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The Mathematical Mind[1]

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I have often been impressed by the interest which non-mathematicians show in mathematics. It is not that they want to learn some mathematics. In fact they have probably tried to do so at some stage and found it unrewarding, but in the words of Professor Newman 'Mathematics enjoys the advantages, and suffers the penalties of being a secret doctrine'. One lady made it clear that she realized that mathematicians had something which they enjoyed in somewhat the same way as she enjoyed the music and the paintings of others, and she did not see why she should not share our enjoyment. Besides the fascination of secrecy, pure mathematics has the distinction of depending more exclusively on pure thought than most other subjects, and perhaps there is a hope that a study of the mathematical mind will show human thought at work uncomplicated by the practical difficulties of its applications. I am not concerned with mathematical physics, but I do not think that we should forget that many of even the most abstract modern theories of pure mathematics have developed from earlier considerations of physical problems.

I hope to let the non-mathematicians share one or two of our jokes, but it seems to me to be impossible for the pure mathematician to share the essential content of his subject with a non-mathematician such as the lady to whom I referred. For the essential content is the chain or interlinked chains of reasoning, and appreciation of the subject depends on the appreciation of the length and complexity of these chains, and statements of results without proofs, or simplifications of proofs, are more misleading than small cheap reproductions of great pictures.

There remains, however, the possibility of talking *about* mathematics. Professor Hardy wrote:

> It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art-critics, and physiologists, physicists, or mathematicians have usually similar feelings; there is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds.

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But it is remarkable how many first-rate minds—Hardy, Poincaré, Hadamard, and others—have descended to it. Among the problems which have fascinated these eminent mathematicians there is one to which I should like to make my contribution. It is the problem of how they obtained their best results and how their minds worked during the process; behind this problem lie two questions. One which every mathematician from the young research student upwards probably asks himself is, 'Should I achieve more if I changed my methods of working?'; the other is, 'What constitutes good mathematics and how does one recognize it?'

I will begin with the last question, but before I try to answer it I must say something about certain convictions or prejudices which most, and probably all, pure mathematicians have about their subject. To most of us something original means the discovery of a new significant result or a completely new proof of a known result, or perhaps the creation of a new theory, that is a collection of definitions and results each by itself scarcely perhaps new in essence, but in combination of considerable importance. The original piece of work is one which has never been done before. In many cases others have tried to do it and failed, and in all cases the originality is, I think, more sharply defined in mathematics than in most other subjects. A paper is only published in a reputable mathematical journal if the editor and referee consider that it contains a clearly recognizable original contribution to mathematics. On the other hand a mathematical book is usually a compilation and rearrangement of works previously published and may be a good book without containing what most mathematicians would consider a new original idea of any importance. Some books certainly do contain important new ideas, for instance Weyl's *Die Idee der Riemannschen Fläche* was packed with new ideas, but it is more usual for important ideas to make their bow in periodicals, and a few substantial papers in periodicals are often better evidence of mathematical originality than any book.

Now I used the word 'discovery', and one of our convictions is that result, proof, or what have you, was there already waiting to be discovered, and is not something new which has been invented or created. Hadamard called his book *The Psychology of Invention in the Mathematical Field*, and he says, 'it would be more correct to speak of discovery . . . discovery concerns a phenomenon, a law, a being which already existed, but had not been perceived'. But he considers the distinction between invention and discovery less evident than appears at first glance. I mentioned new results and new proofs. If the proof of the theorem is known its discovery is complete, but there are cases where the conjecture that a result might be true has had greater influence on the development of mathematics than many complete proofs.

I should like to illustrate this by two historical cases, well known to all mathematicians here. First I should like to quote part of Hadamard's account of Fermat and his Last Theorem.

Pierre de Fermat who lived from 1601–1661 was a magistrate, a counselor at the Parliament of Toulouse. It was a time when life was less complicated than nowadays, and the requirements of his duties apparently did not hamper him in his mathematical researches, which were considerable. . . . Among the ancient mathematicians whose works were in his possession, he owned a translation of the work of Diophantes, a Greek author who had dealt with arithmetical subjects. Now, at Fermat's death, his copy of Diophantes' work was found to bear in the margin the following observation (in Latin): 'I have proved that the relation $x^m+y^m=z^m$ is impossible in integral numbers (x, y, z different from 0; m greater than 2); but the margin does not leave me room enough to inscribe the proof'. Three centuries have elapsed since then, and that proof which Fermat could have written in the margin had the latter been a little broader, is still sought for. However, Fermat does not seem to have been mistaken, for partial proofs have been found, viz., . . . for instance, the proof has been obtained for every m not greater than 100. But the work—an immense one—which made it possible to get these partial results . . . required the help of some important algebraic theories of which no knowledge existed in the time of Fermat *and no conception appears in his writings.*

Before I proceed to my next example I ought to mention another habit of mind acquired by mathematicians. The process of reading and understanding mathematical work is slow and difficult even for the expert, and we do not nowadays expect to understand important new work without a very considerable expenditure of time and energy. We may get a general idea of the sort of thing an expert in another field of mathematics is doing, and of possible repercussions, but he could say that $2+2=5$ in disguised terms, and we should be none the wiser, and, speaking for myself, I feel that my judgement has little value without a more fundamental understanding. For this reason non-mathematicians seem to us to be unduly worried if they cannot understand. After all if I only get a vague impression of similarity to something I once understood, why should they be worried if they don't understand after practically no work at all? In what follows you may replace the mathematical words or symbols by 'abracadabra' and suchlike without losing very much, but I retain them so that perhaps a few may recognize them.

It is almost impossible to explain the significance of the Riemann Hypothesis in an article like this one, but for, shall we say, the inexperienced mathematicians, I will say that complex numbers of the form $x+iy$ are often displayed by their position in the plane relative to axes $x=0$, $y=0$. The Riemann zeta function is defined for all complex numbers, and is used for such things as discussing how many prime numbers there are less than a given number n. Riemann conjectured that the zeta function did not take the value 0 at any point to the right of the line $x=\frac{1}{2}$, and it is called his 'hypothesis' because so many results seem to depend on whether it is true or false, which has not yet been determined.

These two conjectures fulfil one concept of a good piece of mathematics. The result can be stated comparatively simply; there are not a lot of 'ifs' and 'provided thats', and yet it is not easy to prove the conjectures true or false, and the repercussions are immense. A proof of either would be a major event. We admire results and proofs which have an element of unexpectedness, and also those which seem inevitable once stated. We admire a complicated chain of arguments leading to a simply stated result, and also a technique which gives the result with a minimum of effort. The contradictions are always there and I do not think that any formula for good mathematics exists unless it is the old one that mathematics is organized laziness.

Now let us turn to the problem of how to achieve good mathematics. In 1904 *L'Enseignement Mathématique* conducted an inquiry into the working methods of mathematicians. They asked among many other questions:

Have you ever worked in your sleep or have you found in dreams the answers to your problems?

Do you believe that it is beneficial to a mathematician to observe a few special rules of hygiene such as diet, regular meals, time for rest, etc.?

Would you say that a mathematician's work should be interrupted by other occupations or by physical exercises which are suited to the individual's age and strength?

Unfortunately those who had made the really original contributions to mathematics sent no replies. However, we have some records of Professor J. E. Littlewood's experiences as an undergraduate, and I think that all mathematicians will accept him as sufficiently original for his record to be worth consideration. He refers to the first as 'a fatuous experiment'.

> I lived in Bideford (Devon) and decided to spend part of the Easter vacation buried at Hartland Quay. . . . The idea was to give up smoking, concentrate on work in the mornings and late afternoons, and 'relax' on poetry and philosophy (*Principia Ethica*) in the evenings, fortified by strong coffee. . . . My window opened on the sea, which I used as a waste-paper-basket, and on arrival I ceremonially threw my pipes and tobacco into it. Next day I relapsed. . . . The experiment taught me something of the truth that for serious work one does best with a background of familiar routine, and that in the intervals for relaxation one should be relaxed. Much could be said on this theme, but this is not the moment for it: I will say, however, that for me the thing to avoid, for doing creative work, is above all Cambridge life, with the constant bright conversation of the clever, the wrong sort of mental stimulus, all the goods in the front window.

That is the opinion of a great mathematician, and it may be observed that he does not go to conferences and congresses nor serve on many committees; but on the other hand some mathematicians thrive on congresses and colloquia, and it would be tempting to think that different kinds of mathematics require different methods or appeal to different temperaments. Some time ago it was said that in England mathematicians sat down and tried to dig a deep hole whereas in Princeton in the United States they constructed shallow subways from one part of mathematics to another, and certainly some branches of mathematics seem to lend themselves to discussion more than others. However, one has to remember that much of Littlewood's work was done in collaboration with Professor Hardy, who served on the Council of the London Mathematical Society for many years and was much more given to frequenting gatherings of mathematicians. It is of some interest to consider how differences of temperament may have affected the development of mathematics. I feel sure that Professor Hardy's patient care for elegant exposition has had a notable effect on English analysis. The difference between Landau's extreme brevity and the more flowing French style of Valiron or Milloux can be compared in papers dealing with very similar topics, and I think it made the one extend his work in a slightly different direction from the other.

Now let us turn to the actual processes of thought. Hadamard says:

I insist that words are totally absent from my mind when I really think and I shall completely align my case with Galton's in the sense that even after reading or hearing a question, every word disappears at the very moment I am beginning to think it over; words do not reappear in my consciousness before I have accomplished or given up the research, just as happened to Galton; and I fully agree with Schopenhauer when he writes, 'Thoughts die the moment they are embodied by words'.

I think it also essential to emphasize that I behave in this way not only about words, but even about algebraic signs. I use them when dealing with easy calculations; but whenever the matter looks more difficult, they become too heavy a baggage for me. I use concrete representations, but of a quite different nature.

Again Hadamard writes:

Indeed, every mathematical research compels me to build such a schema, which is always and must be of a vague character, so as not to be deceptive. I shall give one less elementary example from my first researches. I had to consider a sum of an infinite number of terms, intending to valuate its order of magnitude. In that case, there is a group of terms which chances to be predominant, all others having a negligible influence. Now, when I think of that question, I see not the formula itself, but the place it would take if written: a kind of ribbon, which is thicker or darker at the place corresponding to the possibly important terms; or (at other moments), I see something like a formula, but by no means a legible one, as I should see it (being strongly long-sighted) if I had no eye-glasses on, with letters seeming rather more apparent (though still *not legible*) at the place which is supposed to be the important one.

Hadamard consulted various mathematicians born or resident in America. 'Practically all of them . . . avoid not only the use of mental words but also the mental use of algebraic or any other precise signs; also as in my case, they use vague images.'

I myself can remember thinking out one group of problems by means of some sort of visual model and it was only after I had obtained a number of results that I recognized to what it corresponded in mathematical terms. Perhaps the vagueness of these mental models is important; they are not nearly so sharp and bright as some of the visual images recorded in Galton's inquiries, and I believe that recent work on the brain tends to rate the excessively visual type of mind rather low. Further I cannot forbear to offer for your consideration Napoleon's dictum, recorded by Galton: 'There are some who, from some physical or moral peculiarity of character form a picture of everything. No matter what knowledge, intellect, courage or good qualities they may have, these men are unfit to command'. Now this article is far too short to go into all the psychological stages which have been recognized by such great men as Helmholz, Poincaré, and Hadamard; there is conscious preparation, unconscious incubation and illumination, but I think that the first two often include the formation of some kind of mental model more or less visual such as Hadamard's with which the thinking is done.

As a record of illumination or inspiration I like best that of Gauss on the construction of a polygon of 17 sides with ruler and compasses.

The day was March 29, 1796, and chance had nothing to do with it. Before this, indeed during the winter of 1796, I had already discovered everything related to the separation of the roots of the equation

$$\frac{x^p-1}{x-1}=0$$

into two groups. . . . After intensive consideration of the relation of all the roots to one another on arithmetical grounds, I succeeded, during a holiday in Braunschweig, on the morning of the day alluded to (before I had got out of bed), in viewing the relation in the clearest way, so that I could immediately make special application to the 17-side.

Poincaré's record of illumination is also noteworthy.

Just at this time, I left Caen, where I was living, to go on a geologic excursion under the auspices of the School of Mines. The incidents of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go to some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those on non-Euclidian geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure.

