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Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER

No. 99

R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We continue this month with the remaining problems proposed to the jury, but not used, at the 29th I.M.O. in Australia. Thanks go again to Peter O'Halloran for relaying a copy to me through Murray Klamkin. Send me your nice solutions!

1. *Proposed by Bulgaria.*

Let n be a positive integer. Find the number of odd coefficients of the polynomial

$$u_n(x) = (x^2 + x + 1)^n.$$

2. *Proposed by Czechoslovakia.*

In a given tetrahedron $ABCD$, let K and L be the centres of edges AB and CD respectively. Prove that every plane that contains the line KL divides the tetrahedron into two parts of equal volume.

3. *Proposed by East Germany.*

The lock on a safe consists of three wheels, each of which may be set in eight different positions. Due to a defect in the safe mechanism the door will open if any two of the three wheels are in the correct position. What is the smallest number of combinations which must be tried if one is to guarantee being able to open the safe (assuming the "right combination" is not known)?

4. *Proposed by France.*

Let $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m$ be m vectors in the plane, each of length at most 1, with zero sum. Show that one can rearrange $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m$ as a sequence $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m$ such that each of the partial sums

$$\mathbf{v}_1,\ \mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\ldots,\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_m\ (= 0)$$

has length less than or equal to $\sqrt{5}$.

5. *Proposed by Korea.*

Let n be the product of two consecutive integers greater than 2. Show that there are *no* integers $x_1,x_2,\ldots,x_n$ satisfying the equation

$$\sum_{i=1}^{n} x_i^2 - \frac{4}{4n+1}\left[\sum_{i=1}^{n} x_i\right]^2 = 1.$$

6. *Proposed by Singapore.*

Let Q be the centre of the inscribed circle of a triangle ABC. Prove that for any point P,

$$a(PA)^2 + b(PB)^2 + c(PC)^2 = a(QA)^2 + b(QB)^2 + c(QC)^2 + (a+b+c)(QP)^2,$$

where $a = BC$, $b = CA$, and $c = AB$.

7. *Proposed by Sweden.*

Let $a_1, a_2, a_3, \ldots$ be a sequence of non–negative real numbers such that

$$a_k - 2a_{k+1} + a_{k+2} \geq 0$$

and

$$\sum_{j=1}^{k} a_j \leq 1$$

for all $k = 1,2,\ldots$. Prove that

$$0 \leq a_k - a_{k+1} < \frac{2}{k^2} \text{ for all } k = 1,2,\ldots .$$

8. *Proposed by the United Kingdom.*

A positive integer is called a *double number* if its decimal representation consists of a block of digits, not commencing with 0, followed by an identical block. So, for instance, 360360 is a double number, but 36036 is not. Show that there are infinitely many double numbers which are perfect squares.

9. *Proposed by the United Kingdom.*

The triangle ABC is acute–angled. L is any line in the plane of the triangle, and u, v, w are the lengths of the perpendiculars from A, B, C respectively to L. Prove that

$$u^2 \tan A + v^2 \tan B + w^2 \tan C \geq 2\Delta,$$

where Δ is the area of the triangle, and determine the lines L for which equality holds.

10. *Proposed by the United Kingdom.*

The sequence $\{a_n\}$ of integers is defined by

$$a_1 = 2, \quad a_2 = 7,$$

and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}, \quad n \geq 2.$$

Prove that a_n is odd for all $n > 1$.

11. *Proposed by the U.S.A.*

A number of signal lights are equally spaced along a one–way railroad track, labelled in order $1,2,\ldots,n$ $(n \geq 2)$. As a safety rule, a train is not allowed to pass a signal if any other train is in motion on the length of track between it and the following signal. However, there is no limit to the number of trains that can be parked motionless at a signal, one behind the other. (That is, assume the trains have zero length.)

A series of k freight trains must be driven from signal 1 to signal n. Each train travels at a distinct but constant speed at all times when it is not blocked by the safety rule. Show that, regardless of the order in which the trains are arranged, the same time will elapse between the first train's departure from signal 1 and the last train's arrival at signal n.

12. *Proposed by the U.S.S.R.*

A point M is chosen on the side AC of the triangle ABC in such a way that the radii of the circles inscribed in the triangles ABM and BMC are equal. Prove that

$$BM^2 = \Delta\cdot\cot(B/2),$$

where Δ is the area of the triangle ABC.

13. *Proposed by the U.S.S.R.*

Around a circular table an even number of persons are seated at a discussion. After a break they sit around the circular table in a different order. Prove that there are at least two people such that the number of participants sitting between them before and after the break is the same.

* * *

The twenty–five problems presented last month and this month, together with the six problems chosen for the 1988 I.M.O. constitute only a short list from the more than ninety problems proposed by the national teams. Many of these were not forwarded to the jury for consideration because they bore too close a resemblance to questions from other years or other national contests, or because of flaws that were perceived. To give some idea of the remaining problems I have selected a sample of five of them, but I will suppress their countries of origin.

1. Let k be a positive integer and M_k the set of all the integers that are between $2k^2 + k$ and $2k^2 + 3k$, both included. Is it possible to partition M_k into two subsets A and B such that

$$\sum_{x\in A} x^2 = \sum_{x\in B} x^2?$$

2. Find the total number of different integer values the function

$$f(x) = [x] + [2x] + [5x/3] + [3x] + [4x]$$

takes for $0 \le x \le 100$. (Of course $[x]$ is the integer part of x.)

3. The polynomial $x^{2k} + 1 + (x + 1)^{2k}$ is not divisible by $x^2 + x + 1$. Find the possible values of k.

4. Let S be the set of all sequences $\{a_i: 1 \leq i \leq 7,\ a_i = 0 \text{ or } 1\}$. The distance between two elements $\{a_i\}$ and $\{b_i\}$ of S is defined as

$$d(\{a_i\}, \{b_i\}) = \sum_{i=1}^{7} |a_i - b_i|.$$

Suppose T is a subset of S in which any two elements have a distance apart greater than or equal to three. Prove that T contains at most 16 elements. Give an example of such a subset with 16 elements.

5. Does there exist a number α $(0 < \alpha < 1)$ such that there is an infinite sequence $\{a_n\}$ of positive numbers satisfying

$$1 + a_{n+1} \leq a_n + \frac{\alpha}{n} a_n, \quad n = 1,2,\ldots?$$

* * *

Before turning to solutions received for problems posed in 1987, I give the following "induction-free" variant to a solution given in the May number.

2. [1986: 97; 1988: 133] *1985 Spanish Mathematical Olympiad — 1st Round.*
Let n be a natural number. Prove that the expression

$$(n + 1)(n + 2)\ldots (2n - 1)(2n)$$

is divisible by 2^n.

Solution by David Vaughan, Wilfrid Laurier University, Waterloo, Ontario.

$$\frac{(n+1)\ldots(2n-1)(2n)}{2^n} = \frac{n!(n+1)\ldots(2n-1)(2n)}{n!2^n} = \frac{(2n)!}{(2n)!!} = (2n-1)!!\,,$$

where $k!!$ is the "skip factorial", i.e. $k!! = k(k-2)\ldots j$, where $j = 2$ if k is even, and $j = 1$ if k is odd. Note that 2^n is the largest power of 2 that divides the expression as $(2n - 1)!!$ is the product of odd integers only.

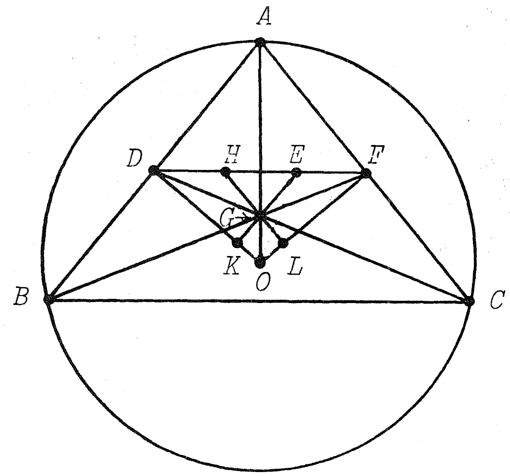
* * *

Now we turn to problems from the *Second Balkan Mathematical Olympiad 1985.*

1. [1987: 71] *Proposed by Bulgaria.*
Let O be the centre of the circle through the points A, B, C and let D be the midpoint of AB. Let E be the centroid of the triangle ACD. Prove that the line CD is perpendicular to the line OE if and only if $AB = AC$.

Solution by Dan Sokolowsky, Williamsburg, Virginia.

Let DE meet AC at F. Then F is the midpoint of AC, so DF is parallel to BC (1). Also, $EF = DF/3$. Let H be the midpoint of DE (2). Then $DH = HE = EF = DF/3$ (3). Now



let G be the centroid of ΔABC, so $FG = FB/3$, $DG = DC/3$, whence by (^{3}) EG is parallel to AB and GH is parallel to AC (^{4}). Together with (^{1}) this implies ΔEGH is similar to ΔBAC (^{5}). Also since $OD \perp AB$ and $OF \perp AC$, with (^{4}) we have $EG \perp OD$ and $HG \perp OF$ (^{6}).

Suppose first that $AB = AC$. Then by obvious symmetry $OA \perp BC$ so $OA \perp DF$, OA passes through G, and so by (^{6}) G is the orthocentre of ΔOED and thus $CD \perp OE$.

Conversely, suppose $CD \perp OE$. Let EG meet OD at K, and HG meet OF at L. From (^{6}), G is the orthocentre of ΔOED, and therefore $OG \perp DF$. Hence by (^{6}) $\angle FOG = 90^\circ - \angle OFH = \angle FHL$ (^{7}). Since $\angle GKO = \angle GLO = 90^\circ$, $GKOL$ is a cyclic quadrilateral. Then $\angle GKL = \angle FOG$. Now by (^{7}), $\angle GKL = \angle FHL$. Thus $EHKL$ is cyclic (^{8}). By (^{2}), in the right triangle EKD, $KH = DH = DF/3$. Similarly in the right triangle HLF, E is the midpoint of HF, so $EL = EF = DF/3 = KH$. Now by (^{8}), we have $\angle EHG = \angle HEG$. From (^{5}) we get $\angle B = \angle C$ and $AB = AC$.

