Crux Mathematicorum

VOLUME 43, NO. 8 October / Octobre 2017

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Founding Editors / Rédacteurs-fondateurs: Léopold Sauvé & Frederick G.B. Maskell
Former Editors / Anciens Rédacteurs: G.W. Sands, R.E. Woodrow, Bruce L.R. Shawyer, Shawn Godin

Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

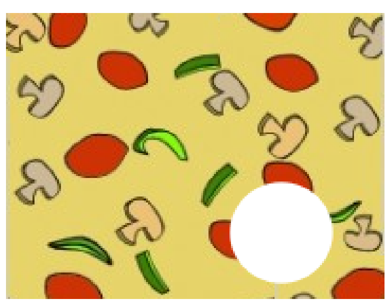
EDITORIAL

Once a problem solver, always a problem solver. While you can give up regular practice of problem solving techniques, it is hard (if not impossible) to not marvel at nice problems that we as mathematicians come across in our "regular" lives. One too many occurrences of meeting mathematical gems, and you are right back where you started flipping through the pages of ***Crux*** or some other problem solving journal.

It is my distinct pleasure to welcome Shawn Godin back to ***Crux*** as a regular contributor. Starting with this issue, Shawn's new column *Problem Solving 101* will target our more mathematically junior audiences with materials suitable for high school students. Both Shawn and I felt the need to close the gap left by the departure of Mayhem, so I am happy to be offering such readings on a regular basis. I hope that this column will attract not only high schoolers and their teachers, but also those who like playing around with problems that require more enthusiasm than technical knowledge.

Speaking of nice problems, I recently went to the seminar presentation by Anna Kyczynska (a past ***Crux*** editor) who spoke about her teaching methods. Not surprisingly, interesting problems make their way into her teaching of upgrade courses. But these problems are not as elementary as one might expect and often require some ingenuity. I leave you with one example from her presentation (the problem is originally due to Peter Liljedahl). You can easily share this one with your friends next time you enjoy pizza for dinner:

> Two friends go to a pizza place that serves rectangular pizza. The restaurant is called "The Hole in One Pizza" because the chef always cuts out a circular piece of some radius randomly from the rectangular pizza. How can two friends equally divide the pizza with one cut?



Kseniya Garaschuk

THE CONTEST CORNER

No. 58

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC286. On définit une fonction f par $f(n) = an + b$, a et b étant des entiers, de manière que pour entier n, $f(3n+1)$, $f(3n)+1$ et $3f(n)+1$ soient trois entiers consécutifs dans un ordre quelconque. Déterminer toutes les expressions possibles pour $f(n)$.

CC287. Aucun élève n'a réussi à résoudre tous les problèmes d'un concours. Chaque problème a été résolu par exactement trois élèves et chaque paire de problèmes a été résolue par exactement un élève. Quel était le nombre maximal de problèmes dans ce concours?

CC288. Les longueurs des côtés d'un triangle ABC sont des entiers positifs consécutifs. Soit D le milieu de BC. Sachant que AD est perpendiculaire à la bissectrice de l'angle C, déterminer le produit des longueurs des trois côtés.

CC289. On dit qu'un entier strictement positif est *spécial* si on peut le représenter comme la somme d'un carré parfait et d'un nombre premier. Par exemple, 101 est spécial, car $101 = 64 + 37$. En effet, 64 est le carré de 8 et 37 est un nombre premier.

a) Démontrer qu'il existe un nombre infini de nombres spéciaux.
b) Démontrer qu'il existe un nombre infini d'entiers strictement positifs qui ne sont pas spéciaux.

CC290. À partir d'un point A, on trace un nombre de demi-droites de manière que tous les angles formés aient pour mesures des multiples de 10°. Quel est le nombre maximal de demi-droites que l'on peut tracer de manière que tous les angles de sommet A aient des mesures différentes, y compris les angles formés par des demi-droites non adjacentes?

. . . .

CC286. The function $f(n) = an + b$, where a and b are integers, is such that for every integer n, $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such $f(n)$.

CC287. In a contest, no student solved all problems. Each problem was solved by exactly three students and each pair of problems was solved by exactly one student. What is the maximum number of problems in this contest?

CC288. The lengths of the sides of triangle ABC are consecutive positive integers. D is the midpoint of BC and AD is perpendicular to the bisector of angle C. Determine the product of the lengths of the three sides.

CC289. A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because $101 = 64 + 37$. Here 64 is the square of 8 and 37 is a prime number.

a) Show that there are infinitely many positive integers which are special.

b) Show that there are infinitely many positive integers which are not special.

CC290. Randy plots a point A. Then he starts drawing some rays starting at A, so that all the angles he gets are integral multiples of $10°$. What is the largest number of rays he can draw so that all the angles at A between the rays are unequal, including all angles between non-adjacent rays?

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(8), p. 332–333.

CC236. A *rhumb line* or *loxodrome* is a curve on the earth's surface that follows a constant direction relative to true (not magnetic) north. Find the maximum possible length of a rhumb line directed northeastward (a bearing of 45° true), or show that it can be arbitrarily long. You may assume the earth to be a sphere of circumference 40,000 km.

Originally question 6 from the 2016 Science Atlantic Mathematics Competition.

We received three correct submissions (two by one author) and we present two of them here.

Solution 1, by Missouri State University Problem Solving Group.

We first consider a rhumb line on the unit sphere following the constant bearing Λ, where the angle Λ is measured clockwise away from true north. For example, $\Lambda = 0°$ corresponds to north and $\Lambda = 45°$ corresponds to northeast.

For any point P on this curve, we represent the vector from the sphere's center to P by

$$\mathbf{P} = \langle x(z), y(z), z\rangle = \langle r\cos\theta, r\sin\theta, z\rangle.$$

Taking r and θ as functions of z, the velocity of P is

$$\mathbf{V} = \langle x'(z), y'(z), 1\rangle.$$

On the other hand, from point P, east is the unit vector

$$\mathbf{E} = \langle -\sin\theta, \cos\theta, 0\rangle,$$

and hence, by the right hand rule, the vector

$$\mathbf{N} = \mathbf{P}\times\mathbf{E} = \langle -z\cos\theta, -z\cos\theta, r\rangle$$

points north on the unit sphere, $r^2 + z^2 = 1$, hence $\mathbf{N}$ is also a unit vector. Let $\mathbf{U}$ be a unit vector that points in the direction of the bearing Λ. Since $\mathbf{N}$ and $\mathbf{E}$ are orthogonal unit vectors, we know

$$\mathrm{proj}_{\mathbf{N}}\mathbf{U} = (\cos\Lambda)\mathbf{N} \quad \text{and} \quad \mathrm{proj}_{\mathbf{E}}\mathbf{U} = (\sin\Lambda)\mathbf{E}$$

and therefore,

$$\mathbf{U} = (\cos\Lambda)\mathbf{N} + (\sin\Lambda)\mathbf{E}.$$

Note that the third component of $\mathbf{U}$ is $r\cos\Lambda$. Since P is following the rhumb line in the same direction, the vectors $\mathbf{V}$ and $\mathbf{U}$ are parallel. Therefore, there exists a scalar $s(z)$ such that

$$s(z)\mathbf{V} = \mathbf{U}.$$

Equating the third components, we have

$$s(z) = r\cos\Lambda = \cos\Lambda\sqrt{1-z^2}.$$

It follows that

$$||\mathbf{V}|| = \frac{||\mathbf{U}||}{|s(z)|} = \frac{|\sec\Lambda|}{\sqrt{1-z^2}}.$$

Since $z=-1$ corresponds to the south pole and $z=1$ corresponds to the north pole, the length of the rhumb line from pole to pole is

$$L = \int_{-1}^{1} ||\mathbf{V}||dz = \int_{-1}^{1} \frac{|\sec\Lambda|}{\sqrt{1-z^2}}dz = |\sec\Lambda| \arcsin z|_{-1}^{1} = \pi|\sec\Lambda|.$$

Note that this is undefined for $\Lambda = 90^\circ$ or $\Lambda = 270^\circ$, but in those cases the rhumb line is a circle, which could be considered as a repeating path of infinite length.

In general, for a sphere of radius R, the length of a rhumb line with bearing Λ from pole to pole is

$$L = \pi R|\sec\Lambda|.$$

For the sphere of circumference 40000 km, the length of a rhumb line bearing northeastward is

$$L = \pi\frac{40000}{2\pi}|\sec 45^\circ| = 20000\sqrt{2} \text{ km}.$$

Solution 2, by Ivko Dimitrić.

Some general facts about loxodromes and relevant formulas can be found in [1]–[5]. It is known, for example, that starting from an initial point on the equator at longitude λ_0 and moving northeastward and southwestward following a rhumb course at constant bearing $0 < \beta < \pi/2$, the path will spiral around as it approaches the North and the South pole, reaching them only in the limit. For a given bearing, all loxodromes through various points of the equator are congruent. We compute the length of a loxodrome using its known parametrization in terms of azimuthal parameter λ (the longitude, adjusted by adding or subtracting the required number of revolutions as the curve spirals around the globe approaching the poles so that $-\infty < \lambda < \infty$). Namely, if R is the radius of a sphere, β the angle of constant bearing with respect to meridians, and $m = \cot\beta$ then, as shown in [1], [3], [5] :

$$x = R\cos\lambda\,\mathrm{sech}[m(\lambda-\lambda_0)],\quad y = R\sin\lambda\,\mathrm{sech}[m(\lambda-\lambda_0)],\quad z = R\tanh[m(\lambda-\lambda_0)],$$

where λ_0 is the longitude of an initial point on the equator which can be taken to be 0. In our case $m = \cot 45^\circ = 1$ and taking $\lambda_0 = 0$ and replacing λ with parameter t, $-\infty < t < \infty$ we get the following parametrization of a loxodrome for $\beta = 45^\circ$:

$$x = R\cos t \operatorname{sech} t, \quad y = R\sin t \operatorname{sech} t, \quad z = R\tanh t.$$

We recall

$$(\tanh t)' = \operatorname{sech}^2 t,\ (\operatorname{sech} t)' = -\operatorname{sech} t\ \tanh t,\ \tanh^2 t + \operatorname{sech}^2 t = 1,$$

so that we compute

$$\begin{aligned}\frac{dx}{dt} &= -R\sin t \operatorname{sech} t - R\cos t \operatorname{sech} t \tanh t\\ \frac{dy}{dt} &= R\cos t \operatorname{sech} t - R\sin t \operatorname{sech} t \tanh t\\ \frac{dz}{dt} &= R \operatorname{sech}^2 t.\end{aligned}$$

Then,

$$\begin{aligned} ds^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 &= R^2 \operatorname{sech}^2 t + R^2 \operatorname{sech}^2 t \tanh^2 t + R^2 \operatorname{sech}^4 t\\ &= R^2 \operatorname{sech}^2 t (1 + \tanh^2 t + \operatorname{sech}^2 t)\\ &= 2R^2 \operatorname{sech}^2 t.\end{aligned}$$

Therefore, $ds = \sqrt{2}\, R \operatorname{sech} t$ and the maximum length l of a loxodrome, from the South pole to the North pole, reached in the limit as $t \to \pm\infty$ is

$$\begin{aligned} l = \int_S^N ds = \sqrt{2}R \int_{-\infty}^{\infty} \operatorname{sech} t\, dt &= \sqrt{2}R \int_{-\infty}^{\infty} \frac{2e^t}{e^{2t}+1}\, dt\\ = 2\sqrt{2}R \tan^{-1}(e^t)\Big|_{-\infty}^{\infty} &= 2\sqrt{2}R\left(\frac{\pi}{2} - 0\right) = \sqrt{2}R\pi,\end{aligned}$$

which is $\sqrt{2}$ times the length of a meridian and for the Earth with circumference of $40,000$ km equals $l = \sqrt{2} \cdot 20,000 \approx 28,284$ km.

A heuristic method computing the length of a loxodrome is found in [3].

[1] J. Alexander, *Loxodromes: A rhumb way to go*, Mathematics Magazine, **77** (2004), 349-356.

[2] T. G. Feeman, *Portraits of the Earth. A Mathematician Looks at Maps*, Mathematical World, Vol. 18, American Mathematical Society, Providence, RI, 2002.

[3] J. Nord and E. Miller, *Mercator's rhumb lines: A multivariable application of arc length,* The College Math. J. Vol. **27**, No. 5 (1996), 384-387.

[4] J. F. Queiró, *Pedro Nunes e as linhas de rumo*, Gazeta de Matemática, **143** (July 2002), 42-47.

