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CONTENTS

THE OLYMPIAD CORNER: 78

M.S. KLAMKIN

All communications about this column should be sent to M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

At the Fifth International Congress on Mathematical Education (I.C.M.E.-5) held in Adelaide, Australia during August 24-30, 1984, one of the symposiums given was "A Kaleidoscope of Competitions". This session was devoted to a look at some of the mathematical competitions that are given in various parts of the world. Very brief summaries of these competitions appeared fairly recently in the Proceedings of I.C.M.E.-5 (Birkhäuser Boston, Inc., 1986, pp.249-250). My presentation at this symposium was on the William Lowell Putnam Mathematical Competition; in particular, I contrasted its time constraints with other competitions. Since this criticism with respect to the time constraint and other matters was not published, I now give the paper (re-edited) which I had submitted. Due to time constraints on the symposium, not all the material in this paper was given in my talk.

The William Lowell Putnam Mathematical Competition

The Putnam competition began in 1938 and was designed to stimulate a healthy rivalry in mathematical studies among the colleges and universities of the U.S.A. and Canada. Its impact on the study of mathematics by gifted college students has been considerable since its inception. Quite a number of the top contestants have become prominent mathematicians. In this respect, it rivals the classic Tripos in Cambridge and the influential Eötvös competition in Hungary.

The competition is given yearly and is open officially only to regularly enrolled undergraduates, in colleges and universities of the U.S.A. and Canada, who have not yet received a college degree. Also, no individual may participate in the competition more than four times. There are individual and team prizes. Teams consist of three members and must be designated a priori. This helps reduce the advantage of a university being large. All contestants, including team members, work independently on the problems. There were 2079 individual contestants from 348 colleges and universities in the 1985 competition. Teams were entered by 264 institutions.

The competition is given in two parts on the first Saturday of December. The first part is given in the morning, 9:00-12:00 A.M., and the second part in the afternoon, 2:00-5:00 P.M. with each part consisting of six problems. The character of the problems now is different from that of the earlier years. For examples, see the two Putnam Problem Books mentioned subsequently. The problems are prepared by a rotating three member committee in which one member is replaced by a new one each year. Some of the most distinguished mathematicians of the U.S.A. and Canada have served on this committee. In the earlier years, these have included Professors Tibor Rado, George Polya, Mark Kac, Irving Kaplansky, Orrin Frink Jr., Bancroft H. Brown, Emory P. Starke, Ralph G. Stanger, Andrew M. Gleason, Leroy M. Kelly, R.E. Greenwood, Leo Moser, W.R. Scott, Richard E. Bellman, Ivan Niven, D.E. Richmond, John M. Olmsted, Gian-Carlo Rota, H.S.M. Coxeter, and A.M. Garsia.

The book *The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1938-1964* by A.M. Gleason, R.E. Greenwood, and L.M. Kelly, M.A.A., 1980, contains the first twenty-five competitions with multiple detailed solutions and extensions. Also reprinted in the book are four articles by G. Birkhoff, L.E. Bush, L.J. Mordell, and L.M. Kelly dealing with the early history, the later history and a summary of results, and an examination in depth of both some of the weaknesses and some of the strengths of the competition. A second book, published in 1985 and edited by G.L. Alexanderson, L.F. Klosinski, and L.C. Larson, contains the Problems and Solutions: 1965-1984. This book, as noted by the editors themselves, is not as scholarly or as comprehensive as the earlier one.

Each year, the problems and solutions and a listing of the top ranking contestants and teams are published in the American Mathematical Monthly. The five highest-ranking contestants are designated as Putnam Fellows and each one receives a cash award of $500. Also, one of these five contestants is now awarded a scholarship of up to $12,000 plus tuition at Harvard University. The next five highest-ranking contestants each receive $250. The team prize for the first ranking team is $5000 to their department of mathematics and $250 to each team member. The corresponding cash awards to the second, third, fourth and fifth ranking teams are $2500, $200; $1500, $150; $1000, $100; and $500, $50; respectively.

I have been involved in the Putnam competition in almost all respects: (1) I participated in the 2nd, 3rd, 4th and 5th competitions. My school was Cooper Union, a very small college consisting solely (apart from the art school) of some 400 engineering majors. We received a 3rd place plus three honorable mentions in 4 successive years. (2) I have assisted in the grading of the Putnam papers several times. (3) I coached students for the Putnam at the Polytechnic Institute of New York (formerly of Brooklyn), the University of Waterloo and the University of Alberta, the former two schools winning 1st places. (4) I have been on the Putnam Examination Committee for the usual 3 year period. (5) (Pre-Putnam) I have been on the Canadian Olympiad Committee for 10 years, chairman of the U.S.A. Mathematical Olympiad Examination Committee for 14 years and a coach of the U.S.A. Mathematical Olympiad Team for 10 years. Since in the coaching we stressed problem write-ups as well as giving selected Putnam problems for practice, it is not too surprising that the top contestants of the U.S.A.M.O. became the top contestants of the Putnam, a number of them even in their first college year.

In view of the above participation, it should not be surprising that despite the continued long and successful impact of the Putnam Competition, I still have a few criticisms of it. The first one concerns the originality of the problems. Since there are so many different mathematical competitions around the world, it has become increasingly difficult to come up with new, appropriate, and challenging problems each year. (Incidentally, Professor Arthur Engel and I gave talks on this very point in another I.C.M.E.-5 session.) A number of the Putnam problems have appeared previously in books or other prior competitions, e.g., the International Mathematical Olympiad (I.M.O.) for secondary school students. One possible remedy is for Examination Committees not to use problems simply taken from problem collections and to carefully check other mathematical competition problems on a periodic basis. Another way is to try to originate "new" problems. However, this does not always guarantee that the problem has not been discovered independently and used previously.

In the Putnam as well as some other competitions (e.g., I.M.O., U.S.A.M.O.) quite a number of the total scores of individuals have been "complete zeros". Because of this, there have been pressures to eliminate these zero scores by introducing one or two "easy" problems. I can go along with this provided the competition is a less demanding one, e.g., the American High School Mathematics Examination (A.H.S.M.E.) which is given to some 400,000 students and is the first step to becoming eligible for the U.S.A.M.O. However, the Putnam, the I.M.O. and the U.S.A.M.O. are high level competitions and even the "easy" problems should still present "some challenge". Examples of several problems which in my view offered too little challenge are the following:

#B-1 (Putnam, 1983). *Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.*

COMMENT: I can see using this on a secondary school competition.

#1 (I.M.O., 1984). *Prove that*

$$0 \leq yz + zx + xy - 2xyz \leq 7/27,$$

where x, y, z *are non-negative real numbers for which* $x + y + z = 1$.

COMMENT: As leader of the U.S.A. team and consequently a member of the I.M.O. jury, I tried to have the left hand side inequality thrown out since it follows immediately from the well known inequality

$$(x + y + z)(1/x + 1/y + 1/z) \geq 9.$$

Even more elementarily, it follows immediately from $yz, zx, xy \geq xyz$. The jury voted overwhelmingly to keep it in. This part was worth 2 of the 7 points for the problem. I suppose that it is too much to expect that there will not be non-mathematical reasons involved in the choice of the I.M.O. problems used.

#1 (U.S.A.M.O., 1984). *The product of two of the four roots of the quartic equation* $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ *is* -32. *Determine the value of* k.

COMMENT: As chairman of the U.S.A.M.O. Committee, I was partially responsible for the inclusion of this problem. Since in retrospect practically all the contestants got this problem, it should not have been used. It served little purpose in the ranking of the contestants.

