# Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

# From the Editor

## The Irrationals

JULIAN HAVIL

# THE IRRATIONALS

### A Story of the Numbers You Can't Count On
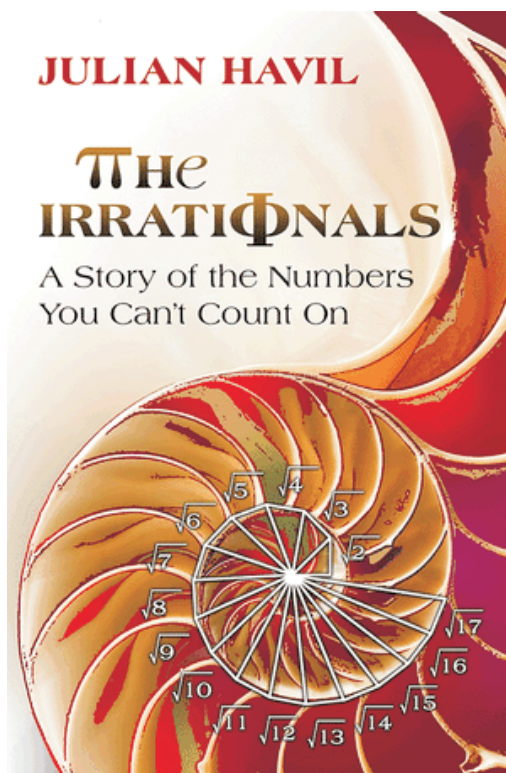
The ancient Greek mathematicians came across what they termed 'incommensurable magnitudes' in their geometry. For example, a right-angled triangle with opposite and adjacent sides of unit measure has a hypotenuse which is of incommensurable measure $x$ when compared to the other two sides. By Pythagoras' theorem, $x^2 = 2$, and there is no rational number $p/q$, where $p$ and $q$ are positive integers, satisfying this. For then $p^2 = 2q^2$, and, looking at factorization into primes on each side of this equation, there are an even number of factors 2 on the left-hand side but an odd number on the right-hand side, which contradicts unique factorization into primes, which was known to the Greeks. Nowadays we say that $\sqrt{2}$ is an irrational number, i.e. a real number which is not the ratio of two integers. A similar argument will show the $\sqrt{n}$ is irrational whenever $n$ is a positive integer which is not a perfect square. The book pictured above (see reference) explores the history of irrational numbers from their Greek beginnings.

Chapter 2 takes us to the Hindus and Arabs, leapfrogging the Romans 'who did nothing for the pure mathematician', Fibonacci in Italy in the thirteenth century, Pierre de Fermat, and Descartes amongst others.

In Chapter 3 we meet the great 18th century Swiss methematician Euler and a new irrational number, $e$. We define $e$ as the sum of an infinite series

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

and suppose that $e$ is rational, say, $e = m/n$, where $m$ and $n$ are positive integers. Then

$$n!\,e = n!\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + n!\left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right),$$

so that

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$$

is a positive integer. But it is smaller than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n+1}\frac{1}{1-1/(n+1)} = \frac{1}{n} \leq 1,$$

which would give a positive integer smaller than 1, which is impossible. This argument was given by Fourier in 1815.

The other famous irrational number is $\pi$. It was Hermite in 1873 who proved that $\pi^2$, and therefore also $\pi$, is irrational. Havil tells the story of Dame Mary Cartwright who posed the problem to prove the irrationality of $\pi^2$ in a Cambridge University Preliminary Examination. He does not tell us whether any candidate solved the problem!

Chapter 5 introduces 'a very special irrational' $\zeta(3)$, the sum of the infinite series

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots,$$

proved to be irrational by the French mathematician Roger Apéry in 1978. Should you wish to see Apéry's tomb in Paris, with a plaque with the engraving

$$\frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots \neq \frac{p}{q},$$

Havil supplies detailed directions, including at which Metro station to alight. You can visit Oscar Wilde's grave at the same time!

It could be said that some irrationals are 'more irrational than others'. For example, $\sqrt{2}$ is the solution of the quadratic equation $x^2 - 2 = 0$ with integer coefficients; it is said to be 'algebraic'. But $e$ and $\pi$ are not the solutions of algebraic equations, proved by Hermite in 1873 and Lindeman in 1882. Such irrational numbers are said to be 'transcendental'. In fact, even though there are an infinite number of algebraic irrational numbers and an infinite number of transcendental ones, there are more transcendentals than algebraic ones in that algebraic numbers are countable but transcendental ones are not. You will need Cantor's revolutionary work on countability and uncountability to make sense of that. An intriguing 'pathological' transcendental number is

$$0.1234567891011121314\ldots,$$

where the positive integers are concatenated ('stuck together'). This is called Champernowne's constant. (The editor of Mathematical Spectrum as a student went to the first of Professor Champernowne's lectures on Statistics, but only the first!)

There is lots more to explore in Havil's book, including more pathological numbers, the question of how random the decimal expansion of a given irrational number might be, how to approximate irrationals by rationals, and there are lots of questions where the answers are not known. I certainly found parts of it challenging, but there is no law that says you must read and understand every formula. There are some errors and misprints to trip up the unwary. But it does present the big picture of what after all is the basis of mathematics, namely numbers. It even asks and answers the question: what exactly is a real number? Not as obvious as you might think!

**Reference**

1  J. Havil, *The Irrationals* (Princeton University Press, New Jersey, 2012).

---

**The Area of a Lune**

The three agitos (asymmetric lunes) logo of the Paralympic Games led me to the following problem: prove that the area of the shaded lune is equal to the area of the shaded triangle in the two overlapping semicircles below.



*O*
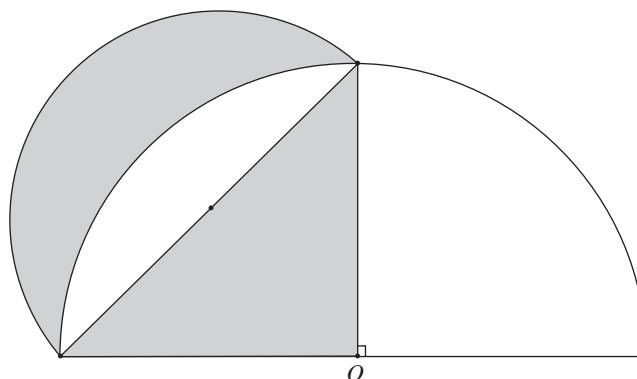
Midsomer Norton, Bath                                         **Bob Bertuello**

---

A question spotted in the novel *Dead Heat* by Dick and Felix Francis (Michael Joseph, 2007):

> If five men can build five houses in five months, how long will it take six men to build six houses?

# On the Trail of Reverse Divisors: 1089 and All that Follow

## ROGER WEBSTER and GARETH WILLIAMS

*We determine all natural numbers that divide their reverses.*

This is an account of the authors' adventures in tracking down, capturing, and cataloguing a rare species of whole number, known as a *reverse divisor*. So scarce are they that there are only 6 of them under a million, 16 under a billion, and 38 under a trillion. As the name suggests, a reverse divisor is a decimal integer that divides the number obtained by reversing its digits. It should be emphasized here that our whole discussion in the first two sections is based on the *decimal* expansion of a number. To avoid the commonplace, we exclude palindromic numbers, those that are unchanged by reversal of digits, in our formal definition: a *non-palindromic* natural number in decimal form $ab \ldots cd$ ($a \neq 0$) which divides its *reverse* $dc \ldots ba$ is called a *reverse divisor*. For such a number, the quotient of the reverse by the original is called the *quotient of the reverse divisor*. Reverse divisors must have at least two digits, *cannot* end in 0, and the quotient of a reverse divisor is one of the numbers $2, 3, \ldots, 9$.

The first sighting of a reverse divisor is hard to come by. Single-digit numbers, being palindromic, do not qualify. A few minutes' mental arithmetic shows there are no two-digit reverse divisors, and even longer on a pocket calculator shows there are no three-digit reverse divisors. Rather than baldly announce the first reverse divisor, we invite the eagle-eyed amongst you to take stock and make an inspired stab in the dark at it, without lifting a finger. For those of you who were successful, and those who were not, please read on and join us on a mathematical journey, from knowing nothing about reverse divisors to knowing everything! We found it exciting, we hope that you will do so too.

The *only* reverse divisors having four or fewer digits are 1089 and its double 2178, with respective quotients 9 and 4, an observation that G. H. Hardy, the greatest English number theorist of the twentieth century, alludes to in his delightful *A Mathematician's Apology* (see pages 104–105 of reference 1) as *non-serious* mathematics. The *smallest reverse divisor* 1089 has over recent years become something of a *nombre célèbre* in recreational mathematics (see page 9 of reference 2 and page 163 of reference 3) on account of its following remarkable property, which you should try out for yourself, if you have not already done so.

> *Subtract from* any *three digit number, whose first digit exceeds the last, its reverse. Add this difference to its own reverse. Then (in three-digit arithmetic) this last sum is always* 1089.

The result generalizes to four or more digit numbers (reference 4). A best-selling paperback by David Acheson (reference 5; see illustration overleaf) *actually* bears the title **1089 and All That**, despite not mentioning that 1089 is the *smallest reverse divisor*! Lewis Carroll entertained his child-friends with this arithmetical curiosity, and may even have been its discoverer (see pages 158–159 of reference 6). It appears under the heading ABRACADABRA in the News Chronicle's *I-SPY Annual* for 1956, whilst Johnny Ball in his fun maths book *Think of a Number* (see page 48 of reference 7) exploits it in a *mind-boggling* conjuring trick!
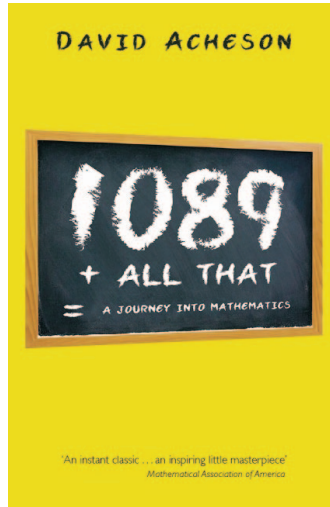
## Guidance for the first-time adventurer

The *official* account of our adventure that is presented here is both detailed and formal, which can be off-putting for the beginner. We suggest, therefore, that the reader whose only interest is in discovering the facts should study the statements of the lemmas, theorems, and corollaries, but omit their proofs. Even such a cursory reading will reveal the intricacy of the sequence of arguments designed to pin down reverse divisors in Theorem 2.10. For the more adventurous amongst you, and those eager to learn what is happening behind the scenes, we encourage you to read through the proofs, attempting to understand them. They are in truth, much of a muchness, depending on basic arithmetic and argument by contradiction. For those of you, even more ambitious yet, we challenge you to state and supply proofs for all the results in Section 2, suitably modified for reverse divisors with quotient 4, and to investigate the ideas mooted in Section 3.

This article may be likened to the log of a mountain ascent. The lemmas, theorems, and corollaries mark the high points on the trek upwards, the proofs record the paths traversed. It tells nothing of the days of despair when, confined to camp, no progress was possible. Nor does it tell of the thrill of an unexpected thrust forward, climbing from one perch to another higher up, closer to a summit we could not see or even know was there. The adventure began amidst uncertainty, for we knew to begin with 1089 and 2178, but had no idea of where to go from there! That is the thrill of the chase, the challenge of mathematics. Finally, the elation of reaching the summit, a fitting climax to a memorable adventure.

## 1. The fundamental theorem

**Theorem 1.1** *The quotient of a reverse divisor is either* 4 *or* 9. *The two* $n$-*digit* ($n \geq 4$) *numbers*

$$11 \times (10^{n-2} - 1) = \mathbf{10}99\ldots99\mathbf{89} \quad and \quad 22 \times (10^{n-2} - 1) = \mathbf{21}99\ldots99\mathbf{78},$$

*each with* $n - 4$ *central digits* 9, *are reverse divisors with respective quotients* 9 *and* 4. *No reverse divisor has* fewer *than four digits.*

*Proof* Suppose that $a \ldots d$ $(a \neq 0)$ is a reverse divisor with quotient $q$. Then

$$q \times (a \ldots d) = d \ldots a. \qquad (1)$$

We show that each of the cases $q = 2, 3, 5, 6, 7, 8$ is untenable, drawing on the observation that $qa \leq 9$ to restrict possible choices for $a$. This will establish the first assertion of the theorem.

$q = 2$.  Then $a$ is one of $1, 2, 3, 4$. But (1) is even, so $a$ is 2 or 4. For the units digit of (1) to be 2, $d$ would have to be 1 or 6, but it is 4 or 5. For the units digit of (1) to be 4, $d$ would have to be 2 or 7, but it is 8 or 9. Thus, 2 is *never* a quotient of a reverse divisor.

$q = 3$.  Then $a$ is one of $1, 2, 3$. For the units digit of (1) to be 1, $d$ would have to be 7, but it is one of $3, 4, 5$. For the units digit of (1) to be 2, $d$ would have to be 4, but it is one of $6, 7, 8$. For the units digit of (1) to be 3, $d$ would have to be 1, but it is 9. Thus, 3 is *never* a quotient of a reverse divisor.

$q = 5$.  Then $a$ is 1, which is impossible for (1) is divisible by 5.

$q = 6, 8$. Then $a$ is 1, which is impossible for (1) is even.

$q = 7$.  Then $a$ is 1. For the units digit of (1) to be 1, $d$ would have to be 3, but it is one of $7, 8, 9$.

The numbers $\mathbf{10}99 \ldots 99\mathbf{89}$ and $\mathbf{21}99 \ldots 99\mathbf{78}$ are reverse divisors with respective quotients 9 and 4, since

$$9899 \ldots 9901 = 9 \times (\mathbf{10}99 \ldots 99\mathbf{89}) \quad \text{and} \quad 8799 \ldots 9912 = 4 \times (\mathbf{21}99 \ldots 99\mathbf{78}).$$

Finally, we observe that none of the equations $9 \times (ab) = ba$, $4 \times (ab) = ba$, $9 \times (abc) = cba$, $4 \times (abc) = cba$ has a solution in which $a \neq 0$. Thus, no reverse divisor has *fewer* than four digits. $\qquad\square$

The *two $n$-digit* reverse divisors displayed in Theorem 1.1, from which all others can be constructed (see Theorem 2.10 below), are called *basic* reverse divisors. For $n = 4, 5, 6, 7$, they are the *only* reverse divisors. For *all* higher values of $n$, there are non-basic reverse divisors—for example, 10891089 and 108901089, both with quotient 9.

