Mathematical Spectrum

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Mathematical Spectrum is a magazine for the instruction and entertainment of student mathematicians in schools, colleges, and universities. It is published by the Applied Probability Trust, a non-profit making organization established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

This number 2 completes Volume 5 of *Mathematical Spectrum* (1972/73); the first issue in the Volume was published in the Autumn of 1972.

Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There is also a section devoted to problems. The copyright of all published material is vested in the Applied Probability Trust.

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Articles are normally commissioned by the Editors; the Editorial Committee also welcomes the submission of suitable material, including correspondence, queries and solutions to problems, for publication in *Mathematical Spectrum*. All correspondence about the contents should be sent to:

The Managing Editor, *Mathematical Spectrum*,
Hicks Building, The University, Sheffield S3 7RH.

Mathematical Spectrum—1973 Questionnaire

Please fill in and return to

Editorial Office,
Mathematical Spectrum,
Hicks Building,
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READER'S NAME: ....

ADDRESS: ....

....

OCCUPATION: ....

Please place crosses in appropriate squares. You may wish to read the Survey before answering.

PART I—READER SATISFACTION

1. How well do you think that the Editorial Board has met its avowed aim of entertaining and instructing mathematics students?
   - Well ☐
   - Moderately ☐
   - Badly ☐

2. Do you consider the average article in *Mathematical Spectrum* is:
   - Too hard ☐
   - About right ☐
   - Easy ☐

3. Do you think the balance of topics discussed is:
   - Good ☐
   - Average ☐
   - Bad ☐

4. Rank the topics below in order of their interest to you. Write 1 for the most interesting, up to 10 for the least interesting.

Pure Mathematics	Statistics	Book Reviews	Computing Science	Applied Mathematics
☐	☐	☐	☐	☐
Operational Research	**Mathematical Problems**	**Letters**	**Careers and Miscellaneous**	**History**
☐	☐	☐	☐	☐

5. What would make *Mathematical Spectrum* a magazine of wider readership?

Simpler articles	More puzzles and entertainments	Closer relationship to current syllabuses	Other
☐	☐	☐	☐

If you have marked "Other", specify your suggestions:

PART 2—COSTS AND CIRCULATION

6. Increased printing costs will result in a price of 50p in the U.K. and £1·00 abroad for Volume 6 of *MS* (70–80 pages) in 1973/74. Do you think your library will take out an order?

   Yes ☐
   No ☐

7. Would you recommend a personal copy of *Mathematical Spectrum* to your fellow students or colleagues as a good buy?

   Yes ☐
   No ☐

8. Would you prefer a cheaper form of printing with a saving in price to the present high quality printing?

   Yes ☐
   No ☐

9. If you think a cheaper form of printing is desirable, would you accept cyclostyled typing?

   Yes ☐
   No ☐

PART 3—PERSONAL COMMENTS

If you have any suggestions for articles, or general ideas on how to improve *Mathematical Spectrum*, please give details of these below.

Mathematical Spectrum 1968–1973—A Survey of the First 5 Years

J. GANI

University of Sheffield

Mathematical Spectrum (*MS*) has now been appearing for a full five years; during these it has established itself as a popular student magazine in mathematics, instructive but not without its lighter side. It has suffered from publishing and distribution problems, benefited from discussions between authors and readers, and occasionally been rewarded with a truly outstanding article. As magazines go, *MS* has come of age: the time has arrived to take stock of its position, and plan for the future.

The policy of *MS* has always been to maintain the closest possible contact with its readers; to make certain that we (the Editors) are receiving adequate feed-back, we are asking all of you (readers) to fill in the attached QUESTIONNAIRE and mail it to us with your answers. We should be glad to have your views, however abrasive; your response will largely determine the way *MS* develops over the next few years. Before filling in the QUESTIONNAIRE, you may find the following facts of interest.

1. How *MS* was started

Early in 1967 a small group of mathematical enthusiasts in Britain decided that a student magazine in mathematics for schools, colleges of education and universities would be desirable. While a great deal of mathematics is taught in such educational institutions, we felt that students often missed the main ideas of mathematics—its spirit, as against its techniques. It was precisely this spirit which we intended the magazine to foster.

When the Organizing Committee was formed, it was surprised to find that there were no established facts about potential demand on which to base its planning. It was therefore agreed that, with the assistance of Oxford University Press, 2,000 copies of a preliminary announcement together with a mockup of *MS* should be sent out to 1,100 schools and other institutions in Britain, the U.S.A., and Australia. The mockup contained four short articles

The irrationality of π,
Meteoric dust and noctilucent clouds,
The spread of an epidemic, and
G. H. Hardy and British mathematics.

These were similar in style and content to articles projected for the new magazine. A few problems and their solutions were also included. A short questionnaire was mailed out to readers for information on the likely number of school and university subscribers, and their interests in historical, expository, and elementary research articles, as well as vocational advice, information on university courses, and mathematical problems.

The response to the questionnaire was excellent; replies from students and teachers confirmed the Committee's belief that the publication of *MS* would be a worthwhile venture. At this stage in 1968, the Committee set out to commission papers from practising mathematicians on various mathematical topics. These included subjects in pure and applied mathematics, statistics and biomathematics, operational research and computing science. Correspondence was also encouraged, and it was made clear to readers that their active participation was an important function of the magazine. What the Editors hoped for was active feed-back which would 'help to make *MS* a periodical tuned in to the wavelength of its readers'.

2. The structure of *MS*

In order to carry out the aims they had set themselves, the Organizing Committee appointed an Editorial Committee consisting of a Managing Editor, a Consulting Editor, and Editors in Operational Research, Pure Mathematics, Applied Mathematics, Statistics and Biomathematics, Computing Science and Numerical Analysis, Mathematical Problems, and Book Reviews.

Each of the Editors was to commission articles in his field from mathematicians with a flair for expository writing. The Editor for Mathematical Problems was to collect problems in various areas of mathematics, and correspond with readers sending in their solutions to him. The Book Reviews Editor would select appropriate books for his panel of reviewers and prepare their comments for publication. The Consulting Editor, a mathematician working in education, was to read each of the contributions critically and pronounce on their suitability for students, while the Managing Editor co-ordinated the activities of all other Editors and supervised the business management of the magazine.

It is this Editorial Committee which has been responsible for the commissioning of articles and refereeing of contributed papers for *MS*. Over the past five years, they have been responsible for the submission and revision of some 61 articles, 43 problems and 58 book reviews. The table below gives a breakdown of articles by fields, and the numbers of papers by category in each of the five volumes of *MS*.

The spread of articles has been very wide; the largest number of articles published has been in Pure Mathematics, with articles in Statistics, Applied Mathematics, and Careers and Miscellaneous next, and somewhat smaller numbers in the remaining fields. So far as the spread of mathematical topics is concerned, the Editorial Board has achieved a reasonable coverage of the currently active fields; it has also provided a lively service with its mathematical problems and the stimulating comments of its book reviews.

MS	Careers and Miscellaneous	History and Philosophy	Pure Mathematics	Applied Mathematics	Statistics and Biomathematics	Operational Research	Computing Science	Mathematical Problems	Book Reviews	Letters
Vol 1 No 1	1	1	2	1	1	1	1	6		2
Vol 1 No 2		1	1	1		1		5	3	2
Vol 2 No 1	1		2	1	1		1	4	2	
Vol 2 No 2	1	1	3		1	1		4	5	1
Vol 3 No 1	2		3		1			3	13	
Vol 3 No 2	1		2	1	2			5	6	1
Vol 4 No 1		1	3	2	1			4	7	1
Vol 4 No 2			2	1	1	1	1	4	8	
Vol 5 No 1	1		2	1	1			4	9	1
Vol 5 No 2	1		2			1	2	4	5	1
TOTALS	8	4	22	8	9	5	5	43	58	9

3. Difficulties and their resolution

When a magazine reaches its readers, its neat printing, attractive diagrams or photographs, and colourful cover may give the impression that its production has been all too simple a task. In fact, the opposite is the case.

The greatest difficulty is perhaps the commissioning of articles and encouragement of contributions for the magazine. The Editors have found it difficult to persuade practising mathematicians to contribute material with any regularity. While many mathematicians are willing to write articles at the research level, their interest in educational writing is often not quite as pronounced. Even those who are interested may find that they cannot spare the considerable time required to prepare an article suitable for non-specialists.

The situation is quite different in such countries as Hungary, the German Democratic Republic, and Russia, where a tradition has been established of research mathematicians writing for the lay public and for students. There is at least one popular magazine in each of these countries, entirely devoted to student mathematics and each enjoying a very large circulation. This is not true of Western countries; although mathematical magazines exist for teachers and a few are produced for students, none is designed for students with the aims, or at the level of *MS*. It has therefore proved difficult to start a fashion in student mathematics in the English-speaking countries to rival that of the Communist countries.

Some articles submitted to *MS* have been too difficult; others have dealt with topics of insufficiently broad interest. So authors have at times been asked to revise their contributions and occasionally articles have had to be declined. The Editors have felt that their greatest successes have been articles submitted by students themselves; of these, there have unfortunately been only two out of the 61 so far published.

A second problem is the level of difficulty of the published articles themselves. Here the Editors are faced with a dilemma: *MS* is designed for students in the higher classes in school as well as those in colleges of education and universities. The range of articles must therefore be such that they span levels of difficulty suitable for both the sixth former in school, and the beginning undergraduate in university. It has not proved easy to achieve this span; the Editors have attempted to include in each issue at least one or two elementary articles as well as a few of greater difficulty. In recent issues it has been the practice to order these, starting with the simplest at the beginning, and working up to the most difficult. Despite these efforts, the Editors have been aware of continuing criticism on the level of difficulty of *MS* articles. Their answer has always been that they strive valiantly to present articles at levels of difficulty suitable to the wide range of training of the students they are trying to reach. Nevertheless, it is clear that they will occasionally fail to select articles of the right levels in any single issue; the Editors remain aware of the complexity of the situation, and constantly bear in mind the need for simplicity in selecting papers.

A further difficulty has been the lack of adequate feed-back from our readers; the smallness of our postbag is an indication that readers are not overkeen to correspond with us. Why this should be so is a mystery to the Editors; possibly both students and teachers are too busy during the academic year to react to our pleas for response. Another possibility is that *MS* does not yet possess the mass readership which its Editors would have liked to command. Though *MS* copies are probably read by more than one reader, the magazine has never sold more than 3,000 copies per issue. Many of these copies go to libraries where the magazine will be seen by anything from 10–100 students, but it is likely that only students with a personal copy of the magazine feel sufficiently involved to wish to write to us about their problems and interests.

Lastly, finance has always been a serious consideration. The Editors have never known with any degree of certainty the sort of price that students are prepared to pay for a magazine such as *MS*. Our guess was that possibly 40–50p per annual volume for students in Britain, and about double this amount abroad might be acceptable, but information has never been forthcoming from students on this topic. One thing is certain: in this age of expensive printing, a magazine must have a large mass readership if it is to maintain prices of this order. *MS* has succeeded in weaving its uneasy way through financial difficulties up to now, but it is doubtful whether it can continue to function indefinitely on its present small budget.

4. The mechanics of publishing *MS*

After articles are received at the Editorial Office they are prepared for the printer by our Technical Editor. This involves underlining all letters which are to be italicised ($\underline{e}$ for *e*), marking those which are to appear in bold type ($\underset{\sim}{e}$ for **e**), stating the size of diagrams and sometimes redrawing them for the printer, deciding on layout and ensuring overall consistency of expression and of formulae. The articles are then sent to the printer who returns galley proofs of the papers (proofs running continuously without pagination); these are carefully corrected at the Editorial Office. Second page proofs similar to the final magazine copy are returned later, and once these have been corrected again, the magazine is ready for its final printing. It is run off in thousands of copies, and these are now ready for distribution. In its earlier years, *MS* was distributed by Oxford University Press (OUP); orders were sent in by customers to OUP and distribution was arranged entirely by them. This unfortunately met with serious delays and difficulties, as OUP was not efficiently set up for the distribution of a small magazine. This was taken over in 1972 by the Applied Probability Trust, and most of the problems have now been happily resolved.

