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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

C R U X   M A T H E M A T I C O R U M

Vol. 12, No. 3
March 1986

Published by the Canadian Mathematical Society/
Publié par la Société Mathématique du Canada

The support of the University of Calgary Department of Mathematics and Statistics is gratefully acknowledged.

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CRUX MATHEMATICORUM is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural, or recreational reasons.

It is published monthly (except July and August). The yearly subscription rate for ten issues is $22.50 for members of the Canadian Mathematical Society and $25 for nonmembers. Back issues: $2.75 each. Bound volumes with index: Vols. 1 & 2 (combined) and each of Vols. 3-10: $20. All prices quoted are in Canadian dollars. Cheques and money orders, payable to CRUX MATHEMATICORUM, should be sent to the Managing Editor.

All communications about the content of the journal should be sent to the Editor. All changes of address and inquiries about subscriptions and back issues should be sent to the Managing Editor.

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ISSN 0705 - 0348.

Second Class Mail Registration No. 5432. Return Postage Guaranteed.

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CONTENTS

THE OLYMPIAD CORNER: 73

M.S. KLAMKIN

All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada, T6G 2G1.

I dedicate this Corner to Léo Sauvé who has recently retired as editor after starting this journal 11 years ago. I will miss his editing of this corner greatly. For those colleagues who wish to drop him a note, his address is 2206 E, Halifax Drive, Ottawa, Ontario, K1G 2W6.

I give one new problem set this month: the Third Round of the 1982 Leningrad High School Olympiad (my thanks to Alex Merkujev for its transmittal and to Larry Glasser for its translation). As usual, I solicit from all readers, especially secondary school students, elegant solutions with possible generalizations or extensions to all these problems. Readers submitting solutions or comments should include their institution as well as clearly identify the problems by giving their numbers and the year and page number of the issue where they appear. The solutions need not be typewritten, but in any case they should be *easily legible*.

I am continually grateful to receive problem sets from various national olympiads around the world. It would be preferred if these were received in English translation together with the official (or nonofficial) solutions since this would make the editing job easier. In a number of cases in which I did not have the translation and the solutions, I have had difficulty in getting the problems down correctly. Two such examples occur in problems #6 and #12 following even though the problems were translated a second time by another person. Consequently, if anyone comes up with the "correct" formulation of these problems, I would be very appreciative to receive them.

1982 Leningrad High School Olympiad (Third Round).

1. P_1, P_2, and P_3 are quadratic trinomials with positive leading coefficients and real roots. Show that if each pair of them has a common root, then the trinomial $P_1 + P_2 + P_3$ also has real roots. (Grade 8.)

2. If in triangle ABC, $C = 2A$ and $AC = 2BC$, show that it is a right triangle. (Grade 8, 9.)

3. Write a sequence of digits, the first four of which are 1, 9, 8, 2 and such that each following digit is the last digit of the sum of the preceding four digits (in base 10). Does this sequence contain 3, 0, 4, 4 as consecutive digits? (Grade 8, 9.)

4. If the angle between any two diagonals of a convex polygon of 180 sides is an integral number of degrees, show that the polygon is regular. (Grade 8, 9.)

5. The cells in a 5×41 rectangular grid are two-colored. Prove that three rows and three columns can be selected so that the nine cells in their intersection have the same color. (Grade 8, 10.)

6.† The plane is divided into regions by $2n$ straight lines ($n > 1$) no two of which are parallel and no three of which pass through the same point. Prove that these regions form no more than $2n - 1$ different angles. (Grade 8.)

7. If

$$\frac{b^2 + c^2 - a^2}{2bc} + \frac{c^2 + a^2 - b^2}{2ca} + \frac{a^2 + b^2 - c^2}{2ab} = 1,$$

show that among these three fractions two are equal to 1 and one is equal to -1. (Grade 9, 10.)

8. Prove that for any natural number k, there is an integer n such that

$$\sqrt{n + 1981^k} + \sqrt{n} = (\sqrt{1982} + 1)^k.$$

(Grade 9.)

9. A number of points are given in a plane, not all lying on a straight line. One is allowed to move any point to a point centrosymmetric to it with respect to any other given point. Show that after a finite number of moves, the points will lie at the vertices of a convex polygon. (Grade 9.)

10. In a given tetrahedron $ABCD$, $\angle BAC + \angle BAD = 180^\circ$. If AK is the bisector of $\angle CAD$, determine $\angle BAK$. (Grade 10.)

11. Show that it is possible to place non-zero numbers at the vertices of a given regular n-gon P so that for any set of vertices of P which are vertices of a regular k-gon ($k \leq n$), the sum of the corresponding numbers equals zero. (Grade 10.)

12.† $4n$ points are marked on a circle and colored red or blue. The points of each color are paired and the points in each pair are

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†See introduction regarding these problems.

connected by a line segment of the same color (no three segments intersect at a point). Show that the red lines and the blue lines intersect in at least n points. (Grade 10.)

*

SOLUTIONS

West German Mathematical Olympiad 1981 [1982: 12].

First Round.

1. Let a and n be positive integers and $s = a + a^2 + \ldots + a^n$. Prove that $s \equiv 1 \pmod{10}$ if and only if $a \equiv 1 \pmod{10}$ and $n \equiv 1 \pmod{10}$.

Solution by Paul Wagner, Chicago, Illinois.

If $a \equiv 1 \pmod{10}$ and $n \equiv 1 \pmod{10}$, it is easy to see that $s \equiv 1 \pmod{10}$.

Suppose $s \equiv 1 \pmod{10}$. Clearly a and n must both be odd and $a \not\equiv 5 \pmod{10}$. Also, $a \equiv 1 \pmod{10}$ implies $n \equiv 1 \pmod{10}$. Letting $n = 2m - 1$, we have

$$s = \frac{a(a^{2m-1} - 1)}{a - 1}$$

and thus we must have

$$a^{2m} - a \equiv a - 1 \pmod{10}. \tag{1}$$

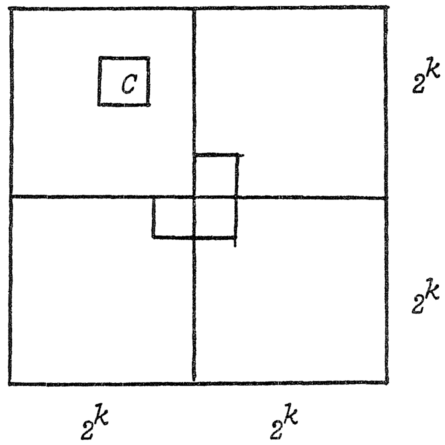
We now try $a = 3, 7, 9$ successively. If $a = 3$, (1) becomes $3^{2m} - 5 \equiv 0 \pmod{10}$ which is clearly impossible. If $a = 7$, (1) becomes $7^{2m} - 13 \equiv 0 \pmod{10}$. This is impossible since $7^{2m} = (50 - 1)^m \equiv \pm 1 \pmod{10}$. Finally, if $a = 9$, (1) becomes $9^{2m} - 7 \equiv 0 \pmod{10}$. This is also impossible since $9^{2m} = (80 + 1)^m \equiv 1 \pmod{10}$.

2. See [1982: 171].

3. A cell is removed from a $2^n \times 2^n$ checkerboard. Prove that the remaining surface can be tiled with L-shaped trominoes.

Solution.

Here is a proof by induction, due to Golomb (Checkerboards and polyominoes, *Amer. Math. Monthly* 61 (1954) 675-682). Clearly the case $n = 1$ is valid. Assume the result is valid for $n = k$. Now consider the case $n = k + 1$, with the indicated cell C missing. Without loss of generality it



is in the top left quadrant of the $2^{k+1} \times 2^{k+1}$ square. Remove the 3 cells in the interior corners of the other three quadrants. Tile each of the four $2^k \times 2^k$ quadrants by induction. We now fill in the missing tromino in the center, and so the result is valid for $n = k + 1$ and thus for all n.

