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## MASS POINTS<sup>1</sup>

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and

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### DEFINITIONS:

$D_1$ : A *mass point* is a pair consisting of a positive number  $n$  (called its *weight*) and a point  $P$ . It is designated  $(n,P)$  or simply  $nP$ .

$D_2$ : Two mass points  $(a,A)$  and  $(b,B)$  are called *equal* if and only if both  $a = b$  and  $A = B$ .

$D_3$ : The *sum* of two mass points is given by

$$(a,A) + (b,B) = (a+b,C)$$

and is called their *centroid*, where  $C$  is in  $\overline{AB}$  such that  $AC:CB = b:a$  (note the reversal of numbers).

### POSTULATES OF ADDITION:

$P_1$ : Addition produces a unique sum.

Hence *subtraction* is possible as follows:

$$(a,X) - (b,Y) = (a-b,Z),$$

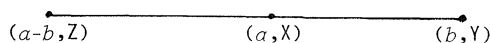


Figure 1

<sup>1</sup>This article was originally written for the N.Y.C. Senior "A" Mathletes. It is published here, with the permission of the authors, in slightly revised form.

where  $a > b$  and  $X$  is in  $\overline{YZ}$  such that  $ZX:XY = b:(a-b)$  (see Figure 1).

$P_2$ : Addition is *commutative*, that is,

$$(a,A) + (b,B) = (b,B) + (a,A).$$

$P_3$ : Addition is *associative*, that is,

$$(a,A) + (b,B) + (c,C) = [(a,A) + (b,B)] + (c,C) = (a,A) + [(b,B) + (c,C)].$$

This sum is also called the *centroid of the system*. (Associativity is here an assumption which is equivalent to Ceva's Theorem.)

*EXAMPLE 1: Basics.*

In  $\triangle ABC$ ,  $D$  is the midpoint of  $\overline{BC}$ , and  $E$  is the trisection point of  $\overline{AC}$  nearer  $A$  (see Figure 2). Assign weight 1 to each of  $B$  and  $C$ ; then

$$(1,B) + (1,C) = (2,D).$$

Assign weight 2 to  $A$ ; then

$$(2,A) + (1,C) = (3,E).$$

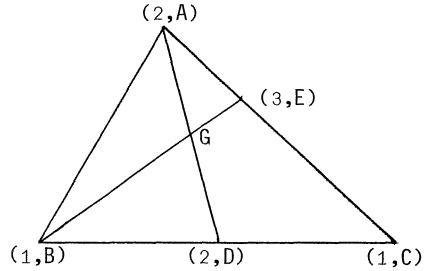


Figure 2

By  $P_3$ ,

$$(1,B) + (1,C) + (2,A) = (2,D) + (2,A) = (1,B) + (3,E) = (4,G),$$

where  $G$  is in  $\overline{AD}$  such that  $AG:GD = 2:2 = 1:1$ , and  $G$  is also in  $\overline{BE}$  such that  $BG:GE = 3:1$ . Now

$$(4,G) - (1,C) = (3,X).$$

Show that  $X$  is in  $\overline{AB}$ , that  $(1,B) + (2,A) = (3,X)$ , and that  $CG:GX = 3:1$ .

*Exercise:* Discuss Example 1 when  $E$  is the trisection point nearer  $C$  and  $D$  is still the midpoint of  $\overline{BC}$ .

*EXAMPLE 2: Medians.*

In  $\triangle ABC$ , let  $D$ ,  $E$ , and  $F$  be respectively the midpoints of  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CA}$  (see Figure 3). Assign weight 1 to each of  $A$ ,  $B$ , and  $C$ . By associativity and commutativity,

$$\begin{aligned} (1A + 1B) + 1C &= 2D + 1C = 3G \\ &= (1B + 1C) + 1A = 2E + 1A = 3H \\ &= (1C + 1A) + 1B = 2F + 1B = 3K. \end{aligned}$$

Why does  $G = H = K$ ? Note that for medians equal weights are assigned to the vertices.

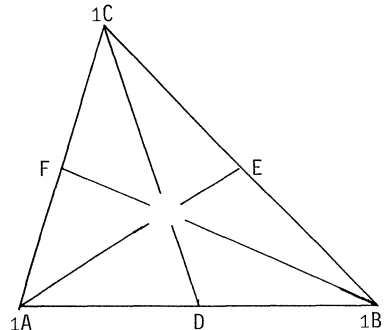


Figure 3

*Exercise:* Prove the converse of the theorem of Example 2 using mass points, then try it conventionally.

*EXAMPLE 3: Quadrilaterals.*

In quadrilateral ABCD, E, F, G, H are the trisection points of  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ ,  $\overline{DA}$  nearer A, C, C, A respectively (see Figure 4). Let  $\overline{EG}$  intersect  $\overline{HF}$  in K. Study the figure to show what weights are assigned to A, B, C, D, how E, G are obtained through addition and used to get the centroid (6,K), then how (6,K) may also be obtained with H, F through other additions. How does the conclusion that EFGH is a parallelogram follow?

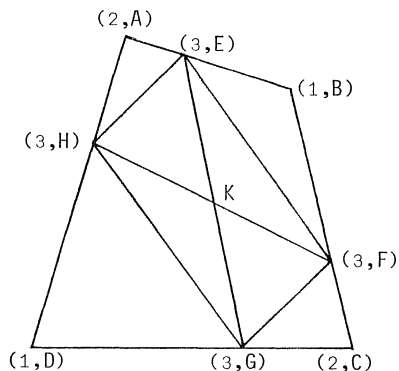


Figure 4

*Exercise:* If  $AE:EB = 1:2$ ,  $BF:FC = 2:1$ ,  $CG:GD = 1:1$ , and  $DH:HA = 1:1$ , find the numerical values of  $GK:KE$  and  $HK:KF$ . (This exercise is to be done in quadrilateral ABCD with G and H midpoints of  $\overline{CD}$  and  $\overline{DA}$  respectively.)

*EXAMPLE 4: Angle bisectors.*

In the  $\triangle ABC$  of Figure 5,  $\overline{BE}$  bisects  $\angle B$  and  $\overline{AD}$  is a median. Let  $AB = c$  and  $BC = a$ . Since  $AE:EC = c:a$ , assign weight  $c$  to C and  $a$  to A; also assign weight  $c$  to B. Then

$$(c,B) + (c,C) = (2c,D)$$

and

$$(a,A) + (c,C) = (a+c,E).$$

If  $\overline{AD}$  intersects  $\overline{BE}$  in K, find  $BK:KE$  and  $AK:KD$ .

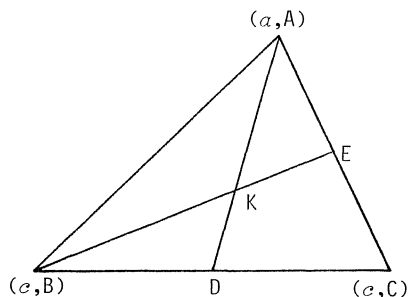


Figure 5

*Exercise 1:* For angle bisectors, we assign weights to vertices proportional to the lengths of opposite sides, or to the sines of the angles. In what ratios would angle bisectors  $\overline{BE}$  and  $\overline{AD}$  cut each other?

*Exercise 2:* Prove that the angle bisectors of a triangle are concurrent.

*EXAMPLE 5: Non-concurrency.*

In the  $\triangle ABC$  of Figure 6, D, E, F are the trisection points of  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CA}$  nearer A, B, C respectively. Let  $\overline{BF}$  intersect  $\overline{AE}$  in J, and assign weights 1,

2, 4 to A, C, B respectively. Show that  $BJ:JF = 3:4$  and  $AJ:JE = 6:1$ . Let  $\overline{CD}$  intersect  $\overline{AE}$  in K and  $\overline{BF}$  in L. Using the above results, show that

$$DK:KL:LC = 1:3:3 = EJ:JK:KA = FL:LJ:JB.$$

(This is difficult.)

*Exercise:* (a) Repeat Example 5 when D, E, F are such that

$$AD:DB = BE:EC = CF:FA = 1:3.$$

(b) Generalize for  $1:n$ . (This is difficult.)

*EXAMPLE 6: Mass points in space.*

Let ABCD be a tetrahedron (triangular pyramid) as in Figure 7. Assume the same definitions and properties of addition of mass points in space as for a plane. Assign weights 1, 1, 1, 1 to A, B, C, D. Let  $1A + 1B + 1C = 3G$  and  $3G + 1D = 4F$ . Show that  $DF:FG = 3:1$ . What is G? What is F?

*Exercise 1:* Show that the four segments joining vertices to the centroid of the opposite face meet in the point F of Example 6.

*Exercise 2:* Let E be in  $\overline{AB}$  such that  $AE:EB = 1:2$ , let H be in  $\overline{BC}$  such that  $BH:HC = 1:2$ , and let  $\overline{AH}$  intersect  $\overline{CE}$  in K. Let M be the midpoint of  $\overline{DK}$  and let ray HM intersect  $\overline{AD}$  in L. Show that  $AL:LD = 7:4$ .

*Exercise 3:* Compose a problem like Exercise 2. This is your theorem.

*Exercise 4:* Show that the three segments joining the midpoints of opposite edges of a tetrahedron bisect each other. (Opposite edges have no vertex in common.)

*Exercise 5:* Let P-ABCD be a pyramid on convex base ABCD with E, F, G, H the respective midpoints of  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ , and  $\overline{DA}$ ; and let E', F', G', H' be the respective centroids of  $\triangle PCD$ ,  $\triangle PDA$ ,  $\triangle PAB$ , and  $\triangle PBC$ . Show that  $\overline{EE'}$ ,  $\overline{FF'}$ ,  $\overline{GG'}$ , and  $\overline{HH'}$  are concurrent in a point K which divides each of the latter segments in the ratio 2:3.

*EXAMPLE 7: Splitting mass points.*

*POSTULATE:*  $(a+b)P = aP + bP$ .

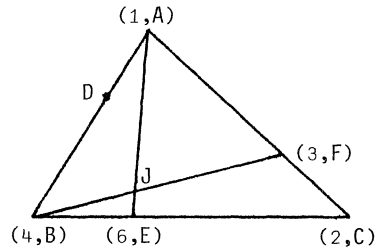


Figure 6

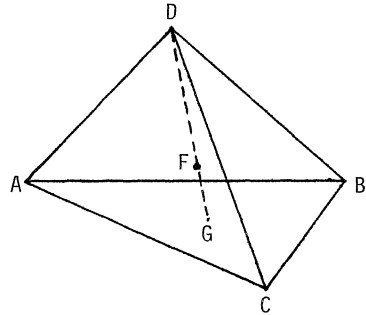


Figure 7

Splitting mass points is useful when working with transversals, such as  $\overline{EF}$  in Figure 8.

In  $\triangle ABC$ , let  $E$  be in  $\overline{AC}$  such that  $AE:EC = 1:2$ ,  $F$  in  $\overline{BC}$  such that  $BF:FC = 2:1$ , and let  $G$  be in  $\overline{EF}$  such that  $EG:GF = 1:2$ . Finally, let ray  $CG$  meet  $\overline{AB}$  in  $D$ . To find  $CG:GD$ , express  $E$  and  $F$  as mass points, consistent with the given data concerning ratios. Start with

$$xC + 2xA = 3xE \quad \text{and} \quad 2yC + yB = 3yF.$$

Since the weight at  $E$  must be twice that at  $F$ , we have  $3x = 6y$ . Let  $x = 2$ ,  $y = 1$ . Then  $C$  has been assigned a *total* weight of 4 and  $4C$  is *split* as  $2C + 2C$ . Continuing,

$$6E + 3F = 9G \quad \text{and} \quad 4C + nD = 9G.$$

Thus  $n = 5$ ,  $CG:GD = 5:4$ , and  $AD:DB = 1:4$ .

