Mathematical Spectrum

Volume 11 1978/79 *Number* 3

A Magazine of *Contemporary Mathematics*

Published by the *Applied Probability Trust*

Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

Volume 11 of *Mathematical Spectrum* consists of three issues of which this is the third. The first issue was published in September 1978, and the second in January 1979.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems.

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The Four-Colour Theorem

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On 22 July 1976, Kenneth Appel and Wolfgang Haken, two mathematicians at the University of Illinois, announced the solution (see reference 1) of what was probably the best-known unsolved problem in the whole of mathematics: the four-colour map problem. This problem asks whether the regions of a map can always be coloured with four colours in such a way that no two neighbouring regions have the same colour. ('Neighbouring' here means 'having a length of common border'. We do not insist on giving two regions different colours if they meet only at a finite number of points, like regions D and F in Figure 1. We do, however, insist on giving a colour to the 'outside' region (or 'sea'), region G, which is considered to be a perfectly normal region of the map.)

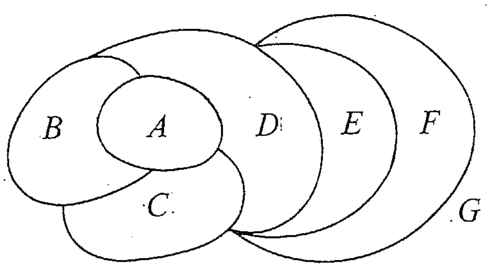


Figure 1

This problem was first proposed in 1852 by a London student, Francis Guthrie, who is reported to have thought of it while colouring a map of the counties of England. He noticed that four colours are sometimes needed (for example, for regions A, B, C and D in Figure 1), and conjectured that four colours always suffice, but was unable to prove this. He mentioned the problem to his brother Frederick, who passed it on to his professor, Augustus De Morgan (of 'De Morgan's laws'). De Morgan wrote to Hamilton about it, who said that he wasn't interested. But De Morgan was much taken with the problem, and told all his friends about it. He gave it its first mention in print, in an unsigned book review in the *Athenaeum* in 1860. In 1878, after De Morgan's death, Cayley raised the problem at a meeting of the London Mathematical Society, asking whether it had been solved, and making it fairly clear that he had no idea how to solve it himself. This brought it to the attention of A. B. Kempe, a barrister and keen amateur mathematician who was

Treasurer, and later President, of the London Mathematical Society. In 1879, he published a 'proof' (reference 4) in the *American Journal of Mathematics*, which seems to have been generally accepted. But in 1890 P. J. Heawood, Professor of Mathematics at Durham University, pointed out (reference 3) that the 'proof' contained a flaw. For some years after that, the flaw seems not to have been recognised as serious, and the theorem was regarded as 'essentially proved'. However, as the years went by and nobody found a satisfactory way round the difficulty, it gradually became realized that the problem was much deeper than had been supposed. Since then, almost every mathematician of repute has probably dabbled with the problem at some time or other, so Appel and Haken's achievement in solving it (in the affirmative) is a very fine one.

As might perhaps be expected of such a refractory problem, the proof is long. In typescript, it runs to 100 pages of summary, 100 pages of detail and a further 700 pages of back-up work, plus about 1200 hours of computer time. For comparison, the average proof presented in undergraduate lectures probably does not last more than one or two pages. In the published literature, I would regard a 20-page proof as quite long.

In retrospect—and it is easy to be wise after the event—it is clear that in the early days people grossly underestimated the difficulty of the problem—perhaps none more so than the Headmaster of Clifton College, the boys' public school. He was in the habit of setting a Challenge Problem to the school each term, and in the Autumn Term of 1886 he decided to set the four-colour problem. His 'challenge' ends, optimistically: 'No solution may exceed one page, 30 lines of MS., and one page of diagrams.' (The complete wording of the 'challenge', and much more information about the history of the problem, can be found in the excellent historical survey *Graph Theory* 1736–1936 by Biggs, Lloyd and Wilson (reference 2).)

The Headmaster's challenge was published in the *Journal of Education*, with an editorial comment which suggested: 'Perhaps it will tempt some of our correspondents to send us a solution'. One of their correspondents was Frederick Temple, then Bishop of London and later Archbishop of Canterbury. On his own admission, he allowed his mind to wander while attending a meeting, and dashed off a quick 'proof' of the four-colour theorem. This 'proof' is interesting, in that it reveals a common misunderstanding about the four-colour problem. What Temple actually set out to prove was that it is not possible to have five regions in any map in the plane, each of which touches each of the others (along a length of common boundary). This problem was introduced, around 1840, by Möbius, and has often been confused with the four-colour problem. There is certainly a connection between the two problems: if one could find five regions, each touching each of the others, then clearly one would need five colours to colour them, and so the four-colour theorem could not be true. On the other hand, proving that one can *not* find five such regions does not prove that the four-colour theorem *is* true. Figure 2 shows a map that requires at least four colours to colour it (try it!), and yet does not contain four regions each of which touches each of the others. So the number of colours needed in order to colour a map may be larger than the largest number of regions in the map all

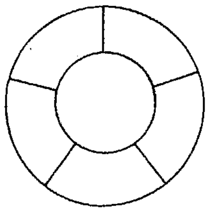


Figure 2

of which touch each other, and proving that this latter number is at most four does not prove that the map is four-colourable.

An outline of the method

In common with most other recent workers, Appel and Haken tackled the problem in the form: 'Show that the vertices of every plane graph can be coloured with four colours in such a way that no two adjacent vertices have the same colour.' (A plane graph is a graph (=network) drawn in the plane without edges crossing: see Figure 3.) It is not difficult to show that this version of the problem is equivalent to that described for maps. (One way round is easy: take a vertex inside each region, and join two vertices by an edge whenever the corresponding two regions are neighbours. If we can four-colour the vertices of the resulting graph, then we can certainly four-colour the regions of the original map. This does not quite show that the two problems are actually equivalent: we still have to show that every graph can be obtained in this way from some map; but this is not very difficult.) It is also easy to see that it suffices to consider *plane triangulations*; that is, graphs that divide the plane into regions bordered by exactly three edges. As an illustration, Figure 3 shows the graph corresponding to the map of Figure 1, and the same graph made into a triangulation: if we can four-colour the vertices of the latter, then the same colouring will do for the former. Note that we may end up with more than one edge joining a given pair of vertices (E and G in this case).

Kempe's proof. In order to understand Appel and Haken's method, it will be helpful to translate Kempe's attempted proof into the language of plane triangulations. Kempe started with Euler's polyhedron formula, which shows that a plane triangulation T satisfies the formula $V - E + F = 2$, where V is the number of

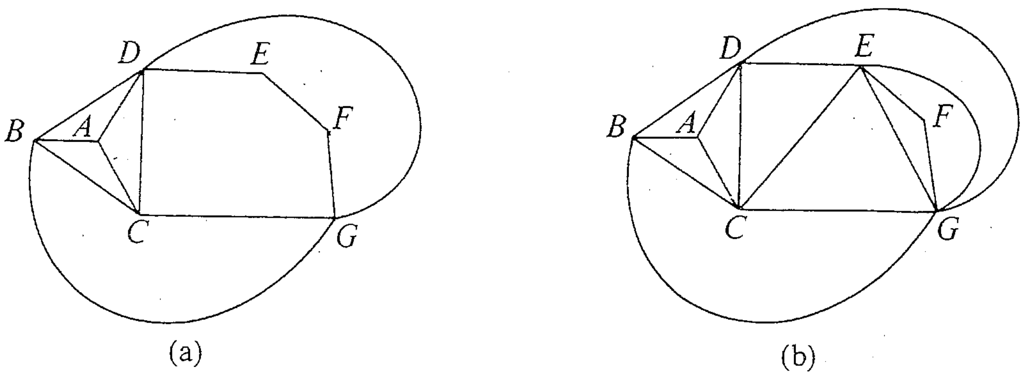


Figure 3

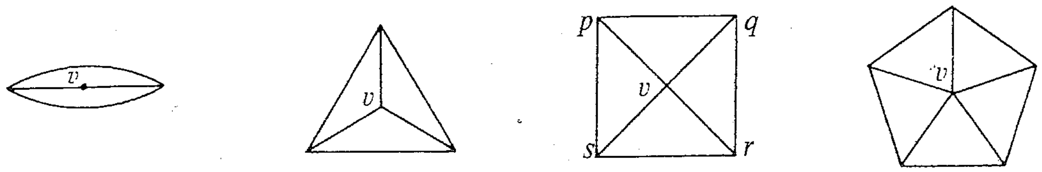
vertices, E the number of edges and F the number of faces (= regions, including the outside region) of T. Since every face of a triangulation is bordered by three edges, and every edge borders two faces, $2E = 3F$ (by counting in two different ways the number of 'sides of edges': every edge has two sides, and every face contains three sides of edges). If V_i denotes the number of vertices with valency i (the valency of a vertex is the number of edges incident to it), then clearly $\sum V_i = V$ (the total number of vertices of T) and $\sum iV_i = 2E$ (since every edge has two ends). Substituting these in Euler's formula now gives

$$\sum_i (6 - i)V_i = 12;$$

that is,

$$4V_2 + 3V_3 + 2V_4 + V_5 - V_7 - 2V_8 - 3V_9 - \cdots = 12.$$

(In a triangulation there are no vertices of valency 0 or 1, so we do not need terms $6V_0$ and $5V_1$ in the above sum.) It follows immediately from this that at least one of V_2, V_3, V_4 and V_5 is positive; that is, that T must contain at least one of the four configurations in Figure 4. (Note that Figure 3(b) contains (somewhat distorted) copies of *all* these configurations, with v equal to F, A, B and C, respectively—but the third of these is 'inside out' in Figure 3(b), and so is difficult to recognize.)



(a) (b) (c) (d)

Figure 4

Let us now suppose that there exists a counterexample to the four-colour conjecture; we try to prove that this is impossible by obtaining a contradiction. Choose a counterexample with the smallest possible number of vertices. If this is not already a triangulation, then we can certainly add extra edges to turn it into a triangulation T without increasing the number of vertices, so that every graph with fewer vertices than T *is* four-colourable but T itself is *not*.

If T contains the configuration of Figure 4(a) or 4(b), we need only remove v from T, colour what is left with four colours, and restore v; since v is adjacent to at most three other vertices, we can certainly find a colour for it. Thus we have four-coloured T, which contradicts our supposition that T is not four-colourable. This contradiction shows that T cannot in fact contain configuration 4(a) or 4(b).

In the case of configuration 4(c) we can try the same thing, but this time we are in trouble if p, q, r and s all have different colours; in this case we cannot colour v. However, we can then reason as follows. Suppose that p, q, r and s are coloured blue, green, red and yellow, respectively. Consider the subset of T consisting of all the blue and red vertices, and all the edges joining pairs of these vertices. Each connected piece of this subgraph is called a (blue–red) *Kempe chain* (and the type of argument we are about to use is now referred to as a *Kempe-chain argument*). If p and r are in different (blue–red) Kempe chains, then we interchange the colours blue and red on all the vertices in the same chain as p, without changing any of the other colours. This gives us a new (valid) colouring in which p and r are both coloured red, and so we can now colour v blue. Similarly, if q and s are in different green–yellow Kempe chains, then we can interchange the colours green and yellow in the chain to which q belongs to make q yellow, without changing the colour of s; and we can now colour v green. So the only problem would arise if we had *both* a blue–red chain connecting p to r, *and* a green–yellow chain connecting q to s. But it is easy to see that this cannot happen, since these two chains would have to cross somewhere, at a vertex that would have to be *both* blue or red *and* green or yellow, and this is clearly impossible. Thus, whatever happens, we have shown that we can four-colour T, and this contradiction shows that T cannot contain configuration 4(c).

Thus Kempe showed that T cannot contain configuration 4(a), 4(b) or 4(c). If he had been able to show that T cannot contain configuration 4(d) either, then he would have completed his proof (by contradiction) that the definition of T (as a minimum counterexample to the conjecture) is impossible, since we know that if T exists it *must* contain one of configurations 4(a) to 4(d). He would therefore have proved that no counterexample to the conjecture can exist; in other words, he would have proved the conjecture. Unfortunately, he tried to use the same trick with configuration 4(d) that he used with 4(c), and it was here that he made his mistake. Nevertheless, he made a very fine contribution towards the solution of the problem, which has often been underestimated by later writers. Although his proof was fallacious (and hence technically worthless), the slightest modification to his argument yields a perfectly valid demonstration that *five* colours will suffice for every map, and his arguments have formed the foundation for most of the subsequent work on the problem.

The two main steps. To summarize Kempe's argument briefly in modern terminology, he attempted to exhibit a set U of configurations ((a) to (d) of Figure 4) such that

(i) the set U is *unavoidable*, meaning that every plane triangulation contains at least one configuration in the set, and
(ii) all of the configurations in the set U are *reducible*, meaning that they cannot be contained in a minimum counterexample to the conjecture. (That is, any counterexample containing any of them can be *reduced* to a smaller counterexample.)

