THE ACADEMY CORNER

No. 13

Bruce Shawyer

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Mathematics Competitions

There are many mathematical competitions at many different levels. There is in fact a journal devoted to this. It is called **Mathematics Competitions**, and is the Journal of the World Federation of National Mathematics Competitions. It is published by the Australian Mathematics Trust, and information on it can be obtained by contacting them at PO Box 1, Belconnen, ACT 2616, Australia.

The journal covers all sorts of mathematical competitions, and articles not only give problems and solutions, but also cover the philosophy of competition and alternative ways of stimulating talented students.

The World Federation also makes two awards, the David Hilbert International Award (which recognises contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the international level and which have been a stimulus for mathematical learning), and the Paul Erdös National Award (which recognises contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the national level and which have been a stimulus for the enrichment of mathematics learning).

THE OLYMPIAD CORNER

No. 184

R.E. Woodrow

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We begin this number with the problems of the Mock Test for the International Mathematical Olympiad team of Hong Kong. My thanks go to Richard Nowakowski, Canadian Team Leader, for collecting them at the 35^{th} IMO in Hong Kong.

INTERNATIONAL MATHEMATICAL OLYMPIAD 1994

Hong Kong Committee — Mock Test, Part I

Time: 4.5 hours

1. In $\triangle ABC$, we have $\angle C = 2\angle B$. P is a point in the interior of $\triangle ABC$ satisfying $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

2. In a table-tennis tournament of 10 contestants, any two contestants meet only once. We say that there is a winning triangle if the following situation occurs: i^{th} contestant defeated j^{th} contestant, j^{th} contestant defeated k^{th} contestant, and k^{th} contestant defeated i^{th} contestant. Let W_i and L_i be respectively the number of games won and lost by the i^{th} contestant. Suppose $L_i + W_j \geq 8$ whenever the i^{th} contestant beats the j^{th} contestant. Prove that there are exactly 40 winning triangles in this tournament.

3. Find all the non-negative integers x, y, and z satisfying that

$$7^x + 1 = 3^y + 5^z.$$

Mock Test, Part II

Time: 4.5 hours

1. Suppose that $yz + zx + xy = 1$ and x, y, and $z \geq 0$. Prove that

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq \frac{4\sqrt{3}}{9}.$$

2. A function $f(n)$, defined on the natural numbers, satisfies:

$$f(n) = n - 12 \text{ if } n > 2000, \text{ and } f(n) = f(f(n+16)) \text{ if } n \leq 2000.$$

(a) Find $f(n)$.

(b) Find all solutions to $f(n) = n$.

3. Let m and n be positive integers where m has d digits in base ten and $d \leq n$. Find the sum of all the digits (in base ten) of the product $(10^n - 1)m$.

As a second Olympiad set we give the problems of the Final Round of the 45^{th} Mathematical Olympiad written in April, 1994. My thanks go to Marcin E. Kuczma, Warszawa, Poland; and Richard Nowakowski, Canadian Team leader to the 35^{th} IMO in Hong Kong, for collecting them.

45th MATHEMATICAL OLYMPIAD IN POLAND

Problems of the Final Round — April 10–11, 1994

First Day — Time: 5 hours

1. Determine all triples of positive rational numbers (x, y, z) such that $x + y + z$, $x^{-1} + y^{-1} + z^{-1}$ and xyz are integers.

2. In the plane there are given two parallel lines k and l, and a circle disjoint from k. From a point A on k draw the two tangents to the given circle; they cut l at points B and C. Let m be the line through A and the midpoint of BC. Show that all the resultant lines m (corresponding to various points A on k) have a point in common.

3. Let $c \geq 1$ be a fixed integer. To each subset A of the set $\{1, 2, \ldots, n\}$ we assign a number $w(A)$ from the set $\{1, 2, \ldots, c\}$ in such a way that

$$w(A \cap B) = \min(w(A), w(B)) \quad \text{for} \quad A, B \subset \{1, 2, \ldots, n\}.$$

Suppose there are $a(n)$ such assignments. Compute $\lim_{n\to\infty} \sqrt[n]{a(n)}$.

Second Day — Time: 5 hours

4. We have three bowls at our disposal, of capacities m litres, n litres and $m + n$ litres, respectively; m and n are mutually coprime natural numbers. The two smaller bowls are empty, the largest bowl is filled with water. Let k be any integer with $1 \leq k \leq m + n - 1$. Show that by pouring water (from any one of those bowls into any other one, repeatedly, in an unrestricted manner) we are able to measure out exactly k litres in the third bowl.

5. Let $A_1, A_2, \ldots, A_8$ be the vertices of a parallelepiped and let O be its centre. Show that

$$4(OA_1^2 + OA_2^2 + \cdots + OA_8^2) \leq (OA_1 + OA_2 + \cdots + OA_8)^2.$$

6. Suppose that n distinct real numbers $x_1, x_2, \ldots, x_n$ $(n \geq 4)$ satisfy the conditions $x_1 + x_2 + \cdots + x_n = 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that one can choose four distinct numbers a, b, c, d from among the x_i's in such a way that

$$a + b + c + nabc \leq x_1^3 + x_2^3 + \cdots + x_n^3 \leq a + b + d + nabd.$$

We now give three solutions to problems given in the March 1996 *Corner* as the Telecom 1993 Australian Mathematical Olympiad [1996: 58].

TELECOM 1993 AUSTRALIAN MATHEMATICAL OLYMPIAD
Paper 1

Tuesday, 9th February, 1993

(Time: 4 hours)

6. In the acute-angled triangle ABC, let D, E, F be the feet of altitudes through A, B, C, respectively, and H the orthocentre. Prove that

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

Solution by Mansur Boase, student, St. Paul's School, London, England.

$$\begin{aligned}
\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} &= 3 - \left(\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF}\right) \\
&= 3 - \left(\frac{[BHC]}{[ABC]} + \frac{[CHA]}{[ABC]} + \frac{[AHB]}{[ABC]}\right) \\
&= 3 - \frac{[ABC]}{[ABC]} = 2.
\end{aligned}$$

7. Let n be a positive integer, $a_1, a_2, \ldots, a_n$ positive real numbers and $s = a_1 + a_2 + \cdots + a_n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{s - a_i} \geq \frac{n}{n-1} \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{s - a_i}{a_i} \geq n(n-1).$$

Solutions by Mansur Boase, student, St. Paul's School, London, England and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Boase.

$$\sum_{i=1}^{n} \frac{s-a_i}{a_i} = \sum_{i=1}^{n} \left(\frac{s}{a_i} - 1\right) = \sum_{i=1}^{n} \frac{s}{a_i} - n$$

$$\sum_{i=1}^{n} \frac{a_i}{s} = 1 \quad \text{and} \quad \sum_{i=1}^{n} \frac{s}{a_i} \sum_{i=1}^{n} \frac{a_i}{s} \geq \left(\sum 1\right)^2 = n^2,$$

by the Cauchy–Schwarz inequality.

Thus

$$\sum_{i=1}^{n} \frac{s}{a_i} \geq n^2.$$

Hence $\displaystyle\sum_{i=1}^{n} \frac{s-a_i}{a_i} \geq n^2 - n = n(n-1)$.

To prove the first inequality, first note that

$$\sum_{i=1}^{n} 1 \sum_{i=1}^{n} a_i^2 \geq \left(\sum_{i=1}^{n} a_i\right)^2 = s^2.$$

Hence $\displaystyle\sum_{i=1}^{n} a_i^2 \geq \frac{s^2}{n}$.

By the Cauchy–Schwarz inequality,

$$\sum_{i=1}^{n} a_i(s-a_i) \sum_{i=1}^{n} \frac{a_i}{s-a_i} \geq \left(\sum_{i=1}^{n} a_i\right)^2 = s^2.$$

Therefore

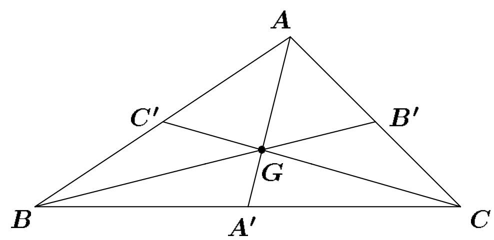
$$\begin{aligned}\sum_{i=1}^{n} \frac{a_i}{s-a_i} &\geq \frac{s^2}{\sum_{i=1}^{n} a_i(s-a_i)} = \frac{s^2}{s\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_2^2} \\ &\geq \frac{s^2}{s^2 - \frac{s^2}{n}} = \frac{1}{1-\frac{1}{n}} = \frac{n}{n-1}.\end{aligned}$$

So, both inequalities are proved.

8. [1996: 58] *Telecom 1993 Australian Mathematical Olympiad.*

The vertices of triangle ABC in the xy–plane have integer coordinates, and its sides do not contain any other points having integer coordinates. The interior of ABC contains only one point, G, that has integer coordinates. Prove that G is the centroid of ABC.

Solution by Mansur Boase, student, St. Paul's School, London, England.



By Pick's Theorem

$$\begin{aligned} [ABC] &= 1 + 3\left(\frac{1}{2}\right) - 1 = \frac{3}{2}, \\ [ABG] &= 0 + 3\left(\frac{1}{2}\right) - 1 = \frac{1}{2}, \\ [BCG] &= \frac{1}{2} \quad \text{and} \\ [CAG] &= \frac{1}{2}. \end{aligned}$$

Therefore

$$\frac{[ABG]}{[ABC]} = \frac{[BCG]}{[ABC]} = \frac{[CAG]}{[ABC]} = \frac{1}{3}.$$

Hence

$$\frac{GA'}{AA'} = \frac{GB'}{BB'} = \frac{GC'}{CC'} = \frac{1}{3}.$$

The unique point satisfying this above is well-known to be the centroid.

Next we give one solution from the Japan Mathematical Olympiad 1993 given in the March 1996 Corner.

2. [1996: 58] *Japan Mathematical Olympiad 1993.*

Let $d(n)$ be the largest odd number which divides a given number n. Suppose that $D(n)$ and $T(n)$ are defined by

$$D(n) = d(1) + d(2) + \cdots + d(n),$$

$$T(n) = 1 + 2 + \cdots + n.$$

Prove that there exist infinitely many positive numbers n such that $3D(n) = 2T(n)$.

Solutions by Mansur Boase, student, St. Paul's School, London, England, and Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Boase.

$$T(n) \;=\; \frac{n(n+1)}{2}.$$

Thus we need to prove that there are infinitely many n for which

$$D(n) = \frac{n(n+1)}{3} \qquad \text{so that} \qquad 3D(n) = 2T(n) \qquad \text{holds.}$$

Consider

$$\begin{aligned} D(2^n) &= d(1) + d(3) + \cdots + d(2^n - 1) + d(2) + d(4) + \cdots + d(2^n) \\ &= 1 + 3 + \cdots + (2^n - 1) + d(1) + d(2) + \cdots + d(2^{n-1}) \\ &= 1 + 3 + \cdots + (2^n - 1) + D(2^{n-1}). \end{aligned}$$

Now

$$\begin{aligned} 1 + 3 + \cdots + (2^n - 1) &= \frac{2^n(2^n+1)}{2} - 2\frac{2^{n-1}(2^{n-1}+1)}{2} \\ &= 2^{n-1}(2^n - 2^{n-1}) \\ &= 2^{2n-2}. \end{aligned}$$

Thus $D(2^n) = D(2^{n-1}) + 2^{2n-2}$.

Now, $D(2^1) = 2$ and we shall prove by induction that $D(2^n) = \dfrac{2^{2n}+2}{3}$ for $n \geq 0$.

This holds for $n = 0$ and for $n = 1$. Suppose it holds for $n = k$.

Thus $D(2^k) = \dfrac{2^{2k}+2}{3}$.

Then

$$\begin{aligned} D(2^{k+1}) = D(2^k) + 2^{2k} &= \frac{2^{2k}+2}{3} + 2^{2k} \\ &= \frac{4(2^{2k})+2}{3} \\ &= \frac{2^{2k+2}+2}{3} \end{aligned}$$

and the result follows by induction. Now consider $D(2^n - 2)$.

$$\begin{aligned} D(2^n - 2) &= D(2^n) - d(2^n - 1) - d(2^n) \\ &= \frac{2^{2n}+2}{3} - (2^n - 1) - 1 \\ &= \frac{2^{2n}+2}{3} - 2^n \\ &= \frac{2^{2n} - 3(2^n) + 2}{3} = \frac{(2^n - 1)(2^n - 2)}{3}. \end{aligned}$$

Thus $D(x) = \dfrac{x(x+1)}{3}$ for $x = 2^n - 2$, and there are infinitely many such x.

Next we turn to comments and solutions from the readers to problems from the April 1996 number of the Corner where we gave the selection test for the Romanian Team to the 34th IMO as well as three contests for the Romanian IMO Team [1996: 107–109].

SELECTION TESTS FOR THE ROMANIAN TEAM, 34th IMO.
Part II — First Contest for IMO Team

1st June, 1993

1. Find the greatest real number a such that

$$\frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}} > a$$

is true for all positive real numbers x, y, z.

Solutions by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We claim that $a = 2$. Let

$$f(x,y,z) \;=\; \frac{x}{\sqrt{y^2+z^2}} + \frac{y}{\sqrt{z^2+x^2}} + \frac{z}{\sqrt{x^2+y^2}}.$$

We show that $f(x,y,z) > 2$. Since $f(x,y,z) \to 2$ as $x \to y$ and $z \to 0$, the lower bound 2 is sharp. Without loss of generality, assume that $x \geq y \geq z$. Since by the arithmetic–harmonic–mean inequality, we have

$$\frac{\sqrt{z^2+x^2}}{\sqrt{y^2+z^2}} + \frac{\sqrt{y^2+z^2}}{\sqrt{z^2+x^2}} \;\geq\; 2,$$

it suffices to show that

$$f(x,y,z) \;>\; \frac{\sqrt{z^2+x^2}}{\sqrt{y^2+z^2}} + \frac{\sqrt{y^2+z^2}}{\sqrt{z^2+x^2}}$$

or equivalently,

$$\frac{z}{\sqrt{x^2+y^2}} \;>\; \frac{\sqrt{z^2+x^2}-x}{\sqrt{y^2+z^2}} + \frac{\sqrt{y^2+z^2}-y}{\sqrt{z^2+x^2}}.$$

By simple algebra, this is easily seen to be equivalent to

$$\frac{z}{\sqrt{y^2+z^2}(\sqrt{z^2+x^2}+x)} + \frac{z}{\sqrt{z^2+x^2}(\sqrt{y^2+z^2}+y)} \;<\; \frac{1}{\sqrt{x^2+y^2}}. \qquad (1)$$

Since $\sqrt{y^2+z^2} \geq \sqrt{2z^2} = \sqrt{2}\,z$, $\sqrt{z^2+x^2} > x$ and $\sqrt{2}\,x \geq \sqrt{x^2+y^2}$, we have

$$\frac{z}{\sqrt{y^2+z^2}(\sqrt{z^2+x^2}+x)} < \frac{z}{\sqrt{2}\,z(x+x)} = \frac{1}{2\sqrt{2}\,x} \leq \frac{1}{2\sqrt{x^2+y^2}}.$$

Thus to establish (1), it remains to show that

$$\frac{z}{\sqrt{z^2+x^2}(\sqrt{y^2+z^2}+y)} < \frac{1}{2\sqrt{x^2+y^2}}$$

or equivalently

$$\frac{2z}{\sqrt{y^2+z^2}+y} < \sqrt{\frac{z^2+x^2}{x^2+y^2}}.$$

Since

$$\frac{z^2+x^2}{x^2+y^2} = 1 - \frac{y^2-z^2}{x^2+y^2},$$

which is an non-decreasing function of x, we have

$$\frac{z^2+x^2}{x^2+y^2} \geq \frac{z^2+y^2}{2y^2},$$

and thus it suffices to show that

$$\frac{\sqrt{z^2+y^2}}{\sqrt{2}\,y} > \frac{2z}{\sqrt{y^2+z^2}+y},$$

or equivalently

$$y^2+z^2+y\sqrt{y^2+z^2} > 2\sqrt{2}\,yz. \qquad (2)$$

Since $y^2+z^2 \geq 2yz$, we have

$$\begin{aligned} y^2+z^2+y\sqrt{y^2+z^2} &\geq 2yz + y\sqrt{2z^2} \\ &= (2+\sqrt{2})yz > 2\sqrt{2}\,yz \end{aligned}$$

and thus (2) holds. This completes the proof.

