Crux Mathematicorum

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Crux Mathematicorum

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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER

No. 98

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with twelve of the problems proposed to the jury but not used at the 29th I.M.O. in Australia. Thanks go to Peter O'Halloran for relaying a copy to me through Murray Klamkin. Send me your nice solutions!

1. *Proposed by Bulgaria.*

An integer sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 0$, $a_1 = 1$, and

$$a_n = 2a_{n-1} + a_{n-2} \quad (n > 1).$$

Prove that 2^k divides a_n if and only if 2^k divides n.

2. *Proposed by Canada.*

The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B and C meet the circle again at A', B' and C' respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of triangle ABC.

3. *Proposed by Czechoslovakia.*

The squares of an $n \times n$ chessboard ($n \geq 2$) are numbered by the numbers $1,2,\ldots,n^2$ (and every number occurs). Prove that there exist two abutting (with common edge) squares such that their numbers differ by at least n.

4. *Proposed by East Germany.*

Let $N = \{1,2,\ldots,n\}$, $n \geq 2$. A collection $\mathscr{A} = \{A_1,\ldots,A_t\}$ of subsets of N is said to be *separating* if, for every pair $\{x,y\} \subseteq N$, there is a set $A_i \in \mathscr{A}$ so that $A_i \cap \{x,y\}$ contains just one element. $\mathscr{A}$ is said to be *covering* if every element of N is contained in at least one set $A_i \in \mathscr{A}$. What is the smallest value $f(n)$ of t so that there is a collection $\mathscr{A} = \{A_1,\ldots,A_t\}$ which is simultaneously separating and covering?

5. *Proposed by France.*

Let a be the greatest positive root of the equation $x^3 - 3x^2 + 1 = 0$. Show that $[a^{1788}]$ and $[a^{1988}]$ are both divisible by 17. ($[x]$ denotes the integer part of x.)

6. *Proposed by Greece.*

In a triangle ABC, choose any points $K \in BC$, $L \in AC$, $M \in AB$, $R \in LM$, $S \in MK$ and $T \in KL$. If E_1, E_2, E_3, E_4, E_5, E_6, and E denote the areas of the triangles AMS,

CKS, BKT, ALT, BMR, CLR and ABC respectively, show that

$$E \geq 8(E_1E_2E_3E_4E_5E_6)^{1/6}.$$

7. *Proposed by Hungary.*

For what values of n does there exist an $n \times n$ array of entries, each -1, 0 or 1, such that the $2n$ sums obtained by summing the elements of the rows and the columns are all different?

8. *Proposed by Iceland.*

Let ABC be an acute–angled triangle. Three lines L_a, L_b and L_c are constructed through the vertices A, B and C respectively as follows. Let H be the foot of the altitude drawn from the vertex A to the side BC and let the circle with diameter AH meet the sides AB and AC at M and N respectively, where M and N are distinct from A. Then L_a is the line through A perpendicular to MN. The lines L_b and L_c are constructed similarly. Prove that L_a, L_b and L_c are concurrent.

9. *Proposed by Israel.*

In the convex pentagon $ABCDE$, the sides BC, CD, DE are equal. Moreover each diagonal of the pentagon is parallel to a side (AC is parallel to DE, BD is parallel to AE, etc.). Prove that $ABCDE$ is a regular pentagon.

10. *Proposed by Mexico.*

Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose that

$$f(f(n) + f(m)) = n + m$$

for all positive integers n, m. Find all possible values for $f(1988)$.

11. *Proposed by Morocco.*

Find the largest natural number n such that, if the set $\{1,2,\ldots,n\}$ is arbitrarily divided into two non–intersecting subsets, then one of the subsets contains three distinct numbers such that the product of two of them equals the third.

12. *Proposed by Poland.*

Forty–nine students solve a set of three problems. The score for each problem is a whole number of points from 0 to 7. Prove that there exist two students A and B such that, for each problem, A will score at least as many points as B.

* * *

We now return to solutions submitted for problems posed in the 1987 numbers of *Crux*.

2. [1987: 38] *Final Round of the Bulgarian Olympiad 1985.*

Find all values of the parameter a such that all roots of the equation
$$x^6 + 3x^5 + (6-a)x^4 + (7-2a)x^3 + (6-a)x^2 + 3x + 1 = 0$$
are real numbers.

Solution by J.T. Groenman, Arnhem, The Netherlands.

On division by x^3 we have
$$x^3 + \frac{1}{x^3} + 3(x^2 + \frac{1}{x^2}) + (6-a)(x + \frac{1}{x}) + 7 - 2a = 0.$$
Thus
$$(x + \frac{1}{x})^3 - 3(x + \frac{1}{x}) + 3(x + \frac{1}{x})^2 - 6 + (6-a)(x + \frac{1}{x}) + 7 - 2a = 0.$$
This becomes
$$(x + \frac{1}{x})^3 + 3(x + \frac{1}{x})^2 + (3-a)(x + \frac{1}{x}) + (1-2a) = 0.$$

Let $u = x + 1/x$. (Notice that x is real just in case $|u| \geq 2$ since we have $x^2 - ux + 1 = 0$ which has discriminant $u^2 - 4$.) In terms of u we have
$$u^3 + 3u^2 + (3-a)u + 1 - 2a = 0.$$
Let $u = v - 1$. Now u and v are both real or both complex. Also $|u| \geq 2$ becomes $v \leq -1$ or $v \geq 3$. In terms of v the equation is thus
$$(v-1)^3 + 3(v-1)^2 + (3-a)(v-1) + (1-2a) = 0$$
or
$$v^3 - av - a = 0.$$
Hence $v \neq -1$ and $a = v^3/(v+1)$.

Thus we consider the image of
$$f(v) = \frac{v^3}{v+1}, \quad v \in (-\infty,-1) \cup [3,\infty).$$
We claim $f(v) \geq 27/4$ on its domain. Now
$$f'(v) = \frac{3v^2(v+1) - v^3}{(v+1)^2} = \frac{v^2(2v+3)}{(v+1)^2}.$$
From this we see that $f(v)$ is decreasing on $(-\infty,-3/2)$ and increasing on $(-3/2,-1)$. The minimum value on this interval is then $f(-3/2) = 27/4$. On the interval $[3,\infty)$, $f'(v)$ is positive so $f(v)$ is increasing. Its minimum is then $f(3) = 27/4$ on $[3,\infty)$. Thus the image of f is $[27/4,\infty)$.

Hence the roots are all real just if $a \geq 27/4$.

[*Editor's note*: This problem was also solved by George Evagelopoulos, Athens, Greece.]

4. [1987: 39] *Final Round of the Bulgarian Olympiad 1985.*

Let $P_1, P_2, \ldots, P_7$ be seven points in space, no four of which lie on a plane. Color each of the line segments P_iP_j, $i < j$, with one of two colors, red or black. Prove that there are two monochromatic triangles that have no common side. (We say that a triangle is monochromatic if all its sides are colored with the same color.)

Is the analogous statement true for six points?

Solutions by George Evagelopoulos, Athens, Greece, and René Schipperus, The University of Calgary.

We label the points $A,B,\ldots$. First observe that there must be at least one monochromatic triangle, since this much is well known to be true even for six points. (Consider the edges from one point A to five other points B, C, D, E, F. At least three have the same colour, say AB, AC and AD are red. If any of the edges BC, CD, BD are also red, we are done. Otherwise BCD is a black triangle.)

Let ABC be a monochromatic triangle, coloured, say, red. Removing A we have six remaining points and so we must again have a monochromatic triangle, not containing A as a vertex. If this triangle does not contain the edge BC, we are done. So we have a fourth point A' such that $A'BC$ is also a red triangle. Similarly by removing B and C we obtain B' and C' such that $AB'C$ and ABC' are red triangles.

We first consider the possibility that A', B', and C' are not distinct. In this case one of them forms a red tetrahedron with ABC. There are three remaining points. If the edges between any one of these remaining points and two vertices of the tetrahedron are red we have two edge disjoint red triangles, one using the new point and one edge of the tetrahedron, and the other on the tetrahedron. Otherwise, each point has at most one red edge to the tetrahedron. Thus there is at least one vertex of the tetrahedron joined to all three remaining vertices by black edges. If any pair of the three are joined by a black edge we have a black and a red triangle. Otherwise we obtain two vertex disjoint red triangles.

Therefore we may suppose that A', B' and C' are distinct. If any two, say A' and B', are joined by a red edge we have two red triangles ABC and $A'B'C$. Otherwise ABC and $A'B'C'$ give a red and a black triangle, respectively. This completes the proof.

The statement is not true for six points. For a counterexample, let $P_1P_2P_3P_4$ form a red tetrahedron, let P_5P_6 be also red, but P_iP_j be black for $1 \le i \le 4$, $5 \le j \le 6$.

5. [1987: 39] *Final Round of the Bulgarian Olympiad 1985.*

Let ABC ($AC \neq BC$) be a triangle with $\gamma = \angle ACB$ a given acute angle, and let M be the midpoint of AB. Point P is chosen on the line segment CM so that the bisectors of the angles PAC and PBC meet at a point Q on CM. Find the angles APB and AQB.

Solution by George Evagelopoulos, Athens, Greece and by J.T. Groenman, Arnhem, The Netherlands.