*Exercise 1:* Let  $\overline{CD}$  be a median, let  $CE:EA = 1:x$  and  $CF:FB = 1:y$ . Show that  $DG:GC = (x+y):2$  and  $EG:GF = (1+y):(1+x)$ . What must be true for  $\overline{EF}$  and  $\overline{CG}$  to bisect each other? to have  $CG:GD = m:n$ ? to have  $EG:GF = p:q$ ?

*Exercise 2:* Discuss Example 7 for the data  $CE = EA$ ,  $CF:FB = 2:1$ , and  $EG:GF = 2:3$ . Show that  $CG:GD = 17:13$  and that  $AD:DE = 8:9$ .

*Exercise 3:* Discuss Example 7 for the data  $CE = EA$ ,  $CF:FB = 1:2$ , and  $AD = DB$ . Show that  $CG:GD = 2:3$  and  $EG:GF = 3:2$ .

#### EXAMPLE 8: Altitudes.

For altitudes, say  $\overline{AD}$  in  $\triangle ABC$  (see Figure 9), assign weights *proportional* to  $\cot B$  and  $\cot C$  to  $C$  and  $B$  respectively (since  $BD:DC = \cot B:\cot C$ ). For example, let  $\angle B = 45^\circ$ ,  $\angle C = 60^\circ$ , and let  $\overline{BF}$  bisect  $\angle B$ . We must assign to  $B$  and  $C$  weights proportional to  $\cot 60^\circ = 1/\sqrt{3}$  and  $\cot 45^\circ = 1$  respectively; and by Example 4 we must assign to  $C$  and  $A$  weights proportional to  $\sin 60^\circ = \sqrt{3}/2$  and  $\sin 75^\circ = (\sqrt{6} + \sqrt{2})/4$  respectively. To satisfy both purposes, we assign weights  $\cot 45^\circ \sin 60^\circ = \sqrt{3}/2$  to  $C$ ,  $\sin 75^\circ$  to  $A$ , and  $\cot 60^\circ \sin 60^\circ = 1/2$  to  $B$ . Then

$$AE:ED = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right):\sin 75^\circ \quad \text{and} \quad BE:EF = \left(\sin 75^\circ + \frac{\sqrt{3}}{2}\right):\frac{1}{2}.$$

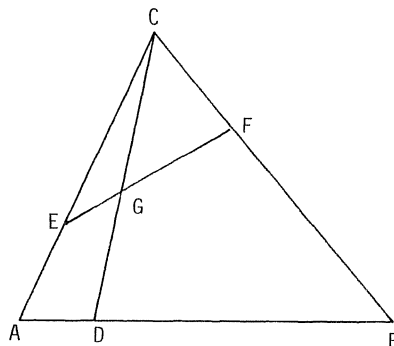


Figure 8

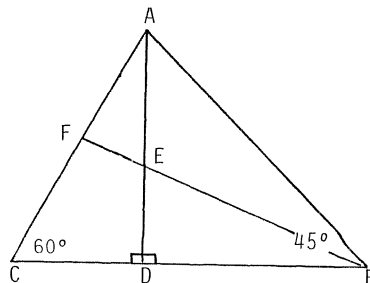


Figure 9

*Exercise 1:* In Example 8, change only the datum " $\overline{BF}$  is an angle bisector" to " $\overline{BF}$  is a median." Show that

$$AE:ED = \left(1 + \frac{1}{\sqrt{3}}\right) : \frac{1}{\sqrt{3}} = (\sqrt{3} + 1):1 \quad \text{and} \quad BE:EF = \frac{2}{\sqrt{3}}:1 = 2:\sqrt{3} = 2\sqrt{3}:3.$$

*Exercise 2:* Prove that the altitudes of a triangle are concurrent.

*EXAMPLE 9: Parallelism.*

Sometimes equal ratios of lengths of segments, stemming from parallelism, aid in assigning weights. For example, let  $ABCD$  be a trapezoid, as in Figure 10, with  $CD = 2AB$ . Let  $E$  be in  $\overline{BC}$  such that  $BE:EC = 1:3$ ,  $F$  the intersection of the diagonals, and let ray  $EF$  intersect  $\overline{AD}$  in  $G$ . Since  $AB:DC = 1:2$ , we have  $BF:FD = AF:FC = 1:2$ . So we can assign weights  $2x$  to  $B$ ,  $x$  to  $D$ ,  $2y$  to  $A$ , and  $y$  to  $C$ , so that  $(2x)(1) = (3)(y)$ . Let  $x = 3$  and  $y = 2$ . We then find  $EF:FG = 7:8$  and  $AG:GD = 3:4$ .

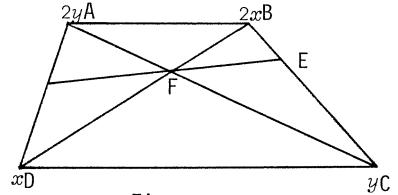


Figure 10

*Exercise:* For  $BE:EC = 2:5$ , find the ratios  $EF:FG$  and  $AG:GD$ .

*EXAMPLE 10: Ceva's Theorem.*

A *cevian* of a triangle is a segment whose endpoints are a vertex of the triangle and a point (not a vertex) which lies on the opposite side. Suppose cevians  $\overline{AE}$  and  $\overline{CD}$  meet in  $G$  (see Figure 11), with  $AD = p$ ,  $BD = q$ ,  $BE = r$ , and  $CE = s$ . Assign weights  $sq$ ,  $sp$ ,  $rp$  to  $A$ ,  $B$ ,  $C$  respectively. Adding, we get

$$sqA + spB = (sq + sp)D$$

and

$$spB + rpC = (sp + rp)E.$$

Now  $F$  is collinear with  $B$  and  $G$  if and only if

$$sqA + rpC = (sq + rp)F.$$

But this implies  $CF:FA = sq:rp$ , and we observe that the products of the alternating segments on the sides of  $\triangle ABC$  are equal, that is,

$$AD \cdot BE \cdot CF = DB \cdot EC \cdot FA.$$

Thus we have proved Ceva's Theorem and its converse: *three cevians of a triangle*

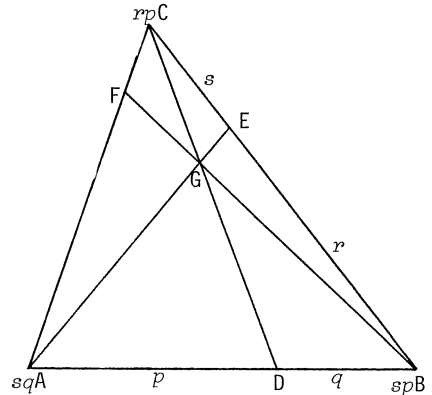


Figure 11

are concurrent if and only if the products of the lengths of the non-adjacent segments of the sides are equal.

*Exercise 1:* In  $\triangle ABC$ ,  $AD:DB = 3:1$  and  $BE:EC = 4:1$ . Find  $CF:FA$ . Prove that

$$\frac{CF}{FA} \cdot \frac{AD}{DB} \cdot \frac{BE}{EC} = 1.$$

This is an alternate statement of Ceva's Theorem.

*Exercise 2:* From the data in the proof of Ceva's Theorem, prove that

$$\frac{GE}{AE} + \frac{GF}{BF} + \frac{GD}{CD} = 1.$$

*Exercise 3:* The converse of Ceva's Theorem can be used to prove that the cevians joining the vertices of a triangle to the points of tangency of the inscribed circle are concurrent. Prove this.

*EXAMPLE 11: Menelaus' Theorem.*

The theorem states that, if a transversal is drawn across the three sides of a triangle (extended, if necessary), the products of the nonadjacent segments are equal. Written in more convenient form, it says (see Figures 12 and 13):

$$\frac{AD}{DB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = 1.$$

The formula is best remembered by going around the triangle clockwise. The first segment goes from a vertex to an intersection point of the transversal, and the second goes from the intersection point to the second vertex of the side considered. Continue until the first vertex is reached again.

A proof using Figure 12 follows. Assign weights  $a, b, x$  to  $A, B, F$  in  $\triangle BAF$ ; then  $C$  has weight  $x+b$ . Since  $D, E, F$  are collinear, we have  $(x+b)d = ca$ , or  $x = (ac - bd)/d$ . Thus

$$\frac{BC}{CF} = \frac{ac - bd}{bd}, \quad \frac{BF}{CF} = \frac{BC + CF}{CF} = \frac{ac}{bd}$$

and so

$$\frac{AD}{DB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = \frac{b}{a} \cdot \frac{ac}{bd} \cdot \frac{d}{c} = 1.$$

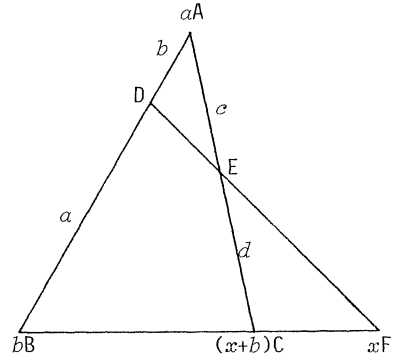


Figure 12

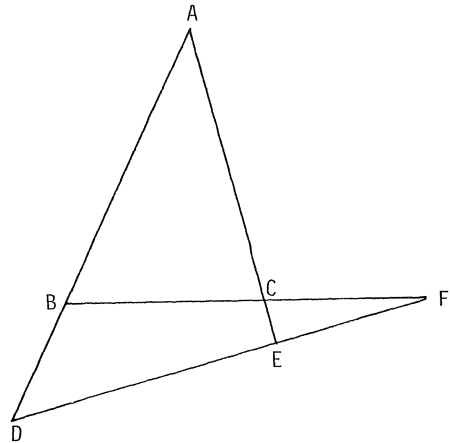


Figure 13



*Exercise 1:* Find a proof using Figure 13.

*Exercise 2:* The converse of this theorem is often used to prove that three points are collinear. Prove the converse of the theorem for each of the figures.

*Exercise 3:* In  $\triangle ABC$ , D is in  $\overline{AB}$  such that  $AD:DB = 2:3$ , E is in  $\overline{BC}$  such that  $BE:EC = 2:3$ , and  $\overline{AE}$  meets  $\overline{DC}$  in F. Show, by using Menelaus' Theorem first in  $\triangle BCD$  then in  $\triangle EAB$ , that

$$\frac{DF}{FC} + \frac{EF}{FA} = \frac{4}{15} + \frac{9}{10} = \frac{7}{6}.$$

Now do the same problem using mass points in  $\triangle ABC$ .

Further reading on mass points can be found in Melvin Hausner's article "The Center of Mass and Affine Geometry," *American Mathematical Monthly*, Vol. 69 (1962), pp. 724-737, where an axiomatic treatment of the topic can be found.

*Editor's note.*

Harry Sitomer is (and has been for the past fifteen years or so) retired from active teaching. He was the mathematics chairman at New Utrecht High School, Brooklyn, N.Y., until about 1960. He then spent a year or two teaching at C.W. Post College. He is the co-author of two textbooks in geometry and one in linear algebra. He has, in addition, collaborated with his wife on a series of children's books on mathematical ideas. Playing in string quartets is now his major interest.

