Crux Mathematicorum

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How To Write A *Crux* Article

Robert Dawson

Longtime ***Crux*** readers will notice that the dates on your magazines are getting closer to the dates on your calendar. This is good news, and is largely thanks to the hard work of our dedicated editor-in-chief (stand up and take a bow, Kseniya!) Once we catch up, we do not plan to keep charging into the future at the same rate – production will slow to one year per year. This means that our current demand for articles may slow somewhat, though we'll still need plenty.

This seems like a good time to remind people what a ***Crux*** article is, and isn't.

(1) ***Crux*** is accessible. ***Crux*** is read by university professors and graduate students. It's also read by undergraduates, school teachers, school students, and amateurs whose day jobs have nothing to do with mathematics. We ask prospective writers to write for a very clever high school student. Assume high intelligence but not a lot of specialized knowledge. That said:

(2) ***Crux*** is primarily for problem solvers, and by "problems" I mean the sort of thing that might appear on a regional or national math contest. We may run puzzles and alphametics, but they aren't our main topic, and we probably wouldn't run articles about them. The natural home for articles on those topics used to be the late and much lamented *Journal of Recreational Mathematics*. Right now perhaps the best place, if the article has some mathematical content, is probably *Mathematics Gazette*, the *Mathematical Intelligencer*, or the *College Math Journal*.

You should assume that most ***Crux*** readers know the standard tricks of the trade. Don't stop and explain mathematical induction or double counting unless you're explaining something new or unusual about those topics.

A good ***Crux*** article is not one where readers watch you solve problems. It's one that tells a significant proportion of them something they didn't know before about how to solve problems.

(3) ***Crux*** is short. Our articles don't usually run more than five or six pages, and we're more likely to run it if it's three or four. That's the right length for the sort of thing we publish. We do like well-done illustrations, nice examples, and interesting asides – but please keep it all brief.

(4) ***Crux*** is not a research journal. If you've written a research paper, please send it somewhere else, where it has a chance of getting published. We publish a dozen or so articles per year – the College Math Journal, at a similar level, publishes almost that many in a month. None of our articles are research articles in the conventional sense.

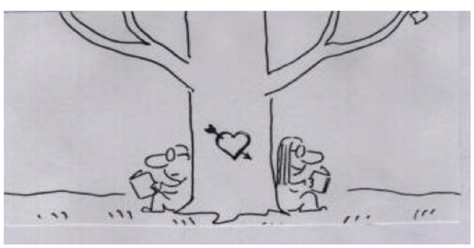
Crux is definitely not a "research journal of last resort." We don't even publish good research papers, and we have no need at all for "research" that other journals won't print.

*You cannot trisect the angle, square the circle, or duplicate the cube using classical tools. If you understand Wantzel's classic 1837 proof, you won't try. If you don't understand it, you have not done your preparatory work and you have no business trying.

*You probably cannot prove the Riemann Conjecture, Fermat's Last Theorem, the ABC conjecture, or the Four Color Theorem in five pages. If you could, you would not be sending your proof to ***Crux*** : you would be sending it to one of the very top math journals. Finally, please don't use a theorem-proof-lemma form. It's not our style.

Okay? Now you know what we're looking for. Write it – get somebody to check it over to make sure the math and English (or French) is correct – and send it to us.

Mathematics Books for Sale



... for book lovers ...

... and book hunters

Over 1500 used books from all areas of pure and applied mathematics, including some computer science, statistics, and mathematical physics, are for sale at

`http://www.mathstat.dal.ca/~dilcher/oldbooks.html`

Of particular interest to ***Crux*** readers will be the sections on Problem Solving, Puzzles and Games, Biographies, History, and General and Popular Mathematics.

This is a fundraiser, and prices are moderate. All proceeds go, in equal parts, to the Canadian Mathematical Society and to the Dalhousie Department of Mathematics and Statistics.

(Illustrations by Vladimír Jiránek.)

THE CONTEST CORNER

No. 39

John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC191. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

CC192. Let M be a 3×3 matrix with all entries drawn randomly (and with equal probability) from $\{0, 1\}$. What is the probability that $\det M$ will be odd?

CC193. Consider the set of numbers $\{1, 2, \ldots, 10\}$. Let $\{a_1, a_2, \ldots, a_{10}\}$ be some permutation of these numbers and compute

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

What is the maximum possible value of the above sum over all possible permutations and how many permutations give you this maximum value?

CC194. At a strange party, each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the party that they both knew. For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew. How many people were at the party?

CC195. A bisecting curve is one that divides a given region into two subregions of equal area. The shortest bisecting curve of a circle is clearly a diameter. What is the shortest bisecting curve of an equilateral triangle?

CC191. Il y a 32 concurrents dans un tournoi et il n'y a pas deux concurrents de force égale. Dans n'importe quel match entre deux concurrents, le plus fort l'emporte toujours. Démontrer qu'il est possible de déterminer les récipiendaires des médailles d'or, d'argent et de bronze après 39 parties.

CC192. Soit M une matrice 3×3 dont les éléments sont tous choisis de façon aléatoire dans l'ensemble $\{0, 1\}$. Quelle est la probabilité pour que $\det M$ soit impair?

CC193. On considère l'ensemble $\{1, \ldots, 10\}$ et une permutation $\{a_1, \ldots, a_{10}\}$ de cet ensemble. On calcule

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

Parmi toutes les permutations de l'ensemble, quelle est la valeur maximale de cette somme et combien de permutations donnent cette valeur maximale?

CC194. Lors d'une drôle de fête, chaque personne connaît 22 autres personnes. Pour chaque paire de personnes X et Y qui se connaissent l'une l'autre, il n'y a aucune autre personne à la fête que X et Y connaissent tous les deux. Pour chaque paire de personnes X et Y qui ne se connaissent pas l'une l'autre, il y a 6 personnes à la fête que X et Y connaissent tous les deux. Combien y a-t-il de personnes à la fête?

CC195. Une courbe bissectrice d'une surface est une courbe qui coupe la surface en deux régions de même aire. La courbe bissectrice d'un cercle est évidemment un diamètre. Quelle est la courbe bissectrice la plus courte d'un triangle équilatéral?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(9), p. 368–369. All problems are from 42nd Ural tournament of young mathematicians, as printed in Kvant 2014(1).

CC141. Alice writes down 100 consecutive natural numbers. Bob multiplies 50 of them: 25 smallest ones and 25 largest ones. He then multiplies the remaining 50 numbers. Can the sum of the two products be equal to $100! = 1 \cdot 2 \cdot \ldots \cdot 100$?

Problem 2 of grade 7 level of 42nd Ural tournament of young mathematicians.

We received three solutions, two of which were correct. We present a composite of the two correct solutions from Konstantine Zelator and Fernando Ballesta Yagüe.

Write the 100 consecutive natural numbers as $n+1$, $n+2$, $\ldots$, $n+100$, where n is a natural number. Let $P_1 = (n+1)(n+2)\cdots(n+25)$ be the product of the 25 smallest numbers, $P_2 = (n+26)(n+27)\cdots(n+75)$ be the product of the 50 middle numbers, and $P_3 = (n+76)(n+77)\cdots(n+100)$ be the product of the largest 25 numbers. We want to know if it's possible to have $P_1 \cdot P_3 + P_2 = 100!$.

The factors in the products P_1, P_2 and P_3 contain 100 consecutive natural numbers. Thus, if $P_1 \cdot P_3 + P_2 = 100!$, the greatest prime number smaller than 100, which is 97, must be a divisor of each of $P_1 \cdot P_3$ and P_2.

If 97 is a divisor of P_2, that means it is a divisor of one of its factors $n+k$, $k \in 26, \ldots, 75$. The neighbouring multiples of 97 are $n+k-97$ and $n+k+97$. Both of these fall outside the range of the natural numbers in P_1 (which, for all options of $n+k$ in P_2, must fall within $n+k-74$ to $n+k-1$) and P_3 (which must fall within $n+k+1$ to $n+k+74$). Thus if 97 is a divisor of P_2, then it cannot be a divisor of either P_1 or P_3. We cannot have $P_1 \cdot P_3 + P_2 = 100!$.

CC142. Roboto writes down a number. Every minute, he increases the existing number by the double of the number of its natural divisors (including 1 and itself). For example, if he started with 5, the sequence would be $5, 9, 15, 23, \ldots$. What is the maximum number of perfect squares that appears on the board within 24 hours?

Problem 3 of grade 7 level of 42nd Ural tournament of young mathematicians.

We received one correct solution. We feature the solution by Konstantine Zelator.

We show that such a sequence contains at most one perfect square. We make two observations. First, if n is a perfect square, then $n \equiv 0$ or $1 \pmod 4$. Second, the number of positive divisors of an integer is odd if and only if that integer is a perfect square.

Suppose such a sequence contains a perfect square k. Then k has $2m+1$ divisors for some integer m, and twice the number of divisors of k is $4m+2$. Since k is a perfect square $k \equiv 0$ or $1 \pmod 4$ and $k+4m+2 \equiv 2$ or $3 \pmod 4$. We notice that numbers of these forms cannot be perfect squares, and so have an even number of divisors. Thus, twice the number of their divisors is a multiple of 4. So after this point, the terms in the sequence will be the same modulo 4 and hence can never be perfect squares. Thus, there can be at most one perfect square in the sequence.

CC143. Summer Camp has attracted 300 students this year. On the first day, the students discovered (as mathematicians would) that the number of triples of students who mutually know each other is greater than the number of pairs of students who know each other. Prove that there is a student who knows at least 5 other students.

Problem 9 of grade 7 level of 42nd Ural tournament of young mathematicians.

We received no solutions to this problem.

CC144. Year 2013 is the first one since Middle Ages that uses 4 consecutive digits in its base 10 representation. How many other years like this will there be before year 10,000?

Problem 1 of grade 8 level of 42nd Ural tournament of young mathematicians.

We received three solutions, of which two were correct and complete. We present the most succinct, by Andrea Fanchini.

We have seven possible groups of four consecutive digits:

$$\{0123, 1234, 2345, 3456, 4567, 5678, 6789\}.$$

For each group we have $4! = 24$ possible permutations. In total, this makes $7 \cdot 24 = 168$ possibilities. But from these we have to cancel those that start with 0, those that start with 1, and the year 2013.

The possibilities that start with 0 are $3! = 6$ permutations (0 is present only in the first group). The possibilities that start with 1 are $3! + 3! = 12$ permutations (1 is present in the first and second groups).

Finally, we remain with $168 - 6 - 12 - 1 = 149$ years with the property requested.

CC145. Can a natural number be divisible by all numbers between 1 and 500 except for some two consecutive ones? If so, find these two numbers (show all possible cases).

Problem 3 of grade 8 level of 42nd Ural tournament of young mathematicians.

We received two correct solutions. We present the solution of Titu Zvonaru below.

Let n be a natural number less than 500 satisfying the statement of the problem. Let p be a number such that n is not divisible by p. If $p = a \cdot b$ with $a, b > 1$ relatively prime, we deduce that n is not divisible by a or is not divisible by b. Since a (or b) and p are not consecutive, we obtain a contradiction. This yields that p is a power of a prime. If $p \leq 250$ then n is not divisible by $2p$. It follows that $p > 250$.

By the same reasoning, $p + 1$ must be a power of a prime, and $p + 1 > 250$. One of them is even, meaning it must be a power of 2 (the only even prime). It is not hard to see that $256 = 2^8$ is the only number that fits our criteria. So either $p = 256$ or $p + 1 = 256$. If $p = 256$ then $p + 1 = 257$ which is easily seen to be prime. If $p + 1 = 256$ then $p = 255$ which factors as $255 = 3 \cdot 5 \cdot 17$, not a power of a prime. As a conclusion, the number

$$n = \operatorname{lcm}(1, 2, \ldots, 254, 255, 258, 259, \ldots, 500)$$

is divisible by all numbers between 1 and 500, except for consecutive numbers 256 and 257.

