Mathematical Spectrum

A magazine for students and teachers of mathematics in schools, colleges and universities, and for everyone interested in mathematics

Volume 42 2009/2010 Number 1

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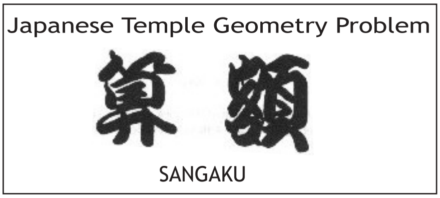
Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

One volume of *Mathematical Spectrum* is published in each British academic year and consists of three issues, which appear in September, January and May.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor



Bob Bertuello has written about an intriguing appearance of mathematics. Here is his letter:

> Sangaku is the name given to colourful tablets which posed mathematical problems. These were offered in Japanese shinto shrines during the seventeenth to the nineteenth centuries. The tradition of Sangaku consisted of a wide variety of geometrical problems, inscribed on delicately coloured wooden tablets and hung in religious buildings. Sangaku literally means mathematical tablet.
>
> For the most part, Sangaku deals with ordinary Euclidean geometry. Most of the problems are different from those found in a typical geometry course, circles and ellipses playing a prominent part. The exercises vary greatly in difficulty.
>
> For a comprehensive treatment of the subject, go to www.cut-the-knot.com/sangaku, where many examples can be found as well as some wonderful applets that allow the user to control the geometry of the diagrams. Many interesting links are available from this website.

As a sampler of what is available, Bob has picked out a problem which we have included in the problems section (Problem 42.1) and invite the reader to try.

So, when you are next in Japan, look out for mathematics in surprising places. Come to think of it, you may not need to go to Japan!

Reference

1 http://www.wasan.jp/english/

$$\begin{aligned} 100 &= 12 + 3 - 4 + 5 + 67 + 8 + 9 \\ &= 123 + 4 - 5 + 67 - 89 \\ &= 123 - 45 - 67 + 89. \end{aligned}$$

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A Simple Geometric Proof of Morley's Trisector Theorem

BRIAN STONEBRIDGE

Morley's theorem is one of the most surprising and attractive twentieth century results in plane geometry. Its simplicity is part of its beauty, but could easily lead us to expect an equally simple proof. No known proof shows the desirable properties of being purely geometric, concise, and transparent. The lack of such a proof may be a reason why the result is not more widely known.

We provide a simple geometric proof, which relies only on the angle sums of triangles, and the properties of similar triangles and of tangents to a circle. This elementary approach makes the derivation of the result more easily accessible.

Theorem 1 (Morley's Theorem (1899).) *The points of intersection of adjacent trisectors of any triangle form an equilateral triangle.*

Proof Define α, β, and γ such that the angles of $\Delta \boldsymbol{ABC}$ are 3α, 3β, and 3γ. See figure 1. Then, we have the identity

$$\alpha + \beta + \gamma = \frac{\pi}{3} \quad \text{(the angle sum of the triangle)}. \tag{1}$$

Start with an *arbitrary* equilateral triangle, ΔXYZ, as shown in figure 2.

1. Let P, Q, and R be points on the altitudes (produced) of ΔXYZ such that

$$\angle XPY (= \angle XPZ) = \alpha + \frac{\pi}{6}, \qquad \angle YQZ (= \angle YQX) = \beta + \frac{\pi}{6},$$
$$\angle ZRX (= \angle ZRY) = \gamma + \frac{\pi}{6}.$$

(If the theorem were indeed true, these would be the values taken by the corresponding angles. We are free to choose them here, and they give (2).)

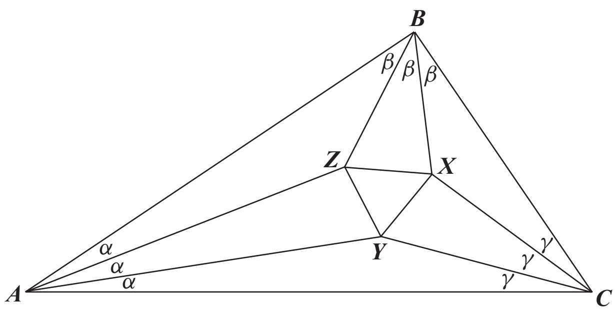

Figure 1 Given any $\Delta \boldsymbol{ABC}$ and its trisectors, we prove that $\Delta \boldsymbol{XYZ}$ is equilateral.

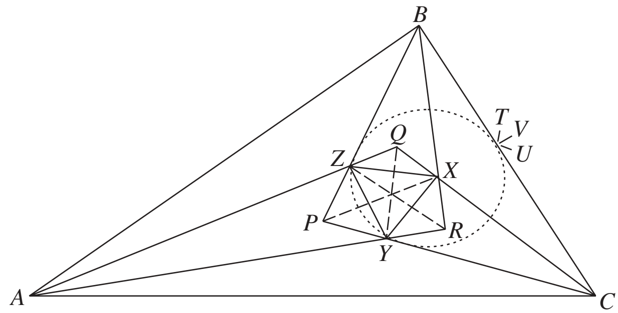

Figure 2 From the equilateral triangle XYZ, construct ΔABC, similar to the given $\Delta \boldsymbol{ABC}$.

2. Let QZ and RY meet at A, let RX and PZ meet at B, and let PY and QX meet at C. Then
$$\angle ZAY = \alpha, \qquad \angle XBZ = \beta, \qquad \angle YCX = \gamma \tag{2}$$
(by the angle sums of $XRAQ$, $YPBR$, $ZQCP$ (equal to 2π)).

3. Draw a circle with centre X touching PB and, hence, also PC (PX bisects $\angle BPC$).

4. Draw tangents BT and CU, and let them meet at V. Then
$$\angle XBT = \angle XBZ = \beta \quad \text{and} \quad \angle XCU = \angle XCY = \gamma \quad \text{(by (2))}. \tag{3}$$

5. The sum of the angles at P, B, and C in the quadrilateral $PBVC$ is
$$2\alpha + \frac{\pi}{3} + 2\beta + 2\gamma = \pi \quad \text{(by (1))}.$$
Therefore, $\angle BVC = \pi$ (the angle sum of $PBVC$), and the points T, V, and U coincide.

Therefore, $\angle XBC = \beta$ and $\angle XCB = \gamma$ (by (3)); thus, the angles of ΔXBC are determined.

Similarly, the angles of ΔYCA and ΔZAB are determined by drawing circles with centres at Y and Z.

The above shows that the constructed ΔABC has the same angles as the original $\Delta \boldsymbol{ABC}$, and the trisectors of ΔABC form an equilateral triangle, ΔXYZ. Hence, the same is true of the original $\Delta \boldsymbol{ABC}$, since it is similar to ΔABC.

Previous proofs

One of the shortest proofs of Morley's theorem is that attributed to Penrose (reference 5). Longer proofs are given by Lyness (reference 4), Coxeter (reference 2), and Naraniengar (1909) (which appears in Coxeter and Greitzer (reference 1) and Honsberger (reference 3)). Sastry (reference 6) referred to the latter for a proof, indicating that amongst methods employing simple Euclidean geometry, the method of Naraniengar had not been bettered. This opinion is reinforced in the historical background of the theorem, provided by Guy (reference 7).

We refer the reader to http://www.cut-the-knot.org/triangle/Morley/ for twelve links to a variety of proofs, allowing their merits to be compared.

Acknowledgements

Sincere thanks are due to Jim Gowers, Bill Millar, and John Shepherdson who very patiently moderated the sequence of contortions which led to this article.

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6 K. R. S. Sastry, Morley's theorem, *Math. Spectrum* **23** (1990/91), p. 1.
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Added in proof

The above proof is a 'backwards' proof, its main virtue being its brevity. I am pleased to say that I have subsequently produced a 'forwards' proof, which is also Euclidean, though longer and less transparent. This can be found at http://www.cut-the-knot.org/triangle/Morley/sb.shtml.

Brian Stonebridge *spent over 30 years as a lecturer in Mathematics and Computer Science at the University of Bristol, doing research in optimization, combinatorics, and graph theory. Now, in retirement, his main relaxations are hill walking, choral singing, and tackling dormant mathematical challenges*

Modular arithmetic

In arithmetic modulo 6, how many solutions does the quadratic equation

$$x^2 + 3x + 2 = 0$$

have?

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Bob Bertuello

Simson's Envelope

G. T. VICKERS

1. Simson's line and Simson's envelope

Let P be any point on the circumcircle of the triangle ABC. Let D, E, and F be the feet of the perpendiculars from P onto the lines BC, AC, and AB, respectively. Then, as shown in figure 1, the classical result of Simson says that the points D, E, and F are collinear. The line DEF is called the *Simson's line* of the point P. The first problem considered here is to find the envelope of the Simson's lines as P moves around the circle while the ΔABC is fixed. As an aside, Robert Simson (1687–1768) was trained in medicine before becoming a Professor of Mathematics at the University of Glasgow. His book *Elements of Euclid* was published in 1756 and (according to C. B. Boyer, *A History of Mathematics*) was in its twenty-fourth edition by 1834 and strongly influenced all subsequent versions of Euclid.

The reader is warned that the writer is retired and, especially when it comes to searching for background information, lazy. Very probably this is a classical problem which has been considered many times before. My excuse for presenting it is that I found it fun to do, partly because the answer is unexpected and also because it leads on to several interesting properties and questions, not all of which I am able to answer. If any reader can provide more information and references to other approaches, then I would be most interested to hear from them.

For simplicity, let the radius of the circumcircle be unity. Choose axes centred upon O, the centre of the circumcircle. The position of a point P on this circle is determined by the angle that OP makes with the x-axis. For brevity, 'the point θ' will mean the point P on the circumcircle for which $\angle xOP = \theta$. Let A, B, and C be the (fixed) points θ_1, θ_2, and θ_3, respectively. Now consider the effect of rotating the axes. If they are rotated through an angle

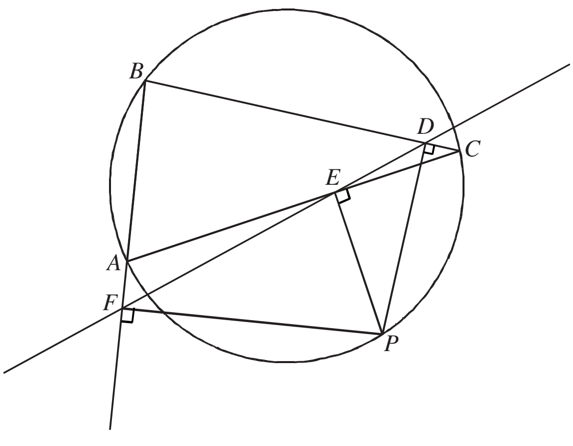

Figure 1 Illustrating the classical result of Simson.

ω then $\theta_1+\theta_2+\theta_3$ will change by 3ω. Thus, we may choose ω so as to make

$$\theta_1+\theta_2+\theta_3=0. \tag{1}$$

It will be found that this simple condition greatly simplifies the algebra and has quite a profound effect upon the following discussion. A curious point to note is that these new axes cannot be constructed by classical ruler and compass methods since they involve trisecting an angle.

Let P be the point ϕ_1. The line AB makes an angle $(\pi+\theta_1+\theta_2)/2$ with Ox and so, with F the foot of the perpendicular from P onto AB, the line PF has slope

$$m=\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}\right)=-\tan\left(\frac{\theta_3}{2}\right).$$

The equations of the lines AB and PF are

$$-m(y-\sin\theta_1)=x-\cos\theta_1 \qquad \text{and} \qquad y-\sin\phi_1=m(x-\cos\phi_1),$$

respectively, and they meet where $x=(-m\sin\phi_1+m\sin\theta_1+\cos\theta_1+m^2\cos\phi_1)/(1+m^2)$. The coordinates of F can now be found using this expression, our knowledge of m, (1), and some perseverance with trigonometric identities. The result is

$$\left(\tfrac{1}{2}[\cos\theta_1+\cos\theta_2+\cos\phi_1-\cos(\theta_3+\phi_1)],\ \tfrac{1}{2}[\sin\theta_1+\sin\theta_2+\sin\phi_1+\sin(\theta_3+\phi_1)]\right).$$

The coordinates of D may now be written down (by a cyclic permutation $\theta_1,\theta_2,$ and θ_3) and, having first found that the slope of DF is $-\tan(\phi_1/2)$, the equation of the line DF is found to be

$$2x\sin\left(\frac{\phi_1}{2}\right)+2y\cos\left(\frac{\phi_1}{2}\right)=\sin\left(\theta_1+\frac{\phi_1}{2}\right)+\sin\left(\theta_2+\frac{\phi_1}{2}\right)+\sin\left(\theta_3+\frac{\phi_1}{2}\right)+\sin\left(\frac{3\phi_1}{2}\right). \tag{2}$$

This equation being symmetrical in $\theta_1,\theta_2,$ and θ_3 implies that D, E, and F are collinear; so the classical result has been proved and (2) is the equation of the Simson's line of P.

If Q is the point ϕ_2 then the Simson's line of Q is found by replacing ϕ_1 by ϕ_2 in (2), and these two Simson's lines meet at

$$x=x_0+\tfrac{1}{2}[\cos\phi_1+\cos\phi_2+\cos(\phi_1+\phi_2)], \tag{3}$$

$$y=y_0+\tfrac{1}{2}[\sin\phi_1+\sin\phi_2-\sin(\phi_1+\phi_2)], \tag{4}$$

where

$$x_0=\tfrac{1}{2}(\cos\theta_1+\cos\theta_2+\cos\theta_3) \tag{5}$$

$$\text{and}\quad y_0=\tfrac{1}{2}(\sin\theta_1+\sin\theta_2+\sin\theta_3). \tag{6}$$

The Simson's lines form a one-parameter family of curves. When the parameter is changed (by a small amount), the two members will intersect at a point and, letting the change tend to zero, we obtain a point on each line. The totality of these points is called the *envelope* of the family. Each member of the family will be a tangent to the envelope. (This is usually the case but is not always true.) Figure 2 shows several Simson's lines and, it is hoped, makes it believable that the complete collection of such lines defines a curve. Replacing both ϕ_1 and ϕ_2 by ϕ in

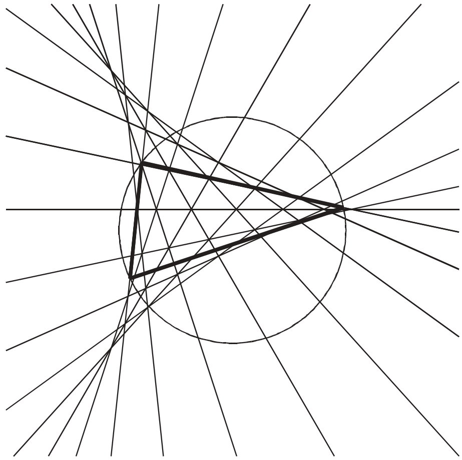

Figure 2 A collection of Simson's lines and their associated triangle (shown with thick lines).

