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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

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Some Prime Results

MANSUR S. BOASE

1998 was the 50th anniversary of a fascinating result about prime numbers.

1. Introduction

Everyone knows the definition of a prime: A prime number is a natural number larger than 1 whose only divisors are 1 and the number itself. The first few prime numbers are $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11, \ldots$. There are infinitely many primes and this was first proved by Euclid in the third century BC using the following very simple argument:

Suppose 2, 3, 5, ..., p are all the primes. Consider the number

$$(2 \cdot 3 \cdot 5 \cdot \ldots \cdot p) + 1.$$

It is not divisible by any of the primes we have listed and hence must be either a new prime or a multiple of primes all greater than p, a contradiction.

Since then, hundreds of different proofs of this fundamental result have appeared. The prolific Swiss mathematician Leonhard Euler actually proved a more general result. In this article we shall prove Euler's result, generalise it, and use it to prove a theorem about the differences between consecutive primes first proved in 1948, exactly 50 years ago, by Erdös and Turán.

2. A theorem of Euler

It is well known that the harmonic series diverges. This can be seen quickly and easily as follows.

Suppose

$$\sum \frac{1}{n} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots = S,$$

where S is finite. Then

$$\begin{aligned} S &> \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{6} + \ldots \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \\ &= S. \end{aligned}$$

We therefore have a contradiction and no such S exists.

Problem 1. By approximating the area under the graph of $y = 1/x$ from 1 to $k + 1$ by a set of rectangles of width 1 above the graph, show that

$$\sum_{n=1}^{k} \frac{1}{n} > \ln(k+1),$$

and hence deduce that the harmonic series diverges.

What Euler discovered was that if instead of taking the reciprocals of all the natural numbers, we only take the reciprocals of the primes, then the sum of the resulting series

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \ldots$$

will also be divergent.

Theorem 1. $\sum_{i=1}^{\infty}(1/p_i)$ *diverges where* p_n *denotes the* n*th prime.*

Proof. First we show that

$$\lim_{k\to\infty} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{10}{11}\right)\cdots$$

is equal to zero. To do this, we will look at the reciprocal

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1},$$

and show that it tends to infinity as $k \to \infty$.

$$\left(1 - \frac{1}{p_i}\right)^{-1} = 1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \ldots,$$

from the formula for the sum of an infinite geometric series, so

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \ldots\right).$$

If we multiply out this last expression, then note that we get all numbers of the form $1/n$, where n does not have any prime factor greater than p_k. For if n is not divisible by the ith prime, we must choose 1 from the ith bracket of the above product. Otherwise we must choose the reciprocal of the highest power of p_i which divides n.

There is therefore a unique way of obtaining each term of the form $1/n$ where n does not have any prime factor greater than p_k. Let us denote the set of such n by $\mathbb{N}_k$. Then

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{-1} = \sum_{n\in\mathbb{N}_k} \frac{1}{n} > \sum_{n=1}^{p_{k+1}-1} \frac{1}{n}.$$

Note that if there were only k primes, then p_{k+1} would not exist and all natural numbers belong to the set $\mathbb{N}_k$. The right-hand side of the equation would then be the harmonic series which is infinite, whereas the left-hand side would be finite, thus resulting in a contradiction.

We can therefore let k tend to infinity in the above equation to find that

$$\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)^{-1}$$

must also tend to infinity, as required.

Next we prove that

$$\lim_{k\to\infty}\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)\to\infty.$$

We know that

$$\lim_{k\to\infty}\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)=\lim_{k\to\infty}\prod_{i=1}^{k}\left(\frac{p_i-1}{p_i}\right)=0,$$

so

$$\lim_{k\to\infty}\prod_{i=1}^{k}\left(\frac{p_i}{p_i-1}\right)\to\infty,\quad \lim_{k\to\infty}\prod_{i=2}^{k+1}\left(\frac{p_i}{p_i-1}\right)\to\infty.$$

Now the function $x/(x-1)$ is decreasing in x, $x>1$. Since $p_i\geqslant p_{i-1}+1$, we therefore have

$$\frac{p_i}{p_i-1}\leqslant\frac{p_{i-1}+1}{p_{i-1}}.$$

Taking the product of these inequalities for $i=2,3,\ldots,k+1$, we have

$$\prod_{i=2}^{k+1}\left(\frac{p_i}{p_i-1}\right)\leqslant\prod_{i=1}^{k}\left(\frac{p_i+1}{p_i}\right)=\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right).$$

As k tends to infinity, the left-hand side of the above inequality becomes arbitrarily large, so the right-hand side must also become arbitrarily large, as we wanted to prove.

Now suppose for a contradiction that $\sum_{i=1}^{\infty}1/p_i$ converges and has limit L. The arithmetic–geometric (AM-GM) inequality states that if $a_1, a_2, \ldots, a_n$ are positive numbers, then

$$\frac{a_1+a_2+\ldots+a_n}{n}\geqslant\sqrt[n]{a_1a_2\ldots a_n}$$

or

$$a_1a_2\ldots a_n\leqslant\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n$$

(e.g. $4\times5\leqslant((4+5)/2)^2$, $20\leqslant 81/4$). For a proof, see pp. 20–26 of reference 4.

Applying this with $n=k$ and $a_i=(1+(1/p_i))$ for $i=1,2,\ldots k$, we have

$$\begin{aligned}\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right) &\leqslant \left(\frac{\left(1+\frac{1}{p_1}\right)+\ldots+\left(1+\frac{1}{p_k}\right)}{k}\right)^k\\ &= \left(\frac{k+\left(\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}\right)}{k}\right)^k\\ &< \left(1+\frac{L}{k}\right)^k\\ &< \mathrm{e}^L\end{aligned}$$

(for a proof of this last inequality, see section 215 in chapter XI of reference 2).

This contradicts the fact that the left-hand side tends to infinity as k tends to infinity. Therefore no such L exists and the sum of the reciprocals of the primes must diverge.

We can in fact with only a little more work prove a much stronger result.

Theorem 2.

$$\sum_{i=1}^{k}\frac{1}{p_i}>\ln(\ln p_{k+2})-\ln 2$$

$(\ln 2=0.693\ldots)$.

Proof. If we let L_k denote $\sum_{i=1}^{k}1/p_i$, then by the same argument as above,

$$\mathrm{e}^{L_k}\geqslant\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right),$$

and since

$$\begin{aligned}\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right) &\geqslant \prod_{i=2}^{k+1}\left(1-\frac{1}{p_i}\right)^{-1}\\ &> \tfrac{1}{2}\sum_{n=1}^{p_{k+2}-1}\frac{1}{n},\\ \mathrm{e}^{L_k} &> \tfrac{1}{2}\sum_{n=1}^{p_{k+2}-1}\frac{1}{n}.\end{aligned}$$

Therefore, from the result in problem 1, we have

$$\begin{aligned}\mathrm{e}^{L_k} &> \tfrac{1}{2}\ln p_{k+2}\\ L_k &> \ln(\ln p_{k+2})-\ln 2.\end{aligned}$$

A proof of the weaker inequality that for any positive integer y,

$$\sum_{p\leqslant y}\frac{1}{p}>\ln(\ln y)-1,$$

can be found on pp. 26–28 of reference 5.

Problem 2. What is wrong with the following ‘proof’, that

$$\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i}\right)=0\,?$$

The probability that a randomly chosen number is divisible by a given prime p_i is $1/p_i$. So $1-(1/p_i)$ is the probability that it is not divisible by p_i. The above product is then the probability that our randomly chosen number is not divisible by any of the primes and this is clearly zero.

Problem 3. A number is said to be square-free if it has no square divisor. We proved that L did not exist by showing that

$$\lim_{k\to\infty}\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)\to\infty.$$

By expanding out this product for a finite k and then letting k tend to infinity, show that it implies the divergence of the sum of the reciprocals of the square-free numbers.

Thus prove the curious fact that the divergence of the sum of the reciprocals of the primes and the divergence of the sum of the reciprocals of the square-free numbers are equivalent statements!

3. An Erdös–Turán theorem

We shall now use the fact that $\sum 1/p$ diverges to help prove the following theorem where d_i denotes the difference between consecutive primes p_i and p_{i+1}.

Theorem 3. *(a) $d_{n+1} > d_n$ for infinitely many values of n, and (b) $d_{n+1} \leqslant d_n$ for infinitely many values of n.*

This can alternatively be phrased as: there are infinitely many 'early' primes and infinitely many primes that are not 'early'.

First we give a proof of the following lemma.

Lemma 1. *If s and t are natural numbers, $s > 1$ and if the s terms of the arithmetical progression $m, m+t, \dots, m+(s-1)t$ are odd prime numbers, then the common difference t is divisible by every prime less than s.*

Proof. We must have $m \geqslant s$ since otherwise the composite number $m+mt = m(1+t)$ would be a term of the arithmetical progression. Let p denote an arbitrary prime less than s and let $r_0, r_1, \dots, r_{p-1}$ be the remainders on dividing the numbers $m, m+t, \dots, m+(p-1)t$ by p. If any of the remainders is zero, then $p < s \leqslant m$ must divide one of the prime numbers which is impossible.

Therefore the p remainders can only take the $p-1$ values $1, 2, \dots, p-1$, so $r_k = r_l$ for some two integers k and l, such that $0 \leqslant k < l \leqslant p-1$. Consequently, p divides

$$(m+lt)-(m+kt) = t(l-k).$$

But $(l-k)$ cannot be divisible by p so p divides t. Since p was an arbitrary prime less than s, the result follows.

Proof of (a). Suppose that $d_{n+1} > d_n$ for only finitely many n. Then there must exist a K, such that for all $n \geqslant K$, $d_{n+1} \leqslant d_n$. Therefore d_n is a decreasing sequence in n for $n \geqslant K$.

Therefore there must exist some K' such that for all $n \geqslant K'$, d_n is constant. Hence all the primes greater than or equal to $p_{K'}$ must be in arithmetical progression, so from our lemma the common difference of this progression must be divisible by all the primes, a contradiction.

Therefore $d_{n+1} > d_n$ for infinitely many n.

Problem 4. Find an alternative proof for part (a) of theorem 3 by proving that $n!+r$ must be composite for $2 \leqslant r \leqslant n$ and deducing that gaps between consecutive primes must become arbitrarily large.

Before proceeding to our proof of part (b) of theorem 3, we shall briefly discuss the triangular numbers. The nth triangular number is defined by

$$t_n = \frac{n(n+1)}{2},$$

so that $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, and so on. The difference between consecutive triangular numbers is initially 2 and increases by one at each step; a little algebra shows that $t_{n+1} - t_n = n+1$. The reciprocal of the nth triangular number can be written as

$$\frac{1}{t_n} = \frac{2}{n(n+1)} = 2\left(\frac{1}{n}-\frac{1}{n+1}\right),$$

so the sum of the reciprocals of the first k triangular numbers telescopes

$$\begin{aligned}\sum_{n=1}^{k}\frac{1}{t_n} &= 2\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}\dots+\frac{1}{k}-\frac{1}{k+1}\right)\\ &= 2\left(1-\frac{1}{k+1}\right).\end{aligned}$$

Therefore, $\lim_{k\to\infty}\sum_{n=1}^{k} 1/t_n = 2$.

Proof of (b). Suppose that $d_{n+1} \leqslant d_n$ for only finitely many n. Then $d_{n+1} > d_n$ for all $n \geqslant S$ for some $S > 1$. ($S \neq 1$; consider for example the consecutive primes 7, 11, 13.) Thus $d_{n+1} \geqslant d_n + 1$, $p_S > t_1 = 1$ and $d_S \geqslant 2$. It can now be proved by induction that for all $r \geqslant 0$, $p_{S+r} > t_{r+1}$, since this is initially true and the gaps in the primes are increasing at least as fast as those in the triangular numbers.

Consequently

$$\frac{1}{p_{S+r}} < \frac{1}{t_{r+1}} \quad \text{for all } r \geqslant 0,$$
$$\sum_{r=0}^{\infty}\frac{1}{p_{S+r}} < \sum_{r=1}^{\infty}\frac{1}{t_r} = 2\,.$$

contradicting the divergence of $\sum 1/p$.

Therefore $d_{n+1} \leqslant d_n$ for infinitely many n.

The actual result proved by Erdös and Turán at the beginning of their 1948 paper (see reference 1) is actually slightly stronger than theorem 3. (Even more general results than the one below were proved, but the approach was not elementary.) We have $d_{n+1} \leqslant d_n$ for infinitely many n, but even better, we have the following result.

Theorem 3. *(b)** $d_{n+1} < d_n$ *for infinitely many* n.

First we prove a nice little lemma, making use of the ideas of Euclid's proof of the infinitude of the primes.

Lemma 2. *The number of primes which are consecutive elements in an arithmetic progression with common difference* t *is not more than* $t+1$.