THE OLYMPIAD CORNER

No. 337

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

OC251. Let a, b, c and d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value of the product

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1).$$

OC252. In an obtuse triangle ABC ($AB > AC$), O is the circumcentre and D, E and F are the midpoints of BC, CA and AB respectively. Median AD intersects OF and OE at M and N respectively and BM meets CN at point P. Prove that $OP \perp AP$.

OC253. Prove that there exist infinitely many positive integers n such that $3^n + 2$ and $5^n + 2$ are all composite numbers.

OC254. Find all non-negative integers k, n which satisfy

$$2^{2k+1} + 9 \cdot 2^k + 5 = n^2.$$

OC255. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be positive reals. Show that there are numbers $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ such that the following holds:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \geq (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2.$$

. . . .

OC251. Soit a, b, c et d des nombres réels tels que $b - d \geq 5$ et tels que les racines x_1, x_2, x_3 et x_4 du polynôme $P(x) = x^4 + ax^3 + bx^2 + cx + d$ sont toutes réelles. Déterminer la plus petite valeur possible pour le produit

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1).$$

OC252. Pour un triangle obtus ABC ($AB > AC$), soit O le centre du cercle circonscrit et soit D, E et F les mi points de BC, CA et AB respectivement. La médiane AD intersecte OF et OE à M et N respectivement; BM rencontre CN au point P. Démontrer que $OP \perp AP$.

OC253. Démontrer qu'il existe un nombre infini d'entiers positifs n tels que $3^n + 2$ et $5^n + 2$ sont composés.

OC254. Déterminer tous les entiers non négatifs k et n satisfaisant

$$2^{2k+1} + 9 \cdot 2^k + 5 = n^2.$$

OC255. Soit n un entier positif et soit $x_1, x_2, \ldots, x_n$ des nombres réels positifs. Démontrer qu'il existe des nombres $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ tels que l'inégalité suivante tient:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \geq (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2.$$

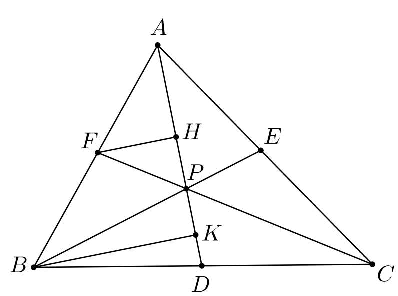
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(7), p. 282–283.

OC191. Let P be a point in the interior of triangle ABC. Extend AP, BP, and CP to meet BC, AC, and AB at D, E, and F, respectively. If triangle APF, triangle BPD and triangle CPE have equal areas, prove that P is the centroid of triangle ABC.

Originally problem 2 from the 2013 Philippines Mathematical Olympiad.

We received four correct submissions. We present the solution by Michel Bataille.



Let $P = \alpha A + \beta B + \gamma C$ where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. Since the point $P - \alpha A = \beta B + \gamma C$ is on AP and on BC, we have

$$(1-\alpha)D = P - \alpha A = \beta B + \gamma C$$

and so $\frac{AP}{PD} = \frac{1-\alpha}{\alpha}$. Similarly,

$$(1-\gamma)F = P - \gamma C = \alpha A + \beta B$$

and so $\frac{AF}{AB} = \frac{\beta}{1-\gamma}$.

Since the altitudes FH and BK in triangles AFP and BPD, respectively, satisfy $\frac{FH}{BK} = \frac{AF}{AB}$ (note that ΔAFH and ΔABK are similar), we obtain

$$1 = \frac{[APF]}{[BPD]} = \frac{FH \cdot AP}{BK \cdot PD} = \frac{AF.AP}{AB.PD} = \frac{\beta(1-\alpha)}{\alpha(1-\gamma)}$$

where $[XYZ]$ denotes the area of ΔXYZ. It follows that $\alpha(\alpha+\beta) = \beta(\beta+\gamma)$; in the same way, we have $\beta(\beta+\gamma) = \gamma(\gamma+\alpha)$.

Now, let $k = \alpha^2 + \alpha\beta = \beta^2 + \beta\gamma = \gamma^2 + \gamma\alpha$. Then, recalling that $\alpha + \beta + \gamma = 1$, we successively obtain

$$k = k\beta + k\gamma + k\alpha = \alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$$

and

$$k = k\gamma + k\alpha + k\beta = \alpha^2\gamma + \alpha\beta\gamma + \beta^2\alpha + \alpha\beta\gamma + \gamma^2\beta + \alpha\beta\gamma.$$

As a result, we have

$$\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha = 3\alpha\beta\gamma = 3\sqrt[3]{\alpha^2\beta \cdot \beta^2\gamma \cdot \gamma^2\alpha}.$$

From the case of equality in AM-GM, this means that $\alpha^2\beta = \beta^2\gamma = \gamma^2\alpha$. This easily yields $\alpha^3 = \alpha\beta\gamma = \beta^3 = \gamma^3$ and so $\alpha = \beta = \gamma$. Thus, P is the centroid of ΔABC.

OC192. Find all possible values of a positive integer n for which the expression $S_n = x^n + y^n + z^n$ is constant for all real x, y, z with $xyz = 1$ and $x + y + z = 0$.

Originally problem 2 from the 2013 Spain Mathematical Olympiad.

We received two correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We claim that the only solutions are $n = 1$ and $n = 3$ (and $n = 0$ if non-negative integers are allowed). Let $\sigma_1 = x + y + z$, $\sigma_2 = xy + xz + yz$, and $\sigma_3 = xyz$ (the elementary symmetric functions in x, y, z). It is straightforward to verify that for any x, y, z,

$$S_n = \sigma_1 S_{n-1} - \sigma_2 S_{n-2} + \sigma_3 S_{n-3}.$$

In our case, this gives

$$S_n \quad = \quad -\sigma_2 S_{n-2} + S_{n-3}. \tag{1}$$

We have $S_0 = 3$, $S_1 = \sigma_1 = 0$, and $S_2 = \sigma_1^2 - 2\sigma_2 = -2\sigma_2$. By recurrence (1), we have $S_3 = -\sigma_2 S_1 + S_0 = 0 + 3 = 3$.

We claim that for $n > 3$, S_n is never constant. We first note that σ_2 is not constant. The values $x = -1, y = (1+\sqrt{5})/2, z = (1-\sqrt{5})/2$ satisfy the conditions of the problem and here $\sigma_2 = -2$. On the other hand, $x = 2, y = (-2+\sqrt{2})/2, z = (-2-\sqrt{2})/2$ also satisfy the conditions, but $\sigma_2 = -7/2$. Since σ_2 is not constant, the following lemma will suffice to prove our claim.

Lemma. For $k \geq 1$, S_{2k} is a polynomial of degree k in σ_2 with leading coefficient $(-1)^k \cdot 2$ and S_{2k+1} is a polynomial of degree $k-1$ in σ_2 with leading coefficient $(-1)^{k-1}(2k+1)$.

Proof. We have $S_2 = -2\sigma_2$ and $S_3 = 3$, so the result follows when $k = 1$. Assuming the result holds for all $k < N$, we have

$$S_{2N} = -\sigma_2 S_{2N-2} + S_{2N-3}.$$

Since by the induction hypothesis, S_{2N-2} is polynomial in σ_2 of degree $N-1$ with leading coefficient $(-1)^{N-1} \cdot 2$ and S_{2N-3} is a polynomial in σ_2 of degree $N-2$, S_{2N} is a polynomial of degree N with leading coefficient $(-1)(-1)^{N-1} \cdot 2 = (-1)^N \cdot 2$. Similarly,

$$S_{2N+1} = -\sigma_2 S_{2N-1} + S_{2N-2}$$

with $-\sigma_2 S_{2N-1}$ being a polynomial of degree $N-1$ with leading coefficient

$$-(-1)^{N-2}(2N-1) = (-1)^{N-1}(2N-1)$$

and S_{2N-2} is a polynomial of degree $N-1$ with leading coefficient $(-1)^{N-1} \cdot 2$, so their sum is a polynomial of degree $N-1$ with leading coefficient

$$(-1)^{N-1}(2N-1) + (-1)^{N-1} \cdot 2 = (-1)^{N-1}(2N+1).$$

OC193. Let $\{a_n\}$ be a positive integer sequence such that $a_{i+2} = a_{i+1} + a_i$ for all $i \geq 1$. For positive integer n, define $\{b_n\}$ as

$$b_n = \frac{1}{a_{2n+1}} \sum_{i=1}^{4n-2} a_i.$$

Prove that b_n is a positive integer, and find the general form of b_n.

Originally problem 4 from day 1 of the 2013 Korea National Olympiad.

We present the solution by Ángel Plaza. There were no other submissions.

From the definition of $\{a_n\}$ it is deduced that $a_n = F_{n-2}a_1 + F_{n-1}a_2$ for $n > 2$ where F_n is the n-th Fibonacci number beginning with $F_1 = F_2 = 1$. This fact can be proved easily by induction.

Then, using well known facts about Fibonacci numbers,

$$\sum_{i=1}^{4n-2} a_i = \left(1 + \sum_{i=1}^{4n-4} F_i\right) a_1 + \left(\sum_{i=1}^{4n-3} F_i\right) a_2 = F_{4n-2}a_1 + (F_{4n-1} - 1)a_2.$$

On the other hand, $a_{2n+1} = F_{2n-1}a_1 + F_{2n}a_2$. Finally, since $L_{2n-1}F_{2n-1} = F_{4n-2}$ and $L_{2n-1}F_{2n} = F_{4n-1} - 1$, if follows that $b_n = L_{2n-1}$, where L_n is the n-th Lucas number beginning with $L_1 = 2$ and $L_2 = 1$ and the problem is done.

OC194. Let $\mathbb{Q}^+$ be the set of all positive rational numbers. Let $f : \mathbb{Q}^+ \to \mathbb{R}$ be a function satisfying the following three conditions:

1. for all $x, y \in \mathbb{Q}^+$, $f(x)f(y) \geq f(xy)$;
2. for all $x, y \in \mathbb{Q}^+$, $f(x + y) \geq f(x) + f(y)$;
3. there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}^+$.

Originally problem 5 from day 2 of the 2013 International Mathematical Olympiad.

There were no submitted solutions.

OC195. Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

Originally problem 5 from the 2013 Canadian Mathematical Olympiad.

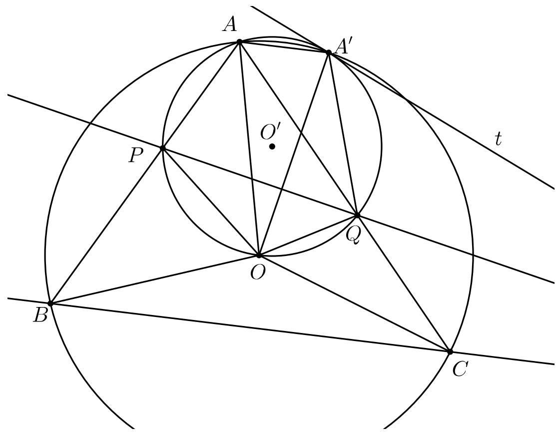
We received two correct submissions. We present the solution by Michel Bataille.

Let Γ and Γ' be the circumcircles of ΔABC and ΔAPQ, respectively. Let O' be the centre of Γ' and let $\alpha = \angle BAC, \beta = \angle CBA$ and $\gamma = \angle BCA$. Since ΔABC is acute-angled, $\angle BOC = 2\alpha$ and so

$$\angle POQ = 360^\circ - 2\alpha - \beta - \gamma = 180^\circ - \alpha.$$

It follows that O lies on Γ'.

Let A' be the second point of intersection of Γ and Γ'. From Focus On... No 12, the spiral similarity with centre A' transforming Γ into Γ' transforms B into P and C into Q and so $\angle(BC, PQ) = \angle(A'O, A'O')$. (Here and in what follows, $\angle(\ell, \ell')$ denotes the directed angle of lines from the line ℓ to the line ℓ' and we suppose that the orientation is such that $\angle(AB, AC) = \alpha, \angle(BC, BA) = \beta, \angle(CA, CB) = \gamma$.)