2. Show that if x, y, z are positive integers such that $x^2+y^2+z^2 = 1993$, then $x+y+z$ is not a perfect square.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

We show that the result holds for *nonnegative* integers x, y, and z. Without loss of generality, we may assume that $0 \leq x \leq y \leq z$. Then

$$3z^2 \geq x^2+y^2+z^2 = 1993$$

implies that

$$z^2 \geq 665, \quad z \geq 26.$$

On the other hand $z^2 \leq 1993$ implies that $z \leq 44$ and thus $26 \leq z \leq 44$.

Suppose that $x + y + z = k^2$ for some nonnegative integer k. By the Cauchy–Schwarz Inequality we have

$$k^4 = (x + y + z)^2 \leq (1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) = 5979$$

and so $k \leq \lfloor \sqrt[4]{5979} \rfloor = 8$. Since $k^2 \geq z \geq 26$, $k \geq 6$. Furthermore, since $x^2 + y^2 + z^2$ is odd, it is easily seen that $x + y + z$ must be odd, which implies that k is odd. Thus $k = 7$ and we have $x + y + z = 49$.

Let $z = 26 + d$, where $0 \leq d \leq 18$. Then

$$\begin{aligned} x + y = 23 - d \Rightarrow y \leq 23 - d \Rightarrow x^2 + y^2 &\leq 2y^2 \\ &= 2(23 - d)^2 = 1058 - 92d + 2d^2. \end{aligned} \tag{1}$$

On the other hand, from $x^2 + y^2 + z^2 = 1993$ we get

$$x^2 + y^2 = 1993 - z^2 = 1993 - (26 + d)^2 = 1317 - 52d - d^2. \tag{2}$$

From (1) and (2), we get

$$1317 - 52d - d^2 \leq 1058 - 92d + 2d^2$$

or

$$3d^2 - 40d \geq 259$$

which is clearly impossible since $3d^2 - 40d = d(3d - 40) \leq 18 \times 14 = 252$. This completes the proof.

Remark: It is a well-known (though by no means easy) result in classical number theory that a natural number n is the sum of three squares (of nonnegative integers) if and only if $n \neq 4^l(8k + 7)$ where l and k are nonnegative integers. Since $1993 \equiv 1 \pmod 8$ it can be so expressed and thus the condition given in the problem is not "vacuously" true. In fact, 1993 can be so expressed in more than one way; for example,

$$\begin{aligned} 1993 &= 0^2 + 12^2 + 43^2 \\ &= 2^2 + 15^2 + \leq 42^2 \\ &= 2^2 + 30^2 + 33^2 \\ &= 11^2 + 24^2 + 36^2. \end{aligned}$$

These representations also show that the conclusion of the problem is *false* if we allow x, y, and z to be negative integers; e.g. if $x = -2$, $y = -30$, $z = 33$ then $x^2 + y^2 + z^2 = 1993$ and $x + y + z = 1^2$; and if $x = -11$, $y = 24$, $z = 36$ then $x^2 + y^2 + z^2 = 1993$ and $x + y + z = 49 = 7^2$.

4. Show that for any function $f : \mathcal{P}(\{1, 2, \ldots, n\}) \to \{1, 2, \ldots, n\}$ there exist two subsets, A and B, of the set $\{1, 2, \ldots, n\}$, such that $A \neq B$ and $f(A) = f(B) = \max\{i \mid i \in A \cap B\}$.

Comment by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The problem, as stated, is clearly incorrect since for $\max\{i : i \in A \cap B\}$ to make sense, we must have $A \cap B \neq \emptyset$. For $n = 1$ clearly there are no subsets A and B with $A \neq B$ and $A \cap B \neq \emptyset$. A counterexample when $n = 2$ is provided by setting $f(\emptyset) = f(\{1\}) = f(\{2\}) = 1$ and $f(\{1, 2\}) = 2$. This counterexample stands if $\max$ is changed to $\min$. The conclusion is still incorrect if $A \cap B$ is changed to $A \cup B$. A counterexample would be $f(\emptyset) = 2$ and $f(\{1\}) = f(\{2\}) = f\{(1, 2)\} = 1$.

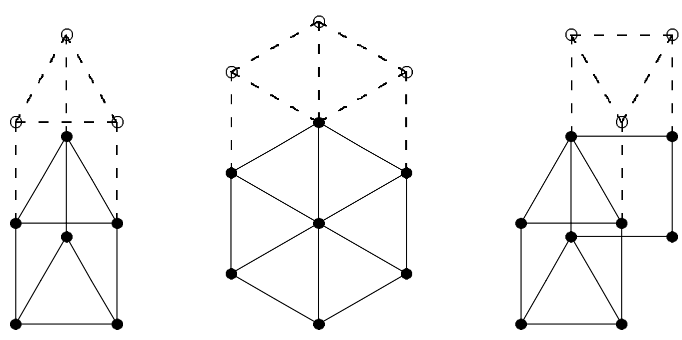
Part III — Second Contest for IMO Team

2nd June, 1993

3. Prove that for all integer numbers n, with $n \geq 6$, there exists an n–point set M in the plane such that every point P in M has at least three other points in M at unit distance to P.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The three diagrams displayed below illustrate the existence of such a set. M_1 is for all $n = 3k$, M_2 is for all $n = 3k+1$ and M_3 is for all $n = 3k+2$, where $k = 2, 3, 4, \ldots$. In each diagram, the solid lines connecting two points all have unit length and the dotted lines, also of unit length, indicate how to construct an $(n + 3)$–point set with the described property from one with n points.



M_1 $(n = 6, 9, 12, \ldots)$ M_2 $(n = 7, 10, 13, \ldots)$ M_3 $(n = 8, 11, 14, \ldots)$

Part IV — Third Contest for IMO Team

3rd June, 1993

1. The sequence of positive integers $\{x_n\}_{n\geq 1}$ is defined as follows: $x_1 = 1$, the next two terms are the even numbers 2 and 4, the next three terms are the three odd numbers 5, 7, 9, the next four terms are the even numbers 10, 12, 14, 16 and so on. Find a formula for x_n.

Solutions by Mansur Boase, student, St. Paul's School, London, England; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Shan and Wang.

We arrange the terms of the sequence $\{x_n\}$ in a triangular array according to the given rule so that the 1^{st} level consists of a single 1 and for all $k \geq 2$, the k^{th} level consists of the k consecutive even (odd) integers that follow the last odd (even) integer in the $(k-1)^{\text{st}}$ level:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & & 4 & & & \\
 & & 5 & & 7 & & 9 & & \\
 & 10 & & 12 & & 14 & & 16 & \\
17 & & 19 & & 21 & & 23 & & 25
\end{array}
$$

For any given $n \in N$, let l_n denote the unique integer such that

$$\binom{l_n}{2} < n \leq \binom{l_n+1}{2};$$

that is,

$$\frac{(l_n-1)l_n}{2} < n \leq \frac{l_n(l_n+1)}{2}. \tag{1}$$

Note that l_n is simply the number of the level which x_n is on. We claim that

$$x_n = 2n - l_n. \tag{2}$$

When $n = 1$, clearly $l_1 = 1$ and thus $2n - l_n = 1 = x_1$. Suppose (2) holds for some $n \geq 1$. There are two possible cases:

Case (i): If $n + 1 \leq \binom{l_n+1}{2}$; that is, if x_{n+1} and x_n are on the same level, then $l_{n+1} = l_n$ and hence

$$x_{n+1} = x_n + 2 = 2n - l_n + 2 = 2(n+1) - l_{n+1}.$$

Case (ii): If $n+1 > \binom{l_n+1}{2}$ then x_n is the last number on the l_n^{th} level and x_{n+1} is the first number on the l_{n+1}^{th} level. Thus

$$l_{n+1} = l_n + 1$$

and

$$x_{n+1} = x_n + 1 = 2n - l_n + 1 = 2(n+1) - l_{n+1}.$$

Hence, by induction, (2) is established. From (1) we get

$$l_n^2 - l_n < 2n \le (l_n+1)^2 - (l_n+1).$$

Solving the inequalities, we easily obtain

$$l_n < \frac{1+\sqrt{1+8n}}{2} \le l_n + 1.$$

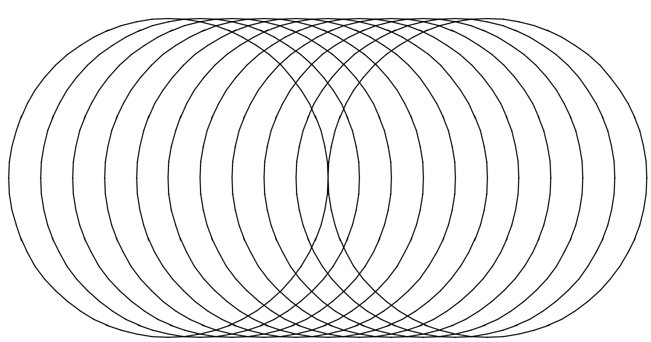
Hence

$$l_n = \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil - 1 \qquad (3)$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x (that is, the ceiling function). From (2) and (3) we conclude that

$$x_n = 2n + 1 - \left\lceil \frac{1+\sqrt{1+8n}}{2} \right\rceil .$$

That completes this number of the *Corner*. Send me your nice solutions as well as Olympiad contests.



BOOK REVIEWS

Edited by ANDY LIU

A Mathematical Mosaic, by **Ravi Vakil**,
published by Brendan Kelly Publishing Inc., 1996,
2122 Highview Drive, Burlington, ON L7R3X4,
ISBN# 1-895997-04-6, softcover, 253+ pages, US$16.95 plus handling.
Reviewed by ***Jim Totten****, University College of the Cariboo.*

So, you have a group of students who have decided they want the extra challenge of doing some mathematics competitions. You want a source of problems which will pique the students' interest, and which also lead to further exploration. The problem source should lend itself well to independent work. The question is: where do you find the appropriate level enrichment material? Many of us have already tried to answer this question and have a collection of such problem books. Well, here is a book to be added to your collection!

It is certainly a problem book, but it is much more than that. The author at one moment guides the reader through some very nice mathematical developments, and throws out problems as they crop up in the development, and in the next moment uses a problem as a starting point for some interesting mathematical development.

With a few exceptions the problems in this book are not new, nor are the solutions. They are, however, well organized, both by topic and by level of mathematical maturity needed. Answers are NOT always provided; instead there is often simply a solution strategy or hint given, and occasionally there is simply a reference to some other source for a full-blown treatment. Even when answers are provided, they are not tucked away at the end of the book, but rather they are worked into another topic (usually later in the book, but not always), where they become part of the development of another topic or problem.

The author is a PhD candidate in pure mathematics at Harvard University (at the time the book was written). Being still very young, he knows how to speak to today's teenagers. His sense of humour and general puckishness is present throughout: just when you are lulled into some serious computation in probability, he deviously throws a trick question at you, that has a totally non-obvious answer (non-obvious, that is, until you CAREFULLY re-read the question).

Many mathematics books published today include short biographies on famous mathematicians through history, especially those whose names come up in the theory developed in the book. This book is no exception. But what is unique about this book is the inclusion of Personal Profiles of young mathematicians from several countries that he has met at International Mathematical Olympiads (IMOs) in the past. The profiles are quite diverse, which means that most bright students could find one to identify with and to use as a role model. The author and those he profiles have taken a risk in doing this: they have tried to predict some of the important mathematicians in the early part of the next century. It should be interesting to follow their careers and see if those predictions can come true, or if by placing them in the spotlight, they find too much pressure to deal with.

The problems range from puzzles that elementary school children can do to problems that provide training for Putnam candidates (toward the end of the book). There are many cross-references and connections between seemingly unrelated problems from different areas of mathematics, connections that most students would be unable to make. Many of these connections are new to this reviewer. However, once made these connections are quite clear.

As for his credentials, Ravi Vakil placed among the top five in the Putnam competition in all four of his undergraduate years at the University of Toronto. Before that he won two gold medals and a silver medal in IMOs and coached the Canadian team to the IMO from 1989 to 1995.

Tests for Divisibility

Some tests for divisibility are well known, such as the test for divisibility by 3: find the sum of the digits of the whole number n — if that sum is divisible by 3, then the original whole number n is divisible by 3.

Many people know this test, but do not know why it works. So, why does it work?

The basic principle is that the remainder obtained when 10^k is divided by 3, is 1. This is easy to check, since $10^k = 3 \times 3 \times 10^{k-1} + 10^{k-1}$. This enables one to step down one power at a time. So, if d is a digit, the "remainder" obtained when $d \times 10^k$ is divided by 3 is d. (NB: this is not the true remainder one gets when dividing by 3!)

If $n = abc\ldots def$ is a whole number, then we write it as

$$n = a \times 10^k + b \times 10^{k-1} + c \times 10^{k-2} + \ldots + d \times 100 + e \times 10 + f.$$

When we divide by 3, we get a "remainder" of $a + b + c + \ldots + d + e + f$.

If this is divisible by 3, the the true remainder is 0, and so n is divisible by 3.

Now, do you know, or can you construct, a test for divisibility by 7?

Packing Boxes with N–tetracubes

Andris Cibulis
Riga, Latvia

Introduction

With the popularity of the video game *Tetris*, most people are aware of the five connected shapes formed by four unit squares joined edge to edge. They are called the $\boldsymbol{I}$–, $\boldsymbol{L}$–, $\boldsymbol{N}$–, $\boldsymbol{O}$– and $\boldsymbol{T}$–tetrominoes, after the letter of the alphabet whose shapes they resemble. They form a subclass of the polyominoes, a favourite topic in research and recreational mathematics founded by Solomon Golomb.

Here is a problem from his classic treatise, *Polyominoes*. Is it possible to tile a rectangle with copies of a particular tetromino? Figure 1 shows that the answer is affirmative for four of the tetrominoes but negative for the $\boldsymbol{N}$–tetromino, which cannot even fill up one side of a rectangle.

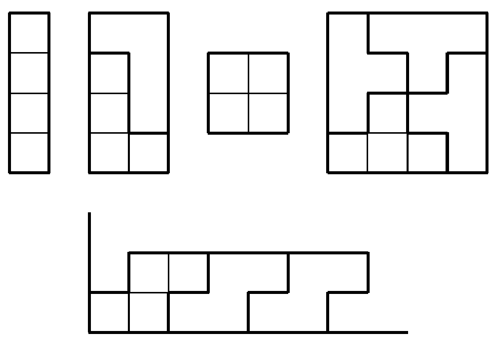


Figure 1

Getting off the plane into space, we can join unit cubes face to face to form polycubes. By adding unit thickness to the tetrominoes, we get five tetracubes, but there are three others. They are shown in Figure 2, along with the $\boldsymbol{N}$–tetracube.

Is it possible to pack a rectangular block, or box, with copies of a particular tetracube? The answer is obviously affirmative for the $\boldsymbol{I}$–, $\boldsymbol{L}$–, $\boldsymbol{O}$– and $\boldsymbol{T}$–tetracubes, and it is easy to see that two copies of each of the three tetracubes not derived from tetrominoes can pack a $2 \times 2 \times 2$ box. Will the $\boldsymbol{N}$–tetracube be left out once again? Build as many copies of it as possible and experiment with them.

If the $k \times m \times n$ box can be packed with the $\boldsymbol{N}$–tetracube, we call it an $\boldsymbol{N}$–**box**. Are there any such boxes? Certain types may be dismissed immediately.

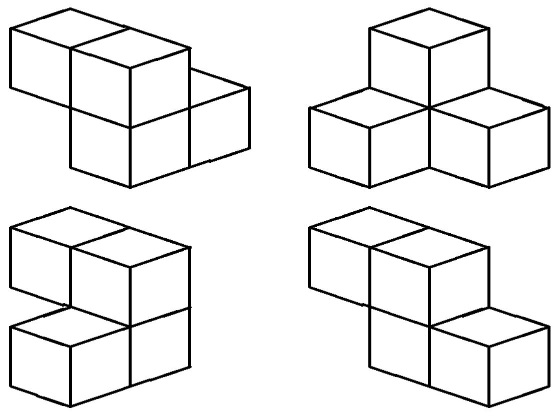


Figure 2

Observation 1.
The $k \times m \times n$ box cannot be an N–box if it satisfies at least one of the following conditions:

(a) one of k, m and n is equal to 1;
(b) two of k, m and n are equal to 2;
(c) kmn is not divisible by 4.

It follows that the $2 \times 3 \times 4$ box is the smallest box which may be an N–box. Figure 3 shows that this is in fact the case. The box is drawn in two layers, and two dominoes with identical labels form a single N–tetracube.

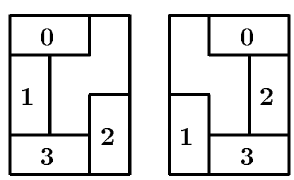


Figure 3

So there is life in this universe after all! The main problem is to find all N–boxes.