Proof. If we have s primes, $s \geqslant 3$, in an arithmetical progression with common difference t, then from lemma 1, t is divisible by every prime less than or equal to s. Thus if p_k is the largest prime less than s, then $p_1 p_2 \dots p_k$ must divide t, so $t \geqslant p_1 p_2 \dots p_k$. Now $p_1 p_2 \dots p_k + 1$ is either a prime or a product of primes all greater than p_k, so

$$s \leqslant p_{k+1} \leqslant p_1 p_2 \dots p_k + 1 \leqslant t+1.$$

Therefore the number of primes in an arithmetical progression with common difference t is at most $t+1$.

Proof of (b).* Suppose to the contrary $d_{n+1} \geqslant d_n$ for all $n \geqslant S'$, for some $S' > 1$ (again S' cannot equal 1).

With the result of lemma 2 in mind, we construct the following sequence u_n with $u_0 = 2$:

$$2, 4, 6, 8, 11, 14, 17, 20, 24, 28, 32, 36, 40, \dots,$$

where the consecutive differences are

$$2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, \dots$$

so that for each q there are $q+2$ terms u_n such that the $q+1$ differences between them take the value q.

The sum of the reciprocals of this sequence is:

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)+$$
$$\left(\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}\right)+$$
$$\left(\frac{1}{20}+\frac{1}{24}+\frac{1}{28}+\frac{1}{32}+\frac{1}{36}\right)+\dots.$$

Note that the reciprocals of the terms at the beginning of each bracket are 2, 8, 20, ... and, in general, the ith term in this sequence is

$$(1)(2)+(2)(3)+(3)(4)+\dots+(i)(i+1),$$

adding up the consecutive differences along the sequence and this equals $i(i+1)(i+2)/3$.

Therefore

$$\sum_{n=0}^{\infty}\frac{1}{u_n} < \sum_{i=1}^{\infty}\frac{i+2}{(i(i+1)(i+2)/3)},$$

since the ith bracket contains $(i+2)$ terms each less than the first term (except of course for the first term itself). Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{u_n} &< \sum_{i=1}^{\infty}\frac{3}{i(i+1)}\\ &= \frac{3}{2}\sum_{i=1}^{\infty}\frac{1}{t_i},\\ \sum_{n=0}^{\infty}\frac{1}{u_n} &< \frac{3}{2}\cdot 2 = 3.\end{aligned}$$

We can prove by induction that u_n is such that $p_{S'+r} > u_r$ for all $r \geqslant 0$ because we have constructed u_n in such a way that this is initially true and because of lemma 2 the gaps between consecutive primes $\geqslant p_S$ increase at least as fast as those between consecutive terms of u_n. Thus,

$$\frac{1}{p_{S'+r}} < \frac{1}{u_r} \quad \text{for all } r \geqslant 0,$$

so

$$\sum_{r=0}^{\infty}\frac{1}{p_{S'+r}} < \sum_{r=0}^{\infty}\frac{1}{u_r} < 3.$$

This contradicts the divergence of $\sum 1/p$, so we can conclude that $d_{n+1} < d_n$ for infinitely many n.

Problem 5. It is quite easy to prove that the sum of the reciprocals of the set of integers not containing a given digit in any base b converges (see theorem 144 of reference 3). Derive from this fact a wonderfully simple and elementary proof that given any finite sequence of digits, e.g. '1998' (or the first billion digits of π), there exist infinitely many primes containing that sequence of digits in their decimal expansion.

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Mansur Boase *was 16 years old when this article was written and a John Colet Scholar at St Paul's School, London. He won a silver medal at the 1997 International Mathematical Olympiad in Argentina and became a student at Trinity College, Cambridge, in October 1998, where he is studying Mathematics.*

The Sequence $p^n - 1$

K. PRAKASH

This sequence goes back to Catalan.

In my article 'Powerless Sequences' (*Mathematical Spectrum* **30**, pp. 39–40), I showed that, for an odd prime p, if the sequence $(p^n + 1)$ contains a power N^q $(q > 1)$, then p must be a Mersenne prime, i.e. a prime of the form $2^k - 1$. Here I prove that $p^n - 1$ is not expressible as N^q, where N is an integer and q is an odd prime, except for the case

$$3^2 - 1 = 2^3.$$

Further I prove that, if $p^n - 1$ is a square, then n is odd.

Assume that $p^n - 1 = N^q$, where N is an integer and q is an odd prime. Then

$$\begin{aligned} p^n &= N^q + 1 \\ &= (N+1)(N^{q-1} - N^{q-2} + \cdots + 1) \\ &= (N+1)S_q \end{aligned}$$

say. Further,

$$\begin{aligned} (N+1)S_q &= (N+1-1)^q + 1 \\ &= (N+1)^q - \binom{q}{1}(N+1)^{q-1} + \cdots + q(N+1), \end{aligned} \tag{$*$}$$

so

$$\begin{aligned} S_q &= (N+1)^{q-1} - q(N+1)^{q-2} + \cdots + q \\ &\equiv q \bmod (N+1). \end{aligned}$$

Let $d = (S_q, N+1)$, the highest common factor of S_q and $N+1$. Then $d|q$ so, since q is a prime, $d = 1$ or q. Now

$$p^n = (N+1)S_q,$$

so $(N+1) = p^i$ and $S_q = p^j$ for some integers i and j with $i + j = n$. If $d = 1$ then i or $j = 0$. If $i = 0$ then $N+1 = 1$, giving $N = 0$, which is impossible. If $j = 0$ then $i = n$ and $N + 1 = p^n$, whence $p^n - 1 = N$, again not so. Therefore $d = q$ and so $q = p$.

We now observe that $q^2 \nmid S_q$. From $(*)$ we have

$$S_q \equiv (N+1)^{q-1} + q \bmod (N+1)q. \tag{$**$}$$

Since $d = q$, $q^2 \mid (N+1)q$ and, as $q > 2$, $q^2 \mid (N+1)^{q-1}$. If $q^2 \mid S_q$ then, from (**) we obtain $q^2 \mid q$, which is not so. Hence $S_q = q$ and $(N+1) = q^{n-1}$.

For $n > 2$, $S_q = q < N + 1$, so that

$$\frac{N^3+1}{N+1} \leqslant \frac{N^q+1}{N+1} < N+1,$$

leading to $N < 2$, i.e. $N = 1$. But this is impossible as $p^n - 1 \geqslant 2^3 - 1 = 7$. For n=2, $S_q = q = (N+1)$ so that $N^q + 1 = (N+1)^2$. Then $N^q = N(N+2)$, so that $N^{q-1} = N + 2$, $N|2$ and hence $N = 1$ or 2. When $N = 1$, $N^q + 1 = 2 < (N+1)^2$ which is not so. When $N = 2$, $(N+1)^2 = 9 = 2^q + 1$, so $q = 3$. When $n = 1$ we have $N + 1 = q^{n-1} = 1$, i.e. $N = 0$, which is not so. Thus the only solution is

$$3^2 = 2^3 + 1.$$

We now show that, if $p^n - 1 = N^2$ for some integer N, then n is odd. Suppose that n is even, say $n = 2k$ for some positive integer k. Then

$$p^{2k} - 1 = N^2$$

gives

$$1 = (p^{2k} - N^2) = (p^k - N)(p^k + N),$$

a factorization of 1 as a product of two integers. Both terms must be $+1$ or -1 giving $N = 0$ and $k = 0$, which is not so.

Although $p-1$ is frequently a square, e.g. $p = 5, 17, 37$, I know of no example where

$$p^n - 1 = N^2$$

and $n > 1$, nor have I been able to show that $p^n - 1 \neq N^2$ for $n > 1$.

Editorial comment

Catalan conjectured in the 19th century that 8 and 9 are the only pair of consecutive integers which are both powers of integers, i.e.

$$x^n - 1 = y^m$$

has the single solution $x = 3$, $y = 2$, $n = 2$, $m = 3$. This conjecture is still open, although it is known that there can only be finitely many solutions. (See T. N. Storey & R. Tydeman, *Exponential Diophantine Equations* (CUP, 1986), Chapter 12).

K. Prakash *teaches in Jawahar Navodaya Vidyalaya, Bellasy Dist, India.*

The Josephus Problem and Ahrens Arrays

I. M. RICHARDS

The Josephus Problem is one that turns up very often. It is frequently rediscovered as a field of investigation by mathematicians. The most important text on the Josephus Problem is rarely available to investigators. Here its results are summarised and extended.

I wrote in *Mathematical Spectrum* (reference 1), about the Josephus Problem. This is the age-old problem about counting out in a circle by using a counting rhyme. I explained in that article how to deduce formulae for deciding who would be eliminated at what stage when one counted out by threes. That is, one takes a group of n persons arranged in a circle and removes every third one in turn, working around the circle. This happens to be the method proposed by Bachet in the best-known formulation of the problem. I have been able to find out more about previous solutions of the Josephus Problem which I shall look at in this article. The solutions throw up new connections which I had not thought possible before. The problem of counting out by twos has arisen most recently in 'The Daughter's Dilemma', *Mathematical Spectrum* (reference 2).

Ahrens–Schubert solution

Inquirers after the solution to the Josephus Problem are usually referred to Ahrens (reference 3). It can be very difficult to obtain Ahrens' article, in which he gives a solution of the problem and an equivalent and similar solution by Schubert. I must thank Tamara Curnow for finding me a copy after some years of my searching. I think it is worth restating Ahrens' result here if only to have a modern statement of these results which is more widely available. Before giving Schubert's version of the solution, I should comment that these solutions do not give simple formulae for the solution of the Josephus Problem; the method of 'solution' is still algorithmic, although the solution algorithms are usually much quicker than finding the desired person by counting out directly. It would appear that the only solutions by formula are those in my previous article, dealing with counting by threes, and some other formulae I have since deduced dealing with counting by fours.

The Ahrens–Schubert solution begins with the generation of certain sequences called *Oberreihen* by Ahrens, but which I shall refer to as *rounded sequences*.

$\lfloor x \rfloor$ is to denote the smallest integer greater than or equal to x, or informally, x rounded up. The rounded sequence (a, q) is defined as follows:

$$a_1 = \lfloor a \rfloor$$
$$a_{n+1} = \lfloor q \times a_n \rfloor \quad \text{for } n \geqslant 1$$

The rounded sequence behaves much like a geometric progression with ratio q, but with rounding up of each term before the next one is generated from it.

Example. The rounded sequence (5, 4/3) begins $5, 7, 10, 14, 19, 26, 35, \ldots$. Each term is generated by taking the previous term, multiplying by 4/3 and rounding up.

The Ahrens–Schubert solution to the Josephus Problem may now be stated in terms of these rounded sequences.

Ahrens–Schubert Solution. *If we arrange n persons in a circle and eliminate by circular counting, removing every dth person, then the number of the eth person to be eliminated is the difference between $dn + 1$ and the largest term of the rounded sequence $(d(n-e)+1,\ d/(d-1))$ which is less than $dn + 1$.*

Let us see how this works. We follow the example in reference 1. We begin with 41 persons. We count out by threes. Who is the last one left? In terms of the variables above, $n = 41$, $d = 3$, and $e = n = 41$ (we are concerned with the 41st person to be eliminated). We calculate $d(n-e)+1$ which is, of course, equal to 1. We use therefore the rounded sequence (1, 3/2). This sequence begins $1, 2, 3, 5, 8, 12, 18, 27, 41, 62, 93, 140, \ldots$. We now calculate $dn + 1 = 124$ and see where it lies relative to this sequence. We find that the largest term of the sequence that is less than 124 is 93. We find the difference $124 - 93 = 31$. The last person to be eliminated is therefore the person in the 31st place.

Rounded sequences

These rounded sequences seem to be worthy of study themselves and Ahrens does show an interesting property of these sequences.

- Take a fixed value of q, say $q = 3/2$. Construct the rounded sequence $(1, q)$ $1, 2, 3, 5, 8, 12, 18, 27, \ldots$.
- The smallest integer missing from this list is 4. Construct the rounded sequence $(4, q)$ $4, 6, 9, 14, 21, 32, 48, 72, \ldots$.
- The smallest integer missing from these two lists is 7. Construct the rounded sequence $(7, q)$ $7, 11, 17, 26, 39, 59, 89, \ldots$.

We write such sequences in a grid which I shall call an *Ahrens Array*. Table 1 shows the first eleven rows of the first ten columns of the Ahrens Array for $q = 3/2$. This array encodes the solutions of the problem of elimination by threes and forms an alternative solution to the problem in reference 1. The first column is used to find the last to be eliminated

Table 1.

Ahrens Array $q = 3/2$									
1	4	7	10	13	16	19	22	25	28
2	6	11	15	20	24	29	33	38	42
3	9	17	23	30	36	44	50	57	63
5	14	26	35	45	54	66	75	**86**	**95**
8	21	39	53	68	81	**99**	**113**	129	143
12	32	59	80	**102**	**122**	149	170	194	215
18	48	**89**	**120**	153	183	224	255	291	323
27	72	134	180	230	275	336	383	437	485
41	**108**	201	270	345	413	504	575	656	728
62	162	302	405	518	620	756	863	984	1092
93	243	453	608	777	930	1134	1295	1476	1638

Table 2.

