THE ACADEMY CORNER

No. 47

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Here, we present the official solutions of the 8$^{\text{th}}$ International Mathematics Competition, held at the Charles University, Prague, Czech Republic, on 19–25 July 2001. See [2002 : 3]. This competition is for university students completing up to their fourth year, and consists of two sessions, each of five hours. Thanks to Moubinool Omarjee for sending them to us.

8$^{\text{th}}$ International Mathematics Competition

Day 1 Problems

Problem 1. Let n be a positive integer. Consider an $n \times n$ matrix with entries 1, 2, $\ldots$, n^2 written in order starting top left and moving along each row in turn left to right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Solution. Since there are exactly n rows and n columns, the choice is of the form

$$\{(j,\sigma(j))\ :\ j=1,\,\ldots,\,n\}\,,$$

where $\sigma \in S_n$ is a permutation. Thus, the corresponding sum is equal to

$$\begin{aligned}\sum_{j=1}^{n} n(j-1)+\sigma(j) &= \sum_{j=1}^{n} nj - \sum_{j=1}^{n} n + \sum_{j=1}^{n}\sigma(j)\\ &= n\sum_{j=1}^{n} j - \sum_{j=1}^{n} n + \sum_{j=1}^{n} j\\ &= (n+1)\frac{n(n+1)}{2} - n^2 \;=\; \frac{n(n^2+1)}{2}\,,\end{aligned}$$

which shows that the sum is independent of σ.

Problem 2. Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.

Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Solution.

1. There exist integers u and v such that $us + vt = 1$. Since $ab = ba$, we obtain

$$\begin{aligned} ab &= (ab)^{us+vt} &&= (ab)^{us}\left((ab)^t\right)^v &&= (ab)^{us}e \\ &= (ab)^{us} &&= a^{us}\left(b^s\right)^u &&= a^{us}e &&= a^{us}. \end{aligned}$$

Therefore, $b^r = eb^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e$. Then

$$b = b^{xr+ys} = (b^r)^x (b^s)^y = e.$$

It follows similarly that $a = e$ as well.

2. This is not true. Let $a = (123)$ and $b = (34567)$ be cycles of the permutation group S_7 of order 7. Then, $ab = (1234567)$ and $a^3 = b^5 = (ab)^7 = e$.

Problem 3. Find $\lim_{t\nearrow 1}(1-t)\sum\limits_{n=1}^{\infty}\left(\frac{t^n}{1+t^n}\right)$, where $t \nearrow 1$ means that t approaches 1 from below.

Solution.

$$\begin{aligned} \lim_{t\nearrow 1}(1-t)\sum_{n=1}^{\infty}\left(\frac{t^n}{1+t^n}\right) &= \lim_{t\nearrow 1}\left(\frac{1-t}{-\ln t}\right)\cdot(-\ln t)\sum_{n=1}^{\infty}\left(\frac{t^n}{1+t^n}\right) \\ &= \lim_{t\nearrow 1}(-\ln t)\sum_{n=1}^{\infty}\left(\frac{1}{1+e^{-n\ln t}}\right) \\ &= \lim_{h\searrow 0} h\sum_{n=1}^{\infty}\left(\frac{1}{1+e^{nh}}\right) = \int_0^{\infty}\frac{dx}{1+e^x} = \ln 2. \end{aligned}$$

Problem 4. Let k be a positive integer. Let $p(x)$ be a polynomial of degree n, each of whose coefficients is -1, 1 or 0, and which is divisible by $(x-1)^k$. Let q be a prime such that $\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$. Prove that the complex q^{th} roots of unity are roots of the polynomial $p(x)$.

Solution. Let $p(x) = (x-1)^k r(x)$ and $\epsilon_j = e^{2\pi i j/q}$ ($j = 1, 2, \ldots, q-1$). As is well known, the polynomial

$$x^{q-1} + x^{q-2} + \cdots + x + 1 = (x-\epsilon_1)\cdots(x-\epsilon_{q-1})$$

is irreducible. Thus, all $\epsilon_1, \ldots, \epsilon_{q-1}$ are roots of $r(x)$, or none of them are.

Suppose that none of them are roots of $r(x)$. Then $\prod_{j=1}^{q-1} r(\epsilon_j)$ is a rational integer, which is not 0, and

$$
\begin{aligned}
(n+1)^{q-1} \quad &\geq \quad \prod_{j=1}^{q-1} |p(\epsilon_j)| \;=\; \left|\prod_{j=1}^{q-1} (1-\epsilon_j)^k\right| \cdot \left|\prod_{j=1}^{q-1} r(\epsilon_j)\right| \\
&\geq \quad \left|\prod_{j=1}^{q-1} (1-\epsilon_j)\right|^k \;=\; \left(1^{q-1}+1^{q-2}+\cdots+1^1+1\right)^k \;=\; q^k .
\end{aligned}
$$

This contradicts the condition $\dfrac{q}{\ln q} < \dfrac{k}{\ln(n+1)}$.

Problem 5 Let A be an $n \times n$ matrix such that $A \neq \lambda I$ for all $\lambda \in \mathbb{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Solution. The statement will be proved by induction on n. For $n = 1$, there is nothing to do.

In the case $n = 2$, write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $b \neq 0$ and $c \neq 0$, or if $b = c = 0$, then A is similar to

$$
\begin{bmatrix} 1 & 0 \\ a/b & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a/b & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ c-ad/b & a+d \end{bmatrix},
$$

or

$$
\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b-ad/c \\ c & a+d \end{bmatrix},
$$

respectively. If $b = c = 0$ and $a \neq d$, then A is similar to

$$
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & d-a \\ 0 & d \end{bmatrix},
$$

and we can perform the step seen in the case $b \neq 0$ again.

Assume now that $n > 3$ and that the problem has been solved for all $n' < n$. Let $A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix}_n$, where A' is an $(n-1) \times (n-1)$ matrix. Clearly, we may assume that $A' \neq \lambda' I$, so that the induction provides a P with, say, $P^{-1}A'P = \begin{bmatrix} 0 & * \\ * & \alpha \end{bmatrix}_{n-1}$. But then, the matrix

$$
B = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \begin{bmatrix} P & o \\ o & 1 \end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta \end{bmatrix}
$$

is similar to A, and its diagonal is $(0, 0, \ldots, 0, \alpha, \beta)$. On the other hand, we may also view B as $\begin{bmatrix} 0 & * \\ * & C \end{bmatrix}_n$, where C is an $(n-1) \times (n-1)$ matrix with diagonal $(0, 0, \ldots, 0, \alpha, \beta)$. If the inductive hypothesis is applicable to C, we would have $Q^{-1}CQ = D$, with $D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_{n-1}$, so that, finally, the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} B \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}$$

is similar to A, and its diagonal is $(0, 0, \ldots, 0, 0, \gamma)$ as required.

The inductive argument can fail only when $n - 1 = 2$, and the resulting matrix applying P has the form

$$P^{-1}AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix},$$

where $d \neq 0$. The numbers a, b, c, e cannot be 0 at the same time. If, say, $b \neq 0$, then A is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ c & d & 0 \\ e-b-d & a & b+d \end{bmatrix}.$$

Performing the first half of the induction step again, the diagonal of the resulting matrix will be $(0, d-b, d+b)$ [the trace is the same] and the induction step can be finished. The cases $a \neq 0$, $c \neq 0$ and $e \neq 0$ are similar.

Problem 6. Suppose that the differentiable functions a, b, f, $g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0, \quad f'(x) \geq 0, \quad g'(x) > 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x\to\infty} a(x) = A > 0, \quad \lim_{x\to\infty} b(x) = B > 0, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 2\frac{B}{A+1}.$$

Solution. Let $0 < \epsilon < A$ be an arbitrary real number. If x is sufficiently large, then $f(x) > 0$, $g(x) > 0$, $|a(x) - A| < \epsilon$, $|b(x) - B| < \epsilon$, and

$$\begin{aligned} B - \epsilon &< b(x) = \frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} < \frac{f'(x)}{g'(x)} + (A+\epsilon)\frac{f(x)}{g(x)} \\ &< \frac{(A+\epsilon)(A+1)}{A} \cdot \frac{f'(x)\,(g(x))^A + A\,f(x)\,\,(g(x))^{A-1}\,g'(x)}{(A+1)\,(g(x))^A\,g'(x)} \\ &= \frac{(A+\epsilon)(A+1)}{A} \cdot \frac{\left(f(x)\,(g(x))^A\right)'}{\left((g(x))^{A+1}\right)'}. \end{aligned}$$

Thus,

$$\frac{\left(f(x)\,(g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} \;>\; \frac{A(B-\epsilon)}{(A+\epsilon)(A+1)}\,.$$

Similarly, it can be obtained that, for sufficiently large x,

$$\frac{\left(f(x)\,(g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} \;<\; \frac{A(B+\epsilon)}{(A-\epsilon)(A+1)}\,.$$

From letting $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0}\frac{\left(f(x)\,(g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} \;=\; \frac{B}{A+1}\,.$$

By l'Hôspital's Rule, this implies that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} \;=\; \lim_{x\to\infty}\frac{f(x)\,(g(x))^A}{(g(x))^{A+1}} \;=\; \frac{B}{A+1}\,.$$

Day 2 Problems

Problem 1 Let r, $s \geq 1$ be integers and a_0, a_1, $\ldots$, a_{r-1}, b_0, b_1, $\ldots$, b_{s-1} be real non-negative numbers such that

$$(a_0+a_1x+a_2x^2+\cdots+a_{r-1}x^{r-1}+x^r)(b_0+b_1x+b_2x^2+\cdots+b_{s-1}x^{s-1}+x^s)$$

$$= \;1+x+x^2+\cdots+x^{r+s-1}+x^{r+s}\,.$$

Prove that each a_i and each b_j equals either 0 or 1.

Solution. Multiply the left hand side polynomials. We obtain the following equalities:

$$a_0b_0 = 1\,, \quad a_0b_1 + a_1b_0 = 1\,, \quad \ldots\,.$$

Amongst them, one can find the equations $a_0 + a_1b_{s-1} + a_2b_{s-2} + \cdots = 1$ and $b_0 + b_1a_{r-1} + b_2a_{r-2} + \cdots = 1$. From these equations, it follows that a_0, $b_0 \leq 1$.

Using $a_0b_0 = 1$, we can see that $a_0 = b_0 = 1$.

Now, looking at the following equations, we notice that all a's must be less than or equal to 1. The same holds for the b's. It follows from the equation $a_0b_1 + a_1b_0 = 1$ that one of the numbers a_1, b_1 equals 0, while the other must be 1.

The rest of the proof is by induction.

Problem 2 Let $a_0 = \sqrt{2}$, $b_0 = 2$, and $a_{n+1} = \sqrt{2-\sqrt{4-a_n^2}}$, $b_{n+1} = \dfrac{2b_n}{2+\sqrt{4+b_n^2}}$.

(a) Prove that the sequences $\{a_n\}$, $\{b_n\}$ are decreasing and converge to 0.

(b) Prove that the sequence $\{2^n a_n\}$ is increasing, the sequence $\{2^n b_n\}$ is decreasing and that these two sequences converge to the same limit.

(c) Prove that there is a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \dfrac{C}{8^n}$.

Solution. Clearly $a_1 = \sqrt{2-\sqrt{2}} < \sqrt{2}$. Since the function $f(x) = \sqrt{2-\sqrt{4-x^2}}$ is increasing on the interval $[0,2]$, the inequality $a_1 > a_2$ implies that $a_2 > a_3$. Simple induction ends the proof of the monotonicity of $\{a_n\}$. In the same way, we can prove that $\{b_n\}$ decreases $\left(\text{just notice that } g(x) = \dfrac{2x}{2+\sqrt{4+x^2}} = \dfrac{2}{\frac{2}{x}+\sqrt{1+\frac{4}{x^2}}}\right)$.

It is a matter of simple manipulation to prove that $2f(x) > x$ for all $x \in (0,2)$. This implies that the sequence $\{2^n a_n\}$ is strictly increasing. The inequality $2g(x) < x$ for all $x \in (0,2)$ implies that the sequence $\{2^n b_n\}$ is strictly decreasing. By an easy induction, one can show that $a_n^2 = \dfrac{4b_n^2}{4+b_n^2}$ for positive integers n. Since the limit of the *decreasing* sequence $\{2^n b_n\}$ of *positive* numbers is finite, we have

$$\lim 4^n a_n^2 = \lim \frac{4\cdot 4^n b_n^2}{4+b_n^2} = \lim 4^n b_n^2 .$$

Thus, we know that the limits $\lim 2^n a_n$ and $\lim 2^n b_n$ are equal. The first of the two is positive because the sequence of positive numbers, $\{2^n a_n\}$, is strictly increasing. The existence of a number C follows from the equalities

$$2^n b_n - 2^n a_n = \frac{4^n b_n^2 - \dfrac{4^{n+1} b_n^2}{4+b_n^2}}{2^n b_n + 2^n a_n} = \frac{(2^n b_n)^4}{4+b_n^2}\cdot\frac{1}{4^n}\cdot\frac{1}{2^n(b_n+a_n)}$$

and from the existence of positive limits $\lim 2^n b_n$ and $\lim 2^n a_n$.

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of the sine and cosine functions. It is sufficient to notice that $a_n = \sin\left(\frac{\pi}{2^{n+1}}\right)$ and $b_n = \tan\left(\frac{\pi}{2^{n+1}}\right)$.

Problem 3. Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater that $\sqrt{2}$.

Solution. The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} x_k^2 = 1 \right\} .$$

The distance, $d(X,Y)$ between the points $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ is given by

$$d^2(X,Y) = \sum_{k=1}^{n} (x_k - y_k)^2 .$$

We have

$$\begin{aligned} d(X,Y) > \sqrt{2} &\iff d^2(X,Y) > 2 \\ &\iff \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 - 2\sum_{k=1}^{n} x_k y_k > 2 \\ &\iff \sum_{k=1}^{n} x_k y_k < 0 . \end{aligned} \tag{1}$$

Because of the symmetry of the sphere, we may suppose that

$$A_1 = (-1, 0, \ldots, 0) .$$

For $X = A_1$, condition (1) implies that $y_1 > 0 \ \forall \ Y \in M_n$.

Let $X = (x_1, \overline{X})$, $Y = (y_1, \overline{Y}) \in M_n \setminus \{A_1\}$, $\overline{X}$, $\overline{Y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n} x_k y_k < 0 \implies x_1 y_1 + \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0 \implies \sum_{k=1}^{n-1} x'_k y'_k < 0 ,$$

where

$$x'_k = \frac{\overline{x}_k}{\sqrt{\sum \overline{x}_k^2}} , \qquad y'_k = \frac{\overline{y}_k}{\sqrt{\sum \overline{y}_k^2}} .$$

Therefore, $\qquad (x'_1, \ldots, x'_{n-1}) , \ (y'_1, \ldots, y'_{n-1}) \in S_{n-2} ,$

and this verifies condition (1).

If a_n is the number of points in $\mathbb{R}^n$ sought, we have $a_n \leq 1 + a_{n-1}$, and $a_1 = 2$ implies that $a_n \leq n + 1$.

We show that $a_n = n+1$, giving an example of a set in M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$\begin{aligned}
A_1 &= (-1, 0, 0, 0, \dots, 0, 0) \\
A_2 &= \left(\tfrac{1}{n}, -c_1, 0, 0, \dots, 0, 0\right) \\
A_3 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1}c_1, -c_2, 0, \dots, 0, 0\right) \\
A_4 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1}c_1, \tfrac{1}{n-2}c_2, -c_3, \dots, 0, 0\right) \\
&\ \vdots \\
A_{n-1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1}c_1, \tfrac{1}{n-2}c_2, \tfrac{1}{n-3}c_3, \dots, -c_{n-2}, 0\right) \\
A_n &= \left(\tfrac{1}{n}, \tfrac{1}{n-1}c_1, \tfrac{1}{n-2}c_2, \tfrac{1}{n-3}c_3, \dots, \tfrac{1}{2}c_{n-2}, -c_{n-1}\right) \\
A_{n+1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1}c_1, \tfrac{1}{n-2}c_2, \tfrac{1}{n-3}c_3, \dots, \tfrac{1}{2}c_{n-2}, c_{n-1}\right),
\end{aligned}$$

where

$$c_k = \sqrt{\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n-k+1}\right)}, \qquad k = 1, 2, \dots, n-1.$$

We have

$$\sum_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0 \quad \text{and} \quad \sum_{k=1}^{n} x_k^2 = 1 \qquad \forall\, X, Y \in \{A_1, \dots, A_{n+1}\}.$$

These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

$$d = \sqrt{2}\sqrt{1+\frac{1}{n}} > \sqrt{2}\,.$$

Remark. For $n = 2$, the points form an equilateral triangle in the unit circle; for $n = 3$, the four points form a regular tetrahedron; and in $\mathbb{R}^n$, the points form an n–dimensional regular simplex.

Problem 4. Let $A = (a_{k,t})_{k,t=1,\dots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \dots, n\}$ and $1 \le j_1 < \dots < j_m \le n$ the determinant of the matrix $(a_{j_k,j_t})_{k,t=1,\dots,m}$ is zero. Prove that (1) $A^n = 0$ and (2) that there exists a permutation $\sigma \in S_n$ such that the matrix

$$(a_{\sigma(k),\sigma(t)})_{k,t=1,\dots,n}\,.$$

has all of its non-zero elements above the diagonal.

Solution. We shall prove only (2), since it implies (1).

Consider a directed graph G with n vertices $V_1, \dots, V_n$, and a directed edge from V_k to V_t, where $a_{k,t} \neq 0$. We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length m. Let $j_1 < \cdots < j_m$ be the vertices that the cycle goes through, and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k, j_{\sigma_0(k)}} \neq 0$ for $k = 1, \ldots, m$. Observe that for any other $\sigma \in S_n$, we have $a_{j_k, j_{\sigma(k)}} = 0$ for some $k \in \{1, \ldots, m\}$; for otherwise, we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally,

$$\begin{aligned} 0 &= \det\left(a_{j_k, j_t}\right)_{k,t=1,\ldots,m} \\ &= (-1)^{\operatorname{sgn}\sigma_0} \prod_{k=1}^{m} a_{j_k, j_{\sigma_0}(k)} + \sum_{\sigma \neq \sigma_0} (-1)^{\operatorname{sgn}\sigma} \prod_{k=1}^{m} a_{j_k, j_\sigma(k)} \neq 0\,, \end{aligned}$$

which is a contradiction.

Since G is acyclic, there exists a topological ordering; that is, a permutation $\sigma \in S_n$ such that $k < t$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(t)}$. It is easy to see that this permutation solves the problem.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x+y) \geq f(x) + yf(f(x)) \text{ for all } x, y \in \mathbb{R}\,.$$

Solution. Suppose that there exists a function satisfying the inequality. If $f(f(x)) \leq 0$ for all x, then f is a decreasing function because of the inequalities $f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$ for any $y \leq 0$. Since $f(0) > 0 \geq f(f(x))$, we have $f(x) > 0$ for all x, which is a contradiction.

Hence, there is a z such that $f(f(z)) > 0$. Then, the inequality $f(z+x) \geq f(z) + xf(f(z))$ shows that $\lim\limits_{x\to\infty} f(x) = +\infty$, and, therefore, $\lim\limits_{x\to\infty} f(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that $f(x) \geq 0$, $f(f(x)) > 1$, $y \geq \dfrac{x+1}{f(f(x))-1}$ and $f(f(x+y+1)) \geq 0$. Then, $f(x+y) \geq f(x) + yf(f(x)) \geq x + y + 1$ and hence,

$$\begin{aligned} f(f(x+y)) &\geq f(x+y+1) + (f(x+y) - (x+y+1))\, f(f(x+y+1)) \\ &\geq f(x+y+1) \geq f(x+y) + f(f(x+y)) \\ &\geq f(x) + yf(f(x)) + f(f(x+y)) > f(f(x+y))\,. \end{aligned}$$

This contradiction completes the solution of the problem.

Problem 6. For each positive integer n, let $f_n(\theta) = \sin\theta \cdot \sin(2\theta) \cdot \sin(4\theta) \cdots \sin(2^n\theta)$.

For all real θ and all n, prove that

$$|f_n(\theta)| \leq \frac{2}{\sqrt{3}}\,|f_n(\pi/3)|\,.$$

Solution. We first prove that $g(\theta) = |\sin\theta|\,|\sin(2\theta)|^{\frac{1}{2}}$ attains it maximum value $\left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}$ at points $2^k\frac{\pi}{3}$ (where k is a positive integer). This can be seen by using derivatives of a classical bound like

$$\begin{aligned} |g(\theta)| &= |\sin\theta|\,|\sin(2\theta)|^{\frac{1}{2}} \\ &= \frac{\sqrt{2}}{\sqrt[4]{3}}\left(\sqrt[4]{|\sin\theta|\,|\sin\theta|\,|\sin\theta|\,\left|\sqrt{3}\cos\theta\right|}\right)^2 \\ &\le \frac{\sqrt{2}}{\sqrt[4]{3}}\,\frac{3\sin^2\theta + 3\cos^2\theta}{4} \;=\; \left(\frac{\sqrt{3}}{2}\right)^{\frac{3}{2}}. \end{aligned}$$

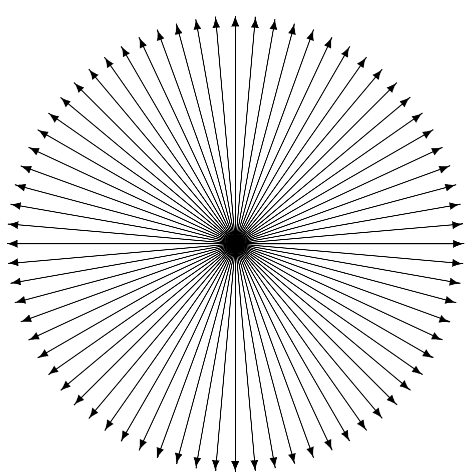
Hence,

$$\begin{aligned} \left|\frac{f_n(\theta)}{f_n\left(\frac{\pi}{3}\right)}\right| &= \left|\frac{g(\theta)\,g(2\theta)^{\frac{1}{2}}\,g(4\theta)^{\frac{3}{4}}\ldots g(2^{n-1}\theta)^E}{g\left(\frac{\pi}{3}\right)\,g\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}\,g\left(\frac{4\pi}{3}\right)^{\frac{3}{4}}\ldots g\left(\frac{2^{n-1}\pi}{3}\right)^E}\right| \left|\frac{\sin(2^n\theta)}{\sin\left(\frac{2^n\pi}{3}\right)}\right|^{1-\frac{E}{2}} \\ &\le \left|\frac{\sin(2^n\theta)}{\sin\left(\frac{2^n\pi}{3}\right)}\right|^{1-\frac{E}{2}} \;\le\; \left(\frac{1}{\frac{\sqrt{3}}{2}}\right)^{1-\frac{E}{2}} \;\le\; \frac{2}{\sqrt{3}}, \end{aligned}$$

where $E = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. This is exactly the bound that we were required to obtain.

Correction: 2001 APIC Math Competition [2002 : 2].

4. Show that $\lim\limits_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(2n)!}{n!}} = \frac{4}{e}$.



THE OLYMPIAD CORNER

No. 222

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

As a problem selection this issue we feature the first half of a selection of problems from the St. Petersburg Contests 1965–1984. The problems were selected (and translated) by two students and forwarded to me by Andy Liu, University of Alberta, Edmonton, Alberta. I hope you enjoy the selections by Oleg Ivrii and by Robert Berrington Leigh.

ST. PETERSBURG CONTESTS 1965–1984
Problems from Various Contests

1. There are n glasses each big enough to hold all the water. Initially, all glasses contain the same amount of water. It is allowed to pour from any glass to any other glass as much water as in the second glass. For which values of n is it possible to collect all water into one glass?

2. The point C is on the segment AB. A straight line through C intersects the circle with diameter AB at E and F, the circle with diameter AC again at M and the circle with diameter BC again at N. Prove that $MF = EN$.

3. A game starts with a pile of 25 markers. Two players alternatively remove 1, 2 or 3 of them. When all the markers are taken, the winner is the player who has an even number of markers. Assuming perfect play, does the first player or the second have a sure win?

4. The sides of a heptagon $A_1A_2A_3A_4A_5A_6A_7$ have equal length. From a point O inside, perpendiculars are dropped to the sides A_1A_2, A_2A_3, $\ldots$, A_7A_1, meeting them, and not their extensions, at H_1, H_2, $\ldots$, H_7, respectively. Prove that

$$A_1H_1 + A_2H_2 + \cdots + A_7H_7 \;=\; H_1A_2 + H_2A_3 + \cdots + H_7A_1\,.$$

5. There are $2N$ people at a party. Each knows at least N others. Prove that one can always choose four people and place them at a round table so that each person knows both neighbours.

6. Prove that any non-negative even integer can be uniquely represented as $(x+y)^2 + 3x + y$ where x and y are non-negative integers.

7. In triangle ABC, the sides satisfy $AB + AC = 2BC$. Prove that the bisector of $\angle A$ is perpendicular to the line segment joining the incentre and circumcentre of ABC.

8. In a town there are 10 parallel streets, and the other 10 intersect them at right angles. What is minimal number of turns in a closed path which passes through all street-intersections?

9. Four pedestrians were moving at uniform velocities along four straight roads in general positions. Two of them met each other as well as the other two. Prove that the other two also met.

10. At King Arthur's Court, $2n$ knights gathered at the Round Table. Each has at most $n-1$ enemies among the others. Prove that Merlin the wizard can devise a seating arrangement such that no knight will be next to any of his enemies.

11. Construct a set of circles with non-zero radii such that exactly one of them passes through each point of three-dimensional space.

12. Does there exist a positive integer n such that

$$27^n + 84^n + 110^n + 133^n \;=\; 144^n\,?$$

13. On an infinite one-dimensional board, n black markers and n white markers are placed in an alternating pattern in $2n$ adjacent squares, starting with a black marker. In each move, we can transfer two adjacent markers to any two adjacent empty squares. What is the minimum number of moves to reach the arrangement with n adjacent black markers followed immediately by n adjacent white markers?

14. Prove that

$$\sum_{i,j=1}^{\infty} \frac{a_i a_j}{i+j} \;\le\; \pi \sum_{k=1}^{\infty} a_k^2\,.$$

15. A circle with radius 10 centimetres is divided into regions by 32 lines. Prove that a circle with radius 3 millimetres can be placed inside one of the regions.

16. Decompose $235^2 + 972^2$ into two factors.

17. Students in a school go for ice cream in groups of at least two. No two students will go together more than once. After $k > 1$ groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at least k.

18. We choose 2^{p-1} subsets from a set with p elements such that any three have a common element. Prove that they all have a common element.

19. Let a, b and c be real numbers with sum 0. Prove that

$$\frac{a^7+b^7+c^7}{7} = \left(\frac{a^5+b^5+c^5}{5}\right)\left(\frac{a^2+b^2+c^2}{2}\right).$$

20. An inverted number triangle is constructed as follows. There are k 0's and a 1 in the top row. Each number in a subsequent row is 0 if the two numbers over it are equal, or 1 if they are not. For which k is it true that the single number in the last row is independent of the position of the 1 in the top row?

21. Segments AC and BD intersect at point E. Points K and M, on segments AB and CD respectively, are such that the segment KM passes through E. Prove that $KM \leq \max\{AC, BD\}$.

22. Prove that

$$\sum_{k=0}^{n}\binom{n}{k}(a+k)^{k-1}(b+n-k)^{n-k-1} = (a+b+n)^{n-1}\left(\frac{1}{a}+\frac{1}{b}\right).$$

23. The plane is divided into regions by n lines in general positions. Prove that at least $n-2$ of the regions are triangles.

Now we turn to solutions for X Form, Georgian Mathematical Olympiad 1997 given [2000 : 133].

1. Find all triples (x,y,z) of integers satisfying the inequality:

$$x^2+y^2+z^2+3 < xy+3y+2z.$$

Solutions by Pierre Bornsztein, Pontoise, France; by Christopher J. Bradley, Clifton College, Bristol, UK; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Klamkin's solution.