A *curious and interesting* cancellation property of basic reverse divisors is exhibited below:

$$\frac{1}{9} = \frac{1089}{9801} = \frac{10989}{98901} = \frac{109989}{989901} = \cdots \quad \text{and} \quad \frac{1}{4} = \frac{2178}{8712} = \frac{21978}{87912} = \frac{219978}{879912} = \cdots .$$

Since the quotients 4 and 9 of reverse divisors are *squares*, reverse divisors enjoy the following property.

**Corollary 1.2** *The product of a reverse divisor and its reverse is a square.* $\qquad\square$

The question as to when the product of a natural number and its reverse is a *square* appears from time to time in recreational literature. Ogilvy and Anderson in their *Excursions in Number Theory* (see pages 88–89 of reference 8) mention that the product of a two-digit number and its reverse is *never* a square unless the number is palindromic, and remark that this is *not* the

case for *three or more digit* numbers, citing $169 \times 961 = 403^2$ and $1089 \times 9801 = 3267^2$. Such examples led to the conjecture (see page 434 of reference 9) that, *when an integer and its reverse are unequal, their product can only be a square when both numbers are*. Corollary 1.2 provides an abundance of reverse-divisor counterexamples to the conjecture, the smallest being 2178.

## 2. Reverse divisors with quotient 9

We show that each reverse divisor with quotient 9 both begins and ends with the *same* basic reverse divisor with quotient 9. This leads us to a complete description of all reverse divisors with quotient 9 in terms of the basic ones.

**Theorem 2.1** *Let $ab \ldots cd$ ($a \neq 0$) be a reverse divisor with quotient 9. Then it has the form $10 \ldots 89$.*

*Proof* Since $9 \times (ab \ldots) = dc \ldots$, $a = 1$, $d = 9$, and $b$ is 0 or 1. The case $b = 1$ would imply that $c = 9$, and hence that $9 \times (\ldots 99) = \ldots 11$, which is false. Thus, $b = 0$ and $9 \times (\ldots c9) = \ldots 01$, showing that $c = 8$. □

**Corollary 2.2** *The only* four-*digit reverse divisor with quotient 9 is 1089.* □

**Theorem 2.3** *Let $10ab \ldots cd89$ be an $n$-digit ($n \geq 5$) reverse divisor with quotient 9. Then*

$$9 \times (99 \ldots 99 - ab \ldots cd) = 99 \ldots 99 - dc \ldots ba,$$

*where $99 \ldots 99$ is the $(n-4)$-digit number comprising solely of 9s.*

*Proof* By Theorem 1.1, the $n$-digit ($n \geq 5$) number $1099 \ldots 9989$, all of whose $n - 4$ central digits are 9s, is a reverse divisor with quotient 9, whence $9 \times (1099 \ldots 9989) = 98(99 \ldots 99)01$. By hypothesis, $9 \times (10ab \ldots cd89) = 98dc \ldots ba01$. Subtracting these equalities, then dividing by 100, gives the result. □

Theorem 2.3 can be used to show, even without a calculator, that the *only* 5-, 6-, 7-digit reverse divisors with quotient 9 are the basic ones: 10989, 109989, 1099989. A routine computer search, using the theorem, shows that the only *non-basic* 8- and 9-digit reverse divisors with quotient 9 are 10891089 and 108901089. Thus, the total number of reverse divisors with quotient 9 under a billion is eight.

Our study of the beginnings and endings of reverse divisors below culminates in key Theorem 2.8.

**Theorem 2.4** *A reverse divisor with quotient 9 ends either 089 or 989.*

*Proof* A reverse divisor with quotient 9 that ends $b89$ ($b > 0$) begins $(98b \ldots) \div 9 = 109 \ldots$. Hence, $9 \times (\ldots b89) = \ldots 901$, which shows that $b = 9$. □

**Theorem 2.5** *A reverse divisor with quotient 9 ending 089 both begins and ends with 1089.*

*Proof* A reverse divisor with quotient 9 that ends $b089$ begins $(980b \ldots) \div 9 = 108a \ldots$, where $a = 8$ if $b = 0$, but 9 otherwise. Also, $9 \times (\ldots b089) = \ldots a801$, which shows that $9b$ has units digit $a$. Hence, $b \neq 0$, so $a = 9$ and $b = 1$. □

For each non-negative integer $r$, respectively denote by $I_r$ and $0_r$ the string of $r$ consecutive 9s and 0s, and by $S_r$ the basic reverse divisor $10I_r89$ with quotient 9. Thus, $S_0$ is 1089 itself.

**Lemma 2.6** *A reverse divisor with quotient* $9$ *ending* $989$ *begins* $10I_r8$ *and ends* $0I_r89$ *for some* $r \geq 1$.

*Proof* A reverse divisor with quotient $9$ ending $989$ must end in $bI_r89$ for some $b \neq 9$ and $r \geq 1$. It begins $(98I_rb\ldots) \div 9 = 10I_ra\ldots$, where $a = 8$ if $b = 0$, but $9$ otherwise. Since $9 \times (\ldots bI_r89) = \ldots aI_r01$, $a$ is the units digit of $9b + 8$, so $a \neq 9$, for $a = 9$ would imply that $b = 9$. Thus, $b = 0$ and $a = 8$. $\qquad\square$

**Theorem 2.7** *A reverse divisor with quotient* $9$ *ending* $989$ *both begins and ends* $10I_r89$ *for some* $r \geq 1$.

*Proof* By Lemma 2.6, a reverse divisor with quotient $9$ ending $989$, ends $b0I_r89$ for some $b$ and $r \geq 1$. It begins $(98I_r0b\ldots) \div 9 = 10I_r8a\ldots$, where $a = 8$ if $b = 0$, but $9$ otherwise. Since $9 \times (\ldots b0I_r89) = \ldots a8I_r01$, $a$ is the units digit of $9b$, whence $b \neq 0$. Thus, $a = 9$ and $b = 1$. $\qquad\square$

Theorems 2.4, 2.5, and 2.7 together yield Theorem 2.8 below, the pivotal result in decomposing a general reverse divisor with quotient $9$ into basic ones.

**Theorem 2.8** *Each reverse divisor with quotient* $9$ *both begins and ends with the* same *basic reverse divisor with quotient* $9$.

Below we denote by $Q^R$ the reverse of a string of digits $Q$. Thus, the condition for such a $Q$, with leading digit non-zero, to be a reverse divisor with quotient $9$ becomes $9 \times Q = Q^R$.

**Lemma 2.9** *Let* $M$ *be a* non-basic *reverse divisor with quotient* $9$. *Then* $M = S_r 0_s V 0_s S_r$ *for some* $r, s \geq 0$, *where* $V$ *is either* *a reverse divisor with quotient* $9$ *or* $0_t$ *for some* $t \geq 0$.

*Proof* Since $M$ is *non-basic*, Theorem 2.8 shows that $M = S_r P S_r$ for some $r \geq 0$ and some string of digits $P$. If $P$ is empty or consists only of zeros, then $M$ is in the desired form. Suppose, then, that $P$ contains some non-zero digits.

Since $S_r P S_r$ is a reverse divisor with quotient $9$,

$$9 \times (S_r P S_r) = (S_r P S_r)^R = (S_r)^R P^R (S_r)^R = (9 \times S_r) P^R (9 \times S_r).$$

Comparing this result with that of the direct multiplication of $S_r P S_r$ by $9$ shows that $9 \times P = P^R$. If $P$'s first digit is non-zero, then $P$ is a reverse divisor with quotient $9$, and $M$ is in the desired form. Suppose, then, that $P$ has the form $0_s a \ldots$ for some $a, s > 0$. Then

$$9 \times (0_s a \ldots) = (0_s a \ldots)^R = \ldots a 0_s,$$

which shows that $P$ has the form $0_s a \ldots b 0_s$ for some $b$, and that $9 \times (a \ldots b) = b \ldots a$. Thus, $a \ldots b$ is a reverse divisor with quotient $9$. Finally, we note that $M = S_r 0_s (a \ldots b) 0_s S_r$. $\qquad\square$

We now arrive at our main result. It states, in essence, that *all* reverse divisors with quotient $9$ can be formed from *basic* ones by concatenating these with strings of zeros in a symmetric and alternating way.

**Theorem 2.10** (Structure theorem for reverse divisors with quotient 9) *Non-basic reverse divisors with quotient* $9$ *are precisely those natural numbers of the form*

$$S_{a_1} 0_{b_1} S_{a_2} 0_{b_2} \ldots S_{a_n} 0_{b_n} V 0_{b_n} S_{a_n} \ldots 0_{b_2} S_{a_2} 0_{b_1} S_{a_1},$$

*where* either $V = S_{a_0}$ or $0_{b_0}$, *for some non-negative integers* $a_0, a_1, \ldots, a_n$ *and* $b_0, b_1, \ldots, b_n$.

*Proof* Every number of the given form is certainly a non-basic reverse divisor with quotient 9. That every non-basic reverse divisor with quotient 9 has this form follows by repeated application of Lemma 2.9. □

The structure theorem enables us to write down at will all reverse divisors with quotient 9 having a given number of digits. Just for fun, we list all 19 that are under a trillion, **boldfacing** the basic ones.

**1089**, **10989**, **109989**, **1099989**, 10891089, **10999989**, 108901089, **109999989**,
1089001089, 1098910989, **1099999989**, 10890001089, 10989010989, **10999999989**,
108900001089, 108910891089, 109890010989, 109989109989, **109999999989**.

**Postscript** All the results and proofs in this section generalize to the quotient 4 case (see reference 10).

## CHALLENGE: What is the thirty-ninth reverse divisor?

## 3. (retsbeW regoR) sgnittoJ

These jottings were intended for my co-author, *not* for publication. The editor, however, chanced upon them and was moved to comment '*I would like to see the extra, more informal section included. It helps tone down the formal presentation and stimulate readers to think for themselves.*' Who am I to disagree?

1. My interest in *reverse divisors* stems from 11's remarkable property that whenever it divides a number, it divides its reverse. This helped me solve Victor Bryant's *Sunday Times* (13 March 2011) TEASER 2529:

   > A letter to *The Times* concerning the inflated costs of projects read: '*When I was a financial controller, I found that multiplying original cost estimates by pi used to give an excellent indication of the final outcome.*' Interestingly, I used this process (using 22/7 as an approximation for pi). On one occasion, the original estimate was a whole number of pounds (under 100,000), and this method for the probable final outcome gave a number of pounds consisting of the same digits, but in reverse order. What was the original estimate?

   The problem is to express 22/7 as the quotient of a number and its reverse, both under 100 000. See reference 10. This led to the question: What numbers can be written as the ratio of a natural number to its reverse?

2. The only numbers *I* know with the property that, if they divide a number, they divide its reverse are 1, 3, 9, 11, 33, 99. Are there others? If so, they must be palindromic or reverse divisors. Hence this study!

3. The fractions (suggested by David Huitson) below, in which the denominator is the reverse of the numerator, exhibit the same *curious and interesting* cancellation property that we saw earlier for basic reverse divisors.

$$\frac{2}{3} = \frac{4356}{6534} = \frac{43956}{65934} = \frac{439956}{659934} = \cdots \quad \text{and} \quad \frac{3}{8} = \frac{27}{72} = \frac{297}{792} = \frac{2997}{7992} = \cdots.$$

Scope for pleasurable investigations! The second example is a special case of the rule: if $a + b = 9$ then

$$\frac{ab}{ba} = \frac{a9b}{b9a} = \frac{a99b}{b99a} = \frac{a999b}{b999a} = \cdots .$$

4. Is it *obvious* why the basic reverse divisor 2178 should be twice the basic reverse divisor 1089? Does this occur in other bases?

5. What of reverse divisors in non-decimal bases? There are *none* in binary. In base $b$ ($b \geq 3$), the four-digit number $1\,0\,(b-2)\,(b-1)$ is a reverse divisor, quotient $b-1$. Thus, reverse divisors in ternary (quaternary, quinary) are $1012\,(1023, 1034)$, quotients $2\,(3, 4)$. In quinary, $143$ is a reverse divisor, quotient $2$.

   **HOT OFF THE PRESS:** the **1089** trick holds in base $b$ with denouement **reverse divisor** $1\,0\,(b-2)\,(b-1)$.

6. The links below (found by David Cuddington) touch on reverse divisors, but are meagrely presented.

   http://mathworld.wolfram.com/Reversal.html    and    http://oeis.org/A008919.

7. Students seem much taken with reverse divisors. They eagerly track them down on calculators and computers, make wild guesses, build large ones from small ones, and research them online. A suitable topic for the classroom and undergraduate projects, with conjectures, discoveries, calculations, and wall charts!

**References**

1  G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1992).
2  W. W. Rouse Ball, *Mathematical Recreations & Essays* (11th edn., Macmillan, London, 1939).
3  D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers* (Penguin Books, London, 1986).
4  R. Webster, A combinatorial problem with a Fibonacci solution, *Fibonacci Quart.* **33** (1995), pp. 26–31.
5  D. Acheson, *1089 and All That: A Journey into Mathematics* (Oxford University Press, 2002).
6  R. Wilson, *Lewis Carroll in Numberland* (Penguin Books, London, 2008).
7  J. Ball, *Think of a Number* (Dorling Kindersley, London, 2005).
8  C. S. Ogilvy and J. T. Anderson, *Excursions in Number Theory* (Dover Publications, New York, 1988).
9  Conjecture on reversals, *Amer. Math. Monthly* **64** (1957), p. 434, E-1243.
10  R. Webster and G. Williams, Reverse divisors with quotient 4, http://users.mct.open.ac.uk/gw3285/publications/rdq4.pdf.

**Roger Webster** lectures on the history of mathematics at Sheffield University. As a twelve-year old, his party piece was to perform a £ s d version of the 1089 trick! Even *today, he remembers well that final answer: £12 18s 11d.*

**Gareth Williams** is a staff tutor in mathematics at The Open University. He is a pure mathematician specializing in topology. His interest in the number 1089 was first aroused by a gift of 1089 and All That *from his co-author.*

# Mathematicians Prefer Cake to pi

## PAUL GLAISTER

An alternative approach to cake-cutting leads to some interesting mathematics and an optical illusion as to who will get the largest slice.

