Mathematical Spectrum

A magazine for students and teachers of mathematics
in schools, colleges and universities,
and for everyone interested in mathematics

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- Athletic Performance Trends in Olympics
- A Geometric Gambling Game
- Avoiding the Flames
- Paradoxes of Infinity

Mathematical Spectrum is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom and to computing.

From the Editor

The lady with the lamp

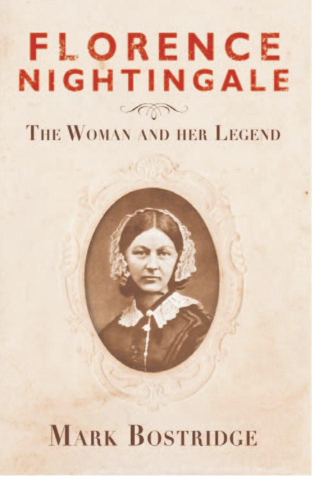
Florence Nightingale died on 13 August 1910, aged ninety. She was a tireless campaigner to improve nursing conditions for soldiers and the general population in Britain in the nineteenth century. The story of her time in the Crimea during the Crimean war is the stuff of legend.

But why should *Mathematical Spectrum* be featuring the century of her death? She was not primarily a nurse, despite the popular picture of the lady with the lamp, but rather an organizer and campaigner. To quote Mark Bostridge, her biographer (see reference 1):

> Florence 'has taken to mathematics', her sister reported in July 1840, 'and like everything she undertakes she is deep in them and working very hard'. She was quick to pass on her knowledge to her cousins, 'doing a little Algebra' with the young Nicholsons, Laura and Lothian, and later coaching their older brother William for his Sandhurst exams, through Mr Nicholson impressed on her the importance of remaining discreet about this, as William would become a laughing stock if it became known that he was receiving tuition from a woman.

The distinguished mathematician J. J. Sylvester was friendly with Florence's uncle and may have tutored Florence, although Mark Bostridge points out that there is no documentary evidence for this. Charles Babbage, whose engines were the forerunners of today's computers, was a guest in her home when she was a child.

But Florence Nightingale's great interest was Statistics. Her mentor was Alphonse Quetelet, an astronomer and meteorologist at the Royal Observatory in Belgium, whose international reputation rested on his work as a statistician and sociologist. She was a pioneer in the collection and use of statistics, including visual presentation in pie charts, in her reports to government and health officials. Today it is the norm for reports to be backed up by statistical analysis, but that was not the case in the nineteenth century.



To quote Karl Pearson, reproduced in the *Oxford Dictionary of Scientific Quotations* (see reference 2):

> Her statistics were more than a study, they were indeed a religion…. Florence Nightingale believed – and in all the actions of her life acted upon that belief – that the administrator could only be successful if he were guided by statistical knowledge…. To understand God's thoughts, she held we must study statistics, for those are the measure of his purpose. Thus the study of statistics was for her a religious duty.

Florence Nightingale revoked a bequest of £2000 in her will to the University of Oxford to go to the funding of a Chair in Statistics when the university authorities showed reluctance to establish a chair in a subject for which there was no Final Honours School. She feared that the money would be used to sponsor a statistical essay prize rather than to teach statistics.

So, if you are unfortunate and find yourself in hospital, you can thank Florence Nightingale for pioneering for high standards of nursing and cleanliness. You can also thank statistics!

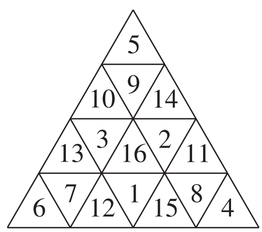
References

1 M. Bostridge, *Florence Nightingale: The Woman and Her Legend* (Penguin, London, 2008).

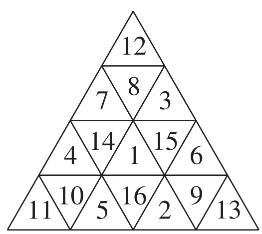
2 W. F. Bynum and R. Porter (eds), *Oxford Dictionary of Scientific Quotations* (Oxford University Press, 2005), p. 465.

Magic triangles

$$6 + 12 + 13 + 7 = 38$$
$$4 + 11 + 15 + 8 = 38$$
$$5 + 10 + 14 + 9 = 38$$
$$6 + 7 + 12 + 1 + 15 + 8 + 4 = 53$$
$$4 + 8 + 11 + 2 + 14 + 9 + 5 = 53$$
$$5 + 9 + 10 + 3 + 13 + 7 + 6 = 53$$



$$11 + 5 + 4 + 10 = 30$$
$$13 + 6 + 2 + 9 = 30$$
$$12 + 7 + 3 + 8 = 30$$
$$11 + 10 + 5 + 16 + 2 + 9 + 13 = 66$$
$$13 + 9 + 6 + 15 + 3 + 8 + 12 = 66$$
$$12 + 8 + 7 + 14 + 4 + 10 + 11 = 66$$



10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Rooholamini Gugheri

A Pouring Problem

PRITHWIJIT DE

Often at parties guests order half a glass of drink (it need not be a hard drink!) and they expect half of the volume of the glass to be filled up with their desired drink. It is a simple task for the waiter if the glass is cylindrical. In that case, the radius of the cross-section is independent of the height of the glass and so he has to pour the drink only up to the halfway mark from the base. Now, what should he do if the shape of the glass is like an inverted frustum of a cone? Let us try to figure it out.

An inverted frustum of a cone is an inverted decapitated cone. Figure 1 shows a two-dimensional section of the glass with the drink poured up to a height, t (say). Assume that the radii of the top and the base are r_1 and r_2 units and the height is h, with $r_1 > r_2$.

The volume of the glass is $V_{\mathrm{G}} = \frac{1}{3}\pi(r_1^2 + r_1 r_2 + r_2^2)h$. This may be found by taking the difference of the volumes of the cones ABC and ADE (see figure 2). If h_1 is the height of

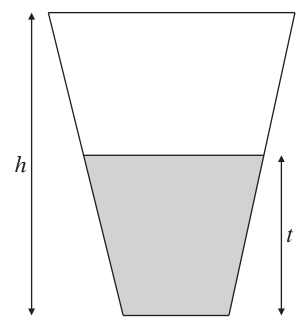


Figure 1 The glass with some amount of the drink in it.

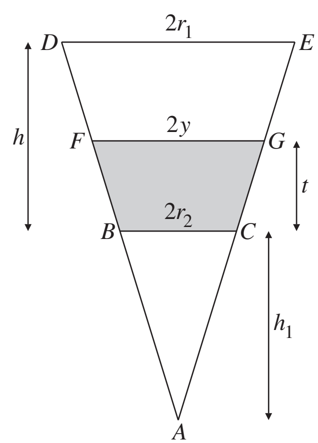


Figure 2 The glass $BDEC$ and the truncated cone ABC. Line FG is the top layer of the drink.

the truncated part of the cone then its volume is $\frac{1}{3}\pi r_2^2 h_1$. Also, $\triangle ABC$ is similar to $\triangle ADE$ and hence $h_1 : (h+h_1) = r_2 : r_1$, which yields $h_1 = hr_2/(r_1 - r_2)$. Therefore the volume is

$$\frac{\pi}{3}(r_1^2(h+h_1) - r_2^2 h_1) = \frac{\pi h}{3}\frac{r_1^3 - r_2^3}{r_1 - r_2} = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2).$$

To find the volume of the liquid in the glass, assume that the radius of the cross-section of the top-most layer of the liquid is y when the height is t.

Now from similar triangles ABC and AFG we have

$$\frac{h_1}{h_1 + t} = \frac{r_2}{y} \qquad \Longrightarrow \qquad h_1 = \frac{r_2 t}{y - r_2}.$$

But $h_1 = hr_2/(r_1 - r_2)$. Therefore,

$$\frac{r_2 t}{y - r_2} = \frac{hr_2}{r_1 - r_2} \qquad \Longrightarrow \qquad y = r_2 + \frac{t(r_1 - r_2)}{h}$$

and the volume of the drink in the glass is equal to

$$V_{\mathrm{L}} = \frac{\pi}{3}(y^2(h_1 + t) - r_2^2 h_1) = \frac{\pi}{3}(a^2 t^3 + 3ar_2 t^2 + 3r_2^2 t),$$

where $a = (r_1 - r_2)/h$. Hence,

$$\frac{V_{\mathrm{L}}}{V_{\mathrm{G}}} = \frac{a^2 t^3 + 3ar_2 t^2 + 3r_2^2 t}{h(r_1^2 + r_1 r_2 + r_2^2)}. \tag{1}$$

Now,

$$\begin{aligned} \frac{V_{\mathrm{L}}}{V_{\mathrm{G}}} = \frac{1}{2} \qquad &\Longleftrightarrow \qquad 2(a^2 t^3 + 3ar_2 t^2 + 3r_2^2 t) = h(r_1^2 + r_1 r_2 + r_2^2) \\ &\Longleftrightarrow \qquad \left(t + \frac{r_2}{a}\right)^3 = \frac{2r_2^3 + ah(r_1^2 + r_1 r_2 + r_2^2)}{2a^3} \\ &\Longleftrightarrow \qquad t = \frac{\sqrt[3]{(r_1^3 + r_2^3)/2} - r_2}{r_1 - r_2} h. \end{aligned}$$

It is obvious that $t > h/2$ when $V_{\mathrm{L}} = V_{\mathrm{G}}/2$. To see this from our calculated value of t, we need to show that

$$\frac{\sqrt[3]{(r_1^3 + r_2^3)/2} - r_2}{r_1 - r_2} > \frac{1}{2}.$$

This is equivalent to showing that

$$\frac{r_1^3 + r_2^3}{2} > \left(\frac{r_1 + r_2}{2}\right)^3$$

and this is true by virtue of the convexity of $f(x) = x^3$ on the interval $(0, \infty)$.

Table 1 Values of t for a glass with $r_1 = 10$, $r_2 = 5$, and $h = 15$.

y	t	p/q	t	y	t
$\frac{1}{10}$	2.902 248	$\frac{3}{8}$	8.042 376	$\frac{7}{10}$	12.104 53
$\frac{1}{9}$	3.171 206	$\frac{2}{5}$	8.407 361	$\frac{5}{7}$	12.256 81
$\frac{1}{8}$	3.496 591	$\frac{3}{7}$	8.811 016	$\frac{3}{4}$	12.630 24
$\frac{1}{7}$	3.898 816	$\frac{4}{9}$	9.029 478	$\frac{7}{9}$	12.913 85
$\frac{1}{6}$	4.409 84	$\frac{1}{2}$	9.764 454	$\frac{4}{5}$	13.136 66
$\frac{1}{5}$	5.082 989	$\frac{5}{9}$	10.458 22	$\frac{5}{6}$	13.464 41
$\frac{2}{9}$	5.507 83	$\frac{4}{7}$	10.649 64	$\frac{6}{7}$	13.693 97
$\frac{1}{4}$	6.015 295	$\frac{3}{5}$	10.987 17	$\frac{7}{8}$	13.863 76
$\frac{2}{7}$	6.633 744	$\frac{5}{8}$	11.275 49	$\frac{8}{9}$	13.994 45
$\frac{3}{10}$	6.871 496	$\frac{2}{3}$	11.742 41	$\frac{9}{10}$	14.098 16

If drink is poured up to the halfway mark along the straight line joining the centre of the base with the centre of the top, then $V_{\mathrm{L}} = \pi h(r_1^2 + 4r_1r_2 + 7r_2^2)/24$ and

$$\frac{V_{\mathrm{L}}}{V_{\mathrm{G}}} = \frac{r_1^2 + 4r_1r_2 + 7r_2^2}{8(r_1^2 + r_1r_2 + r_2^2)}.$$

Since $r_1 > r_2$, a simple calculation gives that $V_{\mathrm{L}}/V_{\mathrm{G}} < \frac{1}{2}$, as must obviously be the case.

The next step is to extend this concept to any value of the ratio $V_{\mathrm{L}}/V_{\mathrm{G}}$. That is, we have to find the height to which the drink has to be poured when we desire $V_{\mathrm{L}}/V_{\mathrm{G}} = y$, for any real number y between 0 and 1. From (1) we obtain

$$a^2t^3 + 3ar_2t^2 + 3r_2^2t = yh(r_1^2 + r_1r_2 + r_2^2).$$

Solving for t we obtain

$$t = \frac{\sqrt[3]{yr_1^3 + (1-y)r_2^3} - r_2}{r_1 - r_2}h. \tag{2}$$

Note that if $f(x)$ is defined on $[r_1, r_2]$ as $f(x) = (\sqrt[3]{x} - r_2)h/(r_1 - r_2)$ then the expression for t in (2) can be rewritten as

$$t = f(yr_1^3 + (1-y)r_2^3). \tag{3}$$

The value of t may be computed easily from the functional equation in (3). In table 1 we present the values of t for a few values of y.