My last criticism is my major one. It concerns the amount of time alloted for the competition. Certainly "time" is a real constraint in our world but it is not quite the constraint which a number of competition and school examination makers make it out to be. In all the competitions that I have been associated with (A.H.S.M.E., A.I.M.E., U.S.A.M.O., Canadian Math. Olympiad, and the Putnam), I have endeavored for a long time to get more time per problem. In this regard, I and like minded colleagues have been successful in all of these competitions except the Putnam. Since in the Putnam, there are two sets of six problems to be done in two 3-hour time periods with a 2-hour break in between, this puts a premium on being fast as well as being good. To me a more appropriate timing for the Putnam would be a two day competition with four 3-hour sessions with 3 problems per session. A less drastic change would be to reduce the two sets of 6 problems to 5 or even 4 problems each.

The actual Putnam timing should be contrasted with that of the U.S.A.M.O., the I.M.O. and the Hungarian Schweitzer Competition[1]. In the U.S.A.M.O. there were originally 5 problems to be done in 3 hours. Subsequently, I argued for a change from 3 hours to 4 hours and this was accepted by the U.S.A.M.O. Committee at the time. However, due to difficulties in getting secondary school teachers to proctor this exam, the time alloted became 3-1/2 hours. In the I.M.O., there is more adequate time. There are two 4-1/2 hour sessions on consecutive days, each consisting of 3 problems (the extra 1/2 hour each day is for possible translational difficulties). A number of countries which participate in the I.M.O. use the same format in their National Olympiads. Perhaps the U.S.A. should follow suit. To me, the best timing setup of any mathematical competition is in the Schweitzer which is open to high school students, undergraduates, and those students who have just received their college degrees. There are ten problems to be done usually in seven to ten days. One can use any books but no personal help. This more nearly approximates the way mathematical research is done and may give some indication why Hungary has a disproportionately high number of top level mathematicians.

At the opposite end of the timing spectrum to the Schweitzer is the secondary school competition "Number Sense" which is given in Texas. This contest consists of 80 elementary problems in numerics, algebra, geometry, analytic geometry and elementary calculus to be done mentally in 10 minutes. That is 7-1/2 seconds per problem!! You are scored 5 points for each correct answer and -4 points for each problem solved incorrectly or skipped. Problems coming after the last attempted problem are not considered as skipped. There are a number of starred problems in which you only have to get within 5% of the correct numerical answer. Personally, I think the amount of time alloted

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[1] For a further description and copies of the problems and solutions of the U.S.A.M.O., the I.M.O., and the Schweitzer, see *International Mathematical Olympiads*, 1959-1977, compiled with solutions by S.L. Greitzer, NML vol. 27, M.A.A., Washington, D.C., 1978; G. Szasz et al (eds.), *Contests in Higher Mathematics, Hungary* 1949-1961, Akademiai Kiado, Budapest, 1968; and selected issues of *Crux Mathematicorum* and *Mathematics Magazine*.

for this contest is absurd. Nevertheless, I have been informed that there is much student enthusiasm for this contest and that the top scores are in the 250-300 range. I am rather skeptical concerning these high scores.

Unfortunately, in many universities these unrealistic timing considerations extend to final examinations. In my university, finals are mainly of 2-hour duration. Consequently, all one can really give are essentially routine type questions. Hopefully in the future more time will be alloted per problem in the Putnam Competition as well as in the final examinations in my university.

*

I conclude this corner with some new problems and I am grateful to Gregg Patruno and Cecil Rousseau for their transmission. These were proposed by students at the 1986 U.S.A.M.O. Training Session.

1. *Proposed by Daniel Lee, Houghton, Michigan.*

Prove that $(a^a b^b)^{2ab} \le (a^{a^2} b^{b^2})^{a+b}$ where $a,b > 0$.

2. *Proposed by Michael Zieve, Midlothian, Virginia.*

Determine the maximum value of

$$x^3 + y^3 + z^3 - x^2y - y^2z - z^2x$$

for $0 \le x,y,z \le 1$.

3. *Proposed by Jeremy Kahn, New York, N.Y.*

Can a solid figure have a nonzero even number of axes of "reflection"? (A rotation of 180° about an axis of reflection takes the body into itself.)

4*. *Proposed by Andrew Yeh, Binghamton, New York.*

The rational functions $F(x) = a + b - a^2/(x - b)$, $a \ne 0$, have the property that $F(F(F(x))) = x$ on $\mathbb{R} - \{b, a + b\}$. Is $F(x) = x$ the only real-valued function that has this property on all of $\mathbb{R}$?

5*. *Proposed by Thomas Chung, Apple Valley, Minnesota.*

Points O_1, O_2 are centers of disjoint circles C_1, C_2, respectively. Show how to construct points A_1, A_2, in or on C_1, C_2, respectively, such that segments A_1A_2 and O_1O_2 have the same length and intersect at a maximum angle.

6*. *Proposed by John Overdeck, Columbia, Maryland.*

Johnny is playing on the ground with three coins of unit radii. On each turn, he shoots one coin between the other two, but he is not allowed to shoot the same coin twice in a row. What is the radius of the smallest circular boundary within which Johnny can play this game forever?

*　　　　　*　　　　　*

GEOFFREY JAMES BUTLER, 1944-1986

It is with the deepest regret that I inform you of the passing away of Professor Geoffrey James Butler of the University of Alberta. He was well known to the readers of this Corner in his various positions as the Executive Secretary, former Chairman and long-time member of the Alberta High School Mathematics Prize Examination Board, Chairman of the Canadian Mathematical Olympiad Committee (1981-1983) and the Canadian International Mathematical Olympiad Committee (1984-1986) as well as the Leader of the Canadian Team in the International Mathematical Olympiads (1981-1984).

Geof Butler was born March 4, 1944 in Gillingham, England. He grew up and received his early schooling in Bournemouth. In 1965 he earned a B.Sc. (Special) degree and in 1969 his Ph.D., both at University College, London, England, the latter under the tutelage of C.A. Rogers.

In 1968 he came to the University of Alberta as a post-doctoral fellow, was appointed to the academic staff in 1971, promoted to Associate Professor in 1974 and Professor in 1980.

His research activities included convexity, ordinary differential equations, and modeling in population biology. He supervised three Ph.D. students, J. Chapin, G.S.K. Wolkowicz and J. Roessler.

He has presented many papers in several continents. In 1982 he was awarded a McCalla Professorship for excellence in research. Just before his illness he was appointed Chairman of the Mathematics Department.

Geof died on July 13, 1986, peacefully in his sleep, only 70 days after being diagnosed as having cancer. He is sorely missed by those who knew and loved him, but his kind nature, his good thoughts and his marvelous ideas will be with us always.

THE GEOFFREY JAMES BUTLER MEMORIAL FUND

The Mathematics Department, University of Alberta wishes to announce the establishment of the Geoffrey James Butler Memorial Fund. Moneys from this fund will be used for student scholarships at all levels, as well as to fund the Geoffrey James Butler Memorial Lectures, the first of which will be given at a conference dedicated to Dr. Butler during the summer of 1988 at the University of Alberta, details of which will be announced in due course.

Tax deductible contributions payable to the University of Alberta should be sent to

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Edmonton, Alberta
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* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1987, although solutions received after that date will also be considered until the time when a solution is published.

1171*. *Proposed by D.S. Mitrinovic and J.E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)

(i) Determine all real numbers λ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b+c)^\lambda\ ,\ (c+a)^\lambda\ ,\ (a+b)^\lambda .$$

(For $\lambda = -1$ we have *Crux* 14 [1975: 28].)

(ii) Determine all pairs of real numbers λ, μ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b+c+\mu a)^\lambda\ ,\ (c+a+\mu b)^\lambda\ ,\ (a+b+\mu c)^\lambda .$$

1172. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Show that for any triangle ABC, and for any real $\lambda \geq 1$,

$$\Sigma\ (a+b)\sec^{\lambda}\frac{C}{2} \geq 4(2/\sqrt{3})^{\lambda}s,$$

where the sum is cyclic over ΔABC and s is the semiperimeter.

