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1990

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GENERAL INFORMATION

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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER
No. 117
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first item is the 31st IMO, Beijing, China. I would like to thank Andy Liu of the University of Alberta, who acted as a trainer for the Chinese Team, and Richard Nowakowski of Dalhousie University, a Canadian Team leader, for kindly sending me the information which I relay.

This year a record 308 students from 54 countries took part in the competition. (The number of countries invited was originally 58, but some did not appear.) For the first time the number of European countries participating (23) was exceeded by the non–European contingent (31). Algeria, Cyprus, Iceland, Kuwait, Luxembourg, Morocco, and Tunisia did not field complete six–person teams, the maximum team size that has become standard for the competition. Two first–time entries were Bahrain and Macau.

The six problems of the competition were assigned equal weights of seven points each (the same as in the last nine IMO's) for a maximum possible individual score of 42 (and a maximum possible team score of 252). For comparison see the last nine IMO reports in [1981: 220], [1982: 223], [1983: 205], [1984: 249], [1985: 202], [1986: 169], [1987: 207], [1988: 193] and [1989: 193].

This year first place (gold) medals were awarded to the 23 students with scores in the range 34–42. There were 4 perfect papers written, two from China and one each from France and the USSR. The 15 year old Soviet girl, Evgenia Malinnikova, who wrote a perfect paper also wrote a perfect paper last year. She was awarded a special prize at the Closing Ceremony. Second place (silver) medals were awarded to the 56 students with scores in the range 23–33, and third prize (bronze) medals were awarded to the 76 students in the range 16–22. These compare to 20 gold (with 10 perfect papers), 59 silver and 68 bronze medals awarded last year. This year no special prizes were awarded for particularly elegant solutions.

Congratulations to the gold medalists, and in particular to those scoring a perfect paper.

Zhou Tong	China	42
Vincent Lafforgue	France	42

Evgenia Malinnikova	USSR	42
Wang Jianhua	China	42
Wang Song	China	41
Alexandr Stoyauovski	USSR	40
Alexandr Perlin	USSR	40
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Here are the problems from this year's IMO Competition. Solutions to these problems, along with those of the 1990 USA Mathematical Olympiad, will appear in a booklet entitled *Mathematical Olympiads 1990* which may be obtained for a small charge from:

Dr. W.E. Mientka
Executive Director
MAA Committee on H.S. Contests
917 Oldfather Hall
University of Nebraska
Lincoln, Nebraska, USA 68588

31ST INTERNATIONAL MATHEMATICAL OLYMPIAD

Beijing, China

First Day: July 12, 1990

Time: 4.5 hours

1. Two chords AB, CD of a circle intersect at point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, M intersects the lines BC, AC at F, G respectively. If $AM/AB = t$, find EG/EF in terms of t.

2. Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be coloured black. Such a colouring is good if there is at least one pair of black points such that the interior

of one of the arcs contains exactly n points from E. Find the smallest value of k so that every such colouring of k points of E is good.

3. Determine all integers $n > 1$ such that $(2^n + 1)/n^2$ is an integer.

Second Day: July 13, 1990
Time: 4.5 hours

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f:\mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(xf(y)) = f(x)/y$ for all x, y in $\mathbb{Q}^+$.

5. Given an initial integer $n_0 > 1$, two players A and B choose integers $n_1, n_2, n_3, \ldots$ alternately according to the following rules; knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \le n_{2k+1} \le n_{2k}^2$. Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that n_{2k+1}/n_{2k+2} is a positive power of a prime. Player A wins the game by choosing the number 1990, player B wins by choosing the number 1. For which n_0 does

(a) A have a winning strategy,
(b) B have a winning strategy,
(c) neither player have a winning strategy?

6. Prove that there exists a convex 1990–gon with the following two properties:
(a) all angles are equal,
(b) the lengths of the sides are the numbers $1^2, 2^2, 3^2, \ldots, 1989^2, 1990^2$ in some order.

*

As the IMO is officially an individual event, the compilation of team scores is unofficial, if inevitable. Team scores are obtained by adding up the individual scores of the members. These totals, as well as a breakdown of the medals awarded by country, is given in the following table. Congratulations once again to China on a convincing victory.

Rank	Country	Score (Max. 252)	Prizes 1st	Prizes 2nd	Prizes 3rd	Total Prizes
1	China	230	5	1	–	6
2	U.S.S.R.	193	3	2	1	6
3	U.S.A.	174	2	3	–	5
4	Romania	171	2	2	2	6
5	France	168	3	1	–	4
6	Hungary	161	1	3	2	6
7	East Germany	158	–	4	2	6

8	Czechoslovakia	153	–	5	1	6	
9	Bulgaria	152	1	4	1	6	
10	U.K.	141	2	–	2	4	
11	Canada	139	–	3	1	4	
12	West Germany	138	–	2	4	6	
13	Italy	131	1	1	4	6	
14	Iran	122	–	4	–	4	
15–16	Australia	121	–	2	4	6	
15–16	Austria	121	–	1	4	5	
17	India	116	1	1	2	4	
18	Norway	112	–	3	1	4	
19	North Korea	109	–	1	3	4	
20	Japan	107	–	2	1	3	
21	Poland	106	–	2	1	3	
22	Hong Kong	105	–	–	4	4	
23	Vietnam	104	–	1	3	4	
24	Brazil	102	1	–	2	3	
25	Yugoslavia	98	–	1	2	3	
26	Israel	95	–	1	3	4	
27	Singapore	93	–	–	2	2	
28	Sweden	91	–	1	2	3	
29	Netherlands	90	–	1	2	3	
30	Colombia	88	–	1	2	3	
31	New Zealand	83	–	–	2	2	
32	South Korea	79	–	1	1	2	
33–34	Thailand	75	–	–	2	2	
33–34	Turkey	75	–	–	1	1	
35	Spain	72	–	–	–	–	
36	Morocco	71	–	1	–	1	(Team of 5)
37	Mexico	69	–	–	1	1	
38–39	Argentina	67	–	–	1	1	
38–39	Cuba	67	–	–	1	1	
40–41	Bahrain	65	–	–	–	–	
40–41	Ireland	65	–	–	1	1	
42	Greece	62	–	–	1	1	
43	Finland	59	–	–	1	1	
44	Luxembourg	58	1	–	1	2	(Team of 2)
45	Tunisia	55	–	1	1	2	(Team of 4)
46	Mongolia	54	–	–	–	–	
47	Kuwait	53	–	–	1	1	(Team of 4)
48–49	Cyprus	46	–	–	1	1	(Team of 4)
48–49	Philippines	46	–	–	1	1	
50	Portugal	44	–	–	–	–	
51	Indonesia	40	–	–	–	–	
52	Macau	32	–	–	–	–	
53	Iceland	30	–	–	1	1	(Team of 3)
54	Algeria	28	–	–	–	–	(Team of 4)

*

This year the Canadian team improved from 19th to 11th place, with the team score higher on what was evidently a more difficult contest. The team members, scores and the leaders of the Canadian team were

Danny Brown	32	Silver
Ian Goldberg	29	Silver
Jeffrey Grossman	28	Silver
Etsuko Amano	20	Bronze
Andrew Chow	15	
Hugh Thomas	15	

Team leaders: Georg Gunther, Sir Wilfred Grenfell College
Richard Nowakowski, Dalhousie University.

Readers may be interested that János Csirik, who visited Canada last year and finished fourth in the 1990 USAMO and (unofficially) second in the 1990 Canadian Mathematical Olympiad, represented the Hungarian team for the third time, missing a gold medal by one point.

At least three students sustained injuries during the IMO, two with broken feet. The third, Canada's Hugh Thomas, injured his writing hand walking through a glass door the day before the competition.

The next few Olympiads are:

1991	Sweden
1992	USSR
1993	Turkey
1994	?
1995	Canada

* * *

Next we give the "official solutions" for the 1990 Canadian Mathematical Olympiad, which we published in the June number of the Corner.

1990 CANADIAN MATHEMATICAL OLYMPIAD

1. A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1,2,3,\ldots,n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n,k) for which this is possible.

Solution. The total number of points awarded to all players over all days is $kn(n + 1)/2 = 26n$, so that $k(n + 1) = 52$. For $(n,k) = (3,13)$ we have a possible allocation given by

$$(26,26,26) = (1,2,3) + 2(2,3,1) + 2(3,1,2) + 3(1,3,2) + 2(3,2,1) + 3(2,1,3) .$$

For $(n,k) = (12,4)$, we have

$$(26,26,\ldots,26,26) = 2(1,2,\ldots,11,12) + 2(12,11,\ldots,2,1) .$$

For $(n,k) = (25,2)$, we have

$$(26,26,\ldots,26,26) = (1,2,\ldots,24,25) + (25,24,\ldots,2,1) .$$

However, $(n,k) = (51,1)$ is not realizable. Hence (n,k) must be one of $(3,13)$, $(12,4)$, $(25,2)$.

2. A set of $n(n+1)/2$ distinct numbers is arranged at random in a triangular array:

$$\begin{array}{ccccccccc} & & & & \times & & & & \\ & & & \times & & \times & & & \\ & & \times & & \times & & \times & & \\ & & \vdots & & \vdots & & \vdots & & \\ \times & \times & \cdot & & \cdot & & \cdot & \times & \times \end{array}$$

Let M_k be the largest number in the kth row from the top. Find the probability that $M_1 < M_2 < M_3 < \cdots < M_n$.

