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CONTENTS

THE NUMEROLOGY OF PERIODIC DECIMAL EXPANSIONS

SOLOMON W. GOLOMB

Consider the following decimal expansions, each repeating with period *six*:

$$\frac{1}{7} = .142857142857\ldots$$

$$\frac{1}{13} = .076923076923\ldots$$

$$\frac{1}{77} = .012987012987\ldots$$

$$\frac{1}{91} = .010989010989\ldots$$

$$\frac{1}{143} = .006993006993\ldots$$

$$\frac{1}{1001} = .000999000999\ldots$$

to which we may also adjoin

$$\frac{1}{11} = .090909090909\ldots$$

which has a "true" period of *two*, but for present purposes may also be regarded as having a period of *six*.

All of these six-digit periods have many remarkable properties in common. Suppose we denote the first three digits in the expansion of $1/n$ as the number A_n, and the next three digits as the number B_n. Thus, in the expansion of $1/7$, we have $A_7 = 142$ and $B_7 = 857$. We see that $A_7 + B_7 = 999$. This is true for all the other examples as well:

$$\begin{aligned} A_{13} + B_{13} &= 76 + 923 = 999 \\ A_{77} + B_{77} &= 12 + 987 = 999 \\ A_{91} + B_{91} &= 10 + 989 = 999 \\ A_{143} + B_{143} &= 6 + 993 = 999 \\ A_{1001} + B_{1001} &= 0 + 999 = 999 \end{aligned}$$

and also

$$A_{11} + B_{11} = 90 + 909 = 999.$$

In each case, also, $A_n = 1001/n - 1$ and $B_n = 999 - A_n$. Since $1001 = 7 \cdot 11 \cdot 13$, this makes it very easy to "remember" all of these expansions:

To get the decimal version of 1/7, take the *other* two factors of 1001, namely 11 and 13. Their product is 143. One less is A_7. That is, $A_7 = 142$, and B_7 is its "9's complement", $B_7 = 857$.

Similarly, to expand 1/77, the remaining factor of 1001 is 13. Thus $A_{77} = 012$, and $B_{77} = 987$, so that $1/77 = .012987\ldots$ repeating with period *six*.

This can all be proved by observing that

$$\frac{1}{1001} = \frac{1}{10^3} - \frac{1}{10^6} + \frac{1}{10^9} - \frac{1}{10^{12}} + \frac{1}{10^{15}} - \frac{1}{10^{18}} + \ldots,$$

so that, for example,

$$\begin{aligned}\frac{1}{7} = \frac{143}{1001} &= \frac{143}{10^3} - \frac{143}{10^6} + \frac{143}{10^9} - \frac{143}{10^{12}} + \ldots \\ &= \frac{142}{10^3} + \frac{(1000-143)}{10^6} + \frac{142}{10^9} + \frac{(1000-143)}{10^{12}} + \ldots \\ &= \frac{142}{10^3} + \frac{857}{10^6} + \frac{142}{10^9} + \frac{857}{10^{12}} + \ldots \\ &= .142857142857\ldots .\end{aligned}$$

Another remarkable identity is

$$B_n = (n-1)A_n + (n-2),$$

which may also be stated as: A_n can be divided into B_n to give a quotient of $n-1$ and a remainder of $n-2$. Caution: This will not always be the *least* remainder. Thus:

for $n = 7$, $\quad \begin{array}{r} 6 \\ 142\overline{)857} \\ \underline{852} \\ 5 \end{array}$

and

for $n = 13$, $\quad \begin{array}{r} 12 \\ 76\overline{)923} \\ \underline{912} \\ 11 \end{array}$

we get a "proper" quotient of $n-1$ and a "proper" remainder of $n-2$; but

for $n = 77$, $\quad \begin{array}{r} 76 \\ 12\overline{)987} \\ \underline{912} \\ 75 \end{array}$

or

for $n = 143$, $\quad \begin{array}{r} 142 \\ 6\overline{)993} \\ \underline{852} \\ 141 \end{array}$

we get n-1 and n-2 as an "improper" quotient and remainder, respectively.

We can even consider $n = 1001$, where

$$\begin{array}{r} 1000 \\ 0\overline{)\ 999} \\ \underline{0} \\ 999 \end{array}$$

gives us an "improper" quotient of n-1 and an "improper" remainder of n-2. This says that, contrary to popular opinion, it *is* possible to divide by zero, *provided* that you are willing to accept a positive remainder!

The reader may wish to generalize the results and examples in this paper to cover decimal expansions whenever the period is an even number (or, at least, twice an odd number), and expansions in bases other than ten.

Paradox: Expansions are always in base 10, but not always in base ten.

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* * *

INTERNATIONAL CAMPAIGN-MASSERA

We have received the following communiqué, dated February 22, 1982 and signed by Pierre Berton, from Israel Halperin, Director of International Campaign-Massera, 39 Elm Ridge Drive, Toronto, Ontario, Canada M6B 1A2:

On behalf of the Canadian Committee of Scientists and Scholars I wish to protest in the strongest possible terms the outrageous imprisonment and torture of a distinguished mathematician by the Uruguayan authorities.

The scientist in question is Jose Luis Massera. He is his country's leading mathematician, a scientist of world rank, internationally respected for his research in differential equations.

When the military junta in Uruguay seized power in 1975, Professor Massera was arrested and savagely beaten. He is still in prison at the age of 66. One leg is shorter than the other as a result of torture that fractured his hip.

For six years scientific and scholarly societies, together with thousands of individuals, have protested his incarceration and demanded his release. Amnesty International has named him a prisoner of conscience. The Human Rights committee of the United Nations has confirmed that his human rights have been denied.

The Canadian Committee of Scientists and Scholars joins with the International Campaign-Massera in urging the release of this distinguished scientist. The Canadian committee is composed of more than thirty leading public figures in the field of the arts, sciences and culture -- ranging from Professor Northrop Frye and Senator Eugene Forsey to Moderator Lois Wilson of the United Church and Nobel Laureate Gerhard Herzberg.

The International Campaign is a determined, united attempt to force the Uruguayan government to act.

The Campaign plans to focus world wide publicity on Uruguay regarding Professor Massera, the prison Libertad and the Uruguayan government's policy of torture.

Letters are being dispatched to fifty leading personalities in Uruguay detailing the international revulsion felt over Massera's incarceration. Regular bulletins and telephone calls to the local Uruguayan embassies will follow.

Since world wide support for Massera's release has been voiced over the past six years the case has become a test case: can decent people all over the world stop oppression by a brutal military regime?

Everyone who cares about political freedom is asked to make his voice heard by the Uruguayan government.

PIERRE BERTON
21 Sackville Street
Toronto, Ontario M5A 3E1

* * *

THE PUZZLE CORNER

Puzzle No. 10: Jigsaw Puzzle

An 8-cm cube has the word TOP penciled on one face. The opposite face will be designated as "bottom", and the remaining four faces as "sides".

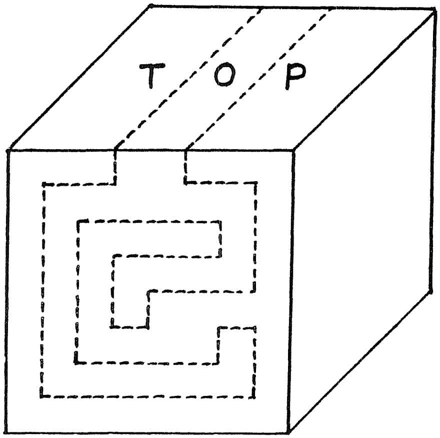


Figure 1

Place the cube with one side on the table of a jig saw and make a single rather circuitous cut with the jig saw, as shown in Figure 1. The cube may now be separated into two pieces, but *do not do so*. Next, place one uncut side on the table of the jig saw and make the same rather circuitous single cut with the jig saw. The top of the cube now looks like Figure 2. The bottom is uncut.