[5] Rhumb line. From Wikipedia, *https://en.wikipedia.org/wiki/Rhumb_line*

Editor's comments. Other ways of studying loxodromes and computing their lengths, as was done e. g. in [1], use the cylindrical projection of the sphere and Mercator's map (a conformal map preserving angles), on which meridians and the circles of latitude form a rectangular grid of vertical and horizontal lines, and loxodromes appear as straight lines making a constant angle with vertical meridians.

CC237. Find the volume of the portion of a unit cube that is at distance at most $\sqrt{2}$ from a specified corner.

Originally question 3 from the 2016 Science Atlantic Mathematics Competition.

We received three correct solutions. We present the solution of the Missouri State Problem Solving Group.

If we place the specified corner at the origin in three-space, the volume we seek is one eighth of the volume of the portion of a sphere of radius $\sqrt{2}$ centered at the origin that lies inside a cube of sidelength 2 centered at the origin. The portion of this sphere lying outside the cube is six (one for each face) congruent spherical caps of height $\sqrt{2}-1$.

The volume of a spherical cap of height h in a sphere of radius r is known to be

$$V = \frac{\pi h(3r-h)}{3}.$$

In our case,

$$V = \frac{\pi(\sqrt{2}-1)^2(3\cdot\sqrt{2}-(\sqrt{2}-1))}{3} = \frac{(4\sqrt{2}-5)\pi}{3}.$$

Therefore, the volume of the portion of the sphere lying inside the cube is

$$\frac{4}{3}\pi(\sqrt{2})^3 - 6\left(\frac{(4\sqrt{2}-5)\pi}{3}\right) = \frac{2(15-8\sqrt{2})\pi}{3}.$$

Thus the volume we want is

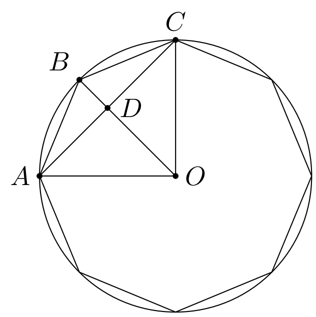
$$\frac{1}{8}\cdot\frac{2(15-8\sqrt{2})\pi}{3} = \frac{(15-8\sqrt{2})\pi}{12}.$$

CC238. For your sister's 8th birthday, you decided to make her a cake in the shape of a regular octagon. Since you couldn't find a cake tin in this shape, you used an 8 inch diameter round cake tin and trimmed off the sides in such a way that you achieved the largest regular octagon possible. The area you cut off to form this octagonal cake is of the form $a\pi - b$ where a and b are real numbers. What is $\frac{b}{a}$?

Originally question 6 from the UNM-PNM Statewide Mathematics Contest XLIX, first round.

We received five correct solutions. We present the solution of Ivko Dimitrić.

Clearly, the largest regular octagon (octagonal cake) is an octagon which is inscribed in the circle (circular tin) of radius $r = 4$. Join the center O of the circle with each of the eight vertices of the inscribed regular octagon. These radii divide the octagon into eight isosceles triangles with vertex angle at the center equal to $\pi/4 = 45°$.



Two of these neighboring triangles OAB and OBC form a kite with perpendicular diagonals that intersect at D as on the picture. The legs of isosceles right triangles OAD and OCD are easily computed to be $2\sqrt{2}$, each equal to half of the segment AC. The area of the kite is twice the area of $\triangle OAB$ and hence is equal to $4\cdot 2\sqrt{2} = 8\sqrt{2}$. The area of the trimmings is the area of the circle minus the sum of the areas of four kites and equals

$$\pi \cdot 4^2 - 4 \cdot 8\sqrt{2} = 16\pi - 32\sqrt{2},$$

so that

$$\frac{b}{a} = 32\sqrt{2}/16 = 2\sqrt{2}.$$

CC239. Find all integers $n > 1$ such that $4n + 9$ and $9n + 4$ are both perfect squares.

Originally question 8 from the UNM-PNM Statewide Mathematics Contest XLIX, First round.

We received fourteen correct solutions. Most were along the lines of the first solution. The second solution was provided by the Crux gnome.

The answer is $n = 28$.

Solution 1.

Let $4n + 9 = a^2$ and $9n + 4 = b^2$ with a and b positive. Elimination of n yields

$$65 = 9a^2 - 4b^2 = (3a - 2b)(3a + 2b).$$

Since $a > 3$ and $b > 2$, then $3a + 2b > 13$. The only possibility is that $3a + 2b = 65$ and $3a - 2b = 1$, from which $(a, b) = (11, 16)$ and $n = 28$.

Solution 2.

$4n+9$ is odd and so equal to $(2t+3)^2$ for some positive integer t. Hence $n = t(t+3)$ and there is a positive integer w for which

$$w^2 = 4(9n+4) = 36t^2 + 108t + 16 = (6t+9)^2 - 65.$$

Hence $65 = (6t+9)^2 - w^2 = (6t+9-w)(6t+9+w)$. Since the second factor must be 65 and the first 1, we are led to $t = 4$ and $n = 28$.

Editor's Comments. Konstantine Zelator considered the more general system $u^2n + v^2 = a^2$, $v^2n + u^2 = b^2$ where $u > v > 0$ is given. This leads to

$$u^4 - v^4 = u^2b^2 - v^2a^2 = (ub - va)(ub + va).$$

If $ub - va = u^2 - v^2$ and $ub + av = u^2 + v^2$, we get the trivial solution $n = 0$.

The choice $ub - va = 1$, $ub + va = u^4 - v^4$ yields

$$n = \frac{(u^4 - v^4 - 1)^2 - 4v^4}{4u^2v^2} = \frac{(u^4-1)^2 + (v^4-1)^2 - (2u^4v^4+1)}{4u^2v^2}.$$

When $u = v + 1$, this turns out to be an integer and we find that

$$n = 4(v^2+v+1), \qquad (v+1)^2n+v^2 = (2v^2+3v+2)^2, \qquad v^2n+(v+1)^2 = (2v^2+v+1)^2.$$

Zelator looked at the particular situation when u^2-v^2 and u^2+v^2 are both prime, in which case $u = v + 1$ and the solution for positive integer n is unique.

CC240. Big Sandy MacDonald and Little Sandy MacDonald take turns choosing positive integers to be the coefficients of a sixteenth degree polynomial

$$a_0 + a_1z + \cdots + a_{16}z^{16} .$$

The same coefficient may be used more than once. Little Sandy moves first, and wins the game if, at the end, the polynomial has a repeated root (real or complex), or two distinct real or complex roots ζ_1, ζ_2 with $|\zeta_1 - \zeta_2| \leq 1$. Find a winning strategy for Little Sandy and show that it works.

Originally question 8 from the 2016 Science Atlantic Mathematics Competition.

No solutions were submitted.

Editor's comments. The judges' solution was that Little Sandy should play first, putting any number as a_8, and thereafter play to make the polynomial a palindrome. Such a polynomial has roots in pairs $\{\zeta, 1/\zeta\}$, and hence eight roots within the open unit disc or at least nine on the closed unit disc. A geometric pigeonhole principle argument finishes the proof. However, we suspect that there may be better strategies that reduce the minimax distance! Consider this as an open-ended problem.

THE OLYMPIAD CORNER

No. 356

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC346. On considère une suite arithmétique non constante $(a_n)_{n\in\mathbb{N}}$ de nombres réels et une suite géométrique non constante $(g_n)_{n\in\mathbb{N}}$ de nombres réels, de manière que $a_1 = g_1 \neq 0$, $a_2 = g_2$ et $a_{10} = g_3$. Démontrer que pour tout entier strictement positif p, il existe un entier strictement positif m tel que $g_p = a_m$.

OC347. On considère le système suivant de 10 équations en 10 variables réelles $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Déterminer tous les 10-uplets $(v_1, v_2, \ldots, v_{10})$ qui sont les solutions du système.

OC348. On considère un triangle acutangle isocèle ABC ($AB = AC$) et une hauteur CD du triangle. Soit C_1 le cercle de centre B et de rayon BD et C_2 le cercle de centre C et de rayon CD. Sachant que C_2 coupe AC en K, le prolongement de AC en Z et C_1 en E et que la droite DZ coupe C_1 en M, démontrer que:

a) $\widehat{ZDE} = 45°$,

b) les points E, M et K sont alignés,

c) $BM \parallel EC$.

OC349. Déterminer toutes les fonctions f ($f : \mathbb{R} \to \mathbb{R}$) telles que

$$f(yf(x) - x) = f(x)f(y) + 2x$$

pour tous réels x et y.

OC350. Deux joueurs, A et B, jouent à un jeu à tour de rôle en commençant par A. Il y a 2016 jetons au départ et le joueur dont c'est le tour doit retirer s

jetons de la pile, $s \in \{2, 4, 5\}$. Le joueur qui ne peut plus jouer (c.-à-d. qui ne peut pas retirer un nombre réglementaire de jetons) perd la partie. Lequel des deux joueurs peut avoir une stratégie gagnante?

. . . .

OC346. Two real number sequences are given, one arithmetic $(a_n)_{n\in\mathbb{N}}$ and another geometric $(g_n)_{n\in\mathbb{N}}$, neither of them constant. These sequences satisfy $a_1 = g_1 \neq 0$, $a_2 = g_2$ and $a_{10} = g_3$. Prove that, for every positive integer p, there is a positive integer m, such that $g_p = a_m$.

OC347. Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

OC348. Triangle ABC is an acute isosceles triangle ($AB = AC$) and CD one altitude. Circle $C_2(C, CD)$ meets AC at K, AC produced at Z and circle $C_1(B, BD)$ at E. Line DZ meets circle (C_1) at M. Show that:

a) $\widehat{ZDE} = 45^\circ$.

b) Points E, M, K lie on a line.

c) $BM \parallel EC$.

OC349. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x) - x) = f(x)f(y) + 2x$$

for all $x,\ y \in \mathbb{R}$.

OC350. Two players, A (first player) and B, take alternate turns in playing a game using 2016 chips as follows: the player whose turn it is, must remove s chips from the remaining pile of chips, where $s \in \{2, 4, 5\}$. No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Which of the two players has a winning strategy?

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(6), p. 246–247.

OC286. There are four basketball players A, B, C, D. Initially the ball is with A. The ball is always passed from one person to a different person. In how many ways can the ball come back to A after **seven** moves? (For example, A passes to C who passes to B who passes to D who passes to A who passes to B who passes to C who passes to A.)

Originally problem 4 of the 2015 India National Olympiad.

We received 4 correct solutions and 1 incorrect submission. We present the solution by Gabriel Wallace.

We claim there are 546 ways. First, we examine the range of A. It is trivial to show that the minimum number of times person A has the ball is 2. Since no one can pass the ball to themselves, there has to be a space between each person, so the max number of times for A is 4. Then we have three cases.

Case 1: $A = 4$. Here we have three possibilities, as follows:

$$\begin{array}{cccccccc} A & - & A & - & A & - & - & A \\ A & - & - & A & - & A & - & A \\ A & - & A & - & - & A & - & A \end{array}$$

A blank space directly following an A can take 3 values: B, C, or D. A space after that would have two possibilities, whatever two letters that were not selected prior. So a 1-blank will have 3 possibilities and a 2-blank will have 6. Notice in all 3 permutations, there are two 1-blanks and one 2-blank. Summing everything, we have $3(3 \cdot 3 \cdot 3 \cdot 2) = 162$ ways for 4 A's.

Case 2: $A = 3$. Here we have four possibilities, as follows:

$$\begin{array}{cccccccc} A & - & A & - & - & - & - & A \\ A & - & - & A & - & - & - & A \\ A & - & - & - & A & - & - & A \\ A & - & - & - & - & A & - & A \end{array}$$

Directly following an A we still have 3 possible values. Since we have already fixed the A's, then any other space will have two possible values, whatever was not selected directly before. So here we have two lines with one 1-blank and one 4-blank, and two lines with one 2-blank and one 3 blank. Thus we have $2(3 \cdot 3 \cdot 2^3) + 2(3 \cdot 2 \cdot 3 \cdot 2^2) = 288$ ways for 3 A's.

Case 3: $A = 2$. Here we have one possibility, as follows:

$$\begin{array}{cccccccc} A & - & - & - & - & - & - & A \end{array}$$

Following the same logic as above, for this one 6-blank we have $3 \cdot 2^5 = 96$ ways for 2 A's.

Summing all the cases we have a total of $162 + 288 + 96 = 546$ ways.