If his attempt had succeeded, it would clearly have been enough to prove the conjecture. In fact it failed, because he did not show satisfactorily that configuration 4(d) is reducible. Appel and Haken have been successful with exactly the same approach. However, whereas Kempe's unavoidable set contained only four configurations, Appel and Haken's contains about 1900. (Their original proof used 1939 configurations. It has now been shown that this can be reduced to just over

1400.) The proof that this number of configurations is reducible involves massive reliance on the computer. One of their configurations is shown in Figure 5. This one is bordered by a ring of 12 vertices. All of their configurations are bordered by rings of 14 or fewer vertices. If they had had to use configurations much larger than this, they would probably not have been able to prove them all reducible with the present generation of computers.

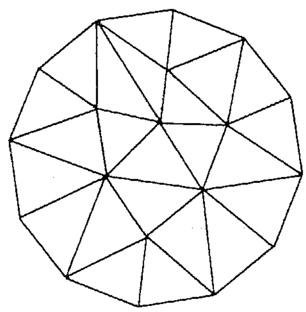


Figure 5

Appel and Haken's proof thus seems to involve two main steps: the construction of an unavoidable set of configurations, and the proof that all of these configurations are reducible. Each of these steps is comparatively straightforward on its own; it is the interplay between them that is sophisticated, and in which Appel and Haken's work goes qualitatively, and not just quantitatively, way beyond anything that has been done before.

Conclusion

The length of Appel and Haken's proof is unfortunate, for two reasons. The first is that it makes it difficult to verify. Clearly a proof has no value unless it can be checked by other mathematicians, and ideally one would like a proof to be verifiable by as many people, and as quickly, as possible. A very long proof is likely to take a long time to check, and may be intellectually accessible to only comparatively few people. (Of course, a short proof may have these defects as well.) This is particularly true of a proof that uses the computer. Before the introduction of computers into pure mathematics, every proof could be checked by anyone possessing the necessary mental apparatus. Now a further requirement is added, namely, access to a high-speed computer. Appel estimated that to check all the details of their proof would take about 300 hours on a big machine. There must be few mathematicians who have access to that sort of computer time.

The other big disadvantage of a long proof is that it tends not to give very much understanding of why the result is true. This is particularly true of a proof that involves looking at a large number of separate cases, whether or not it uses the computer. I am sure that many mathematicians would agree that proving theorems is only a means to the true end of pure mathematics, which is to understand what is going on. Sometimes a proof is so illuminating that one feels immediately that it explains the 'real reason' for the result being true. It may be unreasonable to expect every theorem to have a proof of this sort, but it seems nonetheless to be a goal worth aiming for. So undoubtedly much work will be done in the next few years in an attempt to shorten Appel and Haken's proof, and possibly to find a more illuminating method of proof altogether. (It seems unlikely that their method of proof can be shortened sufficiently to avoid massive reliance on the computer.) None of this, of course, detracts in any way from Appel and Haken's magnificent achievement in proving the four-colour theorem.

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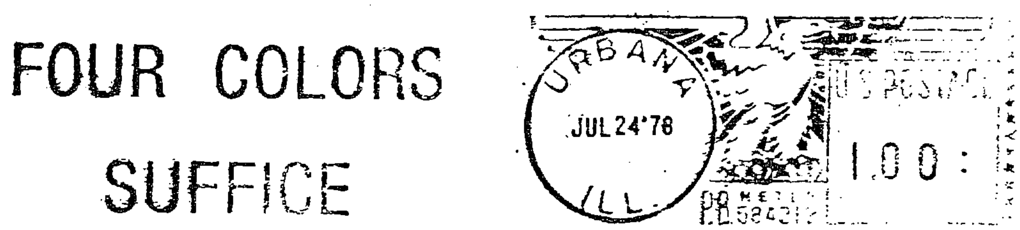

Postmark issued by the University of Illinois at Urbana-Champaign to celebrate the proof of the four-colour conjecture.

Computerised Anthropology—Finding and Settling Polynesian Islands

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There are some scientific problems which seem permanently destined to hold a favoured place in the public imagination. One such is the settlement of Polynesia. Books are written proving that the strange stone heads of Easter Island could not exist without the intervention of Beings from Outer Space. Anthropologists with a practical bent sail balsa wood rafts from (close to) South America to the centre of the Pacific, proving that the islands could have been settled from the East, provided someone helped give the pioneer mariners a push away from the coast. And Western anthropologists of a theoretical turn of mind wonder how this vast ocean could have been settled so completely 3000 years ago, when their own ancestors could barely manage to sail to Ireland.

One might feel that this is the sort of descriptive, almost romantic, area of exploration in which mathematics has no part. But in fact it may be that mathematics holds out one of the few real hopes of gaining insight into the possible origins of Polynesian settlements, short of the Beings arriving to tell us they did colonise Easter Island, or some fortunate anthropologist discovering the lost Treasure of Quetzlcoatl conveniently buried in the Gilbert and Ellice Islands.

For the interesting questions today, so long after the fact of settlement, are mainly ones of probability. How likely is it that, sailing from Island A, one would reach Island B by chance? What possibility is there that this would happen before one's food supply ran out, especially if the voyage was unintentional? What number of passengers must a boat hold to give a settlement some chance of succeeding? How long would such a settlement take to reach a reasonable size?

In this article we look at two different investigations recently carried out in Canberra which attempt to give answers to such questions using models which can be analysed by computer.

The first of these is concerned with the problem of island finding. The mathematical model involved here is very simple. One takes a map of the relevant part of the Pacific Ocean and places a very fine grid on it; then one imagines that one is sailing, not continuously, but in straight lines from the centre of a grid square to one of the adjoining grid square centres (see Figure 1). The choice of which centre to

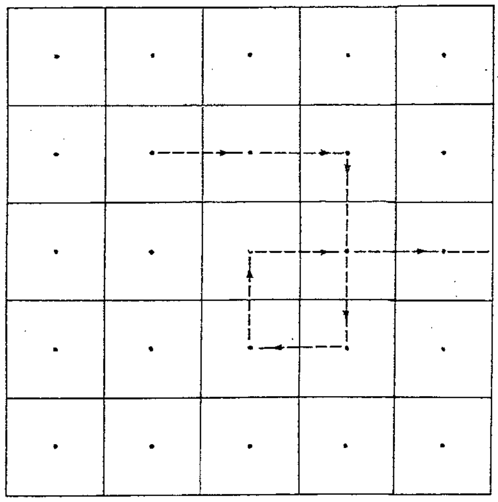

Figure 1. Possible path for Polynesian sailor.

sail to next, and the speed of the voyage, are probabilistic: one takes the meteorological data on currents and windspeeds in each grid square, and if one is sailing randomly then one of the four directions is chosen with probability proportional to the relative 'thrust' in that direction. If desired, of course, one can also superimpose a probability of going in a preferred direction, corresponding to some navigational effect. Finally, after each 'leg' of the journey, one can decide according to some 'survival probability' whether the crew has remained alive or not.

Such a model is clearly very suitable for computer simulation, and Levison, Ward and Webb (reference 2) first approached it in that way. They used a computer to run 'random voyages' from a large number of starting points both on islands in

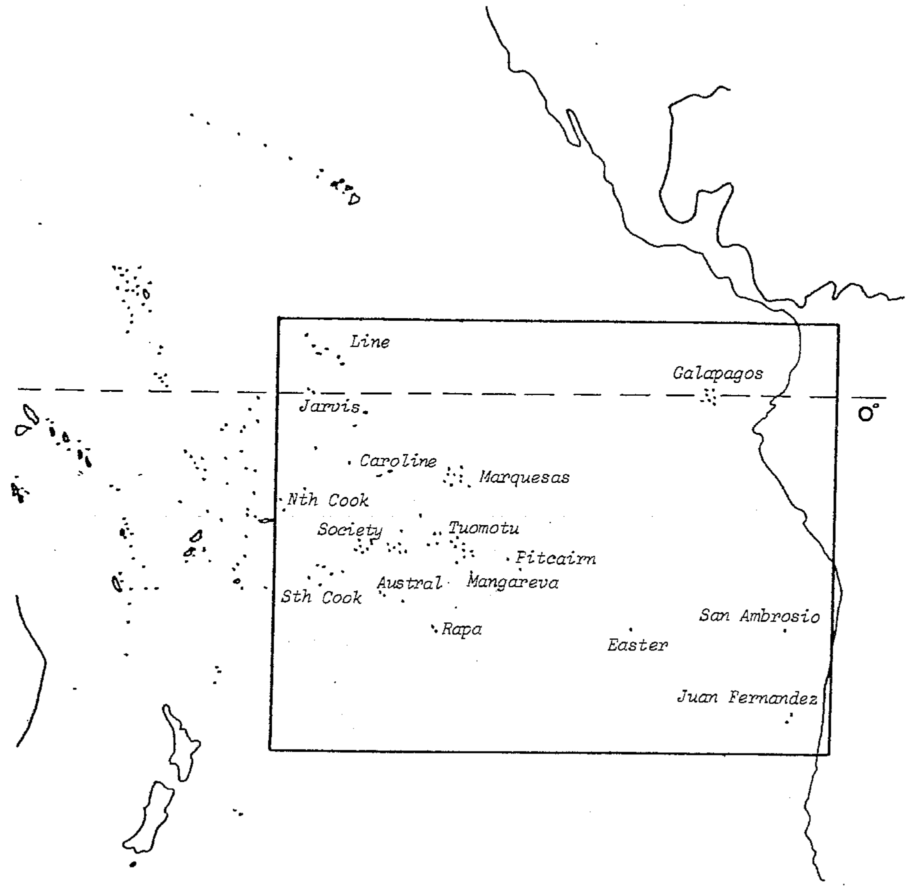

Figure 2. Pacific region studied by Bover and others (from reference 1).

Polynesia and from more remote areas, such as South America or Melanesia (see Figure 2). From these they were able to estimate the probabilities $P(A, B)$ of reaching Island B from Starting Point A as the proportion of simulated voyages from A which actually did reach B.

However, to estimate such probabilities accurately one needs a large number of voyages from each starting place; and this is very costly. If possible, it would be better to calculate $P(A, B)$ explicitly from the known values of the weather and navigation probabilities; in his paper on ocean-drift migration, David Bover (reference 1) showed how to do this. He first realised that known techniques for the numerical solution of the boundary problem for diffusion, of which the model for voyages above is a particular case, enabled one to find $P(A, B)$ provided B was the only island in the ocean. Even for mathematical convenience, however, such simplification is a little drastic; but this 'single island' solution (which we will call $P_s(A, B)$) can be utilised rather elegantly as follows. Suppose we have k islands $A_1, \ldots, A_{k-1}$ and B. We want the probability $P(A, B)$ that B is reached before any of the other islands $A_1, \ldots, A_{k-1}$; what we have is $P_s(A, B)$, which may be interpreted as the probability of ever reaching B at all, since in the single-island problem we sail 'through' the other islands if we reach them, rather than stopping at the first one.

But now *one* of the islands (A_j, say) must be reached first if one is to reach B, and from that island the probability of going on to B is $P_s(A_j, B)$. The probability of reaching B can thus be calculated as the sum of the probabilities of the mutually exclusive events 'A_j reached first and then B reached from A_j'; that is

$$P_s(A, B) = P(A, B) + \sum_{j=1}^{k-1} P(A, A_j)P_s(A_j, B). \tag{1}$$

Equations such as (1) can be derived for each of the islands $A_1, \ldots, A_{k-1}$ as well as for the island B; the coefficients $P_s(A, B)$, $P_s(A_j, B)$ are known from the single-island problem; and hence one can solve the k equations in the k unknowns $P(A, A_1), \ldots, P(A, A_{k-1})$, $P(A, B)$ and the multi-island problem is solved once the single-island solution is known.

To take into account the possibility of death whilst sailing, one needs to take $P(A, B)$ as the probability of reaching B before dying. Now (1) no longer holds in general, because the event 'reach A_j first then reach B' no longer factorises conveniently as $P(A, A_j)P_s(A_j, B)$, for after 'landing' upon A_j one has already used up some of the 'lifetime' of the crew. This does not matter if the lifetime is infinite, or if it has an exponential distribution, when the probability of living longer than t is $e^{-\lambda t}$ for some $\lambda > 0$. Note that here, if we arrive on A_j at time s, the probability of living longer than $s + t$ *given* we have lived until s is $e^{-\lambda(t+s)}/e^{-\lambda s}$ or just $e^{-\lambda t}$ again. However, neither of these distributions is realistic, and Bover also shows how the

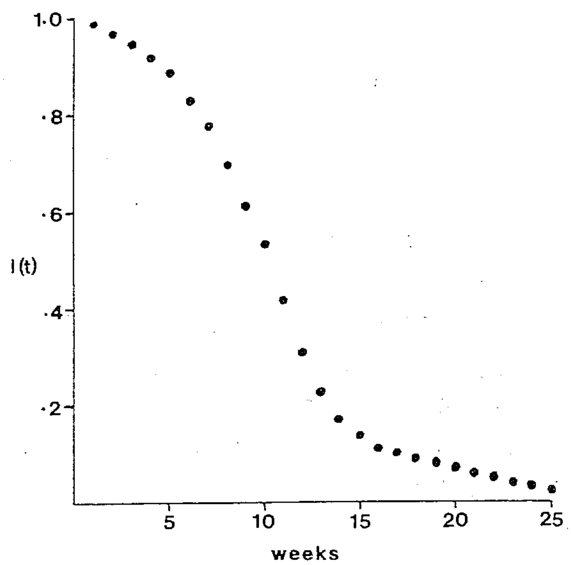

Figure 3. Life expectancy curve in Pacific conditions: $l(t)$ = Probability of surviving more than time t (from reference 1).

problem can be solved using a survival curve such as Figure 3, which is estimated from data collected on U.S. servicemen adrift in the Pacific during World War II. From these models one obtains the sort of map shown in Figure 4 for probabilities of live contact from island to island.

