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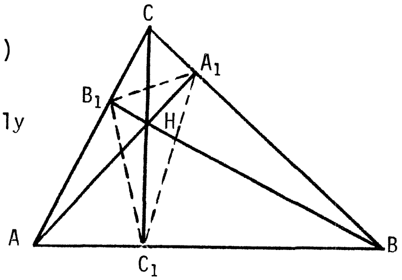
CONTENTS

A NEW INEQUALITY FOR THE ANGLES OF A TRIANGLE

O. BOTTEMA

Let ABC be a triangle with sides a,b,c and angles α,β,γ in the usual order, and circumradius R. We will assume that ABC is *not* right-angled, so that its orthic triangle $A_1B_1C_1$ is nondegenerate, and the orthocenters H and H_1 of ABC and $A_1B_1C_1$ are well-defined points. We will first find an expression for $d = HH_1$ in terms of elements of triangle ABC. We will then use this expression (which is interesting in its own right) to deduce a new inequality for the angles of *any* triangle.

We will assume that triangle ABC has been labeled so that γ is the largest angle. (The figure shows a triangle with $\gamma < \pi/2$, but it can be verified that everything that follows remains valid if $\gamma > \pi/2$.) We introduce a Cartesian frame OXY with the origin O at C_1 and X-axis along AB. We easily find that



$$H = (0,\ 2R\cos\alpha\cos\beta). \qquad (1)$$

Furthermore, $A_1C = b\cos\gamma$ and $BA_1 = c\cos\beta$; hence

$$A_1 = (b\cos\beta\cos\gamma,\ c\cos\beta\sin\beta)$$

and similarly

$$B_1 = (-a\cos\alpha\cos\gamma,\ c\cos\alpha\sin\alpha).$$

Now B_1C_1 is antiparallel to BC; hence $\angle B_1C_1A = \gamma$ and the equation of B_1C_1 is

$$x\sin\gamma + y\cos\gamma = 0.$$

From this it follows that the line through A_1 perpendicular to B_1C_1 (an altitude of $A_1B_1C_1$) is

$$x\cos\gamma - y\sin\gamma + k_1 = 0, \qquad (2)$$

where

$$\begin{aligned} k_1 &= c\cos\beta\sin\beta\sin\gamma - b\cos\beta\cos^2\gamma \\ &= -R\sin 2\beta\cos 2\gamma. \end{aligned} \qquad (3)$$

In the same way, we find that the line through B_1 perpendicular to A_1C_1 has the equation

$$x \cos\gamma + y \sin\gamma + k_2 = 0, \tag{4}$$

where

$$k_2 = R \sin 2\alpha \cos 2\gamma. \tag{5}$$

Solving (2) and (4) simultaneously, we find that the coordinates (x_1, y_1) of H_1 are given by

$$x_1 = -\frac{k_1+k_2}{2\cos\gamma}, \qquad y_1 = \frac{k_1-k_2}{2\sin\gamma}.$$

In view of (3) and (5), we therefore have

$$H_1 = (R\cos 2\gamma \sin(\alpha-\beta),\ -R\cos 2\gamma \cos(\alpha-\beta)). \tag{6}$$

Now, from (1) and (6), the value of $d = HH_1$ is given by

$$d^2 = R^2\cos^2 2\gamma + 4R^2\cos\alpha\cos\beta\{\cos 2\gamma \cos(\alpha-\beta) + \cos\alpha\cos\beta\}. \tag{7}$$

Obviously, it must be possible to write d^2 as a symmetrical function of α,β,γ. We reduce (7) as follows:

$$\begin{aligned} d^2 &= R^2\cos^2 2\gamma + 4R^2\cos\alpha\cos\beta\{\cos\alpha\cos\beta(1+\cos 2\gamma) + \sin\alpha\sin\beta\cos 2\gamma\} \\ &= R^2\cos^2 2\gamma + 4R^2\cos\alpha\cos\beta\{2\cos\alpha\cos\beta\cos^2\gamma + \sin\alpha\sin\beta\cos 2\gamma\} \\ &= 8R^2\cos^2\alpha\cos^2\beta\cos^2\gamma + R^2\cos 2\gamma(\sin 2\alpha\sin 2\beta + \cos 2\gamma) \\ &= 8R^2\cos^2\alpha\cos^2\beta\cos^2\gamma + R^2\cos 2\gamma\{\sin 2\alpha\sin 2\beta + \cos(2\alpha+2\beta)\} \\ &= R^2(8\cos^2\alpha\cos^2\beta\cos^2\gamma + \cos 2\alpha\cos 2\beta\cos 2\gamma), \end{aligned} \tag{8}$$

and this yields the desired expression for $d = HH_1$.

Since $d^2 \geq 0$, we conclude from (8) that

$$8\cos^2\alpha\cos^2\beta\cos^2\gamma + \cos 2\alpha\cos 2\beta\cos 2\gamma \geq 0 \tag{9}$$

holds for every triangle in which $\gamma \neq \pi/2$. But it holds also if $\gamma = \pi/2$, for then (9) reduces to

$$\cos^2 2\alpha \ (= \cos^2 2\beta) \geq 0. \tag{10}$$

So (9) holds for every triangle.

If triangle ABC is equilateral, it is easy to see that H coincides with H_1, so $d = 0$ and equality holds in (9); and it follows from (10) that equality also holds in (9) when $\alpha = \beta = \pi/4$ and $\gamma = \pi/2$.

Finally, we note that (9) can be expressed more symmetrically in the form

$$(1+\cos 2\alpha)(1+\cos 2\beta)(1+\cos 2\gamma) + \cos 2\alpha\cos 2\beta\cos 2\gamma \geq 0.$$

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* * *

SOME MAJORIZATION INEQUALITIES

M.S. KLAMKIN and GEORGE TSINTSIFAS

In a recent note in this journal [1982: 162], the second author came up with still another proof of the A.M.-G.M. inequality. He showed that, if $x_i > 0$ for $i = 1,2,\ldots,n$, then (sums and products throughout are for $i = 1,2,\ldots,n$)

$$\frac{\Pi(x + x_i)}{\{\Sigma(x + x_i)\}^n}$$

increases monotonically for $x \geq 0$, and thus

$$\frac{\Pi(x + x_i)}{\{\Sigma(x + x_i)\}^n} \geq \frac{\Pi x_i}{\{\Sigma x_i\}^n}, \tag{1}$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$. The A.M.-G.M. inequality is then obtained by letting $x \to \infty$ in (1). We will relate (1) to other inequalities.

If we set $x/x_i = a_i$ for $x > 0$, then (1) is equivalent to

$$\Pi(1 + a_i) \geq (1 + H)^n, \tag{2}$$

where H is the harmonic mean $n/\Sigma(1/a_i)$. If we now let

$$A = \frac{\Sigma a_i}{n} \quad \text{and} \quad G = (\Pi a_i)^{1/n},$$

then (2) can be extended to

$$(1 + A)^n \geq \Pi(1 + a_i) \geq (1 + G)^n \geq (1 + H)^n. \tag{3}$$

The first inequality in (3) follows from the A.M.-G.M. inequality applied to the n numbers $1+a_i$; for a proof of the second, see [1]; and the third follows from $G \geq H$.

Now we prove a stronger result in which the $\geq$ signs in (3) are replaced by majorization signs $\succ$.

First, let T_r, $r = 0,1,\ldots,n$, denote the elementary symmetric functions of the a_i, which are defined by

$$\Pi(z + a_i) \equiv T_0 z^n + T_1 z^{n-1} + \ldots + T_{n-1} z + T_n.$$

We say that $(1+A)^n$ *majorizes* $\Pi(1+a_i)$ and write $(1+A)^n \succ \Pi(1+a_i)$ if, termwise,

$$\binom{n}{r} A^r \geq T_r, \qquad r = 0,1,\ldots,n.$$

To show that

$$(1 + A)^n \succ \Pi(1 + a_i) \succ (1 + G)^n \succ (1 + H)^n,$$

all we need to prove is that, for $r = 0,1,\ldots,n$,

$$\binom{n}{r}A^r \geq T_r \geq \binom{n}{r}G^r \geq \binom{n}{r}H^r. \tag{4}$$

For $r = 0$, equality clearly holds throughout in (4). For $r = 1,2,\ldots,n$, we must show equivalently that

$$A \geq p_r^{1/r} \geq G \geq H, \tag{5}$$

where $p_r = T_r/\binom{n}{r}$. The last inequality in (5) goes without saying, and the first two follow from the Maclaurin inequalities [2], according to which we have

$$A = p_1 \geq p_2^{1/2} \geq p_3^{1/3} \geq \ldots \geq p_n^{1/n} = G.$$

REFERENCES

1. D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, Freeman, San Francisco, 1962, Problem 305, solution on pp. 410-411.