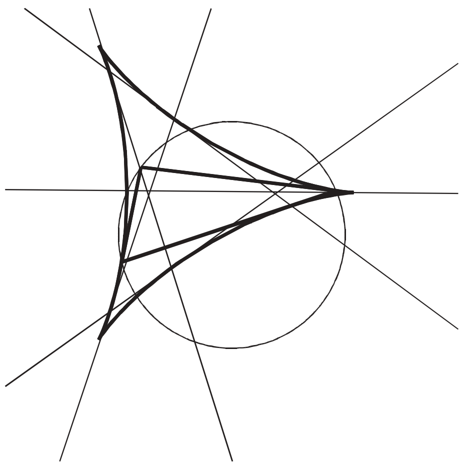

Figure 3 A few Simson's lines and Simson's envelope. Note that not only is each of these lines a tangent to the envelope but the three sides of ΔABC are also tangents.

(3) and (4) shows that a point on the envelope is

$$x = x_0 + \cos\phi + \tfrac{1}{2}\cos 2\phi, \tag{7}$$

$$y = y_0 + \sin\phi - \tfrac{1}{2}\sin 2\phi. \tag{8}$$

Figure 3 shows this curve, which is referred to here as *Simson's envelope*, and it appears to have a three-fold symmetry. This property is far from obvious from the parametric equation given above. With $X = x - x_0$ and $Y = y - y_0$, the equation of the envelope becomes (after some manipulation)

$$2(X^2 + Y^2)^2 + 9(X^2 + Y^2) + 24XY^2 - 8X^3 = \tfrac{27}{8},$$

and, with $X = r\cos\psi$ and $Y = r\sin\psi$,

$$\cos 3\psi = \frac{16r^4 + 72r^2 - 27}{64r^3},$$

which confirms the three-fold symmetry. The condition that $\cos 3\psi$ shall be at most unity implies that

$$16r^4 - 64r^3 + 72r^2 - 27 = (2r+1)(2r-3)^3 \leq 0,$$

and so $r \leq \frac{3}{2}$. A corresponding argument for the minimum value of $\cos 3\psi$ gives $r \geq \frac{1}{2}$.

Before we discuss this remarkable curve, it is worth pointing out a consequence of the above analysis. With $\phi_1 = \theta_1$ and $\phi_2 = \theta_2$ (so that $\phi_1 + \phi_2 = -\theta_3$), P is the point A and its Simson's line is the altitude of ΔABC which passes through A. Likewise for Q and B. The two Simson's lines will meet at the orthocentre, H, of ΔABC. This shows that the three altitudes are concurrent and the common point, H, has coordinates $(2x_0, 2y_0)$.

Returning to the envelope given by (7) and (8), it is remarkable that the shape of the original triangle is irrelevant to the Simson's envelope. Changing the ΔABC can change the

- scale,
- position, and
- orientation

but not the shape of the Simson's envelope. Put another way, Simson's envelope is essentially unique. Perhaps we may even describe it as a universal curve. It is this fact which makes this investigation interesting; if the envelope had varied in shape for different triangles then the result would have far less significance. Each of the three bullet points listed above is now investigated. The less enthusiastic reader may wish to take much of what follows on trust. The main point of this investigation has already been achieved; the rest is just for fun.

The scale of the curve is simply determined by the circumcircle of ΔABC. Specifically, the size of the curve is proportional to the radius of the circumcircle.

The position of the curve is specified by the point N with coordinates (x_0, y_0) given by (5) and (6). It has already been established that N is the midpoint of the line segment OH. To people of my generation, this says 'N is the centre of the nine-point circle of ΔABC'; Boyer refers to 'the British predilection for pure geometry'. For those unfortunates who were not brought up on *Modern Geometry* by Durell, for any triangle, the three midpoints of the sides, the feet of the three altitudes, and the midpoints of AH, BH, and CH all lie on a circle (the nine-point circle). Furthermore, the centre of this circle is the midpoint of OH. A result that is easier to see from the current analysis is that the coordinates of the centroid, G, of ΔABC are $(2x_0/3, 2y_0/3)$. This shows that $O, G, N,$ and H all lie on a straight line (the Euler line) and that $OG = OH/3$.

Of more immediate relevance is another classical result that the radius of the nine-point circle is one half of the radius of the circumcircle. Thus, the centre of the Simson's envelope is the centre of the nine-point circle of ΔABC and this circle touches the curve internally at three points. This is illustrated in figure 4.

The orientation of the Simson's envelope is determined by the requirement that the angles of the three vertices of the triangle shall satisfy condition (1). However, there is another interesting property of triangles which is closely related. For any ΔABC, let the three interior trisectors of the angles meet in order at $X, Y,$ and Z, as shown in figure 5. Then it is perhaps not so

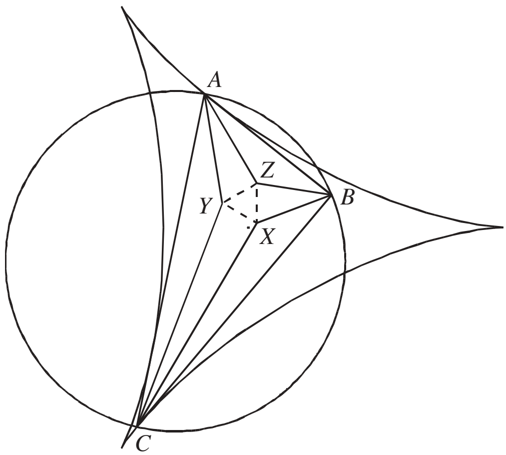

Figure 4 A triangle with its Simson's envelope, its nine-point circle, and the circle with the same centre and radius $\frac{3}{2}$.

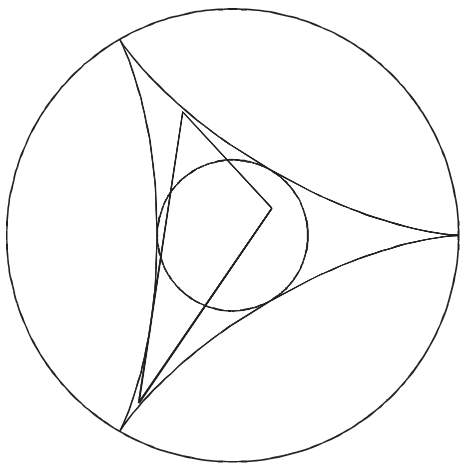

Figure 5 This shows the equilateral triangle formed by the angle trisectors. Note the orientation of this triangle relative to the Simson's envelope.

well known that ΔXYZ is always an equilateral triangle. I say 'not so well known' since this was not in Durell's *Modern Geometry*. However, even now I remember well the feeling of satisfaction on completing an A-level question which required the derivation of all of the results necessary to prove this curious result. If you are feeling in the mood for some trigonometry, then you might like to verify that $\angle AZY = (\pi + B)/3$. In my derivation, intermediate results were

$$AZ = 8\sin\left(\frac{B}{3}\right)\sin\left(\frac{C}{3}\right)\sin\left(\frac{\pi}{3}+\frac{C}{3}\right) \quad \text{and} \quad YZ = 8\sin\left(\frac{A}{3}\right)\sin\left(\frac{B}{3}\right)\sin\left(\frac{C}{3}\right).$$

This will enable you to show that the orientation of Simson's envelope (which is a sort of curly equilateral triangle) and the equilateral triangle formed from the angle trisectors differ by $\pi/3$. Unfortunately, the construction of ΔXYZ is not possible by ruler and compass methods. This trisection result is usually associated with the name of Morley (1860–1937) and so, by the standards of mathematics taught in schools (and much in universities), is truly modern; see reference 1 for an attractive, short proof.

If you have managed to convince yourself of the truth of most of the statements so far then reward yourself (chocolate biscuit?). Maybe a better reward is to quit while ahead! At any rate, the most noteworthy of the remaining results are items 3 and 8 in the following list. Just reading that may well be enough for most people. But if you like challenges….

1.1. Further properties

The points P and Q referred to here are always on the circumcircle of ΔABC. Recall that 'the point θ' means the point P on the circumcircle for which $\angle xOP = \theta$. Also A', B', and C' denote the points diametrically opposite A, B, and C.

1. The angle between the Simson's lines of P and Q is equal to half of $\angle POQ$.

2. The fact that each Simson's line is a tangent to the Simson's envelope follows from the standard properties of envelopes. The Simson's line of A' is BC. Thus, each side of ΔABC touches the Simson's envelope as shown in figure 3. Furthermore, BC touches the envelope at the foot of the perpendicular from A' to BC.

3. (Our esteemed editor refers to the following as 'heading for the stratosphere'. I prefer to say that it is worth some comfort food—my preference is for seed cake.)

   (a) The equation of the tangent at the point θ to the curve given parametrically by
   $$X = \cos\phi + \tfrac{1}{2}\cos 2\phi \quad \text{and} \quad Y = \sin\phi - \tfrac{1}{2}\sin 2\phi,$$
   is
   $$2X\sin\left(\frac{\theta}{2}\right) + 2Y\cos\left(\frac{\theta}{2}\right) = \sin\left(\frac{3\theta}{2}\right).$$
   Also, $\mathrm{d}s/\mathrm{d}\phi = 2|\sin(3\phi/2)|$, where s is the arc length. Thus, the total length of Simson's envelope is eight times the radius of the circumcircle of ΔABC.

   (b) The Simson's line for the point θ touches the Simson's envelope at the point $\phi = \theta$ (in the parametric form of (7) and (8)), and meets it at the points U and V with parameters $\phi = -\theta/2$ and $\phi = \pi - \theta/2$.

   (c) The length UV is equal to the diameter of the circumcircle of ΔABC.

   (d) The midpoint of UV lies on the nine-point circle.

   (e) The tangents to the envelope at U and V are perpendicular and

   (f) meet at the point on the nine-point circle which is diametrically opposite to the midpoint of UV (use (3) and (4)).

4. If P is the point ϕ and Q is the point $\phi+\pi$ (so that they are diametrically opposite one another), then their Simson's lines intersect at right angles at a point W on the nine-point circle. Also, if the Simson's line of P or Q touches the envelope at P^* or, respectively, Q^* then P^*Q^* is the Simson's line for the point -2ϕ and the midpoint of the line segment P^*Q^* is the point on the nine-point circle diametrically opposite W. (This is just the converse of the previous item.)

5. The Simson's lines of any points P and Q intersect at a point on or in the Simson's envelope.

6. Precisely three Simson's lines pass through each point within the Simson's envelope.

7. Let P be the point α and let Q be the point β. The Simson's lines of P and Q will intersect on the nine-point circle if β has one of the following four values:
$$\frac{\pi-\alpha}{2}, \qquad \frac{3\pi-\alpha}{2}, \qquad \pi+\alpha, \qquad \pi-2\alpha.$$

Triangles and Simson's lines

Let P, Q, and R be the points on the circumcircle of the (fixed) ΔABC with angles ϕ_1, ϕ_2, and ϕ_3, respectively. Set $\phi = \phi_1 + \phi_2 + \phi_3$.

8. The three Simson's lines of P, Q, and R form a triangle which is similar to ΔPQR. Furthermore, the ratio of similitude is $|\sin(\phi/2)|$.

9. If $\phi = 0$ (or any multiple of 2π) then the three Simson's lines are concurrent. Likewise, the three Simson's lines of A, B, and C with respect to ΔPQR are concurrent. Furthermore, the two points of concurrency coincide (at W) and this point has position vector $\overrightarrow{ON} + \overrightarrow{OM}$, where N and M are the centres of the nine-point circles of ΔABC and ΔPQR.

10. If P, Q, and R coincide with A, B, and C, respectively, then M and N coincide and W is the orthocentre H of the triangle. This gives the classical result that N is the midpoint of OH.

11. The triangle formed by the Simson's lines of A', B', and C' (with respect to ΔABC) is ΔABC, and the triangle formed by the Simson's lines of A, B, and C (with respect to $\Delta A'B'C'$) is $\Delta A'B'C'$.

2. Challenge

The Simson's envelope is bounded internally by the nine-point circle. This circle has radius $\frac{1}{2}$ (when the radius of the circumcircle is taken as unity) and is an extremely significant circle for ΔABC. Now the Simson's envelope is bounded externally by a circle of radius $\frac{3}{2}$. It is reasonable to conjecture that this circle also has significance for ΔABC. I am unable to find any (other) connection between the triangle and the circle. Can you?

3. Above the stratosphere and into orbit

For those fortunate readers who have access to (or those, such as the writer, especially blessed who have their own copy of) *An Elementary Course of Infinitesimal Calculus* by Horace Lamb, now is the time to look at figure 86 on page 301. It is reproduced here as figure 6 (with the addition of the inner, nine-point circle). Simson's envelope is in fact an example of ahypocycloid

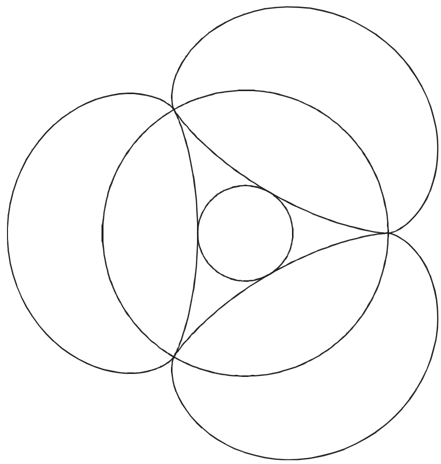

Figure 6 Showing the epicycloid and hypocycloid.

and is that part of the curve inside the outer circle. Its friend on the outside is the corresponding epicycloid and has parametric equation

$$x = 2\cos\phi - \tfrac{1}{2}\cos 4\phi, \qquad y = 2\sin\phi - \tfrac{1}{2}\sin 4\phi.$$

The hypocycloid is the path traced out by a point on the circumference of a circle (radius $\frac{1}{2}$) as it rolls (without slipping) on the inside of the large circle (radius $\frac{3}{2}$). Likewise, the epicycloid is the result of rolling the same circle on the outside.

It is left for the very keen to investigate how the tangent and midpoint properties extend to this extra curve. Suffice to say that it all matches up very neatly. The only surprise (apart from the whole bundle of results being a big surprise) is that the tangents to the outer curve meet on a yet bigger circle at a point collinear with O and the new midpoint. All very bizarre. Can anyone relate the epicycloid directly to the original triangle? I cannot.