1173. *Proposed by Jordi Dou, Barcelona, Spain.*

A knight is placed at random on a square of a chessboard. It then makes a sequence of legal moves, each chosen randomly from all possible legal moves. Find the average number of moves it has made when it first makes a move which is the reverse of its preceding move.

1174. *Proposed by Clark Kimberling, University of Evansville, Evansville, Indiana.*

Suppose ABC is an acute triangle. Prove that there is a point P inside ABC and points D, E on BC; F, G on CA; and H, I on AB such that GPH, IPD, and EPF are congruent equilateral triangles.

1175. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Prove that if α, β, γ are the angles of a triangle,

$$-2 < \sin 3\alpha + \sin 3\beta + \sin 3\gamma \leq \frac{3}{2}\sqrt{3}.$$

1176. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.* (Dedicated to Léo Sauvé.)

Let n be squarefree such that

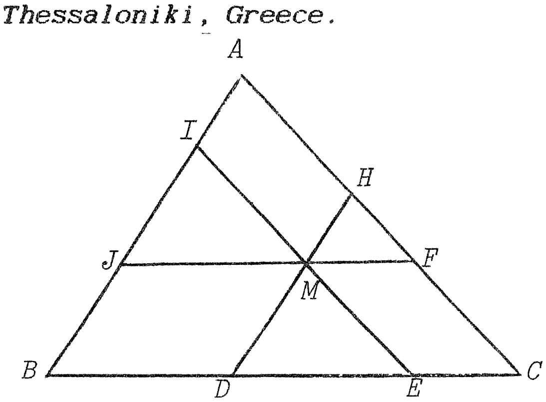
$$n = r^2 + s^2 = t^2 + u^2$$

where r, s, t, u are positive integers. Prove that

$$2n(n + rt + su)$$

is a square if and only if $r = t$ and $s = u$.

1177. *Proposed by George Tsintsifas, Thessaloniki, Greece.*



ABC is a triangle and M an interior point with barycentric coordinates $(\lambda_1,\lambda_2,\lambda_3)$. Lines HMD, JMF, EMI are parallel to AB, BC, CA respectively as shown. The centroids of triangles DME, FMH, IMJ are denoted G_1, G_2, G_3 respectively. Prove that

$$[G_1G_2G_3] = \frac{(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)[ABC]}{3},$$

where $[X]$ stands for the area of figure X.

1178. *Proposed by Gary Gislason, University of Alaska, Fairbanks, Alaska, and M.S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

Determine pairs of functions (F,G) such that

$$(F \circ G)' = F \circ G' + F' \circ G$$

where $\circ$ denotes composition and $'$ denotes differentiation.

1179. *Proposed by Jack Garfunkel, Flushing, New York.*

Squares are erected outwardly on each side of a quadrilateral $ABCD$.

(a) Prove that the centers of these squares are the vertices of a quadrilateral $A'B'C'D'$ whose diagonals are equal and perpendicular to each other.

(b)* If squares are likewise erected on the sides of $A'B'C'D'$, with centers A'', B'', C'', D'', and this procedure is continued, will quadrilateral $A^{(n)}B^{(n)}C^{(n)}D^{(n)}$ tend to a square as n tends to infinity?

1180. *Proposed by J.R. Pounder, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)

(a) It is well known that the Simson line of a point P on the circumcircle of a triangle T envelopes a deltoid ("Steiner's hypocycloid") as P varies. Show that this is true for an oblique Simson line as well. (An *oblique* Simson line of $T = ABC$ is the line passing through the points A_1, B_1, C_1 chosen on edges BC, CA, AB respectively so that the lines PA_1, PB_1, PC_1 make equal angles (say θ), in the same sense of rotation, with BC, CA, AB respectively. The usual Simson line occurs when $\theta = 90^\circ$.)

(b)* Given such an "oblique" deltoid for T, locate a triangle T' similar to T such that the "normal" deltoid for T' and the oblique deltoid for T coincide.

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1033*. [1985: 121] *Proposed by W.R. Utz, University of Missouri-Columbia.*

Let D_n be any symmetric determinant of order n in which the elements in the principal diagonal are all 1's and all other elements are either 1's or -1's, and let $\overline{D}_n$ be the determinant obtained from D_n by replacing the non-principal-diagonal elements by their negatives. It is easy to show that $D_2\overline{D}_2 = 0$ for all D_2 and $D_3\overline{D}_3 = 0$ for all D_3. For which $n > 3$ is it true that $D_n\overline{D}_n = 0$ for all D_n?

I. *Partial solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

We show that when $n = 4$, $D_n\overline{D}_n = 0$ for all D_n and give examples to show that this is not true when $n = 5$, 6, or 7.

Let $n = 4$ and let D_4 denote any 4×4 determinant satisfying the given conditions. Since multiplying a row and the corresponding column of D_4 by -1 will not change the determinant's value, we may assume without loss of generality that the first row and first column consist of only 1's, i.e.

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & a & b \\ 1 & a & 1 & c \\ 1 & b & c & 1 \end{vmatrix}.$$

- If $a = b = 1$, then clearly $D_4 = 0$.
- If $a = 1$, $b = -1$, then

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & c \\ 1 & -1 & c & 1 \end{vmatrix}$$

which is 0 since the 1st row and the 3rd row are the same when $c = 1$, and the 2nd row and the 3rd row are the same when $c = -1$.

- If $a = -1$, $b = 1$, then

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & c \\ 1 & 1 & c & 1 \end{vmatrix}$$

which is 0 since the 4th row equals the 1st or 2nd row depending on whether $c = 1$ or -1.

- Finally, if $a = b = -1$, then

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & c \\ 1 & -1 & c & 1 \end{vmatrix},$$

so

$$\overline{D}_4 = \begin{vmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -c \\ -1 & 1 & -c & 1 \end{vmatrix} = 0$$

since the first two rows are proportional.

For $n = 5$, consider

$$D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{vmatrix}.$$

Then it is easily verified that

$$D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 & 0 \end{vmatrix} = (-2)^4 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{vmatrix} = 2^4 \neq 0.$$

On the other hand,

$$\overline{D}_5 = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{vmatrix} = (-2)^4 \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 2^4 \neq 0.$$

For $n = 6$, consider

$$D_6 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 \end{vmatrix}.$$

Then

$$D_6 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \end{vmatrix} = (-2)^5 \begin{vmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{vmatrix}$$

$$= -2^5 \begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} = -2^5 \left[- \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \right]$$

$$= -2^5 \left[\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \right] = -2^6 \neq 0.$$

On the other hand,

$$\overline{D_6} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \end{vmatrix}$$

$$= (-2)^5 \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 \end{vmatrix} = -2^5 \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

$$= -2^5 \left[- \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \right]$$

$$= -2^5 \left[\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \right]$$

$$= -2^6 \neq 0.$$

For $n = 7$, we use a similar construction as in the case $n = 5$, thus:

$$D_7 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 & 0 & -2 & 0 \end{vmatrix} = -2^6 \neq 0,$$

$$\overline{D}_7 = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 & 0 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 0 & 0 \end{vmatrix}$$

$$= (-2)^6 \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{vmatrix} = 2^6 \cdot 3 \neq 0.$$

In general for odd n, we can try to construct D_n in the same way as D_5 and D_7 above. In fact, note that if (i) all the entries on the main diagonal are 1, (ii) all the entries on and above the "secondary" diagonal (the diagonal that runs from upper right to lower left) are 1, and (iii) the $n - 1$ entries just below the secondary diagonal are -1, then regardless of the values of the remaining entries,

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 & 1 \\ 1 & 1 & 1 & \cdots & & & 1 & -1 \\ 1 & 1 & 1 & \cdots & & 1 & -1 & \\ \vdots & \vdots & \vdots & & 1 & -1 & & \\ & & & & -1 & 1 & & \\ & & 1 & & & & & \\ 1 & 1 & -1 & & & & 1 & \\ 1 & -1 & & & & & & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & & & 1 & 1 \\ 0 & 0 & 0 & \cdots & & & 0 & -2 \\ 0 & 0 & & \cdots & & 0 & -2 & \\ \vdots & & & & & \cdot & & \\ & & & & \cdot & & & \\ & & & \cdot & & & & \\ & & \cdot & & & & & \\ 0 & -2 & & & & & & \end{vmatrix}$$

$$= (-1)^{(n-1)/2}\, 2^{n-1}$$
$$\neq 0.$$

Since it is highly unlikely that $\overline{D}_n = 0$ for *all* such D_n's, this suggests that the condition of the problem will fail for all odd $n > 7$. Unfortunately a simple way of evaluating $\overline{D}_n$ seems to elude me for the time being.