The distribution is carried out by sending adhesive labels with the names and addresses of the subscribers to the printer. Each label is attached to an envelope, into which a copy of the magazine is slipped, and the printer then posts these for the Trust. It is in this way that the magazine finally reaches its readers in schools, colleges of education and universities all over the world.

Orders are currently sent in by customers to the Editorial Office, where a card index system of their names and addresses is kept, together with a record of the status of their subscriptions. The labels for the printers are typed up from these index cards, and sent out to the printer twice a year in autumn and spring when the magazine is ready for distribution. As readers will realise, the entire process is complex, and delays sometimes occur, but the staff at the Editorial Office are prompt and accurate and have met complaints with speed and understanding.

5. Plans for the future

How does the Editorial Board view the future? Does it expect more of the same, or hope for radical changes? The answers to these questions are not very simple. Briefly, while the Editors feel they have *defined* the problems which arise in making mathematics interesting to students, they have not yet *solved* them. It has become clearer what role a magazine such as *MS* can play in mathematical education: students do not always want to read the details of a proof in order to become acquainted with an important mathematical idea. It is precisely here that *MS*, with its mixture of broad mathematical instruction and entertainment, can be of use.

However, it has also become obvious that *MS* has raised a variety of problems to which easy solutions cannot be found. The most important is perhaps the encouragement of a steady flow of articles of a suitable level written by practising mathematicians as well as student readers. The Editors certainly wish to attract contributions from readers; the latter must not, however, be too sensitive if their offerings are sometimes declined. It is the Editors' duty always to select the very best articles available to them for any particular issue of *MS*, and this may involve some unpleasant decisions. The second most important task is to reach a large mass readership; the Editors feel that this can best be achieved by encouraging feed-back, and assisting students to participate in *MS* much more actively than in the past. To some degree, these problems are interrelated: a steadier flow of lively articles may well lead to mass readership, while the knowledge of a readership of thousands of students may encourage more mathematicians to write for *MS*.

Whether such plans will in fact help to shape *MS* largely depends on you, the readers. The Editors hope that their evangelistic zeal, their wish to spread their interest in mathematical ideas and history, will be met by the equal enthusiasm of students to whom *MS* is devoted.

The Perpetual Calendar

H. P. ROGOSINSKI

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1. Historical background

Our civil calendar has its origin in Roman times. It is based on the Julian calendar which was introduced in 45 B.C. by Julius Caesar. On the advice of the Alexandrian astronomer Sosigenes, Caesar abolished the use of the cumbersome lunar Roman Republican calendar and decreed that henceforth the year should be regulated by the sun. Astronomical evidence at the time suggested that the period taken by the earth to complete one revolution in its orbit about the sun was $365\frac{1}{4}$ days, and consequently this was taken as the length of the solar year. Since in civil reckoning it was impossible to accommodate an odd quarter day, it was decided to adopt a four-year cycle in which the first three years should have 365 days each, whilst the fourth year should have 366 days. The procedure of inserting additional days in a standard calendar year is called *intercalation*. The ancient Roman Republican calendar, being governed by the moon, required an elaborate system of intercalations so that its months could bear a proper relationship to the seasons. This calendar began in 'Martius' and intercalations were made at the end of the year. It is for this historical reason that, to this day, the intercalation in a leap-year is still carried out at the end of February.

Since Roman times, more accurate astronomical observations showed that in supposing the solar year to be $365\frac{1}{4}$ days, the Julian reckoning had overestimated this length by about 11 minutes 14 seconds; an error amounting to approximately one day in every 128 years. Astronomers predicted that eventually the effect of this error would result in a noticeable displacement of the seasons relative to the months. By the 16th century the discrepancy between the true vernal equinox and its 'allotted' calendar date of 21 March had amounted to ten days and, after considerable pressure from the astronomers, the Vatican decided to reform matters.

In 1582 Pope Gregory XIII directed the suppression of ten days in the calendar, and ordered that the Julian intercalations be omitted in all centenary years excepting those which are multiples of 400. In this way the next vernal equinox fell correctly on 21 March instead of on 11 March, and the error of three days in 384 years incurred by the Julian estimation of the solar year was approximately remedied. The Gregorian calendar was adopted forthwith by the principal states of the Holy Roman Empire, but was not adopted in Protestant Great Britain until 1752. Russia retained the Julian calendar until 1917, and religious and political anniversaries there are still to-day celebrated according to the old-style calendar.

The Gregorian calendar is in error by about one day in 3323 years. It has been proposed to correct this by omitting the Gregorian intercalation in the year 4000 and all its multiples!

2. The week-day problem

Since neither 365 nor 366 are multiples of 7, a given calendar date never falls on the same week-day in consecutive years. This suggests the problem of determining the week-day on which a given date occurs. A solution was given by Zeller in 1882, and his method is the following. Assign a characteristic number to each week-day as indicated in Table 1.

TABLE 1

Week-day	Sun	Mon	Tues	Wed	Thur	Fri	Sat
Characteristic number	1	2	3	4	5	6	0

Let

$$f(q, m, n, J) = q + \left[\frac{26(m+1)}{10}\right] + n + \left[\frac{n}{4}\right] + \left[\frac{J}{4}\right] - 2J, \tag{1}$$

where $[x]$ denotes the integer part of x. Then the remainder left on dividing $f(q, m, n, J)$ by 7 is the characteristic number attached to the week-day of the qth day of the mth month of the year $100J+n$. On applying (1) however the convention is made that the January and February of any given year are to be regarded respectively as the 13th and 14th month of the preceding year. As an illustration consider the following problem.

Problem 1. *On which week-day does* 18 *January* 1973 *fall*?

Here $q = 18$, $m = 13$, $n = 72$ and $J = 19$. Hence,

$$f(q, m, n, J) = 18 + \left[\frac{26\times 14}{10}\right] + 72 + \left[\frac{72}{4}\right] + \left[\frac{19}{4}\right] - 38$$

$$= 18+36+72+18+4-38 = 110.$$

Since 110 leaves the remainder 5 on division by 7, the required week-day is Thursday.

Knowledge of the elementary theory of congruences (see, e.g., reference 1, Chapter 1, Section 9) enables us to solve problems of the following type.

Problem 2. *In which years of this century does February have* 5 *Sundays*?

Suppose this occurs in the year $1900+N$, where $1 \leqslant N \leqslant 99$. Clearly the year must be a leap-year in which 1 February falls on a Sunday. Thus N is a multiple of 4, and so

$$N \equiv 0 \pmod{4}. \tag{2}$$

Applying (1) with $q = 1$, $m = 14$, $n = N-1$ and $J = 19$ gives

$$6+(N-1)+[\tfrac{1}{4}(N-1)] \equiv 1 \pmod{7}. \tag{3}$$

Since N is a multiple of 4, $[\frac{1}{4}(N-1)] = \frac{1}{4}N-1$, and so (3) reduces to

$$5(\tfrac{1}{4}N)+4 \equiv 1 \pmod{7}.$$

Multiplication by 12 gives

$$15N+48 \equiv 12 \pmod{7}$$

and this reduces to

$$N \equiv 6 \pmod{7}. \tag{4}$$

The standard procedure for solving the simultaneous congruence equations (2) and (4) yields the solution

$$N \equiv 20 \pmod{28}. \tag{5}$$

It now follows from (5) that the requisite years are 1920, 1948 and 1976.

As a similar exercise, the reader can determine in which years of this century his birthday falls on a Sunday!

It is easy to establish the validity of Zeller's method. Identify each date with a quadruple (q, m, n, J), and define a *simple* transformation of a date (q, m, n, J) as one in which one of the co-ordinates is altered by ± 1, whilst the three remaining co-ordinates are left fixed. Clearly any given date (q_1, m_1, n_1, J_1) can be transformed into any other given date (q_2, m_2, n_2, J_2) by a finite sequence of such simple transformations. If the characteristic of the week-day of the date (q, m, n, J) is denoted by $\chi(q, m, n, J)$, the formula (1) can be written as follows:

$$f(q, m, n, J) - \chi(q, m, n, J) \equiv 0 \pmod{7}.$$

Problem 1 together with a 1972 calendar verify that this holds for the date (18, 13, 72, 19). It therefore suffices to show that $f-\chi$ remains constant modulo 7 when the date undergoes a simple transformation. Such transformations fall into eight classes, namely two corresponding to each co-ordinate which is altered. Each class is considered as a separate case. Suppose, for example, that $3 \leqslant m \leqslant 13$ and that m is *increased* by 1. Then f increases by $d(m)$, where

$$d(m) = \left[\frac{26(m+2)}{10}\right] - \left[\frac{26(m+1)}{10}\right],$$

and the date advances by the number of days in the month 'm'. It is easily verified that this number is $28+d(m)$, and consequently χ increases by $d(m)$, modulo 7. Hence $f-\chi$ remains constant modulo 7. The remaining types of simple transformations are dealt with analogously. In checking through them, the reader will find that the terms $[\frac{1}{4}n]$ and $[\frac{1}{4}J]$ occurring in (1) take care respectively of the intercalations in leap-years and the Gregorian correction to these intercalations.

3. The determination of Easter

At a convention of the Council of Nicaea in A.D. 325 it was decreed that Easter should be celebrated on the Sunday following the first full moon which occurs on or after the vernal equinox (21 March). The celebrated German mathematician Gauss devised a simple numerical rule for determining the occurrence of Easter Sunday in a given year. This rule, specialised for the current century, is given below.

Determination of Easter Sunday in the year $1900+N$.

(i) Divide N by 19, 4 and 7. Let a, b and c be the respective remainders.
(ii) Divide $19a+24$ by 30. Let d be the remainder.
(iii) Divide $2b+4(c-1)+6d$ by 7. Let e be the remainder.

Easter Sunday occurs on $(22+d+e)$ March, when $d+e \leqslant 9$, and otherwise on $(d+e-9)$ April. There are two exceptions: if $d=29$ and $e=6$ then Easter occurs on 19 April (not 26 April), and if $d=28$ and $e=6$ then Easter occurs on 18 April (not 25 April).

Problem 3. *Determine Easter Sunday for* 1973.

Here $N=73$. By (i), $a=16$, $b=1$ and $c=3$. Hence $19a+24=328$ and by (ii), $d=28$. Finally, $2b+4(c-1)+6d=178$ and by (iii), $e=3$. Thus $d+e=31$, and Easter Sunday falls on 22 April.

Gauss' rule has been stated here without proof. The reader, however, is challenged to demonstrate for himself that the date so determined must necessarily fall on a Sunday!

It is an immediate consequence of Gauss' rule that Easter occurs within the period 22 March to 25 April. We solve the following problem.

Problem 4. *When in this century does Easter Sunday fall on* 25 *April*?

For this to occur it is necessary that $d=29$ and $e=5$. Since $d=29$, it follows from (ii) that $19a \equiv 5 \pmod{30}$, and so, on multiplying both sides by 19, $a \equiv 5 \pmod{30}$. Since $0 \leqslant a < 19$, it follows that $a=5$. Consequently by (i),

$$N \equiv 5 \pmod{19}. \tag{6}$$

Since $e=5$, it follows from (iii) that

$$2b+4(c-1)+6\times 29 \equiv 5 \pmod{7}.$$

This simplifies to

$$4b+c \equiv 6 \pmod{7}.$$

However by (i), $c \equiv N \pmod 7$ and so,

$$4b+N \equiv 6 \pmod 7. \tag{7}$$

From (6) it is seen that the possible values for N are 5, 24, 43, 62 or 81. By writing down the appropriate corresponding values for b, it appears that the only combination of N and b satisfying (7) is $N = 43$ and $b = 3$. Thus in this century Easter Sunday occurs on 25 April only in 1943.