Assume that $A'O \perp PQ$ has been proved. Then, since the tangent t to Γ' at A' is perpendicular to $A'O'$, we have $\angle(A'O, A'O') = \angle(PQ, t)$. It follows that $\angle(BC, PQ) = \angle(PQ, t)$, hence the reflection of BC in PQ is t. Therefore, it is sufficient to prove the assumption $A'O \perp PQ$.

We observe that $OB = OA$ and $\angle(BC, BO) = 90^\circ - \alpha$, hence

$$\angle(BO, BA) = \angle(AB, AO) = \angle(AP, AO) = \beta - (90^\circ - \alpha) = 90^\circ - \gamma.$$

Similarly,

$$\angle(CA, CO) = \angle(CQ, CO) = \angle(AO, AQ) = 90^\circ - \beta.$$

Since O, A, A' and Q are concyclic, we obtain

$$\angle(A'O, A'Q) = \angle(AO, AQ) = \angle(CQ, CO).$$

Thus, the circle Γ' is the locus of all points M such that $\angle(MO, MQ) = 90^\circ - \beta$ and since $\angle(CO, CQ) = -\angle(AO, AQ)$, C is on the reflection of Γ' in the line OQ. Since Γ is its own reflection in its diameter OQ, the reflection of C in OQ is on Γ and Γ', hence is A' (not A, unless $CQ \perp OQ$ but then $\gamma = \angle COQ = 90^\circ - \angle OCQ = \beta$, hence $AB = AC$, in which case it is easily seen that $A = A'$). As a result, $\angle(OQ, OA') = \angle(OC, OQ) = \gamma$ and so

$$\begin{aligned}\angle(PQ, A'O) &= \angle(QP, QO) + \angle(OQ, OA')\\ &= \angle(AP, AO) + \angle(OC, OQ)\\ &= 90^\circ - \gamma + \gamma = 90^\circ.\end{aligned}$$

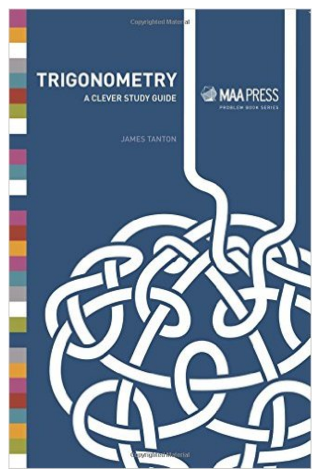
and we are done.

BOOK REVIEWS

Robert Bilinski

As a first for my present tenure as book reviewer for ***Crux***, I present multiple books in this column. The two books are radically different, one appealing to the brain and the other the heart. What links them is that I see both as possible gifts.

Trigonometry: A clever study guide by James Tanton
ISBN 978-088385-836-3, 212 pages
Published by MAA Press: Problem book series, 2015



James Tanton is an education consultant, who used to teach in high school and university. He worked on several outreach programs for the MAA and serves as an ambassador for the MAA in Washington D.C. This is his third book for the MAA.

Trigonometry problems in the book are split in 20 categories, each occupying its own chapter, each following the same pattern : a list of Common Core State Standards covered in the chapter, a brief synopsis of the chapter with, where possible, some historical notes, a simple calculation example, a worked out example with comments and, in most cases, a series of problems to solve. All the proposed problems come from either the American Mathematics Competitions (AMC) or American High School Mathematics Examination (AHSME) contests ranging from 1952 to 2013, but most coming from the 1980–2010 period. The 20 chapters take up about three quarters of the book and cover mostly classical sections of a standard trigonometry course from Chapter 1 "The Pythagorean Theorem" to Chapter 17 "Heron's formula". The last three chapters stand out for several reasons : their originality and the fact they do not come with practise problems. This seems to be at odds with the aim of the book, which is to train the reader for math competitions, but they do round out the knowledge a true mathlete should have about the subject. The first of these is "Fitting trigonometric functions to periodic data". In my opinion, the name of the chapter is misleading as you might think that this is a chapter on statistics, namely trigonometric regression, and you would be wrong. The following non-trivial and fun problem illustrates the chapter well:

> For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$? (#24, AMC 12A, 2007)

Not an easy cookie to crack, but far from the only interesting math morsel in the book! Sadly, there are no practise problems of the same kind to hone one's skills. The last two chapters, "Polar Coordinates" and "Polar Graphs", are even sparser, but they are clearly labelled as extra chapters. After all, they do not correspond to any Common Core State Standards, but maybe, that is the mistake.

The book is a small paperback and can easily be carried everywhere, even to that coffeeshop around the corner where you can dabble in math problems while sipping a latte. The book is abundantly illustrated and the problems are definitely worthy of the attention of ***Crux*** readers. Here are two more:

> All three vertices of an equilateral triangle lie on a parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of m/n, where m and n are relatively prime positive integers. What is the value of $m + n$? (#24, AMC 12B, 2005)
>
> Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52 and 60 taken consecutively. What is the diameter of this circle? (#25, AHSME, 1972)

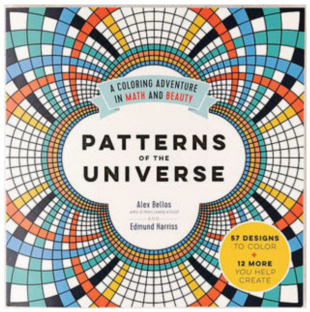
The book is clearly not a standalone resource on the subject and has no real introductory level exercises. The reader must have a basic comprehension of basic trigonometry and can use this book as a further study guide. An interesting feature is the presence of links to webpages, although I must admit I did not explore them. The last 50 pages of the book are filled with complete solutions to the problems, which are labelled from 1 to 100 in the order they appear and independent of the chapter they are in. This provides for an easy way to find the solutions. A little warning : the solutions proposed in the book are bareboned. But if you are working on a solution, it should be enough to see if you are right or help you find your mistake if you took the same approach. At the end, the author also presents a 10-step strategy to conquer math contest problems, though I do not really see its use since the book itself is aimed at problem solvers who should already have developed and practiced these strategies.

This book has a well defined readership, namely all prospective mathletes and various honours students who need extra stimulation to keep them interested. ***Crux*** readers and problem solvers of all ages and strengths will also find this book interesting. I hear a lot of people reminisce about the good old days when they had time to play around with math, especially trigonometry. Maybe this book might make a good gift for that nostalgic engineer friend of yours? In any case, good reading!

Patterns of the universe : a coloring adventure in math and beauty by Alex Bellos and Edmund Harriss
ISBN 978-1-61519-323-3, 144 pages
Published by The experiment publishing, 2015

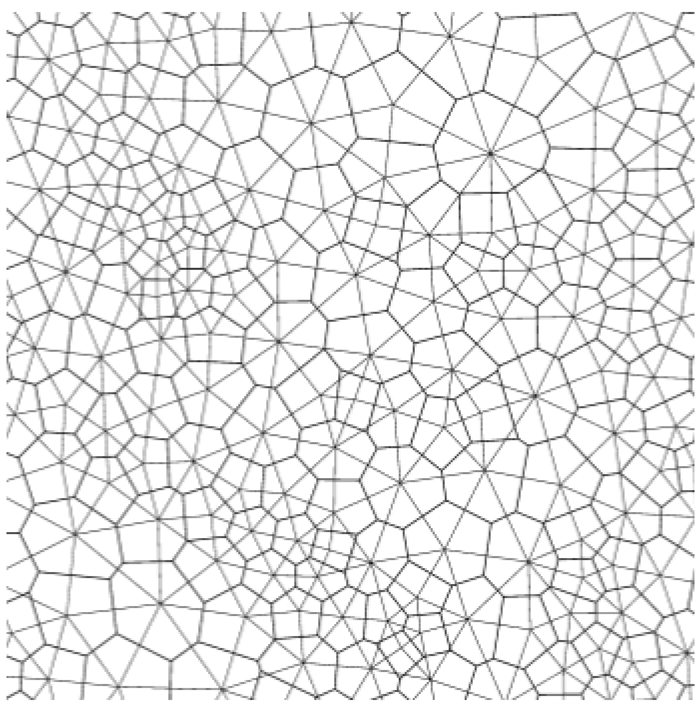
Alex Bellos wears many hats : he is math and puzzles blogger for *The Guardian*, he is a science presenter on the BBC and he has written a few mathematics general interest books. Edmund Harriss is a mathematics professor at the University of

Arkansas who actively participated in many math outreach programs. Together, they created the most mathematical colouring I've ever seen.



This book is made to stimulate artistic beauty and creativity while subtly enticing mathematical curiosity. Each drawing is accompanied by a small description of the the mathematical object to be drawn. The book is separated into 2 sections, which are further divided into subcategories : coloring (Voronoi diagrams, transformations, fractals, periodic tiling, non-periodic tiling, knots, mechanical curves, polyhedra, proportions, space-filling curves, ...) and creating (randomness, latin squares, cellular automata, ...).

Naturally, any artistically inclined mathematician would be interested in such a book. It would make a good gift for artists who say they were "never good at math". Maybe, they would be surprised how much of mathematics has been discovered by artists or how much good artistic representations helped advance mathematics. Good reading!



Colour the above Voronoi diagram (image is from *Patterns of the universe : a coloring adventure in math and beauty* courtesy of Amazon).

FOCUS ON...

No. 19

Michel Bataille

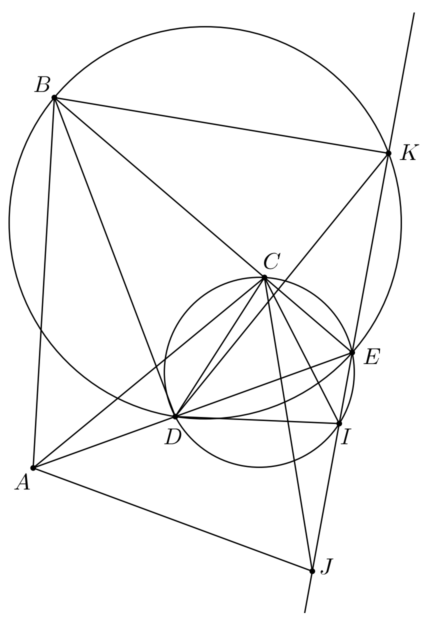
Solutions to Exercises from Focus On... No. 12 – 16

From Focus On... No. 12

Let $ABCD$ be a quadrilateral such that AD and BC intersect at E. Suppose that $ID = IC, JA = JC, KB = KD$ and $\angle(\overrightarrow{ID}, \overrightarrow{IC}) = \angle(\overrightarrow{JA}, \overrightarrow{JC}) = \angle(\overrightarrow{KD}, \overrightarrow{KB}) = \angle(\overrightarrow{EA}, \overrightarrow{EB})$. Show that E, I, J, K are collinear.

If $E = I = J$, we are done, so we may suppose that E is distinct from I or J, say $E \neq I$. From the hypothesis about the angles of the figure, E lies on the circles (DIC), (AJC) and (BKD). Let σ be the spiral similarity with centre D transforming the circle (DKB) into the circle (DIC). Since these two circles intersect at D and E and B, C, E are collinear, we have $\sigma(B) = C$.

Now, ΔBKD and ΔCID are isosceles at K and I, respectively, and $\angle(\overrightarrow{ID}, \overrightarrow{IC}) = \angle(\overrightarrow{KD}, \overrightarrow{KB})$, hence $\angle(\overrightarrow{DB}, \overrightarrow{DK}) = \angle(\overrightarrow{DC}, \overrightarrow{DI})$ and so $\angle(\overrightarrow{DK}, \overrightarrow{DI}) = \angle(\overrightarrow{DB}, \overrightarrow{DC})$. Since K is on (DKB) and I is on (DIC), it follows that $\sigma(K) = I$.



Thus, E, K, I are collinear. In a similar way, we obtain that E, J, I are collinear and conclude that J, K are on the line EI.

From Focus On... No. 13

(a) *Let the angle bisectors of triangle ABC meet its sides at D, E, F. Show that ΔDEF is right-angled if and only if one of the angles of ΔABC equals 120°.*

We shall use the familiar notations for the sides and the angles of ΔABC and we suppose that D, E, F are on sides BC, CA, AB, respectively. We show that ΔDEF is right-angled at D if and only if $A = 120^\circ$.

