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*Editor: Léo Sauvé, Architecture Department, Algonquin College, 281 Echo Drive, Ottawa, Ontario, K1S 1N3.*

*Managing Editor: F.G.B. Maskell, Mathematics Department, Algonquin College, 200 Lees Ave., Ottawa, Ontario, K1S 0C5.*

*Typist-compositor: Lillian Marlow.*

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CONTENTS

# PRIME ARITHMETIC PROGRESSIONS

CHARLES W. TRIGG

Every prime greater than 3 has one of the forms $6k-1$ or $6k+1$.

If $a$ and $b$ are the first two terms of an arithmetic progression, then the third term is $2b-a$. If $6m-1$ and $6n+1$ are the first two terms of an arithmetic progression, then the third term is the composite $3(4n-2m+1)$. If the first two terms of an arithmetic progression are $6p+1$ and $6q-1$, then the third term is the composite $3(4q-2p-1)$. Hence, in any arithmetic progression with prime terms greater than 3, all terms must be of the same form, and the common difference is a multiple of 6. (More generally, as reported earlier in this journal [1979: 237-238], in any $n$-term ($n \geq 3$) prime arithmetic progression, the common difference is divisible by every prime less than $n$.)

It follows that any prime arithmetic progression with 3 as a first term cannot have more than three terms. Furthermore, since $3+6+6=15$, and integers greater than 5 and ending in 5 are composite, the prime arithmetic progression cannot have a common difference terminating in 6, nor can the middle term of the progression end in 9. Other arithmetic progressions beginning with 3 and another prime may also have composite third terms; for example: 3, 47, $91(=7 \cdot 13)$. The twenty-five smallest prime arithmetic progressions beginning with 3 are:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 7 | 3 | 23 | 43 | 3 | 53 | 103 | 3 | 101 | 199 | 3 | 157 | 311 |
| 3 | 7 | 11 | 3 | 31 | 59 | 3 | 67 | 131 | 3 | 107 | 211 | 3 | 167 | 331 |
| 3 | 11 | 19 | 3 | 37 | 71 | 3 | 71 | 139 | 3 | 113 | 223 | 3 | 181 | 359 |
| 3 | 13 | 23 | 3 | 41 | 79 | 3 | 83 | 163 | 3 | 127 | 251 | 3 | 191 | 379 |
| 3 | 17 | 31 | 3 | 43 | 83 | 3 | 97 | 191 | 3 | 137 | 271 | 3 | 193 | 383 |

The first nineteen terms of the sequence of primes that cannot be the second term of a prime arithmetic progresssion beginning with 3 are: 19, 29, 47, 59, 61, 73, 79, 89, 103, 109, 131, 139, 149, 151, 163, 173, 179, 197, 199, ... .

"We do not know if there are infinitely many arithmetic progressions of three different primes of which the first term is 3," says Wacław Sierpiński in *A Selection of Problems in the Theory of Numbers* (Macmillan, New York, 1964, page 47). However, we do know that

$$3, \quad 5003261, \quad 10006519$$

and

3, 5003303, 10006603

are prime arithmetic progressions.

In another direction, as reported earlier in this journal [1979: 244], it is not known if there are arbitrarily long arithmetic progressions of primes. The longest known one, discovered by S. Weintraub in 1977, has 17 terms, a common difference of

$$87297210 = 2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19,$$

and first term 3430751869.

2404 Loring Street, San Diego, California 92109.

* * *

# ON EQUIANGULAR POLYGONS

M.S. KLAMKIN and A. LIU

In a previous paper in this journal [1981: 2-5], it was conjectured that a planar convex $n$-gon $V_1V_2\ldots V_nV_1$ (with $n \geq 5$ and odd) is regular if

$$\angle V_{i-1}V_iV_{i+1} = \text{constant}$$

and

$$\angle V_{i-2}V_iV_{i+2} = \text{constant}$$

for $i = 1,2,\ldots,n$, where $V_{i+n} = V_i$. Equivalently, if a convex odd $n$-gon and its inscribed star $n$-gon

$$V_1V_3V_5\ldots V_{n-1}V_1$$

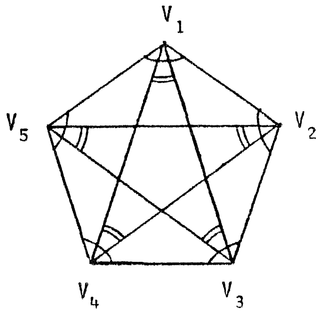

Figure 1

(usually denoted by $\{\frac{n}{2}\}$) are each equiangular, then the given $n$-gon and its inscribed star $n$-gon $\{\frac{n}{2}\}$ are both regular. (Figure 1 illustrates the case $n = 5$.) In this paper, we prove this conjecture. Additionally, we establish other related results.

We will find it convenient to say that a planar convex $n$-gon $P$ has property

$S_1$: if the edges of $P$ are congruent,

$S_2$: if the edges of the inscribed star $n$-gon $\{\frac{n}{2}\}$ are congruent,

$A_1$: if the angles of $P$ are congruent,

$A_2$: if the angles of the inscribed star $n$-gon $\{\frac{n}{2}\}$ are congruent.

The polygon $P$ is obviously regular if it has properties $S_1$ and $A_1$, or $S_1$ and $S_2$. We now show that $P$ is also regular if $n$ is odd and $P$ has any of the remaining pairs of properties:

(I) $S_2$, $A_2$; (II) $S_2$, $A_1$; (III) $S_1$, $A_2$; (IV) $A_1$, $A_2$.

That it is necessary for $n$ to be odd follows from Figures 2 and 3 for hexagons, and one can give similar counterexamples for larger even $n$. Figure 2 is a counterexample for (I), (II), and (IV), while Figure 3 is a counterexample for (III).

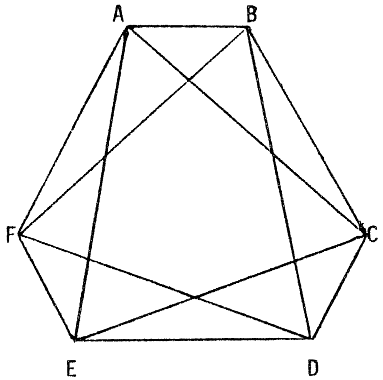


Figure 2

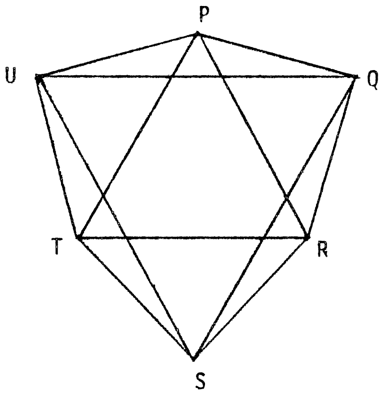


Figure 3

In Figure 2, ACE and BDF are congruent equilateral triangles and ABCDEFA is equiangular. In Figure 3, PRT and QSU are noncongruent equilateral triangles and PQRSTUP is equilateral.

*Proof of* (I): $S_2, A_2 \Rightarrow P$ *is regular.*

It follows easily that the inscribed star $n$-gon $\{\frac{n}{2}\}$ has a circumcircle and then that $P$ also has property $S_1$.

*Proof of* (II): $S_2, A_1 \Rightarrow P$ *is regular.*

Here, for any four consecutive vertices A, B, C, D of $P$ (Figure 4), we have AC = BD, BC = BC, and $\angle$ ABC = $\angle$ BCD, these angles being obtuse since $n \geq 5$. Hence AB = CD, or every other edge of $P$ has the same length. Since $n$ is odd, $P$ has property $S_1$ and is thus regular.

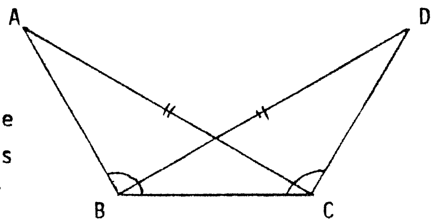


Figure 4

*Proof of* (III): $S_1, A_2 \Rightarrow P$ *is regular.*

We will first prove a result using only property $A_2$. (We will also appeal to this result in the proof of (IV).) Let A, B, C, D be four consecutive vertices of $P$ (see Figure 5). Let $\angle$ AZ'A' = $l$ and $\angle$ AA'Z' = $m$. We will show that $l = m$. Note that

$$\angle BA'B' = m,$$

$$\angle BB'A' = \pi - a - m = l,$$

$$\angle CB'C' = l,$$

$$\angle CC'B' = \pi - a - l = m,$$

and so on. This generates a sequence of $n$ pairs of angles: $l,m$; $m,l$; $l,m$;... . Since $n$ is odd, the last pair is also $l,m$. This leads to a contradiction ($\angle AZ'A' = m$) unless $l = m$ as claimed.

