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CONTENTS

THE OLYMPIAD CORNER: 79

M.S. KLAMKIN

All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

This month I give three sets of problems.

The first set is a number of "Quickies" proposed by me and whose solutions appear at the end of this corner. Mathematical Quickies were initiated by C.W. Trigg in 1950 as then editor of the Problems and Questions Department of *Mathematics Magazine*. These are problems which can be solved laboriously, but with proper insight and knowledge can be disposed of quickly. Knowing a problem is a Quickie is a decided hint toward solving it.

Q-1. Determine a two parameter integral solution (x,y,z,w) of the Diophantine equation

$$w^2 = 2(x^4 + y^4 + z^4).$$

Q-2. Determine a three parameter integral solution (x,y,z,w) of the Diophantine equation

$$(xyz + yzw + zwx + wxy)^2 = (yz - xw)(xz - yw)(xy - zw).$$

Q-3. Determine the number of real solutions (x,y,z,w) of the simultaneous set of equations

$$x^5 + x = 2y^2z, \quad y^5 + y = 2z^2w, \quad z^5 + z = 2w^2x, \quad w^5 + w = 2x^2y.$$

Q-4. For a, b positive determine the maximum value of

$$y = (a - x)(x + \sqrt{x^2 + b^2})$$

over all real x (no calculus please!).

Q-5. It is known that if A, B, C are the angles of a triangle, then

$$\cot A/2 + \cot B/2 + \cot C/2 \geq 3\sqrt{3}$$

and

$$\cot A + \cot B + \cot C \geq \sqrt{3}.$$

Prove that in fact

$$3(\cot A + \cot B + \cot C) \geq \cot A/2 + \cot B/2 + \cot C/2.$$

Q-6. Determine the minimum value of

$$S \equiv \frac{\cot(\pi - A)/4 + \cot(\pi - B)/4 + \cot(\pi - C)/4}{\sin A + \sin B + \sin C}$$

where A, B, C are the angles of a triangle.

Q-7. Prove that the diagonals of a quadrilateral are perpendicular if and only if the sum of the squares of one pair of opposite sides equals the sum of the squares of the other pair of opposite sides.

Q-8. If a, b, c and $a - x$, $b - y$, $c - z$ are non-negative, prove that
$$\{abc - az(b - y) - bx(c - z) - cy(a - x)\}^2 \geq 4xyz(a - x)(b - y)(c - z).$$

Q-9. By factoring
$$(a^2 + b^2 + c^2)(a + b + c)(b + c - a)(c + a - b)(a + b - c) - 8a^2b^2c^2$$
or otherwise, show that a triangle is acute, right, or obtuse, according as $a^2 + b^2 + c^2$ is $\geq$, $=$, or $\leq 8R^2$. Here a, b, c are the sides and R is the circumradius of the triangle.

Q-10. Find the shortest distance between two non-intersecting face diagonals lying in adjacent faces of a unit cube.

*

The next two sets of problems are by courtesy of Walther Janous, and as usual we invite elegant solutions for them.

17th Austrian Mathematical Olympiad 1986

1st day, June 4, 1986 (time allowed 4 1/2 hours)

1. Show that a square can be inscribed in any regular n-gon.

2. For $s,t \in \mathbb{N}$, let
$$M = \{(x,y) \mid 1 \leq x \leq s,\ 1 \leq y \leq t,\ x,y \in \mathbb{N}\}$$
be a given set of points in a plane. Determine the number of rhombuses whose vertices belong to M and whose diagonals are parallel to the x,y coordinate axes.

3. Determine the set of all values of $x_0, x_1 \in \mathbb{R}$ such that the sequence defined by
$$x_{n+1} = \frac{x_{n-1}x_n}{3x_{n-1} - 2x_n}, \quad n \geq 1$$
contains infinitely many natural numbers.

2nd day (4 1/2 hours)

4. Determine the maximum value of n such that there exists a number N with base 10 representation $d_1d_2\ldots d_n$ having different digits and such that $n! \mid N$. Furthermore, for this maximum n, determine all the possible numbers N.

5. Show that for every convex n-gon ($n \geq 4$), the arithmetic mean of the lengths of all the sides is less than the arithmetic mean of the lengths of all the diagonals.

6. For n a given positive integer, determine all functions $F: \mathbb{N} \longrightarrow \mathbb{R}$ such that $F(x + y) = F(xy - n)$ for all $x,y \in \mathbb{N}$ with $xy > n$.

*

9th Austrian-Polish Mathematical Competition 1986 (Warsaw)

1st day, June 25, 1986 (time allowed 4 1/2 hours)

1. $A_1A_2A_3$ is a given non-right-angled triangle. Three circles C_1, C_2, C_3 are mutually tangent and they pass through the vertices such that $A_2,A_3 \in C_1$; $A_3,A_1 \in C_2$; and $A_1,A_2 \in C_3$. Determine the possible angles for $A_1A_2A_3$ such that the triangle determined by the centers of the three circles is similar to the given triangle.

2. For $n > 1$, let

$$P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$$

be a polynomial having n distinct negative real roots. Prove that $a_1P(1) > 2n^2a_0$.

3. If every point in space is either blue or red, show that there exists at least one unit square whose number of blue vertices is 0, 1, or 4.

2nd day (4 1/2 hours)

4. Determine all triplets (x,y,z) of natural numbers such that

$$x^{z+1} - y^{z+1} = 2^{100}.$$

5. Determine all quadruples (x,y,u,v) of real numbers satisfying the simultaneous equations

$$x^2 + y^2 + u^2 + v^2 = 4,$$

$$xu + yv = -xv - yu,$$

$$xyu + yuv + uvx + vxy = -2,$$

$$xyuv = -1.$$

6. Determine the extreme values of the ratio of the circumradius to the inradius of a tetrahedron whose circumcenter and incenter coincide.

3rd day (4 1/2 hours) - Team Competition

7. Show that $n!$ is divisible by k whenever n and k are natural numbers such that $n^2 \geq 4k > 0$ and k has no prime factor $> n$.

8. An $m \times n$ matrix of distinct real numbers is given. The elements of each row are rearranged (in the same row) such that the elements are increasing from left to right. Next the elements of each column are rearranged (in the same column) such that the elements are increasing from top to bottom. Show that now the elements in each row are still increasing from left to right.

9. Determine all continuous and monotone functions $F: \mathbb{R} \longrightarrow \mathbb{R}$ such that $F(1) = 1$ and $F(F(x)) = (F(x))^2$ for all $x \in \mathbb{R}$.

*

I now give solutions to some previous corner problems as well as the "Quickies" at the beginning of this corner.

5. [1981: 43] *Competition in Mersch, Luxembourg (Belgium, Great Britain, Luxembourg, The Netherlands, and Yugoslavia).*

Ten gamblers started playing each with the same amount of money. Each in turn threw five dice. At each stage the gambler who had thrown paid to each of his nine opponents $1/n$ times the amount which that opponent owned at that moment, where n is the total shown by the dice. They threw and paid one after the other. At the tenth throw the dice showed a total of 12, and after payment it turned out that every gambler had the same sum as he had at the beginning. Determine if possible the totals shown by the dice at each of the other throws.

Solution by Andy Liu, University of Alberta, Edmonton, Alberta.

Suppose M was one of the gamblers and N was the next gambler. Before M's play, both M and N had received exactly the same amounts from each previous gambler and so still had the same amount x of money. Suppose M's die-roll was m; then M's total payment to the other gamblers amounted to $\frac{T - x}{m}$ where T was the total wealth of all the gamblers and $T - x$ was the wealth of all the gamblers other than M. After M's payment, M had

$$x - \frac{T - x}{m} = x(1 + \frac{1}{m}) - \frac{T}{m}$$

while N had $x(1 + \frac{1}{m})$. Now suppose N's die-roll was n. Then, in the same manner, after N's payment M had

$$(x(1 + \frac{1}{m}) - \frac{T}{m})(1 + \frac{1}{n}) = x(1 + \frac{1}{m})(1 + \frac{1}{n}) - \frac{T}{m}(1 + \frac{1}{n})$$

and N had

$$x(1 + \frac{1}{m}) - \frac{T - x(1 + \frac{1}{m})}{n} = x(1 + \frac{1}{m})(1 + \frac{1}{n}) - \frac{T}{n}.$$

These two amounts must be equal, since M and N had equal amounts at the end of the game. Thus

$$\frac{1}{m}(1 + \frac{1}{n}) = \frac{1}{n},$$

that is, $m = n + 1$.

Since in our problem, we have 10 players with the last die-roll being 12, the previous die-roll must have been 12+1 and then the die-roll before that 12+1+1, etc. Hence the ten die-rolls are 21, 20, 19, 18, 17, 16, 15, 14, 13, and 12 in that order. It is to be noted that the information about the five dice being used, though consistent with the answers, is in fact redundant.

