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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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SOME SUMS ARE NOT RATIONAL FUNCTIONS OF R, r, AND s

Stanley Rabinowitz

Let R, r, and s denote the circumradius, inradius, and semiperimeter of a triangle with angles A, B, and C. In problem 652 of this journal [2], W.J. Blundon pointed out the well–known formulae

$$\sum \sin A = \frac{s}{R}$$

$$\sum \cos A = \frac{R+r}{R}$$

$$\sum \tan A = \frac{2rs}{s^2 - 4R^2 - 4Rr - r^2}$$

$$\sum \tan \frac{A}{2} = \frac{4R+r}{s}$$

where the sums are cyclic over the angles of the triangle. He asked if there were similar formulae for $\sum \sin A/2$ and $\sum \cos A/2$. All the solutions received were very complicated. Murray Klamkin [3] pointed out that Anders Bager in [1] tacitly implied that there were no known simple $R - r - s$ representations for the following symmetric triangle functions:

$$\begin{array}{cccc} \sum \sin \frac{A}{2}, & \sum \sin \frac{B}{2} \sin \frac{C}{2}, & \sum \csc \frac{A}{2}, & \sum \csc \frac{B}{2} \csc \frac{C}{2}, \\ \sum \cos \frac{A}{2}, & \sum \cos \frac{B}{2} \cos \frac{C}{2}, & \sum \sec \frac{A}{2}, & \sum \sec \frac{B}{2} \sec \frac{C}{2}. \end{array} \tag{1}$$

Klamkin went on to conjecture that these sums cannot be expressed as rational functions of R, r, and s. (A *rational function* is the quotient of two polynomials.) This conjecture is made plausible by the fact that compendiums of such formulae (such as chapter 4 of [4]) do not include values for these particular sums. In this note, we will prove Klamkin's conjecture.

THEOREM. $\sum \sin A/2$ and $\sum \cos A/2$ can not be expressed as rational functions of R, r, and s.

Proof. Consider a triangle ABC with sides $BC = 13$, $CA = 14$, and $AB = 15$. This triangle has area 84, semiperimeter 21, inradius 4, and circumradius 65/8.

From the Law of Cosines, we can easily compute the cosines of the angles, finding

$$\cos A = \frac{3}{5} \qquad\qquad \sin A = \frac{4}{5}$$
$$\cos B = \frac{33}{65} \qquad \text{and} \qquad \sin B = \frac{56}{65}$$
$$\cos C = \frac{5}{13} \qquad\qquad \sin C = \frac{12}{13}\,.$$

From the half–angle formulae, we find that

$$\sin\frac{A}{2} = \frac{1}{\sqrt{5}} \qquad \sin\frac{B}{2} = \frac{4}{\sqrt{65}} \qquad \sin\frac{C}{2} = \frac{2}{\sqrt{13}}$$
$$\cos\frac{A}{2} = \frac{2}{\sqrt{5}} \qquad \cos\frac{B}{2} = \frac{7}{\sqrt{65}} \qquad \cos\frac{C}{2} = \frac{3}{\sqrt{13}}$$
$$\sec\frac{A}{2} = \tfrac{1}{2}\sqrt{5} \qquad \sec\frac{B}{2} = \tfrac{1}{7}\sqrt{65} \qquad \sec\frac{C}{2} = \tfrac{1}{3}\sqrt{13}$$
$$\csc\frac{A}{2} = \sqrt{5} \qquad \csc\frac{B}{2} = \tfrac{1}{4}\sqrt{65} \qquad \csc\frac{C}{2} = \tfrac{1}{2}\sqrt{13}\,.$$

We thus see that

$$\sum \sin\frac{A}{2} = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{65}} + \frac{2}{\sqrt{13}}$$

which is irrational. This shows that $\sum \sin A/2$ cannot be a rational function of R, r, and s, for if it were, then in this particular case, its numerical value would be rational (since R, r, and s are rational in this case), a contradiction. Similarly, $\sum \cos A/2$ cannot be a rational function of R, r, and s, because that would be contradicted by this particular case, in which

$$\sum \cos\frac{A}{2} = \frac{2}{\sqrt{5}} + \frac{7}{\sqrt{65}} + \frac{3}{\sqrt{13}}$$

is also irrational. □

A similar calculation and argument shows that none of the expressions in display (1) can be expressed as rational functions of R, r, and s. In fact, the same argument shows further than none of these expressions can be expressed as rational functions of R, r, s, a, b, c, and K, where a, b, and c are the lengths of the sides of the triangle and K is its area.

In many cases, similar results can be shown using simpler examples. For example, let m_a, m_b, m_c denote the lengths of the medians of a triangle. Using a 3–4–5 right triangle, I showed in [5] that there is no rational function, M, of a, b, and c such that each of m_a, m_b, m_c can be expressed as rational functions of a, b, c, and $\sqrt{M}$.

As an exercise, the reader can prove that sin $x/2$ and cos $x/2$ can not be expressed as rational functions of sin x and cos x. It is well known that tan $x/2$ can be so expressed, namely

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} .$$

References:

[1] Anders Bager, A family of goniometric inequalities, *Publikacije Elektrotehničkog Fakulteta Univerziteta U Beogradu, Serija: Matematika i Fizika* Nos. 338–352 (1971) 5–25.

[2] W.J. Blundon, Problem 652, *Crux Mathematicorum* 7 (1981) 179.

[3] M.S. Klamkin, Comment on Problem 652, *Crux Mathematicorum* 8 (1982) 189.

[4] D.S. Mitrinović, J.E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Boston, 1989.

[5] Stanley Rabinowitz, On the computer solution of symmetric homogeneous triangle inequalities, *Proceedings of the ACM–SIGSAM 1989 International Symposium on Symbolic and Algebraic Computation* (ISSAC '89), 272–286.

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* * *

THE OLYMPIAD CORNER
No. 111
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

Another year has passed and it is time to thank the readership for its continued interest and participation. One of the projects for this year will be to present the results of the challenges thrown out in 1988 to fill in the gaps with solutions to problems posed in the Corner. The response was very good and I'm sure some of you are beginning to wonder what happened to the solutions you sent me. But I also want to continue with solutions to more current problems so as to ensure that our new solvers' efforts are recognized. Beginning next month the solutions will be split in two parts, "archival" and "current" problem solutions. I would particularly like to thank those who have sent in problem sets, solutions, and comments. Among those whose efforts have helped this column appear during the year are Seung–Jin Bang, Francisco Bellot, Len Bos, Curtis Cooper, Nicos Diamantis, George Evagelopoulos, Douglass L. Grant, the late J.T. Groenman, H.N. Gupta, R.K. Guy, Denis Hanson, Walther Janous, Gy. Karoly, Murray S. Klamkin, Andy Liu, Robert Lyness, Stewart Metchette, Dave McDonald, David Monk, John Morvay, Gillian Nonay, Richard Nowakowski, J. Pataki, Bob Prielipp, M.A. Selby, Zun Shan, Bruce Shawyer, Dimitris Vathis, and Edward T.H. Wang.

* * *

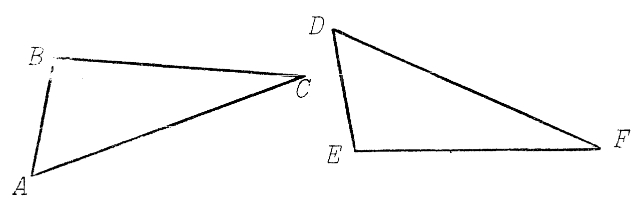
The problem sets that we present this month all come to us from Andy Liu, The University of Alberta, who also translated the Chinese I.M.O. Selection Test questions for us.

SINGAPORE MATHEMATICAL SOCIETY
INTERSCHOOL MATHEMATICAL COMPETITION 1988
Part B (2 hours)

[*Editor's note*: Part A consisted of 10 multiple–choice questions, to be answered in one hour.]

1. Let $f(x)$ be a polynomial of degree n such that $f(k) = \frac{k}{k+1}$ for each $k = 0,1,2,\ldots,n$. Find $f(n+1)$.

2. Suppose ΔABC and ΔDEF in the figure are congruent. Prove that the perpendicular bisectors of AD, BE, and CF intersect at the same point.



3. Find all positive integers n such that P_n is divisible by 5, where $P_n = 1 + 2^n + 3^n + 4^n$. Justify your answer.

4. Prove that for any positive integer n, any set of $n+1$ distinct integers chosen from the integers $1,2,\ldots,2n$ always contains 2 distinct integers such that one of them is a multiple of the other.

5. Find all positive integers x, y, z satisfying the equation

$$5(xy + yz + zx) = 4xyz.$$

*

The next two sets were forwarded to Andy Liu by Professor Zun Shan from Hefei in Anhui province of the People's Republic of China.

FIRST SELECTION TEST OF THE CHINESE I.M.O. TEAM

May 3, 1988 ($4\frac{1}{2}$ hours)

1. What necessary and sufficient conditions must real numbers A, B, C satisfy in order that

$$A(x - y)(x - z) + B(y - z)(y - x) + C(z - x)(z - y)$$

is non–negative for all real numbers x, y and z?

2. Determine all functions f from the rational numbers to the complex numbers such that

(i) $f(x_1 + x_2 + \cdots + x_{1988}) = f(x_1)f(x_2)\cdots f(x_{1988})$

for all rational numbers $x_1,x_2,\cdots,x_{1988}$, and

(ii) $\overline{f(1988)}f(x) = f(1988)\overline{f(x)}$ for all rational numbers x, where $\bar{z}$ denotes the complex conjugate of z.