2. G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1952, p. 52.

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* * *

MATHEMATICAL CLERIHEWS

David Hilbert,	Karl Weierstrass[2]
W.S. Gilbert[1] —	Enjoyed a glass
Each "problemist"	Of Rhine wine for drinking.
Compiled "a list".	He hated fuzzy thinking.

ALAN WAYNE, Holiday, Florida

[1]W.S. Gilbert, in *The Mikado*, Act I:

As some day it may happen that a victim must be found,
I've got a little list — I've got a little list
Of society offenders who might well be underground,
And who never would be missed — who never would be missed!

[2]Solomon Bochner in *The Role of Mathematics in the Rise of Science*, Princeton University Press, 1966, p. 369: "He [Weierstrass] liked his Rhine wine rich; but this somehow did not prevent him from becoming a leading enemy of fuzziness in mathematical thinking in the 19th century."

* * *

THE OLYMPIAD CORNER: 40

M.S. KLAMKIN

A few months ago in this column [1982: 70], I gave the problems set in the first round of the 1982 West German Olympiad. Now, through the courtesy of Bernhard Leeb, I present the problems set in the second round, for which I solicit elegant solutions.

WEST GERMAN MATHEMATICAL OLYMPIAD 1982

Second Round

1. Max divides the positive integer p by the positive integer q, where $q \le 100$. In the decimal expansion of the quotient p/q, Max finds somewhere after the decimal point the digit-block 1982. Prove that Max's division is wrong.

2. Decide whether every triangle ABC can be transformed by orthogonal projection on a certain plane into an equilateral triangle.

3. The nonnegative real numbers $a_1, a_2, \ldots, a_n$ satisfy $a_1+a_2+\ldots+a_n = 1$. Show that the sum

$$\frac{a_1}{1+a_2+a_3+\ldots+a_n} + \frac{a_2}{1+a_1+a_3+\ldots+a_n} + \ldots + \frac{a_n}{1+a_1+a_2+\ldots+a_{n-1}}$$

has a minimum and compute it.

4. If $4^n + 2^n + 1$ is a prime number for the positive integer n, then n is a power of 3.

*

I now present a set of problems proposed by students attending the 1982 U.S.A. Mathematical Olympiad Practice Session held last summer at the U.S. Naval Academy in Annapolis, Maryland. I may later publish the best solutions received from readers. (Starred problems were submitted without a solution.) For earlier sets of such problems, see [1980: 210 ; 1982: 99].

1. *Proposed by Jeremy Kahn and Steven Newman.*

For a positive integer $n > 1$, let $s(n)$ denote the sum of all the divisors of n, excluding n itself. The integer n is said to be *deficient*, *perfect*, or *abundant* according as $s(n) < n$, $s(n) = n$, or $s(n) > n$.

(a) Determine the maximal length of a block of consecutive integers each of which is deficient.

(b) Determine the maximal length of a block of consecutive integers each of which is abundant.

2.* *Proposed by Steven des Jardins.*

Given is a sphere of integer radius r. Find the maximum number of points which can be placed on the surface of the sphere in such a way that all pairs of points are separated by an integer distance

(a) if chordal distances are used;

(b) if distances are measured along the minor arcs of great circles of the sphere.

3.* *Proposed by Andy Tikofsky.*

For a positive integer $n > 3$, does there exist an integer $m(n)$ such that $x^{2^n} + m(n)$ has $2(n-1)$ nonconstant polynomial factors with integral coefficients?

4. *Proposed by Noam Elkies.*

Find all solutions (x,y,z) of the Diophantine equation

$$x^3 + y^3 + z^3 + 6xyz = 0.$$

5. *Proposed by Noam Elkies.*

Three disjoint spheres whose centers are not collinear are such that there exist eight planes each tangent to all three spheres. The points of tangency of each of these planes are the vertices of a triangle. Prove that the circumcenters of these eight triangles are collinear.

6. *Proposed by Noam Elkies.*

Let p be an odd prime, and let n be a factor of

$$N = \frac{p^{2p}+1}{p^2+1}.$$

Prove that $n \equiv 1 \pmod{8p}$.

7. *Proposed by Brian Hunt.*

In R^n let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, and, for $p \in (0,1)$, define

$$F_p(X,Y) \equiv \left(\left|\frac{x_1}{p}\right|^p \left|\frac{y_1}{1-p}\right|^{1-p}, \left|\frac{x_2}{p}\right|^p \left|\frac{y_2}{1-p}\right|^{1-p}, \ldots, \left|\frac{x_n}{p}\right|^p \left|\frac{y_n}{1-p}\right|^{1-p}\right).$$

Prove that

$$\| X \|_m + \| Y \|_m \geq \| F_p(X,Y) \|_m,$$

where

$$\| X \|_m = (|x_1|^m + |x_2|^m + \ldots + |x_n|^m)^{1/m}.$$

8. *Proposed by John Steinke.*

Determine all six-digit integers n such that n is a perfect square and the number formed by the last three digits of n exceeds the number formed by the first three digits of n by 1. (n might look like 123124.)

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before May 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

791. *Proposed by Alan Wayne, Holiday, Florida.*

At whom might this be shouted?

$$\begin{array}{r} \text{SO} \\ \text{SCAT} \\ \underline{\text{SCAT}} \\ 5032 \end{array}$$

The sum of this base ten cryptarithm will answer the question.

792. *Proposed by E.J. Barbeau, University of Toronto.*

The number 144 has the remarkable property that, with only the three exceptions

$$2448 = 17\cdot 144, \quad 2736 = 19\cdot 144, \quad 3312 = 23\cdot 144,$$

each of its first 50 multiples differs from the first perfect square not less than it by a perfect square.

Show that, for any positive integer n, there is a number m such that n consecutive positive multiples

$$km,\ (k+1)m,\ \ldots,\ (k+n-1)m$$

have the property that each differs from the smallest perfect square not less than it by a perfect square.

793. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Consider the following double inequality for the Riemann Zeta function: for $n = 1,2,3,\ldots,$

$$\frac{1}{(s-1)(n+1)(n+2)\ldots(n+s-1)} + \zeta_n(s) < \zeta(s) < \zeta_n(s) + \frac{1}{(s-1)n(n+1)\ldots(n+s-2)}, \quad (1)$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad \text{and} \qquad \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}.$$

Go as far as you can in determining for which of the integers $s = 2,3,4,\ldots$ the inequalities (1) hold.

(N.D. Kazarinoff asks for a proof that (1) holds for $s = 2$ in his *Analytic Inequalities*, Holt, Rinehart & Winston, 1964, page 79; and Norman Schaumberger asks for a proof or disproof that (1) holds for $s = 3$ in *The Two-Year College Mathematics Journal*, 12 (1981) 336.)

794. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Determine the positive integers n for which

(a) $\displaystyle\lim_{x\to 0} \frac{\exp(\tan^n x) - \exp(\sin^n x)}{x^{n+2}} = 1$;

(b) $\displaystyle\lim_{x\to 0} \frac{\exp(x^n) - \exp(\sin^n x)}{x^{n+2}} = 1$;

(c) $\displaystyle\lim_{x\to 0} \frac{\exp(\tan^n x) - \exp(x^n)}{x^{n+2}} = 1$.

795. *Proposed by Jack Garfunkel, Flushing, N.Y.*

Given a triangle ABC, let t_a, t_b, t_c be the lengths of its internal angle bisectors, and let T_a, T_b, T_c be the lengths of these bisectors extended to the circumcircle of the triangle. Prove that

$$T_a + T_b + T_c \geq \frac{4}{3}(t_a + t_b + t_c).$$

796. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

For an integer $n > 1$, let $f(n)$ be the product of all the positive divisors of n other than n itself. Characterize the fixed points of f, that is, characterize the integers $n > 1$ such that $f(n) = n$. (For example, $f(8) = 1\cdot 2\cdot 4 = 8$.)

797. *Proposed by H. Kestelman, University College, London, England.*

Show that the trace of a real $n \times n$ matrix A is equal to x^TAx for some real column vector x with $x^Tx = n$.

798*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

For a nonnegative integer n, evaluate

$$I_n \equiv \int_0^1 \binom{x}{n} dx.$$

799. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Can the 25 consecutive primes 1327, ..., 1523 be rearranged into a fifth-order magic square?

800. *Proposed by Charles W. Trigg, San Diego, California.*

Find triads of consecutive triangular numbers whose products are square numbers.

* * *

THE HOUSE NUMBERS OF EUROPE

Georg Freiherr von Vega was born on 23 March 1754 in Zagorica (Krain), in the Austrian Empire, and murdered on 17 September 1802 near Vienna. In this computer age, it may be permitted to recall that he published his *Thesaurus Logarithmorum Completus* in Leipzig in 1794 (to ten decimal places). Its 94th edition (seven decimal places, no misprints, *no errors*) was published in Berlin in 1935. It had become ever more reliable and was still widely used after a century and a half. Numerous checks and calculations to many decimal places (e.g.,

$$\log \cos 24^\circ 55' 30'' = -1 + 0.95754\,03499\,9866$$

shows that, rounded to seven decimal places, the last digit is 3, not 4) had made of Vega's Tables a computer's "Bible".