The inequality can be rewritten in the form

$$\left(x-\frac{y}{2}\right)^2+\frac{3(y-2)^2}{4}+(z-1)^2 < 1.$$

Hence, we have the following bounds:

$$-1 < z-1 < 1, \quad \frac{-2}{\sqrt{3}} < y-2 < \frac{2}{\sqrt{3}}, \quad -1 < x-\frac{y}{2} < 1,$$

so that we can only have $z=1$, $y=1$, 2, or 3 and $x=0$, 1, or 2. Finally, the only solution is

$$(x,y,z) = (1,2,1).$$

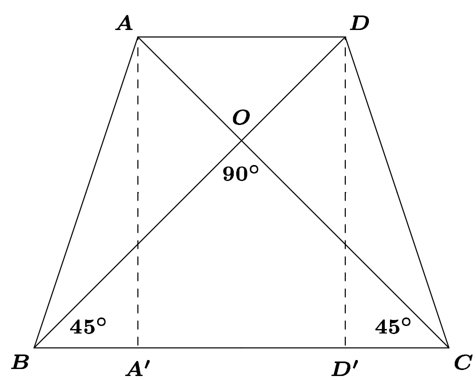
If the inequality sign $<$ in the given problem were to be changed to the equal sign $=$, we would have more solutions. Here we can only have $z = 0$, 1, or 2, $y = 1$, 2, or 3, and $x = 0$, 1, 2, or 3. Then the only solutions are $(x, y, z) = (0,1,1)$, $(0,2,1)$, $(1,2,0)$, $(1,1,1)$, $(1,3,1)$, $(1,2,2)$, $(2,2,1)$ and $(2,3,1)$.

4. The area of a given trapezoid is 2 cm^2 and the sum of its diagonals equals 4 cm. Find the altitude of the trapezoid.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give the solution of Amengual Covas.

It follows from the solution to a problem of the 44$^{\text{th}}$ Lithuanian Mathematical Olympiad given in the *Corner* in the October 1999 number [1999 : 337] that the diagonals of the given trapezoid ($ABCD$, $AD \| BC$, $AD < BC$, say) are equal and mutually perpendicular.

Now, it is not difficult to see that trapezoid $ABCD$ is isosceles. Draw perpendiculars AA' and DD' from A and D to the line BC. Right triangles $AA'C$ and $DD'B$ have two sides respectively equal, making them congruent, and giving $A'C = BD'$.



Subtracting $A'D'$ from each side gives $D'C = BA'$.

Hence, $AB = DC$, because they are hypotenuses in congruent right triangles $AA'B$ and $DD'C$.

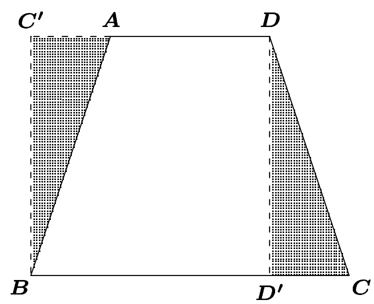
Let O be the intersection of the diagonals. Since angle BOC is a right angle, we have immediately that BOC is an isosceles right-angled triangle. The right triangle $DD'B$ has a $45°$ angle and therefore, is also an isosceles right-angled triangle. Hence,

$$BD' = DD'. \tag{1}$$

Moreover, since $ABCD$ is an isosceles trapezoid, we have the special relation that

$$\text{area of } ABCD = \overline{BD'} \cdot \overline{DD'}. \tag{2}$$

This is clear from the following figure.



From (1) and (2) it follows that

$$\overline{DD'} \;=\; \sqrt{\text{area of } ABCD}$$

so that the required altitude is equal to $\sqrt{2}$.

5. Prove that in any triangle the following inequality holds: $pR \geq 2S$, where p, R, S are respectively the semiperimeter, the radius of circumcircle and the area of the triangle.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's write-up.

Let a, b, c be the sides of the triangle. Since $R = \frac{abc}{4S}$, the inequality to be proved is equivalent to

$$p \cdot abc \;\geq\; 8S^2$$

or

$$abc \;\geq\; (a+b-c)(a+c-b)(b+c-a) \tag{1}$$

[via Heron's formula $S = \sqrt{p(p-a)(p-b)(p-c)}$].

Now denote by x, y, z the positive real numbers $\frac{-a+b+c}{2}$, $\frac{a-b+c}{2}$, $\frac{a+b-c}{2}$ (respectively). Then we have $a = y+z$, $b = z+x$, $c = x+y$ and (1) becomes

$$(y+z)(z+x)(x+y) \;\geq\; 8xyz\,.$$

But this inequality is certainly true since $y+z \geq 2\sqrt{yz}$, $z+x \geq 2\sqrt{zx}$ and $x+y \geq 2\sqrt{xy}$, so that we are done.

Next we look at reader solutions to problems of the 6$^{\text{th}}$ ROC (Taiwan) Mathematical Olympiad given [2000 : 134–135].

Part I

1. Let a be a rational number, b, c, d be real, and the function $f : R \to [-1, 1]$ satisfying

$$f(x + a + b) - f(x + b) = c \cdot \lfloor x + 2a + \lfloor x\rfloor - 2\lfloor x + a\rfloor - \lfloor b\rfloor\rfloor + d$$

for each $x \in R$, where $\lfloor t\rfloor$ denotes the largest integer that is less than or equal to t. Show that f is a periodic function (that is, there is a positive number p such that $f(x + p) = f(x)\ \forall x \in R$).

Solution by Mohammed Aassila, Strasbourg, France.

If we replace x by $x - b + n$, where n is any integer, we obtain

$$\begin{aligned} & f(x + n + a) - f(x + n) \\ &\quad = \ c\lfloor x - b + n + 2a + \lfloor x - b + n\rfloor - 2\lfloor x - b + n + a\rfloor - \lfloor b\rfloor\rfloor + d \\ &\quad = \ c\lfloor x - b + 2a + \lfloor x - b\rfloor - 2\lfloor x - b + a\rfloor - \lfloor b\rfloor\rfloor + d \\ &\quad = \ f(x + a) - f(x)\,. \end{aligned}$$

If $a = \frac{p}{q}$ with $q \neq 0$ and $(p, q) = 1$, then we claim that $f(x + aq) = f(x)$ for all $x \in \mathbb{R}$. Indeed, by the relation $f(x+n+a) - f(x+n) = f(x+a) - f(x)$ we have for all integers m

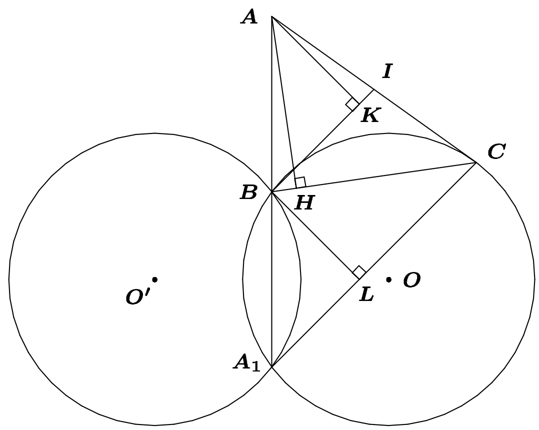
$$\begin{aligned} f(x + maq) - f(x) &= m\sum_{i=1}^{q}(f(x + ai) - f(x + a(i - 1))) \\ &= m(f(x + aq) - f(x))\,. \end{aligned}$$

But, since $|f(x)| \leq 1$, we deduce that $f(x + aq) = f(x)$.

2. Let AB be a given line segment. Find all possible points C in the plane such that in $\triangle ABC$, the height from the vertex A and the length of the median from the vertex B are equal.

Solutions by Mohammed Aassila, Strasbourg, France; and by Michel Bataille, Rouen, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give Bataille's solution.

Let A_1 be symmetric to A about B. If C is any point not on the line AB, denote by I the mid-point of AC and by H, K, L the projections of A onto BC, A onto BI, B onto CA_1 respectively (see figure). Note that $AK = BL$.



The relation $[ABC] = 2[ABI]$ yields $AH \cdot BC = 2AK \cdot BI$ or $AH \cdot BC = 2BL \cdot BI$. It follows that $AH = BI$ if and only if $\dfrac{BL}{BC} = \dfrac{1}{2}$; that is, if and only if $\angle BCA_1 = 30^\circ$ or 150°. Therefore, the locus of points C such that $AH = BI$ is the union of the two circles passing through B and A_1 and centred at O and O' defined by

$$\angle BOA_1 = \angle BO'A_1 = 60^\circ$$

[points B and A_1 must be deleted for $\triangle ABC$ to be non-degenerate].

3. Let $n \geq 3$. Suppose that the sequence $a_1, a_2, \ldots, a_n$ of positive real numbers satisfies $a_{i-1} + a_{i+1} = k_i a_i$, $\forall i = 1, 2, \ldots, n$, where each k_i is a positive integer, $a_0 = a_n$, $a_{n+1} = a_1$. Show that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n .$$

Solution by Mohammed Aassila, Strasbourg, France.

Since $k_i = \dfrac{a_{i-1} + a_{i+1}}{a_i} = \dfrac{a_{i-1}}{a_i} + \dfrac{a_{i+1}}{a_i}$ for all $i = 1, 2, \ldots, n$, we have

$$k_1 + k_2 + \cdots + k_n = \sum_{i=1}^{n} \left(\frac{a_i}{a_{i+1}} + \frac{a_{i+1}}{a_i} \right) \geq n \times 2 = 2n .$$

To prove the other inequality, we use induction on n: for $n = 3$ the result is easily verified. Assume that $k_1 + k_2 + \cdots + k_{n-1} \leq 3n - 3$ and let us prove the following

$$k_1 + k_2 + \cdots + k_n \leq 3n .$$

If all the a_i are equal, then it is true. Otherwise, there exists i such that ($a_i \geq a_{i-1}$ and $a_i > a_{i+1}$) or ($a_i > a_{i-1}$ and $a_i \geq a_{i+1}$). Hence, $a_{i-1} + a_{i+1} < 2a_i$ and thus, $k_i = 1$, and consequently the sequence $\{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$ satisfies the same condition with $\{k_1, k_2, \ldots, k_{i-2}, k_{i-1} - 1, k_{i+1} - 1, k_{i+2}, \ldots, k_n\}$. Since

$$k_1 + k_2 + \cdots + k_{i-2} + k_{i-1} - 1 + k_{i+1} - 1 + k_{i+2} + \cdots + k_n \leq 3n - 3 ,$$

then we obtain

$$k_1 + k_2 + \cdots + k_n \;\le\; 3n - 3 + 2 + 1 \;=\; 3n\,.$$

Part II

1. Let $k = 2^{2^n} + 1$ for some positive integer n. Show that k is a prime if and only if k is a factor of $3^{(k-1)/2} + 1$.

Comments by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Pontoise, France.

This result is known as Pépin's Theorem which states:

A necessary and sufficient condition for the Fermat number $F_n = 2^{2^n} + 1$, $n \ge 1$ to be prime is that

$$3^{2^{2^n - 1}} \;\equiv\; -1 \pmod{F_n}\ .$$

References:

[1] H. Riesel, *Prime Numbers and Computer Methods for the Factorization*, Birkhäuser, Boston, 1994 (Theorem 4.5, pp. 100–101).

[2] P. Ribenboim, *The Book of Prime Number Records*, Springer, p. 71–72.

2. Let $ABCD$ be a tetrahedron. Show that

(i) if $AB = CD$, $AD = BC$, $AC = BD$, then $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are acute triangles;

(ii) if the area of $\triangle ABC$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$ are the same, then $AB = CD$, $AD = BC$, $AC = BD$.

Solution by Michel Bataille, Rouen, France. Comments by Mohammed Aassila, Strasbourg, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We given Klamkin's "backgrounder".

These are classic results about isosceles tetrahedra which are defined as having their opposite sides equal. Many of their properties are given in [1]. Since more and more of these kinds of problems are appearing in this section and many students are not aware of these results, I am just listing a number of them as given in order in [1] except for a proof of (i).

Theorem 293. The faces of an isosceles tetrahedron are congruent triangles.

Theorem 294. The four altitudes of an isosceles tetrahedron are equal.

Theorem 295. In an isosceles tetrahedron the medians are equal, and conversely, if the medians are equal, the tetrahedron is isosceles.

Theorem 296. The bimedians of an isosceles tetrahedron coincide with the bialtitudes and form a trirectangular trihedral angle.

Theorem 297. If two of the bimedians of a tetrahedron coincide with the respective bialtitudes, the tetrahedron is isosceles.

Theorem 298. In an isosceles tetrahedron the circumcentre and the centroid coincide, and conversely, if in a tetrahedron the circumcentre and the centroid coincide, the tetrahedron is isosceles.

Theorem 299. The Monge point of an isosceles tetrahedron coincides with the centroid, and conversely, if the Monge point coincides with the centroid, the tetrahedron is isosceles.

Theorem 300. In an isosceles tetrahedron, the incentre coincides with the centroid.

Corollary I 301. The inradius of an isosceles tetrahedron is equal to one fourth of the altitude of the tetrahedron.

Corollary II 302. The points of contact of the faces of an isosceles tetrahedron with its inscribed sphere are the circumcentres of these faces.

Converse Theorem 303. If the incentre of a tetrahedron coincides with the centroid, the tetrahedron is isosceles.

Theorem 304. If the circumcentre and the incentre of a tetrahedron coincide, the tetrahedron is isosceles.

Theorem 306. If the altitudes of a tetrahedron are equal, the tetrahedron is isosceles.

It is to be noted that part (ii) follows from Theorem 306. For (i), we use a more general result given in a problem, also in [1]; that is, in a tetrahedron the product of any pair of opposite edges is smaller than the sum of the products of the other two pairs of opposite edges. It now follows that if the three coterminal edges of the given isosceles tetrahedron are a, b, c, then the sum of any two of a^2, b^2, c^2 is greater than the third and thus a triangle of sides a, b, c is acute. The proof that we now give for this more general result involves ideas which can be useful in obtaining other geometric inequalities.

In tetrahedron $DABC$, let P be the foot of the altitude from D and let $DP = h$. Also let $DA = a$, $DB = b$, $DC = c$, $BC = a'$, $CA = b'$, $AB = c'$, $AP = x$, $BP = y$ and $CP = z$. We first obtain the two-dimensional version of our desired result; that is, the sum of any two of xa', yb', zc' is greater than the third. Now let u, v, w denote complex numbers from origin P to the respective vertices A, B, C. Since $u(v-w)+v(w-u)+w(u-v)=0$, it follows by the triangle inequality that

$$|u(v-w)|+|v(w-u)| > |w(u-v)| \quad \text{or} \quad xa'+yb' > zc', \text{etc.}$$

We now use $a^2 = x^2 + h^2$, etc., and Minkowski's Inequality to give

$$\begin{aligned} & a'\sqrt{x^2+h^2}+b'\sqrt{y^2+h^2} \\ & \quad \geq \sqrt{(a'x+b'y)^2+(a'+b')^2h^2} > \sqrt{(c'z)^2+(c'h)^2}\,, \end{aligned}$$

or $a'a + b'b > c'c$, etc.

Reference

[1] N. Altshiller-Court, Modern Pure Solid Geometry, Macmillan, N.Y., 1935, pp. 94–98, 250.

Part III

1. Determine all the possible integers k such that there is a function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ such that

(i) $f(1997) = 1998$,

(ii) $f(ab) = f(a) + f(b) + k \cdot f(d(a,b))$, $\forall a,\, b \in \mathbb{N}$, where $d(a,b)$ denotes the greatest common divisor of a and b.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; and by Pierre Bornsztein, Pontoise, France. We give Bataille's solution.

Such a function exists if and only if $k = 0$ or $k = -1$.

First suppose that such an f exists. Then, for any prime number p, we have

$$f(p^2) \;=\; f(p) + f(p) + kf(p) \;=\; (k+2)f(p)$$

and

$$f(p^3) \;=\; f(p) + f(p^2) + kf(p) \;=\; (2k+3)f(p)\,.$$

Hence, we may write $f(p^4) = f(p^2) + f(p^2) + kf(p^2) = (k+2)^2 f(p)$ as well as $f(p^4) = f(p) + f(p^3) + kf(p) = (3k+4)f(p)$.

Taking $p = 1997$, for which $f(p) \neq 0$, yields $(k+2)^2 = 3k+4$; hence $k = 0$ or $k = -1$.

Conversely, set $f(1) = 0$ and for $a > 1$ with standard factorization into primes $a = p_1^{s_1} \cdots p_r^{s_r}$ where $s_1, \ldots, s_r$ are positive integers and $p_1, \ldots, p_r$ are distinct prime numbers, set

$$\begin{array}{rcl} f(a) & = & s_1(p_1+1) + \cdots + s_r(p_r+1) \\ \left[\text{respectively, } f(a)\right. & = & (p_1+1) + \cdots + (p_r+1) \;=\; \left. p_1 + \cdots + p_r + r\right]. \end{array}$$

Then, f satisfies conditions (i) (obviously) and (ii) with $k = 0$ [respectively, $k = -1$].

Indeed, (ii) is true when a or $b = 1$ and if $a = p_1^{s_1} \cdots p_r^{s_r}$, $b = q_1^{t_1} \cdots q_m^{t_m}$ are the standard factorization into primes of a and $b > 1$.

When a, b are coprime, we readily have $f(ab) = f(a) + f(b)$ in both cases, as desired (since $f(d(a,b)) = f(1) = 0$).

When $d(a,b) > 1$, we may suppose that $p_1 = q_1, \ldots, p_n = q_n$ ($n \leq \min(r,m)$) are the common prime factors of a and b. Then

$$ab \;=\; p_1^{s_1+t_1} \cdots p_n^{s_n+t_n} p_{n+1}^{s_{n+1}} \cdots p_r^{s_r} q_{n+1}^{t_{n+1}} \cdots q_m^{t_m}$$

so that

$$\begin{aligned}f(ab) &= (s_1+t_1)(p_1+1)+\cdots+(s_n+t_n)(p_n+1)+s_{n+1}(p_{n+1}+1)\\ &\qquad +\cdots+s_r(p_r+1)+t_{n+1}(q_{n+1}+1)+\cdots+t_m(q_m+1)\\ &= f(a)+f(b) \;=\; f(a)+f(b)+0\cdot f(d(a,b))\end{aligned}$$

[respectively,

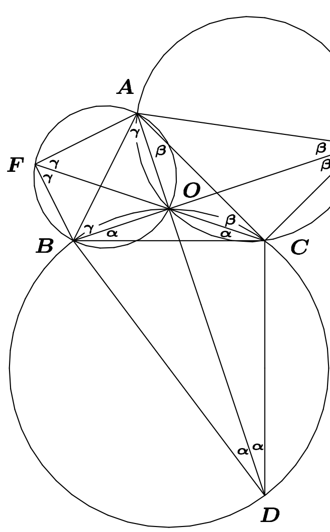
$$\begin{aligned}f(ab) &= p_1+\cdots+p_n+p_{n+1}+\cdots+p_r+q_{n+1}+\cdots+q_m+r+m-n\\ &= (p_1+\cdots+p_r+r)+(q_1+\cdots+q_m+m)-(q_1+\cdots+q_n+n)\\ &= f(a)+f(b)-f(d(a,b))\big]\,.\end{aligned}$$

But, from $f(a^2)=(k+2)f(a)$, we also obtain $f(a^4)=(k+2)f(a^2)=(k+2)^2f(a)$. Hence, $(k+2)^2=3k+4$ (we can take $a=1997$), so that $f(a)\neq 0$, and such a function f exists for $k=0$ and $k=-1$.

2. Let $\triangle ABC$ be an acute triangle with circumcentre O and circumradius R. Show that if AO meets the circle OBC again at D, BO meets the circle OCA again at E, and CO meets the circle OAB again at F, then $OD\cdot OE\cdot OF\geq 8R^3$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Christopher J. Bradley, Clifton College, Bristol, UK. We give the comment and solution of Aassila (adapted slightly by the editor).

This problem was proposed to the jury but not used at the 37th IMO at Mumbai, India. A solution appeared in this journal [1999 : 8–9]. We propose here a new solution.



First note the following similar triangles: $\triangle BCE\sim\triangle DCA$, $\triangle CAF\sim\triangle EAB$, $\triangle ABD\sim\triangle FBC$.

Hence,

$EC:CB:BE=AC:CD:DA$, etc.

Use Ptolemy's Theorem in the three cyclic quadrilaterals that share O as a vertex. In these cyclic quadrilaterals, we have

$$\begin{aligned}OD\cdot BC &= OC\cdot BD+OB\cdot DC\\ OD &= R\left(\frac{BD}{BC}+\frac{DC}{BC}\right),\ \text{etc.}\\ &= R\left(\frac{AD}{CF}+\frac{AD}{BE}\right),\ \text{etc.}\end{aligned}$$

Therefore, $OD\cdot OE\cdot OF\;=$

$$R^3\left[\left(\frac{AD}{CF}+\frac{CF}{AD}\right)+\left(\frac{AD}{BE}+\frac{BE}{AD}\right)+\left(\frac{BE}{CF}+\frac{CF}{BE}\right)+2\right]$$

$$\geq\; R^3(2+2+2+2) \;=\; 8R^3\,.$$

3. Let $X = \{1, 2, 3, \ldots, n\}$, $n \geq k \geq 3$, and let F_k be a family of subsets of X with k elements, so that any two subsets in F_k have at most $k-2$ common elements. Show that for each $k \geq 3$ there exists a subset M_k of X with at least $\lfloor \log_2 n \rfloor + 1$ elements such that it does not contain any subset in F_k.

Solution by Mohammed Aassila, Strasbourg, France.

First, notice that if $k > \log_2 n$, then there is nothing to prove. Otherwise, since each $(k-1)$–element subset of X lies in at most one subset of F_k, and each element of F_k contains k $(k-1)$–element subsets, we have

$$\mathrm{card}(F_k) \;\leq\; \frac{1}{k}\binom{n}{k-1} \;=\; \frac{1}{n-k+1}\binom{n}{k},$$

where $\mathrm{card}(F_k)$ is the number of elements of F_k.

Now, for a randomly chosen $(\lfloor \log_2 n \rfloor + 1)$–element subset of X, the number of elements of F_k that it contains is

$$\begin{aligned}
\binom{\lfloor \log_2 n \rfloor + 1}{k}\frac{\mathrm{card}(F_k)}{\binom{n}{k}} &\leq \frac{1}{n-k+1}\binom{\lfloor \log_2 n \rfloor + 1}{k} \\
&\leq \frac{1}{n-k+1} 3 \cdot 2^{\lfloor \log_2 n \rfloor - 2} \\
&\leq \frac{3n}{4(n-k+1)} \\
&< 1 .
\end{aligned}$$

Hence, some $\lfloor \log_2 n \rfloor + 1$-element subset must contain no element of F_k.

To complete this number of the *Corner*, we give solutions from our readers to problems of the 11^{th} Iberoamerican Mathematical Olympiad, 1996 — Costa Rica given [2000 : 136–137].

1. (Brazil): Let n be a natural number. A cube of side n can be split into 1996 cubes. The sides of these cubes are, also, natural numbers. Determine the minimum possible value of n.

Solution by Mohammed Aassila, Strasbourg, France.

n is certainly greater than or equal to 13 because $12^3 = 1728 < 1996$. But, in fact, 13 is the minimum possible value of n since we can place one cube of edge 5, one cube of edge 4, and two cubes of edge 2, in the cube of edge 13. The total number of cubes used is: $13^3-(5^3-1)-(4^3-1)-2(2^3-1) = 1996$.

2. (Spain): Let M be the mid-point of the median AD of the triangle ABC (D belongs to the side BC). The line BM meets the side AC at the point N. Show that AB is tangent to the circumcircle of the triangle NBC if and only if the equality

$$\frac{BM}{MN} = \frac{BC^2}{BN^2}$$

holds.

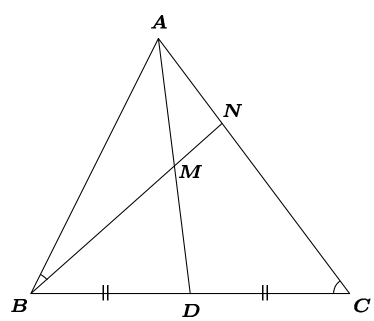
Solutions by Michel Bataille, Rouen, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.

By Menelaus' Theorem for $\triangle BCN$ we get

$$\frac{BM}{MN} \cdot \frac{NA}{AC} \cdot \frac{CD}{DB} = 1 .$$

Since $BD = DC$ we have

$$\frac{BM}{MN} = \frac{AC}{AN} . \qquad (1)$$



(a) If AB is tangent to the circumcircle of $\triangle NBC$, then we have

$$\frac{BM}{MN} = \frac{BC^2}{BN^2} .$$

Since $\angle ABN = \angle ACB$, we get $\triangle ABN \sim \triangle ACB$, so that

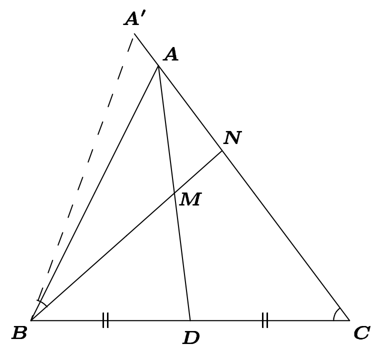
$$\frac{AN}{AB} = \frac{AB}{AC} = \frac{BN}{CB} .$$

Thus, $\frac{AN}{AB} \times \frac{AB}{AC} = \left(\frac{BN}{CB}\right)^2$; that is

$$\frac{AC}{AN} = \frac{BC^2}{BN^2} .$$

Therefore, we obtain from (1)

$$\frac{BM}{MN} = \frac{BC^2}{BN^2} .$$



(b) If $\dfrac{BM}{MN} = \dfrac{BC^2}{BN^2}$, then BA is tangent to the circumcircle of $\triangle NBC$.

From (1) we get

$$\frac{AC}{AN} = \frac{BM}{MN} = \frac{BC^2}{BN^2}. \tag{2}$$

Let the tangent to the circumcircle of $\triangle NBC$ at B meet AC at A'. Then we have

$$\frac{A'C}{A'N} = \frac{BC^2}{BN^2}. \tag{3}$$

Thus, we obtain from (2) and (3)

$$\frac{AC}{AN} = \frac{A'C}{A'N}.$$

Hence, A coincides with A', so AB is tangent to the circumcircle of $\triangle NBC$.

Therefore, AB is tangent to the circumcircle of $\triangle NBC$ if and only if the equality $\dfrac{BM}{MN} = \dfrac{BC^2}{BN^2}$ holds.

Comment. As shown in the proof, the condition that M is the mid-point of AD is not necessary.

3. (Spain): We have a chessboard of size $(k^2 - k + 1) \times (k^2 - k + 1)$, with $k = p + 1$, p being a prime number.

For each prime number p, give a method of distribution of the numbers 0 and 1, one number in each square of the chessboard, in such a way that in each row there are exactly k 0's; in each column, there are exactly k 0's; and moreover, no rectangle with sides parallel to the sides of the chessboard has a number 0 on [all four of] the vertices.

Comment by Pierre Bornsztein, Pontoise, France.

The solution of this problem and related results (in particular about links between the problem and the existence of projective planes with given order) may be found in: N.S. Mendelsohn, *Packing a square lattice with a rectangle-free set of points*, Math. Magazine (1987), pp. 229–233.

4. (Brazil): Given a natural number $n \geq 2$, all the fractions of the form $\frac{1}{ab}$, with a and b natural numbers, coprime and such that

$$a < b \leq n, \qquad a + b > n,$$

are considered. Show that the sum of all these fractions equals $\frac{1}{2}$.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's presentation.