Editor's note: J.T. Groenman, Arnhem, The Netherlands, also submitted a solution based on trigonometry.

<u>2</u>. [1987: 71] *Proposed by Romania.*

Let a, b, c, d be real numbers in the interval $[-\pi/2,\pi/2]$ such that

$$\sin a + \sin b + \sin c + \sin d = 1$$

and

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Prove that a, b, c, d actually must belong to the interval $[0,\pi/6]$.

Solution by M. Selby, Department of Mathematics, The University of Windsor, Ontario.

Let $x = \sin a$, $y = \sin b$, $z = \sin c$, and $w = \sin d$. The given conditions can now be rewritten

$$x + y + z + w = 1$$
$$x^2 + y^2 + z^2 + w^2 \leq \frac{1}{3}.$$

The second condition is obtained by using $\cos 2\alpha = 1 - 2\sin^2\alpha$. Using the Cauchy–Schwarz inequality,

$$|(1,1,1)\cdot(x,y,z)| \leq \sqrt{3}(x^2 + y^2 + z^2)^{1/2},$$

that is,

$$|x + y + z| \le \sqrt{3}(x^2 + y^2 + z^2)^{1/2}.$$

But $x + y + z = 1 - w$. Therefore $(1 - w)^2 \le 3(x^2 + y^2 + z^2)$. Using $x^2 + y^2 + z^2 \le 1/3 - w^2$ we obtain $(1 - w)^2 \le 3(1/3 - w^2)$ or $2w^2 - w \le 0$. Hence $0 \le w \le 1/2$.

Similarly $0 \le x \le 1/2$, $0 \le y \le 1/2$, and $0 \le z \le 1/2$. Since $x = \sin a$, $y = \sin b$, $z = \sin c$, and $w = \sin d$ with a, b, c, d being in $[-\pi/2,\pi/2]$ and their sines between 0 and 1/2 we must have that a, b, c, and d lie in $[0,\pi/6]$.

* * *

Next we turn to problems from the Bulgarian Winter Competition from the April 1987 number.

1. [1987: 112] *Bulgarian Winter Competition, 1986.*

The equation $x^2 + px + q = 0$ has real roots x_1 and x_2. Determine p and q, if $\frac{1}{1+x_1}$ and $\frac{1}{1+x_2}$ are also roots of the same equation. (Grade 8)

Solution by Bob Prielipp, University of Wisconsin–Oshkosh, U.S.A.

We shall show that $p = 1$ and $q = -1$. Since x_1 and x_2 are roots of $x^2 + px + q = 0$ and $\frac{1}{1+x_1}$ and $\frac{1}{1+x_2}$ are also roots of the same equation,

$$x_1 = \frac{1}{1 + x_1} \text{ and } x_2 = \frac{1}{1 + x_2} \tag{1}$$

or

$$x_1 = \frac{1}{1 + x_2} \text{ and } x_2 = \frac{1}{1 + x_1}. \tag{2}$$

In case (1) the desired equation is $x = \frac{1}{1 + x}$ or $x^2 + x - 1 = 0$, making $p = 1$ and $q = -1$. In case (2), the desired equation is

$$x = \cfrac{1}{1 + \cfrac{1}{1 + x}}$$

again giving $x^2 + x - 1 = 0$ and making $p = 1$ and $q = -1$.

2. [1987: 112] *Bulgarian Winter Competition, 1986.*

Solve the equation $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\}$ where $[x]$ denotes the greatest integer which does not exceed x, and $\{x\} = x - [x]$. (Grade 9)

Solution by Bob Prielipp, The University of Wisconsin–Oshkosh, U.S.A.

We shall denote the given equation by (#). If $x < 0$, then (#) has no solution because its left side is negative, while the right side is non–negative. For $0 \le x < 1$ the left side of (#) is undefined. If $1 \le x < 2$ the left side is greater than 1 while the right side is less than 1. In the remainder we shall assume that $x \ge 2$. Also, no integer is a solution of (#) since the equation $\frac{1}{x} + \frac{1}{2x} = 0$ has no solution.

Since the left side of (#) is a rational number, a solution of (#) must be of the form $x = n + a/b$ where n is an integer, $n \geq 2$, and $0 < a/b < 1$, where a and b are integers with $b > 0$.

Case 1. $0 < a/b < 1/2$.

Then $n + a/b$ is a solution of (#) if and only if

$$\frac{3}{2n} = \frac{1}{n} + \frac{1}{2n} = \frac{a}{b}.$$

Now $\frac{3}{2n} < \frac{1}{2}$ iff $3 < n$. This case gives solutions of the form $n + \frac{3}{2n}$ for $n > 3$.

Case 2. $1/2 \leq a/b < 1$.

Then $n + a/b$ is a solution of (#) if and only if

$$\frac{3n+1}{2n^2+n} = \frac{1}{n} + \frac{1}{2n+1} = \frac{a}{b}.$$

Now

$$\frac{1}{2} \leq \frac{3n+1}{2n^2+n} < 1$$

is equivalent to the system

$$2n^2 - 5n - 2 \leq 0, \quad 0 < 2n^2 - 2n - 1.$$

The first inequality entails $n \leq 2$, since the roots of $2x^2 - 5x - 2 = 0$ are $x = (5 \pm \sqrt{41})/4$, and

$$2 < \frac{5+\sqrt{41}}{4} < 3.$$

This case produces the solution $2 + 7/10$. The solutions are then

$$\{n + \tfrac{3}{2n} \colon n \text{ is an integer with } n > 3\} \cup \{27/10\}.$$

<u>4</u>. [1987: 112] *Bulgarian Winter Competition, 1986.*

Given that $a \geq 1$ and b are real numbers, prove that the system

$$y = x^3 + ax + b$$
$$z = y^3 + ay + b$$
$$x = z^3 + az + b$$

has exactly one real solution. (Grade 11)

Solution by George Evagelopoulos, Law Student, Athens, Greece.

Since the system is symmetrical, we will first examine if it has a solution with $x = y = z$. If $x = y = z$, then the system reduces to $x = x^3 + ax + b$ or equivalently

$$x^3 + (a-1)x + b = 0. \tag{1}$$

Since this third degree polynomial has real coefficients, it will also have at least one real root, x_0, say.

We next examine whether equation (1) may have other real solutions. Suppose that x_1 is another solution and $x_1 \neq x_0$. Then

$$x_0^3 + (a-1)x_0 + b = 0,$$
$$x_1^3 + (a-1)x_1 + b = 0.$$

This gives

$$(x_0^3 - x_1^3) + (a-1)(x_0 - x_1) = 0$$

and

$$(x_0 - x_1)(x_0^2 + x_0x_1 + x_1^2 + a - 1) = 0.$$

Then since $x_0 \neq x_1$,

$$x_0^2 + x_0x_1 + x_1^2 + a - 1 = 0,$$

or

$$(x_0 + x_1/2)^2 + 3x_1^2/4 + a - 1 = 0,$$

which is absurd, because the inequalities $a - 1 \geq 0$ and $(x_0 + x_1/2)^2 + 3x_1^2/4 > 0$ give the strict inequality

$$(x_0 + x_1/2)^2 + 3x_1^2/4 + a - 1 > 0.$$

Consequently, the equation (1) has a unique real solution, x_0, and the original system has at least the solution $x = y = z = x_0$.

We now argue that the system has no other solution but (x_0,x_0,x_0).

Let us suppose there is also another solution, (x,y,z), with $x \leq y$ (with no loss of generality, because of the symmetry of the system). Then

$$y = x^3 + ax + b \leq y^3 + ay + b = z$$

so $y \leq z$. Now

$$z = y^3 + ay + b \leq z^3 + az + b = x$$

and we obtain $z \leq x$. Thus $x \leq y \leq z \leq x$ giving $x = y = z$. Consequently, the given system has only one real solution (x,y,z), namely (x_0,x_0,x_0) where x_0 is the real root of $x^3 + (a-1)x + b = 0$.

Editor's note: Solutions were also submitted by Nicos Diamantis, Department of Mathematics, University of Patrus, Greece, and by M. Selby of the University of Windsor, Ontario, Canada. These two solutions were also very nice, but used tools from the Calculus.

* * *

Oops. I misfiled a solution and failed to credit D.J. Smeenk of Zaltbommel, The Netherlands with a solution to 5 [1987: 39], *which I discussed in last month's column* [1988: 228]. *In the same file we also had Dr. Smeenk's solution to* 5 [1987: 38] *which was discussed in* [1988: 173]. *My apologies.*

* * *

Next we turn to problems from the May 1987 number, where the above were erroneously filed.

2. [1987: 138] *Bulgarian Spring Competition–Kasanlik, 1985.*

Let x_1, x_2 and x_3, x_4 be the roots of the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$, respectively, where p, p' and q, q' are non–zero real numbers. Prove that if $x_1x_4 = x_2x_3$ then $\left[\frac{p}{p'}\right]^2 = \frac{q}{q'}$. (Grade 8)

Solution by George Evagelopoulos, Law Student, Athens, Greece.