Ahrens Array $q = 7/3$									
1	2	4	6	8	9	11	13	15	16
3	5	10	14	19	21	26	31	35	38
7	12	24	33	45	49	61	73	82	89
17	28	56	77	105	115	143	171	192	208
40	66	131	180	245	269	334	399	448	486
94	154	306	420	572	628	780	931	1046	1134
220	360	714	980	1335	1466	1820	2173	2441	2646
514	840	1666	2287	3115	3421	4247	5071	5696	6174
1200	1960	3888	5337	7269	7983	9910	11833	13291	14406
2800	4574	9072	12453	16961	18627	23124	27611	31013	33614

and the second column determines the last but one to be eliminated, the third column determines the last but two to be eliminated and so on.

To show this we can extend the example given above. If $n = 41$, the last person to be eliminated was found by the location of the number 124 relative to the rounded sequence which is the first column of this table. If we wish to find who is last but one to be eliminated then we set $n = 41$, $d = 3$ and $e = 40$. In doing this we find the 40th to be eliminated. We calculate $d(n - e) + 1 = 4$. We consider the rounded sequence (4, 3/2). This is the second column of our array. We calculate $dn + 1 = 124$, as before. We look where 124 occurs relative to the second column of the array and find that the largest term less than 124 is 108. The difference between 124 and 108 is 16. It is the 16th person that is eliminated last but one. If we carry on the example further we will find that the second from last to be eliminated will be determined by the location of 124 relative to the rounded sequence (7, 3/2). That rounded sequence is the third column of the array. And so it continues, column by column.

To emphasise their significance I have put the key terms of the array in bold type. The order of elimination when we count out by threes from among 41 persons is, running from last to first, $124 - 93$, $124 - 108$, $124 - 89$, $124 - 120$, $124 - 102$, $124 - 122$, $124 - 99$, ... In this remarkable way the array encodes the entire solution to the problem of elimination by threes.

We notice certain features of the array, as far as we can judge from the part of the array shown:

1. Every integer occurs.
2. No integer occurs more than once.
3. The top row is an arithmetic progression.

That the first of these should be true is obvious from the way in which the array has been constructed. Are the others generally true? Table 2 shows the first ten rows of the first ten columns of the Ahrens Array for the case $q = 7/3$.

Unlike the array for $q = 3/2$, which is is appropriate for the solution of the problem of counting out by threes, the array for $q = 7/3$ has no such application as yet. Yet, as far as we can judge from the part of the array shown in Table 2, these features can still be seen:

1. Every integer occurs.
2. No integer occurs more than once.
3. The top row is not now an arithmetic progression, but it is a union of four arithmetic progressions. These have first terms 1, 2, 4 and 6 and difference 7.

Ahrens had noticed these patterns for a general array of this type and has proved the following.

Table 3.

Ahrens Array $q = (1+\sqrt{5})/2$									
1	**3**	6	**8**	11	14	16	19	**21**	24
2	**5**	10	**13**	18	23	26	31	**34**	39
4	9	17	22	30	38	43	51	56	64
7	15	28	36	49	62	70	83	91	104
12	25	46	59	80	101	114	135	148	169
20	41	75	96	130	164	185	219	240	274
33	67	122	156	211	266	300	355	389	444
54	109	198	253	342	431	486	575	630	719
88	177	321	410	554	698	787	931	1020	1164
143	287	520	664	897	1130	1274	1507	1651	1884

Theorem (Ahrens). *Let q (> 1) be a fraction that is equal in its lowest terms to m/n. Then the Ahrens Array for q contains every integer once and once only. The top row comprises $m - n$ arithmetic progressions each with a difference of m.*

This result, like the previous result of Ahrens, I shall leave unproved, in order to keep this article short. If we accept this result we can go on to ask what an Ahrens array for an irrational q looks like.

Irrational Ahrens arrays

Table 3 shows the Ahrens Array for the value $q = (1+\sqrt{5})/2$. The general impression is still the same. Each integer seems to appear once and once only. But what is the sequence that is in the top row? What initial values are required to generate the other integers by rounded sequences? The answer seems to me to be remarkable.

I wrote, in *Mathematical Spectrum* (reference 4), an article entitled 'Reflection Problems'. In that I referred to a theorem of S. Beatty that I find particularly elegant. Now there is a slight difference between Ahrens and Beatty in terms of notation. Beatty's Theorem is expressed in terms of the standard $[x]$, that is, x rounded down to the nearest integer, so I am restating Beatty's Theorem in a notation to make it comparable with Ahrens' results.

Beatty's Theorem. *Let x be a positive irrational number. Set $a = 1 + x$ and $b = 1 + 1/x$. The two sequences*

$$\lfloor a\rfloor, \lfloor 2a\rfloor, \lfloor 3a\rfloor, \lfloor 4a\rfloor, \dots,$$
$$\lfloor b\rfloor, \lfloor 2b\rfloor, \lfloor 3b\rfloor, \lfloor 4b\rfloor, \dots,$$

are complementary, that is, these sequences, if combined, list the positive integers greater than one exactly, each integer occurring once.

What we have in the top row of the Ahrens Array for $q = (1+\sqrt{5})/2$ is the sequence 1, followed by a Beatty sequence, $\lfloor a\rfloor, \lfloor 2a\rfloor, \lfloor 3a\rfloor, \lfloor 4a\rfloor, \dots$, where $a = q/(q-1) = (3+\sqrt{5})/2$. The rest of the array is the complementary sequence divided into infinitely many subsequences. That is, the rows other than the first are the elements of the sequence, $\lfloor b\rfloor, \lfloor 2b\rfloor, \lfloor 3b\rfloor, \lfloor 4b\rfloor, \dots$, where $b = (1+\sqrt{5})/2$, but this sequence has been broken up and reordered into infinitely many rounded sequences, running vertically down the Ahrens Array.

I am always surprised that what are apparently different areas of mathematics can be connected. I had never imagined that there could be any connection between these two articles that I had written for *Mathematical Spectrum*, and yet the Beatty sequences are significant in both cases!

Other features of Ahrens arrays

- All Ahrens Arrays for irrational numbers q, have a curious 'self-referring' property. I alluded above to the fact that, apart from the top row, the entries, when put in order, form the sequence $\lfloor b\rfloor, \lfloor 2b\rfloor, \lfloor 3b\rfloor, \lfloor 4b\rfloor, \dots$, where $b = q$. Where does any entry in the array occur in this sequence? Its position is given by the entry in the array that is immediately above it!
  For example, where does 38 occur in the Beatty sequence $\lfloor b\rfloor, \lfloor 2b\rfloor, \lfloor 3b\rfloor, \lfloor 4b\rfloor, \dots$, where $b = q$? We simply locate 38 on the Ahrens Array for q and read off the value above it, which is, in the case $q = (1+\sqrt{5})/2$, equal to 23. Therefore 38 is 23rd in the Beatty sequence.

- Each Ahrens Array, for an irrational q, has what might be called its dual. If we take the Ahrens Array for q, we can construct then the Ahrens Array for $q/(q-1)$. The first row of the Ahrens Array for q, apart from the initial 1, is now spread out into infinitely many rows, while all the other rows of the Ahrens Array for q have now compacted into the first row of the Ahrens Array for $q/(q-1)$.

  To illustrate this, take $q = (1+\sqrt{5})/2$. The Ahrens Array for q is shown in part in Table 3; we construct $q/(q-1) = (3+\sqrt{5})/2$. Table 4 is part of the Ahrens array for $q/(q-1) = (3+\sqrt{5})/2$. We see that the first row of one of the tables is, with the exception of the first term, spread over all the rows, other than the first, of the other table. This is the effect of constructing the dual array.

Table 4.

Ahrens Array $q = (3+\sqrt{5})/2$									
1	2	4	5	7	9	10	12	13	15
3	6	11	14	19	24	27	32	35	40
8	16	29	37	50	63	71	84	92	105
21	42	76	97	131	165	186	220	241	275
55	110	199	254	343	432	487	576	631	720
144	288	521	665	898	1131	1275	1508	1652	1885
377	754	1364	1741	2351	2961	3338	3948	4325	4935
987	1974	3571	4558	6155	7752	8739	10336	11323	12920
2584	5168	9349	11933	16114	20295	22879	27060	29644	33825
6765	13530	24476	31241	42187	53133	59898	70844	77609	88555

The properties given above are general properties for all arrays generated by irrationals. We chose the particular value $q = (1+\sqrt{5})/2$ to illustrate the irrational array because it has a bearing on the Fibonacci numbers. In Table 3, the Ahrens Array for $q = (1+\sqrt{5})/2$, we have put the Fibonacci numbers in bold type. Notice where they occur. If we combine the top two rows to get a single sequence, we get the Fibonacci numbers occurring as a particular subsequence. When we form the dual array, in Table 4, the Fibonacci numbers are scattered around. Readers may wish to consider where the Fibonacci numbers are to be found now.

In conclusion, I feel it is likely that Ahrens Arrays have other uses. Some Ahrens Arrays form a solution of the Josephus Problem. So what do the others do?

References

1. I. M. Richards, The Josephus Problem, *Mathematical Spectrum* **24** (1991/92), pp. 97–104.
2. D. W. Sharpe, The Daughters Dilemma *Mathematical Spectrum* **29** (1996/97), p. 29.
3. W. Ahrens, *Mathematische Unterhaltungen und Spiele*, Vol. 2 (Teubner, Leipzig, 1901), pp. 286–301.
4. I. M. Richards, Reflection Problems, *Mathematical Spectrum* **27** (1994/95), pp. 35–37.

*The author, **Ian Richards**, being rather isolated from academic resources, is very pleased with the decision of Exeter University to build a new campus near Penzance, Cornwall, about three miles from his home. He hopes earnestly that the Millennium Commission will support the project. He is an active supporter of the Young Archaeologists Club and is currently retraining for a career in computing.*

Mathematical Spectrum Awards for Volume 30

Prizes have been awarded to the following student readers for contributions in Volume 30:

Dave Hills and **Sam Webster**
for their article 'An estimate for π from a one-dimensional random walk' (pages 30–31);

Isaac Vun
for his contribution to the article 'Catalan numbers' (pages 3–5);

Mansur Boase, Toby Gee and **Andrew Lobb**
for various contributions.

The editors remind readers that prizes are available anually for student contributions as follows: up to the value of £50 for articles, and up to £25 for letters, solutions to problems and other items.

The 1999 puzzle

Our annual puzzle is to express the numbers 1 to 100 with the digits of the year in order using only the operations of $+, -, \times, \div, \sqrt{\ }$, !, brackets and concatenation (e.g. putting the digits 1 and 9 together to make 19). Powers are not allowed. Thus, for example,

$$1 = 19 - 9 - 9$$

or $$1 = 1 + 9 - \sqrt{9} - (\sqrt{9})!$$

Can any reader evaluate the integral

$$\int x^x \, \mathrm{d}x \ ?$$

David Benjamin

Some Integrals Evaluated

FRANK CHORLTON

Let us first consider the evaluation of the indefinite integral

$$I = \int f(x-a, b-x)\,\mathrm{d}x, \quad (a < x < b),$$

where we assume that the integrand is continuous in the stated range. The identity

$$(x-a) + (b-x) \equiv b - a (> 0)$$

means that, for all x satisfying $a \leqslant x \leqslant b$, we can find θ satisfying the two relations

$$x - a = (b-a)\sin^2\theta, \quad b - x = (b-a)\cos^2\theta.$$

From either of these forms

$$\mathrm{d}x = 2(b-a)\sin\theta\cos\theta\mathrm{d}\theta.$$

and I reduces to

$$\begin{aligned} I &= 2(b-a) \\ &\times \int f\left[(b-a)\sin^2\theta,\ (b-a)\cos^2\theta\right]\sin\theta\cos\theta\,\mathrm{d}\theta. \end{aligned}$$

We now consider a number of cases which are either illustrations of this theme or slight variants or extensions of it.

