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CONTENTS

A 1984 FARRAGO

CHARLES W. TRIGG

$$\boxed{1 + 9 - 8 - \sqrt{4} = 0}$$

Although 4 divides 1984, no man should leap to the conclusion that he needs to protect himself against the wiles of those of the opposite sex from 19 to 84. They might feel that such an attitude goes against the grain. Rather, we would like to go to our farrago for nourishment and even the possible yeasty stimulation gained when either grain or thoughts ferment. Or, you may prefer the productivity of controlled thought.

$$\boxed{-1\cdot 9 + 8 + \sqrt{4} = 1}$$

Each digit of 1984 is a power; that is,

$$1 = 1^1, \quad 9 = 3^2, \quad 8 = 2^3, \quad 4 = 2^2.$$

The sum of the bases is 8, and so is the sum of the exponents.

Also, $1984 = 12^3 + 16^2$.

$$\boxed{-1 - 9 + 8 + 4 = 2}$$

$1 - \sqrt{9} + 8 - 4 = 2$ and $19 + 8 + \sqrt{4} = 29$, representing that non-leapable date in every leap year, 2/29.

$$\boxed{1\cdot(-9+8) + 4 = 3}$$

$1984 = 4\cdot 4\cdot 4\cdot 31$, and $3 + 1 = 4$.

$1984 = 2^6\cdot 31 = 1\cdot 32\cdot 2\cdot 31 = 4\cdot 2\cdot 1\cdot 31\cdot 2\cdot 4$, two palindromic arrays.

The sum of the prime factors of 1984 is 43, the 13th odd prime.

$1 + 9 + 8 + 4 = 2\cdot 11$, and $1 + 1 = 2$.

$1\cdot 9\cdot 8\cdot 4 = 4\cdot 3\cdot 2\cdot 3\cdot 4$.

$$\boxed{1 - 9 + 8 + 4 = 4}$$

The reverse of 1984 is $4891 = 67\cdot 73$, the product of alternate primes; and $73 - 67 = 6$, the smallest perfect number.

The overlapping blend of 1984 and its reverse is 1984891, a prime whose rank in the sequence of primes is 147881, a prime whose rank is 13669, a prime whose rank is 1614, and the litany must sadly end here. The other overlapping blend, $4891984 = 1 \cdot 2 \cdot 8 \cdot 305749$ in which the only missing digit is 6.

$$\boxed{(1 + 9)/(8/4) = 5}$$

$1984 = 1857024/936$, in which all ten digits are present. [Stewart Metchette, *Journal of Recreational Mathematics*, 13 : 1 (1980-81) 27.]

$$\boxed{1 + 9 - 8 + 4 = 6}$$

$\pi = 3.1415926\ldots1984\ldots$, where the year ends in the 3361st decimal place, and $31415926 \equiv 1984 \equiv 3361 \pmod 9$.

$$\boxed{19 - 8 - 4 = 7}$$

$$1 = -\sqrt{9} + 8 - 4, \qquad -1 + \sqrt{9} = 8/4, \qquad -1 - \sqrt{9} + 8 = 4.$$

$$\boxed{(1 + 9 - 8) \cdot 4 = 8}$$

$84 - 19 = 65 = 5 \cdot 13$, so 19 and 84 are terminal terms of arithmetic progressions with common differences of 5 and 13. The longer one, with $d = 5$, contains four primes and two squares. It is

$$\underline{19} \quad 24 \quad \underline{29} \quad 34 \quad 39 \quad 44 \quad \overline{49} \quad 54 \quad \underline{59} \quad \overline{64} \quad 69 \quad 74 \quad \underline{79} \quad 84.$$

In the six-term A.P., with $d = 13$, the units' digits are distinct, as are the tens' digits. No 0 or 6 appears in any term of

$$\underline{19} \quad \overline{32} \quad 45 \quad 58 \quad \underline{71} \quad 84,$$

but two of the terms are primes and one is a fifth power.

$$\boxed{19 - 8 - \sqrt{4} = 9}$$

$1/1984 = 0.000504032\ldots$, wherein the nonzero digits are in descending order of magnitude.

$$\boxed{1 - \sqrt{9} + 8 + 4 = 10}$$

$$\begin{array}{rcrcrcrcrcrcrcrcr}
1 & + & 9 & + & 8 & + & 4 & = & 22 & = & 4 & + & 8 & + & 9 & + & 1 \\
19 & + & 98 & + & 84 & + & 41 & = & 242 & = & 14 & + & 48 & + & 89 & + & 91 \\
198 & + & 984 & + & 841 & + & 419 & = & 2442 & = & 914 & + & 148 & + & 489 & + & 891 \\
1984 & + & 9841 & + & 8419 & + & 4198 & = & 24442 & = & 8914 & + & 9148 & + & 1489 & + & 4891
\end{array}$$

$$\boxed{1 \cdot 9 + 8/4 = 11}$$

$$\begin{aligned}
19^2 + 98^2 + 84^2 + 41^2 &= 14^2 + 48^2 + 89^2 + 91^2 \\
198^2 + 984^2 + 841^2 + 419^2 &= 914^2 + 148^2 + 489^2 + 891^2 \\
1984^2 + 9841^2 + 8419^2 + 4198^2 &= 8914^2 + 9148^2 + 1489^2 + 4891^2
\end{aligned}$$

$$\boxed{1 + 9 + 8/4 = 12}$$

Two palindromic arrays for 1984:

$$1^9 \cdot 8^4 = 1 \cdot 2 \cdot 4 \cdot 2 \cdot 32 \cdot 4 \cdot 2 \cdot 1,$$

$$1 \cdot 9^8 \cdot 4 = 9 \cdot 3 \cdot 2 \cdot 1 \cdot 9 \cdot 9 \cdot 3 \cdot 1 \cdot 3 \cdot 9 \cdot 9 \cdot 1 \cdot 2 \cdot 3 \cdot 9.$$

$$\boxed{1 \cdot 9 + 8/\sqrt{4} = 13}$$

In a Schram sequence [*Journal of Recreational Mathematics*, 14:2 (1981-82) 141], each succeeding term is generated from its predecessor by taking the absolute difference between the sum of the squares of the odd digits and the sum of the squares of the even digits. For example:

1984 '2 4 16 35 34 7 49 65 11' 2 ... .

Thus 1984 is a charm attached to a bracelet of nine terms, including three squares, one repunit, and three primes. The sum of the loop members is 223, the 48th prime. The sum of the loop members and the charm is 2207, the 329th prime.

$$\boxed{1 + 9 + 8/\sqrt{4} = 14}$$

In the reiterative operation wherein an integer and its reverse are added, N_0 is the starting integer, and the kth versum, N_k, is the sum resulting from the kth addition. If $N_0 = 1984$, then N_1, a permutation of consecutive digits, is an antipalindrome in that each digit differs from the corresponding digit of its reverse. The first palindrome encountered in the sequence is N_{13}.

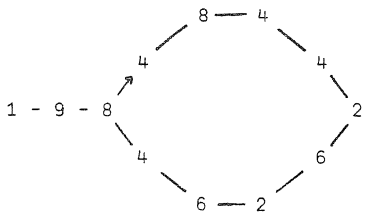
$$\begin{array}{rrrcrr}
N_0 = 1984 & 6875 & 12661 & \ldots & 16222151 & 31344412 \\
\underline{4891} & \underline{5786} & \underline{16621} & \ldots & \underline{15122261} & \underline{21444313} \\
N_1 = 6875 & 12661 & 29282 & \ldots & 31344412 & 52788725 = N_{13}
\end{array}$$

Other antipalindromes in this sequence are

$N_5 = 105149$, $N_{10} = 10474750$, $N_{18} = 1059424960$, and $N_{26} = 102618981520$.

$$\boxed{1 \cdot 9 + 8 - \sqrt{4} = 15}$$

The recursive operation $u_{n+4} = u_n u_{n+1} u_{n+2} u_{n+3}$, with each product reduced modulo 10, when applied to 1984 produces a 10-digit bracelet with a 2-digit charm attached, namely:



The charm digits and those at the extremities of all but one diameter sum to 10.

$$\boxed{1 + 9 + 8 - \sqrt{4} = 16}$$

1984 is part of a 1560-digit additive bracelet wherein each element is the units' digit of the sum of the four preceding digits, namely:

19842 37684 53020 57462 91808 ... .

The complete bracelet is included in "A Digital Bracelet for 1968", *Journal of Recreational Mathematics*, 1 (April 1968) 108-111.

$$\boxed{19 - 8/4 = 17}$$

Nonagonal numbers have the form $N(n) = n(7n-5)/2$. All nonagonal numbers of the forms

$$N(1920 \pm 768 + 20000k) \quad \text{and} \quad N(3795 \pm 768 + 20000k), \quad k = 0,1,2,\ldots,$$

terminate in 1984. The digits of $N(3027) = 32061984$ are distinct. Some that start with 1984 are

$$N(7530) = 198434325, \qquad N(7531) = 198487036,$$
$$N(23810) = 1984146825, \qquad N(23815) = 1984980250.$$

$$\boxed{1 \cdot 9 \cdot (8/4) = 18}$$

The digits of 1984 can be distributed on the 12 mid-edges of a regular octahedron in two ways (shown on the Schlegel diagrams of Figure 1) so that each digit appears on the 4 edges issuing from each vertex.

