

CruX Mathematicorum

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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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EDITORIAL

New Year, New Volume!

This Volume 44 is my first palindromic one of ***Cruz***, so I asked my dear friend Marco Buratti (who is symmetry-addicted and palindrome-obsessed) to provide me with some appropriate ways to wish ***Cruz*** readers Happy New Year 2018. He did not disappoint, so pick your favourite!

Happy New Year

$$8 + 2002 + 8$$

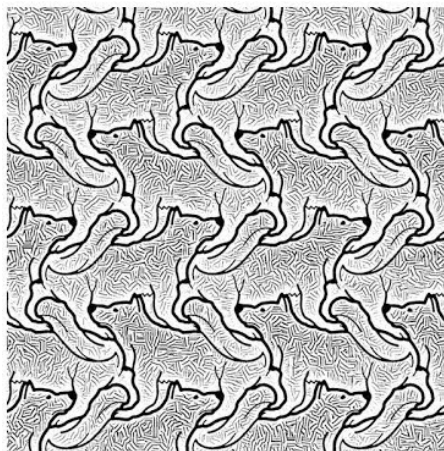
$$666 + 686 + 666$$

$$767 + 22 \cdot 22 + 767$$

!

(The last palindrome is courtesy of Ranganathan Padmanabhan.) If you find other palindromic representations of 2018, do let me know.

Mathematically, it is the new number of the year that excites me. Culturally, it is the Zodiac sign of the new year that carries special meaning to me: in the Chinese calendar, 2018 is the Year of the Dog. Combining this fact with symmetry, please enjoy this dog plane tessellation sent to me by Andy Liu:



Whatever you choose to celebrate for the start of 2018, have a great year and enjoy Volume 44.

Kseniya Garaschuk

THE CONTEST CORNER

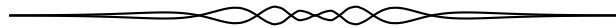
No. 61

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

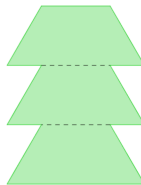
*To facilitate their consideration, solutions should be received by **June 1, 2018**.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.



CC301. All natural numbers are coloured using 100 different colours. Prove that you can find several (no less than 2) different numbers, all of the same colour, that have a product with exactly 1000 different natural divisors.

CC302. Nikolas used construction paper to make a regular tetrahedron (a pyramid consisting of equilateral triangles). Then he cut it in some ingenious way, unfolded it and this resulted in the following Christmas tree-like shape consisting of three halves of a regular hexagon:



How did Nikolas do this?

CC303. Consider two convex polygons M and N with the following properties: polygon M has twice as many acute angles as polygon N has obtuse angles; polygon N has twice as many acute angles as polygon M has obtuse angles; each polygon has at least one acute angle; at least one of the polygons has a right angle.

- a) Give an example of such polygons.
- b) How many right angles can each of these polygons have? Find the complete set of all the possibilities and prove that no others exist.

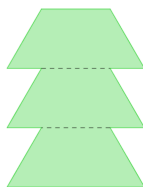
CC304. Consider a natural number n greater than 1 and not divisible by 10. Can the last digit of n and second last digit of n^4 (that is, the digit in the tens position) be of the same parity?

CC305. Can you arrange n identical cubes in such a way that each cube has exactly three neighbours (cubes are considered to be neighbours if they have a common face)? Solve the problem for $n = 2016, 2017$ and 2018 .

.....

CC301. On a colorié tous les entiers strictement positifs en utilisant 100 couleurs distinctes. Démontrer qu'il est possible de trouver au moins deux nombres différents, tous de la même couleur, dont le produit admet exactement 1000 diviseurs différents.

CC302. Nicolas a utilisé du papier de bricolage pour construire un tétraèdre régulier (une pyramide dont les quatre faces sont des triangles équilatéraux). Il a ensuite découpé le tétraèdre de façon ingénieuse et l'a déplié de manière à obtenir la forme suivante composée de trois moitiés d'un hexagone régulier:



Comment Nicolas s'est-il pris?

CC303. On considère deux polygones convexes, M et N , qui satisfont aux conditions suivantes: le nombre d'angles aigus du polygone M est deux fois le nombre d'angles obtus du polygone N ; le nombre d'angles aigus du polygone N est deux fois le nombre d'angles obtus du polygone M ; chaque polygone a au moins un angle aigu; au moins un des polygones a un angle droit.

- a) Donner un exemple de deux tels polygones.
- b) Combien d'angles droits chacun de ces polygones peut-il avoir? Déterminer l'ensemble complet de toutes les possibilités et démontrer qu'il n'en existe aucune autre.

CC304. On considère un entier n supérieur à 1 qui n'est pas divisible par 10. Est-il possible que le chiffre des unités de n et le chiffre des dizaines de n^4 aient la même parité?

CC305. Est-il possible de placer n cubes identiques de manière que chaque cube ait exactement trois voisins (deux cubes sont voisins si un cube a une face qui touche au complet à une face de l'autre cube)? Résoudre le problème pour n égal à 2016, 2017 et 2018.



CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2017: 43(1), p. 5–7.

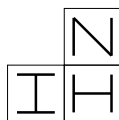
CC251. The six faces of a cube are labeled F, H, I, N, X and Z. Three views of the labelled cube are shown. Note that the H and the N on the die are indistinguishable from the rotated I and Z, respectively. The cube is then unfolded to form the lattice shown, with F shown upright. What letter should be drawn upright on the shaded square?



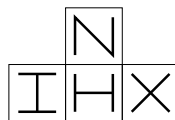
Originally Problem 10, Junior Round part A, of the 2015 BC Secondary Math Contest.

We received two complete solutions. We present a solution loosely based on that by the Missouri State University Problem Solving Group.

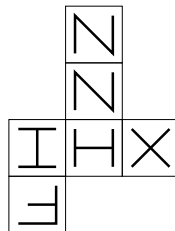
We start with the view of the cube containing both the H and the I . Partially unfolded, it looks as follows:



Now we try to align the second view of the cube, containing the X . The face labelled H or I in this view must be one of the two faces in the bottom row in the figure above. Of the four possible ways to overlay them, only one works with the remaining faces:



The third view, containing the F , has only one way to match with this, which is



By rotating the perspective by 180° we see that the answer to the question is H .

CC252. There are ten coins, each blank on one side and numbered on the other side with numbers 1 through 10. All ten coins are tossed and the sum of the numbers landing face up is calculated. What is the probability that this sum is at least 45?

Originally Problem 4, Senior Round part B, of the 2014 BC Secondary Math Contest.

We received 4 solutions; 1 was correct and complete. Two of the four were very close, but forgot to count one outcome in their arguments. We present the solution by Ivko Dimitrić.

Let us label each coin by the number that appears on its numbered side (the one that is not blank). We compute the required probability as the ratio of the number of favourable outcomes (when the sum of the numbers that turn up is at least 45) to the total number of all possible outcomes, each outcome being a string of ten symbols composed of numbers and blanks. The number of all outcomes is 2^{10} , since every one of the ten coins can turn blank side up or the numbered side up when the coins are flipped (two possibilities for each coin, independently of one another).

The sum of all ten numbers is $10 \cdot 11/2 = 55$ and a favourable outcome is any one for which the sum of numbers that turn up is at least 45, that is, when the sum of numbers that did not turn up is at most 10. Thus, no more than 4 blank sides could possibly turn up when the coins are flipped since, otherwise, the largest sum obtained would be less than $55 - (1 + 2 + 3 + 4) = 45$. Therefore, we make our count of favourable outcomes for each case, depending on the number $k \in \{0, 1, 2, 3, 4\}$ of blank faces that turn up.

(0) If there is no blank face turned up and all the numbers appear, the sum is 55, so such (single) outcome is favourable,

(1) If there is exactly one blank that turned up, the outcome will be always favourable since the other side of that coin with the blank that turned up would have a number on it that is at most 10, so that the sum of all other numbers that turned up is at least $55 - 10 = 45$. The number of these outcomes is 10, since the blank side can turn up with any (but only one) of the ten coins.

(2) If there are exactly two blanks up, the outcome is favourable when the sum of two numbers on downsides of these coins is at most 10. That happens for the following 20 pairs in order:

(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (2, 3), (2, 4),
(2, 5), (2, 6), (2, 7), (2, 8), (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6),

where each number also refers to the label of the coin in which the blank turned up with that number on the down side. Thus we have 20 favourable outcomes of this kind.

(3) If there are three blank sides up, for a favourable outcome to occur the sum of three numbers on downsides should be at most 10. There are 11 such triples:

(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 2, 7),
(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 5), (2, 3, 4), (2, 3, 5).

(4) Finally, if there are four blanks that turn up, there is only one favourable outcome, namely when the numbers on down sides are (1, 2, 3, 4) in the same numbered coins. Any other quadruple would have the sum of numbers that did not turn up greater than 10, causing the sum of numbers that did turn up less than $55 - 10 = 45$.

Hence, we have $1 + 10 + 20 + 11 + 1 = 43$ favourable outcomes out of total of $2^{10} = 1024$ possible outcomes so that the required probability equals

$$\frac{43}{1024} \approx 0.042.$$

CC253. Let $A(n)$ represent the number of ways n pennies can be arranged in any number of rows, where each row starts at the same position as the row below it and has fewer pennies than the row below it. For example, $A(6) = 4$, as shown below:



1. Show that $A(9) = 8$.

2. Find the smallest number k which is not equal to $A(n)$ for any n .

Originally Problem 5, Junior Round part B, of the 2015 BC Secondary Math Contest.

We received 3 correct solutions. Solution by Dan Daniel.

Let $\Lambda(n)$ denote the set of allowable arrangements of n pennies and $|\Lambda(n)|$ its size. Then

1. We have $\Lambda(9) = \{(9), (8, 1), (7, 2), (6, 3), (5, 4), (6, 2, 1), (5, 3, 1), (4, 3, 2)\}$, so $A(9) = |\Lambda(9)| = 8$.

2. Easy calculation gives the following values for the first several values of $A(n)$.

n	1	2	3	4	5	6	7	8	9
$A(n)$	1	1	2	2	3	4	5	6	8

Hence $A(n) \neq 7$ for $n \leq 9$. If $(k_1, k_2, \dots) \in \Lambda(n)$, then $(k_1 + 1, k_2, \dots) \in \Lambda(n + 1)$ so that $|\Lambda(n + 1)| \geq |\Lambda(n)|$. Hence $A(n) \geq 8$ for $n \geq 10$, and $A(n) \neq 7$ for any n .

CC254. Hayden has a lock with a combination consisting of two 8s separated by eight digits, two 7s separated by seven digits, all the way down to two 1s separated by one digit. Unfortunately, Hayden spilled coffee on the paper that the combination was written on, and all that can be read of the combination is

584***

Determine all the possible combinations of the lock.

Originally Problem 4, Junior Round part B, of the 2015 BC Secondary Math Contest.

We received three correct solutions and one incorrect solution. We present the solution of Ivko Dimitrić.

There are only two possible combinations:

4635843765121827 and 1615847365432872.

