Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing. The copyright of all published material is vested in the Applied Probability Trust.

From the Editor

Divisors, maps, and interest

It isn't often that I get a response from my students – a yawn, the occasional snore, not much else. So I was a bit taken aback when David came to see me after a lecture to ask about perfect numbers. A *perfect number* is a positive integer which is the sum of its positive divisors, excluding the number itself. Thus the first four perfect numbers are

$$6 = 1 + 2 + 3,$$
$$28 = 1 + 2 + 4 + 7 + 14,$$
$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248,$$
$$8128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1016 + 2032 + 4064.$$

What David did was to multiply these divisors instead of adding them, i.e.

$$1 \times 2 \times 3 = 6,$$
$$1 \times 2 \times 4 \times 7 \times 14 = 784 = 28^2.$$

Similarly, the product of the divisors of 496 is 496^4 and the product of the divisors of 8128 is 8128^6. David wanted to know whether this product is always a power of the number. A good question!

Coincidentally, a letter arrived from a reader, V. Tyagi of Shyam Lal College, Delhi University, posing the same problem, but with no reference to perfect numbers. He asked for a formula for the product of the positive divisors of a number.

Perfect numbers turn out to be a red herring. Start with any positive integer n; take $n = 24$ as an example. Its positive divisors pair off as follows so that their product is 24:

$$24 = 1 \times 24 = 2 \times 12 = 3 \times 8 = 4 \times 6.$$

Thus, the product of the positive divisors of 24, including 24 itself, is 24^4, and the power is half the number of positive divisors of 24. This works for all numbers which are not perfect squares. But take 36, for example, a perfect square. This time we obtain

$$36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6,$$

and the product of the positive divisors of 36 is $36^4 \times 6$ (6 can only be included in the product once), or $36^{9/2}$. Again the power is half the number of positive divisors; this works for all perfect squares.

It remains to work out how many positive divisors a number n has. First factorize n into its prime factors, say

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

(for example $24 = 2^3 \times 3^1$ or $36 = 2^2 \times 3^2$). The positive divisors of n are the numbers

$$p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r},$$

where $l_i = 0, 1, \ldots, k_i$ for $i = 1, \ldots, r$, so there are $k_i + 1$ possibilities for l_i and the number of positive divisors of n is

$$(k_1 + 1)(k_2 + 1) \cdots (k_r + 1).$$

For example, 24 has $4 \times 2 = 8$ positive divisors, and 36 has $3 \times 3 = 9$. Hence, the product of the positive divisors of n is

$$n^{\frac{1}{2}(k_1+1)(k_2+1)\cdots(k_r+1)}.$$

Simple, but pretty!

Another of my students, Lloyd, found an interesting item on the web. (This sudden bout of interest must be catching!) We had been talking about the famous four-colour theorem. Given any map, how many colours are needed to colour the countries so that every two countries with a common boundary are coloured differently? The map shown in the figure below needs four colours, labelled 1 to 4; you can even colour the outside using colour 1 again.
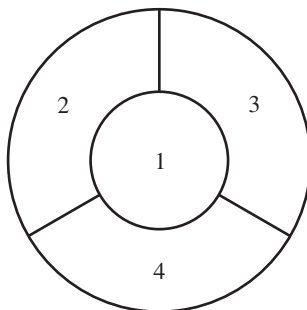
(I have to insist that students do not literally colour the maps because I am colour-blind. Strange for a colour-blind person to be teaching map-colouring!) It was conjectured way back in the 19th century that four colours suffice for any map. This was finally proved in 1976 by Ken Appel and Wolfgang Haken at the University of Illinois, but their proof requires the checking of a large number of cases which can only realistically be done by computer. How do we know that the computer got it right? Now Georges Gonthier at Microsoft's research laboratory in Cambridge, UK, and Benjamin Werner at INRIA in France have translated the proof into a language called *Coq*, used to represent logical propositions, and have created logic-checking software to check the proof. I quote from the *New Scientist* website, but more information can be found on links there, including Georges Gonthier's website. Maybe someone will explain it to me sometime.

To bring us down to earth, one of our readers, whose anonymity had better be preserved to avoid begging letters, has written about his monthly savings scheme, in which he saves £250 per month at a rate of interest of 7% per annum. The bank provided table 1 showing the interest earned after a year using a very precise way of calculating depending on the number of days in each month.

Table 1

Period of calculation		Number of	Cleared account	Interest	Interest
From	To	days	balance	rate	earned
04/04/2005	03/05/2005	30	£250	7.00%	£1.44
04/05/2005	03/06/2005	31	£500	7.00%	£2.97
04/06/2005	03/07/2005	30	£750	7.00%	£4.32
04/07/2005	03/08/2005	31	£1000	7.00%	£5.95
04/08/2005	03/09/2005	31	£1250	7.00%	£7.43
04/09/2005	03/10/2005	30	£1500	7.00%	£8.63
04/10/2005	03/11/2005	31	£1750	7.00%	£10.40
04/11/2005	03/12/2005	30	£2000	7.00%	£11.51
04/12/2005	03/01/2006	31	£2250	7.00%	£13.38
04/01/2006	03/02/2006	31	£2500	7.00%	£14.86
04/02/2006	03/03/2006	28	£2750	7.00%	£14.77
04/03/2006	03/04/2006	31	£3000	7.00%	£17.84
				Total	£113.49

He was told that this was equivalent to an annual equivalent rate (AER) of interest of 7.07%. Being a curious sort of guy, he asked the bank where this figure came from. After three letters from the bank, he gave up and asked us. Our best attempt is to say that the first payment after a year becomes

$$£250\left(1 + \tfrac{7}{100}\right).$$

The second payment, using the compound interest formula, becomes

$$£250\left(1 + \tfrac{7}{100}\right)^{335/365},$$

the third

$$£250\left(1 + \tfrac{7}{100}\right)^{304/365},$$

and so on. Adding up the twelve figures, I got £3113.27, 21 pence out. Given possible calculator errors on my part, is this close enough? And is it right?

Finally, another reader, Mr Sastry of Bangalore, India, has sent us a couple of problems which you might like to try.

1. A Pythagorean triangle, i.e. a right-angled triangle with integer sides, has area 451 350. What is its semi-perimeter?

2. The sides of a triangle are 6324, 7493, and 9805. What is the radius of its incircle?

As a clue, both problems are connected with the year. Happy solving!

The Life and Work of Augustus De Morgan

SCOTT H. BROWN

2006 marks the bicentenary of the birth of Augustus De Morgan

Augustus De Morgan was one of Britain's notable mathematicians of the 19th century. He made significant contributions to Logic, Algebra, Analysis, and the History of Mathematics. Known for his great sense of humour, he had a passion for collecting and posing problems, puzzles, and riddles. De Morgan was a talented teacher and made a lasting impression on many of his students.

De Morgan was born in Madura, India, on 27 June 1806. He was the fifth child of Colonel John De Morgan, who was serving with the East India Company. His mother was a descendant of John Dodson, who was known for his Table of Anti-logarithms. During De Morgan's infancy, he lost the sight in his right eye and, at seven months old, his father took the family to England. De Morgan had 'painful remembrance' of most of his early school days. He did not participate in the usual sports. Often he was the victim of pranks played by the other boys.

Figure 1 Augustus De Morgan (UCL Library Services: Special Collections).

He received his early education in private schools, where he studied Greek, Latin, and Mathematics. His talent in Mathematics became quite evident by the age of 14. According to the American mathematician George Halstead, 'he read Algebra "like a novel", and pricked out equations on the school pew instead of listening to the sermon'. De Morgan entered Trinity College, Cambridge, in 1823 and established a lifelong friendship with two of his professors, George Peacock and William Whewell. Peacock would influence De Morgan's interest in Algebra, while Whewell would foster his interest in Logic.

During his college years, De Morgan showed little interest in athletics, but was an active member of the musical club CAMUS (Cambridge Amateur Musical Union Society). He competed in The Mathematical Tripos and took fourth place which, according to many of his contemporaries, did not reflect his true mathematical abilities. In 1827 De Morgan received his BA degree. Due to his strong religious convictions, he objected to signing the theological tests and as a result was not able to obtain an MA degree or fellowship from Cambridge.

His opportunity to teach at Cambridge had been basically closed. Fortunately, University College London had just been founded and, in 1828, De Morgan was elected the first Chair of Mathematics at the age of 21. With the exception of a five-year hiatus, De Morgan served in this position for 30 years. During his career, De Morgan not only become an influential mathematician, but he established a reputation as a brilliant and inspiring teacher.

Incorporating a natural display of good humour with his mathematical prowess, De Morgan delivered clear and systematic lectures that captured his students' attention. In addition to his lectures, De Morgan developed handwritten notes, which were located in the library, for his students to supplement their studies. He was a firm believer in assigning numerous homework problems, usually at the end of each lecture, designed to foster critical thinking among his students. Several of his students who later became successful themselves include Stanely Jevons, Isaac Todhunter, E. J. Routh, and J. J. Sylvester.

De Morgan's 'unrivalled' ability as a teacher paralleled his ability in writing a tremendous number of books and articles. In 1828, he published a translation of the first three chapters of Pierre Louis Marie Bourdon's (1779–1854) *Élémens d'Algèbre*. Bourdon's treatise on algebra was extremely popular during this period in France. Likewise, De Morgan believed Bourdon's book contained material 'well adapted' for teaching the fundamental concepts of algebra.

De Morgan published *Elements of Arithmetic* in 1831. This book was primarily written because he believed most educators at the time did not see the relevance of teaching the young mind arithmetic through reason and demonstration. Modelling his approach to teaching, De Morgan's book stressed establishing 'the foundation of principles' and then developing the concepts. According to one of De Morgan's past students, Richard Hutton, 'the publication of his *Arithmetic*, a book which has not unnaturally been more useful to masters than to scholars, began a new era in the history of elementary teaching in England'.

During the summer of 1831, an administrative action by University College London on a fellow colleague affected De Morgan's relationship with the institution. The University Council dismissed the Professor of Anatomy 'without any fault of his own' (according to De Morgan). Having strong opinions of his own regarding the sanctity of professorship, De Morgan, in turn, sent a letter of resignation to the Council and left the College.

After leaving the College, De Morgan became heavily involved with the Astronomical Society, and was elected as honorary secretary in 1831. He established close relationships

with several notable members, including Sir John Herschel. De Morgan was particularly fond of Herschel, corresponding regularly with him for the next 40 years. The Astronomical Society provided De Morgan with the opportunity to establish notoriety as an historian. His lack of eyesight prevented him from experimenting with instruments, so he turned to writing articles on Astronomy and related subjects. Many of his articles were published in the *Penny Cyclopedia*, which accounted for about one-sixth of the journal's publications. During his hiatus from the College, he also wrote articles on mathematics education. These articles were published in the *Quarterly Journal of Education* beginning with the first volume in 1831. Both journals were published by the Society for the Diffusion of Useful Knowledge, of which he later became a committee member.

De Morgan returned to University College London in 1836, and shortly thereafter married Sophie Elizabeth Frend. Sophie was the daughter of William Frend who was known for his radical beliefs about religion. Frend and De Morgan had established a friendship, which lasted until Frend's death in 1841. De Morgan had his own religious convictions which Sophie respected, and the two were married in the register office rather than in a church. The first of their seven children, Elizabeth Alice, was born one year later. De Morgan's professional career was also beginning to take shape again. He was given the opportunity to present the 'introductory lecture' at the opening of the Faculty of Arts at the University.

De Morgan continued his prolific writing during this time. He published the *Connexion of Number and Magnitude* in 1836, which was devoted to using algebraic notation to express the concepts of proportion found in Euclid's fifth book. The complexity of the task was expressed by De Morgan as follows:

> The subject is one of some real difficulty arising from the limited character of the symbols of Arithmetic considered as representatives of ratios, and the consequent introduction of incommensurable ratios, that is, of ratios which have no arithmetical representation.

His next work, *The Differential and Integral Calculus*, was published in 1842 by the Useful Knowledge Society, and consisted of over 770 pages that covered a broad range of subjects. An important aspect of this book was De Morgan's 'rejection of the whole doctrine of series' in developing the foundation of both areas of Calculus. Instead, he preferred the 'theory of limits' as 'the sole foundation of the science'. De Morgan stressed the importance of number and magnitude as he discussed the use of the introduction of limits by stating 'the ideas attached to the words nothing and infinite do not permit the application of many rules in the strict and direct sense in which they are applied to numbers'. The book illustrated De Morgan's devotion to ensuring that the reader developed a 'conceptual understanding' of Calculus.

By this time, De Morgan's second son, George Campbell, had been born. Like his father, George was an extremely talented mathematician. He achieved the top prizes at University College London in 'Mathematics and Natural Philosophy'. George and Arthur Couper, in 1864, proposed the formation of the 'University College Mathematical Society', which later became 'The London Mathematical Society'. During the last two years of his short life, George was a Mathematics teacher at The University College School. He died in 1867 at the age of 26. According to Sophie, 'his father had a high opinion of the power of George's mind, which in some ways resembled his own'. Only one other child, William, achieved success somewhat comparable to his father. William became an inventor, creating ceramic tiles, and after retiring wrote a best-selling novel.

De Morgan's most significant work that brought him lasting fame was in Logic. He published *Formal Logic* in 1847, which focused on the theory of syllogism. In his work, he pointed out how the Aristotelian syllogism was limited by 'two distinct principles of exclusion'. The first exclusion suggests 'that in every syllogism the middle term must be universal in one of the premises, in order that we may be sure that the affirmation or denial in the other premise may be made of some or all of the things which affirmation or denial has been made in the first'. The other exclusion exists according to De Morgan because 'Aristotle will have no contrary terms: not-man, he says is not the name of anything'.