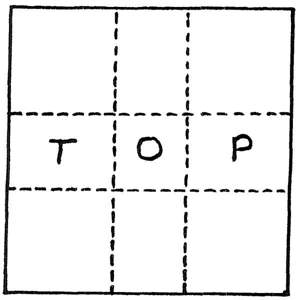


Figure 2

(a) Into how many pieces can the original cube now be separated?

(b) Does the answer to (a) depend upon which uncut side was placed on the jig saw table to make the second cut?

RICHARD V. ANDREE
University of Oklahoma

Answer to Puzzle No. 8 [1982: 5]: hearten; hear ten.

Answer to Puzzle No. 9 [1982: 5]: a gross exaggeration.

FRACTIONS ON A CALCULATOR

CLAYTON W. DODGE

Although specialized calculators do yield the results of fraction computations in fraction form (even some very inexpensive ones, like the Radio Shack EC-8497), it is possible to perform simple additions and multiplications of fractions on a standard calculator having 8 or more digits in its display. The process is delightfully simple and easy to establish.

Recall that

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd},\qquad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd},$$

and that

$$(1000a+b)(1000c+d) = 1000000ac + 1000(ad+bc) + bd.$$

In this last computation, we see the ac, $ad+bc$, and bd that appear in the fraction computations above, and we have the following algorithm.

For fractions having relatively small numerators and denominators, enter the first numerator in the thousands' place and its denominator in the units' place, enter times, enter the same for the second fraction, and enter equals. For 3/5 and 4/7, for example, we have

$$3005\cdot 4007 = 12\,041\,035.$$

We read the product 12/35 as the two (or more) digits in the millions' place divided by the three digits in the units' group. The sum 41/35 is read as the three digits in the thousands' group divided by the units' group.

One is limited to three digits in each place and, with an 8-digit calculator, to 2 digits in the numerator of the product. Nonetheless, the technique can be a convenience upon occasion and is certainly a useful teaching device for high schools and even elementary classes.

The sum and product may not be in lowest terms (they are on the Radio Shack EC-8497), but a reducing algorithm is easily devised. For the fractions 7/15 and 5/12, for example, we obtain

$$7015 \cdot 5012 = 35\,159\,180.$$

The sum 159/180 can be reduced since 3 is a factor of both numerator and denominator. So we store the 35 159 180 in memory or write it on paper, subtract 35 000 000 to display just the 159 180, and then divide by 3 to get 53 060. Thus we see that

$$\frac{7}{15} + \frac{5}{12} = \frac{159}{180} = \frac{53}{60}.$$

To reduce the product, we recall the 35 159 180 from memory and subtract the 159 000 to display 35 000 180. Now a factor of 5 can be divided out of both 35 and 180, so divide by 5 to get 7 000 036. With no memory, just enter 35 180 and divide by 5 to get 7 036. Thus

$$\frac{7}{15} \cdot \frac{5}{12} = \frac{35}{180} = \frac{7}{36}.$$

Since division is readily reduced to multiplication, quotients can easily be found on the calculator. Just use the formula

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Thus, for $(7/15)\div(5/12)$, we calculate

$$7015 \cdot 12005 = 84\,215\,075 \quad \text{and} \quad 84\,000\,075 \div 3 = 28\,000\,025,$$

so we find that

$$\frac{7}{15} \div \frac{5}{12} = \frac{84}{75} = \frac{28}{25}.$$

Here, then, we have a technique for performing addition, multiplication, and division of fractions on a calculator. The procedure should be especially helpful to children having trouble with fraction arithmetic. The calculator is an excellent motivational tool, and a child who can use one to solve his problems may eventually learn to get along without this crutch. Whether or not he overcomes this dependence, he has learned to solve his fraction problems and he is better off than beforehand.

Since we have found a delightful algorithm for addition and multiplication and a reasonable modification for division, we naturally ask how complicated subtraction is. One can devise calculator techniques for subtracting fractions, but they are not as convenient and easily remembered as our earlier algorithm. For example, if $a/b > c/d$, then a relatively simple method is derived from the equation

$$(1000a-b)(1000c+d) = 1000000ac + 1000(ad-bc) - bd.$$

Adding $2bd$ yields

$$1000000ac + 1000(ad-bc) + bd,$$

and we read the difference

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

as the three digits in the thousands' group divided by the three digits in the units' group. The product ac/bd is also read as before. Thus one would enter, for $2/3 - 3/7$,

$$(2000-3)\cdot 3007 + 42 = 6\,005\,021.$$

Hence

$$\frac{2}{3} - \frac{3}{7} = \frac{5}{21} \quad \text{and} \quad \frac{2}{3} \cdot \frac{3}{7} = \frac{6}{21}.$$

If $\frac{a}{b} < \frac{c}{d}$, then use the formula

$$\begin{aligned}(1000a-b)(1000c+d) - 1000000ac &= 1000(ad-bc) - bd \\ &= -1000(bc-ad) - bd.\end{aligned}$$

Thus, for $2/7 - 5/9$ we have

$$(2000-7)\cdot 5009 - 10000000 = -17\,063,$$

so

$$\frac{2}{7} - \frac{5}{9} = -\frac{17}{63}.$$

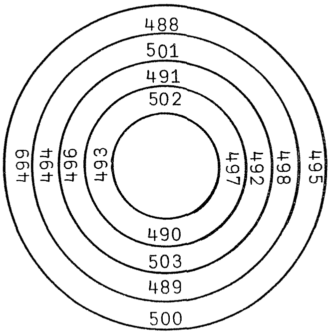
One need not decide beforehand which fraction is larger: if the number in the millions' place equals ac, then $1000(ad-bc) - bd$ is positive, so $a/b > c/d$ and one should add $2bd$. If the number in the millions' place is $ac - 1$, then $1000(ad-bc) - bd$ is negative, so $a/b < c/d$ and it is necessary to subtract $1000000ac$.

For students having difficulties with fractions the subtraction technique may well be too complicated; one must remember to subtract the first denominator when entering the numbers and then decide whether to add $2bd$ or subtract $1000000ac$. Of course, any class would enjoy investigating how to perform the four basic operations on fractions using an ordinary calculator.

Mathematics Department, University of Maine, Orono, Maine 04469.

* * *

A MAGIC CIRCLE FOR 1982



The adjoining magic circle contains all the consecutive integers from 488 to 503. The magic constant, 1982, can be obtained in 32 different ways. It is presented to *Crux* readers, with her best wishes for 1982, by KAMALA KUMARI (age 16), daughter of Shree Ram Rekha Tiwari, Radhaur, Bihar, India.

THE OLYMPIAD CORNER: 32

M.S. KLAMKIN

It is now definite: the 1982 International Mathematical Olympiad will take place in Hungary. The countries invited to participate have received an official invitation an extract of which is given below:

> The 23rd International Mathematic Olympiad for Students will be held in Hungary, between 5 and 14 July 1982. The Hungarian Ministry of Culture and Education has the honour to invite hereby a delegation also from your country.
>
> The delegation should consist of 4 competitors, i.e. secondary school students in the academic year 1981/82, who were born later than 1 July 1962. The delegation is supposed to be completed with two accompanying teachers, one of whose is meant to be the leader of the delegation and the other to be his deputy. Both of the teachers have to speak at least one of the following languages: English, French, German or Russian.
>
> The leader of the delegation must arrive in Budapest on 5 July, 1982, while his deputy and the four competitors are expected to arrive on 7 July, 1982, in Budapest. The delegations are supposed to leave together on 14 July. During their stay the Hungarian Ministry of Culture and Education will provide for board, lodging and pocket-money. Travel costs to and from Budapest are to be covered by your country.

In the past, a country's team could consist of as many as 8 students. This year, no doubt for economic reasons, each participating country is limited to a 4-student team. This will probably result in an improvement in the overall caliber of each team. I do not yet have a list of all the countries which have been invited to compete.