Here epicycles have been defined by the 'rolling circle' construction. A slightly different approach is to think of a fairground ride in which an arm (with one end fixed) rotates (at constant angular velocity) and the moving end of the arm is attached to the midpoint of a bar (with seats at each end). This bar rotates relative to the arm; all the motion taking place in a horizontal plane. The (sickening) path traced out by the seats is a hypocycloid. The special case of Simson's envelope occurs when the lengths of the bar and arm are equal and the bar rotates at three times the rate of the arm and in the opposite sense. With this construction, the two ends of the bar perform congruent paths which are not coincident. An alternative (but perhaps not so practical realisation) is to make the bar four times longer than the arm and make it rotate at 1.5 times that of the arm (again in the opposite sense). In this case the two ends of the bar follow exactly the same path in space. More information on fairgrounds will be found in reference 2. There are also more fascinating facts about hypocycloids and epicycloids (and a brief mention of their cousins pericycloids, epitrochoids, and hypotrochoids—I could not resist name dropping) in Lamb's book.

Finally, epicycles were the basis of the construction of Hipparchus and Ptolemy in their description of planetary motion. So the subject is not so modern after all!

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$$
\begin{aligned}
1 \times 9 + 2 &= 11,\\
12 \times 9 + 3 &= 111,\\
123 \times 9 + 4 &= 1111,\\
1234 \times 9 + 5 &= 11111,\\
12345 \times 9 + 6 &= 111111,\\
123456 \times 9 + 7 &= 1111111,\\
1234567 \times 9 + 8 &= 11111111,\\
12345678 \times 9 + 9 &= 111111111.
\end{aligned}
$$

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$$
\begin{aligned}
512 &= (5 + 1 + 2)^3,\\
4\,913 &= (4 + 9 + 1 + 3)^3,\\
5\,832 &= (5 + 8 + 3 + 2)^3,\\
17\,576 &= (1 + 7 + 5 + 7 + 6)^3,\\
19\,683 &= (1 + 9 + 6 + 8 + 3)^3.
\end{aligned}
$$

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Conjectures from a Historic Table by John Wallis

LEE COLLINS and THOMAS J. OSLER

Introduction

The first higher transcendental function that most of us meet is the gamma function. We usually define it by the definite integral

$$\Gamma(x) = \int_0^x \mathrm{e}^{-t} t^{x-1} \mathrm{d}t, \tag{1}$$

which is valid for $x > 0$. Integrating by parts we can derive the functional equation

$$\Gamma(x)x = \Gamma(x+1). \tag{2}$$

(Since (1) is an improper integral, the usual care must be exercised when performing this integration by parts. First evaluate the integral from m to M, then carefully pass to the limit as $m \to 0$ and $M \to \infty$.) The functional equation (1) along with $\Gamma(1) = 1$ results in $\Gamma(n+1) = n!$ when n is a positive integer. For fractional values of x, the gamma function enables us to extend the meaning of 'factorials' to noninteger values. This has many applications. For example, the binomial coefficients defined for nonnegative integers n and r by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,n!}$$

can be extended to arbitrary values of n and r by

$$\binom{n}{r} = \frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)}. \tag{3}$$

A surprising result is

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \tag{4}$$

and from (2) we have $\Gamma(\frac{1}{2})(\frac{1}{2}) = \Gamma(\frac{3}{2}) = \sqrt{\pi}/2$. Thus, we have the remarkable result that $\frac{1}{2}$ factorial is $\sqrt{\pi}/2$. The usual derivation of (4) involves a few tricks that are not obvious. In this article we will not prove (4) but rather we will show how to 'conjecture' the truth of (4) from simple observations of some very old and historically important tables by John Wallis.

Another important result involving the gamma function is the 'beta integral':

$$\int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{5}$$

Like (4), derivations of this integral are also nontrivial. We will use the same tables of Wallis to conjecture the truth of (5), again without giving a rigorous proof. These relatively easy conjectures are examples of 'experimental mathematics'. Often, important new mathematical theorems are conjectured before they are formally proved. They give us another way of probing the unknown.

Brief history of Wallis's tables

In 1656 John Wallis published his *Arithmetica Infinitorum* (see reference 4), in which he displayed many ideas that were to lead to the integral calculus of Newton. In this work we find the celebrated infinite product for π,

$$\frac{2}{\pi} = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdots \tag{6}$$

which we now call the 'Wallis product'.

Using modern notation, we can say that Wallis knew the integration formula

$$\int_0^c x^p \mathrm{d}x = \frac{c^{p+1}}{p+1}, \tag{7}$$

and could use it for values of p that were both integers and fractions. Wallis wanted to find some convenient expression for the area bound by the unit circle 'in terms of integers', and (again in modern notation) he wanted to evaluate the integral

$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x. \tag{8}$$

If Wallis had known the binomial theorem in the form

$$(1+t)^p = 1 + \frac{p}{1}t + \frac{p}{1} \cdot \frac{p-1}{2}t^2 + \frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3}t^3 + \cdots,$$

he could have integrated term by term with $p = \frac{1}{2}$ and $t = -x^2$. (Modern mathematicians will first check the validity of term-by-term integration of an infinite series, but such care was usually ignored by mathematicians before 1800.) At the time of Wallis, the binomial theorem for fractional exponents had not yet been discovered, and so knowing only (7), Wallis had no direct way of evaluating the integral in (8). Instead, Wallis used an ingenious method of interpolation. He reasoned that the value of the integral in (8) was between the two integrals $\int_0^1 (1 - x^2)^0 \mathrm{d}x$ and $\int_0^1 (1 - x^2)^1 \mathrm{d}x$, and, of course, he could evaluate both of these. To achieve this interpolation, he created a table of values of the reciprocal integrals $1/\int_0^1 (1 - x^{1/Q})^P \mathrm{d}x$ for values of P and Q that he could evaluate. (The reason for the reciprocal was to obtain more integer values in the table, as will be revealed.) A very careful and ingenious study of this table led Wallis to tease out his product (6).

Constructing Wallis's table

Following Wallis, we invite the reader to compute the reciprocal integral $1/\int_0^1 (1 - x^{1/Q})^P \mathrm{d}x$ for integers P and Q. In table 1 we show the results of this calculation. Note also that we leave space in the table for fractional values of P and Q. We will use the symbol $\{Q, P\}$ to denote the entry in a cell of the table where Q is the row and P is the column. Thus, we have

$$\{Q, P\} = \frac{1}{\int_0^1 (1 - x^{1/Q})^P \mathrm{d}x}. \tag{9}$$

Note that $\{\frac{1}{2}, \frac{1}{2}\} = 4\pi$ since we know that the area under the circle of unit radius $y = \sqrt{1 - x^2}$ in the first quadrant is $\int_0^1 \sqrt{1 - x^2}\, \mathrm{d}x = \pi/4$. Note that the values obtained are symmetric. The number $\{P, Q\}$ is the same as the number $\{Q, P\}$.

Table 1 The reciprocal integral $1/\int_0^1 (1 - x^{1/Q})^P \mathrm{d}x$ for integers P and Q.

Q	$P=0$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	1		1		1		1
$\frac{1}{2}$		$\frac{4}{\pi}$					
1	1		2		3		4
$\frac{3}{2}$							
2	1		3		6		10
$\frac{5}{2}$							
3	1		4		10		20

Note While the above observation suffices for our 'conjectures', the modern reader can show analytically that $\{P, Q\} = \{Q, P\} (P, Q > 0)$ without difficulty. Note that, with the change of variables $u = x^{1/Q}$, we obtain $x = u^Q$ and $\mathrm{d}x = Qu^{Q-1}\mathrm{d}u$, and we have

$$\frac{1}{\{P, Q\}} = \int_0^1 (1 - x^{1/Q})^P \mathrm{d}x = Q \int_0^1 (1 - u)^P u^{Q-1} \mathrm{d}u. \tag{10}$$

Next integrate this last integral by parts to obtain

$$\begin{aligned} \frac{1}{\{P, Q\}} &= Q \int_0^1 (1 - u^P) u^{Q-1} \mathrm{d}u \\ &= [(1-u)^P u^Q]_0^1 + P \int_0^1 (1-u)^{P-1} u^Q \mathrm{d}u \\ &= P \int_0^1 (1-u)^{P-1} u^Q \mathrm{d}u. \end{aligned}$$

Now make the change of variables $u = 1 - v$ and convert this last integral to

$$\frac{1}{\{P, Q\}} = P \int_0^1 (1 - v)^Q v^{P-1} \mathrm{d}v. \tag{11}$$

Comparing (10) and (11) we see that $\{P, Q\} = \{Q, P\}$.

Recognizing binomial coefficients

We recognize the numbers in table 1 as the numbers in the famous 'Pascal's triangle'. No doubt Wallis also made this observation. However, it was of little use to him since the binomial theorem for fractional exponents was not known at the time. The formula

$$\binom{n}{k} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-k+1}{k}$$

was also unknown, but Wallis discovered this expression without making the connection to the binomial theorem (see table 3, below). The binomial theorem for fractional exponents would have to wait for Newton, who credited reading Wallis for his discovery.

Table 2 Values expressed as binomial coefficients.

Q	P: 0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	$\binom{0}{0}$		$\binom{1}{0}$		$\binom{2}{0}$		$\binom{3}{0}$
$\frac{1}{2}$		$\frac{4}{\pi}$					
1	$\binom{1}{1}$		$\binom{2}{1}$		$\binom{3}{1}$		$\binom{4}{1}$
$\frac{3}{2}$							
2	$\binom{2}{2}$		$\binom{3}{2}$		$\binom{4}{2}$		$\binom{5}{2}$
$\frac{5}{2}$							
3	$\binom{3}{3}$		$\binom{4}{3}$		$\binom{5}{3}$		$\binom{6}{3}$

Note that in the Qth row and Pth column we find the entry

$$\{Q, P\} = \binom{P+Q}{Q}; \tag{12}$$

see table 2.

First conjecture, $\Gamma(\frac{1}{2})$

From (3) and (12), we have at once

$$\{Q, P\} = \binom{P+Q}{Q} = \frac{\Gamma(P+Q+1)}{\Gamma(Q+1)\Gamma(P+1)}.$$

Of course, this result is valid only for integer values of P and Q, but we can conjecture that it might be true for fractional values also. Recall from table 1 that $\{\frac{1}{2}, \frac{1}{2}\} = 4/\pi$ and so, using fractional values $P = Q = \frac{1}{2}$, we obtain

$$\frac{4}{\pi} = \frac{\Gamma(2)}{\Gamma(3/2)^2} = \frac{1}{\Gamma(3/2)^2}.$$

Therefore, we can write the important unexpected conjecture

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

From the factorial property of the gamma function (2), $\Gamma(x)x = \Gamma(x+1)$, we can write this last result as

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Thus, we have our important conjecture

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

Second conjecture, the beta integral

For our second conjecture, we start with (9) and (12) and write

$$\frac{1}{\int_0^1 (1 - x^{1/Q})^P \mathrm{d}x} = \binom{P+Q}{Q}. \tag{13}$$

Using (3), we can write (13) in terms of the gamma function, i.e.

$$\frac{1}{\int_0^1 (1-x^{1/Q})^P \mathrm{d}x} = \frac{\Gamma(P+Q+1)}{\Gamma(Q+1)\Gamma(P+1)},$$

or, taking reciprocals,

$$\int_0^1 (1-x^{1/Q})^P \mathrm{d}x = \frac{\Gamma(Q+1)\Gamma(P+1)}{\Gamma(P+Q+1)}. \tag{14}$$

Again, this has been proved only for integer values of P and Q, but we will assume that it is true for other values as well.

Making the substitution $u = x^{1/Q}$ we obtain $x = u^Q$ and $\mathrm{d}x = Qu^{Q-1}\mathrm{d}u$. Now the integral in (14) becomes

$$\int_0^1 (1-x^{1/Q})^P \mathrm{d}x = Q\int_0^1 (1-u)^P u^{Q-1} \mathrm{d}u. \tag{15}$$

From (14) and (15), we obtain

$$\int_0^1 (1-u)^P u^{Q-1} \mathrm{d}u = \frac{\Gamma(Q+1)\Gamma(P+1)}{Q\Gamma(P+Q+1)},$$

and since $\Gamma(Q+1) = Q\Gamma(Q)$, we have

$$\int_0^1 (1-u)^P u^{Q-1} \mathrm{d}u = \frac{\Gamma(Q)\Gamma(P+1)}{\Gamma(P+Q+1)}.$$

Writing $P = x - 1$ and $Q = y$, this last equation becomes our important conjecture known as the *beta integral*

$$\int_0^1 (1-u)^{x-1} u^{y-1} \mathrm{d}u = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

It can be shown that this is valid for $x > 0$ and $y > 0$.

Final remarks

We end this article by showing the final form in which Wallis could write his table. These beautiful results took several years of brilliant work. Table 3 proved to be seminal for further research into the gamma function, beta integral, and continued fractions, especially the second row in which $Q = \frac{1}{2}$. Lord Brouncker (see reference 3), a colleague of Wallis, after seeing this table was able to convert it into the amazing continued fraction

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cdots}}}}}.$$

The papers by Nunn (see references 1 and 2) give an excellent introduction to this work by Wallis.

Table 3 Wallis's complete table of the reciprocal integral $1/\int_0^1 (1-x^{1/Q})^P \, \mathrm{d}x$.

Q	P						
	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
0	1	$1\cdot\frac{1}{1}$	$1\cdot\frac{2}{2}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}$	$1\cdot\frac{1}{1}\cdot\frac{3}{3}\cdot\frac{5}{5}$	$1\cdot\frac{2}{2}\cdot\frac{4}{4}\cdot\frac{6}{6}$
$\frac{1}{2}$	1	$\frac{2}{\pi}\cdot\frac{2}{1}$	$1\cdot\frac{3}{2}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}$	$\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}$	$1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}$
1	1	$\frac{1}{2}\cdot\frac{3}{1}$	$1\cdot\frac{4}{2}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}$	$\frac{1}{2}\cdot\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}$	$1\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}$
$\frac{3}{2}$	1	$\frac{4}{3\pi}\cdot\frac{4}{1}$	$1\cdot\frac{5}{2}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}$	$\frac{4}{3\pi}\cdot\frac{4}{1}\cdot\frac{6}{3}\cdot\frac{8}{5}$	$1\cdot\frac{5}{2}\cdot\frac{7}{4}\cdot\frac{9}{6}$
2	1	$\frac{3}{8}\cdot\frac{5}{1}$	$1\cdot\frac{6}{2}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}$	$\frac{3}{8}\cdot\frac{5}{1}\cdot\frac{7}{3}\cdot\frac{9}{5}$	$1\cdot\frac{6}{2}\cdot\frac{8}{4}\cdot\frac{10}{6}$
$\frac{5}{2}$	1	$\frac{16}{15\pi}\cdot\frac{6}{1}$	$1\cdot\frac{7}{2}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}$	$\frac{16}{15\pi}\cdot\frac{6}{1}\cdot\frac{8}{3}\cdot\frac{10}{5}$	$1\cdot\frac{7}{2}\cdot\frac{9}{4}\cdot\frac{11}{6}$
3	1	$\frac{5}{8}\cdot\frac{7}{1}$	$1\cdot\frac{8}{2}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}$	$\frac{5}{8}\cdot\frac{7}{1}\cdot\frac{9}{3}\cdot\frac{11}{5}$	$1\cdot\frac{8}{2}\cdot\frac{10}{4}\cdot\frac{12}{6}$

If the reader studies table 3 carefully, he/she will see how the numbers evolve from cell to cell. An interesting challenge is to extend this table for larger values of P and Q.