De Morgan ameliorated these limitations regarding 'limiting the inferences used' and the 'ambiguity of negation', by introducing terms such as 'whole' and 'universe'. Most importantly, De Morgan introduced the 'Numerically Definite Syllogism', which essentially quantified the terms in the propositions.

Although De Morgan did not truly discover the following complementation laws of set theory, he is given credit for officially introducing them as they are shown, which is why they are named after him:

$$(A \cup B)^C = A^C \cap B^C,$$
$$(A \cap B)^C = A^C \cup B^C.$$

The first law demonstrates that negating an OR makes it an AND, while the second law shows that negating an AND makes it an OR. Today, these laws are frequently used in circuit design, modern proof theory, and in software programming.

Early in the course of preparing *Formal Logic*, De Morgan wrote to Sir William Hamilton at the University of Edinburgh requesting information about 'the Aristotelian theory of syllogism'. Soon thereafter, Sir Hamilton sent a letter inferring that De Morgan's work was not original. In his letter of reply, De Morgan displays his temper and obvious wit as follows:

> I will not allude to the hasty manner in which you have expressed your suspicions of an odious charge, except to state that it does not diminish the sincere respect with which I subscribe myself.

Thus, the controversy began and continued until 1852. After the feud ceased, the two would exchange letters and books until Sir Hamilton's death. An obituary notice of him was written by De Morgan and published in the *Athenaeum*. De Morgan influenced several mathematicians and their work in the field of Logic, including George Boole and one of De Morgan's past students, William Stanley Jevons.

During the 1850's De Morgan spent a portion of his time supporting the adoption of a decimal currency in Great Britain. The idea of decimal coinage was introduced by Sir John Wrottesley to the House of Commons in 1824. De Morgan advocated the advantages and adoption of the system in articles published in *The Penny Cyclopedia* (1833) and *The Companion to the Almanac* (1841). In 1854, the Parliamentary Committee announced a favourable report regarding the decimal plan. As a result, the Decimal Association was established, of which De Morgan became a member. After a course of several meetings, the proceedings of the Decimal Association were published. De Morgan wrote an introduction stressing the major points regarding decimal coinage. The issue of a 'cents-and-

mile system' was presented again to parliament in 1855. This time, the resolution to 'address her Majesty' about finalizing the scale with the issue of a 'cents-and-mile system' was removed.

A Royal Commission was established to inquire into the subject of decimal coinage. A report was published in 1857. The debate continued as to whether the pound-and-mile system should be adopted or the present system retained. De Morgan's article, which was published in the *Literarium* in 1857, spoke to those who were in favour of the present system. He stated:

> If this school be a logical one, it ought to be prepared to maintain that a country with a decimal system already established ought to abandon its coinage, and to introduce the succession of 4, 12, 20.

This method of 'reckoning and payment' has not changed in the last 150 years. Nevertheless, De Morgan's writings on the subject were voluminous, providing information to the reader that is both instructive and historical.

In January 1865, at the age of 58, De Morgan became the first President of the 'University College Mathematical Society', which would soon thereafter become the London Mathematical Society. During his inaugural speech De Morgan spoke of the proper goals of a mathematical society. He then addressed one of his pet peeves – the Cambridge examination.

De Morgan was not fond of the typical 'hard ten-minute conundrums' as he called them, which comprised of the examination. Instead, he preferred an examination that would take the student 'two or three hours to solve', though he agreed this was not practical. Perhaps most important was De Morgan's reflection on the importance and requirement for interweaving logic and mathematics into the research function of the society. De Morgan served a two-year term as president and some of his original papers appeared in the earliest reports. In 1884, the De Morgan medal was established in his memory. The first recipient was Arthur Cayley for his contributions to mathematics. The medal has been given every third year when the year is divisible by three.

While President of the London Mathematical Society, De Morgan faced similar circumstances as he had in the past, when University College London failed to appoint Reverend James Martineau, a well-respected Unitarian Minister, for the Chair of Mental Philosophy and Logic. Upon hearing of the Council's decision not to appoint the Unitarian Minister, De Morgan resigned from University College London in November 1866. After De Morgan's resignation, the university and his colleagues did not acknowledge all of the work he had done during his 30 years of service. However, several of his past students and friends petitioned the government to give De Morgan a monetary compensation. Although he was not in favour of accepting the money, De Morgan received a pension within a year of his death.

During the last few years of his life, De Morgan divided his time between studying the *Greek Testament*, writing a history of his family and himself, and working on his *Budget of Paradoxes*. The *Budget* was a compendium of essays, which are for the most part entertaining accounts by De Morgan about 'paradoxes' written by individuals known for their 'undisciplined intellect'. According to De Morgan, 'a paradox is something which is apart from general opinion'. A portion of the book focused on material pertaining to the 'quadrature problem', 'trisecting angles', and 'squaring circles'. De Morgan was very candid, yet would mix some humour with

his remarks regarding the work of these individuals. For example, the work by Hobbes on the quadrature problem is admonished by De Morgan in the following passage:

> Hobbes, who began in 1655, was very wrong in his quadrature; but, though not a Gregory St. Vincent, he was not the ignoramus in geometry that he is sometimes supposed.

Other portions of the *Budget* were written about 'parodoxes' regarding various subjects such as astronomy, religion, and science. The book was later edited and published by De Morgan's wife after his death.

The bitter break from the college was beginning to take a toll on De Morgan. This event combined with the loss of his son, George, in 1867 and his daughter, Christiania, in 1870, marked the start of the decline of De Morgan's health. He was also suffering from 'nervous prostration' according to his wife. On 18 March 1871, he died at the age of 64.

References

1 A. De Morgan, *A Budget of Paradoxes* (Longmans, Green, and Co., London, 1872).

2 A. De Morgan, *Elements of Arithmetic* (Taylor, Walton and Maberly, London, 1848).

3 A. De Morgan, *Formal Logic* (Taylor and Walton, London, 1847).

4 A. De Morgan, *The Differential and Integral Calculus* (Baldwin and Cradock, London, 1842).

5 S. E. De Morgan, *Memoir of Augustus De Morgan* (Longmans, Green, and Co., London, 1882).

6 I. Grattan-Guinness, An eye for method: Augustus De Morgan and mathematics education, *Paradigm* **9** (1992).

7 B. Halsted, Biography: De Morgan, *Amer. Math. Monthly* **4** (1897), pp. 1–5.

8 B. S. Hawkins, De Morgan, Victorian syllogistic and relational logic, *Modern Logic* **5** (1995), pp. 131–166.

9 A. Macfarlane, *Lectures on Ten British Mathematicians of the Nineteenth Century* (New York, 1916), pp. 19–33.

10 B. H. Neumann, Augustus De Morgan, *Bull. London Math. Soc.* **16** (1984), pp. 575-589.

11 J. J. O'Connor and E. F. Robertson, Augustus De Morgan (http://www-history.mcs.st-andrews.ac.uk/history/Biographies/De_Morgan.html).

12 H. M. Pycior, Augustus De Morgan's algebraic work: the three stages, *Isis* **74** (1983), pp. 211–226.

13 A. Rice, Augustus De Morgan (1806–1871), *Math. Intelligencer* **18** (1996), pp. 40–43.

14 A. Rice, What makes a great mathematics teacher? The case of Augustus De Morgan, *Amer. Math. Monthly* **106** (1999), pp. 534–552.

15 J. L. Richards, Augustus De Morgan, the history of mathematics, and the foundations of algebra, *Isis* **78** (1987), pp. 7–30.

16 M. A. Rothman, Two 19th-century skeptics, Augustus De Morgan and John Fiske. *Skeptical Inquirer* **16** (1992), pp. 292-297.

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Is Anyone Sitting in the Right Place?

FEI YAO and PAUL BELCHER

A young mathematician is travelling by air to the International Mathematics Olympiad Competition. Just after check in there is a power cut and the airline computer loses data. This does not affect the passengers that much, except that no one was issued with a boarding card. The passengers therefore sit at random in the plane. To keep her brain active, ready for the competition, the young mathematician decides to calculate the probability that none of the passengers are sitting in their correct seat.

Let there be n passengers and n seats in the plane and let the number of arrangements of these n passengers, where no one sits in their correct seat, be u_n. The probability, p_n, that none of the passengers are sitting in their correct seat is given by

$$p_n = \frac{u_n}{n!}.$$

For $n = 1$, the young mathematician is the only passenger and so she must sit in the correct seat, giving $u_1 = 0$. For $n = 2$ we have $u_2 = 1$. Direct counting gives us the values in table 1.

We now attempt to obtain an iterative formula for u_n. Consider one of the n passengers, A, whose correct seat is a. Let passenger A sit in seat x, belonging to passenger X. There are $n - 1$ choices for seat x. We will now consider the following two possibilities:

  (i) passenger X sits in seat a,

  (ii) passenger X does not sit in seat a.

In case (i) we will have $n - 2$ remaining passengers to seat, where they all have to sit in the wrong seat, which can be done in u_{n-2} ways. In case (ii) passenger X has a seat (seat a) that they are not allowed to sit in and the other $n - 2$ passengers also have a seat (their own)

Table 1

n	u_n	p_n
1	0	0
2	1	$\frac{1}{2}$
3	2	$\frac{1}{3}$
4	9	$\frac{3}{8}$
5	44	$\frac{11}{30}$
6	265	$\frac{53}{144}$

that they are not allowed to sit in. So the number of ways of seating these $n-1$ passengers, including passenger X, is u_{n-1}. This gives us the iterative formula

$$u_n = (n-1)(u_{n-1} + u_{n-2}), \tag{1}$$

which confirms the values in table 1 and allows it to be extended indefinitely.

It would be better if we could obtain an implicit formula for u_n in terms of n. Manipulating (1) creates

$$u_n - nu_{n-1} = -(u_{n-1} - (n-1)u_{n-2}),$$

repeated use of this gives

$$u_n - nu_{n-1} = (-1)^{n-2}(u_2 - 2u_1) = (-1)^{n-2} = (-1)^n.$$

We can now use

$$u_n = nu_{n-1} + (-1)^n, \quad \text{where } u_1 = 0, \tag{2}$$

to build up the desired formula for u_n. Let us define $u_0 = 1$ which fits in with (1) and (2). We then obtain

$$u_1 = 1 - 1 = 1!\left(1 - \frac{1}{1!}\right),$$

$$u_2 = 2\left(1 - \frac{1}{1!}\right) + 1 = 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right),$$

$$u_3 = 3!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) - 1 = 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right),$$

which leads to

$$u_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{3}$$

Now that we have the formula (3) for u_n, an alternative proof for this could be done by using induction with the iterative formula (1), $u_0 = 1$, and $u_1 = 0$.

Equation (3) also looks like the type of formula that is obtained using the inclusion–exclusion counting principle. Let A_i, $i = 1, \ldots, n$, be the set of all arrangements of the n passengers onto the n seats so that passenger i sits in their correct seat. Then using the inclusion–exclusion formula we obtain

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}\left|\bigcap_{i=1}^n A_i\right|$$

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n-1}$$

$$= \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}$$

$$= n!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1}\frac{1}{n!}\right).$$

Now the set $\bigcup_{i=1}^{n} A_i$ has at least one passenger in the correct seat, therefore

$$u_n = n! - \left| \bigcup_{i=1}^{n} A_i \right|$$

$$= n! - n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1} \frac{1}{n!} \right)$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right),$$

which confirms (3).

From (3) we have

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}, \tag{4}$$

which, we realise, upon recalling that $e^x = \sum_{i=0}^{\infty} (x^i / i!)$, is a truncated version of the Maclaurin expansion for e^{-1}. Hence we obtain the following result:

$$p_n \to e^{-1} \quad \text{as } n \to \infty.$$

So, for a large number of passengers, the probability that nobody is in their correct seat is approximately e^{-1}.

We can also use (3) to obtain a neater way of calculating u_n. We have

$$|u_n - n! \, e^{-1}| = n! \left| \frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \frac{(-1)^{n+3}}{(n+3)!} + \cdots \right|.$$

The expression in the modulus signs on the right-hand side is an alternating series that, in absolute value, is monotonically decreasing and so is less than the modulus of the first term. Hence,

$$|u_n - n! \, e^{-1}| < \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{2},$$

since $n \geq 1$. So to find u_n we only need to calculate $n!/e$ and then round this to the nearest integer. Starting with $n = 1$ we will be alternately rounding down and then up, and $n!/e$ will become closer and closer to an integer as n increases. We illustrate this in table 2.

An alternative approach to obtain (4) is to use a generating function. From (1) the iterative formula for the probabilities is

$$n p_n = (n - 1) p_{n-1} + p_{n-2}, \quad \text{where } p_0 = 1 \text{ and } p_1 = 0. \tag{5}$$

We define the generating function as follows:

$$G(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots + p_r x^r + \cdots.$$

Then we obtain

$$G'(x) = p_1 + 2p_2 x + 3p_3 x^2 + \cdots + r p_r x^{r-1} + \cdots,$$

$$x G'(x) = \qquad p_1 x + 2p_2 x^2 + 3p_3 x^3 + 4p_4 x^4 + \cdots + r p_r x^r + \cdots, \tag{6}$$

$$x^2 G'(x) = \qquad p_1 x^2 + 2p_2 x^3 + 3p_3 x^4 + \cdots + (r-1) p_{r-1} x^r + \cdots, \tag{7}$$

$$x^2 G'(x) = \qquad p_0 x^2 + p_1 x^3 + p_2 x^4 + \cdots + p_{r-2} x^r + \cdots. \tag{8}$$

Table 2 The third column is rounded to two decimal places

n	u_n	$n!/e$
1	0	0.37
2	1	0.74
3	2	2.21
4	9	8.83
5	44	44.15
6	265	264.87
7	1 854	1 854.11
8	14 833	14 832.90
9	133 496	133 496.09
10	1 334 961	1 334 960.92

So, using (5), (6) minus (7) minus (8) gives

$$xG'(x) - x^2 G'(x) - x^2 G(x) = 0$$

and, hence, we obtain the first-order linear differential equation

$$G'(x) + \frac{x}{x-1} G(x) = 0.$$

The integrating factor is $\exp(\int x/(x-1)\,dx) = e^x(x-1)$, so multiplying by this we obtain

$$e^x(x-1)G'(x) + e^x x G(x) = 0,$$

and hence $D[e^x(x-1)G(x)] = 0$, giving $e^x(x-1)G(x) = k$, where k is a constant. Putting $x = 0$ shows that $k = -1$, so we get

$$G(x) = \frac{e^{-x}}{1-x} = \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)(1 + x + x^2 + x^3 + \cdots).$$

So the coefficient of x^n in $G(x)$, which is p_n, is equal to

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!},$$

which confirms (4).

