Mathematical Spectrum

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A magazine for students and teachers of mathematics
in schools, colleges and universities

MATHEMATICAL SPECTRUM

This is a magazine for students and teachers in schools, colleges and universities, as well as the general reader interested in mathematics. It is published by the Applied Probability Trust, a non-profit-making organisation established in 1963 with the support of the London Mathematical Society. The object of the Trust is the encouragement of study and research in the mathematical sciences.

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Articles published in *Mathematical Spectrum* deal with the entire range of mathematical disciplines (pure mathematics, applied mathematics, statistics, operational research, computing science, numerical analysis, biomathematics). Both expository and historical material may be included, as well as elementary research and information on educational opportunities and careers in mathematics. There are also sections devoted to problems, to mathematics in the classroom, and to computing.

From the Editor

Puzzles and problems

It is always good to hear from students of mathematical items that they have come across. One of my former students, who now teaches mathematics, saw the following puzzle on a calendar published by Pomegranate Communications in California, which in turn originated from Clifford Pickover (http://www.pickover.com), who has at times contributed to *Mathematical Spectrum*:

> Construct the largest number you can from the digits 1, 2, 3, 4, each used once, using powers and any number of minus signs and decimal points and brackets you like. (For example, 31^{42} has an amazing 63 digits, but can you do better?)

A nice problem for you to while away the time in a boring lesson.

In the last issue, I gave a problem sent to me by a former student of mine. Here it is again:

> The sequence (u_n) is defined by
>
> $$u_0 = 3\,, \qquad u_1 = 0\,, \qquad u_2 = 2\,,$$
> $$u_n = u_{n-2} + u_{n-3} \quad \text{for } n > 2\,,$$
>
> and the problem is to find those n for which n divides u_n.

Nick Lord of Tonbridge School has written to tell me that this is a problem with a long pedigree, and refers me to the March 1998 issue of *Mathematical Gazette*, where it appears as Problem 81G. There, John Chapman submitted three proofs of the fact that, when p is prime, p divides u_p. Here is one of these proofs. The given recurrence relation has 'auxiliary equation' $x^3 - x - 1 = 0$. Denote the roots of this equation by α, β, γ. Then the elementary symmetric functions in α, β, γ are

$$\begin{aligned} e_1 &= \alpha + \beta + \gamma = 0\,, \\ e_2 &= \alpha\beta + \alpha\gamma + \beta\gamma = -1\,, \\ e_3 &= \alpha\beta\gamma = 1\,. \end{aligned}$$

Write $\alpha^n + \beta^n + \gamma^n = s_n$ for $n = 0, 1, 2, \ldots$. Then

$$\begin{aligned} s_0 &= 3 = u_0\,, \\ s_1 &= \alpha + \beta + \gamma = 0 = u_1\,, \\ s_2 &= \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\ &= 2 = u_2\,. \end{aligned}$$

Moreover, $\alpha^3 - \alpha - 1 = 0$ so, for $n \geq 3$,

$$\alpha^n - \alpha^{n-2} - \alpha^{n-3} = 0\,,$$

and similarly for β, γ. If we add the three equations, we obtain

$$s_n - s_{n-2} - s_{n-3} = 0\,.$$

Thus, s_n and u_n satisfy the same recurrence relation, with the same initial values, so $s_n = u_n$ for all n.

Now let p be a prime. Then

$$\begin{aligned} 0 &= (\alpha + \beta + \gamma)^p \\ &= \sum_{r+s+t=p} \frac{p!}{r!\,s!\,t!} \alpha^r \beta^s \gamma^t \\ &= \alpha^p + \beta^p + \gamma^p + pF(\alpha, \beta, \gamma) \\ &= s_p + pF(\alpha, \beta, \gamma) \\ &= u_p + pF(\alpha, \beta, \gamma)\,, \end{aligned}$$

where $F(\alpha, \beta, \gamma)$ is a symmetric function with integer coefficients. This is true because, when r, s, t are less than p, $r!\,s!\,t!$ is not divisible by p but $p!$ is; and p is prime. We now need the result that a symmetric function in α, β, γ with integer coefficients is a polynomial in the elementary symmetric functions e_1, e_2, e_3 with integer coefficients. Since e_1, e_2, e_3 are integers in this case, this means that $F(\alpha, \beta, \gamma)$ is an integer. Hence u_p $(= -pF(\alpha, \beta, \gamma))$ is divisible by p. Very clever!

But are there any non-primes n for which n divides u_n? Yes! According to the *Gazette* article, the smallest such non-prime is $n = 521^2$. Such a non-prime is called a *Perrin pseudoprime*. The sequence in the problem is called the *Perrin sequence* and was first considered by R. Perrin in 1899. There is a lot more fascinating information in the *Gazette* article, with references to follow up. Quite a problem!

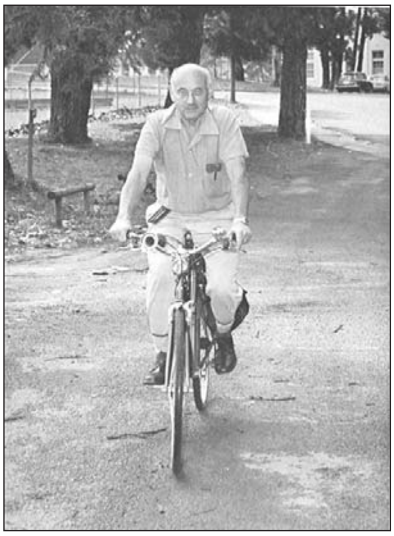
Another of our readers, Alastair Summers of Stamford School, has also sent in a proof that p divides u_p when p is prime in the Perrin sequence. In the last issue, I also gave the sequence (u_n) defined by

$$\begin{aligned} u_1 &= 1, \\ u_n &= (n-1)u_{n-1} + 1 \quad \text{for } n > 1. \end{aligned}$$

Again the problem is to find those n for which n divides u_n. Alastair Summers has shown that when p and q are coprime, $p \mid u_p$ and $q \mid u_q$ implies that $pq \mid u_{pq}$, so that it is only necessary to consider primes and powers of primes. He says that the only primes p under 300 (found using EXCEL®) for which $p \mid u_p$ are $p = 2$, 5, 13 and 37. Also, $p = 4$ works. The only integers under 300 which work are 2, 4, 5, 10, 13, 20, 26, 37, 52, 65, 74, 130, 148, 185 and 260. Further enlightenment is needed!

An Ancient Mathematician Remembers

B. H. NEUMANN



Bernhard Neumann, *c.* 1979.

A note from the Editor

Many of our readers are students of mathematics at schools who may be wondering whether to study mathematics at a university. Others are already at university and are wondering what to do afterwards. Should they even do postgraduate study in mathematics and become Mathematicians with a capital M! In view of this, we asked one of our distinguished consulting editors, Professor Bernhard Neumann, if he would write something for us reminiscing on his life as a professional mathematician.

Bernhard Neumann was born in Berlin on 15 October 1909. He recalls his formative mathematical years in this article. He emigrated to Britain in 1933 to continue his studies at Cambridge. After two years of unemployment, he taught at University College, Cardiff, until war broke out. He was briefly interned as an enemy alien, but was released to join the Pioneer Corps, the Royal Artillery, and later the Intelligence Corps in which he served between 1940 and 1945, one of a number of mathematicians whose skills were invaluable to intelligence during the war.

At the end of the war, Bernhard and his wife Hanna taught mathematics at the University of Hull and then in Manchester, where Bernhard taught at the University of Manchester and Hanna at the University of Manchester Institute of Science and Technology (UMIST). Bernhard was elected to a Fellowship of the Royal Society in 1959. They had five children, and the whole family were cycling enthusiasts, enjoying frequent cycling holidays together. Peter, their second child, followed in his parents' footsteps and is a distinguished mathematician, working in the same general area of group theory as his parents.

In 1962 a new chapter began for Bernhard and Hanna when Bernhard emigrated to Australia and Hanna followed the next year. Your Editor recalls attending a seminar at UMIST shortly before Hanna left. Surrounded by packing cases, we enjoyed omelettes cooked by Hanna! Bernhard became Head of the Department of Mathematics in the Institute of Advanced Studies at the Australian National University (ANU) and Hanna Head of the Department of Pure Mathematics in the School of General Studies at ANU. They made major contributions to the growing mathematical scene in Australia. Hanna died in 1971 and Bernhard retired in 1974; he married Dorothea in 1973.

Following his retirement, Bernhard continued to be active in mathematical research and in mathematical affairs in Australia and abroad. He helped to start the New Zealand Mathematical Society and pursued his interest in chess, music and cycling. Bernhard and Hanna are in many ways role models for today's aspiring young mathematicians.

What can a nonagenarian mathematician contribute to *Mathematical Spectrum* that is of relevance today? Clearly, nothing. However, some memories of the time before most of you were born may still be of a little interest.

I spent the first two semesters as a university student at the University of Freiburg, in the south of Germany. Professor Lothar Heffter taught me projective geometry, which was an eye-opener for me. He was in his mid-sixties at the time, had recently married the young widow of a colleague, and had a baby daughter. I kept in touch with him for years after I had left Freiburg: he was an athletic type and went on working, and walking and swimming until he was well into his 90s. When his 100th birthday approached, I wanted to dedicate a paper to him. I sent it to a journal that was based in Freiburg, and had the reply that the evening before my paper arrived they had heard on the radio that Geheimrat [Privy Councillor] Heffter had just died: he was $99\frac{1}{2}$ years old. So I had to dedicate that paper to his memory.

When I left Freiburg, I continued at the University of Berlin. At the time it was quite usual for students to move from university to university in Germany; their higher school leaving certificate, called the *Abitur*, entitled them to study at all universities. In Berlin there were four famous professors of mathematics at the time, plus a number of lecturers (*Privatdozenten*), among them Johann Neumann von Margitta, later very well known as John von Neumann (no relation of mine). Thus I had an exciting mathematical life. John von Neumann introduced me in his lectures to Gödel's first famous theorem, when it was still red hot. I also learnt the foundations of set theory from him, and much admired him.

One of the lecturers was Robert Remak, who one semester announced a course on introductory group theory. He started off with some 30 students, but soon there were only three of us left, and in the middle of the semester one left, so there were only two. The other one also left before the end of the semester, so Robert Remak lectured to me alone. He talked about some of his still unpublished work, which I found exciting, though, having never heard of groups before, I missed some details. In the end Robert Remak asked me how I had found his lectures, so I admitted to having had some difficulties, because groups had been so new to me; whereupon he said that if he had known that, he would have given a quite different course; I was glad he hadn't.

One of the lecturers at the time was the topologist Heinz Hopf, and in a student seminar he ran with the geometer Georg Feigl I had to read a paper on infinite groups. It set me thinking, and I much reduced the number of generators of the groups concerned. I wrote this up as a short paper and showed it to Heinz Hopf, who asked me whether I wanted to use it as my doctoral dissertation. I said 'no', as I was only in my third undergraduate year and the paper was, I thought, too slight. Heinz Hopf accepted that and showed the paper to the senior professor in algebra, Issai Schur. Nothing happened until near the end of that semester, when Professor Schur asked me, through his assistant Alfred Brauer, whether I wanted to use it as my doctoral dissertation. I gave the same reply. At that time Heinz Hopf moved to the ETH (Eidgenössische Technische Hochschule) in Zürich as professor, and he stayed there for the rest of his life. He moved at the beginning of the Easter holidays, and at the end of those holidays I visited him in Zürich, and we remained friends to the end of his life. When I returned to the University of Berlin after the holidays, Alfred Brauer told me: 'but Professor Schur wants you to take your doctorate with that paper'; so I had to agree. However, a little later Professor Schur told me he agreed it was a little light as a dissertation; could I perhaps use the same methods to deal with what later became known as the wreath product of two symmetric groups? It did not take me long to do this, and it more than doubled the length of the paper: I took my Dr.phil. with it, becoming the youngest Herr Doktor in mathematics in Berlin. It is now 70 years ago that the degree was conferred on me.

Not long after that, Adolf Hitler came to power in Germany, and I moved to England. Many other Jewish mathematicians remained too long. Issai Schur was stripped of his professorship and all other privileges, and eventually migrated to Switzerland, where his daughter still lives, and finally to Israel, where he died on his 66th birthday. Alfred Brauer migrated to the USA, where he eventually died in his 90s. Robert Remak was caught by the Nazis in the Netherlands and died in a concentration camp. Another famous professor, Richard von Mises, eventually migrated to Turkey and then to the USA. The ones who left early, like Kurt Hirsch, Richard Rado, Hans Heilbronn, all had satisfactory careers in the end, and made their contribution to the mathematical life of the country in which they settled.

To Be or Not 2^b — That Other Classic

P. GLAISTER

The inequality $2^b \geq 1 + b$ for positive integers b, with equality only for $b = 1$, is readily proved using the binomial theorem in the form

$$2^b = (1+1)^b = \binom{b}{0} + \binom{b}{1} + \cdots + \binom{b}{b}$$
$$= 1 + b + \binom{b}{2} + \cdots + \binom{b}{b} \begin{cases} > 1 + b & \text{for } b \geq 2\,, \\ = 1 + b = 2 & \text{for } b = 1. \end{cases}$$

Although I knew this result, it wasn't until recently that I used it for the first time.