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Construction of Magic Knight's Towers

AWANI KUMAR

Introduction

Traditional study of the knight's tour on a chessboard has mostly been confined to two-dimensions and is almost as old as the game itself. According to Jelliss (see reference 1), the earliest known knight's tour is believed to be the tour by al-Adli ar-Rumi in the year 840. A knight's tour is called a *magic tour* if all row and column sums are the same; this sum is known as the *magic constant*. Figure 1, constructed by Beverley (see reference 2), is the first example of a magic knight's tour on a conventional chessboard. Readers can verify that each row and column sum is 260. Using powerful computers and intelligent programming, Stertenbrink (see reference 3) and his team proved that there are only 280 magic tours on an 8×8 board, but none of them are diagonally magic; 12×12 is the smallest board size in which a diagonally magic tour is possible, as shown by Kumar (see reference 4). Here, in addition to rows and columns, we also have 'pillars' consisting of cells over one another. Figure 2 is an example of a three-dimensional knight's tour. The reader can visualize it in three dimensions

1	30	47	52	5	28	43	54
48	51	2	29	44	53	6	27
31	46	49	4	25	8	55	42
50	3	32	45	56	41	26	7
33	62	15	20	9	24	39	58
16	19	34	61	40	57	10	23
63	14	17	36	21	12	59	38
18	35	64	13	60	37	22	11

Figure 1 A magic knight's tour on a chessboard.

B

22	13	4	11
5	10	15	20
14	21	12	3

A

1	8	23	18
16	19	6	9
7	2	17	24

Figure 2 A $3 \times 4 \times 2$ knight's tour.

by stacking the layers, one above the other, in alphabetical order. Note that each row sum is 50. Such tours are known as *semi-magic tours*. Another interesting feature is that the sum of the numbers of each layer which reflect to one another in the centre is 25. Basically, the knight is a three-dimensional galloping piece and its move (0, 1, 2) has an aesthetic appeal!, being the first three whole numbers. Readers can visualize the three-dimensional knight's move with the moves (0, 1, 2) in three directions at right angles. This increases the mobility of the knight tremendously. In two dimensions, the knight can move up to eight squares from a given square, but in three-dimensional space it can move up to 24 cells. It is generally believed that Schubert (see reference 5) was the first person to study a knight's tour in three-dimensional space, as quoted by Ahrens (see reference 6). However, perusal of the literature reveals that Vandermonde (see reference 7) was the first person to construct a knight's tour in a $4 \times 4 \times 4$ cube. Later, Gibbins (see reference 8), Stewart (see reference 9), Jelliss and Marlow (see reference 10), and others have found simple knight's tours of various sizes in three-dimensional space. These authors have not looked into their magical properties. More recently, Kumar (see reference 11) has looked into knight's tours having magical properties, but they are mostly confined to knight's tours in cubes. Can there be towers of knight's tours having magic properties? Is it possible that all the pillars of the tower sum up to a magic constant? How small can its base or height be? Can it have a 'shaft', that is, a vertical opening? What about magic towers having 'bimagic' properties too? The author plans to look at these questions.

Magic knight's towers

We say that a knight's tour in a tower is magic if all its pillars sum up to the same number called the magic constant, and we refer to such a tower as a *magic tower*. The smallest size of the base for which a knight's tour tower is possible is 2×3. For a tower (cuboid) of base $m \times n$ and height k, the total number of cells will be equal to mnk. The sum of the numbers from 1 to mnk is $mnk(mnk + 1)/2$. As there are mn pillars, the magic constant will be equal to $k(mnk + 1)/2$. The squares on a chessboard are alternately black and white, and a knight always moves from a square of one colour to a square of the other colour. Thus, in a knight's tour, all the black squares will have even numbers and all the white squares odd numbers, or vice versa. A similar thing happens in three dimensions. This means that a magic tower cannot be of odd height because, if it were, the pillar sums of adjacent pillars would be even and odd, so could not be equal. Furthermore, a magic tower cannot have a height of two, because then the magic constant would be $2mn + 1$ and the numbers in the mn pillars must be $(1, 2mn)$, $(2, 2mn - 1)$, $(3, 2mn - 2)$, $\dots$, $(mn, mn + 1)$. But this would mean that the knight moved from square mn to adjacent square $mn + 1$, which is impossible. *Hence, the height of a magic tower must be even and at least four.* As we shall see later, four is a sufficient height for a magic tower, and by stacking them, magic towers of any even height can be constructed.

Construction of magic knight's towers

It is easy to see that 2×2 is too small a base size for the knight to manoeuvre. Therefore, the smallest base size is 2×3; see, for example, figure 3(a). Its magic constant is 50. By stacking this tower onto itself, it can be extended to any height that is a multiple of four. Figure 3(b) shows the tower stacked three times with magic constant equal to 438. Figure 4(a) is an example of a magic tower with a 2×4 base size; its magic constant is 66. Figure 4(b) is an example of a magic tower with a 2×5 base size; its magic constant is 82. A magic tower with a base size of 2×6 can be constructed by choosing and stacking $2 \times 3 \times 4$ magic towers. Figure 5 is one such example; its magic constant is 388. It can be raised to any height that is a multiple

(a)

D	17	2	9
	8	15	24
C	10	21	18
	23	12	7
B	19	16	3
	6	1	14
A	4	11	20
	13	22	5

(b)

L	65	50	57
	56	63	72
K	58	69	66
	71	60	55
J	67	64	51
	54	49	62
I	52	59	68
	61	70	53
H	41	26	33
	32	39	48
G	34	45	42
	47	36	31
F	43	40	27
	30	25	38
E	28	35	44
	37	46	29
D	17	2	9
	8	15	24
C	10	21	18
	23	12	7
B	19	16	3
	6	1	14
A	4	11	20
	13	22	5

Figure 3 (a) $2 \times 3 \times 4$ and (b) $2 \times 3 \times 12$ magic towers.

of eight. Similarly, magic towers of larger base size, such as 2×7, 2×8, etc. can also be constructed using towers of smaller base size.

Magic towers with a bigger base size

The smallest magic tower with a square base is of the size $3 \times 3 \times 4$; see, for example, figure 6(a). Here, the magic constant is 74. Hundreds of such towers can be constructed. We know that a

D	28	13	22	3
	19	6	29	12
C	23	2	25	16
	32	9	18	7
B	14	27	4	21
	5	20	11	30
A	1	24	15	26
	10	31	8	17

(a)

D	38	15	22	7	28
	21	4	39	16	31
C	23	6	33	14	9
	40	17	30	5	12
B	20	37	8	27	32
	3	26	11	36	29
A	1	24	19	34	13
	18	35	2	25	10

(b)

Figure 4 (a) $2 \times 4 \times 4$ and (b) $2 \times 5 \times 4$ magic towers.

H	78	93	86	57	62	71
	87	80	95	72	59	64
G	85	90	77	70	53	66
	96	75	88	65	68	51
F	92	79	84	61	58	55
	81	94	73	56	63	60
E	83	76	91	54	69	50
	74	89	82	49	52	67
D	18	7	22	35	44	27
	21	16	19	42	33	48
C	23	2	13	28	39	36
	14	11	4	47	30	41
B	8	17	6	37	34	45
	5	20	15	32	43	26
A	1	24	9	46	29	38
	10	3	12	25	40	31

Figure 5 A $2 \times 6 \times 8$ magic tower.

two-dimensional knight's tour is not possible in a 3×3 square because the knight can neither go into nor come out of the central cell. However, we can have a closed tour in a 3×3 square if the central cell is left vacant. By stacking such tours, we get a tower with a hollow central

(a)

D	20	33	36
	29	12	31
	32	21	28
C	35	22	19
	24	7	26
	27	34	23
B	18	15	2
	13	30	11
	10	3	14
A	1	4	17
	8	25	6
	5	16	9

(b)

D	24	21	18
	19		23
	22	17	20
C	27	32	29
	30		26
	25	28	31
B	14	9	12
	11		15
	16	13	10
A	1	4	7
	6		2
	3	8	5

(c)

F	38	33	46
	45		37
	34	47	44
E	41	48	39
	36		42
	43	40	35
D	32	29	26
	27		31
	30	25	28
C	19	24	11
	12		20
	23	10	13
B	16	9	18
	21		15
	14	17	22
A	1	4	7
	6		2
	3	8	5

Figure 6 (a) $3 \times 3 \times 4$ magic tower. (b) $3 \times 3 \times 4$ and (c) $3 \times 3 \times 6$ magic towers with a shaft.

shaft. Figure 6(b) is an example of such a magic tower with magic constant equal to 66. This tour is also closed. Figure 6(c) is an example of a magic tower of height six with a shaft; its magic constant is 147. By judiciously selecting these two types of towers, we can get magic towers of 'singly even' height, that is, 10, 14, 18, etc. Constructing magic towers of singly even height is not that simple.

Conclusion

The author has many examples of larger magic towers and even of pyramids with magic properties. Many problems remain. For example, we have seen that a height of four is sufficient to construct a magic tower with a small base size. Is four sufficient for any base size? Enumeration of knight's tours is notoriously difficult. How many knight's tours and magic knight's tours are there in an $m \times n \times k$ tower?

Acknowledgement

The author is grateful to Takaya Iwamoto for providing references 8 and 9.

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Using the same digits

$$15 \times 93 = 1395, \qquad 321 \times 975 = 312\,975,$$
$$35 \times 41 = 1435, \qquad 681 \times 759 = 516\,879,$$
$$21 \times 87 = 1827, \qquad 843 \times 876 = 738\,468,$$
$$27 \times 81 = 2187, \qquad 902 \times 875 = 789\,250.$$

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Excavations and Integrations –A Note of Caution

J. D. MAHONY

An exercise involving a topographical map and the tools of the integral calculus has been carried out to estimate quantities of soil removed in a building site excavation process. A real life scenario involving costs associated with the removal of soil provoked interest in this activity. The associated sums serve to illustrate a very practical application of some very simple but fundamental mathematics that should be of interest to the enterprising sixth former/undergraduate mathematician or engineer.

Introduction

The process of buying a new house in the Antipodes is somewhat different to that in the United Kingdom. Typically, the intended homeowner firstly purchases a plot of land that is suitable for a build and then approaches an architect to produce an approved design. Lastly, a builder is approached to implement the build scheme and, in the first instance, this process will result in site excavation to produce a level build platform, assuming of course that the proposed house is not going to be built on stumps. The excavation process can be time consuming and expensive, and most builders allow for only a provisional sum to cover the perceived costs. The reality might be that both builder and excavator underestimate the amount of soil to be removed, particularly for a build on a sloping or undulating site, leaving the prospective homeowner to face an unanticipated increase in costs. It is possible to avoid such embarrassments by making an a-priori estimate for the amount of soil to be removed and it is the purpose of this note to show just how this might be achieved, armed only with a relevant topographical map and the tools of integral calculus. The sums, as they relate to a specific topography (the author's), are discussed below but they apply equally well to other build scenarios involving excavations.

The build topography

A topographical map and axis set for the intended site is shown in figure 1. In this figure, contour lines (dashed) are shown at intervals of one metre and the site is to be reduced to a datum (reference level) at 259.5 m—see node number 0 at the axis origin. Also shown in the figure are other nodes distributed arbitrarily around the site boundary and along the datum contour level. In order to estimate the amount of soil that has to be removed to reach the datum level, a triangular lattice (or mesh) is imposed on the topographical map at the various nodes and then the problem is essentially one of considering the volume under a series of variously inclined and interconnected triangular patches. If the site altitudes vary smoothly then a fairly coarse mesh should suffice, otherwise a finer one must be envisaged. To appreciate the problem of now determining the volume in a typical triangular column that terminates in an inclined triangular patch, a schematic such as that shown in figure 2 must be envisaged and the tools of

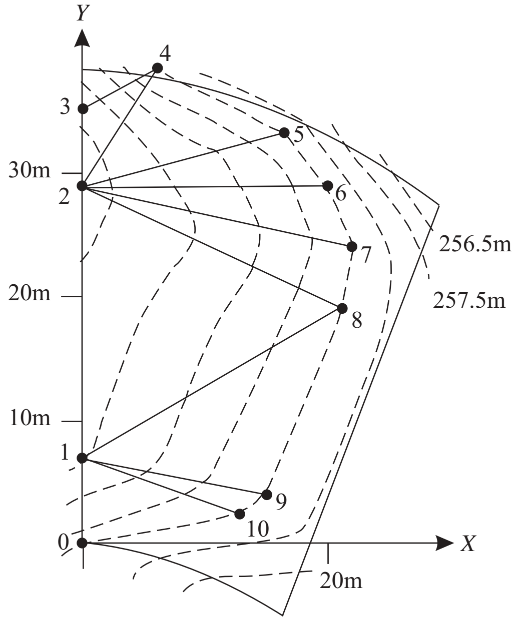

Figure 1 Site topography, coordinate axis set, and nodes.

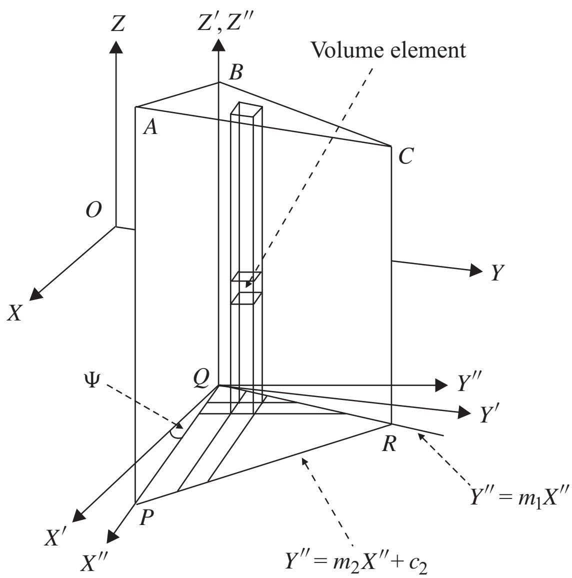

Figure 2 Reference systems and schematic of a triangular column.

integral calculus should then suffice to determine the volume under such a column and, ergo, by summation, the total excavated volume. First though, a computable expression for the volume in a typical column must be obtained and this is discussed below.