We know that $(b+c)D = bB + cC$, $\quad (c+a)E = aA + cC$, $\quad (a+b)F = aA + bB$, and deduce the following vectorial relations:

$$(a+b)(b+c)\overrightarrow{FD} = b(a-c)\overrightarrow{AB} + c(a+b)\overrightarrow{AC}$$

$$(a+c)(b+c)\overrightarrow{ED} = b(a+c)\overrightarrow{AB} + c(a-b)\overrightarrow{AC}.$$

Now, ΔDEF is right-angled at D if and only if $\overrightarrow{FD} \cdot \overrightarrow{ED} = 0$, which is successively equivalent to

$$b^2c^2(a^2-c^2) + b^2c^2(a^2-b^2) + bc(\overrightarrow{AB} \cdot \overrightarrow{AC})((a-c)(a-b) + (a+c)(a+b)) = 0,$$
$$2a^2 - b^2 - c^2 + 2(a^2+bc)\cos A = 0,$$
$$1 + 2\cos A = 0 \qquad (\text{since} \quad b^2 + c^2 = a^2 + 2bc\cos A),$$
$$A = 120^\circ.$$

(b) *We have n distinct points $A_1, \ldots, A_n$ in the plane. To each point A_i a real number $\lambda_i \neq 0$ is assigned, in such a way that $A_iA_j^2 = \lambda_i + \lambda_j$ for all i, j with $i \neq j$. Show that $n \leq 4$ and that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 0$ if $n = 4$.*

Assume that $n \geq 5$. For distinct i, j, k we have

$$\begin{aligned}\lambda_j + \lambda_k = A_jA_k^2 = (\overrightarrow{A_jA_i} + \overrightarrow{A_iA_k})^2 &= A_jA_i^2 + A_iA_k^2 + 2\overrightarrow{A_jA_i} \cdot \overrightarrow{A_iA_k} \\ &= \lambda_j + \lambda_i + \lambda_i + \lambda_k + 2\overrightarrow{A_jA_i} \cdot \overrightarrow{A_iA_k}\end{aligned}$$

and so $\lambda_i = \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k}$.

From $\lambda_1 = \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3} = \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_4} = \overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_5}$, we deduce (by difference)

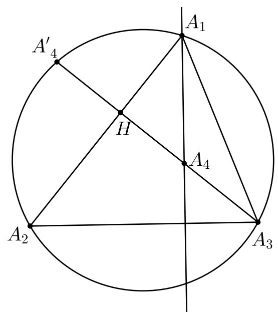
$$\overrightarrow{A_1A_2} \cdot \overrightarrow{A_4A_3} = \overrightarrow{A_1A_2} \cdot \overrightarrow{A_5A_3} = 0.$$

Therefore the points A_3, A_4, A_5 lie on a perpendicular to the line A_1A_2. In the same way, A_1, A_2, A_3 must lie on a perpendicular to A_4A_5 and consequently A_3 must be the point of intersection of the lines A_1A_2 and A_4A_5. However, this implies $A_1A_4^2 = A_1A_3^2 + A_3A_4^2$, which leads to $\lambda_3 = 0$, in contradiction with the hypothesis. As a result, we must have $n \leq 4$.

From now on, we suppose that $n = 4$. As in the first part, we see that $A_2A_3 \perp A_1A_4$, $A_3A_4 \perp A_1A_2$, $A_1A_3 \perp A_2A_4$ and deduce that $A_1A_2A_3A_4$ is formed by the vertices of a triangle together with its orthocenter.

Let H be the point of intersection of A_3A_4 and A_1A_2. Then, $\lambda_1 = \overrightarrow{A_1A_2}\cdot\overrightarrow{A_1H}$ and $\lambda_2 = \overrightarrow{A_1A_2}\cdot\overrightarrow{HA_2}$ (since $\overrightarrow{HA_3}$ is orthogonal to $\overrightarrow{A_1A_2}$). Since A_1, H, A_2 are collinear, this yields $\lambda_1\lambda_2 = A_1A_2^2(\overrightarrow{HA_2}\cdot\overrightarrow{A_1H})$. Similarly, we find $\lambda_3\lambda_4 = A_3A_4^2(\overrightarrow{HA_4}\cdot\overrightarrow{A_3H})$ and so

$$\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2) = A_1A_2^2.A_3A_4^2(\overrightarrow{HA_2}\cdot\overrightarrow{A_1H}+\overrightarrow{HA_4}\cdot\overrightarrow{A_3H}). \quad (1)$$



We may consider A_4 as the orthocenter of $\Delta A_1A_2A_3$ and we know that its reflection A_4' in A_1A_2 is on the circumcircle Γ of $\Delta A_1A_2A_3$. It follows that the power of H with respect to Γ is $\overrightarrow{HA_2}\cdot\overrightarrow{HA_1}$ as well as $\overrightarrow{HA_3}\cdot\overrightarrow{HA_4'} = -\overrightarrow{HA_3}\cdot\overrightarrow{HA_4}$. From (1), $\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2) = 0$, and the desired equality is derived at once.

From Focus On... No. 15

1. *Show that for each integer $n \geq 2$*

$$\sum_{k=1}^{n-1}\binom{n}{k}\frac{(-1)^{k-1}}{k}\left(1-\frac{k}{n}\right)^n = \sum_{k=1}^{n-1}\frac{1}{k+1}.$$

With the help of the binomial theorem, we rewrite the left-hand side L_n as

$$L_n = \sum_{j=0}^{n}\frac{(-1)^j}{n^j}\binom{n}{j}a_j, \qquad \text{where} \qquad a_j = \sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\cdot k^j\binom{n}{k}.$$

Let $H_n = \sum_{k=1}^{n}\frac{1}{k}$ denote the nth harmonic number. Recalling the well-known equality $H_n = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k}$ (easily proved by induction), we see that

$$a_0 = H_n + \frac{(-1)^n}{n}.$$

Then, we calculate a_1 as follows:

$$\begin{aligned} a_1 = \sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k} &= -\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k} - 1 - (-1)^n\right) \\ &= -(0-1-(-1)^n) = 1+(-1)^n. \end{aligned}$$

Lastly, if $2 \leq j \leq n$, we obtain

$$a_j = (-1)^{n+1} \sum_{k=1}^{n-1} (-1)^{n-k} \binom{n}{k} k^{j-1} = (-1)^{n+1} \left(S(n, j-1) - \binom{n}{n} n^{j-1} \right)$$
$$= (-1)^n n^{j-1}.$$

(Note that $S(n, j-1) = 0$ since $1 \leq j-1 < n$.)

Back to L_n, all this leads to

$$\begin{aligned} L_n &= a_0 - \frac{1}{n} \cdot n \cdot a_1 + \sum_{j=2}^{n} \frac{(-1)^j}{n^j} \binom{n}{j} (-1)^n n^{j-1} \\ &= H_n + \frac{(-1)^n}{n} - 1 - (-1)^n + \frac{(-1)^n}{n}(-1+n). \end{aligned}$$

Thus

$$L_n = H_n - 1 = \sum_{k=1}^{n-1} \frac{1}{k+1}.$$

2. *For nonnegative integer n, evaluate in closed form*

$$\sum_{k=0}^{n} \frac{(-1)^k}{(k!)^2} \cdot \frac{(n+k+2)!}{(n-k)!}.$$

Let S_n be the given sum. We show that $S_n = \frac{(-1)^n}{2} \cdot (n+1)^2(n+2)^2$.

Consider $T_n = \frac{(-1)^n S_n}{(n+1)(n+2)}$. We calculate:

$$\begin{aligned} T_n &= \frac{(-1)^n (S_n) n!}{(n+2)!} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{n+k+2}{n+2} \\ &= \frac{1}{(n+2)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k+n+2)(k+n+1) \cdots (k+2)(k+1) \\ &= \frac{1}{(n+2)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k^{n+2} + ak^{n+1} + bk^n + P(k)) \end{aligned}$$

where $P(x)$ is a polynomial whose degree is less than n, $a = \sum_{j=1}^{n+2} j = \frac{(n+2)(n+3)}{2}$ and

$$b = \sum_{1 \leq i < j \leq n+2} i \cdot j = \frac{1}{2} \left(\left(\sum_{j=1}^{n+2} j \right)^2 - \sum_{j=1}^{n+2} j^2 \right) = \frac{(n+1)(n+2)(n+3)(3n+8)}{24}$$

using $\sum_{j=1}^{n+2} j^2 = \frac{(n+2)(n+3)(2n+5)}{6}$.
It follows that

$$T_n = \frac{1}{(n+2)!}\left(S(n, n+2) + aS(n, n+1) + bS(n, n)\right).$$

Now, $S(n, n+1) = \frac{n(n+1)!}{2}$ (as seen in the number) and in a similar way, using

$$x^n(1 + \frac{x}{2} + \frac{x^2}{6} + \cdots)^n = x^n + \frac{n}{2}x^{n+1} + n \cdot \frac{x^{n+2}}{6} + \frac{n(n-1)}{2} \cdot \frac{x^{n+2}}{4} + \cdots,$$

we readily find $S(n, n+2) = \frac{n(3n+1)(n+2)!}{24}$.
Finally, T_n is equal to

$$\frac{1}{(n+2)!}\left(\frac{n(3n+1)(n+2)!}{24} + \frac{(n+2)(n+3)}{2} \cdot \frac{n(n+1)!}{2} + \frac{(n+1)(n+2)(n+3)(3n+8)}{24} \cdot (n!)\right)$$

that is, to

$$\frac{(n+1)(n+2)}{2}$$

and the announced value of S_n follows.

From Focus On... No. 16

1. *Let P be an arbitrary point in the plane of a triangle ABC with sidelengths a, b, c. Prove that*

$$PA^2 + PB^2 + PC^2 \geq \frac{a^2 + b^2 + c^2}{3}.$$

Let G be the centroid of ΔABC. Leibniz's relation gives

$$PA^2 + PB^2 + PC^2 = 3PG^2 + GA^2 + GB^2 + GC^2. \tag{2}$$

Now, if m_a, m_b, m_c denote the medians from A, B, C, respectively, we have

$$\begin{aligned} GA^2 + GB^2 + GC^2 &= \frac{4}{9} \cdot (m_a^2 + m_b^2 + m_c^2) \\ &= \frac{1}{9} \cdot (2b^2 + 2c^2 - a^2 + 2c^2 + 2a^2 - b^2 + 2a^2 + 2b^2 - c^2) \\ &= \frac{a^2 + b^2 + c^2}{3}. \end{aligned}$$

From (2), we then deduce $PA^2 + PB^2 + PC^2 \geq \frac{a^2+b^2+c^2}{3}$ (with equality if and only if $P = G$).

2. *Let A, B, C, D be four points on a line ℓ in this order and let M not on ℓ be such that $\angle AMB = \angle CMD$. Prove that*

$$\frac{MA^2}{MC^2} > \frac{AB}{CD} > \frac{MB^2}{MD^2}.$$

Let us apply Stewart's relation to points A, B, D and M:

$$\overline{AB} \cdot MD^2 + \overline{BD} \cdot MA^2 + \overline{DA} \cdot MB^2 + \overline{AB} \cdot \overline{BD} \cdot \overline{DA} = 0.$$

It follows that $AB \cdot MD^2 + BD \cdot MA^2 - AD \cdot MB^2 = AB \cdot BD \cdot AD > 0$, or, since $AD = AC + CD$,

$$(AB \cdot MD^2 - CD \cdot MB^2) + (BD \cdot MA^2 - AC \cdot MB^2) > 0. \tag{3}$$

The inequality $\frac{MB^2}{MD^2} < \frac{AB}{CD}$ clearly holds if $AB \cdot MD^2 - CD \cdot MB^2 > 0$. But otherwise, from (3) we must have

$$BD \cdot MA^2 - AC \cdot MB^2 > 0 \tag{4}$$

However, since $\angle AMB = \angle CMD$, we have

$$\frac{MA \cdot MB}{MC \cdot MD} = \frac{AB}{CD} \left(= \frac{\text{area}(MAB)}{\text{area}(MCD)} \right) \quad \text{and} \quad \frac{MA \cdot MC}{MB \cdot MD} = \frac{AC}{BD}$$

hence $MA^2 = MD^2 \cdot \frac{AB \cdot AC}{CD \cdot BD}$. Inequality (4) now yields

$$MD^2 \cdot \frac{AB \cdot AC}{CD} - AC \cdot MB^2 > 0$$

and again $\frac{MB^2}{MD^2} < \frac{AB}{CD}$.