The German philosopher Kuno Fischer (1824-1907) tells from his school days (the late 1830s) that two of his classmates had a rather simple-minded uncle. He sometimes looked over their homework, although he rarely grasped what it was all about. Once he found them doing mathematics, and for the first time in his life he saw Vega's Log Tables. The big book containing numbers, nothing but numbers, fascinated him and aroused his curiosity. He asked what it was. One of the boys, his face suitably despondent, in keeping with the enormous task, answered: "It is the house numbers of Europe." The uncle said nothing, but did not doubt his word. After all, why shouldn't numbers be house numbers? That evening, he told friends that in his own youth school had been difficult enough, but that was nothing compared to the present. "Why, my poor nephews are sitting at home, and what do they learn? The house numbers of Europe!" Well, it might come in handy, he mused, if the Germans ever conquer Paris again. Wouldn't it be helpful to know all the house numbers right away?

HAYO AHLBURG
Benidorm, Alicante, Spain

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problem.

687. [1981: 276] *Proposed jointly by J. Chris Fisher, University of Regina; and Roger Servranckx, University of Saskatchewan at Saskatoon.*

(a) Show that there exists a number γ such that the equation

$$c^{c^{c^x}} = x$$

has three solutions whenever $0 < c < \gamma$.

(b) How many solutions does the equation

$$c^{c^{\cdot^{\cdot^{\cdot^{c^x}}}}} = x$$

have when there are $2n$ c's in the ladder and $0 < c < \gamma$?

Solution by the proposers.

(a) For a fixed $c \in (0,1)$, let

$$f(x) := c^{c^x} :$$

then $f(0) = c > 0$, $f(\infty) = 1$, and $f(x)$ is strictly increasing. Thus, if we find a solution to the equation $f(x) = x$ for which $f'(x) > 1$, then there will be *at least* three solutions. Every c in $(0,1)$ can be written uniquely as $c = a^{-a}$ for some $a > 1$. Then $c^{1/a} = 1/a$ and $f(1/a) = 1/a$. Furthermore,

$$f'(x) = (\ln c)^2 c^{c^x} c^x,$$

and so $f'(1/a) = (\ln a)^2 > 1$ for $a > e$. Note that $a > e$ if and only if $c < e^{-e}$, so we let $\gamma = e^{-e}$. To see that there are at most three solutions, note that

$$f''(x) = (\ln c)^3 c^{c^x} c^x (1 + c^x \ln c)$$

has exactly one zero, $x = \{-\ln(-\ln c)\}/\ln c$, and therefore $f(x)$ has only one inflection point.

(b) With $f(x)$ as defined above, let

$$f^1(x) = f(x), \qquad f^{n+1}(x) = f(f^n(x)), \qquad n = 1,2,3,\ldots .$$

We show that $f^n(x) = x$ has three solutions for all n when $c \in (0,\gamma)$. Let $x = s_1$ and $x = s_2$ be the first and third solutions, respectively, of $f(x) = x$. (The middle solution is always $x = 1/a$ according to part (a).) We have $0 < f'(s_1) < 1$, so s_1 can be approximated by the iteration process: $f^m(x) \to s_1$ as $m \to \infty$ for any $x < 1/a$. Even more, $0 < f(x) < s_1$ for $x < s_1$, so $f^m(x)$ increases to s_1; consequently, $f^m(x) = x$ for no $x < s_1$. Similarly, $f^m(x)$ decreases to s_1 as $m \to \infty$ for $x \in (s_1, 1/a)$, so that $f^m(x) = x$ for no $x \in (s_1, 1/a)$. The analogous arguments for s_2 imply that, for any n, $f^n(x) = x$ can have no solution except for s_1, $1/a$, and s_2. In fact, as $n \to \infty$, $f^n(x)$ approaches the step function

$$g(x) = \begin{cases} s_1, & \text{for } x < 1/a, \\ s_2, & \text{for } x > 1/a. \end{cases}$$

A comment was received from LEROY F. MEYERS, The Ohio State University.

Editor's comment.

Meyers noted that the results of this problem, and a great deal of related material, can be found in a recent article by Knoebel [1]. In that reference, note particularly Figure 4, which shows that $f(x) = x$ has three solutions if $0 < c < e^{-e}$, one solution if $e^{-e} \le c \le 1$, and two solutions if $1 < c < e^{1/e}$.

REFERENCE

1. R. Arthur Knoebel, "Exponentials Reiterated", *American Mathematical Monthly*, 88 (1981) 235-252.

* * *

688. [1981: 276] *Proposed by Robert A. Stump, Hopewell, Virginia.*

Let $\circ$ denote a binary operation on the set of all real numbers such that, for all real numbers a,b,c,

(i) $0\circ a = -a$; (ii) $a\circ(b\circ c) = c\circ(b\circ a)$.

Show that $a\circ(b\circ c) = (a\circ b)\circ(-c)$.

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

For all real numbers a,b,c,

$$\begin{aligned} a\circ(b\circ c) &= \big(0\circ(-a)\big)\circ(b\circ c) = \big(0\circ(0\circ a)\big)\circ(b\circ c) = \big(a\circ(0\circ 0)\big)\circ(b\circ c) \\ &= \big(a\circ(-0)\big)\circ(b\circ c) = (a\circ 0)\circ(b\circ c) = c\circ\big(b\circ(a\circ 0)\big) \\ &= c\circ\big(0\circ(a\circ b)\big) = (a\circ b)\circ(0\circ c) = (a\circ b)\circ(-c). \quad \square \end{aligned}$$

When doing a problem like this, I always like to know if there can be such an operator, just to make sure that I am not playing the null game. In this case, at least one such operator exists, namely, that defined by $a\circ b = a-b$.

Also solved by JAMES J. BOWE, Erskine College, Due West, South Carolina; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CURTIS COOPER, Central Missouri State University; CLAYTON W. DODGE, University of Maine at Orono; JOEL ERICKSON, student, Fort Lewis College, Durango, Colorado; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; J.D. HISCOCKS, University of Lethbridge, Alberta; PETER A. LINDSTROM, Genesee Community College, Batavia, N.Y.; ANDY LIU, University of Alberta; LEROY F. MEYERS, The Ohio State University; DAN SOKOLOWSKY, California State University at Los Angeles; DAVID R. STONE, Georgia Southern College; ANDREAS STRIEBICH, student, Goethe Gymnasium, Gaggenau, West Germany; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; and the proposer.

Editor's comment.

Nearly all solutions received for this easy problem were essentially equivalent, so presentation was the deciding factor in selecting one for publication. In the editor's opinion, the one we have featured won by a whisker.

Meyers noted that the restriction to real numbers is unnecessary, since the desired conclusion holds for the elements of any group, or, more generally, for the elements of any set in which a unary operation "-" (such that $-0 = 0$ for some element 0 and $-(-a) = a$ for all elements a) and a binary operation "$\circ$" are defined which satisfy (i) and (ii).

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689. [1981: 276] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Let m_a, m_b, m_c denote the lengths of the medians to sides a,b,c, respectively, of triangle ABC, and let M_a, M_b, M_c denote the lengths of these medians extended to the circumcircle of the triangle. Prove that

$$M_a/m_a + M_b/m_b + M_c/m_c \geq 4.$$

I. *Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let G be the centroid of the triangle, and let AG, BG, CG meet the circumcircle again in A', B', C', respectively. It is known (see [1] or [2]) that

$$\frac{AG}{GA'} + \frac{BG}{GB'} + \frac{CG}{GC'} = 3. \tag{1}$$

It is easily shown (or see [3]) that, for positive p,q,r,

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{9}{p+q+r}$$

with equality just when $p = q = r$. Applying this to (1), we obtain

$$\frac{GA'}{AG} + \frac{GB'}{BG} + \frac{GC'}{CG} \geq 3,$$

whence, since $M_a/\mathrm{AG} = 1 + \mathrm{GA'}/\mathrm{AG}$, etc.,

$$M_a/\mathrm{AG} + M_b/\mathrm{BG} + M_c/\mathrm{CG} \geq 6.$$

Finally, since $\mathrm{AG} = (2/3)m_a$, etc., we have

$$M_a/m_a + M_b/m_b + M_c/m_c \geq 4.$$

II. *Solution by W.J. Blundon, Memorial University of Newfoundland.*

From $m_a^2 = (2b^2+2c^2-a^2)/4$ and two similar results, we get

$$a^2 = \frac{8}{9}m_b^2 + \frac{8}{9}m_c^2 - \frac{4}{9}m_a^2$$

and two similar results. From $m_a(M_a-m_a) = (a/2)^2$, we now get

$$M_am_a = m_a^2 + \frac{a^2}{4} = \frac{8}{9}m_a^2 + \frac{2}{9}m_b^2 + \frac{2}{9}m_c^2,$$

and so

$$M_a/m_a = \frac{8}{9} + \frac{2}{9}(m_b^2/m_a^2) + \frac{2}{9}(m_c^2/m_a^2).$$

With this and two similar results, we obtain, with sums cyclic over a,b,c,

$$\Sigma M_a/m_a = \frac{8}{3} + \frac{2}{9}\Sigma(m_b^2/m_c^2 + m_c^2/m_b^2) \geq 4,$$

since $x^2 + 1/x^2 \geq 2$, with equality just when $m_a = m_b = m_c$, that is, just when the triangle is equilateral.