The reader is invited to determine when in this century Easter Sunday occurs on 22 March. He may find the result surprising!

Reference

1. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, (Macmillan, New York, 1947).

Calculating Square Roots of Perfect Square Numbers by Inspection

P. N. MEHTA

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A number of approaches are known for finding the square roots of numbers. In this article we explain how to find the square root of a perfect square number by the method of inspection.

Considering the squares of numbers from 1 to 9, it is seen from Table 1 that the digit in the unit place for the square numbers is symmetrically repeated in the order 1, 4, 9, 6, 5, 6, 9, 4, 1.

TABLE 1

Number	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81

Thus, for a perfect square number only one of 1, 4, 9, 6, 5, or 0 occurs in the unit place.

A square number with 1 in the unit place will have either 1 or 9 in the unit place of its square root. More generally, if a_1 be the digit in the unit place of the square number, there will be two possible digits, say a and b, in the unit place of its square root. Let $b \geqslant a$. We show the possible digits, a and b, corresponding to a_1 in the unit place of the square number in Table 2.

TABLE 2

a_1	1	4	9	6	5	0
a	1	2	3	4	5	0
b	9	8	7	6	5	0

If $a_1 = 9$, a and b will be respectively 3 and 7, but when $a_1 = 5$ or 0, $a = b$ and each will be equal to 5 or 0 according to whether a_1 is 5 or 0.

Let any perfect square number A be expressed in the form

$$A = a_3 t^2 + a_2 t + a_1,$$

where $t = 10$ and a_1, a_2 are digits but a_3 may be any number whatever. Thus, a_1, a_2 will be the digits in the units place and tens place respectively and a_3 will be the number of 100's in A.

Corresponding to a_1 in the unit place of the square number A, $\sqrt{A}$ will be equal to $10n+a$ or $10n+b$, where n is so chosen that

$$n^2 \leqslant a_3 < (n+1)^2.$$

To determine a or b consider the two inequalities

$$a_3 - n^2 < n \tag{1}$$

and

$$(n+1)^2 - a_3 \leqslant (n+1). \tag{2}$$

We see that either

$$a_3 < n(n+1) \tag{3}$$

or

$$a_3 \geqslant n(n+1). \tag{4}$$

If n satisfies the inequality (3)

$$\sqrt{A} = 10n + a,$$

while if n satisfies the inequality (4)

$$\sqrt{A} = 10n + b.$$

Illustrative examples

(i) Find the square root of 110224.

Let $110224 = 1102t^2 + 2t + 4$, where $t = 10$; then $a_3 = 1102$, $a_2 = 2$ and $a_1 = 4$. As $a_1 = 4$, we have $a = 2$ and $b = 8$. Now

$$n^2 \leqslant a_3 < (n+1)^2$$

leads to

$$(33)^2 < 1102 < (34)^2,$$

whence $n = 33$ and $n(n+1) = 33 \times 34 = 1122$. Further, the inequality

$$1102 < 33(33+1)$$

holds, or $a_3 < n(n+1)$ is satisfied. Hence $a = 2$ is the correct digit in the unit place and

$$\sqrt{110224} = 10\times 33+2 = 332.$$

(ii) Find the square root of 1296.

In this example the inequality (4) is satisfied with the equality sign, so that $\sqrt{1296} = 10n+b = 36$.

(iii) Find the square root of 65536.

Here also the inequality (4) is true and $\sqrt{65536} = 256$.

(iv) Find the square root of 93025.

In this case also the inequality (4) holds, again with the equality sign. As $a = b = 5$, we obtain $\sqrt{93025} = 10\times 30+5 = 305$.

Solving Cubic Equations Numerically by Recurrence Formulae

JOAN M. HOLLAND

1. Recurrence sequences

In a recurrence sequence each term depends in a clearly defined way on a specified number of the terms immediately preceding it. Each term a_{n+2} of the Fibonacci sequence, for example, is the sum of the preceding two terms a_{n+1} and a_n so that with $a_0 = 0$, $a_1 = 1$ the sequence is

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots.$$

The Fibonacci sequence is known to be related to the quadratic equation

$$x^2+x-1 = 0 \tag{1}$$

and its reciprocal

$$x^2-x-1 = 0. \tag{2}$$

One way of establishing the connection is to define α as one of the roots of Equation (1) and then to reduce each term of the sequence α^3, α^4, α^5, ... to a linear form by repeated use of the equation in the form

$$\alpha^2 = 1-\alpha.$$

Thus

$$\alpha^3 = \alpha\times\alpha^2 = \alpha(1-\alpha) = \alpha-\alpha^2$$
$$= \alpha-(1-\alpha) = 2\alpha-1.$$

Similarly $\alpha^4 = 2-3\alpha$, $\alpha^5 = 5\alpha-3$, $\alpha^6 = 5-8\alpha$ and, in general,

$$\alpha^n = (-1)^n(a_{n-1}-a_n\,\alpha) \tag{3}$$

where the a_n form a Fibonacci sequence.

Now the roots of Equation (1) are $\frac{1}{2}(-1\pm\sqrt{5})$ or approximately 0·6180 and $-1{\cdot}6180$ and the roots of (2) are approximately the reciprocals of these values, namely 1·6180 and $-0{\cdot}6180$. If in Equation (3) we assume that α is the smaller numerical value, 0·6180, then α^n tends to zero as n increases to infinity. Hence

$$(-1)^n(a_{n-1}-a_n\,\alpha)\to 0 \quad \text{as} \quad n\to\infty$$

or

$$a_{n-1}/a_n\to\tfrac{1}{2}(-1+\sqrt{5}) \quad \text{as} \quad n\to\infty,$$

which may be recognised as a well-known property of the Fibonacci sequence.

2. A particular cubic equation

Recurrence formulae can be used to solve cubic equations. We begin by examining the equation

$$x^3-x^2-2x+1 = 0, \tag{4}$$

which has interesting features of its own and is discussed elsewhere by the writer (reference 1).

There is a cyclic relation between its three roots α, β, γ,

$$\beta = 1-\alpha^{-1},$$
$$\gamma = 1-\beta^{-1},$$
$$\alpha = 1-\gamma^{-1},$$

from which it also follows that

$$\gamma = (1-\alpha)^{-1}.$$

The reciprocals of α, β, γ are the roots of the reciprocal equation of (4)

$$x^3-2x^2-x+1 = 0. \tag{5}$$

The roots of (4) and (5) are also associated with the regular heptagon. It can be shown that α is the length of a chord subtending $\pi/7$ at the centre of a unit circle while $-\beta$ and γ are the lengths of chords subtending $3\pi/7$ and $5\pi/7$ respectively. From this it follows that

$$\alpha = 2\sin(\pi/14)\simeq 0{\cdot}4450,$$
$$\beta = -2\sin(3\pi/14)\simeq -1{\cdot}247,$$
$$\gamma = 2\sin(5\pi/14)\simeq 1{\cdot}802.$$

3. Solution of the equation by recurrence formulae

Since α is a root of (4)

$$\alpha^3 = \alpha^2+2\alpha-1 \tag{6}$$

and this relation may be used to develop a sequence of powers of α reducing each in turn to a quadratic expression:

$$\alpha^4 = \alpha\times\alpha^3 = \alpha(\alpha^2+2\alpha-1) = \alpha^3+2\alpha^2-\alpha$$
$$= (\alpha^2+2\alpha-1)+2\alpha^2-\alpha = 3\alpha^2+\alpha-1.$$

By the same method $\alpha^5 = 4\alpha^2+\alpha-1$, $\alpha^6 = 9\alpha^2+5\alpha-4$ and, in general,

$$\alpha^n = a_n\alpha^2+b_n\alpha+c_n \quad (n = 3, 4, \ldots). \tag{7}$$

Because $|\alpha| < 1$ the left-hand side, α^n converges rapidly to zero as n increases so that for large n the *residual quadratic expression* on the right is very nearly zero. If we equate it to zero and change α to x we get the *residual quadratic equation*

$$a_n x^2+b_n x+c_n = 0. \tag{8}$$

One of its two roots α_n must lie very close to α. The question arises whether the second root β_n has any significance. Do its successive values tend to any limit and if so can we attach any meaning to this? Some laborious calculations showed that it did appear to be converging, though somewhat slowly, possibly towards the second root β whose approximate value is $-1{\cdot}247$. This was rather surprising because if we write β instead of α in Equation (6), as we are entitled to do, it is quite certain that the left-hand side, β^n, does not diminish towards zero as n increases but on the contrary increases its value steadily.

At this stage we include a table (Table 1) of values compiled by a computer confirming the calculations and making it easier for the reader to follow the argument.*

The symbols α_n and β_n stand for the successive roots of the residual quadratic equation and a column for a_{n+1}/a_n has been included because it is needed later.

TABLE 1

n	a_n	b_n	c_n	α_n	β_n	a_{n+1}/a_n
3	1	2	−1			
4	3	1	−1			
5	4	5	−3			
6	9	5	−4	0·4444	−1·000	
7	14	14	−9	0·4449	−1·445	
8	28	19	−14	0·4450	−1·124	
9	47	42	−28	0·4450	−1·339	
10	89	66	−47	0·4450	−1·187	1·782
11	155	131	−89	0·4450	−1·290	1·846
12	286	221	−155	0·4450	−1·220	1·773
13	507	417	−286	0·4450	−1·267	1·823
14	924	728	−507	0·4450	−1·230	1·788
15	1652	1349	−924	0·4450	−1·256	1·812

*Thanks are due to Mr A. Wragg for his help with the computer at the Royal College of Advanced Technology, Salford (now the University of Salford).

We shall now show that, subject to an important proviso, the second root β_n of Equation (8) does converge towards β.

Since β satisfies (4) it also satisfies (7):

$$\beta^n/a_n = \beta^2+(b_n/a_n)\beta+c_n/a_n \tag{9}$$

and, for $n+1$,

$$\beta^{n+1}/a_{n+1} = \beta^2+(b_{n+1}/a_{n+1})\beta+c_{n+1}/a_{n+1}.$$

We may write this as

$$\frac{\beta^n}{a_n}\times\frac{\beta}{a_{n+1}/a_n} = \beta^2+(b_{n+1}/a_{n+1})\beta+c_{n+1}/a_{n+1}. \tag{10}$$

If it can be shown that for large n a_{n+1}/a_n is always numerically greater than β then as the process is repeated many times the left-hand side must converge to zero though possibly rather slowly. It can be seen from Table 1 that a_{n+1}/a_n also appears to be converging towards a value of about 1·8. Is it perhaps converging towards the third root γ of the cubic? If so this is another unexpected bonus. Fortunately we can short-circuit some of the remaining steps of the investigation by referring to a well-known result obtained by Daniel Bernoulli in 1728. But first we shall summarise our results obtained so far, not yet firmly established but clearly illustrated in Table 1.

(a) Each residual quadratic equation $a_n x^2+b_n x+c_n = 0$, associated with the cubic (4) has one root, α_n, which converges to α as n increases.

(b) The ratio a_{n+1}/a_n appears to be converging towards γ.

(c) Assuming this to be true then the other root β_n is converging to β because $a_{n+1}/a_n \to \gamma$ and $\gamma > \beta$.

Bernoulli's result states that if a polynomial equation of the form

$$x^n = px^{n-1}+qx^{n-2}+rx^{n-3}+\ldots$$

has a root γ which is numerically larger than any of the other roots then γ is the limit of a_{n+1}/a_n where a_n and a_{n+1} are successive terms of the sequence developed by the recurrence formula

$$a_n = pa_{n-1}+qa_{n-2}+ra_{n-3}+\ldots.$$

That is to say, the equation determines a recurrence formula and the formula then determines the largest root as the limit of the ratio of two successive terms (reference 2). Hence we can now regard the statement in (b) above as true.