A current paper of Chu and Johnsonbaugh (Tiling deficient boards with trominoes, *Math. Mag.* 59 (1986) 34-40) contains more general results.

4. Prove that if p is a prime, then $2^p + 3^p$ is not of the form n^k, where n and k are integers greater than 1.

Solution by H. Abbott, University of Alberta, Edmonton, Alberta.

The statement is true for $p = 2$.

Since $2^p + 3^p = 5(2^{p-1} - 3\cdot 2^{p-2} + \ldots + 3^{p-1})$ for $p > 2$, if there is a solution we must have $5|n$ and, since $k > 1$, $5^2|(2^p + 3^p)$. Thus it will suffice to show that

$$2^p + 3^p \equiv 0 \pmod{25} \tag{1}$$

has no solutions. Let $p = 20s + r$, $0 < r < 20$, $(r,20) = 1$. Since $\varphi(25) = 20$, we have $2^{20} \equiv 1 \pmod{25}$ and $3^{20} \equiv 1 \pmod{25}$. Thus,

$$2^p + 3^p \equiv 2^r + 3^r \pmod{25}.$$

Hence, if p is to satisfy (1), we must have

$$2^r + 3^r \equiv 0 \pmod{25}. \tag{2}$$

Checking each of the values $r = 1, 3, 7, 9, 11, 13, 17$, and 19, we find no solutions for (2).

Editorial note: More generally, K. Szymiczek (On the equations $a^x \pm b^x = c^y$, *Amer. J. Math.* 87 (1965) 262-266) shows that each of the equations in the title has at most one solution (x,y) aside from $x = y = 1$. Also, the author determines the only possible solution rather explicitly.

Second Round

1. See [1983: 238].

2. A bijective projection of the plane onto itself projects every circle onto a circle. Prove that it projects every straight line onto a straight line.

Solution.

Our proof is an indirect one. Assume that three collinear points A, B, C project into three noncollinear points A', B', C'. Let D' be a point on the circle through A', B', C' and consider its preimage D. It must be on the line ABC or otherwise the two circles DAB and DBC would project into the same circle $A'B'C'$ which is contrary to the hypothesis. Thus the image of the line ABC is a circle. Now consider a circle tangent to line ABC say at A. The projection of this configuration must be two circles tangent at A'. Consider any line through A' intersecting the two circles in points P', Q'. The preimages A, P, Q of A', P' and Q' are two points on line ABC and one off the line. This gives a contradiction since the circle through A, P, Q must project into a circle through A', P', Q'.

3. Let n be a positive power of 2. Prove that from any set of $2n - 1$ positive integers, one can choose a subset of n integers such that their sum is divisible by n.

Solution.

Letting $n = 2^m$, our proof is by induction on m. Clearly the result is valid for $m = 1$. Assume the result is valid for $n = 2^m$. Now consider the case $n = 2^{m+1}$. Since $2^{m+2} - 1 = 2^m + 2^m + (2^{m+1} - 1)$, by the inductive hypothesis we can always select three disjoint subsets, each of 2^m numbers, from $2^{m+2} - 1$ numbers such that the sum of each subset is divisible by 2^m. Letting the three sums be $a\cdot 2^m$, $b\cdot 2^m$, $c\cdot 2^m$, at least two of the numbers a, b, c have the same parity. By selecting the two sets corresponding to these numbers, we obtain 2^{m+1} numbers whose sum is divisible by 2^{m+1}. Consequently, the result is valid for all positive integers m by induction.

More generally, any set S of $2n - 1$ integers contains a subset of n integers whose sum is divisible by n. This is a known result of Erdös, Ginsburg and Ziv (Theorem in additive number theory, *Bull. Res. Coun. Israel* 10 (1961)). Two further proofs of this result appeared in the *Mathematical Intelligencer*, one by Ron Graham [1 (1979) 250], and the other by Tim Redmond and Charles Ryavec [2 (1980) 106].

A related result appears in H.B. Mann and J.E. Olson, Sums of sets in the elementary Abelian group of type (p,p), *J. Combinatorial Theory* 2 (1967) 275-284. The main result of this paper is as follows: Let p be a prime, let H be the additive group of residues modulo p and let G be the direct sum

$G = H \oplus H$. If S is any set of $2p - 1$ non-zero elements of G, then every element of G can be expressed as the sum over some subset of S. For $p > 3$, this is best possible in the sense that $2p - 1$ cannot be replaced by $2p - 2$.

4. Let M be a nonempty set of positive integers such that $4x$ and $[\sqrt{x}]$ both belong to M whenever x does. Prove that M is the set of all positive integers.

Solution.

Since M is nonempty it contains at least one positive integer n. By considering the sequence $a_1, a_2, \ldots$ where $a_0 = n$ and $a_{j+1} = [\sqrt{a_j}]$, $j \geq 1$, it is easy to see that $1 \in M$. Then, by multiplying repeatedly by 4 it follows that $4^j \in M$, and by taking square roots that $2^j \in M$, for $j = 0, 1, 2, \ldots$. It also follows that if any integer in the interval $[k^2, k^2 + 2k]$ is in M, then also $k \in M$. Similarly, $k \in M$ if any integer in the following intervals is in M : $[k^4, k^4 + 4k^3]$, $[k^8, k^8 + 8k^7]$, $\ldots$, $[k^{2^r}, k^{2^r} + 2^r k^{2^r - 1}]$, $\ldots$. Since $2k^{2^r} \leq k^{2^r} + 2^r k^{2^r - 1}$ if $2^r \geq k$, for every integer $k > 2$ one of the above intervals will contain a power of two. Thus M is the set of all positive integers.

*

THE 1981 JÓZSEF KÜRSCHÁK COMPETITION [1982: 13]

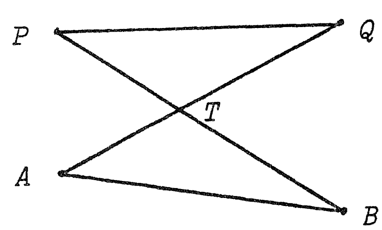
The solutions given here are the English translations (with small changes) of the Hungarian solutions by L. Csirmaz.

1. The points A, B, P, Q, R lie in a plane. Prove that

$$AB + PQ + QR + RP \leq AP + AQ + AR + BP + BQ + BR$$

where XY denotes the distance between points X and Y.

Solution.



Case A. If there are two of the points P, Q, R, say P and Q, such that segments AQ and BP intersect (at T, say), then we have by the triangle inequality:

$$AQ + PB = AT + TQ + PT + TB \geq AB + PQ. \qquad (1)$$

Again by the triangle inequality,

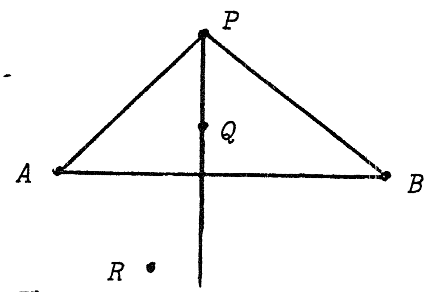
$$AP + AR \geq PR \qquad (2)$$

and

$$BR + BQ \geq RQ, \qquad (3)$$

and by adding inequalities (1), (2), (3), we get the desired result.

Case B. The other possible configuration is given in the figure. Without loss of generality, P and Q lie on the same side of AB extended. It then follows that PQ extended must cut segment AB. Now, without loss of generality, Q will be contained in triangle BPR. Then,



$$PQ + QR \leq PB + BR,$$
$$AB \leq AQ + QB,$$
$$RP \leq AP + AR.$$

Adding, we again get the desired result.

