Crux Mathematicorum

Published by the Canadian Mathematical Society.

http://crux.math.ca/

The Back Files

The CMS is pleased to offer free access to its back file of all issues of Crux as a service for the greater mathematical community in Canada and beyond.

Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

CRUX Mathematicorum

VOLUME 15 # 4

APRIL / AVRIL 1989

CONTENTS

The Olympiad Corner: No. 104 R.E. Woodrow 97

Problems: 1431-1440 109

Solutions: 1318-1331 111

Canadian Mathematical Society
Société Mathématique du Canada

ISSN 0705-0348 Second Class Mail Registration Number 5432

GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

G.W. Sands
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4

SUBSCRIPTION INFORMATION

Crux is published monthly (except July and August). The 1989 subscription rate for ten issues is $17.50 for members of the Canadian Mathematical Society and $35.00 for non-members. Back issues: $3.50 each. Bound volumes with index: volumes 1 & 2 (combined) and each of volumes 3, 4, 7, 8, 9 and 10: $10.00. (Volumes 5 & 6 are out-of-print). All prices quoted are in Canadian dollars. Cheques and money orders, payable to the CANADIAN MATHEMATICAL SOCIETY, should be sent to the Managing Editor:

Graham P. Wright
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario
Canada K1N 6N5

ACKNOWLEDGEMENT

The support of the Departments of Mathematics and Statistics of the University of Calgary and Carleton University, and of the Department of Mathematics of the University of Ottawa, is gratefully acknowledged.

Published by the Canadian Mathematical Society
Printed at Carleton University

THE OLYMPIAD CORNER
No. 104
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

The first problems we present are from the 7th annual American Invitational Mathematics Examination (A.I.M.E.) written March 21, 1989. The time allowed was three hours. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. The numerical solutions only will be published next month. Full solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322.

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
Tuesday, March 21, 1989

1. Compute $\sqrt{(31)(30)(29)(28)+1}$.

2. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

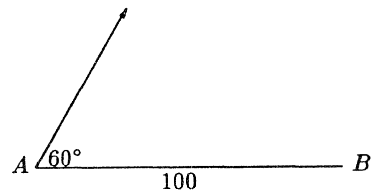
3. Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots\ .$$

4. If $a < b < c < d < e$ are consecutive positive integers such that $b + c + d$ is a perfect square and $a + b + c + d + e$ is a perfect cube, what is the smallest possible value of c?

5. When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let i/j, in lowest terms, be the probability that the coin comes up heads exactly 3 times out of 5. Find $i + j$.

6. Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second along a straight line that makes an angle of 60° with AB, as shown. At the same time that Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight line path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?



7. If the integer k is added to each of the numbers 36, 300 and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k.

8. Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 = 1$$
$$4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 = 12$$
$$9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 = 123.$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

9. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed that there is a positive integer n such that

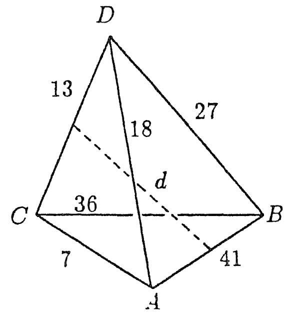
$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

Find the value of n.

10. Let a, b, c be the three sides of a triangle, and let α, β, γ, respectively, be the angles opposite them. If $a^2 + b^2 = 1989c^2$, find

$$\frac{\cot \gamma}{\cot \alpha + \cot \beta}.$$

11. A sample of 121 integers is given, each between 1 and 1000 inclusive, with repetitions allowed. The sample has a unique mode (most frequent value). Let D be the difference between the mode and the arithmetic mean of the sample. If D is as large as possible, what is $\lfloor D \rfloor$? (For real x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)



12. Let $ABCD$ be a tetrahedron with $AB = 41$, $AC = 7$, $AD = 18$, $BC = 36$, $BD = 27$, and $CD = 13$, as shown in the figure. Let d be the distance between the midpoints of edges $\overline{AB}$ and $\overline{CD}$. Find d^2.

13. Let S be a subset of $\{1,2,3,\ldots,1989\}$ such that no two members of S differ by 4 or 7. What is the largest number of elements S can have?

14. Given a positive integer n, it can be shown that every complex number of the form $r + si$, where r and s are integers, can be uniquely expressed in the base $-n + i$ using the integers $0,1,2,\ldots,n^2$ as "digits". That is, the equation

$$r + si = a_m(-n + i)^m + a_{m-1}(-n + i)^{m-1} + \cdots + a_1(-n + i) + a_0$$

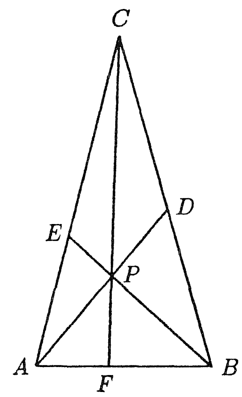
is true for a unique choice of non-negative integer m and digits $a_0,a_1,\ldots,a_m$ chosen from the set $\{0,1,2,\ldots,n^2\}$, with $a_m \neq 0$. We then write

$$r + si = (a_m a_{m-1} \cdots a_1 a_0)_{-n+i}$$

to denote the base $-n + i$ expansion of $r + si$. There are only finitely many integers $k + 0i$ that have four-digit expansions

$$k = (a_3a_2a_1a_0)_{-3+i} \qquad a_3 \neq 0.$$

Find the sum of all such k.



15. Point P is inside ΔABC. Line segments $\overline{APD}$, $\overline{BPE}$ and $\overline{CPF}$ are drawn with D on $\overline{BC}$, E on $\overline{CA}$, and F on $\overline{AB}$ (see the figure at the right). Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$ and $CF = 20$, find the area of ΔABC.

* * *

This month's Olympiad is the 1987 Hungarian National Olympiad. The questions were collected and translated by Gy. Karolyi and J. Pataki of the János Bolyai Mathematical Society. They were forwarded to me by Bruce Shawyer, of the Mathematics Department of Memorial University of Newfoundland.

1987 HUNGARIAN NATIONAL OLYMPIAD (ages 17–18)

1st Round

1. The surface area and the volume of a cylinder are equal to each other. Determine the radius and the altitude of the cylinder if both values are even integers.

2. Cut the regular (equilateral) triangle AXY from rectangle $ABCD$ in such a way that the vertex X is on side BC and the vertex Y is on side CD. Prove that among the three remaining right triangles there are two, the sum of whose areas equals the area of the third.

3. Determine the minimum of the function

$$f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2}$$

where a, b, c are positive numbers.

4. Consider points A and B on given rays (semilines) starting from C, such that the sum $CA + CB$ is a given constant. Show that there is a point $D \neq C$ such that for each position of A and B the circumcircle of triangle ABC passes through D.

5. Show that

(a) there exist 1324 different positive integers less than 1987 such that there are no three pairwise relatively prime ones among them;

(b) any set of 1325 different positive integers less than 1987 has three elements which are pairwise relatively prime.

2nd Round, Basic Level

6. N is a 4–digit perfect square all of whose decimal digits are less than seven. Increasing each digit by three we obtain a perfect square again. Find N.

7. Let a, b, c be the sides and α, β, γ be the opposite angles of a triangle. Show that if

$$ab^2\cos\alpha = bc^2\cos\beta = ca^2\cos\gamma$$

then the triangle is equilateral.

8. Let u and v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove or disprove that uv is rational.

2nd Round, Advanced Level

9. The lengths of the sides of a triangle are 3, 4 and 5. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

10. Let us build congruent regular pyramids on the faces of a $2 \times 2 \times 2$ cube, and call the solid obtained a "thorny cube". Determine the maximum of V/S where V and S denote the volume and the surface area of the thorny cube, respectively. Is it possible to tesselate 3–space with copies of the thorny cube realizing this maximum ratio?

11. The domain of function f is $[0,1]$, and for any $x_1 \neq x_2$

$$|f(x_1) - f(x_2)| < |x_1 - x_2|. \qquad (1)$$

Moreover, $f(0) = f(1) = 0$. Prove that for any x_1, x_2 in $[0,1]$,

$$|f(x_1) - f(x_2)| < 1/2.$$

2nd Round, Superior Level

12. We call a natural number *unary* if each digit of its decimal representation is 1. For what natural numbers m do there exist m unary numbers such that any two of them give different remainders on division by m?

13. Let f be a continuous function on the interval $[0,1]$ such that for each x with $0 < x < 1$ there is h such that

$$0 \leq x - h < x + h \leq 1$$

and

$$f(x) = \frac{f(x-h) + f(x+h)}{2}.$$

Show that $f(x) = x$ for all x in $[0,1]$.

14. The centroid of triangle $A_1A_2A_3$ is S, and its circumcircle is c. The second intersection of A_iS and c is B_i for $i = 1,2,3$. Show that

$$SB_1 + SB_2 + SB_3 \geq SA_1 + SA_2 + SA_3.$$

* * *

As promised we now begin the solutions to problems proposed for the 1987 I.M.O. in Havana, Cuba, and given in the November 1987 column. We have not received as complete a set of solutions as for the October 1987 problems, and readers are encouraged to send in nice solutions to fill the gaps.

Belgium 1. [1987: 277]

Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

Independent solutions by Curtis Cooper, Central Missouri State University; George Evagelopoulos, Law student, Athens, Greece; Douglass L. Grant, University College of Cape Breton, Sydney, N.S.; John Morvay, Dallas, Texas; and M.A. Selby, Department of Mathematics, University of Windsor, Ontario.

The value of n is such that in the prime factorization of $n!$, the factor 5 appears 1987 times, since the even numbers between multiples of 5 give ample powers of 2 to give a factor of 10^{1987}. Let $[x]$ denote the greatest integer in x. The largest power of 5 which divides $n!$ is well–known to be given by

$$h(n) = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \cdots + \left[\frac{n}{5^k}\right] ,$$

where k is largest such that $5^k \le n$.

The least n for which $h(n) = 1987$ is clearly a multiple of 5, and so it may be written in the form

$$n = \sum_{i=1}^{k} a_i 5^i$$

where each $a_i \in \{0,1,2,3,4\}$. Now

$$h(n) = \sum_{i=1}^{k} a_i h(5^i)$$

where

$$h(5^m) = 1 + 5 + 5^2 + \cdots + 5^{m-1} = \frac{1}{4}(5^m - 1).$$

Thus $h(5) = 1$, $h(5^2) = 6$, $h(5^3) = 31$, $h(5^4) = 156$, $h(5^5) = 781$. Since

$$1987 = 2\cdot781 + 2\cdot156 + 3\cdot31 + 3\cdot6 + 2\cdot1$$

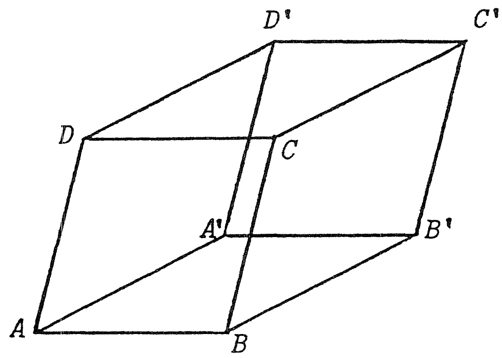
we conclude that

$$n = 2\cdot5^5 + 2\cdot5^4 + 3\cdot5^3 + 3\cdot5^2 + 2\cdot5 = 7960$$

satisfies $h(n) = 1987$. Clearly 7960 is the least such n.