In both these cases the next stages of stating the results precisely and verifying them followed easily, but, as Hadamard observes, effective calculations require discipline, attention, volition, and therefore a second period of conscious work. Both Poincaré and Hadamard recognized that inspiration may be deceptive, according to Poincaré especially ideas which come in bed. My own experience is a little peculiar. I particularly remember an idea which came to me in my bath of using what are called normal families in connexion with p-valent functions. It seemed at the time the key to the solution of an important problem, but when the work was polished for publication, normal families were not needed, and a better result was obtained without them. However, the idea gave me conviction, and conviction may be very important as an examination experience recorded by Littlewood shows. He writes:

I began on a question on elementary theory of numbers, in which I felt safe in my school days. It did not come out, nor did it on a later attack. I had occasion to fetch more paper; when passing a desk my eye lit on a heavy mark against the question. The candidate was not one of the leading people, and I half-unconsciously inferred that I was making unnecessarily heavy weather; the question then came out fairly easily.

The conviction is partly a matter of determining the point of attack, whether to try to prove the result true or false, and which method is most hopeful. As Sherlock Holmes says, 'When you have excluded the impossible, whatever remains, however improbable, must be the truth'; and the mathematician tries one method after another, rejecting them as impossible often when they are in fact right, but if you have rejected all other approaches as *quite impossible* it adds greatly to the strength of your attack on a problem by the remaining method.

I question very much whether inspired ideas in mathematics are really as evanescent as Rosamund Harding in her *Anatomy of Inspiration* seems to think that they are for poets and musicians. The ideas go, but in my experience they force themselves back again if one thinks much about the subject. At certain stages in mathematics, results have been discovered simultaneously by two or more mathematicians. The time was ripe for them. Further, I strongly suspect that those ideas which have come to some in inspired moments have come to a number of others in moments of no particular uplift or inspiration, but the few were able to make effective use of them owing to previous preparation and others could not. Returning to my work on p-valent functions, the idea which was vital to the final proof was not that which came to me in my bath of using normal families, but was that of using Ahlfors' Distortion Theorem, and this came to me, so far as I remember, with practically no preparation immediately I heard Mr Collingwood lecture on it; but for a year or perhaps two I was unable to make effective use of it owing to inadequate technique for thinking and writing about the subject.

I wonder very much whether the mathematical work produced as the result of sudden flashes of inspiration is really better than that produced entirely as the cumulative result of deliberate conscious effort. In my more recent topological work I had a number of experiences of apparent inspiration or illumination which was either wrong or could not be verified owing to lack of technique, and, so far as I remember, the final valid proofs did not owe anything directly to inspiration, although perhaps the false inspirations assisted the development of the technique which led to success. So far as I know, Professor Hardy left no record of the thought processes by which he achieved his greatest work, but I remember his saying that he preferred analytical arguments to geometrical ones because in analytical arguments one's pen guided one. This suggested to me that most of his work was done on the level of conscious deliberate effort, and owed comparatively little to incubation and sudden flashes.

I often wonder whether I am really doing better work when things go smoothly or whether I have failed to see the snags that will worry me another day. If I begin a page five or six times, is it because I am not warmed up, or because I am too tired, or because I am just beginning to be conscious that I overlooked or fudged something at an earlier stage? It constitutes progress to be even vaguely aware that there is an imperfection.

This brings us to another problem: when is a mistake important and when is it unimportant? For there is no doubt that even the greatest mathematicians make plenty. In fact it is said of one eminent professor that he says *a*, he means *b*, he writes *c*, but it *should* be *d*. If the proof is easily rectified and the result holds, we do not worry, but I have known good mathematicians who read my own work and passed over a mistake in it which was so fundamental that the results were not true without modification, and I think it was because there were sufficiently important new ideas in the main proofs to overshadow the particular cases where modification was needed. But there are mistakes and mistakes. When a self-respecting mathematician discovers what appears to be a proof of Fermat's Last Theorem, he settles down to look for the mistake because he knows that after so many have tried to prove it and failed it is extremely improbable that he should have found a proof, and if he produces a proof with a mistake in it he will be a laughing-stock.

Let us return to the problem of how the best results are obtained and approach it from another angle. Hardy said: 'No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game. . . . Newton gave up mathematics at fifty, and had lost his enthusiasm long before'. Now although one may dispute this and find counter-examples, there is some truth in it, and I believe that one reason for it is that major advances are usually obtained by approaching a subject from a slightly different angle from that previously adopted; the ideas often come in the course of approaching, that is learning, the subject. When it is fully understood interest often wanes, and ideas become fixed. The younger person has a slightly different background; he is trained in the new methods in allied subjects, and above all he finds it easier to learn a completely new subject. In general it is said that there are only two kinds of mathematical problem, one is so easy that it isn't worth doing and the other so difficult that it can't be done. Approaching a subject for the first time one has to classify the problems and occasionally we find one which is of a third kind. But I have only shifted the problem in saying that older people find it less easy to learn. Speaking for myself I feel that my brain is less persistently active than it used to be and I fear that declining mental energy is the real answer.

Finally we may ask why we do mathematics. In the main it is because we like it and we do it better than anything else, and sometimes better than other people. For our bread and butter we prefer fiddling with numbers, formulae, and figures to history or classics, or medicine, or politics, or business accounts and sales returns. Some of us would like to be useful if we could, but those original theorems give us immense satisfaction, and we regret it when they cease to be obtainable either because we no longer have the ability or because we have let our time become filled with other interests. One eminent mathematician of my generation said to me a few years ago that what he yearned for was to prove 'just one more nice juicy theorem'. Now it must be remembered that we mathematicians are all in our own eyes to some extent failures. We have *not* proved Fermat's Last Theorem *nor* the Riemann Hypothesis, however many other juicy theorems we have achieved. We have been up against something so hard and difficult that when we attempt work in another field we may fail to recognize the less concrete obstacles, and so we plunge into talking about mathematics. As I said earlier, many eminent mathematicians have done so, and it may be asked why we do it. It may be the urge for an audience at all costs, but I fear that Barry Pain describes the phenomenon only too well in *Confessions of Alphonse*. I may say that I quote at second-hand from the heading to Appendix C, 'The relation of mathematical physics to geology', in *The Earth*, by Sir Harold Jeffreys:

> If you make some compliments to a man about that which he knows he does well, that is no good. But if you praise him for what he do pretty dambad, then you give him a great pleasure. Suppose you go to a painter and say to him: 'Your painting is exquisite. Your pictures are of a sublimity'. Well, you do yourself no good. He laugh and change the subject. But if you go to that same painter and say: 'But what a ravishing voice! Why are you not in Grand Opera?' . . . then you may perhaps borrow half-a-crown till next Thursday.

I can only hope that this article has not been such a painful experience as the painter's voice probably was.

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Sums of Sets of Integers

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1. The problem and its solution

In this first section we will state a problem about sets of integers, and describe the solution. In the next section we will prove that it is indeed the full solution of the problem. Then we will close the paper by stating two very similar problems that as yet remain unsolved. Although the problems under discussion are not simple, they are easy to describe. So Sections 1 and 3 are not difficult to follow, and with perhaps some effort the proof in Section 2 is within reach of the reader. The line of argument in the proof is a little out of the ordinary, and so quite interesting.

Let N be the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$. Two subsets A and B of N are said to be complementing sets if given any element n belonging to N, written $n \in N$, there exist unique elements $a \in A$ and $b \in B$ such that $a+b = n$. A simple example (Example 1) of complementing sets is given by $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{0, 10, 20, 30, 40, 50, \ldots\}$, with every non-negative multiple of 10 a member of B. In general, if A and B are complementing sets then it must be the case that $0 \in A$ and $0 \in B$. Furthermore no element of N other than 0 can belong to A and B. For example suppose $3 \in A$ and $3 \in B$. Then taking $n = 3$ in the definition, the elements a and b can be chosen in two ways, $a = 3$, $b = 0$ and $a = 0$, $b = 3$, contrary to the requirement of uniqueness.

Another example (Example 2) of complementing sets A and B is as follows. Let A contain 0 and every positive integer whose decimal representation has zeros in all the even-numbered positions counting from the right, for example, such an integer as 809070305. Let B contain 0 and every positive integer whose decimal representation has zeros in all the odd-numbered positions counting from the right, for example, such an integer as 60305040. Note that such an integer as 70005 is in A; that is, the integers in A *can* have zeros in the odd-numbered positions. All that is required is that an integer in A *must* have zeros in the even-numbered positions. Similarly such an integer as 500000 is in B. It is easy to verify that any integer $n \in N$ is uniquely expressible as a sum $a+b$ with $a \in A$ and $b \in B$. For instance consider $n = 146205$; taking $a = 40205$ and $b = 106000$ we see that $a+b = n$, and a and b are unique.

The central question is this: can we characterize all complementing sets A and B? That is, can we describe a simple rule or procedure that leads to all complementing sets? The answer is yes, and it turns out that Example 2 in a more general form almost provides such a description, at least for all cases where A and B are both infinite sets. First let it be noted that the sets A and B in Example 2 can be described in another way. For any non-negative integer n let $a_0, a_2, a_4, a_6, \dots, a_{2n}$ be non-negative integers, each less than 10, and form the sum

$$a_0 + a_2 10^2 + a_4 10^4 + a_6 10^6 + \dots + a_{2n} 10^{2n}. \tag{1}$$

Any integer in the set A in Example 2 can be obtained in this way; for example the integer 809070305 arises by taking $n = 4$ and $a_0 = 5$, $a_2 = 3$, $a_4 = 7$, $a_6 = 9$ and $a_8 = 8$. The integer 70005 is obtained by taking $n = 2$ and $a_0 = 5$, $a_2 = 0$, $a_4 = 7$. (The integer 70005 can also be obtained by taking $n = 3$ and $a_0 = 2$, $a_2 = 0$, $a_4 = 7$, $a_6 = 0$, and in other ways with larger values of n, but this lack of uniqueness of representation does not matter.) Note that the integer 0 is obtained from (1) by taking $n = 0$ and $a_0 = 0$.

Similarly the set B consists of every integer that can be expressed in the form

$$a_1 10^1 + a_3 10^3 + a_5 10^5 + \dots + a_{2n+1} 10^{2n+1}, \tag{2}$$

where n is any non-negative integer, and $a_1, a_3, a_5, \dots, a_{2n+1}$ are non-negative integers less than 10.

By generalizing this we now outline a characterization of all complementing sets A and B, a characterization established by N. G. de Bruijn in a short paper 'On number systems' in the *Nieuw Archiev voor Wiskunde*, **4** (1956), pp. 15–17. One obvious generalization of the expressions (1) and (2) comes from replacing the powers of 10 by the powers of any positive integer m; this amounts to switching from base ten to base m in the arabic notation for numbers. But a much more sweeping generalization comes from using the arabic notation with the units digit to one base, the 'tens' digit to another base, the 'hundreds' digit to yet another base, and so on. To spell this out in detail let $m_1, m_2, m_3, \dots$ be any infinite sequence of integers, each > 1, and define

$$M_1 = m_1, \quad M_2 = m_1 m_2, \quad M_3 = m_1 m_2 m_3, \quad M_4 = m_1 m_2 m_3 m_4, \dots . \tag{3}$$

(The ordinary decimal system of arabic notation is just the special case where $m_1, m_2, m_3, \dots$ is just 10, 10, 10,... and $M_1, M_2, M_3, \dots$ is 10, 10^2, $10^3, \dots$.) Then corresponding to expressions (1) and (2) we write

$$a_0 + a_2 M_2 + a_4 M_4 + \dots + a_{2n} M_{2n}, \tag{4}$$

$$a_1 M_1 + a_3 M_3 + a_5 M_5 + \dots + a_{2n+1} M_{2n+1}. \tag{5}$$

Now any complementing sets A and B are of two types, first with A and B both infinite sets (as in Example 2), and second with only one infinite set (as in Example 1). Note that in both cases the integer 1 is in one and only one of the sets A and B; by interchanging the labels on the two sets we can presume without loss of generality that $1 \in A$. Then all pairs of infinite complementing sets can be characterized in this way: take any infinite sequence of integers $m_1, m_2, m_3, \dots$ each >1; define $M_1, M_2, M_3, \dots$ as in (3); take any non-negative integers $a_0, a_1, a_2, \dots, a_{2n}, a_{2n+1}$ satisfying

$$a_0 < m_1, \quad a_1 < m_2, \quad a_2 < m_3, \quad \dots, \quad a_{2n} < m_{2n+1}, \quad a_{2n+1} < m_{2n+2}. \tag{6}$$

Let A consist of all integers that can be written in the form (4), and B those that can be written in the form (5).

