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CHAÎNES DE FRACTIONS IRRÉDUCTIBLES

LÉO SAUVÉ, Collège Algonquin

L'étude qui va suivre ne contient rien de bien neuf ou de bien profond, mais elle nous conduira à un résultat intéressant et trop peu connu. Elle a été suggérée par le Problème 91, dont la solution apparaît à la page 44, et par un problème analogue dans Bréard [1]. Ayant donné deux fractions positives $\frac{a}{b}$ et $\frac{a'}{b'}$, nous verrons comment on peut, dans certains cas, intercaler en ordre entre $\frac{a}{b}$ et $\frac{a'}{b'}$ *toutes* les fractions irréductibles dont le dénominateur ne dépasse pas un certain nombre fixé d'avance.

Comme point de départ, nous supposons que a, a', b, b' sont quatre nombres naturels et que les fractions $\frac{a}{b}$ et $\frac{a'}{b'}$ vérifient l'égalité

$$ab' - ba' = -1. \tag{1}$$

Plusieurs conclusions découleront de cette hypothèse.

(i) On a toujours l'inégalité $\frac{a}{b} < \frac{a'}{b'}$.

Il suffit de diviser les deux membres de (1) par bb' pour obtenir

$$\frac{a}{b} - \frac{a'}{b'} = -\frac{1}{bb'}, \tag{2}$$

et l'inégalité désirée est une conséquence évidente.

(ii) Les fractions $\frac{a}{b}$ *et* $\frac{a'}{b'}$ *sont irréductibles.*

Les nombres a et b sont premiers entre eux puisque, d'après (1), tout diviseur commun de a et b doit diviser 1. Il en est de même pour a' et b'.

(iii) Si l'on pose $c = a + a'$ *et* $d = b + b'$*, la fraction* $\frac{c}{d}$ *est irréductible et comprise entre* $\frac{a}{b}$ *et* $\frac{a'}{b'}$.

L'égalité (1) pouvant s'écrire diversement

$$a(b+b') - b(a+a') = -1, \quad (a+a')b' - (b+b')a' = -1,$$

c'est-à-dire

$$ad - bc = -1, \quad cb' - da' = -1, \tag{3}$$

les inégalités $\frac{a}{b} < \frac{c}{d}$ et $\frac{c}{d} < \frac{a'}{b'}$ sont garanties par *(i)* et l'irréductibilité de $\frac{c}{d}$ découle de *(ii)*.

(iv) $\frac{c}{d}$ *est la seule fraction de dénominateur* d *située entre* $\frac{a}{b}$ *et* $\frac{a'}{b'}$.

S'il existe une fraction $\frac{\lambda}{d}$ telle que $\frac{a}{b} < \frac{\lambda}{d} < \frac{c}{d}$, alors $ad < b\lambda < bc$, d'où, en vertu de (3),

$$-1 = ad - bc < b(\lambda - c) < 0,$$

ce qui est impossible. De même, l'existence supposée d'une fraction $\frac{\mu}{d}$ telle que $\frac{c}{d} < \frac{\mu}{d} < \frac{a'}{b'}$ mènerait à l'impossibilité

$$-1 = cb' - da' < b'(c - \mu) < 0\,.$$

(v) Il n'existe entre $\frac{a}{b}$ *et* $\frac{a'}{b'}$ *aucune fraction de dénominateur inférieur à* d.

La démonstration de cette propriété est un peu plus délicate. Supposons qu'il existe une fraction $\frac{\gamma}{n}$, où $n < d = b + b'$, telle que

$$\frac{a}{b} < \frac{\gamma}{n} < \frac{a'}{b'};$$

nous avons alors

$$\frac{na}{b} < \gamma < \frac{na'}{b'}. \tag{4}$$

Si

$$na = bq + r\,, \quad 0 \le r \le b - 1\,, \tag{5}$$

$$na' = b'q' + r'\,, \quad 0 \le r' \le b' - 1\,, \tag{6}$$

alors (4) peut s'écrire

$$q + \frac{r}{b} < \gamma < q' + \frac{r'}{b'}, \tag{7}$$

et l'on a aussi

$$bq \leq na < b(q+1), \tag{8}$$
$$b'q' \leq na' < b'(q'+1) .$$

Donc

$$q' + 1 > \frac{na'}{b'}$$

et

$$-q \geq -\frac{na}{b},$$

de sorte que, en vertu de (2),

$$q' - q + 1 > n\left(\frac{a'}{b'} - \frac{a}{b}\right) = \frac{n}{bb'},$$

d'où

$$q' - q > -1 + \frac{n}{bb'} > -1 .$$

Donc $q' - q \geq 0$ et

$$q' \geq q . \tag{9}$$

Multiplions maintenant (5) par b' et (6) par b; on obtient

$$nab' = bb'q + rb',$$
$$nba' = bb'q' + br',$$

d'où

$$n(ab' - ba') = -bb'(q' - q) + rb' - br',$$

et enfin, compte tenu de (1),

$$bb'(q' - q) = n + rb' - br' . \tag{10}$$

Considérons maintenant deux cas:

(a) Si n n'est pas un multiple de b', alors $r' \neq 0$ dans (6) puisque a' et b' sont premiers entre eux; donc

$$1 \leq r' \leq b' - 1 .$$

Si dans (10) on majore n par $b + b'$, r par $b - 1$, et si on minore r' par 1, on obtient

$$bb'(q' - q) < b + b' + (b-1)b' - b = bb' ;$$

donc $q' - q < 1$, d'où $q' < q + 1$ et $q' \leq q$. Ce résultat avec (9) donne $q' = q$. Revenant maintenant à (7), on a

$$q + \frac{r}{b} < \gamma < q + \frac{r'}{b'},$$

et, *a fortiori*,

$$q < \gamma < q + 1 ,$$

ce qui est impossible.