Inscribe the triangle ABC in the circle Γ. Since AQ bisects $\angle PAC$, an application of the law of sines gives

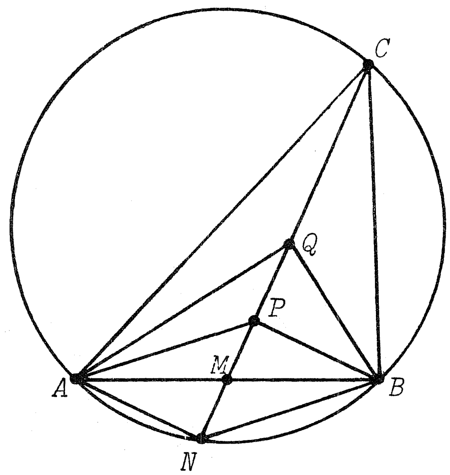
$$\frac{|QP|}{|QC|} = \frac{|AP|}{|AC|}.$$

Similarly,

$$\frac{|QP|}{|QC|} = \frac{|BP|}{|BC|}.$$

Thus $|AP|/|AC| = |BP|/|BC|$ and

$$\frac{|AP|}{|BP|} = \frac{|AC|}{|BC|} = \frac{b}{a}. \qquad (1)$$



Let us denote $|MP| = k|MC|$ and set $\angle BMC = \theta$. Using the law of cosines in the triangles BMP and AMP, we find that

$$|BP|^2 = \frac{c^2}{4} + k^2|MC|^2 - ck|MC|\cos\theta \qquad (2)$$

and

$$|AP|^2 = \frac{c^2}{4} + k^2|MC|^2 - ck|MC|\cos(\pi - \theta) = \frac{c^2}{4} + k^2|MC|^2 + ck|MC|\cos\theta. \qquad (3)$$

Now from (1), (2), and (3) we have that

$$\frac{a^2}{b^2} = \frac{c^2/4 + k^2|MC|^2 - ck|MC|\cos\theta}{c^2/4 + k^2|MC|^2 + ck|MC|\cos\theta}.$$

Equivalently,

$$\frac{a^2c^2}{4} + a^2k^2|MC|^2 + a^2ck|MC|\cos\theta = \frac{b^2c^2}{4} + b^2k^2|MC|^2 - b^2ck|MC|\cos\theta.$$

This becomes

$$\frac{c^2}{4}(a^2 - b^2) + k^2|MC|^2(a^2 - b^2) + ck|MC|\cos\theta(a^2 + b^2) = 0. \qquad (4)$$

But using the law of cosines in the triangles CMA and CMB we find that

$$b^2 = \frac{c^2}{4} + |MC|^2 + c|MC|\cos\theta$$

and

$$a^2 = \frac{c^2}{4} + |MC|^2 - c|MC|\cos\theta \qquad (5)$$

so that

$$b^2 - a^2 = 2c|MC|\cos\theta. \qquad (6)$$

Thus from (5)

$$c^2 + 4|MC|^2 - 4a^2 = 4c|MC|\cos\theta = 2(b^2 - a^2),$$

giving

$$4|MC|^2 = 2b^2 + 2a^2 - c^2. \qquad (7)$$

Substituting (7) and (6) into (4), we get that

$$k^2(2a^2 + 2b^2 - c^2)(a^2 - b^2) + 2k(b^2 - a^2)(a^2 + b^2) + c^2(a^2 - b^2) = 0$$

and thus, since $a \neq b$,

$$k^2(2a^2 + 2b^2 - c^2) - 2k(a^2 + b^2) + c^2 = 0$$

which can be written

$$(k - 1)[k(2a^2 + 2b^2 - c^2) - c^2] = 0.$$

The case $k = 1$ corresponds to choosing $P = C$ which we disallow, so

$$k = \frac{c^2}{2a^2 + 2b^2 - c^2} = \frac{c^2}{4|MC|^2} \quad \text{(by (7))}.$$

Therefore $|MP| \cdot |MC| = k|MC|^2 = c^2/4$.

Let N be the point where CM meets Γ. Then

$$|MN| \cdot |MC| = |MB| \cdot |MA| = c^2/4,$$

calculating the power of the point M with respect to Γ. From $|MP| \cdot |MC| = |MN| \cdot |MC|$ we conclude that $|PM| = |MN|$. Thus $ANBP$ is a parallelogram and consequently

$$\angle APB = \angle ANB = \pi - \gamma.$$

Also,

$$\begin{aligned}
\angle AQB &= \pi - (\angle BAQ + \angle ABQ) \\
&= \pi - (\angle BAP + \angle PAQ + \angle ABP + \angle PBQ) \\
&= \pi - [(\angle BAP + \angle ABP) + \tfrac{1}{2}(\angle PAC + \angle PBC)] \\
&= \pi - [(\pi - \angle APB) + \tfrac{1}{2}(\angle APB - \angle ACB)] \\
&= \pi - [\gamma + \tfrac{1}{2}(\pi - 2\gamma)] = \tfrac{\pi}{2}.
\end{aligned}$$

6. [1987: 39] *Final Round of the Bulgarian Olympiad 1985.*

For every positive integer a let α_a be the greatest odd divisor of a. Also set

$$s_b = \sum_{a=1}^{b} \frac{\alpha_a}{a}.$$

Prove that the sequence $\{s_b/b\}_{b=1}^{\infty}$ is convergent and find its limit.

Solution by George Evagelopoulos, Athens, Greece.

Let $a \in \mathbb{N}^+$. Let its unique prime factorization be

$$a = 2^{n_1(a)}3^{n_2(a)}5^{n_3(a)} \cdots p_k^{n_k(a)}$$

where $n_1(a), n_2(a), n_3(a), \ldots, n_k(a)$ are nonnegative integers and p_k is the kth prime number. Then

$$\alpha_a = 3^{n_2(a)} \cdots p_k^{n_k(a)}$$

which gives

$$s_b = \sum_{a=1}^{b} \frac{\alpha_a}{a} = \sum_{a=1}^{b} \frac{1}{2^{n_1(a)}}. \tag{1}$$

Let $[x]$ stand for the greatest integer not exceeding x. Now among the numbers $1,2,\ldots,b$ there are $b - [b/2]$ odd numbers, $[b/2] - [b/4]$ numbers which are divisible by 2 but not by 4, and in general $[b/2^k] - [b/2^{k+1}]$ numbers which are divisible by 2^k but not by 2^{k+1}. Regrouping the sum (1) we then have

$$s_b = (b - [\tfrac{b}{2}]) + \tfrac{1}{2}([\tfrac{b}{2}] - [\tfrac{b}{4}]) + \cdots + \frac{1}{2^k}\left[\left[\frac{b}{2^k}\right] - \left[\frac{b}{2^{k+1}}\right]\right] + \cdots$$

$$= b - \tfrac{1}{2}[\tfrac{b}{2}] - \frac{1}{2^2}\left[\frac{b}{2^2}\right] - \cdots - \frac{1}{2^k}\left[\frac{b}{2^k}\right] - \cdots . \qquad (2)$$

But we always have

$$\frac{b}{2^k} - 1 < \left[\frac{b}{2^k}\right] \le \frac{b}{2^k}$$

for $k = 1,2,\ldots$. Substituting successively $k = 1,2,\ldots$ in these inequalities and substituting in (2) we obtain

$$b - \sum_{k=1}^{\infty}\left[\frac{1}{2^k}\cdot\frac{b}{2^k}\right] \le s_b \le b - \sum_{k=1}^{\infty}\frac{1}{2^k}\left[\frac{b}{2^k} - 1\right] .$$

Thus

$$b - \tfrac{1}{3}b \le s_b \le b - \tfrac{1}{3}b + 1$$

so

$$\tfrac{2}{3} \le \tfrac{s_b}{b} \le \tfrac{2}{3} + \tfrac{1}{b}$$

and

$$\lim_{b\to\infty} \frac{s_b}{b} = \frac{2}{3} .$$

*

1. [1987: 70] *6th Brazilian Mathematical Olympiad, 1984.*

Find all natural numbers n and k such that $(n + 1)^k - 1 = n!$

Editor's comments.

George Evagelopoulos points out that the problem is number 4 of the 1983 Australian Olympiad and its solution is in *Crux* [1986: 23]. Solutions were also submitted by Edward Wang of Wilfrid Laurier University, Waterloo, and by Josep Rifa i Coma of the Institut "Jaume Callis", Barcelona, Spain. Wang points out that many such diophantine problems occur. One that was first posed in the *American Mathematical Monthly* in 1942 (E534, Vol. 49, #7, p.475) is to show that the only solutions of the diophantine equation $n! + 1 = k^2$ are $(n,k) = (4,5)$, $(5,11)$ and $(7,71)$.

2. [1987: 70] *6th Brazilian Mathematical Olympiad, 1984.*

The 289 students enrolled in a course are to be distributed into 17 classes each having 17 students, for each of several units of instruction. No two students may be assigned to the same class for different units. What is the largest number of units for which an assignment is possible following this rule?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We consider the general case when n^2 students enrolled in a course are to be distributed into n classes each having n students with the stipulated rule. Let l_n denote the largest number of units for which the described assignment is possible. Then for each of these units the number of *unordered* pairs of students in each of the n classes is $\begin{bmatrix} n \\ 2 \end{bmatrix}$. Hence the total number of pairs for the l_n units of instruction is $\begin{bmatrix} n \\ 2 \end{bmatrix} \cdot n \cdot l_n$ which cannot exceed $\begin{bmatrix} n^2 \\ 2 \end{bmatrix}$. This yields the upper bound $l_n \le n + 1$. If n is a prime, then from the general theory of finite geometry, the affine plane $AP(F)$ determined by a finite field F with n elements has $n + 1$ parallel classes of n lines each. Each line contains exactly n points and each point is on exactly $n + 1$ lines. (See, for example, R.A. Brualdi, *Introductory Combinatorics*, §9.2.) Each of these parallel classes can be identified with a partition of the n^2 students into n classes each having n students. Thus $l_n = n + 1$ is possible where n is prime. For this problem $n = 17$ and so the largest number of units possible is 18.

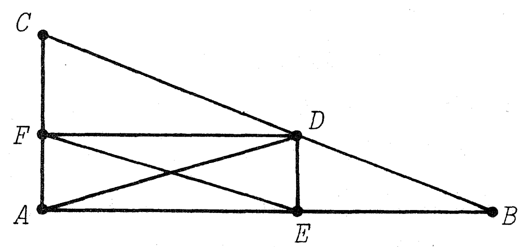
Comment. Using the terminology in combinatorics, the general problem discussed above is equivalent to the existence of a resolvable BIBD (balanced incomplete block design) with parameters $(v,b,r,k,\lambda) = (n^2, n^2 + n, n + 1, n, 1)$. See for example A.P. Street and W.D. Wallis, *Combinatorial Theory: An Introduction*, Theorems 3 and 4, pp. 183–184.

4. [1987: 70] *6th Brazilian Mathematical Olympiad, 1984.*

Let D be a point on the hypotenuse BC of a right triangle ABC. Suppose points E and F, on the legs of the triangle, are such that DE and DF are perpendicular to the corresponding leg. Find the position of D that minimizes the length EF.

Solution by J.T. Groenman, Arnhem, The Netherlands.

Notice that the length of EF is equal to that of AD; they are diagonals of a rectangle. Obviously we want D to be the foot of the perpendicular from A to BC to minimize the length.



6. [1987: 71, 140] *6th Brazilian Mathematical Olympiad, 1984.*

Figure 1 shows a board used in a game called "One Left". The game starts with one marker on each square of the board except for the central square which has no marker. Let A, B, C (or C, B, A) be three adjacent squares in a horizontal row or a vertical column. If A and B are occupied and C is not, then the marker at A may be jumped to C, and the marker at B removed (see Figure 2). Is it possible to end the game with one marker in the position shown in Figure 3?