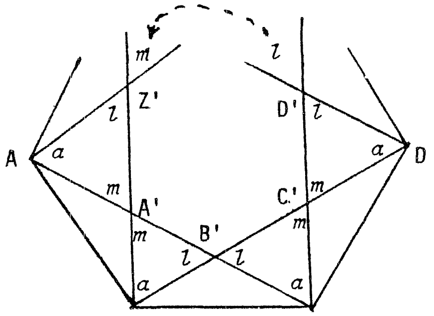


Figure 5

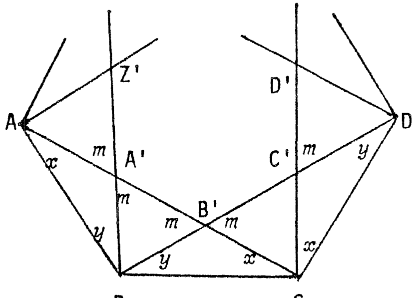


Figure 6

Now let $\angle A'AB = x$ and $\angle A'BA = y$, as shown in Figure 6. Then

$$\angle B'CB = x, \qquad \text{since } AB = BC;$$

$$\angle B'BC = y, \qquad \text{since } y = m - x;$$

$$\angle C'DC = y, \qquad \text{since } BC = CD;$$

$$\angle C'CD = x, \qquad \text{since } x = m - y;$$

and so on. This generates a sequence of $n$ pairs of angles: $x,y$; $y,x$; $x,y$;... . As before, we must have $x = y$, and the regularity of $P$ follows easily.

*Proof of* (IV): $A_1, A_2 \Rightarrow P$ *is regular.*

Let $V_i$ and $V_{i+1}$ be any two consecutive vertices of $P$, where $1 \le i \le n$, with $V_{i+n} = V_i$ (see Figure 7). Let the angles of $P$ be equal to $q$. That the angles marked $m$ are indeed equal has been proved in (III). Now

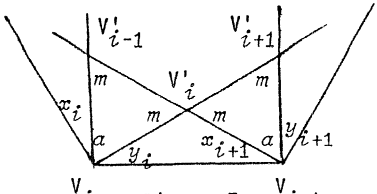


Figure 7

$$x_i + y_i = q - a$$

and

$$y_i + x_{i+1} = m.$$

Thus $x_i - x_{i+1} = q - a - m$, a constant, for all values of $i$. It follows that all the $x_i$'s are equal (to $x$, say) and all the $y_i$'s are equal (to $y$). Suppose $x \geq y$. Then we have

$$V_1V_1' \geq V_1'V_2 = V_2V_2' \geq V_2'V_3 = V_3V_3' \geq \ldots \geq V_n'V_1 = V_1V_1'.$$

It follows that all these edges are equal, and hence $P$ has property $S_1$. This completes the proof.

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

* * *

# THE OLYMPIAD CORNER: 23

MURRAY S. KLAMKIN

I start off this month with the problems given at the 42nd Moscow Olympiad (1979) for students of Grades 7 to 10 (Russian schools go from Grade 1 to Grade 10). Some of the problems were reserved for students of only one grade, but others were given to students of two successive grades. Of the students participating in the Olympiad, 634 were from Grade 7, 497 from Grade 8, 363 from Grade 9, and 353 from Grade 10. I am grateful to Mark E. Saul who obtained the problems in Russian and translated them into English. As usual, I solicit solutions from *all* readers (particularly, but not exclusively, from secondary school students, who should give their grade and the name of their school). The number of successful solvers appears after the grade at the end of each problem.

## FORTY-SECOND MOSCOW OLYMPIAD (1979)

1. A point A is chosen in a plane.

(a) Is it possible to draw (i) 5 circles, (ii) 4 circles, none of which cover point A, and such that any ray with endpoint A intersects at least two of the circles? (Gr. 7 - (i) 370, (ii) 21)

(b) As a variant, is it possible to draw (i) 7 circles, (ii) 6 circles, none of which cover point A, and such that any ray with endpoint A intersects at least three of the circles? (Gr. 8 - (i) 62, (ii) 45)

2. A (finite) set of weights is numbered 1, 2, 3, ... . The total weight of the set is 1 kilogram. Show that, for some number $k$, the weight numbered $k$ is heavier than $1/2^k$ kilogram. (Gr. 7 - 120; Gr. 8 - 129)

3. A square is dissected into several rectangles. Show that the sum of the areas of the circles circumscribing the rectangles is no less than the area of the circle circumscribing the square. (Gr. 7 - 25; Gr.8 - 55)

4. Karen and Billy play the following game on an infinite checkerboard. They take turns placing markers on the corners of the squares of the board. Karen plays first. After each player's turn (starting with Karen's second turn), the markers placed on the board must lie at the vertices of a convex polygon. The loser is the first player who cannot make such a move. For which player is there a winning strategy? (Gr. 7 - 74)

5. Quadrilateral ABCD is inscribed in a circle with center O. The diagonals AC and BD are perpendicular. If OH is the perpendicular from O to AD, show that $OH = \frac{1}{2}BC$. (Gr.8 - 75; Gr. 9 - 97)

6. A scientific conference is attended by $k$ chemists and alchemists, of whom the chemists are in the majority. When asked a question, a chemist will always tell the truth, while an alchemist may tell the truth or may lie. A visiting mathematician has the task of finding out which of the $k$ members of the conference are chemists and which are alchemists. He must do this by choosing a member of the conference and asking him: "Which is So-and-So, a chemist or an alchemist?" In particular, he can ask a member: "Which are you, a chemist or an alchemist?" Show that the mathematician can accomplish his investigation by asking

(a) $4k$ questions;

(b) $2k - 2$ questions;

(c) $2k - 3$ questions. (Gr. 8 - (a) 8, (b) 6; Gr.9 - (a) 7, (b) 7; Gr.10-(a) 7,(b) 7)

(d) [After the Olympiad, it was announced that the minimum number of questions is no greater than $[(3/2)k] - 1$. Prove it.]

7. Each of a set of stones has a mass of less than 2 kg., and the total mass of the stones is more than 10 kg. A subset of the stones is chosen whose total mass is as close as possible to 10 kg. Let the (positive) difference between the total mass of the subset and 10 kg. be $D$. If we start out with different sets of stones, what is the largest possible $D$? (Gr. 9 - 10; Gr. 10 - 5)

8. Can we represent (three-dimensional) space as the union of an infinite set of lines any two of which intersect? (Gr.9 - 10)

9. Does there exist an infinite sequence $\{a_1, a_2, a_3, \ldots\}$ of natural numbers such that no element of the sequence is the sum of any number of other elements and such that, for all $n$, (a) $a_n \le n^{10}$; (b) $a_n \le n\sqrt{n}$? (Gr.9 - (a) 0, (b) 53)

10. A number of intervals are chosen along the line segment $[0,1]$. The distance between two points belonging to any one interval, or even belonging to two different intervals, is never equal to $1/10$. Show that the sum of the lengths of the chosen intervals is not greater than $\frac{1}{2}$. (Gr.10 - 45)

11. The sum of the areas of a set of circles is 1. Show that a subset of these circles may be chosen such that no two of the chosen circles intersect, and such that the sum of the areas of the chosen circles is no greater than $1/9$. (Gr.10 - 28)

12. The function $f$ is defined on the interval $[0,1]$ and is twice differentiable at each point. The absolute value of the derivative is never greater than 1 on the interval, and $f(0) = f(1) = 0$. What is the greatest possible maximum value that $f(x)$ can have on the interval? (Gr. 10 - 32)

*

I now give the problems given in four selection tests held in April and June 1978 to select the Romanian team members for the International Mathematical Olympiad. As I reported last month, these problems were obtained through the courtesy of the Romanian Ministry of Education. Solutions are solicited from all readers.

First Selection Test - 9 April 1978 - 4 hours

1. Consider the set $X = \{1,2,3,4,5,6,7,8,9\}$. Show that, for any partition of $X$ into two subsets, one of these contains three numbers such that the sum of two of them equals twice the third.