Yet another approach would be to generalise the problem and let $u(n, r)$ be the number of ways of letting n people sit in n aircraft seats with exactly r people sitting in their correct seats. So $u(n, 0)$ corresponds to u_n. Let $p(n, r) = u(n, r)/n!$ be the corresponding probability, with $p(n, 0)$ corresponding to p_n. To seat exactly r people in their correct seats we could choose r seats from the n seats and get the correct people to sit in these and then the remaining $n - r$ people will not be allowed to sit in their own seats. This shows that

$$u(n, r) = \binom{n}{r} u(n-r, 0) = \frac{n!}{(n-r)!\, r!} u(n-r, 0) \quad \text{and} \quad p(n, r) = \frac{1}{r!} p(n-r, 0).$$

Now $\sum_{r=0}^{n} p(n, r) = 1$ so $\sum_{r=0}^{n} (1/r!) p(n-r, 0) = 1$.

We now use the convolution theorem for generating functions. If sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are connected by the formula $a_n = \sum_{r=0}^{n} b_r \times c_{n-r}$, then

generating function $\{a_n\} = $ generating function $\{b_n\} \times$ generating function $\{c_n\}$.

So, in this case,

generating function $\{1\} = $ generating function $\left\{\dfrac{1}{n!}\right\} \times$ generating function $\{p(n,0)\}$,

giving $1/(1-x) = e^x G(x)$, using our earlier notation. So $G(x) = e^{-x}/(1-x)$, as before.

This problem can appear in a variety of different guises, for example the secret Santa problem or a boy putting n letters to n girlfriends in n envelopes.

*This article is an adapted version of the Extended Essay that **Fei Yao** an 18-year-old Atlantic College student from China wrote as part of her assessment for the International Baccalaureate. **Paul Belcher** was her Extended Essay tutor.*

Throwing a die

In seven throws of a fair die, what is the probability that there will be exactly three 2s?

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Guido Lasters

How many solutions?

How many solutions does the equation

$$x_1 + x_2 + \cdots + x_{100} = 2006$$

have in positive integers?

Mathematics of Sudoku I

BERTRAM FELGENHAUER and FRAZER JARVIS

1. Introduction

Sudoku puzzles have been extremely popular in Britain since late 2004. *Sudoku*, or *Su Doku*, is a Japanese word (or phrase) meaning something like *Number Place*. The idea of the puzzle is extremely simple; the solver is faced with a 9×9 grid, divided into nine 3×3 blocks.
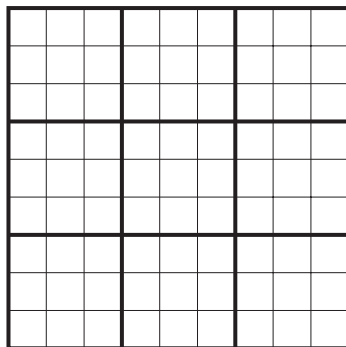
In some of these boxes, the setter puts some of the digits $1, \ldots, 9$; the aim of the solver is to complete the grid by filling in a digit in every box in such a way that each row, each column, and each 3×3 box contains each of the digits $1, \ldots, 9$ exactly once.

It is a very natural question to ask how many Sudoku grids there can be. That is, in how many ways can we fill in the grid above so that each row, column, and box contains the digits $1, \ldots, 9$ exactly once.

In this article, we explain how we first computed this number. This was the first computation of this number; we should point out that it has been subsequently confirmed using other (faster!) methods.

First, let us notice that Sudoku grids are simply special cases of Latin squares; recall that a *Latin square of order n* is an $n \times n$ square containing each of the digits $1, \ldots, n$ in every row and column. The calculation of the number of Latin squares is itself a difficult problem, with no general formula known. The numbers of Latin squares of sizes up to 11×11 have been worked out (see references 1, 2, and 3 for the 9×9, 10×10, and 11×11 squares), and the methods are broadly brute-force calculations, much like the approach we sketch for Sudoku grids below. (Brute-force calculations are those where part of the calculation involves constructing all possible answers, and seeing which ones really do work.) It is known that the number of 9×9 Latin squares is $5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \approx 5.525 \times 10^{27}$. Since this answer is enormous, we are going to have to be clever about how we do the brute-force counting, in order to be able to get an answer in a sensible amount of computing time.

2. Initial observations

Our aim is to compute the number N_0 of valid Sudoku grids. In the discussion below, we will refer to the blocks as B1, ..., B9, where these are labelled as follows.

	B1			B2			B3	
	B4			B5			B6	
	B7			B8			B9	

We first observe that we can simplify the counting we need to do by *re-labelling*. We could, for example, exchange all of the 1s and 2s in a valid Sudoku grid, and get another valid grid. We call this *re-labelling* the 1s and 2s. If we like, we can re-label all of the digits; in particular, we can re-label any grid so that the top left block (B1) is as follows.

1	2	3
4	5	6
7	8	9

We will call a grid with this top-left-hand block a grid in *standard form*. This re-labelling procedure reduces the number of grids by a factor of $9! = 362\,880$. The problem is reduced to counting the number $N_1 = N_0/9!$ of Sudoku grids in this standard form.

Our strategy is to consider all possible ways to fill in blocks B2 and B3, given that B1 is in the standard form above. Then, for each of these possibilities for blocks B1, B2, and B3, we work out all possible complete grids with this given collection of blocks.

We will see that it happens that, given blocks B2 and B3, there are other possibilities, $B2'$ and $B3'$ say, such that the number of ways of completing blocks B1, B2, and B3 to full grids is *the same* as the number of ways of completing blocks B1, $B2'$, and $B3'$. This means that we only need to count the number of ways to complete B1, B2, and B3 to a full grid, and we *do not need* to count the number of ways to complete B1, $B2'$, and $B3'$, as the answer will be the same. In fact, we will find that we only need to work out the number of ways of completing a grid for remarkably few pairs B2 and B3.

3. Blocks B2 and B3

Here we try to catalogue efficiently the possibilities for blocks B2 and B3. We first work out how many possibilities there are for B2 and B3, and then explain how we can reduce the size of the list we need to count.

3.1. Top row of blocks

We want to list all the possible configurations for the top three blocks, given that the first block is of standard form. We now think about the top row of the second block. Either it consists (in some order) of the three numbers on row 2 of block B1, or on row 3 of block B1, or a mixture of the two. The possible top rows of blocks B2 and B3 are given by

$$
\begin{array}{ll}
\{4, 5, 6\} \,|\, \{7, 8, 9\}, & \{7, 8, 9\} \,|\, \{4, 5, 6\}, \\
\{4, 5, 7\} \,|\, \{6, 8, 9\}, & \{6, 8, 9\} \,|\, \{4, 5, 7\}, \\
\{4, 5, 8\} \,|\, \{6, 7, 9\}, & \{6, 7, 9\} \,|\, \{4, 5, 8\}, \\
\{4, 5, 9\} \,|\, \{6, 7, 8\}, & \{6, 7, 8\} \,|\, \{4, 5, 9\}, \\
\{4, 6, 7\} \,|\, \{5, 8, 9\}, & \{5, 8, 9\} \,|\, \{4, 6, 7\}, \\
\{4, 6, 8\} \,|\, \{5, 7, 9\}, & \{5, 7, 9\} \,|\, \{4, 6, 8\}, \\
\{4, 6, 9\} \,|\, \{5, 7, 8\}, & \{5, 7, 8\} \,|\, \{4, 6, 9\}, \\
\{5, 6, 7\} \,|\, \{4, 8, 9\}, & \{4, 8, 9\} \,|\, \{5, 6, 7\}, \\
\{5, 6, 8\} \,|\, \{4, 7, 9\}, & \{4, 7, 9\} \,|\, \{5, 6, 8\}, \\
\{5, 6, 9\} \,|\, \{4, 7, 8\}, & \{4, 7, 8\} \,|\, \{5, 6, 9\},
\end{array}
$$

where $\{a, b, c\}$ indicates the numbers a, b, and c *in any order*.

The top row $\{4, 5, 6\} \,|\, \{7, 8, 9\}$ can be completed as follows.

1	2	3	{4,	5,	6}	{7,	8,	9}
4	5	6	{7,	8,	9}	{1,	2,	3}
7	8	9	{1,	2,	3}	{4,	5,	6}

This gives $(3!)^6$ possible configurations (each set of three numbers can be written in $3! = 6$ different ways). The same is true for its reversal $\{7, 8, 9\} \,|\, \{4, 5, 6\}$. However, the other 18 possibilities behave differently; here, the top row of B2 consists of a mixture of some of the second row and some of the third row of B1. For example, the top row $\{4, 5, 7\} \,|\, \{6, 8, 9\}$ can be completed as follows.

1	2	3	{4,	5,	7}	{6,	8,	9}
4	5	6	{8,	9,	a}	{7,	b,	c}
7	8	9	{6,	b,	c}	{4,	5,	a}

Here, a, b, and c stand for 1, 2, and 3, in some order, giving $3 \times (3!)^6$ possible configurations (b and c are interchangeable).

In total, we therefore have

$$
2 \times (3!)^6 + 18 \times 3 \times (3!)^6 = 56 \times (3!)^6 = 2\,612\,736
$$

possible completions of the top three rows.

Note that this means that the number of possibilities for the top three rows of a Sudoku grid is $9! \times 2\,612\,736 = 948\,109\,639\,680$.

3.2. Reduction

At this stage, we have a list of all possibilities for blocks B2 and B3. For each of these possibilities, we will try to fill in the remaining blocks to form valid Sudoku grids (the 'brute force' part of the method). However, to run through all 2 612 736 possibilities for B2 and B3 would be extremely time consuming. We need some way to reduce the number of possibilities which we need to consider. We will identify configurations of numbers in these blocks which give the same number of ways of completing to a full grid.

Luckily, there are a lot of things that we can do to the top three rows which preserve the number of completions to a full grid. We have already seen the re-labelling operation. But there are others; for example, if we exchange B2 and B3, then every way of completing B2 and B3 to a complete grid gives us a unique way to complete B3 and B2 to a complete grid (just exchange B5 and B6, and B8 and B9). Indeed, we can permute B1, B2, and B3 in any way we choose. Although this changes B1, we can then re-label to put B1 back into standard form.

Furthermore, we can permute the columns within any block in any way we wish, performing the same operation to the columns in a completed grid. We can even permute the three rows of B1, B2, and B3, and again re-label to put B1 into standard form.

3.2.1. Lexicographical reduction

Taking all of the 2 612 736 possibilities mentioned above, we catalogue them first as follows.

1. We begin by permuting the columns within B2 and B3 so that the top entries are in increasing order.

2. We then exchange B2 and B3 if necessary, so that the top-left-hand entry of B2 is smaller than that of B3.

The first step gives six ways to permute the columns in each block B2 and B3, so that, given any grid, there are $6^2 = 36$ grids derived from it with the same number of ways of completing; then the second essentially doubles this number. Overall, we are reducing the number of possibilities we need to consider by a factor of 72, giving 36 288 possibilities for our catalogue. This is becoming more practicable, although more reductions are desirable.

3.2.2. Permutation reduction

In fact, we haven't really made full use of all of the permutation and re-labelling possibilities. As mentioned above, for each of the 36 288 possibilities, we can consider all six permutations of the *three* blocks B1, B2, and B3, and all six permutations of columns within each block, making $6^4 = 1296$ possibilities in total. Having done this, our first block will not be in standard form, but we can re-label so that it is (re-labelling B2 and B3 in the same way), and then use the lexicographical reduction on the result. Each of the 1 296 permutations gives a new pair, B2 and B3, which has the same number of completions to a full grid. This provides a huge improvement again: computer calculations showed that this reduced the size of the list we need to test to just 2 051 possible pairs, B2 and B3. (The huge majority of these 2 051 pairs arise from exactly $18 = 6^4/72$ of the 36 288 possibilities. Some, however, arise from fewer, so it is necessary to store exactly how many of the 36 288 possibilities give rise to the given blocks.)

But this is not all – we can do the same for the six permutations of the three *rows* of the configuration. That is, we can choose any permutation of these rows, and then re-label to put

B1 back into standard form. It turns out that this gives a further reduction to testing just 416 possibilities for blocks B2 and B3.

We illustrate the procedure with an example. The top three rows might look something like the following.

1	2	3	4	5	8	6	7	9
4	5	6	1	7	9	2	3	8
7	8	9	2	3	6	1	4	5

Now, we permute the columns of B1, by exchanging the first two columns, say. (It is similarly possible to permute the three rows of all three boxes simultaneously, or to change the order of the boxes.) This gives the following grid.

2	1	3	4	5	8	6	7	9
5	4	6	1	7	9	2	3	8
8	7	9	2	3	6	1	4	5

Of course, the first block is no longer in standard form. But we can put it back in standard form by re-labelling; that is, we re-label 1 as 2 and vice versa, 4 as 5 and vice versa, and 7 as 8 and vice versa.

1	2	3	5	4	7	6	8	9
4	5	6	2	8	9	1	3	7
7	8	9	1	3	6	2	5	4

Now the first block is back in standard form, but blocks B2 and B3 aren't lexicographically reduced. We therefore sort the columns of B2 and B3, and then exchange B2 and B3 if necessary.

