Crux Mathematicorum

VOLUME 40, NO. 4 APRIL / AVRIL 2014

Editorial Board

Published by
Canadian Mathematical Society
209 - 1725 St. Laurent Blvd.
Ottawa, Ontario, Canada K1G 3V4
FAX: 613–733–8994
email: subscriptions@cms.math.ca

Publié par
Société mathématique du Canada
209 - 1725 boul. St. Laurent
Ottawa (Ontario) Canada K1G 3V4
Téléc : 613–733–8994
email : abonnements@smc.math.ca

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

Dear ***Crux*** readers,

With this issue, I am happy to announce our cooperation with Kvant, a popular science magazine in physics and mathematics for high school students and teachers. It has been published in Russian since 1970 and I am excited to be able to publish selected materials from it in English on the pages of ***Crux***. You can find Kvant issues dating back to 1976 here: `http://kvant.mccme.ru/`

I would also like to welcome Carmen Bruni who will be joining ***Crux*** as the Olympiad Corner Editor alongside Nicolae.

Now, let me address one housekeeping issue. Unfortunately, the process of transferring files and emails from one person to another is never perfect and complete. In our case, this means that some solutions submitted before I started in this position did not quite make it into my file system. My sincere apologies to any of you who have submitted solutions and did not see them acknowledged. I believe we have managed to catch most of these missing attributions by releasing ***Crux*** online before it goes to print, but we have missed some. Do not hesitate to contact me to let me know if your solution did not appear when you think it should have.

Happy reading!

Kseniya Garaschuk

4 and 100

Starting with number 1 in the top left corner, draw a path to the number 9 in the bottom right corner. You can move only down and to the right. Your path has to turn exactly 4 times and the total sum of the numbers on the path has to be exactly 100.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

Puzzle by L. Mochalov, Kvant, 1976 (12).

THE CONTEST CORNER

No. 24

Robert Bilinski

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-contest@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

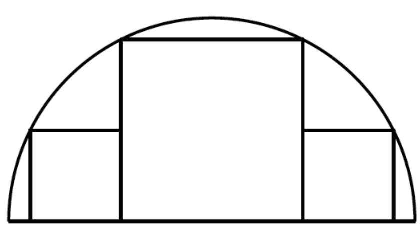
Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

CC116. Est-ce que n^2 a plus de diviseurs $1 \pmod 4$ ou $3 \pmod 4$?

CC117. Dans un triangle ABC avec $BC = 3$, choisissez D sur BC pour que $BD = 2$. Trouver la valeur de $AB^2 + 2AC^2 - 3AD^2$.

CC118. Si 2 petits carrés de côté 2 et un plus gros carré sont inscrits dans un demi-cercle, déterminer le côté du plus gros carré.



CC119. Si on efface les dizaines d'un nombre à 3 chiffres abc, on a un nombre à 2 chiffres ac. Combien de nombres abc sont tels que $abc = 9ac + 4c$? (Par exemple, $245 = 9 \times 25 + 4 \times 5$.)

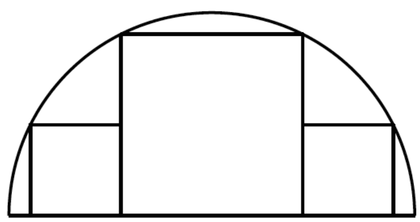
CC120. Si A est un nombre à 2 chiffres et B est un nombre à 3 chiffres tel que A augmenté de $B\%$ donne B réduit de $A\%$, trouver toutes les paires (A, B).

. . . .

CC116. Does n^2 have more divisors 1 (mod 4) or 3 (mod 4) ?

CC117. In a triangle ABC with $BC = 3$, choose a point D on BC such that $BD = 2$. Find the value of $AB^2 + 2AC^2 - 3AD^2$.

CC118. If 2 small squares of side 2 and a bigger square are inscribed into a semi-circle, find the side of the larger square.



CC119. When the tens digit of a three digit number abc is deleted, a two digit number ac is formed. How many numbers abc are there such that $abc = 9ac + 4c$? (For example, $245 = 9 \times 25 + 4 \times 5$.)

CC120. Suppose A is a 2-digit number and B is a 3-digit number such that A increased by $B\%$ equals B reduced by $A\%$. Find all possible pairs (A, B).

CONTEST CORNER SOLUTIONS

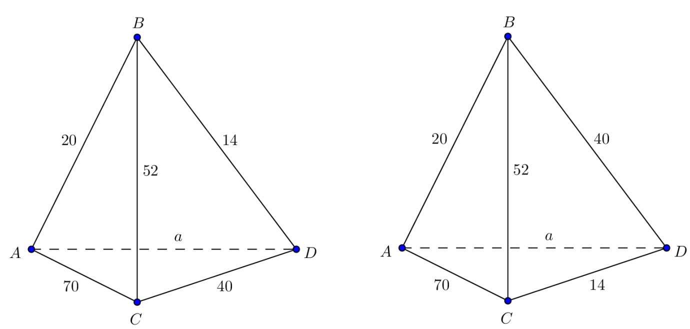
CC66. The lengths of all six edges of a tetrahedron are integers. The lengths of five of the edges are 14, 20, 40, 52, and 70. Determine the number of possible lengths for the sixth edge.

Originally problem 25 from 2002 Fermat Contest.

Solved by R. Hess ; and N. Stanciu and T. Zvonaru. We present a hybrid solution.

Let $ABCD$ be the given tetrahedron and let a be the length of the sixth edge, which we assume is AD. The edges of $\triangle ABC$ and $\triangle BCD$ must satisfy the triangle inequality, and the two triangles have exactly one edge in common. Therefore, we may assume, after a few simple calculations, that $\triangle ABC$ has edge lengths $20, 52$, and 70 and $\triangle BCD$ has edge lengths $14, 40$, and 52, and that $|BC| = 52$.

This yields two possibilities for the edge lengths of the faces, $\triangle ABC$, $\triangle BCD$, $\triangle ABD$, and $\triangle ACD$. They are either : Case 1, $(20, 52, 70)$, $(52, 40, 14)$, $(20, 14, a)$, and $(70, 40, a)$, respectively (on the left below) ; or Case 2, $(20, 52, 70)$, $(52, 14, 40)$, $(20, 40, a)$, and $(70, 14, a)$, respectively (on the right below).



The triangle inequality gives the necessary bounds on a.

Case 1. The inequalities $20 + 14 > a$, $20 + a > 14$, $14 + a > 20$, $70 + 40 > a$, $40 + a > 70$, and $70 + a > 40$ imply that $30 < a < 34$, of which integer solutions are $a \in \{31, 32, 33\}$.

Case 2. The inequalities $20 + 40 > a$, $20 + a > 40$, $40 + a > 20$, $70 + 14 > a$, $14 + a > 70$, and $70 + a > 14$ imply that $56 < a < 60$, of which integer solutions are $a \in \{57, 58, 59\}$.

So, there are exactly six possible lengths for the sixth edge : 31, 32, 33, 57, 58, 59.

CC67. The twenty volumes, clearly numbered 1 to 20, of an encyclopedia are to be arranged on a shelf. If ten volumes have blue covers, six have red covers, and the remainder have green covers, determine in how many ways the books can be arranged so that no two books of the same colour are side by side.

Originally 1978 Descartes Contest, problem 7.

Solved by Edward Wang, whose solution we present below.

First, permute the 10 blue books. This can be done in 10! ways. For convenience, call each of the empty spaces between two adjacent blue books a slot. By the given conditions, each of these 9 slots must be filled with at least one, but at most two non-blue books. Let's denote the leftmost book L and the rightmost book R. It is clear that L and R cannot both be non-blue. Considering the colours of L and R leads us to two separate cases.

Case (i). Either L or R (but not both) is occupied by a non-blue book. Clearly there are $C(6,1) \times 2 + C(4,1) \times 2 = 20$ ways of choosing and placing this book. Then the 9 remaining non-blue books can be arranged in the 9 slots in 9! ways. Hence, there are $20 \times 9!$ such arrangements.

Case (ii). Both L and R are blue. In this case we must insert two books, one red and one green, in one of the 9 slots and then exactly one non-blue book in each of the 8 remaining slots. This can be done in $C(9,1) \times C(6,1) \times C(4,1) \times 2 \times 8! = 48 \times 9!$ ways.

Combining our two cases we conclude that the total number of permissible arrangements is $10!(20 \times 9! + 48 \times 9!) = 68 \times 10! \times 9! = 89543688192000$.

CC68. A family of straight lines is determined by the condition that the sum of the reciprocals of the x and y intercepts is a constant k for each line in the family. Show that all members of the family are concurrent.

Originally 1977 Descartes Contest, problem 8.

Solved by M. Coiculescu; R. Hess; K. Zelator; and N. Stanciu and T. Zvonaru. We present a solution based on those of Konstantine Zelator and Matei Coiculescu (done independently).

The equation of any line can be written as $ax+by = 1$ for some constants a, b. The x- and y-intercepts of such a line are $1/b$ and $1/a$, respectively. Then the defining condition is that $a + b = k$, so $a = k - b$, and any line in the family can be written as

$$(k - b)x + by = 1$$

for some b. Given another such line

$$(k - c)x + cy = 1,\ c \neq b,$$

we subtract the second equation from the first to find the point of intersection :

$$\begin{aligned}(c - b)x + (b - c)y &= 0\\ (b - c)y &= (b - c)x\end{aligned}$$

and hence $x = y$. So $(k - b)x + bx = 1$, and $kx = 1$.

Thus, $x = 1/k$, provided that $k \neq 0$, which implies $y = 1/k$. Since this is independent of b and c, every line in the family passes through $(1/k, 1/k)$.

If $k = 0$, then $b = -a$, so all the lines in the family have slope 1 and are parallel.

CC69. The Fibonacci sequence is defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. A Pythagorean triangle is a right-angled triangle with integer side lengths. Prove that f_{2k+1} is the hypotenuse of a Pythagorean triangle for every positive integer k with $k \geq 2$.

Originally from 2010 Sun Life Financial Rêpechage Competition, problem 5.

Solved by Š. Arslanagić ; M. Bataiile ; D. E. Manes ; H. Ricardo ; T. Zvonaru and N. Stanciu. We present the solution by Henry Ricardo.

We will use the identity $f_{2k+1} = f_k^2 + f_{k+1}^2$. We get

$$\begin{aligned}
f_{2k+1}^2 &= (f_k^2 + f_{k+1}^2)^2 \\
&= f_k^4 + f_{k+1}^4 + 2f_k^2 f_{k+1}^2 \\
&= (f_k^4 - 2f_k^2 f_{k+1}^2 + f_{k+1}^4) + 4f_k^2 f_{k+1}^2 \\
&= (f_k^2 - f_{k+1}^2)^2 + (2f_k^2 f_{k+1}^2)^2.
\end{aligned}$$

Thus f_{2k+1} is the hypotenuse of a Pythagorean triangle with legs

$$f_{k+1}^2 - f_k^2 \text{ and } 2f_k f_{k+1}.$$

CC70. The game of Square Meal is played with a heap of peanuts, initially containing N nuts. The two players take it in turns to eat a positive square number $(1, 4, 9, \dots)$ of nuts. Whoever eats the last nut wins. For which values of N can the first player always win ?

Originally Question 6 from 2004 APICS Math Competition.

One incorrect solution was received. This is, in fact, an open problem.

THE OLYMPIAD CORNER

No. 322

Nicolae Strungaru and Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-olympiad@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC176. Résoudre l'équation

$$y = 2x^2 + 5xy + 3y^2,$$

où x et y sont entiers.

OC177. Pour chaque entier positif a, définissons $M(a)$ comme étant le nombre d'entiers positifs b tels que $a + b$ divise ab. Déterminer tout entier(s) a tel(s) que $1 \leq a \leq 2013$ et tel(s) que $M(a)$ atteint la plus grande valeur possible pour a dans ce domaine.