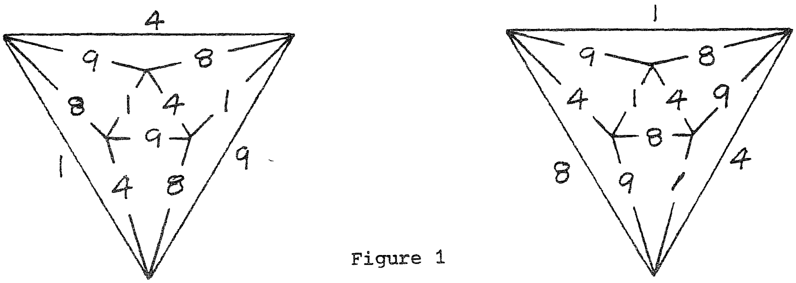


Figure 1

In each distribution the orders around the vertices constitute the six cyclic arrangements of four distinct elements, with the orders around opposite vertices being opposite. Also, the digits on the perimeters of the eight faces are the combinations of the four digits taken three at a time, with the same combination appearing on opposite faces, but in reverse order.

$$\boxed{1 + 9\cdot(8/4) = 19}$$

There is but one way to distribute the digits of 1984 on the faces of a regular octahedron so that different digits appear on the faces meeting at each vertex. This distribution is shown on the Schlegel diagram of Figure 2, the isolated digit being the one on the hidden face. The same digit appears on opposite faces. The orders of digits around opposite faces are opposite.

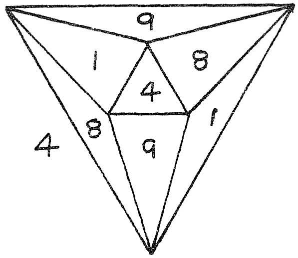


Figure 2

$$\boxed{1 + 9 + 8 + \sqrt{4} = 20}$$

$$445\cdot 446 = \underline{1984}\,\underline{70}$$
$$4454\cdot 4455 = \underline{1984}\,25\,\underline{70}$$
$$44545\cdot 44546 = \underline{1984}\,3015\,\underline{70}$$
$$445454\cdot 445455 = \underline{1984}\,297115\,\underline{70}$$

$$\boxed{1\cdot 9 + 8 + 4 = 21}$$

In the following determinants the 2-by-2 center is stationary, while the 198419841984 and its reverse rotate around the perimeter, and the value of each is given in palindromic form.

$$\begin{vmatrix} 1 & 9 & 8 & 4 \\ 4 & 1 & 9 & 1 \\ 8 & 4 & 8 & 9 \\ 9 & 1 & 4 & 8 \end{vmatrix} = -828 \qquad \begin{vmatrix} 8 & 4 & 1 & 9 \\ 9 & 1 & 9 & 8 \\ 1 & 4 & 8 & 4 \\ 4 & 8 & 9 & 1 \end{vmatrix} = 3 \cdot 4 \cdot 7 \cdot 43$$
$$= -2 \cdot 3 \cdot 2 \cdot 3 \cdot 23$$

$$\begin{vmatrix} 4 & 1 & 9 & 8 \\ 8 & 1 & 9 & 4 \\ 9 & 4 & 8 & 1 \\ 1 & 4 & 8 & 9 \end{vmatrix} = 0 = \begin{vmatrix} 9 & 8 & 4 & 1 \\ 1 & 1 & 9 & 9 \\ 4 & 4 & 8 & 8 \\ 8 & 9 & 1 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 8 & 9 & 1 \\ 1 & 1 & 9 & 4 \\ 9 & 4 & 8 & 8 \\ 8 & 4 & 1 & 9 \end{vmatrix} = 35 \cdot 4 \cdot 5 \cdot 3 = -\begin{vmatrix} 9 & 1 & 4 & 8 \\ 8 & 1 & 9 & 9 \\ 4 & 4 & 8 & 1 \\ 1 & 9 & 8 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 8 & 9 & 1 & 4 \\ 4 & 1 & 9 & 8 \\ 1 & 4 & 8 & 9 \\ 9 & 8 & 4 & 1 \end{vmatrix} = 2112 \qquad \begin{vmatrix} 1 & 4 & 8 & 9 \\ 9 & 1 & 9 & 1 \\ 8 & 4 & 8 & 4 \\ 4 & 1 & 9 & 8 \end{vmatrix} = 84 \cdot 8$$
$$= 8 \cdot 33 \cdot 8 \qquad = 4 \cdot 21 \cdot 2 \cdot 4$$

$$\boxed{-1 - 9 + 8 \cdot 4 = 22}$$

The circulant

$$\begin{vmatrix} 1 & 9 & 8 & 4 \\ 4 & 1 & 9 & 8 \\ 8 & 4 & 1 & 9 \\ 9 & 8 & 4 & 1 \end{vmatrix} = -(1 + 9 + 8 + 4) \cdot 296.$$

$$\boxed{-1 + (\sqrt{9})! \cdot (8/\sqrt{4}) = 23}$$

On pages 134 and 138 of his *History of Binary and Other Nondecimal Numeration*, Tomash Publishers, 1981, Anton Glasser gives the expressions for 1984 in four computer codes, namely in:

Binary coded decimal, 1984 = 0001 1001 1000 0100;

Pure binary, 1984 = 11111000000:

Excess three, 1984 = 0100 1100 1011 0111; and

Biquinary, 1984 = 0100010 1010000 1001000 0110000.

$$\boxed{(-1 + \sqrt{9}) \cdot (8 + 4) = 24}$$

The smallest power of 2 that ends in 1984 is 2^{54}, and the next one is 2^{554}. Since, for all positive integers n,

$$2^{n+500} \equiv 2^n \pmod{10^4},$$

a consequence of the fact that $2^{500} - 1 \equiv 9375 = 3 \cdot 5^5 \pmod{10^4}$, it follows that

$$2^{54},\quad 2^{554},\quad 2^{1054},\quad 2^{1554},\quad 2^{2054},\quad \ldots$$

all end in 1984, and they are the only powers of 2 that do.

$$\boxed{19 + 8 - \sqrt{4} = 25}$$

And how was *your* year?

2404 Loring Street, San Diego, California 92109.

*　　　　*　　　　*

WHY NEWTON *AND* LEIBNIZ *HAD* TO DO IT

The following quotation is taken from N. Piskunov, *Differential and Integral Calculus* (translated from the Russian by George Yankovsky), second edition, Mir Publishers, Moscow, 1977, Vol. I, page 14. (The italics are in the text.)

"As we shall see throughout this course, the concept of a variable quantity is the basic concept of differential and integral calculus. In 'Dialectics of Nature', Friedrich Engels wrote: 'The turning point in mathematics was Descartes' variable magnitude. With that came *motion* and hence *dialectics* in mathematics, and *at once*, too, *of necessity* the differential and integral calculus.'"

Friedrich Engels gives compelling reasons why the calculus *had* to be invented (or discovered, depending on your philosophical point of view). But why did it have to be done by Gottfried Wilhelm von Leibniz *and* Sir Isaac Newton?

Because Leibniz was Gottfriedrich and Newton was Engelish.

*　　　　*　　　　*

THE PUZZLE CORNER

Answer to Puzzle No. 49 [1983: 311]:

$$\begin{array}{r} 314 \\ \underline{198} \\ 62172 \end{array} \quad \text{or} \quad \begin{array}{r} 138 \\ \underline{379} \\ 52302 \end{array}, \quad \text{so } E = 2.$$

Answer to Puzzle No. 50 [1983: 311]:

$$\begin{array}{r} 683 \\ \underline{73} \\ 49859 \end{array}, \quad \text{with NIECE} = 20959 \text{ and MAN} = 182.$$

*　　　　*　　　　*

THE OLYMPIAD CORNER: 51

M.S. KLAMKIN

In recent years, the mathematical preparation of students entering post-secondary institutions in Canada and the United States has been steadily declining, and the same thing can be said for the standards used in assessing this preparation. The elementary and secondary schools are equally to blame for this decline. The parlous state of our precollege mathematical education has finally received international exposure in a number of recent reports from the following prestigious groups:

National Commission on Excellence in Education;

National Science Board's Commission on Precollege Education in Mathematics, Science, and Technology;

Twentieth-Century Fund Task Force on Federal Elementary and Secondary Education Policy;

National Task Force on Education for Economic Growth;

Carnegie Foundation for the Advancement of Teaching.

Preparation in mathematics and the physical sciences is much more demanding almost everywhere abroad. As one example, I give below the 1981 matriculation examination in mathematics for Grade 11 students in Hong Kong (for which I am grateful to K.P. Shum, The Chinese University of Hong Kong). It is worth noting that, for a Hong Kong student to enter college, he or she must first complete Grade 12 (2 years) and then take university examinations which include mathematics at a still higher level. In contrast, the University of Alberta gives a diagnostic (only) test to entering students enrolled in calculus. Considering the low level of this test and the low grades achieved, it is likely that the vast majority of these students would be wiped out if they had to pass the following Hong Kong Grade 11 test. The present-day economic situation and the competition for jobs may soon result in students being better prepared in the secondary schools for admittance to postsecondary institutions. If so, it will not be a moment too soon!

1981 HONG KONG GRADE 11 MATRICULATION EXAMINATION

Time: 3 hours

1. (a) Let P and Q be nonsingular square matrices of the same order. Simplify

(i) $PQ(Q^{-1}P^{-1})$,

(ii) $(Q^{-1}P^{-1})PQ$. (2 marks)

(b) Let $M = PQ$, where

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Find a, b, and c.

(ii) Find Q^{-1} and P^{-1}; hence, or otherwise, find M^{-1}. (11 marks)

(c) Using the result of (b)(ii), or otherwise, solve the following system of linear equations:

$$\begin{aligned} 2x \quad\;\; + z &= 7 \\ 4x + y + 2z &= 12 \\ 3y + z &= -3. \end{aligned}$$

(3 marks)

2. (a) Prove by mathematical induction that, for any positive integer n,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

(6 marks)

(b) The set of positive integers is grouped as follows:

(1), (2,3), (4,5,6), (7,8,9,10), ...,

so that the nth group, G_n, consists of n consecutive positive integers.

(i) Show that the first term in G_k is $\frac{k^2-k+2}{2}$.

(ii) Calculate the sum of all the terms in G_k.