2. Let $k,l \geq 1$ be fixed natural numbers. Show that if $(11m - 1,k) = (11m - 1,l)$ for any natural number $m$ (where $(x,y)$ means the greatest common divisor of $x$ and $y$), then there is an integer $n$ such that $k = 11^n \cdot l$.

3. Let $P(x,y)$ be a polynomial in $x,y$ of degree at most 2. Let A, B, C, A', B', C' be six distinct points in the $xy$-plane such that A,B,C are not collinear, A' lies on BC, B' on CA, and C' on AB. Prove that if $P$ vanishes at these six points, then $P \equiv 0$.

4. Let ABCD be a convex quadrilateral and O the intersection of the diagonals AC and BD. Show that if the triangles OAB, OBC, OCD, and ODA all have the same perimeter, then ABCD is a rhombus. Does this assertion remain true if O is another interior point?

5. Show that there is no square whose vertices are located on four concentric circles whose radii are in arithmetic progression.

6. Show that there is no polyhedron whose orthogonal projection on every plane is a nondegenerate triangle.

7. $P$, $Q$, and $R$ are three polynomials of degree 3 with real coefficients such that $P(x) \le Q(x) \le R(x)$ for all real $x$. Moreover, there is a real number $a$ such that $P(a) = R(a)$. Prove that there is a constant $k$, $0 \le k \le 1$, such that $Q = kP + (1-k)R$. Does this property still hold if $P$, $Q$, and $R$ are of degree 4?

8. Let $A$ be an arbitrary set. Two maps $f,g\colon A \to A$ are called *similar* if there is a bijective map $h : A \to A$ such that $f \circ h = h \circ g$.

(a) If $A$ has only three elements, construct functions $f_1, f_2, \ldots, f_k\colon A \to A$ such that $f_i$ is similar to $f_j$ whenever $i \ne j$ and such that any function $f\colon A \to A$ is similar to one of the functions $f_i$, $1 \le i \le k$.

(b) If $A$ is the set of all real numbers, show that the functions sin and -sin are similar.

9. Let $\{x_0, x_1, x_2, \ldots\}$ be a sequence of real numbers defined by

$$x_0 = \alpha > 1, \qquad x_{n+1}(x_n - [x_n]) = 1, \qquad n = 0,1,2,\ldots .$$

Show that, if the sequence $\{[x_n]\}$ is periodic, then $\alpha$ is a root of a quadratic equation with integral coefficients. Study the converse.

Second Selection Test - 10 April 1978 - 3 hours

1. Consider in the $xy$-plane the infinite network defined by the lines $x = h$ and $y = k$, where $h$ and $k$ range over the set $Z$ of all integers. With each node $(h,k)$ we try to associate an integer $a_{h,k}$ which is the arithmetic mean of the integers associated with the four lattice points nearest to $(h,k)$, that is,

$$a_{h,k} = \tfrac{1}{4}(a_{h-1,k} + a_{h+1,k} + a_{h,k-1} + a_{h,k+1})$$

for any $h,k \in Z$.

(a) Prove that there is a network for which the *nodal numbers* $a_{h,k}$ are not all equal.

(b) Given a network with at least two distinct nodal numbers $a_{h,k}$, show that, for any natural number $n$, there are in the network nodal numbers greater than $n$ and nodal numbers less than $-n$.

2. With $N$ the set of natural numbers, let the function $f\colon N \to N$ be defined by $f(n) = n^2$. Prove that there is a function $F\colon N \to N$ such that $F \circ F = f$.

3. Given in a plane are the $3n$ points $A_i$, $i = 1,2,\ldots,3n$, such that triangle $A_1A_2A_3$ is equilateral and

$$A_{3k+1}, \quad A_{3k+2}, \quad A_{3k+3}, \qquad k = 1,2,\ldots,n-1$$

are the midpoints of the sides of triangle $A_{3k-2}A_{3k-1}A_{3k}$. Each of the $3n$ points is coloured either red or blue.

(a) Show that if $n \geq 7$ there exists at least one monochromatic isosceles trapezoid (i.e., with four vertices of the same colour).

(b) Does the conclusion still hold if $n = 6$?

4. Consider a set $M$ of $3n$ distinct points in a plane such that the maximum distance between any pair of points is $1$. Show that:

(a) for any four points of $M$, there are at least two whose distance apart is at most $1/\sqrt{2}$;

(b) if $n = 2$, for any $\varepsilon > 0$ there is a configuration of the six points such that $12$ of the $15$ distances between them belong to the interval $(1-\varepsilon, 1]$, but there is no configuration such that at least $13$ of the distances belong to the interval $(1/\sqrt{2}, 1]$;

(c) there is a circle of radius at most $\sqrt{3}/2$ containing all $3n$ points of $M$;

(d) there are two points of $M$ whose distance apart is at most $4/(3\sqrt{n} - \sqrt{3})$.

### Third Selection Test - 22 June 1978 - 4 hours

1. Let ABCD be a quadrilateral and A',B' the respective orthogonal projections of A,B on CD.

(a) Assume that $BB' \leq AA'$ and that the area of ABCD equals $\frac{1}{2}(AB + CD) \cdot BB'$. Does this imply that ABCD is a trapezoid?

(b) The same question if $\angle$ BAD is obtuse.

2. On the edges SA, SB, SC of a triangular pyramid S-ABC, points A', B', C', respectively, are chosen in such a way that the planes ABC and A'B'C' intersect in a line $d$. If the plane A'B'C' is now made to rotate about line $d$, show that the lines AA', BB', CC' remain concurrent and find the locus of their point of intersection.

3. Let $D_1$, $D_2$, $D_3$ be three straight lines any of two of which are skew. Through each point $P_2$ of $D_2$, there exists a *common secant* which meets $D_1$ in $P_1$ and $D_3$ in $P_3$.

(a) If coordinate systems are introduced in $D_2$ and $D_3$, with origins $O_2$ and $O_3$, respectively, establish the relation between the abscissas $x_2$ and $x_3$ of $P_2$ and $P_3$ (with respect to $O_2$ and $O_3$).

(b) Show that there exist four straight lines, any two of which are skew and which are not all parallel to the same plane, which have exactly two common secants.

Do the same problem for only *one* secant and for *no* secant.

(c) Let $F_1$, $F_2$, $F_3$, $F_4$ be any four common secants of $D_1$, $D_2$, $D_3$. Show that $F_1$, $F_2$, $F_3$, $F_4$ have infinitely many common secants.

4. For a natural number $n \geq 1$, solve the equation

$$\sin x \sin 2x \ldots \sin nx + \cos x \cos 2x \ldots \cos nx = 1.$$

5. Find the locus of points M in the interior of equilateral triangle ABC such that

$$\angle MBC + \angle MCA + \angle MAB = \frac{\pi}{2}.$$

6. (a) Show that in the set

$$\{x\sqrt{2} + y\sqrt{3} + z\sqrt{5} \mid x, y, z \in Z;\ x^2 + y^2 + z^2 \neq 0\}$$

there are nonzero numbers that are arbitrarily close to zero.

(b) Show that, if $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are replaced by rational approximations $a$, $b$, and $c$, respectively, then the expression

$$|xa + yb + zc|$$

equals zero for infinitely many distinct integer triples $(x, y, z)$ but cannot be made arbitrarily close but not equal to zero.

7. Let $M$ be a set of points, no three collinear, in a Cartesian plane (rectangular axes), and consider the following assertion:

(A) *The barycentre of any finite subset of $M$ has integral coordinates.*

Prove that:

(a) for every $n \geq 1$, there exists a set $M$ of $n$ points for which assertion (A) is true;

(b) assertion (A) is false if $M$ is an infinite set.

8. Solve the following problem, first reformulating it in set-theoretic language:

A certain number of boys and girls are at a party, and it is known which boys are acquainted with which girls. This acquaintance relationship is such that, for any subset $M$ of the boys, the subset of the girls acquainted with at least one boy of $M$ is at least as large as $M$. Prove that, simultaneously, every boy can dance with a girl of his acquaintance.

Fourth Selection Test - 24 June 1978 - 4 hours

1. Show that, for any natural number $a \geq 3$, there are infinitely many natural

numbers $n$ such that $a^n - 1$ is divisible by $n$. Does the same property hold if $a = 2$?