France 2. [1987: 277]

Let $ABCDA'B'C'D'$ be an arbitrary parallelepiped as shown.



Establish the inequality

$$AB' + AD' + AC \le AB + AD + AA' + AC'.$$

When is there equality?

Solution by George Evagelopoulos, Law student, Athens, Greece and Murray S. Klamkin, Mathematics Department, The University of Alberta.

We set $\mathbf{X}_1 = \overrightarrow{AB}$, $\mathbf{X}_2 = \overrightarrow{AD}$, and $\mathbf{X}_3 = \overrightarrow{AA'}$, giving three vectors in 3–space. We must prove that

$$|\mathbf{X}_1+\mathbf{X}_2| + |\mathbf{X}_2+\mathbf{X}_3| + |\mathbf{X}_3+\mathbf{X}_1| \le |\mathbf{X}_1| + |\mathbf{X}_2| + |\mathbf{X}_3| + |\mathbf{X}_1+\mathbf{X}_2+\mathbf{X}_3|. \quad (1)$$

Let $<u|v>$ denote the usual inner (dot) product. Then, with summations cyclic over $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$,

$$\begin{aligned}\Big(\sum|\mathbf{X}_1|\Big)^2 - \Big|\sum\mathbf{X}_1\Big|^2 &= 2\sum(|\mathbf{X}_1||\mathbf{X}_2| - <\mathbf{X}_1|\mathbf{X}_2>)\\ &= \sum\{(|\mathbf{X}_1| + |\mathbf{X}_2|)^2 - |\mathbf{X}_1 + \mathbf{X}_2|^2\}\\ &= \sum(|\mathbf{X}_1|+|\mathbf{X}_2|+|\mathbf{X}_1+\mathbf{X}_2|)(|\mathbf{X}_1|+|\mathbf{X}_2|-|\mathbf{X}_1+\mathbf{X}_2|). \quad (2)\end{aligned}$$

Now

$$|\mathbf{X}_i| + |\mathbf{X}_j| - |\mathbf{X}_i + \mathbf{X}_j| \ge 0 \quad (3)$$

and

$$|\mathbf{X}_i| + |\mathbf{X}_j| + |\mathbf{X}_i + \mathbf{X}_j| \le \sum|\mathbf{X}_1| + \Big|\sum\mathbf{X}_1\Big|. \quad (4)$$

Thus, substituting into (2) we obtain

$$\begin{aligned}&\Big(\sum|\mathbf{X}_1|\Big)^2 - \Big|\sum\mathbf{X}_1\Big|^2\\ &\qquad\le \Big(\sum|\mathbf{X}_1| + \Big|\sum\mathbf{X}_1\Big|\Big)\cdot\sum(|\mathbf{X}_1| + |\mathbf{X}_2| - |\mathbf{X}_1 + \mathbf{X}_2|).\end{aligned}$$

Cancelling

$$\sum|\mathbf{X}_1| + \Big|\sum\mathbf{X}_1\Big|$$

from both sides gives

$$\sum |X_1| - \left|\sum X_1\right| \le \sum (|X_1| + |X_2| - |X_1 + X_2|)$$
$$= 2\sum |X_1| - \sum |X_1 + X_2|$$

which immediately gives the desired inequality (1).

In the case that some $X_i = 0$ or the sum $X_1 + X_2 + X_3 = 0$, we have obvious equality in (1). We thus assume below that $X_1 + X_2 + X_3 \neq 0$ and that no $X_i = 0$.

From the above proof, inequality (1) is strict unless equality holds in (3) for any distinct pair X_i, X_j such that strict inequality holds in (4). Now equality in (3) is equivalent to X_i and X_j being collinear and in the same direction. Equality in (4) is equivalent to the third vector X_k being collinear with, and in the opposite direction as, $X_1 + X_2 + X_3$.

Notice that equality in (3) for *two* pairs of vectors from $\{X_1,X_2,X_3\}$ implies, since no $X_i = 0$, that all three vectors are collinear and in the same direction. In this case equality in (1) is again clear. Thus we now assume equality *fails* in (3) for at least two pairs of vectors, say for (X_1,X_2) and (X_1,X_3). Then for equality in (1) to hold, equality in (4) must hold for these same pairs of vectors. This gives that X_2 and X_3 are both collinear with, and in the opposite direction as, $X_1 + X_2 + X_3$, which entails (since $X_1 + X_2 + X_3 \neq 0$) that X_1 is collinear with X_2 and X_3, in the opposite direction as them, and $|X_1| \ge |X_2 + X_3|$. Moreover, since equality holds in (3) for the pair (X_2,X_3), equality in (1) holds in this case.

In summary, equality holds in (1) precisely in the following cases:

(i) some $X_i = 0$;

(ii) $X_1 + X_2 + X_3 = 0$;

(iii) X_1, X_2, X_3 all lie in the same direction;

(iv) X_i and X_j lie in the same direction, X_k in the opposite direction, and $|X_k| \ge |X_i + X_j|$ (i, j, k being some permutation of 1, 2, 3).

Comments by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Alberta.

We use the notation from the above solution. The equivalent inequality (1) is known as Hlawka's inequality [1]. In [2] it is shown that an equivalent form for (1) is

$$|\mathbf{u}+\mathbf{v}+\mathbf{w}| + |\mathbf{v}+\mathbf{w}-\mathbf{u}| + |\mathbf{w}+\mathbf{u}-\mathbf{v}| + |\mathbf{u}+\mathbf{v}-\mathbf{w}| \ge 2(|\mathbf{u}| + |\mathbf{v}| + |\mathbf{w}|).$$

Also, the generalization (due to Levi)

$$\sum |\pm \mathbf{A}_1 \pm \mathbf{A}_2 \pm \cdots \pm \mathbf{A}_r| \geq 2\binom{r-1}{t} \sum_{i=1}^{r} |\mathbf{A}_i| ,$$

where the $\mathbf{A}_i$ are vectors in E^m, $t = [(r-1)/2]$, and the sum on the left hand side is taken over all combinations of $+$ and $-$ signs, is proven by an application of Levi's theorem [1]. Levi's theorem applied here states that to establish this inequality, it is sufficient to prove the result for all vectors in E^1 (i.e. for all real numbers).

A much wider generalization follows by applying Levi's theorem to the following result of Stankovic [3]:

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} (P_{i_1} + \cdots + P_{i_k}) f\left(\frac{P_{i_1}x_{i_1} + \cdots + P_{i_k}x_{i_k}}{P_{i_1} + \cdots + P_{i_k}}\right)$$

$$\leq \binom{n-2}{k-2}\left[\frac{n-k}{k-1}\sum_{i=1}^{n} P_i f(x_i) + \left(\sum_{i=1}^{n} P_i\right) f\left(\frac{P_1x_1 + \cdots + P_nx_n}{P_1 + \cdots + P_n}\right)\right].$$

Here $2 \leq k \leq n-1$, $n \geq 3$; $P_1, P_2, \ldots, P_n$ are positive numbers; f is a convex function from $[a,b]$ to $\mathbb{R}$; and the x_i's are positive numbers in $[a,b]$.

For Levi's theorem to apply we also assume that f is convex over all real numbers, as for example if $f(x) = |x^s|$, $s \geq 1$. Then the Stankovic result extends to the x_i being vectors in E^m.

References:

[1] D.S. Mitrinovic, *Analytic Inequalities*, Springer–Verlag, Heidelberg, 1970, pp. 171–172, 175.

[2] M.S. Klamkin, *International Mathematical Olympiads 1979–1985*, M.A.A., Washington, 1986, problem G.I./5, pp. 88–91.

[3] Lj.R. Stankovic, Prilozi teorji analitickih nejednakost, *Univ. Beograd Publ. Electrotehn. Fak. Ser. Mat. Fiz,* No. 543 (1976) pp. 1–47.

<u>Great Britain 1</u>. [1987: 277]

Find, with proof, the point P in the interior of an acute-angled triangle ABC for which $(BL)^2 + (CM)^2 + (AN)^2$ is a minimum, where L, M, N are the feet of the perpendiculars from P onto BC, CA, AB, respectively.

Solution by George Evagelopoulos, Law student, Athens, Greece and Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Alberta.

Since $(BL)^2 - (LC)^2 = (PB)^2 - (PC)^2$, etc.,

$$(BL)^2 + (CM)^2 + (AN)^2 = (LC)^2 + (MA)^2 + (NB)^2 = s,$$

say. Now

$$2s = ((BL)^2 + (LC)^2) + ((CM)^2 + (MA)^2) + ((AN)^2 + (NB)^2)$$
$$\geq 2\left[\left(\frac{BL + LC}{2}\right)^2 + \left(\frac{CM + MA}{2}\right)^2 + \left(\frac{AN + NB}{2}\right)^2\right].$$

Thus

$$s \geq \left(\frac{BC}{2}\right)^2 + \left(\frac{CA}{2}\right)^2 + \left(\frac{AB}{2}\right)^2$$

and equality obtains just if $BL = LC$, $CM = MA$, and $AN = NB$. As the perpendicular bisectors meet at the circumcentre, the desired point P is the circumcentre of the triangle.

<u>Great Britain 2</u>. [1987: 278]

Prove that if x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$ then $x + y + z \leq xyz + 2$.

Solutions by George Evagelopoulos, Law student, Athens, Greece and by Dr. David Monk (forwarded by Robert Lyness of Suffolk, England).

Write p for $x + y + z$ and r for xyz. It is enough to show that

$$E = 4 - (p - r)^2 \geq 0.$$

Now using $x^2 + y^2 + z^2 = 2$,

$$\begin{aligned} 4E &= 2^3 - 2^2(p^2 - 2) + 2(4pr) - 4r^2 \\ &= 2^3 - 2^2(2yz+2zx+2xy) + 2(4x^2yz+4xy^2z+4xyz^2) - 8x^2y^2z^2 + 4r^2 \\ &= (2 - 2yz)(2 - 2zx)(2 - 2xy) + 4r^2. \end{aligned}$$

Since

$$\begin{aligned} 2 - 2yz &= x^2 + (y - z)^2, \\ 2 - 2zx &= y^2 + (z - x)^2, \\ 2 - 2xy &= z^2 + (x - y)^2, \end{aligned}$$

the above quantities are non–negative. Thus, so also is E, completing the proof.