Further, since

$$\frac{MA \cdot MB}{MC \cdot MD} \cdot \frac{MB \cdot MD}{MA \cdot MC} = \frac{AB}{CD} \cdot \frac{BD}{AC},$$

we have

$$\frac{MB^2}{MC^2} = \frac{AB \cdot BD}{CD \cdot AC}$$

and the inequality $MD^2 > MB^2 \cdot \frac{CD}{AB}$ already obtained gives

$$MA^2 \cdot \frac{CD \cdot BD}{AB \cdot AC} > \frac{CD}{AB} \cdot \frac{AB \cdot BD}{CD \cdot AC} \cdot MC^2.$$

The desired inequality $MA^2 > MC^2 \cdot \frac{AB}{CD}$ follows.

Alternately, the latter inequality can be obtained from Stewart's relation applied to A, C, D and M in the same way as above. The details are left to the reader.

A Mathematical Performance

Hee-Joo Nam, Giavanna Valacco and Ling-Feng Zhu

Two of us girls were members of the *S.M.A.R.T.* Circle in Edmonton. The acronym stands for *Saturday Mathematical Activities, Recreations & Tutorials*. In 2010, the Circle sent two teams of four to the Junior High Division of the *International Mathematics Competition (IMC)*. Five Circle members, reinforced by three students elsewhere from Alberta, went to the host city Iuncheon for the event. Hee-Joo, the third girl, came from Calgary. She was particularly excited since she was originally from South Korea. We had the highest percentage of female representation in the history of the competition, except that Iran, not fully understanding the rules in their first year of participation, sent nine all-girl teams!

The individual and team contest papers are given in the second part of this article, which will appear in Volume 41, issue 10 of ***Crux***. For further details, see their website `http://www.imc-official.org/en\_US/`; for solutions, see the book [1]. Our lack of international experience showed in that Giavanna was the only one who managed a Bronze Medal in the individual contest.

Apart from academic matters, the *I.M.C.* emphasized international friendship and understanding. The most wonderful feature was the Cultural Evening, when teams from various countries gave short performances. These are usually in the form of songs and dances, but there had been innovative presentations as well.

The activity we chose was *Platonic Metamorphosis*. We used six coloured strings to construct the skeleton of each of the five Platonic solids, in six steps in a continuous transformation. The clever idea came from Karl Schaffer. See his paper [2]. However, we made our own adaptation. Ten students were required, preferably in a six-to-four gender mix. Fortunately, Giavanna's younger brother and sister came along on a family holiday, and we had the perfect combination.

The official languages for the *I.M.C.* were English and Chinese, along with Korean for 2010. During our performace, narration was done by Giavanna in English, Ling-Feng in Chinese and Hee-Joo in Korean. This was extremely well-received by the audience.

Step 1. Construction of the Tetrahedron

Start off with the four girls identified as N(orth), S(outh), E(ast) and W(est). Each designates one hand as the U(pper) hand and the other hand as the L(ower) hand. N and S hold out their U hands while E and W hold out their L hands. String 1 is held between UN and LW, string 2 between UN and LE, string 3 between LW and LE, string 4 between LW and US, string 5 between LE and US, and string 6 between UN and US. The completed tetrahedron is shown in Figure 1, with string 6 drawn in such a way to facilitate the description of the next step.

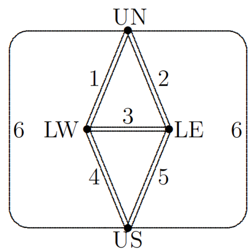


Figure 1

Step 2. Transformation into the Cube

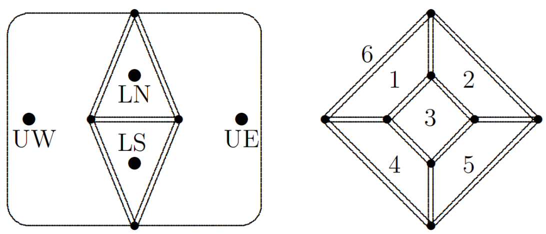


Figure 2

Each of the four girls holds out the other hand and places it at the center of one of the four faces of the tetrahedron, as shown on the left side of Figure 2. Each of these hands will grab the three sides of the triangular face. The end result is a cube, as shown on the right side of Figure 2. Each string forms a face of the cube.

Step 3. Transformation into the Dodecahedron

First, the cube is redrawn as shown in Figure 3.

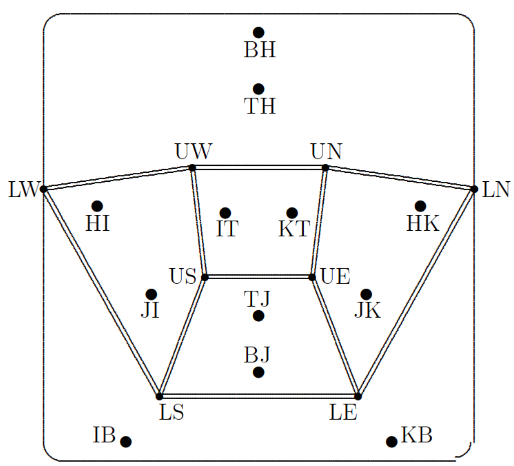


Figure 3

Now the six boys enter the picture. They are identified as T(op face), B(ottom face), H (northwest face), I (southwest face), J (southeast face) and K (northeast face). Each of them holds out both hands and places them symmetrically about the center of the assigned face of the cube. The line segment joining the two hands of each student is parallel to a side of the cube, and the segments on adjacent faces are perpendicular to each other.

Each pair of these hands will grab the two sides of the square face parallel to the segment they form. Each hand will also grab the nearer one of the remaining two sides of the square face. **It should be emphasized that while each face of the cube is formed of one string, no part of this string is to be grabbed by the hands assigned to this face.** Instead, the other four strings joining adjacent pairs of vertices of the face are grabbed, as illustrated in Figure 4.

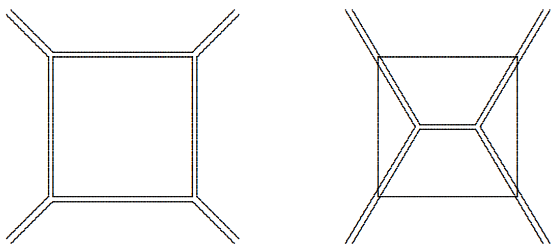


Figure 4

The end result is a dodecahedron, as shown in Figure 5. Failure to observe the caution in the preceding paragraph will still produce a dodecahedron, but the whole structure will then fall apart in Step 4.

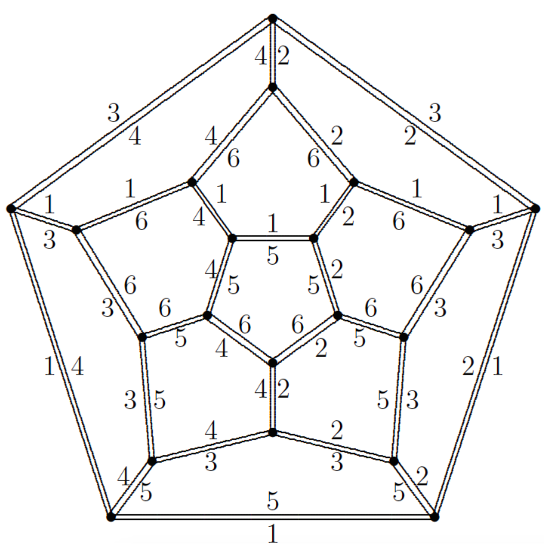


Figure 5

Step 4. Transformation into the Icosahedron

The four girls let go of their strings. The end result is an icosahedron, as shown in Figure 6.

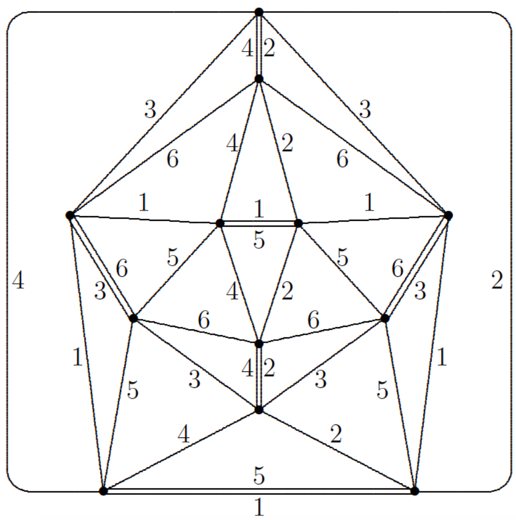


Figure 6

Step 5. Transformation into the Octahedron

Each of the six boys slides both hands together. The end result is an octahedron, as shown in Figure 7. Two strings which are opposite sides of the original tetrahedron now form the same square cross-section of the octahedron.

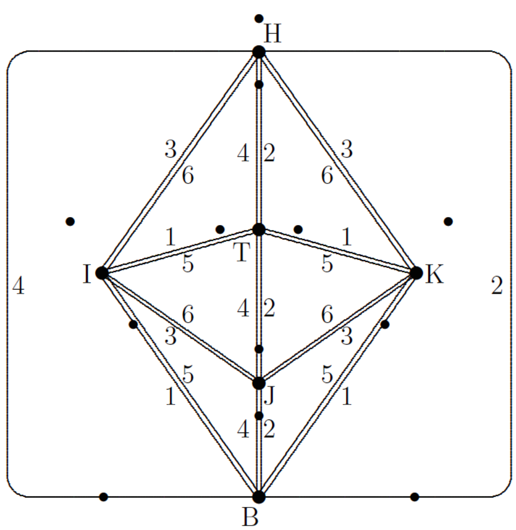


Figure 7

Step 6. Return to the Tetrahedron

The four girls N, S, E and W re-enter the picture. N puts the U hand in triangle HKT (north and top), S puts the U hand in triangle IJT (south and top), E puts the L hand in triangle JKB (bottom and east) and W puts the L hand in triangle HIB (bottom and west). This is shown in Figure 8.

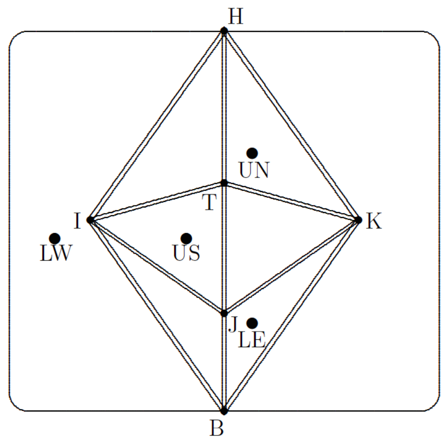


Figure 8

Each hands grabs the three strings it originally holds, and then the six boys let go of theirs. The end result is once again a tetrahedron, as shown in Figure 9.

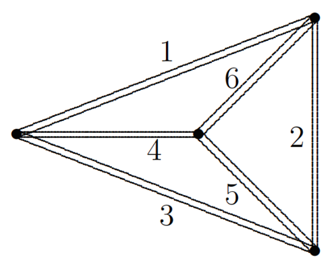


Figure 9

Bibliography:

[1] Wen-Hsien Sun, Huan Zheng and Hua-Wei Zhu, *An Indepth Study of the International Mathematics Competition*, Chiu Chang Math. Publ., Taipei, (2014) 204–211, 325–333.