Case 1. First consider

$$J = \int_a^b \left(\frac{x-a}{b-x}\right)^{1/2} \mathrm{d}x.$$

Here the integrand is real and continuous in $a \leqslant x < b$ and we have

$$\begin{aligned} J &= 2\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}(b-a)\sin\theta\cos\theta\,\mathrm{d}\theta \\ &= (b-a)\int_0^{\pi/2}(1-\cos 2\theta)\,\mathrm{d}\theta \\ &= (b-a)\frac{\pi}{2}. \end{aligned}$$

Case 2. Let us next consider

$$J = \int_2^3 \frac{\mathrm{d}x}{\sqrt{(5-x)} + \sqrt{(x-1)}}.$$

Here the identity

$$(5-x) + (x-1) \equiv 4$$

means that for each x in [1,5] we can find θ such that

$$5 - x = 4\cos^2\theta, \quad x - 1 = 4\sin^2\theta.$$

As $x = 2$ and $x = 3$ are both in [1,5], J is found to reduce to

$$J = \int_{\pi/6}^{\pi/4} \frac{4\sin\theta\cos\theta}{\cos\theta + \sin\theta}\,\mathrm{d}\theta.$$

Now $2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2 - 1$ and so

$$J = 2\int_{\pi/6}^{\pi/4}\left\{(\sin\theta + \cos\theta) - \frac{1}{\sin\theta + \cos\theta}\right\}\mathrm{d}\theta.$$

Further,

$$\int_{\pi/6}^{\pi/4}(\sin\theta + \cos\theta)\,\mathrm{d}\theta = \tfrac{1}{2}(\sqrt{3} - 1);$$

and, putting $\phi = \theta + \frac{1}{4}\pi$, we have

$$\begin{aligned} \int_{\pi/6}^{\pi/4} \frac{\mathrm{d}\theta}{\sin\theta + \cos\theta} &= \frac{1}{\sqrt{2}}\int_{5\pi/12}^{\pi/2}\frac{\mathrm{d}\phi}{\sin\phi} \\ &= \frac{1}{\sqrt{2}}\left[\ln\tan\frac{\phi}{2}\right]_{5\pi/12}^{\pi/2} \\ &= -\frac{1}{\sqrt{2}}\ln\tan\frac{5\pi}{24}. \end{aligned}$$

Thus

$$J = \sqrt{3} - 1 + \sqrt{2}\ln\tan\frac{5\pi}{24}.$$

Case 3. We now consider the integral

$$J = \int_0^2 \left(\frac{3-x}{2-x}\right)^{1/2}\mathrm{d}x.$$

Here the integrand is real and continuous for $0 \leqslant x < 2$. Also the relationship $(3-x) - (2-x) = 1$ means that for all x in $0 \leqslant x < 2$, $3 - x > 0$, $2 - x > 0$ and so we can take

$$3 - x = \cosh^2 t, \quad 2 - x = \sinh^2 t.$$

At $x = 0$, $\cosh t = \sqrt{3}$ and $\sinh t = \sqrt{2}$ and so $\mathrm{e}^t = \sqrt{3} + \sqrt{2}$, or $t = \ln(\sqrt{3} + \sqrt{2}) = c$, say. At $x = 2$, $\cosh t = 1$, $\sinh t = 0$, i.e. $t = 0$. One finds

$$\begin{aligned} J &= \int_0^c 2\cosh^2 t\,\mathrm{d}t \\ &= \int_0^c (\cosh 2t + 1)\,\mathrm{d}t \\ &= \left[\tfrac{1}{2}\sinh 2t + t\right]_0^c \\ &= \tfrac{1}{2}\sinh 2c + c, \end{aligned}$$

where

$$2c = 2\ln(\sqrt{3}+\sqrt{2}) = \ln(5+2\sqrt{6})$$

and so

$$\tfrac{1}{2}\sinh 2c$$
$$= \tfrac{1}{4}\left(e^{\ln(5+2\sqrt{6})} - e^{-\ln(5+2\sqrt{6})}\right)$$
$$= \tfrac{1}{4}\left(5+2\sqrt{6} - \frac{1}{5+2\sqrt{6}}\right)$$
$$= \frac{12+5\sqrt{6}}{5+2\sqrt{6}} = \sqrt{6}.$$

Hence $J = \sqrt{6} + \ln(\sqrt{3}+\sqrt{2})$.

*Now retired from university teaching, **Frank Chorlton** was Senior Lecturer in Mathematics for many years at Aston University. In retirement he endeavours to maintain an interest in mathematics and frequently publishes in the learned journals. He is also interested in music and used to belong to local choral societies. He is especially fond of Bach.*

Reversing Digits

ROGER COOK

7641 - 1467 = 6174

The following problem came to me through the 'Ask A Librarian' scheme, and the question was asked originally by Anisa Visram. Since the librarians didn't know the answer, they asked me; and I am not sure that I know the complete answer.

Take a 4-digit number '*abcd*' where the digits are arranged in descending order, e.g. 6542, reverse it to give '*dcba*', i.e. 2456 for our example, subtract from the original, to give 4086, and arrange the digits in descending order again: 8640. Repeating this process we can construct a sequence of 4-digit numbers

$$6542, 8640, 8721, 7443, 9963, 6642, 7641, 7641, \ldots .$$

Of course, once the sequence starts to repeat itself it will go on doing so; but why does this happen? In one sense this is easy to answer. We are constructing an infinite sequence

$$n_1 = 6542, \quad n_2 = 8640, \quad \ldots ,$$

on a finite set (the 4-digit numbers). The sequence must eventually repeat itself and once it does that it just goes on and on repeating the same loop of values. But rather more may be going on here:

1. Rather than entering a loop of repeated values we have a single repeated value of the sequence — a 'fixed point' for the sequence. When can fixed points exist?
2. Starting from different 4-digit numbers we might expect to end the sequence in different repeating loops, or do they all end up in the same terminating loop?

To answer the second question first, numbers such as 1111 end up at 0000 (which we regard as a 'trivial' fixed point) so the final loop is not unique. However, we shall see that most 4 digit numbers do end up at the fixed point 7641, so why is this fixed point so popular?

Let us start by considering simpler cases. The case of 1-digit numbers is rather trivial, so think about 2-digit numbers '*ab*' where, for simplicity, we will assume $a > b \geqslant 0$.

To perform the subtraction

$$\begin{array}{rcc} & a & b \\ - & b & a \end{array}$$

we have to perform a 'borrow', because $a > b$,

$$\begin{array}{rcc} & a-1 & b+10 \\ - & b & a \end{array}$$

giving

$$(a-b)-1 \quad 10-(a-b)\,.$$

Since $a - b - 1 < a$, the only possibility for a fixed point would come from

$$a-b-1 = b, \quad 10-(a-b) = a,$$

and these equations have no integer solution so there can be no fixed points in this case.

We can also see the importance of the difference $(a-b)$. If we write $x = (a-b)$ then the next member of the sequence is '$(x-1)(10-x)$' when $x \geqslant 6$, and '$(10-x)(x-1)$' when $x < 6$. For example, starting with 95, $x = 4$ so that $10 - x = 6$, $x - 1 = 3$ and the next term in the sequence is 63. Thus the final loops are determined not by the 99 possible values of '*ab*' but by the 9 possible values of $x = a - b$ and the only values taken are

$$90, 81, 72, 63, 54.$$

It is now easy to check that the sequence must enter a loop

$$90 \to 81 \to 63 \to 72 \to 54 \to 90 \ldots .$$

In the case of 3-digit numbers 'abc', where now we will suppose $a \geqslant b \geqslant c \geqslant 0$ with $a > c$, something unexpected happens. When we are in a repeating loop, a must be 9. Consider the subtraction with the required 'borrows':

$$\begin{array}{rccc} & a-1 & b+9 & c+10 \\ - & c & b & a \end{array}$$

giving

$$(a-c)-1 \qquad 9 \quad 10-(a-c),$$

so the largest digit must be 9. Also, the 'length' of the number, $a - c$, is important so now we write $x = a - c$. The digits are $9, x - 1$ and $10 - x$ (in some order) so the only possibilities are

$$990, 981, 972, 963, 954,$$

and the sequence must end up at the single fixed point 954.

In the case of a 4-digit number '$abcd$' (where now we suppose that $a \geqslant b > c \geqslant d \geqslant 0$), at the next stage we have the 4 digits

$$a-d, \quad (b-c)-1, \quad 9-(b-c), \quad 10-(a-d)$$

in some order, or

$$x, \quad 10-x, \quad y-1, \quad 9-y,$$

where now $x = a - d$ and $y = b - c$ with $x \geqslant y > 0$. The order in which these 4 digits occur depends on the exact values of x and y but as x goes from 1 to 9 we get only 5 pairs of digits (9,1), (8,2), (7,3), (6,4) and (5,5) for $(x, 10 - x)$, and varying y gives the 5 pairs (8,0), (7,1), (6,2), (5,3) and (4,4) for $(y - 1, 9 - y)$. Combining these gives 25 possible 4-digit numbers in our repeating loop and it is straightforward to work through the sequence of possible values to see that they all end up at the fixed point 7641.

The 4-digit numbers $abbc$ with $a > c$ also end up at 7641. The next term in the sequence has the digits $9, 9, x - 1, 10 - x$ where $x = a - c$ so it is one of the numbers 9990, 9981, 9972, 9963, 9954, all of which end up at 7641.

Taking the idea a step further, any 5-digit number '$abcde$' (where for simplicity we suppose $a \geqslant b \geqslant c \geqslant d \geqslant e \geqslant 0$ with $b > d$) will lead at the next stage to the digits

$$x, \quad y-1, \quad 9, \quad 9-y, \quad 10-x$$

with $x = a - e$ and $y = b - d$. (In general, when we are looking at numbers with an odd number of digits, the largest digit must be 9 in the final loops.)

The possibilities that occur for the pairs $(x, 10 - x)$ and $(y-1, 9-y)$ are precisely the same as those for 4-digit numbers, but now we have the additional digit 9. This radically changes the behaviour in the terminating loops. Instead of a single fixed point we now have three separate loops:

$$97731 \to 98532 \to 97443 \to 96642 \to 97731 \ldots ,$$

$$98622 \to 97533 \to 96543 \to 97641 \to 98622 \ldots ,$$

and

$$95553 \to 99954 \to 95553 \ldots .$$

Even when $b = d$ the sequences will enter one of these three loops provided that $a > e$. The subtraction becomes

$$\begin{array}{rccccc} & a-1 & b+9 & b+9 & b+9 & e+10 \\ - & e & b & b & b & a \end{array}$$

to give

$$(a-e)-1 \qquad 9 \qquad 9 \qquad 9 \quad 10-(a-e)$$

and writing x for $a - e$ there are 5 possibilities for the pair $(x - 1, 10 - x)$. It is then easy to check that each of them ends up in either the first or the third of the three loops listed above.

By now it is clear how to tackle the more general cases. When we have $2k$ digits we get k pairs of digits at the next stage. One pair $(x, 10 - x)$ runs through the values (9,1), (8,2), (7,3), (6,4) and (5,5). One pair is of the form $(y - 1, 9 - y)$ and runs through the values (8,0), (7,1), (6,2), (5,3) and (4,4). The remaining $k - 2$ pairs are of the form $(z, 9 - z)$ and run through the 5 values (9,0), (8,1), (7,2), (6,3) and (5,4). (You might like to check the case of 6-digit integers for yourself to see what is going on.) The condition $b > c$ we used for 4-digit numbers $abcd$ should be replaced by a condition requiring strict inequality between the two central digits (i.e. the kth and $(k + 1)$th digit). This ensures that the sequence of borrows ends at the midpoint of the number.

For $2k + 1$ digits we get the same k pairs of numbers together with the extra digit 9. The condition $b > d$ for $abcde$ should be replaced by a condition requiring strict inequality between the kth and $(k + 2)$th digit. In both cases, of $2k$ and $2k + 1$ digits, we get 5^k numbers to consider in the terminating loops and it would not be difficult to run through all these numbers on a computer for small values of k.

I have not conducted an exhaustive search of these cases but here are a few observations from particular numbers. In the case of 6-digit numbers a computer search revealed the 7 member loop

$$\begin{aligned} &866322 \to 665442 \to 876420 \to 875421 \\ &\qquad \to 875430 \to 885420 \to 886320 \ldots . \end{aligned}$$

A computer check on 12-digit numbers revealed a 7-member loop, including the number 866(666333)322 and the loop was formed by inserting three 6's and three 3's in each member of the 6-digit loop. Knowing this it is then straightforward to check that including k 6's and k 3's in each element of the 6-digit loop provides a 7-member loop on (6+2k)-digit numbers. One amusing feature of the 8-digit loop is that it includes the number 87654321.

When we look at numbers with an odd number of digits the situation is more complicated. There is an 8-element loop on 7-digit numbers including 977(63)31 and a 14-element loop on 9-digit numbers including 977(6633)31. Because the length of the loop is increasing with the number of digits it is more difficult to prove precisely what is going on.

In both cases the pair (63) acts as a 'seed' for creating longer numbers. Clearly $6 - 3 = 3$ and, when we are in a sequence of 'borrows', the subtraction of 6 from 3 becomes the subtraction of 6 from 12, to give 6 and moves the 'borrow' along one place. If they are positioned to meet each other when the number is reversed, long matching sequences of 6's and 3's may remain unchanged by the subtraction provided that other digits present can repair the changes caused at the edges of the sequences of 6's and 3's. For our original problem with 7641, longer fixed points of the form 76(6...6)4(3...3)1 can be generated, demonstrating that in general both fixed points and loops occur; and that the terminating loop is not unique. The uniqueness for 4 digits occurs because there are not many cases for the two parameters x and y.

The way in which the loops form as we increase the lengths of the integers seems interesting. For example, returning to the 6-digit loop we saw that we can increase the length of the integers using the 'seed' (63) but if we remove (63) from the starting point 866322 we get the sequence

$$8622 \to 6543 \to 8730 \to 8532 \to 7641.$$

You might also think about what happens when the arithmetic is in a base other than 10. For example, we can have a 2-digit fixed point 'ab' in a base B if and only if B is of the form $3t + 2$ and the fixed point is then given by $a = 2t + 1$, $b = t$; 3-digit fixed points abc exist if and only if $B = 2t$, and are given by $a = B - 1, b = t, c = t - 1$.