Solution. Let p_n be the probability when there are n rows. We claim that $p_n = \frac{2^n}{(n+1)!}$. Clearly, $p_1 = 1$ and $p_2 = 2/3$. In general, the largest number must go in the last row, the probability for which is

$$\frac{n}{n(n+1)/2} = \frac{2}{n+1}.$$

There is no restriction on the remaining members of the last row. The probability that the numbers of the first $n-1$ rows are suitably placed is p_{n-1}. Hence by induction

$$p_n = \left(\frac{2}{n+1}\right)p_{n-1} = \frac{2^n}{(n+1)!}.$$

3. Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'|.$$

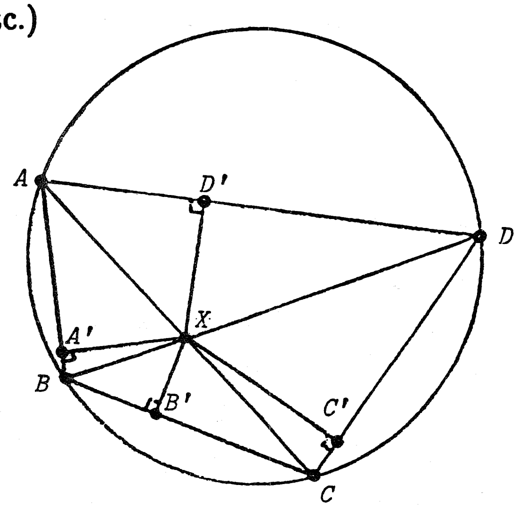
($|A'B'|$ is the length of line segment $A'B'$, etc.)

Solution. Let d be the length of the diameter of the circle $ABCD$. $A'XB'B$ is concyclic and the angle subtended by $A'B'$ at B equals the angle subtended by AC in circle $ABCD$. Therefore

$$\frac{|A'B'|}{|BX|} = \frac{|AC|}{d},$$

since BX is a diameter of circle $A'XB'B$. Similarly

$$\frac{|C'D'|}{|DX|} = \frac{|AC|}{d}.$$



Hence

$$|A'B'| + |C'D'| = \frac{|AC|}{d}(|BX| + |DX|) = \frac{|AC|\,|BD|}{d} .$$

Likewise,

$$|A'D'| + |B'C'| = \frac{|AC|\,|BD|}{d} .$$

4. A particle can travel at speeds up to 2 metres per second along the x–axis, and up to 1 metre per second elsewhere in the plane. Provide a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

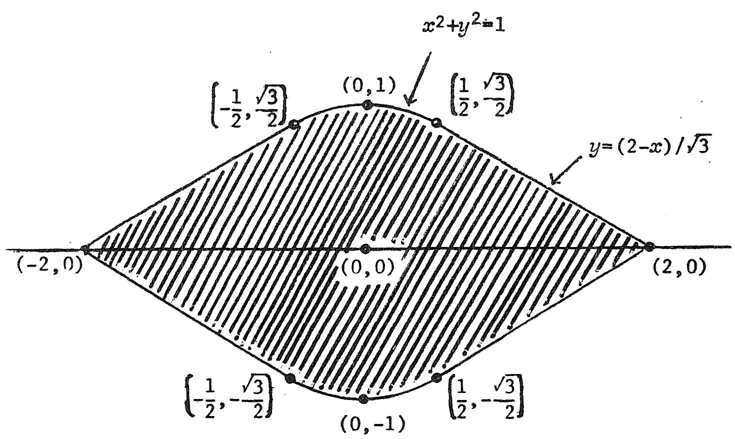
Solution. Because of symmetry, we can restrict attention to the positive quadrant. The most efficient path to an accessible point follows the x–axis to a point $(t,0)$ and then moves directly to the point (where $0 \le t \le 2$). We consider paths of this type.

Fix the abscissa of an accessible point at x and look for possible values of y. A particle leaving the x–axis at $(t,0)$ can move a further distance of at most $(2 - t)/2$. Then $(x - t)^2 + y^2 \le ((2 - t)/2)^2$, so that (after some rearrangement)

$$y^2 \le \frac{3}{4}\left[\frac{4}{9}(2 - x)^2 - \left(t - \frac{2(2x - 1)}{3}\right)^2\right] .$$

If $1/2 \le x \le 2$, y^2 is maximized when $t = \frac{2}{3}(2x - 1)$ and so $y \le (2 - x)/\sqrt{3}$. If $0 \le x \le 1/2$, y^2 is maximized when $t = 0$ and so $y^2 \le 1 - x^2$.

Remark. The set of points accessible from $(t,0)$ on the x–axis lie in a circle of radius $(2 - t)/2$ and centre $(t,0)$. These circles are similar, under a dilation with centre $(2,0)$, to the circle $x^2 + y^2 = 1$. The lines $y = \pm(2 - x)/\sqrt{3}$ are the envelopes of this family of circles.



5. Suppose that a function f defined on the positive integers satisfies

$$f(1) = 1 , \qquad f(2) = 2 ,$$

$$f(n + 2) = f(n + 2 - f(n + 1)) + f(n + 1 - f(n)) \qquad (n \ge 1) .$$

(a) Show that

(i) $0 \le f(n + 1) - f(n) \le 1$;

(ii) if $f(n)$ is odd, then $f(n + 1) = f(n) + 1$.

(b) Determine, with justification, all values of n for which

$$f(n) = 2^{10} + 1 .$$

Solution. (a) (i) Since

$$f(3) = f(3 - f(2)) + f(2 - f(1)) = f(1) + f(1) = 2$$

and

$$f(4) = f(4 - f(3)) + f(3 - f(2)) = f(2) + f(1) = 3 ,$$

the result holds for $n = 1$, 2 and 3. Suppose we have shown for $n = 1,2,\dots, m - 1$ that $0 \le f(n + 1) - f(n) \le 1$, where $m \ge 4$. Then for each such n,

$$[n + 2 - f(n + 1)] - [n + 1 - f(n)] = 1 - [f(n + 1) - f(n)] \in \{0,1\} ,$$

so that

$$f(n + 2 - f(n + 1)) - f(n + 1 - f(n)) \in \{0,1\} \qquad (*)$$

by the induction hypothesis.

[*Editor's note.* By the statement of the problem, we can assume that $f(n + 1 - f(n))$ is defined for all $n \ge 1$, that is, $n + 1 - f(n) \ge 1$ and so $f(n) \le n$ for all $n \ge 1$. It is also clear from the problem that $f(n) \ge 1$ for all $n \ge 1$. Thus

$$1 \le n + 1 - f(n) \le m - 1 \qquad \text{for all} \quad 1 \le n \le m - 1 ,$$

and the induction hypothesis can be applied to obtain (*). Note that this proof *does not show* that f exists!]

From the induction hypothesis with $n = m - 1$, there are two cases.

First case: $f(m) = f(m - 1) + 1$. Then

$$\begin{aligned} f(m + 1) - f(m) &= f(m + 1 - f(m)) + f(m - f(m - 1)) \\ &\qquad - f(m - f(m - 1)) - f(m - 1 - f(m - 2)) \\ &= f(m + 1 - f(m)) - f(m - 1 - f(m - 2)) \\ &= f(m - f(m - 1)) - f(m - 1 - f(m - 2)) \\ &\in \{0,1\} \end{aligned}$$

by (*) with $n = m - 2$.

Second case: $f(m) = f(m - 1)$. Then

$$\begin{aligned} &f(m - f(m - 1)) + f(m - 1 - f(m - 2)) \\ &\qquad = f(m - 1 - f(m - 2)) + f(m - 2 - f(m - 3)) , \end{aligned}$$

so that

$$f(m - f(m - 1)) = f(m - 2 - f(m - 3)) .$$

By (*), each member of this equation equals $f(m - 1 - f(m - 2))$. Hence

$$\begin{aligned} f(m + 1) - f(m) &= f(m + 1 - f(m)) - f(m - 1 - f(m - 2)) \\ &= f(m + 1 - f(m)) - f(m - f(m - 1)) \\ &\in \{0,1\} \end{aligned}$$

by (*) with $n = m - 1$. Thus, (a) (i) holds for $n = m$, and the required result follows by induction.

(a) (ii) The result holds for $n = 1$. Suppose the result has been established for $n = 1,2,\ldots,m - 1$. Let $f(m)$ be odd. Then $f(m - 1)$ is even (since otherwise $f(m) = f(m - 1) + 1$ is even by the induction hypothesis), so that $f(m) = f(m - 1) + 1$ by (a) (i). Hence

$$\begin{aligned} f(m + 1) &= f(m + 1 - f(m)) + f(m - f(m - 1)) \\ &= 2f(m - f(m - 1)) . \end{aligned}$$

Therefore, $f(m + 1)$ is even and equals $f(m) + 1$, by (a) (i).

(b) We will prove by induction that, for each integer k, $n = 2^k$ is the unique solution of $f(n) = 2^{k-1} + 1$, so that, in particular, $n = 2^{11}$ is the unique solution for $f(n) = 2^{10} + 1$.