*

The 1982 Alberta High School Prize Examination in Mathematics took place on 16 February 1982. Part I (not given here) consists of 20 questions with multiple choice answers to be done in 60 minutes. Students are then allowed 110 minutes to do the 5 problems of Part II, which are given below. Solutions to these 5 problems will appear soon in this column. The problems and solutions were prepared by a joint committee of the Departments of Mathematics of the University of Alberta and the University of Calgary. The members of the committee were G. Butler, M.S. Klamkin, Andy Liu, and J. Pounder from the University of Alberta; and W. Sands and R. Woodrow from the University of Calgary. (Liu was also a member of the 1981 Canadian Mathematics Olympiad Committee. His name was unfortunately omitted from last year's report [1981: 139].)

1982 ALBERTA HIGH SCHOOL

PRIZE EXAMINATION IN MATHEMATICS

PART II

16 February 1982 - 110 minutes

1. A 9×12 rectangular piece of paper is folded so that a pair of diagonally opposite corners coincide. What is the length of the crease?

2. Let $a = \sin A$, $b = \sin B$, and $c = \sin(A+B)$. Determine $\cos(A+B)$ as a quotient of two polynomials in a,b,c with integral coefficients.

3. A cylindrical tank with diameter 4 feet and open top is partially filled with water. A cone 2 feet in diameter and 3 feet in height is suspended (vertex up) above the water so that the bottom of the cone just touches the surface of the water. The cone is then lowered at a constant rate of 10 feet per minute into the water. How long does it take until the cone is completely submerged, given that the water does not overflow?

4. John added the squares of two positive integers and found that his answer was the square of an integer. He subtracted the squares of the same two positive integers and again found that his answer was the square of a positive integer. Show that John must have made an error in his calculation.

5. Twenty-five Knights gather at the Round Table for a jolly evening. They belong to various Orders, with every two Orders having at least one common member. Members of the same Order occupy consecutive seats at the Round Table.

(a) If each Order has at most nine members, prove that there is a Knight who belongs to no Order and there is a Knight who belongs to every Order.

(b) Without the restriction on the sizes of the Orders, prove that there are two Knights such that between them they hold membership to every Order.

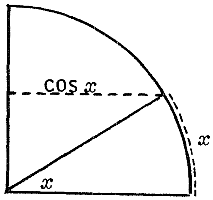
*

I now give solutions to some problems published earlier in this column. Several of the solutions are by Noam D. Elkies, who obtained a perfect score as a member of the winning U.S.A. team in the 1981 International Mathematical Olympiad.

J-8. [1980: 146; 1981: 108] Prove that $x\cos x < 0.71$ for all $x \in [0,\pi/2]$.

II. *Solution by Basil Rennie, James Cook University of North Queensland, Australia.*

The figure shows a quarter of a unit circle. It is clear that the length of



the dotted curve is $x + \cos x \le \pi/2$. Hence

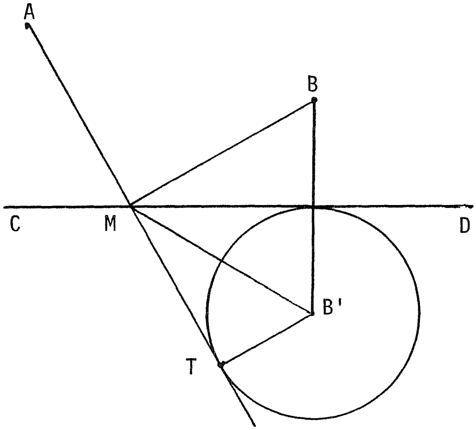
$$x \cos x = \tfrac{1}{4}\{(x+\cos x)^2 - (x-\cos x)^2\}$$
$$\le \tfrac{1}{4}(x+\cos x)^2 \le \pi^2/16 \approx 0.617$$
$$< 0.71.$$

*

J-33. [1981: 144, 299] A straight line CD and two points A and B not on the line are given. Locate the point M on this line such that $\angle$AMC = 2$\angle$BMD.

II. *Solution by J.D.E. Konhauser, Macalester College, Saint Paul, Minnesota.*

As in solution I [1981: 299], we may assume that A and B are on the same side of CD. Let B' be the reflection of B in line CD and draw the circle with center B' tangent to CD. If AT is the tangent to this circle such that A and T are on the same side of BB', as shown in the figure, then the intersection of lines AT and CD is the required point M.



The proof is obvious.

Comment by M.S.K.

The above solution jogged my memory, and I then found two simple solutions to this problem, one similar to the above, in Yaglom [1]. It would be interesting to know if there is a Euclidean construction for a point M such that $\angle$AMC = $k\angle$BMD for some values of k other than 1 and 2.

REFERENCE

1. I.M. Yaglom, *Geometric Transformations I*, Random House, New York, 1962 (now available from the M.A.A.), pp. 43, 98-99.

*

2. [1981: 236] Find the minimum value of

$$\max\{a+b+c,\ b+c+d,\ c+d+e,\ d+e+f,\ e+f+g\}$$

subject to the constraints

(a) $a,b,c,d,e,f,g \ge 0$;

(b) $a+b+c+d+e+f+g = 1$.

Solution by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.

Let M and m be the largest and smallest numbers, respectively, in the set

$$\{a+b+c,\ b+c+d,\ c+d+e,\ d+e+f,\ e+f+g\},$$

subject to the stated constraints.

We have

$$\begin{aligned} M &\geq \max\{a+b+c,\ c+d+e,\ e+f+g\} \\ &\geq \frac{(a+b+c)+(c+d+e)+(e+f+g)}{3} \\ &= \frac{c+e}{3} + \frac{a+b+c+d+e+f+g}{3} \\ &= \frac{c+e+1}{3} \\ &\geq \frac{1}{3}, \end{aligned}$$

and min $M = 1/3$ is attained for

$$(a,b,c,d,e,f,g) = (\tfrac{1}{3},0,0,\tfrac{1}{3},0,0,\tfrac{1}{3}). \quad \square$$

Also

$$\begin{aligned} m &\leq \frac{(a+b+c)+(b+c+d)+(d+e+f)+(e+f+g)}{4} \\ &= \frac{2-a-g}{4} \\ &\leq \frac{1}{2}, \end{aligned}$$

and max $m = \frac{1}{2}$ is attained for

$$(a,b,c,d,e,f,g) = (0,\tfrac{1}{4},\tfrac{1}{4},0,\tfrac{1}{4},\tfrac{1}{4},0).$$

*

5. [1981: 236] Let P and Q be polynomials over the complex field, each of degree at least 1. Let

$$P_k = \{z \in C \mid P(z) = k\}, \qquad Q_k = \{z \in C \mid Q(z) = k\},$$

and assume that $P_0 = Q_0$ and $P_1 = Q_1$. Prove that $P = Q$.

Comment by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.

This problem is equivalent to Problem 7 (of the Afternoon Session) of the 1956 William Lowell Putnam Mathematical Competition.

Comment by M.S.K.

This is one of the problems proposed (by Cuba) but unused at the 1981 I.M.O. I noted at the time that this was a former Putnam problem, and this is one of the reasons why the problem was rejected.

*

6. [1981: 236] A sequence $\{a_n\}$ is defined by means of the recursion formula

$$a_1 = 1; \quad a_{n+1} = \frac{1}{16}\left(1 + 4a_n + \sqrt{1+24a_n}\right), \qquad n = 1,2,3,\ldots .$$

Find an explicit formula for a_n.

Solution by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.

The first four terms of the sequence are 1, 5/8, 15/32, 51/128. Motivated by the fact that 1/3 is the fixed point of the mapping F defined by

$$F(x) = \frac{1 + 4x + \sqrt{1+24x}}{16},$$

we subtract 1/3 from each term to obtain $4/6$, $7/6\cdot 4$, $13/6\cdot 4^2$, $25/6\cdot 4^3$, whose nth term appears to be $(3\cdot 2^{n-1}+1)/6\cdot 4^{n-1}$. Thus we conjecture that

$$a_n = \frac{1}{3} + \frac{3\cdot 2^{n-1}+1}{6\cdot 4^{n-1}}.$$

[Elkies then goes on to establish his conjecture by a straightforward induction.]