4. Recurrence formulae for the general cubic equation

It is not difficult to prove that if the same method—reducing powers of x to quadratic forms—is applied to the general cubic

$$x^3 = px^2+qx+r$$

then the coefficients of the residual quadratic equation

$$a_n x^2+b_n x+c_n = 0$$

are given by the following recurrence formulae:

$$a_n = pa_{n-1}+qa_{n-2}+ra_{n-3},$$
$$b_n = qa_{n-1}+ra_{n-2},$$
$$c_n = ra_{n-1},$$

taking 0, 0, 1 as the initial terms of the sequence $\{a_n\}$.

5. Application to the cubic $x^3 = 3x^2+4x-12$

This equation has integral roots 2, -2, and 3. Initial values of a_n, b_n, and c_n are given in Table 2.

TABLE 2

n	a_n	b_n	c_n
1	0		
2	0		
3	1		
4	3	4	-12
5	13	0	-36
6	39	16	-156
7	133	0	-468
8	399	64	-1596
9	1261	0	-4788
10	3783	256	-15132
11	11065	0	
12	34815	1024	
13	105464	0	

Three trends may be observed.

(i) The ratio a_{n+1}/a_n is either exactly 3 or a number approximately 3 as n increases.

(ii) The b_n sequence alternates between zero and a number relatively small compared with a_n, so that b_n/a_n approaches zero as n increases.

(iii) The ratio c_n/a_n is either exactly -4 or a number which approaches -4 as n increases.

These trends suggest that the residual quadratic equation

$$x^2+(b_n/a_n)\,x+c_n/a_n = 0$$

may be approaching the equation $x^2-4 = 0$, whose roots are 2 and -2. The process in fact yields all three roots, the quadratic equation providing the two smaller roots and the limiting ratio of a_n/a_{n+1} the largest.

6. Convergence of a_{n+1}/a_n

The proviso of Section 3 was that a_{n+1}/a_n should converge to a limit. This does not always happen. We noticed that if the limit exists it is equal to the root greatest in absolute value, while the two smaller ones emerge as roots of the residual quadratic equation. If we start with a cubic equation having two roots with equal numerical values greater than the third we can hardly expect convergence of a_{n+1}/a_n.

For example, the equation

$$x^3 = x^2+2x-2$$

produces the sequence $\{a_n\}$ with values

$$0, 0, 1, 1, 3, 3, 7, 7, 15, 15, \ldots$$

and a_{n+1}/a_n oscillates between 1 and a value which approaches 2. The roots are 1, $\sqrt{2}$ and $-\sqrt{2}$. However, with a cubic equation the difficulty can be overcome by considering the reciprocal equation which in this case is

$$y^3 = y^2+\tfrac{1}{2}y-\tfrac{1}{2}.$$

Then a_{n+1}/a_n would approach the largest root, 1, and the other two roots, $1/\sqrt{2}$ and $-1/\sqrt{2}$, would appear as roots of the residual quadratic equation.

7. Conclusion

We have seen that recurrence formulae can help us to solve cubic equations; the method sketched is particularly useful as a quick way of obtaining numerical solutions. A variant of the method can be presented more elegantly using matrix methods but this will be left to a later article.

8. Problems

(1) Use recurrence methods to obtain approximate values for the roots of the equations below and confirm your results by other methods:

(a) $x^3-2x^2+1 = 0$,

(b) $x^3-6x^2+7x+4 = 0$.

(2) Let

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate $F^2, F^3, F^4, F^8, F^{16}$. Show that F generates a recurrence sequence $\{a_n\}$ where $a_0 = 0$, $a_1 = 1$ and give the values of a_8, a_{16} and a_{32} in this sequence.

References

1. Joan M. Holland, *Studies in Structure* (Macmillan, London, 1972), 168–172.
2. R. Wooldridge, *An Introduction to Computing* (Oxford University Press, London, 1962), 125–130.

Strategy for Life—A Guide to Decision Making

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1. Introduction

The influence of gambling on the development of probability theory has been discussed by Gani in *Mathematical Spectrum*, Volume 4, pages 9–14. An understanding of probability can, in turn, help decisions in gambling and many rather more practical situations. In this article we consider a strategy for action in a gamble involving rather high stakes.

You are one of three rivals in a contest which is a fight to be the lone survivor. As you are the worst shot you are allowed to fire first, followed by the next worst shot and finally the third contestant who is a 'dead' shot. Contestants then continue to shoot in rotation until only the winner is left alive. For each shot there is an option: to shoot at an opponent or to fire in the air. The dead shot is not allowed the latter option. It is assumed that if a shot is fired in the air it does not hit an opponent. What is your best strategy, having been given the dubious honour of shooting first? We shall see that the intuitively obvious one of firing at the dead shot is not always superior.

To solve the problem of the shooting contest we need two basic theorems of probability, the addition and multiplication theorems. The addition theorem states that if E_1 and E_2 are two mutually exclusive events (i.e., only one of them can occur at a time), then

$$\Pr\{E_1 \text{ or } E_2\} = \Pr\{E_1\}+\Pr\{E_2\},$$

where $\Pr\{E\}$ represents the probability of the event E occurring.

The multiplication theorem states that if E_1 and E_2 are independent events (i.e., the occurrence of E_1 does not affect the probability of occurrence of E_2 and vice versa), then

$$\Pr\{E_1 \text{ and } E_2\} = \Pr\{E_1\}\times\Pr\{E_2\}.$$

2. A strategy for the shooting contest

Let the probabilities of the contestants A, B, and C killing the person they aim at be p_1, p_2, and 1, respectively (with $0 < p_1 < p_2 < 1$). A's first action is either to aim in the air, to shoot at C or to shoot at B. We will find the probability of A's winning in each instance, and thus determine his optimal strategy.

(i) *A shoots in the air.* The possible variations of the outcome of the contest for this case such that A wins are shown in Table 1. From these we derive the probability of A winning, as

$$\Pr\{A \text{ wins} \mid \text{fires in air}\} = P_1$$

$$= p_2 p_1 + p_2 q_1 q_2 p_1 + p_2 (q_1 q_2)^2 p_1 + \ldots + p_2 (q_1 q_2)^r p_1 + \ldots + q_2 p_1$$

$$= \frac{p_1(1-q_1 q_2^2)}{1-q_1 q_2},$$

where $q_i = 1-p_i$ (for $i = 1, 2$), and $\Pr\{E_1 \mid E_2\}$ is the probability of E_1 occurring given that E_2 has already occurred.

TABLE 1

Different ways in which A can win if he fires in the air on his first shot

	A fires in air *and* B kills C
	and A kills B
or	A fires in air *and* B kills C
	and A misses B
	and B misses A
	and A kills B
or ...	A fires in air *and* B kills C
	and A misses B } repeated any number
	and B misses A } of times
	...
	and A kills B
... *or*	A fires in air *and* B misses C
	and C kills B
	and A kills C

(ii) *A shoots at C*. For this strategy there are two possible situations after the first shot. First, if A misses C the contest is as in (i). If he kills C, however, there are an extra series of possible outcomes shown in Table 2, and the chance of A

TABLE 2

Different ways in which A can win if he kills C on his first shot

	A kills C *and* B misses A
	and A kills B
or	A kills C *and* B misses A
	and A misses B
	and B misses A
	and A kills B
or ...	A kills C *and* B misses A
	and A misses B } repeated any number
	and B misses A } of times
	...
	and A kills B

winning is now

$$\Pr\{A \text{ wins} \mid \text{fires at } C\} = P_2$$

$$= q_1 \Pr\{A \text{ wins} \mid \text{fires in air}\}$$

$$+p_1 q_2 p_1 + p_1 q_2 q_1 q_2 p_1 + \ldots + p_1 q_2 (q_1 q_2)^r p_1 + \ldots$$

$$= \frac{p_1 q_1 (1 - q_1 q_2^2) + p_1^2 q_2}{1 - q_1 q_2}.$$

(iii) *A shoots at B*. In this case after the initial shot there are again two possible situations. First, if A kills B then he is killed immediately by C and so cannot win. Secondly, if he misses B then we are in a state identical to (i) after A's first shot in the air. So that here

$$\Pr\{A \text{ wins} \mid \text{fires at } B\} = P_3$$

$$= q_1 \Pr\{A \text{ wins} \mid \text{fires in air}\}.$$

Clearly strategy (iii) is inferior to (i) whatever the values of p_1 and p_2, since $q_1 < 1$, and so can be ruled out. Strategy (i) will be better than (ii) if $P_1 > P_2$, and vice versa. Now,

$$P_1 - P_2 = \frac{p_1^2(p_2 - q_1 q_2^2)}{1 - q_1 q_2} > 0$$

if

$$p_1 > 1 - \frac{p_2}{(1-p_2)^2},$$

i.e., if

$$p_2 > q_1 q_2^2,$$

and this inequality is the basis of our solution.

The decision space can be represented diagrammatically as in Figure 1, the curve representing the case of equality, $P_1 = P_2$. It is clear that firing in the air is the better strategy for most values of p_1 and p_2. When $p_2 > 0{\cdot}382$ or $p_2 < 0{\cdot}318$, the value of p_1 is immaterial to A's decision; it is a relevant consideration only when $0{\cdot}318 < p_2 < 0{\cdot}382$ (the limits of the inequality for p_2 are given by the intersections of the curve $p_2 = q_1 q_2^2$ with $p_1 = p_2$ and $p_1 = 0$, respectively). Thus we can say that, if p_2 is large let one of B or C kill the other, and wait to get the first shot at the survivor in a two man contest. But, on the other hand, if p_2 is small C is unlikely to be killed by B and A must help in doing this.

Having established A's strategy we can now calculate his actual chances of winning, and compare them to those of B and C. To do this we need, in the same manner as for A, to find the probabilities of B and C winning when they also adopt their optimal strategy.

(i) *A shoots in air.* B must shoot at C, otherwise he clearly loses to the next shot, and can only win if he kills him. Thus

$$\Pr\{B \text{ wins} | A \text{ fires in air}\} = p_2 q_1 p_2 + p_2 q_1^2 q_2 p_2 \ldots$$
$$= \frac{p_2^2 q_1}{1 - q_1 q_2},$$

and

$$\Pr\{C \text{ wins} | A \text{ fires in air}\} = q_1 q_2.$$

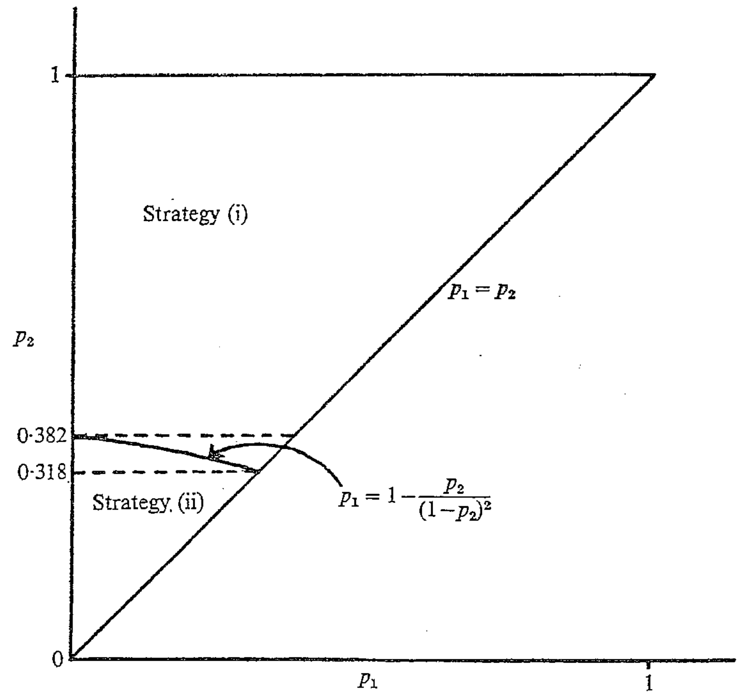

Figure 1. The best strategy for A for different values of p_1 and p_2.

