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CONTENTS

EVERY LINEAR, DIFFERENTIAL OPERATOR IS A LINEAR DIFFERENTIAL OPERATOR

LEROY F. MEYERS

In many texts on ordinary differential equations, such as [1, pp. 82-83], the following theorem is proved, either explicitly or as part of a theorem on solutions of homogeneous linear differential equations. (See below for definitions.)

Let $a_0, a_1, \ldots, a_n$ *be ordinary functions defined on the real interval* J, *and let* L *be the operator defined so that, on* J,

$$Lf = a_0 f^{(n)} + a_1 f^{(n-1)} + \ldots + a_{n-1} f' + a_n f. \tag{1}$$

Then L *is a linear operator.*

Definitions. The domain and range of an *ordinary function* consist of real numbers; the domain and range of an *operator* (or *superfunction*) consist of ordinary functions. The operator L is a *linear operator* on the interval J just when

$$L(f+g) = Lf + Lg \qquad \text{and} \qquad L(\alpha \cdot f) = \alpha \cdot Lf$$

on J (or, equivalently, $L(\alpha \cdot f + \beta \cdot g) = \alpha \cdot Lf + \beta \cdot Lg$ on J) for all ordinary functions f and g in the domain of L and all real numbers α (and β). (It is assumed that $f+g$ and $\alpha \cdot f$ are (or $\alpha \cdot f + \beta \cdot g$ is) in the domain of L whenever f and g are in the domain of L.) If f and g are ordinary functions and S is a set of real numbers, then the statement "$f = g$ on S" means "$f(x) = g(x)$ for all x in S". Thus (1) may be expanded to read:

$$(Lf)(x) = a_0(x) f^{(n)}(x) + a_1(x) f^{(n-1)}(x) + \ldots + a_{n-1}(x) f'(x) + a_n(x) f(x) \tag{2}$$

for all x in J, for all ordinary functions f which are differentiable n times on J. An operator L of the form specified by (1) or (2) is a *linear differential operator* of order n on J.

There are many linear operators which are not linear differential operators, such as the *shift operator* E and the Laplace transform (not the Laplace operator) $\mathcal{L}$, defined by the equations

$$(Ef)(x) = f(x+1) \qquad \text{and} \qquad (\mathcal{L}f)(x) = \int_0^{+\infty} e^{-tx} f(t)\,dt$$

for all appropriate f and x.

But must every *differential* operator which is linear be of the form specified for L in (1)? An answer cannot be given until the term "differential operator" has been defined. (Such a definition is not normally given in texts on differential equations.)

Definition. The operator L is a *differential operator* of order n, where n is a nonnegative integer, on the real interval J just when there is an ordinary function F of $n+2$ arguments such that, for every ordinary function f (of 1 argument) which is n times differentiable on J, we have

$$(Lf)(x) = F(x, f(x), f'(x), \ldots, f^{(n)}(x)) \tag{3}$$

for all x in J.

Now an operator of the form specified for L in (1) *is* a differential operator of order n on J, since we can take $F(x, t_0, t_1, \ldots, t_n)$ to be

$$a_0(x)t_n + a_1(x)t_{n-1} + \ldots + a_{n-1}(x)t_1 + a_n(x)t_0$$

for all x in J and all real $t_0, \ldots, t_n$. Thus, the theorem customarily proved in differential equations texts (slightly extended) can be stated more concisely:

Every linear differential operator (of order n on an interval J) is a linear, differential operator.

The converse is true (and its truth is the reason for this paper).

THEOREM. *Every linear, differential operator (of order n on an interval J) is a linear differential operator.* (Cf. [2, v. 3, p. 32, Satz 674].)

Proof. Let L be a linear, differential operator of order n on the interval J. To show that L is a linear differential operator of order n on J, we have to find ordinary functions $a_0, \ldots, a_n$ defined on J so that (1) holds. Such functions are determined by their values at each x_0 in J. Let now x_0 be a given (and fixed) number in J. Let f be any function which is differentiable n times on J, and let P be the Taylor polynomial of degree at most n which approximates f in the neighborhood of x_0; i.e.,

$$P(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for all x. Now P can be written as a linear combination of the *reduced power functions* $p_0, \ldots, p_n$ centered at x_0:

$$P = f(x_0) \cdot p_0 + f'(x_0) \cdot p_1 + \ldots + f^{(n)}(x_0) \cdot p_n \tag{4}$$

on J, where $p_0(x) = 1$ for all x, and $p_j(x) = (x - x_0)^j/j!$ for all x if j is a positive integer. Each of f, P, p_0, $p_1, \ldots, p_n$ is defined and differentiable n times on J, and so belongs to the domain of L.

Let the function F be defined for L so that (3) holds. Since, by Taylor's formula,

$$P(x_0) = f(x_0),\ P'(x_0) = f'(x_0), \ldots,\ P^{(n)}(x_0) = f^{(n)}(x_0),$$

we see that

$$\begin{aligned}(Lf)(x_0) &= F(x_0,\ f(x_0),\ f'(x_0), \ldots,\ f^{(n)}(x_0)) \\ &= F(x_0,\ P(x_0),\ P'(x_0), \ldots,\ P^{(n)}(x_0)) \\ &= (LP)(x_0).\end{aligned}$$

But since L is a linear operator, we have, from (4),

$$LP = f(x_0) \cdot Lp_0 + f'(x_0) \cdot Lp_1 + \ldots + f^{(n)}(x_0) \cdot Lp_n,$$

and so

$$(Lf)(x_0) = f(x_0)(Lp_0)(x_0) + f'(x_0)(Lp_1)(x_0) + \ldots + f^{(n)}(x_0)(Lp_n)(x_0).$$

We now define:

$$a_0(x_0) = (Lp_n)(x_0),\ a_1(x_0) = (Lp_{n-1})(x_0), \ldots,\ a_n(x_0) = (Lp_0)(x_0).$$

Since the numbers $a_0(x_0)$, $a_1(x_0), \ldots, a_n(x_0)$ are independent of f, we have shown that, for each x_0 in J,

$$(Lf)(x_0) = a_0(x_0)f^{(n)}(x_0) + \ldots + a_n(x_0)f(x_0)$$

for all functions f which are n times differentiable on J, as desired.

It may be noted that since

$$p_j^{(k)}(x_0) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k\ , \end{cases}$$

the values of $a_0(x_0), \ldots, a_n(x_0)$ can be given directly in terms of F and x_0. In fact,

$$\begin{aligned} a_0(x_0) &= F(x_0, 0, 0, \ldots, 0, 1), \\ a_1(x_0) &= F(x_0, 0, 0, \ldots, 1, 0), \\ &\cdots\cdots\cdots\cdots, \\ a_n(x_0) &= F(x_0, 1, 0, \ldots, 0, 0). \end{aligned}$$

Although the theorem above was stated and proved for real-valued functions defined on a real interval, the proof can be carried over without change to complex-valued functions defined on a real interval (with differentiation understood in the real sense) and to complex-valued functions defined on a connected open subset of the complex plane (with differentiation understood in the complex sense). A similar theorem can be proved for partial differential operators.

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* * *

THE TWO OCTADELTAHEDRA

CHARLES W. TRIGG

Polyhedra with equilateral triangles only as faces were designated by Cundy [1] as *deltahedra*, presumably because of the triangular shape of the capital Greek letter delta, Δ. Since each face has three edges and each edge is common to two faces, Euler's formula, $V+F = E+2$, reduces to $V-2 = F/2$ for deltahedra. Thus every deltahedron has an even number of faces.

The tetradeltahedron or regular tetrahedron has 4 faces, 4 vertices, and 6 edges, with 3 faces and 3 edges at each vertex. In the regular tetrahedron A-BCD of Figure 1, with edge e, two medians of the faces are $AM = MC = e\sqrt{3}/2$. The foot of the altitude from A is G, the centroid of face BCD. From the right triangle AGM, the altitude is

$$AG = \sqrt{AM^2 - MG^2} = (e\sqrt{3}/2)\sqrt{1^2 - (1/3)^2} = e\sqrt{2/3}.$$

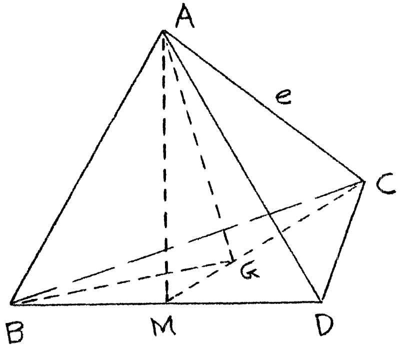

Figure 1

Also, the magnitude of a dihedral angle is

$$\angle AMG = \arccos GM/AM = \arccos 1/3 \approx 70^\circ 32'.$$

The volume of the tetrahedron is $(1/3)(e/2)(e\sqrt{3}/2)(e\sqrt{2/3}) = e^3\sqrt{2}/12$.