OC178. Déterminer tous les ensembles S d'entiers tels que $3m - 2n \in S$ pour tout $m, n \in S$.

OC179. Déterminer la valeur maximale de

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|,$$

où a, b, c sont des nombres réels dans $[-2, 2]$.

OC180. Dans un triangle aigü ABC, soient O le centre du cercle circonscrit, G le centroïde et H l'orthocentre. Soit D un point sur BC tel que OD est perpendiculaire à BC, et soit E un point sur CA tel que HE est perpendiculaire à CA. Soit F le mi point de AB. Si les triangles ODC, HEA et GFB ont la même surface, déterminer toute les valeur possible de l'angle $\angle C$.

. . . .

OC176. Solve the equation

$$y = 2x^2 + 5xy + 3y^2$$

for x and y integers.

OC177. For any positive integer a, define $M(a)$ to be the number of positive integers b for which $a + b$ divides ab. Find all integer(s) a with $1 \leq a \leq 2013$ such that $M(a)$ attains the largest possible value in the range of a.

OC178. Find all nonempty sets S of integers such that $3m - 2n \in S$ for all $m, n \in S$.

OC179. Find the maximum value of

$$|a^2 - bc + 1| + |b^2 - ac + 1| + |c^2 - ba + 1|$$

where a, b, c are real numbers in $[-2, 2]$.

OC180. In an acute triangle ABC, let O be its circumcentre, G be its centroid and H be its orthocentre. Let D be a point on BC with OD perpendicular to BC and E a point on CA with HE perpendicular to CA. Let F be the midpoint of AB. If triangles ODC, HEA and GFB have the same area, find all possible values of the angle $\angle C$.

OLYMPIAD SOLUTIONS

OC116. Find all positive integers n which are 300 times the sum of their digits.

Originally question 2 from Italian Math Olympiad 2012.

Solved by O. Geupel, R. Hess; S. Muralidharan; D. Văcaru; T. Zvonaru and N. Stanciu; and K. Zelator. We give the solution by S. Muralidharan.

We show that 2700 is the only number with the property.

Let N be a $k-$digit number such that N equals 300 times the sum of its digits. As N is divisible by 300, the last two digits must be 00, and hence the sum of the digits of N is at most $9(k-2)$. Thus we must have

$$10^{k-1} < N = 300 \times \text{sum of digits} \leq 300 \times 9(k-2)\,.$$

The above inequality yields $k \leq 4$. Clearly $k \geq 3$.

If $k = 3$, then, as 300 divides N, we must have $N = 300, 600$ or 900 and it is easy to see that none of these values work.

Let $k = 4$. Then $N = \overline{a_1a_200} = 1000a_1 + 100a_2$. Thus we must have $10a_1 + a_2 = 3(a_1 + a_2)$ and hence $7a_1 = 2a_2$. The only possibility is $a_1 = 2$ and $a_2 = 7$. Thus 2700 is the only 4$-$digit number with the required property.

OC117. Find the smallest positive integer m such that for all prime numbers $p > 3$, we have

$$105|9^{p^2} - 29^p + m\,.$$

Originally question 1 from China Western Olympiad 2012.

Solved by N. Evgenidis; O. Geupel; D. E. Manes; D. Văcaru; K. Zelator. We give the solution of Nikolaos Evgenidis.

Since $105 = 3 \cdot 5 \cdot 7$ we must have

$$9^{p^2} - 29^p + m \equiv 0 \pmod{3, 5, 7}\ .$$

Since $3 \mid 9^{p^2}$, we have

$$0 \equiv -29^p + m \equiv -(-1)^p + m \equiv 1 + m \pmod 3\ .$$

This shows that $m \equiv -1 \pmod 3$.

Modulo 5 we also have

$$0 \equiv 9^{p^2} - 29^p + m \equiv (-1)^{p^2} - (-1)^p_m \equiv m \pmod 5\ .$$

This shows that $m \equiv 0 \pmod 5$.

Finally, modulo 7 we have

$$0 \equiv 9^{p^2} - 29^p + m \equiv 2^{p^2} - 1^p + m \equiv 2^{p^2} - 1 + m \pmod 7 .$$

As $p^2 \equiv 1 \pmod 3$ and $2^3 \equiv 1 \pmod 7$ we have $2^{p^2} \equiv 2 \pmod 7$.

Therefore

$$0 \equiv 2 - 1 + m = m + 1 \pmod 7 .$$

Now $m \equiv -1 \pmod{3,7}$ is equivalent to $m \equiv -1 \equiv 20 \pmod{21}$. Also, we know that we must have $m \equiv 0 \equiv 20 \pmod 5$. By the Chinese Remainder Theorem, this is equivalent to

$$m \equiv 20 \pmod{105} .$$

The smallest m is therefore $m = 20$.

OC118. Find all functions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfying

$$f(x) + f(y) \leq \frac{f(x+y)}{2} \quad \text{and} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y} ,$$

for all $x, y \in (0, \infty)$.

Originally question 4 from Indian IMO training camp 2012, day 3.

Solved by Oliver Geupel whose solution we present below.

It is straightforward to verify that the functions $f(x) = cx^2$ with $c \leq 0$ are solutions of the problem. We prove that there are no other solutions.

Let f be any solution and write $g(x) = f(x)/x$. The inequalities rewrite as

$$2(xg(x) + yg(y)) \leq (x+y)g(x+y) , \tag{1}$$

$$g(x+y) \leq g(x) + g(y) . \tag{2}$$

By (1) and (2),

$$2(xg(x) + yg(y)) \leq (x+y)g(x+y) \leq (x+y)(g(x) + g(y)) , \tag{3}$$

whence $(x-y)(g(x) - g(y)) \leq 0$, that is, the function g is nonincreasing.

Setting $x = y$ in (3), we obtain $g(2x) = 2g(x)$. A straightforward inductive argument yields $g(2^n x) = 2^n g(x)$ for every natural number n.

We prove by induction on n that $g(nx) = ng(x)$. We have seen that the assertion holds for n being a power of 2. Assume that $n = 2^q + r$ where $0 \leq r < 2^q$ and that for every $k < n$ it holds $g(kx) = kg(x)$.

By induction hypothesis and (2), it holds

$$\begin{aligned} 2^{q+1}g(x) &= g(2^{q+1}x) \leq g((2^q + r)x) + g((2^q - r)x) = g(nx) + (2^q - r)g(x) \\ &\leq (2^q + r)g(x) + (2^q - r)g(x) = 2^{q+1}g(x) . \end{aligned}$$

Thus, $g(nx) + (2^q - r)g(x) = 2^{q+1}g(x)$, that is, $g(nx) = ng(x)$, which completes the induction.

We deduce that

$$g\left(\frac{m}{n}\right) = mg\left(\frac{1}{n}\right) = m \cdot \frac{1}{n}g(1) = \frac{m}{n}g(1)$$

holds for natural numbers m and n. Thus, for every positive rational number x it holds that $g(x) = x \cdot g(1)$. Since g is nonincreasing, we have $g(1) \leq 0$.

Finally, let x be any positive real number. Let (y_k) and (z_k) be sequences of rational numbers such that $y_k < x < z_k$ for $k = 1, 2, 3, \ldots$ as well as

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k = x.$$

Since g is nonincreasing, we have $g(z_k) \leq g(x) \leq g(y_k)$. Because

$$\lim_{k\to\infty} g(y_k) = \lim_{k\to\infty} g(z_k) = x \cdot g(1),$$

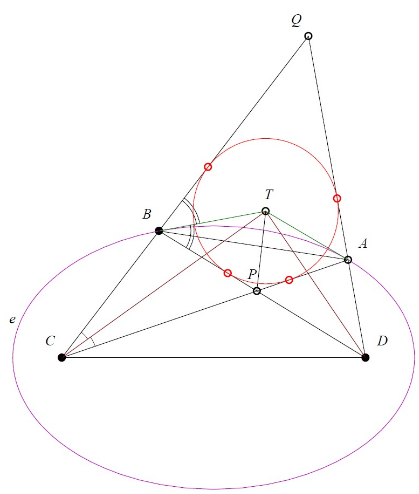
we deduce $g(x) = x \cdot g(1)$. Consequently, $f(x) = x \cdot g(x) = x^2 \cdot g(1)$.

OC119. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB, \angle ADB$ and $\angle APB$ meet at a common point.

Originally question 3 from Canadian Math Olympiad 2012.

Solved by O. Geupel; and M. Dincă and M. Miculiţa. We give the common solution of Marian Dincă and Mihai Miculiţa.

Consider the ellipse passing through A and B having the foci at C and D. The tangents to the ellipse at A respectively B meet at some point T. Denote by Q the intersection of AD and BC. We will show that the three bisectors meet at T.



To show this, it suffices to prove that T is equidistant from the lines AQ, AP, BQ respectively BP.

Since TB is tangent to the ellipse, we have $\angle TBD \equiv \angle TBQ$, which shows that TB is the bisector of $\angle QBD$. This shows that T is equidistant from BQ and BP.

In the same way, since TA is tangent to the ellipse, T is equidistant from AP and AQ.

Moreover, as T is the point of intersection of the tangents at A and B, it follows that TC and TD are the angle bisectors of BCA and BDA. This implies that T is equidistant from AP, BQ and that T is equidistant from BP, AQ.

Therefore T is equidistant from the lines AQ, AP, BQ.

OC120. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where n is a positive integer and r is a rational number. (a, b, c) is called a nice triple if a, b are positive rationals, c is a positive integer and

$$S_a(n) = (S_b(n))^c$$

for all positive integers n. Find all nice triples.

Originally question 1 from Turkey team selection test 2012, day 3.

Solved by Oliver Geupel.

It is straightforward to verify that the following triples are nice: $(q, q, 1)$ for every positive rational number q, and $(3, 1, 2)$. We show that there are no other nice triples.

Note that, for $r > 0$,

$$\frac{n^{r+1}}{r+1} = \int_0^n x^r \,\mathrm{d}x < S_r(n) < \int_0^n (x+1)^r \,\mathrm{d}x = \frac{(n+1)^{r+1}}{r+1}\,.$$

Suppose that (a, b, c) is nice.

Then, for every $n \in \mathbb{N}$,

$$\frac{n^{a+1}}{a+1} \cdot \frac{(b+1)^c}{(n+1)^{(b+1)c}} < \frac{S_a(n)}{(S_b(n))^c} = 1 < \frac{(n+1)^{a+1}}{a+1} \cdot \frac{(b+1)^c}{n^{(b+1)c}}\,.$$

We argue that $a + 1 = (b + 1)c$. For, assume that $d = |a + 1 - (b + 1)c| > 0$. If $a + 1 - (b + 1)c > 0$, then

$$\frac{n^{a+1}}{a+1} \cdot \frac{(b+1)^c}{(n+1)^{(b+1)c}} = \frac{(b+1)^c}{a+1} \cdot \left(\frac{n}{n+1}\right)^{(b+1)c} \cdot n^d \to \infty$$

for $n \to \infty$, a contradiction.

On the other hand, if $a+1-(b+1)c<0$, then

$$\frac{(n+1)^{a+1}}{a+1}\cdot\frac{(b+1)^c}{n^{(b+1)c}}=\frac{(b+1)^c}{a+1}\cdot\left(\frac{n+1}{n}\right)^{a+1}\cdot\frac{1}{n^d}\to 0\,,$$

for $n\to\infty$, a contradiction. Consequently, $a+1=(b+1)c$.

Moreover, for every $n\in\mathbb{N}$,

$$\left(\frac{n}{n+1}\right)^{a+1}<\frac{a+1}{(b+1)^c}<\left(\frac{n+1}{n}\right)^{a+1},$$

hence $a+1=(b+1)^c$.

If $c=1$ then obviously $a=b$.