Before going on to give a proof that all infinite complementing sets A and B are included in the above description, we characterize the complementing sets having not both A and B infinite. This can be done with a slight modification of (3), (4) and (5). First we take only a finite sequence $m_1, m_2, m_3, \dots, m_r$ so that (3) gives only a finite sequence $M_1, M_2, M_3, \dots, M_r$. If r is odd we define n by $2n+1 = r$, and define A and B in terms of (4) and (5) as before but with one change: the last inequality in (6) is now meaningless because m_{r+1} does not exist, and this inequality is removed to permit a_r to be any non-negative integer. Thus if r is odd, A is a finite set and B is an infinite set. Example 1 at the start of this paper is an illustration of this, with $r = 1$, $n = 0$, $m_1 = 10$. It is to be understood that in this case the expression (4) shrinks to the single term a_0 and (5) shrinks to the single term $a_1 m_1$ or $10\, a_1$.

On the other hand if r is even we define n by $2n = r$ and again we define A in terms of (4). But (5) needs to be shortened to

$$a_1 M_1 + a_3 M_3 + a_5 M_5 + \dots + a_{2n-1} M_{2n-1}. \tag{7}$$

Also the last two inequalities in (7) are deleted, and again we allow a_r to be any non-negative integer.

2. A proof of the characterization

First we establish a simple basic result that will be useful in the argument.

Lemma 1. Let A and B be complementing sets, and let t be any positive integer. If $t \in A$ or $t \in B$ then there are no positive integers $a \in A$ and $b \in B$ such that $t = a+b$. Conversely, if there are positive integers $a \in A$ and $b \in B$ satisfying $t = a+b$, then $t \notin A$ and $t \notin B$.

We have already seen that t cannot belong to both A and B. If $t \in A$ then $t+0$ is a representation of t as a sum of an element in A and the element $0 \in B$. But this representation is unique, so there cannot be positive integers $a \in A$ and $b \in B$ such that $t = a+b$. A similar argument applies in case $t \in B$. Conversely suppose there are positive integers a and b in A and B respectively such that $t = a+b$. Then $t \notin A$ because if $t \in A$ we see that $t = t+0$ is a second representation of a sum of an element in A and an element in B. Similarly $t \notin B$, and the proof of the lemma is complete.

Now if A and B are complementing sets we have seen that $0 \in A$ and $0 \in B$, and as before we presume that $1 \in A$. If B has no elements except the element 0, then A must be the set of all non-negative integers. Setting aside this trivial case, let m denote the least positive integer in B, so that A must contain the integers $1, 2, 3, \ldots, m-1$. Thus $m \geqslant 2$ and we define the sets $S_0, S_1, S_2, \ldots$, in general S_k for any non-negative integer k, as follows:

$$S_k = \{mk, mk+1, mk+2, \ldots, mk+m-1\}. \tag{8}$$

The sets $S_0, S_1, S_2, \ldots$ are disjoint (i.e., no two have an element in common), and their union is the set of all non-negative integers.

Lemma 2. For any non-negative integer k the set S_k is contained completely in A or not at all; the set S_k either has no element in B or just the one element mk. In the notation of set theory,

$$S_k \cap A = S_k \text{ or } \emptyset, \quad S_k \cap B = \{mk\} \text{ or } \emptyset,$$

where $\emptyset$ denotes the empty set.

The proof is by mathematical induction on k. Note that for $k = 0$ the lemma holds because S_0 is contained in A. Suppose the lemma holds for all sets up to S_{k-1}; we prove the result for S_k. The proof is in three parts, according as the positive integer mk is in B, or is in A, or is in neither A nor B.

Case 1. $mk \in B$. Then if j is any one of the integers $1, 2, 3, \ldots, m-1$, we see that $j+mk$ is a sum of an element $j \in A$ and an element $mk \in B$. By Lemma 1, $mk+j \notin B$ and $mk+j \notin A$. Hence we conclude that

$$S_k \cap A = \emptyset, \quad S_k \cap B = \{mk\},$$

and Lemma 2 is proved in this case.

Case 2. $mk \in A$. We must prove that $S_k \cap A = S_k$ and $S_k \cap B = \emptyset$. Suppose that this is not so. Let j be the smallest integer among $1, 2, 3, \ldots, m-1$ such that $mk+j \notin A$. Now there are two possibilities, $mk+j \in B$ or $mk+j \notin B$, and we take them up in turn. If $mk+j \in B$ then we note that there are two representations of $mk+m$ as a sum of elements in A and B, namely

$$(mk)+(m) = (m-j)+(mk+j).$$

This contradicts the uniqueness condition in the definition of complementing sets.

On the other hand if $mk+j \notin B$, then since $mk+j \notin A$ we apply Lemma 1 with $t = mk+j$ to write $mk+j = a+b$ with $a \in A$ and $b \in B$, and $a > 0$, $b > 0$. Now $b < mk$ (since $mk, \ldots, mk+j-1$ belong to A), so b is a member of some set S_h, where h is a positive integer and $h < k$. By the induction hypothesis b must be mh and so $a = (k-h)m+j$, where $k-h$ is also a positive integer satisfying $k-h < k$. Again by the induction hypothesis we see that $(k-h)m$ belongs to A. But now we have two representations of mk as a sum of an element in A and an element in B, namely

$$(mk)+(0) = (km-hm)+(hm),$$

and this is a contradiction. Thus the proof of Case 2 is complete.

Case 3. $mk \notin A$, $mk \notin B$. By Lemma 1 there are positive integers $a \in A$ and $b \in B$ such that $mk = a+b$. Now $b < mk$ and so b belongs to some set S_h with $h < k$. Also h is positive and by the induction hypothesis we know that $b = mh$. Hence $a = (k-h)m$ with the positive integer $k-h < k$. So if j is any one of the integers 1, 2, 3,..., $m-1$ we see that $(k-h)m+j$ belongs to A by the induction hypothesis. Thus we have a representation of $mk+j$ as sum of the positive integers $(k-h)m+j$ in A and hm in B. By Lemma 1 we see that $mk+j \notin A$, $mk+j \notin B$. If follows that $S_k \cap A = \emptyset$ and $S_k \cap B = \emptyset$ and so the proof of Lemma 2 is complete.

Now by Lemma 2 every element of B is a multiple of m. Take the set B and divide each element by m; let B_2 be the resulting set. Consider Example 1 as a case in point; here $m = 10$ and B_2 is the set N of all non-negative integers. We write $B = mB_2$ to mean that if every element of B_2 is multiplied by m the resulting set is B.

Similarly consider every multiple of m in A; divide each such multiple by m and let A_2 be the resulting set of elements. Then we write

$$A = mA_2 + \{0, 1, 2, \ldots, m-1\} = mA_2 + S_0, \tag{9}$$

meaning that every element in A can be obtained by adding some element in mA_2 to some element of S_0.

Now the important point is that A_2 and B_2 are also complementing sets. To prove this we observe first that $0 \in A_2$ and $0 \in B_2$. Next let t be any positive integer; we must prove that t is expressible uniquely as a sum of an element in A_2 and an element in B_2. Consider the expression for mt as a sum of an element in A and an element in B, say $mt = a+b$, $a \in A$, $b \in B$. By Lemma 2, b is a multiple of m and so is a by the equation $mt = a+b$. Let a_2 and b_2 satisfy $a = ma_2$, $b = mb_2$ and observe that $a_2 \in A_2$ and $b_2 \in B_2$. Also $t = a_2+b_2$. To show that this representation of t is unique, suppose $t = a_2^*+b_2^*$ with $a_2^* \in A_2$ and $b_2^* \in B_2$. Then $mt = ma_2^*+mb_2^*$ with $ma_2^* \in A$ and $mb_2^* \in B$. But this representation of mt must be the same as $mt = ma_2+mb_2$ and so $a_2^* = a_2$, $b_2^* = b_2$.

Thus A_2 and B_2 are complementing sets, and now $1 \in B_2$ because $m \in B$. So let m_2 be the least positive element of A_2, if there is a least positive element; in case there is not the process stops, a situation we shall deal with in a moment. As before with the argument leading up to (9) we get sets A_3, B_3 satisfying

$$A_2 = m_2 A_3, \quad B_2 = m_2 B_3 + \{0, 1, 2, \ldots, m_2-1\}. \tag{10}$$

Continuing we now have $0 \in A_3$, $0 \in B_3$, $1 \in A_3$ and if m_3 is the least positive element of A_3 then we get sets A_4 and B_4 satisfying

$$B_3 = m_3 B_4, \quad A_3 = m_3 A_4 + \{0, 1, 2, \ldots, m_3-1\}. \tag{11}$$

Now we have $0 \in A_4$, $0 \in B_4$, $1 \in B_4$. To illustrate the situation where the process of moving from sets A, B to A_2, B_2 to A_3, B_3 and so on is finite, let us suppose that A_4 has no positive elements so we cannot find any element m_4. That means that 0 is the only element of A_4, thus $A_4 = \{0\}$, and so $B_4 = N$, the set of all non-negative integers. Then from the previous equations we can determine the sets A_3, B_3, then A_2, B_2, and then A, B. Let us now write m_1 for m to conform to the pattern in equations (3). We see that A is the set of all integers of the form $a_0+a_2M_2$ where a_0 and a_2 are any non-negative integers satisfying $a_0 < m_1$ and $a_2 < m_3$. Also B is the set of all integers of the form $a_1M_1+a_3M_3$ where a_3 is any non-negative integer but a_1 satisfies $a_1 < m_2$. Thus we have equations (4) and (5) with $n = 1$. (Note that if the process had stopped at A_5, B_5, with $B_5 = \{0\}$, then A and B would be as in (4) and (7), with $n = 2$.)

On the other hand, if the process giving equations (9), (10) and (11) continues indefinitely we see that the sets A and B are as given in (4) and (5) with no limit to be imposed on the possible size of n. Thus we see that Lemma 2 applied repeatedly gives us the characterization of A and B in (4), (5), (6), and (7). The reader will find it instructive to carry through this process in Examples 1 and 2 given at the beginning of the paper.

3. Some unsolved problems

If instead of beginning with the set N of all non-negative integers we had started with the set Z of all integers, positive, negative and zero, the problem would be more difficult. The problem in this case is to find a characterization of all pairs of subsets A, B of Z such that every integer can be written as a unique sum $a+b$ with $a \in A$, $b \in B$. An example of such a pair of sets is readily given: take

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and B to be the infinite set of all multiples of 10, thus

$$B = \{\ldots, -30, -20, -10, 0, 10, 20, 30, \ldots\}.$$

Other examples can be given, but the problem of characterizing all such sets A and B is as yet unsolved.

Here is another problem of a different sort: let Z^+ denote the set of all positive integers. Characterize all subsets A and B of Z^+ such that any positive integer can be uniquely represented as a product ab with $a \in A$ and $b \in B$. This might be called the problem of factoring the integers, not one at a time, but as a collection. Again it is easy to give examples; for instance take

$$A = \{1, 2, 2^2, 2^3, 2^4, \ldots\}, \quad B = \{1, 3, 5, 7, 9, \ldots\},$$

that is, A is the set of all non-negative powers of 2, and B is the set of all positive odd integers. But to describe or characterize all sets A, B with this property is an unsolved problem. Perhaps someday one of the readers of this article will settle such an unsolved question!

Plane and Coloured

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Most readers will have heard of the following puzzle: Three families live in three detached houses in the country. They must use the same water, electricity and gas supply, but they want to keep out of each other's way and insist that their paths to the controls of these utilities must not cross. How can such paths be traced?