3. In triangle ABC, angle C is 30°. D is a point on AC and E is a point on BC such that $AD = BE = AB$. Prove that $OI = DE$ and OI is perpendicular to DE, where O and I are respectively the circumcentre and incentre of triangle ABC.

4. Let k be a positive integer. Let $S_k = \{(a,b)\colon a, b = 1,2,\cdots,k\}$. Two elements (a,b) and (c,d) of S_k are said to be indistinguishable if and only if $a - c \equiv 0$ or ±1 mod k and $b - d \equiv 0$ or ±1 mod k. Let r_k denote the maximum number of pairwise indistinguishable elements in S_k.

(i) Determine r_5, with justification.

(ii) Determine r_7, with justification.

(iii) Determine r_k in general. (Justification is not required.)

SECOND SELECTION TEST OF THE CHINESE I.M.O. TEAM

May 4, 1988 ($4\frac{1}{2}$ hours)

1. Define $x_n = 3x_{n-1} + 2$ for all positive integers n. Prove that an integer value can be chosen for x_0 such that 1988 divides x_{100}.

2. $ABCD$ is a trapezoid with AB parallel to DC. M and N are fixed points on AB with M closer to A than N is. P is a variable point on CD. Let DN cut AP at E and CM at F, and let CM cut BP at G. For which point P is the area of $PEFG$ maximized?

3. A polygon in the xy-plane has area greater than n. Prove that it contains points (x_i,y_i), $1 \le i \le n+1$ such that $x_i - x_j$ and $y_i - y_j$ are integers for all i and j.

4. With numbers u and v as input, a machine generates the number $uv + v$ as output. In the first operation, the only numbers that can be used for input are 1, –1 and a fixed number c. In later operations, numbers generated by the machine as output in preceding operations may also be used. Prove that for any polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ with integer coefficients, the machine can generate $f(c)$ as output after a finite number of operations.

* * *

We now continue with solutions for problems posed in the 1988 numbers of the Corner. The first is a solution to a problem from the January number.

1985.3 [1988: 3] *Kürschak Competition.*

Let each vertex of a triangle be reflected across the opposite side. Prove that the area of the triangle determined by the three points of reflection is less than 5 times the area of the original triangle.

Solution by Murray S. Klamkin, Mathematics Department, University of Alberta, Edmonton.

Let F and F' denote the respective areas of the original triangle ABC and the new triangle $A'B'C'$. We extend the problem by finding both upper and lower bounds for F'/F for the case when ABC is non–obtuse, and also for several cases when ABC is obtuse. The five cases to be considered are

(i). ABC is non–obtuse;

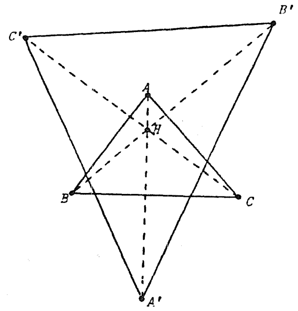
(ii). ABC is obtuse with $A > 90^\circ$, $A - C < 90^\circ$, $A - B < 90^\circ$;

(iii). ABC is obtuse with $A > 90^\circ$, $A - C > 90^\circ$, $A - B > 90^\circ$;

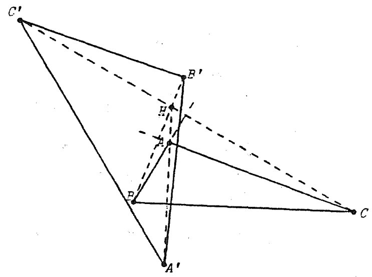
(iv). ABC is obtuse with $120^\circ \ge A > 90^\circ$, $A - C < 90^\circ$, $A - B > 90^\circ$;

(v). ABC is obtuse with $135^\circ > A > 120^\circ$, $A - C < 90^\circ$, $A - B > 90^\circ$.

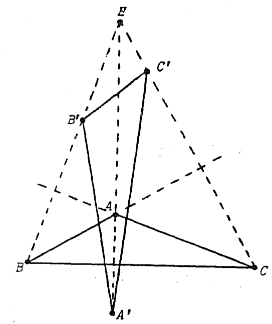
As indicated by the five associated figures, these cases lead to different representations of F'/F. Actually, this difference is one of sign only, but it affects the upper and lower bounds.



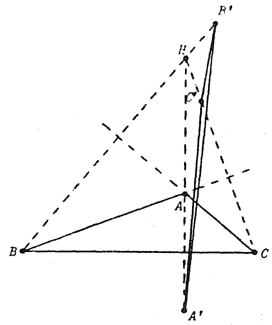
Case (i)



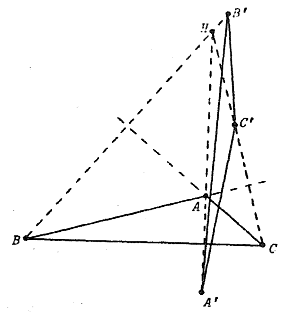
Case (ii)



Case (iii)



Case (iv)



Case (v)

For all cases

$$h_b = c \sin A\,, \qquad h_c = b \sin A\,, \qquad h_a = c \sin B\,,$$
$$BH = 2R \cos B\,, \qquad CH = 2R \cos C\,, \qquad AH = \pm 2R \cos A$$

(according to whether A is non–obtuse or obtuse). Here h_a is the altitude to side a, etc., R is the circumradius, and H is the orthocenter. Also,

$$F = 2R^2 \sum \cos B \cos C \sin A. \tag{1}$$

The summations here and subsequently are symmetric over the elements of ABC. We also write $[X]$ for the area of figure X.

Case (i). $\quad F' = [B'HC'] + [C'HA'] + [A'HB']$

$$= \sum \frac{HB' \cdot HC'}{2} \sin\angle B'HC'$$
$$= 2 \sum (h_b - R \cos B)(h_c - R \cos C)\sin A\,. \tag{2}$$

Since

$$\sum h_b h_c \sin A = \sum bc \sin^3 A = 2F \sum \sin^2 A$$

and

$$\begin{aligned} 2R \sum (h_c \cos B + h_b \cos C)\sin A &= 2R \sum (\sin B \cos C + \sin C \cos B) h_a \\ &= 2R \sum h_a \sin A \\ &= \sum a h_a = 6F\,, \end{aligned}$$

we have from (1) and (2)

$$\frac{F'}{F} = 4 \sum \sin^2 A - 5 .$$

It is a known result (see for example [1]) that for non–obtuse triangles

$$2 \leq \sum \sin^2 A \leq \frac{9}{4} .$$

The upper bound is obtained only for the equilateral triangle and the lower bound for right triangles. Thus for the non–obtuse case,

$$3 \leq F'/F \leq 4 .$$

Case (ii). For this case we have

$$2h_b > 2R \cos B \quad \text{and} \quad 2h_c > 2R \cos C .$$

To see this note that

$$\begin{aligned} 2h_b &= 2c \sin A = 4R \sin C \sin A \\ &= 2R \cos(A - C) - 2R \cos(A + C) \\ &> -2R \cos(A + C) = 2R \cos B \end{aligned}$$

since $\cos(A - C) > 0$. Similarly $2h_c > 2R \cos C$ follows from $\cos(A - B) > 0$.

Proceeding as before,

$$\frac{F'}{F} = 4 \sum \sin^2 A - 5.$$

Now $A > 90^\circ$ and since $2A - B - C < 180^\circ$, A is also less than 120°. Here, due to restrictions on the angles,

$$0 < F'/F < 3 .$$

The upper bound of 3, for example, corresponds to $\sum \sin^2 A < 2$. To see the latter inequality, we keep A fixed and minimize

$$\cos 2B + \cos 2C = 2 \cos(\pi - A)\cos(C - B).$$

This gives $B = 0$ and $C = \pi - A$. These bounds are best possible since we can get arbitrarily close to them by choosing triangles with angles $(120^\circ - 2\epsilon, 30^\circ + \epsilon, 30^\circ + \epsilon)$ for the lower bound and angles $(90^\circ + \epsilon, 2\epsilon, 90^\circ - 3\epsilon)$ for the upper bound. Note that for the case where the angles are $(120^\circ, 30^\circ, 30^\circ)$, $F' = 0$ since B', C', and H coincide.

Case (iii). Here $A - C > 90^\circ$, $B - C > 90^\circ$ so that $A > 120^\circ$ and

$$F' = [C'HA'] + [A'HB'] - [B'HC'] .$$

Proceeding as before we get the same expression except for a change of sign, i.e.

$$\frac{F'}{F} = 5 - 4 \sum \sin^2 A .$$

Hence

$$0 < F'/F < 5 .$$

Again these bounds are best possible since we can get arbitrarily close to them by choosing triangles with angles $(120^\circ + 2\epsilon, 30^\circ - \epsilon, 30^\circ - \epsilon)$ for the lower bound and angles $(180^\circ - 2\epsilon, \epsilon, \epsilon)$ for the upper bound.

Case (iv). In this case we have $A - C < 90^\circ$, $A - B > 90^\circ$ and

$$F' = [A'HB'] - [B'HC'] - [C'HA'].$$

As before this leads to the same expression and bounds for F'/F as in case (*ii*). To get arbitrarily close to the bounds we use triangles of the slightly different angles $(120^\circ, 30^\circ - \epsilon, 30^\circ + \epsilon)$ and $(90^\circ + 2\epsilon, \epsilon, 90^\circ - 3\epsilon)$ (to satisfy the angle hypothesis).