(b) Si n est un multiple de b', on a $r' = 0$ dans (6) et donc

$$\frac{a'}{b'} = \frac{q'}{n}. \tag{11}$$

D'autre part, puisque $n < b+b'$, $r' = 0$, et $r \le b-1$, on obtient de (10)

$$bb'(q'-q) < b+b'+(b-1)b' = bb'+b,$$

d'où

$$q'-q < 1+\frac{1}{b'};$$

donc $q'-q \le 1$ et $q' \le q+1$. Compte tenu de (9), on peut donc avoir

$$q' = q \quad \text{ou} \quad q' = q+1.$$

Si on suppose $q' = q$, on obtient de (11) et (8)

$$\frac{a'}{b'} = \frac{q}{n} \le \frac{a}{b},$$

en contradiction avec *(i)*. Si on suppose $q' = q+1$, on obtient de (7)

$$q+\frac{r}{b} < \gamma < q+1$$

et, *a fortiori*,

$$q < \gamma < q+1,$$

une impossibilité. La démonstration de *(v)* est donc complète.

Résumons. Si les fractions $\frac{a}{b}$ et $\frac{a'}{b'}$ vérifient (1), on a

$$\frac{a}{b} < \frac{a+a'}{b+b'} < \frac{a'}{b'},$$

chacune de ces trois fractions est irréductible, $\frac{a+a'}{b+b'}$ est la seule fraction entre $\frac{a}{b}$ et $\frac{a'}{b'}$ dont le dénominateur ne dépasse pas $b+b'$, et on peut recommencer le procédé avec les paires de fractions $\frac{a}{b}$, $\frac{a+a'}{b+b'}$ et $\frac{a+a'}{b+b'}$, $\frac{a'}{b'}$ puisque, d'après (3), chacune de ces paires vérifie (1).

Comme application, proposons-nous de trouver, en ordre, toutes les fractions irréductibles entre $\frac{2}{3}$ et $\frac{5}{7}$ dont le dénominateur est inférieur à 36. Puisque $\frac{a}{b} = \frac{2}{3}$ et $\frac{a'}{b'} = \frac{5}{7}$ vérifient bien (1), on trouve successivement

$$\frac{2}{3} < \frac{5}{7},$$

$$\frac{2}{3} < \frac{7}{10} < \frac{5}{7},$$

$$\frac{2}{3} < \frac{9}{13} < \frac{7}{10} < \frac{12}{17} < \frac{5}{7},$$

et continuant de la sorte on aboutit finalement à

$$\frac{2}{3} < \frac{23}{34} < \frac{21}{31} < \frac{19}{28} < \frac{17}{25} < \frac{15}{22} < \frac{13}{19} < \frac{24}{35} < \frac{11}{16} < \frac{20}{29} < \frac{9}{13} < \frac{16}{23} < \frac{23}{33} < \frac{7}{10} < \frac{19}{27} < \frac{12}{17} < \frac{17}{24} < \frac{22}{31} < \frac{5}{7}.$$

Si on commence avec les fractions $\frac{a}{b} = \frac{0}{1}$ et $\frac{a'}{b'} = \frac{1}{1}$, qui vérifient bien (1), la suite de fractions de dénominateurs ne dépassant pas n qui résulte est appelée *suite de Farey* d'ordre n.

RÉFÉRENCE

1. C. Bréard, *Mathématiques élémentaires*, Éditions de L'École, Paris, 1963, Tome 1, p. 449.

PROBLEMS--PROBLÈMES

Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 37.

For the problems given below, solutions, if available, will appear in EUREKA Vol.2, No.6, to be published around June 25, 1976. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than June 15, 1976.

121. *Proposed by Léo Sauvé, Algonquin College.*

For which n is there a convex polyhedron having exactly n edges?

122. *Proposed by Jeremy Wheeler, British Railways, Melbourne, Derby, England.*

I had leant my ladder up against the side of the house to paint my bedroom window and found that it just reached the bottom of the window. My son was pushing a box around and was just able to get it under the ladder. The box was a 1-metre cube and the ladder was 4 metres long. How high was the bedroom window off the ground?

123. *Proposed by Walter Bluger, Department of National Health and Welfare.*

By means of only three weighings on a two-pan balance, you are to find among 13 dimes the one counterfeit coin and be able to tell whether it is heavier or lighter than a true coin. You are given the 13 coins and a balance, and you may bring anything you like with you that may help you in solving the problem.

124. *Proposé par Bernard Vanbrugghe, Université de Moncton.*

Calculer: $\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt$.

125. *Proposé par Bernard Vanbrugghe, Université de Moncton.*

A l'aide d'un compas seulement, déterminer le centre inconnu d'un cercle donné.

(Ce problème est bien connu sons le nom de *problème de Napoléon*.)

126. *Proposed by Viktors Linis, University of Ottawa.*

Show that, for any triangle ABC,

$$|OA|^2\sin A + |OB|^2\sin B + |OC|^2\sin C = 2K ,$$

where O is the centre of the inscribed circle and K is the area of ΔABC.

127. *Proposed by Viktors Linis, University of Ottawa.*

A, B, C, D are four distinct points on a line. Construct a square by drawing two pairs of parallel lines through the four points.

128. *Proposé par Paul Khoury, Collège Algonquin.*

Déterminer les nombres réels a, b, c, ayant donné que l'équation $az^2+bz+c = 0$ admet comme une de ses racines $v+v^2+v^4$, oû v est une racine imaginaire de $z^7-1 = 0$.