In base 4, 3210 is a fixed point but we also have a 2 member loop consisting of 3321 and 2220; demonstrating that all numbers need not end up in a unique loop even for 4-digit numbers. When is the final loop unique? (By unique here, I mean that it is the only loop other than the 'trivial' fixed point 0000.) Returning to our original question, for what bases B do all 4-digit numbers end up in a unique fixed point?

Finally we remember that in base 10 the pair (6 3) acts as a seed for creating bigger numbers, and notice that in any base B of the form $3t + 1$ there is a corresponding seed pair $(2t, t)$, since $2t - t = t$ and $(t + B - 1) - 2t = 2t$. For example, in base 4 the number 32 (2...2 1...1) 10 is a fixed point with $4 + 2k$ digits since

$$\begin{array}{rll} & 32(2...2) & (1...1)10 \\ - & 01(1...1) & (2...2)23 \end{array}$$

gives

$$31(1...10) \quad (2...2)21\,.$$

Roger Cook *is a number theorist in the Department of Pure Mathematics at Sheffield University. Outside mathematics his interests include photography, watching ice hockey, travel and hunting in second hand book and junk shops.*

Solution to Braintwister 6

(Square round robin)

Answer: The totals consisted of three of 1, thirteen of 9 and one of 16.

Solution:
Assume that there are A totals of 1, B totals of 4, C totals of 9, and D totals of 16. Then

$$A + B + C + D = 17 \tag{1}$$

$$\begin{aligned} A + 4B + 9C + 16D &= \text{number of games played} \\ &= \tfrac{1}{2} \cdot 17 \cdot 16 \\ &= 136. \end{aligned} \tag{2}$$

Also $D \leqslant 1$ (since two players cannot both win all their games) and $A \leqslant 3$ (since if four players only won 1 game each that would not even account for the 6 games they play amongst themselves).
Now

$$9 \times (1) - (2) \Rightarrow 8A + 5B - 7D = 17.$$

So $D = 0$ is soon ruled out because $8A + 5B = 17$ has no solution with A =0,1,2 or 3. Hence $D = 1$ and $8A + 5B = 24$. This makes B divisible by 8 and leads to the unique solution $A = 3$, $B = 0$, $C = 13$ and $D = 1$.

VICTOR BRYANT

Calculus Unequalled

P. GLAISTER

The amazing versatility of calculus is investigated further with inequalities

I never cease to be amazed at the wide applications of calculus, particularly in its simplest form. A favourite of mine is the use of the derivative to prove an inequality. If a function $f(x)$ satisfies $f(x_0) \geqslant 0$ for some x_0, and $f'(x) \geqslant 0$ for $x \geqslant x_0$, then $f(x) \geqslant 0$ for $x \geqslant x_0$. In other words, a smooth function which is non-negative at $x = x_0$, and has non-negative slope for $x \geqslant x_0$, is necessarily non-negative itself for $x \geqslant x_0$. Similarly, if $f(x_0) \geqslant 0$ for some x_0, and $f'(x) \leqslant 0$ for $x \leqslant x_0$, then $f(x) \geqslant 0$ for $x \leqslant x_0$. In this article we describe some simple applications of this idea, one of which leads to an interesting problem for further consideration.

We begin with the inequality

$$x^p - 1 \geqslant p(x-1), \quad p \geqslant 1,$$

for $x \geqslant 0$. If $f(x) = x^p - 1 - p(x-1)$ then $f(1) = 0$ and $f'(x) = p(x^{p-1} - 1) \geqslant 0$ for $x \geqslant 1$. Similarly $f'(x) \leqslant 0$ for $0 < x \leqslant 1$. Thus by the argument described above, $f(x) \geqslant 0$ for all $x > 0$. Moreover, since $f(0) = p - 1 \geqslant 0$, we have $f(x) \geqslant 0$ for $x \geqslant 0$. Other examples for readers to consider include the corresponding one to that above, namely

$$x^p - 1 \leqslant p(x-1), \quad 0 < p \leqslant 1,$$

for $x \geqslant 0$, as well as

$$\begin{aligned} \tan(x) &\geqslant x && \text{for } x \geqslant 0, \\ x &\geqslant \sin(x) && \text{for } x \geqslant 0, \\ \cos(x) &\geqslant 1 - \tfrac{1}{2}x^2 && \text{for } x \geqslant 0, \end{aligned}$$

(for which the previous inequality is required),

$$\begin{aligned} \ln(x) &\leqslant x - 1 && \text{for } x > 0, \\ e^x &\geqslant 1 + x && \text{for } x \geqslant 0, \end{aligned}$$

and the well-known inequality relating the arithmetic and geometric means

$$\tfrac{1}{2}(1+x) \geqslant \sqrt{x} \quad \text{for } x \geqslant 0.$$

One less well-known inequality is $\sinh^{-1}(x) \geqslant \tan^{-1}(x)$ for $x \geqslant 0$. With $f(x) = \sinh^{-1}(x) - \tan^{-1}(x)$, we see from figure 1 that f is indeed non-negative for $x \geqslant 0$. To prove this we note first that $f(0) = 0$ and

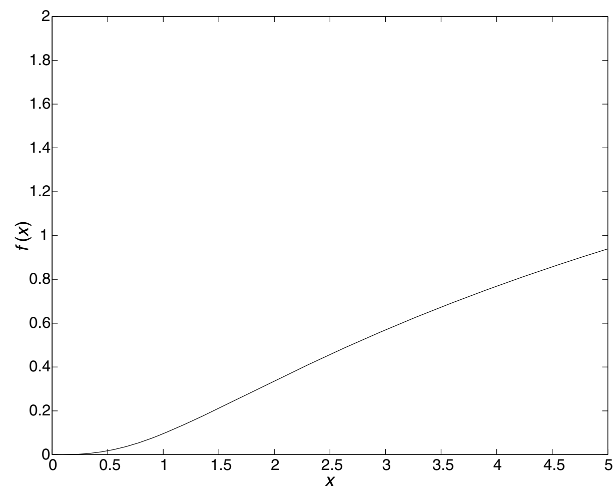


Figure 1. Graph of $f(x) = \sinh^{-1}(x) - \tan^{-1}(x)$.

$$f'(x) = \frac{1}{\sqrt{1+x^2}} - \frac{1}{1+x^2} \quad = \frac{\sqrt{1+x^2} - 1}{1+x^2} \geqslant 0$$

for $x \geqslant 0$, and hence $f(x) \geqslant 0$ for $x \geqslant 0$. Interestingly, f' increases to a maximum value and then decreases, although it is always positive, and this transition point corresponds to a point of inflexion where $f''(x) = 0$ and $f'(x) \neq 0$. This is easily proved since

$$f''(x) = \frac{x(2 - \sqrt{1+x^2})}{(1+x^2)2} \begin{cases} > 0 & \text{for } 0 < x < \sqrt{3}, \\ = 0 & \text{for } x = \sqrt{3}, \\ < 0 & \text{for } x > \sqrt{3}. \end{cases}$$

This last example is not an abstract one, however. A well-known problem in mechanics is to consider the time of flight of a ball projected vertically upwards and subject to a resistance proportional to the square of the speed. For this problem it can be shown that the time up to the highest point, T_u, and the time down from the highest point back to the point of projection, T_d, are given by

$$T_u = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}\, U\right)$$

and

$$T_d = \frac{1}{\sqrt{gk}} \sinh^{-1}\left(\sqrt{\frac{k}{g}}\, U\right),$$

where g is the acceleration due to gravity, k is the constant of proportionality in the resistance and U is the speed of projection. The difference

$$T_d - T_u = \frac{1}{\sqrt{gk}}\left(\sinh^{-1}\left(\sqrt{\frac{k}{g}}\,U\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\,U\right)\right)$$

is therefore positive by the result above. Thus, although what goes up must come down, it takes longer to come down than to go up!

An inequality whose generalisation has many interesting features is $\mathrm{e}^x \geqslant x$ for $x \geqslant 0$. This is readily proved as follows. Consider the function $g(x) = x/\mathrm{e} - \ln(x)$ for which $g(\mathrm{e}) = 0$ and $g'(x) = 1/\mathrm{e} - 1/x \geqslant 0$ for $x \geqslant \mathrm{e}$ so that $g(x) \geqslant 0$ for $x \geqslant \mathrm{e}$. Similarly, $g'(x) \leqslant 0$ for $0 < x \leqslant \mathrm{e}$ so that $g(x) \geqslant 0$ for $0 < x \leqslant \mathrm{e}$, and hence $g(x) \geqslant 0$ for all $x > 0$. Therefore $x \geqslant \mathrm{e}\ln(x) = \ln(x^{\mathrm{e}})$, i.e. $\mathrm{e}^x \geqslant x^{\mathrm{e}}$ for $x \geqslant 0$. (Note that $\mathrm{e}^0 = 1$ and $0^{\mathrm{e}} = 0$.)

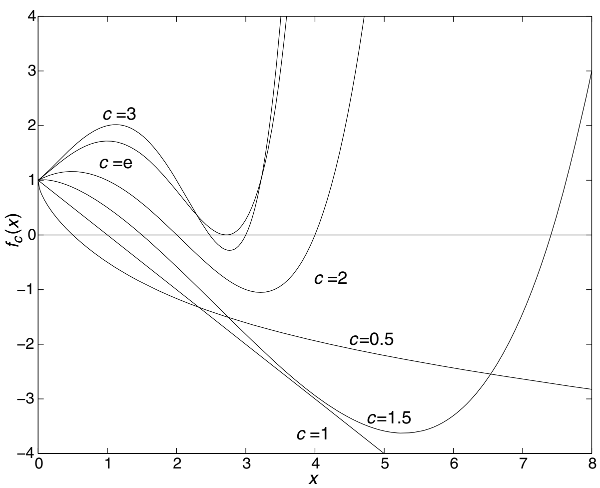


Figure 2. Graph of $f_c(x) = c^x - x^c$ for various c.

Following on from this, a natural question to ask is how does the function $f_c(x) = c^x - x^c$, $x \geqslant 0$, behave as c varies? Figure 2 shows the graph of f_c for $c = 0.5$, 1, 1.5, 2, e and 3, from which we observe that $f_{\mathrm{e}}(x) \geqslant 0$ for all $x \geqslant 0$, whereas for other values of c, for example the case $c = 2$, there are intervals in which $f_c(x) < 0$. In fact, for $0 < c \leqslant 1$, $f_c(x) \leqslant 0$ in the infinite interval $[c, \infty)$ which shrinks to $[1, \infty)$ as c increases from 0 to 1. For $c > 1$, $f_c(x) \leqslant 0$ in a finite interval only; moreover, this interval shrinks to a point (namely e) as c increases from 1 to e, and then expands again as c increases beyond e. This is a remarkable property of the number e.

Further theoretical properties, suggested by the graphs, can be established by the use of calculus, and depend on the value of c. For example, for all $c > 0$,

$$\begin{aligned} f_c(x) &= c^x - x^c, \quad \text{so } f_c(c) = 0,\\ f_c'(x) &= \ln(c)c^x - cx^{c-1},\\ f_c''(x) &= (\ln(c))^2 c^x - c(c-1)x^{c-2}. \end{aligned}$$

Thus, with $c = \mathrm{e}$,

$$\begin{aligned} f_{\mathrm{e}}'(x) &= \mathrm{e}^x - \mathrm{e}x^{\mathrm{e}-1},\\ f_{\mathrm{e}}'(1) &= f_{\mathrm{e}}'(\mathrm{e}) = 0,\\ f_{\mathrm{e}}''(x) &= \mathrm{e}^x - \mathrm{e}(\mathrm{e}-1)x^{\mathrm{e}-2},\\ f_{\mathrm{e}}''(1) &= \mathrm{e}(2-\mathrm{e}) < 0 \quad \text{and } f_{\mathrm{e}}''(\mathrm{e}) = \mathrm{e}^{\mathrm{e}-1} > 0. \end{aligned}$$

Hence f_e has a maximum of amount $\mathrm{e} - 1$ at 1 and a minimum of amount 0 at e. As a special case, if $c = 1$, then $f_1(x) = 1 - x$.

For $0 < c < 1$, since $\ln(c) < 0$, $f_c'(x) < 0$ for $x > 0$ and so f_c decreases for $x \geqslant 0$. It follows that $x = c$ is the only root of $f_c(x) = 0$. As $x \to \infty$, $c^x \to 0$ and $x^c \to \infty$. Hence $f_c(x) \to -\infty$ as $x \to \infty$ and, for large x, $f_c(x)$ behaves like $-x^c$. Also $f_c'(x) \to 0$ as $x \to \infty$ since c^x and $x^{c-1} \to 0$ as $x \to \infty$. Collectively, the graphs in figure 2 show the various shapes of f_c for all positive c.