Prithwijit De *teaches at the Institute of the Chartered Financial Analyst of India Business School, Kolkata. He received his PhD in Statistics from National University of Ireland, Cork. He loves to write articles on recreational mathematics.*

Cantor's Rationals in Closed Form

M. A. NYBLOM

A striking property of the set of positive rationals is that, despite the fact that they cannot be arranged in order of increasing magnitude, the rationals can still be put in one-to-one correspondence with the natural numbers. This countability property of the rationals was first established by Cantor using an ingenious visual proof which consisted of an infinite-dimensional matrix, having as entries in the nth row and mth coloumn the fraction m/n (see figure 1). The essence of Cantor's argument lay with the development of a sequence starting with the fraction $1/1$ in the top-left-hand corner of figure 1 and traversing successive diagonals to produce a list containing every positive rational number infinitely often which begins like this:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{4}, \ldots. \tag{1}$$

The construction of such a sequence then allows us to conclude the countability of the positive rationals. Since Cantor studied this, many other enumerations of the rationals have been

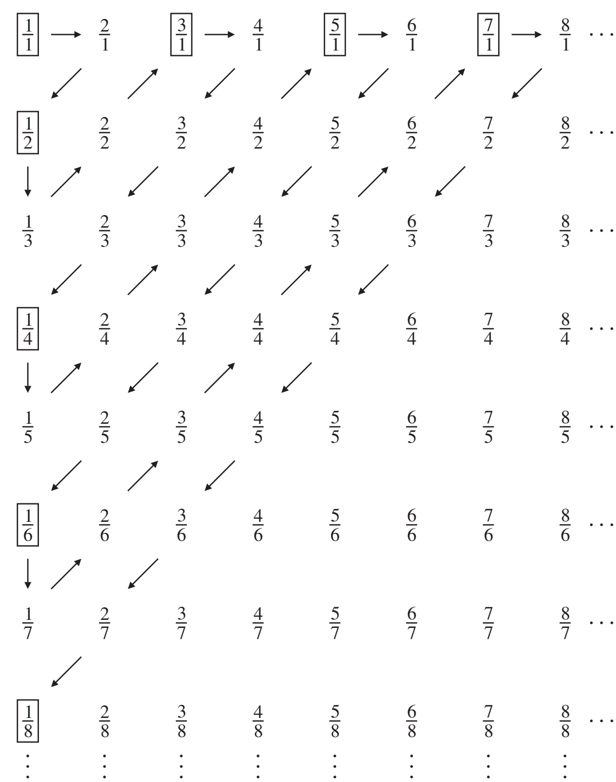


Figure 1 Infinite-dimensional matrix of positive rationals.

constructed (see reference 1), of which perhaps the most elegant is due to Calkin and Wilf who produced a listing of all positive rationals without duplication using a systematic procedure related to the hyperbinary partitions of the integers (see reference 2). In this article we return to Cantor's original enumeration to produce a closed form expression for the nth term of the sequence in (1) as it appears in nonreduced form. In particular, if for an integer $n \geq 1$ we let $s_n = \lceil (1+\sqrt{8n+1})/2 \rceil - 1$, where $\lceil x \rceil = \min\{n \in \mathbb{N} \colon x \leq n\}$, then we show that the nth term of the sequence in (1), denoted by C_n, is given by

$$C_n = \begin{cases} \dfrac{s_n - s_n(s_n+1)/2 + n}{1 + s_n(s_n+1)/2 - n} & \text{if } s_n \text{ is odd,} \\[2ex] \dfrac{1 + s_n(s_n+1)/2 - n}{s_n - s_n(s_n+1)/2 + n} & \text{if } s_n \text{ is even.} \end{cases} \tag{2}$$

As an illustration of (2), consider the case $n = 9$. Then $s_9 = \lceil (1+\sqrt{73})/2 \rceil - 1 = 4$, so

$$C_9 = \frac{1 + (4 \times 5)/2 - 9}{4 - (4 \times 5)/2 + 9} = \frac{2}{3}.$$

If $n = 12$, then $s_{12} = \lceil (1+\sqrt{97})/2 \rceil - 1 = 5$, so

$$C_{12} = \frac{5 - (5 \times 6)/2 + 12}{1 + (5 \times 6)/2 - 12} = \frac{2}{4}.$$

To begin our demonstration of (2) it will first be necessary to observe the following key properties of the matrix of rational numbers in figure 1.

1. Counting from the top-left-hand corner of the matrix, the mth diagonal, where $m = 0, 1, 2, \ldots$, contains $m+1$ consecutive terms of the sequence in (1) which, when written in order as specified by the indicated direction, are either of the form

$$\frac{m+1}{1}, \frac{m}{2}, \frac{m-1}{3}, \frac{m-2}{4}, \ldots, \frac{1}{m+1},$$

   when m is odd, or

$$\frac{1}{m+1}, \frac{2}{m}, \frac{3}{m-1}, \frac{4}{m-2}, \ldots, \frac{m+1}{1},$$

   when m is even. (Note that the diagonals are counted from zero.)

2. In the first row of the matrix, the rational number $(2m-1)/1$ located at the top of the $2(m-1)$th diagonal, as indicated in a box, is the $(1 + 2 + \cdots + (2m-1))$ term of the sequence (1), that is of the form $C_{(2m-1)2m/2}$. Similarly, in the first column of the matrix, the rational number $1/2m$ located at the bottom of the $(2m-1)$th diagonal is the $(1 + 2 + \cdots + 2m)$ term of the sequence (1), that is of the form $C_{2m(2m+1)/2}$. Consequently, a sequence element C_n located on a given diagonal has an index n which is either a triangular number, that is of the form $s(s+1)/2$ for some $s \in \mathbb{N}$, or is bounded between two consecutive triangular numbers. Moreover the largest value an index variable n can attain on a given diagonal occurs when n is a triangular number.

Equipped with these properties consider now the following argument. An integer $n \geq 1$ is either a triangular number or is bounded between two consecutive triangular numbers, so we may find an $s_n \in \mathbb{N}$ such that

$$\frac{(s_n - 1)s_n}{2} < n \leq \frac{s_n(s_n + 1)}{2}. \tag{3}$$

Thus, regardless of whether s_n is even or odd we deduce from property 2 that C_n must be located in diagonal $s_n - 1$ of figure 1. Moreover, by property 1 observe that the i th rational number along diagonal $s_n - 1$, where $i = 1, 2, \ldots, s_n$, is either given by $(s_n - i + 1)/i$ if $s_n - 1$ is odd or $i/(s_n - i + 1)$ if $s_n - 1$ is even. Now as the s_n th entry of diagonal $s_n - 1$ is $C_{s_n(s_n+1)/2}$, the $(s_n - j)$ th entry on the same diagonal is $C_{s_n(s_n+1)/2-j}$, where $j = 0, 1, \ldots, s_n - 1$. Consequently, if $s_n(s_n+1)/2 - j = n$, then C_n must be the i th entry of diagonal $s_n - 1$, where $i = s_n - s_n(s_n+1)/2 + n$. By substituting $i = s_n - s_n(s_n+1)/2 + n$ into the expressions for the i th rational number on diagonal $s_n - 1$, we find that

$$C_n = \begin{cases} \dfrac{s_n - s_n(s_n+1)/2 + n}{1 + s_n(s_n+1)/2 - n} & \text{if } s_n \text{ is odd,} \\[2ex] \dfrac{1 + s_n(s_n+1)/2 - n}{s_n - s_n(s_n+1)/2 + n} & \text{if } s_n \text{ is even.} \end{cases}$$

Finally, all that is required now is to express s_n explicitly in terms of n. To this end, consider the inequality in (3) and observe that $(2s_n - 1)^2 < 8n + 1 \leq (2s_n + 1)^2$, from which it follows that

$$s_n < \frac{1 + \sqrt{8n+1}}{2} \leq s_n + 1.$$

However, as $s_n \in \mathbb{N}$, this means that $s_n + 1 = \lceil (1 + \sqrt{8n+1})/2 \rceil$, as required.

References

1 D. M. Bradley, Counting the positive rationals: a brief survey, Technical Reprint.

2 N. Calkin and H. Wilf, Recounting the rationals, *Amer. Math. Monthly* **107** (2000), pp. 360–363.

***Michael Nyblom** is a lecturer at the Department of Mathematics at RMIT University. His general research interests include Number Theory, Combinatorics, and Analysis.*

Not Pascal's triangle

Arrange the numbers 1 to 10 in the form of Pascal's triangle in four rows such that each number (other than the four in the bottom row) is the positive difference of the two to the left and right below it.

12 Pinewood Road, Midsomer Norton, Bath BA3 2RG, UK **Bob Bertuello**

Revisiting a Problem of Diophantus

KONSTANTINE ‘HERMES’ ZELATOR

1. Introduction

In David Burton’s book *The History of Mathematics: An Introduction* (see reference 1), the following problem can be found.

Problem 1 Find three (rational) numbers such that the sum of the product of any two of them added to the square of the third one equals the square of a rational number.

This problem is listed as an exercise in reference 1; it can also be found in Book III, problem 14, of Diophantus’ *Arithmetica* (see the historical note in Section 4). The hint given is to take, as Diophantus did, two rationals in the form x and $4x+4$, the other rational being 1. Then we observe that $x(4x+4)+1^2=(2x+1)^2$, which is satisfied for any rational value of x. Also, $(4x+4)\cdot 1+x^2=(x+2)^2$, again an identity. Then, we may conditionally set

$$(4x+4)^2+x\cdot 1=(4x-5)^2,$$

an equation which is linear, since the quadratic terms on either side of the equation are both equal to $16x^2$. We solve for x and find $x=\frac{9}{73}$; and consequently $4x+4=\frac{328}{73}$.

The answer provided by the text is the triple of numbers 9, 328, and 73, instead of the expected $\frac{9}{73}$, $\frac{328}{73}$, and 1. However, if three rationals r_1, r_2, r_3 satisfy

$$r_1r_2+r_3^2=\text{rational square},$$

then the three rationals cr_1, cr_2, cr_3 also have the same property for any choice of the rational number c. So, in our case, if we take $c=73$, the answer listed in reference 1 becomes clear. What is not entirely clear, however, is why we should choose $4x-5$ to form the above conditional equation (as compared to an identity). Instead, we should rather take $4x+k$ and form the equation

$$(4x+4)^2+x\cdot 1=(4x+k)^2, \tag{1}$$

where k is an integer to be determined, so that the rational solution in x is positive (as Diophantus’ approach would require). In Section 2 we determine all the rational values of k for which the resulting solution to (1) is a positive rational. We also show that no rational value of k in (1) can produce a nonzero integer value of x. In Section 3, we address our main question: how can we choose five integers a_1, a_2, b_1, b_2, c with $a_1a_2c\neq 0$ and such that

$$(a_1x+a_2)(b_1x+b_2)+c^2=\text{(identically) the square of a linear polynomial with integer coefficients}$$

and

$$(a_1x+a_2)^2+(b_1x+b_2)\cdot c=\text{(identically) the square of a linear polynomial with integer coefficients?}$$

By doing this, we would generalize the above method of Diophantus and would be able to solve Diophantus' problem in the most general way. As it turns out, a complete answer to this question is $a_1 = \lambda$, $a_2 = \nu$, $b_1 = 4\lambda$, $b_2 = 4\mu$, $c = \mu - \nu$ where μ, ν, λ can be any integers such that $\lambda \neq 0$ and $\mu \neq \nu$. We introduce property P via definition 1, below, for practical reasons and convenience, so that we can refer to it throughout this article.

Definition 1 Let $L_1(x)$, $L_2(x)$ be linear polynomials with integer coefficients and c a nonzero integer. We say that the ordered triple $(L_1(x), L_2(x), c)$ has the property P if there exist linear polynomials $\ell_1(x)$, $\ell_2(x)$, with integer coefficients such that $L_1(x) \cdot L_2(x) + c^2 = [\ell_1(x)]^2$ and $[L_1(x)]^2 + c \cdot L_2(x) = [\ell_2(x)]^2$, identically.

The ordered triple $(x, 4x+4, 1)$ has property P since $x \cdot (4x+4) + 1 = (2x+1)^2$ and $x^2 + 1 \cdot (4x+4) = (x+2)^2$.

In Section 4 we offer a brief historical note on Diophantus.

2. More solutions

If we solve for x in (1) we obtain

$$x = \frac{k^2 - 16}{33 - 8k}; \quad \text{or, equivalently,}$$
$$x = \frac{(k-4)(k+4)}{8(\frac{33}{8} - k)}. \tag{2}$$

We easily see from (2) that x will be positive precisely when either $k < -4$ or alternatively $4 < k < \frac{33}{8}$. This then shows that, since there is no integer in the open interval $(4, \frac{33}{8})$, the integer values of k for which x is a positive rational are $k \leq -5$. This explains the value $k = -5$ mentioned in Section 1, which produces the solution $x = \frac{9}{73}$.

Conclusion Any rational value of k in the union of open intervals $(-\infty, -4) \cup (4, \frac{33}{8})$, will produce a positive rational value of x. Consequently, for such a value of k, the three rational numbers x, $4x+4$, and 4 will constitute a solution to the problem of Diophantus.

We now show that no rational value of k in (2) can produce a nonzero integer value for x. We argue by contradiction. Suppose, on the contrary, that there exists a rational number $k = m/n$, for which (2) produces a nonzero integer value $x = j$. Here we assume that m, n are integers with $n > 0$ and in lowest terms, i.e. they are relatively prime. Substituting for $x = j$ and $k = m/n$ in (2) gives

$$j = \frac{m^2 - 16n^2}{n(33n - 8m)}. \tag{3}$$

Since j is an integer, $n(33n - 8m)$ is a divisor of the numerator $m^2 - 16n^2$. In particular, n must be a divisor of $m^2 - 16n^2$; and since $n | -16n^2$, it follows that n must be a divisor of m^2. However, n is relatively prime to m, and so to m^2 as well. Hence, $n = 1$. Setting $n = 1$ in (3) leads to the quadratic equation

$$m^2 + 8jm - (16 + 33i) = 0$$

with a rational (integer) solution, so its discriminant must be a perfect square, i.e.

$$64j^2 + 4(33j + 16) = \text{integer square}.$$

Obviously this integer square must be an even square:

$$64j^2 + 4(33j + 16) = (2K)^2 \quad \text{for some integer } K \geq 0$$

or, equivalently,

$$16j^2 + 33j + 16 - K^2 = 0. \tag{4}$$

Equation (4) is quadratic in j with a rational (integer) solution, so its discriminant must be a perfect square, i.e. $(33)^2 - 4(16)(16 - K^2) = N^2$ for some integer $N \geq 0$. Thus,

$$65 = (N - 8K)(N + 8K). \tag{5}$$

Since $N \geq 0$, $K \geq 0$, $N > 8K$.

The positive divisors of 65 are 1, 5, 13, and 65. Thus, according to (5), and since $1 \leq N - 8K < N + 8K$, there are only two possibilities:

$$\left\{ \begin{array}{l} N + 8K = 13 \\ N - 8K = 5 \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{l} N + 8K = 65 \\ N - 8K = 1 \end{array} \right\}.$$

In the first possibility $K = \frac{1}{2}$ and $N = 9$, obviously a contradiction since K is an integer. The second possibility gives $K = 4$, $N = 33$ and from $j = (-33 \pm N)/32 = (-33 \pm 33)/32 = 0$ or $\frac{-66}{32}$. Only one integer value for j emerges; the value $j = 0$. It is now clear that no choice of rational value of k in (2) can produce a nonzero integer value of x.

3. Precise conditions and conclusions

We finish this article by determining all the triples $(L_1(x), L_2(x), c)$ which have property P (see definition 1). We have $L_1(x) = a_1x + a_2$, $L_2(x) = b_1x + b_2$, c, where a_1, a_2, b_1, b_2, c are integers with $a_1b_1c \neq 0$. First,

$$\begin{aligned} L_1(x)L_2(x) + c^2 &= (a_1x + a_2)(b_1x + b_2) + c^2 \\ &= a_1b_1x^2 + (a_1b_2 + a_2b_1)x + a_2b_2 + c^2. \end{aligned}$$

Since we require $L_1(x)L_2(x) + c^2$ to be identically equal to the square of a linear polynomial with integer coefficients, the discriminant of this quadratic expression in x must be zero, and its coefficient of x^2 must be the square of an integer, i.e.

$$(a_1b_2 + a_2b_1)^2 - 4a_1b_1(a_2b_2 + c^2) = 0 \quad \text{and} \quad a_1b_1 = \text{integer square;}$$

or, equivalently,

$$(a_1b_2 - a_2b_1)^2 = 4a_1b_1c^2 \quad \text{and} \quad a_1b_1 = \text{integer square.} \tag{6}$$

Next, we require that $[L_1(x)]^2 + c \cdot L_2(x)$ be identically equal to the square of a linear polynomial with integer coefficients. We have

$$\begin{aligned} [L_1(x)]^2 + c \cdot L_2(x) &= (a_1x + a_2)^2 + c \cdot (b_1x + b_2) \\ &= a_1^2x^2 + (2a_1a_2 + cb_1)x + a_2^2 + cb_2. \end{aligned}$$

This quadratic expression in x must also have zero discriminant, that is

$$(2a_1a_2 + cb_1)^2 - 4a_1^2 \cdot (a_2^2 + cb_2) = 0,$$

i.e.

$$4a_1^2a_2^2 + 4a_1a_2cb_1 + c^2b_1^2 - 4a_1^2a_2^2 - 4a_1^2cb_2 = 0;$$

and since $c \neq 0$ we obtain

$$4a_1(a_1b_2 - a_2b_1) = cb_1^2. \tag{7}$$

By squaring both sides of (7) and in conjunction with (6) we obtain

$$16a_1^2 \cdot 4a_1b_1c^2 = c^2b_1^4 \quad \Longleftrightarrow \quad (4a_1)^3 = b_1^3 \quad \Longleftrightarrow \quad b_1 = 4a_1. \tag{8}$$

By (8) and (7) we have

$$4a_1(a_1b_2 - 4a_1a_2) = 16a_1^2 \cdot c.$$

Since $a_1 \neq 0$, this gives

$$c = \frac{b_2}{4} - a_2.$$

Now b_2 must be a multiple of 4 since both c and a_2 are integers. Putting $a_1 = \lambda$, $a_2 = \nu$, $b_1 = 4\lambda$ (by (8)) and $b_2 = 4\mu$, we arrive at the following conclusion.