III. *Solution by M.S. Klamkin, University of Alberta.*

Since $m_a(M_a-m_a) = a^2/4$, etc., the proposed inequality is equivalent to

$$\sum\frac{a^2}{4m_a^2} = \sum\frac{a^2}{2b^2+2c^2-a^2} \geq 1, \tag{2}$$

where the sums, here and later, are cyclic over a,b,c. It is a known result (see [4] or my comment following my solution of Crux 589 [1981: 308]) that if

$$I(a,b,c,m_a,m_b,m_c) \geq 0$$

is a valid inequality, then so is the dual inequality

$$I(m_a,m_b,m_c,\tfrac{3}{4}a,\tfrac{3}{4}b,\tfrac{3}{4}c) \geq 0,$$

and conversely. The dual of (2) is

$$\sum\frac{4m_a^{\ 2}}{9a^2} = \sum\frac{2b^2+2c^2-a^2}{9a^2} \geq 1,$$

and this is equivalent to

$$\left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right) + \left(\frac{c^2}{a^2} + \frac{a^2}{c^2}\right) + \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) \geq 6,$$

which follows immediately from $x^2 + 1/x^2 \geq 2$. There is equality if and only if the triangle is equilateral. □

It is known [5] that, if a,b,c are the sides of a triangle, then so are a^n, b^n,c^n for any n such that $0 < n < 1$. With $n = \frac{1}{2}$, it therefore follows from (2) that

$$\sum\frac{a}{2b+2c-a} \geq 1,$$

which is not so easy to prove directly.

More generally, I show that, for $k \geq 1$,

$$\sum\frac{a}{k(b+c)-a} \geq \frac{3}{2k-1}, \tag{3}$$

with equality if and only if $a = b = c$.

Proof. Let $x = s-a$, $y = s-b$, $z = s-c$, where s is the semiperimeter; then

$$a = y+z, \qquad b = z+x, \qquad c = x+y,$$

and (3) is equivalent to

$$2(k-1)T_1^3 \geq (5k-7)T_1T_2 + 9(k+1)T_3,$$

where $T_1 = \Sigma x$, $T_2 = \Sigma yz$, $T_3 = xyz$, with sums cyclic over x,y,z. Since

$$T_3 \leq T_1^3/27 \qquad \text{and} \qquad T_1T_2 \leq T_1^3/3,$$

we have

$$(5k-7)T_1T_2 + 9(k+1)T_3 \leq \frac{(5k-7)T_1^3}{3} + \frac{9(k+1)T_1^3}{27} = 2(k-1)T_1^3,$$

and so (3) is established. □

Finally, I leave as open problems to find all fixed $k \geq 2$ such that

$$\sum\frac{a^2}{k(b^2+c^2)-a^2} \geq \frac{3}{2k-1},$$

and, even more generally, to determine all fixed k and n such that

$$\sum\frac{a^n}{k(b^n+c^n)-a^n} \geq \frac{3}{2k-1}.$$

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; GALI SALVATORE, Perkins, Québec; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer. A comment was sent by S.C. CHAN, Singapore; and one incorrect solution was received, wherein it is "proved" that the inequality of the problem must be reversed!

REFERENCES

1. Thu Pham, solution to Problem 1119 (proposed by K.R.S. Sastry), *Mathematics Magazine*, 55 (1982) 180-182.

2. George Tsintsifas, Problem 723 (b), this journal, 8 (1982) 77.

3. S. Barnard and J.M. Child, *Higher Algebra*, Macmillan, London, 1936 (reprinted many times), p. 224, Ex. 4.

4. Aufgabe 677, solution by M.S. Klamkin, *Elemente der Mathematik*.

5. Solutions to Problem E 1366 (proposed by V.E. Hoggatt, Jr.), *American Mathematical Monthly*, 67 (1960) 82-84.

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690. [1981: 276] *Proposé par Hippolyte Charles, Waterloo, Québec.*

n étant un entier positif donné, soit

$$S_k = n^{n+k} - \binom{n}{1}(n-1)^{n+k} + \binom{n}{2}(n-2)^{n+k} - \ldots + (-1)^{n-1}\binom{n}{n-1}.$$

Calculer S_k pour $k = 0,1,2,3$.

I. *Solution by Gali Salvatore, Perkins, Québec.*

Let

$$f(x) = (e^x - 1)^n, \tag{1}$$

that is,

$$f(x) = e^{nx} - \binom{n}{1}e^{(n-1)x} + \binom{n}{2}e^{(n-2)x} - \ldots + (-1)^{n-1}\binom{n}{n-1}e^x + (-1)^n.$$

Differentiating $n+k$ times, we get

$$f^{(n+k)}(x) = n^{n+k}e^{nx} - \binom{n}{1}(n-1)^{n+k}e^{(n-1)x} + \binom{n}{2}(n-2)^{n+k}e^{(n-2)x} - \ldots + (-1)^{n-1}\binom{n}{n-1}e^x,$$

from which it is clear that $S_k = f^{(n+k)}(0)$. Therefore $S_k = (n+k)! \times$ the coefficient of x^{n+k} in the Maclaurin expansion of (1). This expansion, as far as we need, is

$$f(x) = x^n + \frac{n}{2}x^{n+1} + \frac{n(3n+1)}{24}x^{n+2} + \frac{n^2(n+1)}{48}x^{n+3} + \ldots,$$

and therefore

$$\begin{cases} S_0 = n!, \\ S_1 = \dfrac{(n+1)!n}{2}, \\ S_2 = \dfrac{(n+2)!n(3n+1)}{24}, \\ S_3 = \dfrac{(n+3)!n^2(n+1)}{48}. \end{cases} \tag{2}$$

II. *Solution by V.N. Murty, Pennsylvania State University, The Capitol Campus.*

Let n distinguishable boxes and $n+k$ distinguishable balls be given, and let E_k be the event that no box is empty after the $n+k$ balls have been distributed randomly among the n boxes. The evaluation of $\Pr(E_k)$ is a classical occupancy problem. Feller has shown [1] that

$$n^{n+k}\Pr(E_k) = S_k.$$

We easily (but tediously) find that

$$\Pr(E_0) = \frac{n!}{n^n}, \quad \Pr(E_1) = \frac{(n+1)!n}{2n^{n+1}}, \quad \Pr(E_2) = \frac{(n+2)!n(3n+1)}{24n^{n+2}}, \quad \Pr(E_3) = \frac{(n+3)!n^2(n+1)}{48n^{n+3}},$$

from which the results (2) follow.

III. *Comment by M.S. Klamkin, University of Alberta.*

These results are known and can be obtained by means of Stirling Numbers of the second kind. The values required here, and more, are given in a table in Gould [2].

Also solved by G.P. HENDERSON, Campbellcroft, Ontario; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

REFERENCES

1. William Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1, Third Edition, Wiley, New York, 1968, p. 60 (Equation 11.6) and p. 65 (Problems 16 and 17).