The triangular patch column volume

With reference to figure 2, points A, B, and C are the vertices of a typical, inclined triangular patch on the site surface and the points P, Q, and R are their respective projections onto a plane at the required datum level. O is the origin of some arbitrary coordinate system $(OXYZ)$ at the required datum level (the Z-axis denotes the altitude) and a parallel system $(O'X'Y'Z')$ is shown with its origin at the point Q. Also shown is another coordinate system, $(O''X''Y''Z'')$, which is simply a rotation of the latter system through some angle, ψ say, as shown, so that QP lies along the X''-axis.

Data drawn from the topographical map will enable us to write down the coordinates (X_A, Y_A, Z_A), (X_B, Y_B, Z_B), and (X_C, Y_C, Z_C) of A, B, and C, respectively, in the $(OXYZ)$ system. It is then a simple matter of inspection to write down the coordinates of P, Q, and R in this system. The (X, Y)–coordinates of P, Q, and R will be the same as those of A, B, and C, respectively—the Z-coordinates for these points P, Q, and R will be the datum value (without loss of generality, this can be chosen to be 0). Finally at this juncture, it is a further straightforward exercise involving coordinate translations and rotations to write down the coordinates of P, Q, and R, first in the $(O'X'Y'Z')$ system and ultimately in the $(O''X''Y''Z'')$ system, as follows:

$$X''_P = (Y_A - Y_B)\sin(\psi) + (X_A - X_B)\cos(\psi), \qquad Y''_P = 0, \qquad Z''_P = 0, \tag{1}$$
$$X''_Q = 0, \qquad Y''_Q = 0, \qquad Z''_Q = 0, \tag{2}$$

and

$$\begin{aligned} X''_R &= (Y_C - Y_B)\sin(\psi) + (X_C - X_B)\cos(\psi), \\ Y''_R &= (Y_C - Y_B)\cos(\psi) + (X_C - X_B)\sin(\psi), \qquad Z''_R = 0. \end{aligned} \tag{3}$$

The angle ψ above is determined from

$$\tan(\psi) = \frac{Y_A - Y_B}{X_A - X_B}. \tag{4}$$

Thus, the coordinates of the base points in the triangular column have been secured in terms of known quantities. In order to carry the argument further and write down an expression for the column volume as a triple integral, it is necessary, with reference to figure 2, to write down expressions (equations) for the lines PR and QR, and the inclined patch plane ABC in the $(O''X''Y''Z'')$ system. For the moment, it will be assumed that these equations are as follows:

$$PR: \quad Y'' = m_2 X'' + c_2, \tag{5}$$
$$QR: \quad Y'' = m_1 X'', \tag{6}$$
$$\text{plane } ABC: \quad LX'' + MY'' + NZ'' = p. \tag{7}$$

The parameters $L, M, N, p, m_1, m_2,$ and c_2 in (5)–(7) will be required in the volume integration process, and the manner in which they may be determined is discussed later.

Now, with reference to the volume element in figure 2, it is possible to write down the column volume, V say, as a triple integral in the form

$$V = \int_{Y''_{\mathrm{L}}}^{Y''_{\mathrm{U}}} \left(\int_{X''_{\mathrm{L}}}^{X''_{\mathrm{U}}} \left(\int_{Z''_{\mathrm{L}}}^{Z''_{\mathrm{U}}} \mathrm{d}Z'' \right) \mathrm{d}X'' \right) \mathrm{d}Y''.$$

The upper and lower integration limits in these integrals may be found from the figure. Specifically, the integration in Z'' is performed first, whilst X'' and Y'' are fixed (i.e. held constant with respect to the Z'' integration), so that the volume element runs from the datum plane, where $Z''=0$, to the plane ABC, where Z'' assumes a value determined from (7). Thus, $Z''_{\mathrm{L}}=0$ and $Z''_{\mathrm{U}}=(p-LX''-MY'')/N$. Having performed the Z'' integration and evaluated the ensuing function at the upper and lower limits in the usual way, we are left to perform a double integration of a function involving only X'' and Y''. The integration with respect to X'' may be carried out next, whilst Y'' is fixed. The limits of integration for X'' are, with reference to figure 2, determined from (5) and (6). Specifically, $X''_{\mathrm{L}}=Y''/m_1$ and $X''_{\mathrm{U}}=(Y''-c_2)/m_2$. The remaining integration now involves only the variable Y'' and the limits of integration for this variable are, from figure 2, given by its values at Q and R. That is to say, $Y''_{\mathrm{L}}=0$ and $Y''_{\mathrm{U}}=Y''_R$, the upper limit being determined from equation set (3).

On performing the above integrations, it is a tedious but straightforward exercise to determine the column volume in the form

$$V=\frac{1}{N}\left(pY''_R\left[\frac{1}{m_2}(0.5Y''_R-c_2)-0.5\frac{Y''_R}{m_1}\right]+\frac{L}{6}(Y''_R)^3\left[\left(\frac{1}{m_1}\right)^2-\left(\frac{1}{m_2}\right)^2\left(1-\frac{c_2}{Y''_R}\right)^3\right]\right.$$
$$\left.+\frac{M}{3}(Y''_R)^3\left[\frac{1}{m_1}-\frac{1}{m_2}\left(1-\frac{1.5c_2}{Y''_R}\right)\right]-\frac{(L/6)(c_2)^3}{(m_2)^2}\right). \tag{8}$$

This expression for the column volume lends itself well to the business of spreadsheet computations, once input parameters have been specified. These are discussed in the next section.

Determination of the input parameters $Y''_R, L, M, N, p, m_1, m_2,$ and c_2

As mentioned before, the input parameter Y''_R is determined from equation set (3), which involves known parameters. The remaining input parameters can also be expressed in terms of known quantities. For example, the parameter m_1 is determined by the slope of the line QR in the $(O''X''Y''Z'')$ system and is given by

$$m_1=\frac{Y''_R}{X''_R}, \tag{9}$$

where the parameters on the right-hand side are known (see equation set (3)). In a not dissimilar fashion, it may be seen that, since P and R satisfy (5),

$$m_2=\frac{Y''_R}{X''_R-X''_P}\quad\text{and}\quad c_2=-m_2X''_P. \tag{10}$$

The quantities on the right-hand side of these equations are essentially known (see equation sets (1) and (3)). For the remaining quantities, involving $L, M, N,$ and p, it must be appreciated that $L, M,$ and N are simply the direction cosines of the normal to the plane containing $A, B,$ and C and, thus, there is the normalising property $L^2+M^2+N^2=1$. Moreover, the coordinates of the points $A, B,$ and C as seen in the $(O''X''Y''Z'')$ system are shown in table 1.

These points can be forced to satisfy (7). The ensuing results, together with the normalising property for the direction cosines of the plane enable to determine $L, M, N,$ and p in terms

Table 1 The coordinates of the surface points of the triangular patch in the $(O''X''Y''Z'')$ system.

	X''	Y''	Z''
A	X''_P	0	Z_A
B	0	0	Z_B
C	X''_R	Y''_R	Z_C

of known parameters as follows:

$$N = \frac{1}{\sqrt{(1+\alpha^2+\beta^2)}}, \tag{11}$$

where

$$\alpha = \frac{Z_B - Z_A}{X''_P} \quad \text{and} \quad \beta = \frac{Z_B - Z_C - \alpha X''_R}{Y''_R}. \tag{12}$$

For the other terms,

$$p = Z_B N, \qquad L = \alpha N, \quad \text{and} \quad M = \beta N. \tag{13}$$

Thus, required parameters have been determined in terms of known inputs and it is appropriate to recall the steps now necessary to compute the column volume in a schematic.

- Input the coordinates of the surface points A, B, and C as seen in the $(OXYZ)$ system.
- Determine ψ from (4).
- Determine X''_P, Y''_P, Z''_P and X''_R, Y''_R, Z''_R using equation sets (1) and (3), respectively.
- Determine m_1, m_2, and c_2 from (9) and (10).
- Determine L, M, N, and p using (11), (12), and (13).
- The required volume V is then determined using (8).

The stage is now set to implement this schematic several times over in relation to the excavation scenario of figure 1.

Excavation scenario

The triangular mesh in figure 1 and the reference coordinate system have been chosen arbitrarily, but the choice has nonetheless been driven by the particular topography. Other equally valid mesh configurations (with internal nodes if necessary) and reference systems can be envisaged, and a more accurate result can be secured if the triangular patches are made small enough. For the present purpose, the number of triangular patches used was nine and the nodes of the lattice have therefore been numbered from 0 to 10 (as per the figure) and the vertices of each triangle are therefore numbered rather than lettered. It is important to appreciate that in using the above formulae for each triangle, the coordinates of the vertices are not input in some arbitrary order, but are listed as one moves around the triangle in a clockwise fashion. The chosen triangles, the coordinates of their vertices, and the associated volumes calculated using the above formulae are shown in table 2. Also shown in the table is the overall volume of excavated soil.

Table 2 The coordinates (in metres) of the triangular patch vertices that cover the excavated area and the associated volumes in cubic metres. (The Z-coordinate denotes the height above the datum level of 259.5 m.)

Triangle	Nodes	X	Y	Z
1	0	0	0	0.5
	1	0	7	3
	10	12.8	2.8	0
		Volume = 52.3		
2	10	12.8	2.8	0
	1	0	7	3
	9	14.8	4.1	0
		Volume = 12.5		
3	9	14.8	4.1	0
	1	0	7	3
	8	20.9	19.4	0
		Volume = 122.1		
4	8	20.9	19.4	0
	1	0	7	3
	2	0	29	4.25
		Volume = 555.6		
5	8	20.9	19.4	0
	2	0	29	4.25
	7	21.9	24.3	0
		Volume = 79.3		
6	7	21.9	24.3	0
	2	0	29	4.25
	6	19.8	29.3	0
		Volume = 70.6		
7	6	19.8	29.3	0
	2	0	29	4.25
	5	16.2	33.5	0
		Volume = 59.7		
8	5	16.2	33.5	0
	2	0	29	4.25
	4	6	38.5	0
		Volume = 89.9		
9	4	6	38.5	0
	2	0	29	4.25
	3	0	35.2	3.5
		Volume = 48.1		
		Total volume = 1090		

Discussion and conclusion

The total excavated volume under these patches was found to be about 1090 cubic metres, which was significantly more than the provisional estimate of 500 cubic metres proposed originally by the builder and excavator who could only suggest that '… to start with, it always looks less than it really is …'. It is not necessary to pursue accuracy to the nth decimal place because, at the end of the day, builders and excavators determine the amount of soil excavated by the number of trucks required for its removal. In this instance, based on a truck count, it was reckoned that about 1000 cubic metres of soil had been removed, which is not significantly different from the figure produced by the sums.

The preference here has been to cover the topography of interest with a simple, triangular lattice. It would of course have been possible to envisage lattices with cells of other shapes, but this would most likely have resulted in more cumbersome mathematics. For those involved in similar house building programmes, where large amounts of material might have to be excavated and removed, do not rely on the well-intentioned advice of builders and excavators but do the sums and be convinced that the advice is sound, or otherwise. After all, the problem is not difficult; essentially, it has been broken down into two problems—one that is concerned with finding the volume of a triangular prism with the top sliced off in some triangular plane not parallel to the XY-plane, and the other which is concerned with moving the XYZ coordinates of a triangulated area with respect to a given coordinate system to ones that make the required calculations easier.

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Using all the digits

$$
\begin{aligned}
3 \times 6819 &= 20\,457,\\
4 \times 3907 &= 15\,628,\\
4 \times 7039 &= 28\,156,\\
6 \times 5817 &= 34\,902,\\
7 \times 9403 &= 65\,821.
\end{aligned}
$$

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A Generalization of a Coin-Sliding Problem

PETER DERLIEN

Here is a puzzle that you might have encountered. You have four touching coins showing tails (denoted by a T) followed by four touching coins showing heads (denoted by an H). The problem is to create a touching sequence ordered HTHTHTHT from TTTTHHHH in just four slides of adjacent coin pairs. Note that you must retain the left–right order of the two moved coins as you slide.

You might like to stop reading now and try to solve this puzzle in order to familiarize yourself with the problem and gauge the difficulty of solving it.

If it begins to annoy, you could use some of the following observations and hints. The initial sequence is

$$\mathrm{T}^4\mathrm{H}^4 = \mathrm{T}_1\mathrm{T}_2\mathrm{T}_3\mathrm{T}_4\mathrm{H}_1\mathrm{H}_2\mathrm{H}_3\mathrm{H}_4.$$

In order to get $(\mathrm{HT})^4$ = HTHTHTHT our moves must certainly break:

$$\begin{array}{ll} & \text{three like-pair T bonds: } \mathrm{T}_1\text{-}\mathrm{T}_2, \mathrm{T}_2\text{-}\mathrm{T}_3, \mathrm{T}_3\text{-}\mathrm{T}_4 \\ \text{and} & \text{three like-pair H bonds: } \mathrm{H}_1\text{-}\mathrm{H}_2, \mathrm{H}_2\text{-}\mathrm{H}_3, \mathrm{H}_3\text{-}\mathrm{H}_4. \end{array}$$

There are six like-pair bonds to be broken and we only have four moves to arrive at a solution. Some moves must therefore break two like-pair bonds without creating any new like-pair bonds on arrival. The number of such moves must equal the excess of the number of bonds to be broken over the total number of moves allowed. The excess here is $6 - 4 = 2$, so at least two moves must each achieve a net break of two bonds.

Hints.

1. As the first move, try moving $\mathrm{T}_2\mathrm{T}_3$ to the right end to touch H_4.
2. Track the bond-break count.
3. All moves, except the last, leave a single pair of gaps internal to the sequence.
4. All moves, except the last, leave the leftmost coin and the rightmost coin as Ts.
5. Shift like pairs (TT or HH) before unlike pairs (TH or HT).
6. By the end, the whole sequence will have been shifted two places to the right.

Now it is time to give a solution. Table 1 shows the four moves needed to create the sequence HTHTHTHT from TTTTHHHH. The *overline* marks the pair about to be shifted in the upcoming move. The *underline* marks the pair that has just been deposited by a completed move. If you use this notation to record solution attempts for the puzzles involving more coins, you may like to omit the letters which remain in the same position during a move. That is, you could just record the start sequence, the gaps, and the underlined pairs.