It all began with the divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ and the corresponding convergent alternating series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$. The well-known series for $\ln(1+x)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\,, \tag{1}$$

converges for $-1 < x \leq 1$, and so with $x = 1$

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots\,, \tag{2}$$

which is the usual series that is quoted for $\ln 2$. However, with $x = -\frac{1}{2}$ we obtain, after simplification,

$$\ln 2 = \left(1 \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{1}{2^2}\right) + \left(\frac{1}{3} \cdot \frac{1}{2^3}\right) + \cdots\,, \tag{3}$$

which is a less-commonly quoted result. An interesting observation is that the expression in (3) contains the terms of the series in (2) multiplied by powers of $\frac{1}{2}$, but with all negative signs becoming positive. Moreover, the series of these powers of $\frac{1}{2}$ is a geometric one whose sum is

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1\,.$$

Thus,

$$\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right)\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots\right)$$
$$= \left(1 \cdot \frac{1}{2}\right) + \left(\frac{1}{2} \cdot \frac{1}{2^2}\right) + \left(\frac{1}{3} \cdot \frac{1}{2^3}\right) + \cdots\,, \tag{4}$$

or

$$1 \times \ln 2 = \ln 2\,.$$

This begs the question as to whether there are series for which

$$\begin{aligned}(a_1 + a_2 + a_3 + \cdots)(b_1 - b_2 + b_3 - \cdots)\\ = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots \end{aligned} \tag{5}$$

other than the one above with $a_i = 1/2^i$, $b_i = 1/i$.

One obvious starting point is to try for a more general form of (4), such as

$$\begin{aligned}\Big(\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\Big)\Big(1 - \frac{1}{2} + \frac{1}{3} - \cdots\Big)\\ = \Big(1 \cdot \frac{1}{p}\Big) + \Big(\frac{1}{2} \cdot \frac{1}{p^2}\Big) + \Big(\frac{1}{3} \cdot \frac{1}{p^3}\Big) + \cdots\,. \end{aligned} \tag{6}$$

Now, the first term on the left-hand side of (6) is

$$\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{1/p}{1 - 1/p} = \frac{1}{p-1}\,, \tag{7}$$

provided that $-1 < 1/p < 1$, and from (1) with $x = -1/p$

$$\ln\Big(1 - \frac{1}{p}\Big) = -\Big(1 \cdot \frac{1}{p}\Big) - \Big(\frac{1}{2} \cdot \frac{1}{p^2}\Big) - \Big(\frac{1}{3} \cdot \frac{1}{p^3}\Big) - \cdots\,,$$

provided that $-1 \le 1/p < 1$, i.e.

$$\ln\Big(\frac{p}{p-1}\Big) = \Big(1 \cdot \frac{1}{p}\Big) + \Big(\frac{1}{2} \cdot \frac{1}{p^2}\Big) + \Big(\frac{1}{3} \cdot \frac{1}{p^3}\Big) + \cdots\,. \tag{8}$$

Combining (2), (7) and (8) with (6), we seek p such that

$$\frac{1}{p-1} \ln 2 = \ln\Big(\frac{p}{p-1}\Big)\,,$$

so, with $1/(p-1) = b$, we have

$$b \ln 2 = \ln(1 + b), \quad \text{that is,} \quad 2^b = 1 + b\,.$$

However, as we have already seen, $2^b \ge 1 + b$ with equality only for $b = 1$, which corresponds to $p = 2$, and hence (4) is the only solution of the kind in (6).

So, it wasn't to be after all. I haven't given up hope of finding other examples of the expression in (5), or similar ones, and leave readers to turn their minds to it. But there was an interesting spin-off.

Returning to the series in (3), this can be viewed as the divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ where the terms are scaled by the corresponding one for a geometric series, which is essentially what (1) says. Thus, with $1/2^i$ replaced by $1/3^i$, the result corresponding to (3) is obtained from (8) with $p = 3$ as

$$\ln\frac{3}{2} = \Big(1 \cdot \frac{1}{3}\Big) + \Big(\frac{1}{2} \cdot \frac{1}{3^2}\Big) + \Big(\frac{1}{3} \cdot \frac{1}{3^3}\Big) + \cdots\,, \tag{9}$$

and so on with $p = 4, 5, \ldots$. Now adding (3) and (9) gives

$$\begin{aligned}\ln 3 &= \ln\Big(2 \times \frac{3}{2}\Big) = \ln 2 + \ln\frac{3}{2}\\ &= \Big(1 \cdot \frac{1}{2}\Big) + \Big(\frac{1}{2} \cdot \frac{1}{2^2}\Big) + \Big(\frac{1}{3} \cdot \frac{1}{2^3}\Big) + \cdots\\ &\quad + \Big(1 \cdot \frac{1}{3}\Big) + \Big(\frac{1}{2} \cdot \frac{1}{3^2}\Big) + \Big(\frac{1}{3} \cdot \frac{1}{3^3}\Big) + \cdots\\ &= 1\Big(\frac{1}{2} + \frac{1}{3}\Big) + \frac{1}{2}\Big(\frac{1}{2^2} + \frac{1}{3^2}\Big) + \frac{1}{3}\Big(\frac{1}{2^3} + \frac{1}{3^3}\Big) + \cdots\,. \end{aligned} \tag{10}$$

More generally, for any integer $n \ge 2$,

$$\begin{aligned}\ln n &= \ln\Big(\frac{n}{n-1}\frac{n-1}{n-2}\cdots\frac{3}{2}\frac{2}{1}\Big)\\ &= \ln\Big(\frac{n}{n-1}\Big) + \ln\Big(\frac{n-1}{n-2}\Big) + \cdots + \ln\Big(\frac{3}{2}\Big) + \ln\Big(\frac{2}{1}\Big)\\ &= 1\Big(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big) + \frac{1}{2}\Big(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\Big)\\ &\quad + \frac{1}{3}\Big(\frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3}\Big) + \cdots \end{aligned} \tag{11}$$

following some rearrangement, and (11) can be viewed as the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ where the terms have been scaled by the terms of a series whose individual terms are $1/2^i + 1/3^i + \cdots + 1/n^i$.

I think many readers will not be familiar with the expression for $\ln n$ in (11). Indeed, the usual series expressions for $\ln x$ which can be derived from (1) are

$$\ln x = 2\Big[\frac{x-1}{x+1} + \frac{1}{3}\Big(\frac{x-1}{x+1}\Big)^3 + \frac{1}{5}\Big(\frac{x-1}{x+1}\Big)^5 + \cdots\Big]\,, \quad x > 0\,, \tag{12}$$

$$\ln x = \frac{x-1}{x} + \frac{1}{2}\Big(\frac{x-1}{x}\Big)^2 + \frac{1}{3}\Big(\frac{x-1}{x}\Big)^3 + \cdots\,, \quad x \ge \tfrac{1}{2}\,, \tag{13}$$

which we leave as an exercise. From (12),

$$\ln 2 = 2\Big(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \cdots\Big)$$

and

$$\ln 3 = 2\Big(\frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2^3} + \frac{1}{5} \cdot \frac{1}{2^5} + \cdots\Big)\,,$$

which are different from the series in (3) and (10), whereas from (13) we have

$$\ln 2 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} + \cdots$$

and

$$\ln 3 = \frac{2}{3} + \frac{1}{2} \cdot \Big(\frac{2}{3}\Big)^2 + \frac{1}{3} \cdot \Big(\frac{2}{3}\Big)^3 + \cdots\,,$$

and the former is the series in (3), but the latter is also different from the series in (10).

Finally, returning to (11) with $n = 4$ we have

$$\begin{aligned}\ln 4 &= 1\Big(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\Big) + \frac{1}{2}\Big(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\Big)\\ &\quad + \frac{1}{3}\Big(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3}\Big) + \cdots\,,\end{aligned}$$

whereas, also from (11),

$$\ln 4 = \ln 2^2 = 2\ln 2$$
$$= 2\left(\left(1\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2^2}\right) + \left(\frac{1}{3}\cdot\frac{1}{2^3}\right) + \cdots\right),$$

so that

$$1\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right) + \cdots$$
$$= 2\left(\left(1\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2^2}\right) + \left(\frac{1}{3}\cdot\frac{1}{2^3}\right) + \cdots\right). \quad (14)$$

For me, this expression has an aesthetic quality. Although it would be satisfying if it had some use, I suspect that it has none at all. A more general expression of this kind is obtained from (11) with $n = N^M$, and other results with $n = NM$, where in both cases N, M are integers. A particularly intriguing set of results is obtained with $n = 2^N$, generalising that in (14) where $N = 2$, as

$$1\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^N}\right) + \frac{1}{2}\left(\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{2^{2N}}\right) + \cdots$$
$$= N\left(\left(1\cdot\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{1}{2^2}\right) + \left(\frac{1}{3}\cdot\frac{1}{2^3}\right) + \cdots\right). \quad (15)$$

To conclude, we note that, in contrast to the slow convergence of the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ for $\ln 2$, the series in (3) converges much more quickly. For example, only ten terms of $1\cdot(1/2) + \frac{1}{2}\cdot(1/2^2) + \frac{1}{3}\cdot(1/2^3) + \cdots$ are required to evaluate $\ln 2$ to four decimal place accuracy as 0.6931. Similarly, with $N = 3$, the left-hand side of (15) gives $\ln 8$ to four decimal place accuracy as 2.0794 using eleven terms. Readers are invited to write a program that allows numerical experimentation of the formulae in (15) and other formulae derived from (11).

The author lectures in mathematics at Reading University. His research interests include computational fluid dynamics, numerical analysis, perturbation methods as well as mathematics and science education. He realised recently how little mathematics he knew when he was unable to help his eleven-year-old daughter tackle her first piece of homework from her new secondary school. The topic was magic squares. However, a little more progress was made with his eight-year-old son's permutations and combinations homework!

A Computer Hunt for Apéry's Constant

THOMAS J. OSLER and BRIAN SEAMAN

1. Introduction

The zeta function $\zeta(z)$, given by the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{for } \mathrm{Re}(z) > 1\,,$$

is one of the most important functions in mathematics. As early as 1734, Euler solved a long-standing problem by finding the 'closed-form' expression $\zeta(2) = \frac{1}{6}\pi^2$. The problem of finding $\zeta(2)$ had been worked on, without success, by many of the best mathematical minds of the time, including John Wallis, John, James and Daniel Bernoulli, Christian Goldbach and Gottfried Wilhelm Leibniz. Not only did Euler find $\zeta(2)$, but he also showed that $\zeta(4) = \frac{1}{90}\pi^4$, $\zeta(6) = \frac{1}{945}\pi^6$; in fact, he summed $\zeta(z)$ for every even positive integer z (see the next section for more details). Naturally, Euler looked for a closed form for $\zeta(3)$ as well as for $\zeta(z)$ whenever z is odd and positive, but all such attempts failed. (See reference 1 for an interesting historical summary of Euler's work on this and other topics.) To this day, we do not know if a nice closed-form expression for these values of the zeta function exists. Only recently, Apéry (see references 2, 3 and 4) proved that $\zeta(3)$ is irrational and since then $\zeta(3)$ has been known as 'Apéry's constant'. It is not known if $\zeta(z)$ is irrational for odd z greater than 3.

In this article, we will hunt for a closed-form expression for $\zeta(3)$ with the help of simple computer programs. Deep-sea divers who hunt for sunken treasure ships from Spanish galleons do not just dive anywhere in the wide ocean. They first look for old maps and other evidence that suggest where the ships perished. In the same way, we will seek evidence suggesting the probable appearance of the closed form for $\zeta(3)$. For example, the evidence listed above suggests that $\zeta(3)$ is possibly $(N/D)\pi^3$, where N and D are natural numbers. Using a program like MATHEMATICA® we can get a numerical approximation to any desired precision, for example,

$$\zeta(3) \approx 1.202056903159594285393816151145\,.$$

We can let the computer try many values of N and D until we get very close to this number with $(N/D)\pi^3$. If we can repeat all 30 digits of the above approximation with relatively small values of N and D, we might have struck gold! We could then check more digits in the numerical expansion of Apéry's constant. If the digits always checked exactly, our confidence would increase, but we would never be completely convinced of success. This is because we can never actually check all the digits. It would remain to be proved mathematically that our result was correct, and the closed form we obtained might help in finding such a proof.

Most likely, our search will fail. However, even if we do not get the exact value of $\zeta(3)$, we will have obtained at least two items of information:

1. Our computer search would reveal that $\zeta(3)$ is *not* of a particular form for a given range of parameters. For example, if we search as above for the form $(N/D)\pi^3$ by allowing the computer to try all positive integer values of N and D less than 100 000, then we will be able to say that Apéry's constant is *not* of this form for these values.
2. Our computer search will produce some values of the selected form that are very good *approximations* of Apéry's constant. These simple approximations are sometimes of interest in themselves. For example, $\frac{22}{7}$ and $\frac{355}{113}$ are very good rational approximations for π that are used frequently.

2. The evidence

There are infinitely many possible expressions that a 'simple closed form' for Apéry's constant $\zeta(3)$ might assume. For example, Apéry proved that it is *not* of the form N/D, but it could take the form $(N/D)\pi^3$, or $(N\sqrt{2}/D)\pi^3$, or $(N\sqrt{3}/D)\pi^3$, or $(N/D)\pi^3 + P/Q$, or $(N/D)\pi^3 + (P/Q)\log 2, \ldots,$ (where N, D, P and Q are integers). We begin by looking at known closed-form expressions for series related to $\zeta(3)$.