OC287. Let
$$P(x) = ax^3 + (b-a)x^2 - (c+b)x + c$$
and
$$Q(x) = x^4 + (b-1)x^3 + (a-b)x^2 - (c+a)x + c$$
be polynomials of x with a, b, c non-zero real numbers and $b > 0$. If $P(x)$ has three distinct real roots x_0, x_1, x_2 which are also roots of $Q(x)$, then:

1. Prove that $abc > 28$,
2. If a, b, c are non-zero integers with $b > 0$, find all their possible values.

Originally problem 2 of the 2015 Greece National Olympiad.

We received 4 correct submissions. We present the solution by Ali Adnan.

Clearly the fourth root of $Q(x)$ must also be real. Let that root be x_3. Then from Viete's relations,

$$x_0x_1x_2 = -\frac{c}{a}, \qquad x_0x_1x_2x_3 = c,$$
$$x_0 + x_1 + x_2 = 1 - \frac{b}{a}, \qquad x_0 + x_1 + x_2 + x_3 = 1 - b.$$

From the first set of equations, $x_3 = -a$ and using this in the second set, we get $\frac{b}{a} = b - a$. Again from Viete's relation

$$x_0x_1 + x_1x_2 + x_2x_0 = -\frac{b+c}{a},$$
$$x_3(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_2x_0 = a - b.$$

The above two relations imply that

$$(-a) \cdot \left(1 - \frac{b}{a}\right) + \frac{-(b+c)}{a} = a - b \Rightarrow \frac{b}{a} + \frac{c}{a} = 2(b-a) \Rightarrow c = b.$$

So $abc = ab^2$, while

$$\frac{b}{a} = b - a \Rightarrow b = \frac{a^2}{a-1}.$$

(Note that the above relation implies that $a > 1$).

Thus $ab^2 = a^5(a-1)^{-2}$ ($= f(a)$, say). It is routine calculus to obtain $f'(a) = 0 \Rightarrow a = 5/3$ and check that $a = 5/3$ gives a minimum indeed for $f(a)$, $a > 1$. After that it is easily obtained that $f(5/3) > 28$, which proves part 1.

For part 2, it is seen that $b = \frac{a^2}{a-1}$ implies that

$$(a-1)|a^2 \Rightarrow (a-1)|(a^2-(a^2-1)) \Rightarrow a-1 = \pm 1.$$

But we also see that $a > 1$ and so $a - 1 = 1 \Rightarrow a = 2$. Thus $b = c = 4$. So in summary the only possible triplet of non-zero integral values of (a, b, c) with $b > 0$ is $(2, 4, 4)$.

OC288. Find all positive integers n such that for any positive integer a relatively prime to n, $2n^2 \mid a^n - 1$.

Originally problem 6 from day 2 of the 2015 Turkey National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

If $n \equiv 1 \pmod 2$, then since there are infinitely many primes, there exists integer a relatively prime to n such that $a \equiv 0 \pmod 2$, so $2n^2 \nmid a^n - 1$, which is a contradiction.

Let n_1 be the integer such that $2n_1 = n$. Therefore

$$2n^2 \mid a^n - 1 \implies 2(2n_1)^2 \mid a^{2n_1} - 1 \implies 2^3 n_1^2 \mid (a^{n_1}+1)(a^{n_1}-1).$$

Let $2^k \mid\mid n_1$. From Dirichlet's Theorem, there exists positive integer a relatively prime to $n = 2n_1$ such that $a \equiv 5 \pmod 8$, so

$$\begin{aligned} 2^3 n_1^2 \mid (a^{n_1}+1)(a^{n_1}-1) &= \left(\prod_{j=0}^{k}\left(a^{\frac{n_1}{2^j}}+1\right)\right)\left(a^{\frac{n_1}{2^k}}-1\right) \\ \implies 3+2k \le (k+1)+2 \implies k \le 0. \end{aligned}$$

Therefore $k = 0$.

If n_1 is not divisible by a prime, then $n_1 = 1$ and $n = 2$. For all positive integers a relatively prime to $2 = n$, $2n^2 = 2(2)^2 \mid a^2 - 1 = a^n - 1$, so this works.

Otherwise, n_1 is divisible by a prime.

If there exists a prime $p \ge 3$ and an integer m such that $p^m \mid\mid n_1$, then

$$p^{2m} \mid n_1^2 \mid a^{2n_1} - 1 \implies a^{2n_1} \equiv 1 \left(\bmod\ p^{2m}\right),$$

so since p^{2m} has primitive roots (well known), we have

$$p^{2m-1} \mid p^{2m-1}(p-1) = \phi(p^{2m}) \mid 2n_1 \implies 2m-1 \le m \implies m \le 1.$$

Therefore n is square free.

Let p_1 be the least prime such that $p_1 \mid n_1$. Since $k = 0$, $p_1 \ge 3$. By definition, $p_1 - 1 \nmid n_1$, and from above, $p_1(p_1-1) \mid 2n_1$, so $p_1 = 3$.

If $n = (2)(3)$, then from Euler's Theorem, for any positive integer a relatively prime to n, $a^n = a^{(3)(2)} = a^{\phi(3^2)} \equiv 1 \pmod{3^2}$, and since 2 satisfies for n, $2n^2 \mid a^n - 1$, so this works.

Otherwise, let $p_2 > p_1 = 3$ be the least prime such that $p_2 \mid n_1$. From primitive roots, $p_2(p_2 - 1) = \phi(p_2^2) \mid 2n_1$, so by the definition of p_2, $p_2 - 1 \mid 2p_1 = 2(3)$, so $p_2 = 7$.

If $n = (2)(3)(7)$, then from Euler's Theorem, for any positive integer a relatively prime to n, $a^n = a^{(7)(6)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2}$, and since $(2)(3)$ satisfies the condition for n, $2n^2 \mid a^n - 1$, so this works.

Otherwise, let $p_3 > p_2 = 7$ be the least prime such that $p_3 \mid n_1$. From primitive roots, $p_3(p_3 - 1) = \phi(p_3^2) \mid 2n_1$, so by the definition of p_3, $p_3 - 1 \mid 2p_1p_2 = 2(3)(7)$, so $p_3 = 43$.

If $n = (2)(3)(7)(43)$, then from Euler's Theorem, for any positive integer a relatively prime to n, $a^n = a^{(43)(42)} = a^{\phi(7^2)} \equiv 1 \pmod{7^2}$, and since $(2)(3)(7)$ satisfies the condition for n, $2n^2 \mid a^n - 1$, so this works.

If there exists a least prime $p_4 > p_3 = 43$ such that $p_4 \mid n_1$, then from primitive roots $p_4(p_4 - 1) = \phi(p_4^2) \mid 2n_1$, so by the definition of p_4,

$$p_4 - 1 \mid 2p_1p_2p_3 = 2(3)(7)(43),$$

so $p_4 \in \{44, 87, 130, 259, 302, 603, 904, 1807\}$, so p_4 is not prime which is a contradiction.

Therefore all possible n are

$$n \in \{2, (2)(3), (2)(3)(7), (2)(3)(7)(43)\} = \{2, 6, 42, 1806\}.$$

OC289. Let a, b, c, d, e be distinct positive integers such that $a^4 + b^4 = c^4 + d^4 = e^5$. Show that $ac + bd$ is a composite number.

Originally problem 5 from day 2 of the 2015 USAMO.

No submitted solutions.

OC290. Let $\triangle ABC$ be a scalene triangle and X, Y and Z be points on the lines BC, AC and AB, respectively, such that $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. The circumcircles of BXZ and CXY intersect at P. Prove that P is on the circle whose diameter has ends in the orthocenter H and in the barycenter G of $\triangle ABC$.

Originally problem 6 from day 2 of the 2015 Brazil National Olympiad.

We present the solution by Andrea Fanchini. There were no other submissions.

We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC.

Then the generic points X, Y and Z have absolute coordinates

$$X(0, v, 1-v), \qquad Y(1-w, 0, w), \qquad Z(u, 1-u, 0)$$

where u, v and w are parameters.

Equation of line AX is $(v-1)y + vz = 0$, then the $\angle AXB$ gives

$$S_{AXB} = S \cot AXB = S_C - a^2 v$$

Equation of line BY is $wx + (w-1)z = 0$, then the $\angle BYC$ gives

$$S_{BYC} = S \cot BYC = S_A - b^2 w$$

Equation of line CZ is $(u-1)x + uy = 0$, then the $\angle CZA$ gives

$$S_{CZA} = S \cot CZA = S_B - c^2 u$$

Now if $\angle AXB = \angle BYC = \angle CZA$ we have the system

$$\begin{cases} S_C - a^2 v = S_B - c^2 u \\ \\ S_A - b^2 w = S_B - c^2 u \end{cases} \quad \Rightarrow \quad \begin{cases} v = \frac{b^2 - c^2 + c^2 u}{a^2} \\ \\ w = \frac{b^2 - a^2 + c^2 u}{b^2} \end{cases}$$

Therefore points X and Y have coordinates that depend only on the parameter u:

$$X\left(0, \frac{b^2 - c^2 + c^2 u}{a^2}, \frac{2S_B - c^2 u}{a^2}\right), \qquad Y\left(\frac{a^2 - c^2 u}{b^2}, 0, \frac{b^2 - a^2 + c^2 u}{b^2}\right).$$

Equation of a generic circle is

$$a^2 yz + b^2 zx + c^2 xy - (x+y+z)(px + qy + rz) = 0.$$

If this circle passes through B, X, Z we obtain the three conditions

$$q = 0, \qquad r = b^2 - c^2 + c^2 u, \qquad p = c^2(1-u),$$

so this circle has equation

$$a^2 yz + b^2 zx + c^2 xy - (x+y+z)\left(c^2(1-u)x + (b^2 - c^2 + c^2 u)z\right) = 0$$

and then the circumcircle CXY has equation

$$a^2 yz + b^2 zx + c^2 xy - (x+y+z)\left((b^2 - a^2 + c^2 u)x + (2S_B - c^2 u)y\right) = 0$$

so the radical axis is the line r

$$r : 2(S_B - c^2 u)x + (c^2 u - 2S_B)y + (b^2 - c^2 + c^2 u)z = 0.$$

Point P (and X) is the intersection between the circle CXY and the radical axis

$$\begin{cases} b^2x^2 + a^2y^2 + 2S_Cxy + (c^2u - a^2)x + (c^2 - b^2 - c^2u)y = 0, \\ \\ y = \dfrac{(3S_B - S_C - 3c^2u)x + S_C - S_B + c^2u}{a^2}. \end{cases}$$

Solving it we obtain the coordinates of P:

$$\begin{aligned} &x - \text{coord} : 4S_B^2 + uc^2(S_C - 7S_B + 3uc^2) \\ &y - \text{coord} : a^2c^2 - 2S_AS_B + 2S_B^2 + uc^2(S_A - 6S_B - S_C + 3uc^2) \\ &z - \text{coord} : 2c^2S_B + uc^2(3uc^2 - 5S_B - S_A). \end{aligned}$$

This circle has as center the midpoint between $H(S_BS_C : S_CS_A : S_AS_B)$ and $G(1 : 1 : 1)$, that is

$$M_{HG}(3S_BS_C + S^2 : 3S_CS_A + S^2 : 3S_AS_B + S^2).$$

The radius ρ is the distance GM_{HG}

$$\rho^2 = \frac{S^2(S_A + S_B + S_C) - 9S_AS_BS_C}{36S^2},$$

so the circle with diameter HG has equation

$$a^2yz + b^2zx + c^2xy - \frac{2}{3}(x + y + z)(S_Ax + S_By + S_Cz) = 0.$$

Now we put the coordinates of P in this last equation and with a bit of algebra we can verify that this point is on the circle.

PROBLEM SOLVING 101

No. 1

Shawn Godin

In this new column we will look at solutions to problems that may be instructive to people interested in problem solving, but not necessarily experts. I am aiming at providing some interesting problems for secondary and undergraduate students to play with. The problems may also be of interest to high school teachers for their own recreation or for use in their classrooms. Problems that we will examine will not require knowledge beyond the high school level. If there is a topic that isn't usually in the pre-university curricula, but is understandable to a high school student and of interest to a problem solver, it will be introduced. I hope you enjoy this new column, and I welcome any feedback at `Shawn.Godin@ocdsb.ca`.

. . . .

In the Ross Honsberger Commemorative issue, I presented the problems, and a solution to problem #5, from Assignment #1 of the course C & O 380 that I took with Professor Honsberger back in 1986 (***Crux*** 43(4), p. 151–153). This month, we will look at Assignment #2.

C&O 380 — **Assignment #2** — **Due: February 26, 1986**

#1. P is a point inside the square $ABCD$ such that $AP = 1$, $BP = 2$, and $CP = 3$. Determine the size of angle APB.

#2. P is a point chosen at random inside an equilateral triangle ABC. What is the probability that the three perpendiculars from P to the sides of ΔABC can be arranged to form a triangle?