19. [1985: 4] *From the 1984 All-Union Olympiad (proposed by A.B. Balotov).*

Find $xy + 2yz + 3zx$ if x, y, z are positive numbers for which

$$x^2 + xy + y^2/3 = 25, \quad y^2/3 + z^2 = 9, \quad \text{and} \quad z^2 + zx + x^2 = 16.$$

Solution by George Evagelopoulos, Law student, Athens, Greece.

Let

$$A = x^2 + xy + y^2/3,$$
$$B = y^2/3 + z^2,$$
$$C = z^2 + zx + x^2,$$
$$D = xy + 2yz + 3zx.$$

Since $A = B + C$, we get $xy = z(x + 2z)$. Also,

$$D = y(x + 2z) + 3zx = y(xy)/z + 3xz = (y^2/3 + z^2)3x/z = 27x/z.$$

Furthermore,

$$D = xy + 2yz + 3zx = 2z^2 + zx + 2yz + 3zx = 2z(2x + y + z),$$

so that

$$2x + y + z = D/2z = 27x/2z^2.$$

Next,

$$A - B + C = 32 = x(2x + y + z) = 27x^2/2z^2.$$

Since $x,y > 0$, $x/z = 8\sqrt{3}/9$. Finally, $D = 27x/z = 24\sqrt{3}$.

This problem appears in *KVANT*, November 1984, p.61.

23. [1985: 5] *From the 1984 All-Union Olympiad (proposed by Yu. Mikheyev).*

For what integers m and n do we have

$(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$ and $(a + b\sqrt{d})^m = (b + a\sqrt{d})^n$,

where a and b are relatively prime integers, while d is a natural number not divisible by the square of a prime?

Solution by George Evagelopoulos, Law student, Athens, Greece.

We will show that, with the assumption (implicit in the problem) that a, b, and d are all greater than one, the only solution to either equation is the obvious one $m = n = 0$.

Suppose that

$$(a + b\sqrt{d})^m = (b + a\sqrt{d})^n$$

and that one of m and n is nonzero; then both are, and in fact they both must have the same sign. Without loss of generality, we may take m and n positive and $a > b$. Then since

$$1 < a + b\sqrt{d} < b + a\sqrt{d},$$

it follows that $m > n$.

By expanding $(a + b\sqrt{d})^m$ and $(b + a\sqrt{d})^n$ by the binomial theorem and equating rational and irrational parts, it also follows that

$$(a - b\sqrt{d})^m = (b - a\sqrt{d})^n$$

and thus

$$\left[\frac{a - b\sqrt{d}}{a + b\sqrt{d}}\right]^m = \left[\frac{b - a\sqrt{d}}{b + a\sqrt{d}}\right]^n . \tag{1}$$

We will now show that (1) cannot hold. Let

$$u = \left|\frac{a - b\sqrt{d}}{a + b\sqrt{d}}\right| \quad \text{and} \quad v = \left|\frac{b - a\sqrt{d}}{b + a\sqrt{d}}\right| .$$

Then since $a > b$, $b - a\sqrt{d} < 0$, so

$$v = \frac{a\sqrt{d} - b}{a\sqrt{d} + b} .$$

We will use the fact that for positive p and q,

$$\frac{p - q}{p + q} = 1 - \frac{2q}{p + q} = 1 - \frac{2}{(p/q) + 1}$$

increases with p/q. Now if $a - b\sqrt{d} > 0$, then

$$u = \frac{a - b\sqrt{d}}{a + b\sqrt{d}}$$

and

$$\frac{a}{b\sqrt{d}} < \frac{a\sqrt{d}}{b} ,$$

and thus $u < v$. On the other hand if $a - b\sqrt{d} < 0$, then

$$u = \frac{b\sqrt{d} - a}{b\sqrt{d} + a}$$

and

$$\frac{b\sqrt{d}}{a} < \frac{a\sqrt{d}}{b} ,$$

and again it follows that $u < v$. In either case $0 < u, v < 1$ and $m > n$ implies that $u^m < v^n$, contradicting (1).

<u>24</u>. [1985: 5] *From the 1984 All-Union Olympiad (proposed by O.R. Musin).*

If $n > 3$ natural numbers are written around a circle so that the ratio of the sum of the two neighbours of any number to itself is a natural number, prove that the sum of all such ratios is

(a) not less than $2n$,

(b) not more than $3n$.

Solution by George Evagelopoulos, Law student, Athens, Greece.

Let the n natural numbers be denoted by $a_1, a_2, \ldots, a_n$ and let

$$S_n = \frac{a_n + a_2}{a_1} + \frac{a_1 + a_3}{a_2} + \ldots + \frac{a_{n-1} + a_1}{a_n}.$$

(a) Since

$$x/y + y/x \geq 2$$

for $x,y > 0$, it follows that

$$S_n = \left[\frac{a_2}{a_1} + \frac{a_1}{a_2}\right] + \left[\frac{a_3}{a_2} + \frac{a_2}{a_3}\right] + \ldots + \left[\frac{a_1}{a_n} + \frac{a_n}{a_1}\right] \geq 2n.$$

(b) We will show that $S_n < 3n$ for $n \geq 3$ by mathematical induction.

For $n = 3$, we consider two cases.

(i) All the numbers a_i are equal. In this case $S_3 = 6 < 3\cdot 3$.

(ii) At least two of a_1, a_2, a_3 are not equal. Suppose $a_1 > a_2$ and $a_1 \geq a_3$. Then

$$a_1 > \frac{a_2 + a_3}{2} \quad \text{so} \quad 2 > \frac{a_2 + a_3}{a_1} .$$

Since the latter ratio equals a natural number, $a_1 = a_2 + a_3$. Consequently,

$$S_3 = 1 + \frac{a_3 + a_1}{a_2} + \frac{a_1 + a_2}{a_3} = 3 + \frac{2a_3}{a_2} + \frac{2a_2}{a_3} ,$$

where $k = 2a_3/a_2$ and $\ell = 2a_2/a_3$ are natural numbers. Since $k\ell = 4$, we can only have the following possibilities:

$$k = 1,\ \ell = 4, \text{ then } S_3 = 8 < 9;$$
$$k = 2,\ \ell = 2, \text{ then } S_3 = 7 < 9;$$
$$k = 4,\ \ell = 1, \text{ then } S_3 = 8 < 9.$$

We assume now that the inequality is valid for S_{n-1} and we will show that $S_n < 3n$.

If all the numbers a_i are equal, then $S_n = 2n < 3n$. Otherwise, we choose the largest of the n numbers. If there is more than one such number, we take one of them which is greater than at least one of its nearest neighbors. Let this number be a_n. Then

$$a_n > \frac{a_{n-1} + a_1}{2},$$

so

$$2 > \frac{a_{n-1} + a_1}{a_n}.$$

Since the latter ratio equals a natural number, $a_n = a_{n-1} + a_1$ so that

$$\frac{a_{n-2} + a_n}{a_{n-1}} = 1 + r, \qquad \frac{a_n + a_2}{a_1} = 1 + s,$$

where

$$r = \frac{a_{n-2} + a_1}{a_{n-1}}$$

and

$$s = \frac{a_{n-1} + a_2}{a_1}$$

are natural numbers. Consequently the sequence $a_1, a_2, \ldots, a_{n-1}$ satisfies the inductive hypothesis and so $S_{n-1} < 3(n - 1)$. Finally,

$$\begin{aligned} S_n &= \frac{a_1 + a_3}{a_2} + \cdots + \frac{a_{n-2} + a_n}{a_{n-1}} + \frac{a_{n-1} + a_1}{a_n} + \frac{a_n + a_2}{a_1} \\ &= \frac{a_1 + a_3}{a_2} + \cdots + \left[\frac{a_{n-2} + a_1}{a_{n-1}} + 1\right] + 1 + \left[1 + \frac{a_{n-1} + a_2}{a_1}\right] \\ &= S_{n-1} + 3 < 3n. \end{aligned}$$

Editorial comment. By letting $a_i = i$ for all i, we obtain

$$\begin{aligned} S_n &= \frac{1+3}{2} + \dots + \frac{(n-2)+n}{n-1} + \frac{(n-1)+1}{n} + \frac{n+2}{1} \\ &= 2 + \dots + 2 + 1 + (n+2) \\ &= 2(n-2) + n + 3 \\ &= 3n - 1, \end{aligned}$$

which is the least upper bound.

*

Solutions to the Quickies

Q-1. Letting $z = x + y$, we find that $w = \pm 2(x^2 + xy + y^2)$.

Q-2. Expanding out the left and right sides, we obtain

$$\Sigma\, x^2y^2z^2 + 2xyzw\, \Sigma\, xy = \Sigma\, x^2y^2z^2 - xyzw\, \Sigma\, x^2$$

or

$$xyzw(x + y + z + w)^2 = 0.$$

Thus all solutions are given by $xyzw = 0$ or $x + y + z + w = 0$.