The answer is, of course, that such paths are impossible. For instance, imagine that B and C have constructed their paths, before the third had arrived, as follows:

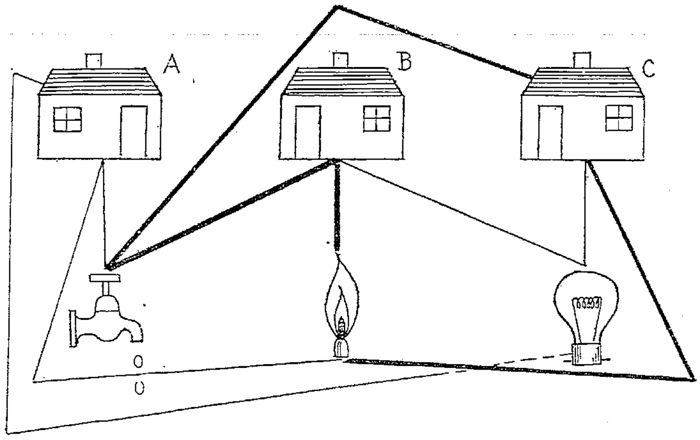

Figure 1

Then A cannot possibly find a way to electricity because of the two points to be connected, one is inside, and the other outside, the closed heavy curve shown.

This does not show the impossibility of doing better, since the first two families might have constructed their respective paths differently. To prove the impossibility rigorously, and generally, we must introduce a few concepts.

A collection of points ('nodes') and lines ('edges') connecting some or all pairs of nodes is called a graph. We consider, in particular, graphs in which no two lines intersect. If in such a graph there are at least two edges to each node, then such a graph has an inside and an outside. The latter is the region of those points which can be reached from a point far away, without crossing any of the edges of the graph. We call regions of the plane within which there is no node a 'face', and count the whole of the outside also as a face. Then the following formula due to Euler holds:

$$N+F-E=2,$$

where N, E and F are, respectively, the numbers of nodes, edges and faces. This can be proved as follows:

An n-sided non-intersecting polygon has n nodes, n edges and two faces (do not forget the outside!). Every other non-intersecting graph with an inside and an outside can be made up by starting from a polygon, for which Euler's formula holds, and adding others, neighbouring ones, to the graph already obtained. A rigorous proof of this possibility is lengthy, but the result is plausible and we leave it at that. Whenever we add such a polygon, with m sides, say, we add one face, $m-1$ edges, and $m-2$ nodes. Thus the formula remains valid after each addition.

Now imagine that the graph of our original puzzle, with its houses and control stations as nodes, and the paths as edges could be drawn without crossings. Then we would have a graph with 6 nodes and 9 edges. Euler's formula would then tell us that there are $9+2-6$, i.e., 5 faces. We shall now show that this leads to a contradiction.

Every face of the graph has at least 4 edges; this is so, because if it were a triangle, then 2 of its vertices would be either both houses, or both stations, but two such places are not connected. Now every edge belongs to 2 faces which it separates, so that there must be at least $4F/2$ edges on the graph, and if $F=5$, this would be 10. But there are only 9 of them. Thus the puzzle cannot be solved by a graph of non-intersecting edges.

Let us apply these ideas to another graph, the pentagon with all its 5 diagonals.

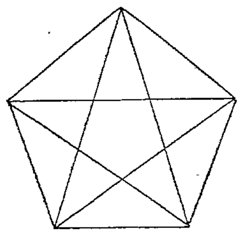

Figure 2

Could we replace the diagonals by (curved) lines connecting the same vertices, but in such a way that no two lines intersect?

We would then have 5 vertices and 10 edges, so that Euler's formula produces $F=7$. But each face has at least 3 edges, and each of these belongs to precisely 2 faces. So there must be at least $7 \times 3/2 = 10\frac{1}{2}$ edges, but there cannot be. Once more we see that the answer to our question is: no.

We call graphs which can be redrawn so that no 2 edges intersect 'planar'. The property of the pentagon with all its diagonals of not being planar enables us to prove a very simple case of the celebrated Four-Colour Conjecture. This theorem states that a map of connected countries (i.e., not consisting of parts separated by regions of another country, as for instance Pakistan) can always be coloured by not more than 4 colours in such a way that no two neighbouring countries have

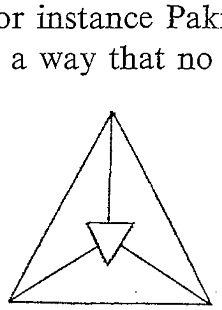

Figure 3

the same colour. (Less than 4 colours will not always suffice. For instance, on this map every one of the 4 countries borders on every other, so that we need 4 colours in this case.)

Actually, the Four-Colour Theorem is only a conjecture, and many ingenious mathematicians have tried but not succeeded to prove it. It has, however, been proved when there are not more than 35 countries on the map.[1] We shall now prove it for the simplest case, that of 5 countries.

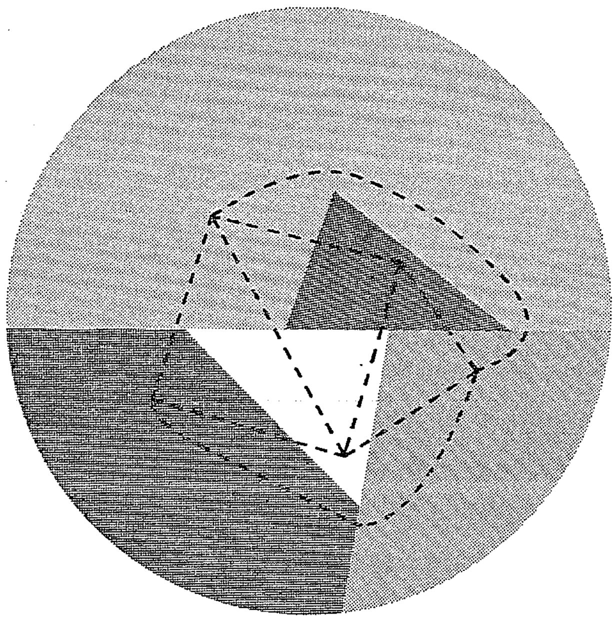

Figure 4

If 5 colours were ever required in this case, then every country would be adjacent to every other. We could then choose a point in every country, and connect points in adjacent countries by lines (not necessarily straight) across their borders. No two of such lines would intersect, and we would then have a planar graph corresponding to a pentagon with all its diagonals. But we know that such a graph does not exist.

[1] According to O. Ore, *The Four-Color Problem* (Academic Press, New York, 1967).

A Problem on Convex Quadrilaterals

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Given a set S of $n > 4$ points in a plane such that no three are collinear, prove that one can find at least ${}_{n-3}C_2$ convex quadrilaterals whose vertices are four of the given points.

This problem was one of those posed to competitors in the International Mathematical Olympiad[1] held in Bucarest in July 1969. The problem and its solutions present some features which may be of interest, and I hope to use it to illustrate some general strategies of problem solving. It is not realised as generally as it might be that the way in which mathematicians express themselves publicly, for example when writing for one another in Journals, is not the way in which they *think* about problems when solving them. The schoolboy beginning geometry may sometimes wonder how it was that Pythagoras proved his theorem. Did he write it out straight off, beginning with 'In △'s *ABK*, *PBC*....'? (See note 1.) More experienced pupils realise that the solution of a geometrical problem usually involves a certain amount of trial and error: we first stumble on a method and *then* begin to write it out formally. Much the same is true in all mathematical problem solving. The ease and elegance of a solution may conceal a lot of false starts and hard work: one must not expect to produce a polished proof at the first attempt.

Schoolboys and undergraduates, when first exposed to the icy winds of 'rigour', may become excessively timid; this seems to them to be the language of *real* mathematics. I do not wish to deny the importance of rigour—but merely to emphasise that its place is *after* a proof or solution has been found. The initial steps may be tentative, illogical even—one is casting around for a method, following 'hunches', exploring the problem. As George Pólya wrote in his article 'On Learning, Teaching, and Learning Teaching' in the *American Mathematical Monthly*, **70** (1963), p. 605 (see note 2): "Mathematical thinking is not purely 'formal'; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it, generalising from observed cases, inductive arguments, arguments from analogy, recognising a mathematical concept in, or extracting it from, a concrete situation.... Let us teach proving by all means, but let us us also teach guessing."

Perhaps the most profitable course will be not to read the present note straight through but to use pencil and paper at each stage to examine each of the sections (separated by lines) without looking ahead.

[1] For further details of the International Olympiads, see
Mathematical Gazette, **52** (May 1968), p. 130.
The New Science Teacher, **11** (Oct. 1967), p. 30; **12** (Dec. 1968), p. 31; **13** (Dec. 1969), p. 59.
American Mathematical Monthly, **71** (1964), p. 308.
Mathematics Teaching, No. 43 (Summer 1968), p. 4.

(i) Try to solve the problem as stated above.

Hints: Why does $(n-3)$ occur in the result? Is the idea of 'choosing 2 from $(n-3)$' involved? Is the stated result correct for $n = 5$? (It ought to be, but this simple case may give us some ideas.)

(ii) The solution submitted with the proposed problem was as follows:

Consider the ${}_nC_2$ distances between pairs of the points of S and select the least (or one of the least) of these. Let the corresponding points be A and B. Then we can find a point C such that $\triangle\ ABC$ contains no points of S. (Proof?) Given any 5 points, we can choose from them at least one set of 4 points which form a convex quadrilateral (C.Q.). (Proof?) Thus given any pair of points X, Y of S outside $\triangle\ ABC$ the set $\{A, B, C, X, Y\}$ gives rise to at least one C.Q., and so there are at least ${}_{n-3}C_2$ such.

This proof is invalid. Can you see why? (If not, see note 3.)

(iii) A correct proof follows:

We can find 3 points ABC such that all the other points of S lie within the angle ABC. Let X, Y be a pair of points of S, distinct from A, B, C. Now XY must cut at least one side of $\triangle\ ABC$ externally and then X, Y and the 2 points which define that side form a C.Q. Hence there are at least as many C.Q.'s as there are pairs X, Y—i.e., at least ${}_{n-3}C_2$. (Why does this proof avoid the difficulty encountered in (ii)?)

(iv) Have we now finished? Certainly we have solved the problem, but it may be worth looking further, and asking some questions. Is there any significance in the expression ${}_{n-3}C_2$? It is the *answer*. Is it *the* answer? Is there a *better* answer? Let us rephrase the question: 'what is the least number of C.Q.'s whose vertices . . . ?', i.e., 'find a *lower bound* for the number of C.Q.'s . . . '. Now we may be led to ask: 'Is there an *upper* bound?' 'What can be said about the number of C.Q.'s . . . ?'

One way of improving the lower bound is as follows:

After finding at least ${}_{n-3}C_2$ C.Q.'s as in (ii) we may delete A, B, C from the set and . . . . . . (?) Use this method to find a lower bound 'of order n^3'—i.e., which, when n is large, behaves like the expression n^3 (note that the expression ${}_{n-3}C_2$ behaves for large n like n^2, i.e., is 'of order n^2'). (See note 4.)

(v) The problem as posed illustrates the danger of being too inflexible in one's approach, of acquiring a so-called 'mental set'. The proposer expected a proof similar to that given in (ii), and the expression ${}_{n-3}C_2$ leads one to pick out a triangle from among the points S. In fact a much stronger result is possible in which ${}_{n-3}C_2$ is replaced by ${}_nC_5/(n-4)$. This was found by several of the students, despite the unintentional misdirection caused by stating the result. Two proofs produced by members of the British team are outlined below.

D. J. Aldous. Let $F(n)$ be the minimum number of C.Q.'s obtainable from n points. We are asked to show $F(n) \geqslant {}_{n-3}C_2$. Remove an arbitrary point from S.

The number of C.Q.'s obtainable from the remainder is at least $F(n-1)$. By repeating with a different point of S we obtain at least $n.F(n-1)$ C.Q.'s (including repetitions). But each repetition occurs at most $(n-4)$ times. Hence

$$F(n) \geqslant \frac{n}{n-4} F(n-1),$$

i.e.,

$$\frac{F(n)}{F(n-1)} \geqslant \frac{n}{n-4}.$$

Also $F(5) = 1$. Let $G(n) = {}_{n-3}C_2$. Then

$$\frac{G(n)}{G(n-1)} = \frac{n-3}{n-5}$$

and $G(5) = 1$. Thus $F(n) \geqslant G(n)$. But in fact

$$F(n) \geqslant \frac{n(n-1)(n-2)\dots 6}{(n-4)(n-5)(n-6)\dots 2} \quad F(5) = \frac{{}_nC_5}{(n-4)}.$$

N. S. Wedd. Given a set of 5 points we can select at least one sub-set of 4 points forming a C.Q. There are ${}_nC_5$ such sets of 5 points in S, though two non-identical sets of 5 points may lead to the *same* C.Q. Each C.Q. can arise from at most $(n-4)$ sets of 5 points. Thus there are at least ${}_nC_5/(n-4)$ C.Q.'s whose vertices are points of S.