The hexadeltahedron or triangular dipyramid has 6 faces, 5 vertices, and 9 edges. There are 3 faces at each of 2 vertices, and 4 faces at each of the other 3 vertices. This dipyramid consists of two regular tetrahedra joined at a common face [2]. Two connected altitudes of the two tetrahedra form the single space diagonal of the hexahedron, whose length is $2e\sqrt{2/3} \approx 1.633e$.

An octadeltahedron, with edge e, has 8 faces, 6 vertices, and 12 edges. It cannot have 6 faces at one vertex, since this would require a minimum of 7 vertices.

The Convex Octadeltahedron

If 4 faces are at a vertex, the faces have 4 free edges. With an additional face on each of these edges, the deltahedron can be closed to form a polyhedron having 4 faces at each of its 6 vertices. This is the *regular* octahedron, one of the five Platonic solids. It may be viewed as (1) a truncated regular tetrahedron; or (2) a triangular antiprism, a type of prismoid; or as assemblages of (3) eight congruent triangular pyramids, or of (4) eight congruent trirectangular tetrahedra, or of (5) two congruent square pyramids [3]. (See Figure 2.) Viewed in this last guise, it is evident that the diagonals of the square base common to the pyramids have a length of $e\sqrt{2}$ and are two of the three mutually perpendicular space diagonals of the octahedron, and that the diagonals bisect each other. These diagonals are axes of symmetry. There are nine planes of symmetry: three determined by the space diagonals taken two at a time, and two passing through each of the three space diagonals and bisecting a pair of opposite edges.

It follows immediately that the circumradius, R, of the octahedron is $e/\sqrt{2}$ and the radius, t, of the sphere touching the edges is $e/2$. The insphere touches the faces at their centroids, so its radius is $r = \sqrt{(e/2)^2 - (e\sqrt{3}/6)^2} = e/\sqrt{6}$. Thus

$$R : t : r : : 1 : (1/\sqrt{2}) : (1/\sqrt{3}).$$

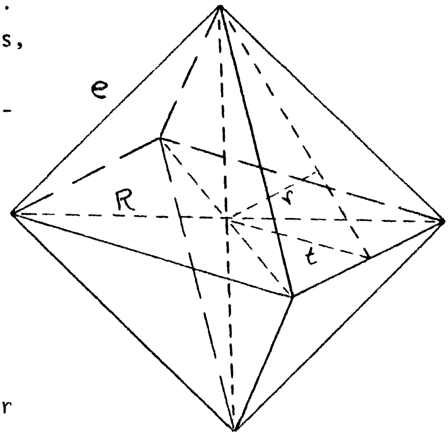

Figure 2

The surface of a regular tetrahedron can be flattened into an equilateral triangle of edge $2e$ after slitting the three edges at a vertex. Two such triangles cover the surface of a regular octahedron of edge e without overlapping [4].

A plane through the midpoints of the three edges issuing from one vertex will cut off a smaller regular tetrahedron with a volume $1/2^3$ that of the similar larger one (since their edges are in the ratio of $1 : 2$). It follows that the four tetrahedra formed by similarly cutting off each vertex of the original tetrahedron will have a combined volume of $4(1/8)$ or $1/2$ that of the larger tetrahedron. The residual solid after the truncation has eight equilateral triangular faces (Figure 3), and a volume equal to half that of the larger tetrahedron. It follows that a regular octahedron, with edge e, has a volume equal to four times that of a regular tetrahedron with edge e, namely, $e^3\sqrt{2}/3$. It also follows that the dihedral angle of the octahedron, being the supplement of the dihedral angle of the tetrahedron [5], is 109°28'.

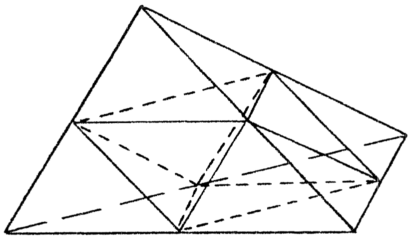

Figure 3

The Concave Octadeltahedron

With 4 faces at a vertex, a fifth face can be attached to two adjacent free edges to form a ditetrahedron with one open face. This leaves 3 free edges to which a tetrahedron with one open face can be attached. The resulting polyhedron is a concave octadeltahedron which can be viewed as (1) a hexadeltahedron with an attached tetrahedron (in two ways); or (2) three tetrahedra with a common edge and faces in contact; or (3) more imaginatively, as a dugout, a bicorne, an elbow, or a tent.

With 5 faces at a vertex, there are 5 free edges to which there must be attached 3 faces to close the deltahedron. Every possible order of attachment produces the aforesaid concave octahedron of Figure 4.

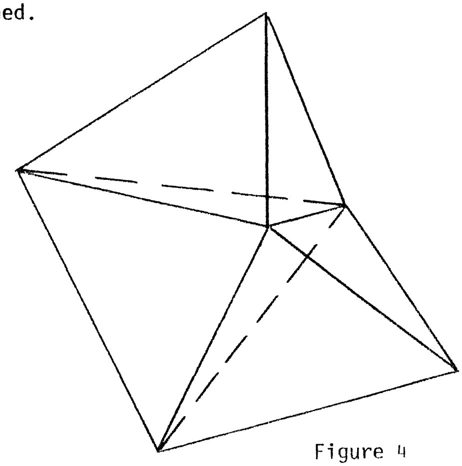

Figure 4

Thus, there are exactly two octahedra with equilateral triangular faces, as has been shown in a different manner in [6].

The concave octadeltahedron has two adjacent vertices at each of which there are 5 faces, two adjacent vertices at which there are 4 faces each, and two remote vertices at which there are 3 faces each. It has 12 dihedral angles: 7 of 70°32', 4 of 141°04', and 1 of 211°36'.

As with its convex counterpart, the concave octahedron has three space diagonals. The two internal diagonals are $2e\sqrt{2/3}$ long (see the hexadeltahedron). They intersect and are coplanar with the exterior diagonal. The plane containing the diagonals is one plane of symmetry; the only other is that plane through the common edge of the three tetrahedra that bisects the opposite edge of the middle

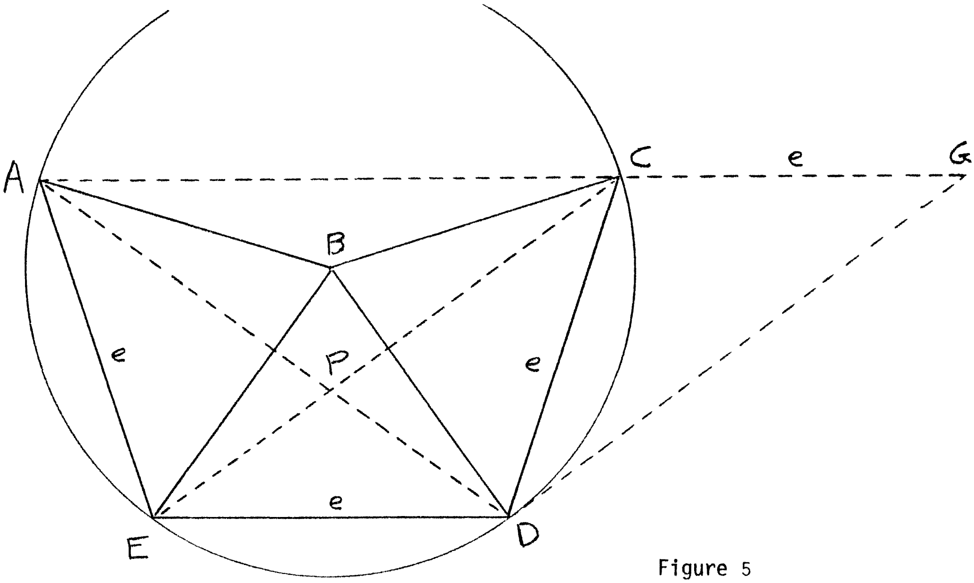

Figure 5

tetrahedron.

