the worst quality that we are prepared to accept.

An alternative to acceptance sampling is 100 per cent inspection whereby every item in the batch is examined. However, this often proves to be an uneconomical proposition; if the test involves destruction of the item, as in the case of ammunition, 100 per cent inspection is clearly out of the question!

The simplest acceptance-sampling scheme is the single-sampling plan in which n items are selected at random from the batch; if the number of defectives found is more than c, say, the batch is rejected. Thus for a given batch a plan is specified by the two numbers n and c. For instance, $n = 20$, $c = 1$ means that a sample of 20 items is selected, and that the batch is rejected if more than 1 item in the sample is defective. Note that this practical sampling problem is in principle the same as the urn problem which gave rise to the hypergeometric probabilities; the white balls represent the good items, the black balls the defective ones.

A 'perfect' plan would accept every batch with a p value not more than p^*, and reject all others. If $P_A(p)$ is the probability of accepting a batch of quality p, then we would have

$$P_A(p) = 1 \quad \text{if} \quad p \leqslant p^*,$$

$$P_A(p) = 0 \quad \text{if} \quad p > p^*.$$

A graph of $P_A(p)$ against p is called the operating characteristic (o.c.) curve for the plan. Figure 1 gives the o.c. curve for the 'perfect' plan.

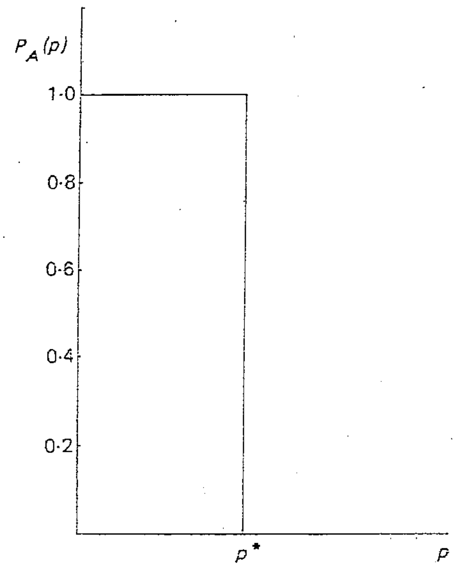


Figure 1. O.c. curve for the 'perfect' plan.

In practice, the value of p^* would not be known exactly; we might feel that $p = 0{\cdot}02$ represented good quality, and $p = 0{\cdot}03$ represented bad, but it would be very difficult to choose a value in the range $0{\cdot}02 \leqslant p \leqslant 0{\cdot}03$ which represented a clear division between good and bad quality. In short, the 'perfect' plan is too rigid. However we may 'round off the corners' of its o.c. curve by saying that there is a value p^* representing an approximate boundary between good and bad quality. Batches having p values much less than p^* we would like to accept with high probability, those with p values much more than p^* we would like to reject with high probability (i.e. accept with low probability).

In the general case, a batch will be accepted if 0, 1, . . ., c defectives are found, so that

$$\begin{aligned} P_A(p) &= \Pr\{0 \text{ or } 1 \text{ or } \ldots \text{ or } c \text{ defectives in sample}\} \\ &= p_0 + p_1 + \cdots + p_c \\ &= \sum_{x=0}^{c} p_x, \end{aligned}$$

where p_x is given by (1). Often N is large relative to n, and n and p are such that the much simpler Poisson formula can be used and we get

$$P_A(p) \simeq \sum_{x=0}^{c} \frac{\exp(-np)(np)^x}{x!}.$$

An important consequence is that this Poisson approximation enables us to make a good choice of the parameters n and c.

3.1 *Choice of n and c*

As an example, suppose that a good quality level can be represented by $p = 0{\cdot}01$ (1 per cent defectives) and a poor one by $p = 0{\cdot}10$ (10 per cent defectives). In this case we wish to choose n and c so that

$$P_A(p) = \exp(-np) \sum_{x=0}^{c} \frac{(np)^x}{x!} \quad \text{is near} \begin{cases} 1 \text{ for } 0 \leqslant p \leqslant 0{\cdot}01, \\ 0 \text{ for } 0{\cdot}10 \leqslant p \leqslant 1. \end{cases}$$

In order to economise on inspection effort we might decide to try a small sample, say $n = 20$, and accept the batch if no defectives are found, i.e. $c = 0$. This gives

$$P_A(p) = \exp(-20p),$$

which is plotted in Figure 2. We can readily see from this figure that the plan $n = 20$, $c = 0$ is a poor one in this case, since the o.c. curve is steep in the neighbourhood of $p = 0$, and 'good' batches have only an 81 per cent chance of being accepted.

The Poisson approximation tells us that the o.c. curve for any plan with $c = 0$ is given by $P_A(p) = \exp(-np)$, so that if we want to accept 'good' batches with a probability of at least 0·95, say, we require

$$\exp[-n(0{\cdot}01)] \geqslant 0{\cdot}95,$$

which is only satisfied when $n \leqslant 5$. This means that unless we are prepared to inspect very small samples only, the smallest value of c we should consider is $c = 1$.

Returning to our example, if we allow just 1 defective in the sample without rejecting the batch, we have $n = 20$, $c = 1$, which gives

$$P_A(p) = \exp(-20p)(1 + 20p). \tag{5}$$

From (5) we see that $P_A(0{\cdot}01) = 0{\cdot}98$ and $P_A(0{\cdot}10) = 0{\cdot}41$. This latter probability is too high since it means that we would be accepting about 2 in every 5 bad batches, but we can reduce it by increasing n to say 40, and keeping c fixed at 1. Then

$$P_A(p) = \exp(-40p)(1 + 40p)$$

and so

$$P_A(0{\cdot}01) = 0{\cdot}93,$$
$$P_A(0{\cdot}10) = 0{\cdot}09.$$

By doing this we achieve a substantial reduction in the proportion of bad batches accepted (from 41 per cent to 9 per cent) without too much increase in the proportion of good ones rejected (from 2 per cent to 7 per cent). Figure 3 gives the o.c. curve for this plan.

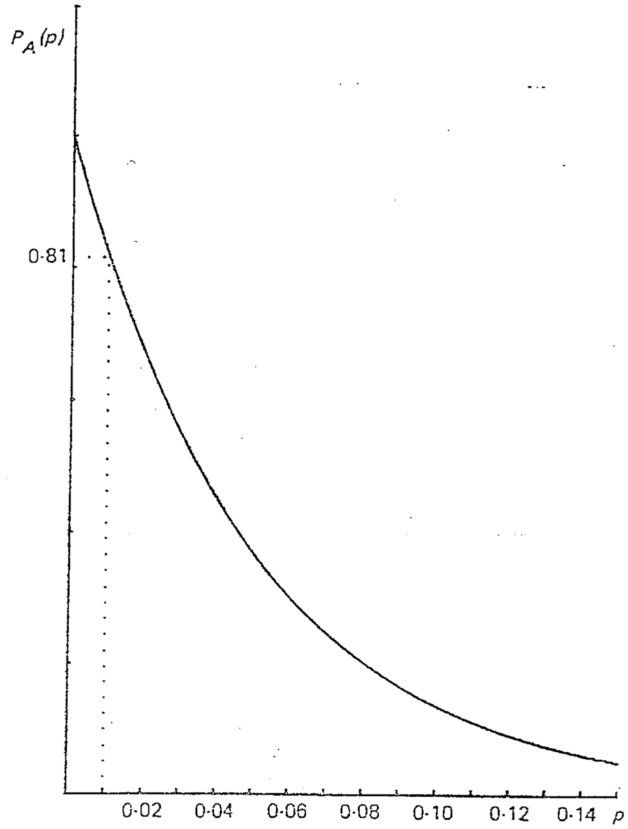


Figure 2
O.c. curve for the plan $n = 20$, $c = 0$.