1	2	3	4	5	7	6	8	9
4	5	6	8	2	9	1	3	7
7	8	9	3	1	6	2	5	4

This configuration has the same number of completions as the one we started with, because all operations we applied leave that number invariant. This means that we can consider the two as equivalent for the purpose of the enumeration.

3.2.3. Column reduction While this improvement is extremely useful, the calculation will be even faster if we can cut down our list further. Luckily, there are still more possibilities for improving our list. Here is a possible arrangement for the top three rows in a Sudoku grid.

1	2	3	4	5	8	6	7	9
4	5	6	1	7	9	2	3	8
7	8	9	2	3	6	1	4	5

Consider the positions marked for the numbers 8 and 9.

1	2	3	4	5	(8)	6	7	(9)
4	5	6	1	7	(9)	2	3	(8)
7	8	9	2	3	6	1	4	5

It is easy to see that any grid completing these three blocks also completes the following grid.

1	2	3	4	5	(9)	6	7	(8)
4	5	6	1	7	(8)	2	3	(9)
7	8	9	2	3	6	1	4	5

It follows that this has the same number of completions to a full grid as the original arrangement did. Consequently, the number of ways to extend the original three rows is the same as the number of ways of extending these three rows, and so we should only compute this once. Note that the same can be done for the pairs of numbers $(1,2)$ in columns 4 and 7, $(1,4)$ in columns 1 and 4, $(5,8)$ in columns 2 and 9, and also $(6,9)$ in columns 3 and 6. (It is necessary to re-label to get B1 back into standard form in all but the first of these examples.) We require that there is a 2×2 subrectangle in the grid with the same entries on the bottom row as on the top (but in the other order, of course).

An extension of this method allows us to identify any two configurations (the first with a subrectangle of size $k \times 2$ whose entries consist of two columns, and the second with a subrectangle of size $2 \times k$ whose entries consist of two rows), with the same numbers. Using this trick with just the 2×2 subrectangles reduced the list of 416 to 174. Using 2×3, 3×2, and 4×2 rectangles as well reduced this list to just 71.

In fact, all that we really need is that the *columns* of two configurations consist of the same numbers, in any order, to guarantee that two configurations can be completed in the same number of ways. The first arrangement below has the same number of completions to a full grid as the second arrangement. (The second arrangement needs to be re-labelled to put it back into standard form, and then lexicographically reduced.)

(1)	(2)	3	(4)	6	7	(5)	8	9
(4)	(5)	6	8	(1)	9	3	(2)	7
7	8	9	(2)	(5)	3	(1)	(4)	6

(4)	(5)	3	(2)	6	7	(1)	8	9
(1)	(2)	6	8	(5)	9	3	(4)	7
7	8	9	(4)	(1)	3	(5)	(2)	6

In fact, we already had sufficiently few configurations that we didn't need to implement these further reductions. But, having made an exhaustive search of ways to complete our 71 representatives, we found that there were 44 distinct answers. Subsequent work by Ed Russell showed that these further reductions really do reduce the 71 classes to 44 classes.

3.3. Summary

At this stage, we have explained that every one of the original possibilities for blocks B1, B2, and B3 is equivalent (in the sense of having the same number of completions to a full grid) to one of a collection $\mathcal{C}$ of just 44. We just need to count the number of ways each of these 44 can be completed to a full grid. If C is one of the 44 possibilities, we will need to know the number of ways n_C that we can complete C to a full grid, and also the number of possibilities m_C for B1, B2, and B3 which are equivalent to C. Then N_0, the total number of Sudoku grids, will be the sum of all of the $m_C n_C$ for C in $\mathcal{C}$.

4. The brute-force counting

The brute force part of the calculation comes next; for each of the possible 44 configurations of B2 and B3, we have to find how many ways we can complete B1, B2, and B3 to a full grid. Essentially, we just try all possible ways to fill the remaining boxes and see which are Sudoku grids, but we can again make our lives a little easier by insisting that the first columns of blocks B4 and B7 are lexicographically ordered (that is, by permuting the middle three rows we can assume that the entries on the left-hand side of B4 are in numerical order, and similarly for B7; we can also exchange B4 and B7). This speeds up our calculation by a factor of 72 once again.

   We could have reduced the search a little further by doing more of the reduction steps given above. However, the predicted running time at this point was sufficiently low that the possible benefit gained by going through a catalogue of possibilities for all of blocks B4 and B7, reduced by some of the methods listed above, was hardly worth implementing.

   Recall that at this stage, the top three blocks B1, B2, and B3, and also the first columns of blocks B4 and B7 are filled. Now we just try each of the possible ways to complete this information to a full Sudoku grid, and count how many of them work. This was programmed very efficiently by the first author using a *backtracking algorithm*. This proved to be a very efficient method, exhausting the possibilities for a given configuration of blocks B2 and B3 to just under two minutes on a single PC.

4.1. The result

At this stage, we have a list of 44 configurations for the top three rows. We know that *every* possibility for the top three rows can be completed to a full grid in the same number of ways as one of these 44. For example, we have computed that 178 848 of the 2 612 736 possibilities for the top three rows in standard form can be completed to a full grid in the same number of ways as the following grid.

1	2	3	4	7	8	5	6	9
4	5	6	1	3	9	2	7	8
7	8	9	2	5	6	1	3	4

   We have also computed that this particular choice of top three rows can be completed to a full grid in $72 \times 97\,961\,464 = 7\,053\,225\,408$ ways. The same calculations have been made for all of the 44 possibilities on our list. This then works out, all at once, how many ways there are to complete 178 848 of the 2 612 736 possibilities for the top rows. We do the same for the other 43, to find exactly how many ways we can complete all of the top three rows to a full

grid. This computes the number of grids with B1 in standard form; we need to multiply by $9! = 362\,880$ to get the total number.

In total, there are $N_0 = 6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \times 10^{21}$ valid Sudoku grids. Taking out the factors of $9!$ and 72^2, coming from re-labelling and the lexicographical reduction of the top row of blocks B2 and B3, and of the left column of blocks B4 and B7, this leaves $3\,546\,146\,300\,288 = 2^7 \times 27\,704\,267\,971$ arrangements, the last factor being prime. Subsequently, Ed Russell verified the result; it has now been verified by several other people as well. Although our original method requires a couple of hours of computer time, Guenter Stertenbrink and Kjell Fredrik Pettersen have subsequently developed a method which completes the entire calculation in less than a second!

Appendix. A heuristic argument

Here is a simple heuristic argument which happens to give almost exactly the right answer. It is due to Kevin Kilfoil.

Firstly, it is clear that there are $N = (9!)^9$ ways to fill in each of the blocks B1, ..., B9 in such a way that each block has the digits $1, \ldots, 9$.

We also know the number of ways to fill in the three blocks B1, B2, and B3 so that each block has the digits $1, \ldots, 9$ and also each row has the digits $1, \ldots, 9$; we calculated this above as $948\,109\,639\,680$. The same will be true for blocks B4, B5, and B6 and B7, B8, and B9. So the number of ways to fill in each of the blocks B1, ..., B9 in such a way that each block has digits $1, \ldots, 9$, and each row has digits $1, \ldots, 9$, is $948\,109\,639\,680^3$.

It follows that the proportion of all of the N possibilities which also satisfy the row property is

$$k = \frac{948\,109\,639\,680^3}{(9!)^9}.$$

This will also be the proportion of the N grids satisfying the column property. A Sudoku grid is just one of the N grids that has both the row and column property. *Assuming that these are independent*, this would give the total number of Sudoku grids as

$$Nk^2 = \frac{948\,109\,639\,680^6}{(9!)^9} \approx 6.6571 \times 10^{21}.$$

In fact, this answer cannot be correct (it is not even an integer), and the problem is that the row and column probabilities are *not* quite independent. However, this prediction is really very close to the actual answer we found above: the difference from our exact value is just 0.2%.

The programs and data are available at http://www.afjarvis.staff.shef.ac.uk/sudoku/.

References

1 S. F. Bammel and J. Rothstein, The number of 9×9 Latin squares, *Discrete Math.* **11** (1975), pp. 93–95.
2 B. D. McKay and E. Rogoyski, Latin squares of order 10, *Electron. J. Combin.* **2** (1995), Note 3, 4pp.
3 B. D. McKay and I. M. Wanless, The number of Latin squares of order eleven, *Ann. Combin.* **9** (2005), pp. 335–344.

*The authors began this collaboration through the Mathematics of Sudoku thread on the forum at http://www.sudoku.com/. At the time this work was done, **Bertram Felgenhauer** was a student at the Technical University of Dresden in Germany, where he was studying computer science. **Frazer Jarvis** is a member of the Department of Pure Mathematics at the University of Sheffield; he usually works on problems of a more number-theoretic nature.*

Modelling SARS

J. GANI

The mathematical modelling of an epidemic can be very useful: we can make predictions about the spread of the disease and, in some cases, use the model to develop a strategy for containing it, whether by quarantine, immunisation, or some other method. The case of SARS (Severe Acute Respiratory Syndrome), which developed mostly but not exclusively in Southeast Asia (Hanoi, Singapore, Hong Kong, and mainland China, as well as Toronto) in 2003, is an example of containment by quarantine. Patients diagnosed with SARS were promptly isolated in hospitals, and their health carers were also quarantined once it was discovered that they were potential carriers of the disease. In this way, SARS was rapidly brought under control. A simple model of the spread of the disease is presented, mainly as an illustration of epidemic modelling, without any claim that it is a close reflection of reality.

On 31 July 2003, the World Health Organization (WHO) stated in their Report (see reference 1) that there had been 8098 cases of SARS worldwide, of which 774 had been fatal. Because of the enforced quarantine measures, we may regard the initial number of susceptibles to have been 8097, with a single initial infective, so that the total population involved is $N = 8098$.

A simplistic model of the epidemic may be constructed as follows, with $x(t)$ susceptibles at time t, $x(0) = 8097$, $y(t)$ infectives, $y(0) = 1$, and $u(t)$ deaths, $u(0) = 0$, with time $t \geq 0$.

$x(t)$		$y(t)$		$u(t)$
Susceptibles		Infectives		Deceased
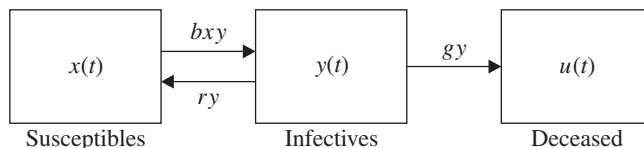
The system may be described deterministically by differential equations of the following type first proposed by Kermack and McKendrick in 1927 in their paper (reference 2) on the mathematical theory of epidemics:

$$\frac{dx}{dt} = -bxy + ry, \qquad \frac{dy}{dt} = bxy - (r+g)y, \qquad \frac{du}{dt} = gy, \tag{1}$$

where b, r, and g are the infection, recovery, and death rates respectively.

We may easily find on dividing dx/dt by du/dt that

$$\frac{dx}{du} = -\frac{b}{g}x + \frac{r}{g}.$$

The solution to this can be obtained by first writing

$$\frac{dx}{du} + \frac{b}{g}x = \frac{r}{g},$$

and using an integrating factor $e^{(b/g)u}$, so that

$$\frac{dx}{du}e^{(b/g)u} + \frac{b}{g}xe^{(b/g)u} = \frac{r}{g}e^{(b/g)u}.$$
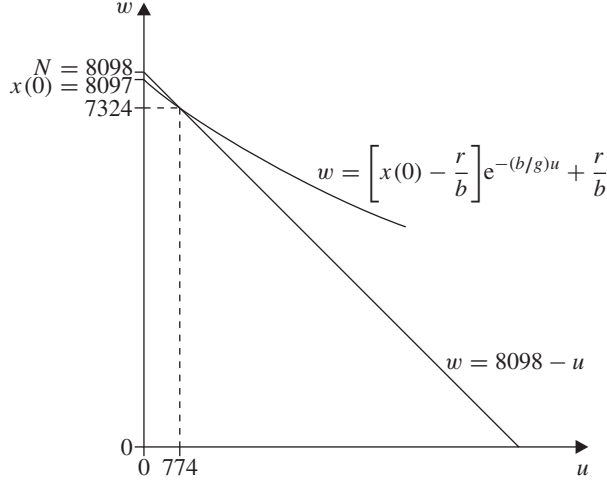
Figure 1 Plot of $w = x(u)$ and $w = N - u$.

We can see that this is in fact

$$\frac{\mathrm{d}}{\mathrm{d}u}[xe^{(b/g)u}] = \frac{r}{g}e^{(b/g)u}$$

which, on integration, yields the solution

$$xe^{(b/g)u} = \frac{r}{b}e^{(b/g)u} + C,$$

where C is a constant. Note that, when $t = 0$, u is also equal to 0, so that setting $t = 0$, we have

$$x(0) = \frac{r}{b} + C,$$

and, hence, x in terms of u is given by

$$x(u) = \left[x(0) - \frac{r}{b}\right]e^{-(b/g)u} + \frac{r}{b}.$$

Since the recovery rate may be assumed to be smaller than the infection rate, we have

$$1 > \frac{r}{b} > 0.$$

The graphs of $w = x(u)$ and $w = N - u$, where $N = 8098$, are plotted in figure 1. We shall see that the final value of $x(t)$ as $t \to \infty$ is given by the intersection of $w = x(u)$ and $w = N - u$.