N–cubes

If $k = m = n$, the $k \times m \times n$ box is called the k–cube, and a cube which can be packed by the N–tetracube is called an N–**cube**. We can easily assemble the 12–cube with the $2 \times 3 \times 4$ N–box, which makes it an N–cube. This is a special case of the following result.

Observation 2.

Suppose the $k \times m \times n$ and $\ell \times m \times n$ boxes are N–boxes. Let a, b and c be any positive integers. Then the following are also N–boxes:

(a) $(k + \ell) \times m \times n$;
(b) $ak \times bm \times cn$.

The 12–cube is not the smallest N–cube. By Observation 1, the first candidate is $k = 4$. It turns out that this is indeed an N–cube. It can be assembled from the $2 \times 4 \times 4$ N–box, whose construction is shown in Figure 4.

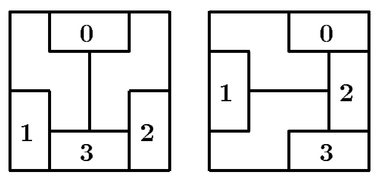


Figure 4

The next candidate, the 6–cube, is also an N–cube, but a packing is not that easy to find. In Figure 5, we begin with a packing of a $2 \times 6 \times 6$ box, with a $1 \times 2 \times 4$ box attached to it. To complete a packing of the 6–cube, add a $2 \times 3 \times 4$ N–box on top of the small box, flank it with two $2 \times 4 \times 4$ N–boxes and finally add two more $2 \times 3 \times 4$ N–boxes.

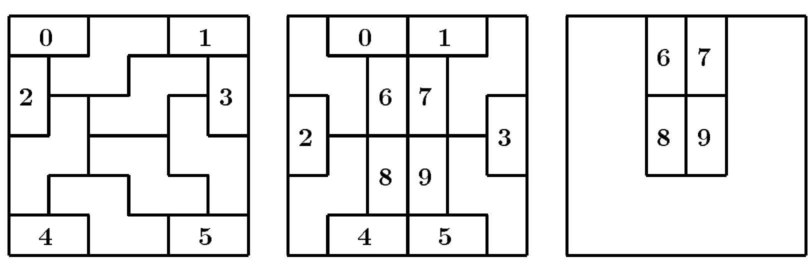


Figure 5

Can we pack the 8–cube, the 10–cube, or others? It would appear that as size increases, it is more likely that we would have an N–cube. However, it is time to stop considering one case at a time. We present a recursive construction which expands N–cubes into larger ones by adding certain N–boxes.

Theorem 1.
The k–cube is an N–cube if and only if k is an even integer greater than 2.

Proof:

That this condition is necessary follows from Observation 1. We now prove that it is sufficient by establishing the fact that if the k–cube is an N–cube, then so is the $(k+4)$–cube. We can then start from either the 4–cube or the 6–cube and assemble all others.

From the $2\times3\times4$ and $2\times4\times4$ N–boxes, we can assemble all $4\times m\times n$ boxes for all even $m,n\geq 4$, via Observation 2. By attaching appropriate N–boxes from this collection, we can enlarge the k–cube first to the $(k+4)\times k\times k$ box, then the $(k+4)\times(k+4)\times k$ box and finally the $(k+4)$–cube. This completes the proof of Theorem 1.

Further Necessary Conditions

Observation 1 contains some trivial necessary conditions for a box to be an N–box. We now prove two stronger results, one of which supersedes (c) in Observation 1.

Lemma 1.
The $k\times m\times n$ box is not an N–box if at least two of k,m and n are odd.

Proof:
We may assume that m and n are odd. Place the box so that the horizontal cross-section is an $m\times n$ rectangle. Label the layers L_1 to L_k from bottom to top. Colour the unit cubes in checkerboard fashion, so that in any two which share a common face, one is black and the other is white. We may assume that the unit cubes at the bottom corners are black. It follows that L_i has one more black unit cube than white if i is odd, and one more white unit cube than black if i is even.

Suppose to the contrary that we have a packing of the box. We will call an N–tetracube **vertical** if it intersects three layers. Note that the intersection of a layer with any N–tetracube which is not vertical consists of two or four unit cubes, with an equal number in black and white. The intersection of a vertical N–tetracube with its middle layer consists of one unit cube of each colour.

Since L_1 has a surplus of one black unit cube, it must intersect ℓ_1 vertical N–tetracubes in white and ℓ_1+1 vertical N–tetracubes in black, for some non-negative integer ℓ_1. These N–tetracubes intersect L_3 in ℓ_1 black unit cubes and ℓ_1+1 white ones. Hence the remaining part of L_3 has a surplus of two black. They can only be packed with ℓ_3 vertical N–tetracubes intersecting L_3 in white, and ℓ_3+2 vertical N–tetracubes in black, for some non-negative integer ℓ_3. However, the surplus in black unit cubes in L_5 is now three, and this surplus must continue to grow. Thus the $k\times m\times n$ box cannot be packed with the N–tetracube. This completes the proof of Lemma 1.

Lemma 2.
The $k\times m\times n$ box is not an N–box if kmn is not divisible by 8.

Proof:

Suppose a $k \times m \times n$ box is an N–box. In view of Lemma 1, we may assume that at least two of k, m and n are even. Place the packed box so that the horizontal cross-section is an $m \times n$ rectangle, and label the layers L_1 to L_k from bottom to top. Define vertical N–tetracubes as in Lemma 1 and denote by t_i the total number of those which intersects L_i, L_{i+1} and $L_{i+2}, 1 \leq i \leq k-2$.

Since at least one of m and n is even, each layer has an even number of unit cubes. It follows easily that each t_i must be even, so that the total number of vertical N–tetracubes is also even. The same conclusion can be reached if we place the box in either of the other two non-equivalent orientations. Hence the total number of N–tetracubes must be even, and kmn must be divisible by 8. This completes the proof of Lemma 2.

N–Boxes of Height 2

We now consider $2 \times m \times n$ boxes. By Observation 1, $m \geq 3$ and $n \geq 3$. By Lemma 2, mn is divisible by 4. We may assume that m is even. First let $m = 4$. We already know that the $2 \times 4 \times 3$ and $2 \times 4 \times 4$ boxes are N–boxes. Figure 6 shows that so is the $2 \times 4 \times 5$ box. If the $2 \times 4 \times n$ box is an N–box, then so is the $2 \times 4 \times (n+3)$ box by Observation 2. It follows that the $2 \times 4 \times n$ box is an N–box for all $n \geq 3$.

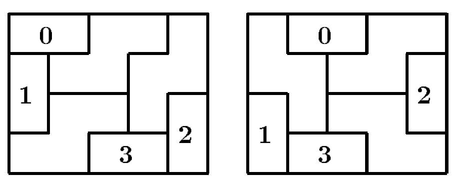


Figure 6

Now let $m = 6$. Then n is even. We already know that the $2 \times 6 \times 4$ box is an N–box. However, the $2 \times 6 \times 6$ box is not. Our proof consists of a long case-analysis, and we omit the details. On the other hand, the $2 \times 6 \times 10$ box is an N–box. In Figure 7, we begin with the packing of a $2 \times 3 \times 6$ box with a $2 \times 2 \times 3$ box attached to it. We then build the mirror image of this solid and complete the packing of the $2 \times 6 \times 10$ box by adding a $2 \times 4 \times 3$ N–box.

If the $2 \times 6 \times n$ box is an N–box, then so is the $2 \times 6 \times (n+4)$ box by Observation 2. It follows that the $2 \times 6 \times n$ box is an N–box for $n = 4$ and all even $n \geq 8$.

Theorem 2.
The $2 \times m \times n$ box is an N–box if and only if $m \geq 3, n \geq 3$ and mn is divisible by 4, except for the $2 \times 6 \times 6$ box.

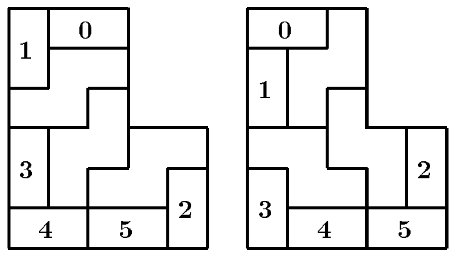


Figure 7

Proof:

If m is divisible by 4, the result follows immediately from Observation 2. Let $m = 4\ell + 2$ for some positive integer ℓ. We already know that the $2 \times 10 \times 6$ box is an N–box. If the $2 \times (4\ell + 2) \times n$ box is an N–box, then so is the $2 \times (4\ell + 6) \times n$ box by Observation 2. This completes the proof of Theorem 2.

The Main Result

Theorem 3.

The $k \times m \times n$ box, $k \leq m \leq n$, is an N–box if and only if it satisfies all of the following conditions:

(a) $k \geq 2$;
(b) $m \geq 3$;
(c) at least two of k, m and n are even;
(d) kmn is divisible by 8;
(e) $(k, m, n) \neq (2, 6, 6)$.

Proof:

Necessity has already been established, and we deal with sufficiency. We may assume that $k \geq 3$, since we have taken care of N–boxes of height 2. Consider all $3 \times m \times n$ boxes. By (c), both m and n must be even. By (d), one of them is divisible by 4. All such boxes can be assembled from the $3 \times 2 \times 4$ box.

Consider now the $k \times m \times n$ box. We may assume that m and n are even. If k is odd, then one of m and n is divisible by 4. Slice this box into one $3 \times m \times n$ box and a number of $2 \times m \times n$ boxes. Since these are all N–boxes, so is the $k \times m \times n$ box.

Suppose k is even. Slice this box into a number of $2 \times m \times n$ boxes, each of which is an N–box unless $m = n = 6$. The $4 \times 6 \times 6$ box may be assembled from the $4 \times 2 \times 3$ box, and we already know that the 6–cube is an N–cube. If the $k \times 6 \times 6$ box is an N–box, then so is the $(k + 4) \times 6 \times 6$ box. This completes the proof of Theorem 3.

Research Projects

Problem 1.
Try to prove that the $2 \times 6 \times 6$ box is not an N–box. It is unlikely that any elegant solution exists.

Problem 2.
An N–box which cannot be assembled from smaller N–boxes is called a *prime* N–box. Find all prime N–boxes.

Problem 3.
Prove or disprove that an N–box cannot be packed if we replace one of the N–tetracubes by an O–tetracube.

Problem 4.
For each of the other seven tetracubes, find all boxes which it can pack.

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Acknowledgement

This article has been published previously in a special edition of *delta-k, Mathematics for Gifted Students II*, Vol. 33, **3** 1996, a publication of the Mathematics Council of the Alberta Teachers' Association and in *AGATE*, Vol. 10, **1**, 1996, the journal of the Gifted and Talented Education Council of the Alberta Teachers' Association. It is reprinted with permission of the author and *delta-k*.

THE SKOLIAD CORNER

No. 24

R.E. Woodrow

This number we give the problems of the 1995 Concours Mathématique du Québec. This contest comes to us from the organizers via the Canadian Mathematical Society which gives partial support to the contest. My thanks go to Thérèse Ouellet, secretary to the contest. The contest was written by over 2000 students February 2, 1997 over a three hour period. This will also test your French! We will, of course, accept solutions in either English or French.

CONCOURS MATHÉMATIQUE DU QUÉBEC 1995

February 2, 1995

Time: 3 hours

1. LA FRACTION SIMPLIFIÉE

Simplifier la fraction

$$\frac{1\ 358\ 024\ 701}{1\ 851\ 851\ 865}.$$

2. LA FORMULE MYSTÈRE

Considérons les équations suivantes

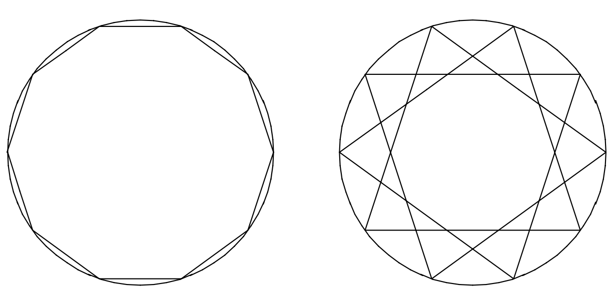
$$xy = p, \quad x + y = s, \quad x^{1993} + y^{1993} = t, \quad x^{1994} + y^{1994} = u.$$

En faisant appel aux lettres p, s, t, u mais pas aux lettres x, y, donnez une formule pour la valeur

$$x^{1995} + y^{1995}.$$

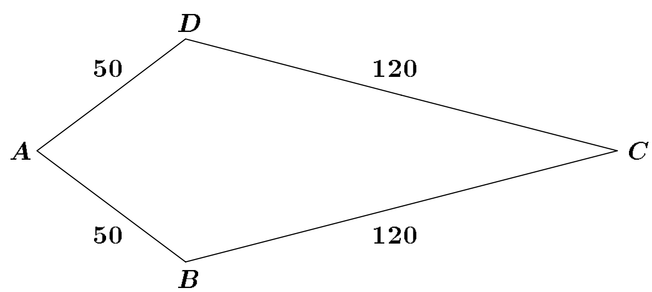
3. LA DIFFÉRENCE ÉTONNANTE

Lorsque la circonférence d'un cercle est divisée en dix parties égales, les cordes qui joignent les points de division successifs forment un décagone régulier convexe. En joignant chaque point de division au troisième suivant, on obtient un décagone régulier étoilé. Montrer que la différence entre les côtés de ces décagones est égale au rayon du cercle.



4. LE TERRAIN EN FORME DE CERF-VOLANT

Abel Belgrillet est membre du *Club des aérocervidophiles* (amateurs de cerfs-volants) *du Québec*. Il dispose de quatre tronçons de clôture rectilignes AB, BC, CD, DA pour délimiter un terrain (en forme de cerf-volant, voir figure) sur lequel il s'adonnera à son activité favorite cet été. Sachant que les tronçons AB et DA mesurent 50 m chacun et que les tronçons BC et DA mesurent 120 m chacun, déterminer la distance entre les points A et C qui maximisera l'aire du terrain.



5. L'INÉGALITÉ MODIFIÉE D'AMOTH DIEUFUTUR

(a) (2 points) L'inégalité $x^2 + 2y^2 \geq 3xy$ est-elle vraie pour tous les entiers?

(b) (8 points) Montrer que l'inégalité $x^2 + 2y^2 \geq \frac{14}{5}xy$ est valide pour tous réels x et y.

6. L'ÉCHIQUIER COQUET

Trouver l'unique façon de colorier les 36 cases d'un échiquier 6×6 en noir et blanc de sorte que chacune des cases soit voisines d'un nombre impair de cases noires.

(*Note:* deux cases sont voisines si elles se touchent par un côté ou par un coin). On ne demande pas de démontrer que la solution est unique.

7. LA FRACTION D'ANNE GRUJOTE

En base 10, $\frac{1}{3} = 0.333\ldots$. Écrivons $0.\overline{3}$ pour ces décimles répétées. Comment écrit-on $\frac{1}{3}$ dans une base b, où b est de la forme

(i) $b = 3t$ (trois points),

(ii) $b = 3t + 1$ (trois points),

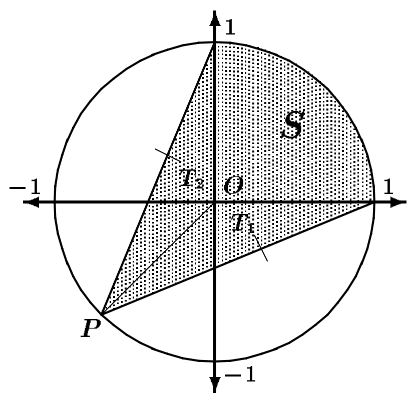
(iii) $b = 3t + 2$ (quatre points),

où t est un entier positif quelconque?

To finish this number of the Skoliad Corner we give the official solutions to the 1995 P.E.I. Mathematics Competition given last issue [1997: 278]. My thanks go to Gordon MacDonald, University of Prince Edward Island for forwarding the materials.

1995 P.E.I. Mathematics Competition

1. Find the area of the shaded region inside the circle in the following figure.



Solution. First determine the coordinates of the point P. Since P lies on the line $y = x$, $P = (a, a)$ for some value of a. Since P lies on the circle $x^2 + y^2 = 1$, $a^2 + a^2 = 1$ and so $a = \frac{-1}{\sqrt{2}}$. Hence $P = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$. Partition the shaded region as shown. Then the area of the quarter circle S is $\frac{\pi}{4}$ and the area of the triangle T_1 is $\frac{1}{2}$(base)(height). The base is 1 and the height is $\frac{1}{\sqrt{2}}$. (Turn the page upside down.) So the area of T_1 is $\frac{1}{2\sqrt{2}}$. The triangle T_2 has the same area so the total area of the shaded region is

$$\frac{\pi}{4} + \frac{1}{\sqrt{2}}.$$

2. "I will be n years old in the year n^2", said Bob in the year 1995. How old is Bob?

