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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX MATHEMATICORUM

Volume 16 #2

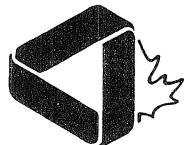
February / Février

1990

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THE OLYMPIAD CORNER
No. 112
R.E. WOODROW

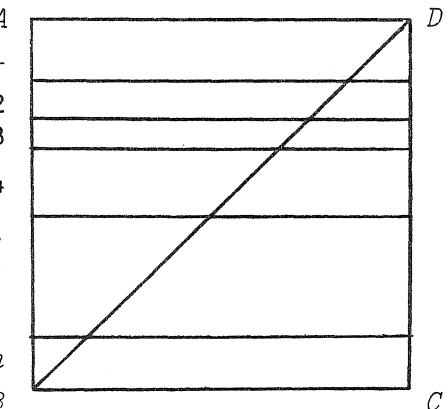
All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

First a correction and apology. The problems for the 24th Spanish Olympiad which we presented in the December 1989 number of the Corner were forwarded to us from Willie Yong of Singapore. Willie Yong also sent us the following problems from the XIV "All Union" Mathematical Olympiad of the U.S.S.R. The translation was done by Mark Saul. We give ten problems this month and the remaining ten next month.

XIV "ALL UNION" MATHEMATICAL OLYMPIAD (U.S.S.R.)

1. All the two-digit numbers from 19 to 80 are written in a row. The result is read as a single integer 19202122...787980. Is this integer divisible by 1980?

2. Side AB of a square $ABCD$ is divided into n segments in such a way that the sum of lengths of the even-numbered segments equals the sum of the lengths of the odd-numbered segments. Lines parallel to AD are drawn through each point of division, and each of the n "strips" thus formed is divided by diagonal BD into a left region and a right region. Show that the sum of the areas of the left regions with odd numbers is equal to the sum of the areas of the right regions with even numbers.



3. A payload, packed into containers, is to be delivered to the orbiting space station "Salyut". There are at least 35 containers, and the total payload weighs exactly 18 tons. Seven "Progress" transport ships are available, each of which can deliver a 3-ton load. It is known that these ships altogether can (at least) carry any 35 of the containers at once. Show that in fact they can carry the entire load at once.

4. Points M and P are the midpoints of sides BC and CD of convex quadrilateral $ABCD$. If $AM + AP = a$, show that the area of the region $ABCD$ is less than $a^2/2$.
5. Does the equation $x^2 + y^3 = z^4$ have solutions for prime numbers x , y and z ?
6. Given point E on diameter AC of a circle, construct chord BD passing through E and such that the quadrilateral $ABCD$ has the largest possible area.
7. Several points are located along the shore of a circular lake. Some of these are linked by ferry service. Points A and B are linked by a ferry if and only if the next two adjacent points A' and B' (proceeding clockwise around the lake) are not linked. Show that one can go by ferry from any point to any other point, changing boats not more than twice.
8. A number is written (in base 10 notation) using six distinct non-zero digits, and it is divisible by 37. Show that by permuting the digits one can obtain at least 23 different new numbers, each divisible by 37.

9. Solve simultaneously:

$$\begin{aligned}\sin x + 2 \sin(x + y + z) &= 0 \\ \sin y + 3 \sin(x + y + z) &= 0 \\ \sin z + 4 \sin(x + y + z) &= 0.\end{aligned}$$

10. A set of 1980 vectors are given in the plane, not all of which are collinear. It is known that the sum of any 1979 of these vectors is collinear with the one vector not included in the sum. Prove that the sum of all 1980 given vectors is equal to the vector 0.

*

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As I mentioned in last month's number, it is now time to begin giving the results to some of the challenges to solve remaining problems. The response has been very good. The first of these are to problems from Volume 7 (1981). Several of these will be attributed to Anonymous. A reader sent in a package of solutions but did not indicate his/her name on the sheets. Unfortunately, the envelope has disappeared and short of calling in handwriting experts I don't see how to properly attribute these solutions. I would be delighted if Anonymous would write to reestablish contact!

1. [1981: 15] *36th Moscow Olympiad (1973)*.

A square island consists of several estates. Can one divide these estates into smaller estates in such a way that no new intersection points of boundaries are introduced, and so that the entire map of the island can be coloured with only two colours (estates with a common boundary being coloured differently)?

Solution by Anonymous.

In graph theoretic language, we have a planar map where the estates are the faces, the boundaries are the edges, and the intersection points of boundaries are the vertices. The answer depends on whether multiple edges are permitted. If not, no further subdivision of the map in Figure 1 is possible, and this map is clearly not two-colourable. If multiple edges are allowed, the answer is trivially affirmative. One can simply double up every inland edge. Colour all the crescent-shaped new faces created in one colour, and the remaining faces in the other colour. The result of applying this process to Figure 1 is Figure 2.

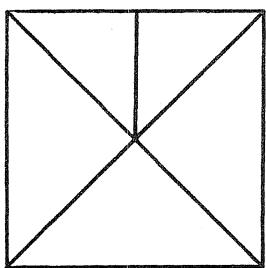


Figure 1

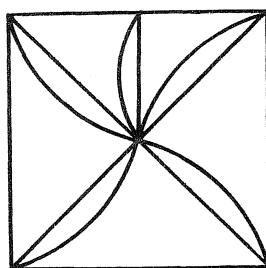


Figure 2

8. [1981: 15] *36th Moscow Olympiad (1973)*.

The faces of a cube are numbered 1,2,...,6 in such a way that the sum of the numbers on opposite faces is always 7. We have a chessboard of 50×50 squares, each square congruent to a face of the cube. The cube "rolls" from the lower left-hand corner of the chessboard to the upper right-hand corner. The "rolling" of the cube consists of a rotation about one of its edges so that one face rests on a square of the chessboard. The cube may roll upward and to the right (never downward or to the left). On each square of the chessboard that was occupied during the trip is written the number of the face of the cube that rested there. Find the largest and the smallest sum that these numbers may have.

Solution by George Evangelopoulos, Law student, Athens, Greece, and by Anonymous.

The mean value of the sum of the numbers written on the chessboard is equal to the number $3.5 \times 99 = 346.5$. If during the rolling of the cube we meet twice

the number a , then between these two times we will also meet the number $7 - a$. Thus for each a the numbers of times we will meet the numbers a and $7 - a$ will either be equal or will differ by 1. Consequently, the sum of the written numbers can differ from the mean value of the sum by at most $0.5 + 1.5 + 2.5 = 4.5$ so the largest value for the sum is 351 and the smallest is 342.

9. [1981: 16] *36th Moscow Olympiad (1973).*

On a piece of paper is an inkblot. For each point of the inkblot, we find the greatest and smallest distances from that point to the boundary of the inkblot. Of all the smallest distances we choose the maximum and of all the greatest distances we choose the minimum. If these two chosen numbers are equal, what shape can the inkblot have?

Solution by Anonymous and by George Evangelopoulos, Law student, Athens, Greece.

Let A be a point such that its greatest distance from the boundary of the inkblot is r and that r is minimum among all points. Then the inkblot is enclosed by the circle γ with center A and radius r . Let B be a point such that its smallest distance from the boundary of the inkblot is r where r is maximum among all points, by hypothesis. Suppose A and B are distinct. Then there is a boundary point C of the inkblot such that B lies on the segment AC . Now we have $BC \geq r$ and $AC \leq r$, a contradiction. Hence A and B coincide. If there were a boundary point D inside of γ , then $BD < r$ is a contradiction. Hence the inkblot is the disc enclosed by γ .

13. [1981: 16] *36th Moscow Olympiad (1973).*

The following operation is performed on a 100 digit number: a block of 10 consecutively-placed digits is chosen and the first five are interchanged with the last five (the 1st with the 6th, the 2nd with the 7th,..., the 5th with the 10th). Two 100-digit numbers which are obtained from each other by repeatedly performing this operation will be called *similar*. What is the largest number of 100-digit integers, each consisting of the digits 1 and 2, which can be chosen so that no two of the integers will be similar?

Solution by Anonymous.

A digit in the n th position is said to belong to the i th orbit, $0 \leq i \leq 4$, if $n \equiv i \pmod{5}$. In a 100 digit number, each orbit consists of 20 digits. For numbers consisting of 1's and 2's only, two numbers are said to be *alike* if they have the same number of 1's in each pair of corresponding orbits. Since there are 21 possible

numbers (from 0 to 20) of 1's in each orbit, the maximum number of numbers no two of which are alike is 21^5 . Since the operation allowed does not take any digit out of its orbit, two numbers that are similar must be alike. We claim that, conversely, two numbers that are alike are also similar. Our basic transformation consists of the following sequence of operations:

First	$a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8 \ a_9 \ a_{10} \ a_{11} \dots$
Second	$a_6 \ a_7 \ a_8 \ a_9 \ a_{10} \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_{11} \dots$
Second	$a_6 \ a_2 \ a_3 \ a_4 \ a_5 \ a_{11} \ a_7 \ a_8 \ a_9 \ a_{10} \ a_1 \dots$

Notice this permutes the digits in one orbit only, which allows us to fix up the number one orbit at a time. Notice, too, that in each orbit three consecutive digits may be permuted cyclically. Consider a given arrangement b_1, b_2, \dots, b_{20} of 1's and 2's in an orbit and a permutation $b_1\sigma, b_2\sigma, \dots, b_{20}\sigma$ of them. Either this permutation or $b_1\sigma, b_2\sigma, \dots, b_{18}\sigma, b_{20}\sigma, b_{19}\sigma$ is even and so is produced via successive applications of the basic operations. Now if $b_{19}\sigma$ and $b_{20}\sigma$ are both 1's or both 2's we are done. Thus we may suppose that $b_{19}\sigma = 1$ and $b_{20}\sigma = 2$, while we have produced the reversed arrangement. If $b_{18}\sigma = 1$ the desired orbit ends 112 while we produce 121. If $b_{18}\sigma = 2$ the desired orbit ends 212 while we see 221. In either case a cyclic permutation of these three positions will produce the desired arrangement. It follows that the maximum number of pairwise dissimilar numbers is 21^5 .