Moral. Problems are easier if tackled in the right way! But remember that the ease and elegance of a solution may conceal a lot of false starts and hard work.

(vi) *In conclusion.* We may learn from looking back on what we have done. The numbering refers to the stages above.

(i) A problem may seem inaccessible—we have to 'try around', explore a little, ask a few questions, fix our ideas. In these early attempts we are getting our bearings, understanding what the problem is about.

(ii) 'One hole in a bucket is enough to let the water out', that is, one error in a proof is enough to render it invalid; hence the need for rigour. This was a promising idea, but it didn't work.

(iii) 'Holes can sometimes be mended', i.e., a proof can sometimes be salvaged. With a little adaptation the idea can be made to work.

(iv) Don't always stop at 'Q.E.D.'; there may be more we can do, extensions, or generalisations of the problem. A second look is often rewarding.

(v) It may be unhelpful to fix ideas *too* firmly too early; keep an open mind. Try several approaches. If one doesn't work, another may.

I have tried to stress the informal, tentative, uncertain steps by which one sets about solving an unfamiliar problem, in contrast with the assured, concise way in which the solution, when arrived at, is finally written up. If one knows at the outset the appropriate method of solution, the problem is no longer a problem, it becomes merely a routine exercise. Don't be afraid to make conjectures, to try unproved assumptions, to reason from the particular to the general, to give partial solutions: these may all help in reaching the goal.

The eminent French mathematician, Laplace, was well aware that 'Even in the mathematical sciences, our principal instruments to discover the truth are induction and analogy' (see note 5). Having discovered the solution, we must then verify, rigorously, that our method is valid. But we must not expect to produce a polished proof at the first attempt.

Notes

1. (For the teacher.) It seems unlikely that he used a proof by 'shearing' since by definition, modern mathematics had not yet been invented.

2. An excellent book on strategies for solving problems in mathematics is G. Pólya, *Induction and Analogy in Mathematics* (Princeton University Press, 1954 (reprinted 1965)). See also his *Mathematical Discovery* (2 Vols.) (Wiley, New York, 1962 and 1965).

3. The C.Q.'s associated with $\{A, B, C, X, Y\}$ and $\{A, B, C, X, Z\}$ may be identical, viz. $ABCX$.

4. Lower bound $\geqslant {}_{n-3}C_2 + {}_{n-6}C_2 + {}_{n-9}C_2 + \dots$ while the terms remain defined. The sum can be shown to be

$$\frac{(n-1)^2(n-4)}{18}, \quad \frac{(n-5)(n^2-n+4)}{18}, \quad \frac{(n-3)^2 n}{18}$$

for n of form $3k+1$, $3k+2$, $3k$, respectively. Each is greater than $(n-4)^3/18$ for $n > 5$.

5. Quoted by Pólya in *Induction and Analogy in Mathematics*, page 35.

A Probabilistic Triangle Problem

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1. Introduction

In this article some problems involving both geometry and probability will be treated; basic ideas in probability theory will be introduced as they are needed in the course of the exposition.

Geometry has a very long history starting more than 2000 years ago. Probability theory, on the other hand, is a rather young product of human thought; its history starts essentially about 300 years ago. Both fields are important parts of mathematics, and both are indispensable tools when one tries to describe the real world in mathematical language.

The problems discussed in this article may seem artificial, but they are in fact very important; we usually refer to them as 'geometric probability problems'.

2. A simple problem

Let us start with the following simple example: in Figure 1 a stick of length 5 units is pictured, with $A_0A_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_3A_5$ forming the equal units.

Figure 1

Two of the 4 'interior points' A_1, A_2, A_3, A_4 are chosen at random and the stick is broken at these points. The problem is to find the probability that a triangle can be constructed with these 3 sticks as edges.

In order to solve this problem we must clarify what is meant by 'at random', and specify how the probability of an event is calculated. We first introduce the *outcome set* of the experiment, namely the set of all its possible outcomes. This description is loose, and can lead to different outcome sets, depending on what we mean by 'possible outcome'. One may in this case use the following set of doublets, or sets with two elements, as the outcome set:

$$\{\{A_1, A_2\}, \{A_1, A_3\}, \{A_1, A_4\}, \{A_2, A_3\}, \{A_2, A_4\}, \{A_3, A_4\}\}.$$

Here, for instance, the outcome $\{A_1, A_3\}$ means that the stick is broken at the points A_1 and A_3. Note that there are 6 elements in the outcome set.

Now from elementary geometry, it is known that a triangle can be formed from 3 line segments if and only if the length of each segment is less than the sum of the lengths of the other two. For instance, if A_1 and A_3 are chosen as breaking points, the sticks of length 1, 2 and 2 can form a triangle; but if points A_1 and A_2 are chosen, a triangle cannot be constructed from the sticks of length 1, 1 and 3.

In the following table we have indicated, those outcomes for which a triangle can be formed by an asterisk (*).

Outcome	Lengths of sticks
$\{A_1, A_2\}$	1, 1, 3
$\{A_1, A_3\}$*	1, 2, 2
$\{A_1, A_4\}$	1, 3, 1
$\{A_2, A_3\}$*	2, 1, 2
$\{A_2, A_4\}$*	2, 2, 1
$\{A_3, A_4\}$	3, 1, 1

Thus, of the 6 possible outcomes, there are 3 for which a triangle can be formed. The event that a triangle can be formed is thus determined by the subset $\{\{A_1, A_3\}, \{A_2, A_3\}, \{A_2, A_4\}\}$ of the outcome set.

In probability theory one usually *defines* an event precisely as a subset of the outcome set. Thus if T is the event that a triangle can be formed we may identify T as the set $T = \{\{A_1, A_3\}, \{A_2, A_3\}, \{A_2, A_4\}\}$.

It now remains to find its probability $P(T)$. It was previously stated that the breaking points were chosen at random. By this is meant that each of the 6 mutually exclusive outcomes is equally likely, and since the event T consists of three of these, the probability of T will be 3/6 or 1/2.

In general if A is an arbitrary event in an outcome set Ω, that is $A \in \Omega$, where the (mutually exclusive) elementary outcomes in Ω are equally likely, then the probability of A is given by

$$P(A) = \frac{n(A)}{n(\Omega)},$$

where $n(A)$ and $n(\Omega)$ are the numbers of elements in A and Ω respectively. While this is not the most general method of defining probabilities, it is the method we shall use in this article. In such a case the outcome set is usually said to have a *uniform probability distribution.*

3. Another solution

We shall now discuss another solution to our problem using a different outcome set in the standard coordinate system.

As outcomes we shall use the ordered pairs of the lengths of the first two sticks (from the left in Figure 1). Such an ordered pair determines the breaking points uniquely. Our new outcome set, easily obtained from the table above, is then

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}.$$

The event T that a triangle can be constructed is now the set $\{(1,2), (2,1), (2,2)\}$, and for a uniform probability distribution $P(T)$ is 3/6 or 1/2. In Figure 2 this solution is illustrated by representing the outcome set and the event T in a coordinate system.

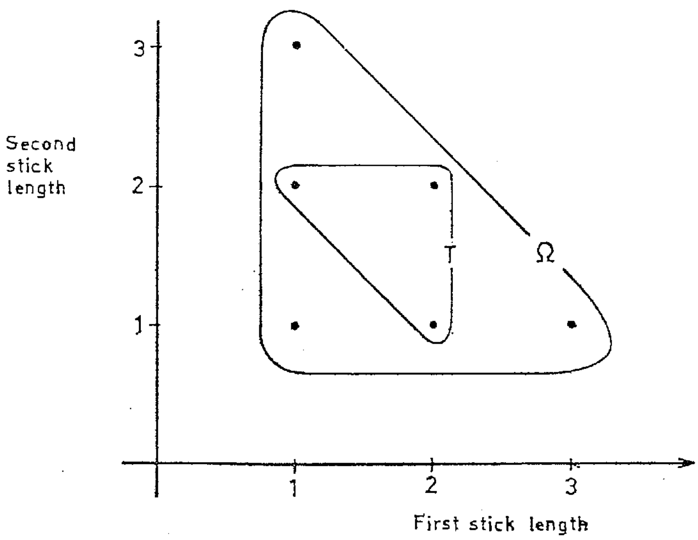


Figure 2

The solution of the corresponding triangle problem for a stick of length 8 (with 7 equally spaced breaking points) can readily be found. Here, a diagram similar to Figure 2 rapidly indicates that the outcome set has 21, and the event T 3 elements, namely (2, 3), (3, 2), (3, 3). Thus the probability that a triangle can be formed in this case is 3/21 or 1/7. The industrious reader may wish to draw more diagrams to illustrate the solution of the corresponding triangle problem for sticks of length 6 and 7. The probabilities in these cases are 1/10 and 2/5 respectively.

4. A more general problem

In this section we solve the general problem for a stick of length n, on which are marked $n+1$ equidistant points $A_0, A_1, \ldots, A_n$. Two of the interior points are chosen at random, and at these the stick is broken. Let P_n be the probability that a triangle can be constructed from the 3 sticks obtained; we derive an expression for P_n.

From the preceding sections we know that

$$P_5 = 1/2, \quad P_6 = 1/10, \quad P_7 = 2/5 \quad \text{and} \quad P_8 = 1/7.$$

We now want to find a general expression for P_n, of which these are special cases.

As our outcome set let us use the set of ordered pairs (x, y), where x, y are the lengths of the first and second line segments respectively, starting from the left. Using set theoretic language we can thus define the outcome set Ω as

$$\Omega = \{(x, y)\colon x,\ y \in N \text{ and } x+y < n\}.$$

Here N is the set of all natural numbers 1, 2, 3,... of which x and y are elements such that $x+y < n$. The outcome set Ω is illustrated in Figure 3.

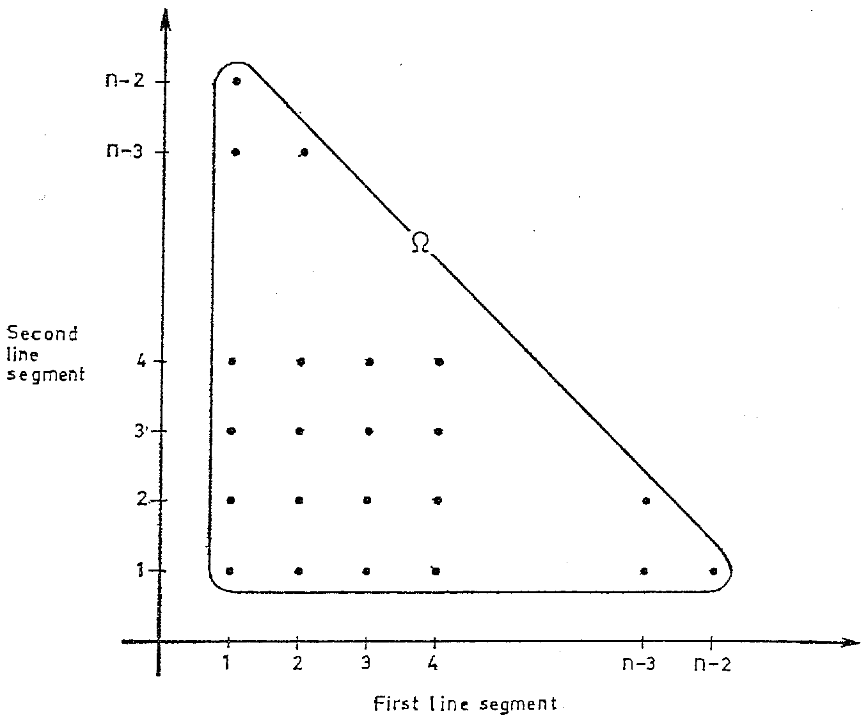


Figure 3

Let us now find an expression for the number of elements $n(\Omega)$ in Ω; this is clearly

$$n(\Omega) = 1+2+3+\ldots+(n-3)+(n-2) = \tfrac{1}{2}(n-1)(n-2).$$

Now let T be the event that a triangle can be formed from the 3 segments of lengths, x, y and $n-(x+y)$. Thus T is the set of ordered pairs in Ω such that the following three conditions

$$x+y > n-x-y,$$
$$x+(n-x-y) > y,$$
$$y+(n-x-y) > x$$

are satisfied, or equivalently that

$$x+y > n/2, \quad y < n/2, \quad x < n/2.$$

Observe that the necessary and sufficient conditions for constructing a triangle can be formulated as 'All three line segments must be shorter than half the stick', or more formally

$$T = \{(x,y)\colon x,\ y \in N \text{ and } x < n/2 \text{ and } y < n/2 \text{ and } x+y > n/2\}.$$