Solution. Assuming Bob's age is an integer, Bob must have a reasonable chance to be alive in a year that is a perfect square. Since $44^2 = 1936$, $45^2 = 2025$ and $46^2 = 2116$, Bob must be 45 years old in the year 2025. Hence Bob is now 15 years old (or 14 years old if he hasn't had his birthday yet this year).

3. Draw the set of points (x, y) in the plane which satisfy the equation $|x| + |x - y| = 4$.

Solution. Consider the following four possible cases:

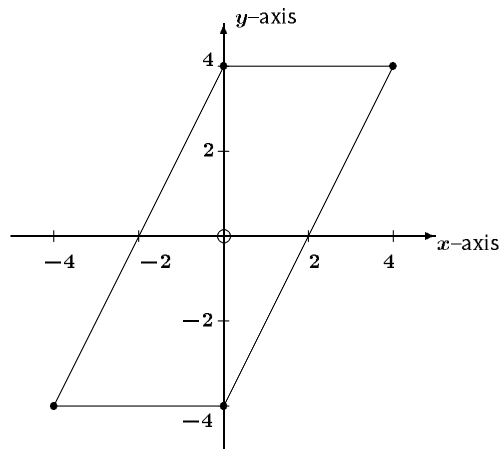
Case 1: When $x \geq 0$ and $x \geq y$. Then $x + x - y = 4$ so $2x - y = 4$.

Case 2: When $x \geq 0$ and $x < y$. Then $x - (x - y) = 4$ so $y = 4$.

Case 3: When $x < 0$ and $x \geq y$. Then $-x + x - y = 4$ so $y = -4$.

Case 4: When $x < 0$ and $x < y$. Then $-x - (x - y) = 4$ so $-2x + y = 4$.

Thus the set of points is made up of four line segments. When these line segments are drawn we obtain the following parallelogram:



4. An *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, $6,210,001,000$ is an autobiographical number. Find the smallest autobiographical number and prove that it is the smallest.

Solution. When you add the digits of an autobiographical number, you count the total number of digits. (For example the digits of the above 10 digit autobiographical number must sum to 10.) Using this fact and the process of elimination we can find the smallest autobiographical number.

The only possible one-digit number whose digits sum to 1 is 1 and it is not autobiographical so there are no one digit autobiographical numbers.

The only possible two digit numbers whose digits sum to two are (in increasing order) 11 and 20 and they are not autobiographical so there are no two digit autobiographical numbers.

The only possible three digit numbers whose digits sum to three are (in increasing order) $102, 111, 120, 201, 210$ and 300 and they are not autobiographical so there are no three digit autobiographical numbers.

The possible four digit numbers whose digits sum to four are (in increasing order) $1003, 1012, 1021, 1030, 1102, 1111, 1120, 1201, \ldots$.

Checking these numbers we find that the first autobiographical number in this list, and hence the smallest autobiographical number is 1201.

5. A solid cube of radium is floating in deep space. Each edge of the cube is exactly 1 kilometre in length. An astronaut is protected from its radiation if she remains at least 1 kilometre from the nearest speck of radium. Including the interior of the cube, what is the volume (in cubic kilometres) of space that is forbidden to the astronaut? (You may assume that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$.)

Solution. Along with the cube of radium, on each of the six faces of the cube there will be a cube with sides of length 1 km of forbidden space. Along each of the 12 edges of the cube of radium there will be a quarter of a cylinder of radius 1 km and length 1 km of forbidden space, and at each of the eight corners of the cube of radium there will be an eighth of a sphere of radius 1 km of forbidden space.

Putting it all together, there are 7 cubes (with sides of length 1 km), 3 cylinders of radius 1 km and length 1 km, and 1 sphere of radius 1 km of forbidden space. So the volume of forbidden space is:

$$7 + 3\pi + \frac{4}{3}\pi \text{ km}^3 = 7 + \frac{16}{3}\pi \text{ km}^3.$$

6. Which is greater, $999!$ or 500^{999}? (Where $999!$ denotes 999 factorial, the product of all the natural numbers from 1 to 999 inclusive.) Explain your reasoning.

Solution. Write

$$999! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 995 \cdot 996 \cdot 997 \cdot 998 \cdot 999$$

and successively pair off numbers from the left with numbers from the right so

$$999! = (1 \cdot 999)(2 \cdot 998)(3 \cdot 997)(4 \cdot 996) \cdots (449 \cdot 501)(500).$$

(Note that no number is paired with 500 and there are 499 such pairs.)

Now use the fact that $(500 - k)(500 + k) = 500^2 - k^2 < 500^2$ for each value of k from 1 to 499 so

$$999! < (500^2)(500^2)(500^2)(500^2) \cdots (500^2)(500)$$

where 500^2 is repeated 499 times. Hence

$$999! < (500^2)^{499}(500) = 500^{2(499)+1} = 500^{999},$$

so 500^{999} is greater.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

Shreds and Slices

Invariants of Inscribed Regular n-gons

There are two unexpected (in their simultaneity) invariants of inscribed regular n-gons.

Let $P_1P_2\cdots P_n$ be a regular n-gon, and let P be a point on its circumcircle. Then both

$$PP_1^2+PP_2^2+\cdots+PP_n^2 \quad\text{and}\quad PP_1^4+PP_2^4+\cdots+PP_n^4$$

are independent of the position of P.

Proof. We will show the result using complex numbers. Without loss of generality, assume the radius of the circumcircle is 1, and that P_k is at the complex number ω^{k-1} in the complex plane, where $\omega = \operatorname{cis}(\frac{2\pi}{n})$. Let z be the complex number corresponding to P, so $|z|=1$, and $z\overline{z}=|z|^2=1$.

Note

$$\begin{aligned} PP_{k+1}^2 &= |z-\omega^k|^2 = (z-\omega^k)(\overline{z}-\overline{\omega^k}) \\ &= z\overline{z}-\omega^k\overline{z}-\overline{\omega^k}z+\omega^k\overline{\omega^k} = 2-\omega^k\overline{z}-\omega^{n-k}z, \end{aligned}$$

since $\overline{\omega^k}=\frac{|\omega^k|^2}{\omega^k}=\frac{1}{\omega^k}=\frac{\omega^n}{\omega^k}=\omega^{n-k}$. Hence,

$$\begin{aligned} PP_1^2+PP_2^2+\cdots+PP_n^2 &= \sum_{k=0}^{n-1}|z-\omega^k|^2 = \sum_{k=0}^{n-1}(2-\omega^k\overline{z}-\omega^{n-k}z) \\ &= 2n-\left(\sum_{k=0}^{n-1}\omega^k\right)\overline{z}-\left(\sum_{k=0}^{n-1}\omega^{n-k}\right)z \\ &= 2n, \end{aligned}$$

since

$$\sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega} = 0.$$

Therefore, the sum is indeed independent of z. Similarly,

$$\begin{aligned} PP_{k+1}^4 &= (2-\omega^k\overline{z}-\omega^{n-k}z)^2 \\ &= 4+\omega^{2k}\overline{z^2}+\omega^{2(n-k)}z^2-4\omega^k\overline{z}-4\omega^{n-k}z+2 \\ &= 6+\omega^{2k}\overline{z^2}+\omega^{2(n-k)}z^2-4\omega^k\overline{z}-4\omega^{n-k}z, \end{aligned}$$

so

$$\begin{aligned} & PP_1^4+PP_2^4+\cdots+PP_n^4 \\ = \; & \sum_{k=0}^{n-1}(6+\omega^{2k}\overline{z^2}+\omega^{2(n-k)}z^2-4\omega^k\overline{z}-4\omega^{n-k}z) \\ = \; & 6n+\left(\sum_{k=0}^{n-1}\omega^{2k}\right)\overline{z^2}+\left(\sum_{k=0}^{n-1}\omega^{2(n-k)}\right)z^2 \\ & \quad -4\left(\sum_{k=0}^{n-1}\omega^k\right)\overline{z}-4\left(\sum_{k=0}^{n-1}\omega^{n-k}\right)z \;=\; 6n. \end{aligned}$$

Four Large Spheres and a Small Sphere

Wai Ling Yee

Problem: Given four identical spheres of radius R which are mutually tangent, find the radius r of the sphere that will fit in the hole at the centre of the tetrahedron that they form, in terms of R.

When I first tried this problem, I looked at several triangles and, after many calculations, produced an answer. I came across this problem again in chemistry when calculating the size of tetrahedral holes in ion lattices. Try this:

Construct a cube so that the centres of the spheres are at alternate corners of the cube. The length of a face diagonal of the cube is $2R$. Therefore, the length of a body diagonal of the cube is $\sqrt{6}R$. Note that we can also express the length of the body diagonal as $2(R+r)$. Hence,

$$2(R+r)=\sqrt{6}R \quad \text{implies that} \quad r=\frac{\sqrt{6}-2}{2}R.$$

Rider. This is the radius of the sphere that is externally tangent to all four. What is the radius of the sphere that is internally tangent to all four?

IMO REPORT

Adrian Chan

Two weeks of training at St. Mary's University in Halifax in early July kicked off the month-long journey the 1997 Canadian IMO team would endure. Four flights later, the team found itself in Mar del Plata, Argentina, ready to participate in the 37^{th} International Mathematical Olympiad.

This year's team members included: Adrian "Terrible Taco" Birka, Sabin "Cursed Corner" Cautis, Adrian "Flash Flood" Chan, Jimmy "Fruits and Veggies" Chui, Byung Kyu "Argentinian Polite" Chun, and Mihaela "My Dearest" Enachescu. We gave our team leader Dr. Richard "Richard" Nowakowski and our deputy leader Naoki "Raspberry" Sato all they could handle. Special thanks goes out to our coaches Dr. Bruce Shawyer and leader observer Dr. Chris "Bob's Your Uncle" Small.

This year's contest was not as difficult as last year's killer in India, but still challenging. It was the largest IMO ever, with a record 82 countries competing. Canada was well represented, bringing back 2 silver, 2 bronze, and an honourable mention. Our team's scores were as follows:

CAN1	Adrian Birka	6	
CAN2	Sabin Cautis	16	Bronze Medal
CAN3	Adrian Chan	25	Silver Medal
CAN4	Jimmy Chui	10	Honourable Mention
CAN5	Byung Kyu Chun	29	Silver Medal
CAN6	Mihaela Enachescu	21	Bronze Medal

As a team, Canada finished 29^{th} out of the 82 countries. Best of luck to the graduating members of the team, Sabin and Byung, as they take their math skills to the University of Waterloo next year. The remaining four members of the team are eligible to return to Taiwan for next year's IMO. However, it seems that the team should focus more in the area of geometry!

Special thanks must also go out to Dr. Graham Wright of the Canadian Mathematical Society for paying all our bills, and Professor Ed Barbeau of the University of Toronto for conducting a year-long correspondence program for all IMO hopefuls.

Visiting Argentina was a new experience for many of us, and was definitely a memorable one to all. The food was great, especially the beef and the chocolate alfajores, and the IMO was very well-organized and run. Best of luck to all IMO hopefuls who are working hard to be on the 1998 IMO team, which will compete in Taipei, Taiwan.

Unique Forms

Naoki Sato

For the mathematician, there is something re-assuring about being able to put an object into a unique form, or a canonical form, such as the prime factorization of a positive integer, or a matrix in Jordan Canonical Form. Such forms make working with these objects much easier and more tractable, as well as allowing a way to get a handle on them. Here, we explore the idea of the unique, canonical form.

1. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, and let c be a real number. Show that there exist unique constants b_n, b_{n-1}, ... , b_1, b_0, such that

$$p(x) = b_n(x-c)^n + b_{n-1}(x-c)^{n-1} + \cdots + b_1(x-c) + b_0.$$

   Can you find the b_i explicitly?

What may seem like a lot of (linear) algebra to untangle, comes easily undone with one idea: consider $p(x+c)$. This is a polynomial of degree at most n, so there exist unique b_i such that

$$p(x+c) \;=\; b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0,$$

which is equivalent to

$$p(x) \;=\; b_n(x-c)^n + b_{n-1}(x-c)^{n-1} + \cdots + b_1(x-c) + b_0.$$

The expression may look familiar, because it is the n^{th} approximant to a power series around $x = c$, a central object of calculus. Suppose we take the k^{th} derivative of both sides. The "constant term" is then $k!b_k$, so evaluating both sides at $x = c$ leads to

$$p^{(k)}(c) = k!b_k \quad \text{so that} \quad b_k = \frac{p^{(k)}(c)}{k!}.$$

A quick corollary: If $p(x)$ is a polynomial, then $(x-c)^k \mid p(x)$ if and only if $p(c) = p'(c) = p''(c) = \cdots = p^{(k-1)}(c) = 0$. (Both conditions are equivalent to $b_0 = b_1 = \cdots = b_{k-1} = 0$.)

2. Let N be a non-zero integer. Show that there exist unique positive integers k and $a_1 < a_2 < \cdots < a_k$ such that

$$N = (-2)^{a_k} + (-2)^{a_{k-1}} + \cdots + (-2)^{a_1}.$$

   (In other words, show that there is a unique representation of N in base -2.)

We will use strong induction. The statement is true for $N = 1 = (-2)^0$ and for $N = -1 = (-2)^1 + (-2)^0$. Now, assume there is a positive integer n such that the statement is true for $N = 1, 2, \ldots, n$ and $N = -1, -2, \ldots, -n$.

Let $N = n + 1$. If N is odd, then n is even, and $n/(-2)$ is an integer between -1 and $-n$, so by the induction hypothesis, for some unique distinct a_i,

$$n/(-2) \;=\; (-2)^{a_k} + (-2)^{a_{k-1}} + \cdots + (-2)^{a_1},$$

which is equivalent to

$$N \;=\; n + 1 = (-2)^{a_k+1} + (-2)^{a_{k-1}+1} + \cdots + (-2)^{a_1+1} + (-2)^0.$$

Note that the exponents are still distinct. The same trick works if N is even, and for $N = -(n + 1)$. Subtracting off $1 = (-2)^0$ if necessary, and then dividing by -2, we obtain a form which is simultaneously seen to exist and be unique, by the induction hypothesis. Hence, by strong induction, the statement is proved for all N. Note that this also provides an algorithm for finding the exponents.

One may be tempted to isolate the higher exponents, but this tends to get very messy. Instead, one can start with the lower exponents - after seeing if $(-2)^0$ is in the sum, they can then be picked off inductively.

3. Let r be a positive rational number. Show that there exist unique positive integers k and $a_1 \le a_2 \le \cdots \le a_k$ such that

$$r = \frac{1}{a_1} + \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} + \cdots + \frac{1}{a_1a_2a_3\cdots a_k}.$$

Note that

$$r = \frac{1}{a_1} + \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} + \cdots + \frac{1}{a_1a_2a_3\cdots a_k},$$

which is equivalent to

$$a_1r - 1 = \frac{1}{a_2} + \frac{1}{a_2a_3} + \cdots + \frac{1}{a_2a_3\cdots a_k}.$$

This suggests finding the right a_1, reducing, and applying the same argument to $a_1r - 1$.

Define the sequences $\{a_i\}$ and $\{r_i\}$ as follows: set $r_1 = r$, $a_i = \lceil 1/r_i \rceil$, $r_{i+1} = a_ir_i - 1$, $i \ge 1$, and we terminate when $r_{i+1} = 0$. For example, if $r_1 = 3/7$, then we obtain $a_1 = \lceil 7/3 \rceil = 3$, $r_2 = 3 \cdot 3/7 - 1 = 2/7$, $a_2 = 4$, $r_2 = 1/7$, $a_3 = 7$, and $r_3 = 0$. And we see

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{3\cdot 4} + \frac{1}{3\cdot 4\cdot 7}.$$

We will show that in general, this is the only possible sequence which works.

Consider the choice of a_1. Since $a_1 r \geq 1$, a_1 must be at least $\lceil 1/r \rceil$. However, if a_1 was set to $\lceil 1/r \rceil + 1$, then

$$\begin{aligned} r &= \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots + \frac{1}{a_1 a_2 a_3 \cdots a_k} \\ &< \frac{1}{a_1} + \frac{1}{a_1^2} + \frac{1}{a_1^3} + \cdots \\ &= \frac{1}{a_1 - 1} = \frac{1}{\lceil 1/r \rceil} \leq \frac{1}{1/r} \\ &= r, \end{aligned}$$

contradiction. Hence, there is only one choice for a_1, the one we have chosen above. By the above reduction, there is only one choice for a_2, a_3, and so on. Hence, we have uniqueness.

