Mathematical Spectrum

2000/2001 Volume 33 Number 1

- Niels Henrik Abel
- More than a match!
- The three Ws of West Indian cricket

A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year consisting of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

Niels Henrik Abel: 1802–1829

H. BURKILL

A biography of one of the earliest 'modern' mathematicians.

Although his country's history and his parents' temperaments conspired to inhibit Abel's genius, they could not stifle it. He died after accomplishing marvels during a short life that could have been so much more fulfilling, both personally and mathematically. Who knows what heights he might have scaled if only fate had been a little kinder to him?

When Abel was born, Norway had for centuries been the junior partner in the dual monarchy Denmark–Norway. The country was still primitive in many ways, largely ruled by Danish officials and lacking, as yet, institutions such as universities and banks. However, increasing trade, particularly with England, gave rise to a reawakening of national consciousness, from 1809 fostered by the Society for a Thriving Norway (STN).

During the Napoleonic Wars, British sea power initially kept Denmark–Norway in the Allied camp, but in 1807 a threatening French army forced the country to change sides; and in response Britain formed an alliance with Sweden. At that time the Swedish royal line was seen to be dying out, and Jean Bernadotte, one of Napoleon's marshals, was invited to become Crown Prince. The Norwegian nationalists had hoped to win independence as a result of this development and had, in anticipation, even drafted a democratic constitution for their country. However, the Allied powers permitted Bernadotte to subdue the nationalists, which he did in a short campaign, and in 1814 the sovereignty of Norway passed from the Danish to the Swedish crown. Norwegian independence did not, in fact, come about until 1905.

Niels Henrik's father, Sören Georg Abel (1772–1820) was a Lutheran pastor and the son of a pastor. He was educated in Denmark, first at a school in Elsinore, then at the University of Copenhagen. Talented, and endowed with all the social graces, he seemed destined for a brilliant career. Upon graduation in theology and philology, he was sent as curate to his father's parish of Gjerstad in south-west Norway. When he married, he was given his own parish of Finnöy, in the same area, which was much more fatiguing as parts of it could only be reached by boat. Undismayed, he threw himself enthusiastically into his work: to his pastoral duties he added a deep interest in education, as well as service as Justice of the Peace and Public Vaccinator. When his father died in 1803 he was pleased to succeed to the wealthier parish of Gjerstad, since by then he had a growing family (eventually to comprise six sons and a daughter). By this time he was also an ardent nationalist, and when the STN was formed he became the local chairman.

In 1811 the King agreed to the foundation of a university at Oslo and the STN raised nearly 1m daler for the purpose. Pastor Abel himself contributed very generously. In recognition of his public service he was knighted in 1813. Abel was also a member of the 1814 Storting (assembly) which drew up a constitution for the longed-for independent Norway; but sadly, as his national involvement increased, he grew ever more impatient with his parish duties. There were domestic strains as well and, as a result, both he and his wife began to drink to excess. Although Abel continued to do good work in the Storting, he now became increasingly quarrelsome. In 1818 he initiated a particularly unpleasant dispute and refused to tender the apology which would have saved his reputation. Instead he returned to his parish, unconsoled by its continuing loyalty, and drank himself to an early death. His widow, also an alcoholic, received a small pension from the parish, while the new pastor saw to it that the family did not become entirely destitute.

Initially Pastor Abel himself had taught his sons, but in 1815, by which time Niels Henrik, his second son, was 13, conditions at home were no longer ideal. The two eldest sons were therefore sent to the Cathedral School in Oslo which had a long and distinguished history. Unfortunately just then the School was suffering from the competition posed by the new University. The University had admitted its first students in 1813 and had given professorships to the best teachers in the School. Two of these were Sören Rasmussen, Professor of Pure Mathematics, and his former pupil, Christoffer Hansteen, now Professor of Astronomy and Applied Mathematics. Rasmussen, a capable administrator, also spent a good deal of time setting up the new Bank of Norway. The Cathedral School suffered physically too: when the Storting was in session it borrowed the School's auditorium and many of its classrooms, which had to be replaced by rented accommodation in the town.

Pastor Abel paid school fees for the first year, but subsequently the boys were supported by school scholarships. Niels Henrik's marks, at first very satisfactory, later sharply deteriorated, perhaps because he was upset by the public controversies that surrounded his father. He, as well as all other boys in the school, also had to put up with the sadism of the mathematics teacher, Hans Peder Bader, who had replaced the excellent Sören Rasmussen. Bader viciously assaulted the boys whether they had transgressed in some way or not, and one day in 1817 a boy was so severely injured that he died a week later. Bader was dismissed (though never prosecuted), and the 22-year-old Bernt Michael Holmboe was appointed in his place.

Holmboe, who was already assistant to Professor Hansteen, believed in letting his pupils work at their own pace. Niels Henrik flourished under this system and soon outstripped all other boys. In fact, it was not long before, in their exploration of mathematics, Niels Henrik became the leader and Holmboe the follower. A lesser man than Holmboe might have suffered some pangs of jealousy, but he showed nothing but delight at having discovered a genius. So, during the rest of his school career, Abel spent as much of his time as possible not only voraciously reading mathematics, but also working at ever more challenging problems.

Figure 1. Abel in 1826.

Figure 2. Bernt Holmboe.

At the age of 18, when in his last year at school, Abel turned his attention to very serious mathematics indeed, namely the solution of polynomial equations by radicals, which means that the only permitted operations on the coefficients are addition, subtraction, multiplication, division and the extraction of nth roots. It had been known since the 16th century that equations up to degree four could be solved under this constraint, but the quintic had resisted all attacks for 300 years. Abel now believed that he had devised the desired method. Since no-one in Oslo was able to follow his argument, Hansteen asked the Danish mathematician Carl Ferdinand Degen to check the paper and to submit it to the Danish Academy. Degen could find no mistake, but suggested that a numerical example would aid readers' understanding. Curiously, he found the subject matter 'sterile' and commended to Abel another topic that he judged to have more research potential. Both suggestions were adopted, eventually with spectacular consequences. However, at first there was disappointment: the numerical example asked for by Degen showed Abel that his method for the solution of the quintic was invalid.

Having passed the entrance examination, Abel was admitted to the University in the autumn of 1821. Since he had no means of support, the University allotted to him a free room in Regensen, the student residence; but there were no scholarships, and so some of the professors generously paid for his subsistence. Niels Henrik, in turn, helped the fourth of the Abel brothers, Peder, who showed some ability and was therefore sent to the Cathedral School: he asked for and got permission for Peder to share his room in Regensen. Sadly, the eldest Abel boy, who had started at school very promisingly, experienced a gradual clouding of his mind and had to be withdrawn in 1820.

The University required all students to take a preliminary examination at the end of the first year, but, once this was out of the way, Abel could devote himself entirely to mathematics. In 1823 Hansteen, the most distinguished scientist in Oslo, founded a Norwegian-language popular science magazine. It gave space to several articles by Abel, a welcome experience for him as previously he had had no outlet for his work. However, this facility was of strictly limited utility, for most of the subscribers could not cope with a diet too rich in mathematics. In the most interesting of these papers Abel solves, for ϕ, the equation

$$f(x) = \int_0^x (x-t)^{-\alpha} \phi(t)\, \mathrm{d}t \qquad (x > 0)$$

which arises from a physical problem in which f is given and $0 < \alpha < 1$. This is the first occurrence in the literature of a so-called *integral equation* and its solution; since then the subject has grown prodigiously.

The senior members of the University had already shown by their financial support of Abel how much they valued his rare talent. Now, in 1823, Professor Rasmussen gave Abel 100 daler to enable him to visit Degen and other mathematicians in Copenhagen. The trip was mathematically very fruitful; moreover during it he met Christine Kemp, soon to be his fiancée.

Figure 3. The University of Oslo, 1820–1852.

After his return from Copenhagen Abel once more attacked the problem of the quintic, but he no longer believed in the existence of a solution by radicals of the general quintic or higher degree polynomial equation. Intense labour now paid a rich dividend; around Christmas 1823 he finally proved that the general quintic could not be solved by radicals. Without access to suitable journals in which to publish this remarkable result, Abel decided to pay for its production in pamphlet form and to send copies to selected mathematicians in Norway and elsewhere. To help foreign mathematicians he wrote the text in French; but also, to save production costs, he condensed his argument to such an extent that the paper became virtually incomprehensible. At any rate, no mathematician reacted to it.

When Abel wrote another substantial paper he applied to the University for a grant to pay for its publication.

The eventual response to this request was unexpectedly favourable: he was to be given a small grant to support him whilst he improved his language skills, and then 600 daler a year for two years of study abroad. In the summer of 1825 Abel was ready for his Continental travel, but first he wished to attend to some of his family's problems. His brother Peder caused him much anxiety for he seemed to be in danger of succumbing to alcoholism; but he was most worried about his 15-year-old sister, for whom the Abel home was highly unsuitable. Fortunately Mrs Hansteen came to the rescue by taking the girl into her house and by undertaking to administer a fund of 50 daler that Niels Henrik left with her in case Peder got into difficulties. He also left some money for his mother.

Abel left Oslo for the Continent in September 1825. Being very sociable he was delighted that five of his friends were also engaging in foreign study, but their itineraries tended to pull them in different directions. Abel was to spend most of his time in Paris, then the centre of the mathematical world, while his friends were to winter in or near Berlin. Unable to face months of loneliness in Paris, he disregarded the terms of his fellowship and also made for Berlin. Among the letters of introduction he brought with him was one addressed to August Leopold Crelle, a distinguished engineer with a passionate interest in mathematics. Once they had overcome the language barrier between them, they became very good friends. Moreover, Crelle, quick to recognise Abel's genius, cordially invited him to contribute to the new mathematical journal that he was about to launch. In fact, Volume 1 of *Crelle's Journal*, which appeared in 1826, contained seven papers by Abel, one of them, at Crelle's suggestion, a greatly expanded version of his paper on the impossibility of solving the general quintic by radicals. Incidentally, from its beginning *Crelle's Journal* has been published almost uninterruptedly; the year 2000 begins with Volume 518.

During his stay in Berlin Abel became very conscious of the flimsy foundations on which most of the mathematics of his time was built, for rigour, the necessary binding material, was missing from the work of all mathematicians other than Gauss in Göttingen, Cauchy in Paris, and now himself. As part of his offensive against sloppiness he wrote his celebrated paper on power series in which, among many results, he obtained precise conditions under which the binomial identity

$$(1+x)^\alpha = \sum_{n=0}^{\alpha} \binom{\alpha}{n} x^n$$

holds.

Abel and his friends did not work every hour of the day. They frequented the theatre, becoming authorities on their favourite actresses; and they also went to balls, especially at carnival time. Unfortunately for Abel this turned out to be a time of worry as well as of pleasure. Rasmussen decided that his work for the Bank of Norway took precedence over his mathematics and resigned from his professorship. His former teaching duties were then assigned to Holmboe, whom the University appointed to a lectureship in mathematics. Knowing how poor the University was, Abel realised this meant that, in the foreseeable future, there would be no permanent position for him in Oslo, and that he could only look forward to his accustomed financial worries. In spite of a feeling of injustice, Abel wrote Holmboe a warm letter of congratulation, and their friendship continued unchanged.

In the spring of 1826 Abel and his friends made plans for further travel. Abel was due to visit first Gauss and then as many of the Paris mathematicians as practicable. He did not relish facing Gauss who could be unwelcoming to visitors, but Crelle offered to accompany him to Göttingen and possibly to Paris, thus making the trip less daunting. So, when Crelle had to withdraw his offer, Abel simply cancelled the visit to Gauss and fell in with his friends' plans.

For the next few months the young men's travels were, on the whole, devoted to sightseeing. Abel had some qualms, but salved his conscience with the thought that, if he did not now take the opportunity of seeing the world, it would never recur. To judge from their letters home the friends certainly gained much from their experience. Meeting in Dresden they travelled to Prague, Vienna and Trieste by coach and then by steamer to Venice. The party largely broke up in Verona; and, after some final sightseeing in Switzerland, Abel with one companion only, reached Paris in July 1826.

Abel's arrival in Paris was, unfortunately, ill-timed, for the city had emptied during the high summer. However, initially he had plenty to do: he was polishing the great paper that he had brought with him for presentation to the French Academy and which he hoped would establish his reputation. The paper was received at the Academy's meeting of 30 October 1826; A. L. Cauchy (1789–1857) and the veteran analyst A. M. Legendre (1752–1833) were appointed referees, the former chief referee. Abel naturally hoped for a quick decision and, having met several young mathematicians who had made their mark in this way, he was doubly disappointed that the only reaction to his paper was silence. No doubt he was also lonely and so, after Christmas, he spent his remaining resources on the fare to Berlin. He would have preferred to return to Norway, but felt obliged to remain abroad until his fellowship had expired. It was at this time that he consulted a doctor on account of a persistent cough and was told that he had tuberculosis.

Back in 1821, when he was still a student, Abel had begun to think about elliptic integrals and functions, the research area that Professor Degen had recommended to him. For a sketch of the basic ideas we begin with a rational function $F(x, y)$ and a polynomial $g(x)$. It was commonplace to Abel's contemporaries that, when g was of the first or second degree, then

$$u(x) = \int_0^x F\big(s, \sqrt{g(s)}\,\big)\,\mathrm{d}s \tag{1}$$

was an elementary function, whereas a g without a repeated factor and of degree 3 or 4 led to a non-elementary integral. The latter is called an *elliptic integral* because the arc length of an ellipse (which is not a circle) is an integral of this kind. To see this, note first that the length of the ellipse

$(x^2/a^2) + (y^2/b^2) = 1$ is

$$4a \int_0^1 \sqrt{\frac{1 - e^2 s^2}{1 - s^2}}\, \mathrm{d}s, \tag{2}$$

where e is the eccentricity given by $a^2 - b^2 = a^2 e^2$. If, now, $F(s, t) = (1 - e^2 s^2)/t$ and $g(s) = (1 - s^2)(1 - e^2 s^2)$, then $F(s, \sqrt{g(s)}\,) = \sqrt{(1 - e^2 s^2)/(1 - s^2)}$, so that (2) is of the form (1).