Q-3. Multiplying the equations together, we obtain

$$xyzw\{\Pi(1 + x^4) - 16x^2y^2z^2w^2\} = 0.$$

Since $1 + x^4 \geq 2x^2$ with equality if and only if $x = \pm 1$, etc., all the solutions are given by

$$(x,y,z,w) = (0,0,0,0) \text{ or } (\pm 1,\pm 1,\pm 1,\pm 1)$$

provided x and z have the same sign and the same for y and w. Thus there are five real solutions.

Q-4. Letting $x = b \sinh \theta$,

$$\begin{aligned} 2y &= 2(a - b\sinh\theta)(b\sinh\theta + b\sqrt{\sinh^2\theta + 1}) \\ &= (2a + b(e^{-\theta} - e^{\theta}))be^{\theta} \\ &= b^2 + be^{\theta}(2a - be^{\theta}). \end{aligned}$$

Thus y takes on its maximum value for $be^{\theta} = a$. Finally,

$$y_{\max} = \frac{b^2 + a^2}{2}$$

and is taken on for

$$x = \frac{b(a/b - b/a)}{2} = \frac{a^2 - b^2}{2a}.$$

Q-5. Let a, b, c be the sides of the triangle, s the semiperimeter, and r the inradius. Since

$$\cot A/2 = \frac{s - a}{r}, \quad \text{etc.}$$

and

$$2 \cot A = \cot A/2 - \frac{1}{\cot A/2}, \quad \text{etc.},$$

the inequality is equivalent to

$$\frac{3}{2}\left[\left[\frac{s - a}{r} - \frac{r}{s - a}\right] + \left[\frac{s - b}{r} - \frac{r}{s - b}\right] + \left[\frac{s - c}{r} - \frac{r}{s - c}\right]\right] \geq \frac{s - a}{r} + \frac{s - b}{r} + \frac{s - c}{r}$$

or

$$\frac{1}{2}\left[\frac{s - a}{r} + \frac{s - b}{r} + \frac{s - c}{r}\right] \geq \frac{3}{2}\left[\frac{r}{s - a} + \frac{r}{s - b} + \frac{r}{s - c}\right]$$

or

$$s = 3s - (a + b + c) \geq 3r^2\left[\frac{1}{s - a} + \frac{1}{s - b} + \frac{1}{s - c}\right] . \qquad (1)$$

Since it is known that

$$r^2 s = (s - a)(s - b)(s - c),$$

(1) is equivalent successively to

$$s^2 \geq 3[(s - b)(s - c) + (s - c)(s - a) + (s - a)(s - b)],$$

$$s^2 \geq 3[3s^2 - 2s(a + b + c) + bc + ca + ab],$$

$$(a + b + c)^2 \geq 3(bc + ca + ab),$$

and finally

$$(b - c)^2 + (c - a)^2 + (a - b)^2 \geq 0,$$

which is true, with equality if and only if the triangle is equilateral.

Q-6. For any triangle ABC, there is a non-obtuse triangle $A'B'C'$ with angles

$$A' = (\pi - A)/2, \quad B' = (\pi - B)/2, \quad C' = (\pi - C)/2.$$

So equivalently, we want the minimum of

$$S' \equiv \frac{\cot A'/2 + \cot B'/2 + \cot C'/2}{\sin 2A' + \sin 2B' + \sin 2C'}$$

for non-obtuse triangles. It will turn out that we can drop the primes in this expression and consider all triangles, since the minimum will occur only for equilateral triangles.

Since the area F of a triangle ABC is given by

$$F = R^2\{\sin 2A + \sin 2B + \sin 2C\}/2 = r^2\{\cot A/2 + \cot B/2 + \cot C/2\}$$

where R and r are the circumradius and inradius, respectively, S' will equal $(R/r)^2/2$. It is well known that $(R/r)_{\min} = 2$ and is taken on only for equilateral triangles. Finally, the minimum values of S and S' are then both equal to 2.

Q-7. This problem appeared in the 1912 Eotvös Competition. Two solutions are given in the *Hungarian Problem Book II*, New Mathematical Library, MAA, Washington, D.C., 1963, pp.50-54 where it is assumed that the figure is coplanar. We give a vectorial solution which is simpler and which does not assume the figure is coplanar.

Denoting the vertices of the quadrilateral by vectors A, B, C, D from a common origin, the result follows immediately from the identity

$$(A - D)^2 + (B - C)^2 - (A - B)^2 - (D - C)^2 = 2(A - C)\cdot(B - D).$$

Q-8. The result follows immediately from the equivalent inequality

$$\{\sqrt{xyz} - \sqrt{(a - x)(b - y)(c - z)}\}^2 \geq 0,$$

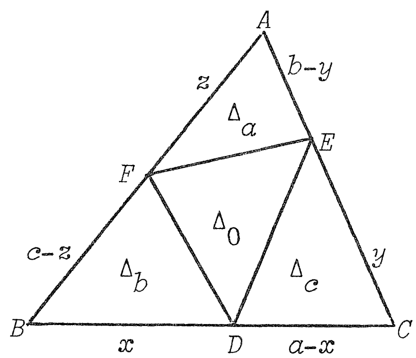
since this inequality implies

$$[xyz + (a - x)(b - y)(c - z)]^2 \geq 4xyz(a - x)(b - y)(c - z)$$

and the left side reduces to

$$[abc - az(b - y) - bx(c - z) - cy(a - x)]^2.$$

The given inequality is related to a result of J.F. Rigby, "Inequalities concerning the areas obtained when one triangle is inscribed in another", *Math. Mag.* 45 (1972) 113-116. Let a, b, c denote the sides of a triangle ABC with an inscribed triangle DEF where $BD = x$, $CE = y$, $AF = z$, and let Δ, Δ_a, Δ_b, Δ_c, and Δ_0 denote the areas of ABC, AFE, BFD, CDE, and DEF, respectively. Then



$$\frac{\Delta_a}{\Delta} = \frac{(b - y)z}{bc}, \quad \frac{\Delta_b}{\Delta} = \frac{(c - z)x}{ca}, \quad \frac{\Delta_c}{\Delta} = \frac{(a - x)y}{ab},$$

so dividing the given inequality by $(abc)^2$ gives

$$\left[1 - \frac{\Delta_a}{\Delta} - \frac{\Delta_b}{\Delta} - \frac{\Delta_c}{\Delta}\right]^2 \geq 4\,\frac{\Delta_a}{\Delta}\cdot\frac{\Delta_b}{\Delta}\cdot\frac{\Delta_c}{\Delta}$$

or

$$\Delta_0^2\Delta \geq 4\Delta_a\Delta_b\Delta_c$$

with equality if and only if the three cevians AD, BE, and CF are concurrent. Thus

$$\Delta_0^3 + \Delta_0^2\Delta_a + \Delta_0^2\Delta_b + \Delta_0^2\Delta_c \geq 4\Delta_a\Delta_b\Delta_c$$

and it follows quickly that

$$\Delta_0 \geq \min(\Delta_a, \Delta_b, \Delta_c),$$

with equality if and only if the three cevians are medians.

Q-9. Letting $S = a^2 + b^2 + c^2$, we have

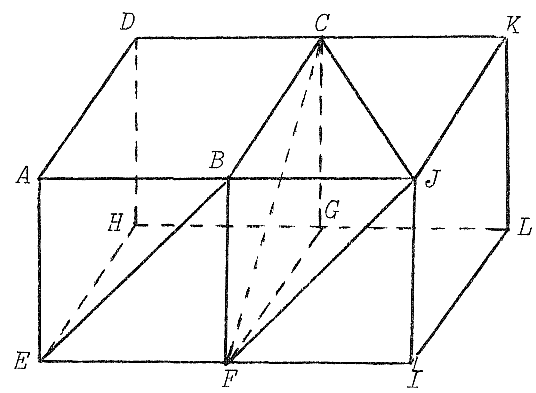
$$\begin{aligned}
&(a^2 + b^2 + c^2)(a + b + c)(b + c - a)(c + a - b)(a + b - c) - 8a^2b^2c^2 \\
&\quad = S[2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4)] - 8a^2b^2c^2 \\
&\quad = S[4(b^2c^2 + c^2a^2 + a^2b^2) - S^2] - 8a^2b^2c^2 \\
&\quad = 4S(b^2c^2 + c^2a^2 + a^2b^2) - S^3 - 8a^2b^2c^2 \\
&\quad = S^3 - 2S^2(a^2 + b^2 + c^2) + 4S(b^2c^2 + c^2a^2 + a^2b^2) - 8a^2b^2c^2 \\
&\quad = (S - 2a^2)(S - 2b^2)(S - 2c^2) \\
&\quad = (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2). \qquad (1)
\end{aligned}$$

Also, since $abc = 4RF$ and

$$(a + b + c)(b + c - a)(c + a - b)(a + b - c) = 16F^2$$

where R and F are the circumradius and area, respectively, of the triangle, (1) is also equal to $16F^2(a^2 + b^2 + c^2 - 8R^2)$. Finally, since (1) is positive, 0, or negative according to whether the triangle is acute, right, or obtuse, the desired result follows.