There are 16 places, numbered 1 through 16 from left to right, for 16 digits d_1, d_2, \dots, d_{16} of the combination lock of which three digits are known, $d_4 = 5, d_5 = 8, d_6 = 4$. For each digit that has been determined, the other one in the pair of the same digits may occur to the left or to the right of the known digit (provided there is room to place that digit) separated by the required number of places. Since $d_5 = 8$, the other digit 8 must be to the right of the given one (there is no room to the left) separated by eight places, so $d_{14} = 8$. Also, $d_4 = 5$ is given in the fourth place so the other digit five, separated by five places from the first one also has to be to the right, i.e. $d_{10} = 5$. One digit 4 is given in the sixth place, $d_6 = 4$, so the other digit 4 is either in the first position or in the 11th position, that is $d_1 = 4$ or $d_{11} = 4$.

The digit in the second position could be one of the remaining digits 1, 2, 3, 6, 7. However, all other digits, except 6, cannot occupy that position since there is no room to the right (or left) for the other digit of the pair to be placed. For example, if $d_2 = 7$ the other digit 7 would have to be to the right of the first one separated by 7 places i. e. in the position occupied by digit $d_{10} = 5$. Similarly, the impossibility for digits 1, 2, 3 to be placed in the second position is also obvious. Thus $d_2 = 6$ and consequently $d_9 = 6$.

Regarding the possibilities for the position of second digit 4, we distinguish between two possible arrangements:

1. If $d_1 = 4$, then digits 4, 5, 6, and 8 have been used to create the following portion of the combination:

$$46 * 584 * * 65 * * * 8 * *.$$

The third digit could be only 3 or 7 on account of the available room to place the other digit of the pair. If $d_3 = 3$ then $d_7 = 3$ and $d_8 = d_{16} = 7$, producing 4635843765 * * * 8 * 7. There remain now only two pairs of digits 1 and 2 to be placed. Digit d_{11} cannot be 2, otherwise the position for the second digit 2 would be blocked by 8 on the right and 7 on the left. Thus $d_{11} = d_{13} = 1$ and finally $d_{12} = d_{15} = 2$ to complete one of the possible combinations:

$$4635843765121827.$$

If $d_3 = 7$, then $d_{11} = 7$, which means $d_8 = d_{12} = 3$ or $d_{12} = d_{16} = 3$. Either way, we cannot complete the combination.

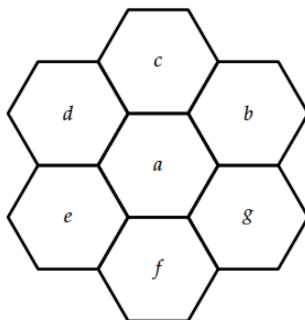
2. If $d_{11} = 4$ then the recovered portion of the combination reads as

$$*6 * 584 * * 654 * * * * *.$$

In this arrangement, we must have $d_1 = 1$ since for the other possibilities 2, 3, 7 for the first position, the place for the second digit of the pair would have been already taken by other digits. Hence $d_3 = 1$. Digit d_7 cannot be 2 or 3, otherwise there would be no room to the right (or left) to place the second digit of the same pair. Thus $d_7 = 7$ and then $d_{15} = 7$. Then $d_8 = 3$ (that position cannot be taken by 2 since there would be no room for the other 2) and hence $d_{12} = 3$. The remaining two places are taken by digits 2, $d_{13} = d_{16} = 2$, for the other possible combination:

$$1615847365432872.$$

CC255. Antonino is instructed to colour the honeycomb pattern shown, which is made up of labelled hexagonal cells:



If two cells share a common side, they are to be coloured with different colours. Antonino has four colours available. Determine the number of ways he can colour the honeycomb, where two colourings are different if there is at least one cell that is a different colour in the two colourings.

Originally Problem 10, Senior Round part A, of the 2014 BC Secondary Math Contest.

We received 3 solutions, all correct and complete. We present 2 solutions.

Solution 1, by Ivko Dimitrić.

There are 264 different colourings.

Denote four colours by letters W, X, Y and Z . Once the colour for the central cell a is chosen, say W (and we write $a = W$) then the remaining 6 cells are to be coloured by three remaining colours (X, Y, Z) according to the rules. The pattern cannot be coloured just by two colours since for three neighbouring cells a, b, c one would need three different colours. Three colours would suffice where the cells b, d, f must be one colour (other than the colour of a) and cells c, e, g the third colour. Given the colour of a (4 choices for that) we can choose two additional colours for cells $b - g$ out of the remaining three in $\binom{3}{2} = 3$ ways so that each can be used for cells b, d, f (2 ways for that) and the remaining one for cells c, e, g . Therefore, we get $4 \cdot 3 \cdot 2 = 4! = 24$ ways to colour the cells if only three out of four colours are used.

Now consider the number of ways if all 4 colours are used. Once the choice for the colour of a has been made (4 choices), for example $a = W$, there are choices for diagonal cells b and e that involve same or different colours.

1. There are three ways to colour b and e by the same colour X, Y or Z . If that colour is X then according to the rule that no two neighbours are the same colour, there are two choices for colouring f and g and independently two choices for colouring d and e , namely $de, fg \in \{YZ, ZY\}$ which involves 4 combinations. That makes $4 \cdot 3 = 12$ possibilities for such an arrangement.

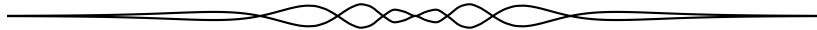
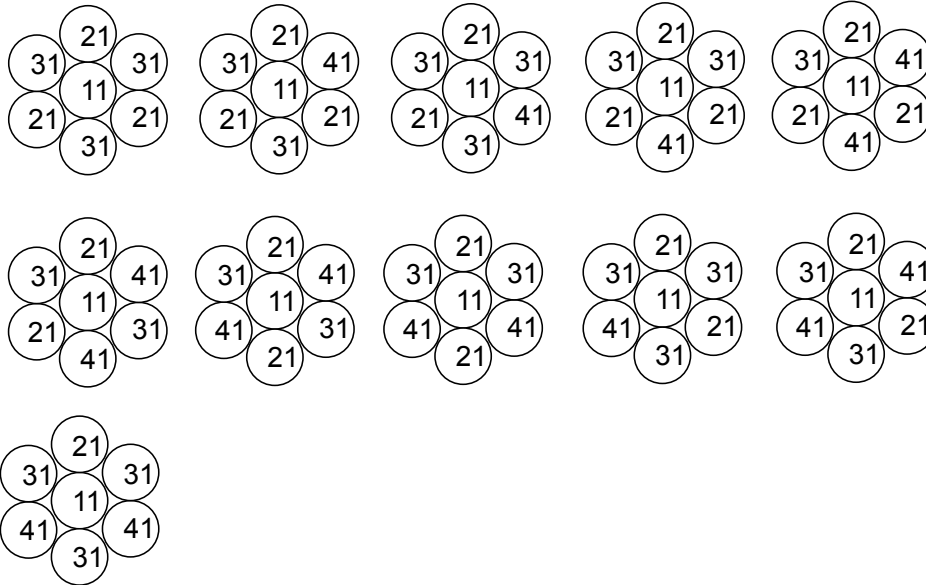
2. If different colours are used for b and e , then there are 6 possibilities

$$be = XY, YX, XZ, ZX, YZ, ZY$$

for the choice of these colours (as above we assume that the colour of a has been chosen to be W .) Let $b = X, e = Y$. Then the possible (mutually different) colours for f and d are X and Z , in any order, and colours for e and g are Y and Z . This makes 3 choices for the pair $fg \in \{XY, XZ, ZY\}$ and the same three choices independently for the pair de . That makes $3 \cdot 3 = 9$ possibilities for colouring d, e, f, g once the colours for b and e have been chosen in this arrangement. That amounts to $9 \cdot 6 = 54$ different colourings of cells $b - g$ in this arrangement once the colour for cell a has been selected. Add the previous count in 1. to the count in 2. to get $12 + 54 = 66$ ways to do the colouring once the colour of cell a has been fixed. If we now vary the colour of a (among 4 choices) that will produce $4 \cdot 66 = 264$ ways in total.

Solution 2, by Richard Hess.

The diagrams below demonstrate 11 basic ways to distribute the colours. Each of these admits to 24 cases given by permuting the four colours. Thus there are a total of 264 ways to colour the pattern.



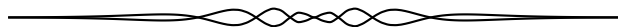
THE OLYMPIAD CORNER

No. 359

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

*To facilitate their consideration, solutions should be received by **June 1, 2018**.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.



OC361. Let $n \geq 2$ be a positive integer and define k to be the number of primes less than or equal to n . Let A be a subset of $S = \{2, \dots, n\}$ such that $|A| \leq k$ and no two elements in A divide each other. Show that one can find a set B of cardinality k such that $A \subseteq B \subseteq S$ and no two elements in B divide each other.

OC362. Given a positive prime number p , prove that there exists a positive integer α such that $p|\alpha(\alpha - 1) + 3$ if and only if there exists a positive integer β such that $p|\beta(\beta - 1) + 25$.

OC363. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(y)f(x + f(y)) = f(x)f(xy)$$

for all positive real numbers x and y .

OC364. Consider an acute triangle ABC . Suppose $AB < AC$, let I be the incenter, D the foot of perpendicular from I to BC , and suppose that altitude AH meets BI and CI at P and Q , respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L and suppose that the circumcircle of $\triangle AIL$ meets BC again at N . Prove that $\frac{BD}{CD} = \frac{BN}{CN}$.

OC365. A square $ABCD$ is divided into n^2 equal small (fundamental) squares by drawing lines parallel to its sides. The vertices of the fundamental squares are called vertices of the grid. A rhombus is called *nice* when:

1. it is not a square;
2. its vertices are points of the grid;
3. its diagonals are parallel to the sides of the square $ABCD$.

Find (as a function of n) the number of nice rhombuses (n is a positive integer greater than 2).

.....

OC361. Soit n un entier ($n \geq 2$) et soit k le nombre de nombres premiers inférieurs ou égaux à n . Soit A un sous-ensemble de $S = \{2, \dots, n\}$ tel que $|A| \leq k$ et que A ne contienne pas un élément qui est un diviseur d'un autre élément. Démontrer qu'il existe un ensemble B tel que $|B| = k$, $A \subseteq B \subseteq S$ et que B ne contienne pas un élément qui est un diviseur d'un autre élément.

OC362. Soit p un nombre premier. Démontrer qu'il existe un entier strictement positif α tel que $p|\alpha(\alpha-1)+3$ si et seulement si il existe un entier strictement positif β tel que $p|\beta(\beta-1)+25$.

OC363. Déterminer toutes les fonctions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ telles que

$$f(y)f(x+f(y)) = f(x)f(xy)$$

pour tous réels x et y strictement positifs.