The nature of the diagonals can be examined readily with the aid of Figure 5, an orthogonal projection of the octahedron onto the plane of the diagonals AD, EC, and AC. Now $BA = BE = BD = BC = e\sqrt{3}/2$, so $AE = ED = DC = e$ are equal chords of the circle (B), hence subtend equal angles: $\angle EAD = \angle EDA = \angle DEP$. Therefore, triangles PED and EAD are similar, with $PD/ED = ED/AD$, so $PD = ED^2/AD$. Then $AD/PD = AD^2/ED^2$, and

$$\frac{AP}{PD} = \frac{AD - PD}{PD} = \frac{AD^2 - ED^2}{ED^2} = \frac{(2e\sqrt{2/3})^2 - e^2}{e^2} = \frac{5}{3}.$$

That is, the internal diagonals divide each other in the ratio 5 : 3.

The arcs $AE = ED = DC$. Draw a tangent to (B) at D intersecting AC extended at G. Then

$$\angle CDG = \tfrac{1}{2}\,\text{arc}\,DC = \tfrac{1}{2}\,\text{arc}\,ED = \angle ECD\,,$$

so $EC \parallel DG$. Now $ACG \parallel ED$, so the quadrilateral ECGD is a parallelogram and $CG = ED = e$. In triangle DAG, we have $AC/CG = AP/PD = 5/3$. Thus the exterior diagonal $AC = 5e/3$. Or, the length of the exterior diagonal can be computed by Ptolemy's Theorem. Since the trapezoid ACDE is inscriptible, we have $ED \cdot AC + AE \cdot CD = AD \cdot CE$. That is,

$$e \cdot AC + e^2 = (2e\sqrt{2/3})^2, \quad \text{so} \quad AC = 5e/3 \approx 1.667e.$$

Since the concave octahedron is composed of three tetrahedra, its volume is $3(e^3\sqrt{2}/12)$ or $e^3/2\sqrt{2}$. This is 3/4 the volume of the companion convex octahedron, a fact hard to believe upon casual inspection of models of the two.

If two of the edges at a single vertex of each of two regular tetrahedra are slit to form flaps, these two objects can be assembled into a concave octadeltahedron. This is equivalent to covering without overlapping the surface of the concave octahedron with two equilateral triangles developed from tetrahedral surfaces as was done on the convex octahedron. So it is not surprising that a net of the regular octahedron can be folded to give the surface of a concave octahedron, as pointed out by Ordman [7]. Indeed, 7 of the 11 different nets of the regular octahedron are also nets of the concave octahedron.

Conversion of the surface of one of the octadeltahedra into the surface of the other can be accomplished without first developing its surface into a plane net.

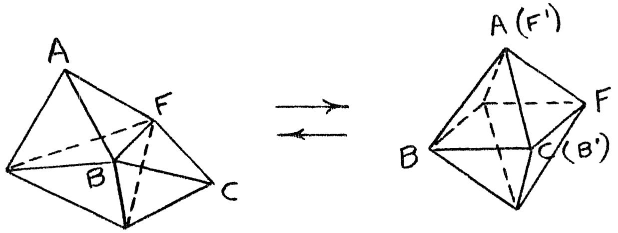

Figure 6

Consider the model of the concave octahedron in Figure 6. The edges AB, BF (the edge of the reflex dihedral angle), and FC can be slit to form two flaps. Then vertex F on a flap can be brought into coincidence with A, and flap vertex B into coincidence with C. The rigidity of the surface having been destroyed by the slitting operation, the dihedral angles formed by the other faces adjust so that a model of the surface of a regular octahedron results.

It requires 4 colors to color the faces of a regular tetrahedron in such a way that no two adjacent faces have the same color. The regular octahedron can be colored with 2 colors, whereas the concave octahedron requires 3, as indicated on its distorted projection in Figure 7. The colors outside the

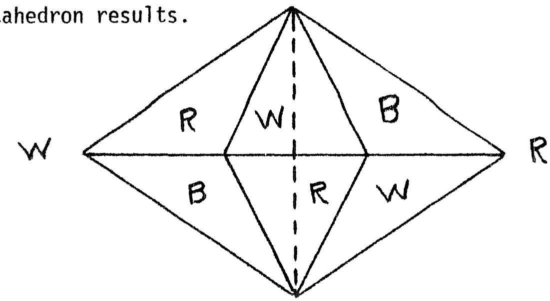

Figure 7

figure belong to the two lower (hidden) faces.

The edges of the regular octahedron can be colored with 4 colors in such a way that there are no duplicate colors at any vertex. The concave octahedron requires 5 colors.

The framework of the regular octahedron has 6 even nodes; hence it can be completely traversed in a continuous closed path. The framework of the concave octahedron has 2 even and 4 odd nodes, so requires two separate paths to cover all the edges.

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THE OLYMPIAD CORNER: 6

MURRAY S. KLAMKIN

First a correction. The *Journal of Recreational Mathematics* is published *four* times a year, not twice a year as announced earlier in this column [1979: 104]. The individual subscription price is $10 per volume (4 issues).

In addition to the solutions for Practice Set 5, I give this month the questions posed at the Eleventh Canadian Mathematics Olympiad, which took place on May 2, 1979.

The problems and solutions were prepared by the Canadian Mathematics Olympiad Committee, consisting of

Paul Arminjon, Université de Montréal
W.J. Blundon, Memorial University of Newfoundland (Chairman)
J.H. Burry, Memorial University of Newfoundland (Acting Chairman)
R.H. Eddy, Memorial University of Newfoundland (Treasurer)
M.S. Klamkin, University of Alberta
M.M. Parmenter, Memorial University of Newfoundland
P. Stewart, Dalhousie University
J. Wilker, University of Toronto
E.R. Williams, Memorial University of Newfoundland (Executive Secretary)

The solutions prepared by the committee will appear in this column in the next (August - September) issue.

ELEVENTH CANADIAN MATHEMATICS OLYMPIAD (1979)

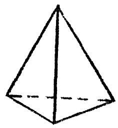
(5 *questions* - 3 *hours*)

1. Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1A_2 \ge G_1G_2.$$

2. It is known in Euclidean geometry that the sum of the angles of a triangle is constant. Prove, however, that the sum of the dihedral angles of a tetrahedron is *not* constant.

NOTE: (i) A tetrahedron is a triangular pyramid.
(ii) A dihedral angle is the interior angle between a pair of faces.



3. Let a, b, c, d, e be integers such that $1 \le a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \le \frac{15}{16},$$

where $[m,n]$ denotes the least common multiple of m and n (e.g. $[4,6] = 12$).

4. A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

5. A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is self-avoiding if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding

walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

SOLUTIONS TO PRACTICE SET 5

5-1. A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original position?

Solution.

Consider first the same problem with 5 cards. A shuffle is equivalent to a permutation of the 5 cards and can be represented by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix},$$

which means that card r is replaced by card a_r. For example, consider the shuffle represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}.$$

In the following table, line 0 gives the original order of the cards and line i, $1 \leq i \leq 6$, gives the result of the ith shuffle.

0	1	2	3	4	5
1	2	1	4	5	3
2	1	2	5	3	4
3	2	1	3	4	5
4	1	2	4	5	3
5	2	1	5	3	4
6	1	2	3	4	5

In this case, it takes 6 shuffles to get the cards back to their original position.

Now every permutation can be "factored" into a product of cycles. Here we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix} = (1,2) \cdot (3,4,5),$$

where the cycle $(3,4,5)$, for example, means $3 \to 4$, $4 \to 5$, $5 \to 3$. Since the periods of the two cycles are 2 and 3, respectively, the period of the permutation is 2×3, which is the least common multiple (L.C.M.) of 2 and 3.

For our original problem, we have to partition 13 into a sum of positive integers whose L.C.M. is a maximum. A little trial and error yields

$$13 = 5+4+3+1,$$

giving a maximum period of $5\times 4\times 3 = 60$.

For 52 cards, the maximum period 180180 can be found from

$$52 = 13+11+9+7+5+4+2+1.$$

As a research problem, the reader may wish to consider devising an algorithm to determine the maximum cycling period for n cards.