Steven R. Conrad is president of the New York City Interscholastic Mathematics League. He and Harry Sitomer write many of the problems for contests sponsored by the League. Conrad also writes for the Suffolk County, New York IML, the Bergen County, New Jersey IML, the New York State ML, the Atlantic Region ML, the New England Association of Mathematics Leagues, the Massachusetts Association of Mathematics Leagues, and the Atlantic Pacific Canada Mathematics League. He would like to exchange contest materials with any readers of EUREKA who might know of similar competitions. His address is 39 Arrow St., Selden, N.Y., 11784, U.S.A.

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## NUMBER CURIOSITIES

The following equations are true for  $n = 0, 1, 2$ :

$$1239^n + 5615^n + 6424^n = 2428^n + 7613^n + 3237^n$$

$$\begin{array}{l} 1^n + 5^n + 6^n = 2^n + 7^n + 3^n \\ 239^n + 615^n + 424^n = 428^n + 613^n + 237^n \end{array}$$

$$\begin{array}{l} 12^n + 56^n + 64^n = 24^n + 76^n + 32^n \\ 39^n + 15^n + 24^n = 28^n + 13^n + 37^n \end{array}$$

$$\begin{array}{l} 123^n + 561^n + 642^n = 242^n + 761^n + 323^n \\ 9^n + 5^n + 4^n = 8^n + 3^n + 7^n \end{array}$$

Such relationships between sets of numbers and their powers are called *multigrades*. Many can be found on pp. 173-175 in Joseph S. Madachy's *Mathematics on Vacation*, Charles Scribner's Sons, N.Y., 1966. Here is one from that source which holds for  $n = 0, 1, 2, 3, 4, 5$ :

$$1^n + 9^n + 18^n + 38^n + 47^n + 55^n = 3^n + 5^n + 22^n + 34^n + 51^n + 53^n.$$

# ON CIRCUMSCRIBABLE QUADRILATERALS

LÉO SAUVÉ, Algonquin College

## 1. Introduction.

A quadrilateral is said to be *circumscribable* if its four sides are tangent to the same circle. Nathan Altshiller Court [3] says that the circumscribable quadrilateral was first considered in the thirteenth century by Jordanus Nemorarius, a contemporary of Fibonacci. With the inevitable accretions of the centuries, it turns out that by 1846 much of the information contained in the following two theorems was known, although it was disseminated in bits and pieces in various books and journals.

**THEOREM 1.** *The following four properties are equivalent:*

(a) *A circle is internally tangent to the four sides of a complete quadrilateral, as shown in Figure 1.*

$$(b) AB - BF = AD - DF.$$

$$(c) AC - CF = AE - EF.$$

$$(d) DC - CB = BE - ED.$$

**THEOREM 2.** *The following four propositions are equivalent:*

(e) *A circle is externally tangent to the four sides of a complete quadrilateral, as shown in Figure 2.*

$$(f) AB + BF = AD + DF.$$

$$(g) AC + CF = AE + EF.$$

$$(h) DC + CB = BE + ED.$$

As far as I know, these theorems were first formulated into essentially the form given above by Grossman [8]. The main purpose of this article is to investigate the history of various parts of these theorems and to show how these parts eventually came to be recognized as belonging to the beautiful organic whole represented by Theorems 1 and 2.

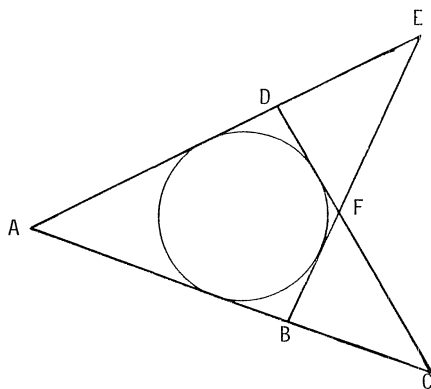


Figure 1

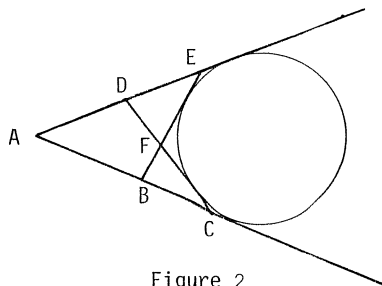


Figure 2

2. *Who proved what and when?*

I have no information as to what parts of these two theorems, if any, Jordanus Nemorarius succeeded in proving.

F.G.-M. [7] calls the implication (a) $\Rightarrow$ (b) *Pitot's Theorem*. He claims it was proved in 1725, in the equivalent form

$$(a) \Rightarrow AB + DF = AD + BF,$$

by Pitot (1695-1771), who was an engineer in Languedoc and author of *Théorie de la manoeuvre des vaisseaux*.

The converse proposition (b)  $\Rightarrow$  (a) was proved in 1815 by J.-B. Durrande (1797-1825) [5], who was a professor in Cahors. He also proved that it holds as well for a skew quadrilateral and for a spherical quadrilateral. This leads to

*THEOREM 3. If four circles in a plane, or on a sphere, are tangent in pairs, then the four points of contact lie on one circle.*

The proofs of Pitot and Durrande which are given in [7] are for a quadrilateral such as ABFD in Figure 1, but they apply equally well, with minor variations, to quadrilaterals like AEFC and BEDC, which shows that the four propositions of Theorem 1 are equivalent. But I have seen no evidence that anyone thought of making this *rapprochement* until Grossman did it in [8].

The equivalences (e)  $\Leftrightarrow$  (f) and (e)  $\Leftrightarrow$  (g) were proved in 1846 by Jakob Steiner (1796-1863) [11]. Steiner's proof, as given in [7], is for quadrilaterals such as ABFD and AEFC in Figure 2. Again, I have seen no evidence that anyone before Grossman [8] thought of adapting the proof to quadrilaterals such as BEDC, and thus showing that the four propositions of Theorem 2 are equivalent.

G. Darboux [4] made a study of the theorems of Pitot and Steiner, and arrived at the following

*THEOREM 4. If the circumscribable quadrilateral ABCD is deformed in such a way that vertices A and B remain fixed and the lengths of all sides remain invariant, then the locus of the centre of the inscribed circle is a circle whose diameter is the segment which divides harmonically the diagonals AC, BD of the quadrilateral, when the latter is brought into the position where its four vertices are collinear.*

For detailed studies of Pitot's Theorem and its converse, see also M. Coissard [2] and Th. Caronnet [1].

3. *The most "elementary" theorem of Euclidean geometry.*

The results described above were peacefully slumbering in obscure libraries when suddenly a minor earthquake with epicentre in Australia caused them to shake off the dust of centuries. It was caused by Malcolm L. Urquhart (1902-1966), Australian mathematical physicist. He discovered the following theorem which he called "the most elementary theorem of Euclidean geometry":

*URQUHART'S THEOREM.* In Figure 3, if  $AB + BF = AD + DF$ , then  $AC + CF = AE + EF$ .

Of course, this is simply the implication  $(f) \Rightarrow (g)$  of our Theorem 2.

David Elliott described the event in [6, p. 132] and Dan Pedoe [9] quotes Elliott as follows: "Urquhart considered this to be 'the most elementary theorem' since it involves only the concepts of straight line and distance. The proof of this theorem by purely geometrical methods is not elementary. Urquhart discovered this result when considering some of the fundamental concepts of the theory of special relativity." George Szekeres [12, p. 159] gives a proof by Basil Rennie<sup>1</sup> which is based on the following lemma:

*RENNIE'S LEMMA.* Given a circle and two points A and C outside the circle, the line AC will be a tangent if the distance AC equals the difference between (or the sum of) the length of a tangent from A and the length of a tangent from C.

Grossman [8] came next. He wrapped it all up in the beautiful package represented by Theorems 1 and 2.

Pedoe says in [9] that he attempted to find a proof of Urquhart's Theorem which does not involve circles, and he suggests in [10] that readers of EUREKA may be amused to try. If any reader should find such a proof, I am sure the editor would be pleased to publish it as a postscript to the present article.

4. *Pedoe's two-circle theorem.*

Pedoe says in [9] that Urquhart's Theorem is equivalent to the following (see Figure 4):

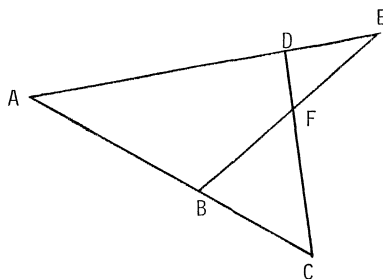


Figure 3

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<sup>1</sup>Professor Basil Rennie edits a journal similar to EUREKA. His address is Mathematics Department, James Cook University of North Queensland, Post Office, James Cook University, 4811, Townsville, North Queensland, Australia.

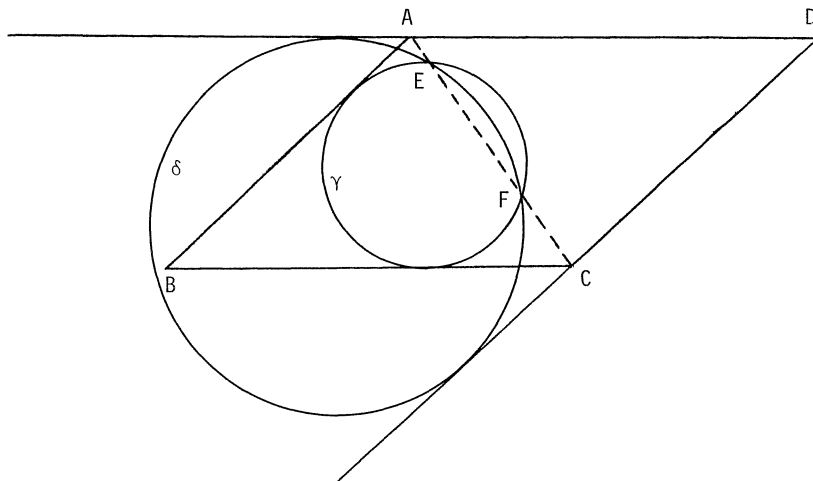


Figure 4

*PEDOE'S TWO-CIRCLE THEOREM.*  $ABCD$  is a parallelogram, and a circle  $\gamma$  touches  $AB$  and  $BC$  and intersects  $AC$  in the points  $E$  and  $F$ . Then there exists a circle  $\delta$  which passes through  $E$  and  $F$  and touches  $AD$  and  $DC$ .

Pedoe then proved the theorem directly with the help of Rennie's Lemma. Later, in [10], he said that there is a simpler proof of his two-circle theorem which does not use Rennie's Lemma. In Problem 139 on p. 68, he invites the readers of EUREKA to find such a proof.

#### 5. Acknowledgments.

I wish to thank Professor Dan Pedoe, University of Minnesota, for his communication [10], which provided the inspiration for this article. I obtained references 6, 8, 12 from [9]; reference 9 came from [10]; references 3, 7, 10 are my own; all the remaining references are from [7].

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3. Nathan Altshiller Court, *College Geometry*, Barnes and Noble, 1952, p. 300.
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7. F. G.-M., *Exercices de géométrie*, Mame et Fils, Tours, 1907, pp. 318-320.
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9. Dan Pedoe, The most elementary theorem of Euclidean geometry, *Mathematics Magazine*, Vol. 49 (1976), pp. 40-42.
10. Dan Pedoe, communication to the editor.
11. Jakob Steiner, *Crelle's Journal*, 1846.
12. George Szekeres, Kinematic geometry, M.L. Urquhart in memoriam, *Journal of the Australian Mathematical Society*, Vol. 8 (1968), pp. 134-160.