[2] Karl Schaffer, A Platonic sextet in strings, *College Mathematics Journal*, **43** (2012) 64–69.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **November 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

4081. *Proposed by Daniel Sitaru.*

Determine all $A, B \in M_2(\mathbb{R})$ such that:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix}, \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix}. \end{cases}$$

4082. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a right-angle triangle with $\angle A = 90^\circ$ and $BC = a$, $AC = b$ and $AB = c$. Consider the Fibonacci sequence F_n with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all non-negative integers n. Prove that

$$\frac{F_m^2}{(bF_n + cF_p)^2} + \frac{F_n^2}{(bF_p + cF_m)^2} + \frac{F_p^2}{(bF_m + cF_n)^2} \geq \frac{3}{2a^2}$$

or all non-negative integers m, n, p.

4083. *Proposed by Ovidiu Furdui.*

Calculate

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \int_0^n \frac{x}{1 + n\cos^2 x} \mathrm{d}x.$$

4084. *Proposed by Michel Bataille.*

In the plane, let Γ be a circle and A, B be two points not on Γ. Determine when $\frac{MA}{MB}$ is not independent of M on Γ and, in these cases, construct with ruler and compass I and S on Γ such that

$$\frac{IA}{IB} = \inf\left\{\frac{MA}{MB} : M \in \Gamma\right\} \quad \text{and} \quad \frac{SA}{SB} = \sup\left\{\frac{MA}{MB} : M \in \Gamma\right\}.$$

4085. *Proposed by José Luis Díaz-Barrero. Correction.*

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

4086. *Proposed by Daniel Sitaru.*

Let be $f:[0,1]\to\mathbb{R}$; f twice differentiable on $[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$. Prove that

$$25\int_{\frac{1}{5}}^{1} f(x)dx\geq 16\int_{0}^{1} f(x)dx+4f(1).$$

4087. *Proposed by Lorian Saceanu.*

If S is the area of triangle ABC, prove that

$$m_a(b+c)+2m_a^2\geq 4S\sin A,$$

where b and c are the lengths of sides that meet in vertex A, and m_a is the length of the median from that vertex; furthermore, equality holds if and only if $b=c$ and $\angle A=120^\circ$.

4088. *Proposed by Ardak Mirzakhmedov.*

Let a,b and c be positive real numbers such that $a^2b+b^2c+c^2a+a^2b^2c^2=4$. Prove that

$$a^2+b^2+c^2+abc(a+b+c)\geq 2(ab+bc+ca).$$

4089. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a,b,c and d be real numbers with $0<a<b<c<d$. Prove that

$$\frac{b}{a}+\frac{c}{b}+\frac{d}{c}>3+\ln\frac{d}{a}.$$

4090. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a,b and c be non-negative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{3}{8}.$$

. . . .

4081. *Proposé par Daniel Sitaru.*

Déterminer toutes les matrices $A, B \in M_2(\mathbb{R})$ telles que:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix}, \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix}. \end{cases}$$

4082. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit ABC un triangle rectangle tel que $\angle A = 90^\circ$, $BC = a$, $AC = b$ et $AB = c$. Soit F_n la suite de Fibonacci définie par $F_0 = F_1 = 1$ et $F_{n+2} = F_{n+1} + F_n$ pour tout entier non négatif n. Démontrer que

$$\frac{F_m^2}{(bF_n + cF_p)^2} + \frac{F_n^2}{(bF_p + cF_m)^2} + \frac{F_p^2}{(bF_m + cF_n)^2} \geq \frac{3}{2a^2}$$

pour tout entier non négatif m, n et p.

4083. *Proposé par Ovidiu Furdui.*

Évaluer

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \int_0^n \frac{x}{1 + n\cos^2 x} \mathrm{d}x.$$

4084. *Proposé par Michel Bataille.*

Dans le plan, soit un cercle Γ et deux points, A et B, qui ne sont pas sur Γ. Déterminer les conditions qui font que $\frac{MA}{MB}$ n'est pas indépendant d'un point M sur Γ. De plus, construire avec compas et règle deux points I et S sur Γ tels que

$$\frac{IA}{IB} = \inf\left\{\frac{MA}{MB} : M \in \Gamma\right\} \quad \text{et} \quad \frac{SA}{SB} = \sup\left\{\frac{MA}{MB} : M \in \Gamma\right\}.$$

4085. *Proposé par José Luis Díaz-Barrero. Correction.*

Soit ABC un triangle acutangle. Démontrer que

$$\sqrt[4]{\sin(\cos A) \cdot \cos B} + \sqrt[4]{\sin(\cos B) \cdot \cos C} + \sqrt[4]{\sin(\cos C) \cdot \cos A} < \frac{3\sqrt{2}}{2}.$$

4086. *Proposé par Daniel Sitaru.*

Soit une fonction $f : [0,1] \to \mathbb{R}$ qui est deux fois dérivable sur $[0,1]$ et telle que $f''(x) < 0$ pour tout $x \in [0,1]$. Démontrer que

$$25 \int_{\frac{1}{5}}^1 f(x)dx \geq 16 \int_0^1 f(x)dx + 4f(1).$$

4087. *Proposé par Lorian Saceanu.*

Soit S l'aire d'un triangle ABC, $b = AC$, $c = AB$ et m_a la longueur de la médiane issue de A. Démontrer que

$$m_a(b + c) + 2m_a^2 \geq 4S \sin A$$

et démontrer qu'il y a égalité si et seulement si $b = c$ et $\angle A = 120°$.

4088. *Proposé par Ardak Mirzakhmedov.*

Soit a, b et c des réels strictement positifs tels que $a^2b + b^2c + c^2a + a^2b^2c^2 = 4$. Démontrer que

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq 2(ab + bc + ca).$$

4089. *Proposé par Daniel Sitaru et Leonard Giugiuc.*

Soit a, b, c et d des réels tels que $0 < a < b < c < d$. Démontrer que

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > 3 + \ln \frac{d}{a}.$$

4090. *Proposé par Nermin Hodžić et Salem Malikić.*

Soit a, b et c des réels non négatifs tels que $a^2 + b^2 + c^2 = 3$. Démontrer que

$$\frac{a}{3b^2 + 6c - bc} + \frac{b}{3c^2 + 6a - ca} + \frac{c}{3a^2 + 6b - ab} \geq \frac{3}{8}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014: 40(9), p. 391–394.

3981. *Proposed by José Luis Díaz-Barrero.*

Let a, b, c be three positive numbers such that $ab+bc+ca = 6abc$. For all positive integers $n \geq 2$, show that

$$\frac{bc}{a^n(b+c)} + \frac{ca}{b^n(c+a)} + \frac{ab}{c^n(a+b)} \geq 3 \cdot 2^{n-2}.$$

We received 16 submissions, of which 15 were correct and complete. We present the solution by Wolfgang Remmel, slightly modified by the editor.

We make the substitutions $x = \frac{1}{a}$, $y = \frac{1}{b}$ and $z = \frac{1}{c}$. Dividing both sides of $ab + bc + ac = 6abc$ by abc we get the condition $x + y + z = 6$. For the first summand of the inequality we have

$$\frac{bc}{a^n(b+c)} = \frac{\frac{1}{a^n}}{\frac{b+c}{bc}} = \frac{x^n}{z+y};$$

applying a similar treatment to the other two summands, the inequality we have to prove becomes

$$\frac{x^n}{z+y} + \frac{y^n}{z+x} + \frac{z^n}{x+y} \geq 3 \cdot 2^{n-2}.$$

To start with, by the Cauchy-Schwarz Inequality,

$$\begin{aligned}&2(x+y+z)\cdot\left(\tfrac{z^n}{x+y} + \tfrac{y^n}{x+z} + \tfrac{x^n}{y+z}\right)\\ &= \left((\sqrt{x+y})^2 + (\sqrt{x+z})^2 + (\sqrt{y+z})^2\right)\left(\left(\tfrac{z^{\frac{n}{2}}}{\sqrt{y+x}}\right)^2 + \left(\tfrac{z^{\frac{n}{2}}}{\sqrt{y+x}}\right)^2 + \left(\tfrac{z^{\frac{n}{2}}}{\sqrt{y+x}}\right)^2\right)\\ &\geq \left(z^{\frac{n}{2}} + y^{\frac{n}{2}} + x^{\frac{n}{2}}\right)^2.\end{aligned}$$

Using the fact that $x + y + z = 6$, we can divide both sides by 12 to get

$$\frac{z^n}{x+y} + \frac{y^n}{x+z} + \frac{x^n}{y+z} \geq \frac{1}{12}\cdot\left(z^{\frac{n}{2}} + y^{\frac{n}{2}} + x^{\frac{n}{2}}\right)^2. \tag{1}$$

If $n = 2$ we are done. Else, by Hölder's inequality,

$$\left(z^{\frac{n}{2}} + y^{\frac{n}{2}} + x^{\frac{n}{2}}\right)^{\frac{2}{n}} \cdot (1+1+1)^{\frac{n-2}{n}} \geq z + y + x = 6.$$

Raising both sides to the power n and then dividing by 3^{n-2} gives us

$$\left(z^{\frac{n}{2}} + y^{\frac{n}{2}} + x^{\frac{n}{2}}\right)^2 \geq 2^n \cdot 3^2. \tag{2}$$

Combining (1) and (2) we get

$$\frac{z^n}{x+y} + \frac{y^n}{x+z} + \frac{x^n}{y+z} \geq \frac{1}{12} \cdot (2^n \cdot 3^2) = 3 \cdot 2^{n-2},$$

which is what we needed to prove. Note that equality holds if and only if

$$x = y = z = 2,$$

which in terms of the original variables is

$$a = b = c = \frac{1}{2}.$$

3982. *Proposed by Michel Bataille.*

Let $n \in \mathbb{N}$, $u > 0$ and for $k = 0, 1, \ldots, n-1$, let a_k be such that $0 < a_k \leq \sinh(u)$. Prove that if $x \geq e^u$, then

$$a_{n-1}x^{n-1} - a_{n-2}x^{n-2} + \cdots + (-1)^{n-2}a_1 x + (-1)^{n-1}a_0 < \frac{x^n}{2}.$$

There were 3 submitted solutions for this problem, all of which were correct. We present the solution by Joel Schlosberg.

Since the natural exponential, natural logarithm, and hyperbolic sine functions are strictly increasing, $x \geq e^u > e^0 = 1$ and

$$0 < a_k \leq \sinh(u) \leq \sinh[\ln(x)] = \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{x - x^{-1}}{2}.$$

By the formula for the sum of an infinite geometric series with common ratio $x^{-2} \in (0, 1)$,

$$\begin{aligned} a_{n-1}x^{n-1} - a_{n-2}x^{n-2} &+ \cdots + (-1)^{n-2}a_1 x + (-1)^{n-1}a_0 \\ &< \frac{x - x^{-1}}{2} \cdot x^{n-1} + \frac{x - x^{-1}}{2} \cdot x^{n-3} + \frac{x - x^{-1}}{2} \cdot x^{n-5} + \cdots \\ &= \frac{x - x^{-1}}{2} \cdot \frac{x^{n-1}}{1 - x^{-2}} = \frac{x^n}{2}. \end{aligned}$$

Editor's Comments. The other two solutions involve a little more effort; induction, or a bit of case work between odd and even m. The weakness of the inequality is shown in the featured solution. In fact, the solution shows that if we have a sequence of positive a_i, all of which are less than or equal to $\sinh(u)$, and $x \geq e^u$, then we can bound an infinite series:

$$\sum_{i=1}^{\infty} a_i x^{n+1-2i} \leq \frac{x^n}{2}.$$

3983. *Proposed by Marcel Chiriţă.*

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x^2 - y^2)\max\left(f'(x), f'(y)\right)$$

for all real numbers x, y.

We received three solutions, all of which were correct and complete. We present two solutions.

Solution 1, by Michel Bataille.

It is readily checked that the functions $x \mapsto kx$ ($k \in \mathbb{R}$) are solutions. We show that there are no others. To this aim, let f satisfy the conditions and let $a \neq 0$. We show that $f'(a) \leq f'(0)$. Assume that $f'(0) < f'(a)$. Let $\alpha \in (f'(0), f'(a))$. Since a derivative satisfies the Darboux property, we have $\alpha = f'(u)$ for some $u \in (0, a)$ or $(a, 0)$.

