Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Harry Burkill
1925–2007.

It is sad when one loses a valued friend. This happened to *Mathematical Spectrum* in 2007 on the death of Harry Burkill. Harry was one of the founding fathers of *Mathematical Spectrum* way back in 1968, and served on the editorial committee until his death, and between 1974 and 1979 as editor. He gave his time generously to the magazine, editing many contributions with the care for which he was renowned and writing biographical articles on H. J. E. Littlewood, G. H. Hardy, N. H. Abel, and G. F. B. Riemann, which well illustrate his skill as a writer – see the references. His choice of subjects for biographies demonstrated his primary mathematical interest in classical analysis.

A refugee from the Nazi regime, Harry was a student a Trinity College, Cambridge. Throughout his professional life he taught at the University of Sheffield, generations of students benefitted from his precise teaching, clear thinking and understanding. He served with distinction for a period as Head of the Department of Pure Mathematics.

All who knew him and worked with him, whether in *Mathematical Spectrum* or in the wider mathematical community, will miss him and remember him with gratitude.

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Poincaré and his Infamous Conjecture

GARETH WILLIAMS

Disclaimer *Topology is a deep and beautiful subject. To provide precise statements and proofs assumes a certain degree of sophistication in the discipline. The current article makes no such assumption. Consequently, technicalities are omitted. Experts may shudder, but it is hoped that an audience with little background in topology will be grateful. For a more comprehensive account, the reader may consult references 1 or 2.*

1. Topology

Elementary geometry was well established in ancient Greece by 300 BC, but there is a newer idea called *topology*. Imagine doing your homework not on paper, but on a sheet of stretchy rubber. Once you have finished, what would happen if you stretched the rubber? Concepts such as *straight* and *parallel* become meaningless – even *size* is challenged. Yet many interesting properties *do* make sense after stretching.

Let us illustrate this with an example. Picture a map of the London Underground. Does it tell you how far it is from King's Cross to Euston, or the direction from one to the other? The answers are no, but millions of travellers navigate everyday with this *topological* map, which conveys how the stations are linked together. If you draw the Underground map on a rubber sheet, and then stretch the rubber, it will still tell you how the stations are configured.

In topology, we imagine that everything is made from rubber, and **we allow ourselves to stretch and squash, but not to cut or glue**. In terms of the rubber London Underground map, this means that you can distort the picture smoothly. Cutting the map might break links between stations, whilst gluing two Underground lines adds extra links. The confusion this would cause London's population is why topologists forbid cutting and gluing!

If one rubber object can be manipulated to look just like another, we say that the two are *topologically the same*. For example, a rubber football could be squashed to look like a rugby ball, so they are topologically the same. Similarly, a handkerchief and a spinnaker sail are topologically the same. Can you suggest further pairs of objects which are topologically the same?

Note that when we talk about a football, we mean the skin or the *surface* of the ball, and not the air inside. Similarly, we distinguish between a *circle* and a *solid circle* or *disc*; see figure 1.

You may like to ask if the following phrase, common with French school children, is accurate from a topologist's point of view. How about a geometer's?

Qu'est-ce qu'un cercle? Ce n'est point carré.
(What is a circle? It's not a square.)

Figure 1 A circle and a solid circle (disc).

Figure 2 Topologically removing an elastic loop from a football.

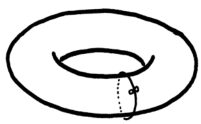

Figure 3 An elastic loop that can never be topologically removed from the torus.

1.1. Elastic loops

Think again of a football. Tie a loop around it with a thread. Topology allows us to imagine an *elastic loop*. We can stretch and squash until it forms a smaller and smaller loop on the football. Eventually it becomes so small that it is a mere speck! We then say that we have *topologically removed* the elastic loop from the football, as in figure 2. Notice as we manipulate the elastic, we never pull it away from the football. All points of the loop remain in contact with the football throughout.

Consider now the surface of the earth – ignoring the rock and magma inside (as we did with the air inside the football). The surface of the earth is topologically the same as the football, so should we expect to be able to repeat our experiment with an elastic loop tied around the equator? The answer is provided by the following result.

Theorem *If two objects X and Y are topologically the same, then we have precisely one of the following situations.*

(i) *You can topologically remove any elastic loop from X and any elastic loop from Y.*

(ii) *You can find an elastic loop λ on X and an elastic loop μ on Y so that neither λ nor μ can be topologically removed.*

As a contrast, consider a ring donut – its mathematical name is a *torus*. (Make certain to distinguish between a torus and a solid torus.) You can find an elastic loop which *cannot* be topologically removed from your torus – no matter how much you stretch or squash it; see figure 3.

One goal of modern topology is to determine if two given objects are topologically the same. Our Theorem is a powerful tool in this pursuit. Try to use it to decide if a football is topologically the same as a torus.

2. Poincaré and manifolds

2.1. The 'father of topology'

Henri Poincaré (1854–1912) came from a distinguished French family: his cousin Raymond was President of the French Republic during World War I. But Henri Poincaré preferred mathematics and physics over politics – highlights of his career include studying the relative motion of a system of three planets, *special relativity* (a brilliant subject attributed to Einstein), and the *Poincaré–Hopf* theorem. The latter is commonly known as the *hairy ball* theorem, saying that you cannot continuously comb a dog's hair without either a parting or a bald spot! His 1895 masterpiece *Analysis Situs* earned Poincaré the title 'father of topology'.

Mathematicians like to characterise things. For example, the integers which are divisible by 5 can be characterised as those with final digit either 0 or 5. Poincaré wished to characterise 'high-dimensional' footballs, in terms of topologically removing elastic loops. He never found this characterisation, but he made a guess as to what it might be. This guess became known as the *Poincaré conjecture*, and engaged mathematicians until very recently. Before we state it, we need to discuss what we mean by a 'high-dimensional' football!

2.2. Manifolds

Topologists work with many different objects. Among the nicest types are *manifolds*. You may have an intuitive idea of what a *surface* is, perhaps the surface of a football or the surface of the earth. Manifolds generalise this notion to other dimensions. We describe a manifold as an n-manifold if it is a generalisation to dimension n (a positive integer).

Most people have an intuitive idea about the following statements.

- A straight line is one-dimensional.
- A plane is two-dimensional.
- Space is three-dimensional.

Once you feel happy with these statements, manifolds are not much harder.

- A 1-manifold looks one-dimensional (like part of a straight line) if you zoom in at any point.
- A 2-manifold looks two-dimensional (like part of a plane) if you zoom in at any point.
- A 3-manifold looks three-dimensional (like a piece of space) if you zoom in at any point.

For example, pick a point on a circle (remember the difference between a circle and a solid circle). Zoom in on that point, at great magnification. As in figure 4, the result will look like a straight line segment, i.e. one dimensional. So a circle is a 1-manifold.

Consider what happens when you zoom in closely to the earth's surface, or a football, or a torus. You should convince yourself that each of these objects is a 2-manifold. Try also to find

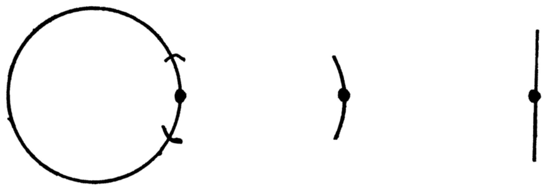

Figure 4 A small portion of a circle looks like a straight line segment when we zoom in closely enough.

some objects which are *not* manifolds. For example, what happens when you zoom in at the centre of a letter X?

Let us now list a few manifolds. The circle and football are examples of what topologists call *spheres*. We use the notation S^1 for a circle – 'S' for sphere and '1' because a circle is a 1-manifold. The football is known as S^2. In Euclidean space, they can be described symbolically as follows:

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\}, \qquad S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

Can you spot a pattern and guess how we might describe something called S^3? The answer is

$$S^3 = \{(w, x, y, z) \mid w^2 + x^2 + y^2 + z^2 = 1\},$$

and this object is a 3-manifold, called the 3-sphere. (Naturally enough, S^2 is called the 2-sphere and S^1 the 1-sphere. What is less obvious is that there is also a 0-sphere and an n-sphere for each positive integer n.)

Picturing S^3 is tricky, and takes practice, but 3-manifolds are worthwhile objects. Pieces of the universe appear three-dimensional when we look closely at them, so the universe can be modelled as a 3-manifold. A natural question to ask is *which* 3-manifold? The answer is not known, but see reference 3 for a more comprehensive discussion.

3. The Poincaré conjecture

We now rephrase Poincaré's goal as *a characterisation of S^3 in terms of topologically removing elastic loops*. In 1900, Poincaré made a first guess, which he proved wrong in 1904 (the breakthrough arose when he considered something now called *Poincaré dodecahedral space*). Inspired, this motivated Poincaré to probe further, leading to the following conjecture.

The Poincaré conjecture (1904) *If a 3-manifold has the property that any elastic loop can be topologically removed, then it is topologically the same as S^3.*

Remark Recall our initial disclaimer. The Poincaré conjecture is a precise statement, whereas the statement above is more suited to this article.

In other words, Poincaré asked if S^3 is characterised by being the only 3-manifold with the property that all elastic loops can be topologically removed from it. (You may have thought of Euclidean 3-space as a 3-manifold in which any elastic loop can be topologically removed. This is *not* a counter-example to the Poincaré conjecture, since a technicality, omitted in the spirit of the remark, precludes it.)

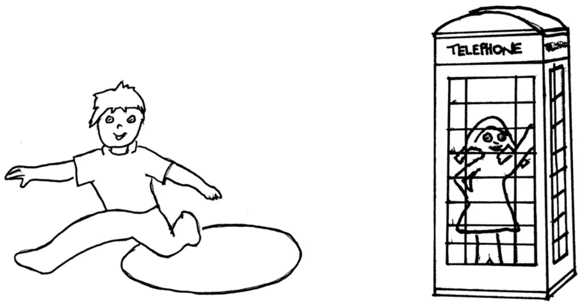

Figure 5 Escaping from S^1 (a circle) is easy; escaping from S^2 (topologically the same as a telephone box with the door glued shut) is not!

3.1. Early attempts at proof

It became apparent that the Poincaré conjecture would be difficult to settle, and mathematicians began to despair. Characterisations of S^4, S^5, and so on were considered, and by 1961 the analogous result was known for S^n with $n \geq 5$. In 1982, S^4 was dealt with, but S^3 remained an unsolved mystery.

We might expect that, as n increases, questions about S^n become harder to answer, but beyond $n = 2$ the opposite is true! In higher dimensions, the topologist has more space, more freedom, in which to manoeuvre. Consider the following illustration.

Suppose that someone asks you to vacate a telephone box, *without* opening the door. Naturally, you can't do it. Now, suppose that on the beach someone draws a circle in the sand around you. You can easily escape without breaking the circle; see figure 5.

The two situations are analogous – escape from within a sphere – but the spheres (S^1 and S^2) are of different dimensions. It is the third dimension that we inhabit that allows you to jump over the circle in the sand. If we had an extra, fourth, dimension available, we could use it to escape from the telephone box. Likewise, in a 3-manifold there is not enough 'room' to do all that we can in higher dimensions. For a delightful account of dimension-related phenomena, reference 4 cannot be recommended too highly.

So the Poincaré conjecture remained a conjecture, but it grew in stature as a holy grail for mathematicians. In 2000, The Clay Mathematics Institute listed seven problems, judged to be the most difficult and important mathematical problems of the time, and offered a prize of \$1 000 000 for the first solution to each. (Prospective prize-winners can find out about the *Millennium Problems* in reference 1.) The inclusion of the Poincaré conjecture thrust the quest for a proof, or counter-example, into the limelight. The race was on, and the conclusion does not disappoint!

3.2. Grigory Perelman and the first proof

In 2002 and 2003, almost one hundred years after the Poincaré conjecture was formulated, the Russian mathematician Grigory Perelman uploaded three papers onto the internet. This was peculiar, because papers are usually submitted to learned journals, but Perelman himself is somewhat mysterious too.

Perelman lives in St. Petersburg, where he has resigned from his job, and has not made attempts to publish his three papers formally. As a child, he won the International Mathematical

Olympiad with a perfect score, and went on to earn a reputation as a mathematician who thought deeply and seldom made mistakes.

Perelman's three papers did not mention the Poincaré conjecture. They talked about the *Geometrization conjecture*. In 1983, the American mathematician William Thurston conjectured that any 3-manifold can be decomposed into pieces, each with one of eight possible structures. Crucially, the Poincaré conjecture is a consequence of the Geometrization conjecture, and mathematicians started to believe that Perelman's papers provided the final steps of a proof of the Geometrization conjecture. (The method of proof was devised by Richard Hamilton, another American mathematician. The key idea is called *Ricci flow,* which 'smoothes out' structures on manifolds until they resemble one of the eight specified in the Geometrization conjecture.)

Despite no formal publication, most of the mathematical community have now accepted that Perelman's work is correct, if lacking details. In 2006, Perelman was awarded a Fields Medal (the most prestigious prize in mathematics), but the delightful Russian responded with the words 'I refuse', making him the first person ever to turn down such an honour.