Table 1 A solution for $n = 4$.

Move	Position									
	1	2	3	4	5	6	7	8	9	10
0	T	T	T	T	H	H	H	H	.	.
1	T	.	.	T	H	H	H	H	T	T
2	T	H	H	T	.	.	H	H	T	T
3	T	H	H	T	H	T	H	.	.	T
4			H	T	H	T	H	T	H	T

An even more compact notation is to use the following short code. Using 1, 2, 3,… for the addresses where the action takes place, write down the sequence indicating the addresses of the leftmost member of the moving pair on each move. The destinations can remain implicit provided we adopt the convention that the pair must move into the only available pair of gaps.

I find it helpful to include in the code a suffix of a minus sign if a move is to the left. In the case of the T^4H^4 to $(HT)^4$ problem (henceforth, known as P_4), the short code gives the solution as $(2, 5-, 8-, 1)$.

Note, in passing, that the above solution shifts the eight coins two places to the right. There is a variant solution which shifts the coins two places to the left. It involves moving pairs from addresses reflected about the T_4H_1 boundary, and is given by the short code $(6-, 3, 0, 7-)$. In this article, I shall use right-shifting versions throughout.

Suppose that we had P_5 to solve, i.e. TTTTTHHHHH to be sorted into HTHTHTHTHT. Can this be done in five moves? If you want to attempt it, you can use the same hints given for P_4. A numbered strip of paper above the coins will be helpful if you want to record your solution in the short code. The solution for P_5 is presented in table 2.

Table 2 A solution for $n = 5$.

Move	Position											
	1	2	3	4	5	6	7	8	9	10	11	12
0	T	T	T	T	T	H	H	H	H	H	.	.
1	T	.	.	T	T	H	H	H	H	H	T	T
2	T	H	H	T	T	H	H	.	.	H	T	T
3	T	H	H	T	.	.	H	T	H	H	T	T
4	T	H	H	T	H	T	H	T	H	.	.	T
5			H	T	H	T	H	T	H	T	H	T
Code	2, 8−, 5, 10−, 1											

Now we want to consider the general problem of rearranging n tails and n heads in n moves. More precisely, the problem to solve is P_n, that is:

$$\mathrm{T}^n\mathrm{H}^n \text{ to } (\mathrm{HT})^n \text{ in } n \text{ moves of adjacent coin pairs } (n \geq 3).$$

Can it be done? How can it be done? Apparently, a French mathematician, Henri–Auguste Delannoy, proved in *La Nature* (June 1887) that it can always be done. I do not know what his proof was, but I assume that it was a constructive proof, showing an actual method of solution, rather than a mere existence proof. I set out to find such a solution method, and, after much pondering over red (equal to head) and white counters, this article is the result.

Leaving aside the rather special case of P_3, let us see if we can make any progress with P_6 and P_7; see tables 3 and 4.

Table 3 A solution for $n = 6$.

Move	Position													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	T	$\overline{\mathrm{T}}$	$\overline{\mathrm{T}}$	T	T	T	H	H	H	H	H	H	.	.
1	T	.	.	T	T	T	H	$\overline{\mathrm{H}}$	$\overline{\mathrm{H}}$	H	H	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{T}}$
2	T	$\underline{\mathrm{H}}$	$\underline{\mathrm{H}}$	$\overline{\mathrm{T}}$	$\overline{\mathrm{T}}$	T	H	.	.	H	H	H	T	T
3	T	H	H	.	.	T	H	$\underline{\mathrm{T}}$	$\underline{\overline{\mathrm{T}}}$	$\overline{\mathrm{H}}$	H	H	T	T
4	T	H	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{H}}$	T	H	T	.	.	H	$\overline{\mathrm{H}}$	$\overline{\mathrm{T}}$	T
5	$\overline{\mathrm{T}}$	$\overline{\mathrm{H}}$	H	T	H	T	H	T	$\underline{\mathrm{H}}$	$\underline{\mathrm{T}}$	H	.	.	T
6			H	T	H	T	H	T	H	T	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{H}}$	T
Code	2, 8−, 4, 9−, 12−, 1													

Table 4 A solution for $n = 7$.

Move	Position															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	T	$\overline{\mathrm{T}}$	$\overline{\mathrm{T}}$	T	T	T	T	H	H	H	H	H	H	H	.	.
1	T	.	.	T	T	T	T	H	H	H	$\overline{\mathrm{H}}$	$\overline{\mathrm{H}}$	H	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{T}}$
2	T	$\underline{\mathrm{H}}$	$\underline{\mathrm{H}}$	T	$\overline{\mathrm{T}}$	$\overline{\mathrm{T}}$	T	H	H	H	.	.	H	H	T	T
3	T	H	H	T	.	.	T	H	H	$\overline{\mathrm{H}}$	$\underline{\overline{\mathrm{T}}}$	$\underline{\mathrm{T}}$	H	H	T	T
4	T	H	H	T	$\underline{\mathrm{H}}$	$\underline{\mathrm{T}}$	$\overline{\mathrm{T}}$	$\overline{\mathrm{H}}$	H	.	.	T	H	H	T	T
5	T	H	H	T	H	T	.	.	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{H}}$	T	H	$\overline{\mathrm{H}}$	$\overline{\mathrm{T}}$	T
6	$\overline{\mathrm{T}}$	$\overline{\mathrm{H}}$	H	T	H	T	$\underline{\mathrm{H}}$	$\underline{\mathrm{T}}$	H	T	H	T	H	.	.	T
7			H	T	H	T	H	T	H	T	H	T	H	$\underline{\mathrm{T}}$	$\underline{\mathrm{H}}$	T
Code	2, 11−, 5, 10−, 7, 14−, 1															

I do not think it is obvious how to dispose of $n = 8, 9, 10, 11, 12, \ldots$. However, now we have everything we need for the $n > 3$ general solution. The solutions for $n = 4, 5, 6,$ and 7 all start with a move of two coins into the pair of spaces immediately to the right of the starting configuration and finish with the full set of coins having been shifted two places to the right.

Consider P_8. If it occurs to us, we could parse the starting sequence as

$$\mathrm{TTTT|TTTTHHHH|HHHH.}$$

Two observations

(a) The central eight coins form a P_4 puzzle, which we could solve if we could create a pair of gaps immediately to their right.

(b) The flanking leftmost and rightmost four coins form a split P_4 puzzle.

Now look back at the solution for P_4 and you will discover something inviting about the positions of the coins after the completion of move two. It is that there is a pair of gaps where the first pair of heads used to be. This is just what we need. Now we can solve P_8 with the strategy: half solve the split P_4, solve the central P_4, then complete the solution of the split P_4.

Explicitly, we create a gap to the right of the central block with

move 0: $\mathrm{T\,\overline{TT}\,T|TTTTHHHH|HHHH\;.\;.}$,

move 1: $\mathrm{T\;.\;.\;T|TTTTHHHH|\overline{HH}HH\underline{TT}}$,

move 2: $\mathrm{T\underline{HH}T|TTTTHHHH|\;.\;.\;HHTT.}$

Then, after the solution of the central block of eight with moves three to six, we have

$$\mathrm{THHT\;.\;.\;|HTHTHTHT|H\,\overline{HT}T.}$$

There is still an internal gap, albeit moved to the left of the central block, so we can complete the solution with

move 7: $\mathrm{\overline{TH}\,HT\underline{HT}|HTHTHTHT|H\;.\;.\;T}$,

move 8: $\mathrm{HTHT|HTHTHTHT|H\underline{TH}T.}$

Thus, as before, the whole set of coins has been shifted two places to the right. Hence, if we keep prefixing sets with 4Ts and suffixing with 4Hs, we can also solve for $n = 12$, 16, 20, …, in fact for any $n = 4k$, where $k = 2, 3, 4, 5, \ldots$. (If needed, the short code for P_{12} is $(2, 21-, 6, 17-, 10, 13-, 16-, 9, 20-, 5, 24-, 1)$.)

What of P_9? Parse P_9 as

$$\mathrm{TTTT|TTTTTHHHHH|HHHH,}$$

which requires the use of solution P_5 for the central block of ten coins, but is otherwise the same as for P_8. We can use this approach for $n = 9, 13, 17, \ldots$, in fact for $n = 4k+1$, where $k = 2, 3, 4, \ldots$.

Now it should be clear how to handle $n = 4k + 2$ and $n = 4k + 3$.

If you want the satisfaction of completing the proof for all $n \geq 3$, you should now try to solve the special case of P_3. To shield you from an accidental glimpse of the solution, I shall simply tell you that the digits of its short code are given by the square root of 26 569.

When P_3 has been dealt with, you might reflect on how far we have come from the puzzle with which we started.

Peter Derlien *works part time in Sheffield University's Department of Probability and Statistics as a programmer and tutorial assistant. He has an enduring interest in the psychology of problem solving in all its varied aspects, but particularly as it relates to the teaching and learning of mathematics.*

Mathematics in the Classroom

Generalization of Ceva's theorem to polygons with an odd number of sides

1. Introduction

Giovanni Ceva (1647–1734) was an Italian mathematician whose main interest was geometry. His famous theorem was published in 1678 in a book named *De lineis rectis*. Although this theorem is surprising and has various applications, it is usually not included in school textbooks. In this article we present Ceva's theorem and two of its many proofs. Then we present a generalization to this theorem, which we discovered through using dynamic geometrical software.

2. Ceva's theorem

In a triangle ABC, three lines, AD, BE, and CF, intersect at a single point G if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \quad \text{(see figure 1).}$$

The following two proofs require basic high school geometry knowledge.

2.1. First proof

Suppose that AD, BE, and CF in the triangle ABC intersect at a single point G.

Through A, line l parallel to BC is constructed (see figure 2). Lines BE and CF are extended to meet line l at I and H, respectively.

Consideration of similar triangles gives

$$\frac{AF}{FB} \cdot \frac{CE}{EA} \cdot \frac{BD}{DC} = \frac{AH}{BC} \cdot \frac{BC}{AI} \cdot \frac{AI}{AH} = 1.$$

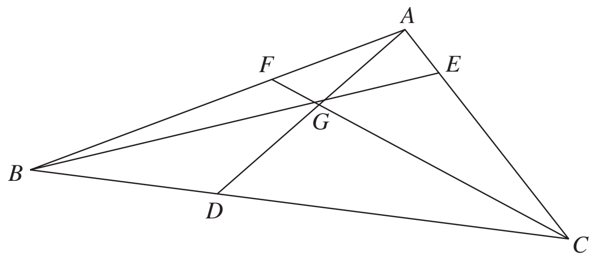

Figure 1

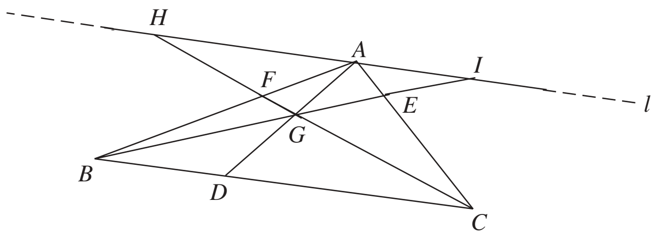

Figure 2

Conversely, suppose that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Denote by G the point of intersection between lines BE and CF. Draw line AG and extend it so that D' is its intersection point with BC. Then

$$\frac{AF}{FB} \cdot \frac{CE}{EA} \cdot \frac{BD'}{D'C} = 1.$$

Hence,

$$\begin{aligned}
\frac{BD'}{D'C} = \frac{BD}{DC} \quad &\Longrightarrow \quad \frac{BD'}{D'C} + 1 = \frac{BD}{DC} + 1 \\
&\Longrightarrow \quad \frac{BD' + D'C}{D'C} = \frac{BD + DC}{DC} \\
&\Longrightarrow \quad \frac{BC}{D'C} = \frac{BC}{DC} \\
&\Longrightarrow \quad D'C = DC.
\end{aligned}$$

This implies that D' and D are the same point, so that AD, BE, and CF are concurrent.

2.2. Second proof

Suppose that AD, BE, and CF intersect at a single point G.

Triangles BGD and CGD have a common altitude h from vertex G (see figure 3). Hence,

$$\frac{BD}{DC} = \frac{\text{Area}(\triangle BGD)}{\text{Area}(\triangle CGD)}.$$

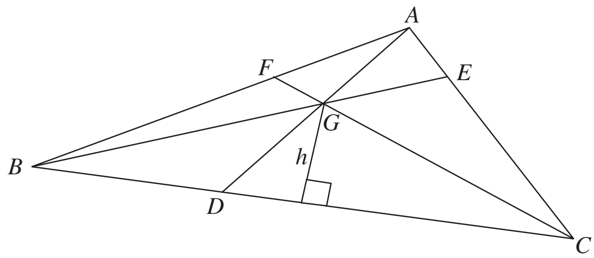

Figure 3

Similarly,

$$\frac{BD}{DC} = \frac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle ACD)}.$$

From $a/b = c/d$, it follows that $a/b = (a-c)/(b-d)$. Therefore,

$$\frac{BD}{DC} = \frac{\text{Area}(\triangle BGD)}{\text{Area}(\triangle CGD)} = \frac{\text{Area}(\triangle ABD) - \text{Area}(\triangle BGD)}{\text{Area}(\triangle ACD) - \text{Area}(\triangle CGD)} = \frac{\text{Area}(\triangle AGB)}{\text{Area}(\triangle AGC)}.$$

Similarly,

$$\frac{AF}{FB} = \frac{\text{Area}(\triangle AGC)}{\text{Area}(\triangle BGC)} \quad \text{and} \quad \frac{CE}{AE} = \frac{\text{Area}(\triangle BGC)}{\text{Area}(\triangle AGB)}.$$

Hence,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

For the converse, we use the previous proof.

3. Generalization of Ceva's theorem

Using dynamic geometrical software, we examined the existence of Ceva's theorems for polygons with an odd number of sides (clearly, the theorem does not hold for polygons with an even number of sides). To our surprise, we discovered a similar (although not identical) property for such polygons. The following is a proof for a pentagon. This proof can be easily applied to other polygons with an odd number of sides.

'Almost' Ceva's theorem for a pentagon

If in a pentagon $ABCDE$ the five Cevians $AF, BG, CH, DI,$ and EJ are concurrent, then

$$\frac{AH}{HE} \cdot \frac{EG}{GD} \cdot \frac{DF}{FC} \cdot \frac{CJ}{JB} \cdot \frac{BI}{IA} = 1 \quad \text{(see figure 4).}$$

The proof for the case of a pentagon is similar to the proof presented in Section 2.2 for the triangle. As for the converse, there is a certain restriction, to which we will refer later.