If $c=2$ then $2(b+1)=(b+1)^2$, which implies $b=1$ and $a=3$.

Finally assume $c\geq 3$. Then, the number $c=(b+1)^{c-1}$ is an integer. It follows that b itself is an integer, so that $b\geq 1$. But now $c=(b+1)^{c-1}\geq 2^{c-1}>c$, which is impossible. Consequently, our list of nice triples is complete.

Math Quotes

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks, of which the logical paradoxes constitute the most important part. For, if mathematics is to advance securely and confidently it must first set its affairs in order at home.

Kasner, E. and Newman, J., "Mathematics and the Imagination", New York: Simon and Schuster, 1940.

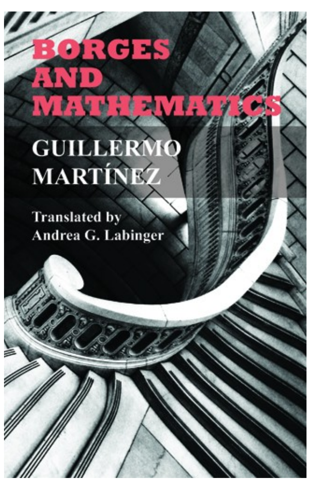
BOOK REVIEWS

Robert Bilinski

Borges and Mathematics by Guillermo Martinez
ISBN: 9781557536327
Purdue University Press, 2012, 140 pages, $16.95 (paperback)

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

The book under review is a study of the work of Argentinian author Jorge Luis Borges, whose life spanned the 20th century. His work transcends borders and is recognized as pivotal in world literature by its unique style and its variety. As such, he has won many prizes, even sharing one with Beckett and being on the Nobel's short list many times (without, sadly, ever winning it). This book's author, Guillermo Martinez, is an Argentinian mathematician who has a PhD in mathematics and writes successful mystery novels, one of which has been adapted to the big screen (*The Oxford Murders*, 2008). Mathematics plays an integral role in his plots and he recognizes, like a lot of people, that mathematics plays an important role in the plots of Borges' work. Yet he goes beyond this and conjectures that Borges' work IS mathematics. He believes that the structure of his work itself is a proof.



He starts by expounding on the common observation that there is mathematics in a lot of the novels and short stories of Borges. There's actually a large number of articles on this and some editors have published works of Borges annotated by mathematicians (OUP's "The Unimaginable Mathematics of Borges' Library of Babel" from 2008 is only one example). In appendix A of the book, Martinez groups Borges' works by mathematical ideas present in the plot: there are 19 themes from infinity to Occam's Razor. We also learn that Borges has mastered a high level of mathematics even if he has not practised mathematics, and we learn he has either written or has claimed to have read the works of around 40 mathematicians. So it is an established fact that mathematics in Borges' work is not "intuitive" as in Escher's work (who has no mathematical training) but "constructed by design". Then the main thesis of Martinez is presented with a mix of his ideas backed by excerpts of Borges' work and interviews he has given. I won't spoil the book more than I already have, but I will just say that I was enthralled by the dissertation. The whole approach is completely different from pretty much all the other books I've read about mathematics (maybe an exception would be Denis Guedj's "La gratuité ne vaut plus rien") and it pushes one to think outside the box about what we do as mathematicians. Which is a bit what we try to achieve in ***Crux*** , with our problem solving slant to mathematics.

And then, suddenly, you notice that this book is not only about Borges or Martinez, but about mathematical thought and culture, and most of all, the different kinds of passions the subject can waken in us. Apart from a few examples of mathematical reasoning to illustrate that Borges writes his works as a mathematician writes his proofs (a few of Cantor's proofs), this book is surprisingly devoid of Mathematics. Yet, it is full of mathematical history, sociology, psychology even. Ironically, about that point, the book itself stops treating Borges exclusively and expands a bit to other authors (Fermat, Oliver Sacks, Hans Enzensberger, Stephen Hawking, etc). In Chapter 9, Martinez even recounts an interview he did with Gregory Chaitin where they expound on the foundations and the nature of mathematics.

This book of Martinez's has many other levels. In its own right, it resembles his mystery novels, but the characters battling in it are the ghosts of history and the plot centers around Borges' writing. It is as though Borges' work constitutes the proof that mathematics exists and that great minds were called to stand witness. The text is full of anecdotes about mathematicians like Fermat, Andrew Wiles, Beppo Levi, Hawking, Chaitin. Martinez seems to have been at the right time and the right place to meet some of the great mathematicians of the end of the 20th century.

The book is a quick read at 140 pages from cover to cover. It is organized into 13 chapters. The first 4 of them treat Borges' works and take up half the book. The rest is a mix of everything I mentioned with chapters unequal in length. In sum, this book is full of goodies for the mathematician and the passionate about mathematics who want to learn about the culture, the history, the problems facing real-life mathematicians, the dilemmas, the biology, the limits and the beauty of mathematics. A fascinating little paperback book that I am sure to reread when I am on vacation and a bit more rested and, above all, a bit freer to explore all the side dishes of mathematics that are served up to the reader. Reading the book, I often had flashes about subjects or problems I wanted to explore, mathematicians I wanted to read about; mostly, it gave me a thirst to read more Borges. Guillermo Martinez truly has a passion for mathematics and the skills to communicate it.

Good reading!

P.S. You can find a sizeable excerpt at `http://guillermomartinezweb.blogspot.ca/2011/06/borges-and-mathematics.html`

UNSOLVED CRUX PROBLEMS

*As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Chris Fisher published a list of unsolved problems from **Crux** [2010 : 545, 547]. Below is a sample of two of these unsolved problems.*

527. [1980: 78; 1981: 88-89]

Proposed by Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.

a) You stand at a corner in a large city of congruent square blocks and intend to take a walk. You flip a coin – tails, you go left; heads, you go right – and you repeat the procedure at each corner you reach. What is the probability that you will end up at your starting point after walking n blocks?

b)$\star$ Same question, except that you flip the coin twice: TT, you go left; HH, you go right; otherwise, you go straight ahead.

714$\star$. [1982: 48; 1983: 58]

Proposed by Harry D. Ruderman, Hunter College Campus School, New York.

Prove or disprove that for every pair (p, q) of nonnegative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer.

(This problem was suggested by ***Crux*** problem 556 [1981: 282] proposed by Paul Erdös.)

Ramsey's Theory Through Examples, Exercises, and Problems: Part II

Veselin Jungić

1 Introduction

In the first part of this two-part article (appearing in ***Crux***, Volume 40 (2)), we introduced, for natural numbers s and t, $s, t \geq 2$, *the Ramsey number* $R(s,t)$ as the minimum number n for which any edge 2-colouring of K_n in red and blue, contained a red K_s or a blue K_t. We also established that $R(3,3) = 6$ and that $R(2,t) = R(t,2) = t$.

In this sequel, we continue our investigation of Ramsey numbers.

2 Ramsey Numbers $R(4,3)$, $R(3,5)$, and $R(4,4)$

In Exercises 1 – 4, we will establish that $R(4,3) \leq 9$.

Suppose that the edges of the complete graph K_9 are coloured with two colours, red and blue. We need to prove that this edge 2-colouring of K_9 yields a red K_4 or a blue K_3.

Since each vertex in K_9 is incident with 8 edges we notice that there are three possible cases:

1. There is a vertex incident with at least 6 red edges.
2. There is a vertex incident with at least 4 blue edges.
3. Each vertex is incident with exactly 5 red edges and 3 blue edges.

Exercise 1 *Suppose that there is a vertex incident with at least 6 red edges. Use the fact that $R(3,3) = 6$ to conclude that in this case the given edge 2-colouring of K_9 yields a red K_4 or a blue K_3. See Figure 1.*

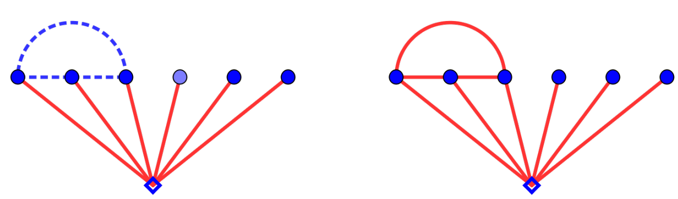


Figure 1: At least six red (solid) edges incident to the fixed vertex

Exercise 2 *Suppose that there is a vertex incident with at least 4 blue edges. Prove that in this case the given edge 2-colouring of K_9 yields a red K_4 or a blue K_3. See Figure 2.*

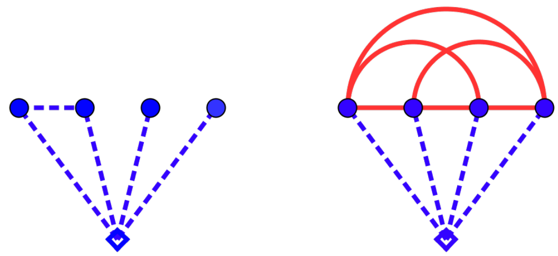


Figure 2: At least four blue (dashed) edges incident to the fixed vertex

Exercise 3 *Suppose that every vertex is incident with 5 red edges and with 3 blue edges. How many blue edges are there altogether?*

Exercise 4 *Based on Exercises 1 – 3, conclude that $R(4,3) \leq 9$. Justify your answer, i.e., explain what else you need to conclude that $R(4,3) \leq 9$.*

Problem 1 *Prove that $R(4,3) = R(3,4) = 9$.*

Problem 2 *Using $R(2,5) = 5$ and $R(3,4) = 9$, prove that $R(3,5) \leq 14$.*

Problem 3 *Prove that $R(4,4) \leq 18$.*

Actually it is known that $R(3,5) = 14$ and $R(4,4) = 18$. There is only one edge 2-colouring of K_{17} that avoids a monochromatic K_4!

Note that $R(4,4) = 18$ means that at any party with at least 18 people there would be either four mutual acquaintances or four mutual strangers. Interestingly enough, if one asks for the size of a party that would guarantee either five mutual acquaintances or five mutual strangers the best that we can say is that there should be no less than 43 and no more than 49 people at the party.

3 General Ramsey Numbers

We have established Ramsey numbers in a few special cases. In the rest of this note we will consider the following three questions:

Question 1 *Does the Ramsey number $R(s,t)$ exist for any choice of natural numbers $s \geq 2$ and $t \geq 2$?*

Question 2 *If the Ramsey number $R(s,t)$ exists, is its exact value known?*

Question 3 *If $R(s,t)$ exists and if we do not know its exact value, what are the known bounds for $R(s,t)$?*

We examine Question 1 through Exercises 5 – 7.

Exercise 5 *Observe that if $s, t \in \mathbb{N}\backslash\{1\}$ are such that*

$$s + t = 4 \text{ or } s + t = 5 \text{ or } s + t = 6$$

then $R(s, t)$ exists.

Exercise 6 *Suppose that $s, t \geq 3$ are such that $R(s-1, t)$ and $R(s, t-1)$ exist. To prove that $R(s, t)$ exists it is enough to prove that any 2-colouring of a complete graph K_M where*

$$M = R(s-1, t) + R(s, t-1)$$

yields a monochromatic K_s or a monochromatic K_t. Why?

Exercise 7 *Suppose that $n \geq 6$ is such that for any $u, v \geq 3$ such that $u + v = n$ the Ramsey number $R(u, v)$ exists. Suppose that $s, t \geq 3$ are such that $s+t = n+1$.*

1. *Conclude that $R(s-1, t)$ and $R(s, t-1)$ exist.*
2. *Prove that any 2-colouring of a complete graph K_M where*

   $$M = R(s-1, t) + R(s, t-1)$$

   yields a monochromatic K_s or a monochromatic K_t.
   (For a hint, see Problems 1 – 3 and Figure 3.)