(iii) Hence, or otherwise, find the sum of all the terms in the n groups $G_1, G_2, G_3, \ldots, G_n$. (10 marks)

3. (a) Solve the inequality $\log_2\{\log_3(\log_4 x)\} \geq 0$. (4 marks)

(b) Solve the equation $2^{2x+3} + 7\cdot 2^x - 1 = 0$. (5 marks)

(c) Solve the system of equations

$$\begin{cases} \log_2 x - \log_4 y = 4 \\ \log_2(x-2y) = 5. \end{cases}$$

(7 marks)

4. (a) Let $f(x) = x^2+px+q$ and $g(x) = x^2-9x+r$, where p,q,r are constants; and let α,β be the roots of $f(x) = 0$ and $\alpha+2\beta$, $2\alpha+\beta$ the roots of $g(x) = 0$.

(i) Find the value of p.

(ii) When $f(x)$ is divided by $x-2$, the remainder is -10. Find the values of q and r. (8 marks)

(b) Factorize $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$. Hence evaluate $\begin{vmatrix} 1 & 2 & 8 \\ 1 & 3 & 27 \\ 1 & 5 & 125 \end{vmatrix}$. (8 marks)

5. Let $f(\theta) = 4\cos^2\theta - 2\sin\theta\cos\theta + 2\sin^2\theta$.

(a) Express $f(\theta)$ in the form $A + B\cos(2\theta+\alpha)$, where A, B, and α are constants and $0 < \alpha < \pi/2$. What are the values of these constants? (6 marks)

(b) Find the maximum and minimum values of $f(\theta)$. What are the general values of θ for which $f(\theta)$ is a maximum? (5 marks)

(c) Figure 1 shows the graph of $y = \sqrt{2}\cos 2\theta$ for $-\pi \le \theta \le \pi$.

(i) Sketch the graph of $y = \sqrt{2}\cos(2\theta+\frac{\pi}{4})$ on Figure 1 for $-\pi \le \theta \le \pi$.

(ii) Sketch the graph of $y = f(\theta)$ on Figure 1 for $-\pi \le \theta \le \pi$. (5 marks)

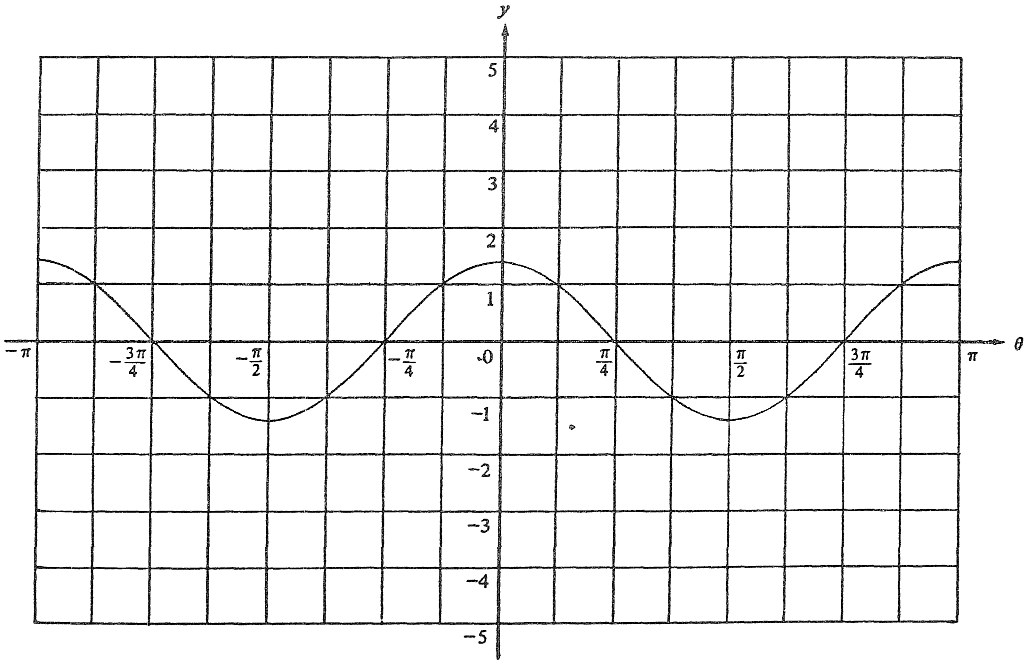


Figure 1

6. Given:

the line (L): $y = mx + c$,

the circle (C): $x^2 + y^2 = 5$,

and the ellipse (E): $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

(a) If the line (L) touches the circle (C), find the relation between c and m. (4 marks)

(b) If the line (L) touches the ellipse (E), find the relation between c and m. (4 marks)

(c) If the line (L) touches both the circle (C) and the ellipse (E), determine the values of m. Hence find the equations of all the common tangents of the circle (C) and the ellipse (E). (8 marks)

7. (a) Given:

$$A = \{1, 2, 3, 4, 5\} \quad \text{and} \quad B = \{3, 4, 5, 6\}.$$

Let $X = \{x: x = 100a + 10b + c, \text{ where } a,b,c \in A\}$. (In other words, X consists of all three-digit numbers where each digit is chosen from the set A.)

Let $Y = \{y: y = 100a + 10b + c, \text{ where } a,b,c \in B\}$.

Find the number of elements in each of the following sets:

(i) X,

(ii) Y,

(iii) $X \cap Y$,

(iv) $X \cup Y$,

(v) $X - Y$ (i.e., the set of elements in X but not in Y). (8 marks)

(b) (i) Let $P = \{a, b, c, d, e\}$,

$Q = \{d, e, f, g\}$,

$R = \{a, b, f, g, h, i\}$.

Find $P \cup (Q \cap R)$ and $(P \cup Q) \cap R$.

(ii) Prove that, for any three sets A, B, and C, $A \subset C$ if and only if

$$A \cup (B \cap C) = (A \cup B) \cap C.$$ (8 marks)

8. (a) In a throw of a die, the probability of getting the number n is $P(n)$. It is known that

$$P(1) = \frac{1}{12} \text{ and } P(2) = P(3) = P(4) = P(5) = \frac{1}{6}.$$

(i) What is the probability of getting an odd number in one throw of the die?

(ii) Find $P(6)$.

(iii) If the die is thrown twice, what is the probability that an odd number is obtained in the first throw and an even number is obtained in the second throw? (6 marks)

(b) The 115 employees of a firm are classified into three groups A, B, and C according to their monthly salaries (see the table below).

Group	Monthly Salary	Number of Employees
A	below $1000	8
B	$1000 – $3000	100
C	above $3000	7

The distribution of the monthly salaries of the 100 employees in Group B is shown in the histogram in Figure 2.

Distribution of the Monthly Salaries of the 100 Employees in Group B

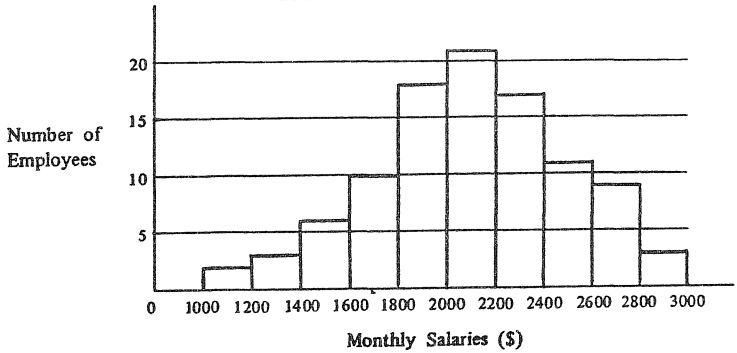


Figure 2

(i) Let σ and σ_B be the standard deviations of the monthly salaries of all the 115 employees and the 100 employees in Group B, respectively. Without calculating the numerical values of σ and σ_B, determine which standard deviation is greater. Explain briefly.

(ii) It is known that the mean and standard deviation of the monthly salaries of the 100 employees in Group B are $2100 and $400, respectively.

(1) From June 1981 onwards, the salaries of all the employees will be increased by 10%. Calculate the mean and standard deviation of the monthly salaries of the 100 employees in Group B in June 1981.

(2) If the firm has decided on a second salary revision giving an increase of $300 per month for every employee from September 1981 onwards, calculate the mean and standard deviation of the monthly salaries of the 100 employees in Group B in September 1981. (10 marks)

END OF PAPER

*

Readers of this Corner are certainly aware of the International Mathematical Olympiads (I.M.O.). But many readers may not be aware that there are also international olympiads in physics and chemistry. For those student and teacher readers who are also interested in physics, I will give some information about the International Physics Olympiad (I.P.O.). More information can be obtained from the journal *Out-of-School Scientific and Technical Education*, issue 18-19, obtainable for $1.25 US from I.C.C., Vekemanstraat 71-73, 1120 Brussels, Belgium.

The first I.P.O. was organized in Warsaw, Poland, in 1967. The Rules and Regulations are given below. They are very similar to those of the I.M.O.

1. Aims of the Competition.

International Physics Olympiads are conducted for the purpose of widening contacts in the field of Physics education at school level, for promotion of extra-curricula activities of talented students in Physics and exchange of experience gained through national Physics competitions (Olympiads). International Physics Olympiads help to establish friendly relations between young people of different countries and strengthen understanding and cooperation between nations. International Physics Olympiads help their participants to develop a creative approach to Physics studies and to master skills in application of the knowledge gained at school. They promote participants' individual thinking and Physics abilities.

2. Organizer of the XII IPO.

The Organizer of the IPO sends out official invitations not later than 20 January of the year of the Olympiad and provides equal conditions for all participating delegations.

3. Participants.

Each participating country may send a delegation of five high school students to the IPO, as well as students of professional training schools. At the beginning of the competition the age of the participants should not exceed 20 years.

4. Supervisors.

The team of students is accompanied by two supervisors, one of whom is an official representative of the delegation. At least one of the supervisors should be able to translate the problems from one of the working languages into the mother tongue of the delegations and to take part in evaluation of competition papers. On their arrival, the representatives of the delegations hand over to the Organizer the following documents:

a) list of participants with their names, dates of birth, home address, name of school and its address;

b) list of supervisors with their names, occupation and home address.