2. A function $f\colon \{x_1, x_2, \ldots, x_k\} \to R$, defined on a finite set of real numbers, is said to be *additive* if, for any integers $n_1, n_2, \ldots, n_k$ such that

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k = 0,$$

it is true that

$$n_1f(x_1) + n_2f(x_2) + \ldots + n_kf(x_k) = 0.$$

Show that, for any such function $f$ and for any real numbers $y_1, y_2, \ldots, y_p$, there is an additive function

$$F\colon \{x_1, \ldots, x_k, y_1, \ldots, y_p\} \to R$$

such that $F(x_i) = f(x_i)$ for $i = 1, 2, \ldots, k$.

3. Let $M$ be a set (of $|M|$ elements) and let

$$\{A_1,\ A_2,\ \ldots,\ A_p\} \qquad \text{and} \qquad \{B_1,\ B_2,\ \ldots,\ B_p\}$$

be two partitions of $M$ such that.

$$A_i \cap B_j = \emptyset \Rightarrow |A_i| + |B_j| \geq p, \qquad\qquad 1 \leq i, j \leq p.$$

Show that $|M| \geq \frac{1}{2}(p^2+1)$. Can the equality hold?

4. Consider a set $M$ of $n$ points in a plane, no three of which are collinear. With any segment whose endpoints are in $M$ is associated one of the numbers 1, -1; and a triangle with vertices in $M$ is called *negative* if the product of the numbers associated with its sides is -1.

If the number -1 is associated with $p$ of the segments and if $n$ is even [resp. odd], then the number of negative triangles is even [resp. of the same parity as $p$].

5. Given a fixed triangle, determine the set of all interior points M such that there is a straight line $d$ passing through M which divides the triangle into two regions, the symmetric of one region with respect to $d$ being included in the other region.

6. Can 20 regular tetrahedra of edge 1 be placed in a sphere of radius 1 in such a way that no two of them have interior points in common?

*Editor's note.* All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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# PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before September 1, 1981, although solutions received after that date will also be considered until the time when a solution is published.

**612*** *Proposed by G.C. Giri, Midnapore College, West Bengal, India.*

(a) A sequence $\{x_n\}$ has the $n$th term

$$x_n = \sum_{j=1}^{(n-1)^2} \frac{1}{\sqrt{n^2-j}}, \qquad n = 2,3,4,\ldots .$$

Does the sequence converge? If so, to what limit?

(This problem was reported to me by students of my college as having been set in a Public Examination.)

(b) Do the same problem with the $j$ under the square root replaced by $j^2$.

**613.** *Proposed by Jack Garfunkel, Flushing, N.Y.*

If $A+B+C = 180^\circ$, prove that

$$\cos\tfrac{1}{2}(B-C) + \cos\tfrac{1}{2}(C-A) + \cos\tfrac{1}{2}(A-B) \geq \frac{2}{\sqrt{3}}(\sin A + \sin B + \sin C).$$

(Here A, B, C are not necessarily the angles of a triangle, but you may assume that they are if it is helpful to achieve a proof without calculus.)

**614.** *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Given is a triangle with sides of lengths $a,b,c$. A point P moves inside the triangle in such a way that the sum of the squares of its distances to the three vertices is a constant $(=k^2)$. Find the locus of P.

**615.** *Proposed by G.P. Henderson, Campbellcroft, Ontario.*

Let $P$ be a convex $n$-gon with vertices $E_1, E_2, \ldots, E_n$, perimeter $L$ and area $A$. Let $2\theta_i$ be the measure of the interior angle at vertex $E_i$ and set $C = \Sigma\cot\theta_i$. Prove that

$$L^2 - 4AC \geq 0$$

and characterize the convex $n$-gons for which equality holds.

616. *Proposed by Alan Wayne, Holiday, Florida.*

Find all solutions $(x,y)$, where $x$ and $y$ are nonconsecutive positive integers, of the equation

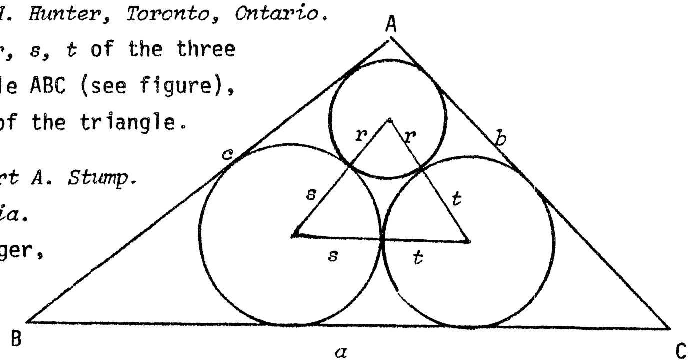
$$x^2 + 10^{340} + 1 = y^2 + 10^{317} + 10^{23}.$$

617. *Proposed by Charles W. Trigg, San Diego, California.*

The sum of two positive integers is 5432 and their least common multiple is 223020. Find the numbers.

618. *Proposed by J.A.H. Hunter, Toronto, Ontario.*

Given the radii $r$, $s$, $t$ of the three Malfatti circles of a triangle ABC (see figure), calculate the sides $a$, $b$, $c$ of the triangle.




619. *Proposed by Robert A. Stump. Hopewell, Virginia.*

If $k$ is a positive integer, find the value of

$$\sum_{i=1}^{\infty}\frac{1}{i(i+k)},$$

620. *Proposed by Fred A. Miller, Elkins, West Virginia.*

Using the digits 0 and 1, express each of the following in the negabinary system of notation (base -2):

$$\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ -\frac{1}{2},\ -\frac{1}{3},\ -\frac{1}{4},\ -\frac{1}{5},\ -\frac{1}{6}.$$

621. *Proposed by Herman Nyon, Paramaribo, Surinam.*

For the adjoining alphametic, there is unfortunately no solution in which SQUARE is a square, but there is one in which the digital sum of SQUARE is, very appropriately, the square

```
 THREE
 THREE
 THREE
 EIGHT
 EIGHT
SQUARE
```

$$3 + 3 + 3 + 8 + 8 = 25.$$

Find this solution.

*　　　　*　　　　*

MAMA-THEMATICS

Mother, about son Pythagoras: "At first he had trouble with his arithmetic and geometry, but once he got past 3, 4, 5, he went to town."

CHARLES W. TRIGG

# SOLUTIONS

*No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.*

521. [1980: 77] *Proposed by Sidney Kravitz, Dover, New Jersey.*

No visas are needed for this alphametical Cook's tour:

$$\begin{array}{r} \text{SPAIN} \\ \underline{\text{SWEDEN}} \\ \text{POLAND} \end{array}.$$

*Solution by Charles W. Trigg, San Diego, California.*

From the units' column D is even, and from the hundreds' column $D = 0$ or $9$. Hence $D = 0$ and $N = 5$. Now the tens' column gives $I + E = 4$, so that $\{I,E\} = \{1,3\}$; and the last column gives $S + 1 = P$, from which $(S,P) = (6,7)$, $(7,8)$, or $(8,9)$. It is easy to verify that

$$(I,E,S,P) = \begin{cases} (3,1,6,7) \Rightarrow L = 8 \text{ and no solution for } 2,\ 4,\ 9; \\ (3,1,7,8) \Rightarrow L = 9 \text{ and no solution for } 2,\ 4,\ 6; \\ (3,1,8,9) \Rightarrow L = 0 = D; \\ (1,3,6,7) \Rightarrow L = 0 = D; \\ (1,3,7,8) \Rightarrow L = 1 = I. \end{cases}$$

Hence we must have $(I,E,S,P) = (1,3,8,9)$ from which $L = 2$, $W = 7$, $O = 6$, $A = 4$, and we have the unique solution

$$\begin{array}{r} 89415 \\ \underline{873035} \\ 962450 \end{array}.$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; EDGAR LACHANCE, Ottawa, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; HYMAN ROSEN, student, The Cooper Union, New York, N.Y.; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka; Kansas; and the proposer.

* * *

522. [1980: 77] *Proposed by William A. McWorter, Jr. and Leroy F. Meyers, The Ohio State University.*

A recent visitor to our department challenged us to prove the following result, which is not new. We pass on the challenge.

Prove that

$$\binom{a}{b} \equiv \binom{a_1}{b_1}\binom{a_2}{b_2}\cdots\binom{a_m}{b_m} \pmod{p},$$

where $p$ is a prime and the nonnegative integers $a$ and $b$ have base-$p$ representations $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_m$, respectively, with initial zeros permitted.

The standard conventions for binomial coefficients, namely

$$\binom{r}{0} = \binom{r}{r} = 1 \quad \text{if} \quad r \geq 0 \quad \text{and} \quad \binom{r}{s} = 0 \quad \text{if} \quad r < s,$$

are assumed.