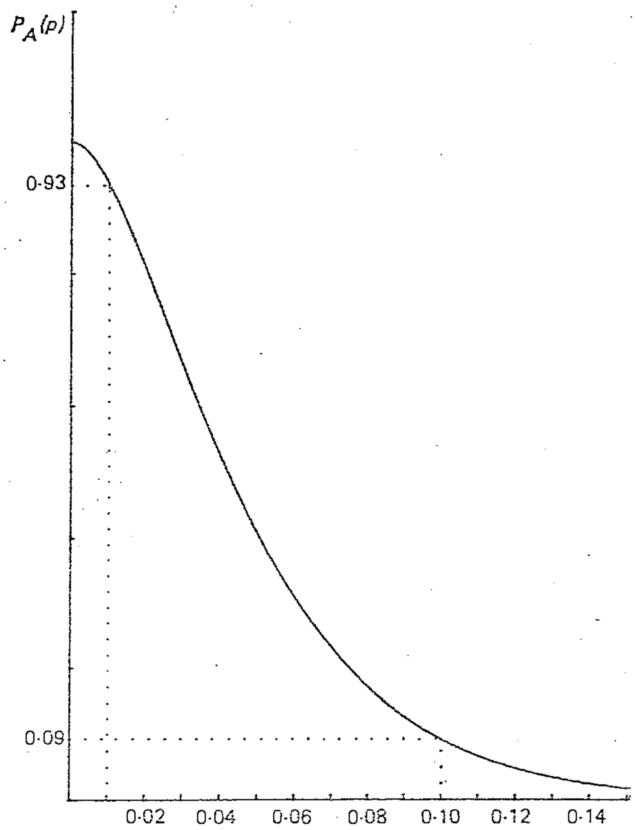


Figure 3
O.c. curve for the plan $n = 40$, $c = 1$.

Typical values for $P_A(p)$ at the 'good' and 'poor' quality levels should be around 0·95 and 0·10 respectively, so the plan $n = 40$, $c = 1$ would not be an unreasonable one. Remember that, since our variables n and c must be integers, by varying them we cause $P_A(p)$ to jump from one value to another, so that we cannot guarantee to find values of n and c which will make the graph of $P_A(p)$ pass through any predetermined points.

Suppose that in general p_0 represents a 'good' quality level and p_1 a 'poor' one, where $p_0 < p_1$. Suppose also that we wish $P_A(p_0)$ to be near 0·95 and $P_A(p_1)$ to be near 0·10. Then Table 1 may be used to find reasonable values of n and c.

TABLE 1. Values of p_1/p_0 for single sampling plans with $P_A(p_0) = 0{\cdot}95$ and $P_A(p_1) = 0{\cdot}10$

c	$\mu_0 = np_0$	$\mu_1 = np_1$	$\mu_1/\mu_0 = p_1/p_0$
0	0·051	2·30	45·10
1	0·355	3·89	10·96
2	0·818	5·32	6·50
3	1·366	6·68	4·89
4	1·970	7·99	4·06
5	2·613	9·28	3·55

In Table 1 μ_0 and μ_1 are the solutions of

$$\exp(-\mu_0) \sum_{x=0}^{c} \frac{\mu_0^x}{x!} = 0{\cdot}95, \quad \exp(-\mu_1) \sum_{x=0}^{c} \frac{\mu_1^x}{x!} = 0{\cdot}10$$

respectively, and at the solution points $\mu_1/\mu_0 = (np_1)/(np_0) = p_1/p_0$. Thus to find n and c we simply look down the p_1/p_0 column until we find the required ratio, which gives c directly. To find n we divide μ_0, μ_1 by p_0, p_1 respectively.

Example. Suppose that $N = 2000$, $p_0 = 1$ per cent, $p_1 = 7$ per cent; then $p_1/p_0 = 0{\cdot}07/0{\cdot}01 = 7$. From Table 1 we see that the nearest value to 7 in the p_1/p_0 column is 6·5, which occurs when $c = 2$. From the μ_0 column $np_0 = 0{\cdot}818$, so that $n = 0{\cdot}818/0{\cdot}01 = 81{\cdot}8$. Similarly $np_1 = 5{\cdot}32$ which gives $n = 5{\cdot}32/0{\cdot}07 = 76{\cdot}0$. In practice we would use the plan $n = 80$, $c = 2$ for which $P_A(0{\cdot}01) = 0{\cdot}953$ and $P_A(0{\cdot}07) = 0{\cdot}82$. You will appreciate the great advantage of the Poisson approximation if you try to solve this problem by using either the hypergeometric or binomial probability laws.

4. Remarks

You will not be surprised to know that there are other plans in existence besides the single-sampling plans. When a new plan is developed a natural yardstick by which to judge it is provided by a single-sampling plan with a similar o.c. curve. Two plans with similar o.c. curves are often said to be equivalent plans.

The main advantage of the single-sampling plan is its simplicity. However, an alternative is the sequential plan, where items are selected 1 (or k) at a time, and after each selection one of the three following decisions is made:

(i) accept the batch,
(ii) reject the batch,
(iii) continue sampling.

For batches of extreme quality (i.e. p near 0 or 1) sequential plans can offer substantial improvement in terms of the amount of sampling performed.

As an exercise you might like to try to find the equation of the o.c. curve of the following *double* sampling plan.

Take a sample of 20 items, let r be the number of defectives. Then

(i) if $r = 0$ accept the batch,
(ii) if $r \geqslant 2$ reject the batch,
(iii) if $r = 1$ take a further sample of 40 items, and if it contains 0 or 1 defectives, accept the batch.

How do you think this plan would perform if $p_0 = 0{\cdot}01$ and $p_1 = 0{\cdot}1$? Test your intuition by calculating $P_A(p_0)$ and $P_A(p_1)$.

Suggestions for further reading

J. H. Durran, *SMP Statistics and Probability* (Cambridge University Press, 1970).
C. S. Smith, *Quality and Reliability: An Integrated Approach* (Sir Isaac Pitman and Sons, London, 1969).