We also know terms will be generated, but how do we know the sequence always terminates? Let $r_i = p_i/q_i$, as a reduced fraction. Then $a_i = \lceil q_i/p_i \rceil$, so

$$r_{i+1} = \frac{p_i}{q_i} \left\lceil \frac{q_i}{p_i} \right\rceil - 1 = \frac{p_i \left\lceil \frac{q_i}{p_i} \right\rceil - q_i}{q_i}.$$

However,

$$p_i \left\lceil \frac{q_i}{p_i} \right\rceil - q_i < p_i \left(\frac{q_i}{p_i} + 1 \right) - q_i = p_i.$$

Hence, the numerators of the r_i are strictly decreasing. But as they are all positive integers, the sequence must terminate at some point.

Finally, we must show the sequence $\{a_i\}$ is non-decreasing. We have

$$a_{i+1} = \left\lceil \frac{q_i}{p_i \left\lceil \frac{q_i}{p_i} \right\rceil - q_i} \right\rceil \geq \frac{q_i}{p_i \left\lceil \frac{q_i}{p_i} \right\rceil - q_i} > \frac{q_i}{p_i} > \left\lceil \frac{q_i}{p_i} \right\rceil - 1 = a_i - 1.$$

Therefore, $a_{i+1} \geq a_i$.

Rider. Does an analogous result hold for positive irrational numbers? What is the sequence for $\sqrt{2}$? For e?

4. Let γ be a root of $x^3 - x - 1 = 0$, and let p, q be polynomials with rational coefficients, such that $q(\gamma) \neq 0$. Show that there exist unique rationals a, b, and c such that

$$\frac{p(\gamma)}{q(\gamma)} = a\gamma^2 + b\gamma + c.$$

First, let us express $p(\gamma)/q(\gamma)$ as a polynomial in γ. Note that x^3-x-1 is irreducible over the rationals. Hence, there exist polynomials $u(x)$ and $v(x)$ with rational coefficients such that

$$u(x)q(x) + v(x)(x^3 - x - 1) = \gcd(q(x), x^3 - x - 1) = 1.$$

Then by substituting $x = \gamma$,

$$u(\gamma)q(\gamma) = 1 \quad \text{implies that} \quad \frac{p(\gamma)}{q(\gamma)} = p(\gamma)u(\gamma) = r(\gamma),$$

where $r(x) = p(x)u(x)$, a polynomial with rational coefficients.

Now, by the division algorithm for polynomials,

$$r(x) = (x^3 - x - 1)w(x) + ax^2 + bx + c,$$

for some rationals a, b, and c (dividing $r(x)$ by $x^3 - x - 1$ leaves, at most, a quadratic remainder). Hence, $r(\gamma) = a\gamma^2 + b\gamma + c$, which shows existence.

To see uniqueness, suppose $r(\gamma) = a'\gamma^2 + b'\gamma + c'$ implies that $(a - a')\gamma^2 + (b - b')\gamma + (c - c') = 0$; that is, γ is a root of the quadratic $(a - a')x^2 + (b - b')x + (c - c') = 0$. But as indicated above, the cubic $x^3 - x - 1$ is irreducible; hence, all the coefficients of the quadratic must be zero, and $a = a'$, $b = b'$, $c = c'$, showing uniqueness.

5. Let a and b be distinct positive integers, and let $d = \gcd(a, b)$. Consider the Diophantine equation $ax + by = d$. It is well-known that all solutions are given by $(x, y) = (u + bt/d, v - at/d)$, where t is an integer and (u, v) is some solution.

   (i) Show that there exists a unique solution satisfying $|x| \leq b/2d$, $|y| \leq a/2d$.

   (ii) Show that the solution in (i) is given by the Euclidean Algorithm. For example, for $a = 4$ and $b = 7$, we have

   $$\begin{aligned} 7 &= 1 \cdot 4 + 3, \\ 4 &= 1 \cdot 3 + 1, \\ 3 &= 1 \cdot 3, \end{aligned}$$

   so gcd(4,7) = 1, and

   $$1 = 4 - 3 = 4 - (7 - 4) = 2 \cdot 4 - 7.$$

   Hence, the Euclidean Algorithm gives the solution $(x, y) = (2, -1)$, which satisfies $|x| \leq 7/2$ and $|y| \leq 4/2$,

We can assume $d = 1$, as a way of normalizing the equation, since $ax + by = d$ is equivalent to $(a/d)x + (b/d)y = 1$. It is easy to check that d may be factored back in.

(i) Uniqueness is immediate, since the solutions in x form an arithmetic progression with common difference $b/d = b$, and similarly for y, with difference $a/d = a$.

If $b = 1$, then $(x, y) = (0, 1)$ serves as a solution. If $b = 2$, then a is odd, and $(x, y) = (1, (1 - a)/2)$ serves as a solution. Assume that $b > 2$.

Because the solutions in x form an arithmetic progression with difference b, there exists a solution (x, y) with $|x| \le b/2$, or $-b/2 \le x \le b/2$. Then $y = (1 - ax)/b$, so

$$y \ge \frac{1 - a \cdot \frac{b}{2}}{b} = \frac{1}{b} - \frac{a}{2} > -\frac{a}{2},$$

and

$$y \le \frac{1 + a \cdot \frac{b}{2}}{b} = \frac{1}{b} + \frac{a}{2} < \frac{a+1}{2},$$

so $y \le a/2$ implies that $|y| \le a/2$.

(ii) We must qualify the statement somewhat — if either a or b is equal to 1, then the Euclidean Algorithm does not explicitly provide a solution. We may stipulate now that the solution is $(x, y) = (1, 0)$ or (0,1), if a or b is 1, respectively (they cannot both be 1, since they must be distinct).

We will use strong induction on $\max(a, b)$. The statement is easily verified for $\max(a, b) \le 2$. Assume the statement is true for $\max(a, b) \le n$, for some positive integer n. We will prove the statement for $\max(a, b) = n + 1$. Assume $b = n + 1$.

We seek solutions of $ax + (n + 1)y = 1$. By the division algorithm, there exist unique non-negative integers q and r such that $n + 1 = aq + r$, $0 \le r \le a - 1$. Then re-arranging, the equation becomes $ax + (aq + r)y = a(x + qy) + ry = 1$. Since a, $r \le n$, by the induction hypothesis, there exist solutions satisfying $|x + qy| \le r/2$, $|y| \le a/2$. By the triangle inequality, $|x| \le r/2 + |qy| \le r/2 + qa/2 = (r + qa)/2 = (n+1)/2$. Hence, by strong induction, the statement is proved for all values of $\max(a, b)$.

1996 Balkan Mathematical Olympiad

1. Let O and G be the circumcentre and centroid of a triangle ABC respectively. If R is the circumradius and r is the inradius of ABC, show that
$$OG \le \sqrt{R(R - 2r)}\,.$$

2. Let $p > 5$ be a prime number and $X = \{p - n^2 | n \in \mathbb{Z}^+, n^2 < p\}$. Prove that X contains two distinct elements x, y, such that $x \ne 1$ and x divides y.

3. Let $ABCDE$ be a convex pentagon. Denote by M, N, P, Q, R the midpoints of the segments AB, BC, CD, DE, EA respectively. If the segments AP, BQ, CR, DN have a common point of intersection, prove that this point also belongs to the segment EN.

4. Show that there exists a subset A of the set $\{1, 2, 3, \ldots, 2^{1996} - 1\}$ having the following properties:

   (a) $1 \in A$ and $2^{1996} - 1 \in A$,

   (b) Every element of $A \backslash \{1\}$ is the sum of two (not necessarily distinct) elements of A, and

   (c) The number of elements of A is at most 2012.

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
Ravi Vakil	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 November 1997, for publication in the issue 5 months ahead; that is, issue 4 of 1998. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems — Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H211. *Proposed by Richard Hoshino, grade OAC, University of Toronto Schools*

Let p, q, and r be the roots of the cubic equation $ax^3 - bx^2 + cx + d = 0$, where a, b, c, and d are real coefficients.

Given that $\mathrm{Arctan}\, p + \mathrm{Arctan}\, q + \mathrm{Arctan}\, r = \frac{\pi}{4}$, where Arctan denotes the principal value, prove that $a = b + c + d$.

Solution by Miguel Carrión Álvarez, Universidad Compluteuse de Madrid, Spain.

Let θ = Arctan p, ϕ = Arctan q, and ψ = Arctan r. Then we have $\theta + \phi + \psi = \frac{\pi}{4}$ and

$$\begin{aligned} 1 &= \tan(\theta+\phi+\psi) = \frac{\tan(\theta+\phi)+\tan\psi}{1-\tan(\theta+\phi)\tan\psi} = \frac{\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}+\tan\psi}{1-\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}\tan\psi} \\ &= \frac{\tan\theta+\tan\phi+\tan\psi-\tan\theta\tan\phi\tan\psi}{1-\tan\theta\tan\phi-\tan\theta\tan\psi-\tan\theta\tan\psi} = \frac{p+q+r-pqr}{1-pq-pr-qr}\,. \end{aligned}$$

Finally, $a(x-p)(x-q)(x-r) = a[x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr] = ax^3 - bx^2 + cx + d$. From the relationship between the coefficients and the roots of this cubic, we get $p+q+r = \frac{b}{a}$, $pq+pr+qr = \frac{c}{a}$, and $pqr = -\frac{d}{a}$. Using these relations in the above equation, we have

$$1 = \frac{\frac{b}{a}+\frac{d}{a}}{1-\frac{c}{a}} \;\Rightarrow\; 1 - \frac{c}{a} = \frac{b}{a} + \frac{d}{a} \;\Rightarrow\; a = b + c + d\,.$$

Also solved by BOB PRIELIPP, University of Wisconsin-Oshkosh, WI, USA. Prielipp also points out that this problem is similar to problem E2299 (pp. 520-521 of the May 1972 issue of The American Mathematical Monthly.)

H212. Given two sequences of real numbers of length n, with the property that for any pair of integers (i, j) with $1 \le i < j \le n$, the i^{th} term is equal to the j^{th} term in at least one of the two sequences. Prove that at least one of the sequences has all its values the same.

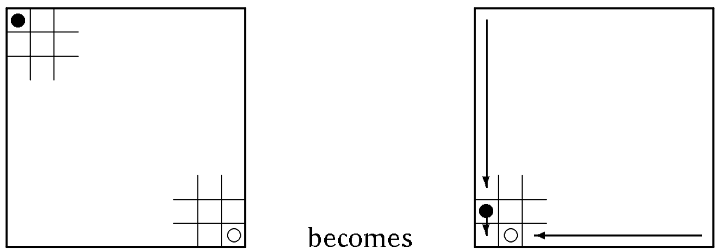
Solution by Miguel Carrión Álvarez, Universidad Compluteuse de Madrid, Spain.

Let the two sequences be $\{a_i\}$ and $\{b_i\}$. Assume that $\{a_i\}$ does not have all its values the same. Then, there is a pair (i, j) such that $a_i \neq a_j$, and hence $b_i = b_j$. Now, it is impossible that both $a_k = a_i$ and $a_k = a_j$. Hence, either $a_k \neq a_i$ or $a_k \neq a_j$. In either case, $b_k = b_i = b_j$. This is true for all k, so the sequence of b_i's has all its values the same.

H213. Two players begin with one counter each, initially on opposite corners of an $n \times n$ chessboard. They take turns moving their counter to an adjacent square. A player wins by being the first to reach the row opposite from their initial starting row, or landing on the opponent's counter. Who has the winning strategy? (Generalize to an $m \times n$ board.)

Solution by Miguel Carrión Álvarez, Universidad Compluteuse de Madrid, Spain.

On a square board, the first player always wins by crossing the board to the opposite row. In $n - 2$ moves,



and the first player wins.

If there are more columns than rows, the same strategy works for the first player.

If there are more rows than columns, one of the players wins by "capturing" the opponent. If the parity of both dimensions of the board is the same, both players start from squares of the same colour. In this case, the first player always moves to the opposite colour and cannot capture the second player. The second player wins by "closing in" on the first player and capturing the first player. [Ed: Why is this always possible?] Now if the parity of both dimensions of the board is opposite, the first player moves towards the colour of the second player, and so can capture the second player using the same strategy.

Advanced Problems — Solutions

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A190. Find all positive integers $n > 1$ such that $ab \equiv -1 \pmod{n}$ implies that $a + b \equiv 0 \pmod{n}$.

Solution by Wai Ling Yee, University of Waterloo, Waterloo, Ontario.

Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ where the p_i are distinct primes. By the Chinese Remainder Theorem, the given property holds for n if and only if it holds for each $p_i^{\alpha_i}$; that is, $ab \equiv -1 \pmod{p_i^{\alpha_i}}$, implies that $a + b \equiv 0 \pmod{p_i^{\alpha_i}}$. Also, both a and b must be relatively prime with n.

Let S_i be the set of positive integers both relatively prime with and less than $p_i^{\alpha_i}$. Take x, u, and $v \in S_i$. Note that $ux \equiv vx \bmod p_i^{\alpha_i}$ is equivalent to $x(u-v) \equiv 0 \pmod{p_i^{\alpha_i}}$, which, in turn, is equivalent to $u = v$, since x is relatively prime with $p_i^{\alpha_i}$ and u and v are less than $p_i^{\alpha_i}$. Hence, each element of $\{x, 2x, \ldots, (p_i-1)x, (p_i+1)x, \ldots, (p_i^{\alpha_i}-1)x\}$ leaves a unique residue modulo $p_i^{\alpha_i}$. Also note that the product of two numbers in S_i is itself in S_i. Therefore, the set $\{x, 2x, \ldots, (p_i-1)x, (p_i+1)x, \ldots, (p_i^{\alpha_i}-1)x\}$ taken modulo $p_i^{\alpha_i}$ is a permutation of S_i. Since $-1 \in S_i$, for each $x \in S_i$, there exists a unique $y \in S_i$ such that $xy \equiv -1 \pmod{p_i^{\alpha_i}}$.

Case I: p_i is odd.

Let $a = 2$. Then there exists a b such that $2b \equiv -1 \pmod{p_i^{\alpha_i}}$. Thus, $2 + b \equiv 0 \pmod{p_i^{\alpha_i}}$, so that $4 \equiv 1 \pmod{p_i^{\alpha_i}}$. Therefore $3 \equiv 0 \pmod{p_i^{\alpha_i}}$, from which it follows that $p_i = 3$, $\alpha_i = 1$. Therefore, the only possible odd prime factor of n is 3, and only one such factor may appear. By a simple check, $n = 3$ satisfies the given property.

Case II: $p_i = 2$.

Let $a = 3$. Then there exists a b such that $3b \equiv -1 \pmod{2^{\alpha_i}}$. Thus $3 + b \equiv 0 \pmod{2^{\alpha_i}}$, so that $9 \equiv 1 \pmod{2^{\alpha_i}}$. Therefore $2^{\alpha_i} \mid 8$, from which it follows that $\alpha_i \leq 3$. Thus n can have 0, 1, 2, or 3 factors of 2, and again by a simple check, all satisfy the given property.

From the above two cases, the solutions for n are 2, 3, 4, 6, 8, 12, and 24.

This problem, which generalizes problem B-1 on the 1969 Putnam Competition, is equivalent to finding all $n > 1$, such that a relatively prime to n implies that $a^2 \equiv 1 \pmod{n}$.

A191. Taken over all *ordered* partitions of n, show that

$$\sum_{k_1+k_2+\cdots+k_m=n} k_1k_2\cdots k_m = \binom{m+n-1}{2m-1}.$$

Solution.

The problem is particularly suitable for generating functions. As is convention, let $[x^n]f(x)$ denote the coefficient of x^n in $f(x)$. Consider each k_i being contributed from a factor of $x + 2x^2 + 3x^3 + \cdots$. Then the number we seek is:

$$\begin{aligned}
& [x^n]\underbrace{(x+2x^2+\cdots)(x+2x^2+\cdots)\cdots(x+2x^2+\cdots)}_{m \text{ factors}} \\
= \;& [x^n]\frac{x^m}{(1-x)^{2m}} = [x^{n-m}]\frac{1}{(1-x)^{2m}} \\
= \;& [x^{n-m}]\left(\binom{2m-1}{2m-1} + \binom{2m}{2m-1}x + \binom{2m+1}{2m-1}x^2 + \cdots\right) \\
= \;& \binom{m+n-1}{2m-1}.
\end{aligned}$$

A192. Let ABC be a triangle, such that the Fermat point F lies in the interior. Let $u = AF$, $v = BF$, $w = CF$. Derive the expression

$$u^2 + v^2 + w^2 - uv - uw - vw = \frac{a^2 + b^2 + c^2}{2} - 2\sqrt{3}K.$$

Solution by Wai Ling Yee, University of Waterloo, Waterloo, Ontario.