Recently, and unexpectedly, I was presented with a (circular) birthday cake in a meeting with a group of my students. It became clear that I was going to have to cut it to distribute it evenly to the gathering, something I have never done before. This caused some amusement, as you might guess! Being a novice at this, I did not attempt to cut the cake in traditional wedges, but instead first cut the cake in half through the centre, and then attempted to cut in a series of parallel lines, as shown in figure 1, giving the eight pieces we needed. The students unanimously agreed to let me have one of the end pieces as they were clearly the largest; or were they? Had I cleverly arranged for all the pieces to be of the same size, except the end ones?

Consider an $x$-axis through the centre perpendicular to the initial cut with the origin at the centre of the cake, as shown in figure 2. Suppose that the cake is divided into $2n$ pieces of equal area $A_{2n} = \pi/2n$, where $n \geq 2$ and taking the radius of the circle to be unity. Consider further, two cuts at $x = x_i$ and $x = x_{i+1}$, both parallel to the first cut, as shown in figure 2. A little calculus shows that the area of the shaded piece is

$$2 \int_{x_i}^{x_{i+1}} \sqrt{1 - x^2}\, dx = \arcsin(x_{i+1}) - \arcsin(x_i) + x_{i+1}\sqrt{1 - x_{i+1}^2} - x_i\sqrt{1 - x_i^2}.$$
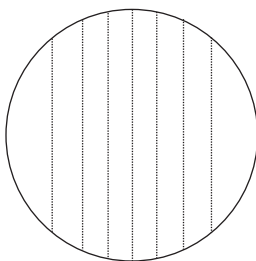


**Figure 1** A cake cut into eight pieces.
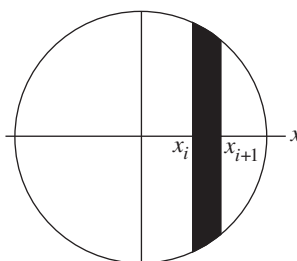


**Figure 2** A typical slice.

If we set this equal to the (fixed) area $A_{2n} = \pi/2n$ and rearrange, we have

$$x_{i+1} = \sin\left(\frac{\pi}{2n} + \arcsin(x_i) - x_{i+1}\sqrt{1 - x_{i+1}^2} + x_i\sqrt{1 - x_i^2}\right). \tag{1}$$

Now if we take $x_0 = 0$, i.e. at the origin, then the value of $x_1$ satisfying

$$x_1 = \sin\left(\frac{\pi}{2n} - x_1\sqrt{1 - x_1^2}\right) \tag{2}$$

gives the location of the first cut. A simple iteration based on (2), taking as an initial estimate $x_1 \approx x_0 = 0$, will give the solution to any desired accuracy. In general, starting with $x_0 = 0$ and performing an iteration based on (1) for each $i = 0, 1, \ldots, n$, using the previous (converged) value of $x_i$ as an initial estimate for $x_{i+1}$, we obtain the other locations to make the cuts. Note that $x_n = 1$, the edge of the cake. The other half of the cake is cut in a similar manner.

Returning to figure 1, these are precisely the cuts required to divide the cake into eight equally-sized pieces. It is simply an optical illusion that the end pieces are larger. Indeed, figure 3 has a grid placed on the cake, and a simple count reveals that the end pieces have an area of approximately 40 squares. This is the same value as all the other pieces (which are easier to count) and which all look of roughly equal width, but of course are obviously not.

Further examples for different numbers of pieces, $2n$, are shown in figure 4 and, in case readers ever need to use this method to cut a cake, we have calculated in table 1 the corresponding values of $x_i$, and hence the location of the cuts. We see, for example, that for four pieces a cut is made at $x = 0.4040$, and for numbers of pieces which are multiples of this, e.g. eight and 12 pieces, then there remains a cut at this location, but with further cuts on either side of it.
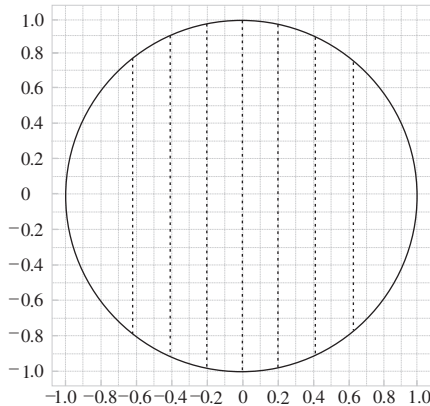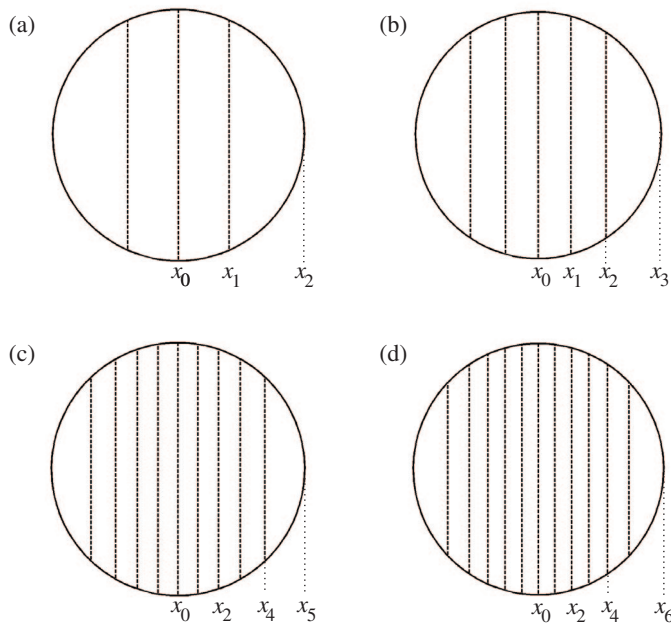
Now, what if an odd number of pieces are required...



**Figure 3**

**Figure 4** Examples of equitably cutting a cake into (a) four pieces, (b) six pieces, (c) 10 pieces, and (d) 12 pieces (see table 1 for the exact locations of the cuts).

**Table 1**

| Number of pieces, $2n$ | Area of each piece, $A_{2n}$ | Position of cut | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| 4 | 0.7854 | 0 | 0.4040 | 1 | | | | |
| 6 | 0.5236 | 0 | 0.2649 | 0.5533 | 1 | | | |
| 8 | 0.3927 | 0 | 0.1976 | 0.4040 | 0.6347 | 1 | | |
| 10 | 0.3142 | 0 | 0.1577 | 0.3197 | 0.4919 | 0.6870 | 1 | |
| 12 | 0.2618 | 0 | 0.1313 | 0.2649 | 0.4040 | 0.5533 | 0.7240 | 1 |

***Paul Glaister*** *lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, perturbation methods, as well as mathematics and science education. His life has been taken over recently by becoming Head of Mathematics and Statistics at Reading, which might possibly explain why his students are presenting him with cake.*

# Coins and Fractions

## G. T. VICKERS

Problems in which the answer is required to be an integer are often more challenging than their real number counterparts. For example, insisting upon integer lengths for the sides of a right-angled triangle is quite different from just Pythagoras' theorem. Here two problems are presented (each with only a partial solution) where integers are involved. The first, concerning coins, is a much-studied problem but perhaps not too familiar. The second is about finding rational numbers between a given pair of real numbers. They may appear to be unrelated, but a result from the first proves to be useful in the second.

## 1. Coins

In the financially fatuous land of Fiscalia, the only coins that exist have the values 5 and 7 florins. A peculiar pecuniary prohibition imposed by the charming chancellor is that shopkeepers are not allowed to give change for any purchase. This imposes severe constraints upon the permitted costs of small-value articles. It is a small challenge to show that 23 florins is not allowed (because no combination of 5 and 7 can result in 23) but that every value greater than 23 is allowed.

This problem, which I have seen in several guises in various puzzle columns of newspapers, has obvious extensions which give severe difficulty. When there are only two coins, the solution is quite easy, but (except for some special cases) if there are more than two it is a classical problem upon which much has been written (see reference 1) but rather little progress made.

### 1.1. The mathematical problem

We will use $\mathbb{Z}^0$ for the set $\{0, 1, 2, \dots\}$. Let $a, b$ be a pair of positive integers with no common factor (commonly written as $(a, b) = 1$). This note is concerned with those numbers (necessarily integers) which can be expressed in the form

$$xa + yb \ (x, y \in \mathbb{Z}^0).$$

Such numbers are said to be *attainable* and they form the *attainable set*. Not surprisingly, the *unattainable set* consists of those integers which are not attainable. It will be shown that the unattainable set always has a largest element and it is this number, written as $M(a, b)$ and referred to as the *largest unattainable number*, upon which interest will centre.

The main result is the following

**Result 1** *With* $(a, b) = 1$, $M(a, b) = ab - a - b$. *Furthermore, if* $0 \le n \le M = M(a, b)$ *then just one of* $n$, $M - n$ *is attainable.*

*Proof* Let $n \in \mathbb{Z}^0$ be unattainable. Since $(a, b) = 1$, the numbers $b, 2b, \dots, (a-1)b \mod a$ are a permutation of $1, 2, \dots, a - 1$. Hence, there is an integer $\beta$ for which

$$n \equiv \beta b \mod a, \qquad 1 \le \beta \le a - 1,$$

and $n - \beta b = -\alpha a$ for some integer $\alpha \geq 1$ (since $\alpha \leq 0$ would imply that $n$ was attainable). Also, any number $m$ of the form $\beta b - \alpha a$ ($1 \leq \beta \leq a - 1$ and $\alpha \geq 1$) is unattainable. For if $m$ were attainable then

$$m = \gamma a + \delta b = \beta b - \alpha a \quad \Longrightarrow \quad a(\gamma + \alpha) = b(\beta - \delta) \quad \Longrightarrow \quad \alpha + \gamma = rb, \quad \beta - \delta = ra$$

(where $\gamma$, $\delta$ and $r$ belong to $\mathbb{Z}^0$) which would imply that $\beta \geq a$. Thus the maximum value of $m$ occurs when $\alpha = 1$ and $\beta = a - 1$ which implies that

$$M(a, b) = (a - 1) \times b - 1 \times a = ab - a - b.$$

Also,

$$M - n = ab - a - b - (\beta b - \alpha a) = a(\alpha - 1) + b(a - \beta - 1).$$

Since $\alpha \geq 1$ and $\beta \leq a - 1$, this shows that $M - n$ is attainable.

Finally, it is not possible for both $r$ and $M - r$ to be attainable (for then their sum, $M$, would also be attainable). This completes the proof.

As very special cases we have

$$M(1, a) = -1, \qquad M(5, 7) = 23, \quad \text{and} \quad M(2, a) = a - 2 \quad \text{if } a \text{ is odd.}$$

The function $M(a, b)$ has only been defined when $(a, b) = 1$. But the definition may be extended by requiring that

$$M(a, b) = hM\left(\frac{a}{h}, \frac{b}{h}\right) \quad \text{whenever } h = (a, b).$$

This is sensible since all attainable numbers must be divisible by $h$. Having made this definition, it follows that

$$M(ka, kb) = kM(a, b)$$

whatever the highest common factor of $a$ and $b$ might be. Also our main result generalises to

$$M(a, b) = \frac{ab}{(a, b)} - a - b.$$

## 1.2. Tricky stuff

Our problem with two coins can easily be extended to any number of coins. Unfortunately, the problems are much easier to state than to solve. Let $S = \{a, b, c, \ldots\}$ be a finite set of positive integers. The attainable set consists of those integers expressible in the form

$$xa + yb + zc + \ldots, \quad \text{where } x, y, x, \ldots \in \mathbb{Z}^0,$$

and, of course, other integers are unattainable. It is left as a challenge to show that if the highest common factor of the members of $S$ is unity then there is a largest unattainable number, written $M(S)$. As before, if the highest common factor is $h$ then $M(S)$ is defined by

$$M(S) = hM(S'), \quad \text{where } S' = \left\{\frac{a}{h}, \frac{b}{h}, \frac{c}{h}, \ldots\right\}.$$

Here are three results that are fairly difficult to establish even though their proofs do not require anything that has not already been used. In what follows, $\lfloor x \rfloor$ is used for the largest integer which does not exceed $x$ and $\lceil x \rceil$ will later be used for the smallest integer which is not less than $x$.

**Result 2**  *If $a$ and $b$ are positive integers with $(a, b) = 1$ and*

$$S = \{a, a + b, a + 2b, \ldots, a + rb\},$$

*then*

$$M(S) = b(a - 1) + a \left\lfloor \frac{a - 2}{r} \right\rfloor.$$

**Result 3**  *Let $S_h = \{ah, bh, c\}$, where $(h, c) = 1$. If $M_h$ is the largest number unattainable by $S_h$ then*

$$M_h = hM_1 + (h - 1)c.$$

**Result 4**  *Let $S = \{a^n, a^{n-1}b, a^{n-2}b^2, \ldots, b^n\}$, where $(a, b) = 1$. Then*

$$M(S) = \frac{(a - 1)b^{n+1} - (b - 1)a^{n+1}}{b - a}.$$

From result 3, it is easy to show that if $(a, b) = 1 = (b, c) = (c, a)$ then

$$M(bc, ca, ab) = 2abc - (bc + ca + ab).$$

Result 3 is also useful in proving result 4.

Finding $M(a, b, c)$ (i.e. the largest unattainable number for a set of only three numbers) is, in general, difficult. By that I mean that there is no 'nice' formula for it. Reference 1 does provide algorithms for sets of any size (and some bounds for $M$) but it is, in my opinion, all rather messy. A result which seems to promise rather more than it actually delivers is the following.

**Result 5**  *If $M(m, n, n+1)$ is known for all $m$ and $n$ then $M(a, b, c)$ is known for all $a, b, c$.*

## 2. Fractions

Let $x$ and $y$ be real numbers with $1 > x > y > 0$ and let $k$ be the smallest integer which is greater than or equal to $1/(x - y)$ (so $k = \lceil 1/(x - y) \rceil$). Because the interval $[x, y]$ is at least as long as $1/k$ there will certainly exist integers $p$ and $q$ with $x \geq p/q \geq y$ whenever $q \geq k$. This may be expressed as follows.