Q-10. In the following figure we have two adjacent unit cubes touching face to face and we will determine the shortest distance h between EB and FC.



Since $FJ \| EB$, h is the distance from B to the plane of FJ and FC. The volume V of tetrahedron $BFCJ = \frac{1}{6} = (\frac{h}{3})\cdot\text{Area}(FCJ)$. Since $\text{Area}(FCJ) = \frac{\sqrt{3}}{2}$, $h = \frac{1}{\sqrt{3}}$.

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PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

* * *

1181. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

Let x, y, z be real numbers such that

$$xyz(x + y + z) > 0,$$

and let a, b, c be the sides, m_a, m_b, m_c the medians and F the area of a triangle. Prove that

(a) $|yza^2 + zxb^2 + xyc^2| \geq 4F\sqrt{xyz(x + y + z)}$

(b) $|yzm_a^2 + zxm_b^2 + xym_c^2| \geq 3F\sqrt{xyz(x + y + z)}$.

1182. *Proposed by Peter Andrews and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.* (Dedicated to Léo Sauvé.)

Let $a_1,a_2,\ldots,a_n$ denote positive reals where $n \geq 2$. Prove that

$$\frac{\pi}{2} \leq \tan^{-1}\frac{a_1}{a_2} + \tan^{-1}\frac{a_2}{a_3} + \ldots + \tan^{-1}\frac{a_n}{a_1} \leq \frac{(n - 1)\pi}{2}$$

and for each inequality determine when equality holds.

1183. *Proposed by Roger Izard, Dallas, Texas.*

Let $ABCD$ be a convex quadrilateral and let points E, G lie on BD and F, H lie on AC such that AE, BF, CG, DH bisect angles DAB, ABC, BCD, CDA respectively. Suppose that $AE = CG$ and $BF = DH$. Prove that $ABCD$ is a parallelogram.

1184. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a nonequilateral triangle and let O, I, H, F denote the circumcenter, incenter, orthocenter, and the center of the nine-point circle, respectively. Can either of the triangles OIF or IFH be equilateral?

1185*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Determine the set M of all integers $k \geq 2$ such that there exists a positive real number u_k satisfying $[u_k^n] \equiv n \pmod k$ for all natural numbers n, where $[x]$ denotes the greatest integer $\leq x$.

(Problem A-5 in the 1983 Putnam Competition is equivalent to showing that $2 \in M$.)

1186. *Proposed by Svetoslav Bilchev, Technical University and Emilia Velikova, Mathematicalgymnasium, Russe, Bulgaria.*

If a, b, c are the sides of a triangle and s, R, r the semiperimeter, circumradius, and inradius, respectively, prove that

$$\Sigma(b + c - a)\sqrt{a} \geq 4r(4R + r)\sqrt{\frac{4R + r}{3Rs}}$$

where the sum is cyclic over a, b, c.

1187. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Find a polynomial with integer coefficients that has $2^{1/5} + 2^{-1/5}$ as a root.

1188. *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Given a circle K and distinct points A, B in the plane of K, construct a chord PQ of K such that B lies on the line PQ and $\angle PAQ = 90^\circ$.

1189*. *Proposed by Kee-wai Lau, Hong Kong.*

Find the surface area and volume of the solid formed by the intersection of eight unit spheres whose centres are located at the vertices of a unit cube.

1190*. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Prove or disprove that

$$\lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{(-1)^k(n-k)^k e^{n-k}}{k!} - 2n\right] = \frac{2}{3}\ .$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1010*. [1985: 17; 1986: 113] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Are there integers $k \neq 1$ such that the sequence $\{3n^2k + 3nk^2 + k^3\}$, n an integer, contains infinitely many squares? If the answer is yes, determine all such k. (The case $k = 1$ is dealt with in *Crux* 873 [1984: 335].)

I. *Comment by Richard K. Guy, University of Calgary, Calgary, Alberta.*

We show that k must be of the form $p_1 \ldots p_m \ell^2$ or $3p_1 \ldots p_{m+1}\ell^2$ where $m \geq 0$ and each p_i is a prime $\equiv 1 \bmod 6$; that is, the condition conjectured by Hess [1986: 115] is at least necessary for k to be a solution of the problem. (In the comment following this one, however, G. Walsh shows that it is *not* sufficient.)

If we put $k = k_1k_2^2$ where k_1 is squarefree, and write

$$\begin{aligned} z^2 &= 3n^2k + 3nk^2 + k^3 \\ &= 3n^2k_1k_2^2 + 3nk_1^2k_2^4 + k_1^3k_2^6\ , \end{aligned} \qquad (1)$$

then $k_1k_2^2 | z^2$, and it follows that $k_1k_2 | z$ since k_1 is squarefree. Putting $z = k_1k_2z_1$, (1) becomes

$$\begin{aligned} k_1^2k_2^2z_1^2 &= 3n^2k_1k_2^2 + 3nk_1^2k_2^4 + k_1^3k_2^6 \\ k_1z_1^2 &= 3n^2 + 3nk_1k_2^2 + k_1^2k_2^4. \end{aligned} \qquad (2)$$

Hence $k_1 | 3n^2$, so (as in Kierstead's argument) either $k_1 | n$ or $3 | k_1$. Assuming first $3 \nmid k_1$, put $n = k_1n_1$. Then (2) becomes

$$\begin{aligned} k_1z_1^2 &= 3k_1^2n_1^2 + 3k_1^2k_2^2n_1 + k_1^2k_2^4 \\ z_1^2 &= 3k_1n_1^2 + 3k_1k_2^2n_1 + k_1k_2^4 \end{aligned}$$

so we can put $z_1 = k_1z_2$, and then

$$k_1z_2^2 = 3n_1^2 + 3k_2^2n_1 + k_2^4$$

so

$$4k_1z_2^2 = 3(2n_1 + k_2^2)^2 + k_2^4. \qquad (3)$$

If $3|k_1$, we put $k_1 = 3k_3$, and with similar calculations (2) can be written in the form

$$4k_3z_2^2 = (2n_1 + 3k_2^2)^2 + 3k_2^4. \tag{4}$$

Thus we need to show that if (3) (resp. (4)) has infinitely many solutions for a given k_1 (resp. k_3) then k_1 (resp. k_3) is a product of primes all $\equiv 1 \bmod 6$ (and $k_3 \neq 1$).

First we show that $2\nmid k_1$ in (3) and that $2\nmid k_3$ in (4). Since $k_1 = 3k_3$ and thus both k_1 and k_3 are squarefree, we need to show that both

$$3(2n_1 + k_2^2)^2 + k_2^4$$

and

$$(2n_1 + 3k_2^2)^2 + 3k_2^4$$

have even powers of 2 in their prime factorizations. It is enough to show this for expressions of the form $3X^2 + Y^2$. But this is obvious if X and Y have opposite parity, and follows by induction if X and Y are both even. Finally, if X and Y are both odd, then $3X^2 + Y^2 \equiv 4 \bmod 8$, so $3X^2 + Y^2$ is divisible by 4 but not 8.

Next we show that $k_3 \neq 1$ in (4). Otherwise, (4) becomes

$$4z_2^2 = (2n_1 + 3k_2^2)^2 + 3k_2^4,$$

or

$$3k_2^4 = (2z_2 + 2n_1 + 3k_2^2)(2z_2 - 2n_1 - 3k_2^2)$$

which obviously has only a finite number of solutions for each k_2.

It is clear since k_1 is squarefree that $3\nmid k_3$. Thus we suppose that some $p \equiv 2 \bmod 3$ is a divisor of k_1 in (3) or k_3 in (4), and it remains to show this is impossible. Since k_1 and k_3 are squarefree, this means that either

$$3(2n_1 + k_2^2)^2 + k_2^4$$

or

$$(2n_1 + 3k_2^2)^2 + 3k_2^4$$

has an odd power of p in its prime factorization. We show that no integer of the form $3X^2 + Y^2$ can have this property. This is clear if p divides just one of X and Y, and follows by induction if p divides both of X and Y. So suppose that $p\nmid Y$. Then

$$3X^2 + Y^2 \equiv 0 \bmod p$$

may be written

$$9X^2Y^{-2} + 3 \equiv 0 \bmod p,$$

or

$$(3XY^{-1})^2 \equiv -3 \bmod p.$$

However it is known that -3 is a quadratic non-residue of such primes p. This contradiction completes the proof.

II. *Comment by Gary Walsh, student, University of Calgary, Calgary, Alberta.*

It was conjectured by Hess [1986: 115] that for any number k of the form $p_1 \cdots p_m \ell^2$ or $3p_1 \cdots p_{m+1} \ell^2$, where $m \geq 0$ and each p_i is a prime $\equiv 1 \bmod 6$, there are infinitely many integers n such that $(n + k)^3 - n^3 = z^2$ for some integer z. We give a constructive proof that there are infinitely many counterexamples to this conjecture.