Many other mathematicians were involved in the work, notably the Chinese mathematician Shing-Tung Yau. Drama followed when *The New Yorker* magazine alleged that Yau denounced Perelman's contribution in an attempt to divert credit to himself (see reference 5). Yau responded with formal steps for defamation.

Now that the dust is settling, the consensus is that Perelman's work proves the Geometrization conjecture, and hence the Poincaré conjecture. Others have expanded the details, resulting in a 327 page paper which gives a complete account. But the problem which tortured mathematicians for almost a century was ultimately laid to rest by a stroke of genius from a Russian recluse.

And as for the \$1 000 000 prize? Officially, a solution should be published in a 'refereed mathematics publication of worldwide repute', something which Grigory Perelman seems to have no interest in doing. Controversy exists as to whether the rules should be reinterpreted in this electronic age, and speculation continues over what Perelman might do if he is ever offered the million dollar prize.

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The Reflecting Property of Parabolas

THEODOROS M. VALAHAS and ANDREAS BOUKAS

Introduction

Greek mathematicians knew the reflecting property of the parabola (i.e. the fact that all light rays parallel to the axis of symmetry, after reflection, pass through the same point) from the time of Archimedes. This property is discussed in most calculus books, but the fact that this property is unique to the parabola is usually not mentioned. In this article we give an easy proof of this property using basic calculus techniques.

The problem

Find the equation of a curve (representing a mirror) with the property that all light rays parallel to the x-axis, after reflection, pass through the same point (called the focus of the mirror) on the x-axis.

The solution

Our solution uses the fact that the angle of reflection of a light ray is equal to its angle of incidence; both angles are measured from the normal (see figure 1). Let l be a light ray parallel

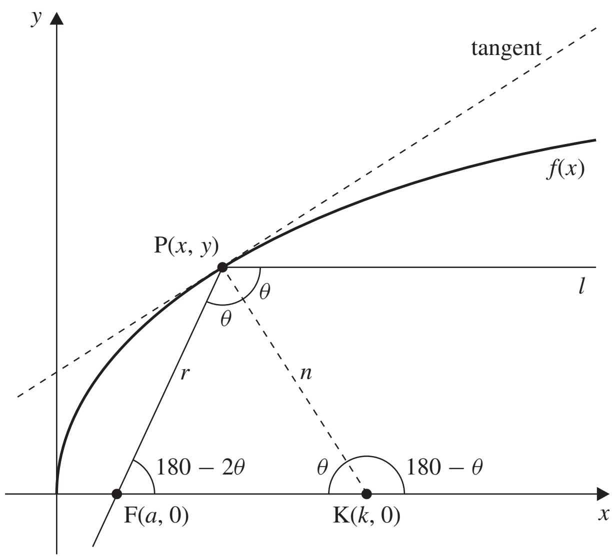

Figure 1

to the x-axis, $P(x, y)$ be the point of incidence on the curve, $F(a, 0)$ be the common point on the x-axis through which reflected rays pass, and $K(k, 0)$ be the point where the normal to the point of incidence meets the x-axis.

The slope of the tangent at P is $\mathrm{d}y/\mathrm{d}x$, so the slope m_1 of the normal line PK is

$$\frac{-1/}{\mathrm{d}y/\mathrm{d}x},$$

we assume that $\mathrm{d}y/\mathrm{d}x \neq 0$. If we denote by θ the angle of incidence of the light ray at P, then the slope m_2 of PF is $\tan(180 - 2\theta)$; see Figure 1. Also $m_1 = \tan(180 - \theta)$, so that

$$m_1 = -\tan\theta = -\frac{1}{\mathrm{d}y/\mathrm{d}x}.$$

Thus,

$$\begin{aligned} m_2 &= \tan(180 - 2\theta) \\ &= -\tan(2\theta) \\ &= \frac{2\tan\theta}{\tan^2\theta - 1}, \end{aligned}$$

so that

$$m_2 = \frac{2(\mathrm{d}y/\mathrm{d}x)}{1 - (\mathrm{d}y/\mathrm{d}x)^2}.$$

Also $m_2 = y/(x - a)$, so we obtain the differential equation

$$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(x - a)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) - y = 0,$$

which can be solved as a quadratic equation in $\mathrm{d}y/\mathrm{d}x$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(x - a) \pm \sqrt{(x - a)^2 + y^2}}{y}.$$

To solve this we make the substitution $y = u(x - a)$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u + (x - a)\frac{\mathrm{d}u}{\mathrm{d}x}.$$

This gives

$$\frac{u\,\mathrm{d}u}{-1 \pm \sqrt{1 + u^2} - u^2} = \frac{\mathrm{d}x}{x - a}.$$

We make the further substitution $1 + u^2 = t^2$ to give

$$\frac{\mathrm{d}t}{\pm 1 - t} = \frac{\mathrm{d}x}{x - a},$$

which integrates to

$$\ln|\pm 1 - t| = \ln|x-a| - \ln c,$$

where c is a constant. This gives

$$(x-a)(\pm 1 - t) = c.$$

If we substitute back we obtain

$$\pm(x-a) - \sqrt{(x-a)^2 + y^2} = c$$

or

$$\sqrt{(x-a)^2 + y^2} = \pm(x-a) - c.$$

This squares to give

$$y^2 = \mp 2c(x-a) + c^2$$

or (writing $\mp c$ as c)

$$y^2 = 2c(x-a) + c^2,$$

i.e. a family of parabolas. The member of this family which passes through the origin is given by

$$-2ca + c^2 = 0 \quad \text{or} \quad c = 2a,$$

giving the familiar equation $y^2 = 4ax$.

Theodoros M. Valahas *received his PhD from the Massachusetts Institute of Technology in 1969 and is now Head of the Mathematics and Natural Sciences Department of The American College of Greece.*

Andreas Boukas *received his PhD in Mathematics from Southern Illinois University in 1988. He is now Professor of Mathematics at The American College of Greece.*

A computer is to be programmed to select the smaller of two given unequal positive integers m and n if $m+n$ is even and the larger of the two if $m+n$ is odd. Can you find a simple formula which will do this?

C/o A. A. Khan, Regional Office, Indian Overseas Bank, Ashok Marg, Lucknow, India

M. A. Khan

Fibonacci Numbers from a Long Division Formula

ABBAS ROOHOL AMINI

The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

where $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, has properties too numerous to list, and each property has almost as many ways of proving it. As a last resort, there is always induction! Our starting point here is long division. We ask: what is the remainder when x^n $(n \geq 2)$ is divided by $x^2 - x - 1$? This can be done by writing out a long division, in which case it turns out that the quotient starts like

$$x^{n-2} + x^{n-3} + 2x^{n-4} + 3x^{n-5} + 5x^{n-6} + \cdots .$$

In other words, the coefficients are the terms of the Fibonacci sequence. In fact,

$$x^n = (x^2 - x - 1)(F_1 x^{n-2} + F_2 x^{n-3} + F_3 x^{n-4} + \cdots + F_{n-2} x + F_{n-1}) + (F_n x + F_{n-1}). \quad (1)$$

This is easily verified by expanding the right-hand side and using the recurrence formula

$$F_n - F_{n-1} - F_{n-2} = 0, \quad \text{for } n \geq 3.$$

From this long division we can deduce a number of properties of the Fibonacci sequence.

For example, put $x = 1$ in (1). Then we obtain

$$1 = (1 - 1 - 1)(F_1 + F_2 + \cdots + F_{n-1}) + F_n + F_{n-1},$$

so that

$$F_1 + F_2 + \cdots + F_{n-2} = F_n - 1, \quad \text{for } n \geq 3,$$

or, raising n by 2,

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1, \quad \text{for } n \geq 1.$$

Now, put $x = -1$ in (1) and make n even, say $n = 2k$ for some positive integer k. Then we obtain

$$1 = (1 + 1 - 1)(F_1 - F_2 + F_3 - \cdots - F_{2k-2} + F_{2k-1}) + (-F_{2k} + F_{2k+1}),$$

so that

$$F_1 - F_2 + F_3 - \cdots - F_{2k-2} + F_{2k-1} = 1 + F_{2k-2}, \quad \text{for } k \geq 2,$$

because

$$F_{2k} - F_{2k-1} = F_{2k-2}.$$

The roots of the polynomial $x^2 - x - 1$ are $\frac{1}{2}(1 \pm \sqrt{5})$. We write

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

This is the famous *golden ratio*. Thus, the roots of $x^2 - x - 1$ are ϕ and $1 - \phi$, and

$$\phi^2 - \phi - 1 = 0.$$

If we put $x = \phi$ in (1), we obtain

$$\phi^n = \mathrm{F}_n \phi + \mathrm{F}_{n-1}, \quad \text{for } n \geq 2. \tag{2}$$

If we put $x = 1 - \phi$ in (1), we obtain

$$(1 - \phi)^n = \mathrm{F}_n(1 - \phi) + \mathrm{F}_{n-1}, \quad \text{for } n \geq 2. \tag{3}$$

If we multiply (2) and (3), we obtain

$$(\phi - \phi^2)^n = \mathrm{F}_n^2(\phi - \phi^2) + \mathrm{F}_n\mathrm{F}_{n-1}\phi + \mathrm{F}_n\mathrm{F}_{n-1}(1 - \phi) + \mathrm{F}_{n-1}^2,$$

so that

$$\begin{aligned}(-1)^n &= -\mathrm{F}_n^2 + \mathrm{F}_n\mathrm{F}_{n-1} + \mathrm{F}_{n-1}^2 \\ &= -\mathrm{F}_n^2 + \mathrm{F}_{n+1}\mathrm{F}_{n-1},\end{aligned}$$

whence

$$\mathrm{F}_{n+1}\mathrm{F}_{n-1} - \mathrm{F}_n^2 = (-1)^n,$$

yet another well-known formula.

Let $m, n \geq 2$. Then we have

$$\phi^{m+n} = \phi^m \phi^n.$$

From (2), this gives

$$\begin{aligned}\mathrm{F}_{m+n}\phi + \mathrm{F}_{m+n-1} &= (\mathrm{F}_m\phi + \mathrm{F}_{m-1})(\mathrm{F}_n\phi + \mathrm{F}_{n-1}) \\ &= \mathrm{F}_m\mathrm{F}_n\phi^2 + (\mathrm{F}_m\mathrm{F}_{n-1} + \mathrm{F}_{m-1}\mathrm{F}_n)\phi + \mathrm{F}_{m-1}\mathrm{F}_{n-1} \\ &= (\mathrm{F}_m\mathrm{F}_n + \mathrm{F}_m\mathrm{F}_{n-1} + \mathrm{F}_{m-1}\mathrm{F}_n)\phi + (\mathrm{F}_m\mathrm{F}_n + \mathrm{F}_{m-1}\mathrm{F}_{n-1}),\end{aligned}$$

because $\phi^2 = \phi + 1$. Because ϕ is irrational, we can equate coefficients of 1 and ϕ to give

$$\mathrm{F}_{m+n-1} = \mathrm{F}_m\mathrm{F}_n + \mathrm{F}_{m-1}\mathrm{F}_{n-1}. \tag{4}$$

If we put $m = n + 1$, this gives

$$\begin{aligned}\mathrm{F}_{2n} &= \mathrm{F}_{n+1}\mathrm{F}_n + \mathrm{F}_n\mathrm{F}_{n-1} \\ &= \mathrm{F}_{n+1}(\mathrm{F}_{n+1} - \mathrm{F}_{n-1}) + \mathrm{F}_n\mathrm{F}_{n-1} \\ &= \mathrm{F}_{n+1}^2 - (\mathrm{F}_{n+1} - \mathrm{F}_n)\mathrm{F}_{n-1} \\ &= \mathrm{F}_{n+1}^2 - \mathrm{F}_{n-1}^2.\end{aligned} \tag{5}$$

If we put $m = n + 2$ in (4), we have

$$\begin{aligned} F_{2n+1} &= F_{n+2}F_n + F_{n+1}F_{n-1} \\ &= (F_{n+1} + F_n)F_n + F_{n+1}F_{n-1} \\ &= F_{n+1}(F_n + F_{n-1}) + F_n^2 \\ &= F_{n+1}^2 - F_n^2. \end{aligned} \tag{6}$$

If we put $m = 2n + 1$ in (4), we obtain

$$\begin{aligned} F_{3n} &= F_{2n+1}F_n + F_{2n}F_{n-1} \\ &= (F_{n+1}^2 + F_n^2)F_n + (F_{n+1}^2 - F_{n-1}^2)F_{n-1} \qquad \text{from (5) and (6)} \\ &= F_{n+1}^2(F_n + F_{n-1}) + F_n^3 - F_{n-1}^3 \\ &= F_{n+1}^3 + F_n^3 - F_{n-1}^3. \end{aligned}$$

Readers may like to look for formulae for F_{4n} and F_{5n}, for example, in a similar way. The possibilities seem endless!

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Mathematical Spectrum Awards for Volume 39

Prizes have been awarded to the following student readers for contributions in Volume 39:

Jonathan Smith

for the article ‘80% Incan’

Fei Yao

for the article ‘Is Anyone Sitting in the Right Place?’ (with Paul Belcher);

Vicky Bird

for the review ‘When Computers were Human’

The editors remind readers that prizes are available annually for student contributions as follows: up to the value of £50 for articles, and up to £50 for letters, solutions to problems and other items.