Using the law of cosines on $\triangle BCF$, $\triangle ACF$, and $\triangle ABF$:

$$\begin{aligned} a^2 &= v^2 + w^2 - 2vw\cos 120^\circ = u^2 + v^2 + vw, & (1)\\ b^2 &= u^2 + w^2 - 2uw\cos 120^\circ = u^2 + v^2 + uw, & (2)\\ c^2 &= u^2 + v^2 - 2uv\cos 120^\circ = u^2 + v^2 + uv. & (3)\end{aligned}$$

Adding the areas of $\triangle BCF$, $\triangle ACF$, and $\triangle ABF$:

$$\begin{aligned} K &= \frac{1}{2}uv\sin 120^\circ + \frac{1}{2}uw\sin 120^\circ + \frac{1}{2}vw\sin 120^\circ \\ &= \frac{\sqrt{3}}{4}(uv + vw + uw). & (4)\end{aligned}$$

Therefore, $u^2 + v^2 + w^2 - uv - uw - vw = \frac{a^2+b^2+c^2}{2} - 2\sqrt{3}K$, from $\frac{1}{2}[(1) + (2) + (3)] - 2\sqrt{3}(4)$.

Since $u^2 + v^2 + w^2 - uv - uw - vw \geq 0$ for all real u, v, w, it follows from this problem that $a^2 + b^2 + c^2 \geq 4\sqrt{3}K$.

Challenge Board Problems — Solutions

Editor: Ravi Vakil, Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544–1000 USA <vakil@math.princeton.edu>

C70. Prove that the group of automorphisms of the dodecahedron is S_5, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is A_5.

Solution.

Assign a number to each of the dodecahedron's edges as shown in Figure 1. (Alternatively, think of it as painting each of the edges one of five different colours.) Notice that if an edge on a face F is numbered n, then the edge from the opposite vertex in F (that isn't an edge of F) has the same number. Thus the entire numbering can be recovered from the numbering of the edges of a simple face. Notice also that the edges around each face are numbered 1 through 5 (in some order).

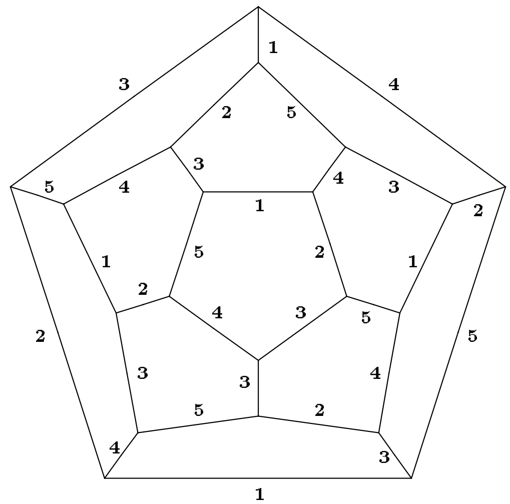


Figure 1.

By observation, if $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix}$ is an element of A_5, then there is a unique face where the edges are numbered (clockwise) a-b-c-d-e. In this way each element $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix}$ of A_5 induces a rotation of the dodecahedron (by sending the face 1-2-3-4-5 to the face a-b-c-d-e, with edge 1 of the first face sent to edge a of the second). Conversely, each rotation induces an element of A_5. This set map is clearly a group homomorphism, so the group of rotations of the dodecahedron is A_5.

Essentially the same argument shows that the group of automorphisms of the dodecahedron (including reflections) is S_5.

Comments.

1. What are the rotation and automorphism groups of the icosahedron? How about the other platonic solids?

2. The group A_5 comes up often in higher mathematics. For example, because of a simple property of A_5, there are quintic polynomials with integer co-efficients whose roots can't be written in terms of radicals. (This is not true of polynomials of degree less than 5. The proof involves Galois theory.) Surprisingly, the description of A_5 as the automorphism group of the dodecahedron comes up in a variety of contexts.

C72. A finite group G acts on a finite set X transitively. (In other words, for any $x, y \in X$, there is a $g \in G$ with $g \cdot x = y$.) Prove that there is an element of G whose action on X has no fixed points.

(The problem as stated is incorrect. We need to assume that X has more than one element.)

Solution.

Fix any element x_0 of X. The number of elements of G sending x_0 to y is independent of $y \in X$: if y_1 and y_2 are any two elements of X, and

$$S_i = \{g \in G \mid g \cdot x_0 = y_i\},\ i = 1, 2,$$

and z is an element of G sending y_1 to y_2 (z exists by the transitivity of the group action), then $zS_1 = S_2$. In other words, the elements of S_2 are obtained from the elements of S_1 by multiplication on the left by z. In particular, $|S_1| = |S_2|$.

Let $f(g, x) = 1$ if $g \cdot x = x$ and 0 otherwise. Then by the previous comment, for any fixed $x \in X$,

$$\frac{\sum_{g\in G} f(g, x)}{|G|} = \frac{1}{|X|}.$$

The average number of fixed points (over all elements of the group) is 1:

$$\begin{aligned}
\frac{1}{|G|} \sum_{g\in G} \left(\sum_{x\in X} f(g, x) \right) &= \sum_{x\in X} \left(\frac{\sum_{g\in G} f(g, x)}{|G|} \right) \\
&= \sum_{x\in X} \frac{1}{|X|} \\
&= 1.
\end{aligned}$$

One element of the group (the identity) has more than the average (as $|X| > 1$), so there is an element with less than the average number (and hence zero) fixed points.

Comments.

1. How does this relate to problem 1 on the 1987 IMO?:

   Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

   $$\sum_{k=0}^{n} k p_n(k) = n!.$$

   (Hint: Let X be the set $\{1, \ldots, n\}$ and G the group of permutations on X, acting on X in the natural way.)

2. Can the result be salvaged if the group action is not transitive?

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 April 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

Solutions submitted by FAX

There has been an increase in the number of solutions sent in by FAX, either to the Editor-in-Chief's departmental FAX machine in St. John's, Newfoundland, or to the Canadian Mathematical Society's FAX machine in Ottawa, Ontario. While we understand the reasons for solvers wishing to use this method, we have found many problems with it. The major one is that hand-written material is frequently transmitted very badly, and at times is almost impossible to read clearly. We have therefore adopted the policy that we will no longer accept submissions sent by FAX. We will, however, continue to accept submissions sent by email or regular mail. We do encourage email. Thank you for your cooperation.

2263. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle, and the internal bisectors of $\angle B$, $\angle C$, meet AC, AB at D, E, respectively. Suppose that $\angle BDE = 30^\circ$.

Characterize $\triangle ABC$.

2264. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a right angled triangle with the right angle at A. Points D and E are on sides AB and AC respectively, such that $DE \| BC$. Points F and G are the feet of the perpendiculars from D and E to BC respectively.

Let I, I_1, I_2, I_3 be the incentres of $\triangle ABC$, $\triangle ADE$, $\triangle BDF$, $\triangle CEG$ respectively. Let P be the point such that $I_2P \| I_1I_3$, and $I_3P \| I_1I_2$.

Prove that the segment IP is bisected by the line BC.

2265. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Given triangle ABC, let ABX and ACY be two variable triangles constructed outwardly on sides AB and AC of $\triangle ABC$, such that the angles $\angle XAB$ and $\angle YAC$ are fixed and $\angle XBA + \angle YCA = 180^\circ$. Prove that all the lines XY pass through a common point.

2266. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

$BCLK$ is the square constructed outwardly on side BC of an acute triangle ABC. Let CD be the altitude of $\triangle ABC$ (with D on AB), and let H be the orthocentre of $\triangle ABC$. If the lines AK and CD meet at P, show that

$$\frac{HP}{PD} = \frac{AB}{CD}.$$

2267. *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA and Peter Yff, Ball State University, Muncie, IN, USA.*

In the plane of $\triangle ABC$, let F be the Fermat point and F' its isogonal conjugate.

Prove that the circles through F' centred at A, B and C meet pairwise in the vertices of an equilateral triangle having centre F.

2268. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let x, y be real. Find all solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2}.$$

2269. *Proposed by Cristóbal Sánchez–Rubio, I.B. Penyagolosa, Castellón, Spain.*

Let $OABC$ be a given parallelogram with $\angle AOB = \alpha \in (0, \pi/2]$.

A. Prove that there is a square inscribable in $OABC$ if and only if

$$\sin\alpha - \cos\alpha \leq \frac{OA}{OB} \leq \sin\alpha + \cos\alpha$$

and

$$\sin\alpha - \cos\alpha \leq \frac{OB}{OA} \leq \sin\alpha + \cos\alpha .$$

B. Let the area of the inscribed square be S_s and the area of the given parallelogram be S_p. Prove that

$$2S_s = \tan^2\alpha \left(OA^2 + OB^2 - 2S_p\right) .$$

2270. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given $\triangle ABC$ with sides a, b, c, a circle, centre P and radius ρ intersects sides BC, CA, AB in A_1 and A_2, B_1 and B_2, C_1 and C_2 respectively, so that

$$\frac{\overline{A_1A_2}}{a} = \frac{\overline{B_1B_2}}{b} = \frac{\overline{C_1C_2}}{c} = \lambda \geq 0.$$

Determine the locus of P.

2271. *Proposed by F.R. Baudert, Waterkloof Ridge, South Africa.*

A municipality charges householders per month for electricity used according to the following scale:

first 400 units — 4.5¢ per unit;

next 1100 units — 6.1¢ per unit;

thereafter — 5.9¢ per unit.

If E is the total amount owing (in dollars) for n units of electricity used, find a closed form expression, $E(n)$.

2272★. *Proposed by no name on proposal – please identify!*

Write $r \precsim s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n ($n \geq 2$), the least positive integer k satisfying

$$\frac{k}{n} \precsim \frac{k}{n-1} \precsim \frac{k}{n-2} \precsim \cdots \precsim \frac{k}{2} \precsim k.$$

[The proposer has not seen a proof, and has verified the conjectured solutions for $2 \leq n \leq 600$.]

2273. *Proposed by Tim Cross, King Edward's School, Birmingham, England.*

Consider the sequence of positive integers: $\{1,\ 12,\ 123,\ 1\,234,\ 12\,345,\ \ldots\ \}$, where the next term is constructed by lengthening the previous term at its right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying" occurring as in addition: thus the ninth and tenth terms of the sequence are $123\,456\,789$ and $1\,234\,567\,900$ respectively.

Determine which terms of the sequence are divisible by 7.

2274. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

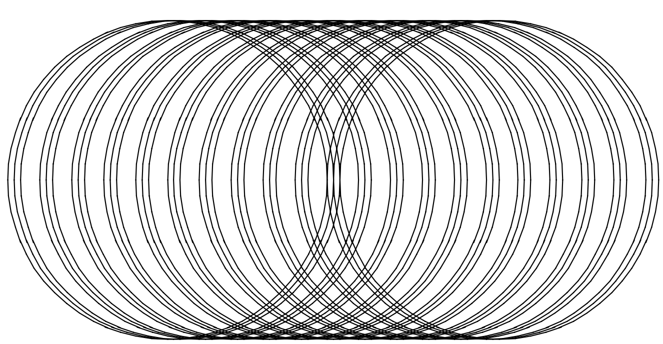
A. Let m be a non-negative integer. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)$.

B. Let $m \in \{1,2,3,4\}$. Find a closed form for $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^2$.

C$\star$. Let m and α_j $(j = 0, 1, \ldots, m)$ be non-negative integers. Prove or disprove that $\displaystyle\sum_{k=1}^{n}\prod_{j=0}^{m}(k+j)^{\alpha_j}$ is divisible by $\displaystyle\prod_{j=0}^{m+1}(n+j)$.

2275. *Proposed by M. Perisastry, Vizianagaram, Andhra Pradesh, India.*

Let $b > 0$ and $b^a \geq ba$ for all $a > 0$. Prove that $b = e$.



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1940. [1994: 108; 1995: 107, 205; 1996: 321; 1997: 170] *Proposed by Ji Chen, Ningbo University, China.*

Show that if $x, y, z > 0$,

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}\,.$$

Comment by Vedula N. Murty, Andhra University, Visakhapatnam, India.

In response to Marcin Kuczma's comment [1997: 170], I present the details of my proof of the assertion:

$$bc(b-c)^2(2a^2-bc) + ca(c-a)^2(2b^2-ca) \;>\; 0$$

when

$$2a^2 - bc \;<\; 0 \;<\; 2b^2 - ca \;\leq\; 2c^2 - ab,$$

where a, b, c are the side lengths of a triangle satisfying $0 < a \leq b \leq c$.

We have

$$a + b \;>\; c. \tag{1}$$

So, $2\left(a^2+b^2\right) \;\geq\; (a+b)^2 \;>\; c(a+b)$ implies that $\dfrac{bc-2a^2}{2b^2-ca} \;<\; 1$, and further that

$$\frac{bc}{ac}\,\frac{bc-2a^2}{2b^2-ca} \;<\; \frac{bc}{ac}. \tag{2}$$

Thus $\dfrac{c-a}{c-b} \;\geq\; 1$ implies that $\dfrac{(c-a)^2}{(c-b)^2} \;\geq\; \dfrac{c-a}{c-b} \;\geq\; \dfrac{b}{a}$, and further that $\dfrac{(c-a)^2}{(c-b)^2} \;\geq\; \dfrac{bc}{ac} \;>\; \dfrac{bc}{ac}\left(\dfrac{bc-2a^2}{2b^2-ca}\right)$. So we have

$$bc(b-c)^2(2a^2-bc) + ca(c-a)^2(2b^2-ca) \;>\; 0.$$

2158. [1996: 218] *Proposed by P. Penning, Delft, the Netherlands.*

Find the smallest integer in base eight for which the square root (also in base eight) has 10 immediately following the 'decimal' point.
In base ten, the answer would be 199, with $\sqrt{199} = 14.10673\ldots$.

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let n be a positive integer with the given property. Let m be the integer part of $\sqrt{n}$ and $n = m^2 + p$. The base 8 representation of a real number x is denoted by $(x)_8$. By hypothesis,

$$m + \frac{1}{8} = m + (0.10)_8 \leq \sqrt{n} < m + (0.11)_8 = m + \frac{9}{64},$$

which is equivalent to $m^2 + \frac{1}{4}m + \frac{1}{64} \leq n < m^2 + \frac{9}{32}m + \frac{9^2}{64^2}$.
It is easy to see that these inequalities can be replaced by the following stronger ones by taking into account that the expressions on both sides cannot be integers:

$$m^2 + \frac{m+1}{4} \leq n \leq m^2 + \frac{9m}{32}.$$

[For example, any integer greater than or equal to $m^2 + \frac{m}{4} + \frac{1}{64}$ must be greater than or equal to $m^2 + \frac{m+1}{4}$, since m is an integer.—*Ed.*] So

$$\frac{m+1}{4} \leq p \leq \frac{9m}{32}\,.$$

The lower limit has to be less than the upper limit, hence $m \geq 8$. For $8 \leq m \leq 10$ the lower limit is greater than 2 and the upper limit less than 3 [so no integer value of p exists]. If $m = 11$ then $p = 3$; therefore the smallest m is 11 and so the smallest n is $m^2 + p = \mathbf{124} = (174)_8$, where

$$\sqrt{(174)_8} = (13.10531\ldots)_8.$$

Also solved by CHARLES ASHBACHER, Hiawatha, Iowa, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; MIHAI CIPU, Institute of Mathematics, Romanian Academy, Bucharest, Romania; GEORGI DEMIREV, Varna, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; ROBERT GERETSCHLÄGER, Bundesrealgymnasium, Graz, Austria; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, Oregon, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

The curious fact that the smallest solution for base 8 is $124 = 11^2 + 3$ and for base 10 is $199 = 14^2 + 3$ (both written in base 10) is no coincidence! Geretschläger, Godin, and the proposer consider other bases as well, and *show that the smallest (base 10) integer whose square root when written in base b has 10 immediately following the "decimal" point is*

$$\begin{cases} \left(\frac{b-1}{2}\right)^2 + 1 & \text{if } b \text{ is odd} \\ \left(\frac{3b-2}{2}\right)^2 + 3 & \text{if } b \text{ is even.} \end{cases}$$

Janous lists all (base 10) positive integers < 10000 which satisfy the condition of the problem, and gets 144 such integers, starting 124, 229, 329, ... From this list one can find, for example, that: the first occurrence of a consecutive pair *is 1860, 1861 (also pointed out by Engelhaupt); the last integer* not *part of a consecutive pair appears to be 3616; the first consecutive triple is 5644, 5645, 5646; and so on. Can anyone find any general patterns here for arbitrary bases?*

Tsaoussoglou reports that the smallest integer whose square root in base 10 has 100 immediately following the decimal point is $99^2 + 20 = 9821$ ($\sqrt{9821} = 99.100958\ldots$), and that the corresponding answer for base 8 is $71^2 + 18 = 5059$ ($\sqrt{5059} = (107.100657\ldots)_8$). What if we want 1000 following the decimal point? Or 1 followed by n zeros?