We have that $x_1x_4 = x_2x_3$ implies

$$\frac{x_1 + x_2}{x_2} = \frac{x_3 + x_4}{x_4}.$$

(Note that x_1, x_2, x_3, $x_4 \neq 0$ as q, $q' \neq 0$.) Hence $-p/x_2 = -p'/x_4$ and so

$$\frac{p}{p'} = \frac{x_2}{x_4} = \frac{x_1}{x_3}.$$

Now

$$\frac{q}{q'} = \frac{x_1x_2}{x_3x_4} = \left[\frac{p}{p'}\right]^2.$$

3. [1987: 138] *Bulgarian Spring Competition–Kazanlik, 1985.*

Six different natural numbers less than 108 are given. Prove that it is possible to choose three of them, a, b and c, so that $a < bc$, $b < ca$ and $c < ab$. (Grade 8)

Solution by George Evagelopoulos, Law Student, Athens, Greece.

We consider six distinct natural numbers $a_1, a_2, \ldots, a_6$ less than 108, such that

$$1 \leq a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < 108.$$

Then it is obvious that $a_2 \geq 2$ and $a_3 \geq 3$.

For any three numbers p, q, r such that $1 \leq p < q < r$ we obviously have $p < qr$ and $q < pr$. If there were not p, q, r among $a_1, a_2, \ldots, a_6$ satisfying $p < q < r$ and $r < pq$ we would have

$$a_4 \geq a_3 \cdot a_2 \geq 6,$$
$$a_5 \geq a_4 \cdot a_3 \geq 6 \cdot 3 = 18,$$

and so

$$a_6 \geq a_5 \cdot a_4 \geq 18 \cdot 6 = 108,$$

contrary to the hypothesis that $a_6 < 108$.

Editor's comment. John Morvay, Dallas, Texas, also solved this problem, pointing out that the interpretation of "natural" number is crucial. If $a_1 = 0$ is allowed, (0,1,2,3,6,18) gives a counterexample.

4. [1987: 138] *Bulgarian Spring Competition–Kazanlik, 1985.*

Find all positive values of the parameter a for which the common solutions of the inequalities $x^2 - 2x \leq a^2 - 1$ and $x^2 - 4x \leq -a - 2$ form an interval of length 1 on the real axis. (Grade 9)

Solution by George Evagelopoulos, Law Student, Athens, Greece.

We are given that $x^2 - 2x \le a^2 - 1$, or equivalently $(x-1)^2 \le a^2$, giving (since $a > 0$)

$$1 - a \le x \le 1 + a.$$

Also $x^2 - 4x \le -a - 2$ becomes $(x-2)^2 \le 2 - a$, and thus

$$2 - \sqrt{2 - a} \le x \le 2 + \sqrt{2 - a}.$$

From this we conclude that $0 < a < 2$. Now, consider the following four cases.

Case 1.

$$2 - \sqrt{2 - a} \le 1 - a < 1 + a \le 2 + \sqrt{2 - a},$$
$$1 + a - (1 - a) = 1.$$

Then $a = 1/2$ and

$$2 - \sqrt{2 - a} = 2 - \sqrt{3/2} > 1/2 = 1 - a,$$

contradicting the hypothesis.

Case 2.

$$1 - a \le 2 - \sqrt{2 - a} < 2 + \sqrt{2 - a} \le 1 + a,$$
$$2 + \sqrt{2 - a} - (2 - \sqrt{2 - a}) = 1.$$

Then $2\sqrt{2 - a} = 1$ and we get the solution $a = 7/4$.

Case 3.

$$2 - \sqrt{2 - a} < 1 - a < 2 + \sqrt{2 - a} < 1 + a,$$
$$2 + \sqrt{2 - a} - (1 - a) = 1.$$

Then $\sqrt{2 - a} = -a$, which is impossible since $a > 0$.

Case 4.

$$1 - a < 2 - \sqrt{2 - a} < 1 + a < 2 + \sqrt{2 - a},$$
$$1 + a - (2 - \sqrt{2 - a}) = 1.$$

This gives $\sqrt{2 - a} = 2 - a$, and since $0 < a < 2$, we have that $a = 1$.

Therefore the only two positive values for a are $a = 7/4$ and $a = 1$.

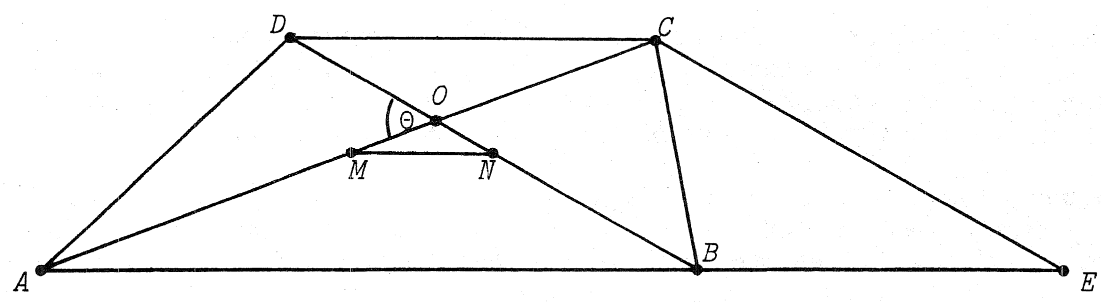
5. [1987: 138] *Bulgarian Spring Competition–Kazanlik, 1985.*

The diagonals AC and BD of a trapezoid $ABCD$ intersect at the point O. Let the bases of this trapezoid be $AB = a$ and $CD = b$ and let $\angle AOD = \theta$. Prove that the circles with diameters AC and BD are tangent exactly when the area of $ABCD$ equals $ab\tan(\theta/2)$. (Grade 10)

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Set $AC = p$ and $BD = q$, and let M and N be the midpoints of AC and BD respectively. Then $|a - b|/2 = MN$ is the distance between the centres of the circles with diameters AC and BD which have lengths p and q respectively. Thus the circles are (internally) tangent just when

$$\frac{1}{2}|a - b| = \frac{1}{2}|p - q|. \tag{1}$$



Draw CE parallel to DB with E on the line through AB. Then $\angle ACE = \pi - \theta$, and $CE = q$ and $BE = b$ since $DBEC$ is a parallelogram. The law of cosines for ΔAEC gives

$$\begin{aligned}(a+b)^2 &= p^2 + q^2 + 2pq\cos\theta \\ &= p^2 + q^2 + 2pq[2\cos^2(\theta/2) - 1] \\ &= p^2 + q^2 - 2pq + 4pq\cos^2(\theta/2). \qquad (2)\end{aligned}$$

Now the area of $ABCD$ equals the area of ΔAEC which is

$$\frac{1}{2}pq\sin(\pi - \theta) = \frac{1}{2}pq\sin\theta.$$

Thus the area of $ABCD$ equals $ab\tan(\theta/2)$ just when

$$\frac{1}{2}pq\sin\theta = ab\tan(\theta/2),$$

equivalently

$$ab = pq\cos^2(\theta/2).$$

From (2) this is equivalent to

$$(a+b)^2 = (p-q)^2 + 4ab,$$

then

$$(a-b)^2 = (p-q)^2,$$

and finally

$$|a-b| = |p-q|$$

which by (1) is equivalent to the tangency of the circles.

Editor's note. Solutions were also submitted by George Evagelopoulos, Athens, Greece, and J.T. Groenman, Arnhem, The Netherlands.

* * *

Please keep sending your solutions as well as copies of National Olympiads!

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **June 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

1381. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

ABC is a triangle with circumcircle Ω. AA_1, BB_1, CC_1 are three parallel chords of Ω, and A_2, B_2, C_2 are the feet of the perpendiculars from A_1 to BC, B_1 to AC, C_1 to AB, respectively. Prove that

(a) A_1A_2, B_1B_2, C_1C_2 intersect in a point on Ω;

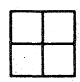
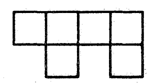
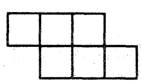
(b) A_2, B_2, C_2 all lie on a line parallel to AA_1.

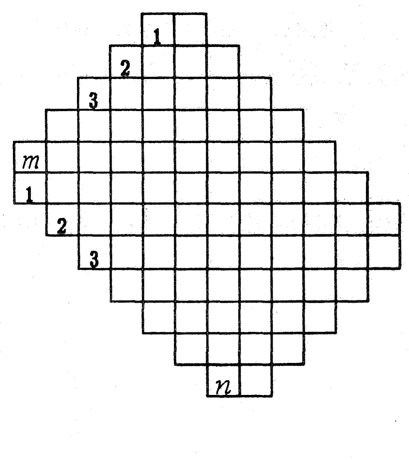
(This problem is not new. References will be given when a solution is published.)

1382. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A be a nonconstant arithmetic sequence of n terms. Characterize, in terms of n only, those positive integers k such that A can be partitioned into k subsets of equal size having equal sums of their elements.

1383. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

A polyomino (rook-wise connected set of squares from a chessboard) is called *square-free* if it does not contain a 2×2 square . For example,  is square-free while  isn't. Consider the jagged $m \times n$ "rectangle" R illustrated at the right, where m and n are each at least 2. Obviously R can be tiled by $m + n$ pairwise nonoverlapping square-free polyominoes; just use the rows (or the columns). Prove that R cannot be tiled by less than $m + n$ pairwise nonoverlapping square-free polyominoes.



1384. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

If the center of curvature of every point on an ellipse lies inside the ellipse, prove that the eccentricity of the ellipse is at most $1/\sqrt{2}$.

1385. *Proposed by Murray S. Klamkin, University of Alberta.*

Show that the sides of the pedal triangle of any interior point P of an equilateral triangle T are proportional to the distances from P to the corresponding vertices of T.

1386. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2...A_n$ be a polygon inscribed in a circle and containing the centre of the circle. Prove that

$$n - 2 + \frac{4}{\pi} < \sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} \le \frac{n^2}{\pi} \sin\frac{\pi}{n},$$

where a_i is the side A_iA_{i+1} and $\hat{a}_i$ is the arc A_iA_{i+1}.

1387. *Proposed by Ravi Vakil, student, University of Toronto.*

Show that for all positive integers d, a, n such that $3 \le d \le 2^{n+1}$, d does not divide into $a^{2^n} + 1$.