2. H.W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972, p. 3, where the reference is to "A remarkable combinatorial formula of Heselden", *Proc. W. Va. Acad. of Sci.*

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691. [1981: 301] *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

Here is an alphametic about the man who coined the word "alphametic", the well-known author of the syndicated column *Fun With Figures*, J.A.H. Hunter:

```
    H E'S
  T H A T
    F U N
M A T H S.
---------
H U N T E R
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The apostrophe has no mathematical significance and the answer, like the man himself, is unique.

Solution by Clayton W. Dodge, University of Maine at Orono.

It is immediately apparent that H = 1, M = 9, and U = 0. Now

$$2S + T + N = R + 10\alpha, \qquad \alpha = 1 \text{ or } 2,$$
$$\alpha + A + 1 = 10,$$
$$3 + F = 10,$$

and

$$1 + T + A = N + 10.$$

Thus F = 7, A = 8, and $\alpha = 1$. Now T = N + 1; hence 2S + 2N = R + 9 and R is odd, that is, R = 3 or 5 and S + N = 6 or 7, respectively. But R = 5 must be rejected, since then {S,N} = {3,4} and there is no value for T = N + 1. Thus R = 3, S = 2, N = 4, T = 5, which leaves E = 6. The unique solution is

$$\begin{array}{r} 162 \\ 5185 \\ 704 \\ \underline{98512} \\ 104563 \end{array}.$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.A.H. HUNTER, Toronto, Ontario; W.C. IGIPS, Danbury, Connecticut; ALLAN WM. JOHNSON JR., Washington, D.C.; STANLEY RABINOWITZ, Digital Equipment Corp.,Merrimack, New Hampshire; DONVAL R. SIMPSON, Fairbanks, Alaska; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and the proposer.

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692. [1981: 301] *Proposed by Dan Sokolowsky, California State University at Los Angeles.*

S_n is a set of n distinct objects. For a fixed $k \geq 1$, $2k$ subsets of S_n are denoted by A_i, B_i, $i = 1,\ldots,k$. Find the largest possible value of n for which the following conditions (a)-(d) can hold simultaneously for $i = 1,\ldots,k$.

(a) $A_i \cup B_i = S_n$.

(b) $A_i \cap B_i = \emptyset$.

(c) For each pair of distinct elements of S_n, there exists an i such that the two elements are either both in A_i or both in B_i.

(d) For each pair of distinct elements of S_n, there exists an i such that one of the two elements is in A_i and the other is in B_i.

Solution by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.

Let $k \geq 1$ be a fixed integer and let $S_n = \{1,2,\ldots,n\}$. We will show that the answer to our problem is $n = 2^{k-1}$. But first we must set up some machinery.

Let E be the set of vertices of the k-dimensional unit "cube" (more precisely, E is the Cartesian product $\{0,1\}^k$). For all $x,y \in E$, we define

$$x \sim y \iff x = y \text{ or } x + y = (1,1,\ldots,1),$$

where the addition in E is, as usual, coordinatewise. Then $\sim$ is an equivalence relation on E which partitions E into 2^{k-1} equivalence classes. Number these classes from 1 to 2^{k-1} and pick a representative P_i from the ith equivalence class.

The desired result is a consequence of the following theorem:

(i) If $n \le 2^{k-1}$, then there exist $2k$ subsets of S_n which satisfy (a)-(d).

(ii) If $n > 2^{k-1}$, then no collection of $2k$ subsets of S_n satisfy all of (a)-(d).

The theorem is clearly true if $k = 1$, so we assume that $k > 1$.

Proof of (i). Suppose $n \le 2^{k-1}$ and, for $i = 1,2,\ldots,k$, let

$$A_i = \{j \mid 1 \le j \le n \text{ and the } i\text{th coordinate of } P_j \text{ is } 0\}$$

and

$$B_i = \{j \mid 1 \le j \le n \text{ and the } i\text{th coordinate of } P_j \text{ is } 1\}.$$

Clearly, (a) and (b) hold for $i = 1,2,\ldots,k$. To show that (c) and (d) hold, let $1 \le i,j \le n$ with $i \ne j$. Since P_i and P_j are in different equivalence classes, there exist integers i_0 and j_0, $1 \le i_0,j_0 \le k$, such that

$$i_0\text{th coordinate of } P_i = i_0\text{th coordinate of } P_j$$

and

$$j_0\text{th coordinate of } P_i \ne j_0\text{th coordinate of } P_j.$$

Thus i and j are either both in A_{i_0} or both in B_{i_0}, and one of i and j is in A_{j_0} and the other is in B_{j_0}.

Proof of (ii). Suppose $n > 2^{k-1}$ and let A_j,B_j, $j = 1,2,\ldots,k$, be $2k$ subsets of S_n. If the subsets A_j,B_j satisfy (a) and (b), then there is a function $f\colon S_n \to E$ defined by

$$j\text{th coordinate of } f(i) = \begin{cases} 0, & \text{if } i \in A_j, \\ 1, & \text{if } i \in B_j, \end{cases}$$

for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$. Since $n > 2^{k-1}$, there exist distinct integers i_0 and j_0, $1 \le i_0,j_0 \le n$, such that $f(i_0) \sim f(j_0)$, and either (c) or (d) is false.

Also solved by J.D. HISCOCKS, University of Lethbridge, Alberta; and the proposer. Comments were received from PAUL R. BEESACK, Carleton University, Ottawa; and DAVID R. STONE, Georgia Southern College, Statesboro, Georgia.

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693*. [1981: 301] *Proposed by Ferrell Wheeler, student, Texas A & M University.*

On a 4×4 tick-tack-toe board, a winning path consists of four squares in a row, column, or diagonal. In how many ways can three X's be placed on the board, not all on the same winning path, so that if a game is played on this partly-filled board, X going first, then X can absolutely force a win?

Editor's comment.

No solution was received for this problem, which therefore remains open.

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694. [1981: 302] *Proposed by Jack Garfunkel, Flushing, N.Y.*

Three congruent circles with radical center R lie inside a given triangle with incenter I and circumcenter O. Each circle touches a pair of sides of the triangle. Prove that O, R, and I are collinear.

(This generalizes Problem 5 of the 1981 International Mathematical Olympiad [1981: 223], where it was specified that the three circles had a common point.)

Solution by Roland H. Eddy, Memorial University of Newfoundland.

This proposal is essentially the same as the original International Olympiad problem. The radical center R, irrespective of the position of the three circles, is simply the circumcenter of the triangle formed by joining the centers of the three circles. This triangle and the original one are homothetic with homothetic center I. Hence the circumcenters O and R, which are corresponding points in this homothecy, are collinear with the homothetic center I.

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland (second solution); J.T. GROENMAN, Arnhem, The Netherlands; F.G.B. MASKELL, Algonquin College, Ottawa; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and GEORGE TSINTSIFAS, Thessaloniki, Greece.

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695. [1981: 302 ; 1982: 30] (Corrected) *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

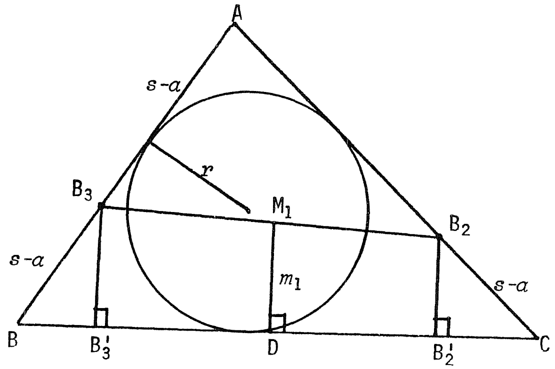
For $i = 1,2,3$, A_i are the vertices of a triangle with sides a_i and excircles with centers I_i touching a_i in B_i. For $j,k \neq i$, M_i are the midpoints of B_jB_k; and m_i are the lines through M_i perpendicular to a_i. Prove that the m_i are concurrent.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

We will for typographical convenience rename the triangle ABC and its sides a,b,c in the usual order (with r, R, and s having their usual meanings), but leave the rest of the notation unchanged. Suppose m_1,m_2,m_3 meet a,b,c in D,E,F, respectively.

It is well known [1] that the m_i are concurrent if and only if

$$(BD^2 - DC^2) + (CE^2 - EA^2) + (AF^2 - FB^2) = 0.$$



Referring now to the figure, we have

$$\begin{aligned} BD^2 - DC^2 &= (BD + DC)(BD - DC) \\ &= a(BB_3' - B_2'C) \\ &= a(s-a)(\cos B - \cos C) \\ &= \left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(r\cot\frac{A}{2}\right)(\cos B - \cos C) \\ &= 4Rr\cos^2\frac{A}{2}(\cos B - \cos C) \\ &= 2Rr(1 + \cos A)(\cos B - \cos C). \end{aligned}$$

With this and two similar results, we have, with sums cyclic over A,B,C,

$$\frac{\Sigma(BD^2 - DC^2)}{2Rr} = \Sigma(\cos B - \cos C) + \Sigma\cos A(\cos B - \cos C) = 0,$$

