Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- Vertical-Axis Wind Turbines
- Abundant Numbers
- The Divergence of {cos(n)}
- Pascal's Triangle Modulo 3

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing. The copyright of all published material is vested in the Applied Probability Trust.

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From the Editor

Books! Books! Books!

'Of making many books there is no end, and much study wearies the body' is a sentiment by the writer of *Ecclesiastes* in the *Bible*, with which students may well agree. Yet it seems that the book may be on the way out, to be replaced by iPads, smartphones, Kindles, and the like. Which of us has not consulted Wikipedia for a theorem? Bookshops are disappearing from our high streets, or are becoming pleasant surroundings for drinking coffee. The novelist Fay Weldon recently spoke on the radio (a medium that was also thought to have had its day) about a marvellous invention made of sheets of paper bound together down a spine which could be held in the hand, the sheets could be turned over and covered with symbols. There are many homes where not a book is to be seen. Yet look at the delight on the face of a child who brings his or her favourite book to you and demands that you read it to them yet again.

The book *The Best Writing on Mathematics 2013* has come our way from Princeton University Press, edited by Mircea Pitici. The title suggests that this is an annual event and we did review the 2011 volume. Whether it lives up to its title I must leave other readers to judge. It starts with an interesting personal foreword from Roger Penrose, and includes articles on mathematics in the multimedia world, fearful symmetry looking at the markings of a tiger, how far it is possible to make universal statements about complex systems, the prime numbers being a case in point, degrees of separation starting with the statement 'every person in the world is no more than six people away from every other person on Earth', the three laws of randomness, randomness in music, the odds in games of chance, the complexity of problems in which there is an intriguing diagram of the best route between all 24 978 cities, towns, and villages in Sweden, and a new bridge in Jerusalem is used as an example of a modelling problem. An article looks at the mathematics and, in particular, the conic sections in the work of the German artist Albert Durer, there is a description of how topology influenced fashion in the 2010 Paris Fashion Week, with pictures of real models, and an article on the Jordan curve theorem with some beautiful drawings. An article asks the question 'Why teach mathematics and what is it anyway?', including the problem of anxiety about mathematics amongst students, there is plenty of jargon here, e.g. what does 'activation in a frontparietal network' mean? An article asks how old are the Platonic solids, examining the claim that the five Platonic solids were known before Plato, with photographs of artefacts from The National Museum of Scotland. There is a rather tedious article (to me at least) on the development of mathematical instruments in the 16th to 18th centuries, a not very clear article on the revolution that took place in mathematics a century ago, an article on the errors that took place in the historical development of probability, an article which seems to be claiming that probability is not the right tool in the real market, if I have understood correctly amidst all the jargon, with the final article on the Shinichi Mochizuki's proof of the *abc* conjecture (not given) which is 'too tough for mathematicians'. The style of the articles is geared to the professional mathematician rather than to the student, which is a pity, for there is a lot here to interest students.

Finally, a bit of light relief. Have you read the novel *The Curious Incident of the Dog in the Night-Time* by Mark Haddon? I won't give the game away. Suffice it to say that you will find the Monty Hall problem in Chapter 101. Here it is.

> You are on a game show on television. On this game the idea is to win a car as a prize. The game show host shows you three doors. He says that there is a car

> behind one of the doors and there are goats behind the other two doors. He asks you to pick a door. You pick a door but door is not opened. Then the game show host opens one of the other doors you didn't pick to show a goat (because he knows what is behind the doors). Then he says you have one final chance to change your mind before the doors are opened and you get a car or a goat. So he asks you if you want to change your mind and pick the other unopened door instead. What should you do?

Oh, I forgot to tell you that the chapters are numbered with the prime numbers rather than boring 1, 2, 3, and so on. Alternatively, at the time of writing this there is a dramatization currently in production at The National Theatre which is to be screened in cinemas worldwide. You see, books ARE on the way out, aren't they!

References

1 M. Pitici (ed.), *The Best Writing on Mathematics 2013* (Princeton University Press, 2014).
2 M. Haddon, *The Curious Incident of the Dog in the Night-Time* (Jonathan Cape, London, 2003).

Tom Moore

Thomas Eugene Moore (1944–2014) had a productive and spectacular professional career that spanned 44 years, all at Bridgewater State University (BSU), Massachusetts, USA. He delighted in teaching abstract algebra, number theory, mathematics for liberal arts, and courses for prospective teachers, and inspired many during his career. An active member of the Mathematical Association of America (MAA), Professor Moore was instrumental in organizing the Fall meeting of the Northeast Section of the MAA at BSU in 2001 and in 2012. He was the first recipient of Section's Distinguished College or University Teaching Award, and was posthumously inducted into the Association of Teachers of Mathematics in June 2014.

Professor Moore published over 100 original mathematics problems in various mathematics journals, including *Mathematics Magazine*, *Math Horizons*, *The Pi Mu Epsilon Journal*, *The Pentagon*, *School Science and Mathematics Journal*, *Mathematical Spectrum*, *Bulletin of the Irish Mathematical Society*, and the *Newsletter of the Ramanujan Mathematical Society*. More than half of his problems were created since his retirement in December 2012. In addition, he authored over two dozen scholarly articles in both national and international mathematics journals.

After a long and valiant battle with cancer and surrounded by his family, Professor Moore died on June 8, 2014.

Framingham State University **Thomas Koshy**

Blade Shape of a Troposkein-Type of Vertical-Axis Wind Turbine

SEAN MCGINTY and SEAN MCKEE

In this article we discuss a mathematical model of the required blade shape for a troposkein-type of vertical-axis wind turbine. We consider the case where the blade rotation speed is constant. The problem is an example of how elliptic integrals emerge in practice.

1. Introduction

For as long as man has roamed the Earth, he has been fascinated by wind. Whether it be observing the brute force of hurricanes and tornadoes, or watching a leaf being blown across a field, wind has intrigued and mystified for thousands of years. This has led to man harnessing the wind for his own purposes. The use of wind power, however, is not relatively recent: in fact, it is steeped in history.

The earliest known application of wind power is, of course, in driving ships. But whilst our early ancestors may not have had the resources available to us nowadays, nor the understanding of the physics involved, they were more advanced than one might expect. Ancient sailors, for example, knew about and used the force acting perpendicular to the motion of their ships, a force we now know as lift. This knowledge was crucial in their everyday lives.

Wind turbines date back many centuries with the earliest recorded English turbine in 1191 (see reference 1). Early wind turbines in the western world were horizontal-axis wind turbines (HAWTs), and were generally used to grind grain and pump water. Indeed, the word windmill itself comes from the fact that these machines were used to mill grain.

These examples are not, however, the earliest wind turbines: windmill design was developed in Persia about 500–900 AD (see reference 2). However, it may come as a surprise that this design was not of a horizontal-axis system, like those described above. In fact, these early machines made use of a vertical axis. The Chinese too have used vertical-axis wind turbines (VAWT) for many centuries, and claim that China is their birthplace. But what do we mean by HAWT and VAWT? Essentially, a HAWT has the axis of its rotor parallel to the wind direction, whereas a VAWT has its rotor axis perpendicular to the wind stream. We can distinguish between these two types of turbine further by introducing the idea of the angle of attack as the angle between the blade and the direction of the wind. For the horizontal-axis case, the angle of attack is approximately constant at any point along the blade, whereas with VAWTs the angle of attack at any point on the blade varies with time (see reference 3).

It is known that HAWTs do have their drawbacks. One of the main problems is that the turbine has to be turned into the wind (see reference 4). When working with large machines this can prove very costly. On the other hand, VAWTs do not require realignment when the wind changes direction. Furthermore, the blades for a HAWT are very costly, whereas vertical-axis designs can make use of light inexpensive blades. Another advantage of VAWTs is that the generator and main bearings are located at ground level, which makes for easy maintenance. However, VAWT do have their disadvantages. The primary disadvantage is the longer blade length required (see reference 3). Also, these turbines usually have low or insignificant starting torque, so that the rotor must be brought up to speed either by using the generator as a motor or by means of a small secondary rotor.

The government's growing concern about the environment and global warming means that more and more wind farms are inevitable, and so designing and building efficient wind turbines is crucial. While there has been a considerable amount of research into vertical-axis turbine technology (see e.g. references 1–5), this has not come close to matching the rate of development of horizontal-axis systems. As a result, HAWTs are still regarded as superior in terms of efficiency and performance and are the machine of choice for most medium- to large-scale wind farms. But small scale vertical-axis turbines are increasingly being used in urban environments due to their silent operation and lower risk associated with slower rates of rotation.

Wind is one of the most promising of the renewable energy sources, particularly since, as an energy source, wind is free. There is an enormous resource of wind in the UK, particularly in Scotland. In 2011 renewable energy, as a whole, accounted for 3.8% of energy consumption in the UK: the amount of electricity generated from renewable sources in 2011 was 34 410 GWh (see reference 6). Wind generation saw the largest increase from the previous year. A major factor in this was the low windiness of the unusually cold winter of 2010. This only serves to demonstrate that a mixture of different renewable sources are required and wind alone is not a viable option. The UK has agreed to a target of 15% of energy consumption being accounted for by energy from renewable sources by 2020.

The idea for a VAWT was originally patented by Darrieus in 1931. Blackwell and Reis produced a Sandia Laboratories Research Report in 1974 (see reference 5) describing the blade shape of a troposkein-type of VAWT. The word *troposkein* (from the Greek: $\tau\rho o\pi os$ (turning) and $\sigma\chi o\iota\nu\iota o\nu$ (rope)) will be used to describe the shape assumed by a perfectly flexible cable of uniform density and cross-section if its ends are attached at two points on a vertical axis and it is then spun at constant angular velocity about the vertical axis. Figure 1 displays an example of a Darrieus (troposkein), or 'eggbeater'-type VAWT. Of course, turbines make use of blades rather than rope and wind speed is clearly highly variable. Nevertheless, the blades of the troposkein are extremely light and flexible laminae, so approximation by a rope is not too unrealistic. Furthermore, despite variable wind speeds, a constant blade rotational speed can be achieved through the use of appropriate control systems (see reference 4). Thus, the calculation of the shape of the blades subject to a constant rotational speed is a problem of real

Figure 1 Darrieus-type vertical-axis wind turbine (see reference 2).

practical interest in blade design. This type of problem is highly amenable to mathematical modelling. Moreover, as we shall see, it is a practical example of the use of elliptic integrals.

Darrieus-type VAWTs are essentially lift force driven wind turbines. The lift forces are created by a set of airfoils, which are the actual blades of the turbine. These allow the turbine to reach speeds that are higher than the actual speed of the wind, which makes them well suited to generating electricity. As the airfoils move forward through the air, the air flow creates an angle of attack that generates a force that gives a positive torque to the shaft of the turbine and helps it move in the direction it is already rotating. The energy coming from the resultant torque and the speed of the airfoils is then converted into electricity.

This article is concerned with a re-examination of the analysis of Blackwell and Reis. It is readily seen that the blade shape they developed is dependent on knowing the maximum horizontal blade displacement. We show how to obtain the blade shape directly from the underlying two-point boundary-value problem, without recourse to the maximum blade displacement. The analysis can be extended to more general blade shapes. We conclude by indicating how the 'cable length' may be obtained in the above cases.

2. Blade shape for the symmetric case

In figure 2 we present a diagram of a perfectly flexible cable rotating at a constant angular velocity ω about a vertical axis z. The r-axis represents the displacement of the cable from the z-axis. The origin of this coordinate system is chosen to be on the z-axis at a point corresponding to the maximum horizontal displacement of the cable from the z-axis. The tension of the cable at the point of maximum horizontal displacement, T_0, is directed vertically downwards. The centrifugal force, C, acts on the length of cable between the point of maximum horizontal displacement and the point P, as do G, the force due to gravity, and the two tensions, T and T_0.

Considering the horizontal forces acting at the point P, we observe that

$$T \sin\theta = C, \tag{1}$$

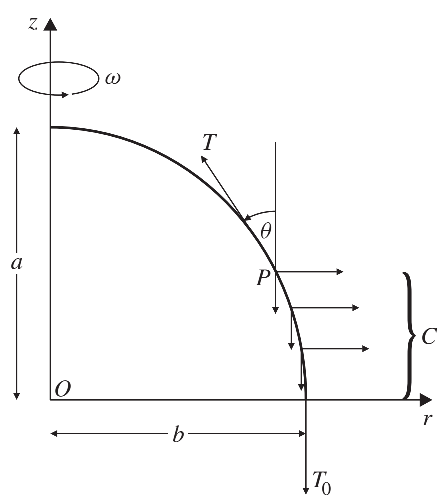

Figure 2 Diagram of a perfectly flexible cable rotating about a vertical axis.

where the centrifugal force can be written as

$$C = \int_0^s \sigma\omega^2 r \, \mathrm{d}s, \tag{2}$$

σ is the mass of the cable per unit length, and s is the length of the cable between the point of maximum horizontal displacement and P. Resolving in the vertical direction, the balance of forces at the point P requires that

$$T \cos\theta = T_0 + G, \tag{3}$$

where G is the force due to gravity and can be written as

$$G = \int_0^s \sigma g \, \mathrm{d}s. \tag{4}$$

Here, g is the acceleration due to gravity. It is assumed that σ and g remain constant, and that the centrifugal force per unit length increases linearly with increasing distance from the axis of rotation. Then, dividing (1) by (3), we see that

$$\tan\theta = \frac{C}{T_0 + G} = -\frac{\mathrm{d}r}{\mathrm{d}z}. \tag{5}$$

Substituting (2) and (4) into (5) gives

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{\sigma\omega^2 \int_0^s r \, \mathrm{d}s}{T_0 + \sigma g s}.$$

The boundary conditions for this problem are as follows: $r = 0$ at $z = a$ and $\mathrm{d}r/\mathrm{d}z = 0$ at $z = 0$. It is now convenient to make the assumption that at large rotational speeds, T_0 and $\sigma\omega^2$ will be significantly larger than $\sigma g s$. Thus, if the gravitational forces are relatively small, we may neglect $\sigma g s$ resulting in the following integro–differential equation describing the cable shape:

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{\sigma\omega^2}{T_0} \int_0^s r \, \mathrm{d}s. \tag{6}$$

The solution of (6) is developed as follows. By Pythagoras' theorem, the arc length $\mathrm{d}s$ is related to the cable slope by

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2} \, \mathrm{d}z. \tag{7}$$

Now, substituting (7) into (6) gives

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\frac{\sigma\omega^2}{T_0} \int_0^z r \sqrt{1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2} \, \mathrm{d}z. \tag{8}$$

For convenience, all lengths are now normalised by a, that is, $r' = r/a$ and $z' = z/a$. Hence, (8) becomes

$$\frac{\mathrm{d}r'}{\mathrm{d}z'} = -\Omega^2 \int_0^{z'} r' \left(1 + \left(\frac{\mathrm{d}r'}{\mathrm{d}z'}\right)^2\right)^{1/2} \mathrm{d}z', \quad \text{where } \Omega^2 = \frac{\sigma\omega^2 a^2}{T_0}. \tag{9}$$