(ii) *A shoots at C.* The chance of B winning when A adopts this strategy is composed of two parts: (a) when A misses C, the probability of B winning is q_1 multiplied by the probability of B winning in (i) above, and (b) when A kills C, the probability of B winning is

$$p_1(p_2 + q_2 q_1 p_2 + \ldots) = \frac{p_1 p_2}{1 - q_1 q_2}.$$

So

$$\Pr\{B \text{ wins} | A \text{ fires at } C\} = \frac{p_2(p_1 + q_1^2 p_2)}{1 - q_1 q_2},$$

and clearly,

$$\Pr\{C \text{ wins} | A \text{ fires at } C\} = q_1^2 q_2.$$

Many developments of the shooting contest are possible, and make for interesting probability problems. For example, we can remove the restriction that A goes first, and allow a fair draw between the three contestants for the initial shot. Also the introduction of one or more further contestants will make the problem more complicated. Further, the cyclic order of shooting may be changed, and the constraint that C be a dead shot relaxed.

The problem described here can be posed on less bloodthirsty lines for those who feel a degree of repugnance. For example, the targets are not the contestants themselves, but three balloons marked A, B, and C at which darts are thrown.

Suggestions for further reading

A special case of the shooting contest, along with many other interesting problems, is included in Martin Gardner's books on *Mathematical Puzzles and Diversions*, published by Penguin Books.

The Parallel Axiom

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1. Introduction

Try to imagine that you live in a world where triangles have angles which add up to less than two right-angles and straight lines have infinitely many lines parallel to them, passing through a given point. You will probably find this quite difficult because it violates your intuition. But it might surprise you to learn that such a world does exist and is entirely free from contradictions. It is called *non-Euclidean geometry.*

The discovery of this strange new geometry was made quite independently, but at roughly the same time, by three nineteenth century mathematicians: Gauss (1777–1855), who was German; Bolyai (1802–1860), who was Hungarian and Lobachevski (1793–1856), who was Russian. However, it all began much earlier, in the fourth century B.C., when Euclid made the first attempt to formalise geometry.

2. The debate on Euclid's geometry

Euclid's idea was to exhibit all known geometric theorems as logical consequences of certain basic, intuitively obvious facts which he called axioms. The axioms he chose fell into two categories: logical and non-logical. The logical axioms were meant to be absolute in the sense that they were common to all sciences and mathematical theories:

(i) things which are equal to the same thing are also equal to one another;

(ii) the whole is greater than the part.

There were three other common notions in this category. The non-logical axioms were peculiar to geometry and were chosen on the basis that they were the simplest established principles upon which one could hope to build the entire structure. They are as follows.

1. Given any two points, there exists a straight line through them.
2. A terminated straight line may be produced to any length in a straight line.
3. A circle may be described with any radius and any centre.
4. All right-angles are equal.
5. (*Parallel Axiom*) Through a given point, only one parallel can be drawn to a given straight line.

In addition to these lists of axioms, Euclid tried to define all the terms he used in them. It is curious that although he clearly realised the necessity for unproved premises in his system he saw no similar need for undefined, primitive terms. If you do not include such notions you end up dizzy or regressing rapidly into infinity! For instance:

'a point is that which has no part';

'a line is breadthless length'.

Some of the definitions (e.g., circle, right-angle), on the other hand, have withstood criticism.

Let us now stop to consider what we should require from a reasonable system of axioms. The greatest crime in mathematics is inconsistency. If our system is to make any sense at all, therefore, we should not include axioms which contradict each other, or which lead to contradictions. Mathematicians test for consistency by trying to discover a world (a model) in which, once the primitives have been assigned meanings, only those propositions which arise honestly from the axioms are the true ones. Thus, one sure criterion for consistency is the truth, in our physical world, of all the axioms. And since truth was based on (rather vague) intuitions, no-one doubted that Euclid's system was consistent. However, this myth was exploded by Cantor when he proved that every infinite set is similar to a part of itself. (Reference 2, in particular the corollary on page 14.) Consider an infinite set of points in the plane and you will see immediately that this invalidates Euclid's logical axiom (ii). Nonetheless, this inconsistency is easily removed and it is likely that, if Euclid had stated the logic he was using, it would have been spotted at the time.

A second requirement is completeness. In other words, there must be sufficient axioms so that every geometric problem can be solved. On this count also, Euclid's system fails. Many of his 'proofs' use intuitive assumptions which cannot be deduced from the axioms. In his very first proposition, Euclid assumes, with no right, that two constructed circles will meet in a point. Additional axioms must be added to ensure this. Other 'proofs' make use of undefined notions such as 'between' and 'congruent' although none of the axioms indicate how they are to be interpreted.

Now the obvious way for us to achieve completeness is to carry out the development of the theorems of the system, adding new axioms as we need them, carefully preserving consistency as we go. An even simpler method would be to add every theorem as an axiom! However, to combat such extravagant behaviour, mathematicians insist that the axiom system they choose is independent, i.e., they are prevented from including redundant propositions which could be deduced from the others. Independence is not an essential restriction, one of aesthetic importance really, so it is uncanny that the one requirement which, it was believed, must surely fail turned out to be the only one which Euclid's system did meet. The very complexity of the Parallel Axiom by comparison with the remaining axioms suggested at first that it should be a theorem instead of an assumption. Euclid thought so too and tried very hard to prove it. A further reason for this belief, which helped sustain over twenty centuries of frustrated effort, was that a theorem's converse could usually be proved without using any additional axioms. And the converse of the Parallel Axiom was known to be derivable from the remaining axioms.

All attempts at proving the Parallel Axiom or finding a simpler substitute for it failed. However, in 1733, Saccheri, an Italian Jesuit priest, tried the following new, indirect method of argument. If we accept Euclid's definition of axiom (and one, incidentally, which is still given by dictionaries today), then the axioms and their consequences are *a priori* true. They arose out of basic considerations of the ideal relationship between distances and objects on the earth's surface. Thus all geometric statements are either true or false. Nature is not fickle! Consequently, the negation of a true statement must be a false one, and so, if we assume the negation holds, we must eventually reach a contradiction. Now, if Saccheri had been able to find the contradiction he was looking for in assuming the negation of the Parallel Axiom, he would have proved it dependent on the other axioms. As it happened, although in the course of his investigations he proved a number of theorems which later became part of non-Euclidean geometry, he did not understand the implication of his findings. So the credit for proving the independence of the Parallel Axiom does not rest with him.

What Bolyai and Co. succeeded in doing was to show that *no* contradiction arose if one assumed the negation of the Parallel Axiom. So great was the intellectual prejudice of the time in favour of the geometry of Euclid, that their discovery was greeted with derision and disbelief. Nevertheless, the fact remained that the negation of the Parallel Axiom was consistent with the other axioms. Thus, there are two theories of geometry, each consistent within itself but incompatible with the other. Furthermore, although measurements seemed to indicate that Euclidean geometry gave the more accurate description of the physical world, there was no mathematical reason for assuming that either was superior to the other. The idea that Euclidean geometry was *a priori* true was thus shown to be incorrect. In the words of Gauss: 'I keep coming closer to the conviction that the necessary truth of our geometry [i.e., Euclidean] cannot be proved, at least, *by* the human intellect *for* the human intellect.'

The influence of the growth of non-Euclidean geometry on the development of modern mathematics is tremendous. The possibility of such a strange geometry, and the inconsistency of Euclid's original axioms were a blow to the belief in intuition as a criterion for truth. For centuries the negation of the Parallel Axiom had seemed false to anyone who had studied it. Now it seemed that truth itself could be relative and ideas on the very nature of axioms had to change. Although intuition has a very important role to play in designing an axiom system it must not be the only consideration. What the axiom system is actually about (i.e., the truth of the individual axioms) is irrelevant when one's interest is in deriving the consequences of the given axioms by valid argument. Thus, whenever the axioms are true, any conclusion of such a deduction will also be true. It may be that the axioms are never true—not because they are false but because they are not capable of being either true or false and are purely abstract. Such opinions have led to the creation of a logic which is much more adequate for the rigours of modern mathematics than the classical Aristotelian variety.

We shall return to the problem of independence later but first we shall give a brief description of a modern axiomatisation of Euclidean geometry in which all the objections lodged against Euclid have been accounted for. At this point it is only fair to mention that mathematicians hold Euclid in very high esteem. The main reason for this is that it has taken them over two thousand years to comprehend something which he saw fit to do in a lifetime!

3. The axiomatic system of David Hilbert (1862–1943)

Primitive terms: point, line, plane.

Primitive relations: 'are situated', 'between', 'congruent'.

Notice first that no term has been defined. We are therefore free to interpret them in any way we please, provided that it is consistent with the axioms—the precise nature of the existing mutual relations between the terms will then follow as a result of the relevant axioms.

Axioms. These are divided into five groups.

Group 1. (Connection) Seven axioms in this group specify the first relation. For example: any two points are situated on a unique line.

Group 2. (Order) Here the properties of 'between' are described: of any three points situated on a line, there is exactly one which is between the other two. There are five axioms in this group.

Group 3. (Congruence) One axiom expresses the infinite extent of the line: given a segment AB and a point C, on every line through C it is possible to find at least two points D and D', on either side of C, such that CD and CD' are both congruent to AB. Another axiom states that, if two sides and the included angle of a triangle are congruent respectively to two sides and the included angle of another (a property known as SAS), then the remaining angles are congruent in pairs. (The first congruence theorem then follows easily from two of the remaining four axioms in this group.)

Group 4. (Continuity) The axiom in this group removes the logical gap in the proof of Euclid's first proposition.

Group 5. Inevitably—the Parallel Axiom.

Logic. A statement of the logical rules of inference by which new propositions may be deduced from old ones.

Theorems. A demonstration of all known results developed from the given axioms by the rules prescribed.

The alert reader will have observed a few terms and relations in addition to the primitive ones which have not been defined (e.g., segment, 'on either side of', angle, triangle). Definitions can be supplied which involve only these primitives.

Hilbert's system conforms to all three of our requirements. We do not propose, nor do we have the space, to defend this claim here. However, we shall prove the independence of the Parallel Axiom from the remaining axioms in Groups 1–4 (often referred to collectively as *absolute geometry*).

To do this, we shall employ the method first developed by Saccheri, i.e., we show that it is possible for absolute geometry to be true and the Parallel Axiom to be false. Now this is achieved by describing a world, or a model, where the axioms of absolute geometry and the negation of the Parallel Axiom are all valid. Remember that all consequences of these axioms will then also be true in the model. Now, since the negation of the Parallel Axiom is true in the model, the Parallel Axiom itself must be false. Thus it cannot possibly be a consequence of absolute geometry (for otherwise, it would be both true and false in the same model). Poincaré's model for non-Euclidean geometry is the one we shall describe because it is easy to visualise.

4. Poincaré's model for non-Euclidean geometry

The first thing we have to do is to describe the inhabitants: there is one *-plane and there are infinitely many *-lines and *-points. The * distinguishes these animals from their Euclidean counterparts; we shall observe this convention throughout. (Incidentally, it would help considerably in what follows if you could stop being narrow-minded about what you mean by a point, a line and a plane. Try to think of these words, instead, as dummy variables. We are free to give them any meaning we wish which preserves the consistency of the axioms.)