1388. *Proposed by Jordi Dou, Barcelona, Spain.*

Given four lines a, b, c, d in general position in the plane, show that there is a unique line x cutting a, b, c, d in the respective points A, B, C, D, and in that order, such that $AB = BC = CD$.

1389. *Proposed by Derek Chang, California State University, Los Angeles, and Raymond Killgrove, Indiana State University, Terre Haute.*

Find

$$\max_{\pi \in S_n} \sum_{i=1}^{n} |i - \pi(i)|,$$

where S_n is the set of all permutations of $\{1,2,...,n\}$.

1390. *Proposed by H. Fukagawa, Aichi, Japan.*

A, B, C are points on a circle Γ such that CM is the perpendicular bisector of AB. P is a point on CM and AP meets Γ again at D. As P varies over segment CM, find the largest radius of the inscribed circle tangent to segments PD, PB, and arc DB of Γ, in terms of the length of CM.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1230. [1987: 87; 1988: 151] *Proposed by Jordi Dou, Barcelona, Spain.*

Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let $P = AC \cap BD$ and let s be a line through P cutting AD at E and BC at F. Prove that there exists a conic tangent to AD at E, to BC at F, and twice tangent to Ω.

II. *Solution by G.R. Veldkamp, De Bilt, The Netherlands.*

A short analytic solution using general homogeneous projective point coordinates x_1, x_2, x_3 with respect to ΔPAD could run as follows. The equations of AC, BD and AD are $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ respectively. The equation of BC is

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

where $a_1a_2a_3 \neq 0$. Hence we get

$$\lambda x_1x_2 + x_3(a_1x_1 + a_2x_2 + a_3x_3) = 0$$

$(\lambda \neq 0)$ as the equation of Ω. Without loss of generality the equation of EF may be taken as $x_1 - x_2 = 0$. A conic C touching AD at E and BC at F can be represented by

$$x_3(a_1x_1 + a_2x_2 + a_3x_3) + \mu(x_1 - x_2)^2 = 0.$$

The intersections of Ω and C are on the conic

$$\lambda x_1x_2 = \mu(x_1 - x_2)^2$$

or

$$\mu x_1^2 - (2\mu + \lambda)x_1x_2 + \mu x_2^2 = 0,$$

and this is a double line if and only if

$$(2\mu + \lambda)^2 - 4\mu^2 = 0,$$

i.e. $\mu = -\lambda/4$. Therefore the conic

$$4x_3(a_1x_1 + a_2x_2 + a_3x_3) - \lambda(x_1 - x_2)^2 = 0$$

answers the question.

[*Editor's note*: Veldkamp also pointed out a slight discrepancy in the published solution [1988: 151] in that as written it only applies to conics Ω which are ellipses. This can be repaired by replacing the two occurrences of the word "affine" by the word "projective"; the rest of the proof then proceeds as before.]

*   *   *

1269*. [1987: 217] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a non–obtuse triangle with circumcenter M and circumradius R. Let u_1, u_2, u_3 be the lengths of the parts of the cevians (through M) between M and the sides opposite to A, B, C respectively. Prove or disprove that

$$\frac{R}{2} \leq \frac{u_1 + u_2 + u_3}{3} < R.$$

I. *Solution by G. Tsintsifas, Thessaloniki, Greece.*

The lower bound has already been proved in *Crux* 718 [1983: 82]. Also, the upper bound can be lowered. The inequality to be proved becomes

$$\frac{3R}{2} \leq u_1 + u_2 + u_3 \leq 2R.$$

We know that

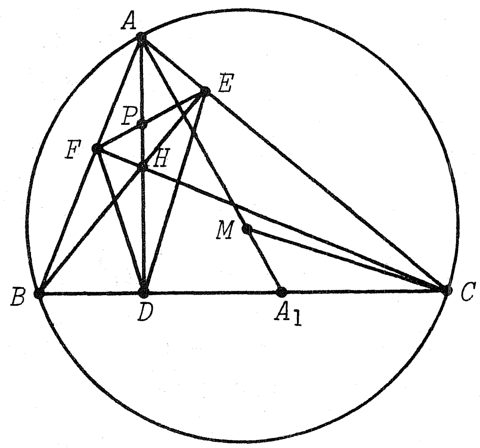
(a) the altitudes of ΔABC are the angle bisectors of the orthic triangle DEF,

and

(b) the radii MA, MB, MC are perpendicular to the sides of ΔDEF.

Thus, letting $P = AD \cap EF$, we get by (b) that

$$\Delta DEP \sim \Delta CMA_1.$$



Therefore by (a)

$$\frac{MA_1}{MC} = \frac{EP}{ED} = \frac{FP}{FD}$$

and thus

$$\frac{MA_1}{MC} = \frac{EF}{DE + DF},$$

or

$$u_1 = MA_1 = \left[\frac{d}{e + f}\right]R$$

where d, e, f are the sides of ΔDEF. Similarly,

$$u_2 = \left[\frac{e}{d + f}\right]R, \quad u_3 = \left[\frac{f}{d + e}\right]R,$$

so that

$$u_1 + u_2 + u_3 = R\left[\frac{d}{e + f} + \frac{e}{d + f} + \frac{f}{d + e}\right].$$

But it is known (1.16 of Bottema et al, *Geometric Inequalities*) that

$$\frac{3}{2} \leq \frac{d}{e + f} + \frac{e}{d + f} + \frac{f}{d + e} \leq 2,$$

so our inequality has been proved.

II. *Solution by Murray S. Klamkin, University of Alberta.*

We generalize the problem and also give a sharp upper bound.

Using the notation of [1987: 274–275], let $A_0, A_1, \ldots, A_n$ be the vertices of an n–dimensional simplex and P an arbitrary point not in the exterior. Also, let the line A_jP intersect the $(n - 1)$–dimensional face opposite A_j in the point K_j. Using barycentric coordinates, P will have a vector representation

$$\mathrm{P} = \sum_{j=0}^{n} x_j \mathrm{A}_j\,,\quad \sum_{j=0}^{n} x_j = 1\,,\quad x_j \geq 0 \text{ for all } j.$$

Geometrically, x_j corresponds to the ratio of the volume of the simplex determined by P and the face opposite A_j to the volume of the given simplex. We then have

$$\frac{PK_j}{A_jP} = \frac{x_j}{1 - x_j}.$$

Now let $F(t)$ be a non–decreasing convex function. Since $t/(1 - t)$ is also a convex function in $[0,1)$, so is $\phi(t) = F(t/(1 - t))$. Then by Jensen's inequality,

$$\begin{aligned} \sum_{j=0}^{n} F\left[\frac{PK_j}{A_jP}\right] &= \sum_{j=0}^{n} \phi(x_j) \geq (n+1)\phi\left[\sum_{j=0}^{n} x_j/(n+1)\right] \\ &= (n+1)F\left[\frac{\sum_{j=0}^{n} x_j/(n+1)}{1 - \sum_{j=0}^{n} x_j/(n+1)}\right] \\ &= (n+1)F\left[\frac{1}{n}\right]. \end{aligned} \tag{1}$$

There is equality if and only if P is the centroid so that all the x_j's are equal.

There is no upper bound for

$$\sum_{j=0}^{n} F\left[\frac{PK_j}{A_jP}\right]$$

in general since one of the x_j's could equal 1. However there is an upper bound if we restrict P to be the circumcenter M. To show this we first prove that the altitude from M to the face opposite any A_j is at most one–half the altitude from A_j to this face. This follows since M lies in each hyperplane which bisects and is orthogonal to an edge of the simplex, and thus lies on the opposite side from A_j of the hyperplane passing through the midpoints of the edges emanating from A_j. Thus max $x_j \leq 1/2$. We now use the majorization inequality. Since

$$(1/2,1/2,0,0,\ldots,0) \succ (x_0,x_1,\ldots,x_n),$$

we have

$$\phi(1/2) + \phi(1/2) + \phi(0) + \cdots + \phi(0) \geq \sum_{j=0}^{n} \phi(x_j)$$

or from (1), letting R be the circumradius,

$$2F(1) \geq \sum_{j=0}^{n} F\left[\frac{MK_j}{R}\right] \geq (n+1)F(1/n), \tag{2}$$

with equality in the first inequality if and only if M coincides with the midpoint of an edge.

If we let $F(t) = t$ in (2) and put $u_j = MK_j$, we obtain

$$2R \geq \sum_{j=0}^{n} u_j \geq \frac{(n+1)R}{n}. \tag{3}$$

For $n = 2$ we obtain

$$2R \geq u_1 + u_2 + u_3 \geq \frac{3R}{2}, \tag{4}$$

which gives the required result. There is equality on the left if and only if the triangle is a right triangle and equality on the right if and only if the triangle is equilateral.

Also solved by PAUL ERDOS and GEORGE SZEKERES, University of New South Wales, Kensington, Australia; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; and TOSIO SEIMIYA, Kawasaki, Japan.

Most solvers discovered the sharp upper bound given in the above solutions. Erdos and Szekeres in fact also established Klamkin's inequality (3). *Garfunkel also pointed out that the lower bound is just* Crux 718.

* * *

<u>1271</u>. [1987: 256] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated in memoriam to Léo Sauvé.)