Let $\mathcal{I}_n$ be the set of all pairs (a, b) of coprime integers satisfying $1 \leq a < b \leq n$ and $a + b > n$. We observe:

- if $(a, b) \in \mathcal{I}_n$, then $(a, b) \in \mathcal{I}_{n+1}$ except when $a + b = n + 1$.
- if $(a, b) \in \mathcal{I}_{n+1}$, then $(a, b) \in \mathcal{I}_n$ except when $b = n + 1$.

From these observations, denoting by S_n the sum $\sum\limits_{(a,b)\in\mathcal{I}_n} \frac{1}{ab}$, we get $S_{n+1} - S_n = s_1 - s_2$, where s_1 is the sum of all $\frac{1}{ab}$ with $(a, b) \in \mathcal{I}_{n+1}$ and $b = n + 1$ and s_2 is the sum of all $\frac{1}{ab}$ with $(a, b) \in \mathcal{I}_n$ and $a + b = n + 1$.

Using $\gcd(a, n + 1) = 1 \iff \gcd(n + 1 - a, n + 1) = 1$, we may write

$$\begin{aligned} s_1 &= \sum_{1 \leq a < n+1}^{*} \frac{1}{(n+1)a} \\ &= \sum_{1 \leq a < \frac{n+1}{2}}^{*} \frac{1}{n+1}\left(\frac{1}{a} + \frac{1}{n+1-a}\right) = \sum_{1 \leq a < \frac{n+1}{2}}^{*} \frac{1}{a(n+1-a)} \end{aligned}$$

where the $*$ indicates that we keep only terms for which $\gcd(a, n + 1) = 1$.

Now, if $(a, b) \in \mathcal{I}_n$ and $a + b = n + 1$, we have $b = n + 1 - a$ and $a < \frac{n+1}{2}$ (because $n + 1 = a + b > 2a$), so that

$$s_2 = \sum_{\gcd(a,n+1-a)=1}^{**} \frac{1}{a(n+1-a)} = \sum_{\gcd(a,n+1)=1}^{**} \frac{1}{a(n+1-a)}$$

where the $**$ indicates that we keep only terms for which $1 \leq a < \dfrac{n+1}{2}$.

Thus, $s_2 = s_1$ and $S_{n+1} = S_n$. It follows that $S_n = S_1 = \frac{1}{2}$ for all n.

5. (Peru): Three coins, A, B and C are situated one at each vertex of an equilateral triangle of side n. The triangle is divided in little equilateral triangles of side 1 by lines parallel to the sides.

At the beginning, all the lines of the figure are blue. The coins move along the lines, painting in red their trajectory, following the two following rules:

(i) First coin to move is A, then B, then C, then again A, and so on. At each turn, each coin paints exactly one side of one of the little triangles.

(ii) No one coin can move along a side of a triangle which is already painted red; but that coin can stay at the end of a painted segment, alone or with another coin waiting its turn at moving.

Show that, for all integers $n > 0$, it is possible to paint all the sides of all the little triangles red.

Solution by Pierre Bornsztein, Pontoise, France.

For $n \in \mathbb{N}^*$ denote by T_n the main equilateral triangle, with side n and P_n the following claim "It is possible to paint all the sides of all the little triangles in T_n red, in such a way that at the end of the process of the colouring, each vertex of T_n is occupied by a coin".

(i) First we prove that, if for a given n, P_n is true then P_{n+3} is true: Let $n > 0$ be given such that P_n is true. The coins are initially one at each vertex of T_{n+3}. The A, B, C move to A', B', C' respectively (see figure 1).

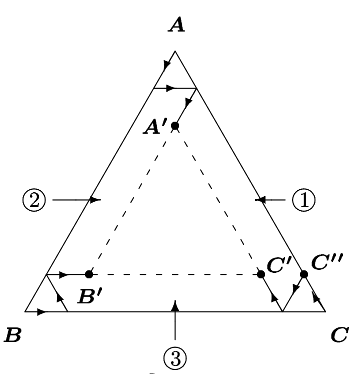


fig. 1

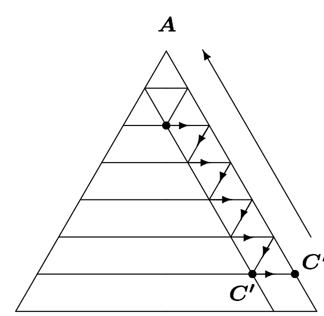


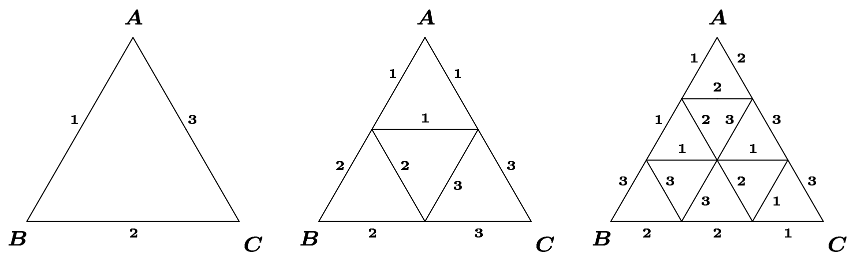
fig. 2

Since A', B', C' are the vertices of an equilateral T_n with side n, and since P_n is true, we may paint all the sides of all the little triangles of T_n red. Obviously this colouring does not go out of T_n. Moreover, (from P_n), the one situated in A' goes to A by following the path in the area ①:

- It moves alternatively horizontally and descending from right to left, until it reaches C'.
- It then goes from C' to C''.
- Then it goes from C'' to A along the side of T_{n+3} (see figure 2, for $n = 4$).

The coins situated at B' and C' move analogously in ② and ③. Thus P_{n+3} is true.

(ii) It follows from the above that we only have to find a good colouring in the cases $n = 1, 2, 3$.



In each case, A, B, C move along the edges numbered 1, 2, 3 respectively, completing the proof.

6. (Spain): We have n distinct points $A_1, \ldots, A_n$ in the plane. To each point A_i a real number $\lambda_i \neq 0$ is assigned, in such a way that

$$\overline{A_iA_j}^2 = \lambda_i + \lambda_j, \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Show that

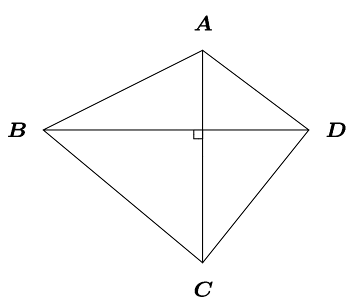
(a) $n \leq 4$.

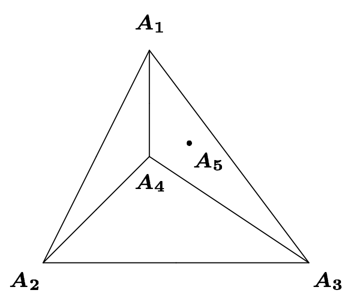
(b) If $n = 4$, then
$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0.$$

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Pontoise, France; and by Toshio Seimiya, Kawasaki, Japan. We give the solution by Seimiya.

The following lemma is a well-known theorem.

Lemma. A, B, C and D are four distinct points.
If $AB^2 + CD^2 = BC^2 + AD^2$, then $AC \perp BD$.





(a) We assume that $n \geq 5$.

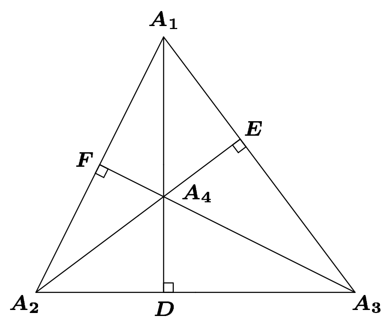
Since $\overline{A_1A_2}^2 + \overline{A_3A_4}^2 = (\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) = (\lambda_1 + \lambda_3) + (\lambda_2 + \lambda_4) = \overline{A_1A_3}^2 + \overline{A_2A_4}^2$, we have by, the lemma, $A_2A_3 \perp A_1A_4$.

Similarly, we get

$$\begin{aligned} \overline{A_1A_2}^2 + \overline{A_3A_4}^2 &= (\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) \\ &= (\lambda_1 + \lambda_4) + (\lambda_2 + \lambda_3) \;=\; \overline{A_1A_4}^2 + \overline{A_2A_3}^2 . \end{aligned}$$

Hence, we have $A_1A_3 \perp A_2A_4$. Therefore, A_4 is the orthocentre of $\triangle A_1A_2A_3$.

Similarly, A_5 is the orthocentre of $\triangle A_1A_2A_3$. Thus, A_5 coincides with A_4. This contradicts $A_4 \neq A_5$. Therefore, $n < 5$; that is, $n \leq 4$.



(b) By the above argument A_4 is the orthocentre of $\triangle A_1A_2A_3$. We may assume, without loss of generality, that A_4 is the orthocentre of acute triangle $A_1A_2A_3$, A_1A_4, A_2A_4 and A_3A_4 meet A_2A_3, A_3A_1 and A_1A_2 at D, E and F respectively, and we have $A_1D \perp A_2A_3$, $A_2E \perp A_3A_1$ and $A_3F \perp A_1A_2$. Since

$$\begin{aligned} 2\lambda_1 &= (\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_4) - (\lambda_2 + \lambda_4) \;=\; \overline{A_1A_2}^2 + \overline{A_1A_4}^2 - \overline{A_2A_4}^2 \\ &= 2\overline{A_1A_2} \cdot \overline{A_1A_4} \cos \angle A_2A_1A_4 \;=\; 2\overline{A_1A_2} \cdot \overline{A_1F} , \end{aligned}$$

so that
$$\lambda_1 = \overline{A_1A_2} \cdot \overline{A_1F} . \tag{1}$$

Similarly, we have

$$\begin{aligned} \lambda_2 &= \overline{A_2A_1} \cdot \overline{A_2F} , & (2) \\ \lambda_3 &= \overline{A_3A_2} \cdot \overline{A_3A_4} \cos \angle A_2A_3A_4 \;=\; \overline{A_3A_4} \cdot \overline{A_3F} & (3) \\ \lambda_4 &= \overline{A_4A_2} \cdot \overline{A_4A_3} \cos \angle A_2A_4A_3 \;=\; -\overline{A_3A_4} \cdot \overline{A_4F} . & (4) \end{aligned}$$

It follows from (1) and (2)

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2} = \frac{1}{\overline{A_1A_2}\cdot\overline{A_1F}}+\frac{1}{\overline{A_1A_2}\cdot\overline{A_2F}} = \frac{\overline{A_2F}+\overline{A_1F}}{\overline{A_1A_2}\cdot\overline{A_1F}\cdot\overline{A_2F}}$$

$$= \frac{\overline{A_1A_2}}{\overline{A_1A_2}\cdot\overline{A_1F}\cdot\overline{A_2F}} = \frac{1}{\overline{A_1F}\cdot\overline{A_2F}}. \qquad (5)$$

Since $\angle FA_1A_4 = \angle FA_3A_2$ we get $\triangle A_1FA_4 \sim \triangle A_3FA_2$, we have

$$\overline{A_1F} : \overline{A_3F} = \overline{A_4F} : \overline{A_2F}\text{; that is } \overline{A_1F}\cdot\overline{A_2F} = \overline{A_3F}\cdot\overline{A_4F}. \qquad (6)$$

Thus, we have from (3), (5) and (6),

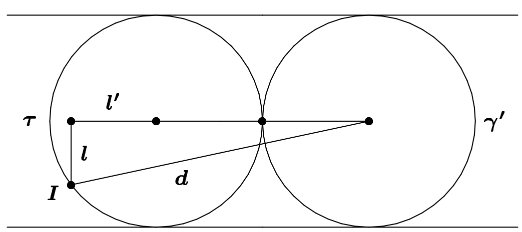
$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3} = \frac{1}{\overline{A_3F}\cdot\overline{A_4F}}+\frac{1}{\overline{A_3A_4}\cdot\overline{A_3F}}$$

$$= \frac{\overline{A_3A_4}+\overline{A_4F}}{\overline{A_3F}\cdot\overline{A_4F}\cdot\overline{A_3A_4}} = \frac{\overline{A_3F}}{\overline{A_3F}\cdot\overline{A_4F}\cdot\overline{A_3A_4}} = \frac{1}{\overline{A_3A_4}\cdot\overline{A_4F}}. \qquad (7)$$

Hence, we obtain from (4) and (7)

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4} = \frac{1}{\overline{A_3A_4}\cdot\overline{A_4F}}-\frac{1}{\overline{A_3A_4}\cdot\overline{A_4F}} = \mathbf{0}.$$

Finally, we give a correction to a diagram [2002 : 141] — the ℓ' was misplaced last month.



That wraps up our selection of problems and solutions to keep you busy over the summer hiatus. I look forward to receiving your Olympiad Contest problems as well as your solutions, comments and generalizations on the problems we have presented so far this year. Have a great summer vacation!

BOOK REVIEWS

JOHN GRANT McLOUGHLIN

π *unleashed*, by Jörg Arndt, Christoph Haenel,
(Translated from the German by Catriona and David Lischka) with CD-ROM
Springer-Verlag, 2001; xii+270 pp. ISBN 3-540-66572-2
Reviewed by ***Edward J. Barbeau****, University of Toronto, Toronto, Ontario.*

Likely, every ancient civilization had to address the question of determining either the circumference or the area of a circle with a given diameter, either by direct measurement or some rule-of-thumb. As people gained more sophistication with numbers or geometry, penetrating questions emerged that turned out to be deep and intractable. How good an approximation can we obtain? Can we construct within some framework a square whose area equals that of a given circle? What sort of number is π — rational or irrational, algebraic or transcendental? Along the way, π not only entered the popular culture but also turned out to be significant in broad areas of theoretical mathematics, with the result the story of π includes chapters ranging from the bizarre to the profoundly serious.

The task for the writer of a book on π is to sort through a vast and chaotic mound of material and come up with a concise work that gives the flavour of the mathematical treatment of this constant without skipping significant points nor trivializing the story. The authors of this volume have succeeded admirably. It is a beautiful book, gracefully written with enough solid mathematics to indicate important developments without getting bogged down in technicalities. It helps that the book is accompanied by a compact disc, so that readers can perform their own computations, draw their own conclusions about the digits of π, review source codes and embark on their own investigations.

The opening chapters provide an overview of the terrain, with a judicious mix of trivia, historical reference, big questions and sample computations. During the period from about 250 BCE until 1650, π was approximated geometrically. From 1650 to around 1980, infinite series and products predominated, especially those based on the arctangent function; a typical formula is $\pi = 4[4\arctan(1/5) - \arctan(1/239)]$. At the end of this era, intrepid π–calculators could bag the first million or so digits. However, the progress of the last twenty years has been almost beyond belief. High speed computers, more efficient algorithms and a completely new set of strategies have boosted the number of known digits of π to over 200 billion and allowed for the sampling of the digits of π beyond that. It is clear that the vitality of π research is greater than ever, and for this we have, among others, Canada's own Jonathan and Peter Borwein to thank. "Above all," say the authors, "we owe it to the Borweins that π research is today held in high esteem — even among mathematicians, of whom not a few believed that the whole topic had been laid to rest since Lindemann's proof of the transcendence of π." They suggest that there might be separate branches, e–mathematics and π–mathematics, the former being "linear, explicit, open to generalisation, easily approachable, highly algebraic" while the latter would be "non-linear, chthonic, barely generalisable, highly analytical and, via modular functions and Ramanujan identities, would have a significant impact on function theory, number theory and combinatorial logic".

The authors have structured the book carefully to keep the reader engaged. The pace is varied. There is lots of history, both ancient and current. Brief descriptions of oddities and tidbits of trivia offset more sustained discussion of significant progress; yet this is done without damage to the coherence of the story, so that a feeling of superficiality is avoided. Just about everything you have ever heard of — mnemonics, distribution of digits, the intuitionist example, Buffon's needle, representations by series and products, the irrationality and transcendence of π — is referred to somewhere. A bibliography of over 100 items helps the reader probe further; an interesting feature is that each entry lists the pages in the book that refer to it.

Separate chapters treat arctangent formulae, spigot algorithms (digits appear in sequence during rather than at the end of the computation), the arithmetic-geometric mean, Ramanujan's formulae, the Bailey-Borwein-Plouffe and similar series, fast multiplication algorithms and the internet project. The final two chapters tabulate formulae and algorithms, as well as digits and continued fraction elements of the expansions of π.

The authors hope, not just to interest readers, but to draw them into participating in further investigations of π. Schemata for algorithms and **C**–programs appear in the text; an appendix gives the documentation for the *h-float* library package for the accompanying disc so that you can mess around yourself.

This disc, however, presented a problem to me. My local computer technician was unable to get it up and running on our *Linux* system. However, a colleague, Stewart Craven, at the Toronto District Board of Education, enlisted the help of the Board's guru to get it up and running on his *Microsoft Explorer* system, and we spent a morning sampling its delights. Apart from containing files packed with information, including the digits of π, representations of π and programs, it allowed one to run the programs and it was quite interesting to compare the running times and how these times varied with the number of digits sought. Like the book, the disc appeared to be very well-organized and it was easy to find one's way around.

This book is at once accessible to and inspirational for mathematically-inclined high school students and serious mathematicians. It is a good acquisition for school, college and personal libraries.

Remark on the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+m)}$

Kenneth S. Williams

Let m be a positive integer and set

$$S(m) = \sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+m)}.$$

In a recent article in ***Crux with Mayhem***, it was shown [1, Corollary 5] that

$$S(m) = \sum_{k=1}^{m} \frac{(-1)^{m-k} \sum_{l=m-k+1}^{m} \frac{1}{l}}{(m-k)!k!}.$$

We note that

$$S(m) = \frac{1}{m\,m!}.$$

This is easily seen by summing the identity

$$\frac{m}{k(k+1)\cdots(k+m)} = \frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)(k+2)\cdots(k+m)}$$

over all positive integers k to obtain

$$mS(m) = S(m-1) - \left(S(m-1) - \frac{1}{m!}\right).$$

References

1. Z. Mashreghi and J. Mashreghi, *On the closed form of power series*, Crux Math. **27** (2001), 436–439.

Kenneth S. Williams
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario K1S 5B6
Canada
williams@math.carleton.ca

An Inequality for a Product of Logarithms

Erhard Braune

In this short article we prove an inequality for a product of logarithms. An example is then given where we use this result to get a lower bound for a product of two logarithms of tangent functions.

Theorem. Let β_1, γ_1, β_2, γ_2 be real positive numbers with $\beta_1 > \gamma_1$ and $\beta_2 > \gamma_2$. Let $U := \dfrac{\beta_1+\gamma_1}{\beta_1-\gamma_1}$ and $V := \dfrac{\beta_2+\gamma_2}{\beta_2-\gamma_2}$ and suppose $U \neq V$. Then

$$\big(\log U\big)\big(\log V\big) \;\geq\; \log^2\left(\frac{\beta_{12}+\gamma_{12}}{\beta_{12}-\gamma_{12}}\right) , \tag{1}$$

where $\beta_{12} = \beta_1\gamma_2 + \beta_2\gamma_1$, $\gamma_{12} = 2\gamma_1\gamma_2$, $\beta_{12} > \gamma_{12}$.

Proof. We have, using [1], p. 361, 3.551.6,

$$\log\left(\frac{\beta_0+\gamma_0}{\beta_0-\gamma_0}\right) \;=\; 2\int_0^\infty \frac{e^{-\beta_0 t}}{t}\sinh\left(\gamma_0 t\right)\,dt \tag{2}$$

for all real numbers β_0, γ_0 with $\beta_0 > |\gamma_0|$, and, where, by definition,

$$\sinh x \;=\; \frac{e^x - e^{-x}}{2} .$$

Since $\beta_1 > \gamma_1$ and $\beta_2 > \gamma_2$ we may apply (2) to the functions

$$\log\left(\frac{\beta_1+\gamma_1}{\beta_1-\gamma_1}\right) \qquad \text{and} \qquad \log\left(\frac{\beta_2+\gamma_2}{\beta_2-\gamma_2}\right) .$$

We have to show that

$$\left(2\int_0^\infty \frac{e^{-\beta_1 t}}{t}\sinh\left(\gamma_1 t\right)\,dt\right)\left(2\int_0^\infty \frac{e^{-\beta_2 t}}{t}\sinh\left(\gamma_2 t\right)\,dt\right)$$
$$\geq\; \log^2\left(\frac{\beta_{12}+\gamma_{12}}{\beta_{12}-\gamma_{12}}\right) .$$

If we substitute $t_1 := t\gamma_1$, then

$$\int_0^\infty \frac{e^{-\beta_1 t}}{t}\sinh\left(\gamma_1 t\right)\,dt \;=\; \int_0^\infty \frac{e^{-\frac{\beta_1}{\gamma_1}t_1}}{t_1}\sinh\left(t_1\right)\,dt_1 ,$$

and a similar expression with subscript 2 also holds. Hence we have to show that

$$4\left(\int_0^\infty f_1^2(t)dt\right)\left(\int_0^\infty f_2^2(t)dt\right) \;\geq\; \log^2\left(\frac{\beta_{12}+\gamma_{12}}{\beta_{12}-\gamma_{12}}\right),$$

where

$$f_1 := f_1(t) \;=\; \left(\frac{e^{-\frac{\beta_1}{\gamma_1}t}}{t}\sinh t\right)^{1/2}$$

and

$$f_2 := f_2(t) \;=\; \left(\frac{e^{-\frac{\beta_2}{\gamma_2}t}}{t}\sinh t\right)^{1/2}.$$

Note that, for all positive real numbers β_1, γ_1, β_2, γ_2 with $\beta_1 > \gamma_1$ and $\beta_2 > \gamma_2$, the integrals $\int_0^\infty f_1^2 dt$, $\int_0^\infty f_2^2 dt$, and $\int_0^\infty f_1 f_2 dt$, exist because of (2). Hence, applying the Cauchy-Schwarz inequality to $\int_0^\infty f_1^2 dt$ and $\int_0^\infty f_2^2 dt$ and noting that $\frac{\beta_1}{\gamma_1} > 1$ and $\frac{\beta_2}{\gamma_2} > 1$ imply $\frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right) > 1$, we obtain

$$\begin{aligned}
4\left(\int_0^\infty f_1^2 dt\right)\left(\int_0^\infty f_2^2 dt\right) &\geq 4\left(\int_0^\infty f_1 f_2 dt\right)^2 \\
&= 4\left(\int_0^\infty e^{-\frac{t}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)}\frac{\sinh t}{t}dt\right)^2 \\
&= \log^2\frac{\frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)+1}{\frac{1}{2}\left(\frac{\beta_1}{\gamma_1}+\frac{\beta_2}{\gamma_2}\right)-1} \\
&= \log^2\frac{\beta_{12}+\gamma_{12}}{\beta_{12}-\gamma_{12}},
\end{aligned}$$

and (1) is proved.

Example. We now use the Theorem to obtain a lower bound for the product of two logarithms of tangent functions. Suppose $0 < x,\ y < \pi/2$, $x \neq y$. Then

$$\left(\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)\left(\log\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right) \;\geq\; \frac{1}{4}\log^2\left(\frac{x_0}{y_0}\right), \tag{3}$$

where

$$x_0 \;=\; \frac{1}{2}\left(\sqrt{1+\cot^2 x}+\sqrt{1+\cot^2 y}\right)+1$$

and

$$y_0 \;=\; \frac{1}{2}\left(\sqrt{1+\cot^2 x}+\sqrt{1+\cot^2 y}\right)-1\,.$$

Proof. In (1) put $\beta_1 = 1$, $\gamma_1 = \sin x$, $\beta_2 = 1$, and $\gamma_2 = \sin y$. By [2], p. 163, Bemerkung 1, we obtain, for $0 < t < \pi/2$,

$$\frac{1+\sin t}{1-\sin t} \;=\; \tan^2\left(\frac{t}{2}+\frac{\pi}{4}\right) . \tag{4}$$

Then, using $0 < \sin x < 1$ and $0 < \sin y < 1$, it follows from (1) and (4) that

$$\left(\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)\left(\log\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right) \;\geq\; \frac{1}{4}\log^2\left(\frac{\widetilde{\beta}_{12}+\widetilde{\gamma}_{12}}{\widetilde{\beta}_{12}-\widetilde{\gamma}_{12}}\right) ,$$

where

$$\widetilde{\beta}_{12}+\widetilde{\gamma}_{12} \;=\; \sin x + \sin y + 2\sin x \sin y$$

and

$$\widetilde{\beta}_{12}-\widetilde{\gamma}_{12} \;=\; \sin x + \sin y - 2\sin x \sin y .$$

Hence

$$\frac{\widetilde{\beta}_{12}+\widetilde{\gamma}_{12}}{\widetilde{\beta}_{12}-\widetilde{\gamma}_{12}} \;=\; \frac{\frac{1}{2}\left(\frac{1}{\sin x}+\frac{1}{\sin y}\right)+1}{\frac{1}{2}\left(\frac{1}{\sin x}+\frac{1}{\sin y}\right)-1} .$$

Finally, using the fact that

$$\frac{1}{\sin t} \;=\; \sqrt{1+\cot^2 t}\,,$$

we obtain (3).

The author would like to thank the referee for suggesting some minor corrections.

References

[1] I.S. Gradshteyn, I.M. Ryzhik. Table of integrals, series, and products. Academic Press, New York, 1965.

[2] Strubecker, K., Einführung in die höhere Mathematik, Volume 2, R. Oldenburg, München, 1967.

Erhard Braune
A - 4020 Linz
Lannergasse 16
Austria

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N'oubliez pas d'inclure à toute correspondance votre nom, votre année scolaire, le nom de votre école, ainsi que votre ville, province ou état et pays. Nous sommes surtout intéressés par les solutions d'étudiants du secondaire. Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le *1er novembre 2002*. Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Pour être admissibles au DÉFI MAYHEM de ce mois-ci, les solutions doivent avoir été postées *avant le* 1er septembre 2002, cachet de la poste faisant foi.

M45. *Proposé par un douanier canadien, Aéroport International Pearson de Toronto, Ontario.*

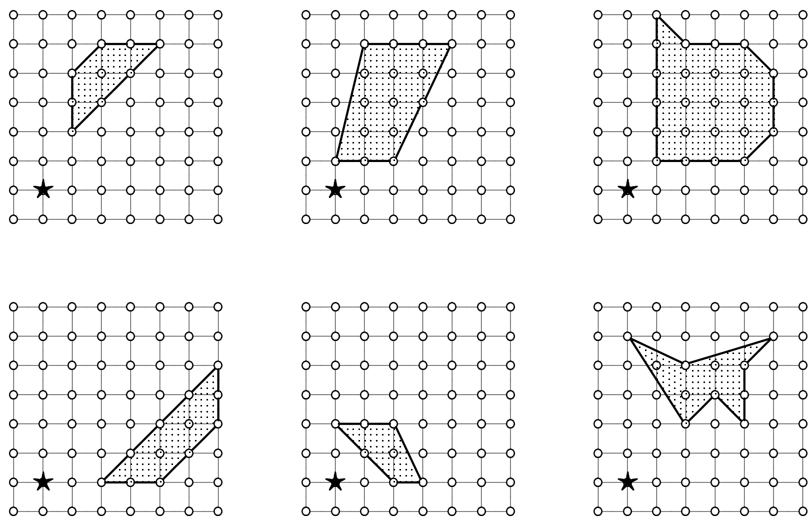
On appuye une échelle de dix mètres à la fois contre une paroi et contre l'arête d'une boîte cubique mesurant $2\ m^3$, elle-même posée contre la paroi. A quelle hauteur au-dessus du sol se trouve le sommet de l'échelle?

. . . .

A 10 metre long ladder is leaning upright against a wall, touching the edge of a cubic box. The box itself is put against the wall and measures 2 cubic metre. What is the height of the top of the ladder from the ground?

M46. *Proposé par Eckard Specht, Otto-von-Guericke-University Magdeburg, Allemagne.*

Les polygones des réseaux de la ligne supérieure sont caractérisés par une propriété commune. Ceux de la ligne inférieure par son inverse. Quelle est cette propriété ?