2. Let $n > 2$ be an even number. The squares of an $n \times n$ chessboard are colored with $n^2/2$ colors in such a way that every color is used for coloring exactly two of the squares. Prove that one can place n rooks on squares of different colors in such a way that no two of the rooks can capture each other.

Solution.

One can place the n rooks on the chessboard in $n!$ different ways such that no two rooks are attacking each other. Now we estimate the number of these placings which do not satisfy the color condition of the problem. Choose one of the $n^2/2$ pairs of squares of the same color. Put two rooks into these squares. If these rooks are not attacking each other, this placing can be completed in $(n - 2)!$ different ways. Therefore there are at most $n^2(n - 2)!/2$ wrong placings, and this number is strictly smaller than $n!$ for $n \geq 4$.

3. For a natural number n, $r(n)$ denotes the sum of the remainders of the divisions

$$n \div 1,\ n \div 2,\ n \div 3,\ \ldots\ ,\ n \div n.$$

Prove that $r(k) = r(k - 1)$ for infinitely many natural numbers k.

Solution.

The remainder of the division $n \div d$, $1 < d < n$ is one more than the remainder in $(n - 1) \div d$, except if d is a divisor of n when it is $d - 1$ less. Consequently,

$$r(k) - r(k - 1) = (k - 2) - (\text{sum of the proper divisors of } k).$$

Thus we wish to find infinitely many natural numbers k such that the sum of the proper divisors of k is $k - 2$. This is the case if $k = 2^m$: the proper divisors of 2^m are $2, 2^2, \dots, 2^{m-1}$, whose sum is $2^m - 2$.

*

West German Mathematical Olympiad 1982
First Round [1982: 70]

1. See [1983: 310].

2. In a quadrilateral $ABCD$, the sides AB and CD are each divided into m equal parts, the points dividing AB being labeled (in order from A) $S_1, S_2, \dots, S_{m-1}$, and those dividing CD being labeled (in order from D) $T_1, T_2, \dots, T_{m-1}$. Similarly, the sides BC and AD are each divided into n equal parts by the points $U_1, U_2, \dots, U_{n-1}$ (in order from B) and $V_1, V_2, \dots, V_{n-1}$ (in order from A), respectively. Prove that each of the segments $S_iT_i (1 \le i \le m - 1)$ is divided into n equal parts by the segments $U_jV_j (1 \le j \le n - 1)$.

Editorial note: A more general result with extensions has appeared in a problem in *Amer. Math. Monthly* 81 (1974) 666-668. For the convenience of the reader, I include it here.

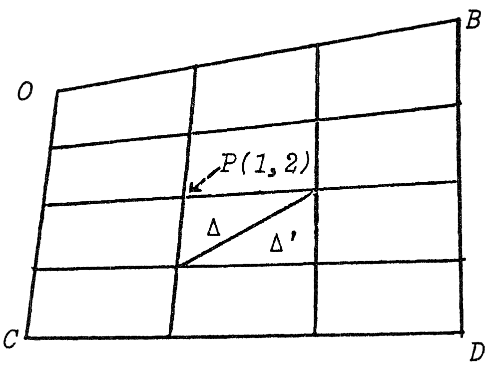
"Area Summations in Partitioned Convex Quadrilaterals

E 2423 [1973, 691]. *Proposed by Lyles Hoshek, Monterey Park, California, and B.M. Stewart, Michigan State University.*

Let there be given a plane convex quadrilateral of area A. Divide each of its four sides into n equal segments and join the corresponding points of division of opposite sides, forming n^2 smaller quadrilaterals. Prove: (a) the n smaller quadrilaterals in any diagonal (ordinary or broken) have a composite area equal to A/n; (b) the composite area of any row of smaller quadrilaterals and its complementary row (row i and row $n + 1 - i$) is equal to $2A/n$. (In particular, if n is odd this implies that the composite area of the middle row is A/n.)

Solution by Donald Batman, M.I.T. Lincoln Laboratory, and M.S. Klamkin, Ford Motor Company.

We obtain more general results by dividing one pair of opposite sides into n equal segments and the other pair of sides into m equal segments, as shown in the figure.



Denote the given quadrilateral by $OBDC$, where O is the origin. If X is a point in the plane, then we make the usual identification of X with the vector X from the origin to the point X. Define p, q by

$$\mathrm{D} = (p + 1)\mathrm{B} + (q + 1)\mathrm{C}.$$

Note that p, $q > -1$ and also $p + q > -1$ since the quadrilateral is convex. The points of division will be denoted by $P(r,s)$, with $r = 0,1,\ldots,m$ and $s = 0,1,\ldots,n$; e.g., $P(0,0) = O$ and $P(m,n) = D$. Let $Q(r,s)$ denote the small quadrilateral whose upper left-hand vertex is $P(r,s)$ and partition $Q(r,s)$ into the two triangles $\Delta(r,s)$ and $\Delta'(r,s)$ as shown in the figure.

One can show that for suitable scalars x and y

$$\mathrm{P}(r,s) = \frac{r}{m}\mathrm{B} + x\left\{\mathrm{C} + \frac{r}{m}(\mathrm{D} - \mathrm{C} - \mathrm{B})\right\} = \frac{s}{n}\mathrm{C} + y\left\{\mathrm{B} + \frac{s}{n}(\mathrm{D} - \mathrm{B} - \mathrm{C})\right\}.$$

Since B and C are linearly independent, we find that $x = s/n$ and $y = r/m$. Thus

$$\mathrm{P}(r,s) = \frac{r}{m}\left\{1 + \frac{sp}{n}\right\}\mathrm{B} + \frac{s}{n}\left\{1 + \frac{rq}{m}\right\}\mathrm{C}. \tag{1}$$

Since $\mathrm{P}(r + 1,s) - \mathrm{P}(r,s)$ and $\mathrm{P}(r,s + 1) - \mathrm{P}(r,s)$ are independent of r and s respectively, each segment of the figure is divided into equal parts - m for the "horizontal" segments and n for the "vertical" segments (as shown in the figure).

For the area $|\Delta(r,s)|$ of $\Delta(r,s)$ we have

$$\begin{aligned} 2|\Delta(r,s)| &= |\{\mathrm{P}(r + 1,s) - \mathrm{P}(r,s)\} \times \{\mathrm{P}(r,s + 1) - \mathrm{P}(r,s)\}| \\ &= \frac{1}{mn}\left\{1 + \frac{sp}{n} + \frac{rq}{m}\right\}|\mathrm{B} \times \mathrm{C}|, \end{aligned} \tag{2}$$

and similarly

$$2|\Delta'(r,s)| = \frac{1}{mn}\left\{1 + \frac{(s + 1)p}{n} + \frac{(r + 1)q}{n}\right\}|\mathrm{B} \times \mathrm{C}|. \tag{3}$$

Note also that if A is the area of $OBDC$, then

$$2A = (p + q + 2)|\mathrm{B} \times \mathrm{C}|. \tag{4}$$

Look now at any $\Delta(r,s)$ and its centro-symmetric $\Delta'(m-1-r, n-1-s)$. From (2), (3) and (4) we have

$$|\Delta(r,s)| + |\Delta'(m-1-r, n-1-s)| = \frac{A}{mn}. \tag{5}$$

For m, n odd it follows from this that the central small quadrilateral has area A/mn. (The special case $m = n = 3$ was established using a long synthetic proof by B. Greenberg, *That area problem*, Math. Teacher 64 (1971), 79-80.)