> The ordered triples which have property P of definition 1 are precisely those of the form $(L_1(x), L_2(x), c)$ where $L_1(x) = \lambda x + \nu$, $L_2(x) = 4\lambda x + 4\mu$, $c = \mu - \nu$, where λ, ν, μ can be any integers such that $\lambda \neq 0$ and $\mu \neq \nu$.

The triple $(x, 4x + 4, 1)$, discussed in Section 1 is obtained by taking $\lambda = 1$, $\nu = 0$, and $\mu = 1$.

It is now clear that we can solve Diophantus' problem in the most general way. That is, in order to find three positive rationals with the property that the product of any two of them added to the square of the third produces a rational square, we may pick any three positive integers λ, μ, ν with $\mu \neq \nu$ and try to determine a positive rational x such that $\lambda x + \nu$, $4\lambda x + 4\mu$, $\mu - \nu$ are three such rationals. A straightforward calculation verifies the following two identities:

$$(\lambda x + \nu)(4\lambda x + 4\mu) + (\mu - \nu)^2 = (2\lambda x + \mu + \nu)^2,$$
$$(4\lambda x + 4\mu)(\mu - \nu) + (\lambda x + \nu)^2 = [\lambda x - \nu + 2\mu]^2.$$

Thus, to find such a rational x, we may just solve the conditional equation

$$(4\lambda x + 4\mu)^2 + (\lambda x + \nu)(\mu - \nu) = (4\lambda x + \rho)^2,$$

where ρ is some integer. Solving for x yields

$$x = \frac{\rho^2 - \nu(\mu - \nu) - 16\mu^2}{33\lambda\mu - \lambda(\nu + 8\rho)}.$$

To ensure that x is positive, all we have to do is choose an integer ρ in an appropriate open interval or union of open intervals; something which depends on the given values of λ, μ, and ν.

4. An historical note

Diophantus was the first historically-known person to have studied, somewhat systematically, equations whose solutions were sought in the set of positive rationals; or sometimes more narrowly, in the set of integers. Hence, an entire branch of number theory, Diophantine equations, was named after him.

Practically nothing is known of Diophantus as an individual, save that he lived in Alexandria sometime between the years 150 and 250. Although his works were written in Greek, and he displayed the Greek genius for abstraction, Diophantus was, according to reference 1, more likely Hellenized Babylonian. What personal particulars we have of his career come from the wording of an epigram problem (apparently dating from the 4th century):

> His boyhood lasted for $\frac{1}{6}$ of his life; his beard grew after $\frac{1}{12}$ more; after $\frac{1}{7}$ more he married, and his son was born five years later. His son lived to half his father's age and the father died four years after his son.

If x was the age at which Diophantus died, we obtain the equation

$$\tfrac{1}{6}x + \tfrac{1}{12}x + \tfrac{1}{7}x + 5 + \frac{1}{2}x + 4 = x,$$

whose solution is $x = 84$. But in what year, or even what century, is not certain. His reputation rests in his great work, *Arithmetica*, which can be thought of as the earliest treatise devoted to algebra. Only six of the original thirteen books have been preserved. Of the other works attributed to Diophantus, we know little except for the titles. Fragments of a tract on polygonal numbers are extant. There is also the *Porisms*, but this is lost in its entirety. For more details, see reference 1

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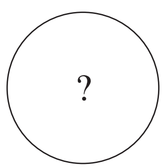
Konstantine Zelator *has taught mathematics at various colleges and universities in the United States. He was a faculty member with the University of Toledo from 2005 to 2008 and is currently an assistant professor of mathematics at Rhode Island College.*

Given a circle, how would you locate its centre using only a straight edge, a pair of compasses, and a pencil?



Athletic Performance Trends in Olympics: Part II

A. TAN and SONYA LYATSKAYA

Introduction

Two decades ago, an article appeared in *Mathematical Spectrum* (see reference 1) which analysed the athletic performance trends in the Track and Field events of the Olympic Games. It was observed that in any particular event, the winning performance betrayed a relentless trend towards better results. It was therefore boldly predicted that this trend was likely to continue well into the 21st century (see reference 1, p. 82). However, that prediction has turned out to be quite incorrect. In the past five Olympics, the winning performances have levelled off in nearly all events. The trend lines over the last five Olympics show that in only 10 out of the 22 Track and Field events, were there marginal overall improvements, whereas in the remaining 12 events, the overall winning performance actually showed nominal decline. Stated otherwise, the athletic performances in all Track and Field events have nearly reached saturation levels. This article addresses the problems of predicting the asymptotic value of the winning performance in any particular event, and modelling the overall historical trend.

Models

We begin with the assertion that all Track and Field events can be classified into two categories. The jumping and throwing events belong to the first category, where the value of the jump or throw is a direct measure of the contestant's performance. For over three quarters of a century since the inception of the modern Olympics, the winning performances (given by the winning

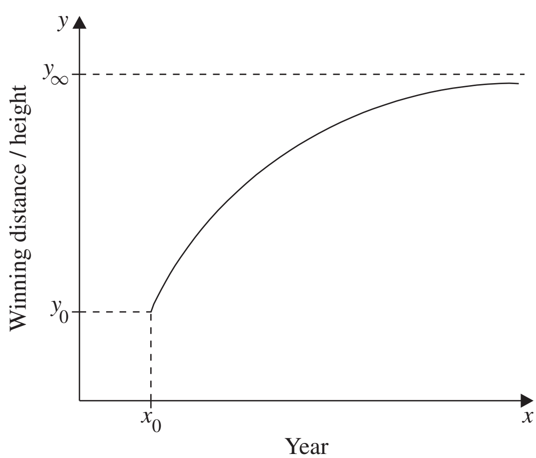


Figure 1 Winning distances/heights in the jumping/throwing events in the Olympic Games.

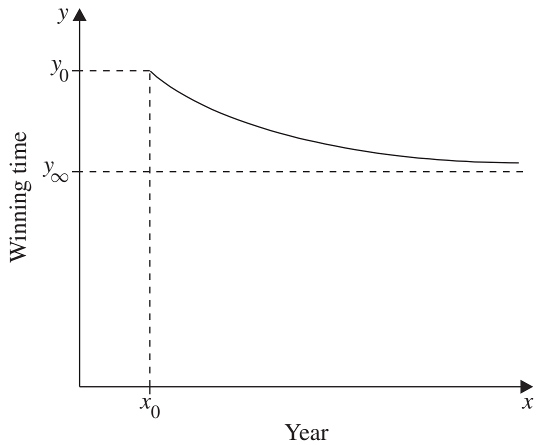


Figure 2 Winning times in the running events in the Olympic Games.

distances or heights) exhibited nearly rectilinear trends towards higher values (cf reference 1). However, in the past five Olympics, this trend has become nearly horizontal in all events. Figure 1 is a model curve of the winning distance/height versus time (in years). In this figure, x_0 is the year of the first Olympics (1896); y_0 is the value of the ordinate at x_0; and y_∞ is the limiting value of the ordinate. The appropriate empirical equation representing the first category of events is given by

$$y = y_\infty - (y_\infty - y_0)\mathrm{e}^{-\alpha(x-x_0)}. \tag{1}$$

The running events belong to the second category. Here, the value of the running time is the reciprocal of the measure of performance. However, here too, the running times of the winning performances have approached their limiting values during the last five Olympics. Figure 2 shows a representative curve of the winning time in a running event. In this case, the appropriate empirical equation is as follows:

$$y = y_\infty + (y_0 - y_\infty)\mathrm{e}^{-\alpha(x-x_0)}. \tag{2}$$

Here the symbols have similar meaning as in (1), with the winning time representing the ordinate.

In this article, we illustrate a method for modelling an Olympics running event as described by (2). An analogous method can be found to model the throwing and jumping events given by (1).

Method

In (2) (as well as in (1) for that matter), x_0 and y_0 are regarded as known constants, and y_∞ and α are the unknown parameters to be determined. To simplify the task, first transfer the origin of the coordinates to $(x_0; y_0)$. In the new coordinate system, (2) assumes the following simpler form:

$$y = y_\infty - y_\infty \mathrm{e}^{-\alpha x}. \tag{3}$$

The task is to determine the two model parameters $\hat{y}_\infty$ and $\hat{\alpha}$. After that is accomplished, we may transform back to the original coordinates to obtain the appropriate parameters relevant to (2).

We employ the method of least squares for the error to model the expected trend. The sum of the squares of the errors of the data points from the model curve of (3) is given by

$$E(\hat{y}_\infty, \hat{\alpha}) = \sum (y - \hat{y})^2 = \sum (y - \hat{y}_\infty + \hat{y}_\infty \mathrm{e}^{-\hat{\alpha} x})^2,$$

where $\sum$ indicates the sum over the data points. By expansion,

$$E(\hat{y}_\infty, \hat{\alpha}) = \sum y^2 + n\hat{y}_\infty^2 + \hat{y}_\infty^2 \sum \mathrm{e}^{-2\hat{\alpha}x} - 2\hat{y}_\infty \sum y - 2\hat{y}_\infty^2 \sum \mathrm{e}^{-\hat{\alpha}x} + 2\hat{y}_\infty \sum y\mathrm{e}^{-\hat{\alpha}x}, \quad (4)$$

where n is the number of data points. The partial derivatives of E in (4) with respect to $\hat{y}_\infty$ and $\hat{\alpha}$ are

$$\frac{\partial E}{\partial \hat{y}_\infty} = 2n\hat{y}_\infty + 2\hat{y}_\infty \sum \mathrm{e}^{-2\hat{\varepsilon}x} - 2\sum y - 4\hat{y}_\infty \sum \mathrm{e}^{-\hat{\alpha}x} + 2\sum y\mathrm{e}^{-\hat{\alpha}x}$$

and

$$\frac{\partial E}{\partial \hat{\alpha}} = -2\hat{y}_\infty^2 \sum x\mathrm{e}^{-2\hat{\alpha}x} - 2\hat{y}_\infty \sum xy\mathrm{e}^{-\hat{\alpha}x} + 2\hat{y}_\infty^2 \sum x\mathrm{e}^{-\hat{\alpha}x}$$

respectively. To minimise E, set the derivatives equal to zeros to obtain

$$\hat{y}_\infty \Big[n + \sum \mathrm{e}^{-2\hat{\alpha}x} - 2\sum \mathrm{e}^{-\hat{\alpha}x} \Big] = \sum y - \sum y\mathrm{e}^{-\hat{\alpha}x} \quad (5)$$

and

$$-\hat{y}_\infty \Big[\sum x\mathrm{e}^{-2\hat{\alpha}x} - \sum x\mathrm{e}^{-\hat{\alpha}x} \Big] = \sum xy\mathrm{e}^{-\hat{\alpha}x}. \quad (6)$$

Eliminating $\hat{y}_\infty$ between (5) and (6) gives

$$\begin{aligned} &\sum xy\mathrm{e}^{-\hat{\alpha}x} \Big[n + \sum \mathrm{e}^{-2\hat{\alpha}x} - 2\sum \mathrm{e}^{-\hat{\alpha}x} \Big] \\ &\quad = \Big[\sum y - \sum y\mathrm{e}^{-\hat{\alpha}x} \Big] \Big[\sum x\mathrm{e}^{-\hat{\alpha}x} - \sum x\mathrm{e}^{-2\hat{\alpha}x} \Big]. \end{aligned} \quad (7)$$

Equation (7) is a transcendental equation in the parameter $\hat{\alpha}$, which may be solved numerically by the graphical method. Let

$$\begin{aligned} f(\hat{\alpha}) = &\sum xy\mathrm{e}^{-\hat{\alpha}x} \Big[n + \sum \mathrm{e}^{-2\hat{\alpha}x} - 2\sum \mathrm{e}^{-\hat{\alpha}x} \Big] \\ &- \Big[\sum y - \sum y\mathrm{e}^{-\hat{\alpha}x} \Big] \Big[\sum x\mathrm{e}^{-\hat{\alpha}x} - \sum x\mathrm{e}^{-2\hat{\alpha}x} \Big], \end{aligned} \quad (8)$$

$f(\hat{\alpha})$ can be calculated for various values of $\hat{\alpha}$ and plotted against $\hat{\alpha}$. The intersection of the curve $f(\hat{\alpha})$ and the $\hat{\alpha}$-axis gives the desired $\hat{\alpha}$. We can then determine $\hat{y}_\infty$ from either (5) or (6), e.g.

$$\hat{y}_\infty = \frac{\sum xy\mathrm{e}^{-\hat{\alpha}x}}{\sum x\mathrm{e}^{-\hat{\alpha}x} - \sum x\mathrm{e}^{-2\hat{\alpha}x}}. \quad (9)$$

The initial approximation of $\hat{\alpha}$ can be obtained by expanding the exponential functions in (7) in powers of $\hat{\alpha}x$ and retaining the lowest powers in the expansions. Retaining terms up to the quadratic powers of $\hat{\alpha}x$, for example, we obtain

$$\hat{\alpha} \approx \frac{2\sum x^2 \sum x^2 y - 6\sum x^3 \sum xy}{2\sum x^2 \sum x^3 y - 3\sum x^3 \sum x^2 y} \quad (10)$$

for an initial estimate of $\hat{\alpha}$. We owe this elucidation to an anonymous referee.

Table 1 Winning times in the 400 metres run in the Olympics.