The lattice polygons in the upper row of the figure are characterized by a common property, the lower ones by the reverse. Which property is it?

M47. *Proposé par Bill Sands, University of Calgary, Calgary, Alberta.*

(a) Trouver tous les polynômes unitaires quadratiques $x^2 + ax + b$ avec 1, a, b en progression arithmétique et possédant des racines entières.

(b) Montrer qu'il n'existe pas de nombres réels a, b, c avec 1, a, b, c en progression arithmétique et tels que toutes les racines de $x^3 + ax^2 + bx + c$ soient réelles.

(a) Find all monic quadratic polynomials x^2+ax+b with integer roots, where 1, a, b is an arithmetic progression.

(b) Prove that there are no real numbers a, b, c such that 1, a, b, c is an arithmetic progression and $x^3 + ax^2 + bx + c$ has all real roots.

M48. *Proposé par J. Walter Lynch, Athens, GA, USA.*

Décrire comment faire un bouchon pouvant à la fois boucher un trou carré, un trou rond et un trou triangulaire et aussi passer à travers chacun d'eux.

A titre indicatif, notons qu'une pyramide peut boucher un trou carré et un trou triangulaire, ou encore qu'un cylindre peut boucher un trou carré et un trou rond.

Tell how to make a single stopper that will stop a square hole, a round hole, and a triangular hole, and will pass through each.

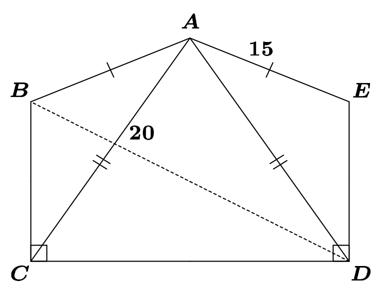
If one wanted to give a hint, he might point out that a pyramid will stop a square hole and a triangular hole, and a cylinder will stop a square hole and a round hole.

M49. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

La figure montre un pentagone de Héron dans lequel les côtés, les diagonales et l'aire sont des nombres naturels.

(a) $AB = AE = 15$, $AC = AD = 20$ et $BCDE$ est un rectangle. Trouver la longueur de BD.

(b) Trouver un ensemble d'expressions générales pour les côtés, les diagonales et l'aire pouvant engendrer une famille infinie de pentagones de Héron $ABCDE$ pareils à celui donné dans la figure.



The figure shows a Heron pentagon in which the sides, the diagonals and the area are natural numbers.

(a) $AB = AE = 15$, $AC = AD = 20$ and $BCDE$ is a rectangle. Find the length of BD.

(b) Give a set of general expressions for the sides, the diagonals and the area to generate an infinite family of such Heron pentagons $ABCDE$ as in the figure.

M50. *Proposé par l'équipe de Mayhem.*

Ceci est une variante d'un problème bien connu et souvent exploité. On demande d'engendrer une liste de valeurs aussi longue que possible en utilisant des opérations impliquant, par exemple, 4 fois le nombre 4, comme

$$\frac{4+4}{4+4} = 1, \qquad 4+4-\sqrt{4}-4 = 2,$$

et ainsi de suite. Dans ce genre de problème, on utilise souvent les chiffres de l'année courante (bien qu'on soit maintenant forcé de se contenter de quelques zéros pendant un certain temps).

Le problème ici est de faire la plus longue liste possible de nombres en utilisant **au maximum** cinq fois le nombre π, comme par exemple:

$$\frac{\pi+\pi+\pi}{\pi} = 3, \qquad \left\lfloor\sqrt{\pi^\pi} - \pi + \frac{\pi}{\pi}\right\rfloor ! = 6.$$

.·.

This question is a bit of a variation of a well known and used problem. There are forms of the question where you want to use four 4's and some operations to make as long a list of values as possible. Thus

$$\frac{4+4}{4+4} = 1, \qquad 4+4-\sqrt{4}-4 = 2,$$

and so on. It is popular to use the digits of the year in such a problem (although, we will have to deal with a couple of zeros for a while).

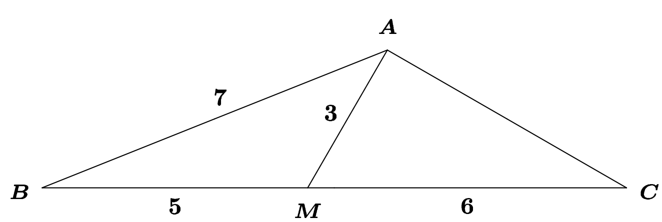
The problem is to make as many numbers as possible using **up to** five π's. Thus some acceptable results would be:

$$\frac{\pi+\pi+\pi}{\pi} = 3, \qquad \left\lfloor\sqrt{\pi^\pi} - \pi + \frac{\pi}{\pi}\right\rfloor ! = 6.$$

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. In $\triangle ABC$, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



(1998 Euclid, Problem 6b)

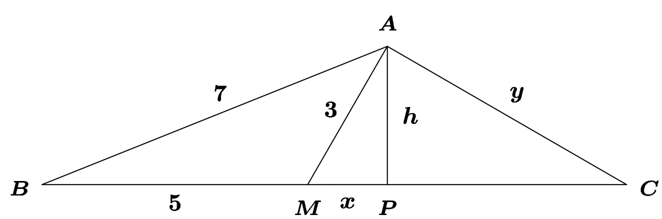
Solution 1. From the Cosine Law, $\cos\angle AMB = \dfrac{5^2+3^2-7^2}{2\cdot5\cdot3} = -\dfrac{1}{2}$. Also, $\angle AMB$ and $\angle AMC$ are supplementary, and thus their cosines are negatives of each other.

Finally, $AC^2 = 3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cdot \cos \angle AMC = 9 + 36 - 36 \cdot \frac{1}{2} = 27$ and it follows that $AC = 3\sqrt{3}$.

Solution 2. Let $AC = x$. By the Cosine Law applied to $\triangle ACM$, $\cos \angle C = \dfrac{6^2 + x^2 - 3^2}{2 \cdot 6 \cdot x}$; applied to $\triangle ABC$, $\cos \angle C = \dfrac{11^2 + x^2 - 7^2}{2 \cdot 11 \cdot x}$.

Equating these, and realizing that x must be a positive number, we have $11\left(6^2 + x^2 - 3^2\right) = 6\left(11^2 + x^2 - 7^2\right)$. Solving, we get $x^2 = 27$ and thus, $x = 3\sqrt{3}$.

Solution 3. For the Pythagoreans in the crowd! Drop a perpendicular from A to BC at P. Let $x = MP$, $h = PA$ and $y = AC$ as in the diagram. Clearly, $\angle AMB$ is obtuse since $7^2 > 3^2 + 5^2$, so that P is located between M and C as in the diagram.



Thus, we have $h^2 = 3^2 - x^2$ from $\triangle APM$. Similarly in $\triangle APB$ we have:

$$\begin{aligned} 7^2 &= (5+x)^2 + h^2 = 25 + 10x + x^2 + 9 - x^2 \\ &= 34 + 10x\,, \end{aligned}$$

which yields $x = 1.5$. Thus, if we go to $\triangle APC$ we get:

$$\begin{aligned} y^2 &= (6-x)^2 + h^2 = 36 - 12x + x^2 + 9 - x^2 \\ &= 45 - 12x = 27\,. \end{aligned}$$

Therefore, again, we get $AC = 3\sqrt{3}$.

Advanced Solutions

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A249. *Proposed by Mohammed Aassila, Strasbourg, France.*

A circle is circumscribed around $\triangle ABC$ with sides a, b, c. Let A', B', C' denote the mid-points of the arcs BC, CA, AB, respectively. The straight lines $A'B'$, $B'C'$, $C'A'$ intersect BC and AC, AC and AB, AB and BC, in P, Q, R, S, T, U, respectively. Prove that

$$\frac{[PQRSTU]}{[ABC]} \;=\; \frac{(a+b)^2+(b+c)^2+(c+a)^2}{2(a+b+c)^2}\,,$$

where $[X]$ denotes the area of the polygon X.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

Let $\mathcal{H} = [PQRSTU]$, and $\mathcal{T} = [ABC]$. Let D be the circum-diameter of $\triangle ABC$, and s be its semi-perimeter.

$\angle AC'B' = \frac{1}{2}\widehat{AB'} = \frac{1}{2}\cdot\frac{1}{2}\widehat{AC} = \frac{1}{2}B$; similarly, $\angle AB'C' = \frac{1}{2}C$, $\angle C'AB = \frac{1}{2}C$ and $\angle B'AC = \frac{1}{2}B$.

Therefore, $\angle ASR = \angle ARS = \frac{1}{2}(B+C) = 90^\circ - \frac{1}{2}A$. By the Law of Sines applied to $\triangle AC'R$,

$$\frac{AR}{AC'} \;=\; \frac{\sin\frac{1}{2}B}{\sin\left(90^\circ - \frac{1}{2}A\right)}\,.$$

Hence,

$$AR \;=\; \frac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A}\cdot D\sin\tfrac{1}{2}C \;=\; AS\,.$$

Then,

$$\begin{aligned}
\frac{[ASR]}{\mathcal{T}} &= \frac{AR\cdot AS}{bc} \\
&= \frac{D^2\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C}{bc\cos^2\frac{1}{2}A} \;=\; \frac{D^2}{s(s-a)}\cdot\frac{(s-a)(s-c)}{ac}\cdot\frac{(s-a)(s-b)}{ab} \\
&= \frac{D^2\mathcal{T}^2}{s^2abc}\cdot\frac{1}{a}\,;
\end{aligned}$$

that is, $\dfrac{[ASR]}{\mathcal{T}} = \dfrac{bc}{4s^2}$. Similarly, $\dfrac{[BTU]}{\mathcal{T}} = \dfrac{ac}{4s^2}$ and $\dfrac{[CPQ]}{\mathcal{T}} = \dfrac{ab}{4s^2}$. Adding these three equations together, we obtain

$$\frac{\mathcal{T}-\mathcal{H}}{\mathcal{T}} \;=\; \frac{bc+ac+ab}{4s^2}\,;$$

that is,

$$\frac{\mathcal{H}}{\mathcal{T}} \;=\; 1 - \frac{bc+ac+ab}{4s^2}$$

The expression on the right is easily seen to be equal to the one given in the statement of the problem.

Also solved by MICHEL BATAILLE, Rouen, France.

A250. *Proposed by the Mayhem Staff.*

Suppose polynomial $P(x)$ has integer coefficients such that for any integer m, $P(m)$ is a perfect square. Show that the degree of P is even.

Solution by Michel Bataille, Rouen, France.

Suppose, for purpose of contradiction, that the degree of P is a positive odd integer $2n+1$. Then, $\lim\limits_{x\to-\infty} P(x) = \lim\limits_{x\to-\infty} x^{2n+1} = -\infty$ and $\lim\limits_{x\to+\infty} P(x) = \lim\limits_{x\to+\infty} x^{2n+1} = +\infty$. [Ed. This is true only if the coefficient of the term of highest power of $P(x)$ is positive. If it is negative, the argument still holds for $-P(x)$, which has the same degree as $P(x)$.] Since P is a continuous function on $\mathbb{R}$, $P(x)$ has at least one real root . Let x_0 be the smallest of the real roots of $P(x)$ so that $P(x) \neq 0$ when $x \in (-\infty, x_0)$. Since P is continuous and $\lim\limits_{x\to\infty} P(x) = -\infty$, we even have $P(x) < 0$ for $x \in (-\infty, x_0)$. Now, take $m = \lfloor x_0 \rfloor - 1$. We have $m \in (-\infty, x_0)$ so that $P(m) < 0$, and m is an integer so that $P(m)$ must be a perfect square; hence, $P(m) \geq 0$. We have reached our contradiction and it follows that the degree of P is even.

Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

The following more general result is known [1]:

Theorem: Let $P(x)$ and $Q(x)$ be polynomials which are integer-valued at the integers, of degrees p and q, respectively. If $P(n)$ is of the form $Q(m)$ for all n, or even for infinitely many blocks of consecutive integers of length greater than or equal to $\frac{p}{q}+2$, then there is a polynomial $R(x)$ such that $P(x) = Q[R(x)]$.

Reference

1. H.S. Shapiro, The range of an integer-valued polynomial, Amer. Math. Monthly, 64 (1957), 424–425.

A251. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

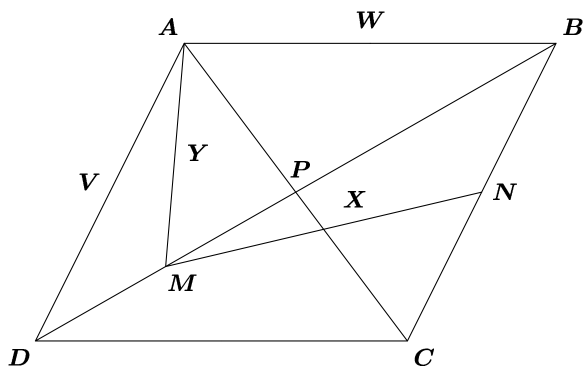
In a parallelogram $ABCD$, let P be the intersection of AC with BD. Let M, N be the mid-points of PD, BC, respectively. Prove that the following two statements are equivalent:

(i) $\triangle AMN$ is a (non-degenerate) right-angled triangle such that $AM = MN$.

(ii) Quadrilateral $ABCD$ is a square.

I. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $\boldsymbol{R} \mapsto \boldsymbol{R}'$ be the linear transformation that rotates each non-zero vector counterclockwise by $90°$ while preserving its length.



Let $\overrightarrow{AD} = \boldsymbol{V}$, $\overrightarrow{AB} = \boldsymbol{W}$, $\overrightarrow{MN} = \boldsymbol{X}$ and $\overrightarrow{MA} = \boldsymbol{Y}$. We have

$$\boldsymbol{X} = \overrightarrow{MD} + \overrightarrow{DC} + \overrightarrow{CN} = \tfrac{1}{4}(\boldsymbol{V} - \boldsymbol{W}) + \boldsymbol{W} - \tfrac{1}{2}\boldsymbol{V} = \tfrac{1}{4}(-\boldsymbol{V} + 3\boldsymbol{W}),$$

$$\boldsymbol{Y} = -\overrightarrow{AM} = -\left(\tfrac{3}{4}\overrightarrow{AD} + \tfrac{1}{4}\overrightarrow{AB}\right) = -\tfrac{1}{4}(3\boldsymbol{V} + \boldsymbol{W}).$$

These two equations can be inverted to give:

$$\boldsymbol{V} = -\tfrac{2}{5}(\boldsymbol{X} + 3\boldsymbol{Y}) \quad \text{and} \quad \boldsymbol{W} = \tfrac{2}{5}(3\boldsymbol{X} - \boldsymbol{Y}).$$

Suppose that (i) holds. Then $\boldsymbol{X}' = \boldsymbol{Y}$ and $\boldsymbol{Y}' = -\boldsymbol{X}$. Therefore, $\boldsymbol{V}' = -\frac{2}{5}(\boldsymbol{X}' + 3\boldsymbol{Y}') = -\frac{2}{5}(\boldsymbol{Y} - 3\boldsymbol{X}) = \boldsymbol{W}$, hence, (ii).

Suppose (ii) holds. Then $\boldsymbol{V}' = \boldsymbol{W}$, and $\boldsymbol{W}' = -\boldsymbol{V}$. Therefore, $\boldsymbol{X}' = \frac{1}{4}(-\boldsymbol{V}' + 3\boldsymbol{W}') = \frac{1}{4}(-\boldsymbol{W} - 3\boldsymbol{V}) = \boldsymbol{Y}$, hence, (i).

II. *Solution by Michel Bataille, Rouen, France.*

Choose a system of axes with origin at P and let a, b be the complex affixes of A, B respectively.

Observing that C, D, M, N have respective affix $-a$, $-b$, $-\frac{b}{2}$, $\frac{b-a}{2}$, we have:

$$\begin{aligned} \text{(i)} \iff a + \frac{b}{2} = i\left(b - \frac{a}{2}\right) &\iff a = ib \\ \iff PA \perp PB \text{ and } PA = PB &\iff \text{(ii)}. \end{aligned}$$

A252. *Proposed by Mohammed Aassila, Strasbourg, France.*

For every positive integer n, prove that there exists a polynomial of degree n with integer coefficients of absolute value at most n, which admits 1 as a root with multiplicity at least $\lfloor\sqrt{n}\rfloor$.

No solution available yet.

A253. *Proposed by Mohammed Aassila, Strasbourg, France.*

Does there exist a polynomial $f(x, y, z)$ with real coefficients, such that $f(x, y, z) > 0$ if and only if there exists a non-degenerate triangle with side lengths $|x|$, $|y|$, and $|z|$?

Solution by Mihály Bencze, Brasov, Romania.

For the polynomial $f(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \cdots + x_{n-1} - x_n)$ $(x_2 + x_3 + \cdots + x_n - x_1) \ldots (x_n + x_1 + \cdots + x_{n-2} - x_{n-1})$ with real coefficients, we have $f(x_1, x_2, \ldots, x_n) > 0$ if and only if there exists a non-degenerate polygon with n sides for which the sides are x_1, x_2, $\ldots$, x_n.

A254. In the acute triangle ABC, the bisectors of $\angle A$, $\angle B$, and $\angle C$ intersect the circumcircle again at A_1, B_1 and C_1, respectively. Let M be the point of intersection of AB and B_1C_1, and let N be the point of intersection of BC and A_1B_1. Prove that MN passes through the incentre of triangle ABC.

(1997 Baltic Way)

Solution by Michel Bataille, Rouen, France.

From the arcs of the circumcircle that the involved angles intercept, we get: $\angle B_1C_1A = \frac{B}{2}$, and $\angle BAC_1 = \frac{C}{2}$. Let U be the point of intersection of AA_1 and B_1C_1. Thus, we have $\angle UC_1A = \frac{B}{2}$ and $\angle C_1AU = \frac{C}{2} + \frac{A}{2}$. Since $A + B + C = 180°$, it follows that $\angle AUC_1 = 90°$. If I denotes the incentre, we see that $\triangle IC_1A$ is isosceles (C_1U is the altitude from C_1 as well as the bisector of $\angle IC_1A$). Since M lies on C_1U, $\triangle AMI$ is also isosceles so that $\angle MIA = \frac{A}{2}$. Observing that $\angle C_1IA = \angle C_1AI = \frac{C}{2} + \frac{A}{2}$, we deduce $\angle MIC_1 = \frac{C}{2}$. Reasoning in the same way at vertex C instead of A, we obtain in particular $\angle NIC = \frac{C}{2}$. Now, MI and NI make the same angle $\frac{C}{2}$ with the line CC_1; hence, M, I, N are collinear. This completes the proof.

A255. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Define $A = (\sum_{i=1}^{n} a_i)/n$, $G = \sqrt[n]{\prod_{i=1}^{n} a_i}$, and $H = n/(\sum_{i=1}^{n} 1/a_i)$ for positive real numbers a_1, a_2, $\ldots$, a_n. It is known that $A \geq G \geq H$, from which it follows that $0 \geq \log(G/A)$ and $0 \geq 1 - A/H$. Prove that $0 \geq \log(G/A) \geq 1 - A/H$, and determine when equality holds.

Solution by Michel Bataille, Rouen, France.

Equivalently, we show that $0 \leq \log(A/G) \leq (A/H) - 1$.

Since the function $\log x$ is strictly concave, the curve $y = \log(x)$ is below its tangent at $(1, 0)$ (whose equation is $y = x - 1$). It follows that $\log(x) \leq x - 1$ for all positive x, with equality only when $x = 1$. Taking $x = A/G \geq 1$, we get $0 \leq \log(A/G) \leq (A/G) - 1$. But $A/G \leq A/H$. Hence,

$$0 \leq \log(A/G) \leq (A/G) - 1 \leq (A/H) - 1\,,$$

and the required inequality is obtained.

Clearly equality $0 = \log(A/G)$ holds if and only if $A = G$. As for equality $\log(A/G) = (A/H) - 1$, it will hold if and only if $\log(A/G) = (A/G) - 1$; that is, $A/G = 1$ or $A = G$ again. Thus, either equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Also solved by Mihály Bencze, Brasov, Romania.

A256. *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that for any positive integer n, there exist $n+1$ points $M_1, M_2, \ldots, M_{n+1}$ in $\mathbb{R}^n$ such that for any integers i and j for which $1 \le i < j \le n+1$, the Euclidean distance between M_i and M_j is 1.

Solution by Michel Bataille, Rouen, France.

In a Euclidean space let a subset be called a $(n+1)$–*admissible set* if it has $n+1$ elements whose mutual distance is 1. We answer the question by proving the following property by induction on the positive integer n: in an n–dimensional Euclidean space E_n, there exists a $(n+1)$–admissible set S which is contained in a sphere with radius less than $\frac{\sqrt{2}}{2}$. [Ed: the sphere with centre A and radius $r > 0$ is the set of all points M such that $d(A, M) = r$.]

The property is obvious in E_1 : just take $S = \{0, 1\}$, which is (contained in) the sphere with centre $\frac{1}{2}$ and radius $r = \frac{1}{2} < \frac{\sqrt{2}}{2}$.

Assume now that the property holds in any n–dimensional Euclidean space and consider in E_{n+1} a fixed hyperplane H. From the assumption, there exists a $(n+1)$–admissible set $S = \{M_1, M_2, \ldots, M_{n+1}\}$ contained in a sphere of H with radius $r < \frac{\sqrt{2}}{2}$ and centre $A \in H$. On the line D orthogonal to H at A and on the same side of H, let M_{n+2} be such that $d(A, M_{n+2}) = \sqrt{1-r^2}$ and B be such that $d(A, B) = \dfrac{1-2r^2}{2\sqrt{1-r^2}}$. (Note that A, B, M_{n+2} are in this order since $0 < \dfrac{1-2r^2}{2\sqrt{1-r^2}} < \sqrt{1-r^2}$.) Then, $d(M_{n+2}, M_i) = \sqrt{[d(A, M_{n+2})]^2 + [d(A, M_i)]^2} = 1$ $(i = 1, 2, \ldots, n+1)$ so that $\{M_1, M_2, \ldots, M_{n+1}, M_{n+2}\}$ is a $(n+2)$–admissible set in E_{n+1}.

Moreover, easy calculations yield

$$d(B, M_i) \;=\; \sqrt{[d(B, A)]^2 + [d(A, M_i)]^2} \;=\; \frac{1}{2\sqrt{1-r^2}}$$

and

$$d(B, M_{n+2}) \;=\; d(A, M_{n+2}) - d(A, B) \;=\; \frac{1}{2\sqrt{1-r^2}}\,.$$

Hence, $\{M_1, M_2, \ldots, M_{n+1}, M_{n+2}\}$ is contained in the sphere of E_{n+1} with centre B and radius $\dfrac{1}{2\sqrt{1-r^2}}$. Since $\dfrac{1}{2\sqrt{1-r^2}} < \dfrac{\sqrt{2}}{2}$ when $r < \frac{\sqrt{2}}{2}$, the induction is now complete and the conclusion follows.

A Variation of the Ass and Mule Problem

David Singmaster

The following appeared in The Skoliad Corner of December 2001 [1], taken from the 2001 Maritime Mathematics Contest.

> Alice and Bob were comparing their stacks of pennies. Alice said "If you gave me a certain number of pennies from your stack, then I'd have six times as many as you, but if I gave you that number, you'd have one-third as many as me." What is the smallest number of pennies that Alice could have had?

This is a variation of the classic Ass and Mule Problem attributed to Euclid. I discussed the history of this problem in [2] and gave a criterion for the general problem to have an integral solution when the parameters are integral. I also discussed a different variation of the problem. Here I show that similar reasoning gives a similar, but simpler, criterion for the general version of the present variation.

The classic Ass and Mule Problem has these two animals carrying sacks. The mule says to the ass: "If you give me one of your sacks, I will have as many as you." The ass responds: "If you give me one of your sacks, I will have twice as many as you." How many sacks did they have?

The general version of the problem (for two individuals) is the situation where the first says: "If I had a from you, I'd have b times you," and the second responds: "And if I had c from you, I'd have d times you." It is traditional for the parameters a, b, c, d and the solutions, say x, y, to be integers. This leads to the equations:

$$x + a \;=\; b(y - a)\,; \quad y + c \;=\; d(x - c)\,. \tag{1}$$

The general form of our variation is the situation where the first says: "If I had a from you, I'd have b times you, but if I gave c to you, I'd have d times you." This leads to the equations:

$$x + a \;=\; b(y - a)\,; \quad d(y + c) \;=\; x - c\,. \tag{1$'$}$$

The integral d of the classic problem has been changed to $1/d$ with d integral. Note that $b > d$ for reasonable solutions. In the particular problem given, we have $b = 6$, $d = 3$ and $a = c$ is an unknown, but our analysis turns out to deal with this easily.

The solutions of (1) are readily computed as:

$$x \;=\; c + \frac{(b+1)(a+c)}{(bd-1)}\,; \quad y \;=\; a + \frac{(d+1)(a+c)}{(bd-1)}\,. \tag{2}$$

Thus, x and y are integers if and only if the second terms in (2) are integers. One can see from (2), and it is obvious from (1), that x is an integer if and only if y is an integer.

In [2], I gave the following analysis. The values of x and y are integers if and only if $bd-1$ divides both $(b+1)(a+c)$ and $(d+1)(a+c)$, which is if and only if $bd-1$ divides $\mathrm{GCD}[(b+1)(a+c), (d+1)(a+c)] = (a+c)(b+1, d+1)$, where $(b+1, d+1)$ denotes the GCD (= Greatest Common Divisor) of $b+1$ and $d+1$.

Now consider $g = (b+1, d+1)$. This g also divides

$$(b+1)(d+1) - (b+1) - (d+1) \;=\; bd-1\,.$$

Hence, the last statement of the previous paragraph can be divided by g to give us that x and y are integers if and only if

$$\frac{(bd-1)}{(b+1, d+1)} \quad \text{divides} \quad a+c\,. \tag{3}$$

This seems to be as simple a criterion for integrality as one could expect. The criterion allows us to pick arbitrary b and d, assuming $bd \neq 1$, and then determines which values of a and c give integral solutions. I find it particularly striking that a and c only enter via the sum $a+c$. I am also intrigued to see that any b and any d can be used, assuming $bd \neq 1$, which I would not have predicted.

We can solve (1′) directly or use (2), obtaining

$$x \;=\; c + \frac{d(b+1)(a+c)}{(b-d)} \qquad y = a + \frac{(d+1)(a+c)}{(b-d)}\,. \tag{2′}$$

One can see from (2′), and it is obvious from (1′), that x is an integer if y is an integer, but the converse does not hold.

The value of y is an integer if and only if $b-d$ divides $(d+1)(a+c)$, which is if and only if $(b-d, d+1)$ divides $a+c$.

Now consider $g = (b-d, d+1)$. This is equal to $(b+1, d+1)$. Hence, the last statement of the previous paragraph can be divided by g to give us that x and y are integers if and only if

$$\frac{(b-d)}{(b+1, d+1)} \quad \text{divides} \quad a+c\,. \tag{3′}$$

Again, this seems to be as simple a criterion for integrality as one could expect. This criterion allows us to pick arbitrary b and d, assuming $b > d$, and then it determines which values of a and c give integral solutions. I again find it particularly striking that a and c only enter via the sum $a+c$. It is also pleasantly surprising that this variant has an easier condition for integrality than the original, though (1′) is less symmetric than (1) so that I expected a more complex criterion to appear.

In the given problem, $a = c$, $b = 6$, $d = 3$ and (3′) simply becomes $3 \mid 2a$ or $3 \mid a$, with the simplest positive answer being $a = 3$, $x = 45$, $y = 11$. When $a = c$, the solutions are all multiples of the simplest case. Inspection of small values of b and d shows that the given problem is the smallest 'interesting' situation in some sense.

References

1. 2001 Maritime Mathematics Contest, no. 1. *Crux Mathematicorum with Mathematical Mayhem* 27 : 8 (Dec 2001) 521.