Now consider the integral (1) in which $F(s, t) = t/(1 - s^2)$ and $g(s) = 1 - s^2$. Then (1) is simply

$$u = \int_0^x \frac{1}{\sqrt{1 - s^2}}\, \mathrm{d}s,$$

i.e. $u = \sin^{-1} x$, a relation most simply investigated in the form $x = \sin u$. With the insight of genius Abel divined that it would be rewarding to carry out this kind of inversion in (1), i.e. to study x as a function $J(u)$ of u (rather than u as a function of x), also when $g(s)$ was of degree 3 or 4 and without a repeated factor. The extensions $J(z)$ of these functions to the whole complex plane turned out to be altogether remarkable and Abel called them *elliptic functions.* One of their spectacular properties is *double periodicity*: to each elliptic function $J(z)$ there correspond complex numbers ω_1, ω_2, with ω_1/ω_2 non-real, such that, for all z,

$$J(z + \omega_1) = J(z) = J(z + \omega_2).$$

This implies that $J(z)$ need only be studied in the fundamental parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$. (See figure 4.)

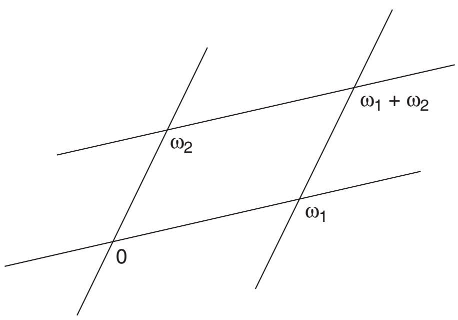


Figure 4. The fundamental parallelogram arising from ω_1, ω_2.

Abel took yet another leap forward when he initiated the study of *hyperelliptic integrals,* namely integrals (1) in which g is of degree greater than 4; and of the much more general *'Abelian integrals'*

$$\int_0^x F(s, t)\, \mathrm{d}s$$

in which F is again a rational function of s and t, while t is now an *algebraic function* of s. Abel had defined elliptic functions from elliptic integrals, but his short life prevented him from obtaining *hyperelliptic functions* and *Abelian functions* in an analogous manner. The formidable technical difficulties were eventually overcome by some of the foremost mathematicians of the first half of the 19th century. These classes of functions are employed in much of pure and applied mathematics. It has been remarked that they turn up in all manner of places, usually unexpected ones. Their literature is therefore vast.

Abel's return to Oslo in May 1827 was not a happy one. As ever, his problems were financial: a small grant from the University could only be supplemented by a small income from tutoring schoolboys. Fortunately, though, things improved early in 1828, for Hansteen had obtained a grant for a two-year scientific expedition into Siberia, and Abel was to take over his teaching duties. Meanwhile the first part of his revolutionary paper on elliptic functions had appeared in *Crelle's Journal* and this generated enormous interest. The new approach led to rapid advances in the subject with Abel in the van, but with the slightly younger K. G. J. Jacobi (1804–1851) at his heels. At this stage in the development of the theory the key ideas were certainly Abel's. Nevertheless, Abel himself had to some extent been anticipated by Gauss who, however, chose not to publish his discoveries.

Two years after having submitted his great Academy paper Abel had still not heard its fate. Lest it should sink entirely into oblivion he wrote a brief note on it late in 1828. On the other hand, some idea of its contents must, by then, have become known, for F. W. Bessel wrote to his friend Gauss on 2 January 1829:

> I am very surprised by Abel's Theorem which enables one to obtain properties of integrals without first evaluating them. It is extraordinary that this theorem was presented to the Paris Academy and yet remains unknown. What do you think? Do I over-rate the theorem?

There is no known reply, but Legendre had no doubts; he called it, in Horace's words, *'monumentum aere perennis'* (a memorial longer lasting than bronze).

Anxious to devote as much of his time as possible to elliptic functions, Abel at one time suspended his work on the solution by radicals of polynomial equations of arbitrary degree. He had proved that, if all roots of an equation can be expressed as rational functions of one of them, say x, and if any two of the roots, say $\theta(x)$ and $\phi(x)$, are such that

$$\theta\big(\phi(x)\big) = \phi\big(\theta(x)\big),$$

then the equation can be solved by radicals. Owing to this early and extensive use of commutativity by Abel, groups in which the operation is commutative came, in his honour, to be known as *Abelian.*

While Abel was concentrating on research into elliptic functions, another tragic young genius, the French mathematician Évariste Galois (1811–1832), created an elegant new theory giving conditions for the solvability by radicals of polynomial equations of arbitrary degree. To some extent the researches of Abel and of Galois intertwine. For instance the insolvability of the general quintic follows from some of Galois' work, while Galois uses a result of Abel's, and both independently formulate and employ the notion of a field.

By the beginning of 1828 Abel's genius was gaining recognition among the Continental mathematicians, while his contrasting poverty was emphasised by Crelle who made great efforts to secure for his friend a position worthy of his talents. Four distinguished members of the French Academy even petitioned Bernadotte, the French-born King of Norway–Sweden, to come to Abel's aid, but their intervention was unavailing. However, in spite of the distracting uncertainty, a stream of papers continued to flow from his pen.

Although there could be no question of marriage as long as Abel did not have a permanent post, he saw much more of his fiancée, Christine Kemp, after she had moved from Denmark to the town of Arendal on the Norwegian coast, about 120 miles south of Oslo. She was now governess in a household long friendly with the Abel family, and Niels Henrik was therefore able to spend the 1828 summer vacation with her. Obliged to return to Oslo for the new academic year, he insisted on going back to Arendal for Christmas though this involved a bitterly cold sledge journey lasting several days. While waiting for the sledge that was to take him back to Oslo he suffered a severe lung haemorrhage; and, despite some initial signs of recovery, he died three months later on 6 April 1829, at the age of 26.

Figure 5. Christine Kemp in 1835.

Though destined not to marry Christine Kemp, Abel was instrumental in bringing about her marriage. After his death one of the special friends with whom he had travelled on the Continent wrote to her, without ever having met her, proposed to her, and was accepted. No doubt Abel had extolled to each the virtues of the other.

Two days after Abel's death, Crelle wrote triumphantly to report that he had at last succeeded in securing a suitable position for him. Abel's standing was now beyond doubt, but the official seal of approval came a year later when the French Academy awarded its Grand Prix jointly to Abel and Jacobi. The sheer bulk of Abel's research output is also amazing; in the space of seven years he produced 29 papers, totaling 619 pages.

In March 1829 Jacobi noticed a reference to Abel's great Paris paper of 1826, as yet unpublished. It emerged that both the referees, Cauchy and Legendre, had been negligent, but they quickly recommended publication once they had been reminded of the paper's existence. The manuscript then got caught up in the revolution of 1830 and was not printed until 1841. Incredibly, the manuscript again disappeared and only resurfaced in 1952 in Florence.

Figure 6. Monument erected by Abel's friends at his grave.

Acknowledgements

Figures 1–3, 5 and 6 are reprinted from *Niels Henrik Abel, Mathematician Extraordinary*, by Oystein Ore (© University of Minnesota Press, 1957).

References

1. N. H. Abel, *Oeuvres complètes,* Nouvelle édition publiée aux frais de l'état norvégien par M. M. L. Sylow et S. Lie, 2 vol. (Christiania, 1881).
2. E. T. Bell, *The Development of Mathematics,* 2nd edn (Dover Publications Inc., New York, 1945).
3. *Dictionary of Scientific Biography*, C. C. Gillespie, ed. (Charles Scribner & Sons, New York, 1970).
4. *Encyclopædia Britannica* (Chicago, 1967).
5. Oystein Ore, *Niels Henrik Abel, Mathematician Extraordinary* (Chelsea, New York, 1957).
6. B. L. van der Waerden, *A History of Algebra* (Springer, Berlin, 1980).

Before retiring, ***Harry Burkill*** *was a reader in pure mathematics at the University of Sheffield; he edited* Mathematical Spectrum *from 1974 to 1979.*

More than a Match!

DAVID YATES

How good are you at guessing?

One year, my usual family Christmas quiz took this form: match 10 painters with their paintings. At least a few correct answers would be expected just by chance, so I worked out the odds, finding the general solution for the probability $\mathrm{P}(n, a)$ of at least n painters and paintings correctly matching where there are a painters and paintings. I thought that, for example, P(4, 10), P(4, 100), P(4, 1000) would be appreciably different probabilities, but, as you will see, I got a surprise!

I was able to find a recursion to generate the sequence of the number of ways in which there are no correct matches (i.e. the derangements) for successive numbers of painters and paintings, reasoning as follows. Let the number of ways that there are no correct matches for $n-1$ painters and paintings be D_{n-1}. Add another painter correctly assigned to a painting (the nth of each) to one of the D_{n-1} derangements of the $n-1$ painters and paintings. Now we have n painters and paintings, with exactly one correctly matched. If the correctly assigned painter is exchanged with any of the incorrectly assigned painters, then we would have a derangement of the n painters and paintings. This can be done $n-1$ ways. But there are D_{n-1} possible original derangements of the $n-1$ painters and paintings. Thus, there are $(n-1)D_{n-1}$ ways to produce derangements among the n paintings and painters this way. But, if just one of the original $n-1$ painters had been correctly assigned, and the other $n-2$ deranged (which could be in any of D_{n-2} ways), there would be the possibility of exchanging the new (nth) correctly assigned painter with that one, producing some derangements. The number of derangements it would be possible to produce this way would be $(n-1)D_{n-2}$. There are evidently no other ways to produce derangements among the n paintings and painters besides these two methods. The total possible number of derangements among the n painters and paintings is thus given by the recursion

$$D_n = (n-1)(D_{n-1} + D_{n-2}). \tag{1}$$

Then, the numbers of ways in which there are no correct matches with increasing numbers of painters and paintings form the sequence (given that D_1 and D_2 are obviously 0, 1)

$$0, 1, 2, 9, 44, 265, 1854, \ldots. \tag{2}$$

From (1) we have

$$\begin{aligned} D_n - nD_{n-1} &= -\big(D_{n-1} - (n-1)D_{n-2}\big) \\ &= D_{n-2} - (n-2)D_{n-3} \\ &= -(D_{n-3} - (n-3)D_{n-4}) \\ &= \cdots \\ &= (-1)^n (D_2 - 2D_1), \end{aligned}$$

i.e.

$$D_n = nD_{n-1} + (-1)^n. \tag{3}$$

Now

$$\begin{aligned} D_n &= n[(n-1)D_{n-2} + (-1)^{n-1}] + (-1)^n \\ &= n(n-1)[(n-2)D_{n-3} + (-1)^{n-2}] \\ &\qquad + n(-1)^{n-1} + (-1)^n \\ &= \cdots \\ &= n(n-1)\ldots 3D_2 + n(n-1)\ldots 4(-1)^3 \\ &\quad + \cdots + (-1)^n \\ &= n!\Big(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\Big) \\ &= n!\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\Big). \end{aligned}$$

Then, since there are $a!/(a-n)!\,n!$ ways of picking out n things from a things, the number of ways for exactly n painters and paintings to be incorrectly matched where there are a painters and paintings is given by

$$\begin{aligned} &\Big[\frac{a!}{(a-n)!n!}\Big]n!\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big) \\ &\qquad = \Big[\frac{a!}{(a-n)!}\Big]\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big). \end{aligned}$$

Since the total number of possible matchings is $a!$, the probability of exactly n painters and paintings being incorrectly matched where there are a painters and paintings is given by the expression

$$\Big[\frac{1}{(a-n)!}\Big]\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big). \tag{4}$$

Then, the probability that at least n are correctly matched is 1 minus the sums of the probabilities that exactly $a-n+1$, $a-n+2, \ldots, a$ are incorrectly matched, i.e.

$$\begin{aligned} \mathrm{P}(n, a) = 1 - \Bigg\{ &\frac{1}{(n-1)!}\Big[\frac{1}{0!} - \frac{1}{1!} + \cdots + \frac{(-1)^{a-n+1}}{(a-n+1)!}\Big] \\ &+ \frac{1}{(n-2)!}\Big[\frac{1}{0!} - \frac{1}{1!} + \cdots + \frac{(-1)^{a-n+2}}{(a-n+2)!}\Big] \\ &+ \cdots + \frac{1}{0!}\Big[\frac{1}{0!} - \frac{1}{1!} + \cdots + \frac{(-1)^a}{a!}\Big]\Bigg\} \end{aligned}$$

which can be rearranged to give

$$\begin{aligned}\mathrm{P}(n,a)=1-\Big\{&\Big[\frac{1}{0!}-\frac{1}{1!}+\cdots+\frac{(-1)^{a-n+1}}{(a-n+1)!}\Big]\\&\times\Big[\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+\cdots+\frac{1}{0!}\Big]\\&+\frac{(-1)^{a-n+2}}{(a-n+2)!}\Big[\frac{1}{(n-2)!}+\cdots+\frac{1}{0!}\Big]\\&+\cdots+\frac{(-1)^a}{a!}\Big[\frac{1}{0!}\Big]\Big\}.\end{aligned}\tag{5}$$

But $1/0!-1/1!+\cdots+(-1)^{a-n+1}/(a-n+1)!$ are the first few terms in the expansion of e^{-1}, so as a gets bigger this converges to

$$1-\mathrm{e}^{-1}\Big[\frac{1}{(n-1)!}+\frac{1}{(n-2)!}+\cdots+\frac{1}{0!}\Big].\tag{6}$$

After working this out, I was surprised to find that the convergence can be very fast. For example, using (5), the probability of correctly matching at least four painters and paintings by chance can be seen to lie between $\frac{1}{52}$ and $\frac{1}{53}$ from only eight painters and paintings upwards, converging to $1-\left(\frac{8}{3}\right)\mathrm{e}^{-1}$ by (6) with $n=4$.