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## P R O B L E M S - - P R O B L È M E S

*Problem proposals, preferably accompanied by a solution, should be sent to the editor, whose name appears on page 55.*

*For the problems given below, solutions, if available, will appear in EUREKA Vol. 2, No. 7, to be published around Sept. 15, 1976. To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should be mailed to the editor no later than Sept. 1, 1976.*

131. *Proposé par André Bourbeau, École Secondaire Garneau.*

Soit  $p$  un nombre premier  $\geq 7$ . Si  $p^{-1} = 0.\dot{a}_1\dot{a}_2\dots\dot{a}_k$ , montrer que l'entier  $N = \overline{a_1a_2\dots a_k}$  est divisible par 9.

132. *Proposed by Léo Sauvé, Algonquin College.*

If  $\cos \theta \neq 0$  and  $\sin \theta \neq 0$  for  $\theta = \alpha, \beta, \gamma$ , prove that the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of eccentric angles  $\alpha, \beta, \gamma$  are concurrent if and only if

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

133. *Proposed by Kenneth S. Williams, Carleton University.*

Let  $f$  be the operation which takes a positive integer  $n$  to  $\frac{1}{2}n$  (if  $n$  even) and to  $3n+1$  (if  $n$  odd). Prove or disprove that any positive integer can be reduced to 1 by successively applying  $f$  to it.

Example:  $13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$ .

(This problem was shown to me by one of my students.)

134. *Proposed by Kenneth S. Williams, Carleton University.*

ABC is an isosceles triangle with  $\angle ABC = \angle ACB = 80^\circ$ . P is the point on AB such that  $\angle PCB = 70^\circ$ . Q is the point on AC such that  $\angle QBC = 60^\circ$ . Find  $\angle PQA$ .

(This problem is taken from the 1976 Carleton University Mathematics Competition for high school students.)

135. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

How many  $3 \times 5$  rectangular pieces of cardboard can be cut from a  $17 \times 22$  rectangular piece of cardboard so that the amount of waste is a minimum?

136. *Proposed by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y.*

In  $\triangle ABC$ ,  $C'$  is on  $\overline{AB}$  such that  $AC':C'B = 1:2$  and  $B'$  is on  $\overline{AC}$  such that  $AB':B'C = 4:3$ . Let P be the intersection of  $\overline{BB'}$  and  $\overline{CC'}$ , and let  $A'$  be the intersection of  $\overline{BC}$  and ray AP. Find  $AP:PA'$ .

137. *Proposed by Viktors Linis, University of Ottawa.*

On a rectangular billiard table ABCD, where  $AB = a$  and  $BC = b$ , one ball is at a distance  $p$  from AB and at a distance  $q$  from BC, and another ball is at the centre of the table. Under what angle  $\alpha$  (from AB) must the first ball be hit so that after the rebounds from AD, DC, CB it will hit the other ball?

138. *Proposé par Jacques Marion, Université d'Ottawa.*

Soit  $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$  un polynôme non constant tel que  $|p(z)| < 1$  sur  $|z| = 1$ . Montrer que  $p(z)$  a un zéro sur  $|z| = 1$ .

139. *Proposed by Dan Pedoe, University of Minnesota.*

ABCD is a parallelogram, and a circle  $\gamma$  touches AB and BC and intersects AC in the points E and F. Then there exists a circle  $\delta$  which passes through E and F and touches AD and DC.

Prove this theorem without using Rennie's Lemma (see p. 65).

140. *Proposed by Dan Pedoe, University of Minnesota.*

THE VENESS PROBLEM. A paper cone is cut along a generator and unfolded into a plane sheet of paper. What curves in the plane do the originally plane sections of the cone become? (This problem is due to J.H. Veness<sup>1</sup>.)

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If a man who cannot count finds a four-leaf clover, is he entitled to happiness?

STANISLAW JERZY LEC (1909- )

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<sup>1</sup>J.H. Veness is the editor of *The Australian Mathematics Teacher*. His address is 4/69 Cremorne Road, Cremorne, New South Wales, Australia 2090.

# SOLUTIONS

18. [1975: 8, 31; 1976: 42] *Proposé par Jacques Marion, Université d'Ottawa.*

Montrer que, dans un triangle rectangle dont les côtés ont 3, 4 et 5 unités de longueur, aucun des angles aigus n'est un multiple rationnel de  $\pi$ .

IV. *Comment by Léo Sauvé, Algonquin College.*

I am now able to fill in the gap in my solution I [1975: 31] reported in my comment III [1976: 42]. In fact, I prove the following

*THEOREM. Let  $\theta$  be an acute angle such that  $\tan \theta \neq 1$  is rational. Then  $\theta$  is not a rational multiple of  $\pi$ .*

*Proof.* Suppose, on the contrary, that  $\theta$  is a rational multiple of  $\pi$ ; then  $2\theta$  and

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \quad (1)$$

are both rational. It now follows from the theorem of Olmsted quoted by the proposer in his solution II [1975: 32] that the only values (1) can assume are  $\pm \frac{1}{2}$ , and so  $\tan^2 \theta = 3$  or  $\frac{1}{3}$ , and  $\tan \theta = \sqrt{3}$  or  $\frac{1}{\sqrt{3}}$ , contradicting the assumed rationality of  $\tan \theta$ .

65. [1975: 57, 100; 1976: 7] *Correction.*

An unfortunate transcription error occurred in the solution given in [1975: 100]. The last two lines should read:

and it is clear that the tens digit of  $n^2$  is of the same parity as the tens digit of  $y^2$ , which is even in every case.

F.G.B. Maskell, Algonquin College, spotted the error and brought it to the attention of the editor, who is solely responsible for it.

91. [1975: 97; 1976: 44] *Late solution: André Bourbeau, École Secondaire Garneau.*

92. [1975: 97; 1976: 44] *Late solution: André Bourbeau.*

97. [1975: 97; 1976: 48] *Late solution: André Bourbeau.*

99. [1975: 98; 1976: 50] *Proposed by H.G. Dworschak, Algonquin College.*

If  $a$ ,  $b$ , and  $n$  are positive integers, prove that there exist positive integers  $x$  and  $y$  such that

$$(a^2 + b^2)^n = x^2 + y^2. \quad (1)$$

*Application:* If  $a = 3$ ,  $b = 4$ , and  $n = 7$ , find at least one pair  $\{x, y\}$  of positive integers which verifies (1).

III. *Solution by Kenneth S. Williams, Carleton University.*

This is an answer to the question raised at the end of the editor's



comments [1976: 52] about the number of distinct solutions of (1).

Let  $a, b, n$  be positive integers and set  $k = (a^2 + b^2)^n$ . We determine the number  $N(k)$  of ordered pairs  $(x, y)$  of integers with

$$x^2 + y^2 = k, \quad 0 < x \leq y,$$

that is, the number of representations of  $k$  in the form  $x^2 + y^2$ , where  $x$  and  $y$  are *positive integers* with  $x \leq y$ . In order to do this, we define  $r(k), r_1(k), r_2(k), r_3(k)$  as follows.

Let  $r(k)$  denote the number of ordered pairs  $(x, y)$  of integers with  $x^2 + y^2 = k$ .

Let  $r_1(k)$  denote the number of ordered pairs  $(x, y)$  of integers with  $x^2 + y^2 = k$  and either  $x = 0$  or  $y = 0$ .

Let  $r_2(k)$  denote the number of ordered pairs  $(x, y)$  of integers with  $x^2 + y^2 = k$  and either  $x = y$  or  $x = -y$ .

Let  $r_3(k)$  denote the number of ordered pairs  $(x, y)$  of integers with  $x^2 + y^2 = k$  and  $0 < x < y$ .

Clearly  $r(k) - r_1(k) - r_2(k)$  is the number of ordered pairs  $(x, y)$  of integers with

$$x^2 + y^2 = k, \quad x \neq 0, \quad y \neq 0, \quad x \neq y, \quad x \neq -y. \quad (2)$$

Now any solution  $(x, y)$  of (2) gives rise to 8 distinct such solutions, namely,

$$(x, y), (x, -y), (-x, y), (-x, -y), (y, x), (y, -x), (-y, x), (-y, -x),$$

and amongst these there is exactly one satisfying  $0 < x < y$ . Hence we have

$$r(k) - r_1(k) - r_2(k) = 8r_3(k).$$

Next we note that

$$N(k) = r_3(k) + \frac{1}{4}r_2(k)$$

so that, on eliminating  $r_3(k)$ , we obtain

$$N(k) = \frac{1}{8} \{r(k) - r_1(k) + r_2(k)\}. \quad (3)$$

Thus there remains only to compute each of  $r(k), r_1(k)$ , and  $r_2(k)$ . To do this, we write  $a^2 + b^2$  ( $\geq 2$ ) in its prime power decomposition, say

$$a^2 + b^2 = 2^d p_1^{d_1} \dots p_r^{d_r} q_1^{e_1} \dots q_s^{e_s}, \quad (4)$$

where  $d, d_1, \dots, d_r, e_1, \dots, e_s$  are nonnegative integers not all zero, the  $p_i$  are distinct primes congruent to 1 (mod 4) and the  $q_j$  are distinct primes congruent to 3 (mod 4). A well-known theorem in elementary number theory [1, p. 126, Theorem 7-3] tells us that, as the right side of (4) is a sum of two squares, the  $e_j$  are all even, say  $e_j = 2f_j$  ( $j = 1, 2, \dots, s$ ). Hence

$$a^2 + b^2 = 2^d p_1^{d_1} \dots p_r^{d_r} q_1^{2f_1} \dots q_s^{2f_s}$$

and

$$k = 2^{nd} p_1^{nd_1} \dots p_r^{nd_r} q_1^{2nf_1} \dots q_s^{2nf_s}.$$

Then another well-known theorem of elementary number theory [1, p. 132, Theorem 7-8] gives

$$r(k) = 4(nd_1 + 1) \dots (nd_r + 1). \quad (5)$$

Next we have

$$\begin{aligned} r_1(k) &= 4 \times (\text{number of positive integers } \ell \text{ such that} \\ &\quad \ell^2 = k = 2^{nd} p_1^{nd_1} \dots p_r^{nd_r} q_1^{2nf_1} \dots q_s^{2nf_s}) \\ &= \begin{cases} 4, & \text{if } nd, nd_1, \dots, nd_r \text{ are all even,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (6)$$

and

$$\begin{aligned} r_2(k) &= 4 \times (\text{number of positive integers } \ell \text{ such that} \\ &\quad 2\ell^2 = k = 2^{nd} p_1^{nd_1} \dots p_r^{nd_r} q_1^{2nf_1} \dots q_s^{2nf_s}) \\ &= \begin{cases} 4, & \text{if } nd \text{ odd, } nd_1, \dots, nd_r \text{ all even,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (7)$$

Putting (3), (5), (6), (7) together, we obtain

$$N(k) = \begin{cases} \frac{1}{2} ((nd_1 + 1) \dots (nd_r + 1) - 1), & \text{if } nd \text{ even; } nd_1, \dots, nd_r \text{ all even,} \\ \frac{1}{2} ((nd_1 + 1) \dots (nd_r + 1) + 1), & \text{if } nd \text{ odd; } nd_1, \dots, nd_r \text{ all even,} \\ \frac{1}{2} (nd_1 + 1) \dots (nd_r + 1), & \text{if } nd_1, \dots, nd_r \text{ not all even.} \end{cases} \quad (8)$$