129. *Proposed by Léo Sauvé, Algonquin College.*

It has been known since Weierstrass that there exist functions continuous over the whole real axis but differentiable nowhere. Describe a function which is continuous over the whole real axis but differentiable only at (a) $x = 0$; (b) a finite number of points; (c) a countable number of points.

130. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit A l'anneau $\{z : r \le |z| \le R\}$. Montrer que la fonction $f(z) = \frac{1}{z}$ n'est pas limite uniforme de polynômes sur A.

* * *

SOLUTIONS

18. [1975: 8, 31] *Proposé par Jacques Marion, Université d'Ottawa.*

Montrer que, dans un triangle rectangle dont les côtés ont 3 4, et 5 unités de longueur, aucun des angles aigus n'est un multiple rationnel de π.

III. *Comment by Léo Sauvé, Algonquin College.*

Let θ be an acute angle in a right triangle. In my solution I, I attempted to prove that if $\tan\theta = \frac{b}{a}$, an irreducible rational, then θ is not a rational multiple of π. In looking over the proof again, I realized that I have not succeeded in proving quite so much. What I did succeed in proving is that if $\tan\theta = \frac{b}{a}$, an irreducible rational, *and if a and b are of different parity*, then θ is not a rational multiple of π.

Thus my proof is valid for the 3-4-5 triangle in the statement of the problem. More generally, it is valid for every Pythagorean triangle (three sides of integral length) since every such triangle is similar to another in which the sides form a primitive Pythagorean triple, and it is well-known that, in all such triples (a,b,c), a and b are necessarily of different parity. But it does not hold, as I claimed, for every right triangle in which an acute angle θ has a rational tangent. For example, in an isosceles right triangle we have $\tan\theta = 1$, an irreducible rational, and $\theta = \frac{\pi}{4}$, a rational multiple of π.

If $\tan\theta = \frac{b}{a} \neq 1$, where a and b are both odd, can θ ever be a rational multiple of π? That question is still open. I mention here, for whatever help it might be, that certain linear combinations of such angles *are* rational multiples of π. For example, there is the well-known relation

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \frac{\pi}{4}.$$

58. [1975: 49, 91] *Proposé par Jacques Marion, Université d'Ottawa.*

Soit $f : \{z : \operatorname{Re} z = 0\} \to R$ une fonction continue et bornée. Si l'on définit $\mu : \{z : \operatorname{Re} z > 0\} \to R$ par

$$\mu(z) = \mu(x+iy) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{xf(it)}{x^2+(y-t)^2}\,dt,$$

montrer que $f(ic) = \lim\limits_{z\to ic} \mu(z)$.

Solution du proposeur.

Notons

$$M = \sup_{t\in R} |f(it)|, \qquad G = \{z : \operatorname{Re} z > 0\}.$$

Nous aurons besoin plus bas du résultat suivant

$$|\mu(z)| = |\mu(x+iy)| \le M, \qquad z\in G,$$

que nous établirons d'abord. On a, pour tout $z \in G$,

$$|\mu(z)| = \frac{1}{\pi}\left|\int_{-\infty}^{\infty} \frac{xf(it)}{x^2+(y-t)^2}\,dt\right| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{xM}{x^2+(y-t)^2}\,dt$$

$$= \frac{xM}{\pi}\int_{-\infty}^{\infty} \frac{dt}{x^2+(y-t)^2} = \frac{xM}{\pi}\cdot\frac{\pi}{x} = M\,.$$

Montrons maintenant que $f(ic) = \lim\limits_{z\to ic} \mu(z)$. Dans l'intégrale qui définit $\mu(z)$, faisons le changement de variable $w = \dfrac{t-y}{x}$; puis, pour la commodité de la notation, remplaçons w de nouveau par t. Il résulte

$$\mu(z) = \mu(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f[i(y+tx)]}{1+t^2}\,dt.$$

On a maintenant

$$|\mu(x+iy) - f(ic)| = \left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f[i(y+tx)]}{1+t^2}\,dt - \frac{f(ic)}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}\right|$$

$$\le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|f[i(y+tx)] - f(ic)|}{1+t^2}\,dt.$$

Soit $\epsilon > 0$. Il existe $\xi > 0$ tel que

$$\int_{|t|>\xi} \frac{dt}{1+t^2} < \frac{\epsilon}{2M}\,.$$

La fonction f étant continue sur $\{iy : y \in R\}$, il existe $\delta > 0$ tel que

$$|y+tx-c| < \delta \implies |f[i(y+tx)] - f(ic)| < \frac{\epsilon}{2}.$$

Vu que

$$x+iy \to ic \iff x\to 0 \quad \text{et} \quad y\to c,$$

il existe $\eta > 0$ tel que

$$|(x+iy)-ic| < \eta \Longrightarrow |c-y| < \frac{\delta}{2} \quad \text{et} \quad |x| < \frac{\delta}{2\xi},$$

de sorte que, pout $|t| \le \xi$,

$$|y+tx-c| < |c-y|+|x||t| < |c-y|+\xi|x| < \frac{\delta}{2}+\frac{\delta}{2} = \delta.$$

Par conséquent, si $|(x+iy)-ic| < \eta$, on a

$$|\mu(x+iy)-f(ic)| \le \frac{1}{\pi}\int_{|t|\le\xi}\frac{|f[i(y+tx)]-f(ic)|}{1+t^2}\,dt+\frac{1}{\pi}\int_{|t|>\xi}\frac{|f[i(y+tx)]-f(ic)|}{1+t^2}\,dt$$

$$\le \frac{1}{\pi}\int_{|t|\le\xi}\frac{\epsilon/2}{1+t^2}\,dt+\frac{1}{\pi}\int_{|t|>\xi}\frac{2M}{1+t^2}\,dt \le \frac{\epsilon}{2}+\frac{2M}{\pi}\cdot\frac{\epsilon}{2M} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

On peut donc conclure que $f(ic) = \lim\limits_{z\to ic}\mu(z)$.