15. [1981: 16] *36th Moscow Olympiad (1973).*

On each endpoint of a line segment the number 1 is placed. As a first step, between these two numbers is placed their sum, 2. For each successive step, between every two adjacent numbers we place their sum. (After the second step we have 1, 3, 2, 3, 1; after the third we have 1, 4, 3, 5, 2, 5, 3, 4, 1; and so on.) How many times will the number 1973 appear after a million steps?

Solution by R.K. Guy, The University of Calgary, and by Anonymous.

1. Consecutive numbers after any step are prime to one another, because this is true for the first few steps, and $\gcd(a, b) = 1$ implies $\gcd(a, a + b) = 1 = \gcd(a + b, b)$.

2. Every pair of coprime integers will appear at some stage, for (a, b) appears after $(a - b, b)$ or $(a, b - a)$ has appeared, according as $a > b$ or $a < b$, and these last will have appeared by induction.

3. 1973 appears in the step after (a, b) where $a + b = 1973$ and $\gcd(a, b) = 1$, i.e. whenever $1 \leq a < 1973$, and $\gcd(a, 1973 - a) = 1$, that is $\varphi(1973)$ times, where φ is Euler's totient function, the number of numbers not exceeding 1973 and prime to it. As 1973 is prime, $\varphi(1973) = 1972$, which is the number of times

1973 occurs, provided there have been at least 1972 steps, which is less than one million.

4. Notice that the numbers will all eventually occur, and (as a subsequence) in the same order, even if we don't insert every possible number at each step. For example, if at step n we only insert $n + 1$ (and no larger sums):

$$\begin{array}{cccc} 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 & 3 & 1 \\ 1 & 4 & 3 & 2 & 3 & 4 & 1 \\ 1 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 1 \\ \vdots & & & & & & & & & & & \end{array}$$

Then *all* the occurrences, $\varphi(n + 1)$ of them, of $n + 1$ occur at the same step, and the sequence is that of the denominators of the *Farey series* of order $n + 1$ [Hardy and Wright, *Introduction to Theory of Numbers*, 4th Edition, Oxford Univ. Press, 1959, Ch. 3], which has many interesting properties.

* * *

Now we return to solutions to problems from the April 1988 number of the Corner. Some readers sent in solutions to problems from the AIME, which we shall not reproduce here because of the solutions booklet available from the M.A.A. But thank you for sending them along. The solutions we do give are to problems of the *Tenth Undergraduate Mathematics Competition* of the Atlantic Provinces Council on the Sciences.

1. [1988: 101] *Tenth Atlantic Provinces Mathematics Competition.*

Find all different right-angled triangles with all sides of integral length whose areas equal their perimeters.

Solution by the late J.T. Groenman, Arnhem, The Netherlands.

Let the two legs have lengths x and y and the hypotenuse be z . Then

$$x^2 + y^2 = z^2 \quad \text{and} \quad \frac{xy}{2} = x + y + z.$$

This gives

$$x^2 + y^2 = \left(\frac{xy}{2} - x - y\right)^2 = \frac{x^2y^2}{4} + x^2 + y^2 - x^2y - xy^2 + 2xy.$$

From this

$$xy(xy - 4x - 4y + 8) = x^2y^2 - 4x^2y - 4xy^2 + 8xy = 0.$$

Since we rule out degenerate triangles, $xy - 4x - 4y + 8 = 0$ and so

$$(x - 4)(y - 4) = 8.$$

Supposing $x \geq y$, we have either $x - 4 = 8$, $y - 4 = 1$ and so $x = 12$, $y = 5$, $z = 13$, or $x - 4 = 4$, $y - 4 = 2$ and so $x = 8$, $y = 6$, $z = 10$. Therefore the only triangles with the given properties are the 5, 12, 13 and 6, 8, 10 right triangles.

2. [1988: 101] *Tenth Atlantic Provinces Mathematics Competition.*

Sketch the graph and find the measure of the area bounded by the relation

$$|x - 60| + |y| = |x/4|.$$

Solution by the late J.T. Groenman, Arnhem, The Netherlands.

Note

$$|x/4| - |x - 60| = |y| \geq 0.$$

If $x \geq 60$ we have

$$\frac{x}{4} - (x - 60) = 60 - \frac{3}{4}x \geq 0$$

giving $x \leq 80$ and $y = \pm(60 - 3x/4)$. If

$0 \leq x \leq 60$ we have

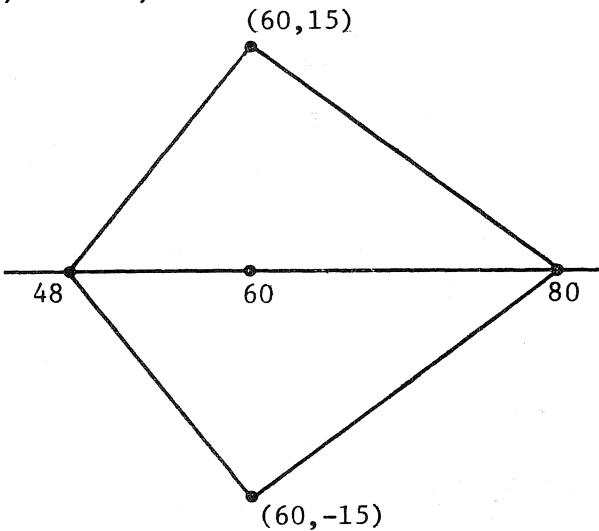
$$\frac{x}{4} - (60 - x) = \frac{5}{4}x - 60 \geq 0,$$

and $x \geq 48$, $y = \pm(5x/4 - 60)$. Finally if

$x \leq 0$ we have

$$-\frac{x}{4} - (60 - x) = \frac{3}{4}x - 60 \geq 0$$

giving $x \geq 80$, which is impossible. The area is $(30 \cdot 32)/2 = 480$.



4. [1988: 102] *Tenth Atlantic Provinces Mathematics Competition.*

Given a function f and a constant $c \neq 0$ such that

- (i) $f(x)$ is even,
- (ii) $g(x) = f(x - c)$ is odd,

prove that f is periodic and determine its period.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

Let x be an arbitrary element of the domain of f . Then

$$\begin{aligned}f(x - 4c) &= g(x - 3c) = -g(3c - x) && [\text{because } g(x) \text{ is odd}] \\&= -f(2c - x) = -f(x - 2c) && [\text{since } f(x) \text{ is even}] \\&= -g(x - c) = g(c - x) = f(-x) = f(x).\end{aligned}$$

It follows that f is periodic with a period of $4|c|$.

The prescribed conditions do *not* guarantee that f has a smallest positive period. When f is the zero constant function, g is also the zero constant function. The zero constant function is both odd and even. Each positive real is a period, so

there is no smallest positive period.

[Editor's note: this partially answers *Crux* 1497 [1989: 298].]

5. [1988: 102] *Tenth Atlantic Provinces Mathematics Competition.*

Show that

$$\sqrt{3} \leq \exp \left[\int_{\pi/6}^{\pi/2} \frac{\sin x}{x} dx \right] \leq 3 .$$

Solution by Robert E. Shafer, Berkeley, California.