Comment by M.S.K.

Another way of finding an explicit formula for a_n is to obtain and solve a linear recurrence equation. Solving the given equation for a_n in terms of a_{n+1}, we obtain

$$2a_n = 1 + 8a_{n+1} - \sqrt{1+24a_{n+1}}.$$

In this, we replace n by $n-1$ and add the result to the given equation to obtain the linear recurrence equation

$$16a_{n+1} = 2 + 12a_n - 2a_{n-1},$$

which can be solved in a standard way.

In a similar fashion, we can determine integer constants a,b,c,d,α such that the sequence $\{x_n\}$, where

$$x_1 = \alpha; \qquad x_{n+1} = a + bx_n + \sqrt{c+dx_n}, \qquad n = 1,2,3,\ldots,$$

contains only rational terms for all n. See also [1].

REFERENCE

1. M.S. Klamkin, "Perfect Squares of the Form $(m^2-1)a_n^2 + t$", *Mathematics Magazine*, 42 (1969) 111-113.

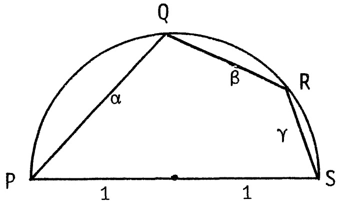
*

8. [1981: 236] On a semicircle with unit radius four consecutive chords AB, BC, CD, DE with lengths a, b, c, d, respectively, are given. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

Solution by George Tsintsifas, Thessaloniki, Greece.

We will assume that AE is a diameter of the given semicircle, a fact which is not stated unequivocally in the proposal, since the desired inequality, if true when AE is a diameter, will hold *a fortiori* if AE is not a diameter.



Let α,β,γ be the consecutive shorter sides of a convex quadrilateral PQRS inscribed in a semicircle of unit radius, as shown in the figure. Then $\gamma = 2\cos S = -2\cos Q$ and we have

$$\gamma^2 + \alpha^2 + \beta^2 - 2\alpha\beta\cos Q = \gamma^2 + \mathrm{PR}^2 = 4,$$

or

$$\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta\gamma = 4. \qquad (1)$$

If we apply (1) successively to quadrilaterals ABCE and ACDE, we get

$$a^2 + b^2 + \mathrm{CE}^2 + ab\mathrm{CE} = 4,$$
$$c^2 + d^2 + \mathrm{AC}^2 + cd\mathrm{AC} = 4,$$

and adding these two results gives

$$a^2 + b^2 + c^2 + d^2 + ab\mathrm{CE} + cd\mathrm{AC} = 4. \qquad (2)$$

Finally, since $c < \mathrm{CE}$ and $b < \mathrm{AC}$, the desired inequality

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$$

follows from (2). □

Using $c+d > \mathrm{CE}$ and $a+b > \mathrm{AC}$ in (2), we obtain the related inequality

$$a^2 + b^2 + c^2 + d^2 + bcd + cda + dab + abc > 4.$$

Comment by M.S.K.

If the diameter of the semicircle in the figure is D instead of 2, we have the well-known result (see [1] and references therein)

$$D^3 - (\alpha^2+\beta^2+\gamma^2)D - 2\alpha\beta\gamma = 0,$$

from which (1) follows when $D = 2$.

REFERENCE

1. Solutions to Problem 880, *Mathematics Magazine*, 48 (January 1975) 53.

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9, [1981: 236] Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1}, \qquad k = 0,1,\ldots,n.$$

Determine $P(n+1)$.

Solution by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.

We consider two cases.

If n is even, let $Q(x) \equiv P(x) - P(n+1-x)$, so that $Q(x)$ is a polynomial of degree at most $n-1$. Since

$$\binom{n+1}{k} = \binom{n+1}{n+1-k},$$

we have $Q(k) = 0$ for $k = 1,2,\ldots,n$. Thus $Q(x) \equiv 0$ and $Q(n+1) = P(n+1) - P(0) = 0$, from which we get

$$P(n+1) = P(0) = 1.$$

If n is odd, let $R(x) \equiv (x+1)P(x) - (n+1-x)P(n-x)$, so that $R(x)$ is a polynomial of degree at most n. Since, for $k = 0,1,\ldots,n$,

$$R(k) = \frac{(k+1)k!(n+1-k)!}{(n+1)!} - \frac{(n+1-k)(n-k)!(k+1)!}{(n+1)!} = 0,$$

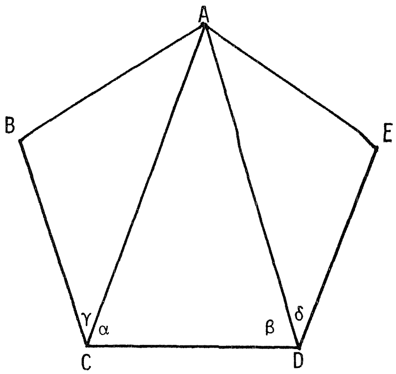
we conclude that $R(x) \equiv 0$. Now $R(-1) = 0 - (n+2)P(n+1) = 0$ gives $P(n+1) = 0$.

The required answer is thus

$$P(n+1) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

*

10, [1981: 237] Prove that a convex pentagon ABCDE with equal sides and for which the interior angles satisfy A≥B≥C≥D≥E is a regular pentagon.



Solution by Noam D. Elkies, Stuyvesant H.S., New York, N.Y.

Since the sides of the pentagon are all equal and

$$A \ge B \ge C \ge D \ge E, \qquad (1)$$

consideration of the isosceles triangles EAB, ABC, BCD, CDE, and DEA gives

$$EB \geq AC \geq BD \geq CE \geq DA.$$

Let $\alpha,\beta,\gamma,\delta$ be the angles shown in the figure and suppose $B > E$. Then we have

$$AC \geq DA \implies \alpha \leq \beta \quad \text{and} \quad B > E \implies \gamma < \delta,$$

from which

$$C = \alpha+\gamma < \beta+\delta = D,$$

a contradiction. Hence we must have $B = E$ and the last four terms in (1) are equal. Similarly, the assumption $A > D$ leads to a contradiction, and so the first four terms in (1) are equal. Thus equality holds throughout in (1), and the pentagon is regular.

*

1. [1981: 267] For which natural numbers n is $2^8 + 2^{11} + 2^n$ a perfect square?

Solution by J.T. Groenman, Arnhem, The Netherlands.

Suppose $2^8 + 2^{11} + 2^n \equiv 48^2 + 2^n = m^2$; then $2^n = (m+48)(m-48)$ and we have

$$m + 48 = 2^k$$
$$m - 48 = 2^{n-k}.$$

Subtracting these equations yields

$$3 \cdot 2^5 = 2^{n-k}(2^{2k-n}-1),$$

from which we get $2k-n = 2$, $n-k = 5$, $k = 7$, and $n = 12$. The only solution is $n = 12$, since

$$2^8 + 2^{11} + 2^{12} = 80^2.$$

*

2. [1981: 269] Given are the positive integers m and n. S_m is the sum of m terms of the series

$$(n+1) - (n+1)(n+3) + (n+1)(n+2)(n+4) - (n+1)(n+2)(n+3)(n+5) + \ldots,$$

where the terms alternate in sign and each, after the first, is the product of consecutive integers with the last but one omitted.

Prove that S_m is divisible by $m!$ but not necessarily by $m!(n+1)$.

Solution by J.T. Groenman, Arnhem, The Netherlands.

A straightforward induction shows that

$$S_m = (-1)^{m+1}(n+1)(n+2)\ldots(n+m), \qquad m = 1,2,3,\ldots .$$

Thus

$$S_m = (-1)^{m+1}\binom{n+m}{n}\cdot m!,$$

so S_m is divisible by $m!$.

The counterexample $(m,n) = (3,2)$ shows that S_m is not necessarily divisible by $m!(n+1)$. For then $S_m = 60$, $m!(n+1) = 18$, and 60 is not divisible by 18.

Editor's note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1982, although solutions received after that date will also be considered until the time when a solution is published.