Note finally that the above determinants all contain powers of 2 as factors. This is, of course, no accident, for it is well-known and easy to prove that if A is an $n \times n$ $(1,-1)$-matrix, then $\det A \equiv 0 \pmod{2^{n-1}}$. For instance, see problem 526, page 102 of D.K. Faddeev and I.S. Sominskii, *Problems in Higher Algebra* (translated by J.L. Brenner), Freeman, San Francisco, 1965.

II. *Partial solution by the proposer.*

For $n = 5$, let

$$D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{vmatrix};$$

then $D_5\overline{D}_5 \neq 0$. In general when $n = 2k + 1$, let $D_{2k+1} = (a_{ij})$, where $a_{ij} = a_{ji}$, $a_{1j} = 1$ for all j, and for $2 \leq i \leq j$,

$$a_{ij} = \begin{cases} -1 & \text{if } k + 2 \leq j \leq 2k + 3 - i \\ 1 & \text{otherwise.} \end{cases}$$

Then $D_{2k+1} \neq 0$ is easy to show. I expect also that $\overline{D}_{2k+1} \neq 0$.

For $n = 6$, let

$$D_6 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \end{vmatrix};$$

then $D_6\overline{D}_6 \neq 0$.

The proposer also solved the case $n = 4$.

This problem remains mostly unsolved but is surely do-able. How about it, dear readers?

* * *

<u>1034</u>. [1985: 121] *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Find all positive integers n such that $n^4 + 1$ has a divisor of the form $dn - 1$, where d is a positive integer.

Solution by the proposer.

Let

$$n^4 + 1 = (dn - 1)D$$

for some positive integers n, d, and D. Reducing this equation to a congruence modulo n, we have

$$D \equiv -1 \pmod{n}.$$

Hence there is a positive integer r such that $D = rn - 1$, so

$$n^4 + 1 = (dn - 1)(rn - 1). \tag{1}$$

We consider the set T of all triples (d,n,r) which satisfy (1). Clearly, if (d,n,r) is in T, so is (r,n,d).

We first show that if (d,n,r) is in T, then there is a (unique) positive integer x such that (x,d,n) is in T. (d,n,r) being in T means that

$$\frac{n^4 + 1}{dn - 1}$$

is an integer, and hence

$$\frac{n^4 + 1}{dn - 1} + \frac{(dn)^4 - 1}{dn - 1} = n^4\left[\frac{d^4 + 1}{dn - 1}\right]$$

is an integer. Also, $(n^4, dn - 1) = 1$ and thus

$$\frac{d^4 + 1}{nd - 1}$$

is an integer as well. Hence, as above,

$$d^4 + 1 = (nd - 1)(xd - 1)$$

for some positive integer x, so (x,d,n) is in T. It is clear that x is unique. Also, of course, we have that for every triple (d,n,r) in T there is a unique positive integer y such that (n,r,y) is in T; for if (d,n,r) is in T then so is (r,n,d), hence so is (y,r,n) for some y, and thus so is (n,r,y).

For $n = 1$, (1) has the trivial solution

$$1^4 + 1 = (2\cdot1 - 1)(3\cdot1 - 1), \tag{2}$$

so $(2,1,3)$ is in T. Let (d,n,r) be a nontrivial triple in T (i.e. $n > 1$) where (we may assume) $d \leq r$. Note that

$$\left[\sqrt{n^4 + 1}\right] = n^2$$

and thus from $d \leq r$ and (1),

$$dn - 1 \leq n^2.$$

Since $n > 1$, this means $d \leq n$. Also, from

$$n^4 + 1 = (n^2 - 1)(n^2 + 1) + 2$$

and $n > 1$, $n^2 - 1$ does not divide into $n^4 + 1$, and hence $d < n$. It follows from (1) that $d < n < r$.

From the above, there is a positive integer x so that (x,d,n) is in T. Moreover, if $d > 1$ then we must have $x < d < n$. By continuing this process and applying Fermat's method of infinite descent, we will, starting from any nontrivial solution of (1), eventually reach a solution of the form $(1,n,r)$. But when $d = 1$, (1) becomes

$$rn - 1 = \frac{n^4 + 1}{n - 1} = \frac{n^4 - 1}{n - 1} + \frac{2}{n - 1}$$

which is an integer only if $n = 2$ or 3. Thus the solutions of (1) with $d = 1$ are precisely

$$\begin{aligned} 2^4 + 1 &= (1\cdot2 - 1)(9\cdot2 - 1) \\ 3^4 + 1 &= (1\cdot3 - 1)(14\cdot3 - 1) \end{aligned} \tag{3}$$

corresponding to the ordered triples $(1,2,9)$ and $(1,3,14)$ respectively. Conversely, all nontrivial solutions of (1) may be found by starting from the solutions (3) and ascending. Thus the integers n answering the problem are precisely those integers in the sequence

$$n_1 < n_2 < n_3 < \ldots$$

where any three consecutive terms satisfy

$$n_i^4 + 1 = (n_{i-1}n_i - 1)(n_{i+1}n_i - 1) \tag{4}$$

and where the initial terms are either $n_1 = 1$, $n_2 = 2$ or $n_1 = 1$, $n_2 = 3$.

Now (4) is equivalent to

$$n_{i-1}n_{i+1} = n_i^2 + \frac{n_{i-1} + n_{i+1}}{n_i},$$

and thus for any i there is a positive integer k_i such that

$$n_{i-1} + n_{i+1} = k_i n_i. \tag{5}$$

We claim that $k_i = 5$ for all i. Note that this holds for the initial cases (3) (and even for the trivial solution (2)). So to prove the claim, we need only show that

$$\frac{n_{i-1} + n_{i+1}}{n_i} = \frac{n_i + n_{i+2}}{n_{i+1}}$$

for all i. But from (4),

$$n_{i-1} = \frac{1}{n_i}\left[\frac{n_i^4 + 1}{n_{i+1}n_i - 1} + 1\right] = \frac{n_i^3 + n_{i+1}}{n_{i+1}n_i - 1},$$

and similarly from

$$n_{i+1}^4 + 1 = (n_i n_{i+1} - 1)(n_{i+2}n_{i+1} - 1)$$

we have

$$n_{i+2} = \frac{n_{i+1}^3 + n_i}{n_i n_{i+1} - 1}.$$

Thus

$$\begin{aligned}
\frac{n_{i-1} + n_{i+1}}{n_i} &= \frac{\dfrac{n_i^3 + n_{i+1}}{n_{i+1}n_i - 1} + n_{i+1}}{n_i} \\
&= \frac{n_i^2 + n_{i+1}^2}{n_{i+1}n_i - 1} \\
&= \frac{n_i + \dfrac{n_{i+1}^3 + n_i}{n_i n_{i+1} - 1}}{n_{i+1}} \\
&= \frac{n_i + n_{i+2}}{n_{i+1}}
\end{aligned}$$

as claimed.