Suppose that a line (an ordinary one) is fixed in the Euclidean plane—we shall call it λ (the Greek lambda). This divides the plane into two half-planes—the top one (not including λ) shall be the *-plane.† So the *-plane is half of the ordinary plane. All the points in the *-plane are also *-points; no point of λ is therefore a *-point. As for the *-lines, we have two types: (a) straight lines in the *-plane perpendicular to λ; and (b) semicircles in the *-plane having their centres on λ.

Figure 1 shows two *-lines l_1 and l_2; a line l_3 and part of a circle l_4 which are not *-lines. (Why?)

Having thus defined the primitive terms we are ready to verify that the axioms of absolute geometry are true in the model. If 'are situated' and 'between' are interpreted intuitively, it is not too difficult to show that Groups 1 and 2 are valid. Take, for example, the axiom mentioned in Group 1.

† The reason for excluding λ is given later.

Let A and B be any two fixed *-points. We have to produce a unique *-line through them. So, draw the unique line in the Euclidean sense determined by A and B. If this line is perpendicular to λ, then its upper half is the unique *-line we are after (see l_2 in Figure 1). If not, construct the perpendicular bisector of the segment AB and let it cut λ at C (see l_1 in Figure 1). Clearly, there is only one circle with centre C and radius AC (or BC). The top half of this circle is the unique *-line in this case.

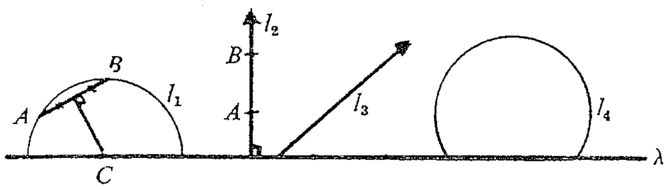

Figure 1

Consider now the Parallel Axiom, and forget about Groups 3 and 4 for a moment. Following Euclid and Hilbert, we define two lines to be parallel if they have no points in common. Thus *-lines are parallel if they have no *-points in common. A glance at Figure 2 should convince you that every *-line passing through

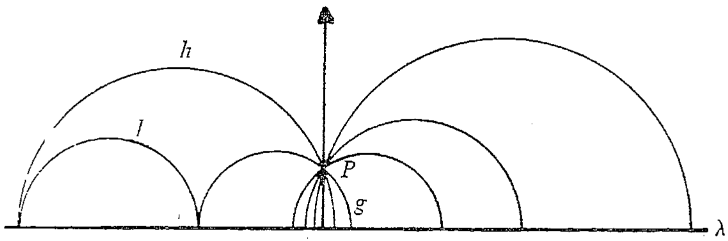

Figure 2

the *-point P is parallel to the *-line l. Even g and h are parallel to l (points on λ are not *-points). In fact, there are infinitely many other *-lines through P that we could have drawn, all parallel to l. Thus the Parallel Axiom has been violated.

To verify that Groups 3 and 4 hold in our model we need the concept 'congruence'. The obvious way to look at it is that two *-segments (defined in the natural way) are congruent if they have the same length. But will our normal ideas on length be good enough for this far fiom normal world? Consider Figure 3 in the light of the axiom which expresses the infinite extent of the line.

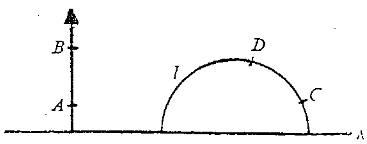

Figure 3

It is obvious that we can find only one *-point D on l such that CD and AB are congruent—the only point on the other side of C with this property is not a *-point. Every *-line of the second kind will have finite length, and every *-line of the first will be infinite in one direction only if we adhere to the usual definition of length. We clearly need a new one!

The following is somewhat unnatural, but it works! We shall denote the *-length of the *-segment AB by AB^*, and its ordinary length by AB as usual. In case A and B lie on a *-line of the second type (b), the ordinary length AU we refer to (see below) is that of the straight line joining A and U. The definition is in two parts depending on whether A and B lie on a *-line of type (a) or (b).

Definition.

(a) In Figure 4,

$$AB^* = \log\frac{BV}{AV}.$$

(b) In Figure 5,

$$AB^* = \log\left(\frac{AU}{BU}\cdot\frac{BV}{AV}\right).$$

Some of you may recognise the term in the brackets in (b) as the cross-ratio $(A, B;\ U, V)$.

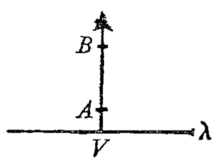

Figure 4

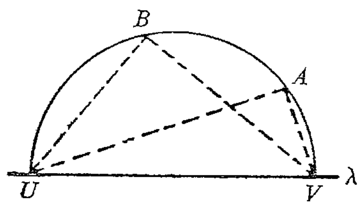

Figure 5

The first thing to observe is that, in both cases,

$$AB^* = -BA^*.$$

Thus the *-lengths of the *-segments AB, BA have the same absolute value. Furthermore, in the Euclidean sense, A is nearer to λ than B, so that AU, BV are greater respectively than BU, AV. Since this entails that $AU/BU, BV/AV$ and consequently $(A, B;\ U, V)$ are all $\geqslant 1$, it follows that $AB^* \geqslant 0$, with $AB^* = 0$ if and only if A and B coincide. This all accords very nicely with our intuition.

Now let A move along the *-line towards V. The ratio AU/BU will grow a little, but nothing in comparison with BV/AV. This ratio increases indefinitely (because AV decreases indefinitely), so that AB^* tends to infinity. This means that the point V is infinitely far from any *-point on the *-line. In a similar way, if we let B move towards U, we can show that U also is 'a point at infinity'. Incidentally, in case any of you have been wondering, this is the reason for not including the points of λ among our *-points!

The task of proving our new concept of congruence consistent with the axioms of Groups 3 and 4 is not too difficult, but it is technical and relies rather heavily on some acquaintance with various transformations in the *-plane (notably reflection) which preserve *-length. Anyone who is interested in going through the details is recommended to the excellent account by Meschkowski in reference 1. We do not intend going through them here.

A few of the strange results which can be proved in this geometry are worth a mention. We have promised that *-triangles (triangles whose sides are *-lines) have angle-sum less than two right-angles. Some of them have hardly any sum at all! Such a *-triangle is shown in Figure 6. A *-circle, i.e., a set of *-points whose

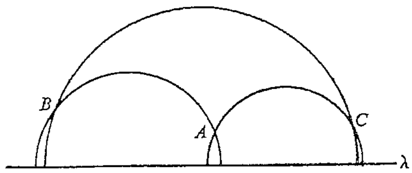

Figure 6

*-distance from a given *-point is constant, turns out to be an ordinary Euclidean circle, but the *-centre is not where you might expect to find it! There exist *-triangles which have no circumscribing *-circle. This is because the circumscribing *-circle will be an ordinary circle through the three vertices. By a suitable choice of these vertices, it is possible to make this circle lie partly outside the *-plane. It is therefore not a *-circle. This situation is shown in Figure 7. Poincaré's

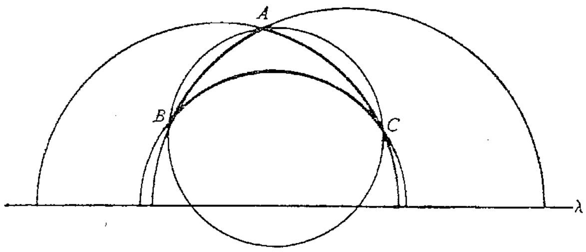

Figure 7

model is also rich in congruence theorems. We have an extra one! Two *-triangles with pair-wise congruent *-angles (i.e., equal angles) are not just similar—they are congruent too. So AAA implies SSS!

5. Remark

We leave non-Euclidean geometry with one final remark. There is another assertion which we have not considered, but which is also a negation of the Parallel Axiom. It is that there are no parallel lines—or, equivalently, all lines intersect. In 1854, Bernhard Riemann, a pupil of Gauss, delivered a lecture to the faculty of the University of Göttingen in which he described a model for absolute geometry (with one modification to the axioms of Order) where there are no parallel lines. Thus, there is a third geometry! Now, if one of these geometries deserves superiority, it is this one, for, by Einstein's Theory of Relativity, a generalised version of Riemann's geometry is true of physical space!

References

1. H. Meschkowski, *Non-Euclidean Geometry* (Academic Press, New York, 1964).
2. Catherine Smallwood, Infinities. *Mathematical Spectrum* **5** (1972/73), 11–15.

Suggestion for further reading

L. R. Lieber, *Infinity* (Rinehart, New York, 1953). I recommend this little book of free prose and art for pure enjoyment. Although mainly concerned with infinite sets, it has a good section on non-Euclidean geometry.

Letter to the Editor

Dear Editor,

Irrational rectangles—dozenal

I should like to add a few remarks to the article by Mr Fletcher on 'Irrational Rectangles', *Mathematical Spectrum*, Volume 3, pages 12–17.

As secretary of the Duodecimal Society of Great Britain (DSGB) I spend much time considering suitable schemes and standards for use with duodecimal numeration, including the question of standard paper sizes. There is a need for standardisation of paper sizes, but whether the series based on the *metre* provides a suitable scheme is debatable. This is not to suggest that the *standard rectangle* discussed by Mr Fletcher is unsound; the principle is mathematically sound, if not aesthetically satisfying. The *standard rectangle* can be based on any linear unit of measurement; it does not *have* to be the metre.

I am not entirely happy with the actual sheets in the A series; the A4, lying as it does between our Quarto and Foolscap, seems to have the advantages of neither; if a different length standard were chosen—say the yard or the foot—then the sheets would approximate better what we have already. (For the yard sheet A4 comes out at about $8'' \times 11''$.)

The DSGB has received several schemes of paper sizes, based on existing length-units (reference 1) or new ones (references 2 and 3). The following is extracted from reference 1; but before you wonder why the figures look peculiar, let me explain the notation.

The duodecimal system, or, as we prefer to call it, the *dozenal* system, is the proper title for base-twelve numeration. In this system everything is grouped by twelves instead of, as in the decimal, by tens. To distinguish dozenal numbers from decimal we usually affix a star (*) to integers in dozenal notation, and use a distinctive *dozenal semicolon* (;) to provide 'decimal-form' fractions for the system. Thus, for example, sixteen is *14 (a dozen-and-four) and 1·5 in the decimal system now becomes 1 ; 6 (one and six-twelfths). Since the base is twelve we must invent two numerals for ten and eleven; we have chosen Ƨ for ten and Ɛ for eleven. Our first integers are 1, 2, 3, 4, 5, 6, 7, 8, 9, Ƨ, Ɛ, 10, 11, 12, ... and so on. This is sufficient for one to understand what follows; more information can be obtained from the DSGB (reference 1).

Let the sheet A0 have an area of 1 square yard. Then the series of sheets becomes:

A0	26 ; 36 × 36 ; 96 ins.	A4	7 ; 6Ɛ × Ƨ ; 85 ins.	
A1	19 ; 49 × 26 ; 36	A5	5 ; 42 × 7 ; 6Ɛ	
A2	13 ; 19 × 19 ; 49	A6	3 ; 95 × 5 ; 42	and so on.
A3	Ƨ ; 85 × 13 ; 19			

Recent research by Mr T. Pendlebury, while working on a dozenal dynamic system of measurements, TGM (reference 3), has uncovered the interesting fact that the proposed metric series corresponds to a dozenal series based on the *grafut* or gravity-foot of his TGM (reference 2). The metric A series, already discussed, corresponds to his B series, and vice versa. The B series, by the way, has a width of 1 metre, length of $\sqrt{2}$ metre for sheet B0. For a yard system sheet B0 would be 1 yard by $\sqrt{2}$ yard, i.e., $1 \times 1;4\mathcal{E}792\ldots$ yds, say $1 \times 1;5$ or $30;0 \times 42;\mathcal{E}$ ins.