Year (x)	Time, s (y)	$x - x_0$ (x in (3))	$y - y_0$ (y in (3))
1896 (x_0)	54.2 (y_0)	0	0.0
1900	49.4	4	−4.8
1904	49.2	8	−5.0
1906	53.2	10	−1.0
1908	50.0	12	−4.2
1912	48.2	16	−6.0
1920	49.6	24	−4.6
1924	47.6	28	−6.6
1928	47.8	32	−6.4
1932	46.2	36	−8.0
1936	46.5	40	−7.7
1948	46.2	52	−8.0
1952	45.9	56	−8.3
1956	46.7	60	−7.5
1960	44.9	64	−9.3
1964	45.1	68	−9.1
1968	43.86	72	−10.34
1972	44.66	76	−9.54
1976	44.26	80	−9.94
1980	44.60	84	−9.60
1984	44.27	88	−9.93
1988	43.87	92	−10.33
1992	43.50	96	−10.70
1996	43.49	100	−10.71
2000	43.84	104	−10.36
2004	44.00	108	−10.20

Example

As an illustrative example, we model the winning times in the 400 metres running event as recorded by the gold medal performance in each Olympics. The year of the Olympics (x) and the winning times (y) are taken from reference 2 and entered in table 1. Also shown in table 1 are the $x - x_0$ and $y - y_0$ values. The data clearly show that since the inception of the modern Olympics in 1896, the 400 metres running times had steadily declined until 1988 but have levelled off since then. In a span of 108 years, the winning time has dropped by over 10 seconds.

The overall trend of the 400 metres winning time is now modelled using the method described above. The summations of the various quantities can be conveniently carried out using an EXCEL Spreadsheet or any other similar software. Equation (10) gives a starting value of α in the search for $\hat{\alpha}$ and $\hat{y}_\infty$. Next, α is varied and the function $f(\alpha)$ computed using (8) until the function changes sign. In figure 3 we plot $f(\alpha)$ against α. The intersection of the curve $f(\alpha)$ with the abscissa gives the desired value of α as $\hat{\alpha} = 0.0354$. Equation (9) then provides $\hat{y}_\infty = -10.37$. Adding y_0 to the latter gives the limiting value of winning time in (2) as $y_\infty = 43.83$ (seconds). Thus, the resulting trend line of the 400 metres running event in

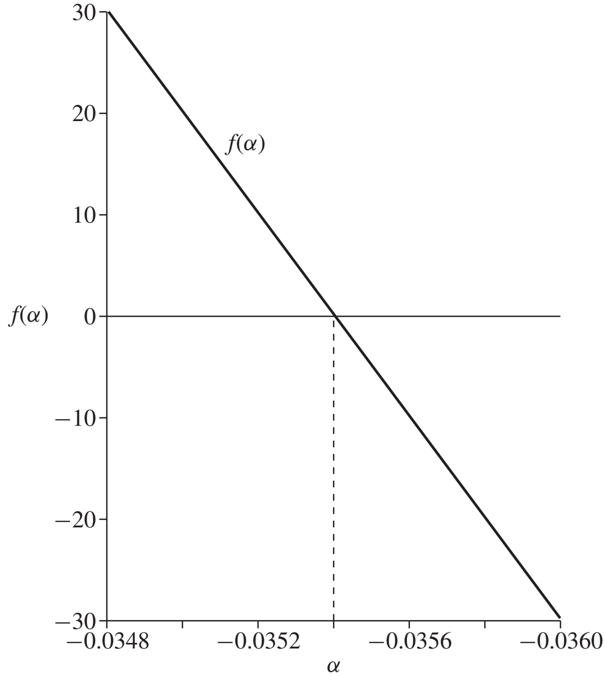


Figure 3 Graphical solution for the parameter $\hat{\alpha}$. The intersection of $f(\alpha)$ and the α-axis gives the desired value.

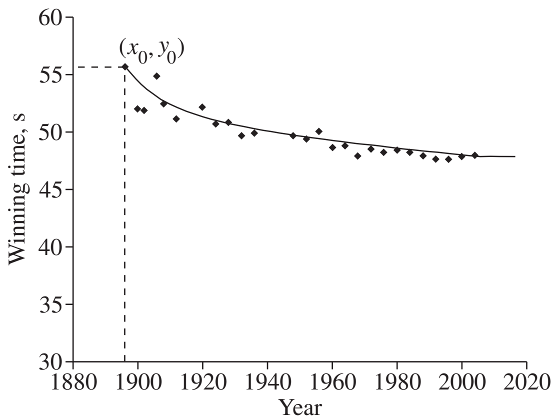


Figure 4 Winning times in the 400 metres run in the Olympics. Also shown is the model curve obtained by the least-squares error method.

the Olympics as expressed by (2) is given by

$$y = 43.83 + 10.37\mathrm{e}^{-0.0354(x-x_0)}.$$

In figure 4, the data points (x, y) as well as the model curve are plotted for comparison. We must caution that we should not be led to believe that $y_\infty = 43.83$ seconds, being the asymptotic value of the timing, can never be bettered. In fact, it has already been bettered twice (in 1992 and 1996). The proper meaning of the limiting value is that it represents a constant asymptotic value, around which the future timings will be scattered. The fact that this has already been breached twice indicates that we are very near this limiting value and that the performance levels may well have reached stability.

Trend lines for other running events can be similarly determined. A similar method can also be applied to the throwing and jumping events, if we use (1) instead of (2).

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2 D. Wallechinsky, *The Complete Book of the Olympics* (Sport Classic Books, Toronto, 2008).

A. Tan *is a Professor of Physics at Alabama Agricultural and Mechanical University. He has a special interest in Applied Mathematics and has published articles in Mathematical Spectrum on a regular basis.*

Sonya Lyatskaya *has a Master's degree in Computer Science and is finishing her PhD in Physics at Alabama Agricultural and Mechanical University.*

How many pages?

Given any sheet of newspaper, consisting of four pages, the sum of the two odd-page numbers is the number of pages of the newspaper.

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Bob Bertuello

Using all the digits

$$\begin{aligned}
100 &= 67^2 - 4385 - 1 - \sqrt{9}, \\
\tfrac{1}{2} &= (123 - 45) \div (67 + 89), \\
1 &= 1 + 2 - 3 + 4 - 5 - 6 + 7 - 8 + 9, \\
1 &= 1 + 23 - 45 - 67 + 89, \\
12 \times 483 &= 5796,
\end{aligned}$$

$$\begin{aligned}
&12\,345\,678\,987\,654\,321 \\
&\quad \times (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1) \\
&\quad = 999\,999\,999^2.
\end{aligned}$$

10 Shahid Azam Lane,
Makki Abad Avenue, Sirjan, Iran

Abbas Rooholamini Gugheri

A Geometric Gambling Game

DAVID K. NEAL and FRANK POLIVKA

Two players simultaneously make independent attempts on separate games of chance with player 1 having probability p_1 of success and player 2 having probability p_2 of success. The players bet on who can succeed first, with the winner being the player who succeeds in the minimum number of attempts, or player 1 in the event of a tie. We derive the probability of player 1 winning and the fair payoff for the bet. We also analyze a side bet of having a winner within the average number of attempts needed for the game to end.

Introduction

Suppose that you and a friend both roll a pair of dice at the same time. You (the house) are trying to roll a sum of seven, while your opponent is trying to roll a sum of nine, and you wager on who can succeed in the minimum number of attempts. Your opponent bets £b that he can succeed first, and if it takes each of you the same number of attempts, then you win as part of your house advantage. Because your opponent has the lesser probability of success on each attempt, you will have a greater chance of the winning the bet, but what exactly is the probability of winning? And for his bet of £b, what kind of payoff can you give that will entice your opponent but still give you an average net gain?

In this article we shall analyze such a gambling game between two players, where player 1 (the house) has probability $p_1 > 0$ of success on any attempt and player 2 (the opponent) has probability $p_2 > 0$ of success. Simultaneous attempts are made, and the opponent wins by succeeding in fewer attempts. Otherwise the house wins. Under these conditions, we shall derive the house's probability $\tilde{p}$ of winning the bet which will allow us to determine the fair payoff. We shall also determine the average number of attempts needed for this game to end, as well as the conditional average for just the games won by the house. Furthermore, we shall determine the general conditions on p_1 and p_2 that make $\tilde{p} > \frac{1}{2}$. Using this condition, we shall design a game that could possibly entice an opponent by having $p_2 > p_1$, yet still give the house a huge hidden advantage by having $\tilde{p} > \frac{1}{2}$.

The geometric distribution

In our game, a player rolls two dice over and over until the desired sum is rolled. In general, a geometric distribution $Y \sim \text{geo}(p)$ is used to count the number of independent attempts needed for the first success, where p is the probability of success on any attempt and $q = 1 - p$ is the probability of failure. For $k \geq 1$, the event that it takes exactly k attempts for the first success is attained by having $k - 1$ initial failures followed by a success, and has probability $\mathrm{P}(Y = k) = q^{k-1}p$.

The event of k failures in a row is equivalent to having the first success in more than k attempts, and this event has probability $\mathrm{P}(Y > k) = q^k$. The complement is the event of succeeding *within* k attempts and its probability is given by

$$\mathrm{P}(Y \leq k) = 1 - q^k, \tag{1}$$

which is known as the cumulative distribution function and which also characterizes the geometric distribution.

Lastly, the average number of attempts needed for the first success is

$$\mathrm{E}[Y] = \frac{1}{p}. \tag{2}$$

(see reference 1).

We now let $X_1 \sim \text{geo}(p_1)$ denote the number of attempts needed for the house to succeed, and let $X_2 \sim \text{geo}(p_2)$ denote the number of attempts needed for the opponent to succeed, with $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$.

Length of the game

The house and the opponent roll independently of each other, but roll at the same time while playing this game. We shall let $X = \min(X_1, X_2)$ denote the number of such simultaneous rolls needed for the game to conclude by having a success from the house, the opponent, (or both). Then X also has a geometric distribution.

To see this result, we let A_1 be the event that the house succeeds and let A_2 be the event that the opponent succeeds. Then the probability of having a winner on any simultaneous roll is

$$p = \mathrm{P}(A_1 \cup A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2) - \mathrm{P}(A_1 \cap A_2) = p_1 + p_2 - p_1 p_2.$$

Because successive simultaneous rolls are independent of each other, we have $X \sim \text{geo}(p)$ by the definition of a geometric distribution.

Alternatively, we can let $q = q_1 q_2$ and $p = 1-q = 1-(1-p_1)(1-p_2) = p_1+p_2-p_1 p_2$. Then the probability that it takes more than k simultaneous rolls to have a winner is given by

$$\begin{aligned}
\mathrm{P}(X > k) &= \mathrm{P}(X_1 > k \cap X_2 > k) \\
&= \mathrm{P}(X_1 > k)\,\mathrm{P}(X_2 > k) \\
&= q_1^k q_2^k \\
&= (q_1 q_2)^k.
\end{aligned}$$

Thus the cumulative distribution function of X is $\mathrm{P}(X \leq k) = 1 - (q_1 q_2)^k = 1 - q^k$, which means that $X \sim \text{geo}(p)$ with $p = 1 - q = p_1 + p_2 - p_1 p_2$.

By (2), the average length of a single game is then given by

$$\mathrm{E}[X] = \frac{1}{p_1 + p_2 - p_1 p_2}. \tag{3}$$

The probability of the house winning

The house wins the wager by succeeding in less than or equal to the number of attempts needed by the opponent. Thus the house wins if and only if $X_1 \leq X_2$, and the probability of this event is

$$\begin{aligned}
\tilde{p} &= \mathrm{P}(X_1 \le X_2) \\
&= \sum_{k=1}^{\infty} \mathrm{P}(X_1 = k \cap X_2 \ge k) \\
&= \sum_{k=1}^{\infty} \mathrm{P}(X_1 = k)\,\mathrm{P}(X_2 \ge k) \\
&= \sum_{k=1}^{\infty} q_1^{k-1} p_1 q_2^{k-1} \\
&= p_1 \sum_{k=0}^{\infty} (q_1 q_2)^k \\
&= \frac{p_1}{1 - q_1 q_2} \\
&= \frac{p_1}{p_1 + p_2 - p_1 p_2}.
\end{aligned} \tag{4}$$

For example, suppose that the house is trying to roll a sum of seven with two dice, while the opponent is trying to roll a sum of nine. Then $p_1 = \frac{1}{6}$ and $p_2 = \frac{1}{9}$. On average, it takes the house six rolls to succeed and the opponent nine rolls. But by rolling at the same time, the average number of simultaneous rolls needed for a winner is

$$\mathrm{E}[X] = \frac{1}{\frac{1}{6} + \frac{1}{9} - \frac{1}{54}} = \frac{27}{7} \approx 3.857 \text{ rolls,}$$

and the probability of the house winning is

$$\tilde{p} = p_1\,\mathrm{E}[X] = \tfrac{1}{6}\tfrac{27}{7} = \tfrac{9}{14}.$$

It is easy to show that $\tilde{p} > \frac{1}{2}$ if $p_1 \ge p_2$. However, we shall see later that it is not necessary to have $p_1 \ge p_2$. We also note that $\tilde{p} > 0$ because we assume that $p_1 > 0$, and $\tilde{p} = 1$ if and only if $p_1 = 1$.

An interesting side-bet

The average duration of a game given in (3) will always be less than the averages $\mathrm{E}[X_1]$ and $\mathrm{E}[X_2]$ of the individual players. Indeed, because $p = \mathrm{P}(A_1 \cup A_2) > \mathrm{P}(A_1) = p_1$, we have $\mathrm{E}[X] = 1/p < 1/p_1 = \mathrm{E}[X_1]$. But this average is often deceptively much lower than either individual average as seen in the above example where $\mathrm{E}[X] \approx 3.857$. Now suppose that we round $\mathrm{E}[X]$ *down* to the nearest integer n and bet that there will be a winner within n simultaneous rolls. We assert that the probability $\tilde{s}$ of winning this side bet is always at least $\frac{1}{2}$.

From (1), the probability of having a winner within k simultaneous rolls is $1 - q^k$, which is an increasing function of k. So if $p \ge \frac{1}{2}$, then $n \ge 1$ and $\tilde{s} = 1 - q^n \ge 1 - q = p \ge \frac{1}{2}$.

But suppose that $p < \frac{1}{2}$. By rounding $\mathrm{E}[X] = 1/p$ down to the nearest integer n, we have $q/p = 1/p - 1 < n \le 1/p$. Thus, $1 - q^n > 1 - q^{q/p}$. We will show that $1 - q^{q/p} \ge \frac{1}{2}$, which is equivalent to $q^{q/p} \le \frac{1}{2}$.

To see this result, consider the differentiable function $f(q) = (q/2)^q$ for $\frac{1}{2} \le q \le 1$. Then $f(\frac{1}{2}) = \frac{1}{2} = f(1)$, and $f'(q) = (q/2)^q(\ln(q/2) + 1)$ shows that the only critical point is at $q = 2/e \in [\frac{1}{2}, 1]$ with $f(2/e)$ being a minimum value. Thus, $(q/2)^q \le \frac{1}{2}$ for $\frac{1}{2} \le q \le 1$.