5. Financial Arrangements.

The country of origin covers the cost of transportation of a delegation to the place of the Olympiad and back. The expenditures connected with the stay of delegations in the host country and the conducting of the competition are covered by the Organizer of the IPO.

6. Working Languages.

The working languages of the Olympiad are Russian, English, French and German.

7. Schedule.

The competition is conducted in two rounds. On the first day of competition (first round) participants take part in solving theory problems, during the second day of competition (second round) they do experimental tasks. The participants may use slide rules, electronic calculators and drawing accessories. Those things are supposed to be brought at choice from their own country.

8. International Jury.

The representatives of delegations compose the International Jury. The Chairman of the Jury is appointed by the Organizer. When voting on issues concerning the competition each country as well as the Chairman has one vote. Decisions are taken but the decisive vote is that of the Chairman.

9. International Jury Responsibilities.

The members of the Jury are supposed to:

a) ensure that the present Rules and Regulations are strictly followed;

b) look after the accuracy of evaluation and equal application of evaluation criteria to all participants;

c) keep secret problems and tasks during the competition and restrain from offering help to the participants;

d) finalize the results of the Olympiad.

10. Competition Tasks.

The problems and tasks for the competition are prepared by the Organizer. They are taken from the following Physics topics:

— Mechanics (including Oscillations and Wave Motion).

— Thermodynamics and Molecular Physics.

— Electricity and Magnetism.

— Optics and Atomic Physics.

11. Evaluation.

The criteria for the evaluation are set by the Organizer. The highest score for solving theory problems is 10 points, for experimental tasks—20 points. Checking of the competition papers is done by the members of a working group appointed by the Organizer. The representatives of delegations are given an opportunity to look through their participants' papers and make remarks to the members of the working group if there are any.

12. Prizes.

Final results of the participants are decided at the International Jury meeting on the basis of the evaluation marks. The guidelines for the prize distribution are as follows: 1st Degree Diploma is awarded to those with more than 90% of that score; 2nd Degree Diploma is awarded to participants having 78% to 90% of the highest score; and 3rd Degree Diploma—to those having 65% to 77% of it. Participants with scores of 50% to 64% get an Encouragement Diploma. Those having scores less than 50% are given a certificate stating their participation in the Olympiad. There is no team winner in the IPO.

13. Organizer's Responsibilities.

In addition to abovementioned responsibilities, the Organizer of the IPO should:

a) guarantee the course of the competition in accordance with the present Rules and Regulations;

b) work out the schedule for the competition and other events to be organized during the Olympiad;

c) ascertain on arrival that the participants meet the requirements of the Olympiad;

d) provide the translation of problems, tasks and cues into the working languages;

e) provide interpreters for delegations and the International Jury meetings;

f) ensure carrying out of the experimental tasks in accordance with the safety rules;

g) organize closing of the IPO and presenting the diplomas and prizes to the participants of the Olympiad.

I now give as a sample a set of three theory problems and one experimental problem in physics. They are taken from the reference given above, where solutions and a scoring scheme can also be found. The Russian journal *Kvant* and the Hungarian journal *Középiskolai Matematikai Lapok* both have in every issue sets of practice problems (in English) in both mathematics and physics.

1. A test tube of mass M is placed in vacuum. A barrier of mass m whose thickness is negligible divides the volume of the test tube in two equal parts. The closed part of the test tube contains a mole of ideal monatomic gas of molar mass μ at a temperature T. The barrier is released and, moving without friction, leaves the test tube. Then the gas flows out.

What is the final velocity of the test tube if it is at rest at the moment when the barrier starts its motion? Neglect the momentum of the gas until the barrier leaves the test tube and the heat exchange between the gas and both the test tube and the barrier. The change of the temperature after the barrier has left the test tube is neglected too. Gravitational pull is not taken into consideration as well.

2. An electric bulb of resistance $R_0 = 2.0\ \Omega$ operating at a nominal voltage $U_0 = 4.5$ V is fed from an accumulator with an E.M.F. $E = 6.0$ V, its internal resistance is negligible.

(i) Let the nominal voltage be applied to the bulb through a slide-wire rheostat connected potentiometrically. What is the rheostat resistance and to what maximum current should the rheostat stand so that the efficiency of the system should be not less than $\eta_0 = 0.6$?

(ii) What is the maximum possible efficiency for the given light bulb (at its nominal voltage) and accumulator, and how are they to be connected through a suitably chosen rheostat so that the efficiency η should reach its maximum value?

3. A radiowave receiver of the radioastronomical observatory is placed on the sea coast at a height $h = 2$ m above sea level. When a radiostar radiating electromagnetic waves of wavelength $\lambda = 21$ cm rises above the horizon the receiver registers alternating maxima and minima. The receiver registers only those electromagnetic waves whose electric vector $\vec{E}$ vibrates parallel to the water surface. The registered signal is proportional to E^2. Define:

(i) The angles of altitude of the star above the horizon along the celestial sphere at which maxima and minima are registered (in general form).

(ii) Will the signal in the receiver be increased or decreased immediately when the radiostar rises? Explain why.

(iii) The ratio of signals in the first maximum and the next minimum. When the electromagnetic wave is reflected by the water surface, the ratio of amplitudes of the registered electric field of the reflected (E_{refl}) and incident (E_{inc}) waves is described by the relation

$$\frac{E_{refl}}{E_{inc}} = \frac{n - \cos\phi}{n + \cos\phi},$$

where n is the refractive index and ϕ is the incident angle of the electromagnetic wave. For air-water boundary at λ = 21 cm, n = 9.

(iv) Will the ratio of the signals received at adjacent maxima and minima be increased or decreased with the rising of the star above the horizon?

Assume the sea surface to be smooth while solving the problem.

4. *Experimental Task.*

You have at your disposal an elastic rubber cord suspended vertically on a stand. The initial length of the cord is I_0 = 150 mm, its initial mass together with a pan is 5 g. A set of loads with masses from 10 to 100 g, a stopwatch, a measurement ruler, a drawing curve and plotting paper are also at your disposal. The acceleration due to gravity is assumed to be 10 m/s^2.

1. Load consecutively the elastic cord with masses from 15 to 105 g. Tabulate and plot graphically in an appropriate scale the experimental dependence of the extension ΔI on the force of strain F.

2. Using the results of measurements performed in accordance to Point 1, calculate and write in a table the values of the volume of the cord loaded with masses from 35 to 95 g. Perform the calculations consecutively for each two adjacent loading values within the given interval. Write the formulae used in the calculations. Express by formulae your assumption about the dependence of the volume on the loading. For the Young's modulus the tabulated value $E = 2\cdot10^6$ N/m^2 is accepted. While discussing the results you should bear in mind that at given loadings Hooke's law for rubber holds approximately and the deviations from this law can reach up to 10%.

3. Measure the rubber cord volume, using a stopwatch and loading the cord with a mass of 60 g. Write the formulae you have used.

Instruction. Do not extend the rubber cord without need and do not let it stay loaded longer than necessary for the measurements. Do not extend the rubber cord to such extent that the amplitudes of its vibration to be larger than 15-20 mm.

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before June 1, 1984, although solutions received after that date will also be considered until the time when a solution is published.

872. [1983: 241] (Corrected) *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let T be a given triangle ABC with sides a,b,c and circumradius R, and let P be any point in the plane of T. It is known (M.S. Klamkin, "Triangle inequalities from the triangle inequality", *Elemente der Mathematik*, Vol. 34 (1979), No. 3) that there exists a triangle $T_0 = T_0(\mathrm{P})$, possibly degenerate, with sides $a\cdot\mathrm{PA}$, $b\cdot\mathrm{PB}$, and $c\cdot\mathrm{PC}$. Find the locus of all the points P for which

$$\mathrm{PA}\cdot\mathrm{PB}\cdot\mathrm{PC} \le R\cdot R_0,$$

where $R_0 = R_0(\mathrm{P})$ is the circumradius of T_0. When does equality occur?

901. *Proposed by Charles W. Trigg, San Diego, California.*

In

$$\mathrm{BOXER} = \mathrm{HITS},$$

the X doubles as a multiplication sign. Find (a) the fewest HITS, and (b) the most HITS, that the BOXER can deliver.

902. *Proposed by J. Chris Fisher, Université Libre de Bruxelles, Belgique.*

(a) For any point P on a side of a given triangle, define Q to be that point on the triangle for which PQ bisects the area. What is the locus of the midpoint of PQ?

(b) Like the curve in part (a), the locus of the midpoints of the perimeter-bisecting chords of a triangle (see Crux 674 [1982: 256]) has an orientation that is opposite to that of the given triangle. Is this a general principle? More precisely, given a triangle and a family of chords joining $\mathrm{P}(t)$ to $\mathrm{Q}(t)$, where (i) $\mathrm{P}(t)$ and $\mathrm{Q}(t)$ move counterclockwise about the triangle as t increases and (ii) $\mathrm{P}(t) \ne \mathrm{Q}(t)$ for any t, does the midpoint of PQ always trace a curve that is clockwise-oriented?

903. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*

Let ABC be an acute-angled triangle with circumcenter O and orthocenter H.

(a) Prove that an ellipse with foci O and H can be inscribed in the triangle.

(b) Show how to construct, with straightedge and compass, the points L,M,N where this ellipse is tangent to the sides BC,CA,AB, respectively, of the triangle.

(c) Prove that AL,BM,CN are concurrent.

904. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let M be any point in the plane of a given triangle ABC. The cevians AM,BM,CM intersect the lines BC,CA,AB in A',B',C', respectively. Find the locus of the points M such that

$$[MCB'] + [MAC'] + [MBA'] = [MC'B] + [MA'C] + [MB'A],$$

where the square brackets denote the signed area of a triangle.

905. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle that is not right-angled at B or C. Let D be the foot of the perpendicular from A upon BC, and let M and N be the feet of the perpendiculars from D upon AB and AC, respectively.