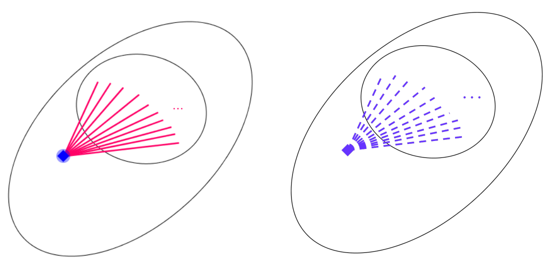


Figure 3: At least $R(s-1, t)$ red edges (left) and at least $R(s, t-1)$ blue edges (dashed, on the right).

Problem 4 *Use mathematical induction on the sum $s+t$ to prove that for $s, t \geq 3$*

$$R(s, t) \leq R(s-1, t) + R(s, t-1).$$

Hence, Problem 4 establishes an affirmative answer to Question 1.

It turns out that finding the exact values for $R(s, t)$ is a challenging task. In Erdős's words:

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

For an extensive account of the currently known exact values and bounds for Ramsey numbers see Section 2 in [2].

Finally, we address Question 3 through the following problem.

Problem 5 *Use mathematical induction on the sum $s+t$ to prove that for $s, t \geq 2$*

$$R(s,t) \leq \binom{s+t-2}{t-1}.$$

We note that Problem 5 implies Ramsey's theorem, as it was stated in Part I, in the case $r = \mu = 2$ and $n = s$.

4 Two Problems

Problem 6 *Prove that for $s \geq 3$,*

$$R(s,s) > 2^{s/2}.$$

(See [1].)

Problem 7 *Show that $R(3,3,3) \leq 17$, that is show that every 3-colouring of the edges of K_{17} gives a monochromatic K_3.*

References

[1] Graham, R., Rothschild, B., and Spencer, J.H., *Ramsey Theory* (2nd ed.), New York: John Wiley and Sons, 1990.

[2] Radziszowski, S.P., *Small Ramsey Numbers*, Electronic Journal of Combinatorics, Dynamic Survey DS1, revision #14 (2014), `http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1`http://www.combinatorics.org.

Veselin Jungić
Department of Mathematics, Simon Fraser University
Burnaby, BC, Canada
vjungic@sfu.ca

Excerpt from The Math Olympian

Richard Hoshino

Continued from ***Crux***, Volume 40 (2).

We left off as Bethany was working on problem #3 of The Canadian Mathematical Olympiad which asked to determine the value of:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\cdots+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

. . . .

Calm down, Bethany, calm down. There's lots of time left. You can do this.

I think about the soothing words of Mr. Collins, and am reminded of another important problem-solving strategy I learned from him: simplify the problem by breaking it into smaller and easier parts, in order to find a pattern.

I can do that.

I don't want to deal with the horrible expression given in the problem, a complicated sum of nearly one thousand fractions. I've seen enough contest problems to know that the number 1000 is a distracter, and that it has nothing to do with the question. By making the number big, the problem looks a lot more intimidating than it actually is.

For example, in the addition question that Mr. Collins posed to me that day, as soon as I realize that the series telescopes, it doesn't matter whether there are nine fractions or nine thousand fractions. In the former the answer is $\frac{1}{1}-\frac{1}{10}=\frac{9}{10}$ and in the latter the answer is $\frac{1}{1}-\frac{1}{9001}=\frac{9000}{9001}$. The final answer is different, but at its heart, it's the exact same problem.

I'm sure the same is true with this Olympiad problem. Especially being the first problem, I know there has to be a short and elegant solution. Remembering the advice of Mr. Collins, I decide to simplify the problem in order to discover a pattern, which will then allow me to solve the actual problem.

I change the denominator from 1000 to 4, to have just a few terms to play with. So now, instead of the exponents ranging from $\frac{1}{1000}$ to $\frac{999}{1000}$ I only have to consider $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$.

Instead of adding 999 ugly terms as in the actual problem, I only have three terms in the simplified problem. By making the expression easier, I am hopeful that I'll discover something interesting.

So my simplified problem is to determine the value of

$$\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3}+\frac{9^{\frac{2}{4}}}{9^{\frac{2}{4}}+3}+\frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3}.$$

This looks much more reasonable. The middle expression is easy — I figured this out ten minutes earlier.

$$\frac{9^{\frac{2}{4}}}{9^{\frac{2}{4}}+3} = \frac{9^{\frac{1}{2}}}{9^{\frac{1}{2}}+3} = \frac{\sqrt{9}}{\sqrt{9}+3} = \frac{3}{3+3} = \frac{3}{6} = \frac{1}{2}.$$

As I ponder how to calculate the values of $\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3}$ and $\frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3}$, a few ideas occur to me. I scribble some calculations on my notepad, add up the two fractions, and am surprised that the sum is exactly one.

$$\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3} + \frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3} = 1.$$

Interestingly, the first and last terms of my simplified problem add up to 1. I have a hunch that this might also be true in the more complicated Olympiad problem with 999 terms.

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3} + \frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3} + \frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3} + \cdots + \frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3} + \frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

To my delight, the hunch is correct.

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3} + \frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3} = 1.$$

I run through the calculations one more time, double-checking that I haven't made any mistakes. Yes, the terms in the numerator perfectly match the terms in the denominator, and the sum is indeed one.

I wonder whether this pattern continues, and am shocked to discover that

$$\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3} + \frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3} = 1.$$

I suddenly feel a lump in my throat. I know how to solve the Olympiad problem.

The key insight is staring me in the face.

All I need to do is apply the technique I discovered in Mrs. Ridley's class seven years ago, when I was in Grade 5. I can't believe it.

It's the Staircase.

"The Math Olympian" was published by FriesenPress in January 2015. For more information, please visit `www.richardhoshino.com`.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-psol@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays.*

Comment soumettre un problème. *Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er août 2015***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction remercie André Ladouceur d'avoir traduit les problèmes.

3931. *Proposé par Bill Sands.*

Deux épreuves de mathématiques ont été présentées aux élèves d'une classe. Chaque élève devait subir l'épreuve 1 ou l'épreuve 2, mais chacun pouvait subir les deux épreuves. Or, un quart des élèves qui ont subi l'épreuve 1 ont obtenu un A et un tiers des élèves qui ont subi l'épreuve 2 ont obtenu un A. Le nombre d'élèves ayant obtenu un A dans l'épreuve 1 est le même que le nombre d'élèves ayant obtenu un A dans l'épreuve 2. De plus, la moitié des élèves de la classe ont obtenu un A dans au moins une des deux épreuves. Démontrer que

a) chaque élève de la classe a subi l'épreuve 1 et

b) aucun élève a obtenu un A dans les deux épreuves.

3932. *Proposé par Arkady Alt.*

Soit x et y deux entiers strictement positifs qui vérifient l'équation $x^2 - 14xy + y^2 - 4x = 0$. Déterminer le PGCD$(x, y)$ en fonction de x et de y.

3933. *Proposé par Dragoljub Milošević.*

Soit $ABCDEFG$ un heptagone régulier. Démontrer que

$$\frac{AD^3}{AB^3} - \frac{AB + 2AC}{AD - AC} = 1.$$

3934. *Proposé par George Apostolopolous.*

Soit a, b et c les longueurs des côtés d'un triangle. Démontrer que

$$\frac{a}{\sqrt[3]{4b^3 + 4c^3}} + \frac{b}{\sqrt[3]{4a^3 + 4c^3}} + \frac{c}{\sqrt[3]{4a^3 + 4b^3}} < 2.$$

3935. *Proposé par Michel Bataille.*

Pour tout entier strictement positif n, soit $P_n(x) = x^n + \sum\limits_{k=1}^{n} (-1)^k (n-k+1) x^{n-k}$.

a) Démontrer que si $n \geq 3$, le polynôme P_n admet un seul zéro x_n dans l'intervalle $(1, \infty)$ et déterminer des réels α, β tels que $\lim\limits_{n\to\infty} (x_n - \alpha n) = \beta$.

b) Démontrer que pour tout entier n, $n \geq 2$,

$$1 - \frac{1}{4n^2} < x_{2n+1} - x_{2n} < 1 + \frac{1}{2n+1} \quad \text{et} \quad 1 - \frac{1}{2n-1} < x_{2n} - x_{2n-1} < 1 + \frac{1}{4n^2}.$$

3936. *Proposé par Paul Bracken.*

Soit p un entier, $p \geq 1$, et soit $\{x_k\}_{k=1}^{n} \in (0, 1)$. Démontrer que

$$\prod_{k=1}^{n} (1 - x_k^p) \leq e^{-(p+1)^{1/p} \sum_{k=1}^{n} x_k^{p+1}}.$$

3937. *Proposé par Marcel Chiriţă.*

Les sommets d'un triangle sont représentés par les nombres complexes a, b et c. Démontrer que si

$$\frac{a-b}{c-b} + \frac{c-a}{b-a} = 2\frac{b-c}{a-c},$$

le triangle est équilatéral.

3938. *Proposé par Francisco Javier García Capitán.*

Soit un triangle ABC et un cercle O. Déterminer un point P sur O pour lequel l'expression $PA^2 + PB^2 + PC^2$ a une valeur minimale et un autre point P sur O pour lequel l'expression a une valeur maximale.

3939. *Proposé par George Apostolopolous.*

Soit a, b et c des réels strictement positifs tels que $a^2 + b^2 + c^2 = 27$. Démontrer que

$$\sum_{\text{cycl}} \frac{a}{\sqrt{a^2 - 3a + 9}} \leq 3.$$

3940. *Proposé par Michał Kremzer.*

Déterminer des réels strictement positifs a et b tels que $\dfrac{a+b}{a(\tan a + \tan b)} = 2015.$

. . . .

3931. *Proposed by Bill Sands.*

A class is given two math tests. Each student in the class must write either Test 1 or Test 2 , but could write both tests. It turned out that one-quarter of the students who wrote Test 1 got an A, that one-third of the students who wrote Test 2 got an A, and that the same number of students got A on the two tests. Also, one-half of all the students in the class got an A on at least one of the two tests. Prove that

a) every student wrote Test 1, and

b) no student got A on both tests.

3932. *Proposed by Arkady Alt.*

Let x and y be natural numbers satisfying equation $x^2 - 14xy + y^2 - 4x = 0$. Find $\gcd(x, y)$ in terms of x and y.

3933. *Proposed by Dragoljub Milošević.*

Let $ABCDEFG$ be a regular heptagon. Prove that

$$\frac{AD^3}{AB^3} - \frac{AB + 2AC}{AD - AC} = 1.$$

3934. *Proposed by George Apostolopolous.*

Let a, b and c be the side lengths of a triangle. Prove that

$$\frac{a}{\sqrt[3]{4b^3 + 4c^3}} + \frac{b}{\sqrt[3]{4a^3 + 4c^3}} + \frac{c}{\sqrt[3]{4a^3 + 4b^3}} < 2.$$

3935. *Proposed by Michel Bataille.*

For positive integers n, let $P_n(x) = x^n + \sum\limits_{k=1}^{n}(-1)^k(n-k+1)x^{n-k}$.

a) Prove that if $n \geq 3$, the polynomial P_n has a unique zero x_n in $(1, \infty)$ and find real numbers α, β such that $\lim\limits_{n\to\infty}(x_n - \alpha n) = \beta$.

b) Prove that for all integers $n \geq 2$:

$$1-\frac{1}{4n^2} < x_{2n+1}-x_{2n} < 1+\frac{1}{2n+1} \quad \text{and} \quad 1-\frac{1}{2n-1} < x_{2n}-x_{2n-1} < 1+\frac{1}{4n^2}.$$

3936. *Proposed by Paul Bracken.*

Let $p \geq 1$ and suppose $\{x_k\}_{k=1}^n \in (0,1)$. Prove that

$$\prod_{k=1}^{n}(1-x_k^p) \leq e^{-(p+1)^{1/p}\sum_{k=1}^n x_k^{p+1}}.$$

3937. *Proposed by Marcel Chiriţă.*

If the vertices of a triangle are represented by the complex numbers a, b, c, and these numbers satisfy

$$\frac{a-b}{c-b} + \frac{c-a}{b-a} = 2\frac{b-c}{a-c},$$

then prove that the triangle is equilateral.