Prove that

$$\sqrt{3}\sum \sin\frac{A_1}{2} \geq 4\sum \sin B_1 \sin\frac{A_2}{2}\sin\frac{A_3}{2},$$

where $A_1A_2A_3$ and $B_1B_2B_3$ are two triangles and the sums are cyclic over their angles.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

I guess the absolute shortcut is as follows. Take inequality (5) of [1], i.e.,

$$\sum w_2w_3a_1 \leq R\sum w_1\sqrt{\sum w_2w_3},$$

where a_1, a_2, a_3, R are the sides and circumradius of a triangle and w_1, w_2, w_3 are nonnegative numbers. Put

$$w_1 = \sin\frac{A_1}{2}, \text{ etc.}$$

and

$$a_1 = 2R\sin B_1, \text{ etc.}$$

Then

$$2\sum \sin B_1 \sin\frac{A_2}{2}\sin\frac{A_3}{2} \le \sum \sin\frac{A_1}{2}\sqrt{\sum \sin\frac{A_2}{2}\sin\frac{A_3}{2}}.$$

Using

$$\sum \sin\frac{A_2}{2}\sin\frac{A_3}{2} \le \frac{3}{4}$$

(item 2.15 of [1]) we get immediately the stated inequality.

We now show the following more general result. *Let t, x_1, x_2, x_3 be real numbers with $0 < t \le 2$, and let B_1, B_2, B_3 be angles of a triangle. Then*

$$\sum x_2x_3 \sin^t B_1 \le \frac{1}{3}\left[\frac{\sqrt{3}}{2}\right]^t\left[\sum x_1\right]^2. \qquad (1)$$

Indeed, if we put

$$\sin B_1 = \frac{b_1}{2R}, \text{ etc.,}$$

where b_1, b_2, b_3, R are the sides and circumradius, then (1) becomes

$$\sum b_1^t x_2x_3 \le \frac{(R\sqrt{3})^t}{3}\left[\sum x_1\right]^2. \qquad (2)$$

Now [1], item 14.1, tells us

$$\sum b_1^2 x_2x_3 \le R^2\left[\sum x_1\right]^2. \qquad (3)$$

(Of course, (3) also follows from Murray's polar moment of inertia inequality, given in [2], by putting $R_1 = R_2 = R_3 = R$!) Further, in [3] it is shown that for $p > 1$, $b_1^{1/p}$, $b_2^{1/p}$, $b_3^{1/p}$ also form a triangle whose circumradius R_p satisfies

$$(R_p\sqrt{3})^p \le R\sqrt{3}.$$

Hence we get from (3)

$$\sum b_1^{2/p}x_2x_3 \le R_p^2\left[\sum x_1\right]^2 \le \frac{(R\sqrt{3})^{2/p}}{3}\left[\sum x_1\right]^2.$$

Putting $t = 2/p$ we have (2).

Applications of (1):

(i) Putting

$$x_1 = \sin\frac{A_1}{2}, \text{ etc.,}$$

and using

$$\sum \sin\frac{A_1}{2} \le \frac{3}{2}$$

([1], item 2.9), we get

$$\sum \sin^t B_1 \sin\frac{A_2}{2}\sin\frac{A_3}{2} \le \frac{1}{2}\left[\frac{\sqrt{3}}{2}\right]^t \sum \sin\frac{A_1}{2}.$$

For $t = 1$ we have the given inequality.

(ii) Put

$$x_1 = \sin^k A_1, \text{ etc.}$$

where

$$k \le \frac{\log 9 - \log 4}{\log 4 - \log 3};$$

then from [1], item 5.28, follows immediately

$$\sum \sin^k A_1 \le 3\left[\frac{\sqrt{3}}{2}\right]^k,$$

and thus from (1) we get

$$\sum \sin^t B_1 \sin^k A_2 \sin^k A_3 \le \left[\frac{\sqrt{3}}{2}\right]^{t+k} \sum \sin^k A_1.$$

(iii) Putting

$$x_1 = \frac{\sqrt{\tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_1}{2}}, \text{ etc.}$$

and using

$$\sum \frac{\sqrt{\tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_1}{2}} \le 2$$

([1], item 2.64) yields

$$\sum \frac{\tan \frac{A_1}{2} \sqrt{\tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_2}{2} \cos \frac{A_3}{2}} \sin^t B_1 \le \frac{2}{3}\left[\frac{\sqrt{3}}{2}\right]^t \sum \frac{\sqrt{\tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_1}{2}},$$

then

$$\frac{\sqrt{\tan \frac{A_1}{2} \tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_1}{2} \cos \frac{A_2}{2} \cos \frac{A_3}{2}} \sum \sqrt{\sin A_1} \sin^t B_1 \le \frac{2\sqrt{2}}{3}\left[\frac{\sqrt{3}}{2}\right]^t \sum \frac{\sqrt{\tan \frac{A_2}{2} \tan \frac{A_3}{2}}}{\cos \frac{A_1}{2}},$$

and finally the curious result

$$\sum \sin^t B_1 \sqrt{\sin A_1} \le \frac{2\sqrt{2}}{3}\left[\frac{\sqrt{3}}{2}\right]^t \sum \frac{\cos \frac{A_2}{2} \cos \frac{A_3}{2}}{\sqrt{\tan \frac{A_1}{2}}}.$$

(iv) Putting

$$x_1 = \cos A_2 \cos A_3, \text{ etc.}$$

and using

$$\sum \cos A_2 \cos A_3 \le \frac{3}{4}$$

([1], item 2.22) leads to

$$\sum \sin^t B_1 \cos A_1 \le \frac{1}{4}\left[\frac{\sqrt{3}}{2}\right]^t \sum \sec A_1$$

if $\Delta A_1A_2A_3$ is acute, and the opposite inequality if $\Delta A_1A_2A_3$ is obtuse.

References:

[1] O. Bottema et al, *Geometric Inequalities*, Groningen, 1968.

[2] M.S. Klamkin, On a triangle inequality, *Crux* 10 (1984) 139–140.

[3] A. Oppenheim, Inequalities involving elements of triangles, quadrilaterals or tetrahedra, *Univ. Beograd Publ. El. Fak. Ser. Mat. Fiz.* No.461–497 (1974) 257–263.

Also solved by the proposer, who gave as special cases

$$\sum \sin B_1 \le \frac{3\sqrt{3}}{2}$$

([1], item 2.1), *obtained by letting* $\Delta A_1A_2A_3$ *be equilateral; and*

$$\sum \sin \frac{A_1}{2} \ge 2 \sum \sin \frac{A_2}{2} \sin \frac{A_3}{2},$$

obtained by letting $\Delta B_1B_2B_3$ *be equilateral. Noting that*

$$\frac{3}{4} \le \sum \sin \frac{A_1}{2} - \sum \sin \frac{A_2}{2} \sin \frac{A_3}{2} < 1$$

was the subject of Crux 1154 [1987: 267], *he asks for upper and lower bounds for*

$$\sum \sin \frac{A_1}{2} - k \sum \sin \frac{A_2}{2} \sin \frac{A_3}{2}$$

where $k > 0$ *is constant.*

* * *

<u>1273</u>. [1987: 256] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle, M an interior point, and $A'B'C'$ its pedal triangle. Denote the sides of the two triangles by a, b, c and a', b', c' respectively. Prove that

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} < 2.$$

I. *Solution by the proposer.*

We have

$$\frac{a'}{a} + \frac{b'}{b} + \frac{c'}{c} = \frac{AM\cdot\sin A}{2R\sin A} + \frac{BM\cdot\sin B}{2R\sin B} + \frac{CM\cdot\sin C}{2R\sin C} = \frac{AM + BM + CM}{2R},$$

so we need to prove

$$AM + BM + CM < 4R. \qquad (1)$$

Suppose $a \le b \le c$. Draw line DME parallel to BC with D on AB, E on AC. Then since $c \ge b$,

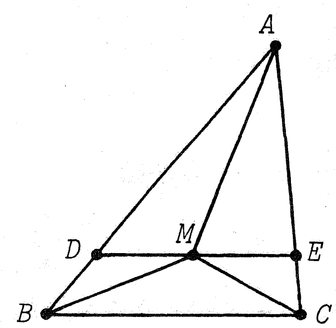
$$AM \le \max\{AD, AE\} = AD. \qquad (2)$$

Also

$$BM + CM \le BD + DE + EC, \qquad (3)$$

and, since $BC \le AC$,

$$DE \le AE. \qquad (4)$$



Thus by (2), (3), (4),

$$AM + BM + CM \le AD + BD + AE + EC = AB + AC = 2R(\sin B + \sin C) < 4R.$$

[*Editor's comment*: All solvers of this problem began by reducing it to inequality (1). We pick up the next two solutions from this point.]

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let O be the circumcenter of ΔABC. Then M has to be contained in (the interior or the boundary of) one of the triangles OAB, OBC, OCA, let's say in OAB. Then

$$AM + BM \leq AO + BO = 2R.$$

Furthermore, $CM \leq 2R$ since C and M are both in or on the circumcircle. Done.

Taking right triangles ABC ($\angle B = 90^\circ$) such that $C \approx B$ and choosing $M \approx A$, we have $AM \approx 0$, $BM \approx CM \approx 2R$. Thus the constant 4 in (1) (and so the constant 2 in the problem) is sharp.

Via the process of reciprocation (see Klamkin's abstract [1] extending earlier work of Oppenheim [2]), we can apply the transformations

$$R_1 \to r_2r_3\,, \quad R \to \frac{R_1R_2R_3}{4R}\,, \quad \text{etc.},$$

where $R_1 = AM$ and r_1 is the distance from M to BC, etc., to (1) and get the new inequality

$$r_2r_3 + r_3r_1 + r_1r_2 < \frac{R_1R_2R_3}{R}\,.$$

References:

[1] M.S. Klamkin, Triangle inequalities via transforms, *Notices A.M.S.*, January 1972, A–103, 104.

[2] A. Oppenheim, The Erdös inequality and other inequalities for a triangle, *Amer. Math. Monthly* 68 (1961) 226–230.

III. *Solution by P. Penning, Delft, The Netherlands.*

The sum

$$S = AM + BM + CM$$

is minimum in the point of Torricelli (where AM, BM, CM make equal angles of 120° with each other, or in oblique triangles the vertex with the obtuse angle). The sum S rises monotonically with distance from the point of Torricelli. Hence S is maximum in the vertex with the smallest angle and then equal to the sum of the two largest sides. Since the triangle fits in the circle with radius R, any side is equal to or smaller than $2R$. Hence the sum S is equal to or smaller than $4R$. The equality is obtained for the triangle degenerated into the diameter of the circumcircle.