Having solved the problem, I wondered how others had tackled matching problems. I found plenty of material on the Internet; see for example Torsten Sillke's extensive annotated references (reference 1), one of which, L. Takács (reference 2), gives a history of the problem and various methods important for the development of statistical theory invented to tackle it. Recursion was the first method used in problems of this sort, beginning with Pierre de Montmort in 1708, who corresponded with Johann and Nikolaus Bernoulli on what was called the *'problème des rencontres'*. They were trying to determine the probabilities of winning a game of chance (*Jeu du Treize*), the analysis of games of chance figuring greatly in the early development of probability theory. Euler, in 1779 (see reference 2), was the first to present the derangements form of recursions derived here. Takács, in 1946 (see reference 2), proved Euler's recursions using determinants, and Remmel, in 1983 (reference 3), gave proofs of them from combinatorics.

But there are many more approaches to the problem than recursion. From Nikolaus Bernoulli's methods (see reference 2) can be derived the principle of 'inclusion and exclusion' (an account of inclusion–exclusion is given in reference 4, for example). The probability that at least one of two events (A and B) happens is the probability of A, i.e. P(A), plus the probability of B, i.e. P(B), minus the probability of both, since P(A) includes both the probability of A alone and the probability of A with B, and similarly for B. Hence, adding P(A) and P(B) includes the probability of 'both A and B' twice, so then subtracting one of them leaves just the probability of 'A alone or B alone or both'. Thus

$$\mathrm{P}(\mathrm{A}\cup\mathrm{B})=\mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})-\mathrm{P}(\mathrm{A}\cup\mathrm{B}).$$

This extends further. Thus, for three events,

$$\begin{aligned}\mathrm{P}(\mathrm{A}\cup\mathrm{B}\cup\mathrm{C})=\mathrm{P}(\mathrm{A})+\mathrm{P}(\mathrm{B})+\mathrm{P}(\mathrm{C})-\mathrm{P}(\mathrm{A}\cap\mathrm{B})\\-\mathrm{P}(\mathrm{A}\cap\mathrm{C})-\mathrm{P}(\mathrm{B}\cap\mathrm{C})+\mathrm{P}(\mathrm{A}\cap\mathrm{B}\cap\mathrm{C}).\end{aligned}$$

Generally, the probability of at least one of a number of events happening is the probability of each one, minus the probability of each pair plus the probability of each treble (since in subtracting the pairs, the trebles upwards would also be subtracted and they need adding back in), and so on.

Now the number of ways for n items of a to match a given ordering of the a items correctly, ignoring for now whether more than n match, is the number of ways of choosing n of the a multiplied by the number of ways for the rest to be distributed, namely

$$\Big[\frac{a!}{n!(a-n)!}\Big](a-n)!=\frac{a!}{n!}$$

and the probability of this is $1/n!$.

Then the probability of at least one of a correctly matching the ordering, by inclusion–exclusion, is

$$\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+\frac{(-1)^a}{a!},$$

from which we can obtain that the probability there are no matches of the ordering is

$$1-\Big(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+\frac{(-1)^a}{a!}\Big),$$

which is simply (4) with $n=a$.

I was very gratified to learn that my matching problem, in Takács' words, 'had great influence on the development of probability theory'!

References

1. Torsten Sillke, Average Number of Matches (last update 20.4.98), http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/fixpoint-expected.
2. L. Takács, The Problem of Coincidences, *Arch. Hist. Exact Sci.* **21**(3) (1980), pp. 229–244.
3. J. B. Remmel, A Note on a Recursion for the Number of Derangements, *Europ. J. Combin.* **4** (1983), pp. 371–374.
4. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 3rd edn (John Wiley, New York, 1968).

David Yates *has a Ph.D. in psychology and does independent research in philosophical psychology. He obtained an A level in mathematics over 30 years ago and likes to dabble in mathematics for fun!*

Random Selection of 3-digit Numbers

EDWARD ALLEN

If 3-digit numbers are selected so that they differ from each other in two or more positions, then the total possible number selected is not unique but ranges from 50 to 100.

1. Introduction

This article examines some problems in the random selection of 3-digit numbers; the study originated in the following manner. In a regional mathematics contest, students were to be examined so that their names remained anonymous. Each student was assigned a unique random 3-digit identification number to be written, in place of his or her name, on the answer sheet. To reduce the possibility of a mix-up associated with any student putting down the incorrect identification number, the numbers were randomly selected so that each identification number differed from every other one in at least two positions. A computer program was written that randomly selected 3-digit numbers, each of which differed from the others in at least two positions. The computations in the program continued until no more numbers could be selected satisfying the required condition. However, each time the program was run, a different number of identification numbers was generated. This surprising result motivated the present study. Of particular interest was the minimum number, the average number, and the maximum number of identification numbers that could be generated.

After some notation is defined, an analysis of the problem is presented. Additional research, suggested by this study, is accessible to undergraduate students.

2. Notation and Analysis

Let $S = \{000, 001, 002, \dots, 998, 999\}$ be the set of all 3-digit numbers. Notice that S has 1000 elements. If $x \in S$, then x will be written as $x = x_1x_2x_3$ where $0 \le x_j \le 9$ is the digit in the jth position of the number x. Two elements x and y of S differ in at least two positions if $x_j \ne y_j$ for at least two values of $j = 1, 2, 3$. For example, 134 and 035 differ in two positions, 134 and 717 differ in three positions, and 134 and 135 differ in one position.

Let $T \subset S$ be constructed in the following manner. First, T is set equal to the empty set. Then, one at a time, an element x of S is randomly selected. If x differs from each element in T in at least two positions, then x is included in T. This process is continued until no more elements from S can be included in T. Hence, the final set T satisfies the following condition.

Condition on *T*. *The elements of T differ from each other in at least two positions and no additional element of S can be included in T without violating this restriction.*

Let N be the number of elements or the size of T, $N = |T|$. The number N is of interest in this study. First, as a result of the random selection procedure for T, N is a random variable. Indeed, the elements of T and the size of T depend on the order in which the elements from S are selected. For example, if 135 and 234 are selected first, then 134 cannot be included in T. However, if 134 is selected first, then neither 135 nor 234 can be included in T. The probability distribution of N is unknown. However, by performing the above procedure computationally many times, the probability distribution for N can be estimated. The computational results, when the above procedure was performed 10,000 times, are summarized in table 1 and displayed in figure 1. For these 10,000 sets, the value of N ranges from 78 to 94. The average value of N is 86 and over 50% of the values of N are either 85, 86, or 87. A listing of one randomly selected set T, when $N = 90$, is given in table 2.

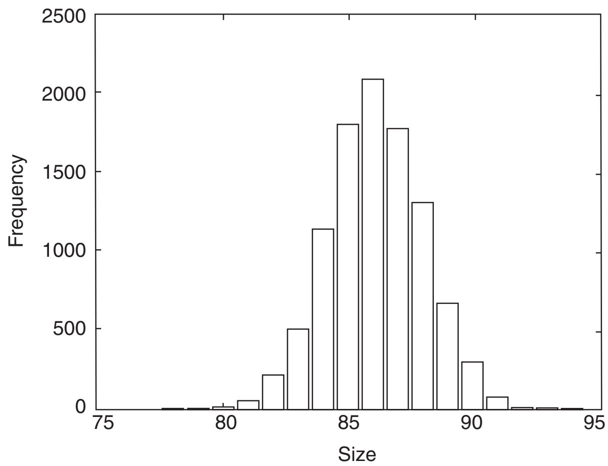


Figure 1. Frequencies of sizes for 10,000 sets.

Based on these results, one might guess that the maximum size of set T is 100 and the minimum size is 75. As will be seen shortly, the maximum size is indeed 100. However, by using symmetry properties, a set of size 50 can be constructed. To prove that 50 is indeed the minimum size is interesting. First, however, we show that T cannot have over 100 elements.

Proposition 1. *The maximum size of T is 100.*

Table 1. Frequencies of sizes N for 10,000 randomly selected sets T

Size	Frequency	Size	Frequency
76	0	86	2094
77	0	87	1782
78	1	88	1314
79	1	89	675
80	10	90	301
81	53	91	82
82	212	92	12
83	508	93	4
84	1142	94	1
85	1808	95	0

Table 2. Several examples of set T

A randomly selected set of size 90									
463	595	428	312	146	101	588	736	122	689
539	937	373	540	954	842	645	027	032	368
715	285	326	231	879	824	405	164	506	357
066	758	902	567	048	175	551	472	217	981
774	993	886	344	456	259	782	769	978	000
696	399	204	925	480	270	419	638	623	514
610	130	243	497	262	916	652	890	071	703
187	335	083	811	853	298	441	677	808	720
791	113	747	055	094	949	434	865	960	661
A set of minimum size 50									
000	011	022	033	044	101	112	123	134	140
202	213	224	230	241	303	314	320	331	342
404	410	421	432	443	555	566	577	588	599
656	667	678	689	695	757	768	779	785	796
858	869	875	886	897	959	965	976	987	998
A set of maximum size 100									
000	011	022	033	044	055	066	077	088	099
101	112	123	134	145	156	167	178	189	190
202	213	224	235	246	257	268	279	280	291
303	314	325	336	347	358	369	370	381	392
404	415	426	437	448	459	460	471	482	493
505	516	527	538	549	550	561	572	583	594
606	617	628	639	640	651	662	673	684	695
707	718	729	730	741	752	763	774	785	796
808	819	820	831	842	853	864	875	886	897
909	910	921	932	943	954	965	976	987	998

Proof. First, a set T is constructed that has exactly 100 elements. Let $x_{m,j}$ be the jth digit for the mth element of T. Let $x_{m,1} = [(m-1)/10]$, $x_{m,2} = (m-1) \pmod{10}$, and $x_{m,3} = (x_{m,1} + x_{m,2}) \pmod{10}$ for $m = 1, 2, \ldots, 100$ where $[a]$ is the greatest integer less than or equal to a. This set T, listed in table 2, can be verified to satisfy the required condition on T.

To see that a set T cannot have more than 100 elements, notice that the first and second positions for elements of T have at most 100 possibilities, namely, $00, 01, 02, \ldots, 99$. If a set T has more than 100 elements, then some element of T is identical in the first two positions with some other element of T and thus, this set does not satisfy the required condition. Hence, the maximum size of set T is 100.

Consider now the minimum number of elements that T can have. Based on the computational results summarized in table 1, it is difficult to believe that the minimum size of T could be 50. To see how a set T of 50 elements can be constructed, notice that any element of S must have either two or more digits between 0 and 4 inclusive or two or more digits between 5 and 9 inclusive. Also, it is straightforward to find a set of 25 3-digit numbers with digits 0, 1, 2, 3 and 4 such that any element of S that has two digits between 0 and 4 must agree with one of these 25 numbers in two positions. (Indeed, this set can be constructed in a manner analogous to that used for the set of maximum size 100 but using 5 digits instead of 10. Specifically, letting $x_{m,j}$ be the jth digit for the mth element of this set, then $x_{m,1} = [(m-1)/5]$, $x_{m,2} = (m-1) \pmod 5$, and $x_{m,3} = (x_{m,1} + x_{m,2}) \pmod 5$ for $m = 1, 2, \ldots, 25$.) Similarly, 25 3-digit numbers can be found using digits 5, 6, 7, 8 and 9 such that any element of S with two digits between 5 and 9 must agree with one of these 25 numbers in two positions. Thus, the union of these two sets of 25 numbers forms a set T of size 50. One such set of 50 elements is listed in table 2. The following proposition shows that 50 is the minimum size of T.

Proposition 2. *The minimum size of T is 50.*

Proof. Table 2 presents a set with 50 elements that satisfies the required condition on T. Suppose now that there is a set T of size N that satisfies the required condition. It will be shown that N cannot be less than 50 which implies that the minimum size of T is 50.

Let $S_{12} \subset S$ be all the numbers in S such that if $x \in S_{12}$ then x agrees in the first two positions with some $y \in T$. That is, if $x = x_1x_2x_3$ then there exists a $y = y_1y_2y_3$ in T with $x_1 = y_1$ and $x_2 = y_2$. In a similar manner, define S_{13} and S_{23}. (That is, S_{ij} is the set of all elements in S that agree in the ith and jth positions with some element in T.) Notice that each set S_{ij} has exactly $10N$ elements. To see this, notice that if $y \in T$ has the form $y = y_1y_2y_3$, then $y_1y_2z \in S_{12}$ for $z = 0, 1, 2, \ldots, 9$. Thus, as each y_1y_2 is distinct in T and T has N elements, there are exactly $10N$ elements in each set S_{12}, S_{13}, and S_{23}.

Now let $R = S_{12} \cup S_{13} \cup S_{23}$. If $R = S$, then any $x \in S$ must be an element of S_{12}, S_{13}, or S_{23} and therefore must have at least two digits in common with some element of T. Hence, no element of S can be added to T and T is as large as possible. However, if the size of R is less than that of S, then additional elements of S can be added to T and so T does not satisfy the required condition.