Let us now turn to the complementary problem of finding the probability of viability of an initial population once it has managed to reach an island. Various

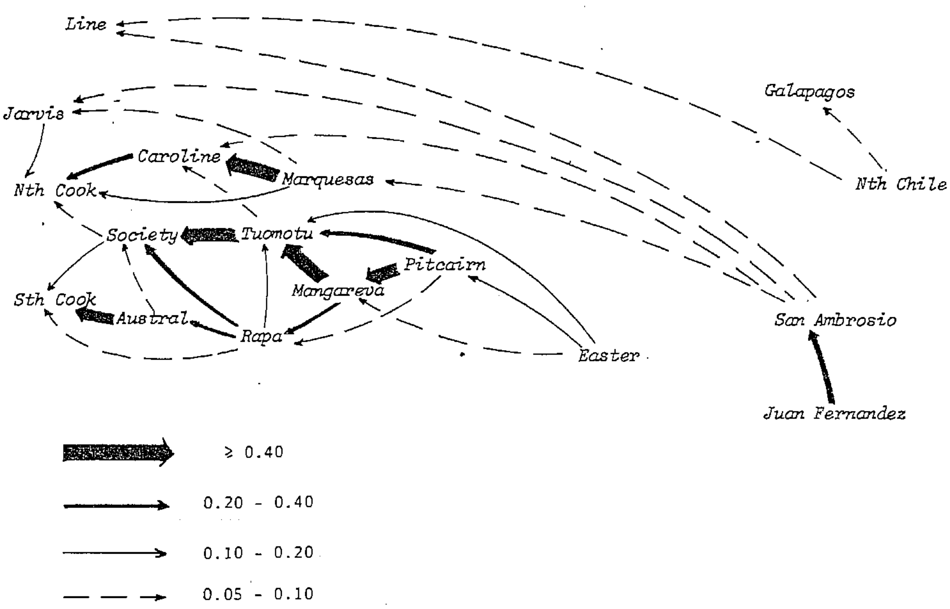

Figure 4. Contact probabilities in ocean drift models (from reference 1).

anthropologists have given estimates of population sizes needed for viability: some have suggested 50 is the minimum viable size, which would mean that all successful colonizations must have been the result of deliberate expeditions rather than the chance blowing-away-from-home of a small boatload, whilst other authors claim still higher numbers, implying even greater expeditionary organization on the part of the Polynesian settlers.

In two recent papers, Norma McArthur, Ian Saunders and I (references 3 and 4) have studied a simple simulation method designed to answer the probabilistic question of viability. The basic data consist of some numerical age-specific death and fertility rates, and a certain amount of knowledge of social customs: for example, Polynesians (despite popular misconceptions) are essentially monogamous, and close incest is taboo.

It is these latter aspects which make realistic analytic models difficult. For example, the most widely used population model in probability theory, the branching process, normally demands that all members of the population reproduce with identical offspring probabilities (i.e. have the same fertility rates) and also that they reproduce independently of each other. Even if we consider only the female population, the latter constraint fails to hold if only married women can give birth.

The model we studied uses a technique known as micro-simulation. The computer is used to store all relevant information (age, sex, whether married, etc.) for each individual in the entire population. Numbers are generated at the end of each 'year' to determine whether a female will give birth or whether a person will die, all with the probabilities relevant to their age or sex. By storing information on parents or grandparents as well, any desired level of incest can be prohibited when 'arranging' marriage.

Perhaps the most interesting part of such a model is the marriage rule used. In the sociological literature there are various quite complex rules involving probabilities of meetings between persons of various ages, probabilities of attraction given such meetings, probabilities of marrying the first, second, or subsequent attractive person one meets. For our purpose such rules proved much too sophisticated, especially when considering very small populations where the number of available mates is very small. Instead of adopting any probabilistic rules, we used a simple patriarchal rule: females became eligible for marriage at the age of 14, males at 18, and each 'year' the oldest unmarried eligible male married the youngest unmarried eligible female, the second oldest married the second youngest female, and so on. In simulations where incest was taboo this rule was altered in the obvious way. As well as being simple, such a rule ensured that all females were married (and hence fulfilling their potential for childbearing) as early as possible, which is obviously realistic when a small population is attempting to survive and grow.

The constitution of initial populations varied in size and age structure, and some of the results for populations consisting initially of 3, 5, 7 or 10 married couples with ages of males averaging 22, and ages of females averaging 19, are given in Table 1.

TABLE 1. Probability of extinction of Polynesian groups: average age of females = 19, average age of males = 22.

Initial size	Incest taboo	Incest not taboo
3 couples	0·65	0·70
5 couples	0·30	0·15
7 couples	0·00	0·15
10 couples	0·00	0·05

Here the populations are classified as successful if they either (a) 'exploded' in the sense that the total number of males or females who had ever lived exceeded 500 (a constraint which reflected the storage capacity of the computer), or (b) reached a time limit of 500 years and then achieved a population size of more than 30. Otherwise they are classified as failed (mostly by actually dying out completely before 500 years was reached), unless at 500 years they had a population size between 10 and 30, when they were allocated to successes or failures with probability $\frac{1}{2}$ each time. It is clear from Table 1 that 30 is a conservative value, and that even a population of half that number should be classified as more likely to be viable than not.

The overall study, for other starting ages, did not contain too many surprises. It was clear that the older the starting groups, the less viable their colony, and that in any age group the bigger colonies had a better chance of success. Somewhat unexpectedly, the influence of incest prohibitions was not strong, although it had some effect on the viability of the smallest starting groups; however, it seemed clear that if a population was going to survive it generally grew rapidly enough for a reasonable supply of non-related marriage partners to become available.

The micro-simulation approach made it clear that a group with constant mortality and fertility rates is fairly likely to survive even if its initial size is rather small; this result is in contrast to the frequently accepted anthropological view. Taken with the findings of Bover (reference 1) in Figure 4, which also show a high probability that random drifting will succeed as a method of island-hopping, these results indicate that much of the spread of the Polynesian population through their scattered island groups could be accounted for by the hypothesis of accidental voyages in relatively small canoes.

The detailed interpretation of computer simulations is always dubious, and not all factors can be taken into account. Nonetheless, the detailed model described here incorporates far more realistic conditions that could be considered in an analytic model, and throws considerable light on what could have happened in a 'random' Polynesian population.

Moreover, negative conclusions from such simulations can also serve as a warning about the detailed interpretation of real situations. For example, our simulated Polynesian populations, all growing from common sizes in common conditions, became highly diverse in size and composition as time passed, and this made it clear that any estimate as to their time of origin must be very inaccurate. It is not uncommon for anthropologists to infer, from the present size of an island population, the date of settlement of the island; since real populations fluctuate much less predictably than our simulated ones, such inferences seem to be extremely dubious.

As computers become more powerful, the ease of similar computer based studies will become greater. They should never replace analytic models when these are available, although in some cases analysis can reinforce the results of simulation, as in the models for migration. But there is no doubt that we shall see mathematics entering many more non-traditional fields through the use of computers; computing is becoming as necessary a tool for the mathematician of the future as algebra and calculus have been for the mathematician of the past.

Acknowledgements

I am grateful to Norma McArthur, Ian Saunders and David Bover for comments, and to David Bover for permission to reproduce the figures from reference 1 here.

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An Old Fable with a New Twist

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There is a well-known fable of the simple but numerate peasant who, when offered any gift he desired by his King, chose what he described as 'the reasonable quantity of wheat, Sire, amassed by placing one grain of wheat on the first square of a chess board, two grains on the second, four on the third, eight on the fourth, and so on till all the squares are filled'; the final number of grains to be placed on the sixty-fourth square is of course 2^{63} and the value of the wheat enough either to impoverish the King or, more likely on reflection, to lose the peasant his head.

Consider now an infinite sequence of squares as indicated below, each containing a number $\frac{1}{n}$.

1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}\ldots$

Had the peasant chosen as his gift 1 oz of gold on the first square, $\frac{1}{2}$ oz on the second, $\frac{1}{3}$ oz on the third, and so on, and also the privilege of saying how long the procedure shall be carried out, then again the request would be looked upon with displeasure by the King since the series $S = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$ has an 'infinite sum'; i.e. the partial sums can be made as large as we please by taking a large enough number of terms (see for example *Mathematical Spectrum* Vol. 9, page 33, where it is shown that if n is any integer greater than 1 and $m = (3^n - 1)/2$, then $1 + \frac{1}{2} + \cdots + \frac{1}{m} > n$).

Now if the King, not wishing to disappoint his loyal serf, agrees to this second request with the proviso that each square with a digit 7 (or, indeed, any one of the digits $1, 2, \ldots, 9$) appearing in the denominator be omitted, has he made a royal blunder or can he meet the peasant's demands?

Solution. We consider the following system of inequalities:

$$1 + \frac{1}{2} + \cdots + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} < 8$$

$$\left.\begin{array}{l}
\dfrac{1}{10} + \dfrac{1}{11} + \cdots + \dfrac{1}{16} + \dfrac{1}{18} + \dfrac{1}{19} < \dfrac{9}{10} \\
\cdots \\
\dfrac{1}{60} + \dfrac{1}{61} + \cdots + \dfrac{1}{66} + \dfrac{1}{68} + \dfrac{1}{69} < \dfrac{9}{10} \\
\dfrac{1}{80} + \dfrac{1}{81} + \cdots + \dfrac{1}{86} + \dfrac{1}{88} + \dfrac{1}{89} < \dfrac{9}{10} \\
\dfrac{1}{90} + \dfrac{1}{91} + \cdots + \dfrac{1}{96} + \dfrac{1}{98} + \dfrac{1}{99} < \dfrac{9}{10}
\end{array}\right\} = 8\left(\frac{9}{10}\right).$$

$$\left.\begin{array}{l}
\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{106}+\frac{1}{108}+\cdots+\frac{1}{199}<\frac{81}{100}\\
\cdots\\
\frac{1}{600}+\frac{1}{601}+\cdots+\frac{1}{606}+\frac{1}{608}+\cdots+\frac{1}{699}<\frac{81}{100}\\
\frac{1}{800}+\frac{1}{801}+\cdots+\frac{1}{806}+\frac{1}{808}+\cdots+\frac{1}{899}<\frac{81}{100}\\
\frac{1}{900}+\frac{1}{901}+\cdots+\frac{1}{906}+\frac{1}{908}+\cdots+\frac{1}{999}<\frac{81}{100}
\end{array}\right\} = 8\left(\frac{9}{10}\right)^2$$

$$\left.\begin{array}{l}
\frac{1}{1000}+\frac{1}{1001}+\cdots+\frac{1}{1006}+\frac{1}{1008}+\cdots+\frac{1}{1999}<\frac{729}{1000}\\
\cdots\\
\frac{1}{6000}+\frac{1}{6001}+\cdots+\frac{1}{6006}+\frac{1}{6008}+\cdots+\frac{1}{6999}<\frac{729}{1000}\\
\frac{1}{8000}+\frac{1}{8001}+\cdots+\frac{1}{8006}+\frac{1}{8008}+\cdots+\frac{1}{8999}<\frac{729}{1000}\\
\frac{1}{9000}+\frac{1}{9001}+\cdots+\frac{1}{9006}+\frac{1}{9008}+\cdots+\frac{1}{9999}<\frac{729}{1000}
\end{array}\right\} = 8\left(\frac{9}{10}\right)^3$$

and so on.

Therefore the sum of any finite number of terms of the (depleted) series is less than

$$8+8\left(\frac{9}{10}\right)+8\left(\frac{9}{10}\right)^2+8\left(\frac{9}{10}\right)^3+\cdots=80.$$

So our peasant friend has to settle for rather less than 80 oz of gold. However, he invests it wisely, forms his own institute of mathematics, and devotes his life to the study of series. Perhaps we can keep the story of his further exploits for a future article.

Fictitious Forces

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1. Introduction

My College has a turntable for cars in a small quadrangle so that more cars can be squeezed in than could otherwise park there. Actually the Boat Club think the turntable is for their benefit and after a grand dinner to celebrate some success on the river they put their Captain on the turntable and then rotate it. Experiment shows that the Captain comes off the turntable at high speed. Why? The usual answer is 'centrifugal force', but many sixth-formers nowadays tell me that centrifugal force does not exist—that it is 'fictitious'. How would any such sixth-former persuade the Captain of Boats that his high-speed departure from the turntable was not caused by a real force?