Editor's comment.

This problem, with the additional requirement to prove that $\mu(z)$ is harmonic on the right half-plane (which the proposer also proved), forms what is known as *Dirichlet's problem for the half-plane*. The proposer noted that it can be found, without solution, on p. 265 of John B. Conway's *Functions of One Complex Variable* (Springer-Verlag, 1973).

91. [1975: 97] *Proposé par Léo Sauvé, Collège Algonquin.*

Si a, a', b, et b' sont des entiers positifs, montrer qu'une condition suffisante pour que la fraction $\frac{a+a'}{b+b'}$ soit irréductible est que

$$|ab'-ba'| = 1.$$

Cette condition est-elle aussi nécessaire?

Solution by G.D. Kaye, Department of National Defence.

The given condition is sufficient since when it is satisfied we have

$$1 = |ab'-ba'| = |b'(a+a')-a'(b+b')|,$$

and the only possible divisor common to $a+a'$ and $b+b'$ is 1.

The condition is, however, not necessary, as can be seen from the counter-example $\frac{13}{19} = \frac{1+12}{3+16}$.

Also solved by Walter Bluger, Department of National Health and Welfare; Sheila Gribble, Picton, Ont.; André Ladouceur, École Secondaire De La Salle; F.G.B. Maskell, Algonquin College; and the proposer.

92. [1975: 97] *Proposé par Léo Sauvé, Collège Algonquin.*

Si a est un entier positif, montrer que la fraction

$$\frac{a^3+2a}{a^4+3a^2+1}$$

est irréductible.

I. *Solution de F.G.B. Maskell, Collège Algonquin.*

Si $a=1$ la fraction donnée égale $\frac{3}{5}$, qui est irréductible. Si $a>1$, écrivons

$$\frac{a^3+2a}{a^4+3a^2+1} = \frac{a(a^2+2)}{a^2(a^2+3)+1} \qquad (1)$$

$$= \frac{a(a^2+2)}{(a^2+2)(a^2+1)-1}\ . \qquad (2)$$

Il découle alors de (1) que 1 est le seul facteur de a commun au numérateur et au dénominateur, et de (2) que 1 est le seul facteur de a^2+2 commun au numérateur et au dénominateur. *La fraction donnée est donc irréductible.*

II. *Comment by the proposer.*

The problem can be found in [1], with the following interesting solution based on the evident fact that a fraction is irreducible if and only if its reciprocal is irreducible. Each of the following fractions except the last is irreducible if and only if the next one is:

$$\frac{a^3+2a}{a^4+3a^2+1},\ \frac{a^4+3a^2+1}{a^3+2a} = a+\frac{a^2+1}{a^3+2a},\ \frac{a^2+1}{a^3+2a},\ \frac{a^3+2a}{a^2+1} = a+\frac{a}{a^2+1},\ \frac{a}{a^2+1},$$

$$\frac{a^2+1}{a} = 1+\frac{1}{a},\ \frac{1}{a}\ .$$

Since $\frac{1}{a}$ is irreducible for all a, so is the given fraction.

Also solved by G.D. Kaye, Department of National Defence. One incorrect solution was received.

REFERENCE

1. D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Co., 1962, pp. 18, 146-147.

93. [1975: 97] *Proposed by H.G. Dworschak, Algonquin College.*

Is there a convex polyhedron having exactly seven edges?

I. *Solution by F.G.B. Maskell, Algonquin College.*

Suppose there exists a convex polyhedron having exactly seven edges. Since each edge is the meet of two polygonal faces as well as the join of two polygonal vertices, there are 14 polygonal sides and 14 polygonal vertices.

But each polygonal face has at least three sides and at least three vertices. Therefore, if E, F, V represent respectively the number of edges, faces, and vertices of the polyhedron, we must have $F<5$ and $V<5$. Since clearly $F>3$ and $V>3$ in every polyhedron, it follows that $F=4$ and $V=4$, so that $V-E+F = 1$, which contradicts Euler's formula $V-E+F = 2$, and we conclude that there is no polyhedron having exactly seven edges.

II. *Solution by Léo Sauvé, Algonquin College.*

In any convex polyhedron, the number of vertices (V), of edges (E), and of faces (F) must satisfy Euler's formula $V-E+F = 2$. Suppose $E=7$, so that

$$V+F = 9. \qquad (1)$$

Since at least three edges meet at each vertex and each edge is incident with exactly two vertices, we have

$$3V \le 2E = 14. \qquad (2)$$

Since each face is bounded by at least three edges and each edge belongs to exactly two faces, we have also

$$3F \le 2E = 14. \qquad (3)$$

Now $3V + 3F = 27$ from (1); hence from (2) and (3), either

$$3V = 14 \quad \text{and} \quad 3F = 13$$

or

$$3V = 13 \quad \text{and} \quad 3F = 14.$$

Neither possibility is acceptable since 13, for example, is not divisible by 3.

Thus there is no convex polyhedron having exactly seven edges.

Also solved by Sheila Gribble, Picton, Ont.; and G.D. Kaye, Department of National Defence. One incorrect solution was received.

Editor's comment.

It is obvious that there exist n-sided convex polygons for every $n \ge 3$. But it is not true that there exist n-edged convex polyhedra for every $n \ge 6$, since this problem shows that $n = 7$ is an exception. The following question now instantly springs to mind: are there other exceptional values besides $n = 7$? Problem 121 in this issue solicits your answer to this question.