Now for $p < \frac{1}{2}$, we have $\frac{1}{2} < q \le 1$, which gives $(q/2)^q \le \frac{1}{2}$. Multiplying by 2^q gives $q^q \le 2^{q-1} = 2^{-p}$, and thus $q^{q/p} \le 2^{-1} = \frac{1}{2}$. Therefore, $\tilde{s} = 1 - q^n > 1 - q^{q/p} \ge \frac{1}{2}$.

For example, if it takes you six rolls on average to succeed and your opponent nine rolls on average, then it takes on average about 3.857 simultaneous rolls for there to be a winner. So you can bet that there will be a winner within three rolls. Here, $p = 1/\operatorname{E}[X] = \frac{7}{27}$. So $q = \frac{20}{27}$, and the actual probability of winning the side bet is $\tilde{s} = 1 - (\frac{20}{27})^3 \approx 0.5935$.

Conditional averages

Interestingly enough, the average duration of a game is independent of who wins. That is, if we consider just the games won by the house, then the average length of these games will still be $\operatorname{E}[X] = 1/(p_1 + p_2 - p_1 p_2)$. To see this result, we note that $X = X_1$ when the house wins. Then, because X_1 and X_2 are independent, the conditional average length of games won by the house is obtained by

$$\begin{aligned}
\operatorname{E}[X \mid X_1 \le X_2] &= \operatorname{E}[X_1 \mid X_1 \le X_2] \\
&= \sum_{k=0}^{\infty} k \operatorname{P}(X_1 = k \mid X_1 \le X_2) \\
&= \sum_{k=1}^{\infty} k \frac{\operatorname{P}(X_1 = k \cap k \le X_2)}{\operatorname{P}(X_1 \le X_2)} \\
&= \frac{1}{\tilde{p}} \sum_{k=1}^{\infty} k \operatorname{P}(X_1 = k) \operatorname{P}(X_2 \ge k) \\
&= \frac{1}{\tilde{p}} \sum_{k=1}^{\infty} k q_1^{k-1} p_1 q_2^{k-1} \\
&= \frac{p_1}{\tilde{p} q_1 q_2} \sum_{k=1}^{\infty} k (q_1 q_2)^k \\
&= \frac{p_1}{\tilde{p} q_1 q_2} \frac{q_1 q_2}{(1 - q_1 q_2)^2} \\
&= \frac{p_1}{\tilde{p}} \frac{1}{(p_1 + p_2 - p_1 p_2)^2} \\
&= \frac{1}{p_1 + p_2 - p_1 p_2} \\
&= \operatorname{E}[X].
\end{aligned}$$

It quickly follows that, if $p_1 < 1$, then the average duration of just the games won by the opponent must also equal $\operatorname{E}[X]$. To see this result, we write $\operatorname{E}[X]$ using a weighted average as follows:

$$\begin{aligned}
\operatorname{E}[X] &= \operatorname{E}[X \mid X_1 \le X_2] \operatorname{P}(X_1 \le X_2) + \operatorname{E}[X \mid X_1 > X_2] \operatorname{P}(X_1 > X_2) \\
&= \operatorname{E}[X] \tilde{p} + \operatorname{E}[X \mid X_1 > X_2](1 - \tilde{p});
\end{aligned}$$

or

$$\mathrm{E}[X](1-\tilde{p}) = \mathrm{E}[X \mid X_1 > X_2](1-\tilde{p}).$$

Because $1-\tilde{p} > 0$ when $p_1 < 1$, we obtain $\mathrm{E}[X \mid X_1 > X_2] = \mathrm{E}[X]$.

Choosing a payoff

If $\tilde{p}$ is much higher than $\frac{1}{2}$, then the opponent should expect a payoff higher than £b for a £b bet. But games are designed for the house to make a profit, so we must determine a payoff of £a that entices your opponent but is still unfair.

With a payoff of £a on a bet of £b, the house's average gain per bet is given by $b\tilde{p} - a(1-\tilde{p})$. For the house to make a profit in the long run, we must have $0 < b\tilde{p} - a(1-\tilde{p})$, which is satisfied with a payoff of

$$a < \frac{\tilde{p}}{1-\tilde{p}} b. \tag{5}$$

The ratio $\tilde{p} : 1-\tilde{p}$ gives the fair payoff, and the ratio $\tilde{s} : 1-\tilde{s}$ gives the fair payoff for the side bet. In the example with $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{9}$, and $\tilde{p} = \frac{9}{14}$, the fair payoff ratio is $9:5$, or £1.80 for every £1 bet. So the house could have a profitable game by paying £1.50 for every £1 bet. Likewise, the fair payoff for the side bet of having a winner within three rolls is $(1-(\frac{20}{27})^3) : (\frac{20}{27})^3$, or about 1.46 to 1. So even money side bets would be very profitable in the long run.

Even if $\tilde{p} < \frac{1}{2}$, the house can profit by paying less than $1:1$. For instance, if the house is attempting to roll a sum of nine while the opponent is trying to roll a sum of seven, then $p_1 = \frac{1}{9}$, $p_2 = \frac{1}{6}$, and now $\tilde{p} = p_1/(p_1+p_2-p_1p_2) = \frac{3}{7}$, which gives a fair payoff ratio of $3:4$. So a payoff of £0.50 for every £1 bet will produce a long-term profit for the house, but may not entice many players.

Guaranteeing a house advantage

It is generally assumed that the house will have the advantage in any gambling game. This advantage can always be created with an unfair payoff regardless of the chances of winning. However, it is also guaranteed with an even payoff when the house has a greater than 50% chance of winning. For our game, we can easily determine the conditions that make $\tilde{p} > \frac{1}{2}$. From (4), we need

$$\frac{p_1}{p_1+p_2-p_1p_2} > \frac{1}{2}.$$

Solving for p_1, we find that $\tilde{p} > \frac{1}{2}$ if and only if

$$p_1 > \frac{p_2}{1+p_2}. \tag{6}$$

This condition is easy to check given the probabilities of success p_1 and p_2 of each player. However, by taking reciprocals in (6), we also find that $\tilde{p} > \frac{1}{2}$ if and only if

$$\mathrm{E}[X_1] = \frac{1}{p_1} < \frac{1}{p_2} + 1 = \mathrm{E}[X_2] + 1. \tag{7}$$

Thus we also can compare the average number of attempts needed to succeed for each player. For example, suppose that the opponent has $p_2 = \frac{1}{6}$ chance of success and therefore averages six attempts for the first success. Then the house would need to average less than seven attempts for the first success, which is to have $p_1 > \frac{1}{7}$, in order to have $\tilde{p} > \frac{1}{2}$. But (7) shows that it is possible to have $\tilde{p} > \frac{1}{2}$ even though $p_1 < p_2$. As a final example, we shall describe a simple party game that provides a dramatic advantage to the house.

An advantageous party game

You and a friend simultaneously draw a single card from your own shuffled decks of 52 cards. You are trying to draw any card from 2 to 7 inclusive, so that $p_1 = \frac{6}{13}$ and $\mathrm{E}[X_1] = \frac{13}{6} \approx 2.17$, while your friend is trying to draw any card from 8 to Ace inclusive, so that $p_2 = \frac{7}{13}$ and $\mathrm{E}[X_2] = \frac{13}{7} \approx 1.857$. Whoever draws a desired card first wins with ties going to you because your opponent has the slight advantage. You bet even money on the outcome. You also make a side bet paying your opponent $2:1$ that there will be a winner on the first draw.

Because $\mathrm{E}[X_1] < \mathrm{E}[X_2] + 1$, you have a greater than 50% chance of winning and therefore will make a profit in the long run on an even money bet. But in fact, from (4), your chance of winning is $\tilde{p} = \frac{78}{127} \approx 0.614$, which provides a huge advantage on an even money bet. By (5), a fair payoff to your opponent would be $78 : 49$, or about 1.59 to 1. For an even payoff, your average gain for a £1 bet would be $\tilde{p} - (1 - \tilde{p}) = £\frac{29}{127}$, or about £2.28 average profit for every ten bets of £1.

The side bet is also a profit maker. In this case, the probability of a winner on any draw is $p = p_1 + p_2 - p_1 p_2 = \frac{127}{169} \approx 0.75$, which also gives the probability $\tilde{s}$ of winning the side bet of having a winner on the first draw. The average number of draws needed for a winner is in fact $1/p \approx 1.33$. Even paying your opponent $2:1$, your average gain on the £1 side bet is $\tilde{s} - 2(1 - \tilde{s}) = £\frac{43}{169}$, or about £2.54 average profit for every ten bets of £1. This simple party game provides a nice way to win money (and lose friends) in the long run.

Reference

1 S. M. Ross, *A First Course in Probability*, 7th edn. (Prentice Hall, Upper Saddle River, NJ, 2006).

David Neal *is a professor or mathematics at Western Kentucky University and his main interest is probability theory. When not teaching, he enjoys caring for rescued cats, physical fitness activities, and classic rock music.*

Frank Polivka *is a graduate of Western Kentucky University where he majored in mathematics. He is now preparing for a career as an actuary.*

Triangular numbers

The nth triangular number t_n is given by

$$t_n = 1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1).$$

An alternative definition of a triangular number t is one such that $8t + 1$ is a perfect square. Can you see that these two definitions amount to the same thing?

Lucknow, India **M. A. Khan**

Avoiding the Flames

STUART SIMONS and ANDREW TWORKOWSKI

The whole shop front is ablaze, from ground level up to height h. The fire fighters have a ladder of length l, which is greater than h, for evacuating the occupants of the building or for fighting the fire, but they need to know the optimum angle θ it should make with the horizontal in order that the nearest approach to the flames from anywhere on the ladder should be as large as possible. Despite its obvious deficiencies we model the situation very simply by supposing that the flames are confined to the front of the building, rather than extending outwards, and that their effects are independent of time. This is represented in figure 1 where the fire extends from A to B ($AB = h$) and CD is the ladder of length l ($l > h$). Now, from any point P on the ladder below the level of B, the minimum distance to the fire is PP', while for points Q above the level of B this minimum distance is QB which, as Q varies, will be minimised when BQ is perpendicular to CD. That is, for a specified value of θ, the nearest approach to the flames from anywhere on the ladder is the distance s ($s = BE$) of B from CD. This distance s will depend on θ and we therefore want to calculate the value of θ which maximises $s(\theta)$. Now, $BE = BD\cos\theta$ and $BD = l\sin\theta - h$. Thus

$$s(\theta) = (l\sin\theta - h)\cos\theta = l\left(\tfrac{1}{2}\sin 2\theta - \beta\cos\theta\right), \tag{1}$$

where $\beta = h/l$ ($\beta < 1$). Now, $s(\theta)$ is maximised by θ satisfying $\mathrm{d}s/\mathrm{d}\theta = 0$, corresponding to

$$\cos 2\theta + \beta\sin\theta = 0, \tag{2}$$

and this in turn yields $2\sin^2\theta - \beta\sin\theta - 1 = 0$ with solution for θ in the interval $[0, 90°]$ being given by

$$\theta = \varphi(\beta) = \sin^{-1}\left(\frac{(8+\beta^2)^{1/2}+\beta}{4}\right), \tag{3}$$

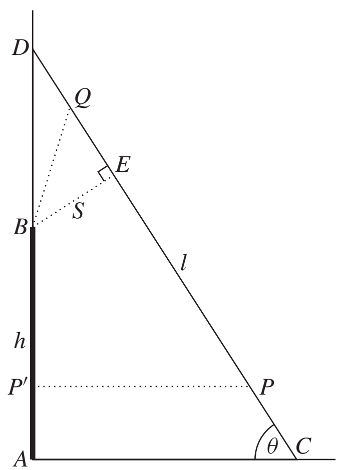

Figure 1

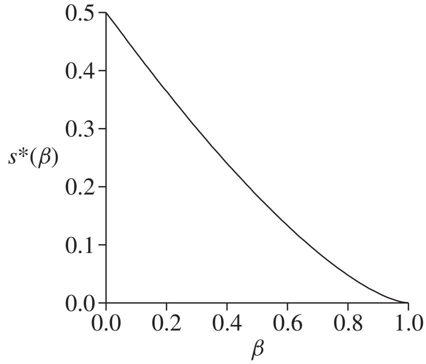

Figure 2

which corresponds to a maximum value of $s(\theta)$ since $\mathrm{d}^2 s/\mathrm{d}\theta^2|_{\theta=\varphi} < 0$. The maximum value of $s(\theta)$ is $s(\varphi)$ and on introducing the corresponding dimensionless separation $s^*(\beta) = s(\varphi)/l$, we readily obtain from (1) and (3)

$$s^*(\beta) = \frac{1}{8\sqrt{2}}((8+\beta^2)^{1/2} - 3\beta)(4 - \beta^2 - \beta(8+\beta^2)^{1/2})^{1/2}.$$

In figure 2 we give a graph of s^* as a function of β, showing that as β increases from 0 to 1 (since $0 < h < l$), $s^*(\beta)$ decreases monotonically from $\frac{1}{2}$ to 0. As a consequence of this, if the value of h is specified, then (since $\beta = h/l$ and $s = ls^*$) increase in the ladder length l by a factor X leads to an increase in the value of s by a factor $Y > X$. As a numerical example of this, let us suppose that in a particular case the value of l doubles ($X = 2$), with the value of h being such that β decreases from 0.8 to 0.4. It then transpires from the data depicted in figure 2 that the corresponding value of s^* increases by a factor of about 5 and thus $Y \approx 10$. As another example of the use of figure 2, consider the following question. If the intensity of the fire is such that we require $s \geq 10$ feet and the length of the only available ladder is $l = 100$ feet, what is the maximum height h of a fire that can be tackled? Here the values of s and l correspond to $s^* \geq 0.1$, which from figure 2 means that $\beta \leq 0.67$, and thus the maximum value of h is 67 feet.