If we take any small quadrilateral $Q(r,s)$ and its centro-symmetric quadrilateral $Q(m-1-r, n-1-s)$ we see from (5) that

$$\begin{aligned}|Q(r,s)| + |Q(m-1-r, n-1-s)| &= |\Delta(r,s)| + |\Delta'(r,s)| \\ &\quad + |\Delta(m-1-r, n-1-s)| + |\Delta'(m-1-r, n-1-s)| \\ &= \frac{A}{mn} + \frac{A}{mn} = \frac{2A}{mn},\end{aligned}$$

which proves part (b).

From (2), (3), and (4) we have

$$\begin{aligned}|Q(r,s)| &= |\Delta(r,s)| + |\Delta'(r,s)| \\ &= \frac{A}{mn(p+q+2)}\left\{2 + \frac{(2s+1)p}{n} + \frac{(2r+1)q}{m}\right\}.\end{aligned} \tag{6}$$

Let $m = n$; we can now show that part (a) follows from this formula. In fact we can show that the result holds not only for broken diagonals, but for "generalized diagonals", i.e., for selections of n smaller quadrilaterals with one from each row and each column, as in the individual terms of a matrix expansion. More precisely, let σ be a permutation of $(0,1,\ldots,n-1)$; an easy computation shows that

$$\sum_{r=0}^{n-1} |Q(r, r\sigma)| = \frac{A}{n},$$

giving the result.

We note that Problem E 1548 [1963, 892] and its generalizations follow from the above results."

3. A 1982-gon is given in the plane. Let S be the set of all triangles whose vertices are also vertices of the 1982-gon. A point P lies on none of the sides of these triangles. Prove that the number of triangles in S that contain P is even.

Solution.

Since only some of the triangles will contain P, a direct count is difficult. So we consider all quadrilaterals formed by four vertices. For each of these quadrilaterals P will lie in 0 or 2 of the four subtriangles of the quadrilateral. Consequently this total count (with duplications) is an even number. Each triangle of S will be contained in exactly $1982 - 3 = 1979$ such quadrilaterals. Thus the number of triangles which contain P is an even number divided by 1979 and thus must be an even number.

4. A set of real numbers is called *simple* if it contains no elements x, y, z such that $x + y = z$. Find the maximum size of a simple subset of $\{1,2,\ldots,2n + 1\}$.

Solution.

Clearly, the two subsets $\{1,3,5,\ldots,2n + 1\}$ and $\{n + 1,n + 2,\ldots,2n + 1\}$ are simple. Thus the maximum size of a simple set is $\geq n + 1$. We now show that the maximum size is $n + 1$ by an indirect proof. Assume the maximum size is at least $n + 2$ and let any $n + 2$ elements of this simple set be listed in ascending order $a_1 < a_2 < \ldots < a_{n+2} = M$. We will obtain a contradiction by showing some a_i and a_j add to M. Consider all the pairs from the initial set whose sums are equal to M. For $M = 2N + 1$ these are

$$(1,2N)\ ,\ (2,2N - 1)\ ,\ \ldots\ ,\ (N,N + 1),$$

while for $M = 2N$ they are

$$(1,2N - 1)\ ,\ (2,2N - 2)\ ,\ \ldots\ ,\ (N - 1,N + 1).$$

Since $M \leq 2n + 1$, $N \leq n$. By the pigeonhole principle some pair of the $n + 1$ elements a_1, a_2, $\ldots$, a_{n+1} must fall into one of the above pairings since there are at most n of them.

Editorial note: An equivalent problem was proposed by Erwin Just, *Pi Mu Epsilon J.* 6 (1976) 315-316.

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Here are the answers to the 1985 Dutch Mathematical Olympiad, First Round [1986: 2]. I am grateful to Andy Liu for providing them.

A. (1) 1585, (2) 1156, (3) 3971, (4) 320, (5) 1985, (6) 1/2.

B. (1) 168, (2) 5/2, (3) (3,7,42), (3,8,24), (3,9,18), (3,10,15), (4,5,20), and (4,6,12), (4) $a = 56$, $b = 334$, $c = 18704$.

C. (1) $ab/13$, (2) 3456, (3) 286.

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To conclude this corner, I note that the 27th International Mathematical Olympiad is to be held in Warsaw, Poland from July 4 to July 15, 1986. The maximum team size will be six students, the same as last year (I am grateful to Cecil Rousseau for passing on this information). Also, Greg Patruno notes that Peter Yu's proposed problem [1985: 271, #5] appears as a special case in J.P. Hoyt's Quickie Problem Q694, *Math. Mag.* (1984) 239, 242.

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PROBLEMS

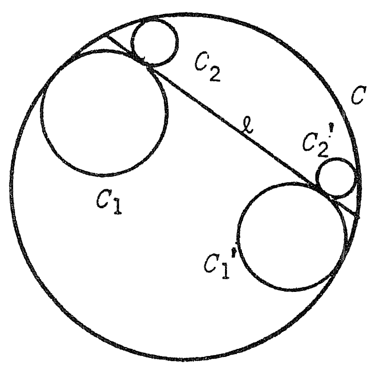
Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before October 1, 1986, although solutions received after that date will also be considered until the time when a solution is published.

1121. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

Let ℓ be a chord of a circle C. Let C_1 and C_2 be circles of radii r_1 and r_2 respectively, interior to C and tangent to C, and on opposite sides of ℓ and tangent to ℓ at a common point. Let C_1' and C_2' be another such pair of circles, of radii r_1' and r_2' respectively, with C_1 and C_1' on the same side of ℓ. Show that $\frac{r_1}{r_2} = \frac{r_1'}{r_2'}$.



1122. *Proposed by Richard K. Guy, University of Calgary, Calgary, Alberta.*

Find a dissection of a 6 × 6 × 6 cube into a small number of connected pieces which can be reassembled to form cubes of sides 3, 4, and 5, thus demonstrating that $3^3 + 4^3 + 5^3 = 6^3$. One could ask this in at least four forms:

(a) the pieces must be bricks, with integer dimensions;

(b) the pieces must be unions of 1 × 1 × 1 cells of the cube;

(c) the pieces must be polyhedral;

(d) no restriction.

1123. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle with sides a, b, c and angles α, β, γ such that $a \neq b$. Let the interior bisectors of α and β intersect the opposite sides at D and E respectively, and find D_1 and E_1 on AC and BC respectively such that $AD_1 = AD$ and $BE_1 = BE$. Suppose that $D_1E_1 \parallel AB$. Find γ.

1124. *Proposed by Stanley Rabinowitz and Peter Gilbert, Digital Equipment Corp., Nashua, New Hampshire.*

If $1 < a < 2$ and k is an integer, prove that

$$[a[k/(2-a)] + a/2] = [ak/(2-a)]$$

where $[x]$ denotes the greatest integer not larger than x.

1125*. *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A, B, C are the angles of an acute triangle ABC, prove that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \le \frac{3}{2}(\csc 2A + \csc 2B + \csc 2C)$$

with equality when triangle ABC is equilateral.

1126. *Proposed by Péter Ivády, Budapest, Hungary.*

For $0 < x \le 1$, show that

$$\sinh x < \frac{3x}{2 + \sqrt{1 - x^2}} < \tan x.$$

1127*. *Proposed by D.S. Mitrinovic, University of Belgrade, Belgrade, Yugoslavia.*

(a) Let a, b, c and r be real numbers > 1. Prove or disprove that

$$(\log_a bc)^r + (\log_b ca)^r + (\log_c ab)^r \ge 3 \cdot 2^r.$$

(b) Find an analogous inequality for n numbers $a_1, a_2, \ldots, a_n$ rather than three numbers a, b, c.