*Solution by the proposers.*

The result is trivial if $m = 1$. Suppose, then, that the result has been proved for $(m-1)$-digit representations. Let $a = rp + \alpha$ and $b = sp + \beta$ have $m$-digit representations, where $r$ and $s$ have $(m-1)$-digit representations and $0 \leq \alpha < p$ and $0 \leq \beta < p$. We wish to deduce that

$$\binom{a}{b} \equiv \binom{r}{s}\binom{\alpha}{\beta} \pmod{p}.$$

The congruence obviously holds if $a < b$, for then either $r < s$ or $\alpha < \beta$, and so $\binom{a}{b}$ and at least one of $\binom{r}{s}$ and $\binom{\alpha}{\beta}$ are zero. We now assume that $a \geq b$. Then $r \geq s$. Now, by definition

$$\binom{a}{b} = \frac{a(a-1)\ldots(a-b+1)}{b(b-1)\ldots\ 3\cdot 2\cdot 1}\,.$$

We partition the factors in the numerator and denominator into $s$ blocks of $p$ consecutive integers, from the left, together with a "remainder" block of $\beta$ consecutive integers. The $j$th full block in the numerator then contains the number $(r-j+1)p$; the other numbers in the block form a reduced residue system modulo $p$, and so their product is congruent to $-1$ modulo $p$, by Wilson's Theorem. Similarly, the $j$th full block in the denominator contains $(s-j+1)p$, together with a reduced residue system modulo $p$. After dividing the product of each full block by $-p$ and reducing modulo $p$, we find that

$$\binom{a}{b} = \frac{r(r-1)\ldots(r-s+1)}{s(s-1)\ldots\quad 1}\cdot\frac{((r-s)p+\alpha)\ldots((r-s)p+\alpha-\beta+1)}{\beta\quad\ldots\quad 1} \equiv \binom{r}{s}\binom{\alpha}{\beta} \pmod{p},$$

since $\beta!$ is relatively prime to $p$.

*Special cases.* If $\beta = 0$, then there is no "remainder" block, and so

$$\binom{a}{b} \equiv \binom{r}{s}\cdot 1 \equiv \binom{r}{s}\binom{\alpha}{\beta} \pmod{p}$$

anyway. If $s = 0$, then there is *only* the "remainder" block, and

$$\binom{a}{b} \equiv 1 \cdot \binom{\alpha}{\beta} \equiv \binom{r}{s}\binom{\alpha}{\beta} \pmod{p}$$

also, whether or not $\beta = 0$. □

The result is stated and illustrated in Lucas [1]. But we are not convinced that a proof is given there. Lucas makes an obviously incorrect reference to Augustin Cauchy, "Mémoire sur la théorie des nombres, présenté à l'Académie des Sciences le 31 mai 1830" [2]. A proof similar to our own is found in Glaisher [3]. A proof is also found in Fine [4], and the result is stated (not proved) and then generalized in Roberts [5]. A recent exposition of a special case ($a_m = b_m = 0$) is found in Hillman et al. [6]. A related problem appeared earlier in this journal [1975: 85; 1976: 34].

*Editor's comment.*

In a subsequent letter, the second proposer (Meyers) noted that our problem is given (and proved) as a lemma in J.G. Mauldon's solution to a problem in the *American Mathematical Monthly* [7].

Our problem is also given (without proof) in [8] in a comment following the solution of a problem in the 1956 Putnam Competition.

REFERENCES

1. Édouard Lucas, *Théorie des nombres,* t. 1 (reprint), Paris, 1961, pp. 417-420, with further remarks on pp. 503-505.

2. *Mémoires de l'Académie des Sciences,* (2) 17 (1840), pp. 249-768; reprinted in Cauchy's *Oeuvres complètes,* (1) 3 (entire volume), Paris, 1911.

3. J.W.L. Glaisher, "On the residue of a binomial-theorem coefficient with respect to a prime modulus," *Quarterly Journal of Mathematics,* 30 (1899) 150-156.

4. N.J. Fine, "Binomial coefficients modulo a prime," *American Mathematical Monthly,* 54 (1947) 589-592, especially Theorem 1.

5. J.B. Roberts, "On binomial coefficient residues," *Canadian Journal of Mathematics,* 9 (1957) 363-370.

6. A.P. Hillman, G.L. Alexanderson, and L.F. Klosinski, "The William Lowell Putnam Mathematical Competition [for 1977]," *American Mathematical Monthly,* 86 (1979) 171, 173.

7. J.G. Mauldon, Solution to Problem E 2775, *American Mathematical Monthly,* 87 (1980) 578-579.

8. A.M. Gleason, R.E. Greenwood, and L.M. Kelly, *The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938-1964,* Mathematical Association of America, 1980, p. 425.

* * *

523. [1980: 77] *Proposed by James Gary Propp, student, Harvard College, Cambridge, Massachusetts.*

Find all zero-free decimal numbers $N$ such that both $N$ and $N^2$ are palindromes.

*Solution by Gali Salvatore, Perkins, Québec.*

If

$$N = a_0 + a_1x + \dots + a_nx^n + \dots,$$

then, formally,

$$N^2 = A_0 + A_1x + \dots + A_{2n}x^{2n} + \dots,$$

where

$$A_i = \sum_{j+k=i} a_ja_k, \qquad i = 0,1,\dots,2n,\dots .$$

Suppose now that $x$ is a natural number and that $N$ is an $(n+1)$-digit zero-free palindrome in base $x$, that is, that $0 < a_i = a_{n-i} < x$ for $i = 0,1,\dots, n$ and that $a_{n+1} = a_{n+2} = \dots = 0$. Then we also have $A_i = A_{2n-i} > 0$ for $i = 0,1,\dots,n$ and $A_{2n+1} = A_{2n+2} = \dots = 0$, and $N^2$ will be a ($(2n+1)$-digit) palindrome in base $x$ if and only if $A_i < x$ for $i = 0,1,\dots,n$. We will now assume that the base is $x = 10$, as required by our problem, but the method we use is applicable to all bases. We will use only necessary conditions to find the possible solutions. That these conditions are also sufficient can be verified later by actually squaring the possible solutions obtained.

$n = 0 \Rightarrow A_0 = a_0^2 < 10 \Rightarrow a_0 = 1, 2, 3 \Rightarrow N = 1, 2, 3.$

$n = 1 \Rightarrow A_1 = 2a_0^2 < 10 \Rightarrow a_0 = 1, 2 \Rightarrow N = 11, 22.$

$n = 2 \Rightarrow A_2 = 2a_0^2 + a_1^2 < 10 \Rightarrow (a_0,a_1) = (1,1), (1,2), (2,1)$
$\Rightarrow N = 111, 121, 212.$

$n = 3 \Rightarrow A_3 = 2(a_0^2+a_1^2) < 10 \Rightarrow (a_0,a_1) = (1,1) \Rightarrow N = 1111.$

$n = 4 \Rightarrow A_4 = 2(a_0^2+a_1^2) + a_2^2 < 10 \Rightarrow (a_0,a_1,a_2) = (1,1,1), (1,1,2)$
$\Rightarrow N = 11111, 11211.$

$n = 5 \Rightarrow A_5 = 2(a_0^2+a_1^2+a_2^2) < 10 \Rightarrow (a_0,a_1,a_2) = (1,1,1) \Rightarrow N = 111111.$

$n = 6 \Rightarrow A_6 = 2(a_0^2+a_1^2+a_2^2) + a_3^2 < 10 \Rightarrow (a_0,a_1,a_2,a_3) = (1,1,1,1)$
$\Rightarrow N = 1111111.$

$n = 7 \Rightarrow A_7 = 2(a_0^2+a_1^2+a_2^2+a_3^2) < 10 \Rightarrow (a_0,a_1,a_2,a_3) = (1,1,1,1)$
$\Rightarrow N = 11111111.$

$n = 8 \Rightarrow A_8 = 2(a_0^2+a_1^2+a_2^2+a_3^2) + a_4^2 < 10 \Rightarrow (a_0,a_1,a_2,a_3,a_4) = (1,1,1,1,1)$
$\Rightarrow N = 111111111.$

$n > 9 \Rightarrow A_n =$ (sum of 10 or more terms) $< 10 \Rightarrow$ no solution.