Two Derivations of a Higher-Order Newton-Type Method

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In this article, we present two different derivations of a higher-order Newton-type rootfinding method. We provide a numerical example implementing Newton's method and the higher-order method. The results are previously known; undergraduate co-authors Chris and Nathan each re-discovered parts of these ideas as they participated in their co-author professor's numerical methods course, and we present the progression of their ideas here.

The Phillips idea

'What if we include another term', Chris asked as we looked at the standard derivation of Newton's method involving a two-term Taylor expansion.

Here is that standard derivation. We want to find a root $x = r$ of $f(x) = 0$. We expand $f(x)$ in a two-term Taylor expansion, i.e. $f(x) \approx f(a) + f'(a)(x-a)$. We replace x with r and a with x_n, obtaining

$$\begin{aligned} f(r) &\approx f(x_n) + f'(x_n)(r - x_n), \\ 0 &\approx f(x_n) + f'(x_n)(r - x_n), \end{aligned}$$

where we have used the fact that $f(r) = 0$ since r is a root of $f(x) = 0$. Solving for r, we get

$$r \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

This result suggests the Newton's method iteration scheme, given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

Now, to address Chris's question, we will expand $f(x)$ in a three-term Taylor expansion and again replace x with r and a with x_n, obtaining

$$\begin{aligned} f(r) &\approx f(x_n) + f'(x_n)(r - x_n) + \frac{f''(x_n)}{2}(r - x_n)^2, \\ 0 &\approx f(x_n) + f'(x_n)(r - x_n) + \frac{f''(x_n)}{2}(r - x_n)^2. \end{aligned}$$

This is quadratic in r. Using the quadratic formula to solve for r (or, more elegantly, to solve first for $r - x_n$), we get

$$r \approx x_n + \frac{-f'(x_n) \pm \sqrt{(f'(x_n))^2 - 2f''(x_n)f(x_n)}}{f''(x_n)}.$$

This result suggests a modified Newton's method iteration scheme, given by

$$x_{n+1} = x_n + \frac{-f'(x_n) \pm \sqrt{[f'(x_n)]^2 - 2f''(x_n)f(x_n)}}{f''(x_n)}. \tag{2}$$

See, for example, reference 1 for this result.

The Olson idea, part I

Nathan noticed that it is necessary in (2) to evaluate a square root, which in general also requires an iterative method. He decided instead to approximate the square root itself by a two-term Taylor expansion, i.e.

$$\sqrt{1+u} \approx 1 + \tfrac{1}{2}u, \qquad \text{for } |u| << 1.$$

The results are as follows (where we have assumed that $f(x_n)$ is close to 0 so that the Taylor approximation for the square root is good):

$$\begin{aligned} x_{n+1} &= x_n + \frac{-f'(x_n) \pm \sqrt{[f'(x_n)]^2 - 2f''(x_n)f(x_n)}}{f''(x_n)} \\ &= x_n + \frac{-f'(x_n) \pm |f'(x_n)|\sqrt{1 - 2f''(x_n)f(x_n)/[f'(x_n)]^2}}{f''(x_n)} \\ &\approx x_n + \frac{-f'(x_n) \pm |f'(x_n)|[1 - f''(x_n)f(x_n)/[f'(x_n)]^2]}{f''(x_n)} \\ &\approx x_n + \frac{-f'(x_n) \pm |f'(x_n)|}{f''(x_n)} - \frac{\pm f(x_n)}{|f'(x_n)|}. \end{aligned} \tag{3}$$

Suppose that $f'(x) > 0$ throughout some interval about the root r, that we choose an initial guess x_0 in this interval, and that at every iteration we remain in this interval. Then, taking the plus signs in (3), the formula reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Note that Nathan thus has retrieved the original Newton's method scheme given in (1). Taking the minus signs in (3), the formula reduces to

$$x_{n+1} = x_n - \frac{2f'(x_n)}{f''(x_n)} + \frac{f(x_n)}{f'(x_n)},$$

which numerical experimentation has shown us does not converge to the root in general. Similarly, if we suppose that $f'(x) < 0$ throughout some interval about the root r, choose an initial guess x_0 in this interval, and remain in this interval for every iteration, then we obtain the same two results given above, this time taking the minus signs to obtain Newton's method and the plus signs to obtain the nonconvergent result.

To get a result different from Newton's method, Nathan included a third term in his Taylor expansion for the square root, i.e.

$$\sqrt{1+u} \approx 1 + \tfrac{1}{2}u - \tfrac{1}{8}u^2, \quad \text{for } |u| << 1.$$

The modified results are shown as follows (where again we have taken the plus sign):

$$\begin{aligned} x_{n+1} &= x_n + \frac{-f'(x_n) \pm f'(x_n)\sqrt{1 - 2f''(x_n)f(x_n)/[f'(x_n)]^2}}{f''(x_n)} \\ &\approx x_n + \frac{1}{f''(x_n)}\left(-f'(x_n) \pm f'(x_n)\left(1 - \frac{f''(x_n)f(x_n)}{[f'(x_n)]^2} - \frac{[f''(x_n)]^2[f(x_n)]^2}{2[f'(x_n)]^4}\right)\right) \\ &\approx x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)[f(x_n)]^2}{2[f'(x_n)]^3}. \end{aligned} \tag{4}$$

See, for example, reference 2, where this method is given and it is stated that the method yields, in general, cubic convergence. This means that, roughly speaking, if we cut the error in the nth approximation x_n in half, the error in the resulting $(n+1)$th approximation x_{n+1} will decrease by approximately a factor of 8. For comparison, the standard Newton's method yields, in general, quadratic convergence; cutting the error in the nth approximation x_n in half, the error in the resulting $(n+1)$th approximation x_{n+1} would decrease by approximately a factor of 4.

The Olson idea, part II

Nathan also wondered what would happen if we worked instead with inverse functions. Again, he re-discovered some interesting previously known results. See, for example, reference 3. We will start with $y = f(x)$ and assume that $f(x)$ has an inverse function (at least if we restrict the domain of $f(x)$). Let $x = f^{-1}(y) = g(y)$. Note that since

$$y = f(x) = f(g(y)),$$

we can differentiate with respect to y on both sides and obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}[y] &= \frac{\mathrm{d}}{\mathrm{d}y}[f(g(y))], \\ 1 &= f'(g(y))g'(y), \\ 1 &= f'(x)g'(y), \end{aligned} \tag{5}$$

which gives

$$y_n = f(x_n) \quad \text{and} \quad g'(y_n) = \frac{1}{f'(x_n)}.$$

We proceed by expanding $g(y)$ in a two-term Taylor expansion as follows:

$$\begin{aligned} x &= f^{-1}(y) \\ &= g(y) \\ &\approx g(y_n) + g'(y_n)(y - y_n). \end{aligned}$$

Evaluating this at $y = 0$ to approximate a root $x = r$ of $y = f(x)$, we obtain

$$\begin{aligned} r &\approx g(y_n) - g'(y_n)y_n \\ &\approx x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{6}$$

From (6) we have once more retrieved the standard Newton's method formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

What if we proceed as in the Olson idea, part I, and use a three-term Taylor expansion of the inverse function? Since

$$g'(y) = \frac{1}{f'(g(y))},$$

we can differentiate with respect to y and obtain

$$\begin{aligned} g''(y) &= -\frac{1}{[f'(g(y))]^2} f''(g(y))g'(y) \\ &= -\frac{f''(x)}{[f'(x)]^3}, \end{aligned}$$

where we have again used (5). We proceed by expanding $g(y)$ in a three-term Taylor expansion as follows:

$$\begin{aligned} x &= g(y) \\ &\approx g(y_n) + g'(y_n)(y - y_n) + \frac{g''(y_n)}{2}(y - y_n)^2. \end{aligned}$$

Again, evaluating this at $y = 0$ to approximate a root $x = r$, we obtain

$$\begin{aligned} r &\approx g(y_n) - g'(y_n)y_n + \frac{g''(y_n)}{2}y_n^2 \\ &\approx x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)[f(x_n)]^2}{2[f'(x_n)]^3}. \end{aligned}$$

We have thus retrieved the result (4) in The Olson idea, part I:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)[f(x_n)]^2}{2[f'(x_n)]^3}.$$

A numerical example

We conclude with a numerical example of Newton's method and the modified method given by (4). We apply each method to approximate $\sqrt[3]{2}$ as the root of $f(x) = x^3 - 2 = 0$. We use an initial value $x_1 = 10$. The results are shown in table 1, where we can see the more rapid convergence provided by the modified method in (4).

Note, in conclusion, that the methods discussed here can be extended further provided that we are willing to compute higher-order derivatives.

Table 1 First ten iterations in approximating $\sqrt[3]{2}$ as the root of $f(x) = x^3 - 2 = 0$, using Newton's method and (4).

Iteration n	Newton's method x_n	Magnitude of error
1	10	8.740 078 950
2	6.673 333 333	5.413 412 283
3	4.463 858 934	3.203 937 884
4	3.009 363 019	1.749 441 969
5	2.079 855 871	0.819 934 821
6	1.540 684 640	0.280 763 590
7	1.307 977 495	0.048 056 446
8	1.261 665 070	0.001 744 020
9	1.259 923 460	0.000 002 409 68
10	1.259 921 050	0.000 000 000 004 608 54
Iteration n	Modified method x_n	Magnitude of error
1	10	8.740 078 950 105 130 0
2	5.566 662 222 222 22	4.306 741 172 327 350 0
3	3.128 363 465 224 16	1.868 442 415 329 290 0
4	1.850 029 647 067 40	0.590 108 597 172 530 0
5	1.331 924 782 038 58	0.072 003 732 143 709 9
6	1.260 253 211 587 05	0.000 332 161 692 172 3
7	1.259 921 049 933 32	0.000 000 000 038 447 5
8	1.259 921 049 894 87	
9	1.259 921 049 894 87	
10	1.259 921 049 894 87	

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How Much Money do You Need?

IAN SHORT

We examine strategies for paying for shopping that minimise the number of coins and notes in your pocket.

The other day I was shopping in Aldi and my bill came to £10.46. Without the appropriate money to pay exactly that amount, I gave the woman serving £20.57. She returned in change a ten pound note, two twenty pence pieces and a penny. I said 'that's not right', and she agreed and replaced the two twenty pence pieces with a fifty pence piece. This was even less right, and I thought to pursue the matter, but she had moved swiftly on to the next customer; no time is wasted in Aldi. I left with my winnings.

These events led me to consider two possibilities. Firstly, I could make some cash by paying for shopping with confusing quantities of money. Secondly, I wondered what strategy I should employ in paying for shopping to minimise the number of coins and notes that I leave with. For example, if I have a fifty pence piece, a ten pence piece, and a penny, and I owe thirty-six pence, then (if I pay with all three coins) I am given in change a twenty pence piece and a five pence piece, and I could not possibly leave with any fewer coins than that. This article is about strategies for minimising the number of coins and notes in your pocket.

If you are really desperate to minimise the number of coins and notes in your keeping, then you can chuck them all away. We assume that you definitely do want to keep all the money, and that you are returned the correct change from every transaction. Another necessary assumption is that you are returned the minimal amount of change. By 'amount of change' we refer to the number of coins and notes given to you, rather than the total value of those coins and notes. Indeed, it occasionally happens that the shop assistant is without five pound notes and returns in change numerous one and two pound coins. Clearly such occurrences disrupt our problem as they are not predictable, so we exclude them.

This is a stronger assumption than it first appears, for if the shop assistant returns the minimal amount of change, an optimum strategy for minimising your collection of coins and notes is to give the shop assistant *all* of your money. They do the optimising for you! This observation is not without reasonable practical use: there are an increasing number of supermarkets that have self-service tills. You scan your own shopping, then put money into a machine which gives you change. Rather than wasting time thinking 'how best should I pay for my shopping to minimise the number of coins and notes in my pocket?', just fill the machine with all your money and it returns the least possible amount of change. Make sure you work up with denominations though – put pennies in first, then two pennies, and so on up to fifty pound notes. If you do it in the reverse order, then the machine will return change as soon as it acknowledges that you have inserted sufficient funds.

This strategy of emptying the entire contents of your wallet on to the shop assistant's desk seems practically flawed, in that you will annoy the shop assistant and queue, and more than likely confuse everyone sufficiently that you do not leave with the minimum number of coins and notes. Yet after the first time you apply this scheme (you could, for example, first visit one of the aforementioned change sorting machines), you are left with a sufficiently small quantity of money that thereafter it is not so unreasonable to pay with all your money.

In this article, we describe practical measures for implementing this scheme of paying for shopping with all your money. The notation £N is used to refer to a quantity of money £$d_1d_2.d_3d_4$, where the d_i are digits (hence, $d_1d_2.d_3d_4$ is the decimal expansion of N, with pounds as units). We assume throughout that the cost of your shopping is less than £100, and we recall the assumption that following any purchase you are returned the minimal amount of change. We identify a collection Ω of coins and notes that has two particular properties. Firstly, given any amount £N less than £100, we can choose a subcollection A of Ω whose total value is £N. Moreover, any other collection of coins and notes that also has this feature cannot contain fewer coins and notes than Ω. Secondly, provided that you replenish your wallet with money such that your money is always a subcollection of Ω, and provided that you pay for goods with all your cash, then your collections of coins and notes will never extend beyond Ω. Many other collections of coins and notes have these properties, but all these other collections contain Ω. The collection Ω will be defined later. We first look at the simpler case of quantities of money less than 10p.