Case (v). In this case $A - B > 90^\circ > A - C$, and $3A/2 > 180^\circ$ so that

$$F' = [B'HC'] + [C'HA'] - [A'HB'].$$

This leads to the same expression as in case (*iii*), i.e.

$$\frac{F'}{F} = 5 - 4 \sum \sin^2 A.$$

We can get arbitrarily close to the lower bound $F'/F > 0$ by choosing angles $(120^\circ + \epsilon, 30^\circ - 3\epsilon, 30^\circ + 2\epsilon)$. The upper bound here is 1 and is obtained for angles $(135^\circ - \epsilon, \epsilon, 45^\circ)$. To see this, holding A fixed we want to maximize $\cos 2B + \cos 2C$. Here we can take

$$C = 90^\circ - A/2 + x \quad \text{and} \quad B = 90^\circ - A/2 - x$$

where

$$90^\circ - A/2 > x > 3A/2 - 180^\circ.$$

Since

$$\cos 2B + \cos 2C = 2 \cos(180^\circ - A)\cos 2x ,$$

x must be as small as possible, i.e. $x = 3A/2 - 180^\circ$. (Subsequently we will add an ϵ to it.) Then $C = A - 90^\circ$ and $B = 270^\circ - 2A$. Then

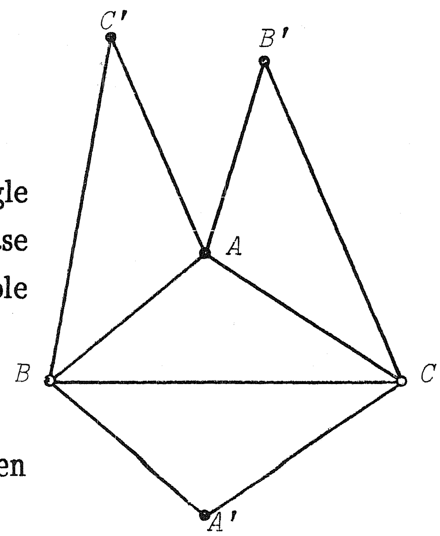
$$\sum \sin^2 A = \sin^2 A + \cos^2 A + \cos^2 2A$$

is a minimum for $A = 135^\circ$.

In summary, for all triangles

$$0 \leq F'/F < 5.$$

Another way of proceeding is to reflect the triangle across each side as in the figure. Although the case corresponding to B, $C \leq 60^\circ$, $A \leq 120^\circ$ is rather simple and leads easily to



$$\frac{F'}{F} = 4 \sum \sin^2 A - 5,$$

there are different, less simple, cases to be considered when $A \geq 120^\circ$.

Reference:

[1] O. Bottema et al, *Geometric Inequalities*, Wolters–Noordhoff, Groningen, 1969, p. 18, #2.3.

*

The second is a solution to a problem from the February number.

9. [1988: 37] *10th Austrian–Polish Mathematics Competition.*

Consider the set M of all points in the plane whose coordinates (x,y) are both whole numbers that satisfy $1 \le x \le 12$, $1 \le y \le 13$.

(i) Show that every subset of M containing at least 49 points must contain the four vertices of a rectangle having its sides parallel to the coordinate axes.

(ii) Construct a counterexample to (i) if the subset is allowed to consist of only 48 elements.

Solution by Curtis Cooper, Central Missouri State University.

(i) Let $S \subset M$ be such that $|S| = 49$. Let $y_i = |\{x: (x,i) \in S\}|$ for $i = 1,2,\ldots,13$. Let

$$T = \{(t_1,t_2,\cdots,t_{13}):\ t_i \in \mathbb{Z}\ ,\quad 0 \le t_i \le 12 \text{ for } i = 1,2,\ldots,13 \quad\text{and}\quad \sum_{i=1}^{13} t_i = 49\}.$$

It follows that $(y_1,y_2,\cdots,y_{13}) \in T$. Finally, let

$$P_i = \{\{r,s\}:\ r \ne s \text{ and } (r,i),(s,i) \in S\}$$

for $i = 1,2,\ldots,13$. Now

$$\sum_{i=1}^{13} |P_i| = \sum_{i=1}^{13} \binom{y_i}{2} \ge \min_T \sum_{i=1}^{13} \binom{t_i}{2} = 3\cdot\binom{3}{2} + 10\cdot\binom{4}{2} = 69$$

(see the claim below). However, there are $\binom{12}{2} = 66$ ways to pick integers r and s such that $1 \le r,\ s \le 12$ and $r \ne s$. Hence, there exist $j \ne k$ such that

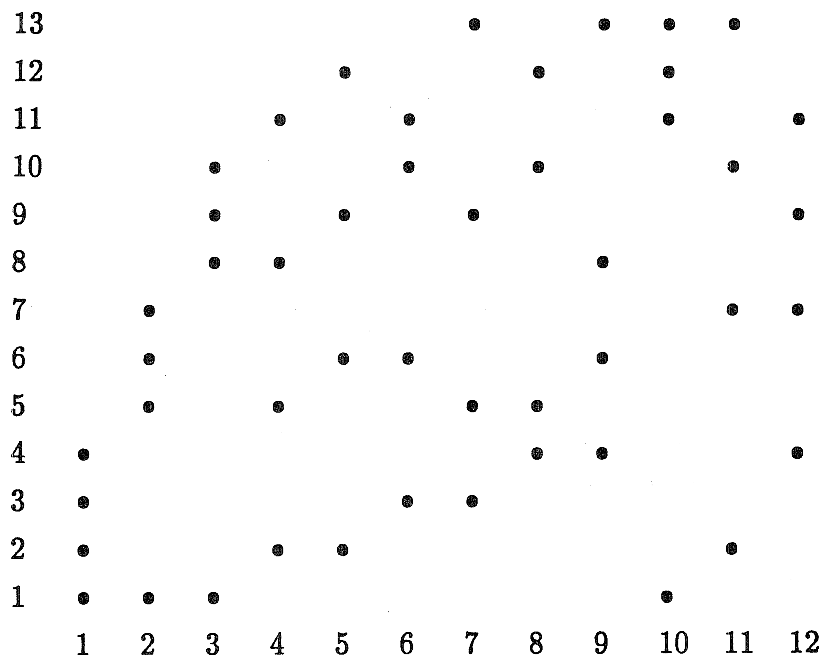
$$\{a,b\} \in P_j \qquad\text{and}\qquad \{a,b\} \in P_k\ .$$

Thus (a,j), (b,j), (a,k), $(b,k) \in S$ form a rectangle having its sides parallel to the coordinate axes, establishing (i).

Claim. $$\min_T \sum_{i=1}^{13} \binom{t_i}{2} = 3\cdot\binom{3}{2} + 10\cdot\binom{4}{2}\ .$$

To see the claim, notice first that $\binom{0}{2} = \binom{1}{2} = 0$ and $\binom{k+1}{2} - \binom{k}{2} = k$ for $k \geq 0$. Now assume that $(z_1,\cdots,z_{13})$ achieves the minimum. In this case each $z_i \leq 4$, for if say $z_i \geq 5$ then, as the remaining twelve terms sum to at most 44, one must have $z_j \leq 3$ for some j. Decreasing z_i by 1 and augmenting z_j by 1 would decrease the sum. Next we argue that in fact $z_i \geq 3$ for all i. Otherwise suppose some $z_i \leq 2$. The remaining terms must sum to at least 47, but with the observation above this gives $z_i = 2$, eleven 4's and one 3, or $z_i = 1$ and twelve 4's. The values of the corresponding expressions are 70 and 72, respectively, which are not the minimum. Consequently $3 \leq z_i \leq 4$ for $i = 1,2,\ldots,13$. The only possibility is to use three 3's and ten 4's.

(ii)



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The next solutions that we give are to problems posed in the March 1988 number of the Corner. The first are for the three problems from the Entrance Examination of the Polytechnic of Athens (1962).

1. [1988: 65] *Polytechnic of Athens Entrance Exam.*

Prove that

$$\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) < x^2 < \left(x - \frac{23}{48}\right)\left(x + \frac{25}{48}\right)$$

for all real $x \geq 6$. Then prove that

$$\frac{2}{11} > \sum_{i=6}^{\infty} \frac{1}{i^2} > \frac{48}{265}$$

and calculate $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2}$ accurate to three decimal places.

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Noting that

$$\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) < x^2 < \left(x - \frac{23}{48}\right)\left(x + \frac{25}{48}\right)$$

is equivalent to

$$x^2 - \frac{1}{4} < x^2 < x^2 + \frac{x}{24} - \frac{575}{48^2},$$

the first inequality is evident, and for $x \geq 6$ the second follows since

$$\frac{x}{24} - \frac{575}{48^2} \geq \frac{1}{4} - \frac{575}{48^2} > 0.$$

Taking reciprocals

$$\frac{1}{(x - 1/2)(x + 1/2)} > \frac{1}{x^2} > \frac{1}{(x - 23/48)(x + 25/48)} .$$

Equivalently

$$\frac{1}{x - 1/2} - \frac{1}{x + 1/2} > \frac{1}{x^2} > \frac{1}{x - 23/48} - \frac{1}{x + 25/48} .$$

Now, summing all three quantities over all $x \geq 6$, the outside series telescope with

$$\sum_{i=6}^{\infty} \left(\frac{1}{i - 1/2} - \frac{1}{i + 1/2}\right) = \lim_{N\to\infty} \left(\frac{2}{11} - \frac{1}{N + 1/2}\right) = \frac{2}{11}$$

and

$$\sum_{i=6}^{\infty} \left(\frac{1}{i - 23/48} - \frac{1}{i + 25/48}\right) = \lim_{N\to\infty} \left(\frac{48}{265} - \frac{1}{N + 25/48}\right) = \frac{48}{265} .$$

As each inequality is strict,

$$\frac{2}{11} > \sum_{i=6}^{\infty} \frac{1}{i^2} > \frac{48}{265} .$$

Now we have

$$\sum_{i=1}^{5} \frac{1}{i^2} \simeq 1.4636 , \qquad \frac{2}{11} \simeq 0.1818 \quad \text{and} \quad \frac{48}{265} \simeq 0.1811 .$$

This gives $1.6447 < \sum_{i=1}^{\infty} \frac{1}{i^2} < 1.6454$. Therefore, to three decimal places $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is 1.645.