94. [1975, 97: 1976, 25] *Proposed by H.G. Dworschak, Algonquin College.*

If, in a tetrahedron, two pairs of opposite edges are orthogonal, is the third pair of opposite edges necessarily orthogonal?

I. *Solution by the proposer.*

The answer is yes. For let S-ABC be a tetrahedron (see Figure 1). The desired property follows at once from the following identity,

$$\overrightarrow{SA}\cdot\overrightarrow{BC} + \overrightarrow{SB}\cdot\overrightarrow{CA} + \overrightarrow{SC}\cdot\overrightarrow{AB} = 0 \qquad (1)$$

which, as we will show below, is valid for any four points S, A, B, C in space; for if two of the scalar products in (1) vanish, the third must vanish also.

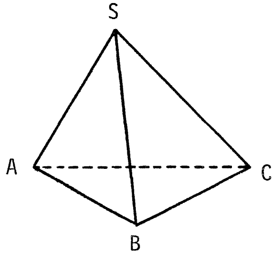

Figure 1

To prove (1), let

$$\overrightarrow{SA} = \vec{a}, \qquad \overrightarrow{SB} = \vec{b}, \qquad \overrightarrow{SC} = \vec{c};$$

then the left side of (1) equals

$$\vec{a}\cdot(\vec{c} - \vec{b}) + \vec{b}\cdot(\vec{a} - \vec{c}) + \vec{c}\cdot(\vec{b} - \vec{a}) = 0.$$

II. *Solution d'André Ladouceur, École Secondaire De La Salle.*

Si, dans la figure 1, on suppose $SA \perp BC$ et $SB \perp CA$, de sorte que

$\vec{SA}\cdot\vec{BC} = 0$ et $\vec{SB}\cdot\vec{CA} = 0$, on a alors

$$\begin{aligned}\vec{SC}\cdot\vec{AB} &= (\vec{SA}+\vec{AC})\cdot(\vec{AS}+\vec{SB})\\ &= \vec{AS}\cdot(\vec{SA}+\vec{AC}) + \vec{SA}\cdot\vec{SB} + 0\\ &= \vec{AS}\cdot\vec{SC} + \vec{AS}\cdot\vec{BS}\\ &= \vec{AS}\cdot(\vec{BS}+\vec{SC})\\ &= \vec{AS}\cdot\vec{BC}\\ &= 0,\end{aligned}$$

et on peut conclure que $SC \perp AB$.

III. *Solution by F.G.B. Maskell, Algonquin College.*

Suppose we have $SB \perp CA$ and $SC \perp AB$ in tetrahedron S-ABC (see Figure 2). Let the altitudes AD, BE, and CF of ΔABC meet in the orthocentre H.

Since $CA \perp SB$ and BE, then $CA \perp$ plane SBE, and therefore $CA \perp$ line SH in the plane. Similarly, $AB \perp SH$. Since $SH \perp CA$ and AB, then $SH \perp$ plane ABC, and therefore $SH \perp$ line BC in the plane. Since $BC \perp SH$ and AD, then $BC \perp$ plane SAD, and therefore $\perp$ line SA in the plane.

Thus the answer to the question is "yes".

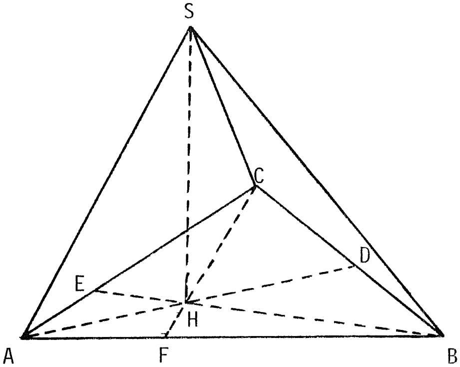

Figure 2

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College (two additional solutions); Léo Sauvé, Algonquin College; and the proposer (one additional solution).

95. [1975: 97] *Proposed by Walter Bluger, Department of National Health and Welfare.*

Said a math teacher, full of sweet wine:
"Your house number's the exact square of mine."
—"You are tight and see double
Each digit. That's your trouble!"
These two-digit numbers you must divine.

I. *Solution d'André Ladouceur, Ecole Secondaire De La Salle.*

On a $(\overline{xy})^2 = \overline{mmnn}$, où x, y, m, n sont les chiffres, c'est-à-dire

$$(10x+y)^2 = 1100m + 11n.$$

Puisque le carré est divisible par 11, $\overline{xy}$ l'est aussi. On doit donc avoir $x = y$. Essayant successivement $\overline{xy} = 11, 22, \ldots$, on trouve la solution unique $88^2 = 7744$.

Les adresses sont donc 88 et 74.

II. *Gracious answer by Sheila Gribble, Picton, Ont.*

An abode, eighty-eight on its door,
Could be squared, and its rooms be much more.
Would then it be double,
Or without much trouble
Would thus it have sev'n sev'n forty-four?

Also solved by G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College, and Mrs. Elizabeth Sinclair, Inuvik, N.W.T. (jointly); and the proposer.

96. [1975: 97] *Proposed by Viktors Linis, University of Ottawa.*

By Euclidean methods divide a $13°$ angle into 13 equal parts.

I. *Essence of the solutions submitted independently by Walter Bluger, Department of National Defence; and the proposer.*

$7 \times 13° = 90° + 1°$.

II. *Solution by Sheila Gribble, Picton, Ont.*

$45° - 13° = 32°$. Bisect $32°$ angle and resulting angles repeatedly until a $1°$ angle is left.