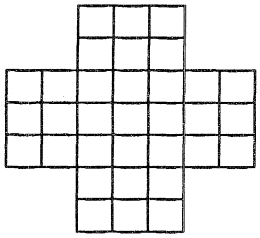


Figure 1

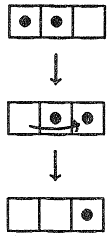


Figure 2

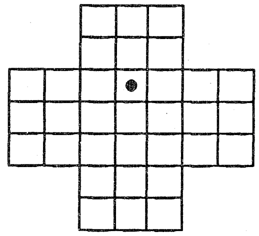


Figure 3

Solutions by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario and Andy Liu, The University of Alberta, Edmonton, Alberta.

Color the 33 squares using the three colors blue (B), green (G), and red (R) as shown in the figure. (Squares on the same diagonal from north-east to south–west are given the same color.) Notice that there are exactly 11 squares of each color. By a green marker we mean a marker situated on a green square, etc. At the beginning of the game, since the center square is empty, the number of green markers is 10 while the number of blue and red markers are both 11. Each time a jump is made, the number of markers in *each* color group must change by one. For example, a jump by a green marker over a blue marker onto an unoccupied red square would increase the number of red markers by one, while decreasing the number of green and blue markers each by one. Therefore, the parity of the number of blue markers is always the same as that of the number of red markers throughout the game, while the parity of the number of green markers will be different. If the last marker were to land on the square above the center square, then the numbers of blue and red markers remaining, being one and zero respectively, will have opposite parity, a contradiction.

		B	G_1	R		
		G	R	B		
B	G	R	B	G	R	B
G_2	R	B	G_3	R	B	G_4
R	B	G	R	B	G	R
		R	B	G		
		B	G_5	R		

Editor's comments.

E.T.H. Wang adds that with the original empty square at the center, the last marker must land on a green square, and that with a further analysis it is possible to show that it must land on one of G_1, G_2, G_3, G_4 or G_5. The game has been called "The French Puzzle" or "Solitaire" and a complete analysis of the possible finishing positions for each of the 33 initial vacant squares can be found in *Mathematical Games* by C. Lukács and E. Tarján, Ch.

6 (translated by John Dabai, published by Walker). A. Liu extracted the solution from J.D. Beasley, *The Ins and Outs of Peg Solitaire*, Oxford University Press, 1985 and E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways, Vol.* 2, Academic Press, 1982.

* * *

Send in your solutions and copies of national contests.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **May 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

1371*. *Proposed by Murray S. Klamkin, University of Alberta.*

In *Math. Gazette* 68 (1984) 222, P. Stanbury noted the close approximation $\pi^9/e^8 \approx 9.999838813 \approx 10$. Are there positive integers l, m such that π^ℓ/e^m is closer to a positive integer than for the case given? (See *Crux* 1213 [1988: 116] for a related problem.)

1372. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC has circumcentre O and median point G, and the lines AG and BG intersect the circumcircle again at A_1 and B_1 respectively. Suppose the points A, B, O and G are concyclic. Show that

(a) $AA_1 = BB_1$;

(b) ΔABC is acute angled.

1373. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Evaluate

$$\sum_{n=1}^{\infty} \frac{9^n - 6^{n-1}}{(3^n - 2^n)(3^{n+1} - 2^{n+1})(3^{n+2} - 2^{n+2})}.$$

1374. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $A_1A_2...A_9$ be a regular 9–gon. Show that A_1, A_2 lie on one branch of a hyperbola and A_5, A_8 lie on the other, and determine the ratio of the axes of this hyperbola.

1375. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Evaluate

$$\int_0^{\pi/4} \frac{e^{\sec x}\sin(x+\pi/4)}{\cos x\,(1-\sin x)}\,dx\,.$$

1376. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let $ABCD$ be a quadrilateral with an inscribed circle of radius r and a circumscribed circle of radius R. Let $AC = p$ and $BD = q$ be the diagonals. Prove that

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = 1.$$

1377. *Proposed by Colin Springer, student, Waterloo, Ontario.*

In right triangle ABC, hypotenuse AC has length 2. Let O be the midpoint of AC and let I be the incentre of the triangle. Show that $\overline{OI} \geq \sqrt{2} - 1$.

1378*. *Proposed by J. Walter Lynch, Georgia Southern College, Statesboro, Georgia.*

Suppose $a_0, a_1, a_2, ...$ is a sequence of positive real numbers such that $a_0 = 1$ and $a_n = a_{n+1} + a_{n+2}$, $n \geq 0$. Find a_n.

1379. *Proposed by P. Penning, Delft, The Netherlands.*

Given are an arbitrary triangle ABC and an arbitrary interior point P. The pedal-points of P on BC, CA, and AB are D, E, and F respectively. Show that the normals from A to EF, from B to FD, and from C to DE are concurrent.

1380. *Proposed by Kee–Wai Lau, Hong Kong.*

Prove the inequality

$$\sin(\tan x) < \tan(\sin x)$$

for $0 < x < \pi$, $x \neq \pi/2$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1224. [1987: 86; 1988: 145] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

$A_1A_2A_3$ is a triangle with circumcircle Ω. Let $x_1 < X_1$ be the radii of the two circles tangent to A_1A_2, A_1A_3, and arc A_2A_3 of Ω. Let x_2, X_2, x_3, X_3 be defined analogously. Prove that:

(a) $$\sum_{i=1}^{3} \frac{x_i}{X_i} = 1;$$

(b) $$\sum_{i=1}^{3} X_i \geq 3 \sum_{i=1}^{3} x_i \geq 12r,$$

where r is the inradius of $\Delta A_1A_2A_3$.

III. *Solution by G.R. Veldkamp, De Bilt, The Netherlands.*

Pondering the question of how to construct the circles Γ_k (with radius X_k) and γ_k (with radius x_k), $k = 1,2,3$, I got the following solution of the problem itself.

The inversion $(A_1,(s - a_1)^2)$ transforms Ω into a straight line ℓ_1 perpendicular to OA_1, i.e. antiparallel to A_2A_3, and meeting A_1A_2 and A_1A_3 at points S (between A_1 and A_2) and T (between A_1 and A_3) respectively such that $A_1S = (s - a_1)^2/a_3$. Hence $\Delta A_1ST \sim \Delta A_1A_3A_2$, where the ratio of similitude is obviously

$$f = \frac{(s - a_1)^2}{a_2a_3} = \frac{s - a_1}{s}\cdot\frac{s(s - a_1)}{a_2a_3} = \frac{s - a_1}{s}\cos^2\frac{\alpha_1}{2}.$$

The circle Γ_1 is now the image by inversion of the incircle c of ΔA_1ST. The radius of c is fr, r being the inradius of $\Delta A_1A_2A_3$. The power of A_1 with respect to c is $f^2(s - a_1)^2$. It follows that

$$X_1 = fr\cdot\frac{(s - a_1)^2}{f^2(s - a_1)^2} = \frac{r}{f} = \frac{rs}{(s - a_1)\cos^2(\alpha_1/2)} = \frac{r_1}{\cos^2(\alpha_1/2)},$$

where r_1 is the exradius of $\Delta A_1A_2A_3$ on A_2A_3 [see for example p.50–51 of R.A. Johnson, *Advanced Euclidean Geometry*]. Similarly, γ_1 is the image by inversion of the excircle c_1 of ΔA_1ST, the radius of c_1 is fr_1, and the power of A_1 with respect to c_1 is f^2s^2. We find that

$$x_1 = fr_1\cdot\frac{(s - a_1)^2}{f^2s^2} = \frac{r_1(s - a_1)}{s\cos^2(\alpha_1/2)} = \frac{r}{\cos^2(\alpha_1/2)},$$

so that

$$\frac{x_1}{X_1} = \frac{r}{r_1},$$

which is (3) on [1988: 146]. The proof now proceeds easily as before.

* * *

1260. [1987: 181] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokaisi, Aichi, Japan.*

Let ABC be a triangle with angles B and C acute, and let H be the foot of the perpendicular from A to BC. Let O_1 be the circle internally tangent to the circumcircle O of ΔABC and touching the segments AH and BH. Let O_3 be the circle similarly tangent to O, AH and CH. Finally let O_2 be the incircle of ΔABC, and denote the radii of O_1, O_2, O_3 by r_1, r_2, r_3, respectively.

(a) Show that $r_2 = \frac{r_1 + r_3}{2}$.

(b) Show that the centers of O_1, O_2, O_3 are collinear.

Editor's comment.

This problem was taken from the 1841 Japanese book *Sanpo Jojutu.* It turns out that a stronger result is known; Francisco Bellot has kindly contributed the following information.

A very elegant generalization (see below) was proposed by V. Thébault in 1938 in the *American Mathematical Monthly* (problem 3887, Vol. 45, p.482), and not solved until 1983, by K.B. Taylor, whose long argument is summarized in the *Monthly* in Vol. 90, pp.486–487. A much shorter proof, due to Gerhard Turnwald, later appeared (in German) in *Elemente der Mathematik* Vol. 41 (1986) 11–13. For the convenience of the reader we now give the theorem and Turnwald's proof. The notation differs slightly from the original problem.

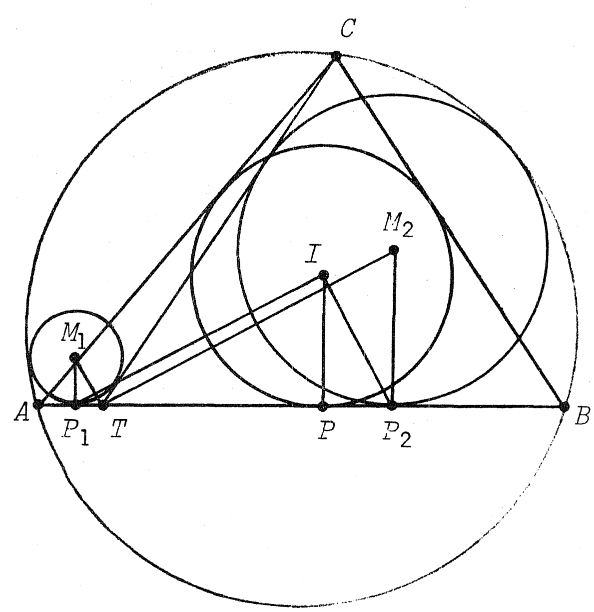
Theorem. T is an arbitrary point on the side c of triangle ABC. (M_1,r_1) and (M_2,r_2) are the circles tangent respectively to sides AT, CT and to sides BT, CT, and also internally tangent to the circumcircle of ΔABC. (I,r) is the incircle of ΔABC. Then M_1, M_2, and I are collinear. Moreover, if θ denotes half the angle ATC, then

$$\overline{M_1I} : \overline{IM_2} = \sin^2\theta : \cos^2\theta \qquad (1)$$

and

$$r_1\cos^2\theta + r_2\sin^2\theta = r. \qquad (2)$$

Proof. To start we shall only assume as given ΔABC (and its incircle (I,r) and circumcircle (U,R)) and the point T on AB. The angles and sides of ΔABC will be denoted α, β, γ and a, b, c as usual. Then we shall *construct* two points M_1 and M_2, show that they lie on a line through I, that they satisfy (1) and (2), and finally prove that they are the centres of the two circles described in the statement of the theorem.