5-2. In a non-recent edition of Ripley's *Believe It Or Not*, it was stated that the number

$$N = 526315789473684210$$

is a *persistent* number, that is, if multiplied by any positive integer the resulting number always contains the ten digits 0, 1, 2,..., 9 in some order with possible repetitions.

(a) Prove or disprove the above statement.

(b) Are there any persistent numbers smaller than the above number?

Solution.

(a) The statement is false. Observe that $2N = 1052631578947368420$. Thus

$$19N = 20N - N = 9999999999999999990,$$

which does not contain all the ten digits.

(b) There are no persistent numbers! Our proof is indirect. Assume that N is persistent. We can express N in the form $N = 2^a5^bM$, where M is relatively prime to both 2 and 5. Since

$$2^b5^aN = 10^{a+b}M$$

must also be persistent, all multiples of M must contain the nine nonzero digits. We now invoke Euler's generalization of Fermat's Theorem: *if a is relatively prime to n, then*

$$a^{\phi(n)} - 1 = kn,$$

where $\phi(n)$ (Euler's ϕ-function) is the number of positive integers less than or equal to n that are relatively prime to n. Now we have

$$10^{\phi(M)} - 1 = kM, \tag{1}$$

and this gives a contradiction, for kM should contain all nine nonzero digits, but the left side of (1) contains only nines.

For an alternate, more elementary, solution, assume again that N is persistent and consider the remainders obtained by dividing the following N numbers by N:

$$1, \quad 11, \quad 111, \quad \ldots, \quad 111\ldots1$$

where the last number has N digits. Since at most $N-1$ different nonzero remainders can result, either one of the above numbers is divisible by N, in which case N is not persistent, or else two of them, say

$$R = 11\ldots1 \ (r \text{ ones}) \quad \text{and} \quad S = 11\ldots1 \ (s \text{ ones}), \quad s > r$$

give the same remainder, in which case their difference

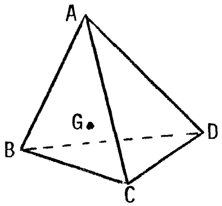
$$S - R = 11\ldots100\ldots0 \quad (s - r \text{ ones and } r \text{ zeros})$$

is divisible by N and N is not persistent.

5-3. In a regular (equilateral) triangle, the circumcenter O, the incenter I, and the centroid G all coincide. Conversely, if any two of O, I, G coincide, the triangle is equilateral. Also, for a regular tetrahedron, O, I, and G coincide. Prove or disprove the converse result that if O, I, and G all coincide for the tetrahedron, the tetrahedron must be regular.

Solution.

The converse result is false. We will show that if a tetrahedron is merely isosceles (opposite edges congruent in pairs), hence not necessarily regular, then O, I, and G coincide.



Let A-BCD be an isosceles tetrahedron with centroid G (see figure) and let

$$\vec{a} = \overrightarrow{GA}, \quad \vec{b} = \overrightarrow{GB}, \quad \vec{c} = \overrightarrow{GC}, \quad \vec{d} = \overrightarrow{GD}.$$

Because the tetrahedron is isosceles, we have (using the notation $\vec{v}^2 = \vec{v}\cdot\vec{v} = |\vec{v}|^2$)

$$(\vec{a}-\vec{b})^2 = (\vec{c}-\vec{d})^2, \quad (\vec{a}-\vec{c})^2 = (\vec{b}-\vec{d})^2, \quad (\vec{a}-\vec{d})^2 = (\vec{b}-\vec{c})^2; \tag{1}$$

and because G is the centroid, so that $\vec{a}+\vec{b}+\vec{c}+\vec{d} = \vec{0}$, we have

$$(\vec{a}+\vec{b})^2 = (\vec{c}+\vec{d})^2, \quad (\vec{a}+\vec{c})^2 = (\vec{b}+\vec{d})^2, \quad (\vec{a}+\vec{d})^2 = (\vec{b}+\vec{c})^2; \tag{2}$$

Expanding the squares and adding corresponding equations in (1) and (2) yield

$$\vec{a}^2 + \vec{b}^2 = \vec{c}^2 + \vec{d}^2, \quad \vec{a}^2 + \vec{c}^2 = \vec{b}^2 + \vec{d}^2, \quad \vec{a}^2 + \vec{d}^2 = \vec{b}^2 + \vec{c}^2,$$

from which $\vec{a}^2 = \vec{b}^2 = \vec{c}^2 = \vec{d}^2$. Thus $|\overrightarrow{GA}| = |\overrightarrow{GB}| = |\overrightarrow{GC}| = |\overrightarrow{GD}|$ and G coincides with the circumcenter O.

It is obvious that in an isosceles tetrahedron the four faces are congruent and hence so are the four altitudes (consider the volume!). Since G divides each median (segment from a vertex to the centroid of the opposite face) in the ratio $3:1$, the distance of G from any face is 1/4 the corresponding altitude. Hence G is equidistant from the four faces and thus coincides with the incenter I.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

LETTER TO THE EDITOR

Dear Editor:

In the interest of completeness, I have expanded the reference list appended to the editor's comment to Problem 365 [1979: 116] by providing further details for the items cited.

1. Problem 862, proposed by K.R.S. Sastry, *National Mathematics Magazine*, 46 (1973) 103; solved by Charles W. Trigg, 47 (1974) 52-53.

2a. William E. Heal, [Discussion] Relating to a demonstration of a geometrical theorem, *American Mathematical Monthly*, 24 (1917) 344-345.

2b. Problem E 15, proposed by Pearl C. Miller, *American Mathematical Monthly*, 39 (1932) 607; solved by Roy MacKay, 40 (1933) 423.

2c. Problem E 305, proposed by D.L. MacKay, *American Mathematical Monthly*, 44 (1937) 659; solved by W.E. Buker, 45 (1938) 480.

3. Problem 224, proposed by Dewey C. Duncan, *National Mathematics Magazine*, 12 (1937-38) 311; solved by C.W. Trigg, 14 (1939-40) 51-52.

4a. Problem 1148, proposed by Walter Carnahan, *School Science and Mathematics*, 30 (1930) 1067; "no solutions have been received" by editor, 31 (1931) 347-348; solution by Lu Chin-Shih, 31 (1931) 465-466.

4b. David L. MacKay, The Lehmus-Steiner theorem, *School Science and Mathematics*, 39 (1939) 561-572, esp. 561-563.

4c. J.J. Corliss, If two external bisectors are equal is the triangle isosceles?, *School Science and Mathematics*, 39 (1939) 732-735.

4d. David L. MacKay, The pseudo-isosceles triangle, *School Science and Mathematics*, 40 (1940) 464-468.

5. G. Cesàro, Sur la division de la circonférence en neuf parties égales, Académie Royale de Belgique, *Bulletins de la Classe des Sciences*, 5e série, 11 (1925) 126-130, esp. 129-130.

6. [unchanged].

7. Question 129, proposée par Alauda [pseud.], *L'Intermédiaire des Mathématiciens*, 1 (1894) 70; réponse par H. Dellac, avec commentaire du proposeur, 2 (1895) 101-102.

8. J. Neuberg, *Bibliographie des triangles spéciaux*, Mémoires de la Société Royale des Sciences de Liége, 3e série, 12 (1924) [Mémoire 13] 52 pp., esp. pp. 9-11. [Each *mémoire* is paginated separately.]

LEROY F. MEYERS,
The Ohio State University.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before November 1, 1979, although solutions received after that date will also be considered until the time when a solution is published.

451. *Proposed by Herman Nyon, Paramaribo, Surinam.*

Solve the doubly-true alphametic

TWENTY + TWENTY + THIRTY = SEVENTY,

in which THIRTY is divisible by 30.

452. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Precocious Percy wrote a polynomial on the blackboard and told his mathematics professor: "This polynomial has my age as one of its zeros." The professor looked at the blackboard and thought to himself: "This polynomial is monic, quintic, has integral coefficients, and is truly an odd function. If I try 10, I get -29670."

Find Percy's age and the polynomial.

453. *Proposed by Viktors Linis, University of Ottawa.*

In a convex polyhedron each vertex is of degree 3 (i.e. is incident with exactly 3 edges) and each face is a polygon which can be inscribed in a circle. Prove that the polyhedron can be inscribed in a sphere.

454*. *Proposed by Ram Rekha Tiwari, The Belsund Sugar Co., P.O. Riga, Bihar, India.*

(a) Is there a Euclidean construction for a triangle ABC given the lengths of its internal angle bisectors t_a, t_b, t_c?