III. *Solution by G.D. Kaye, Department of National Defence; and André Ladouceur, École Secondaire De La Salle (independently).*

$\frac{1}{4} \cdot 60° = 15°$; $15° - 13° = 2°$; bisect $2°$ angle to get $1°$ angle.

Also solved by F.G.B. Maskell, Algonquin College; H.G. Dworschak, Algonquin College; and Léo Sauvé, Collège Algonquin.

97. [1975: 97] *Proposed by Viktors Linis, University of Ottawa.*

Find all primes p such that $p^3 + p^2 + 11p + 2$ is a prime.

I. *Solution by Sheila Gribble, Picton, Ont.*

Let $f(p) = p^3 + p^2 + 11p + 2$. Since

$$f(3k+1) = 3(9k^3 + 12k^2 + 16k + 5)$$

and

$$f(3k+2) = 9(3k^3 + 7k^2 + 9k + 4)$$

are not primes, the only prime p for which $f(p)$ might be a prime is $p = 3$; and indeed $f(3) = 71$, a prime.

II. *Solution by G.D. Kaye, Department of National Defence.*

Since $p^2 + 11$ is divisible by 3 when $p = 3k+1$, and $p+1$ is divisible by 3 when $p = 3k+2$, it follows that

$$f(p) = p^3 + p^2 + 11p + 2 = (p^2 + 11)(p+1) - 9$$

is divisible by 3 in these two cases. Hence $p = 3$ is the only prime for which $f(p)$ might be prime. Since $f(3) = 71$, a prime, $p = 3$ is the only solution.

Also solved by Walter Bluger, Department of National Health and Welfare; F.G.B. Maskell, Algonquin College; Léo Sauvé, Collège Algonquin; and the proposer.

Editor's comment.

One reader asks whether $f(x) = x^3 + x^2 + 11x + 2$ generates only primes when x is a multiple of 3. This is certainly not true when x is even, since then every term of $f(x)$ is even. For odd multiples of 3, $f(x)$ is prime for $x = 3$, 9, 15, but $f(21) = 9935 = 5\cdot 1987$.

98. [1975: 97] *Proposed by Viktors Linis, University of Ottawa.*

Prove that, if $0 < a < b$, then $\ln\frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}$.

I. *Solution de Bernard Vanbrugghe, Université de Moncton.*

La démonstration dépend de l'inégalité bien connue: $x < \sinh x$ pour $x > 0$. On a

$$\ln\frac{b^2}{a^2} = 2\ln\frac{b}{a} < 2\sinh(\ln\frac{b}{a}) = e^{\ln\frac{b}{a}} - e^{-\ln\frac{b}{a}} = e^{\ln\frac{b}{a}} - e^{\ln\frac{a}{b}} = \frac{b}{a} - \frac{a}{b}.$$

II. *Solution by John A. Tierney, United States Naval Academy.*

Let $\frac{b}{a} = p > 1$ and let $f(p) = \ln p^2 - p + p^{-1}$; then $f'(p) = -p^{-2}(p-1)^2 < 0$ for $p > 1$. Since f is continuous on $[1,p]$ and differentiable on $(1,p)$, and $f(1) = 0$, the mean value theorem yields

$$f(p) = 0 + [-c^{-2}(c-1)^2](p-1) < 0, \quad 1 < c < p.$$

Thus $\ln p^2 < p - p^{-1}$ or $\ln\frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}$.

III. *Solution by Léo Sauvé, Algonquin College.*

If $s \neq 1$, $(1 - \frac{1}{s})^2 > 0$ implies $\frac{2}{s} < 1 + \frac{1}{s^2}$; hence, for $x > 1$,

$$\ln x^2 = 2\ln x = \int_1^x \frac{2}{s}\,ds < \int_1^x (1 + \frac{1}{s^2})\,ds = x - \frac{1}{x},$$

and setting $x = \frac{b}{a}$ yields $\ln\frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}$.

Also solved by F.G.B. Maskell, Algonquin College; and Bernard Vanbrugghe, Université de Moncton (second solution).

Editor's comments.

1. The second solver, John A. Tierney, is the author of one of the most widely adopted calculus textbooks, now in its third edition [1]. Since we have been using his textbook at Algonquin College for the past three or four years, we consider him an old friend. He wrote to the editor:

> I thought it might be appropriate to let you know that I have been enjoying EUREKA very much. It sounds as if you not only have an excellent department, but have a lot of fun at the same time. I am always on the lookout for calculus problems and would think that a good student might come up with something like this on Problem 98.

His solution given above then followed.

2. It is easy to show that the inequality in this problem is equivalent to

$$\ln\frac{b}{a} < \frac{1}{2}(b-a)\left(\frac{1}{b}+\frac{1}{a}\right), \qquad 0 < a < b.$$

It can be shown that, if $b - a$ is small compared with b, then

$$\ln\frac{b}{a} \approx \frac{1}{2}(b-a)\left(\frac{1}{b}+\frac{1}{a}\right), \tag{1}$$

the error being approximately $\dfrac{(b-a)^3}{6b^3}$ and certainly less than $\dfrac{(b-a)^3(3b-2a)}{6b^3a}$. For a discussion and proof of these statements, see [2]. Formula (1) was used by Napier in calculating logarithms. It is usually referred to as *Napier's formula*.