It is well known that $|R| = |S_{12} \cup S_{13} \cup S_{23}| = |S_{12}| + |S_{13}| + |S_{23}| - |S_{13} \cap S_{23}| - |S_{12} \cap S_{13}| - |S_{12} \cap S_{23}| + |S_{12} \cap S_{13} \cap S_{23}|$. If we let $r = \min(|S_{13} \cap S_{23}|, |S_{12} \cap S_{13}|, |S_{12} \cap S_{23}|)$ and note that $r \geq |S_{12} \cap S_{13} \cap S_{23}|$, it is clear that $|R| \leq |S_{23}| + |S_{12}| + |S_{13}| - 2r = 3(10N) - 2r$. However, the number of elements in $S_{12} \cap S_{23}$ (or in $S_{12} \cap S_{13}$ or in $S_{13} \cap S_{23}$) is greater than or equal to $\frac{1}{10}N^2$. To see this, recall that S_{12} is the set of numbers in S that have the same first two digits as some element of T and S_{23} is the set of numbers in S that have the same second two digits as some element of T. Let n_0 be the number of times that a zero occurs in the second position for elements of T. Similarly, let n_k be the number of times that the digit k occurs in the second position for elements of T for $k = 0, 1, \ldots, 9$. Then, clearly, $\sum_{k=0}^{9} n_k = N$. Now consider S_{12} and S_{23}. The number of times that 0 is the second digit in either S_{12} or S_{23} is exactly $10n_0$ and the number of elements shared by both S_{12} and S_{23} with 0 as the second digit is exactly n_0^2. For example, if $n_0 = 4$, then S_{12} and S_{23} each have 40 elements with 0 as the second digit and they share $n_0^2 = 16$ of these elements. Considering $0, 1, 2, \ldots, 9$ in the second position, successively, the total number of elements shared by S_{12} and S_{23} is $\sum_{k=0}^{9} n_k^2$. That is, $|S_{12} \cap S_{23}| = \sum_{k=0}^{9} n_k^2$. However, by the Cauchy–Schwarz[1] inequality,

$$\sum_{k=0}^{9} n_k^2 \geq \frac{1}{10}\left(\sum_{k=0}^{9} n_k\right)^2 = \frac{1}{10}N^2.$$

This implies that $r \geq \frac{1}{10}N^2$. Hence, the size of R satisfies the inequality $|R| \leq 30N - \frac{1}{5}N^2$. But $30N - \frac{1}{5}N^2 < 1000$ when $N < 50$ as $30N - \frac{1}{5}N^2 - 1000 = -\frac{1}{5}(N-50)(N-100) < 0$ when $N < 50$. Thus, the size of set R is strictly less than 1000 if $N < 50$. This implies that more elements of S can be added to T if $N < 50$. It follows that 50 is the minimum size of T.

3. Summary

We have considered the random selection of 3-digit numbers so that each number selected differs from the others in at least two positions. For 3-digit numbers, this is the most interesting case. If the numbers were selected to differ in at least one position or in at least three positions, then the number generated would be exactly 1000 or 10, respectively. However, when the numbers are chosen to differ in at least two positions, the total number generated ranges from 50 to 100 with an average number of 86. An investigation of M-digit numbers with $M > 3$ is suggested by this study. Besides being interesting, the results of such an investigation may turn out to be useful in certain applications.

__Edward Allen__ is a professor of mathematics at Texas Tech University. His research interests include numerical analysis and the development of numerical methods for deterministic and stochastic problems. He enjoys the outdoors, especially the plains of West Texas.

Braintwister

12. Magic magic square

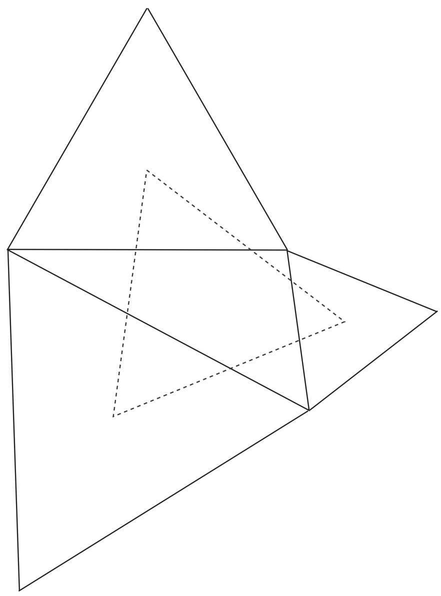
	FIFTEEN	

As is traditional with magic squares, write in words a different whole number in the other eight boxes so that the sum of each row, each column, and each main diagonal is the same. But do it in such a way that the number of **letters** in each of the nine squares is different and so that the total number of letters in each row, each column, and each main diagonal is the same.

What is the highest number in the magic magic square?

Victor Bryant

The following result is attributed to Napoleon Bonaparte.
Starting with any triangle, if we attach an equilateral triangle to each of its sides, then the centroids of the equilateral triangles will form the vertices of an equilateral triangle.
It is not clear whether Napoleon proved this. Can you?



[1] The Cauchy–Schwarz Inequality for real numbers is $(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$; we have $n = 10$ and $y_1 = \cdots = y_{10} = 1$.

The Three Ws of West Indian Cricket — A Statistical Analysis

A. TAN

1. The three Ws of West Indian cricket

In successive years in the tiny island of Barbados were born three of the greatest West Indian cricketers, named Frank Worrell, Everton Weekes and Clyde Walcott, in reverse alphabetical order. Popularly known as the three Ws, they became three leading contemporary batsmen in the West Indies team. The three Ws were unrelated and quite different in physical makeup. Their batting styles were also different. Yet all three scored runs with great efficiency and determination, compiling lofty numbers upon retirement. In fact their achievements were often so close that debate raged and still continues today as to who was the greatest cricketer among them.

2. Ranking the three Ws

To address the above question, we restrict our attention to the batting aspect of the game, since only Worrell bowled moderately and Walcott kept wicket in the early part of his career and bowled sparingly. Traditional cricket records are archived under the Test and First-class categories. Since the Test figures are included in the First-class figures, we subtract the former from the latter to obtain Other First-class (OFC) figures. The figures for the three Ws are obtained from references 1–3 and compiled in table 1.

The three quantities by which a batsman is most often judged are the total runs, average and centuries made. It is evident from table 1 that as far as Test cricket is concerned, Weekes (leader in runs and average and co-leader in centuries) was the clear winner, with Walcott second and Worrell (trailer in average and centuries) third. A different story emerges from the OFC matches. Here Worrell (leader in runs and centuries) was the winner, with Walcott second and Weekes (trailer in runs, average and centuries) third.

It is more difficult to judge the overall ranking from the career figures which are heavily weighted towards the OFC numbers. If one gives equal weights to the Test and OFC rankings, Worrell (third place in Tests and first place in OFC matches), Weekes (first and third places) and Walcott (two second places) would have to be judged about equal. Curiously, their respective career averages of 54+, 55+ and 56+ are in a near-perfect arithmetic progression with a common difference of about 1. Their career centuries plus double and triple centuries of 47, 46 and 45 are again in a perfect arithmetic progression, this time with a common difference of -1. The sum of the above statistics gives 101+, 101+ and 101+ — a perfect tie!

Table 1. Batting statistics

	Worrell	Weekes	Walcott
Test matches	51	48	44
Test innings	87	81	74
Not outs	9	5	7
Test runs	3 860	4 455	3 798
Test average	49.48	58.61	56.68
Test centuries	9	15	15
Test 200	2	2	1
OFC runs	11 165	7 555	8 022
OFC average	56.10	53.58	56.49
OFC centuries	30	21	25
OFC 200	5	7	3
First class 300	1	1	1
Career runs	15 025	12 010	11 820
Career average	54.24	55.34	56.55
Career centuries	39	36	40
Career 100 + 200 + 300	47	46	45

3. Statistical analysis

Regardless of which of the Ws was the greatest, there is no doubt that they easily rounded up the top three positions of West Indies batting. As a mathematical exercise, one might ponder this question: what is the probability that the top three batsmen in a West Indies team would be born in Barbados and would have surnames beginning with W? First, the vast majority of the West Indian players came from four regions: Jamaica, Barbados, Trinidad and British Guiana (now Guyana). Despite its small size, Barbados contributed about 30% of the Test players (reference 3). Thus the probability of any player hailing from Barbados is 3/10. Second, according to the 1998–99 Barbados telephone book, names beginning with W account for just under 7% of the subscribers. Assuming that the same percentage was reflected in the general population of Barbados at the time of the three Ws and that the events discussed are independent, the probability of the number 1 batsman in the West Indies team hailing from Barbados and having a surname beginning with W is (3/10)(7/100) or 0.021. The same probability applies to the number 2 and number 3 batsmen. Hence the probability that the top three batsmen in the West Indies team were born in Barbados and had surnames starting with W is $(0.021)^3$ or 9 in a million! The assumption of independence is unverifiable, so this may be smaller than the true probability, though it is likely to be a ballpark figure. One has to wonder whether the appearance of the three Ws in cricket history is purely coincidental. To many, W signified the very team they played for — West Indies.

What is the probability of all three Ws scoring centuries in the same Test innings? From table 1, we see that, among the three Ws, Walcott scored centuries at the fastest rate — 15 in 74 innings batted or 1 in every 4.933 innings. The probability of his scoring a century in any innings was 15/74 or 20.27%. Likewise, Weekes scored a century in every 5.4 innings batted, giving a probability of 18.52% per innings. Worrell, whose Test career extended beyond those of Weekes and Walcott, scored 8 centuries from 57 innings in the company of the other two. During this period, his frequency of scoring a century was 1 in 57/8 or 7.125 innings with a probability of 14.035% per innings. Assuming independence, the probability of all three Ws scoring a century in the same Test innings is the product of the individual probabilities, i.e., (8/57)(15/81)(15/74) or 0.527%. This translates to a frequency of 1 in 190 innings or 2 in 380 innings. Interestingly, this actually happened twice in the only 50 times all three Ws batted in the same Test innings: once, against India in 1953 (Worrell 237, Weekes 109, Walcott 118) and then against England in the following year (Worrell 167, Weekes 206, Walcott 124). Thus this rare event actually occurred far more frequently than the number of times predicted by theory.

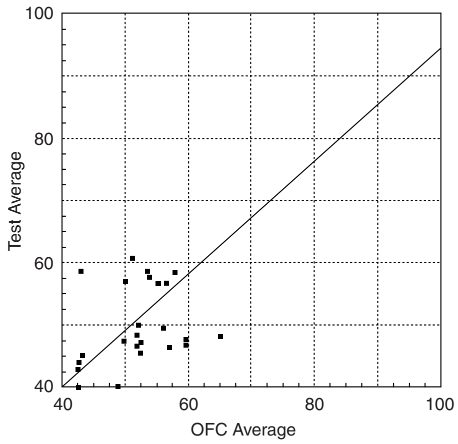


Figure 1. Test average v. OFC average of leading players and least-squares straight line showing positive correlation.

In order to study the correlation between two independent sets of data x and y, one needs to calculate the correlation coefficient defined by

$$r_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2\sum(y-\bar{y})^2}},$$

where the summation runs from 1 to n (the number of data points) and the bar indicates the average value of the variable. We first calculate the correlation coefficient between the OFC and Test averages. The OFC and Test averages of leading batsmen who had both measures greater than 40 and who retired by 1980 are taken from reference 1 and plotted in figure 1. Also shown in the figure is the least-squares fit straight line whose equation is given by

$$y = mx + c,$$

with

$$m = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2},$$

and

$$c = \frac{\sum y \sum x^2 - \sum x \sum xy}{n\sum x^2 - (\sum x)^2}.$$

As expected, there is a positive correlation between the OFC and Test averages, the correlation coefficient being 0.7726. One might additionally expect the Test averages to be less than the OFC averages given the standards of the Test and OFC opponents. The slope of the least-squares straight line is 0.8884 (41.62°), which is slightly less than that of the 45° line (1.0) on which the Test and OFC averages are equal. Thus, on average, the OFC averages slightly exceed the Test averages. However, it is surprising that nearly half of the leading batsmen actually had greater Test averages than OFC averages. One is led to believe that many great batsmen are able to elevate their performances in Test matches.

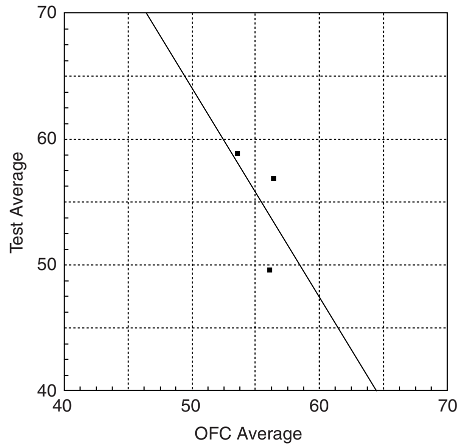


Figure 2. Test average v. OFC average of the three Ws and least-squares straight line showing negative correlation.

In figure 2, the OFC and Test averages of the three Ws are plotted. Interestingly, it shows a negative correlation between the two, the correlation coefficient being −0.566. The least-squares straight line has a slope of −1.7242 or −59.9°. The reason for this inverse correlation can be traced to the data. First, Worrell's Test average was well below his OFC average and Weekes' Test average was well above his OFC average, while Walcott's OFC and Test averages were nearly identical. Second, the career averages of all three were about equal. This caused the Worrell and Weekes data points to move in opposite directions along a line of negative slope, thus creating an inverse correlation between Test and OFC averages.