The primes on r' and z' will now be dropped for reasons of clarity, with the understanding that r and z have both been normalised by a. Hence, we can rewrite (9) as

$$\frac{\mathrm{d}r}{\mathrm{d}z} = -\Omega^2 \int_0^z r\left(1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2\right)^{1/2} \mathrm{d}z, \tag{10}$$

with the boundary conditions (after normalisation) given by $\mathrm{d}r/\mathrm{d}z = 0$ at $z = 0$ and $r = 0$ at $z = 1$. Now, if we differentiate (10) we obtain

$$\frac{\mathrm{d}^2 r}{\mathrm{d}z^2} = -\Omega^2 r\left(1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2\right)^{1/2},$$

which can be rearranged as follows so that both sides of the equation are exact differentials:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2\right]^{1/2} = -\frac{\Omega^2}{2}\frac{\mathrm{d}}{\mathrm{d}z}r^2. \tag{11}$$

Upon integrating (11) with respect to z, we obtain

$$\left[1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2\right]^{1/2} = -\frac{\Omega^2}{2}r^2 + c, \tag{12}$$

where $c = 1 + \Omega^2 b^2/2a^2$, since, when $z = 0$, $\mathrm{d}r/\mathrm{d}z = 0$ and $r = b/a$, where b denotes the maximum (dimensional) displacement. Equation (12) may be written as

$$\left[1 + \left(\frac{\mathrm{d}r}{\mathrm{d}z}\right)^2\right]^{1/2} = 1 - \frac{\Omega^2}{2}(r^2 - \beta^2), \quad \text{where } \beta = \frac{b}{a}. \tag{13}$$

After a little manipulation, (13) may be solved for $\mathrm{d}r/\mathrm{d}z$ as follows:

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \pm\frac{\Omega^2\beta^2}{2}\left(1 + \frac{4}{\Omega^2\beta^2}\right)^{1/2}\left[\left(1 - \frac{r^2}{\beta^2}\right)\left(1 - \frac{r^2}{\beta^2(1 + 4/\Omega^2\beta^2)}\right)\right]^{1/2}. \tag{14}$$

At this point we note that we have derived two differential equations, one for the positive z-axis and one for the negative z-axis. To focus on the part of the troposkein corresponding to the positive z-axis, we choose the minus sign since $\mathrm{d}r/\mathrm{d}z$ is negative for $z \in (0, a]$. If we let $t = r/\beta$ and

$$k^2 = \frac{1}{1 + 4/\Omega^2\beta^2}, \tag{15}$$

(14) may be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}z} &= -\frac{\Omega^2\beta}{2}\left(1 + \frac{4}{\Omega^2\beta^2}\right)^{1/2}\left[(1 - t^2)\left(1 - \frac{t^2}{1 + 4/\Omega^2\beta^2}\right)\right]^{1/2} \\
&= -\frac{\Omega^2\beta}{2}\left(1 + \frac{4}{\Omega^2\beta^2}\right)^{1/2}\left[(1 - t^2)(1 - t^2k^2)\right]^{1/2},
\end{aligned}$$

or, after further manipulation, as

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{\Omega}{(1 - k^2)^{1/2}}[(1 - t^2)(1 - t^2k^2)]^{1/2}. \tag{16}$$

Integrating (16) gives

$$\frac{\Omega}{(1-k^2)^{1/2}}\int_0^z \mathrm{d}z = -\int_1^t \frac{\mathrm{d}t}{[(1-t^2)(1-t^2k^2)]^{1/2}}. \tag{17}$$

If we make the substitution $t = \sin\phi$, the right-hand side of (17) becomes

$$\begin{aligned}
-\int_{\pi/2}^{\phi} \frac{\cos\phi\,\mathrm{d}\phi}{[(1-\sin^2\phi)(1-k^2\sin^2\phi)]^{1/2}} &= -\int_{\pi/2}^{\phi} \frac{\mathrm{d}\phi}{(1-k^2\sin^2\phi)^{1/2}} \\
&= \int_0^{\pi/2} \frac{\mathrm{d}\phi}{(1-k^2\sin^2\phi)^{1/2}} - \int_0^{\phi} \frac{\mathrm{d}\phi}{(1-k^2\sin^2\phi)^{1/2}} \\
&= F\left(\frac{\pi}{2};k\right) - F(\phi;k),
\end{aligned}$$

where $F(\phi;k)$ is the elliptic integral of the first kind defined by

$$F(\phi;k) = \int_0^{\phi} \frac{\mathrm{d}\xi}{(1-k^2\sin^2\xi)^{1/2}}.$$

Hence, recalling that z and r have been normalised by a, (17), when expressed in the original dimensional variables, becomes

$$\frac{z}{a} = \frac{1}{\Omega}(1-k^2)^{1/2}\left[F\left(\frac{\pi}{2};k\right) - F(\phi;k)\right]. \tag{18}$$

Recall that β is the ratio of the maximum horizontal displacement to the maximum vertical displacement, i.e.

$$\beta = \frac{b}{a} = \frac{2k}{\Omega\sqrt{1-k^2}}. \tag{19}$$

Now, when $z = a$, we have $r = 0$, and we may therefore deduce an expression for Ω in terms of k, i.e.

$$\Omega = \sqrt{1-k^2}\,F\left(\frac{\pi}{2};k\right); \tag{20}$$

hence, on eliminating Ω from (18), we obtain

$$\frac{z}{a} = 1 - \frac{F(\phi;k)}{F(\pi/2;k)}. \tag{21}$$

It should be noted that (21) describes only the blade shape for the upper half, i.e. the positive z-axis. If, instead of choosing the negative sign in (14), we had chosen the positive sign, then we would have obtained the equation for that part of the blade corresponding to the negative z-axis, i.e.

$$\frac{z}{a} = \frac{F(\phi;k)}{F(\pi/2;k)} - 1. \tag{22}$$

Hence, (21) and (22) now completely specify the blade shape for a given height a and maximum horizontal displacement b. A plot of z/a against r/b for five different values of k is displayed in figure 3. This demonstrates a clear dependence on k. But for a given height a, we do not know *a priori* the maximum horizontal displacement b. It is interesting to ask if similar expressions for the blade shape may be obtained that do not involve specifying b.

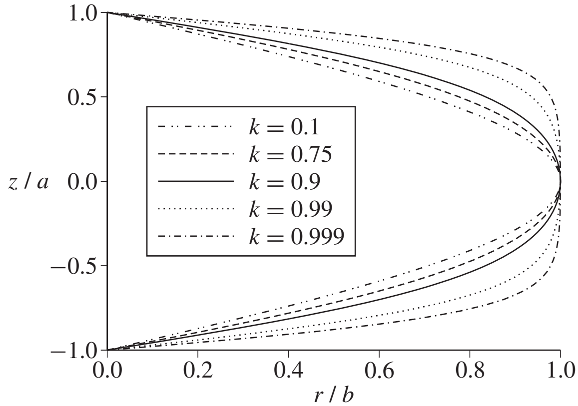

Figure 3 Nondimensional cable shape as a function of the parameter k.

3. Blade shape for the symmetric case without specification of the maximum horizontal displacement

So far, we have followed the analysis of Blackwell and Reis (see reference 5) where the maximum horizontal displacement, b, has been required. In fact, the problem as described in section 2 has been over-specified. Equation (20) clearly shows that there is a relationship between Ω and k. From (15) it is evident that when the turbine is operational (i.e. $\Omega \neq 0$, $b \neq 0$) the parameter k must lie in the range $0 < k < 1$. A plot of Ω against k is displayed in figure 4. Now, making use of (19) in the definition of ϕ, we obtain

$$\phi = \arcsin\left(\frac{r\Omega\sqrt{1-k^2}}{2k}\right). \tag{23}$$

Further, if we specify a value of Ω, we can use (20) to find k and substitute this value into (23). The resulting expression for ϕ (as a function of r) can then be substituted into (21) and (22) to obtain the equations governing the blade shape. Figure 5 displays the blade shape for $\Omega = 1.5$ and height $a = 10$. For this case, k and b turn out to be 0.4083 and 5.9641, respectively. Thus, it is clear that we need not specify b initially.

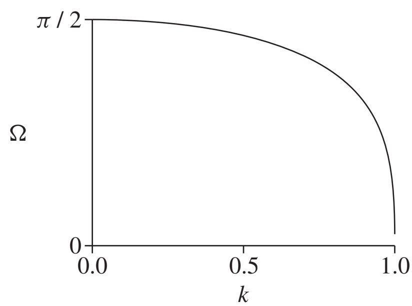

Figure 4 Variation of rotational parameter Ω with k.

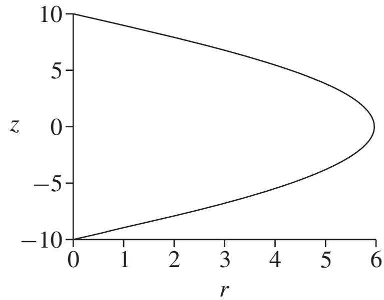

Figure 5 Cable shape for $a = 10$ and $\Omega = 1.5$.

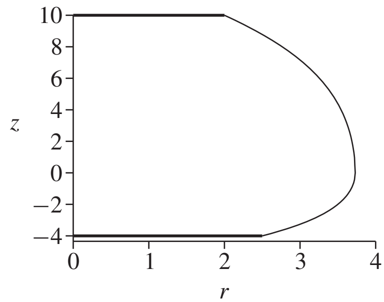

Figure 6 Cable shape for the asymmetric case. Bold lines indicate struts.

4. Asymmetric case and cable length

So far, we have restricted our attention to blade shapes that are symmetrical about the r-axis. But, of course, that need not be the case. The analysis can be extended to the asymmetric case where there are two horizontal struts of different lengths, with each end of the cable attached to one of the struts. Figure 6 displays the blade shape for the case where $\Omega = 0.75$, the distance between the struts is 14 metres, and the upper and lower struts are of length 2 and 2.5 metres, respectively. It can also be shown that, for both the symmetric and asymmetric cases, if we specify the total cable length then we can determine the blade shape without prior knowledge of Ω or k (see reference 7).

5. Conclusions

This article is concerned with investigating mathematical models for obtaining the blade shapes for Darrieus-type VAWTs. Analytical solutions are derived which involve elliptic integrals. The analysis may be extended to asymmetric configurations and knowledge of the cable length can be used to determine the blade shape.

References

1 G. L. Johnson, *Wind Energy Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1985).

2 http://www.telosnet.com/wind/early.html.

3 H. J. Sutherland, D. E. Berg and T. D. Ashwill, A retrospective of VAWT technology, Res. Rep. SAND2012-0304, Sandia Laboratories, 2012.

4 P. Cooper and O. C. Kennedy, Development and analysis of a novel vertical axis wind turbine. In *Life, The Universe and Renewables* (Australian and New Zealand Solar Energy Society, Perth, 2004), pp. 1–9.
5 B. F. Blackwell and G. E. Reis, Blade shape for a troposkein type of vertical-axis wind turbine, Res. Rep. SLA74-0154, Sandia Laboratories, 1974.
6 Renewable Energy in 2011. Available at http://www.gov.uk.
7 S. McGinty and S. McKee, Windmills of your mind, Res. Rep. 10, Department of Mathematics, University of Strathclyde, 2006.

Dr Sean McGinty *is an applied mathematician with a passion for using mathematics to solve real-world problems. His main research interests lie broadly in the area of mathematical biology. He has a background in the the Wind Industry. He worked for two years as an Energy and Project Development Engineer for GL Garrad Hassan, the world's largest Renewable Energy Consultancy. Dr McGinty is currently a Research Associate at the University of Strathclyde working on a project entitled 'Optimal design of drug-eluting stents'.*

Professor Sean McKee *has 40 years experience working on numerical methods, mathematical modelling, and analysis of real industrial problems. In 1986 he founded the European Consortium for Mathematics in Industry (ECMI). He has received a DSc from the University of Oxford, a Homenagem and a Medal of Honour from the University of Sao Paulo, and has been elected Fellow of the Royal Society of Edinburgh.*

The year 2015

Why not have a go at our annual puzzle? The idea is to express a number using the digits of the year in order and the operations $+$, $-$, $\times$, $\div$, $\sqrt{}$, !, and concatenation. For example, $100 = -20 + (1 \times 5)!$. Your Editor tried on the back of an envelope and was successful with the numbers 1 to 30 but came to a halt with 31.

Correction: Lazar Karno's theorem

In Volume 47, Number 1, pp. 41–42, Abbas Rouhol Amini gave a proof of Lazar Karno's theorem. Guido Lasters has written to point out that the proof and the result are only true for an acute-angled triangle. For an obtuse-angled triangle, with the obtuse angle at C, the proof can be amended to give the following result:

$$OA' + OB' - OC' = R + r,$$

with the same notation as in Abbas' letter.

A conjecture concerning regular polygons

Let $A_0A_1 \cdots A_8$ be a regular nine-sided polygon. Then we have

$$A_0A_1 + A_0A_2 = A_0A_4.$$

This is because

$$\sin\frac{\pi}{9} + \sin\frac{2\pi}{9} = \sin\frac{4\pi}{9}.$$

Now consider a regular n-sided polygon $A_0A_1 \cdots A_{n-1}$. We ask whether there exist positive integers a, b, c such that $1 \le a < b < c$, $c < n/2$, and

$$A_0A_a + A_0A_b = A_0A_c, \tag{1}$$

or, equivalently,

$$\sin\frac{a\pi}{n} + \sin\frac{b\pi}{n} = \sin\frac{c\pi}{n}. \tag{2}$$

When n is a multiple of 3 and a, b are such that $1 \le a < b$, $(a+b) = n/3$, and $(2a+b) < n/2$, then we have

$$\begin{aligned}\sin\frac{a\pi}{n} + \sin\frac{b\pi}{n} &= 2\sin\frac{(a+b)\pi}{2n}\cos\frac{(b-a)\pi}{2n}\\ &= 2\sin\frac{\pi}{6}\sin\left(\frac{\pi}{2} - \frac{(b-a)\pi}{2n}\right)\\ &= \sin\left(\frac{n\pi - (b-a)\pi}{2n}\right)\\ &= \sin\left(\frac{(2a+b)\pi}{n}\right).\end{aligned}$$

A few cases when (1) is true are given in the following table.

Table $A_0A_a + A_0A_b = A_0A_c$.

n	a	b	c
9	1	2	4
12	1	3	5
15	1	4	6
	2	3	7
18	1	5	7
	2	4	8

We conjecture that, if n is not a multiple of 3, then it is not possible to find a, b, c satisfying (2).

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A Brief Study of Abundant Numbers not Divisible by any of the First *n* Primes

JAY L. SCHIFFMAN

In this article we embark on a study of the initial square-free abundant numbers that are not divisible by any of the first *n* primes. In addition, we will furnish the smallest abundant numbers that are not divisible by each of the first seven primes.