2. David Singmaster, *Some diophantine recreations*, in: *The Mathemagician and Pied Puzzler — A Collection in Tribute to Martin Gardner*, ed. Elwyn R. Berlekamp & Tom Rodgers, A. K. Peters, Natick, Massachusetts, 1999, pp. 219–235.

David Singmaster
87 Rodenhurst Road
London, SW4 8AF, UK

SKOLIAD No. 62

Shawn Godin

Solutions may be sent to Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5, or emailed to

`mayhem-editors@cms.math.ca`.

Please include on any correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 September 2002*. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to the pre-university reader(s) who send in the best set of solutions before the deadline. The decision of the editor is final.

Our item this issue is the 2001 Canadian Open Mathematics Challenge. My thanks go out to Ian VanderBurgh and Peter Crippen of The University of Waterloo for forwarding the material to me.

2001 Canadian Open Mathematics Challenge

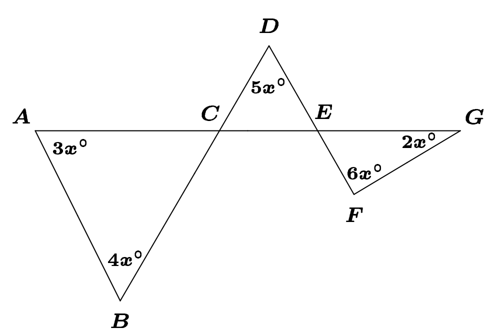
PART/PARTIE A

1. An operation "$\triangledown$" is defined by: $a \triangledown b = a^2 + 3^b$. What is the value of $(2 \triangledown 0) \triangledown (0 \triangledown 1)$?

On définit une opération "$\triangledown$" comme suit : $a \triangledown b = a^2 + 3^b$. Quelle est la valeur de $(2 \triangledown 0) \triangledown (0 \triangledown 1)$?

2. In the given diagram, what is the value of x?

Dans le diagramme ci-contre, quelle est la valeur de x ?



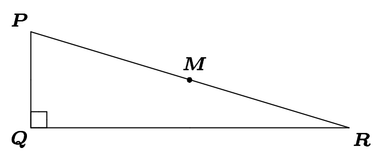
3. A regular hexagon is a six-sided figure which has all of its angles equal and all of its side lengths equal. If P and Q are points on a regular hexagon which has a side length of 1, what is the maximum possible length of the line segment PQ?

Un hexagone régulier est un polygone à 6 côtés dont tous les angles sont congrus et tous les côtés sont congrus. Soient P et Q des points sur un hexagone régulier dont les côtés ont une longueur de 1. Quelle est la longueur maximale possible du segment PQ ?

4. Solve for x: $2\left(2^{2x}\right) = 4^x + 64$.

Résoudre l'équation suivante : $2\left(2^{2x}\right) = 4^x + 64$.

5. Triangle PQR is right-angled at Q and has side lengths $PQ = 14$ and $QR = 48$. If M is the mid-point of PR, determine the cosine of $\angle MQP$.



Le diagramme illustre un triangle rectangle PQR, dans lequel $PQ = 14$ et $QR = 48$. M est le milieu du côté PR. Déterminer le cosinus de l'angle MQP.

6. The sequence of numbers t_1, t_2, t_3, $\ldots$ is defined by $t_1 = 2$ and $t_{n+1} = \dfrac{t_n - 1}{t_n + 1}$, for every positive integer n. Determine the numerical value of t_{999}.

On définit une suite de nombres, t_1, t_2, t_3, $\ldots$, comme suit : $t_1 = 2$ et $t_{n+1} = \dfrac{t_n - 1}{t_n + 1}$, pour tout entier strictement positif n. Déterminer la valeur numérique de t_{999}.

7. If a can be any positive integer and

$$\begin{aligned} 2x + a &= y\,, \\ a + y &= x\,, \\ x + y &= z\,, \end{aligned}$$

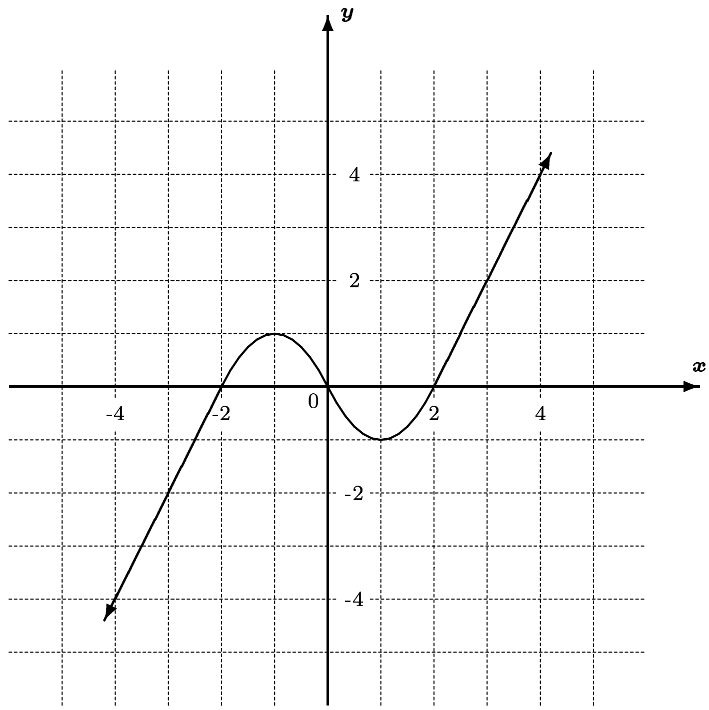
determine the maximum possible value for $x + y + z$.

Sachant que a peut prendre la valeur de n'importe quel entier strictement positif et que

$$\begin{aligned} 2x + a &= y\,, \\ a + y &= x\,, \\ x + y &= z\,, \end{aligned}$$

déterminer la valeur maximale possible de l'expression $x + y + z$.

8. The graph of the function $y = g(x)$ is shown. Determine the number of solutions of the equation $\Big||g(x)| - 1\Big| = \frac{1}{2}$.



La représentation graphique de la fonction définie par $y = g(x)$ est donnée ci-dessus. Déterminer le nombre de solutions de l'équation $\left||g(x)| - 1\right| = \frac{1}{2}$.

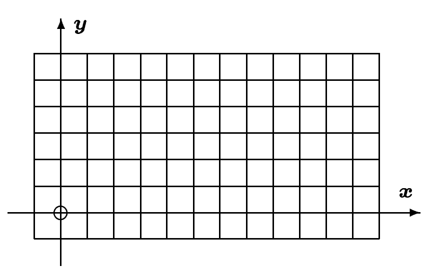
PART /PARTIE B

1. The triangular region T has its vertices determined by the intersections of the three lines $x + 2y = 12$, $x = 2$ and $y = 1$.

(a) Determine the coordinates of the vertices of T, and show this region on the grid provided.

(b) The line $x + y = 8$ divides the triangular region T into a quadrilateral Q and a triangle R. Determine the coordinates of the vertices of the quadrilateral Q.

(c) Determine the area of the quadrilateral Q.



Les sommets d'une région triangulaire T sont les points d'intersection des droites définies par $x + 2y = 12$, $x = 2$ et $y = 1$.

(a) Déterminer les coordonnées des sommets de T et tracer la région dans le plan cartésien fourni à cet effet.

(b) La droite d'équation $x + y = 8$ divise la région triangulaire T en deux, formant ainsi un quadrilatère Q et un triangle R. Déterminer les coordonnées des sommets du quadrilatère Q.

(c) Déterminer l'aire du quadrilatère Q.

2. (a) Alphonse and Beryl are playing a game, starting with a pack of 7 cards. Alphonse begins by discarding at least one but not more than half of the cards in the pack. He then passes the remaining cards in the pack to Beryl. Beryl continues the game by discarding at least one but not more than half of the remaining cards in the pack. The game continues in this way with the pack being passed back and forth between the two players. The loser is the player who, at the beginning of his or her turn, receives only one card. Show, with justification, that there is always a winning strategy for Beryl.

(b) Alphonse and Beryl now play a game with the same rules as in (a), except this time they start with a pack of 52 cards, and Alphonse goes first again. As in (a), a player on his or her turn must discard at least one and not more than half of the remaining cards from the pack. Is there a strategy that Alphonse can use to be guaranteed that he will win? (Provide justification for your answer.)

(a) Alphonse et Béatrice jouent aux cartes. Au début, le jeu compte 7 cartes. Alphonse commence en retirant du jeu au moins une des cartes, mais pas plus de la moitié des cartes. Il remet ensuite le reste du jeu de cartes à Béatrice. Celle-ci retire du jeu au moins une des cartes, mais pas plus de la moitié des cartes qui sont encore dans le jeu. Elle remet ensuite le reste du jeu de cartes à Alphonse. Ils continuent de la sorte, à tour de rôle. La perdante ou le perdant est celui ou celle qui reçoit une seule carte lorsqu'on lui remet le jeu. Démontrer qu'il existe toujours une stratégie gagnante pour Béatrice. Justifier son raisonnement.

(b) Alphonse et Béatrice jouent aux cartes selon les règlements de la partie (a), mais en commençant avec un jeu de 52 cartes. Alphonse joue premier. Comme dans la partie (a), la personne qui reçoit le jeu doit retirer du jeu au moins une des cartes, mais pas plus de la moitié des cartes qui sont encore dans le jeu. Y a-t-il une stratégie qu' Alphonse peut suivre pour garantir qu'il gagnera ? Justifier son raisonnement.

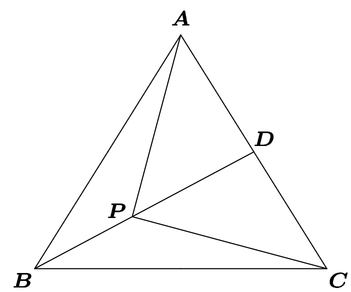
3. (a) If $f(x) = x^2 + 6x + c$, where c is an integer, prove that $f(0) + f(-1)$ is odd.

(b) Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(-1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

(a) Soit $f(x) = x^2 + 6x + c$, c étant un entier. Démontrer que $f(0) + f(-1)$ est impair.

(b) Soit $g(x) = x^3 + px^2 + qx + r$, p, q et r étant des entiers. Démontrer que si $g(0)$ et $g(-1)$ sont impairs tous les deux, alors l'équation $g(x) = 0$ ne peut admettre trois racines entières.

4. Triangle ABC is isosceles with $AB = AC = 5$ and $BC = 6$. Point D lies on AC and P is the point on BD so that $\angle APC = 90°$. If $\angle ABP = \angle BCP$, determine the ratio $AD : DC$.



Le diagramme illustre un triangle isocèle ABC dans lequel $AB = AC = 5$ et $BC = 6$. Le point D est situé sur le côté AC et le point P est situé sur le segment BD de manière que $\angle APC = 90°$. Déterminer le rapport $AD : DC$, sachant que $\angle ABP = \angle BCP$.

Finally, we move on to the answers to the 2001 Concours De Mathématiques Du Nouveau-Brunswick from the December 2001 issue [2001 : 516].

UNIVERSITÉ DE MONCTON
et
UNIVERSITY OF NEW BRUNSWICK

20e CONCOURS DE MATHÉMATIQUES DU NOUVEAU-BRUNSWICK

le vendredi 11 mai 2001

9e année

1. C	2. B	3. D	4. C	5. C	6. D	7. C	8. A	9. B
10. B	11. D	12. B	13. C	14. C	15. B	16. D	17. D	18. E
19. C	20. B	21. B	22. D	23. C	24. C	25. C	26. E	

PROBLEMS

Faire parvenir les propositions de problèmes et les solutions à Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (Terre-Neuve), Canada, A1C 5S7. Les propositions de problèmes doivent être accompagnées d'une solution ainsi que de références et d'autres indications qui pourraient être utiles à la rédaction. Si vous envoyez une proposition sans solution, vous devez justifier une solution probable en fournissant suffisamment d'information. Un numéro suivi d'une astérisque ($\star$) *indique que le problème a été proposé sans solution.*

Nous sollicitons en particulier des problèmes originaux. Cependant, d'autres problèmes intéressants pourraient être acceptables s'ils ne sont pas trop connus et si leur provenance est précisée. Normalement, si l'auteur d'un problème est connu, il faut demander sa permission avant de proposer un de ses problèmes.

Pour faciliter l'étude de vos propositions, veuillez taper ou écrire à la main (lisiblement) chaque problème sur une feuille distincte de format $8\frac{1}{2}''\times 11''$ *ou* A4, *la signer et la faire parvenir au rédacteur en chef. Les propositions devront lui parvenir au plus tard le* **1er octobre 2002**. *Vous pouvez aussi les faire parvenir par courriel à* **crux-editors@cms.math.ca**. (*Nous apprécierions de recevoir les problèmes et solutions envoyés par courriel au format* LaTeX). *Les fichiers graphiques doivent être de format « epic » ou « eps » (encapsulated postscript). Les solutions reçues après la date ci-dessus seront prises en compte s'il reste du temps avant la publication. Veuillez prendre note que nous n'acceptons pas les propositions par télécopieur.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

2739. *Proposé par Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Soit a, b et c des nombres réels positifs. Montrer que

$$\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c}+\frac{\sqrt{a+b+c}+\sqrt{b}}{c+a}+\frac{\sqrt{a+b+c}+\sqrt{c}}{a+b} \;\geq\; \frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}\,.$$

. . . .

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c}+\frac{\sqrt{a+b+c}+\sqrt{b}}{c+a}+\frac{\sqrt{a+b+c}+\sqrt{c}}{a+b} \;\geq\; \frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}\,.$$

2740. *Proposé par Victor Oxman, University of Haifa, Haifa, Israel.*

On donne trois ellipses E_1, E_2 et E_3 dans un plan. Les points A, B et C satisfont les conditions suivantes:

> A et B sont les foyers de E_1, B et C sont les foyers de E_2, C et A sont les foyers de E_3, C est sur E_1, A est sur E_2, B est sur E_3.

Muni d'une règle sans graduation, construire le centre du cercle inscrit au triangle ABC.

. . . .

In the plane are given three ellipses, E_1, E_2 and E_3. The points A, B and C satisfy the following conditions:

> A and B are the foci of E_1, B and C are the foci of E_2, C and A are the foci of E_3, C is on E_1, A is on E_2, B is on E_3.

With only an unmarked straight-edge, construct the incentre of $\triangle ABC$.

2741. *Proposé par Victor Oxman, University of Haifa, Haifa, Israel.*

Dans le plan, on donne deux points A et B sur une ellipse de foyers M et N, de sorte que AB soit parallèle à MN. Muni d'une règle sans graduation, construire un diamètre du cercle passant par $ABNM$.

. . . .

In the plane are given an ellipse with its two foci M and N, and two points, A and B, on it, so that $AB \parallel MN$.

With only an unmarked straight-edge, construct a diameter of the circle $ABNM$.

2742. *Proposé par Manuel Murillo Tsijli, Instituto Tecnológico de Costa Rica, Cartago, Costa Rica.*

On considère la function $f : \mathbb{R} \to \mathbb{R}$ definie par

$$f(x) \;=\; \begin{cases} 5x & \text{if} \quad x \leq \frac{1}{2}\,, \\ 5-5x & \text{if} \quad x > \frac{1}{2}\,. \end{cases}$$

Pour $n \geq 2$, soit $f^1(x) = f(x)$ et $f^n(x) = f\left(f^{n-1}(x)\right)$. Calculer la valeur exacte de $f^{1998}\left(\dfrac{4}{5^{16}-1} + \dfrac{4}{5^{125}-1}\right)$.

. . . .

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) \;=\; \begin{cases} 5x & \text{if} \quad x \leq \frac{1}{2}\,, \\ 5-5x & \text{if} \quad x > \frac{1}{2}\,. \end{cases}$$

Let $f^1(x) = f(x)$ and $f^n(x) = f\left(f^{n-1}(x)\right)$ for $n \geq 2$. Calculate the exact value of $f^{1998}\left(\dfrac{4}{5^{16}-1} + \dfrac{4}{5^{125}-1}\right)$.

2743. *Proposé par Péter Ivády, Budapest, Hungary.*
Montrer que, si x, $y \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{x}{\sin x} + \frac{y}{\sin y}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) \ < \ 2\,.$$

. . . .

Show that, for x, $y \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{x}{\sin x} + \frac{y}{\sin y}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) \ < \ 2\,.$$

2744. *Proposé par K.R.S. Sastry, Bangalore, India.*
Soit D et E deux points choisis respectivement sur les côtés opposés aux sommets A et B d'un triangle ABC. Les droites AD et BE se coupent en un point intérieur K. On suppose que $\frac{AK}{KD} = \frac{BK}{KE} = \lambda \neq \pm 1$.

Montrer que $AD^2 + BE^2 = \left(\frac{2}{\lambda}\right) AB^2 + \left(\frac{\lambda-1}{\lambda}\right)^2 \left(BC^2 + CA^2\right)$.

. . . .

The cevian AD and BE of $\triangle ABC$ intersect at an interior point K. Assume that $\frac{AK}{KD} = \frac{BK}{KE} = \lambda \neq \pm 1$.

Show that $AD^2 + BE^2 = \left(\frac{2}{\lambda}\right) AB^2 + \left(\frac{\lambda-1}{\lambda}\right)^2 \left(BC^2 + CA^2\right)$.

2745. *Proposé par K.R.S. Sastry, Bangalore, India.*
Soit ABC un triangle pythagoricien primitif (c.-à-d. que le pgcd des côtés est égal à 1) avec C le sommet de l'angle droit. Soit D un point sur AB et E un point sur AC de sorte que DE est à la fois perpendiculaire à AB et tangente au cercle inscrit du triangle.

Montrer que BE est de longueur rationnelle si et seulement si la longueur de AB est le carré d'un entier.

. . . .

Let ABC be a primitive Pythagorean triangle (that is, the gcd of the sides is 1) in which $\angle ACB$ is the right angle. Let D be a point in AB and E a point in AC such that DE is perpendicular to AB and also tangent to the incircle of $\triangle ABC$.

Prove that BE has rational length if and only if the length of AB is the square of an integer.

2746. *Proposé par K.R.S. Sastry, Bangalore, India.*
Dans un triangle ABC, on suppose que les côtés sont en progression arithmétique, avec $AB + BC = 2AC$. La médiane AD coupe le segment de Gergonne (c.-à-d. le segment allant de B au point de contact E du cercle incrit avec AC) au point S.

Montrer que le triangle ABC est semblable à un triangle de Héron rationnel (c.-à-d. ayant des côtés et une aire rationnels) si et seulement si $\frac{AS}{SD}$ est égal à un sixième d'un carré rationnel.

. . . .

In triangle ABC, the sides are in arithmetic progression with $AB + BC = 2AC$. The median AD intersects the Gergonne cevian BE (that is, the line segment from B to the contact point E of the incircle with AC) at the point S.

Prove that $\triangle ABC$ is similar to a rational Heron triangle (that is, one with rational sides and area) if and only if $\frac{AS}{SD}$ is one-sixth of a rational square.

2747. *Proposé par K.R.S. Sastry, Bangalore, India.*

Montrer que l'orthocentre d'un triangle se trouve dans ou sur le cercle inscrit si et seulement si le rayon de celui-ci est une moyenne proportionnelle des segments déterminés sur une des hauteurs par l'orthocentre.

. . . .

Prove that the orthocentre of a triangle lies inside or on the incircle if and only if the inradius is a mean proportional to the two segments of an altitude, sectioned by the orthocentre.

2748. *Proposé par Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Soit a_1, a_2, $\ldots$, a_n $(n \geq 1)$ des nombres réels non négatifs tels que $a_1 \leq a_2 \leq \ldots \leq a_n$ et $\sum\limits_{k=1}^{n} a_k = 1$.

Determiner l'infimum de $a_n \sum\limits_{k=1}^{n} (n+1-k)a_k$.

. . . .

Let a_1, a_2, $\ldots$, a_n $(n \geq 1)$ be non-negative real numbers such that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $\sum\limits_{k=1}^{n} a_k = 1$.

Determine the least upper bound of $a_n \sum\limits_{k=1}^{n} (n+1-k)a_k$.

2749. *Proposé par Christopher J. Bradley, Clifton College, Bristol, UK.*

Soit P un point intérieur du triangle ABC. La droite par P parallèle à AB coupe BC en L et CA en M'. La droite par P parallèle à BC coupe CA en M et AB en N'. La droite par P parallèle à CA coupe AB en N et BC en L'.

Montrer que

(a) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right)\left(\frac{BL'}{L'C}\right)\left(\frac{CM'}{M'A}\right)\left(\frac{AN'}{N'B}\right) = 1;$

(b) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right) \leq \frac{1}{8}$;

(c) $[LMN] = [L'M'N']$; [Note: $[XYZ]$ désigne l'aire du triangle XYZ.]

(d) $[LMN] \leq \frac{[ABC]}{3}$.

Situer le point P lorsqu'il y a égalité dans les parties (b) et (d).

....

Suppose that P is an interior point of $\triangle ABC$. The line through P parallel to AB meets BC at L and CA at M'. The line through P parallel to BC meets CA at M and AB at N'. The line through P parallel to CA meets AB at N and BC at L'.

Prove that

(a) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right)\left(\frac{BL'}{L'C}\right)\left(\frac{CM'}{M'A}\right)\left(\frac{AN'}{N'B}\right) = 1$;

(b) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right) \leq \frac{1}{8}$;

(c) $[LMN] = [L'M'N']$; [Note: $[XYZ]$ denotes the area of $\triangle XYZ$.]

(d) $[LMN] \leq \frac{[ABC]}{3}$.

Locate the point P when equality holds in parts (b) and (d).

2750 *Proposé par Paul Bracken, CRM, Université de Montréal, Montréal, Québec.*

Un triangle ABC possède un angle droit en C, et le produit des longueurs des côtés AB et BC est constant.

Si $\lambda > 2\sqrt{2}$, montrer que la quantité $AC + \lambda BC$ possède un minimum lorsque $AC = \left(\frac{\lambda+\sqrt{\lambda^2-8}}{2}\right) BC$, et un maximum lorsque $AC = \left(\frac{\lambda-\sqrt{\lambda^2-8}}{2}\right) BC$.

....

A triangle ABC has a right angle at C, and the product of the lengths of the sides AB and BC is constant.

If $\lambda > 2\sqrt{2}$, show that the quantity $AC + \lambda BC$ has a minimum when $AC = \left(\frac{\lambda+\sqrt{\lambda^2-8}}{2}\right) BC$, and a maximum when $AC = \left(\frac{\lambda-\sqrt{\lambda^2-8}}{2}\right) BC$.

CORRECTION to / pour 2702 — $\lambda > 0$ is required. / est nécessaire.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2609. [2001 : 49] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

A convex polygon P_n ($n \geq 4$) has the following property:

> the $n-3$ diagonals emanating from each of the n vertices of P_n divide the corresponding angle of P_n into $n-2$ equal parts.

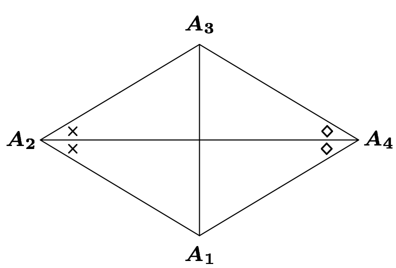
Determine the shape of P_n.

2636. [2001 : 215] was withdrawn, since it is essentially same as 2609. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that $A_1A_2\ldots A_n$ is a convex n–gon with $n \geq 5$, and that the angle at each vertex is divided into $(n-2)$ equal angles by the $(n-3)$ diagonals through that vertex. Prove that $A_1A_2\ldots A_n$ is a regular n–gon.

a. *The case $n = 4$. Solution by the majority of solvers.*

$\triangle A_4A_1A_2 \cong \triangle A_4A_3A_2$ by ASA (as in the figure).

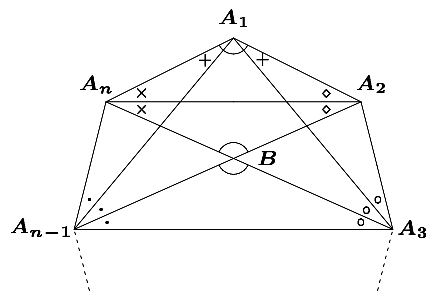


Thus, $A_4A_1 = A_4A_3$ and $A_2A_1 = A_2A_3$. Using $\triangle A_1A_2A_3 \cong \triangle A_1A_4A_3$ we see that $A_4A_1 = A_2A_1$, which proves that $A_1A_2A_3A_4$ is a rhombus. Conversely, if $A_1A_2A_3A_4$ is a rhombus, it satisfies the given conditions (namely, the diagonal at each vertex cuts the angle at that vertex in half).

b. *The case $n \geq 5$.*

I. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that A_2A_{n-1} intersects A_3A_n at B. Let $\alpha_i = \dfrac{\angle A_i}{n-2}$ for each i. Since $\triangle A_1A_2A_n \cong \triangle BA_2A_n$ by ASA (as in the case $n = 4$ — see the figure below), we have $\angle A_1 = \angle A_2BA_n = \angle A_3BA_{n-1}$.



Considering the angle sums of $\triangle A_1A_3A_{n-1}$ and $\triangle BA_3A_{n-1}$ we get

$$(n-4)\alpha_1 + 2\alpha_3 + 2a_{n-1} \;=\; (n-2)\alpha_1 + \alpha_3 + \alpha_{n-1}\,.$$

This implies that $2(2\alpha_1 - \alpha_3 - \alpha_{n-1}) = 2\alpha_1 - \alpha_3 - \alpha_{n-1}$, and therefore, that $2\alpha_1 - \alpha_3 - \alpha_{n-1} = 0$, whence,

$$\alpha_1 \;=\; \frac{\pi}{n}\,.$$

It follows that $\alpha_i \;=\; \dfrac{\pi}{n}$ and $A_iA_{i+1} = A_iA_{i-1}$ for each i (taken modulo n).

II. *Outline of solutions submitted independently by Christopher J. Bradley, Clifton College, Bristol, UK and Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

A somewhat stronger result will be proved:

> *For $n \geq 5$ let the convex n–gon $A_0A_1A_2\cdots A_{n-1}$* (with subscripts reduced modulo n) *have vertex angles $(n-2)x_i = \angle A_{i-1}A_iA_{i+1}$, $i = 0, 1, \ldots, n-1$. If the angles at each vertex between the adjacent sides and the shortest diagonals* (namely, $\angle A_{i+1}A_iA_{i+2}$ and $\angle A_{i-2}A_iA_{i-1}$) *both equal x_i, then the polygon is regular.*

By considering the angle sum of triangles $A_0A_1A_2$, $A_1A_2A_3$, $A_2A_3A_4$, $\ldots$, we get the following set of simultaneous equations for the angles x_i:

$$x_k + (n-2)x_{k+1} + x_{k+2} \;=\; \pi\,, \quad k = 0, 1, \ldots, n-1.$$

It follows that one solution of these equations is that all the x_i are equal. To show that the solution is unique, it suffices to show that the determinant of the coefficient matrix is non-zero. We have an $n \times n$ circulant matrix whose first row is $1\ \ n-2\ \ 1\ \ 0\ \ 0\ \ \ldots\ \ 0$. It is easily seen that the determinant is zero when $n = 4$ (in agreement with case (a) above), and it is a known result that the determinant is non-zero for $n \geq 5$. [Bradley provides the historic reference Wolstenholme, *Mathematical Problems*, Macmillan, 1867, problem 1631; Klamkin refers to P.H. Davis, *Circulant Matrices*, Wiley, 1979, p. 81, number 26. Each gives the determinant explicitly. In fact it is simpler than this: the Gershgorin circle theorem applied to the n by n circulant matrix whose first row is $n-2\ 1\ 0\ \ldots\ 0\ 1$ says that the real part of the eigenvalues is not less than $n-4$; hence, the determinant is non-zero for $n \geq 5$. See, for example, Gilbert Strang, *Linear Algebra and Its Applications*, 3rd ed. p. 386, exercise 7.4.2.] Finally, since all the x_i are equal, all the triangles $A_0A_1A_2$, $A_1A_2A_3$, ... are isosceles; thus, the sides of the n–gon are equal as are its vertex angles, so that it is regular.