From (5) then, the n_i's satisfy the recurrence

$$n_{i+1} = 5n_i - n_{i-1}$$

with either $n_1 = 1$, $n_2 = 2$ or $n_1 = 1$, $n_3 = 3$. Thus the values of n solving the problem are those in the sequences

$$1, 2, 9, 43, 206, 987, \ldots$$

and

$$1, 3, 14, 67, 321, 1538, \ldots .$$

Also solved partially by FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; and J.A. MCCALLUM, Medicine Hat, Alberta (they each found all the above values of n but didn't show there are no others). Two incorrect solutions were received.

* * *

1035*. [1985: 122] *From a Trinity College, Cambridge, examination paper dated December 6, 1901.*

If the equations

$$axy + bx + cy + d = 0,$$
$$ayz + by + cz + d = 0,$$
$$azw + bz + cw + d = 0,$$
$$awx + bw + cx + d = 0,$$

are satisfied by values of x, y, z, w which are all different, show that

$$b^2 + c^2 = 2ad.$$

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Number the above equations (1) - (4). Then (1) and (2) can be written

$$(ax + c)y + (bx + d) = 0$$

and

$$(az + b)y + (cz + d) = 0,$$

and eliminating y from these two equations produces

$$(ax + c)(cz + d) = (bx + d)(az + b),$$

or

$$a(b - c)zx + (b^2 - ad)x + (ad - c^2)z + (b - c)d = 0. \tag{5}$$

Similarly, eliminating w from (3) and (4) produces

$$a(b - c)xz + (b^2 - ad)z + (ad - c^2)x + (b - c)d = 0. \tag{6}$$

Subtracting (6) from (5) produces

$$(b^2 + c^2 - 2ad)(x - z) = 0.$$

Similarly, by eliminating x and z we can get

$$(b^2 + c^2 - 2ad)(w - y) = 0.$$

Thus, if *either* $x \neq z$ *or* $w \neq y$, then $b^2 + c^2 = 2ad$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; J.A. McCALLUM, Medicine Hat, Alberta; M. PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; BASIL RENNIE, James Cook University of North Queensland, Townsville, Australia; and DAN SOKOLOWSKY, College of William and Mary, Williamsburg, Virginia.

* * *

1036. [1985: 122] *Proposed by Gali Salvatore, Perkins, Québec.*

Find sets of positive numbers $\{a,b,c,d,e,f\}$ such that, simultaneously,

$$\frac{abc}{def} < 1\ ,\quad \frac{a+b+c}{d+e+f} < 1\ ,\quad \frac{a}{d} + \frac{b}{e} + \frac{c}{f} > 3\ ,\quad \frac{d}{a} + \frac{e}{b} + \frac{f}{c} > 3\ ,$$

or prove that there are none.

I. *Solution by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

There are infinitely many such sets. One way to obtain them is to choose $d/a \geq 3$ and $c/f \geq 3$, which ensures the last two inequalities. The first two inequalities are then easily satisfied, for example, by $\{a,b,c,d,e,f\} = \{1,b,3,3,e,1\}$ where $b < e$.

II. *Solution by the proposer.*

Let ABC be a triangle with incentre I, and let lines AI, BI, CI meet the circumcircle of the triangle again in D, E, F, respectively. Put $a = AI$, $b = BI$, $c = CI$, $d = ID$, $e = IE$, $f = IF$. Then by *Crux* 644 and Comments III and IV of its solution [1982: 156], the first three of the above inequalities hold (strictly, if ABC is not equilateral). Also, by *Crux* 778 [1982: 246],

$$4\,\frac{def}{abc} = \frac{d}{a} + \frac{e}{b} + \frac{f}{c} + 1$$

from which, since $abc < def$, the last of the above inequalities holds as well.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; and FRIEND H. KIERSTEAD, Cuyahoga Falls, Ohio.

* * *

1037. [1985: 122] *Proposed by (the late) H. Kestelman, University College, London, England.*

If A and B are Hermitian matrices of the same order and A is positive definite, show that AB is similar to a Hermitian matrix and that the latter is positive definite if B is so. Show that if A is assumed only to be positive semidefinite then it may happen that the matrix AB is not similar to any diagonal matrix.

Solution by the proposer.

Since A is positive definite, $A = MM^*$ for some invertible M, and then

$$AB = MM^*B = M(M^*BM)M^{-1}.$$

This means that AB is similar to the matrix M^*BM, which is Hermitian since B is Hermitian. If B is positive definite, put $B = N^*N$ where N is invertible; then

$$M^*BM = (M^*N^*)(NM) = (NM)^*(NM)$$

and so, since NM is invertible, M^*BM is positive definite.

If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then A is positive semidefinite and A and B are Hermitian, and

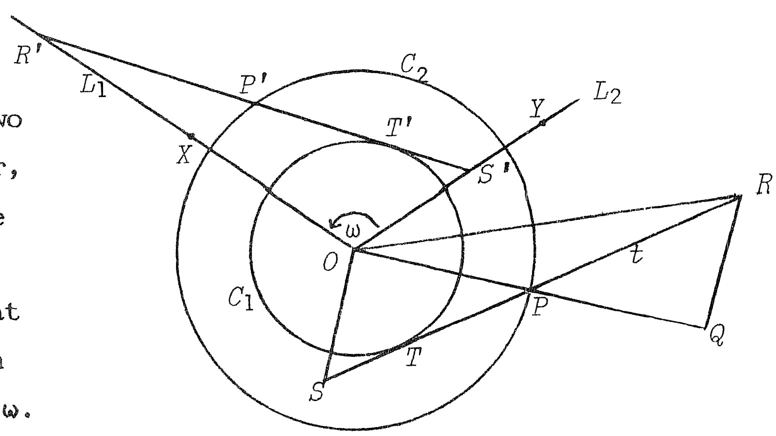
$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus 0 is the only eigenvalue of AB, and so AB would have to be the zero matrix if it were similar to a diagonal matrix.

1038. [1985: 122] *Proposed by Jordi Dou, Barcelona, Spain.*

Given are two concentric circles and two lines through their centre. Construct a tangent to the inner circle such that one of its points of intersection with the outer circle is the midpoint of the segment of the tangent cut off by the two given lines.

Solution by Dan Sokolowsky, College of William and Mary, Williamsburg, Virginia.



Let C_1 be the smaller of the two concentric circles and C_2 the larger, let O be their center, let L_1, L_2 be the two lines through O, and let $\omega = \angle XOY$ be one of the four angles at O formed by L_1 and L_2, where X is on L_1 and Y is on L_2. Put $\omega' = 180^\circ - \omega$.

Let P be an arbitrary point on C_2, and extend ray OP to Q with $OP = PQ$. Let t denote a line through P and tangent to C_1, say at T.

Construct the locus L of points Z on the side of OQ opposite T such that $\angle OZQ = \omega'$. (L is the arc of a circle whose center can be found.) Let R denote the intersection of L and t. Then $\angle ORQ = \omega'$.

Let the line through O parallel to RQ meet t at S. Then $\angle ROS = 180^\circ - \angle ORQ = \omega$.

Now, on OX (or OY) lay off $OR' = OR$, and on OY (or OX) lay off $OS' = OS$. Then $t' = R'S'$ is a solution to our problem. Since $\Delta OR'S' \cong \Delta ORS$, it is easy to show that $R'S'$ is tangent to C_1, say at T', and that the midpoint P' of $R'S'$ is the intersection of $R'S'$ with C_2.

We note that for the same $\angle XOY$ there is also a solution $t'' = R''S''$ obtained by laying off $OR'' = OR$ on OY and $OS'' = OS$ on OX. Moreover $t'' \neq t'$, since $t'' = t'$ implies $OR = OS$ and thus $OP \perp RS$, which is impossible. Since $\angle XOY$ is any one of the four angles formed by L_1 and L_2, it follows that there are always eight distinct solutions to the problem.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; and the proposer.