Set

$$\theta = \frac{1}{2}\angle ATC,$$

and let P be the foot of the perpendicular from I to AB. Find points P_1, P_2 on AB such that P_1 is between A and P, P_2 is between P and B, and

$$\angle PIP_1 = \theta = \angle PP_2I.$$

(This is possible: since

$$\alpha + \angle ATC < 180^\circ$$

we have

$$\angle PAI + \theta < 90^\circ,$$

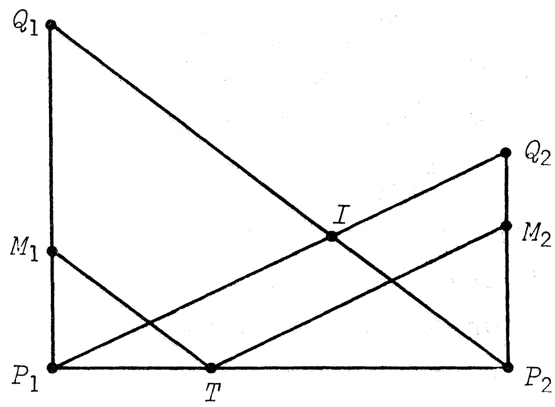
and thus $\angle PIA > \theta$, showing that P_1 exists; and from $\beta < \angle ATC$ we have $\angle PBI < \theta$, and P_2 exists.) Let M_1 and M_2 be the intersections of the normals to AB at P_1 and P_2 with the bisectors of $\angle ATC$ and $\angle BTC$, respectively. Then

$$\angle M_1TP_1 = \theta = \angle IP_2T$$

and

$$\angle M_2TP_2 = 90^\circ - \theta = \angle IP_1T,$$

so $M_1T \| IP_2$ and $M_2T \| IP_1$.



To prove that M_1, M_2, and I are collinear, we put

$$Q_1 = P_1M_1 \cap P_2I,\ Q_2 = P_2M_2 \cap P_1I.$$

Then

$$\frac{\overline{P_1M_1}}{\overline{M_1Q_1}} = \frac{\overline{P_1T}}{\overline{TP_2}} = \frac{\overline{Q_2M_2}}{\overline{M_2P_2}},$$

and since $Q_1P_1 \| Q_2P_2$ it follows that M_1, I and M_2 are collinear.

Now

$$\overline{P_1P} = r\tan\theta \text{ and } \overline{PP_2} = r\cot\theta, \tag{3}$$

so

$$\overline{M_1I} : \overline{IM_2} = \overline{P_1P} : \overline{PP_2} = \tan\theta : \cot\theta = \sin^2\theta : \cos^2\theta,$$

which is (1). Also, putting $r_1 = \overline{P_1M_1}$ and $r_2 = \overline{P_2M_2}$ we have

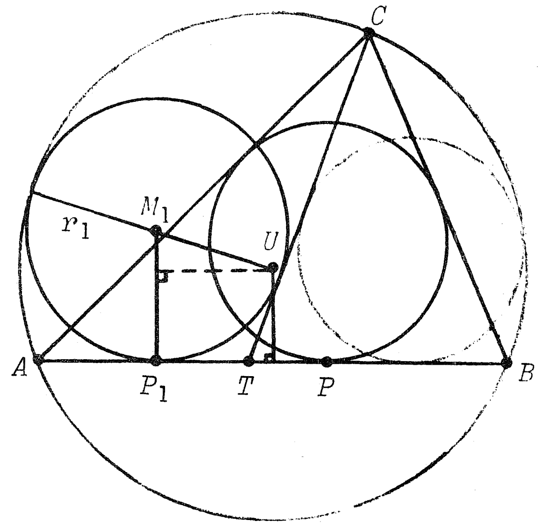
$$r_1 = \overline{P_1T}\cdot\tan\theta \text{ and } r_2 = \overline{TP_2}\cdot\cot\theta, \tag{4}$$

which with (3) gives

$$\begin{aligned} r_1\cos^2\theta + r_2\sin^2\theta &= (\overline{P_1T} + \overline{TP_2})\sin\theta\cos\theta \\ &= (\overline{P_1P} + \overline{PP_2})\sin\theta\cos\theta \\ &= r\sin^2\theta + r\cos^2\theta = r, \end{aligned}$$

i.e., (2).

Consider the circles $O_1 = (M_1, r_1)$ and $O_2 = (M_2, r_2)$. O_1 is tangent to AB and, since M_1 is on the bisector of $\angle ATC$, O_1 is tangent to TC as well. Similarly, O_2 is tangent to AB and



TC. It remains to prove that O_1 and O_2 are tangent to the circumcircle (U,R). We prove this for O_1; the proof for O_2 is similar. We need to show that

$$\overline{UM_1} = R - r_1.$$

The distance from U to AB is

$$\frac{c}{2}\cot\gamma,$$

so we wish to prove that

$$(\overline{P_1B} - \tfrac{c}{2})^2 + (r_1 - \tfrac{c}{2}\cot\gamma)^2 = (R - r_1)^2$$

which simplifies to

$$\overline{P_1B}^2 - c\overline{P_1B} + \frac{c^2}{4}\csc^2\gamma - cr_1\cot\gamma = R^2 - 2Rr_1. \tag{5}$$

Using

$$R = \frac{c}{2}\csc\gamma,$$

(5) is equivalent to

$$\overline{P_1B}^2 - c\overline{P_1B} - cr_1\cot\gamma = -cr_1\csc\gamma,$$

or

$$\frac{1-\cos\gamma}{\sin\gamma}cr_1 = \overline{P_1B}(c - \overline{P_1B}). \tag{6}$$

From (3),

$$\overline{P_1B} = \overline{P_1P} + \overline{PB} = r\tan\theta + s - b \tag{7}$$

where s is the semiperimeter of ΔABC, and so

$$c - \overline{P_1B} = b + c - s - r\tan\theta = s - a - r\tan\theta.$$

Also,

$$\frac{1-\cos\gamma}{\sin\gamma} = \tan\frac{\gamma}{2} = \frac{r}{s-c}.$$

Thus (6) is equivalent to

$$\begin{aligned} r_1 &= \frac{s-c}{rc}(r\tan\theta + s - b)(s - a - r\tan\theta) \\ &= \frac{s-c}{rc}[(s-a)(s-b) - r(a-b)\tan\theta - r^2\tan^2\theta] \\ &= \frac{s-c}{c}\left[\frac{rs}{s-c} - (a-b)\tan\theta - r\tan^2\theta\right], \end{aligned} \tag{8}$$

where we used

$$r^2s = (s-a)(s-b)(s-c).$$

It therefore remains to prove (8). By (4) and (7),

$$\begin{aligned} r_1 &= \overline{P_1T}\tan\theta = (\overline{P_1B} - \overline{TB})\tan\theta \\ &= (r\tan\theta + s - b - \overline{TB})\tan\theta, \end{aligned} \tag{9}$$

and by the law of sines applied to ΔTBC,

$$TB = \frac{a\ \sin(2\theta - \beta)}{\sin 2\theta} = a\left[\cos\beta - \frac{\sin\beta\cos 2\theta}{\sin 2\theta}\right]$$
$$= a[\cos\beta + \tfrac{1}{2}\sin\beta(\tan\theta - \cot\theta)]. \qquad (10)$$

It follows from (9) and (10) that

$$r_1 = \frac{a}{2}\sin\beta + (s - b - a\cos\beta)\tan\theta + (r - \frac{a}{2}\sin\beta)\tan^2\theta. \qquad (11)$$

Finally, using

$$\frac{ac\ \sin\beta}{2} = \text{area } \Delta ABC = rs$$

and

$$2ac\cos\beta = a^2 + c^2 - b^2$$

we conclude from (11) that

$$cr_1 = rs + \tfrac{1}{2}[(a - b + c)c - (a^2 + c^2 - b^2)]\tan\theta + (rc - rs)\tan^2\theta$$
$$= rs - (a - b)(s - c)\tan\theta - r(s - c)\tan^2\theta,$$

which is equivalent to (8). □

This theorem, with a somewhat longer proof, was also found by Dan Sokolowsky.

Solved by J.T. GROENMAN, Arnhem, The Netherlands; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer. The proposer also sent a translation of a solution by S. IWATA.

* * *

<u>1261</u>*. [1987: 215] *Proposed by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

The insphere of the tetrahedron $ABCD$ with equilateral base BCD touches the faces ACD, ADB, ABC at points E, F, G respectively. If $BE = CF = DG$, must the tetrahedron be regular? (This is an extension of *Crux* 1133 [1986: 77; 1987: 225].)

Editor's confession.

In transcribing this problem for publication, the editor unwisely substituted the word "tetrahedron" for the proposer's word "pyramid". This led two readers, naturally enough, to conclude that the answer to the problem is an easy NO, as any tetrahedron with $AB = AC = AD$ will have an insphere with the right property. Two other readers, however, treated the problem as the proposer intended, showing that $AB = AC = AD$ must occur. One of them proved a somewhat stronger result, and his solution appears below. All's well that ends well? Maybe, but the editor still apologizes for messing up.

Solution by G.P. Henderson, Campbellcroft, Ontario.

If a tetrahedron $ABCD$ has an equilateral base BCD, if $AB = AC = AD$, and if a sphere (not necessarily the insphere) inscribed inside the trihedral angle at A touches the faces opposite B, C, D at E, F, G respectively, then $BE = CF = DG$. This suggests the converse: if $BC = CD = DB$ and $BE = CF = DG$, then $AB = AC = AD$. We will show that this is true for a certain set of spheres which includes the insphere.