Alternate solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton, Canada.

Although not elementary, Lagrange multipliers provide a straightforward solution. Here the Lagrangian is

$$\mathscr{L} = x + y + z - xyz - \frac{\lambda(x^2 + y^2 + z^2)}{2}.$$

Then we set

$$\frac{\partial \mathscr{L}}{\partial x} = \frac{\partial \mathscr{L}}{\partial y} = \frac{\partial \mathscr{L}}{\partial z} = 0$$

to obtain

$$\begin{aligned} 1 - yz &= \lambda x \\ 1 - zx &= \lambda y \\ 1 - xy &= \lambda z. \end{aligned}$$

On subtraction, we get

$$(x - y)(z - \lambda) = 0 = (y - z)(x - \lambda).$$

Thus the critical points are $x = y = z$ and $x = y$, $z = (1 - x^2)/x$ and permutations thereof. The maximum value corresponds to the critical point $x = y$, $z = (1 - x^2)/x$. Since $x^2 + y^2 + z^2 = 2$ this leads to $(3x^2 - 1)(x^2 - 1) = 0$. Finally, the critical point (1,1,0) and permutations of it give the maximum value of $x + y + z - xyz$ to be 2.

[*Editor's note.* A third (and elementary) solution which involved consideration of cases was submitted by Zun Shan and Edward T.H. Wang of the Department of Mathematics, Wilfrid Laurier University.]

Greece 2. [1987: 278]

Show that if a, b, c are the lengths of the sides of a triangle and if $2s = a + b + c$, then

$$\frac{a^n}{b + c} + \frac{b^n}{a + c} + \frac{c^n}{a + b} \geq \left(\frac{2}{3}\right)^{n-2} s^{n-1}, \quad n \geq 1.$$

First solution by George Evagelopoulos, Law student, Athens, Greece, and by Bob Prielipp, University of Wisconsin–Oshkosh.

This solution only uses that a, b, c are positive. Without loss of generality we may assume that $a \leq b \leq c$, then

$$\frac{1}{b + c} \leq \frac{1}{a + c} \leq \frac{1}{a + b}$$

and we may apply the following known inequalities.

(i) Chebyshev's inequality: If $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \leq b_2 \leq \cdots \leq b_m$, then

$$\sum_{i=1}^{m} a_i b_i \geq \frac{1}{m}\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{i=1}^{m} b_i\right).$$

(ii) If a, b, c are positive numbers and n is a positive integer, then

$$(a + b + c)^n \leq 3^{n-1}(a^n + b^n + c^n).$$

(iii) If a, b, c are positive numbers, then

$$\frac{1}{b + c} + \frac{1}{a + c} + \frac{1}{a + b} \geq \frac{9}{2(a + b + c)}.$$

[For a proof of (i), see pp. 16–17 of *Elementary Inequalities* by Mitrinovic et al,

P. Noordhoff Ltd., Groningen, The Netherlands, 1964. For (ii) see p.17 of the same book. Finally (iii) follows from the arithmetic mean–harmonic mean inequality.]

Let $a_1 = a^n$, $a_2 = b^n$, $a_3 = c^n$ and

$$b_1 = \frac{1}{b+c}\,, \quad b_2 = \frac{1}{a+c}\,, \quad b_3 = \frac{1}{a+b}\,.$$

By (i)–(iii),

$$\frac{a^n}{b+c} + \frac{b^n}{a+c} + \frac{c^n}{a+b} \ge \frac{1}{3}(a^n + b^n + c^n)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right)$$

$$\ge \frac{1}{3^n}(a+b+c)^n \cdot \frac{9}{2(a+b+c)} = \left(\frac{2}{3}\right)^{n-2} s^{n-1}.$$

Second solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

We prove more generally that if $x_1,\ldots,x_m$ are positive then

$$\frac{x_1^n}{S-x_1} + \frac{x_2^n}{S-x_2} + \cdots + \frac{x_m^n}{S-x_m} \ge \frac{m}{m-1}\left(\frac{S}{m}\right)^{n-1} \tag{1}$$

where $S = x_1 + x_2 + \cdots + x_m$.

This rests on the power mean inequality,

$$\frac{\sum x_i^{n-1}\cdot x_i/(S-x_i)}{\sum x_i/(S-x_i)} \ge \left\{\frac{\sum x_i\cdot x_i/(S-x_i)}{\sum x_i/(S-x_i)}\right\}^{n-1}. \tag{2}$$

(The sums here and subsequently are all from $i = 1$ to $i = m$.) We also show that

$$\sum \frac{x_i}{S-x_i} \ge \frac{m}{m-1} \tag{3}$$

and

$$\frac{\sum x_i^2/(S-x_i)}{\sum x_i/(S-x_i)} \ge \frac{S}{m}\,. \tag{4}$$

Note that (1) is immediate from (2), (3) and (4).

Let $S - x_i = a_i$ so that $x_i = S - a_i$ and $a_1 + a_2 + \cdots + a_m = S(m-1)$. Rewrite (3) and (4) as

$$\sum \frac{S-a_i}{a_i} = S\sum \frac{1}{a_i} - m \ge \frac{m}{m-1} \tag{5}$$

and

$$\frac{\sum (S-a_i)^2/a_i}{\sum (S-a_i)/a_i} \ge \frac{S}{m} \tag{6}$$

respectively. On expanding (6), we get

$$S^2 \sum \frac{1}{a_i} - 2mS + (m-1)S \geq \frac{S}{m}\left(S \sum \frac{1}{a_i} - m\right)$$

or

$$(m-1)S \sum \frac{1}{a_i} \geq m^2,$$

which is also equivalent to (5). By Cauchy's inequality,

$$S \sum \frac{1}{a_i} = \frac{1}{m-1} \sum a_i \cdot \sum \frac{1}{a_i} \geq \frac{m^2}{m-1} .$$

This gives (5) and (6) and the result follows.

*　　　　　*　　　　　*

Next issue we continue the November 1987 I.M.O. solutions. Keep sending in your national contests and your nice solutions.

*　　　　　*　　　　　*

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **November 1, 1989,** *although solutions received after that date will also be considered until the time when a solution is published.*

1431. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Show that, for prime $p \equiv 1$, 3 or 5 mod 8,

$$\sum_{k=1}^{(p-1)/2} \tan \frac{k^2\pi}{p} = 3 \cdot \sum_{k=1}^{(p-1)/2} \cot \frac{k^2\pi}{p} . \tag{1}$$

(b)* Prove that (1) fails for all primes $p \equiv 7$ mod 8.

1432. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

If the Nagel point of a triangle lies on the incircle, prove that the sum of two of the sides of the triangle equals three times the third side.

1433. *Proposed by G.R. Veldkamp, De Bilt, The Netherlands.*

Let ABC be a triangle with sides $a > b > c$ and circumcircle Ω. Its internal angle–bisectors meet BC, CA and AB at D, E and F respectively. The line through B parallel to EF meets Ω again at Q, and P is on Ω such that $QP \| AC$. Prove that $\overline{PC} = \overline{PA} + \overline{PB}$.

1434. *Proposed by Harvey Abbott and Murray S. Klamkin, University of Alberta.*

It is known that

$$\frac{(3m)!(3n)!}{m!n!(m+n)!(n+m)!}, \quad \frac{(4m)!(4n)!}{m!n!(2m+n)!(2n+m)!}, \quad \frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

are all integers for positive integers m, n.

(i) Find positive integers m, n such that

$$I(m,n) = \frac{(6m)!(6n)!}{m!n!(4m+n)!(4n+m)!}$$

is *not* an integer.

(ii) Let A be the set of pairs (m,n), with $n \le m$, for which $I(m,n)$ is not an integer, and let $A(x)$ be the number of pairs in A satisfying $1 \le n \le m \le x$. Show that A has positive lower density in the sense that

$$\liminf_{x\to\infty} \frac{A(x)}{x^2} > 0.$$

1435. *Proposed by J.B. Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Find all pairs of integers x, y such that

$$(xy-1)^2 = (x+1)^2 + (y+1)^2.$$

1436. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

A point P lies on the circumcircle Γ of a triangle ABC, P not coinciding with one of the vertices. Circles Γ_1 and Γ_2 pass through P and are tangent to AB at B, and to AC at C, respectively. Γ_1 and Γ_2 intersect at P and at Q.

(a) Show that Q lies on the line BC.

(b) Show that as P varies over Γ the line PQ passes through a fixed point on Γ.

1437. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be an equilateral triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. If

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B},$$

prove that ABC is equilateral.

1438. *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville.*

AB is a fixed chord in a given circle, and C is a variable point on the circle, other than A and B. Prove that

$$\frac{\sin A + \cos B}{\sin\left(\frac{A-B}{2}\right) + \cos\left(\frac{A-B}{2}\right)}$$

is independent of C, where the angles A and B are the angles of ΔABC.

1439. *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University.*

Prove that

$$\frac{1}{6}[(2+\sqrt{3})^{2n-3} + (2-\sqrt{3})^{2n-3} + 2]$$

is a perfect square for all positive integers n.

1440*. *Proposed by Jack Garfunkel, Flushing, New York.*

Prove or disprove that if A, B, C are the angles of a triangle,

$$\frac{\sin A}{\sqrt{\sin A + \sin B}} + \frac{\sin B}{\sqrt{\sin B + \sin C}} + \frac{\sin C}{\sqrt{\sin C + \sin A}} \le \frac{3}{2}\cdot\sqrt[4]{3}.$$

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1318. [1988: 46] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Find, without calculus, the largest possible value of

$$\frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x}.$$

I. *Solution by L.J. Hut, Groningen, The Netherlands.*

Let

$$T = \frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x} = \frac{\sin 5x + \sin(90^\circ + 3x)}{\sin 4x + \sin(90^\circ + 4x)}$$

$$= \frac{2\sin\left(\frac{8x + 90^\circ}{2}\right)\cos\left(\frac{90^\circ - 2x}{2}\right)}{2\sin\left(\frac{8x + 90^\circ}{2}\right)\cos\left(\frac{90^\circ}{2}\right)}$$

$$= \frac{\cos(45^\circ - x)}{\cos 45^\circ} = \sqrt{2}\cos(45^\circ - x).$$

Then $\cos(45^\circ - x)$ is maximal (and equals 1) when $x = 45^\circ$. Thus $T_{\max} = \sqrt{2}$.

With calculus it takes much longer!