#3. Two people agree to meet for lunch at their favourite restaurant. Each agrees to wait 15 minutes for the other, after which time he will leave. If each chooses his time of arrival at random between noon and 1 o'clock, what is the probability of a meeting taking place?

#4. P is a point inside equilateral ΔABC such that $AP = 3$, $BP = 4$, and $CP = 5$. Determine the length of the side of ΔABC.

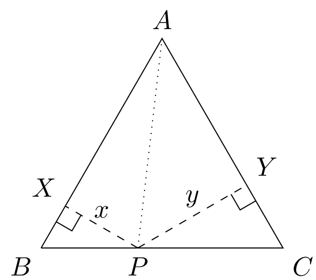
#5. Prove the following theorem (known as Archimedes' Theorem of the Broken Chord):

M is the midpoint of the arc AB of a circle that is cut off by chord AB and C is an arbitrary point on this arc. D is the foot of the perpendicular from M upon the longer of AC and BC. Then the point D bisects the broken path ACB.

We will look at problem #2. To start, we will prove the following lemma:

Lemma: If a point, P, is chosen on one side of an equilateral triangle and perpendiculars are dropped from P to the other sides, then the sum of the lengths of the two perpendiculars is equal to the height of the triangle.

Proof: Let P be a point on side BC of equilateral ΔABC. The perpendiculars from P to AB and AC meet in X and Y respectively. Let s and h be the side length and the height of triangle ABC, and let x and y be the lengths of PX and PY, respectively.

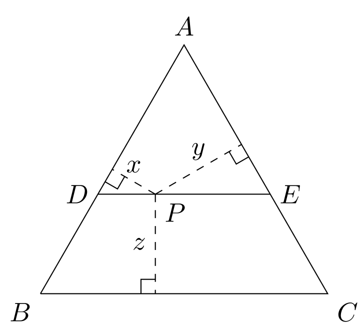


Then

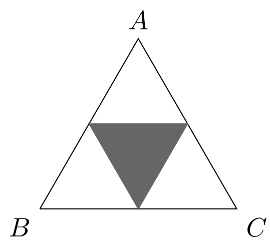
$$\begin{aligned}[ABC] &= [ABP] + [APC] \\ \frac{sh}{2} &= \frac{sx}{2} + \frac{sy}{2} \\ \frac{s}{2}(h) &= \frac{s}{2}(x+y) \\ h &= x + y\end{aligned}$$

where $[ABC]$ represents the area of ΔABC. $\square$

Returning to the original problem, let x, y and z represent the distances from P to AB, AC and BC respectively. Draw the segment through P, parallel to BC meeting AB and AC at D and E respectively, as shown in the diagram below.

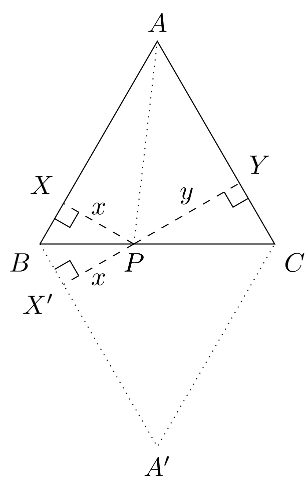


Thus x, y, and z form a triangle if the triangle inequalities are satisfied. In particular, we need $x + y > z$. Since $x + y$ is the height of ΔADE by our lemma, and z is the distance from DE to BC, we need DE to be *below* the segment joining the midpoints of AB and AC. Since we also need $x + z > y$ and $y + z > x$, we can use the symmetry of an equilateral triangle to show that x, y and z will form a triangle only if P is inside the triangle formed by connecting the midpoints of the sides of ΔABC as shown in the diagram below.



We can easily show that the four small triangles are congruent, so the desired probability is $\frac{1}{4}$.

In this problem, as in many geometry problems, we benefited by adding a line segment, DE, to our diagram. We also benefitted by considering a seemingly unrelated measurement: the area. We could have proved the lemma by reflecting in BC. The point X' is the image of X on the image $A'B$ of AB, and $X'Y = XP + PY$ is the distance between the parallel lines $A'B$ and AC. We can readily see that the length of $X'Y$ is equal to the heights of both ΔABC and $\Delta A'BC$.



My thanks to *Crux* editor Chris Fisher for suggesting the reflection. Originally, when I had solved this problem I had used some similarity arguments to give the same result. That solution was more algebraic, but the solutions presented above seems more satisfying to me. Always be on the lookout for those solutions that, as Paul Erdős used to say, were from "The Book". Enjoy the rest of the problems from assignment #2.

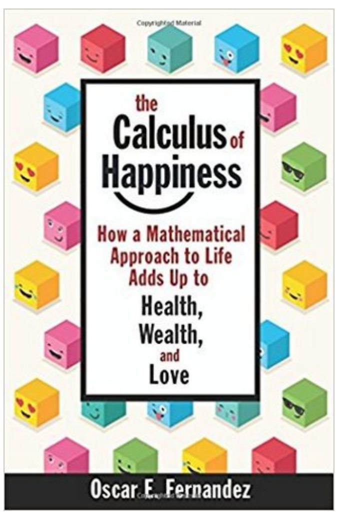
BOOK REVIEWS

Robert Bilinski

The Calculus of Happiness by O.E. Fernandez
ISBN 978-0-691-16863-0, 159 pages
Published by Princeton University Press, 2014

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

O.E. Fernandez teaches mathematics at Wellesley College, a higher education institution near Boston. His research interest is in geometric mechanics. As a teacher, he has started outreach programs which probably helped him create material for his two general interest books, with *The Calculus of Happiness* being his most recent.

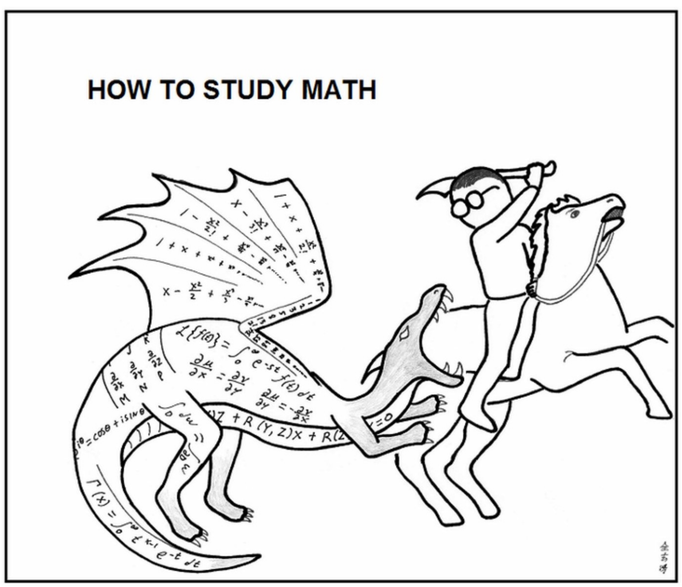


The name of the game here is modelling the world and finding insight through math. The models are taken from research papers in different fields pertaining to our lives: nutrition, finance and psychology to name the inspirations for each of the three parts of this book (each subdivided into chapters). The first chapter is longer since it contains an explanation, in laymen's terms, of how mathematical algebraic models are built and validated using statistical methods. Here we find mathematical problem solving of a different kind than we usually find in ***Crux*** , but I invite our readers on a trip through common interests. After all, not only is Fernandez a relatively new voice in the field of popular mathematics, he has also chosen a path less trodden which makes the attraction of his book that much greater for its novelty.

Why name this book the "Calculus of Happiness"? Each chapter tries to show that mathematics permits us to make better health choices, earn more money or understand how to make a relationship last, which ultimately should lead to a higher level of happiness. The models vary in complexity, multiple linear models in the first part, annuities, logarithmic and exponential models in the second part and a mix of multiplicative, differential and dynamical models in the third part. All models are given with references to the articles from which they are taken. A few high school level calculations are made (mostly in the first part with the multiple linear regression models) as this is a general interest book. A key point is that the author is up to date in his scientific literature, citing models built in 1999 and after in his book.

The problem solving gives life choices as solutions; namely, concrete choices one can make in their consumption of food, in their interaction with others or the management of their finances. I do not want to expand too much on the conclusions as it would entail too many spoilers. Naturally, if one has taken a financial mathematics course, they probably saw the entire second part of the book in the first few minutes of the course. I doubt any mathematics major has ever seen the contents of the other two parts in their studies or even in a mathematical conference.

An interesting read, approachable to anyone with a high school education and, above all, food for thought. How should we to make informed lifestyle choices? A deep question with implications for each of us. Here's a look at the work of people trying to find a reasoned answer to the question with some of the conclusions we can gather from their findings discussed by the author. What will ***Crux*** readers who venture into those same equations find? I am curious to find out and hope you share some of your thoughts on this and other books I've covered in the last few years. Good reading!



Don't just read it; fight it!

--- Paul R. Halmos

Quadratic Congruences in Olympiad Problems

Paul Stoienescu and Tudor-Dimitrie Popescu

In this note, we will present some Olympiad problems which can be solved using quadratic congruence arguments.

1 Definitions and Properties

1.1 The Legendre Symbol

Given a prime number p and an integer a, Legendre's symbol $\left(\frac{a}{b}\right)$ is defined as:

$$\left(\frac{a}{b}\right) = \begin{cases} 0 & \text{if } a \text{ is divisible by } p; \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1 & \text{otherwise;} \end{cases} \tag{1}$$

Property 1: If $a \equiv b \pmod{p}$ and ab is not divisible by p, then $\left(\frac{a}{b}\right) = \left(\frac{b}{p}\right)$.

Property 2: Legendre's symbol is multiplicative, *i.e.* $\left(\frac{ab}{p}\right) = \left(\frac{a}{b}\right)\left(\frac{b}{p}\right)$ for all integers a, b and prime numbers $p > 2$.

Property 3: If $p \neq 2$, then $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

Property 4: If $p \neq 2$, then $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

Property 5 ***(Euler's Criterion)****:* If $p \nmid a$, $p \neq 2$, then $a^{\frac{p-1}{2}} = \left(\frac{a}{b}\right)$.

Property 6 ***(Quadratic Reciprocity Law or Gauss's Law)****:* If p, q are distinct odd prime numbers, then

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) \cdot (-1)^{\frac{(p-1)\cdot(q-1)}{4}}.$$

Gauss's law, combined with the properties of the Legendre symbol, proves that any Legendre symbol can be calculated. This makes it possible to determine whether the quadratic equation $x^2 \equiv a \pmod{p}$, where p is an odd prime, has a solution. Moreover, the solution can be found using quadratic residues.

Lemma. Let p be an odd prime. There are $\frac{p-1}{2}$ quadratic residues in the set $\{1, 2, 3 \ldots, p-1\}$.

A quick application: Find $\left(\frac{30}{211}\right)$.

Solution. We have
$$\left(\frac{30}{211}\right) = \left(\frac{2}{211}\right)\left(\frac{3}{211}\right)\left(\frac{5}{211}\right).$$

Since $211 \equiv 3 \pmod 8$, we get $\left(\frac{2}{211}\right) = -1$.

For $\left(\frac{3}{211}\right)$, we apply quadratic reciprocity law to obtain
$$\left(\frac{3}{211}\right) = \left(\frac{211}{3}\right)(-1)^{105} = -\left(\frac{211}{3}\right) = -\left(\frac{1}{3}\right) = -1.$$

Finally, we have $\left(\frac{5}{211}\right) = \left(\frac{211}{5}\right)(-1)^{105\cdot 2} = \left(\frac{211}{5}\right) = \left(\frac{1}{5}\right) = 1.$

Combining all of the above, we obtain $\left(\frac{30}{211}\right) = (-1)(-1)(+1) = 1.$ □

1.2 Quadratic Congruences with Composite Moduli

Let a be an integer and b an odd number, and let $b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ be the factorization of b into primes. Jacobi's Symbol $\left(\frac{a}{b}\right)$ is defined as:
$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)^{\alpha_1}\left(\frac{a}{p_2}\right)^{\alpha_2} \dots \left(\frac{a}{p_n}\right)^{\alpha_n} \tag{2}$$

Jacobi's Symbol has almost the same properties as Legendre's with a few changes: it does not have Property 5, while in Properties 3 and 4, p can be an odd integer and in Property 6, p, q can be distinct odd integers with no common divisors.

It is easy to see that $\left(\frac{a}{b}\right) = -1$ implies that a is a quadratic nonresidue $\pmod p$. Indeed, if $\left(\frac{a}{b}\right) = -1$, then by definition $\left(\frac{a}{p_i}\right) = -1$ for at least one $p_i | b$; hence a is a quadratic nonresidue modulo p_i. The converse is false as we can see in the following example:
$$\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) = (-1)(-1) = 1,$$

but 2 is not a quadratic residue modulo 15. As such we have the following.