Question 1 What is the minimum number of coins that you need to be able to pay with exact money any value less than 10p, and which collections of coins have this property?

The phrase 'pay with exact money' in Question 1 will be used again. It means that we pay with coins and notes whose total value is equal to the cost of our transaction.

Suppose that X is a collection of fewer than five coins that has the property described in Question 1. There can be no coins in X of value 10p or greater. There must be a coin of value 1p in X, else the total 1p cannot be achieved. There are coins of values both 2p and 5p in X, else the total 9p cannot be achieved with fewer than five coins. These three coins with values 1p, 2p, and 5p do not yield the total 4p, so we must adjoin at least one more coin. The addition of coins with values of either 1p or 2p suffice to generate a collection of four coins with the desired property, but a coin with value 5p does not suffice. We summarise these conclusions in Solution 1, the solution to Question 1.

Solution 1 The least number of coins that are needed to ensure that you can pay with exact money any value less than 10p is four. There are two such collections of four coins, and we represent these two solution collections by the 4-tuples [1, 1, 2, 5] (pence) and [1, 2, 2, 5] (pence).

The notation $[a_1, \dots, a_m]$, where $a_i \in \mathbb{N}$ is used for a collection of coins and notes with values $a_1, \dots, a_m$. Two collections $[a_1, \dots, a_m]$ and $[b_1, \dots, b_n]$ are considered to be equal if and only if upon listing each collection of numbers in ascending order, the two resulting lists are identical. If the m-tuple $[a_1, \dots, a_m]$ is followed by a bracketed monetary unit, such as '(pence)', '(pounds)', or '(ten pence)', then we are referring to a collection of coins of values $a_1, \dots, a_m$ in the stated monetary unit. If no bracketed expression is present, then this is a deliberate omission so that our calculations apply in any unit. In Solution 1 these values a_i were quantities in pence; later we apply Solution 1 when the quantities a_i are multiples of ten pence, pounds, and multiples of ten pounds. We do not have recourse to any units in the working of the next paragraph, although we may consider the development in terms of 1p, 2p, and 5p coins if that aids understanding.

For any given value $N \in \{0, 1, \dots, 9\}$ we can find a collection X of coins with values either 1, 2, or 5 such that the sum of the values of the coins in X is N. Let $\Phi(N)$ denote the size of the smallest possible collection X with this property. We determine the function Φ. Given a collection Y of coins with values either 1, 2, or 5, consider the following two replacement strategies that can be applied to subcollections of Y.

(R1) Replace $[1, 1]$ with $[2]$.

(R2) Replace $[2, 2, 2]$ with $[1, 5]$.

If either of these replacement strategies are applied to the collection Y, then the resulting collection Y' is smaller than Y, but the sum of the values in Y and Y' is the same. Carry these observations over to a specific collection X of $\Phi(N)$ coins, where the total value of the coins in X is N, and we see that neither (R1) nor (R2) may be applied to X because the total N cannot be made with fewer than $\Phi(N)$ coins. Notice also that X cannot contain two coins of value 5 since then the total sum would exceed 9. Hence, X contains no more than one coin of value 1, no more than two coins of value 2, and no more than one coin of value 5. Thus, X is contained within $[1, 2, 2, 5]$.

With the $[1, 2, 2, 5]$ collection, we can pay for any of the totals $0, 1, 2, \dots, 10$ with exact money. The two subcollections $[1, 2, 2]$ and $[5]$ of X both yield the same total, but no other pair of subcollections yields the same total. Thus,

$$\begin{gathered} \Phi(1) = \Phi(2) = \Phi(5) = 1, \\ \Phi(3) = \Phi(4) = \Phi(6) = \Phi(7) = 2, \qquad \Phi(8) = \Phi(9) = 3 \end{gathered} \tag{1}$$

(and obviously $\Phi(0) = 0$). It may be instructive for the reader to calculate the unique collection X of $\Phi(N)$ coins with total value N, for each $N = 1, 2, \dots, 9$. We have now established the following proposition.

Proposition 1 *Given $N \in \{0, 1, \dots, 9\}$, the smallest collection X of coins with values 1, 2, and 5 such that the total value of all the coins in X is N, has size $\Phi(N)$, where Φ is given by (1). Moreover, this smallest collection X is unique and is a strict subcollection of* $[1, 2, 2, 5]$.

The UK monetary system has a neat symmetry to it (as does the Euro system). The denominations are 1p, 2p, and 5p; 10p, 20p, and 50p; £1, £2, and £5; £10, £20, and £50. Solution 1 may be applied to show that any of the quantities $1, 2, \dots, 9$ (pence) can be made up from a subcollection of a collection of four coins with values 1p, 2p, 2p, and 5p. Likewise, Solution 1 may be applied to show that any of the quantities £10, £$20, \dots,$ £90 can be made up from a subcollection of a collection of four notes with values £10, £20, £20, and £50. We mirror the content and methods of Question 1 and Proposition 1 in Question 2 and Proposition 2, below. These are comparable results which apply to quantities of money less than £100 (rather than quantities less than 10p). Our approach is to consider separately the digits d_i in the quantity £N, where $N = d_1 d_2 . d_3 d_4$.

Question 2 What is the minimum number of coins and notes that you need to be able to pay with exact money any value less than £100, and which collections of coins and notes have this property?

Suppose that X is a minimal collection of coins and notes as described in Question 2. To make the values 1p$, \dots, 9$p, the collection X must include one of the 4-tuples $[1, 1, 2, 5]$ (pence) or $[1, 2, 2, 5]$ (pence) of Solution 1. Suppose that X contains two further coins with values $a, b \in \{1, 2, 5\}$ (pence). The collection $[1, 2, 2, 5, 10]$ (pence) is smaller than the collection $[1, 2, 2, 5, a, b]$ (pence) and any price that can be paid with exact money from the latter collection can also be paid with exact money from the former collection. This replacement strategy contradicts the minimal nature of X, therefore X cannot contain the two coins with values a and b (like comments apply with $[1, 1, 2, 5]$ replacing $[1, 2, 2, 5]$).

Thus, there are no more than five coins with values 1, 2, or 5 (pence). Notice that these no more than five coins together sum to a total of at most 15p.

Let A_k denote a subcollection of X, consisting of coins with total value k, where $k = 19, 29, \dots, 99$ (pence). The 10p, 20p, and 50p coins in A_k sum to pence values that are multiples of 10, for each k. Therefore, we must be able to make at least one of the values $9, 19, 29, \dots$ with the 1p, 2p, and 5p coins in A_k, for each k. But the full collection of 1p, 2p, and 5p coins in X has a total value of no more than 15p, so the 1p, 2p, and 5p coins in A_k must have a total value of exactly 9p. Therefore, the 10p, 20p, and 50p coins in A_1 have total value 10p, the 10p, 20p, and 50p coins in A_2 have total value 20p, and so forth. Hence, we must be able to construct values $10, 20, \dots, 90$ (pence) with the 10p, 20p, and 50p coins in X. Solution 1 may now be applied to show that X contains either $[10, 10, 20, 50]$ (pence) or $[10, 20, 20, 50]$ (pence). This argument may be repeated to see that X also contains $[1, 1, 2, 5]$ (pounds) or $[1, 2, 2, 5]$ (pounds), and $[10, 10, 20, 50]$ (pounds) or $[10, 20, 20, 50]$ (pounds). We have so far shown that X must contain a collection of sixteen coins and notes, although there has been some freedom in choosing these sixteen coins and notes.

Explicitly, let Ω_1, Ω_2, Ω_3, Ω_4 denote the four subcollections, each of size four, that we list in order as follows:

$$\text{1p, 2p, 2p, 5p,} \qquad \text{10p, 20p, 20p, 50p,} \qquad \text{£1, £2, £2, £5,} \qquad \text{£10, £20, £20, £50;} \tag{2}$$

and let Δ_1, Δ_2, Δ_3, Δ_4 denote the four subcollections, each of size four, that we list in order as follows:

$$\text{1p, 1p, 2p, 5p,} \qquad \text{10p, 10p, 20p, 50p,} \qquad \text{£1, £1, £2, £5,} \qquad \text{£10, £10, £20, £50.} \tag{3}$$

Then X contains one of the collections

$$Z = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where Γ_i is either Ω_i or Δ_i, for each $i = 1, 2, 3, 4$. There are sixteen such collections Z; the collection

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$$

will be seen to be of particular importance. These collections Z do have the property described in Question 2 – that you can pay with exact money any value less than £100. Specifically, given a value £N less than £100, where $N = d_1 d_2 . d_3 d_4$ for digits d_i, then coins from Γ_1 can be used to pay with exact money the total d_4 (pence), coins from Γ_2 can be used to pay with exact money the total d_3 (times ten pence), coins and notes from Γ_3 can be used to pay with exact money the total d_2 (pounds), and notes from Γ_4 can be used to pay with exact money the total d_1 (times ten pounds). The full collection of all coins and notes from the previous sentence has value £N. Given that X is a smallest possible collection satisfying the properties described in Question 2, it is equal to one of these collections Z. A formal statement of the solution to Question 2 follows.

Solution 2 The least number of coins and notes that are needed to ensure that you can pay with exact money any quantity of money less than £100 is sixteen. There are sixteen combinations of sixteen coins and notes with which you can achieve this goal, as described by (2), (3), and the comments succeeding those lists.

Question 2 was about the smallest collection of coins and notes with which we could create any value less than £100. We now specialise to looking at particular quantities of money £N and ask what is the least number of coins and notes needed to pay exactly £N. In this way Proposition 2 extends Proposition 1, just as Question 2 extends Question 1. With our assumption that shop assistants return the least amount of change, Proposition 2 tells us exactly which coin and note denominations we should expect to receive. For a quantity £N of money we let $\Phi(N)$ be the size of the smallest collection of coins and notes that has total value £N (this extends our earlier definition of Φ to cash values above 9p).

Proposition 2 *Let £N be a quantity of money less than £100, where $N = d_1 d_2.d_3 d_4$ for digits d_i, $i = 1, 2, 3, 4$. Then*

$$\Phi(N) = \Phi(d_1) + \Phi(d_2) + \Phi(d_3) + \Phi(d_4).$$

Moreover, there is only one collection X of $\Phi(N)$ coins and notes with total value N and this collection is a subcollection of Ω.

Proof Let X be a minimal collection of coins and notes with total value £N, and let X_1, X_2, X_3, X_4 be subcollections of X consisting of the coins and notes in X with values 1p, 2p, and 5p; 10p, 20p, and 50p; £1, £2, and £5; and £10, £20, and £50 respectively. We define a new replacement strategy as follows.

(R3) Replace $[5, 5]$ with $[10]$.

We consider applying (R1), (R2), and (R3) to each of the collections X_1, X_2, X_3, and X_4 in turn. Since the successful application of any of these replacement strategies to any X_i would result in a collection X' smaller than X, yet with the same total value, none of the strategies may be applied. We can deduce from this that $X_i \subseteq \Omega_i$ for $i = 1, 2, 3, 4$. (The rule (R3) does not make sense for X_4, but X_4 cannot contain more than one £50 note anyway.) Moreover, $X_i \neq \Omega_i$ for any i, because a subcollection $[1, 2, 2, 5]$ of X could be replaced by $[10]$, contradicting minimality of X (and X cannot contain $[10, 20, 20, 50]$ (pounds) since this collection does not have total value less than £100). This shows that $X \subsetneq \Omega$ and that the total value of X_1 is d_4 pence, the total value of X_2 is d_3 times ten pence, the total value of X_3 is d_2 pounds, and the total value of X_4 is d_1 times ten pounds. Hence,

$$\Phi(N) = \Phi(d_1) + \Phi(d_2) + \Phi(d_3) + \Phi(d_4),$$

and the uniqueness clause follows from the uniqueness facet of Proposition 1.

We observed early on that the number of coins and notes that you hold is minimised by paying for shopping with all your money. Of course, there is no point handing the shop assistant both your twenty pound note and fifty pence piece when purchasing a first class stamp (the twenty pound note will be returned straight back to you). More generally, there is no point paying with coins and notes that will immediately be returned to you. Proposition 2 allows us to calculate precisely which coins and notes we receive in change for paying for shopping with all our money. Thus, we can determine those coins and notes that we both pay and receive in change, and not bother handing them over in the first place.

Provided that you replenish the money in your wallet with money that does not take your collection beyond Ω, and provided that you pay for each item (of value less than £100) with all of your cash, Proposition 2 shows that your money will always be a subcollection of Ω. In fact, whilst sixteen items of currency are needed to pay with exact money any value between 1p and £100; for any specific value you need no more than twelve items (three corresponding to each of d_1, d_2, d_3, and d_4 of the value £N, where $N = d_1d_2.d_3d_4$). Should a shop assistant violate our assumption and not return to you the minimal amount of change, then you may temporarily carry a collection of coins and notes that is not a subcollection of Ω until the next time you pay for goods, from someone with better optimisation skills.