For $k = 2$, this is evident from (a) and small values of $f(n)$. Suppose we have shown that $n = 2^m$ is the unique solution for $f(n) = 2^{m-1} + 1$. From (a) (ii) we deduce that there is a unique number u such that $f(u) = 2^m + 1$. (If there is no such u, then $f(n)$ would assume a constant value from some point on and for large n this would contradict the recursion equation satisfied by $f(n + 2)$.) Note that $f(u - 1) = 2^m$ by part (a). Now

$$2^m + 1 = f(u) = f(u - f(u - 1)) + f(u - 1 - f(u - 2)) ,$$

and so $f(u - f(u - 1))$ and $f(u - 1 - f(u - 2))$ are of opposite parity. Since $f(u - f(u - 1)) - f(u - 1 - f(u - 2)) \in \{0,1\}$, we have

$$f(u - f(u - 1)) = f(u - 1 - f(u - 2)) + 1$$

and thus

$$f(u - f(u - 1)) = 2^{m-1} + 1 .$$

Therefore, by the induction hypothesis, $u - f(u - 1) = 2^m$ and hence

$$u = 2^m + f(u - 1) = 2^m + 2^m = 2^{m+1} .$$

*

The winners of this year's Olympiad are:

First	Jeffrey Grossman, Northern Secondary School, Toronto
Second	Ian Goldberg, University of Toronto Schools
Third	Hugh Thomas, Kelvin High School, Winnipeg
Fourth	Andrew Chow, Albert Campbell Collegiate, Scarborough
Fifth	Daniel Brown, Earl Haig Secondary School, Willowdale Peter Jong, Earl Haig Secondary School, Willowdale Eric Lai, Toronto French School Andre Chang, Woburn Collegiate Institute, Scarborough Gregory Ward, Sarnia Northern C.I. & V.S., Sarnia

* * *

Finally as promised we give the solutions to Murray Klamkin's Quickies from the June number of the Corner.

1. Completely factor the polynomial

$$(y + z + w - x)(z + w + x - y)(w + x + y - z)(x + y + z - w) - 16xyzw\ .$$

Solution. First note that the given polynomial is symmetric and homogeneous in x, y, z, w. Since the first term on the left can be rewritten as

$$(x+y-z-w+2z)(z+w-x-y+2y)(z+w-x-y+2x)(x+y-z-w+2w)\ ,$$

it follows immediately that $(x + y - z - w)$ is a factor. Then by symmetry, two other factors must be $(x + z - y - w)$ and $(x + w - y - z)$. This leaves one factor left of degree 1. By symmetry and homogeneity, this must be $(x + y + z + w)$.

2. Determine all solutions (a,b,c) in natural numbers satisfying the simultaneous Diophantine equations

$$a^2 = 2(b + c) \qquad \text{and} \qquad a^6 = b^6 + c^6 + 31(b^2 + c^2)\ .$$

Solution. From the first equation, a is even; while from the second equation $a > b$, c. Thus $4a > 2(b + c) = a^2$, so that a can only be 2. Then $b = c = 1$. (This problem is based on an idea of the late S.L. Greitzer.)

3. If a, b, c are the sides of a triangle of area F, prove that

$$[a^2 + (b - c)^2]^2 + [b^2 + (c - a)^2]^2 + [c^2 + (a - b)^2]^2 \geq 16F^2$$

and determine when there is equality.

Solution. Using the representation $a = y + z$, $b = z + x$, $c = x + y$, where $x = s - a \geq 0$, etc., s being the semiperimeter (see [1984: 46–48]), the inequality becomes

$$(y^2 + z^2)^2 + (z^2 + x^2)^2 + (x^2 + y^2)^2 \geq 4xyz(x + y + z)$$

or

$$(x^2 - yz)^2 + (y^2 - zx)^2 + (z^2 - xy)^2 \geq 0\ .$$

There is equality if and only if $x = y = z$, i.e. the triangle is equilateral.

Second solution. It suffices to show that

$$a^4 + b^4 + c^4 \geq 16F^2 = 2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4$$

or

$$(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0\ .$$

4. Prove that $re^{sx} + se^{-rx} \geq 1$ where $r + s = 1$ and r, $s \geq 0$.

Solution. The inequality is equivalent to

$$\frac{(e^{sx} - 1)}{s} \geq \frac{(1 - e^{-rx})}{r}$$

or

$$\int_0^x e^{st}dt \geq \int_0^x e^{-rt}dt .$$

The latter follows since $e^{st} \geq e^{-rt}$ for $t \geq 0$.

5. A sphere is tangent to the six edges of a given tetrahedron. Prove that the three segments joining pairs of opposite points of tangency are concurrent.

Solution. Let A_1, A_2, A_3, A_4 denote the vertices of the tetrahedron and let A_{ij} denote the points of tangency of the sphere with the edges A_iA_j. Now let $a_1 = A_1A_{12} = A_1A_{13} = A_1A_{14}$, $a_2 = A_2A_{23} = A_2A_{24} = A_2A_{21}$, $a_3 = A_3A_{34} = A_3A_{31} = A_3A_{32}$, and $a_4 = A_4A_{41} = A_4A_{42} = A_4A_{43}$ (note that tangents from an external point to a sphere are equal). Our solution is by centroids so we place weights $1/a_i$ at vertex A_i. The centroid of the weights at A_1 and A_2 is A_{12} and the centroid of the weights at A_3 and A_4 is at A_{34}. Hence the centroid of the four weights must lie on the segment joining A_{12} and A_{34}. Similarly, it must lie on the segments joining A_{13} to A_{24} and A_{14} to A_{23}. Since the centroid is unique, the three segments must be concurrent.

* * *

BOOK REVIEW

Japanese Temple Geometry Problems, by H. Fukagawa and D. Pedoe. The Charles Babbage Research Centre (P.O. Box 272, St. Norbert Postal Station, Winnipeg, Manitoba, Canada R3V 1L6), 1989. ISBN 0–919611–21–4. xvi + 206 pp., softcover. *Reviewed by Bill Sands, University of Calgary.*

Here is a book heartily recommended for any devotee of elementary geometry, or for anyone who enjoys a beautiful mathematical fact. It contains over 250 illustrated geometrical problems, divided into categories such as "circles", "circles and triangles", "ellipses", "spheres", etc., together with solutions for some of them. The problems were originally inscribed on wooden tablets and hung in temples in Japan, during a period of over two hundred years ending late in the last century. Readers of *Crux* will remember that several such problems have appeared in this journal in the last few years as proposals of Hidetosi Fukagawa, who has been engaged for some time in collecting them. The present book, a collaboration of Fukagawa and Professor Dan Pedoe of Minnesota, contains some problems which may be familiar for this reason. (And how nice it is to come across them in browsing!) The book also contains many others, however, of equal beauty. At the back, there are some photographs of tablets and temples, reproductions of the original proofs (in Japanese characters) for a couple of the problems, and two maps. There is also an appendix, a mind–blowing list of 100 figures, each a unit square containing at least one circle of radius 1/16.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **April 1, 1991,** *although solutions received after that date will also be considered until the time when a solution is published.*

1561. *Proposed by Murray S. Klamkin, University of Alberta.*

Determine an infinite class of integer triples (x,y,z) satisfying the Diophantine equation

$$x^2 + y^2 + z^2 = 2yz + 2zx + 2xy - 3.$$

1562. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let M be the midpoint of BC of a triangle ABC such that $\angle B = 2\angle C$, and let D be the intersection of the internal bisector of angle C with AM. Prove that $\angle MDC \le 45^\circ$.

1563*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $N \ge 2$. For each positive integer n the number $A_n > 1$ is implicitly defined by

$$1 = \sum_{k=1}^{n} \frac{1}{A_n k^N - 1}.$$

Show that the sequence $A_1, A_2, A_3, \cdots$ converges.

1564. *Proposed by Jordi Dou, Barcelona, Spain.*

Given three pairs of points (P,P'), (Q,Q'), (R,R'), each pair isogonally conjugate with respect to a fixed unknown triangle, construct the isogonal conjugate X' of a given point X.

1565. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

From the set of vertices of the n-dimensional cube choose three at random. Let p_n be the probability that they span a right-angled triangle. Find the asymptotic behavior of p_n as $n \to \infty$.

1566. *Proposed by K.R.S. Sastry, Addis Ababa, Ethiopia.*

Find all Heronian triangles ABC (i.e. with integer sides and area) such that the lengths OA, AH, OH are in arithmetic progression, where O and H are respectively the circumcenter and the orthocenter of ΔABC.

1567. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*

Let

$$f(x_1,x_2,\cdots,x_n) = \frac{x_1\sqrt{x_1 + \cdots + x_n}}{(x_1 + \cdots + x_{n-1})^2 + x_n}.$$

Prove that $f(x_1,x_2,\cdots,x_n) \leq \sqrt{2}$ under the condition that $x_1 + \cdots + x_n \geq 2$.

1568. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Show that

$$\sum \sin A \geq \frac{2}{\sqrt{3}}\left(\sum \cos A\right)^2$$

where the sums are cyclic over the angles A, B, C of an acute triangle.

1569. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Evaluate

$$\int_0^a \frac{\ln(1 + ax)}{1 + x^2}dx$$

where a is a constant.

1570. *Proposed by P. Penning, Delft, The Netherlands.*

In n-dimensional space it is possible to arrange $n + 1$ n-dimensional solid spheres of unit radius in such a way that they all touch one another. Determine the radius of the small solid sphere that touches all $n + 1$ of these spheres.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1439. [1989: 111] *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University.*

Prove that

$$\frac{1}{6}[(2+\sqrt{3})^{2n-3}+(2-\sqrt{3})^{2n-3}+2]$$

is a perfect square for all positive integers n.