1. Introduction and basic ideas

Positive integers are classified as *abundant*, *deficient*, or *perfect*. If $\sigma(n)$ denotes the sum of the positive divisors of n, then n is abundant when $\sigma(n) > 2n$. Thus, 12 is abundant since

$$\sigma(12) = \sigma(2^2 \times 3) = \sigma(2^2)\sigma(3) = \frac{2^{2+1} - 1}{2 - 1}(3 + 1) = 7 \times 4 = 28 > 24 = 2 \times 12.$$

The first odd abundant and square-free odd abundant numbers are 945 and 15 015 respectively, while 5 391 411 025 is the first abundant number not divisible by either 2 or 3. (An integer is *square-free* if it is not divisible by the square of any prime and hence all of its prime factors are raised to the first power in the prime factorization.) The function $\sigma(n)/n$ yields the ratio of the sum of the positive divisors of an integer to the integer itself and is known as the *abundancy* of the number. If $\sigma(n)/n$ is an integer, we classify n as *multiply-perfect*. For example, deficient, perfect, and abundant numbers have abundancies less than two, equal to two, and greater than two respectively. If the abundancy of a number is a positive integer greater than or equal to two, the integer is classified as multiply-perfect. The integers 120 and 30 240 can be shown to be 3-perfect and 4-perfect respectively.

2. Verification of the assertions in section 1

We first show that 945, 15 015, and 5 391 411 025 are abundant numbers. We have

$$\begin{aligned}
945 &= 3^3 \times 5 \times 7, \\
\sigma(945) &= \sigma(3^3 \times 5 \times 7) \\
&= \sigma(3^3)\sigma(5)\sigma(7) \\
&= \frac{3^{3+1} - 1}{3 - 1}(5 + 1)(7 + 1) \\
&= 40 \times 6 \times 8 \\
&= 1920, \\
\frac{\sigma(945)}{945} &= \frac{1920}{945} = 2.031\,746\,031\,75,
\end{aligned}$$

$$\begin{aligned}\sigma(15\,015) &= \sigma(3 \times 5 \times 7 \times 11 \times 13)\\ &= \sigma(3)\sigma(5)\sigma(7)\sigma(11)\sigma(13)\\ &= (3+1)(5+1)(7+1)(11+1)(13+1)\\ &= 4 \times 6 \times 8 \times 12 \times 14\\ &= 32\,256,\end{aligned}$$

$$\frac{\sigma(15\,015)}{15\,015} = \frac{32\,256}{15\,015} = 2.148\,251\,748\,25,$$

$$\begin{aligned}\sigma(5\,391\,411\,025) &= \sigma(5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29)\\ &= \sigma(5^2)\sigma(7)\sigma(11)\sigma(13)\sigma(17)\sigma(19)\sigma(23)\sigma(29)\\ &= 31 \times 8 \times 12 \times 14 \times 18 \times 20 \times 24 \times 30\\ &= 10\,799\,308\,800,\end{aligned}$$

$$\frac{\sigma(5\,391\,411\,025)}{5\,391\,411\,025} = \frac{10\,799\,308\,800}{5\,391\,411\,025} = 2.003\,057\,965\,7.$$

We next verify that 30 is a square-free abundant number; we have

$$\sigma(30) = \sigma(2 \times 3 \times 5) = \sigma(2)\sigma(3)\sigma(5) = (2+1)(3+1)(5+1) = 3 \times 4 \times 6 = 72,$$
$$\frac{\sigma(30)}{30} = \frac{72}{30} = 2.4.$$

Finally, we verify that 120 and 30 240 are 3-perfect and 4-perfect respectively; we have

$$\sigma(120) = \sigma(2^3 \times 3 \times 5) = \sigma(2^3)\sigma(3)\sigma(5) = 15 \times 4 \times 6 = 360,$$
$$\frac{\sigma(120)}{120} = \frac{360}{120} = 3,$$
$$\sigma(30\,240) = \sigma(2^5 \times 3^3 \times 5 \times 7) = \sigma(2^5)\sigma(3^3)\sigma(5)\sigma(7) = 63 \times 40 \times 6 \times 8 = 120\,960,$$
$$\frac{\sigma(30\,240)}{30\,240} = \frac{120\,960}{30\,240} = 4.$$

The computations only show the numbers are abundant, but do not imply that 30 and 15 015 are the smallest square-free abundant and square-free odd abundant numbers respectively. While 5 391 411 025 is indeed the smallest abundant number not divisible by the initial two primes 2 and 3, the computation does not illustrate this. In order to achieve our desired goal in the case of square-free abundant numbers, it is necessary to examine the product of the abundancies of the required primes. Abundant numbers that are not square-free will be discussed later. We first observe that $\sigma(p)/p \to 1$ as p becomes large. In more precise terms,

$$\frac{\sigma(p)}{p} = \frac{p+1}{p} = 1 + \frac{1}{p} \to 1 \quad \text{as } p \to \infty.$$

It is now essential to note that we are considering consecutive primes in securing the smallest square-free abundant numbers. If p and q are primes such that $p < q$, then $\sigma(p)/p > \sigma(q)/q$. Hence, if you have a product of nonconsecutive primes, then the larger primes in this product could be replaced by 'missing primes' to produce a larger abundancy number. Hence, the smallest abundant numbers will consist of consecutive primes.

It should be noted that if we consider positive integers such as 10, 14, 15, 21, 22, and 26, all of which are the products of two distinct primes, none is abundant. In addition, it is easily seen that any prime and any power of a prime is deficient. To prove that any prime is deficient it suffices to note that $\sigma(p) = 1 + p < p + p < 2p$, since $p > 1$. To prove that any power of a prime is deficient, observe that

$$\begin{aligned} 2p^n - \sigma(p^n) &= p^n - (1 + p + p^2 + \cdots + p^{n-1}) \\ &= p^n - \frac{p^n - 1}{p - 1} \\ &= \frac{p^{n+1} - 2p^n + 1}{p - 1} \\ &= \frac{p^n(p-2) + 1}{p - 1} \\ &> 0. \end{aligned}$$

A second easily proven result shows that any positive integer that is the product of two distinct primes p and q is deficient, apart from $p = 2$ and $q = 3$ which yields the initial perfect number 6. For primes p, q with $p < q$, we have

$$\begin{aligned} 2p^n - \sigma(p^n) &= p^n - (1 + p + p^2 + \cdots + p^{n-1}) \\ &= p^n - \frac{p^n - 1}{p - 1} \\ &= \frac{p^{n+1} - 2p^n + 1}{p - 1} \\ &= \frac{p^n(p-2) + 1}{p - 1} \\ &> 0, \end{aligned}$$

except when $p = 2$, $q = 3$, when it is zero. Hence, $6 = 2 \times 3$ is perfect and all others are deficient. It follows easily from this that 30 is the smallest square-free abundant number. The integers

$$\begin{gathered} 33\,426\,748\,355, \\ 1\,357\,656\,019\,974\,967\,471\,687\,377\,449, \\ 7\,105\,630\,242\,567\,996\,762\,185\,122\,555\,313\,528\,897\,845\,637\,444\,413\,640\,621 \end{gathered}$$

are, respectively, the first square-free abundant numbers that are not divisible by the first two, three, and four primes. Initially note the following prime factorizations:

$$\begin{aligned} &33\,426\,748\,355 \\ &\qquad = 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31, \\ &1\,357\,656\,019\,974\,967\,471\,687\,377\,449 \\ &\qquad = 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times \cdots \times 73, \\ &7\,105\,630\,242\,567\,996\,762\,185\,122\,555\,313\,528\,897\,845\,637\,444\,413\,640\,621 \\ &\qquad = 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times \cdots \times 149. \end{aligned}$$

Table 1 Starting and ending primes required to furnish square-free abundant numbers not divisible by the first n primes, $0 \le n \le 11$.

n	Starting prime	Ending prime
0	$p_1 = 2$	$p_3 = 5$
1	$p_2 = 3$	$p_6 = 13$
2	$p_3 = 5$	$p_{11} = 31$
3	$p_4 = 7$	$p_{21} = 73$
4	$p_5 = 11$	$p_{35} = 149$
5	$p_6 = 13$	$p_{51} = 233$
6	$p_7 = 17$	$p_{73} = 367$
7	$p_8 = 19$	$p_{98} = 521$
8	$p_9 = 23$	$p_{130} = 733$
9	$p_{10} = 29$	$p_{167} = 991$
10	$p_{11} = 31$	$p_{204} = 1\,249$
11	$p_{12} = 37$	$p_{249} = 1\,579$

Moreover, the abundancies of the products of consecutive primes required to form these abundant numbers satisfying the desired criteria are shown as follows:

$$\tfrac{6}{5}\,\tfrac{8}{7}\,\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29} = 1.938\,443\,192\,61, \quad \text{meanwhile,}$$
$$\tfrac{6}{5}\,\tfrac{8}{7}\,\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29}\,\tfrac{32}{31} = 2.000\,973\,618\,18;$$
$$\tfrac{8}{7}\,\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29}\,\tfrac{32}{31}\cdots\tfrac{72}{71} = 1.987\,111\,626\,59, \quad \text{meanwhile,}$$
$$\tfrac{8}{7}\,\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29}\,\tfrac{32}{31}\cdots\tfrac{72}{71}\,\tfrac{74}{73} = 2.014\,332\,333\,81;$$
$$\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29}\,\tfrac{32}{31}\cdots\tfrac{140}{139} = 1.992\,499\,563\,91, \quad \text{meanwhile,}$$
$$\tfrac{12}{11}\,\tfrac{14}{13}\,\tfrac{18}{17}\,\tfrac{20}{19}\,\tfrac{24}{23}\,\tfrac{30}{29}\,\tfrac{32}{31}\cdots\tfrac{140}{139}\,\tfrac{150}{149} = 2.005\,872\,044\,2.$$

Table 1 provides the starting and ending primes needed to produce square-free abundant numbers not divisible by each of the first eleven primes. In between, we consider the products of the abundancies of consecutive primes to obtain the least product which exceeds two (30 is considered the minimal square-free abundant number not divisible by any of the first zero primes and is the product of the initial three primes). With the aid of MATHEMATICA®, I was able to extend this table to $n = 300$.

To cite an example, if we want to secure the initial square-free abundant number that is not divisible by any of the initial eleven primes (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31), we begin at the twelfth prime (37) and form the product of the consecutive primes from 37 to the 249th prime (1 579). The minimal square-free abundant number will be formed with an abundancy of 2.000 429 058 662 386. If we drop the ending prime, 1 949, the product of the consecutive primes from 37 to 1 571 would yield an abundancy of 1.999 162 964 321 460 and hence a deficient number.

3. Square-free abundant numbers do not produce the smallest possible abundant numbers

Unfortunately square-free abundant numbers do not produce the smallest possible abundant numbers. For example, 30 is the smallest square-free abundant number while 12 is the smallest abundant number. In the same fashion, 15 015 is the smallest square-free odd abundant number while 945 is indeed the smallest odd abundant number. In the next section, we discuss a procedure to secure the minimal abundant number not divisible by any of the initial n primes.

4. Minimal abundant numbers not divisible by the first *n* primes

By raising the lower primes to higher powers, we can truncate some of the ending primes required in the square-free case that do not contribute much to the number being abundant and obtain the minimal positive integers not divisible by any of the first few primes. For example, the smallest abundant integers not divisible by the first 0, 1, 2, 3, and 4 primes are

$$\begin{gathered} 12, \\ 945, \\ 5\,391\,411\,025, \\ 20\,169\,691\,981\,106\,018\,776\,756\,331, \\ 49\,061\,132\,957\,714\,428\,902\,152\,118\,459\,264\,865\,645\,885\,092\,682\,687\,973 \end{gathered}$$

respectively. We first note the prime factorization of these integers:

$$\begin{aligned} 12 &= 2^2 \times 3, \\ 945 &= 3^3 \times 5 \times 7, \\ 5\,391\,411\,025 &= 5^2 \times 7 \times 11 \\ &\quad \times 13 \times 17 \times 19 \\ &\quad \times 23 \times 29, \\ 20\,169\,691\,981\,106\,018\,776\,756\,331 &= 7^2 \times 11^2 \\ &\quad \times 13 \times 17 \\ &\quad \times \cdots \times 67, \\ 49\,061\,132\,957\,714\,428\,902\,152\,118\,459\,264\,865\,645\,885\,092\,682\,687\,973 &= 11^2 \times 13^2 \\ &\quad \times 17 \times 19 \\ &\quad \times \cdots \times 137. \end{aligned}$$

Based upon our earlier calculations we observe that the initial square-free abundant numbers not divisible by the first 0, 1, 2, 3, and 4 primes are

$$\begin{gathered} 30, \\ 15\,015, \\ 33\,426\,748\,355, \\ 1\,357\,656\,019\,974\,967\,471\,687\,377\,449, \\ 710\,563\,024\,256\,799\,676\,218\,512\,255\,313\,528\,897\,845\,637\,444\,413\,640\,621 \end{gathered}$$

respectively.

In table 2, we present the smallest abundant and square-free abundant numbers not divisible by each of the first 0, 1, 2, 3, 4, 5, 6, and 7 primes respectively. We utilized MATHEMATICA to

Table 2 The smallest abundant and square-free abundant numbers not divisible by the first k primes, $0 \le k \le 7$.

k	Smallest abundant number	Smallest square-free abundant number
0	$12 = 2^2 \times 3$	$30 = 2 \times 3 \times 5$
1	$945 = 3^3 \times 5 \times 7$	$15\,015 = 3 \times 5 \times 7 \times 11 \times 13$
2	$5\,391\,411\,025 = 5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$	$33\,426\,748\,355 = 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31$
3	$7^2 \times 11^2 \times 13 \times 17 \times \cdots \times 59 \times 61 \times 67$	$7 \times 11 \times 13 \times 17 \times \cdots \times 59 \times 61 \times 67 \times 71 \times 73$
4	$11^2 \times 13^2 \times 17 \times 19 \times 23 \times 29 \times \cdots \times 127 \times 131 \times 137$	$11 \times 13 \times 17 \times 19 \times 23 \times 29 \times \cdots \times 127 \times 131 \times 137 \times 139 \times 149$
5	$13^2 \times 17^2 \times 19 \times 23 \times 29 \times \cdots \times 211 \times 223 \times 227$	$13 \times 17 \times 19 \times 23 \times 29 \times \cdots \times 211 \times 223 \times 227 \times 229 \times 233$
6	$17^2 \times 19^2 \times 23^2 \times 29 \times 31 \times 37 \times \cdots \times 337 \times 347 \times 349$	$17 \times 19 \times 23 \times 29 \times 31 \times 37 \times \cdots \times 337 \times 347 \times 349 \times 353 \times 359 \times 367$
7	$19^2 \times 23^2 \times 29^2 \times 31 \times 37 \times 41 \times \cdots \times 487 \times 491 \times 499$	$19 \times 23 \times 29 \times 31 \times 37 \times 41 \times \cdots \times 487 \times 491 \times 499 \times 503 \times 509 \times 521$

verify these results. In lieu of the fact that the last few of these integers in question have a very large number of digits, only the prime factorizations will be shown beyond the third. Moreover, trial and error shows that these large integers are indeed the smallest abundant numbers and square-free abundant numbers with the required property.

We now explain why 12, 945, and 5 391 411 025 are the smallest abundant numbers not divisible by the first 0, 1, and 2 primes respectively. (As we proceed further, the analysis becomes more involved.) If we examine the prime factorization of the first square-free abundant number not divisible by the first 0 primes (30), we notice that 2, 3, and 5 are the prime factors. If we truncate the ending prime 5 to seek a smaller abundant number we might try to raise the prime 2 to the second power. Upon doing this, we arrive at the abundant number 12 which is indeed the smallest.