1128. *Proposed by Roger Izard, Dallas, Texas.*

Triangles CBA and ADE are so placed that B and D lie inside ADE and CBA respectively. DE and CB intersect at O. Angle CAB is equal to angle DAE. $AD = AB$ and $CO = OE$. Prove that triangles ADE and CBA are congruent.

1129. *Proposed by Donald Cross, Exeter, England.*

(a) Show that every positive whole number ≥ 84 can be written as the sum of three positive whole numbers in at least four ways (all twelve numbers different) such that the sum of the squares of the three numbers in any group is equal to the sum of the squares of the three numbers in each of the other groups.

(b) Same as part (a), but with "three" replaced by "four" and "twelve" by "sixteen".

(c)* Is 84 minimal in (a) and/or (b)?

1130. *Proposed by George Tsintsifas, Thessaliniki, Greece.*

Show that

$$a^{3/2} + b^{3/2} + c^{3/2} \leq 3^{7/4}R^{3/2}$$

where a, b, c are the sides of a triangle and R is the circumradius.

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SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

990. *Proposed by Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin.*

Find all pairs (u,v) of positive integers such that

$$u^3 + (u + 1)^3 = v^2.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The given equation can also be written as

$$(2u + 1)^3 + 3(2u + 1) = (2v)^2,$$

that is,

$$a(a^2 + 3) = b^2 \tag{1}$$

where $a = 2u + 1$ and $b = 2v$. Since $g = \gcd(a,a^2 + 3) = \gcd(a,3) \in \{1,3\}$, we have to consider the following two cases.

(i) $g = 1$. By (1), $a^2 + 3 = s^2$ with s a positive integer. Hence $a = 1$ and $u = 0$, which is not possible since u must be positive.

(ii) $g = 3$. By (1), $a = 3k^2$ and $a^2 + 3 = 3\ell^2$, with ℓ even (a odd!), and thus

$$\ell^2 = 3k^4 + 1. \tag{2}$$

By Theorem 9, page 270 of L.J. Mordell, *Diophantine Equations*, Academic Press, 1969, the only solutions of (2) are $\ell = 2$, $k = 1$ and $\ell = 7$, $k = 2$. The first yields $u = 1$, $v = 3$. (As ℓ has to be even, the second is of no interest.) So the only solution is $(u,v) = (1,3)$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; ROGER CUCULIERE, Paris, France; J. SUCK, Essen, Federal Republic of Germany; and the proposer. Two partial solutions and two incorrect solutions were received.

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991. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that the synonymical addition

$$\begin{array}{r} \text{STAIN} \\ \text{SPOT} \\ \hline \text{TAINT} \end{array}$$

has a unique solution in base seven and in base eight, and at least one solution in every base $b \geq$ nine.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Clearly N = 0 and S + 1 = T, with I + O = b where b is the base of numeration. Also T + S $\geq b$ by column 4 (counting from the right).

Now for $b = 7$ we must have:

1. (S,T) = (3,4) whence A = 1, (I,O) = (2,5) in some order, leaving no value for P; or

2. (S,T) = (4,5) whence (I,O) = (1,6) in some order, and thus A = 2, P = 3, I = 6 and O = 1 which yields

$$\begin{array}{r} 45260 \\ 4315 \\ \hline 52605 \end{array} \text{; or}$$

3. (S,T) = (5,6) whence A = 4 leaving no combination for (I,O).

Hence case 2 provides the only solution base 7. This solution can be rewritten as

$$\begin{array}{ccccc} b-3 & b-2 & b-5 & b-1 & 0 \\ & b-3 & 3 & 1 & b-2 \\ \hline b-2 & b-5 & b-1 & 0 & b-2 \end{array}$$

which is also valid for all $b \geq 9$.

Finally, for $b = 8$ we have

N	T	S	I,O	unused	A	P
0	5	4	6,2	1,3,7	1	-
			1,7	2,3,6	2	6
	6	5	7,1	2,3,4	3	-
	7	6	5,3	1,2,4	-	-

Hence the unique solution base 8 is

$$\begin{array}{r} 45210 \\ 4675 \\ \hline 52105 \end{array}.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; MARK KANTROWITZ, Maimonides School, Brookline, MA (for b = 7,8, *and even integers* $\geq$ *10 only); J.A. McCALLUM, Medicine Hat, Alberta* (b = 7 *and 8 only); and the proposer.*

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992. *Proposed by Harry D. Ruderman, Bronx, N.Y.*

Let $\alpha = (a_1, a_2, \ldots, a_{mn})$ be a sequence of positive real numbers such that $a_i \leq a_j$ whenever $i < j$, and let $\beta = (b_1, b_2, \ldots, b_{mn})$ be a permutation of α. Prove that

(a) $\displaystyle\sum_{j=1}^{n} \prod_{i=1}^{m} a_{m(j-1)+i} \geq \sum_{j=1}^{n} \prod_{i=1}^{m} b_{m(j-1)+i}$;

(b) $\displaystyle\prod_{j=1}^{n} \sum_{i=1}^{m} a_{m(j-1)+i} \leq \prod_{j=1}^{n} \sum_{i=1}^{m} b_{m(j-1)+i}$.

Solution by the proposer.

(a) We first establish the following "switch" lemma.

Lemma. Let x, y, u, $v \geq 0$, $xy \leq uv$, and $u \leq y$. Then $xy + uv \leq xu + yv$. Briefly, this says that when $xy \leq uv$ and $u \leq y$, we may switch factors y and u without decreasing the sum $xy + uv$.

Proof. Since $u \leq y$ and $xy \leq uv$ we must have $x \leq v$. But then $(y - u)(v - x) \geq 0$, which implies $xy + uv \leq xu + yv$.

We now prove (a) by induction on n. When $n = 1$ there is nothing to do. Assume (a) is true for $n = r$ addends, and take $n = r + 1$. Choose j so that

$$B = \prod_{i=1}^{m} b_{m(j-1)+i}$$

is largest among all addends of the right side of (a). Without loss of generality we can take $j = r + 1$. If

$$\left\{b_{mr+1}, b_{mr+2}, \ldots, b_{m(r+1)}\right\} = \left\{a_{mr+1}, a_{mr+2}, \ldots, a_{m(r+1)}\right\} \qquad (1)$$

then

$$\prod_{i=1}^{m} a_{mr+i} = \prod_{i=1}^{m} b_{mr+i},$$

and

$$\sum_{j=1}^{r} \prod_{i=1}^{m} a_{m(j-1)+i} \geq \sum_{j=1}^{r} \prod_{i=1}^{m} b_{m(j-1)+i}$$

is true by hypothesis, so (a) holds for $n = r + 1$. Thus assume (1) is not true; then there exists $b = b_{mr+s}$, $1 \leq s \leq m$, so that

$$b \notin \left\{a_{mr+1}, \dots, a_{m(r+1)}\right\}$$

and there exists $a = a_{mr+t}$, $1 \leq t \leq m$, so that

$$a \notin \left\{b_{mr+1}, \dots, b_{m(r+1)}\right\},$$

since the two sets in (1) have the same size. It follows that $b \leq a$, since the a_i's are in nondecreasing order. Also, a must equal some $b_{m(j-1)+i}$ for $j \leq r$. Without loss of generality, take $a = b_{mr}$. Let

$$B' = \prod_{i=1}^{m} b_{m(r-1)+i} ;$$

then

$$\frac{B'}{b_{mr}} \cdot b_{mr} = B' \leq B = \frac{B}{b} \cdot b$$

and $b \leq b_{mr}$, so by the lemma we have

$$B' + B \leq \frac{B'}{b_{mr}} \cdot b + \frac{B}{b} \cdot b_{mr} .$$

In other words, by interchanging $b_{mr} = a$ and b we do not decrease the right side of (a). By continuing this kind of switching we arrive at (1), and induction again takes over.