We use the following facts, where m and n denote squarefree odd integers greater than 1 (for the first, see D.T. Walker, The equation $mX^2 - nY^2 = \pm 1$, *Amer. Math. Monthly* 74 (1967) 504-513; the second follows from the first by a simple induction):

(i) If a positive solution (X,Y) to $mX^2 - nY^2 = 1$ exists, then there is a smallest positive solution (A,B), and all positive solutions (A_k,B_k) are obtained from

$$A_k\sqrt{m} + B_k\sqrt{n} = (A\sqrt{m} + B\sqrt{n})^{2k+1}$$

with $k = 0,1,2,\ldots$.

(ii) If (A,B) is the smallest positive solution to $mX^2 - nY^2 = 1$, then A and B are of opposite parity, and if A (resp. B) is even, then A_k (resp. B_k) is even for every k, and B_k (resp. A_k) is odd for every k.

As well, we need the result proved in [1986: 113], that for $k = p_1 \cdots p_m$ with each $p_i \equiv 1 \bmod 6$, the problem of finding infinitely many n such that $(n + k)^3 - n^3$ is a square is equivalent to the solvability of the equation

$$4ka^2 - 3b^2 = 1$$

for some integers a, b.

Suppose that x, y, k are positive integers with k squarefree and $k > 1$, and suppose $12x^2 + 1 = ky^2$. We claim that the equation $4ka^2 - 3b^2 = 1$ is not solvable. We have that the equation $kX^2 - 3Y^2 = 1$ is solvable by $(X,Y) = (y,2x)$. Thus

$$y\sqrt{k} + 2x\sqrt{3} = A_i\sqrt{k} + B_i\sqrt{3} = (A\sqrt{k} + B\sqrt{3})^{2i+1}$$

for some integer i, where (A,B) is the smallest positive solution to $kX^2 - 3Y^2 = 1$. Since $B_i = 2x$, B_j is even and A_j is odd for every j. Now if

a, b are integers such that $4ka^2 - 3b^2 = 1$, then $(2a,3) = (A_\ell,B_\ell)$ for some ℓ, contradiction.

We now have that any squarefree integer $k > 1$ for which $12x^2 + 1 = ky^2$ is solvable is not a solution to the original problem. Next we show that any integer k of this form is a product of primes each congruent to 1 mod 6. If $p|k$, then

$$12x^2 + 1 \equiv 0 \bmod p$$

so clearly $p \neq 2$ or 3. Also

$$(6x)^2 \equiv -3 \bmod p$$

so $p \not\equiv 2 \bmod 3$ since -3 is a quadratic nonresidue modulo such primes. Thus $p \equiv 1 \bmod 6$.

Thus any squarefree integer $k > 1$ for which $12x^2 + 1 = ky^2$ holds is a counterexample of Hess's conjecture; for instance,

$$x = 1,\quad k = 13,$$
$$x = 3,\quad k = 109,$$
$$x = 4,\quad k = 193,$$
$$x = 5,\quad k = 301,$$

and so on. For $n = 1,2,\dots$ put $12n^2 + 1 = k_n y_n^2$ where each $k_n \geq 1$ is squarefree. It remains to show that the set $K = \{k_n | n = 1,2,\dots\}$ is infinite. We suppose for a contradiction that K is finite, say $|K| = s$. For each $k \in K$, the equation $12n^2 + 1 = ky^2$ can be written $ky^2 - 3(2n)^2 = 1$, and thus its positive solutions are all of the form $(y,2n) = (A_i,B_i)$ where

$$A_i\sqrt{k} + B_i\sqrt{3} = (A\sqrt{k} + B\sqrt{3})^{2i+1}$$

and (A,B) is the smallest positive solution. Since

$$\begin{aligned} A_{i+1}\sqrt{k} + B_{i+1}\sqrt{3} &= (A\sqrt{k} + B\sqrt{3})^{2i+3} \\ &= (A_i\sqrt{k} + B_i\sqrt{3})(A\sqrt{k} + B\sqrt{3})^2 \\ &= \{A_i(kA^2 + 3B^2) + 6ABB_i\}\sqrt{k} + \{B_i(kA^2 + 3B^2) + 2kABA_i\}\sqrt{3}, \end{aligned}$$

we have

$$B_{i+1} = B_i(kA^2 + 3B^2) + 2kABA_i,$$

and in particular $B_{i+1} > 4B_i$. Thus $|B_i - B_j| > 3s$ whenever $i \neq j$ and $B_i,B_j \geq s$. Now consider the equations

$$12s^2 + 1 = k_s y_s^2$$

$$12(s+1)^2 + 1 = k_{s+1}y_{s+1}^2$$
$$\vdots$$
$$12(2s)^2 + 1 = k_{2s}y_{2s}^2.$$

Then from $|K| = s$, two of $\{k_s, k_{s+1}, \ldots, k_{2s}\}$ must be equal, say $k_{s+i} = k_{s+j} = k$ with $i \neq j$, and so

$$ky_{s+i}^2 - 3[2(s+i)]^2 = 1$$

and

$$ky_{s+j}^2 - 3[2(s+j)]^2 = 1.$$

But $2(s+i),\ 2(s+j) \geq 2s > s$ and

$$|2(s+i) - 2(s+j)| = 2|i-j| \leq 2s < 3s,$$

which is a contradiction.

* * *

1048. [1985: 147] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

In base ten, $361 = 19^2$. Find at least three other bases in which 361 is a perfect square.

Solution by Hayo Ahlburg, Benidorm, Alicante, Spain.

It is well known[1] that Euler [1] solved the following problem: find all triangular numbers $N = \frac{k^2+k}{2}$ which are also pentagonal numbers $N = \frac{3\ell^2 - \ell}{2}$. This leads to

$$\ell = \frac{1 \pm \sqrt{12k^2 + 12k + 1}}{6},$$

where the expression R under the square root should be a square to make ℓ an integer. With $k = \frac{x-1}{2}$ we obtain the Fermat-Pell equation

$$R = 3x^2 - 2 = y^2.$$

Euler gives recursion formulas, in our notation

$$x_0 = y_0 = 1, \quad x_{n+1} = 2x_n + y_n, \quad y_{n+1} = 3x_n + 2y_n.$$

Following methods developed by Lagrange [2], we can also give x and y in closed form:

$$x_n = \left[\frac{1+\sqrt{3}}{2\sqrt{3}}(2+\sqrt{3})^n\right] + 1, \tag{1}$$

[1] perhaps by few people?

$$y_n = \left[\frac{1 + \sqrt{3}}{2}(2 + \sqrt{3})^n\right],$$

where $n = 0,1,2,\ldots$ and $[a]$ is the greatest integer $\leq a$. The triangular-pentagonal numbers N are now found from

$$N = \frac{k^2 + k}{2} = \frac{3\ell^2 - \ell}{2}$$

with

$$k = \frac{x - 1}{2}, \quad \ell = \frac{1 \pm y}{6},$$

that is,

$$N = \frac{x^2 - 1}{8} = \frac{y^2 - 1}{24}.$$

The first values are

n	0	1	2	3	4	5	6	7
x	1	3	11	41	153	571	2131	7953
y	1	5	19	71	265	989	3691	13775
N	0	1	15	210	2926	40755	567645	7906276

Now, I hope the patient reader who has stayed with me thus far will be as enchanted as I am to discover cross-connections between seemingly unrelated problems. In *Crux* 1048 we are looking for those positive integers b (base) and y which are solutions of

$$3b^2 + 6b + 1 = y^2.$$

Substituting $b = x - 1$, this becomes

$$3x^2 - 2 = y^2.$$

Voilà! $b = 2, 10, 40, 152, 570, 2130, 7952$, etc. (The base b in closed form is, according to (1),

$$b = \left[\frac{1 + \sqrt{3}}{2\sqrt{3}}(2 + \sqrt{3})^n\right], \quad n = 1,2,3,\ldots .)$$

References:

[1] Leonhard Euler, Vollständige Anleitung zur Algebra, St. Petersburg, 1770, Part II, Section 2, Chapter 6, §91, Problem V.

[2] Joseph Louis Lagrange, Additions to the French translation of Euler's Algebra, Lyon, 1774, §VII, #75.

Also solved by SAM BAETHGE, San Antonio, Texas; C. FESTRAETS-HAMOIR, Bruxelles, Belgique; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; J.A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, *Culver City, California*; *LEROY F. MEYERS, The Ohio State University, Columbus, Ohio*; *BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin*; *STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire*; *J. SUCK, Essen, Federal Republic of Germany*; *KENNETH M. WILKE, Topeka, Kansas*; *and the proposer. There was one partial solution received.*

* * *

1049*. [1985: 148] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let ABC and $A'B'C'$ be two nonequilateral triangles such that $A \geq B \geq C$ and $A' \geq B' \geq C'$. Prove that

$$A - C > A' - C' \Leftrightarrow \frac{s}{r} > \frac{s'}{r'},$$

where s, r and s', r' are the semiperimeter and inradius of triangles ABC and $A'B'C'$, respectively.