I. *Solution by David Poole, Trent University, Peterborough, Ontario.*

The result is true for all integers n. Let $a = 2+\sqrt{3}$. Then $a^{-1} = 2-\sqrt{3}$ and we must show that

$$\frac{1}{6}(a^{2n-1}+a^{1-2n}+2) = \frac{(a^{2n-1}+1)^2}{6a^{2n-1}}$$

is a perfect square for all $n \in \mathbb{Z}$. Equivalently,

$$\frac{a^{2n-1}+1}{\sqrt{6}a^{n-1/2}} = \frac{a^{n-1/2}+a^{1/2-n}}{\sqrt{6}} \in \mathbb{Z} \qquad \forall n \in \mathbb{Z}\,. \tag{1}$$

Set

$$\beta = \sqrt{a} = \sqrt{2+\sqrt{3}} = \frac{\sqrt{6}+\sqrt{2}}{2}\,.$$

Then $\beta^{-1} = (\sqrt{6}-\sqrt{2})/2$ and (1) becomes

$$\frac{\beta^{2n-1}+(\beta^{-1})^{2n-1}}{\sqrt{6}} \in \mathbb{Z} \qquad \forall n \in \mathbb{Z}\,. \tag{2}$$

We claim that, for each $n \ge 1$, there exist integers a and b such that

$$\beta^{2n-1} = \frac{a\sqrt{6}+b\sqrt{2}}{2} \qquad \text{and} \qquad (\beta^{-1})^{2n-1} = \frac{a\sqrt{6}-b\sqrt{2}}{2}\,.$$

This is easily established by induction on n. For $n = 1$, it is clear. Assuming the statement to be true for n, we have

$$\beta^{2n+1} = \beta^{2n-1}a = \frac{1}{2}(a\sqrt{6}+b\sqrt{2})(2+\sqrt{3}) = \frac{(2a+b)\sqrt{6}+(3a+2b)\sqrt{2}}{2}\,.$$

Similarly,

$$(\beta^{-1})^{2n+1} = (\beta^{-1})^{2n-1}a^{-1} = \frac{1}{2}(a\sqrt{6}+b\sqrt{2})(2-\sqrt{3}) = \frac{(2a+b)\sqrt{6}-(3a+2b)\sqrt{2}}{2}\,.$$

This proves the claim.

It follows that for each $n \in \mathbb{Z}$,

$$\beta^{2n-1}+(\beta^{-1})^{2n-1} = a\sqrt{6}$$

for some integer a, which establishes (2).

Remark. Since

$$\beta = 2\cos\frac{\pi}{12} \qquad \text{and} \qquad \beta^{-1} = 2\sin\frac{\pi}{12},$$

(2) is equivalent to the statement that, for all *odd* integers k,

$$\frac{2^k}{\sqrt{6}}\left(\cos^k\left(\frac{\pi}{12}\right) + \sin^k\left(\frac{\pi}{12}\right)\right) \in \mathbb{Z} .$$

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We show more generally: *Let* $a \geq 2$ *and let* n *be an integer. Then*

$$x_n = \frac{1}{2(a+1)}[(a+\sqrt{a^2-1})^{2n+1} + (a-\sqrt{a^2-1})^{2n+1} + 2]$$

is a perfect square.

Case (i): $n \geq 0$. Then, because of

$$(a+\sqrt{a^2-1})(a-\sqrt{a^2-1}) = 1$$

and

$$a \pm \sqrt{a^2-1} = \left(\sqrt{\frac{a+1}{2}} \pm \sqrt{\frac{a-1}{2}}\right)^2$$

we get

$$\begin{aligned} x_n &= \left[\frac{(a+\sqrt{a^2-1})^{n+1/2} + (a-\sqrt{a^2-1})^{n+1/2}}{\sqrt{2(a+1)}}\right]^2 \\ &= \left[\frac{\sqrt{a+1}+\sqrt{a-1}}{2\sqrt{a+1}}(a+\sqrt{a^2-1})^n + \frac{\sqrt{a+1}-\sqrt{a-1}}{2\sqrt{a+1}}(a-\sqrt{a^2-1})^n\right]^2 \\ &= w_n^2 . \end{aligned}$$

Thus the sequence $(w_n)_{n\geq 0}$ satisfies the recursion with characteristic equation

$$(t-a)^2 - (a^2-1) = [t-(a+\sqrt{a^2-1})][t-(a-\sqrt{a^2-1})] = 0 ,$$

i.e., the recursion

$$w_{n+2} = 2aw_{n+1} - w_n , \qquad n \geq 0 .$$

Since

$$w_0 = \frac{\sqrt{a+1}+\sqrt{a-1}}{2\sqrt{a+1}} + \frac{\sqrt{a+1}-\sqrt{a-1}}{2\sqrt{a+1}} = 1$$

and

$$\begin{aligned} w_1 &= \frac{1}{2\sqrt{a+1}}[a\sqrt{a+1} + a\sqrt{a-1} + (a+1)\sqrt{a-1} + (a-1)\sqrt{a+1} \\ &\qquad + a\sqrt{a+1} - a\sqrt{a-1} - (a+1)\sqrt{a-1} + (a-1)\sqrt{a+1}] \\ &= 2a-1 , \end{aligned}$$

all w_n's are integral.

Case (ii): $n < 0$. Let $n = -k$, $k > 0$. Then, because of

$$(a \pm \sqrt{a^2 - 1})^{2n+1} = (a \pm \sqrt{a^2 - 1})^{-2k+1} = (a \mp \sqrt{a^2 - 1})^{2k-1} ,$$

we obtain $x_{-k} = x_{k-1}$.

III. *Solution by Hayo Ahlburg, Benidorm, Spain.*

In Euler's *Algebra*, part II, section 2, chapter 6, paragraph 91, problem V, the Fermat–Pell equation

$$3x^2 - 2 = y^2$$

is solved. The first values are

x	1	3	11	41	$\cdots$
y	1	5	19	71	$\cdots$

The general form of the solution according to Lagrange's procedure yields the *integers*

$$x = \frac{1}{2\sqrt{3}}[(2 + \sqrt{3})^m(1 + \sqrt{3}) - (2 - \sqrt{3})^m(1 - \sqrt{3})] ,$$

and so, with $m = n - 2$,

$$x^2 = \frac{1}{6}[(2 + \sqrt{3})^{2n-3} + (2 - \sqrt{3})^{2n-3} + 2] ,$$

q.e.d. See also my solution of *Crux* 1048 [1986: 247]. (Another cross–connection!)

Also solved by HARRY ALEXIEV, Zlatograd, Bulgaria; SEUNG–JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; GINGER BOLTON, Swainsboro, Georgia; JOEL L. BRENNER, Palo Alto, California, and LORRAINE L. FOSTER, California State University, Northridge; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; GUO–GANG GAO, student, Université de Montréal; RICHARD I. HESS, Rancho Palos Verdes, California; JILL HOUGHTON, Sydney, Australia; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; SAM MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; TAT Y. NGAI, University College of Cape Breton, Sydney, Nova Scotia; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; ANTONIO LUIZ SANTOS, Rio de Janeiro, Brazil; SHAILESH SHIRALI, Rishi Valley School, India; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; EICHI WATANABE, Rio de Janeiro, Brazil; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; and the proposers. There was one partial solution submitted.

The generalization in solution II *was also given by Penning. Several solvers pointed out that the given result is true for all integers n.*

The problem arose as a result of trying to solve problem 4 of the 1987 *Bulgarian Olympiad* [1989: 33].

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1440*. [1989: 111] *Proposed by Jack Garfunkel, Flushing, New York.*

Prove or disprove that if A, B, C are the angles of a triangle,

$$\frac{\sin A}{\sqrt{\sin A + \sin B}} + \frac{\sin B}{\sqrt{\sin B + \sin C}} + \frac{\sin C}{\sqrt{\sin C + \sin A}} \le \frac{3}{2}\cdot\sqrt[4]{3}\ .$$

Solution by Marcin E. Kuczma, Warszawa, Poland.

True.

By the Cauchy–Schwarz inequality,

$$\left(\sum \frac{\sin A}{\sqrt{\sin A + \sin B}}\right)^2 \le \left(\sum \frac{\sin A}{\sin A + \sin B}\right)\left(\sum \sin A\right)$$

(in cyclic sum notation), so it suffices to show that the product on the right does not exceed $9\sqrt{3}/4$. This can be restated as

$$2\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right) \le \frac{9\sqrt{3}R}{a+b+c} \tag{1}$$

with a, b, c and R the sides and circumradius of the triangle.

The difficulty lies in that the expression on the left of (1) is wildly asymmetric. The idea of the proof is to insert in between the left and right sides of (1) an expression which is easily comparable with the left side, and is symmetric, so that the task of comparing it with the right side can be handled in a rather standard way.

Let

$$x = \frac{3a-2s}{4s}\ , \qquad y = \frac{3b-2s}{4s}\ , \qquad z = \frac{3c-2s}{4s}\ ,$$

i.e.

$$a = \tfrac{2}{3}(1+2x)s\ , \qquad b = \tfrac{2}{3}(1+2y)s\ , \qquad c = \tfrac{2}{3}(1+2z)s\ ,$$

where $s = (a+b+c)/2$ is the semiperimeter. Since a, b, c are side lengths of a triangle, the numbers x, y, z belong to the interval $(-1/2,1/4)$, and of course $x+y+z = 0$. We shall prove

$$2\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right) \le 3 + x^2 + y^2 + z^2 \tag{2}$$

and

$$3 + x^2 + y^2 + z^2 \le \frac{9\sqrt{3}R}{a+b+c}\ . \tag{3}$$

In x, y, z – notation, the left hand side of (2) takes on the form

$$\mathrm{LHS}(2) \;=\; \frac{1 + 2x}{1 - z} + \frac{1 + 2y}{1 - x} + \frac{1 + 2z}{1 - y}\,.$$

For $x < 1/4$ we have

$$\frac{1}{1 - x} \;\le\; 1 + x + \frac{4x^2}{3}$$

and hence (in view of $y < 1/4$)

$$\begin{aligned}\frac{1 + 2y}{1 - x} &\le 1 + x + 2y + 2xy + \tfrac{4}{3}x^2 + \tfrac{8}{3}x^2y \\ &\le 1 + x + 2y + 2xy + 2x^2.\end{aligned}$$

By cyclic shift $x \to y \to z \to x$ we obtain two analogous inequalities. Summing, we get (on account of $x + y + z = 0$)

$$\begin{aligned}\mathrm{LHS}(2) &\le 3 + 2xy + 2yz + 2zx + 2x^2 + 2y^2 + 2z^2 \\ &= 3 + x^2 + y^2 + z^2 + (x + y + z)^2 = 3 + x^2 + y^2 + z^2\,,\end{aligned}$$

proving (2).