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<u>1040</u>. [1985: 122] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

ABC being a given triangle, describe the locus of all points P such that $\angle ACP = \angle ABP$.

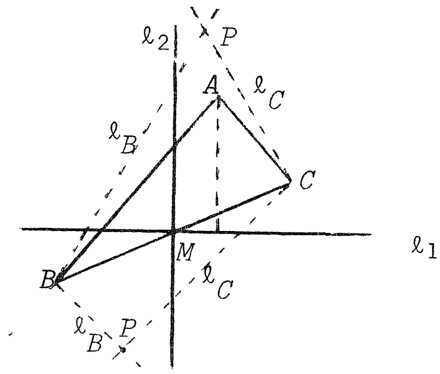
Composite of solutions by Jordi Dou, Barcelona, Spain, and Jordan B. Tabov, Sofia, Bulgaria.

Consider two rays ℓ_B and ℓ_C with endpoints B and C respectively, which rotate with equal constant angular speeds around the points B and C, and at some moment coincide with BA and CA respectively.

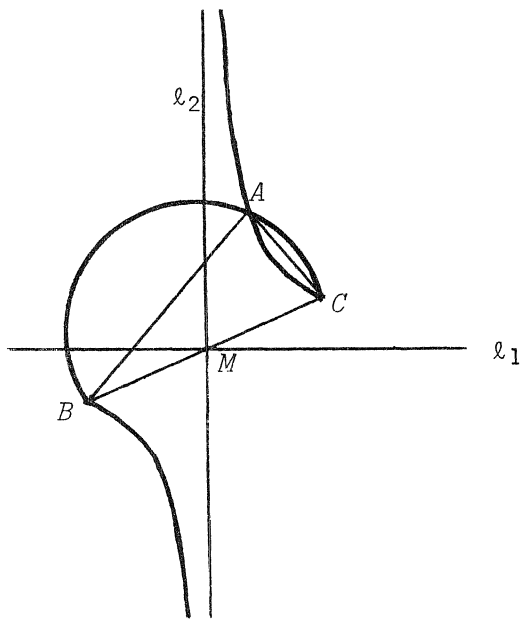
When ℓ_B and ℓ_C rotate in the same direction, the angle between them is a constant, and therefore their common point P describes an arc of the circumcircle Γ of ΔABC, namely that portion of the circumcircle between B and C which contains A.

When ℓ_B and ℓ_C rotate in opposite directions, then their common point P describes part of a rectangular hyperbola $\mathcal{H}$, which passes through A, B and C

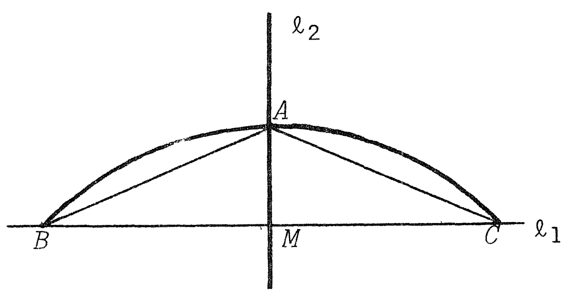
and whose asymptotes are the two mutually perpendicular lines ℓ_1 and ℓ_2, passing through the midpoint M of BC, with ℓ_2 parallel to the bisector of $\angle BAC$. In order to prove this known fact, consider a Cartesian system of coordinates with axes ℓ_1 and ℓ_2. Let B: $(-1,-m)$, C: $(1,m)$, and ℓ_C: $y = k(x - 1) + m$. Then ℓ_B: $y = -k(x + 1) - m$. Note that when ℓ_B and ℓ_C become parallel, they are vertical. It follows that the locus of intersection of ℓ_B and ℓ_C lies between the vertical lines $x = -1$ and $x = 1$.



We first consider the case $m \neq 0$. From the equations of ℓ_B and ℓ_C we find that their common point P has coordinates $(-\frac{m}{k},-k)$. Thus as k varies ($k \neq 0$), P describes the curve $y = \frac{m}{x}$, i.e. the hyperbola $\mathcal{H}$ mentioned above. It is clear that A, B, C will all lie on this hyperbola. The entire locus of P in this case consists of the union of the portions of the circumcircle and hyperbola as described above, and is illustrated in the diagram at right. Note that every nonvertical line passing through A intersects the locus in exactly one other point. Also note that the points on $\Gamma \cup \mathcal{H}$ not on the locus are exactly those points P satisfying $\angle ABP = \pi - \angle ACP$.



Finally, when $m = 0$ (i.e. when $AB = AC$), the locus of the common point of ℓ_B and ℓ_C consists of the vertical axis ℓ_2 (together with the arc of the circumcircle mentioned above).



Also solved (at least partially) by J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. Janous interpreted the angles in the problem as being directed *angles and so only found the circumcircle. The proposer only found the hyperbola.*

* * *

<u>1041</u>. [1985: 146] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

The deepest mine in the world is Western Deep Levels near Carletonville, Transvaal, South Africa. It is both

$$\begin{array}{r}
A \\
GOLD \\
\hline
** \\
*4 \\
** \\
* \\
\hline
LODE
\end{array}
\qquad \text{and} \qquad
\begin{array}{r}
A \\
GOLD \\
\hline
** \\
*4 \\
** \\
* \\
\hline
LOAD
\end{array}\ .$$

Solve these homophonic decimal multiplications independently.

Solution.

$$\begin{array}{r}
2 \\
3975 \\
\hline
10 \\
14 \\
18 \\
6 \\
\hline
7950
\end{array}
\qquad \text{and} \qquad
\begin{array}{r}
6 \\
1594 \\
\hline
24 \\
54 \\
30 \\
6 \\
\hline
9564
\end{array}\ .$$

Found by JIM FARE, student, Eastview Secondary School, Barrie, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California; JACK LESAGE, student, Eastview Secondary School, Barrie, Ontario; J.A. MCCALLUM, Medicine Hat, Alberta; FRED A. MILLER, Elkins, West Virginia; GLEN E. MILLS, Valencia Community College, Orlando, Florida; J. SUCK, Essen, Federal Republic of Germany; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

<u>1042</u>. [1985: 146] *Proposed by Clark Kimberling, University of Evansville, Indiana.*

Let P be a point in the plane of a given triangle ABC; let $A'B'C'$ be the cevian triangle of the point P for the triangle ABC (with A' on line BC, etc.); and let the circumcircle of triangle $A'B'C'$ meet the lines BC, CA, AB again in A'', B'', C'', respectively. Prove that the lines AA'', BB'', CC'' are concurrent.

Solution by several readers.

By Ceva's theorem,

$$\overline{AC'}\cdot\overline{BA'}\cdot\overline{CB'} = \overline{B'A}\cdot\overline{C'B}\cdot\overline{A'C}.$$

From the circumcircle of $\Delta A'B'C'$,

$$\overline{AC'}\cdot\overline{AC''} = \overline{AB'}\cdot\overline{AB''}$$
$$\overline{BA'}\cdot\overline{BA''} = \overline{BC'}\cdot\overline{BC''}$$
$$\overline{CA'}\cdot\overline{CA''} = \overline{CB'}\cdot\overline{CB''}.$$

It follows that

$$\overline{AC''}\cdot\overline{BA''}\cdot\overline{CB''} = \overline{B''A}\cdot\overline{C''B}\cdot\overline{A''C},$$

and so AA'', BB'', CC'' are concurrent by Ceva's theorem.

Solved by W.J. BLUNDON, Memorial University of Newfoundland, St. John's, Newfoundland; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland, St. John's, Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRED A. MILLER, Elkins, West Virginia; DAN PEDOE, University of Minnesota, Minneapolis, Minnesota; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; MALCOLM A. SMITH, Georgia Southern College, Statesboro, Georgia; and the proposer.