As in the second proof of Ceva's theorem,

$$\frac{AH}{HE} = \frac{\text{Area}(\triangle ACP)}{\text{Area}(\triangle ECP)}; \qquad \frac{EG}{GD} = \frac{\text{Area}(\triangle EBP)}{\text{Area}(\triangle DBP)}; \qquad \frac{DF}{FC} = \frac{\text{Area}(\triangle DAP)}{\text{Area}(\triangle CAP)};$$

$$\frac{CJ}{JB} = \frac{\text{Area}(\triangle CEP)}{\text{Area}(\triangle BEP)}; \qquad \frac{BI}{IA} = \frac{\text{Area}(\triangle BDP)}{\text{Area}(\triangle ADP)}.$$

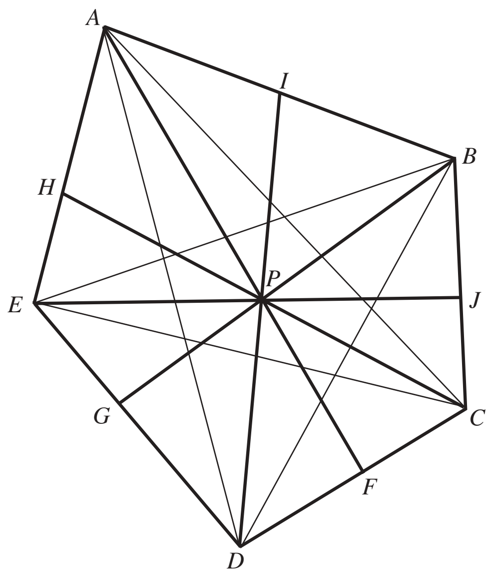

Figure 4

Therefore,

$$
\begin{aligned}
&\frac{AH}{HE}\cdot\frac{EG}{GD}\cdot\frac{DF}{FC}\cdot\frac{CJ}{JB}\cdot\frac{BI}{IA}\\
&\quad=\frac{\text{Area}(\triangle ACP)}{\text{Area}(\triangle ECP)}\cdot\frac{\text{Area}(\triangle EBP)}{\text{Area}(\triangle DBP)}\cdot\frac{\text{Area}(\triangle DAP)}{\text{Area}(\triangle CAP)}\cdot\frac{\text{Area}(\triangle CEP)}{\text{Area}(\triangle BEP)}\cdot\frac{\text{Area}(\triangle BDP)}{\text{Area}(\triangle ADP)}\\
&\quad=1.
\end{aligned}
$$

For the converse, suppose that

$$\frac{AH}{HE}\cdot\frac{EG}{GD}\cdot\frac{DF}{FC}\cdot\frac{CJ}{JB}\cdot\frac{BI}{IA}=1.$$

We will show that if four of the segments $AF, BG, CH, DI,$ and EJ meet at a point, then the fifth segment also passes through this point.

Let P be the intersection point of $AF, BG, CH,$ and DI. We draw segment EP and extend it to meet BC at J'. Then

$$\frac{AH}{HE}\cdot\frac{EG}{GD}\cdot\frac{DF}{FC}\cdot\frac{CJ'}{J'B}\cdot\frac{BI}{IA}=1.$$

Hence,

$$\frac{CJ}{JB}=\frac{CJ'}{J'B} \implies \frac{CJ}{JB}+1=\frac{CJ'}{J'B}+1$$
$$\implies \frac{CJ+JB}{JB}=\frac{CJ'+J'B}{J'B}$$
$$\implies \frac{CB}{JB}=\frac{CB}{J'B}$$
$$\implies JB=J'B.$$

Hence, J' and J are the same point, so that EJ also passes through P.

The idea of the proof can be generalized to every polygon with an odd number of sides. Consequently, it is possible to formulate the following general theorem.

Let $A_1, A_2, A_3, \ldots, A_n$ be an n-sided polygon, where n is odd. Let B_1 be a point on A_1A_2, let B_2 be a point on $A_2A_3, \ldots,$ let B_n be a point on A_nA_1.

If the n Cevians of the polygon concur, then

$$\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdots\frac{A_nB_n}{B_nA_1}=1.$$

If $(A_1B_1/B_1A_2)\cdot(A_2B_2/B_2A_3)\cdots(A_nB_n/B_nA_1)=1$ and $n-1$ of the Cevians concur, then the nth Cevian also passes through the same point of concurrence.

4. Comments

Students are often presented with special cases of theorems that can be generalized, without being aware of these possible generalizations.

For example, in a triangle, each median divides it into two triangles having the same area. Obviously, this theorem is only a special case of dividing side BC into two segments. If we divide the side so that $BD/DC = m/n$, we will obtain

$$\frac{\text{Area}(\triangle ABD)}{\text{Area}(\triangle ACD)}=\frac{m}{n}.$$

Another known example is Pythagoras' theorem, according to which in a right-angled triangle the area of the square built on the hypotenuse equals the sum of the areas of the squares that are built on the other two sides. This is a special case of the broader theorem in which other similar regular polygons (as well as semicircles), that are built on the sides of the right-angled triangle have the same relation.

We believe that discovering a generalized phenomenon and examining its special cases is a much more interesting and educative process than being satisfied merely with special cases.

After completing our work it was brought to our attention that in 1995 a generalization to Ceva's theorem was published by Grunbaum and Shephard (see reference 1). We believe, however, that the proof presented in their work is beyond school level. In this article we have adapted a proof provided for a triangle to all other polygons with an odd number of sides.

Reference

1 B. Grünbaum and G. C. Shephard, Ceva, Menelaus, and the area principle, *Math. Magazine* **68** (1995), pp. 254–268.

Oranim Academic College of Education, Israel **Atara Shriki**

Pre-Academic Studies, Haifa University, Israel **Clara Ziskin**

The Max Stern Academic College of Emek Yezreel, Israel **Ilana Lavy**

Letters to the Editor

Dear Editor,

Lottery combinatorics

I very much enjoyed reading Ian McPherson's and Derek Hodson's article *Lottery Combinatorics* (*Math. Spectrum*, Volume 41, Number 3, pp. 110–115). It gives useful statistical information to use in the classroom. However, I am not sure if they had 'tongue in cheek' when they wrote the final sentence: 'Even more obviously it would be a bad idea to have a run of six'. Of course, the lottery is random, and irrespective of your choice of six numbers each choice has an equally likely chance to turn up (or not turn up, if you prefer!). Indeed, the last two sentences are nonsense, and could seriously mislead.

Yours sincerely,

Alan Williamson
(26 Junction Close
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[*Reply by the authors:* Mr Williamson is, of course, quite correct on the two counts:

(i) each set of six numbers has the same likelihood of turning up,

(ii) tongue was firmly in cheek, hinted at by the question mark in parenthesis after the word 'obviously'. A stronger/additional hint would have been an exclamation mark at the end of the final sentence.

The misleading remarks were, in a way, a recognition of popular erroneous beliefs, often couched in statistical or scientific terms.]

Dear Editor,

Simson's triangle

Let P be any point in the plane of the triangle ABC. Let D, E, and F be the feet of the perpendiculars from P onto the lines BC, CA, and AB, respectively. The classical result of Simson says that the points D, E, F are collinear if (and only if) P lies on the circumcircle of ΔABC.

If P does not lie on the circumcircle of ΔABC then it is reasonable to enquire how nearly do D, E, F form a straight line. A possible measure of 'non-straight-line-ness' is the area of ΔDEF. It is left as a challenge to show that, if

δ is the area of ΔABC,

δ' is the area of ΔDEF,

R is the circumradius of ΔABC,

and O is the circumcentre of ΔABC, then

$$4\delta' = \delta\left|1 - \frac{OP^2}{R^2}\right|.$$

While definitely not new, this result seems to be a lot less well known than Simson's line, which it obviously includes. One might guess that it was discovered by Simson and if anyone can confirm (or refute) this suggestion then I am sure that the Editor would be pleased to hear from them.

A few results follow from the above.

1. Choosing P to be H, the orthocentre, gives

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C) = 9R^2 - (a^2 + b^2 + c^2),$$

where a, b, c are the lengths of the sides and A, B, C are the angles of ΔABC.

2. Choosing P to be I, the incentre (the centre of the inscribed circle), gives

$$OI^2 = R^2 - 2Rr,$$

where r is the inradius (the radius of the inscribed circle). This is known as Euler's formula.

3. Choosing P to be I_i, the centre of one of the three escribed circles, gives

$$OI_i^2 = R^2 + 2Rr_i,$$

where r_i is the radius of the corresponding escribed circle. (Perhaps not everyone on a typical Clapham omnibus is familiar with the fact that the inscribed and three escribed circles touch the nine-point circle.)

Yours sincerely,

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Dear Editor,

Sums of powers

In Volume 40, Number 2, pp. 73–76, Robertson and Osler gave a method to calculate $S_r = \sum_{k=1}^{n} k^r$ using Euler's little summation formula. An alternative way is to use properties of the binomial coefficient. Thus,

$$S_1 = 1 + 2 + \cdots + n = \binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}.$$

For S_2, we write

$$k^2 = k^2 - k + k = 2\binom{k}{2} + \binom{k}{1},$$

so that

$$S_2 = 2\left\{\binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right\} + \left\{\binom{1}{1} + \binom{2}{1} + \cdots + \binom{n}{1}\right\}$$
$$= 2\binom{n+1}{3} + \binom{n+1}{2}$$
$$= \binom{n+1}{3} + \binom{n+2}{3}$$
$$= \tfrac{1}{6}n(n+1)(2n+1).$$

For S_3, we write

$$k^3 = k(k-1)(k-2) + 3k(k-1) + k = 6\binom{k}{3} + 6\binom{k}{2} + \binom{k}{1},$$

so that, in a similar way,

$$S_3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = \frac{1}{4}n^2(n+1)^2.$$

Yours sincerely,

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Dear Editor,

Sumlines of sides of regular n-gons

I enjoyed reading Jonny Griffiths' article *Sumlines* (*Math. Spectrum*, Volume 41, Number 2, pp. 50–56). His conjecture that the sumline of the sides of a regular n-gon (for odd n) always touches the incircle is quite correct.

As outlined in his article, we may suppose that the n-gon is centred at the origin of co-ordinates and inscribed in a circle of radius 1 with vertices

$$\left(\cos\left(\theta + \frac{2k\pi}{n}\right), \sin\left(\theta + \frac{2k\pi}{n}\right)\right), \qquad 0 \le k \le n-1.$$

Jonny's algebra for equilateral triangles on p. 55 readily extends to show that the sumline of the sides has the equation

$$ny = -\left(\sum_{k=0}^{n-1} \cot\left(\theta + \frac{\pi}{n}(2k+1)\right)\right)x + \cos\frac{\pi}{n}\left(\sum_{k=0}^{n-1} \operatorname{cosec}\left(\theta + \frac{\pi}{n}(2k+1)\right)\right)$$

or $ny = -Ax + \cos(\pi/n)B$, say.

This sumline is a perpendicular distance $|\cos(\pi/n)B|/\sqrt{n^2 + A^2}$ from the origin and, thus, touches the incircle (of radius $\cos(\pi/n)$) if

$$\frac{B}{\sqrt{n^2 + A^2}} = 1 \quad \text{or} \quad (B - A)(B + A) = n^2.$$

Since

$$\operatorname{cosec}\phi - \cot\phi = \frac{1-\cos\phi}{\sin\phi} = \tan\frac{\phi}{2} \quad \text{and} \quad \operatorname{cosec}\phi + \cot\phi = \frac{1+\cos\phi}{\sin\phi} = \cot\frac{\phi}{2},$$

we see that

$$(B-A)(B+A) = \left(\sum_{k=0}^{n-1}\tan\left(\frac{\theta}{2}+\frac{\pi}{2n}+\frac{k\pi}{n}\right)\right)\left(\sum_{k=0}^{n-1}\cot\left(\frac{\theta}{2}+\frac{\pi}{2n}+\frac{k\pi}{n}\right)\right).$$

Consider the equation $\tan nx - \tan(n\theta/2+\pi/2) = 0$ with roots $x = \theta/2+\pi/2n+k\pi/n$ (where k is any integer).

When n is odd, the expansion of $\tan nx$ in terms of $t = \tan x$ (given, for example, by equating real and imaginary parts of $\cos nx + \mathrm{i}\sin nx = (\cos x + \mathrm{i}\sin x)^n$) takes the form

$$\tan nx = \frac{\binom{n}{1}t - \binom{n}{3}t^3 + \cdots + (-1)^{(n-1)/2}t^n}{1-\binom{n}{2}t^2+\cdots+(-1)^{(n-1)/2}\binom{n}{n-1}t^{n-1}}.$$

Equating this to $\tan(n\theta/2+\pi/2) = -\cot(n\theta/2)$ shows that $t = \tan(\theta/2+\pi/2n+k\pi/n)$ for $k = 0, 1, \ldots, n-1$ give the n roots of

$$(-1)^{(n-1)/2}t^n + (-1)^{(n-1)/2}\binom{n}{n-1}\cot\frac{n\theta}{2}t^{n-1} + \cdots + \binom{n}{1}t + \cot\frac{n\theta}{2} = 0.$$

By standard relations between the roots and coefficients, we then see that

$$(B-A)(B+A) = \text{"}\sum \text{roots} \sum \frac{1}{\text{roots}}\text{"} = -n\cot\frac{n\theta}{2}\,\frac{-n}{\cot(n\theta/2)} = n^2,$$

as required.

Finally, it is perhaps worth remarking that we can give an alternative proof that the sumline always goes through the origin when n is even. Here, we need to show that $B = 0$, which is immediate from pairing the terms of the sum:

$$\sum_{k=0}^{n-1}\operatorname{cosec}\left(\theta+\frac{\pi}{n}(2k+1)\right) = \sum_{k=0}^{n/2-1}\left(\operatorname{cosec}\left(\theta+\frac{\pi}{n}(2k+1)\right)+\operatorname{cosec}\left(\theta+\frac{\pi}{n}(2k+1+n)\right)\right) = 0,$$

since $\operatorname{cosec} x + \operatorname{cosec}(x+\pi) = 0$.