Note

$$\int_{\pi/6}^{\pi/2} \frac{\sin x}{x} dx = \int_{\pi/6}^{\pi/2} \frac{4}{x} \cos \frac{x}{2} \cos \frac{x}{4} \sin \frac{x}{4} dx .$$

Now $f(t) = (\sin t)/t$ has

$$f'(t) = \frac{t \cos t - \sin t}{t^2}$$

and so $f'(t) < 0$ on $(0, \pi/2]$, since $g(t) = \tan t - t$ has $g'(t) = \sec^2 t - 1 \geq 0$ giving a minimum of 0 at $t = 0$. Thus

$$\frac{\sin(x/4)}{x/4}$$

is decreasing on $[\pi/6, \pi/2]$ giving that

$$\frac{\sin(\pi/24)}{\pi/24} \int_{\pi/6}^{\pi/2} \cos \frac{x}{2} \cos \frac{x}{4} dx \quad \text{and} \quad \frac{\sin(\pi/8)}{\pi/8} \int_{\pi/6}^{\pi/2} \cos \frac{x}{2} \cos \frac{x}{4} dx$$

are respectively an upper bound and a lower bound to

$$\int_{\pi/6}^{\pi/2} \frac{\sin x}{x} dx .$$

But

$$\begin{aligned} \int_{\pi/6}^{\pi/2} \cos \frac{x}{2} \cos \frac{x}{4} dx &= \frac{1}{2} \int_{\pi/6}^{\pi/2} \left(\cos \frac{x}{4} + \cos \frac{3x}{4} \right) dx \\ &= \frac{2}{3} \sin \frac{3\pi}{8} + \frac{4}{3} \sin \frac{\pi}{8} - 2 \sin \frac{\pi}{24} . \end{aligned}$$

Thus, the lower bound is

$$L = \frac{8}{\pi} \left[\frac{2}{3} - \frac{5}{6\sqrt{2}} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2\sqrt{2}} \right] = \frac{8}{\pi} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right) \left(\frac{1 + \sqrt{2} + \sqrt{3}}{\sqrt{2}} \right)$$

while the upper bound is

$$U = \frac{24}{\pi} \left[-\frac{5}{6} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{2\sqrt{3}}{3\sqrt{2}} \right] = \frac{24}{\pi} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right) .$$

(Not too pretty, perhaps, but reasonably precise, as the estimates are within 3% of

each other.) Now $L \geq 0.84$ and $e^{0.84} > 2.3 > \sqrt{3}$ while $U \leq 0.87$ and $e^{0.87} < 2.4 < 3$. This gives the result.

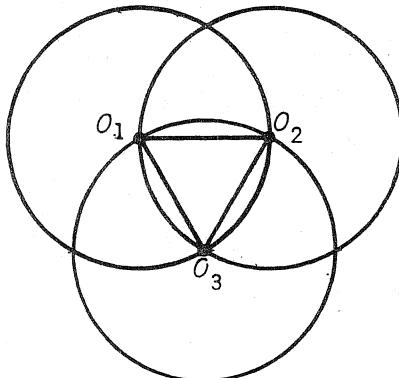
6. [1988: 102] *Tenth Atlantic Provinces Mathematics Competition.*

Three equal circles each pass through the centres of the other two. What is the area of their common intersection?

Solution by the late J.T. Groenman, Arnhem, The Netherlands.

Let the centres of the circles be O_1 , O_2 , O_3 and let the radius be r . Let Δ be the area of the triangle $O_1O_2O_3$ and let S be the area of the sector of the circle with centre O_1 subtended by chord O_2O_3 . Then the required area is

$$\begin{aligned}\Delta + 3(S - \Delta) &= 3S - 2\Delta \\ &= 3 \cdot \frac{1}{6}\pi r^2 - 2 \cdot \frac{1}{4}r^2\sqrt{3} \\ &= \frac{1}{2}r^2(\pi - \sqrt{3}) .\end{aligned}$$



7. [1988: 102] *Tenth Atlantic Provinces Mathematics Competition.*

Prove that

$$\frac{(1987^2)!}{(1987!)^{1988}}$$

is an integer.

Comment by Bob Prielipp, University of Wisconsin-Oshkosh.

The following more general result is known:

$$\frac{(ab)!}{a!(b!)^a}$$

is an integer where a and b are non-negative integers. [For a proof of this see p. 103 of Niven and Zuckerman, *An Introduction to the Theory of Numbers* (4th Edition).]

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This exhausts the solutions sent in to problems from the April 1988 column.

Don't forget to send me the contests from around the world as well as your solutions to problems posed!

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MINI-REVIEWS

by
ANDY LIU

OXFORD UNIVERSITY PRESS SERIES ON RECREATIONS IN MATHEMATICS

This hardcover series is edited by David Singmaster of the Polytechnic of the South Bank in London. He is well-known for his writing on recreational mathematics, and is the leading expert on the history of the subject. The four volumes published so far are new, but the series may contain translations and reprints of classic works.

Volume 1. *Mathematical Byways in Ayling, Beeling and Ceiling*, H. ApSimon, 1984. (97 pp.)

This book contains eleven chapters, each built around a central problem, with solutions and generalizations. Each problem is in the form of an event in the three villages named in the title, though there are occasional wanderings off to Dealing and beyond. The titles of the problems are ladder-box, meta-ladder-box, complete quadrilateral, bowling (as in cricket) averages, centre-point, counting sheep, transport, alley ladder, counterfeit coins, wrapping a parcel and sheep-dog trials.

Volume 2. *The Ins and Outs of Peg Solitaire*, J.D. Beasley, 1985. (275 pp.)

This book deals mainly with peg solitaire on the standard "English" board with 33 holes, each of which can hold one man. Each move consists of a jump by one man over an adjacent one into an empty hole, or a continuous sequence of such jumps. The normal objective is to convert a starting configuration into a target position with fewer men (as those jumped over are removed). The book presents a balanced treatment of the mathematical ideas behind the puzzle as well as the actual techniques in solving it.

Volume 3. *Rubik's Cube Compendium*, E. Rubik, T. Varga, G. Kéri, G. Marx and T. Vekerdy, 1986. (225 pp.)

This book consists of six articles, and an informative update by David Singmaster. The lead article is written by the inventor of this mathematical phenomenon, giving some insight into how the idea was originally conceived. The other authors are Rubik's compatriots, and their articles deal with the mathematics and the techniques of "cubing". The book has many brightly coloured illustrations.

Volume 4. *Sliding-Piece Puzzles*, L.E. Hordern, 1986. (249 pp.)

The classic example of a sliding-piece puzzle is the 14-15 puzzle of Sam Loyd. This and 271 other puzzles are catalogued by class in this book. In almost all cases, there is enough information for home-made copies to be constructed. There is also a colourful history of the subject.

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P R O B L E M S

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1990, although solutions received after that date will also be considered until the time when a solution is published.

1511. Proposed by J.T. Groenman, Arnhem, The Netherlands.

Evaluate

$$\lim_{n \rightarrow \infty} \prod_{k=3}^n \left(1 - \tan^4 \frac{\pi}{2^k}\right).$$

1512^{*}. Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Given $r > 0$, determine a constant $C = C(r)$ such that

$$(1 + z)^r (1 + z^r) \leq C(1 + z^2)^r$$

for all $z > 0$.

1513. Proposed by M.S. Klamkin, University of Alberta.

(a) A planar centrosymmetric polygon is inscribed in a strictly convex planar centrosymmetric region R . Prove that the two centers coincide.

(b) Do part (a) if the polygon is circumscribed about R .

(c) Do (a) and (b) still hold if the polygon and region are n -dimensional for $n > 2$?

1514. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle with sides a, b, c and let P be a point in the same plane. Put $AP = R_1, BP = R_2, CP = R_3$. It is well known that there is a triangle with sides aR_1, bR_2, cR_3 . Find the locus of P so that the area of this triangle is a given constant.

1515. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

We are given a finite collection of segments in the plane, of total length 1. Prove that there exists a straight line l such that the sum of lengths of projections of the given segments to line l is less than $2/\pi$.

1516. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle in which $AB = AC$ and $\angle A < 90^\circ$. Let D be any point on segment BC . Draw CE parallel to AD meeting AB produced in E . Prove that $CE > 2CD$.

1517^{*}. *Proposed by Bill Sands, University of Calgary.*

[This problem came up in a combinatorics course, and is quite likely already known. What is wanted is a nice answer with a nice proof. A reference would also be welcome.]

Imagine you are standing at a point on the edge of a half-plane extending infinitely far north, east, and west (say, on the Canada-USA border near Estevan, Saskatchewan). How many walks of n steps can you make, if each step is 1 metre either north, east, west, or south, and you never step off the half-plane? For example, there are three such walks of length 1 and ten of length 2.

1518. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

O and H are respectively the circumcenter and orthocenter of a triangle ABC in which $\angle A \neq 90^\circ$. Characterize triangles ABC for which ΔAOH is isosceles. Which of these triangles ABC have integer sides?

1519. *Proposed by P. Penning, Delft, The Netherlands.*

Find all prime numbers which, written in the number system with base b , contain each digit $0, 1, \dots, b-1$ exactly once (a leading zero is allowed).

1520. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

A point is said to be *inside* a parabola if it is on the same side of the parabola as the focus. Given a finite number of parabolas in the plane, must there be some point of the plane that is not inside any of the parabolas?

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1393. [1988: 302] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let $A_1A_2A_3$ be a triangle with incenter I , excenters I_1, I_2, I_3 , and median point G . Let H_1 be the orthocenter of $\triangle I_1A_2A_3$, and define H_2, H_3 analogously. Prove that A_1H_1, A_2H_2, A_3H_3 are concurrent at a point collinear with G and I .

Solution by P. Penning, Delft, The Netherlands.

We note that A_3 is located on the straight line I_1I_2 , with A_3 the pedal point of I , since both I_2A_3 and I_1A_3 are perpendicular to IA_3 (outer bisectrix perpendicular to inner bisectrix). So $\triangle A_1A_2A_3$ is the pedal triangle of I in $\triangle I_1I_2I_3$. In fact, for any triangle $I_1I_2I_3$, an arbitrary point I inside this triangle, and $A_1A_2A_3$ its pedal triangle, the property given in this problem holds.