(b) Find formulas for the sides a, b, c in terms of t_a, t_b, t_c.

455. *Proposé par Hippolyte Charles, Waterloo, Québec.*

Calculer l'intégrale

$$I = \int_0^{\frac{\pi}{2}} \frac{x \cos x \, \sin x}{\cos^4 x + \sin^4 x} \, dx.$$

456. *Proposed by Orlando Ramos, Havana, Cuba.*

Let ABC be a triangle and P any point in the plane. Triangle MNO is determined by the feet of the perpendiculars from P to the sides, and triangle QRS is determined by the cevians through P and the circumcircle of triangle ABC. Prove that triangles MNO and QRS are similar.

457*. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Here are examples of two n-digit squares whose juxtaposition forms a $2n$-digit square:

$$\begin{array}{rcrcrl} 4 & \text{and} & 9 & \text{form} & 49 = & 7^2, \\ 16 & \text{and} & 81 & \text{form} & 1681 = & 41^2, \\ 225 & \text{and} & 625 & \text{form} & 225625 = & 475^2. \end{array}$$

Is there at least one such juxtaposition for each $n = 4, 5, 6, \ldots$?

458. *Proposed by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

Let $\phi(n)$ denote the Euler function. It is well known that, for each fixed integer $c > 1$, the equation $\phi(n) = n - c$ has at most a finite number of solutions for the integer n. Improve this by showing that any such solution, n, must satisfy the inequalities $c < n \le c^2$.

459. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*

If n is a positive integer, prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \le \frac{\pi^2}{8} \cdot \frac{1}{1 - 2^{-2n}}.$$

460. *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Problem 124 in the *MATYC Journal* (12 (Fall 1978) 254), proposed by C.W. Trigg, asks: "Can two consecutive odd integers be the sides of a Pythagorean triangle?"

It is easy to show that the answer is "Yes" provided the two consecutive odd integers are a leg and the hypotenuse and their mean is an even square integer. Answer the more difficult question: Can two consecutive even integers be the sides of a Pythagorean triangle? Show how to find all such Pythagorean triangles.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

355. [1978: 160; 1979: 78] *Proposed by James Gary Propp, Great Neck, N.Y.*

Given a finite sequence $A = (a_n)$ of positive integers, we define the family of sequences

$$A_0 = A; \qquad A_i = (b_r), \quad i = 1,2,3,\ldots ,$$

where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}.

For example, if $A = A_0 = (2,4,2,2,4,5)$, then $A_1 = (3,2,1)$, $A_2 = (1,1,1)$, $A_3 = (3)$, and $A_4 = (1) = A_5 = A_6 = \ldots$.

The *degree* of a sequence A is the smallest i such that $A_i = (1)$.

(a) Prove that every sequence considered has a degree.

(b) Find an algorithm that will yield, for all integers $d \geq 2$, a shortest sequence of degree d.

(c) Let $\Lambda(d)$ be the length of the shortest sequence of degree d. Find a formula, recurrence relation, or asymptotic approximation for $\Lambda(d)$.

(d) Given sequences A and B, define C as the concatenation of A and B. Find sharp upper and lower bounds on the degree of C in terms of the degrees of A and B.

IV. *Comment on part (b) by Leroy F. Meyers, The Ohio State University.*

The published solution II [1979: 79-80] appears to be incomplete. In the lemma, $M = (m_n)$ is the canonical sequence of degree d derived by the algorithm and $A = (a_n)$ is an arbitrary canonical sequence of degree d. Note now that $A' = A_1$ is not necessarily canonical. For example,

$$A = (3,3,2,2,1,1,1,\bar{0}) \implies A_1 = (3,2,2,\bar{0}),$$

and A is canonical but A_1 is not. Thus, the statement "From now on, all sequences will be assumed canonical." causes difficulty at the two places where the induction hypothesis is invoked in the proof of the lemma, since the sequence A' need not be canonical. The proof can be completed, however, by showing that, given M and A, the unique canonical sequence $B=(b_n)$ such that B_1 is the canonicalization of A_1 satisfies $m_n \le b_n \le a_n$. But to show this seems to require going through a proof similar to that of the lemma two more times. The following rewriting of the solution seems to be somewhat shorter, although the idea is essentially the same.

Any sequence considered can be put in *normal form* as follows: first, we rename the terms so that 1 is the smallest term, 2 is the second smallest term, and so on, where equal terms are renamed equal terms; second, we arrange the terms in decreasing order; and third, we tack an unending string of zeros at the end, with the stipulation that the length of such a sequence is the number of nonzero terms it contains. Thus

$$(2,4,2,2,4,5) \longrightarrow (1,2,1,1,2,3) \longrightarrow (3,2,2,1,1,1) \longrightarrow (3,2,2,1,1,1,\bar{0}).$$

Notice that the process preserves length and degree as long as the degree is not 1. A nonnormal sequence is also assumed to have an unending string of zeros tacked on at the end.

Here is the desired algorithm. Suppose M' is a shortest normal sequence of degree $d-1$ and let M be the unique normal sequence such that $M_1 = M'$. Then M is the shortest normal sequence of degree d. This procedure, applied repeatedly to

$(1,1,\bar{0})$, yields $(2,1,\bar{0})$, $(2,1,1,\bar{0})$, $(3,2,1,1,\bar{0})$, $(4,3,2,2,1,1,1,\bar{0}),\ldots$.

We show that these are in fact shortest sequences of their respective degrees.

LEMMA 1. Let $A=(a_n)$ be any normal sequence, let $B'=(b'_n)$ be the normalization of $A'=A_1=(a'_n)$, and let $B=(b_n)$ be the unique normal sequence such that $B_1 = B'$. Then for all positive integers n we have $b_n \le a_n$.

Proof. Let u be the length of A' (and of B'). Then

$$b'_u + b'_{u-1} + \ldots + b'_i \le a'_u + a'_{u-1} + \ldots + a'_i, \qquad 1 \le i \le u,$$

since the left side of the inequality does not exceed the sum of the $u+1-i$ lowest nonzero terms of A' (repetitions allowed), which in turn does not exceed the sum of the last $u+1-i$ nonzero terms of A'. If $n > b'_u + b'_{u-1} + \ldots + b'_1$, then $b_n = 0 \le a_n$, as desired. On the other hand, if

$$b'_u + b'_{u-1} + \ldots + b'_{i+1} < n \le b'_u + b'_{u-1} + \ldots + b'_i,$$

where $1 \le i \le u$ and the left side is taken to be 0 if $i=u$, then $n \le a'_u + a'_{u-1} + \ldots + a'_i$, and so $a_n \ge u+1-i = b_n$, as desired.

LEMMA 2. For $d \ge 2$, let $M=(m_n)$ be the normal sequence of degree d derived by the algorithm given above, and let $A=(a_n)$ be any normal sequence of degree d. Then *for all positive integers n we have* $m_n \le a_n$.

An immediate consequence of the inequalities $m_n \le a_n$ is that $\lambda M \le \lambda A$ [in the notation of solution I, p. 78], and this validates our algorithm.

Proof. It is clear that $(1,1,\bar{0})$ is the shortest normal sequence of degree 2, so the lemma is true for $d = 2$. We assume the lemma holds for $d = q-1$ and show by contradiction that it must hold for $d = q$. Suppose A is a normal sequence of degree q for which the conclusion of the lemma is false, and let $B = (b_n)$ be a normal sequence such that $B' = B_1$ is the normalization of $A' = A_1$. Since $b_n \le a_n$ for all n (by Lemma 1) and the degree of B is also q, the conclusion of the lemma is *a fortiori* false for B. Let, then, j be the least integer such that $m_j > b_j$. If $j = 1$, then $m_1 > b_1$; but the first term of a normal sequence is the number of distinct nonzero terms it contains, which equals the length of the next sequence in its family. Thus $M_1 = M'$ contains more nonzero terms than B', and there exists an n such that $m'_n > 0$ and $b'_n = 0$. But since M' and B' are normal sequences of degree $q-1$, our inductive hypothesis has been contradicted. If $j > 1$, then

$$b_j < m_j \le m_{j-1} \le b_{j-1} \le b_j + 1;$$

therefore $b_j = b_{j-1} - 1$ and $m_j = m_{j-1} = b_{j-1} = v$, say. Since $m_1 \ge m_2 \ge \ldots \ge m_j = v$, there are at least j terms of M that are greater than or equal to v, and so

$$m'_v + m'_{v+1} + m'_{v+2} + \ldots \ge j.$$

Also, since $b_j = v - 1$, there are at most $j-1$ terms of A that are greater than or equal to v, and so

$$b'_v + b'_{v+1} + b'_{v+2} + \ldots \le j - 1.$$

Combining these inequalities, we get

$$m'_v + m'_{v+1} + m'_{v+2} + \ldots > b'_v + b'_{v+1} + b'_{v+2} + \ldots .$$

Yet by the inductive hypothesis $m'_n \le b'_n$ for all $n \ge v$. This contradiction completes the proof of the lemma.