Also solved by PIERRE BORNSZTEIN, Pontoise, France; BRYAN DAWSON, Union University, Jackson, TN, USA, and KENNETH M. WILKE, Topeka, KS, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; HENRY LIU, student, University of Memphis, Memphis, TN, USA; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and by both proposers. There was one incorrect submission.

2611. [2001 : 49] and **2647**. [2001 : 269] *Proposed by Michel Bataille, Rouen, France.*

Let O, H and R denote the circumcentre, the orthocentre and the circumradius of $\triangle ABC$, and let Γ be the circle with centre O and radius $\rho = OH$. The tangents to Γ at its points of intersection with the rays $[OA)$, $[OB)$ and $[OC)$ form a triangle.

Express the circumradius of this triangle as a function of R and ρ.

Solution de C. Festraets-Hamoir, Brussels, Belgium.

Soit $A'B'C'$ ce triangle, avec $B'C' \perp AO$, $C'A' \perp BO$ et $A'B' \perp CO$, et soit R' le rayon du cercle circonscrit à $A'B'C'$.

$$\left\{\begin{array}{lcl} \angle B'A'C' & = & 180^\circ - \angle BOC \ = \ 180^\circ - 2A\,, \\ \angle A'B'C' & = & 180^\circ - 2B\,, \\ \angle B'C'A' & = & 180^\circ - 2C\,. \end{array}\right.$$

Dans le triangle $A'B'C'$, la relation qui lie les rayons des cercles inscrit et circonscrit est

$$\begin{aligned} \rho &= 4R' \sin\left(\frac{180^\circ - 2A}{2}\right) \sin\left(\frac{180^\circ - 2B}{2}\right) \sin\left(\frac{180^\circ - 2C}{2}\right) \\ &= 4R' \cos A \cos B \cos C\,. \end{aligned} \tag{1}$$

D'autre part, dans le triangle ABC

$$OH^2 \ = \ \rho^2 \ = \ R^2(1 - 8\cos A \cos B \cos C)$$

d'où $4\cos A \cos B \cos C \ = \ \dfrac{R^2-\rho^2}{2R^2}$, ce qui donne dans (1) : $\rho \ = \ R'\dfrac{R^2-\rho^2}{2R^2}$ et $R' \ = \ \dfrac{2\rho R^2}{R^2-\rho^2}$.

Editor's comment: the editor apologizes for printing the same problem twice. However, Li Zhou commented that "a lovely problem like this deserves to appear twice"!

We have highlighted the case of an acute-angled triangle. The case of an obtuse-angled triangle needs some modifications, as most solvers noted.

Also solved by BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; VINAYAK GANESHAN, student, University of Waterloo, Waterloo, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, Berkhamsted, UK; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2616★. [2001 : 137] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

The following are three known properties of parabolas:

1. The area of the parabolic segment upon any chord as base is equal to $\frac{4}{3}$ times the area of the triangle having the same base and height (the tangent at a vertex of the triangle is parallel to the chord). [Due to Archimedes.]
2. The area of the parabolic segment cut off by any chord is $\frac{2}{3}$ times the area of the triangle formed by the chord and the tangents at its extremities.
3. The area of a triangle formed by three tangents to a parabola is $\frac{1}{2}$ times the area of the triangle whose vertices are the points of contact.

Are there any other smooth curves having any one of the above properties?

Editor's comment.

To date, we have two submissions which give contradictory answers. We are still trying to resolve this issue.

2623★. [2001 : 138] *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x_1, x_2, ..., $x_n > 0$. Let $x_{n+1} = x_1$, $x_{n+2} = x_2$, etc. For $k = 0, 1, \ldots, n-1$, let

$$S_k = \sum_{j=1}^{n} \left(\frac{\sum\limits_{i=0}^{k} x_{j+i}}{\sum\limits_{i=0}^{k} x_{j+1+i}} \right).$$

Prove or disprove that $S_k \geq S_{k+1}$.

No solutions have been received so far. The problem remains open.

2625. *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

If R denotes the circumradius of triangle ABC, prove that

$$18R^3 \;\geq\; \left(a^2+b^2+c^2\right)R+\sqrt{3}abc\,.$$

Editor's Remark.

Most of the solutions were along the same lines as the one featured below, which was chosen for its succinctness.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

The required inequality follows immediately from the well-known inequalities

$$a^2+b^2+c^2 \;\leq\; 9R^2 \qquad \text{item 5.13 on p. 52 in [1]}$$

and

$$abc \;\leq\; \left(R\sqrt{3}\right)^3 \qquad \text{item 5.27 on p. 55 in [1]}$$

in which there is equality if and only if the triangle is equilateral.

Reference

[1] O. Bottema *et al.*, *Geometric Inequalities*, Groningen, 1968.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SCOTT BROWN, Auburn University, Montgomery, AL, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, Berkhamsted, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFLER, student, Cotham School, Bristol, UK; KEE-WAI LAU, Hong Kong, China; HENRY LIU, student, University of Memphis, TN, USA; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2626★. [2001 : 213] *Proposed by Achilleas Sinefakopoulos, student, University of Athens, Greece.*

Let $\alpha_n = 2n + \lfloor n\sqrt{2}\rfloor$ for $n = 1, 2, \ldots$. Suppose that k and m are positive integers such that α_m is a multiple of 10 and $\alpha_k = \alpha_m + 10j$ for some positive integer j. Prove or disprove that if $j \leq 4$, then $k = m + 3j$.

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina..

For $n = 1, 2, \ldots$, let δ_n be the fractional part of $n\sqrt{2}$. It is easy to see that

$$0 \;<\; \delta_n \;=\; (2+\sqrt{2})n - \alpha_n \;<\; 1\,.$$

Hence, $\alpha_k = \alpha_m + 10j$ implies

$$-1 \;<\; \delta_k - \delta_m \;=\; (2+\sqrt{2})(k-m) - 10j \;<\; 1\,,$$

whence it follows that

$$\frac{10j-1}{2+\sqrt{2}} \;<\; k-m \;<\; \frac{10j+1}{2+\sqrt{2}}\,.$$

From this we get

$$\begin{array}{lllll} j=1 & \Longrightarrow & 2.636\cdots < k-m < 3.221\cdots & \Longrightarrow & k-m = 3\cdot 1 \\ j=2 & \Longrightarrow & 5.564\cdots < k-m < 6.150\cdots & \Longrightarrow & k-m = 3\cdot 2 \\ j=3 & \Longrightarrow & 8.493\cdots < k-m < 9.079\cdots & \Longrightarrow & k-m = 3\cdot 3 \\ j=4 & \Longrightarrow & 11.422\cdots < k-m < 12.008\cdots & \Longrightarrow & k-m = 3\cdot 4 \end{array}$$

as we wanted to show.

Remarks: The hypothesis that α_m is a multiple of 10 is unnecessary. On the other hand, $k - m = 3j$ is false if $j = 5$, because in such a case $14.35\cdots < k-m < 14.93\cdots$.

Also solved by HENRY LIU, student, University of Memphis, Memphis, TN, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There were three incorrect solutions.

2627. [2001 : 214] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x_1, \ldots, x_n$ be positive real numbers and let $s_n = x_1 + \cdots + x_n$ ($n \geq 2$). Let $a_1, \ldots, a_n$ be non-negative real numbers. Determine the optimum constant $C(n)$ such that

$$\sum_{j=1}^{n} \frac{a_j(s_n - x_j)}{x_j} \;\geq\; C(n)\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}\,.$$

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

Equivalently, we want the minimum value of

$$S \;=\; \sum \frac{b_1 x_2}{x_1} + \sum \frac{b_1 x_3}{x_1} + \cdots + \sum \frac{b_1 x_n}{x_1}\,,$$

where the sums are cyclic over the indices $1, 2, \ldots, n$, and

$$b_j \;=\; \frac{a_j}{\left(\prod\limits_{j=1}^{n} a_j\right)^{\frac{1}{n}}}\,.$$

Applying the AM–GM Inequality to each of these sums, we get

$$S \;\geq\; (n-1)+(n-1)+\cdots+(n-1) \;=\; n(n-1) \;=\; C(n)\,.$$

There is equality if and only if all the a_j's are equal and all the x_j's are equal.

Also solved by MICHEL BATAILLE, Rouen, France; PIERRE BORNSZTEIN, Pontoise, France; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Klamkin also notes that if we replace $\left(\dfrac{s_n - x_j}{x_j}\right)$ *in the given sum by* $\left(\left(\dfrac{s_n}{x_j}\right)^r - 1\right)$*, then we can show in a similar fashion that we now have* $C(n) = n(n^r - 1)$*, where* r *is a positive integer.*

The other solvers had almost identical methods, again using the AM–GM Inequality. Since Klamkin's solution was quite different, we decided to highlight it.

2628. [2001 : 214] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

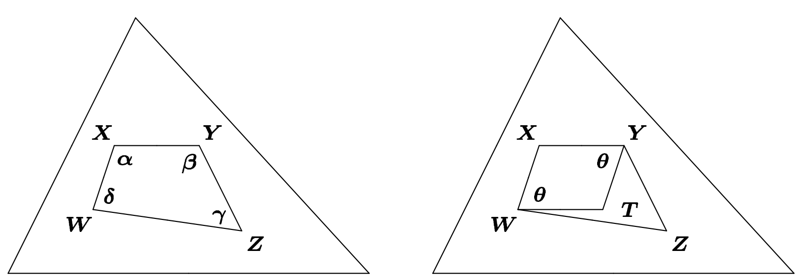
Four points, X, Y, Z and W are taken inside or on triangle ABC. Prove that there exists a set of three of these points such that the area of the triangle formed by them is less than $\frac{3}{8}$ of the area of the given triangle.

Solution by Pierre Bornsztein, Pontoise, France.

We will prove that the bound "less than $\frac{3}{8}$" can be replaced by "not greater than $\frac{1}{3}$". The notation $[F]$ will be used for the area of the figure F. Let X, Y, Z and W be four points inside or on the triangle ABC. The result is trivial if three of the four points are collinear. From now on, we assume that no three points are collinear. Let $\mathcal{C}$ be the convex hull of the set $\{X, Y, Z, W\}$. Clearly, $\mathcal{C}$ is either a convex quadrilateral or a triangle. We will use the following well-known result [1].

Lemma If a parallelogram $PQRS$ is covered by a triangle TUV, then $[PQRS] \leq \frac{1}{2}[TUV]$.

We now show that if $XYZW$ is a convex quadrilateral covered by the triangle ABC, then at least one of the triangles XYZ, XYW, XZW and YZW has area not exceeding $\frac{1}{4}[ABC]$.



Let α, β, γ and δ be the interior angles of the quadrilateral $XYZW$, in radians. Since $\alpha+\beta+\gamma+\delta=2\pi$, then either $\alpha+\beta\geq\pi$ or $\gamma+\delta\geq\pi$. Similarly, either $\alpha+\delta\geq\pi$ or $\beta+\gamma\geq\pi$. Without loss of generality, we can assume that $\alpha+\beta\geq\pi$ and $\alpha+\delta\geq\pi$. Let T be the point such that $XYTW$ is a parallelogram and let $\theta=\angle XWT=\angle XYT$. Then $\alpha+\theta=\pi\leq\alpha+\beta$ and $\alpha+\theta=\pi\leq\alpha+\delta$. It follows that $\theta\leq\beta$ and $\theta\leq\delta$, from which we deduce that the parallelogram $XYTW$ is covered by the quadrilateral $XYZW$, and therefore, by the triangle ABC. By the lemma, $[XYW]=\frac{1}{2}[XYTW]\leq\frac{1}{4}[ABC]$, which completes the proof.

Thus, if $\mathcal{C}$ is a convex quadrilateral, then three of its vertices form a triangle of area not exceeding $\frac{1}{4}[ABC]$.

Let $\mathcal{C}$ be a triangle. Without loss of generality, we may assume that the point W is inside the triangle XYZ. Then, trivially, one of the triangles XYW, YZW and ZXW has area not exceeding $\frac{1}{3}[ABC]$. If X, Y, Z and W are the vertices A, B, C and the centroid of the $\triangle ABC$, correspondingly, then $[XYW]=[YZW]=[ZXW]=\frac{1}{3}[ABC]$. This shows that the bound $\frac{1}{3}$ cannot be replaced by a smaller one.

[1] A.M. Yaglom, I.M. Yaglom, *Challenging mathematical problems with elementary solutions*, vol. II, Dover, ex. 121.b, 83–85.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. The improved bound was also found by Zhou.

2629. [2001 : 214] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

In triangle ABC, the symmedian point is denoted by S. Prove that

$$\frac{1}{3}\left(AS^2+BS^2+CS^2\right)\ \geq\ \frac{BC^2AS^2+CA^2BS^2+AB^2CS^2}{BC^2+CA^2+AB^2}\,.$$

Solution by Joel Schlosberg, student, New York University, NY, USA.

It is known that

$$AS\ =\ \frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2}\,,\quad BS\ =\ \frac{ca\sqrt{2c^2+2a^2-b^2}}{a^2+b^2+c^2}\,,$$

$$\text{and}\quad CS\ =\ \frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2}\,.$$

[Ed. some of the solvers proved this result.]

From these formulas it follows that

$$\begin{aligned}&AS^2+BS^2+CS^2\\ &=\ \frac{b^2c^2\left(2b^2+2c^2-a^2\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{c^2a^2\left(2c^2+2a^2-b^2\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{a^2b^2\left(2a^2+2b^2-c^2\right)}{\left(a^2+b^2+c^2\right)^2}\\ &=\ \frac{2\left(a^2b^4+a^4b^2+b^2c^4+b^4c^2+c^2a^4+c^4a^2\right)-3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2}\,.\end{aligned}$$

Also, we compute that

$$\begin{aligned}
&BC^2AS^2 + CA^2BS^2 + AB^2CS^2 \\
&= \frac{a^2b^2c^2\left(2b^2+2c^2-a^2\right)}{\left(a^2+b^2+c^2\right)^2} + \frac{a^2b^2c^2\left(2c^2+2a^2-b^2\right)}{\left(a^2+b^2+c^2\right)^2} \\
&\qquad + \frac{a^2b^2c^2\left(2a^2+2b^2-c^2\right)}{\left(a^2+b^2+c^2\right)^2} \\
&= \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)}\,.
\end{aligned}$$

Then, the inequality

$$\frac{AS^2+BS^2+CS^2}{3} \ \geq\ \frac{BC^2AS^2+CA^2BS^2+AB^2CS^2}{BC^2+CA^2+AB^2}$$

becomes

$$\frac{2\left(a^2b^4+a^4b^2+b^2c^4+b^4c^2+c^2a^4+c^4a^2\right)-3a^2b^2c^2}{3\left(a^2+b^2+c^2\right)^2} \ \geq\ \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2}\,,$$

which is equivalent to

$$a^2b^4+a^4b^2+b^2c^4+b^4c^2+c^2a^4+c^4a^2 \ \geq\ \frac{9a^2b^2c^2+3a^2b^2c^2}{2} \ =\ 6a^2b^2c^2\,,$$

which is a straightforward application of the Arithmetic-Geometric Mean Inequality for the numbers a^2b^4, a^4b^2, b^2c^4, b^4c^2, c^2a^4 and c^4a^2.

Solutions submitted by: ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; DAVID LOEFFLER, student, Cotham School, Bristol, UK; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The majority of the submitted solutions were a variation of the solution given above. Several solvers used the Tchebychev inequality; some others used areal co-ordinates.

Two solvers, Heuver and Janous, noted that this problem is essentially a result found in Recent Advances in Geometric Inequalities, *D.S. Mitrinovic, J.E. Pearic and V. Volenec, Kluwer Academic Publishers, 1989, p. 279.*

2630. [2001 : 214] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}\sum_{j=1}^{n}\frac{1}{j} = \frac{\pi^2}{12}$.

Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n2^n}\sum_{j=1}^{n}\frac{1}{j} \\
&= \sum_{j=1}^{\infty}\frac{1}{j}\sum_{n=j}^{\infty}\frac{1}{n2^n} = \sum_{j=1}^{\infty}\frac{1}{j}\int_0^1\sum_{n=j}^{\infty}\frac{y^{n-1}}{2^n}\,dy = \sum_{j=1}^{\infty}\frac{1}{j}\int_0^1\frac{2^{-j}y^{j-1}}{1-\frac{1}{2}y}\,dy \\
&= \int_0^1\frac{1}{1-\frac{1}{2}y}\sum_{j=1}^{\infty}\frac{y^{j-1}}{j2^j}\,dy = \int_0^1\frac{2}{y(2-y)}\sum_{j=1}^{\infty}\frac{y^j}{j2^j}\,dy \\
&= \int_0^1\frac{2}{y(2-y)}\int_0^y\frac{1}{2-t}\,dt\,dy = \int_0^1\frac{2}{y(2-y)}\ln\left(\frac{2}{2-y}\right)\,dy \\
&= \int_0^1\frac{\ln(u+1)}{u}\,du \qquad \text{where } u+1=\frac{2}{2-y} \\
&= \int_0^1\sum_{k=1}^{\infty}\frac{(-1)^{k+1}u^{k-1}}{k}\,du = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.
\end{aligned}$$

The last equality is well known $\left(\text{it can be derived from the very well-known identity } \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}\right)$. Also, whenever summation and integration are interchanged, it is easy to check that the sum concerned is uniformly convergent.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; JOSHUA GREEN, University of Arizona, Tucson, AZ, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; DAVID LOEFFLER, student, Cotham School, Bristol, UK; CHRIS WILDHAGEN, Rotterdam, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Janous remarks that we can get a much stronger result: Let $w \in \left(-1, \frac{1}{2}\right]$ be an arbitrary real number. Then the following identity is valid:

$$\sum_{n=1}^{\infty}\frac{w^n}{n}\sum_{j=1}^{n}\frac{1}{j} = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^2}\left(\frac{w}{1-w}\right)^j.$$

2631. [2001 : 214] *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Find the exact value of $\displaystyle\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n-k}{k}2^{n-2k}$.

Solution by Pierre Bornsztein, Pontoise, France.

Let n be a non-negative integer. With the convention $\binom{n}{k}=0$ when $k>n$, we have

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n-k}{k}2^{n-2k} \;=\; \sum_{k\geq 0}\binom{n}{k}\binom{n-k}{k}2^{n-2k}\,. \tag{1}$$

Let P be the polynomial defined by $P(x)=(1+x)^{2n}$. By the Binomial Expansion Theorem, we have

$$P(x) \;=\; \sum_{j\geq 0}\binom{2n}{j}x^j\,.$$

On the other hand,

$$\begin{aligned} P(x) &= \left((1+2x)+x^2\right)^n \;=\; \sum_{k\geq 0}\binom{n}{k}x^{2k}(1+2x)^{n-k} \\ &= \sum_{k\geq 0}\left(\binom{n}{k}x^{2k}\sum_{j\geq 0}\binom{n-k}{j}2^{n-k-j}x^{n-k-j}\right)\,. \end{aligned}$$

Comparing the coefficients of x^n in the two expansions of $P(x)$ and using the equality (1), we obtain

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}\binom{n-k}{k}2^{n-2k} \;=\; \binom{2n}{n}\,,$$

which completes the solution.

References

[1] M.J. Erickson and J. Flowers *Principles of Mathematical Problem Solving*, Prentice Hall, 1999.

[2] J. Riordan, *Combinatorial identities*, Krieger, New York, 1979.

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HENRY LIU, student, University of Memphis, Memphis, TN, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands (two solutions); LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. One solver submitted the correct answer without a proof.

Approximately half of the solvers suggested nice combinatorial solutions. This editor was surprised to see twelve different solutions from the eleven solvers! The problem seems to be well known: Guersenzvaig located it in [1] (*p. 114, problem 10.14*). *Lau refers to problem 4 in chapter 2 of* [2]. *The following generalization was found by Seiffert.*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n}{k}\binom{n-k}{k}}{\binom{k+m}{m}} 2^{n-2k} = \frac{\binom{2m+2n}{n}}{\binom{n+m}{m}} .$$

2632. [2001 : 214] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $S_m = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le j\le k\le n} \cos\left(\frac{(j+(-1)^m k)x}{n}\right)$, where $x \in \mathbb{R}$.

Find exact expressions for S_0 and S_1.

Solution by Heinz-Jürgen Seiffert, Berlin, Germany.

It is easily verified that $S_0 = S_1 = \frac{1}{2}$ if $x = 0$. Suppose that x is real and non-zero. From known trigonometric relations, we have

$$\cos\left(\frac{(j-k)x}{n}\right) + \cos\left(\frac{(j+k)x}{n}\right) = 2\cos\left(\frac{jx}{n}\right)\cos\left(\frac{kx}{n}\right)$$

and

$$\cos\left(\frac{(j-k)x}{n}\right) - \cos\left(\frac{(j+k)x}{n}\right) = 2\sin\left(\frac{jx}{n}\right)\sin\left(\frac{kx}{n}\right) .$$

Since

$$2\sum_{1\le j\le k\le n} \cos\left(\frac{jx}{n}\right)\cos\left(\frac{kx}{n}\right) = \left(\sum_{j=1}^{n}\cos\left(\frac{jx}{n}\right)\right)^2 + \sum_{j=1}^{n}\cos^2\left(\frac{jx}{n}\right) ,$$

$$2\sum_{1\le j\le k\le n} \sin\left(\frac{jx}{n}\right)\sin\left(\frac{kx}{n}\right) = \left(\sum_{j=1}^{n}\sin\left(\frac{jx}{n}\right)\right)^2 + \sum_{j=1}^{n}\sin^2\left(\frac{jx}{n}\right)$$

and $\cos^2 y \le 1$ and $\sin^2 y \le 1$ for real y, we find that

$$S_1 + S_0 = \left(\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\cos\left(\frac{jx}{n}\right)\right)^2 = \left(\int_0^1 \cos(tx)\,dt\right)^2 = \left(\frac{\sin x}{x}\right)^2$$

and

$$S_1 - S_0 = \left(\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\sin\left(\frac{jx}{n}\right)\right)^2 = \left(\int_0^1 \sin(tx)\,dt\right)^2 = \left(\frac{1-\cos x}{x}\right)^2.$$

These equations and the relation $\sin^2 x + \cos^2 x = 1$ imply that

$$S_0 \;=\; \frac{\cos x - \cos^2 x}{x^2} \quad\text{and}\quad S_1 \;=\; \frac{1-\cos x}{x^2}\,.$$

Also solved by BENJAMIN ARMBRUSTER, student, University of Arizona, Tucson, AZ, USA; MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solutions submitted were similar to the above one. However, not everyone noted that when $x = 0$, both S_0 and S_1 are $\frac{1}{2}$.

2633. [2001 : 213] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that $\displaystyle \frac{n(n+1)}{2e} \;<\; \sum_{k=1}^{n}(k!)^{1/k} \;<\; \frac{31}{20}+\frac{n(n+1)}{4}\,.$

Amalgamated Solutions of Michel Bataille, Rouen, France and Heinz-Jürgen Seiffert, Berlin, Germany (adapted by the Editor).

Let $\displaystyle S_n \;=\; \sum_{k=1}^{n}(k!)^{1/k}$. Since $\displaystyle e^x \;=\; \sum_{n=0}^{\infty}\frac{x^n}{n!}$, we have $\displaystyle e^x \;>\; \frac{x^k}{k!}$ for $x > 0$ and $k \ge 1$. Hence, $\displaystyle k! > \left(\frac{k}{e}\right)^k$ or $\displaystyle (k!)^{1/k} \;>\; \frac{k}{e}$ for all $k \ge 1$.

Hence, $\displaystyle S_n > \sum_{k=1}^{n}\frac{k}{e} = \frac{n(n+1)}{2e}$.

To establish the right-hand inequality, we note first that the inequality $\displaystyle S_n < \frac{31}{20}+\frac{n(n+1)}{4}$ can be easily verified for $n = 1$, 2, 3, 4, 5. In particular,

$$\begin{aligned} S_5 \;&=\; \sum_{k=1}^{5}(k!)^{1/k} \;<\; 1 + 1.41422 + 1.81713 + 2.21337 + 2.60518 \\ &=\; 9.0499 \;<\; 9.05 \;=\; 1.55 + 7.5 \;=\; \frac{31}{20} + \frac{5\times 6}{4}\,. \end{aligned}$$

We now prove by induction that, for all $k \ge 6$,

$$k! \;<\; \left(\frac{k}{2}\right)^k\,. \tag{1}$$

This is clearly true for $k = 6$, since $6! = 720 < 729 = 3^6$.

Suppose that (1) holds for some $k \ge 6$. Then,

$$(k+1)! \;=\; (k+1)k! < (k+1)\left(\frac{k}{2}\right)^k\,. \tag{2}$$

Since it is well known that $2 \leq (1+\frac{1}{k})^k$ for all $k \geq 1$, we have $2k^k \leq (k+1)^k$, or $2(k+1)k^k < (k+1)^{k+1}$. Hence,

$$(k+1)\left(\frac{k}{2}\right)^k < \left(\frac{k+1}{2}\right)^{k+1}. \qquad (3)$$

From (2) and (3), we have $(k+1)! < \left(\frac{k+1}{2}\right)^{k+1}$, completing the induction.

From (1) we then have that $(k!)^{1/k} < \frac{k}{2}$ for all $k \geq 6$. Therefore, for $n \geq 6$, we obtain

$$\begin{aligned} S_n &= S_5 + \sum_{k=6}^{n} (k!)^{1/k} < \frac{31}{20} + \frac{5\times 6}{4} + \frac{1}{2}\sum_{k=6}^{n} k \\ &= \frac{31}{20} + \frac{1}{2}\sum_{k=1}^{5} k + \frac{1}{2}\sum_{k=6}^{n} k = \frac{31}{20} + \frac{n(n+1)}{4} \end{aligned}$$

and the proof is complete.

Also solved by BENJAMIN ARMBRUSTER, student, University of Arizona, Tucson, AZ, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Most solvers either quoted the double inequalities $k/e < (k!)^{1/k} < k/2$ *(for* $k \geq 6$*) directly from some texts or simply stated without any proof that they are consequences of the fact that* $2 \leq \left(1+\frac{1}{k}\right)^k < e$ *"by induction".*

Using the known inequalities $\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1}$*, Seiffert actually refined the left-hand inequality to* $\frac{e-2}{e} + \frac{n(n+3)}{2e} < S_n$*. Loeffler remarked that it could be proved that* $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^{n}(k!)^{1/k}}{n(n+1)} = \frac{1}{2e}$*. This answers a query raised by Janous.*

2634. [2001 : 214] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $P(x) = 1 + \sum_{k=1}^{n} a_k x^k$, where $a_k \in [0,2]$ $(k = 1, 2, \ldots, n)$.

Prove that $P(x)$ is never zero in $(1-\sqrt{2}, 0]$.

I. *Essentially the same solution by Joe Howard, Portales, NM, USA; Kee-Wai Lau, Hong Kong, China; Henry Liu, student, University of Memphis, Memphis, TN, USA; and Robert P. Sealy, Mount Allison University, Sackville, New Brunswick.*

Clearly, $P(0) = 1 > 0$. Hence, it suffices to show that $P(x) > 0$ for $x \in (1-\sqrt{2}, 0)$. Let m be the greatest odd integer, not exceeding n.