(a) Prove that, if $\angle A = 90^\circ$, then $\angle BMC = \angle BNC$. (This problem is given without proof in M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*, Dover, New York, 1968, p. 95, Ex. 8.)

(b) Prove or disprove the converse of part (a).

906.* *Proposed by E.J. Eckert and P.D. Vestergaard, Institute of Electronic Systems, Aalborg University Centre, Denmark.*

Let P denote the set of rational points on the unit circle C, that is, the set of all points $(r/t,\ s/t)$ where r,s,t are integers, $t > 0$, and $r^2+s^2 = t^2$. It is known that P is dense in C. Let T be the subset of P for which t is prime. Is T dense in C?

907. *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

The four consecutive positive integers 76, 77, 78, 79 are such that no one of them is expressible as the sum of two squares. Prove that there are infinitely many such quadruples of consecutive integers.

908. *Proposed by M.S. Klamkin, University of Alberta.*

Determine the maximum value of

$$P \equiv \sin^\alpha A \cdot \sin^\beta B \cdot \sin^\gamma C,$$

where A,B,C are the angles of a triangle and α,β,γ are given positive numbers.

909*. *Proposed by Stan Wagon, Smith College, Northampton, Massachusetts.*

For which positive integers n is it true that, whenever an integer's decimal expansion contains only zeros and ones, with exactly n ones, then the integer is not a perfect square?

910. *Proposed by O. Bottema, Delft, The Netherlands.*

Determine the locus of the centers of the conics through the incenter and the three excenters of a given triangle.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

777. [1982: 246] *Proposed by O. Bottema, Delft, The Netherlands; and J.T. Groenman, Arnhem, The Netherlands.*

Let Q = ABCD be a convex quadrilateral with sides AB = a, BC = b, CD = c, DA = d, and area $[Q]$. The following theorem is well known: If Q has both a circumcircle and an incircle, then $[Q] = \sqrt{abcd}$.

Prove or disprove the following converse: If Q has a circumcircle and $[Q] = \sqrt{abcd}$, then there exists a circle tangent to the four lines AB, BC, CD, and DA.

Solution by George Tsintsifas, Thessaloniki, Greece (revised and extended by the editor).

We will say that the convex quadrilateral Q is *cyclic* if it has a circumcircle, and that it is *circumscribable* if it has an incircle or an excircle. Consider the following three theorems:

THEOREM 1. If Q is cyclic and circumscribable, then $[Q] = \sqrt{abcd}$.

THEOREM 2. If $[Q] = \sqrt{abcd}$ and Q is circumscribable, then Q is cyclic.

THEOREM 3. If $[Q] = \sqrt{abcd}$ and Q is cyclic, then Q is circumscribable.

The well-known theorem mentioned in the proposal is a weakened form of Theorem 1, and Theorem 3 is a restatement of the proposed problem. If Q is a nonsquare rectangle, then the hypothesis of Theorem 3 is true and its conclusion false. We will prove these three theorems with the additional assumption, for Theorem 3 only, that Q is not a nonsquare rectangle. We will need the following lemmas, in which $s = (a+b+c+d)/2$.

LEMMA 1. Q is cyclic if and only if B+D $= \pi$.

LEMMA 2. Q has an incircle if and only if $s-b-d = 0$.

LEMMA 3. If Q does not have a pair of opposite sides parallel (that is, if *its sides produced form a complete quadrilateral), then Q has an excircle if and only if* $(s-a-d)(s-c-d) = 0$.

LEMMA 4. For any Q,

$$[Q]^2 = \Pi(s-a) - abcd\cos^2\frac{B+D}{2},$$

where the product is cyclic over a,b,c,d.

LEMMA 5. For all numbers a,b,c,d,

$$\Pi(s-a) = s(s-a-d)(s-b-d)(s-c-d) + abcd.$$

Lemmas 1 and 4 are classical results. The conditions of Lemmas 2 and 3 are equivalent to $a+c = b+d$, and $b+c = a+d$ or $a+b = c+d$, respectively. These conditions are given, with numerous historical references, in Sauvé [1]. The identity of Lemma 5 is easily verified, but we do not recall seeing it before in the literature. Our proofs of the three theorems will be based on Lemmas 1,2,3 and on the formula

$$[Q]^2 = s(s-a-d)(s-b-d)(s-c-d) + abcd - abcd\cos^2\frac{B+D}{2}, \tag{1}$$

valid for any Q, which is an immediate consequence of Lemmas 4 and 5.

Proof of Theorem 1. It follows from the hypothesis that

$$\cos\frac{B+D}{2} = 0 \quad \text{and} \quad (s-a-d)(s-b-d)(s-c-d) = 0,$$

and (1) reduces to $[Q]^2 = abcd$.

Proof of Theorem 2. It follows from the hypothesis that

$$[Q]^2 = abcd \quad \text{and} \quad (s-a-d)(s-b-d)(s-c-d) = 0.$$

Thus (1) is equivalent to $\cos(B+D)/2 = 0$, so $B+D = \pi$ and Q is cyclic.

Proof of Theorem 3. It follows from the hypothesis that $[Q]^2 = abcd$ and $\cos(B+D)/2 = 0$, so (1) is equivalent to

$$(s-a-d)(s-b-d)(s-c-d) = 0.$$

If $s-b-d = 0$, then Q has an incircle. Suppose $(s-a-d)(s-c-d) = 0$. If Q does not have a pair of opposite sides parallel, then it has an excircle. If it has a pair of opposite sides parallel, then the other two opposite sides are also parallel, so Q is a rectangle, hence a square, and it has an incircle. In any case, Q is circumscribable provided it is not a nonsquare rectangle.

Also solved by ELWYN ADAMS, Gainesville, Florida; LEON BANKOFF, Los Angeles, California; HIPPOLYTE CHARLES, Waterloo, Québec; JORDI DOU, Barcelona, Spain; GALI SALVATORE, Perkins, Québec; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposers (two solutions).

Editor's comment.

Three of the above solvers proved to be faint-hearted: they simply gave the nonsquare rectangle counterexample (a technical knockout of the problem) and then prudently retired from the field without achieving honour.

Theorem 2 appeared as Problem B-6 in the thirty-first William Lowell Putnam Mathematical Competition held on December 5, 1970. A solution was published in [2] with the following comment: "This elegant solution was presented by Robert Oliver. He was the only student to receive a perfect score on this problem." Readers will find it interesting to compare Oliver's solution with the one given here.

REFERENCES

1. Léo Sauvé, "On Circumscribable Quadrilaterals", this journal, 2 (1976) 63-67.

2. J.H. McKay, "The William Lowell Putnam Mathematical Competition", *American Mathematical Monthly*, 78 (1971) 763-770.

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779. [1982: 246] *Proposed by H. Kestelman, University College, London, England.*

Suppose $A = X + iY$, where X and Y are real square matrices. If A is invertible, show that

$$A^{-1} = \bar{A}(X^2+Y^2)^{-1}$$

if and only if X and Y commute.

If A is singular, can $X^2 + Y^2$ be invertible? If A is invertible, can $X^2 = Y^2 = 0$ (the zero matrix)?

Solution by the proposer.

If A is invertible, so are $\bar{A}$ and

$$A\bar{A} = (X+iY)(X-iY) = X^2 + Y^2 + i(YX-XY). \qquad (1)$$

Suppose $XY = YX$. Then X^2+Y^2 is invertible by (1) and $A\bar{A}(X^2+Y^2)^{-1} = I$, from which follows

$$A^{-1} = \bar{A}(X^2+Y^2)^{-1}. \qquad (2)$$

Conversely, suppose X^2+Y^2 is invertible and (2) holds. Then

$$I = AA^{-1} = A\bar{A}(X^2+Y^2)^{-1} = I + i(YX-XY)(X^2+Y^2)^{-1},$$

and $XY = YX$ follows.

If

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{then} \quad A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Here A is singular and $X^2+Y^2 = 2I$ is invertible, so the answer to the first question is yes.

If

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{then} \quad A = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}.$$

Here A is invertible and $X^2 = Y^2 = 0$, and the answer to the second question is also yes.

Also solved by KENT D. BOKLAN, student, Massachusetts Institute of Technology; and partially solved by MARCO A. ETTRICK, New York Technical College.

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780. [1982: 247] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Prove that one can take a walk on Pascal's triangle, stepping from one element only to one of its nearest neighbors, in such a way that each element $\binom{m}{n}$ gets stepped on exactly $\binom{m}{n}$ times.

Solution adapted from those submitted by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio; and the proposer.

Let P_m be a finite Pascal triangle of $m+1$ rows, so that the last row contains the coefficients of $(a+b)^m$. The walker will not work up much of a sweat if $m = 0$, so we assume that $m \geq 1$. We will show by induction that there exists a walk on P_m starting at $\binom{0}{0}$ in which each number $\binom{j}{i}$, $i = 0,1,\ldots,j$, $j = 0,1,\ldots,m$, gets stepped on exactly $\binom{j}{i}$ times, with the walker going at all times from a number to one of its nearest neighbors. If after his walk is completed the walker were to head directly back to $\binom{0}{0}$, then the extended walk would constitute an Eulerian circuit in a connected directed graph whose nodes are the numbers in P_m and whose directed arcs are the segments of the walk.

Before setting up the induction we explain some notation. We will denote by r_j the row containing the coefficients of $(a+b)^j$. The *midnodes* of r_j are the two nodes closest to the center if j is odd or the two nodes flanking the central node if j is even. We will let W_m denote a walk on P_m starting at $\binom{0}{0}$ in which each node $\binom{j}{i}$ gets stepped on exactly $\binom{j}{i}$ times, with the walker going at all times from a node to one of the nearest nodes, and ending at a midnode of r_m if m is odd or at the central node of r_m if m is even. Whenever such a walk W_m exists, we will denote by r_m the node of r_m last stepped on, and by N_{m+1} a midnode of r_{m+1} nearest to N_m.