Define as origin I , and the vectors

$$\mathbf{A}_1 = \overrightarrow{IA_1}, \text{ etc.}$$

Now

$$\mathbf{H}_1 = \mathbf{A}_2 + \mathbf{A}_3, \quad \mathbf{H}_2 = \mathbf{A}_3 + \mathbf{A}_1,$$

$$\mathbf{H}_3 = \mathbf{A}_1 + \mathbf{A}_2,$$

because the quadrangles $IA_2H_1A_3$, etc. are all three parallelograms. Since

$$\frac{\mathbf{A}_1 + \mathbf{H}_1}{2} = \frac{\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3}{2}, \text{ etc.}$$

it is now easy to see that the point S such that

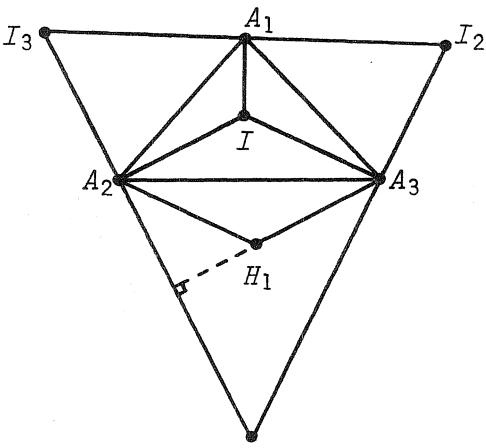
$$\mathbf{S} = \frac{\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3}{2}$$

is the midpoint of all three straight lines A_1H_1, A_2H_2, A_3H_3 . Since the median point G is located at $(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3)/3$ it follows that S lies on the line through I and G with

$$\overrightarrow{IG} = 2\overrightarrow{GS}.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSION LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; L.J. HUT, Groningen, The Netherlands; T. SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; JOSE YUSTY PITA, Madrid, Spain; and the proposer.

The above generalization was also proved by Hut.



1394. [1988: 302] *Proposed by Murray S. Klamkin, University of Alberta.*

If $x, y, z > 0$, prove that

$$\sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} + \sqrt{x^2 + xy + y^2} \geq 3\sqrt{yz + zx + xy} .$$

I. *Solution by Jack Garfunkel, Flushing, New York.*

Note that

$$\sqrt{3}(x + y + z) \geq 3\sqrt{xy + yz + zx}$$

holds, since it is known that

$$(x + y + z)^2 \geq 3(xy + yz + zx) .$$

Thus we will prove the stronger inequality

$$\sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} + \sqrt{x^2 + xy + y^2} \geq \sqrt{3}(x + y + z) . \quad (1)$$

Now

$$\sqrt{y^2 + yz + z^2} \geq \frac{\sqrt{3}}{2}(y + z) \quad (2)$$

holds, since it is equivalent to

$$4y^2 + 4yz + 4z^2 \geq 3y^2 + 6yz + 3z^2$$

or

$$(y - z)^2 \geq 0 .$$

Similarly

$$\sqrt{z^2 + zx + x^2} \geq \frac{\sqrt{3}}{2}(z + x) \quad \text{and} \quad \sqrt{x^2 + xy + y^2} \geq \frac{\sqrt{3}}{2}(x + y) ,$$

which with (2) proves (1).

II. *Solution by Sa'ar Hersonsky, student, Technion, Haifa, Israel.*

We show that

$$\prod (x^2 + xy + y^2) \geq (xy + yz + zx)^3 , \quad (3)$$

from which the given inequality follows by the AM-GM inequality:

$$\left(\frac{1}{3} \sum \sqrt{x^2 + xy + y^2} \right)^3 \geq \prod \sqrt{x^2 + xy + y^2} \geq (\sqrt{xy + yz + zx})^3 .$$

(The above sum and products are cyclic over x, y, z .) When multiplied out, (3) becomes

$$\begin{aligned} \sum x^4yz + \sum x^4y^2 + 2 \sum x^3y^2z + \sum x^3y^3 + 3x^2y^2z^2 \\ \geq \sum x^3y^3 + 3 \sum x^3y^2z + 6x^2y^2z^2 \end{aligned}$$

(each sum over all symmetric versions of its summand), or

$$\sum x^4yz + \sum x^4y^2 \geq \sum x^3y^2z + 3x^2y^2z^2 .$$

By the AM-GM inequality,

$$\sum x^4yz \geq 3 \sqrt[3]{x^6y^6z^6} = 3x^2y^2z^2,$$

so it is enough to prove

$$\sum x^4y^2 \geq \sum x^3y^2z.$$

But this is just

$$\frac{1}{2} \sum (x^2y - y^2z)^2 \geq 0,$$

which is obvious.

[Editor's note. Upon being told of Hersonsky's solution, Murray Klamkin offered the following very short proof of inequality (3), using Hölder's inequality:

$$\begin{aligned} xy + yz + zx &= (xy)^{1/3}(y^2)^{1/3}(x^2)^{1/3} + (y^2)^{1/3}(yz)^{1/3}(z^2)^{1/3} + (x^2)^{1/3}(z^2)^{1/3}(zx)^{1/3} \\ &\leq (xy + y^2 + x^2)^{1/3}(y^2 + yz + z^2)^{1/3}(x^2 + z^2 + zx)^{1/3} \\ &= \prod (x^2 + xy + y^2)^{1/3}, \end{aligned}$$

and (3) follows.]

III. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We show more generally: if $\alpha, \beta, \gamma > 0$ such that $\alpha + \beta + \gamma = 2\pi$, then

$$\sum \sqrt{x^2 - 2xy \cos \gamma + y^2} \geq \sqrt{6\sqrt{3} \left| \sum xy \sin \gamma \right|}. \quad (4)$$

Let P be a point of the plane and consider the triangle ABC generated by $x, y, z, \alpha, \beta, \gamma$ as indicated by the figure. Then

$$AB = \sqrt{x^2 - 2xy \cos \gamma + y^2}, \text{ etc.}$$

By item 4.2 of [1],

$$s \geq \sqrt{3\sqrt{3}F} \quad (5)$$

where s is the semiperimeter and F the area of $\triangle ABC$. Since

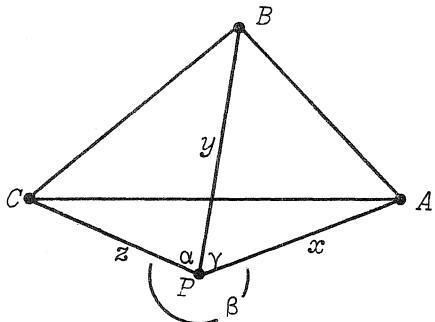
$$F = \frac{1}{2} \left| \sum xy \sin \gamma \right|,$$

inequality (5) reduces to (4). The case $\alpha = \beta = \gamma = 2\pi/3$ yields the stated result.

It should be noted that any triangle inequality $I(a, b, c, F, \dots) \geq 0$ can be transformed into an algebraic inequality $I(x, y, z, \alpha, \beta, \gamma) \geq 0$ via the $x - y - z - \alpha - \beta - \gamma$ transformation as indicated above (i.e.

$$a = \sqrt{y^2 - 2yz \cos \alpha + z^2}, \text{ etc.})$$

Especially,



$$x = a, \quad y = b, \quad z = c, \quad \alpha = \pi - A, \quad \beta = \pi - B, \quad \gamma = \pi - C$$

leads to the new triangle with sides

$$\sqrt{b^2 - 2bc \cos(\pi - A) + c^2} = \sqrt{2b^2 + 2c^2 - a^2} = 2m_a, \text{ etc.}$$

where m_a, m_b, m_c are the medians of ΔABC . Thus the above defined class of transformations contains as a special case the median-duality (cf. [2], pp. 109–112).

We now give three applications of the transformation in case $x, y, z > 0$ and $\alpha = \pi - A, \beta = \pi - B, \gamma = \pi - C$.

(i) The known inequality

$$a^2 + b^2 + c^2 \geq 4F\sqrt{3}$$

([1], item 4.4) becomes

$$\sum x^2 + \sum yz \cos A \geq \sqrt{3} \sum yz \sin A,$$

i.e.

$$\sum x^2 \geq 2 \sum yz \sin\left(A - \frac{\pi}{6}\right).$$

(ii) Putting $x = y = z$ in (4) we get

$$\sum \sqrt{1 + \cos A} \geq \sqrt{3\sqrt{3} \sum \sin A},$$

i.e.

$$\sum \cos \frac{A}{2} \geq \sqrt{\frac{3\sqrt{3}s}{2R}},$$

where s is the semiperimeter and R the circumradius of ΔABC .

(iii) Starting from the Neuberg–Pedoe two-triangle inequality

$$\sum a'^2(-a^2 + b^2 + c^2) \geq 16FF'$$

([1], item 10.8), we get upon setting

$$a'^2 = y^2 + 2yz \cos A + z^2, \text{ etc.}$$

that

$$\sum (y^2 + z^2 + 2yz \cos A)bc \cos A \geq 4F \sum yz \sin A,$$

or, since $2F = bc \sin A$, etc.,

$$\sum x^2a(b \cos C + c \cos B) + 2 \sum yzbc \cos^2 A \geq 2 \sum yzbc \sin^2 A.$$

Collecting terms and noting $b \cos C + c \cos B = a$ and $\sin^2 A - \cos^2 A = -\cos 2A$ we get

$$\sum a^2x^2 \geq -2 \sum yzbc \cos 2A,$$

or equivalently (via $a = 2R \sin A$, etc.)

$$\sum x^2 \sin^2 A \geq -2 \sum yz \sin B \sin C \cos 2A .$$

References:

- [1] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.
- [2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Dordrecht, 1989.