Many people have seen the Foucault pendulum swinging in the stairwell of the Science Museum in London. Most of those I ask tell me that the plane the pendulum swings in rotates at the rate of once a day (some even tell me they mean a sidereal not a solar day). Actually it would take about 30 hours for a complete revolution (if friction did not bring it to rest sooner).

Many students believe that objects which are dropped fall, initially at least, towards the centre of the Earth. Actually they fall vertically, i.e. perpendicular to the horizontal plane, which, since the Earth is not a perfect sphere, is not perpendicular to the Earth's radius. Indeed we may ask why the Earth is not spherical and why the observed value of the gravitational acceleration varies.

How would you solve the following problem?

A pendulum is suspended freely from a point which is given a constant acceleration f at a constant angle α below the horizontal. If the pendulum is initially vertical, show that it just reaches the horizontal if $g = f(\sin \alpha + \cos \alpha)$.

Everyone has heard about weightlessness in spacecraft, but nowadays few have heard about Coriolis† and centrifugal forces—and those who have, dismiss them as 'fictitious'.

2. Moving frames of reference

Let us look at a spacecraft far away from any matter. A scientist inside does experiments. He knows from Newton's Laws that $\boldsymbol{F} = m(d^2/dt^2)\boldsymbol{r}$, where $\boldsymbol{r}$ is measured relative to some fixed point in space. Because the spacecraft is moving he cannot measure $\boldsymbol{r}$ directly; instead he measures from some reference point in the spacecraft. If his reference point is $\boldsymbol{R}$ from the fixed point and the particle is $\boldsymbol{\rho}$ from the reference point, clearly

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{\rho}$$

and

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{R}}{dt^2} + m\frac{d^2\boldsymbol{\rho}}{dt^2}$$

or

$$m\frac{d^2\boldsymbol{\rho}}{dt^2} = \boldsymbol{F} - m\frac{d^2\boldsymbol{R}}{dt^2}.$$

Now suppose the spacecraft is going along at uniform speed so that $\boldsymbol{R} = \boldsymbol{v}t$,

† Gaspard Gustave Coriolis (1792–1843) was a French engineer with a strong mathematical interest. In 1835 he published a paper in which he investigated the laws of motion relative to a rotating frame of reference. This is where the force, later named after him, was first postulated.

where $\boldsymbol{v}$ is constant. Then $m(d^2\boldsymbol{\rho}/dt^2) = \boldsymbol{F}$, so Newton's Second Law is true with local co-ordinates. Of course this is our experience on trains or aeroplanes travelling level at constant speeds.

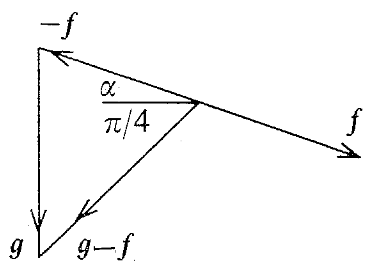
What if the spacecraft accelerates steadily, at $\boldsymbol{f}$ say, because it fires its rockets? Then $\boldsymbol{R} = \frac{1}{2}\boldsymbol{f}t^2$, where $\boldsymbol{f}$ is constant, so $m(d^2\boldsymbol{\rho}/dt^2) = \boldsymbol{F} - m\boldsymbol{f}$. This equation looks like that for ordinary motion under gravity, $m(d^2\boldsymbol{r}/dt^2) = \boldsymbol{F} + m\boldsymbol{g}$. An object dropped in this spacecraft will accelerate (relative to the spacecraft) with acceleration $-\boldsymbol{f}$. Rather like gravity! Any result we calculate with ordinary gravity will carry over into apparent gravity.

What if the spacecraft is in orbit around the Earth so that $\boldsymbol{F} = \boldsymbol{F}_{\text{non-grav}} + m\boldsymbol{g}$ ($\boldsymbol{F}_{\text{non-grav}}$ is the non-gravitational force), but its motors are not firing? Then it is accelerating at $\boldsymbol{g}$, so $d^2\boldsymbol{R}/dt^2 = \boldsymbol{g}$ and $m(d^2\boldsymbol{\rho}/dt^2) = \boldsymbol{F}_{\text{non-grav}} + m\boldsymbol{g} + m(-\boldsymbol{g}) = \boldsymbol{F}_{\text{non-grav}}$. The gravitational term has vanished. Bodies in free fall are weightless. If you want to experience weightlessness, jump. If you jump off a building 64 feet high you will have 2 seconds of weightlessness but there are disadvantages. However jump off a 4-foot-high table and you will have half a second of weightlessness and reasonable safety. You will be in orbit!

Let us return to the problem in the introduction. If we imagine accelerating with the point of suspension we have

$$m\frac{d^2\boldsymbol{\rho}}{dt^2} = \boldsymbol{F}_{\text{non-grav}} + m\boldsymbol{g} - m\boldsymbol{f},$$

so in this frame bodies behave as if gravity were $\boldsymbol{g} - \boldsymbol{f}$ instead of $\boldsymbol{g}$, i.e. $\boldsymbol{g} - \boldsymbol{f}$ is 'apparent gravity'. How does a pendulum swing? Equal angles on either side of gravity. In this case it swings from vertical to horizontal and so apparent gravity is at $45°$, i.e. $g - f\sin\alpha = f\cos\alpha$ or $g = f(\sin\alpha + \cos\alpha)$.



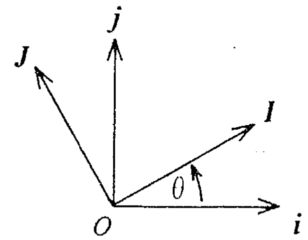
Uniform acceleration can be experienced on a train. Inside a train, can you distinguish between accelerating and going uphill?

3. Rotating frames of reference

Of course our turntable does not behave like a train. We are now forced to think about rotating axes. If unit perpendicular vectors along OX and OY are $\boldsymbol{I}, \boldsymbol{J}$ and these rotate relative to fixed vectors $\boldsymbol{i}, \boldsymbol{j}$ along Ox, Oy, then

$$\boldsymbol{I} = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta,$$

$$\boldsymbol{J} = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta.$$



Hence

$$\frac{d\boldsymbol{I}}{dt} = -\boldsymbol{i}\dot{\theta}\sin\theta + \boldsymbol{j}\dot{\theta}\cos\theta = \dot{\theta}\boldsymbol{J},$$

where $\dot{\theta}$ represents $d\theta/dt$ (and, similarly, $\ddot{\theta} = d^2\theta/dt^2$). Also

$$\frac{d\boldsymbol{J}}{dt} = -\boldsymbol{i}\dot{\theta}\cos\theta - \boldsymbol{j}\dot{\theta}\sin\theta = -\dot{\theta}\boldsymbol{I}.$$

So any vector $\boldsymbol{a} = \alpha\boldsymbol{I} + \beta\boldsymbol{J}$ satisfies the equation

$$\frac{d\boldsymbol{a}}{dt} = \alpha\frac{d\boldsymbol{I}}{dt} + \frac{d\alpha}{dt}\boldsymbol{I} + \beta\frac{d\boldsymbol{J}}{dt} + \frac{d\beta}{dt}\boldsymbol{J} = (\dot{\alpha} - \beta\dot{\theta})\boldsymbol{I} + (\dot{\beta} + \alpha\dot{\theta})\boldsymbol{J},$$

and, if we apply this twice,

$$\begin{aligned}\frac{d^2\boldsymbol{a}}{dt^2} &= \left[\frac{d}{dt}(\dot{\alpha} - \beta\dot{\theta}) - \dot{\theta}(\dot{\beta} + \dot{\alpha}\theta)\right]\boldsymbol{I} + \left[\frac{d}{dt}(\dot{\beta} + \alpha\dot{\theta}) + \dot{\theta}(\dot{\alpha} - \beta\dot{\theta})\right]\boldsymbol{J}\\ &= (\ddot{\alpha} - 2\dot{\beta}\dot{\theta} - \beta\ddot{\theta} - \alpha\dot{\theta}^2)\boldsymbol{I} + (\ddot{\beta} + 2\dot{\alpha}\dot{\theta} + \alpha\ddot{\theta} - \beta\dot{\theta}^2)\boldsymbol{J}.\end{aligned}$$

Or, if $\boldsymbol{r} = r\boldsymbol{I}(\alpha = r, \beta = 0)$,

$$\frac{d^2\boldsymbol{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{I} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{J},$$

which gives the radial and transverse components of acceleration.

If we have uniform motion in a circle with $r = |\boldsymbol{r}| = \text{constant}$ and $\dot{\theta} = \text{constant}$,

$$\frac{d^2\boldsymbol{r}}{dt^2} = -r\dot{\theta}^2\boldsymbol{I} = -\boldsymbol{r}\dot{\theta}^2,$$

which is an acceleration directed towards the origin. This is called the centripetal acceleration. Consider a stone rotating at the end of a string; the string has a tension $mr\dot{\theta}^2$ giving the stone an acceleration $r\dot{\theta}^2$ inwards. If you sat on the stone you would not see any acceleration. But you would see the tension in the string; if you thought Newton's Second Law applied you would have to invent a force to balance the tension. This force would be called the centrifugal force as it would be directed outwards. The point is that the true acceleration is $-\boldsymbol{r}\dot{\theta}^2$ and the true force is $-m\boldsymbol{r}\dot{\theta}^2$; the apparent acceleration is zero, so the apparent total force must be zero, and we postulate a centrifugal force $m\boldsymbol{r}\dot{\theta}^2$.

Let us consider, again, a general rotation about an axis $\boldsymbol{K}$ perpendicular to $\boldsymbol{I}$ and $\boldsymbol{J}$. If the cartesian co-ordinates in the two frames are (x, y, z) and (X, Y, Z), then

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = X\boldsymbol{I} + Y\boldsymbol{J} + Z\boldsymbol{K}.$$

But, as $\boldsymbol{i}$ and $\boldsymbol{j}$ lie in the same plane as $\boldsymbol{I}$ and $\boldsymbol{J}$, $\boldsymbol{k} = \boldsymbol{K}$ and $z = Z$, so that $d^2z/dt^2 = d^2Z/dt^2$. Now, if the rotation is at the correct rate to keep the particle in the $\boldsymbol{I}, \boldsymbol{K}$ plane, then $\boldsymbol{r} = X\boldsymbol{I} + Z\boldsymbol{K}$, or, writing $r = \sqrt{(x^2 + y^2)}$ for X and z for Z, we have $\boldsymbol{r} = r\boldsymbol{I} + z\boldsymbol{K}$; and thus

$$m\frac{d^2\boldsymbol{r}}{dt^2} = m\{(\ddot{r} - r\dot{\theta}^2)\boldsymbol{I} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{J} + \ddot{z}\boldsymbol{K}\} = \boldsymbol{F} + m\boldsymbol{g},$$

where $m\boldsymbol{g}$ is the true gravitational force and $\boldsymbol{F}$ is the sum of the other forces acting. But the apparent acceleration, as seen by someone moving round with the rotation, will be $\ddot{r}\boldsymbol{I} + \ddot{z}\boldsymbol{K}$. Thus

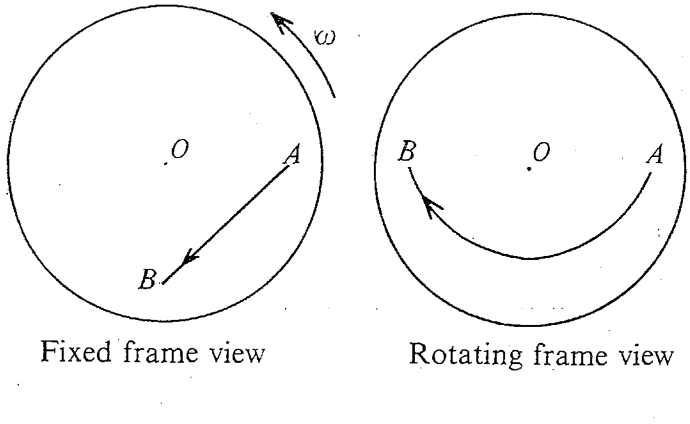
$$m(\ddot{r}\boldsymbol{I} + \ddot{z}\boldsymbol{K}) = \boldsymbol{F} + m\boldsymbol{g} + mr\dot{\theta}^2\boldsymbol{I} - 2m\dot{r}\dot{\theta}\boldsymbol{J} - mr\ddot{\theta}\boldsymbol{J}.$$

We have mass times apparent acceleration equals true forces ($\boldsymbol{F}$ and $m\boldsymbol{g}$) plus fictitious forces ($mr\dot{\theta}^2\boldsymbol{I}$ the centrifugal force, $-2m\dot{r}\dot{\theta}\boldsymbol{J}$ a velocity-dependent term known as the Coriolis force, and $-mr\ddot{\theta}\boldsymbol{I}$ the angular acceleration force).

How can the Captain of Boats survive on the turntable if it is rotating uniformly ($\ddot{\theta} = 0$)? If he is at the centre he should stay there. Otherwise he should keep still and lean in, so that $\dot{r} = \ddot{r} = 0$ and $\boldsymbol{F} = -m\boldsymbol{g} - mr\dot{\theta}^2\boldsymbol{I}$. An outside observer would say that the frictional force between his feet and the turntable pushes him inwards, just enough to give him the centripetal acceleration he needs to stay at rest relative to the turntable. He would say the centrifugal force tries to push him out.