711*. *Proposed by J.A. McCallum, Medicine Hat, Alberta.*

Find all the solutions of the adjoining alphametic, which I have worked on from time to time but never carried to completion:

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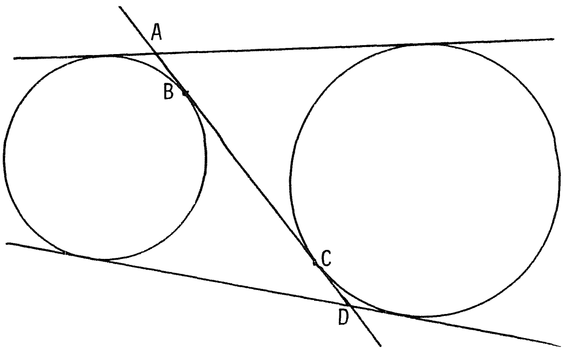
712. *Proposed by Donald Aitken, Northern Alberta Institute of Technology, Edmonton, Alberta.*

Prove that AB = CD in the figure below.



713. *Proposed jointly by Hartmut Maennel and Bernhard Leeb, West German team members, 1981 International Mathematical Olympiad.*

Consider the series

$$\sum_{i=1}^{\infty} \frac{1}{p_i}\{\prod_{j=1}^{i}(1 - \frac{1}{p_j})\}.$$

(a) Show that the series converges if $\{p_i\}$ is the sequence of primes.

(b) Does it still converge if $\{p_i\}$ is a real sequence with each $p_i \geq 1$?

714*. *Proposed by Harry D. Ruderman, Hunter College, New York, N.Y.*

Prove or disprove that for every pair (p,q) of nonnegative integers there is a positive integer n such that

$$\frac{(2n-p)!}{n!(n+q)!}$$

is an integer. (This problem was suggested by Problem 556 [1981: 282] proposed by Paul Erdös.)

715. *Proposed by V.N. Murty, Pennsylvania State University, Capitol Campus.*

Let k be a real number, n an integer, and A,B,C the angles of a triangle.

(a) Prove that

$$8k(\sin nA + \sin nB + \sin nC) \leq 12k^2 + 9.$$

(b) Determine for which k equality is possible in (a), and deduce that

$$|\sin nA + \sin nB + \sin nC| \leq 3\sqrt{3}/2.$$

716. *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

A student has been introduced to common logarithms and is wondering how their values can be calculated. He decides to obtain their binary representations (perhaps to see how a computer would do it). Help him by finding a simple algorithm to generate numbers $b_n \in \{0,1\}$ such that

$$\log_{10} x = \sum_{n=1}^{\infty} b_n \cdot 2^{-n}, \qquad 1 \leq x < 10.$$

717. *Proposed jointly by J.T. Groenman, Arnhem, The Netherlands; and D.J. Smeenk, Zaltbommel, The Netherlands.*

Let P be any point in the plane of (but not on a side of) a triangle ABC. If H_a, H_b, H_c are the orthocenters of triangles PBC, PCA, PAB, respectively, prove that $[ABC] = [H_aH_bH_c]$, where the brackets denote the area of a triangle.

718. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

ABC is an acute-angled triangle with circumcenter O. The lines AO, BO, CO intersect BC, CA, AB in A_1, B_1, C_1, respectively. Show that

$$OA_1 + OB_1 + OC_1 \geq \frac{3}{2}R,$$

where R is the circumradius.

719. *Proposed by Noam D. Elkies, student, Stuyvesant H.S., New York, N.Y.*

Are there positive integers a,b,c such that

$$(c-a-b)^3 - 27abc = 1?$$

720*. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

On the sides AB and AC of a triangle ABC as bases, similar isosceles triangles ABE and ACD are drawn outwardly. If BD = CE, prove or disprove that AB = AC.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

556. [1980: 184; 1981: 189, 241, 282] *Proposed by Paul Erdös, Mathematical Institute, Hungarian Academy of Sciences.*

Every baby knows that

$$\frac{(n+1)(n+2)\ldots(2n)}{n(n-1)\ldots2.1}$$

is an integer. Prove that for every k there is an integer n for which

$$\frac{(n+1)(n+2)\ldots(2n-k)}{n(n-1)\ldots(n-k+1)} \tag{1}$$

is an integer. Furthermore, show that if (1) is an integer, then $k = o(n)$, that is, $k/n \to 0$.

III. *Comment by Wojciech Komornicki, Hamline University, St. Paul, Minnesota.*

We prove a stronger statement: that for every positive integer k there is an integer $n > k$ such that

$$\frac{(n+1)(n+2)\ldots(2n-2k+1)}{n(n-1)\ldots(n-k+1)} \tag{1'}$$

is an integer.

Let $n(k)$ be the smallest integer $n > k$ such that (1') is an integer. Thus

$n(1) = 6$ and $n(2) = 9$. We show that, for $k \geq 3$, (1') is always an integer for $n = (k+1)! - 2$, which implies that $n(k)$ exists for every k.

Consider the chain of inequalities

$$n+3 \leq \frac{k+2}{k+1}(n-k+1) \leq \frac{i+3}{i+2}(n-i) \leq \frac{3n}{2} \leq 2n-2k+1, \tag{2}$$

where $i \in \{0,1,\ldots,k-1\}$. For $n > 0$, $f(i) \equiv (n-i)(i+3)/(i+2)$ is a monotonic decreasing function of i, and the two middle inequalities in (2) simply state the now obvious fact that

$$f(k-1) \leq f(i) \leq f(0).$$

The first inequality in (2) holds whenever $k \leq -2+\sqrt{n+3}$, the last holds whenever $k \leq (n+2)/4$, and both of these conditions are satisfied when $n = (k+1)! - 2$ and $k \geq 3$.

For $i \in \{0,1,\ldots,k-1\}$ and $n = (k+1)! - 2$, we have $n-i = (k+1)!-(i+2)$, so $i+2 | n-i$. Hence

$$\frac{i+3}{i+2}(n-i) \in \{n+3,\ n+4,\ \ldots,\ 2n-2k+1\}$$

and

$$\prod_{i=0}^{k-1} \frac{i+3}{i+2}(n-i) = \left(\prod_{i=0}^{k-1} \frac{i+3}{i+2}\right) \cdot n(n-1)(n-2)\ldots(n-k+1)$$

divides $(n+3)(n+4)\ldots(2n-2k+1)$. Now

$$\prod_{i=0}^{k-1} \frac{i+3}{i+2} = \frac{k+2}{2}, \qquad \text{so} \qquad \left(\prod_{i=0}^{k-1} \frac{i+3}{i+2}\right)(n+2) = \frac{k+2}{2}\cdot(k+1)!$$

is an integer. We conclude that

$$\frac{(n+1)(n+2)\ldots(2n-2k+1)}{n(n-1)\ldots(n-k+1)} = \frac{(n+1)\left(\prod_{i=0}^{k-1} \frac{i+3}{i+2}\right)(n+2)(n+3)\ldots(2n-2k+1)}{\left(\prod_{i=0}^{k-1} \frac{i+3}{i+2}\right)n(n-1)\ldots(n-k+1)}$$

is an integer when $k \geq 3$ and $n = (k+1)! - 2$. □

We have shown that $n(1) = 6$, $n(2) = 9$, and $n(k) \leq (k+1)! - 2$ for $k \geq 3$. For large k, we have the better upper bound

$$n(k) \leq \text{lcm}\{2,3,\ldots,k+1\} - 2, \tag{3}$$

which ensures that $i+2 | n-i$ if k is large enough. In fact, we conjecture that (3) holds for all $k \geq 4$. Certainly a necessary condition on n is that the numbers n, $n-1$, $\ldots$, $n-k+1$ all be composite.

This solution of course implies a solution of the original Erdös problem.

Since the value of $n(k)$ in our problem is never less than the value of $n(k)$ in Erdös's problem, the result $\lim k/n(k) = 0$ as $k \to \infty$, already established for the Erdös problem, also holds for our problem.