II. *Solution by J. Suck, Essen, Federal Republic of Germany.*

Par les théorèmes d'addition et le théorème de Pythagore,

$$\left(\frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x}\right)^2 = \left(\frac{\sin 4x\cos x + \cos 4x\sin x + \cos 4x\cos x + \sin 4x\sin x}{\sin 4x + \cos 4x}\right)^2$$

$$= (\sin x + \cos x)^2$$

$$= 1 + 2\sin x\cos x = 1 + \sin 2x \le 2.$$

On a l'égalité pour $x = \pi/4$, par exemple. Ainsi, la valeur cherchée est $\sqrt{2}$.

Also solved by SEUNG–JIN BANG, Seoul, Korea; DUANE M. BROLINE, Eastern Illinois University; CURTIS COOPER, Central Missouri State University; HANS ENGELHAUPT, Gundelsheim, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; H. FUKAGAWA, Aichi, Japan; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, Winnenden, Federal Republic of Germany; RICHARD I. HESS, Rancho Palos Verdes, California; ERIC HOLLEMAN, student, Memorial University of Newfoundland; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; SIDNEY KRAVITZ, Dover, New Jersey; KEE–WAI LAU, Hong Kong; Z.F. LI, University of Regina; J.A. MCCALLUM, Medicine Hat, Alberta; VEDULA N. MURTY, Penn State University at Harrisburg; M.A. SELBY, University of Windsor; COLIN SPRINGER, student, University of Waterloo; EDWARD T.H. WANG, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Janous and Klamkin generalized to the maximum value of

$$\frac{\sin(a + b)x + \cos(a - b)x}{\sin ax + \cos ax}$$

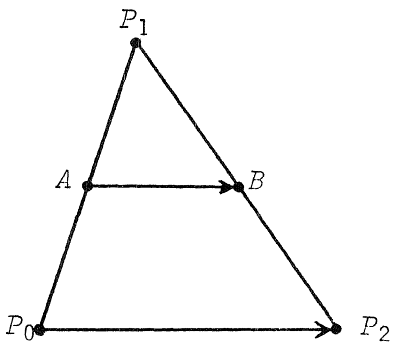
for a, b real.

* * *

1319. [1988: 46] *Proposed by Geng-zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.*

Let A, B, C be fixed points in the plane. An ant starts from a certain point P_0 of the plane and crawls in a straight line through A to a point P_1 such that $P_0A = AP_1$. It then crawls from P_1 through B to P_2 such that $P_1B = BP_2$, then through C to P_3, and so on. Altogether it repeats the same action 1986 times successively through the points A, B, C, A, B, C, A, ..., finally stopping in exhaustion at P_{1986}. Where is the ant now? (Compare with problem 2 of the 1986 I.M.O. [1986: 173].)

I. *Solution by Jordi Dou, Barcelona, Spain.*



The result of two successive symmetries with respect to the points A and B is a translation equal to $2\cdot\overrightarrow{AB}$. The result of the six symmetries with respect to A, B, C, A, B, C will be the translation

$$2\cdot\overrightarrow{AB} + 2\cdot\overrightarrow{CA} + 2\cdot\overrightarrow{BC},$$

which is equal to the identity. Thus the result of the $1986 = 6\cdot 331$ symmetries will be the identity. The ant finishes at P_0.

Note: The same result holds for $2k(2n + 1)$ successive symmetries through $2n + 1$ ordered points. If there are $2n$ points the result of $k\cdot 2n$ symmetries is $k\mathbf{T}$ where

$$\mathbf{T} = 2(\overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{2n-1}A_{2n}}).$$

II. *Solution by Eric Holleman, student, Memorial University of Newfoundland.*

Set up a Cartesian coordinate system with P_0 at the origin. Let $A = (a_1,a_2)$, $B = (b_1,b_2)$, $C = (c_1,c_2)$.

If the ant is at a point (x_0,y_0) and crawls in a straight line through another point (x_1,y_1) to a final point (x_2,y_2), such that the point (x_1,y_1) is equidistant from the other two, then

$$x_2 = x_1 - (x_0 - x_1) = 2x_1 - x_0,$$
$$y_2 = y_1 - (y_0 - y_1) = 2y_1 - y_0.$$

Thus the ant's journey starts as follows:

$$(0,0) \text{ through } A \text{ to } (2a_1,2a_2);$$

through B to $(2b_1 - 2a_1, 2b_2 - 2a_2)$;
through C to $(2a_1 - 2b_1 + 2c_1, 2a_2 - 2b_2 + 2c_2)$;
through A again to $(2b_1 - 2c_1, 2b_2 - 2c_2)$;
through B again to $(2c_1, 2c_2)$;
through C again to $(0,0)$.

After six legs of the journey the ant returns to his starting point. Since $1986 \equiv 0 \bmod 6$, after 1986 legs the ant will be at P_0.

Also solved by HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; Z.F. LI, University of Regina; P. PENNING, Delft, The Netherlands; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Festraets–Hamoir's proof was the same as Dou's. The other proofs were similar to solution II. Janous and Penning (like Dou) generalized to any odd number of points in n–dimensional space. Penning also considered the situation when a crawl after passing through a point extends beyond only a fraction $\lambda(< 1)$ of the distance to that point.

* * *

1320. [1988: 46] *Proposed by Themistocles M. Rassias, Athens, Greece.*

Assume that $a_1, a_2, a_3, \ldots$ are real numbers satisfying the inequality

$$|a_{m+n} - a_m - a_n| \le C$$

for all $m,n \ge 1$ and for some constant C. Prove that there exists a constant k such that

$$|a_n - nk| \le C$$

for all $n \ge 1$.

Solution by Jim Braselton, student, The Ohio State University.

For $n,m \in \mathbb{N}$ we have that

$$|a_{nm} - na_m| = \left| \sum_{i=0}^{n-2} (a_{(n-i)m} - a_{(n-i-1)m} - a_m) \right|$$

$$\le \sum_{i=0}^{n-2} |a_{(n-i)m} - a_{(n-i-1)m} - a_m|$$

$$\leq (n-1)C.$$

Therefore

$$\left| \frac{a_{nm}}{nm} - \frac{a_m}{m} \right| \leq \frac{(n-1)C}{nm} \leq \frac{C}{m}$$

so that

$$\begin{aligned}\left|\frac{a_n}{n} - \frac{a_m}{m}\right| &= \left|\frac{a_n}{n} - \frac{a_{nm}}{nm} + \frac{a_{nm}}{nm} - \frac{a_m}{m}\right| \\ &\leq \left|\frac{a_n}{n} - \frac{a_{nm}}{nm}\right| + \left|\frac{a_{nm}}{nm} - \frac{a_m}{m}\right| \\ &\leq C\left(\frac{1}{n} + \frac{1}{m}\right) . \qquad (1)\end{aligned}$$

Consequently $\{a_n/n\}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges. Let

$$k = \lim_{n\to\infty} \frac{a_n}{n} .$$

Fix $n \in \mathbb{N}$ and $\epsilon > 0$, and choose M such that

$$\left|\frac{a_m}{m} - k\right| < \frac{\epsilon}{2n} \text{ for } m \geq M$$

and also

$$\frac{nC}{M} < \frac{\epsilon}{2} .$$

Then for $m \geq M$,

$$\begin{aligned}|a_n - nk| &\leq n\left|\frac{a_n}{n} - \frac{a_m}{m}\right| + n\left|\frac{a_m}{m} - k\right| \\ &< nC\left(\frac{1}{n} + \frac{1}{m}\right) + n\cdot\frac{\epsilon}{2n} \\ &< C + \frac{\epsilon}{2} + \frac{\epsilon}{2} = C + \epsilon.\end{aligned}$$

Since ϵ was arbitrary, this establishes that $|a_n - nk| \leq C$.

Also solved by WALTER JANOUS, Ursulinengymnasium, Innsbruck, Austria; C. WILDHAGEN, Breda, The Netherlands; and the proposer. Two other readers sent in incorrect solutions.

Janous and Wildhagen noted that inequality (1) (*with* $C = 1$) *occurs as problem* 6 *of the* 1980 *Austrian–Polish Mathematics Competition* [1986: 5], *with a proof different from the above. Janous and the proposer also gave the above proof.*

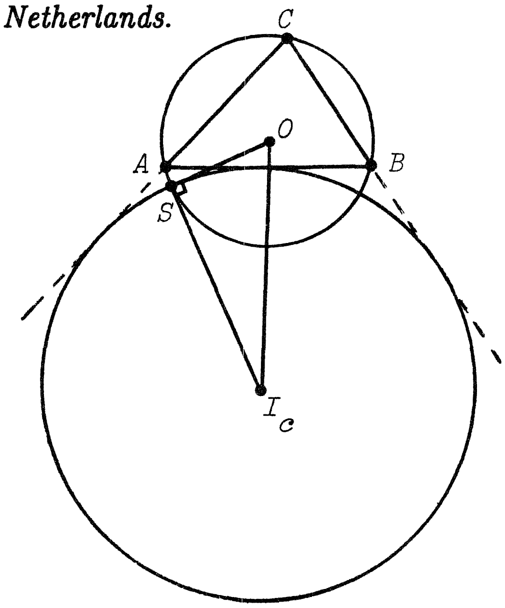
Len Bos, University of Calgary, pointed out the paper "On the stability of the linear functional equation", by D.H. Hyers, Proc. Nat. Acad. Sci. U.S.A. 27 (1941) 222–224, *in which the problem is credited to Ulam and solved in a more general setting.*

*　　　　　*　　　　　*

1321. [1988: 76] *Proposed by Jordi Dou, Barcelona, Spain.* (Dedicated in memoriam to Léo Sauvé.)

The circumcircle of a triangle is orthogonal to an excircle. Find the ratio of their radii.

Solution by J.T. Groenman, Arnhem, The Netherlands.



Let ABC be the triangle, with circumcircle (O,R) and excircle (I_c,r_c) meeting at a point S, where $\angle OSI_c = 90^\circ$. Then

$$R^2 + r_c^2 = \overline{OI_c}^2. \qquad (1)$$

It is known (p.187 of R.A. Johnson, *Advanced Euclidean Geometry*) that

$$\overline{OI_c}^2 = R^2 + 2Rr_c. \qquad (2)$$

Hence from (1) and (2)

$$r_c^2 = 2Rr_c,$$

so

$$R/r_c = 1/2.$$

Also solved by SVETOSLAV JOR. BILCHEV, Technical University, Russe, Bulgaria; CLAYTON W. DODGE, University of Maine, Orono; C. FESTRAETS–HAMOIR, Brussels, Belgium; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer.