Theorem. *Let a be an integer and b be a positive integer, and let $b = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ be the factorization of b into primes. Then a is a quadratic residue modulo b if and only if a is a quadratic residue modulo $p_i^{\alpha_i}$, for each $i = 1, 2, \dots, n$.*

2 Warm-Up Problems

Problem 1. Let a and b be positive integers such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares? (IMO 1996.)

Solution. Let $15a+16b=k^2$ and $16a-15b=l^2$. Then

$$a=\frac{15k^2+16l^2}{481},\ b=\frac{16k^2-15l^2}{481},\quad k,l\in N^*.$$

Since $481=13\cdot 37$, we have

$$15k^2+16l^2\equiv 0\pmod{13},\ 2k^2\equiv -3l^2\pmod{13},\ k^2\equiv 5l^2\pmod{13}.$$

We then obtain $\left(\frac{5}{13}\right)=-1$, which implies that $13|l$ and $13|k$. Note that

$$\begin{aligned}15k^2+16l^2&\equiv 0\pmod{37},\\ 32l^2&\equiv -30k^2\pmod{37},\\ -5l^2&\equiv -30k^2\pmod{37},\\ l^2&\equiv 6k^2\pmod{37}.\end{aligned}$$

Combined with the fact that $\left(\frac{6}{37}\right)=-1$, we get that $37|k$ and $37|l$. The least possible value for l is $13\cdot 37=481$. We can take $k=l=481$ and thus we will get $a=31\cdot 481$, $b=481$. □

Problem 2. Prove that 2^n+1 has no prime factors of the form $8k+7$. (Vietnam Team Selection Test 2004.)

Solution. Assume that there exists a prime p such that $p|2^n+1$ and $p\equiv 7\pmod 8$.

If n is even, $\left(\frac{-1}{p}\right)=1$. But

$$\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}=-1,$$

since $p\equiv 3\pmod 4$, so we have reached a contradiction.

If n is odd, we get that $2^{n+1}\equiv -2\pmod p$, so -2 is a quadratic residue modulo p, since $n+1$ is even, so $\left(\frac{-2}{p}\right)=1$. But

$$\left(\frac{-2}{p}\right)=(-1)^{\frac{p^2-1}{8}}\cdot(-1)^{\frac{p-1}{2}}=-1,$$

which yields a contradiction. □

Problem 3. Let p be a prime number such that $p \equiv 1 \pmod 4$. Calculate

$$S = \sum_{k=1}^{\frac{p-1}{2}} \left(\left[\frac{2k^2}{p} \right] - 2 \cdot \left[\frac{k^2}{p} \right] \right).$$

Solution. Let $r_1, r_2, \ldots, r_{\frac{p-1}{2}}$ be the quadradic residues $\pmod p$. First, observe that the sum is equivalent to

$$\sum_{i=1}^{\frac{p-2}{2}} 2 \left(\left\{ \frac{r_i}{p} \right\} - \left\{ \frac{2r_i}{p} \right\} \right).$$

Each term $2\{\frac{r_i}{p}\} - \{\frac{2r_i}{p}\}$ is 0 if $r_i \leq \frac{p-1}{2}$, and 1 if $r_i > \frac{p-1}{2}$. So S is the number of quadratic residues which are greater than $\frac{p-1}{2}$. Since $p \equiv 1 \pmod 4$, if r_i is a quadratic residue, then so is $p - r_i$, so half of the integers greater than $\frac{p-1}{2}$ are quadratic residues $\Rightarrow S = \frac{p-1}{4}$. $\square$

Problem 4. Let $m,\ n \geq 3$ be positive odd integers. Prove that $2^m - 1$ doesn't divide $3^n - 1$.

Solution. Here we will use Jacobi's Symbol. Suppose that $2^m - 1$ divides $3^n - 1$. Let $x = 3^{\frac{n-1}{2}}$. We have that $3x^2 \equiv 1 \pmod{2^m - 1}$, so $(3x)^2 \equiv 3 \pmod{2^m - 1}$ and hence $\left(\frac{3}{2^m - 1}\right) = 1$. Using quadratic reciprocity, we get that

$$1 = \left(\frac{3}{2^m - 1}\right) = \left(\frac{2^m - 1}{3}\right)(-1)^{\frac{2^m - 2}{2}} \Rightarrow \left(\frac{2^m - 1}{3}\right) = -1,$$

contradiction due to the fact that $2^m - 1 \equiv 1 \pmod 3$. $\square$

3 Harder Problems

Problem 5. For a positive integer a, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \ldots, y_k$ are all prime. (2013 Romanian Masters of Mathematics.)

Solution. We will prove that the answer is 2. Suppose that there exists a such that $k \geq 3$. The numbers $2^a - 1$, $2^{2a+1} - 1$, $2^{4a+3} - 1$ are primes, so the numbers $a, 2a+1, 4a+3$ are primes (this is because of the fact that if $2^M - 1$ is prime, then M is also a prime; otherwise, if there existed a natural number d such that $d|M$, then $2^d - 1$ would divide $2^M - 1$). Let's use Euler's Criterion:

$$2^{\frac{4a+3-1}{2}} \equiv \left(\frac{2}{4a+3}\right) \pmod{4a+3} \Rightarrow 2^{2a+1} \equiv \left(\frac{2}{4a+3}\right) \pmod{4a+3}.$$

Since $2^{2a+1}-1$ is prime, then $2^{2a+1} \not\equiv 1 \pmod{4a+3}$, otherwise $2^{2a+1} = 4a+4$ and that will lead to $a=1$, false. Hence we have

$$\left(\frac{2}{4a+3}\right) = -1 \Rightarrow -1 = (-1)^{\frac{(4a+2)(4a+4)}{8}} = (-1)^{(2a+1)(a+1)},$$

which implies that $a+1$ is odd. But a is prime so $a=2$. If $a=2$, we have that $2^{11}-1 = 23 \cdot 87$ is not prime, contradiction. So we get that the answer is 2 and it is achieved for $a=2$. □

Problem 6. Prove that there are infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$. (IMO 2008.)

Solution. Let p be a prime, $p=8k+1$. Note that $4^{-1} \equiv 6k+1 \pmod p$. Choose $n = 4k-a$, $0 \le a < 4k$. Then

$$\left(\frac{p-1}{2}-a\right)^2 + 1 \equiv 0 \pmod p \qquad \Longleftrightarrow \qquad 4^{-1}+a+a^2+1 \equiv 0 \pmod p\,,$$

so

$$a(a+1) \equiv -6k-2 \equiv 2k-1 \pmod p\,.$$

But $a(a+1)$ is even and positive, so $a(a+1) \ge 10k$. We have that

$$(a+1)^2 > a(a+1) \ge 10k > p,$$

so

$$n = \frac{p+1}{2} - (a+1) < \frac{p+1}{2} - \sqrt{p} < \frac{p+1}{2} - \sqrt{2n},$$

so $2n+2\sqrt{2n}-1 > p$. Note that this result is a bit stronger than the initial inequality. □

Problem 7. Suppose that the positive integer a is not a perfect square. Then $\left(\frac{a}{b}\right) = -1$ for infinitely many primes p.

Solution. Let us assume that the claim is false. This means that there exists a number r such that for every prime $q > r$, $\left(\frac{a}{q}\right) = 1$. Because a is not a perfect square, we can write $a = x^2 p_1 p_2 \dots p_k$, where $p_1, p_2 \dots p_k$ are primes in increasing order. Take a prime $p > r$, $p \equiv 5 \pmod 8$. We have that

$$\left(\frac{a}{b}\right) = \left(\frac{p_1}{p}\right)\left(\frac{p_2}{p}\right)\dots\left(\frac{p_k}{p}\right).$$

If p_i is odd, then $\left(\frac{p_i}{p}\right) = \left(\frac{p}{p_i}\right)$, by the Quadratic Reciprocity Law.

If $p_1 = 2$, $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = -1$. So

$$\left(\frac{a}{b}\right) = \left(\frac{p}{p_1}\right)\dots\left(\frac{p}{p_k}\right) \quad \text{or} \quad \left(\frac{a}{b}\right) = -\left(\frac{p}{p_2}\right)\dots\left(\frac{p}{p_k}\right).$$

We can take $r_2, r_2, \ldots, r_k$ residues $\pmod{p_2, p_3, \ldots, p_k}$ such that $\left(\frac{r_2}{p_2}\right) \ldots \left(\frac{r_k}{p_k}\right)$ is 1 or -1 as we wish. By the Chinese Remainder Theorem, there are infinitely many numbers t with

$$t \equiv 5 \pmod 8, \ t \equiv r_i \pmod{p_i}, \ 2 \leq i \leq k.$$

Now we look at the progression $t + l8p_2p_3 \ldots p_k$. By Dirichlet's Theorem, there are infinitely many primes q in this sequence and we take $q > r$. Note that we have $\left(\frac{a}{q}\right) = 1$, but as already discussed, we can select $r_2, r_3, \ldots, r_k$ such that $\left(\frac{a}{q}\right) = -1$, a contradiction. $\square$

Problem 8. Let S be the set of all rational numbers expressible in the form

$$\frac{(a_1^2 + a_1 - 1)(a_2^2 + a_2 - 1) \ldots (a_n^2 + a_n - 1)}{(b_1^2 + b_1 - 1)(b_2^2 + b_2 - 1) \ldots (b_n^2 + b_n - 1)}$$

for some positive integers $n, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. Prove that there is an infinite number of primes in S. (2013 Romanian IMO Team Selection Test.)

Solution. Clearly, S is closed under multiplication and division: if r and s are in S, so are rs and $\frac{r}{s}$. Any prime number which is 0, 1 or 4 (mod 5) is in S. Since $2^2 + 2 - 1 = 5$, we know that 5 is in S. Now we will prove by induction that every prime number which is 1 or 4 $\pmod 5$ is in S. Of course, $11 = 3^2 + 3 - 1$ and $19 = 4^2 + 4 - 1$ are in S. Denote $p_1, p_2, \ldots$ the sequence of primes of this form in increasing order, and suppose that $p_1, p_2, \ldots, p_{n-1}$ are in S. We will show that p_n is also in S. Because 5 is a quadratic residue $\pmod{p_n}$, there exists a number x such that

$$p_n | (2x + 1)^2 - 5 \Rightarrow p_n | x^2 + x - 1$$

and we can choose x such that $2x + 1 < p_n$. Hence, p_n^2 does not divide $x^2 + x - 1$. Note that any prime which divides $x^2 + x - 1$ is $0, 1, 4 \pmod 5$ (if q is a prime such that $q | x^2 + x - 1$, then $q | (2x + 1)^2 - 5$, so 5 is a quadratic residue $\pmod q$, so using Gauss's law we get that q is $0, 1, 4 \pmod 5$).

Also, if $q | x^2 + x - 1$, then $q < p_n$, so $x^2 + x - 1$ is a product of primes which are among $p_1, p_2, \ldots, p_n$. Let $x^2 + x - 1 = tp_n$, where t is in S and so p_n is in S (since $p_n = \frac{x^2+x-1}{t}$ and $\frac{x^2+x-1}{1^2+1-1} = x^2 + x - 1$ is in S). The induction step is complete. $\square$

4 Exercises

Exercise 1. Let $p \geq 3$ be a prime number. Prove that the least quadratic nonresidue $\pmod p$ is less than $\sqrt{p} + 1$.

Exercise 2. Let p be a prime number such that $p \equiv 1 \pmod 4$. Prove that the equation $x^p + 2^p = p^2 + y^2$ doesn't have any solutions in natural numbers.

Exercise 3. Prove that there are only finitely many positive integers n such that

$$\left(\frac{n}{1}+1\right)\left(\frac{n}{2}+2\right)\dots\left(\frac{n}{n}+n\right)$$

is an integer. (2016 Danube Mathematical Competition, proposed by Adrian Zahariuc.)

Exercise 4. Let p be a prime of the form $4k+1$ such that $2^p \equiv 2 \pmod{p^2}$. Prove that there is a prime number q, divisor of 2^p-1, such that $2^q > (6p)^p$. (Mathematical Reflections.)

Exercise 5. If m is a positive integer, show that 5^m+3 has neither a prime divisor of the form $p=30k+11$ nor of the form $p=30k-1$. (Mathematical Reflections.)

Exercise 6. Solve the equation $p^2-pq-q^3=1$ in prime numbers. (2013 Tuymaada International Olympiad, Junior League, proposed by A. Golovanov.)