OC364. On considère un triangle acutangle ABC où $AB < AC$. Soit I le centre du cercle inscrit dans le triangle et D le pied de la perpendiculaire abaissée de I jusqu'à BC . La hauteur AH coupe BI et CI aux points respectifs P et Q . Soit O le centre du cercle circonscrit au triangle IPQ . On prolonge AO jusqu'au point L sur BC . Le cercle circonscrit au triangle AIL coupe BC de nouveau au point N . Démontrer que $\frac{BD}{CD} = \frac{BN}{CN}$.

OC365. On divise un carré $ABCD$ en n^2 petits carrés en traçant des segments parallèles à ses côtés, formant ainsi un quadrillage. Les points d'intersection du quadrillage sont appelés des points de treillis. On dit qu'un losange est *plaisant* lorsque:

1. il n'est pas un carré;
2. ses sommets sont des points de treillis;
3. ses diagonales sont parallèles aux côtés du carré $ABCD$.

Déterminer, en fonction de n , le nombre de losanges plaisants, n étant un entier supérieur à 2.



OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(9), p. 385 – 386.

OC301. Solve the following Diophantine equation for integers x and y :

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

Originally Day 1 Problem 1 of the 2015 United States of America Mathematical Olympiad.

We received four correct submissions and we present the solution by Prithwijit De.

Since $3|x+y$, we set $x+y = 3u$, where $u \in \mathbb{Z}$. Then $3(x+y)^2 + (x-y)^2 = 4(x^2 + xy + y^2)$ gives

$$\begin{aligned}(x-y)^2 &= 4(x^2 + xy + y^2) - 3(x+y)^2 \\ &= 4(u+1)^3 - 27u^2 \\ &= 4u^3 - 15u^2 + 12u + 4 \\ &= (u-2)^2(4u+1).\end{aligned}$$

Hence, $4u+1$ is an odd perfect square. Let $4u+1 = (2k+1)^2$ where $k \in \mathbb{Z}$. Then $u = k(k+1)$. Therefore

$$x+y = 3k(k+1) = 3k^2 + 3k \tag{1}$$

and

$$\begin{aligned}x-y &= \sqrt{(u-2)^2(4u+1)} = \pm(u-2)\sqrt{4u+1} = \pm(k^2+k-2)(2k+1) \\ &= \pm(2k^3+3k^2-3k-2).\end{aligned} \tag{2}$$

Solving (1) and (2), we conclude that the solutions are given by the following set S of all unordered pairs:

$$S = \{(x, y) = (k^3 + 3k^2 - 1, -k^3 + 3k + 1) | k \in \mathbb{Z}\}$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

Originally Day 1 Problem 2 of the 2015 Turkey Mathematical Olympiad

We received six correct submissions and we present a composite of the solutions by Mohammed Aassila and Michel Bataille, expanded slightly by the editor.

Let $2a = x + y$ and $b^2 = xy$. Then by straightforward computations we have

$$\begin{aligned}(x^2 + y)(x + y^2) &= x^3 + y^3 + x^2y^2 + xy \\ &= (x + y)^3 - 3xy(x + y) + x^2y^2 + xy \\ &= 8a^3 - 6ab^2 + b^4 + b^2,\end{aligned}\tag{1}$$

and

$$\begin{aligned}(x + y - 1)^2(x + y - 1/4) &= (2a - 1)^2(2a - 1/4) \\ &= (4a^2 - 4a + 1)(2a - 1/4) \\ &= 8a^3 - 9a^2 + 3a - 1/4.\end{aligned}\tag{2}$$

From (1) and (2) we obtain

$$\begin{aligned}(x^2 + y)(x + y^2) - (x + y - 1)^2(x + y - 1/4) \\ = 9a^2 + b^4 + b^2 - 6ab^2 - 3a + 1/4 = (1/4)(6a - 2b^2 - 1)^2 \geq 0,\end{aligned}$$

so

$$\frac{(x^2 + y)(x + y^2)}{(x + y - 1)^2} \geq x + y - 1/4.$$

Hence,

$$\frac{(x^2 + y)(x + y^2)}{(x + y - 1)^2} + \frac{(y^2 + z)(y + z^2)}{(y + z - 1)^2} + \frac{(z^2 + x)(z + x^2)}{(z + x - 1)^2} \geq 2(x + y + z) - \frac{3}{4}.$$

Since

$$6a - 2b^2 - 1 = 0 \iff 3(x + y) - 2xy - 1 = 0 \iff 3(x + y) = 2xy + 1,$$

we see that equality holds in the given inequality if and only if

$$3(x + y) = 2xy + 1, \quad 3(y + z) = 2yz + 1, \quad \text{and} \quad 3(z + x) = 2zx + 1.$$

Subtracting, we have

$$3(x - z) = 2y(x - z) \quad \text{or} \quad (x - z)(2y - 3) = 0.$$

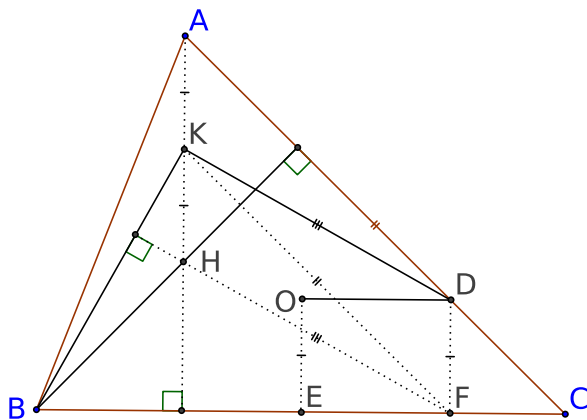
But $2y - 3 \neq 0$ since $y = 3/2$ implies $3(x + 3/2) = 3x + 1$, a contradiction. Hence, $x = z$. Similarly, $y = z$, so $x = y = z$. Then solving $2x^2 - 6x + 1 = 0$ we get $x = \frac{3 \pm \sqrt{7}}{2}$. In conclusion, equality holds if and only if

$$x = y = z = \frac{3 + \sqrt{7}}{2} \quad \text{or} \quad x = y = z = \frac{3 - \sqrt{7}}{2}.$$

OC303. Let ABC be a triangle with orthocenter H and circumcenter O . Let K be the midpoint of AH . Point P lies on AC such that $\angle BKP = 90^\circ$. Prove that $OP \parallel BC$.

Originally Problem 2 of the 2015 Iranian Mathematical Olympiad (Geometry).

We received eight submissions, all of which were correct and complete, although one was completed by a computer, so maybe it should not count. We present a composite of the similar solutions submitted by Daniel Dan and Oliver Geupel.



Define D to be the point where the parallel to the side BC through O meets the line AC , and let E and F denote the feet of the perpendiculars from the points O and D onto the line BC . We are to prove that D coincides with P ; that is, we must show that KD is perpendicular to BK . Because $AH = 2OE$ and both lines are perpendicular to BC , it follows that the line segments AK , KH , OE , and DF are parallel and have equal lengths. Hence $AD \parallel KF$ and $HF \parallel KD$. Since $BH \perp AD$, we deduce that $BH \perp KF$. Moreover, we have $KH \perp BF$. Thus, H is the orthocenter of $\triangle BFK$. We obtain $HF \perp BK$ and, therefore, $KD \perp BK$, as desired.

Editor's comment. Most of the other solutions used some form of coordinates. It turned out that almost any approach — Cartesian or triangular coordinates, vectors, or complex numbers — leads to an attractive solution. There is no need for help from *MAPLE*; the use of a computer seems counter to the spirit of Olympiad problem solving.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer greater than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

Originally Day 1 Problem 2 of the 2015 Serbian National Mathematical Olympiad.

Two correct solutions were received. The solution below mainly follows that of Kathleen Lewis.

Suppose, if possible, that $m < n$ and $F(m) = F(n) = u$. Let $m = ra^2$ and $n = sb^2$, where r and s are both squarefree. Since um is square, u must be divisible by r , so that $u = rc^2$ for some integer c . Likewise, $u = sd^2$ for some integer d . Since $(rc)^2 = rsd^2$, rs is square and so $r = s$. Thus, also, $c = d$.

Since $m < n$, then $a < b$ so that $b - a \geq 1$. Since $k \geq 1$, then $b\sqrt{k} - a\sqrt{k} \geq 1$ and there exists a positive integer v for which $a\sqrt{k} < v \leq b\sqrt{k}$. Therefore

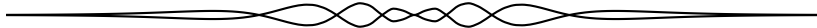
$$km = rka^2 < rv^2 \leq rkb^2 = ksb^2 = kn < F(n) = F(m).$$

But this is a contradiction, since $mr v^2 = (rav)^2$ is square so that rv^2 should be not less than $F(m)$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p . Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

Originally Problem 5 of the 2015 Canadian Mathematical Olympiad.

No correct solutions were received.



FOCUS ON...

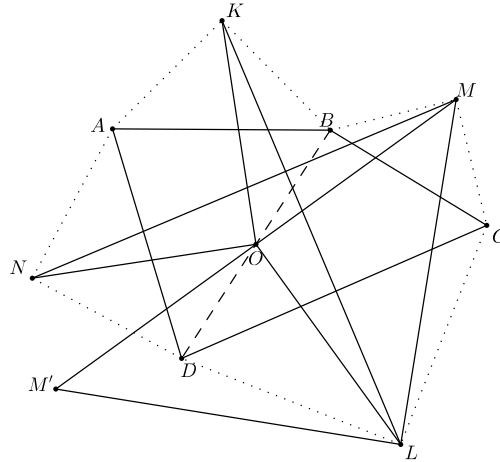
No. 29

Michel Bataille

Solutions to Exercises from Focus On... No. 22–26

From Focus On... No. 22

1. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA , construct isosceles triangles KAB, MBC, LCD, NDA external to $ABCD$ such that the angles at K, L, M, N are right angles. Show that if O is the midpoint of BD , then one of the triangles MON or LOK is a 90° rotation of the other around O .



Without loss of generality, we assume that $ABCD$ is clockwise oriented as on the diagram above. Let \mathbf{r}_X denote the rotation with positive right angle and centre X . The isometry $\mathbf{r}_L \circ \mathbf{r}_M$ is a 180° rotation and

$$\mathbf{r}_L \circ \mathbf{r}_M(B) = \mathbf{r}_L(C) = D.$$

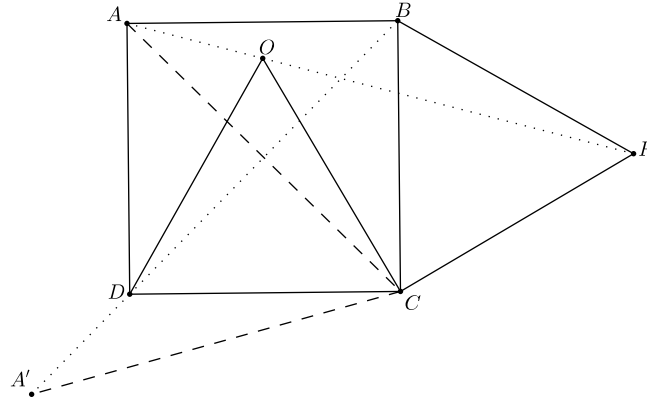
Hence $\mathbf{r}_L \circ \mathbf{r}_M$ is the symmetry \mathbf{s}_O about the mid-point O of BD . Let $M' = \mathbf{s}_O(M)$. Then

$$M' = \mathbf{r}_L \circ \mathbf{r}_M(M) = \mathbf{r}_L(M)$$

so that $\triangle MLM'$ is an isosceles right-angled triangle with right angle at L and $\mathbf{r}_O(L) = M$. Similarly, $\mathbf{r}_O(K) = N$ and therefore the triangle MON is the image of LOK under the 90° rotation \mathbf{r}_O .