Most solvers used equation (2), *Janous referring to p. 43 of E. Donath,* Die Merkwürdigen Punkte und Linien des Ebenen Dreiecks, *Berlin, 1969.*

* * *

1322. [1988: 76] *Proposed by M.S. Klamkin, University of Alberta, and D.J. Newman, Temple University.*

Show that closed form expressions exist for the two integrals

$$\int t^{p-1}(1-t)^{q-1}dt \qquad \text{and} \qquad \int t^{p-1}(1+t)^{q-1}dt$$

for the special case $p + q = 1$, p rational.

I. *Solution by Colin Springer, student, University of Waterloo.*

Let $p = a/b$, a, b integers, so that $q - 1 = -a/b$. Then

$$\int t^{p-1}(1 - t)^{q-1}dt = \int \frac{1}{t}\left(\frac{t}{1 - t}\right)^{a/b} dt.$$

Substitute

$$u^b = \frac{t}{1 - t};$$

then

$$t = \frac{u^b}{u^b + 1}, \qquad dt = \frac{bu^{b-1}}{(u^b + 1)^2}du,$$

so we have

$$\int \frac{1}{t}\left(\frac{t}{1 - t}\right)^{a/b} dt = \int \frac{bu^{a-1}}{u^b + 1}\, du.$$

It is well known that a closed–form expression exists for the integral of any rational function (see any calculus text). Thus a closed–form expression exists for the first given integral.

We proceed similarly for the second integral:

$$\int t^{p-1}(1 + t)^{q-1}dt = \int \frac{1}{t}\left(\frac{t}{1 + t}\right)^{a/b} dt, \quad \text{where } p = a/b;$$

$$u^b = \frac{t}{1 + t} \implies t = \frac{u^b}{1 - u^b} \quad \text{and} \quad dt = \frac{bu^{b-1}}{(1 - u^b)^2}du;$$

then

$$\int \frac{1}{t}\left(\frac{t}{1 + t}\right)^{a/b} dt = \int \frac{bu^{a-1}}{1 - u^b}du,$$

and again a closed–form expression exists for this integral.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

These two integrals are special cases of the following

Theorem. ([1], p.25; or [2], 2.202, p.71) Integrals of type

$$\int x^r(ax^t + b)^s\, dx, \qquad r,\ s,\ t, \text{ rational},$$

can be represented as a combination of finitely many elementary functions if and only if at least one of

$$s, \quad \frac{r + 1}{t}, \quad \frac{r + 1}{t} + s$$

is an integer.

References:

[1] A.P. Prudnikov, Ju. A. Brychkov, and O.I. Marichev, *Integrals and Series* (*Elementary Functions*), Moscow, 1981. (in Russian)

[2] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*,

Academic Press, 1980.

Also solved by KEE–WAI LAU, Hong Kong; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

Wildhagen remarked that the problem is a special case of a result of Tschebyscheff and referred to G.M. Fichtenholz, Differential– und Integralrechnung, II, §279.

* * *

1323. [1988: 76] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be the right triangle with $\angle A = 1$ radian and $\angle C = \pi/2$. Let I be the incenter, O the midpoint of AB, and N the midpoint of OC. Is ΔNIO an acute, obtuse, or right triangle?

Solution by Jordi Dou, Barcelona, Spain.

Suppose $\overline{AB} = 4$. Then $\overline{OA} = 2$, so $\overline{ON} = \overline{CN} = 1$. Also

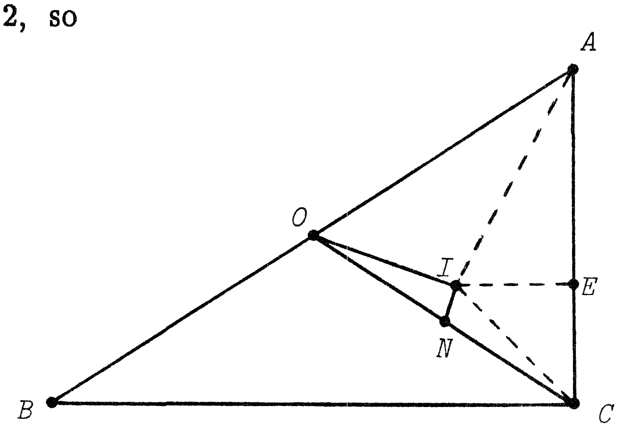
$$\angle ICN = \frac{\pi}{4} - \angle B = 1 - \frac{\pi}{4},$$

and the inradius r of ΔABC is

$$r = \overline{IE} = \overline{CE} = s - \overline{AB} = 2(\cos 1 + \sin 1 - 1)$$

where s is the semiperimeter. Hence

$$\overline{IN}^2 = \overline{CN}^2 + \overline{CI}^2 - 2\overline{CN}\cdot\overline{CI}\cdot\cos\angle ICN = 1 + 2r^2 - 2r\sqrt{2}\cos(1 - \pi/4) \approx 0.055910219$$



and

$$\overline{IO}^2 = \overline{AI}^2 + \overline{AO}^2 - 2\overline{AI}\cdot\overline{AO}\cdot\cos\angle OAI = \left(\frac{r}{\sin 1/2}\right)^2 + 4 - \frac{4r}{\tan 1/2} \approx 0.945813674,$$

so

$$\overline{IN}^2 + \overline{IO}^2 \approx 1.001723894 > 1 = \overline{NO}^2.$$

Therefore ΔNIO is an acute triangle. In fact $\angle NIO \approx 89^\circ 47'7''$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; and the proposer. There was one incorrect solution submitted.

* * *

1324. [1988: 76] *Proposed by Len Bos, University of Calgary.*

Let $S_i(x_1,x_2,\ldots,x_n)$ be the ith elementary symmetric function, i.e. $S_0 = 1$, $S_1 = x_1 + x_2 + \cdots + x_n$, $S_2 = \sum_{i<j} x_ix_j$, etc. Let J be the $n \times n$ Jacobian matrix of $S_1,\ldots,S_n$ with respect to $x_1,\ldots,x_n$; i.e.

$$J_{ij} = \frac{\partial S_i}{\partial x_j} .$$

Show that

$$\det J = \prod_{i<j} (x_j - x_i),$$

the Vandermonde determinant.

Solution by Colin Springer, student, University of Waterloo.

We see that if $x_i = x_j$ for $i \neq j$ then column i and column j of J are identical, so $\det J = 0$. Therefore (since $\det J$ is a polynomial) $x_i - x_j$ is a factor of $\det J$ for all $i \neq j$. Thus (since the degree of $\det J$ is clearly $1 + 2 + \cdots + (n - 1)$)

$$\det J = k \prod_{i<j} (x_i - x_j)$$

for some constant k.

Consider the term

$$kx_1^{n-1}x_2^{n-2} \cdots x_{n-1}$$

arising from multiplying the first term of each $x_i - x_j$ in the above product. In $\det J$ it is necessarily the result of multiplying terms

$$1\cdot x_1(x_1x_2)(x_1x_2x_3)\ldots(x_1x_2\cdots x_{n-1}).$$

Now, since no term involving x_k exists in column k of J,

term $x_1x_2\cdots x_{n-1}$ comes only from column n, row n,

thus

term $x_1x_2\cdots x_{n-2}$ comes only from column $n - 1$, row $n - 1$,

$\vdots$

term x_1 comes only from column 2, row 2,

and finally

term 1 comes only from column 1, row 1;

i.e. $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ arises only in multiplying together the main diagonal. Therefore it has coefficient 1, so $k = 1$ and

$$\det J = \prod_{i<j} (x_i - x_j).$$

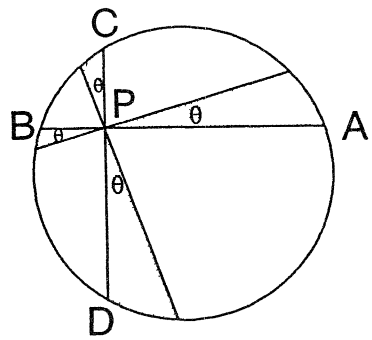
Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; M.S. KLAMKIN, University of Alberta; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

All solvers, including the proposer, noticed the minor correction to the original statement of the problem.

* * *

1325. [1988: 77] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

Let P be any point inside a unit circle. Perpendicular chords AB and CD pass through P. Two other chords passing through P form four angles of θ radians each as shown in the figure. Prove that the area of the shaded region is 2θ.

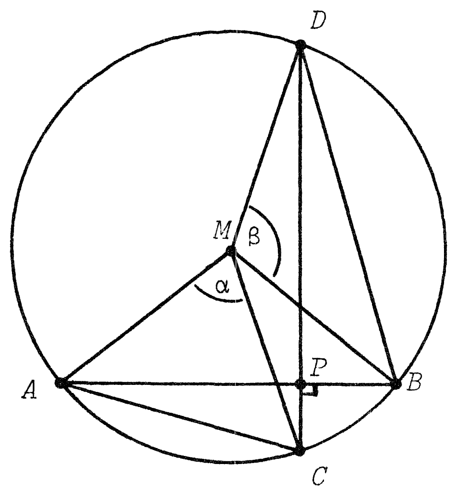


I. *Solution by Jörg Härterich, student, University of Stuttgart.*

I'm first going to prove

Lemma. If AB and CD are two perpendicular chords of a circle of radius r and the intersection of AB and CD is called P, then

$$\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = 4r^2.$$



Proof. Referring to the diagram, we have

$$\begin{aligned}\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 &= \overline{AC}^2 + \overline{BD}^2\\ &= 2r^2 - 2r^2\cos\,\alpha + 2r^2 - 2r^2\cos\,\beta\\ &= 4r^2 - 2r^2(\cos\,\alpha + \cos\,\beta).\end{aligned}$$

The lemma is proved if we can show that $\alpha + \beta = 180^\circ$. But this follows from

$$\frac{\alpha}{2} + \frac{\beta}{2} + 90^\circ = \angle ABC + \angle DBC + 90^\circ = 180^\circ. \qquad \square$$

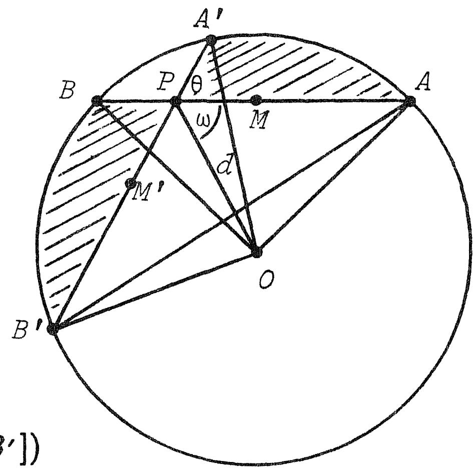
We now use this lemma to solve the problem. Choose polar coordinates with centre P and denote by s the distance from P to the circle. Then the area of the shaded region is

$$A = \frac{1}{2}\left(\int_0^\theta s^2 d\varphi + \int_{\pi/2}^{\pi/2+\theta} s^2 d\varphi + \int_\pi^{\pi+\theta} s^2 d\varphi + \int_{3\pi/2}^{3\pi/2+\theta} s^2 d\varphi\right).$$

After substitution, and by using the lemma, we get

$$A = \frac{1}{2}\int_0^{\theta}(s^2(\varphi) + s^2(\varphi + \tfrac{\pi}{2}) + s^2(\varphi + \pi) + s^2(\varphi + 3\pi/2))d\varphi$$
$$= \frac{1}{2}\int_0^{\theta}4d\varphi = 2\theta.$$

II. *Solution by Shiko Iwata, Gifu, Japan.* (Translated by Hidetosi Fukagawa.)



Let P be any point inside a unit circle with centre O and let AB and $A'B'$ be two chords through P. We first show that the area of the shaded region is

$$S_{ab}(\theta) = \theta + d^2\sin\theta\cos(2\omega + \theta),$$

where $\overline{OP} = d$, $\omega = \angle APO$, and $\theta = \angle APA'$.