Without loss of generality, let the coordinates of the vertices of the tetrahedron be A: (x_1,y_1,z_1), B: $(2,0,0)$, C: $(-1,\sqrt{3},0)$, D: $(-1,-\sqrt{3},0)$, where $z_1 > 0$, and let the sphere have centre (x_0,y_0,z_0) and radius r. Suppose $BE = CF = DG$. Then we prove:

Theorem. If

$$\frac{z_0}{r} \leq \frac{-5\sqrt{5}}{9} \quad \text{or} \quad \frac{z_0}{r} \geq 0.40941,$$

then $AB = AC = AD$. If

$$\frac{-5\sqrt{5}}{9} < \frac{z_0}{r} < 0.40941,$$

then we do not necessarily have $AB = AC = AD$. In particular, if the sphere is the insphere then $z_0 = r$, so $AB = AC = AD$ must hold.

Proof. The plane ACD is

$$z_1 x - (x_1 + 1)z + z_1 = 0.$$

The vector from (x_0,y_0,z_0) to E is of length r along the normal to this plane. Therefore E is

$$E\colon \left[x_0 - \frac{r z_1}{\sqrt{(x_1+1)^2+z_1^2}}\,,\ y_0\,,\ z_0 + \frac{r(x_1+1)}{\sqrt{(x_1+1)^2+z_1^2}}\right].$$

We will see in a moment that the correct sense has been chosen for the normal. Since E is in ACD,

$$x_0 z_1 - z_0(x_1 + 1) - r\sqrt{(x_1+1)^2+z_1^2} + z_1 = 0.$$

Thus

$$x_0 z_0 - z_0(x_1 + 1) + z_1 > 0,$$

and since z_1 was chosen positive, it follows at once that (x_0,y_0,z_0) and B are on the same side of ACD, thus verifying the sign of the normal.

Similarly, the plane ABD is

$$-z_1 x + z_1 y\sqrt{3} + (x_1 - y_1\sqrt{3} - 2)z + 2z_1 = 0,$$

and F is

$$F\colon \left[x_0 + \frac{r z_1}{\sqrt{(x_1-y_1\sqrt{3}-2)^2+4z_1^2}}\,,\ y_0 - \frac{r z_1\sqrt{3}}{\sqrt{(x_1-y_1\sqrt{3}-2)^2+4z_1^2}}\,,\ z_0 - \frac{r(x_1 - y_1\sqrt{3} - 2)}{\sqrt{(x_1-y_1\sqrt{3}-2)^2+4z_1^2}}\right]$$

where

$$-x_0 z_1 + y_0 z_1\sqrt{3} + z_0(x_1 - y_1\sqrt{3} - 2) - r\sqrt{(x_1-y_1\sqrt{3}-2)^2+4z_1^2} + 2z_1 = 0.$$

ABC and G are as above with $\sqrt{3}$ replaced by $-\sqrt{3}$.

Introducing new variables

$$u = \frac{x_1 + 1}{z_1}, \quad v = \frac{-x_1 + y_1\sqrt{3} + 2}{2z_1}, \quad w = \frac{-x_1 - y_1\sqrt{3} + 2}{2z_1},$$

the coordinates of E, F, G become

$$E: \left[x_0 - \frac{r}{\sqrt{1 + u^2}}, \ y_0, \ z_0 + \frac{ru}{\sqrt{1 + u^2}} \right],$$

$$F: \left[x_0 + \frac{r}{2\sqrt{1 + v^2}}, \ y_0 - \frac{r\sqrt{3}}{2\sqrt{1 + v^2}}, \ z_0 + \frac{rv}{\sqrt{1 + v^2}} \right],$$

$$G: \left[x_0 + \frac{r}{2\sqrt{1 + w^2}}, \ y_0 + \frac{r\sqrt{3}}{2\sqrt{1 + w^2}}, \ z_0 + \frac{rw}{\sqrt{1 + w^2}} \right],$$

where

$$\left.\begin{aligned} x_0 - uz_0 - r\sqrt{1 + u^2} + 1 &= 0, \\ -x_0 + \sqrt{3}y_0 - 2vz_0 - 2r\sqrt{1 + v^2} + 2 &= 0, \\ -x_0 - \sqrt{3}y_0 - 2wz_0 - 2r\sqrt{1 + w^2} + 2 &= 0. \end{aligned}\right\} \tag{1}$$

Also, $z_1 > 0$ is equivalent to

$$u + v + w > 0. \tag{2}$$

Since $BE = CF = DG$,

$$\left[x_0 - \frac{r}{\sqrt{1 + u^2}} - 2 \right]^2 + y_0^2 + \left[z_0 + \frac{ru}{\sqrt{1 + u^2}} \right]^2$$

$$= \left[x_0 + \frac{r}{2\sqrt{1 + v^2}} + 1 \right]^2 + \left[y_0 - \frac{r\sqrt{3}}{2\sqrt{1 + v^2}} - \sqrt{3} \right]^2 + \left[z_0 + \frac{rv}{\sqrt{1 + v^2}} \right]^2$$

$$= \left[x_0 + \frac{r}{2\sqrt{1 + w^2}} + 1 \right]^2 + \left[y_0 + \frac{r\sqrt{3}}{2\sqrt{1 + w^2}} + \sqrt{3} \right]^2 + \left[z_0 + \frac{rw}{\sqrt{1 + w^2}} \right]^2.$$

Simplifying these, we get

$$\frac{-2x_0 r}{\sqrt{1 + u^2}} - 6x_0 + \frac{4r}{\sqrt{1 + u^2}} + \frac{2z_0 ru}{\sqrt{1 + u^2}}$$

$$= \frac{x_0 r}{\sqrt{1 + v^2}} + \frac{4r}{\sqrt{1 + v^2}} - \frac{y_0 r\sqrt{3}}{\sqrt{1 + v^2}} - 2y_0\sqrt{3} + \frac{2z_0 rv}{\sqrt{1 + v^2}}$$

$$= \frac{x_0 r}{\sqrt{1 + w^2}} + \frac{4r}{\sqrt{1 + w^2}} + \frac{y_0 r\sqrt{3}}{\sqrt{1 + w^2}} + 2y_0\sqrt{3} + \frac{2z_0 rw}{\sqrt{1 + w^2}},$$

which can be written

$$\frac{-2r}{\sqrt{1 + u^2}}(x_0 - uz_0) - 6x_0 + \frac{4r}{\sqrt{1 + u^2}} = \frac{-r}{\sqrt{1 + v^2}}(-x_0 + y_0\sqrt{3} - 2vz_0) + \frac{4r}{\sqrt{1 + v^2}} - 2y_0\sqrt{3}$$

$$= \frac{-r}{\sqrt{1 + w^2}}(-x_0 - y_0\sqrt{3} - 2wz_0) + \frac{4r}{\sqrt{1 + w^2}} + 2y_0\sqrt{3}.$$

Using (1), this becomes

$$\frac{-2r}{\sqrt{1+u^2}}(r\sqrt{1+u^2}-1) - 6x_0 + \frac{4r}{\sqrt{1+u^2}} = \frac{-r}{\sqrt{1+v^2}}(2r\sqrt{1+v^2}-2) + \frac{4r}{\sqrt{1+v^2}} - 2y_0\sqrt{3}$$

$$= \frac{-r}{\sqrt{1+w^2}}(2r\sqrt{1+w^2}-2) + \frac{4r}{\sqrt{1+w^2}} + 2y_0\sqrt{3}$$

which simplifies to

$$-3x_0 + \frac{3r}{\sqrt{1+u^2}} = -y_0\sqrt{3} + \frac{3r}{\sqrt{1+v^2}} = y_0\sqrt{3} + \frac{3r}{\sqrt{1+w^2}}.$$

Solving for x_0 and y_0, we get

$$x_0 = \frac{r}{2}\left[\frac{2}{\sqrt{1+u^2}} - \frac{1}{\sqrt{1+v^2}} - \frac{1}{\sqrt{1+w^2}}\right],$$

$$y_0 = \frac{r\sqrt{3}}{2}\left[\frac{1}{\sqrt{1+v^2}} - \frac{1}{\sqrt{1+w^2}}\right].$$

Substituting in (1):

$$\left.\begin{aligned}
r\left[\frac{2u^2}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+v^2}} + \frac{1}{\sqrt{1+w^2}}\right] + 2uz_0 &= 2,\\
r\left[\frac{1}{\sqrt{1+u^2}} + \frac{2v^2}{\sqrt{1+v^2}} + \frac{1}{\sqrt{1+w^2}}\right] + 2vz_0 &= 2,\\
r\left[\frac{1}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+v^2}} + \frac{2w^2}{\sqrt{1+w^2}}\right] + 2wz_0 &= 2.
\end{aligned}\right\} \qquad (3)$$

Eliminating r and z_0:

$$\begin{vmatrix}
\frac{2u^2}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+v^2}} + \frac{1}{\sqrt{1+w^2}} & u & 1\\
\frac{1}{\sqrt{1+u^2}} + \frac{2v^2}{\sqrt{1+v^2}} + \frac{1}{\sqrt{1+w^2}} & v & 1\\
\frac{1}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+v^2}} + \frac{2w^2}{\sqrt{1+w^2}} & w & 1
\end{vmatrix} = 0.$$

Multiplying the third column by

$$\frac{1}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+v^2}} + \frac{1}{\sqrt{1+w^2}}$$

and subtracting from the first column, we get

$$\begin{vmatrix} f(u) & u & 1 \\ f(v) & v & 1 \\ f(w) & w & 1 \end{vmatrix} = 0, \tag{4}$$

where (for later convenience) we use

$$f(x) = -\frac{1}{2}\left[\frac{2x^2 - 1}{\sqrt{1 + x^2}}\right] = \frac{1 - 2x^2}{2\sqrt{1 + x^2}}.$$

If we choose u, v and w satisfying (2) and (4) and such that (3) determines a positive r, we will have a configuration for which $BE = CF = DG$. We find that

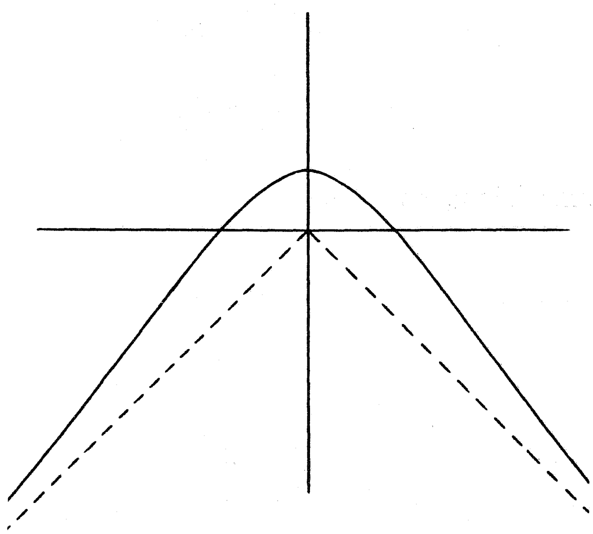
$$AB = AC = AD \Leftrightarrow x_1 = y_1 = 0 \Leftrightarrow u = v = w.$$

This is certainly one way to get $BE = CF = DG$. Just choose arbitrary $u = v = w > 0$ and arbitrary $r > 0$, and determine z_0 from (3).