Note that if $ABCD$ is a parallelogram, then O is also the mid-point of AC and, as above, $\mathbf{r}_O(M) = K$ and $\mathbf{r}_O(N) = L$. It follows that $\mathbf{s}_O(M) = N$ and $\mathbf{s}_O(K) = L$. Then the triangles MON and LOK are degenerate and $LMKN$ is a square with centre O .

2. Let $ABCD$ be a square and O, P be such that DOC and BCP are equilateral triangles with O inside $ABCD$ and P external to $ABCD$. Show that A, O, P are collinear. (A possible solution follows from the value of $\angle AOB$ found in problem **3458** [2009 : 326 ; 2010 : 347]; preferably solve the problem with the help of a well-chosen rotation.)



We introduce the rotation \mathbf{r} with centre C transforming P into B . Let $\mathbf{r}(A) = A'$. Since $\triangle ACA'$ is equilateral, we have $A'C = A'A$, hence A', B, D are collinear (on the perpendicular bisector of AC). As a result, the points

$$A = \mathbf{r}^{-1}(A'), \quad P = \mathbf{r}^{-1}(B), \quad O = \mathbf{r}^{-1}(D)$$

are collinear as well.

From Focus On... No. 23

1. Given that the polynomial $X^3 - 5X + m$ has two roots z_1, z_2 such that $z_1 + z_2 = 2z_1z_2$, find the value of m and all the roots.

Denoting by z_3 the third root of $X^3 - 5X + m$, Vieta's formulas give

$$z_1 + z_2 + z_3 = 0, \quad z_1z_2 + z_2z_3 + z_3z_1 = -5, \quad z_1z_2z_3 = -m.$$

Now, $z_1z_2 - (z_1 + z_2)^2 = -5$ and, if $z_1 + z_2 = 2z_1z_2$ holds, we obtain

$$z_1z_2 - 4(z_1z_2)^2 = -5$$

so that $z_1z_2 = -1$ or $z_1z_2 = \frac{5}{4}$. In the former case,

$$z_3 = -(z_1 + z_2) = -2z_1z_2 = 2 \quad \text{and} \quad m = -z_1z_2z_3 = 2$$

and, similarly, $z_3 = -\frac{5}{2}$ and $m = \frac{25}{8}$ in the latter case.

Conversely, if $m = 2$ the polynomial writes as $X^3 - 5X + 2$ whose roots are

$$z_3 = 2, \quad z_2 = -1 + \sqrt{2}, \quad z_1 = -1 - \sqrt{2}.$$

If $m = \frac{25}{8}$, the roots of $X^3 - 5X + \frac{25}{8}$ are

$$z_3 = -\frac{5}{2}, \quad z_2 = \frac{5 + \sqrt{5}}{4}, \quad z_1 = \frac{5 - \sqrt{5}}{4}.$$

In both cases, it is readily checked that $z_1 + z_2 = 2z_1z_2$ holds.

2. Let $Q(x) \in \mathbb{R}[x]$ and $P(x) = a + bx + cx^2 + x^3Q(x)$ where a, b, c are real numbers and $ac \neq 0$. Prove that if all the roots of P are real, then $b^2 > 2ac$. (Hint: if n is the degree of P , consider $x^n P(1/x)$.)

We first note that the conclusion $b^2 > 2ac$ certainly holds if $ac < 0$, so we suppose that $ac > 0$ from now on.

If $Q(x)$ is the zero polynomial, then $P(x) = a + bx + cx^2$ and we must have $b^2 \geq 4ac$, hence $b^2 > 2ac$ holds. Otherwise, let $n > 2$ be the degree of $P(x)$ so that $Q(x)$ is of degree $n - 3$, say $Q(x) = q_0 + q_1x + \cdots + q_{n-3}x^{n-3}$. Then, the polynomial

$$U(x) = x^n \cdot P(1/x) = ax^n + bx^{n-1} + cx^{n-2} + q_0x^{n-3} + \cdots + q_{n-2}x + q_{n-3}$$

also has n real roots (the reciprocals of the roots of $P(x)$), say x_1, x_2, \dots, x_n . From Vieta's formulas, we have

$$x_1 + x_2 + \cdots + x_n = -\frac{b}{a}, \quad \text{and} \quad \sum_{1 \leq i < j \leq n} x_i x_j = \frac{c}{a}.$$

Using the arithmetic-geometric mean inequality, we deduce

$$\frac{2c}{a} \leq \sum_{1 \leq i < j \leq n} (x_i^2 + x_j^2) = (n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) = (n-1) \left(\frac{b^2}{a^2} - \frac{2c}{a} \right).$$

It follows that $2nac \leq (n-1)b^2$, that is, $b^2 \geq \frac{n}{n-1} \cdot (2ac)$ and $b^2 > 2ac$ holds.

From Focus On... No. 25

1. Find all complex numbers λ such that the product of two roots of $x^4 - 2x^2 + \lambda x + 3$ is 1.

The polynomial $a(x) = x^4 - 2x^2 + \lambda x + 3$ has two roots whose product is 1 if and only if $a(x)$ is divisible by some polynomial $b(x)$ of the form $x^2 - \mu x + 1$. The long division of $a(x)$ by $b(x)$ gives

$$a(x) = b(x)(x^2 + \mu x + \mu^2 - 3) + (\lambda + \mu^3 - 4\mu)x + 6 - \mu^2.$$

The remainder is the zero polynomial if and only if $\mu^2 = 6$ and $\mu^3 - 4\mu + \lambda = 0$ for some complex number μ . The elimination of μ is immediate and provides the condition $(\lambda + 2\sqrt{6})(\lambda - 2\sqrt{6}) = 0$ on λ . Thus the suitable values of λ are $2\sqrt{6}$ and $-2\sqrt{6}$.

It is easy to obtain the two roots with product 1: they are $\frac{\sqrt{6}+\sqrt{2}}{2}$ and $\frac{\sqrt{6}-\sqrt{2}}{2}$ if $\lambda = -2\sqrt{6}$, and their opposites in the case when $\lambda = 2\sqrt{6}$.

2. Find real numbers a_k, b_k ($k = 1, 2, \dots, 2017$) such that

$$\frac{3x^5 - 3x^4 - 2x^2 + 2x + 4}{(x^2 + x + 1)^{2017}} = \sum_{k=1}^{2017} \frac{a_k x + b_k}{(x^2 + x + 1)^k}.$$

Let $f(x)$ denote the rational fraction on the left-hand side. The problem is to calculate the decomposition of $f(x)$ into partial fractions in $\mathbb{R}(x)$. Multiplying both sides by $(x^2 + x + 1)^{2017}$, we obtain

$$3x^5 - 3x^4 - 2x^2 + 2x + 4 = \sum_{k=1}^{2017} (a_k x + b_k)(x^2 + x + 1)^{2017-k}, \quad (1)$$

which is nothing else than the long division of the numerator of $f(x)$ by $x^2 + x + 1$:

$$3x^5 - 3x^4 - 2x^2 + 2x + 4 = (x^2 + x + 1)q_1(x) + a_{2017}x + b_{2017}.$$

Substituting into (1) and dividing by $x^2 + x + 1$, we see that $a_{2016}x + b_{2016}$ is the remainder in the division of $q_1(x)$ by $x^2 + x + 1$:

$$q_1(x) = (x^2 + x + 1)q_2(x) + a_{2016}x + b_{2016}.$$

Here $q_2(x)$ is of degree 1 and so $q_2(x) = a_{2015}x + b_{2015}$ and $a_k = b_k = 0$ if $k \leq 2014$.

An easy calculation successively provides

$$\begin{aligned} a_{2017}x + b_{2017} &= -2x + 3, & q_1(x) &= 3x^3 - 6x^2 + 3x + 1, \\ a_{2016}x + b_{2016} &= 9x + 10, & q_2(x) &= 3x - 9 = a_{2015}x + b_{2015}. \end{aligned}$$

Therefore the decomposition of $f(x)$ is

$$f(x) = \frac{-2x + 3}{(x^2 + x + 1)^{2017}} + \frac{9x + 10}{(x^2 + x + 1)^{2016}} + \frac{3x - 9}{(x^2 + x + 1)^{2015}}.$$

From Focus On... No. 26

1. Let $p(x) \in \mathbb{R}[x]$ with $\deg(p(x)) \geq 2$. Prove that the graph of the function p cannot have more than one centre of symmetry.

For the purpose of a contradiction, assume that (x_1, y_1) and (x_2, y_2) are two distinct centres of symmetry. Then for any real number x , we have

$$p(x_i + x) + p(x_i - x) = 2y_i \quad (i = 1, 2),$$

from which we readily deduce that $p(x + h) = p(x) + k$ where $h = 2(x_1 - x_2)$ and $k = 2(y_1 - y_2)$. Note that $y_i = p(x_i)$ so that $h \neq 0$. Differentiating, we get $p'(x + h) = p'(x)$ so that whenever the polynomial $p'(x)$ has a complex root z , it has infinitely many roots, namely the numbers $z + nh$, $n = 0, 1, 2, \dots$. It follows that $p'(x)$ is either the zero polynomial or a nonzero constant polynomial, in contradiction with $\deg(p(x)) \geq 2$.

2. Let n be a positive integer and let $a(x) \in \mathbb{R}[x]$ with $\deg(a(x)) = n$. Find $a(n+1)$ given that $a(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \dots, n$,

a) using the polynomial $(x+1)a(x) - x$;

b) using a Lagrange interpolation polynomial.

a) By hypothesis, the $n+1$ numbers $0, 1, 2, \dots, n$ are roots of the polynomial

$$b(x) = (x+1)a(x) - x$$

whose degree is $n+1$, hence $b(x) = \rho \cdot x(x-1)(x-2) \cdots (x-n)$ for some nonzero constant ρ .