A Complex-Number Slide Rule

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As the logarithm of a complex number has both a real and an imaginary part (if $r = |z|$ and $\theta = \arg z$, then $z = re^{i\theta}$, so $\log z = \log r + i\theta$), a slide rule for multiplying (or dividing) complex numbers must be essentially two-dimensional. However, as increasing a logarithm by $2\pi i$ does not alter the number it represents, the imaginary axis must be closed to form a circle: and thus the numbers to be multiplied must be represented on the surface of a cylinder.

It is then easy to see how to construct a slide rule for complex numbers given in the modulus-argument form. A cylindrical stock is provided with a rectangular grid, the logarithm of the modulus being measured along the length of the cylinder and the argument being measured around its axis; and a slide, which is a transparent cylinder carrying the same grid, is fitted over the stock.

However, complex numbers are more usually given in the form $a + ib$ (a, b real) and, if the slide rule is to be simple to operate in this situation, the rectangular grid has to be replaced by a different set of curves.

Let $z = a + ib$. If $a > 0$, then $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,

$$a = r \cos \theta,$$

and

$$\log a = \log r + \log \cos \theta,$$

or

$$\rho = \log a - \log \cos \theta, \tag{1}$$

where $\rho = \log r$. We now draw the graphs (1) with $a = 1, 2, \ldots, 9$ (for $\rho \leqslant \log 10$), putting the ρ-axis horizontal and the θ-axis vertical. These are the solid curves in Figure 1.

If $z = a + ib$ and $b > 0$, then $0 < \theta < \pi$,

$$b = r \sin \theta = r \cos (\theta - \tfrac{1}{2}\pi)$$

and therefore

$$\rho = \log b - \log \cos (\theta - \tfrac{1}{2}\pi). \tag{2}$$

These curves, for $b = 1, 2, \ldots, 9$ ($\rho \leqslant \log 10$), are shown dotted in Figure 1. Of course, this set of curves is simply the first set translated by the amount $\frac{1}{2}\pi$ upwards. The point of intersection of the solid curve corresponding to a and the dotted curve corresponding to b has the co-ordinates (ρ, θ), where $\rho = \log |a + ib|$ and $\theta = \arg (a + ib)$; the point therefore represents the complex number $a + ib$. The complex numbers $a + i0$ ($a > 0$) are represented by the points on the line $\theta = 0$, and the complex numbers $0 + ib$ ($b > 0$) correspond to the points on the line $\theta = \frac{1}{2}\pi$.

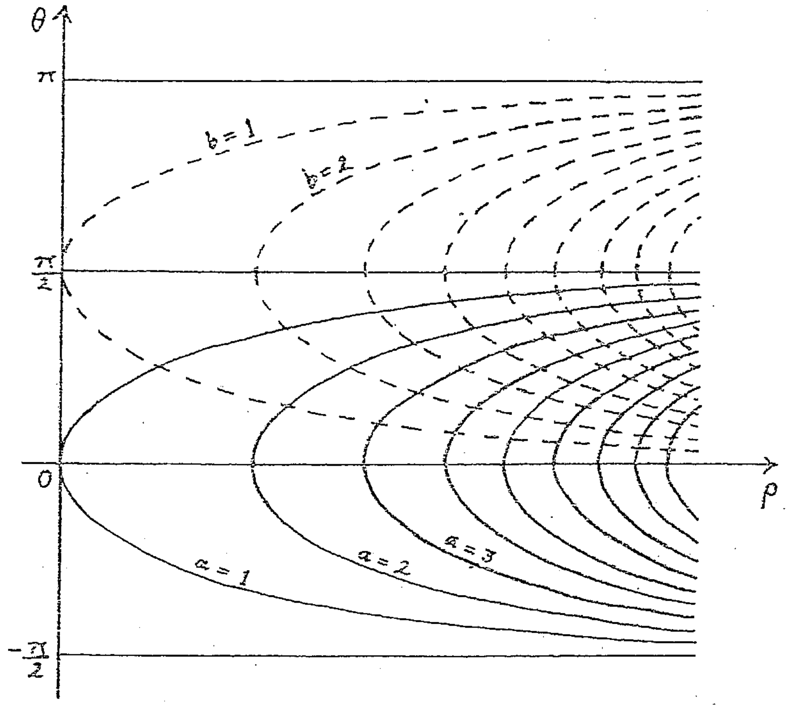

Figure 1

If $a < 0$, then

$$a = r\cos\theta = -r\cos(\theta - \pi),$$

so that

$$\rho = \log(-a) - \log\cos(\theta - \pi). \tag{3}$$

These curves, with $a = -1, -2, \ldots, -9$, are the curves (1) translated by an amount π.

If $b < 0$, then

$$b = r\sin\theta = -r\cos(\theta + \tfrac{1}{2}\pi)$$

and

$$\rho = \log(-b) - \log\cos(\theta + \tfrac{1}{2}\pi). \tag{4}$$

The curves (4), corresponding to $b = -1, -2, \ldots, -9$, are the curves (1) translated by an amount $-\frac{1}{2}\pi$.

The sets of curves (1)–(4), together with the lines $\theta = 0, \pm\frac{1}{2}\pi, \pi$, enable us to represent all complex numbers $z = a + ib$ with $1 \leqslant |z| \leqslant 10$. These curves, which are now labelled with the real or imaginary part that they represent (e.g., the curve $b = 2$ is labelled $2i$), are drawn on a sheet of paper. This sheet of paper is now mounted on a cardboard tube whose perimeter is equal to the distance along the θ-axis between the points 0 and 2π. This forms the stock of the slide rule and is shown in Figure 2. An identical set of curves is drawn on a piece of tracing paper which is then joined into a cylinder around the stock to form the slide.

To find the product of two numbers z_1 and z_2, turn the slide so that the origin, representing $1 + i0$, is over the point representing z_1 on the stock; the product is then found under the point representing z_2 on the slide.

The procedures for division and for dealing with numbers outside the range $1 \leqslant |z| \leqslant 10$ are similar to those for ordinary slide rules.

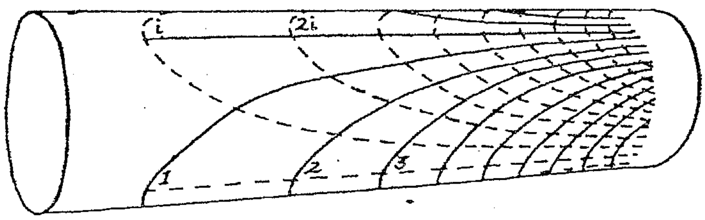

Figure 2

Notes

1. Since all the curves have the same shape, only one need be plotted by coordinates: the rest can be copied by a suitable method of tracing.
2. A scale of θ values drawn around one end of the stock, together with the scale of modulus values provided by the points representing $1 + i0, 2 + i0, 3 + i0$, etc., will enable numbers in the $a + ib$ form to be converted to the modulus-argument form (and *vice versa*).
3. If the scales of ρ (to base e) and θ (in radians) can be made equal, the two sets of curves will intersect at right angles, thus making numbers with non-integral parts easier to estimate. However, this requires a cylinder with an inconveniently large diameter.