3938. *Proposed by Francisco Javier García Capitán, modified by the editor.*

Let ABC be a triangle, O a circle and P a point on O. Find two points on O for which the sum $PA^2 + PB^2 + PC^2$ reaches its minimum and its maximum.

3939. *Proposed by George Apostolopolous.*

Let a, b and c be positive real numbers such that $a^2 + b^2 + c^2 = 27$. Prove that

$$\sum_{\text{cyclic}} \frac{a}{\sqrt{a^2 - 3a + 9}} \leq 3.$$

3940. *Proposed by Michał Kremzer.*

Find positive a and b so that $\dfrac{a+b}{a(\tan a + \tan b)} = 2015.$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the following solvers whose solutions were overlooked: Arkady Alt for problems 3813, 3815, 3816 and 3818; Paolo Perfetti for 3800, 3815, 3818 and 3820. The editor apologizes sincerely for the oversight.

3831. *Proposed by George Apostolopoulos.*

Let a, b, c, d be positive real numbers with $abcd = 16$. Prove that

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{d^3} + \frac{d^3}{a^3} + 4 \geq a + b + c + d\,.$$

Solved by M. Bataille; M. Benito, Ó.Ciaurri, E. Fernández and L. Roncal; O. Geupel; S. Malikić; T. K. Parayiou; P. Perfetti; T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Oliver Geupel; and Titu Zvonaru and Neculai Stanciu.

Each sum is cyclic with four terms. Using the arithmetic-geometric means inequality, we obtain that

$$\begin{aligned}\left[\sum \frac{a^3}{b^3}\right] + 4 &= \left[\sum \frac{1}{6}\left(\frac{3a^3}{b^3} + \frac{2b^3}{c^3} + \frac{c^3}{d^3}\right)\right] + 4 \\ &\geq \left[\sum \frac{a^2}{\sqrt{abcd}}\right] + 4 = \left[\sum \frac{a^2}{4}\right] + 4 \\ &= \frac{1}{4}[a^2 + b^2 + c^2 + d^2 + 16] = a + b + c + d + \frac{1}{4}\sum (a-2)^2 \\ &\geq a + b + c + d,\end{aligned}$$

with equality if and only if $a = b = c = d = 2$.

Editor's comment. Perfetti followed a similar strategy. Benito, Ciaurri, Fernández and Roncal applied the arithmetic geometric means inequality to four terms to obtain

$$\frac{3a^3}{b^3} + \frac{2b^3}{c^3} + \frac{c^3}{d^3} + 6 \geq \frac{12a}{(abcd)^{1/4}}.$$

This added to its three analogues leads to

$$\sum 6a^3/b^3 + 24 \geq 12(a + b + c + d)(abcd)^{-1/4}$$

and the result follows.

Zvonaru and Stanciu generalized the method of the solution to obtain

$$\sum_{i=1}^{m} \frac{a_i^{n-1}}{a_{i+1}^{n-1}} + n \geq a_1 + a_2 + \cdots + a_n,$$

where $n \geq 3$, $a_i > 0$ for each i, $a_{n+1} = a_1$ and $a_1 a_2 \dots a_n = 2^n$.

3832. *Proposed by Marcel Chiriţă.*

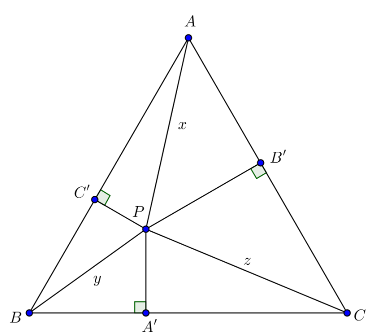
Let P be a point inside the equilateral triangle ABC with side length equal to 1, and let $x = PA$, $y = PB$, $z = PC$. Prove that:

$$(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 = 3(x^2y^2 + y^2z^2 + z^2x^2)\,.$$

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; M. Benito, Ó.Ciaurri, E. Fernández and L. Roncal; P. De; M. Dinca; O. Kouba; S. Malikić; T. K. Parayiou; P. Perfetti; J. Hawkins and D. R. Stone; D. Văcaru; P. Woo; and the proposer. We present two solutions.

Solution 1, by Peter Woo.

Let PA', PB' and PC' be the perpendiculars from P to BC, CA and AB, respectively. Let $a = PA', b = PB'$ and $c = PC'$.



Since $Q = PA'CB'$ is a cyclic quadrilateral, PC is the diameter of the circumcircle of Q, so by the Law of sines and the Law of cosines we have:

$$\begin{aligned} z = PC &= \frac{A'B'}{\sin\angle A'PB'} = \frac{A'B'}{\sin\frac{2\pi}{3}} \\ &= \frac{2}{\sqrt{3}}A'B' = \frac{2}{\sqrt{3}}\sqrt{a^2 + b^2 - 2ab\cos\frac{2\pi}{3}} = \frac{2}{\sqrt{3}}\sqrt{a^2 + b^2 + ab}. \end{aligned}$$

Hence,

$$z^2 = \frac{4}{3}(a^2 + b^2 + ab)$$

and, similarly,

$$x^2 = \frac{4}{3}(b^2 + c^2 + bc) \text{ and } y^2 = \frac{4}{3}(a^2 + c^2 + ac). \qquad (1)$$

It is a well known fact that $a + b + c = \frac{\sqrt{3}}{2}$, the height of $\triangle ABC$. Using this and (1), we have:

$$x^2 - y^2 = \frac{4}{3}(b^2 - a^2 + c(b - a)) = \frac{4}{3}(b - a)(b + a + c) = \frac{2}{\sqrt{3}}(b - a)$$

and, similarly,

$$y^2 - z^2 = \frac{2}{\sqrt{3}}(c - b) \text{ and } z^2 - x^2 = \frac{2}{\sqrt{3}}(a - c). \qquad (2)$$

Also,

$$\begin{aligned} x^2 + y^2 + z^2 &= \frac{4}{3}[2(a^2 + b^2 + c^2 + ab + bc + ca)] \\ &= \frac{4}{3}[2(a + b + c)^2 - 3(ab + bc + ca)] \\ &= \frac{4}{3}\left[\frac{3}{2} - 3(ab + bc + ca)\right] = 2 - 4(ab + bc + ca). \end{aligned}$$

Using this and (2), we then have:

$$\begin{aligned} &(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 - 3(x^2y^2 + y^2z^2 + z^2x^2) \\ &\quad = x^4 + y^4 + z^4 - (x^2y^2 + y^2z^2 + z^2x^2) - (x^2 + y^2 + z^2) + 1 \\ &\quad = \frac{1}{2}\left((x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2\right) - (2 - 4(ab + bc + ca)) + 1 \\ &\quad = \frac{2}{3}\left((b - a)^2 + (c - b)^2 + (a - c)^2\right) + 4(ab + bc + ca) - 1 \\ &\quad = \frac{4}{3}\left((a + b + c)^2 - 3(ab + bc + ca)\right) + 4(ab + bc + ca) - 1 \\ &\quad = \frac{4}{3}\left(\frac{3}{4}\right) - 1 = 0, \end{aligned}$$

which completes the proof.

Solution 2, by M. Benito, Ó.Ciaurri, E. Fernández and L. Roncal slightly expanded by the editor.

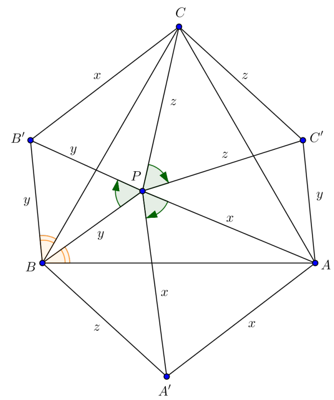
After expanding all the factors, the identity to be proved is equivalent to

$$x^4 + y^4 + z^4 + 1 - (x^2 + y^2 + z^2) + 2(x^2y^2 + y^2z^2 + z^2x^2) = 3(x^2y^2 + y^2z^2 + z^2x^2)$$

or

$$1 + x^4 + y^4 + z^4 = x^2 + y^2 + z^2 + x^2y^2 + y^2z^2 + z^2x^2. \qquad (3)$$

To establish (3), we rotate the lines PA, PB, and PC clockwise through an angle of $\pi/3$ radians so A, B, and C are mapped to the points A', B', and C', respectively:



Then clearly, $AA' = x$, $BB' = y$, and $CC' = z$. Since $BC = BA$, $BB' = BP = y$ and $\angle B'BC = \angle PBA = \frac{\pi}{3} - \angle PBC$, we have that $\triangle BB'C \sim \triangle BPA$. Hence, $B'C = PA = x$. Similarly, $C'A = y$ and $A'B = z$.

We now consider the area of the hexagon $H = AA'BB'CC'$. Clearly,

$$\text{Area}(H) = 2 \cdot \text{Area}(\triangle ABC) = 2\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}.$$

On the other hand,

$$\begin{aligned}\text{Area}(H) = {} & \text{Area}(\triangle PAA') + \text{Area}(\triangle PA'B) + \text{Area}(\triangle PBB') \\ & + \text{Area}(\triangle PB'C) + \text{Area}(\triangle PCC') + \text{Area}(\triangle PC'A).\end{aligned}$$

We also have that

$$\text{Area}(\triangle PAA') = \frac{\sqrt{3}}{4}x^2, \quad \text{Area}(\triangle PBB') = \frac{\sqrt{3}}{4}y^2, \ \text{Area}(\triangle PCC') = \frac{\sqrt{3}}{4}z^2.$$

By Heron's formula, we have $\text{Area}(\triangle PA'B) = \text{Area}(\triangle PB'C) = \text{Area}(\triangle PC'A)$, which equals to

$$\sqrt{\frac{x+y+z}{2} \cdot \frac{x+y-z}{2} \cdot \frac{x-y+z}{2} \cdot \frac{-x+y+z}{2}}.$$

Combining all of the above, we have

$$\begin{aligned}\text{Area}(H) &= \frac{\sqrt{3}}{4}(x^2+y^2+z^2) + \frac{3}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)} \\ &= \frac{\sqrt{3}}{2},\end{aligned}$$

or

$$(x^2+y^2+z^2-2)^2 = 3(x+y+z)(x+y-z)(x-y+z)(-x+y+z),$$

which, after expanding and simplifying, becomes

$$\begin{aligned}&x^4+y^4+z^4+4-4(x^2+y^2+z^2)+2(x^2y^2+y^2z^2+z^2x^2)\\&=3((x+y)^2-z^2)(z^2-(x-y)^2)\\&=3(x^2+y^2-z^2+2xy)(z^2-x^2-y^2+2xy)\\&=-3(x^4+y^4+z^4)+6(x^2y^2+y^2z^2+z^2x^2)\end{aligned}$$

or

$$4(x^4+y^4+z^4)+4=4(x^2+y^2+z^2+x^2y^2+y^2z^2+z^2x^2),$$

which establishes (3) and completes the proof.

Editor's comment. Bailey, Campbell, and Diminnie proved that the result holds for any point P in the plane of A, B, and C. De gave a proof based on the following identity, for which no proof or reference was given:

$$3(x^4+y^4+z^4+s^4)=(x^2+y^2+z^2+s^2)^2,$$

where s denotes the semiperimeter of $\triangle ABC$. If we set $s=1$, then the required identity follows readily after some simple calculations.

It has been pointed out that this is Problem 250 from *Revista Escolar de la Olimpiada Iberoamericana de Mathemática*, whose solutions has been published here: `http://www.oei.es/oim/revistaoim/numero51/250_Bruno.pdf`.