To find configurations for which AB, AC and AD are not all equal, we assume from now on that u, v and w are not all equal. From (4) we see that u, v and w correspond to collinear points on the graph of $f(x)$. Assuming $u \neq v$ say, and subtracting the second equation of (3) from the first, we get that

$$\frac{z_0}{r} = \frac{\dfrac{1 - 2u^2}{2\sqrt{1 + u^2}} - \dfrac{1 - 2v^2}{2\sqrt{1 + v^2}}}{u - v} = \frac{f(u) - f(v)}{u - v},$$

that is, the slope of the line L through such a triple of collinear points is z_0/r.



The graph of $y = f(x)$ has a maximum at (0,1) and crosses the x-axis at $(\pm 1/\sqrt{2},0)$. It approaches the asymptote $y = -x$ from above as $x \to \infty$ and the asymptote $y = x$ as $x \to -\infty$.

First we consider the case in which the slope of L is positive or zero. We can assume $u \leq v \leq w$. Then (2) implies $w > 0$.

We want to show that $v = w$. If $v < w$, we see from the graph that $v < 0$. From (2), $w > -v$. Since $f(x)$ is a decreasing function for $x \geq 0$, $f(w) < f(-v) = f(v)$. The slope of L is then

$$\frac{f(w) - f(v)}{w - v} < 0,$$

a contradiction. Therefore $v = w > 0$ and $-2v < u \leq 0$.

For v fixed, the slope of L is a decreasing function of u and is therefore less than the slope of the chord through $(v, f(v))$ and $(-2v, f(-2v))$. That is,

$$\frac{z_0}{r} < \frac{f(v) - f(-2v)}{3v} = \frac{1}{6}\left[\frac{1 - 2v^2}{v\sqrt{1 + v^2}} - \frac{1 - 8v^2}{v\sqrt{1 + 4v^2}}\right].$$

By calculus, the maximum value, m_2, of this latter expression occurs at the positive root v_2 of

$$256v^8 + 160v^6 - 75v^4 - 65v^2 - 5 = 0.$$

The numerical values are

$$v_2 \approx 0.78300 \quad \text{and} \quad m_2 \approx 0.40941.$$

Thus if the slope $z_0/r \geq m_2$, then $AB = AC = AD$. For any given z_0/r in $[0, m_2)$ there exists a tetrahedron with AB, AC, AD not all equal. It can be obtained by choosing $v = w = v_2$ and a suitable u in $(-2v_2, -v_2]$. To see that $r > 0$, we use the first two equations of (3) to get

$$r = \frac{2(v - u)}{\dfrac{2u^2v - u}{\sqrt{1 + u^2}} + \dfrac{2v - u(2v^2 + 1)}{\sqrt{1 + v^2}}}$$

$$= \frac{2}{\dfrac{1 - 2uv}{v - u}\left[\dfrac{v}{\sqrt{1 + v^2}} - \dfrac{u}{\sqrt{1 + u^2}}\right] + \dfrac{1}{\sqrt{1 + v^2}}} \tag{5}$$

which is positive since $u < 0 < v$.

We now consider the case $z_0/r < 0$. The graph of $f(x)$ has a point of inflection at $x = \sqrt{5}/2$ and the slope of the curve at this point is

$$m_1 = \frac{-5\sqrt{5}}{9} \approx -1.24226.$$

Any chord must have a slope greater than this. Hence if $z_0/r \leq m_1$, then $AB = AC = AD$. For any given value of z_0/r in $(m_1, 0)$ a tetrahedron with AB, AC, AD not all equal can be obtained by choosing $v = w$ in $(0, \sqrt{5}/2)$ such that $f'(v)$ is slightly less than z_0/r and by choosing u close to v so that the slope of the chord is z_0/r. Then since

$$\frac{d}{dx}\left[\frac{x}{\sqrt{1 + x^2}}\right] = \frac{1}{(1 + x^2)^{3/2}},$$

by letting $u \to v$ in (5) we have

$$r \to \frac{2}{\dfrac{1 - 2v^2}{(1 + v^2)^{3/2}} + \dfrac{1}{\sqrt{1 + v^2}}} = \frac{2(1 + v^2)^{3/2}}{2 - v^2},$$

which is positive since $v < \sqrt{5}/2$. Thus r is positive for u and v close enough together. □

Note: Similar results hold in the two–dimensional case. Let ABC be a triangle. Suppose a circle of radius r is tangent to AB at E and AC at D, and suppose that $BD = CE$. Let y_0 be the distance of the centre of the circle from BC, where $y_0 > 0$ for points inside ΔABC and < 0 for points outside. Then $AB = AC$ if

$$\frac{y_0}{r} \leq -\sqrt{2} \quad \text{or} \quad \frac{y_0}{r} \geq 0.$$

AAGE BONDESEN, Royal Danish School of Educational Studies, Copenhagen, also solved the proposer's original problem in the affirmative. RICHARD I. HESS, Rancho Palos Verdes, California and P. PENNING, Delft, The Netherlands solved the editor's version in the negative.

* * *

<u>1262</u>. [1987: 215] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.* (Dedicated to Léo Sauvé.)

Pick a random n–digit decimal integer, leading 0's allowed, with each integer being equally likely. What is the expected number of distinct digits in the chosen integer?

Solution by Hans Engelhaupt, Gundelsheim, Federal Republic of Germany.

For every n–digit decimal integer N I define the random variable X_i, $i = 0,1,\ldots,9$, by

$$X_i = \begin{cases} 1 & \text{if the digit } i \text{ doesn't occur in } N, \\ 0 & \text{otherwise.} \end{cases}$$

Then the probability

$$P(X_i = 1) = (0.9)^n,$$

so the expected value of X_i is

$$E(X_i) = (0.9)^n.$$

Now the random variable

$$X: = \text{"number of digits which don't occur in } N\text{"}$$

satisfies

$$X = X_0 + X_1 + \cdots + X_9,$$

so

$$E(X) = 10(0.9)^n.$$

Thus the expected number of distinct digits in an n–digit decimal integer is

$$10 - 10(0.9)^n.$$

Also solved by JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer. One other reader misinterpreted the problem.

Several solvers pointed out the natural generalization to n–digit numbers in base b, where the expected number of distinct digits comes out to be

$$b - b\left[\frac{b-1}{b}\right]^n.$$

* * *

1263. [1987: 215] *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokaisi, Aichi, Japan.*

Find all triangles whose sides are three consecutive integers and whose area is an integer.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let the sides be n, $n+1$, $n+2$. Then the semiperimeter

$$s = \frac{3}{2}(n+1),$$

so by Heron's formula

$$A = \frac{1}{4}\sqrt{3(n+1)^2(n+3)(n-1)} = \frac{n+1}{4}\sqrt{3(n+3)(n-1)}.$$

For A to be an integer, n must be odd, say $n = 2k-1$. Then

$$A = k\sqrt{3(k^2-1)}.$$

Now

$$3(k^2-1) = m^2$$

for some integer m, and $m = 3q$ for some integer q, so we get

$$k^2 - 1 = 3q^2.$$

This is a Pell–Fermat equation with the following set of solutions:

q	k	n	$n+1$	$n+2$	A
1	2	3	4	5	6
4	7	13	14	15	84
15	26	51	52	53	1170
56	97	193	194	195	16296
⋮	⋮	⋮	⋮	⋮	⋮

In general, the k's satisfy the recurrence

$$k_i = 4k_{i-1} - k_{i-2}$$

so the n's satisfy

$$n_i = 4n_{i-1} - n_{i-2} + 2.$$

A closed form solution is

$$k_i = \frac{(2+\sqrt{3})^i + (2-\sqrt{3})^i}{2},$$

so that

$$n_i = (2+\sqrt{3})^i + (2-\sqrt{3})^i - 1.$$

Also solved by SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, *Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; M.A. SELBY, University of Windsor, Windsor, Ontario; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer. Two other readers sent in incomplete solutions.*

The problem was taken from an 1825 Japanese sangaku.

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1264. [1987: 216] *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Define the *character* of a finite set of real numbers to be the sum of its maximum element and minimum element. Evaluate the average of the characters of all the nonempty subsets of $\{1,2,\ldots,n\}$.

I. *Solution by Leroy F. Meyers, The Ohio State University.*

Let S be a nonempty subset of $\{1,2,\ldots,n\}$. If

$$\max S + \min S = n + 1,$$

then S contributes $n + 1$ to the sum of characters. If

$$\max S + \min S \neq n + 1,$$

then let

$$S' = \{x : n + 1 - x \in S\},$$

so that S' is a nonempty subset of $\{1,2,\ldots,n\}$. Also,

$$\begin{aligned}\max S + \min S + \max S' + \min S' &\\ &= \max S + \min S + (n + 1 - \min S) + (n + 1 - \max S)\\ &= 2(n + 1).\end{aligned}$$

Thus $S' \neq S$, and, since $(S')' = S$, $\{S,S'\}$ is a pair of "complementary" sets which contributes an average of $n + 1$ to the sum of characters. Hence the average of the characters of all $2^n - 1$ nonempty subsets of $\{1,2,\ldots,n\}$ is $n + 1$.