* * *

584. [1980: 283; 1981: 290; 1982: 16] *Proposed by F.G.B. Maskell, Algonquin College, Ottawa.*

If a triangle is isosceles, then its centroid, circumcentre, and the centre of an escribed circle are collinear. Prove the converse.

Editor's comment.

In solution II [1982: 16], a qualification should have been added by the solver (or the editor). The claim "$\angle HAI_a = \angle OAI_a = 0$" is meaningful and true just when angle A is acute. The claim is meaningless if angle A is a right angle and false if A is obtuse, and the proof would have to be modified in these cases.

* * *

600. [1981: 19] *Proposed by Jordi Dou, Escola Tecnica Superior Arquitectura de Barcelona, Spain.*

(Propuesta para CRUX dedicada al Prof. Léo Sauvé.)

En una urna hay 4 bolas señaladas [marked] con las letras C, R, U, X. Se extraen sucesivamente n bolas con devolución [with replacement]. Sea P_n la probabilidad de que aparezca CRUX en 4 extracciones consecutivas.

(a) Calcular el valor mínimo de n para que $P_n > 0.99$.

(b) Hallar una fórmula explícita de P_n en función de n.

Solución del proponente.

Entre las 4^n variaciones de orden n de los 4 elementos de {C,R,U,X}, sea A_n el número de las que no contienen CRUX. $A_1 = 4$, $A_2 = 16$, $A_3 = 64$, $A_4 = 255$, $A_5 = 1016$, ... y en general $A_n = 4A_{n-1} - A_{n-4}$, ya que entre las A_{n-1} variaciones de orden $n-1$ hay A_{n-4} que terminan en CRU y por tanto

$$A_n = 4(A_{n-1}-A_{n-4}) + 3A_{n-4}.$$

(a) Llamando $a_n = A_n/A_{n-1}$, se tiene

$$4 = a_2 = a_3 > a_4 > a_5 > \dots > a_n > a$$

siendo $a \neq 0$ el valor que satisface $a^n = 4a^{n-1} - a^{n-4}$. ($a \approx 3.984188231$.)

Sea $\bar{P}_n = 1 - P_n$. Se tendrá $\bar{P}_n = A_n \cdot 4^{-n}$. Sea $p_n = \bar{P}_n/\bar{P}_{n-1}$, tenemos

$$1 = p_2 = p_3 > p_4 > \dots > p_n > p = a/4 = 0.996047057.$$

$$\bar{P}_{10} = A_{10}\cdot 4^{-10} = 1020000\cdot 4^{-10} = 0.9727478027,$$

$$p_{11} = \frac{A_{11}}{4A_{10}} = \frac{4063872}{4\cdot 1020000} = 0.996047058.$$

Tendremos $\bar{P}_{10}\cdot p^{n-10} < \bar{P}_n < \bar{P}_{10}\cdot p_{11}^{n-10}$. Poniendo $\bar{P}_n = 0.01$,

$$n-10 > \frac{\log 0.01 - \log \bar{P}_{10}}{\log p} \approx 1155.7.$$

Vemos que

$$\bar{P}_{1165} > 0.972747\cdot 0.996047^{1155} \approx 0.0100277,$$

y que

$$\bar{P}_{1166} < 0.972748\cdot 0.996048^{1156} \approx 0.0099997,$$

por tanto $P_{1165} < 0.99 < P_{1166}$. Luego $n = 1166$.

(b) Sea

$$\phi(n) = 4^n - \binom{n-3}{1}\cdot 4^{n-4} + \binom{n-6}{2}\cdot 4^{n-8} - \dots = \sum_{0\le i\le n/4} (-1)^i\binom{n-3i}{i}\cdot 4^{n-4i}.$$

Para $n = 1,2,3,4,5$, se tiene $\phi(n) = A_n$. Si suponemos $A_i = \phi(i)$ para $i < n$ se tiene

$$A_n = 4A_{n-1} - A_{n-4} = 4\phi(n-1) - \phi(n-4),$$

y siendo

$$4(-1)^i\binom{n-1-3i}{i}\cdot 4^{n-1-4i} - (-1)^{i-1}\binom{n-4-3(i-1)}{i-1}\cdot 4^{n-4-4(i-1)} = (-1)^i\binom{n-3i}{i}\cdot 4^{n-4i},$$

tendremos $4\phi(n-1)-\phi(n-4) = \phi(n) = A_n$. $P_n = 1-\bar{P}_n = (4^n-A_n)\cdot 4^{-n}$, luego

$$P_n = \sum_{1\le i\le n/4} (-1)^{i+1}\binom{n-3i}{i}\cdot 4^{-4i}. \quad \square$$

En una sucesión A_i: $A_1, A_2, A_3, A_4, A_i = 4A_{i-1}-A_{i-4}$, tal que $a_2 \ge a_3 \ge a_4 \ge a$ (que es el caso del problema), claro que las a_i son decrecientes.

El resultado $n = 1166$ de (a) puede obtenerse facilmente de la expresión de P_n hallada en (b). Para el calculo de P_{1166} con error menor de 10^{-6} basta calcular los 20 prímeros términos. Claro que el método utilizado en la solución, basado en la rápida convergencia de a_n ó p_n, es más simple.

Editor's comment.

The linguistic policy of this journal is to publish in French and English only. This time, exceptionally, we decided to honour our distinguished proposer, who recently retired from the Escola Tecnica Superior Arquitectura de Barcelona after a lifetime of service to mathematics and architecture, by publishing his solution in the original Castilian.

* * *

612. [1981: 79] *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

(a) A sequence $\{x_n\}$ has the nth term

$$x_n = \sum_{j=1}^{(n-1)^2} \frac{1}{\sqrt{n^2-j}}, \qquad n = 2,3,4,\ldots .$$

Does the sequence converge? If so, to what limit?

(This problem was reported to me by students of my college as having been set in a Public Examination.)

(b) Do the same problem with the j under the square root replaced by j^2.

Editor's comment.

As several readers pointed out, both parts of this problem seem to have been transmitted incorrectly somewhere in the chain

Public Examination $\to$ students $\to$ proposer $\to$ editor $\to$ *Crux*.

In part (a), every one of the $(n-1)^2$ terms of x_n is greater than $1/n$, so, trivially,

$$x_n > \frac{(n-1)^2}{n} = n - 2 + \frac{1}{n} \to +\infty,$$

which is not much of a problem, even for a Public Examination. In part (b), x_n contains terms that are undefined (for $j = n$) and imaginary (for $n+1 \le j \le (n-1)^2$).

We give below solutions to some modified versions of the problem.

I. *Solution by Leroy F. Meyers, The Ohio State University.*

(a) We assume that

$$x_n = \sum_{j=1}^{n-1} \frac{1}{\sqrt{n^2-j}}, \qquad n = 2,3,4,\ldots .$$

Since

$$\frac{n-1}{\sqrt{n^2-1}} \le x_n \le \frac{n-1}{\sqrt{n^2-n+1}}$$

and both estimates of x_n approach 1 as $n\uparrow$, it follows that $\lim x_n = 1$.

(b) Here we assume that

$$x_n = \sum_{j=1}^{n-1} \frac{1}{\sqrt{n^2-j^2}}, \qquad n = 2,3,4,\ldots .$$

Since x_n can be rewritten as

$$\sum_{j=0}^{n-1} \frac{1}{n} \cdot \frac{1}{\sqrt{1-(j/n)^2}} - \frac{1}{n},$$

which (except for the subtracted term $1/n$, which obviously goes to 0) can be recognized as a Riemann sum, using a partition into n equal subintervals and evaluating the function at the left endpoint of each subinterval, of the improper integral

$$\int_0^1 \frac{1}{\sqrt{1-y^2}} dy = \text{Arcsin}\, 1 - \text{Arcsin}\, 0 = \frac{\pi}{2},$$

it follows that $\lim x_n = \pi/2$.