REFERENCES

1. John A. Tierney, *Calculus and Analytic Geometry*, 3rd Edition, Allyn and Bacon, 1975.

2. Clement V. Durell, *Advanced Algebra*, G. Bell and Sons, London, 1959, Vol. I, p. 119.

99. [1975: 98] *Proposed by H.G. Dworschak, Algonquin College.*

If a, b, and n are positive integers, prove that there exist positive integers x and y such that

$$(a^2+b^2)^n = x^2+y^2. \tag{1}$$

Application: If $a = 3$, $b = 4$, and $n = 7$, find at least one pair $\{x,y\}$ of positive integers which verifies (1).

I. *Solution by Léo Sauvé, Algonquin College.*

The problem is not quite correct as stated. The following restriction should be added at the end: *except possibly when $a = b$ and n is even.*

If $a = b = 1$ and $n = 2$, for example, then (1) gives

$$4 = x^2+y^2,$$

and it is clearly impossible to express 4 as the sum of two nonzero integral squares; while if $a = b = 5$ and $n = 2$ we have

$$(5^2+5^2)^2 = 30^2+40^2.$$

If $a = b$ and n is odd, say $n = 2k+1$, there is always the solution

$$x = y = a\cdot(2a^2)^k.$$

However, the restriction mentioned above clearly becomes unnecessary if we ask for *nonnegative* integers x and y, thus allowing x or y to be zero; for then, when n is even, there is always at least the solution

$$(a^2+b^2)^n = [(a^2+b^2)^{n/2}]^2+0^2.$$

We adopt the second alternative and prove the existence of nonegative integers x and y which satisfy (1) for arbitrary positive a, b, and n.

If we set $u+vi = (a+bi)^n$, then

$$u = \operatorname{Re}(a+bi)^n \quad\text{and}\quad v = \operatorname{Im}(a+bi)^n$$

are surely integers. Since $u - vi = (a - bi)^n$ from properties of conjugates, we have

$$\begin{aligned}(a^2 + b^2)^n &= (a + bi)^n(a - bi)^n\\ &= (u + vi)(u - vi)\\ &= u^2 + v^2,\end{aligned}$$

and it follows that one solution to our problem is

$$x = |u| = |\mathrm{Re}\,(a + bi)^n|, \quad y = |v| = |\mathrm{Im}\,(a + bi)^n|. \tag{2}$$

If $a = 3$, $b = 4$, and $n = 7$, then $(3 + 4i)^7 = 76443 + 16124i$, and (2) gives the solution

$$(3^2 + 4^2)^7 = 76443^2 + 16124^2.$$

This is not the only solution. To find other solutions, we first note that

$$\begin{aligned}(a^2 + b^2)(c^2 + d^2) &= (a + bi)(a - bi)(c + di)(c - di)\\ &= [(a + bi)(c - di)][(a - bi)(c + di)] = (ac + bd)^2 + (ad - bc)^2 \qquad (3)\\ &= [(a + bi)(c + di)][(a - bi)(c - di)] = (ac - bd)^2 + (ad + bc)^2. \qquad (4)\end{aligned}$$

We now multiply each of $3^2 + 4^2$ and $5^2 + 0^2$ successively by $3^2 + 4^2$ and write the products in the two forms described by (3) and (4); this gives three distinct forms for $(3^2 + 4^2)^2$. Each of these three forms is then multiplied by $3^2 + 4^2$, again using (3) and (4); this yields four distinct forms for $(3^2 + 4^2)^3$. Continuing thus, we obtain the following results:

$$\begin{aligned}(3^2 + 4^2)^2 &= 7^2 + 24^2\\ &= 15^2 + 20^2\\ &= 25^2 + 0^2\end{aligned}$$

$$\begin{aligned}(3^2 + 4^2)^3 &= 35^2 + 120^2\\ &= 75^2 + 100^2\\ &= 117^2 + 44^2\\ &= 125^2 + 0^2\end{aligned}$$

$$\begin{aligned}(3^2 + 4^2)^4 &= 175^2 + 600^2\\ &= 375^2 + 500^2\\ &= 527^2 + 336^2\\ &= 585^2 + 220^2\\ &= 625^2 + 0^2\end{aligned}$$

$$\begin{aligned}(3^2 + 4^2)^5 &= 237^2 + 3116^2\\ &= 875^2 + 3000^2\\ &= 1875^2 + 2500^2\\ &= 2635^2 + 1680^2\\ &= 2925^2 + 1100^2\\ &= 3125^2 + 0^2\end{aligned}$$

$$\begin{aligned}(3^2 + 4^2)^6 &= 1185^2 + 15580^2\\ &= 4375^2 + 15000^2\\ &= 9375^2 + 12500^2\\ &= 11753^2 + 10296^2\\ &= 13175^2 + 8400^2\\ &= 14625^2 + 5500^2\\ &= 15625^2 + 0^2\end{aligned}$$

$$\begin{aligned}(3^2 + 4^2)^7 &= 5925^2 + 77900^2\\ &= 21875^2 + 75000^2\\ &= 46875^2 + 62500^2\\ &= 58765^2 + 51480^2\\ &= 65875^2 + 42000^2\\ &= 73125^2 + 27500^2\\ &= 76443^2 + 16124^2\\ &= 78125^2 + 0^2\end{aligned}$$

II. *Comment by Viktors Linis, University of Ottawa.*

The solution given by (2) can be represented in various ways. For example, using the polar representation of complex numbers, we have

$$a^2 + b^2 = (a^2 + b^2)(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi),$$

where

$$\cos\phi = a(a^2+b^2)^{-1/2}, \quad \sin\phi = b(a^2+b^2)^{-1/2}.$$

Then

$$(a^2+b^2)^n = (a^2+b^2)^n(\cos n\phi + i\sin n\phi)(\cos n\phi - i\sin n\phi),$$

and we can put

$$x = (a^2+b^2)^{n/2}\cos n\phi, \quad y = (a^2+b^2)^{n/2}\sin n\phi,$$

It is not immediate that these numbers will be integers unless one uses again the representation (2).