Also solved by SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria; KEE-WAI LAU, Hong Kong; and the proposer.

Bilchev and Velikova, and the proposer, used a method similar to solution III.

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- 1395.** [1988: 302] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given an equilateral triangle ABC , find all points P in the same plane such that $(PA)^2, (PB)^2, (PC)^2$ form a triangle.

Solution by Murray S. Klamkin, University of Alberta.

Let the vertices of the equilateral triangle be given by the points $1, \omega$ and ω^2 in the complex plane, where $\omega (\neq 1)$ is a cube root of unity. It follows by Ptolemy's inequality that PA, PB, PC are sides of a triangle (possibly degenerate) for all P in the plane or out of the plane [e.g. see *Crux* 1214 (1988: 118)]. The conditions in the current problem require that PA, PB, PC are sides of a *non-obtuse* triangle (possibly degenerate, e.g. if $P = A, B$ or C).

If P lies in the domain of $\angle BOC$ (O the origin), then $PA \geq PB$ and $PA \geq PC$, and it suffices to require $(PB)^2 + (PC)^2 \geq (PA)^2$. If P has the complex representation $z = x + iy$, then this condition becomes

$$|z - \omega|^2 + |z - \omega^2|^2 \geq |z - 1|^2 ,$$

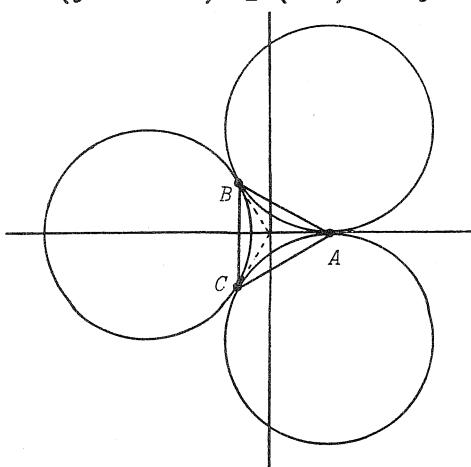
or

$$(x - \cos 120)^2 + (y - \sin 120)^2 + (x - \cos 240)^2 + (y - \sin 240)^2 \geq (x - 1)^2 + y^2 .$$

The latter reduces to

$$(x + 2)^2 + y^2 \geq 3 .$$

Thus, for this case, P must lie on or outside a circle of center $(-2, 0)$ and radius $\sqrt{3}$. This circle is tangent to lines OB and OC at B and C respectively. By symmetry, P must also lie on or outside the two circles obtained by rotating the above circle 120° and 240° about O as shown in the figure.



If we allow P to lie outside the plane it follows easily that P must lie on or outside three spheres whose centers and radii are the same as for the three previous circles. Also, the triangle formed is degenerate only when P lies on the boundary of the three spheres.

For a general triangle ABC of sides a, b, c and any point P in or out of the plane of ABC , it again follows from Ptolemy's inequality that $a \cdot PA, b \cdot PB, c \cdot PC$ are sides of a triangle (possibly degenerate). In order for the latter triangle to be non-obtuse, P will have to lie on or outside three spheres if ABC is acute; on or outside two spheres and on or on one side of a plane if ABC is right-angled; and on or outside two spheres and inside one sphere if ABC is obtuse. To see this let the vertices of ABC be given by $A = (0,0)$, $B = (c,0)$, and $C = (u,v)$ where

$$u^2 + v^2 = b^2 \quad \text{and} \quad (u - c)^2 + v^2 = a^2.$$

If $P = (x,y,z)$, then

$$\begin{aligned} (PA)^2 &= x^2 + y^2 + z^2, \\ (PB)^2 &= (x - c)^2 + y^2 + z^2, \\ (PC)^2 &= (x - u)^2 + (y - v)^2 + z^2, \end{aligned}$$

and our conditions become

$$\begin{aligned} a^2(x^2 + y^2 + z^2) + b^2[(x - c)^2 + y^2 + z^2] - c^2[(x - u)^2 + (y - v)^2 + z^2] &\geq 0, \\ a^2(x^2 + y^2 + z^2) - b^2[(x - c)^2 + y^2 + z^2] + c^2[(x - u)^2 + (y - v)^2 + z^2] &\geq 0, \\ -a^2(x^2 + y^2 + z^2) + b^2[(x - c)^2 + y^2 + z^2] + c^2[(x - u)^2 + (y - v)^2 + z^2] &\geq 0. \end{aligned}$$

The above result follows.

The given problem can be generalized another way by considering an n -dimensional regular simplex whose vertices A_0, A_1, \dots, A_n are the endpoints of $n+1$ concurrent unit vectors $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_n$ and determining all points P such that $(PA_0)^2, (PA_1)^2, \dots, (PA_n)^2$ are sides of an $(n+1)$ -gon. We claim that if $n \geq 3$, any point P will do. Here we must satisfy

$$(P - \mathbf{A}_0)^2 + (P - \mathbf{A}_1)^2 + \dots + (P - \mathbf{A}_n)^2 \geq 2(P - \mathbf{A}_k)^2$$

for $k = 0, 1, \dots, n$. Squaring out and using the fact that $\sum \mathbf{A}_k = 0$, we obtain the following $n+1$ conditions on P :

$$(n-1)(P^2 + 1) + 4P \cdot \mathbf{A}_k \geq 0, \quad k = 0, 1, \dots, n. \quad (1)$$

The case for $n = 2$ reduces to the given problem which was solved above. For $n = 3 + m$ ($m \geq 0$), (1) becomes

$$m(P^2 + 1) + 2(P + \mathbf{A}_k)^2 \geq 0,$$

and this is clearly satisfied for all P . Furthermore, since

$$\left(\sum_{k=1}^n |\mathbf{P} - \mathbf{A}_k| \right)^2 \geq \sum_{k=1}^n |\mathbf{P} - \mathbf{A}_k|^2 \geq |\mathbf{P} - \mathbf{A}_0|^2, \text{ etc.,}$$

it follows that PA_0, PA_1, \dots, PA_n are also sides of an $(n + 1)$ -gon for all $n > 2$ and for all P . The case for $n = 2$ was taken care of previously using Ptolemy's inequality. More generally, let $F(x)$ be any nonnegative, nondecreasing concave function with $F(0) = 0$. Then

$$\sum_{k=1}^n F(|\mathbf{P} - \mathbf{A}_k|^2) \geq F\left(\sum_{k=1}^n |\mathbf{P} - \mathbf{A}_k|^2\right) \geq F(|\mathbf{P} - \mathbf{A}_0|^2), \text{ etc.}$$

As an example, we can take $F(x) = x^r$ for $0 \leq r \leq 1$. It then follows that

$$|\mathbf{P} - \mathbf{A}_k|^{2r}, \quad k = 0, 1, \dots, n$$

are sides of an $(n + 1)$ -gon.

Also solved by JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; P. PENNING, Delft, The Netherlands; and the proposer.

The extension of the problem to an equilateral triangle ABC and a point P in 3-space (or even n -space) was also done by the proposer.

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1396. [1988: 302] *Proposed by Colin Springer, student, Waterloo, Ontario.*

Evaluate

$$\prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n}\right)$$

where n is a positive integer, $n \geq 2$.

I. *Solution by C. Wildhagen, Breda, The Netherlands.*

We begin with the obvious identity

$$z^n - 1 = (z - 1) \prod_{k=1}^{n-1} (z - e^{2k\pi i/n}).$$

Dividing both sides by $z - 1$ and putting $z = 1$ gives

$$n = \prod_{k=1}^{n-1} (1 - e^{2k\pi i/n}). \tag{1}$$

Take the complex conjugate of both sides of (1) to obtain

$$n = \prod_{k=1}^{n-1} (1 - e^{-2k\pi i/n}) . \quad (2)$$

Multiplying (1) and (2) yields

$$n^2 = \prod_{k=1}^{n-1} (2 - e^{2k\pi i/n} - e^{-2k\pi i/n}) = 2^{n-1} \prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n}\right) ,$$

hence

$$\prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n}\right) = \frac{n^2}{2^{n-1}} .$$

II. Solution by Murray S. Klamkin, University of Alberta.

This is a special case of a well known result given in good trigonometry books (see [1]). "The equation

$$x^{2n} - 2x^n \cos n\alpha + 1 = 0$$

is a quadratic in x^n , with roots $\cos n\alpha \pm i \sin n\alpha$; thus the $2n$ values of x are

$$\cos \left(\alpha + \frac{2k\pi}{n}\right) \pm i \sin \left(\alpha + \frac{2k\pi}{n}\right) ,$$

for $k = 0, 1, \dots, n-1$." This leads to the factorization

$$x^{2n} - 2x^n \cos n\alpha + 1 = \prod_{k=0}^{n-1} \left(x^2 - 2x \cos \left(\alpha + \frac{2k\pi}{n}\right) + 1\right) .$$

Many results can be obtained from this identity by specializing x and α . In particular, let $\alpha = 0$ to give

$$\frac{(x^n - 1)^2}{(x - 1)^2} = \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{2k\pi}{n} + 1\right) .$$

Now let $x \rightarrow 1$ to give

$$n^2 = 2^{n-1} \prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n}\right) = 2^{2n-2} \prod_{k=1}^{n-1} \sin^2 \left(\frac{k\pi}{n}\right) .$$

Reference:

- [1] C.V. Durell, A. Robson, *Advanced Trigonometry*, G. Bell, London, 1953, p. 227.