2159. [1996: 218] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let the sequence $\{x_n, n \geq 1\}$ be given by

$$x_n = \frac{1}{n}(1 + t + \ldots + t^{n-1})$$

where $t > 0$ is an arbitrary real number.

Show that for all k, $l \geq 1$, there exists an index $m = m(k, l)$ such that $x_k \cdot x_l \leq x_m$.

Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, Oregon, USA.

Since $x_n = 1$ for all n if $t = 1$, any index m will suffice. We distinguish two other cases.

First, let $t > 1$. By the AM–GM inequality,

$$\begin{aligned} x_n &= \frac{1 + t + \cdots + t^{n-1}}{n} > \sqrt[n]{1 \cdot t \cdot \ldots \cdot t^{n-1}} \\ &= \sqrt[n]{t^{1+\cdots+(n-1)}} = \sqrt[n]{t^{n(n-1)/2}} \\ &= t^{(n-1)/2}. \end{aligned}$$

This last quantity increases without bound as $n \to \infty$, so there must be an index m for which $x_k x_l \leq x_m$ for given indices k, l.

Second, let $0 < t < 1$. Then

$$x_n = \frac{1 + t + \cdots + t^{n-1}}{n} < \frac{1 + 1 + \cdots + 1}{n} = 1,$$

so $x_k x_l < x_k$ and $x_k x_l < x_l$, and we may choose the index m to equal either k or l.

Also solved by THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; MIHAI CIPU, Institute of Mathematics, Romanian Academy, Bucharest, Romania; GEORGI DEMIREV, Varna, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; THOMAS C. LEONG, City College of City University of New York, New York, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Some solvers noted that since $x_1 = 1$ for all t, $m = 1$ will work for all k and l whenever $t < 1$. Cipu and the proposer found that $m = k + l - 1$ works when $t > 1$.

2160. [1996: 218] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$\triangle ABC$ is a triangle with $\angle A < 90°$. Let P be an interior point of ABC such that $\angle BAP = \angle ACP$ and $\angle CAP = \angle ABP$. Let M and N be the incentres of $\triangle ABP$ and $\triangle ACP$ respectively, and let R_1 be the circumradius of $\triangle AMN$. Prove that

$$\frac{1}{R_1} = \frac{1}{AB} + \frac{1}{AC} + \frac{1}{AP}.$$

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

Let a, b, c denote the sides BC, CA, AB and α, β, γ the angles of the triangle. Then

$$\angle APB = 180° - \angle ABP - \angle BAP = 180° - \angle ABP - \alpha + \angle CAP = 180° - \alpha$$

and by the cosine law in $\triangle APB$

$$c^2 = AP^2 + BP^2 + 2AP \cdot BP \cos\alpha.$$

$\triangle ABP$ and $\triangle CAP$ are by hypothesis similar, whence $BP = \frac{c \cdot AP}{b}$ and

$$c^2 = \frac{AP^2}{b^2}(b^2 + c^2 + 2bc\cos\alpha).$$

By setting $x^2 = b^2 + c^2 + 2bc\cos\alpha$

$$AP = \frac{bc}{x} \qquad \text{and} \qquad BP = \frac{c \cdot AP}{b} = \frac{c^2}{x}.$$

Let X be the point where the incircle of $\triangle APB$ touches AP. Then

$$\begin{aligned} PX &= \frac{AP+BP-AB}{2} = \frac{c\cdot(b+c-x)}{2x} \\ &= \frac{c\cdot((b+c)^2-x^2)}{2x\cdot(b+c+x)} \\ &= \frac{2bc^2\sin^2\frac{\alpha}{2}}{x\cdot(b+c+x)} \end{aligned}$$

and since $\angle XPM = \frac{1}{2}\angle APB = 90^\circ - \frac{\alpha}{2}$ we get

$$PM = \frac{PX}{\sin\frac{\alpha}{2}} = \frac{2bc^2\sin\frac{\alpha}{2}}{x\cdot(b+c+x)}.$$

Since $\triangle NPM$ and $\triangle CPA$ are similar, $MN = \frac{b\cdot PM}{AP}$, and by using the fact that $\angle MAN = \frac{\alpha}{2}$, we have

$$\begin{aligned} \frac{1}{R_1} &= \frac{2\sin\frac{\alpha}{2}}{MN} \\ &= \frac{2\cdot\frac{bc}{x}\sin\frac{\alpha}{2}}{b\cdot\frac{2bc^2\sin\frac{\alpha}{2}}{x\cdot(b+c+x)}} \\ &= \frac{b+c+x}{bc} \\ &= \frac{1}{AB}+\frac{1}{AC}+\frac{1}{AP}. \end{aligned}$$

Also solved by MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2161. [1996: 219] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Evaluate

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(3n-1)}.$$

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA (modified slightly be the editor).

Let S denote the given summation. We show that

$$S = 2\ln 2 - \frac{3}{2}\ln 3 + \frac{\sqrt{3}\pi}{6} \approx 0.64527561.$$

Using partial fractions, we have

$$\begin{aligned} S &= \sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{3}{3n-1}\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{2}{2n-1}-\frac{2}{2n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{3}{3n-1}\right) \\ &= 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}-3\sum_{n=1}^{\infty}\left(\frac{1}{3n-1}-\frac{1}{3n}\right) \\ &= 2\ln 2-3\sum_{n=1}^{\infty}\left(\frac{1}{3n-1}-\frac{1}{3n}\right). \end{aligned} \tag{1}$$

Since the sequence of functions $\{f_n(x)\}$, where $f_n(x) = x^{3n+1} - x^{3n+2}$, is uniformly convergent on $[0, 1]$,

$$\begin{aligned} \sum_{n=1}^{\infty}\left(\frac{1}{3n-1}-\frac{1}{3n}\right) &= \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\cdots \\ &= \int_0^1 (x-x^2+x^4-x^5+x^7-x^8+\cdots)\,dx \\ &= \int_0^1 x(1-x)(1+x^3+x^6+\cdots)\,dx \\ &= \int_0^1 \frac{x(1-x)}{1-x^3}\,dx \;=\; \int_0^1 \frac{x\,dx}{1+x+x^2} \\ &= \int_0^1 \frac{x+\frac{1}{2}}{1+x+x^2}\,dx-\frac{1}{2}\int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\ &= \left[\frac{1}{2}\ln(1+x+x^2)-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]\Bigg|_0^1 \\ &= \frac{1}{2}\ln 3-\frac{\sqrt{3}}{3}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{2}\ln 3-\frac{\sqrt{3}}{3}\cdot\frac{\pi}{6}. \end{aligned} \tag{2}$$

Substituting (2) into (1), we find $S = 2\ln 2 - \frac{3}{2}\ln 3 + \frac{\sqrt{3}\pi}{6}$, as claimed.

Also solved by THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria (jointly); RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Valley Catholic High School, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. Two incorrect solutions were also received.

About half of the solvers used the non–elementary approach of considering the Euler psi-function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *where* $\Gamma(x)$ *denotes the gamma function. Both Hess and Seiffert pointed out that the evaluation of the more general sum* $\sum_{n=1}^{\infty} \frac{1}{(n+p)(n+f)}$ $(p > -1,\ f > -1,\ p \neq f)$ *can be found in* Tables of Integrals, Series, and Products *by I.S. Gradsteyn and I.M. Ryzhik (Academic Press, 1994, 5th edition). Specifically, the following formulas can be found on p. 952 and p. 954, respectively:*

$$\begin{aligned}
\psi(x) &= -\gamma - \sum_{n=1}^{\infty} \left(\frac{1}{n-1+x} - \frac{1}{n} \right), \quad x > 0, \\
\psi\left(\frac{1}{2}\right) &= -\gamma - 2\ln 2, \\
\psi\left(\frac{2}{3}\right) &= -\gamma + \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ln 3.
\end{aligned}$$

(Here, γ *denotes Euler's constant.)*

Since $S = \sum_{n=1}^{\infty} \left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{3}} \right)$, *one obtains easily from the above formulas that* $S = \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{2}\right) = 2\ln 2 - \frac{3}{2}\ln 3 + \frac{\sqrt{3}\pi}{6}$ *as in the solution above.*

2162. [1996: 219] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, the Cevian lines AD, BE, and CF concur at P. $[XYZ]$ is the area of $\triangle XYZ$. Show that

$$\frac{[DEF]}{2[ABC]} = \frac{PD}{PA} \cdot \frac{PE}{PB} \cdot \frac{PF}{PC}$$

I. *Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain (slightly shortened by the editor).*

It is a standard exercise (for example, #13.5.6 in H.S.M. Coxeter, *Introduction to Geometry*) to show that

$$\frac{[DEF]}{[ABC]} = \frac{\lambda\mu\nu + 1}{(\lambda+1)(\mu+1)(\nu+1)}$$

with $\lambda = \dfrac{AF}{FB}$, $\mu = \dfrac{BD}{DC}$, $\nu = \dfrac{CE}{EA}$.

Since in our case, according to Ceva's Theorem, $\lambda\mu\nu = 1$, all we have to prove is that

$$\frac{PA}{PD} \cdot \frac{PB}{PE} \cdot \frac{PC}{PF} = (\lambda+1)(\mu+1)(\nu+1).$$

But it is true because

$$
\begin{aligned}
\frac{PA}{PD} &= \frac{[PCA]}{[PDC]}, \qquad \frac{PB}{PE} = \frac{[PAB]}{[PEA]},\\
\frac{PC}{PF} &= \frac{[PBC]}{[PFB]},\\
\lambda+1 &= \frac{AF}{FB}+1 = \frac{[PAF]}{[PFB]}+1 = \frac{[PAB]}{[PFB]},\\
\mu+1 &= \frac{BD}{DC}+1 = \frac{[PBD]}{[PDC]}+1 = \frac{[PBC]}{[PDC]},
\end{aligned}
$$

and

$$
\nu+1 = \frac{CE}{EA}+1 = \frac{[PCE]}{[PEA]}+1 = \frac{[PCA]}{[PEA]}.
$$

II. *Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

The corresponding more general problem for a simplex is given in [2] and also with a slightly modified proof in [3]. It is shown that if $V_0, V_1, \ldots, V_n$ denote the $n+1$ vertices of a simplex S in n-dimensional Euclidean space and if $V_0', V_1', \ldots, V_n'$ denote the $n+1$ vertices of an inscribed simplex S' such that the cevians V_iV_i' are concurrent at a point P within S, then

$$
\frac{\text{VOL } S'}{\text{VOL } S} = \frac{n\lambda_0\lambda_1\ldots\lambda_n}{(1-\lambda_0)(1-\lambda_1)\ldots(1-\lambda_n)}.
$$

Here the barycentric representation for P is $P = \lambda_0V_0+\lambda_1V_1+\ldots+\lambda_nV_n$ where $\lambda_0+\lambda_1+\ldots+\lambda_n = 1$ and $\lambda_i \geq 0$. It is also shown that the volume of S' is a maximum if P is the centroid.

Also $PV'/PV = \lambda_i/(1-\lambda_i)$. For this and other metric properties of concurrent cevians of a simplex see [1987: 274–275].

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Janous refers to [4, p. 342, item 1.3] *for a closely related identity.*

Bellot quotes Victor Thébault who reported an 1881 reference for the exercise that began our featured solution. This exercise is often attributed *to Routh (1891). For those who read Bulgarian, he also provides a reference concerning interesting properties of a cevian triangle DEF:* Hristo Lesov, Projections of noteworthy points of the triangle following the respective cevians (in Bulgarian). *Matematyka & Informatyka,* **5** (1994) 42–49. *He further mentions* [1, p. 78] *where a problem of Langendonck is solved:* Given a $\triangle ABC$, find the probability of selecting a point P inside the triangle such that it is possible to form a triangle whose sides equal the respective distances from P to the sides of the $\triangle ABC$. The probability turns out to be $[DEF]/[ABC]$.

[1] Heinrich Dörrie, *Mathematische Miniaturen*, F. Hirt, Breslau, 1943.

[2] M.S. Klamkin, A volume inequality for simplexes, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* No. 357–380 (1971) 3–4.

[3] M.S. Klamkin, *International Mathematical Olympiads 1978–1985*, Math. Assoc. Amer., Washington, D.C., 1986, pp. 87–88.

[4] D.S. Mitrinović et. al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

2163. [1996: 219] *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Prove that if $n, m \in \mathbb{N}$ and $n \geq m^2 \geq 16$, then $2^n \geq n^m$.

I. *Solution by Florian Herzig, student, Perchtoldsdorf, Austria.*
Let $x = \sqrt{n} \geq m \geq 4$. Then we first prove that $x^2 \leq 2^x$. This inequality is equivalent to

$$\frac{x}{\ln x} \geq \frac{2}{\ln 2}.$$

Define $f(x)$ as $x/(\ln x)$. Then

$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}$$

which is positive for all $x \geq 4$. Moreover $f(x)$ is continuous and differentiable in the interval $[4, \infty)$ and $f(4) = f(2)$. Hence $x^2 \leq 2^x$, and so, $n^m \leq (x^2)^x \leq (2^x)^x = 2^n$.

II. *Solution by D. Kipp Johnson, Valley Catholic High School, Beaverton, Oregon.*

We first show that the inequality is true for any $m \geq 4$ with the smallest allowable n, namely $n = m^2$, and then we induct on n.

For the first step we let $m \geq 4$ and $n = m^2$ and consider the following equivalent statements:

$$\begin{aligned} 2^{m^2} \geq (m^2)^m \quad &\iff \quad 2^{m^2} \geq 2^{2m\log_2 m} \\ &\iff \quad m^2 \geq 2m \log_2 m \\ &\iff \quad m/2 \geq \log_2 m. \end{aligned}$$

We establish this last statement by noticing that, for $m \geq 4$, we have

$$\begin{aligned} \frac{1}{m-1} &\leq 2^{1/2} - 1 \\ 1 + \frac{1}{m-1} &\leq 2^{1/2} \\ \log_2\left(1 + \frac{1}{m-1}\right) &\leq \frac{1}{2} \\ \log_2 m - \log_2(m-1) &\leq \frac{1}{2}. \end{aligned}$$

This permits us to write the following $m - 4$ inequalities:

$$\begin{aligned} \log_2 5 - \log_2 4 &\leq \frac{1}{2} \\ \log_2 6 - \log_2 5 &\leq \frac{1}{2} \\ &\vdots \\ \log_2(m) - \log_2(m-1) &\leq \frac{1}{2}, \end{aligned}$$

which, when added, telescope to produce the desired $\log_2 m \leq m/2$.

For the second step we induct on n. We shall assume that for some $n \geq m^2 \geq 16$ we have $2^n \geq n^m$. Multiplying each side by 2 we obtain $2^{n+1} \geq 2 \cdot n^m$. It will suffice to show that $2 \cdot n^m \geq (n+1)^m$, which is equivalent to $2 \geq (1 + 1/n)^m$. The following sequence of inequalities does the trick:

$$2 \geq \sqrt{e} \geq \sqrt{\left(1 + \frac{1}{m^2}\right)^{m^2}} \geq \left(1 + \frac{1}{m^2}\right)^m \geq \left(1 + \frac{1}{n}\right)^m,$$

and the proof is finished.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHAI CIPU, Romanian Academy, Bucharest, Romania; GEORGI DEMIZEV, Varna, Bulgaria, and MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, Missouri; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; FABIAN MARTIN HERCE, student, Universidad de La Rioja, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, *Mount Royal College, Calgary, Alberta; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. There was 1 incomplete solution.*

Hess actually proves a somewhat stronger result by replacing the condition $n \geq m^2 \geq 16$ by the 2 conditions $n \geq 16$ and $n \geq m^2$.