Comment by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

Since the perimeter of a triangle circumscribed about a given circle can be unbounded if either two angles of the triangle are close to 90° or else close to 0°, the stated result is clearly false. For a specific counterexample, just let

$$A = 120^\circ, \quad B = 30^\circ, \quad C = 30^\circ,$$
$$A' = 89^\circ, \quad B' = 89^\circ, \quad C' = 2^\circ.$$

Then $A - C > A' - C'$, but

$$\frac{s}{r} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \approx 8$$

while

$$\frac{s'}{r'} = \cot\frac{A'}{2} + \cot\frac{B'}{2} + \cot\frac{C'}{2} \approx 59.$$

One can also find counterexamples involving only acute triangles, for example

$$A = 89^\circ, \quad B = 86.5^\circ, \quad C = 4.5^\circ,$$
$$A' = 88^\circ, \quad B' = 88^\circ, \quad C' = 4^\circ.$$

Nevertheless, since $\cot(x/2)$ is convex in $(0,\pi)$, we can obtain a positive result by the Majorization Inequality (e.g. see p.30 of E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Heidelberg, 1965). Suppose that

$$A \geq B \geq C, \quad A' \geq B' \geq C',$$

and also that

$$A \geq A' \quad \text{and} \quad A + B \geq A' + B' ;$$

then

$$\frac{s}{r} = \Sigma \cot A/2 \geq \Sigma \cot A'/2 = \frac{s'}{r'} .$$

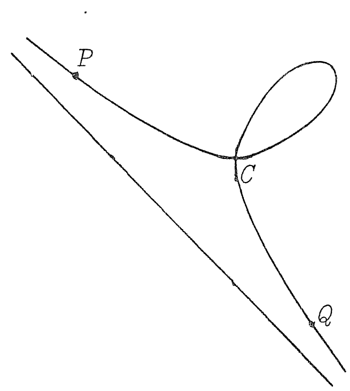
* * *

1050. [1985: 148] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

In the plane, you are given the curve known as the folium of Descartes. Show how to construct the asymptote to this curve using straightedge and compasses only.

Solution by J.A. McCallum, Medicine Hat, Alberta.

We assume that the folium is in general position. Then the only definite point in the plane is the point where the folium intersects itself. Call this C. With C as centre and any convenient radius draw a circle. This circle will cut the two tails of the folium in points P and Q. (It may also cut the loop of the folium but we ignore these points if they occur.) Join PQ and find its midpoint D. Join DC and produce it to cut the loop of the folium at A. A is then the apex of the folium. Trisect AC, and produce AC to E so that $CE = \frac{1}{3}AC$. Finally, at E erect a line perpendicular to ECA, and extend this line indefinitely far in both directions. This is the required asymptote to the folium. The construction works because of the known symmetry of the folium and the known fact that the apex of the folium is three times as far from point C as the asymptote is (easy to prove from the usual equation $x^3 + y^3 = 3axy$ of the folium).



Also solved by ELWYN ADAMS, Gainesville, Florida; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; and the proposer. Adams, however, assumed that the coordinate axes were given. Two other readers submitted solutions assuming as given not only the axes, but also the equation of the folium.

The proposer wonders if asymptotes to other familiar curves, for example $y = \tan x$ or $y = \log x$, can be constructed with straightedge and compasses.

* * *

1051. [1985: 187] *Proposed by George Tsintsifas, Thessaloniki,* *Greece.*

Let a, b, c be the side lengths of a triangle of area K, and let u, v, w be positive real numbers. Prove that

$$\frac{ua^4}{v+w}+\frac{vb^4}{w+u}+\frac{wc^4}{u+v}\geq 8K^2.$$

When does equality occur? Some interesting triangle inequalities may result if we assign specific values to u, v, w. Find a few.

I. *Generalization by D.S. Mitrinovic and J.E. Pečaric, University of Belgrade, Belgrade, Yugoslavia.*

Let $r_1,\ldots,r_n$ be real numbers such that

$$R_k=\sum_{\substack{i=1\\ i\neq k}}^{n} r_i > 0,\qquad k=1,\ldots,n,$$

and let $x_1,\ldots,x_n$ be positive numbers. We will prove that

$$\sum_{i=1}^{n}\frac{r_i}{R_i}x_i^2\geq\frac{1}{n-1}\left[\sum_{i=1}^{n}x_i\right]^2-\sum_{i=1}^{n}x_i^2\,. \tag{1}$$

Indeed,

$$\sum_{i=1}^{n}\frac{r_i}{R_i}x_i^2=\sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{n}r_j-R_i}{R_i}\cdot x_i^2\right]$$

$$=\left(\sum_{j=1}^{n}r_j\right)\left(\sum_{i=1}^{n}\frac{x_i^2}{R_i}\right)-\sum_{i=1}^{n}x_i^2$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}R_i\right)\left(\sum_{i=1}^{n}\frac{x_i^2}{R_i}\right)-\sum_{i=1}^{n}x_i^2$$

$$\geq\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2$$

where we used Cauchy's inequality in the last step. Equality is valid in (1) if and only if

$$\frac{x_1}{R_1}=\cdots=\frac{x_n}{R_n}\,.$$

Putting $n=3$, $r_1=u$, $r_2=v$, $r_3=w$ and $x_1=a^2$, $x_2=b^2$, $x_3=c^2$, (1) becomes

$$\frac{ua^4}{v+w} + \frac{vb^4}{w+u} + \frac{wc^4}{u+v} \geq \frac{1}{2}(a^2+b^2+c^2)^2 - (a^4+b^4+c^4)$$
$$= \frac{1}{2}[2a^2b^2 + 2b^2c^2 + 2c^2a^2 - (a^4+b^4+c^4)]$$
$$= \frac{1}{2}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$= 8K^2,$$

with equality if and only if

$$\frac{a^2}{v+w} = \frac{b^2}{w+u} = \frac{c^2}{u+v}\ .$$

II. *Generalization by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

More generally, we show that for $0 \leq n \leq 1$,

$$\frac{ua^{4n}}{v+w} + \frac{vb^{4n}}{w+u} + \frac{wc^{4n}}{u+v} \geq \frac{3}{2}\left[\frac{4K}{\sqrt{3}}\right]^{2n}. \tag{2}$$

The proposed inequality corresponds to the case $n = 1$.

Letting S be the left side of (2), we have

$$2[S + (a^{4n}+b^{4n}+c^{4n})] = 2\left[\frac{u+v+w}{v+w}\cdot a^{4n} + \frac{u+v+w}{w+u}\cdot b^{4n} + \frac{u+v+w}{u+v}\cdot c^{4n}\right]$$
$$= [(v+w)+(w+u)+(u+v)]\left[\frac{a^{4n}}{v+w} + \frac{b^{4n}}{w+u} + \frac{c^{4n}}{u+v}\right]$$
$$\geq (a^{2n}+b^{2n}+c^{2n})^2 \tag{3}$$

by Cauchy's inequality. Equality holds in (3) if and only if

$$\frac{a^{2n}}{v+w} = \frac{b^{2n}}{w+u} = \frac{c^{2n}}{u+v}\ .$$

(3) is equivalent to

$$2S \geq (a^{2n}+b^{2n}+c^{2n})^2 - 2(a^{4n}+b^{4n}+c^{4n})$$
$$= 2b^{2n}c^{2n} + 2c^{2n}a^{2n} + 2a^{2n}b^{2n} - (a^{4n}+b^{4n}+c^{4n})$$
$$= (a^n+b^n+c^n)(b^n+c^n-a^n)(c^n+a^n-b^n)(a^n+b^n-c^n). \tag{4}$$

Since a, b, c are the edges of a triangle, $\{a^n, b^n, c^n\}$ also form a triangle whose area we denote by F_n. Then (4) becomes

$$S \geq 8F_n^2$$

by Heron's formula. We now obtain (2) by applying Oppenheim's generalization of the Finsler-Hadwiger inequality, i.e.,

$$\frac{4F_n}{\sqrt{3}} \geq \left[\frac{4F}{\sqrt{3}}\right]^n, \quad 0 \leq n \leq 1$$

with equality (for $n \neq 1$) if and only if $a = b = c$. (See A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Publications de la Faculté d'Electrotechnique de l'Université à Belgrade*, No.461 - No.497, (1974) 257-263.)

Note also that the case $n = \frac{1}{2}$ gives the proposer's problem E3150 in the *American Mathematical Monthly* (May 1986, p.400).

It is of interest to compare the given inequality with the not so easily proven, and incomparable, inequality

$$\frac{ub^2c^2}{v + w} + \frac{vc^2a^2}{w + u} + \frac{wa^2b^2}{u + v} \geq 8K^2.$$

(See M.S. Klamkin, Two non-negative quadratic forms, *Elem. der Math.*, 28 (1973) 141-146.)