and the desired result follows.

Also solved by DAN SOKOLOWSKY, California State University at Los Angeles; and the proposer. Comments (pointing out the error in the original formulation of the problem) were received from JORDI DOU, Barcelona, Spain; ROLAND H. EDDY, Memorial University of Newfoundland; and STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire.

REFERENCE

1. H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, Wiley, 1969, p. 16, Ex. 6.

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606. [1981: 302] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be a triangle; a,b,c its sides; and s,r,R its semiperimeter, inradius and circumradius. Prove that, with sums cyclic over A,B,C,

(a) $\frac{3}{4} + \frac{1}{4}\Sigma\cos\frac{1}{2}(B-C) \geq \Sigma\cos A$;

(b) $\Sigma a\cos\frac{1}{2}(B-C) \geq s(1 + 2r/R)$.

Solution by V. N. Murty, Pennsylvania State University, Capitol Campus.

(a) From

$$\cos\frac{B-C}{2} = \cos\frac{B+C}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2}$$
$$= \sin\frac{A}{2} + \left(2\Pi\sin\frac{A}{2}\right)\csc\frac{A}{2}$$

and two similar results (sums and products throughout are cyclic over A,B,C), we get

$$\Sigma\cos\frac{B-C}{2} = \Sigma\sin\frac{A}{2} + \left(2\Pi\sin\frac{A}{2}\right)\Sigma\csc\frac{A}{2}; \qquad (1)$$

and from

$$\cos B + \cos C = 2\sin\frac{A}{2}\cos\frac{B-C}{2} \leq 2\sin\frac{A}{2} \qquad (2)$$

and two similar results, we get

$$\Sigma\sin\frac{A}{2} \geq \Sigma\cos A. \qquad (3)$$

We will also use the well-known inequality [1, p. 31]

$$\Sigma\csc\frac{A}{2} \geq 6. \qquad (4)$$

Now, from (1), (3), and (4), we obtain

$$\Sigma\cos\frac{B-C}{2} \geq \Sigma\cos A + 12\Pi\sin\frac{A}{2}. \qquad (5)$$

Finally, we apply to (5) the classical result

$$\Sigma\cos A = 1 + 4\Pi\sin\frac{A}{2}$$

and obtain

$$\Sigma\cos\frac{B-C}{2} \geq \Sigma\cos A + 3(\Sigma\cos A - 1) = 4\Sigma\cos A - 3, \qquad (6)$$

which is equivalent to the inequality proposed in part (a). Equality holds in (3) and (4), and hence in (6), just when the triangle is equilateral.

(b) It is easy to show (or see [2]) that

$$\Sigma a\cos\frac{B-C}{2} = \Sigma(b+c)\sin\frac{A}{2},$$

and then, from (2), we have

$$\Sigma(b+c)\sin\frac{A}{2} \geq \tfrac{1}{2}\Sigma(b+c)(\cos B + \cos C).$$

Now, from $a = b\cos C + c\cos B$, etc., we have

$$\tfrac{1}{2}\Sigma(b+c)(\cos B + \cos C) = s + \Sigma a\cos A,$$

where s is the semiperimeter. Thus

$$\Sigma a\cos\frac{B-C}{2} \geq s + \Sigma a\cos A.$$

Now, with K and R the area and circumradius of the triangle, respectively, we have

$$\Sigma a\cos A = R\Sigma\sin 2A = 4R\Pi\sin A = \frac{2K}{R} = \frac{2rs}{R},$$

and therefore

$$\Sigma a\cos\frac{B-C}{2} \geq s\left(1 + \frac{2r}{R}\right). \tag{7}$$

In all the inequalities we have used, and hence in (7), equality holds just when the triangle is equilateral. □

If we compare

$$\Sigma\cos\frac{B-C}{2} \geq 4\Sigma\cos A - 3 \tag{6}$$

and Garfunkel's inequality (Crux 613 [1982: 138])

$$\Sigma\cos\frac{B-C}{2} \geq \frac{2}{\sqrt{3}}\Sigma\sin A, \tag{8}$$

it is natural to ask which inequality is sharper. With

$$x = \frac{r}{R} = \Sigma\cos A - 1 \qquad \text{and} \qquad y = \frac{s}{R} = \Sigma\sin A,$$

(6) and (8) can be expressed as

$$\Sigma\cos\frac{B-C}{2} \geq 1 + 4x \tag{9}$$

and

$$\Sigma\cos\frac{B-C}{2} \geq \frac{2y}{\sqrt{3}}. \tag{10}$$

With the inequalities in this form, our question is easily answered.

It is known [1, p. 22] that $0 < x \leq \frac{1}{2}$. For the notation and other results I will now use, see my article recently published in this journal [1982: 62-68]. For all triangles of type I, we have $y \geq \sqrt{3}(1+x)$, so

$$\frac{2y}{\sqrt{3}} \geq 2(1+x) \geq 1 + 4x,$$

and (10) is sharper than (9). The line

$$y = \frac{\sqrt{3}}{2}(1 + 4x)$$

divides the type II region (shaded with horizontal lines in the figure on page 65) into two parts, in one of which we have

$$\frac{\sqrt{3}}{2}(1+4x) \leq y \leq \sqrt{3}(1+x) \tag{11}$$

and in the other

$$0 < y < \frac{\sqrt{3}}{2}(1 + 4x). \tag{12}$$

Inequality (10) is sharper than (9) for the type II triangles which satisfy (11), and (9) is sharper than (10) for the type II triangles which satisfy (12).

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and by the proposer.

REFERENCES

1. O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

2. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1959, p. 3, Ex. 36.

* * *

697. [1981: 302] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

Let

$$a = \tan\theta + \tan\phi,$$
$$b = \sec\theta + \sec\phi,$$
$$c = \csc\theta + \csc\phi.$$

If the angles θ and ϕ are such that the requisite functions are defined and $bc \neq 0$, show that $2a/bc < 1$.

Solution by W.J. Blundon, Memorial University of Newfoundland.

Let $u = \tan(\theta/2)$ and $v = \tan(\phi/2)$. It follows from the hypothesis that neither u nor v can equal any of -1, 0, 1. Now

$$a = \frac{2u}{1-u^2} + \frac{2v}{1-v^2} = \frac{2(u+v)(1-uv)}{(1-u^2)(1-v^2)},$$

$$b = \frac{1+u^2}{1-u^2} + \frac{1+v^2}{1-v^2} = \frac{2(1+uv)(1-uv)}{(1-u^2)(1-v^2)},$$

$$c = \frac{1+u^2}{2u} + \frac{1+v^2}{2v} = \frac{(u+v)(1+uv)}{2uv}.$$

Therefore

$$\frac{2a}{bc} = \frac{4uv}{(1+uv)^2} = 1 - \frac{(1-uv)^2}{(1+uv)^2} < 1.$$

The inequality is strict, since $uv = 1$ implies $b = 0$, contrary to the hypothesis.

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; W.C. IGIPS, Danbury, Connecticut; V.N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. A comment was received form STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire.

Editor's comment.

Several solvers proved instead that $bc > 2a$, which they claimed was equivalent to $2a/bc < 1$. This would have been true if the proposal had read: "Prove that $bc > 2a$ if b and c are of the same sign." (This, in fact, is exactly the wording the proposer had used, and Rabinowitz found the problem, in the exact notation and wording of the proposer, in Hobson [1].) But the proposal as modified by the editor is slightly more general and allows the possibility $bc < 0$, so $bc > 2a$ is *not* equivalent to $2a/bc < 1$.

REFERENCE

1. E.W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover, New York, 1957, p. 101, Ex. 64.

* * *

698. [1981: 302] *Proposé par Hippolyte Charles, Waterloo, Québec.*

Les sommes partielles de la série harmonique (laquelle, on le sait bien, est divergente) sont définies par

$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

La série $\sum_{n=1}^{\infty} \frac{1}{s_n}$ est-elle convergente ou divergente?

Solution de Leroy F. Meyers, The Ohio State University.

On voit aisément que

$$s_n = \sum_{j=1}^{n} \frac{1}{j} \le \sum_{j=1}^{n} 1 = n.$$

Donc $1/s_n \geq 1/n$, et la série proposée diverge par comparaison avec la série harmonique.

Also solved by BERNARD BAUDIFFIER, Collège de Sherbrooke, Québec; PAUL R. BEESACK, Carleton University, Ottawa; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; YVES GALIPEAU, Collège de Sherbrooke, Québec; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; J.D. HISCOCKS, University of Lethbridge, Alberta; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa; V.N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; DAVID R. STONE, Georgia Southern College; KENNETH S. WILLIAMS, Carleton University, Ottawa; and DAVID ZAGORSKI, student, Massachusetts Institute of Technology.

Editor's comment.

Several (but by no means all) of the solutions to this easy problem were equivalent to the trivial one given above. We chose to feature one submitted in French precisely because the problem was easy, to enable our readers to improve their French, if not their mathematics.

*   *   *

699. [1981: 302] *Proposed by Charles W. Trigg, San Diego, California.*

A quadrilateral is inscribed in a circle. One side is a diameter of the circle and the other sides have lengths of 3, 4, and 5. What is the length of the diameter of the circle?

Solution by the proposer.