Some solvers commented that the problem appears as Theorem 227, page 151 of R.A. Johnson's *Advanced Euclidean Geometry* (Dover, 1960) and in other places. Also, some solvers noted that the result of the problem remains true if the circumcircle of $\Delta A'B'C'$ is replaced by any conic through A', B', C'.

* * *

1043. [1985: 146] *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Prove that the projections of a point P on the faces of a tetrahedron T are coplanar if and only if P lies on a particular cubic surface which passes through the edges of T.

(This is an extension to three dimensions of the Wallace-Simson theorem, which states that the projections of a point P on the sides of a triangle are collinear if and only if P lies on the circumcircle of the triangle.)

Solution by J.T. Groenman, Arnhem, The Netherlands.

Let the tetrahedron $T = ABCD$ and let $P(a_1,a_2,a_3)$ be an arbitrary point. The equation of the plane through B, C, D will be of the form

$$u_1x + u_2y + u_3z + u_4 = 0,$$

where we may assume $u_1^2 + u_2^2 + u_3^2 = 1$, and thus the perpendicular line from P to this plane will be of the form

$$x = a_1 + \lambda u_1\ ,\ y = a_2 + \lambda u_2\ ,\ z = a_3 + \lambda u_3.$$

Hence the projection P_1 of P onto this plane will satisfy

$$u_1(a_1 + \lambda u_1) + u_2(a_2 + \lambda u_2) + u_3(a_3 + \lambda u_3) + u_4 = 0$$

$$\lambda(u_1^2 + u_2^2 + u_3^2) + u_1a_1 + u_2a_2 + u_3a_3 + u_4 = 0$$

$$\lambda = -(u_1a_1 + u_2a_2 + u_3a_3 + u_4),$$

so P_1 will have coordinates

$$x_1 = a_1 - u_1(u_1a_1 + u_2a_2 + u_3a_3 + u_4)$$

$$y_1 = a_2 - u_2(u_1a_1 + u_2a_2 + u_3a_3 + u_4)$$

$$z_1 = a_3 - u_3(u_1a_1 + u_2a_2 + u_3a_3 + u_4).$$

Note that all three coordinates are linear in a_1, a_2, a_3.

Similarly we obtain the projections $P_2 = (x_2,y_2,z_2)$, $P_3 = (x_3,y_3,z_3)$, and $P_4 = (x_4,y_4,z_4)$. Then P_1, P_2, P_3, P_4 are coplanar if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0. \tag{1}$$

As all coordinates x_i, y_i, z_i are linear in a_1, a_2, a_3, this equation has degree 3 in a_1, a_2, a_3, and so defines a cubic surface.

Finally we note that if P is chosen on an edge of the tetrahedron, at least two of the projections P_i will coincide so the projections will be coplanar. Thus P satisfies (1) and lies on the cubic surface.

Also solved by the proposer.

* * *

<u>1044</u>. [1985: 146] *Proposed by Peter Messer, M.D., Mequon, Wisconsin.*

Find a simple expression for the positive root of the equation

$$x^3 - 3x^2 - x - \sqrt{2} = 0.$$

Solution by Kenneth M. Wilke, Topeka, Kansas.

By making the change of variable $y = x - 1$, the given equation becomes

$$y^3 - 4y = 3 + \sqrt{2}. \tag{1}$$

But since $y^3 - 4y = y(y + 2)(y - 2)$ and $(\sqrt{2} + 1)(\sqrt{2} - 1) = 1$, equation (1) can be written

$$y(y + 2)(y - 2) = (\sqrt{2} + 1)(\sqrt{2} - 1)(3 + \sqrt{2}).$$

Then since the factors on each side of this equation form arithmetic progressions having a common difference of 2, we find that $y = \sqrt{2} + 1$ is a solution. Thus

$$x = 2 + \sqrt{2}$$

is a positive root of the original equation, which by Descartes' Rule of Signs has only one positive real root.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, San Antonio, Texas; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; J.A. MCCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; J. SUCK, Essen, Federal Republic of Germany; KENNETH S. WILLIAMS, Carleton University, Ottawa, Ontario; and the proposer.

Janous and the proposer remark that by Cardano's formula for the roots of a cubic polynomial,

$$\sqrt[3]{\frac{3+\sqrt{2}}{2} + \frac{1}{6}\sqrt{\frac{41+162\sqrt{2}}{3}}} + \sqrt[3]{\frac{3+\sqrt{2}}{2} - \frac{1}{6}\sqrt{\frac{41+162\sqrt{2}}{3}}} = 1 + \sqrt{2}\ .$$

* * *

1045. [1985: 147, 189] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let P be an interior point of triangle ABC; let x, y, z be the distances of P from vertices A, B, C, respectively; and let u, v, w be the distances of P from sides BC, CA, AB, respectively. The well-known Erdős-Mordell inequality states that

$$x + y + z \geq 2(u + v + w).$$

Prove the following related inequalities:

(a) $\dfrac{x^2}{vw} + \dfrac{y^2}{wu} + \dfrac{z^2}{uv} \geq 12,$

(b) $\dfrac{x}{v+w} + \dfrac{y}{w+u} + \dfrac{z}{u+v} \geq 3,$

(c) $\dfrac{x}{\sqrt{vw}} + \dfrac{y}{\sqrt{wu}} + \dfrac{z}{\sqrt{uv}} \geq 6.$

I. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

With the notation of the problem, 12.28 of [1] reads

$$\prod \frac{x}{v + w} \geq \frac{1}{8S}$$

where $S = \prod \sin A/2$, and thus

$$\prod \left[\frac{x}{v + w}\right]^k \geq \frac{1}{(8S)^k}$$

for any $k > 0$. By [1], 2.12 we have $0 < 8S \leq 1$, and thus

$$\prod \left[\frac{x}{v + w}\right]^k \geq 1. \qquad (1)$$

The A.M.-G.M. inequality now tells us that

$$\sum \left[\frac{x}{v + w}\right]^k \geq 3.$$

Putting $k = 1$ yields (b). Using $v + w \geq 2\sqrt{vw}$, we get

$$\sum \left[\frac{x}{\sqrt{vw}}\right]^k \geq 3\cdot 2^k,$$

which yields (a) and (c) when $k = 2$ and $k = 1$ respectively.

Similarly, from (1) we also get

$$\sum \left[\frac{x}{u + v}\right]^k \geq 3$$

and

$$\sum \left[\frac{x}{u + w}\right]^k \geq 3$$

for $k > 0$.

II. *Generalization by M.S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let w_1, w_2, w_3 be the lengths of the angle bisectors of $\angle BPC$, $\angle CPA$, $\angle APB$ respectively, and let α, β, γ, d, e, f, i, j, k, ℓ, m, n be non-negative numbers such that

$$d + e + f = i + j + k + \ell + m + n.$$

Then by the A.M.-G.M. inequality,

$$S \equiv \frac{\alpha^3 x^{d+e+f}}{w_1^i w_2^j w_3^k (w_1 + w_2)^\ell (w_2 + w_3)^m (w_3 + w_1)^n} + \frac{\beta^3 y^{d+e+f}}{w_1^k w_2^i w_3^j (w_1 + w_2)^n (w_2 + w_3)^\ell (w_3 + w_1)^m}$$

$$+ \frac{\gamma^3 z^{d+e+f}}{w_1^j w_2^k w_3^i (w_1 + w_2)^m (w_2 + w_3)^n (w_3 + w_1)^\ell}$$

$$\geq 3\alpha\beta\gamma \left\{ \frac{(xyz)^{d+e+f}}{(w_1 w_2 w_3)^{i+j+k}[(w_1 + w_2)(w_2 + w_3)(w_3 + w_1)]^{\ell+m+n}} \right\}^{1/3}$$