When finding an expression for the number of elements $n(T)$ in T we treat the cases where n is even and odd separately. In Figure 4 the event T for even n is illustrated. Using reasoning similar to that above, we conclude that

$$n(T) = 1+2+\ldots+\left(\frac{n}{2}-2\right) = \frac{(n-2)(n-4)}{8}.$$

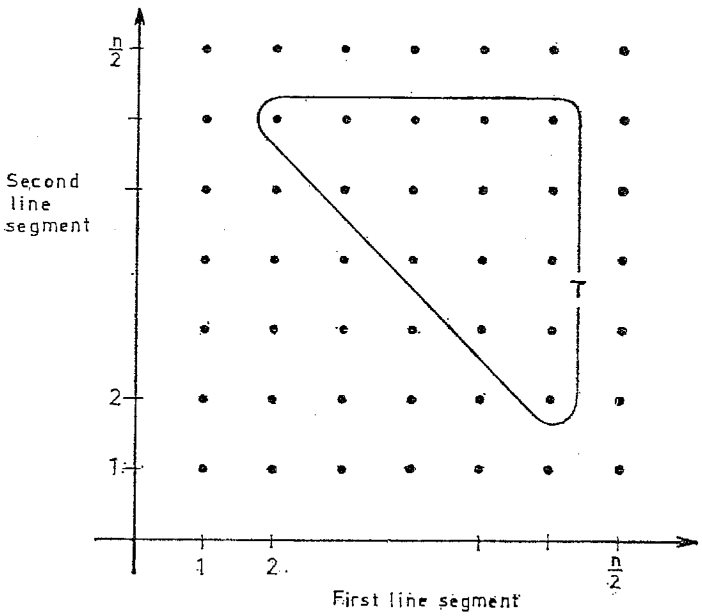


Figure 4

For the case when n is odd the reader can easily verify that

$$n(T) = 1+2+\ldots+\frac{n-1}{2} = \frac{n^2-1}{8}.$$

We can now easily derive a simple expression for $P_n = n(T)/n(\Omega)$; we get

$$P_n = \begin{cases} \dfrac{n-4}{4(n-1)} & \text{for } n \text{ even,} \\[2ex] \dfrac{n+1}{4(n-2)} & \text{for } n \text{ odd.} \end{cases}$$

5. A limiting process

It is easy to verify that the above formulae for P_n give the probabilities we obtained before as special results; in Figure 5 we have graphed P_n as a function of n.

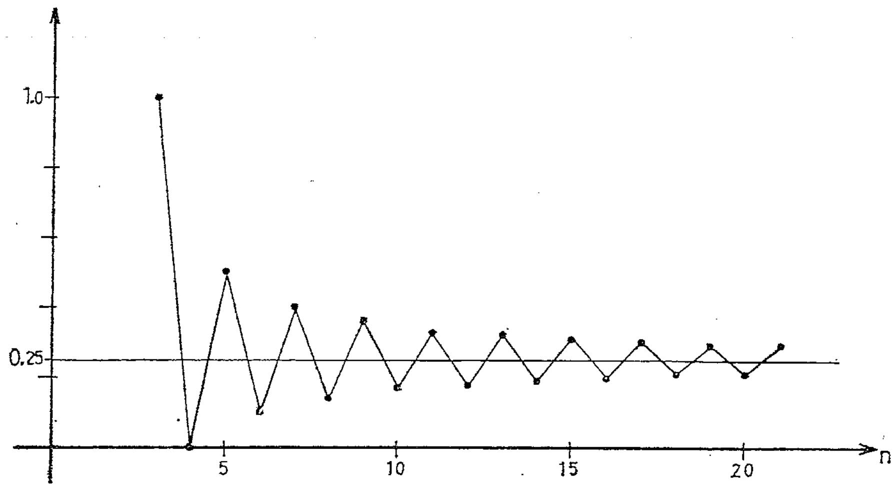


Figure 5

It is clear that for all odd numbers n, P_n is greater than 1/4 while for all even numbers n it is less than 1/4. There is thus always a greater probability of constructing a triangle with an even than an odd number of breaking points.

From the graph we can also see that for increasing n the probabilities come closer and closer to 1/4. More precisely, this is expressed as the limit

$$\lim_{n\to\infty} P_n = 1/4.$$

It seems hard to interpret this result concretely since as n tends to infinity the stick becomes infinitely long. However, instead of considering a stick of length n with possible breaking points at unit intervals, we can consider a stick of length 1 with possible breaking points spaced $1/n$ apart. The probability that we can form a triangle depends only on the number of possible breaking points and thus the identical formulae will apply for P_n. Here n tending to infinity means that we take the possible breaking points closer and closer on the same stick, with the limiting case being that where all points on the stick are possible breaking points. The probability of constructing a triangle in this case is 1/4. In the next section we show that this is in fact the case.

6. The continuous triangle problem

Consider a stick of unit length. Two points at distances x and $x+y$ from the origin are chosen at random on it and the stick is broken at these. We shall determine the probability that a triangle can be constructed from the three parts of lengths x, y, $1-x-y$ obtained.

The triangle problems we treated in the last sections were discrete: the number of possible outcomes was finite. In this case, as all points on the stick are possible breaking points, we have an infinite number of outcomes. Clearly we cannot calculate probabilities as before; there is for instance no finite number to replace $n(\Omega)$ in the expression $n(T)/n(\Omega)$. We must find a new method of specifying probabilities.

Let us start by defining a suitable outcome set. As outcomes we can use ordered pairs (x,y), where x, y are the lengths of the first and second sticks or line segments. This means that x and y are non-negative real numbers whose sum is less than 1. For the outcome set Ω we have (see Figure 6a)

$$\Omega = \{(x,y): x,\ y \in R^+ \text{ and } x+y < 1\},$$

where R^+ is the set of non-negative real numbers. Let T be the event that we can make a triangle of the three segments of lengths x, y and $1-x-y$. We have previously seen that T is the set of ordered pairs (x,y) such that all line segments have length less than $1/2$; thus

$$T = \{(x,y): x,\ y \in R^+ \text{ and } x < 1/2 \text{ and } y < 1/2 \text{ and } x+y > 1/2\}.$$

The event T is illustrated in Figure 6b.

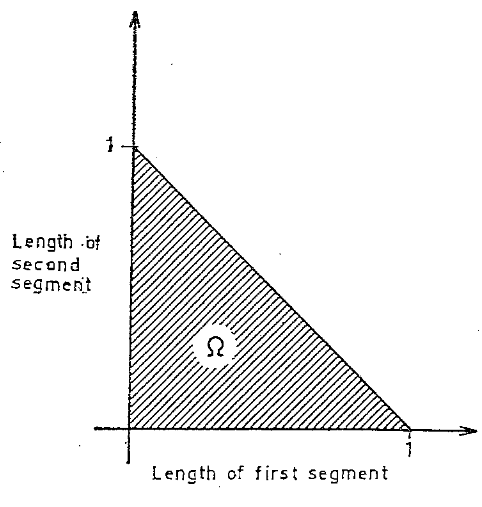


Figure 6a

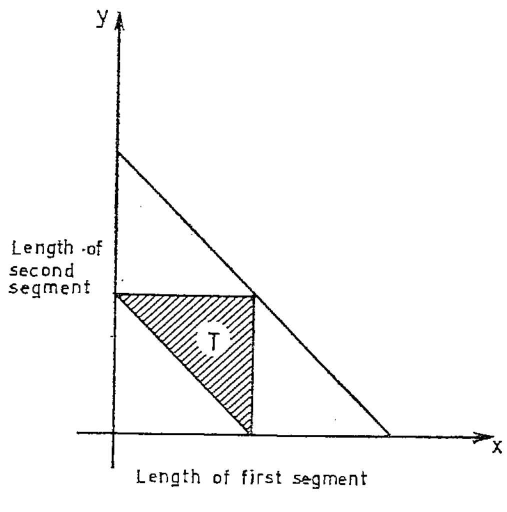


Figure 6b

We must now re-define the probabilities in the case, again assuming that the two breaking points were chosen at random. In Figure 6 we see from elementary geometry that the area of the triangle (event) T is $1/4$ of the area of Ω (the outcome set). Thus it seems reasonable to give the probability $P(T)$ of T as $1/4$. In general, in situations similar to the one studied here, this means that if A is a subset of Ω

the probability $P(A)$ of A is found as

$$P(A) = \frac{a(A)}{a(\Omega)},$$

where $a(A)$ and $a(\Omega)$ are the areas of A and Ω respectively.

7. Some concluding remarks

We have now solved some probability problems by first establishing an outcome set, then defining events as subsets of the outcome set and finally devising methods for calculating the probabilities of events. Problems in probability are usually solved in this way; however, the methods we have used are very special. We have in all cases relied on a *uniform probability distribution.* If for instance the breaking points were chosen in such a way that there was a tendency to prefer points near the ends of the stick, our methods would not be valid without some modifications. We have also made a distinction between continuous and discrete probability problems. This distinction is important in probability theory as well as in other parts of mathematics. Observe that we have reached the solution of the continuous problem from that of the analogous discrete problem by a limiting process.

An important question, not discussed so far, is the practical significance of a probability. What does it mean to state that the probability of an event is 1/4? One possible interpretation (there are others) is the following. If we perform a practical experiment in which a certain event can occur, then the probability of the event is a prediction of the relative frequency of its occurrence in a large number of trials. For instance in the continuous triangle problem above the probability $P(T)$ was found to be 1/4. The author has performed the corresponding practical experiment by tossing matches on a lined plane such that only one line could cross the match when it landed on the plane. Each match was tossed twice to find two breaking points. In 100 such trials it was found that a triangle could be formed in 29 cases. The relative frequency was thus 29/100 or 0·29; the theoretical probability of 0·25 is not a bad prediction of this.

For readers interested in testing their skill at solving similar problems in geometric probability, the following two may be of interest:

1. Three sticks of length 1 are broken each at a point chosen at random. If one piece is taken from each stick, what is the probability that a triangle can be formed from these three sticks?

2. On a stick of length 1 two points are chosen at random and the stick is broken at these points. Find the probability that at least one of the three pieces obtained has a length of at most 0·2. Solve the same problem when three, instead of two, points are chosen.

The Promotion Problem

D. J. BARTHOLOMEW

University of Kent

Most people spend their working lives in jobs which occupy a place in a hierarchy of some kind. They begin with their foot on the bottom rung of the promotion ladder and 'work their way up'. In some organizations such as the armed forces the grades or ranks are very precisely defined and are there for all to see on the uniform. Even in civilian life there are subtle distinctions and status symbols which are sometimes more conspicuous than the salary ranges or job descriptions which lie behind them. People not only move from one job to another within an organization but between organizations so that in reality there is a complex network of movements which people can make. One of the objects of the rapidly growing activity of manpower planning is to quantify these flows of employees so that their inter-relationships can be studied. When this has been done the way is open to forecast the future pattern of movement and to anticipate bottlenecks and shortages.

In this article we shall consider what we have termed the 'promotion problem' as it arises in a single firm. The problem has two sides. On the one hand the individual employee expects reasonable chances of promotion so that he can rise to a level at which his abilities are fully used. The management share the same concern but have the added responsibility of ensuring an adequate supply of trained people able to move into new jobs as they occur.

The problem of investigating career prospects by mathematical methods is not new. Jones (reference) describes an early article in which the writer attempted to calculate various chances of promotion in the Royal Marines. However, it is only recently, with the pressing need to make full use of scarce manpower, that a mathematical theory of manpower flows has begun to emerge.

In order to illustrate a part of this theory, let us imagine a firm which employs 750 managers divided into grades. The grades are numbered 1, 2 and 3 from the bottom to the top. The present numbers in the grades are 500, 200 and 50 respectively. Once a year each man in grades 1 and 2 is considered for promotion and as a result he either stays where he is or moves up one grade. Alternatively he may decide to leave. In order to keep the total size constant all leavers are replaced by new entrants who all enter grade 1.