Now for the induction. It is clear that there exists a walk W_1 on P_1 (there are in fact two). Suppose there exists a walk W_m on P_m for some $m \geq 1$. For $i = 0,1,\ldots,m$, between the consecutive nodes $\binom{m+1}{i}$ and $\binom{m+1}{i+1}$ of r_{m+1} we place $2\binom{m}{i}$ directed arcs, half in one direction and half in the other, and then replace one of the arcs ending at N_{m+1} by an arc from N_m to N_{m+1}. If the walker, having completed walk W_m, now follows the arc from N_m to N_{m+1}, heads away from the center to the end of r_{m+1}, and then weaves back and forth along the directed arcs, going as far as he can each time before reversing direction, then, when he has traversed all the arcs, node $\binom{m+1}{0}$ will have been stepped on exactly once; for $i = 1,2,\ldots,m+1$, node $\binom{m+1}{i}$ will have been stepped on exactly

$$\binom{m}{i-1} + \binom{m}{i} = \binom{m+1}{i}$$

times; and the walk will end either at the central node or at a midnode of r_{m+1}, depending on the parity of m; so a walk W_{m+1} will have been accomplished. The induction is complete.

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781. [1982: 276] *Proposed by Alan Wayne, Holiday, Florida.*

The Philosophic Father

Your brother's first to bathe, my son,
So do not look so W R Y .
You'll have to use the T U B in turn,
There's room for only O N E .

There's very little room, I guess.
Aye, there's the R U B! But try.
Use soap on cloth to R U B. Don't spurn
My counsel; answer "Y E S".

In time, you'll find it's true—
Much later, by the B Y E—
As ashes resting in an U R N,
There will be room for T W O.

Regard the preceding three patterns of three capitalized words as interdependent arithmetic additions in the decimal system. Restore the digits.

Solution by the proposer.

From the outside columns of the three additions, none of W, T, O, R, Y, B, U, E, or N can be zero, so S = 0. Then B = 5, so E is odd and Y is even. This implies that 2U+1 = E. Thus

$$(U, E) = (1, 3), (3, 7), \text{ or } (4, 9).$$

If (U, E) = (3, 7), then Y = 2, R = 1, N = 4, and O = 1 = R. If (U, E) = (4, 9), then Y = 4 = U. Hence (U, E) = (1, 3), and there follow immediately Y = 8, R = 4, N = 6, O = 9, W = 2, and T = 7.

The unique solution is

$$\begin{array}{r} 248 \\ \underline{715} \\ 963 \end{array}, \quad \begin{array}{r} 415 \\ \underline{415} \\ 830 \end{array}, \quad \begin{array}{r} 583 \\ \underline{146} \\ 729 \end{array}.$$

Also solved by MEIR FEDER, Haifa, Israel; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; ADAM MILLING, Grade 11 student, A.B. Lucas Secondary School, London, Ontario; STANLEY RABINOWITZ, Digital Equipment Corp., Nashua, New Hampshire; RAM REKHA TIWARI, Radhaur, Bihar, India; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and ANNELIESE ZIMMERMANN, Bonn, West Germany.

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782. [1982: 276] *Proposed by H.S.M. Coxeter, University of Toronto.*

(a) Sketch the plane cubic curve given by the parametric equations

$$x = \alpha(\beta-\gamma)^2, \quad y = \beta(\gamma-\alpha)^2, \quad z = \gamma(\alpha-\beta)^2, \quad \alpha+\beta+\gamma = 0,$$

where (x,y,z) are barycentric (or areal, or trilinear) coordinates, referred to an equilateral triangle. In what respect do its asymptotes behave differently from those of a hyperbola?

(b) Eliminate the parameters α,β,γ so as to obtain a single equation

$$x^3+y^3+z^3 + a(x^2y+x^2z+y^2z+y^2x+z^2x+z^2y) + bxyz = 0$$

for certain numbers a and b.

(c) What equation does the curve have in terms of polar coordinates?

Solution by G.P. Henderson, Campbellcroft, Ontario.

Let O be the centre and u the inradius of the equilateral triangle. We represent any point P of the plane by its *trilinear coordinates* (x,y,z) referred to the equilateral triangle, that is, x,y,z are the directed perpendicular distances from P to the sides of the triangle, the positive direction for each coordinate being from O to the corresponding side. Thus the coordinates of O are (u,u,u) and the coordinates of any point P satisfy

$$x + y + z = 3u.$$

In the sequel, all sums will be cyclic over x,y,z or α,β,γ.

Set

$$\Sigma yz = v, \quad xyz = w, \quad \Sigma\beta\gamma = B, \quad \alpha\beta\gamma = C. \tag{1}$$

Then, since $\Sigma\alpha = 0$,

$$\Sigma\alpha^2 = -2B, \quad \Sigma\alpha^3 = 3C, \quad \Sigma(\beta^2\gamma+\beta\gamma^2) = -3C.$$

As

$$3u = \Sigma x = \Sigma\alpha(\beta-\gamma)^2 = \Sigma(\beta^2\gamma+\beta\gamma^2) - 6\alpha\beta\gamma = -9C,$$

we have

$$u = -3C. \tag{2}$$

Similarly,

$$v = B^3 + 27C^2 \quad \text{and} \quad w = -C(4B^3 + 27C^2). \tag{3}$$

Eliminating B and C from (2) and (3), we obtain

$$27u^3 - 12uv + 9w = 0, \tag{4}$$

that is,

$$(\Sigma x)^3 - 4(\Sigma x)(\Sigma yz) + 9xyz = 0,$$

which reduces to

$$\Sigma x^3 - \Sigma(y^2z+yz^2) + 3xyz = 0. \tag{5}$$

This has the form required in part (b), with $a = -1$ and $b = 3$.

Using $\Sigma x = 3u$, (5) can be written in the form

$$(3x - 4u)(3y - 4u)(3z - 4u) = -u^3. \tag{6}$$

This is a necessary condition for (x,y,z) to be on the locus. We will show that it is also a sufficient condition, so we can take (6) to be the equation of the locus. It follows that

$$x = \frac{4u}{3}, \quad y = \frac{4u}{3}, \quad z = \frac{4u}{3}$$

are asymptotes of the locus.

Conversely, let x,y,z satisfy (6) and $\Sigma x = 3u$. We have to show that there exist α,β,γ such that $\Sigma\alpha = 0$ and $x = \alpha(\beta-\gamma)^2$, etc. We claim that, for a suitable choice of cube roots (to be specified below), satisfactory values of α,β,γ are

$$\alpha = \sqrt[3]{x - 4u/3}, \quad \beta = \sqrt[3]{y - 4u/3}, \quad \gamma = \sqrt[3]{z - 4u/3}. \tag{7}$$

For any choice of cube roots in (7), we have

$$\Sigma\alpha^3 = \Sigma x - 4u = -u \tag{8}$$

and, from (6), $\alpha^3\beta^3\gamma^3 = -(u/3)^3$. We first choose cube roots $\alpha_0,\beta_0,\gamma_0$ in (7) such that

$$\alpha_0\beta_0\gamma_0 = -\frac{u}{3}. \tag{9}$$

Then, from (8) and (9), $\Sigma\alpha_0^3 - 3\alpha_0\beta_0\gamma_0 = 0$, or

$$(\alpha_0+\beta_0+\gamma_0)(\alpha_0+\omega\beta_0+\omega^2\gamma_0)(\alpha_0+\omega^2\beta_0+\omega\gamma_0) = 0,$$

where ω is a primitive cube root of unity. So, for one of the choices

$$(\alpha,\beta,\gamma) = (\alpha_0,\beta_0,\gamma_0) \text{ or } (\alpha_0,\omega\beta_0,\omega^2\gamma_0) \text{ or } (\alpha_0,\omega^2\beta_0,\omega\gamma_0),$$

we have $\alpha+\beta+\gamma = 0$ and $\alpha\beta\gamma = -u/3$. Finally,

$$x = \alpha^3 + \frac{4u}{3} = \alpha^3 - 4\alpha\beta\gamma = \alpha(\alpha^2 - 4\beta\gamma) = \alpha\{(\beta+\gamma)^2 - 4\beta\gamma\} = \alpha(\beta-\gamma)^2,$$

with similar expressions for y and z. For real x,y,z, we can choose real cube roots in (7) except for the point $O(u,u,u)$. To obtain this point we can take for α,β,γ the three cube roots of $-u/3$.

For our system of polar coordinates (r,θ), we take the pole to be the point O and the initial ray parallel to the line $x = 0$. Then

$$x = u + r\sin\theta, \quad y = u + r\sin(\theta+\frac{2\pi}{3}), \quad z = u + r\sin(\theta+\frac{4\pi}{3}).$$

To get the polar equation of the locus, we eliminate x,y,z from these and (4). Since x,y,z are the roots of

$$t^3 - 3ut^2 + vt - w = 0,$$

$r\sin\theta$, $r\sin(\theta+2\pi/3)$, $r\sin(\theta+4\pi/3)$ are the roots of

$$(t+u)^3 - 3u(t+u)^2 + v(t+u) - w = 0,$$

that is,

$$t^3 + (v-3u^2)t + u^3 - \frac{uv}{3} = 0,$$

where we have used (4) to eliminate w. The sum of the products of the roots taken two at a time is

$$\begin{aligned} v - 3u^2 &= r^2\{\sin\theta\sin(\theta+\frac{2\pi}{3}) + \sin\theta\sin(\theta+\frac{4\pi}{3}) + \sin(\theta+\frac{2\pi}{3})\sin(\theta+\frac{4\pi}{3})\} \\ &= \frac{r^2}{2}\{\cos\frac{2\pi}{3} - \cos(2\theta+\frac{2\pi}{3}) + \cos\frac{4\pi}{3} - \cos(2\theta+\frac{4\pi}{3}) + \cos\frac{2\pi}{3} - \cos 2\theta\} \\ &= -\frac{3r^2}{4}. \end{aligned} \tag{10}$$