From (1), recalling that $x(t) + y(t) + u(t) = N = 8098$, we may rewrite $\mathrm{d}u/\mathrm{d}t$ as

$$\frac{\mathrm{d}u}{8098 - u - x} = g \, \mathrm{d}t = \frac{\mathrm{d}u}{8098 - u - [x(0) - r/b]e^{-(b/g)u} - r/b},$$

so that integrating over t, it is clear that

$$gt = \int_0^{u(t)} \frac{\mathrm{d}u}{8098 - u - [x(0) - r/b]e^{-(b/g)u} - r/b}.$$
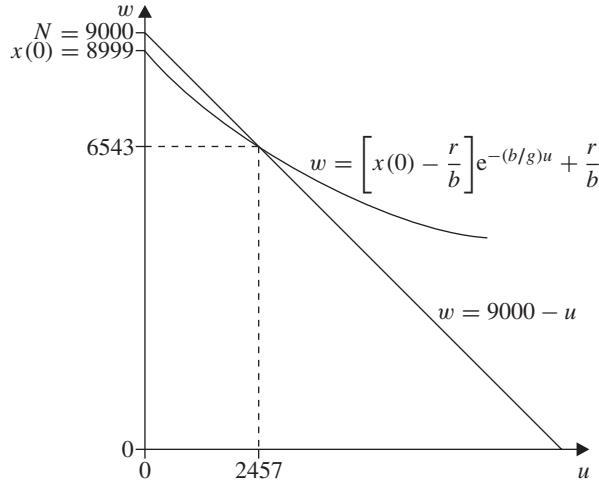
Figure 2 Plot of $w = x(u)$ and $w = N - u$.

When $t = 0$, $u(0) = 0$; as $t \to \infty$, since $u(\infty)$ is finite and cannot exceed $N = 8098$, the denominator must tend to 0, so that, for $u(\infty) = 774$,

$$8098 - 774 - \left[8097 - \frac{r}{b} \right] e^{-(b/g)774} - \frac{r}{b} = 0,$$

or

$$7324 - 8097 e^{-(b/g)774} = \frac{r}{b}[1 - e^{-(b/g)774}].$$

Since $0 < r/b < 1$, we can readily see that

$$1 > 7324 - 8097 e^{-(b/g)774} > 0,$$

so that

$$7324 > 8097 e^{-(b/g)774} > 7323.$$

Taking logarithms, we find that

$$\ln \frac{7324}{8097} = -0.10033699 > -774 \frac{b}{g} > -0.10047354 = \ln \frac{7323}{8097},$$

so that b/g is approximately $0.000\,129\,72$. The estimate of this ratio of parameters would enable us to model future SARS epidemics if and when they occur.

Thus, on the basis of this model, and assuming conditions remain unchanged in the future, we might wish to predict the number of deaths to be expected in a SARS epidemic where the total population infected is $N = 9000$ with one initial infective. We now need to solve the approximate equation (see figure 2)

$$9000 - u(\infty) = 8999 e^{-0.00012972u(\infty)},$$

and find that the number of deaths expected is $u(\infty) = 2457$ or 27.3% of those infected, as against 9.56% in the actual 2003 epidemic. We note that the number of deaths appears much

larger than is reasonable, an indication that our oversimplified model is inadequate, except as an illustrative example. The actual models used for SARS, which are far more complicated, subdivide the population at risk into several categories and yield more realistic results.

Further details of the SARS epidemic of 2003 may be found in accessible articles in *Significance* (see reference 3) and *Science* (see reference 4).

References

1 World Health Organization, Cumulative number of reported probable cases of SARS (World Health Organization, Geneva, 2003).

2 W. O. Kermack and A. G. McKendrick, A contribution to the mathematical theory of epidemics, *Proc. R. Soc. London A* **115** (1927), pp. 700–721.

3 C. Donnelly and A. Ghani, Real-time epidemiology: understanding the spread of SARS, *Significance* **1** (2004), pp. 176–179.

4 S. Riley, C. Fraser, C. A. Donnelly, A. C. Ghani, L. J. Abu-Raddad, A. J. Hedley, G. M. Leung, L.-M. Ho, T.-H. Lam, T. Q. Thach, P. Chau, K.-P. Chan, S.-V. Lo, P.-Y. Leung, T. Tsang, W. Ho, K.-H. Lee, E. M. C. Lau, N. M. Ferguson and R. M. Anderson, Transmission dynamics of the etiological agent of SARS in Hong Kong: impact of public health interventions, *Science* **300** (2003), pp. 1961–1966.

Joe Gani is a statistician who has worked for many years on the modelling of epidemics. He is now retired, but holds a Visiting Fellowship in the Mathematical Sciences Institute of the Australian National University in Canberra.

The largest possible number

James Whiteman spotted the following problem, which he sent to us.

Construct the largest possible number using only the eight symbols

$$1 \quad 2 \quad 3 \quad 4 \quad (\quad) \quad . \quad -$$

The numbers can only be used once, but the brackets, the decimal point, and the minus sign can be used any number of times.

Reference

1 www.pickover.com

The Areas Problem

ATARA SHRIKI

1. Introduction

Most of the mathematical problems that are presented in common textbooks are 'local' in the sense that they deal with specific cases. Many of these problems can be extended, and sometimes a general pattern of behaviour can be found. In such circumstances, the original problem becomes merely a point on a graph, and not the whole situation. Obviously, an isolated point on a graph cannot provide the entire picture. In this article, I would like to demonstrate this idea.

2. The areas problem

2.1. Introducing the areas problem

I was looking for an interesting calculus problem for one of my classes when, in a book by Honsberger (see reference 1), I found the following problem.

> Let P be a point on the graph $y = x^3$. The tangent at P crosses the curve at Q, and A is the area between the curve and the segment PQ. Similarly, the tangent at Q meets the curve again at R, and B is the area between the curve and QR. Prove that the measure of area B is always sixteen times greater than the measure of area A for every choice of the point P (see figure 1).
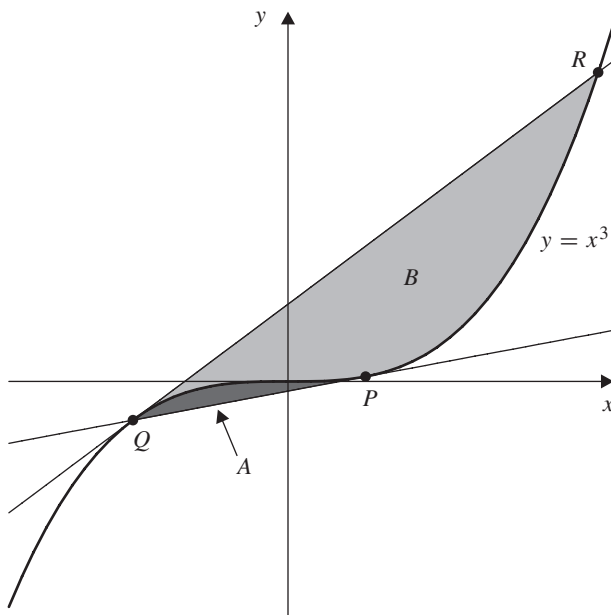
Figure 1

The problem seems to be very surprising and unexpected. I now present a proof of the areas problem.

The equation of a tangent PQ at $P(x_P, x_P^3)$ is $y = 3x_P^2 x - 2x_P^3$. The solution of $3x_P^2 x_Q - 2x_P^3 = x_Q^3$ is $x_Q = -2x_P$. Similarly, we obtain $x_R = -2x_Q = 4x_P$. The ratio of the measures of areas A and B is

$$\frac{m(B)}{m(A)} = \frac{\left| \int_{-2x_P}^{4x_P} (3x_Q^2 x - 2x_Q^3 - x^3) \, dx \right|}{\left| \int_{-2x_P}^{x_P} (3x_P^2 x - 2x_P^3 - x^3) \, dx \right|} = \frac{108x_P^4}{27x_P^4/4} = 16.$$

Thus the statement is proved.

From the above proof it can be seen that a constant ratio also exists between the x-rates of points P, Q, and R. This ratio is equal to -2.

2.2. Extending the areas problem

Looking at the results, I wondered whether there was more to the problem. I asked myself if there was anything special in the function $y = x^3$ and its graph, or could it be that the result was only one instance of a broader regularity that could be found for the family of functions $f(x) = x^n$, where n is an odd number greater than or equal to 3. Since the graphs of that family share many properties, and actually 'look alike', it seemed reasonable to me to wonder whether similar findings would emerge for other members of that family.

I began with the function $f(x) = x^5$. Since algebraic manipulation on this function is time consuming, I studied the cases in which $x_P = \pm 1, \pm 2, \pm 3$. I was glad to find that, in each case, the ratio between the x-rates of points P, Q, and R was identical ($-1.650\,62$), as well as the ratio between the measures of areas A and B ($20.224\,7$). I repeated the process for other members of the family, using the same specific values for x_P. For each member of the family, I obtained a constant ratio between the x-rates of points P, Q, and R (denoted by r_1) and a constant ratio between the measures of the areas (denoted by r_2). The results are summarized in table 1. At this phase I was quite confident that r_1 and r_2 were constants for each member of the family. Though I did not yet have a proof, I kept searching for some other numerical connections. Looking at table 1, I asked myself whether there might be any connection between r_1 and r_2, or between r_1 and n, or between r_2 and n? Except for the case in which $n = 3$, it was hard to see any connection. For $n = 3$, I could see that $r_1^4 = r_2$,

Table 1

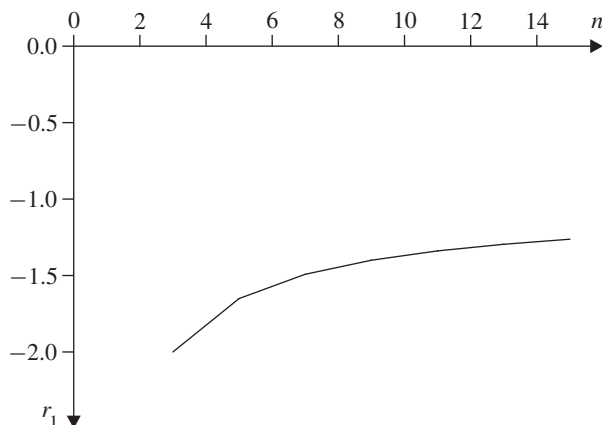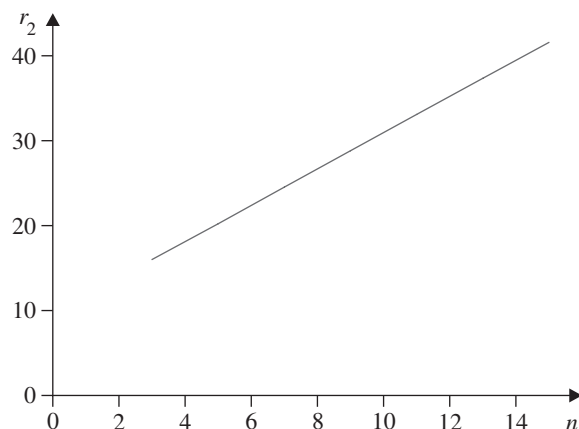| n	$r_1 = x_Q/x_P = x_R/x_Q$	$r_2 = m(B)/m(A)$	$r_2 = f(r_1)$
3	-2	16	$(-2)^4 = 16$
5	$-1.650\,62$	20.224 7	$(-1.650\,62)^6 = 20.224\,7$
7	$-1.491\,79$	24.527 9	$(-1.491\,79)^8 = 24.527\,9$
9	$-1.399\,5$	28.822 3	$(-1.399\,5)^{10} = 28.822\,3$
11	$-1.338\,59$	33.095 6	$(-1.338\,59)^{12} = 33.095\,6$
13	$-1.295\,09$	37.342 1	$(-1.295\,09)^{14} = 37.342\,1$
15	$-1.262\,34$	41.573 9	$(-1.262\,34)^{16} = 41.573\,9$

Figure 2

Figure 3

namely $(-2)^4 = 16$ (which is obviously not the only connection between the two numbers). I examined, analogously, this connection between r_1 and r_2 for $n = 5$. I was surprised to discover that indeed $r_1^6 = r_2$. As can be seen from the third column of table 1, for each n that was examined the connection is $r_2 = r_1^{n+1}$.

In order to determine whether there is any connection between r_1 and n or between r_2 and n, I drew two graphs using the points that are given in table 1, see figures 2 and 3. (Obviously the graphs are based on a finite number of points, but they are represented as continuous curves.) The linear function surprised me the most! I believe linear connections are always surprising. In order to find the exact algebraic pattern I used the computer software DERIVE®, which also enabled me to find the rate of deviation between the algebraic pattern and the exact values. The linear function which links r_2 and n is $y = 2.13436x - 9.58882$, with a minor deviation of 0.0231. The function that links r_1 and n is $y = -2.76039/x - 1.08819$, with a minor deviation of 0.00766.

I now present a proof of the extended areas problem.

The tangent to $f(x) = x^n$ at point $P(x_P, x_P^n)$ is $y = nx_P^{n-1}x - (n-1)x_P^n$. We have to solve the equation $x_Q^n = nx_P^{n-1}x_Q - (n-1)x_P^n$, or

$$(x_Q - x_P)(x_Q^{n-1} + x_Q^{n-2}x_P + x_Q^{n-3}x_P^2 + \cdots + x_Q x_P^{n-2} + x_P^{n-1}) = nx_P^{n-1}(x_Q - x_P).$$

Given that n is an odd number, this equation has a solution, x_Q, such that $x_Q \neq x_P$. Thus, the equation can be divided by $x_Q - x_P$. We obtain

$$x_Q^{n-1} + x_Q^{n-2}x_P + x_Q^{n-3}x_P^2 + \cdots + x_Q x_P^{n-2} + x_P^{n-1} = nx_P^{n-1}.$$

Since there is no general algorithm for solving such an equation, we will find an equivalent equation, in which the unknown is the ratio between x_Q and x_P. Knowing that $x_P \neq 0$, we obtain

$$\left(\frac{x_Q}{x_P}\right)^{n-1} + \left(\frac{x_Q}{x_P}\right)^{n-2} + \left(\frac{x_Q}{x_P}\right)^{n-3} + \cdots + \frac{x_Q}{x_P} + 1 = n.$$

This equation contains the ratio between the x-rate of points Q and P.