*Examples.*

$$(i) \ a = 1, b = 2, n = 2; a^2 + b^2 = 5, k = 25; d = 0, d_1 = 1.$$

$$N(25) = \frac{1}{2} ((2 \cdot 1 + 1) - 1) = 1$$

$$(25 = 3^2 + 4^2)$$

$$(ii) \ a = 1, b = 1, n = 2; a^2 + b^2 = 2, k = 4; d = 1, d_1 = 0.$$

$$N(4) = \frac{1}{2} ((2 \cdot 0 + 1) - 1) = 0$$

$$(4 \neq x^2 + y^2 \text{ with } 0 < x \leq y)$$

$$(iii) \ a = 3, b = 3, n = 1; a^2 + b^2 = 18, k = 18; d = 1, d_1 = 0, f_1 = 1.$$

$$N(18) = \frac{1}{2} ((1 \cdot 0 + 1) + 1) = 1$$

$$(18 = 3^2 + 3^2)$$

$$(iv) \ a = 1, b = 2, n = 3; a^2 + b^2 = 5, k = 125; d = 0, d_1 = 1.$$

$$N(125) = \frac{1}{2} (3 \cdot 1 + 1) = 2$$

$$(125 = 2^2 + 11^2 = 5^2 + 10^2)$$

(v)  $a = 3, b = 4, n = 7; a^2 + b^2 = 25, k = 5^{14}; d = 0, d_1 = 2.$

$$N(5^{14}) = \frac{1}{2} [(7 \cdot 2 + 1) - 1] = 7.$$

Example (v) shows that the algorithm used in solution I did indeed generate all possible solutions for  $a = 3, b = 4, n = 7.$

As a consequence of (8), we have  $(a^2 + b^2)^n = x^2 + y^2$  where  $x > 0$  and  $y > 0$  (that is,  $N(k) > 0$ ) *except* when

$$nd \text{ even, } nd_1 = \dots = nd_r = 0,$$

that is, when the exact power of 2 dividing  $k$  is *even* and no prime congruent to 1 (mod 4) divides  $k.$

*Example.*

$$a = 3, b = 3, n = 2; k = 324 = 2^2 \cdot 3^4; N(k) = 0.$$

Indeed  $324 = 0^2 + 18^2$  only.

*Also solved by André Bourbeau, École Secondaire Garneau.*

*Editor's comment.*

The remarkable formula (8) completely answers the questions of existence and number of solutions of (1). That only leaves the following question: how do we find those solutions? Will the algorithm used in solution I [1976: 51] always generate *all* the solutions? Can some more efficient algorithm be devised that will give all the solutions *directly*, without first having to find the solutions for  $n = 1, 2, 3, \dots$ ?

#### REFERENCE

1. William J. LeVeque, *Topics in Number Theory*, Vol. I, Addison-Wesley, 1956.

101. [1976: 5] *Proposé par Léo Sauvé, Collège Algonquin.*

Montrer que le cube de tout nombre rationnel est égal à la différence des carrés de deux nombres rationnels.

*I. Solution by Walter Bluger, Department of National Health and Welfare; and G.D. Kaye, Department of National Defence (independently).*

$$\text{If } x \text{ is rational, we have } x^3 = \left(\frac{x^2 + x}{2}\right)^2 - \left(\frac{x^2 - x}{2}\right)^2.$$

*II. Comment by the proposer.*

My original intention was to ask for a proof that the cube of every positive integer equals the difference of squares of two integers, an easy but interesting problem with the following mildly startling solution:

$$\begin{aligned} n^3 &= (1^3 + 2^3 + \dots + n^3) - (1^3 + 2^3 + \dots + (n-1)^3) \\ &= \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{(n-1)n}{2}\right]^2. \end{aligned}$$

Then I thoughtlessly decided to put a little more "meat" on it by asking it for rational numbers instead of positive integers, not realizing at the time that I

was thereby rendering the problem not only even more trivial, but also completely uninteresting.

The editor should have been more discerning and pitched the problem in the wastebasket.

Also solved by André Ladouceur, École Secondaire De La Salle; Jacques Marion, Université d'Ottawa; F.G.B. Maskell, Algonquin College; and Leroy F. Meyers, Ohio State University.

102. [1976: 5] *Proposé par Léo Sauvé, Collège Algonquin.*

Si, dans un  $\triangle ABC$ , on a  $a = 4$ ,  $b = 5$ , et  $c = 6$ , montrer que  $C = 2A$ .

I. *Solution d'André Ladouceur, École Secondaire De La Salle.*

On trouve facilement  $\cos C = \frac{1}{8}$  et  $\cos A = \frac{3}{4}$  par la loi des cosinus.

Puisque  $\cos 2A = 2 \cos^2 A - 1 = \frac{1}{8}$ , on a donc  $\cos C = \cos 2A$ . Or on a  $C < 180^\circ$  et

$$a < c \Rightarrow A < C \Rightarrow A < 90^\circ \Rightarrow 2A < 180^\circ.$$

La fonction cosinus étant injective entre  $0^\circ$  et  $180^\circ$ , on peut conclure que  $C = 2A$ .

II. *Solution without words by Steven R. Conrad, B.N. Cardozo High School, Bayside, N.Y. (see Figure 1).*

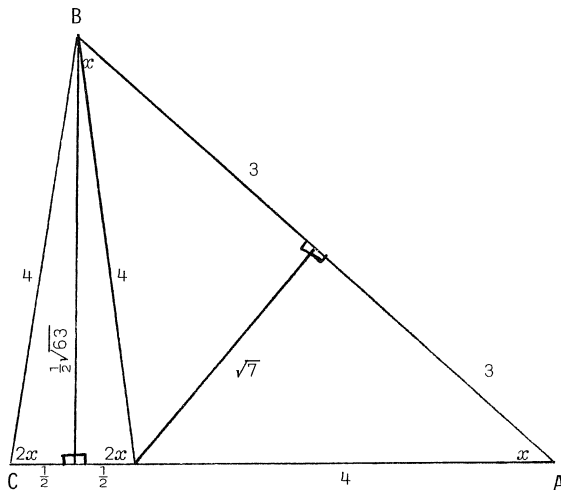


Figure 1

III. *Solution de Leroy F. Meyers, Ohio State University.*

Une conséquence immédiate de la loi des sinus et de celle des cosinus pour un triangle quelconque est que

$$\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{a \sin C}{c} \cdot \frac{b^2 + c^2 - a^2}{2bc}.$$

Si  $a = 4$ ,  $b = 5$ , et  $c = 6$ , cette relation se réduit à

$$\sin 2A = \sin C,$$

d'où on obtient: ou bien  $2A = C$ , ou bien  $2A + C = 180^\circ$ . Or l'inégalité  $A < B$  implique

$$180^\circ = 2A + C < A + B + C = 180^\circ,$$

ce qui est impossible. Par conséquent,  $2A = C$ .

#### IV. Solution du proposeur.

Plus généralement si, comme dans le cas présent, on a  $c^2 - a^2 = ab$ , alors

$$\sin^2 C - \sin^2 A = \sin(C+A) \sin(C-A) = \sin A \sin B.$$

Or  $\sin(C+A) = \sin B \neq 0$ ; donc  $\sin(C-A) = \sin A$  et  $C - A = A$ , car  $C - A \neq 180^\circ - A$ . Donc  $C = 2A$ .

#### V. Second solution by the proposer.

Produce BC to D, so that  $CD = 5$  (see Figure 2). Then  $BD = 9$  and  $\overline{BA}^2 = BC \cdot BD$ . Hence BA is tangent to the circumcircle of  $\triangle ACD$ , and so  $\angle BAC = \angle ADC = \angle DAC = \frac{1}{2}\angle ACB$ . Thus  $C = 2A$ .

The same proof holds, and  $C = 2A$ , whenever  $c^2 = a(a+b)$ .

Also solved by Walter Bluger, Department of National Health and Welfare; H.G. Dworschak, Algonquin College; G.D. Kaye, Department of National Defence; F.G.B. Maskell, Algonquin College; and Ken H. Williams, Algonquin College.

Editor's comment.

One solver proved that  $C = 2A$  by calculating C and A by the law of cosines and verifying that  $C = 2A$  was true to three decimals of degrees. This is an example of "engineering equality" which is all right in its place, but is not to be confused with true mathematical equality.

103. [1976: 5] Proposed by H.G. Dworschak, Algonquin College.

If  $\frac{\cos \alpha}{\cos \beta} + \frac{\sin \alpha}{\sin \beta} = -1$ , prove that

$$\frac{\cos^3 \beta}{\cos \alpha} + \frac{\sin^3 \beta}{\sin \alpha} = 1.$$

#### I. Solution by the proposer.

From the hypothesis

$$\frac{\cos \alpha}{\cos \beta} + \frac{\sin \alpha}{\sin \beta} = -1, \quad (1)$$

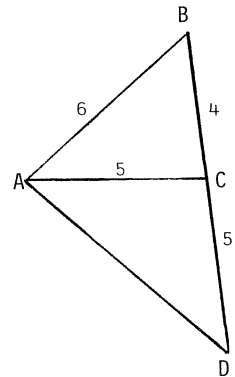


Figure 2

we have  $\cos \beta \neq 0$  and  $\sin \beta \neq 0$ . Since, from (1),

$$\cos \alpha = 0 \Rightarrow \sin \alpha = \pm 1 \Rightarrow \sin \beta = \pm 1 \Rightarrow \cos \beta = 0,$$

we conclude that  $\cos \alpha \neq 0$  and similarly  $\sin \alpha \neq 0$ . Furthermore, if

$$\frac{\cos \alpha}{\cos \beta} = \frac{\sin \alpha}{\sin \beta} = -\frac{1}{2}, \quad (2)$$

then  $\sin(\alpha - \beta) = 0$ ,  $\alpha = \beta + n\pi$ , and  $\frac{\cos \alpha}{\cos \beta} = \pm 1$ , contradicting (2).