Suppose the Captain of Boats and his Coxswain play ball—tossing it from one to the other as they rotate on the turntable, say at opposite ends of a diameter. If, while the ball is in the air, the turntable rotates a quarter-turn, an observer looking from above would see the ball moving in a vertical plane. But the Captain would see the ball moving along a curved path.

The two diagrams show the two views of the ball's path from the Captain (A) to the Coxswain (B). On the left the path seen by fixed observers who regard the turntable as rotating, on the right the path of the ball relative to the turntable, i.e. the Captain's view. In polar co-ordinates, the path $(r(t), \theta(t))$ relative to fixed observers



Fixed frame view   Rotating frame view becomes $(r(t), \theta(t) - \omega t)$ relative to the turntable, where ω is the constant angular velocity of the turntable.

4. Motion on the rotating Earth

This example is not as artificial as it may appear because we all live on a rotating Earth. How should we predict motion? Clearly it depends on the observer. A Martian might work in a proper Newtonian frame. But, when school students measure acceleration, they use trolleys and ticker-tape moving relative to the Earth. It is the relative motion which we measure.

However, if we ought to put all these corrections into our equations, why do text-books ignore them? Let us examine the magnitudes of the terms that are usually omitted. The centrifugal force on a particle due to the rotation of the Earth is $m\boldsymbol{r}\dot{\theta}^2$, where $|\boldsymbol{r}|$ is the distance from the earth's axis to the particle (say 5×10^6 metres in England) and θ is the angular velocity of the Earth (about 7×10^{-5} radians per second). So the centrifugal force is equivalent to a gravitational acceleration of $2{\cdot}5 \times 10^{-2}\,\text{m/s}^2$, about $0{\cdot}25\,\%$ of ordinary gravity.

Consider how we weigh a body with a spring balance. We have the body at rest ($\ddot{\boldsymbol{r}} = \dot{\boldsymbol{r}} = \boldsymbol{0}$) and the Earth has a constant spin ($\ddot{\theta} = 0$). Hence, if the true gravitational force is $m\boldsymbol{g}_{\text{true}}$, the equation

$$m(\ddot{r}\boldsymbol{I} + \ddot{z}\boldsymbol{K}) = \boldsymbol{F} + m\boldsymbol{g}_{\text{true}} + mr\dot{\theta}^2\boldsymbol{I} - 2m\dot{r}\dot{\theta}\boldsymbol{J} - mr\ddot{\theta}\boldsymbol{J}$$

becomes

$$\boldsymbol{0} = \boldsymbol{F} + m\boldsymbol{g}_{\text{true}} + mr\dot{\theta}^2\boldsymbol{I}.$$

Here F is the reading on the spring balance and we have

$$-\boldsymbol{F} = m(\boldsymbol{g}_{\text{true}} + r\dot{\theta}^2\boldsymbol{I}).$$

We may well call this reading $m\boldsymbol{g}_{\text{apparent}}$, the apparent weight.

So when we measure $\boldsymbol{g}$ on the Earth we measure a mixture of true gravity and centrifugal force.

What about the Coriolis force $-2m\dot{r}\dot{\theta}\boldsymbol{J}$? Unless the body has a high speed (e.g. a cannon shell) this will be small, so this force can usually be ignored. But if it acts for a long time (as in the Foucault pendulum) it will have an effect.

Weather maps show areas of high pressure and areas of low pressure and you may have noticed that the winds do not blow directly from one to the other, but rotate anticlockwise around the low-pressure areas in the northern hemisphere. This is caused by the Coriolis forces.

Of course the Earth rotates about a line between the poles, not about a vertical line like my College turntable. As angular velocity is a vector along the axis of the Earth, you may think it fairly natural that the effective rate of rotation about the vertical is just $\omega \sin \lambda$ (where ω is the angular velocity about the poles and λ is the apparent latitude, the angle of the Pole Star above the horizon). So we would expect the Foucault pendulum to rotate, relative to the Earth, at $-\omega \sin \lambda$ (i.e. clockwise in the northern hemisphere and anti-clockwise in the southern hemisphere). (Incidentally if you go to the Botanic Gardens in Capetown and look at the old sundial there you will understand why clocks go round the way they do.)

Note that we need the epithet 'apparent' to describe the latitude, because the true latitude, which is the angle between the radius vector and the equatorial plane, would only equal the apparent latitude everywhere if the Earth were a perfect sphere—and we know it is not. And we can now see why it is not. The sea will always try to lie flat (the tides are small compared with the Earth's radius) and the horizontal is therefore perpendicular to the vertical direction of apparent gravity. The direction of apparent gravity $\boldsymbol{g}_{\text{apparent}} = \boldsymbol{g}_{\text{true}} + r\dot{\theta}^2$ is (for a point in the northern hemisphere) to the south of the direction of true gravity, but lies in the plane of the axis of the Earth and the point being considered. (Even though the Earth is not a sphere it is a solid of revolution about its axis.) 'Apparent latitude' is thus slightly larger than 'true latitude'. And the opposite direction (vertically upwards) is slightly nearer the north than the outward radius vector (if we assume that true gravity is along the inward radius). This has the effect of pushing in the poles, so that the Earth's polar radius is about 21·5 km shorter than the equatorial radius.

5. The Foucault pendulum

A more convincing proof that the rate of rotation of a Foucault pendulum is $-\omega \sin \lambda$ can be obtained by using the techniques of multiplication of vectors, and I will develop these techniques for those readers who have not yet studied them.

If we have two vectors $\boldsymbol{a} \equiv (a_x, a_y, a_z)$ and $\boldsymbol{b} \equiv (b_x, b_y, b_z)$, the scalar product $\boldsymbol{a} \,.\, \boldsymbol{b}$ is defined as the scalar $a_x b_x + a_y b_y + a_z b_z$. Clearly $\boldsymbol{a} \,.\, \boldsymbol{b} = \boldsymbol{b} \,.\, \boldsymbol{a}$ and $(\boldsymbol{a} + \boldsymbol{c}) \,.\, \boldsymbol{b} = (\boldsymbol{a} \,.\, \boldsymbol{b}) + (\boldsymbol{c} \,.\, \boldsymbol{b})$. Moreover $(\alpha \boldsymbol{a}) \,.\, \boldsymbol{b} = \alpha(\boldsymbol{a} \,.\, \boldsymbol{b})$ when α is a real number. Also $(d/dt)(\boldsymbol{a} \,.\, \boldsymbol{b}) = (d\boldsymbol{a}/dt) \,.\, \boldsymbol{b} + \boldsymbol{a} \,.\, (d\boldsymbol{b}/dt)$.

The vector product $\boldsymbol{a} \times \boldsymbol{b}$ (sometimes written as $\boldsymbol{a} \wedge \boldsymbol{b}$) is defined as the vector $(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$, which is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. Now $\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$, but $(\boldsymbol{a} + \boldsymbol{c}) \times \boldsymbol{b} = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \boldsymbol{b})$, $(\alpha \boldsymbol{a}) \times \boldsymbol{b} = \alpha(\boldsymbol{a} \times \boldsymbol{b})$, and $(d/dt)(\boldsymbol{a} \times \boldsymbol{b}) = (d\boldsymbol{a}/dt) \times \boldsymbol{b} + \boldsymbol{a} \times (d\boldsymbol{b}/dt)$. Also, from the definitions,

$$\boldsymbol{a} \,.\, (\boldsymbol{a} \times \boldsymbol{b}) = a_x(a_y b_z - a_z b_y) + a_y(a_z b_x - a_x b_z) + a_z(a_x b_y - a_y b_z) = 0.$$

Suppose we have three mutually perpendicular unit vectors $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ forming a set of moving axes. Then $\boldsymbol{I} \times \boldsymbol{J} = \boldsymbol{K}$ and $\boldsymbol{I} \,.\, \boldsymbol{J} = 0$ while $\boldsymbol{I} \times \boldsymbol{I} = \boldsymbol{0}$ and $\boldsymbol{I} \,.\, \boldsymbol{I} = 1$.

Hence differentiating the identities $\boldsymbol{I} \,.\, \boldsymbol{J} = 0$ and $\boldsymbol{I} \,.\, \boldsymbol{I} = 1$, we have $(d\boldsymbol{I}/dt) \,.\, \boldsymbol{J} + \boldsymbol{I} \,.\, (d\boldsymbol{J}/dt) = 0$ and $\boldsymbol{I} \,.\, (d\boldsymbol{I}/dt) = 0$.

Now consider the vector $\boldsymbol{\omega}$ defined by

$$\boldsymbol{\omega} = \left(\frac{d\boldsymbol{J}}{dt} \cdot \boldsymbol{K}\right)\boldsymbol{I} + \left(\frac{d\boldsymbol{K}}{dt} \cdot \boldsymbol{I}\right)\boldsymbol{J} + \left(\frac{d\boldsymbol{I}}{dt} \cdot \boldsymbol{J}\right)\boldsymbol{K}.$$

Then

$$\begin{aligned}\boldsymbol{\omega} \times \boldsymbol{I} &= \left(\frac{d\boldsymbol{J}}{dt} \cdot \boldsymbol{K}\right)(\boldsymbol{I} \times \boldsymbol{I}) + \left(\frac{d\boldsymbol{K}}{dt} \cdot \boldsymbol{I}\right)(\boldsymbol{J} \times \boldsymbol{I}) + \left(\frac{d\boldsymbol{I}}{dt} \cdot \boldsymbol{J}\right)(\boldsymbol{K} \times \boldsymbol{I}) \\ &= \left(\frac{d\boldsymbol{J}}{dt} \cdot \boldsymbol{K}\right)\boldsymbol{0} + \left(\frac{d\boldsymbol{K}}{dt} \cdot \boldsymbol{I}\right)(-\boldsymbol{K}) + \left(\frac{d\boldsymbol{I}}{dt} \cdot \boldsymbol{J}\right)\boldsymbol{J} \\ &= \left(-\boldsymbol{K} \cdot \frac{d\boldsymbol{I}}{dt}\right)(-\boldsymbol{K}) + \left(\frac{d\boldsymbol{I}}{dt} \cdot \boldsymbol{J}\right)\boldsymbol{J},\end{aligned}$$

since

$$\frac{d\boldsymbol{I}}{dt}\cdot\boldsymbol{K}+\boldsymbol{I}\cdot\frac{d\boldsymbol{K}}{dt}=\frac{d}{dt}(\boldsymbol{I}\,.\,\boldsymbol{K})=0.$$

But $d\boldsymbol{I}/dt$ is a vector. Hence, if its components along $\boldsymbol{I}, \boldsymbol{J}$ and $\boldsymbol{K}$ are α, β, γ, i.e. if $d\boldsymbol{I}/dt = \alpha\boldsymbol{I} + \beta\boldsymbol{J} + \gamma\boldsymbol{K}$, then $\boldsymbol{I}\,.\,(d\boldsymbol{I}/dt) = \alpha$, $\boldsymbol{J}\,.\,(d\boldsymbol{I}/dt) = \beta$ and $\boldsymbol{K}\,.\,(d\boldsymbol{I}/dt) = \gamma$, since $\boldsymbol{I}\,.\,\boldsymbol{I} = 1$ and $\boldsymbol{I}\,.\,\boldsymbol{J} = 0 = \boldsymbol{I}\,.\,\boldsymbol{K}$ etc.