Exercise 7. Solve in natural numbers: $10^n+89=x^2$.

Exercise 8. Let p be an odd prime such that $p \equiv 3 \pmod 8$. Find all pairs of integers (x,y) that satisfy $y^2=x^3-p^2x$, where x is even.

Exercise 9. Let a be a positive integer such that for each positive integer n the number $a+n^2$ can be written as a sum of two squares. Prove that a is a square. (Mathematical Reflections.)

Exercise 10. Let $p=4k+3$ be a prime number. Find the number of different residues $\pmod p$ of $(x^2+y^2)^2$, where $(x,p)=(y,p)=1$. (2007 Bulgarian IMO Team Selection Test.)

Exercise 11. Let p be an odd prime congruent to 2 modulo 3. Prove that at most $p-1$ members of the set $\{m^2-n^3-1 \mid 0<m,\ n<p\}$ are divisible by p. (1999 Balkan Mathematical Olympiad.)

Exercise 12. Let q be an odd prime and r a positive integer such that q does not divide r, $r \equiv 3 \pmod 4$ and $\left(\frac{-r}{q}\right)=1$. Prove that $4qk+r$ does not divide q^n+1 for any k,n positive integers.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er avril 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4271. *Proposé par Hung Nguyen Viet, avec variation venant de l'éditeur.*

a) Soient a, b et c des nombres réels non nuls tels que

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1.$$

Démontrer que

$$\sqrt{\frac{(b+c)^2}{a^4}+\frac{(c+a)^2}{b^4}+\frac{(a+b)^2}{c^4}}$$

est une fonction rationnelle de a, b et c.

b) (*Ajout suggéré par l'éditeur.*) Démontrer que l'équation

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$$

n'a aucune solution rationnelle ou démontrer le contraire.

4272. *Proposé par Václav Konečný.*

Soit A un point sur le cercle unitaire et soit r un nombre réel tel que $0 < r < \frac{1}{2}$. Soient $\alpha = \angle BAC$ et $\alpha' = \angle B'AC'$, où ABC et $AB'C'$ sont deux triangles isocèles à l'intérieur du cercle unitaire, avec sommet A et dont les rayons de cercles inscrits ont la même valeur r. Déterminer $\sin(\alpha/2) + \sin(\alpha'/2)$.

4273. *Proposé par Ion Nedelcu et Leonard Giugiuc.*

Soit n un entier plus grand ou égal à 3. Démontrer que pour tous nombres réels $a_i \geq 1, i = 1, 2, \ldots, n$ l'inégalité suivante tient

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \leq n^2 + \sum_{1\leq i<j\leq n}|a_i-a_j|.$$

4274. *Proposé par Michel Bataille.*

Soit P un point à l'intérieur du triangle ABC et soient L, M et N des points à l'intérieur des segments PA, PB et PC respectivement. Définissons α, β, γ par $\overrightarrow{PL} = \alpha\overrightarrow{LA}$, $\overrightarrow{PM} = \beta\overrightarrow{MB}$, $\overrightarrow{PN} = \gamma\overrightarrow{NC}$. Supposant qu'au moins un de α, β, γ est différent de 1, construire le centre de masse de A, B et C avec poids α, β, γ, à l'aide d'une règle rectifiée.

4275. *Proposé par Leonard Giugiuc.*

Soient a, b et c des nombres réels non négatifs tels que $a + b + c = 3$. Déterminer la meilleure valeur possible de k pour laquelle l'inégalité suivante tient

$$\left(\frac{ab + bc + ca}{3}\right)^k (ab + bc + ca - abc) \le 2.$$

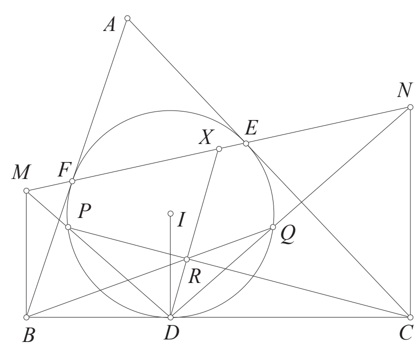
4276. *Proposé par Daniel Sitaru.*

Soit P un point à l'intérieur du triangle ABC et soient $PA = x, PB = y$ et $PC = z$. Démontrer que

$$27(ax + by - cz)(by + cz - ax)(cz + ax - by) \le (ax + by + cz)^3.$$

4277. *Proposé par Tran Quang Hung.*

Soit ABC un triangle dont le cercle inscrit (I) touche BC, CA et AB en D, E et F respectivement. De plus, M et N se trouvent sur EF de façon à ce que BM et CN sont perpendiculaires à BC. Aussi, DM et DN intersectent (I) de nouveau en P et Q. Enfin, BQ intersecte CP en R. Démontrer que DR bissecte MN.



4278. *Proposé par Lorian Saceanu.*

Pour x, y et z des nombres réels positifs, démontrer que

$$\sqrt{\frac{y+z}{x}} + \sqrt{\frac{x+y}{z}} + \sqrt{\frac{z+x}{y}} = \sqrt{\frac{(y+z)(z+x)(x+y)}{xyz}} + \frac{2(x+y+z)}{\sqrt{x(y+z)} + \sqrt{y(x+z)} + \sqrt{z(x+y)}}.$$

4279. *Proposé par Leonard Giugiuc.*

Soit ABC un triangle aigu. Démontrer que

$$\tan A + \tan B + \tan C \geq 4\left(\frac{1}{\sqrt{3}+\cot A} + \frac{1}{\sqrt{3}+\cot B} + \frac{1}{\sqrt{3}+\cot C}\right).$$

4280. *Proposé par Mihaela Berindeanu.*

Déterminer les solutions entières de $x^3 + y^3 - 6xy = 2^z - 8$.

. . . .

4271. *Proposed by Hung Nguyen Viet, supplemented by the Editorial Board.*

a) Let a, b, c be nonzero real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1.$$

Prove that

$$\sqrt{\frac{(b+c)^2}{a^4} + \frac{(c+a)^2}{b^4} + \frac{(a+b)^2}{c^4}}$$

is a rational function of a, b, c.

b) (*Suggested by the Editorial Board*). Prove or disprove that the equation

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$$

has no rational solution.

4272. *Proposed by Vaćlav Konečný.*

Let A be a fixed point on the unit circle and let r be a real number such that $0 < r < \frac{1}{2}$. Let $\alpha = \angle BAC$ and $\alpha' = \angle B'AC'$, where ABC and $AB'C'$ are two isosceles triangles with apex A and inradius r that are inscribed in the unit circle. Find $\sin(\alpha/2) + \sin(\alpha'/2)$.

4273. *Proposed by Ion Nedelcu and Leonard Giugiuc.*

Let n be an integer greater than or equal to 3. Prove that for any real numbers $a_i \geq 1, i = 1, 2, \ldots, n$ we have

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \leq n^2 + \sum_{1 \leq i < j \leq n} |a_i - a_j|.$$

4274. *Proposed by Michel Bataille.*

Let P be a point interior to a triangle ABC and L, M, N be points interior to the line segments PA, PB, PC, respectively. Define α, β, γ by $\overrightarrow{PL} = \alpha\overrightarrow{LA}$, $\overrightarrow{PM} = \beta\overrightarrow{MB}$, $\overrightarrow{PN} = \gamma\overrightarrow{NC}$. Assuming that at least one of α, β, γ is different from 1, construct with straightedge alone the center of mass of A, B, C with respective masses α, β, γ.

4275. *Proposed by Leonard Giugiuc.*

Let a, b and c be nonnegative real numbers such that $a + b + c = 3$. Find the best possible k for which the following inequality holds:

$$\left(\frac{ab + bc + ca}{3}\right)^k (ab + bc + ca - abc) \leq 2.$$

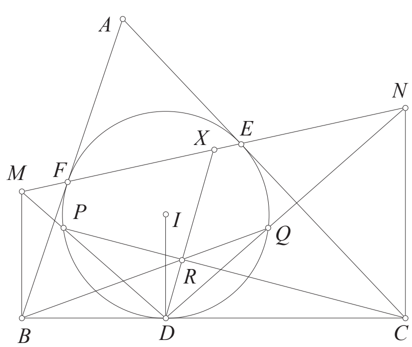
4276. *Proposed by Daniel Sitaru.*

Let P be a point on the interior of a triangle ABC and let $PA = x, PB = y$ and $PC = z$. Prove that

$$27(ax + by - cz)(by + cz - ax)(cz + ax - by) \leq (ax + by + cz)^3.$$

4277. *Proposed by Tran Quang Hung.*

Let ABC be a triangle with incircle (I) touches BC, CA and AB at D, E and F, respectively. Suppose M and N lie on EF such that BM and CN are perpendicular to BC. Finally, suppose DM and DN intersect (I) again at P and Q, respectively, and that BQ cuts CP at R. Prove that DR bisects MN.



4278. *Proposed by Lorian Saceanu.*

For any positive real numbers x, y and z, show that

$$\sqrt{\frac{y+z}{x}} + \sqrt{\frac{x+y}{z}} + \sqrt{\frac{z+x}{y}} = \sqrt{\frac{(y+z)(z+x)(x+y)}{xyz}} + \frac{2(x+y+z)}{\sqrt{x(y+z)} + \sqrt{y(x+z)} + \sqrt{z(x+y)}}.$$

4279. *Proposed by Leonard Giugiuc.*

Let ABC be an acute angled triangle. Show that

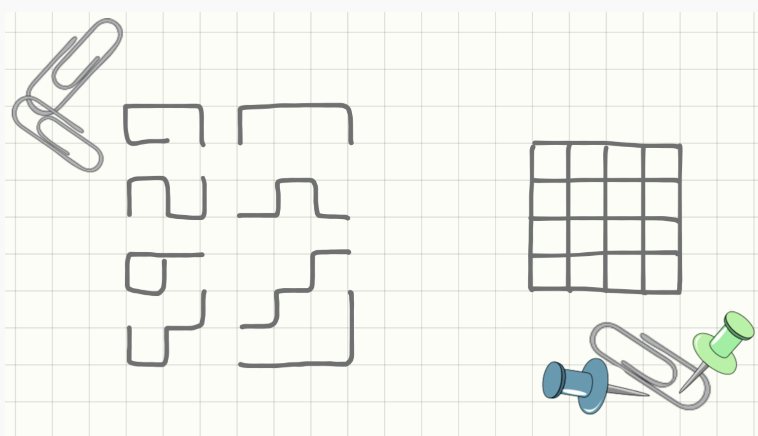
$$\tan A + \tan B + \tan C \geq 4\left(\frac{1}{\sqrt{3}+\cot A} + \frac{1}{\sqrt{3}+\cot B} + \frac{1}{\sqrt{3}+\cot C}\right).$$

4280. *Proposed by Mihaela Berindeanu.*

Solve the following equation in integers: $x^3 + y^3 - 6xy = 2^z - 8$.

Square lattice

Take eight pieces of wire of length five units each and bend them as follows (you can use regular paper clips):



Your mission, should you choose to accept it, is to put them all together to form a 4×4 square lattice without gaps and overlaps.

Can you do it with five bent pieces of length eight units each?

Puzzle by Nikolai Avilov (first appeared on the 17th All-Russian Olympiad in 1983).

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 42(8), p. 355–358.

4171. *Proposed by J. Chris Fisher.*

In triangle ABC, if L and M are the feet of the perpendiculars dropped from vertex A to the internal and external bisectors of the angle at B, while L' and M' are the feet of the perpendiculars from the orthocenter to those lines, prove that the lines LM and $L'M'$ intersect at a point of AC.

We received 12 submissions of which ten were correct and complete, one was incomplete, and one interpreted the statement in a way that turned it into a different, but interesting problem. We feature one of the solutions and some observations by the proposer. The editor's comments at the end address the alternative interpretation.

Solution by Michel Bataille.

First, we prove the following lemma: *Let m, m' be perpendicular lines intersecting at P and let X be a point in their plane, with $X \neq P$. Then, if Y, Y' are the orthogonal projections of X onto m, m', respectively, YY' is parallel to the reflection of PX in m (or in m').*
Proof. Let X' be the reflection of X in m (see Figure 1). We have

$$\overrightarrow{X'Y} = \overrightarrow{YX} = \overrightarrow{PY'}$$

so that $PY'YX'$ is a parallelogram and $YY' \parallel PX'$ follows. $\square$

Turning to the problem itself, we will show that each of LM and $L'M'$ passes through the midpoint B' of AC (see Figure 2).