We shall next consider the more important correlation — that between different batsmen. Since the three Ws were unrelated individuals, their numbers may be assumed to be independent variables. The data belonging to the three (table 1) are so close that the correlation coefficient for any set of quantities comes out very high and meaningless. The quantities chosen should have at least the same order of magnitude in order to arrive at any meaningful correlation. For that reason, the total number of runs is a quantity that must be left out. The average and the number of centuries are the other two measures by which a batsman is most frequently judged. In the case of the three Ws, even these measures generate extremely high correlation coefficients. The problem here is to seek the least correlation coefficient in order to differentiate the batsmen. The quantities chosen are the Test averages, OFC averages, Test centuries doubled and OFC centuries doubled. We denote the subscripts 1, 2 and 3 to correspond to Worrell, Weekes and Walcott respectively. Thus r_{12} is the correlation coefficient between Worrell and Weekes, r_{13} that between Worrell and Walcott and r_{23} the same between Weekes and Walcott. The results of the calculations are as follows: $r_{12} = 0.701$; $r_{13} = 0.876$; and $r_{23} = 0.957$. Thus there was maximum correlation between the Weekes and Walcott figures and (relatively) the least correlation between the Worrell and Weekes figures.

We can also calculate the multiple correlation coefficients between the three Ws (cf. reference 4). We first construct the correlation matrix, which is given by

$$(r) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}.$$

Here $r_{11} = r_{22} = r_{33} = 1$; $r_{21} = r_{12}, r_{31} = r_{13}$ and $r_{32} = r_{23}$ which means that the matrix is symmetric. The determinant of the correlation matrix R is found to be 0.000715464. Geometrically speaking, this determinant is the volume of the parallelopiped formed by the Worrell, Weekes and Walcott vectors, which are, respectively (r_{11}, r_{12}, r_{13}), (r_{21}, r_{22}, r_{23}) and (r_{31}, r_{32}, r_{33}). The smallness of the determinant indicates that the three vectors are very nearly parallel. Following reference 4, we can finally calculate the multiple correlation coefficient between the three Ws. The multiple correlation coefficient of Worrell with respect to Weekes and Walcott is, for example,

$$r_{1\text{–}23} = \sqrt{1 - \frac{R}{R_{11}}},$$

where R_{11} is the minor of r_{11}. We have, similarly,

$$r_{2\text{–}31} = \sqrt{1 - \frac{R}{R_{22}}},$$

and

$$r_{32\text{–}12} = \sqrt{1 - \frac{R}{R_{33}}}.$$

On carrying out the calculation, we obtain: $r_{1\text{–}23} = 0.9957$; $r_{2\text{–}31} = 0.9985$; and $r_{3\text{–}12} = 0.9993$. This shows that Walcott's numbers have the highest correlation with the other two whereas Worrell's numbers have (relatively) the least correlation with the others.

4. Conclusions

In summary, the appearance of the three Ws in cricket history was a truly remarkable event. Their batting statistics showed a high degree of correlation. Statistically, Weekes' and Walcott's numbers had the highest correlation and Worrell's and Weekes' numbers had (relatively) the least correlation. In Test cricket, Weekes and Walcott excelled above Worrell but in OFC matches Worrell had a better record than the other two. Overall, their batting performances were about equal. All three Ws received Wisden's award and all three were eventually crowned with knighthoods (cf. reference 3). Finally, all three were inducted into Cricket's first Hall of Fame at Lord's recently. In a nutshell, the three Ws were almost inseparable!

References

1. E. W. Swanton, *Barclays World of Cricket* (Collins Publishers, London, 1980).
2. B. Frindall, *The Wisden Book of Cricket Records* (Headline, London, 1993).
3. C. Martin-Jenkins, *World Cricketers: A Biographical Dictionary* (Oxford University Press, 1996).
4. J. F. Kenney and E. S. Keeping, *Mathematics of Statistics* (Van Nostrand Co., Princeton, 1959).

A. Tan *is a faculty member at Alabama A & M University. He frequently publishes articles with applied mathematical interest in* Mathematical Spectrum *and elsewhere.*

Suppose that we form a sequence of independent zeroes and ones, starting with 0, where $P(0) = P(1) = \frac{1}{2}$. Is the probability of the first occurrence in a sequence of length n (n greater than or equal to 3) of the pattern 01 the same as that of the pattern 10?

J. Gani

The Steps Problem and the Fibonacci Sequence: an Interesting Connection

ANAND M. NARASIMHAMURTHY

The steps problem

There is a flight of n steps. A person standing at the bottom climbs either 1 or 2 steps at a time. In how many ways can he get to the top?

Let us consider the case of 10 steps. One way of climbing 10 steps is the sequence 1, 2, 2, 1, 1, 1, 2. Other possible sequences are 2, 2, 2, 1, 2, 1 and 1, 2, 1, 2, 2, 2 and so on.

We provide two solutions to this problem and then link them.

The first solution

Let x be the number of times he climbs one step and y be the number of times he climbs two steps. The total number of steps is a constant (10 in our case, but n in general). However, the number of climbs can vary from 5 to 10. In general the number of climbs can vary from $(n+1)/2$ to n for odd n and $n/2$ to n for even n.

The possible values of x and y are:

$$\begin{array}{cccccc} x = 10 & x = 8 & x = 6 & x = 4 & x = 2 & x = 0 \\ y = \ \ 0 & y = 1 & y = 2 & y = 3 & y = 4 & y = 5 \end{array}$$

For each pair (x, y), we must find the number of ways of distributing the x single steps among the y double steps, namely

$$^{x+y}C_x = \frac{(x+y)!}{x!y!},$$

the familiar binomial coefficient.

The total number of possible ways is just the sum of the individual values. Hence the number of ways is

$$1 + 9 + 28 + 35 + 15 + 1 = 89.$$

The corresponding number for n steps is, for odd n,

$$^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + {}^{n-3}C_3 + \cdots + {}^{(n+1)/2}C_{(n-1)/2}$$

and, for even n,

$$^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + {}^{n-3}C_3 + \cdots + {}^{n/2}C_{n/2}.$$

A second solution

Let the number of ways in which he can climb n steps be $y(n)$. Since he can climb either 1 or 2 steps at a time, he could have come to the nth step from either the $n-1$th step or the $n-2$th step only. The total number of ways in which he can get to the nth step is thus the sum of the ways in which he can get to the $n-1$th and $n-2$th steps, i.e.

$$y(n) = y(n-1) + y(n-2), \qquad n > 2.$$

It is easily seen that $y(1) = 1$ since he can climb one step in only one way, and $y(2) = 2$ since he can climb two steps in two ways (sequence 1, 1 or 2).

Thus $y(n)$ is a sequence whose terms are 1, 2, 3, 5, 8 . . . , and $y(n)$ can be expressed as

$$\begin{aligned} &y(n) = y(n-1) + y(n-2) \qquad n > 2 \\ &y(1) = 1 \\ &y(2) = 2. \end{aligned}$$

It is evident that $y(n)$ is indeed the Fibonacci sequence displaced by a term. In particular

$$y(n) = F(n+1)$$

where $F(k)$ represents the kth term of the Fibonacci sequence.

Combining the two solutions we can find an expression for $F(n)$ (which is equal to $y(n-1)$). We have the identities

$$\begin{aligned} F(n) = {}&^{n-1}C_0 + {}^{n-2}C_1 + {}^{n-3}C_2 + {}^{n-4}C_3 \\ &+ \cdots + {}^{n/2}C_{(n/2)-1} \end{aligned}$$

when n is even, and

$$\begin{aligned} F(n) = {}&^{n-1}C_0 + {}^{n-2}C_1 + {}^{n-3}C_2 + {}^{n-4}C_3 \\ &+ \cdots + {}^{(n-1)/2}C_{(n-1)/2} \end{aligned}$$

when n is odd.

Anand M. Narasimhamurthy *is studying for a PhD degree in Computer Vision at Pennsylvania State University, USA. The 'steps problem' was given to a group of students at the Indian Institute of Science, Bangalore, India.*

Mathematics in the Classroom

Curves and graph sketching

In a recent TES article (reference 1), Jo Gatoff looked at the life of Maria Agnesi (1718–1799), Italian daughter of a professor of mathematics, and a great mathematician in her own right. We learn that, as testimony to her abilities, she was fluent in French by the age of five and had mastered Latin, Greek and Hebrew by the age of nine, delivering a speech in defence of higher education for women entirely in Latin at this tender age to a group of her father's friends. Even so, although Agnesi herself had had aspirations for the convent life, she found instead that it fell to her to look after her father and 20 brothers and sisters on the death of her mother. Yet she still found time to develop her own understanding of mathematics and published a book in 1748 (*Instituzioni Analitiche* — Foundations of Analysis) which led to her subsequent fame.

Gatoff describes Agnesi's devotion to the poor and the way in which she sold gifts from influential people, given in recognition of her mathematical work, to fund her charitable work. At her death, she was buried in a common grave, as she had given all she had to meet the needs of the impoverished. But she is still remembered in Milan, place of her birth, where streets and a school have been named after her.

In mathematical terms, Agnesi's name is attached to a special curve which Gatoff describes as a symmetrical wave which fits around the top of a circle. Gatoff goes on to give details of how the curve can be produced on a spreadsheet by plotting two curves:

$$y = \frac{216}{36 + x^2} \tag{1}$$

and the circle with centre (0, 3) and radius 3, i.e.

$$x^2 + (y - 3)^2 = 9. \tag{2}$$

It can be seen that the circle fits precisely under the curve; see figure 1. The name widely given to this graph is the *Witch of Agnesi*. Gatoff explains that this name occurred following a translation of Agnesi's book, where the word used to describe the curve was *versiera* — coming from the Latin, meaning to turn. But this word was also an abbreviation for another Italian word meaning the wife of the devil, and the translator chose the wrong translation, which has since become the accepted terminology.

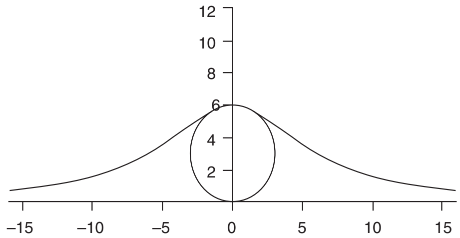


Figure 1.

Curve-sketching at A-level

A-level mathematicians are expected to be able to sketch the graphs of circles such as that given in equation (2) and rational functions, including equations similar to equation (1) above. An interesting exercise, left to the reader, is to investigate the shape of the graph

$$y = \frac{ax^2 + bx + c}{Ax^2 + Bx + C}$$

under various conditions relating to the roots of the numerator and denominator. Try the three cases when

(i) $b^2 < 4ac$ and $B^2 = 4AC$,
(ii) $b^2 \geq 4AC$ and $B^2 < 4AC$,
(iii) $b^2 < 4ac$ and $B^2 > 4AC$

and very different-looking graphs will result from this apparently straightforward rational function.

It is at this point that it becomes clear that graphs can take on very unusual and different shapes. Take a look at the graph of

$$y^2 = x^2(x + 2)$$

and you will see loops appearing. On the other hand, a sketch of a similar function,

$$y^2 = x^3$$

will produce a cusp (this is known as the *semi-cubical parabola*).

The introduction of polar co-ordinates leads to many beautiful flower-like shapes from relatively simple functions, but that's another story.

Carol Nixon

References

1. Jo Gatoff, Magic Circle, *Times Educational Supplement*, 19 May 2000.

Computer Column

A JavaScript calculator

There are several ways of adding interactivity to a Web page. The most powerful methods, such as CGI programs or Active Server Pages, require you to have certain rights on servers; unfortunately, not all ISPs provide their users with such rights. If you wish to add just a small measure of interactivity to your Web page, and you do not have access to server-side technologies, then one solution is to use client-side JavaScript.

JavaScript is a simple interpreted programming language, which has grown in popularity as part of the general growth of the World Wide Web. The success of the similarly named Java, another programming language that is used widely on the Web, probably also helped the take-up of JavaScript. (Note that Java and JavaScript have relatively little in common. Java is a fully featured and powerful language; JavaScript is much simpler.) JavaScript code can be inserted into an HTML file using a simple text editor, or whichever tool you use to prepare Web pages. When a browser opens the file, the JavaScript interpreter inside the browser acts upon any code it finds there. This is why it is called client-side JavaScript: the processing is done by the client, i.e. the browser, rather than the server. (Note that there are some differences in the way that the two most popular browsers — Netscape Navigator and Microsoft Internet Explorer — interpret JavaScript.)

As an illustration of the possibilities that JavaScript opens up for you, consider a very simple (and not very useful!) programming task: the creation of an on-line desk-style calculator. The idea is that a visitor to your Web page sees an image of a calculator; the visitor clicks on the appropriate buttons with a mouse, and sees the answer to an arithmetical calculation presented on screen.

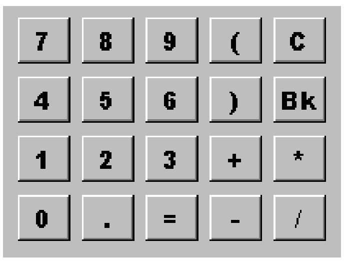


Figure 1. A GIF image of a calculator keyboard.

There are three stages to consider. First, you need to obtain an image — preferably in GIF format — of a calculator pad; this is the image that will appear on your Web page. You may be able to draw a calculator using a paint package; alternatively, you may be able to use clip-art. A typical example is shown in figure 1. The buttons are those found on a simple calculator, with the `Bk` button allowing the visitor to backspace one character and the `C` button allowing the visitor to clear the current entry.

In addition to inserting the graphic into your Web page, you also need a form field in which numbers can be entered and results presented. If you put both the form field and the graphic into a table, and use a suitably coloured background, you can obtain a pleasant result; typical HTML code for this is shown in appendix 1.

Note the `usemap` attribute of the `<img>` tag. It indicates the use of a client-side image map. This brings us to the second stage of the solution: the creation of an image map. You want each of the calculator buttons to be a clickable region. The process of creating an image map is essentially one of determining the coordinates of the various clickable regions on the graphic. Several graphics packages allow you to do this (in this example I used the proprietary Adobe PhotoShop, but you could use one of several freeware packages for this purpose). Once you have the coordinates of the clickable regions, you need to put them in your HTML file within a `<map>` tag (see appendix 2).