Since $b(-1) = 1$, the constant ρ satisfies $1 = \rho \cdot (-1)^{n+1}((n+1)!)$ and therefore

$$b(x) = \frac{(-1)^{n+1}}{(n+1)!} \cdot x(x-1)(x-2) \cdots (x-n).$$

We first deduce

$$(n+2)a(n+1) - (n+1) = b(n+1) = (-1)^{n+1}$$

and finally

$$a(n+1) = \frac{n+1 + (-1)^{n+1}}{n+2}.$$

b) The polynomial $a(x)$ is the Lagrange interpolation polynomial associated with

$$x_1 = 0, x_2 = 1, \dots, x_{n+1} = n \quad \text{and} \quad y_1 = 0, y_2 = \frac{1}{2}, \dots, y_{n+1} = \frac{n}{n+1}.$$

Thus,

$$a(x) = \sum_{k=0}^n \frac{k}{k+1} \cdot \frac{x(x-1) \cdots \widehat{(x-k)} \cdots (x-n)}{k! \cdot (-1)^{n-k}(n-k)!},$$

where the factor with the hat is omitted. A short calculation then gives

$$a(n+1) = \sum_{k=1}^n (-1)^{n-k} \frac{k}{k+1} \binom{n+1}{k}$$

or, using $k \binom{n+1}{k} = (n+1) \binom{n}{k-1}$,

$$a(n+1) = (-1)^{n+1}(n+1) \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k-1} = (-1)^{n+1}(n+1)f(1),$$

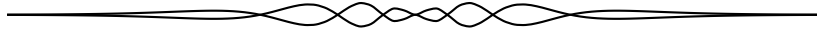
where

$$f(x) = \sum_{k=1}^n \frac{(-1)^{k+1} x^{k+1}}{k+1} \binom{n}{k-1}.$$

We readily obtain the derivative $f'(x) = x((1-x)^n - (-1)^n x^n)$ so that

$$\begin{aligned} f(1) &= \int_0^1 x(1-x)^n dx + \frac{(-1)^{n+1}}{n+2} \\ &= \int_0^1 (1-u)u^n du + \frac{(-1)^{n+1}}{n+2} \\ &= \frac{1}{(n+1)(n+2)} + \frac{(-1)^{n+1}}{n+2}. \end{aligned}$$

As a result, $a(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}$, in accordance with part (a).



Math Quotes

There's a touch of the priesthood in the academic world, a sense that a scholar should not be distracted by the mundane tasks of day-to-day living. I used to have great stretches of time to work. Now I have research thoughts while making peanut butter and jelly sandwiches. Sure it's impossible to write down ideas while reading *Curious George* to a two-year-old. On the other hand, as my husband was leaving graduate school for his first job, his thesis advisor told him, "You may wonder how a professor gets any research done when one has to teach, advise students, serve on committees, referee papers, write letters of recommendation, interview prospective faculty. Well, I take long showers."

Susan Landau, "In Her Own Words: Six Mathematicians Comment on Their Lives and Careers." Notices of the AMS, V. 38, no. 7 (September 1991).

Application of Hadamard's Theorems to inequalities

Daniel Sitaru and Leonard Giugiuc

In this note we will develop a technique based on Hadamard's Theorems for solving some inequalities.

1 Introduction

This method is based on identifying an inequality member as being a possible squared value of a determinant. We will call it *the determinant of a key matrix*. After applying one of Hadamard's theorems, one of the inequality members will be the squared key matrix determinant, and the other one will be the sum of the squared elements on the key matrix columns, or one of the inequality members will be the determinant value and the other one will be the product of the elements on the main diagonal of the key matrix. The trick, of course, is identifying the key matrix.

In 1893, Hadamard published Theorems 1 and 2 in [1]. The consequences of those theorems are extensively analysed in [2]. A recent proof of Theorem 1 can be found in [3]. An innovative approach to proving Theorem 2 can be found in [4].

Theorem 1 ([1, 2, 3]) If $A \in M_n(\mathbb{R})$ then

$$(\det A)^2 \leq \prod_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2 \right). \quad (1)$$

Definition 1 ([4]) A square matrix A is positive if $\det A \geq 0$ and $a_{ii} \geq 0$ for $i = 1, \dots, n$.

Theorem 2 ([4]) If $A \in M_n(\mathbb{R})$ is positive, then

$$\det A \leq \prod_{i=1}^n a_{ii}, \quad (2)$$

with equality if and only if A is a diagonal matrix.

2 Examples

Example 1 Prove that if a, b, c and d are real numbers, then

$$(\sin 2a - \sin 2b)^2 \leq 4(\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 a).$$

Proof. Substituting $\sin x = 2 \sin x \cos x$ on the left, and cancelling the 4's, this inequality is equivalent to:

$$(\sin a \cos a - \sin b \cos b)^2 \leq (\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 a).$$

We recognize the left side as a possible square of a determinant, specifically that of

$$A = \begin{pmatrix} \sin a & \sin b \\ \cos b & \cos a \end{pmatrix} = (a_{ij})_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2}}.$$

Now using (1), we get:

$$(\sin a \cos a - \sin b \cos b)^2 \leq (\sin^2 a + \cos^2 b)(\sin^2 b + \cos^2 a).$$

□

Example 2 Prove that if a, b and c are real numbers, then

$$(3abc - a^3 - b^3 - c^3)^2 \leq (a^2 + b^2 + c^2)^3.$$

Proof. As we can see, the expression $3abc - a^3 - b^3 - c^3$ is plausibly the sum of the threefold products each involving some mixture of a, b, c . This structure reminds us of a determinant. The three summands are positive, suggesting each forward extended diagonal has one each of a, b, c . The three cubes are negative, suggesting the back diagonals with repeated values. Let

$$A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

Then $\det A = 3abc - a^3 - b^3 - c^3$. From Hadamard's first theorem, we have

$$(\det A)^2 \leq \prod_{j=1}^3 \left(\sum_{i=1}^3 a_{ij}^2 \right).$$

□

Example 3 Prove that if $0 < a \leq b \leq c$, then

$$(b-a)(c-a)(c-b) < bc^2.$$

Proof. As we can see, the expression $(b-a)(c-a)(c-b)$ is plausibly the value of a Vandermonde determinant

$$V(a, b, c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

Since A is a positive matrix, so $\det A > 0, b > 0, c^2 > 0$. From Hadamard's second theorem,

$$\det A \leq \prod_{i=1}^n a_{ii} = bc^2$$

It follows that $\det A \leq 1 \cdot b \cdot c^2$ and $(b-a)(c-a)(c-b) < bc^2$.

□

3 Proposed problems

Problem 1. Prove that if a, b, c and d are real numbers, then

$$(ad - bc)^2 \leq (a^2 + c^2)(b^2 + d^2).$$

Problem 2. Prove that if $a, b, c, d \in (1, \infty)$, then

$$(e^a \ln b - e^b \ln a)^2 \leq (e^{2a} + e^{2b})(\ln^2 a + \ln^2 b).$$

Problem 3. Prove that if a, b and c are real numbers, then

$$(2 - a - b - c + abc)^2 \leq (a^2 + 2)(b^2 + 2)(c^2 + 2).$$

Problem 4. Prove that if a, b, c and d are positive real numbers, then

$$(abc - ac - bc - ac)^2 \leq 4(1 + a^2)(1 + b^2)(1 + c^2).$$

Problem 5. Prove that if n is a positive natural number and $a > 1$, then

$$(n + a - 1)(a - 1)^{n-1} \leq a^n.$$

Hint. The key matrices are, in order:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e^a & \ln a \\ e^b & \ln b \end{pmatrix}, \begin{pmatrix} a & a & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \dots & 1 \\ 1 & a & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & a \end{pmatrix}.$$

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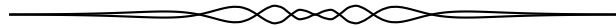
PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

*To facilitate their consideration, solutions should be received by **June 1, 2018**.*

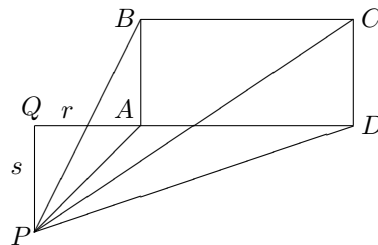
The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

An asterisk () after a number indicates that a problem was proposed without a solution.*



4301. *Proposed by Bill Sands.*

Four trees are situated at the corners of a rectangle $ABCD$. You are standing at a point P outside the rectangle, the nearest point of the rectangle to you being its corner A . To you in this position, the four trees, in the order B, A, C, D as in the diagram, appear to be equally spaced apart. Let Q be the foot of the perpendicular from P to line AD , and set $r = QA$, $s = PQ$.



- Find the lengths of the sides of the rectangle in terms of r and s .
- Find the range of $\angle APQ$.

4302. *Proposed by Martin Lukarevski.*

Let A be a $m \times n$ matrix with $m \geq n$ and X be any $n \times m$ matrix such that XA is invertible. Find the eigenvalues of the matrix $A(XA)^{-1}X$.

4303. *Proposed by Tung Hoang.*

Find the following limit

$$\lim_{n \rightarrow \infty} \left\{ (6 + \sqrt{35})^n \right\},$$

where $\{x\} = x - [x]$ and $[x]$ is the greatest integer function.

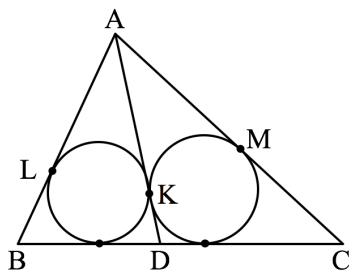
4304. *Proposed by Michel Bataille.*

Evaluate

$$\cot \frac{\pi}{7} + \cot \frac{2\pi}{7} + \cot \frac{4\pi}{7} + \cot^3 \frac{\pi}{7} + \cot^3 \frac{2\pi}{7} + \cot^3 \frac{4\pi}{7}.$$

4305. *Proposed by Moshe Stupel and Avi Sigler.*

Find a nice description of the point D on side BC of a given triangle ABC so that the incircles of the resulting triangles ABD and ADC are tangent to one another at a point of their common tangent line AD .



4306. *Proposed by Marius Drăgan.*

Prove that

$$\left[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \right] = \left[\sqrt{16n+20} \right]$$

for all $n \in \mathbb{N}$.

4307. *Proposed by Adnan Ibric and Salem Malikic.*

In a non-isosceles triangle ABC , let H and M denote the orthocenter and the midpoint of side BC , respectively. The internal angle bisector of $\angle BAC$ intersects BC and the circumcircle of triangle ABC at points D and E , $E \neq A$. If K is the foot of the perpendicular from H to AM and S is the intersection (other than E) of the circumscribed circles of triangles ABC and DEM , prove that quadrilateral $ASDK$ is cyclic.

4308. *Proposed by Leonard Giugiuc and Sladjan Stankovic.*

Let a, b and c be positive real numbers. Prove that

$$27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2.$$

4309. *Proposed by Daniel Sitaru.*

Let a, b and c be real numbers such that $a + b + c = 3$. Prove that

$$2(a^4 + b^4 + c^4) \geq ab(ab + 1) + bc(bc + 1) + ca(ca + 1).$$

4310. *Proposed by Steven Chow.*

Let $\triangle A_1B_1C_1$ be the incentral triangle with respect to $\triangle ABC$, i.e., A_1 is the point of intersection of \overline{BC} and \overleftrightarrow{AI} where I is the incentre of $\triangle ABC$, with B_1 and C_1 similarly defined. Let r be the inradius of $\triangle ABC$.