Can You Contribute to Time Series Research?

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1. Introduction

Certain mildly mathematical problems of interest are currently being tackled by time series analysts. On the whole, these researchers do not have a very extensive mathematical background, but they are concerned with questions of practical importance in real life.

A time series is a sequence of historical data whose individual values are not independent, but interrelated, such as for example the monthly balance of payments series since the war. The motivation for the subject arises largely from the desire to forecast such a series. I have recently published a number of papers presenting new results, all of which could have been discovered by a mathematically competent sixth-former or undergraduate. In other words, this particular area of study could, in my opinion, be opened up to school research.

In this article, I propose to give some basic terminology for discrete time series, and then outline a problem whose solution has recently been obtained. The final section will list some other problems for readers to test themselves on, and I shall welcome solutions. Should the response be encouraging, a later article will pose problems whose solutions have not yet been attempted. It will be interesting to see whether, as I suspect, undergraduates and students at school can contribute to research in this field.

2. Moving average processes

Denote a time series by $z_1, \ldots, z_N$, a set of observations on the set of random variables $Z_1, \ldots, Z_N$. A moving average process of general order q, abbreviated to MA(q), is defined by

$$Z_i = A_i + \theta_1 A_{i-1} + \cdots + \theta_q A_{i-q}$$

where the random variables A_i to A_{i-q} are independent and identically distributed with zero means.

Such a process may help to explain, in an intuitively plausible way, how an economic series results from the buffeting effects of unpredictable events, such as strikes, policy changes or outbreaks of exceptional weather. These effects take time to be fully assimilated, and hence A_{10}, the value of A_1 at time 0, will have unit effect on Z_{10}, the value of Z_1 at time 0; however, it will have only θ_1 effect on Z_{11}, the value of Z_1 at time 1, θ_2 effect on Z_{12}, the value of Z_1 at time 2, and so on. The model was originally introduced to explain the behaviour of a pendulum subjected to random disturbance.

The theoretical autocorrelations for any stable process (not necessarily MA(q)) are defined by

$$\rho_j - \mathrm{Cov}[Z_i, Z_{i-j}]/\mathrm{Var}[Z_i] \quad j > 1,$$

where Cov denotes covariance, and Var variance.

One approach to time series analysis is to study the sampled autocorrelations r_j, and hence hope to recognise any underlying process ρ_j. To do this, the theoretical properties of the ρ_j must first be investigated.

It has long been known, and is easy to show, that for an MA(q)

$$\rho_j = 0 \quad j > q.$$

Thus, at a first guess, a set

$$\rho_1 = \cdots = \rho_q = \tfrac{2}{3}$$
$$\rho_j = 0 \quad j > q$$

could be compatible with an MA(q).

However, on further thought, this can be shown to be false. For instance, for $q = 1$, since the A's have zero mean and are independent,

$$\begin{aligned}\mathrm{Cov}[Z_i, Z_{i-1}] &= E[A_i + \theta A_{i-1}, A_{i-1} + \theta A_{i-2}]/E[(A_i + \theta A_{i-1})^2] \\ &= \theta\sigma_A^2/(1 + \theta^2)\sigma_A^2.\end{aligned}$$

In no way can $\theta/(1 + \theta^2)$ attain the value $\frac{2}{3}$, since its maximum is clearly $\frac{1}{2}$. Thus, for MA(1)

$$|\rho_1| \leqslant \tfrac{1}{2}. \tag{1}$$

Can this result be generalised? The reader can easily show that for MA(2)

$$|\rho_1| \leqslant \frac{1}{\sqrt{2}}. \tag{2}$$

Further, the general formula for ρ_1 of MA(q) is

$$\rho_1 = \sum_{r=0}^{q-1} \theta_r\theta_{r+1} \Big/ \sum_{r=0}^{q} \theta_r^2$$

and the greatest attainable magnitude for this is $\cos\{\pi/(q + 2)\}$, so that

$$|\rho_1| \leqslant \cos\{\pi/(q + 2)\}. \tag{3}$$

If the reader cannot obtain this independently, the details can be found in Anderson (reference 1). Note that this agrees with (1) and (2), and for an MA(0) such as the trivial process

$$Z_i = A_i$$

our result correctly gives $\rho_1 = 0$.

3. Problems for solution

(a) Further generalise (3) to obtain the result

$$|\rho_k| \leqslant \cos\left\{\frac{\pi}{[q/k] + 2}\right\}$$

for all k, where $[x]$ denotes 'the integer part of' x.

(b) Given two processes $\{X_i\}$ and $\{Y_i\}$, which are respectively MA(q_1) and MA(q_2), what can be said of the process $\{X_i + Y_i\}$?

(c) Given an MA(2) process, under what conditions can it be decomposed into the sum of two processes

$$X_i = B_i + \beta B_{i-1}$$
$$Y_i = C_i + \gamma C_{i-2}$$

where the B's and C's are unrelated, and each set consists of independent and identically distributed zero-mean random variables? Can you generalise this result?

References

O. D. Anderson, An inequality with a time series application. *Journal of Econometrics* 2 (1974), 189–193.

O. D. Anderson, *Time Series Analysis and Forecasting—the Box-Jenkins Approach* (Butterworth, London, 1974).

Factorisation of Quadratic Forms

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1. Introduction

Often in the study of the analytical geometry of conics and conicoids one encounters second-degree equations whose quadratic forms resolve into a pair of linear factors so that the equation represents a pair of lines or planes. The necessary and sufficient condition for this to happen is well known: it involves the vanishing of a certain third-order determinant of coefficients. Thus, for example, the problem arises in three-dimensional coordinate geometry in showing that the equation

$$15x^2 - 36y^2 - 8z^2 - 7xy + 19xz + 41yz + 24x - 62y + 50z - 12 = 0$$

represents a pair of planes and in finding the separate equations of the planes. In this paper such an investigation is considered from a new point of view: one which, it is believed, facilitates the task of finding the linear factors of a quadratic form which has such factors.

2. Necessary and sufficient conditions for a homogeneous quadratic form in three variables to have a pair of linear homogeneous factors

2.1. *Statement of problem.* We investigate the factorisation, if possible, of the quadratic form

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

a function of the three variables x, y, z for which the constant coefficients $a, \ldots, h$ may be real or complex. For the moment we suppose that at least one of a, b, c is not zero but the degenerate case $a = b = c = 0$ is subsequently discussed. Without loss of generality we suppose in the following that $a \neq 0$.

If we write

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2,$$

then the necessary and sufficient conditions for $Q(x, y, z)$ to have a pair of linear homogeneous factors is $\Delta = 0$. The proof of this is preceded by two lemmas.

2.2. *Lemma* I. A homogeneous quadratic form in two variables, such as $Q(x, y, 0) = ax^2 + 2hxy + by^2$, always has a pair of linear homogeneous factors, real or complex.