Problems of particular interest include the determination of the roots of $f_c(x)$. The cases $c = 2$ and 3 are worth considering first. (Note that $x = c$ is one root since $f_c(c) = 0$ as observed above.) The equation $c^x = x^c$ suggests two numerical iterations:

$$x_{n+1} = \mathrm{e}^{px_n}$$

and

$$x_{n+1} = \frac{1}{p}\ln(x_n),$$

where $p = \ln(c)/c$. The first of these converges to the smallest root from starting values x_0 below the larger root, whilst the second converges to the larger root from starting values x_0 between the roots. For example, the roots of $3^x = x^3$ are $x = 3$ and $x = 2.478\,0506$. As well as the intersection of two graphs for different c, the determination of the maxima, minima and oblique points of inflexion is also of interest.

Finally, figure 2 suggests that $2^4 = 4^2$ gives the only integer solution of $c^x = x^c$ for $x \neq c$. There are clearly non-integer solutions of this equation; however, what are the rational solutions? For example, $(9/4)^{(27/8)} = (27/8)^{(9/4)}$.

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, and perturbation methods, as well as mathematics and science education. With both his children firmly established at school, he is taking the opportunity, through them, to engage in those areas of the curiculum that he neglected, regrettably, first time round!

Cigarette Butts

DAVID SINGMASTER

A tramp collects cigarette butts. From b butts, he can make one cigarette. If he has N butts, how many smokes can he get?

This is a fairly classic problem, dating from at least the 1930s, but it is usually given with specific numbers, e.g. $N = 25$, $b = 5$, and it is often given as a 'catch' problem. In our example, the obvious answer is 25/5 = 5 — but these five smokes yield five further butts which yield one further smoke! Once the catch is seen, it is easy to compute S, the number of smokes. This gives a somewhat elaborate expression, but some thought allows one to simplify this to an easy expression. But, as commonly happens, an easy expression for an answer indicates that there must be some more direct way to find the result, and indeed there is.

The straightforward computation proceeds as follows. Divide b into N, obtaining a quotient q_1 and remainder r_1, with $0 \leqslant r_1 < b$:

$$N = bq_1 + r_1. \tag{1}$$

After q_1 smokes, there are q_1+r_1 butts remaining. We repeat the division process:

$$q_1 + r_1 = bq_2 + r_2. \tag{2}$$

We continue until step s:

$$q_{s-1} + r_{s-1} = bq_s + r_s, \tag{3}$$

where $q_{s-1} + r_{s-1} \geqslant b$ but $q_s + r_s < b$, so that no more smokes can be obtained. Then:

$$S = \sum_{i=1}^{s} q_i. \tag{4}$$

(There is some irregularity for $N = 0$, so we assume that $N > 0$.)

The previous paragraph certainly solves the problem, but it takes some calculation. Is there any easier way to express S? Some experimentation shows that it depends on $b - 1$. We can rewrite (1) and (2) as

$$N = (b-1)q_1 + (q_1 + r_1); \tag{1$'$}$$

$$q_1 + r_1 = (b-1)q_2 + (q_2 + r_2). \tag{2$'$}$$

Hence

$$N = (b-1)(q_1 + q_2) + (q_2 + r_2). \tag{5}$$

Continuing, we have

$$\begin{aligned} N &= (b-1)(q_1 + \cdots + q_s) + (q_s + r_s) \\ &= (b-1)S + (q_s + r_s). \end{aligned} \tag{6}$$

Now we assumed that $q_{s-1} + r_{s-1} \geqslant b$, so $q_s \geqslant 1$ and so $q_s + r_s \geqslant 1$. We also have $q_s + r_s \leqslant b - 1$, so $q_s + r_s \in [1, b-1]$. Hence (6) is not exactly the formula for dividing N by $b - 1$, but we can adjust this by subtracting one from it:

$$N - 1 = (b-1)S + (q_s + r_s - 1), \tag{7}$$

where $0 \leqslant q_s + r_s - 1 \leqslant b - 1$. Thus, we have

$$S = [(N-1)/(b-1)], \tag{8}$$

where $[x]$ is the integer part of x.

Equation (8) is obviously much easier to use than equation (4). Indeed, equation (8) is so much easier that it should inspire us to hunt for an easier derivation of it. Considering the process of making and smoking a cigarette, we see that it uses $b-1$ butts. For each $b-1$ butts, we get a single smoke — provided there are at least b butts. That is, we divide N by $b - 1$, but with remainder $\in [1, b-1]$, and the quotient is S. This is precisely the relation found in (6) above! The case $b = 2$ is equivalent to the well-known 'aha' problem of determining the number of matches required in a knock-out tournament with N players. This 'aha' viewpoint has the further advantage of showing that one cannot obtain more smokes by some ingenious alternative method of combining the butts.

David Singmaster *is a part-time lecturer at South Bank University, London.*

A woman says:
'I have two children of whom at least one is a girl.'
What is the probablility that both are girls?
(Taken from the Canberra Times.)

Which number is bigger,

100^{101} or 101^{100}?

Mathematics in the Classroom

Proof in Mathematics

At A-level, the elementary ideas involved in mathematical proofs can appear to be trying to prove the obvious. But the media has recently played its part in raising awareness of the long and involved processes that rigorous mathematical proof can demand (as with the much-publicised proving of Fermat's Last Theorem, for example, a problem whose solution had eluded mathematicians for years). Indeed, there are instances where proofs are still waiting to be found, so this is not an area where all the answers are known. But at this opening level, two methods of proof are usually introduced.

Proof by contradiction

Euclid of Alexandria used this method as early as 300 BC to prove his theorem on prime numbers:

Theorem. *There are infinitely many prime numbers*

In the proof of this he assumed the opposite, i.e. he assumed that there were a finite number of primes, n say, and these were denoted $p_1, p_2, \ldots, p_n$. From these, he formed the number $X = p_1 p_2 \ldots p_n + 1$, and reasoned as follows:

If X is prime then it is larger than any of the other primes, so there must be more than n primes.

If, on the other hand, X is not prime, then it must be divisible by a prime. But this is not p_1 nor $p_2 \ldots$ nor p_n because each of these leaves a remainder of 1 when they divide X. Hence there must be another prime divisor and again the assumption of only n primes is false.

Hence the original assumption of a finite number of primes is false, so there must be infinitely many primes. Thus the theorem is proved by contradiction.

Applying this same technique, it is relatively straightforward to prove other theorems such as:

Theorem. $\sqrt{2}$ *is irrational.*

Proof. Assume $\sqrt{2}$ is rational, i.e. $\sqrt{2} = p/q$, where p and q are integers. Then $2q^2 = p^2$. As the right-hand side must have an even number of prime factors and the left-hand side must have an odd number of prime factors, a contradiction is achieved. Hence $\sqrt{2}$ cannot be rational and so must be irrational.

Can you use the same method to prove that

rational + irrational = irrational?

Proof by counterexample

If you can find one particular case for which a statement is not true, then that general statement is proved to be false. For example, is the statement irrational + irrational = irrational true or false?

Although there are lots of examples for which this statement is true, we can readily find a counterexample to show that this is a false statement. Such an example is simply $(4 - \sqrt{2}) + (\sqrt{2}) = 4$, which is certainly not irrational.

Can you use the same method to disprove the conjecture (thought to be made in 1556 by Tartaglia (see reference 1)) that the numbers

$$1 + 2 + 4, \quad 1 + 2 + 4 + 8, \quad 1 + 2 + 4 + 8 + 16, \ldots$$

are alternately prime and composite (i.e. not prime)?

Proof by induction

This method involves a conjecture relating to a positive integer n. If it can be shown that

- if the conjecture is true when $n = k$, then it is also true when $n = k + 1$; and that
- the conjecture is true when $n = 1$,

then the conjecture is taken as true for all positive integral n. For example:

Conjecture. $9^n - 1$ *is divisible by* 8 *for all positive integers* n.

Proof by induction. Assume the conjecture is true for $n = k$, i.e. $9^k - 1 = 8N$. Then

$$\begin{aligned} 9^{k+1} - 1 &= 9.9^k - 9 + 8 \\ &= 9(9^k - 1) + 8 \\ &= 9.8N + 8 \\ &= 8(9N + 1)\,, \end{aligned}$$

which is clearly divisible by 8. Hence if the result is true for $n = k$, it is also true for $n = k + 1$.

Now all we need to do is to check the case $n = 1$: the left-hand side $= 9 - 1 = 8$, which is also clearly divisible by 8 and hence the proof is complete.

See if you can use this method to prove the conjecture that for all integral values of n, $2^{n+2} + 3^{2n+1}$ is exactly divisible by 7.

Getting stuck

It is all very well successfully solving a problem when you have access to a similar example, but not always so easy when you are approaching the problem with just a blank sheet of paper in front of you. John Mason (reference 1) offers some helpful strategies for when you find yourself in this position. He suggests that you should try reminding yourself of what you *know* and what you *want* to achieve. Then operate the strategy of

- specializing, i.e. looking at particular cases of a general statement, and then
- generalizing, i.e. abstracting features common to several particular examples,
- exercising, i.e. practising examples and techniques to reach mastery.

As an example, he offers the following exercise.

Exercise. By investigating the sequence of numbers

$$\sqrt{2}, \quad \sqrt{2}^{\sqrt{2}}, \quad \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}, \ldots,$$

show that it is possible to find an irrational number which can be raised to an irrational power, and yet yield a rational number as the result. Generalize.

Mathematically thinking it through.

We *know* that $\sqrt{2}$ is irrational.

We *know* that

$$\sqrt{2}^{\sqrt{2}}$$

is an irrational raised to an irrational power.

We *want* an irrational raised to an irrational to be rational, but we do not know if $\sqrt{2}^{\sqrt{2}}$ is rational or irrational, although we suspect it is irrational.

So *continue specializing:*

We know that

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is certainly rational.

Hence, *either* $\sqrt{2}^{\sqrt{2}}$ is *rational*, in which case we have what we wanted, *or*, it is *irrational*, in which case, itself raised to $\sqrt{2}$ is rational, which is what we wanted.

Hence, in either case, we get what we want and the claim is justified. Although we do not know which example is the one we want, we do know that it must be one of them! To generalize try the same idea with cube roots etc.

There are a lot more exercises where this came from, all of them recommended for building up confidence and familiarity with approaches and ability to think mathematically.

Carol Nixon

Reference

1. John Mason, *Learning and Doing Mathematics* (MacMillan Education Ltd, 1988).

Computer Column

Mathematics on the Web

The explosive growth of the World Wide Web means that everyone now has the opportunity to publish documents electronically. It *should* be an exciting new way of presenting mathematics. For example, teachers could make lecture notes available for pupils to read at home; students' project work could develop into a useful shared resource; one could imagine clicking on an expression and automatically firing up a symbolic manipulation package to draw a graph of the expression; there are many interesting possibilities. Unfortunately, there's a problem: Web pages are written in hypertext markup language (HTML) and at present HTML does not support complicated mathematical notation.

There are several reasons why it is not straighforward for computers to represent mathematics. Mathematicians use special symbols (Greek letters, for instance) that are not in the standard ASCII character set. And lots of mathematics notation (like matrix notation, for example) is 'two dimensional', whereas computer keyboards produce a 'one-dimensional' input stream. These and other problems mean that most of the Web sites you visit — even those devoted to mathematics — will consist simply of text and diagrams. But what if you really want your Web page to contain mathematics? Unfortunately there is as yet no universally accepted Web-standard format for publishing mathematics. This column is intended to give you some ideas.

The options open to you depend greatly upon how you authored the mathematics in the first place. Many professional mathematicians and computer scientists use TeX, or one of its flavours such as LaTeX, to write mathematics. A TeX file is just a plain ASCII file, so that it is easy to key into a computer. After processing the file, however, you generate typeset output of the highest quality. (Note that TeX is freely available in the public domain, and runs on nearly all computer platforms!) In LaTeX, a 'two-dimensional' object like

$$\begin{pmatrix}\beta & 1\\ 1 & 0\end{pmatrix}$$

is obtained by the 'one-dimensional' string

```
$\begin{pmatrix}\beta&1\\1&0\end{pmatrix}$
```

A piece of LaTeX code like this is of little use in a Web page, since the Web browser would display the ASCII code rather than the typeset matrix. However, there is a public-domain program called LaTeX2html that takes LaTeX code and, whenever it comes across a mathematical expression, automatically generates a transparent gif image of that expression. In other words, it generates a 'picture' of an equation.

Even if you do not have access to TeX, you can still use this idea in your own Web pages. (The details of how to generate a transparent gif image of an equation will vary, depending on how you authored the equation and on the graphics software available to you.) The advantage to this approach is that, whichever method you use to produce the image, visitors to your site should see your mathematics: transparent gifs are the standard way of embedding an image in a Web page. There are several disadvantages to this approach, though. If you have a lot of mathematics and you have to generate each gif image ‘by hand’, it will be a time-consuming task. Downloading your page may be slow. Furthermore, although the resulting Web page might be readable, it often will not look good and the print quality will almost certainly be poor. The moral is: beware of Greek-bearing gifs!