For a proof of (3) we write

$$u = 3\left(1 - \frac{a}{s}\right)\,, \qquad v = 3\left(1 - \frac{b}{s}\right)\,, \qquad w = 3\left(1 - \frac{c}{s}\right)\,,$$

$$p = vw + wu + uv\,, \qquad q = uvw\,.$$

Then u, v, $w > 0$, $u + v + w = 3$,

$$x = \frac{1 - u}{4}\,, \qquad y = \frac{1 - v}{4}\,, \qquad z = \frac{1 - w}{4}\,,$$

$$a = \left(1 - \frac{u}{3}\right)s\,, \qquad b = \left(1 - \frac{v}{3}\right)s\,, \qquad c = \left(1 - \frac{w}{3}\right)s\,,$$

and

$$\begin{aligned}x^2 + y^2 + z^2 &= \frac{1}{16}[(1 - 2u + u^2) + (1 - 2v + v^2) + (1 - 2w + w^2)] \\ &= \frac{1}{16}(3 - 6 + u^2 + v^2 + w^2) \\ &= \frac{1}{16}[-3 + (u + v + w)^2 - 2(vw + wu + uv)] \\ &= \frac{3 - p}{8}\,.\end{aligned}$$

Hence

$$\mathrm{LHS}(3) = \frac{27 - p}{8}\,.$$

Further,

$$\begin{aligned}R &= \frac{abc}{4\sqrt{s(s - a)(s - b)(s - c)}} = \frac{(1 - u/3)(1 - v/3)(1 - w/3)s^3}{4\sqrt{s(us/3)(vs/3)(ws/3)}} \\ &= \frac{(3 - u)(3 - v)(3 - w)s}{4\sqrt{27uvw}} = \frac{(3^3 - 3^2\cdot 3 + 3p - q)s}{4\sqrt{27q}}\end{aligned}$$

$$= \frac{(3p - q)s}{4\sqrt{27q}} .$$

Thus

$$\text{RHS}(3) = \frac{3(3p - q)}{8\sqrt{q}} .$$

The inequality (3), which is to be proved, becomes

$$(27 - p)\sqrt{q} \le 3(3p - q) ,$$

or equivalently

$$3\sqrt{q}(9 + \sqrt{q}) \le p(9 + \sqrt{q}) ,$$

i.e.

$$\sqrt{q} \le \tfrac{1}{3}p . \tag{4}$$

In the well-known inequality

$$\sqrt{\frac{lm + mk + kl}{3}} \le \frac{k + l + m}{3} \qquad (k,l,m > 0)$$

set

$$k = \sqrt{\frac{vw}{u}} , \qquad l = \sqrt{\frac{wu}{v}} , \qquad m = \sqrt{\frac{uv}{w}}$$

(so that $lm = u$, $mk = v$, $kl = w$ and thus $lm + mk + kl = 3$), and multiply both sides of the resulting inequality by $\sqrt{uvw}$. The result is (4).

Thus (2) and (3) are settled, and (1) follows.

Remark. It is seen from the proof that each of the inequalities (2), (3), (4) becomes an equality in the case of an equilateral triangle alone. Hence, the same concerns (1) and the inequality posed in the problem.

[*Editor's comment.* This inequality was inspired by Walther Janous's inequality *Crux* 1366 [1988: 202], and with it yields item 2.5 of Bottema et al, *Geometric Inequalities.*]

* * *

1441*. [1989: 147] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let

$$S_n = (x - y)^n + (y - z)^n + (z - x)^n.$$

It is easy to see that if p is a prime, S_p/p is a polynomial with integer coefficients. Prove that

$$\frac{S_2}{2} \mid S_{2+6k} , \quad \frac{S_3}{3} \mid S_{3+6k} ,$$

$$\frac{S_5}{5} \mid S_{5+6k} , \quad \frac{S_7}{7} \mid S_{7+6k} ,$$

for all $k = 1,2,3,\cdots$, where $\mid$ denotes polynomial divisibility.

Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany.

Since $S_1 = 0$,

$$S_{n+3} = \frac{1}{3}S_3S_n + \frac{1}{2}S_2S_{n+1} \tag{1}$$

(see equation (1) in solution II of *Crux* 639 [1982: 145]). Furthermore

$$\frac{1}{5}S_5 = \frac{1}{3}S_3\cdot\frac{1}{2}S_2\ , \tag{2}$$

$$\frac{1}{7}S_7 = \frac{1}{5}S_5\cdot\frac{1}{2}S_2\ , \tag{3}$$

$$S_4 = \frac{1}{2}S_2\cdot S_2\ . \tag{4}$$

From (1) follows that

$$\frac{1}{2}S_2 \mid S_n \implies \frac{1}{2}S_2 \mid S_{n+3}\ .$$

Since from (4)

$$\frac{1}{2}S_2 \mid S_2 \quad \text{and} \quad \frac{1}{2}S_2 \mid S_4\ ,$$

we have that

$$\frac{1}{2}S_2 \mid S_{1+3m} \quad \text{and} \quad \frac{1}{2}S_2 \mid S_{2+3m}\ , \qquad m = 1,2,\cdots\ . \tag{5}$$

From (1) follows that

$$\frac{1}{3}S_3 \mid S_n \implies \frac{1}{3}S_3 \mid S_{n+2}\ . \tag{6}$$

Thus

$$\frac{1}{3}S_3 \mid S_{1+2m}\ , \qquad m = 0,1,2,\cdots\ .$$

From (1) follows that

$$\begin{aligned} S_{n+6} &= \frac{1}{3}S_3S_{n+3} + \frac{1}{2}S_2S_{n+4} \\ &= \left(\frac{1}{3}S_3\right)^2S_n + \frac{1}{3}S_2S_3S_{n+1} + \left(\frac{1}{2}S_2\right)^2S_{n+2}\ . \end{aligned} \tag{7}$$

Thus from (7), using (2) and (6),

$$\frac{1}{5}S_5 \mid S_n \implies \frac{1}{5}S_5 \mid S_{n+6}\ .$$

Therefore

$$\frac{1}{5}S_5 \mid S_{5+6k}.$$

From (2) and (3),

$$\frac{1}{7}S_7 = \left(\frac{1}{2}S_2\right)^2\cdot\frac{1}{3}S_3\ .$$

Thus from (6),

$$\frac{1}{7}S_7 \mid S_n \implies \frac{1}{7}S_7 \;\Big|\; \left(\frac{1}{2}S_2\right)^2S_{n+2}\ ,$$

and from (5), for $n \not\equiv 2 \bmod 3$,

$$\tfrac{1}{7}S_7 \;\Big|\; \tfrac{1}{3}S_2S_3S_{n+1} \;.$$

Therefore by (7), for $n \not\equiv 2 \bmod 3$,

$$\tfrac{1}{7}S_7 \mid S_n \;\Rightarrow\; \tfrac{1}{7}S_7 \mid S_{n+6} \;,$$

and so

$$\tfrac{1}{7}S_7 \mid S_{1+6k} \;.$$

[*Editor's note.* Equation (1), which Engelhaupt found in a published solution (by Klamkin) of *Crux* 639, follows from *Crux* 514 [1981: 56]. Equation (2) is in fact the statement of *Crux* 639, and equations (3) and (4) are also mentioned in the same solution referred to above. Actually, all three equations have been known for over a hundred years; an 1887 reference is given in the solutions of both *Crux* 514 and *Crux* 639. In his solution to *Crux* 639, Klamkin mentions a related problem he put on the 1982 U.S.A. Mathematical Olympiad (2, [1982: 165]). See pp. 25–28 of M.S. Klamkin, *U.S.A. Mathematical Olympiads 1972–1986*, New Mathematical Library Vol. 33, M.A.A., 1988, for a solution and further discussion of this problem. Here Klamkin also points out another related problem, *Crux* 769 [1983: 283].]

Also solved by HARRY ALEXIEV, Zlatograd, Bulgaria; MURRAY S. KLAMKIN, University of Alberta; ROBERT E. SHAFER, Berkeley, California; and KENNETH M. WILKE, Topeka, Kansas.

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<u>**1443**</u>. [1989: 148] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Given an integer $n \geq 2$, determine the minimum value of

$$\sum_{\substack{1\le i,\, j\le n \\ i\ne j}} \left(\frac{x_i^2}{x_j}\right)$$

over all positive real numbers $x_1,\cdots,x_n$ such that $x_1^2 + \cdots + x_n^2 = 1$.

Solution by Vedula N. Murty, Pennsylvania State University at Harrisburg.