The well known exact value of $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is $\frac{\pi^2}{6}$ which agrees with this approximation.

2. [1988: 65] *Polytechnic of Athens Entrance Exam.*

Determine those odd natural numbers n such that the common roots of

$$f(x) = (x + 1)^n - x^n - 1$$

and

$$h(x) = (x + 1)^{n-1} - x^{n-1}$$

contain the roots of $x^2 + x + 1$.

Solution by George Evagelopoulos, Law student, Athens, Greece.

The roots of the polynomial $x^2 + x + 1$ are the complex cube roots of 1, namely

$$\omega_1 = \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \omega_2 = \frac{-1 - i\sqrt{3}}{2}$$

where

$$\omega_1^3 = \omega_2^3 = 1$$
$$\omega_1 = \omega_2^2 , \quad \omega_2 = \omega_1^2$$
$$\omega_1 + \omega_2 = -1 , \quad \omega_1\omega_2 = 1 .$$

In order to find which odd natural numbers n have the property that ω_1 and ω_2 are roots of both $f(x)$ and $h(x)$, first observe that

$$f(\omega_1) = (\omega_1 + 1)^n - \omega_1^n - 1 = (-\omega_1^2)^n - \omega_1^n - 1 = (-1)^n\omega_1^{2n} - \omega_1^n - 1$$

and

$$h(\omega_1) = (\omega_1 + 1)^{n-1} - \omega_1^{n-1} = (-\omega_1^2)^{n-1} - \omega_1^{n-1} = (-1)^{n-1}\omega_1^{2n-2} - \omega_1^{n-1}.$$

Next distinguish the following cases for values of n:

(i) $n = 6k + 1$, $k \in \mathbb{N}$. Then

$$f(\omega_1) = -\omega_1^{12k+2} - \omega_1^{6k+1} - 1 = -\omega_1^2 - \omega_1 - 1 = -(\omega_1^2 + \omega_1 + 1) = 0,$$
$$h(\omega_1) = \omega_1^{12k} - \omega_1^{6k} = 1 - 1 = 0.$$

(ii) $n = 6k + 3$, $k \in \mathbb{N}$. Then

$$f(\omega_1) = -\omega_1^{12k+6} - \omega_1^{6k+3} - 1 = -1 - 1 - 1 = -3 \neq 0,$$
$$h(\omega_1) = \omega_1^{12k+4} - \omega_1^{6k+2} = \omega_1 - \omega_1^2 \neq 0 .$$

(iii) $n = 6k + 5$, $k \in \mathbb{N}$. Then

$$f(\omega_1) = -\omega_1^{12k+10} - \omega_1^{6k+5} - 1 = -\omega_1 - \omega_1^2 - 1 = 0,$$

$$h(\omega_1) = \omega_1^{12k+8} - \omega_1^{6k+4} = \omega_1^2 - \omega_1 \neq 0.$$

We have the same results for ω_2, and consequently the roots of $x^2 + x + 1$ are also roots of both $f(x)$ and $h(x)$ if and only if $n = 6k + 1$ for some integer k.

3. [1988: 65] *Polytechnic of Athens Entrance Exam.*

Prove that the polynomial

$$f_n(x) = x \sin a - x \sin(na) + \sin(n-1)a$$

is exactly divisible by

$$h(x) = x^2 - 2x \cos a + 1$$

where a is a real number and n is a natural number ≥ 2.

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

There is an obvious misprint in the statement of the problem: the function $f_n(x)$ should be $x^n \sin a - x \sin(na) + \sin(n-1)a$. By the quadratic formula the roots of $h(x)$ are easily found to be $\cos a \pm i \sin a$. Thus to show $h(x)$ divides $f_n(x)$ we must show that $f_n(\cos a \pm i \sin a) = 0$. Now

$$f_n(\cos a + i \sin a) = (\cos(na) + i \sin(na))\sin a - (\cos a + i \sin a)\sin(na)$$
$$+ (\sin(na)\cos a - \cos(na)\sin a) = 0 .$$

A similar calculation gives

$$f(\cos a - i \sin a) = 0.$$

[*Editor's note*: E.T.H. Wang continues with more on exact division.]

To see that $h(x)$ divides $f_n(x)$ *exactly* consider

$$f_n'(x) = nx^{n-1}\sin a - \sin(na).$$

Now

$$f_n'(\cos a \pm i \sin a) = n[\cos(n-1)a \pm i \sin(n-1)a]\sin a - \sin(na) \neq 0$$

for $n > 1$, whence $h(x)$ divides $f_n(x)$ but $(h(x))^2$ does not.

Since $f_1(x) = 0$, all powers of $h(x)$ divide it.

*

The next solutions are from the 1986 Spanish Olympiad in Valladolid, Spain.

2. [1988: 67] *23rd Spanish Mathematical Olympiad.*

Find all x, y, z (real numbers) such that

$$xyz = \frac{x^3 + y^3 + z^3}{3} .$$

Solutions by Seung-Jin Bang, Seoul, Korea; George Evagelopoulos, Law student, Athens, Greece; Bob Prielipp, University of Wisconsin, Oshkosh; M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario; and *by Edward T.H. Wang, Wilfrid Laurier University, Waterloo.*

From

$$x^3 + y^3 + z^3 - 3xyz = \frac{(x + y + z)}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] ,$$

the given equality holds if and only if $x + y + z = 0$ or $x = y = z$.

[*Editor's note*: At the bottom of the page of Bang's solution was a solution to Kürschak Competition problem 1983.1 [1988: 2] which should have been mentioned among the alternate solutions found in [1989: 229]. My apologies.]

4. [1988: 67] *23rd Spanish Mathematical Olympiad.*

Consider the equations

$$x^2 + bx + c = 0 \qquad \text{and} \qquad x^2 + b'x + c' = 0$$

where b, c, b' and c' are integers such that

$$(b - b')^2 + (c - c')^2 > 0.$$

Show that if the equations have a common root, then the second roots are distinct integers.

Solutions by Bob Prielipp, University of Wisconsin, Oshkosh, and by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Suppose that r is a common root, and let s and s' denote the other roots of $x^2 + bx + c = 0$ and $x^2 + b'x + c' = 0$, respectively. It follows that

$$r + s = -b,\ r + s' = -b',\ rs = c \text{ and } rs' = c'.$$

If $s = s'$ then $b = b'$ and $c = c'$ contradicting $(b - b')^2 + (c - c')^2 > 0$. Thus $s \neq s'$. Thus $b \neq b'$. Because r is a common root,

$$r^2 + br + c = 0 \qquad \text{and} \qquad r^2 + b'r + c' = 0$$

so

$$(b - b')r = c' - c ,$$

and

$$r = \frac{c' - c}{b - b'}$$

is rational. As $x^2 + bx + c$ is a monic polynomial r must be an integer. It follows that $s = -b - r$ and $s' = -b' - r$ are both integers as well.

6. [1988: 67] *23rd Spanish Mathematical Olympiad.*

Let $a_1, a_2, \ldots, a_n$ be n distinct real numbers. Calculate those points x on the real line which minimize $\sum_{i=1}^{n} |x - a_i|$, the sum of the distances from x to the a_i, in the cases $n = 3$ and $n = 4$.

Comment by Seung–Jin Bang, Seoul, Korea.

This problem is a special known case of a problem I posed on p. 95, vol. 16, 1987 of the *Mathematical Chronicle.*

Solution by M.A. Selby, Department of Mathematics and Statistics, The University of Windsor, Ontario.

Assume $a_i < a_j$ for $i < j$. Define

$$f(x) = \sum_{i=1}^{n} |x - a_i|.$$

Then $f(x)$ is continuous on $(-\infty,\infty)$ and differentiable everywhere except at $x = a_i$. Clearly $f(x)$ has an absolute minimum since $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. This minimum occurs on $[a_1,a_n]$. The only critical points are the a_i, and hence the minimum must occur at one of them (or an end point).

For $n = 3$,

$$f(a_1) = |a_2 - a_1| + |a_3 - a_1| = a_2 - a_1 + a_3 - a_1 ,$$
$$f(a_2) = a_2 - a_1 + a_3 - a_2 ,$$
$$f(a_3) = a_3 - a_1 + a_3 - a_2 .$$

Now

$$a_2 - a_1 + a_3 - a_1 > a_2 - a_1 + a_3 - a_2$$

and

$$a_3 - a_1 + a_3 - a_2 > a_2 - a_1 + a_3 - a_2 .$$

Thus the minimum occurs at $x = a_2$ and $f(a_2) = a_3 - a_1$.

For $n = 4$,

$$f(a_1) = a_2 - a_1 + a_3 - a_1 + a_4 - a_1 ,$$
$$f(a_2) = a_2 - a_1 + a_3 - a_2 + a_4 - a_2 ,$$
$$f(a_3) = a_3 - a_1 + a_3 - a_2 + a_4 - a_3 ,$$
$$f(a_4) = a_4 - a_1 + a_4 - a_2 + a_4 - a_3 .$$

Now

$$f(a_1) > f(a_2) = a_4 - a_1 + a_3 - a_2$$

and

$$f(a_4) > f(a_3) = a_4 - a_1 + a_3 - a_2 .$$

Hence the minimum occurs at $x = a_2$ or $x = a_3$. Actually $f(x) = a_4 - a_1 + a_3 - a_2$ for $a_2 \le x \le a_3$.