The coding in appendix 2 is almost self-explanatory; note that the hyperlinks are to JavaScript functions called `arith`, `clearit`, `backspace` and `calcit`. Writing these JavaScript functions is the third and final stage of the process. Appendix 3 gives the relevant code, which needs to be inserted between the `<head>` and `</head>` tags of your HTML file.

A commentary on these functions requires more space than is available in this column; however, any introductory book on JavaScript will provide a full explanation of code such as this.

It is easy to think of ways to extend this simple example. You could provide some error checking on the input, for instance; or you could develop a fully functioned calculator. Once you have gained some experience of JavaScript, you might wish to script some more challenging applications. For instance, you could use your knowledge of mathematics and programming to present visitors to your Web site with the 8-Queens problem; or you could develop a simple game or quiz. With JavaScript your Web pages can be a little less boring!

Appendix 1. HTML code for the presentation of the form and the calculator graphic

```
<table border="1" bgcolor="#C0C0C0">
 <form name="calc">
   <tr>
     <td width="210">
       <center>
         <input name="shownumber" size="25">
       </center>
     </td>
   </tr>
   <tr>
     <td width="210">
       <center>
         <p>
         <img src="calculator.gif" border="0" width="206"
                    height="153" usemap="#CalculatorMap">
       </center>
     </td>
   </tr>
 </form>
</table>
```

Appendix 2. A map tag

```
<map name="CalculatorMap">
<area shape="RECT" coords="10,7,40,34"
                        href="javascript:arith(’7’)">
<area shape="RECT" coords="50,7,80,34"
                        href="javascript:arith(’8’)">
<area shape="RECT" coords="90,7,120,34"
                        href="javascript:arith(’9’)">
<area shape="RECT" coords="130,7,160,34"
                        href="javascript:arith(’(’)">
<area shape="RECT" coords="170,7,200,34"
                        href="javascript:clearit()">
<area shape="RECT" coords="10,44,40,71"
                        href="javascript:arith(’4’)">
<area shape="RECT" coords="50,44,80,71"
                        href="javascript:arith(’5’)">
<area shape="RECT" coords="90,44,120,71"
                        href="javascript:arith(’6’)">
<area shape="RECT" coords="130,44,160,71"
                        href="javascript:arith(’)’)">
<area shape="RECT" coords="170,44,200,71"
                        href="javascript:backspace()">
<area shape="RECT" coords="10,81,40,108"
                        href="javascript:arith(’1’)">
<area shape="RECT" coords="50,81,80,108"
                        href="javascript:arith(’2’)">
<area shape="RECT" coords="90,81,120,108"
                        href="javascript:arith(’3’)">
<area shape="RECT" coords="130,81,160,108"
                        href="javascript:arith(’+’)">
<area shape="RECT" coords="170,81,200,108"
                        href="javascript:arith(’*’)">
<area shape="RECT" coords="10,118,40,145"
                        href="javascript:arith(’0’)">
<area shape="RECT" coords="50,118,80,145"
                        href="javascript:arith(’.’)">
<area shape="RECT" coords="90,118,120,145"
                        href="javascript:calcit()">
<area shape="RECT" coords="130,118,160,145"
                        href="javascript:arith(’-’)">
<area shape="RECT" coords="170,118,200,145"
                        href="javascript:arith(’/’)">
</map>
```

Appendix 3. JavaScript code for the image map

```
<script LANGUAGE="JavaScript">

function arith(anystr)
{
      document.calc.shownumber.value += anystr
}

function calcit()
{
      document.calc.shownumber.value
      = eval(document.calc.shownumber.value)
}

function clearit()
{
      document.calc.shownumber.value = ""
}

function backspace()
{
      curvalue = document.calc.shownumber.value
      curlength = curvalue.length
      curvalue = curvalue.substring(0,curlength-1)
      document.calc.shownumber.value = curvalue
}

</script>
```

Stephen Webb

Solution to Braintwister 11
(A Millennium sum)

Answer: 7 was torn up; either $63 + 529 + 1408 = 2000$ or $80 + 324 + 1596 = 2000$.
Solution: The digits 0–9 add up to 45, which is a multiple of 9. Therefore the special property of 9 means that any addition sum using all ten digits once will give an answer whose digits add up to a multiple of 9 (e.g. $473\,251 + 809 + 6 = 474\,066$ whose digits add up to 27). Since the sum in the puzzle adds to 2000, its digit sum is 7 short of being a mutiple of 9: hence it is the digit 7 that is missing from the sum.
So now we are looking for

$$\begin{array}{cccc} & & - & - \\ & - & - & - \\ 1 & - & - & - \\ \hline 2 & 0 & 0 & 0 \\ \hline \end{array}$$

Therefore the three-figure square (using three different digits) must not use a 1 or 7. Nor can it begin with a 2 because that would require the third number to begin 17. The only possibilities are 324, 529 and 625. But with 625 it is impossible to choose the units digits in the other two numbers. Hence we are left with

$$\begin{array}{cccc} & & - & - \\ & 5 & 2 & 9 \\ 1 & 4 & - & - \\ \hline 2 & 0 & 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{cccc} & & - & - \\ & 3 & 2 & 4 \\ 1 & - & - & - \\ \hline 2 & 0 & 0 & 0 \\ \hline \end{array}$$

In the former case, the units digits must now be 3/8 and the tens digits 0/6. In the latter case, the units digits must be 0/6 and the tens digits 8/9. The only ones giving one multiple of 7 are

$$\begin{array}{cccc} & & 6 & 3 \\ & 5 & 2 & 9 \\ 1 & 4 & 0 & 8 \\ \hline 2 & 0 & 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{cccc} & & 8 & 0 \\ & 3 & 2 & 4 \\ 1 & 5 & 9 & 6 \\ \hline 2 & 0 & 0 & 0 \\ \hline \end{array}$$

VICTOR BRYANT

Letters to the Editor

Dear Editor,

The annual puzzle

As a subscriber since 1988, I have always been interested in the 'year' problem, and my son wrote a program in 1988 to deal with all years. As you remark, the year 2000 is pretty barren for solutions. The numbers of solutions for the next nine years are:

2001: 17	2002: 23	2003: 41	2004: 47	2005: 42
2006: 32	2007: 41	2008: 37	2009: 48	

But, if you can wait that long, in 2099, 88 are possible!

There are, of course, other calendars, and I have tried the Jewish years 5759, 5760, 5761 and the Muslim year 1378 AH. I suggest that a 'temporary' switch from the Christian calendar for the next hundred or so years might be made!

The program does not always give the simplest nor the most elegant solution, but it is sometimes very ingenious. For example, in the year 5759: $2 = -57+59$ or $5-7-5+9$. But I like the program's solution: $2 = \sqrt{}\sqrt{}\sqrt{}\Big(\big((5!+7)+5!\big)+9\Big)$.

Yours sincerely,

K. W. Williams
(314 Chester Road,
Streetly,
Sutton Coldfield,
West Midlands B74 3ED)

Dear Editor,

A 'prime-rich' sequence?

In Volume 32, Number 2, Bob Bertuello asks whether the sequence 2, 3, 7, 43, 1807, 3263443 ... is 'prime-rich'; each term being constructed from the product of the previous terms plus 1. I would guess the answer is 'no more than similarly constructed sequences'. The primality of the first four terms is, *a priori*, almost inevitable. Calling the nth term u_n, $u_5 = 13 \times 139$ and u_6 is prime as given by Mr Bertuello but $u_7 = 10\,650\,056\,950\,807 = 547 \times 607 \times 1033 \times 31051$. Also, using $a \mid b$ to stand for a divides (into) b, 181 and $1987 \mid u_{10}$, $2287 \mid u_{11}$, $73 \mid u_{12}$, $1171 \mid u_{30}$, $1861 \mid u_{43}$, $2029 \mid u_{60}$, $1459 \mid u_{65}$ and $2437 \mid u_{81}$. I discovered these using a programmable calculator by investigating each possible (prime) factor p in turn, taking all calculations modulo p and going as far as 3000.

If we let $p_n = \prod_{i=1}^{n} u_i$ then $u_{n+1} = p_n + 1$ and so $p_{n+1} = p_n(p_n + 1)$. To find a u_n divisible by p, we can look for the first p_n divisible by p. For example, if $p = 13$, working modulo 13: $2^2 + 2 = 6$, $6^2 + 6 = 3$, $3^2 + 3 = 12 = -1$, $(-1)^2 + (-1) = 0 \bmod 13$ and so $13 \mid u_5$. For 73 we get the sequence 2, 6, 42, 54, 50, 68, 20, 55, 14, 64, 72, 0, i.e. 12 terms in all, so $73 \mid u_{12}$.

Apart from the primes mentioned so far, there are no others below 3000 dividing any u_n. Clearly no prime divides more than one u_n.

This sequence also yields a proof that there are an infinite number of primes of the form $4n + 3$ ($n \in \mathbb{Z}^+$), since for $n > 1$, each $u_n = 3 \mod 4$ (each $= 2 \times$ an odd number $+ 1 = 2(2m + 1) + 1 = 4m + 3$, say), so it must have a prime factor $= 3 \bmod 4$. Also, as already stated, no two different u_n have a common prime factor, so each u_n introduces a new prime $= 3 \bmod 4$, either u_n itself or one of its factors.

It also suggests an investigation into periodicity of the recurrence relation $p_{n+1} = p_n(p_n+1)$ modulo p. In practice it was observation of the repetition of some value of p_n for different n which led to the rejection of a particular p as a possible divisor of some u_n. Those with period 2 are interesting. By Fermat's theorem on the sum of two squares, they can only occur if $p = 1 \bmod 4$, and when they do the oscillating values of p_n sum to $-2 \bmod p$ (easily proved). For example, $p = 17$ yields the sequence 2, 6, 8, 4, 3, 12, 3, 12, 3.

So thank you Mr Bertuello for an interesting problem. It could make a good excuse to introduce modular arithmetic to a bright class in school, perhaps using Euclid's proof of the infinity of primes as a starting point (see the letter from A. A. K. Majumdar in Volume 29, Number 2, p. 41, where incidentally $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$ should be given as 59×509).

Yours sincerely,

Alastair Summers
(57 Conduit Road,
Stamford,
Lincs PE9 1QL)

Dear Editor,

Jim Whiteman's question, Volume 32, Number 2, p. 42

Perhaps the following remarks will be of help with this question.

Take $s > 0$ and $k \geq 2$ an integer, and consider maximizing the product p of k positive numbers, $p = a_1 \dots a_k$, subject to the condition of their sum being s, $a_1 + \cdots + a_k = s$. The geometric mean–arithmetic mean inequality (and in particular the case when equality occurs) shows that the maximum occurs when $a_1 = \cdots = a_k = a$, say. Then of course $a = s/k$ and $p = a^k = (s/k)^k$.

Now we want to choose k so that $(s/k)^k$ is a maximum. Simple calculus shows that $f(x) = (s/x)^x$ has a maximum at $x = s/\mathrm{e}$, with value $P = \mathrm{e}^{s/\mathrm{e}}$.

If k could take the value s/e, only possible of course if $s = n\mathrm{e}$ for some positive integer n, then the common value is $a = \mathrm{e}$, a value hinted at in the letter.

What we have to do then if k is an integer is find the integer n such that $n-1 < s/\mathrm{e} < n$; and then look at the cases $k = n-1, n$ and the $a_1, \dots, a_k$ as equal as possible.

Let $s = 13$, as in the letter; then $s/\mathrm{e} = 13/\mathrm{e} \approx 4.78$ and $P = \mathrm{e}^{13/\mathrm{e}} \approx 119$. So $n-1 = 4$, $n = 5$.

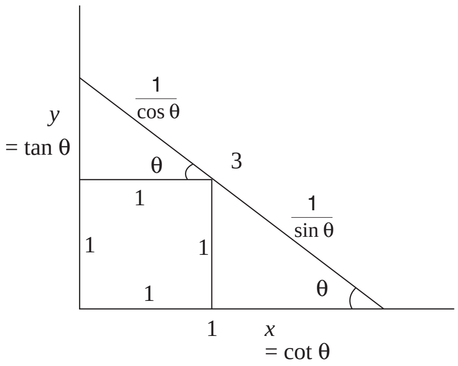
In the case of four terms, the best choice is 3, 3, 3, 4, giving $p = 108$; allowing fractions and taking the terms all equal, 3.25, gives $p = 111$. In the case of 5 terms the best choice is 3, 3, 3, 2, 2, again giving $p = 108$; allowing fractions and taking the terms all equal, 2.6, gives $p = 118.8$. It is not too surprising that this value of p is closer to P, as 5 is nearer to 13e than is 4.

Yours sincerely,
PETER BULLEN
(Department of Mathematics,
University of British Columbia,
Vancouver BC,
Canada V6T 1Z2)

Dear Editor,

Problem 31.9

Readers may be interested to see an alternative solution to Problem 31.9. Thus:



$$\frac{1}{\cos\theta} + \frac{1}{\sin\theta} = 3$$

$$\sin\theta + \cos\theta = 3\sin\theta\cos\theta = \tfrac{3}{2}\sin 2\theta.$$

Squaring gives

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \tfrac{9}{4}\sin^2 2\theta$$

i.e. $$1 + \sin 2\theta = \tfrac{9}{4}\sin^2 2\theta.$$

Thus $9\sin^2 2\theta - 4\sin 2\theta - 4 = 0$ whence

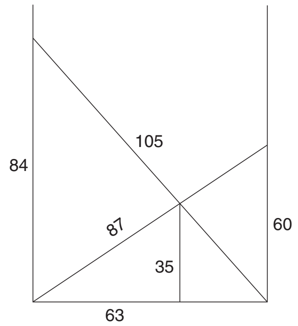
$$\sin 2\theta = \frac{2+\sqrt{40}}{9} \quad \text{(discarding the } (-)\text{ve value of } \sin 2\theta)$$

and $$\theta = \tfrac{1}{2}\sin^{-1}\left(\frac{2+\sqrt{40}}{9}\right)$$

$$\therefore\ x = \cot\left(\tfrac{1}{2}\sin^{-1}\left(\frac{2+\sqrt{40}}{9}\right)\right)$$

and the foot of the ladder is distant $1 + x$ from the wall.