It is easily verified that

$$\sum_{\substack{1\le i,\, j\le n \\ i\ne j}} \left(\frac{x_i^2}{x_j}\right) = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} x_i \;. \qquad (1)$$

Also, $x_1^2 + \cdots + x_n^2 = 1$ implies

$$x_1 + x_2 + \cdots + x_n \leq \sqrt{n} \;,$$

with equality when $x_1 = x_2 = \cdots = x_n = 1/\sqrt{n}$. From the Arithmetic Mean–Harmonic Mean inequality

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} \frac{1}{x_i} \geq n^2 ,$$

we obtain

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n^2}{x_1 + \cdots + x_n} \geq \frac{n^2}{\sqrt{n}} = n\sqrt{n} ,$$

and therefore

$$\sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} x_i \geq (n-1)\sqrt{n} .$$

Thus the minimum is $(n-1)\sqrt{n}$, and is attained when $x_1 = x_2 = \cdots = x_n = 1/\sqrt{n}$.

Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; and the proposer.

Klamkin also observed (1), *and used the power mean inequality to obtain the minimum value of*

$$\sum_{i=1}^{n} x_i^r \sum_{i=1}^{n} x_i^{-s} - \sum_{i=1}^{n} x_i^t$$

over all positive real numbers $x_1,\cdots,x_n$ *subject to*

$$\sum_{i=1}^{n} x_i^p = na^p ,$$

where $r \geq p \geq t > 0$, $s > 0$, $a > 0$ *are given.*

* * *

<u>1444</u>. [1989: 148] *Proposed by Jordi Dou, Barcelona, Spain.*

Given the centre O of a conic γ and three points A, B, C lying on γ, construct those points X on γ such that XB is the bisector (interior or exterior) of $\angle AXC$.

Solution by Dan Pedoe, Minneapolis, Minnesota.

We immediately obtain three more points A', B' and C' on γ, where $OA = OA'$, $OB = OB'$, and $OC = OC'$. This is one more point than we need for the construction. We use the notion of projective ranges on a conic, and make use especially of the Pascal Theorem. For all this see [1].

We first find the point D on γ such that the cross–ratio $[A,C;B,D] = -1$, so that the four points A, C, B, D form a harmonic range. If they do, then AC is conjugate to BD, and the pole of AC lies on BD. Pascal's Theorem enables us to find, using only an unmarked ruler, the further intersection with γ of any line which passes through a point on γ, and in particular enables us to find the tangent at A to γ, and the tangent at C to γ. If these tangents intersect at P, then BP intersects γ again in the required point D, which we can find, and $[A,C;B,D] = -1$. If we join the four points A, C, B, D to any point of γ, we obtain a harmonic pencil.

If we can now find a point X on γ such that $XB \perp XD$, then, the pencil of rays $X\{A,C;B,D\}$ being harmonic, XB and XD will be bisectors of $\angle AXC$ (see p. 319 of [1]).

We draw the circle Γ on BD as diameter, and show how to find the points X, X' where Γ intersects the conic γ again, using only an unmarked ruler. For simplicity, we change the notation.

The five points P_i $(i = 1,\cdots,5)$ lie on a conic γ, and Γ is the circle on P_4P_5 as diameter. We want to find where Γ intersects γ again.

If we join P_4 and P_5 respectively to a point P on γ, and the further intersection of P_4P with Γ is Q, and the further intersection of P_5P with Γ is R, then $Q = R$ if and only if Q is an intersection of Γ and γ outside the points P_4 and P_5.

We join P_4 and P_5 respectively to the points P_1, P_2 and P_3 on γ, obtaining points Q_1, Q_2, Q_3 on Γ from P_4, and R_1, R_2, R_3 on Γ from P_5, and we have a projectivity defined by $\{Q_1,Q_2,Q_3,\cdots\} \barwedge \{R_1,R_2,R_3,\cdots\}$ on Γ. The fixed points of this projectivity are X and X', the further intersections, outside P_4 and P_5, of Γ and γ.

The cross–axis of this projectivity intersects Γ in X and X', and this cross–axis is the Pascal line on which the three points

$$Q_1R_2 \cap R_1Q_2 \ , \qquad Q_2R_3 \cap R_2Q_3 \ , \qquad Q_3R_1 \cap R_3Q_1$$

lie, so that with the exception of the use of a compass for drawing Γ, the whole construction is possible with the help of an unmarked ruler.

Reference:

[1] Dan Pedoe, *Geometry: A Comprehensive Course*, Dover Publications, 1988.

Also solved (in a similar way) by the proposer.

1445. [1989: 148] *Proposed by M.S. Klamkin and Andy Liu, University of Alberta.*

Determine the minimum value of

$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{z^3}{1-z^8}$$

where $x, y, z \geq 0$ and $x^4+y^4+z^4=1$.

Solution by the proposers.

More generally, we will determine the minimum value of

$$\frac{x_1^r}{1-x_1^s}+\frac{x_2^r}{1-x_2^s}+\cdots+\frac{x_n^r}{1-x_n^s}, \qquad (1)$$

where $x_1,\cdots,x_n \geq 0$, $x_1^m+\cdots+x_n^m=1$, and

$$m=\alpha(n^k-1)>r=m-k\alpha>0\ , \qquad s=km>0\ ,$$

α and k being arbitrary positive parameters. It will turn out that in every such case the minimum of the above expression occurs when $x_1=x_2=\cdots=x_n$.

It follows easily (by differentiation) that

$$x^{m-r}(1-x^s)$$

takes on its maximum in [0,1] when

$$m-r=(m-r+s)x^s\ ,$$

i.e. for

$$x^{km}=x^s=\frac{m-r}{m-r+s}=\frac{k\alpha}{k\alpha+km}=\frac{\alpha}{\alpha n^k}=n^{-k}\ ,$$

or

$$x=n^{-1/m}.$$

Thus for this value of x,

$$x_1^{m-r}(1-x_1^s)\leq x^{m-r}(1-x^s)\ , \qquad \text{etc.,}$$

i.e.

$$\frac{x_1^r}{1-x_1^s}\geq\frac{x^{r-m}}{1-x^s}\cdot x_1^m\ , \qquad \text{etc.,}$$

so that

$$\sum_{i=1}^{n}\frac{x_i^r}{1-x_i^s}\geq\frac{x^{r-m}}{1-x^s}\sum_{i=1}^{n}x_i^m=\frac{x^{r-m}}{1-x^s}\ , \qquad (2)$$

equality holding when $x_i=x$ for all i.

The given problem corresponds to the special case $n=3$, $k=2$, $\alpha=1/2$, so that $m=4$, $r=3$, $s=8$, and with $x=3^{-1/4}$ (2) yields

$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{z^3}{1-z^8}\geq\frac{3^{1/4}}{1-3^{-2}}=\frac{9\sqrt[4]{3}}{8}\ .$$

The special case corresponding to $m = s = r + 1 = 2$ (i.e. $k = \alpha = 1$) appeared as problem 1067 in *Kvant* (1988, p.29), proposed by V.E. Matizen.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and KEE–WAI LAU, Hong Kong. These solutions were all much more complicated than the proposers' solution!

Janous wonders for which r, s, m the minimum of (1), *under the condition* $x_1^m + \cdots + x_n^m = 1$, *occurs when the* x_i*'s are all equal. Some times when this happens are given in the above solution: are there others?*

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<u>1446</u>. [1989: 148] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be an equilateral triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. We denote by G', G the centroids, by O', O the circumcenters, by I', I the incenters and by H', H the orthocenters of triangles $A'B'C'$ and ABC respectively. Prove that in each of the four cases

(a) $G = G'$, (b) $O = O'$, (c) $I = I'$, (d) $H = H'$,

ABC must be equilateral.

I. *Solution to* (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Since A', B', C' are on BC, CA, AB respectively, there exist $0 \le \lambda, \mu, \nu \le 1$ such that

$$\mathbf{A}' = \lambda\mathbf{B} + (1-\lambda)\mathbf{C}\ , \qquad \mathbf{B}' = \mu\mathbf{C} + (1-\mu)\mathbf{A}\ , \quad \mathbf{C}' = \nu\mathbf{A} + (1-\nu)\mathbf{B}$$

(where $\mathbf{X}$ stands for the vector from the origin to the point X). Therefore

$$\begin{aligned}\mathbf{G}' &= \tfrac{1}{3}(\mathbf{A}' + \mathbf{B}' + \mathbf{C}') \\ &= \tfrac{1}{3}[\mathbf{A}(1+\nu-\mu) + \mathbf{B}(1+\lambda-\nu) + \mathbf{C}(1+\mu-\lambda)]\ ,\end{aligned}$$

and $G' = G$ is equivalent to

$$\mathbf{A}(\nu-\mu) + \mathbf{B}(\lambda-\nu) + \mathbf{C}(\mu-\lambda) = \mathbf{0}\ . \tag{1}$$

If at least two of λ, μ, ν are equal, say $\nu = \mu$, then (1) yields

$$\mathbf{C}(\mu-\lambda) = \mathbf{B}(\mu-\lambda)\ .$$

From this we get (since $B \neq C$) that $\lambda = \mu$, i.e. A', B', C' divide the respective sides of ΔABC in the same ratio. By *Crux* 1437 [1990: 187] we infer ΔABC is equilateral.