For $n = 3$,

$$f(x) = \begin{cases} a_2 - a_1 + a_3 - x , & a_1 \le x \le a_2 , \\ x - a_1 + a_3 - a_2 , & a_2 \le x \le a_3 , \end{cases}$$

and for $n = 4$,

$$f(x) = \begin{cases} a_2 - a_1 + a_3 + a_4 - 2x\ , & a_1 \le x \le a_2\ , \\ a_4 - a_1 + a_3 - a_2\ , & a_2 \le x \le a_3\ , \\ 2x - a_1 - a_2 - a_3 + a_4\ , & a_3 \le x \le a_4\ . \end{cases}$$

We can now conclude that for $n = 3$ the minimum occurs at $x = a_2$, while for $n = 4$ it occurs on the interval $a_2 \le x \le a_3$.

7. [1988: 68] *23rd Spanish Mathematical Olympiad.*

In a triangle ABC with opposite sides a, b, c respectively, show that if
$$a + b = (a \tan A + b \tan B)\tan(C/2)$$
then the triangle is isosceles.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh.

Let ABC be a triangle such that neither angle A nor angle B is a right angle, and let R be the circumradius. Then we have the following equivalent equalities:

$$a + b = (a \tan A + b \tan B)\tan(C/2)\ ,$$

$$\tan\left(\frac{A+B}{2}\right) = \frac{a}{a+b}\cdot\tan A + \frac{b}{a+b}\cdot\tan B\ ,$$

$$\begin{aligned}\frac{\sin A + \sin B}{\cos A + \cos B} &= \frac{2R\sin A\tan A}{2R\sin A + 2R\sin B} + \frac{2R\sin B\tan B}{2R\sin A + 2R\sin B} \\ &= \frac{\sin A\tan A + \sin B\tan B}{\sin A + \sin B}\ ,\end{aligned}$$

$$\begin{aligned}\sin^2 A + 2\sin A\sin B + \sin^2 B &= \sin^2 A + \cos A\sin B\tan B \\ &\quad + \cos B\sin A\tan A + \sin^2 B\ ,\end{aligned}$$

$$2\sin A\sin B = \frac{\cos A\sin^2 B}{\cos B} + \frac{\cos B\sin^2 A}{\cos A}\ ,$$

$$\sin^2 A\cos^2 B - 2\sin A\cos B\cos A\sin B + \cos^2 A\sin^2 B = 0\ ,$$

$$\sin^2(A - B) = 0\ ,$$

and thus $A = B$. It follows that triangle ABC is isosceles.

[*Editor's note*: This problem was also solved by George Evagelopoulos, Law student, Athens, Greece, by a somewhat different sequence of identities.]

8. [1988: 68] *23rd Spanish Mathematical Olympiad.*

In a right triangle ABC, with centroid G, consider the triangles AGB, GBC, CGA. If the *sides* of ABC are all natural numbers, show that the *areas* of AGB, BGC, CGA are even natural numbers.

Solution by Bob Prielipp, University of Wisconsin, Oshkosh.

By hypothesis triangle ABC is a Pythagorean triangle. It suffices to establish the required result when triangle ABC is a primitive Pythagorean triangle. In the remainder we assume this is the case. Let a and b be the lengths of the legs of triangle ABC where b is an even positive integer, and let c be the length of the hypotenuse. Then there are positive integers r and s, $r > s$, such that $a = r^2 - s^2$, and $b = 2rs$, where r and s are of opposite parity and are relatively prime. [See for example, Theorem 5.1, p. 139 of *An Introduction to the Theory of Numbers*, 4th Edition, by Niven and Zuckerman, John Wiley & Sons, New York, 1980.]

Lemma. The area of triangle ABC is divisible by 6.

Proof. Because r and s are of opposite parity, 4 divides b. If 3 divides r or 3 divides s, then 3 divides b. Otherwise $r^2 \equiv 1 \bmod 3$ and $s^2 \equiv 1 \bmod 3$, and so $r^2 \equiv s^2 \bmod 3$. Thus 3 divides $r^2 - s^2$, and so 3 divides a. It now follows that 12 divides ab, giving the lemma. □

It is known that triangles AGB, BGC, and CGA all have the same area, L, say [see Theorem 23, p. 47 of *Modern College Geometry*, by Davis, Addison–Wesley, Reading, MA, 1949]. Then $3L = K$, the area of triangle ABC. But as 6 divides K, we have that L is a positive even integer.

* * *

Olympiad Season is fast approaching. Remember to send me your national and regional contests!

* * *

P R O B L E M S

Problem proposals and solutions should be sent to the editor, B. Sands, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada, T2N 1N4. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **August 1, 1990,** *although solutions received after that date will also be considered until the time when a solution is published.*

1501. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Two circles K and K_1 touch each other externally. The equilateral triangle ABC is inscribed in K, and points A_1, B_1, C_1 lie on K_1 such that AA_1, BB_1, CC_1 are tangent to K_1. Prove that one of the lengths $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ equals the sum of the other two. (The case when the circles are internally tangent was a problem of Florow in *Praxis der Mathematik* 13, Heft 12, page 327.)

1502. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

AB is a chord, not a diameter, of a circle with centre O. The smaller arc AB is divided into three equal arcs AC, CD, DB. Chord AB is also divided into three equal segments AC', $C'D'$, $D'B$. Let CC' and DD' intersect in P. Show that $\angle APB = \frac{1}{3}\angle AOB$.

1503. *Proposed by M.S. Klamkin, University of Alberta.*

Prove that

$$1 + 2\cos(B+C)\cos(C+A)\cos(A+B) \geq \cos^2(B+C) + \cos^2(C+A) + \cos^2(A+B),$$

where A, B, C are nonnegative and $A + B + C \leq \pi$.

1504. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $A_1A_2\cdots A_n$ be a circumscribable n-gon with incircle of radius 1, and let $F_1,F_2,\dots,F_n$ be the areas of the n corner regions inside the n-gon and outside the incircle. Show that

$$\frac{1}{F_1} + \cdots + \frac{1}{F_n} \geq \frac{n^2}{n\tan(\pi/n) - \pi}\,.$$

Equality holds for the regular n-gon.

1505. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $x_1 = 1$ and

$$x_{n+1} = \frac{1}{x_n}\left(\sqrt{1 + x_n^2} - 1\right)\,.$$

Show that the sequence $(2^n x_n)$ converges and find its limit. $\to \frac{\pi}{2}$

1506. *Proposed by Jordi Dou, Barcelona, Spain.*

Let A and P be points on a circle Γ. Let l be a fixed line through A but not through P, and let x be a variable line through P which cuts l at L_x and Γ again at G_x. Find the locus of the circumcentre of ΔAL_xG_x.

1507. *Proposed by Nicos D. Diamantis, student, University of Patras, Greece.*

Find a real root of

$$y^5 - 10y^3 + 20y - 12 = 0.$$

1508. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $a \le b < c$ be the lengths of the sides of a right triangle. Find the largest constant K such that

$$a^2(b + c) + b^2(c + a) + c^2(a + b) \ge Kabc$$

holds for all right triangles and determine when equality holds. It is known that the inequality holds when $K = 6$ (problem 351 of the *College Math. Journal*; solution on p. 259 of Volume 20, 1989).

1509. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*

Professor Chalkdust teaches two sections of a mathematics course, with the same material taught in both sections. Section 1 runs on Mondays, Wednesdays, and Fridays for 1 hour each day, and Section 2 runs on Tuesdays and Thursdays for 1.5 hours each day. Normally Professor Chalkdust covers one unit of material per hour, but if she is teaching some material for the second time she teaches twice as fast. The course began on a Monday. In the long run (i.e. after N weeks as $N \to \infty$) will one section be taught more material than the other? If so, which one, and how much more?

1510*. *Proposed by Jack Garfunkel, Flushing, New York.*

P is any point inside a triangle ABC. Lines PA, PB, PC are drawn and angles PAC, PBA, PCB are denoted by α, β, γ respectively. Prove or disprove that

$$\cot \alpha + \cot \beta + \cot \gamma \ge \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2},$$

with equality when P is the incenter of ΔABC.

* * *

SOLUTIONS

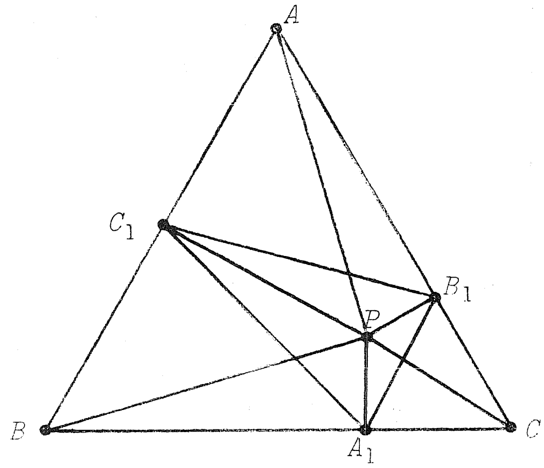
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1385. [1988: 269] *Proposed by Murray S. Klamkin, University of Alberta.*

Show that the sides of the pedal triangle of any interior point P of an equilateral triangle T are proportional to the distances from P to the corresponding vertices of T.

Solution by Wilson da Costa Areias, Rio de Janeiro, Brazil.