III. Solution by Stanley Rabinowitz, Westford, Massachusetts.

For $n \geq 2$, let

$$P = \prod_{k=1}^{n-1} \left(1 - \cos \frac{2k\pi}{n}\right).$$

Applying the substitution $1 - \cos 2x = 2 \sin^2 x$ yields $P = 2^{n-1} Q^2$ where

$$Q = \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}, \quad (3)$$

a result which can be found in Landau ([6], p. 211). Thus

$$P = \frac{n^2}{2^{n-1}}.$$

This result can also be obtained from the identity

$$\prod_{k=1}^{n-1} \left(1 - \cos \left(\theta + \frac{2k\pi}{n}\right)\right) = \frac{1}{2^{n-1}} \cdot \frac{1 - \cos n\theta}{1 - \cos \theta}$$

(given in [2], p. 119) by taking the limit as θ approaches 0 (using l'Hôpital's Rule twice). A related result can be obtained from the identity

$$\prod_{k=1}^{n-1} \left(\cos \theta - \cos \frac{k\pi}{n}\right) = \frac{1}{2^{n-1}} \cdot \frac{\sin n\theta}{\sin \theta}$$

(given in [1], p. 167, problem 11) by taking the limit as θ approaches 0 (using one application of l'Hôpital's Rule). This gives

$$\prod_{k=1}^{n-1} \left(1 - \cos \frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

We mention a few related results. Baica has stated in [3] that, for $n \geq 3$,

$$\prod_{k=1}^{n-2} \left(1 - \cos \frac{2k\pi}{n}\right)^{n-1-k} = \frac{n^{n-2}}{2^{(n-1)(n-2)/2}}.$$

His remark "Any proof avoiding cyclotomic fields could be very difficult, if not insoluble" should represent a challenge to *Crux* readers. Baica also proved (in [4]) that for $n \geq 2$,

$$\prod_{k=1}^{n-1} \left(2 \sin \frac{k\pi}{n}\right)^{k-1} = n^{(n-2)/2}.$$

Evans [5] has stated that for odd $n \geq 3$,

$$\prod_{k=1}^{n-1} \left(2 \cos \frac{k\pi}{n}\right)^{k-1} = (-1)^{\binom{n^2-1}{2}}$$

and

$$\prod_{k=1}^{n-1} \left(2 \cos \frac{k\pi}{n}\right) = (-1)^{\binom{n-1}{2}}.$$

References:

- [1] J. Heading, *Mathematics Problem Book I: Algebra and Complex Numbers*, Pergamon Press, New York, 1964.
- [2] E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover Publications Inc., New York, 1957.
- [3] Malvina Baica, Trigonometric identities, *Abstracts of the A.M.S.* 7 (1986) 249.
- [4] Malvina Baica, Trigonometric identities, *International Journal of Mathematics and Mathematical Science* 9 (1986) 705-714.
- [5] R.J. Evans, Review of [4], *Math. Reviews* 88c #11046.
- [6] E. Landau, *Elementary Number Theory*, Second Edition, Chelsea, New York, 1966.

Also solved by SEUNG-JIN BANG, Seoul, Republic of Korea; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; SA'AR HERSONSKY, student, Technion, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; DAVID G. POOLE, Trent University, Peterborough, Ontario; M.A. SELBY, University of Windsor; EDWARD T.H. WANG, Wilfrid Laurier University; KENNETH M. WILKE, Topeka, Kansas; KENNETH S. WILLIAMS, Carleton University; and the proposer. One other reader gave the correct answer without proof.

Several readers solved the problem as in solution III by using identity (3). Wang, for instance, referred to exercise 52, p. 25 of Complex Variables in the Schaum's Outline Series, while Williams gave the reference p. 173 of T. Nagell's Introduction to Number Theory. Other readers used the identity

$$\prod_{k=1}^{m-1} \sin \frac{k\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}};$$

among the references cited was problem 143, p. 24 of Yaglom and Yaglom, Challenging Mathematical Problems with Elementary Solutions Vol. II, Dover, New York, 1967.

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1397. [1988: 302] *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Find the equation of the circle passing through the points other than the origin which are common to the two conics

$$x^2 + 6xy + 10x - 2y = 0 ,$$
$$x^2 + 3y^2 - 7x + 5y = 0 .$$

I. *Solution by Clayton W. Dodge, University of Maine, Orono.*

The desired circle has the equation

$$39x^2 + 39y^2 - 267x - 59y = 144.$$

To obtain this circle we derive an equation that has a factor of x and that passes through the four points of intersection of the two conics. Then we divide out the factor of x , removing the point at the origin from this equation, then juggle this equation with the original two equations to get an equation of a circle that passes through the remaining three points common to the two equations. To this end, solve the first given equation for $2y$, double the second equation and substitute selectively for $2y$, obtaining

$$2x^2 + 3y(x^2 + 6xy + 10x) - 14x + 5(x^2 + 6xy + 10x) = 0 ,$$

which reduces to

$$x(7x + 3xy + 18y^2 + 60y + 36) = 0 .$$

Divide by x and then double the resulting equation to get

$$14x + 6xy + 36y^2 + 120y + 72 = 0 .$$

Next subtract the first given equation to eliminate the xy term:

$$4x - x^2 + 36y^2 + 122y + 72 = 0 .$$

Finally double this last equation and subtract the result from 37 times the second given equation to get equal coefficients for the x^2 and y^2 terms.

II. *Solution by the proposer.*

We write the equations of the given conics as follows:

$$x(x + 10) + y(6x - 2) = 0 ,$$

$$x(x - 7) + y(3y + 5) = 0 .$$

The common points other than O are therefore on the conic represented by

$$(x + 10)(3y + 5) - (6x - 2)(x - 7) = 0 ,$$

or

$$6x^2 - 3xy - 49x - 30y - 36 = 0 .$$

The circle we are looking for has therefore an equation of the form

$$k(x^2 + 6xy + 10x - 2y) + m(x^2 + 3y^2 - 7x + 5y) + n(6x^2 - 3xy - 49x - 30y - 36) = 0 .$$

This is actually the equation of a circle if and only if

$$6k - 3n = 0 \quad \text{and} \quad k + m + 6n = 3m,$$

i.e.

$$n = 2k \quad \text{and} \quad 13k = 2m.$$

Hence $k:m:n = 2:13:4$. The equation of the circle is therefore

$$39x^2 + 39y^2 - 267x - 59y - 144 = 0.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong, and P. PENNING, Delft, The Netherlands. Two incorrect solutions were sent in.

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1398. [1988: 302] Proposed by Ravi Vakil, student, University of Toronto.

Let p be an odd prime. Show that there are at most $(p + 1)/4$ consecutive quadratic residues mod p . For which p is this bound attained?

Solution by Kee-Wai Lau, Hong Kong.

We need the following well-known results.

I) There are

$$\frac{p - 4 - (-1)^{(p-1)/2}}{4}$$

numbers a such that both a and $a + 1$ are quadratic residues mod p .

II) There are

$$\frac{p - 2 + (-1)^{(p-1)/2}}{4}$$

numbers b such that both b and $b + 1$ are quadratic nonresidues mod p .

III) For $p = 4m + 3$, g is a quadratic residue mod p if and only if $p - g$ is a quadratic nonresidue mod p .

IV) i) The product of two quadratic residues or nonresidues is a residue mod p .

ii) The product of a quadratic residue and a quadratic nonresidue is a quadratic nonresidue mod p .

Hence by I there are at most $(p - 1)/4$ consecutive quadratic residues if $p = 4m + 1$, and at most $(p + 1)/4$ consecutive quadratic residues if $p = 4m + 3$.

We now proceed to find those primes $p = 4m + 3$ for which the bound $(p + 1)/4$ is attained. By direct calculations we find that the bound is attained for $p = 3, 7$ and 11 , but not attained for $p = 19$. From now on we assume that $p > 19$, i.e. $m \geq 5$.

Suppose that the bound is attained for some such $p = 4m + 3$. Let $S = \{1, 2, \dots, 2m + 1\}$, $T = \{2m + 2, 2m + 3, \dots, 4m + 2\}$, R the set of $(p + 1)/4 = m + 1$ consecutive quadratic residues and (by III) N the corresponding set of $m + 1$ consecutive quadratic nonresidues. By I and II there are no other consecutive residues or nonresidues. Hence by III either (a) $R \subset S$, $N \subset T$ or (b) $N \subset S$, $R \subset T$. Denote residues by r and nonresidues by n .