We now consider improving the model used in the previous paragraph by taking into account that the flames of the fire extend outwards from the front of the building. We model this effect very simply by supposing that the conflagration, of height h, extends horizontally towards the ladder by a distance k (throughout the height of the fire), and thus fills a volume whose vertical cross section is the shaded rectangle $AGFB$ in figure 3, with $AB = h$ and $BF = k$. We now wish to calculate θ in order to maximise the minimum separation between any point along the line CD and any point inside the rectangle $AGFB$. This minimum separation is now $S(\theta) = FH$, the perpendicular distance of F from CD, and it immediately follows from the geometry of figure 3 that

$$S(\theta) = JE = BE - BJ = s(\theta) - k\sin\theta = l\left(\tfrac{1}{2}\sin 2\theta - \beta\cos\theta - \gamma\sin\theta\right),$$

where $\beta = h/l$ and $\gamma = k/l$ ($\gamma < 1$). We now maximise $S(\theta)$ by considering $\mathrm{d}S/\mathrm{d}\theta = 0$, corresponding to

$$\cos 2\theta + \beta\sin\theta - \gamma\cos\theta = 0. \tag{4}$$

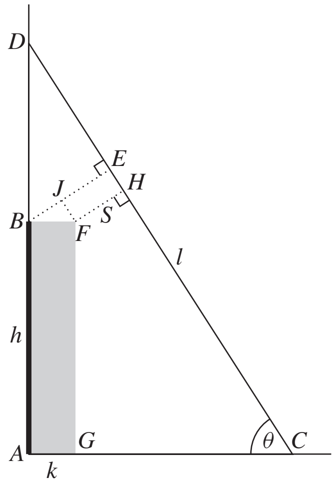

Figure 3

This equation differs from (2) by the addition of the term '$\gamma \cos\theta$', and as a consequence we cannot now obtain an explicit solution of (4), similar to result (3), since (4) cannot be reduced to a quadratic equation in a single trigonometrical ratio as (2) could. The best we can do in the direction of transforming (4) into an algebraic equation is to introduce $t = \tan(\theta/2)$ when we obtain a quartic equation in t, whose solution will then require a numerical approach for specified values of β and γ. Alternatively we could tackle (4) directly by a numerical technique (for specified β and γ) by, for example, using the Newton–Raphson method. Now, our improved model with suitable values for β and γ is, despite its defects, presumably a better description of the true situation than was our first model (with $\gamma = 0$). However, any additional accuracy that the introduction of a nonzero γ may yield could well be offset by the additional effort required to deal numerically with the solution of (4) over a range of values of both β and γ. It was therefore felt that in the absence of a particular situation with specified values for β and γ there was little point in considering any further the solution of (4). Our general conclusion is therefore that our earlier semiquantitative discussion of the implications of (3) is probably the best we can currently give, based on the very simple modelling ideas that we are now using.

Stuart Simons *was Reader in Applied Mathematics at the University of London before his retirement with main interests in transport theory and the mathematical theory of aerosols. His current interests include mathematical modelling together with developing novel approaches to relatively elementary problems.*

Andrew Tworkowski *is currently a member of the School of Mathematical Sciences, Queen Mary, University of London. He has published numerous papers on solar magneto hydrodynamic physics and also on dissipative dynamical systems theory. He is also a Principal Examiner for GCE and IGCSE mathematics.*

An Unusual Look at Primitive Pythagorean Triples

JONATHAN WEISBROD

Primitive Pythagorean triples are expressed in terms of two parameters for which a geometrical interpretation is possible.

Introduction

A *Pythagorean triple* (*PT*) is a sequence of three positive integers (a, b, c) such that $a^2+b^2 = c^2$. A *primitive Pythagorean triple* (*PPT*) is a PT such that a, b, and c have no common factors greater than one. If (a, b, c) is a PPT, then one of a, b is even and the other is odd, and c is odd. Since a, b can be interchanged, a can be taken to be even.

Mathematicians from various times and civilizations throughout history have developed ways to generate PPTs. Euclid's formula is the most commonly used method of generating PPTs. It requires two coprime positive integers, m and n, of opposite parity such that $m > n$. Then, $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$ generates all PPTs (a, b, c). Pythagoras and Plato (see reference 1, pp. 109–111), are two of the more renowned mathematicians who have derived other methods.

In this article, we will show how to derive and use an unusual method to generate PPTs. This method requires two parameters α and β, as does Euclid's formula. Unlike Euclid, our parameters have geometric meanings, as we shall see.

Theorem showing how our parameters make PPTs

Theorem 1 *(a, b, c) is a primitive Pythagorean triple with a even if and only if there exist coprime positive integers α and β such that α is an odd square number, β is twice a square, $a = \beta + \sqrt{2\alpha\beta}$, $b = \alpha + \sqrt{2\alpha\beta}$, and $c = \alpha + \beta + \sqrt{2\alpha\beta}$.*

Proof Suppose that a, b, and c are given by the formulae. Then

$$\begin{aligned} a^2 + b^2 &= \beta^2 + 2\beta\sqrt{2\alpha\beta} + 2\alpha\beta + \alpha^2 + 2\alpha\sqrt{2\alpha\beta} + 2\alpha\beta \\ &= (\alpha + \beta + \sqrt{2\alpha\beta})^2 \\ &= c^2. \end{aligned}$$

Clearly, a is even. If d divides a, b, c, then $d|(c - a)$, $d|(c - b)$, i.e. $d|\alpha, \beta$, so $d = 1$ and (a, b, c) is a primitive Pythagorean triple.

Conversely, suppose that (a, b, c) is a PPT with a even. Put $\alpha = c - a$ and $\beta = c - b$. Then $a = c - \alpha$, $b = c - \beta$, and $(c - \alpha)^2 + (c - \beta)^2 = c^2$ so that

$$c^2 - 2(\alpha + \beta)c + \alpha^2 + \beta^2 = 0$$

and

$$c = \alpha + \beta \pm \sqrt{(\alpha+\beta)^2 - (\alpha^2+\beta^2)} = \alpha + \beta \pm \sqrt{2\alpha\beta}.$$

Hence, $a = c - \alpha = \beta \pm \sqrt{2\alpha\beta}$ and $b = c - \beta = \alpha \pm \sqrt{2\alpha\beta}$.

If $\alpha \geq \beta$, then $\sqrt{2\alpha\beta} \geq \sqrt{2}\beta > \beta$, so we must take the plus sign since $a > 0$. If $\alpha < \beta$, then $\sqrt{2\alpha\beta} > \sqrt{2}\alpha > \alpha$ and again we need the plus sign since $b > 0$. Any common factor of α, β also divides a, b, and c, so α, β are coprime. Also, $c^2 = a^2 + b^2$ so

$$(c-a)(c+a) = b^2. \tag{1}$$

If d is a common factor of $c - a$ and $c + a$, then $d|2a$ and $d|2c$. Since $c - a$ is odd, d is odd. Hence, $d|a$ and $d|c$, so d must be 1. It follows from (1) that $c - a$, i.e. α, is an odd

Table 1 Sides generated by parameters.

α	1	9	25	49	81	121	169	225	289	361
$\beta = 2$	4 3 5	8 15 17	12 35 37	16 63 65	20 99 101	24 143 145	28 195 197	32 255 257	36 323 325	40 399 401
$\beta = 8$	12 5 13	20 21 29	28 45 53	36 77 85	44 117 125	52 165 173	60 221 229	68 285 293	76 357 365	84 437 445
$\beta = 18$	24 7 25		48 55 73	60 91 109		84 187 205	96 247 265		120 391 409	132 475 493
$\beta = 32$	40 9 41	56 33 65	72 65 97	88 105 137	104 153 185	120 209 241	136 273 305	152 345 377	168 425 457	184 513 545
$\beta = 50$	60 11 61	80 39 89		120 119 169	140 171 221	160 231 281	180 299 349		220 459 509	240 551 601
$\beta = 72$	84 13 85		132 85 157	156 133 205		204 253 325	228 325 397		276 493 565	300 589 661
$\beta = 98$	112 15 113	140 51 149	168 95 193		224 207 305	252 275 373	280 351 449	308 435 533	336 527 625	364 627 725
$\beta = 128$	144 17 145	176 57 185	208 105 233	240 161 289	272 225 353	304 297 425	336 377 505	368 465 593	400 561 689	432 665 793
$\beta = 162$	180 19 181		252 115 277	288 175 339		360 319 481	396 403 565		468 595 757	504 703 865
$\beta = 200$	220 21 221	260 69 269		340 189 389	380 261 461	420 341 541	460 429 629		540 629 829	580 741 941

square. Finally, $c^2 - b^2 = a^2$ so $(c-b)(c+b) = a^2$ or

$$\frac{c-b}{2}\frac{c+b}{2} = \left(\frac{a}{2}\right)^2 \tag{2}$$

and $(c-b)/2$, $(c+b)/2$, and $a/2$ are positive integers. Any common factor of $(c-b)/2$ and $(c+b)/2$ must divide their sum and difference, namely c and b, so must be 1. Hence, $(c-b)/2$ and $(c+b)/2$ are coprime. It follows from (2) that $(c-b)/2$, i.e. $\beta/2$, is a square, so that β is twice a square.

The least possible values for α and β are 1 and 2 respectively. When we substitute these values, we get

$$\begin{aligned} a &= 2 + \sqrt{2 \times 1 \times 2} = 4, \\ b &= 1 + \sqrt{2 \times 1 \times 2} = 3, \\ c &= 1 + 2 + \sqrt{2 \times 1 \times 2} = 5. \end{aligned}$$

Examples

Table 1 displays the triples generated by the first ten possible values for each parameter. Notice that the columns are values of α and rows are values of β. When the parameters are not coprime, the generated triple is shaded black. These are PTs; however, they are not primitive. The black cells represent triangles similar to a PPT with each side's length longer by a factor of $\gcd\{a, b, c\}$. We could continue this table infinitely down and to the right.

Geometrical interpretation of the parameters

Since $\alpha = c - a$, it is the difference between the length of the hypotenuse and the length of the even adjacent side of the right-angled trangle of sides (a, b, c). Also, $\beta = c - b$ so is the difference between the length of the hypotenuse and the length of the odd adjacent side. We derive a general formula for the radius of the incircle of a right-angled triangle. Let figure 1 show any right-angled triangle with sides a, b, c, and inradius r.

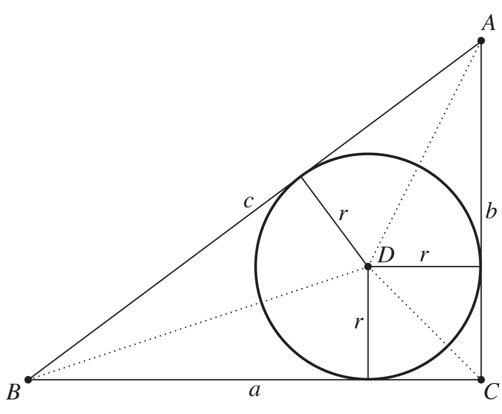

Figure 1 The inradius of a triangle.

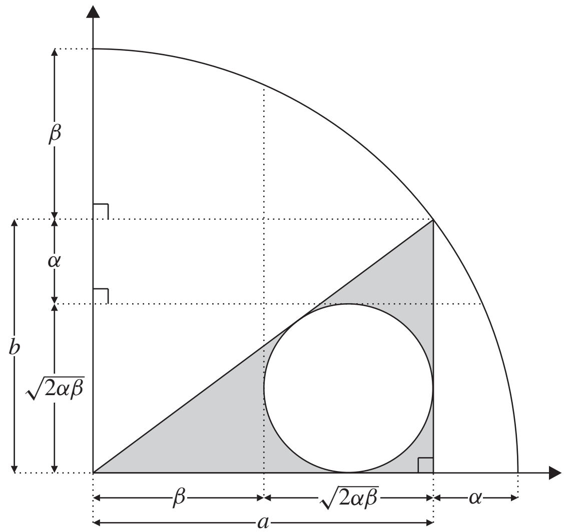

Figure 2 Geometry of the parameters.

Notice that the area of the whole triangle is the sum of the areas of $\triangle ADB$, $\triangle BDC$, and $\triangle ADC$, so that

$$\frac{1}{2}ab = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2},$$

from which we get

$$r = \frac{ab}{a+b+c}.$$

Hence,

$$r = \frac{(\beta + \sqrt{2\alpha\beta})(\alpha + \sqrt{2\alpha\beta})}{2(\alpha+\beta) + 3\sqrt{2\alpha\beta}} = \frac{(\alpha+\beta)\sqrt{2\alpha\beta} + 3\alpha\beta}{2(\alpha+\beta) + 3\sqrt{2\alpha\beta}} = \frac{\sqrt{2\alpha\beta}}{2},$$

so the diameter of the inscribed circle of the right-angled triangle with sides a, b, c is $\sqrt{2\alpha\beta}$. Figure 2 shows how the parameters α, β give various measurements connected with the triangle.

Final remark

In reference 2 results identical to theorem 1 are shown, but the details of the geometry and the incircle are not given.

Acknowledgements

The author would like to acknowledge Dr Thomas J. Osler and Dr Marcus Wright for their help in developing and organizing this article.

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Jonathan Weisbrod *is an undergraduate student at Rowan University; he is planning on studying for a PhD after graduation. Other than being a student, he spends a lot of his time as a soccer (football) official in the northeastern United States.*

Some continued radicals

$$2+\sqrt{2}=\sqrt{2+2\sqrt{2^2+2^{3/2}\sqrt{2^3+2^2\sqrt{2^4+2^{5/2}\sqrt{2^5+\cdots}}}}}\,,$$

$$\frac{\sqrt{3}}{2}(3+\sqrt{13})=\sqrt{3+3\sqrt{3^2+3^{3/2}\sqrt{3^3+3^2\sqrt{3^4+3^{5/2}\sqrt{3^5+\cdots}}}}}\,,$$

$$\frac{\sqrt{2}}{8}(1+\sqrt{17})=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2^2}+\frac{1}{2^{3/2}}\sqrt{\frac{1}{2^3}+\frac{1}{2^2}\sqrt{\frac{1}{2^4}+\frac{1}{2^{5/2}}\sqrt{\frac{1}{2^5}+\cdots}}}}}\,,$$

$$\frac{5}{4}=\sqrt{1+\frac{1}{2}\sqrt{1+\frac{1}{4}\sqrt{1+\frac{1}{8}\sqrt{1+\frac{1}{16}\sqrt{1+\cdots}}}}}\,,$$

$$\frac{5}{2\sqrt{2}}=\sqrt{2+\frac{1}{2}\sqrt{4+\frac{1}{2^{3/2}}\sqrt{8+\frac{1}{4}\sqrt{16+\frac{1}{2^{5/2}}\sqrt{32+\cdots}}}}}\,,$$

$$\frac{33}{256}=\frac{1}{4}\sqrt{\frac{1}{4}+\frac{1}{16}\sqrt{\frac{1}{16}+\frac{1}{64}\sqrt{\frac{1}{64}+\frac{1}{256}\sqrt{\frac{1}{256}+\cdots}}}}\,,$$

$$3=\sqrt{4+\sqrt{4^2+\sqrt{4^3+\sqrt{4^4+\sqrt{4^5+\cdots}}}}}\,,$$

$$24=2\sqrt{16+2^2\sqrt{16^2+2^3\sqrt{16^3+2^4\sqrt{16^4+\cdots}}}}\,,$$

$$288=4\sqrt{64+4^2\sqrt{64^2+4^3\sqrt{64^3+4^4\sqrt{64^4+\cdots}}}}\,,$$

$$4224=8\sqrt{256+8^2\sqrt{256^2+8^3\sqrt{256^3+8^4\sqrt{256^4+\cdots}}}}\,.$$

Student in Fundamental Physics,
Université Pierre et Marie Curie, Paris, France

Yasar Atas

Paradoxes of Infinity[1]

PHILL SCHULTZ

Continuous and discrete infinity

Is the Universe infinite?

Are there infinitely many stars?

Questions like these no doubt puzzled our ancestors in ancient days; nowadays cosmologists have settled them both negatively with the theories of the Big Bang and General Relativity.