But, since $2\boldsymbol{I}\,.\,(d\boldsymbol{I}/dt) = d/dt(\boldsymbol{I}\,.\,\boldsymbol{I}) = 0$, $\alpha = 0$ and

$$\frac{d\boldsymbol{I}}{dt}=\left(\boldsymbol{J}\cdot\frac{d\boldsymbol{I}}{dt}\right)\boldsymbol{J}+\left(\boldsymbol{K}\cdot\frac{d\boldsymbol{I}}{dt}\right)\boldsymbol{K}=\boldsymbol{\omega}\times\boldsymbol{I}.$$

Similarly

$$\frac{d\boldsymbol{J}}{dt}=\boldsymbol{\omega}\times\boldsymbol{J}\qquad\text{and}\qquad\frac{d\boldsymbol{K}}{dt}=\boldsymbol{\omega}\times\boldsymbol{K}.$$

Hence, if we have a general vector $\boldsymbol{a}$ with co-ordinates (a_x, a_y, a_z) relative to the moving axes, so that $\boldsymbol{a} = a_x\boldsymbol{I} + a_y\boldsymbol{J} + a_z\boldsymbol{K}$, then the rate of change of $\boldsymbol{a}$ (relative to a fixed space) is

$$\begin{aligned}\frac{d\boldsymbol{a}}{dt}&=\frac{da_x}{dt}\boldsymbol{I}+a_x\frac{d\boldsymbol{I}}{dt}+\frac{da_y}{dt}\boldsymbol{J}+a_y\frac{d\boldsymbol{J}}{dt}+\frac{da_z}{dt}\boldsymbol{K}+a_z\frac{d\boldsymbol{K}}{dt}\\&=\frac{da_x}{dt}\boldsymbol{I}+\frac{da_y}{dt}\boldsymbol{J}+\frac{da_z}{dt}\boldsymbol{K}+a_x(\boldsymbol{\omega}\times\boldsymbol{I})+a_y(\boldsymbol{\omega}\times\boldsymbol{J})+a_z(\boldsymbol{\omega}\times\boldsymbol{K})\\&=\frac{da_x}{dt}\boldsymbol{I}+\frac{da_y}{dt}\boldsymbol{J}+\frac{da_z}{dt}\boldsymbol{K}+\boldsymbol{\omega}\times\boldsymbol{a}.\end{aligned}$$

Now if we measure $\boldsymbol{a}$ relative to the moving axes $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ then the apparent rate of change of $\boldsymbol{a}$, which we will write as $\partial\boldsymbol{a}/\partial t$, is

$$\frac{da_x}{dt}\boldsymbol{I}+\frac{da_y}{dt}\boldsymbol{J}+\frac{da_z}{dt}\boldsymbol{K}.$$

So the total rate of change

$$\frac{d\boldsymbol{a}}{dt}=\frac{\partial\boldsymbol{a}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{a}.$$

The true rate of change is the apparent rate of change plus a rotation term $\boldsymbol{\omega} \times \boldsymbol{a}$. If $\partial\boldsymbol{a}/\partial t = 0$, $d\boldsymbol{a}/dt = \boldsymbol{\omega} \times \boldsymbol{a}$, which shows that $\boldsymbol{\omega}$ is the angular velocity. For if $d\boldsymbol{a}/dt = 0$ and $\partial\boldsymbol{a}/\partial t = 0$, the vector $\boldsymbol{a}$ is fixed in both frames of reference and must be parallel to $\boldsymbol{\omega}$. (Solve $\boldsymbol{\omega} \times \boldsymbol{a} = \boldsymbol{0}$ for given $\boldsymbol{\omega}$.)

If $\boldsymbol{a} = \boldsymbol{r}$, then

$$\frac{d\boldsymbol{r}}{dt}=\frac{\partial\boldsymbol{r}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{r}.$$

If $\boldsymbol{a} = \boldsymbol{\omega}$, then

$$\frac{d\boldsymbol{\omega}}{dt}=\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{\omega}\times\boldsymbol{\omega}=\frac{\partial\boldsymbol{\omega}}{\partial t}.$$

And if $\boldsymbol{a} = (d\boldsymbol{r}/dt)(=(\partial \boldsymbol{r}/\partial t) + \boldsymbol{\omega} \times \boldsymbol{r})$, then

$$\frac{d}{dt}\frac{d\boldsymbol{r}}{dt} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}\right) + \boldsymbol{\omega} \times \left(\frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{r}\right),$$

so that

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + \frac{\partial \boldsymbol{\omega}}{\partial t} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

(You can see that this equation corresponds exactly to

$$\frac{d^2(r\boldsymbol{I})}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{I} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{J},$$

which was obtained in the case of rotation of the axes about $\boldsymbol{K}$.) For the Earth, $\boldsymbol{\omega}$ is constant, so we just have

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2} + 2\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

The Newtonian equation of motion

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F} + m\boldsymbol{g}_{\text{true}}$$

(where $m\boldsymbol{g}_{\text{true}}$ is the true force attracting a mass m towards the Earth), becomes

$$m\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \boldsymbol{F} + m\boldsymbol{g}_{\text{true}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2m\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t}.$$

On the left we have the apparent acceleration (which we measure with ticker-tape); on the right to balance the equation we have the apparent forces—the sum of the true forces $\boldsymbol{F}$ and $m\boldsymbol{g}_{\text{true}}$, and fictitious forces $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ and $-2m\boldsymbol{\omega} \times (\partial \boldsymbol{r}/\partial t)$.

If we consider a spring balance (with $\partial^2\boldsymbol{r}/\partial t^2 = \boldsymbol{0} = \partial \boldsymbol{r}/\partial t$), we are led to putting $\boldsymbol{g}_{\text{true}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{g}_{\text{apparent}}$ (the apparent gravity, which is naturally vertical since that is the direction of a plumb line). Therefore in general we have

$$m\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \boldsymbol{F} + m\boldsymbol{g}_{\text{apparent}} - 2m\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t}$$

which is exactly what we are used to except for the Coriolis force $-2m\boldsymbol{\omega} \times (\partial \boldsymbol{r}/\partial t)$.

Now consider a Foucault pendulum of length l. If the bob (of mass m) is displaced, say x eastwards, y northwards and z up from its position of equilibrium which we will call the origin $(0, 0, 0)$ (so that the point of suspension is at $(0, 0, l)$), the tension T in the string is along the string, i.e. along the vector $(-x, -y, l - z)$ (which has, of course, length l). Hence

$$\boldsymbol{F} = \left(\frac{-x}{l}, \frac{-y}{l}, 1 - \frac{z}{l}\right)T.$$

Now $\boldsymbol{g}_{\text{apparent}} = (0, 0, -g)$, since apparent gravity acts downwards. Also, as $\boldsymbol{\omega}$ is in the direction of the Pole Star and so lies in the plane of the vertical and the north, $\boldsymbol{\omega} = (0, \omega\cos\lambda, \omega\sin\lambda)$, where λ is the apparent latitude (i.e. angle of the Pole Star above the horizon). Therefore, if $\boldsymbol{r} = (x, y, z)$,

$$\boldsymbol{\omega} \times \frac{\partial \boldsymbol{r}}{\partial t} = (\omega\dot{z}\cos\lambda - \omega\dot{y}\sin\lambda, \omega\dot{x}\sin\lambda, -\omega\dot{x}\cos\lambda)$$

and we have

$$m\ddot{x} = \frac{-xT}{l} + 2m\omega\dot{y}\sin\lambda - 2m\omega\dot{z}\cos\lambda,$$

$$m\ddot{y} = \frac{-yT}{l} - 2m\omega\dot{x}\sin\lambda,$$

$$m\ddot{z} = T - \frac{zT}{l} - mg + 2m\omega\dot{x}\cos\lambda.$$

Now, for small oscillations, z is small compared with $(x^2+y^2)^{1/2}$ and l. Since $x^2 + y^2 + (l-z)^2 = l^2$, $z^2 - 2lz + (x^2+y^2) = 0$ giving $z = l \pm (l^2 - x^2 - y^2)^{1/2}$. The $+$ sign gives the z corresponding to the pendulum bob above its point of suspension and so $z = l - (l^2 - x^2 - y^2)^{1/2}$ for a small oscillation in which $(x^2+y^2) \ll l^2$. ($\ll$ means 'is much less than'.) Hence

$$z = l - l\left(1 - \frac{x^2+y^2}{l^2}\right)^{1/2} = l - l\left(1 - \frac{x^2+y^2}{2l^2} + \text{smaller terms}\right).$$

Thus

$$z \simeq \frac{x^2+y^2}{2l} \qquad \text{and} \qquad \dot{z} \simeq \frac{x\dot{x} + y\dot{y}}{l}.$$

Hence $\dot{z}$ is also very small compared with the larger $\dot{x}$ and $\dot{y}$. Similarly $\ddot{z}$ is very small. Now ω is small, so in the third equation $\omega\dot{x}$, z and $\ddot{z}$ are all small compared with the other terms and

$$T = mg + O(z, \ddot{z}, \omega\dot{x})$$

where the small terms are shown in the bracket following the O.

If we multiply the second equation by $i(\sqrt{-1})$ and add it to the first equation, we have

$$m(\ddot{x} + i\ddot{y}) = -\frac{T}{l}(x + iy) + 2m\omega(\dot{y} - i\dot{x})\sin\lambda - 2m\omega\dot{z}\cos\lambda.$$

Also $|z| = |x + iy|^2/2l$ and so, writing $x + iy = \zeta$ and substituting for T, we get

$$m\ddot{\zeta} = -\frac{\zeta}{l}\{mg + O(z, \ddot{z}, \omega\dot{x})\} - 2m\omega i\dot{\zeta}\sin\lambda - O(\omega z).$$

Thus

$$\ddot{\zeta} + 2i\omega\dot{\zeta}\sin\lambda + \frac{g\zeta}{l} = O(\zeta z, \zeta\ddot{z}, \omega\zeta\dot{x}, \omega\dot{z})$$

where all the terms on the right-hand side are very small compared with the terms on the left and may therefore be neglected.

The equation

$$\ddot{\zeta} + 2i\omega\dot{\zeta}\sin\lambda + \frac{g\zeta}{l} = 0$$

has the standard solution

$$\zeta = Ae^{\alpha_1 t} + Be^{\alpha_2 t},$$

where A and B are arbitrary complex constants and α_1, α_2 are the roots of

$$\alpha^2 + 2i\omega\alpha\sin\lambda + \frac{g}{l} = 0,$$

i.e.

$$\alpha_1, \alpha_2 = -i\omega\sin\lambda \pm \left(-\omega^2\sin^2\lambda - \frac{g}{l}\right)^{1/2}.$$

Now, since ω is very small, we can neglect $l\omega^2$ compared with g. Hence

$$\alpha_1, \alpha_2 = -i\omega\sin\lambda \pm i\sqrt{g/l}$$

and

$$\zeta = x + iy = e^{-i\omega t\sin\lambda}(C\cos\sqrt{g/l}t + D\sin\sqrt{g/l}t),$$

where C and D are arbitrary complex constants.

A particular solution for the Foucault pendulum could have C real and $D = 0$, so that

$$x = C\cos(\omega t\sin\lambda)\cos\sqrt{g/l}t$$

and

$$y = -C\sin(\omega t\sin\lambda)\cos\sqrt{g/l}t.$$

In circular polar co-ordinates $r = C\cos\sqrt{g/l}t$ and $\theta = -\omega t\sin\lambda$ (at least when $\cos\sqrt{g/l}t$ is positive), which means that the plane of the Foucault pendulum is rotating clockwise at the angular rate $\omega\sin\lambda$. So it will take about 30 hours for a complete rotation in London.

6. Zero gravity in spacecraft

I have shown that a frame which is moving with uniform acceleration f can be treated as a fixed frame with an extra gravitational acceleration $-f$. This may have suggested that gravity is as artificial as a fictitious force and could be removed from a spacecraft by allowing the craft to fall freely.

Many scientific and engineering experiments and activities would be easier without gravity. For example, liquids could be heated without convection currents; metals could be melted in a furnace without touching the sides of the furnace. How nearly zero might we want gravity to be? An object under an apparent gravity of 10^{-4} normal gravity (i.e. 10^{-3} m/s^2) will fall 1 cm in $4\frac{1}{2}$ seconds, which gives very little time for heating. To get something to stay within 1 mm for an hour, gravity must be reduced to $1{\cdot}6 \times 10^{-10}$ m/s^2. A spacecraft in orbit 200 km above the Earth will be slowed down by the drag of the very thin atmosphere and so will not provide a zero-gravity environment. This slowing down would eventually cause the space laboratory to fall back to earth. So to prevent this the motors must fire from time to time (to keep them firing steadily will be too difficult, as the slowing down is so slight)—so again zero-gravity will be absent. Anyone on the spacecraft who walks around will also accelerate the spacecraft.

But there is the further difficulty that gravity is not uniform. The appropriate acceleration at the centre of the spacecraft will not be appropriate at other points. For example, at a point in the spacecraft 10 m further from the Earth than the centre there is a lower true gravity with a difference of about 3×10^{-5} m/s^2.

Of course on a journey to Mars or Venus or out of the Solar System, we could approach more closely to zero-gravity. But such journeys are much more expensive than going into orbit 200 km above the Earth.

So zero-gravity is difficult to attain; gravity cannot easily be eliminated with fictitious forces. But, even if these forces are fictitious, they are useful in explaining why bodies move as they do on the surface of the earth.

7. A few questions

1. What do you think the sundial in Capetown looks like?
2. If a man (of mass 70 kg) starts walking with acceleration 1 m/s^2 on a spacecraft which has mass 100 tons (10^5 kg), what is the consequent acceleration of the spacecraft?
3. A smooth puck is placed on a perfectly smooth rotating table. If the puck is at rest with respect to a fixed reference frame (so moves in a circle with respect to the table), explain what 'force' in the table's frame of reference causes the circular motion.

The editor would be pleased to receive from readers answers to this third question; the best answer will be published in the next issue but one.