Lemma II. Suppose that $Q(x, y, 0)$ factorises as

$$Q(x, y, 0) = a(x + \lambda y)(x + \mu y)$$

and that $Q(x, 0, z)$ factorises as

$$Q(x, 0, z) = a(x + \rho z)(x + \sigma z).$$

Then if $Q(x, y, z)$ is known to have linear factors, they must assume one of the following alternative forms:

either

$$Q(x, y, z) = a(x + \lambda y + \rho z)(x + \mu y + \sigma z),$$

or

$$Q(x, y, z) = a(x + \lambda y + \sigma z)(x + \mu y + \rho z).$$

Of the two alternatives, the one which gives $2f$ as the coefficient of yz is selected. When neither gives such a coefficient, $Q(x, y, z)$ cannot be factorised.

Lemma II indicates how to obtain the factors in a numerical case.

2.3. *Necessity*. We start with the supposition that $Q(x, y, z)$ has a pair of linear homogeneous factors and prove that $\Delta = 0$.

By comparing coefficients of y^2 we see that

$$Q(x, y, 0) = a\left(x + \frac{h + \alpha}{a} y\right)\left(x + \frac{h - \alpha}{a} y\right)$$

if and only if α satisfies the relation

$$a\left(\frac{h^2 - \alpha^2}{a^2}\right) = b,$$

i.e.,

$$h^2 - ab = \alpha^2; \tag{1}$$

and similarly

$$Q(x, 0, z) = a\left(x + \frac{g + \beta}{a} z\right)\left(x + \frac{g - \beta}{a} z\right)$$

if and only if

$$g^2 - ac = \beta^2. \tag{2}$$

When α and β are chosen to satisfy (1) and (2) respectively, then, by Lemma II, either

$$Q(x, y, z) = a\left(x + \frac{h + \alpha}{a} y + \frac{g + \beta}{a} z\right)\left(x + \frac{h - \alpha}{a} y + \frac{g - \beta}{a} z\right) \tag{3}$$

or

$$Q(x, y, z) = a\left(x + \frac{h + \alpha}{a} y + \frac{g - \beta}{a} z\right)\left(x + \frac{h - \alpha}{a} y + \frac{g + \beta}{a} z\right). \tag{4}$$

The first alternative holds if

$$a\left(\frac{h+\alpha}{a}\cdot\frac{g-\beta}{a}+\frac{h-\alpha}{a}\cdot\frac{g+\beta}{a}\right)=2f,$$

i.e.,

$$\frac{1}{a}(2gh-2\alpha\beta)=2f,$$

or

$$gh-af=\alpha\beta; \tag{5}$$

and the second if

$$a\left(\frac{h+\alpha}{a}\cdot\frac{g+\beta}{a}+\frac{h-\alpha}{a}\cdot\frac{g-\beta}{a}\right)=2f,$$

i.e.,

$$\frac{1}{a}(2gh+2\alpha\beta)=2f,$$

or

$$gh-af=-\alpha\beta. \tag{5'}$$

Therefore, whether (5) or (5′) holds,

$$(gh-af)^2=(h^2-ab)(g^2-ac)$$

and (since $a\neq 0$) this is equivalent to the required condition

$$\Delta=0. \tag{6}$$

2.4. *Sufficiency.* We now suppose that (6) is given and prove that $Q(x, y, z)$ has a pair of linear homogeneous factors.

The numbers α, β are taken to satisfy (1) and (2) respectively, and consequently

$$Q(x,y,0)=a\left(x+\frac{h+\alpha}{a}y\right)\left(x+\frac{h-\alpha}{a}y\right),$$

$$Q(x,0,z)=a\left(x+\frac{g+\beta}{a}z\right)\left(x+\frac{g-\beta}{a}z\right).$$

We establish either (5) or (5′) from (1), (2) and (6). For these equations give

$$0=-a\Delta=(gh-af)^2-(h^2-ab)(g^2-ac)=(gh-af)^2-\alpha^2\beta^2,$$

i.e.,

$$gh-af=\pm\alpha\beta. \tag{7}$$

Hence we obtain either (3) or (4) according as the (+) or (−) sign holds in (7).

Thus, when $\Delta=0$, $Q(x, y, z)$ has linear homogeneous factors.

2.5. *Case* $a=b=c=0$. Now $Q(x,y,z)=2(fyz+gzx+hxy)$ and $\Delta=2fgh$.

If $f\neq 0$, the problem reduces to resolving into linear factors $yz+\lambda zx+\mu xy$, where $\lambda=g/f$, $\mu=h/f$. These factors can only assume the form $(y+\lambda x)(z+\mu x)$ and this requires $\lambda\mu=0$, i.e., either $g=0$ or $h=0$. Thus, if Q factorises, then $\Delta=0$ as before.

Conversely, $\Delta = fgh = 0$ implies that f, g or $h = 0$ and Q may then be factorised.

Thus the necessary and sufficient condition $\Delta = 0$ for Q to have linear factors also holds for the case $a = b = c = 0$.

3. Examples

1. Suppose

$$Q(x, y, z) = 6x^2 + 35y^2 + 24z^2 - 58yz + 24zx - 29xy.$$

Then

$$Q(x, y, 0) = 6x^2 - 29xy + 35y^2 = (3x - 7y)(2x - 5y);$$

$$Q(x, 0, z) = 6x^2 + 24xz + 24z^2 = (3x + 6z)(2x + 4z).$$

Hence

$$Q(x, y, z) = (3x - 7y + 6z)(2x - 5y + 4z),$$

since this form gives the correct coefficient of yz. Also $\Delta = 0$.

2. Consider

$$\begin{aligned} Q(x, y, z, w) = {} & 15x^2 - 36y^2 - 8z^2 - 12w^2 \\ & - 7xy + 19xz + 24xw + 41yz - 62yw + 50zw. \end{aligned}$$

We have successively

$$Q(x, y, 0, 0) = 15x^2 - 7xy - 36y^2 = (3x + 4y)(5x - 9y),$$

$$Q(x, 0, z, 0) = 15x^2 + 19xz - 8z^2 = (3x - z)(5x + 8z),$$

$$Q(x, 0, 0, w) = 15x^2 + 24xw - 12w^2 = (3x + 6w)(5x - 2w).$$

In each case the coefficients of x in corresponding brackets have been matched. This suggests the combination

$$Q(x, y, zw) = (3x + 4y - z + 6w)(5x - 9y + 8z - 2w).$$

It is easily seen that the factorisation is correct.

3. To factorise the non-homogeneous form

$$15x^2 - 36y^2 - 8z^2 - 7xy + 19xz + 41yz + 24x - 62y + 50z - 12$$

construct the homogeneous form $Q(x, y, z, w)$ of Example 2, proceed as in that Example and finally take $w = 1$.