Euler (reference 5, pp. 137–153) found closed-form expressions for $\zeta(z)$ when z is an even natural number (see also references 6–9). He showed that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\,,\quad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}\,,\quad \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}\,,$$

and in general

$$\sum_{n=1}^{\infty}\frac{1}{n^{2p}}=\frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!}\,\pi^{2p}$$

(see reference 8, p. 237). Here the numbers B_n are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0=1\,,\quad B_1=-\tfrac{1}{2}\,,\quad B_2=\tfrac{1}{6}\,,\quad B_4=-\tfrac{1}{30}\,,\quad B_6=\tfrac{1}{42}\,,$$

and $B_3=B_5=B_7=\cdots=0$. These can all be calculated recursively by starting with $B_0=1$ and using

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0$$

for $n=2,3,4,\ldots.$

Euler also found alternating series related to $\zeta(z)$. These include

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^1}=\frac{\pi}{4}\,,$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32}\,,$$
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}=\frac{\pi^5}{1536}\,,$$

and in general

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2p+1}}=\frac{(-1)^pE_{2p}}{2^{2p+2}(2p)!}\,\pi^{2p+1}$$

(see reference 8, p. 240). Here the E_n are called Euler's numbers. They are all integers and the first few are $E_0=1$, $E_2=-1$, $E_4=5$, $E_6=-61$, $E_8=1385,\ldots,$ and $E_1=E_3=E_5=\cdots=0$. The E_{2n} can all be calculated recursively by starting with $E_0=1$ and then using

$$E_{2n}+\binom{2n}{2}E_{2n-2}+\binom{2n}{4}E_{2n-4}+\cdots+E_0=0$$

for $n=1,2,3,\ldots.$

Some of the series shown above suggest the possible closed form $(N/D)\pi^3$ for Apéry's constant, but other series suggest more complex possibilities. For example, $\zeta(1)$ is undefined because the series $\sum_{n=1}^{\infty}1/n^1$ (the harmonic series) diverges. However, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}n^1$ is defined, with sum $\log 2$. Euler suggested that Apéry's constant might involve $\log 2$, and this series might be the motivation for this idea.

A series of the form $\sum_{n=1}^{\infty}a_n/n^z$ is known as a *Dirichlet series* (see references 10 and 11). Examples of other Dirichlet series of interest include those given recently by Balanzario (reference 12). These series are periodic in the sense that the given numerators repeat. Below are three such series:

$$\frac{1}{1^5}+\frac{0}{2^5}+\frac{-1}{3^5}+\frac{0}{4^5}+\cdots=\frac{5\pi^5}{1536}\quad\text{(period 4)},$$

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{-1}{3^3}+\frac{-1}{4^3}+\frac{0}{5^3}+\cdots=\frac{4\pi^3}{625}\sqrt{25+2\sqrt{5}}\quad\text{(period 5)},$$

and

$$\frac{1}{1^3}+\frac{0}{2^3}+\frac{1}{3^3}+\frac{0}{4^3}-\frac{1}{5^3}+\frac{0}{6^3}-\frac{1}{7^3}+\frac{0}{8^3}+\cdots=\frac{3\pi^3}{64\sqrt{2}}\quad\text{(period 8)}.$$

For example, in the first series the sequence of numerators 1, 0, -1, 0 is to be repeated again and again yielding

$$\sum_{n=0}^{\infty} \frac{1}{(4n+1)^5} + \frac{0}{(4n+2)^5} + \frac{-1}{(4n+3)^5} + \frac{0}{(4n+4)^5} = \frac{5\pi^5}{1536}.$$

From the evidence gathered above, we conclude that our computer search can begin profitably by examining the following possible expressions for Apéry's constant:

$$\frac{N}{D}\pi^3, \qquad \frac{N\sqrt{2}}{D}\pi^3, \qquad \frac{N\sqrt{3}}{D}\pi^3,$$
$$\frac{N}{D}\log 2, \qquad \frac{N}{D}\sqrt{3}\log 2, \qquad \frac{N}{D}\pi^3\log 2,$$

where N and D are natural numbers. We also notice that the numbers N and D are likely to be small, say under 500, based on the integers that are shown in specific closed-form expressions above. Many other possible closed forms are of interest, but we will concentrate on the above simple forms in this article.

3. Good rational approximations

Suppose that we are testing the possible closed-form expression $(N/D)\pi^3$ for $\zeta(3)$. Our computer search will then try to identify rational numbers N/D which are very close to $\zeta(3)/\pi^3$. This raises the question of 'what is close'? By taking the denominator D to be large enough, we can get as close as we like. But how close can we get if the denominator is not large? In more precise terms, if a real number x and the denominator D are given, how close can we expect the rational number N/D to be to x? How small can we expect the error $|N/D - x|$ to be? A little thought shows that we can always find N so that $|N/D - x| \le 1/2D$. But can we do better? The theory of continued fractions reveals that, if we let D vary over all the positive integers, then for an irrational number x there are infinitely many values of D such that

$$\left|\frac{N}{D} - x\right| < \frac{1}{2D^2}.$$

We refer readers to reference 13 for details. Other good books on continued fractions are references 14 and 15.

If x is rational, then only a finite number of such fractions exist. We will call rational approximations 'good' if they satisfy this last inequality.

4. A computer search

As in most mathematical problems, in this instance, there are several ways to solve our problem. We will look at the simplest approach first. In our case, we are using a computer program to find a closed-form expression for $\zeta(3)$. The following is pseudo-code for a scan for possible integers for the form $(N/D)\pi^3$:

```
max = 1000              (max is the largest value of N
                          and D that will be scanned)
D = 1                   (starts the denominator at 1)
while (D ≤ max,         (begin scan of denominator)
  N = 1                 (sets the numerator to 1)
  while (N ≤ max,       (begin scan of numerator at
                          given denominator)
    error = 1/2D^2      (a 'good' error)
    if |ζ(3)/π^3 − N/D| ≤ error, output values of N and D
    N = N + 1           (increases N)
  )                     (end of loop for the numerator)
D = D + 1               (increases D)
)                       (end of loop for denominator)
```

This program will scan all combinations of the numerator and denominator up to values of 1000 and outputs the combinations that are within a given margin of error. The program can be adjusted in line 7 to fit any of the possible forms for $\zeta(3)$ that are to be tested.

Table 1. Approximations of $\zeta(3)$.

Form of $\zeta(3)$	Close fraction	Absolute error	Error in terms of $1/2D^2$
$\frac{N}{D}$	$\frac{119}{99}$	3.6×10^{-5}	0.7194
$\frac{N}{D}\pi^3$	$\frac{5}{129}\pi^3$	2.6×10^{-4}	0.2824
$\frac{N}{D}\sqrt{2}\pi^3$	$\frac{2}{73}\sqrt{2}\pi^3$	1.5×10^{-5}	0.1702
$\frac{N}{D}\sqrt{3}\pi^3$	$\frac{3}{134}\sqrt{3}\pi^3$	2.8×10^{-4}	0.1872
$\frac{N}{D}\log 2$	$\frac{137}{79}\log 2$	1.6×10^{-5}	0.3034
$\frac{N}{D}\sqrt{3}\log 2$	$\frac{806}{805}\sqrt{3}\log 2$	6.2×10^{-7}	0.6693
$\frac{N}{D}\pi^3\log 2$	$\frac{8}{143}\pi^3\log 2$	2.8×10^{-4}	0.5478

This is a rather tedious method but the easiest to understand. More advanced algorithms that can be used include continued fractions. Table 1 gives small number fractions that can be used to approximate each possible form of $\zeta(3)$, along with its absolute error and its error in terms of $1/2D^2$. The absolute error is given by $|\zeta(3) - (N/D)\pi^3|$ for the case where we assume that $\zeta(3) = (N/D)\pi^3$. The second form of error gives us an idea of how good the approximation is relative to the size of the denominator and is determined by $|\zeta(3)/\pi^3 - N/D|\, 2D^2$. It should be noted that no exact match was found.

Using QUICK BASIC we were able to determine that no values of the denominator less than 10^{15} will work for any of the above forms. With the help of MATHEMATICA, it appears that no values for the denominator less than 10^{50} will work for any of the above forms and that the denominator must be larger than a googol, 10^{100}, for the simplest form, $(N/D)\pi^3$, to work. In this case, we find that $\zeta(3)$ can be approximated to within an absolute error of 10^{-215} by the following fraction, whose denominator is of the order of 10^{108}:

$$\frac{\begin{matrix}5278556927339203124949764069429306213320750823467183992 6\\ 9972309376166540072594474500689399528764284756502 98\end{matrix}}{\begin{matrix}1361569457582181828763524127902452783117500534325528461 02\\ 7612903759448566621244154757958584796796472071383357\end{matrix}}\,\pi^3.$$

5. Final remarks

It is interesting to see that our computer search for a closed form for Apéry's constant is closely related to the subject of simple continued fractions. The program of section 4, when searching for rational approximations to the number $\zeta(3)/\pi^3 = 0.0387681796029168\ldots$, found the following sequence of numbers:

$$\frac{1}{26},\ \frac{5}{129},\ \frac{34}{877},\ \frac{107}{2760},\ \frac{141}{3637},\ \frac{248}{6397},\ \frac{885}{22828},\ \frac{9487}{244711},\ \ldots.$$

When the same number is expanded in a continued fraction we get

$$\frac{\zeta(3)}{\pi^3} = 0 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{1 + \cdots}}}}}}}.$$

This continued fraction generates the sequence of convergents:

$$\frac{1}{25},\ \frac{1}{26},\ \frac{4}{103},\ \frac{5}{129},\ \frac{34}{877},\ \frac{107}{2760},\ \frac{141}{3637},\ \frac{248}{6397},\ \frac{637}{16431},\ \frac{885}{22828},\ \frac{9487}{244711},\ \ldots.$$

Notice that the previous sequence is a subsequence of the above.

We have only begun to explore possible closed forms for Apéry's constant. The reader might want to continue the search.

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Drawing Network Graphs with DERIVE™ 5

P. SCHOFIELD

A system for specifying and drawing network graphs in a users' file of DERIVE™ 5.

These days, most A-level syllabuses contain a module (or modules) which includes activities on network graphs (for example, AQA modules D1 and D2 and EDEXCEL module D1 in discrete mathematics). In my previous article (reference 1) I mentioned that, in DERIVE™ 5, there is now a users' file for specifying and drawing a wide selection of network graphs. Here, I will explain how this file can be used in combination with DERIVE's algebraic and two-dimensional plotting facilities to generate some interesting graph drawings (you can download a 30-day free trial version of DERIVE™ 5 from the DERIVE website (reference 2)). To work through the following examples you will need to run the DERIVE users' file ngraphs5.dfw. This file contains a list of utilities (and demonstrations) for drawing network graphs. Although basically a 'fun file', it nevertheless develops an understanding of many of the mathematical features of network graphs.

1. Drawing simple graphs

The specification for drawing network graphs with ngraphs5 arises from the following definition of a network graph or digraph.

Definition 1. A *network (di)graph* is a pair (V, E) where V is a non-empty set of *vertices* and E is a set of *edges*. Each edge $e \in E$ joins a pair of vertices, which can be directed (ordered) or undirected. Vertices joined by an edge are called *adjacent*.

Example 1. (*Drawing a graph of a pyramid.*) Load the file ngraphs5.dfw and select the following DERIVE two-dimensional plot window settings: Options>Simplify Before Plotting (On); Options>Change Plot Colors (Off); Options>Display>Points>Connect (Yes), Large; Options>Display>Grids (Off); Options>Display>Cross (Off); Window>Tile Vertically. To draw the pyramid graph of figure 2, first work out some vertex positions (figure 1). Note that there is a ring (of radius 3) of four vertices (with the first vertex offset from the direction of the x-axis by $\frac{1}{4}\pi$) surrounding a single vertex at the origin. In ngraphs5.dfw this information can be assigned to a variable **pv** as follows:

pv:=append([[0,0]],3ring(4,π/4))

The next stage is to work out the edges. These could be specified by the vector of pairs of positive integers $[1, 2; 1, 3; 1, 4; 1, 5; 2, 3; 3, 4; 4, 5; 5, 1]$. However, they can also be constructed using two trails (starting and finishing at odd vertex corners). In ngraphs5.dfw we can assign:

pe:=trails[2,1,3;4,1,5,2,3,4,5]

We now have specifications for both the vertex and edge properties of the graph, and so to draw the graph plot **draw[pv,pe]**.

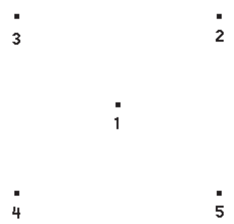


Figure 1.

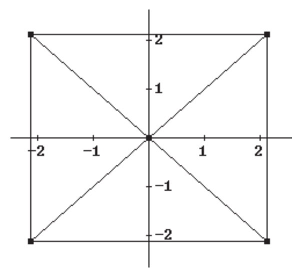


Figure 2.

The axes have been left on in figure 2 to help with positioning the vertices. These can be turned off in the plot window using Options>Axes, etc. That's about all there is to it, except that in ngraphs5 there are instructions for specifying evenly spaced rows of vertices between two points, parallelogram grids of vertices, and vertices evenly spaced along any given section of a two-dimensional parametric curve. Also, ngraphs5 has many ways of specifying edges including: walks, trials, tours, paths, cycles, complete (sub)graphs, edges joining all pairs of vertices from two sets of vertices.

$$\left[\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}, [[2, 3, 4, 5], [1, 3, 5], [1, 2, 4], [1, 3, 5], [1, 2, 4]]\right]$$

Figure 3. An adjacency matrix and list.

As well as the edge set, ngraphs5 uses two other methods of specifying the vertex/edge relationship of a graph or digraph. These are the adjacency matrix and the adjacency list. There are also instructions for converting between these. For example, using the edge set **pe** of example 1, simplify

[amatpe, alistmatamatpe]

for figure 3.

Example 2. (*Graphs with multiple edges and loops.*) An adjacency matrix can be used to draw a graph or digraph with multiple edges and loops. For example, in ngraphs5, assign:

mv:=2.5ring(3, π /2)
ma:= [2, 1, 3; 1, 2, 2; 3, 2, 2]

Then plotting **draw[mv,ma]** will draw the graph in figure 4, and plotting **drawdi[mv,ma]** will draw the digraph in figure 4 (in the plot parameter window, select ‘Apply parameters to rest of plot list (ON)’).