Yours sincerely,

Nick Lord
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Dear Editor,

An imaginary, and imaginative, approach to an identity

Consider

$$(a+b\mathrm{i})(a-b\mathrm{i}) = a^2+b^2, \qquad (c+d\mathrm{i})(c-d\mathrm{i}) = c^2+d^2,$$

where $\mathrm{i} = \sqrt{-1}$. Then,

$$\begin{aligned}(a^2+b^2)(c^2+d^2) &= (a+b\mathrm{i})(a-b\mathrm{i})(c+d\mathrm{i})(c-d\mathrm{i})\\ &= (a+b\mathrm{i})(c-d\mathrm{i})(a-b\mathrm{i})(c+d\mathrm{i})\\ &= \{ac+ad+\mathrm{i}(bc-ad)\}\{ac+bd-\mathrm{i}(bc-ad)\}\\ &= (ac+bd)^2+(bc-ad)^2.\end{aligned}$$

Thus, when multiplying two numbers, each the sum of two squares, the resulting product will itself be the sum of two squares.

Replacing $\sqrt{-1}$ by $\sqrt{-n}$, it follows in the same way that

$$(a^2+nb^2)(c^2+nd^2) = (ac+nbd)^2+n(bc-ad)^2.$$

Furthermore, if we replace $\sqrt{-n}$ by $\sqrt{n}$, we obtain

$$(a^2-nb^2)(c^2-nd^2) = (ac-nbd)^2-n(bc-ad)^2.$$

In these identities, $a, b, c,$ and d may be integers, rationals, or reals; n would generally be a natural number.

Yours sincerely,

Bob Bertuello
(12 Pinewood Road
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Bath BA3 2RG)

Dear Editor,

Calculation of s_5

We write $s_k = 1^k+2^k+\cdots+n^k$. The method that M. A. Khan used in his letter published in Volume 40, Number 1, p. 40 to show that $s_3 = s_1^2$ can be used to calculate s_5 from s_3. We write $T_r = r^3(r+1)^3$, so that

$$T_r - T_{r-1} = r^3(r+1)^3-(r-1)^3r^3 = r^3(6r^2+2).$$

Hence,

$$(T_1-T_0)+(T_2-T_1)+\cdots+(T_n-T_{n-1}) = 6s_5+2s_3,$$

so that

$$6s_5 = T_n - 2s_3 = n^3(n+1)^3 - 2\tfrac{1}{4}n^2(n+1)^2,$$

which gives

$$s_5 = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1).$$

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

42.1 A square sits in a right-angled triangle with circles of radii R, r and t touching as shown in figure 1. Find the relative dimensions of the right-angled triangle and the relationship between R and t.

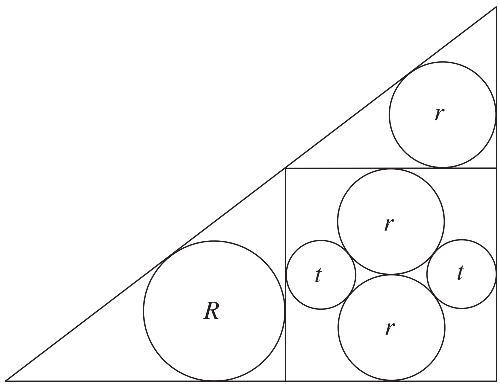


Figure 1

(Submitted by Bob Bertuello, Midsomer Norton, Bath)

42.2 The normals to the curve $y = \sin nx$ at the points P and Q with x-coordinates $\pi/2n$ and $\pi/2n + h$, respectively, meet at M. Find the limiting position of M as $Q \to P$.

(Submitted by Guido Lasters, Tienen, Belgium)

42.3 Show that, given any partition into two subsets of the set of positive rational numbers smaller than 1 with denominators not exceeding 6, one of the subsets contains a, b, c such that $ab = c$.

(Submitted by Joshua Lam, Year 10, The Leys School, Cambridge)

42.4 (i) Determine the number of permutations ϕ of $\{1, \ldots, n\}$ such that $\phi^2 = \mathrm{Id}$.

(ii) Determine the number of permutations ϕ of $\{1, \ldots, n\}$ such that $\phi^3 = \mathrm{Id}$.

(iii) Determine all permutations ϕ of $\{1, 2, 3, 4, 5, 6\}$ such that $\phi(i) + \phi^{-2}(i) = 7$ for all i

(Submitted by M. N. Deshpande, Kavita Laghate and M. R. Modak*, Institute of Science, Nagpur, *S. P. College, Pune, India)

Solutions to Problems in Volume 41 Number 2

41.5 Prove that, for all positive integers n,

$$n^{2n-1} \geq (2n-1)!.$$

Solution by Henry Ricardo, Medgar Evers College, New York

By the arithmetic mean-geometric mean inequality,

$$\sqrt[2n-1]{(2n-1)!} \leq \frac{1+2+\cdots+(2n-1)}{2n-1} = \frac{n(2n-1)}{2n-1} = n,$$

from which the inequality follows.

Alternative solution by Dorin Mărghidanu, who proposed the problem

For $n > 1$ and $1 \leq k \leq n-1$, we have $(n-k)^2 > 0$, whence $n^2 > k(2n-k) > 0$. If we multiply these inequalities, we obtain

$$n^{2(n-1)} > 1 \times 2 \times \cdots \times (n-1) \times (2n-1) \times (2n-2) \times \cdots \times (n+1),$$

whence $n^{2n-1} > (2n-1)!$. Hence, for $n \geq 1$, $n^{2n-1} \geq (2n-1)!$.

41.6 Determine all sequences of consecutive Fibonacci numbers which are in arithmetic progression.

Solution

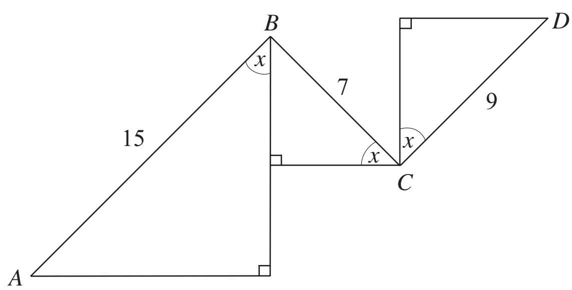
Suppose that f_n, f_{n+1}, f_{n+2} are in arithmetic progression, so that

$$f_{n+2} - f_{n+1} = f_{n+1} - f_n \quad \text{or} \quad f_n = f_{n-1}.$$

The only two equal consecutive Fibonacci numbers are $f_1 = f_2 = 1$, so $n = 2$, and the only three consecutive Fibonacci numbers in arithmetic progression are $f_2 = 1,\ f_3 = 2,\ f_4 = 3$.

41.7 A is 15 km south x degrees west of B, B is 7 km west x degrees north of C, and D is 9 km north x degrees east of C. How far is A from D?

Solution by Bob Bertuello, who proposed the problem



$$\begin{aligned} AD^2 &= (15\sin x + 7\cos x + 9\sin x)^2 + (15\cos x - 7\sin x + 9\cos x)^2 \\ &= 24^2 + 7^2 \\ &= 25^2, \end{aligned}$$

so $AD = 25$ km.

41.8 The vertices of a triangle lie on the rectangular hyperbola $xy = 1$. Show that the orthocentre also lies on the hyperbola.

Solution by J. A. Scott, who proposed the problem

Let the vertices of the triangle be $A(a, 1/a)$, $B(b, 1/b)$, $C(c, 1/c)$. The slope of AB is

$$\frac{1/a - 1/b}{a - b} = -\frac{1}{ab},$$

so the line through C perpendicular to AB has the equation

$$y - \frac{1}{c} = ab(x - c).$$

Similarly, the line through A perpendicular to BC has the equation

$$y - \frac{1}{a} = ab(x - a),$$

and these meet at the orthocentre H with the x-coordinate given by

$$ab(x - c) + \frac{1}{c} = bc(x - a) + \frac{1}{a} \quad \text{or} \quad b(a - c)x = \frac{1}{a} - \frac{1}{c} \quad \text{or} \quad x = -\frac{1}{abc}.$$

The y-coordinate of H is thus

$$\frac{1}{c} + ab\left(-\frac{1}{abc} - c\right) = -abc.$$

Hence, H has coordinates $(-1/abc, -abc)$ and so lies on the rectangular hyperbola.

Reviews

Is Mathematics Inevitable? By Underwood Dudley (ed.). Spectrum, 2007. Hardback, 160 pages, \$45.50 (ISBN 978-0-88385-566-9).

This book consists of 26 articles on mathematics that have caught the author's eye. It is very wide-ranging, varied, and eclectic. There is no particular theme or connection between the articles, other than the fact that the author enjoyed reading them, thinks that you will as well, and also believes that they deserve a wider readership. The title is taken from the third article by Nathan Altshiller Court. I enjoyed reading the book and felt that it was time well spent. You can take your time over it, reading an article whenever you feel like it. There are articles on Mathematical Philosophy, Mathematical History, Mathematical Teaching, Recreational Mathematics, and Mathematicians and Mathematics itself. Different readers will put different values on the various articles. My favourite was 'There are three times as many obtuse-angled triangles as there are acute-angled ones' by Richard K. Guy, due to his clarity of style, sense of humour, and ability to look at the same problem from different perspectives.

Paul Belcher

Sink or Float? Thought Problems in Math & Physics. By Keith Kendig. The Mathematical Association of America , 2008. Hardback, 392 pages, \$59.95 (ISBN 978-0-88385-339-9).

This is a terrific book. It begins with the author's (successful) attempt to turn his dentist onto maths; a dangerous manoeuvre in itself, one might think, as your student brandishes a drill in the direction of your mouth. Professor Kendig argues that it was his use of a compelling example that won his dentist round (exploring $(x+1)(x-1) = x^2 - 1$ for various values of x) and this book carries that philosophy forwards, collecting together 200 pages of 'examples in the guise of problems'. And rather wonderful problems they are too, ranging from the purest of pure mathematics to the most applied of practical physics. Some are one-liners, some take a full page, but they always engendered in me that feeling of 'Hang on a minute!' followed by an urgent search for a pencil and paper.

But I have still only scratched the surface of this book's attributes. It is beautifully written; the language is evocative and controlled, and the style aims for wry understatement rather than being disconcertingly comic. There are also helpful historical allusions throughout, all naturally motivated by the problem in hand. The presentation is excellent, with attractive yet quirky diagrams that lend the book a happy consistency. The same can be said of the choice of problems—they 'ring true', so to speak, as a statement of Professor Kendig's mathematical credo. (Every problem creator develops a style, and here you will find the notion of dipping-things-into-water repeated as a pleasing leitmotif.) My guess based upon examining this book is that those who find themselves in Professor Kendig's classes at Cleveland State University must count themselves fortunate indeed.

Jonny Griffiths

Modern Regression Methods. By Thomas P. Ryan. John Wiley, Chichester, 2nd edition, 2009. Hardback, 642 pages, £67.95 (ISBN 0-470-08186-0).

Modern Regression Methods, Second Edition maintains the accessible organization, breadth of coverage, and cutting-edge appeal that earned its predecessor the title of being one of the top five books for statisticians by an *Amstat News* book editor in 2003. This new edition has been updated and enhanced to include all-new information on the latest advances and research in the evolving field of regression analysis.

Piano-Hinged Dissections: Time to Fold! By Greg N. Frederickson. A. K. Peters, Wellesley, MA, 2006. Hardback, 303 pages, \$49.00 (ISBN 1-56881-299-X).

From traditional puzzles such as tangrams to the challenges of Sam Loyd and the polyominoes of Solomon Golomb, puzzlers as well as geometers have been interested in dissections for centuries: Can we cut one shape into a finite number of pieces so that when rearranged the pieces form a second shape? Traditional dissections involve pieces that are not attached to each other—add hinges between the pieces, and you are looking at a whole new set of rules and challenges. This book showcases a new type of hinged dissection that generates even more challenges.

A piano hinge is a long, narrow hinge that runs the full length of the joint—like on the top of a grand piano—so that one piece flaps on top of or under the other piece. This mechanism can be simulated by folding a piece of paper, so you can test and experiment with piano-hinged dissections without needing special materials: just paper and scissors–and some intuition and creativity! The author provides over 150 dissections and outlines methods for discovering them. The videos on the CD provide demonstrations for creating your own dissections.

Visual Group Theory By Nathan Carter. MAA, Washington, DC, 2009. Hardback, 334 pages, \$71.95 (ISBN 978-0-88385-757-1).

An attractive introduction to Group Theory with a long lead-in, the formal definition does not come until page 51! It would be interesting to know how helpful this long introduction to the subject is to those who do not know what a group is. Despite the leisurely start, it reaches Galois Theory, the crowning achievement of the subject which makes it all worthwhile. Lots of attractive diagrams and numerous exercises help a lot.

David Sharpe

Probability with R: An Introduction with Computer Science Applications By Jane M. Horgan. John Wiley, Chichester, 2009. Hardback, 393 pages, £58.95 (ISBN 0-470-28073-7).

Probability with R serves as a comprehensive and introductory book on probability with an emphasis on computing-related applications. Real examples show how probability can be used in practical situations, and the freely available and downloadable statistical programming language R illustrates and clarifies the book's main principles.

Other books received

Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers By D. W. Jordan and P. Smith. Oxford University Press, 4th edition, 2007. Paperback, 560 pages, £30.00 (ISBN 978-0-19-920825-8).

Nonlinear Ordinary Differential Equations: Problems and Solutions: A Sourcebook for Scientists and Engineers By D. W. Jordan and P. Smith. Oxford University Press, 2007. Paperback, 450 pages, £25.00 (ISBN 978-0-19-921203-3).

Crosspacks & Cross Hypersolids: Being the Seventh Part of Several Comprising the Complete? Polyhedra By Patrick Taylor. Nattygrafix, Ipswich, 2007. Paperback, 80 pages, £6.00 (ISBN 0-9516701-8-2).

Reliability and Risk: A Bayesian Perspective By Nozer D. Singpurwalla. John Wiley, Chichester, 2006. Hardback, 371 pages, £65.00 (ISBN 0-470-85502-9).

International Migration in Europe: Data, Models and Estimates Edited by James Raymer and Frans Wiilekens. John Wiley, Chichester, 2008. Hardback, 385 pages, £65.00 (ISBN 0-470-03233-6).

From Quantum Cohomology to Integrable Systems By Martin A. Guest. Oxford University Press, 2008. Hardback, 336 pages, £45.00 (ISBN 978-0-19-856599-4).

Introduction to Algebra By Peter J. Cameron. Oxford University Press, 2007. Paperback, 342 pages, £28.50 (ISBN 978-0-19-852793-0).

Mathematical Asset Management By Thomas Höglund. John Wiley, Chichester, 2008. Hardback, 222 pages, £47.50 (ISBN 0-470-23287-3).

Mathematical Spectrum

Volume 42 2009/2010 Number 1

ISSN 0025-5653

http://www.appliedprobability.org

Published by the Applied Probability Trust

Printed by Berforts South East Ltd, Stevenage, Hertfordshire, UK