Let $t = x_Q/x_P$; thus we obtain

$$t^{n-1} + t^{n-2} + t^{n-3} + \cdots + t - (n-1) = 0.$$

This equation does not depend on x_P (the 'starting point') but on the ratio between the x-rate of points Q and P. Since for each graph from the family $f(x) = x^n$, $n \geq 3$ odd, all the tangents to the graph intersect it at two points (of which the tangency point is one), the equation has two real roots. One of the roots is obviously $t_1 = 1$ (because of the tangency point), the other root is $t_2 < 0$. The other roots are complex. Thus we obtain

$$(t - 1)(t^{n-2} + 2t^{n-3} + 3t^{n-4} + \cdots + (n-2)t + (n-1)) = 0.$$

In order to find the solution for $t \neq 1$, we have to solve

$$t^{n-2} + 2t^{n-3} + 3t^{n-4} + \cdots + (n-2)t + (n-1) = 0.$$

To find solutions for various values of n, $n \geq 3$ odd, I used DERIVE. The solutions obtained are exactly those in the second column of table 1.

It is left to find the ratio $m(B)/m(A)$ of the areas. According to the connection between the x-rates of points P and Q, we will find the pattern for the measure of area A. Using $x_Q = tx_P$, we obtain

$$\left| \int_{tx_P}^{x_P} (x^n - nx_P^{n-1}x + (n-1)x_P^n) \, dx \right|$$

$$= \left| x_P^{n+1} \left[(1-t) \left(\frac{1 + t + t^2 + \cdots + t^n}{n+1} - \frac{n}{2}(1+t) + n - 1 \right) \right] \right|.$$

The expression in the squared parenthesis depends on t and n. We shall denote it by $K(t, n)$. Thus

$$m(A) = |x_P^{n+1} K(t, n)| \quad \text{and} \quad m(B) = |x_Q^{n+1} K(t, n)|$$

implies that

$$\frac{m(B)}{m(A)} = \left(\frac{x_Q}{x_P}\right)^{n+1} = t^{n+1}.$$

Thus the findings described in table 1 are proved, and the extended areas problem is solved.

3. Summary

As was noted in Section 1, the connection that is presented in the original areas problem can be described as points on figures 2 and 3.

A careful look at textbooks can reveal a wonderful source of problems that can serve as a starting point for an elaborate inquiry.

Acknowledgement

Reference

1 R. Honsberger, *More Mathematical Morsels* (Mathematical Association of America, Washington, DC, 2001), pp. 114–118.

> ***Atara Shriki*** *teaches pre-service teachers in Oranim Academic College and works in professional development programmes for teachers in the Israel National Centre for Mathematics Education. She is interested in finding new rules and generalizations in algebra, calculus, and geometry.*

Mathematics in the Classroom

Maxima and minima through geometry

The main goal of mathematics competitions around the world is to present a good mathematical challenge for brilliant young minds. Problems based on concepts of inequalities are a vital source of this challenge. In this column, we provide examples of inequalities which succumb to a geometrical approach.

Problem 1 The real numbers x and y satisfy the condition

$$x^2 + y^2 - 2x - 6y - 6 = 0.$$

Find the maximum value of $4x + 3y$.

Solution The equation $x^2 + y^2 - 2x - 6y - 6 = 0$ represents a circle of radius 4 and centre, O, $(1, 3)$. Let k be a possible value of $4x + 3y$ for (x, y) on the circle. Then the straight line $4x + 3y = k$ must intersect the circle in at least one point. Hence, the required maximum value will occur when the straight line $4x + 3y = k$ is a tangent to the circle as in figure 1. The tangent has slope $-\frac{4}{3}$, so OP will have slope $\frac{3}{4}$, and P is the point $(1 + 4\cos\theta, 3 + 4\sin\theta)$, where $\tan\theta = \frac{3}{4}$. Hence, $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$, and P is the point $(\frac{21}{5}, \frac{27}{5})$. Hence, $k = 4(\frac{21}{5}), +3(\frac{27}{5}) = 33$ and this is the required maximum value.
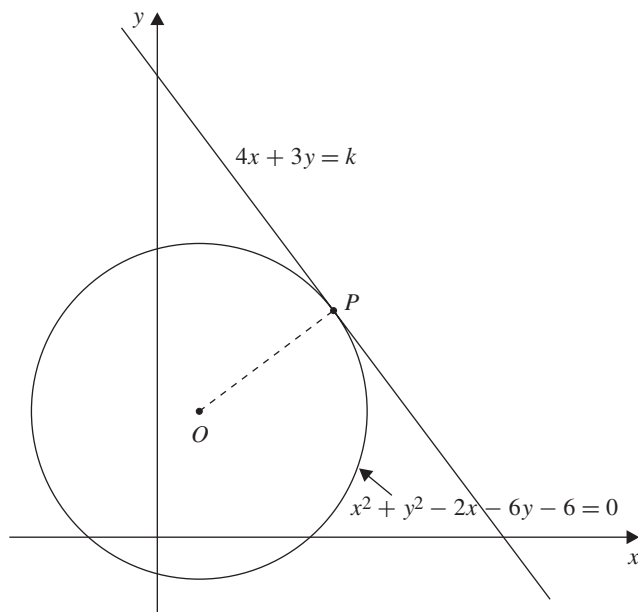
Figure 1
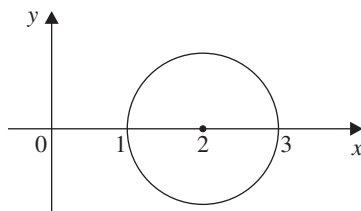
Figure 2

Problem 2 Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, what is the value of $M - m$?

Solution The given equation $(x - 2)^2 + y^2 = 1$ represents a circle of radius 1 with centre $(2, 0)$, and $x^2 + y^2$ is the square of distance of the point (x, y) from the origin (see figure 2).

For points on the circle, the maximum distance to the origin is from the point $(3, 0)$ and the minimum distance is from the point $(1, 0)$. Thus $M - m = 9 - 1 = 8$.

Problem 3 In any triangle ABC, determine the maximum value of

$$\sin^2 A + \sin B \sin C \cos A.$$

Solution This is a trigonometric problem but here we are interested in its solution using coordinate geometry. By using the sine and cosine formulae, we have

$$\sin^2 A + \sin B \sin C \cos A = \frac{a^2}{4R^2} + \frac{bc}{4R^2} \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 + b^2 + c^2}{8R^2},$$
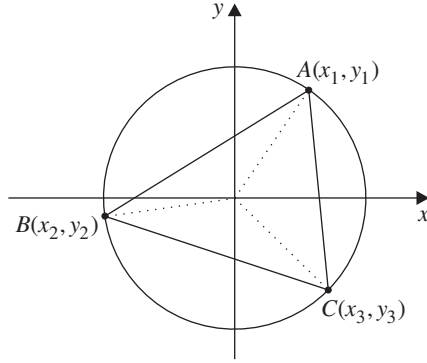
Figure 3

where R is the radius of the circumcircle. Let the vertices of triangle ABC be (x_1, y_1), (x_2, y_2), and (x_3, y_3), and let the equation of the circumcircle be $x^2 + y^2 = R^2$ (see figure 3).

Since all three vertices of the triangle ABC lie on the circle $x^2 + y^2 = R^2$, we have $x_i^2 + y_i^2 = R^2$ for $i = 1, 2, 3$. We now obtain

$$\frac{a^2 + b^2 + c^2}{8R^2}$$

$$= \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 + (x_3 - x_1)^2 + (y_3 - y_1)^2}{8R^2}$$

$$= \frac{3(x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2) - (x_1 + x_2 + x_3)^2 - (y_1 + y_2 + y_3)^2}{8R^2}$$

$$= \frac{9R^2 - (x_1 + x_2 + x_3)^2 - (y_1 + y_2 + y_3)^2}{8R^2}.$$

The maximum value of this expression is $\frac{9}{8}$, and it occurs when

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 0,$$

i.e. when the centroid of the triangle ABC,

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right),$$

coincides with its circumcentre $(0, 0)$. This occurs if the triangle is equilateral.

Problem 4 For a real number x, find the maximum value of the function

$$f(x) = \sqrt{x^4 - 3x^2 - 6x + 13} - \sqrt{x^4 - x^2 + 1}.$$

Solution It could be difficult to solve this problem using calculus or algebra. We again use geometrical concepts for the solution. The given expression can be rewritten as

$$\sqrt{(x - 3)^2 + (x^2 - 2)^2} - \sqrt{x^2 + (x^2 - 1)^2}$$
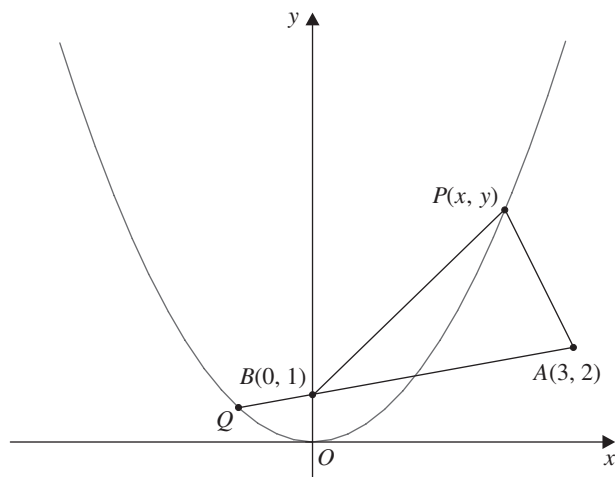
Figure 4

or

$$\sqrt{(x-3)^2 + (y-2)^2} - \sqrt{(x-0)^2 + (y-1)^2},$$

where $y = x^2$. Geometrically, we have to determine the maximum value of $PA - PB$, where A and B are the points $(3, 2)$ and $(0, 1)$ respectively, and $P(x, y)$ is any point on the parabola $y = x^2$ (see figure 4).

By the triangle inequality, we have

$$PA \le PB + AB$$

or

$$PA - PB \le AB = \sqrt{10}.$$

Moreover, if Q is the point where AB produced meets the parabola, we have

$$QA - QB = AB = \sqrt{10},$$

so the maximum value is $\sqrt{10}$.

References

1 A. Liu, *Chinese Mathematics Competitions and Olympiads* (Australian Mathematics Trust, 1981–1993).

2 G. Kessler and L. Zimmerman, *NYSML-ARML Contests* (National Council of Teachers of Mathematics, 1983–1988).

3 E. J. Barbeau, M. S. Klamlin and W. O. J. Moser, *Five Hundred Mathematical Challenges* (Mathematical Association of America, Washington, DC, 1995).

Ramanujan School of Mathematics,
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Letters to the Editor

Dear Editor,

A geometrical solution to the problem in Mathematical Spectrum,
Volume 38, Number 2, pp. 83–84
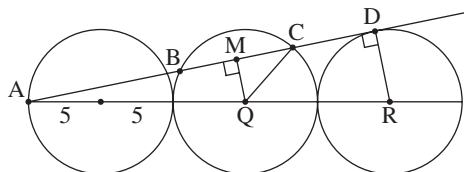
Let Q and R be the centres of the two right-most circles. The perpendicular from Q intersects AD at M, so that M is the midpoint of BC. Then $AR = 25$, $DR = 5$, and $AQ = 15$. Since $\triangle ARD$ and $\triangle AQM$ are similar, we have

$$\frac{AR}{AQ} = \frac{RD}{QM} \iff \frac{25}{15} = \frac{5}{QM} \iff QM = 3.$$

In the right-angled triangle $\triangle QMC$ we see that $QC = 5$ (radius of the circle) so, using Pythagoras' theorem, we obtain

$$QM^2 + MC^2 = QC^2 \iff 3^2 + MC^2 = 5^2 \iff MC = 4.$$

Since M is the midpoint of BC we must have $BC = 2 \times MC = 8$.

Yours sincerely,
Jens Carstensen
(Taarnby Gymnasium
Kastrup
Copenhagen
Denmark)

[Similar solutions have been sent in by John MacNeill and M. A. Khan – Ed.]

Dear Editor,

Another solution to the problem in Mathematical Spectrum,
Volume 38, Number 2, pp. 83–84

We have

$$\sin\theta = \frac{5}{25} = \frac{1}{5}, \qquad \cos\theta = \sqrt{1 - \frac{1}{25}} = \frac{2\sqrt{6}}{5}.$$

By the cosine formula for $\triangle EAB$ and $\triangle EAC$, x_1 and x_2 satisfy

$$5^2 = x^2 + 15^2 - 2 \times 15x \cos\theta,$$
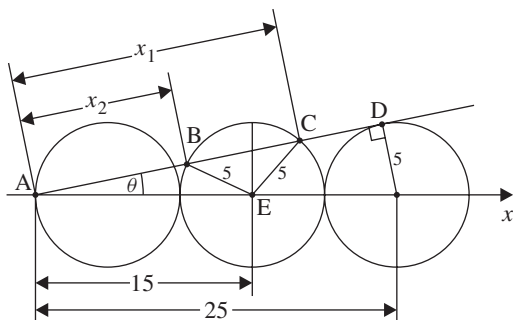
i.e.

$$x^2 - 12\sqrt{6}x + 200 = 0.$$

Hence, $x_1 + x_2 = 12\sqrt{6}$ and $x_1 x_2 = 200$, so that

$$(x_1 + x_2)^2 = 144 \times 6 = 864 \quad \text{and} \quad 4x_1 x_2 = 800.$$

Hence, $(x_1 - x_2)^2 = 64$, so that

$$x_1 - x_2 = 8 = \text{BC}.$$

<div align="center">
Yours sincerely,
Bob Bertuello
(12 Pinewood Road
Midsomer Norton
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UK)
</div>

Dear Editor,

<div align="center">

Gardiner's three circles problem

</div>

At the end of Carsten Ghedia's solution to Gardiner's three circles problem (*Math. Spectrum*, Volume 38, Number 2), which he solved by coordinate geometry, is a request for a solution by pure geometry. This is possible by similar triangles and Pythagoras' theorem.