AC_1PB_1, BA_1PC_1 and CA_1PB_1 are inscribed quadrilaterals in the circles whose diameters are respectively PA, PB and PC. Using the law of sines in ΔAB_1C_1, we have



$$\frac{B_1C_1}{\sin\, 60^\circ} = \frac{B_1A}{\sin\, \angle AC_1B_1} = \frac{B_1A}{\sin\, \angle APB_1} = PA\ ,$$

so

$$\frac{B_1C_1}{PA} = \frac{\sqrt{3}}{2}\ .$$

Similarly

$$\frac{A_1C_1}{PB} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \frac{A_1B_1}{PC} = \frac{\sqrt{3}}{2}\ ,$$

therefore

$$\frac{B_1C_1}{PA} = \frac{A_1C_1}{PB} = \frac{A_1B_1}{PC}\ .$$

Also solved by SEUNG–JIN BANG, Seoul, Republic of Korea; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Several readers, including the proposer, pointed out that the formulae

$$B_1C_1 = PA\ \sin\, A\ , \quad etc.$$

for the sides of a pedal triangle are known (e.g. Theorem 190 *p.* 136 *of R.A. Johnson,* Advanced Euclidean Geometry), *and that the result follows easily, as above. The proposer's original solution was more involved, though.*

* * *

<u>1386</u>. [1988: 269] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2 \cdots A_n$ be a polygon inscribed in a circle and containing the centre of the circle. Prove that

$$n - 2 + \frac{4}{\pi} < \sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} \le \frac{n^2}{\pi}\sin\frac{\pi}{n}\ ,$$

where a_i is the side A_iA_{i+1} and $\hat{a}_i$ is the arc A_iA_{i+1}.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

Let

$$2\varphi_i = \angle A_i O A_{i+1}\ , \qquad i = 1,\ldots,n \qquad (A_{n+1} = A_1)\ ,$$

where O is the center of the circle. Then, with R the radius,

$$a_i = 2R\sin\varphi_i\ , \qquad \hat{a}_i = 2R\varphi_i\ ,$$

whence

$$\frac{a_i}{\hat{a}_i} = \frac{\sin\varphi_i}{\varphi_i}\ , \qquad i = 1,\ldots,n\ .$$

Now it is either known or not hard to check that

$$f(\varphi) = \frac{\sin\varphi}{\varphi}\ , \qquad 0 < \varphi \le \frac{\pi}{2}\ ,$$

is concave. (Indeed,

$$f'(\varphi) = \frac{\varphi\cos\varphi - \sin\varphi}{\varphi^2}\ ,$$

$$f''(\varphi) = \frac{-\varphi^2\sin\varphi - 2\varphi\cos\varphi + 2\sin\varphi}{\varphi^3} =: \frac{d(\varphi)}{\varphi^3}\ ,$$

and from

$$d'(\varphi) = -\varphi^2\cos\varphi \le 0$$

and $d(0) = 0$ we infer $f''(\varphi) \le 0$ for $0 < \varphi \le \pi/2$.) Thus

$$\sum_{i=1}^{n}\frac{a_i}{\hat{a}_i} = \sum_{i=1}^{n} f(\varphi_i) \le nf\left(\sum_{i=1}^{n}\frac{\varphi_i}{n}\right) = nf\left(\frac{\pi}{n}\right) = \frac{n^2}{\pi}\sin\frac{\pi}{n}\ .$$

Since the polygon contains O, all φ_i's are $\le \pi/2$. This and the concavity of f yield (cf. [1], p.22)

$$\sum_{i=1}^{n} f(\varphi_i) > f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + (n-2)f(0) = \frac{4}{\pi} + n - 2\ .$$

[*Editor's note*: Other solvers noted that

$$(\varphi_1,\varphi_2,\cdots,\varphi_n) \prec \left(\frac{\pi}{2},\frac{\pi}{2},0,\cdots,0\right)$$

and applied the majorization inequality to obtain this last inequality.]

Reference:

[1] D.S. Mitrinović, *Analytic Inequalities*, Springer–Verlag, 1970.

Also solved (in the same way) by MURRAY S. KLAMKIN, University of Alberta; MARCIN E. KUCZMA, Warszawa, Poland; COLIN SPRINGER, student, University of Waterloo; and the proposer.

The given bounds are best possible, as is evident from the above proof. The proposer conjectures that if the inscribed polygon is not required to contain the centre, *then the best bounds are*

$$n - 1 < \sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} \le \frac{n^2}{\pi}\sin\frac{\pi}{n} .$$

Any comments?

* * *

1387. [1988: 269] *Proposed by Ravi Vakil, student, University of Toronto.*

Show that for all positive integers d, a, n such that $3 \le d \le 2^{n+1}$, d does not divide into $a^{2^n} + 1$.

Solution by Sverrir Thorvaldsson, student, University of Iceland, Reykjavik.

Let $d \ge 3$ be a number that divides into $a^{2^n} + 1$; we show that $d > 2^{n+1}$. Since a^{2^n} is a perfect square, it is clear that 4 does not divide into $a^{2^n} + 1$. Therefore d has an odd prime divisor p, and $a^{2^n} \equiv -1 \bmod p$. Let m be the smallest natural number such that $a^m \equiv 1 \bmod p$. Now

$$a^{2^{n+1}} = \left(a^{2^n}\right)^2 \equiv (-1)^2 = 1 \bmod p ,$$

so $m|2^{n+1}$. If $m < 2^{n+1}$ write $m = 2^k$ with $k \le n$. But then

$$1 \equiv (a^{2^k})^{2^{n-k}} = a^{2^n} \equiv -1 \bmod p ,$$

a contradiction since p is odd. Thus $m = 2^{n+1}$. But according to Fermat's small theorem, $a^{p-1} \equiv 1 \bmod p$, and therefore $p - 1 \ge 2^{n+1}$, i.e.

$$d \ge p \ge 2^{n+1} + 1 ,$$

which is exactly what we set out to prove. As a byproduct we also get that $p \equiv 1 \bmod 2^{n+1}$.

Also solved by ANDREW CHOW, student, Albert Campbell C.I., Scarborough, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

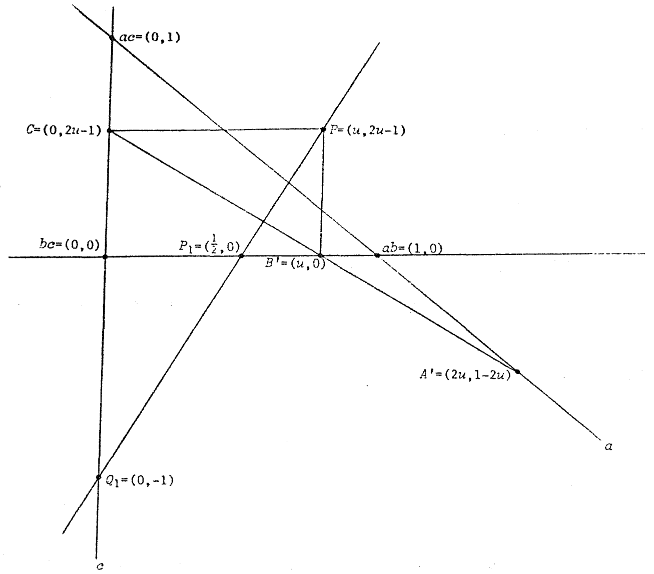
* * *

1388. [1988: 269] *Proposed by Jordi Dou, Barcelona, Spain.*

Given four lines a, b, c, d in general position in the plane, show that there is a unique line x cutting a, b, c, d in the respective points A, B, C, D and in that order, such that $AB = BC = CD$.

I. *Solution by J. Chris Fisher, University of Regina.*

Let P_1 be the midpoint of the segment joining bc (short for $b \cap c$) to ab. Let Q_1 be chosen so that bc is the midpoint of the segment joining it to ac. Then we observe: *for any $P \in P_1Q_1$, if B' is the point of b where it meets the parallel to c through P, C' is the point of c where it meets the parallel to b through P, and $A' = B'C' \cap a$, then $A'B' = B'C'$.* The proof of this claim comes from the figure using the affine coordinates shown there.

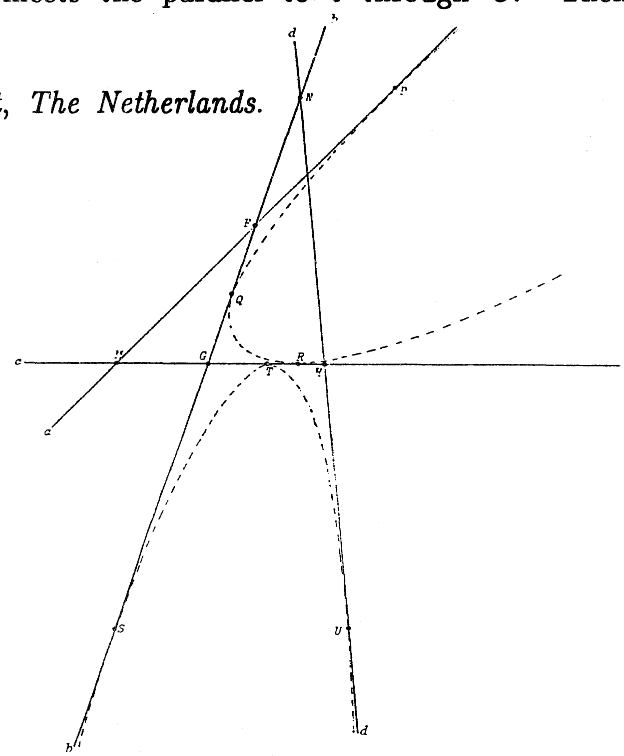


Similarly let Q_2 be the midpoint of the segment joining bc to cd, and let P_2 be chosen so that bc is the midpoint of the segment joining it to bd. Then given any $P \in P_2Q_2$, the corresponding statement that $B'C' = C'D'$ follows by replacing a by d and interchanging the roles of b and c in the above figure.