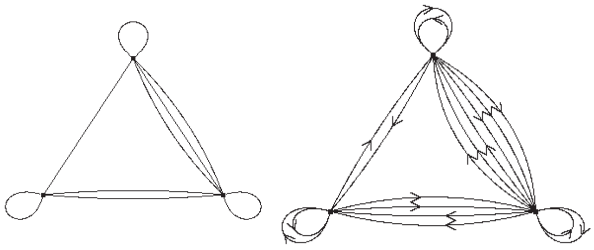


Figure 4. Graph and digraph with multiple edges.

Note that any square matrix can represent a digraph, but to represent a graph, the matrix must be symmetric with even entries on its main diagonal.

2. Manipulating graphs

A selection of ‘standard graphs’ has been defined in ngraphs5, including: null, circuit and complete graphs; star and wheel graphs; complete bipartite graphs; platonic graphs; the Petersen graph. Graph drawings can be enlarged or contracted by multiplying by a scale factor. They can also be transformed using a 2 × 2 matrix, or displaced using a coordinate vector.

Example 3. (*Transforming a dodecahedron graph.*) First assign a dodecahedron graph to a variable by:

d:=drawdodecahedron

Plotting this assignment will draw the basic dodecahedron in figure 5. Next, define a positive rotation about the origin through an angle α by:

rotate(g,α):=tran_g([cosα,–sinα;sinα,cosα],g)

Now, plotting

dd:=vector(rotate(d,α),α,0,2π/5,π/10)

will rotate and copy the basic dodecahedron through step angles of $\frac{1}{10}\pi$, starting at 0 and finishing at $\frac{2}{5}\pi$. See figure 5.

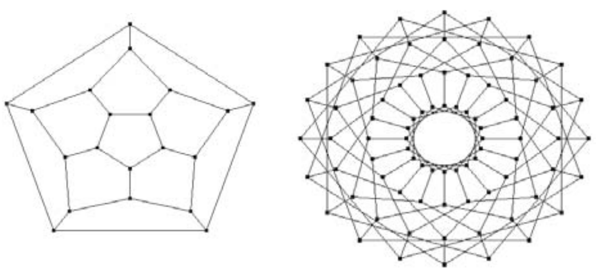


Figure 5. Rotating a dodecahedron.

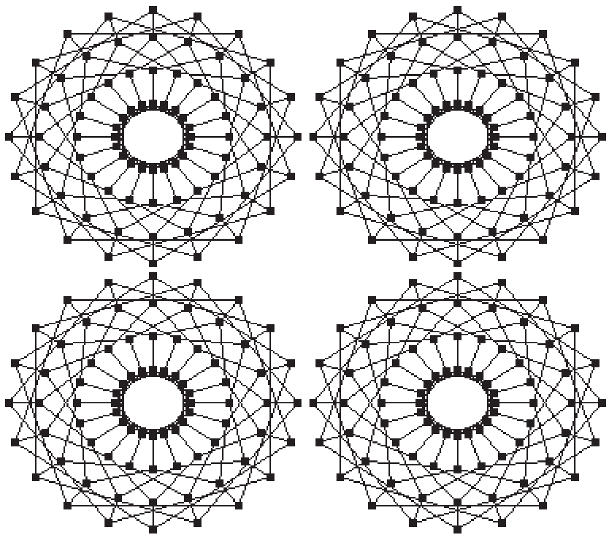


Figure 6. Four displaced copies of figure 5.

Finally, four copies of the right-hand side of figure 5 can be generated, displaced and reduced by plotting

[dis_gs(dd, [–4, –4]), dis_gs(dd, [–4, 4]), dis_gs (dd, [4,–4]), dis_gs(dd, [4, 4])]/2

See figure 6.

The possibilities for making attractive patterns and designs by transforming basic network graphs are endless. Try repeating example 3 with **d:=drawpetersen**.

3. Using DERIVE 5’s other facilities

The powerful algebraic manipulation facilities of DERIVE 5 are also available for use in constructing graphs. Using these, it is possible to start from simple graph drawings and build up to complex graph structures.

Example 4. (*Graph of a dream catcher.*) First construct the graph drawing of a single spiral arm using:

rr:=vector(r[cos(rπ/6),sin(rπ/6)],r,0,3.5,.5)
arm:=draw[rr,path[1,2,3,4,5,6,7,8]]

Graph drawings **g** and **h** can be combined by appending vertices and edges as follows:

combine(g,h):=[append(g↓1,h↓1),append(g↓2,h↓2)]

It is now possible to combine **arm** and its reflection in the y-axis using the assignment:

arms:=combine(arm,tran_g([–1,0;0,1],arm))

See figure 7.

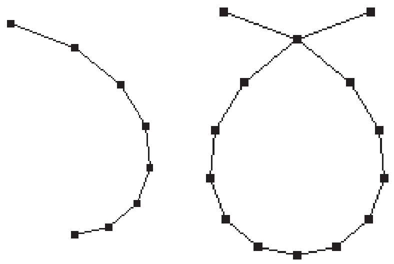


Figure 7. arms for a dream catcher design.

The final stage is to rotate and copy these spiral arms through steps of a full turn, and then add an outer circle to complete a dream catcher. This can be accomplished by, for example, plotting:

[vector(rotate(arms, nπ/6),n,0,11),x^2+y^2=49/4]

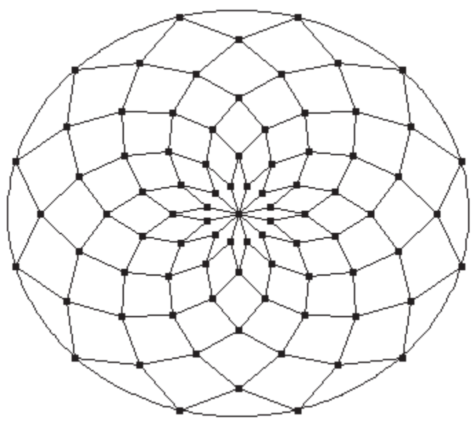


Figure 8. Graph of a dream catcher.

Once you have constructed the dream catcher of figure 8, you might also think about drawing variations. See figure 9 for some examples.

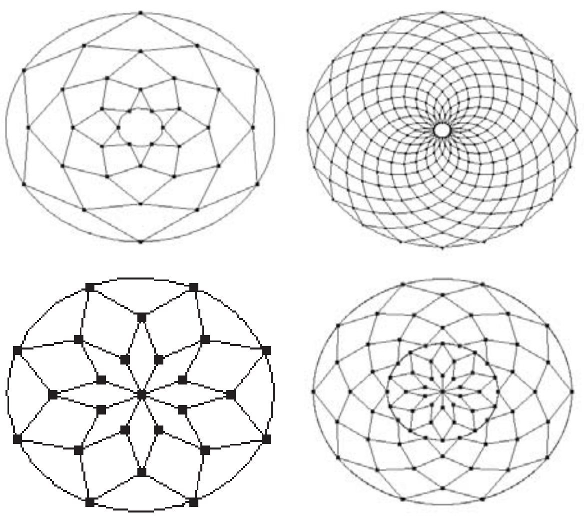


Figure 9. Examples of various dream catchers.

Example 5. (*Graphs of hypercubes.*) Hypercubes are fundamental objects in n-dimensional Euclidean spaces. Although we cannot see these objects 'in the round' beyond three dimensions, it is still possible to construct their two-dimensional networks (up to a point). Suppose we denote an n-dimensional cube by **qn**, then, starting from **q2** (a square), this example describes a method for using ngraphs5 and DERIVE 5 to construct graphs of **q3**, **q4**, **q5** and **q6**.

Start by specifying the graph drawing of a square and assigning this to **q2**:

q2:=draw[[2,1;2,5;–2,5; –2,1],cycle[1,2,3,4]]

(plot **q2** and zoom out once to see the full square.)

To construct the graph of **q3** (a cube) we require two distinct copies of **q2**, with corresponding vertices joined by edges. To combine a graph **g** with a copy of itself displaced by a vector **p** define:

co(g,p):=combine(g,dis_g(g,p))

To join the corresponding vertices of graphs **g** and **dis_g(g,p)** define:

pairs(g):= vector([i,i+dimg↓1],i,1,dimg↓1)
li(g,p):=draw[append(co(g,p)↓1),pairsg]

It is now possible to construct a single instruction to construct successive cube graphs as follows:

link(g,p):=[co(g,p)↓1,append(co(g,p)↓2,li(g,p)↓2)]

The remaining challenge is to select the displacement vectors to draw each hypercube to its full effect. For example, to draw **q3** assign and plot:

q3:=link(q2,[3,–3])

Figure 10 shows the plots of **q2** and **q3**.

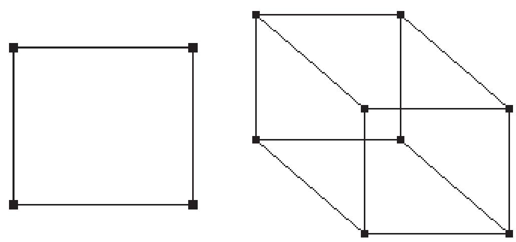


Figure 10. Square and cube graphs.

Once you have constructed **q3**, assign and plot:

q4:=link(q3,[–3,–3])

to obtain the (well-known) beautifully symmetric graph of **q4** in figure 11.

The graphs **q5** and **q6** can now be built successively by, for example,

q5:=link(q4,[3.5,–2.5])
q6:=link(q5,[–3.5,–2.5])

When you first plot these graphs, they will extend beyond the plot window. This can easily be rectified in DERIVE by zooming out (until you see the whole graph) and then framing

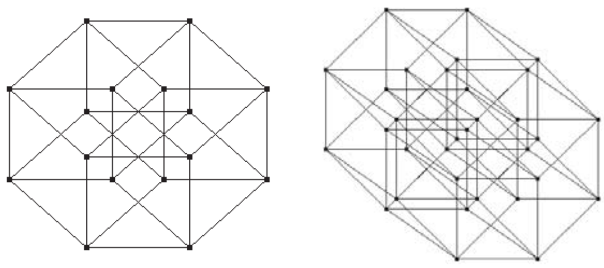


Figure 11. Graphs of **q4** and **q5**.

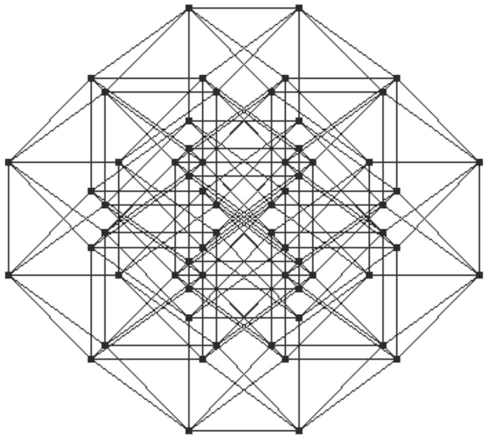


Figure 12. Graph of **q6**.

so that the whole graph fits into the window (see figures 11 and 12).

Because of the 'symmetrical view' of **q6** some of the edges overlap. I would be interested if anyone has a better perspective of this remarkable object.

4. Conclusion

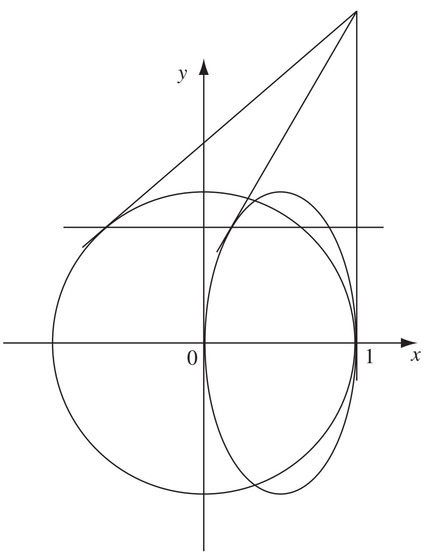
The basic methods of specifying and drawing graphs and digraphs are simple, yet combined with the powerful algebraic and plotting facilities of DERIVE 5 the outcomes are very productive. The potential of these techniques appears to be unlimited.

For more examples of using ngraphs5 to draw and manipulate graphs, consult reference 3. DERIVE 5 is also becoming a useful tool for other investigations into network graphs. For example, for using DERIVE to model an algorithmic method for finding Hamiltonian cycles of network graphs see reference 4.

References

1. P. Schofield, What's new in DERIVE™ 5, *Math. Spectrum* (2000/2001), **33,** pp. 59–62.
2. DERIVE website, http://www.derive.com.
3. P. Schofield, Drawing network graphs with DERIVE 5, in *Proc. 4th Internat. DERIVE & TI-89/92 Conf.,* Liverpool, 2000 (bk teachware, Hagenberg, 2001 (CD-ROM)).
4. P. Schofield, Using DERIVE to interpret an algorithmic method for finding Hamiltonian circuits (and rooted paths) in network graphs, in *Proc. 4th Internat. DERIVE & TI-89/92 Conf.,* Liverpool, 2000 (bk teachware, Hagenberg, 2001 (CD-ROM)).

The author has recently retired from the post of Head of Mathematics at Trinity and All Saints College, University of Leeds, where he taught mathematics for over 30 years. Dr Schofield made extensive use of DERIVE while working with college students.



Curves in perspective

The circle with equation $x^2 + y^2 = 1$ and the ellipse with equation $4x^2 - 4x + y^2 = 0$ have the property that tangents to each curve at points with the same y-coordinate meet on the line $x = 1$. This is true more generally of curves with equations $y = f(x)$ and $y = f(2x - 1)$, where f is any differentiable function.

GUIDO LASTERS
Tienen, Belgium

Lecturer Makes History in Maths

KAREN GOLD

Susan Howson.