Case (a). If the pattern

$$\begin{array}{ccccccccc} 1 & 2 & 3 & \cdots & m+1 & m+2 & m+3 & m+4 & m+5 & m+6 & \cdots \\ r & r & r & \cdots & r & n & r & n & r & n \end{array}$$

holds, we deduce from IV (i) (since $m > 2$) that $m + 2$, $m + 4$ and $m + 6$ are prime numbers, which is impossible. Otherwise, the pattern

$$\begin{array}{ccccccccc} 1 & 2 & \cdots & t-2 & t-1 & t & t+1 & t+2 & t+3 & \cdots & t+m & t+m+1 & \cdots \\ r & n & & r & n & r & r & r & r & & r & n \end{array}$$

holds for some odd t , $3 \leq t \leq m + 1$. Let $s \geq 0$ be the greatest integer such that $s(s + 1) < t$. Then

$$t \leq (s + 1)(s + 2) \leq t + m ,$$

since

$$\begin{aligned} (s + 1)(s + 2) &= s(s + 1) + 2(s + 1) < t + 2(s + 1) \\ &\leq \begin{cases} t + s(s + 1) \leq 2t - 1 \leq t + m & \text{if } s \geq 2 , \\ t + 4 \leq t + m & \text{if } s = 0 \text{ or } 1 . \end{cases} \end{aligned}$$

Also

$$s + 2 \leq \begin{cases} t & \text{if } s = 0 \text{ or } 1 , \\ s(s + 1) < t & \text{if } s \geq 2 . \end{cases}$$

Since $(s + 1)(s + 2)$ is a quadratic nonresidue by IV (ii) we arrive at a contradiction.

Case (b). We have the pattern

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & \cdots & t-1 & t & t+1 & \cdots & t+m & \cdots \\ r & n & r & n & & r & n & n & & n \end{array}$$

for some even t , $2 \leq t \leq m + 1$. If $t = 2$, then $6 = 2 \cdot 3$ is a residue by IV (i), which is a contradiction since $m \geq 4$. If $t \geq 4$, then $t + 1$, $t + 3$ and $t + 5$ are odd and (since $m \geq 5$) nonresidues. By IV (i) they must be prime numbers, which is a contradiction.

Thus equality is attained only for $p = 3, 7$ and 11 .

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and by PAT SURRY, student, University of Western Ontario, and the proposer. The first part only was solved by KENNETH S. WILLIAMS, Carleton University.

The fact I) in Lau's solution was mentioned by both Janous and Williams, the latter giving the reference I.M. Vinogradov, Elements of Number Theory, p. 97.

It is known (D.A. Burgess, The distribution of quadratic residues and non-residues, Mathematika 4(1957) 106–112) that the maximum number $R(p)$ of consecutive quadratic residues modulo the prime p satisfies

$$R(p) < p^{1/4 + \delta} \text{ times a constant}$$

for any $\delta > 0$. For more information on the problem of estimating $R(p)$, see p. 136 of R.K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, 1981.

* * *

1399. [1988: 302] Proposed by Sydney Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Prove that

$$\sigma(n!) \leq \frac{(n+1)!}{2}$$

for all natural numbers n and determine all cases when equality holds. (Here $\sigma(k)$ denotes the sum of all positive divisors of k .)

Solution by Robert E. Shafer, Berkeley, California.

It can be shown directly that the inequality holds for $n \leq 7$, with equality for $n = 1, 2, 3, 4, 5$. Thus assume $n \geq 8$. Let

$$n! = 2^{a_1} 3^{a_2} 5^{a_3} \cdots p,$$

where p is the largest prime less than or equal to n . Then

$$\begin{aligned} \sigma(n!) &= \frac{2^{a_1+1}-1}{1} \cdot \frac{3^{a_2+1}-1}{2} \cdot \cdots \cdot \frac{p^2-1}{p-1} \\ &= n! \left(\frac{2^{a_1+1}-1}{2^{a_1}} \cdot \frac{3^{a_2+1}-1}{2 \cdot 3^{a_2}} \cdot \cdots \cdot \frac{p^2-1}{(p-1)p} \right) \\ &< n! \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{p}{p-1} \right) \\ &\leq n! \left(\frac{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n}}{\frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{9}{8}} \right) \\ &= \frac{(n+1)!}{72/35} < \frac{(n+1)!}{2}. \end{aligned} \tag{1}$$

Thus the equality holds for $1 \leq n \leq 5$ and for no other values of n .

The reader may recall the problem of estimating the product

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right)$$

in the computation of

$$\sum_{2 \leq p \leq n} \frac{1}{p}.$$

Also solved by SA'AR HERSONSKY, student, Technion, Haifa, Israel; RICHARD I. HESS, Rancho Palos Verdes, California; KEE-WAI LAU, Hong Kong; C. WILDHAGEN, Breda, The Netherlands; and the proposers. Two other readers sent in short induction arguments that the editor found suspect.

The proposers' proof was similar to Shafer's (but not quite as elegant). They remark that a much sharper upper bound for $\sigma(n!)$ is known. Namely, there exists a constant $c > 0$ such that for all $n \geq 2$,

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right) > \frac{c}{\log n}$$

(e.g. Theorem 9.1A.1, p. 324 of H.N. Shapiro, Introduction to the Theory of Numbers), and hence (from (1) in Shafer's proof)

$$\sigma(n!) < \frac{n! \log n}{c}.$$

They say, however, that the proof does not seem to be easy. The problem was motivated by the easy lower bound

$$\sigma(n!) \geq n! \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

(e.g. problem 10(b) in Chapter 6 of D.M. Burton, Elementary Number Theory).

* * *

1400. [1988: 302; 1989: 177] Proposed by Robert E. Shafer, Berkeley, California.

In a recent issue of the *American Mathematical Monthly* (June-July 1988, page 551), G. Klambauer showed that if $x^s e^{-x} = y^s e^{-y}$ ($x, y, s > 0$, $x \neq y$) then $x + y > 2s$. Show that if $x^s e^{-x} = y^s e^{-y}$ where $x \neq y$ and $x, y, s > 0$ then $xy(x + y) < 2s^3$.

I. *Solution by the proposer (slightly modified by the editor).*

Suppose on the contrary that

$$xy(x + y) = 2\lambda s^2$$

where $\lambda \geq 1$. Assuming without loss of generality that $x > y$, we put

$$\begin{aligned} yx^2 &= \lambda s^3(1 + t) \\ xy^2 &= \lambda s^3(1 - t) \end{aligned}$$

where $0 < t < 1$. Then

$$\left(\frac{x}{y}\right)^s = \left(\frac{x^2y}{y^2x}\right)^s = \left(\frac{1+t}{1-t}\right)^s = e^{s\log[(1+t)/(1-t)]},$$

while

$$xy = \sqrt[3]{yx^2 \cdot xy^2} = s^2 \cdot \sqrt[3]{\lambda^2(1 - t^2)}.$$

Thus

$$x - y = \frac{yx^2 - xy^2}{xy} = \frac{2\lambda s^3 t}{s^2 \cdot \sqrt[3]{\lambda^2(1 - t^2)}} = 2st \cdot \sqrt[3]{\frac{\lambda}{1 - t^2}}.$$

Now recall from the proof of *Crux* 1247 [1988: 191] that for $0 < t < 1$,

$$\frac{2t}{(1 - t^2)^{1/3}} > \log\left(\frac{1+t}{1-t}\right),$$

from which we find (since $\lambda \geq 1$)

$$\begin{aligned} \left(\frac{x}{y}\right)^s &= e^{s\log[(1+t)/(1-t)]} \\ &< e^{2st/(1-t^2)^{1/3}} \\ &< e^{2st[\lambda/(1-t^2)]^{1/3}} \\ &= e^{x-y}, \end{aligned}$$

or $x^s e^{-x} < y^s e^{-y}$, a contradiction.

II. Solution by Murray S. Klamkin, University of Alberta.

Note that if $x^s e^{-x} = y^s e^{-y}$ then

$$s = \frac{x - y}{\ln x - \ln y},$$

the so-called logarithmic mean of x and y , and that the Klambauer result alluded to goes back at least to Ostle and Terwilliger [3]. A stronger inequality

$$s < \left(\frac{\sqrt[3]{x} + \sqrt[3]{y}}{2}\right)^3$$

is due to Lin [2]. For an inequality going the other way, Carlson [1] has shown that

$$\frac{\sqrt[4]{xy}(\sqrt{x} + \sqrt{y})}{2} < s.$$

Unfortunately, the latter inequality is not comparable with the proposed problem here.

The log mean of x and y is not defined for $x = y$ but its definition can be extended to be x for this case. Then this function would be continuous for all $x, y > 0$.