Part (b) is proved in a very similar fashion, using a similar switching lemma: if $x, y, u, v \geq 0$, $x + y \leq u + v$, and $u \leq y$ then $(x + y)(u + v) \geq (x + u)(y + v)$.

Part (a) (and also part (b)) implies the Arithmetic-Geometric Mean Inequality. Given positive numbers $x_1, x_2, \dots, x_n$, put $m = n$ and

$$\begin{aligned} a_{n(j-1)+i} &= x_j^{1/n} \\ b_{n(j-1)+i} &= x_i^{1/n} \end{aligned} \qquad (1 \leq i,j \leq n).$$

Then (a) says

$$\sum_{j=1}^{n} x_j \geq n \left\{\prod_{j=1}^{n} x_j\right\}^{1/n} .$$

For a companion result, see H.D. Ruderman, Two new inequalities, *Amer. Math. Monthly* 59 (1952), 29-32.

There was one incorrect solution received.

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993. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let P be the product of the $n + 1$ positive real numbers $x_1, x_2, \ldots, x_{n+1}$. Find a lower bound (as good as possible) for P if the x_i satisfy

(a) $\displaystyle\sum_{i=1}^{n+1} \frac{1}{1 + x_i} = 1$;

(b)* $\displaystyle\sum_{i=1}^{n+1} \frac{a_i}{b_i + x_i} = 1$, where the a_i and b_i are given positive real numbers.

Solution to part (a) by M.S. Klamkin, University of Alberta, Edmonton, Alberta.

By letting $a_i = 1/(1 + x_i)$, we obtain the equivalent problem: determine the greatest lower bound for

$$\prod_{i=1}^{n+1} \frac{1 - a_i}{a_i}$$

given $\Sigma a_i = 1$, $0 < a_i < 1$. This is a known result. In [1] it is shown that

$$\prod_{i=1}^{n+1} \frac{1 - a_i}{a_i} \geq n^{n+1} \tag{1}$$

with equality if and only if $a_i = 1/n+1$ for all i. A simple proof follows from the Arithmetic-Geometric Mean Inequality. Since

$$\frac{1 - a_i}{n} = \frac{a_1 + a_2 + \ldots + a_{n+1} - a_i}{n} \geq \left\{\frac{a_1 a_2 \ldots a_{n+1}}{a_i}\right\}^{1/n} ,$$

we obtain

$$\frac{\prod_{i=1}^{n+1} (1 - a_i)}{n^{n+1}} \geq \prod_{i=1}^{n+1} \left\{\frac{a_1 a_2 \ldots a_{n+1}}{a_i}\right\}^{1/n} = \prod_{i=1}^{n+1} a_i ,$$

equivalent to (1).

Here is a related result due to Ky Fan [2]. Let $a_1, a_2, \ldots, a_n$ be such that $\sum_{i=1}^{n} a_i = \lambda$ with $0 < a_i \le 1/2$ for all i. Since $\ell n\left[\frac{1-x}{x}\right]$ is convex for $0 < x \le 1/2$, it follows by Jensen's inequality that

$$\sum_{i=1}^{n} \ell n\left[\frac{1-a_i}{a_i}\right] \ge n\ \ell n\left[\frac{1-\Sigma a_i/n}{\Sigma a_i/n}\right] ,$$

or

$$\prod_{i=1}^{n} \frac{1-a_i}{a_i} \ge \left[\frac{n}{\lambda}-1\right]^n$$

with equality if and only if $a_i = \lambda/n$ for each i. As above, this is equivalent to: if $\sum_{i=1}^{n} \frac{1}{1+x_i} = \lambda$ and $x_i \ge 1$ for all i , then

$$\prod_{i=1}^{n} x_i \ge \left[\frac{n}{\lambda}-1\right]^n$$

with equality if and only if $x_i = \frac{n}{\lambda} - 1$ for each i.

Similarly, by using the fact that $\ell n\left[\frac{1-x}{x}\right]$ is concave for $x \ge 1/2$, we obtain: if

$$\sum_{i=1}^{n} \frac{1}{1+x_i} = \lambda \quad \text{and} \quad 0 \le x_i \le 1 \quad \text{for all } i ,$$

then

$$\prod_{i=1}^{n} x_i \le \left[\frac{n}{\lambda}-1\right]^n$$

with equality if and only if $x_i = \frac{n}{\lambda} - 1$ for each i.

Finally, problem (a) is also equivalent to a previously proposed problem of J. Berkes (*Elem. der Math.* 14 (1959), 132 where the solution of C. Bindschedler is given), namely; if $x_i > 0$ for $i = 1,2,\ldots,n$ then

$$\sum_{i=1}^{n+1} \frac{1}{1+x_i} \ge n \Longrightarrow \prod_{i=1}^{n+1} \frac{1}{x_i} \ge n^{n+1}$$

(just replace x_i by $1/x_i$). Also see M.S. Klamkin, Extensions of the Weierstrass product inequalities II, *Amer. Math. Monthly* 82 (1975), 741-742.

References:

[1] M.S. Klamkin and D.J. Newman, Extensions of the Weierstrass product inequalities, *Math. Mag.* 43 (1970), 137-141.

[2] E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, New York, 1965, p.5.

Comment on part (b) by M.S. Klamkin.

By dividing top and bottom of the i^{th} summand by b_i, we see there is no loss of generality in letting $b_i = 1$ for all i. Then, if any a_i is less than 1, the product $x_1x_2 \ldots x_{n+1}$ can be made arbitrarily small by choosing the corresponding x_i to be sufficiently small. If all a_i are greater than 1, the problem appears difficult.

Part (a) was also solved by MARK KANTROWITZ, Maimonides High School, Brookline, MA; and the proposer. The proposer noted that in part (b) it may be assumed that $a_i = 1$ *for each* i. *Readers are invited to contribute further to part (b).*

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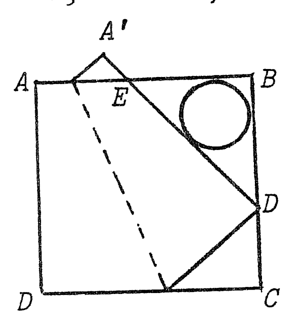
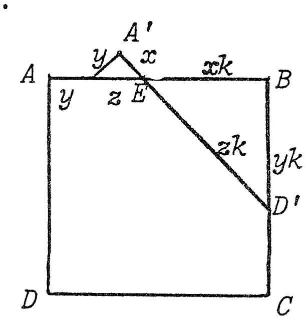
995. *Proposed by Hidetosi Fukagawa, Yokosuka High School, Tokai-City, Aichi, Japan.*

A square sheet of paper *ABCD* is folded as shown in the figure, with *D* falling on *D'* along *BC*, *A* falling on *A'*, and *A'D'* meeting *AB* in *E*. A circle is inscribed in triangle *EBD'*. Prove that the radius of this circle equals *A'E*.



I. *Solution by Sam Baethge, San Antonio, Texas.*

Label the figure as shown with k being a proportionality constant. For a right triangle the diameter of the inscribed circle is the sum of the legs minus the hypotenuse, so we wish to prove $k(x + y - z) = 2x$. Since $y + z + xk = x + zk$, we have

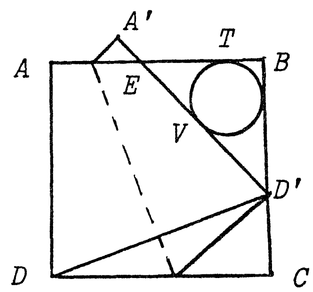


$$k = \frac{y + z - x}{z - x} .$$

Then

$$k(y + x - z) = \frac{y + z - x}{z - x} \cdot (y + x - z)$$
$$= \frac{y^2 - z^2 - x^2 + 2xz}{z - x}$$
$$= \frac{-2x^2 + 2xz}{z - x}$$
$$= 2x.$$

II. *Solution by Leon Bankoff, Los Angeles, California.*



Let s denote the semiperimeter of triangle EBD' and let T, V be the incircle contacts with EB, ED'. Draw DD'.