2164. [1996: 273] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let D be a point on the side BC of triangle ABC, and let E and F be the incentres of triangles ABD and ACD respectively. Suppose that B, C, E, F are concyclic. Prove that

$$\frac{AD+BD}{AD+CD} = \frac{AB}{AC}.$$

Solution by the Con Amore Problem Group, the Royal Danish School of Educational Studies and Informatics, Copenhagen, Denmark.

The intersection points of the line EF with AB, AC, and AD will be called G, H and K, respectively, and we put $\angle B = 2y$, and $\angle C = 2z$, so $\angle GBE = \angle EBD = y$, and $\angle DCF = \angle FCH = z$. Since B, C, E, F are concyclic, $\angle GEB = \angle FCD = z$, and $\angle CFH = \angle EBD = y$; and so

$$\angle AGH = y + z = \angle AHG$$

giving

$$AG = AH. \tag{1}$$

Since AE bisects $\angle A$ in $\triangle GAK$, and AF bisects $\angle A$ in $\triangle KAH$, we have [using (1)]: $\frac{KE}{EG} = \frac{KA}{AG} = \frac{KA}{AH} = \frac{KF}{FH}$, whence $\frac{KE}{EG} + 1 = \frac{KF}{FH} + 1$, or

$$\frac{KG}{EG} = \frac{KH}{FH}. \tag{2}$$

In $\triangle ABD$ let the inradius, the altitude from D, and the area be r_1, h_1 and T_1, respectively; and in $\triangle ACD$ let the corresponding quantities be r_2, h_2 and T_2. In $\triangle AKG$, and $\triangle AKH$ let the altitude from K be k_1 and k_2, respectively. Then $\frac{h_1}{h_2} = \frac{k_1}{k_2}$, whence

$$\frac{h_1}{k_1} = \frac{h_2}{k_2} \tag{3}$$

and [using (2)]

$$\frac{k_1}{r_1} = \frac{KG}{EG} = \frac{KH}{FH} = \frac{k_2}{r_2}, \tag{4}$$

and further, (3) - (4) imply

$$\frac{h_1}{r_1} = \frac{h_2}{r_2}. \tag{5}$$

Now

$$h_1 \cdot AB = 2T_1 = r_1 \cdot (AB + BD + DA),$$

so

$$\frac{h_1}{r_1} = \frac{AB + BD + DA}{AB} = 1 + \frac{BD + DA}{AB}, \tag{6}$$

and similarly,

$$\frac{h_2}{r_2} = \frac{AC + CD + DA}{AC} = 1 + \frac{CD + DA}{AC}, \tag{7}$$

From (5) - (7) it easily follows that

$$\frac{AD + BD}{AD + CD} = \frac{AB}{AC}.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Bellot Rosado notes that this problem is a special case of problem 1206 [*H. Demir, Mathematics Magazine, 58, no. 1, January 1985; solution by V. D. Mascioni, Mathematics Magazine, 59, no. 1, February 1986*]

2165. [1996: 273] *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

Given a triangle ABC, prove that there exists a unique pair of points P and Q such that the triangles ABC, PQC and PBQ are directly similar; that is, $\angle ABC = \angle PQC = \angle PBQ$ and $\angle BAC = \angle QPC = \angle BPQ$, and the three similar triangles have the same orientation. Find a Euclidean construction for the points P and Q.

Solution by the Con Amore Problem Group, the Royal Danish School of Educational Studies and Informatics, Copenhagen, Denmark.

We identify the Euclidean plane with the complex plane, and points with complex numbers. We may assume without loss of generality that

$$B = 0. \tag{1}$$

Suppose such a P and Q exist. Since $\triangle ABC \sim \triangle PQC$, there is a point (complex number) X such that

$$P = XA, \tag{2}$$

and

$$Q = XC, \tag{3}$$

and besides:

$$\frac{P-Q}{C-Q}=\frac{A}{C}. \tag{4}$$

Moreover, if (2) - (4) hold, then P, Q are as desired. Now (2) - (4) imply that

$$\frac{XA-XC}{C-XC}=\frac{A}{C},$$

or

$$XA-XC=A-XA,$$

or

$$X=\frac{A}{2A-C}, \tag{5}$$

and then, by (2) - (3):

$$P=\frac{A^2}{2A-C} \tag{6}$$

and

$$Q=\frac{AC}{2A-C}. \tag{7}$$

On the other hand, if X, P, Q satisfy (5) - (7) then (2) - (4) hold. This proves the existence and uniqueness of P and Q with the desired properties. To find a Euclidean construction for P and Q, define

$$R=2A-C,$$

and note that

$$A=\frac{R+C}{2};$$

that is, R is the image of C by reflection in A. Also note that

$$\frac{Q}{C}=\frac{A}{R};$$

that is, triangles ABR and QBC are similar, and similarly oriented. Finally

$$\frac{A+Q}{2}=\frac{1}{2}\left(\frac{AC}{2A-C}+A\right)=\frac{A^2}{2A-C}=P$$

so P is the midpoint of AQ. All this gives us the following construction:

(A) Reflect C in A to get R.

(B) Construct Q such that A and Q are on opposite sides of the line BC, $\angle CBQ=\angle RBA$, and $\angle QCB=\angle ARB$.

(C) Construct P as the midpoint of AQ.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (two solutions); TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2166. [1996: 273] *Proposed by K.R.S. Sastry, Dodballapur, India.*

In a right-angled triangle, establish the existence of a unique interior point with the property that the line through the point perpendicular to any side cuts off a triangle of the same area.

Solution by Toshio Seimiya, Kawasaki, Japan.

Let the right-angled triangle be ABC, with right angle at A. We may assume without loss of generality that $AB \leq AC$. P is an interior point of $\triangle ABC$. The line through P perpendicular to BC meets BC, AC at D, E respectively (if $AB = AC$, then $E = A$); the line through P perpendicular to CA meets BC, CA at F, G respectively, and the line through P perpendicular to AB meets BC, AB at H, K respectively. We assume that the triangles BHK, CDE, CFG have the same area. Since these triangles are similar, they must be congruent. Since $\triangle CDE \equiv \triangle CGF$, we have $CD = CG$, so that $\triangle CDP \equiv \triangle CGP$. Thus we have $\angle PCD = \angle PCG$. Therefore, CP is the bisector of $\angle C$.

Let M be the intersection of AP with BC (if $AB = AC,\ M = D$). Since $\triangle BHK \equiv \triangle FCG$, we have $BH = FC$, so that

$$BF = HC. \tag{1}$$

Since $PF \| AB$ and $PH \| AC$, we get

$$BF : BM = AP : AM = CH : CM \tag{2}$$

From (1) and (2) we have $BM = CM$. Therefore, AM is the median of $\triangle ABC$. Thus P must be the intersection of the median AM with the bisector of $\angle C$. It is interesting to note that the bisector of $\angle B$ does not pass through P if $AB \neq AC$.

Conversely, let P be the intersection of the median AM with the bisector of $\angle C$. Draw the lines DE, FG, HK through P perpendicular to BC, CA, AB respectively, as shown in the figure. Since $\angle PCD = \angle PCG$, we have $\triangle PCD \equiv \triangle PCG$, so that $CD = CG$. As $\triangle CFG$ is $\triangle CED$ we get $\triangle CFG \equiv CED$. Since $PF \| AB$ and $PH \| AC$, we have $BF : BM = AP : AM = CH : CM$. As $BM = CM$, we get $BF = CH$, so that $BH = FC$. Because $\triangle KBH$ is $\triangle GFC$, we have $\triangle KBH \equiv \triangle GFC$. Since $\triangle CDE, \triangle CGF, \triangle HKB$ are congruent, they have the same area.

A number of solvers used a coordinate approach. If $A = (0, 0)$, $B = (0, c)$, $C = (b, 0)$ and $BC = a$, then $P = \left(\frac{b^2}{2b+a}, \frac{bc}{2b+a}\right)$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; C. DIXON, Royal Grammar School, Newcastle upon Tyne, England; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There were two incorrect solutions.

The proposer comments: "I am unable to establish the existence of such a point in a general triangle. If we replace "perpendicular" by "parallel" then the problem is easy and yields the unique point centroid of a triangle."

2167. [1996: 274] *Proposed by Šefket Arslanagić, Berlin, Germany.*

Prove, **without the aid of the differential calculus**, the inequality, that in a right triangle

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \geq 2+\sqrt{2},$$

where a and b are the legs and c the hypotenuse of the triangle.

Solution by Mihai Cipu, Institute of Mathematics, Romania Academy, Bucharest, Romania, and CICMA, Concordia University, Montreal Quebec.

Since $a^2(b+c)+b^2(a+c) = c(a^2+b^2)+ab(a+b)$ the given inequality is equivalent to

$$c(a-b)^2 \geq (\sqrt{2}c-a-b)ab = \frac{ab(a-b)^2}{\sqrt{2}c+a+b}.$$

In an isosceles triangle this relation holds trivially, while in the general case it is equivalent to $(\sqrt{2}c+a+b)c \geq ab$, a simple consequence of $c^2 \geq ab$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB

PRIELIPP and JOHN OMAN, University of Wisconsin–Oshkosh, Wisconsin, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer. There were four incorrect solutions.

From the proof above it is obvious that equality holds if and only if $a = b$. *Both Konečný and Wang pointed out that a weaker version of this problem with* $2+\sqrt{2}$ *replaced by* π *appeared as problem No. 238 in the January, 1983 issue of the* College Mathematics Journal. *Wang remarked that in the published solution (CMJ 15 (4), 1984; 352–353) Walter Blumberg and Marshall Fraser independently proved the more general result that*

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \;\geq\; 2+\csc(C/2)$$

where a, b, *and* c *are the sides of an arbitrary triangle with* C *being the largest angle. The present problem is the special case when* $C = \pi/2$. *Klamkin gave a proof for this more general result. Romero Márquez asked whether the stronger inequality*

$$\frac{a^2(b+c)+b^2(a+c)}{abc} \;\geq\; \frac{\sqrt{2}c}{a+b-c}$$

holds. The triangle with $a = 3$, $b = 4$, *and* $c = 5$ *provides a simple counterexample. Janous remarked that the given inequality could be considered as the special case (when* $n = 1$*) of the problem of finding the minimum value of*

$$T_n \;=\; \frac{a^{2n}(b^n+c^n)+b^{2n}(a^n+c^n)}{(abc)^n}$$

where a, b, c *are the legs and hypotenuse of a right triangle and* $n > 0$ *is a real number. He proposed the conjecture that* $T_n \geq 2+2^{(2-n)/2}$ *for all* $n \geq n_0$ *where* n_0 *is the solution of the equation* $2^{(n+4)/2}+n = 2$.

2168★. [1996: 274] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be a point inside a regular tetrahedron $ABCD$, with circumradius R and let R_1, R_2, R_3, R_4 denote the distances of P from vertices of the tetrahedron. Prove or disprove that

$$R_1R_2R_3R_4 \leq \frac{4}{3}R^4,$$

and that the maximum value of $R_1R_2R_3R_4$ is attained.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. [Ed: First, we give Klamkin's proof for a triangle, which was not published as a solution to 2073⋆ [1995: 277; 1996: 282]]

Let P be given by the vector $\vec{P} = x_1\vec{A_1} + x_2\vec{A_2} + x_3\vec{A_3}$, where $x_1 + x_2 + x_3 = 1$ and $x_1, x_2, x_3 \geq 0$ (barycentric coordinates). Then

$$\begin{aligned} PA_1^2 &= \left|\vec{P} - \vec{A_1}\right|^2 \\ &= \left|x_2\left(\vec{A_2} - \vec{A_1}\right) + x_3\left(\vec{A_3} - \vec{A_1}\right)\right|^2 \\ &= 3\left(x_2^2 + x_2x_3 + x_3^2\right), \end{aligned}$$

etc. We have taken $R = 1$, so that the side of the triangle $\left|\vec{A_2} - \vec{A_1}\right| = \sqrt{3}$. The given inequality now takes the form

$$\left(x_2^2 + x_2x_3 + x_3^2\right)\left(x_3^2 + x_3x_1 + x_1^2\right)\left(x_1^2 + x_1x_2 + x_2^2\right) \leq \frac{3}{64},$$

or in homogeneous form

$$\begin{aligned} 64\left(x_2^2 + x_2x_3 + x_3^2\right)\left(x_3^2 + x_3x_1 + x_1^2\right)\left(x_1^2 + x_1x_2 + x_2^2\right) & \\ \leq\; 3\left(x_1 + x_2 + x_3\right)^6 & \quad (1) \end{aligned}$$

where (only) $x_1, x_2, x_3 \geq 0$.

We assume without loss of generality that $x_1 \geq x_2 \geq x_3 \geq 0$ so that we can let $x_3 = c$, $x_2 = b + c$ and $x_1 = a + b + c$ with $a, b, c \geq 0$ and $a + b + c > 0$. On substituting back into (1), we get

$$\begin{aligned} &3a^6 + 36a^5b + 54a^5c + 116a^4b^2 + 348a^4bc + 213a^4c^2 + 160a^3b^3 \\ &+816a^3b^2c + 1128a^3bc^2 + 468a^3c^3 + 80a^2b^4 + 864a^2b^3c \\ &+2232a^2b^2c^2 + 2232a^2bc^3 + 765a^2c^4 + 480ab^4c + 2208ab^3c^2 \\ &+3888ab^2c^3 + 3060abc^4 + 918ac^5 + 192b^5c + 1104b^4c^2 \\ &+2592b^3c^3 + 3060b^2c^4 + 1836bc^5 + 459c^6 \quad \geq \quad 0 \end{aligned}$$

since all terms on the left side are non-negative (this expansion was confirmed using **Mathematica** [Ed: and using **DERIVE**]). There is equality only if $a = c = 0$, so that if $x_1 + x_2 + x_3 = 1$, $x_1 = x_2 = 0.5$ and $x_3 = 0$, then the point P must be a mid-point of a side.

For a regular tetrahedron, we proceed as before: let P be given by the vector $\vec{P} = x_1\vec{A_1} + x_2\vec{A_2} + x_3\vec{A_3} + x_4\vec{A_4}$, where $x_1 + x_2 + x_3 + x_4 = 1$ and $x_1, x_2, x_3, x_4 \geq 0$. Then

$$\begin{aligned} PA_1 &= \left|\vec{P} - \vec{A_1}\right| \\ &= \left|x_2\left(\vec{A_2} - \vec{A_1}\right) + x_3\left(\vec{A_3} - \vec{A_1}\right) + x_4\left(\vec{A_4} - \vec{A_1}\right)\right|, \end{aligned}$$

so that

$$PA_1^2 \;=\; a^2\left(x_2^2+x_3^2+x_4^2+x_3x_4+x_4x_2+x_2x_3\right),$$

etc., where a is an edge of the tetrahedron. So if we take $R = 1$, we have $a^2 = \frac{8}{3}$. the desired inequality then becomes (in homogeneous form)

$$9\left(x_1+x_2+x_3+x_4\right)^8 \;\geq\; 2^8 F_1F_2F_3F_4, \tag{2}$$

where

$$F_1 \;=\; x_2^2+x_3^2+x_4^2+x_3x_4+x_4x_2+x_2x_3,$$

and the other F_i's are obtained by cyclic interchange of the indices 1, 2, 3, 4.

Assuming without loss of generality that $x_1 \geq x_2 \geq x_3 \geq x_4$, we take $x_4 = d$, $x_3 = d+c$, $x_2 = d+c+b$, $x_1 = d+c+b+a$, where a, b, c, $d \geq 0$ and $x_1 > 0$. On substituting back into (2) and expanding out (using **Mathematica**), we get a polynomial [Ed: we have omitted this polynomial, since it would cover two whole pages of ***CRUX with MAYHEM***] which is non-negative since all the terms are non-negative. There is equality only if $a = c = d = 0$. Hence $x_1 = x_2 = 0.5$ and so P must be the mid-point of an edge.

I also conjecture for the correponding result for a regular n–dimensional simplex:

the product of the distances from a point within or on a regular simplex to the vertices is a maximum only when the point is a mid-point of an edge.

Analogously to inequality (2), the conjecture is that

$$3^{n-1}\left(x_0+x_1+\ldots+x_n\right)^{2n+2} \;\geq\; 2^{2n+2}F_0F_1\ldots F_n$$

(x_0, x_1, ..., $x_n \geq 0$), where F_i is the complete symmetric homogeneous polynomial of degree 2 with unit coefficients of all the variables x_0, x_1, ..., x_n, except x_i, and there is equality only if two of the variables are equal and the rest are zero.

Also solved by CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; and RICHARD I. HESS, Rancho Palos Verdes, California, USA.