If we examine the prime factorization of the first square-free abundant number not divisible by the first prime 2 (15 015) and hence an odd abundant number, observe that 3, 5, 7, 11, and 13 are the prime factors. If we truncate the ending prime 13 to seek a smaller abundant number we might try to raise the smallest prime 3 to the second power. This gives the factorization $3^2 \times 5 \times 7 \times 11 = 3\,465$; while this is an odd abundant number it is not the smallest. Thus, we might think of deleting the prime 11. This would yield $3^2 \times 5 \times 7 = 315$ which fails to be abundant. Raising both the primes 3 and 5 to the second power yields the abundant number $3^2 \times 5^2 \times 7 = 1\,575$ which is still not the smallest. Undaunted, we raise the smallest prime 3 to the third power yielding the factorization $3^3 \times 5 \times 7 = 945$ which indeed yields the smallest odd abundant number.

Finally, if we examine the prime factorization of the initial square-free abundant number not divisible by the first two primes 2 and 3 (33 426 748 355), we observe that 5, 7, 11, 13, 17, 23, 29, and 31 are the prime factors. Removing the ending prime 31 to seek a smaller abundant number having the desired property might lead us to raise the smallest prime 5 to the second power. This yields the factorization

$$5^2 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 = 5\,391\,411\,025,$$

which turns out to be minimal. Attempting to truncate more primes such as deleting the ending prime 29 would result in a deficient number, not an abundant one, even if we raised several of the beginning primes to higher powers.

5. Next steps

The impetus for this problem was a website puzzle by Carlos Rivera (see reference 1) which sought the smallest abundant number that failed to be divisible by the primes 2 or 3. Presently no minimal abundant numbers that are neither square-free nor divisible by the first n primes are known beyond the initial eight cited. This article nonetheless has aided in generating a method for obtaining square-free abundant numbers not divisible by the first n primes. With the aid of MATHEMATICA, I was able to secure all the starting and ending primes leading to all square-free abundant numbers not divisible by each of the first three hundred primes. For an algorithmic treatment of this topic which is not based on trial and error for integers that are not square-free, the reader is invited to consult reference 2. With the aid of technology, further inroads in the study of abundant numbers with the desired property alluded to in this article should be made.

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References

1 C. Rivera, Puzzle 329, available at http://www.primepuzzles.net/.

2 D. E. Iannucci, On the smallest abundant number not divisible by the first k primes, *Bull. Belgian Math. Soc.* **12** (2005), pp. 39–44.

3 http://mathworld.wolfram.com/.

4 The On-Line Encyclopedia of Integer Sequences (OEIS) https://oeis.org/ (sequences A047802, A007684, A007707, and A007708).

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How to make money

$$1\,\text{p} = £\tfrac{1}{100} = £\tfrac{1}{10} \times £\tfrac{1}{10} = 10\,\text{p} \times 10\,\text{p} = 100\,\text{p} = £1$$

Sharjah, U.A.E. **Muraleedharan K. V.**

The Divergence of $\{\cos(n)\}$

KURT FINK and JAWAD SADEK

In some calculus texts, readers are asked whether $\{\cos(n)\}$ and $\sum_{n=0}^{\infty} \sin(n)$ converge or diverge. The solutions provided (or intended at least) are inadequate. We give several elementary solutions to these problems.

Introduction

In some calculus texts (see, for example, reference 1), readers are asked to determine whether the sequence $\{\cos(n)\}$, where $n \in \mathbb{N}$, or the series $\sum_{n=0}^{\infty} \sin(n)$ converges or diverges. The solution provided (or intended at least) is to notice that $\cos(x)$ and $\sin(x)$ oscillate and so the corresponding sequence and series cannot converge. While the basic idea behind it is correct, this argument is problematic for the astute reader. After all, the function $\cos(2\pi x)$ is oscillatory but the sequence $\{\cos(2\pi n)\}$ converges to 1. Also, a divergent sequence can have convergent subsequences. In fact, there exist subsequences $\{n_k\}$ of $\{1, 2, 3, \ldots\}$ such that $\{\cos(n_k)\}$ converges. Since any irrational number r can be approximated by rational numbers, we can construct two subsequences of natural numbers $\{a_k\}$ and $\{b_k\}$ such that $a_k \to \infty$, $b_k \to \infty$, and $a_k - b_k r \to 0$ (see reference 2, p. 70, for instance). Applying this to $r = 2\pi$, we get $\cos(a_k) = \cos(a_k - 2\pi b_k) \to 1$, as $k \to \infty$.

In this note we provide several solutions to this divergence problem using elementary calculus techniques.

Using trigonometric identities

Assume that $\{\cos(n)\}$ converges to a number L. Then

$$\cos(n+2) - \cos(n) = 2\sin(n+1)\sin(1).$$

Since $\cos(n+2)$ and $\cos(n)$ have to converge to L, letting $n \to \infty$ implies that

$$2\sin(n+1)\sin(1) \to 0.$$

Thus, $\sin(n+1) \to 0$. Using the addition formula for the sine function, we write

$$\sin(n+1) = \sin(n)\cos(1) + \sin(1)\cos(n). \tag{1}$$

Since $\sin(n+1)$ and $\sin(n)$ both tend to 0 as $n \to 0$, $\cos(n) \to 0$. But the conclusion that $\cos(n)$ and $\sin(n)$ both tend to 0 contradicts the identity $\cos^2(n) + \sin^2(n) = 1$.

Now we show that $\sum_{n=0}^{\infty} \sin(n)$ diverges. If it is convergent, then $\sin(n) \to 0$. Using (1), it follows that $\cos(n) \to 0$ as $n \to \infty$. This contradicts the identity $\cos^2(n) + \sin^2(n) = 1$. Note also that the divergence of $\sum_{n=0}^{\infty} \cos(n)$ follows from the fact that $\{\cos(n)\}$ is divergent.

Using de Moivre's formula

Let

$$z_1 = \cos(1) + \mathrm{i}\sin(1).$$

By de Moivre's formula,

$$z_1^n = (\cos(1) + \mathrm{i}\sin(1))^n = \cos(n) + \mathrm{i}\sin(n).$$

The multiplication of two complex numbers amounts to multiplying their moduli and adding their arguments (see reference 1, appendix H). Since $|z_1| = 1$, multiplying z_1 by itself amounts to rotating it an angle of 1 radian round the unit circle. Thus, $\{z_1^n\}$ cannot converge. It follows that neither $\{\cos(n)\}$ or $\{\sin(n)\}$ can converge. In fact, if $\cos(n) \to \alpha$, then the sequence $\{z_1^n\}$ would tend to limiting values

$$\alpha \pm \mathrm{i}\sqrt{1-\alpha^2},$$

which is evidently not the case. Notice that if $\alpha = \pm 1$, the sequence $\{z_1^n\}$ would converge, which is a contradiction.

Using a geometric argument

Consider the intervals

$$E_k = \left(2k\pi,\, (4k+1)\frac{\pi}{2} - \beta\right) \quad \text{and} \quad O_k = \left((2k+1)\pi,\, (4k+3)\frac{\pi}{2} - \beta\right),$$

where k is a nonnegative integer and β is any real number such that $0 < \beta < \pi/2 - 1$. Then the length of each interval is greater than 1 and less than $\pi/2$ (see figure 1). The intervals E_k and O_k each contain integers m_k and n_k, respectively. Thus, $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ are increasing sequences of positive integers. Now,

$$\cos(m_k) > \cos\left(\frac{(4k+1)\pi}{2} - \beta\right) = \sin\beta > 0$$

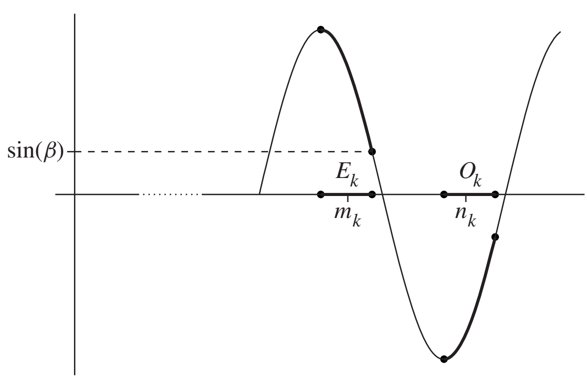

Figure 1

and

$$\cos(n_k) < \cos\left(\frac{(4k+3)\pi}{2} - \beta\right) = -\sin\beta < 0.$$

Therefore, the sequence $\{\cos(n)\}$ diverges, since the subsequences $\{\cos(m_k)\}$ and $\{\cos(n_k)\}$ cannot converge to the same number.

Using the mean value theorem

Let k be an odd positive integer and let n_k be such that

$$n_k < \frac{k\pi}{2} < n_k + 1.$$

Using the mean value theorem for derivatives, there is a θ, $n_k < \theta < n_k + 1$, such that

$$\begin{aligned} |\cos(n_k+1) - \cos(n_k)| &= |\sin(\theta)|\,|n_k + 1 - n_k| \\ &= |\sin(\theta)|. \end{aligned}$$

This implies that

$$\theta \in \left(\frac{k\pi}{2} - 1, \frac{k\pi}{2} + 1\right),$$

and so

$$|\sin(\theta)| \in \left(\left|\sin\left(\frac{k\pi}{2} + 1\right)\right|, 1\right).$$

Hence,

$$|\cos(n_k+1) - \cos(n_k)| > \left|\sin\left(\frac{k\pi}{2} + 1\right)\right| > \left|\sin\left(\frac{k\pi}{2} + \frac{\pi}{3}\right)\right| = \frac{1}{2}.$$

Hence, the subsequence $\{\cos(n_k)\}$ of $\{\cos(n)\}$, and so also the sequence itself, cannot converge.

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References

1 J. Stewart, *Calculus: Early Transcendentals* (Thompson Brooks/Cole, Pacific Grove, CA, 2008), 6th edn.

2 M. C. Reed, *Fundamental Ideas of Analysis* (John Wiley, New York, 1998).

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Fun Rewriting Series

ROBERT NICOLAIDES

The following problem is from the first mathematics competition at the School of Mathematics and Statistics, The University of Sheffield. Here is a solution submitted by students Robert Nicolaides and Lawrence Beesely-Peck.

Problem

The sequence a_n is defined by $a_1 = 1$ and $a_{k+1} = (k+1)(a_k + 1)$ for all $k > 1$. Find

$$\prod_{k=1}^{\infty}\left(1 + \frac{1}{a_k}\right).$$

Solution

We can write each factor as $1 + 1/a_k = (a_k + 1)/a_k = a_{k+1}/(k+1)a_k$. Let

$$P(n) = \prod_{k=1}^{n}\left(1 + \frac{1}{a_k}\right).$$

Then we have

$$\begin{aligned}
P(n) &= \frac{a_2}{2a_1}\,\frac{a_3}{3a_2}\,\frac{a_4}{4a_3}\cdots\frac{a_{n+1}}{(n+1)a_n}\\
&= \frac{a_{n+1}}{(n+1)!}\\
&= \frac{(n+1)(a_n+1)}{(n+1)!}\\
&= \frac{a_n}{n!} + \frac{1}{n!}\\
&= \frac{n(a_{n-1}+1)}{n!} + \frac{1}{n!}\\
&= \frac{a_{n-1}}{(n-1)!} + \frac{1}{(n-1)!} + \frac{1}{n!}\\
&\;\vdots\\
&= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.
\end{aligned}$$

So, as $n \to \infty$, $P(n) = \mathrm{e}$. Hence, $\prod_{k=1}^{\infty}(1 + 1/a_k) = \mathrm{e}$, as required.

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Pascal's Triangle Modulo 3

AVERY WILSON

If you colour the odd entries of Pascal's triangle red and the even entries blue, a beautiful fractal pattern known as *Sierpinski's gasket* appears. A well-known problem is to determine how many odd entries appear in any given row of Pascal's triangle. A natural generalization of this problem is to ask, if we look at the *n*th row of Pascal's triangle modulo any positive integer *m*, how many occurrences of each residue class $0, 1, 2, \ldots, m-1$ will we find? Patterns are hard to come by for composite moduli, but nice formulae can be found for prime moduli. In this article, I derive the solution for the modulo 3 case of the problem using Lucas' theorem on binomial coefficients modulo a prime.

1. Introduction

Pascal's triangle is rife with patterns to explore. This famous triangle – which is generated by beginning with a 1 and letting each successive entry be the sum of the two entries directly above it – has been the subject of the curiosity of many mathematicians, from Chia Hsien in 11th century China to the triangle's namesake, Blaise Pascal, who published *Trait du Triangle Arithmétique* on the fundamental properties of the triangle in 1654. Today, Pascal's triangle is ubiquitous, and it is commonly known that the kth entry of the nth row is the binomial coefficient $\binom{n}{k}$.

A well-known problem is to determine the number of odd entries in the nth row of Pascal's triangle; that is, how many $\binom{n}{k} \equiv 1 \pmod 2$? In general, if we look at Pascal's triangle modulo a prime p by replacing each entry with its modulo p residue class, how many occurrences of each residue class will we find in the nth row? For example, if $p = 3$, how many zeroes, how many ones, and how many twos are there in the nth row? The solution quickly becomes difficult for $p > 3$, not to mention replacing p by a composite number. A general formula for any prime p is given in reference 1. In this article, I derive the formula for the case of $p = 3$.

2. Lucas' theorem

A useful result is *Lucas' theorem*, which gives a congruence for $\binom{n}{k}$ modulo a prime p in terms of the base p digits of n and k.

Theorem 1 (Lucas' theorem) *Let p be a prime number. If n and k are nonnegative integers with base p expansions $n = n_0+n_1p+n_2p^2+\cdots+n_dp^d$ and $k = k_0+k_1p+k_2p^2+\cdots+k_dp^d$, then*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \pmod p.$$

Proof Expand $(1+x)^n$ two different ways: on one side using the binomial theorem as per usual, and, on the other, breaking n up into its base p expansion before applying the binomial theorem. We have

$$\sum_{i=0}^{n}\binom{n}{i}x^i \equiv (1+x)^n \equiv \prod_{i=0}^{d}(1+x)^{n_ip^i} \equiv \prod_{i=0}^{d}(1+x^{p^i})^{n_i} \equiv \prod_{i=0}^{d}\left(\sum_{j=0}^{n_i}\binom{n_i}{j}x^{jp^i}\right) \pmod p.$$

On the left-hand side of the congruence, the coefficient of x^k is $\binom{n}{k}$. Upon expanding the product on the right-hand side of the congruence, the x^k term has coefficient $\binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d}$, if k is written as $k = k_0 + k_1 p + \cdots + k_d p^d$ in base p. Two polynomials are congruent modulo p if and only if their coefficients are congruent modulo p, so this completes the proof.

For a fixed n and $k = 0, 1, 2, \ldots, n$, the number of $\binom{n}{k}$ divisible by p (i.e. the number of zeroes in the nth row of Pascal's triangle modulo p) comes quickly from Lucas' theorem. Using the convention that $\binom{n_i}{k_i} = 0$ for $k_i > n_i$, observe that $p \nmid \binom{n}{k}$ if and only if $0 \le k_i \le n_i$ for all i. Thus, there are $n_i + 1$ bad choices for each k_i, so the total number of $\binom{n}{k}$ *not* divisible by p is $(n_0 + 1)(n_1 + 1)\cdots(n_d + 1)$. Since there are $n + 1$ of the $\binom{n}{k}$ in total, we get the following corollary.