An example illuminates our scheme. Suppose that you have coins of values 1p, 2p, and 50p and you wish to pay for an item of value 28p. Typically, we might hand over the 50p piece, receive 22p in change, and end up with coins of values 1p, 2p, 2p, and 20p (four coins). On the other hand, if you pay with all the 53p in your possession, you receive 25p in change, and end up with coins of values 5p and 20p (two coins). In practice, the author has found that in certain situations, some shop assistants do not return the minimal amount of change. For example, suppose that the above item costs 29p. The shop assistant should return two 2p coins and a 20p coin as change from your 53p payment. Instead, they may immediately return your 1p and 2p coins, not understanding why you have included these copper coins in your payment, and then give you an extra 1p coin and a 20p coin as change from 50p. In this circumstance, there was no point paying with the 2p coin as that was returned straight away. The use of the 2p coin has just confused matters. The comments that immediately follow the proof of Proposition 2 apply, and we recommend that if your mental arithmetic is up to it, you pay for objects with all your money *except* those coins and notes that will be returned immediately.

The answer to the question ‘how much money do you need?’ is now clear: you need 1p, 2p, 2p, 5p, 10p, 20p, 20p, 50p, £1, £2, £2, £5, £10, £20, £20, and £50. Following our strategy of paying for shopping with all your money, you need never carry more currency than these sixteen items.

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$2^{32}+1$ is not prime

From the prime factorization of 640^4, deduce that 641 divides $2^{32}+1$.

10 Shahid Azam Lane,
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Abbas Roohol Amini

Euler's Little Summation Formula and Sums of Powers

ANDREW ROBERTSON and THOMAS J. OSLER

1. Introduction

In this article we introduce an elementary summation formula due to Euler that we call *Euler's little summation formula*. This little summation formula (see reference 1) was found by Euler as an intermediate item in the derivation of his 'big' result (see reference 2, pp. 518–535) that we call today the *Euler–Maclaurin summation formula*. We then apply this formula to the calculation of sums of powers. A recursive method is found to obtain closed forms for the sum $\sum_{n=1}^{N} n^p$, where p is a positive integer.

2. Derivation of Euler's little summation formula

We begin by introducing an unconventional notation for the sum of a series.

Definition 1 Let a, b, and h be constants, and let

$$ {}_aS_{b,h}\, f(x) = f(a) + f(a+h) + f(a+2h) + \cdots + f(b), $$

where $b = a + Nh$, with $N = 0, 1, 2, \ldots$. Here, we read ${}_aS_{b,h}\, f(x)$ as the 'summation of $f(x)$ from a to b with increment h'. Usually we sum with increment $h = 1$ and, in this case, we suppress the h and write ${}_aS_{b,1}\, f(x) = {}_aS_b\, f(x)$.

We use this definition for three reasons. Firstly, a similar notation was used by Euler in his original discussion of this work. Secondly, in the development below, several different summations occur, and having a special notation for the principal sum helps us to identify it in complex expressions. The third reason is that we sum over the increment h rather than the usual increment 1. There are applications to the summation of alternating series where other values of h are needed, but they will not be discussed here.

Theorem 1 *Let $f(x)$ be a function with $M+1$ continuous derivatives for $x > a-1$. Then*

$$ {}_aS_b\, f'(x) = f(b) - f(a-1) + \sum_{n=2}^{M} \frac{(-1)^n\, {}_aS_b\, f^{(n)}(x)}{n!} + R_M, \tag{1} $$

where

$$ R_M = \frac{(-1)^{M+1}}{(M+1)!}\left(f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+1}^*) + \cdots + f^{(M+1)}(x_b^*)\right), $$

for some x_c^* in the interval $c-1 < x_c^* < c$.

Proof From the hypothesis that $f(x)$ is a function of x with $M+1$ continuous derivatives, we can expand $f(x)$ in a Taylor series to get

$$f(x-h) = \sum_{n=0}^{M} \frac{(-1)^n f^{(n)}(x)}{n!} h^n + r_M,$$

where, with $h > 0$,

$$r_M = \frac{(-1)^{M+1} f^{(M+1)}(x_*)}{(M+1)!} h^{M+1}$$

for some x_* in the interval $x - h < x_* < x$. Summing from a to $b = a + Nh$, we have

$$_aS_{b,h} f(x-h) = {}_aS_{b,h} f(x) + \sum_{n=1}^{M} \frac{(-1)^n \, {}_aS_{b,h} f^{(n)}(x)}{n!} h^n + R_M, \tag{2}$$

where R_M is described by

$$R_M = \frac{(-1)^{M+1}}{(M+1)!} h^{M+1} (f^{(M+1)}(x_a^*) + f^{(M+1)}(x_{a+h}^*) + \cdots + f^{(M+1)}(x_b^*))$$

for some x_c^* in the interval $c - h < x_c^* < c$. Notice that

$$_aS_{b,h} f(x-h) = f(a-h) + f(a) + f(a+h) + \cdots + f(b-h)$$

and

$$_aS_{b,h} f(x) = f(a) + f(a+h) + f(a+2h) + \cdots + f(b),$$

so

$$_aS_{b,h} f(x-h) - {}_aS_{b,h} f(x) = f(a-h) - f(b).$$

Now, from (2) we obtain

$$f(b) = f(a-h) - \sum_{n=1}^{M} \frac{(-1)^n \, {}_aS_{b,h} f^{(n)}(x)}{n!} h^n - R_M,$$

and isolating the first term in the sum we get

$$f(b) = f(a-h) + h \, {}_aS_{b,h} f'(x) - \sum_{n=2}^{M} \frac{(-1)^n \, {}_aS_{b,h} f^{(n)}(x)}{n!} h^n - R_M,$$

from which we have

$$_aS_{b,h} f'(x) = \frac{1}{h} f(b) - \frac{1}{h} f(a-h) + \sum_{n=2}^{M} \frac{(-1)^n \, {}_aS_{b,h} f^{(n)}(x)}{n!} h^{n-1} + \frac{1}{h} R_M.$$

Now let $h = 1$ and this last relation becomes (1), and the theorem is proved.

We call (1) Euler's little summation formula.

3. Application of Euler's little summation formula to sums of powers

We now look for closed form expressions for sums of the form

$$ {}_1S_N x^p = \sum_{n=1}^{N} n^p, $$

for $p = 0, 1, 2, \ldots$. Using $f(x) = x^{p+1}/(p+1)$ and $a = 1$ in (1), we get the finite series

$$ {}_1S_b x^p = \frac{b^{p+1}}{p+1} - \frac{0^{p+1}}{p+1} + \sum_{n=2}^{p+1} \frac{(-1)^n}{n!}\, {}_1S_b \frac{p!\, x^{p-n+1}}{(p-n+1)!}. \tag{3} $$

(Note that the series is conveniently finite because the remaining derivatives all vanish.) We start with $p = 0$ and easily see that

$$ {}_1S_N x^0 = \sum_{n=1}^{N} 1 = N. \tag{4} $$

Next we let $p = 1$ and use (3) and (4) to get

$$ {}_1S_N x = \frac{N^2}{2} + \frac{{}_1S_N 1}{2} = \frac{N^2 + N}{2}. \tag{5} $$

Let $p = 2$ and use the same idea to find ${}_1S_N x^2$ as follows:

$$ \begin{aligned} {}_1S_N x^2 &= \frac{N^3}{3} + \frac{{}_1S_N(2x)}{2} - \frac{{}_1S_N(2)}{6} \\ &= \frac{N^3}{3} + {}_1S_N(x) - \frac{{}_1S_N(1)}{3}. \end{aligned} $$

Using (4) and (5), this last result becomes

$$ \begin{aligned} {}_1S_N x^2 &= \frac{N^3}{3} + \frac{N^2 + N}{2} - \frac{N}{3} \\ &= \frac{2N^3 + 3N^2 + N}{6}. \end{aligned} $$

It is now clear that, once we have found the sums ${}_1S_N x^0$, ${}_1S_N x^1, \ldots, {}_1S_N x^{p-1}$, we can use Euler's little summation formula to obtain ${}_1S_N x^p$. We now list the results for p from 1 to 10 for reference:

$$ \begin{aligned} \sum_{n=1}^{N} n &= \frac{N^2 - N}{2}, \\ \sum_{n=1}^{N} n^2 &= \frac{2N^3 + 3N^2 + N}{6}, \\ \sum_{n=1}^{N} n^3 &= \frac{N^4 + 2N^3 + N^2}{4}, \end{aligned} $$

$$\sum_{n=1}^{N} n^4 = \frac{6N^5 + 15N^4 + 10N^3 - N}{30},$$

$$\sum_{n=1}^{N} n^5 = \frac{2N^6 + 6N^5 + 5N^4 - N^2}{12},$$

$$\sum_{n=1}^{N} n^6 = \frac{6N^7 + 21N^6 + 21N^5 - 7N^3 + N}{42},$$

$$\sum_{n=1}^{N} n^7 = \frac{3N^8 + 12N^7 + 14N^6 - 7N^4 + 2N^2}{24},$$

$$\sum_{n=1}^{N} n^8 = \frac{10N^9 + 45N^8 + 60N^7 - 42N^5 + 20N^3 - 3N}{90},$$

$$\sum_{n=1}^{N} n^9 = \frac{2N^{10} + 10N^9 + 15N^8 - 14N^6 + 10N^4 - 3N^2}{20},$$

$$\sum_{n=1}^{N} n^{10} = \frac{6N^{11} + 33N^{10} + 55N^9 - 66N^7 + 66N^5 - 33N^3 + 5N}{66}.$$

For more information on sums of powers, see reference 3.

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The Mathematics Behind a Certain Card Trick

KALYAAN M. RAO, K. P. S. BHASKARA RAO
and M. BHASKARA RAO

Introduction

One of the authors came across the following trick in a popular book on card tricks (see reference 1). As a host, I invite a member of the audience on to the stage and instruct him/her to enact the following steps behind my back.

1. Write the numbers 1 to 9 in three rows, each row containing three numbers, using all the nine numbers. In other words, create three three-digit numbers in any way you like using all the nine numbers. (Cards with numbers 1 to 9 can be used.)

2. Remove one digit from anywhere in any one row and replace it with zero.

3. Sum the resulting three-digit numbers in the usual way.

4. Add the digits of this sum and let me know what it is.

I will tell you which digit you removed.

Reference 1 explains how to identify the number removed. Let x be the final number obtained.

- If $1 \leq x \leq 8$, the number removed is $9 - x$.
- If $9 \leq x \leq 17$, the number removed is $18 - x$.
- If $18 \leq x \leq 26$, the number removed is $27 - x$.
- If $27 \leq x \leq 29$, the number removed is $36 - x$.

As an illustration, suppose that we use the following three three-digit numbers.

$$\begin{array}{c} 912 \\ 483 \\ 576 \end{array}$$

Now, we remove the number 8 and replace it with zero as follows.

$$\begin{array}{r} 912 \\ 403 \\ 576 \\ \hline 1891 \\ \hline \end{array}$$

The sum of the digits of the sum is 19. The number that has been removed is $27 - 19 = 8$, which is correct.

The mathematics behind the trick

Denote the three-digit numbers as follows.

$$x_1x_2x_3$$
$$x_4x_5x_6$$
$$x_7x_8x_9$$

The three numbers are $x_3 + 10x_2 + 10^2x_1$, $x_6 + 10x_5 + 10^2x_4$, and $x_9 + 10x_8 + 10^2x_7$, with sum

$$\begin{aligned} x_3 + x_6 + x_9 &+ 10(x_2 + x_5 + x_8) + 10^2(x_1 + x_4 + x_7) \\ &\equiv (x_3 + x_6 + x_9 + x_2 + x_5 + x_8 + x_1 + x_4 + x_7) \pmod 9 \\ &\equiv (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \pmod 9 \\ &\equiv 45 \pmod 9 \\ &\equiv 0 \pmod 9. \end{aligned}$$

Suppose that x_i is replaced by 0. Now,

$$\begin{aligned} x &\equiv (45 - x_i) \pmod 9 \\ &\equiv -x_i \pmod 9. \end{aligned}$$

Thus, if $1 \le x \le 8$ then $x_i = 9 - x$, if $9 \le x \le 17$ then $x_i = 18 - x$, if $18 \le x \le 26$ then $x_i = 27 - x$, and so on.

We can make the card trick a little bit more sophisticated. After deleting one of the digits in one of the three-digit numbers and replacing it with zero, ask the participant to give you the sum $u_1u_2u_3u_4$ of the resultant three-digit numbers. The participant need not add the digits of the sum. In your head, calculate $a = u_1u_2u_3u_4 \pmod 9 = (u_1 + u_2 + u_3 + u_4) \pmod 9$. The deleted digit is then $9 - a$.

The trick can be generalized without raising its complexity. Ask the participant to create any number of integers of variable lengths using all the digits from $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ exactly once. Remove one digit from any one of the integers created and replace it with zero. Sum the resultant integers in the usual way. Let the participant give you the sum. You can tell precisely which digit was removed. The mathematics involved is identical to that expounded above. For example, suppose that the following integers are created.

$$\begin{array}{r} 23 \\ 8 \\ 456 \\ 91 \\ 7 \end{array}$$

Suppose that we remove the digit 2 and replace it by zero. The resultant integers are as follows.

$$\begin{array}{r} 03 \\ 8 \\ 456 \\ 91 \\ 7 \end{array}$$

The sum of these integers is 565. Then we have $a \equiv 565 \equiv 7 \pmod 9$, and the deleted digit is $9 - 7 = 2$.