The inequalities

$$qx \geq p \geq qy$$

have solutions in positive integers $p$ and $q$ whenever

$$q \geq \left\lceil \frac{1}{x - y} \right\rceil.$$

Clearly, the inequalities have no solution for integer $p$ when $q = 1$ and so it is sensible to define the integer function $Q(x, y)$ by the requirement that the above inequalities shall have solutions for positive integers $p, q$ whenever $q > Q(x, y)$ but not for $q = Q(x, y)$.

It is left to the reader to show that $(x - y)Q(x, y) < 1$. Note that this is slightly better than $Q(x, y) < \lceil 1/(x-y) \rceil$. For example, this last inequality gives $0.3Q(0.8, 0.5) < 0.3\lceil 1/0.3 \rceil = 1.2$ rather than it being less than unity.

The function $Q(x, y)$ has been defined only for $x > y$. It is convenient in what follows to extend the definition by requiring that $Q(x, y) = Q(y, x)$ whenever $x \neq y$. Thus $Q(x, y)$ is defined everywhere within the unit square except along the line $x = y$.

This problem may appear to be unrelated to the coin problem, but look at the proof of the following result.

**Result 6**  *If $x = a/b$ and $y = c/d$ (where $a, b, c, d$ are positive integers with $b > a$ and $ad - bc = 1$) then $Q(x, y) = bd - b - d$.*

*Proof*  Firstly, suppose that $q$ is attainable from $\{b, d\}$. Then there are integers $m$ and $n$ such that

$$q = md + nb \ (m, n \in \mathbb{Z}^0).$$

Define $p$ by $p = na + mc \geq 0$. Then

$$qa - pb = (md + nb)a - (na + mc)b = m(ad - bc) = m \geq 0$$

and

$$pd - qc = (na + mc)d - (md + nb)c = n(ad - bc) = n \geq 0.$$

Hence,

$$qx \geq p \quad \text{and} \quad p \geq qy \quad \implies \quad qx \geq p \geq qy.$$

Now suppose that an integer $p$ exists between $qx$ and $qy$. Then

$$qx \geq p \quad \implies \quad qa \geq pb \quad \text{and} \quad p \geq qy \quad \implies \quad pd \geq qc.$$

With $m, n \in \mathbb{Z}^0$ defined by $m = qa - pb$ and $n = pd - qc$, we have

$$md + nb = (qa - pb)d + (pd - qc)b = q(ad - bc) = q,$$

and so $q$ is attainable from $\{b, d\}$.

So,

$$q > bd - b - d \quad \implies \quad q \text{ is attainable from } \{b, d\}$$
$$\implies \quad p \text{ exists with } qx \geq p \geq qy,$$

and, when $q = bd - b - d$, $q$ is not attainable and no such $p$ exists. Thus $Q(x, y) = bd - b - d$.

**Examples** 1.  $Q(\frac{5}{12}, \frac{2}{5}) = 43$, and so between $\frac{5}{12}$ and $\frac{2}{5}$ there are rational numbers with denominators $44, 45, 46, \ldots$ but not with 43. Note that $\frac{5}{12} - \frac{2}{5} = \frac{1}{60}$ and so the bound provided by $\lceil 1/(x - y) \rceil$ is rather poor.

2.  $Q((9 + 8n)/(10 + 9n), 8/9) = 71 + 72n$.

3.  $Q(1/n, 1/(n + 1)) = n^2 - n - 1$.

4.  $Q(\frac{1}{2}, \frac{2}{5}) = 3$. However, $\lim_{y \to 2/5+} Q(\frac{1}{2}, y) = 5$.

## 2.1. The interval of solution

Suppose that the values $Q$ and $x$ are given and it is required to find the values of $y$ for which $Q(x, y) = Q$ and $x > y$. For $q > Q$ we know that there is an integer $p$ for which $qx \geq p \geq qy$. Clearly, there is no loss in assuming that $p = \lfloor qx \rfloor$, and so we must have

$$\lfloor qx \rfloor \geq qy \quad \Longrightarrow \quad y \leq \frac{\lfloor qx \rfloor}{q}.$$

This last inequality has to be satisfied whenever $q > Q$, and so

$$y \leq \min_{q > Q} \frac{\lfloor qx \rfloor}{q}.$$

Also, there is no integer between $Qy$ and $Qx$. Hence, $Qy > \lfloor Qx \rfloor$ and so $y$ must satisfy

$$\frac{\lfloor Qx \rfloor}{Q} < y \leq \min_{q > Q} \frac{\lfloor qx \rfloor}{q}. \tag{1}$$

Conversely, if $y$ lies in this interval then for $q > Q$, we may set $p = \lfloor qx \rfloor$ and, when $q = Q$, there is no integer between $Qx$ and $Qy$.

In a similar fashion it can be shown that if $Q$ and $y$ are given, then the range of $x$ values for which $Q(x, y) = Q$ is

$$\max_{q > Q} \frac{\lceil qy \rceil}{q} \leq x < \frac{\lceil Qy \rceil}{Q}.$$

**Examples** 1. We have

$$0 < 2y < x < 1 \quad \Longrightarrow \quad Q(x, y) = Q = \left\lceil \frac{1}{x} \right\rceil - 1.$$

From equation (1), $Q(x, y) = Q = \lceil 1/x \rceil$ for $\alpha < y \leq \beta$, where $\alpha Q = \lfloor Qx \rfloor$ and $\beta = \min_{q > Q}(\lfloor qx \rfloor / q)$. Now

$$Qx = \left( \left\lceil \frac{1}{x} \right\rceil - 1 \right) x < \left( \frac{1}{x} \right) x = 1 \quad \Longrightarrow \quad \lfloor Qx \rfloor = 0,$$

and so $\alpha = 0$. The exact value of $\beta$ is not easily found but

$$q > Q \quad \Longrightarrow \quad q \geq Q + 1$$
$$\Longrightarrow \quad qx \geq \left\lceil \frac{1}{x} \right\rceil x \geq 1$$
$$\Longrightarrow \quad 2 \lfloor qx \rfloor \geq qx$$
$$\Longrightarrow \quad \frac{\lfloor qx \rfloor}{q} \geq \frac{x}{2}.$$

Hence, $\beta \geq x/2$ and so $Q(x, y) = Q$ for $y < x/2$.

2. $\lim_{y \to 0+} Q(x, y) = \lceil 1/x \rceil - 1$.

3. $Q(x, \frac{1}{2})$ is always an odd integer and

$$Q\left(x, \frac{1}{2}\right) = 2k + 1 \quad \text{whenever} \quad \frac{k+2}{2k+3} \leq x < \frac{k+1}{2k+1}, \quad k \geq 0.$$

## 2.2. Concluding remarks

The results presented do not solve the complete problem of finding $Q(x, y)$. However, it is not difficult to devise a numerical procedure for calculating $Q$ and figure 1 shows how the unit square is divided up into regions in which $Q$ is a constant and gives the value of $Q$ in each region. Figure 2 concentrates upon just the regions in which $Q$ is 5 and 11. All the regions are L-shaped!! Some more results are to be found in reference 2, but it certainly does not provide a complete solution to the problem of finding $Q(x, y)$.
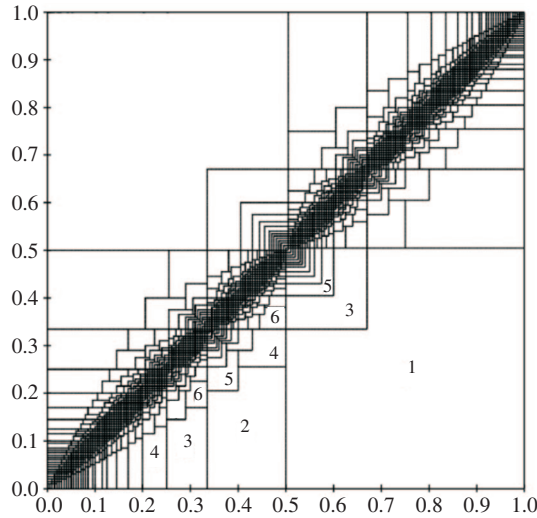


**Figure 1**  Values of $Q(x, y)$ within the unit square and how these values divide up the square.
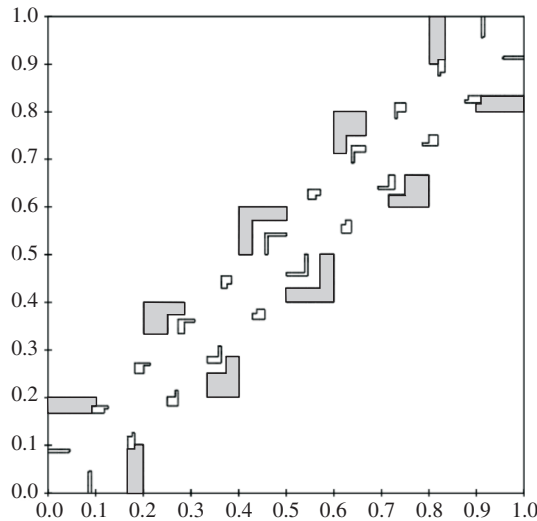


**Figure 2**  Ten regions in which $Q = 5$ *shaded* and 22 smaller regions in which $Q = 11$.

**References**

1  J. L. Ramirez Alfonsin, *The Diophantine Frobenius Problem* (Oxford Lecture Ser. Math. Appl. **30**, Oxford University Press, 2006).

2  G. T. Vickers, Additional thoughts regarding 'fractions', http://g-t-vickers.staff.shef.ac.uk.

***Glenn Vickers*** *graduated from Sheffield University and then gained a PhD from Queen Mary College, London in astrophysics. He is now retired from the School of Mathematics and Statistics in Sheffield (although he still tries to get first year engineering students to respect mathematical notation!) where he taught a wide range of topics including genetics, galactic structure, and evolutionary game theory.*

## A Magic Hexagon
### or
### Aristotle's Number Puzzle

The picture shows a wooden puzzle I received for Christmas. It consists of 19 small hexagons numbered 1 to 19 which fit into a large recessed hexagon. The aim is to make all horizontal and sloping lines add up to the magic number 38. After some research, I found that there is only one solution apart from rotations and relections. I find this amazing.



Midsomer Norton, Bath                                    **Bob Bertuello**

# A Result Concerning Fibonacci Primes

## MARTIN GRIFFITHS and SURAJIT RAJAGOPAL

In this article we employ simple methods to obtain a result concerning the Fibonacci primes. We also indicate how this result, and generalisations of it, follow from a nontrivial piece of mathematics known as Carmichael's theorem on Fibonacci numbers.

## 1. Introduction

There is an air of mystery about the prime numbers appearing in the Fibonacci sequence, known as the *Fibonacci primes*. Indeed, very little about them appears in the literature, and it is not even known whether or not there are infinitely many such numbers. The largest known Fibonacci prime is $F_{81839}$, the decimal representation of which has 17 103 digits (see references 1 and 2). Incidentally, if this was to be written out in full, it would occupy seven or eight pages of this journal!

The Fibonacci sequence itself, denoted $F_n$, is defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, where $F_0 = 0$ and $F_1 = 1$. The initial terms of $F_n$ are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots,$$

from which we see that the first six Fibonacci primes are

$$2, 3, 5, 13, 89, 233.$$

Let us consider the first few Fibonacci numbers that are the product of exactly two primes

$$F_8 = 21 = 3 \times 7,$$
$$F_9 = 34 = 2 \times 17,$$
$$F_{10} = 55 = 5 \times 11,$$
$$F_{14} = 377 = 13 \times 29.$$

Note, thus far at least, that although Fibonacci primes do sometimes appear as factors of these numbers, there is no instance of both prime factors being Fibonacci primes. Of course, this is an extremely flimsy piece of evidence on which to base any sort of conjecture, but we show here, using very simple methods, that no Fibonacci number may be expressed as the product of two Fibonacci primes (distinct or otherwise).

It turns out that knowledge of a result known as *Carmichael's theorem on Fibonacci numbers* enables us to prove our result almost immediately, as is explained in the final section. However, a proof of Carmichael's theorem requires the use of primitive roots of unity and cyclotomic polynomials, mathematical entities that possibly not all readers will be familiar with. The proof we provide here is both simple and elementary, noting that in the field of mathematics these words do not necessarily mean the same thing! Indeed, our proof can be understood by anyone possessing both knowledge of a few basic results concerning the Fibonacci numbers and familiarity with some straightforward notions associated with divisibility.

## 2. Some results we shall need

We collect here a number of results that will be used at certain points in our proof that no Fibonacci number is the product of two Fibonacci primes.

**Result 1** *For any $m, n \geq 2$, it is the case that $F_{mn} > F_m F_n$.*

*Proof* We use the well-known result $F_{a+b} = F_{a-1} F_b + F_a F_{b+1}$ (see references 3 and 4) with $a = m(n-1)$ and $b = m$ to give

$$
\begin{aligned}
F_{mn} &= F_{m(n-1)+m} \\
&= F_{m(n-1)-1} F_m + F_{m(n-1)} F_{m+1} \\
&> F_m (F_{m(n-1)-1} + F_{m(n-1)}) \\
&= F_m F_{m(n-1)+1},
\end{aligned}
$$

where we have used the fact that $F_{m+1} > F_m$ when $m \geq 2$. Then, since $m(n-1) + 1 \geq n$ for $m \geq 1$, our result follows.

**Result 2** *For $3 \leq a \leq b$ it is the case that $a|b$ if and only if $F_a|F_b$.*

*Proof* A straightforward proof is provided in reference 3.

Note in fact that since $F_1 = F_2 = 1$, it is actually the case that $a|b$ implies $F_a|F_b$ for any $a, b \in \mathbb{N}$. However, because $F_a|F_b$ does not imply $a|b$ when $a = 2$ and $b$ is odd, the converse of the statement in the preceding sentence is not true.

**Result 3** *Any Fibonacci prime other than $F_4 = 3$ must have a prime index.*

*Proof* First, we have $F_3 = 2$ and $F_5 = 5$, so the result is certainly true in these special cases. Suppose now that $F_r$ is a Fibonacci number such that $F_r > 5$ and $r = ab$ for some $a, b \geq 2$ (in other words, $r$ is composite). Since $F_r > 5$ implies that $r \geq 6$, it must be the case that either $a \geq 3$ or $b \geq 3$. Let us assume, without loss of generality, that $a \geq 3$. With the conditions on $a$ and $b$, we know that $F_r > F_a$. Furthermore, result 2 tells us that $F_r$ is divisible by $F_a$. Then, since $F_a \geq 2$, it must be the case that $F_r$ is composite. We have thus shown that, for any composite number $r$ other than 4, $F_r$ is composite, which is logically equivalent to the statement of the result.