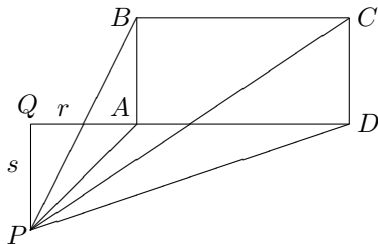
- a) Prove that $AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{3\sqrt{3}}{2} (BC + CA + AB) r^2$.
- b) ★ Prove or disprove that $B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq 3r^3\sqrt{3}$.

Remark. Curiously, this problem was discovered by the proposer when he misread problem 4203. See problem 4203 to appreciate the connection.

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4301. *Proposé par Bill Sands.*

Quatre arbres sont plantés aux coins d'un rectangle $ABCD$. Vous êtes debout à un point P en dehors du rectangle, le point A étant le point du rectangle le plus près de vous. De cette position, les quatre arbres, dans l'ordre B, A, C, D , comme dans le diagramme, semblent également espacés. Soit Q le pied de la perpendiculaire de P vers la ligne AD et posons $r = QA$ puis $s = PQ$.



- a) Déterminer les longueurs des côtés du rectangle en termes de r et s .
- b) Déterminer les valeurs possibles pour $\angle APQ$.

4302. *Proposé par Martin Lukarevski.*

Soit A une matrice $m \times n$ telle que $m \geq n$ et soit X une matrice $n \times m$ telle que XA est inversible. Déterminer les valeurs propres de $A(XA)^{-1}X$.

4303. *Proposé par Tung Hoang.*

Déterminer la limite suivante

$$\lim_{n \rightarrow \infty} \left\{ (6 + \sqrt{35})^n \right\},$$

où $\{x\} = x - [x]$, la fonction $[x]$ étant celle du plus grand entier.

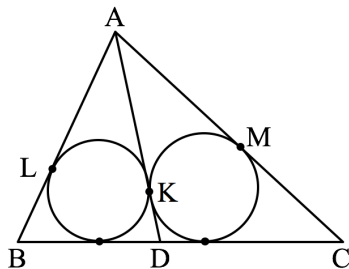
4304. *Proposé par Michel Bataille.*

Évaluer

$$\cot \frac{\pi}{7} + \cot \frac{2\pi}{7} + \cot \frac{4\pi}{7} + \cot^3 \frac{\pi}{7} + \cot^3 \frac{2\pi}{7} + \cot^3 \frac{4\pi}{7}.$$

4305. *Proposé par Moshe Stupel and Avi Sigler.*

Décrire comment déterminer le point D sur le côté BC du triangle ABC , tel que les cercles inscrits des triangles résultants ABD et ADC sont tangents l'un à l'autre à un certain point sur leur tangente en commun AD .



4306. *Proposé par Marius Drăgan.*

Démontrer que

$$\left[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \sqrt{n+3} \right] = \left[\sqrt{16n+20} \right]$$

pour tout $n \in \mathbb{N}$.

4307. *Proposé par Adnan Ibric and Salem Malikic.*

Dans un triangle non isocèle ABC , soient H et M l'orthocentre et le mipoint du côté BC , respectivement. La bissectrice interne de $\angle BAC$ intersecte BC et le cercle circonscrit de ABC aux points D et E , où $E \neq A$. Si K est le pied de la perpendiculaire de H vers AM et S est l'intersection (différente de E) des cercles circonscrits de ABC et DEM , démontrer que le quadrilatère $ASDK$ est cyclique.

4308. *Proposé par Leonard Giugiuc and Sladjan Stankovic.*

Soient a , b et c des nombres réels positifs. Démontrer que

$$27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2.$$

4309. *Proposé par Daniel Sitaru.*

Soient a, b et c des nombres réels tels que $a + b + c = 3$. Démontrer que

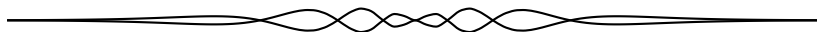
$$2(a^4 + b^4 + c^4) \geq ab(ab + 1) + bc(bc + 1) + ca(ca + 1).$$

4310. *Proposé par Steven Chow.*

Soit I le centre du cercle inscrit de $\triangle ABC$. A_1 est le point d'intersection de BC et AI ; B_1 et C_1 sont définis de façon similaire. Soit r le rayon du cercle inscrit de $\triangle ABC$.

- a) Démontrer que $AA_1 \cdot BB_1 \cdot CC_1 \geq \frac{3\sqrt{3}}{2} (BC + CA + AB) r^2$.
- b) ★ Prouver ou prouver le contraire de l'inégalité $B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq 3r^3\sqrt{3}$.

Remarque. Une mauvaise lecture du problème 4203 a conduit le proposeur à ce problème.



Chaos Train

Let $B(x, y)$ mean “person x accidentally bumps into person y on the train”. Consider the following statements, under the assumption that the quantifiers range over all the people on a particular New York subway train heading down town Monday morning. Match each formal statement with the informal one.

$\exists x \exists y B(x, y)$	Chaos train (or empty train?).
$\exists x \forall y B(x, y)$	Somebody was blocking the only exit.
$\forall x \exists y B(x, y)$	Unfortunately, it happens.
$\forall x \forall y B(x, y)$	Super klutz runs through the train.
$\exists y \forall x B(x, y)$	Universal suffering.
$\forall y \exists x B(x, y)$	Extremely crowded or empty train.

By Joel David Hamkins.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017: 43(1), p. 24–27.



4201. Proposed by Florin Stanescu.

Let M be a point in the interior of a regular polygon $A_1A_2 \dots A_n$ inscribed in the unit circle centered at O , and let A_kB_k be the chord from the vertex A_k through M . Prove that

$$\frac{A_1B_1^2 + A_2B_2^2 + \dots + A_nB_n^2}{n} \geq \frac{4}{1 + OM^2}.$$

All three submissions were correct; we feature the solution by Michele Bataille.

Let $p = 1 - OM^2$, the negation of the power of M with respect to the unit circle. Since $MA_k \cdot MB_k = p$ for $k = 1, 2, \dots, n$, we have

$$A_kB_k^2 = (A_kM + MB_k)^2 = A_kM^2 + \frac{p^2}{A_kM^2} + 2p,$$

and so

$$\sum_{k=1}^n A_kB_k^2 = \sum_{k=1}^n A_kM^2 + p^2 \sum_{k=1}^n \frac{1}{A_kM^2} + 2np. \quad (1)$$

From Leibniz's relation (see Focus On...No 16, Vol. 41(3), p. 110-113), we deduce

$$\sum_{k=1}^n A_kM^2 = nOM^2 + \sum_{k=1}^n OA_k^2 = n(1 + OM^2), \quad (2)$$

and from the Cauchy-Schwarz inequality applied to the vectors $(\frac{1}{A_1M}, \dots, \frac{1}{A_nM})$ and (A_1M, \dots, A_nM) ,

$$\sum_{k=1}^n \frac{1}{A_kM^2} \geq \frac{n^2}{\sum_{k=1}^n A_kM^2}. \quad (3)$$

(Alternatively, (3) compares the harmonic and arithmetic means: $\frac{1}{HM} \geq \frac{1}{AM}$.) Inserting (2) and (3) into (1) yields

$$\sum_{k=1}^n A_kB_k^2 \geq n(1 + OM^2) + \frac{n^2(1 - OM^2)^2}{n(1 + OM^2)} + 2n(1 - OM^2);$$

that is,

$$\sum_{k=1}^n A_kB_k^2 \geq \frac{4n}{1 + OM^2},$$

and the required inequality follows.

Editor's comment. Formula (2) can also be found in “Sums of Squares of Distances” by Tom M. Apostol and Mamikon A. Mnatsakanian [*Math Horizons*, **9**:2 (November 2001)]. It is formula (2) in the proof of their Theorem 1. For the special case of a triangle, it is Theorem 275 on page 174 of Roger A. Johnson's *Advanced Euclidean Geometry*.

4202. *Proposed by Roy Barbara.*

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$f(a^2 + b^2) = f(a)^2 + f(b)^2$$

for all $a, b \in \mathbb{N}$.

We received 8 correct solutions and will feature just one of them here, by Steven Chow.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. By replacing a and b with 0 in the given equation, $f(0) = 2f(0)^2$, so since $f(0) \in \mathbb{N}$, $f(0) = 0$.

By replacing b with 0 in the given equation, $f(a^2) = f(a)^2 + f(0)^2 = f(a)^2$. By replacing a with 1, $f(1) = f(1)^2 \implies f(1) \in \{0, 1\}$.

Using mathematical induction, it shall be proved that for all $x \in \mathbb{N}$, $f(x) = xf(1)$.

By replacing a and b with 1 in the given equation, $f(2) = f(1)^2 + f(1)^2 = 2f(1)$. By replacing a with 2 and b with 0, $f(4) = f(2)^2 + f(0)^2 = 4f(1)$. By replacing a with 2 and b with 1, $f(5) = f(2)^2 + f(1)^2 = 5f(1)$. By replacing a with 4 and b with 3, therefore $f(5)^2 = f(5^2) = f(4)^2 + f(3)^2$, so $f(3) = 3f(1)$. By replacing a and b with 2, $f(8) = f(2)^2 + f(2)^2 = 8f(1)$. By replacing a with 3 and b with 1, $f(10) = f(3)^2 + f(1)^2 = 10f(1)$. By replacing a with 8 and b with 6, therefore $f(10)^2 = f(10^2) = f(8)^2 + f(6)^2$, so $f(6) = 6f(1)$.

Therefore for all integers $0 \leq x \leq 6$, $f(x) = xf(1)$. Assume that for some integer $k \geq 6$, for all integers $0 \leq x \leq k$, $f(x) = xf(1)$.

If $k \equiv 0 \pmod{2}$, then

$$\begin{aligned} (k+1)^2 + \left(\frac{k}{2} - 2\right)^2 &= (k-1)^2 + \left(\frac{k}{2} + 2\right)^2 \\ \implies f\left((k+1)^2 + \left(\frac{k}{2} - 2\right)^2\right) &= f\left((k-1)^2 + \left(\frac{k}{2} + 2\right)^2\right) \\ \implies f(k+1)^2 + f\left(\frac{k}{2} - 2\right)^2 &= f(k-1)^2 + f\left(\frac{k}{2} + 2\right)^2, \end{aligned}$$

and $k \geq 6 \implies k \geq \frac{k}{2} - 2, k-1, \frac{k}{2} + 2$, so from the induction hypothesis, therefore $f(k+1) = (k+1)f(1)$.

If $k \equiv 1 \pmod{2}$, then

$$\begin{aligned} (k+1)^2 + \left(\frac{k-1}{2} - 4\right)^2 &= (k-3)^2 + \left(\frac{k-1}{2} + 4\right)^2 \\ \implies f\left((k+1)^2 + \left(\frac{k-1}{2} - 4\right)^2\right) &= f\left((k-3)^2 + \left(\frac{k-1}{2} + 4\right)^2\right) \\ \implies f(k+1)^2 + f\left(\frac{k-1}{2} - 4\right)^2 &= f(k-3)^2 + f\left(\frac{k-1}{2} + 4\right)^2, \end{aligned}$$

and $k \geq 7 \implies k \geq \frac{k-1}{2} - 4, k-3, \frac{k-1}{2} + 4$, so from the induction hypothesis, therefore $f(k+1) = (k+1)f(1)$.