Mr Fletcher also mentions the *Golden Rectangle*, traditionally the most satisfying and pleasing to look at. It has often been asserted that there is much common sense in our traditional measures and it is therefore interesting to note that many of our present paper sizes approximate to Golden Rectangles (reference 1). But whereas the standard rectangle can be halved and halved again and preserve its proportions (very useful for enlarging and reducing pictures for printing), a Golden Rectangle cannot be halved to produce another, but it can be quartered to do so (reference 1).

The Golden Rectangle is a rectangle such that, given w for width, h for height, $w/h = h/(w+h)$. In dozenals, taking h as 1, $w = 0;75$ (to two places), and taking w as 1, $h = 1;75$. The Golden Rectangle is associated with the *Golden Section* which is featured in many paintings (references 4 and 5).

I hope these remarks will be of interest to your readers. And, having implied that the metre is not quite as perfect as many would have us believe, should any reader care to write to us to defend it, we will be delighted to hear from him.

Yours sincerely,
S. FERGUSON
(Secretary DSGB)

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1. *Duodecimal Review* *23 (1970) (DSGB, 69 Scotby Road, Scotby, Carlisle CA4 8BG).
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4. F. Land, *The Language of Mathematics*, pp. 222–223 (John Murray, London, 1960).
5. A. J. Cameron, *Mathematical Enterprises for Schools*, pp. 170–173 (Pergamon Press, Oxford, 1966).

Problems and Solutions

Readers who have not yet reached the age of 20 on 1 April 1973 are invited to submit solutions to some or all of the problems below: the most attractive solutions will be published in subsequent issues. When writing to the Editorial Office, please state your full name and the postal address of your school, college or university.

Problems

5.5. Show that there exist irrational numbers a, b such that a^b is rational.

5.6. (Submitted by A. J. Douglas, University of Sheffield.) Let p, q be real numbers with

$$\frac{1}{p}-\frac{1}{q}=1 \quad \text{and} \quad 0 \leqslant p \leqslant \tfrac{1}{2}.$$

Show that

$$p+\tfrac{1}{2}p^2+\tfrac{1}{3}p^3+\ldots = q-\tfrac{1}{2}q^2+\tfrac{1}{3}q^3-\ldots.$$

5.7. (Submitted by D. Woodall, University of Nottingham.) A toy boat floats in a bath with a brick as cargo. The brick is taken out of the boat and placed at the bottom of the bath. Does the water level in the bath rise or fall, and by how much?

5.8. Let f be a polynomial with complex coefficients and leading coefficient 1. Show that there exists a complex number z with modulus 1 such that

$$|f(z)| \geqslant 1.$$

Solutions to Problems in Volume 5, Number 1

5.1. (i) Show that the situation consisting of five nodes each joined to the other four is not a 'planger'. (ii) Show that if a planger consisting of m nodes and a total of n joins is constructed with a pen-knife instead of a pencil, then the piece of paper will fall into $n-m+2$ pieces. (See the article in Volume 5, Number 1 by V. W. Bryant.)

Solution

(i) Suppose that the five nodes are such that each node is joined to the other four. Then there must be a loop as shown in Figure 1. The fifth node must be either inside or

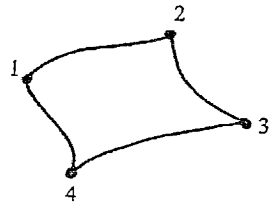


Figure 1

outside this loop. If it is inside, then the joins of 2, 4 and of 1, 3 must both be outside the loop, which is not possible without intersection. If the fifth node is outside the loop, then the join of 1, 3 must be inside the loop and the joins of 2, 4 and of 3, 5 must both be outside the loop, which is again impossible without intersection.

(ii) We use induction on n, the number of joins. If $n = 1$, the planger must be as shown in Figure 2, in which case $m = 1$, and the number of pieces is $2 = n-m+2$, as

Figure 2

required. Now consider $k>1$, and assume the result for all plangers having fewer than k joins. Consider a planger with exactly k joins (and m nodes). On the pencil route for this planger, let the first two nodes be labelled 1, 2 respectively, and construct a new planger by rubbing out an initial segment up to a point somewhere between nodes 1 and 2. If 1 is still a node of this new planger, then in the new planger we have simply decreased the number of joins by one, because $1 \to 2$ is no longer a join. Alternatively, if 1 is no longer a node, then the number of nodes has been reduced by one and the number of joins has been reduced by two ($1 \to 2$ is no longer a join, and the removal of 1 combines two joins into one). Thus, by the inductive hypothesis applied to the new planger, it has

$$(k-1)-m+2 \text{ or } (k-2)-(m-1)+2 \text{ pieces,}$$

i.e., $k-m+1$ pieces in each case. But the number of pieces in the original planger is one more than this, because the effect of removing the initial segment is to glue two pieces together, so the given planger has $k-m+2$ pieces, as required. The result follows for all plangers by induction.

Also solved by Laurence Raphael (Gonville and Caius College, Cambridge).

5.2. In negotiating a sale, the seller S first suggests a price s which the prospective buyer B counters with an offer b ($<s$). S then suggests $\frac{1}{2}(s+b)$, and so they proceed, each offer being the average of the two previous. Find the nth bids made by both S and B, and show that these tend to the same limit (the selling price) as $n \to \infty$. Does it make any difference to the selling price if B takes the initiative by making the first offer, and if so what is the difference?

If S begins by suggesting a price s and B cannot afford to pay more than p, what is the highest offer with which B can counter?

Solution by Laurence Raphael

Let the nth bids made by S and B be s_n and b_n respectively. Then

$$s_n = \tfrac{1}{2}s_{n-1}+\tfrac{1}{2}b_{n-1}, \tag{1}$$

$$b_n = \tfrac{1}{2}s_n+\tfrac{1}{2}b_{n-1}. \tag{2}$$

Eliminating b_{n-1}, we get

$$b_n = \tfrac{3}{2}s_n-\tfrac{1}{2}s_{n-1} \tag{3}$$

or

$$b_{n-1} = \tfrac{3}{2}s_{n-1}-\tfrac{1}{2}s_{n-2}.$$

Substituting in (1), we obtain the difference equation

$$s_n = \tfrac{5}{4}s_{n-1}-\tfrac{1}{4}s_{n-2}.$$

This has solution

$$s_n = A\alpha^n+B\beta^n,$$

where α and β are the roots of the polynomial $x^2-\frac{5}{4}x+\frac{1}{4}$, i.e., we can take $\alpha = 1, \beta = \frac{1}{4}$. Thus

$$s_n = A+\frac{B}{4^n}.$$

From the initial conditions $s_1 = s$, $s_2 = \frac{1}{2}(s+b)$, we obtain

$$s_n = \frac{s+2b}{3}+\frac{2(s-b)}{3.4^{n-1}}. \tag{4}$$

Thus

$$\lim_{n\to\infty} s_n = \frac{s+2b}{3}.$$

From (3) and (4), we obtain

$$b_n = \frac{s+2b}{3}-\frac{s-b}{3.4^{n-1}}$$

and

$$\lim_{n\to\infty} b_n = \frac{s+2b}{3} = \lim_{n\to\infty} s_n.$$

If B makes the first offer, then in place of (1) and (2) we have

$$b_n = \tfrac{1}{2}b_{n-1}+\tfrac{1}{2}s_{n-1},$$

$$s_n = \tfrac{1}{2}b_n+\tfrac{1}{2}s_{n-1}.$$

These are just (1) and (2) with s_n and b_n interchanged. Thus s and b can be interchanged in the whole of the above working (for it did not require $b<s$), and the selling price in this case is $\frac{1}{3}(b+2s)$. The difference between the two selling prices is $\frac{1}{3}(s-b)$.

If, in the former case, B cannot afford to pay more than p, then, when S suggests a price s, the highest offer with which B can counter is b', where

$$p = \frac{s+2b'}{3},$$

which gives

$$b' = \frac{3p-s}{2}.$$

Also solved by Paul R. Rowson (Portsmouth Grammar School), Mangheni Masiga (Makerere University, Kampala).

5.3. A debating society decides to choose the best speech delivered in a given year, so a sequence of scores is allocated to each speech. The first score of a speech is the number of other speeches in the year which have referred back to that speech. The second score of each speech is the sum of the first scores of the speeches which have referred back to that speech. The third score of a speech is the sum of the second scores of the speeches which have referred back to it. And so on. What eventually happens to the scores?

Solution by Laurence Raphael

Let the total number of speeches in the year be n, let the speeches be numbered 1 to n in the order in which they are delivered, and define an $n \times n$ matrix $A = (a_{ij})_{n\times n}$ by

$$a_{ij} = \begin{cases} 1 & \text{if speech } j \text{ referred back to speech } i, \\ 0 & \text{otherwise.} \end{cases}$$

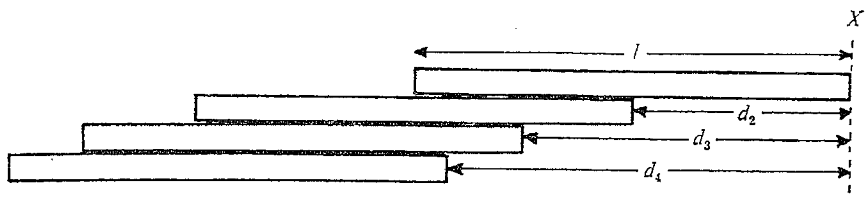
Note that $a_{ij} = 0$ wherever $i \geqslant j$. The first scores of the speeches are given, in order, as the entries in the column vector $v_1 = Av_0$, where v_0 is the column vector with 1 in every place. Note that v_1 has 0 in its nth place. The second scores of the speeches are given, in order, as the entries in the column vector $v_2 = Av_1$. Note that v_2 has 0 in its $(n-1)$th and nth places. Continuing in this way, we see that the nth and all subsequent scores of the speeches are zero.

Also solved by John Barrow (University of Durham).

5.4. There are n identical uniform planks each of length l stacked on top of one another. What is the maximum possible horizontal displacement of the top plank from the bottom one?

Solution by Laurence Raphael

Number the planks from top to bottom and let d_k be the horizontal displacement of the kth plank from the first. This will be maximized when, for $1 \leqslant k \leqslant n$, the centre of gravity of the first k planks occurs directly over the edge of the $(k+1)$th, as shown in the diagram.



In this situation, for $k>1$, d_k will be the horizontal distance of the centre of gravity of the first $k-1$ planks from X; $d_1 = 0$. If, with $k>1$, we take moments about X for the first $k-1$ planks, we obtain

$$(k-1)\, md_k = (k-2)\, md_{k-1} + m(d_{k-1} + \tfrac{1}{2}l),$$

which gives

$$d_k = d_{k-1} + \tfrac{1}{2}l/(k-1).$$

Hence

$$d_n = \frac{1}{2}l\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right),$$

which gives the maximum possible horizontal displacement of the nth plank from the first.

Book Reviews

A First Course in Statistics. By F. N. DAVID. Griffin, London, 1971. Pp. ix+226. £1·90.

This is a reprint of *A Statistical Primer*, first published in 1953, but with the addition of three further chapters on Correlation, Bivariate Grouped Data, and χ^2 and Contingency Tables. It is a work which deserves to be better known in schools than it is, and these additional chapters extend its coverage to most of the A-level syllabuses.

Teachers will find it weak on the mathematical side (deliberately on the part of the author), very little appearing in the way of proof; but very strong on applications, with much clearer discussion of tricky points than normally appears in school text-books. For example, Professor David makes the number of degrees of freedom for χ^2 when using goodness-of-fit tests very clear, pointing out the importance of whether mean and standard deviation are estimated from the sample or are known independently. The examples given are mostly of a fairly trivial biological kind, but it would be unfair to criticise them in a work of this length, for there are a great many of them.

One example, that on pages 97–98, does appear ill-conceived to me. This is a t-test for matched pairs. As, however, there is no correlation between the pairs (it is in fact just below zero), the matching would appear to be unjustified and does not lead to a more sensitive test, in spite of what is said on page 116. This is precisely the sort of difficulty an unwary reader can fall into when a mathematical treatment is avoided.