Since $-1 < 1-\sqrt{2} < x < 0$, and $0 \le a_k \le 2$ for $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} P(x) &\ge 1 + 2x + 2x^3 + \cdots + 2x^m \\ &> 1 + 2x(1 + x^2 + x^4 + \cdots) \\ &> 1 + 2(1-\sqrt{2})(1 + (1-\sqrt{2})^2 + (1-\sqrt{2})^4 + \cdots) \\ &= 1 + \frac{2(1-\sqrt{2})}{1-(1-\sqrt{2})^2} = 0 \end{aligned}$$

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

If $x \in (1-\sqrt{2}, 0)$, then $x^2 - 2x - 1 = (x-(1+\sqrt{2}))(x-(1-\sqrt{2})) < 0$, and hence, $1 - x^2 > -2x > 0$. Together with the condition that $a_k \in [0, 2]$, we then have

$$\begin{aligned} P(x) &\ge 1 + 2x + \sum_{k=2}^{n} a_k x^k > x^2 + a_3x^3 + a_5x^5 + \cdots \\ &\ge x^2 + 2x^3 + 2x^5 + \cdots = x^2 + \frac{2x^3}{1-x^2} \\ &> x^2 + \frac{2x^3}{-2x} = 0\,. \end{aligned}$$

Also solved by BENJAMIN ARMBRUSTER, student, University of Arizona, Tucson, AZ, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ATHANASIOS KALAKOS, Athens, Greece; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

It is clear from the proofs given above that the conclusion is in fact true for the closed interval $[1-\sqrt{2}, 0]$. However, only Janous, Loeffler, and Seiffert pointed this out explicitly. Both Janous and Seiffert obtained the following stronger result:

> *If $c > 0$ and if $P(x) = 1 + \sum_{k=1}^{n} a_k x^k$ is a polynomial such that $0 \le a_k \le 2c$ for all $k = 1, 2, \ldots, n$, then all the real roots of $P(x)$ lie outside the interval $[c - \sqrt{c^2+1}, \infty)$.*

The proposed problem is the special case when $c = 1$. Janous also remarked that the lower bound $c - \sqrt{c^2+1}$ is the best possible.

2635. [2001 : 215] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF, DI and GH.

Prove that $[XYZ] \leq (4-2\sqrt{3})[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

Combination of the solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA and by the proposer.

Let A', B', C' be the centres of squares $BCDE$, $CAFG$, $ABHI$, respectively. We shall use *directed areas* so that, for example, $[A'BC'] > 0$ when $\angle B > 90^\circ$. [*Editor's comment.* Note that we are taking *clockwise* to be the positive direction here.]

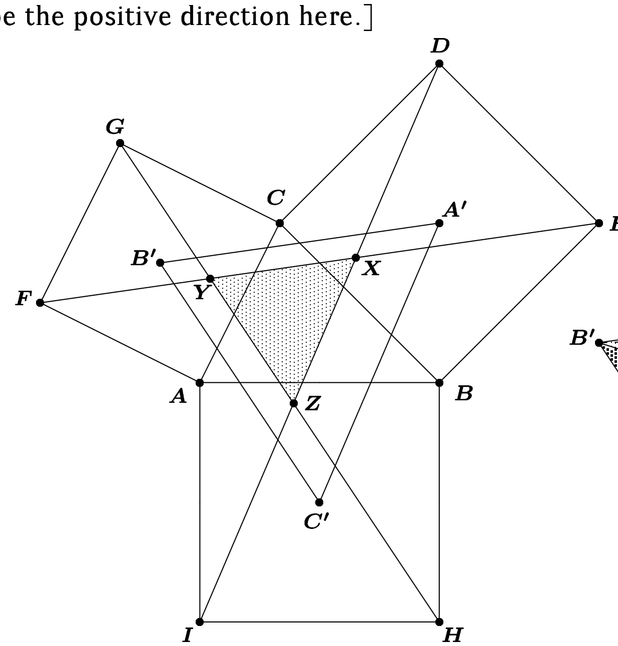


Figure 1.

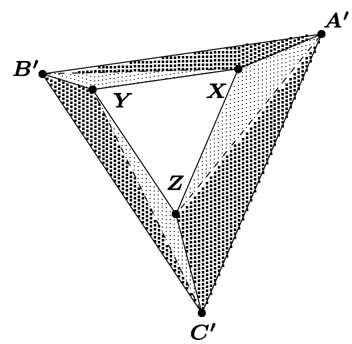


Figure 2.

Step 1. $[AB'C'] + [A'BC'] + [A'B'C] \;=\; \dfrac{a^2+b^2+c^2}{8}$.

Proof. A dilative rotation with centre A and ratio of magnification $\sqrt{2}$ takes $\triangle AB'C'$ to $\triangle AFB$; hence $[AB'C'] = \frac{1}{2}[AFB]$.

Similarly, $[CA'B'] = \frac{1}{2}[CBG]$. Consequently,

$$[AB'C'] + [A'B'C] \;=\; \frac{1}{2}[AFB] + \frac{1}{2}[CBG] \;=\; \frac{1}{4}[CAFG] \;=\; \frac{b^2}{4}\,.$$

[Here we used the Euclidean theorem: For any point X in the plane of parallelogram $PQRS$, $[PQX] + [RSX] = \frac{1}{2}[PQRS]$.] Similarly,

$$[AB'C'] + [A'BC'] \;=\; \frac{c^2}{4}, \quad \text{and} \quad [A'BC'] + [A'B'C] \;=\; \frac{a^2}{4}.$$

Add the three equalities for the result. QED.

[Alternatively, one can use the Cosine Law for step 1:

$$\begin{aligned}[AB'C'] &= \frac{1}{2}AB' \cdot AC' \sin\angle B'AC' \;=\; \frac{1}{2} \cdot \frac{AC}{\sqrt{2}} \cdot \frac{AB}{\sqrt{2}} \sin(90^\circ + A) \\ &= \frac{1}{4}bc\cos A \;=\; \frac{b^2 + c^2 - a^2}{8}\end{aligned}$$

with similar expressions for $[A'BC']$ and $[A'B'C]$.]

Step 2. $[A'B'C'] \;=\; [ABC] + \dfrac{a^2 + b^2 + c^2}{8}$.

Proof.

$$\begin{aligned}[A'B'C'] &= [ABC] + [A'BC] + [AB'C] + [ABC'] \\ &\qquad - \Big([AB'C'] + [A'BC'] + [A'B'C]\Big) \\ &= [ABC] + \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4} - \frac{a^2 + b^2 + c^2}{8} \\ &= [ABC] + \frac{a^2 + b^2 + c^2}{8}\end{aligned}$$

QED.

Step 3. $[XYZ][A'B'C'] = [ABC]^2$.

Proof. Since C' and B' are mid-points of two sides of $\triangle AHG$, we have that GH is parallel to $B'C'$. The analogous statements for $A'B'$ and $A'C'$ imply that $\triangle XYZ \sim \triangle A'B'C'$. Let $XY = \lambda A'B'$; then

$$[XYZ] \;=\; \lambda^2[A'B'C']\,. \tag{1}$$

Hence, the area of the region between $\triangle A'B'C'$ and $\triangle XYZ$ is

$$\begin{aligned}(1 - \lambda^2)[A'B'C'] &= [A'B'C'] - [XYZ] \\ &= \Big([A'B'X] + [XB'Y]\Big) + \Big([B'C'Y] + [YC'Z]\Big) \\ &\qquad + \Big([C'A'Z] + [ZA'X]\Big) \quad \text{(see figure 2)} \\ &= (1 + \lambda)\Big([A'B'X] + [B'C'Y] + [C'A'Z]\Big) \\ &\qquad \text{(using } XY = \lambda A'B'\text{, etc.)} \\ &= (1 + \lambda)\Big([A'CB'] + [B'AC'] + [C'BA']\Big) \\ &\qquad \text{(because } X \text{ and } C \text{ are equidistant} \\ &\qquad \text{from line } A'B'\text{, etc.)}\end{aligned}$$

Therefore, $(1-\lambda)[A'B'C'] = [A'CB'] + [B'AC'] + [C'BA']$, and

$$\lambda = \frac{[A'B'C'] - \left([A'CB'] + [B'AC'] + [C'BA']\right)}{[A'B'C']} = \frac{[ABC]}{[A'B'C']}.$$

Finally, plug this expression for λ back into (1). QED.

Step 4. Proof of the main result.

It is known that $a^2+b^2+c^2 \geq 4\sqrt{3}[ABC]$. (See Bottema et al., *Geometric Inequalities*, p. 42, 4.4.) Hence $\frac{(a^2+b^2+c^2)}{8} \geq \frac{\sqrt{3}}{2}[ABC]$, so that from step 2,

$$[A'B'C'] = [ABC] + \frac{a^2+b^2+c^2}{8} \geq \left(1+\frac{\sqrt{3}}{2}\right)[ABC].$$

Combining this with step 3 we see that

$$[XYZ]\left(1+\frac{\sqrt{3}}{2}\right)[ABC] \leq [XYZ][A'B'C'] = [ABC]^2.$$

Consequently,

$$[XYZ] \leq \left(\frac{1}{1+\frac{\sqrt{3}}{2}}\right)[ABC] = \left(4-2\sqrt{3}\right)[ABC]. \qquad \text{QED.}$$

Also solved by JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Both of the featured solvers used the same four steps. Seimiya shortened his argument by referring to Bottema; Woo managed his entire argument without any trigonometry. Their combined work is really efficient.

Janous proved an interesting generalization:

Consider a triangle ABC and three rectangles $BCDE$, $CAFG$, and $ABHI$ erected outside the triangle such that the side perpendicular to the corresponding side of the triangle is μ times as long. Let XYZ be the triangle enclosed by the lines EF, DI, and GH; then

$$[XYZ] \leq \left(\sqrt{3}\mu - 1\right)^2 [ABC].$$

Moreover, when $\mu \neq 1/\sqrt{3}$ equality holds if and only if $\triangle ABC$ is equilateral; when $\mu = 1/\sqrt{3}$ the three lines EF, DI, and GH are concurrent.

Note that $\mu = 1$ yields Seimiya's original inequality. Janous' proof relies on his computer algebra program.

2636. [2001 : 215] was withdrawn, since it is essentially same as 2609.

See the solution to 2609 above.

2637. [2001 : 267] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is an isosceles triangle with $AB = AC$. Let D be a point on side AB, and let E be a point on AC produced beyond C. The line DE meets BC at P. The incircle of $\triangle ADE$ touches DE at Q.

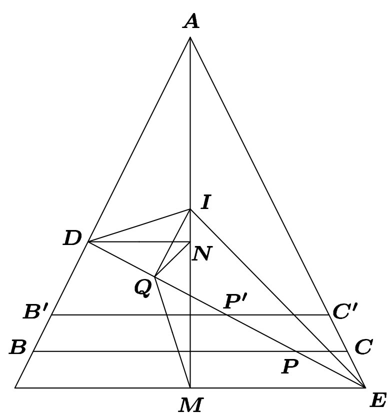
Prove that $BP \cdot PC \le DQ \cdot QE$, and that equality holds if and only if $BD = CE$.

I. *Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let $x = DP$. Then BP and PC are linear functions of x. Hence

$$BP \cdot PC \;=\; f(x) \;=\; ax^2 + bx + c$$

for some real numbers a, b, c. But $f(x) = 0$ when $x = 0$ and when $x = DE$. Hence f is maximum when $x = \frac{DE}{2}$; that is, when P is at the mid-point P' of DE. We shall name BC as $B'C'$ when it passes through the point P'.



Next we shall prove that $B'P' \cdot P'C' = DQ \cdot QE$. Let DN, EM be perpendiculars from D, E, respectively, to the angle bisector AI of $\angle DAE$ where I is the incentre of $\triangle ADE$. Since P' is the mid-point of DE, one can easily see that $DN = P'C'$ and $EM = B'P'$ [Ed. since DN, $B'C'$, EM are all perpendicular to the line AI]. Thus, $B'P' \cdot P'C' = DN \cdot EM$.

Clearly $DINQ$ and $EIQM$ are both cyclic. Therefore,

$$\begin{aligned}\angle DNQ \;=\; \angle DIQ \;&=\; \frac{\pi}{2} - \left(\frac{1}{2}\right)\angle ADE \;=\; \left(\frac{1}{2}\right)\angle DAE + \left(\frac{1}{2}\right)\angle AED \\ &=\; \angle IAE + \angle IEA \;=\; \angle EIM \;=\; \angle EQM\,.\end{aligned}$$

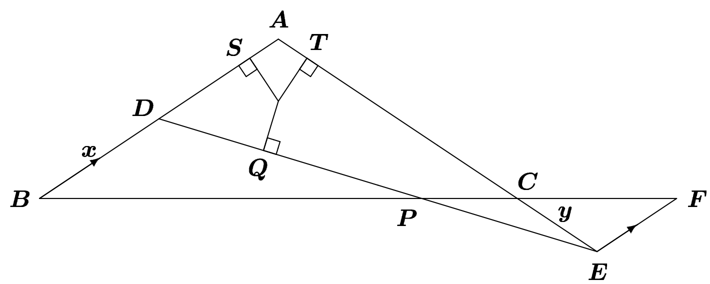
Thus, $\triangle DNQ$ is similar to $\triangle EQM$. Hence, $DN : DQ = EQ : EM$. It then follows that $DN \cdot EM = DQ \cdot QE$, and

$$B'P' \cdot P'C' \;=\; DN \cdot EM \;=\; DQ \cdot EQ\,.$$

[Ed. It follows immediately that in this case we also have $B'D = C'E$.]

II. *Solution by Henry Liu, student, University of Memphis, Memphis, TN, USA.*

Let F be on the extension of BC such that $FE \parallel AB$. Let the incircle of $\triangle ADE$ touch AD and AE at S and T, respectively. Let $a = BC$, $b = AC = AB$, $x = BD$, and $y = CE$ (see figure below).



We have $\angle CEF = \angle A$, $\angle FCE = \angle CFE = \angle B$, and $CE = FE = y$. Thus, $\triangle ABC$ and $\triangle EFC$ are similar, which means that $CF = \frac{ay}{b}$. Also, since $\triangle BDP$ and $\triangle FEP$ are similar, we have

$$\frac{BP}{x} \;=\; \frac{PC+CF}{y} \;=\; \frac{a - BP + \frac{ay}{b}}{y}\,,$$

which implies that

$$BP \;=\; \frac{ax(b+y)}{b(x+y)} \qquad \text{and} \qquad PC \;=\; a - BP \;=\; \frac{ay(b-x)}{b(x+y)}\,.$$

Now, we have that $AS = AT$, $DQ = DS$, and $QE = TE$. Since

$$DS + AS + AT + TE \;=\; 2b - x + y\,,$$

we see that

$$AS \;=\; AT \;=\; b + \frac{1}{2}(y-x) - \frac{1}{2}(DQ+QE)\,.$$

Therefore,

$$\begin{aligned}
DQ \cdot QE &= DS \cdot TE \;=\; (b - x - AS)(b + y - AT) \\
&= \left(b - x - \left(b + \frac{1}{2}(y-x) - \frac{1}{2}(DQ+QE)\right)\right) \\
&\qquad \times \left(b + y - \left(b + \frac{1}{2}(y-x) - \frac{1}{2}(DQ+QE)\right)\right) \\
&= \frac{1}{4}\left((DQ+QE) - (x+y)\right)\left((DQ+QE) + (x+y)\right) \\
&= \frac{1}{4}\left((DQ+QE)^2 - (x+y)^2\right)\,.
\end{aligned}$$

Applying the Law of Cosines to $\triangle ABC$, we have $\cos A = 1 - \frac{a^2}{2b^2}$. Applying the Law of Cosines to $\triangle ADE$ we have

$$\begin{aligned}(DQ+QE)^2 &= (b-x)^2+(b+y)^2-2(b-x)(b+y)\cos A \\ &= (b-x)^2+(b+y)^2-2(b-x)(b+y)+\frac{(b-x)(b+y)a^2}{b^2} \\ &= (x+y)^2+\frac{(b-x)(b+y)a^2}{b^2}\,.\end{aligned}$$

Then, we obtain

$$\begin{aligned}DQ\cdot QE &= \frac{1}{4}\left((x+y)^2+\frac{(b-x)(b+y)a^2}{b^2}-(x+y)^2\right) \\ &= \frac{1}{4}\cdot\frac{(x+y)^2}{xy}\cdot BP\cdot PC \;=\; \frac{1}{4}\left(\frac{x}{y}+\frac{y}{x}+2\right)BP\cdot PC \\ &\geq \frac{1}{4}(2+2)BP\cdot PC \;=\; BP\cdot PC\,.\end{aligned}$$

We applied the AM-GM Inequality to get $\frac{x}{y}+\frac{y}{x}\geq 2$. Equality holds if and only if $\frac{x}{y}=\frac{y}{x}$, or equivalently, $BD=CE$.

III. *Solution by the proposer*.

We will use the same diagram as for Solution II above. Let $DE=a$, $AE=b$, $AD=c$, $BD=x$, $CE=y$, and $a+b+c=2s$. Since $AB=AC$, we have $c+x=b-y$, or

$$x+y \;=\; b-c\,. \tag{1}$$

Let F be the point on PC produced beyond C such that $\angle EFC=\angle EPC$. Then, $PE=EF$. Since $AB=AC$, we get $\angle DBP=\angle ABC=\angle ACB=\angle ECF$, and furthermore, $\angle BPD=\angle EPC=\angle EFC$, we have $\triangle BPD$ similar to $\triangle CFE$. Therefore,

$$\frac{BP}{CF} \;=\; \frac{DP}{EF} \;=\; \frac{BD}{CE} \;=\; \frac{x}{y}\,. \tag{2}$$

Thus, $BP=\frac{x}{y}\cdot CF$, whence $BP\cdot PC=\frac{x}{y}\cdot PC\cdot CF$. Since $EP=EF$ we get

$$PC\cdot CF \;=\; PE^2-CE^2 \;=\; PE^2-y^2\,.$$

Therefore,

$$BP\cdot PC \;=\; \frac{x}{y}\left(PE^2-y^2\right)\,.$$

From (2), we have $\frac{DP}{PE}=\frac{DP}{EF}=\frac{x}{y}$, which means

$$\frac{DP}{x} \;=\; \frac{PE}{y} \;=\; \frac{DP+PE}{x+y} \;=\; \frac{a}{b-c}\,.$$

(The last equality also uses (1)). Thus, we have $PE = \frac{ay}{b-c}$. This implies that

$$\begin{aligned} BP \cdot PC &= \frac{x}{y}\left(\frac{a^2y^2}{(b-c)^2} - y^2\right) = \frac{xy}{(b-c)^2}(a^2 - (b-c)^2) \\ &= \frac{xy}{(b-c)^2}(a-b+c)(a+b-c) = \frac{4xy(s-b)(s-c)}{(b-c)^2}. \end{aligned}$$

Since $DQ = s - b$, $QE = s - c$, and $b - c = x + y$, it follows that

$$BP \cdot PC = \frac{4xy}{(x+y)^2} DQ \cdot QE .$$

Since $(x+y)^2 \geq 4xy$, and equality holds if and only if $x = y$, we have $BP \cdot PC \leq DQ \cdot QE$, and equality holds if and only if $BD = CE$.

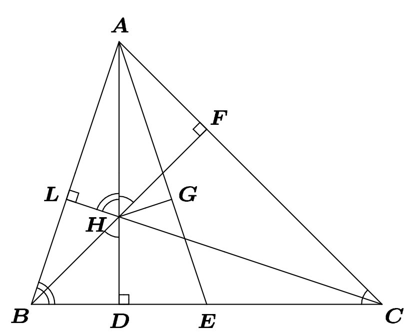
Also solved by BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and G. TSINTSIFAS, Thessaloniki, Greece.

2638. [2001 : 268] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is an acute angled triangle with $AB \neq AC$, and that H and G are the orthocentre and centroid of $\triangle ABC$ respectively. Suppose further that $\frac{1}{[HAB]} + \frac{1}{[HAC]} = \frac{2}{[HBC]}$, where $[PQR]$ denotes the area of $\triangle PQR$.

Prove that $\angle AGH = 90^\circ$.

Solution by Kee-Wai Lau, Hong Kong, China.



Let $a = BC$, $b = CA$, $c = AB$ and let R be the circumradius of $\triangle ABC$. Suppose the altitude AH meets BC at D, BH meets AC at F, CH meets AB in L and the median AG meets BC at E. Since $\angle BHD =$

$\angle AHF = 90^\circ - \angle HAF = 90^\circ - \angle DAC = \angle BCA$ and $BD = c\cos B$, we have

$$\begin{aligned}[HBC] &= \frac{BC \cdot HD}{2} = \frac{BC \cdot BD \cot C}{2} = \frac{ac \cos B \cot C}{2} \\ &= aR \cos B \cos C\,,\end{aligned}$$

where the last equality follows from the Sine Law. Similarly,

$$[HAB] \;=\; cR \cos A \cos B \;\;\text{and}\;\; [HAC] \;=\; bR \cos A \cos C\,.$$

Thus, the given condition on areas is equivalent to the condition $ab\cos C + ac\cos B = 2bc\cos A$, or using the Cosine Law,

$$b^2 + c^2 \;=\; 2a^2\,. \tag{1}$$

Now, $\angle LHA = \angle LBD$, so that $AH = \dfrac{AL}{\sin B} = \dfrac{b\cos A}{\sin B}$. Since $AD = c\sin B$, we obtain

$$AD \cdot AH \;=\; bc\cos A \;=\; \frac{b^2 + c^2 - a^2}{2} \;=\; \frac{a^2}{2}\,,$$

by the Cosine Law and equation (1).

On the other hand, $AE^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$, so that

$$AG \cdot AE \;=\; \frac{2}{3}AE^2 = \frac{1}{6}(2b^2 + 2c^2 - a^2) \;=\; \frac{a^2}{2}\,,$$

by equation (1). Thus, $AD \cdot AH = AG \cdot AE$, and therefore, $\triangle AHG$ is similar to $\triangle AED$. It follows that $\angle AGH = \angle ADE = 90^\circ$, provided that $G \neq H$. By the Cosine Law,

$$\cos AEB \;=\; \frac{AE^2 + BE^2 - AB^2}{2AE \cdot BE} \;=\; \frac{b^2 - c^2}{4AE \cdot BE} \;\neq\; 0\,,$$

so that $\angle AEB \neq 90^\circ$, and therefore, $E \neq D$ and $G \neq H$. This completes the solution.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; JOEL SCHLOSBERG, student, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

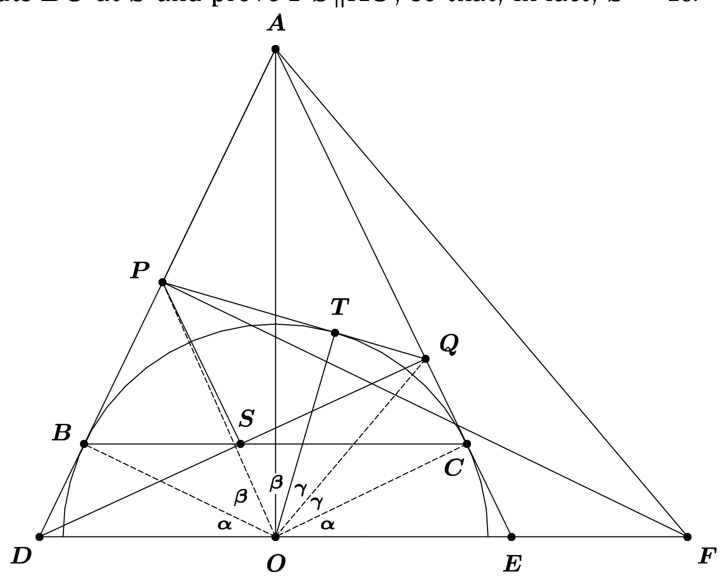
2639. [2001 : 268] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that A is a point outside a circle Γ. The two tangents through A to Γ touch Γ at B and C. A variable tangent to Γ meets AB and AC at P and Q respectively. The line through P parallel to AC meets BC at R.

Prove that the line QR passes through a fixed point.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let O be the centre of the given circle, and let the extension of the diameter that is perpendicular to AO cut AB at D and AC at E. We assert that the variable lines QR always go through D. To prove this, we assume that DQ cuts BC at S and prove $PS \| AC$, so that, in fact, $S = R$.



Let T be the point where the tangent PQ touches the circle, and define the directed angles

$$\alpha = \angle BOD = \angle EOC\,, \quad \beta = \angle POB = \angle TOP\,, \quad \gamma = \angle COQ = \angle QOT\,.$$

Then $2(\alpha+\beta+\gamma) = \angle EOD = \pi$; hence, $\angle AOP = \frac{\pi}{2} - \alpha - \beta = \gamma$. Let F be the point where the perpendicular to AB at P meets the extended diameter DE. Then $APOF$ is cyclic [on diameter AF], which implies $\angle AFP = \angle AOP = \gamma$. Also, $\angle PFD = \alpha$ (because $PF \| BO$). Because corresponding angles are equal, $\triangle AFD \sim \triangle QOE$ while $\triangle PFD \sim \triangle COE$. Therefore $AP : PD = QC : CE$. Since (using $CS \| DE$) $QC : CE = QS : SD$, we therefore get $AP : PD = QS : SD$, which implies that $PS \| AQ$. But AQ is the same line as AC, so that we conclude that $PS \| AC$, as desired.

II. *Solution by Michel Bataille, Rouen, France.*

For every point $Q \neq C$ on AC let t_Q be the second tangent to $\Gamma (\neq AC)$ through Q, and define $t_C = AC$. We denote by π_1, the mapping from AC to AB defined by $\pi_1(Q) = P$, the point where t_Q meets AB. (As usual, we define $\pi_1(A) = B$.) The cross ratio of four distinct points Q_1, Q_2, Q_3, Q_4 of AC and the cross-ratio of their images P_1, P_2, P_3, P_4 on AB are both equal to the cross-ratio of the tangents t_{Q_1}, t_{Q_2}, t_{Q_3}, t_{Q_4} (which is the cross-ratio of their points of contact on Γ). Thus, π_1 preserves the cross-ratio and is therefore a projectivity. By definition, the projection π_2 from AB onto BC by means of lines parallel to AC is also a projectivity. It follows that $\pi = \pi_2 \circ \pi_1$ is a projectivity from AC to BC; moreover, π is a perspectivity since $\pi(C) = C$. As a result, the lines connecting Q and $R = \pi(Q)$ all pass through a fixed point, the centre of the perspectivity π.

Remark. This fixed point is on AB since $B = \pi(A)$. [In his second proof, which exploits the polarity defined by Γ, Bataille shows that the fixed point is the polar of the line joining B to the opposite end of the diameter through C.]

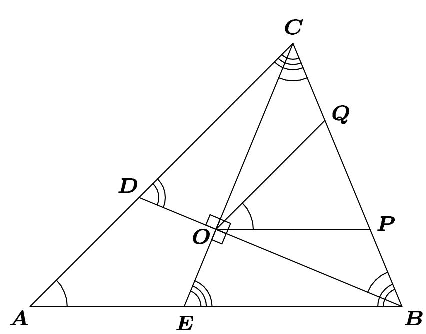
Also solved by MICHEL BATAILLE, Rouen, France (a second solution); BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2640. [2001 : 268] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is an acute angled triangle with $\angle BAC = 45^\circ$. Let O be the circumcentre of $\triangle ABC$, and let D and E be the intersections of BO and CO with AC and AB, respectively. Suppose that P and Q are points on BC such that $OP \| AB$ and $OQ \| AC$.

Prove that $\quad OD + OE = \sqrt{2}PQ$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.



Let R be the circumradius of $\triangle ABC$. Since O is the circumcentre of $\triangle ABC$, $\angle BOC = 2A = 90^\circ$ and $\angle OBC = \angle OCB = 45^\circ = \angle POQ$. From triangles BDC and BEC, we have $\angle BDC = 135^\circ - C = B$ and $\angle BEC = 135^\circ - B = C$, correspondingly. From triangles OPQ and OQB,

we obtain
$$\frac{PQ}{\sin 45^\circ} = \frac{OQ}{\sin B} \quad\text{and}\quad \frac{OQ}{\sin 45^\circ} = \frac{R}{\sin C},$$
by the Sine Law. Therefore,
$$\begin{aligned}\sqrt{2}PQ &= \frac{PQ}{\sin 45^\circ} = \frac{OQ}{\sin B} = \frac{OQ}{\sin 45^\circ}\frac{\sin A}{\sin B} = \frac{R}{\sin C}\frac{\sin A}{\sin B} \\ &= \frac{R\sin(B+C)}{\sin C \sin B} = R(\cot B + \cot C) = OD + OE,\end{aligned}$$
as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; DAN POTTER, student, University of Manitoba, Winnipeg, Manitoba; JOEL SCHLOSBERG, student, New York University, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One solver misinterpreted the condition and solved a different problem.