We end by verifying that the 15 values of $N$ we have obtained are in fact solutions, that is, that $N^2$ is a palindrome for each palindromic $N$. This is done in the following table, where $N \to N^2$.

| | |
|---|---|
| $1 \to 1$ | $1111 \to 1234321$ |
| $2 \to 4$ | $11111 \to 123454321$ |
| $3 \to 9$ | $11211 \to 125686521$ |
| $11 \to 121$ | $111111 \to 12345654321$ |
| $22 \to 484$ | $1111111 \to 1234567654321$ |
| $111 \to 12321$ | $11111111 \to 123456787654321$ |
| $121 \to 14641$ | $111111111 \to 12345678987654321$ |
| $212 \to 44944$ | |

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; HERMAN NYON, Paramaribo, Surinam; BOB PRIELIPP, University of Wisconsin-Oshkosh; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; CHARLES W. TRIGG, San Diego, California; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

*Editor's comment.*

Most of the other solutions received were unsatisfactory to some extent, although none was so bad as to deserve the epithet "incorrect".

Prielipp and Trigg referred to Simmons [1], who has tabulated the 55 palindromic squares less than $2.5 \times 10^{13}$ in the decimal system, and has indicated the 39 cases where both $N$ and $N^2$ are palindromes. Among those are 13 cases where $N$ is zero-free. Such a tabulation provides valuable information but little real understanding.

REFERENCE

1. Gustavus J Simmons, "Palindromic Powers", *Journal of Recreational Mathematics*, 3 (April 1970) 93-98.

* * *

**524.** [1980: 78] *Proposed by Dan Pedoe, University of Minnesota.*

Disproving a "theorem" can be as difficult as proving a theorem. One of my students is very keen to extend the Pascal Theorem to two equal circles, and has come up with the following "theorem", which you are asked to disprove:

*$\gamma$ and $\delta$ are two equal circles.* A,B,C *are distinct points on $\gamma$, and* A',B',C' *are distinct points on $\delta$, with* AA',BB',CC' *concurrent at a point* V. *Show that the three intersections*

$$BC' \cap B'C, \quad CA' \cap C'A, \quad AB' \cap A'B$$

*are collinear, so that the Pascal Theorem holds for the hexagon* AB'CA'BC'.

*Solution by the proposer.*

If the Pascal Theorem holds for the hexagon AB'CA'BC', then these six points lie on a conic $S$. Let VA intersect circle $\gamma$ again in A*. Then, since the theorem must hold for the hexagon A*B'CA'BC', these six points must lie on a conic. But a conic is uniquely determined by five points, so that this second conic coincides with the first conic $S$. Hence this conic $S$ intersects the circle $\gamma$ in the points A, B, C, and A*; and similarly, if VB intersects the circle $\gamma$ again in B*, the conic $S$ also passes through B*, and therefore coincides with the circle $\gamma$. Similarly $S$ coincides with the circle $\delta$. But circles $\gamma$ and $\delta$, although they are equal by hypothesis, are not the same circle. The theorem is therefore untrue.

Also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; BRUCE KING, Western Connecticut State College; and NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.

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525. [1980: 78] *Proposed by G.C. Giri, Midnapore College, West Gengal, India.*

Eliminate $\alpha$, $\beta$, and $\gamma$ from

$$\cos\alpha + \cos\beta + \cos\gamma = a$$

$$\sin\alpha + \sin\beta + \sin\gamma = b$$

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = c$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = d.$$

*Solution by Viktors Linis, University of Ottawa.*

Let $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$; then the given system is equivalent to

$$a + bi = x + y + z \tag{1}$$

$$c + di = x^2 + y^2 + z^2 \tag{2}$$

or to the system obtained by taking complex conjugates

$$a - bi = x^{-1} + y^{-1} + z^{-1} \tag{3}$$

$$c - di = x^{-2} + y^{-2} + z^{-2}. \tag{4}$$

Subtracting (2) from the square of (1), we get

$$(a^2-b^2-c) + (2ab-d)i = 2(yz+zx+xy); \tag{5}$$

and proceeding likewise with (3) and (4) gives

$$(a^2-b^2-c) - (2ab-d)i = 2(y^{-1}z^{-1}+z^{-1}x^{-1}+x^{-1}y^{-1}). \tag{6}$$

Finally, observing that

$$(yz+zx+xy)(y^{-1}z^{-1}+z^{-1}x^{-1}+x^{-1}y^{-1}) = (x+y+z)(x^{-1}+y^{-1}+z^{-1}) = (a+bi)(a-bi),$$

multiplication of (5) and (6) yields the required result:

$$(a^2-b^2-c)^2 + (2ab-d)^2 = 4(a^2+b^2). \tag{7}$$

**Also** solved by W.J. BLUNDON, Memorial University of Newfoundland; J.T. GROENMAN, Arnhem, The Netherlands; ROLF ROSE, Magglingen-Macolin, Switzerland; SANJIB KUMAR ROY, Indian Institute of Technology, Kharagpur, India; and the proposer.

*Editor's comment.*

This problem appears, with two complete solutions, in Briggs and Bryan [1], who stated that they had found it in a College Scholarship paper of the University of Cambridge. Their first solution is an unappetizing mishmash of trigonometric identities. Their second solution, by complex numbers, is more or less equivalent to our featured solution, but it uses complex exponentials, which makes it (at least typographically) less attractive.

The *eliminant* of a system of equations is a necessary and sufficient condition for the equations of the system to have a common solution. What has been shown here (no other solver, or even [1], did any more) is that if the equations of the given system have a common solution $(\alpha,\beta,\gamma)$, then $(a,b,c,d)$ satisfies (7). A proof of the converse would be required before (7) can be called the eliminant of the system. This converse is usually difficult to prove even for algebraic systems, so there is little hope (is there, readers?) that it can be shown for the transcendental system we have here.

REFERENCE

1. William Briggs and G.H. Bryan, *The Tutorial Algebra*, Vol. II (Seventh Edition revised and rewritten by George Walker), University Tutorial Press, London, 1960, pp. 597-598.

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526. [1980: 78] *Proposed by Bob Prielipp, The University of Wisconsin-Oshkosh.*

The following are examples of *chains* of lengths 4 and 5, respectively:

25, 225, 1225, 81225

25, 625, 5625, 75625, 275625.

In each chain, each link is a perfect square, and each link (after the first) is obtained by prefixing a single digit to its predecessor.

Are there chains of length $n$ for $n = 6,7,8,\ldots$?

*Partial solution by Friend H. Kierstead, Jr., Cuyahoga Falls, Ohio.*

It appears to have been the proposer's unstated intention to restrict consideration to chains whose links contain no initial or final zeros. We shall in any case adhere to this restriction, since it eliminates many trivial and uninteresting solutions.

A computer study of all squares less than $10^{14}$ reveals that:

(1) All links end in 25, except in the 2-link chains (1, 81), (4, 64), and (9, 49).

(2) There are no chains of length greater than 5.

(3) The only chain of length 5 is that given by the proposer:

(25, 625, 5625, 75625, 275625).

(4) There are only four chains of maximal length 4 (i.e., that are not subchains of longer chains):

(25, 225, 1225, 81225),

(25, 225, 4225, 34225),

(25, 225, 7225, 27225),

(25, 625, 5625, 15625).

(5) There are only three chains of maximal length 3:

( 3025, 93025, 893025),

(30625, 330625, 3330625),

(50625, 950625, 4950625).

On the basis of (2) above, it is tempting to conjecture that the answer to our problem is NO, but perhaps it would be wise for someone to first extend the search to larger numbers, with a computer more powerful than the one we used.

Partial solutions were also received from LEROY F. MEYERS, The Ohio State University; CHARLES W. TRIGG, San Diego, California; and KENNETH M. WILKE, Topeka, Kansas.

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527. [1980: 78] *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

(a) You stand at a corner in a large city of congruent square blocks and intend to take a walk. You flip a coin - tails, you go left; heads, you go right - and you repeat the procedure at each corner you reach. What is the probability that you will end up at your starting point after walking $n$ blocks?

(b) Same question, except that you flip the coin twice: TT, you go left; HH, you go right; otherwise, you go straight ahead.