It is worth noting that result 3 does not guarantee that a Fibonacci number with a prime index is actually prime. In fact, it does not take long to find a composite Fibonacci number with a prime index: $F_{19} = 4181 = 37 \times 113$.

## 3. The proof

We prove our result by contradiction. That is, we assume that $F_r = F_p F_q$ for some $r \in \mathbb{N}$ and Fibonacci primes $F_p$ and $F_q$, and show that this leads to mathematical nonsense. This in turn implies that our assumption must be false.

Since the case $F_r = F_p^2$ needs slightly different treatment to that in which the Fibonacci primes are distinct, this situation is dealt with separately towards the end of the current section in order to avoid overcomplicating matters. We may thus assume for the time being that $F_p$ and $F_q$, and hence $p$ and $q$, are distinct.

From result 3 we know that $F_4$ is the only Fibonacci prime with a composite index. For the sake of clarity, therefore, let us start by considering this special case. We assume then that $F_r = F_4 F_q = 3F_q$ for some $r \in \mathbb{N}$ and prime $F_q$, where $q$ is a prime such that $q \geq 3$. Since $F_4 | F_r$ we have, using result 2, that $r = 4k$ for some $k \in \mathbb{N}$. Result 2 and the comments following it in turn imply that $F_k | F_r$, and hence $F_k | 3F_q$. Since 3 and $F_q$ are prime, there are only the following possibilities.

$F_k = 1$.   We have $k = 1$ or $k = 2$, so that $F_r = F_4 = 3$ or $F_r = F_8 = 21$, neither of which is a product of two Fibonacci primes.

$F_k = 3$.   Then $k = 4$ and thus $F_r = F_{16} = 987 = 3 \times 7 \times 47$, which is not the product of two Fibonacci primes.

$F_k = F_q$.   Since $F_q$ is a Fibonacci prime, we know that $q \geq 3$ and hence $k = q$, giving $F_4 F_q = F_{4q}$. This contradicts result 1, and is thus false.

$F_k = F_r$.   Then $k = r$ and so $k = 4k$. This in turn implies that $k = 0$, which is not the case.

We now assume that $F_p$ and $F_q$ are distinct Fibonacci primes, neither of which is equal to 3. This condition guarantees that $p$ and $q$ are distinct primes, both of which are at least 3. If $F_r = F_p F_q$ then $r = kp = mq$ for some $k, m \in \mathbb{N}$. Note, bearing in mind that $p \neq q$ by assumption, that $k \neq p$. This is because $k = p$ would imply the false statement $q | p$. Since $F_k | F_{kp}$ we see that $F_k | F_p F_q$. As $F_p$ and $F_q$ are both prime, it follows that there are only the following possible scenarios.

$F_k = 1$.   We have $k = 1$ or $k = 2$. If $k = 1$ then $F_r = F_p$, and hence $F_q = 1$, which contradicts the fact that $F_q$ is prime. If, on the other hand, $k = 2$ then $F_r = F_{2p} = F_p F_q$. This implies, using result 2, that $q | 2p$. Since $p$ and $q$ are distinct primes, this tells us that $q | 2$. This is contradictory to $q$ being at least 3.

$F_k = F_p$.   Then $k = p$. We know, however, that this is not the case.

$F_k = F_q$.   Then $k = q$, so that $r = pq$, leading to $F_{pq} = F_p F_q$, which contradicts result 1.

$F_k = F_r$.   Then $k = kp$ and hence $k = 0$, which is not true.

Finally, let us look at the situation in which $F_r = F_p^2$. Since $F_4^2 = 3^2 = 9$, which is not a Fibonacci number, we may assume that $F_p$ is a prime not equal to 3 (and thus that $p$ is a prime with $p \geq 3$). We have $F_p | F_r$ and hence $p | r$. Therefore $r = kp$, from which it follows that $F_k | F_p^2$. There thus remain the following cases to consider.

$F_k = 1$.   We have $k = 1$ or $k = 2$. If $k = 1$ then $F_p = F_p^2$, which is not possible. If, on the other hand, $k = 2$ then $F_{2p} = F_p^2$. However, on using the result $F_{a+b} = F_{a-1} F_b + F_a F_{b+1}$ once more, this time with $a = b = p$, we obtain

$$F_{2p} = F_{p+p} = F_{p-1} F_p + F_p F_{p+1} > F_p^2$$

for $p \geq 2$, leading to a contradiction.

$F_k = F_p$. Then $k = p$ and hence $r = p^2$. However, from result 1, we know that $F_{p^2} > F_p^2$ when $p \geq 2$, which discounts this case.

$F_k = F_r$. Then $k = kp$ and hence $k = 0$, which is not true.

With all possible situations having been covered, this completes the proof, by contradiction, that no Fibonacci number may be expressed as the product of two Fibonacci primes.

## 4. Closing comments

Our proof would have been much easier if we had allowed ourselves the luxury of assuming Carmichael's theorem on Fibonacci numbers, see references 5, 6, and 7. This theorem tells us that, other than for $n \in \{1, 2, 6, 12\}$, $F_n$ has at least one prime factor that does not divide any earlier Fibonacci number. To take an example, we have $F_{16} = 987 = 3 \times 7 \times 47$. Considering the prime factors in turn, we note that 3 is a factor of $F_4 = 3$ and 7 is a factor of $F_8 = 21$, but 47 does not appear as a factor of $F_n$ for any $n \leq 15$.

To see how this implies our result, suppose that $F_r = F_p F_q$ for some $r \in \mathbb{N}$ and Fibonacci primes $F_p$ and $F_q$, noting that none of the exceptions to Carmichael's theorem may be expressed in this form. From the theorem we know that at least one of $F_p$ or $F_q$ does not divide any $F_n$ for which $n < r$. However, since $p < r$ and $q < r$, this clearly contradicts the fact that both $F_p$ and $F_q$ are Fibonacci numbers (as they certainly divide themselves).

Carmichael's theorem actually allows us to deduce a much stronger result, namely that, for any $m, n \geq 3$, the product $F_m F_n$ is not equal to a Fibonacci number. Taking things even further, it tells us in fact that, apart from the exceptions

$$F_6 = 8 = 2^3 = F_3^3,$$
$$F_{12} = 144 = 2^4 \times 3^2 = F_3^4 F_4^2,$$
$$F_{12} = 144 = 2 \times 3^2 \times 8 = F_3 F_4^2 F_6,$$

no Fibonacci number $F_n$ with $n \geq 3$ may be expressed as a product of two or more Fibonacci numbers, each of which has an index greater than 2. To take an example, without any further checking we are able to state with absolute certainty that

$$F_7^3 F_{10} F_{12} F_{23}^5 F_{102}$$

is not a Fibonacci number.

However, the use of this theorem here might be regarded in some sense as 'cheating', in that it is not totally in keeping with the elementary nature of our article. Indeed, while the proof of our result requires just some simple properties of the integers (as we have shown), proofs of Carmichael's theorem are beyond the scope of this article, and employ mathematical machinery associated with complex numbers, making them somewhat less elementary. Even a so-called simplified proof of this theorem (see reference 7) is not entirely straightforward.

A final point to make here is that, as is proved in reference 8, if $r \in \mathbb{N}$ then $F_r$ is a square number if, and only if, $r$ is equal to either 1, 2, or 12. This result tells us that $F_1 = F_2 = 1 = 1^2$ and $F_{12} = 144 = 12^2$ are the only Fibonacci squares. It follows that there is no Fibonacci number of the form $F_p^2$ for some Fibonacci prime $F_p$. Thus, if we had assumed this result, we could have removed the final set of cases in section 3, thereby shortening our proof. However, although the proof given in reference 8 is elementary, it is a moot point as to whether or not it might be referred to as 'simple', and for this reason we did not assume it.

## References

1 Wikipedia, Fibonacci prime, http://en.wikipedia.org/wiki/Fibonacci_prime.

2 E. W. Weisstein, Fibonacci Prime, http://mathworld.wolfram.com/FibonacciPrime.html.

3 D. M. Burton, *Elementary Number Theory*, (McGraw-Hill, 1998).

4 R. Knott, Fibonacci and Golden Ratio Formulae, http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibFormulae.html.

5 R. D. Carmichael, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, *Ann. Math.* **15** (1913), pp. 30–70.

6 Wikipedia, Carmichael's theorem, http://en.wikipedia.org/wiki/Carmichael's_theorem.

7 M. Yabuta, A simple proof of Carmichael's theorem on primitive divisors, *Fibonacci Quart.* **39** (2001), pp. 439–443.

8 J. H. E. Cohn, Square Fibonacci numbers, *Fibonacci Quart.* **2** (1964), pp. 109–113.

## The Collatz Problem

Lothar Collatz in 1937 posed the following problem. Start with any positive integer. If it is even, divide it by 2; if it is odd, multiply it by 3 and add 1:

$$n \quad \begin{cases} \nearrow \ n/2 & \text{if } n \text{ is even,} \\ \searrow \ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Repeat this process to give a sequence of positive integers, for example, starting with $n = 11$, gives 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. Collatz conjectured that the sequence always reaches 1. This is as yet unproven.

Suppose, instead, that we use the rule

$$n \quad \begin{cases} \nearrow \ n/2 & \text{if } n \text{ is even,} \\ \searrow \ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

e.g. 11, 12, 6, 3, 4, 2, 1. Readers are invited to prove that any such sequence always reaches 1.

University of Delphi, India                                    **Vinod Tyagi**

# Strategy for Identifying a Rogue Ball. II

## MERVYN HUGHES

In this article we continue our quest to find the minimum number of weighings to guarantee the location and identity of a rogue ball among many balls. A rogue ball may be lighter or heavier than the others, but is identical otherwise. We show that we need only five weighings to locate and identify a rogue ball among 120 balls.

This is a follow-up article to reference 1 in which we investigated the problem of identifying a different ball among up to 12 balls which are alike. All balls are similar in shape and colour and all have the same density except one, which is *known* to be either lighter or heavier than the rest (called a rogue ball). We discovered that identifying a heavy rogue is equivalent to identifying a light rogue. We found $f(n)$, the minimum number of weighings to *guarantee* the location of the rogue ball among 12 balls. In the case when we knew there was a rogue but did not know if it was light or heavy, we found $g(n)$, the minimum number of weighings needed to *guarantee* that the rogue ball will not only be located but also classified as light or heavy. We shall consider two cases:

*Case 1:* we know that there is a light rogue (or, equivalently, a heavy rogue),

*Case 2:* we know that there is a rogue, but need to determine whether it is light or heavy.

As the number of balls increases, there are more balls of correct weight, and these come into play more than they do with a lower number of balls. Throughout this article, we use the same notation and definitions as in reference 1, which we recall here for the convenience of the reader.

## 1. Notation and definitions

Consider $n$ balls, labelled $B_1, B_2, \ldots$. Let $S_1 = \{B_1, B_2, \ldots\}$ denote the set of balls. Let $W(S)$ and $W(B)$ be the weights of set $S$ and ball $B$, and, for simplicity, let $W_i$ be the weight of ball $B_i$.

### 1.1. Control balls and $x$-ball plants

We split the set of balls into three subsets. If we weigh a subset of balls against another subset of balls, there are two outcomes. (i) The scales balance, meaning that the rogue ball must be in the third subset. (ii) The scales do not balance. In case 1 this means that the rogue ball is in the lighter side. In case 2 this means that either the light side contains a light rogue or the heavy side contains a heavy rogue. In order to establish which, we use a ball plant.

We shall call a correctly weighted ball a *control* ball. To perform a *1-ball plant*, take a control ball and place it in the left pan, remove a ball from this left pan and place it in the right pan, and then remove a ball from the right pan (see figure 1). By convention, after a weighing we shall always consider the lighter side to be on the left (we can look at the scales from the opposite side, and re-label if necessary). We then re-weigh and consider the outcomes.

By performing a 1-ball plant we can isolate balls B and D in figure 1, and information can be gained about them as well as balls C and E from knowledge about the previous situation.

(a) Before a 1-ball plant



(b) A 1-ball plant



(c) After a 1-ball plant
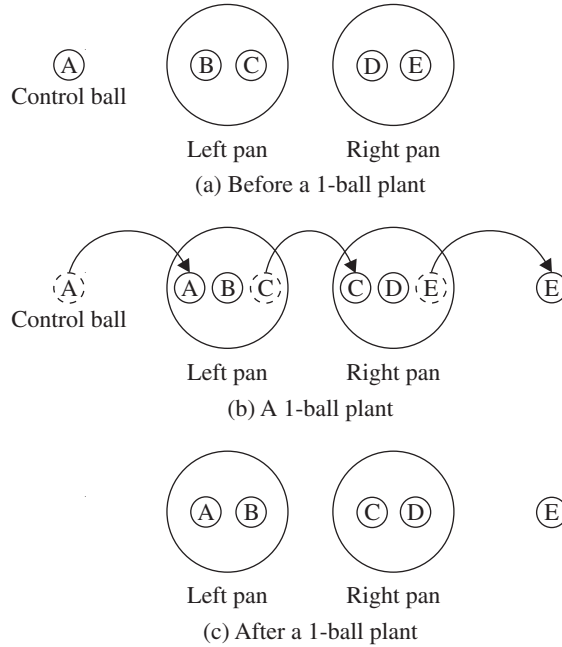
**Figure 1**

As we shall be planting many balls, we shall extend the above definition.

**Definition** Let there be $y$ balls in each pan and $x$ control balls available such that $0 < x < y$.

To perform an $x$-*ball plant*, place $x$ control balls in the left pan whilst removing $x$ balls from the left pan. Then place these $x$ removed balls in the right pan whilst removing $x$ balls from the right pan. Note that we can only perform an $x$-ball plant if we have $x$ control balls available and therefore have at least $3x + 2$ balls in total.

## 2. Up to 12 balls

In reference 1 we developed the following strategy for finding $f(n)$, the minimum number of weighings needed to locate a rogue ball of known weight, and $g(n)$, the minimum number of weighings needed to locate and identify a rogue ball of unknown weight, among $n$ balls for $n \le 12$. As always, the convention will be that, after a first weighing, the lighter side will be taken to be the left side.