Similarly,

$$u^3 - \frac{uv}{3} = \frac{r^3}{4}\sin 3\theta.$$

Eliminating v from the last two equations, we obtain the polar equation

$$r^2(r\sin 3\theta - u) = 0. \qquad (11)$$

This is a necessary condition for (r,θ) to be on the locus, and we will show that the condition is also sufficient. It will then follow that the locus consists of the isolated double point (acnode) $r = 0$ and the curve

$$r\sin 3\theta = u. \qquad (12)$$

We have already seen that $r = 0$ is on the locus, so, to prove the sufficiency of (11), we assume that $r = u\csc 3\theta$ and find α,β,γ such that

$$\begin{cases} \alpha + \beta + \gamma = 0, \\ \alpha(\beta-\gamma)^2 = u(1 + \sin\theta\csc 3\theta), \\ \beta(\gamma-\alpha)^2 = u(1 + \sin(\theta+\frac{2\pi}{3})\csc 3\theta), \\ \gamma(\alpha-\beta)^2 = u(1 + \sin(\theta+\frac{4\pi}{3})\csc 3\theta). \end{cases} \qquad (13)$$

From (1), α,β,γ are the roots of $t^3 + Bt - C = 0$. But from (2), (3), and (10),

$$C = -\frac{u}{3} \quad \text{and} \quad B^3 = -\frac{3}{4}u^2\csc^2 3\theta.$$

The equation for α,β,γ is therefore

$$t^3 - \sqrt[3]{\frac{3}{4}u^2\csc^2 3\theta}\cdot t + \frac{u}{3} = 0.$$

Using Cardan's method to solve this, we find

$$\alpha = \sqrt[3]{\frac{4u}{3}\csc 3\theta}\cdot\sin\theta,$$

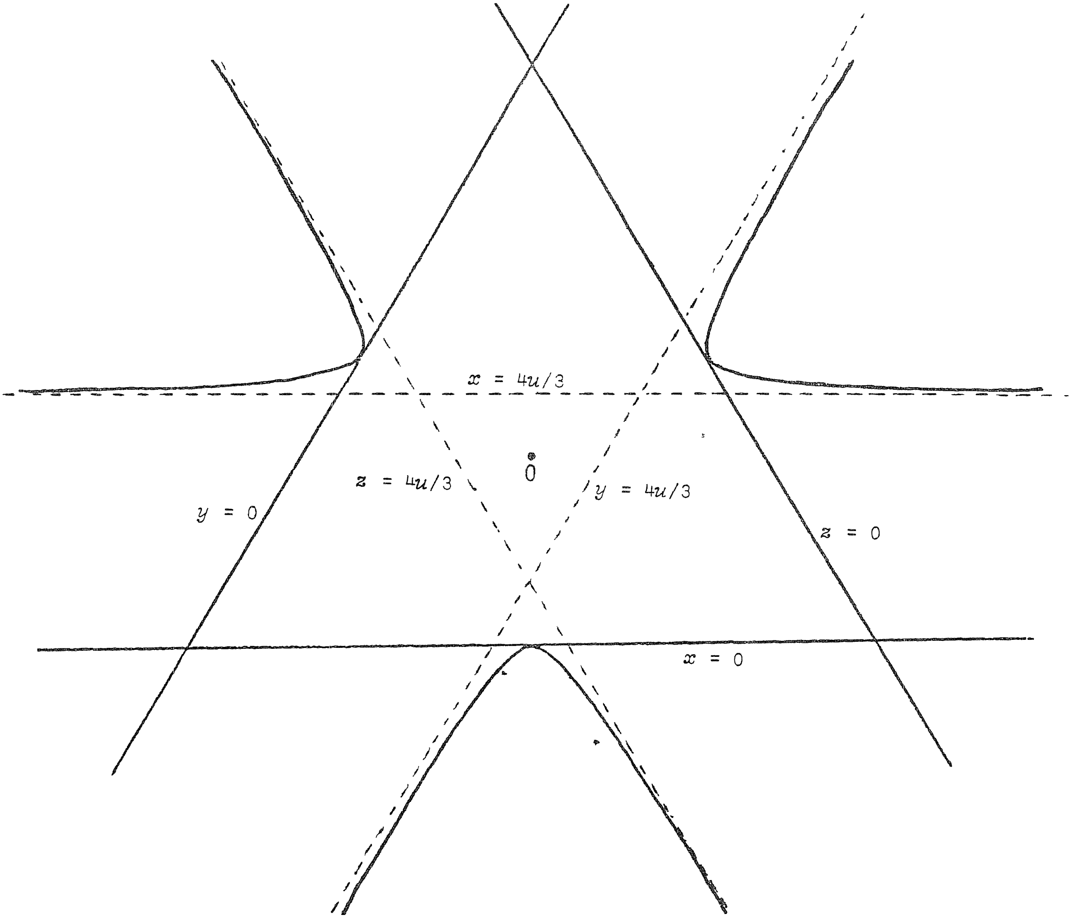
$$\beta = \sqrt[3]{\frac{4u}{3}\csc 3\theta}\cdot\sin(\theta+\frac{2\pi}{3}),$$

$$\gamma = \sqrt[3]{\frac{4u}{3}\csc 3\theta}\cdot\sin(\theta+\frac{4\pi}{3}),$$

and substitution in (13) shows that this actually is a solution.

Using (11) and the fact that $x = 4u/3$, $y = 4u/3$, $z = 4u/3$ are asymptotes, we can sketch the curve (see figure). The asymptotes are *inflectional*, which means that the two branches that approach an asymptote are on the same side of it in both directions, whereas for a hyperbola they are on opposite sides.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; O. BOTTEMA, Delft, The Netherlands; W.L. EDGE, University of Edinburgh, Scotland (parts (a) and (b) only); J.T. GROENMAN, Arnhem, The Netherlands (parts (a) and (b) only); KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.



Editor's comment.

The u in the polar equation (12) is a scale factor that can be chosen at will. Bottema took the circumradius of the reference triangle to be unity and obtained $r \sin 3\theta = \frac{1}{2}$. The proposer did the same but oriented the polar axis differently and obtained $r \cos 3\theta = -\frac{1}{2}$. Some of the other solvers obtained far more complicated expressions for the polar equation. The proposer noted that the curve is drawn and described on page 32 of H.G. Forder's *Geometry* (Hutchinson, London, 1950), and he discussed the curve (among other things) in a recent paper entitled "The Lehmus Inequality" (submitted to *Aequationes Mathematicae*).

The solution we have featured was the only one which could lay any claim to completeness because the solver took the trouble to verify that his equations, in both the trilinear and polar cases, constituted necessary *and* sufficient conditions for a point to lie on the locus.

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783, [1982: 277] *Proposed by R.C. Lyness, Southwold, Suffolk, England.*

Let n be a fixed natural number. We are interested in finding an infinite sequence $(v_0, v_1, v_2, \ldots)$ of strictly increasing positive integers, and a finite sequence $(u_0, u_1, \ldots, u_n)$ of nonzero integers such that, for all integers $m \ge n$,

$$u_0^2 v_m + u_1^2 v_{m-1} + \ldots + u_n^2 v_{m-n} = u_0 v_m^2 + u_1 v_{m-1}^2 + \ldots + u_n v_{m-n}^2. \qquad (1)$$

(a) Prove that (1) holds if

$$u_r = \text{ coefficient of } x^r \text{ in } (1-x)^n$$

and

$$v_r = \text{coefficient of } x^r \text{ in } (1-x)^{-n-1}.$$

(b)* Find other sequences (u_r) and (v_r) for which (1) holds.

Solution to part (a) by Gali Salvatore, Perkins, Québec.

The proposed sequences are

$$u_r = (-1)^r \binom{n}{r}, \qquad r = 0, 1, \ldots, n$$

and

$$v_r = \binom{n+r}{r}, \qquad r = 0, 1, 2, \ldots .$$

We note that (u_r) is a finite sequence of nonzero integers and, since it is easily verified that $v_{r+1} > v_r$, that (v_r) is a strictly increasing infinite sequence of positive integers, as required by the proposal. Our proof will be based on the following identity, given by Bruce in [1], which is valid for all integers $m \ge n$:

$$\binom{m}{n}^2 = u_0^2 \binom{m}{2n} + u_1^2 \binom{m+1}{2n} + \ldots + u_n^2 \binom{m+n}{2n}, \qquad (2)$$

or, since $u_r^2 = u_{n-r}^2$,

$$\binom{m}{n}^2 = u_0^2 \binom{m+n}{2n} + u_1^2 \binom{m+n-1}{2n} + \ldots + u_n^2 \binom{m}{2n}. \qquad (3)$$

We will apply to (3), in two different ways, the difference operator Δ, defined for any sequence (w_m) by

$$\Delta w_m = w_{m+1} - w_m, \quad \Delta^{k+1} w_m = \Delta(\Delta^k w_m), \quad k = 1, 2, 3, \ldots .$$

This operator is known to have the following property (see, e.g., Ferrar [2]):

$$\Delta^n w_m = u_0 w_{m+n} + u_1 w_{m+n-1} + \ldots + u_n w_m.$$

From this property follows

$$\begin{aligned}
\Delta^n\binom{m}{n}^2 &= u_0\binom{m+n}{n}^2 + u_1\binom{m+n-1}{n}^2 + \ldots + u_n\binom{m}{n}^2 \\
&= u_0\binom{n+m}{m}^2 + u_1\binom{n+m-1}{m-1}^2 + \ldots + u_n\binom{n+m-n}{m-n}^2 \\
&= u_0v_m^2 + u_1v_{m-1}^2 + \ldots + u_nv_{m-n}^2. \qquad (4)
\end{aligned}$$

On the other hand, Δ is a linear operator and

$$\Delta\binom{s}{t} = \binom{s+1}{t} - \binom{s}{t} = \binom{s}{t-1}.$$

So if we apply it once to (3), we obtain

$$\begin{aligned}
\Delta\binom{m}{n}^2 &= u_0^2\Delta\binom{m+n}{2n} + u_1^2\Delta\binom{m+n-1}{2n} + \ldots + u_n^2\Delta\binom{m}{2n} \\
&= u_0^2\binom{m+n}{2n-1} + u_1^2\binom{m+n-1}{2n-1} + \ldots + u_n^2\binom{m}{2n-1};
\end{aligned}$$

and if we apply it repeatedly, for a total of n times, the final result is

$$\begin{aligned}
\Delta^n\binom{m}{n}^2 &= u_0^2\binom{m+n}{n} + u_1^2\binom{m+n-1}{n} + \ldots + u_n^2\binom{m}{n} \\
&= u_0^2\binom{n+m}{m} + u_1^2\binom{n+m-1}{m-1} + \ldots + u_n^2\binom{n+m-n}{m-n} \\
&= u_0^2v_m + u_1^2v_{m-1} + \ldots + u_n^2v_{m-n}. \qquad (5)
\end{aligned}$$

Finally, the desired result (1) follows (4) and (5).