Corollary 1 *For a nonnegative integer n with base p digits $n_0, n_1, \ldots, n_d$ and $k = 0, 1, 2, \ldots, n$, the number of $\binom{n}{k}$ divisible by p is $n + 1 - (n_0 + 1)(n_1 + 1)\cdots(n_d + 1)$.*

As an example, take $p = 3$ and $n = 6$. Since $6 = 2 \cdot 3$ in base 3, the number of $\binom{n}{k}$ divisible by three is $6 + 1 - 3 = 4$. They are $\binom{6}{1} = \binom{6}{5} = 6$ and $\binom{6}{2} = \binom{6}{4} = 15$.

The number of odd entries in the nth row of Pascal's triangle is an immediate consequence of corollary 1. If $n_0, n_1, \ldots, n_d$ are the base 2 digits of n, then the number of odd entries in the nth row of Pascal's triangle is $(n_0 + 1)(n_1 + 1)\cdots(n_d + 1) = 2^m$, where m is the number of digits $n_i = 1$. As an example, since $10 = 2 + 2^3$, the number of odd entries in the tenth row of Pascal's triangle is $2^2 = 4$, namely 1 and $\binom{10}{2} = 45$, both twice.

3. Pascal's triangle modulo 3

Using Lucas' theorem and basic properties of the integers modulo 3, we can find the number of zeroes, ones, and twos in the nth row of Pascal's triangle modulo 3.

Fix a nonnegative integer n with base 3 digits $n_0, n_1, \ldots, n_d$, and let $k = 0, 1, 2, \ldots, n$. Denote by $M(r, n)$ the number of $\binom{n}{k}$ congruent to r modulo 3, and denote by N_s the number of digits n_i equal to s. Using corollary 1, we have that $M(0, n) = n + 1 - 2^{N_1}3^{N_2}$. Since $M(0, n) + M(1, n) + M(2, n)$ must add up to the total of $n + 1$ binomial coefficients with which we are concerned, we need only find $M(1, n)$ and then can subtract from the total to get $M(2, n)$ with no further calculation.

Let us find $M(1, n)$. Recalling Lucas' theorem, suppose that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \equiv 1 \pmod 3. \tag{1}$$

Since $3 \nmid \binom{n}{k}$, each $\binom{n_i}{k_i}$ must be nonzero; this occurs if and only if $0 \le k_i \le n_i$ for all i. Now, each $\binom{n_i}{k_i}$ can be replaced by a remainder of either one or two. If we let T be the number of $\binom{n_i}{k_i}$ that are replaced by a two, then we have

$$\binom{n}{k} \equiv 2^T \equiv 1 \pmod 3.$$

This congruence holds if and only if T is even, since $2^2 \equiv 1 \pmod 3$. Thus, to satisfy (1), it is necessary and sufficient that $0 \le k_i \le n_i$ for all i and T is even. Now we count the number of $\binom{n}{k} \equiv 1 \pmod 3$ by considering disjoint cases. For a nonnegative integer j, define the jth case as follows.

Case *j* $\binom{n}{k}$ falls into case j if and only if $0 \le k_i \le n_i$ for all i and $T = 2j$.

To count the number of $\binom{n}{k}$ falling into case j, we suppose that $\binom{n}{k}$ falls into case j and count the number of possibilities for the digits of k. Pick $2j$ of the N_2 digits with $n_i = 2$ to have $k_i = 1$ (and thereby $\binom{n_i}{k_i} \equiv 2 \pmod 3$). Then for the remaining $N_1 + N_2 - 2j$ digits of n that are nonzero, there are two options for each digit: $k_i = 0$ or $k_i = n_i$. Hence, the total number of $\binom{n}{k}$ falling into case j is $\binom{N_2}{2j} 2^{N_1+N_2-2j}$.

Now, j is a nonnegative integer that cannot exceed half of N_2, so j ranges over $0 \le j \le \lfloor N_2/2 \rfloor$. Thus, adding up all possible cases gives us

$$M(1,n) = \sum_{j=0}^{\lfloor N_2/2 \rfloor} \binom{N_2}{2j} 2^{N_1+N_2-2j}.$$

This sum may seem somewhat ugly, but it is just the sum of the even-index terms of a certain binomial expansion. There is in fact a nice closed form for such a sum, given by the following proposition.

Proposition 1 *For a nonnegative integer* m,

$$\sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} x^{2j} = \frac{1}{2}((1+x)^m + (1-x)^m).$$

Proof Apply the binomial theorem to $(1+x)^m$ and $(1-x)^m$:

$$\frac{1}{2}((1+x)^m + (1-x)^m) = \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} x^j + \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} (-x)^j = \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} (x^j + (-x)^j).$$

Notice that the odd-index terms vanish, and we are left with only the even-index terms, i.e.

$$\frac{1}{2}\sum_{j \text{ even}} \binom{m}{j} (x^j + (-x)^j) = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} x^{2j}.$$

Taking $x = \frac{1}{2}$ in proposition 1 gives us a greatly simplified expression for $M(1,n)$:

$$M(1,n) = \sum_{j=0}^{\lfloor N_2/2 \rfloor} \binom{N_2}{2j} 2^{N_1+N_2-2j} = \frac{2^{N_1+N_2}}{2}\left(\left(\frac{3}{2}\right)^{N_2} + \left(\frac{1}{2}\right)^{N_2}\right) = 2^{N_1-1}(3^{N_2}+1).$$

Now, since $M(0,n) + M(1,n) + M(2,n)$ must add up to $n+1$, we can just subtract $M(0,n)$ and $M(1,n)$ from the total to find that

$$\begin{aligned} M(2,n) &= n + 1 - M(0,n) - M(1,n) \\ &= n + 1 - (n + 1 - 2^{N_1} 3^{N_2}) - 2^{N_1-1}(3^{N_2} + 1) \\ &= 2^{N_1-1}(3^{N_2} - 1). \end{aligned}$$

The following theorem compiles these results.

Theorem 2 *If $M(r, n)$ denotes the number of entries congruent to r modulo 3 in the nth row of Pascal's triangle, and N_s denotes the number of base 3 digits of n that are equal to s, then*

(i) $M(0, n) = n + 1 - 2^{N_1}3^{N_2}$,

(ii) $M(1, n) = 2^{N_1-1}(3^{N_2} + 1)$,

(iii) $M(2, n) = 2^{N_1-1}(3^{N_2} - 1)$.

The formulae given in theorem 2 allow us to figure out the composition of the nth row of Pascal's triangle modulo 3, given that we can find the base 3 digits of n. As an example, take $n = 11 = 3^2 + 2$. The eleventh row of Pascal's triangle modulo 3 is

$$1\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 1,$$

so theorem 2 should give us $M(0, 11) = 6$, $M(1, 11) = 4$, and $M(2, 11) = 2$. The number of digits equal to one in the base 3 expansion of 11 is $N_1 = 1$, and the number of digits equal to two is $N_2 = 1$. Reassuringly, when we substitute these numbers into the formulae we get

$$\begin{aligned} M(0, 11) &= n + 1 - 2^{N_1}3^{N_2} = 12 - 2 \cdot 3 = 6, \\ M(1, 11) &= 2^{N_1-1}(3^{N_2} + 1) = 2^0(3 + 1) = 4, \\ M(2, 11) &= 2^{N_1-1}(3^{N_2} - 1) = 2^0(3 - 1) = 2. \end{aligned}$$

4. Further questions

(i) Using corollary 1, show that if p is a prime and the binomial coefficient $\binom{n}{k}$ is picked at random from the first m rows of Pascal's triangle, then the probability that p divides $\binom{n}{k}$ tends to one as m tends to infinity.

(ii) How many occurrences of each residue class are there in the nth row of Pascal's triangle modulo 5?

(iii) Patterns in Pascal's triangle become much more obscure when looked at modulo a composite number. Do the results on Pascal's triangle modulo 2 and 3 tell anything about Pascal's triangle modulo a power of 2 or 3? What about modulo $2 \cdot 3 = 6$?

References

1 E. Hexel and H. Sachs, Counting residues modulo a prime in Pascal's triangle, *Indian J. Math.* **20** (1978), pp. 91–105.

2 N. J. Fine, Binomial coefficients modulo a prime, *Amer. Math. Monthly* **54** (1947), pp. 589–592.

3 http://pages.csam.montclair.edu/~kazimir/history.html.

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Abbas Rouholamini Gugheri's Congruence for an Extended Fibonacci Family

THOMAS KOSHY

Using the well-known Euler's theorem and the addition formula for Fibonacci numbers, Abbas Rouholamini Gugheri recently developed an interesting Fibonacci congruence. We generalize this congruence to an extended Fibonacci family, consisting of Fibonacci, Lucas, Pell, and Pell–Lucas numbers.

Abbas Rouholamini Gugheri's congruence

Recently, Abbas Rouholamini Gugheri published an interesting Fibonacci congruence, which he discovered when he was a teenager: $\mathrm{F}_{\varphi(m)+n} \equiv \mathrm{F}_n \pmod{m/d}$, where φ denotes Euler's phi-function (see reference 1) and $m = a^2 - a - 1$, $d = (2a - 1, m)$, $a \geq 2$ is an integer, and (x, y) denotes the greatest common divisor of the integers x and y (see reference 2). In this article, we extend this congruence to Pell numbers, and then to Lucas numbers and Pell–Lucas numbers, denoted by P_n, L_n, and Q_n, respectively.

Fibonacci, Lucas, Pell, and Pell–Lucas numbers belong to a larger integer family $\{g_n\}$, defined recursively by $g_n = \lambda g_{n-1} + g_{n-2}$, where $g_0 = b$, $g_1 = 1$, λ is a positive integer, and $n \geq 2$ (see references 3–6). Suppose that $\lambda = 1$. If $b = 0$, then $g_n = \mathrm{F}_n$; but if $b = 2$, then $g_n = \mathrm{L}_n$. Suppose that $\lambda = 2$. If $b = 0$, then $g_n = \mathrm{P}_n$; and if $b = 1$, then $g_n = \mathrm{Q}_n$. For the rest of the discussion, we let $g_0 = 0$.

The next result plays a central role in the development of the generalization of Abbas Rouholamini Gugheri's congruence.

Lemma 1 *Let $a \geq 2$ be an integer and $m = a^2 - \lambda a - 1$. Then $a^n \equiv a g_n + g_{n-1} \pmod{m}$, where $n \geq 0$.*

Proof We will establish the congruence using induction. Since $g_{-1} = 1$ and $g_0 = 0$, the statement is clearly true when $n = 0$ and $n = 1$. Now assume that it is true for an arbitrary integer $n \geq 0$. Then we have

$$\begin{aligned} a^{n+1} &\equiv a(a g_n + g_{n-1}) \pmod{m} \\ &\equiv (\lambda a + 1) g_n + a g_{n-1} \pmod{m} \\ &\equiv a(\lambda g_n + g_{n-1}) + g_n \pmod{m} \\ &\equiv a g_{n+1} + g_n \pmod{m}. \end{aligned}$$

So, by induction, the statement holds for every $n \geq 0$.

Since $(\lambda - a)^2 - \lambda(\lambda - a) - 1 = a^2 - \lambda a - 1 = m$, lemma 1 yields the following result.

Corollary 1 *Let $m = a^2 - \lambda a - 1$, where $a \geq 2$ is an integer. Then $(\lambda - a)^n \equiv (\lambda - a) g_n + g_{n-1} \pmod{m}$, where $n \geq 0$.*

It follows by lemma 1 and corollary 1 that

$$\begin{aligned} a^n + (\lambda - a)^n &\equiv (ag_n + g_{n-1}) + [(\lambda - a)g_n + g_{n-1}] \pmod m \\ &\equiv \lambda g_n + 2g_{n-1} \pmod m. \end{aligned}$$

Similarly, $a^n - (\lambda - a)^n \equiv (2a - \lambda)g_n \pmod m$. Thus, we have the following result.

Corollary 2 *Let $m = a^2 - \lambda a - 1$, where $a \geq 2$ is an integer. Then we have*

$$a^n + (\lambda - a)^n \equiv \lambda g_n + 2g_{n-1} \pmod m, \tag{1}$$

$$a^n - (\lambda - a)^n \equiv (2a - \lambda)g_n \pmod m. \tag{2}$$

Next we will develop an addition formula for g_k using matrices.

An addition formula for g_k

Let

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} g_2 & g_1 \\ g_1 & g_0 \end{bmatrix}.$$

It then follows by the recurrence and induction that

$$\boldsymbol{A}^n = \begin{bmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{bmatrix}, \tag{3}$$

where $n \geq 1$. Consequently, we have

$$\boldsymbol{A}^{m+n} = \boldsymbol{A}^m \cdot \boldsymbol{A}^n$$

and

$$\begin{aligned} \begin{bmatrix} g_{m+n+1} & g_{m+n} \\ g_{m+n} & g_{m+n-1} \end{bmatrix} &= \begin{bmatrix} g_{m+1} & g_m \\ g_m & g_{m-1} \end{bmatrix} \begin{bmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} g_{m+1}g_{n+1} + g_m g_n & g_{m+1}g_n + g_m g_{n-1} \\ g_m g_{n+1} + g_{m-1}g_n & g_m g_n + g_{m-1}g_{n-1} \end{bmatrix}. \end{aligned}$$

This yields the desired addition formula:

$$g_{m+n} = g_m g_{n-1} + g_{m+1}g_n.$$

In particular, this gives the following addition formulas for Fibonacci numbers and Pell numbers:

$$\mathrm{F}_{m+n} = \mathrm{F}_m\mathrm{F}_{n-1} + \mathrm{F}_{m+1}\mathrm{F}_n, \qquad \mathrm{P}_{m+n} = \mathrm{P}_m\mathrm{P}_{n-1} + \mathrm{P}_{m+1}\mathrm{P}_n.$$

They correspond to $\lambda = 1$ and $\lambda = 2$, respectively.

Equation (3) has an additional by-product. Since $|\boldsymbol{A}^n| = |\boldsymbol{A}|^n = (-1)^n$, it yields the Cassini's formula for g_n, i.e. $g_{n+1}g_{n-1} - g_n^2 = (-1)^n$, where $|\boldsymbol{M}|$ denotes the determinant of the square matrix $\boldsymbol{M}$.

We need one more lemma before presenting the generalized version of Abbas Rouholamini Gugheri's congruence.

Lemma 2 *Let $a \geq 2$ be an integer, $m = a^2 - \lambda a - 1$, $\lambda = 1$ or $\lambda = 2$, and $d = (2a - \lambda, m)$. Then we have*

$$d = \begin{cases} 1 \text{ or } 5, & \text{if } \lambda = 1, \\ 1 \text{ or } 2, & \text{if } \lambda = 2. \end{cases}$$

Proof Since $d = (2a - \lambda, m)$ and $4m = (2a - \lambda)^2 - (\lambda^2 + 4)$, it follows that $d \mid (\lambda^2 + 4)$.