II. *Solution by Paul R. Beesack, Carleton University, Ottawa, Ontario.*

(a) We assume that

$$x_n = \frac{1}{n} \sum_{j=1}^{(n-1)^2} \frac{1}{\sqrt{n^2-j}}, \qquad n = 2,3,4,\ldots .$$

Since

$$\sqrt{n^2-(j+1)} \le \sqrt{n^2-y} \le \sqrt{n^2-j}$$

for $j \le y \le j+1$, we have

$$x_n \le \frac{1}{n} \sum_{j=1}^{(n-1)^2} \int_j^{j+1} \frac{dy}{\sqrt{n^2-y}} = \frac{1}{n} \int_1^{(n-1)^2+1} (n^2-y)^{-\frac{1}{2}} dy,$$

whence

$$x_n \le 2\left\{\sqrt{1-\frac{1}{n^2}} - \sqrt{\frac{2(n-1)}{n^2}}\right\}. \tag{1}$$

Also,

$$x_n = \frac{1}{n} \sum_{j=0}^{(n-1)^2-1} \frac{1}{\sqrt{n^2-(j+1)}} \ge \frac{1}{n} \sum_{j=0}^{(n-1)^2-1} \int_j^{j+1} \frac{dy}{\sqrt{n^2-y}},$$

which reduces to

$$x_n \ge 2\left\{1 - \sqrt{\frac{2n-1}{n^2}}\right\}. \tag{2}$$

Finally, from (1) and (2) we obtain $\lim x_n = 2$.

The upper and lower bounds in (1) and (2) are quite close even for small n, e.g. 0.677 and 0.712 for $n = 4$.

(b) This time, we assume that

$$x_n = \frac{1}{n} \sum_{j=1}^{n-1} \frac{1}{\sqrt{n^2-j^2}}, \qquad n = 2,3,4,\ldots .$$

Proceeding as in part (a), but now using

$$\sqrt{n^2-(j+1)^2} \le \sqrt{n^2-y^2} \le \sqrt{n^2-j^2}$$

for $j \le y \le j+1$, leads to the inequalities

$$\frac{1}{n}\text{Arcsin}\left(1-\frac{1}{n}\right) \le x_n \le \frac{1}{n}\left(\frac{\pi}{2} - \text{Arcsin}\frac{1}{n}\right),$$

from which we at once obtain $\lim x_n = 0$.

Also solved by CLAYTON W. DODGE, University of Maine at Orono; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; F.G.B. MASKELL, Algonquin College, Ottawa, Ontario; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and DAVID R. STONE, Georgia Southern College, Statesboro, Georgia.

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613. [1981: 79] *Proposed by Jack Garfunkel, Flushing, N.Y.*

If A + B + C = 180°, prove that

$$\cos\tfrac{1}{2}(B-C) + \cos\tfrac{1}{2}(C-A) + \cos\tfrac{1}{2}(A-B) \ge \frac{2}{\sqrt{3}}(\sin A + \sin B + \sin C).$$

(Here A,B,C are not necessarily the angles of a triangle, but you may assume that they are if it is helpful to achieve a proof without calculus.)

A solution was received from VEDULA N. MURTY, Pennsylvania State University, Capitol Campus; and a comment from HAYO AHLBURG, Benidorm, Alicante, Spain. Incomplete solutions were submitted by BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; GALI SALVATORE, Perkins, Québec; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer. In addition, two incorrect solutions were received.

Editor's comment.

Murty has written an interesting article entitled "A New Inequality for R, r, and s". At the end of the article, he derives from his "new inequality" a proof of our problem (when A,B,C are the angles of a triangle). The article will appear in the next issue of this journal.

Our problem may be restated as follows:

Prove that, with sums cyclic over A,B,C,

$$f(A,B,C) \equiv \sqrt{3}\Sigma\cos\tfrac{1}{2}(B-C) - 2\Sigma\sin A \ge 0,$$

where A+B+C = 180° (*and* A,B,C *are not necessarily the angles of a triangle*).

The qualification in parentheses was unwisely added by the editor, not by the proposer. All solvers without exception assumed (wisely) that A,B,C were the angles of a triangle. Most gave no reason for this assumption. Two who did said that since the inequality $f(A,B,C) \ge 0$ does not hold when $(A,B,C) = (-180°,180°,180°)$, therefore it can hold *only* when A,B,C are the angles of a triangle. Well, now, that is jumping to a conclusion without a parachute. Because the inequality certainly holds when $(A,B,C) = (0,0,180^\circ)$ and $(0,90^\circ,90^\circ)$. Of course, one can always weasel out by claiming that these are degenerate triangles. But the inequality also holds when $(A,B,C) = (-10^\circ,95^\circ,95^\circ)$, and no triangle is *that* degenerate. So the inequality holds for some (it would be interesting to know for which), but not all, A,B,C which are not the angles of a triangle.

The proposal had suggested that a proof without calculus would be preferable if A,B,C were assumed to be the angles of a triangle. Our five incomplete solvers all used calculus. They managed to prove that $f(A,B,C)$ attained a (relative) minimum value of 0 when $A = B = C = 60^\circ$. But, as far as the editor could judge from their not always limpid exposition, none of them proved (or even thought of proving, except for one whose proof was unconvincing) that this was an *absolute* minimum for all $A,B,C > 0$ with $A+B+C = 180^\circ$.

Our two incorrect solvers used no calculus. One in fact did not use much of anything. His "solution" consisted of wrapping up, in six lines or so, a couple of truisms into a *non sequitur*. The other one, a highly reputable mathematician, began ingeniously enough, then at a crucial point (all points are crucial in *Crux*) he used the two relations (which he said were "known")

$$\cos\frac{B}{2}\cos\frac{C}{2} \le \cos^2\frac{B+C}{2} = \tfrac{1}{2}(1 + \sin\frac{A}{2}),$$

both of which are patently false. (This is not due to ignorance or stupidity, just to carelessness, and is one reason why editors get gray.) Still, he managed to arrive at the desired conclusion! Whatever happens in midair, a cat always manages to land on its feet.

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614. [1981: 79] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a triangle with sides of lengths a,b,c. A point P moves inside the triangle in such a way that the sum of the squares of its distances to the three vertices is a constant ($= k^2$). Find the locus of P.

Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.

Let G be the centroid of the system of n points $A_1, A_2, \ldots, A_n$ in space, so that $\Sigma\vec{GA}_i = \vec{0}$. (All summations, here and later, are for $i = 1,2,\ldots,n$.) If P is *any* point in space, we have

$$\begin{aligned}\Sigma|\vec{PA}_i|^2 &= \Sigma|\vec{GA}_i - \vec{GP}|^2 = n|\vec{GP}|^2 - 2\vec{GP}\cdot\Sigma\vec{GA}_i + \Sigma|\vec{GA}_i|^2\\ &= n|\vec{GP}|^2 + \Sigma|\vec{GA}_i|^2.\end{aligned}$$

Hence, for a given constant k^2, we have $\Sigma|\vec{PA_i}|^2 = k^2$ if and only if

$$|\vec{GP}|^2 = \frac{1}{n}\{k^2 - \Sigma|\vec{GA_i}|^2\}. \tag{1}$$

Thus the locus of P is a sphere with centre at the centroid G and radius equal to the square root of the right side of (1). The sphere is real if and only if $k^2 \geq \Sigma|\vec{GA_i}|^2$.

When $n = 3$, as in our problem, G is the centroid of some triangle ABC with sides a,b,c, and (1) becomes

$$|\vec{GP}|^2 = \frac{1}{3}(k^2 - |\vec{GA}|^2 - |\vec{GB}|^2 - |\vec{GC}|^2). \tag{2}$$

It is now easy to find the radius of the sphere (or of the circle if we are restricted to the plane of the triangle) in terms of k,a,b,c. For if we substitute

$$|\vec{GA}| = \frac{2}{3}\cdot\frac{1}{2}\sqrt{2b^2+2c^2-a^2}, \quad \text{etc.}$$

in (2), we obtain

$$|\vec{GP}|^2 = \frac{1}{9}\{3k^2 - (a^2+b^2+c^2)\}.$$

The required locus is a sphere (or a circle) with centre G and radius $\frac{1}{3}\sqrt{3k^2-(a^2+b^2+c^2)}$, and the locus is real if $3k^2 \geq a^2+b^2+c^2$.