IV. *Solution by Murray S. Klamkin, University of Alberta.*

We will show more generally that

$$F\left[\frac{a'}{a}\right] + F\left[\frac{b'}{b}\right] + F\left[\frac{c'}{c}\right] \leq 2F(1) + F(0) \leq F(2) + 2F(0),$$

where $F(x)$ is any increasing convex function of x. The proposed inequality corresponds to the special case $F(x) = x$.

Since $a = 2R\sin A$, and $a' = R_1 \sin A$ where $R_1 = AM$, etc., we have

$$F\left[\frac{a'}{a}\right] + F\left[\frac{b'}{b}\right] + F\left[\frac{c'}{c}\right] = F\left[\frac{R_1}{2R}\right] + F\left[\frac{R_2}{2R}\right] + F\left[\frac{R_3}{2R}\right].$$

It now follows by the majorization inequality that

$$F\left[\frac{R_1}{2R}\right] + F\left[\frac{R_2}{2R}\right] + F\left[\frac{R_3}{2R}\right] \le F\left[\frac{R_1 + R_2 + R_3}{2R}\right] + 2F(0).$$

It is known that $R_1 + R_2 + R_3$ is a convex function of the point M (see [1], p.1064). Consequently $R_1 + R_2 + R_3$ takes on its maximum value when M is a vertex of ΔABC. If $a \le b \le c$, then

$$\begin{aligned} F\left[\frac{R_1 + R_2 + R_3}{2R}\right] + 2F(0) &\le F\left[\frac{b + c}{2R}\right] + 2F(0) \\ &= F(\sin B + \sin C) + 2F(0) \\ &\le F(2) + 2F(0). \end{aligned}$$

Hence

$$F\left[\frac{a'}{a}\right] + F\left[\frac{b'}{b}\right] + F\left[\frac{c'}{c}\right] \le F(2) + 2F(0).$$

A stronger inequality follows by noting that

$$F\left[\frac{R_1}{2R}\right] + F\left[\frac{R_2}{2R}\right] + F\left[\frac{R_3}{2R}\right]$$

is also a convex function of the point M (see [2], p. 52). Hence this function takes on its maximum value at vertex A, giving

$$F\left[\frac{a'}{a}\right] + F\left[\frac{b'}{b}\right] + F\left[\frac{c'}{c}\right] \le 2F(1) + F(0).$$

There is equality if and only if the triangle is degenerate with two right angles and the point M coincides with the vertex of the zero angle.

References:

[1] M.S. Klamkin, Vector proofs in solid geometry, *Amer. Math. Monthly* 77 (1970) 1051–1065.

[2] H.G. Eggleston, *Convexity*, Cambridge University Press, Cambridge, 1958.

Also solved by JORG HARTERICH, student, Winnenden, Federal Republic of Germany. Two other readers gave solutions valid for acute triangles only.

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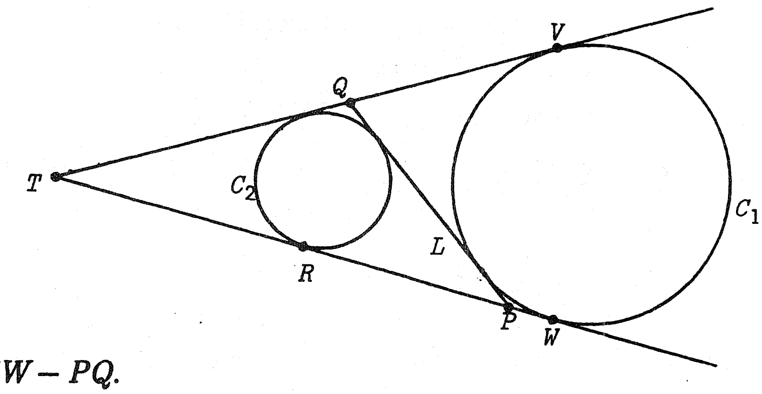
<u>1274</u>. [1987: 256] *Proposed by Dan Sokolowsky, Williamsburg, Virginia.*

Let spheres S_1, S_2 be inscribed in a cone τ. Let a line L, not a lateral element of τ, touch S_1 and S_2, and meet τ at P and Q. Show that the length of PQ is independent of L.

Solution by P. Penning, Delft, The Netherlands.

The plane F, through $L = PQ$ and the top T of τ, meets the spheres in the circles C_1 and C_2, and τ in the straight lines TV and TW. Since C_1 is an excircle and C_2 the incircle of ΔTPQ, $TW = s$, the semi-perimeter of ΔTPQ, and



$$TW - RW = TR = s - PQ = TW - PQ.$$

Hence $PQ = RW$. Now RW is determined only by the radii of the spheres and the distance between them and accordingly is independent of the orientation of F and L.

Also solved by JORDI DOU, Barcelona, Spain; TOSIO SEIMIYA, Kawasaki, Japan; and the proposer.

* * *

1275. [1987: 257] *Proposed by P. Penning, Delft, The Netherlands.*

On a circle C with radius R three points A_1, A_2, A_3 are chosen arbitrarily. Prove that the three circles with radius R, not coinciding with C, and passing through two of the points A_1, A_2, A_3, intersect in the orthocentre of $\Delta A_1A_2A_3$.

Solution by Dan Pedoe, University of Minnesota.

If C_1 is the other circle of radius R through A_2 and A_3, then C and C_1 are mirror images in the line A_2A_3. Let A_1D_1 be the perpendicular from A_1 onto A_2A_3, intersecting the circle C again in H', and let H be the mirror image of H' in A_2A_3. Then it is a known and easily proved theorem that H is the orthocentre of $\Delta A_1A_2A_3$ [see e.g. Theorem 254 of R.A. Johnson, *Advanced Euclidean Geometry*]. Moreover H lies on circle C_1.

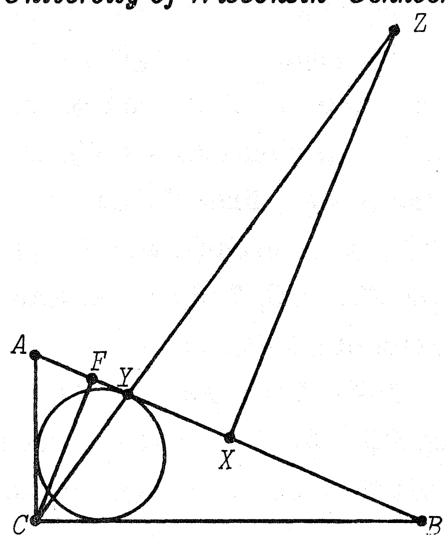
Also solved by JORDI DOU, Barcelona, Spain; J.T. GROENMAN, Arnhem, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer. Most solvers were aware of the above–mentioned theorem from which the problem is immediate.

* * *

1276. [1987: 257] *Proposed by Ernst v. Heydebrand, Heidenheim, Federal Republic of Germany.*

C is the right angle and X the midpoint of the hypotenuse of a nonisosceles right triangle. The incircle of the triangle touches the hypotenuse at Y, and the line CY meets the perpendicular bisector of the hypotenuse at Z. Show that $\overline{XZ} = s$, the semiperimeter of the triangle.

I. *Solution by Bob Prielipp and John Oman, University of Wisconsin–Oshkosh.*



Without loss of generality we may assume that $a > b$. F is the foot of the perpendicular from C onto AB. Since $\overline{AY} = s - a$,

$$\overline{YX} = \frac{c}{2} - (s - a) = \frac{a - b}{2}.$$

Also,

$$\overline{CF} = \frac{ab}{c}.$$

Triangles AFC and BFC are similar, and it follows that

$$\overline{AF} = \frac{b}{a}\cdot\overline{CF} = \frac{b^2}{c}.$$

Now

$$\begin{aligned}\overline{FY} &= \overline{AY} - \overline{AF} = s - a - \frac{b^2}{c} = \frac{(a + b + c)c - 2ac - 2b^2}{2c}\\ &= \frac{(c^2 - 2b^2) - (ac - bc)}{2c} = \frac{(a^2 - b^2) - (ac - bc)}{2c}\\ &= \frac{(a - b)(a + b - c)}{2c}.\end{aligned}$$

Hence, because triangles CFY and XYZ are similar,

$$\overline{XZ} = \frac{\overline{CF}\cdot\overline{YX}}{\overline{FY}} = \frac{ab}{a + b - c} = \frac{ab(a + b + c)}{(a + b)^2 - c^2} = \frac{ab\cdot 2s}{2ab} = s.$$

II. *Comment and solution by Dan Pedoe, University of Minnesota.*

Fairly early in 1987 a visiting professor at the University of Minnesota, Dr. Christopher Pöppe, sent me a note wondering whether a theorem his uncle in Germany had proved would be of any interest. I encouraged him to send it to me, verified the theorem in the manner shown below, and suggested his uncle submit his theorem, now problem 1276, to *Crux.* Léo would have been delighted!

We can extend the problem as follows. If T is any point on AB, and CT intersects the perpendicular bisector of AB in T', the points T and T' are projectively related, and, with respective origins A and X, the relation between the coordinates t of T on AB [i.e. $t = \overline{AT}$] and t' of T' on the perpendicular bisector [i.e. $t' = \overline{XT'}$, signed] is of the form

$$ptt' + qt + rt' + 1 = 0.$$

If we choose T at X, then $t = c/2$ and $t' = 0$; if we take T at the foot of the perpendicular from C onto AB, then t' is infinite and $t = b^2/c$; and if we take T at A, then $t = 0$ and $t' = -ca/2b$. We thus find that the projective relation can be written as

$$2t' = \frac{ab(2t - c)}{b^2 - ct}.$$

If T is Y as in the problem, then $t = s - a$, and with some use of the relation $c^2 = a^2 + b^2$ we find that $t' = \overline{XT} = s$.