Before we can investigate the changes in the system mathematically we must translate this verbal description into precise mathematical language. This involves us in making certain assumptions which, in practice, should be checked against actual data.

Let p_{ij} denote the proportion of people who move from grade i to grade j at the annual review; p_{ii} is, of course, the proportion who remain where they are. For the system we described above these proportions may be set out in an array as follows

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{bmatrix}.$$

This is called the *promotion matrix* or *table.* We assume that these proportions remain the same through time. Next let w_i be the proportion in grade i who leave in the i^{th} year. Then, because everyone must either stay put, be promoted or leave, we see that

$$\begin{aligned} p_{11}+p_{12}+w_1 &= 1, \\ p_{22}+p_{23}+w_2 &= 1, \\ p_{33}+w_3 &= 1. \end{aligned}$$

We can now proceed to write down equations expressing next year's position in terms of this year's. Let $n_i(T)$ be the number in grade i in year T then

$$n_1(T+1) = p_{11}\, n_1(T)+\{w_1\, n_1(T)+w_2\, n_2(T)+w_3\, n_3(T)\}. \tag{1}$$

The first term on the right-hand side is the contribution from those already in the system and the second term is the number of new recruits who go into grade 1. The equations for the other grades are

$$n_2(T+1) = p_{22}\, n_2(T)+p_{12}\, n_1(T) \tag{2}$$

and

$$n_3(T+1) = p_{33}\, n_3(T)+p_{23}\, n_2(T). \tag{3}$$

It is now a simple matter to compute the numbers in each grade as far into the future as we please.

Let us suppose that the numerical values of the p_{ij}'s and the w_i's are as follows:

$$\mathbf{P} = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}3 & 0 \\ 0 & 0{\cdot}6 & 0{\cdot}2 \\ 0 & 0 & 0{\cdot}8 \end{bmatrix}, \quad w_1 = w_2 = w_3 = 0{\cdot}2,$$

and that we start at year zero with the grade sizes 500, 200 and 50. To simplify the calculations we have chosen all the loss proportions to be the same. This means that the last term in (1) is equal to $0{\cdot}2\times(500+200+50) = 150$ whatever the individual $n_i(T)$'s. We can now calculate the grade sizes at year 1 from equations (1), (2) and (3) as follows:

$$\begin{aligned} n_1(1) &= 0{\cdot}5\; n_1(0)+150 = 0{\cdot}5\times 500+150 = 400, \\ n_2(1) &= 0{\cdot}6\; n_2(0)+0{\cdot}3\; n_1(0) = 270, \\ n_3(1) &= 0{\cdot}8\; n_3(0)+0{\cdot}2\; n_2(0) = 80. \end{aligned}$$

The top and middle grades have thus increased at the expense of the lowest. Repeating the calculation with T increased from 0 to 1 we find

$$n_1(2) = 0{\cdot}5 \times 400 + 150 = 350,$$
$$n_2(2) = 0{\cdot}6 \times 270 + 0{\cdot}3 \times 400 = 282,$$
$$n_3(2) = 0{\cdot}8 \times 80 + 0{\cdot}2 \times 270 = 118.$$

There has been a further reduction at the bottom and more growth at the top. If we continue the process for another step it turns out that

$$n_1(3) = 325, \quad n_2(3) = 274{\cdot}2, \quad n_3(3) = 150{\cdot}8.$$

Inspection of the results so far suggests that the $n_i(T)$'s may be approaching limits as T increases. That this is so can be seen by returning to the equation for $n_1(T+1)$ which is

$$n_1(T+1) = 0{\cdot}5\, n_1(T) + 150.$$

It is easy to show by repeated substitution that

$$n_1(T+1) = (0{\cdot}5)^{T+1} n_1(0) + 150\{1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \ldots + (\tfrac{1}{2})^T\}$$
$$= (0{\cdot}5)^{T+1} n_1(0) + 150\left\{\frac{1-(\frac{1}{2})^{T+1}}{1-\frac{1}{2}}\right\}.$$

As $T \to \infty$ the limit of this is

$$n_1(\infty) = 300.$$

A similar argument for $n_2(T)$ and $n_3(T)$ shows that

$$n_2(\infty) = 225 \quad \text{and} \quad n_3(\infty) = 225.$$

Notice that these results do not depend on the original $n_i(0)$'s but only on the p_{ij}'s and w_i's. This is an important practical result. It means that the structure to which our organization will tend is determined by the promotion policy and the leaving rates and not at all by the way in which we set it up initially. Notice also that there has been steady growth at the top. This is what devotees of Professor Parkinson will have expected but there is nothing sinister about it. It is simply the natural consequence of promoting 30 per cent of grade 1 and 20 per cent of grade 2.

Suppose that such growth at the top is unacceptable. We can use our equations in reverse to calculate what the promotion rates would have to be in order to achieve or maintain any desired structure. To illustrate the method suppose that we wish to keep the structure exactly as it was originally. Mathematically this means that we require $n_1(T) = 500$, $n_2(T) = 200$ and $n_3(T) = 50$ for all T. Substituting in equations (1), (2) and (3) it is clear that the p_{ij}'s must now satisfy,

$$\left.\begin{aligned} 500 &= 500p_{11} + 150, \\ 200 &= 200p_{22} + 500p_{12}, \\ 50 &= 50p_{33} + 200p_{23}. \end{aligned}\right\} \qquad (4)$$

There are more unknowns than equations so there is no unique solution. This fact allows us to impose some further restraints of our own choosing. Let us assume therefore that the w's will be unchanged. This means that

$$p_{12} = 0{\cdot}8 - p_{11},$$
$$p_{23} = 0{\cdot}8 - p_{22}.$$

We now have 5 equations in 5 unknowns which are easily solved beginning with (4) to give

$$p_{11} = 0{\cdot}7, \quad p_{12} = 0{\cdot}1, \quad p_{22} = 0{\cdot}75, \quad p_{23} = 0{\cdot}05;$$

$p_{33} = 0{\cdot}8$ as before. This calculation shows that only a drastic cut in the promotion chances can maintain the structure as it is. However desirable the original matrix might appear, it can only be applied if we are prepared to have a top-heavy structure differing greatly from that which we started with. This is the promotion problem. Mathematical analysis does not solve it, but it does demonstrate clearly the dilemma which faces management.

Reference

E. Jones, An actuarial problem concerning the Royal Marines. *Journal of the Institute of Actuaries Students' Society*, **6** (1946), 38–42.

Issoselease Triangles[1]

T. M. FLETT

University of Sheffield

How many different combinations of letters are phonetically correct spellings of *isoscolese*? The question is not primarily mathematical, and only a phonetician could answer it correctly, but we can obtain some idea of the answer by enumerating the possible combinations of letters which might be used to reproduce the different sounds in *isocelles*. Thus we have one possibility for the *i*, four for each of *s* and *sc* (*s*, *ss*, *sc*, *c*), one for the *o*, five for the first *e* (*a*, *e*, *i*, *o*, and *nothing*[2]), two for the *l* (*l* and *ll*), and four for the *es* (*es*, *ese*, *ease*, *ies*), giving the surprising total of $1 \times 4 \times 1 \times 4 \times 5 \times 2 \times 4 = 640$ different combinations.

The inexperienced reader will perhaps feel that some of the combinations obtained in this way are rather unlikely to occur to the average student (though few school-teachers will), but I can refute this charge from my personal collection

[1] This article has previously appeared in the magazine of the Π Society of the University of Liverpool.

[2] According to my dictionary, it is an indeterminate sound.

of *isoceleses*. I began this collection some years ago when I was marking examination scripts. After I had marked two or three hundred scripts, I had acquired the collecting mania together with some thirty versions of *icosoles*. I began to look forward to reading the question which concerned an *isocilies* triangle, and felt quite cheated when a candidate did not attempt it, or when he spelt *iscosceles* correctly. And in a script where, for example, 'respectively' was written 'respectavily', I would read with bated breath until *isosolles* appeared. The longest of my 640 possible combinations unfortunately did not occur, but to make up for that I found *issossallese*, and in a further collection made the following year, I also found one of the shortest possible versions, *isosles*. My greatest disappointment was a script where seven consecutive lines contained 'congrunent', 'squre', 'drectly', 'diangles'[3], and 'radi'; the candidate wrote 'the triangle ABC has equal sides'.

Letter to the Editor

Dear Editor,

I should like to bring the following films to the notice of your readers:

Maths is a Monster—one of the BBC Discovery and Experience Films.

Children learning multiplication tables do not necessarily acquire the ability to solve problems. Here the experiments come first and are directly related to the children's experiences; recording them in mathematical terms follows, when the children really understand what they have done.

Black and white. 30 minutes. Hire charge £1 10*s*.

Children and Mathematics—BBC Television Enterprises.

A series of five films recently shown on BBC television, illustrating new ideas in the teaching of mathematics to children of 5 to 13. The emphasis is on discovery, and new teaching aids. The five films, which were made in collaboration with the Nuffield Foundation Mathematics Teaching Project, are:

We Still Need Arithmetic
Common Sense and New Maths
Freedom to Think
Checking Up
Teachers at the Centre

Black and white. 30 minutes each. Hire charge £1 10*s*. each.

Films are available for review, and our 1969/70 catalogue costs 2*s*. 6*d*. post free.

Yours faithfully,
CYNTHIA JAMES
Concord Films Council, Nacton, Ipswich, Suffolk.

[3] The alternative spelling 'diagnols' is perhaps a little easier to interpret.

Problems and Solutions

Readers who have not yet reached the age of 20 on 1 April 1970 are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

16. Let m and n be natural numbers such that $m \leqslant n$. (a) Show that n can be written as a sum of m natural numbers in $\binom{n-1}{m-1}$ different ways. (b) Determine the number of solutions of the equation

$$x_1+x_2+\ldots+x_m = n$$

when $x_1, x_2, \ldots, x_m$ are restricted to non-negative integers.

17. Peter and Paul agree to meet at their favourite restaurant. Both choose their arrival times at random between 6 p.m. and 7 p.m., and both wait 10 minutes after arriving. When is the probability that they will meet?

18. Show that, if m is a positive rational number, then $m+(1/m)$ is an integer only if $m = 1$.

19. Show that, if m and n are integers such that $m \geqslant 2$, $n \geqslant 3$, then there exist positive integers a, b such that

$$m^n = a^2-b^2.$$

Give an example to show that the result is false when $n = 2$.

Solutions to Problems in Volume 2, Number 1

12. Show that, if z is a complex number such that $-1 \leqslant \mathscr{R}z \leqslant 1$, then

$$|1+z^2| \geqslant 2(\mathscr{R}z)^2.$$

Solution by R. P. Allen (Grammar School for Boys, Cambridge)
Put $z = x+iy$, where x and y are real. Then

$$\begin{aligned} |1+z^2| &= [(1+x^2-y^2)^2+4x^2y^2]^{\frac{1}{2}} \\ &= [1+x^4+y^4+2x^2-2y^2+2x^2y^2]^{\frac{1}{2}} \\ &= [(1-x^2-y^2)^2+4x^2]^{\frac{1}{2}}. \end{aligned}$$

Hence

$$|1+z^2| \geqslant 2|x|.$$

Now $|x| \leqslant 1$, so that $x^2 = |x|^2 \leqslant |x|$. Hence

$$|1+z^2| \geqslant 2x^2 = 2(\mathscr{R}z)^2.$$

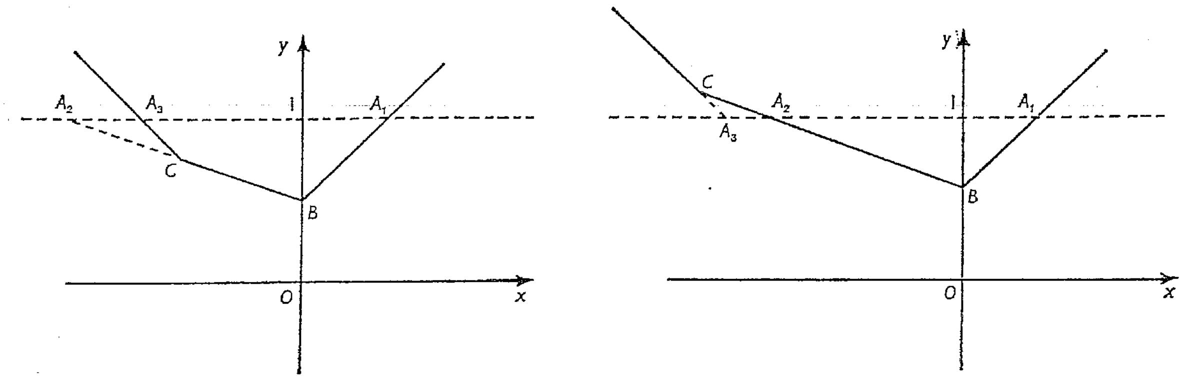
Also solved by M. R. Pooley (Gresham's School, Holt), G. Robertson (Leeds Grammar School).