II. *Solution by Jorg Harterich, student, University of Stuttgart.*

There are $2^n - 1$ nonempty subsets of $\{1,2,\ldots,n\}$. The number of subsets of $\{1,2,\ldots,n\}$ with k being the minimum element is equal to the number of subsets of $\{k + 1,k + 2,\ldots,n\}$, which is 2^{n-k}. Therefore the sum of all minimum elements is

$$s = 1\cdot 2^{n-1} + 2\cdot 2^{n-2} + \cdots + n\cdot 2^{n-n}.$$

Similarly, the number of subsets with maximum element k is equal to the number of subsets of $\{1,2,\ldots,k - 1\}$, which is 2^{k-1}, so the sum of all maximum elements is

$$S = n\cdot 2^{n-1} + (n - 1)2^{n-2} + \cdots + 1\cdot 2^0.$$

Thus the average of the characters of all nonempty subsets of $\{1,2,\ldots,n\}$ is

$$\frac{s + S}{2^n - 1} = \frac{(n+1)(2^{n-1} + 2^{n-2} + \cdots + 2^0)}{2^n - 1} = n + 1.$$

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; ROGER LEE, grade 11, White Plains High School, White Plains, New York; P. PENNING, Delft, The Netherlands; DAVID R. STONE, Georgia Southern College, Statesboro, Georgia; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; KENNETH S. WILLIAMS, Carleton University, Ottawa; and the proposers. There was one incorrect answer submitted.

Many solvers found solution I*; a couple found solution* II*; the rest worked too hard!*

Klamkin points out that a similar result holds for the sum of the mth largest and mth smallest elements of a subset, averaged over all subsets of $\{1,2,\ldots,n\}$ *with at least m elements.*

Stone observes that the result holds for nonempty subsets of any sequence $a_1 < a_2 < \cdots < a_n$ *of real numbers which is symmetric about its midpoint, i.e.*

$$a_1 + a_n = a_2 + a_{n-1} = \cdots ,$$

but that the converse fails: $\{2,13,20,34,43\}$ *has no such symmetry although the average character is* 45 *which equals* 2 + 43.

Stone also notes that the problem asks for the "mean character", and offers two solutions: Ebenezer Scrooge and Darth Vader.

* * *

<u>1265</u>. [1987: 216] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with area F and exradii r_a, r_b, r_c, and let $A'B'C'$ be a triangle with area F' and altitudes h_a', h_b', h_c'. Show that

$$\frac{r_a}{h_a'} + \frac{r_b}{h_b'} + \frac{r_c}{h_c'} \ge 3\cdot\sqrt{\frac{F}{F'}}\,.$$

I. *Generalization by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let λ, μ be nonnegative real numbers. Then we prove that

$$\frac{(r_a)^\lambda}{(h_a')^\mu} + \frac{(r_b)^\lambda}{(h_b')^\mu} + \frac{(r_c)^\lambda}{(h_c')^\mu} \ge 3\cdot 3^{(\lambda-\mu)/4}\sqrt{\frac{F^\lambda}{F'^\mu}}\,. \tag{1}$$

The given inequality follows by putting $\lambda = \mu = 1$.

Applying the arithmetic–geometric mean inequality,

$$\frac{(r_a)^\lambda}{(h_a')^\mu} + \frac{(r_b)^\lambda}{(h_b')^\mu} + \frac{(r_c)^\lambda}{(h_c')^\mu} \ge 3\frac{(r_ar_br_c)^{\lambda/3}}{(h_a'h_b'h_c')^{\mu/3}}\,.$$

By item 6.27 of [1], i.e.

$$(r_a r_b r_c)^{1/3} \geq 3^{1/4} F^{1/2} \geq (h_a h_b h_c)^{1/3},$$

we infer that

$$\frac{(r_a r_b r_c)^{\lambda/3}}{(h_a' h_b' h_c')^{\mu/3}} \geq \frac{F^{\lambda/2}}{F'^{\mu/2}},$$

which gives the result.

Inserting

$$r_a = \frac{F}{s - a}, \quad h_a' = \frac{2F'}{a'}, \quad \text{etc.}$$

into (1), we get after a short simplification

$$\sum a'^{\mu}(-a + b + c)^{-\lambda} \geq 3(4/\sqrt{3})^{(\mu-\lambda)/2} F'^{\mu/2} F^{-\lambda/2}$$

(where the sum is cyclic), which is another restricted generalization of the Neuberg–Pedoe inequality (cf. item 10.8 of Bottema et al, *Geometric Inequalities*, and *Crux* 1114 [1987: 187], inequality (10)).

[*Editor's note*: Janous actually proved a version of (1) involving k unprimed triangles and n primed triangles, with accompanying λ's and μ's. Being somewhat of a notational nightmare, this has been left to the readers' imaginations.]

II. *Generalization by Murray S. Klamkin, University of Alberta.*

Since

$$r_a = \frac{F}{s - a}, \quad h_a' = \frac{2F'}{a'}, \quad \text{etc.},$$

the given inequality reduces to

$$\frac{a'}{s - a} + \frac{b'}{s - b} + \frac{c'}{s - c} \geq 6\sqrt{\frac{F'}{F}}. \tag{1}$$

We now remove the triangle constraints by using the duality relations

$$x = s - a, \quad y = s - b, \quad z = s - c,$$
$$x' = s' - a', \quad y' = s' - b', \quad z' = s' - c'$$

(see [1984: 47]). (1) is then transformed into the equivalent inequality

$$\left[\frac{y' + z'}{x} + \frac{z' + x'}{y} + \frac{x' + y'}{z}\right]^4 \geq 6^4 \cdot \frac{x'y'z'}{xyz} \cdot \frac{x' + y' + z'}{x + y + z},$$

where x, y, z, x', y', z' are arbitrary positive numbers. By using the A.M.–G.M. inequality on the quantity in parentheses on the left hand side, we obtain the stronger inequality

$$(x + y + z)^3 \prod (y' + z')^4 \geq 2^{12} xyz(x'y'z')^3(x' + y' + z')^3.$$

Using the A.M.–G.M. inequality on $(x + y + z)^3$, we end up (after dropping the primes) with the still stronger inequality

$$27(y + z)^4(z + x)^4(x + y)^4 \geq 2^{12}(xyz)^3(x + y + z)^3. \tag{2}$$

The latter inequality will follow from the even stronger, and known, inequality

$$27(y + z)^2(z + x)^2(x + y)^2 \geq 2^6 xyz(x + y + z)^3. \tag{3}$$

Note that if one converts (3) to its dual triangle form using the above duality relations, one obtains

$$27(abc)^2 \geq 64s^3(s-a)(s-b)(s-c),$$

or, letting R be the circumradius,

$$27R^2 \geq 4s^2$$

which is known (item 5.3 of [1]). In geometric terms, this says that the largest perimeter triangle which can be inscribed in a given circle is the equilateral one.

To get (2) from (3), we square (3) to give

$$3^3(y+z)^4(z+x)^4(x+y)^4 \geq 2^{12}(xyz)^2(x+y+z)^6/3^3,$$

and then note that

$$(x+y+z)^3 \geq 3^3(xyz),$$

again by the A.M.–G.M. inequality.

We now give a generalization of (3) using the following inequality for concave functions F, due to T. Popoviciu [2]:

$$2F\left[\frac{y+z}{2}\right] + 2F\left[\frac{z+x}{2}\right] + 2F\left[\frac{x+y}{2}\right] \geq F(x) + F(y) + F(z) + 3F\left[\frac{x+y+z}{3}\right]. \tag{4}$$

Note that (3) immediately follows by letting $F(t) = \log t$. Since

$$3F\left[\frac{x+y+z}{3}\right] \geq F(x) + F(y) + F(z),$$

we have, by first doubling (4) and then substituting,

$$4F\left[\frac{y+z}{2}\right] + 4F\left[\frac{z+x}{2}\right] + 4F\left[\frac{x+y}{2}\right] \geq 3F(x) + 3F(y) + 3F(z) + 3F\left[\frac{x+y+z}{3}\right]. \tag{5}$$

Now letting $F(t) = \log t$ again, we obtain (2). We can convert (4) and (5) into triangle inequalities by using $x = s - a$, etc., to give

$$2F\left[\frac{a}{2}\right] + 2F\left[\frac{b}{2}\right] + 2F\left[\frac{c}{2}\right] \geq F(s-a) + F(s-b) + F(s-c) + 3F\left[\frac{s}{3}\right]$$

and

$$4F\left[\frac{a}{2}\right] + 4F\left[\frac{b}{2}\right] + 4F\left[\frac{c}{2}\right] \geq 3F(s-a) + 3F(s-b) + 3F(s-c) + 3F\left[\frac{s}{3}\right].$$

Finally, by specializing F in (4), we can obtain a number of known inequalities. For instance, with $F(t) = \sin t$, we get that

$$2(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}) \geq \sin A + \sin B + \sin C + \frac{3\sqrt{3}}{2};$$

with $F(t) = \log \sin t$,

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \geq \left[\frac{\sqrt{3}}{2}\right]^3 \sin A \sin B \sin C;$$

with $F(t) = -t^n$ where $n > 1$ or $n < 0$,

$$(s-a)^n + (s-b)^n + (s-c)^n + 3^{1-n}s^n \geq 2^{1-n}(a^n + b^n + c^n)$$

(if $0 < n < 1$, the inequality is reversed).

References:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

[2] T. Popoviciu, Sur certaines inégalités qui caracterisent les fonctions convexes, *An. Sti. Univ. A1.I Cuza Iasi Sect. Ia Mat.* (*N.S.*) 11B (1965) 155–164.

Also solved by GEORGE EVAGELOPOLOUS, Athens, Greece; C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

* * *

1266. [1987: 216] *Proposed by Themistocles M. Rassias, Athens, Greece.*

Let $a_1, a_2, \ldots, a_n$ be distinct odd natural numbers, and let $\prod_{i=1}^{n} a_i$ be divisible by exactly k primes, of which p is the smallest. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{I_{p-2}}{I_{p+2k-2}}$$

where

$$I_{2m+1} = \frac{2m(2m-2)\cdot\ldots\cdot 4\cdot 2}{(2m+1)(2m-1)\cdot\ldots\cdot 3\cdot 1}.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let

$$\prod_{i=1}^{n} a_i = \prod_{j=1}^{k} p_j^{\lambda_j}$$

where $\lambda_j \geq 1$ for each j and $p = p_1 < p_2 < \cdots < p_k$ are all primes. Then

$$\sum_{i=1}^{n} \frac{1}{a_i} < \sum_{\mu_1,\ldots,\mu_k \geq 0}^{\infty} \frac{1}{p_1^{\mu_1} p_2^{\mu_2} \cdots p_k^{\mu_k}}$$

$$= \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right)$$

$$= \prod_{j=1}^{k} \frac{p_j}{p_j - 1}. \tag{1}$$

We note that the function

$$f(t) = \frac{t}{t-1}$$

decreases for $t > 1$ and that $p_j \geq p + 2j - 2$, $j \geq 1$. Thus

$$\prod_{j=1}^{k} \frac{p_j}{p_j - 1} \leq \prod_{j=1}^{k} \frac{p + 2j - 2}{p + 2j - 3} = \frac{I_{p-2}}{I_{p+2k-2}},$$

which with (1) gives the result.