Let $[X]$ denote the area of the region X. Then

$$S_{ab}(\theta) = [OAA'P] + [OB'BP] - ([OAP] + [OPB'])$$
$$= [OAA'] + [OBB'] + ([OPA'] - [OB'P]) + ([OBP] - [OAP])$$
$$= \frac{1}{2}\angle AOA' + \frac{1}{2}\angle BOB' + \frac{1}{2}d\cdot\sin(\theta+\omega)\cdot(\overline{A'P}-\overline{PB'}) + \frac{1}{2}d\cdot\sin\omega\cdot(\overline{PB}-\overline{AP}).$$

Now, letting M and M' be the midpoints of AB and $A'B'$ respectively,

$$\overline{A'P} - \overline{PB'} = (\overline{A'M'} + d\cdot\cos(\omega + \theta)) - (\overline{B'M'} - d\cdot\cos(\omega + \theta))$$
$$= 2d\cdot\cos(\omega + \theta)$$

and

$$\overline{PB} - \overline{AP} = (\overline{BM} - d\cdot\cos\omega) - (\overline{AM} + d\cdot\cos\omega) = -2d\cdot\cos\omega.$$

Thus

$$S_{ab}(\theta) = \angle AB'A' + \angle BAB' + d^2\sin(\omega + \theta)\cos(\omega + \theta) - d^2\sin\omega\cos\omega$$
$$= \theta + \frac{d^2}{2}(\sin 2(\omega + \theta) - \sin 2\omega)$$
$$= \theta + d^2\sin\theta\cos(2\omega + \theta).$$

The same argument works for the chord CD with ω replaced by $\omega + \pi/2$, i.e.

$$S_{cd}(\theta) = \theta + d^2\sin\theta\cos(2\omega + \pi + \theta) = \theta - d^2\sin\theta\cos(2\omega + \theta).$$

Therefore the required area is

$$S_{ab}(\theta) + S_{cd}(\theta) = 2\theta.$$

Also solved by JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; TADASI KAMIYA, Yokosuka High School, Aichi, Japan; M.S. KLAMKIN, University of Alberta; P. PENNING, Delft, The Netherlands; BRUCE SHAWYER, Memorial University of Newfoundland; DAN SOKOLOWSKY, Williamsburg, Virginia; COLIN SPRINGER, *student, University of Waterloo; and the proposer. One incorrect solution was sent in.*

Iwata's solution was the only one received which didn't use calculus. Several solutions, including the proposer's, were similar to solution I.

Klamkin generalized the problem to n chords through P with equal angles of π/n between successive chords. The area swept out when these chords are rotated through an angle θ about P then comes out to be $n\theta$. (This can also be proved using the result in solution II. Can it be proved as in solution I?)

* * *

1326. [1988: 77] *Proposed by R.S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.*

Solve

$$\cos^4 2\theta + 2\sin^2 2\theta = 17(\sin\theta + \cos\theta)^8$$

for $0 \le \theta \le 360°$.

Solution by Hans Engelhaupt, Franz-Ludwig-Gymnasium, Bamberg, Federal Republic of Germany.

The given equation can be written

$$(1 - \sin^2 2\theta)^2 + 2\sin^2 2\theta = 17(1 + \sin 2\theta)^4,$$

which with the substitution $x = \sin 2\theta$ becomes

$$1 + x^4 = 17(1 + x)^4$$

or, dividing by x^2,

$$\frac{1}{x^2} + x^2 = 17\left(\frac{1}{x} + 2 + x\right)^2.$$

The substitution $z = x + 1/x$ yields

$$z^2 - 2 = 17(z + 2)^2,$$

or

$$16z^2 + 68z + 70 = 2(4z + 7)(2z + 5) = 0,$$

so

$$z = -1.75 \quad \text{or} \quad z = -2.5.$$

Case (i): $x + 1/x = z = -1.75$. There is no solution.
Case (ii): $x + 1/x = z = -2.5$. Then $x = -1/2$ or -2. When $x = -2$ there is no solution for θ, so

$$\sin 2\theta = x = -1/2.$$

Thus

$$2\theta = 210° + k\cdot 360° \quad \text{or} \quad 2\theta = 330° + k\cdot 360°$$

so the solutions for θ are

$$105°,\ 285°,\ 165°,\ 345°.$$

Also solved by SEUNG–JIN BANG, Seoul, Korea; GINGER BOLTON, Swainsboro, Georgia; CLAYTON W. DODGE, University of Maine, Orono; C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; JORG HARTERICH, student, University of Stuttgart; RICHARD I. HESS, Rancho Palos Verdes, California; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; KEE–WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; M.A. SELBY, University of Windsor; BRUCE SHAWYER, Memorial University of Newfoundland; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; J. SUCK, Essen, Federal Republic of Germany; EDWARD T.H. WANG, Wilfrid Laurier University; C. WILDHAGEN, Breda, The Netherlands; and the proposer. Four other readers sent in partial solutions.

*   *   *

<u>1327</u>. [1988: 77] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let x_1, x_2, x_3 be the distances of the vertices of a triangle from a point P in the same plane. Let r be the inradius of the triangle, and p be the power of the point P with respect to the circumcircle of the triangle. Prove that

$$x_1x_2x_3 \geq 2rp.$$

Solution by Murray S. Klamkin, University of Alberta.

It is a known result [1] that if one takes a point on each of the three sides (extended or not) of a given triangle, then the circumradius of the triangle determined by the three points is greater than or equal to the inradius of the given triangle. We now show that the proposed inequality is equivalent to this result.

With the more usual notation of $x_i = R_i$, the given inequality is

$$R_1R_2R_3 \geq 2r(R^2 - \overline{OP}^2). \tag{1}$$

Letting $A_1A_2A_3$ be the triangle, we now consider the pedal triangle of P, whose vertices are the feet of the perpendiculars from P to the sides of $A_1A_2A_3$. It is known that the sides of this triangle are

$$R_1 \sin A_1, \quad R_2 \sin A_2, \quad R_3 \sin A_3,$$

and its area is

$$\frac{1}{2}|R^2 - \overline{OP}^2|\sin A_1 \sin A_2 \sin A_3$$

[e.g. see Theorems 190 and 198, pp. 136 and 139 of R.A. Johnson, *Advanced Euclidean Geometry*]. If P is on the circumcircle, the area of the pedal triangle is zero (the three vertices lie on a line called the pedal or Simson line), while if P is outside the circumcircle, the signed area is negative. We now use the result that the product of the sides of a triangle is four times the product of its circumradius and area, so that (1) reduces to $R' \geq r$, where R' is the circumradius of the pedal triangle, as was noted in the first paragraph.

References:

[1] M.S. Klamkin and G.A. Tsintsifas, The circumradius–inradius inequality for a simplex, *Mathematics Magazine* 52 (1979) 20–22.

[*Editor's note*: The proposer's original question had $|p|$ in the inequality rather than p (the absolute value was inadvertently left off by the editor). The above solution handles this stronger inequality.]

II. *Solution by the proposer*.

We consider the inversion $\mathcal{I}$ with pole P and power 1. Let ABC be the given triangle, with sides a, b, c and circumcircle (O,R), and let

$$A'B'C' = \mathcal{I}(ABC), \quad (O',R') = \mathcal{I}((O,R)).$$

Let a', b', c' be the sides of $A'B'C'$, x_1', x_2', x_3' the distances of A', B', C' from P, and p' the power of P with respect to (O',R'). Then it is well-known from inversion theory that

$$x_1 = \frac{1}{x_1'}\,, \quad x_2 = \frac{1}{x_2'}\,, \quad x_3 = \frac{1}{x_3'}\,, \tag{1}$$

$$a = \frac{a'}{x_2'x_3'}\,, \quad b = \frac{b'}{x_3'x_1'}\,, \quad c = \frac{c'}{x_1'x_2'}\,, \tag{2}$$

and

$$R' = |p'|\cdot R \tag{3}$$

[e.g. pp. 48–51 of R.A. Johnson, *Advanced Euclidean Geometry*]. Also, a well-known inequality for the triangle ABC is

$$ax_1 + bx_2 + cx_3 \geq 4F, \tag{4}$$

where F is the area of ABC (e.g. see *Crux* 866 [1984: 327]). From (1)–(4) follows

$$\begin{aligned} a' + b' + c' &= ax_2'x_3' + bx_3'x_1' + cx_1'x_2' = \frac{ax_1 + bx_2 + cx_3}{x_1x_2x_3} \\ &\geq \frac{4F}{x_1x_2x_3} = \frac{abc}{Rx_1x_2x_3} = \frac{a'b'c'}{R'x_1'x_2'x_3'}\cdot|p'| \\ &= \frac{4F'}{x_1'x_2'x_3'}\cdot|p'|, \end{aligned}$$

F' being the area of $A'B'C'$. Hence for the triangle ABC we have

$$2s = a + b + c \geq \frac{4F}{x_1x_2x_3}\cdot|p| = \frac{4sr}{x_1x_2x_3}\cdot|p|\;,$$

or

$$x_1x_2x_3 \geq 2r\cdot|p|.$$

There was one incorrect solution submitted.

* * *

1328. [1988: 77] *Proposed by Sharon Reedyk and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Use a combinatorial argument to establish the (known) identity

$$\binom{n}{0}\binom{n}{m} + \binom{n}{1}\binom{n-1}{m-1} + \cdots + \binom{n}{m}\binom{n-m}{0} = 2^m\binom{n}{m}.$$

Solution by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.

From a jar containing n different pieces of candy, Jimmy will choose m pieces, j of them to be placed in his left hand and $m - j$ of them to be placed in his right hand. How many different such selections are possible for all $j = 0,1,\ldots,m$?