13. Let $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant \frac{1}{2}$. Show that there exists an interval I of length 1 such that

$$|x|+|ax+b| \leqslant 1$$

whenever x belongs to I.

Solution by M. R. Pooley (Gresham's School, Holt)



Put $f(x) = |x|+|ax+b|$. The graph of $y = f(x)$ is as shown.

The line BA_1 has equation $y = (1+a)\,x+b$.

The line BA_2 has equation $y = -(1-a)\,x+b$.

The line CA_3 has equation $y = -(1+a)\,x-b$.

Thus A_1, A_2, A_3 have respective coordinates

$$\left(\frac{1-b}{1+a}, 1\right), \quad \left(-\frac{1-b}{1-a}, 1\right), \quad \left(-\frac{1+b}{1+a}, 1\right).$$

The length A_1A_2 is $\dfrac{1-b}{1+a}+\dfrac{1-b}{1-a} = \dfrac{2(1-b)}{1-a^2} \geqslant \dfrac{1}{1-a^2} \geqslant 1$.

The length A_1A_3 is $\dfrac{1-b}{1+a}+\dfrac{1+b}{1+a} = \dfrac{2}{1+a} \geqslant 1$.

Thus, whether A_2 is to the left or to the right of A_3, there is an interval I of length 1 such that $f(x) \leqslant 1$ wherever x belongs to I.

Also solved by M. J. Mellish (The King's School, Chester), G. Robertson (Leeds Grammar School).

14. Let c be a positive number. The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 0$ and $a_{n+1}-a_n \geqslant c$ for all n. Prove that

$$\frac{a_{n+1}}{(a_{n+1}-a_n)\,a_n^2} \to 0 \quad \text{as} \quad n \to \infty. \qquad (*)$$

Find an increasing sequence $a_1, a_2, a_3, \ldots$ of positive numbers such that $a_n \to \infty$ as $n \to \infty$, but (*) does not hold.

Solution by R. P. Allen (Grammar School for Boys, Cambridge)

We have

$$a_{n+1} \geqslant c+a_n \geqslant 2c+a_{n-1} \geqslant \dots \geqslant nc+a_1 > nc.$$

Thus $a_n \to \infty$ as $n \to \infty$. Now

$$\frac{a_{n+1}}{(a_{n+1}-a_n)\,a_n^2} = \frac{1}{a_n^2}+\frac{1}{(a_{n+1}-a_n)\,a_n} \leqslant \frac{1}{a_n^2}+\frac{1}{ca_n}.$$

Hence

$$\frac{a_{n+1}}{(a_{n+1}-a_n)\,a_n^2} \to 0 \quad \text{as} \quad n \to \infty.$$

The following example was supplied by S. A. E. Briggs (Ebbw Vale Grammar School). Take $a_n = \sqrt{n}$. Then

$$\frac{a_{n+1}}{(a_{n+1}-a_n)\,a_n^2} = \frac{\sqrt{(n+1)}}{[\sqrt{(n+1)}-\sqrt{n}]\,n} = \frac{\sqrt{(n+1)}\,[\sqrt{(n+1)}+\sqrt{n}]}{(n+1-n)\,n}$$

$$= 1+\frac{1}{n}+\sqrt{\frac{n+1}{n}} \to 2 \quad \text{as} \quad n \to \infty.$$

Also solved by M. R. Pooley (Gresham's School, Holt), G. Robertson (Leeds Grammar School).

15. An n-sided die has faces numbered 1, 2, 3,..., n. Peter tosses the die first, and then Paul predicts whether his own toss will be more than, equal to, or less than Peter's in outcome. Assuming that the die is fair, and first taking n to be even, find the rule that Paul should use to maximise his correct-prediction probability, and find that probability. Solve the problem also when n is odd.

Solution. As in the article 'Forecasting Trends' in the last issue, when n is even Paul's best rule is to predict that his toss will exceed Peter's if the latter resulted in any of 1, 2,..., $\tfrac{1}{2}n$. Otherwise, he should predict that his toss will be less than Peter's. The correct-prediction probability is then

$$\frac{1}{n}\left[\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\dots\frac{\frac{1}{2}n}{n}+\frac{\frac{1}{2}n}{n}+\dots+\frac{n-2}{n}+\frac{n-1}{n}\right] = \frac{3}{4}-\frac{1}{2n}.$$

If n were odd, Paul should nominate to beat Peter's toss if the latter results in any one of 1, 2, 3,..., $\tfrac{1}{2}(n-1)$, and to toss something less if Peter's toss is any one of $\tfrac{1}{2}(n+3)$,..., n. In the case when Peter has tossed $\tfrac{1}{2}(n+1)$, the probability that Paul will toss more is $\tfrac{1}{2}(n-1)/n$, that he will toss the same is $1/n$, and that he will toss less is $\tfrac{1}{2}(n-1)/n$. Of these, he might decide always to nominate that he will toss more than Peter, for that is at least as likely as the other alternatives. At any rate, for $n > 3$ he should never nominate to toss the same as Peter. The probability of correct prediction is then

$$\frac{1}{n}\left[\frac{n-1}{n}+\frac{n-2}{n}+\dots+\frac{\frac{1}{2}(n+1)}{n}+\frac{\frac{1}{2}(n-1)}{n}+\frac{\frac{1}{2}(n+1)}{n}+\dots+\frac{n-1}{n}\right] = \frac{3}{4}-\frac{2n+1}{4n^2}.$$

Also solved by M. J. Mellish (The King's School, Chester), M. R. Pooley (Gresham's School, Holt), G. Robertson (Leeds Grammar School), M. D. Mackey (St. Aloysius' College, Glasgow), P. Beverley (King George V Grammar School, Southport).

Book Reviews

Probability Theory. By HENRY E. KYBURG. Prentice Hall, Englewood Cliffs, N.J., 1969. Pp. x+294. 110*s*.

This book claims to be a 'one-semester course, designed primarily for the future critical consumer of statistics'. Unfortunately, the author has been led by his conviction 'that probability needs to be put on as careful a logical foundation as possible' to produce a very unbalanced account. The *definition* of probability does not appear until page 184, and the topics of comparative estimation (including Bayesian, fiducial and confidence intervals) and comparative inference as far as general decision theory, occupy a mere 70 pages. Even if one takes into account that all the matter is at an elementary level, the result is a distortion of the subject. Nor is the accuracy of the work likely to satisfy the 'critical consumer', although spotting the inaccuracies (e.g., in Figure 11.2 on page 242) may prove useful in training him!

University of Birmingham — F. DOWNTON

Mathematical Snapshots. By H. STEINHAUS. Third American Edition. Oxford University Press, New York, 1969. Pp. 311. $7.50.

The previous editions of Professor Steinhaus's book have given great delight to mathematicians, both amateur and professional. The present edition contains some new material, and brings the number of illustrations up to 391. These are an essential feature of the book. Some of them are diagrams, but many are photographs of, for example, surfaces with nets of curves, polyhedra, or knots.

As the title implies, the subject matter is mostly geometrical, but in the course of studying the geometry we meet irrational numbers, continued fractions, and even the musical scale.

This is an admirable book for a library; it is not cheap, but may be just right as a presentation copy on prize day!

University of Leicester — E. J. F. PRIMROSE

Modern Coordinate Geometry. A Wesleyan Experimental Curricular Study. By Pearl Glaubiger and others. Houghton Mifflin Co., Boston, Mass., 1969. Pp. xiv+446. $6.40.

This text has been prepared for American high-school honours classes as an experiment in the initial presentation of formal geometry as a subject of current, as well as of historical, mathematical interest. The aim is to give practice in exact expression and careful reasoning and to develop an appreciation of the nature of proof and of the pattern of logical organization known as the axiomatic method.

A great deal of the material presented would not be essentially new to an English second or third year sixth former, and a student thirsting after bigger and better theorems would hardly be well-advised to drink at this particular fountain. But to one who wishes to explore and consolidate the knowledge he has already gained, the book is to be highly recommended. It would be a useful addition to the sixth form library.

King's College, London — J. A. TYRRELL

Basic Algebraic Concepts. By F. LYNWOOD WREN and JOHN W. LINDSAY. McGraw-Hill, New York, 1969. Pp. 368. 86*s*.

This attractively presented book should be a welcome addition to the sixth form library. In some ways the material is presented in decreasing order of difficulty, the early chapters dealing with simple logic and the algebraic notions of groups, rings, fields and vector spaces, and then later chapters discussing polynomials, systems of equations and finally inequalities. The reader may find, as with *Time* Magazine, that it is more interesting to begin at the end and work backwards.

Many a student at school will find this book very difficult because it demands that he should *think* rather than manipulate formulae. The ideas are quite abstract but they are sympathetically introduced and many examples are provided.

An unusual feature is the paragraph entitled 'Guidelines for Careful Study' at the beginning of each chapter, which includes a searching list of questions. These should be very helpful.

University of Sussex A. J. WEIR

Educational Studies in Mathematics. Vol. 1, Nos. 1/2. D. Reidel, Dordrecht, Holland, May 1968. Subscription D.fl. 80 (about £10) per year, 4 numbers.

This first issue of a new journal is devoted entirely to the report of a colloquium on 'How to Teach Mathematics so as to be Useful', held at Utrecht in August 1967, which may perhaps be regarded as the forerunner of the first International Congress on Mathematical Education, held at Lyons in the summer of 1969. This volume reproduces the twenty main lectures of the colloquium and some discussions of problems in the teaching of mathematics.

The discussions are not of interest to the readers of this periodical, but many of the lectures, based on actual teaching experience, provide fruitful material for sixth form work, such as projects undertaken by the students themselves. These lectures are based on the belief that mathematics in the classroom should reflect mathematical activity in practice, in that it starts with a problem or situation, usually ill-defined, and investigates the relational structure of the situation, making observations, then tentative generalisations (conjectures), then seeking proofs; and that students should themselves take part in all these stages. Thus Engel describes a sixth form level course in Probability arising from a fishing situation (How long must one expect to wait for a catch?); Fletcher describes situations based on airline routes, football league tables, reproductive cycles of species and chemical reactions, all leading to the use of matrices and their eigenvalues; Pollak suggests some problems involving probability—how many packets of (say) tea does one need to buy to have a certain chance of getting a complete set of picture cards, and if cars park along a kerb, is one likely to accommodate more cars with marked spaces, or without? Steiner describes the development with a class of fifteen year olds of a situation concerning voting coalitions, which led to the formulation of a number of definitions and theorems, thus combining application with axiomatics.

This publication will be too expensive for most individuals and even school libraries, but if available at some accessible centre should prove a valuable source of classroom ideas.

Nottingham College of Education A. W. BELL

Notes on Contributors

Dame Mary L. Cartwright DBE, FRS, has recently held several visiting professorships in the U.S.A. and is now at Case Western Reserve University, Cleveland, Ohio. She retired in 1968 from Cambridge where she was Reader in the Theory of Functions in the University, and Mistress of Girton College. She has written one book, and many research papers on the theory of functions and on nonlinear differential equations. She has been President of the Mathematical Association and the London Mathematical Society, and has been awarded honorary doctorates by several English universities and Brown University.

Ivan Niven, who is Professor of Mathematics at the University of Oregon, U.S.A., has written extensively on the theory of numbers and related topics. He devised the first simple proof of the irrationality of π; this may be found in his book *Irrational Numbers* (Carus Mathematical Monograph No. 11). He was for some years an Editor of the *American Mathematical Monthly*.

S. Vajda worked in the Royal Naval Scientific Service until 1965, and thereafter held the Chair of Operational Research at the University of Birmingham until 1968, where he is now a Senior Research Fellow. He is the author of several books on mathematical programming and statistics, among them *An Introduction to Linear Programming and the Theory of Games* and *Planning by Mathematics*.

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