4.
$$\begin{aligned} Q(x, y, z) &= 4x^2 + 10y^2 + z^2 + 4xy + 4xz + 2yz. \\ Q(x, y, 0) &= 4x^2 + 4xy + 10y^2 \\ &= [2x + (1 + 3i)y][2x + (1 - 3i)y]; \\ Q(x, 0, z) &= 4x^2 + 4xz + z^2 = (2x + z)(2x + z). \end{aligned}$$

This suggests

$$Q(x, y, z) = [2x + (1 + 3i)y + z][2x + (1 - 3i)y + z]$$

and this gives the correct coefficient of yz. Also $\Delta = 0$.

5. $$Q(x, y, z) = x^2 + y^2 - z^2.$$

As $\Delta \neq 0$, there are no linear factors. Also

$$Q(x, y, 0) = x^2 + y^2 = (x + iy)(x - iy),$$

$$Q(x, 0, z) = x^2 - z^2 = (x - z)(x + z).$$

But the forms

$$(x + iy - z)(x - iy + z); \quad (x + iy + z)(x - iy - z)$$

do not give the correct term in yz and so neither form is possible.

4. Extension to a homogeneous quadratic form in n variables

Such a quadratic form is

$$\begin{aligned} Q(x_1, x_2, \ldots, x_n) = {} & a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 \\ & + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots + 2a_{n-1,n}x_{n-1}x_n \end{aligned}$$

in the n variables $x_1, x_2, \ldots, x_n$. One may construct homogeneous quadratic functions $Q_{rst\ldots w}$ from $Q(x_1, x_2, \ldots, x_n)$ by taking all x's to be zero save $x_r, x_s, x_t, \ldots, x_w$ where $1 \leqslant r < s < t < \cdots < w \leqslant n$. If $a_{rs} = a_{sr}$, then Q_{rst} is the product of two linear factors if and only if

$$\Delta_{rst} = \begin{vmatrix} a_{rr} & a_{rs} & a_{rt} \\ a_{sr} & a_{ss} & a_{st} \\ a_{tr} & a_{ts} & a_{tt} \end{vmatrix} = 0.$$

One can prove that $Q(x_1, x_2, \ldots, x_n)$ decomposes into a pair of homogeneous linear factors in the n variables if and only if $\Delta_{rst} = 0$ for all admissible third order determinants formed from the coefficients of the x's in $Q(x_1, \ldots, x_n)$. In a practical case, such as Example 2, the procedure for finding the factors, based on an extension of Lemma II, renders such laborious tests unnecessary.

Dear Editor,

$$\textit{The series } \sum_{n=1}^{\infty} 1/n^{\alpha}$$

In his recent article (volume 7, number 1, pp. 9–12) Captain Draim demonstrated the divergence of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \tag{1}$$

in three different ways and asked for some other proofs. Here are two more.

The first approach uses the logarithmic function, but avoids integration. It relies on the fact that, whenever $x > 0$,

$$x > \log(1 + x). \tag{2}$$

This inequality can be proved in many ways. For instance it follows from the relation

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots > 1 + x.$$

Denote by S_n the sum of the first n terms of the series (1) and by T_n the sum of the first n terms of the series

$$\log\frac{2}{1} + \log\frac{3}{2} + \cdots + \log\frac{n+1}{n} + \cdots.$$

Then clearly $T_n = \log(n + 1)$ for every natural number n. But, by (2),

$$\frac{1}{n} > \log\left(1 + \frac{1}{n}\right) = \log\frac{n+1}{n}$$

and therefore $S_n > T_n = \log(n + 1)$. Hence $S_n \to \infty$ as $n \to \infty$, i.e., the series (1) diverges.

The second proof I offer is my favourite one. It has an idea similar to Draim's third proof, but it avoids the summation of double series. It is simple and elementary; and, on top of it, it lends itself to a nice generalisation, as we shall see later. The only general result which is required is that, if

$$s_1, s_2, s_3, \ldots$$

is an increasing sequence which is bounded above (i.e., to which corresponds a real number M such that $s_n \leqslant M$ for all n), then the sequence converges. One can accept the truth of this statement, or one can derive it from a fundamental property of the real number system which is discussed in all introductory texts on analysis. An equivalent statement is that, if

$$a_1 + a_2 + a_3 + \cdots$$

is a series of non-negative terms and the sequence

$$s_n = a_1 + a_2 + \cdots + a_n$$

of partial sums is bounded, then the series converges.

Now let us assume, contrary to what we want to prove, that the series (1) converges. Let its sum be S. Then the series

$$1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots \tag{3}$$

also converges, since its partial sums are evidently less than S. Denote the sum of the series (3) by R. Clearly

$$\begin{aligned} R - \tfrac{1}{2}S &= (1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots) - \tfrac{1}{2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots) \\ &= (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \cdots \\ &> \tfrac{1}{2}. \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned} S &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \\ &= (1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots) + \tfrac{1}{2}(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots) \\ &= R + \tfrac{1}{2}S. \end{aligned}$$

Hence $R = \frac{1}{2}S$. This contradicts (4) and completes the proof.

For any positive real number α, let us now put

$$S_n(\alpha) = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha}$$

and use similar calculations as before.

$$\begin{aligned} S_{2n}(\alpha) &= \left(1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \cdots + \frac{1}{(2n-1)^\alpha}\right) + \frac{1}{2^\alpha}\left(1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{n^\alpha}\right) \\ &= \left(1 + \frac{1}{3^\alpha} + \frac{1}{5^\alpha} + \cdots + \frac{1}{(2n-1)^\alpha}\right) + \frac{1}{2^\alpha} S_n(\alpha) \\ &< \left(1 + \frac{1}{2^\alpha} + \frac{1}{4^\alpha} + \cdots + \frac{1}{(2n-2)^\alpha}\right) + \frac{1}{2^\alpha} S_{2n}(\alpha) \\ &= 1 + \frac{1}{2^\alpha} S_{n-1}(\alpha) + \frac{1}{2^\alpha} S_{2n}(\alpha) \\ &< 1 + \frac{1}{2^\alpha} S_{2n}(\alpha) + \frac{1}{2^\alpha} S_{2n}(\alpha) \\ &= 1 + \frac{1}{2^{\alpha-1}} S_{2n}(\alpha). \end{aligned}$$

If $\alpha > 1$, it follows that

$$S_{2n}(\alpha) < \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}\,. \tag{5}$$

Inequality (5) shows that, when $\alpha > 1$, the partial sums of the series

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} \cdots + \frac{1}{n^\alpha} + \cdots \tag{6}$$

are bounded. Hence the series (6) converges for $\alpha > 1$. We need not investigate the series (6) for α 1,$\leqslant$ since then the divergence of the series (6) follows directly from the divergence of the series (1) and the observation that $1/n^\alpha \geqslant 1/n$.

Yours sincerely,

R. VÝBORNÝ
(University of Queensland, Australia)

Dear Editor,

Divergence of the harmonic series

Captain N. A. Draim (volume 7, number 1, pp. 9–12) invited readers to supply further proofs of the divergence of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots. \tag{1}$$

Here are three more.