The proposed inequality reduces to the form

$$2|x - y|^3 \geq xy(x + y)|\ln^3(x/y)|, \quad x, y > 0,$$

with equality if and only if $x = y$. We can assume without loss of generality that $x \geq y$ and replace x/y by e^t where $t \geq 0$ to give the equivalent inequality

$$F(t) \equiv 2(e^t - 1)^3 - t^3 e^t (e^t + 1) \geq 0. \quad (1)$$

Since $F(0) = 0$, one standard method of establishing inequalities such as (1) is to show that $F'(t) \geq 0$. Since the latter is not obvious and $F'(0) = 0$, we try to show $F''(t) \geq 0$. If necessary, we continue differentiating (it turns out we have to differentiate quite a number of times). Along the way we can discard any non-negative factors for simplification. Here,

$$F'(t) = 6(e^t - 1)^2 e^t - 3t^2(e^t + e^{2t}) - t^3(e^t + 2e^{2t}).$$

We can discard the factor e^t and then consider

$$G(t) \equiv 6(e^t - 1)^2 - 3t^2(1 + e^t) - t^3(1 + 2e^t).$$

Here $G(0) = 0$ and

$$H(t) \equiv G'(t)e^{-t} = 12(e^t - 1) - [6t + 9t^2 + 2t^3 + (6t + 3t^2)e^{-t}].$$

Here $H(0) = 0$ and

$$H'(t)/3 = 4e^t - [2 + 6t + 2t^2 + (2 - t^2)e^{-t}].$$

Here $H'(0) = 0$ and

$$H''(t)/3 = 4e^t - [6 + 4t + (t^2 - 2t - 2)e^{-t}].$$

Here $H''(0) = 0$ and

$$H'''(t)/3 = 4e^t - 4 + (t^2 - 4t)e^{-t} \geq 0.$$

Here $H'''(0) = 0$ and finally

$$H''''(t)/3 = e^{-t}(4e^{2t} - t^2 + 6t - 4) \geq 0$$

by expanding out e^{2t} . Hence $F(t)$ is an increasing function, so $F(t) \geq 0$ for $t \geq 0$.

Another way of proceeding is to expand out (1) and replace e^{3t} , e^{2t} and e^t by their power series. Doing this we end up with the power series

$$\frac{42t^7}{7!} + \frac{504t^8}{8!} + a_9 t^9 + a_{10} t^{10} + \dots,$$

where for $n > 2$

$$n! a_n = 2 \cdot 3^n - 6 \cdot 2^n + 6 - n(n-1)(n-2)(2^{n-3} + 1).$$

To show that $a_n > 0$ for $n \geq 8$, it suffices to show that

$$u_n \equiv 2 - \left[6 + \frac{n(n-1)(n-2)}{8} \right] \left(\frac{2}{3} \right)^n - \frac{n(n-1)(n-2)}{3^n} > 0.$$

But this follows inductively from

$$u_{n+1} - u_n = \left[2 + \frac{n(n-1)(n-8)}{24} \right] \left(\frac{2}{3} \right)^n + \frac{n(n-1)(2n-7)}{3^{n+1}} > 0$$

for $n \geq 8$.

References:

- [1] B.C. Carlson, The logarithmic mean, *Amer. Math. Monthly* 79 (1972) 615–618.
- [2] T.P. Lin, The power mean and the logarithmic mean, *Amer. Math. Monthly* 81 (1974) 879–883.
- [3] B. Ostle and H.L. Terwilliger, A comparison of two means, *Proc. Montana Acad. Sci.* 17 (1957) 69–70.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and KEE-WAI LAU, Hong Kong. There was one incorrect solution sent in.

Janous notes that Klambauer's and the proposer's inequalities combined have the curious interpretation

$$\sqrt[3]{x \cdot y \cdot \frac{x+y}{2}} < s < \frac{1}{3} \left(x + y + \frac{x+y}{2} \right)$$

or

$$M_0 \left(x, y, \frac{x+y}{2} \right) < s < M_1 \left(x, y, \frac{x+y}{2} \right),$$

where as usual

$$M_t(u, v, w) = \begin{cases} \left(\frac{u^t + v^t + w^t}{3} \right)^{1/t}, & t \neq 0 \\ (uvw)^{1/3}, & t = 0 \end{cases}$$

is the t -th mean of the positive numbers u, v, w .

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1401. [1989: 12] *Proposed by P. Penning, Delft, The Netherlands.*

Given are a circle C and two straight lines l and m in the plane of C that intersect in a point S inside C . Find the tangent(s) to C intersecting l and m in points P and Q so that the perimeter of ΔSPQ is a minimum.

Solution by Jordi Dou, Barcelona, Spain.

Let C_1, C_2, C_3, C_4 be the circles tangent to l and m , and externally tangent to C . Let M_i be the point of contact of C_i with m , for $i = 1, 2, 3, 4$, and let SM_k

be the smallest of the segments SM_i . The tangent t which we look for is the tangent common to circles C and C_k at their point of contact.

Notice that C_k is an excircle of the triangle enclosed by lines l , m , t , and hence $2SM_k$ is the perimeter of this triangle. If $t' \neq t$ is a tangent to C which cuts the half-lines Sl and Sm , then the excircle C'_k (opposite S) of the triangle enclosed by l , m , t' will be exterior to and not tangent to C , and therefore its point of contact M'_k with m will be such that $SM'_k > SM_k$.

The construction of circles C_i is known and classical.

Also solved by HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer. All proofs were similar.

Kuczma's solution included a method (unfortunately algebraic) for determining which of the four circles tangent to C yields the correct tangent line.

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1402. [1989: 12] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let M be an interior point of the triangle $A_1A_2A_3$ and B_1, B_2, B_3 the feet of the perpendiculars from M to sides A_2A_3, A_3A_1, A_1A_2 respectively. Put $r_i = B_iM$, $i = 1,2,3$. R' is the circumradius of $\triangle B_1B_2B_3$, and R, r the circumradius and inradius of $\triangle A_1A_2A_3$. Prove that

$$R'Rr \geq 2r_1r_2r_3 .$$

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

In [2], p. 344, 1.4, 6) the following stronger result of Carlitz is given:

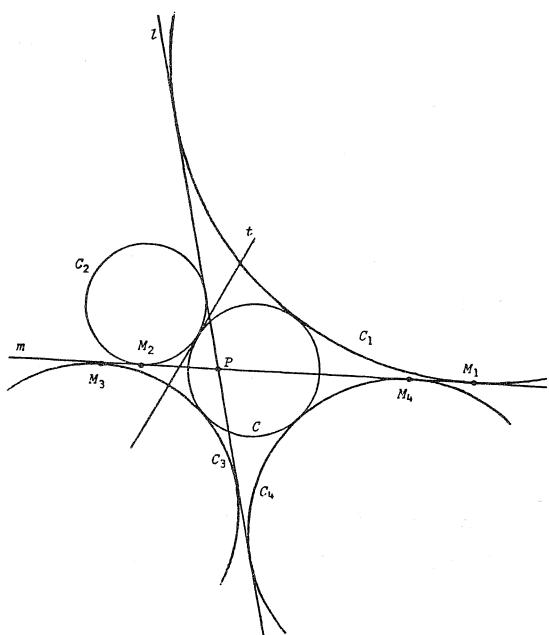
$$2R'R^2 \geq R_1R_2R_3 , \quad (1)$$

where $R_i = A_iM$, $i = 1,2,3$. To see that (1) is better we refer to item 12.26 of [1], i.e.

$$R_1R_2R_3 \geq \frac{r_1r_2r_3}{S} , \quad (2)$$

where

$$S = \prod \sin \frac{A_i}{2} = \frac{r}{4R} . \quad (3)$$



Then (1)–(3) yield

$$2R'R^2 \geq \frac{4Rr_1r_2r_3}{r},$$

i.e. $R'Rr \geq 2r_1r_2r_3$. Done.

It should be noted that (1) is a disguised way of writing

$$F \geq 4F',$$

where F and F' are the areas of triangles $A_1A_2A_3$ and $B_1B_2B_3$, respectively (item 9.5 of [1]).

II. *Solution by Marcin E. Kuczma, Warszawa, Poland.*

This is misleading! The inequality that actually holds is

$$Rr^2 \geq 2r_1r_2r_3. \quad (4)$$

The given inequality hence results automatically, as (trivially) $R' \geq r$. Inequality (4) is an immediate consequence of

$$2s \leq R\sqrt{27}$$

(s the semiperimeter of $\Delta A_1A_2A_3$) and

$$r_1r_2r_3 \leq \frac{2F^2}{27R}$$

($F = rs$ the area of $\Delta A_1A_2A_3$), which are respectively items 5.3 and 12.29 of [1]. Equality holds in (4) only in the case that $\Delta A_1A_2A_3$ is equilateral and M is its center.

References:

- [1] O. Bottema et al, *Geometric Inequalities*.
- [2] D.S. Mitrinović, J.E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Dordrecht, 1989.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; SVETOSLAV J. BILCHEV and EMILIA A. VELIKOVA, Technical University, Russe, Bulgaria; MURRAY S. KLAMKIN, University of Alberta; and the proposer.

Klamkin gave the same improvement (1) to the problem as Janous did, only in the form

$$a_1a_2a_3 \geq 4 \sum a_1r_2r_3.$$

He noted that it appeared earlier in Crux [1987: 260].

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ACKNOWLEDGEMENT

The support of the Departments of Mathematics and Statistics of the University of Calgary and Carleton University, and of the Department of Mathematics of the University of Ottawa, is gratefully acknowledged.

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