Let  $t = \frac{\cos \alpha}{\cos \beta}$ ; then  $\cos \alpha = t \cos \beta$ ,  $\sin \alpha = -(1+t) \sin \beta$ , and it follows from the above discussion that  $t \neq 0$  and  $t \neq -\frac{1}{2}$ . Now

$$1 = \cos^2 \alpha + \sin^2 \alpha = t^2 \cos^2 \beta + (1+t)^2 \sin^2 \beta = t^2 + (1+2t) \sin^2 \beta,$$

and so

$$\sin^2 \beta = \frac{1-t^2}{1+2t}, \quad \cos^2 \beta = \frac{2t+t^2}{1+2t};$$

hence

$$\begin{aligned} \frac{\cos^3 \beta}{\cos \alpha} + \frac{\sin^3 \beta}{\sin \alpha} &= \frac{\cos^2 \beta}{\left(\frac{\cos \alpha}{\cos \beta}\right)} + \frac{\sin^2 \beta}{\left(\frac{\sin \alpha}{\sin \beta}\right)} \\ &= \frac{2t+t^2}{t(1+2t)} - \frac{1-t^2}{(1+t)(1+2t)} = 1. \end{aligned}$$

## II. Solution by Léo Sawé, Algonquin College.

The theorem should be an equivalence rather than a mere implication, that is,

$$\frac{\cos \alpha}{\cos \beta} + \frac{\sin \alpha}{\sin \beta} = -1 \quad (1)$$

if and only if

$$\frac{\cos^3 \beta}{\cos \alpha} + \frac{\sin^3 \beta}{\sin \alpha} = 1. \quad (2)$$

First we note that

$$\cos \beta \neq 0, \sin \beta \neq 0 \Leftrightarrow \cos \alpha \neq 0, \sin \alpha \neq 0.$$

This is easily proved as in solution I. Next we observe that (1) is equivalent to  $\sin(\alpha + \beta) = -\frac{1}{2} \sin 2\beta$ , or to

$$\sin(\beta + \beta) + \sin(\beta + \alpha) + \sin(\alpha + \beta) = 0. \quad (3)$$

Now by Problem 132 (see p. 67 in this issue), (3) holds if and only if the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of eccentric angles  $\alpha, \beta, \beta$  are concurrent; that is, if and only if the centre of curvature for  $\beta$ ,

$$\left( \frac{c^2}{a} \cos^3 \beta, -\frac{c^2}{b} \sin^3 \beta \right), \quad (4)$$

lies on the normal for  $\alpha$ ,

$$(a \sec \alpha)x - (b \csc \alpha)y = c^2, \quad (5)$$

where  $c^2 = a^2 - b^2$ ; and finally, substituting (4) into (5) shows that (3) holds if and only if (2) holds.

*Also solved by Dan Eustice, Ohio State University; G.D. Kaye, Department of National Defence; Leroy F. Meyers, Ohio State University; and F.G.B. Maskell, Algonquin College.*

104. [1976: 5] *Proposed by H.G. Dworschak, Algonquin College.*

Prove that  $\sqrt[3]{5} - \sqrt[4]{3}$  is irrational.

*I. Solution de Jacques Marion, Université d'Ottawa.*

Vu que  $\sqrt[4]{3}$  et  $\sqrt[3]{5}$  sont irrationnels, on peut considérer les extensions du corps de base  $Q$  (le corps des rationnels).

$$\begin{array}{cc} Q(\sqrt[4]{3}) & Q(\sqrt[3]{5}) \\ & \searrow \quad \swarrow \\ & Q \end{array}$$

D'après le critère d'Eisenstein, les polynômes  $X^4 - 3$  et  $X^3 - 5$  sont irréductibles sur  $Q$ . Donc

$$[Q(\sqrt[4]{3}):Q] = 4 \text{ (le degré de } Q(\sqrt[4]{3}) \text{ sur } Q \text{ est 4)}$$

et

$$[Q(\sqrt[3]{5}):Q] = 3 \text{ (le degré de } Q(\sqrt[3]{5}) \text{ sur } Q \text{ est 3)}.$$

Maintenant si  $\sqrt[3]{5} - \sqrt[4]{3} \in Q$ , alors  $Q(\sqrt[4]{3}) = Q(\sqrt[3]{5})$ , une contradiction immédiate. Donc  $\sqrt[3]{5} - \sqrt[4]{3}$  est irrationnel.

On peut démontrer de cette façon le résultat plus général suivant:

Si  $a$  et  $b$  sont des entiers positifs supérieurs à 1 qui ne possèdent aucun facteur carré, et si, pour  $m \neq n$ ,  $\sqrt[n]{a}$  et  $\sqrt[m]{b}$  sont irrationnels, alors les polynômes  $X^n - a$  et  $X^m - b$  sont irréductibles sur  $Q$  (Eisenstein) et les extensions  $[Q(\sqrt[n]{a}):Q]$  et  $[Q(\sqrt[m]{b}):Q]$  sont de degré  $n$  et  $m$  respectivement. Donc  $Q(\sqrt[n]{a}) \neq Q(\sqrt[m]{b})$  et  $\sqrt[n]{a} - \sqrt[m]{b}$  est irrationnel.

*II. Solution by Leroy F. Meyers, Ohio State University.*

Let  $x = \sqrt[3]{5} - \sqrt[4]{3}$ ; then  $x + \sqrt[4]{3} = \sqrt[3]{5}$  and so

$$5 = (x + \sqrt[4]{3})^3 = x^3 + 3\sqrt[4]{3}x^2 + 3\sqrt[4]{9}x + \sqrt[4]{27}.$$

But  $\sqrt[4]{9} = \sqrt{3}$ ; hence

$$5 - x^3 - 3\sqrt{3}x = \sqrt[4]{3}(3x^2 + \sqrt{3}),$$

and squaring both sides yields

$$25 + \dots + x^6 + (\dots)\sqrt{3} = \sqrt{3}(9x^4 + 3 + 6\sqrt{3}x^2),$$

where the dots indicate terms which contain neither the lowest nor the highest powers of  $x$ . Rewriting gives

$$25 + \dots + x^6 = \sqrt{3}(3 + \dots), \quad (*)$$

and squaring again yields

$$625 + \dots + x^{12} = 27 + \dots,$$

that is

$$x^{12} + \dots + 598 = 0,$$

where the missing coefficients are integers.

Now every rational root of this equation must be an integer (since the leading coefficient is 1). But  $-1 < x < 1$ , since  $1 < \sqrt[3]{5} < 2$  and  $1 < \sqrt[4]{3} < 2$ . Hence  $x = 0$ . But this is impossible since  $\sqrt[3]{5} \neq \sqrt[4]{3}$ , for their twelfth powers,  $5^4$  and  $3^3$ , are distinct.

Alternatively, starting from (\*), written out in detail, we may note that, if  $x$  is rational, then  $25 + \dots + x^6$  and  $3 + \dots$  must be 0, since otherwise  $\sqrt{3}$  would be rational. But direct calculation shows that the polynomials  $25 + \dots + t^6$  and  $3 + \dots$  in  $t$  are relatively prime, and so can have no common root at all, in particular no common rational root.

*Also solved by Léo Sauv , Algonquin College.*

*Editor's comment.*

This problem can be found, without solution, on p. 16 in *Advanced Calculus*, by Murray R. Spiegel, in the Schaum's Outline Series.

105. [1976: 5] *Proposed by Walter Bluger, Department of National Health and Welfare.*

INA BAIN declared once at a meeting  
That she'd code her full name (without cheating),  
Then divide, so she reckoned,  
The first name by the second,  
Thus obtaining five digits repeating.

*I. Solution by F.G.B. Maskell, Algonquin College.*

Said the Chairman: "Order please in the meeting!  
It is done while my gavel I'm beating.  
A is 4, I is 3,  
9 is N, 2 is B,  
Sixteen one fifty-four, then repeating."

101  $\overline{\text{BAIN}} = \frac{99,999}{9(41)(271)} = \frac{1}{(41)(2439)} = \frac{1}{2439} = .16154$ , since no other arrangement gives a denominator of four digits.



II. Comment by G.D. Kaye, Department of National Defence.

There is not enough information given to determine INA BAIN's age; but if she is not in her (relative) prime, then there are other solutions. For example,

$$\begin{aligned}\frac{\text{INA}}{\text{BAIN}} &= \frac{650}{4065} = .\dot{1}599\dot{0} \\ &= \frac{840}{1084} = .\dot{7}749\dot{0} \\ &= \frac{764}{1476} = .\dot{5}176\dot{1} \\ &= \frac{750}{3075} = .\dot{2}439\dot{0} \\ &= \frac{250}{1025} = .\dot{2}439\dot{0}.\end{aligned}$$

Also solved by the proposer.

Editor's comment.

This problem has already appeared, in slightly different form, with the same proposer, in the *Journal of Recreational Mathematics*, Vol. 8, No. 1 (1975), p. 47. The proposer was unaware of this prior publication. He believes a friend of his, to whom he mentioned the problem, must have submitted it in his name to the *Journal*.

106. [1976: 6] Proposed by Viktors Linis, University of Ottawa.

Prove that, for any quadrilateral with sides  $a, b, c, d$ ,

$$a^2 + b^2 + c^2 > \frac{1}{3}d^2.$$

I. Solution by M.S. Klamkin, University of Waterloo.

Consider any  $(n+1)$ -gon, coplanar or not, simple or not, with sides  $a_i$ ,  $i=1,2,\dots,n+1$ . Then by Hölder's Inequality (for  $m > 1$ ) (or power mean inequality),

$$(a_1^m + a_2^m + \dots + a_n^m)^{1/m} (1 + 1 + \dots + 1)^{1-1/m} \geq a_1 + a_2 + \dots + a_n,$$

with equality iff  $a_1 = a_2 = \dots = a_n$ . Also, by the triangle inequality,

$$a_1 + a_2 + \dots + a_n \geq a_{n+1};$$

thus

$$a_1^m + a_2^m + \dots + a_n^m \geq a_{n+1}^m / n^{m-1},$$

with equality iff  $a_1 = a_2 = \dots = a_n = a_{n+1}/n$ , i.e. the polygon is degenerate.

The desired result corresponds to the special case  $m=2$ ,  $n=3$ . Also Problem 74 [1975: 71; 1976: 10] corresponds to the special case  $m=2$ ,  $n=2$ .

II. *Solution by F.G.B. Maskell, Algonquin College.*

Expanding and adding  $(b-c)^2 \geq 0$ ,  $(c-a)^2 \geq 0$ ,  $(a-b)^2 \geq 0$ , we obtain

$$2\Sigma a^2 \geq 2\Sigma bc;$$

hence

$$3\Sigma a^2 \geq \Sigma a^2 + 2\Sigma bc = (\Sigma a)^2 > d^2,$$

since in a quadrilateral  $a+b+c > d$ , and the desired inequality follows.

*Also solved by G.D. Kaye, Department of National Defence; and Leroy F. Meyers, Ohio State University.*

107. [1976: 6] *Proposed by Viktors Linis, University of Ottawa.*

For which integers  $m$  and  $n$  is the ratio  $\frac{4m}{2m+2n-mn}$  an integer?

*Solution by G.D. Kaye, Department of National Defence.*

We will find all integral solutions  $(m,n,x)$ , where

$$x = \frac{4m}{2m+2n-mn} = \frac{4m}{4-(m-2)(n-2)}. \quad (1)$$

First there are the following obvious infinite sets of solutions:

$$(0,n,0), \quad \text{all } n \neq 0 \quad (2)$$

$$(m,0,2), \quad \text{all } m \neq 0 \quad (3)$$

$$(2,n,2), \quad \text{all } n, \quad (4)$$

$$(m,2,m), \quad \text{all } m. \quad (5)$$

We now assume  $x \neq 0$  and  $m \neq 2$ . Equation (1) can be rewritten as

$$n = 2 \cdot \frac{m}{m-2} \cdot \frac{x-2}{x}. \quad (6)$$

It is easy to see that, for  $m$  and  $x$  integral,

$$-1 \leq \frac{m}{m-2} \leq 3 \quad \text{and} \quad -1 \leq \frac{x-2}{x} \leq 3;$$

hence, from (6), we must have  $-6 \leq n \leq 18$ . Substituting successively  $n = -6, -5, \dots, 18$  in (1) yields the following additional solutions not included in (2)-(5).

$$n = -6: (1, -6, -1), (3, -6, 1);$$

$$n = -4: (1, -4, -2), (4, -4, 1);$$

$$n = -3: (1, -3, -4), (6, -3, 1);$$

$$n = -1: (1, -1, 4), (-2, -1, 1);$$

$$n = 1: (6, 1, 3), (-1, 1, -4), (-3, 1, 12), (-4, 1, 8), (-6, 1, 6), (-10, 1, 5);$$

$$n = 3: (30, 3, -5), (18, 3, -6), (14, 3, -7), (12, 3, -8), (10, 3, -10), (9, 3, -12), \\ (8, 3, -16), (7, 3, -28), (5, 3, 20), (4, 3, 8), (3, 3, 4), (-2, 3, -1), \\ (-6, 3, -2), (-18, 3, -3);$$

$$n = 4: (12, 4, -3), (8, 4, -4), (6, 4, -6), (5, 4, -10), (3, 4, 6), (-4, 4, -1);$$

$$n = 5: (3, 5, 12), (4, 5, -8), (5, 5, -4), (6, 5, -3), (10, 5, -2);$$

$n = 6: (4,6,-4), (6,6,-2);$   
 $n = 7: (3,7,-12), (14,7,-1);$   
 $n = 8: (3,8,-6), (4,8,-2), (8,8,-1);$   
 $n = 9: (3,9,-4), (6,9,-1),$   
 $n = 10: (3,10,-3), (5,10,-1);$   
 $n = 12: (3,12,-2), (4,12,-1);$   
 $n = 16: (3,18,-1).$

There are thus 53 integral solutions, in addition to the infinite sets of solutions (2)-(5).