Letters to the Editor

Dear Editor,

The circular and hyperbolic functions

On page 26 of Volume 11, Number 1, you published a letter from J. M. H. Peters dealing with the analogy between circular and hyperbolic functions. This analogy can be shown more simply as follows. In the notation of Figure 1 of the earlier letter, for the circle $x^2 + y^2 = 1$, with B at (x, y),

$$\theta = 2\mathscr{A}, \cos\theta = x, \sin\theta = y. \qquad \text{(i)}$$

For the rectangular hyperbola $x^2 - y^2 = 1$ (Figure 2 with P at (x, y)), let us write

$$u = 2\mathscr{H}, \qquad \text{(ii)}$$

where $\mathscr{H}$ is the shaded area. Then

$$\begin{aligned}
\mathscr{H} &= \tfrac{1}{2}xy - \int_1^x \sqrt{(z^2-1)}dz \\
&= \tfrac{1}{2}xy - \tfrac{1}{2}\left\{x\sqrt{(x^2-1)} - \int_1^x \frac{dz}{\sqrt{(z^2-1)}}\right\} \\
&= \tfrac{1}{2}xy - \tfrac{1}{2}\{x\sqrt{(x^2-1)} - \log(x + \sqrt{(x^2-1)})\} \\
&= \tfrac{1}{2}xy - \tfrac{1}{2}(xy - \log(x+y)) \\
&= \tfrac{1}{2}\log(x+y);
\end{aligned}$$

i.e.

$$u = \log(x+y), \qquad \text{or} \qquad e^u = (x+y). \qquad \text{(iii)}$$

Also

$$e^{-u} = \frac{1}{x+y} = \frac{x-y}{x^2-y^2} = x - y. \qquad \text{(iv)}$$

Finally, by addition and subtraction of (iii) and (iv),

$$x = \tfrac{1}{2}(e^u + e^{-u}) = \cosh u, y = \tfrac{1}{2}(e^u - e^{-u}) = \sinh u.$$

Yours sincerely,

E. Holland
(18 Easenby Avenue, Kirkella, Hull, North Humberside)

Dear Editor,

Prime numbers to base 7

One morning last February, Mr John Bickmore, the headmaster and one of the mathematics teaching staff at Yardley Court Preparatory School, Tonbridge, went over a revision prep. This involved, among other things, the writing of various numbers to different bases. Now, bases never interested me much, so after a while I tried to find something better to do. I noticed that, in the case of the examples on the blackboard and several others that I tried, if a prime number was written to a prime base, the sum of the digits was also a prime. When I asked Mr Bickmore about this, he said it was probably a coincidence. After doing some more conversions we found that the theory did not work for base 5, but we could not make it break down for base 7.

Later the head of the Tonbridge School mathematics department heard of my theory. He became most interested and invited me to try out the theory on the school computer. When I got there, the computer had already been programmed to print prime numbers to base 10 in the first column, the same numbers written to base 7 in the second column, and the sum of the digits in base 7 expressions in the third column. After more than two hours the first column of primes had got beyond 4000 and there had been no break-down in my theory. But, when it reached the prime 4801, I saw that my theory was false. This number, written to base 7, is 16666 and $1 + 6 + 6 + 6 + 6 = 25$, not a prime.

Since then I have found that one does not really need a computer to find the number 4801. If a number n is written to base 7, i.e. if

$$n = a_0 + 7a_1 + 7^2a_2 + \cdots + 7^ra_r \qquad (0 \leq a_0, a_1, \ldots, a_r \leq 6),$$

then

$$n - (a_0 + a_1 + \cdots + a_r) = (7 - 1)a_1 + (7^2 - 1)a_2 + \cdots + (7^r - 1)a_r.$$

Since the right-hand side is divisible by 6, n and $a_0 + a_1 + \cdots + a_r$ have the same remainder when divided by 6. Now any prime number greater than 6 can only have remainders 1 and 5 after division by 6. But not all numbers with remainder 1 or 5 are prime. The least non-prime of this kind is 25. So we look for a prime number whose digit sum to base 7 is 25, and the smallest such number is 16666, which is 4801 to base 10.

Yours sincerely,

NEIL NORMAN
(Tonbridge School, Tonbridge, Kent)

Dear Editor,

Heronian triangles

Mr A. R. Pargeter has proved the following theorem (*Mathematical Spectrum* Volume 9, Number 2 (1976/77), pp. 58–59):

If x, y, z are positive integers, the triangle with sides

$$x(y^2 + z^2), y(z^2 + x^2), (x + y)|xy - z^2|$$

is Heronian; and every Heronian triangle is similar to one with sides of this form.

Here is an alternative proof of the first part which shows how these triangles arise.

Consider the two Pythagorean triangles

$$x(y^2 + z^2), 2xyz, x|y^2 - z^2| \qquad \text{and} \qquad y(x^2 + z^2), 2xyz, y|x^2 - z^2|$$

with the same altitude $2xyz$. The first triangle is made to point to left or right according as $y > z$ or $y < z$; and the second to right or left according as $x > z$ or $x < z$. The two triangles are now added along the common altitude if they point in opposite directions and subtracted if they point in the same direction. The resulting Heronian triangle has its base of length

$$|x(y^2 - z^2) + y(x^2 - z^2)| = (x + y)|xy - z^2|.$$

Readers may also be interested in a result on Pythagorean triangles which I have not seen stated elsewhere. It is this:

Given a basic Pythagorean triangle with hypotenuse of length c, there exists a basic Pythagorean triangle with hypotenuse of length c^2.

For the proof, take a basic Pythagorean triangle (a, b, c) with hypotenuse c, so that $a^2 + b^2 = c^2$. Then a, b are coprime, for if they had a common factor, c would have the same factor and (a, b, c) would not be basic. Also it was shown by Mr K. R. S. Sastry (Volume 8, Number 3, p. 78) that one of a, b must be even (and so the other is odd).

Now consider the Pythagorean triangle (A, B, C), where

$$A = |a^2 - b^2|, B = 2ab, C = a^2 + b^2 = c^2.$$

We first note that A and C are odd. Hence, if A, B, C have a common prime factor p, then $p > 2$. But if p divides both A and C, then p divides $A + C = 2a^2$ and $C - A = 2b^2$. It follows that p divides both a and b, which is false. Thus (A, B, C) is a basic Pythagorean triangle.

Yours sincerely,

A. J. GRANVILLE
(Clifton College, Bristol)

Dear Editor,

Vieta's calculation of π

I was lucky enough to be taught computational methods—before the day of the electronic computer—by that superb practitioner of the art, Professor Hartree. He would certainly add to the two principles quoted by Mr Kent in his article in *Mathematical Spectrum* (Volume 11, Number 2) a third, as follows:

> Avoid a process which leads you to calculate a number as the product of a very large number and a very small one.

Mr Kent's method does not accord with this, for it computes the perimeter of each n-gon as the product of the length of the side (which is decreasing to zero) by the number of sides (which is going to infinity). This undesirable state of affairs can be avoided by making a simple transformation, which I think gives a more elegant process and which teaches a useful lesson to anyone learning the basics of numerical computing.

If X_n is the length of the side of a regular n-gon inscribed in the unit circle and P_n its perimeter, then $P_n = nX_n \to 2\pi$ as $n \to \infty$; and the relation we are going to use is

$$X_{2n}^2 = 2 - \sqrt{4 - X_n^2} \qquad \text{with} \qquad X_6 = 1.$$

Calling on the standard method which as a schoolboy I knew as 'multiplying by the conjugate surd', that is, by $2 + \sqrt{4 - X_n^2}$, we obtain

$$X_{2n}^2 = X_n^2/(2 + \sqrt{4 - X_n^2}).$$

As we want to calculate π it is convenient to work with $p_n = \frac{1}{2}P_n = \frac{1}{2}nX_n$; and a little very simple algebra puts the recurrence relation into the form

$$(p_{2n}/p_n)^2 = 2/(1 + \sqrt{1 - (p_n/n)^2}) \qquad \text{with} \qquad P_6 = 3.$$

This is very easy to evaluate on a simple hand calculator (not that I would deny its use as a simple programming exercise). With my Texas TI-30, which has square, square root, reciprocal, sign change and a one-register store, and works to 11-digit precision, I got convergence at $n = 12288$ to $3{\cdot}1415927$, which is correctly rounded to 7 decimals.

Yours sincerely,

JACK HOWLETT
(20b Bradmore Road, Oxford OX2 6QP)

Dear Editor,

The harmonic series

While browsing in our Mathematics common room I came across Volume 7, Number 1 of *Mathematical Spectrum* in which Captain Nicholas Draim invited readers to supply proofs of the divergence of the simple harmonic series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{n}. \tag{1}$$

Though rather late, I should like to mention a proof shown to me by my tutor at university, Dr J. A. Todd, while discussing the magical properties of conditionally convergent series.

In order to present the proof we need two important ideas from infinite series. The first of these is the notion of *absolute convergence*. The series

$$a_1+a_2+a_3+\cdots+a_n+\cdots$$

is absolutely convergent if the series

$$|a_1|+|a_2|+|a_3|+\cdots+|a_n|+\cdots$$

converges, where

$$|a_n|=\begin{cases} a_n & \text{if} \quad a_n \geq 0 \\ -a_n & \text{if} \quad a_n<0. \end{cases}$$

Put rather more informally, a series is absolutely convergent if the corresponding series with all the minus signs replaced by plus signs converges. For example the alternating harmonic series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots+(-1)^{n+1}1/n+\cdots \tag{2}$$

would be absolutely convergent if the simple harmonic series converged (which it does not!). But, for instance, the series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots+\frac{(-1)^{n+1}}{n^2}+\cdots$$

really is absolutely convergent. It is quite easy to prove that an absolutely convergent series actually converges. Any series which converges but which is not absolutely convergent is said to be *conditionally convergent*.

The second idea we need is that of a *rearrangement* of a series. The series

$$b_1+b_2+b_3+\cdots+b_n+\cdots \tag{3}$$

is a rearrangement of the series

$$a_1+a_2+a_3+\cdots+a_n+\cdots, \tag{4}$$

if each term b_m in series (3) appears exactly once as a term a_n in series (4), and, conversely, each a_n in (4) appears exactly once as a b_m in (3).

There is a remarkable pair of theorems involving the ideas we have just introduced. (i) All rearrangements of an absolutely convergent series have the same sum to infinity; while (ii) a conditionally convergent series may be rearranged so as to have any sum to infinity between (and including) ∞ and $-\infty$. Proofs of these results may be found in most books on infinite series.

We are now ready to establish the divergence of the simple harmonic series. In our proof we shall assume that the alternating harmonic series (2) is absolutely convergent (and hence that the simple harmonic series is convergent) and, using (i), we shall obtain a contradiction.

Denote by S_1 and S_2 the sums to infinity of the series (1) and (2) respectively. That series (1) is convergent means that

$$\infty > S_1. \tag{5}$$

Also, comparing the two series term by term, we clearly have

$$S_1 > S_2. \tag{6}$$

Further, taking the terms in series (2) in pairs, thus

$$(1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \cdots$$

we see that

$$S_2 > 0. \tag{7}$$

Hence, from (5), (6) and (7) it follows that

$$\infty > S_2 > 0. \tag{8}$$

We now consider the following rearrangement of the alternating harmonic series (2):

$$(1 - \tfrac{1}{2}) - \tfrac{1}{4} + (\tfrac{1}{3} - \tfrac{1}{6}) - \tfrac{1}{8} + \cdots + \left(\frac{1}{2n+1} - \frac{1}{2(2n+1)}\right) - \frac{1}{4(n+1)} + \cdots$$

$$= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} + \cdots + (-1)^{n+1}\frac{1}{2n} + \cdots = \tfrac{1}{2}S_2,$$

which, since S_2 is neither zero nor infinity, is not the same as S_2. Hence this rearrangement of the alternating series (2) does not have the same sum to infinity as series (2) and so, by (i), the alternating harmonic series cannot be absolutely convergent. We have obtained our contradiction and so conclude that the simple harmonic series is divergent.

Although the alternating harmonic series has been shown not to be absolutely convergent, it can be proved to be conditionally convergent; in fact its sum is $\log_e 2$.

Yours sincerely,

KEVIN GLAZEBROOK
(University of Newcastle upon Tyne)

Problems and Solutions

Sixth formers and students are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

11.7. Show that $e^{kx} + k(1 - e^x) \geq 1$ for every real number x and every integer k.

11.8. Arrange the digits 0 to 9 to form five two-digit numbers in such a way that the product of these five numbers is maximal.

11.9. (Submitted by A. J. Douglas, University of Sheffield.) Let $Z_1Z_2Z_3Z_4$ be a convex quadrilateral in the plane, denote by W_1, W_2, W_3, W_4 the midpoints of the squares, drawn externally to the quadrilateral, with sides Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_1 respectively, and let U_1, U_2, U_3, U_4 be the midpoints of the squares with sides W_1W_2, W_2W_3, W_3W_4, W_4W_1 respectively. Show that (a) W_1W_3, W_2W_4 are equal and perpendicular, (b) U_1Z_2 and U_3Z_4 are perpendicular to Z_1Z_3, and

$$U_1Z_2 = U_3Z_4 = \tfrac{1}{2}Z_1Z_3.$$

Solutions to Problems in Volume 11, Number 1

11.1. The prime factorizations of $r + 1$ positive integers ($r \geq 1$) together involve only r primes. Prove that there is a subset of these integers whose product is a perfect square.