Remark: It is easy to come up with those equations using $a^2 + b^2 = c^2 + d^2 \iff (a+c)(a-c) = (d+b)(d-b)$.

Therefore for all $x \in \mathbb{N}$, $f(x) = xf(1)$, so since $f(1) \in \{0, 1\}$, either $f(x) = 0$ for all x , which satisfies the given conditions, or $f(x) = x$ for all x , which satisfies the given conditions.

Therefore all such f are either $f(x) = 0$ for all $x \in \mathbb{N}$, or $f(x) = x$ for all $x \in \mathbb{N}$.

4203. Proposed by Michel Bataille.

The incircle of a triangle ABC has centre I , radius r and intersects the line segments AI, BI, CI at A', B', C' , respectively. Prove that

$$(a) \quad AA' \cdot BB' \cdot CC' \leq \frac{\sqrt{3}}{18}(AB + BC + CA)r^2;$$

$$(b) \quad A'B' \cdot B'C' \cdot C'A' \leq 3\sqrt{3}r^3.$$

We received 4 correct solutions to part (a) and 5 to part (b). We present the solution by Leonard Giugiuc.

(a) We have

$$AA' = AI - r = \frac{r}{\sin \frac{A}{2}} - r = \frac{r(1 - \sin \frac{A}{2})}{\sin \frac{A}{2}}.$$

Analogously, $BB' = \frac{r(1 - \sin \frac{B}{2})}{\sin \frac{B}{2}}$ and $CC' = \frac{r(1 - \sin \frac{C}{2})}{\sin \frac{C}{2}}$. Hence

$$AA' \cdot BB' \cdot CC' = \frac{r^3(1 - \sin \frac{A}{2})(1 - \sin \frac{B}{2})(1 - \sin \frac{C}{2})}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}.$$

On the other hand,

$$AB + BC + CA = 8R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Hence the required inequality is equivalent to

$$\frac{3\sqrt{3}r^3 \left(1 - \sin \frac{A}{2}\right) \left(1 - \sin \frac{B}{2}\right) \left(1 - \sin \frac{C}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \leq 4Rr^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$\frac{r^3 \left(1 - \sin \frac{A}{2}\right) \left(1 - \sin \frac{B}{2}\right) \left(1 - \sin \frac{C}{2}\right)}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \leq 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

But $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, so that the last inequality becomes

$$3\sqrt{3} \left(1 - \sin \frac{A}{2}\right) \left(1 - \sin \frac{B}{2}\right) \left(1 - \sin \frac{C}{2}\right) \leq \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \quad (1)$$

Set $X = \frac{B+C}{2}$, $Y = \frac{C+A}{2}$, and $Z = \frac{A+B}{2}$. Then X, Y, Z are acute and the sum $X + Y + Z = \pi$. Moreover, (1) becomes

$$3\sqrt{3}(1 - \cos X)(1 - \cos Y)(1 - \cos Z) \leq \sin X \sin Y \sin Z$$

$$3\sqrt{3} \sin^2 \frac{X}{2} \sin^2 \frac{Y}{2} \sin^2 \frac{Z}{2} \leq \sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} \cos \frac{X}{2} \cos \frac{Y}{2} \cos \frac{Z}{2}$$

$$\cot \frac{X}{2} \cot \frac{Y}{2} \cot \frac{Z}{2} \geq 3\sqrt{3}.$$

But since $\frac{X+Y+Z}{2} = \frac{\pi}{2}$,

$$\cot \frac{X}{2} \cot \frac{Y}{2} \cot \frac{Z}{2} = \cot \frac{X}{2} + \cot \frac{Y}{2} + \cot \frac{Z}{2},$$

and since the cotangent function is convex on $(0, \frac{\pi}{2})$, Jensen's inequality gives

$$\cot \frac{X}{2} + \cot \frac{Y}{2} + \cot \frac{Z}{2} \geq 3\sqrt{3}.$$

(b) The triangle $A'B'C'$ is inscribed in a circle of radius r , and we need to prove that $A'B' \cdot B'C' \cdot A'C' \leq 3\sqrt{3}r^3$ or equivalently $4[A'B'C'] \leq 3\sqrt{3}r^2$, which is a well-known inequality.

4204. Proposed by Leonard Giugiuc and Diana Trilescu.

Let ABC be a triangle with $AB \neq AC$ and let I be the incenter of ABC . Suppose the lines AI, BI and CI intersect the sides BC, CA and AB in D, E and F , respectively. If $DE = DF$ and $\angle ABC = 2\angle ACB$, find $\angle ACB$.

We received 9 submissions, all of which were correct; we present the solution by Madhav Modak, with the final paragraph modified by the editor.

We shall see that the required angle is $\angle ACB = \frac{\pi}{7}$. Let a, b, c denote the lengths of the sides BC, CA, AB respectively. We may assume that $a \neq c$, otherwise we would have $\angle A = \angle C = 45^\circ$ and $\angle B = 90^\circ$, which would imply $DE < DF$. Since $\angle B = 2\angle C$, the sine law gives $c \sin B = b \sin C$, or $2c \sin C \cos C = b \sin C$, or

$$b = 2c \cos C. \quad (1)$$

Hence $b = 2c(a^2 + b^2 - c^2)/2ab$ or $(a - c)b^2 = c(a^2 - c^2)$. Therefore, since $a \neq c$, we get $b^2 = c(a + c)$ or

$$a = \frac{b^2 - c^2}{c}. \quad (2)$$

Next, applying the cosine law to the triangles ADE and ADF , the condition $DE = DF$ gives

$$\begin{aligned} AD^2 + AE^2 - 2AD \cdot AE \cos(A/2) &= AD^2 + AF^2 - 2AD \cdot AF \cos(A/2) \iff \\ AE^2 - AF^2 &= 2AD(AE - AF) \cos(A/2) \iff \\ AE + AF &= 2AD \cos(A/2), \end{aligned}$$

using $AE \neq AF$ (because $AE = AF$ would imply $AB = AC$). Squaring, we have

$$\begin{aligned} (AE + AF)^2 &= 2AD^2(1 + \cos A) \iff \\ (AE + AF)^2 &= 2AD^2 \left[\frac{(b + c)^2 - a^2}{2bc} \right]. \end{aligned} \quad (3)$$

Using properties of angle bisectors gives us

$$AE = \frac{bc}{c + a}, \quad AF = \frac{bc}{a + b}, \quad AD^2 = bc \left[1 - \frac{a^2}{(b + c)^2} \right].$$

Substituting in (3) we get

$$\begin{aligned} b^2 c^2 \cdot \frac{(2a + b + c)^2}{(a + b)^2 (a + c)^2} &= 2bc \left[1 - \frac{a^2}{(b + c)^2} \right] \cdot \left[\frac{(b + c)^2 - a^2}{2bc} \right] \iff \\ b^2 c^2 \cdot \frac{(2a + b + c)^2}{(a + b)^2 (a + c)^2} &= \left[\frac{(b + c)^2 - a^2}{b + c} \right]^2. \end{aligned}$$

Taking positive square roots, we get

$$\begin{aligned} bc(2a + b + c)(b + c) &= (a + b)(a + c)[(b + c)^2 - a^2] \iff \\ a^3 + (b + c)a^2 - (b^2 + c^2 + bc)a - (b + c)(b^2 + c^2) &= 0. \end{aligned}$$

Using (2) to eliminate a , we get

$$\begin{aligned} \frac{b(b + c)^2}{c^3} (b^3 - b^2 c - 2bc^2 + c^3) &= 0 \iff \\ x^3 - x^2 - 2x + 1 &= 0, \quad x = b/c = 2 \cos C. \end{aligned} \quad (4)$$

We have seen this equation before in *Cruix*; its zeros are $2 \cos \frac{\pi}{7}$, $-2 \cos \frac{2\pi}{7}$, and $-2 \cos \frac{4\pi}{7}$. It is $P(x) = 0$ where $P(x)$ is essentially the polynomial on page 57 of Michel Bataille's recent article, "About the Side and Diagonals of the Regular Heptagon" [2017 : 55-60]. We conclude that ABC is what has been called the

heptagonal triangle — the scalene triangle that is formed by three vertices of a regular heptagon. Its angles are $\angle C = \frac{\pi}{7}$, $\angle B = \frac{2\pi}{7}$, and $\angle A = \frac{4\pi}{7}$.

Editor's comments. Zvonaru remarked that the converse of our problem is Result 14 on page 17 of Leon Bankoff and Jack Garfunkel, "The Heptagonal Triangle", *Mathematics Magazine*, **46**:1 (Jan.-Feb. 1973) 7-19: *The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle is isosceles.*

Triangle with $\angle B = 2\angle C$ appeared many times before on the pages of ***Crux***; see J. Chris Fisher's "Recurring Crux Configurations 7: Triangles whose angles satisfy $B = 2C$ " [2012: 238-240]. They are characterized by formula (2) above.

4205. *Proposed by Daniel Sitaru.*

Prove that for $0 < a < c < b$, $a, b, c \in \mathbb{R}$, we have

$$\frac{1}{c\sqrt{ab}} \int_a^b x \arctan x dx > \frac{(c-a) \arctan \sqrt{ac}}{\sqrt{bc}} + \frac{(b-c) \arctan \sqrt{bc}}{\sqrt{ac}}.$$

Ten correct solutions were received. They all followed the same strategy, some depending on the Hermite-Hadamard inequality. Our solution is based on that of Paul Bracken.

Let $f(x) = x \arctan x$ for $x > 0$. Since $f(0) = f'(0) = 0$, $f'(x) = \arctan x + x(1+x^2)^{-1}$ and $f''(x) = 2(1+x^2)^{-2}$, then f is positive, strictly increasing and strictly convex. By the Mean Value Theorem, we have that

$$f(p) + f'(p)(x-p) < f(x)$$

for distinct positive x and p . Hence

$$\begin{aligned} (c-a)f(\sqrt{ac}) &< (c-a)f(\sqrt{ac}) + \frac{1}{2}f'(\sqrt{ac})(c-a)(\sqrt{c}-\sqrt{a})^2 \\ &= (c-a)f(\sqrt{ac}) + f'(\sqrt{ac}) \int_a^c (x-\sqrt{ac}) dx \\ &< \int_a^c f(x) dx, \end{aligned}$$

and

$$(b-c)f(\sqrt{bc}) < \int_c^b f(x) dx.$$

Therefore

$$(c-a)\sqrt{ac} \arctan \sqrt{ac} + (b-c)\sqrt{bc} \arctan \sqrt{bc} < \int_a^b x \arctan x dx.$$

Dividing by $(\sqrt{ac})(\sqrt{bc})$ yields the desired inequality.

4206. *Proposed by Gheorghe Alexe and George-Florin Serban.*

Find positive integers p and q that are relatively prime to each other such that $p + p^2 = q + q^2 + 3q^3$.