Decompose the $n$ balls into three sets, $S_1$, $S_2$, and $S_3$, of sizes $(y, y, y - 1)$, $(y, y, y)$, and $(y, y, y + 1)$ for the cases in which $n = 3y - 1$, $n = 3y$, and $n = 3y + 1$, respectively. Weigh $S_1$ against $S_2$.

- If $W(S_1) = W(S_2)$ then neither set contains the rogue ball, which must be in $S_3$.

- If $W(S_1) < W(S_2)$ then the rogue ball is in $S_1$ if it is known to be light, or in either $S_1$ or $S_2$ if we do not know its relative weight, in which case we need to perform ball plants to discover its location and classification.

We give the results for $n < 13$ from reference 1 in table 1.

**Table 1**

| $n$ | $f(n)$ | $g(n)$ |
|-----|--------|--------|
| 2 | 1 | – |
| 3 | 1 | 2 |
| 4 | 2 | 3 |
| 5 | 2 | 3 |
| 6 | 2 | 3 |
| 7 | 2 | 3 |
| 8 | 2 | 3 |
| 9 | 2 | 3 |
| 10 | 3 | 3 |
| 11 | 3 | 3 |
| 12 | 3 | 3 |

The situation is far more complicated as the number of balls increases. There are many control balls available and these come into play. We note that in reference 1 we obtained $g(12) = 3$ and realised that the use of three control balls was crucial when deciding which type of rogue ball was in the third set of four balls, and it is this idea we shall develop in this article. We are therefore able to reduce the number of weighings from a possible four (i.e. $1 + g(4)$) to just three. So, having control balls available can reduce our number of weighings. We introduce some new notation to take into account the use of control balls. It can be shown, by looking at table 1, that having control balls does not reduce the value of $f(n)$.

Let $g(n \mid x)$ be the minimum number of weighings which guarantee the location and classification of the rogue ball among $n$ balls *given* that we have $x$ control balls available.

From table 1, $g(4) = 3$; we will now show that $g(4 \mid 3) = 2$. Weigh the three available control balls against three of the four balls. This will establish whether they balance (in which case the rogue is the fourth ball) or do not balance. If they do not balance, we immediately know whether the rogue is light or heavy. In all three situations, one more weighing locates the rogue, and determines whether it is light or heavy. Thus, $g(4 \mid 3) = 2$.

We need a sufficient number of control balls to reduce the value of $g(n)$. For example, it can easily be established that $g(4 \mid 2) = 3$, which does not reduce $g(4)$. We also note that $g(5 \mid 4) = 3$, showing that having control balls does not always reduce $g(n)$. Indeed, there are certain values which do help and these can be seen from table 1. These are the values of $x$ such that $f(x) + 1 = f(x + 1)$ or $g(x) + 1 = g(x + 1)$, which we shall call *critical* values. We can see that the first critical value is 3 and the next one is 9; we shall see later that $g(13 \mid 9) < g(13)$.

We now introduce some more notation which will be necessary in the later stages. After the initial weighing, we always consider the lighter pan to be on the left, with both pans containing $y$ balls, and $x < y$ control balls available. We then perform an $x$-ball plant.

Let $g((y, y); x)$ denote the minimum number of weighings to *guarantee* the location and classification of the rogue ball in this situation. Now $g((1, 1); 0)$ is not defined since we do not have any control balls to use, and it is easy to see that $g((1, 1); 1) = 1$. We note that in reference 1, when we found $g(12)$, we used the fact that $g((4, 4); 3) = \max\{1 + f(3), 1 + g((1, 1); 1)\} = 2$ to give $g(12) = 3$.

The results of $g(n \mid x)$ and $g((y, y); x)$ for the cases where $n > 12$, for $x < y < n$, will be important.

## 3. More than 12 balls

For $n = 13$, we split the balls into three sets of sizes 4, 4, and 5, respectively, and weigh the first two sets. If

- $\sum_1^4 W_i = \sum_5^8 W_i$, we require $f(5)$ weighings for case 1 and $g(5)$ weighings for case 2,

- $\sum_1^4 W_i < \sum_5^8 W_i$, we require $f(4)$ weighings for case 1 and, for case 2, we need to perform a 3-ball plant, leaving $f(3)$ or $g((1, 1); 1)$ weighings.

Hence, $f(13) = 3$ and $g(13) = 4$. This shows that if we can only use three weighings then we have a maximum of 12 balls to locate and identify a rogue ball of unknown weight. We can continue to find values of $f(n)$ and $g(n)$ as $n$ increases. The methods are very similar, but an interesting case is that of 27 balls, where we split the balls into three sets of size 9. If

- $\sum_1^9 W_i = \sum_{10}^{18} W_i$ then we require $1 + f(9)$ weighings for case 1 and $1 + g(9)$ weighings for case 2,

- $\sum_1^9 W_i < \sum_{10}^{18} W_i$ then we require $1 + f(9)$ weighings for case 1 and, for case 2, we need to perform an 8-ball plant, leaving $f(8)$ or $g((1, 1); 1)$ weighings.

Thus, $f(27) = \max\{1 + f(9), 1 + f(9)\} = 3$ and $g(27) = \max\{1 + g(9), 2 + f(8), 1 + g((1, 1); 1)\} = 4$.

For 28 balls, we split the balls into sets of sizes 9, 9, and 10. If the two sets of nine balls balance, we shall require $f(10)$ weighings for case 1 and $g(10)$ weighings for case 2. If they do not balance then we require $f(9)$ weighings for case 1, and, for case 2, we perform an 8-ball plant and will need $f(8)$ or $g((1, 1); 1)$ weighings. Thus, $f(28) = 4$ and $g(28) = 4$. This shows that the largest number $n$ such that $f(n) = 3$ is 27 and that the smallest number $n$ such that both $f(n)$ and $g(n)$ are equal to 4 is 28. Thus, 27 is the next critical value after 9. Values of $f(n)$ and $g(n)$ for $13 \leq n \leq 27$ are given in table 2.

**Table 2**

| $n$ | $f(n)$ | $g(n)$ |
|---|---|---|
| 13 | 3 | 4 |
| 14 | 3 | 4 |
| 15 | 3 | 4 |
| 16 | 3 | 4 |
| 17 | 3 | 4 |
| 18 | 3 | 4 |
| 19 | 3 | 4 |
| 20 | 3 | 4 |
| 21 | 3 | 4 |
| 22 | 3 | 4 |
| 23 | 3 | 4 |
| 24 | 3 | 4 |
| 25 | 3 | 4 |
| 26 | 3 | 4 |
| 27 | 3 | 4 |

## 4. Up to 40 balls

The methods are again very similar to those of the previous section, using the same strategy and tables 1 and 2 to evaluate $f(x)$ and $g(x)$ for $x < n$. For example, for $n = 30$, we simply split the balls into three sets of size 10 and weigh the first two sets. If they balance then we require $f(10)$ weighings for case 1 and $g(10)$ weighings for case 2 to locate the rogue ball in the third set. If they do not balance then, for case 1, we require $f(10)$ weighings and, for case 2, we perform a 9-ball plant, leaving us with the option of $f(9)$ or $g((1, 1) : 1)$ weighings. Thus, to guarantee the location and identity of the rogue, we require $\max\{1 + f(10), 1 + f(10)\}$ for case 1 and $\max\{1 + g(10), 1 + 1 + f(9), 1 + 1 + g((1, 1); 1)\}$ for case 2. Thus, $f(30) = g(30) = 4$.

When we reach 39 balls, the situation is slightly different and the critical values of $x$ come into play. From table 1, the first critical value is 3, giving $g(4 \mid 3) = 2$. The next critical value is 9, and we now show that $g(13 \mid 9) = 3$, which is less than $g(13)$.

Weigh nine of the 13 balls against the nine control balls, thus establishing a known light or known heavy rogue within the nine balls or an unknown light or heavy rogue in the remaining four. Thus, $g(13 \mid 9) = \max\{1 + f(9), 1 + g(4 \mid 3)\}$, which equals 3.

For 39 balls, we split the balls into three sets of size 13 and weigh the first two sets. If they balance then the rogue ball is in the third set and we require $f(13)$ further weighings for case 1 and $g(13 \mid 9)$ further weighings for case 2 to locate the rogue ball. If they do not balance then, for case 1, we require $f(13)$ weighings and, for case 2, we perform a 9-ball plant, leaving us with the option of $f(9)$ or $g((4, 4); 3)$ weighings. To guarantee the location and identity of the rogue, we require $\max\{1 + f(13), 1 + f(13)\}$ weighings for case 1 and $\max\{1 + g(13 \mid 9), 1 + 1 + f(9), 1 + 1 + g((4, 4); 3)\}$ weighings for case 2. Thus, $f(39) = g(39) = 4$.

It is worth noting that we do not perform a 12-ball plant, but instead perform a 9-ball plant followed by a 3-ball plant. We also note that this is preferable to using a 3-ball plant followed by a 9-ball plant. Thus, the order is important and ball planting is not commutative, so that when performing multiple ball plants, we must ensure that if we perform an $x$-ball plant followed by a $y$-ball plant then $y \leq x$.

The above remarks suggest that there is a link between using control balls and performing a ball plant. It can readily be seen that $g(z \mid x) = g((z, z); x)$ for $x < z$. We can establish that $f(40) = 4$ and $g(40) = 5$, but $g(40 \mid 27) = 3$ using similar techniques. The results for $f(n)$ and $g(n)$, $28 \leq n \leq 39$, are given in table 3.

**Table 3**

| $n$ | $f(n)$ | $g(n)$ |
| --- | --- | --- |
| 28 | 4 | 4 |
| 29 | 4 | 4 |
| 30 | 4 | 4 |
| 31 | 4 | 4 |
| 32 | 4 | 4 |
| 33 | 4 | 4 |
| 34 | 4 | 4 |
| 35 | 4 | 4 |
| 36 | 4 | 4 |
| 37 | 4 | 4 |
| 38 | 4 | 4 |
| 39 | 4 | 4 |

## 5. Five weighings and 120 balls

Finally, we consider locating and identifying a rogue ball among 120 balls. Decompose the 120 balls into three sets of size 40, i.e.

$$S_1 = \{B_1, B_2, \ldots, B_{40}\},$$
$$S_2 = \{B_{41}, \ldots, B_{80}\},$$
$$S_3 = \{B_{81}, \ldots, B_{120}\},$$

and weigh $S_1$ against $S_2$. We will consider case 1 and case 2 separately.

**Case 1**

- If $\sum_1^{40} W_i = \sum_{41}^{80} W_i$ then the rogue ball is in $S_3$ and we require $f(40)$ further weighings.

- If $\sum_1^{40} W_i < \sum_{41}^{80} W_i$ then we also require $f(40)$ further weighings.

Thus, $f(120) = 1 + f(40)$.

**Case 2**

- If $\sum_1^{40} W_i = \sum_{41}^{80} W_i$ then the rogue ball is in $S_3$ and we require $g(40 \mid 27)$ further weighings using 27 of the 80 control balls available.

- If $\sum_1^{40} W_i < \sum_{41}^{80} W_i$ then we perform a 27-ball plant leaving either $f(27)$ further weighings or the situation we met in section 4 in which we have 13 balls in each pan with one side lighter than the other. Simply perform a 9-ball plant followed by a 3-ball plant.

Thus,

$$g(120) = \max\{2 + f(27), 3 + f(9), 4 + f(3), 4 + g((1, 1); 1\} = 5.$$

See figure 2 for a flow diagram to determine case 2 for $n = 120$.

   In reference 1 we discovered that if we are allowed only three weighings then we can locate and identify a rogue of unknown weight among 12 balls. However, in this article we have established that if we increase the number of weighings to five then we can locate and identify a rogue ball among 120.

### Reference

1 M. Hughes, Strategy for identifying a rogue ball. I, *Math. Spectrum* **45** (2013), pp. 63–68.

*Mervyn Hughes* followed a PhD, gained at The University of Aberystwyth, by teaching mathematics in two Sixth Form Colleges, in Hereford and Nuneaton. He has published several articles on Complex Reflection Groups. For over 30 years, he enjoyed teaching all areas of A level mathematics. He is about to retire from teaching and also retire from his other passion of playing cricket.
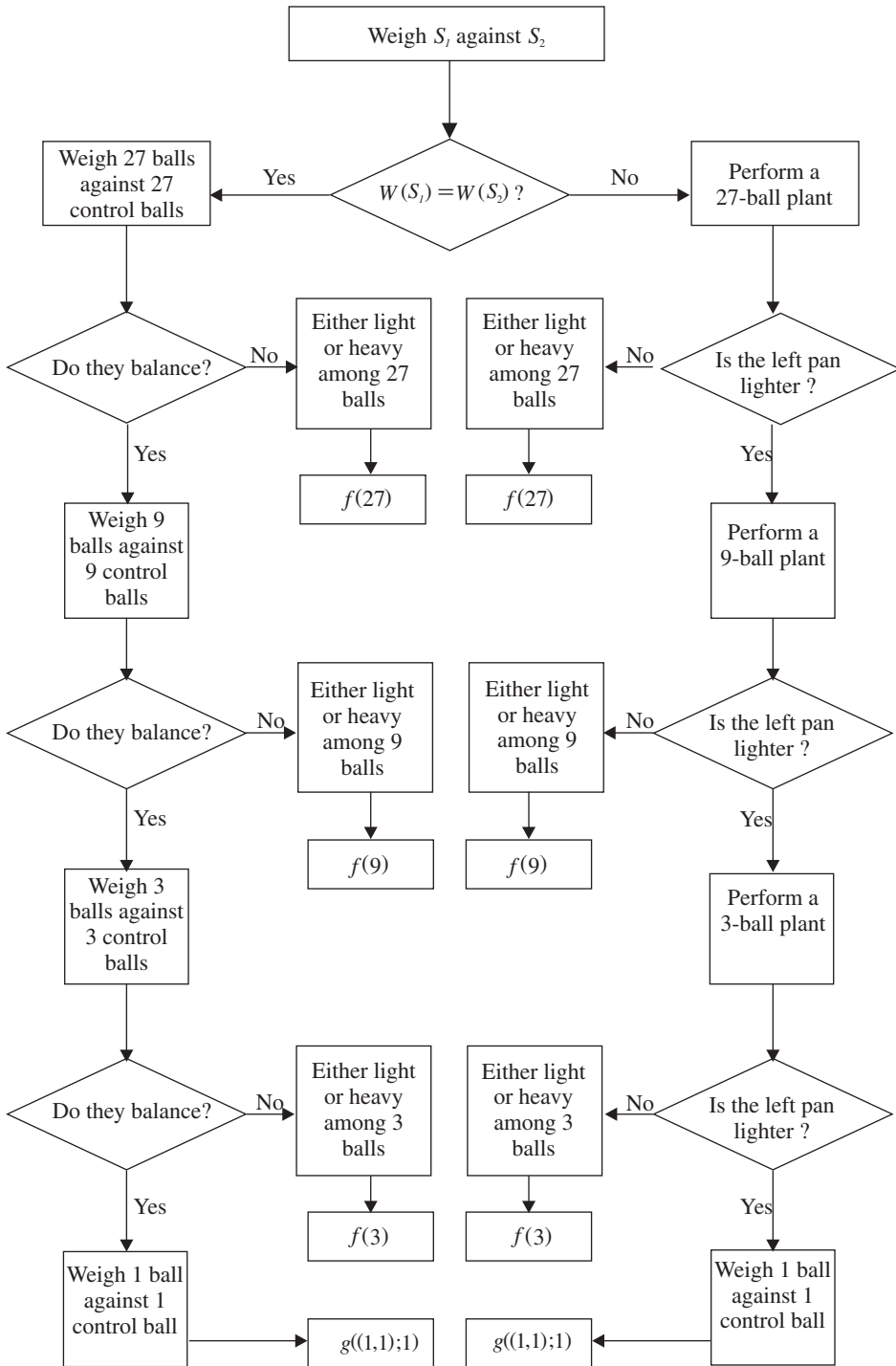
**Figure 2** Schematic of how to calculate $g(120)$.