But their mathematical counterparts persist: Is there a line of infinite length? Are there more whole numbers than primes?

We can describe the first question as concerning *continuous infinity* since it asks 'how much?' and the second *discrete infinity*, since it asks 'how many?'.

A third type of question about infinity, which combines both continuous and discrete aspects, is as follows.

> Are there infinite processes? For example, can you add infinitely many real numbers?

These questions were certainly considered by mathematicians and philosophers in Ancient Greece.

Paradoxes

One of the earliest such considerations that we have records of is due to Zeno (490 BC–430 BC), a mathematical disciple of the philosopher Parmenides (circa 520 BC), who claimed that the world was in a continual state of flux and warned of the danger of confusing perception with reality. Zeno's interpretation of this was to pose the question of whether time and space were infinitely divisible or whether they comprised indecomposable atoms. Whichever you choose to believe could lead to insoluble problems. For example, Zeno proposed a number of paradoxes which would have to be resolved whichever postulate was accepted. Here are two of them.

Paradox 1 (*Achilles*) Let us hypothesise that time (and consequently space) are infinitely divisible and consider a race between Achilles and the tortoise. Since Achilles is the swiftest of the Greeks, he must give the tortoise a start, say 10 m. After 1 second, Achilles has reached the point at which the tortoise started, but at this time the tortoise has already covered 1 m. By the time Achilles has reached that point, the tortoise is still $\frac{1}{10}$ m ahead. And so on. So if time is infinitely divisible, then Achilles can never catch the tortoise.

Paradox 2 (*The arrow*) Let us hypothesise this time that there are indivisible atoms of time. Suppose that an archer shoots an arrow and at a certain instant the arrow is in position A and at the next instant it has reached position B (see figure 1). This is impossible, for when did it get from A to B?

1. A shorter version of this article first appeared in *Vinculum*.

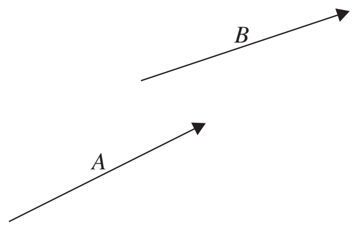

Figure 1

Such were the logical difficulties involved that the greatest philosopher of Ancient Greece, Aristotle (384 BC–322 BC), decided that the concept of actual infinity was meaningless. The most you could hypothesise, he decreed, was potential infinity. That is, in the discrete case you could have sets larger than any given finite set, but not infinite sets. In the continuous case, you could have lines longer than any given line, but not infinite lines.

To see how this works in practice, we turn to the work of Euclid (325 BC–265 BC), the supposed author of the foremost textbook of antiquity, the *Elements of Mathematics*. The first Book of the *Elements*, devoted to *Geometry*, contains the postulate that any given line segment can be extended in the same direction by a line segment of any given magnitude.

The sixth Book of the *Elements*, which is concerned with *Number Theory*, contains a proof that for any given finite set of primes, there is a prime not in that set. Actually, Euclid's Proposition really says that for any three primes there is another prime, but most mathematical historians extend it in the way I stated. Note, however, that in contradiction to statements to be found in many modern Number Theory textbooks, Euclid never claimed that there is an infinite set of primes. That would have been in contradiction to Aristotle's strictures.

Now let us see how infinite processes were achieved using potential but not completed infinity. One remarkable example is Archimedes' Quadrature of the Parabola. Consider a region P of a parabola cut off by a chord (see figure 2). Archimedes discovered and proved that the ratio between the area of P and the area of the largest triangle inscribed in P is 4 to 3.

How, without the aid of calculus, did he do this? The Greeks had no absolute measure of area. Instead they determined areas as multiples of known areas and the area of triangles was well-known. However, before this result of Archimedes, little was known about the area of regions bounded by curved lines.

What is the largest triangle contained in P? It is easy to prove by Euclidean geometry and the known properties of the parabola that the vertices of the largest triangle are the end-points of the chord and the point of tangency of the tangent to the parabola parallel to the chord. This triangle is labelled T in figure 2, so $T < P$.

Archimedes then proceeded as follows. The sides of T are also chords of the given parabola, so the procedure can be repeated to give the small triangles U and V. It can be shown by elementary geometry that the total area $U + V$ is equal to $\frac{1}{4}T$, so that the area of the three triangles adjoined is $\frac{5}{4}T$, which is still less than the area P. Furthermore, the difference between the estimate $\frac{5}{4}T$ and the desired $\frac{4}{3}T$ is exactly $\frac{1}{3}$ of the $\frac{1}{4}T$ that you added.

Now continue to fill in the space between the constructed polygon and P by adjoining small triangles. Each time you do this, you add $\frac{1}{4}$ of what you previously added. The total area is still less than P, and the difference between this area and $\frac{4}{3}T$ is $\frac{1}{3}$ of what you last added.

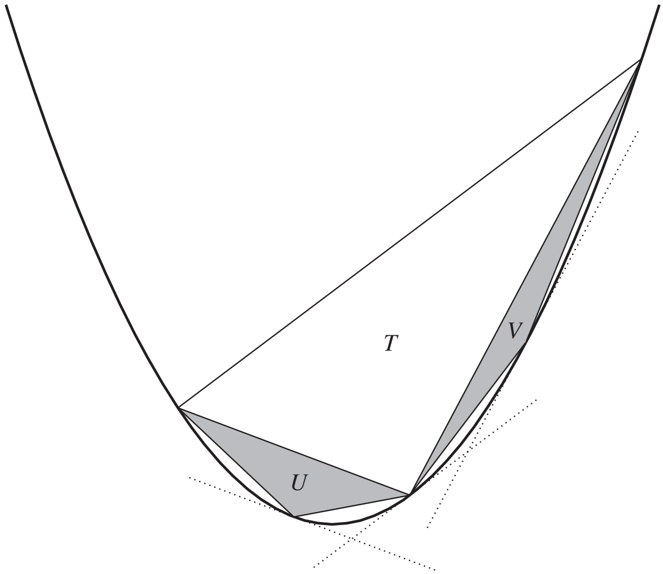

Figure 2

Now Archimedes concludes that $P = \frac{4}{3}T$ as follows. Either $P < \frac{4}{3}T$, $P = \frac{4}{3}T$, or $P > \frac{4}{3}T$. Suppose that $P < \frac{4}{3}T$, say $\frac{4}{3}T - P = \varepsilon$. At every stage n of the construction, the area of the polygon is less than P and differs from $\frac{4}{3}T$ by $T/(3 \times 4^n)$, so as soon as

$$\frac{T}{3 \times 4^n} < \varepsilon,$$

we have a contradiction.

A similar computation shows that a contradiction also arises if you assume that $P > \frac{4}{3}T$. Hence, $P = \frac{4}{3}T$, as required.

In the language of modern calculus, we would say that Archimedes has proved that

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3},$$

though of course he did it without the notions of limits and infinite series. If you examine his proof carefully, you will see that Archimedes made the unstated assumption that the area of the parabola exists, but otherwise his proof is perfectly rigorous, and an example of an infinite process that does not require a notion of completed infinity.

It is interesting to note that none of the Greeks, not even the far-sighted Archimedes, ever came to grips with the notion of instantaneous velocity. If they had, perhaps calculus would have been discovered, or invented, 2000 years earlier than it was.

Mysticism and rationality

Consideration of the notion of infinity in the European tradition next arises in the later Middle Ages when Christian philosophers, known as the Scholastics, wrote voluminously about the existence and properties of God and angels. Critics called their deliberations ‘hairsplitting’, which reminds us of Zeno’s arguments of infinite divisibility of matter. For example, Thomas Aquinas (1226–1274) tried to prove in a purely rational way that God exists, that is without appealing to blind faith. In order to account for properties such as omniscience it was necessary to postulate infinite magnitudes of various sorts. This led the Scholastics to reject the teachings of Aristotle in favour of those of the idealistic philosopher Plato.

Hard though it may be to believe, such considerations actually led to mathematical advances. For example, Nicole Oresme (1323–1382), the bishop of Paris, studied the notion of velocity and more generally rates of change of various magnitudes. By graphical representations using rectangular coordinates, Oresme described uniform motion, uniformly accelerated motion, and even nonuniformly accelerated motion.

What use is infinity?

At this point, it is appropriate to consider the advantages of admitting the notion of infinity to mathematics.

Infinite series

We have already pointed out Archimedes’ use of (what we would call) infinite series in computing the area of a parabolic segment. He also used similar methods to compute the area and circumference of a circle and the surface area and volume of a sphere.

The Indian mathematician Madhava (1350–1425) found infinite series representations of all the trignometric functions, useful not only in astronomy but also in Hindu mysticism.

We are all familiar with the use of infinite series by Leibniz (1646–1716) and Newton (1642–1727) in the foundations of calculus, and of Euler (1707–1783) in many branches of pure and applied mathematics.

Perspective and projective geometry

The artists of the Renaissance discovered and applied the principles of perspective to represent realistically a three-dimensional world on a two-dimensional canvas. These principles included the notions of the vanishing point of a system of parallel lines, that is, the point at an infinite distance to which the lines appear to converge. There is a single horizon on which all the vanishing points lie.

Two of the greatest of these artists were Piero della Francesca (1412–1492), of Florence, and Albrecht Dürer (1471–1528), of Nuremberg. I single them out because both were not only famous artists, but also competent mathematicians who wrote manuals setting out the principles of perspective and the three-dimensional mathematics on which it is based.

Later the mathematician Desargues (1593–1662) introduced projective geometry, a whole new branch of mathematics based on the concept of perspective. Projective geometry freely uses the notion of points and lines at infinity.

Construction of real numbers

After calculus became an integral part, even the major part, of mathematics, it began to be realized that its foundations were not as firmly established as those of the 2000-year-old geometry. This wobbly foundation often produced contradictory or paradoxical results, particularly in the theory of sequences.

One of the first to realize that real numbers must be put on a firm theoretical basis was the Bohemian theologian Bernhard Bolzano (1781–1848). Without actually setting out a system of axioms, he showed that the real numbers are complete in the sense that (what we call) Cauchy sequences always converge. He used this result to prove, for example, the intermediate value theorem and the fact that bounded sets of numbers have least upper bounds.

The actual axiomatic basis of the real and complex numbers was established by Cauchy (1789–1857) and Dedekind (1831–1916) who had no scruples about freely using the concept of completed infinity.

Infinite cardinals

Dealing with the completed infinity of numbers leads to some paradoxical results. For example, Galileo (1564–1643) pointed out that although the squares become scarcer and scarcer among the whole numbers, there is actually exactly one square, n^2, for each whole number n, so in some sense the number of squares and the number of whole numbers is the same.

In his researches on the sets of points on which Fourier series could converge, Georg Cantor (1845–1918) hit upon the idea that there are several 'sizes' of infinity. Two sets, finite or not, are said to have the same *cardinality* if there exists a 1–1 correspondence between them. Cantor proved that the cardinalities of the following sets are equal:

(1) the natural numbers,

(2) the integers,

(3) the primes,

(4) the rationals,

(5) the algebraic integers, that is, the roots of all monic polynomials with integer coefficients.

Call this infinite cardinal $\aleph_0$, $\aleph$ (aleph) being the first letter of the Hebrew alphabet and the subscript 0 indicating that it is the least infinite cardinal.

The cardinalities of the following sets are equal, and strictly greater than $\aleph_0$:

(1) the real numbers,

(2) the interval $[0, 1]$,

(3) the unit cube,

(4) every finite or countable dimensional complex vector space,

(5) the set of all subsets of natural numbers,

(6) the set of all functions from the natural numbers to a set with two elements.

Prompted by (6), we denote this cardinal by $2^{\aleph_0}$.

Cardinal arithmetic

Cantor defined addition, multiplication, and exponentiation of cardinals and proved that, in addition to the commutative and associative laws, they have the following properties.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be infinite cardinals. Then

$$\mathfrak{m} + \mathfrak{n} = \mathfrak{m} \times \mathfrak{n} = \max\{\mathfrak{m}, \mathfrak{n}\}$$

and

$$\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}, \quad \text{if } \mathfrak{m} > \mathfrak{n}, \qquad \text{and} \qquad \mathfrak{m}^{\mathfrak{n}} = 2^{\mathfrak{n}}, \quad \text{if } \mathfrak{m} \leq \mathfrak{n}.$$

Denote the sequence of infinite cardinals by

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots ,$$

where each $\aleph_{i+1}$ is the smallest cardinal larger than $\aleph_i$.

The subscripted indices in this sequence come from another class of numbers discovered by Cantor, called *ordinals*. We don't need to discuss them in this article, other than to say that they are specifically designed to index sequences and that they contain all the cardinals.

Every cardinal lies somewhere in the first sequence, so where does $2^{\aleph_0}$ lie? The Continuum Hypothesis is that $2^{\aleph_0} = \aleph_1$. Gödel showed that this cannot be disproved (see reference 1) while Cohen showed that it cannot be proved (see reference 2). So it is an axiom for 'Cantorian Set Theory', while its negation is an axiom for 'Non-Cantorian Set Theory'. The Generalised Continuum Hypothesis is that $2^{\aleph_i} = \aleph_{i+1}$ for each i.

Aristotle revisited

Finally, we can ask: how many $\aleph$s are there? But this question is meaningless, since logical contradictions in Set Theory arise if you assume that there is a set containing all the $\aleph$s. In other words, for every $\aleph$ there is a larger $\aleph$ but there is no largest $\aleph$.

We have returned full circle to Aristotle's notion of potential infinity!

Literature

This article has barely touched on the manifold ways in which the concept of infinity has influenced the progress of mathematics. A Google search on 'infinity in mathematics' produced half a million hits, but this is a very haphazard way of trying to pursue the subject. A better way is to look in a decent library.

For more on the historical aspects of the subject, I highly recommend reference 3. Other readable sources are references 4 and 5. For a beautiful recent book describing mathematical aspects of Piero's work, see reference 6. For an easy to read introduction to Set Theory and Cantor's cardinals, you can't better reference 7. There are several more advanced treatments, which make clear just how cardinals are constructed and used, see, for example, reference 8. Finally, for a little paperback which discusses many fascinating aspects of infinity neglected in this article, see reference 9.

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***Phill Schultz** is an engineering graduate of Sydney University. After many years in the mining and petroleum industries, he returned to his first love, mathematics, and is currently an Adjunct Associate Professor in the School of Mathematics and Statistics of the University of Western Australia.*

Letters to the Editor

Dear Editor,

Same equation, different application

When I saw the equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$$

in Prithwijit De's article 'Caught up in a Box' (see Volume 42, Number 3, pp. 115–121) my immediate reaction was 'I've seen that equation before somewhere'. But where?