It is a great pity too that the book is now beginning to show its years. Since 1953 many books have been published for those who wish to apply statistics to particular subjects, and the need for a general book is less urgent. The different applications have differing emphases and no one book can any longer serve economically as an introduction to them all. The lack of any consideration other than a passing mention of distribution-free tests (apart from χ^2) is another disadvantage which is becoming apparent; but on the credit side the author does point her reader in the direction of further developments by including briefly a few topics such as analysis of variance.

Other drawbacks are the absence of complete sets of statistical tables, no bibliography, and a complete lack of exercises. The latter will preclude its use as a text-book in schools. Nevertheless, a copy on the library shelf would be a worth-while acquisition and the book might be mentioned to those who are not taking mathematics in the sixth form but will need statistics at university. A teacher who is starting to teach the subject at A-level will find it useful in answering awkward questions; though I would advise the purchase of Durran's SMP book as a corrective—classroom experiments were unheard of in 1953.

University of Nottingham K. E. SELKIRK

Chapters on the Classical Calculus of Probability. By KÁROLY JORDAN. Akadémiai Kiadó, Budapest, 1972. Pp. xviii + 619. £9·40.

This is a translation by Dr P. Medgyessy of the book originally published in Hungarian in 1956. Jordan completed it in 1946 and considered it his greatest work; its publication in Hungarian was delayed for ten years, and this first translation into English comes a further sixteen years later. It is a measure of the book's value that I found it interesting to read, twenty-six years after its initial publication. As Professor B. Gyires, of Debrecen University, writes most aptly in his Preface, 'This book is of particular interest because it sums up what can be achieved by the methods of classical probability theory'.

Before going into the details of the book, it may be relevant to give a short biography of Charles (Károly) Jordan. Most of these facts have been obtained from Professor Takàcs' obituary of Jordan (*Ann. Math. Statist.* **32** (1961), 1–11) and from the Preface by Professor Gyires. Jordan was born in Budapest in 1871, and after matriculating there continued his studies in Paris and Zürich where he graduated in chemical engineering in 1893. After a year at the University of Manchester, he was appointed to the University of Geneva, where he received a higher doctorate in 1895 for his work in physical chemistry. He returned to Hungary in 1899, and continued his work in mathematics, astronomy and geophysics at the University of Budapest. He became Director of the Budapest Institute of Seismology in 1906, and remained in this post until 1913. After a brief period of teaching at a military academy during the First World War, he was appointed to the University of Technical and Economic Sciences in Budapest, and served there from 1920 to 1950, becoming a full Professor in the University in 1933. He was elected to the Hungarian Academy of Sciences in 1947, and awarded the Kossuth Prize for Mathematics in 1956. During his long scientific career he published some ninety works, of which five are books; he died a few days after his eighty-eighth birthday in 1959.

Although Jordan was a scientist of very wide interests, from about 1910 his research increasingly centered on mathematics; the calculus of finite differences, probability and mathematical statistics, lay at the core of his work. This book incorporates many of his own research results in its ten chapters. He had an extensive knowledge of the history of mathematics, and collected many of the mathematical classics in their original editions. Unfortunately, his library of about five thousand volumes was burned down in 1956 during the Hungarian Revolution. Characteristically, Jordan, then 85, having suffered a mild heart attack as a result of the destruction of his home, took this blow with a wise resignation; he promptly set about correcting the errors in the Hungarian edition of the present book. Jordan was an impressive man; his integrity, his broad cultural outlook, and his profound love of his subject shine throughout this book; it is a mine of valuable information, much of it within the reach of sixth formers and most of it within that of their teachers. If the Hungarian School of probability theorists is so flourishing today, this is in no small measure due to the efforts of Jordan.

Let me now turn to the book itself. This consists of Professor Gyires' Preface summarising the main events in Jordan's life, a list of Jordan's ninety publications, and ten chapters of exposition on various problems of probability. The first introduces the principles of classical probability theory, the second outlines its essential tools, while the third lays out the most important theorems of probability. Chapter IV deals with expectations of random variables; Chapters V and VI investigate repeated trials in the case of one and several variables respectively. The following chapter outlines several practical probabilistic problems such as gambler's ruin, poker, roulette, and many others. Chapter VIII deals with geometrical probabilities and considers, among others, Bertrand's problem; this is the problem that any chord of a circle chosen at random will be longer than the side of its inscribed equilateral triangle. The next chapter is concerned with the theory of errors and least squares methods; the principle of least squares is due to Legendre who wrote about it in 1805, but was further developed by Gauss in 1823. Chapter X outlines the kinetic theory of gases and derives, among many other formulae, that for Maxwell's distribution. The book ends with tables of binomial coefficients, followed by author and subject indices.

It is impossible without going into excessive detail to give an idea of the richness of this book. The exposition is careful, and illustrated with numerous examples. Proofs are almost always easy to follow, and frequently elegant. The entire work is illuminated with Jordan's profound knowledge of his subject, his historical insight, and the results of his research. It is a pity that the work is marred by a large number of misprints; fortunately, few of these detract from its intrinsic scientific value. I cannot recommend the book highly enough: every school library could benefit from having a copy of this splendid work, while for university libraries it is a 'must'.

University of Sheffield — J. GANI

Modern Mathematics Revision. By J. CLAY and A. R. TRIPPETT. George Allen & Unwin Ltd, London, 1972. Pp. 174. £1·05.

If it is found that the final year's text-book in use for C.S.E. or O-level does not sufficiently meet revision requirements, and that access to the earlier books is not practicable, then this volume has been designed to fill the void. It could be used as a teacher's vade-mecum for revision classes, or distributed more widely as a text-book.

It covers approximately an S.M.P., M.M.E. or similar course, but its reader must be enabled to distinguish between the requirements of different projects and of different C.S.E. or O-level examinations; a teacher's guidance is therefore advisable. A majority of examples are more suited to the O-level candidate only, and some are quite exacting.

The text generally succeeds in recalling, with some explanations, most of the major points made in the project text-books. It could not, perhaps, do more in a single volume. Its 28 chapters are: Sets, Sets of Numbers, Implication and Indices, Matrices, Relations and Functions, Translations and Vectors, Reflection and Rotation, Enlargement and Shearing, Transformations and Matrices, Graphical Representation of Equations, Equations and Inequalities, Simultaneous Equations, Linear Programming, Trigonometry, Significance in Numbers and Number Bases, Computation, Logarithms, The Slide Rule, Formulae, Proportionality, Calculations from Graphs, Probability, Statistics, Area and Volume, The Circle and the Sphere, Topology, Geometry, Structure.

The order of the chapters does not seem to be important, especially as some cross-reference during revision would naturally occur. Coverage of each topic is generally good and will probe the weak spots in a candidate; there is no escape, however, from the task of supplementing these exercises where a weakness is diagnosed.

But for diagnosis and a quick revision, the student examples are adequate and appear to be original; they are graded, but the gradient in these short exercises is inevitably steep in some chapters. The back-of-book answers are complete in so far as brevity permits; there are no hints, and some further explanation and amplification will be necessary.

This book renders an honest and straightforward account; there is no overt question-spotting, and no advice on examination technique or model answers! But the bones of the projects have been picked clean of all the most probable examination material. For those who feel the need, this book is worthy of consideration.

University of Nottingham — R. L. LINDSAY

Vectors, Transformations and Matrices. By R. S. HERITAGE and J. D. EDGE. Penguin Education, Harmondsworth, Middlesex, 1972. Pp. 126. £0·50.

This is a book of questions, leading the reader to a standard of about O-level in the topics mentioned in the title. All together, there are only three or four pages (out of a hundred) of text or comment on what should have been learnt from questions, together with a number of references to Mr Heritage's related *Learning Mathematics* series. The book is intended to stand alone, and be useful 'for those sixth form members and college of education students to whom the material is new'. We have, therefore, elementary mathematics for relatively mature students, purveyed in a form considered suitable for self-instruction and requiring active participation by the reader. The questions are carefully arranged so that (with judicious use of the answers) progress should be quite smooth. Of course, the format has all the advantages and disadvantages of a programmed text; on the one hand it can be used with the minimum of supervision, while on the other hand it may be found boring since virtually every question must be attempted and little is left to the imagination.

To start with, there are 47 pages on vectors in algebra and geometry (no questions involving velocities, etc.) including the use of scalar products. These three chapters are sadly lacking in motivation, and work is restricted to two dimensions. The next 32 pages cover isometries and the results of combining isometries. The material is possibly a little rushed, but everything is here including an attractive section on finite and infinite groups of symmetries with pictures of all the types of frieze and wallpaper patterns. A mere 20 pages are then devoted to matrices; I cannot imagine anybody finding this treatment sufficient. Contrary to the claim on the cover, links between the three main themes of the book are hardly mentioned, let alone used to encourage increased understanding.

This book may well be found to fulfil a useful role, but I doubt if it will be the one suggested by the authors. More likely, it will *supplement* study of a full modern O-level course.

Marlborough College C. C. GOLDSMITH

A Short Course in Computational Probability and Statistics. By W. FREIBERGER and U. GRENANDER. Springer-Verlag, New York, 1972. Pp. 155. £3·00.

Introductory courses in probability and statistics have gradually evolved, and those provided at university nowadays are the lineal descendants of courses developed before computers became commonplace. Emphasis has therefore been on formulation in terms of the few standard probability models (Bernoulli, Poisson, Normal, etc.) which can be handled conveniently, whether or not they really seem appropriate. This book challenges the traditional approach and suggests that if computing facilities can be assumed, we should teach methods which can be used with any realistically formulated model.

This thesis is of general interest, but the appeal of the book is bound to be limited. It is written for graduate students in mathematical statistics and operational research, and other readers will not get far without comparable background knowledge. Moreover, the computer programs which illustrate each chapter and form a vital part of the book, are written in APL/360 or APL/1130. Users of other high level languages will find these programs fairly intelligible but of little direct use.

The four main themes developed in the book are (i) random number generation, Monte Carlo methods and simulation, (ii) stochastic methods and Markov chains, (iii) analysis of variance and (iv) time series.

Marlborough College C. C. GOLDSMITH

Notes on Contributors

J. Gani is Professor of Statistics and Head of the Department of Probability and Statistics in the University of Sheffield. He has taught in universities in this country as well as in Australia and the U.S.A. He is very interested in mathematical education and has managed *Mathematical Spectrum* since 1968. His other main interest is the application of probability to problems in biology.

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P. N. Mehta graduated in education from Bombay University and then obtained his Master's degree in Science in pure mathematics from Gujarat University, India. He has eighteen years' teaching experience at secondary and college levels. For the past seven years he has been teaching pure mathematics in the B. P. Baria Science Institute, Navsari, where he is in charge of the college Science Student Improvement Programme. He is chiefly interested in mathematical analysis and astronomy.

Joan M. Holland has taught mathematics in schools in St Andrews and Birmingham. She has also held administrative posts with education authorities in Hertfordshire and in Singapore and Kenya. For the final five years before retiring in 1969 she was Senior Lecturer in Mathematics at Bishop Otter College, Chichester. Since retiring she has had a book, *Studies in Structure*, published. Her retirement hobbies are bridge and gardening.

M. J. Gardner is Senior Lecturer in Medical Statistics in Community Medicine at Southampton University. His main interest since graduating as a mathematician has been in the applications of mathematical and statistical techniques in medical studies. Before going to Southampton to teach he worked for a number of years with the Medical Research Council.

Pat Rogers is now at York University, Toronto. After graduating from Oxford she spent a year at Toronto University, and subsequently she was a lecturer at Ealing Technical College and at the Polytechnic of North London. Her research interest lies in mathematical logic.

Contents

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