*Solution of part* (a) *by the proposer (revised by the editor).*

We assume that the streets run north-south and east-west, and that you (the walker) are initially standing at a corner and facing north. Each $n$-block walk can be characterized by an ordered $n$-tuple

$$(t_1, t_2, \ldots, t_n), \tag{1}$$

where $t_i = 1$ if the $i$th flip of the coin (assumed fair) causes you to walk one block east or north, and $t_i = -1$ if it causes you to walk one block west or south. Let

$$O_n = \sum_{i \text{ odd}} t_i \qquad \text{and} \qquad E_n = \sum_{i \text{ even}} t_i.$$

The condition "end up at starting point" is then equivalent to

$$O_n = 0 \qquad \text{and} \qquad E_n = 0, \tag{2}$$

since $O_n$ and $E_n$ measure the net number of blocks walked east-west and north-south, respectively. Suppose we have an $n$-block walk for which $O_n$ contains $k$ positive summands. Then (2) is realized if and only if all of the following are true:

(i) $O_n$ contains $k$ negative summands;

(ii) $E_n$ also contains $k$ positive and $k$ negative summands;

(iii) $n = 4k$.

Since there are $2^n = 2^{4k}$ possible $n$-tuples (1), of which $\binom{2k}{k}^2$ satisfy (i) and (ii), the required probability is

$$P(n) = \begin{cases} \binom{2k}{k}^2/2^{4k}, & \text{if } n = 4k, \\ 0, & \text{if } 4 \nmid n. \end{cases}$$

Part (a) was also solved by JORDI DOU, Escola Tecnica Superior Arquitectura de Barcelona, Spain; and one incorrect solution was received.

*Editor's comment.*

Part (b) remains open. Readers may wish to try instead the following more symmetrical variant of (b) which, just possibly, may be easier to solve: HH, you go right; TT, you go left; HT, you go straight ahead; TH, you go back the way you came. (This variant was in fact suggested by our incorrect solver.)

The proposer wrote that the problem was suggested to him by the editor of *Games* magazine who recounted, in his May-June 1979 issue, how he and a friend would take a walk and, upon reaching each intersection, flip a coin to determine whether to turn left or right. Modifications of the game included the use of a second coin to consider continuing on without turning.

* * *

528. [1980: 78] *Proposed by Kenneth S. Williams, Carleton University, Ottawa.*

Let $k \geq 2$ be a fixed integer. Prove that $\log k$ is the sum of the infinite series

$$1 + \tfrac{1}{2} + \ldots + \frac{1}{k-1} - \frac{k-1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{2k-1} - \frac{k-1}{2k} + \frac{1}{2k+1} + \ldots + \frac{1}{3k-1} - \frac{k-1}{3k} + \ldots .$$

*Solution by the proposer.*

Let $s_n$ be the sum of the first $n$ terms of the given series. For each $n \geq 1$, we have uniquely

$$n = kq_n + r_n, \qquad 0 \leq r_n < k;$$

and since $-(k-1)/tk = 1/tk - 1/t$ for $t = 1,2,\ldots,q_n$, it follows that

$$s_n = (1 + \tfrac{1}{2} + \ldots + \frac{1}{n}) - (1 + \tfrac{1}{2} + \ldots + \frac{1}{q_n}). \tag{1}$$

Now $n = q_n(k + r_n/q_n)$, so $\log n = \log q_n + \log(k + r_n/q_n)$, and (1) can be rewritten as

$$s_n = (1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n) - (1 + \tfrac{1}{2} + \ldots + \frac{1}{q_n} - \log q_n) + \log(k + r_n/q_n)$$

$$\equiv u_n - v_n + w_n.$$

The sequence $\{u_n\}$ converges to $\gamma$, Euler's constant; the sequence $\{v_n\}$ also converges to $\gamma$ since $n \to \infty \Rightarrow q_n \to \infty$; and the sequence $\{w_n\}$ converges to $\log k$ since $r_n$ is bounded. Hence the sequence $\{s_n\}$ converges to $\gamma - \gamma + \log k$, which proves that the given series converges to $\log k$.

Also solved by LEROY F. MEYERS, The Ohio State University. Incomplete solutions were received from G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; J.D. HISCOCKS, University of Lethbridge, Alberta; V.N. MURTY, Pennsylvania State University, Capitol Campus; SANJIB KUMAR ROY, Research Scholar, Indian Institute of Technology, Kharagpur, India; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

*Editor's comment.*

Our incomplete solvers contented themselves with proving that the sequence $\{s_{nk}\}$ converged to $\log k$. This is undoubtedly true but, in the absence of *a priori* knowledge that the series converges, more than this is needed to force the conclusion that the sequence $\{s_n\}$ also converges to $\log k$. For example, for the series

$$-1 + 1 - 1 + \ldots + (-1)^n + \ldots,$$

the sequence of partial sums $\{s_{2n}\}$ converges to 0, but the sequence $\{s_n\}$ does not converge at all.

The proposer wrote that the series in this problem arose in some research he was doing on class numbers of quadratic fields.

* * *

529. [1980: 79] *Proposed by J.T. Groenman, Groningen, The Netherlands.*

The sides of a triangle ABC satisfy $a \le b \le c$. With the usual notation $r$, $R$, and $r_c$ for the in-, circum, and ex-radii, prove that

$$\operatorname{sgn}(2r + 2R - a - b) = \operatorname{sgn}(2r_c - 2R - a - b) = \operatorname{sgn}(C - 90^\circ).$$

*Solution by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India, (revised by the editor).*

We shall for typographical convenience set $A = 2\alpha$, $B = 2\beta$, $C = 2\gamma$, so that $\alpha \le \beta \le \gamma$. From the well-known formula $r/R = 4\sin\alpha\sin\beta\sin\gamma$, we have

$$\begin{aligned}
F &\equiv \frac{1}{2R}(2r + 2R - a - b)\\
&= 4\sin\alpha\sin\beta\sin\gamma + 1 - \sin 2\alpha - \sin 2\beta\\
&= 4\sin\gamma\sin\alpha\cos(\gamma+\alpha) + 1 - \sin(2\gamma+2\alpha) - \sin 2\alpha\\
&= 4\sin\gamma\sin\alpha(\cos\gamma\cos\alpha - \sin\gamma\sin\alpha) + 1 - 2\sin(\gamma+2\alpha)\cos\gamma\\
&= \sin 2\gamma\sin 2\alpha - (1-\cos 2\gamma)(1-\cos 2\alpha) + 1 - 2\sin(\gamma+2\alpha)\cos\gamma\\
&= \cos 2\gamma + \cos 2\alpha - \cos(2\gamma+2\alpha) - 2\sin(\gamma+2\alpha)\cos\gamma\\
&= \cos 2\gamma + 2\sin(\gamma+2\alpha)\sin\gamma - 2\sin(\gamma+2\alpha)\cos\gamma\\
&= (\cos\gamma - \sin\gamma)(\cos\gamma + \sin\gamma - 2\sin(\gamma+2\alpha))\\
&= \cos\gamma(\tan\gamma - 1)(2\sin(\gamma+2\alpha) - \cos\gamma - \sin\gamma)\\
&= \cos\gamma(1+\tan\gamma)\tan(\gamma - 45^\circ)(2\sin(\gamma+2\alpha) - \cos\gamma - \sin\gamma). \qquad (1)
\end{aligned}$$

Now $\gamma + 2\alpha = 90^\circ - (\beta-\alpha)$ and $0 \le \beta-\alpha < \gamma < 90^\circ$, so

$$\sin(\gamma+2\alpha) = \cos(\beta-\alpha) > \cos\gamma\,; \qquad (2)$$

and $\gamma < \gamma + 2\alpha \le 90^\circ$, so

$$\sin(\gamma+2\alpha) > \sin\gamma\,. \qquad (3)$$

It follows from (2) and (3) that the last factor in (1) is strictly positive, so the sign of (1) is that of its third factor. Hence

$$\operatorname{sgn}(2r+2R-a-b) = \operatorname{sgn} F = \operatorname{sgn}(\gamma - 45^\circ) = \operatorname{sgn}(C - 90^\circ).$$

To prove the second equality, we use $r_c/R = 4\cos\alpha\cos\beta\sin\gamma$. (We omit the details since the development is step by step analogous to the preceding one.) We find that

$$\begin{aligned}
F_c &\equiv \frac{1}{2R}(2r_c - 2R - a - b)\\
&= \cos\gamma(1+\tan\gamma)\tan(\gamma-45^\circ)(2\sin(\gamma+2\alpha) + \cos\gamma + \sin\gamma),
\end{aligned}$$

from which follows immediately

$$\operatorname{sgn}(2r_c-2R-a-b) = \operatorname{sgn} F_c = \operatorname{sgn}(\gamma-45^\circ) = \operatorname{sgn}(C-90^\circ).$$

Also solved by NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; and the proposer.