In what sense are the sequences constructed by the algorithm given above unique? Since canonicalization preserves length and degree (if the degree is different from 1), then we may suspect that only sequences whose canonicalization is a sequence constructed by the algorithm are shortest for their degree. In fact, this is true, and is easily verified directly when the degree is low, say up to and including 3. Suppose then that A is a shortest canonical sequence of degree $d \ge 3$ and that M is the corresponding sequence of degree d obtained by the algorithm. With λ and (later)

σ as defined in solution I, we have $\lambda A = \lambda M$, the sequence $M' = M_1$ is canonical (hence normal), and the sequence $A' = A_1$ is nonincreasing (but not necessarily normal). Let B' be the normalization of A'. Then $b_n' \le a_n'$ for all n. By Lemma 2 we have $m_n' \le b_n'$ for all n. Hence

$$\lambda M = \sigma M' \le \sigma B' \le \sigma A' = \lambda A = \lambda M,$$

from which $\sigma M' = \sigma A'$, and so $M' = A'$. Since M and A are determined uniquely from M' and A', we have $M = A$.

Editor's comment.

Readers are reminded that parts (c) and (d) of this problem remain open.

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380. [1978: 226] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let P be a point on the graph of $y = f(x)$, where f is a third-degree polynomial, let the tangent at P intersect the curve again at Q, and let A be the area of the region bounded by the curve and the segment PQ. Let B be the area of the region defined in the same way by starting with Q instead of P. What is the relationship between A and B?

Solution by the proposer.

We may assume, without affecting the areas of interest, that the point of inflection of the cubic is at the origin so that

$$f(x) = ax^3 + bx, \quad a \ne 0.$$

If x_0 is the abscissa of P, the equation of the tangent PQ is

$$y = g(x) = f(x_0) + (x - x_0)f'(x_0).$$

Now the abscissa of Q satisfies $f(x) - g(x) = 0$, an equation with a double root x_0 and sum of roots zero; hence the abscissa of Q is $-2x_0$ and we have (work it out!)

$$A = \left| \int_{-2x_0}^{x_0} (f(x) - g(x))dx \right| = Kx_0^4,$$

where K is independent of x_0. To find B, we start from Q instead of P and obtain the required relation:

$$B = K(-2x_0)^4 = 16A.$$

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; MURRAY S. KLAMKIN, University of Alberta; VIKTORS LINIS, University of Ottawa; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; BASIL C. RENNIE, James Cook University of North Queensland, Australia;

DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; and DAVID R. STONE, Georgia Southern College, Statesboro, Georgia.

Editor's comment.

Stone wrote: "This is an absolutely beautiful problem for first calculus students. With the relation valid for any point of any cubic, who can deny that there is order in the universe!"

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381. [1978: 250] *Proposé par Sidney Kravitz, Dover, N.J.*

Résoudre l'addition légumatique décimale suivante:

$$\begin{array}{r} \text{BETTE} \\ \underline{\text{TOMATE}} \\ \text{OIGNON} \end{array}.$$

Solution by Charles W. Trigg, San Diego, California.

Clearly $T+1 = O$, so from the tens' column $T=1$ and $O=2$. Then from the fifth column $B=8$ and $I=0$. Now from the units' column $E=3$ and $N=6$, whereupon $A=5$ and $M=4$, $G=7$. The digit 9 is missing. The unique solution is

$$\begin{array}{r} 83113 \\ \underline{124513} \\ 207626 \end{array}.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; LOUIS H. CAIROLI, student, Kansas State University, Manhattan, Kansas; CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; FRIEND H. KIERSTEAD Jr., Cuyahoga Falls, Ohio; JACK LeSAGE and the following students of his and Arvon Kyer, Eastview Secondary School, Barrie, Ontario: MARK BRIGHAM, WAYNE COLE, RANDY COREY, JIM ENWRIGHT, DAVE FOXCROFT, DANIALL GARBALLA, STEPHEN JUPP, STEPHEN KOMAR, DOUG LATORNELL, BERNARD MAYOR, PETER MORGAN, and MIKE WALSH; VIKTORS LINIS, University of Ottawa; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; FREDERICK NEIL ROTHSTEIN, New Jersey Department of Transportation, Trenton, N.J.; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

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382. [1978: 250] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let a, b, c, d be positive integers. Evaluate

$$\lim_{n\to\infty} \frac{a(a+b)(a+2b)\ldots(a+(n-1)b)}{c(c+d)(c+2d)\ldots(c+(n-1)d)}.$$

I. *Solution by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*

We adopt the strict lexicographic ordering for the ordered pairs (b,a) and (d,c). We will say, for example, that (b,a) *follows* (d,c) if (i) $b>d$ or (ii) $b=d$ and $a>c$.

As usual, $(b,a) = (d,c)$ means $b = d$ and $a = c$. If L is the required limit, the claim is that, for all positive *reals* a, b, c, d,

$$L = \begin{cases} 1 & \text{if } (b,a) = (d,c), \\ \infty & \text{if } (b,a) \text{ follows } (d,c), \\ 0 & \text{if } (d,c) \text{ follows } (b,a). \end{cases}$$

The case $(b,a) = (d,c)$ is clear. In the following let k be a fixed positive integer such that $c < kd$.

If $b > d$, noting that $c + jd < (j + k)d$, we have

$$\frac{a(a+b)\ldots(a+(n-1)b)}{c(c+d)\ldots(c+(n-1)d)} > \frac{a \cdot (n-1)!b^{n-1}}{k(k+1)\ldots(k+n-1)d^n}.$$

Then for large n the right side of the above equals

$$\frac{a \cdot (k-1)!b^{n-1}}{(k+n-1)(k+n-2)\ldots n \cdot d^n} > \frac{a \cdot (k-1)!}{(k+n-1)^k \cdot d} \cdot \left(\frac{b}{d}\right)^{n-1},$$

which tends to infinity as n increases.

If $b = d$ and $a > c$, then the given ratio equals

$$\prod_{i=0}^{n-1} \frac{a+ib}{c+ib} = \prod_{i=0}^{n-1} \left(1 + \frac{a-c}{c+ib}\right) > \prod_{i=0}^{n-1} \left(1 + \frac{a-c}{b(i+k)}\right) > \frac{a-c}{b} \sum_{i=0}^{n-1} \frac{1}{i+k},$$

and the last sum tends to infinity with n since the harmonic series diverges. This completes the proof when (b,a) follows (d,c).

The claimed result for the last case, when (d,c) follows (b,a), is obtained immediately if we apply the result just proved to the reciprocal of the given ratio.

II. *Solution by Viktors Linis, University of Ottawa.*

The required limit is a special case of infinite products which are expressible in terms of the Gamma function (see [1, pp. 238-239]). If a, b, c, d are real and positive, the required limit, L, can first be written

$$L = \lim_{n\to\infty} \frac{a^n z_2^{n-1}(z_1+1)(z_1+2)\ldots(z_1+n-1)}{c^n z_1^{n-1}(z_2+1)(z_2+2)\ldots(z_2+n-1)},$$

where $z_1 = a/b$ and $z_2 = c/d$. Then, using Euler's formula [1, p. 237]

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!n^z}{z(z+1)\ldots(z+n-1)},$$

we get

$$L = \frac{\Gamma(z_2)}{\Gamma(z_1)} \cdot \lim_{n\to\infty} \left(\frac{b}{d}\right)^n n^{z_1 - z_2}.$$

The desired result is now obvious. If $b \neq d$ then, for any a and c,

$$L = 0 \text{ or } \infty \qquad \text{according as} \qquad b < d \text{ or } b > d.$$

If $b = d$ then

$$L = 0,\ 1, \text{ or } \infty \qquad \text{according as} \qquad a < c,\ a = c, \text{ or } a > c.$$

Also solved by ALLAN Wm. JOHNSON Jr., Washington, D.C.; MURRAY S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. Comments were submitted by HAYO AHLBURG, Benidorm, Spain; and HERMAN NYON, Paramaribo, Surinam.