Generalizing the problem, we let the required diameter be x, the three given sides a,b,c, and the diagonals e,f. As indicated in the figures, there are three possible situations:

I. a convex quadrilateral;

II. a crossed quadrilateral with two vertices on one side of the diameter;

III. a crossed quadrilateral with vertices on opposite sides of the diameter.

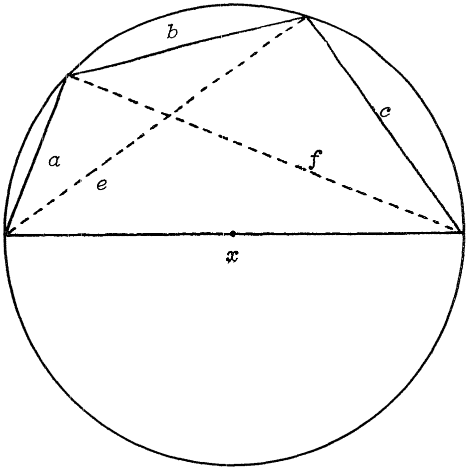
Case I. Applying the theorems of Ptolemy and Pythagoras, we have

$$ef = ac + bx, \qquad a^2 + f^2 = x^2, \qquad e^2 + c^2 = x^2.$$

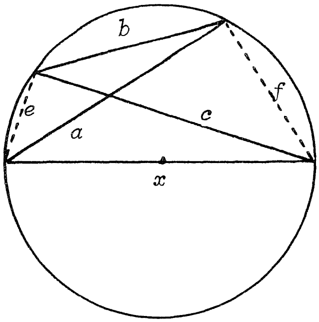
Elimination of e and f leads to $(x^2-c^2)(x^2-a^2) = (ac+bx)^2$, which, since $x \neq 0$, is equivalent to

$$x^3 - (a^2+b^2+c^2)x - 2abc = 0. \tag{1}$$

(This result has appeared frequently in the literature. See, e.g., [1], where more references are given.) With the given values of a,b,c, equation (1) becomes



I



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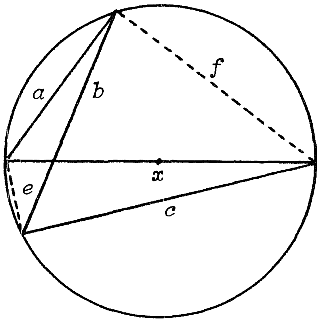
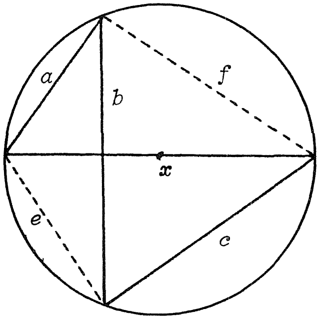


III a



III b

$$x^3 - 50x - 120 = 0. \tag{2}$$

This equation has one positive and two negative roots, the positive root being

$$x = \frac{10\sqrt{6}}{3}\cos\left(\frac{1}{3}\operatorname{Arccos}\frac{9\sqrt{6}}{25}\right) \approx 8.05581036.$$

(For information as to how this root was obtained, see Rausen [2].) We note that, since the central angles subtended by the chords a, b, c are additive on one side of the diameter, the chords may be taken in any of $3! = 6$ orders.

Cases II and III. In case II, we have

$$ac = ef + bx, \qquad e^2 + c^2 = x^2, \qquad f^2 + a^2 = x^2,$$

and case III gives

$$bx = ac + ef, \qquad a^2 + f^2 = x^2, \qquad e^2 + c^2 = x^2.$$

In each of these cases, proceeding as in case I leads to the equation

$$x^3 - (a^2+b^2+c^2)x + 2abc = 0,$$

or, with the given values of a,b,c, to

$$f(x) \equiv x^3 - 50x + 120 = 0.$$

This equation has one negative and two positive roots. Since $f(0)\cdot f(3) < 0$, one of the positive roots is less than 3, an impossible diameter with chords greater than 3. The other positive root, which is acceptable, is

$$x = \frac{10\sqrt{6}}{3}\cos\left(\frac{\pi}{3} - \frac{1}{3}\text{Arccos}\,\frac{9\sqrt{6}}{25}\right) \approx 5.18026779.$$

In a circle with this diameter, there is one crossed quadrilateral (and its reflection) of type II with sides in the order $(4,3,5,x)$, and two crossed quadrilaterals (and their reflections) of type III with sides in the orders $(3,4,5,x)$ and $(3,5,4,x)$.

Also solved by ELWYN ADAMS, Gainesville, Florida; PAUL R. BEESACK, Carleton University, Ottawa; W.J. BLUNDON, Memorial University of Newfoundland; S.C. CHAN, Singapore; CLAYTON W. DODGE, University of Maine at Orono; JORDI DOU, Barcelona, Spain; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; MARIE JANE HAGUEL, Collège de Sherbrooke, Québec; RICHARD I. HESS, Rancho Palos Verdes, California; W.C. IGIPS, Danbury, Connecticut; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; LEROY F. MEYERS, The Ohio State University; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT TRANQUILLE et YVES LACOMBE, Collège de Maisonneuve, Montréal, Québec; DIMITRIS VATHIS, Chalcis, Greece; KENNETH M. WILKE, Topeka, Kansas; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and one anonymous solver from Georgia Southern College.

Editor's comment.

Did the proposal say that the quadrilateral was convex? It did not. Yet, the editor regrets to report, in a rare display of unanimity *each* of the twenty-three other solvers tacitly assumed that the quadrilateral was convex and thus was led to consider only equation (2). Poets may rhapsodize about majority rule (democracy is for the bards), but here we have a proof that you *can* fool all of the people some of the time. Our proposer, an old pro, constitutes an overwhelming majority of one. *Caveat lector*!

REFERENCES

1. Problem 880 (proposed by Richard Corry), *Mathematics Magazine*, 48 (January 1975) 53.

2. John Rausen, "Solving cubic equations on a pocket calculator", this journal, 6 (1980) 5-7.

701. [1982: 14] *Proposed by Alan Wayne, Holiday, Florida.*

Ever since her X-rated contribution to the solution of Crux 411 [1979: 299] shocked the old lady from Dubuque, EDITH ORR (long may she swing!) has maintained a low profile. Yet we all know that she is always THERE at the editor's elbow, like as not blowing in his ear. So we should all be able to solve the subtraction

$$\begin{array}{r} \text{EDITH} \\ -\ \ \text{ORR} \\ \hline \text{THERE} \end{array}.$$

Solution by Sam Baethge, Southwest High School, San Antonio, Texas.

Expressing the problem as an addition,

$$\begin{array}{r} \text{THERE} \\ +\ \ \text{ORR} \\ \hline \text{EDITH} \end{array},$$

we see immediately that H = 9, D = 0, and

$$E = T + 1, \qquad E + R = 9, \qquad 2R = T + \alpha, \tag{1}$$

where α = 0 or 10. Eliminating E and T from (1), we find that $3R = 8 + \alpha$, from which $\alpha = 10$, so R = 6, E = 3, and T = 2. Now

$$4 + O = I + 10, \qquad \text{or} \qquad O - I = 6,$$

for which the remaining digits give O = 7 and I = 1 as the only possible values. The given subtraction therefore has the unique solution

$$\begin{array}{r} 30129 \\ -\ \ 766 \\ \hline 29363 \end{array}.$$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; CLAYTON W. DODGE, University of Maine at Orono; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; MEIR FEDER, Haifa, Israel; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; W.C. IGIPS, Danbury, Connecticut; ALLAN WM. JOHNSON JR., Washington, D.C.; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; RAM REKHA TIWARI, Radhaur, Bihar, India; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

* * *

702. [1982: 14] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are three noncollinear points A_1, B_1, C_1 and three positive real numbers l, m, n whose sum is 1. Show how to determine a point M inside the circumcircle of triangle $A_1B_1C_1$ such that, if A_1M, B_1M, C_1M meet the circumcircle again in

A,B,C, respectively, then we have

$$\frac{[BMC]}{l} = \frac{[CMA]}{m} = \frac{[AMB]}{n} = [ABC], \tag{1}$$

where the brackets denote the area of a triangle.

Partial solution by the proposer (revised by the editor).

Let the given points A_1, B_1, C_1 have affixes $\alpha_1, \beta_1, \gamma_1$, respectively, in the complex plane, and suppose there exists a point M inside the circumcircle such that the resulting triangle ABC satisfies (1). If the affixes of the unknown points M,A,B,C are z, α, β, γ, respectively, then it is well known that

$$z = l\alpha + m\beta + n\gamma. \tag{2}$$

The power of the interior point M with respect to the circle is

$$k \equiv (\overline{z-\alpha})(z-\alpha_1) \neq 0,$$

from which we obtain

$$\frac{lk}{z-\alpha_1} = l\bar{z} - l\bar{\alpha}, \tag{3}$$

and similarly

$$\frac{mk}{z-\beta_1} = m\bar{z} - m\bar{\beta}, \tag{4}$$

$$\frac{nk}{z-\gamma_1} = n\bar{z} - n\bar{\gamma}. \tag{5}$$

Since $l+m+n = 1$ and $k \neq 0$, adding (3), (4), (5), and using (2) result in

$$\frac{l}{z-\alpha_1} + \frac{m}{z-\beta_1} + \frac{n}{z-\gamma_1} = 0,$$

and clearing the fractions gives the equivalent equation

$$z^2 - \{(m+n)\alpha_1 + (n+l)\beta_1 + (l+m)\gamma_1\}z + (l\beta_1\gamma_1 + m\gamma_1\alpha_1 + n\alpha_1\beta_1) = 0. \tag{6}$$

One root of this equation is the affix of the point M, so M and the resulting triangle ABC can be identified.

Editor's comment.

Still unresolved in the above solution of this interesting problem are the following questions:

(a) The editor would like to receive a simple proof or a reference for the "well-known" relation (2), which appears to be correct. It would greatly ease the editor's burden if solvers were to assume that he knows far less than they do, and give proofs or references for results that are "well known" to only a few.

(b) If the point M is assumed to exist, which root z_0 of (6) gives its affix, and what is the geometrical significance, if any, of the other root?

(c) Conversely, if z_0 is the appropriate root of (6), it must be shown that the point M whose affix is z_0, and the resulting triangle ABC, satisfy (1).