*Also solved (partially) by Walter Bluger, Department of National Health and Welfare.*

*Editor's comment.*

The formula in this problem has an unexpected geometrical application.

Each convex regular polyhedron can be characterized by a *Schläfli symbol*  $\{p, q\}$  which means that it has  $p$ -gonal faces,  $q$  at each vertex. If  $V$ ,  $E$ ,  $F$  represent respectively the number of vertices, edges, and faces, then it is a consequence of Euler's formula  $V - E + F = 2$  that we must have

$$V = \frac{4p}{2p + 2q - pq}, \quad E = \frac{2pq}{2p + 2q - pq}, \quad F = \frac{4q}{2p + 2q - pq}.$$

Since these numbers must be positive, we must have  $2p + 2q - pq > 0$  or

$$(p - 2)(q - 2) < 4.$$

Thus  $p - 2$  and  $q - 2$  are two positive integers whose product is less than 4, that is,

$$(p - 2)(q - 2) = 1 \cdot 1 \quad \text{or} \quad 2 \cdot 1 \quad \text{or} \quad 1 \cdot 2 \quad \text{or} \quad 3 \cdot 1 \quad \text{or} \quad 1 \cdot 3.$$

It is an immediate consequence that there are just five convex regular polyhedra; their Schläfli symbols are

$$\{3,3\}, \quad \{4,3\}, \quad \{3,4\}, \quad \{5,3\}, \quad \{3,5\}.$$

The above information, and much more, can be found in H.S.M. Coxeter, *Introduction to Geometry*, Second Edition, John Wiley and Sons, 1969, pp. 152-154.

108. [1976: 6] *Proposed by Viktors Linis, University of Ottawa.*

Prove that, for all integers  $n \geq 2$ ,  $\sum_{k=1}^n \frac{1}{k^2} > \frac{3n}{2n+1}$ .

I. *Solution by L.F. Meyers, Ohio State University.*

We have, for  $n \geq 1$ ,

$$\frac{3n}{2n+1} = \left( \frac{3 \cdot 1}{2 \cdot 1 + 1} - \frac{3 \cdot 0}{2 \cdot 0 + 1} \right) + \left( \frac{3 \cdot 2}{2 \cdot 2 + 1} - \frac{3 \cdot 1}{2 \cdot 1 + 1} \right) + \dots + \left( \frac{3n}{2n+1} - \frac{3(n-1)}{2(n-1)+1} \right).$$

But, for  $k \geq 1$ ,

$$\frac{3k}{2k+1} - \frac{3(k-1)}{2(k-1)+1} = \frac{3}{4k^2-1} \leq \frac{3}{4k^2-k^2} = \frac{1}{k^2},$$

equality holding only if  $k=1$ ; hence, if  $n \geq 2$ ,

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} > \frac{3n}{2n+1}.$$

II. *Solution by F.G.B. Maskell, Algonquin College.*

For  $n \geq 2$ , let

$$a_n = \sum_{k=1}^n \frac{1}{k^2} \quad \text{and} \quad b_n = \frac{3n}{2n+1};$$

then

$$b_n - b_{n-1} = \frac{3}{4n^2-1} < \frac{3+1}{4n^2} = \frac{1}{n^2} = a_n - a_{n-1},$$

and so

$$a_n - b_n > a_{n-1} - b_{n-1}. \quad (1)$$

Since, for  $n=2$ ,  $a_{n-1} - b_{n-1} = 0$ , it follows from (1) and mathematical induction that  $a_n - b_n > 0$  for all  $n \geq 2$ , and this is equivalent to the desired inequality.

*Also solved by G.D. Kaye, Department of National Defence; and André Ladouceur, École Secondaire De La Salle.*

109. [1976: 6] *Proposed by Léo Sauvé, Algonquin College.*

(a) Prove that rational points (i.e. both coordinates rational) are dense on any circle with rational centre and rational radius.

(b) Prove that if the radius is irrational the circle *may* have infinitely many rational points.

(c) Prove that if even one coordinate of the centre is irrational, the circle has at most two rational points.

*Solution by the proposer.*

(a) Let  $C_1$  be a circle with rational centre  $(h, k)$  and rational radius  $r$ . We may without loss of generality assume  $C_1$  to be the unit circle  $x^2 + y^2 = 1$ , since the equations

$$x = h + r \cos \theta, \quad y = k + r \sin \theta \quad (1)$$

show that  $(x, y)$  is a rational point if and only if  $(\cos \theta, \sin \theta)$  is a rational point.

We will show that rational points are dense on the first quadrant arc of the circle; that is, if P and Q are any two distinct points on the first quadrant arc (see Figure 1), then there exists a rational point S between P and Q. The symmetry of the figure will then ensure the density of rational points on the whole circumference.

If A is the point  $(-1, 0)$ , let the lines AP and AQ intersect the line  $x = 1$  in

the points  $(1,p)$  and  $(1,q)$ . Let  $s$  be a rational number between  $p$  and  $q$  (the existence of such a number  $s$  is guaranteed by one of the fundamental properties of the real number system). The line joining  $A$  to the point  $(1,s)$  clearly intersects the circle in a point  $S$  which lies between  $P$  and  $Q$ . We show that  $S$  is a rational point of the circle.

If  $AS$  makes an angle  $\theta$  with the positive direction of the  $x$ -axis, then the coordinates of  $S$  are

$$(\cos 2\theta, \sin 2\theta) = \left( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}, \frac{2 \tan \theta}{1 + \tan^2 \theta} \right).$$

Thus  $S$  is a rational point since  $\tan \theta = s/2$  is rational, and the proof is complete.

(b) Let  $C_2$  be a circle with rational centre  $(h,k)$  and radius  $r$ . We may without loss of generality assume that the centre is at the origin (see Figure 2) since, from (1), the point  $(x,y)$  is rational if and only if the point  $(r \cos \theta, r \sin \theta)$  is rational.

There are circles with irrational radius which contain rational points; for example, the circle  $x^2 + y^2 = 2$  contains the rational point  $(1,1)$ . We will show that in all such cases the rational points are not merely infinite in number, but are also *dense* on the circle.

Suppose  $P(x_0, y_0)$  is a rational point on circle  $C_2$ , whose equation is  $x^2 + y^2 = r^2$ . A straightforward calculation shows that the line through  $P$  with slope  $m$  meets the circle again in the point

$$S = \left( \frac{(m^2 - 1)x_0 - 2my_0}{m^2 + 1}, -\frac{2mx_0 + (m^2 - 1)y_0}{m^2 + 1} \right). \quad (2)$$

Now let  $Q_1$  and  $Q_2$  be two distinct points on the circle, and let the lines  $PQ_1$  and  $PQ_2$  have slopes  $m_1$  and  $m_2$ . If  $m'$  is a rational number between  $m_1$  and  $m_2$ , then the line through  $P$  with slope  $m'$  meets the circle again in a point  $S$  which lies

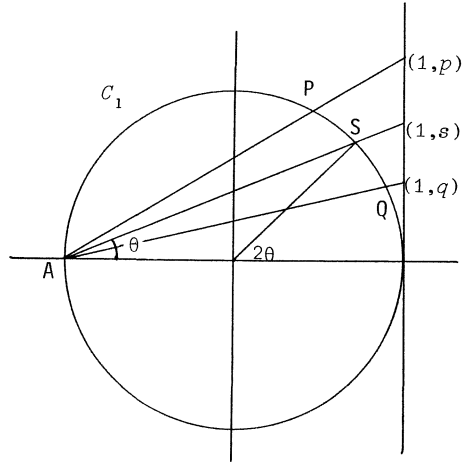


Figure 1

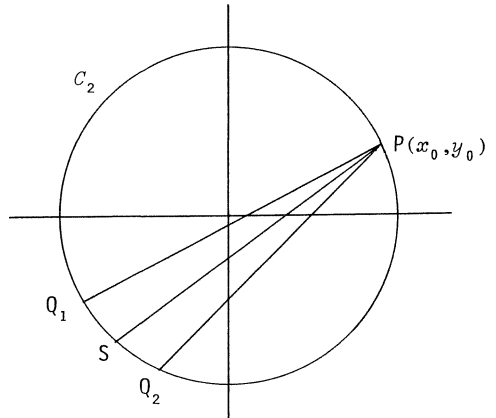


Figure 2

between  $Q_1$  and  $Q_2$ , and  $S$  is a rational point by (2). This completes the proof.

(c) A problem equivalent to part (c) was proposed by V.E. Dietrich in [1], and a solution by C.S. Ogilvy was published in [3]. According to Ogilvy's solution, if a circle contains three distinct rational points, substituting their coordinates successively in the equation

$$x^2 + y^2 + Ax + By + C = 0$$

determines rational values of  $A$ ,  $B$ ,  $C$ , and it follows that the centre  $(-\frac{1}{2}A, -\frac{1}{2}B)$  of the circle has both its coordinates rational, which contradicts the hypothesis.

However, a circle with an irrational centre *can* have rational points, though fewer than three. Ogilvy gives these examples:

$$(x - \sqrt{2})^2 + y^2 = 3 \text{ has } (0,1), (0,-1),$$

$$(x - \sqrt{2})^2 + y^2 = 3 - 2\sqrt{2} \text{ has only } (1,0),$$

$$(x - \sqrt{2})^2 + y^2 = \sqrt{3} \text{ has none.}$$

*Also solved by G.D. Kaye, Department of National Defence; and by Jacques Marion and John Thomas, both from the University of Ottawa (jointly).*

*Editor's comment.*

Let us call a circle a *rational circle* if more than two of its points are rational. It follows from this problem that for every rational circle (1) the centre is rational, (2) it contains infinitely many rational points, and (3) these rational points are dense on the circle.

Penner asks in [4] if there is a point of the plane which is not on any rational circle. Holley replied in [2] by proving that the point  $(e,e)$  is not on any rational circle. More generally, she indicated that if  $t$  is transcendental, and if  $f(t)$  and  $g(t)$  are polynomials in  $t$  with rational coefficients in which the degrees of  $f$  and  $g$  are not both zero, then  $(f(t), g(t))$  does not lie on any rational circle.