REFERENCE

1. E.T. Whittaker & G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1962.

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383. [1978: 250] *Proposed by Daniel Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Let m_a, m_b, m_c be respectively the medians AD, BE, CF of a triangle ABC with centroid G. Prove that

(a) if $m_a : m_b : m_c = a : b : c$; then ΔABC is equilateral;

(b) if $m_b/m_c = c/b$, then either (i) $b = c$ or (ii) quadrilateral AEGF is cyclic;

(c) if both (i) and (ii) hold in (b), then ΔABC is equilateral .

I. *Solution of part (a) by Murray S. Klamkin, University of Alberta.*

We show more generally that if

$$n_a : n_b : n_c = a : b : c, \tag{1}$$

where n_a, n_b, n_c are three cevians dividing the sides BC, CA, AB in the same ratio $u : v$ (with $u + v = 1$), then ΔABC is equilateral. We have

$$\begin{aligned} n_a^2 &= |u\vec{AC} + v\vec{AB}|^2 \\ &= u^2b^2 + v^2c^2 + 2uv\vec{AC}\cdot\vec{AB} \\ &= u^2b^2 + v^2c^2 + uv(b^2 + c^2 - a^2) \\ &= ub^2 + vc^2 - uva^2 \end{aligned}$$

with similar results for n_b and n_c. It now follows from (1) that

$$ub^2 + vc^2 - uva^2 = ka^2, \tag{2}$$
$$uc^2 + va^2 - uvb^2 = kb^2, \tag{3}$$
$$ua^2 + vb^2 - uvc^2 = kc^2, \tag{4}$$

and adding these gives $u + v - uv = k$ or $k = 1 - uv$. Substituting back into (2)-(4) gives

$$ub^2 + vc^2 = a^2, \tag{5}$$
$$uc^2 + va^2 = b^2, \tag{6}$$
$$ua^2 + vb^2 = c^2. \tag{7}$$

Eliminating a^2 from (5) and (6) yields $(1 - uv)b^2 = (u + v^2)c^2$. But since

$$u + v^2 = u + (1 - u)^2 = 1 - u(1 - u) = 1 - uv \neq 0,$$

it follows that $b = c$. Similarly, eliminating b^2 from (6) and (7) gives $c = a$, and ΔABC is equilateral.

II. *Solution of parts (b) and (c) by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

(b) We have

$$m_b/m_c = c/b \implies \frac{(3/2)BG}{(3/2)CG} = \frac{2BF}{2CE} \implies \frac{BG}{BF} = \frac{CG}{CE}.$$

Hence in triangles GBF and GCE, which have equal angles at G, the angles at F and E are either (i) equal or (ii) supplementary. In the first case, trapezoid BFEC is cyclic (hence isosceles) and $b = c$; and in the second case quadrilateral AEGF is cyclic.

(c) If both (i) and (ii) hold in (b), then the angles at F and E are right angles, and it easily follows that ΔABC is equilateral.

Complete solutions were received from W.J. BLUNDON, Memorial University of Newfoundland; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; G.C. GIRI, Research Scholar, Indian Institute of Technology, Kharagpur, India; ALLAN Wm. JOHNSON Jr., Washington, D.C.; MURRAY S. KLAMKIN, University of Alberta; LEROY F. MEYERS, The Ohio State University; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; ORLANDO RAMOS, Instituto Politécnico José Antonio Echevarría, Havana, Cuba; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

Editor's comment.

As a few solvers pointed out, the converses of all three parts of the problem are true. This is obvious for (a) and (c). For (b), we merely observe that all the steps in its proof are reversible.

As our proposer proved earlier [1978: 200], quadrilateral AEGF is cyclic if and only if $2a^2 = b^2 + c^2$, that is, if and only if ABC is an *RMS triangle*, a name due to Leon Bankoff [1978: 14]. See [1978: 194] for Alan Wayne's list of 36 primitive RMS triangles with integral sides and perimters $P < 1000$.

The generalization of (a) proved in solution I can be extended further to

(a)' *If* n_a, n_b, n_c *are three cevians dividing the sides* BC, CA, AB *in the same ratio* $u : v$ *(with* $u + v = 1$ *and* $u \neq 0,1$*) and* (α,β,γ) *is a cyclic permutation of* (a,b,c), *then*

$$n_a : n_b : n_c = \alpha : \beta : \gamma \iff \Delta\,\text{ABC}\ \textit{is equilateral.}$$

(The restriction $u \neq 0,1$ can even be relaxed. If $u = 0$ or 1 then the theorem still holds, but for only two of the three cyclic permutations of (a,b,c).) The proof given above for (a) is easily modified to establish (a)'. This more general result is contained in an unpublished 1978 paper by our proposer. This paper also contains the following generalization of (b), which reduces to (b) when $u = v = \frac{1}{2}$:

(b)' $n_b/n_c = c/b \iff$ (i) $ub^2 = vc^2$ *or* (ii) $vb^2 + uc^2 = a^2$.

Finally, corresponding to (c) we now have

(c)' *If both* (i) *and* (ii) *hold in* (b)', *then* ΔABC *is equilateral if and only if* $u = v = \frac{1}{2}$.

* * *

384. [1978: 250] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Résoudre le système d'équations suivant pour x et y:

$$\frac{(ab+1)(x^2+1)}{x+1} = \frac{(a^2+1)(xy+1)}{y+1}$$

$$\frac{(ab+1)(y^2+1)}{y+1} = \frac{(b^2+1)(xy+1)}{x+1}$$

Solution by W.J. Blundon, Memorial University of Newfoundland (revised by the editor).

The given system is equivalent to the following:

$$(ab+1)(x^2+1)(y+1) = (a^2+1)(xy+1)(x+1), \qquad x,y \neq -1. \tag{1}$$

$$(ab+1)(y^2+1)(x+1) = (b^2+1)(xy+1)(y+1), \tag{2}$$

We give an exhaustive discussion of the nature of the solution sets over the complex field according to the values of the parameters a and b.

(a) If $a = b = \pm i$ (where $i = \sqrt{-1}$), every pair (x,y) with $x \neq -1$ and $y \neq -1$ is a solution.

(b) If $a = b \neq \pm i$, we subtract (1) and (2) and obtain an equation equivalent to

$$(x-y)(x+y-2) = 0.$$

It is easy to verify that if $x+y-2 = 0$ the pair (x,y) is not a solution unless $x=y=1$. So the solutions consist of all the pairs (x,y) with $x = y \neq -1$.

(c) If $a \neq b$ and $ab = -1$, then $a^2+1 \neq 0$, $b^2+1 \neq 0$, and we conclude from (1) or (2) that $xy+1 = 0$. Here the solutions are all the pairs (x,y) with $y = -1/x$ and $x \neq \pm 1$.

(d) We assume from now on that $a \neq b$ and $ab \neq -1$. It follows from (1) and (2) that if $xy+1 = 0$ then $x^2 = -1$ and $y^2 = -1$. This yields the two solutions (i,i) and $(-i,-i)$. The remaining solutions, which we will now find, are those for which $xy+1 \neq 0$. For each such solution, there is a unique nonzero number t such that

$$(ab+1)(x+1)(y+1) = t(xy+1). \tag{3}$$

If we multiply (1) and (2) by the nonzero numbers $x+1$ and $y+1$, respectively, we obtain equations equivalent to

$$t(x^2+1) = (a^2+1)(x+1)^2 \qquad \text{and} \qquad t(y^2+1) = (b^2+1)(y+1)^2. \tag{4}$$

Equations (3) and (4) are equivalent to

$$ab(x+1)(y+1) = t(xy+1) - (x+1)(y+1), \tag{5}$$

$$a^2(x+1)^2 = t(x^2+1) - (x+1)^2, \tag{6}$$

$$b^2(y+1)^2 = t(y^2+1) - (y+1)^2. \tag{7}$$

We now subtract the square of (5) from the product of (6) and (7) and obtain

$$t(t-2)(x-y)^2 = 0.$$

It can be verified from (1) and (2) that $x=y$ would lead to $a=b$ or $xy+1 = 0$, both of which are excluded. Since $t \neq 0$, we must have $t=2$. Substituting this value in equations (4), we have no trouble in getting

$$\left(\frac{x-1}{x+1}\right)^2 = a^2 \qquad \text{and} \qquad \left(\frac{y-1}{y+1}\right)^2 = b^2,$$

from which

$$x = \frac{1+a}{1-a} \text{ or } \frac{1-a}{1+a} \qquad \text{and} \qquad y = \frac{1+b}{1-b} \text{ or } \frac{1-b}{1+b}. \tag{8}$$

It can be verified from (1) and (2) that, of the four pairs (x,y) defined by (8), only two *can* be solutions, these are

$$(x_1,y_1) = \left(\frac{1+a}{1-a}, \frac{1+b}{1-b}\right) \quad \text{and} \quad (x_2,y_2) = \left(\frac{1-a}{1+a}, \frac{1-b}{1+b}\right).$$

And these *are* solutions provided no denominator vanishes. If $a = 1$, for example, then $b \neq -1$ (since $ab \neq -1$) and only (x_2,y_2) is a solution.