A more sophisticated trick

Ask the participant to enact the following series of steps.

(a) Choose r_1 ones, r_2 twos, $\dots, r_9$ nines, where the choice of the nonnegative integers $r_1, r_2, \dots, r_9$ is entirely left to the participant. The participant writes these $(r_1 + r_2 + \dots + r_9)$ single-digit numbers on a blackboard, which I can see clearly.

(b) Build k integers using the ones, twos, $\dots$, nines selected in Step (a). Each of the single-digit numbers selected in Step (a) is used exactly once. The choice of k is entirely left to the participant. The participant writes these k integers on a separate blackboard, which I am not allowed to see.

(c) Remove a digit from one of the k integers and replace it with zero.

(d) Sum the resultant k integers in the usual way.

(e) Let me know what this sum is.

(f) I will tell you precisely which digit you have removed.

As an example, we have the following list of single-digit numbers chosen by the participant.

Two ones:	1	1			
Three twos:	2	2	2		
Zero threes:					
One four:	4				
Five fives:	5	5	5	5	5
Two sixes:	6	6			
One seven:	7				
Three eights:	8	8	8		
Four nines:	9	9	9	9	

Suppose that the following eight integers are created using all these single-digit numbers once.

981, 5, 5245, 67, 92, 815, 2568, 99

Now, we remove 9 from the fifth integer and replace it with zero. Then we sum the integers in the usual way to get

$$981 + 5 + 5245 + 67 + 02 + 815 + 2568 + 99 = 9782.$$

Once the sum 9782 is given by the participant, I can spell out precisely which digit was removed by the participant. The digit can be worked out as follows.

(a) Add all the single digits chosen by the participant. In this example, it is 116. Calculate $a = 116 \pmod 9$, so $a = 8$.

(b) Calculate $b = 9782 \pmod 9$, so $b = 8$.

Then

$$a \equiv \text{the deleted digit} + b \pmod 9,$$

so the deleted digit is $a - b \pmod 9$. In this case this is $0 \pmod 9$, so the deleted digit is 9.

In the basic trick, $a = 0$. In the extended version of the trick, I need to calculate a in my head from the list of single-digit numbers chosen by the participant.

Reference

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Mathematics in the Classroom

Similar triangles and complex numbers

Problems based on similar triangles are frequently set in various competitions. Solving such problems by using simple geometry can be very involved. In this article, we see how effective the role of complex numbers can be in solving such problems.

In terms of complex numbers, let two triangles with the same orientation having vertices z_1, z_2, z_3 and w_1, w_2, w_3 be directly similar to one another (see figure 1). Then the following conditions hold:

(i) corresponding sides of the triangles are proportional,

(ii) corresponding angles are equal.

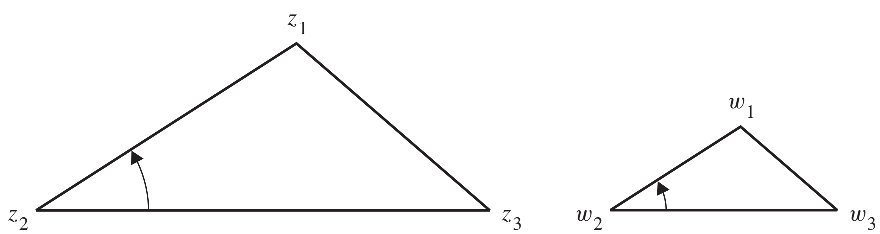

Figure 1

Thus,

$$\left|\frac{z_1 - z_2}{z_3 - z_2}\right| = \left|\frac{w_1 - w_2}{w_3 - w_2}\right| \tag{1}$$

and

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{w_1 - w_2}{w_3 - w_2}\right). \tag{2}$$

Two complex numbers are equal if their modulus and argument are equal. Therefore, it follows from (1) and (2) that

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{w_1 - w_2}{w_3 - w_2}.$$

Conversely, if complex numbers $(z_1 - z_2)/(z_3 - z_2)$ and $(w_1 - w_2)/(w_3 - w_2)$ are equal, then the two triangles having vertices z_1, z_2, z_3 and w_1, w_2, w_3 are directly similar to one another.

Now we are ready to present examples of the usefulness of complex numbers in solving problems of similar triangles.

Problem 1 If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, prove that the two triangles are similar.

Solution We are given that

$$c = (1 - r)a + rb \quad \text{or} \quad r = \frac{c - a}{b - a}, \tag{3}$$

$$w = (1 - r)u + rv \quad \text{or} \quad r = \frac{w - u}{v - u}. \tag{4}$$

It follows from (3) and (4) that $(c - a)/(b - a) = (w - u)/(v - u)$, which establishes the desired result.

Problem 2 On the sides AB, BC, and CA of a triangle ABC we draw similar triangles ADB, BEC, and CFA, having the same orientation. Prove that triangles ABC and DEF have the same centroid.

Solution Let the points A, B, C, D, E, and F be represented by complex numbers a, b, c, d, e, and f respectively. We are given that triangles ADB, BEC, and CFA are similar with the same orientation. Therefore,

$$\frac{d - a}{b - a} = \frac{e - b}{c - b} = \frac{f - c}{a - c} = t, \quad \text{where } t \text{ is a complex number,}$$

or

$$d = a + (b - a)t, \qquad e = b + (c - b)t, \qquad f = c + (a - c)t.$$

The centroid of the triangle DEF is

$$\frac{d + e + f}{3} = \frac{a + (b - a)t + b + (c - b)t + c + (a - c)t}{3} = \frac{a + b + c}{3},$$

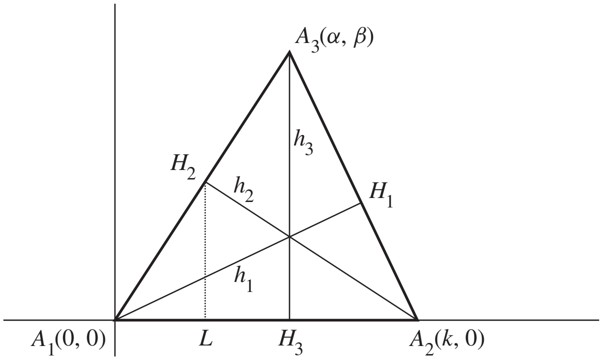

Figure 2

which is the centroid of triangle ABC. Thus, we have proved that triangles ABC and DEF have the same centroid.

Problem 3 Five points A, B, C, U, and V lie in the same plane. Suppose that triangles AUV, VBU, and UVC are similar to one another. Show that triangle ABC is similar to each of them.

Solution Since triangles AUV, VBU, and UVC are similar to one another, therefore (with obvious notation) we have

$$\frac{a-u}{v-u} = \frac{v-b}{u-b} = \frac{u-v}{c-v}$$

or

$$\frac{a-u}{v-u} = \frac{v-b}{u-b} = \frac{u-v}{c-v} = \frac{a-u+v-b+u-v}{v-u+u-b+c-v} = \frac{a-b}{c-b}.$$

This proves that triangle ABC is similar to each of them.

Problem 4 Given an acute-angled triangle $A_1A_2A_3$, let H_1, H_2, and H_3 be the feet of the altitudes from A_1, A_2, and A_3 respectively. Show that each of the triangles $A_1H_2H_3$, $A_2H_3H_1$, and $A_3H_1H_2$ is similar to triangle $A_1A_2A_3$.

Solution For the sake of simplicity, we may assume $A_1(0,0)$, $A_2(k,0)$, and $A_3(\alpha,\beta)$; see figure 2. Now the points A_1, A_2, and A_3 can be represented by complex numbers 0, k, and $\alpha + \mathrm{i}\beta$ respectively.

In the triangle $A_1A_2H_2$, we have

$$\cos A_1 = \frac{A_1H_2}{A_1A_2}.$$

Also, in the triangle $A_3A_1H_3$, we have

$$\cos A_1 = \frac{A_1H_3}{A_1A_3} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}. \tag{5}$$

Hence,

$$A_1H_2 = A_1A_2 \cos A_1 = \frac{k\alpha}{\sqrt{\alpha^2+\beta^2}}. \tag{6}$$

In triangle $A_1A_2A_3$, we draw a perpendicular H_2L from the point H_2 to the side A_1A_2. Now, in the triangle H_2A_1L, we have

$$\cos A_1 = \frac{A_1L}{A_1H_2} \quad \text{or} \quad A_1L = A_1H_2 \cos A_1.$$

With the results (5) and (6), we get

$$A_1L = \frac{k\alpha^2}{\alpha^2+\beta^2}.$$

Also, by using Pythagoras' theorem in triangle H_2A_1L, we get

$$H_2L = \sqrt{(A_1H_2)^2 - (A_1L)^2} = \sqrt{\frac{k^2\alpha^2}{\alpha^2+\beta^2} - \frac{k^2\alpha^4}{(\alpha^2+\beta^2)^2}} = \frac{k\alpha\beta}{\alpha^2+\beta^2}.$$

Now we have the point $H_2(k\alpha^2/(\alpha^2+\beta^2), k\alpha\beta/(\alpha^2+\beta^2))$, so this point can be represented by the complex number $k\alpha^2/(\alpha^2+\beta^2) + \mathrm{i}k\alpha\beta/(\alpha^2+\beta^2)$. Finally, with obvious notation, we obtain

$$\frac{h_2-a_1}{h_3-a_1} = \frac{k\alpha^2/(\alpha^2+\beta^2) + \mathrm{i}k\alpha\beta/(\alpha^2+\beta^2) - 0}{\alpha - 0} = \frac{k(\alpha+\mathrm{i}\beta)}{\alpha^2+\beta^2} = \frac{k}{\alpha-\mathrm{i}\beta} = \frac{\overline{a}_2-\overline{a}_1}{\overline{a}_3-\overline{a}_1}.$$

This shows that triangles $A_1H_2H_3$ and $\overline{A}_1\overline{A}_2\overline{A}_3$ are similar.

Since $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$ are reflected points of A_1, A_2, A_3 with respect to the x-axis, triangles $\overline{A}_1\overline{A}_2\overline{A}_3$ and $A_1A_2A_3$ are similar. Hence, triangle $A_1H_2H_3$ and $A_1A_2A_3$ are similar.

With similar computations, we may prove that triangles $A_2H_3H_1$ and $A_3H_2H_3$ are similar to the triangle $A_1A_2A_3$.

Problem 5 In the equilateral triangle with vertices z_1, z_2, z_3, show that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

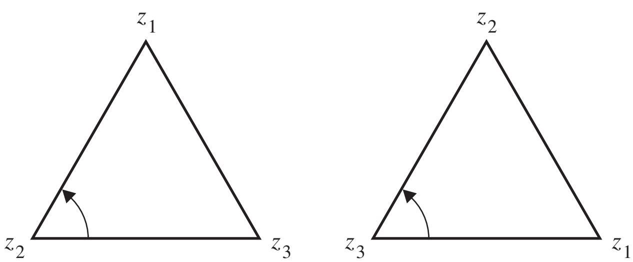

Figure 3

Solution The triangle with vertices z_1, z_2, z_3 is equilateral. To put this another way, triangles with vertices z_1, z_2, z_3 and z_2, z_3, z_1 are similar (see figure 3). Hence,

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{z_2 - z_3}{z_1 - z_3} \quad \text{or} \quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Reference

1 G. Chang and T. W. Sederberg, *Over and Over Again* (MAA, Washington, DC, 1998).

Patna, India **Anand Kumar**

Letters to the Editor

Dear Editor,

Regular polygons

I have discovered a result that as far as I know is original. However, I was wondering if one of your readers has ever heard of it or read about it. This is the result.

Construct a regular polygon of your choice and mark its centre, O. I started with a regular pentagon. Then pick any point within the polygon. Draw circles with their centres at each vertex which pass through the random point. Denote the points of intersection of neighbouring circles (other than the chosen random point) by A, B, C, D, and E. Then I found the centre of gravity of these points; it is the centre of the original regular polygon.

This will work for any regular polygon and for whatever random point you choose even if it is outside, inside, or on one of the sides of the polygon. I have done experiments with equilateral triangles, squares, regular hexagons, and one heptagon to check this.

You can also draw lines from the random point to each side of the original polygon perpendicularly. Denote these points by A', B', C', D', and E'. Mr. Wells, my teacher, spotted that the original intersections of the circles are twice as far from the random point as these new points. Moreover, the centre of gravity of A', B', C', D', and E' is half way between the random point and the centre of the original polygon.

If you have heard of this result before, then please write to *Mathematical Spectrum*.

Yours sincerely,

Daniel Schultz (aged 13)
(C/o David Wells
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UK)

[We have posed this problem for a regular pentagon in our problems section (see Problem 40.5) – Ed.]