Note that there is no reason whatever for restricting P to the interior of (or even to the plane of) triangle ABC.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; O.BOTTEMA, Delft, The Netherlands; HIPPOLYTE CHARLES, Waterloo, Québec; CLAYTON W. DODGE, University of Maine at Orono; BIKASH K. GHOSH, Bombay, India; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; F.G.B. MASKELL, Algonquin College, Ottawa, Ontario (3 solutions); LEROY F. MEYERS, The Ohio State University; J.A. McCALLUM, Medicine Hat, Alberta; FRED A. MILLER, Elkins, West Virginia; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer (2 solutions).

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615. [1981: 79] *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let P be a convex n-gon with vertices $E_1, E_2, \ldots, E_n$, perimeter L and area A. Let $2\theta_i$ be the measure of the interior angle at vertex E_i and set $C = \Sigma\cot\theta_i$. Prove that

$$L^2 - 4AC \geq 0$$

and characterize the convex n-gons for which equality holds.

I. *Comment by O. Bottema, Delft, The Netherlands.*

The inequality $L^2 - 4AC \geq 0$ is proved in my booklet [1] published in 1944.

The proof makes use of the so-called "mixed area" of two convex polygons with parallel sides (a special case of Minkowski's concept for two convex curves). Equality occurs just when the given polygon has an inscribed circle.

II. *Solution by M.S. Klamkin, University of Alberta.*

The proposed inequality is an immediate consequence of the following theorem [2] of Simon Lhuilier (1750-1840), which is itself a consequence of the Brunn-Minkowski inequality:

Among all convex polygons the sides of which have given directions and the perimeters of which are all of length L, that polygon circumscribed about a circle has greatest area.

For we have $A \le A'$, where A' is the area of a polygon of perimeter L circumscribed about a circle (of radius r, say) with sides parallel to those of the given polygon P. Since

$$L = 2r\Sigma\cot\theta_i = 2rC \quad\text{and}\quad A' = r^2\Sigma\cot\theta_i = r^2C,$$

we have

$$L^2 - 4AC \ge L^2 - 4A'C = 0,$$

and equality occurs just when P has an inscribed circle.

III. *Comment by the proposer.*

This problem arose in a generalization of Crux 375 [1979: 142]. It is similar to the classical isoperimetric inequality $L^2 \ge 4\pi A$ [3] but is somewhat stronger since it can be shown that $C > \pi$. However C approaches π if vertices are added in such a way that P approaches a smooth curve.

Also solved by M.S. KLAMKIN, University of Alberta (second solution); GEORGE TSINTSIFAS, Thessaloniki, Greece (two solutions); and the proposer. Partial solutions were received from BIKASH K. GHOSH, Bombay, India; and J.T. GROENMAN, Arnhem, The Netherlands.

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2. L.A. Lyusternik, *Convex Figures and Polyhedra*, Dover, New York, 1963, pp. 118-119.

3. Nicholas D. Kazarinoff, *Geometric Inequalities*, Random House, New York, 1961 (now available from the M.A.A.), p. 63.

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616. [1981: 80] *Proposed by Alan Wayne, Holiday, Florida.*

Find all solutions (x,y), where x and y are nonconsecutive positive integers, of the equation

$$x^2 + 10^{340} + 1 = y^2 + 10^{317} + 10^{23}.$$

Solution by Leroy F. Meyers, The Ohio State University.

The given equation is equivalent to

$$y^2 - x^2 = 10^{340} - 10^{317} - 10^{23} + 1,$$

and so to

$$(y+x)(y-x) = (10^{317} - 1)(10^{23} - 1) = 3^4R_{23}R_{317},$$

where R_n is the n-digit *repunit* 111...11. Since x and y are to be nonconsecutive positive integers, $y+x > y-x > 1$, and so we seek the nontrivial factorizations of $3^4R_{23}R_{317}$. It is claimed [2,5] that R_{23} is prime; it is claimed in [2] that R_{317} is composite, but in [5] that R_{317} is prime. Since I have not verified these statements, and the numbers involved are too large to be tested on my calculator, I can say only that there are *at least* the nine solutions tabulated below. Assuming that R_{23} is prime, these constitute a complete solution set if and only if R_{317} is also prime. (In the third column of the table, the upper sign is used for y and the lower one for x.)

$y - x$	$y + x$	y, x
3	$3^3R_{23}R_{317}$	$\frac{1}{2}\cdot 3(3^2R_{23}R_{317} \pm 1)$
3^2	$3^2R_{23}R_{317}$	$\frac{1}{2}\cdot 3^2(R_{23}R_{317} \pm 1)$
3^3	$3R_{23}R_{317}$	$\frac{1}{2}\cdot 3(R_{23}R_{317} \pm 3^2)$
3^4	$R_{23}R_{317}$	$\frac{1}{2}(R_{23}R_{317} \pm 3^4)$
R_{23}	3^4R_{317}	$\frac{1}{2}(3^4R_{317} \pm R_{23})$
$3R_{23}$	3^3R_{317}	$\frac{1}{2}\cdot 3(3^2R_{317} \pm R_{23})$
3^2R_{23}	3^2R_{317}	$\frac{1}{2}\cdot 3^2(R_{317} \pm R_{23})$
3^3R_{23}	$3R_{317}$	$\frac{1}{2}\cdot 3(R_{317} \pm 3^2R_{23})$
3^4R_{23}	R_{317}	$\frac{1}{2}(R_{317} \pm 3^4R_{23})$

Also solved by HAYO AHLBURG, Benidorm, Alicante, Spain; CLAYTON W. DODGE, University of Maine at Orono; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; BOB PRIELIPP, University of Wisconsin-Oshkosh; and KENNETH M. WILKE, Topeka, Kansas. Incomplete solutions were received from N.ESWARAN, student, Indian Institute of

Technology, Kharagpur, India; BIKASH K. GHOSH, Bombay, India; J.T. GROENMAN, Arnhem, The Netherlands; J.A.H. HUNTER, Toronto, Ontario; and the proposer. A comment was received from HARRY L. NELSON, Livermore, California.

Editor's comment.

This comment contains information and references supplied by Ahlburg, Kierstead, Nelson, Prielipp, and Wilke.

All the references given below agree that R_{23} is prime. Brillhart and Selfridge reported incorrectly in [1] that R_{317} was composite. This misinformation was repeated (at least) in [2], [3], and [4]. Brillhart is said to have admitted to a programming error in his determination of the character of R_{317}. This admission can reportedly be found in *Scientific American*, possibly in Martin Gardner's column "Mathematical Games", but I do not have the exact reference. From 1978 on, as references [5]-[8] show, it is pretty well agreed that R_{317} is prime, so the proposed equation has just the nine solutions given above.

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3. Samuel Yates, "Peculiar Properties of Repunits", *Journal of Recreational Mathematics*, 2 (1969-70) 141.

4. ________, "Extension of Repunit Table", *ibid.*, 9 (1976-77) 279.

5. ________, "The Mystique of Repunits", *Mathematics Magazine*, 51 (1978) 22-28, esp. p. 23.

6. H.C. Williams, "Some Primes With Interesting Digit Patterns", *Mathematics of Computation*, October 1978, pp. 1306-1310.

7. ________ and E. Seah, "Some Primes of the Form $(a^n-1)/(a-1)$", *ibid.*, October 1979, pp. 1337-1342.

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CAVIARE TO THE GENERAL

To generalize is to be an idiot. To particularize is the alone distinction of merit. General knowledges are those knowledges that idiots possess.

WILLIAM BLAKE (1757-1827)

(From his marginalia in Reynold's *Discourses*, ca. 1808, quoted in *The Norton Anthology of English Literature*, W.W. Norton & Co., New York, 1968, Vol. 1, p. 1949.)

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