3833. *Proposed by Ángel Plaza.*

Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{z^3(zx+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\geq\frac{3}{2xyz}\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; M. Bataille; D. M. Bătineţu-Giurgiu and N. Stanciu; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. De; O. Geupel; D. Koukakis; S. Malikić; P. McCartney; M. Modak; T. K. Parayiou; P. Perfetti; C. R. Pranesachar; D. Smith; S. Wagon; and the proposer. We present a composite of similar solutions by Arkady Alt; Salem Malikić; Phil McCartney; and Digby Smith.

The claimed inequality is successively equivalent to

$$\sum_{\text{cyclic}}\frac{x^3yz}{z^3\left(zx+y^2\right)}\geq\frac{3}{2}\quad\text{and}\quad\sum_{\text{cyclic}}\frac{\left(\frac{x}{z}\right)^2}{\frac{z}{y}+\frac{y}{x}}\geq\frac{3}{2},$$

which, with the substitutions $a=\frac{x}{z}$, $b=\frac{y}{x}$, $c=\frac{z}{y}$, becomes

$$\sum_{\text{cyclic}}\frac{a^2}{b+c}\geq\frac{3}{2},$$

with $abc = 1$. The Cauchy-Schwarz inequality applied to $\left\langle \frac{a}{\sqrt{b+c}}, \frac{b}{\sqrt{c+a}}, \frac{c}{\sqrt{a+b}} \right\rangle$ and $\left\langle \sqrt{b+c}, \sqrt{c+a}, \sqrt{a+b} \right\rangle$ gives

$$\sqrt{\sum_{\text{cyclic}} \frac{a^2}{b+c}} \cdot \sqrt{2\,(a+b+c)} \geq a+b+c.$$

Dividing by $\sqrt{2\,(a+b+c)}$, squaring, and applying the AM-GM inequality, we have

$$\sum_{\text{cyclic}} \frac{a^2}{b+c} \geq \frac{a+b+c}{2} \geq \frac{3}{2} \cdot \sqrt[3]{abc} = \frac{3}{2},$$

as claimed. Equality holds if and only if $a = b = c$.

Editor's comment. Bătineţu-Girgiu and Stanciu (jointly) indicated that they proposed the same problem to the College Mathematics Journal and then School Science and Mathematics. Stan Wagon provided a Mathematica solution.

3834. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a parallelogram and E, F be interior points of the sides BC, CD, respectively, such that $\frac{BE}{EC} = \frac{CF}{FD}$. The line segments AE and AF meet the diagonal BD at the points K and L respectively.

(a) Prove that $\text{Area}(\Delta AKL) = \text{Area}(\Delta BKE) + \text{Area}(\Delta DLF)$.

(b) Find the ratio $\dfrac{\text{Area}(\Delta ABCD)}{\text{Area}(\Delta AECF)}$.

Solved by M. Amengual Covas (2 solutions); AN-anduud Problem Solving Group; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. De; P. Deiermann; O. Kouba; T. K. Parayiou; R. Peiró; P. Y. Woo; and the proposer. We present three solutions. In each, we use $[ABC]$ to denote the area of ABC.

Solution 1, by Prithwijit De.

Let $\dfrac{BE}{EC} = \dfrac{CF}{FD} = x$. Observe that

$$\frac{[ABE]}{[ABC]} = \frac{BE}{BC} = \frac{x}{x+1} \quad \text{and} \quad \frac{[AFD]}{[ACD]} = \frac{DF}{CD} = \frac{1}{x+1}.$$

But $[ABC] = [ACD] = \dfrac{1}{2}[ABCD]$. Therefore

$$[ABE] + [AFD] = \frac{1}{2}[ABCD] = [ABD]. \tag{1}$$

Also

$$[ABE] = [BKE] + [AKB],$$

$$[AFD] = [ALD] + [DLF]$$

and

$$[ABD] = [AKB] + [AKL] + [ALD].$$

Using this in the above equation we get

$$[AKL] = [BKE] + [DLF].$$

This proves part (a). For part (b), observe that

$$[ABE] + [AFD] + [AECF] = [ABCD]. \tag{2}$$

Using (1) and (2), we obtain $\dfrac{[ABCD]}{[AECF]} = 2$.

Solution 2, by Omran Kouba.

Let $\vec{i} = \vec{AB}$ and $\vec{j} = \vec{AD}$. Since E is an interior point of the side BC, we have $\vec{BE} = \alpha\vec{j}$ for some $\alpha \in (0,1)$. Now the fact that $\frac{BE}{EC} = \frac{CF}{FD}$ implies that $\vec{DF} = (1-\alpha)\vec{i}$. Now the equation of the line BD in the affine coordinate system $(A;\vec{i},\vec{j})$ is $x+y=1$. So, the point K defined by $\vec{AK} = t\vec{AE} = t\vec{i} + t\alpha\vec{j}$ belongs to BD if and only if $t(1+\alpha) = 1$. Hence

$$\vec{AK} = \left(\frac{1}{1+\alpha}\right)\vec{i} + \left(\frac{\alpha}{1+\alpha}\right)\vec{j}.$$

Similarly, we find that

$$\vec{AL} = \left(\frac{1-\alpha}{2-\alpha}\right)\vec{i} + \left(\frac{1}{2-\alpha}\right)\vec{j}.$$

Now,

$$\begin{aligned}
2\,[AKL] &= \det(\vec{AK},\vec{AL}) = \frac{1-\alpha+\alpha^2}{(1+\alpha)(2-\alpha)} \cdot \det(\vec{i},\vec{j}) \\
2\,[BKE] &= \det(\vec{BE},\vec{BK}) = \det\left(\alpha\vec{j}, \frac{\alpha}{1+\alpha}(\vec{j}-\vec{i})\right) = \frac{\alpha^2}{1+\alpha} \cdot \det(\vec{i},\vec{j}) \\
2\,[DLF] &= \det(\vec{DL},\vec{DF}) = \det\left(\frac{1-\alpha}{2-\alpha}(\vec{i}-\vec{j}), (1-\alpha)\vec{j}\right) = \frac{(1-\alpha)^2}{2-\alpha} \cdot \det(\vec{i},\vec{j}).
\end{aligned}$$

Thus, (a) follows from the fact that

$$\frac{\alpha^2}{1+\alpha} + \frac{(1-\alpha)^2}{2-\alpha} = \frac{1-\alpha+\alpha^2}{(1+\alpha)(2-\alpha)}.$$

On the other hand,

$$\begin{aligned}
[AECF] &= [ABCD] - [ABE] - [AFD] \\
&= [ABCD] - \frac{1}{2}\det(\vec{AB},\vec{BE}) - \frac{1}{2}\det(\vec{DF},\vec{AD}) \\
&= [ABCD] - \frac{1}{2}\det(\vec{i},\alpha\vec{j}) - \frac{1}{2}\det((1-\alpha)\vec{i},\vec{j}) \\
&= [ABCD] - \frac{1}{2}\det(\vec{i},\vec{j}) = \frac{1}{2}\,[ABCD]
\end{aligned}$$

because $[ABCD] = \det(\vec{i}, \vec{j})$. Thus, $\dfrac{[ABCD]}{[AECF]} = 2$, which is the answer to (b).

Solution 3, by Miguel Amengual Covas.

We use analytic geometry. We let the axes be $Ox \equiv BC$ and $Oy \equiv BA$ and give coordinates $B\,(0,0)$, $C\,(a,0)$ and $A\,(0,b)$.

We put $\frac{BE}{EC} = \frac{CF}{FD} = k$. Then the coordinates of D are (a,b), the coordinates of E are $\left(\frac{ka}{k+1}, 0\right)$, and those of F are $\left(a, \frac{kb}{k+1}\right)$. Hence, the coordinates of K are $\left(\frac{ka}{2k+1}, \frac{kb}{2k+1}\right)$ and the coordinates of L are $\left(\frac{(k+1)a}{k+2}, \frac{(k+1)b}{k+2}\right)$.

We put $\angle ABC = \theta$. Since the area of a triangle whose vertices have coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), is given by

$$\frac{\sin\theta}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

provided the three vertices are listed in counterclockwise order around the triangle, we get

$$[AKL] = \frac{\sin\theta}{2}\begin{vmatrix} 0 & b & 1 \\ \frac{ka}{2k+1} & \frac{kb}{2k+1} & 1 \\ \frac{(k+1)a}{k+2} & \frac{(k+1)b}{k+2} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2} \cdot \frac{k^2+k+1}{(k+2)\,(2k+1)}\ ,$$

$$[BKE] = \frac{\sin\theta}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{ka}{k+1} & 0 & 1 \\ \frac{ka}{2k+1} & \frac{kb}{2k+1} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2} \cdot \frac{k^2}{(k+1)\,(2k+1)}\ ,$$

and

$$[DLF] = \frac{\sin\theta}{2}\begin{vmatrix} a & b & 1 \\ \frac{(k+1)a}{k+2} & \frac{(k+1)b}{k+2} & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2} \cdot \frac{1}{(k+1)\,(k+2)}$$

Hence,

$$\begin{aligned}
[BKE] + [DLF] &= \frac{ab\sin\theta}{2(k+1)}\left(\frac{k^2}{2k+1} + \frac{1}{k+2}\right) = \frac{ab\sin\theta}{2\,(k+1)} \cdot \frac{k^2\,(k+2) + 2k + 1}{(2k+1)\,(k+2)} \\
&= \frac{ab\sin\theta}{2\,(k+1)} \cdot \frac{(k+1)\left(k^2+k+1\right)}{(2k+1)\,(k+2)} = \frac{ab\sin\theta}{2} \cdot \frac{k^2+k+1}{(2k+1)\,(k+2)} \\
&= S\,[AKL]
\end{aligned}$$

as desired. For part (b), we have

$$\begin{aligned}
[AECF] &= [AEF]+[FEC] \\
&= \tfrac{\sin\theta}{2}\left(\begin{vmatrix} 0 & b & 1 \\ \frac{ka}{k+1} & 0 & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} + \begin{vmatrix} \frac{ka}{k+1} & 0 & 1 \\ a & 0 & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} \right) \\
&= \tfrac{ab\sin\theta}{2}\left(\tfrac{k^2+k+1}{(k+1)^2} + \tfrac{k}{(k+1)^2} \right) \\
&= \tfrac{1}{2} ab\sin\theta \\
&= \tfrac{1}{2}\,[ABCD]
\end{aligned}$$

that is,

$$\frac{[ABCD]}{[AECF]} = 2.$$

3835. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Determine the functions $f:\mathbb{R}\to\mathbb{R}$, continuous at $x=0$, for which $f(0)=1$ and

$$3f(x)-5f(\alpha x)+2f(\alpha^2 x) = x^2+x\,,$$

for all $x\in\mathbb{R}$, where $\alpha\in(0,1)$ is fixed.

Solved by A. Alt; AN-anduud Problem Solving Group; M. Bataille; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; O. Geupel; O. Kouba; D. Koukakis; J. Ling; S. de Luxán and Á. Plaza; S. Muralidharan; and the proposer. We present 2 solutions.

Solution 1, by many of the solvers.

The required functions are

$$f(x) = \frac{x^2}{(1-\alpha^2)(3-2\alpha^2)} + \frac{x}{(1-\alpha)(3-2\alpha)} + 1.$$

Of course, computation shows that the given functions satisfy the conditions. Let us see that they are the only possible ones.