If $\lambda = 1$, then $d \mid 5$; so $d = 1$ or 5. On the other hand, let $\lambda = 2$. Then $d \mid 8$. If a is even, then $m = a^2 - 2a - 1$ is odd; so $d = 1$. Suppose that a is odd, say $a = 2k + 1$. Then $d = (2a - 2, a^2 - 2a - 1) = (4k, 2(2k^2 - 1)) = 2(2k, 2k^2 - 1) = 2 \cdot 1 = 2$. Thus, $d = 1$ or 2.

As an example, let $a = 13$ and $\lambda = 1$. Then $m = 155$ and $\lambda^2 + 4 = 5 = (25, 155) = d$. On the other hand, let $a = 7$ and $\lambda = 2$. Then $m = 34$ and $d = (12, 34) = 2$.

Suppose that $\lambda = 2$ in lemma 2. Then it follows from the proof that $d = 2$ if and only if a is odd. Hence, m is even if and only if a is odd.

We now present the generalized congruence.

Theorem 1 *Let $a \geq 2$ be an integer, $\lambda = 1$ or $\lambda = 2$, $m = a^2 - \lambda a - 1$, and $d = (2a - \lambda, m)$. Then we have*

$$g_{\varphi(m)+n} \equiv g_n \left(\operatorname{mod} \frac{m}{d}\right).$$

Proof Since $m = a^2 - \lambda a - 1 = -a(\lambda - a) - 1$, $(a, m) = 1 = (\lambda - a, m)$. Therefore, by Euler's theorem (see reference 1), $a^{\varphi(m)} \equiv 1 \equiv (\lambda - a)^{\varphi(m)} \pmod m$. Consequently, by (1) and (2), we have

$$\lambda g_{\varphi(m)} + 2g_{\varphi(m)-1} \equiv 2 \pmod m, \tag{4}$$

$$(2a - \lambda) g_{\varphi(m)} \equiv 0 \pmod m. \tag{5}$$

Since $d = (2a - \lambda, m)$, it follows from (5) that

$$g_{\varphi(m)} \equiv 0 \left(\operatorname{mod} \frac{m}{d}\right). \tag{6}$$

It follows from (4) and (6) that

$$\lambda g_{\varphi(m)} + 2g_{\varphi(m)-1} \equiv 2 \left(\operatorname{mod} \frac{m}{d}\right), \qquad 2g_{\varphi(m)-1} \equiv 2 \left(\operatorname{mod} \frac{m}{d}\right).$$

Suppose that m is odd. Then d and hence m/d are odd. So $g_{\varphi(m)-1} \equiv 1 \pmod{m/d}$. On the other hand, let m be even. Since $(a, m) = 1$, a is odd. Since a is odd and $(\lambda - a, m) = 1$, λ must be even, so $\lambda = 2$. Thus, $d = 2$ by lemma 2. Consequently, by (4) and (6) we have

$$2g_{\varphi(m)} + 2g_{\varphi(m)-1} \equiv 2 \pmod m, \qquad g_{\varphi(m)-1} \equiv 1 \left(\operatorname{mod} \frac{m}{2}\right).$$

Thus, in both cases,

$$g_{\varphi(m)-1} \equiv 1 \left(\operatorname{mod} \frac{m}{d}\right). \tag{7}$$

Congruences (6) and (7), coupled with the recurrence for g_k, imply that

$$g_{\varphi(m)+1} \equiv 1 \left(\operatorname{mod} \frac{m}{d}\right).$$

Using the addition formula for g_k, we then have

$$\begin{aligned} g_{\varphi(m)+n} &\equiv g_{\varphi(m)}g_{n-1} + g_{\varphi(m)+1}g_n \\ &\equiv 0 + g_n \left(\bmod \frac{m}{d}\right) \\ &\equiv g_n \left(\bmod \frac{m}{d}\right), \end{aligned} \tag{8}$$

as desired.

In particular, this congruence yields

$$\mathrm{F}_{\varphi(m)+n} \equiv \mathrm{F}_n \left(\bmod \frac{m}{d}\right), \tag{9}$$

$$\mathrm{P}_{\varphi(m)+n} \equiv \mathrm{P}_n \left(\bmod \frac{m}{d}\right). \tag{10}$$

As an example, let $a = 7$ and $\lambda = 1$. Then $m = 41$, $d = 1$, and $\varphi(41) = 40$. So $\mathrm{F}_{40+n} \equiv \mathrm{F}_n \pmod{41}$. In particular, $\mathrm{F}_{40+10} = 12\,586\,269\,025 \equiv 14 \equiv 55 \equiv \mathrm{F}_{10} \pmod{41}$. On the other hand, let $\lambda = 2$. Then $m = 34$, $d = 2$, and $\varphi(34) = 16$. So $\mathrm{P}_{16+10} = 3\,166\,815\,962 \equiv 15 \equiv 2378 \equiv \mathrm{P}_{10} \pmod{17}$.

Lucas and Pell–Lucas counterparts

Congruences (9) and (10) have counterparts to Lucas and Pell–Lucas numbers. To derive them, we will employ the following addition formulas (see reference 5):

$$\mathrm{L}_{m+n} = \mathrm{F}_m\mathrm{L}_{n-1} + \mathrm{F}_{m+1}\mathrm{L}_n, \qquad \mathrm{Q}_{m+n} = \mathrm{P}_m\mathrm{Q}_{n-1} + \mathrm{P}_{m+1}\mathrm{Q}_n.$$

These can be confirmed using the corresponding Binet's formulas.

Using congruences (6), (7), and (8), it follows that

$$\begin{aligned} \mathrm{L}_{\varphi(m)+n} &= \mathrm{F}_{\varphi(m)}\mathrm{L}_{n-1} + \mathrm{F}_{\varphi(m)+1}\mathrm{L}_n \\ &\equiv 0 + \mathrm{L}_n \left(\bmod \frac{m}{d}\right) \\ &\equiv \mathrm{L}_n \left(\bmod \frac{m}{d}\right). \end{aligned}$$

Similarly, $\mathrm{Q}_{\varphi(m)+n} \equiv \mathrm{Q}_n \pmod{m/d}$.

As an example, let $a = 6$ and $\lambda = 1$. Then $m = 29$, $d = 1$, and $\varphi(29) = 28$. So $\mathrm{L}_{28+n} \equiv \mathrm{L}_n \pmod{29}$. As a special case, $\mathrm{L}_{28+8} = 33\,385\,282 \equiv 18 \equiv 47 \equiv \mathrm{L}_8 \pmod{29}$. On the other hand, let $\lambda = 2$. Then $m = 23$, $d = 1$, $\varphi(23) = 22$, and $\mathrm{Q}_{22+n} \equiv \mathrm{Q}_n \pmod{23}$. In particular, $\mathrm{Q}_{22+6} = 26\,102\,926\,097 \equiv 7 \equiv 99 \equiv \mathrm{Q}_6 \pmod{23}$.

Acknowledgment The author would like to thank the Editor and Z. Gao for helpful comments that improved the original version of this article.

References

1 T. Koshy, *Elementary Number Theory with Applications* (Academic Press, Boston, MA, 2007), 2nd edn.
2 A. Rouholamini Gugheri, Periodicity of the Fibonacci sequence, *Math. Spectrum* **45** (2012/2013), pp. 37–38.
3 M. Bicknell, A primer on the Pell sequence and related sequences, *Fibonacci Quart.* **13** (1975), pp. 345–349.
4 A. F. Horadam, Pell idenitites, *Fibonacci Quart.* **9** (1971), pp. 245–252, 263.
5 T. Koshy, *Fibonacci and Lucas Numbers with Applications* (John Wiley, New York, 2001).
6 T. Koshy, *Pell and Pell–Lucas Numbers with Applications* (Springer, New York, 2014).

Thomas Koshy *received his PhD in Algebraic Coding Theory from Boston University in 1971. He has authored eight books, including 'Fibonacci and Lucas Numbers with Applications', 'Catalan Numbers with Applications', 'Triangular Numbers with Applications', and 'Pell and Pell–Lucas Numbers with Applications'. He is an Emeritus Professor of Mathematics at Framingham State University.*

Letters to the Editor

Dear Editor,

A nice application of the Hermite–Hadamard inequality

In this letter we prove that the following inequality holds:

$$\frac{\mathrm{e}^{\sin x} - \mathrm{e}^{\tan x}}{\sin x - \tan x} > \mathrm{e}^{x}, \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

Applying the Hermite–Hadamard inequality (see my letter 'The history of a famous inequality' (Volume 47, Number 1, pp. 42–43)) for the function e^{x} we have

$$\frac{\mathrm{e}^{b} - \mathrm{e}^{a}}{b-a} = \frac{\int_a^b \mathrm{e}^{t}\,\mathrm{d}t}{b-a} > \mathrm{e}^{(a+b)/2}.$$

Therefore,

$$\frac{\mathrm{e}^{\sin x} - \mathrm{e}^{\tan x}}{\sin x - \tan x} > \mathrm{e}^{(\sin x + \tan x)/2}, \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right). \tag{1}$$

Now we can prove (see reference 1, p. 110) that

$$\frac{\sin x + \tan x}{2} > x, \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right). \tag{2}$$

Indeed, considering the function $f(x) = \tan x + \sin x - 2x$ in $[0, \pi/2)$, we have

$$f'(x) = \frac{1}{\cos^2(x)} + \cos x - 2, \quad f''(x) = \frac{2-\cos^3 x}{\cos^3 x}\sin x > 0, \qquad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

Hence, $f'(x) > f'(0) = 0$ and $f(x) > f(0) = 0$ for all $x \in (0, \pi/2)$. Therefore, (2) is valid. Combining (1) and (2) we get the desired inequality.

References

1 D. S. Mitrinović, *Elementary Inequalities* (Noordhoff, Groningen, 1964).

2 http://www.math.ualberta.ca/pi/issue7/page26-29.pdf.

3 http://en.wikipedia.org/wiki/Hermite%E2%80%93Hadamard_inequality.

Yours sincerely,

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Dear Editor,

Sums of squares and cubes

I was looking for numbers which are the sum of two squares and simultaneously the sum of two cubes and have found the charming sequence 2 000, 16 000, 128 000, 1 024 000, 8 192 000, 65 536 000, ... This can be considered as two interleaved sequences. The first of them, taking every other term, is 2 000, 128 000, 8 192 000, ... and its terms have the form $2^{6k+1} \cdot 10^3$, $k = 0, 1, 2, \ldots$, and the second subsequence is 16 000, 1 024 000, 65 536 000, ... whose terms have the form $2^{6k+4} \cdot 10^3$, $k = 0, 1, 2, \ldots$. To prove these sequences work, I found, for the first subsequence,

$$2^{6k+1} \cdot 10^3 = (2^{2k} \cdot 10)^3 + (2^{2k} \cdot 10)^3 = (2^{3k+1} \cdot 10)^2 + (2^{3k+2} \cdot 10)^2,$$

which I verified as follows:

$$(2^{2k} \cdot 10)^3 + (2^{2k} \cdot 10)^3 = (2^{6k} + 2^{6k})(10^3) = 2 \cdot 2^{6k}(10^3) = 2^{6k+1} \cdot 10^3,$$
$$(2^{3k+1} \cdot 10)^2 + (2^{3k+2} \cdot 10)^2 = (2^{6k+2} + 2^{6k+4})(10^2) = 2^{6k+2}(1+4)(10^2) = 2^{6k+1}(10^3).$$

As to the second subsequence, I observed

$$2^{6k+4} \cdot 10^3 = (2^{2k+1} \cdot 10)^3 + (2^{2k+1} \cdot 10)^3 = (2^{3k+2} \cdot 10)^2 + (3 \cdot 2^{3k+2} \cdot 10)^2.$$

The verification of this is straightforward and similar to the previous demonstration.

Yours sincerely,

Tom Moore

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Dear Editor,

The equations $x^2 + y^2 = z^2 \pm 1$

It is well known that the solutions of Pythagoras' equation $x^2 + y^2 = z^2$ in positive integers can be expressed as $x = a^2 - b^2$, $y = 2ab$, $z = a^2 + b^2$, where a, b are positive integers with $a > b$.

To obtain a family of solutions of the equation

$$x^2 + y^2 = z^2 - 1$$

in positive integers, we can use the identity

$$(2b^2)^2 + (2b)^2 = (2b^2 + 1)^2 - 1.$$

Are there other positive integer solutions?

For the equation

$$x^2 + y^2 = z^2 + 1,$$

we can use the identity

$$(am + bn)^2 + (an - bm)^2 = (an + bm)^2 + (am - bn)^2.$$

Any choice of a, b, m, n such that $am - bn = 1$ will give a solution. For example, $a = 3$, $m = 5$, $b = 2$, $n = 7$ gives

$$29^2 + 11^2 = 31^2 + 1;$$

$a = 4$, $m = 7$, $b = 9$, $n = 3$ gives

$$55^2 + 51^2 = 75^2 + 1.$$

Other families of solutions come from the identities

$$(2a)^2 + (2a^2 - 1)^2 = (2a^2)^2 + 1,$$
$$(2k + 1)^2 + (k^2 + k - 1)^2 = (k^2 + k + 1)^2 + 1.$$

I have also found these solutions involving the Fibonacci numbers F_n and Pell numbers P_n:

$$x = (-1)^n + 2\mathrm{F}_n^2, \qquad y = \mathrm{F}_n^2, \qquad z = \mathrm{F}_{2n}$$

and

$$x = \mathrm{P}_{2n} + 1, \qquad y = \mathrm{P}_{2n} - 1, \qquad z = \mathrm{P}_{2n} + \mathrm{P}_{2n-1}.$$

(See Volume 43, Number 3, p. 129 for a definition of Pell numbers.)

Yours sincerely,

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Dear Editor,

Reverse divisors

A recent article ‘On the trail of reverse divisors: 1089 and all that follow’ (Volume 45, Number 3, pp. 96–102) included a ‘remarkable property’ of 1089 and it prompted me to write about a similar arithmetical phenomenon and a property of the number 6174.

Consider the following transformation of the four-digit number 6174. Use the digits of the original number to create two more numbers. The first is the largest possible creation by taking the digits in descending order (7641); the second is the smallest possible, being the reverse of the first (1467). The second number is then subtracted from the first to give a result; in this case 6174. This is significant because

1. 6174 is the only nonzero four-digit number which transforms to itself, and

2. starting with any four-digit number, if the transformation is applied repeatedly, as necessary, then only multiples of 1111 will result in 0000 being produced; all other starting points will (eventually) repeat 6174, and will do so in eight or fewer transformations.

The investigation can be widened in two directions, considering d $(d > 1)$ digits rather than just four, and by working in number base a $(a > 1)$, instead of base 10. A number of other observations can be made.

1. Apart from numbers with repeated digits, very few numbers with unequal digits are on a path of transformations which terminates with zero. A few show up using two digits; examples include 20_3 and 83_9.

2. Almost all the results of the transformation are not prime. Algebra shows that the result of the transformation is divisible by $a - 1$. Prime results exist for base 2; an example, using six digits, is $011111_2 = 31_{10}$ which occurs as the result of transforming itself (amongst others).