You can get much better quality by producing portable document format (PDF) files. PDF is a version of PostScript that is tailored to electronic publishing. The procedure is to generate a PostScript file containing your typeset mathematics (this need not be from TEX; word processors with equation editors, like Word, work fine). You then run the PostScript file through a program called Adobe Distiller in order to generate the PDF file. The disadvantages to this approach are: (i) you must purchase the software to distill PDF files, and (ii) the person who wants to view your mathematics must first install a PDF viewer on their system. (The viewer is called Acrobat, and is freely available for most platforms.)

Neither of these approaches gives real interactivity to a Web page. Currently, mathematicians are developing the Mathematical Markup Language (MathML). Just as HTML handles text, the goal of MathML is to enable mathematics to be served, received and processed on the Web. The idea is for authors to write mathematics with their favourite systems (TEX, Word, WordPerfect, or whatever) and for conversion programs to generate MathML that browsers can interpret. MathML is still in its infancy, so it is not a realistic option if you want to develop a mathematics-based Web page *now*. But given the incredible rate of Web-technology development, it might well be an option in a year’s time. Point your browswer to `http://www.w3.org/Math/` if you are interested in the current status of MathML.

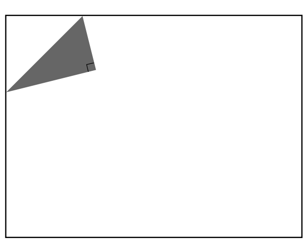
There are several other options available to you, particularly if you are a TEX user (the TTH and techexplorer programs are two more examples). If you know of an interesting site Web site that renders mathematics using a system not mentioned above, why not write and let us know?

Stephen Webb

Braintwister

7. Table tour

I have a piece of cardboard cut in the shape of a right-angled triangle. The lengths of the two shorter sides differ by one centimetre. I slide the triangle around the edge of a large rectangular table-top, as shown, with the two acute-angled vertices always touching the edge of the table:



In one complete trip around the perimeter of the table-top, the right-angled vertex travels 48 cm less far than each of the other two vertices.

What are the lengths of the three sides of the triangle?

(The solution will be published next time.)

VICTOR BRYANT

The 1998 puzzle

This was to express the numbers 1 to 100 using each of the digits of the year in order using only the operations of $+, -, \times, \div, \sqrt{\ }$, !, brackets and concatenation (e.g. putting the digits 1 and 9 together to make 19). A number of readers sent in solutions or partial solutions. Robert Redgrave (Richard Huish College, Taunton) sent in a complete solution, including:

$$59 = -1 + \sqrt{\big((\sqrt{9}!)!\big) \times \big(-\sqrt{9} + 8\big)},$$
$$86 = -1 - \sqrt{9} + \big(((\sqrt{9}!)!) \div 8\big),$$
$$87 = -1 + \big(((\sqrt{9}\,!)!) \div 9\big) + 8.$$

Odd Squares

$3^2 = 4 + 5$	$11^2 = 60 + 61$
$5^2 = 12 + 13$	$13^2 = 84 + 85$
$7^2 = 24 + 25$	$15^2 = 112 + 113$
$9^2 = 40 + 41$	

Is there a general rule?

BABLU CHANDRA DEY,
Calcutta.

Letters to the Editor

Dear Editor,

Problem 30.8

The solution by T. Raine, A. Holland, J. Young (Volume 31, p. 23), unfortunately stops one step short of an important result.

They give the area of the maximal (cyclic) quadrilateral as

$$A = \tfrac{1}{4}\left\{[(a+b)^2-(c-d)^2][(c+d)^2-(a-b)^2]\right\}^{\frac{1}{2}}.$$

This becomes

$$A = \left\{\left[\tfrac{1}{2}(-a+b+c+d)\right]\left[\tfrac{1}{2}(a-b+c+d)\right]\left[\tfrac{1}{2}(a+b-c+d)\right]\left[\tfrac{1}{2}(a+b+c-d)\right]\right\}^{\frac{1}{2}}$$

and, putting $s=\frac{1}{2}(a+b+c+d)$, we have

$$A = \{(s-a)(s-b)(s-c)(s-d)\}^{\frac{1}{2}},$$

a beautiful generalization of the well-known triangle formula.

Johnson (reference 1) attributes this result to W. Fuhrmann, and comments that it and several related theorems seem to be less well known than they merit.

The original problem specified the sides in a definite order. However, because of the symmetry of Fuhrmann's formula, the order is irrelevant. Any four given sides (assuming, of course, that any one is less than the sum of the other three) can be permuted in three ways. The resulting cyclic quadrilaterals are different in shape, but lie within the same circumcircle and are equal in area. This is readily demonstrated thus: given cyclic quadrilateral ABCD, on base BD, and on the same side of it as A, erect $\triangle$A′DB so that A′D=AB and A′B=AD. Then $\triangle$ABD and $\triangle$A′DB are congruent, A′ lies on the circumcircle, and quadrilaterals ABCD and A′BCD are equal in area. A similar construction on diaginal AC produces the third permutation, with similar results.

Yours sincerely,
K. R. Levingston
(P. O. Box 471, Charters Towers,
4820 Queensland, Australia.)

References

1. R. A. Johnson, *Advanced Euclidean Geometry (Modern Geometry),* (Dover, New York, 1960).

Dear Editor,

Unit Fractions

A four-volume treatise, *The World of Mathematics* by James R. Newman, has lain, virtually unopened, on my book shelf for about 35 years. I have just finished reading the first volume.

In it he quotes from a book, *The Great Mathematicians* by H. W. Turnbull. Apparently there is a papyrus in the Rhind collection in the British Museum written by an Egyptian priest Ahmes, who lived sometime before 1700 BC. This gentleman had little better to do with his time than to express fractions such as $2/(2n+1)$ as sums of fractions each of whose numerators is unity. The example quoted is:

$$\frac{2}{29}=\frac{1}{24}+\frac{1}{58}+\frac{1}{174}+\frac{1}{232}.$$

It is difficult to believe that he achieved this by trial and error, even in those days, since many examples are given in the papyrus. Do we know how he did it?

Yours sincerely,
G. L. Turner
(Birmingham Road,
Aldridge, UK.)

Dear Editor,

Adjoining consecutive numbers

Following on from Bob Bertuello's letter (Volume 30, page 64), I have extended the search to 10^{12} for adjoining consecutive numbers mode square, via a Texas T1 85 programmable calculator, yielding these additional examples:

$$1322\,413\,225 = 36\,365^2,$$
$$4049\,540\,496 = 63\,636^2,$$
$$106\,755\,106\,756 = 326\,734^2,$$
$$453\,288\,453\,289 = 673\,267^2,$$

and with the larger numbers first

$$538\,277\,538\,276 = 733\,674^2$$
$$998\,002\,998\,001 = 999\,001^2.$$

A class of examples of this latter type in any base $X>2$ is given by

$$X^2(X^2-2X+1)+(X^2-2X+1)+X^2=(X^2-X+1)^2.$$

Thus, for example,
$X=10$ gives $8281=91^2$,
$X=100$ gives $98\,029\,801=9901^2$,
$X=1000$ gives $998\,002\,998\,001=999\,001^2$,
and so on.

Yours sincerely,
C. M. Christison
(Sheeppark, Whauphill,
Wigtownshire.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

31.5 ABC is a fixed triangle.
(i) P, Q, R are points on the sides BC, CA, AB respectively such that BP/BC = CQ/CA = AR/AB. When is the area of triangle PQR minimal?
(ii) P′, Q′, R′ are points on the sides BC, CA, AB respectively such that P′ is the midpoint of BC and R′Q′ is parallel to BC. When is the area of triangle P′Q′R′ maximal?

(Submitted by J. A. Scott, Chippenham)

31.6 Evaluate $\sum_{k=1}^{n}\left[k2^{1/k}\right]$, where $[\alpha]$ denotes the integer part of α.

(Submitted by Hassan Shah Ali, Tehran)

31.7 Show that the sequence $\{u_n\}$, where $u_n = [n\sqrt{3}]$, contains infinitely many perfect squares.

(Submitted by Hassan Shah Ali, Tehran)

31.8 Discuss the convergence of the series

$$\sum_{n=1}^{\infty}\left(\mathrm{e}-\left(1+\frac{1}{n}\right)^{n}\right)^{\alpha}$$

for different values of the positive real number α.

(Submitted by J. A. Scott, Chippenham)

Solutions to Problems in Volume 30 Number 3

30.9 There are n ballot papers, n_1 of which are coloured 1, n_2 coloured 2, … , n_r coloured r. What is the probability that two ballot papers picked out at random are coloured i and j? (The order of selection is irrelevant.)

Solution by Jeremy Young (Nottingham High School) and independently by Andrew Holland (Nottingham High School).

When $i \neq j$, the probability of choosing i followed by j is

$$\frac{n_i}{n} \times \frac{n_j}{n-1}.$$

The probability of choosing j followed by i is

$$\frac{n_j}{n} \times \frac{n_i}{n-1}.$$

Hence the probability of choosing colours i and j is

$$\frac{2n_i n_j}{n(n-1)}.$$

Note. Neither solution pointed out that, when $i = j$, the probability is

$$\frac{n_i(n_i-1)}{n(n-1)}.$$

30.10 (i) Find all pairs of real numbers whose sum is equal to their product.
(ii) Find all triples of real numbers whose sum is equal to their product.
(iii) For $n > 3$, find all n-tuples of positive real numbers whose sum is equal to their product.

Solution by Ian Glover (Trinity Hall, Cambridge).

(i) Let a, b be real numbers such that

$$a + b = ab.$$

Then $a \neq 1$ (otherwise $1 + b = b$) and

$$b = \frac{a}{a-1}.$$

Conversely, all pairs $(a, a/(a-1))$ with $a \neq 1$ satisfy the condition, so this gives all such pairs.

(ii) Let a, b, c be real numbers such that

$$a + b + c = abc.$$

Then $ab \neq 1$ (otherwise $a + b = 0$ whence $b = -a$ and $-a^2 = 1$, which is impossible) and

$$c = \frac{a+b}{ab-1}.$$

Conversely, all triples $(a, b, (a+b)/(ab-1))$ with $ab \neq 1$ satisfy the condition, so this gives all such triples.

(iii) Let $a_1, \dots, a_n$ $(n > 3)$ be positive real numbers such that $a_1+\cdots+a_n = a_1a_2\dots a_n$. Then $a_1a_2\dots a_{n-1} > 1$ (otherwise $a_1+\cdots+a_{n-1} = (a_1\dots a_{n-1}-1)\,a_n \leqslant 0$, which is impossible since $a_1, \dots, a_{n-1}$ are positive) and

$$a_n = \frac{a_1+\cdots+a_{n-1}}{(a_1a_2\dots a_{n-1})-1}.$$

Conversely, all n-tuples

$$\left(a_1, \dots, a_{n-1}, \frac{a_1+\cdots+a_{n-1}}{a_1a_2\dots a_{n-1})-1}\right)$$

with $a_1a_2\dots a_{n-1} > 1$ satisfy the condition, so this gives all such n-tuples.

Ian Glover extended (iii) to all real numbers. Suppose that

$$a_1 + \cdots + a_n = a_1 a_2 \ldots a_n.$$

Then

$$(a_1 a_2 \ldots a_{n-1} - 1)\, a_n = a_1 + \cdots + a_{n-1}.$$

If $a_1 a_2 \ldots a_{n-1} \neq 1$, then we have all n-tuples of the form

$$\left(a_1, a_2, \ldots, a_{n-1}, \frac{a_1 + \cdots + a_{n-1}}{a_1 a_2 \ldots a_{n-1} - 1} \right),$$

with $a_1 a_2 \ldots a_{n-1} \neq 1$, and, conversely, all such n-tuples satisfy the condition. Now suppose $a_1 a_2 \ldots a_{n-1} = 1$. Then $a_1 + \cdots + a_{n-1} = 0$. Thus

$$a_1 \ldots a_{n-2}\,(a_1 + \cdots + a_{n-2}) = -1\,,$$

so that

$$\begin{aligned}&(a_1 \ldots a_{n-3})\, a_{n-2}^2 \\ &\qquad + (a_1 \ldots a_{n-3})(a_1 + \ldots + a_{n-3}) a_{n-2} + 1 = 0,\end{aligned}$$

so

$$a_{n-2}^2 + (a_1 + \cdots + a_{n-3})\, a_{n-2} + \frac{1}{a_1 \ldots a_{n-3}} = 0.$$

Hence

$$\begin{aligned}a_{n-2} = &-\tfrac{1}{2}(a_1 + \cdots + a_{n-3}) \\ &\pm \tfrac{1}{2}\left\{ (a_1 + \ldots + a_{n-3})^2 - \frac{4}{a_1 \cdots a_{n-3}} \right\}^{1/2}.\end{aligned}$$

Thus the n-tuple is of the form

$$\begin{aligned}\Bigg(a_1, \ldots, a_{n-3}, &-\tfrac{1}{2}(a_1 + \cdots + a_{n-3}) \\ &\pm \tfrac{1}{2}\left\{ (a_1 + \cdots + a_{n-3})^2 - \frac{4}{a_1 \ldots a_{n-3}} \right\}^{1/2}, \\ &- \tfrac{1}{2}(a_1 + \cdots + a_{n-3}) \\ &\mp \tfrac{1}{2}\left\{ (a_1 + \ldots + a_{n-3})^2 - \frac{4}{a_1 \ldots a_{n-3}} \right\}^{1/2}, a_n \Bigg),\end{aligned}$$

where

$$(a_1 + \ldots + a_{n-3})^2 \geqslant \frac{4}{a_1 \ldots a_{n-3}}\,.$$

Conversely, all such n-tuples satisfy the condition.