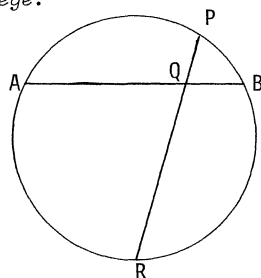
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2. Ann D. Holley, Solution to Problem 22, *The Two-Year College Mathematics Journal*, Vol. 6 (1975), No. 2, p. 32.
3. C.S. Ogilvy, Solution to Problem E832, *American Mathematical Monthly*, Vol. 56 (1949), p. 407.
4. Sidney Penner, Problem 22, *The Two-Year College Mathematics Journal*, Vol. 5 (1974), spring issue.

110. [1976: 6] *Proposed by H.G. Dworschak, Algonquin College.*

(a) Let AB and PR be two chords of a circle intersecting at Q. If A, B, P are kept fixed, characterize geometrically the position of R for which the length of QR is maximal (see figure).

(b) Give a Euclidean construction for the point R which maximizes the length of QR, or show that no such construction is possible.



I. *Solution of (a) by the proposer.*

We will show that the length of QR is maximal when QR is bisected by the diameter perpendicular to AB.

Without loss of generality, we assume the circle to be the unit circle; that AB lies on the line

$$y = k, \quad -1 < k < 1;$$

and that the point  $P(\cos \alpha, \sin \alpha)$  lies on the right half of the upper arc determined by chord AB, so that  $\arcsin k \leq \alpha \leq \frac{\pi}{2}$ . It is easy to see that, for the special cases  $\alpha = \arcsin k$  and  $\alpha = \frac{\pi}{2}$ ,  $QR_{\max}$  does indeed satisfy the characterization given above, so we will in the sequel assume that  $\arcsin k < \alpha < \frac{\pi}{2}$ .

Let  $R(\cos \theta, \sin \theta)$  be any point in the lower arc determined by chord AB, so that

$$-\pi - \arcsin k < \theta < \arcsin k,$$

and let PR intersect AB in  $Q(x, k)$ . Finally let  $S(\cos \theta, k)$  be the foot of the perpendicular from R upon AB. We assume that PR is not vertical (otherwise QR is clearly not maximal), so that  $\cos \alpha, x, \cos \theta$  are all distinct. To simplify the analysis, we make one final assumption, which is geometrically fairly obvious. We assume that QR is maximal only if R lies between the points  $R'$  and  $R''$  indicated in Figure 1. (These are the points considered in the Erdős-Klamkin inequality of Problem 75 [1975: 71; 1976: 10].) The reason for this assumption is that it

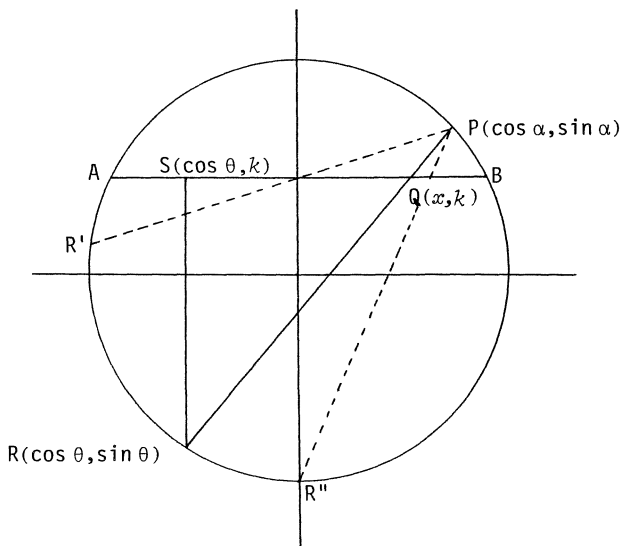


Figure 1

ensures that  $x$  and  $\cos \theta$  are of opposite signs, a fact which will be useful later on.

Equating the slopes of PQ and PR gives

$$\frac{k - \sin \alpha}{x - \cos \alpha} = \frac{\sin \theta - \sin \alpha}{\cos \theta - \cos \alpha},$$

whence

$$x = \frac{(k - \sin \alpha)(\cos \theta - \cos \alpha)}{\sin \theta - \sin \alpha} + \cos \alpha \quad (1)$$

and thus

$$\overline{QR}^2 = (x - \cos \theta)^2 + (k - \sin \theta)^2 \quad (2)$$

$$= \left[ \frac{(k - \sin \alpha)(\cos \theta - \cos \alpha)}{\sin \theta - \sin \alpha} + \cos \alpha - \cos \theta \right]^2 + (k - \sin \theta)^2. \quad (3)$$

Straightforward manipulations reduce (3) to

$$\overline{QR}^2 = (k - \sin \theta)^2 \sec^2 \frac{1}{2}(\theta + \alpha), \quad (4)$$

and so

$$QR = \pm(k - \sin \theta) \sec \frac{1}{2}(\theta + \alpha).$$

Now QR is maximal at an interior point in its domain, at which we must have

$$\frac{d}{d\theta}(QR) = \pm \frac{1}{2} \sec \frac{1}{2}(\theta + \alpha) [(k - \sin \theta) \tan \frac{1}{2}(\theta + \alpha) - 2 \cos \theta] = 0,$$

and thus when

$$\tan \frac{1}{2}(\theta + \alpha) = \frac{2 \cos \theta}{k - \sin \theta}. \quad (5)$$

It is clear from the preceding analysis that (5) has a unique solution within the prescribed range of  $\theta$ . Let  $\theta_0$  denote this solution, and let  $x_0$  be the corresponding value of  $x$  obtained from (1). We now get from (4) and (5)

$$\sec^2 \frac{1}{2}(\theta_0 + \alpha) - \tan^2 \frac{1}{2}(\theta_0 + \alpha) = \frac{\overline{QR}_{\max}^2 - 4 \cos^2 \theta_0}{(k - \sin \theta_0)^2} = 1,$$

and thus

$$\overline{QR}_{\max}^2 = 4 \cos^2 \theta_0 + (k - \sin \theta_0)^2. \quad (6)$$

Comparing (2) and (6) now gives

$$(x_0 - \cos \theta_0)^2 = 4 \cos^2 \theta_0,$$

whence  $x_0 = 3 \cos \theta_0$  or  $x_0 = -\cos \theta_0$ . The first possibility must be rejected since  $x_0$  and  $\cos \theta_0$  are of opposite signs. Thus  $x_0 = -\cos \theta_0$ , SQ is bisected by the diameter perpendicular to AB when QR is maximal, and so  $QR_{\max}$  is also bisected by this diameter, which completes the proof.

II. *Solution by Léo Sauvé, Algonquin College.*

(a) In Figure 2, let PQR be the chord through P such that QR is





the circumcircle of  $\triangle CR'Q'$  is smaller than the circumcircle of  $\triangle ERQ$ , and hence that the *chord*  $Q'R'$  of the first circle is shorter than the *diameter*  $QR$  of the second, which completes the proof.

(b) Introduce a coordinate system in the plane of Figure 2, so that the circle becomes the unit circle  $x^2 + y^2 = 1$ ; the equation of  $AB$  becomes  $y = k$ , with  $-1 < k < 1$ ; and the coordinates of  $P$  become  $(a, b)$ , with  $k < b < 1$ . We assume as in part (a) that the length of  $QR$  is maximal, and hence that  $QR$  is bisected by the  $y$ -axis. Let  $m$  be the slope of  $PR$ . It is clear that  $R$  is constructible by Euclidean means if and only if  $m$  is so constructible from the known values of  $a$ ,  $b$ , and  $k$ .

The equation of  $PR$  is

$$y - b = m(x - a), \quad (7)$$

and substituting  $y = k$  in (7) gives the abscissa of  $Q$ ,

$$x_Q = \frac{ma - b + k}{m}.$$

Substituting the value of  $y$  from (7) into the equation of the circle yields

$$x^2 + [m(x - a) + b]^2 = 1,$$

which reduces to

$$(m^2 + 1)x^2 - 2m(ma - b)x + m^2a^2 - 2abm + b^2 - 1 = 0. \quad (8)$$

Now one root of (8) is  $x = a$ ; and since the sum of the roots is  $\frac{2m(ma - b)}{m^2 + 1}$ , the other root (the abscissa of  $R$ ) is

$$x_R = \frac{2m(ma - b)}{m^2 + 1} - a = \frac{am^2 - 2bm - a}{m^2 + 1}.$$

Now the condition  $x_Q + x_R = 0$  yields after reduction

$$2am^3 + (k - 3b)m^2 + k - b = 0.$$

The slope  $m$ , being the root of a cubic, is in general not constructible by Euclidean means, and hence neither is the point  $R$ .

*Part (b) was also solved (partially) by M.S. Klamkin, University of Waterloo.  
Editor's comment.*

According to information I have received [1,4], the following persons, at least, have worked on this problem, beginning in 1974 or earlier:

Leon Bankoff, Los Angeles, California;  
Paul Erdős, Hungarian Academy of Science;  
Jack Garfunkel, Forest Hills High School, Flushing, N.Y.;  
Murray S. Klamkin, University of Waterloo;  
R. Robinson Rowe, Naubinway, Michigan.

So the problem is not new, as the proposer and I originally thought. Bankoff [1] informs me that the problem was first formulated by Garfunkel.

As far as I can tell, the results obtained so far by the above-mentioned persons were inconclusive or fragmentary.

Klamkin [4] proved that the point R which maximizes QR is not generally constructible in the special case when AB is a diameter, which provides a partial answer to part (b).

Rowe [5] showed that locating the point R depended upon solving a sixth degree equation, which he was unable to reduce.

The Erdős-Klamkin inequality [1975: 71; 1976: 10] appears to be one of the by-products of the research done on this problem.

Garfunkel [2,3] conjectured but did not prove that if QR is maximal then (see Figure 3) PQR lies between the median from P and the bisector of  $\angle P$  in  $\triangle APB$ .

Thus Dworschak's characterization given in solution I appears to be the first definitive answer to part (a).

The solver in solution II (a) was aware of Dworschak's result when he devised his own geometrical proof; he also used this result in his solution to part (b) which shows that locating R depends upon solving a cubic equation.

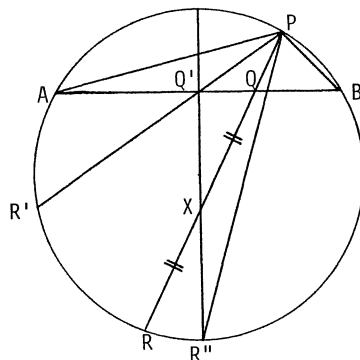


Figure 3

Dworschak's characterization enables us to give an affirmative answer to Garfunkel's conjecture. But this cannot be done from solution I, where the solver *assumed* that R was between R' and R'' (see Figure 3), an assumption equivalent to Garfunkel's conjecture since PR' is the median from P in  $\triangle PAB$  and PR'' is the bisector of  $\angle P$  (since arc AR'' = arc R''B). But no such assumption was made in solution II(a). From that solution and Figure 3, it is clear that when QR is maximal, we have  $0 < Q'X < Q'R''$ , and so PR lies between PR' and PR''.

#### REFERENCES

1. Leon Bankoff, communication to the editor.
2. Jack Garfunkel, communication to Leon Bankoff, used by permission of the recipient.
3. Jack Garfunkel, Problem E2538\*, *American Mathematical Monthly*, Vol. 82 (1975), p. 521.
4. M.S. Klamkin, communication to the editor.
5. R. Robinson Rowe, communication to Leon Bankoff, used by permission of the recipient.