3. Since many numbers share the same digits in a different permutation, there are fewer different subtractions to be done and even fewer different results of the subtraction. If e represents the integer part of $d/2$ then it can be shown that there are ${}^{a+e-1}C_e$ different results.

4. Applying transformations repeatedly will soon repeat one of the results already found, such as 6174 above. Using other bases and other numbers of digits the behaviour varies considerably (although zero is always a termination for numbers with repeated digits). In some cases there may be more than one other termination result; in other cases there may be none but rather a ring of numbers that cycle; some cases have several rings that may not all be the same size, and some cases have both rings and terminations. The longest ring I found consisted of 17 numbers.

5. Searching amongst bases up to 20 and using up to seven digits, I found the maximum of the minimum number of transformations leading to repetition is 41 (try $0009DG_{17}$, where $A_{17} = 10_{10}$, $B_{17} = 11_{10}$, and so on, extending the conventional notation for hexadecimal digits).

6. It is possible to determine the appropriate argument of the transformation which will produce either the smallest or the largest result of the transformation.

   (a) The smallest, nonzero, result of the transformation occurs when the argument of the transformation has only two different digits, and they differ by 1, and these are split as 1 of one and $d - 1$ of the other. An example in base 10 with five digits would be 33323_{10}. The result of the transformation is $a^{d-1} - 1$. If $d > 2$ then there are $2d(a - 1)$ such different arguments, or for $d = 2$ there are $d(a - 1)$ arguments.

(b) The largest result of the transformation occurs when the digits in the argument are equally split between 0 and $a-1$; if there is an extra digit (for odd values of d) it may take any value. An example in base 10 with five digits would be 99070_{10}. The result of the transformation is $(a-1)(a^f-1)(a^{e-1}+a^{e-2}+\cdots+a^0)$, where e is the integer part of $d/2$ and f is the integer part of $(d+1)/2$. The number of arguments giving rise to this maximum increases with the extra digit. For even values of d there are ${}^dC_{d/2}$ arguments, but when d is odd there are ${}^dC_{(d-1)/2}[a(d+1)/2-d+1]$ possible arguments.

Some other questions remain, including

1. Are there numbers with more than two digits which are not all equal, and yet transform to a number with all equal digits, and hence to zero?
2. Using base 2, if any number is the result of a transformation, then does it also further transform to itself?
3. Can the behaviour for other bases and numbers of digits be found without extensive computation?

I had fun with this investigation and I hope it will prove of interest to others also.

Yours sincerely,
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Dear Editor,

Slicing cakes into equal pieces

In a recent article 'Mathematicians prefer cake to pi' (Volume 45, Number 3, pp. 103–105), Paul Glaister considered a circular cake of radius unity to be distributed into $2n$ equal pieces of the same size by cutting it in a series of parallel lines. Let the centre be placed at the origin and let the x-axis be perpendicular to the cuts. The cuts at $x_0=0<x_1<\cdots<x_{n-1}<1:=x_n$ satisfy

$$2\int_{x_i}^{x_{i+1}}\sqrt{1-x^2}\,\mathrm{d}x=\frac{\pi}{2n},\qquad i=0,\dots,n-1.$$

Glaister suggested numerical approximation of the x_i by successive iteration based on the equivalent equation (see equation (1) in Glaister's article), i.e.

$$x_{i+1}=\sin\left(\frac{\pi}{2n}+\arcsin x_i-x_{i+1}\sqrt{1-x_{i+1}^2}+x_i\sqrt{1-x_i^2}\right).$$

Alternatively, we could numerically find the location x_i of the ith cut directly by modifying Glaister's equation (2), for $i=1$, in the form

$$x_i=\sin\left(\frac{i\pi}{2n}-x_i\sqrt{1-x_i^2}\right). \tag{1}$$

We will give a representation of x_i as a power series whose partial sums deliver arbitrary precise approximations of its value. We consider the more general area $0\le y\le f(x)$, $x\in[a,b]$, of size $A = \int_a^b f(x)\,\mathrm{d}x$ with a bounded analytic function f satisfying $f(x) > 0$ on $[a, b)$. For $0 < c < A$, we want to find the location $\bar{x}$ of a cut perpendicular to the x-axis dividing the area into two parts of sizes c and $A - c$, respectively. This means $F(\bar{x}) = c$, where

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Since f is positive on $[a, b)$, F is strictly increasing and possesses an inverse, F^{-1}. It follows that

$$\bar{x} = F^{-1}(c).$$

If F is assumed to be analytic in a region containing the interval $[a, b)$, Lagrange inversion (see, for example, reference 1, equation (3.6.6)) yields

$$\bar{x} = a + \sum_{k=1}^{\infty} \frac{c^k}{k!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-1}\left(\frac{x-a}{F(x)}\right)^k\right]_{|x=a}. \tag{2}$$

Note that the condition $F'(a) = f(a) \neq 0$ is satisfied by assumption.

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, the initial terms of the expansion are given by

$$\begin{aligned}\bar{x} = {} & \frac{1}{a_0}c - \frac{a_1}{2a_0^3}c^2 + \frac{3a_1^2 - 2a_0a_2}{6a_0^5}c^3 + \frac{-15a_1^3 + 20a_0a_1a_2 - 6a_0^2a_3}{24a_0^7}c^4 \\ & + \frac{105a_1^4 - 210a_0a_1^2a_2 + 90a_0^2a_1a_3 + 40a_0^2a_2^2 - 24a_0^3a_4}{120a_0^9}c^5 + \cdots,\end{aligned}$$

where, for the sake of brevity, we restricted ourselves to the special case $a = 0$.

Remark 1 In the case $f(a) = 0$, a similar representation of $\bar{x}$ can be given by a series with rational exponents.

Remark 2 If c is close to A it may happen that the series for $f(a + b - c)$ with $A - c$ in place of c converges faster.

Coming back to the circular cake of radius unity, i.e. the special case

$$f(x) = 2\sqrt{1-x^2} = 2\sum_{k=0}^{\infty}(-1)^k\binom{1/2}{k}x^{2k}, \quad \text{for } |x| < 1,$$

$[a, b] = [0, 1]$, and $0 < c < \pi$, we obtain

$$F(x) = x\sqrt{1-x^2} + \arcsin x = 2x - \tfrac{1}{3}x^3 - \tfrac{1}{20}x^5 - \tfrac{1}{56}x^7 - \tfrac{5}{576}x^9 + \cdots,$$

and, using (2),

$$\bar{x} = F^{-1}(c) = \tfrac{1}{2}c + \tfrac{1}{48}c^3 + \tfrac{13}{3\,840}c^5 + \tfrac{493}{645\,120}c^7 + \tfrac{37\,369}{185\,794\,560}c^9 + \cdots.$$

We finish with some numerical examples for the position of cuts in the case of $2n = 12$ pieces (see table 1). Precise digits are shown in bold face.

For a comparison we mention that 100 iterations using (1) beginning with the starting value 0.1 yield for the position of the first cut an approximate value of $x_1 \approx 0.130\,348$ having a precision of only two exact digits after the decimal point.

Table 1

Terms up to	x_1	x_2
c^3	**0.131 27**3 515 715 3	0.**264** 789 962
c^5	**0.131 277** 679 183 5	**0.264 9**23 193
c^7	**0.131 277 74**3 598 5	**0.264 93**1 438
c^9	**0.131 277 744** 760 5	**0.264 932 0**33
Exact values	0.131 277 744 783 9...	0.264 932 084...
Errors in case c^9	$2.34 * 10^{-11}$	$5.1 * 10^{-8}$

Terms up to	x_3	x_4	x_5
c^3	**0.40**2 792 3	**0.5**47 523 4	**0.7**01 2
c^5	**0.403** 804 0	**0.55**1 786 8	**0.7**14 2
c^7	**0.403** 944 9	**0.55**2 842 1	**0.7**19 3
c^9	**0.403** 967 7	**0.553** 146 7	**0.72**1 5
Exact values	0.403 972 75...	0.553 292 712 3...	0.723 998 67...
Errors in case c^9	$5 * 10^{-6}$	$1.4 * 10^{-4}$	$2.5 * 10^{-3}$

Reference

1 M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U.S. Government Printing Office, Washington, D.C., 1964).

Yours sincerely,

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Dear Editor,

$$\frac{a^2}{r_B r_C} + \frac{b^2}{r_C r_A} + \frac{c^2}{r_A r_B} = 4\left(\frac{R}{r} - 1\right)$$

Problem 40.3 (see *Mathematical Spectrum* Volume 40, Number 1, p. 41) asked readers to prove the following.

> Let a, b, c be the lengths of the sides of a triangle ABC and let r_A, r_B, r_C, be the radii of its escribed circles. Prove that
>
> $$\frac{a^2}{r_B r_C} + \frac{b^2}{r_C r_A} + \frac{c^2}{r_A r_B} \geq 4.$$

In fact,

$$\frac{a^2}{r_B r_C} + \frac{b^2}{r_C r_A} + \frac{c^2}{r_A r_B} = 4\left(\frac{R}{r} - 1\right),$$

where R and r denote the radii of the circumcircle and the inscribed circle, respectively.

We first prove the following three results:

(1) $ab + bc + ca = s^2 + 4Rr + r^2$,

(2) $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$,

(3) $(a+b)(b+c)(c+a) = 2s(s^2 + 2Rr + r^2)$,

where $s = \frac{1}{2}(a+b+c)$.

The area of the triangle is given by

$$\Delta = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = rs$$

and also by

$$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab\frac{c}{2R};$$

whence,

$$abc = 4R\Delta = 4Rrs.$$

By Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. Hence,

$$r^2s^2 = s(s-a)(s-b)(s-c);$$

whence,

$$\begin{aligned} r^2s &= s^3 - s^2(a+b+c) + s(ab+bc+ca) - abc \\ &= s^3 - 2s^3 + s(ab+bc+ca) - 4Rrs \end{aligned}$$

and

$$r^2 = -s^2 + ab + bc + ca - 4Rr,$$

from which (1) follows. Now

$$(a+b+c)^2 = 4s^2,$$

and so

$$a^2 + b^2 + c^2 = 4s^2 - 2(ab+bc+ca) = 2(s^2 - 4Rr - r^2),$$

which proves (2). Since

$$a+b+c = 2s,$$

which gives

$$a+b = 2s-c, \qquad b+c = 2s-a, \qquad c+a = 2s-b,$$

we obtain

$$\begin{aligned} (a+b)(b+c)(c+a) &= (2s-c)(2s-a)(2s-b) \\ &= 8s^3 - 4s^2(a+b+c) + 2s(ab+bc+ca) - abc \\ &= 2s(s^2 + 2Rr + r^2), \end{aligned}$$

which proves (3).

Further,

$$\Delta = \tfrac{1}{2}b\mathrm{r}_A + \tfrac{1}{2}c\mathrm{r}_A - \tfrac{1}{2}a\mathrm{r}_A;$$

whence, $\mathrm{r}_A = \Delta/(s-a)$. Similarly, $\mathrm{r}_B = \Delta/(s-b)$ and $\mathrm{r}_C = \Delta/(s-c)$.

We can now prove our result:

$$\begin{aligned}
&\frac{a^2}{\mathrm{r}_B\mathrm{r}_C} + \frac{b^2}{\mathrm{r}_C\mathrm{r}_A} + \frac{c^2}{\mathrm{r}_A\mathrm{r}_B} \\
&\quad = \frac{a^2(s-b)(s-c) + b^2(s-c)(s-a) + c^2(s-a)(s-b)}{\Delta^2} \\
&\quad = \frac{s^2(a^2+b^2+c^2) - s((a+b)(b+c)(c+a) - 2abc) + abc(a+b+c)}{r^2s^2} \\
&\quad = \frac{s^2 \times 2(s^2 - 4Rr - r^2) - s(2s(s^2+2Rr+r^2) - 2\times 4Rrs) + 4Rrs \times 2s}{r^2s^2} \\
&\quad = \frac{4Rr - 4r^2}{r^2} \\
&\quad = 4\left(\frac{R}{r} - 1\right).
\end{aligned}$$

It follows from this and Problem 40.3 that $R \geq 2r$.

Yours sincerely,

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Dear Editor,

A simple trick in algebraic inequalities

We present a simple trick which is used in some problems. Let a, b, c be three real numbers such that $|a| < 1$, $|b| < 1$, $|c| < 1$. It is well known that

$$a^2 + b^2 + c^2 \geq ab + bc + ac.$$

More generally, we have

$$a^{2n} + b^{2n} + c^{2n} \geq (ab)^n + (bc)^n + (ac)^n. \tag{1}$$

Inequality (1) implies that

$$\sum_{n=0}^{\infty} a^{2n} + \sum_{n=0}^{\infty} b^{2n} + \sum_{n=0}^{\infty} c^{2n} \geq \sum_{n=0}^{\infty}(ab)^n + \sum_{n=0}^{\infty}(bc)^n + \sum_{n=0}^{\infty}(ac)^n$$

or

$$\frac{1}{1-a^2} + \frac{1}{1-b^2} + \frac{1}{1-c^2} \geq \frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ac}.$$

Thus,

$$\frac{1}{1-a^2}-\frac{1}{2}+\frac{1}{1-b^2}-\frac{1}{2}+\frac{1}{1-c^2}-\frac{1}{2}\geq\frac{1}{1-ab}-\frac{1}{2}+\frac{1}{1-bc}-\frac{1}{2}+\frac{1}{1-ac}-\frac{1}{2}$$

or

$$\frac{1+a^2}{1-a^2}+\frac{1+b^2}{1-b^2}+\frac{1+c^2}{1-c^2}\geq\frac{1+ab}{1-ab}+\frac{1+bc}{1-bc}+\frac{1+ac}{1-ac}.$$

This trick appeared in *The Mathematical Intelligencer*, Volume 8 (1986).

Yours sincerely,

Spiros P. Andriopoulos
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Dear Editor,

Triangular oblong numbers

I read with interest Tom Moore's proof that there are infinitely many triangular oblong numbers (see Moore's article 'The infinity of triangular oblong numbers' (Volume 47, Number 1, pp. 40–41)). Underlying the difference equations he used is the number theory behind the solution to the Pell equation $a^2 - 2b^2 = \pm 1$, and this is certainly needed in order to generate the complete sequence of numbers which are simultaneously triangular and oblong.

However, in order just to show that there are an infinite number of solutions, a shorter argument may be given by combining the following two observations.

- There are infinitely many numbers which are simultaneously square and triangular. For if $T_n = \frac{1}{2}n(n+1) = x^2$, then we can create a larger square triangular number from $T_{8T_n} = 4T_n(8T_n+1) = [2x(2n+1)]^2$.
- Given a square triangular number $T_n = \frac{1}{2}n(n+1) = x^2$, we can create a triangular oblong number from

$$\begin{aligned}
T_{n+2x} &= \tfrac{1}{2}(n+2x)(n+2x+1)\\
&= \tfrac{1}{2}[n^2+n(4x+1)+2x^2+2x(x+1)]\\
&= \tfrac{1}{2}[n^2+n(4x+1)+n(n+1)+2x(x+1)]\\
&= n^2+n(2x+1)+x(x+1)\\
&= (n+x)(n+x+1).
\end{aligned}$$

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st July will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

Problems 47.5, 47.6, and 47.7 were submitted by Tom Moore and are dedicated to Tom – see p. 50.