Generalizing slightly to circles of radius r as in the figure, from the similar triangles AMF and ADG we have

$$\frac{\text{FM}}{\text{FA}} = \frac{\text{GD}}{\text{GA}},$$

so that

$$\text{FM} = \frac{3r}{5}.$$
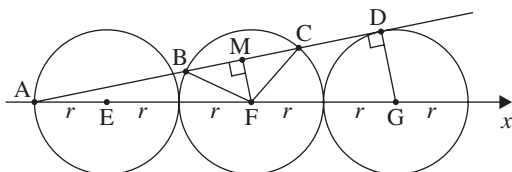
Hence, $BM = 4r/5$ from $\triangle BMF$ and $BC = 8r/5$, which reduces to 8 when $r = 5$.

The problem need not be restricted to three circles. With a chain of n touching circles, each or radius r, the chord of intersection on the mth circle has length

$$\frac{4r}{2n - 1}\sqrt{(n - m)(n + m - 1)}.$$

A further problem is to show that the distance between the mid-points of the chords of adjacent circles along the tangent is

$$\frac{4r}{2n - 1}\sqrt{n(n - 1)}.$$

The general problem can be extended to n touching circles of radii $r_1, r_2, \ldots, r_n$, provided that they are all intersected or touched by the defining tangent. This requires, for the mth circle,

$$\frac{d_m}{2S_m - r_m} = \frac{r_n}{2S_n - r_n},$$

where $d_m \leq r_m$ and $S_k = \sum_{i=1}^{k} r_i$. Hence,

$$r_m S_n \geq r_n S_m.$$

The length of the chord in the mth circle is

$$\frac{A}{2S_n - r_n}\sqrt{(r_m S_n - r_n S_m)(r_m S_n + r_n S_m - r_m r_n)}.$$

Yours sincerely,
Robert J. Clarke
(11 Lansdowne Court
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Stourbridge DY9 0RL
UK)

[Similar solutions have been sent in by Bruce Shawyer (Memorial University of Newfoundland, Canada), Nick Lord (Tonbridge School, Kent, UK), and Leung Chi Kit (The HKTA Ching Chung Secondary School, Kowloon, Hong Kong) – Ed.]

Dear Editor,

Sudoku possibilities

The standard Sudoku game, based on a version created by Euler, involves a 9×9 grid, subdivided into nine 3×3 boxes, with some of the cells already filled, using the numbers from 1 to 9. The object is to complete the grid so that each row, each column, and each 3×3 box contains all the numbers from 1 to 9. How many different ways are there of filling up a 9×9 grid satisfying these constraints? How many different games (trivial or not) can be extracted from any one solution by varying the filled-in cells? What would the answers be to the two above questions for the *general* Sudoku game based on the $n^2 \times n^2$ grid, subdivided into n^2 boxes, each of size $n \times n$, using the numbers 1 to n^2?

Yours sincerely,
Ian McPherson
(9 Gleneagles Gardens
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Dear Editor,

Abundant numbers

I would like to take further some points from Jay Schiffman's very stimulating article *Odd Abundant Numbers* (*Math. Spectrum*, Volume 37, Number 2). I first of all present a more general theorem. Recall that $\sigma(N)$ stands for the sum of the positive divisors of N.

Theorem *There are an infinite number of integers N satisfying $\sigma(N) > kN$, where the following conditions hold:*

1. *k is any real number greater than 1, in particular, if $k = 2$, then we say that such an N is abundant,*

2. *we may insist that N contains any defined set of prime factors raised to any defined powers,*

3. *we may insist that N does not contain any of another defined finite set of prime factors, where all the primes are distinct from those referred to in condition 2.*

Condition 3 may even be extended to many, but not all, infinite sets. For example, we may insist that all the prime factors of N leave a remainder of 1 when divided by 4.

In outline, the proof is as follows. First note that, if p is a prime number, then

$$\frac{\sigma(p^m)}{p^m} = \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^m}\right),$$

so if $N = \prod_{i=1}^{n} p_i^{m_i}$, where all p_i are prime, then

$$\frac{\sigma(N)}{N} = \prod_{i=1}^{n}\left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{m_i}}\right) \geq \prod_{i=1}^{n}\left(1 + \frac{1}{p_i}\right) > \sum_{i=1}^{n}\frac{1}{p_i}.$$

Now, $\sum_{i=1}^{\infty}(1/p_i)$, where p_i is now the ith prime number, is divergent. It follows that, since a divergent sequence is still divergent when a finite number of terms have been removed, the selections required by the theorem can be made.

If we take $k = 2$ and exclude the prime factor 2 we have Schiffman's Theorem 1, i.e. *there exist infinitely many odd abundant numbers*.

I now illustrate a practical use of this theorem by searching for odd abundant numbers containing factors 3^2, 5, and 7. First let $N = 3^2 \times 5 \times 7p = 315p$, where p is a prime greater than 7. We have

$$\frac{\sigma(N)}{N} = \frac{13}{9} \times \frac{6}{5} \times \frac{8}{7}\left(1 + \frac{1}{p}\right) = \frac{208}{105}\left(1 + \frac{1}{p}\right).$$

Then

$$\frac{\sigma(N)}{N} > 2 \iff 1 + \frac{1}{p} > \frac{210}{208} \iff p \leq 103.$$

In this case, any multiple of N is abundant. It is easy to show that $3^3 \times 5 \times 7$, $3^2 \times 5^2 \times 7$, and $3^2 \times 5 \times 7^2$ are abundant and, hence, any multiple of these is abundant.

Putting these results together, we get that $3^2 \times 5 \times 7(3 + 2n) = 945 + 630n$ is abundant when $3 + 2n$ has a prime factor less than or equal to 103. This will be the case if $3 \leq 3 + 2n < 107$, i.e. if $0 \leq n < 52$, but not if $n = 52$, as shown empirically by Schiffman.

Conjecture *There are infinitely many finite arithmetical sequences of abundant numbers containing as many terms as we please.*

This conjecture is suggested by the fact that we may find an N such that $\sigma(N)/N$ is as close to 2 as we please, but less than 2. Can any reader either prove or disprove this?

I now demonstrate how to find longer *finite* sequences of (odd) abundant numbers, and simultaneously find some of the odd abundant numbers with exactly five prime factors (referred to at the end of Schiffman's article). Let $N = 3 \times 5 \times 7 \times 11p$ ($p > 11$ and p prime) be an odd abundant number. Then we require $\sigma(N)/N > 2$, i.e.

$$\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \frac{12}{11}\left(1 + \frac{1}{p}\right) > 2,$$

which leads to $p < 384$. In fact, since the smallest prime number greater than or equal to 384 is 389, we may write $p < 389$.

Now let $N = 3 \times 5 \times 7 \times 11q$, where q is odd but not necessarily prime and $q < 389$. If q has a prime factor, p, greater than 11 (and clearly also less than 389) then N is a multiple of $3 \times 5 \times 7 \times 11p$, which we have already shown to be abundant. If q is a multiple of 11 then N is a multiple of $3 \times 5 \times 7 \times 11^2$. But this is abundant since

$$\frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \frac{11^3 - 1}{11^2(11 - 1)} = \frac{1216}{605} > 2,$$

and so N is abundant. Similarly, $3 \times 5 \times 7^2 \times 11$, $3 \times 5^2 \times 7 \times 11$, and $3^2 \times 5 \times 7 \times 11$ are also all abundant. Hence, if q has a factor 3, 5, 7, or 11 then N is a multiple of an abundant number, and so is abundant. As q is odd, we have now considered all possible prime factors of q less than 389, and have thus established that $N = 3 \times 5 \times 7 \times 11q$ is abundant for all q such that $3 \leq q \leq 387$. Letting $q = 2n + 1$, this is equivalent to

$$u_n = 1155 + 2310n \text{ is abundant, where } 1 < n < 193.$$

The argument can be extended further to obtain even longer sequences. For example, consider $N = 3 \times 5 \times 7 \times 11 \times 389q$. This time $3 \times 5 \times 7 \times 11 \times 389^2$ turns out to be deficient. But otherwise an argument following similar lines leads to N being abundant if $389 < q < 29959$. Letting $q = 2n + 389$, this is equivalent to

$$u_n = 174775755 + 898590n \text{ is abundant, where } 1 < n \leq 14784.$$

<div align="center">

Yours sincerely,
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</div>

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

39.1 For which natural numbers n can the set $\{1, 2, \ldots, n\}$ be partitioned into two subsets, A and B, such that the sum of the numbers in A is equal to the sum of the numbers in B?

(Submitted by H. A. Shah Ali, Tehran, Iran)

39.2 Solve the following equation:

$$2 \log_3 \cot x = \log_2 \cos x.$$

(Submitted by Abbas Roohol Aminy, Sirjan, Iran)

39.3 Two different natural numbers lie strictly between the same successive perfect squares. Prove that their product is not a perfect square.

(Submitted by Anand Kumar, Ramanujan School of Mathematics, Patna, India)

39.4 Let z_1, z_2, z_3, and z_4 be complex numbers whose sum is zero. Prove that

$$(z_1^3 + z_2^3 + z_3^3 + z_4^3)^2 = 9(z_2 z_3 - z_1 z_4)(z_1 z_3 - z_2 z_4)(z_1 z_2 - z_3 z_4)$$

and that

$$8|z_1^3 + z_2^3 + z_3^3 + z_4^3|^2 \leq 9(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)^3.$$

(Submitted by Mihály Bencze, Săcele-Négyfalu, Romania)

Solutions to Problems in Volume 38 Number 2

38.5 Determine all natural numbers n for which there is a permutation $(a_1, \ldots, a_n)$ of $1, \ldots, n$ such that $a_1 + \cdots + a_k$ is divisible by k for $k = 1, \ldots, n$.

Solution by H. A. Shah Ali, who proposed the problem

The number n must be odd because

$$a_1 + \cdots + a_n = 1 + \cdots + n = \tfrac{1}{2}n(n + 1),$$

which is only divisible by n when n is odd. We show that the only possible values of n are 1 and 3. When $n = 1$ the result is trivial; when $n = 3$ the permutation $(1\ 3\ 2)$ satisfies the condition. Now suppose that n is an odd number greater than 3; write it as $n = 2k + 1$ with $k \geq 2$. The number

$$a_1 + \cdots + a_{n-1} = \frac{n(n+1)}{2} - a_n$$

should be divisible by $n - 1 = 2k$. But $1 \leq a_n \leq n$, so that

$$\frac{n(n+1)}{2} - n \leq \frac{n(n+1)}{2} - a_n$$
$$\leq \frac{n(n+1)}{2} - 1,$$

i.e.

$$2k^2 + k \leq a_1 + \cdots + a_{n-1} \leq 2k^2 + 3k.$$

The only possibility is

$$a_1 + \cdots + a_{n-1} = 2k^2 + 2k,$$

so that

$$a_n = \frac{n(n+1)}{2} - (2k^2 + 2k)$$
$$= (2k+1)(k+1) - (2k^2 + 2k)$$
$$= k + 1$$
$$= \frac{n+1}{2}.$$

Also,

$$a_1 + \cdots + a_{n-2} = \frac{n(n+1)}{2} - a_n - a_{n-1}$$
$$= \frac{n^2 - 1}{2} - a_{n-1}$$
$$= \frac{n+1}{2}(n-2) + \frac{n+1}{2} - a_{n-1}$$

must be divisible by $n - 2$, so that

$$\frac{n+1}{2} - a_{n-1}$$

must be divisible by $n - 2$. But

$$0 < \left| \frac{n+1}{2} - a_{n-1} \right| \leq \frac{n-1}{2}$$

and

$$\frac{n-1}{2} < n - 2$$

when $n > 3$, which is impossible.

38.6 Solve the equation

$$(\sqrt{6} - \sqrt{5})^x + (\sqrt{3} - \sqrt{2})^x + (\sqrt{3} + \sqrt{2})^x + (\sqrt{6} + \sqrt{5})^x = 32.$$

Solution by Bar-Yann Chen, University of California, Irvine

Denote the left-hand side minus the right-hand side of the above equation by $f(x)$. Since

$$(\sqrt{3} - \sqrt{2})^{-1} = \sqrt{3} + \sqrt{2} \quad \text{and} \quad (\sqrt{6} - \sqrt{5})^{-1} = \sqrt{6} + \sqrt{5},$$

we have $f(x) = f(-x)$. Now,

$$\begin{aligned}
f'(x) &= (\sqrt{6} - \sqrt{5})^x \ln(\sqrt{6} - \sqrt{5}) + (\sqrt{3} - \sqrt{2})^x \ln(\sqrt{3} - \sqrt{2}) \\
&\quad + (\sqrt{6} + \sqrt{5})^x \ln(\sqrt{6} + \sqrt{5}) + (\sqrt{3} + \sqrt{2})^x \ln(\sqrt{3} + \sqrt{2}) \\
&= ((\sqrt{6} + \sqrt{5})^x - (\sqrt{6} - \sqrt{5})^x) \ln(\sqrt{6} + \sqrt{5}) \\
&\quad + ((\sqrt{3} + \sqrt{2})^x - (\sqrt{3} - \sqrt{2})^x) \ln(\sqrt{3} + \sqrt{2}) \\
&> 0, \quad \text{when } x > 0,
\end{aligned}$$

so $y = f(x)$ is an increasing function when $x > 0$ and is an even function (i.e. $f(x) = f(-x)$ for all x). Hence, it can cross the x-axis, at most, twice. But $f(2) = 0$, so there are precisely two solutions, $x = 2$ and $x = -2$.

A partial solution was also received from Bilesh Ladva, Hills Road Sixth Form College, Cambridge.