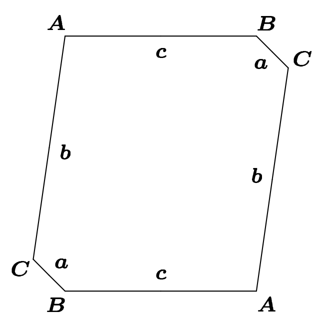
2641. [2001 : 268] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let H be a centrosymmetric convex hexagon, with area h, and let P be its minimal circumscribed parallelogram, with area p.

Prove that $3p \leq 4h$.

Solution by Joel Schlosberg, student, New York University, NY, USA.

Since H is centrosymmetric, the edges and angles can be labelled as follows:



Three different parallelograms, P_a, P_b, P_c, that circumscribe H can be formed by adding a pair of triangles T_a, T_b, T_c, to the sides a, b, c, respectively (form T_a by extending sides b and c; form the other two pairs of triangles similarly).

By the angle-side-angle formula for the area of a triangle, we can see that the pair of triangles T_a has area

$t_a \;=\; a^2\dfrac{\sin B\sin C}{\sin A}$, and similarly $t_b \;=\; b^2\dfrac{\sin C\sin A}{\sin B}$, $t_c \;=\; c^2\dfrac{\sin A\sin B}{\sin C}$, are the areas of T_b, T_c, respectively.

Using the formula for the area of a parallelogram, we obtain

$$\begin{aligned} h &= P_a - t_a \;=\; \sin A\left(b+a\frac{\sin B}{\sin A}\right)\left(c+a\frac{\sin C}{\sin A}\right) - a^2\frac{\sin B\sin C}{\sin A} \\ &= ab\sin C + ca\sin B + bc\sin A\,. \end{aligned}$$

By the Arithmetic-Geometric Mean Inequality,

$$\begin{aligned} \frac{h}{3} &\geq \sqrt[3]{ab\sin C\cdot ac\sin B\cdot bc\sin A} \\ &= \sqrt[3]{a^2\frac{\sin B\sin C}{\sin A}\cdot b^2\frac{\sin C\sin A}{\sin B}\cdot c^2\frac{\sin A\sin B}{\sin C}} \;=\; \sqrt[3]{t_at_bt_c}\,. \end{aligned}$$

Therefore, for some t_k, we have $t_k \leq \frac{h}{3}$. By definition, the minimal circumscribed parallelogram has an area at most equal to any particular circumscribed parallelogram, so that

$$3p \;\leq\; 3\,(t_k+h) \;\leq\; h+3h \;=\; 4h\,.$$

Also solved by BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2642. [2001 : 268] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that k is a positive integer. Prove that $\displaystyle\sum_{n=0}^{\infty}\frac{n+2k}{2^{n+1}\binom{n+k+1}{k}} \;=\; 1\,.$

Essentially the same solution by Joe Howard, Portales, NM, USA; Kee-Wai Lau, Hong Kong, China; and Chris Wildhagen, Rotterdam, the Netherlands (using Wildhagen's notations).

Let $v_n = \dfrac{n+2k}{2^{n+1}\binom{n+k+1}{k}}$. Then

$$\begin{aligned} v_n &= \frac{(n+2k)k!(n+1)!}{2^{n+1}(n+k+1)!} \\ &= \big(2(n+k+1)-(n+2)\big)\frac{k!(n+1)!}{2^{n+1}(n+k+1)!} \\ &= \frac{k!(n+1)!}{2^n(n+k)!} - \frac{k!(n+2)!}{2^{n+1}(n+k+1)!} \\ &= w_n - w_{n+1} \quad \text{where} \quad w_n = \frac{k!(n+1)!}{2^n(n+k)!} \end{aligned}$$

Thus, $\displaystyle\sum_{n=0}^{\infty}(w_n - w_{n+1}) \;=\; w_0 \;=\; 1\,.$

Also solved by MICHEL BATAILLE, Rouen, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

The methods used in various solutions included ordinary integration, the Beta function, Stirling's formula, k–fold integral, and the Laplace transform!

Both Loeffer and the proposer employed induction to show that, for all non-negative integers N, we have $\displaystyle\sum_{n=0}^{N}\frac{n+2k}{2^{n+1}\binom{n+k+1}{k}} = 1 - \frac{N+2}{2^{N+2}\binom{N+k+1}{k}}$, from which the result clearly follows. Seiffert remarked that the following interesting identity is a by-product of his proof:

$$\sum_{n=0}^{\infty}\frac{1}{2^{n+1}\binom{n+k+1}{k}} = 1 + (-1)^k(2k)\sum_{j=1}^{k}\left(\frac{(-1)^{j-1}}{j} - \ln 2\right).$$

2643★. [2001 : 268] *Proposed by Mª Jesús Villar Rubio, Instituto Torres Quevedo Santander, Spain.*

It is known that if a quadrilateral has sides a, b, c and d, then its area is less than or equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semi-perimeter ($2s = a+b+c+d$). What happens for polygons with more than four sides?

NOTE: The editor failed to star the original, due to his lack of good understanding of the original, which was submitted in Spanish.

Comment by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

This is indeed a very delicate question (at least in the form as stated in this problem). Of course, there cannot be estimations of type

$$\sqrt{(s-a_1)(s-a_2)\cdots(s-a_n)}$$

for values of $n \neq 4$. For then we have "dimension-troubles". (Just think of triangles and Hero's area formula!)

Thus, it makes sense to check the literature for inequalities estimating the area $F = F_n$ of a n–gon G_n by a function of its side lengths. (We refer to Chapter 1 of Tóth's classic book [1].)

(i) Suppose that G_n is convex. Let L denote the perimeter of G_n, that r is the inradius of G_n (that is, the radius of the maximal circle contained in G_n, and that f is the area of the n–gon circumscribed about the unit circle whose outer normal directions coincide with the respective directions of the n–gon G_n.

Then, the following (strange) inequality holds:

$$F \leq Lr - fr^2 .$$

This is also

$$L^2 - 4fF \geq (L - 2fr)^2 .$$

A consequence of this (these) inequality(ies) is the inequality of Lhuilier, which states

$$L^2 - 4fF \geq 0 .$$

This in turn yields another inequality stated as item 19 on page 419 of [2]. (As a matter of fact, these inequalities should also be viewed in the context of the powerful Brunn-Minkowski inequality.)

Remark. Item 20 on page 420 of [2] states: If F, L, r and R are the area, perimeter and radii of the incircle and the circumcircle, respectively, of a convex n–gon, then

$$2nr\tan\left(\frac{\pi}{n}\right) \leq 2\sqrt{nF\tan\left(\frac{\pi}{n}\right)} \leq L \leq 2nR\sin\left(\frac{\pi}{n}\right) .$$

As a consequence, we obtain

$$R \geq \frac{r}{\cos\left(\frac{\pi}{n}\right)} .$$

Note that the special case $n = 3$ yields the "good old" Euler inequality $R \geq 2r$.

(ii) It has also been shown (based on results by the recently deceased Professor Santaló) that for non-convex n–gons G_n, the following is always valid:

$$L^2 - 4\pi F \geq (L - 2\pi\rho)^2 ,$$

where ρ is an arbitrary quantity between the inradius and the circumradius of G_n; that is, $r \leq \rho \leq R$. (Here the circumcircle is the smallest circle containing G_n.)

References.

[1] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*. Springer, Berlin 1953.

[2] D.S. Mitrinovic et. al., *Recent Advances in Geometric Inequalities*. Kluwer, Dordrecht 1989.

2644. [2001 : 268] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Find a closed form expression for the sum of the first n terms of the series

$$1+2+4+4+8+8+8+8+16+16+16+16+16+16+16+16+\cdots,$$

where 1 occurs once and, for $k \geq 1$, 2^k occurs 2^{k-1} times.

Solution by Natalio H. Guersenzvaig, Universidad CAECE, Buenos Aires, Argentina..

Let us denote the sum by S_n. It follows by induction that

$$S_{2^m} \;=\; 1+\sum_{k=1}^{m} 2^{k-1}2^k \;=\; 1+2\frac{4^m-1}{3}\,, \qquad m=0,\,1,\,2,\,\ldots.$$

Let $m=\max\{k \in \mathbb{Z} \,:\, 2^k \leq n\}$. Thus, $2^m \leq n < 2^{m+1}$. Hence,

$$S_n \;=\; S_{2^m}+(n-2^m)2^{m+1} \;=\; n2^{m+1}-\frac{4^{m+1}-1}{3}\,.$$

More precisely, let us denote by $d(n)$ the number of digits in the expansion of n in base 2. Then we have the formula:

$$S_n \;=\; n2^{d(n)}-\frac{4^{d(n)}-1}{3}\,,$$

where $d(n)=\lfloor \log_2 n \rfloor + 1$.

Also solved by MICHEL BATAILLE, Rouen, France; BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DUANE BROLINE, Eastern Illinois University, Charleston, IL, USA; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOSHUA GREEN, University of Arizona, Tucson, AZ, USA; KARL HAVLAK, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ATHANASIOS KALAKOS, Athens, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. The proposer's name is not among the list of solvers since the editors changed his original submission, and his accompanying solution was no longer appropriate. However, his solution to his original problem was correct!

2645. [2001 : 269] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{2\left(a^3+b^3+c^3\right)}{abc}+\frac{9(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 33\,.$$

I. *Solution (independently) by Kee-Wai Lau, Hong Kong, China and the Southwest Missouri University Problem Solving Group.*

On multiplying by the common denominator, and collecting on the left, we have to prove that

$$2\left(a^3+b^3+c^3\right)\left(a^2+b^2+c^2\right)+9abc(a+b+c)^2-33abc\left(a^2+b^2+c^2\right) \;\geq\; 0\,.$$

The left side of this is the product of

$$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$$

and

$$(a-b)^2(a+b+3c)+(b-c)^2(b+c+3a)+(c-a)^2(c+a+3b)\,.$$

This product is clearly non-negative.

II. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.*

Using the AM–GM Inequality, we have $a^3+b^3+c^3 \geq 3abc$ and $\frac{1}{3}(a+b+c)^2 \geq a^2+b^2+c^2$ (since AM $\leq$ RMS, where RMS stands for Root-Mean-Square). Hence

$$\frac{2\left(a^3+b^3+c^3\right)}{abc}+\frac{9(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 6\,\frac{abc}{abc}+\frac{9(a+b+c)^2}{\frac{1}{3}(a+b+c)^2} \;=\; 33\,.$$

III *Generalisation by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For a, b, c positive reals, we show the more general inequality:

$$\lambda\frac{a^3+b^3+c^3}{abc}+\frac{(a+b+c)^2}{a^2+b^2+c^2} \;\geq\; 3\lambda+3$$

whenever $\lambda \geq \frac{2}{9}$.

On multiplying by the common denominator, and collecting on the left, we have to prove that the product of

$$\left(a^2+b^2+c^2-ab-bc-ca\right)$$

and

$$\left(\lambda\left(a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)\right)-2abc\right)$$

is non-negative.

The first factor is $\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$, which is non-negative.

For the second, we apply the AM–GM Inequality to get

$$\begin{aligned}
&\lambda\left(a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)\right)\\
&\quad= \lambda\left(a^3+b^3+c^3+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b\right)\\
&\quad\geq 9\lambda\left(a^3\cdot b^3\cdot c^3\cdot a^2b\cdot a^2c\cdot b^2c\cdot b^2a\cdot c^2a\cdot c^2b\right)^{\frac{1}{9}} \;=\; 9\lambda abc\,.
\end{aligned}$$

Since $\lambda \geq \frac{2}{9}$, the proof is complete. The case $\lambda = \frac{2}{9}$ gives the original inequality.

IV. *Generalisation by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

We show that

$$\frac{2S_3}{T_3}+9\frac{S_1^2T_3^n}{S_{2+3n}} \;\geq\; 33\frac{S_2T_3^n}{S_{2+3n}}\,,$$

where $S_n = a^n+b^n+c^n$ and $T_3 = abc$. The given inequality corresponds to the special case $n=0$.

Clearing fractions, the inequality becomes

$$2S_{2+3n}(S_3-3T_s)+6T_3\left(S_{2+3n}-T_3^nS_2\right) \;\geq\; 9T_3^{n+1}\left(3S_2-S_1^2\right)\,.$$

First, $\left(S_{2+3n}-T_3^nS_3\right) \geq 0$, since, by the Power Mean Inequality, we have $\frac{S_{2+3n}}{3} \geq \left(\frac{S_2}{3}\right)^{1+\frac{3n}{2}}$, and then, $\left(\frac{S_2}{3}\right)^{\frac{3n}{2}} \geq \left(\frac{S_1}{3}\right)^{3n} \geq T_3^n$.

Now, since $(S_3-3T_3) = S_1\left(S_2-T_2\right)$ and $\left(3S_2-S_1^2\right) = 2\left(S_2-T_2\right)$, where $T_2 = ab+bc+ca$, and $2\left(S_2-T_2\right) = (a-b)^2+(b-c)^2+(c-a)^2$, we need only show that

$$S_1S_{2+3n} \;\geq\; 9T_3^{n+1}\,.$$

This follows by applying the AM–GM Inequality to both S_1 and S_{2+3n}.

There is equality if and only if $a=b=c$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MARCUS BARNES, student, York University, Toronto, Ontario; MICHEL BATAILLE, Rouen, France; BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; BRIAN BEASLEY, Presbyterian College, Clinton, SC, USA; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PAUL BRACKEN, CRM, Université de Montréal, Montréal, Québec; DUANE M. BROLINE, Eastern Illinois University, Charleston, IL, USA; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CHARLES R. DIMINNIE and DIONNE T. BAILEY; Angelo State University, San Angelo, TX, USA; ZELJKO HANJS, University of Zagreb, Zagreb, Croatia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA (two solutions); VEDULA N. MURTY, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PAULO R. de SOUZA, Rio de Janeiro, Brazil; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece;

G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2646. [2001 : 269] *Proposed by Hojoo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that $p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Show that there exists a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that, for all $n \in \mathbb{N}_0$, we have $f(f(n)) + f(n) = 2n + p$ if and only if $3|p$.

Solution by Chris Wildhagen, Rotterdam, the Netherlands.

First suppose that p is divisible by 3, say $p = 3m$. The function $f : \mathbb{N}_0 \to \mathbb{N}_0$ given by $f(n) = n + m$ clearly satisfies

$$f(f(n)) + f(n) \;=\; 2n + p\,. \tag{1}$$

Note that f is just a translation.

Conversely, let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a function satisfying (1).

Let the functions $f^{(k)} : \mathbb{N}_0 \to \mathbb{N}_0 (k \in \mathbb{N}_0)$ be defined by:

$$f^{(0)}(n) \;=\; n\,, \quad f^{(1)}(n) \;=\; f(n)\,, \quad \text{and} \quad f^{(k+1)}(n) \;=\; f(f^{(k)}(n))\,,$$

for all $n \in \mathbb{N}_0$.

By replacing n by $f^{(k)}(n)$ in (1) we find:

$$f^{(k+2)}(n) + f^{(k+1)}(n) \;=\; 2f^{(k)}(n) + p\,. \tag{2}$$

If we fix n, then (2) can be interpreted as a second-order difference equation in the variable k. Using the standard methods to solve (2), we arrive at the general solution:

$$f^{(k)}(n) \;=\; A(n) + B(n)\cdot(-2)^k + \tfrac{1}{3}kp\,. \tag{3}$$

If $B(n) \neq 0$, then we see from (3) that $f^{(k)}(n) < 0$ for infinitely many values of k, which clearly is impossible since $f^{(k)}(n) \in \mathbb{N}_0$ for all $k \in \mathbb{N}_0$. Therefore, $B(n) = 0$, and thus

$$f^{(k)}(n) \;=\; A(n) + \tfrac{1}{3}kp\,. \tag{4}$$

Taking $k = 0$ in (4) gives $A(n) = n$ and then taking $k = 1$ gives

$$f(n) \;=\; n + \tfrac{1}{3}p\,.$$

This implies (since $f(n) \in \mathbb{N}_0$ for all $n \in \mathbb{N}_0$) that p is divisible by 3.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DUANE M. BROLINE, Eastern Illinois University, Charleston, IL, USA; JOHN

FREMLIN, Erlangen, Germany; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany;LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incomplete solution.

The proposer remarked that the proposal was motivated by problem # 6 of the 20th Austrian Mathematical Olympiad, advanced level, and the published solution given by Seung-Jin Bang (Crux, 1992, Vol. 18, No. 3; pp. 70–71). The statement of that problem was: "Determine all functions $f : N_0 \to N_0$ such that $f(f(n)) + f(n) = 2n + 6$ for all $n \in N_0$".

2648. [2001 : 269] *Proposed by Michel Bataille, Rouen, France.*

Find the smallest positive integer n such that the rightmost digits of 5^{2001+n} reproduce the digits of 5^{2001}. (Here "digit" means "decimal digit", and the order of the digits in 5^{2001+n} and 5^{2001} must be the same.)

Solution by David Loeffler, student, Cotham School, Bristol, UK.

We note that

$$\log_{10}(5^{2001}) \;=\; 2001\frac{\ln 5}{\ln 10} \;=\; 1398.64\ldots\,,$$

which implies that 5^{2001} has 1399 digits. Hence, we require that

$$\begin{aligned} 5^{2001+n} &\equiv 5^{2001} \pmod{10^{1399}} \\ \text{or} \qquad 5^{602+n} &\equiv 5^{602} \pmod{2^{1399}}\,. \end{aligned}$$

We shall prove, by induction, that for $k \geq 3$ the order of $5 \pmod{2^k}$ is 2^{k-2}.

Lemma: $5^{2^{k-3}} \equiv 2^{k-1} + 1 \pmod{2^k}$.

Proof: The lemma is clearly true when $k = 3$. Let us assume it is true for some $k \geq 3$. Set $a = 5^{2^{k-3}}$ and $b = 2^{k-1} + 1$, and note that 2^k divides $a - b$ (by our assumption). Then, in $a^2 - b^2 = (a-b)(a+b)$, we see that 2^k divides the first factor. However, the second factor is also clearly even, so the product is divisible by 2^{k+1}. Hence,

$$5^{2^{k-2}} \;\equiv\; (2^{k-1}+1)^2 \pmod{2^{k+1}}\,.$$

But

$$(2^{k-1}+1)^2 \;=\; 2^{2k-2} + 2^k + 1 \equiv 2^k + 1 \pmod{2^{k+1}}\,.$$

Therefore,

$$5^{2^{k-2}} \;\equiv\; 2^k + 1 \pmod{2^{k+1}}\,,$$

which is the result for $k+1$.

It is clear that the order of $5 \pmod{2^k}$ must divide $\phi(2^k) = 2^{k-1}$, whence it is a power of 2. The lemma tells us that $5^{2^{k-3}} \not\equiv 1 \pmod{2^{k+1}}$. Thus the order of 5 cannot be 2^{k-3}, and since $5^a \equiv 1$ implies that $5^{na} \equiv 1$, we conclude that the order cannot be any smaller power of 2. However, the lemma also shows that

$$5^{2^{k-2}} \equiv 2^k + 1 \equiv 1 \pmod{2^k},$$

implying that the order must be 2^{k-2}.

This immediately implies that the least n satisfying the given equation (that is, the order of $5 \pmod{2^{1399}}$) is 2^{1397}.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; KENNETH M. WILKE, Topeka, KS, USA; and the proposer. There was one incorrect solution.

2649. [2001 : 269] *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

Solve the equations:

$$\sin(2x)\sin(4x)\sin(8x) = \frac{\sqrt{3}}{8}, \tag{1}$$
$$\cos(2x)\cos(4x)\cos(8x) = \frac{1}{8}. \tag{2}$$

See problem 2486 [1999 : 431, 2000 : 510].

Editor's Comments.

Though the original intention of the proposer was to solve equations (1) and (2) independently, almost all the submitted solutions solved the two equations simultaneously. Also, as it turns out, essentially the same question as (2) has actually appeared as problem 2249 [1997 : 245; 1998 : 371], and the solution published was by one of our editors! However, only Howard pointed this out, even though four of the current solvers, including the proposer of the current problem, were also solvers of problem 2249. Specifically, problem 2249 asked for solutions to $\cos(\alpha)\cos(2\alpha)\cos(4\alpha) = \frac{1}{8}$ for $0 \le \alpha < \frac{\pi}{2}$. The answer given in the published solution (for the general problem without the condition that $\alpha \in [0, \frac{\pi}{2}]$) was $\alpha = \frac{2k\pi}{7}$ or $\alpha = \frac{(2k+1)\pi}{7}$, where k denotes any integer such that $\sin\alpha \ne 0$. Replacing α by $2x$, we then see easily that the solutions to (2) are given by $x = \frac{k\pi}{7}$, where $(k,7) = 1$, or $x = \frac{(2k+1)\pi}{18}$, where $k \not\equiv 4 \pmod 9$. This was proved independently and submitted by Wilke. Hence, in the following, we will present two solutions, one for equation (1) and the other, for the simultaneous equations (1) and (2).

I. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK (modified slightly by the editor).*

Put $y = 2x$ and $s = \sin y$.

Then the equation becomes $8s^3(1-s^2)(1-2s^2) = \frac{\sqrt{3}}{8}$. Expanding and rearranging, we have $128s^7 - 192s^5 + 64s^3 - \sqrt{3} = 0$, which factors into $\left(8s^3 - 6s + \sqrt{3}\right)\left(16s^4 - 12s^2 - 2\sqrt{3}s - 1\right) = 0$.

If $8s^3 - 6s + \sqrt{3} = 0$, then $2\left(3\sin y - 4\sin^3 y\right) = \sqrt{3}$, yielding $\sin(3y) = \frac{\sqrt{3}}{2}$, and thus, $\sin(6x) = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$. Hence, $6x = 2k\pi + \frac{\pi}{3}$ or $6x = (2k+1)\pi - \frac{\pi}{3}$; that is, $x = \frac{k\pi}{3} + \frac{\pi}{18}$ or $x = \frac{(2k+1)\pi}{6} - \frac{\pi}{18}$, where $k \in \mathbb{Z}$.

In particular, for $0^\circ \le x \le 360^\circ$, we obtain the following solutions:

$$x = 10^\circ,\ 20^\circ,\ 70^\circ,\ 80^\circ,\ 130^\circ,\ 140^\circ,\ 190^\circ,\ 200^\circ,\ 250^\circ,\ 260^\circ,\ 310^\circ,\ 320^\circ.$$

For $16s^4 - 12s^2 - 2\sqrt{3}s - 1 = 0$, the standard method for solving biquadratics did not provide anything recognizable. Using DERIVE®, we find that there are four solutions: two complex ones, one greater than 1, and one negative. The negative solution yields $\sin(2x) \approx -0.7575007\ldots$, which gives $x \approx (k + 0.636670\ldots)180^\circ$ or $x \approx (k - 0.136670\ldots)180^\circ$, where $k \in \mathbb{Z}$. In the range $0^\circ \le x \le 360^\circ$, we obtain $x \approx 114.6^\circ$, 155.4°, 294.6° or 335.4°, to four significant figures.

II. *Solution to the simultaneous equations* (1) *and* (2) *by Duane M. Broline, Eastern Illinois University, Charleston, IL, USA.*

Since both equations are periodic of period π, it suffices to find solutions in the range $0 \le x \le \pi$. From (2), we have

$$\begin{aligned}\sin(2x) &= 8\sin(2x)\cos(2x)\cos(4x)\cos(8x) \\ &= 4\sin(4x)\cos(4x)\cos(8x) = 2\sin(8x)\cos(8x) = \sin(16x)\,.\end{aligned}$$

Thus, $0 = \sin(16x) - \sin(2x) = 2\cos(9x)\sin(7x)$. Therefore, either $\cos(9x) = 0$ or $\sin(7x) = 0$.

In the second case, $x = \frac{k\pi}{7}$, $k = 0, 1, 2, \ldots, 7$. For each of these angles, by considering the signs of the factors, we find that

$$\sin(2x)\sin(4x)\sin(8x) \le 0\,.$$

Hence, $\cos(9x) = 0$. Since

$$\sin(10x) - \sin(8x) = 2\cos(9x)\sin(x) = 0\,,$$

we have

$$\sin(10x) = \sin(8x)\,. \tag{3}$$

Also,

$$\begin{aligned} 0 &= \cos(6x+3x) = \cos(6x)\cos(3x) - \sin(6x)\sin(3x) \\ &= \cos(6x)\cos(3x) - 2\sin^2(3x)\cos(3x) \\ &= \cos(6x)\cos(3x) - \big(1-\cos(6x)\big)\cos(3x)\,. \end{aligned}$$

Therefore,

$$\big(2\cos(6x)-1\big)\cos(3x) \;=\; 0\,. \tag{4}$$

From (1), we have

$$\begin{aligned} \frac{\sqrt{3}}{2} &= 4\sin(2x)\sin(4x)\sin(8x) \\ &= 2\,\big(-\cos(6x)+\cos(2x)\big)\sin(8x) \\ &= -2\cos(6x)\sin(8x) + \sin(10x) + \sin(6x)\,. \end{aligned}$$

Therefore, by (3), we have

$$\big(1-2\cos(6x)\big)\sin(8x) + \sin(6x) \;=\; \frac{\sqrt{3}}{2}\,. \tag{5}$$

Now, from (4), either $\cos(3x) = 0$ or $\cos(6x) = \frac{1}{2}$.

If $\cos(3x) = 0$, we have $\sin(6x) = 0$. Hence, $6x$ is a multiple of π. It follows that the reference angles of $2x$, $4x$ and $6x$ are either 0 or $\frac{\pi}{3}$. Thus, $\big|\sin(2x)\sin(4x)\sin(8x)\big| = 0$ or $\frac{3\sqrt{3}}{8}$, contrary to (1).

Hence, $\cos(6x) = \frac{1}{2}$. From (5), we get $\sin(6x) = \frac{\sqrt{3}}{2}$. It follows that the solutions set is

$$\left\{\frac{\pi}{18} + \frac{2k\pi}{3} \;:\; k \in \mathbb{Z}\right\}\,.$$

Also solved (both (1) and (2), either independently or simultaneously) by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; KENNETH M. WILKE, Topeka, KS, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

2650. [2001 : 269] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$, let a denote the side BC, and h_a, the corresponding altitude. Let r and R be the radii of the inscribed and circumscribed circles, respectively. Prove that $ra < h_a R$.

I. *Essentially the same solution from most of the solvers listed below.*
Since $ah_a = r(a+b+c) > 2ra$, we have

$$h_a R \;=\; \frac{ah_a}{2\sin A} \;>\; \frac{ra}{\sin A} \;\geq\; ra\,.$$

II. *Solution by Duane M. Broline, Eastern Illinois University, Charleston, IL, USA.*

The diameter of the incircle which passes through the point of tangency with BC is a line segment perpendicular to BC which is contained entirely within the triangle and its boundary. This line segment is clearly shorter than h_a, the longest line segment perpendicular to BC contained entirely within the triangle and its boundary. Therefore, $2r < h_a$.

The line segment BC is contained entirely within the circumcircle and its boundary and thus, the length of this line segment is less than the diameter of the circumcircle; that is, $a \leq 2R$.

Combining these two results, the desired conclusion follows immediately.

Editor's comment.

This latter solution is so nice, because it indicates that you do not have to know any fancy formulae or inequalities, since the result follows immediately from the definitions!

Solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; BAURJAN BEKTEMIROV, student, Aktobe, Kazakstan; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ATHENASIOS KALAKOS, Athens, Greece; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Cotham School, Bristol, UK; VEDULA N. MURTY, Dover, PA, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

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