Dear Editor,

Rational approximation to square roots

We propose an alternative way of looking at the rational approximation to $r = \sqrt{x}$ to the way given by Bob Bertuello in Volume 36, Number 1, p. 20. Bertuello proposed the following recursive relation to find $r = \sqrt{x}$:

$$r_{n+1} = \frac{x - g^2}{r_n + g} + g,$$

where g is the initial trial and $r_0 = g$. This can be written as

$$r_{n+1} = \frac{x + gr_n}{r_n + g}. \tag{1}$$

Suppose that $r_n = y_n/x_n$ is a rational number. Then (1) can be written as

$$\frac{y_{n+1}}{x_{n+1}} = \frac{xx_n + gy_n}{y_n + gx_n}.$$

We can denote this recursive relation in a matrix form as follows:

$$\begin{bmatrix} y_{n+1} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} g & x \\ 1 & g \end{bmatrix} \begin{bmatrix} y_n \\ x_n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} y_{n+1} \\ x_{n+1} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} y_n \\ x_n \end{bmatrix}. \tag{2}$$

Thus, we can obtain a sequence of solutions to (2), namely

$$\begin{bmatrix} y_n \\ x_n \end{bmatrix} = \boldsymbol{A}^n \begin{bmatrix} y_0 \\ x_0 \end{bmatrix},$$

and y_n/x_n gives the nth rational approximation to $\sqrt{x}$. For example, to find $\sqrt{20}$ a first guess could be $g = 4 = r_0$. Then we obtain the successive approximations

$$\begin{bmatrix} y_1 \\ x_1 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 36 \\ 8 \end{bmatrix},$$

$$\begin{bmatrix} y_2 \\ x_2 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 304 \\ 68 \end{bmatrix},$$

$$\begin{bmatrix} y_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2\,576 \\ 576 \end{bmatrix},$$

$$\begin{bmatrix} y_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21\,824 \\ 4\,880 \end{bmatrix},$$

$$\begin{bmatrix} y_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 184\,896 \\ 41\,344 \end{bmatrix},$$

and so on.

Yours sincerely,

Bor-Yann Chen
(National Ilan University
1 Shan-Lung Road
I-Lan 260
Taiwan)

Dear Editor,

Symmetrical Fibonacci products

An early 19th century Cambridge tutor recommended to his tutee that he solve mathematical problems in as many different ways as he can. This man became Senior Wrangler (the top student for the year). I have recommended this approach to many of my sixth form students. However, perhaps unsurprisingly, I cannot remember receiving a homework assignment where more than one method has been offered!

Anyway, this advice has frequently inspired me to look for alternative approaches to mathematical problem solving, and I would like to offer one to prove the results Robert Clarke gave in his letter in Volume 39, Number 3, pp. 129–130. At the same time I was very interested in Stuart Simons' letter on page 130 of the same issue, which is similarly motivated.

Mr Clarke's starting point is the standard Fibonacci definitions: $f_1 = 1 = f_2$ and $f_{n+2} = f_{n+1} + f_n$. All the formulae he derived can be quickly deduced from just *one formula*, namely

$$f_a f_b - f_c f_d = (-1)^{a-1} f_{c-a} f_{d-a}, \quad \text{if } a + b = c + d. \tag{1}$$

This can be proved by means of a double induction. I did it by taking it in the form $f_m f_{n+s} - f_{m+s} f_n = (-1)^{m-1} f_s f_{n-m}$ and using induction on s. I assumed that $f_0 = 0$ and used induction on m to show that $f_m f_{n+1} - f_{m+1} f_n = (-1)^{m-1} f_{n-m}$. For what follows I have also allowed negative indices for f_m, taking $f_{-n} = (-1)^{n-1} f_n$, consistent with $f_{n+2} = f_{n+1} + f_n$. A little more work is required to check that (1) still holds if negative indices are allowed.

(a) Letting $a = n - m$, $b = n + m$, and $c = d = n$ in (1) gives

$$f_{n-m} f_{n+m} - f_n f_n = -(-1)^{n-m} f_m f_m, \quad \text{that is } P_m = f_n^2 - f_m^2 (-1)^{n+m},$$

which is Mr Clarke's equation (2).

(b) Letting $a = 1$, $b = 2n - 1$, and $c = d = n$ in (1) gives

$$f_1 f_{2n-1} - f_n f_n = f_{n-1} f_{n-1}, \quad \text{that is } f_n^2 + f_{n-1}^2 = f_{2n-1},$$

which was given later on in Mr Clarke's letter.

(c) Also, by (1),

$$f_2 f_{4k+1} - f_{2k+2} f_{2k+1} = -f_{2k} f_{2k-1}, \quad \text{that is } f_{4k+1} = f_{2k+2} f_{2k+1} - f_{2k} f_{2k-1}.$$

Summing from $k = 0$ to $k = n$ gives

$$\sum_{k=0}^{n} f_{4k+1} = f_{2n+2} f_{2n+1} - f_0 f_{-1}, \quad \text{that is } \sum_{k=0}^{n} f_{4k+1} = f_{2n+2} f_{2n+1}.$$

Again using (1), we get $f_1 f_{4n+2} - f_{2n+2} f_{2n+1} = f_{2n+1} f_{2n}$. Hence,

$$\begin{aligned} f_{4n+2} &= f_{2n+2} f_{2n+1} + f_{2n+1} f_{2n} \\ &= f_{2n+1}(f_{2n+2} + f_{2n}) \\ &= f_{2n+1}(2 f_{2n+2} - f_{2n+1}), \end{aligned}$$

and so $f_{2n+2}f_{2n+1} = (f_{4n+2} + f_{2n+1}^2)/2$, leading to the result

$$\sum_{k=0}^{n} f_{4k+1} = \frac{f_{4n+2} + f_{2n+1}^2}{2},$$

which was given by Mr Clarke.

Yours sincerely,

Alastair Summers
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Dear Editor,

The power rule for a real exponent

We recall the following well-known result:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^r = rx^{r-1}.$$

For most writers on elementary calculus, proof from first principles of this result is only done for the case $r \in \mathbb{Z}^+$. Perhaps we can consider the following complete proof (from first principles):

$$\frac{\mathrm{d}}{\mathrm{d}x}x^r = rx^{r-1}, \quad \text{where } x > 0 \text{ and } r \in \mathbb{R}.$$

First, we recall that

$$\frac{a^m - 1}{a - 1} = 1 + a + \cdots + a^{m-2} + a^{m-1}, \quad \text{for } m \in \mathbb{Z}^+ \text{ and } a \neq 1,$$

from which we obtain

$$\begin{aligned} \frac{a^{m/n} - 1}{a - 1} &= \frac{a^{m/n} - 1}{a^{1/n} - 1}\frac{a^{1/n} - 1}{a - 1} \\ &= \frac{1 + a^{1/n} + a^{2/n} + \cdots + a^{(m-1)/n}}{1 + a^{1/n} + a^{2/n} + \cdots + a^{(n-1)/n}}, \quad \text{for } m, n \in \mathbb{Z}^+, a > 0, \text{and } a \neq 1. \end{aligned}$$

Thus,

$$\lim_{a \to 1} \frac{a^{m/n} - 1}{a - 1} = \frac{m}{n}.$$

On rewriting, we have

$$\lim_{a \to 1} \frac{a^p - 1}{a - 1} = p, \quad \text{for } p \in \mathbb{Q}^+.$$

Given any positive real number r, let p and q be two positive rational numbers such that

$$p < r < q.$$

Then we obtain

$$a^p - 1 < a^r - 1 < a^q - 1, \quad \text{for } a > 1,$$
$$a^p - 1 > a^r - 1 > a^q - 1, \quad \text{for } 0 < a < 1.$$

Hence,

$$\frac{a^p - 1}{a - 1} < \frac{a^r - 1}{a - 1} < \frac{a^q - 1}{a - 1}, \quad \text{for } a > 0 \text{ and } a \neq 1.$$

On taking limits as $a \to 1$, we have

$$p \leq \lim_{a \to 1} \frac{a^r - 1}{a - 1} \leq q.$$

As p and q can be chosen to be arbitrarily close to r, we conclude that

$$\lim_{a \to 1} \frac{a^r - 1}{a - 1} = r, \quad \text{for } r > 0.$$

Also,

$$\lim_{a \to 1} \frac{a^{-r} - 1}{a - 1} = \lim_{a \to 1} \frac{1/a^r - 1}{a - 1} = \lim_{a \to 1} \frac{1 - a^r}{(a - 1)a^r} = -r, \quad \text{for } r > 0.$$

Thus, we have

$$\lim_{a \to 1} \frac{a^r - 1}{a - 1} = r, \quad \text{for all } r \in \mathbb{R}.$$

Writing $1 + \delta x$ for a, where $0 < 1 + \delta x$, we have

$$\lim_{\delta x \to 0} \frac{(1 + \delta x)^r - 1}{\delta x} = r, \quad \text{for } r \in \mathbb{R}.$$

Now, for $x > 0$, we obtain

$$\begin{aligned} \lim_{\delta x \to 0} \frac{(x + \delta x)^r - x^r}{\delta x} &= \lim_{\delta x \to 0} \frac{x^{r-1}[(1 + \delta x/x)^r - 1]}{\delta x/x} \\ &= x^{r-1} \lim_{\delta x \to 0} \frac{[(1 + \delta x/x)^r - 1]}{\delta x/x} \\ &= r x^{r-1}, \end{aligned}$$

which is the result

$$\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}, \quad \text{for } x > 0 \text{ and } r \in \mathbb{R}.$$

Yours sincerely,

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Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

40.5 Let $P_1P_2P_3P_4P_5$ be a regular pentagon, centre O, and let P be any point in the plane of the pentagon. For $i = 1, 2, 3, 4, 5$, denote by C_i the circle with centre P_i which passes through P, and denote the point of intersection of circles C_i and C_{i+1} other than P by A_i ($C_6 = C_1$). Prove that the centre of gravity of A_1, A_2, A_3, A_4, A_5 is O.

Let A_i' be the foot of the perpendicular from P to P_iP_{i+1} ($P_6 = P_1$). Show that A_i' is the midpoint of A_iP and that the centre of gravity of A_1', A_2', A_3', A_4', A_5' is the midpoint of OP.

(Submitted by Daniel Schultz (aged 13), Southbank International School, Hampstead, London, UK)

40.6 Let a and b be real numbers such that $a \geq b > 0$ and let $g(x)$ be a function such that $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} g'(x) = +\infty$. Compute the limit

$$\lim_{x\to+\infty} \frac{\ln(a\sqrt{\mathrm{e}^{2g(x)}+1} - b\mathrm{e}^{g(x)})}{x}.$$

(Submitted by Spiros P. Andriopoulos, Amaliada City, Eleia, Greece)

40.7 Let $x_1, \dots, x_n$ be positive real numbers and let α be a positive integer. Prove that

$$\sum_{1\leq i<j\leq n} \frac{x_i^{2\alpha} + x_j^{2\alpha}}{x_i^{2\alpha+2} + x_j^{2\alpha+2}} \leq \frac{n-1}{2} \sum_{k=1}^{n} \frac{1}{x_k^2}.$$

(Submitted by José Luis Díaz-Barrero, Barcelona, Spain)

40.8 For a positive integer n, let

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{and} \quad \delta_n = \sum_{k=1}^{n} \frac{2H_k}{k} - H_n^2.$$

Prove that

(a)

$$\lim_{n\to\infty} \delta_n = \zeta(2),$$

(b)

$$\sum_{n=1}^{\infty} \frac{\delta_n}{n(n+1)} = \zeta(3),$$

where $\zeta(s) = \sum_{n=1}^{\infty}(1/n^s)$.

(Submitted by Michel Bataille, Rouen, France)

Solutions to Problems in Volume 39 Number 3

39.9 Prove that the sum of two consecutive odd primes numbers is the product of at least three primes.

Solution by Ian Smith, Massachusetts Institute of Technology

Let p and $p+2$ be consecutive odd primes. Their sum is $2(p+1)$. Now p is odd so $p+1$ is even and greater than 2, say $2k$, where k is either prime or a product of primes. Hence, $2(p+1)$ is the product of 2, 2, and at least one more prime.

39.10 For real numbers a, b, and c such that $0 \le a, b, c \le 1$, prove that

$$a^{17} - a^{10}b^7 + b^{17} - b^{10}c^7 + c^{17} - c^{10}a^7 \le 1.$$

Solution by Alexei Gelbutovski, who proposed the problem

The numbers a, b, and c can by cyclically rearranged, so we can suppose that a is (possibly equal-) smallest. If $0 \le a \le b \le c \le 1$, then

$$a^{17} \le a^{10}b^7, \qquad b^{17} \le b^{10}c^7, \qquad c^{17} \le 1, \qquad 0 \le c^{10}a^7,$$

and we can add these inequalities to give the result. If $0 \le a \le c \le b \le 1$, then

$$a^{17} \le a^7c^{10}, \qquad c^{17} \le b^{10}c^7, \qquad b^{17} \le 1, \qquad 0 \le a^{10}b^7,$$

and we can add these inequalities to give the result.