A woman has won the UK's most prestigious award for a young mathematician, the Adams Prize, for the first time in its 120-year history.

Dr Susan Howson, 29, a Royal Society fellow and lecturer at Nottingham University, was lauded by the judges — an international array of maths professors — for her research on number theory and elliptic curves.

Previous winners of the £12 000 prize, awarded by Cambridge University, include the physicist James Clerk Maxwell and geometrician Sir William Hodge.

Although cryptographers will use Dr Howson's work, she is a pure mathematician, choosing her subject 'because of the beauty of the theorems'.

This sets her apart from most of the other top women mathematicians in the UK, who are almost entirely concentrated in applied maths or statistics.

Dr Howson believes women are deterred from working in pure maths not only by the lack of role models — she had only male teachers as a Cambridge undergraduate — but also by its highly competitive nature.

'I think some girls are put off by that, and female undergraduates seem to drop out because they are not as confident as the men. Competition doesn't usually bother me, but I worked at MIT [Massachusetts Institute of Technology] for a year and I was uncomfortable with the very macho atmosphere: who was working the hardest, staying the latest.

'I think women may be a bit less obsessional and single-minded than men on average, and since those characteristics can help with maths that may make a difference.'

Statistics collected by the London Mathematical Society bear out the story that at every stage beyond school, increasing numbers of women drop out of maths.

In 1998 (the latest available figures) women comprised 38% of maths undergraduates. At postgraduate level that proportion had fallen to 29%, while just 18% of university lectureships were held by women.

Women made up 7% of senior lecturers and 2% — nine individuals — of university professors.

The biggest improvements in almost 10 years have been at the lower levels. In 1990 only 33% of maths undergraduates, and 21% of postgraduates were women.

At the top, the change is fractional: women have gone from four to 7% of senior lecturers, and from one to 2% of professors.

Dr Helen Robinson, senior lecturer in maths at Coventry University, also believes the working atmosphere in pure maths is a major deterrent to some women.

'Women are inclined to get discouraged, and people do discourage them. You still get people, though not as many as you once did, who say that women aren't any good at pure maths,' she said.

'It's not necessarily intentional. Young male research students can seem extremely sure of themselves and women are put off by this. It's partly to do with the style of pure maths: you say something and someone comes back at you and says "you're talking rubbish". It's a very argumentative style in which people put forward opinions without thinking about them a great deal, expecting to see them knocked down. Women are less happy about doing that.'

Dr Howson attributes her success to encouragement at Burley-in-Wharfedale middle school in West Yorkshire and her teacher David Womersley.

'Most of the time at that age we were supposed to be doing long division and fractions, but he took time off from that every week to do investigations and teach us how to really think,' she said.

She has no mathematical background; her father is an electrician, her mother a secretary. 'I was just encouraged at school all the way through to university. When I first arrived at Cambridge I found it quite overwhelming, but Corpus Christi was very encouraging too. I've never really encountered any off-putting attitudes or discrimination.'

Dr Robinson, recalling her experience as the only female in a 45-strong postgraduate cohort at Warwick University 25 years ago, could not say the same. 'I do think attitudes have changed now, but it was very, very lonely.

'There are more women staying on to do research, and gradually they are moving up the ladder. The British Women in Mathematics group and the LMS also organise meetings, aimed at encouraging research students. But when I was doing my MSc, if I gave the right answer to a question in a lecture and some male students had got it wrong before me, the lecturer would say: "Look she can do it, and she's only a girl".'

A Note on Roman Arithmetic

JOE GANI

How to make multiplication difficult.

Introduction

A recent review (reference 1) of Dirk Struik's classical book *A Concise History of Mathematics* encouraged me to re-read the fourth revised edition of this work (reference 2).

In his Chapter II, 'The Ancient Orient', Struik discusses the number systems used by the Mesopotamians, Egyptians and Chinese. The complexity of simple arithmetic operations can hardly be imagined by us today, when the Hindu decimal position system and the use of Arabic numerals are prevalent. The Roman system followed the Egyptian; let us, for example, imagine a Roman child doing the following sum: multiply XIV by IX. To simplify this and avoid the negative values of I implied in IV and IX, we may rewrite this as XIIII by VIIII. Let us assume that Roman children learned their multiplication tables much as we do today, and proceed as follows:

$$\begin{array}{r} \mathrm{XIIII} \\ \underline{\times\ \mathrm{VIIII}} \\ \mathrm{I} \times \mathrm{XIIII} = \mathrm{XIIII} \\ \mathrm{XIIII} \\ \mathrm{XIIII} \\ \mathrm{XIIII} \\ \underline{\mathrm{V} \times \mathrm{XIIII} = \mathrm{LVVVV}} \\ \mathrm{XCXXXVI} = \mathrm{CXXVI} \end{array}$$

where $\mathrm{X} + \mathrm{X} + \mathrm{X} + \mathrm{X} + \mathrm{L} = \mathrm{XC}$, $\mathrm{IIII} + \mathrm{IIII} + \mathrm{IIII} + \mathrm{IIII} + \mathrm{VVVV} = \mathrm{XXXVI}$, and $\mathrm{XCXXX} = \mathrm{CXX}$, since the X before the C cancels the first X after the C. If we compare this with the current calculation, which we set out similarly,

$$\begin{array}{r} 14 \\ \underline{\times\ 9} \\ 36 \\ \underline{90} \\ 126 \end{array}$$

we see how greatly the Hindu–Arabic arithmetic system has simplified our lives.

I cannot imagine how complicated the rules for division must have been, even though many calculations would have been carried out on the abacus. Remarkably, the Hindu–Arabic system was resisted, and Struik writes: 'In the statutes of the "Arte del Cambio" of 1299 and even later the money changers of Florence were forbidden to use Arabic numerals and were obliged to use Roman ones.' It was only in the fourteenth century that Italian merchants began to use the Hindu–Arabic numerals. And just as well!

References

1. D. E. Rowe, Looking back on a bestseller: Dirk Struik's *A Concise History of Mathematics, Notices Amer. Math. Soc.* **48** (2001), pp. 590–592.
2. D. J. Struik, *A Concise History of Mathematics*, 4th edn (Dover, New York, 1987).

Joe Gani *is a retired mathematician who lives in Canberra.*

Changing base

In base 10,

$$\tfrac{5}{7} = 0.\overline{714285}\,.$$

Using other bases,

$$\left(\tfrac{2}{3}\right)_4 = (0.\overline{2})_4\,, \quad \left(\tfrac{2}{3}\right)_5 = (0.\overline{31})_5\,, \quad \left(\tfrac{5}{7}\right)_8 = (0.\overline{5})_8\,,$$

$$\left(\tfrac{7}{8}\right)_9 = (0.\overline{7})_9\,, \quad \left(\tfrac{9}{10}\right)_{11} = (0.\overline{9})_{11}\,, \quad \left(\tfrac{8}{10}\right)_{12} = (0.\overline{9724})_{12}\,,$$

$$\left(\tfrac{2}{3}\right)_{5.5} = (0.\overline{3})_{5.5}\,, \quad \left(\tfrac{1}{3}\right)_2 = (0.\overline{01})_2\,, \quad \left(\tfrac{1}{3}\right)_{\sqrt{2}} = (0.\overline{0001})_{\sqrt{2}}\,.$$

GUIDO LASTERS
Tienen, Belgium

Mathematics in the Classroom

Old favourites and new ideas: inspiring your post-16 students

I have just returned from a conference with the above title run by Villiers Park Educational Trust and hosted by the University of Birmingham. This was a very stimulating event and full of inspiration that I now need to cascade to my students, but they have all finished for the year, so I shall try to impart here some of the ideas that emerged during these two fascinating days.

Origami (courtesy of Tony Gardiner)

Take a sheet of A4 paper (210 mm $\times$ 297 mm; see figure 1(a)) and fold it so that two opposite corners touch; the resulting shape is a pentagon with an axis of symmetry along the diagonal that joined the two vertices that are now touching (figure 1(b)). Fold along this axis of symmetry to produce a crease and then open up again to return to the pentagon. Now take each of the two shortest sides and fold them so that these shortest sides meet along the diagonal crease (figure 1(c)).

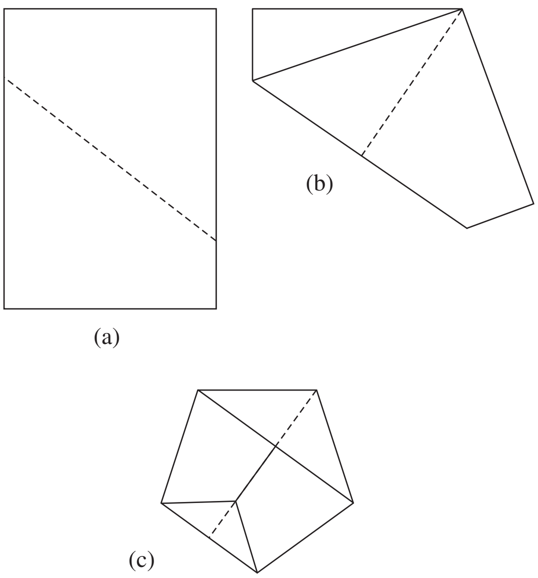


Figure 1.

Is the resulting pentagon regular? Can you provide evidence for your answer? What happens if you start with a square?

Graph sketching (courtesy of Richard Kaye)

Consider the curve in figure 2 (ignore the other lines for now). It clearly has a root at $x = 2$. In order to find the other two roots, draw a tangent to the curve which passes through the root. At the touching point, drop a perpendicular to the x-axis. In general, this meets the x-axis at $x = p$. If the gradient of the tangent is Q and $q = +\sqrt{Q}$, then the other two roots are $p \pm q\mathrm{i}$. Can you say why this is so?

Use this to find the other two roots of the function shown. So what is the equation of the curve?

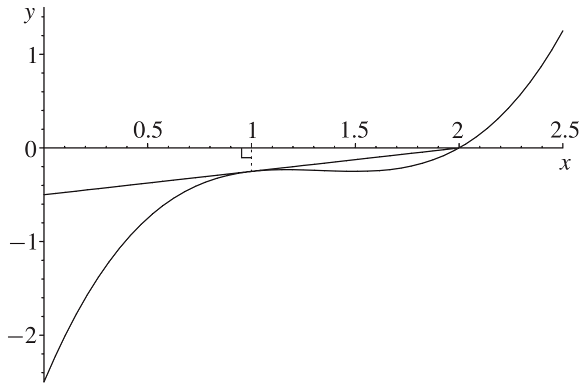


Figure 2.

Proof (courtesy of Joe Kyle)

Proof is back in the A-level mathematics syllabus in a big way. Are the theorem and proof that follow valid?

Theorem.

$$0.\dot{9} = 1.$$

Proof.

$$0.\dot{3} = 0.333\ldots = \tfrac{1}{3},$$

$$\therefore\ 3 \times 0.\dot{3} = 0.999\ldots = 3 \times \tfrac{1}{3} = 1.$$

Do you believe it? Or is there a flaw in the argument?

Now consider the following situation: let x and y be any two integers. Suppose that $x + y = 0$. Then either

(a) $x = -y$ or
(b) $y = -x$.

If (a) holds, then $x + y = (-y) + y = 0$; if (b) holds, then $x + y = x + (-x) = 0$. In either case, $x + y = 0$.

These lines prove that

(A) for all integers x and all integers y, $x + y = 0$,
(B) for each integer x there exists an integer y such that $x + y = 0$,
(C) there exist integers x and y such that $x + y = 0$,
(D) there exists an integer x such that, for all integers y, $x + y = 0$,
(E) something else.

Which of these is valid?

Sequences and series (courtesy of Chris Good)

Clearly

$$\frac{1}{2^n} < \frac{1}{n^2} < \frac{1}{n},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n}.$$

How do we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} = \frac{1}{16} + \cdots$$

converges? Call this sum P. Work out

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

in terms of P. Can you show that P is $\pi^2/6$? (See also the article by Osler and Seaman earlier in this issue.)

Finally

These are just a few of the ideas that challenged us. We also encountered some new websites. Mathletics (named for the similarities between maths and athletics — you don't acquire skills in either by just watching from the sidelines, you need to continually practise!) produced by Brunel University (http://www.brunel.ac.uk/∼mastmmg/) is well worth a look. Similarly, Birmingham University's own website (http://www.mat.bham.ac.uk/) is also a site of special interest.

Carol Nixon

Computer Column

Turing machines

These days, computers are everywhere, growing faster and more powerful all the time. From providing the special effects for the latest blockbuster to making sure that shops keep their stocks up to date, we can't avoid them. We already have the internet-enabled fridge, and the talking toaster cannot be very far away. Have you ever stopped, though, to think about where it all began?

Many would argue that the father of the modern computer was Charles Babbage, who conceived the Analytical Engine in 1833. Although it was never built (it would have been a monumental task), it would have been capable of following simple programs, fed in as a stack of punched cards. Great though this achievement was, however, we will focus instead on Alan Turing, who introduced the theoretical foundations of what is now computer science.

Alan Turing was born in 1912, and was, from the first, the purest of pure mathematicians. He was particularly interested in logic, and became fascinated by the question of whether there could exist, in principle, a method or process by which it could be decided whether any given mathematical conjecture was provable. To answer this question, he devised what has now become known as a *Turing machine*. His idea was that proofs had to proceed by a series of 'mechanical' steps, each of which was known to be logically sound. By breaking these steps down into their most elementary components, he realised that it was possible to conceive of a machine which could be programmed to perform any of these elementary steps, and which could therefore be used, with sufficient ingenuity, to perform any possible proof. A statement could be said to be provable if and only if it could be proved using a Turing machine.

The machine he proposed was surprisingly simple. It consisted of a tape-reader and an infinitely long tape, divided into a series of cells (or squares). Each square could contain one of a finite set of symbols, or be blank. The reader had a finite number of possible states that it could be in (and was always in one or other of them). The only operations it could perform were to read a symbol from the current position on the tape, write one onto the tape, or move one square to the left or right along the tape. It operated according to a *transition function* which stated, for every possible combination of current state and most-recently-read symbol, what the machine should do next. The transition function mapped the machine's current state and symbol to a new state, a new symbol and a movement instruction (left, right or none).

To use the machine, you had to prepare the tape and decide on a transition function. You then moved the reader to the right point on the tape and put the machine into its 'start' state. It then worked its way along the tape as follows.

1. Read the symbol in the current square and apply the transition function.
2. Change to the new state, and write the new symbol onto the tape at the current position.
3. If the new state is the 'halt' state, stop. Otherwise, follow the movement instruction and repeat from 1.

The simplicity of Turing's machine made it ideal for investigating the question of computability, but also made it impractical for use as a real computer. To demonstrate this, a `BASIC` Turing machine simulator is given in appendix A. By changing the `DATA` statements at the end, this can be made to simulate any possible Turing machine. As listed, it implements a machine which simply starts at zero and counts upwards, transforming the tape into a binary representation of each number in turn. Just to achieve this, however, we need three states, four symbols and seven transition matrix elements; Windows® Turing Machine Edition looks unlikely!

However, a Turing machine has much in common with modern computers: the tape head and transition matrix represent a computer's main processor, while the tape represents its memory. In fact, it took just one more step to create the basis for all modern computers: the idea of a *register*, which was also pioneered by Turing, and later developed by John von Neumann. A register is very like a variable in a computer program, in that it is a number whose value can be changed or checked by program instructions. In Turing machine terms, a register could be implemented as a series of states (each representing one value of the register), together with a suitable transition function.

A *register machine* had a set of registers, together with a more intuitive set of possible operations, such as:

```
If X>0 write 5 to Y and jump
               to the next instruction
```

This is exactly what a computer chip is: a hard-wired set of registers and procedures for operating on them. It also has the ability to exchange signals with other devices (such as a monitor or keyboard), but at heart it's just an advanced Turing machine.

Turing, of course, went on to further fame as the man who broke the German Enigma code during the Second World War, using one of the world's first computers to do it. Remember that the next time anyone questions the value of pure mathematics!

Appendix A. Turing machine simulator

```
REM Turing machine simulator
REM First, define variables and read in data
DIM symbols, states, transitions AS INTEGER
DIM state, symbol, cell, n, m, p AS INTEGER
DIM tapelength, haltstate AS INTEGER
READ states, symbols
DIM nextsymbol(states, symbols) AS INTEGER
DIM nextstate(states, symbols) AS INTEGER
DIM move(states, symbols) AS INTEGER
READ transitions
FOR n = 1 TO transitions
   READ m, p
   READ nextsymbol(m, p), nextstate(m, p)
   READ move(m, p)
NEXT n
READ tapelength
DIM tape(tapelength) AS INTEGER
FOR n = 0 TO tapelength - 1
   READ tape(n)
NEXT n
REM Initialise machine
cell = 0 : state = 0
haltstate = states - 1
CLS : PRINT "Turing machine started"
REM Loop until halt state is reached
DO UNTIL state = haltstate
REM Print current tape and machine state,
REM putting bars round the current position
   FOR n = 0 TO tapelength - 1
      IF n = cell THEN
         PRINT USING "|#|"; tape(n);
      ELSE
         PRINT USING " # "; tape(n);
      END IF
   NEXT n
   PRINT " state ="; state
REM Wait ('X' quits, any key to proceed)
   testinput$ = ""
   DO WHILE testinput$ = ""
      testinput$ = UCASE$(INKEY$)
      IF testinput$ = "X" THEN
         PRINT "Turing machine interrupted"
         END
      END IF
   LOOP
REM Update machine
   symbol = tape(cell)
   tape(cell) = nextsymbol(state, symbol)
   cell = cell + move(state, symbol)
   state = nextstate(state, symbol)
LOOP
REM Print final state of machine and exit
FOR n = 0 TO tapelength - 1
   PRINT USING " # "; tape(n);
NEXT n
PRINT "state="; currentstate
PRINT "Turing machine halted"
END
REM Turing machine definition
REM No. of states, symbols and transitions
DATA 3,4,7
REM Transitions: state, symbol, next state,
REM next symbol, tape movement
DATA 0,2,2,1, 1
DATA 0,0,0,0,-1
DATA 0,1,1,0,-1
DATA 1,1,0,1, 1
DATA 1,0,1,0,-1
DATA 0,3,3,2,-1
DATA 1,3,3,2,-1
REM Length of the tape
DATA 10
REM Initial symbols on tape
DATA 2,0,0,0,0,0,0,0,0,3
```

Websites

The Babbage Pages, http://www.ex.ac.uk/BABBAGE/
The Alan Turing Home Page, http://www.turing.org.uk/turing/
Introduction to Turing machines,
http://www.unidex.com/turing/tm_intro.htm

Peter Mattsson

What is the probability that the six numbers in the UK National Lottery for a given draw have a common factor greater than 1? (The winning numbers are an unordered random choice of six distinct numbers from 1 to 49.) For example,

$$2,\quad 6,\quad 10,\quad 14,\quad 18,\quad 44$$

have common factor 2.

Letters to the Editor

Dear Editor,

Triangles whose Euler line is parallel to a side

The circumcentre, centroid and orthocentre of a triangle lie on a straight line; this line is called the *Euler line* ℓ of the triangle. Of course, in an equilateral triangle these three points coincide. The Euler line of an isosceles triangle is the perpendicular bisector of its base, and the Euler line of a right-angled triangle is the median through the hypotenuse. The Euler line of an obtuse-angled triangle intersects its longest side, because its centroid and circumcentre are on opposite sides. Therefore, if we are looking for triangles whose Euler line is parallel to one of its sides, we need a scalene (i.e. unequal sides), acute-angled triangle.

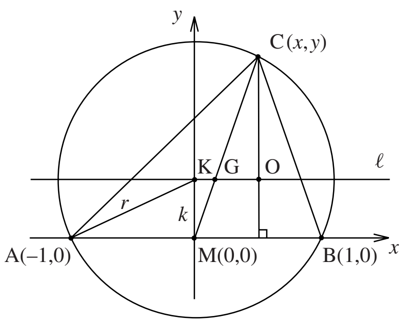


Figure 1.

Let ABC be such a triangle and denote its circumcentre by K, its centroid by G and its orthocentre by O. We assume that the side AB is fixed, and find the locus of C such that the Euler line ℓ is parallel to AB. Choose coordinate axes with the midpoint M of AB as the origin. Let $\mathrm{A} = (-1, 0)$ and $\mathrm{B} = (1, 0)$, so that the length of AB is 2; see figure 1. The coordinates of the circumcentre K will be $(0, k)$ for some k. Let C have coordinates (x, y). The centroid G trisects CM, so G will have coordinates $(\frac{1}{3}x, \frac{1}{3}y)$. Thus, for ℓ to be parallel to AB, we require $k = \frac{1}{3}y$. The circumcircle has radius r, where $r^2 = 1 + k^2$, so its equation is

$$x^2 + (y-k)^2 = 1 + k^2 .$$

Thus, when ℓ is parallel to AB, we have

$$x^2 + \tfrac{1}{3}y^2 = 1 . \qquad (1)$$

Therefore, (1) is the locus of the points C which make the Euler line ℓ parallel to AB. This forms an ellipse whose axes are the coordinate axes, with semi-axis lengths of 1 and $\sqrt{3}$. Of course, C should not be on the y-axis, because then ℓ disappears, and C should not lie on the x-axis, for then the triangle disappears!

We deduce that, when ℓ is parallel to AB, $\angle A < 60^\circ$, $\angle B < 60^\circ$, $\angle C < 90^\circ$, the distance $|k|$ of ℓ from AB is less than $\frac{1}{3}\sqrt{3}$, and $1 < r < \frac{2}{3}\sqrt{3}$. Furthermore, with C having coordinates (x, y), the orthocentre O will have coordinates $(x, \frac{1}{3}y)$, so its locus is the ellipse $x^2 + 3y^2 = 1$. Also, the centroid has coordinates $(\frac{1}{3}x, \frac{1}{3}y)$, so its locus is the ellipse $9x^2 + 3y^2 = 1$; see figure 2.

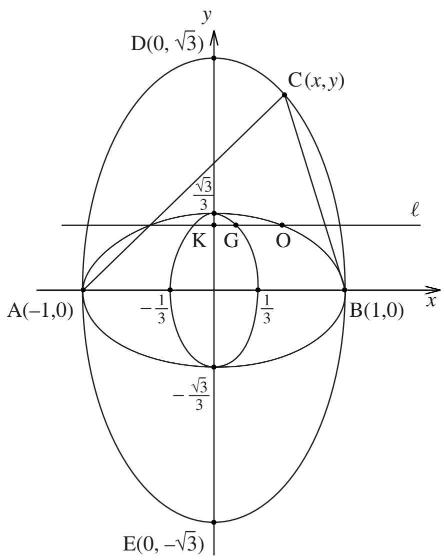


Figure 2.

Reference

1. D. Diemente, Can Euler's line be parallel to a side of a triangle? *Math. Teacher* **93** (2000), 428–431.

Yours sincerely,

Gyula Darvasi
(College of Nyíregyháza,
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H-4400 Nyíregyháza,
Hungary.)

Dear Editor,

The extended crossed ladders theorem

Recently there has been considerable interest shown in *Mathematical Spectrum* in the crossed ladders theorem, see Problem 33.2 in Volume 33, No. 1, 'Mathematics in the Classroom' in Volume 33, No. 2 and K. R. S. Sastry's letter in Volume 34, No. 2. Here is a new property of triangles named the *extended crossed ladders theorem*.

First the crossed ladders theorem and its proof:

Crossed ladders theorem. *If* $\overline{\mathrm{AB}} \| \overline{\mathrm{CD}} \| \overline{\mathrm{FE}}$*, then*

$$\frac{1}{\mathrm{AB}} + \frac{1}{\mathrm{CD}} = \frac{1}{\mathrm{EF}}$$

(see figure 1).

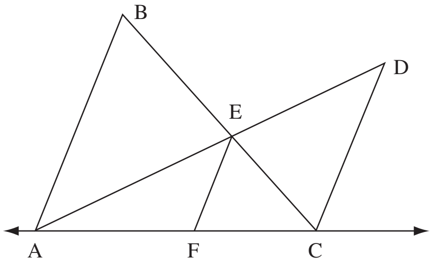

Figure 1.

Proof. Since $\overline{\mathrm{FE}}\|\overline{\mathrm{CD}}$, $\mathrm{AF}/\mathrm{AC} = \mathrm{FE}/\mathrm{CD}$, and since $\overline{\mathrm{FE}}\|\overline{\mathrm{AB}}$, $\mathrm{FC}/\mathrm{AC} = \mathrm{FE}/\mathrm{AB}$. Hence

$$\frac{\mathrm{FE}}{\mathrm{CD}} + \frac{\mathrm{FE}}{\mathrm{AB}} = \frac{\mathrm{AF}}{\mathrm{AC}} + \frac{\mathrm{FC}}{\mathrm{AC}} = \frac{\mathrm{AF}+\mathrm{FC}}{\mathrm{AC}} = 1\,.$$

Hence

$$\frac{1}{\mathrm{CD}} + \frac{1}{\mathrm{AB}} = \frac{1}{\mathrm{FE}}\,.$$

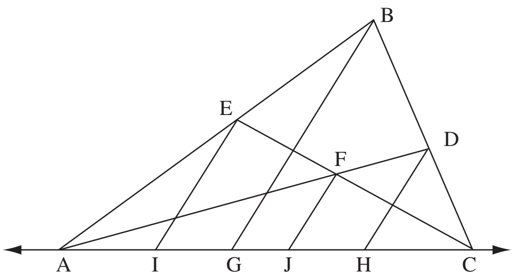

Figure 2.

Extended crossed ladders theorem. *Given a triangle* ABC*, lines* $\overline{\mathrm{AD}}$ *and* $\overline{\mathrm{CE}}$ *intersecting at point* F*, points* G*,* H*,* I*,* J *on the line* $\overline{\mathrm{AC}}$*, as in figure 2, such that* $\overline{\mathrm{EI}}\|\overline{\mathrm{DH}}\|\overline{\mathrm{FJ}}\|\overline{\mathrm{BG}}$*, we have*

$$\frac{1}{\mathrm{EI}} + \frac{1}{\mathrm{DH}} = \frac{1}{\mathrm{FJ}} + \frac{1}{\mathrm{BG}}\,.$$

Proof. Draw a line through A parallel to $\overline{\mathrm{BC}}$ and extend $\overline{\mathrm{CE}}$ such that the lines intersect at K. Draw $\overline{\mathrm{EM}}$ and $\overline{\mathrm{FL}}$ parallel to $\overline{\mathrm{BC}}$ with points M and L on line $\overline{\mathrm{AC}}$ (see figure 3).

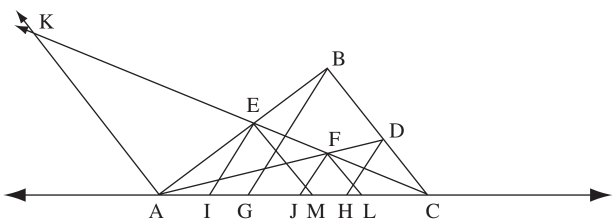

Figure 3.

By the crossed ladders theorem,

$$\frac{1}{\mathrm{AK}} + \frac{1}{\mathrm{BC}} = \frac{1}{\mathrm{EM}} \quad \text{and} \quad \frac{1}{\mathrm{AK}} + \frac{1}{\mathrm{DC}} = \frac{1}{\mathrm{FL}}\,.$$

This implies that

$$\frac{1}{\mathrm{AK}} = \frac{1}{\mathrm{EM}} - \frac{1}{\mathrm{BC}} \quad \text{and} \quad \frac{1}{\mathrm{AK}} = \frac{1}{\mathrm{FL}} - \frac{1}{\mathrm{DC}}\,.$$

Thus

$$\frac{1}{\mathrm{EM}} - \frac{1}{\mathrm{BC}} = \frac{1}{\mathrm{FL}} - \frac{1}{\mathrm{DC}}\,,$$

therefore

$$\frac{1}{\mathrm{EM}} + \frac{1}{\mathrm{DC}} = \frac{1}{\mathrm{FL}} + \frac{1}{\mathrm{BC}}\,.$$

Clearly, the triangles EMI, DCH, FLJ and BCG are similar. This implies that

$$\frac{\mathrm{EI}}{\mathrm{EM}} = \frac{\mathrm{DH}}{\mathrm{DC}} = \frac{\mathrm{FJ}}{\mathrm{FL}} = \frac{\mathrm{BG}}{\mathrm{BC}} = k \quad (\text{say})\,.$$

Therefore $\mathrm{EI} = k\mathrm{EM}$, $\mathrm{DH} = k\mathrm{DC}$, $\mathrm{FJ} = k\mathrm{FL}$, and $\mathrm{BG} = k\mathrm{BC}$. Since

$$\frac{1}{\mathrm{EM}} + \frac{1}{\mathrm{DC}} = \frac{1}{\mathrm{FL}} + \frac{1}{\mathrm{BC}}$$

we have

$$\frac{1}{k\mathrm{EM}} + \frac{1}{k\mathrm{DC}} = \frac{1}{k\mathrm{FL}} + \frac{1}{k\mathrm{BC}}\,,$$

therefore

$$\frac{1}{\mathrm{EI}} + \frac{1}{\mathrm{DH}} = \frac{1}{\mathrm{FJ}} + \frac{1}{\mathrm{BG}}\,.$$

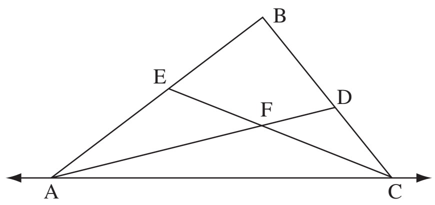

Figure 4.

Corollary 1. *In figure 4,*

$$\frac{1}{\mathrm{area}(\triangle\mathrm{AEC})} + \frac{1}{\mathrm{area}(\triangle\mathrm{ADC})} = \frac{1}{\mathrm{area}(\triangle\mathrm{AFC})} + \frac{1}{\mathrm{area}(\triangle\mathrm{ABC})}\,.$$

Corollary 2. *In figure 4, if* $\mathrm{AE}:\mathrm{EB} = a:b$ *and* $\mathrm{CD}:\mathrm{DB} = c:d$*, then*

$$\frac{\mathrm{area}(\triangle\mathrm{AFC})}{\mathrm{area}(\triangle\mathrm{ABC})} = \frac{ac}{ac+ad+bc}\,.$$

Proofs of these corollaries are left to the reader.

An application. Given a triangle ABC and points D, E and F on each of its sides such that AE : EC = BD : DA = CF : FB = 1 : 2 (see figure 5), find the ratio of the areas of triangles GHI and ABC.

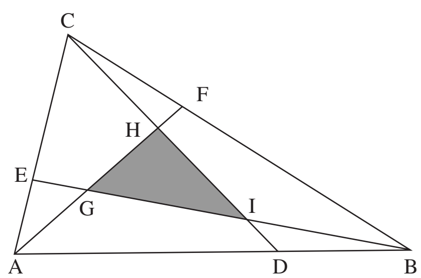

Figure 5.

Solution. By corollary 2, since AE : EC = 1 : 2 and BF : FC = 2 : 1,

$$\frac{\text{area}(\triangle\text{AGB})}{\text{area}(\triangle\text{ABC})} = \frac{1\cdot 2}{1\cdot 2 + 1\cdot 1 + 2\cdot 2} = \frac{2}{7}\,,$$

so area(△AGB) $= \frac{2}{7}$ area(△ABC). Likewise area(△CHA) $= \frac{2}{7}$ area(△ABC) and area(△BIC) $= \frac{2}{7}$ area(△ABC). Hence area(△GHI) $= (1 - \frac{2}{7} - \frac{2}{7} - \frac{2}{7})$ area(△ABC) $= \frac{1}{7}$ area(△ABC). Therefore the ratio of the areas of the triangles GHI and ABC is 1 : 7.

Yours sincerely,
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Dear Editor,

Calculating square roots

With reference to M. E. Hare's letter on square roots in Vol. 34, No. 1, of *Mathematical Spectrum*, it is possible to go a little further with the error calculation given in the Editor's footnote (in which the $3b$ in the denominator should be $2b$). Let

$$\begin{aligned} E &= \frac{-(a-b)^6}{2b(3a^2+b^2)(a^2+3b^2)} \\ &= \frac{-(a^2-b^2)^6}{2b(3a^2+b^2)(a^2+3b^2)(a+b)^6}\,. \end{aligned}$$

Assume first that $a^2 > b^2$, then

$$\begin{aligned} 2b(3a^2+b^2)(a^2+3b^2)(a+b)^6 &> 2b\cdot 4b^2\cdot 4b^2\cdot (2b)^6 \\ &= 2048b^{11}\,, \end{aligned}$$

so

$$|E| < \frac{(a^2-b^2)^6}{2048b^{11}} = 2b\Big(\frac{a^2}{4b^2} - \frac{1}{4}\Big)^6\,.$$

Similarly, if $a^2 < b^2$,

$$|E| < \frac{(a^2-b^2)^6}{2048a^{11}} = 2a\Big(\frac{b^2}{4a^2} - \frac{1}{4}\Big)^6\,.$$

In the example given, $|E| < 3.76\times 10^{-13}$.

We further assume that $100 < a^2 < 10\,000$ (multiply by an even power of 10 to start with to achieve this). Then we need only find a two-digit square root, which is easy enough without the square root key. An upper limit for $|E|$ may be found as follows. If $a > b$, then

$$0 < a^2 - b^2 \le \tfrac{1}{2}\big((b+1)^2 - b^2\big) = b + \tfrac{1}{2} < \tfrac{21}{20}b\,,$$

since b^2 is the nearest square to a^2, so

$$|E| < \frac{1}{1528b^5} < 7\times 10^{-9}\,.$$

If $a < b$, then

$$\begin{aligned} 0 < b^2 - a^2 &\le \tfrac{1}{2}\big(b^2 - (b-1)^2\big) \\ &= b - \tfrac{1}{2} < a + \tfrac{1}{2} < \tfrac{21}{20}a\,, \end{aligned}$$

so

$$|E| < \frac{1}{1528a^5} < 7\times 10^{-9}\,.$$

Mr Hare's method will almost always yield an answer correct to 10 significant figures if adapted as above.

By contrast, I offer the following formula for square roots. If $a^2 < y < b^2$, then interpolating linearly gives a first estimate for $\sqrt{y}$ of

$$\frac{y+ab}{a+b}\,.$$

The error in this is

$$\begin{aligned} E &= \frac{(a+b)\sqrt{y} - y - ab}{a+b} = \frac{(\sqrt{y}-a)(b-\sqrt{y})}{a+b} \\ &\simeq \frac{\big((y+ab)/(a+b) - a\big)\big(b - (y+ab)/(a+b)\big)}{a+b} \\ &= \frac{(y-a^2)(b^2-y)}{(a+b)^3}. \end{aligned}$$

A better estimate for $\sqrt{y}$ is

$$\frac{y+ab}{a+b} + \frac{(y-a^2)(b^2-y)}{(a+b)^3}\,; \tag{1}$$

note that $(y+ab)/(a+b) < \sqrt{y}$. The error in this can be shown to be less than $1/4(a+b)^2$ if $b = a+1$. For Mr Hare's example this is approximately 4.7×10^{-4}; the actual error is about 1.46×10^{-4}. Formula (1) will almost always give an answer correct to five significant figures.

An even better formula is obtained by adding a term

$$\frac{(a^2+b^2-2y)(y-a^2)(b^2-y)}{(a+b)^5}\,,$$

with error less than $1/16(a+b)^3$ and accuracy to seven significant figures.

Yours sincerely,
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Dear Editor,

The Fibonacci finals

The numbers in a Fibonacci sequence are defined by

$$F(n) = F(n-2) + F(n-1) \quad \text{with } F(1) = F(2) = 1\,.$$

Consider the final digits of the Fibonacci numbers. They are: 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1, 2, 3, ... It is known that the sequence repeats after the first 60 digits. The distribution of digits in these 60 final digits is as follows:

Final digit	1 2 3 4 5 6 7 8 9 0
No. of occurrences	8 4 8 4 8 4 8 4 8 4 .

Let $f(n)$ be the final digit of $F(n)$. Then it can be observed that $f(15n) = 0$, for example, $f(45) = f(60) = 0$. Also $f(n) = f(n + 30) = 10$, $f(16) = 7$ and $f(46) = 3$. In addition, $f(15n + 5) = 5$.

It will be seen that $f(n + 60k) = f(n)$, for example, $f(99) = f(39) = 6$, and $f(371) = f(11) = 9$.

Following from the distribution of final digits, we find: for $n = 1, 2, 8, 19, 22, 28, 41, 59$,

$$f(n) = 1 \quad \text{and} \quad \sum n = 180,$$

for $n = 3, 36, 54, 57$,

$$f(n) = 2 \quad \text{and} \quad \sum n = 150,$$

for $n = 4, 7, 13, 26, 44, 46, 47, 53$,

$$f(n) = 3 \quad \text{and} \quad \sum n = 240,$$

for $n = 9, 12, 18, 51$,

$$f(n) = 4 \quad \text{and} \quad \sum n = 90,$$

for $n = 5, 10, 20, 25, 35, 40, 50, 55$,

$$f(n) = 5 \quad \text{and} \quad \sum n = 240,$$

for $n = 21, 39, 42, 48$,

$$f(n) = 6 \quad \text{and} \quad \sum n = 150,$$

for $n = 14, 16, 17, 34, 37, 43, 56$,

$$f(n) = 7 \quad \text{and} \quad \sum n = 240,$$

for $n = 6, 24, 27, 33$,

$$f(n) = 8 \quad \text{and} \quad \sum n = 90,$$

for $n = 11, 29, 31, 32, 38, 49, 52, 58$,

$$f(n) = 9 \quad \text{and} \quad \sum n = 300,$$

for $n = 15, 30, 45, 60$,

$$f(n) = 0 \quad \text{and} \quad \sum n = 150.$$

Query. Why should $\sum n$ be a multiple of 30 in each case?

An unexpected result was the discovery that the sequence is embedded in itself, sometimes in the same order and sometimes in the reverse order. Consider the 60-number cycle repeating itself indefinitely. Then, starting with any of the 60 numbers, say n, take successive mth numbers to form a new sequence, i.e.

$$\text{form } f(n + mi) \quad \text{for } i = 0, \ldots, 59.$$

If m takes any one of the values

$$13,\ 17,\ 29,\ 37,\ 41,\ 49 \text{ or } 53,$$

the sequence is repeated in the same order. If m takes any one of the values

$$7,\ 11,\ 19,\ 23,\ 31,\ 43,\ 47 \text{ or } 59,$$

the sequence is repeated in reverse order. It will be noticed that all these values of m are relatively prime to 60.

Having considered the properties of the final digits of the Fibonacci sequence, we may ask questions about the sequence of the final pair of digits, i.e.

$$01,\ 01,\ 02,\ 03,\ 05,\ 08,\ 13,\ 21,\ 34,\ 55,\ 89,\ 44,\ 33,\ \ldots, 97,\ 02,\ 99,\ 01,\ 00,\ 01,\ 01,\ 02,\ \ldots$$

For the brave-hearted, the sequence repeats after 300 numbers, another multiple of 30.

Alternatively, instead of starting with the pair $(1, 1)$, as in the Fibonacci sequence, we could start with other pairs and consider the final digits. The following results have been found:

- A cycle length of 3 is produced by the pairs $(0, 5)$, $(5, 0)$, $(5, 5)$.
- A cycle length of 4 is produced by $(2, 6)$, $(4, 2)$, $(6, 8)$, $(8, 4)$.
- A cycle length of 12 is produced by $(1, 3)$, $(1, 8)$, $(2, 1)$, $(3, 4)$, $(3, 9)$, $(4, 7)$, $(6, 3)$, $(7, 1)$, $(7, 6)$, $(8, 9)$, $(9, 2)$, $(9, 7)$.
- A cycle length of 20 is produced by $(0, 2)$, $(0, 4)$, $(0, 6)$, $(0, 8)$, $(2, 0)$, $(2, 2)$, $(2, 4)$, $(2, 8)$, $(4, 0)$, $(4, 4)$, $(4, 6)$, $(4, 8)$, $(6, 0)$, $(6, 2)$, $(6, 4)$, $(6, 6)$, $(8, 0)$, $(8, 2)$, $(8, 6)$, $(8, 8)$.

All other starting pairs (except (0,0)) produce a cycle length of 60.

Yours sincerely,
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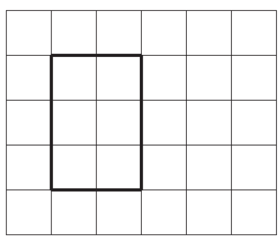
UK.)

[Readers may like to try to prove some of Bob Bertuello's assertions. The well-known result that $F(m)$ divides $F(n)$ when m divides n may help – Ed.]



How many rectangles are there in an $m \times n$ array of equal squares? (The case $m = 5$, $n = 6$ is illustrated.)

Problems and Solutions

Students are invited to submit solutions to some or all of the problems below. The most attractive solutions will be published in subsequent issues and are eligible for annual prizes. When writing to the Editorial Office, please state your full name and also the postal address of your school, college or university.

Problems

35.1 Triangles ABC (with angles denoted by A, B and C) and ABD are similar but not congruent and do not intersect. If AD is parallel to BC and $\theta = \angle\mathrm{ADC}$, express $\cot\theta$ in terms of $\cot \mathrm{A}$, $\cot \mathrm{B}$ and $\cot \mathrm{C}$.

If CD meets the circumcircle of triangle ABC again at P, which is inside the triangle ABD, deduce further that P has a symmetrical relationship with the triangle ABD.

(Submitted by J. A. Scott, Chippenham)

35.2 Let $a_0, a_1, \ldots, a_{2n}$ be positive real numbers. Prove that there exist at least $(n!)^2$ distinct permutations σ of $\{0, 1, \ldots, 2n\}$ such that

$$a_{\sigma(2n)}x^{2n} + \cdots + a_{\sigma(1)}x + a_{\sigma(0)} > 0$$

for every real number x.

(Submitted by Hassan Shah Ali, Tehran)

35.3 Let $\mathrm{A}_1\mathrm{A}_2 \ldots \mathrm{A}_n$ be a regular n-sided polygon. Show that $(\mathrm{PA}_1)^2 + (\mathrm{PA}_2)^2 + \cdots + (\mathrm{PA}_n)^2$ is the same for all points P on the inscribed circle of the polygon.

(Submitted by Zhang Yun, First Middle School of Jinchang City, China)

35.4 Let ABCD be a regular tetrahedron. Show that $(\mathrm{PA})^2 + (\mathrm{PB})^2 + (\mathrm{PC})^2 + (\mathrm{PD})^2$ is the same for all points P of the inscribed sphere of the tetrahedron. (The centre of the inscribed sphere is in the ratio $1:3$ from a face and the opposite vertex.)

(Submitted by Zhang Yun)

Solutions to Problems in Volume 34 Number 2

34.5 A pack of 52 cards is repeatedly shuffled using perfect riffle shuffles. After how many shuffles will every card return to its initial position?

Solution

There are two possible answers to this question depending on what is meant by a perfect riffle or shuffle. If we number the positions of the cards in the pile 1–52, one possibility is that

$$n \to \begin{cases} 2n & \text{for } n = 1, 2, \ldots, 26, \\ 2(n-26)-1 & \text{for } n = 27, 28, \ldots, 52. \end{cases}$$

Then the cards change position according to the 52-cycle

$$\begin{aligned}(&1\ 2\ 4\ 8\ 16\ 32\ 11\ 22\ 44\ 35\ 17\ 34\ 15\ 30\ 7\ 14\ 28\ 3\ 6\ 12\ 24\\ &48\ 43\ 33\ 13\ 26\ 52\ 51\ 49\ 45\ 37\ 21\ 42\ 31\ 9\ 18\ 36\ 19\ 38\ 23\\ &46\ 39\ 25\ 50\ 47\ 41\ 29\ 5\ 10\ 20\ 40\ 27),\end{aligned}$$

so each card will return to its initial position after 52 shuffles.

If we take the shuffle to be

$$n \to \begin{cases} 2n-1 & \text{for } n = 1, 2, \ldots, 26, \\ 2(n-26) & \text{for } n = 27, 28, \ldots, 52, \end{cases}$$

then the cards change position according to the cycles

$$\begin{gathered}(1)\ (2\ 3\ 5\ 9\ 17\ 33\ 14\ 27)\ (4\ 7\ 13\ 25\ 49\ 46\ 40\ 28)\\ (6\ 11\ 21\ 41\ 30\ 8\ 15\ 29)(10\ 19\ 37\ 22\ 43\ 34\ 16\ 31)\\ (12\ 23\ 45\ 38\ 24\ 47\ 42\ 32)(18\ 35)\\ (20\ 39\ 26\ 51\ 50\ 48\ 44\ 36)(52).\end{gathered}$$

These cycles have lengths 1, 2 or 8, so each card will return to its initial position after eight shuffles.

34.6 Sets $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ are such that each of the sets $X_1, \ldots, X_k$ has a non-empty intersection with exactly m of the sets $Y_1, \ldots, Y_k$ and each of the sets $Y_1, \ldots, Y_k$ has a non-empty intersection with exactly n of the sets $X_1, \ldots, X_k$. What is the connection between m and n?

Solution by Hassan Shah Ali, who proposed the problem

For $i, j = 1, \ldots, k$, write

$$a_{ij} = \begin{cases} 0 & \text{if } X_i \cap Y_j = \varnothing, \\ 1 & \text{if } X_i \cap Y_j \neq \varnothing. \end{cases}$$

Then

$$\sum_{j=1}^{k} a_{ij} = m \quad \text{for each } i$$

and

$$\sum_{i=1}^{k} a_{ij} = n \quad \text{for each } j.$$

Hence

$$\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij} = km$$

and

$$\sum_{j=1}^{k}\sum_{i=1}^{k} a_{ij} = kn\,,$$

so $m = n$.

34.7 What is the sum of the series

$$1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{15}+\cdots?$$

Solution by Farshid Arjomandi and James Fulton (University of California, San Diego)

The series is

$$\begin{aligned}
\sum_{k=0}^{\infty}&(-1)^k\Big(\frac{1}{4k+1}+\frac{1}{4k+3}\Big)\\
&=\sum_{k=0}^{\infty}(-1)^k\int_0^1 (y^{4k}+y^{4k+2})\,\mathrm{d}y\\
&=\sum_{k=0}^{\infty}(-1)^k\int_0^1 y^{4k}(1+y^2)\,\mathrm{d}y\\
&=\int_0^1 (1+y^2)(1-y^4+y^8-\cdots)\,\mathrm{d}y\\
&=\int_0^1 \frac{1+y^2}{1+y^4}\,\mathrm{d}y\\
&=\int_0^1 \frac{1+y^2}{(1-y^2)^2+2y^2}\,\mathrm{d}y\\
&=\int_0^1 \frac{(1+y^2)/(1-y^2)^2}{1+\big(y\sqrt{2}/(1-y^2)\big)^2}\,\mathrm{d}y\\
&=\frac{1}{\sqrt{2}}\Big[\tan^{-1}\Big(\frac{y\sqrt{2}}{1-y^2}\Big)\Big]_0^1\\
&=\frac{\pi}{2\sqrt{2}}\,.
\end{aligned}$$

Also solved by Milton Chowdhury (UMIST).

34.8 Given five points A, B, C, D, E in the plane, X_1 is the midpoint of AB, X_2 the midpoint of CD, X_3 the midpoint of X_1X_2, X_4 the midpoint of DE and X_5 the midpoint of X_1X_4. In which point do the lines EX_3 and CX_5 intersect?

Solution

We use lower-case bold letters for the vector of each point relative to any chosen axes. Then

$$\begin{aligned}
\boldsymbol{x}_1 &= \tfrac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\,, & \boldsymbol{x}_2 &= \tfrac{1}{2}(\boldsymbol{c}+\boldsymbol{d})\,,\\
\boldsymbol{x}_3 &= \tfrac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})\,, & \boldsymbol{x}_4 &= \tfrac{1}{2}(\boldsymbol{d}+\boldsymbol{e})\,,\\
\boldsymbol{x}_5 &= \tfrac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{d}+\boldsymbol{e})\,.
\end{aligned}$$

Now

$$\tfrac{1}{5}\Big(\boldsymbol{e}+4\big(\tfrac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})\big)\Big)=\tfrac{1}{5}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e})$$

and

$$\tfrac{1}{5}\Big(\boldsymbol{c}+4\big(\tfrac{1}{4}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{d}+\boldsymbol{e})\big)\Big)=\tfrac{1}{5}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e})\,,$$

so the centre of gravity or mean of A, B, C, D, E lies on both EX_3 and CX_5, so this is the point of intersection of these lines.

Fermat's last theorem

This is the famous result that the equation $x^n+y^n = z^n$ has no solution in positive integers when $n > 2$. It was finally proved by Andrew Wiles in 1994. In the same year, I was studying GCSE Mathematics, and came up with the solution

$$3\,453\,652\,926^3+6\,709\,845\,183^3 = 7\,001\,938\,239^3.$$

How would you demonstrate that I was wrong?

MILTON CHOWDHURY
Student, UMIST

Reviews

The Golden Section. By HANS WALSER. MAA, Washington, DC, 2001. Pp. 132. Paperback \$26.95 (ISBN 0-88385-534-8).

'Since antiquity, the golden section has played a significant role in many parts of geometry, architecture, music, art and philosophy.' That's what the back of this book tells us and inside are oodles of examples. It's a bit like when you were at primary school and your teacher sent you to go and look for right angles, except that those were all man made, whereas the golden section just happens, which makes it much more exciting. I would say there are pages of this book which read a bit like a detective story, looking at places where the suspect ratio might be and digging it out, but there are also pages, especially in the chapter on number sequences, which are a little heavy on the eye. I love maths, but I am still intimidated if I turn a page and find 15 or so equations looking back at me with very little text and no pretty pictures. (For plenty of pretty pictures, see the chapters on golden geometry and, of course, fractals. There is even a chapter, 'Folds and Cuts', where you get to use scissors and paper and make your own constructions of familiar shapes, including the golden rectangle.) This book does have the advantage that each chapter can be read and understood with little or no need for reference to the previous chapters, so you can pick and choose as the fancy takes you. Another good idea is the inclusion of 80 questions scattered throughout, which keep you on your toes and make the book more interactive.

Limonar, Malaga ANNE WILDING

Identification Numbers and Check Digit Schemes. By JOSEPH KIRTLAND. MAA, Washington, DC, 2001. Pp. 184. Paperback $32.95 (ISBN 0-88385-720-0).

This book is a very well written piece of literature. It presents the mathematics which supports a variety of check digit schemes, which does not involve a great amount of A-level mathematics. The book uses simple mathematical methods and step-by-step guides to show the reader how to resolve the more difficult problems.

Identification numbers are used to encode information which can then be sent via the Internet. To avoid the possibility of a transmission error, an extra digit, or check digit, is frequently added to check if an error has occurred. This book is all about the criteria used to judge the reliability of a scheme, such as the ISBN scheme, to catch errors.

I think the book will be enjoyed by those students who relish working outside the classroom, and who are fascinated by analysing and exploring numbers and patterns. I also think the book will have the most appeal to advanced mathematics students and university undergraduates.

Student, Solihull Sixth Form College VINAY SURTI

USA and International Mathematical Olympiads 2000. By TITU ANDREESCU AND ZUMING FENG. MAA, Washington, DC, 2001. Pp. 83. Paperback $16.50 (ISBN 0-88385-804-5).

The USA came third in the 2000 International Mathematical Olympiads (IMO), which is a credit to the thorough selection and training regime the team is subjected to. Hopefuls are required to sit a series of demanding papers before they can qualify for the toughest challenge for high school maths students — the IMO.

This book contains the twelve questions from the selection papers of the USA team, plus the six problems from the IMO itself. This comprises the first chapter of the book, after a brief introduction and information on the notation and symbols used. The next chapter gives hints on how to go about answering the questions, often giving a seemingly inaccessible problem a handle with which to construct a detailed proof. However, the temptation is just to skip straight to this chapter after spending only a few minutes thinking about a question, which is counterproductive, especially as the hint is obvious with a bit of thought anyway.

The main part of the book is concerned with solutions to the problems, often with many different approaches given for completeness. The solutions are well presented and accompanied by relevant diagrams, with any necessary lemmas proved beforehand. The arguments presented in the book are very difficult to read straight off, due to the complexity of the problems, but the clear layout helps the reader, although more spacing would be welcome. Other nice touches include a table of formulae, identities and definitions, as well as an appendix covering the results of the IMO over the four previous years.

This book is really suitable only for those with a fair grasp of the mathematics involved, who will be able follow its content unaided, as it is very easy to get lost mid-proof. As the IMO papers and their solutions are readily obtainable online or through schools and colleges etc., it makes sense to buy this book for the impressive variety of solutions given, rather than just for the questions themselves. In fact, with only eighteen questions in the entire book, it does not take long to lose interest and for the price I would have expected something more substantial.

Student, Trinity College, Cambridge DANIEL LAMY

Symmetry. By HANS WALSER. MAA, Washington, DC, 2001. Pp. 120. Paperback $23.50 (ISBN 0-88385-532-1).

This book aims to make the reader notice all aspects of symmetry in the world around us and recognise its usefulness as a methodological tool. The author uses real life examples of symmetry (e.g. in words, poems and songs, and in repeated images in mirrors). Not surprisingly the book abounds with mathematical diagrams, each accompanied by a clear explanation. The book is made up of six independent chapters, so that the readers can focus on areas of particular interest to them if they wish. These cover topics such as mirrors, circle reflection, centres of gravity, the patterns of parquet floors, looking for the centre of the world and palindromes. All fascinating stuff! I found this book very interesting and fairly easy to read. It is aimed at students and teachers of mathematics, but anyone wanting to expand their mathematical knowledge, or with a special interest in symmetry will enjoy this book.

Student, Solihull Sixth Form College KATIE HALL

Other books received

Problems in Mathematical Analysis II: Continuity and Differentiation. By W. J. KACZOR AND M. T. NOWAK. American Mathematical Society, Providence, RI, 2001. Pp. xiv+398. Paperback £29.95 (ISBN 0-8218-2051-6).

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