If we wish to obtain a further meaningful result, we can take T as the point of contact of AB with the excircle opposite C, so that $t = s - b$ and

$$t' = \frac{-ab}{2s}.$$

III. *Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The same problem by the same author was recently published (with solution) in [1]. Furthermore, in [2] the following extension was given: Situation as above, but take arbitrary, rather than right, triangles. Then $\overline{XZ} = r_c$, the exradius to side c.

References:

[1] E.v. Heydebrand, Eine Geometrie–Aufgabe, *Math. Natwiss. Unterr.* 41 (1988) 48.

[2] Robert Jakobi, Zu "Eine Geometrie–Aufgabe, *ibid* 244.

IV. *Editor's comment.*

Proposers are reminded that no problem should be submitted to more than one journal, or else should be withdrawn from the first before being sent to another. Also, proposers should inform the editor if their problem has appeared, or is to appear, elsewhere (in a research paper, say).

In this case, only one reader seems to have noticed the duplication (the editor thanks him for his sharp eyes). Moreover, no other reader rediscovered Jakobi's seemingly natural generalization, so we wouldn't have known of it without the second appearance of the problem!

Also solved by BENO ARBEL, Tel–Aviv University; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

* * *

<u>1277</u>. [1987: 257] *Proposed by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Determine all possible values of the expression

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1$$

where $n \geq 2$ and $x_i = 1$ or -1 for each i.

I. *Solution by Hans Engelhaupt, Gundelsheim, Federal Republic of Germany.*

Let

$$S = x_1x_2 + x_2x_3 + \cdots + x_nx_1.$$

Then $S_{\max} = n$, occurring when all $x_i = 1$, and

$$S_{\min} = \begin{cases} -n & \text{if } n \text{ is even,} \\ -n+2 & \text{if } n \text{ is odd,} \end{cases}$$

occurring when $x_i = 1$ for i odd and $x_i = -1$ for i even. Moreover, if one x_i changes its value then S either increases by 4, decreases by 4, or doesn't change. Therefore all possible values for S are

$$n\ ,\ n-4\ ,\ n-8\ ,\ \ldots\ ,\ S_{\min}.$$

II. *Solution by Douglass L. Grant, University College of Cape Breton, Sydney, Nova Scotia.*

Let $\mathbf{x} = (x_1,\ldots,x_n)$, $x_i = \pm 1$ for all i, and

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_ix_{i+1}\ ,$$

where $x_{n+1} = x_1$. Within the sequence $\mathbf{x}$, we say a change of sign has occurred at i if $x_ix_{i+1} = -1$, and a repetition has occurred at i if $x_ix_{i+1} = 1$. Let c be the number of changes of sign, r the number of repetitions. Then $n = r + c$, and

$$f(\mathbf{x}) = r - c = n - 2c.$$

Since the determination of c both begins and ends with the sign of x_1, c must be an even integer no larger than n, so $c = 2k$ where $0 \le k \le [n/2]$. All such values can be achieved by c, however; the sequence $\mathbf{x}$ given by

$$x_i = \begin{cases} (-1)^{i+1}, & i = 1,\ldots,2k \\ 1, & i = 2k+1,\ldots,n \end{cases}$$

has exactly $2k$ changes. Therefore the range of f is the set of integers

$$\{n - 4k : 0 \le k \le [n/2]\}.$$

Also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; R.A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; M.M. PARMENTER, Memorial University of Newfoundland, St. John's; KEN ROBERTS, University of Western Ontario, London; M. SELBY, University of Windsor, Windsor, Ontario; DAVID R. STONE, Georgia Southern College, Statesboro; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

The proposers remark that their problem came from the following question in the 1959 U.S.S.R. Math Olympiad:

Suppose $x_1x_2 + x_2x_3 + \cdots + x_nx_1 = 0$, where $x_i = 1$ or -1 for all $i = 1,2,\ldots,n$. Prove that n is divisible by 4.

And of course this follows immediately from the above result.

* * *

<u>1278</u>. [1987: 257] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

(a) Find a non–constant function $f(x,y)$ such that $f(ab + a + b,c)$ is symmetric in a, b, and c.

(b)* Find a non-constant function $g(x,y)$ such that $g(ab(a + b),c)$ is symmetric in a, b, and c.

Solution by Ken Roberts, University of Western Ontario, London.

(a) Let $h(z)$ be any function defined for all real numbers z. Set

$$f(x,y) = h(xy + x + y).$$

Then

$$f(ab + a + b,c) = h(abc + ab + ac + bc + a + b + c)$$

is symmetric in a, b, and c, as desired.

Conversely, any function $f(x,y)$ with the required property must be of the above form. For suppose $f(ab + a + b,c)$ is symmetric in a, b, and c. Then

$$f(ab + a + b,c) = f(ac + a + c,b).$$

Taking $c = 0$ gives

$$f(ab + a + b,0) = f(a,b).$$

Define $h(z) = f(z,0)$ for all z. Then

$$f(x,y) = f(xy + x + y,0) = h(xy + x + y).$$

(b) There is no such function; that is, if $g(x,y)$ satisfies

$$g(ab(a + b),c) = g(ac(a + c),b) = g(bc(b + c),a) \tag{1}$$

for all real numbers a, b, c, then g is constant.

Taking $a = 0$ in (1) gives

$$g(0,c) = g(0,b) = g(bc(b + c),0)$$

for arbitrary b and c. Hence $g(0,y) = H$ for all y, for some constant H. Also, given any x we can find b and c such that $bc(b + c) = x$ (for instance, let $b = c = \sqrt[3]{x/2}$), so $g(x,0) = H$ for all x.

Thus we need only show that $g(x,y) = H$ when both x and y are nonzero. Taking $a = -b$ and $a = b$ respectively in (1) gives

$$H = g(0,c) = g(bc(b - c),b) \tag{2}$$

and

$$g(2b^3,c) = g(bc(b + c),b). \tag{3}$$

Now if $xy < 0$, let

$$b = y\ ,\quad c = \frac{y^2 + \sqrt{y^4 - 4xy}}{2y}\ ,$$

whence $x = bc(b - c)$. Then $g(x,y) = H$ follows from (2). If $xy > 0$, let

$$b = y\ ,\quad c = -\frac{y^2 + \sqrt{y^4 + 4xy}}{2y}\ ,$$

whence $x = bc(b + c)$. Then $g(x,y) = H$ follows from (3) and the fact that $2b^3c < 0$.

Both parts also solved by DUANE M. BROLINE, Eastern Illinois University, Charleston; and LEROY F. MEYERS, The Ohio State University. Part (a) (only) solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; C. WILDHAGEN, Tilburg University, Tilburg, The Netherlands; and the proposer.

Can readers find other functions $h(a,b)$ for which there exists a nonconstant function $f(x,y)$ such that $f(h(a,b),c)$ is symmetric in a, b and c? Maybe even a condition on $h(a,b)$ which ensures that this holds (or fails to hold)?

* * *

1279. [1987: 257] *Proposed by Jordi Dou, Barcelona, Spain.*

Consider a triangle whose orthocentre lies on its incircle.

(a) Show that if one of its angles is given, the others are determined.

(b) Show that if it is isosceles, then its sides are in the proportion 4:3:3.

I. *Editor's apology.*

Your faithful scribe must admit another slip–up! As the proposer knew, and our featured solver (see below) discovered, there are occasions when knowing the value of one of the angles does NOT determine the others. The editor somehow managed to take the proposer's original problem (which, translated literally, reads "given one of its angles, construct the others") and turn it into part (a). Apologies are offered to the proposer, and to those trusting readers, listed below, who otherwise had correct solutions.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) The statement is not quite correct, as will be shown later.

Let H be the orthocentre and I the incentre of the given triangle ABC. In Bottema et al, *Geometric Inequalities* (Groningen 1968), item 5.8, is given the relation

$$HI^2 = 3r^2 + 4rR + 4R^2 - s^2,$$

where r, R, s are the inradius, circumradius, and semiperimeter of the triangle. By assumption $HI = r$, i.e. there has to hold

$$s^2 = 4R^2 + 4Rr + 2r^2. \tag{1}$$

As

$$r = 4R\prod \sin A/2 \quad \text{and} \quad s = 4R\prod \cos A/2,$$

where the products are cyclic, (1) reads equivalently

$$4\prod \cos^2 A/2 = 1 + 4\prod \sin A/2 + 8\prod \sin^2 A/2. \tag{2}$$

Without loss of generality, let A and B be the equal angles in the case that the triangle is isosceles, and put $A \leq B$ in any case. Replacing B by $\pi - C - A$ in (2), we get

$$4\cos^2\frac{C}{2}\cos^2\frac{A}{2}\sin^2\frac{A+C}{2} = 1 + 4\sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{A+C}{2} + 8\sin^2\frac{C}{2}\sin^2\frac{A}{2}\cos^2\frac{A+C}{2},$$

i.e.

$$\begin{aligned}\cos^2\left[\frac{C}{2}\right]\left[\sin\left[A+\frac{C}{2}\right]+\sin\frac{C}{2}\right]^2 &= 1 + 2\sin\frac{C}{2}\left[\sin\left[A+\frac{C}{2}\right]-\sin\frac{C}{2}\right]\\ &\quad + 2\sin^2\frac{C}{2}\left[\sin\left[A+\frac{C}{2}\right]-\sin\frac{C}{2}\right]^2,\end{aligned} \tag{3}$$

where we used

$$2\sin X\cos Y = \sin(X+Y) + \sin(X-Y).$$

Putting now

$$\sin\frac{C}{2} = v > 0, \quad \sin\left[A+\frac{C}{2}\right] = x > 0,$$

we obtain from (3)

$$(1-v^2)(x+v)^2 = 1 + 2v(x-v) + 2v^2(x-v)^2,$$

i.e. after a short simplification

$$x^2(3v^2-1) - 2v^3x + (3v^4 - 3v^2 + 1) = 0,$$

having the solutions

$$x = \frac{v^3 \pm (1-2v^2)^{3/2}}{3v^2 - 1}. \tag{4}$$

This means that (2) is satisfied if and only if

$$\sin\left[A+\frac{C}{2}\right] = \frac{\sin^3 C/2 \pm (\cos C)^{3/2}}{3\sin^2 C/2 - 1}. \tag{4'}$$

Note that (4′) immediately yields $C \leq \pi/2$, i.e. $v \leq 1/\sqrt{2}$. Thus, by symmetry, the triangle must be non–obtuse.