By the condition on f, we have

$$\begin{aligned}
\sum_{k=0}^{n}\left((\alpha^k x)^2+\alpha^k x\right) &= \sum_{k=0}^{n}\left(3f(\alpha^k x)-5f(\alpha^{k+1}x)+2f(\alpha^{k+2}x)\right) \\
&= 3f(x)-2f(\alpha x)-3f(\alpha^{n+1}x)+2f(\alpha^{n+2}x).
\end{aligned}$$

Then,

$$\lim_{n\to\infty}\big(3f(x)-2f(\alpha x)-3f(\alpha^{n+1}x)+2f(\alpha^{n+2}x)\big)=x^2\sum_{k=0}^{\infty}\alpha^{2k}+x\sum_{k=0}^{n}\alpha^k$$

and so

$$3f(x)-2f(\alpha x)-1=\frac{x^2}{1-\alpha^2}+\frac{x}{1-\alpha},$$

because $\lim_{n\to\infty} f(x_n)=1$ for any sequence such that $\lim_{n\to\infty} x_n=0$. Using this equation, we obtain that

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k\left(\frac{(\alpha^k x)^2}{1-\alpha^2}+\frac{\alpha x}{1-\alpha}+1\right)&=\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k\big(3f(\alpha^k x)-2f(\alpha^{k+1}x)\big)\\&=3f(x)-2\left(\frac{2}{3}\right)^n f(\alpha^{n+1}x).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\left(3f(x)-2\left(\frac{2}{3}\right)^n f(\alpha^{n+1}x)\right)\\&\qquad=\frac{x^2}{1-\alpha^2}\sum_{k=0}^{\infty}\left(\frac{2\alpha^2}{3}\right)^k+\frac{x}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{2\alpha}{3}\right)^k+\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k,\end{aligned}$$

which yields, finally,

$$f(x)=\frac{x^2}{(1-\alpha^2)(3-2\alpha^2)}+\frac{x}{(1-\alpha)(3-2\alpha)}+1.$$

Solution 2, by Omran Kouba.

Consider a function f satisfying the proposed conditions. Since

$$3-5\alpha+2\alpha^2=(3-2\alpha)(1-\alpha)\neq 0\quad\text{and}\quad 3-5\alpha^2+2\alpha^4\neq 0,$$

we can define $g:\mathbb{R}\to\mathbb{R}$ by the formula

$$g(x)=f(x)-\frac{x}{3-5\alpha+2\alpha^2}-\frac{x^2}{3-5\alpha^2+2\alpha^4}.$$

Clearly, g is continuous at $x=0$ with $g(0)=1$, and it is straightforward to check that

$$3g(x)-5g(\alpha x)+2g(\alpha^2 x)=0,$$

for all real numbers x.

Now, let t be a fixed nonzero real number, and consider the sequence $\{u_n\}$ defined by $u_n=g(\alpha^n t)$. Using the above relation for g with $x=\alpha^n t$ we see that

$$2u_{n+2}-5u_{n+1}+3u_n=0,\qquad\text{for every } n\geq 0.$$

The characteristic polynomial associated with this linear recursive sequence is $2\lambda^2-5\lambda+3$ and it has two real zeros: 1 and $3/2$. This proves that $u_n = a+b\left(\frac{3}{2}\right)^n$ for some constants a and b. The continuity of g at 0 implies that $\lim_{n\to\infty} u_n = g(0)$, so $b=0$, and the fact that $g(0)=1$ implies that $a=1$. Thus $u_n=1$ for every n. In particular, $g(t)=u_0=1$, but t is arbitrary, so $g\equiv 1$. Going back to the definition of g, we see that, for every $x\in\mathbb{R}$, we have

$$f(x) = 1+\frac{x}{3-5\alpha+2\alpha^2}+\frac{x^2}{3-5\alpha^2+2\alpha^4}.$$

Conversely, it is readily seen that any function f of this form satisfies the conditions of the problem. So, these are all the solutions of the proposed problem.

Editor's comment. Some solutions featured parts of both methods shown above. It has been remarked that this problem appeared in *Mathematics Magazine*, 87 (1), February 2014, as problem 1939 as well as Problem 239 in *Revista Escolar de la Olimpiada Iberoamericana de Mathemática.*

3836. *Proposed by Jung In Lee.*

Determine all triplets (a,b,c) of positive integers that satisfy

$$a!+b^b=c!$$

Solved by B. Beasley; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; S. Malikić; D. Stone and J. Hawkins; and the proposer. We present two solutions.

Solution 1, by the proposer Jung In Lee.

Since $a!<c!$ and $b^b<c!\le c^c$, then $a<c$ and $b<c$. Suppose that $a\le 2$. Since b divides $c!$, it divides $a!$ so that $b\le 2$. The only possible solutions in this situation are $(a,b,c)=(1,1,2),(2,2,3)$. Henceforth, assume that $a\ge 3$.

With $v_2(x)$ denoting the exponent of the largest power of 2 dividing x, we find that $b\le v_2(b^b)=v_2(a!)$ since $b^b=a![(a+1)\dots(c-1)c-1]$, the second factor being odd. Thus

$$b\le v_2(a!)=\sum_{k=1}^{\infty}\left\lfloor\frac{a}{2^k}\right\rfloor<\sum_{k=1}^{\infty}\frac{a}{2^k}=a.$$

Let $\{2=p_1,3=p_2,p_3,\cdots,p_n,\cdots\}$ be the sequence of all primes and suppose that $p_k\le a<p_{k+1}$ so that $p_k\ge 3$. Since $a!$ divides b^b, then b is divisible by $p_1p_2\dots p_k$. Since $p_1p_2\cdots p_k-1$ exceeds 1 and is divisible by none of the first k primes, it is not less than p_{k+1}. Hence

$$p_{k+1}>a>b>p_1p_2\cdots p_k-1\ge p_{k+1},$$

a contradiction. Hence $a\le 2$ and we have found all the solutions.

Editor's comment. The above proposer's solution is the only solution that did not make use of Bertrand's postulate.

Solution 2, by Joseph DiMuro, expanded by the editor.

Note that $c > a$ and $c > b$ since $c! > b^b > b!$. Suppose that p is a prime divisor of b. Then p must divide $b!$, b^b and $c!$, so that p must divide $a!$ and $p \leq a$. Thus, if $a = 1$, then $b = 1$ and we get the solution $(a, b, c) = (1, 1, 2)$. If $a = 2$, then $b \neq 1$ and the only prime divisor of b is 2. But then b^b is a multiple of 4 and $c! \equiv 2 \pmod 4$. The only possibility is $(a, b, c) = (2, 2, 3)$.

Suppose, if possible, that $a \geq 3$; let q be the largest prime that does not exceed a. Then, by Bertrand's postulate that when $m \geq 2$ there is always a prime between m and $2m$, $a < 2q$, so that q^2 cannot divide $a!$. However, since q divides a, it must divide $c!$ and hence divide b. Since $a \geq 3$, $a!$ and $c!$ are both even as is b. Because $b \neq 2$, we must have that $c \geq b \geq 2q$ (whether $q = 2$ or q is odd). Hence q^2 divides $c!$ and b^b and so must divide $a!$, yielding a contradiction.

Therefore, the sole solutions are $(a, b, c) = (1, 1, 2), (2, 2, 3)$.

3837. *Proposed by Arkady Alt.*

Let $(u_n)_{n \geq 0}$ be a sequence defined recursively by

$$u_{n+1} = \frac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4},$$

for $n \geq 3$. Determine $\lim_{n\to\infty} u_n$ in terms of u_0, u_1, u_2, u_3.

Solved by AN-anduud Problem Solving Group; R. Barbara; M. Bataille; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. Deiermann; J. DiMuro; O. Kouba; K. Lewis; Á. Plaza; C. R. Pranesachar; D. Smith; D. Stone and J. Hawkins; R. Zarnowski; and the proposer. We present 2 solutions.

Solution 1, by Joseph DiMuro.

We prove that $\lim_{n\to\infty} u_n = \frac{1}{10}u_0 + \frac{2}{10}u_1 + \frac{3}{10}u_2 + \frac{4}{10}u_3$. The proof is by visual aid.

Put 10 water glasses on a table. Pour u_0 mL of water into one glass. Pour u_1 mL of water into each of 2 glasses, pour u_2 mL into each of 3 glasses, and pour u_3 mL into each of the remaining 4 glasses. Put the glasses into groups based on the amount of water in each glass. (So, the lone glass with u_0 mL is in a group by itself, the 2 glasses with u_1 mL form another group, and so on.)

Now, perform the following operation repeatedly: take one glass from each group. Pour water between those four glasses until they all have the same amount. Then put those four glasses back on the table as a new group. (Each of the old groups will have one fewer glass than before.)

After performing this operation once, you will have 1 glass with u_1 mL, 2 glasses with u_2 mL, 3 glasses with u_3 mL, and 4 glasses with u_4 mL. After performing this operation a second time, you will have 1 glass with u_2 mL, 2 glasses with u_3 mL, 3 glasses with u_4 mL, and 4 glasses with u_5 mL. And so on.

The amount of water in each glass will gradually approach $\lim\limits_{n\to\infty} u_n$. Therefore, $\lim\limits_{n\to\infty} u_n$ must be equal to the average amount of water per glass at the start:

$$\lim_{n\to\infty} u_n = \frac{1}{10}u_0 + \frac{2}{10}u_1 + \frac{3}{10}u_2 + \frac{4}{10}u_3.$$

Editor's comment. This solution, as well the argument above, generalize to any sequence where u_n is defined to be the average of the k previous terms. This solution does assume that $\lim_{n\to\infty} u_n$ exists. Its existence can be proven using the roots of the characteristic polynomial for the recurrence relation, as in the next solution.

Solution 2, by Michel Bataille.

The characteristic equation of the sequence $(u_n)_{n\geq 0}$ is

$$4x^4 - x^3 - x^2 - x - 1 = 0,$$

that is,

$$(x-1)(4x^3 + 3x^2 + 2x + 1) = 0.$$

The function $f : x \mapsto 4x^3 + 3x^2 + 2x + 1$ is continuous and strictly increasing on $\mathbb{R}$ with $f(\mathbb{R}) = \mathbb{R}$, so the equation $f(x) = 0$ has a unique real solution, say r. Noticing that $f(-1) = -2 < 0$ and $f(-\frac{1}{4}) > 0$, we see that

$$-1 < r < -\frac{1}{4} \qquad (1)$$

.

The non real solutions to $f(x) = 0$ are two complex conjugates z_0 and $\overline{z_0}$ and since $r \cdot z_0 \cdot \overline{z_0} = -\frac{1}{4}$, we have $|z_0|^2 = \frac{1}{4|r|}$, hence $|z_0| < 1$ since by (1), $|r| > \frac{1}{4}$.
From the list $1, r, z_0, \overline{z_0}$ of the roots of the characteristic equation, we deduce the form of u_n:

$$u_n = \alpha_1 + \alpha_2 r^n + \alpha_3 z_0^n + \alpha_4 \overline{z_0}^n$$

where the α_j are independent of n and determined from u_0, u_1, u_2, u_3.
Since $|r| < 1$ and $|z_0| = |\overline{z_0}| < 1$, we have $\lim_{n\to\infty} r^n = \lim_{n\to\infty} z_0^n = \lim_{n\to\infty} \overline{z_0}^n = 0$ so that $\lim\limits_{n\to\infty} u_n = \alpha_1$.
Now, the following relations hold

$$u_0 = \alpha_1+\alpha_2+\alpha_3+\alpha_4, \quad u_1 = \alpha_1+\alpha_2 r+\alpha_3 z_0+\alpha_4\overline{z_0}, \quad u_2 = \alpha_1+\alpha_2 r^2+\alpha_3 z_0^2+\alpha_4\overline{z_0}^2$$

and

$$u_3 = \alpha_1 + \alpha_2 r^3 + \alpha_3 z_0^3 + \alpha_4 \overline{z_0}^3.$$

Since $f(r) = f(z_0) = f(\overline{z_0}) = 0$, we obtain

$$u_0 + 2u_1 + 3u_2 + 4u_3 = 10\alpha_1 + \alpha_2 f(r) + \alpha_3 f(z_0) + \alpha_4 f(\overline{z_0}) = 10\alpha_1$$

and we conclude
$$\lim_{n\to\infty} u_n = \frac{u_0 + 2u_1 + 3u_2 + 4u_3}{10}.$$

Editor's comment. Perfetti pointed out that this result appeared in *"On the Solutions of Linear Mean Recurrences"*, American Mathematical Monthly, 121 (6).