39.11 For a triangle ABC, draw through A the line perpendicular to CA, through B the line perpendicular to AB, and through C the line perpendicular to BC, to form a triangle $A'B'C'$. Prove that the ratio of the areas of triangles $A'B'C'$ and ABC is

$$(\cot A + \cot B + \cot C)^2.$$

Solution

Construct BD parallel to $C'B'$ and BE parallel to $A'B'$ as shown in figure 1. From $\Delta A'BC$,

$$A'C = a \cot B.$$

From ΔBCD,

$$CD = a \cot C.$$

From the parallelogram $BDB'E$,

$$DB' = BE.$$

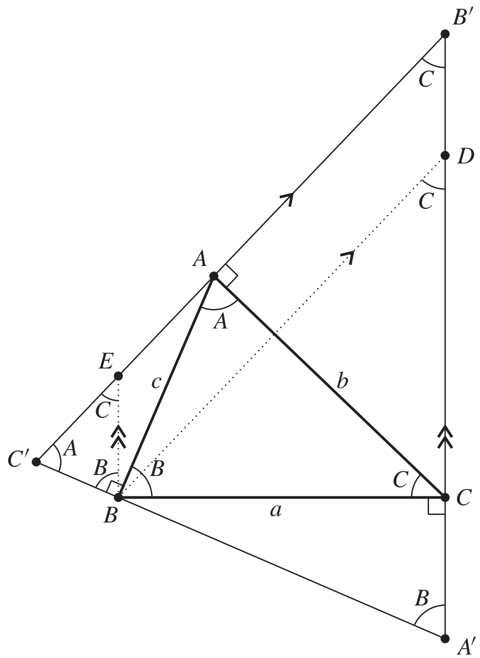

Figure 1

From $\Delta ABC'$, $BC' = c \cot A$. Triangles BEC' and ABC are similar, so that

$$\frac{BE}{BC'} = \frac{BC}{BA};$$

whence,

$$DB' = BE = \frac{ac \cot A}{c} = a \cot A.$$

Hence,

$$\begin{aligned} A'B' &= DB' + A'C + CD \\ &= a \cot A + a \cot B + a \cot C. \end{aligned}$$

Triangles ABC and $A'B'C'$ are similar, so the ratio of their areas is

$$(\cot A + \cot B + \cot C)^2.$$

39.12 The function $f : \mathbb{N} \to \mathbb{N}$ has the properties

1. $f(1) = 1$,
2. $3f(n)f(2n+1) = f(2n)(1 + 3f(n))$ for all $n \in \mathbb{N}$,
3. $f(2n) < 6f(n)$ for all $n \in \mathbb{N}$.

Find all positive integers k and m such that

$$f(k) + f(m) = 2007.$$

Solution by Farshid Arjomandi, who proposed the problem

Since $3f(n)$ and $3f(n)+1$ are coprime, Condition 2 gives that $3f(n) \mid f(2n)$. Hence, from Condition 3,

$$f(2n) = 3f(n).$$

Now Condition 2 gives that

$$f(2n+1) = 3f(n) + 1.$$

It follows from these and Condition 1 that, if n is written in base 2, then $f(n)$ has the same digits as n but in base 3. (For example, $f(7) = f((111)_2) = (111)_3 = 13$.) Now

$$2007 = (2\,202\,100)_3,$$

so to find k and m such that $f(k)+f(m) = 2007$, we need to express $(2\,202\,100)_3$ as a sum of two numbers whose ternary expansions contain only zeros and ones. When two such numbers are added, there is no digit transfer, so the only possibilities for k and m are $(1\,101\,000)_2$ and $(1\,101\,100)_2$, or 104 and 108, in either order.

Reviews

A Garden of Integrals. By Frank Burk. MAA, Washington, DC, 2007. Hardback, 304 pages, \$51.95 (ISBN 0-88385-337-5).

The derivative and the integral are the fundamental notions of calculus. Though there is essentially only one derivative, there is a variety of integrals, developed over the years for a variety of purposes, and this book describes them. No other single source treats all of the integrals of Cauchy, Riemann, Riemann-Stieltjes, Lebesgue, Lebesgue-Stieltjes, Henstock-Kurzweil, Weiner, and Feynman.

Experimental Number Theory. By Fernando Villegas. Oxford University Press, 2007. Paperback, 214 pages, £29.50 (ISBN 0-19-922730-3).

This graduate text is intended for first or second year graduate students in pure mathematics. The main goal of the text is to show how the computer can be used as a tool for research in Number Theory through numerical experimentation. The book contains many examples of experiments in elementary class field theory, binary quadratic forms, sequences, combinatorics, p-adic numbers and polynomials, along with exercises and selected remarks and solutions. The numerous routines used in the examples are written in GP, the scripting language for the computational package PARI, and are available for download from the author's website.

The Genius of Euler: Reflections on His Life and Work. Edited by William Dunham. MAA, Washington, DC, 2007. Hardback, 309 pages, $47.95 (ISBN 0-88385-558-5).
The Early Mathematics of Leonhard Euler. By C. Edward Sandifer. MAA, Washington, DC, 2007. Hardback, 391 pages, $49.95 (ISBN 0-88385-559-3).

These books are being published to commemorate the three-hundredth anniversary of the birth of Euler, arguably the greatest mathematician of the seventeenth century. The first will probably be of wider interest and accessibility. The second is more technical, but will be of interest to historians of mathematics and to anyone wishing to probe deeper into Euler's methods. Both books have facsimiles of Euler's original work, many more being included in the first book, which also has quite a number of pictures of Euler and things relating to Euler, which are listed at the back of the book. Both books contain contents pages and an index. I believe they would both make very good additions to any University library.

The Genius of Euler contains essays by many famous names in mathematics (Glaisher, Ball, Cajori, Finkel, Pólya, Weil, Kline, Erdös and Truesdell), some published over a hundred years ago, and also by many more recent authors. In Part 1 there are 88 fascinating pages of biography, followed by a few poems. Part 2 includes essays on Euler's work on infinite series, the zeta function, the Euler constant γ, differentials, multiple integrals, partial differentiation, number theory, the fundamental theorem of algebra (which Euler didn't quite crack), the famous topological problem of the Königsberg bridges, the theory of numbers, and Euler's mistaken conjecture relating to Graeco–Latin squares. This book is both an excellent read and resource book for the teaching and learning of mathematical history.

The Early Mathematics of Leonhard Euler is much more difficult. The papers are on pure mathematical subjects and are sensibly presented in the order in which they were written, which is only very roughly the order in which they were published. These are designated the numbers E-1, E-2, and so on. The numbering here indicates the order of publication, and all of Euler's 800 or so papers have been given a number in this scheme. The 49 papers run from 1725 when he was 18 (E-1) up to his return to Berlin from St Petersburg in 1741 (E-790), and so cover less than a third of his working life which lasted right up to his death in 1783. For each year one or two world events are given, as well as brief biographical details of Euler's life and his mathematical and other work over the same time period. At the back of the book there is a summary under subject headings of the E-numbers of the articles in order in which they appear. The subjects and numbers of articles on each subject are as follows: Series (18), Geometry and curves (9), Calculus of variations (4), Elliptic integrals (1), Differential equations (8), Number Theory (5), Theory of equations (2), Topology (1), and Philosophy (1). The author indicates which are, in his opinion, the most important papers by a system of one, two, or three asterisks. 13 of the 49 papers are so marked; three asterisks are given to papers on the Königsberg bridges, continued fractions, the product formula for the zeta function, of Riemann hypothesis fame, and finally two fascinating papers on infinite series.

The author gives a précis of each paper containing varying amounts of detail. The reader will have to imagine the missing bits, or in some cases he will be able to fill them in himself, if prepared for the hard graft involved! The articles contain many interesting historical observations relating to both the mathematics and mathematical notation of Euler. It is interesting too to see the development of Euler's own thoughts. Later in life he returned to incompletely solved problems and developed them further; examples of this are given, even though papers written after 1741 are not included in the book. The book makes clear that, to a considerable extent, the history of seventeenth century mathematics is the history of Euler's own mathematical development.

It is fascinating to see how both mathematical principles and mathematical notation we take for granted today developed in the hands of the great man. There are references to the work of others in passing but it is made clear how far ahead, in general, Euler was of his generation.

Reasons for Euler's choice of subject matter for his papers are given. For example, *number theory* was not yet a very prominent subject among the mathematical fraternity in the early eighteenth century; hence, he comes to it relatively late. One fascinating paper deals with the discovery and use of the famous extension of Fermat's little theorem. Two of the other *number theoretic* topics dealt with are Fermat numbers and continued fractions.

I found the *differential equations* hard going because of Euler's perfectly consistent, but no longer familiar, differential notation, and also the assumptions Euler made when ignoring terms of second-order smallness. Also the complexity of some of the calculations is considerable. The papers on the *calculus of variations* I found most fascinating, showing how he derived his results from first principles.

In his work on *infinite series* we observe his lack of rigour! However, it is impressive to learn how often he got things right in spite of this. Here the author helps us out by often writing Euler's working in modern notation. Euler is not always consistent; in one paper he states that $\log\infty$ is the smallest of the infinite values and then later in the same paper says that $\log\log\infty$ is infinitely less than $\log\infty$!

The book ends with a paper on the utility of higher mathematics, the very antithesis of Hardy's *A Mathematician's Apology* (1940). This particular paper, although written in 1741, was not published until 1847, 64 years after Euler's death. In it he uses the applications of mathematics to justify its usefulness. He was well versed in these, though his papers on applied mathematics have not been included in this volume.

Alastair Summers

The Magic Numbers of the Professor. By Owen O'Shea and Underwood Dudley. MAA, Washington, DC, 2007. Hardback, 335 pages, \$39.95 (ISBN 0-88385-557-7).

Let me start with three quotes from the book that sum up the whole demeanour of this publication.

> The word *Apollo*, he said, contains six letters. The professor pointed out that 6 factorial, usually written as $6!$, means $6\cdot5\cdot4\cdot3\cdot2\cdot1$, or 720. The date the first men walked on the moon was 7/20.

> I do know of a number of curiosities concerning the 9/11 attacks. The attacks happened on the 254th day of the year. The sum of these digits is 11. The suspected leader of the group behind the attack is Osama bin Laden. He was said to be based in Afghanistan. The name of that country contains 11 letters. Its capital is Kabul. Its initial letter is the 11th letter of the alphabet.

> Right-thinking people around the world consider the assassinations of Lincoln in 1865 and Kennedy in 1963 as wicked deeds. Of course, one of the best-known numerical symbols for evil is the number 666, which appears very early in $1963^{1865} = 4.112\,724\,666\,3\ldots$.

This is NOT Mathematics. There is no proof to be constructed. There is no underlying pattern. There are no generalisations to consider.

Are they interesting meaningful coincidences? NO; since what is not mentioned is all the other pathetic calculations that the authors did, that they did not consider wonderful, so they did not include them.

This book should be kept (if kept at all) in the numerology, conspiracy theory, and other gobbledegook section, as it certainly does not qualify as Mathematics.

What particularly annoys me is that some proper Mathematical ideas are mentioned, for example Wilson's theorem, but no distinction is made between them and the rest of the dross.

To prove that I can investigate numbers just as well as the authors, 'magic numbers' contains 12 letters as does 'utter rubbish' or, as the book was published (to their shame) by The Mathematical Association of America, so does 'total garbage'.

Atlantic College **Paul Belcher**

How Euler Did It. By C. Edward Sandifer. MAA, Washington, DC, 2007. Hardback, 304 pages, $51.95 (ISBN 0-88385-563-8).

This book is an excellent introduction to the work of Leonhard Euler, the Swiss mathematical genius, and generally recognized as the greatest Western Mathematician of the eighteenth century.

It consists of the 40 articles under the same heading, which appeared on MAA Online between November 2003 and February 2007. The articles should be within the grasp of top grade secondary school students. I found I could read them quickly as I would a novel, but with the pleasurable option of going back later and checking out the mathematics in more detail, which I have done in a few cases already with profit.

As well as describing Euler's mathematics, the articles are full of historical references, discussions, and comments on Euler's methods, thus increasing the interest. Also Sandifer often makes his summaries of Euler's work easier to follow by using modern notation, whilst explaining Euler's. For example, Euler did not use suffix notation, or the summation symbol.

The first article is entitled 'Euler's Greatest Hits', listing the top 10 choice of Euler's results, selected by a group of mathematicians. The top two in the list are the solution to the Basel problem

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

and the even better known result $V - E + F = 2$. In its day the Basel problem had the same kind of mystique as Fermat's last theorem, and Euler's success in cracking it was his gateway to fame at the age of 28 in 1735. There are articles on both of these results, which are typical of many of the articles in that they analyse the defects as well as the genius of Euler's discoveries.

After the first article, the book divides into four parts – Geometry, Number Theory, Combinatorics, and Analysis – the last containing 21 of the 40 articles, and including the only three on Applied Mathematics. Some of the topics are amicable numbers, Pell's equation, the Knight's tour, arc length of an ellipse, and Bernoulli numbers. The last article is an introduction to the Euler Society.

A small defect is the number of typographical errors, for example omission of brackets, inconsistent use of variable names, the writing '$1/x$' instead of 'x', and the misspelling of the Association's website, MAA Online.

To summarize, this is an excellent book for any student or teacher of Mathematics, whether to enhance enjoyment and knowledge of the subject, or as part of a course in the History of Mathematics, in which Euler's contributions are highly significant.

Alastair Summers

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