38.7 Let x, y, z be positive real numbers and let A, B, C be the angles of a triangle. Prove that

$$x^2 + y^2 + z^2 \geq 2yz \cos A + 2zx \cos B + 2xy \cos C.$$

Solution by Abbas Roohol Aminy, who proposed the problem
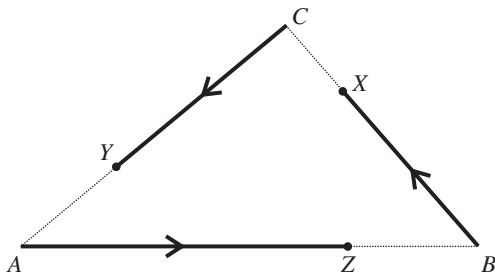
Consider the vectors $\overrightarrow{AZ}$, $\overrightarrow{BX}$, and $\overrightarrow{CY}$ in the directions of the sides of the triangle ABC as shown, of lengths z, x, and y respectively. Now,

$$(\overrightarrow{BX} + \overrightarrow{CY} + \overrightarrow{AZ})^2 \geq 0,$$

so that

$$x^2 + y^2 + z^2 + 2 \overrightarrow{CY} \cdot \overrightarrow{AZ} + 2 \overrightarrow{AZ} \cdot \overrightarrow{BX} + 2 \overrightarrow{BX} \cdot \overrightarrow{CY} \geq 0,$$

i.e.

$$x^2 + y^2 + z^2 + 2yz \cos(180° - A) + 2zx \cos(180° - B) + 2xy \cos(180° - C) \geq 0,$$

which gives the result.

38.8 A set of n points in the plane or in space is partitioned into two non-empty subsets. If P and Q are the mean points of these subsets, show that all such lines PQ are concurrent.

Solution by Guido Lasters, who proposed the problem

Let the points have vectors $x_1, \ldots, x_n$ relative to some origin and axes; let

$$P = \{x_{i_1}, \ldots, x_{i_k}\}, \qquad Q = \{x_{i_{k+1}}, \ldots, x_{i_n}\}.$$

The mean points of P and Q are

$$\frac{x_{i_1} + \cdots + x_{i_k}}{k} \quad \text{and} \quad \frac{x_{i_{k+1}} + \cdots + x_{i_n}}{n - k}$$

respectively, and PQ passes through the point

$$\frac{1}{n}\left(k\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) + (n - k)\left(\frac{x_{i_{k+1}} + \cdots + x_{i_n}}{n - k} \right) \right) = \frac{x_1 + \cdots + x_n}{n},$$

the mean point of the n points.

The National Lottery

In the UK National Lottery, the winning ticket each week consists of six different numbers chosen at random from the numbers 1 to 49.

$$\text{(10)} \ \text{(24)} \ \text{(27)} \ \text{(39)} \ \text{(40)} \ \text{(42)}$$

What is the probability that the winning numbers have greatest common factor greater than 1?

Reviews

Mathematics for Business Decisions: with Interdisciplinary Multimedia Projects, Part 1: Probability and Simulation. By Richard B. Thompson, business projects by Christopher G. Lamoureux and Pamela E. Slaten. MAA, Washington, DC, 2005. CD and paperback, 314 pages, $69.75 (ISBN 0-88385-742-1).

This pack contains a CD together with a student manual and is aimed at first- and second-year undergraduates studying for business and public administration degrees. The prerequisites are algebra and a basic familiarity with *Microsoft Windows* and *Microsoft Office* software. The CD contains an update of an earlier version of the program and so, although one of the business projects is new, others are not. This may explain the slightly dated appearance of some of the screens.

This program utilizes PowerPoint, Excel, and Word files which provide animation, video clips, and links to various websites, thus adding another dimension to the student manual, which mainly displays pages of screen captures. Although this manual can provide a useful study aid when a computer is not accessible, the author advises that this should not replace working with the e-text for the majority of the time. Most of the job of learning these days takes place at a computer, so developing this style of study is essential.

Standard basic probability topics are included; amongst these are queueing, Monte Carlo methods, and exponential growth. These topics are then applied to business projects in which opportunities are provided for students to download real data so that they can apply what they have learned to making business decisions. Specifically, pricing stock options are used in one project – such an example might not be so relevant to a non-USA student.

A variety of learning styles evidently contributes to any student's learning, and the use of IT is an essential skill for all business students. This pack goes some way to facilitating this, but the versatility of computer software has not been fully utilized in the presentation here and the user is left feeling that opportunities have been lost in the revision of this course.

Carol Nixon

USA and International Mathematical Olympiads 2004. Edited by Titu Andreescu, Zuming Feng and Po-Shen Loh. MAA, Washington, DC, 2005. Paperback, 100 pages, $34.95 (ISBN 0-88385-819-3).

Every year, the Mathematical Association of America publishes the problems used in the selection of members of the US team for the International Mathematical Olympiad (IMO) and the IMO questions themselves. Included is advice on how to tackle problems in general, hints for the specific problems, multiple solutions of all problems, and a glossary of the results needed. A valuable resource for all problem-solvers and would-be problem-solvers.

The University of Sheffield   **David Sharpe**

Supporting Assessment in Undergraduate Mathematics. Edited by Lynn Arthur Steen; Case Studies edited by Bonnie Gold, Laurie Hopkins, Dick Jardine and William A. Marion. MAA, Washington, DC, 2005. Paperback. 288 pages, $49.50 (ISBN 0-88385-820-7).

This is the second volume from the Mathematical Association of America on assessment in undergraduate mathematics. The first, *Assessment Practices in Undergraduate Mathematics*, appeared in 1999 and contained a collection of assessment practices that had been tried in a wide variety of schools. These included classroom exercises, groupwork, and assignments.

This later volume offers the findings of a four-year project designed to support mathematics departments in the challenge of assessing student learning.

Case studies cover a variety of courses that range from those involved with teacher training to those delivering to mathematics majors. For those starting on assessment programmes, the kinds of issues they will need to consider are addressed, potential pitfalls raised, and strategies for how these can be avoided discussed. A variety of models are explored in the hope that they can be adapted to an institutions' own needs.

The contributors to this volume have a conviction about the potential benefits of assessment in increased student learning. They hope that by documenting success stories there will be a breakthrough in the perceived resistance to assessment programmes. This book should contribute much to the achievement of this aim.

Carol Nixon

Mathematical Connections. A Companion for Teachers and Others. By Al Cuoco. MAA, Washington, DC, 2005. Hardback, 266 pages, $53.95 (ISBN 0-88385-739-1).

This book is designed to show teachers how mathematics is conceived. It does not intend to be a textbook for students, to follow a particular syllabus, to be a method text, or to follow the definition–theorem–proof–example style of writing. Due to the author's belief in the value of mathematical depth, he wishes to take a few simple themes and pursue them as far as possible.

The book comprises: an introduction, an annotated table of contents, six chapters, a bibliography, an index, and notes about the author. The six chapters are

1. Difference Tables and Polynomial Fits,

2. Form and Function; The Algebra of Polynomials,

3. Complex Numbers, Complex Maps and Trigonometry,

4. Combinations and Locks,

5. Sums of Powers.

The book is written for practising or prospective American high school teachers. The author has picked out ideas that are a joy to his mind and he hopes that the reader will also enjoy them. I think that he has been successful in this; I certainly enjoyed reading the book. I have the impression that the book was a labour of love for him. He wants to get the reader closely involved with the mathematics he is investigating. He also makes very good use of side notes for extra explanation, to make connections, and to evolve several ideas at the same time.

The prerequisites of mathematical knowledge for this book are not too demanding as the author develops what he needs as he goes along. Although the book is not intended for students, I feel that a good British sixth form student taking A-level mathematics or Higher Level in the International Baccalaureate would be able to cope well with reading and understanding this book. What they would need was a love for the subject. The reason for their understanding would be the author's gentle pace and full explanations, as he leads you along. He wants the reader to think about the reasons for taking the next step. The layout of the book is very clear and excellent use is made of tables and diagrams.

The first chapter, on difference tables, sets the tone for the book and its ideas and results are used in subsequent chapters. There are numerous problems at the end of each subsection for the reader to work through and then notes for selected problems at the end of each chapter.

I have the feeling that the author would have had a great reputation as a teacher. Judging from his book he would have the patience and willingness to devote the time to ensure that every single one of his students understood what he was explaining.

I would be happy to recommend this book to any young mathematician who wishes to see how mathematics is developed and to gain insight into connections and ways of thinking. It is not a book that is saying 'look how clever I am at producing these outstanding results which are presented in terse, concise notation' but it is saying 'come with me on a mathematical adventure and I promise I will not lose you along the way'.

Atlantic College **Paul Belcher**

When Computers were Human. By David Alan Grier. Princeton University Press, 2005. Paperback, 424 pages, £22.95 (ISBN 0-691-09157-9).

When Computers were Human is described in its inlay blurb as a 'sad but lyrical story of workers who gladly did the hard labour of research calculation in the hope that they might be part of the scientific community'. However, this does not entirely describe the scope of Grier's book. As an author of several previous historical pieces of mathematics, this may initially sound unrelated to the world of mathematics, but it turns out that there is more of a mathematical element to computing than we might appreciate. Before the electronic computers we have come to rely on, there were the human computers.

Grier starts by focusing on how astronomy kick-started the need for 'computers', and the initial nature of mathematics – a past-time of rich noblemen and, occasionally, noblewomen. He begins with Edmund Halley, whose comet sparked centuries of computational competition, everyone trying to find a calculation of the perihelion as close to the actual date as possible. Grier tells of three French mathematicians who divided work between them, Alexis Clairaut, Nicole-Reine Lepaute, and Joseph Lelande. These estimates were out by a large degree, yet for the time they were done this was still an achievement. The idea of computation spread to governments, who initiated the first computation groups for sets of tables for navigation – the *Nautical Almanacs*.

From navigation, computation is traced to industry. During the 19th century, there were further attempts at computing the perihelion of Halley's Comet, and with it Newton's three-bodied problem of the gravitational pull of two planets and the Sun. Grier follows the work over the Atlantic, settling in America for their attempts at *Nautical Almanacs*, and remains there for the majority of the rest of the book. Amongst the bustle of mathematical and computational activity, he picks up on the beginnings of machinery for computation, namely Charles Babbage's Difference Engine, and Herman Hollerith's initial tabulators. He also extends his text to the lives of several mathematicians, such as Babbage, but places particular emphasis on the women behind computation. Probably overlooked at the time, they provided important input to the work, and Grier recognizes their efforts.

Entering the 20th century, we are returned to the recurring problem of Halley's Comet and Newton's three-bodied problem. However, this quickly gives way to a new area of mathematics in the shadow of impending wars – ballistics. Most noticeably, the author follows the path of the Aberdeen Proving Ground in America, through near-closures and staff issues. Grier shows the full picture over just how many computing offices existed at this point – many major American universities by this time had their own. Increased methods of communication resulted in much more contact between mathematicians, and the sharing of problems between computing offices.

Of all the offices in existence, Grier chooses to focus on an unusual story, that of the Mathematical Tables Project, initially a result of relief work by the Work Projects Administration

(WPA). It emphasizes the change in the nature of computation, from a past time of noblemen and women to a way to create more jobs. Whilst the former had interest in and knowledge of mathematics, the latter largely did not. But one woman, whom Grier also emphasizes, took the project through many tough times from inside and outside. Gertrude Blanch is brought into the story and is rightly highlighted as an important figure in the decline of human computation. Finishing with the fall of the Mathematical Tables Project, Grier touches upon the development of electronic computers, and the unfortunate repercussion that the work of the original human computers is often overlooked.

Grier follows the history of computation in a narrative fashion, leading the reader through the various overlaps, but never heading off on a tangent, despite the many subtopics. From astronomy to statistics, naval to national, individuals to groups, he provides a deep, detailed history of the development of computation via those directly involved and those who were mostly spectators. It is written very much as you would expect a history book to be written, being factual and detailed. Yet Grier does not forget to delve into the lives of the mathematicians, astronomers, and engineers involved in the evolution of computation; instead, their lives are part of the history, providing reasons for what they achieved. As part of their lives, the state of society is also briefly highlighted, enough to make the effects of the culture of the time upon the scientist's work apparent. Grier undertook much research to discover the facts, demonstrated by the extensive bibliography and notes found at the back of the book.

Despite the impressive detail of Grier's work, there are a few areas to be resolved. As willing as he is to explain Babbage's Difference Engine, the same cannot be said for many of the techniques and methods mentioned. The reader is left wondering how computers such as those on the WPA's Mathematical Tables Project, who had little if any mathematical skill, could compute complicated sums. The text also is not always entirely engaging; if the reader has an interest in history, it would be a great read. However, if the reader does not take well to dates or lengthy backgrounds, this book could be daunting and long-winded. If you are intending to read *When Computers were Human* as part of a computer-related course, keep in mind that it is not intended as a book covering the modern use of the word 'computer' – references to this are brief – as it primarily provides the prehistory to their creation. Grier also focuses largely on the perspective of his home country, possibly forsaking the events and people of the remainder of the world.

Overall, this book is well worth the read, especially if you are interested in the origins of computers. Grier's narrative style is engaging, despite the occasional difficulties in reading due to the immense detail provided. By the end of the book you'll find yourself wanting to read it again, to remind yourself of the work before electronic computers, when computers were human.

Further mathematics student, Paston College, Norfolk **Vicky Bird**

Index to Mathematical Problems 1975–1979. Edited by Stanley Rabinowitz and Mark Bowron. MathPro Press, Chelmsford, MA, 1999. Hardback, 528 pages, $69.95 (ISBN 0-9626401-2-3).

This is a comprehensive volume of problems published in various journals and magazines, including *Mathematical Spectrum*, from 1975 to 1979, categorized by subject matter. A problem-solvers' heaven!

Mathematical Spectrum