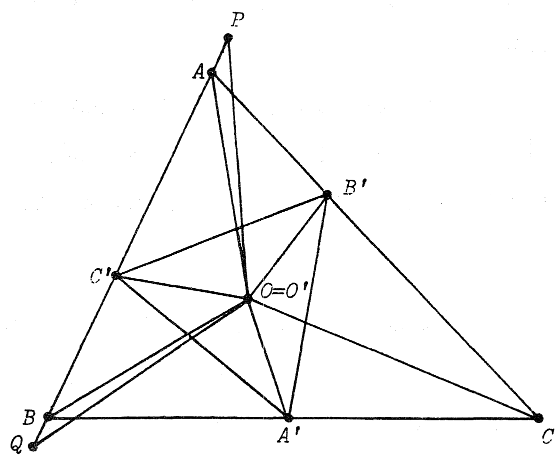
If λ, μ, ν are mutually different, then (1) also reads $\mathbf{C} = \mathbf{A}\eta + \mathbf{B}\xi$, where

$$\eta = \frac{\mu - \nu}{\mu - \lambda}, \qquad \xi = \frac{\nu - \lambda}{\mu - \lambda},$$

and $\eta + \xi = 1$; i.e. C lies on the line through A and B, which obviously is impossible.

II. *Solution to* (b) *and* (d), *and counterexample to* (c), *by Toshio Seimiya, Kawasaki, Japan.*

[Seimiya first proved part (a). –Ed.]



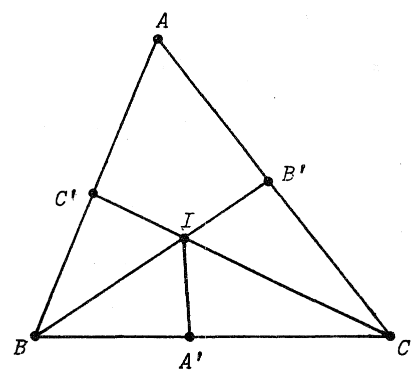
(b) We shall prove this by the indirect method. If ΔABC is not equilateral, then we have $\angle C < 60^\circ$, where without loss of generality we assume that $\angle C$ is the least angle of $\angle A$, $\angle B$, $\angle C$. As O' is an interior point of $\Delta A'B'C'$, $O = O'$ is also an interior point of ΔABC. Then $\angle AOB = 2\angle ACB < 120^\circ$, $\angle OAB = \angle OBA > 30^\circ$. Hence we can take points P, Q on AB produced such that $\angle OPC' = 30^\circ = \angle OQC'$, as shown in the figure. As $\angle OPC' = 30^\circ = \angle OB'C'$, points C', O, B' and P lie on a circle. As A is a point in the chord $C'P$, A is an interior point of this circle, lying on the same side of the line OB' as C', therefore

$$\angle OCA = \angle OAB' > \angle OC'B' = 30^\circ . \tag{2}$$

Similarly, using point Q,

$$\angle OCB > 30^\circ . \tag{3}$$

From (2) and (3), we get $\angle ACB > 60^\circ$, which contradicts the assumption $\angle C < 60^\circ$. Therefore ΔABC must be equilateral.



(c) In this case the conclusion is not correct. We shall give a counterexample. Let ABC be a triangle such that $\angle A = 60^\circ$ and $AB \neq AC$. Let B', C' be the intersections of the bisectors of $\angle B$, $\angle C$, respectively, with the opposite sides. Then I is the intersection of BB' and CC', and

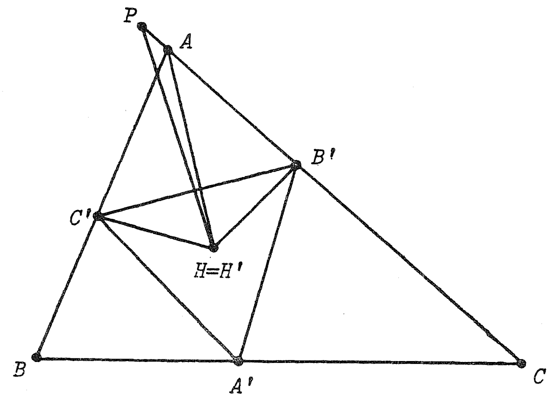
$$\begin{aligned}\angle C'IB = \angle B'IC &= \angle IBC + \angle ICB \\ &= \tfrac{1}{2}(\angle ABC + \angle ACB) = 60^\circ .\end{aligned}$$

Draw the bisector of $\angle BIC$ meeting BC at A'. As $\angle C'IB = 60^\circ$, we get $\angle BIA' = \angle A'IC = 60^\circ$. Then we have $\Delta BIC' \equiv \Delta BIA'$ and $\Delta CIB' \equiv \Delta CIA'$. Therefore we get $IC' = IA' = IB'$ and

$$\angle C'IA' = \angle A'IB' = \angle B'IC' = 120^\circ .$$

Hence $\Delta A'B'C'$ is equilateral, and I is its incenter. But ΔABC is not equilateral.

(d) We shall prove this by the indirect method. If ΔABC is not equilateral, then we have $\angle C < 60^\circ$, where without loss of generality we assume that $\angle C$ is the least angle of $\angle A$, $\angle B$, $\angle C$. As $H' = H$ is an interior point of $\Delta A'B'C'$, H is also an interior point of ΔABC. As $\angle C < 60^\circ$ and $AH \perp BC$, we get $\angle HAC > 30^\circ$. Therefore we can take a point P on AB' produced such that $\angle HPB' = 30^\circ$, as shown in the figure. As $\angle HPB' = 30^\circ = \angle HC'B'$, points C', H, B' and P are concyclic, and A is a point in the chord $B'P$, so we have



$$\angle BAH = \angle C'AH > \angle C'B'H = 30^\circ .$$

Then from $AH \perp BC$ we get

$$\angle ABC = 90^\circ - \angle BAH < 60^\circ . \tag{4}$$

Similarly,

$$\angle BAC < 60^\circ . \tag{5}$$

From $\angle C < 60^\circ$ and (4), (5) above we obtain $\angle C + \angle B + \angle A < 180^\circ$. This is a contradiction. Hence ΔABC must be equilateral.

Parts (a), (b), *and* (d) *also solved by the proposer, part* (a) *as in Solution* I *above. The proposer's "proof" of part* (c) *of course turned out to be incorrect.*

* * *

<u>1447</u>. [1989: 148] *Proposed by Henjin Chi and Raymond Killgrove, Indiana State University, Terre Haute.*

For each natural number n, how many integer–sided right triangles are there such that the area is n times the perimeter? How many of these are *primitive* (the sides have no common factor)?

I. *Solution by the proposers.*

The first question is equivalent to the solution of

$$n(x + y + \sqrt{x^2 + y^2}) = \frac{xy}{2} , \tag{1}$$

where x and y are positive integers. From (1) we obtain

$$\begin{aligned} 4n^2(x^2 + y^2) &= (xy - 2nx - 2ny)^2 \\ &= x^2y^2 + 4n^2x^2 + 4n^2y^2 - 4nx^2y - 4nxy^2 + 8n^2xy , \end{aligned}$$

or

$$y = \frac{4nx - 8n^2}{x - 4n} = 4n + \frac{8n^2}{x - 4n} .$$

Since $x - 4n$ must divide $8n^2$, all divisors d of $8n^2$ will give a solution:

$$x = 4n + d , \qquad y = 4n + \frac{8n^2}{d} .$$

If we only allow $d < \sqrt{8n^2}$ (note $8n^2$ is not a square) then these solutions will all be distinct under exchange of x and y. Thus there are $\tau(8n^2)/2$ integer–sided right triangles with the area n times the perimeter. (Here $\tau(m)$ is the number of divisors of m.)

For such a triangle to be primitive, x and y must be relatively prime. Thus each prime factor of $8n^2$ must only be a factor of x or of y, not of both. Without loss of generality, x is even and y is odd. Then the number of distinct solutions is 2^k, where

$$8n^2 = 2^{j_0}p_1^{j_1}p_2^{j_2}\cdots p_k^{j_k}$$

is the prime factorization of $8n^2$.

II. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

We first consider primitive triangles. Let x and y be the sides and z the hypotenuse. Then the well–known general solution is

$$x = a^2 - b^2 , \qquad y = 2ab , \qquad z = a^2 + b^2 ,$$

where a and b are relatively prime positive integers of opposite parity. The area A and the perimeter P are

$$A = ab(a^2 - b^2) , \qquad P = 2a(a + b) ,$$

and their ratio is

$$\frac{A}{P} = \frac{b(a - b)}{2} = n = 2^r p_1^{s_1} p_2^{s_2}\cdots p_k^{s_k} ,$$

where $p_1, p_2, \cdots, p_k$, and possibly 2, are the prime factors of n. Since $a - b$ is odd and n is an integer, it is clear that b must be even and must contain 2^{r+1} as a factor. The remaining prime factors $p_1, p_2, \cdots$ of n must each be a factor of b or $a - b$, but not both, since they have no common factors. Thus it is clear that the number of primitive triangles with $A/P = n$ is 2^k, where k is the number of distinct odd prime factors of n.