Also solved by HAYO AHLBURG, Benidorm, Spain; H.G. DWORSCHAK, Algonquin College, Ottawa; G.P. HENDERSON, Campbellcroft, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; FRIEND H. KIERSTEAD Jr., Cuyahoga Falls, Ohio; VIKTORS LINIS, University of Ottawa; BASIL C. RENNIE, James Cook University of North Queensland, Australia; FREDERICK NEIL ROTHSTEIN, New Jersey Department of Transportation, Trenton, N.J.; MATS RÖYTER, student, Majornas gymnasium, Gothenburg, Sweden; GALI SALVATORE, Perkins, Québec; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

Editor's comment.

This elementary problem turned out to be unexpectedly challenging, judging from the relatively poor quality of many of the solutions received. The challenge was twofold: (i) to find a systematic and exhaustive way of describing the solution sets according to the values of the parameters a and b; (ii) to find a neat way of solving the general case (part (d) in our featured solution) without getting bogged down in endless calculations and stretching the solution to unconscionable length. Regarding point (ii), some solvers managed to keep their solution to a reasonable length by sweeping acres of calculations under a euphemistic rug: "After some reduction, we get... ." As for point (i), some solvers conveniently forgot to unscramble the various solution sets, or made only a stab at it. A parametric discussion is often a tiresome thing to do—especially when more than one parameter is involved. But if a thing is worth doing, it is worth doing well. Lest this last sentence sound too discouraging to our younger readers, who must learn and practise to walk before they can run, we hasten to add that if a thing is worth doing, it is also worth doing badly.

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385. [1978: 250] *Proposed by Charles W. Trigg, San Diego, California.*

In the decimal system, there is a 12-digit cube with a digit sum of 37. Each of the four successive triads into which it can be sectioned is a power of 3. Find the cube and show it to be unique.

Solution by Hayo Ahlburg, Benidorm, Spain.

The possible triads forming the required number N are

001, 003, 009, 027, 081, 243, 729

with digit sums of

1, 3, 9, 9, 9, 9, 18. (1)

The only way to get a sum of 37 with four numbers from (1) is

$$37 = 1 + 9 + 9 + 18,$$

and the first triad of N is 243 or 729. If it is 243, then

$$243\ 001\ 009\ 729 \le N^3 \le 243\ 729\ 243\ 001 \quad \text{and} \quad 6240 < N < 6247,$$

so $N = 6241$ or 6243. But neither of these numbers has a cube of the proper form. So the first triad of N is 729 and we have

$$729\ 001\ 009\ 009 \le N^3 \le 729\ 243\ 243\ 001 \quad \text{and} \quad 9000 < N < 9002.$$

The only possible solution is $N = 9001$, and it is satisfactory since

$$N^3 = 729\ 243\ 027\ 001$$

is of the proper form.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; G.P. HENDERSON, Campbellcroft, Ontario; ALLAN Wm. JOHNSON Jr., Washington, D.C.; FRIEND H. KIERSTEAD Jr., Cuyahoga Falls, Ohio; VIKTORS LINIS, University of Ottawa; HERMAN NYON, Paramaribo, Surinam; MATS RÖYTER, Majornas gymnasium, Gothenburg, Sweden; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Dodge and the proposer both mentioned that if initial zeros are allowed in N^3, then

$$1009^3 = 001\ 027\ 243\ 729$$

also provides a solution.

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386. [1978: 251] *Proposed by Francine Bankoff, Beverly Hills, California.*

A square PQRS is inscribed in a semicircle (O) with PQ falling along diameter AB (see Figure 1 below). A right triangle ABC, equivalent to the square, is inscribed in the same semicircle with C lying on the arc RB. Show that the incenter I of triangle ABC lies at the intersection of SB and RQ, and that

$$\frac{RI}{IQ} = \frac{SI}{IB} = \frac{1+\sqrt{5}}{2}, \text{ the golden ratio.}$$

I. *Solution by John A. Winterink, Albuquerque Technical Vocational Institute, Albuquerque, New Mexico.*

Let $BS \cap QR = I$. Since $\Delta ROS = \Delta BOC$, we have arc RS = arc BC and

$$\text{arc } CS = \text{arc } BR = \text{arc } SA.$$

Thus BS bisects $\angle ABC$ since S bisects arc CA. Drop perpendicular IM on BC. If

$OR = \rho$, then

$SP = 2\rho/\sqrt{5}$, $QB = \rho(1 - 1/\sqrt{5})$, $PB = \rho(1 + 1/\sqrt{5})$.

We have

$$MC = BC - BM = SP - QB = \frac{\rho(3 - \sqrt{5})}{\sqrt{5}}$$

and, from similar triangles,

$$IM = IQ = QB \cdot \frac{SP}{PB} = \frac{\rho(3 - \sqrt{5})}{\sqrt{5}}.$$

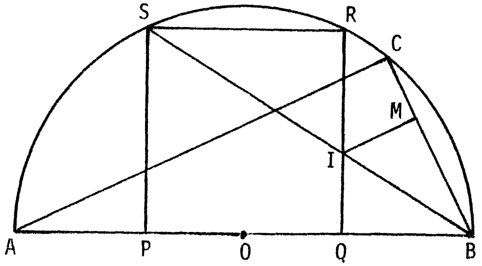

Figure 1

Thus $IM = MC$ and IC bisects $\angle ACB$, which makes I the incenter of ΔABC.

Finally, from similar triangles,

$$\frac{RI}{IQ} = \frac{SI}{IB} = \frac{SR}{QB} = \frac{SP}{QB} = \frac{1 + \sqrt{5}}{2}.$$

II. *Partial solution by Zelda Katz, Beverly Hills, California.*

Meow MEow MeOW (see Figure 2).

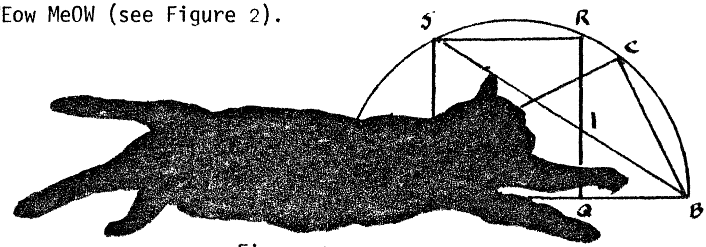

Figure 2

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; CECILE M. COHEN, John F. Kennedy H.S., New York; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; JACK GARFUNKEL, Forest Hills H.S., Flushing, N.Y.; ALLAN Wm. JOHNSON Jr., Washington, D.C.; BRUCE KING, Western Connecticut State College, Danbury, Connecticut; HERMAN NYON, Paramaribo, Surinam; JEREMY D. PRIMER, student, Columbia H.S., Maplewood, N.J.; ORLANDO RAMOS, Instituto Politécnico José Antonio Echevarría, Havana, Cuba; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

This problem is an extension of an earlier problem proposed by Leon Bankoff [1], which asked merely to show that, in the same configuration, the incenter I lies on RQ.

Zelda Katz, who makes her first appearance here, is a frequent contributor to the problem section of *Pi Mu Epsilon Journal*, which is edited by Dr. Bankoff.

REFERENCE

1. Problem 741, *Mathematics Magazine* 43 (May 1970) 167-168.

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