Finally, P_1Q_1 meets P_2Q_2 in a unique point O since we assume the given configuration to be in general position (i.e. no pair of either the given or constructed lines are parallel). Let B be the point of b where it meets the parallel to c through O, and let C be the point of c where it meets the parallel to b through O. Then $x = BC$ is the line we were to construct.

II. *Solution by P. Penning, Delft, The Netherlands.*

Introduce the intersections $a \cap c = M$, $b \cap d = N$, $a \cap b = F$, $b \cap c = G$, $c \cap d = H$. The envelope of the lines for which $AB = BC$ is the parabola P_1, tangent to a, b, c in P, Q, R with $MF = FP$, $FQ = QG$, $MG = GR$. The envelope of the lines for which $BC = CD$ is the parabola P_2, tangent to b, c, d in S, T, U with $NG = GS$, $GT = TH$, $NH = HU$. P_1 and P_2 have common tangents b and c. The third tangent



that P_1 and P_2 have in common is the line we are looking for.

Also solved by L.J. HUT, Groningen, The Netherlands; MARCIN E. KUCZMA, Warszawa, Poland; and the proposer (whose solution was the same as Penning's).

Can someone supply the editor with an accessible reference for the "envelope" property of the parabola, used in Solution II?

*　　　　　*　　　　　*

1389. [1988: 269] *Proposed by Derek Chang, California State University, Los Angeles, and Raymond Killgrove, Indiana State University, Terre Haute.*

Find

$$\max_{\pi \in S_n} \sum_{i=1}^{n} |i - \pi(i)| ,$$

where S_n is the set of all permutations of $\{1,2,\ldots,n\}$.

I. *Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.*

Let

$$T(\pi) = \sum_{i=1}^{n} |i - \pi(i)| .$$

We will show that

$$\max_{\pi \in S_n} T(\pi) = \left[\frac{n^2}{2}\right] , \tag{1}$$

where [ ] is the floor function.

Consider the permutation $\pi = (n,n-1,\ldots,1)$. Then

$$T(\pi) = |1 - n| + |2 - (n-1)| + \cdots + |n - 1| .$$

By expanding the absolute value functions and collecting like terms we obtain

$$T(\pi) = \begin{cases} 2\left\{n+(n-1)+\cdots+\left(\frac{n}{2}+1\right)-1-2-\cdots-\frac{n}{2}\right\} , & n \text{ even} , \\ 2\left\{n+(n-1)+\cdots+\frac{n+3}{2}\right\} + \frac{n+1}{2} - 2\left\{1+2+\cdots+\frac{n-1}{2}\right\} - \frac{n+1}{2} , & n \text{ odd} . \end{cases} \tag{2}$$

It is clear that this is the highest sum that can be obtained with these numbers, since all of the larger numbers appear preceded by a plus sign whereas all of the smaller numbers are preceded by a minus sign. [And the expansion of the sum $T(\pi)$ for any π will always produce each of the numbers $1,\cdots,n$ twice, with n of these $2n$ numbers preceded by a plus sign and n by a minus sign.] There are, of course, other permutations that will produce the same sum. From (2) we obtain

$$T(\pi) = \begin{cases} \frac{n}{2}\left(n + \frac{n}{2} + 1\right) - \frac{n}{2}\left(1 + \frac{n}{2}\right) = \frac{n^2}{2}, & n \text{ even}, \\ \frac{n-1}{2}\left(n + \frac{n+3}{2}\right) - \frac{n-1}{2}\left(1 + \frac{n-1}{2}\right) = \frac{n^2-1}{2}, & n \text{ odd}. \end{cases}$$

These two results may be replaced by a single expression by use of the floor function as in (1).

II. *Generalization by Murray S. Klamkin, University of Alberta.*

Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be nonincreasing sequences; $c_1,c_2,\dots,c_n$ any permutation of the b sequence; $F(t)$ a convex function; and

$$S = \sum_{i=1}^{n} F(a_i - c_i) .$$

Then

$$\max S = F(a_1 - b_n) + F(a_2 - b_{n-1}) + \cdots + F(a_n - b_1) ,$$

the maximum taken over all rearrangements $c_1,c_2,\dots,c_n$.

To prove this we use the majorization inequality [1], [2], i.e. *the conditions* $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$,

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \qquad \text{for } k = 1,2,\dots,n-1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

are necessary and sufficient in order that for every convex function F,

$$\sum_{i=1}^{n} F(x_i) \geq \sum_{i=1}^{n} F(y_i) .$$

(The above conditions are denoted by

$$(x_1,x_2,\dots,x_n) \succ (y_1,y_2,\dots,y_n)$$

and we say that the left hand vector *majorizes* the right hand vector.)

Here it is clear that

$$(a_1 - b_n, a_2 - b_{n-1}, \cdots, a_n - b_1) \succ (a_1 - c_1, a_2 - c_2, \cdots, a_n - c_n)$$

after the components of the latter vector are arranged in nonincreasing order, and the result follows.

The given problem corresponds to the special case $F(x) = |x|$, and $a_i = b_i = i$ for $i = 1,2,\dots,n$. Here the maximum sum is $2m^2$ if $n = 2m$ and $2m^2 + 2m$ if

$n = 2m + 1.$

References:

[1] D.S. Mitrinović, *Analytic Inequalities*, Springer–Verlag, Heidelberg, 1970.

[2] A.W. Marshall, I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, N.Y., 1979.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARCIN E. KUCZMA, Warszawa, Poland; KEE–WAI LAU, Hong Kong; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; PAUL YIU, University of Hong Kong; and the proposers. Two further readers sent in the correct value for the maximum, one with no proof and one with an incorrect proof.

Janous, Lau, and Wang note that the proposal has appeared earlier in stronger form as problem E2424 *of the* American Mathematical Monthly (*co–proposed by Wang, in fact*), *with solution in Vol.* 81 (1974) 668–670. *More recently a related problem also appeared in the* Monthly (E3175, *solution in Vol.* 96 (1989) 59–60).

* * *

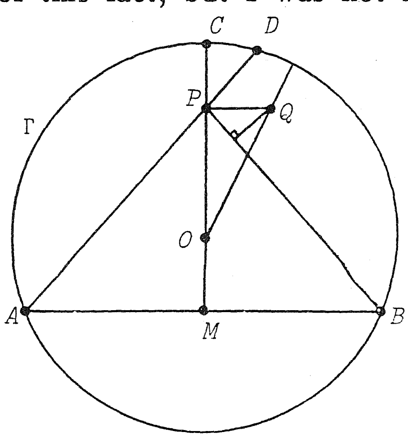
1390. [1988: 269] *Proposed by H. Fukagawa, Aichi, Japan.*

A, B, C are points on a circle Γ such that CM is the perpendicular bisector of AB. P is a point on CM and AP meets Γ again at D. As P varies over segment CM, find the largest radius of the inscribed circle tangent to segments PD, PB, and arc DB of Γ, in terms of the length of CM.

Solution by Marcin E. Kuczma, Warszawa, Poland.

The looked–for maximum equals $\overline{CM}/4$, attained when P is the midpoint of CM. This is amazing! In fact, for *any* P on CM, *the diameter of the incircle under consideration is the harmonic mean of the lengths of CP and PM*; the claim hence follows immediately.

There must exist some nice geometric proof of this fact, but I was not lucky enough to find it. The only solution I was able to work out was a lengthy one, via unpleasant calculations. Here is an outline.



We let (O,r) be the center and radius of Γ, (Q,y) the center and radius of the incircle of the curved triangle PBD (BD the arc of the circle), and put $CM = h$, $MP = x$. Then our claim is

$$y = \frac{x}{h}(h - x) \qquad [= \tfrac{1}{2}\cdot\text{harmonic mean of } x \text{ and } h - x].$$

First, PQ bisects $\angle BPD$, hence $PQ \| AB$ and so

$$\tfrac{1}{2}PQ\cdot x = \text{Area}(PQM) = \text{Area}(PQB) = \tfrac{1}{2}PB\cdot y \ . \qquad (1)$$

Other relevant identities are

$$PQ^2 = OQ^2 - OP^2 = (r - y)^2 - (r + x - h)^2 \qquad (2)$$

and

$$PB^2 = MP^2 + OB^2 - OM^2 = x^2 + r^2 - (h - r)^2. \qquad (3)$$

Putting (2) and (3) into the square of (1) leads, after some manipulation, to the quadratic equation

$$\alpha y^2 + \beta y + \gamma = 0$$

where

$$\alpha = h(2r - h) \ , \qquad \beta = 2rx^2 \ , \qquad \gamma = x^2(h - x)(h - x - 2r) \ .$$

The positive root is

$$y = \frac{x(h - x)}{h}$$

(verification by substitution); the other root is negative since $\alpha > 0$ and $\gamma < 0$. (Of course, these roots can be found without guessing, by the standard algorithm.)

[*Editor's note*: The editor apologizes for an unclear statement of the problem (not in the proposer's original formulation), which misled two other readers into not completely answering the problem. Namely, it should have been said that *point M lies on AB*. *L.J. HUT*, Groningen, The Netherlands, and *P. PENNING*, Delft, The Netherlands, both took M to be the centre of the circle Γ. Hut ended up maximizing the radius for points P on the diameter through C but *outside CM*, and his result also appears to be interesting. Penning's solution (when properly interpreted) is correct, and may be shorter than the one above. The editor invites Hut and Penning, and other *Crux* readers, to find a "nice" algebraic, or perhaps purely geometric, solution.]