We received 19 complete solutions. We present the one by Prithwjit De.

We observe that $p + p^2$ is even for any positive integer p . Therefore in any solution q must be even. By rewriting the given equation as

$$p(1 + p) = q(1 + q + 3q^2)$$

we obtain $p|(1 + q + 3q^2)$ and $q|(p + 1)$. We may also rewrite the equation as

$$(p - q)(p + q + 1) = 3q^3$$

which implies $p > q$. Since $\gcd(p - q, q) = \gcd(p, q) = 1$, we can conclude that $q^3|(p + q + 1)$ and therefore

$$q^3 - q - 1 \leq p \leq 1 + q + 3q^2,$$

which leads to

$$q^3 - 3q^2 - 2q - 2 \leq 0.$$

Thus $q \leq 3$ and since q is positive and even, $q = 2$. We obtain $(p, q) = (5, 2)$ as the only solution.

4207. *Proposed by Mihaela Berindeanu.*

Let x, y and z be real numbers such that $x + y + z = 3$. Show that

$$\frac{1}{1 + 2^{4-3x}} + \frac{1}{1 + 2^{4-3y}} + \frac{1}{1 + 2^{4-3z}} \geq 1.$$

We received 18 solutions. We present 2 solutions.

Solution 1, by AN-anduud Problem Solving Group.

We have $2^{4-3x} \cdot 2^{4-3y} \cdot 2^{4-3z} = 8$, hence there exist a, b, c positive real numbers satisfying the following equalities:

$$2^{4-3x} = \frac{2ab}{c^2}, \quad 2^{4-3y} = \frac{2bc}{a^2}, \quad 2^{4-3z} = \frac{2ca}{b^2}.$$

The given inequality is equivalent to

$$\begin{aligned} & \frac{1}{1 + \frac{2ab}{c^2}} + \frac{1}{1 + \frac{2bc}{a^2}} + \frac{1}{1 + \frac{2ca}{b^2}} \geq 1 \\ \iff & \frac{c^2}{c^2 + 2ab} + \frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} \geq 1 \end{aligned} \quad (1)$$

Using Cauchy-Schwarz inequality, we get

$$\frac{c^2}{c^2 + 2ab} + \frac{a^2}{a^2 + 2bc} + \frac{b^2}{b^2 + 2ca} \geq \frac{(a + b + c)^2}{(c^2 + 2ab) + (a^2 + 2bc) + (b^2 + 2ca)} = 1.$$

Thus inequality (1) is proved. Equality holds if and only if $x = y = z = 1$.

Solution 2, by Arkady Alt.

Let $a = 2^{4-3x}$, $b = 2^{4-3y}$, $c = 2^{4-3z}$. Then $a, b, c > 0$ and

$$abc = 2^{12-3(x+y+z)} = 8.$$

The original inequality becomes

$$\sum_{cyc} \frac{1}{1+a} \geq 1 \iff \sum_{cyc} (1+b)(1+c) \geq (1+a)(1+b)(1+c)$$

The last inequality gives

$$\begin{aligned} 3 + 2(a+b+c) + ab + bc + ca &\geq 1 + a + b + c + ab + bc + ca + abc \\ &= 9 + a + b + c + ab + bc + ca, \end{aligned}$$

so $a + b + c \geq 6$, which is true because by AM-GM Inequality

$$a + b + c \geq 3\sqrt[3]{abc} = 3\sqrt[3]{8} = 6.$$

4208. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x, y and z be positive real numbers such that $x \leq y \leq z$. Prove that for any real number $k > 2$, we have:

$$xy^k + yz^k + zx^k \geq x^2y^{k-1} + y^2z^{k-1} + z^2x^{k-1}.$$

We received 8 solutions. We present the one by Digby Smith.

Since $0 < x \leq y \leq z$ and $k > 2$, we have $0 < x^{k-2} \leq y^{k-2} \leq z^{k-2}$. Thus

$$\begin{aligned} &(xy^k + yz^k + zx^k) - (x^2y^{k-1} + y^2z^{k-1} + z^2x^{k-1}) \\ &= xy(y-x)y^{k-2} + yz(z-y)z^{k-2} + zx(x-z)x^{k-2} \\ &\geq xy(y-x)x^{k-2} + yz(z-y)x^{k-2} + zx(x-z)x^{k-2} \\ &= (xy^2 - x^2y + yz^2 - y^2z + zx^2 - xz^2)x^{k-2} \\ &= (z-y)(y-x)(z-x)x^{k-2}, \end{aligned}$$

where the last line is clearly non-negative. Hence the desired inequality follows, and clearly equality holds if and only if $x = y = z$.

4209. *Proposed by Nguyen Viet Hung.*

Let m and n be distinct positive integers. Evaluate

$$\lim_{x \rightarrow 0} \frac{(1+nx)^m - (1+mx)^n}{\sqrt[n]{1+mx} - \sqrt[n]{1+nx}}.$$

Twenty-one correct solutions were received. The majority followed the approach of Solution 1. Five solvers gave essentially Solution 2, and two had Solution 3.

Solution 1.

Since

$$\begin{aligned} & (1+nx)^m - (1+mx)^n \\ &= [1+mnx + \frac{m(m-1)}{2}n^2x^2 + o(x^2)] - [1+mnx + \frac{n(n-1)}{2}m^2x^2 + o(x^2)] \\ &= -\frac{mn(n-m)}{2}x^2 + o(x^2), \end{aligned}$$

and

$$\begin{aligned} (1+mx)^{1/m} - (1+nx)^{1/n} &= [1+x + \frac{1-m}{2}x^2 + o(x^2)] - [1+x + \frac{1-n}{2}x^2 + o(x^2)] \\ &= \frac{n-m}{2}x^2 + o(x^2), \end{aligned}$$

the desired limit is $-mn$.

Solution 2.

Let $a \equiv a(x) = \sqrt[n]{1+nx}$ and $b \equiv b(x) = \sqrt[m]{1+mx}$. The expression whose limit is sought is equal to

$$-\left[\frac{a^{mn} - b^{mn}}{a - b} \right] = -[a^{mn-1} + a^{mn-2}b + \cdots + ab^{mn-2} + b^{mn-1}].$$

Since $\lim_{x \rightarrow 0} a(x) = \lim_{x \rightarrow 0} b(x) = 1$, the desired limit is $-mn$.

Solution 3.

Using l'Hôpital's Rule twice, we find that

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1+nx)^m - (1+mx)^n}{(1+mx)^{1/m} - (1+nx)^{1/n}} \\ &= mn \lim_{x \rightarrow 0} \frac{(1+nx)^{m-1} - (1+mx)^{n-1}}{(1+mx)^{(1/m)-1} - (1+nx)^{(1/n)-1}} \\ &= mn \lim_{x \rightarrow 0} \frac{n(m-1)(1+nx)^{m-2} - m(n-1)(1+mx)^{n-2}}{(1-m)(1+mx)^{(1/m)-2} - (1-n)(1+nx)^{(1/n)-2}} \\ &= mn \frac{n(m-1) - m(n-1)}{(1-m) - (1-n)} = -mn. \end{aligned}$$

4210. *Proposed by Van Khea and Leonard Giugiuc.*

Let ABC be a triangle in which the circumcenter lies on the incircle. Furthermore, let $BC = a$, $CA = b$ and $AB = c$. For which triangles does the expression $\frac{a+b+c}{\sqrt[3]{abc}}$ attain its minimum?

We received 6 submissions, of which 5 were correct and one was incomplete. We present the solution by Arkady Alt.

Let I , O , r , R , and S denote the incenter, circumcenter, inradius, circumradius, and semiperimeter of $\triangle ABC$, respectively. It is known (Euler's Theorem) that $OI^2 = R^2 - 2Rr$. By assumption, O lies on the incircle of $\triangle ABC$, so $OI = r$. Hence, $R^2 - 2Rr = r^2$ if and only if

$$\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right) - 1 = 0 \iff \frac{R}{r} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2} \iff R = (\sqrt{2} + 1)r,$$

since $R \neq (1 - \sqrt{2})r$. It is known (Emmerich's Inequality, pg. 251 in *Recent Advances in Geometric Inequalities* by D. S. Mitrinović, J. Pečarić, V. Volenec) that for any non-acute triangle T , we have $R \geq (\sqrt{2} + 1)r$ with equality if and only if T is a right isosceles triangle. Hence we may now assume that $\triangle ABC$ is non-obtuse. Then $\cos A \cos B \cos C \geq 0$. Since

$$\cos A \cos B \cos C = \frac{S^2 - (2R + r)^2}{4R^2},$$

we have $S \geq 2R + r$ which by $R = (\sqrt{2} + 1)r$ implies

$$S \geq 2(\sqrt{2} + 1)r + r = (2\sqrt{2} + 3)r \quad \text{or} \quad \frac{S}{r} \geq 2\sqrt{2} + 3.$$

Therefore,

$$\begin{aligned} \frac{(a+b+c)^3}{abc} &= \frac{8S^3}{4RrS} = \frac{2S^2}{Rr} = \frac{2S^2}{(\sqrt{2}+1)r^2} \geq \frac{2(2\sqrt{2}+3)^2}{\sqrt{2}+1} \\ &= 2(17+12\sqrt{2})(\sqrt{2}-1) = 14+10\sqrt{2}. \end{aligned}$$

Since equality holds in $\frac{S}{r} \geq 2\sqrt{2} + 3$ if and only if $\cos A \cos B \cos C = 0$, we conclude that the lower bound $14 + 10\sqrt{2}$ can be attained only for a right angled triangle. Without loss of generality, we may assume that $C = 90^\circ$.

Since $2R = a + b - 2r$, we have

$$a + b = 2(R + r) = 2((\sqrt{2} + 1)r + r) = 2\sqrt{2}(\sqrt{2} + 1)r. \quad (1)$$

Also,

$$ab = 2Sr = 2(2\sqrt{2} + 3)r^2 = 2(\sqrt{2} + 1)^2 r^2. \quad (2)$$

From (1) and (2) we obtain $(a - b)^2 = (a + b)^2 - 4ab = 0$. Hence,

$$a = b = \sqrt{2}(\sqrt{2} + 1)r \quad \text{and} \quad c = \sqrt{a^2 + b^2} = \sqrt{2}a = 2(\sqrt{2} + 1)r.$$

Finally, since

$$\begin{aligned} (\sqrt[3]{2} + \sqrt[6]{32})^3 &= 2 + 3(2^{2/3})(2^{5/6}) + 3(2^{1/3})(2^{5/3}) + 4(2^{1/2}) \\ &= 2 + 3(2^2) + 3(2^{3/2}) + 4(2^{1/2}) = 14 + 10\sqrt{2}, \end{aligned}$$

we have

$$\min \frac{(a+b+c)}{\sqrt[3]{abc}} = \sqrt[3]{14 + 10\sqrt{2}} = \sqrt[3]{2} + \sqrt[6]{32},$$

attained if and only if $\triangle ABC$ is a right angled isosceles triangle.



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