Solution by Hema Murty (Carleton University, Ottawa)

Let $a_1, a_2, \ldots, a_{r+1}$ be the $r + 1$ positive integers. Then there exist r distinct prime numbers $p_1, \ldots, p_r$ such that

$$a_j = {}_1^{\alpha_{1j}}p_2^{\alpha_{2j}} \ldots p_r^{\alpha_{rj}} \qquad \text{for} \qquad 1 \leq j \leq r + 1,$$

where the α_{ij} are non-negative integers. If $\lambda_1, \ldots, \lambda_{r+1}$ are non-negative integers, we can write

$$a_1^{\lambda_1}a_2^{\lambda_2} \ldots a_{r+1}^{\lambda_{r+1}} = p_1^{\mu_1}p_2^{\mu_2} \ldots p_r^{\mu_r},$$

where

$$\mu_i = \alpha_{i1}\lambda_1 + \alpha_{i2}\lambda_2 + \cdots + \alpha_{i,r+1}\lambda_{r+1} \qquad \text{for} \qquad 1 \leq i \leq r.$$

We are looking for $\lambda_1, \ldots, \lambda_{r+1}$ equal to 0 or 1, not all zero, such that $\mu_1, \ldots, \mu_r$ are all even. If we reduce all integers modulo 2, and use a bar to denote such a reduction, this means that the system of r linear equations

$$\begin{array}{c} \bar{\alpha}_{11}x_1 + \bar{\alpha}_{12}x_2 + \cdots + \bar{\alpha}_{1,r+1}x_{r+1} = \bar{0} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \bar{\alpha}_{r1}x_1 + \bar{\alpha}_{r2}x_2 + \cdots + \bar{\alpha}_{r,r+1}x_{r+1} = \bar{0} \end{array}$$

must have a solution in the field $\mathbb{Z}_2$ of integers modulo 2, where $x_1, \ldots, x_{r+1}$ are not all zero. Since there are fewer equations than unknowns, this is always possible.

Also solved by Stephen Ainley (The Open University).

11.2. The positive numbers x, y, z are such that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Show that $(x-1)(y-1)(z-1) \geq 8$.

Solution by Hema Murty

The harmonic mean of x, y, z is less than or equal to their arithmetic mean, i.e.

$$\frac{3}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}} \leq \frac{x+y+z}{3}.$$

Hence $x+y+z \geq 9$. But $yz+zx+xy=xyz$, so that

$$(x-1)(y-1)(z-1)=x+y+z-1 \geq 8.$$

Also solved by Ian Clethero (Winchester College), J. L. Savva (St. Thomas' Medical School, London) and Chris Thornton (Ipswich School).

11.3. The rules for the card game of Clock Patience are as follows:

Shuffle the pack of cards and deal them into thirteen piles of four labelled A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. To play, take away the top card of the K pile (say it is 5), then the top card of the 5 pile (say it is J), then the top card of the J pile, and so on. The game proceeds until the fourth K is taken, and the game is said to 'come out' if, when the fourth K is taken, all the original piles are empty.

(a) What is the probability that a game comes out?

(b) Assume that, after dealing, the bottom cards on the piles form a rearrangement of A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K. Show that this game comes out if and only if this rearrangement is a cyclic rearrangement.

Solution

(a) First shuffle the cards. They can then be distributed among the thirteen named piles in such a way that the cards are taken in this particular ordering. Thus the probability of the game's coming out is the probability that the given shuffling ends in K, i.e. 1/13.

(b) Assume that, initially, the bottom cards of the piles form a rearrangement of A–K, and proceed with the game. Consider the pile whose bottom card is picked up first. It must be the K pile. (For if, say, it was the A pile, then four A's must have been taken prior to picking up this first bottom card. Yet an A occurs at the bottom of some pile.) If K occurs at the bottom of the K pile, then (1) the game does not come out, and (2) the rearrangement of the bottom cards is not cyclic.

Suppose K is not at the bottom of the K pile. Suppose (for the sake of argument), it is 5. Proceed with the game. The next pile to be used up is the 5 pile, because, prior to this, we must have picked up four 5's, one of which must have been at the bottom of a pile. If K is at the bottom of the 5 pile (1) the game does not come out, (2) the rearrangement of the bottom cards is not cyclic.

If the card at the bottom of the 5 pile is (say) J, then the next pile to be used up is the J pile. And so the game proceeds. It is clear from this that the game comes out if and only if the rearrangement formed by the bottom cards is cyclic.

Note: D. Singmaster has pointed out that, since among the 13! rearrangements of A–K there are 12! cyclic rearrangements, the probability that the game comes out in the situation described in (b) is again 1/13.

Linear Algebra: an Introduction. By A. O. Morris. Van Nostrand Reinhold Company Ltd, Wokingham, Berkshire, 1978. Pp. viii + 180. £3·00 paperback; £7·50 hardback.

This is a standard linear algebra book covering the work done in many undergraduate first year honours mathematics courses. There are six chapters; Linear Equations and Matrices; Determinants; Vector Spaces; Linear Transformations on Vector Spaces; Inner Product Spaces; Diagonalization of Matrices and Linear Transformations. These chapter headings themselves give a good idea of the style of development which is very traditional, no attempt being made to recognize that many students in England (the author comes from the University of Wales) have a background which includes a fair amount of experience of linear transformations and matrices. In a book of this nature, I usually look at the section dealing with the effect of changing the basis of a vector space on the matrix of a given linear mapping; I did not think this was easy to read or to follow. On the credit side, the book is very concrete with many clearly laid out worked examples and a large number of exercises. It is a pity there are no answers.

University of Durham — Hugh Neill

First Steps in Numerical Analysis. By R. J. Hosking, D. C. Joyce and J. C. Turner. Hodder and Stoughton, Sevenoaks, Kent, 1978. Pp. 202. £2·95 (paperback); £4·95 (hardback).

This book is intended to provide an introduction to elementary concepts and methods of numerical analysis suitable for Advanced Level Mathematics and introductory courses at polytechnics and universities. The text is presented in a series of 31 short 'steps' covering errors, non-linear equations, systems of linear equations, finite differences, interpolation, curve fitting, numerical differentiation and numerical integration. At the end of each step is a checkpoint, containing two or three pertinent questions on the main points of the text, followed by an exercise which, in many steps, I found brief and unimaginative. For a potential school market a wider selection of examples is required and certainly some that involve more searching applications. On the other hand, solutions to the exercises are admirably presented in full at the back and these are accompanied by an appendix of flow charts illustrating many of the algorithms encountered in the text.

The text starts with a series of five excellent steps on errors. The important section on iterative processes, has its merits but reveals some unevenness in the text where, for example, such statements as '. . . convergence is not so assured as . . .' are not supported by examples and diagrams (which I regard as very important in iterative methods) illustrating possible breakdown of method. I enjoyed the introduction to finite differences and found the section on errors particularly clear. However, I remain bewildered as to why the formulae of Gauss, Stirling, Bessel and Everett are introduced at all, for the only illustrations of their use are in the estimation of intermediate values of trigonometrical or exponential tables.

In general the layout of the book is attractive and the few misprints are immediately obvious. Apart from points mentioned above (and one or two others where justifications are omitted) the style of writing is refreshing in its simplicity, so that the conscientious reader should need little supervision. The technical knowledge required by the reader is fairly elementary—up to Taylor's Theorem. With the advent of universally available calculators and computers there is a place for a book presenting this kind of subject matter. My intelligent students will enjoy this introduction, but it does not fulfil my requirements as a standard textbook.

Monmouth School — M. V. Bradley

Geometric Symmetry. By E. H. LOCKWOOD and R. H. MACMILLAN. Cambridge University Press, London, 1978. Pp. x + 228. £10·50.

The authors remark that, although symmetry is of interest to artists and to mathematicians, and underlies much scientific thought including atomic physics and crystallography, with the exception of Weyl's *Symmetry* and Rosen's *Symmetry Discovered* it has received very little attention in the literature aimed at the general reader.

The object of the book under review is to provide a fairly comprehensive account of symmetry in a form acceptable to readers without much mathematical knowledge who nevertheless wish to understand the basic principles of symmetry. This object is achieved by separating the book into two parts. Part I is descriptive and is written primarily for the non-mathematical reader. Part II is more mathematical, but here the reader is gently led into a symmetrical treatment which enumerates and classifies the symmetry groups of each kind.

Part I consists of 13 short chapters well illustrated by numerous carefully chosen diagrams. It ends with a selection of 14 problems, to which answers are given. Part II contains a further 14 chapters, each clearly written and again provided with many illustrations. At the end of Chapter 25 is a table enumerating the 230 space groups of symmetries.

In Appendix I it is shown that, in the symmetry group of a finite pattern, there is always at least one point that is left fixed by any of the movements of the group. Appendix II shows how earlier calculations in Part II could have been simplified by the use of 3×3 and 4×4 matrices. This is followed by an index of groups, nets and lattices and finally by a useful general index.

I find it impossible to summarize the elegant but detailed treatment of Part II. Suffice it to say that in my opinion the authors have achieved their goal: here, indeed, is a work of considerable interest both to the general reader and also to the professional mathematician. From the point of view of clarity of exposition, this book can have few rivals. I thoroughly commend it.

University of Durham TOM WILLMORE

Geometry and the Liberal Arts. By DAN PEDOE. Penguin Books, Harmondsworth, Middlesex, 1976. Pp. 296. £2·50.

I am not in any sense an expert on geometry, so I read this book hoping to learn more about the subject. I was delighted by much that I learned and also by the way that the subject was presented. The central theme concerns perspective drawing, Euclidean geometry and projective geometry, but Professor Pedoe is prepared to deviate from this theme in order to discuss the ideas which interest him.

The book begins with chapters on three people, Vitruvius, Albrecht Dürer and Leonardo da Vinci and examines briefly the contributions made by them to engineering, art, architecture and geometry. Then follow chapters on form in architecture and the *Optics* of Euclid before an important one on Euclid's *Elements* of geometry in which there is a most readable discussion of the parallel axiom and of the famous Poincaré model for a non-Euclidean geometry. The book then has a short chapter on cartesian and projective geometry before a substantial one on curves and a rather shorter one on space in which a little group theory, the regular solids and the Flatland of Abbott are all discussed.

I must emphasize the very light touch of the author. At no stage does the mathematics become heavy, and thus the book is extremely readable at the sixth-form level in schools. It is the kind of book which will leave the reader wanting to know more. I recommend it with enthusiasm.

University of Durham HUGH NEILL

Student's Guide to Success. By W. Fisher Cassie and T. Constantine. Macmillan Press, London, 1977. Pp. i + 170. £2·95.

The title gives the impression that this book contains yet more well-meaning advice from that older and wiser class of headmasters and parents, who seem out of touch with the attitudes of young people to education. I was pleasantly surprised to find my fears unfounded. The first chapter is a student's confession of failure in the first term at college. Any student in a sixth form or college will be able to identify with the authors' experiences, since each has at some time reached the end of a topic with the discovery that he learned nothing, or has endured an examination, unable to recall facts to mind or express these thoughts in the three-hour period.

The book describes techniques for study, taking lecture notes which make revision easier, organizing time so that evening entertainments are no longer an escape from the backlog of work, but are part of the schedule. The authors give examples of some memory-jogging aids and illustrate their value in the text. Mnemonics, clear headings, witty cartoons and anecdotes impress the facts on one's memory.

Later chapters give valuable advice about using the library, chairing committees, job applications and the like, showing that student life is much more than academic work. Having read this book towards the end of my student career, I recognise its value, but wish it had been published sooner.

Van Mildert College, Durham University — Joyce Pain

Notes on Contributors

Douglas Woodall was an undergraduate at Cambridge and is now Reader in Pure Mathematics at the University of Nottingham. He has contributed to many areas of graph theory and combinatorics. At present he is the Mathematical Consultant to the Oxford English Dictionary. His relaxations include mountain walking and singing in the chorus of the Nottingham Harmonic Society.

Richard Tweedie is currently with the Division of Mathematics and Statistics of CSIRO (Commonwealth Scientific and Industrial Research Organization), where he is Senior Regional Officer of the Victoria/Tasmania region. This division provides statistical and mathematical expertise for diverse groups within CSIRO, and also on occasion for various universities and other groups throughout Australia; it also carries out considerable research in mathematics and statistics. Richard Tweedie's own research interests lie in probability theory, with practical applications in the modelling of various biological populations. His article in this issue was written whilst he was a visiting associate professor at the University of Western Australia.

Alex Russell holds the degrees of B.Sc. and Ph.D. from the University of Dundee and is currently teaching mathematics at George Watson's College, Edinburgh. His main research interest is in the field of integral inequalities.

D. J. Roaf is a Lecturer in Theoretical Physics at Oxford University and Mathematics Fellow of Exeter College. He spent one period of sabbatical leave in the southern hemisphere and another studying inflation accounting in the United States. His hobbies include bell-ringing (see *Mathematical Spectrum* Volume 7, pages 60–66) and politics: he was Liberal parliamentary candidate for Oxford City when this issue went to press.

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ISSN 0025–5653

PRICES (*postage included*)

Prices for Volume 11 (*Issues Nos.* 1, 2, *and* 3):

Subscribers in Britain and Europe: £1·30
Subscribers overseas: £2·60 (US$4.80; $A4.25)
(These prices apply even if the order is placed by an agent in Britain.)
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Editor—*Mathematical Spectrum*,
Hicks Building,
The University,
Sheffield S3 7RH, England.

Printed in England by Galliard (Printers) Ltd, Great Yarmouth