(a) Suppose that the series (1) converges. Denoting its sum by S and halving each term we have

$$\frac{1}{2}S=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}+\cdots. \tag{2}$$

We now subtract (2) from (1) to get

$$\frac{1}{2}S=1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\cdots. \tag{3}$$

Finally we subtract (2) from (3) and obtain

$$0=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\cdots+\frac{1}{(2n-1)2n}+\cdots,$$

which is obviously false. Hence the series (1) cannot converge.

The other two proofs are more geometrical in nature.

(b) We consider two kinds of rectangles, with sides parallel to the axes, associated with the rectangular hyperbola $y = 1/x$ $(x > 0)$.

Rectangles of type A have one vertex at the origin and the opposite one on the hyperbola, and they lie below the line $y = 2$. They are shown dotted in Figure 1. Finite unions of these rectangles form polygons of type A. Any A-polygon can be increased in area by $\frac{1}{2}$ through the addition of an A-rectangle with base equal to twice the base of the widest rectangle in the polygon. It follows that polygons of type A can have an arbitrarily large area.

The sequence of rectangles of type B is shown bordered by solid lines in Figure 1. The first B-rectangle has base vertices (0, 0), (1, 0) and height 2, while the $(n + 1)$th $(n \geqslant 1)$ has base vertices $(n, 0)$, $(n + 1, 0)$ and height $1/n$. In the sequence of B-polygons the $(n + 1)$th polygon is the union of the first $n + 1$ B-rectangles and therefore has area

$$2+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

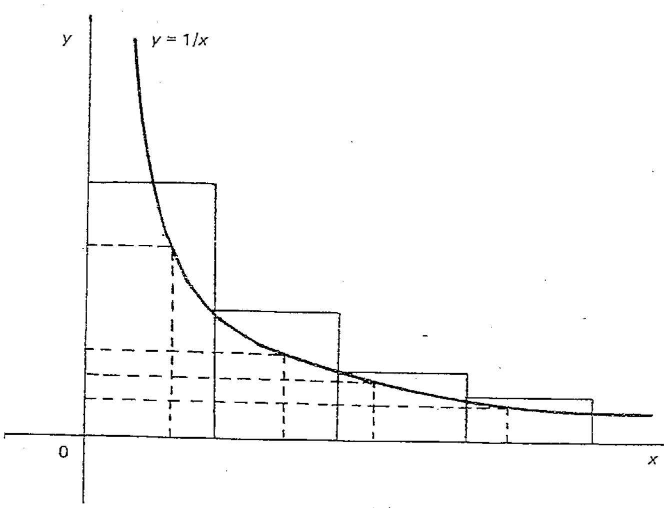

Figure 1

If the series (1) has finite sum S, then all B-polygons have area not exceeding $S + 2$. Since every A-polygon is clearly contained in a B-polygon, the area of every A-polygon is less than $S + 2$. But we have seen that the A-polygons can have arbitrarily large area. The contradiction shows that the series (1) diverges.

(c) On an infinite plane ocean a ship starts one mile north of a lighthouse and steams north-east to a point two miles from the lighthouse. After taking a bearing on the lighthouse the ship sets off in a direction such that the lighthouse is 45° on the starboard quarter (or has relative bearing 135°) and steams in a straight line until she is one mile further away from the lighthouse. This manoeuvre is repeated indefinitely, so that every time the ship is an integral number of miles from the lighthouse she changes course to bring the lighthouse to 45° on the starboard quarter.

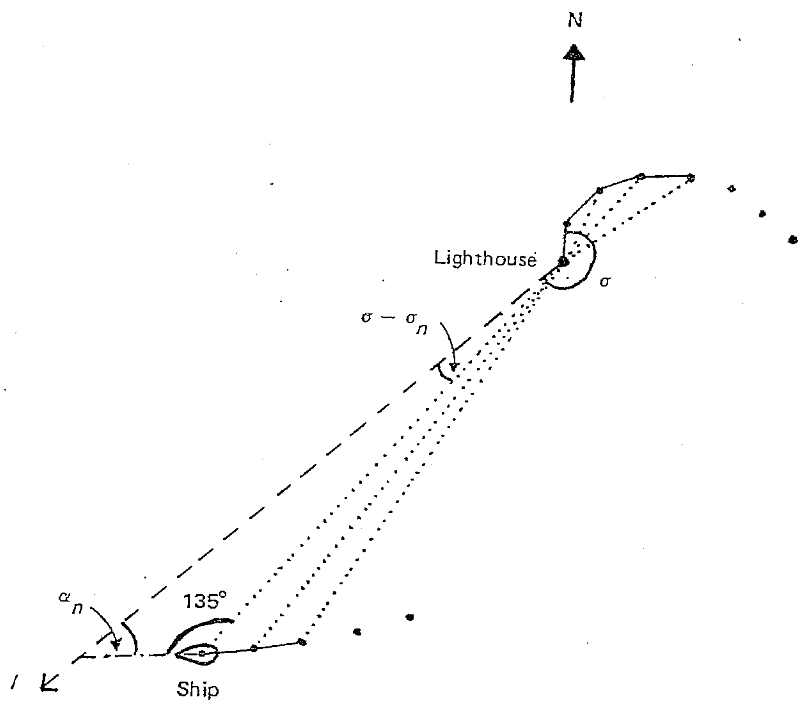

Figure 2

Denote by σ_n the bearing of the ship from the lighthouse when the two are n miles apart. If $\sigma_{n+1} - \sigma_n = \theta_n$, it is easily seen that

$$\cos\theta_n - \sin\theta_n = \frac{n}{n+1}.$$

Hence

$$\sin 2\theta_n = \frac{2n+1}{(n+1)^2} < \frac{2n}{(n^2+1)} = \sin\left(2\tan^{-1}\frac{1}{n}\right),$$

so that

$$0 < \theta_n < \tan^{-1}\frac{1}{n}$$

and

$$0 < \theta_n < \tan\theta_n < \frac{1}{n}.$$

Thus

$$\sigma_{n+1} = \theta_1 + \theta_2 + \cdots + \theta_n < 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \tag{4}$$

But σ_n also increases with n and therefore either tends to ∞ or to a limit σ.[1] In the latter case denote by l the half line whose points have a bearing σ from the lighthouse. When $\sigma - \sigma_n < \pi/8$, which is true for all sufficiently large n, the angle α_n between l and the ship's course satisfies the inequalities $\pi/8 < \alpha_n < \pi/4$ (see Figure 2). Thus the ship actually crosses l. Since this is false, σ_n tends to ∞ and, in view of (4), the harmonic series (1) diverges.

Yours sincerely,

B. C. RENNIE
(James Cook University of North Queensland, Australia)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

8.1. Let n be a positive integer. Show that (a) if $2^n - 1$ is prime, then n is prime, (b) if $2^n + 1$ is prime, then n must be a power of 2. Is the converse of (a) true?

(The Swiss mathematician Euler remarked that $2^{2^5} + 1 = 641 \times 6{,}700{,}417$, so the converse of (b) is false.)