Also solved by the proposer, whose proof (algebraic) also contained the lovely "harmonic mean" relation given by Kuczma.

The problem was taken from the 1840 Japanese mathematics book Sanpo Senmonsho.

* * *

<u>1391</u>. [1988: 301] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

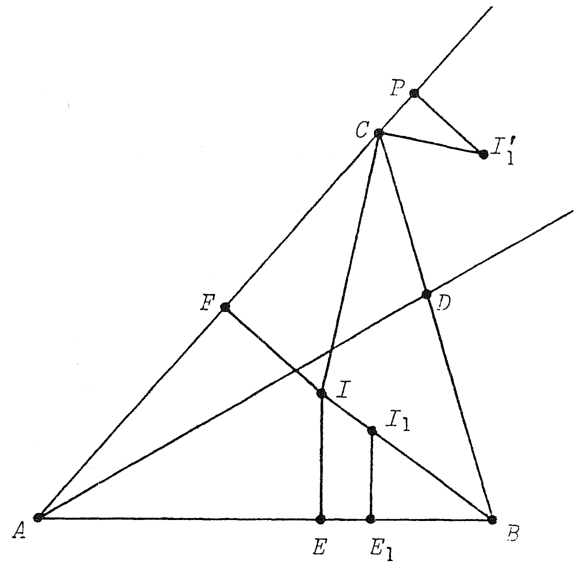
Let ABC be a triangle and D the point on BC so that the incircle of ΔABD and the excircle (to side DC) of ΔADC have the same radius ρ_1. Define ρ_2,

ρ_3 analogously. Prove that

$$\rho_1 + \rho_2 + \rho_3 \geq \frac{9}{4}r ,$$

where r is the inradius of ΔABC.

I. *Solution by Emilia A. Velikova and Svetoslav J. Bilchev, Technical University, Russe, Bulgaria.*



Let the points I, I_1 and I_1' be the centres of the incircles and the excircle in the triangles ABC and ABD as shown, and let E, E_1 be on AB and F, P be on AC so that $IE \perp AB$, $I_1E_1 \perp AB$, $IF \perp AC$, and $I_1'P \perp AC$. Then from the similar triangles $I_1'CP$ and ICF we have

$$\frac{\rho_1}{CP} = \frac{I_1'P}{CP} = \frac{CF}{IF} = \frac{s-c}{r} , \qquad (1)$$

where a, b, c, s are the sides and semiperimeter of ΔABC. It is well–known that

$$CP = AP - AC = \frac{b + AD + CD}{2} - b = \frac{AD + CD - b}{2} , \qquad (2)$$

and from (1) and (2) we get

$$\frac{r\rho_1}{s-c} = \frac{AD + CD - b}{2} . \qquad (3)$$

Further, from the similarity of triangles BIE and BI_1E_1 we obtain

$$\frac{\rho_1}{r} = \frac{I_1E_1}{IE} = \frac{BE_1}{BE} = \frac{c + BD - AD}{2(s-b)} ,$$

i.e.

$$\frac{(s-b)\rho_1}{r} = \frac{c + BD - AD}{2} . \qquad (4)$$

Adding (3) and (4) we get

$$\left(\frac{r}{s-c} + \frac{s-b}{r}\right)\rho_1 = \frac{a + c - b}{2} = s - b ,$$

from where

$$\rho_1 = \frac{r(s-b)(s-c)}{r^2 + (s-b)(s-c)} . \qquad (5)$$

From (5) and

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} ,$$

we obtain the beautiful expression

$$\rho_1 = \frac{rs}{2s-a} = \frac{F}{b+c} ,$$

where F is the area of ΔABC. Analogously we get

$$\rho_2 = \frac{F}{c + a}, \qquad \rho_3 = \frac{F}{a + b}.$$

Then the inequality to be proved is equivalent to

$$F\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq \frac{9}{4}r,$$

i.e.

$$2(a + b + c)\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq 9.$$

But this is equivalent to the well–known inequality

$$[(b + c) + (c + a) + (a + b)]\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq 3^2,$$

so the proof is complete.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We show more generally: *if $t \neq 0$ is a real number, then*

$$\rho_1^t + \rho_2^t + \rho_3^t \begin{cases} \geq 3(3/4)^t r^t & \text{if } t > 0 \text{ or } t < -1, \\ = 4/r & \text{if } t = -1, \\ \leq 3(3/4)^t r^t & \text{if } -1 < t < 0. \end{cases}$$

[*Editor's note*: At this point Janous derived and used the expression

$$\rho_1 = \frac{r}{1 + \tan(B/2)\tan(C/2)},$$

but in view of solution I it is easier to borrow the formula

$$\rho_1 = \frac{rs}{2s - a}$$

proved there. So adapted, Janous's argument continues ...]

Thus

$$\rho_1^{-1} + \rho_2^{-1} + \rho_3^{-1} = \frac{2s - a + 2s - b + 2s - c}{rs} = \frac{4}{r},$$

which is the case $t = -1$. The rest follows by the general means inequality. For $t > -1$, $t \neq 0$ we have $M_t \geq M_{-1}$, i.e.

$$\left(\frac{1}{3}\sum_{i=1}^{3} \rho_i^t\right)^{1/t} \geq \left(\frac{1}{3}\sum_{i=1}^{3} \rho_i^{-1}\right)^{-1} = \frac{3r}{4},$$

whence

$$\sum_{i=1}^{3} \rho_i^t \begin{cases} \geq 3(3/4)^t r^t & \text{if } t > 0, \\ \leq 3(3/4)^t r^t & \text{if } -1 < t < 0. \end{cases}$$

For $t < -1$, $M_t \leq M_{-1}$ and so

$$\left(\frac{1}{3}\sum_{i=1}^{3}\rho_i^t\right)^{1/t} \le \frac{3r}{4},$$

thus

$$\sum_{i=1}^{3}\rho_i^t \ge 3(3/4)^t r^t .$$

Done!

Also solved by L.J. HUT, Groningen, The Netherlands; T. SEIMIYA, Kawasaki, Japan; and the proposer.

* * *

1392. [1988: 301] *Proposed by Angel Dorito, Geld, Ontario.*

An immense spherical balloon is being inflated so that it constantly touches the ground at a fixed point A. A boy standing at a point at unit distance from A fires an arrow at the balloon. The arrow strikes the balloon at its nearest point (to the boy) but does not penetrate it, the balloon absorbing the shock and the arrow falling vertically to the ground. What is the longest distance through which the arrow can fall, and how far from A will it land in this case?

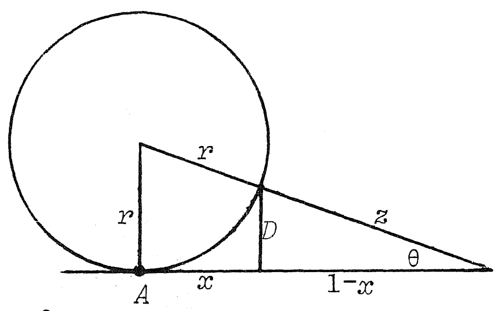
Solution by Friend H. Kierstead Jr., Cuyahoga Falls, Ohio.

From the diagram we easily obtain

$$r = \tan\theta$$

$$\frac{r}{r+z} = \sin\theta$$

$$D = z\sin\theta$$



from which we derive

$$D = \tan\theta(1-\sin\theta) . \qquad (1)$$

In order to find the maximum D, we set $dD/d\theta = 0$:

$$(1-\sin\theta)\sec^2\theta - \tan\theta\cos\theta = 0 .$$

Multiplying by $1 + \sin\theta$ gives

$$1 - \sin\theta(1+\sin\theta) = 0$$

or

$$\sin^2\theta + \sin\theta - 1 = 0 .$$

Solving for $\sin\theta$ gives

$$\sin\theta = \frac{\sqrt{5}-1}{2} .$$

(The other root is ignored because it would make $\sin\theta$ greater than 1.) Substituting this result into (1) gives

$$D_{\max} = \sqrt{\frac{\sqrt{5}-1}{2}} \cdot \frac{3-\sqrt{5}}{2} = 0.300283106\cdots$$

It can easily be shown that $d^2D/d\theta^2$ is negative throughout the region $0 < \theta < \pi/2$, so this is indeed a maximum, not a minimum. [*Editor's note*: or just look at the endpoints $r \to 0$, $r \to \infty$.] The distance from A to the point where the arrow lands is

$$x = 1 - \frac{D}{\tan\,\theta} = \sin\,\theta = \frac{\sqrt{5}-1}{2} = 0.618033988\cdots$$

Also solved by HAYO AHLBURG, Benidorm, Spain; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; M.S. KLAMKIN, University of Alberta; J.A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, The Netherlands; and the proposer.

Not all solvers found simple expressions for the required distances. Ahlburg, on the other hand, noted that, with

$$\phi = \frac{\sqrt{5}+1}{2},$$

the maximum falling distance is $\phi^{-5/2}$, *the radius of the balloon when this occurs is* $\phi^{-1/2}$, *and the distance from A to where the arrow hits the ground is* ϕ^{-1}: *in his words, "a delightful ubiquity of the Golden Section in such a simple figure".*

*　　　　*　　　　*

CRUX COOKIES

A package bearing the above label arrived here over the Christmas holidays from J. SUCK of Essen, and was speedily transformed into the empty set by the editor and others. Sadly, all that remains for the *Crux* readership is the label. The editor thanks Mr. Suck for his thoughtful and appropriate gift.

Hmmm. Does there perhaps exist an automobile manufacturer with a line of *Crux* sports cars?

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