Also solved by the proposer, and (with only a small error) by P. PENNING, Delft, The Netherlands. Two other readers submitted incorrect solutions.

* * *

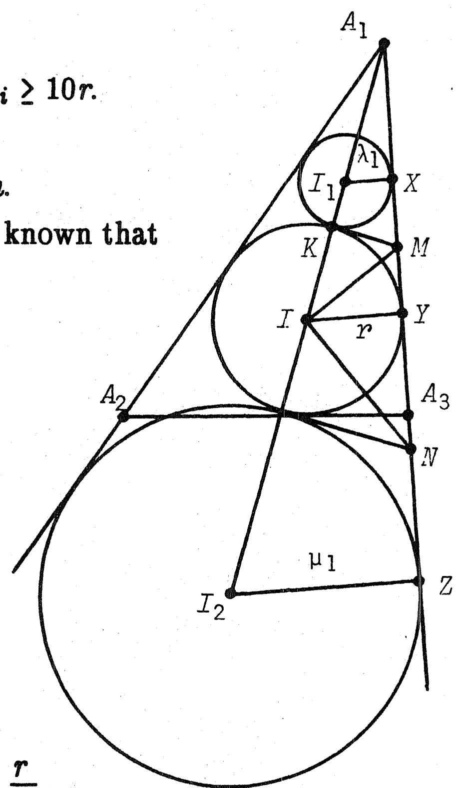
1267. [1987: 216] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with inscribed circle I of radius r. Let I_1 and J_1, of radii λ_1 and μ_1, be the two circles tangent to I and the lines A_1A_2 and A_1A_3. Analogously define circles I_2, J_2, I_3, J_3 of radii λ_2, μ_2, λ_3, μ_3, respectively.

(a) Prove that $\lambda_1\mu_1 = \lambda_2\mu_2 = \lambda_3\mu_3 = r^2$.

(b) Prove that

$$\sum_{i=1}^{3}\lambda_i + \sum_{i=1}^{3}\mu_i \ge 10r.$$



Solution by Tosio Seimiya, Kawasaki, Japan.

With the figure labelled as shown, it is well known that

$$XY = 2\sqrt{r\lambda_1},$$

and thus

$$XM = MK = MY = \sqrt{r\lambda_1}$$

where $MK \perp A_1I$. Similarly,

$$YN = \sqrt{r\mu_1}.$$

Putting

$$\alpha_1 = \angle MIY = \angle INY,$$

from

$$\tan\alpha_1 = \frac{MY}{r}, \quad \tan\alpha_1 = \frac{r}{YN}$$

follows

$$\frac{\lambda_1}{r} = \tan^2\alpha_1 = \frac{r}{\mu_1}. \tag{1}$$

Likewise,

$$\frac{\lambda_2}{r} = \tan^2\alpha_2 = \frac{r}{\mu_2}, \quad \frac{\lambda_3}{r} = \tan^2\alpha_3 = \frac{r}{\mu_3}. \tag{2}$$

By (1) and (2), (a) is clear. Also by (1) and (2),

$$\frac{1}{r}(\lambda_1 + \lambda_2 + \lambda_3) = \tan^2\alpha_1 + \tan^2\alpha_2 + \tan^2\alpha_3, \tag{3}$$

$$\frac{1}{r}(\mu_1 + \mu_2 + \mu_3) = \cot^2\alpha_1 + \cot^2\alpha_2 + \cot^2\alpha_3. \tag{4}$$

Since $\alpha_1 = 45^\circ - A_1/4$ etc., we have $\alpha_1 + \alpha_2 + \alpha_3 = 90^\circ$. Thus

$$\tan^2\alpha_1 + \tan^2\alpha_2 + \tan^2\alpha_3 \ge 1$$

(2.35 of [1]) and

$$\cot^2\alpha_1 + \cot^2\alpha_2 + \cot^2\alpha_3 \ge 9$$

(2.43 of [1]). From (3), (4),

$$\lambda_1 + \lambda_2 + \lambda_3 \geq r, \quad \mu_1 + \mu_2 + \mu_3 \geq 9r,$$

and (b) follows. Equality holds when $\Delta A_1A_2A_3$ is equilateral.

Reference:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969.

Also solved by HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS-HAMOIR, Brussels, Belgium; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, California; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer. One incorrect solution was sent in.

* * *

1268. [1987: 216] *Proposed by Herta T. Freitag, Roanoke, Virginia and Dan Sokolowsky, Williamsburg, Virginia.*

Let N be an arbitrary positive integer. For each divisor d of N let $n(d)$ denote the number of divisors of d. Let S be the set of divisors of N. Prove that

$$\left[\sum_{d\in S} n(d)\right]^2 = \sum_{d\in S} (n(d))^3.$$

Editor's comment.

As some solvers pointed out, this problem is well–known and originally due to Liouville (1857). More recent references are: Ross Honsberger, *Ingenuity in Mathematics*, M.A.A., 1970, pp.72, 75, 76; and William J. LeVeque, *Fundamentals of Number Theory*, Addison–Wesley, 1977, problem 2, p.125. The article "Number patterns – sets with the square–cube property" by R.G. Stein, in the October 1982 *Mathematics Teacher*, contains the same result in another form. I thank those solvers who contributed this information.

The proof of the problem goes as follows. Using the more standard notation $\tau(d)$ instead of $n(d)$, first show that both sides of the required equality

$$\left[\sum_{d\in S} \tau(d)\right]^2 = \sum_{d\in S} (\tau(d))^3 \tag{1}$$

are multiplicative functions (of N). Thus we need only verify (1) when N is a prime power p^k, and this follows from

$$\left[\sum_{d\in S} \tau(d)\right]^2 = \left[\sum_{i=1}^{k+1} i\right]^2 = \sum_{i=1}^{k+1} i^3 = \sum_{d\in S} (\tau(d))^3.$$

Solved by SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; M.A. SELBY, University of Windsor, Windsor, Ontario; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposers.

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<u>1270</u>. [1987: 217] *Proposed by Peter Ivady, Budapest, Hungary.*

Prove the inequality

$$\frac{x}{\sqrt{1 + x^2}} < \tanh x < \sqrt{1 - e^{-x^2}}$$

for $x > 0$.

Solution by Leroy F. Meyers, The Ohio State University.

For the first inequality, note that from $0 < x < \sinh x$, and since the function

$$f(x) = \frac{x}{1 + x}$$

increases for $x > 0$, it follows that

$$\frac{x^2}{1 + x^2} < \frac{\sinh^2 x}{1 + \sinh^2 x} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x,$$

and we may take positive square roots.

The second inequality can be obtained by taking positive square roots of

$$\tanh^2 x = \frac{\cosh^2 x - 1}{\cosh^2 x} < 1 - e^{-x^2},$$

which is a consequence of

$$\cosh^2 x < e^{x^2},$$

which we prove as follows. Now

$$\cosh^2 x = \frac{1}{2} + \frac{1}{2}\cosh(2x) = 1 + x^2 + \sum_{k=2}^{\infty} \frac{(2x)^{2k}}{2\cdot(2k)!}$$

and

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!} = 1 + x^2 + \sum_{k=2}^{\infty} \frac{(2x)^{2k}}{2^{2k}\cdot k!},$$

and so it is sufficient to show that

$$2^{2k} \cdot k! < 2 \cdot (2k)!$$

for $k \geq 2$. But

$$\begin{aligned} 2^{2k} \cdot k! &= 2^k \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \\ &< 2 \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k = 2 \cdot (2k)!. \end{aligned}$$

Also solved by SEUNG–JIN BANG, Seoul, Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; VEDULA N. MURTY, Pennsylvania State University at Harrisburg; ROBERT E. SHAFER, Berkeley, California; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.

* * *

1272. [1987: 256] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

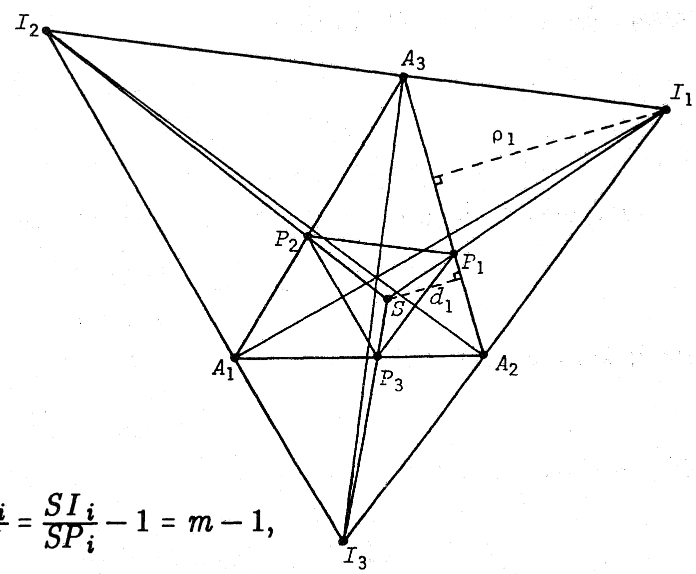
Let $A_1A_2A_3$ be a triangle. Let the incircle have center I and radius ρ, and meet the sides of the triangle at points P_1, P_2, P_3. Let I_1, I_2, I_3 be the excenters and ρ_1, ρ_2, ρ_3 the exradii. Prove that

(a) the lines I_1P_1, I_2P_2, I_3P_3 concur at a point S;

(b) the distances d_1, d_2, d_3 of S to the sides of the triangle satisfy

$$d_1 : d_2 : d_3 = \rho_1 : \rho_2 : \rho_3 .$$

Solution by Jordi Dou, Barcelona, Spain.



We know that I_1A_1, I_2A_2, I_3A_3 are the altitudes of $\Delta I_1I_2I_3$, and that I_1A_1 is perpendicular to P_2P_3, etc. Therefore P_2P_3 is parallel to I_2I_3, etc. Thus the triangles $P_1P_2P_3$ and $I_1I_2I_3$ are homothetic and the point S in (a) is the centre of similitude. Letting

$$\frac{SI_i}{SP_i} = m,$$

it follows that

$$\frac{\rho_i}{d_i} = \frac{P_iI_i}{SP_i} = \frac{SI_i}{SP_i} - 1 = m - 1,$$

showing (b).

Also solved by HIDETOSI FUKAGAWA, Yokosuka High School, Aichi, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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