(i) For $j = 0,1,\ldots,m$, Jimmy selects j pieces from the n in the jar with his left hand, and then selects $m - j$ pieces from the $n - j$ remaining with his right hand. The number of such selections is given by the expression on the left.

(ii) Jimmy chooses m pieces from the n in the jar with his right hand. He then selects a subset of these m pieces with his left hand. The number of all such selections is given by the expression on the right.

Also solved by FRANCISCO BELLOT ROSADO, Emilio Ferrari High School, Valladolid, Spain; JORG HARTERICH, student, University of Stuttgart; G.P. HENDERSON, Campbellcroft, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; PETER ROSS, Santa Clara University, Santa Clara, California; COLIN SPRINGER, student, University of Waterloo; J. SUCK, Essen, Federal Republic of Germany; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

All solutions were of course equivalent, and the editor chose the one that appealed to him the most. Here are two others. Suck counted the number of ways a professor and his student can choose m problems from the n that appear each month in an eminent periodical (actually Suck used n = 10), and then divide the m problems up between them. The proposers counted the number of ways of choosing m shoes from n distinct pairs so that no two shoes chosen form a pair.

* * *

1329. [1988: 77] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Let ABC be a triangle, and let congruent circles C_1, C_2, C_3 be tangent to half lines AB and AC, BA and BC, CA and CB, respectively.

(a) Determine the locus of the circumcentre P of ΔDEF, where D, E, F are the centres of C_1, C_2, C_3.

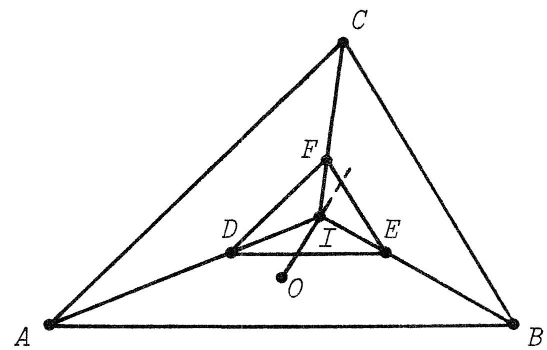
(b) If C_1, C_2, C_3 all pass through the same point, show that their radius ρ satisfies

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{R}$$

where r and R are the inradius and circumradius of ΔABC.

Solution by J.T. Groenman, Arnhem, The Netherlands.

(a) The centers D, E, F have to lie on the angle bisectors of angles A, B, C respectively. As the radii of the three circles are equal, $DE \| AB$, $EF \| BC$ and $FD \| CA$. Thus triangles ABC and DEF are similar and homothetic, with center I, the common point of the bisectors. Thus the locus of the circumcenter is the half-line OI.



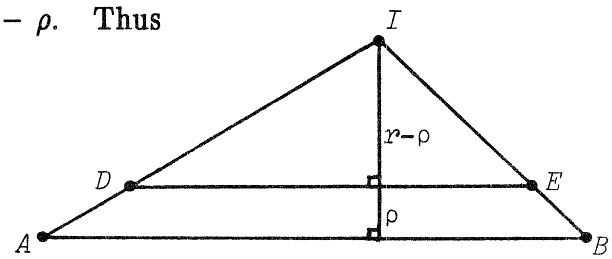
(b) If there is a point Q lying on the three congruent circles, then $QD = QE = QF$ as the radii are equal. Therefore Q is the circumcenter of ΔDEF and so lies on IO. The circumradius of ΔDEF is therefore ρ. I is the incenter of ΔDEF, so the inradius of ΔDEF is $r - \rho$. Thus

$$R : \rho = r : r - \rho,$$

$$Rr - R\rho = \rho r,$$

and so, dividing by $Rr\rho$,

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{R}.$$



Also solved by JORDI DOU, *Barcelona, Spain;* WALTHER JANOUS, *Ursulinengymnasium, Innsbruck, Austria;* P. PENNING, *Delft, The Netherlands;* COLIN SPRINGER, *student, University of Waterloo; and the proposer.*

Penning and Springer apparently assumed that the circles lie inside the triangle and so obtained only the segment OI as the answer. On the other hand, there seems to be another solution for (b), *namely*

$$\frac{1}{\rho} = \frac{1}{r} - \frac{1}{R},$$

for circles which extend outside the triangle, which none of the other solvers noticed!

Janous remarks that the problem extends problem 5 of the 1981 I.M.O.

* * *

1330. [1988: 77] *Proposed by M.A. Selby, University of Windsor, Windsor, Ontario.*

Find all positive integers j such that $[\sqrt{j}]$ divides into j, where $[x]$ is the greatest integer less than or equal to x.

Solution by Ginger Bolton, Swainsboro, Georgia.

Let n, j be positive integers. Then

$$\begin{aligned}[\sqrt{j}] = n &\iff n \le \sqrt{j} < n + 1 \\ &\iff n^2 \le j < (n + 1)^2.\end{aligned}$$

The only positive integers j satisfying $n^2 \le j < (n + 1)^2$ which are divisible by $n = [\sqrt{j}]$ are

$$n^2, \quad n^2 + n, \quad n^2 + 2n,$$

so these for $n = 1,2,\ldots$ are the answers to the problem.

Also solved by SEUNG–JIN BANG, Seoul, Korea; CLAYTON W. DODGE, University of Maine, Orono; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Federal Republic of Germany; C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; M.S. KLAMKIN, University of Alberta; SAM MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; M. PARMENTER, Memorial University of Newfoundland; P. PENNING, Delft, The Netherlands; BOB PRIELIPP, University of Wisconsin–Oshkosh; R.P. SEALY, Mount Allison University, Sackville, New Brunswick; BRUCE SHAWYER, Memorial University of Newfoundland; COLIN SPRINGER, student, University of Waterloo; J. SUCK, Essen, Federal Republic of Germany; C. WILDHAGEN, Breda, The Netherlands; KENNETH M. WILKE, Topeka, Kansas; JÜRGEN WOLFF, Steinheim, Federal Republic of Germany; and the proposer.

Prielipp remembered that the problem appeared as E2491 *of the* American Math. Monthly, *with solution, and generalization to* $[\sqrt[k]{j}]$, *on pp.* 854–855 *of the October* 1975 *issue.* (*Solvers Janous and Klamkin of the current problem also gave this generalization.*) *Prielipp included a solution that he and N.J. Kuenzi had submitted back then.*

<u>1331</u>. [1988: 108] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find the smallest positive integer a so that

$$13 \mid 11^{3n+1} + a \cdot 5^n \qquad \text{and} \qquad 31 \mid 23^{2n+1} + 2^{n+a}$$

both hold for all positive integers n.

Solution by Bob Prielipp, University of Wisconsin–Oshkosh.

Let n be an arbitrary positive integer.

Since $11 \equiv -2 \bmod 13$ and $(-2)^3 = -8 \equiv 5 \bmod 13$,

$$\begin{aligned} 11^{3n+1} + a \cdot 5^n &\equiv -2 \cdot 5^n + a \cdot 5^n \bmod 13 \\ &\equiv (-2 + a)5^n \bmod 13. \end{aligned}$$

It follows that $a \equiv 2 \bmod 13$.

Because $23 \equiv -8 \bmod 31$ and $(-8)^2 = 64 \equiv 2 \bmod 31$,

$$\begin{aligned} 23^{2n+1} + 2^{n+a} &\equiv -8 \cdot 2^n + 2^{n+a} \bmod 31 \\ &\equiv (-8 + 2^a)2^n \bmod 31. \end{aligned}$$

Hence $2^a \equiv 8 \bmod 31$. Since 5 is the smallest positive integer solution of the congruence $2^x \equiv 1 \bmod 31$, $a = 5k + 3$ for some non–negative integer k.

We now have the following collection of equivalent congruences:

$$5k + 3 \equiv 2 \bmod 13,$$
$$5k \equiv -1 \equiv 25 \bmod 13,$$
$$k \equiv 5 \bmod 13.$$

When $k = 5$, $a = 5k + 3 = 28$, so 28 is our answer.

Also solved by C. FESTRAETS–HAMOIR, Brussels, Belgium; RICHARD A. GIBBS, Fort Lewis College, Durango, Colorado; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; SAM MALTBY, student, Calgary; P. PENNING, Delft, The Netherlands; M.A. SELBY, University of Windsor; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; KENNETH M. WILKE, Topeka, Kansas; and the proposer. There was one incorrect solution submitted.

Many solvers remarked that a satisfies the two given properties precisely if $a \equiv 28$ mod 65.

Janous wondered if the number of this problem was assigned because of the appearance of 13 *and* 31 *in the statement. The answer is yes! (However, it was perhaps a lucky coincidence that the proposer sent in such a problem just at the time Crux* 1331 *was to be created.) The editor invites readers to send in more problems designed for a specific problem number.*

CMS SUBSCRIPTION PUBLICATIONS

1989 RATES

CRUX MATHEMATICORUM

Editor: W. Sands
Problem solving journal at the senior secondary and university undergraduate levels. Includes "Olympiad Corner" which is particularly applicable to students preparing for senior contests.
10 issue per year. 36 pages per issues.
Non-CMS Members: $35.00 CMS Members: $17.50

CMS NOTES

Editors: E.R. Williams and P.P. Narayanaswami
Primary organ for the dissemination of information to the members of the C.M.S. The Problems and Solutions section formerly published in the Canadian Mathematical Bulletin is now published in the CMS Notes.
8-9 issues per year.
Non-CMS Members: $10.00 CMS Members FREE

Orders by CMS members and applications for CMS Membership should be submitted using the form on the following page.

Orders by non-CMS members for CRUX MATHEMATICORUM or the CMS NOTES should be submitted using the form below:

Order Form

Canadian Mathematical Society
Société Mathématique du Canada

Please enter subscriptions:

- ☐ Crux Mathematicorum ($35.00)
- ☐ C.M.S. Notes ($10.00)

- ☐ Please bill me
- ☐ I am using a credit card
- ☐ I enclose a cheque made payable to the Canadian Mathematical Society

Visa

☐☐☐☐ ☐☐☐ ☐☐☐ ☐☐☐

MasterCard

☐☐☐☐ ☐☐☐☐ ☐☐☐☐ ☐☐☐☐ ☐☐☐☐

Expiry date Signature

Inquiries and order:
Canadian Mathematical Society
577 King Edward
Ottawa, Ontario K1N 6N5

CMS SUBSCRIPTION PUBLICATIONS

1989 RATES

CANADIAN JOURNAL OF MATHEMATICS

Editor-in-Chief: D. Dawson and V. Dlab
This internationally renowned journal is the companion publication to the Canadian Mathematical Bulletin. It publishes the most up-to-date research in the field of mathematics, normally publishing articles exceeding 15 typed pages. Bimonthly, 256 pages per issue.
Non-CMS Members $250.00 CMS Members $125.00
Non-CMS Members obtain a 10% discount if they also subscribe to the Canadian Mathematical Bulletin. Both subscription must be placed together.