While talking ladders, readers might like to try their hands at a tougher problem, Problem 33.1 in the problems section on page 22. The following, from my scrapbook (source unknown), may throw some light on the problem.



Note that all dimensions are integral and the two road segments as well as the four ladder segments will be found to be rational numbers.

Yours sincerely,
BOB BERTUELLO
(12 Pinewood Road,
Midsomer Norton,
Bath BA3 2RG)

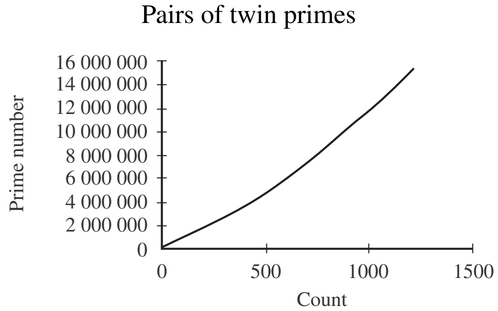
Dear Editor,

Twin primes (Volume 32, Number 2, pp. 37–39)

Following from the article on twin primes by Pantelis A. Damianou, I wonder what investigations have been done on the groups of four primes, or 'pairs of twin primes'. These are the groups of prime numbers such as (11, 13, 17 and 19) and (101, 103, 107 and 109).

I have found 1220 pairs of twins so far, the largest being (15222371, 15222373, 15222377 and 15222379), passing the rather striking groups (22271, 22273, 22277, 22279), (72221, 72223, 72227, 72229) and (1727771, 1727773, 1727777, 1727779) on the way up.

The pairs of twins show no indication of fading out at this stage, as can be seen from the plot.



Yours sincerely,
B. J. HULBERT
(Fourways,
Micklands Road,
Caversham,
Reading RG4, 6LT)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

33.1 Find a formula for the product

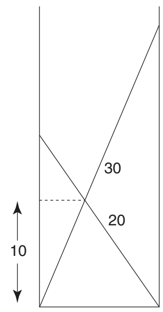
$$\prod_{r=1}^{n} \cos \frac{x}{2^r},$$

and use it to sum the infinite series

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tan \frac{x}{2^r}, \qquad \sum_{r=1}^{\infty} \frac{1}{4^r} \tan^2 \frac{x}{2^r}.$$

(Submitted by J. A. Scott, Chippenham)

33.2 Two old, imperial ladders, 20 ft and 30 ft long, cross 10 ft above the ground when leaning against opposite walls in a passageway, as shown. What is the width of the passageway?



(Submitted by Bob Bertuello, Bath)

33.3 Let $x_1, \ldots, x_n$ ($n \geq 2$) be real numbers such that $x_1 > x_2 > \cdots > x_n$. Prove that

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n - x_1} > 0.$$

(Submitted by Hassan Shah Ali, Tehran)

33.4 Let $\theta_1, \ldots, \theta_n$ ($n \geq 2$) be positive real numbers such that $\theta_1 + \cdots + \theta_n \leq \frac{\pi}{2}$. Prove that

$$\prod_{r=1}^{n} \tan \theta_r \leq 1.$$

(Submitted by Hassan Shah Ali, Tehran)

Solutions to Problems in Volume 32 Number 2

32.5 The point P lies inside the triangle ABC and AP, BP, CP meet the sides BC, CA, AB at D, E, F respectively. Evaluate the expression

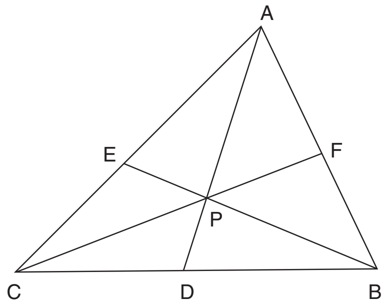
$$\frac{\mathrm{AP}}{\mathrm{AD}} + \frac{\mathrm{BP}}{\mathrm{BE}} + \frac{\mathrm{CP}}{\mathrm{CF}}.$$

By taking P to be the orthocentre of the triangle, deduce that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$

where A, B, C are the angles of the triangle.

Solution



$$\frac{\mathrm{PD}}{\mathrm{AD}} = \frac{\text{area } \triangle \mathrm{PBC}}{\text{area } \triangle \mathrm{ABC}}, \qquad \frac{\mathrm{PE}}{\mathrm{BE}} = \frac{\text{area } \triangle \mathrm{APC}}{\text{area } \triangle \mathrm{ABC}},$$

$$\frac{\mathrm{PF}}{\mathrm{CF}} = \frac{\text{area } \triangle \mathrm{ABP}}{\text{area } \triangle \mathrm{ABC}}.$$

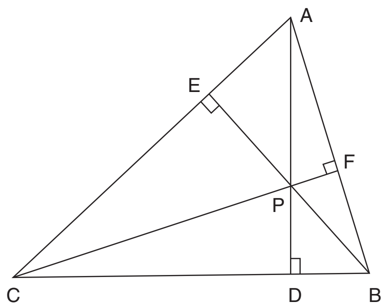
Hence,

$$\frac{\mathrm{PD}}{\mathrm{AD}} + \frac{\mathrm{PE}}{\mathrm{BE}} + \frac{\mathrm{PF}}{\mathrm{CF}} = 1,$$

so

$$\frac{\mathrm{AP}}{\mathrm{AD}} + \frac{\mathrm{BP}}{\mathrm{BE}} + \frac{\mathrm{CP}}{\mathrm{CF}} = \left(1 - \frac{\mathrm{PD}}{\mathrm{AD}}\right) + \left(1 - \frac{\mathrm{PE}}{\mathrm{BE}}\right) + \left(1 - \frac{\mathrm{PF}}{\mathrm{CF}}\right) = 2.$$

Now let P be the orthcentre of $\triangle$ABC.



Then AE $=$ AP $\sin \angle$APE $=$ AP $\sin C$. Also AE $=$ AB $\cos A$ and AD $=$ AB $\sin B$. Hence

$$\frac{\mathrm{AP}}{\mathrm{AD}} = \frac{\mathrm{AE}}{\mathrm{AD}} \cdot \frac{\mathrm{AP}}{\mathrm{AE}} = \frac{\cos A}{\sin B \sin C}.$$

Similarly,

$$\frac{\mathrm{BP}}{\mathrm{BE}} = \frac{\cos B}{\sin C \sin A}, \qquad \frac{\mathrm{CP}}{\mathrm{CF}} = \frac{\cos C}{\sin A \sin B}.$$

Hence

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2,$$

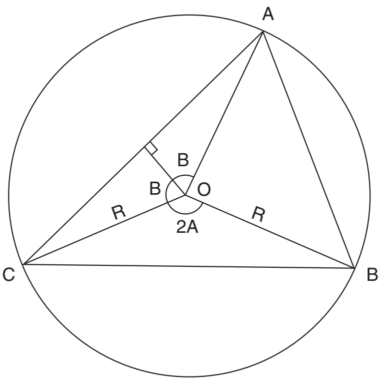
so

$$\begin{aligned}\sin A \cos A + \sin B \cos B + \sin C \cos C \\ = 2 \sin A \sin B \sin C,\end{aligned}$$

and so

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

The first part was solved by Zhan Su (Nottingham High School). Zhan Su solved the second part considering the circumcentre O of the triangle.



$$\text{Area } \triangle \mathrm{OBC} = \tfrac{1}{2}\mathrm{R}^2 \sin 2A, \quad \text{Area } \triangle \mathrm{OCA} = \tfrac{1}{2}\mathrm{R}^2 \sin 2B,$$
$$\text{Area } \triangle \mathrm{OAB} = \tfrac{1}{2}\mathrm{R}^2 \sin 2C,$$

so

$$\begin{aligned}\tfrac{1}{2}\mathrm{R}^2(\sin 2A + \sin 2B + \sin 2C) \\ &= \text{Area } \triangle \mathrm{ABC} \\ &= \tfrac{1}{2}\mathrm{AC} \cdot \mathrm{BC} \sin C \\ &= \tfrac{1}{2}(2\mathrm{R} \sin B)(2\mathrm{R} \sin A) \sin C \\ &= 2\mathrm{R}^2 \sin A \sin B \sin C.\end{aligned}$$

The result follows.

32.6 Prove that

$$1^k + 2^k + \cdots + n^k =$$
$$\frac{n+1}{(k+1)!} \begin{vmatrix} \binom{2}{1} & 0 & \dots & \dots & \dots & 0 & (n+1)-1 \\ \binom{3}{2} & \binom{3}{1} & 0 & \dots & \dots & 0 & (n+1)^2-1 \\ \binom{4}{3} & \binom{4}{2} & \binom{4}{1} & 0 & \dots & 0 & (n+1)^3-1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{k}{k-1} & \binom{k}{k-2} & \dots & \dots & \dots & \binom{k}{1} & (n+1)^{k-1}-1 \\ \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \dots & \dots & \binom{k+1}{2} & (n+1)^k-1 \end{vmatrix}$$

Solution by Seyyed Moosavi, who proposed the problem

Write

$$S_k = 1^k + 2^k + \cdots + n^k.$$

Note that

$$\begin{aligned}(1+1)^{k+1} &= \binom{k+1}{0}1^{k+1} + \binom{k+1}{1}1^k + \binom{k+1}{2}1^{k-1} \\ &\quad + \cdots + \binom{k+1}{k+1}1^0, \\ (2+1)^{k+1} &= \binom{k+1}{0}2^{k+1} + \binom{k+1}{1}2^k + \binom{k+1}{2}2^{k-1} \\ &\quad + \cdots + \binom{k+1}{k+1}2^0, \\ &\dots \\ (n+1)^{k+1} &= \binom{k+1}{0}n^{k+1} + \binom{k+1}{1}n^k + \binom{k+1}{2}n^{k-1} \\ &\quad + \cdots + \binom{k+1}{k+1}n^0.\end{aligned}$$

Adding these together, we have

$$\begin{aligned}2^{k+1} + \cdots + (n+1)^{k+1} &= (1^{k+1} + \cdots + n^{k+1}) + \binom{k+1}{1}S_k \\ &\quad + \binom{k+1}{2}S_{k-1} + \cdots + \binom{k+1}{k}S_1 + n,\end{aligned}$$

so

$$\begin{aligned}(n+1)^{k+1} - (n+1) &= \binom{k+1}{k}S_1 + \binom{k+1}{k-1}S_2 \\ &\quad + \cdots + \binom{k+1}{1}S_k\,.\end{aligned}$$

Hence

$$\begin{gathered}\binom{2}{1}S_1 = (n+1)^2 - (n+1), \\ \binom{3}{2}S_1 + \binom{3}{1}S_2 = (n+1)^3 - (n+1), \\ \dots \\ \binom{k+1}{k}S_1 + \binom{k+1}{k-1}S_2 + \cdots + \binom{k+1}{1}S_k \\ = (n+1)^{k+1} - (n+1).\end{gathered}$$

Hence, by Cramer's Rule,

$$S_k \begin{vmatrix} \binom{2}{1} & 0 & 0 & \dots & 0 \\ \binom{3}{2} & \binom{3}{1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \dots & \binom{k+1}{1} \end{vmatrix} =$$
$$\begin{vmatrix} \binom{2}{1} & 0 & 0 & \dots & 0 & (n+1)^2-(n+1) \\ \binom{3}{2} & \binom{3}{1} & 0 & \dots & 0 & (n+1)^3-(n+1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{k+1}{k} & \binom{k+1}{k-1} & \dots & \binom{k+1}{2} & (n+1)^{k+1}-(n+1) & \end{vmatrix}$$

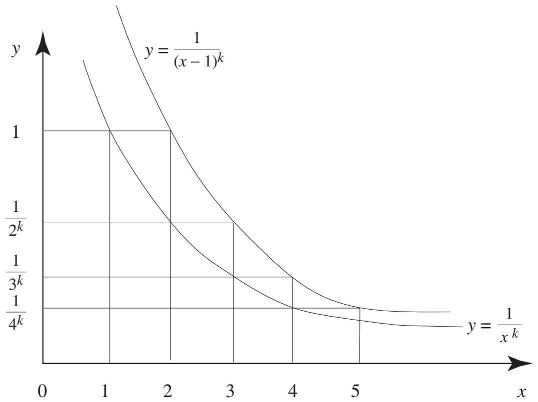
The result follows from this.

32.7 Prove that, for integers $k > 1$ and $n \geq 1$,

$$1 + \frac{1}{k-1}\Big(\frac{1}{2^{k-1}} - \frac{1}{(n+1)^{k-1}}\Big) < \frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{n^k} < 1 + \frac{1}{k-1}\Big(1 - \frac{1}{n^{k-1}}\Big)$$

and hence find lower and upper bounds for $\sum_{k=1}^{\infty} 1/r^k$.