Taking for m, m', and P the internal and external bisectors of the angle at B, the lemma gives $LM \parallel BC$. Since in addition LM passes through the midpoint C' of AB (the centre of the rectangle $ALBM$), LM intersects AC at its midpoint B'. Let H be the orthocentre of ΔABC. The reflection of BH in m, that is, the isogonal conjugate of BH in BA, BC, passes through the isogonal conjugate of H, which is the circumcentre O of ΔABC. Applying the lemma, we obtain that $L'M'$ is parallel to BO. But, if U denotes the point of intersection of BO with the perpendicular to BC at C and A' is the midpoint of BC, we have

$$\overrightarrow{UC} = 2\overrightarrow{OA'} = \overrightarrow{AH}$$

so that $AHCU$ is a parallelogram and therefore B' is also the midpoint of UH. Now, $L'M'$ is parallel to BU and passes through the midpoint V of BH, hence intersects HU at its midpoint, that is, at B'. Thus, $L'M'$ passes through B', as desired.

Note. The line VC' through the midpoints of BH and BA is parallel to AH, hence is perpendicular to BC and to MB'. Since C', V, and B' are on the nine-point circle of ABC, we see that the line $VB' = L'M'$ is a diameter of this nine-point circle.

Figure 1

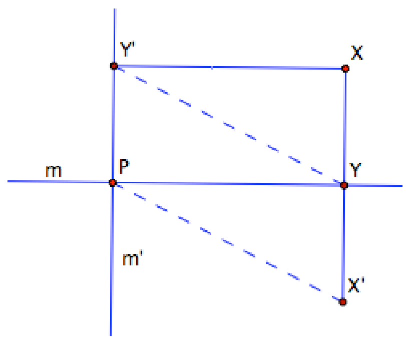
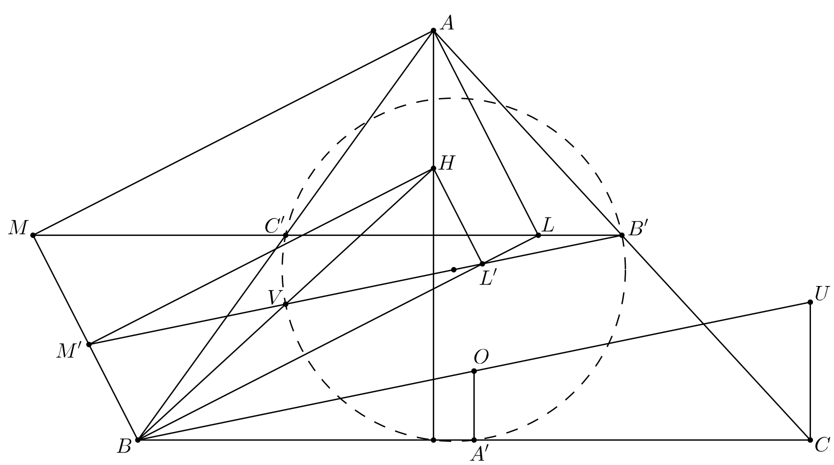


Figure 2



Proposer's comments.

This problem brings together two seemingly unrelated results that have appeared before in ***Crux***: A proof that LM passes through the midpoint B' of the side AC of ΔABC is the subject of the article "Remarkable Bisections" by Bruce Shawyer [2006: 434-435], although it is a much older theorem that can be found, for example, in F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, quatrième édition, J. De Gigord, Paris, 1907, Section 761, pages 327-328. The statement (without a proof or reference) that $L'M'$ also contains B' appeared as problem 20 on page 350 of "Angle Bisectors in a Triangle" by I.F. Sharygin [2015: 346-351].

It turns out that these two problems are indeed related. By interchanging the roles of A and C we see that the line joining the feet of the perpendiculars from C to the angle bisectors at B also passes through the midpoint B' of AC. Of course, the nondegenerate conics that pass through the vertices of triangle ABC as well as through its orthocentre H must all be rectangular hyperbolas, and this suggests a converse:

> Given ΔABC and the midpoint B' of its side AC, for an arbitrary point M on the exterior bisector of angle B define L to be the point where the interior bisector of angle B intersects MB', and define P to be the point where the perpendicular to BL at L intersects the perpendicular to BM at M. Then the locus of P as M moves along the exterior bisector is a conic.

This happens to be an 18th century theorem attributed to both William Braikenridge and Colin MacLaurin, which states more generally that if the sides of a variable triangle pass through three fixed noncollinear points (here those three points are B' together with the points at infinity of the perpendiculars to the angle bisectors of B), while two vertices lie on fixed lines not concurrent with a side of the variable triangle (here, the angle bisectors), then the third vertex describes a conic. See, for example, H.S.M. Coxeter, *Projective Geometry*, Theorem 9.22, page 85. It turns out that given a rectangular hyperbola whose center is B', we can choose any of its points to serve as A, the reflection of A in B' to serve as C, and any third point of the hyperbola as B. This and more was probably known a century ago to school kids; it is easily proved using coordinates: Take B' to be the origin and $xy = 1$ for the conic. With $A = (a, \frac{1}{a})$ and $B = (b, \frac{1}{b})$, C will be $(-a, -\frac{1}{a})$ and the angle bisectors will be $x = b$ and $y = \frac{1}{b}$.

Editor's comments. John Heuver defined his L' and M' to be the feet of the perpendiculars dropped from the orthocenter of the given triangle ABC to the lines AL and AM (where, as in the original problem, L and M are the feet of the perpendiculars dropped from A to the two bisectors of angle B). It turns out Heuver's line $L'M'$ also passes through the midpoint of AC; the lemma from the featured solution turns the proof of this claim into a simple exercise (since the altitude HC is the reflection of the line HA in the mirror HL').

4172. *Proposed by Nathan Soedjak.*

Let S be the set of all n-tuples $a_1, \ldots, a_n \geq 0$ such that $\sum_{i=1}^{n} a_i i = n$. Prove that

$$\sum_{(a_1,\ldots,a_n)\in S} \frac{1}{1^{a_1}2^{a_2}\cdots n^{a_n}(a_1)!(a_2)!\cdots(a_n)!} = 1.$$

We received seven solutions. We present the one by C. R. Pranesachar.

We consider the $n!$ permutations of $\{1, 2, \ldots, n\}$, each expressed as a product of cycles. If there are a_i i-cycles in such a permutation, then it can be associated with an n-tuple $(a_1, a_2, \ldots, a_n)$ in S, since $\sum_{i=1}^n a_i i = n$. The number of permutations associated with any element in S is obtained by first partitioning $\{1, 2, \ldots, n\}$ into a_i i-element blocks, $1 \le i \le n$. This number is

$$\frac{n!}{(1!)^{a_1}(2!)^{a_2}\ldots(n!)^{a_n}a_1!a_2!\ldots a_n!}.$$

Now each i-element block can be converted into $(i-1)!$ distinct i-cycles. Hence each n-tuple in S gives rise to

$$\frac{n!\prod_{i=1}^n((i-1)!)^{a_i}}{\prod_{i=1}^n(i!)^{a_i}a_i!} = \frac{n!}{\prod_{i=1}^n i^{a_i}a_i!}$$

distinct permutations of $\{1, 2, \ldots, n\}$. By summing over all elements of S we therefore get

$$\sum_{(a_1,\ldots,a_n)\in S}\frac{n!}{\prod_{i=1}^n i^{a_i}a_i!} = n!.$$

Dividing both sides by $n!$ gives the desired result.

4173. *Proposed by Dao Thanh Oai, Leonard Giugiuc and Daniel Sitaru.*

Let ABC be an equilateral triangle with circumradius R and centroid O. Let P be an arbitrary point inside ABC and let P_a, P_b and P_c be the feet of the perpendiculars dropped from P onto the sides BC, CA and AB, respectively. Finally, let $OA = R$ and $OP = d$. Prove that

$$\frac{2}{PP_a+r}+\frac{2}{PP_b+r}+\frac{2}{PP_c+r} \le \frac{3}{R-d}+\frac{3}{R+d} \le \frac{1}{PP_a}+\frac{1}{PP_b}+\frac{1}{PP_c}.$$

Of the three submissions we received, two were correct and one was flawed. We feature the solution by Steven Chow.

It is assumed that r (which failed to be identified in the statement of the problem) is the inradius of $\triangle ABC$. We will set $r = 1$ (without loss of generality), so that $R = 2$ and $BC = CA = AB = 2\sqrt{3}$. Using barycentric coordinates we let $(1,0,0) = A$, $(0,1,0) = B$, and $(0,0,1) = C$, and take $(x,y,z) = P$, where $x+y+z = 1$. Therefore $PP_a = 3x$, $PP_b = 3y$, and $PP_c = 3z$, and because $O = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the distance formula says that

$$\begin{aligned} d^2 = OP^2 &= -(2\sqrt{3})^2\sum_{cyc}\left(\left(y-\frac{1}{3}\right)\left(z-\frac{1}{3}\right)\right) \\ &= -12\left(xy+yz+zx-\frac{2}{3}(x+y+z)+\frac{1}{3}\right) = -12(xy+yz+zx)+4. \end{aligned}$$

Therefore

$$\frac{3}{R-d}+\frac{3}{R+d} = \frac{6R}{R^2-d^2} = \frac{6(2)}{2^2-(-12(xy+yz+zx)+4)} = \frac{1}{xy+yz+zx}.$$

The problem is thus reduced to proving that given $x, y, z > 0$ such that $x+y+z = 1$, then

$$\frac{2}{3x+1} + \frac{2}{3y+1} + \frac{2}{3z+1} \le \frac{1}{xy+yz+zx} \le \frac{1}{3x} + \frac{1}{3y} + \frac{1}{3z}. \tag{1}$$

Because the inradius has been fixed, we can use familiar notation for the three-variable elementary symmetric polynomials, namely

$$p = x+y+z = 1, \quad q = xy+yz+zx \quad \text{and} \quad r = xyz.$$

The first inequality in (1) is equivalent, in turn, to

$$\begin{aligned} & 2(xy+yz+zx)\sum_{cyc}((3y+1)(3z+1)) \le (3x+1)(3y+1)(3z+1) \\ \iff\ & 2q(9q+6p+3) \le 27r+9q+3p+1 \\ \iff\ & 27r - 18q^2 - 9q + 4 \ge 0. \end{aligned}$$

But $p^2q + 3pr \ge 4q^2$ (since $p^2q + 3pr - 4q^2 = \sum_{cyc} xy(x-y)^2 \ge 0$), so

$$3r \ge 4q^2 - q,$$

whence

$$27r - 18q^2 - 9q + 4 \ge (36q^2 - 9q) - 18q^2 - 9q + 4 = 18q^2 - 18q + 4 \ge 0$$

(since $0 \le q \le \frac{1}{3}p^2 = \frac{1}{3}$). Thus the first inequality in (1) is true, with equality if and only if $x = y = z$, in which case $P = O$.

The second inequality in (1) becomes $q^2 \ge 3r = 3pr$, which is Newton's inequality. Again, equality holds if and only if $P = O$. Therefore

$$\frac{2}{PP_a + r} + \frac{2}{PP_b + r} + \frac{2}{PP_c + r} \le \frac{3}{R-d} + \frac{3}{R+d} \le \frac{1}{PP_a} + \frac{1}{PP_b} + \frac{1}{PP_c},$$

with equality (in both places) if and only if $P = O$.

4174. *Proposed by Mihaela Berindeanu.*

Let a, b, c be positive real numbers such that $a+b+c = 3$. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a^2 + b^2 + c^2.$$

We received 13 solutions. We present the solution by Titu Zvonaru.

Using $a+b+c=3$,

$$\begin{aligned}\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-3 &= \frac{a^2}{b}-2a+b+\frac{b^2}{c}-2b+c+\frac{c^2}{a}-2c+a\\ &= \frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}, \text{ and}\\ a^2+b^2+c^2-3 &= a^2+b^2+c^2-\frac{(a+b+c)^2}{3}\\ &= \frac{(a-b)^2}{3}+\frac{(b-c)^2}{3}+\frac{(c-a)^2}{3}.\end{aligned}$$

Since $b \leq 3$, we have $\frac{(a-b)^2}{b} \geq \frac{(a-b)^2}{3}$, with equality when $a=b$. We deal similarly with the remaining two terms to conclude that the given inequality holds, with equality when $a=b=c=1$.

4175. *Proposed by George Apostolopoulos.*

Find all pairs of positive integers x, y satisfying the equation

$$4x^2+3y^2-7xy-6x+5y=0.$$

We received 24 *correct solutions. We present two of those, representative of the two most common approaches.*

Solution 1, by Šefket Arslanagić, slightly modified by the editor.