Finally, note that if we put $a = b = c$, we then have $K^2 = \frac{3}{16}a^4$, and so the given inequality reduces to the known inequality

$$\frac{u}{v + w} + \frac{v}{w + u} + \frac{w}{u + v} \geq \frac{3}{2}$$

for any positive real numbers u, v, w.

III. *Further special cases by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(i) Putting $u = v = w$ into the given inequality, we obtain

$$a^4 + b^4 + c^4 \geq 16K^2$$

which is ([1], 4.10).

(ii) Putting $u = a$, $v = b$, $w = c$, we obtain

$$\frac{a^5}{2s - a} + \frac{b^5}{2s - b} + \frac{c^5}{2s - c} \geq 8K^2,$$

where s is the semiperimeter.

(iii) Putting $u = a^{-4}$, $v = b^{-4}$, $w = c^{-4}$, we obtain

$$\frac{b^4c^4}{b^4 + c^4} + \frac{c^4a^4}{c^4 + a^4} + \frac{a^4b^4}{a^4 + b^4} \geq 8K^2,$$

which, because of

$$\frac{b^4c^4}{b^4 + c^4} \leq \frac{b^2c^2}{2}, \text{ etc.},$$

is stronger than ([1], 4.12).

(iv) Putting $u = a^{-2}$, $v = b^{-2}$, $w = c^{-2}$, we obtain

$$a^2b^2c^2\left[\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\right] \geq 8K^2$$

which, using $4KR = abc$ where R is the circumradius, can be written

$$\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2} \geq \frac{1}{2R^2}.$$

Because of

$$\frac{2}{b^2+c^2} \leq \frac{1}{bc}, \quad \text{etc.},$$

this is stronger than the first inequality of ([1], 5.24).

(v) Putting $u = b + c$, $v = c + a$, $w = a + b$, we obtain

$$\left[\frac{2s-a}{2s+a}\right]a^4 + \left[\frac{2s-b}{2s+b}\right]b^4 + \left[\frac{2s-c}{2s+c}\right]c^4 \geq 8K^2.$$

(vi) Since

$$4K^2 = a^2h_a^2 = b^2h_b^2 = c^2h_c^2 ,$$

where h_a, h_b, h_c are the altitudes, we obtain

$$\frac{u}{v+w}\cdot\frac{a^2}{h_a^2} + \frac{v}{w+u}\cdot\frac{b^2}{h_b^2} + \frac{w}{u+v}\cdot\frac{c^2}{h_c^2} \geq 2.$$

Putting $u = h_a^2$, $v = h_b^2$, $w = h_c^2$, we obtain

$$\frac{a^2}{h_b^2+h_c^2} + \frac{b^2}{h_c^2+h_a^2} + \frac{c^2}{h_a^2+h_b^2} \geq 2,$$

which is ([1], 6.7).

Next we apply Klamkin's duality

$$I(a,b,c,m_a,m_b,m_c,K) \geq 0 \Leftrightarrow I\left[m_a,m_b,m_c,\frac{3a}{4},\frac{3b}{4},\frac{3c}{4},\frac{3K}{4}\right] \geq 0$$

where m_a, m_b, m_c are the medians. (This follows immediately from the fact that the medians of a triangle with area K are themselves sides of a triangle with medians $\frac{3a}{4}$, $\frac{3b}{4}$, $\frac{3c}{4}$ and area $\frac{3K}{4}$. See Klamkin's solution to Aufgabe 677, *Elem. der Math.* 28 (1973) 130.) Tsintsifas' inequality then reads

$$\frac{u}{v+w}m_a^4 + \frac{v}{w+u}m_b^4 + \frac{w}{u+v}m_c^4 \geq \frac{9}{2}K^2. \tag{5}$$

We may now use (5) to obtain more or less interesting duals of (i) - (v). For instance, putting $u = m_a^{-4}$, $v = m_b^{-4}$, $w = m_c^{-4}$ into (5) yields

$$\frac{m_b^4m_c^4}{m_b^4+m_c^4} + \frac{m_c^4m_a^4}{m_c^4+m_a^4} + \frac{m_a^4m_b^4}{m_a^4+m_b^4} \geq \frac{9}{2}K^2,$$

which we show is stronger than ([1], 8.6), i.e.

$$m_a^2 + m_b^2 + m_c^2 \geq 3\sqrt{3}K.$$

Squaring, this becomes

$$(m_a^4 + m_b^4 + m_c^4) + 2(m_a^2m_b^2 + m_b^2m_c^2 + m_c^2m_a^2) \geq 27K^2,$$

and thus we have to show that

$$6\sum \frac{m_b^4m_c^4}{m_b^4 + m_c^4} \leq \Sigma\, m_b^4 + 2\,\Sigma\, m_b^2m_c^2 .$$

But this follows because of

$$4\left[\frac{m_b^4m_c^4}{m_b^4 + m_c^4}\right] \leq 2m_b^2m_c^2 , \quad \text{etc.}$$

and

$$2\left[\frac{m_b^4m_c^4}{m_b^4 + m_c^4}\right] \leq \frac{m_b^4 + m_c^4}{2} , \quad \text{etc.}$$

Reference:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

Also solved by C. FESTRAETS-HAMOIR, Bruxelles, Belgique; J.T. GROENMAN, Arnhem, The Netherlands; VEDULA N. MURTY, Pennsylvania State University, Middletown, Pennsylvania; and the proposer. There was a further solution, author unknown, sent to this office by Svetoslav J. Bilchev of Russe, Bulgaria, who received it from the editor of KVANT, *A.B. Sosinskij.*

Janous also obtained Klamkin's generalization (2), using the same method. A couple of Janous' special cases were noticed by other solvers, in particular (i) *by Klamkin and Murty, and* (iv) *by Murty and the proposer.*

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1052*. [1985: 187] *From a Trinity College, Cambridge, examination paper dated June 5, 1901.*

Prove that

$$\frac{1}{1^2\cdot3^3\cdot5^2} - \frac{1}{3^2\cdot5^3\cdot7^2} + \frac{1}{5^2\cdot7^3\cdot9^2} - \ldots = \frac{1}{9} - \frac{\pi}{2^6} - \frac{\pi^3}{2^9} .$$

Solution by George Szekeres, University of New South Wales, Kensington, Australia.

How would a Trinity student of 1901 evaluate this sum? He may choose from a number of different methods.

I. *Partial fractions.*

First,

$$\frac{1}{(n-2)^2n^3(n+2)^2} = \frac{1}{32n} + \frac{1}{16n^3} - \frac{1}{64}\left[\frac{1}{n-2} + \frac{1}{n+2}\right] + \frac{1}{128}\left[\frac{1}{(n-2)^2} - \frac{1}{(n+2)^2}\right]$$

(not very hard to determine the coefficients, the answer in a way gives away the expected form of the partial fraction decomposition).

From here the required sum is

$$\sum_{\substack{n=3\\ n \text{ odd}}}^{\infty} \frac{(-1)^{(n+1)/2}}{(n-2)^2n^3(n+2)^2} = \frac{1}{32} - \frac{1}{32}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right] + \frac{1}{16} - \frac{1}{16}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right]$$

$$+ \frac{1}{64}\left[1 - \frac{1}{3}\right] - \frac{1}{32}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right] + \frac{1}{128}\left[1 - \frac{1}{9}\right]$$

$$= \frac{1}{32} + \frac{1}{16} + \frac{1}{32\cdot 3} + \frac{1}{16\cdot 9} - \frac{1}{16}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right]$$

$$- \frac{1}{16}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right]$$

$$= \frac{1}{9} - \frac{1}{16}\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots\right] - \frac{1}{16}\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right].$$

The answer follows now from

$$1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$$

and

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots = \frac{\pi^3}{32}.$$

The first one is trivial from the arctan series, the second is Euler's. This method is perhaps the most straightforward but assumes that the student remembers the second sum. Of course he can still bluff his way through the problem by working backwards from the given answer.

II. *Theory of residues.*

This is no doubt the most direct method (my own favourite) and best suited for examination conditions as it does not require the student to memorize anything.