Since angles $CD'D$, $D'DA$ and $DD'A'$ are equal, DD' bisects angle $A'D'C$, and since D lies also on the internal bisector of angle ABC, it is the excenter of triangle EBD' relative to vertex B.

Therefore $s = BC = AD = A'D' = A'E + EV + VD'$.

But $s = EV + VD' + TB$.

Hence $A'E = TB$.

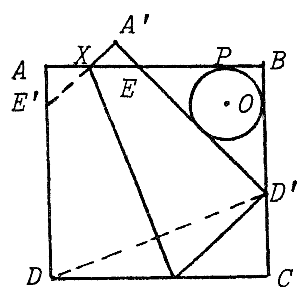
Some interesting sidelights:

1) As D' varies along BC, the perimeter of triangle EBD' remains constant and the quadrantal arc AC of excircle (D) is the envelope of all ED'.

2) The area of triangle $EBD' = EV \cdot VD' = AE \cdot ET = TB \cdot BC$.

3) When D' coincides with B or with C, triangle EBD' degenerates into two coinciding lines, each equal to the side of the square, thus confirming that the perimeter of square $ABCD$ is double that of triangle EBD'.

III. *Solution by Tosio Seimiya, Tokyo, Japan.*



We take the point E' on AD by producing $A'X$. Then, it is clear from $\Delta A'XE \equiv \Delta AXE'$ that $A'E = AE'$, and thus EE' is parallel to DD'. Since $\angle ED'D = \angle ADD' = \angle DD'C$, DD' is the bisector of $\angle ED'C$. Also, BD bisects $\angle EBD'$, so D is an excenter of $\Delta EBD'$. Thus E and D' lie on a circle with diameter DO, since $\angle OED$ and $\angle OD'D$ are right angles.

Since $ED' = E'D$ and $EE' \| DD'$, E' lies on the circumcircle of $\Delta EDD'$. This shows that $\angle DE'O$ is a right angle; that is, $OP = AE' = A'E$. Moreover, when D' runs along BC, the envelope of $A'D'$ is a circle with center D and radius $AD = DC$.

Also solved by GREG BURLILE, student, The Ohio State University, Columbus, Ohio; JORDI DOU, Barcelona, Spain; HERTA T. FREITAG, Roanoke, Virginia; JACK GARFUNKEL, Flushing, N.Y.; RICHARD A. GIBBS, Colorado College, Colorado Springs, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOSKY, Brooklyn, N.Y.; J. SUCK, Essen, West Germany; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

This problem was taken from an extant 1893 Japanese *sangaku*, a wooden tablet on which mathematical problems were written. During the 18^{th} and 19^{th} centuries, *sangaku* had hung beneath roofs of shrines and temples throughout Japan. Most problems recorded on them dealt with geometry, as was typical of *wazan*, the Japanese mathematics of the time. Professor Fukagawa has contributed quite a few of these problems, and they shall be appearing here from time to time. They are likely unfamiliar and certainly beautiful, and seem to be popular with readers. Professor Fukagawa has written a book (in Japanese), *Study of Sangaku*, on this traditional Japanese mathematics; the current problem is on page 138 of this book. He expresses an interest in corresponding with people about geometry and its history. For readers wishing to do so, here is his full address:

Professor Hidetosi Fukagawa
1-121 Higohara, Ogawa
Higasiuracho, Chitagun, Aichiken
470-21 Japan.

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996. *Proposed by Herta T. Freitag, Roanoke, Virginia.*

If $G = (\sqrt{5} + 1)/2$ is the golden ratio, prove that, for every positive integer n,

$$\sum_{k=1}^{n} \sum_{i=0}^{2(k-1)} (-1)^i G^{2(k-i-1)}$$

is a Fibonacci number of even subindex.

I. *Solution by Mark Kantrowitz, student, Maimonides High School, Brookline, Massachusetts.*

Summing the geometric series, we obtain

$$\sum_{i=0}^{2(k-1)} (-1)^i G^{2(k-1-i)} = G^{2(k-1)} - G^{2(k-2)} + \ldots + G^{-2(k-1)}$$

$$= G^{2(k-1)}\left[\frac{1 - (-G^{-2})^{2k-1}}{1 + G^{-2}}\right]$$

$$= \frac{G^{2k} + G^{-2k+2}}{G^2 + 1}$$

from which

$$\sum_{k=1}^{n} \sum_{i=0}^{2(k-1)} (-1)^i G^{2(k-1-i)} = \sum_{k=1}^{n} \frac{G^{2k} + G^{-2k+2}}{G^2 + 1}$$

$$= \frac{1}{G^2 + 1}\left\{G^2\left[\frac{1 - (G^2)^n}{1 - G^2}\right] + \left[\frac{1 - (G^{-2})^n}{1 - G^{-2}}\right]\right\}$$

$$= \frac{G^2}{G^4 - 1}[G^{2n} - G^{-2n}]$$

$$= \frac{1}{\sqrt{5}}[G^{2n} - G^{-2n}]$$

$$= F_{2n},$$

where we used

$$\frac{G^4 - 1}{G^2} = G^2 - G^{-2} = G^2 - (G - 1)^2 = 2G - 1 = \sqrt{5}.$$

II. *Solution by Bob Prielipp, University of Wisconsin, Oshkosh, Wisconsin.*

Let

$$S(n) = \sum_{k=1}^{n} \sum_{i=0}^{2(k-1)} (-1)^i G^{2(k-i-1)}.$$

We shall prove that $S(n) = F_{2n}$ for each n. We begin by establishing the following result.

Lemma. For each positive integer n,

$$\sum_{i=0}^{2n} (-1)^i G^{2(n-i)} = F_{2n+1}.$$

Proof. Letting $\alpha = G$ and $\beta = (1 - \sqrt{5})/2 = -\alpha^{-1}$, we have

$$F_{2n+1} = \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}$$

$$= \frac{G^{2n+1} + (G^{-1})^{2n+1}}{G + G^{-1}}$$

$$= \sum_{i=0}^{2n} (-1)^i G^{2n-i} (G^{-1})^i$$

$$= \sum_{i=0}^{2n} (-1)^i G^{2(n-i)} . \qquad \square$$

Clearly, $S(1) = 1 = F_2$. Assume that $S(n) = F_{2n}$. Then

$$S(n+1) = S(n) + \sum_{i=0}^{2n} (-1)^i G^{2(n-i)}$$

$$= F_{2n} + F_{2n+1}$$

$$= F_{2n+2} ,$$

completing our solution by mathematical induction.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; J. SUCK, Essen, West Germany; and the proposer.

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997. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Suppose that $f(x)$ is bounded in a deleted neighborhood of zero and suppose that a and b are real numbers less than 1 in magnitude such that

$$\lim_{x\to 0} \{f(x) + af(bx)\}$$

exists (and is finite). Prove that $\lim_{x\to 0} f(x)$ exists.

Solution by Jordan B. Tabov, Sofia, Bulgaria.

We shall prove that the stated result is true for any a and b with $|a| \neq 1$ and $b \neq 0$. (Clearly it is also true for $a = b = 1$ and for $b = 0$.)

Let a and b be real numbers with $|a| \neq 1$ and $b \neq 0$. Suppose $\lim_{x\to 0} \{f(x) + af(bx)\}$ exists and equals d. We first do the case $d = 0$.