(d) Can the problem be extended, perhaps by removing the requirement that l,m,n be positive and using signed areas?

Readers are invited to answer these questions, or else to find an entirely different approach to the problem.

* * *

704. [1982: 14] *Proposed by H. Kestelman, University College, London, England.*

Suppose $A_1,A_2,\ldots,A_j$ are square matrices of any orders and no two have a common eigenvalue. If $f_1,f_2,\ldots,f_j$ are any given polynomials, then there is a polynomial F such that $F(A_r) = f_r(A_r)$ for all r.

Solution by the proposer.

Let g_r be the product of the characteristic polynomials of all the A_s with $s \neq r$. Since the g_r are pairwise co-prime, there exist polynomials $\phi_1, \phi_2, \ldots, \phi_j$ such that

$$1 = \sum_{r=1}^{j} \phi_r(t)g_r(t)$$

for all t. Set

$$F(t) = \sum_{r=1}^{j} \phi_r(t)g_r(t)f_r(t).$$

By the Cayley-Hamilton theorem, $g_r(A_s) = 0$ if $r \neq s$; hence

$$F(A_s) = \phi_s(A_s)g_s(A_s)f_s(A_s) \quad \text{and} \quad I = \phi_s(A_s)g_s(A_s)$$

for all s, that is, $F(A_s) = f_s(A_s)$.

* * *

705. [1982: 15] *Proposed by Andy Liu, University of Alberta.*

On page 315 of U.I. Lydna's definitive treatise *Medieval Religion* (published in 1947) is a little piece of irrelevancy entitled "Pagan Island". This island was of circular shape and had along its coast exactly 26 villages whose names, in cyclic order, had the initials A, B, ..., Z. At various times in its history, the island was visited by exactly 26 missionaries whose names all had different initial letters. Each missionary first went to the village which matched his initial. While more than one missionary might be on the island at the same time, no two ever appeared (or disappeared) in the same village at the same time. As the name of the island implies, all the villages were initially pagan. When a missionary arrived at an unconverted village, he converted it and then moved on to the next village along the coast. When a missionary arrived at a converted village, he was promptly devoured and the village became unconverted again.

While the missionaries' fate was never in doubt, how many villages remained converted after their combined effort?

I. *Solution by Leroy F. Meyers, The Ohio State University.*

In the end, Pagan Island remains entirely pagan, since each village receives exactly two visits from missionaries.

Each village is converted at least once, either by the missionary who lands there or by the missionary who first converts the preceding village. (At least one village is converted by the missionary who lands there, so the process can begin.) But in either case there is another visit, by the first missionary to convert the preceding village or by the missionary who lands at the village.

On the other hand, no village is visited more than twice. For suppose that some village is among those tied for first to receive a third visit (for reconversion, after deconversion, after conversion to — what else? — hierophagy, or the devouring of the next missionary to arrive at the village). Then at least two of the three visits must have been by missionaries who had previously (re)converted the preceding village. But a reconversion implies a preceding deconversion. Hence the preceding village must have already had three visits by the time the village under consideration has had its third, so that the village cannot have been tied for first to receive a third missionary. Thus no village can be tied for first, and so no village can have a third visit.

Hence each village receives exactly two visits from missionaries, and so, after the missionary activity has ended, is again pagan.

II. *Incorrect solution by X, who deserves to take on holy orders and receive a one-way ticket to Pagan Island.*

Let u denote the number of unconverted villages on the island (at any particular point in time) and let m denote the number of (live) missionaries on the island at that time. I claim that $u+m$ is always 26.

Proof. Initially, $u = 26$, $m = 0$, so $u+m = 26$. There are only two events that can occur:

Event 1. Missionary arrives at an unconverted village. Here m increases by 1 and u decreases by 1.

Event 2. Missionary arrives at a converted village. Here (alas) m decreases by 1 and u increases by 1.

In all cases, $u+m$ is invariant. □

Thus, when all the missionaries were gone, $m = 0$, and we must have $u = 26$.

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; SAM BAETHGE, Southwest High School, San Antonio, Texas; PAUL R. BEESACK, Carleton University, Ottawa; CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montréal, Québec; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; ANNELIESE ZIMMERMANN, Bonn, West Germany; and the proposer.

Editor's comment.

All solutions credited above were correct, although a few contained more arm-waving than strict application of logic. All solutions assumed that all missionaries travel in the same direction, but Kierstead showed in addition that some villages may remain converted if the missionaries do not all travel in the same direction in going from one village to the next. Beesack proved more generally that if k missionaries visit the island, where $1 \le k \le 26$, then $26-k$ villages remain converted at the end. The debacle at Pagan Island inspired one solver to quote:

A man convinced against his will
Is of the same opinion still.

This is one of the famous misquotations of literary history (another is "a rose is a rose is a rose", about which more next month in the solution of Crux 711). The correct quotation is

He that complies against his will
Is of his own opinion still.

(Samuel Butler (1612-1680), in *Hudibras*, Part III, Canto III, line 547.)

Our first solver's phrase "conversion to ... hierophagy" shows that cannibalism is the religion of the *missionaries*, whose fleeting effects last only until the next meal of long pig. Meyers also sent with his solution a copy of a cartoon from *Monster Rally*, by Charles Addams, showing a group of cannibals seated around a boiling missionary-size kettle. One of them says to another: "Oh, I like missionary, all right, but missionary doesn't like me." To which we may reply: "Well, just leave him there and eat the noodles."

* * *

ONE TWO, BUTTON YOUR SHOE

Good Queen Bess was more than a wonder, she was a Tudor. (Anonymous)

I ate my tenderloin with a fork, then I nine my elevenderloin with a fivek. (Victor Borge)

* * *

THE PUZZLE CORNER

Puzzle No. 26: Alphametic

The following quiz could be plenty fun,
Just remember to work in base twenty-one.

$$\begin{array}{r} \text{NOW} \\ \text{WE} \\ \underline{\text{ARE}} \\ \text{SIX} \end{array}$$ ... a summation of fact;

That ONE divides SIX requires no tact;
That TWO is prime is perfectly valid,
But SIX must be perfect, and thus ends our ballad.

Puzzle No. 27: Alphametic

Prime STEAK from a square COW:
Haven't had it before? At least you've heard of it now.
Once in a while, just as a treat,
Of this I partake 'cause $\begin{array}{r} \text{I} \\ \underline{\text{EAT}} \\ \text{MEAT} \end{array}$.

Puzzle No. 28: Alphametic

SQUARE is a square but Q is not cubic.
There's a cube number of cubes in the form of RUBIK;
Only a cube number of which the first criteria still meet...
All the other solutions you will have to delete.
If I look like a cube it's because of your eyes;
For I've noticed each iris is a square in disguise.

HANS HAVERMANN
Weston, Ontario

Answer to Puzzle No. 23 [1982: 275]: Rhumb, rum.

Answer to Puzzle No. 24 [1982: 275]: Nonplussed (no *N* plussed).

Answer to Puzzle No. 25 [1982: 275]: Iconic.

* * *

A Double Dactyl

THE MOST DEDEKINDEST CUT OF ALL

Upper class, lower class,
Herr Richard Dedekind,
Plagued by real numbers, was
Stuck in a rut.

"How can I deal with their
Irrationality?
Ho! By Eudoxus, I'll
Start with a cut!"

LEROY F. MEYERS
The Ohio State University

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AUTHORS OF ARTICLES AND NOTES

TITLES OF ARTICLES AND NOTES

PROBLEMS AND SOLUTIONS

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