Part (a) was also solved by the proposer.

Editor's comment.

In [1], Bruce gives explicitly only the special cases of identity (2) for $n = 1,2,3,4$, with a plausibility argument for the general case. He adds that "the general result is true and is easily proved by induction". In spite of the fact that the above solution hangs entirely on the validity of (2), our solver did not give a proof of it, presumably because he (or she: is Gali a man's name or a woman's?) felt it was "easy". Our proposer, in his proof of part (a), also used (2), but he proved it only for the special case $n = 3$, and then used it to establish (1), but still only for $n = 3$. He added that "using the methods above the results can be easily generalised". Easy is as easy does, and Bruce doesn't, our proposer doesn't, and Gali doesn't. If none of these three worthies saw fit to give a proof of (2), we suspect it may be because it isn't all that easy. Easy or not, a formal proof of (2) is required to complete part (a). The editor will gladly reopen the problem when such a proof is available.

Another question left dangling is that raised by part (b)*. Are the sequences (u_r) and (v_r) of part (a) the only ones that satisfy (1)? A characterization of all such sequences would be the last word in rounding off this interesting problem.

REFERENCES

1. Ian Bruce, "Binomial Identities", *The Mathematical Gazette*, 65 (December 1981) 282-285.

2. W.L. Ferrar, *Higher Algebra*, Oxford University Press, 1943, p. 29.

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784. [1982: 277] *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let $F_n = (a_i/b_i)$, $i = 1,2,\ldots,m$, be the Farey sequence of order n, that is, the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. (For example,

$$F_5 = (\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}),$$

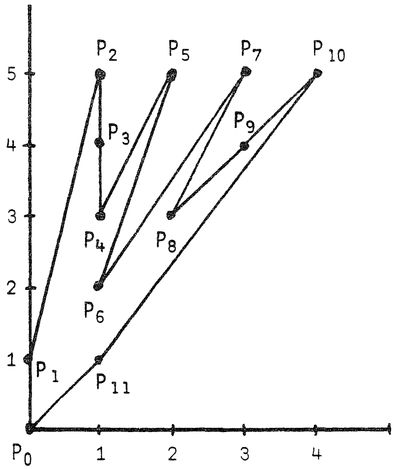
with $m = 11$.) Prove that, if $P_0 = (0,0)$ and $P_i = (a_i,b_i)$, $i = 1,2,\ldots,m$, are lattice points in a Cartesian coordinate plane, then $P_0P_1\ldots P_m$ is a simple polygon of area $(m-1)/2$.

Solution I.

The sequence (a_i/b_i) is strictly increasing. So, as a point P traverses the polygon $P_0P_1\ldots P_i\ldots P_m$ in the sense of increasing i (the figure shows the polygon for the special case $n = 5$), the inclination of the segment P_0P stays at $\pi/2$ as P goes from P_0 to P_1, it decreases strictly from $\pi/2$ to $\pi/4$ as P goes from P_1 to P_m, and it stays at $\pi/4$ as P goes from P_m back to P_0. From this it follows that

(i) no two sides of the polygon can cross, so the polygon is simple;

(ii) the area of the polygon is the sum of the areas Δ_i of the triangles $P_0P_iP_{i+1}$ for $i = 1,2,\ldots,m-1$.



Now, for each i, $\Delta_i = \frac{1}{2}(b_ia_{i+1} - a_ib_{i+1}) = \frac{1}{2}$, since $b_ia_{i+1} - a_ib_{i+1} = 1$ [1, Theorem 28]. Hence the area of the polygon is $\frac{1}{2}(m-1)$.

Solution II.

[As in solution I], the polygon is simple.

Following Hardy & Wright [1, page 29], we call a lattice point $P(x,y)$ *visible* (from P_0) if there is no lattice point on P_0P between P_0 and P. In order that $P(x,y)$ be visible, it is necessary and sufficient that x/y be irreducible. Since each a_i/b_i is irreducible, it follows that all points $P_1,P_2,\ldots,P_m$ are visible.

For each $i = 1,2,\ldots,m-1$, the *parallelogram* with adjacent sides P_0P_i and P_0P_{i+1}, whose area is $b_ia_{i+1}-a_ib_{i+1} = 1$, has no lattice point in its interior [1, Theorem 34].

From the last two paragraphs, it follows that the polygon has only the $m+1$ lattice points $P_0,P_1,\ldots,P_m$ on its boundary and no lattice point in its interior. It now follows from Pick's Theorem (see, e.g., [2]) that the area of the polygon is

$$0 + \tfrac{1}{2}(m+1) - 1 = \tfrac{1}{2}(m-1).$$

Solutions were received from the COPS of Ottawa; HOWARD EVES, University of Maine at Orono; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Of the six solutions received, three were more or less equivalent to solution I, and three to solution II. Solution I is shorter, simpler, and, in this editor's opinion, esthetically more satisfying.

REFERENCES

1. G.H. Hardy & E.M. Wright, *An Introduction to the Theory of Numbers*, Fourth edition, Oxford University Press, 1960.

2. A. Liu, "A Direct Proof of Pick's Theorem", *Crux Mathematicorum*, 4 (1978) 242-244.

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785. [1982: 277] *Proposed by J. Chris Fisher, University of Regina.*

Suppose a closed differentiable curve has exactly one tangent line parallel to every direction. More precisely, suppose that the curve has parametrization $\vec{V}(\theta)$: $[0,2\pi] \to R^2$ for which

(i) $d\vec{V}/d\theta = (r(\theta)\cos\theta,\ r(\theta)\sin\theta)$ for some continuous real-valued function $r(\theta)$, and

(ii) $\vec{V}(\theta) = \vec{V}(\theta+\pi)$.

Prove that the curve is described in the clockwise sense as θ runs from 0 to π.

Solution by the proposer.

Let $v(\theta): [0,2\pi] \to C$ be the complex-valued function corresponding to $\vec{V}(\theta)$, and suppose $r(\theta)$ has the Fourier series

$$r(\theta) = \sum_{k=-\infty}^{\infty} \rho_k e^{ik\theta}.$$

Since $r(\theta)$ is real, $\rho_{-k} = \bar{\rho}_k$ for all k. Furthermore, no term of the series for $v(\theta)$ has a constant first derivative; thus $\rho_{-1} = 0$, and so $\rho_1 = 0$ as well. As can be verified by differentiating, this means that

$$v(\theta) = \frac{\rho_0}{i}e^{i\theta} + \frac{1}{i}\sum_{k=2}^{\infty}\left(\frac{\rho_k}{k+1}e^{i(k+1)\theta} + \frac{\bar{\rho}_k}{1-k}e^{i(1-k)\theta}\right).$$

Since $v(\theta) = v(\theta+\pi)$, only even powers of $e^{i\theta}$ appear, and $v(\theta)$ simplifies to

$$v(\theta) = \frac{1}{i}\sum_{k=2}^{\infty}\left(\frac{\rho_k}{2k}e^{i2k\theta} + \frac{\bar{\rho}_k}{2-2k}e^{i(2-2k)\theta}\right).$$

I now show that the area of the region bounded by $v(\theta)$ is negative, which implies that $v(\theta)$ is described in the clockwise sense. This area equals

$$\frac{1}{2}\,\mathrm{Im}\int_0^{2\pi} v\bar{v}'\,d\theta = \frac{1}{2}\,\mathrm{Im}\sum_{k=2}^{\infty}\frac{1}{i}\int_0^{2\pi}\left(\frac{\rho_k}{2k}e^{i2k\theta} + \frac{\bar{\rho}_k}{2-2k}e^{i(2-2k)\theta}\right)\left(\bar{\rho}_k e^{-i2k\theta} + \rho_k e^{i(2k-2)\theta}\right)d\theta$$

$$= \sum_{k=2}^{\infty}|\rho_k|^2\pi\left(\frac{1}{2k} + \frac{1}{2-2k}\right) = \sum_{k=2}^{\infty}\frac{2\pi|\rho_k|^2}{4k(1-k)} < 0.$$

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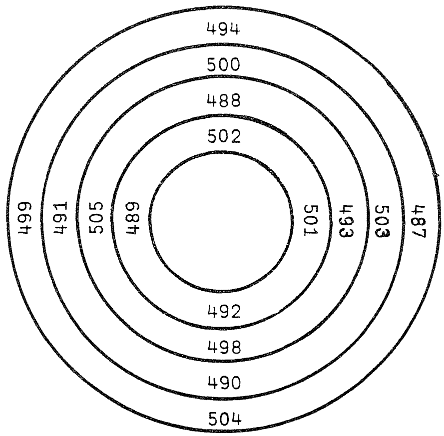
A MAGIC CIRCLE FOR 1984

May the year represented by the magic constant (obtainable in 32 different ways) of the adjoining magic circle, which contains 16 distinct integers from 487 to 494 and 498 to 505, be full of cheer, pleasure, and happiness for *Crux* and its readers.



KAMALA KUMARI
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