Suppose that $f(x)$ is defined in the punctured interval $\Delta = (-c,c) - \{0\}$, and let $|f(x)| \leq M$ for every $x \in \Delta$.

The fact that $\lim_{x\to 0} \{f(x) + af(bx)\} = 0$ means that for all $\epsilon > 0$ there is $\delta > 0$ such that if $x \in \Delta_\epsilon = (-\delta,\delta)$, then

$$|f(x) + af(bx)| < \epsilon . \tag{1}$$

Fix an arbitrary positive integer n. Let x be such that x and $b^n x$ are both in Δ_ϵ. Then $b^k x \in \Delta_\epsilon$ for $k = 1,2,\dots,n$ and, from (1),

$$\begin{array}{rcl}
-\epsilon < & f(x) + af(bx) & < \epsilon \\
-\epsilon|a| < & -af(bx) - a^2 f(b^2 x) & < \epsilon|a| \\
-\epsilon|a|^2 < & a^2 f(b^2 x) + a^3 f(b^3 x) & < \epsilon|a|^2 \\
 & \vdots & \\
-\epsilon|a|^{n-1} < & (-a)^{n-1} f(b^{n-1}x) - (-a)^n f(b^n x) & < \epsilon|a|^{n-1} .
\end{array}$$

Summing, we obtain

$$-\epsilon \frac{1 - |a|^n}{1 - |a|} < f(x) - (-a)^n f(b^n x) < \epsilon \frac{1 - |a|^n}{1 - |a|} . \tag{2}$$

<u>Case 1</u>: $|a| < 1$, $0 < |b| \le 1$. In this case if $x \in \Delta_\epsilon$, then $b^n x \in \Delta_\epsilon$ for every n. Therefore we can conclude that (2) holds for every n. From (2) and the inequality $|f(b^n x)| \le M$ we get

$$|f(x)| < \epsilon(1 - |a|)^{-1} + |a|^n M$$

for every n. Hence

$$|f(x)| \le \epsilon(1 - |a|)^{-1} + \lim_{n\to\infty} |a|^n M = \epsilon(1 - |a|)^{-1} .$$

Since this result is proved for every $x \in \Delta_\epsilon$ and $(1 - |a|)^{-1}$ is a constant, $\lim_{x\to 0} |f(x)| = 0$, so $\lim_{x\to 0} f(x) = 0$.

<u>Case 2</u>: $|a| > 1$, $0 < |b| \le 1$. Denote $b^n x$ by y and $(-|b|^n\delta, |b|^n\delta) - \{0\}$ by Δ_ϵ^n. Note that $\Delta_\epsilon^n \subset \Delta_\epsilon$. Clearly if $y \in \Delta_\epsilon^n$ then $x = b^{-n}y \in \Delta_\epsilon$. Now it follows from (2) that if $y \in \Delta_\epsilon^n$,

$$|a|^n |f(y)| < \epsilon \frac{|a|^n - 1}{|a| - 1} + M$$

and hence

$$|f(y)| < \epsilon(|a| - 1)^{-1} + |a|^{-n} M.$$

Therefore, choosing $\epsilon \le 1$ and $n = [\epsilon^{-1}]$, we get

$$\lim_{y\to 0} |f(y)| \le \lim_{\epsilon\to 0} \{\epsilon(|a| - 1)^{-1} + |a|^{-[\epsilon^{-1}]} M \} = 0.$$

Thus $\lim_{y\to 0} f(y) = 0$.

<u>Case 3</u>: $|a| \neq 1$, $|b| > 1$. If $a = 0$, the stated result is trivial, so we assume $a \neq 0$. Let $a_1 = a^{-1}$, $b_1 = b^{-1}$, and $y = bx$. Then

$$\lim_{y\to 0} \{a_1 f(b_1 y) + f(y)\} = \frac{1}{a} \lim_{x\to 0} \{f(x) + af(bx)\} = 0,$$

and $|a_1| \neq 1$ and $0 < |b_1| < 1$, so by one of the first two cases we get $\lim\limits_{y\to 0} f(y) = 0$.

This completes the proof if $d = 0$. We handle the general case by defining g such that

$$f(x) = g(x) + \frac{d}{1 + a} .$$

Then

$$f(x) + af(bx) = g(x) + ag(bx) + d ,$$

so

$$d = \lim_{x\to 0} \{f(x) + af(bx)\} = \lim_{x\to 0} \{g(x) + ag(bx)\} + d .$$

Hence

$$\lim_{x\to 0} \{g(x) + ag(bx)\} = 0 ,$$

and g is bounded in a deleted neighborhood of zero, so by the above proof $\lim\limits_{x\to 0} g(x) = 0$. Thus $\lim\limits_{x\to 0} f(x)$ exists and equals $\dfrac{d}{1 + a}$.

[*Editor's note*: *We may as well complete the picture by noting that the result is false for* $|a| = 1$ *and* $|b| \neq 0, 1$. *In this case, define*

$$f(x) = \begin{cases} (-a)^n & \text{if } x = b^n \text{ for some } n \in \mathbb{Z} , \\ 0 & \text{otherwise.} \end{cases}$$

Then for all x, $f(x) + af(bx) = 0$, so $\lim\limits_{x\to 0} \{f(x) + af(bx)\} = 0$. However, $\lim\limits_{x\to 0} f(x)$ doesn't exist.]

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; LEROY F. MEYERS, The Ohio State University, Columbus, Ohio; J. SUCK, Essen, West Germany; and the proposer.

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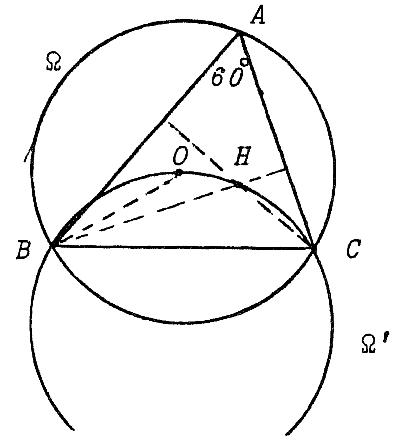
998. *Proposed by Andrew P. Guinand, Trent University, Peterborough, Ontario.*

If just one angle of a triangle is 60°, show that the inverse of the orthocentre with respect to the circumcircle lies on the side (or side produced) opposite that angle.



Solution by Jordi Dou, Barcelona, Spain.

Let Ω be the circumcircle of ABC, O its centre, $\angle A = 60^\circ$, H the orthocentre. Let Ω_1 be the circle BOC. Then $\angle BHC = \angle ABH + 90^\circ = 30^\circ + 90^\circ = 120^\circ$ and $\angle BOC = 2(60^\circ) = 120^\circ$, so H is on Ω_1. Ω_1 and line BC are inverses with respect to Ω. Therefore the inverse of H will lie on BC.

Also solved by HERTA T. FREITAG, Roanoke, Virginia; J.T. GROENMAN, Arnhem, The Netherlands; S. IWATA and H. FUKAGAWA, Gifu and Aichi, Japan; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, University of Minnesota, Minneapolis, Minnesota; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Brooklyn, N.Y.; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. Fukagawa observed that the converse is also true. Groenman and the proposer noted that the same result holds if one angle is 120° *(and can in fact be seen by considering ΔBHC in the diagram).*

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A MESSAGE FROM THE EDITOR

There has been the very good suggestion made that an issue of *Crux Mathematicorum* be dedicated to Léo Sauvé. I suggest the September issue of this year. Accordingly, all readers who would like to propose problems specifically for that issue, or have clerihews, limericks, rebuses, etc. appropriate to the occasion, or would simply like to extend their best wishes, are invited to submit same to the Editor. As many of your contributions as possible will be published in September.

Bill Sands