Matrix representation can also be used by observing that there is a 1-1 correspondence between $a+bi$ and $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and an isomorphism between the two rings.

Also solved by Walter Bluger, Department of National Health and Welfare; G.D. Kaye, Department of National Defence; Viktors Linis, University of Ottawa (two solutions); F.G.B. Maskell, Algonquin College; and the proposer.

Editor's comment.

Several of these solutions were incomplete in one way or another. Some solvers failed to observe that positive solutions $\{x,y\}$ did not always exist when $a=b$, and others concluded wrongly that positive solutions existed only when $a \neq b$.

In one of his solutions, Linis used (3) and mathematical induction to prove the existence of nonnegative solutions $\{x,y\}$ for arbitrary positive a, b, and n.

Three solvers gave only one solution to the numerical example, and two gave two solutions, all of them included among the eight solutions given above.

It would be interesting, but probably not easy, to find the number of distinct solutions $\{x,y\}$ as a function of the positive integers a, b, and n. Would some reader care to try?

100. [1975: 98] *Proposé par Léo Sauvé, Collège Algonquin.*

Soit f une fonction numérique continue et non négative pour tout $x \geq 0$. On suppose qu'il existe un nombre réel $a > 0$ tel que, pour tout $x > 0$,

$$f(x) \leq a\int_0^x f(t)\,dt.$$

Montrer que la fonction f est nulle.

I. *Solution adaptée de celle soumise par Bernard Vanbrugghe, Université de Moncton.*

Choisissons d'abord un nombre b tel que $0 < b < \frac{1}{a}$, et notons M la borne supérieure de f sur l'intervalle $[0,b]$. Puisque f est continue, il existe un nombre $x_0 \in [0,b]$ tel que $f(x_0) = M$. On a alors

$$M = f(x_0) \leq a\int_0^{x_0} f(t)\,dt \leq ax_0M \leq abM.$$

Puisque $ab < 1$, on doit avoir $M = 0$, et la restriction de f à $[0,b]$ est nulle.

Supposons maintenant par récurrence que, pour un entier $n \geq 1$, la restriction de f à $[0,nb]$ soit nulle. Notons M_n la borne supérieure de f sur l'intervalle $[nb,(n+1)b]$. Puisque f est continue, il existe un nombre $x_n \in [nb,(n+1)b]$ tel que $f(x_n) = M_n$. On a alors

$$M_n = f(x_n) \leq a\int_0^{x_n} f(t)dt = a\int_{nb}^{x_n} f(t)\,dt \leq a(x_n - nb)M_n \leq abM_n .$$

Puisque $ab < 1$, on doit avoir $M_n = 0$, et la restriction de f à $[0,(n+1)b]$ est nulle.

Par suite, la fonction f est nulle, ce qu'il fallait prouver.

II. *Remarque de Jacques Marion, Université d'Ottawa.*

En fait, on peut démontrer le résultat plus général suivant: *Soit $F : [c,d] \to R$ une fonction différentiable telle que $F(c) = 0$ et pour laquelle il existe $a > 0$ tel que*

$$|F'(x)| \leq a|F(x)|;$$

alors $F \equiv 0$ sur $[c,d]$.

Also solved by G.D. Kaye, Department of National Defence; Viktors Linis, University of Ottawa; Jacques Marion, Université d'Ottawa; F.G.B. Maskell, Algonquin College; and the proposer.

* * *

THE MATHEMATICIAN IN LOVE

1

A mathematician fell madly in love
With a lady, young, handsome, and charming:
By angles and ratios harmonic he strove
Her curves and proportions all faultless to prove,
As he scrawled hieroglyphics alarming.

2

He measured with care, from the ends of a base,
The arcs which her features subtended;
Then he framed transcendental equations, to trace
The flowing outlines of her figure and face,
And thought the result very splendid.

3

He studied (since music has charms for the fair)
The theory of fiddles and whistles--
Then composed, by acoustic equations, an air,
Which, when 'twas performed, made the lady's long hair
Stand on end, like a porcupine's bristles.

4

The lady loved dancing--he therefore applied,
   To the polka and waltz, an equation;
But when to rotate on his axis he tried,
His center of gravity swayed to one side,
   And he fell, by the earth's gravitation.

5

No doubts of the fate of his suit made him pause,
   For he proved, to his own satisfaction,
That the fair one returned his affection--"because,
"As every one knows, by mechanical laws,
   "Reaction is equal to action.

6

"Let x denote beauty--y, manners well-bred--
   "Z, Fortune--(this last is essential)--
"Let L stand for love"--our philosopher said--
"Then L is a function of x, y, and z,
   "Of the kind which is known as potential."

7

"Now integrate L with respect to dt,
   "(t standing for time and persuasion);
"Then, between proper limits, 'tis easy to see,
"The definite integral *Marriage* must be--
   "(A very concise demonstration)."

8

Said he--"If the wandering course of the moon
   "By Algebra can be predicted,
"The female affections must yield to it soon"
--But the lady ran off with a dashing dragoon,
   And left him amazed and afflicted.

--W.J.M. RANKINE, *Songs and Fables* (1874)[1]

* * *

LETTER TO THE EDITOR

Dear editor:

Keep up the good work. I think EUREKA is a very high quality publication and hope that we can expand its readership in the next few years. I only wish I had more time to work on the many excellent problems.

BILL HIGGINSON,
Faculty of Education,
Queen's University,
Kingston, Ont.

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[1]From *The Mathematical Magpie*, by Clifton Fadiman, Simon and Schuster, 1962.