Suppose that you try to cover a floor with regular polygon tiles, all with sides of the same length, with an x-gon, a y-gon, and a z-gon meeting at each vertex. The interior angles are $\pi - 2\pi/x$, $\pi - 2\pi/y$, and $\pi - 2\pi/z$, so that

$$\left(\pi-\frac{2\pi}{x}\right)+\left(\pi-\frac{2\pi}{y}\right)+\left(\pi-\frac{2\pi}{z}\right)=2\pi.$$

That is,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2},$$

the same as Prithwijit De's equation.

Some, but not all, of the solutions in table 2 of De's article give a tiling which will cover the floor. The most obvious is $(6, 6, 6)$, which produces a tiling using only hexagons in a honeycomb pattern. Other solutions which produce a valid tiling are $(12, 12, 3)$, $(12, 6, 4)$, and $(8, 8, 4)$. Readers may find it interesting to try constructing these for themselves.

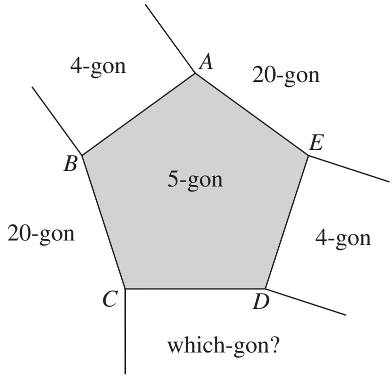

Figure 1

But if one (or more) of x, y, z is odd, then you can only cover the floor if the other two variables are equal. So $(12, 12, 3)$ is possible, since $z = 3$ is odd but both x and y are equal to 12. But, for example, $(20, 5, 4)$ is not possible because $y = 5$ is odd but $x = 20$ and $z = 4$ are unequal.

To see why, begin by putting down one pentagon tile $ABCDE$ and place against this the square and the icosagon tile which have the vertex A in common (see figure 1). Suppose that the square tile shares the side AB, and the icosagon the side AE, with the pentagon.

Now consider the tiles which meet at B. You already have the pentagon and the square, so the tile which shares the side BC with the pentagon must be an icosagon. And for a similar reason, considering the tiles which meet at C, the tile that shares CD with the pentagon must be a square.

But you can also reach CD by going the other way round the pentagon. At E you already have a pentagon and an icosagon, so the tile which shares DE with the pentagon must be a square. Then, considering the tiles which meet at D, the tile that shares CD with the pentagon must be an icosagon. Contradiction!

For similar reasons you can exclude the combinations $(42, 7, 3)$, $(24, 8, 3)$, $(18, 9, 3)$, $(15, 10, 3)$, and $(10, 5, 5)$ in De's table 2.

You can also make tilings with 4, 5, or 6 (but not more; why not?) polygons meeting at each vertex. For example, with a w-gon, x-gon, y-gon, and z-gon at a vertex the corresponding equation is

$$\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

In this application you are not interested in solutions with any variable less than 3. As in Prithwijit De's article, you can simplify the search by listing the possibilities with $w \geq x \geq y \geq z$. Then, using arguments like those in De's article, you can show that $z \leq 4$, so z can only be 3 or 4. Continuing in the same way, you find that, if $z = 4$, then $y = 4$, $x = 4$, and $w = 4$. If $z = 3$, then you can show that $y \leq 4\frac{1}{2}$, so y can only be 3 or 4. The complete set of solutions is $(4, 4, 4, 4)$, $(6, 4, 4, 3)$, $(6, 6, 3, 3)$, $(12, 4, 3, 3)$.

However, reasoning as above but beginning with a triangle rather than a pentagon, the solution $(12, 4, 3, 3)$ does not lead to a tiling which covers the floor. Of the rest, $(4, 4, 4, 4)$ gives a simple pattern of square tiles; and because they have two variables equal, you can use the other two to produce tilings in which the sides of each triangle are shared with either a square or a hexagon.

The most detailed account I have found of such tiling patterns is in reference 1; for a condensed English version of this see reference 2. These both also give examples of non-homogeneous tilings, in which different combinations of polygons meet at different vertices. There is a list of just the homogeneous tilings with diagrams in reference 3.

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Yours sincerely,

Douglas Quadling
(12 Archway Court
Barton Road
Cambridge CB3 9LW
UK)

Dear Editor,

Summing a series of Fibonacci numbers

In Volume 42, Number 2, Page 94, M. A. Khan gave a method of summing a finite series of Fibonacci numbers. The same method can be used to sum the series

$$S_n = f_3 + f_6 + f_9 + \cdots + f_{3n}.$$

We have

$$\begin{aligned} 2S_n &= 2f_3 + 2f_6 + 2f_9 + \cdots + 2f_{3n} \\ &= (f_1 + f_2) + f_3 + (f_4 + f_5) + f_6 + \cdots + (f_{3n-2} + f_{3n-1}) + f_{3n} \\ &= f_{3n+2} - 1, \end{aligned}$$

so that

$$S_n = \tfrac{1}{2}(f_{3n+2} - 1).$$

To sum the series

$$A_n = f_1^2 + f_2^2 + \cdots + f_n^2,$$

we can write

$$\begin{aligned} A_n &= f_1(f_2 - f_0) + f_2(f_3 - f_1) + f_3(f_4 - f_2) + \cdots + f_n(f_{n+1} - f_{n-1}) \\ &= f_n f_{n+1} - f_1 f_0 \\ &= f_n f_{n+1}. \end{aligned}$$

Yours sincerely,

Abbas Rooholamini Gugheri
(10 Shahid Azam Lane
Makki Abad Avenue
Sirjan
Iran)

Dear Editor,

Is π in π ?

In ‘Mathematics in the Classroom’ (Volume 42, Number 3, Page 137), David Benjamin related how he engaged his class with the decimal expansion of π . He raised the question of whether the decimal expansion of π is contained within itself. This is not the case.

If π has decimal expansion $3.1415\ldots$ followed by $31415\ldots$ and so on, this would lead to a repeating decimal expansion, and such numbers are rational. π is an irrational number, and so cannot have its decimal expansion ‘nested’ within itself like this.

However, there are similar questions which would be less easy to resolve. For example, could the digits $31415\ldots$ occur as every *other* digit at some stage in the expansion? (I would guess not.)

Yours sincerely,

Alan Williamson
(Central Sussex College
Harlands Road
Haywards Heath
West Sussex, RH16 1LT
UK)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions received by 1st March will be published in a subsequent issue and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college, or university.

Problems

43.1 *Numerical partial fractions* Given two fractions, it is easy to compute their sum as a single fraction. For example,

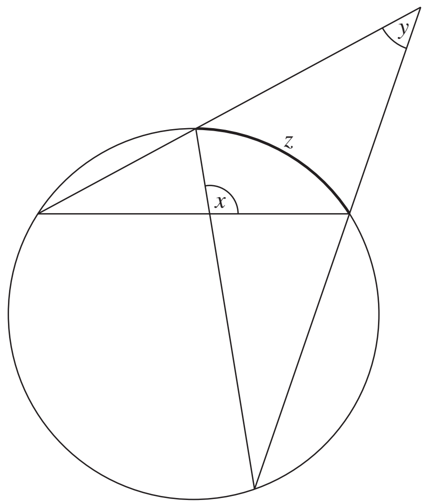
$$\frac{3}{5}+\frac{7}{13}=\frac{(3\times 13)+(7\times 5)}{5\times 13}=\frac{74}{65}.$$

However, the converse problem is not so easy. Find integers a, b, c, d, f such that

$$\frac{325}{1357}=\frac{a}{59}+\frac{b}{23},\qquad \frac{257}{1001}=\frac{c}{7}+\frac{d}{11}+\frac{f}{13}.$$

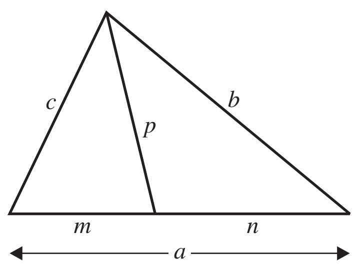
(Submitted by Bob Bertuello, Midsomer Norton, Bath, UK)

43.2 Given a unit circle, angles x, y measured in radians and arc length z as shown, how are x, y, and z related?



(Submitted by Guido Lasters, Tienen, Belgium)

43.3 How are a, b, c, m, n, p related?



(Submitted by Abbas Rooholamini Gugheri, Sirjan, Iran)

43.4 (i) For a fixed positive integer m, determine

$$\sum_{k=1}^{\infty} \frac{1}{k(m+k)}.$$

(ii) For fixed positive integers m, p with $m > p$, determine

$$\sum_{k=1}^{\infty} \frac{1}{(p+k)(p+1+k)\cdots(m+k)}.$$

(Submitted by Ayob Sadeghi Gogheri, Sirjan, Iran)

Solutions to Problems in Volume 42 Number 2

42.5 Which products of numbers in arithmetic progression are the difference of two squares?

Solution

Every odd number is a difference of two squares:

$$2n + 1 = (n + 1)^2 - n^2 \quad (n \in \mathbb{Z}).$$

Every multiple of 4 is a difference of two squares:

$$4n = (n + 1)^2 - (n - 1)^2.$$

If $r, s \in \mathbb{Z}$ are both even, then $r^2 - s^2$ is a multiple of 4; if r, s are both odd, say

$$r = 2m + 1, \qquad s = 2n + 1$$

for some $m, n \in \mathbb{Z}$, then

$$\begin{aligned} r^2 - s^2 &= (2m + 1)^2 - (2n + 1)^2 \\ &= 4(m^2 - n^2 + m - n), \end{aligned}$$

which is a multiple of 4. If one of r, s is even and the other is odd, then $r^2 - s^2$ is odd. Hence, $2 \times$ (an odd number) is not a difference of two squares.

Consider a product of (at least three) integers in arithmetic progression

$$a(a + d)(a + 2d) \cdots .$$

If a is even, then a, $a + 2d$ are both even and this is a multiple of 4 and so is a difference of two squares. If a is odd and d is even, then all the terms are odd so it is a difference of two squares. If a is odd and d is odd, then a is odd, $a + d$ is even, $a + 2d$ is odd, and $a + 3d$ is even. If $a + d$ is a multiple of 4, then it is a difference of two squares. If $a + d$ is not a multiple of 4, then at least four terms in the arithmetic progression are needed to obtain a difference of two squares.

42.6 What are the last six digits of 6249^{6249}?

Solution

The binary representation of 6249 is

$$1100001101001.$$

Now

$$6249^2 \equiv 50001 (\bmod 10^6),$$

so

$$\begin{aligned}
6249^{2^2} &\equiv 100001 (\text{mod } 10^6),\\
6249^{2^3} &\equiv 200001 (\text{mod } 10^6),\\
6249^{2^4} &\equiv 400001 (\text{mod } 10^6),\\
6249^{2^5} &\equiv 800001 (\text{mod } 10^6),\\
6249^{2^6} &\equiv 600001 (\text{mod } 10^6),\\
6249^{2^7} &\equiv 200001 (\text{mod } 10^6),\\
6249^{2^8} &\equiv 400001 (\text{mod } 10^6),\\
6249^{2^9} &\equiv 800001 (\text{mod } 10^6),\\
6249^{2^{10}} &\equiv 600001 (\text{mod } 10^6),\\
6249^{2^{11}} &\equiv 200001 (\text{mod } 10^6),\\
6249^{2^{12}} &\equiv 400001 (\text{mod } 10^6).
\end{aligned}$$

Hence,

$$\begin{aligned}
6249^{6249} &\equiv 6249 \times 200001 \times 800001 \times 600001 \times 200001 \times 400001 (\text{mod } 10^6)\\
&\equiv 6249 \times 200001 (\text{mod } 10^6)\\
&\equiv 806249 (\text{mod } 10^6),
\end{aligned}$$

and the last four digits are 806249.

Alternative solution by Daniel Fretwell, University of Sheffield

We have

$$\begin{aligned}
6249^{6249} &= 6249(6249)^{6248}\\
&\equiv 6249(50001)^{3124} (\text{mod } 10^6)\\
&\equiv 6249(1 + 3124 \times 5 \times 10^4)(\text{mod } 10^6) \quad \text{(by the binomial theorem)}\\
&\equiv 6249 \times 200001 (\text{mod } 10^6)\\
&\equiv 806249 (\text{mod } 10^6).
\end{aligned}$$

42.7 Solve the equation

$$\sqrt{x + 6\sqrt{x} + 8} + \sqrt{x + 3\sqrt{x} + 2} = \sqrt{4x + 18\sqrt{x} + 18}.$$

Solution by Daniel Fretwell, University of Sheffield

Square both sides to give

$$2\sqrt{(x + 6\sqrt{x} + 8)(x + 3\sqrt{x} + 2)} = 2x + 9\sqrt{x} + 8.$$

Square both sides again to give

$$4(x^2 + 9x\sqrt{x} + 28x + 36\sqrt{x} + 16) = 4x^2 + 36x\sqrt{x} + 113x + 144\sqrt{x} + 64,$$

which reduces to $x = 0$. Conversely, $x = 0$ is a solution since

$$\sqrt{8} + \sqrt{2} = 3\sqrt{2} = \sqrt{18}.$$

Hence, $x = 0$ is the only solution.

42.8 Let $\sigma = k + \sqrt{k^2 \pm 1}$, where k is a positive integer. What is $\tan \pi\sigma \mp \tan \pi\sigma^{-1}$?

Solution by Daniel Fretwell, University of Sheffield, and Abbas Rooholamini Gugheri, Sirjan, Iran

Since

$$(k + \sqrt{k^2 \pm 1})(k - \sqrt{k^2 \pm 1}) = k^2 - (k^2 \pm 1) = \mp 1,$$

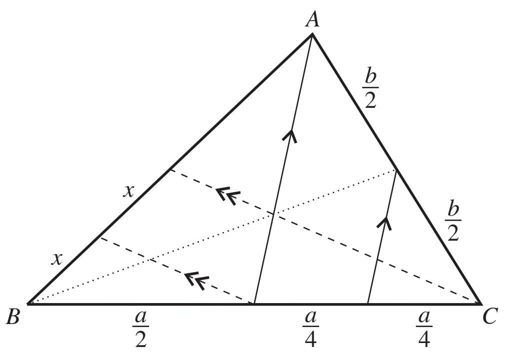
$\sigma^{-1} = \mp(k - \sqrt{k^2 \pm 1})$ and

$$\begin{aligned}\tan \pi\sigma^{-1} &= \tan(\mp\pi(k - \sqrt{k^2 \pm 1})) \\ &= \mp\tan(\pi(k - \sqrt{k^2 \pm 1})) \\ &= \mp\tan(\pi(-k - \sqrt{k^2 \pm 1})) \quad \text{(since tan is of period } \pi) \\ &= \pm\tan(\pi(k + \sqrt{k^2 \pm 1})) \\ &= \pm\tan \pi\sigma.\end{aligned}$$

Hence, $\tan \pi\sigma \mp \tan \pi\sigma^{-1} = 0$.

The medians of a triangle

The medians of a triangle are concurrent and the common point divides each median in the ratio $2 : 1$. But how would you prove it? Here is a diagram to get you going.



Tienen, Belgium **Guido Lasters**

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