Note that the required sum is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^3(2n+3)^2} = \frac{1}{2}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^3(2n+3)^2} + \sum_{n=-2}^{-\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^3(2n+3)^2}\right]$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^3(2n+3)^2} - \frac{1}{2}\left[\frac{(-1)^{-1}}{(-1)^2(1)^3(3)^2} + \frac{(-1)^{-2}}{(-3)^2(-1)^3(1)^2}\right]$$

$$= \frac{1}{2}S + \frac{1}{9},$$

where

$$S = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^3(2n+3)^2}.$$

To evaluate S, we consider the function

$$f(z) = \frac{\pi}{(z-1)^2(z+1)^2z^3\cos \pi z}.$$

Let N be a positive integer and let C be the square in the complex plane having corners $(\pm N, \pm N)$. From the fact that $|\cos \pi z| \geq 1$ for z on C we see that

$$\lim_{N\to\infty} \int_C f(z)dz = 0.$$

Hence the sum of all the residues of $f(z)$ is zero. On the other hand, $f(z)$ has simple poles at $z = k + \frac{1}{2}$ (k and integer), poles of order 2 at $z = \pm 1$, and a pole of order 3 at $z = 0$. The residue of $f(z)$ at $k + \frac{1}{2}$ is

$$\lim_{z\to k+\frac{1}{2}} \frac{\pi\left[z - k - \frac{1}{2}\right]}{(z-1)^2(z+1)^2z^3\cos \pi z} = \frac{\pi}{\left[k - \frac{1}{2}\right]^2\left[k + \frac{3}{2}\right]^2\left[k + \frac{1}{2}\right]^3} \cdot \lim_{z\to k+\frac{1}{2}} \frac{z - k - \frac{1}{2}}{\cos \pi z}$$

$$= \frac{128\pi}{(2k-1)^2(2k+1)^3(2k+3)^2} \lim_{z\to k+\frac{1}{2}} \frac{1}{-\pi \sin \pi z}$$

$$= \frac{(-1)^{k-1}\cdot 128}{(2k-1)^2(2k+1)^3(2k+3)^2}.$$

The residue of $f(z)$ at $z = 1$ is

$$\lim_{z\to 1} \frac{d}{dz}\left[\frac{\pi}{(z+1)^2 z^3 \cos \pi z}\right] = \pi ,$$

the residue of $f(z)$ at $z = -1$ is similarly

$$\lim_{z\to -1} \frac{d}{dz}\left[\frac{\pi}{(z-1)^2 z^3 \cos \pi z}\right] = \pi ,$$

and the residue of $f(z)$ at $z = 0$ is

$$\lim_{z\to 0} \left\{ \frac{1}{2}\cdot\frac{d^2}{dz^2}\left[\frac{\pi}{(z-1)^2(z+1)^2 \cos \pi z}\right] \right\} = \frac{\pi^3}{2} + 2\pi .$$

Thus

$$128S + \pi + \pi + \frac{\pi^3}{2} + 2\pi = 0,$$

$$S = -\frac{\pi}{32} - \frac{\pi^3}{256} ,$$

and so the required sum is

$$\frac{1}{2} S + \frac{1}{9} = \frac{1}{9} - \frac{\pi}{64} - \frac{\pi^3}{512} .$$

III. *Fourier series.*

This is more elaborate than the first two, but gives quite a bit more (definitely not recommended to the run-of-the-mill student).

Let S be as before and define

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{\sin(2n+1)x}{(2n-1)^2(2n+1)^3(2n+3)^2} ; \qquad (1)$$

then clearly

$$S = -F\left(\frac{\pi}{2}\right) .$$

The following two Fourier expansions are well known (any standard textbook, e.g. Knopp, *Unendliche Reihen*, 386-388, or Apostol, *Mathematical Analysis*, 501, #15-8):

$$\sum_{n=-\infty}^{\infty} \frac{1}{2n-1} \sin(2n-1)x = \frac{\pi}{2} \qquad (2)$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x = \frac{\pi^2}{4} - \frac{\pi x}{2} , \qquad (3)$$

both valid for $0 < x < \pi$. There are corresponding formulae for the interval $(\pi, 2\pi)$ but we need not bother about them. Here we go.

First, $F'''(x) = -G(x)$, where

$$G(x) = \sum_{n=-\infty}^{\infty} \frac{\cos(2n+1)x}{(2n-1)^2(2n+3)^2} .$$

Then

$$G''(x) = -\sum_{n=-\infty}^{\infty} \frac{(2n+1)^2\cos(2n+1)x}{(2n-1)^2(2n+3)^2}$$

$$= -\frac{1}{4}\sum_{n=-\infty}^{\infty} \left[\frac{1}{2n-1} + \frac{1}{2n+3}\right]^2 \cos(2n+1)x$$

$$= -\frac{1}{4}\sum_{n=-\infty}^{\infty} \frac{\cos(2n+1)x + \cos(2n-3)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{\cos(2n+1)x}{(2n-1)(2n+3)}$$

$$= -\frac{1}{2}\cos(2x)\sum_{n=-\infty}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}H(x) \tag{4}$$

where

$$H(x) = \sum_{n=-\infty}^{\infty} \frac{\cos(2n+1)x}{(2n-1)(2n+3)}$$

and we used the identity

$$\cos A + \cos B = 2\cos\frac{A-B}{2}\cos\frac{A+B}{2} .$$

Thus

$$H'(x) = -\sum_{n=-\infty}^{\infty} \frac{(2n+1)(\sin(2n+1)x}{(2n-1)(2n+3)}$$

$$= -\frac{1}{2}\sum_{n=-\infty}^{\infty} \left[\frac{1}{2n-1} + \frac{1}{2n+3}\right]\sin(2n+1)x$$

$$= -\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{\sin(2n+1)x + \sin(2n-3)x}{2n-1}$$

$$= -\cos(2x) \sum_{n=-\infty}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

$$= -\frac{\pi}{2} \cos 2x,$$

by (2) and the identity

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} .$$

Integrating, and observing that $H\left(\frac{\pi}{2}\right) = 0$, we get

$$H(x) = -\frac{\pi}{4} \sin 2x,$$

hence from (3) and (4)

$$G''(x) = -\frac{1}{2} \cos(2x) \cdot \left(\frac{\pi^2}{4} - \frac{\pi x}{2}\right) + \frac{\pi}{8} \sin 2x$$

$$= \frac{\pi}{8} \sin 2x - \frac{\pi^2}{8} \cos 2x + \frac{\pi x}{4} \cos 2x.$$

Using

$$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}$$

and

$$\int x \sin 2x \, dx = \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4}$$

and noting $G'(0) = 0$, $G\left(\frac{\pi}{2}\right) = 0$ we get

$$G'(x) = -\frac{\pi^2}{16} \sin 2x + \frac{\pi x}{8} \sin 2x$$

and

$$G(x) = \frac{\pi^2}{32} \cos 2x - \frac{\pi x}{16} \cos 2x + \frac{\pi}{32} \sin 2x.$$

Thus

$$F'''(x) = -\frac{\pi^2}{32} \cos 2x + \frac{\pi x}{16} \cos 2x - \frac{\pi}{32} \sin 2x.$$

Finally, integrating three times and observing that $F''(0) = 0$, $F'\left(\frac{\pi}{2}\right) = 0$, $F(0) = 0$, we obtain successively

$$F''(x) = -\frac{\pi^2}{64} \sin 2x + \frac{\pi x}{32} \sin 2x + \frac{\pi}{32} \cos 2x - \frac{\pi}{32} ,$$

$$F'(x) = \frac{\pi^2}{128} \cos 2x - \frac{\pi x}{64} \cos 2x + \frac{3\pi}{128} \sin 2x - \frac{\pi x}{32} + \frac{\pi^2}{64} ,$$

and

$$F(x) = \frac{\pi^2}{256} \sin 2x - \frac{\pi x}{128} \sin 2x - \frac{\pi}{64} \cos 2x - \frac{\pi x^2}{64} + \frac{\pi^2 x}{64} + \frac{\pi}{64} . \tag{5}$$

Thus

$$S = -F\left[\frac{\pi}{2}\right] = -\frac{\pi}{64} + \frac{\pi^3}{256} - \frac{\pi^3}{128} - \frac{\pi}{64} = -\frac{\pi}{32} - \frac{\pi^3}{256},$$

as required.

Admittedly the Fourier argument is heavier than the other two, but as a bonus, formula (5) evaluates the series (1) for all $0 \le x \le \pi$.

Also solved by C. FESTRAETS-HAMOIR, Bruxelles, Belgique; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE-WAI LAU, Hong Kong; R.C. LYNESS, Southwold, Suffolk, England; and BASIL C. RENNIE, James Cook University, Townsville, Australia.

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1053. [1985: 187] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Exhibit a bijection between the points in the plane and the lines in the plane.

Solution by Leroy F. Meyers, The Ohio State University, Columbus, Ohio.

Every nonvertical line in the (Euclidean) plane has a unique equation of the form $y = mx + b$, where m and b are arbitrary real numbers, and conversely every pair (m,b) of real numbers determines a nonvertical line with equation $y = mx + b$. Since the vertical lines haven't been included in the correspondence, we must modify the correspondence. Thus, to each vertical line with equation $x = b$ we associate the point $(0,b)$; to each nonvertical line with equation $y = mx + b$ we associate the point $(m + 1,b)$ if m is a nonnegative integer and the point (m,b) if m is otherwise. This is the required one-to-one correspondence.

Also solved by J. SUCK, Essen, Federal Republic of Germany; and M.M. PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland. There was one incorrect solution. Two readers submitted solutions for the projective plane (on the surface, an easier problem). I flatly decided to omit these.

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