# Letters to the Editor

Dear Editor,

$$T_a + T_b = T_c$$

In Volume 44, Number 2, page 50, Tom Moore considered this equation, where $T_a$ denotes the $a$th triangular number. Now

$$T_{n-2} + (n-1) + n + (n-1) = T_{n+1}.$$

Try putting $T_k = 3n$. Then $k(k+1) = 6n$, so that $k$ must be one of the forms $6m, 6m+2, 6m+3, 6m+5$. For example, put $k = 11$, when $n = 22$ and

$$T_{20} + T_{11} = T_{23}.$$

Similarly,
$$T_{n-3} + (n-2) + (n-1) + n + (n+1) + (n+2) = T_{n+2}.$$

Try putting $T_k = 5n$. Then $k(k+1) = 10n$, so that $k$ must be one of the forms $10m, 10m+4, 10m+5, 10m+9$. For example, put $k = 29$, when $n = 87$ and

$$T_{84} + T_{29} = T_{89}.$$

I have also found the following solutions for $(a, b, c)$:

$$(3n, 4n+1, 5n+1),$$
$$(3n+2, 4n+2, 5n+3),$$
$$(12n+9, 5n+4, 13n+10),$$
$$(12n+14, 5n+5, 13n+15),$$
$$(15n+5, 8n+3, 17n+6),$$
$$(15n+9, 8n+4, 17n+10),$$
$$(4n^2+3n-1, 4n+1, 4n^2+3n+1),$$
$$(4n^2+5n, 4n+2, 4n^2+5n+2).$$

If $n = T_x + T_y$ then

$$T_{n-1} + T_x + T_y = T_n.$$

If $2n - 1 = T_x + T_y$ then

$$T_{n-2} + T_x + T_y = T_n.$$

Further examples are

$$T_{18} + T_5 + T_9 = T_{21}$$

and

$$T_{29} + T_4 + T_7 + T_{10} = T_{32}.$$

The equation

$$T_1 + T_2 + \cdots + T_x = T_y$$

has $(x, y) = (3, 4), (8, 15), (20, 55)$ as solutions. Are there others?

The equation

$$T_x + T_{x+1} = T_y$$

has $(x, y) = (5, 8), (34, 49)$ as solutions.

As a postscript,

$$T_{20} = 2T_{14}, \qquad T_2 T_5 = T_9.$$

Yours sincerely,
**Abbas Rouholamini Gugheri**
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

$$x^8 + y^8 + z^8 = 2w^8$$

The formulae

$$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2,$$

where $m$ and $n$ are natural numbers with $m > n$, give all the natural number solutions of Pythagoras' equation $x^2 + y^2 = z^2$. It can be verified that, for natural numbers $m$ and $n$ with $m > n$,

$$x = 2mn, \qquad y = m^2 - n^2, \qquad z = m^2 + n^2,$$
$$w = (m^4 - 2m^3 n + 2m^2 n^2 + 2mn^3 + n^4)(m^4 + 2m^3 n + 2m^2 n^2 - 2mn^3 + n^4),$$

satisfy the equation

$$x^8 + y^8 + z^8 = 2w^8.$$

Does this give all the natural number solutions of the equation?

Yours sincerely,
**Muneer Karama**
(Hebron Education Officer
Jerusalem
Box 19149
UNRWA)

Dear Editor,

*Integral triangles with a* $120°$ *angle*

In Volume 43, Number 2, pp 60–64, Kostantine Zelator found all such triangles. Readers may be interested in alternative ways of obtaining a family of such triangles. The cosine rule applied to the triangle in figure 1 gives

$$c^2 = a^2 + b^2 - 2ab \cos 120°,$$

i.e.

$$c^2 = a^2 + b^2 + ab.$$
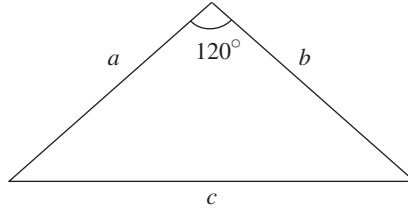
**Figure 1**

Define the 'Fibonacci-like sequence' ($f_n$) by the recurrence relation

$$f_n = 4f_{n-1} - f_{n-2} \quad \text{for } n > 2,$$

with $f_1 = 1$, $f_2 = 4$. Thus, the sequence begins

$$1, 4, 15, 56, 209.$$

For $n > 1$, put $a = f_n - 1$, $b = f_n + 1$. Then

$$c^2 = (f_n - 1)^2 + (f_n + 1)^2 + (f_n - 1)(f_n + 1) = 3f_n^2 + 1.$$

Now $f_n$ can be given by the 'Binet-like formula'

$$f_n = \frac{1}{2\sqrt{3}}\{(2 + \sqrt{3})^n - (2 - \sqrt{3})^n\}.$$

Thus,

$$3f_n^2 + 1 = \tfrac{1}{4}\{(2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n} - 2\} + 1$$
$$= \{\tfrac{1}{2}[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n]\}^2$$

and the term in curly brackets is an integer. Hence, $c$ is an integer. Thus we have an infinite family of such triangles given by figure 2.
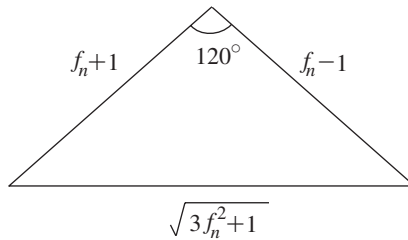


**Figure 2**

When $n$ is odd, the side lengths will have highest common factor 2, so we divide through by 2. The first four such triangles have lengths

$$(3, 5, 7), \qquad (7, 8, 13), \qquad (55, 57, 97), \qquad (104, 105, 181).$$

Another forumlation of a family of solutions is

$$a = r(r + 2s + 2t), \qquad b = (s + t)^2 - r^2, \qquad c = (s + t)^2 + r(r + s + t),$$

where $r, s, t$ are natural numbers. The condition $s + t > r$ is needed to ensure that $a + b > c$.

Table 1 shows families of solutions where $n$ is a positive integer.

**Table 1**

| $a$ | $b$ | $c$ | |
|---|---|---|---|
| $2n + 1$ | $n^2 - 1$ | $n^2 + n + 1$ | $(n > 1)$ |
| $2n + 1$ | $3n^2 + 2n$ | $3n^2 + 3n + 1$ | $(n > 0)$ |
| $4n$ | $n^2 - 2n - 3$ | $n^2 + 3$ | $(n > 3)$ |
| $4n$ | $3n^2 - 2n - 1$ | $3n^2 + 1$ | $(n > 1)$ |
| $8n$ | $n^2 - 4n - 12$ | $n^2 + 12$ | $(n > 6)$ |
| $8n$ | $3n^2 - 4n - 4$ | $3n^2 + 4$ | $(n > 2)$ |
| $8n$ | $4n^2 - 4n - 3$ | $4n^2 + 3$ | $(n > 1)$ |
| $8n$ | $12n^2 - 4n - 1$ | $12n^2 + 1$ | $(n > 0)$ |

Yours sincerely,
**K. S. Bhanu** and **M. N. Deshpande**
(c/o Institute of Science
Nagpur 440001
India)

Dear Editor,

*A Fibonacci similarity*

In Volume 44, Number 2, page 91, Bob Bertuello introduced a Fibonacci-like sequence, $x_n$, and conjectured that the sum of the first $4k - 2$ terms is $r_k x_{2k+1}$, where the sequence $r_n$ is defined by the recurrenece relation

$$r_1 = 1, r_2 = 4, r_k = 3r_{k-1} - r_{k-2} \quad \text{for } k \geq 3.$$

I have proved this and found the formula

$$r_k = F_{2k} + F_{2k-2},$$

where $F_m$ denotes the $m$ th Fibonacci number.

Yours sincerely,
**Abbas Rouholamini Gugheri**
(Students' Investigation House
Shariati Avenue
Sirjan
Iran)

Dear Editor,

*Reversing digits*

I read with interest the article on Reversing Digits in Volume 45, Number 2, pp. 69–71. I recalled some ancient jottings I have, and found the following. The number $865\,281\,023\,607$ is a multiple of $111\,111$ and its reversal $706\,320\,182\,568$ is also a multiple of $111\,111$. Further, the cyclic rotations of the numbers are also divisible by $111\,111$. Thus,

$$865\,281\,023\,607 = 111\,111 \times 7\,787\,537,$$
$$786\,528\,102\,360 = 111\,111 \times 7\,078\,760,$$
$$078\,652\,810\,236 = 111\,111 \times 707\,876,$$
$$607\,865\,281\,023 = 111\,111 \times 5\,470\,793,$$
$$360\,786\,528\,102 = 111\,111 \times 3\,247\,082,$$
$$236\,078\,652\,810 = 111\,111 \times 2\,124\,710,$$
$$023\,607\,865\,281 = 111\,111 \times 212\,471,$$
$$102\,360\,786\,528 = 111\,111 \times 921\,248,$$
$$810\,236\,078\,652 = 111\,111 \times 7\,292\,132,$$
$$281\,023\,607\,865 = 111\,111 \times 2\,529\,215,$$
$$528\,102\,360\,786 = 111\,111 \times 4\,752\,926,$$
$$652\,810\,236\,078 = 111\,111 \times 5\,875\,298,$$

and

$$706\,320\,182\,568 = 111\,111 \times 6\,356\,888,$$
$$870\,632\,018\,256 = 111\,111 \times 7\,835\,696,$$
$$687\,063\,201\,825 = 111\,111 \times 6\,183\,575,$$
$$568\,706\,320\,182 = 111\,111 \times 5\,118\,362,$$
$$256\,870\,632\,018 = 111\,111 \times 2\,311\,838,$$
$$825\,687\,063\,201 = 111\,111 \times 7\,431\,191,$$
$$182\,568\,706\,320 = 111\,111 \times 1\,643\,120,$$
$$018\,256\,870\,632 = 111\,111 \times 164\,312,$$
$$201\,825\,687\,063 = 111\,111 \times 1\,816\,433,$$
$$320\,182\,568\,706 = 111\,111 \times 2\,881\,646,$$
$$632\,018\,256\,870 = 111\,111 \times 5\,688\,170,$$
$$063\,201\,825\,687 = 111\,111 \times 568\,817.$$

Yours sincerely,
**Bob Bertuello**
(Midsomer Norton
Bath)

Dear Editor,

*Prime rationals*

In Volume 44, Number 2, p.90, Guido Lasters raises the question of whether the set of 'prime rationals', $\{\pm p/q : p, q \text{ are prime}\}$, is dense in the set of all real numbers. This is indeed so. It suffices to show that, for all real $\alpha, \beta$ with $0 < \alpha < \beta$, there exist primes $p, q$ such that $\alpha < p/q < \beta$. With $\pi(x)$ denoting the number of primes not exceeding $x$, the prime number theorem shows that

$$\pi(\beta n) - \pi(\alpha n) \sim \frac{\beta n}{\ln(\beta n)} - \frac{\alpha n}{\ln(\alpha n)} = \frac{n \ln(\alpha^\beta \beta^{-\alpha} n^{\beta - \alpha})}{\ln(\alpha n) \ln(\beta n)} \to \infty \quad \text{as } n \to \infty,$$

in particular, there is a prime between $\alpha n$ and $\beta n$ once $n$ is sufficiently large. It follows that, for all sufficiently large primes $q$, there exists a prime $p$ with $\alpha q < p < \beta q$ so that $\alpha < p/q < \beta$ as required.

This is not a new result – there is quite a large literature on it and its extensions. See, for example, the recent article (see reference 1) and the references therein to earlier work.

<div align="center">

Yours sincerely,
**Nick Lord**
(Tonbridge School,
Kent TN9 1JP
UK)
</div>

**Reference**

1  S. R. Garcia, V. Selhorst-Jones, D. E. Poore and N. Simon, Quotient sets and Diophantine equations, *Amer. Math. Monthly* **118** (2011), pp. 704–711.