3838. *Proposed by Jung In Lee.*

Prove that there are no triplets (a, b, c) of distinct positive integers that satisfy the conditions:

- $a + b$ divides c^2, $b + c$ divides a^2, $c + a$ divides b^2, and
- the number of distinct prime factors of abc is at most 2.

Solved by M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; S. Malikić; and the proposer. We present the proposer's solution.

Suppose (a, b, c) is a triplet of distinct positive integers satisfying the given conditions. Let $a = p^{x_1}q^{y_1}$, $b = p^{x_2}q^{y_2}$ and $c = p^{x_3}q^{y_3}$, where p and q are distinct prime numbers and x_i and y_i are nonnegative integers for $i = 1, 2, 3$. Let $i, j, k \in \{1, 2, 3\}$ such that $i \neq j \neq k \neq i$. We consider two cases separately.

Case 1. Suppose $x_i > x_j$ and $y_i > y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j}q^{y_i-y_j} + 1) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j}q^{y_i-y_j} + 1 | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j}q^{y_i-y_j} + 1, p^{2x_k}q^{2y_k}) = 1$.

Case 2. Suppose $x_i > x_j$ and $y_i < y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j} + q^{y_j-y_i}) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j} + q^{y_j-y_i} | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j} + q^{y_j-y_i}, p^{2x_k}q^{2y_k}) = 1$.

By cases 1 and 2, we have $x_i = x_j$ or $y_i = y_j$. It follows that either two or more of the statements $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_1$ are true or two or more of the statements $y_1 = y_2$, $y_2 = y_3$ and $y_3 = y_1$ are true. Hence $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$. Without loss of generality, we assume that $x_1 = x_2 = x_3 = x$. Since the given conditions are homogenous in a, b and c, which are distinct, we may assume that $y_1 > y_2 > y_3$. Then
$$p^xq^{y_1} + p^xq^{y_2} = p^xq^{y_2}(q^{y_1-y_2} + 1),$$

which divides $p^{2x}q^{y_3}$, so $q^{y_1-y_2}+1 = p^l$ for some integer $l > 0$. Similarly, $q^{y_2-y_3}+1 = p^m$ and $q^{y_1-y_3}+1 = p^n$ for some positive integers m and n.

Hence $p^n = q^{y_1-y_3}+1 < (q^{y_1-y_2}+1)(q^{y_2-y_3}+1) = p^{l+m}$, so $n+1 \leq l+m$ and $2p^n \leq p^{l+m}$. But

$$\begin{aligned} 2p^n - p^{l+m} &= 2q^{y_1-y_3}+2-(q^{y_1-y_2}+1)(q^{y_2-y_3}+1) \\ &= q^{y_1-y_3}+1-(q^{y_1-y_2}+q^{y_2-y_3}) \\ &= (q^{y_1-y_2}-1)(q^{y_2-y_3}-1) > 0 \end{aligned}$$

a contradiction and the proof is complete.

Editor's comment. DiMuro gave the triplet

$$(a,b,c) = (90,180,720) = (2\times 3^2\times 5, 2^2\times 3^2\times 5, 2^4\times 3^2\times 5)$$

as an example of three distinct positive integers satisfying the first condition such that abc has three distinct prime factors.

This problem is very similar to OC95, which originally appeared with a typo.

3839. *Proposed by Peter Y. Woo.*

Let ΔABC be an acute triangle, and P any point on the plane. Let AD, BE, CF be the altitudes of ΔABC. Let D', E', F' be the circumcentres of ΔPAD, ΔPBE, ΔPCF respectively. Prove that D', E', F' are collinear.

Solved by M. Bataille; R. Barroso Campos; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; S. Malikić; T. K. Parayiou; N. Stanciu and T. Zvonaru; and the proposer. We present the solution by Ricardo Barroso Campos.

As usual we denote the orthocentre of ΔABC by H. Using similar right triangles, one sees that

$$AH\cdot HD = BH\cdot HE = CH\cdot HF.$$

Let PH intersect the circumcircle of ΔPAD again at Q. Then

$$PH\cdot HQ = AH\cdot HD,$$

which implies that the other two circumcircles, of ΔPBE and ΔPCF, also pass through Q. We conclude that the circumcentres D', E', F' all lie on the perpendicular bisector of the common chord PQ.

Editor's comments. Malikić pointed out that should P be chosen on an altitude or its extension, then a more careful statement of the problem would be required to give rise to a meaningful result. On the other hand, it is clear from the featured solution that there was no need to require ΔABC to be acute. This observation was provided by Bataille and by Benito et al. The latter group observed, moreover, that the point Q (in the featured solution) is known as the *orthoassociate of* P, which is the point $X(5523)$ in Clark Kimberling's *Encyclopedia of Triangle Centers*. The point is discussed further in Bernard Gibert's "Orthocorrespondence and Orthopivotal Cubics," *Forum Geometricorum* **3** (2003), 1–27. Stanciu and Zvonaru noted that in the special case where P is the centroid of ΔABC, then Q is Kimberling's point $X(468)$.

3840$\star$. *Proposed by Šefket Arslanagić.*

Prove or disprove

$$a^3c + ab^3 + bc^3 \geq a^2b^2 + b^2c^2 + c^2a^2 \,,$$

where $a, b, c > 0$.

Solved by A. Alt; AN-anduud Problem Solving Group; R. Barbara; M. Bataille; D. Bailey, E. Campbell, and C. Diminnie; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; O. Geupel; O. Kouba; S. Malikić; P. Perfetti; C. R. Pranesachar; H. Ricardo; D. Smith; S. Wagon; and P. Y. Woo.

All the solvers disproved the given inequality by giving various counterexamples, some of which are $(a, b, c) = (1, 6, 4), (1, 7, 4), (1, 8, 4), (1, 20, 10), (1, 10, 5), (5, 2, 0.5)$. Barbara showed that the inequality is false even if the right-hand side is replaced by $\epsilon(a^2b^2 + b^2c^2 + c^2a^2)$ where $\epsilon > 0$ is arbitrary. Benito, Ciaurri, Fernández and Roncal, showed that $(a, 12a, 6a)$ provides counterexamples for all $a > 0$. Kouba, and Bailey, Campbell and Diminnie pointed out that the symmetric version holds since

$$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3-2(a^2b^2+b^2c^2+c^2a^2) = ab(a-b)^2+bc(b-c)^2+ca(c-a)^2$$

is greater or equal than zero. Malikić and Pranesachar proved that the inequality holds if a, b and c are the side lengths of a triangle. Pranesachar also commented that the inequality holds if $a \leq b \leq c$ or $b \leq c \leq a$ or $c \leq a \leq b$. Bailey, Campbell, and Diminnie actually gave a proof for this as well as the fact that if $a \leq c \leq b$, $c \leq b \leq a$, or $b \leq a \leq c$, then $a^2b^2 + b^2c^2 + c^2a^2 \leq a^3b + b^3c + c^3a$.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Arkady Alt, San Jose, CA, USA: 3932
George Apostolopoulos, Messolonghi, Greece: 3934, 3939
Michel Bataille, Rouen, France: 3935
Paul Bracken, University of Texas, Edinburg, TX, USA: 3936
Marcel Chiriţă, Bucharest, Romainia : 3937
Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain: 3938
Michał Kremzer, Gliwice, Silesia, Poland : 3940
Dragoljub Milošević, Gornji Milanovac, Serbia: 3933
Bill Sands, University of Calgary, Calgary, AB: 3931

Solvers - individuals

Arkady Alt, San Jose, CA, USA: 3813, 3815, 3816, 3818, 3832, 3833, 3835, 3837, 3840
Miguel Amengual Covas, Cala Figuera, Mallorca, Spain : **3834**
George Apostolopoulos, Messolonghi, Greece: 3831, 3832, 3834
Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina: CC69, 3832, 3833
Roy Barbara, Lebanese University, Fanar, Lebanon: 3837, 3840
Ricardo Barroso Campos, University of Seville, Seville, Spain: **3839**
Michel Bataille, Rouen, France: CC69, 3831, 3833, 3834, 3835 , **3837**, 3839, 3840
Brian D. Beasley, Presbyterian College, Clinton, USA: 3836
Marcel Chiritiţă, Bucharest, Romania : 3832, 3835
Matei Coiculescu, East Lyme High School, East Lyme, CT, USA: **CC68**
Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India: 3832, 3833, **3834**
Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA: 3834, 3837
Joseph DiMuro, Biola University, La Mirada, CA, USA: 3835, **3836**, **3837**, 3838, 3840
Marian Dincă, Bucharest, Romania: 3832
Nikolaos Evgenidis, Gymnasium Agia, Thessalia, Greece : **OC117**
Oliver Geupel, Brühl, NRW, Germany: **3831**, 3833, 3835, 3840, OC116, OC117, **OC118**, OC119, **OC120**
Richard I. Hess, Rancho Palos Verdes, CA, USA: CC66, CC68, OC116
Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria: 3832, **3834**, **3835**, 3837, 3840
Dimitrios Koukakis, Kilkis, Greece: 3833, 3835
Jung In Lee, Seoul Science High School, Republic of Korea : **3836**, **3838**
Kathleen E. Lewis, University of the Gambia, Brikama, Gambia: 3837
Joseph M. Ling, The University of Calgary, Calgary, AB : 3835
Salem Malikić, student, Simon Fraser University, Burnaby, BC: 3831, 3832, 3833, 3836, 3838, 3839, 3840
David E. Manes, SUNY at Oneonta, Oneonta, NY, USA: CC69, OC117
Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA: 3833
Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India: 3833

S. Muralidharan, Chennai, India : 3835, **OC116**
Theoklitos K.Parayiou, Agioi Theodoroi Samos Greece : 3831, 3832, 3833, 3834, 3839
Ricard Peirò i Estruch IES "Abastos" Valéncia, Spain : 3834
Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy: 3800, 3815, 3818, 3820, 3831, 3832, 3833, 3840
Angel Plaza, University of Las Palmas de Gran Canaria, Spain: 3833, 3837
C.R. Pranesachar, Indian Institute of Science, Bangalore, India: 3833 3837, 3840
Henry Ricardo, Tappan, NY, USA: **CC69**, 3840
Digby Smith, Mount Royal University, Calgary, AB: 3833, 3837, 3840
Daniel Văcaru, Piteşti, Romania: 3832, OC116, OC117
Stan Wagon, Macalester College, St. Paul, MN, USA: 3833, 3840
Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON : **CC67**
Peter Y. Woo, Biola University, La Mirada, CA, USA: **3832**, 3834, 3839, 3840
Roger Zarnowski, Angelo State University, San Angelo, TX, USA: 3837
Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA: **CC68**, OC116, OC117

Solvers - collaborations

AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia: 3833, 3834, 3835, 3837, 3840
Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA: 3832, 3840
D. M. Bătineţu-Giurgiu, Bucharest, Romania, Neculai Stanciu, and Titu Zvonaru, George Emil Palade Secondary School, Buzău, Romania: 3832, 3833, 3834, 3839, 3840
M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, Logroño, Spain : 3831, **3832**, 3833, 3834, 3835, 3836, 3837, 3838, 3839, 3840
Marian Dinc(a) and Mihai Miculiţa, Bucharest, Romania : **OC119**
Santiago de Luxán, Fraunhofer Heinrich-Hertz-Institute, Berlin, Germany and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain : 3835
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA : 3832
Neculai Stanciu and Titu Zvonaru, George Emil Palade Secondary School, Buzău, Romania: CC66, CC68, CC69, **3831**, OC116
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA: 3836, 3837