Also solved by Jeremy Young.

30.11 Prove that, if p is a prime number, then $L_p \equiv 1 \pmod{p}$, where L_n denotes the nth Lucas number defined by $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$.

Solution (independently given) by Jeremy Young, Andrew Holland, Roger Cueulière (Clichy) and a variant by Harriet Robjent (Gresham's School, Holt).

$L_2 = 3 \equiv 1 \pmod 2$. Now suppose $p > 2$. By Binet's formula,

$$\begin{aligned}L_p &= \frac{1}{2^p}\left\{ \left(1 + \sqrt{5}\right)^p + \left(1 - \sqrt{5}\right)^p \right\} \\ &= \frac{1}{2^p}\left\{ 2 \sum_{k=0}^{(p-1)/2} \binom{p}{2k} 5^k \right\},\end{aligned}$$

so

$$2^{p-1} L_p = \sum_{k=1}^{(p-1)/2} \binom{p}{2k} 5^k + 1\,.$$

For a prime p, $\binom{p}{m} \equiv 0 \pmod p$ for $m = 1, \ldots, p-1$. Also by Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$. It follows that $L_p \equiv 1 \pmod p$.

30.12 Show that no prime number can be written as the sum of two squares in two different ways.

Solution by Jeremy Young.

Suppose that N has two different expressions as a sum of two squares, say $N = p^2 + q^2 = r^2 + s^2$, where p, q, r, s are non-negative integers. We show that N is composite. Clearly $N > 2$. If p, q are both even or both odd, then N is even and so composite. Suppose p is odd and q is even. Similarly, we can take s odd and r even; and $p \neq s$, $q \neq r$. Now

$$s^2 - p^2 = q^2 - r^2$$

so

$$(s + p)(s - p) = (q + r)(q - r)$$

so

$$\frac{q + r}{s - p} = \frac{s + p}{q - r}.$$

Now $q + r$, $s - p$ are both even; put $\operatorname{hcf}(q + r, s - p) = 2d$, where $d \in \mathbb{N}$. Also $s + p, q - r$ are both even; put $\operatorname{hcf}(s + p, q - r) = 2b$ where $b \in \mathbb{N}$. Then

$$\begin{aligned}q + r &= 2da, \quad & s - p &= 2dc\,, \\ s + p &= 2ba, \quad & q - r &= 2bc\,,\end{aligned}$$

for some $a, c \in \mathbb{Z}$, $a, c \neq 0$. Now

$$p = ab - cd,\ q = ad + bc,$$

so

$$N = p^2 + q^2 = (ab - cd)^2 + (ad + bc)^2 = (a^2 + c^2)(b^2 + d^2),$$

which is composite.

Reviews

MEI Mechanics 4: Differential Equations. By JOHN BERRY AND TED GRAHAM. Hodder & Stoughton, London, 1996. Pp. 168 Paperback £12.35. (ISBN 0-340-63085-X).

This is the fourth of the Mechanics books written as part of the MEI Structured Mathematics series, and the following topics are systematically covered: modelling, tangent fields, methods of solution (standard fare, including separation of variables, integrating factors, numerical methods and linear equations with constant coefficients) oscillations (including damped and forced oscillations) and systems of differential equations.

Although part of a *mechanics* course, there are various sections for discussion and investigation, not to mention the exercises, which consider a range of non-mechanical situations — for example, population dynamics (including the Malthusian and Verhulst models, although not referred to as such), law of cooling (with a rough, but entertaining, application to forensic science), radioactive decay, electric circuits, etc. The final chapter gives the student an exciting example of a non-linear differential equation describing fox and rabbit populations. Brief historical notes complement the main text and provide further stimulation.

My main criticism is the lack of really difficult examples, exercises and general detail. Further Mathematicians need a certain amount of tough mathematics; they should not need to have it explained that a certain equation has two sides divided by 2 (as on page 105). Again on page 44, we read the curious note 'the integrating factor method involves several steps that can introduce errors on the way through, so it is important to verify that the function you think is the answer does indeed satisfy the original differential equation'. Further mathematicians deserve better than this. Also, on page 103 is a dubious section about 'dashpots' (a means of describing damping), in which Newton's Third Law is used but not acknowledged (a strange omission for a Mechanics book, and a lost opportunity to reinforce one of the more commonly misunderstood principles among students) and in which '*the amount of travel*' left in the dashpot is denoted by L' (my italics).

However, overall the book is to be commended. The subject matter is, for the most part, very clearly and attractively presented, and I expect that it will rightly gain popularity as an A-level textbook.

Head of Mathematics STEPHEN ROUT
Queen Mary's Grammar School, Walsall

MEI Mathematics 5. By TERRY HEARD AND DAVID MARTIN. Hodder & Stoughton, London, 1997. Pp. 160 Paperback £11.75. (ISBN 0-340-64771-X).

This book, another in the MEI Structured Mathematics series, and aimed at Further Mathematicians, includes chapters on algebra, polar co-ordinates, calculus, complex numbers, conics and hyperbolic functions.

As an A-level text, these topics are dealt with competently and attractively. I was impressed by the wealth and variety of substantial results covered by the exercises and investigation sections, all of which should serve to whet the appetite of any serious-minded student. Geometrical topics are particularly nicely presented: we have references to linkages, Dandelin's spheres, trisection of angles, reflector properties of conics (with an illustration of a lithotriptor, or stone-crusher - surely the kind of inclusion likely to foster an appreciation of the practical relevance of mathematics).

The book has its share of dull ideas. The use of polar co-ordinates to define regions on a dartboard can hardly be uplifting to anyone who is glad to see the back of GCSE mathematics. Likewise, the illustration of a pedal-bin at the start of the chapter on complex numbers is not the best advertisement for the subject. (Why not Joukowski's aerofoil? Or an electric generator?)

The occasional lapse in rigour is unavoidable at A-level, but the question on page 80 surprised me. It concerns a statement, attributed to H. Wronski, which contains expressions such as $(1+i)^{1/\infty}$), and the question asks, 'Was Wronski wrong?' instead of 'What did he mean?'

However, any criticisms are rendered insignificant when compared with the bulk of the book. It is a clear, stimulating A-level text, and is highly commended.

Head of Mathematics STEPHEN ROUT
Queen Mary's Grammar School, Walsall.

The Mathematical Universe: An Alphabetical Journey Through the Great Proofs, Problems and Personalities. By WILLIAM DUNHAM. Wiley, Chichester, UK, 1997. Pp. 320. Paperback £13.99 (ISBN 0-4711-7661-3).

Anyone with an interest in mathematics dipping into this book, whose level of expertise is acquaintance with some later school years' algebra and geometry, will most certainly find their interest stimulated and extended. It can be particularly recommended to young people, and may well encourage many to take up serious further study of the subject!

There is an intentionally strong historical element in Dunham's expositions of his chosen mathematical themes, as he shows both the prowess and the foibles of the many personages who have bestowed upon us such a great legacy of mathematical techniques and results (accompanied by reproductions of their contemporary likenesses). These include: the early Egyptian, Babylonian, Chinese and Indian mathematicians; the Greeks Euclid, Pythagoras and Archimedes; the seventeenth and eighteenth century wizards Fermat, Newton, Leibnitz, the Bernoulli brothers, Jakob and Johann, Euler and Gauss; not forgetting mathematicians who were also female, for example, Sofia Kovalevskaia from the nineteenth century; and Bertrand Russell brings us into our own century.

Topics include the grand trilogy, the fundamental theorems of arithmetic, calculus and algebra; Pythagoras' theorem proved in a number of different ways; Archimedes' determination of the surface area of a sphere; Fermat's method of factorisation; Bernoulli's 'law of large numbers'; Wantzel's proof of the impossibility of the trisection of an angle with ruler and compass; the prime number theorem; Russell's paradox; zero, e, π, i and much more!

His accounts, always clear and technically sound, and peppered with amusing anecdotes and asides, are not all proved, but exemplary 'prooflets' of theorems abound (as well as a chapter 'Justification' on the significance of proof, 'the glue that holds mathematics together'). There are fine accompanying diagrams to the text and even cartoons!

In the Notes are useful citations of a number of other works of varying depth and difficulty, from the simplest to the most advanced (e.g. Descartes' 'Geometrie', said by Dunham to have been a difficult read even by Sir Isaac Newton!).

Although Dunham says in the Preface that his book is 'the response of a single individual to the immense mathematical universe', he acknowledges that the book is similar to John Allen Paulos' 'Beyond Numeracy', also organised in topic sequence following the alphabet, and which has greater breadth of coverage as against Dunham's (successful!) attempt at greater depth.

Wilbraham Street, Preston DAVE YATES

From Erdős to Kiev: Problems of Olympiad Caliber. By R. HONSBERGER. The Mathematical Association of America, Washington DC, 1996. Pp. xii+257. Paperback \$34.00 (ISBN 0-88385-324-8).

This book is a display of ingenuity in problem solving by Ross Honsberger, Professor in the Department of Combinatorics and Optimization at the University of Waterloo, Canada. He is renowned for his lucid expository style and for his choice of mathematical gems. Other books by him in this same style include *Mathematical Gems* (three books), *Mathematical Morsels, More Mathematical Morsels,* and *Mathematical Plums*. If you have read and enjoyed any of the above then the present volume is a must.

The book is a collection of problems, most of which come from national or international Olympiads. It is loosely divided into about fifty short sections, ranging from two problems propsed by the prolific Hungarian mathematician Paul Erdős to a problem from the 1954 Kiev Olympiad; hence the title of the book. There is a useful subject index, separating the problems into three main catagories: combinatorics and combinatorial geometry; algebra, number theory, probability, calculus; and geometry.

The two Erdős problems about infinite series sound simple, but are unusual and 'deceptively deep and subtle.' The first problem is as follows:

Prove that every infinite sequence S of distinct positive integers contains either

(a) an infinite subsequence such that, for every pair of terms, neither term ever divides the other, or

(b) an infinite subsequence such that, for every pair of terms, one alwasys divides the other.

The Kiev Olympiad problems asks the reader to prove that, if a square is circumscribed about the incircle of a triangle, then at least half the perimeter of the square lies inside the triangle.

One of my favourite problems in the book is in the form of a game with two players A and B who alternately pick positive integers less than some agreed limit L such that no integer may be a divisor of a number already used.The first player who is unable to play loses. The reader is asked to prove that if A starts, then he will be able to win, whatever the value of L.

This book is recommended for those interested in problem solving; it is especially useful for students because full solutions are provided to all the problems, often with commentary and discussion by the author.

Student, Trinity College, Cambridge MANSUR BOASE

Other books received

Maths Plus: 1. Student Book. Edited by JILL LANE AND SHEILA BENISTON. Collins Educational, London, 1998. Pp. 80. Paperback. £6.99 (ISBN 0-00-322484-8)

The Maths Plus course and materials have been designed to motivate and reward the least able students. The course aims to prepare them for progress towards GCSE Foundation Tier later on. The scheme is particularly suited to students studying for the NEAB Certificate of Achievement in Mathematics. There are two student books with accompanying teacher's packs. Sample pages of the teacher's packs are available.

Multivariable Calculus. By W. G. MCCALLUM, D. HUGHES-HALLETT, A. M. GLEASON *et al.* Wiley, Chichester, UK, 1998. Pp. 503. Paperback £21.50 (ISBN-0-471-19428-x).

Logic as Algebra. By P. HALMOS AND S. GIVANT The Mathematical Association of America, Washington DC, 1998. Pp. 152. Paperback \$27.00 (ISBN 0-88385-327-2).

Proof, Logic and Conjecture. By R. S. WOLF. Macmillan Distribution Ltd, UK, 1998. Pp. 425. Hardback £31.95 (ISBN 0-7167-3050-2).

Calculus. A New Horizon. By HOWARD ANTON. Wiley, New York, 1999. Pp. xxxi + 758. Hardback. £23.95 (ISBN 0-471-15307-9).

Improve Your Maths. By GORDON BANCROFT AND MIKE FLETCHER. Addison-Wesley, Edinburgh, 1998. Pp. 200. Softback. £11.95 (ISBN 0-201-33130-6).

Foundation Mathematics. 3rd edn. By DEXTER BOOTH. Addison-Wesley, Edinburgh, 1998. Pp. 656. Softback. £19.95 (ISBN-0-201-34294-4).

Discrete Maths. By STEPHEN BARNETT. Addison-Wesley, Edinburgh, 1998. Pp. 427. Softback. £18.95 (ISBN 0-201-34292-8).

A Primer of Abstract Mathematics. By ROBERT B. ASH. The Mathematical Association of America, Washington DC, 1998. Pp. 188. Paperback \$27.95 (ISBN 0-88385-709-X).

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