We now consider triangles whose sides have a common factor K (which may or may not equal 1). The general solution is

$$x = K(a^2 - b^2) , \qquad y = 2Kab , \qquad z = K(a^2 + b^2) , \tag{2}$$

where a and b are again relatively prime and of opposite parity. The area and perimeter are

$$A = K^2ab(a^2 - b^2) , \qquad P = 2Ka(a + b) ,$$

and their ratio is

$$\frac{A}{P} = \frac{Kb(a - b)}{2} = n = 2^r p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} ,$$

so we have

$$Kb(a - b) = 2^{r+1} p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} .$$

In this case the powers of 2 may be distributed between b and K; there are therefore $r + 2$ ways to do this. Furthermore, the powers of each odd prime factor may be distributed between K and b, or between K and $a - b$, but not both. There are $2s_i + 1$ ways of doing so for each prime factor p_i, so the number of triangles with $A/P = n$ is

$$(r + 2)(2s_1 + 1)(2s_2 + 1)\cdots(2s_k + 1) .$$

III. *Editor's comment.*

This was one of those rare occasions when the majority of solvers agreed on an answer which turned out to be wrong! Namely, for the total number of triangles (primitive or not) satisfying the problem most solvers obtained $\tau(2n)$, due to a faulty recollection of the general solution (2) of a Pythagorean triangle. Thus when $n = 3$ most solvers claimed $\tau(6) = 4$ solutions (one solver even listed them), while the correct answer (using the proposers' solution $\tau(72)/2$) is 6, namely:

$$\begin{aligned} x &= 14 , & y &= 48 , & z &= 50 \\ x &= 20 , & y &= 21 , & z &= 29 \\ x &= 16 , & y &= 30 , & z &= 34 \\ x &= 18 , & y &= 24 , & z &= 30 \\ x &= 13 , & y &= 84 , & z &= 85 \\ x &= 15 , & y &= 36 , & z &= 39 . \end{aligned}$$

The only reader to obtain the correct answers to both parts was Kierstead (it is easy to check that solutions I and II agree), but even so he made a small slip in the general case which has been corrected free of charge by the editor. He also (along with most solvers) did not say anything about distinctness of solutions.

Also solved (the primitive case only) by C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario. Two incorrect solutions were also received.

The proposers and Wang mention the article "Right triangles with perimeter and area equal" by W. Parsons, College Math. Journal 15 (1984) 429, *where it is* *suggested that for each n there is at least one primitive Pythagorean triangle whose area equals n times its perimeter. Parsons refers to problem* 3587 *of* School Science and Mathematics 76 (1976) 83–84 *for a proof. Another solution is given by Wiener, Chi, and Poorkarimi in "Involutions and problems involving perimeter and area",* College Math. Journal 19 (1988) 250–252, *an article also cited by the proposers and having much in common with their solution.*

Wang also mentions the old problem of finding all integer–sided triangles with area equal to perimeter (for a history see the solution of his problem E2420 *in the* Amer. Math. Monthly 81 (1974) 662–663).

* * *

<u>1448</u>. [1989: 149] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of a triangle, prove that

$$\frac{2}{3}\left(\sum \sin\frac{A}{2}\right)^2 \ge \sum \cos A\ ,$$

with equality when $A = B = C$.

Comment by M.S. Klamkin, University of Alberta.

By considering the degenerate triangle of angles 0, 0, π, it follows that the inequality is not valid for all triangles. However, it appears to be valid for non–obtuse triangles.

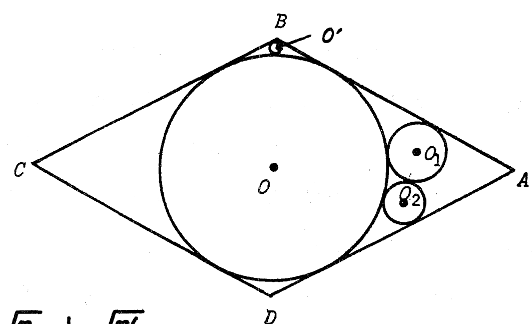
The above counterexample was also found by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incorrect solution sent in, besides of course the proposer's "proof", which employed a false inequality at one step.

Can anyone prove Klamkin's conjecture that the inequality holds for acute triangles?

* * *

<u>1450</u>. [1989: 149] *Proposed by H. Fukagawa, Yokosuka High School, Aichi, Japan.*

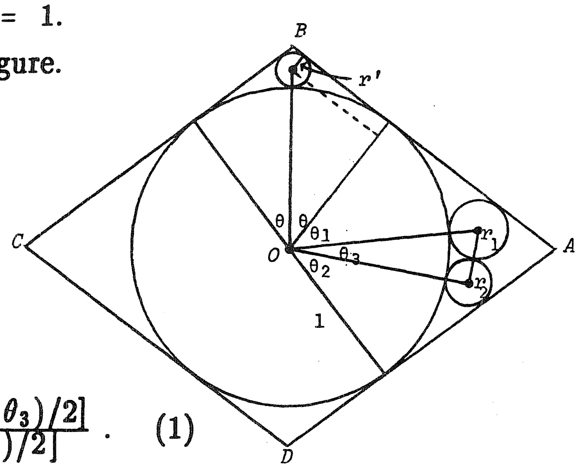
There is a rhombus $ABCD$ and inscribed circles $O(r)$ and $O'(r')$ as shown in the figure. We draw any two circles $O_1(r_1)$ and $O_2(r_2)$ touching the sides AB and AD respectively, and also touching the circle $O(r)$ and each other. Show the simple relation



$$\sqrt{r_1 + r_2 + r} = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r'}\ .$$

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Without loss of generality let $r = 1$. Label angles θ, θ_1, θ_2, θ_3 as in the figure. Then



$$\cos\,\theta = \frac{1 - r'}{1 + r'},$$

so

$$r' = \frac{1 - \cos\,\theta}{1 + \cos\,\theta} = \frac{(1 - \cos\,\theta)^2}{\sin^2\theta},$$

and since $\theta = 90^\circ - \frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$,

$$\sqrt{r'} = \frac{1 - \cos\,\theta}{\sin\,\theta} = \frac{1 - \sin[(\theta_1 + \theta_2 + \theta_3)/2]}{\cos[(\theta_1 + \theta_2 + \theta_3)/2]}. \quad (1)$$

Next

$$\cos\,\theta_1 = \frac{1 - r_1}{1 + r_1} = 1 - \frac{2r_1}{1 + r_1},$$

so

$$\sin\frac{\theta_1}{2} = \sqrt{\frac{r_1}{1 + r_1}}, \qquad \cos\frac{\theta_1}{2} = \sqrt{\frac{1}{1 + r_1}}. \quad (2)$$

Similarly

$$\sin\frac{\theta_2}{2} = \sqrt{\frac{r_2}{1 + r_2}}, \qquad \cos\frac{\theta_2}{2} = \sqrt{\frac{1}{1 + r_2}}. \quad (3)$$

Finally from

$$(r_1 + r_2)^2 = (1 + r_1)^2 + (1 + r_2)^2 - 2(1 + r_1)(1 + r_2)\cos\,\theta_3$$

we get

$$\cos\,\theta_3 = \frac{(1 + r_1)^2 + (1 + r_2)^2 - (r_1 + r_2)^2}{2(1 + r_1)(1 + r_2)} = \frac{1 + r_1 + r_2 - r_1r_2}{(1 + r_1)(1 + r_2)}$$

$$= 1 - \frac{2r_1r_2}{(1 + r_1)(1 + r_2)}$$

and hence

$$\sin\frac{\theta_3}{2} = \sqrt{\frac{r_1r_2}{(1 + r_1)(1 + r_2)}}, \qquad \cos\frac{\theta_3}{2} = \sqrt{\frac{1 + r_1 + r_2}{(1 + r_1)(1 + r_2)}}. \quad (4)$$

Now from (2), (3) and (4),

$$\sin\left(\frac{\theta_1 + \theta_2 + \theta_3}{2}\right) = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$
$$+ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}$$
$$= \frac{(\sqrt{r_1} + \sqrt{r_2})\sqrt{1 + r_1 + r_2} + \sqrt{r_1r_2} - r_1r_2}{(1 + r_1)(1 + r_2)}$$

and

$$\cos\left(\frac{\theta_1 + \theta_2 + \theta_3}{2}\right) = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\theta_3}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\theta_3}{2}$$
$$- \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_3}{2} - \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\theta_3}{2}$$

$$= \frac{(1 - \sqrt{r_1r_2})\sqrt{1 + r_1 + r_2} - r_1\sqrt{r_2} - r_2\sqrt{r_1}}{(1 + r_1)(1 + r_2)} .$$

So from (1),

$$\sqrt{r'} = \frac{1 + r_1 + r_2 + 2r_1r_2 - (\sqrt{r_1} + \sqrt{r_2})\sqrt{1 + r_1 + r_2} - \sqrt{r_1r_2}}{(1 - \sqrt{r_1r_2})\sqrt{1 + r_1 + r_2} - r_1\sqrt{r_2} - r_2\sqrt{r_1}}$$

$$= \sqrt{1 + r_1 + r_2} - \sqrt{r_1} - \sqrt{r_2} ,$$

and the result follows.

Also solved by P. PENNING, Delft, The Netherlands; and the proposer.

The problem was taken from the 1823 *Japanese mathematics book* Zoku Sanagaku Shosen.

* * *

REQUESTS FROM THE EDITOR

An old friend of *Crux*, Dr. Hayo Ahlburg, has recently written to me asking where he might obtain copies of the following three books (all familiar to *Crux* readers):

N. Altshiller–Court, *College Geometry*, Barnes & Noble, 1952

R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960

E.W. Hobson, *Plane Trigonometry*, Cambridge Univ. Press, republished as *A Treatise on Plane and Advanced Trigonometry*, Dover, 1957.

Anyone with suggestions that might be helpful to Dr. Ahlburg could write to him at **Apartado 35, Benidorm (Alicante), Spain**. I'm sure that Dr. Ahlburg would welcome hearing from other *Crux* readers in any case.

Readers are also reminded that *Crux* is always in need of new problems, interesting and accessible, especially in fields like number theory and combinatorics. A solution submitted with a problem aids greatly in judging whether it is suitable.

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