CANADIAN MATHEMATICAL BULLETIN

Editors: J. Fournier and D. Sjerve
This internationally renowned journal is the companion publication to the Canadian Journal of Mathematics. It publishes the most up-to-date research in the field of mathematics, normally publishing articles no longer than 15 pages. Quarterly, 128 pages per issue.
Non-CMS Members $120.00 CMS Members $60.00
Non-CMS Members obtain a 10% discount if they also subscribe to the Canadian Journal of Mathematics. Both subscriptions must be placed together.

Orders by CMS members and applications for CMS membership should be submitted using the form of the following page.

Orders by non-CMS Members for the CANADIAN MATHEMATICAL BULLETIN and the CANADIAN JOURNAL OF MATHEMATICS should be submitted using the form below:

Order Form

Canadian Mathematical Society
Société Mathématique du Canada

- ☐ Please enter my subscription to both the CJM and CMB (Regular institutional rate $250 + $120, combined discount rate $333)
- ☐ Please enter my subscription to the CJM only Institutional rate $250
- ☐ Please enter my subscription to the CMB only Institutional rate $120
- ☐ Please bill me
- ☐ I am using a credit card
- ☐ I enclose a cheque made payable to the University of Toronto Press
- ☐ Send me a free sample of ☐ CJM ☐ CMB

Visa / Bank Americard / Barclaycard

☐☐☐☐ ☐☐☐ ☐☐☐ ☐☐☐

MasterCard / Access / Interbank 4-digit bank no.

☐☐☐☐ ☐☐☐☐ ☐☐☐☐ ☐☐☐☐ ☐☐☐☐

Expiry date Signature

Inquiries and order:
University of Toronto Press
Journals Department, 5201 Dufferin St.
Downsview, Ontario M3H 5T8

Canadian Mathematical Society
Société Mathématique du Canada
577 KING EDWARD OTTAWA, ONT
CANADA K1N 6N5

1989 MEMBERSHIP APPLICATION FORM 1989
(Membership period: January 1 to December 31)

CATEGORY	DETAILS	FEES
1	students and unemployed members	$ 15 per year
2	retired professors, postdoctoral fellows, secondary & junior college teachers	$ 25 per year
3	members with salaries under $30,000 per year	$ 45 per year
4	members with salaries from $30,000 - $60,000	$ 60 per year
5	members with salaries of $60,000 and more	$ 75 per year
10	Lifetime membership for members under age 60	$ 1000 (iii)
15	Lifetime membership for members age 60 or older	$ 500

(i) Members of the AMS and/or MAA WHO RESIDE OUTSIDE CANADA are eligible for a 15% reduction in the basic membership fee.

(ii) Members of the Allahabad, Australian, Brazilian, Calcutta, French, German, Hong Kong, Italian, London, Mexican, Polish of New Zealand mathematical societies, WHO RESIDE OUTSIDE CANADA are eligible for a 50% reduction in basic membership fee for categories 3,4 and 5.

(iii) Payment may be made in two equal annual installments of $500

APPLIED MATHEMATICS NOTES: Reduced rate for members $6.00 (Regular $12.00)
CANADIAN JOURNAL OF MATHEMATICS: Reduced rate for members $125.00 (Regular $250.00)
CANADIAN MATHEMATICAL BULLETIN: Reduced rate for members $60.00 (Regular $120.00)
CRUX MATHEMATICORUM: Reduced rate for members $17.50 (Regular $35.00)

FAMILY NAME ____ FIRST NAME ____ INITIAL ____ TITLE ____

MAILING ADDRESS ____ CITY ____

PROVINCE/STATE ____ COUNTRY ____ POSTAL CODE ____ TELEPHONE ____ ELECTRONIC MAIL ____

PRESENT EMPLOYER ____ POSITION ____

HIGHEST DEGREE OBTAINED ____ GRANTING UNIVERSITY ____ YEAR ____

PRIMARY FIELD OF INTEREST (see list on reverse) ____ MEMBER OF OTHER SOCIETIES (See (i) and (ii)) ____

Membership new ☐ renewal ☐ CATEGORY ____ RECEIPT NO. ____

* Basic membership fees (as per table above) $ ____
* Contribution towards the Work of the CMS ____

Publications requested
Applied Mathematics Notes ($ 6.00) ____
Canadian Journal of Mathematics ($125.00) ____
Canadian Mathematical Bulletin ($ 60.00) ____
Crux Mathematicorum ($ 17.50) ____

TOTAL REMITTANCE: $ ____

CHEQUE ENCLOSED (MAKE PAYABLE TO CANADIAN MATHEMATICAL SOCIETY) - CANADIAN CURRENCY PLEASE

PLEASE CHARGE ☐ VISA ☐ MASTERCARD ☐

ACCOUNT NO. ____ EXPIRY DATE ____

SIGNATURE ____ ( ) BUSINESS TELEPHONE NUMBER ____

(*) INCOME TAX RECEIPTS ARE ISSUED TO ALL MEMBERS FOR MEMBERSHIP FEES AND CONTRIBUTIONS ONLY
MEMBERSHIP FEES AND CONTRIBUTIONS MAY BE CLAIMED ON YOUR CANADIAN TAX RETURN AS CHARITABLE DONATIONS

Canadian Mathematical Society
Société Mathématique du Canada
577 KING EDWARD, OTTAWA, ONT
CANADA K1N 6N5

1989 **FORMULAIRE D'ADHÉSION** 1989

(La cotisation est pour l'année civile: 1 janvier - 31 décembre)

CATÉGORIES	DÉTAILS	COTISATION
1	étudiants et chômeurs	15$ par année
2	professeurs à la retraite, boursiers postdoctoraux, enseignants des écoles secondaires et des collèges	25$ par année
3	revenu annuel brut moins de 30,000$	45$ par année
4	revenu annuel brut 30,000$ - 60,000$	60$ par année
5	revenu annuel brut plus de 60,000$	75$ par année
10	Membre à vie pour membres agés de moins de 60 ans	1000$ (iii)
15	Membre à vie pour membres agés de 60 ans et plus	500$

(i) La cotisation des membres de l'AMS et MAA est réduite de 15% SI CEUX-CI NE RÉSIDENT PAS AU CANADA.

(ii) Suivant l'accord de réciprocité, la cotisation des membres des catégories 3, 4 et 5 des sociétés suivantes: Allahabad, Allemagne, Australie, Brésil, Calcutta, France, Londre, Mexique, Nouvelle Zélande, Pologne, Italie, Hong Kong, est réduite de 50% SI CEUX-CI NE RÉSIDENT PAS AU CANADA.

(iii) Les frais peuvent être réglés en deux versements annuels de 500,00$

NOTES DE MATHÉMATIQUES APPLIQUÉES: Abonnement des membres 6$ (Régulier 12$)
JOURNAL CANADIEN DE MATHÉMATIQUES: Abonnement des membres 125$ (Régulier 250$)
BULLETIN CANADIEN DE MATHÉMATIQUES: Abonnement des membres 60$ (Régulier 120$)
CRUX MATHÉMATICORUM: Abonnement des membres 17,50$ (Régulier 35$)

NOM DE FAMILLE ______ PRÉNOM ______ INITIALE ______ TITRE ______

ADRESSE DU COURRIER ______ VILLE ______

PROVINCE/ÉTAT ______ PAYS ______ CODE POSTAL ______ TÉLÉPHONE ______

ADRESSE ÉLECTRONIQUE ______ EMPLOYEUR ACTUEL ______ POSTE ______

DIPLÔME LE PLUS ÉLEVÉ ______ UNIVERSITÉ ______ ANNÉE ______

DOMAINE D'INTÉRÊT PRINCIPAL (svp voir liste au verso) ______ MEMBRE D'AUTRE SOCIÉTÉ (Voir (i) et (ii)) ______

Membre nouveau ☐ renouvellement ☐ CATÉGORIE:__________ NO. DE REÇU:__________

* Cotisation (voir table plus haut) __________
* Don pour les activités de la Société __________
Abonnements désirés:
Notes de mathématiques appliquées (6.00$) __________
Journal canadien de mathématiques (125.00$) __________
Bulletin canadien de mathématiques (60.00$) __________
Crux Mathematicorum (17.50$) __________

TOTAL DE VOTRE REMISE: __________

CHÈQUE INCLUS (PAYABLE À LA SOCIÉTÉ MATHÉMATIQUE DU CANADA) - EN DEVISES CANADIENNES S.V.P.

PORTER À MON COMPTE ☐ VISA ☐ MASTERCARD ☐

NUMÉRO DE COMPTE ______ DATE D'EXPIRATION ______

SIGNATURE ______ ( ) ______ TÉLÉPHONE D'AFFAIRE

(*) UN REÇU POUR FIN D'IMPÔT SERA ÉMIS À TOUS LES MEMBRES POUR LES DONS ET LES COTISATIONS SEULEMENT
LES FRAIS D'AFFILIATION ET LES DONS SONT DÉDUCTIBLES D'IMPÔT À CONDITION TOUTEFOIS D'ÊTRE INSCRITS DANS LA RUBRIQUE "DON DE CHARITÉ" DES FORMULAIRES D'IMPÔT FÉDÉRAL

PUBLICATIONS

The Canadian Mathematical Society
577 King Edward, Ottawa, Ontario K1N 6N5
is pleased to announce the availability of the following publications:

1001 Problems in High School Mathematics

Collected and edited by E.J. Barbeau, M.S. Klamkin and W.O.J. Moser.

Book I:	Problems 1-100 and Solutions 1-50 58 pages	($5.00)
Book II:	Problems 51-200 and Solutions 51-150 85 pages	($5.00)
Book III:	Problems 151-300 and Solutions 151-350 95 pages	($5.00)
Book IV:	Problems 251-400 and Solutions 251-350 115 pages	($5.00)
Book V:	Problems 351-500 and Solutions 351-450 86 pages	($5.00)

The Canadian Mathematics Olympiads (1968-1978)

Problems set in the first ten Olympiads (1969-1978) together with suggested solutions. Edited by E.J. Barbeau and W.O.J. Moser. 89 pages ($5.00)

The Canadian Mathematics Olympiads (1979-1985)

Problems set in the Olympiads (1979-1985) together with suggested solutions. Edited by C.M. Reis and S.Z. Ditor. 84 pages ($5.00)

Prices are in Canadian dollars and include handling charges.
Information on other CMS publications can be obtained by writing
to the Executive Director at the address given above.