8.2. A projectile is fired upwards from a cliff 45 metres high at an angle of 45° to the horizontal and lands in the sea at a distance 360 metres from the foot of the cliff. The operation is then repeated, but this time a wind of speed 2 metres/sec is blowing on shore. How does this affect the range of the projectile? With the wind blowing, could the range of the projectile be increased by altering the angle of inclination? (You may take the acceleration due to gravity to be 10 metres/sec^2.)

8.3. The real series $\Sigma\, a_n$, $\Sigma\, b_n$ are such that $\Sigma\, a_n$ is convergent, no a_n is zero and $b_n/a_n \to 1$ as $n \to \infty$. Does the series $\Sigma\, b_n$ have to be convergent?

Solutions to Problems in Volume 7, Number 2

7.4. Two players A and B begin with capital of p and q units respectively, and they gamble by tossing a coin; at each toss one unit of capital is transferred from the loser to the winner of that toss. (If, say, A calls 'heads', then heads he wins, tails he loses.) The game continues until one or the other is bankrupt. Compare A's and B's chances of winning.

[1] See the remarks about increasing sequences in the previous letter—Editor.

Solution by Paul Shovlar (King's College, London)

Let the probability that A will win when he has k units of capital be c_k. Since the game now passes to the situations in which A has $k-1$ and $k+1$ units of capital with equal probability 1/2, we have

$$c_k = \tfrac{1}{2}c_{k-1} + \tfrac{1}{2}c_{k+1} \quad (1 \leqslant k \leqslant p+q-1).$$

The general solution of this difference equation is

$$c_k = ak + b,$$

where a, b are arbitrary constants. But $c_0 = 0$ and $c_{p+q} = 1$, so that $b = 0$ and $a = 1/(p+q)$. Hence

$$c_k = \frac{k}{p+q}.$$

Thus the chance of A's winning the game is $p/(p+q)$, so the chance of B's winning is $q/(p+q)$ and their chances of winning are in the ratio $p:q$.

7.5. Two men stand on the edges of two cliffs, the heights of the cliffs above sea level being the same. The cliffs are separated by a deep chasm. The men point loaded pistols directly at each other (the pistols may not be of the same make) and each fires at the same moment. Show that the bullets collide.

Solution by L. S. Hayre (Imperial College)

Consider the vertical motion of the bullets only. They start with zero vertical speed and are acted upon by the same force—gravity. Thus the vertical speeds of the bullets will be the same at all times. Since they are level at the start, they will be level at all times. Since they are moving (horizontally) towards each other, this means that they will collide.

7.6. Show that, among any ten consecutive positive integers, at least one is relatively prime to all the others.

Solution by Paul Shovlar

Suppose we have ten consecutive positive integers such that none of the numbers is relatively prime to all the others. This means that each number has one of the numbers 2, 3, 5, 7 as a prime factor. Consider the odd numbers only. At most two of them can have prime factor 3, so at least three of them must have one of 5, 7 as a prime factor. Thus either two of them must have prime factor 5, or two must have prime factor 7. These will have to be consecutive multiples of 5 (resp. 7), which means that one of them is even. This is not so. The result follows.

Also solved by L. S. Hayre and The Problem Solving Group M 500 (Open University).

Note on problem 7.1. We recall the problem:

Two projectiles P, Q are fired from a point 0 at the same time. Describe how the direction and length of the straight line joining the projectiles vary with time during the subsequent flight. (Air resistance can be neglected.)

James Pretty (Education Centre, New University of Ulster) has pointed out a solution to this problem simpler than the one published in Volume 7, Number 3. The projectile Q has no vertical acceleration relative to P, nor does it have any relative horizontal acceleration. Thus Q moves in a straight line relative to P with no acceleration, so the direction of the straight line joining P and Q is constant during the subsequent flight and the distance between P and Q increases in proportion to the time of the flight.

Michael Kipling (St Bede's Grammar School, Bradford) has pointed out that the same conclusions hold if the paths of the projectiles are not in the same vertical plane, as is clear from the above argument.

The Seven Circles Theorem and Other New Theorems. By C. J. A. EVELYN, G. B. MONEY-COUTTS and J. A. TYRRELL. Stacey International, London, 1974. Pp. viii+68. £2·80.

This book is unusual in a number of ways. Firstly, it includes several theorems which, although they are new (dating from about 1970), can yet be understood quite easily. Secondly, the topics covered are unlikely to have practical application, nor are they relevant to examination syllabuses at school or university: the purpose of the text is evidently simply to give pleasure. And thirdly, a large proportion of the book (44 pages out of 76) is devoted to clear and beautiful diagrams.

There are three sections which are entirely independent of one another. The first is an interesting development of an algebra of vectors by projective methods which does not require ideas of congruence or parallelism. The second gives a number of new theorems which extend and generalise Pascal's theorem. The third and longest section presents some striking results about chains of circles touching one another, including the 'seven circles theorem' of the title.

The results of these theorems should be clear to anyone prepared to spend a few minutes looking at the admirable diagrams. The proofs require more effort, and also a certain amount of knowledge. It is perhaps a pity that the authors assume familiarity with a number of ideas which sadly may well be unknown to sixth formers and undergraduates (and even postgraduates) today. Thus the first section introduces without explanation points at infinity and Desargues' theorem, the second section needs some analytical geometry of conics and the third presumes that the theory of inversion is known. In each case the previous knowledge required is quite limited; the reader who has, say, Maxwell's *Geometry for Advanced Pupils* available for reference will soon be able to fill in the necessary background.

The book is lavishly produced and clearly and accurately printed. Those who think of Euclidean geometry as a thing of the past should be encouraged to look at it and change their minds. It should certainly be bought for the school library, and would make an admirable present for a student or teacher.

University of Birmingham — WILLIAM WYNNE WILLSON

Basic Pure Mathematics II. By A. S.-T. LUE. Van Nostrand Reinhold Company Ltd, London, 1974. Pp. 137. £1·50, paperback; £3·50, cloth.

This volume, which is aimed at honours science and joint honours mathematics students at King's College, London, is a sequel to B.P.M.I (by J. V. Armitage).

The text covers complex numbers, sequences and series, continuous functions of one variable (sic), mean value and Taylor's theorems, ordinary differential equations, functions of several variables (where 'several' means 'two') and Fourier series, all in 135 pages.

The treatment is generally competent though very traditional and uninspiredly turgid. A number of opportunities have been missed to bring liveliness and unity to the text. For example, in B.P.M.I, it is alleged that linear algebra in R^3 is covered; Dr Lue gives us a very solid account of partial derivatives and of the concept of 'differentiability' as a linear map, but then misses the chance to point out how such results as the chain rule can be elegantly and economically derived by matrix multiplication.

A student who thinks he would like to know something about analysis and to get a feel for the subject will be disappointed by this book. Nor is it a very good 'cook-book' for those students who are interested in mathematics simply as a tool for other disciplines. Good students are likely to find themselves frustrated, poor students confused.

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Notes on Contributors

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