Dear Editor,

*Sums of consecutive primes and magic squares*

The first prime number that is the sum of three consecutive primes is $23 = 5 + 7 + 11$. With a table of primes handy it is easy to find other such primes. I have found that the first 25 such primes form the 'semi-magic square' shown in figure 1, in that all row sums and all column sums are equal, namely 1463. The two diagonal sums are not 1463, as would be the case for a magic square.

| 83 | 71 | 829 | 211 | 269 |
|-----|-----|-----|-----|-----|
| 349 | 23 | 311 | 607 | 173 |
| 701 | 251 | 41 | 439 | 31 |
| 131 | 661 | 59 | 109 | 503 |
| 199 | 457 | 223 | 97 | 487 |

<div align="center">

**Figure 1**
</div>

I tried to do a similar thing for primes which are the sum of five consecutive primes, and was able to produce the 'magic L-shape' shown in figure 2. Its three row sums and three column sums are all the same, namely 3141, but I had to miss out 83, 199, 311, 331, 421, 617, and 1151.

When I came to primes which are the sum of seven consecutive primes, I was able to make the square shown in figure 3, in which all row sums are the same, namely 5117, but the column

| 1193 | 373 | 53 | 839 | 683 |
|------|-----|------|-----|------|
| 101 | 991 | 139 | 787 | 1123 |
| 587 | 863 | 1361 | 67 | 263 |
| 811 | 181 | 941 | | |
| 449 | 733 | 647 | | |

**Figure 2**

| 2861 | 223 | 1063 | 251 | 719 |
|------|------|------|------|------|
| 593 | 2659 | 827 | 281 | 757 |
| 991 | 401 | 2347 | 947 | 431 |
| 311 | 463 | 197 | 2393 | 1753 |
| 523 | 659 | 863 | 1171 | 1901 |

**Figure 3**

sums are not all the same. There is one entry which is prime but is not the sum of seven consecutive primes, namely 2659, and I had to miss out 2689, 2731, 2777, and 2819.

The question arises: is it possible to produce $5 \times 5$ or other sized magic or semi-magic squares (or L-shapes) using other (odd) numbers of consecutive primes?

Yours sincerely,
**Anand Prakash**
(s/o Kashi Nath Prasad
Dist-East Champaran
Bihar, Pin-845424
India)

---

**Card Shuffling**

1. Four cards numbered 1 to 4 are shuffled and dealt. What is the probability that the first card is a 1 or the second card is a 2 or the third card is a 3 or the fourth card a 4? What is the probability that exactly two of these occur?

2. Four cards are dealt from a shuffled pack of playing cards. What is the probability that either the first card is an ace or the second is a king or the third is a queen or the fourth is a jack? what is the probability that exactly two of these occur?

**Reference**

1 Hubert Phillips, *Problems Omnibus Volume 1, Word Puzzles, Inference, Mathematics* (ARCO Publications, 1960)

Midsomer Norton, Bath **Bob Bertuello**

# Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st November will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

## Problems

Correction: In Volume 45 Number 2, we wrongly attributed Problem 44.10 and its solution to Tom Moore. It was in fact proposed and solved by Zhang Yun.

**45.9** If $a + b > 0$, find the minimum value of $a \sin^4 x + b \cos^4 x$.

(Submitted by Abbas Rouholamini Gugheri, Sirjan, Iran.)

**45.10** A known inequality for natural logarithms is $\ln x \leq x - 1$ for all $x > 0$. Prove the following refinement of this inequality:

$$\ln x \leq \frac{2}{3}\sqrt{x} - \frac{2}{\sqrt{x}} + \frac{1}{3x} + 1 \leq \sqrt{x} - \frac{1}{\sqrt{x}} \leq x - 1$$

for all $x \geq 1$.

(Submitted by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece)

**45.11** Let $T_n = \frac{1}{2}n(n + 1)$ be the $n$th triangular number and $P_n = \frac{1}{2}n(3n - 1)$ be the $n$th pentagonal number. Prove that among $T_m + P_n$, where $m, n$ are positive integers, there are infinitely many different cubes.

(Submitted by Tom Moore, Bridgewater State University, MA 02325, USA)

**45.12** In the diagram, $ABCD$ is a rectangle and $P, Q$ are the midpoints of the straight lines $CE, DE$. What is the area of the quadrilateral $ABPQ$?

(Submitted by Subramanyam Durbha, Community College of Philadelphia, USA)
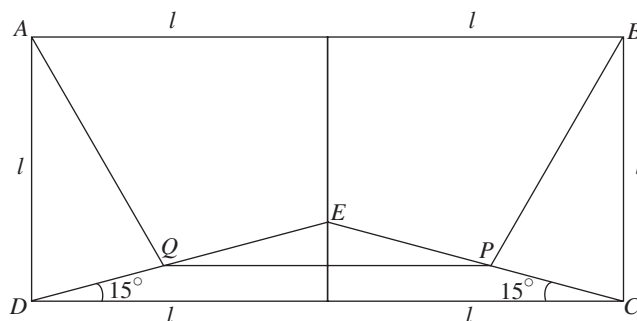
## Solutions to Problems in Volume 45 Number 1

**45.1** For $a > b > 0$, prove that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2ab}.$$

*Solution* by Abbas Rouholamini Gugheri, who proposed the problem

$$0 \leq (a - b - \sqrt{2ab})^2 = (a - b)^2 - 2\sqrt{2ab}(a - b) + 2ab = a^2 + b^2 - 2\sqrt{2ab}(a - b),$$

so that

$$\frac{a^2 + b^2}{a - b} \geq 2\sqrt{2ab}.$$

Also solved by Spiros Andriopoulos, Third High School of Amaliada, Eleia, Greece.

**45.2** The $x$-coordinates of the endpoints of an arc of the circle centre the origin radius $r$ are $a$ and $b$. Express the $x$-coordinate of the midpoint of the arc in terms of $a$, $b$, and $r$.

*Solution* by Gregory and Alex Akulov, who proposed the problem.
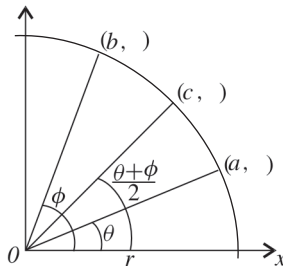


From the diagram,

$$a = r \cos \theta,$$
$$b = r, \cos \phi$$

$$c = r \cos\left(\frac{\theta + \phi}{2}\right)$$

$$= r\left(\cos \frac{\theta}{2} \cos \frac{\phi}{2} - \sin \frac{\theta}{2} \sin \frac{\phi}{2}\right)$$

$$= r\left(\sqrt{\frac{1 + \cos \theta}{2} \frac{1 + \cos \phi}{2}} - \sqrt{\frac{1 - \cos \theta}{2} \frac{1 - \cos \phi}{2}}\right)$$

$$= \frac{r}{2}\left(\sqrt{\left(1 + \frac{a}{r}\right)\left(1 + \frac{b}{r}\right)} - \sqrt{\left(1 - \frac{a}{r}\right)\left(1 - \frac{b}{r}\right)}\right)$$

$$= \tfrac{1}{2}(\sqrt{(r + a)(r + b)} - \sqrt{(r - a)(r - b)}).$$

**45.3** Given a triangle $ABC$ and a point $P$ in the same plane as the triangle, the centroids of triangles $BCP$, $CAP$, $ABP$ are denoted by $L$, $M$, $N$, respectively. What is the connection between the areas of the triangles $LMN$ and $ABC$?

*Solution* by Zhang Yun, who proposed the problem.

Relative to rectangular axes $Oxy$, denote the coordinates of $P$, $A$, $B$, $C$ by $(x_i, y_i)$, where $i = 0, 1, 2, 3$ respectively. Then

$$L = \left( \frac{x_0 + x_2 + x_3}{3}, \frac{y_0 + y_2 + y_3}{3} \right),$$

$$M = \left( \frac{x_0 + x_3 + x_1}{3}, \frac{y_0 + y_3 + y_1}{3} \right),$$

$$N = \left( \frac{x_0 + x_1 + x_2}{3}, \frac{y_0 + y_1 + y_3}{3} \right),$$

and the area of triangle $LMN$ is the absolute value of

$$\frac{1}{2} \begin{vmatrix} \frac{x_0+x_2+x_3}{3} & \frac{y_0+y_2+y_3}{3} & 1 \\ \frac{x_0+x_3+x_1}{3} & \frac{y_0+y_3+y_1}{3} & 1 \\ \frac{x_0+x_1+x_2}{3} & \frac{y_0+y_1+y_2}{3} & 1 \end{vmatrix} = \frac{1}{18} \begin{vmatrix} \frac{x_0+x_2+x_3}{3} & \frac{y_0+y_2+y_3}{3} & 1 \\ x_1 - x_2 & y_1 - y_2 & 0 \\ x_1 - x_3 & y_1 - y_3 & 0 \end{vmatrix}$$

$$= \frac{1}{18} \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_1 - x_3 & y_1 - y_3 \end{vmatrix}$$

$$= \frac{1}{18} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

$$= \frac{1}{18} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \tfrac{1}{9} \times \text{area of } \triangle ABC.$$

**45.4** A cubic curve has equation $y = x^3 + ax^2 + bx + c$, where the polynomial has real coefficients and three real roots. Show that

(a) the $x$-coordinate of the point of inflection is the mean of the roots,

(b) the sum of the slopes of the normals at the roots is zero,

(c) it has two-fold symmetry about its point of inflection.

*Solution* by Michael Wragg, who proposed the problem.

(a) Denote the roots by $\alpha$, $\beta$, $\gamma$, so that the curve has equation

$$y = (x - \alpha)(x - \beta)(x - \gamma).$$

Then

$$\frac{dy}{dx} = (x - \beta)(x - \gamma) + (x - \gamma)(x - \alpha) + (x - \alpha)(x - \beta),$$

$$\frac{d^2 y}{dx^2} = 2(x - \alpha + x - \beta + x - \gamma).$$

The point of inflection occurs when $\frac{d^2 y}{dx^2} = 0$, i.e. when $x = \frac{1}{3}(\alpha + \beta + \gamma)$, which is the mean of the roots.

(b) The slope of the tangent at $x = \alpha$ is $(\alpha - \beta)(\alpha - \gamma)$, so the sum of the slopes of the normals at the roots is

$$-\frac{1}{(\alpha - \beta)(\alpha - \gamma)} - \frac{1}{(\beta - \gamma)(\beta - \alpha)} - \frac{1}{(\gamma - \alpha)(\gamma - \beta)}$$
$$= \frac{(\beta - \gamma) + (\gamma - \alpha) + (\alpha - \beta)}{(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)}$$
$$= 0.$$

(c) The point of inflection has $x$-coordinate $-a/3$, because $\alpha + \beta + \gamma = -a$. If we put $x' = x + a/3$, the equation becomes of the form $y = x'^3 + px' + q$ and the point of inflection is $(0, q)$ in the $0x'y$-axes. If we put $y' = y - q$, the equation becomes $y' = x'^3 + px'$, and we have translated the origin to the point of inflection. The point $(x', y')$ reflects in the origin to $(-x', -y')$ which leaves the equation $y' = x'^3 + px'$ unaltered, so the curve has two-fold symmetry about its point of inflection.

# Reviews

**Foundations and Applications of Statistics: an Introduction using R.** By Randall Pruim. American Mathematical Society, 2011. Hardback, 615 pages, $85.00 (ISBN 978-0-8218-5233-0).

*Foundations and Applications of Statistics* discusses both the mathematical theory underlying statistics and practical applications that make it a powerful tool across disciplines. The book contains ample material for a two-semester course in undergraduate probability and statistics. A one-semester course based on the book will cover hypothesis testing and confidence intervals for the most common situations.

**Guesstimation 2.0: Solving Today's Problems on the Back of a Napkin.** By Lawrence Weinstein. Princeton University Press, 2012. Paperback, 384 pages, $19.95 (ISBN 978-0-6911-5080-2).

How tall is a stack of a trillion one-dollar bills? How far should you be willing to walk to recycle a glass beer bottle? What is the shortest day the Earth could have without flying apart? *Guesstimation 2.0* poses a total of eighty such fun problems, and shows that, after making reasonable assumptions, it is possible to estimate answers to its posed problems. The book is accessible to anyone with a good head for numbers, and the ability to do elementary algebra.

The spirit of the problems encourage readers to model and focus on the core of the question, a practice that is important in everyday problem solving and is under-represented in the standard mathematical curriculum. In fact, there has been a growing call for A-level problems of this

nature, as such questions demonstrate how analytic thinking can extend beyond the abstract to give surprising answers to all sorts of questions.

*Guesstimation*'s problems are fun and engaging in character, and the solutions are intuitive and well explained. Each problem and solution stands independently, and is about four pages long, making the book ideal for passing a quick ten minutes, and easy to pick up and put down. If, like me, you like ill-posed questions to have concrete answers then *Guesstimation* is definitely a good place to hone your estimation skills!

University of Sheffield                                           **Fionntan Roukema**

**Fibonacci and Catalan Numbers – An Introduction.** By Ralph P. Grimaldi. John Wiley, Hoboken, NJ, 2012. Hardback, 380 pages, £66.95 (ISBN 978-0-4706-3157-7).

The exploration of Fibonacci numbers is a fascination shared by many mathematicians of all abilities. It certainly seems that there is no upper bound to the number of places where these numbers can appear. For example, the Fibonacci numbers can be used to tell you how many ways there are to fill a $2 \times n$ rectangle using $1 \times 2$ rectangles. Perhaps less well known are the Catalan numbers, another important set of numbers that have also been found to make numerous appearances.

This book has been produced to induct more people into the exploration of these numbers. Split into two sections, one for Fibonacci and the other for Catalan numbers, the author provides a healthy mix of material. Both kinds of numbers are first introduced via the original problems that created them. After this, the book contains many small chapters, each featuring different aspects of theory and nice "gems". Amongst these are the Binet formula for the $n$th Fibonacci number, tiling problems, recursions, identities, number theoretical properties (including uses in partitions), chess problems, optics, nature, graph theory, chemistry, shortest routes in grids, triangulation of polygons, and so on.

The overall style of writing is engaging and there is something for readers of all abilities, but this book is mainly aimed at advanced A-level students or beginning undergraduates. Only a knowledge of basic combinatorics and mathematical induction is needed to understand most of the book. New ideas are introduced throughout but are explained in a down-to-earth manner. A nice book to have, especially for students competing in the the British Mathematical Olympiad or similar competitions.

University of Sheffield                                             **Daniel Fretwell**

# Index

Volume **43** (2010/11)     Volume **44** (2011/12)     Volume **45** (2012/13)

## Articles

## From the Editor

## Letters to the Editor

## Problems and Solutions

## Reviews

# Mathematical Spectrum

Volume 45    2012/2013    Number 3