Solution by Zhan Su (Nottingham High School)



The area under the graph $y = \frac{1}{x^k}$ between $x = 2$ and $x = n+1$ is

$$\int_2^{n+1} \frac{1}{x^k}\,\mathrm{d}x < \sum_{r=2}^{n} \frac{1}{r^k},$$

so

$$\sum_{r=2}^{n} \frac{1}{r^k} > \frac{1}{k-1}\Big(\frac{1}{2^{k-1}} - \frac{1}{(n+1)^{k-1}}\Big).$$

Hence

$$\sum_{r=1}^{n} \frac{1}{r^k} > 1 + \frac{1}{k-1}\Big(\frac{1}{2^{k-1}} - \frac{1}{(n+1)^{k-1}}\Big).$$

The area under the graph $y = 1/(x-1)^k$ between $x = 2$ and $x = n+1$ is

$$\int_2^{n+1} \frac{1}{(x-1)^k}\,\mathrm{d}x > \sum_{r=2}^{n} \frac{1}{r^k},$$

so

$$\sum_{r=1}^{n} \frac{1}{r^k} < 1 + \frac{1}{k-1}\Big(1 - \frac{1}{n^{k-1}}\Big).$$

As $n \to \infty$, $1/(n+1)^{k-1} \to 0$ and $1/n^{k-1} \to 0$, so

$$1 + \frac{1}{2^{k-1}(k-1)} \leq \sum_{r=1}^{\infty} \frac{1}{r^k} \leq 1 + \frac{1}{k-1}.$$

32.8 What is the largest number of ways of arranging p people in a row, where p is a prime number greater than 2, so that no two people are placed side by side more than once?

Solution by H. A. Shah Ali, who proposed the problem

Denote the people by $1, 2, \ldots, p$. Denote the required number by M, and denote the arrangements which are permutations of $1, \ldots, p$, by $\sigma_1, \ldots, \sigma_M$. Then the $M(p-1)$ unordered pairs $\{\sigma_i(j), \sigma_i(j+1)\}$ for $i = 1, \ldots, M$, $j = 1, \ldots, p-1$, are all different, so

$$M(p-1) \leq \binom{p}{2} = \tfrac{1}{2}p(p-1).$$

Hence

$$M \leq \frac{p}{2}, \quad \text{and so} \quad M \leq \frac{p-1}{2} \quad \text{(since } p \text{ is odd).}$$

We now define permutations $\phi_1, \ldots, \phi_{(p-1)/2}$ of $1, \ldots, p$ as follows:

$$\phi_i(j) \equiv 1 + i(j-1) \pmod{p}$$

and

$$1 \leq \phi_i(j) \leq p$$

for $i = 1, \ldots, \frac{1}{2}(p-1)$ and $j = 1, \ldots, p$. Note that, if $1+i(j-1) \equiv 1+i(k-1) \pmod{p}$, then $i(j-1) \equiv i(k-1) \pmod{p}$ so $j-1 \equiv k-1 \pmod{p}$, since p is prime, so $j \equiv k \pmod{p}$, and so $j = k$. Thus ϕ_i is a permutation of $1, \ldots, p$. Also $\phi_1(2) = 2$, $\phi_2(2) = 3, \ldots, \phi_{(p-1)/2}(2) = \frac{1}{2}(p+1)$, so $\phi_1, \ldots, \phi_{(p-1)/2}$ are all different permutations of $1, \ldots, p$.

Finally, suppose that

$$\{\phi_{i_1}(j_1), \phi_{i_1}(j_1 + i_1)\} = \{\phi_{i_2}(j_2), \phi_{i_2}(j_2 + 1)\}$$

for some i_1, j_1, i_2, j_2 $(1 \leq i_1, i_2 \leq \frac{1}{2}(p-1), 1 \leq j_1, j_2 \leq p-1)$. There are two possibilities. If

$$\phi_{i_1}(j_1) = \phi_{i_2}(j_2) \quad \text{and} \quad \phi_{i_1}(j_1+1) = \phi_{i_2}(j_2+1),$$

then

$$1 + i_1(j_1 - 1) \equiv 1 + i_2(j_2 - 1) \pmod{p}$$

and

$$1 + i_1 j_1 \equiv 1 + i_2 j_2 \pmod{p}$$

so $i_1 \equiv i_2 \pmod{p}$, and so $i_1 = i_2$. The second possibility is that

$$\phi_{i_1}(j_1) = \phi_{i_2}(j_2+1) \quad \text{and} \quad \phi_{i_1}(j_1+1) = \phi_{i_2}(j_2),$$

so

$$1 + i_1(j_1 - 1) \equiv 1 + i_2 j_2 \pmod{p}$$

and

$$1 + i_1 j_1 \equiv 1 + i_2(j_2 - 1) \pmod{p}.$$

Subtracting these, we have $i_1 \equiv -i_2 \pmod{p}$, so $p|(i_1+i_2)$. This is impossible since $1 < i_1 + i_2 < p$. Hence the $\frac{1}{2}(p-1)$ permutations $\phi_1, \ldots, \phi_{(p-1)/2}$ satisfy the condition laid down, thus the largest number of such arrangements is $\frac{1}{2}p - 1$.

Reviews

Strength in Numbers. By Sherman K. Stein. John Wiley & Sons, New York, 1999. Pp. xiii+272. Paperback £16.95 (ISBN 0-471-32974-6).

This book is aimed at those who have either given up or failed to enjoy mathematics, and tries to bring them back into the fold. While this does not preclude mathematicians from enjoying it, they will find much of the material basic is familiar.

The book is in three parts. The first part, 'About Mathematics', deals with the use, abuse and misperceptions of maths. There is a lot of good material on reliable and unreliable statistics, the fallacy of trying to reduce concepts like intelligence to a single number (IQ), and debunking various myths. The work on educational reforms is more pertinent to the American system; but the maths requirements for careers, though USA-based, should also apply in the UK and elsewhere.

The second part deals with arithmetic and algebra for the most part, and their applications to finance, etc. The author explains why the rules of addition, multiplication, exponentiation and fractional arithmetic are as they are. (To many readers with mathematical backgrounds, this is 'old hat'.) He also gives a quick proof of the sum of a geometric series, and an outline of Cantor's diagonal proof, which are easily understood.

The third part is based on trying to show what can be done with limits, and how calculus is useful. His approach is not always rigorous, and may seem frustratingly slow to those who have met basic calculus. (One example of a fallacy: in the start of the chapter he explores the behaviour of $(x^2 - 1)/(x - 1)$ near $x = 1$. He shows that the limit as x tends to 1 is 2. But in part 2 he erroneously states that we defined $0/0$ as 1!! He seems unaware of this contradiction.)

In the last chapters he uses a geometrical argument to give a formula for π: $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$, and provides alternative proofs for earlier statements.

On balance, this book should work well for its target audience. But the style, which makes it accessible to the mathematical 'non-cognoscenti', makes it somewhat dull in parts for mathematicians.

Student, Christ Church, Oxford — Mark Brimicombe

Mathematical Modeling in the Environment. By Charles R. Hadlock. MAA, Washington, 1998. Pp. xiv+302. Paperback £55.00 (ISBN 0-88385-709-X).

The number of environmental concerns is vast, so this book is limited to a discussion of three things that influence how pollution spreads. These are: groundwater flow, determined by Darcy's Law; the diffusion of air, based around Gaussian plume equations; and the behaviour of hazardous materials, for which several submodels are used. In the first half, these topics are introduced in a relatively elementary way, while the second part develops them further using single and multivariable calculus, differential equations, and probability. A spreadsheet for Gaussian plume calculations and a hazardous materials modelling package are included with the book on a PC disk.

Not having any great interest in the subject, I had feared this book would be rather dry. However, its clear explanation and gentle pace made it very easy to read. Typical scenarios are introduced to show the relevance of the theory, and in this context the principles become more interesting. (For example: a small petrol leak is suspected at a service station, near some houses and the wells that supply their water. How should the council evaluate the health risk?) Readers are also exhorted to find examples of the problems in their own neighbourhoods. Thus the book successfully demonstrates the usefulness of maths in many 'real-world' situations.

Student, Nottingham High School — Jeremy Young

A Course in Mathematical Modeling. By Randall Swift and Douglas Mooney. MAA, Washington, 1999. Pp. 400. Paperback £24.95 (ISBN 0-88385-712-X).

This is a good textbook on modelling in probability and statistics. A basic familiarity with calculus is assumed, and while it draws on various other topics — basic probability, linear algebra as applied to matrices, difference equations amongst others — no previous understanding is necessary. (However, such experience could only serve to enhance one's understanding.) It does refer to computer-based models a great deal, and specifically the MATHEMATICA® package; access to such a software package would be an asset for the course.

The course itself is applications-based, and is full of examples, ranging from viewing figures for 'The X-files' to the carbon cycle in an ecosystem. Discrete and continuous, stochastic and deterministic, many types of models are introduced. Throughout, there is an emphasis on how to derive the model, whether from theoretical assumptions or to fit given data.

The book should suit most courses in modelling, and may supplement a course in probability, statistics or difference and differential equations.

Overall, a good textbook for its subject.

Student, Christ Church, Oxford — Mark Brimicombe

The Language of Mathematics. Making the Invisible Visible. By Keith Devlin. W. H. Freeman, New York, 2000. Pp. 350. Paperback £9.99 (ISBN 0716739674).

The Language of Mathematics belongs to the category of popular mathematics books catering for a broad spectrum of people. Keith Devlin places great emphasis on mathematics as the language of patterns and a way of describing the world rather than the science of numbers which is a limiting view but one that is, unfortunately, all too common with non-mathematicians.

Each chapter discusses a branch of mathematics, detailing its progression from Greek times to the modern day. The prose is interspersed with drawings, diagrams and elementary mathematics. The book also contains a colourful element:

two eight-page colour sections referred to in the text. The writing is fluent and the historical anecdotes make for interesting reading. You feel as though you are learning a lot more than in other general popular mathematics books. Without being difficult, more difficult mathematics is presented and easily understood. Keith Devlin certainly possesses a talent for simplifying abstract concepts into everyday language.

Special mention must go to the third chapter, 'Mathematics in Motion', which charts the mathematical description of movement in the real world after the invention of calculus. He gives it a pure mathematical treatment, concentrating on exploring some of the fascinating areas which research in this field has led to: infinite series and the concept of a limit, differential equations and complex numbers. The depth of these topics is remarkable when you consider the number of pages taken to explain them.

Overall, *The Language of Mathematics* paints a general picture of the evolution in mathematics from humble beginnings to complex and abstract theories. The book is *about* mathematics rather than *doing* mathematics; although mathematical knowledge is not assumed, important ideas are introduced and well explained for the layman.

Nottingham High School — ANDREW HOLLAND

Feynman Lectures on Computation. By RICHARD P. FEYNMAN (eds ANTHONY J.G. HEY AND ROBIN W. ALLEN). Penguin Books, London, 1999. Pp. xiv+303. Paperback £18.99 (ISBN 0-14-028451-6).

Not only can modern computers calculate at stunning speed but technology is improving at unprecedented rates. We can only speculate what computers will achieve in the future: is there any limit to their powers? In this book, Feynman discusses six basic restrictions on what machines can do. These are the structure of logic gates, mathematical logic, the unreliability of components, the thermodynamics of computing, quantum mechanics, and the physics of semiconductor devices.

This book is an adaptation by the editors of a lecture course given by Feynman at the California Institute of Technology in the mid-1980s. However, the principles he considers have lost none of their relevance. Apart from the chapter on quantum mechanics, no prior knowledge is assumed. Another benefit is Feynman's style of explanation, which is always direct, clear and readable. This is not a complete textbook of computer science (it has nothing on programming languages or operating systems, for instance, and is not always concerned with detail), but it is an entertaining and approachable introduction to the subject.

Student, Nottingham High School — JEREMY YOUNG

Black Holes, Wormholes and Time Machines. By J. S. AL-KHALILI. The Institute of Physics, Bristol, 1999. Pp. 256. Paperback. £9.99 (ISBN 0-7503-0560-6).

This book was written with the aim of explaining Einstein's theories of relativity and their consequences to the general public; as such, it succeeds very well. The explanations are in general very clear, despite the difficulty of describing four-dimensional theories to a three-dimensional being, and no prior knowledge of the subject is assumed. The book is set out in a logical order, with the idea of a fourth dimension and of Newtonian mechanics being explained before dealing with general relativity, and using virtually no scientific jargon.

I was particularly interested in the use of the fact that it gets dark at night as support for the Big Bang theory as opposed to a static universe theory. Al-Khalili also mentions that the idea of a black hole originated with John Michell in 1783. Michell argued that if light were made up of particles, then a body with a sufficiently large escape velocity would trap light, and become invisible. (Michell calculated that a body 5000 times the size of the Sun but of the same density would be sufficient.) I also enjoyed the surprising observation that if some four-dimensional alien were to flip someone over through the fourth dimension (in the same way that we might turn over a jigsaw piece on a board) then everyone else would see that person as a mirror image of their previous appearance — they would now write backwards, and so on.

Although most of the book is well written, there were a few points I did not like so much. First the section dealing with paradoxes which could be caused if time travel to the past were possible; this section is somewhat confused and relies on one's acceptance of the author's view that we really do have free will. However, the main purpose of this section is to convince us that time travel into the past would cause serious logical difficulties; it succeeds in this. Second, the occasional comment to the effect of 'this is not quite right but I will go into more details later' can be quite irritating.

Overall, I think that the book is well worth buying, especially if you do not know much about relativity, and are willing to learn.

Nottingham High School — PETER ALLEN

Corrections

We regret the incorrect spelling of three names in Volume 32, Number 3. The author of the article 'Circles in the right triangle' on pages 51–53 is Jens Carstensen. François Viète was misspelt twice on the covers. Mr Bablu Chandra Dey's name was also misspelt. We apologise for each of these errors.

Mathematical Spectrum

2000/2001 Volume 33 Number 1

ISSN 0025-5653

Published by the Applied Probability Trust

Printed by Pear Tree Press Ltd, Stevenage, Herts, UK