We can rewrite the given equation as

$$4x^2-(7y+6)x+3y^2+5y=0$$

and therefore

$$x=\frac{7y+6\pm\sqrt{y^2+4y+36}}{8}.$$

For x to be an integer there must exist $k \in \mathbb{N}$ with

$$y^2+4y+36=k^2$$

or

$$(k-y-2)(k+y+2)=32.$$

We need to solve the system of linear equations

$$\begin{aligned}k-y-2&=a\\ k+y+2&=b,\end{aligned}$$

where a, b are integers with $ab=32$. Since $k+y+2>0$ and $k-y-2<k+y+2$, the only options for (a,b) are $(1,32)$, $(2,16)$, and $(4,8)$. The values that we obtain for y from these three cases are $\frac{27}{2}$, 5, and 0 respectively. As y is a positive integer, this leaves us with only one solution, which we calculate to be $(x,y)=(4,5)$.

Solution 2, by I.J.L. Garces.

We first observe that the given equation is equivalent to

$$(4x-3y-2)(x-y-1)=2.$$

We need to solve the system of linear equations

$$\begin{cases} 4x-3y-2=a, \\ x-y-1=b. \end{cases}$$

for the four factorizations of 2 into two integers. Only the case $a=-1$, $b=-2$ gives positive values for both x and y, yielding the unique solution $x=4$ and $y=5$.

4176. *Proposed by Michel Bataille.*

For each integer n with $n \geq 2$, let $\mathcal{P}_n$ be the set of all polynomials of degree n, with rational coefficients, having a root (not necessarily real) of multiplicity greater than 1. For which n is it true that every element of $\mathcal{P}_n$ has a rational root?

There were six correct solutions. We present the solution of Roy Barbara.

The only values of n are $2, 3, 5$. Recall that a polynomial over $\mathbb{Q}$ with a root of multiplicity greater than 1 must be reducible. Thus, if f is a polynomial of degree 2 or 3 over $\mathbb{Q}$, then f must have a linear factor and so a rational root. If the degree of f with a multiple root is 5, then it either has a linear factor and a rational root, or it can be factored $f=gh$ where g and h are irreducible polynomials over $\mathbb{Q}$ of respective degrees 2 and 3. Since the roots of g and h are all simple, they must have a root in common. But then, h, being irreducible, must divide g, an impossibility.

For every other value of n, it is possible to find a polynomial with multiple roots but no rational root. For $n=4$, $(x^2+1)^2$ will do. When $n=4+m$ with $m \geq 2$, an example is $(x^2+1)^2(x^m-3)$.

4177. *Proposed by Daniel Sitaru.*

Prove that if a, b, c and d are nonnegative real numbers such that

$$(a^2+b^2)(c^2+d^2) \neq 0, \text{ then}$$

$$4\Big((ac+bd)^6+(ad-bc)^6\Big) \geq (a^2+b^2)^3(c^2+d^2)^3.$$

We received 14 solutions, all of which were correct. We present a composite of nearly the same solutions by Mihai Bunget; Dionne Bailey, Elsie Campbell, and Charles Diminnie, jointly; Chudamani R Pranesachar; Prithwijit De; and Digby Smith.

By Lagrange's Identity, we have

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2.$$

If we set $x = (ac+bd)^2$ and $y = (ad-bc)^2$, then the given inequality is equivalent, in succession, to

$$\begin{aligned} & 4(x^3+y^3) \geq (x+y)^3 \\ \iff & x^3+y^3 \geq x^2y+xy^2 \\ \iff & (x+y)(x-y)^2 \geq 0, \end{aligned}$$

which is true.

Editor's comments. Daniel Dan pointed out the obvious fact that equality holds if and only if $ac+bd = |ad-bc|$. Digby Smith gave the equivalent condition:

$$(a-b)(a+b)(c-d)(c+d) = 4abcd.$$

Several solvers pointed out the condition $(a^2+b^2)(c^2+d^2) \neq 0$ is superfluous, and Somasundaram Muralidharan remarked that the given inequality holds for all reals a, b, c, d.

4178. *Proposed by Leonard Giugiuc, Dan Marinescu and Marian Cucoanes.*

For $0 \leq a < \dfrac{\pi}{2}$, prove that $\lim_{n\to\infty} \int_a^{\pi-a} x^n \cos x dx = -\infty$.

We received six correct solutions. Solution by Missouri State University Problem Solving Group.

Let $0 < s < t$, with $t > 1$. Then

$$\frac{t^{n+1}-s^{n+1}}{t^n-s^n} = \frac{1-(s/t)^{n+1}}{1/t-(s/t)^n/t} = t \cdot \frac{1-(s/t)^{n+1}}{1-(s/t)^n} > t > 1$$

for every $n \in \mathbb{N}$; hence $\{t^n - s^n\}_1^\infty$ is an increasing sequence. Moreover,

$$t^n - s^n = t^n(1-(s/t)^n) \to \infty.$$

For all $x \in (0, \pi/2)$, $0 < x < \pi - x$ and $\pi - x > 1$; hence it follows from the above that

$$\{(\cos x)\,[(\pi-x)^n - x^n]\}_1^\infty$$

is an increasing family of functions diverging pointwise to infinity a.e. on $[0, \infty]$. Using a suitable change of variable, we see that

$$\int_{\pi/2}^{\pi-a} x^n \cos x\, dx = \int_a^{\pi/2} (\pi-x)^n \cos(\pi-x)\, dx$$

Since $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x = -\cos x$, we have

$$\begin{aligned}
-\int_a^{\pi-a} x^n \cos x\,dx &= \int_a^{\pi/2} -x^n \cos x\,dx + \int_{\pi/2}^{\pi-a} -x^n \cos x\,dx \\
&= \int_a^{\pi/2} -x^n \cos x\,dx - \int_a^{\pi/2} (\pi - x)^n \cos(\pi - x)\,dx \\
&= \int_a^{\pi/2} (\cos x)\left[(\pi - x)^n - x^n\right]\,dx.
\end{aligned}$$

Therefore, by the Lebesgue monotone convergence theorem,

$$\lim_{n\to\infty} \int_a^{\pi-a} -x^n \cos x\,dx = \infty,$$

from which the desired result follows.

4179. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequence (a_n), $n \geq 1$, such that $a_n = \prod_{k=1}^n (k!)^2$. Find $\lim_{n\to\infty} \dfrac{n}{\sqrt[n^2]{a_n}}$.

We received eight solutions. We present the solution by Kee-Wai Lau.

Take the logarithm of the expression whose limit we are trying to calculate:

$$\ln\left(\frac{n}{\sqrt[n^2]{a_n}}\right) = \ln n - \frac{\ln a_n}{n^2}.$$

Note that

$$\ln a_n = 2\sum_{k=1}^n \sum_{j=1}^k \ln j = 2\sum_{k=1}^n (n - k + 1)\ln k = 2(n+1)\sum_{k=1}^n \ln k - 2\sum_{k=1}^n k\ln k. \quad (1)$$

From the Euler-Maclaurin formula for asymptotic expansion of series,

$$\sum_{k=1}^n \ln k = n\ln n - n + O(\ln n) \quad \text{and} \quad \sum_{k=1}^n k\ln k = \frac{n^2 \ln n}{2} - \frac{n^2}{4} + O(n\ln n).$$

Substituting these expressions in (1), we get

$$\ln a_n = 2(n+1)\left(n\ln n - n + O(\ln n)\right) - n^2 \ln n + \frac{n^2}{2} + O(n\ln n);$$

so we can calculate that

$$\ln n - \frac{\ln a_n}{n^2} = \ln n - \frac{2n^2 \ln n - 2n^2 - n^2\ln n + \frac{n^2}{2} + O(n\ln n)}{n^2} = \frac{3}{2} + O\left(\frac{\ln n}{n}\right).$$

Hence $\lim\limits_{n\to\infty} \ln n - \frac{\ln a_n}{n^2} = \frac{3}{2}$; exponentiating, $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n^2]{a_n}} = e^{3/2}$.

4180. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle such that $\angle BAC \geq \dfrac{2\pi}{3}$. Prove that $\dfrac{r}{R} \leq \dfrac{2\sqrt{3}-3}{2}$.

We received 15 submissions, all correct and complete, and feature a composite of the similar solutions from S. Muralidharan and V. Konečný.

We assume that ΔABC is inscribed in a circle whose radius is the fixed positive number R, and investigate how its inradius r varies as the side BC is fixed while the vertex A moves about one arc of the circumcircle. If J is the midpoint of the opposite arc BC then a familiar theorem says that the incenter lies inside the triangle on the circumference of the circle whose center is J and radius is $JB = JC$. It follows that the largest inradius of all such triangles occurs when the triangle is isosceles (when the incenter coincides with the top of that circumference, where it intersects the diameter OJ). For an isosceles triangle, we have $B = C = 90^\circ - \frac{A}{2}$ and

$$\begin{aligned}
\frac{r}{R} &= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
&= 4\sin\frac{A}{2}\sin^2\left(45^\circ - \frac{A}{4}\right) \\
&= 2\sin\frac{A}{2}\left(1-\cos\left(90^\circ - \frac{A}{2}\right)\right) \\
&= 2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right) \\
&= 2\left(\frac{1}{4}-\left(\sin\frac{A}{2}-\frac{1}{2}\right)^2\right).
\end{aligned}$$

As the angle at A of our isosceles triangle increases from 0° to 180°, we observe that the ratio $\frac{r}{R}$ increases monotonically from 0 to its maximum of $\frac{1}{2}$ at 60° (where $\sin\frac{A}{2} = \frac{1}{2}$), then returns monotonically back to 0. Consequently, the maximum value of $\frac{r}{R}$ on any subset of the domain of $\angle BAC$ is achieved for the angle closest to 60°. Turning now to the problem as stated, the domain is restricted to $\angle BAC \geq 120^\circ$, so that the maximum of $\frac{r}{R}$ occurs at $\angle BAC = 120^\circ$; that is,

$$\frac{r}{R} \leq 2\left(\frac{1}{4}-\left(\sin\frac{120}{2}-\frac{1}{2}\right)^2\right) = 2\left(\frac{1}{4}-\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)^2\right) = \frac{2\sqrt{3}-3}{2},$$

as claimed.

Editor's comments. Most of the solutions were shorter than our featured solution — they simply verified the value of the upper bound as requested (without going on to provide a geometric explanation of the result).

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Václav Konečný, Big Rapids, MI, USA : **4180**
Kee-Wai Lau, Hong Kong, China : 4174, 4175, 4178, **4179**, 4180
Kathleen Lewis, University of the Gambia, Brikama, Republic of the Gambia : CC238, CC239
David Manes, SUNY at Oneonta, NY, USA : CC239
Valcho Milchev, Petko Rachov Slaveikov Secondary School, Kardzhali, Bulgaria : OC286
Somasundaram Muralidharan, Chennai, India : 4177, **4180**
Ricard Peiró i Estruch, Valéncia, Spain : 4175
Paolo Perfetti, Dipartimento di Matematica, Universitá degli studi di Tor Vergata Roma, Rome, Italy : CC237, 4174, 4177, 4178, 4179
C.R. Pranesachar, Indian Institute of Science, Bangalore, India : 4171, **4172**, 4174, 4175, **4177**
Landon Pruitt, Auburn University at Montgomery, Piedmont, USA : CC238, CC239
Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA : CC238
George-Florin Serban, Pedagogical High School, Braila, Romania : CC239, 4175, 4177, 4179
Daniel Sitaru, Drobeta Turnu-Severin, Romania : 4177
Digby Smith, Mount Royal University, Calgary, AB : CC239, 4174, 4175, **4177**, 4180
Trey Smith, Angelo State University, San Angelo, TX, USA : 4174
Nathan Soedjak, Rogersville, MO, USA: 4172
Gabriel Wallace, Missouri State University, Springfield, MO, USA : **OC286**
Peter Y. Woo, Biola University, La Mirada, CA, USA : 4171, 4180
Konstantine Zelator, Pittsburgh, Pennsylvania, USA : CC238, CC239
Titu Zvonaru, Cománeşti, Romania : CC239, OC287, **4174**, 4175

Solvers - collaborations

Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, USA : 4175, **4177**
D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania : 4179
Leo Giugiuc, Dan Marinescu and Marian Cucoanes, Romania : 4178
John Hawkins and David R. Stone, Georgia Southern University, Statesboro, USA : 4175
Dao Thanh Oai, Vietnam, Leonard Giugiuc, Romania, and Daniel Sitaru, Romania : 4173
Missouri State University Problem Solving Group, Springfield, MO, USA : **CC236**, **CC237**, CC239, OC286, 4172, 4175, **4178**