$$\equiv I.$$

Now by using the inequalities

$$xyz \geq 8w_1 w_2 w_3$$

$$xyz \geq (w_1 + w_2)(w_2 + w_3)(w_3 + w_1)$$

(12.51 and 12.52 in [1]), we obtain

$$S \geq I \geq 3\alpha\beta\gamma \left\{ \frac{(xyz)^{d+e+f}}{(xyz/8)^{i+j+k}(xyz)^{\ell+m+n}} \right\}^{1/3}$$

$$= 3\alpha\beta\gamma \cdot 2^{i+j+k}. \tag{2}$$

By putting $\alpha = \beta = \gamma = 1$, $d + e + f = 2$, $i = 0$, $j = k = 1$, and $\ell = m = n = 0$, (2) becomes

$$\frac{x^2}{w_2 w_3} + \frac{y^2}{w_3 w_1} + \frac{z^2}{w_1 w_2} \geq 12;$$

by putting $\alpha = \beta = \gamma = 1$, $d + e + f = 1$, $i = j = k = 0$, $m = 1$, and $\ell = n = 0$, (2) becomes

$$\frac{x}{w_2 + w_3} + \frac{y}{w_3 + w_1} + \frac{z}{w_1 + w_2} \geq 3;$$

and by putting $\alpha = \beta = \gamma = 1$, $d + e + f = 1$, $i = 0$, $j = k = \frac{1}{2}$, and $\ell = m = n = 0$, (2) becomes

$$\frac{x}{\sqrt{w_2 w_3}} + \frac{y}{\sqrt{w_3 w_1}} + \frac{z}{\sqrt{w_1 w_2}} \geq 6.$$

Now (a), (b), and (c) follow respectively from these last three inequalities, since w_1, w_2, w_3 are greater than or equal to u, v, w respectively.

Reference:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

Also solved by C. FESTRAETS-HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; BOB PRIELIPP, University of Wisconsin, Oshkosh, Wisconsin; and the proposer.

* * *

1046. [1985: 147] *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*

The *Wallace point* W of any four points A_1, A_2, A_3, A_4 on a circle with center O may be defined by the vector equation

$$\overrightarrow{OW} = \frac{1}{2}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \overrightarrow{OA}_3 + \overrightarrow{OA}_4)$$

(see the article by Bottema and Groenman in this journal [1982: 126]).

Let γ be a cyclic quadrilateral the Wallace point of whose vertices lies inside γ. Let a_i ($i = 1,2,3,4$) be the sides of γ, and let G_i be the midpoint of the side opposite to a_i. Find the minimum value of

$$f(X) = a_1 \cdot G_1X + a_2 \cdot G_2X + a_3 \cdot G_3X + a_4 \cdot G_4X,$$

where X ranges over all the points of the plane of γ.

Solution by the proposer.

We denote with square brackets the signed areas of triangles and quadrilaterals.

Let A_1, A_2, A_3, A_4 and W be respectively the vertices of γ and their Wallace point. We may assume that $A_1A_2A_3A_4 = \gamma$ is oriented positively and that G_i is the midpoint of A_iA_{i+1} ($A_5 \equiv A_1$).

We will use the following inequality from the solution of *Crux* 866 [1984: 327] (the reader may wish to compare these two problems): if P, Q, R, S are any four points in the plane, then

$$PR \cdot QS \geq 2([PRQ] - [PRS]). \tag{1}$$

Equality holds in (1) if and only if $PR \perp QS$ and $[PRQ] \geq [PRS]$.

Let

$$\begin{aligned} f(X) &= a_1 \cdot G_1X + a_2 \cdot G_2X + a_3 \cdot G_3X + a_4 \cdot G_4X \\ &= A_3A_4 \cdot G_1X + A_4A_1 \cdot G_2X + A_1A_2 \cdot G_3X + A_2A_3 \cdot G_4X \end{aligned}$$

where X ranges over the entire plane. Then, from (1),

$$\begin{aligned} f(X) &\geq 2([A_3A_4G_1] - [A_3A_4X] + [A_4A_1G_2] - [A_4A_1X] \\ &\quad + [A_1A_2G_3] - [A_1A_2X] + [A_2A_3G_4] - [A_2A_3X]) \\ &= 2([A_3A_4G_1] + [A_4A_1G_2] + [A_1A_2G_3] + [A_2A_3G_4] - [A_1A_2A_3A_4]). \end{aligned} \tag{2}$$

Since G_i is the midpoint of A_iA_{i+1},

$$[A_3A_4G_1] = \frac{1}{2}([A_3A_4A_1] + [A_3A_4A_2]), \text{ etc.},$$

and hence

$$2([A_3A_4G_1] + [A_4A_1G_2] + [A_1A_2G_3] + [A_2A_3G_4])$$
$$= [A_3A_4A_1] + [A_3A_4A_2] + [A_4A_1A_2] + [A_4A_1A_3]$$
$$+ [A_1A_2A_3] + [A_1A_2A_4] + [A_2A_3A_4] + [A_2A_3A_1]$$
$$= 4[A_1A_2A_3A_4].$$

Thus from (2),

$$f(X) \geq 2[A_1A_2A_3A_4]. \qquad (3)$$

We now show that equality holds in (3) for $X = W$, and thus the minimum value of $f(X)$ is $f(W) = 2[A_1A_2A_3A_4]$.

First, G_1 is the midpoint of A_1A_2, which implies

$$\overrightarrow{OG}_1 = \frac{1}{2}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2).$$

Hence

$$\overrightarrow{OW} = \frac{1}{2}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \overrightarrow{OA}_3 + \overrightarrow{OA}_4)$$
$$= \overrightarrow{OG}_1 + \frac{1}{2}(\overrightarrow{OA}_3 + \overrightarrow{OA}_4),$$

and so

$$\overrightarrow{G_1W} = \frac{1}{2}(\overrightarrow{OA}_3 + \overrightarrow{OA}_4).$$

Since $OA_3 = OA_4$, we conclude that $G_1W \perp A_3A_4$. Also, since $[A_1A_2A_3A_4]$ is positively oriented, and W lies on the perpendicular from G_1 to A_3A_4 and inside A, $[A_3A_4G_1] \geq [A_3A_4W]$. Thus equality holds in (1) with $P = A_3$, $R = A_4$, $Q = G_1$, $S = W$; that is,

$$A_3A_4 \cdot G_1W = 2([A_3A_4G_1] - [A_3A_4W]).$$

Similarly,

$$A_4A_1 \cdot G_2W = 2([A_4A_1G_2] - [A_4A_1W]), \text{ etc.}$$

Hence equality holds in (2) and thus in (3) when $X = W$.

Similar results may be obtained in the same way for cyclic n-gons.

* * *

1047*. [1985: 147] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let p, q, r be three different natural numbers not all even. Prove or disprove that $(x,y,z) = (1,1,1)$ is the only real solution of the system

$$\begin{cases} x^q + y^r + z^p = 3 \\ x^r + y^p + z^q = 3 \\ x^p + y^q + z^r = 3 \end{cases}.$$

Generalize.

Solution by Len Bos, University of Calgary, Calgary, Alberta.

The statement is false whenever the largest of p, q, r is even. Suppose $p > q > r$ and p is even. Then

$$x^p + x^q + x^r - 3 = (x^p - 1) + (x^q - 1) + (x^r - 1)$$

$$=(x - 1)\left[\sum_{i=0}^{p-1} x^i + \sum_{i=0}^{q-1} x^i + \sum_{i=0}^{r-1} x^i\right],$$

and the second factor obviously has a real root $\rho \neq 1$ since its degree is odd and all its coefficients are positive. Thus $(x,y,z) = (\rho,\rho,\rho)$ satisfies the above system.

* * *

A MESSAGE FROM THE CANADIAN MATHEMATICAL SOCIETY

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