Let us study now the choices + and − in (4)/(4′) separately.

If we choose "−" in (4)/(4′), then we have

$$\begin{aligned}\sin\left[A+\frac{C}{2}\right] &= \frac{\sin^3 C/2 - (\sqrt{\cos C})^3}{\sin^2 C/2 - \cos C}\\ &= \frac{\sin^2 C/2 + \sin C/2\sqrt{\cos C} + \cos C}{\sin C/2 + \sqrt{\cos C}}\\ &= \sin C/2 + \frac{\cos C}{\sin C/2 + \sqrt{\cos C}},\end{aligned}$$

where, since $A \leq B$, $A + C/2 \leq \pi/2$. From this, since there has to hold $\sin(A + C/2) \leq 1$, we obtain

$$v + \frac{1 - 2v^2}{v + \sqrt{1 - 2v^2}} \leq 1$$

which simplifies to

$$1 - v - v^2 \leq (1 - v)\sqrt{1 - 2v^2}. \tag{5}$$

If

$$\frac{\sqrt{5} - 1}{2} \leq v \leq \frac{1}{\sqrt{2}},$$

then (5) is okay, since the left side is negative and the right side positive. If

$$0 < v < \frac{\sqrt{5} - 1}{2},$$

then squaring (5) and reducing the obtained expressions we get $v < 2/3$. As

$$\frac{\sqrt{5} - 1}{2} < \frac{2}{3},$$

it follows that (5) holds for all allowed values of v.

If we choose "+" in (4)/(4′), then the numerator is positive, so the denominator must be too. Thus there has to hold $3v^2 - 1 > 0$, i.e. $v > 1/\sqrt{3}$. As above, we get

$$\begin{aligned}\sin(A + C/2) &= \frac{\sin^3 C/2 + (\sqrt{\cos C})^3}{\sin^2 C/2 - \cos C} \\ &= \sin C/2 + \frac{\cos C}{\sin C/2 - \sqrt{\cos C}}\end{aligned}$$

and so we must have

$$v + \frac{1 - 2v^2}{v - \sqrt{1 - 2v^2}} \leq 1,$$

which reduces to

$$v^2 + v - 1 \geq (1 - v)\sqrt{1 - 2v^2}. \tag{6}$$

Squaring yields $v \geq 2/3$. Thus, since $2/3 > 1/\sqrt{3}$, (6) is satisfied if $2/3 \leq v \leq 1/\sqrt{2}$.

In summary, for each v satisfying $2/3 \leq v < 1/\sqrt{2}$, i.e. each C satisfying

$$83.6^\circ \approx 2\arcsin(2/3) \leq C < \pi/2,$$

there can be found two different values of A by formula (4′). For each v, $0 < v < 2/3$, there is only one value of A.

(b) Let $A = B$. Then

$$x = \sin(A + C/2) = \sin \pi/2 = 1,$$

and (4) becomes

$$v^3 - 3v^2 + 1 = \pm(1 - 2v^2)^{3/2}.$$

Thus, after a short calculation,

$$9v^3 - 6v^2 - 3v + 2 = 0,$$
$$(3v - 2)(3v^2 - 1) = 0,$$

and so $v = 2/3$, since $3v^2 - 1 \neq 0$ by (4). This value of v indeed determines by (4) two different possibilities for $x = \sin(C + A/2)$, as stated at the end of part (a), namely $x = 1$ and $x = 7/9$. For $A = B$ the only relevant value is $x = 1$. Via elementary trigonometry we get from $\sin C/2 = 2/3$ and $A + C/2 = 90^\circ$ that $\sin A = \sqrt{5}/3$ and $\sin C = 4\sqrt{5}/9$. Thus

$$a : b : c = \sin A : \sin B : \sin C = \frac{\sqrt{5}}{3} : \frac{\sqrt{5}}{3} : \frac{4\sqrt{5}}{9} = 3 : 3 : 4.$$

Respective angles are $A = B \approx 48.19^\circ$ and $C \approx 83.62^\circ$.

Remark. It could be noted that because of

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

and (1), the condition $HI^2 = r^2$ reads equivalently

$$a^2 + b^2 + c^2 = 8R^2 + 2r^2.$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; MURRAY S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; and the proposer.

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1280. [1987: 257] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let ABC be a triangle and let A_1, B_1, C_1 be points on BC, CA, AB, respectively, such that

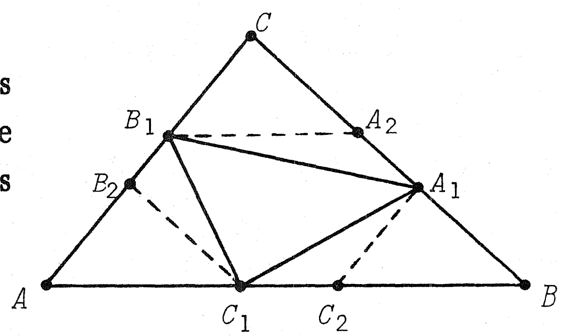
$$\frac{A_1C}{BA_1} = \frac{B_1A}{CB_1} = \frac{C_1B}{AC_1} = k > 1.$$

Show that

$$\frac{k^2 - k + 1}{k(k + 1)} < \frac{\text{perimeter}(A_1B_1C_1)}{\text{perimeter}(ABC)} < \frac{k}{k + 1},$$

and that both bounds are best possible.

Solution by the proposer.



The parallels through A_1, B_1, C_1 to sides CA, AB, BC respectively shall intersect the respective third sides in C_2, A_2, and B_2. It follows at once that

$$\frac{A_1C}{BC} = \frac{B_1A}{CA} = \frac{C_1B}{AB} = \frac{k}{k + 1}$$

and

$$\frac{A_2C}{BC} = \frac{B_1C}{CA} = 1 - \frac{k}{k + 1} = \frac{1}{k + 1}, \text{ etc.,}$$

so that

$$A_2B_1 + B_2C_1 + C_2A_1 = AB\cdot\frac{B_1C}{CA} + BC\cdot\frac{C_1A}{AB} + CA\cdot\frac{A_1B}{BC} = \frac{1}{k+1}\cdot\text{perimeter}(ABC)$$

and

$$A_1A_2 + B_1B_2 + C_1C_2 = A_1C - A_2C + B_1A - B_2A + C_1B - C_2B = \frac{k-1}{k+1}\cdot\text{perimeter}(ABC).$$

Thus

$$\begin{aligned}\text{perimeter}(A_1B_1C_1) &< \text{perimeter}(A_1A_2B_1B_2C_1C_2)\\ &= \frac{1}{k+1}\cdot\text{perimeter}(ABC) + \frac{k-1}{k+1}\cdot\text{perimeter}(ABC)\\ &= \frac{k}{k+1}\cdot\text{perimeter}(ABC),\end{aligned}$$

showing the right–hand inequality. Choosing triangles with $\angle C \to 180^\circ$ and $AC \to 0$ shows that $k/(k+1)$ cannot be decreased for any k.

For the left bound, we note that

$$\overrightarrow{A_1C_1} = \overrightarrow{BC_1} - \overrightarrow{BA_1} = \frac{k}{k+1}\overrightarrow{BA} - \frac{1}{k+1}\overrightarrow{BC}$$

and

$$\begin{aligned}\overrightarrow{A_1B_1} = \overrightarrow{CB_1} - \overrightarrow{CA_1} &= \frac{1}{k+1}\overrightarrow{CA} - \frac{k}{k+1}\overrightarrow{CB}\\ &= \frac{1}{k+1}\overrightarrow{CB} + \frac{1}{k+1}\overrightarrow{BA} - \frac{k}{k+1}\overrightarrow{CB}\\ &= \frac{1}{k+1}\overrightarrow{BA} + \frac{k-1}{k+1}\overrightarrow{BC},\end{aligned}$$

from which follows

$$(k-1)\overrightarrow{A_1C_1} + \overrightarrow{A_1B_1} = \left[\frac{k(k-1)}{k+1} + \frac{1}{k+1}\right]\overrightarrow{BA} = \left[\frac{k^2-k+1}{k+1}\right]\overrightarrow{BA}.$$

Thus

$$\frac{k^2-k+1}{k+1}\cdot AB < A_1B_1 + (k-1)A_1C_1, \text{ etc.}$$

Adding these inequalities we get

$$\begin{aligned}\frac{k^2-k+1}{k+1}\cdot\text{perimeter}(ABC) &< \text{perimeter}(A_1B_1C_1) + (k-1)\text{perimeter}(A_1B_1C_1)\\ &= k\cdot\text{perimeter}(A_1B_1C_1),\end{aligned}$$

which proves the left–hand inequality. Choosing triangles with

$$\angle C \to 180^\circ,\ AB = 1,\ AC \approx 1/k,\ BC \approx (k-1)/k$$

shows that the left bound cannot be increased.

Note: Can anyone generalize to convex polygons?

The right–hand inequality was also solved by M.S. KLAMKIN, University of Alberta.

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