47.5 For positive integers x, y in base 10, $x \sim y$ denotes the concatenation of x and y. For example, $27 \sim 38 = 2738$. Let $T_n = \frac{1}{2}n(n+1)$, $n \geq 1$, be the nth triangular number. Prove that there are infinitely many positive integers n such that $T_n \sim T_{n+1}$ is divisible by 5 and infinitely many n such that $T_n \sim T_{n+1}$ is divisible by 9.

47.6 Prove that every triangular number other than 1 can be written in each of the forms $a^2 + b^2 + a$ and every triangular number can be written in the form $a^2 + b^2 - a$ for some positive integers a and b.

47.7 Find all positive integers n such that

$$3^n < \binom{2n}{n} < 3^{n+1},$$

where $\binom{2n}{n}$ denotes the binomial coefficient.

47.8 For positive integers a and b, prove that $(a-b)^2 = a + b$ if and only if a and b are consecutive triangular numbers.

(Submitted by K. B. Subramaniam, Bhopal, India)

Solutions to Problems in Volume 46 Number 3

46.9 Prove that every number with $6k$ digits, where k is a natural number, which has a repeating block of two or three digits (as, for example, 292 929 and 157 157), is divisible by 13. Prove also that every number with $12k$ digits which has a repeating block of four digits (as, for example, 262 726 272 627) is also divisible by 13.

Solution by Henry Deacy, Ampleforth College, York, UK

The six-digit number $xyxyxy$ has value

$$xy + 100xy + 10\,000xy = 10\,101xy,$$

and

$$10\,101 = 13 \times 777,$$

so it is a multiple of 13. A number of this form with $6k$ digits can be written as

$$xyxyxy + 10^6 xyxyxy + \cdots + 10^{6k} xyxyxy,$$

which is also a multiple of 10 101 and so of 13.

The six-digit number $xyzxyz$ has value

$$xyz + 1000xyz = 1001xyz,$$

and $1001 = 13 \times 77$, so it is a multiple of 13. It follows as before that such a number with $6k$ digits will also be a multiple of 1001 and so of 13.

The 12-digit number $mnopmnopmnop$ has value

$$mnop(1 + 10^4 + 10^8) = 100\,010\,001 mnop,$$

and $100\,010\,001 = 13 \times 7\,693\,077$. Such a number with $12k$ digits will also be a multiple of 100 010 001 and so of 13.

46.10 Given an acute-angled triangle $\triangle$, it is a fact that there is a unique tetrahedron T whose four faces are congruent to $\triangle$. Label the lengths of the sides of $\triangle$ by a, b, c. Prove that the volume of T is

$$\sqrt{\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)(b^2+c^2-a^2)}{72}}.$$

Solution by Fionntan Roukema, who proposed the problem

Consider a cuboid with side-lengths x, y, z, where

$$x = \sqrt{\frac{a^2+b^2-c^2}{2}}, \qquad y = \sqrt{\frac{a^2+c^2-b^2}{2}}, \qquad z = \sqrt{\frac{b^2+c^2-a^2}{2}}.$$

(We note that $a^2+b^2 > c^2$, $a^2+c^2 > b^2$, and $b^2+c^2 > a^2$ because $\triangle$ is an acute-angled triangle.) The face of the cuboid with side-lengths x, y has diagonal-length $\sqrt{x^2+y^2} = a$, and similarly the diagonal-lengths of the two adjacent faces are b and c. The tetrahedron T can be formed from diagonals of the sides of the cuboid as in figure 1. The cuboid is made up from the tetrahedron T and four similar right tetrahedra, each of which will have volume

$$\tfrac{1}{3}(\tfrac{1}{2}xy)z = \tfrac{1}{6}xyz.$$

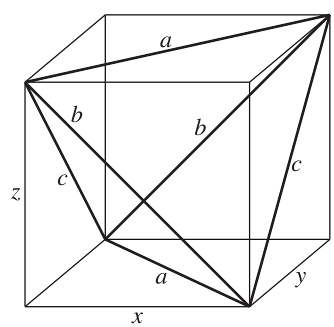

Figure 1

Hence, the volume of the cuboid is

$$xyz = \text{volume of } T + 4 \times \tfrac{1}{6}xyz,$$

so that the volume of T is $\frac{1}{3}xyz$, which gives the required result.

46.11 The three points A, B, C are arbitrary distinct points in the plane. The following are equilateral triangles:

$$\begin{aligned}
&ABD \text{ with } A, B, D \text{ anticlockwise},\\
&BCE \text{ with } B, C, E \text{ anticlockwise},\\
&DEF \text{ with } D, E, F \text{ clockwise},\\
&ABG \text{ with } A, B, G \text{ clockwise},\\
&BCH \text{ with } B, C, H \text{ clockwise}.
\end{aligned}$$

Prove that GHF is an equilateral triangle with G, H, F anticlockwise.

Solution

Choose axes and scale so that A is represented by the complex number 0 (the origin), B by 1, and C by c. Then we have

$$\begin{aligned}
D &= \mathrm{e}^{\pi\mathrm{i}/3},\\
E &= B + (C - B)\mathrm{e}^{\pi\mathrm{i}/3} = c\mathrm{e}^{\pi\mathrm{i}/3} + \mathrm{e}^{-\pi\mathrm{i}/3},\\
F &= D + (E - D)\mathrm{e}^{-\pi\mathrm{i}/3} = D\mathrm{e}^{\pi\mathrm{i}/3} + E\mathrm{e}^{-\pi\mathrm{i}/3} = \mathrm{e}^{2\pi\mathrm{i}/3} + c + \mathrm{e}^{-2\pi\mathrm{i}/3} = c - 1,\\
G &= \mathrm{e}^{-\pi\mathrm{i}/3},\\
H &= B + (C - B)\mathrm{e}^{-\pi\mathrm{i}/3} = \mathrm{e}^{\pi\mathrm{i}/3} + c\mathrm{e}^{-\pi\mathrm{i}/3}.
\end{aligned}$$

Now,

$$\overrightarrow{GH} \times \mathrm{e}^{\pi\mathrm{i}/3} = (\mathrm{e}^{\pi\mathrm{i}/3} + c\mathrm{e}^{-\pi\mathrm{i}/3} - \mathrm{e}^{-\pi\mathrm{i}/3})\mathrm{e}^{\pi\mathrm{i}/3} = c + \mathrm{e}^{2\pi\mathrm{i}/3} - 1 = c - 1 - \mathrm{e}^{-\pi\mathrm{i}/3} = \overrightarrow{GF},$$

so that GHF is an equilateral triangle with G, H, F anticlockwise.

46.12 In a triangle ABC, the medians BD and CE intersect at right angles. Show that $\cos A \geq \frac{4}{5}$. When does equality occur?

Solution by K. S. Bhanu and M. N. Deshpande, who proposed the problem

Consider figure 2. We have

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad BD^2 = a^2 + \left(\frac{b}{2}\right)^2 - ab\cos C,$$

so that

$$c^2 - 2BD^2 = \tfrac{1}{2}b^2 - a^2,$$

giving

$$BD^2 = \tfrac{1}{2}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}b^2.$$

Also,

$$b^2 = a^2 + c^2 - 2ac\cos B, \qquad CE^2 = a^2 + \left(\frac{c}{2}\right)^2 - ac\cos B,$$

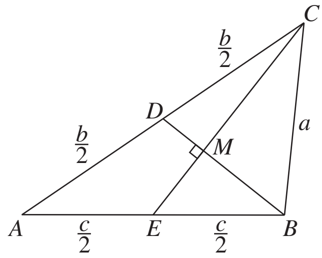

Figure 2

so that

$$b^2 - 2CE^2 = \tfrac{1}{2}c^2 - a^2,$$

giving

$$CE^2 = \tfrac{1}{2}b^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}c^2.$$

Now,

$$BM = \tfrac{2}{3}BD, \qquad CM = \tfrac{2}{3}CE,$$

and

$$BM^2 + CM^2 = a^2,$$

because the medians intersect at right angles. Hence,

$$\tfrac{4}{9}(\tfrac{1}{2}c^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}b^2) + \tfrac{4}{9}(\tfrac{1}{2}b^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}c^2) = a^2,$$

giving

$$a^2 + \tfrac{1}{4}b^2 + \tfrac{1}{4}c^2 = \tfrac{9}{4}a^2 \quad \text{or} \quad b^2 + c^2 = 5a^2.$$

Now,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4(b^2 + c^2)}{10bc} = \frac{2}{5}\left(\frac{b}{c} + \frac{c}{b}\right) = \frac{2}{5}\left(\left(\sqrt{\frac{b}{c}} - \sqrt{\frac{c}{b}}\right)^2 + 2\right) \geq \frac{4}{5}.$$

Equality occurs if and only if $b = c$.

Reviews

A Problem Solver's Handbook. By A. Jobbings. United Kingdom Mathematics Trust, Leeds, 2013. Paperback, 264 pages, £15.00 (ISBN 978-190601193).

A Problem Solver's Handbook is a book primarily aimed at students preparing for junior level Mathematical Olympiads. Namely, the content is based on the mathematics seen by 13–16 year old students at school. The book presents, discusses, and solves a total of twenty examples, and provides the reader with a further 160 problems from past junior Olympiad papers to work through (with solutions to the 160 problems available in an appendix). The problems encountered in this book differ from those encountered in a standard school setting; rather than the application of known methods, these problems require the application of ideas and principles. Namely, the problems require genuine understanding, insight, and ingenuity to be solved.

This book aims to guide the reader along the journey to becoming a mathematical problem solver. The book's presentation of each example comes equipped with a thorough discussion of useful facts, relevant tips on presentation, general principles, and clearly laid out solutions. Importantly, the presentation is clear enough to be accessible to younger readers. The 160 posed problems should be enjoyable for those with a flair and interest in elementary mathematics.

Collecting these junior Olympiad problems together provides a great database of problems. The presentation of examples is very clear, accessible, and instructive. I am particularly happy with the emphasis on the importance of presenting solutions in a clear, readable, and coherent manner. I recommend this book to anyone preparing to take a junior Mathematical Olympiad, to teachers who would like a source of challenging problems for interested students, and to all who enjoy elementary mathematical problems.

University of Sheffield **Fionntan Roukema**

Origins of Mathematical Words: A Comprehensive Dictionary of Latin, Greek, and Arabic Roots. By A. Lo Bello. The Johns Hopkins University Press, Baltimore, MD, 2013. Paperback, 350 pages, £32.00 (ISBN 978-1-4214-1098-2).

Origins of Mathematical Words is a discursive etymological dictionary of mathematical terms with roots in Greek, Latin, or Arabic, the languages in which the larger-than-life author, Professor Anthony Lo Bello, has expertise. He provides Greek, Latin, and Arabic text in its original form to enhance each explanation and avoid doubtful consequences of transliteration. The book is about words, mathematical words, how they were constructed, their history, and how they are used today. A typical entry begins as follows.

> **conchoid** The Greek noun $\kappa o \gamma \chi \eta$ means *mussel* or *shell*, and $\epsilon\iota\delta o \varsigma$ means *shape*. The conchoid is therefore a *shell-shaped curve*. It is a curve first studied by Nicomedes (*circa* 225 B.C.), which, if allowed, permits the trisection of an arbitrary angle.

Then follows a formal definition of the curve, some general comments, and Pascal's generalization of it. A few entries demand an understanding of Greek, Latin, and Arabic grammar, which most readers, including this reviewer, do not have.

This book is not your common-or-garden reference work, where the author seldom graces its pages; unlike here, where the reader is constantly confronted by him, sitting in judgement on a raft of issues, expressing his views in no uncertain terms. He deplores the use of *macaronic* words, those formed from different languages; an infallible sign of a defective education, we are told! Of many examples, he offers **septagon** (a learned mistake for **heptagon**, combining Latin and Greek roots), **nonagon** (an absurd word, used by the unlearned for *enneagon*), and **cohomology** (a bad word, a corruption of the first two letters of a Latin prefix added to a word of Greek origin). Other examples of his red rags are: acronyms; the use of letters to name mathematical objects, e.g. **CW complex**; the regrettable abbreviation **math** (no less silly, than it is natural); and the noncapitalization of adjectives formed from proper names, such as **Abelian** (for otherwise they look ridiculous). Noah Webster is censured for the crime of changing many English spellings in his American dictionary of 1806, thereby severing words from their roots. No one escapes criticism, even the *Oxford English Dictionary* receives a slap on the wrist for including *incommutative* (not a good word, a cautious fellow may call it *rare*, a frank one, *wrong*).

The author's authoritarian exposition can at times be intimidating, so it is comforting to know that even he, like Homer, sometimes nods. For example, the last sentence of the entry for **abundant numbers** should be in that for **amicable numbers**; and the entry for **nonagon** refers to a nonexistent one for *enneagon*. How ironic that a book that is so particular about mathematical vocabulary should pay so little attention to mathematical typography, resulting in such mathematical monstrosities as

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{and} \quad \|ax\| \leq |a|\|x\|$$

in the entry for **pseudonorm**. Whilst the author deplores the ludicrous mixing of upper- and lower-case letters in the word LaTeX, a quality typesetting system, had he employed it, would have given the book the fully professional appearance that it deserves.

This sophisticated, one-of-a-kind dictionary, based on decades of painstaking research by its charismatic author, will delight mathematicians and word lovers alike. It is a treasure chest of rare gems, a godsend to all teachers of the subject and its history. But how to unearth these nuggets, tucked away in the **a** to **z** format of a dictionary? My own solution, to read the book from cover to cover, struck gold immediately, with the feisty first entry **a-**, covering almost three pages – the totality of *all* entries beginning **w**, **x**, **y**, or **z** fills just over a page! Lo Bello's scholarship, combined with his refreshingly personal exposition, brings to mind Briggs' remark on reading Napier's *Mirifici Logarithmorum Canonis Descriptio* in 1614, 'I never saw a book which pleased me better'.

University of Sheffield **Roger Webster**

Bayes' Rule: A Tutorial Introduction to Bayesian Analysis. By J. V. Stone. Sebtel Press, 2013. Paperback, 170 pages, £9.95 (ISBN 978-0-9563728-4-0).

Described as an introductory text aimed at readers who may have little mathematical experience but the willingness to acquire it, this is an ideal book to get someone started on an exploration of Bayesian analysis. Written in an easy-to-read style illustrated with numerous diagrams and examples, it is a very accessible, readable, and novel book. I cannot think of any other statistical text that has grabbed attention by using the two Ronnies' sketch involving fork handles (a.k.a. 'four candles') as an example of Bayesian analysis – truly unique!

I particularly enjoyed the quotations at the start of each of the seven chapters, the final one from John Tukey (1962) 'Far better an approximate answer to the right question …than an exact answer to the wrong question', saying it all!

These chapters, with supporting appendices, comprehensively cover the basics of dichotomous populations and lead on to more general cases of a multivariate nature. The final chapter offers a very interesting discussion on the emotions aroused by Bayes' rule amongst Bayesians and nonBayesians alike with the issues clearly set out – a very fitting conclusion to this delightful book. Definitely recommended reading for anyone from any discipline who wants to grasp the basics of this fascinating topic.

Carol Nixon

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