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530. [1980: 79] *Proposed by Ferrell Wheeler, student, Forest Park H.S., Beaumont, Texas.*

Let $A = (a_n)$ be a sequence of positive integers such that $a_0$ is any positive integer and, for $n \geq 0$, $a_{n+1}$ is the sum of the cubes of the decimal digits of $a_n$. Prove or disprove that $A$ converges to 153 if and only if 3 is a proper divisor of $a_0$.

Solutions were received from CLAYTON W. DODGE, University of Maine at Orono; MICHAEL W. ECKER, Pennsylvania State University, Worthington Scranton Campus; ERNEST W. FOX, Marianopolis College, Montréal, Québec; ALLAN WM. JOHNSON JR., Washington, D.C.; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; BOB PRIELIPP, University of Wisconsin-Oshkosh; FRANCISCO HERRERO RUIZ, Madrid, Spain; CHARLES W. TRIGG, San Diego, California; and KENNETH M. WILKE, Topeka, Kansas. Solution partielle de ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec. A comment was received from M.S. KLAMKIN, University of Alberta.

*Editor's comment.*

There is an extensive literature on this problem and related ones, and proofs can be found in most of the references given below, several of which were sent in by readers (and these, in turn, yielded others). All problems stem from the following two theorems, which are stated and proved in Sierpiński [1]:

THEOREM 1. *For fixed natural numbers $g$ and $s$, let $a_0$ be a natural number written in the scale of $g$ and, for $i = 1,2,3,\ldots$, let $a_i$ denote the sum of the $s$th powers of the digits in the scale of $g$ of the natural number $a_{i-1}$. Then, for any natural number $a_0$, the infinite sequence*

$$A = (a_0,\ a_1,\ a_2,\ \ldots) \qquad (*)$$

*is periodic.*

THEOREM 2. *The period of sequence (*) may begin arbitrarily far.*

The period of sequence (*) is a finite subsequence

$$P(a_0) = (a_j,\ a_{j+1},\ \ldots,\ a_{j+l-1}),$$

which depends upon $a_0$, in which the terms are all distinct and $a_{j+l} = a_j$. It is clear that sequence (*) converges if and only if $P(a_0)$ consists of a single term. From now on in this comment, we will be concerned only with the scale $g = 10$.

Sierpiński's theorems were the culmination of earlier work done by others. The case $s = 2$ had been investigated by Porges [2], who showed that, for any $a_0$, the sequence (*) either converges to 1 (e.g., $P(7) = (1)$) or else diverges with the 8-number period

$$(4, 16, 37, 58, 89, 145, 42, 20).$$

This result was later restated and proved by Steinhaus [3], and Stewart [4] generalized it in several directions.

The case $s = 3$ is the one we are concerned with in this problem. This was first investigated by Iseki ([5] and [6]) who showed that there are 9 possible periods for the sequence (*), 5 of which are 1-number periods (so the sequence converges). Specifically, if $a_0 \equiv 0 \pmod 3$, the sequence converges to 153. This settles the question in our problem (and also shows that the word "proper" in the proposal is *de trop*). If $a_0 \equiv 1 \pmod 3$, the sequence converges to 1 or to 370 or diverges with one of the periods

$$(136, 244), \quad (919, 1459), \quad (55, 250, 133), \quad (160, 217, 352).$$

If $a_0 \equiv 2 \pmod 3$, the sequence converges to 371 or to 407.

For the case $s = 4$, Chikawa et al. [7] showed that the sequence (*) converges to 1, to 1634, to 8208, or to 9474, or else diverges with one of the periods

$$(2178, 6514), \quad (13139, 6725, 4338, 4514, 1138, 4179, 9219).$$

For the case $s = 5$, see Chikawa et al. [8].

Later Harvey and Wetzel [9] asked again to identify the 9 possible periods for $s = 3$, and solutions by Hoffman [10] and Cole [11] were published in due course. In January 1967, C.R.J. Singleton asked again for a proof that, in the case $s = 3$, the sequence (*) converges to 153 when $a_0$ is a multiple of 3, and a solution by Prielipp appeared in [12]. Recently Mohanty and Kumar [13]. after recalling Iseki's results for the case $s = 3$, reversed the calculation and studied powers of sums of digits instead of sums of powers of digits. Then Feser [14], after recalling the narcissistic property of the by now celebrated number 153, investigated not only powers of sums and sums of powers of digits, but also factorials of sums and sums of factorials of digits. And just a few months ago, Hintz [15] gave a complete discussion and proof of the known results for the case $s = 3$.

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531. [1980: 112] *Proposé par Allan Wm. Johnson Jr., Washington, D.C.*

Résoudre la cryptarithmie multiplicative suivante:

```
  CINQ
   SIX
 -----
 ----6
 ---9
------
TRENTE
```
.

*Solution by Charles W. Trigg, San Diego, California.*

From the partial products, Q and S are odd, I is even, and S < I; so S = 1, Q = 9, and I = 4. Then {X,E} = {2,8} or {3,7}. The last three digits of the product are determined for these values of X,E and nonduplicating values of N. Only when X = 2 and N = 0 are the N's in the multiplicand and product the same. For these values, E = 8 and T = 7. Finally, C = 5, and the unique reconstruction of the multiplication is

```
  5409
   142
 10818
21636
5409     .
768078
```

Also solved by JOHN T. BARSBY, St. John's-Ravenscourt School, Winnipeg, Manitoba; N. ESWARAN, Indian Institute of Technology, Kharagpur, India; J.A.H. HUNTER, Toronto, Ontario; LAI LANE LUEY, Willowdale, Ontario; J.A. McCALLUM, Medicine Hat, Alberta; NGO TAN, student, J.F. Kennedy H.S., Bronx, N.Y.; HERMAN NYON, Paramaribo, Surinam; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

* * *

532. [1980: 112] *Proposed by Arun Sanyal, Indian Institute of Technology, Kharagpur, India.*

Let triangles ABP, CDQ be directly similar to a triangle $\alpha$; triangles ACR, BDS directly similar to a triangle $\beta$; and triangle PQT directly similar to $\beta$. Prove that RST is directly similar to $\alpha$.

*Solution by Clayton W. Dodge, University of Maine at Orono.*

In the Gauss plane let triangles $\alpha$ and $\beta$ be (directly similar to) triangles having affixes 0, 1, $u$ and 0, 1, $v$, respectively. With lower-case letters denoting the affixes of the corresponding vertices, the given similarities are equivalent to

$$\frac{p-a}{b-a} = \frac{q-c}{d-c} = u \qquad (1)$$

and

$$\frac{r-a}{c-a} = \frac{s-b}{d-b} = \frac{t-p}{q-p} = v, \qquad (2)$$

and we are required to show that

$$\frac{t-r}{s-r} = u.$$

From (2) we get

$$t = p + v(q-p), \qquad r = a + v(c-a), \qquad s = b + v(d-b);$$

then, using (1) we get

$$\begin{aligned} \frac{t-r}{s-r} &= \frac{(p-a) + v\{(q-c)-(p-a)\}}{(b-a) + v\{(d-c)-(b-a)\}} \\ &= \frac{u(b-a) + v\{u(d-c) - u(b-a)\}}{(b-a) + v\{(d-c) - (b-a)\}} \\ &= u, \end{aligned}$$

as required.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; HOWARD EVES, University of Maine; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, Antioch College, Yellow Springs, Ohio (two solutions); GEORGE TSINTSIFAS, Thessaloniki, Greece (two solutions); JAN VAN DE CRAATS, Leiden University, The Netherlands; and the proposer.

*Editor's comment.*

Most solvers used complex numbers in their proofs. Giri and the proposer used an identity valid in a particular idempotent medial quasigroup, which they found in Merriell [1]. Mechanically using this identity shortens the proof somewhat, but at the cost of some understanding. The identity (equation (6) in [1]) is not an obvious one, and proving it requires going through the essential steps in our featured solution.

REFERENCE

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MATHEMATICS IN THE (NEAR) FUTURE

"You are a slow learner, Winston," said O'Brien gently.

"How can I help it?" he blubbered. "How can I help seeing what is in front of my eyes? Two and two are four."

"Sometimes, Winston. Sometimes they are five. Sometimes they are three. Sometimes they are all of them at once. You must try harder. It is not easy to become sane."

GEORGE ORWELL in *Nineteen Eighty-Four*

* * *