From the condition $xf(x) - yf(y) = (x^2 - y^2)\max(f'(x), f'(y))$, we obtain

$$af(a) = a^2\max(f'(a), f'(0)) = a^2f'(a)$$

and

$$uf(u) = u^2\max(f'(u), f'(0)) = u^2f'(u);$$

that is, $f(a) = af'(a)$ and $f(u) = uf'(u) = \alpha u$. But we also get

$$a^2f'(a) - u^2f'(u) = af(a) - uf(u) = (a^2 - u^2)f'(a)$$

and so $f'(a) = f'(u) = \alpha$, a contradiction with $\alpha \in (f'(0), f'(a))$. It follows that $f'(x) \leq f'(0)$ for any real number x.

Now, the condition gives $xf(x) = x^2f'(0)$ so that $f(x) = kx$ if $x \neq 0$ where $k = f'(0)$. Since f is continuous (even differentiable),

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} kx = 0 = k \cdot 0.$$

Finally, $f(x) = kx$ for any real number x.

Solution 2, by Marcel Chiriţă.

Consider the differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x) - f(0) - xf'(0)$. Clearly $g(0) = 0$; we differentiate to find that $g'(x) = f'(x) - f'(0)$, and so $g'(0) = 0$. We can rewrite the functional equation in terms of g and simplify:

$$\begin{aligned}
x[g(x) + f(0) + xf'(0)] - y[g(y) + f(0) + yf'(0)]& \\
= (x^2 - y^2)\max(g'(x) + f'(0), g'(y) + f'(0))& \\
xg(x) - yg(y) + f(0)(x - y) + (x^2 - y^2)f'(0)& \\
= (x^2 - y^2)\max(g'(x) + f'(0), g'(y) + f'(0))& \\
xg(x) - yg(y) + f(0)(x - y) = (x^2 - y^2)\max(g'(x), g'(y)).&
\end{aligned}$$

If $y = 0$, the above relation reduces to $x(g) + xf(0) = x^2 \max(g'(x), 0)$, which reduces further to $g(x) + f(0) = x \max(g'(x), 0)$ when $x \neq 0$. We take the limit of both sides as x approaches zero:

$$\lim_{x\to 0}(g(x) + f(0)) = \lim_{x\to 0} x \max(g'(x), 0),$$

yielding $g(0) + f(0) = 0$, hence $f(0) = 0$. We then have $g(x) = f(x) - xf'(0)$ and in particular $g(x) = x \max(g'(x), 0)$ for $x \neq 0$.

If $x \geq 0$, then $g(x) \geq 0$ and $g(x) \geq xg'(x)$, or equivalently $g(x) - xg'(x) \geq 0$. On the other hand, if $x \leq 0$, then $g(x) \leq 0$ and $g(x) \leq xg'(x)$, or equivalently $g(x) - xg'(x) \leq 0$.

Define the function $h(x) = \frac{g(x)}{x}$, which is differentiable and non-negative for $x \neq 0$. In fact, $h'(x) = \frac{xg'(x)-g(x)}{x^2}$, which we just showed is non-negative for positive x and non-positive for negative x. Moreover,

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{g(x)}{x} = \lim_{x\to 0} g'(x) = 0.$$

Therefore $h(x)$ is non-negative and non-increasing for negative x; zero for $x = 0$; and non-negative and non-decreasing for positive x. Clearly $h(x) = 0$ for all $x \in \mathbb{R}$. But then $g(x) = f(x) - xf'(0) = 0$ for all $x \in \mathbb{R}$, from which the result follows that $f(x) = kx$, $k = f'(0)$.

3984. *Proposed by Dragoljub Milošević.*

Let ABC be any right-angled triangle with $\angle C = 90°$. Let w_a be the length of the internal bisector of $\angle A$ from A to the side BC; define w_b similarly. If $[ABC]$ is the area of ABC, prove that

$$w_a w_b \geq 4[ABC](2 - \sqrt{2}).$$

We received 22 correct solutions. We present a composite of nearly identical proofs given by Brian Beasley, Miguel Amengual Covas, Cristóbal Sánchez-Rubio, and C. R. Pranesachar (done independently).

Clearly $\cos(A/2) = b/w_a$, $\cos(B/2) = a/w_b$, and $[ABC] = ab/2$. Hence,

$$\begin{aligned} w_a w_b &= \frac{ab}{\cos\frac{A}{2}\cos\frac{B}{2}} = \frac{2ab}{\cos(\frac{A+B}{2}) + \cos(\frac{A-B}{2})} \\ &= \frac{4[ABC]}{\cos\frac{\pi}{4} + \cos\frac{A-B}{2}} \geq \frac{4[ABC]}{\frac{\sqrt{2}}{2} + 1} = \frac{8[ABC]}{2 + \sqrt{2}} \\ &= 4[ABC](2 - \sqrt{2}) \end{aligned}$$

Equality occurs if and only if $A = B$, that is, when ΔABC is an isosceles right-angled triangle.

3985. *Proposed by Mihaela Berindeanu.*

Prove that if a, b, c are positive numbers with sum of 18, then

$$\frac{a}{b^2+36}+\frac{b}{c^2+36}+\frac{c}{a^2+36}\geq\frac{1}{4}.$$

We received 22 solutions of which 21 were correct and complete. We present two solutions.

Solution 1, by AN-anduud Problem Solving Group.

Using AM-GM inequality and using inequality

$$ab+bc+ca\leq\frac{1}{3}(a+b+c)^2,$$

we have

$$\begin{aligned}\sum_{cyc}\frac{a}{b^2+36}=\sum_{cyc}\frac{a(b^2+36)-ab^2}{36(b^2+36)}&=\sum_{cyc}\frac{a}{36}-\frac{1}{36}\sum_{cyc}\frac{ab^2}{b^2+36}\\&\geq\frac{1}{2}-\frac{1}{36}\sum_{cyc}\frac{ab^2}{2\sqrt{b^2\cdot 36}}\\&=\frac{1}{2}-\frac{1}{36}\cdot\frac{1}{12}\sum_{cyc}ab\\&\geq\frac{1}{2}-\frac{1}{36}\cdot\frac{1}{12}\cdot\frac{1}{3}(a+b+c)^2\\&=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}.\end{aligned}$$

Equality holds only when $a=b=c=6$.

Solution 2, by Michel Bataille.

Setting $a=6x,\ b=6y,\ c=6z$, we are led to prove that

$$\frac{x}{y^2+1}+\frac{y}{z^2+1}+\frac{z}{x^2+1}\geq\frac{3}{2}\tag{1}$$

whenever $x, y, z\geq 0$ and $x+y+z=3$.

Now, since $y^2+1\geq 2y$,

$$\frac{x}{y^2+1}=x-\frac{xy^2}{y^2+1}\geq x-\frac{xy^2}{2y}=x-\frac{xy}{2}.$$

Similarly, $\frac{y}{z^2+1}\geq y-\frac{yz}{2}$ and $\frac{z}{x^2+1}\geq z-\frac{zx}{2}$ and so

$$\frac{x}{y^2+1}+\frac{y}{z^2+1}+\frac{z}{x^2+1}\geq x+y+z-\frac{xy+yz+zx}{2}=3-\frac{xy+yz+zx}{2}.\tag{2}$$

But, $9 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx)$, hence $xy+yz+zx \leq 3$. Back to (2), we obtain

$$\frac{x}{y^2+1}+\frac{y}{z^2+1}+\frac{z}{x^2+1} \geq 3-\frac{3}{2}=\frac{3}{2}$$

that is, (1).

Editor's Comments. From Solution 2, it is clear that the given inequality is equivalent to

> If x, y, z are positive real numbers such that $x+y+z=3$, prove that
>
> $$\frac{x}{y^2+1}+\frac{y}{z^2+1}+\frac{z}{x^2+1} \geq \frac{3}{2}.$$

This was observed by several solvers, and Arslanagić and Malikić also noticed that this inequality is problem **2994** part (d) of ***Crux*** 31(8) proposed by Faruk Zejnulahi and Šefket Arslanagić.

Dragojlub Milošević gave the following generalization:

> If a, b, c are positive real numbers such that $a+b+c=3k$ with $k>0$, then
> $$\frac{a}{b^2+k^2}+\frac{b}{c^2+k^2}+\frac{c}{a^2+k^2} \geq \frac{3}{2k}.$$

It is easy to see that if $k=1$ we obtain problem **2994** and if $k=6$ we obtain the problem **3985**.

3986. *Proposed by George Apostolopoulos.*

Let a, b, c be the lengths of the sides of a triangle ABC with circumradius R. Prove that

$$\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)} \geq \frac{1}{4R^2}.$$

There were 19 correct solutions submitted. We present the one below which is both elegant and short. Solution by Joel Schlosberg.

It is well known (O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969, p. 49) that

$$a+b+c \leq 3\sqrt{3}R.$$

By the AM-GM Inequality we have

$$\frac{(a+b)+(b+c)+(c+a)}{3} \geq \sqrt[3]{(a+b)(b+c)(c+a)}.$$

Therefore,

$$\begin{aligned}&\frac{1}{(a+b)(b+c)}+\frac{1}{(b+c)(c+a)}+\frac{1}{(c+a)(a+b)}\\&=\frac{(a+b)+(b+c)+(c+a)}{(a+b)(b+c)(c+a)}\geq\frac{27}{[(a+b)+(b+c)+(c+a)]^2}\\&=\left(\frac{3\sqrt{3}}{2(a+b+c)}\right)^2\geq\frac{1}{4R^2}.\end{aligned}$$

3987. *Proposed by Michel Bataille.*

Let ABC be a triangle with circumcircle Γ and let A' be the point of Γ diametrically opposite to A. The lines AB and AC intersect the tangent to Γ at A' in B' and C', respectively. Prove that the tangents to Γ at B and C intersect at the centroid of $AB'C'$ if and only if $2\cos A = 3\sin B\sin C$.

We received four correct submissions and present the solution by Oliver Geupel.

Consider the problem in the plane of complex numbers where $a, b, c, \ldots$ are the complex numbers representing the points $A, B, C, \ldots$. Assume that Γ with centre O is the unit circle, $a = 1$, and $a' = -1$.

As is easily verified, if P, Q, R, S are points on the unit circle centred at O, the points of intersection of the relevant chords and tangents are represented as in the following table:

The intersection of	*is represented by*
chords PQ and RS	$\frac{pq(r+s)-rs(p+q)}{pq-rs}$
chord PQ and the tangent at R	$\frac{2pqr-r^2(p+q)}{pq-r^2}$
the tangents at P and at R	$\frac{2pr}{p+r}$

Consequently, the intersection T of tangents to Γ at B and C is represented by

$$t=\frac{2bc}{b+c},$$

and the centroid G of triangle $AB'C'$ is represented by

$$g=\frac{1}{3}(1+b'+c')=\frac{1}{3}\left(1+\frac{3b+1}{1-b}+\frac{3c+1}{1-c}\right).$$

(Note that G is always well defined.) Hence the condition $T = G$ is equivalent to $t - g = 0$, which factors as $(2bc - b - c)(3bc + b + c + 3) = 0$. If $2bc - b - c = 0$ then $t = 1 = a$ (that is, $T = A$), which is impossible. Therefore

$$T=G \qquad \Longleftrightarrow \qquad 3bc+b+c+3=0. \tag{1}$$

Next, we show the following:

$$T=G \qquad \Longrightarrow \qquad \cos A>0. \tag{2}$$

If, to the contrary, $\cos A = 0$ then the tangents at B and C would be parallel so that they could not intersect at G. Similarly, if $\cos A < 0$, in which case A would be obtuse, then A would be an interior point of triangle BCT. As a consequence, T would lie on one side of the tangent to Γ at A while B' and C' (and therefore G) would lie on the other, and we would have $T \neq G$. This proves (2).

Note that $\cos A$ is the distance of point O from the line BC, that is $\cos^2 A = \frac{(b+c)^2}{4bc}$; furthermore, $\sin^2 B = \left(\frac{AC}{2}\right)^2 = -\frac{(1-c)^2}{4c}$, and $\sin^2 C = -\frac{(1-b)^2}{4b}$. Hence

$$(2\cos A)^2 = (3\sin B \sin C)^2 \iff (3bc - 7b - 7c + 3)(3bc + b + c + 3) = 0. \quad (3)$$

Next, we show the following:

$$2\cos A = 3\sin B \sin C \implies 3bc + b + c + 3 = 0. \quad (4)$$

The proof is by contradiction. Suppose $3bc - 7b - 7c + 3 = 0$. Then the intersection of chords AA' and BC is represented by $\frac{b+c}{1+bc} = \frac{3}{7}$, which implies that angle A is obtuse, so that $\cos A < 0$. This contradicts the premise of (4).

Finally we are ready to prove that the two conditions

$$T = G \quad \text{and} \quad 2\cos A = 3\sin B \sin C$$

are equivalent. First assume $T = G$. Then by (1) and (3), we have $(2\cos A)^2 = (3\sin B \sin C)^2$. From (2) we obtain $\cos A > 0$. Therefore $2\cos A = 3\sin B \sin C$. The converse is immediate from (4) and (1).

Editor's Comments. Let D be the foot of the altitude from A to BC, and E be the point where the altitude again meets the circumcircle. Then Pranesachar, who used Cartesian coordinates for his solution, observed that $T = G$ if and only if E is the midpoint of AD; furthermore, because D is known to be the midpoint of EH (where H is the orthocentre of ΔABC), he concluded that E and D are points of trisection of the line segment AH if and only if $T = G$, if and only if $2\cos A = 3\sin B \sin C$.

Bataille noted that because the line AT is known to be the symmedian from A of ΔABC, this problem provides an alternative construction of this symmedian as the median from A of $\Delta AB'C'$.

3988*. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Find the maximum and minimum values of the expression

$$\frac{a}{\sqrt{a^2+3b^2}} + \frac{b}{\sqrt{b^2+3c^2}} + \frac{c}{\sqrt{c^2+3a^2}}.$$

We received four correct solutions and three flawed submissions. We present the solution by S. Arslanagić, N. Hodžić and S. Malikić.

Setting (a, b, c) equal to (x^2, x, y) and (x, y^2, y) respectively and letting x tend to infinity and y tend to 0, we obtain the respective limits 2 and 1 for the given expression. We show that these are its supremum and infimum, although neither value is actually attained.

For the upper bound, we exploit an inequality of V. Cîrtoaje and G. Dospinescu: Suppose that $n \geq 3$, that x_i $(1 \leq i \leq n)$ are positive reals with $x_1 x_2 \cdots x_n = 1$, and that $0 < p \leq (2n-1)/(n-1)^2$. Then

$$\frac{1}{\sqrt{1+px_1}} + \frac{1}{\sqrt{1+px_2}} + \cdots + \frac{1}{\sqrt{1+px_n}} \leq \frac{n}{\sqrt{1+p}}.$$

The proof of this can be found online in the book, *Algebraic inequalities: old and new methods*, by V. Cîrtoaje (GIL Publishing House, 2006). See Problem 8 in Sections 4.2 and 4.3, pages 199, 205-206.

Applying this to $n = 3$, $p = 5/4$, $x_1 = b^2/a^2$, $x_2 = c^2/b^2$ and $x_3 = a^2/c^2$, we deduce that

$$\sum_{\text{cyclic}} \sqrt{\frac{a^2}{a^2+3b^2}} = \sum_{i=1}^{3} \sqrt{\frac{1}{1+3x_i}} < \sum_{i=1}^{3} \sqrt{\frac{1}{1+\frac{5}{4}x_i}} \leq 2.$$

For the lower bound, recall a generalization of the Hölder inequality, to wit: Let n be a positive integer, $p, q, r > 1$ with $p^{-1} + q^{-1} + r^{-1} = 1$, and x_i, y_i and z_i $(1 \leq i \leq n)$ be positive reals. Then

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \left(\sum_{i=1}^{n} z_i^r\right)^{1/r} \geq \sum_{i=1}^{n} x_i y_i z_i.$$

Apply this to

$$n = p = q = r = 3,\ x_1 = [a^4(a^2+3b^2)]^{1/3}, y_1 = z_1 = a^{1/3}(a^2+3b^2)^{-1/6},$$

with analogous substitutions for x_2, x_3, y_2, y_3, z_2, z_3, to obtain

$$\begin{aligned}
&\left[\sum a^4(a^2+3b^2)\right]\left[\sum \sqrt{\frac{a^2}{a^2+3b^2}}\right]^2 \\
&= \left[\sum \left(\sqrt[3]{a^4(a^2+3b^2)}\right)^3\right]\left[\sum \left(\sqrt[6]{\frac{a^2}{a^2+3b^2}}\right)^3\right]\left[\sum \left(\sqrt[6]{\frac{a^2}{a^2+3b^2}}\right)^3\right] \\
&\geq \sum \sqrt[3]{a^4(a^2+3b^2)} \cdot \sqrt[6]{\frac{a^2}{a^2+3b^2}} \cdot \sqrt[6]{\frac{a^2}{a^2+3b^2}} \\
&\geq (a^2+b^2+c^2)^3.
\end{aligned}$$

where each sum is cyclic with three terms. Therefore,

$$\begin{aligned}
\left(\frac{a}{\sqrt{a^2+3b^2}} + \frac{b}{\sqrt{b^2+3c^2}} + \frac{c}{\sqrt{c^2+3a^2}}\right)^2 &\geq \frac{(a^2+b^2+c^2)^3}{a^6+b^6+c^6+3(a^4b^2+b^4c^2+c^4a^2)} \\
&> 1.
\end{aligned}$$

We conclude from this and the continuity of the expression that its values fill up the open interval $(0,1)$.

Editor's Comments. A. Alt and K.-W. Lau each avoided an appeal to advanced inequalities by intricate algebraic arguments involving several cases. One submitter used multivariate calculus to identify the critical points of

$$(1+3x)^{-1/2}+(1+3y)^{-1/2}+(1+3/(xy))^{-1/2}$$

in the plane, but failed to provide a comprehensive analysis as $x, y \to \infty$.

3989. *Proposed by Dragoljub Milošević.*

Let h_a, h_b and h_c be the altitudes, r_a, r_b and r_c be the exradii, r the inradius and R the circumradius of a triangle. Prove that

$$\frac{r_a^2}{h_a}+\frac{r_b^2}{h_b}+\frac{r_c^2}{h_c} \geq 3(2R-r).$$

We received 12 solutions. We present the solution by Miguel Amengual Covas.

We will use the following relations:

(i) $r_a+r_b+r_c=4R+r$

(ii) $r_a^2+r_b^2+r_c^2=(r_a+r_b+r_c)^2-2(r_ar_b+r_br_c+r_cr_a)=(4R+r)^2-2s^2$

(iii) $\frac{1}{r}-\frac{1}{r_a}=\frac{2}{h_a}$

(iv) $s^2 \leq 4R^2+4Rr+3r^2$

Equalities (i) and (ii) are well-known. The proof of (iii) is nearly trivial as

$$\frac{1}{r}-\frac{1}{r_a}=\frac{s}{K}-\frac{s-a}{K}=\frac{a}{K}=\frac{2}{h_a},$$

and (iv) is obtained from $IH^2=4R^2+4Rr+3r^2-s^2$. Here, s, K, I, and H denote, respectively, the semiperimeter, area, incenter, and orthocenter.

With sums cyclic over a, b, c, we obtain

$$\begin{aligned}
\sum \frac{r_a^2}{h_a}=\frac{1}{2}\sum r_a^2\left(\frac{1}{r}-\frac{1}{r_a}\right) &= \frac{1}{2}\left(\frac{\sum r_a^2}{r}-\sum r_a\right)\\
&= \frac{1}{2}\left(\frac{(4R+r)^2-2s^2}{r}-(4R+r)\right)=\frac{8R^2+2Rr-s^2}{r}\\
&\geq \frac{8R^2+2Rr-(4R^2+4Rr+3r^2)}{r}=\frac{4R^2-2Rr-3r^2}{r}\\
&= \frac{4(R-2r)+3r(2R-r)}{r} \geq \frac{3r(2R-r)}{r}=3(2R-r).
\end{aligned}$$

Equality holds if and only if $R-2r=0$, if and only if the triangle is equilateral.

3990. *Proposed by Ángel Plaza.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 > a_2 > \ldots > a_n$. Prove that
$$(a_1 - a_n)\left(\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n}\right) \geq (n-1)^2.$$

When does equality hold?

There were 26 submitted solutions for this problem, all of which proved the inequality, 3 of which forgot or missed the equality case, 20 of which stated an equality case that was not as precise as possible, and 3 of which stated the most precise equality case possible.

Solution 1, by Daniel Vacaru (expanded slightly by the editor). By Bergström's Inequality, one has

$$\begin{aligned}
&\frac{1}{a_1 - a_2} + \frac{1}{a_2 - a_3} + \cdots + \frac{1}{a_{n-1} - a_n} \\
&\qquad = \frac{1^2}{a_1 - a_2} + \frac{1^2}{a_2 - a_3} + \cdots + \frac{1^2}{a_{n-1} - a_n} \\
&\qquad \geq \frac{(1 + 1 + \cdots + 1)^2}{(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n)} = \frac{(n-1)^2}{a_1 - a_n}.
\end{aligned}$$

A trivial rearrangement yields the result. For the required equality, one must have
$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n,$$
so
$$a_1 - a_n = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) = \lambda(n-1),$$
and hence the common difference $a_k - a_{k+1}$ is $\lambda = \dfrac{a_1 - a_n}{n-1}$. It follows that:
$$a_i = a_n + (n-i)\left(\frac{a_1 - a_n}{n-1}\right), \qquad i = 1, \ldots, n-1.$$

Solution 2, by Joseph DiMuro. For any such choice of $a_1 > a_2 > \ldots a_n$, let $b_i = a_i - a_{i+1}$ for $1 \leq i < n$ (then $b_i > 0$ for all i). We then have $a_1 - a_n = b_1 + b_2 + \cdots + b_{n-1}$, so we can rewrite the desired inequality as follows:
$$\left(\sum_{i=1}^{n-1} b_i\right)\left(\sum_{j=1}^{n-1} \frac{1}{b_j}\right) \geq (n-1)^2.$$

This can be rewritten as
$$\sum_{1 \leq i,j \leq n-1} \frac{b_i}{b_j} \geq (n-1)^2.$$

And if we break the summation into cases where $i = j$ and cases where $i \neq j$, we obtain

$$(n-1) + \sum_{i \neq j} \frac{b_i}{b_j} \geq (n-1)^2.$$

Now, for $i > j$, let $c_{ij} = \dfrac{b_i}{b_j}$. We can then rewrite the desired inequality as:

$$(n-1) + \sum_{i>j} \left(c_{ij} + \frac{1}{c_{ij}} \right) \geq (n-1)^2.$$

However, for any positive number c, we have $c + \dfrac{1}{c} \geq 2$, with equality if and only if $c = 1$. Thus, we have

$$(n-1) + \sum_{i>j} \left(c_{ij} + \frac{1}{c_{ij}} \right) \geq (n-1) + 2 \cdot \frac{(n-1)(n-2)}{2} = (n-1)^2,$$

as desired. And the only way to have equality is to have $c_{ij} = 1$ for all $i > j$. That means $b_i = b_j$ for all i and j, which means that the a_i's form an arithmetic sequence.

Editor's Comments. There are a multitude of solution methods for this one. Only Vacaru's solution used Bergström's inequality; three other solutions used the direct computation in DiMuro's solution. 5 solutions used the AM-GM inequality, 6 used the AM-HM inequality, and 6 used the Cauchy-Schwarz inequality. S. Malikić listed six different ways to obtain the inequality, five of which are completely general (the three just listed, the direct method, and Jensen's inequality), and Chebyshev's sum inequality, which requires the difference of the terms to either increase or decrease. DiMuro and J. Schlosberg both remarked that we don't actually need the original terms to be positive.

AUTHORS' INDEX

Solvers and proposers appearing in this issue
(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations