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Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek, Shawn Godin

Ross Honsberger
1929–2016

Ross Honsberger: In Memoriam

Ross Honsberger was born June 2, 1929 in Toronto. Ross attended Bloor Collegiate Institute, graduating in 1947. After high school, Ross entered the University of Toronto, graduating in 1950 with a BA degree. He decided to become a high school mathematics teacher. He obtained his teaching certification at the Ontario College of Education in Toronto and taught at various high schools for 10 years.

Among the students influenced by Ross are James Stewart and Michael Feldstein. Stewart, who became a mathematician and calculus textbook author, was asked in an interview what got him interested in mathematics. He responded by describing Ross, saying that he was "not your typical high school math teacher". Stewart was fascinated by Ross's "digressions"; for example, he proved in his grade 11 class that the rationals are countable and the reals are not. Feldstein, who went on to an academic career as a biostatistician, considers Ross his "first real mentor in mathematics and, more importantly, in teaching". He remembers especially the humour that Ross brought to his teaching, and his empathy for the students. Notably, while he took pains to obtain student feedback, Ross never embarrassed students by singling them out.

The year 1963-64 was a pivotal one for Ross. He took a sabbatical from high school teaching and enrolled in a master's degree program at the University of Waterloo. Near the end of that year, Ralph Stanton offered Ross a position as a Lecturer in Mathematics. During the next three years Ross carried a full teaching load while completing his master's degree. In 1967 he was promoted to Assistant Professor. In 1971 when he was promoted to Associate Professor, Ross wrote "My main area of endeavour is prospective high school teachers and those already in service". At the time of promotion, a department colleague wrote in a supporting letter that Ross was "a research mathematician in the best sense of the word, i.e., a contributor to fundamental and original thinking about the nature of the subject as a whole".

In the sixties and seventies, there was a high demand for qualified high school mathematics teachers. Programs were needed for undergraduate mathematics majors at Waterloo interested in teaching as a career. In addition, many teachers already in service wanted to upgrade their qualifications. Ross played a leading role in addressing these needs. He developed several courses aimed at current and prospective teachers. Two notable ones were *History of Mathematics* and *Mathematical Discovery and Invention*. The latter became widely known as the *One Hundred Problems Course*. While their regular offerings were aimed at prospective teachers, they attracted many other students, partly due to Ross's reputation as an inspiring lecturer.

Ross's energy and ambition led him in another direction, too. From the beginning of his career, he had been searching out fascinating mathematical problems having elegant solutions, polishing them, and describing them in short essays. Ross's first book, *Ingenuity in Mathematics*, appeared in 1970. Ross eventually produced thirteen books, all published through the MAA. Ross's approach in his books can be summarized as follows (paraphrasing the introduction to one of his books). Too often, the study of mathematics is undertaken with an air of such seriousness, that it is not fun at the time. However, it is amazing how many beautiful parts of mathematics one can appreciate with a high school background. Ross's goal in his writing was to describe such gems, not as an attempt to instruct, but as a reward for the reader's concentration. The success that Ross enjoyed was due to the combination of his superb taste in the choice of topics, and his talent and care in exposition. Ross also gave credit to his wife Nancy for helping him to improve his writing.

Ross was a wonderful positive influence on those around him. He enthusiastically shared his latest elegant solution with everyone. Former colleagues recall Ross asking for a few minutes of their time, leading to much longer discussions. "Everyone" also included friends of his children that happened to come to the house. Accessibility to such friends was aided by the fact that, as his daughter Sandy recalls, "his office was our living room".

As time passed, Ross stepped back from teaching to concentrate on his writing. He took a reduced load appointment in 1990, and early retirement in 1991. Ross did not teach in retirement, but his occasional special lectures drew large and enthusiastic audiences. Meanwhile, he devoted a lot of energy to his writing. Six of his books appeared after his retirement, the last in 2004.

Throughout his life Ross had wide interests, including reading, music, gardening, handball, darts, billiards, and poker. He always had a dog. And of course, there was his family. Ross died April 3, 2016. In addition to his daughter Sandy, he is survived by three grandchildren and seven great grandchildren. He will be long remembered by his family, his friends, his colleagues, his students, and his many readers.

This piece is adapted from the biography appearing on the University of Waterloo website: `https://uwaterloo.ca/combinatorics-and-optimization/about/ross-honsberger`

EDITORIAL

Welcome to the Ross Honsberger Commemorative Issue. I am very glad that Shawn Godin suggested an idea of doing this commemorative issue, the idea that was supported by the Board and whose product you now hold in your hands. Inside you will find tributes, articles written in Honsberger's memory and some of his own material.

Although I never met the man, I feel that I know him through his wonderful books (some of which reside on my bookshelf), as his spirit jumps off their pages. The ease with which he writes, his obvious passion for the subject and his sense of humour make the material come to life. Here is an excerpt from the first page of his first book *Ingenuity in Mathematics*:

> I can remember reading years ago that the probability of two positive integers, chosen at random, being relatively prime is $6/\pi^2$. It seems that one R. Chartres, in about 1904, tested this mathematical result experimentally by having each of fifty students write down at random five pairs of positive integers. Out of the 250 pairs thus obtained, he found 154 pairs were relatively prime, giving a probability of 154/250. Calling this $6/x^2$, he found $x = 3.12$, while $\pi = 3.14159\cdots$.
>
> This simply astounded me! How a random choice of pairs of positive integers could have anything to do with π was beyond my imagination. The prospect of actually determining the value of π through an experiment of repeated trials—in which the producer of the pairs of integers has no idea what they are to be used for—seemed utterly incredible.

Even if the material is not new to you, the author's enthusiasm is genuine and contagious, which just makes you wonder what he will do next. We are lucky in that, through Shawn, Honsberger passed along some of his unpublished essays to ***Crux***, and they will be appearing in future issues of this journal. The article *The Lucas Circles of a Triangle* in this issue is the first of those essays. Also special in this issue are the Contest Corner and the Honsberger Corner, which contain a collection of Ross's favourite problems. Articles written for our special issue give further insight into Honsberger's influence on his students and readers.

So grab some paper and a pencil and enjoy the mathematical elegance of Honsberger-inspired materials.

Kseniya Garaschuk

THE CONTEST CORNER

No. 54

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er décembre 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC266. On considère un treillis 5×5 de points. Tracer un ensemble de cercles de manière que chaque point de treillis soit situé sur exactement un cercle.

CC267. Dix personnes sont assises en cercle. Chaque personne choisit un nombre et le révèle à son voisin de droite et à son voisin de gauche. Donc, chaque personne révèle un nombre et reçoit deux nombres. Chaque personne révèle ensuite la moyenne des deux nombres qu'elle a reçus. Or, les moyennes annoncées, dans l'ordre autour du cercle, sont $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. Quel est le nombre choisi par la personne qui a révélé une moyenne de 6?

CC268. Déterminer le plus petit terme de la suite

$$\sqrt{\frac{7}{6}}+\sqrt{\frac{96}{7}},\ \sqrt{\frac{8}{6}}+\sqrt{\frac{96}{8}},\ldots,\sqrt{\frac{n}{6}}+\sqrt{\frac{96}{n}},\ldots,\sqrt{\frac{95}{6}}+\sqrt{\frac{96}{95}}.$$

CC269. Étant donné les entiers $1, 2, 3, \ldots, n$, on doit les utiliser pour écrire une suite ordonnée qui satisfait aux critères suivants: deux éléments consécutifs ne peuvent être égaux et la séquence $\ldots a \ldots b \ldots a \ldots b \ldots$ ne peut pas paraître dans la suite. Par exemple, si $n = 5$, la suite 123524 satisfait aux critères, mais la suite 1235243 ne satisfait pas aux critères, car la séquence 2323 paraît dans la suite.

a) Démontrer qu'une suite qui satisfait à ces critères doit contenir un élément x qui ne paraît qu'une fois dans la suite.

b) Écrire une telle suite de longueur $2n - 1$ dans laquelle tous les éléments sauf un paraissent plus d'une fois.

CC270. Il est possible de tracer des droites qui passent au point $(3, -2)$ de manière que pour chaque droite, la somme de l'abscisse à l'origine et de l'ordonnée à l'origine est égale à trois fois la pente. Déterminer la somme des pentes de toutes ces droites.

. . . .

CC266. Given a 5×5 array of lattice points, draw a set of circles that collectively pass through each of the lattice points exactly once.

CC267. Each of ten people around a circle chooses a number and tells it to the neighbour on each side. Thus each person gives out one number and receives two numbers. The players then announce the average of the two numbers they received.Remarkably, the announced numbers, in order around the circle, were $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. What was the number chosen by the person who announced number 6?

CC268. Find the smallest term in the sequence

$$\sqrt{\frac{7}{6}} + \sqrt{\frac{96}{7}},\ \sqrt{\frac{8}{6}} + \sqrt{\frac{96}{8}}, \dots, \sqrt{\frac{n}{6}} + \sqrt{\frac{96}{n}}, \dots, \sqrt{\frac{95}{6}} + \sqrt{\frac{96}{95}}.$$

CC269. One is given integers $1, 2, 3, \dots, n$ and is required to write an ordered sequence with the following properties: no two adjacent elements are the same and within the ordered sequence the subsequence $\dots a \dots b \dots a \dots b \dots$ may not appear. For example, if $n = 5$, the sequence 123524 obeys the properties, but 1235243 does not, since the subsequence 2323 occurs.

a) Prove that any sequence obeying these rules must contain an element x which appears only once in the sequence.

b) Write down such a sequence of length $2n - 1$ in which all elements but one occur more than once.

CC270. It is possible to draw straight lines through the point $(3, -2)$ so that for each line the sum of the intercepts is equal to three times its slope. Find the sum of the slopes of all such lines.

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(4), p. 144–145.

CC216. Starting with a list of three numbers, the "changesum" procedure creates a new list by replacing each number by the sum of the other two. For example, from $\{3, 4, 6\}$ "changesum" gives $\{10, 9, 7\}$ and a new "changesum" leads to $\{16, 17, 19\}$. If we begin with $\{20, 1, 3\}$, what is the maximum difference between two numbers of the list after 2014 consecutive "changesums"?

Originally question 23 of Irish Junior Maths Competition Final 2014.

We received four correct solutions. We present the solution of Doddy Kastanya.

Suppose we have a list of numbers $\{a, b, c\}$ where $a < b < c$. The maximum difference between the largest and the smallest number is $c-a$. The "changesum" operation on this list will create a new list $\{b + c, a + c, a + b\}$. Since $a < b$, we know that $a + c < b + c$. Since $b < c$, we also know that $a + b < a + c$. Combining these two inequalities, we can write $a + b < a + c < b + c$.

The maximum difference between any number is $(b+c)-(a+b)$ or $c-a$. So, after the first "changesum" operation, the maximum difference stays the same. Since the maximum difference is independent of the order of $\{a, b, c\}$ in the original list, the difference will always be $c - a$.

Equipped with this knowledge, for the initial list of $\{20, 1, 3\}$, the maximum difference will always be $20 - 1$ or 19. So, after 2014 consecutive "changesum" operations, the maximum difference between two numbers will be 19.

CC217. A right triangle ABC has its hypotenuse AB trisected at M and N. If $CM^2 + CN^2 = k \cdot AB^2$, then what is the value of k?

Originally question 27 in the Indiana State Mathematics Contest 2009 (Comprehensive Test).

We received eight correct submissions. We present a solution by Doddy Kastanya.

Let K and L be the two points on CB such that $MK \perp BC$ and $NL \perp BC$. This yields

$$CM^2 = CK^2 + MK^2 \tag{1}$$
$$CN^2 = CL^2 + NL^2. \tag{2}$$

Using similar triangles MKB and ACB, we have $CK = \frac{1}{3}BC$ and $MK = \frac{2}{3}AC$.

Substituting this into (1), we get

$$CM^2 = \frac{1}{9}BC^2 + \frac{4}{9}AC^2. \tag{3}$$

Using similar triangles NLB and ACB, we have $CL = \frac{2}{3}BC$ and $NL = \frac{1}{3}AC$. Substituting this into (2), we get

$$CN^2 = \frac{1}{9}AC^2 + \frac{4}{9}BC^2. \tag{4}$$

Adding (3) and (4) together, we get

$$CM^2 + CN^2 = \frac{5}{9}(AB^2 + BC^2) = \frac{5}{9}AB^2.$$

So the value of k is $\frac{5}{9}$.

Editor'c Comments. Zelator presented (and proved) the following generalization of the problem. (His solution uses the relation between a median and the sides of a triangle as well as the cosine law.)

Let r_1 and r_2 be distinct fixed positive real numbers, $0 < r_1 < r_2 < 1$. Let ABC be a right triangle with the right angle at C, hypotenuse AB and with lengths $AB = c, BC = a$ and $AB = c$. Let M and N be points on the hypotenuse AB, such that $AM = r_1c$ and $AN = r_2c$. Finally, let $CM = x$ and $CN = y$.

a) Show that $x^2 = r_1^2a^2 + (1-r_1)^2b^2$ and $y^2 = r_2^2a^2 + (1-r_2)^2b^2$.

b) Suppose that $r_1 + r_2 = 1$ and $x^2 + y^2 = kc^2$. Show that $k = r_1^2 + r_2^2$.

c) Suppose that $r_1 + r_2 \neq 1$ and $x^2 + y^2 = kc^2$. Express k in terms of r_1, r_2 and the ratio $R = \frac{b}{a}$.

CC218. Solve the following system of equations:

$$\begin{cases} 3^{\ln x} = 4^{\ln y}, \\ (4x)^{\ln 4} = (3y)^{\ln 3}. \end{cases}$$

Originally question 10 from the 2014 Texas A&M High School Mathematics Contest.

We received seven correct submissions. We present a solution by Šefket Árslanagić, modified by the editor, and a generalization.

Using the fact that a logarithmic function is one-to-one and employing properties of logarithms, we get

$$\begin{cases} 3^{\ln x} = 4^{\ln y}, \\ (4x)^{\ln 4} = (3y)^{\ln 3}, \end{cases} \iff \begin{cases} \ln x \ln 3 = \ln y \ln 4, \\ (\ln 4 + \ln x)\ln 4 = (\ln 3 + \ln y)\ln 3. \end{cases}$$

From the first equation we see that $\ln x = \frac{\ln y \ln 4}{\ln 3}$ and substituting into the second equation, we have

$$\left(\ln 4 + \frac{\ln y \ln 4}{\ln 3}\right) \ln 4 = (\ln 3 + \ln y) \ln 3$$

and so

$$(\ln^2 4 - \ln^2 3)\left(1 + \frac{\ln y}{\ln 3}\right) = 0.$$

From here, since $\ln^2 3 - \ln^2 4 \neq 0$, we get $\ln y = -\ln 3$, so $y = \frac{1}{3}$. Plugging this back into the first equation, we have

$$\ln x = \frac{\ln y \ln 4}{\ln 3} = \frac{-\ln 3 \ln 4}{\ln 3} = -\ln 4,$$

so $x = \frac{1}{4}$. Therefore, the solution to the system is $(x, y) = (\frac{1}{4}, \frac{1}{3})$.

Editor's Comments. Sitaru and Zelator (independently) both noticed that the use of numbers 3 and 4 was arbitrary and gave the following generalized version: the system

$$\begin{cases} a^{\ln x} = b^{\ln y}, \\ (bx)^{\ln b} = (ay)^{\ln a}, \end{cases}$$

has solution $(x, y) = (\frac{1}{b}, \frac{1}{a})$. Natural logarithms can also be replaced with logarithms in base $c > 0$.

CC219. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. Find the ratio of the number of cubes with exactly two green faces to the number of cubes with exactly three green faces.

Originally question 18 of the 2014 Texas A&M High School Mathematics Contest.

We received three correct solutions. We present the solution of Fernando Ballesta.

There are 8 cubes with three faces coloured (which are the 8 corners), and there are

$$4(6-2) + 4(5-2) + 4(4-2) = 16 + 12 + 8 = 36$$

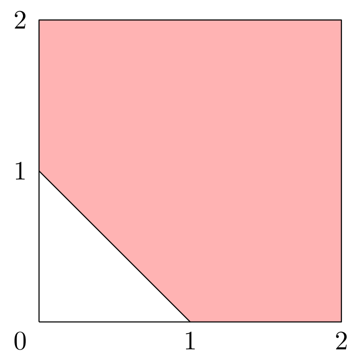
cubes with two faces coloured (which are the ones on the edges and are not corners). So the ratio is $36 : 8 = 9 : 2$, that is, for every two faces with three faces coloured there are nine with two faces coloured.

CC220. Two random numbers x and y are drawn independently from the closed interval $[0, 2]$. What is the probability that $x + y > 1$?

Originally question 13 of the 2014 Texas A&M High School Mathematics Contest.

We received three correct solutions. We present the solution of Doddy Kastanya.

We can draw on the Cartesian plane the area (shaded in the figure below) represented by $x + y > 1$.



The probability of $x + y > 1$ is simply the ratio between the shaded area and the area of the overall square. The area of the non-shaded part of the square is $\frac{1}{2}$. The overall area of the square is $2 \cdot 2 = 4$. So, the area of the shaded part of the square is $4 - \frac{1}{2} = \frac{7}{2}$.

Therefore, the probability that $x + y > 1$ is $\frac{7}{2}/4$ or $\frac{7}{8}$.

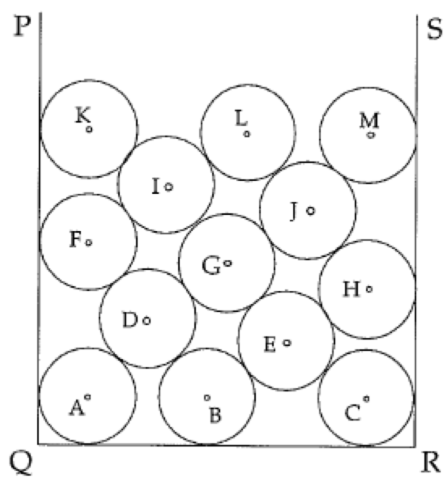
THE HONSBERGER CORNER

Problems in this section include some of Ross Honsberger's favourites. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your solutions to crux-editors@cms.math.ca.

To facilitate their consideration, solutions should be received by **December 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

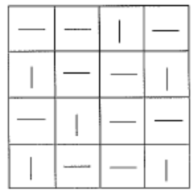
H1. On considère un casier à bouteilles de forme rectangulaire $PQRS$. Dans la première rangée du casier, il y a un peu plus de place que pour trois bouteilles A, B et C, mais pas suffisamment de place pour une quatrième bouteille (voir la figure ci-dessous). Les bouteilles A et C touchent aux côtés du casier et dans la rangée suivante de deux bouteilles, D et E, la bouteille B est maintenue en place quelque part entre les bouteilles A et C. On place une troisième rangée de trois bouteilles, F, G et H, de manière que F et H touchent aux côtés du casier. On place ensuite une quatrième rangée de deux bouteilles, I et J, puis une cinquième rangée de trois bouteilles, K, L et M.



Sachant que toutes les bouteilles dont de mêmes dimensions, démontrer que peu importe l'espacement entre les bouteilles A, B et C dans la première rangée, la cinquième rangée de bouteilles est toujours parfaitement horizontale.

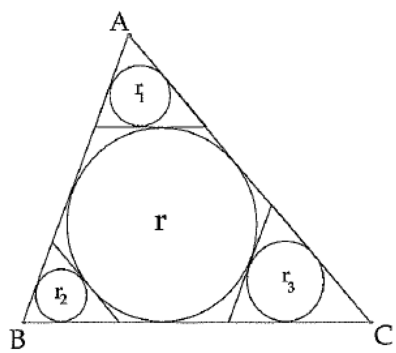
H2. On vous présente un coffre-fort dont le système de déverrouillage consiste en un arrangement 4×4 de clés. Chacune des 16 clés peut être placée en position horizontale ou verticale. Pour ouvrir le coffre-fort, toutes les clés doivent être en position verticale. Lorsqu'on tourne une clé, toutes les clés de la même rangée et de la même colonne changent de position. On peut tourner une clé plus d'une fois.

a) Étant donné la configuration dans la figure suivante, comment fait-on pour ouvrir le coffre-fort?



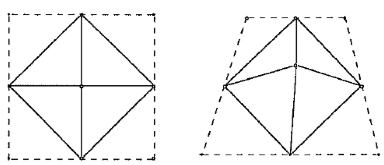
b) Si on vous permet de tourner un maximum de 2002 clés, quel est le plus grand arrangement $2n \times 2n$ de clés qui permet de toujours ouvrir le coffre-fort?

H3. Soit r le rayon du cercle inscrit dans le triangle ABC. Trois tangentes au cercle sont tracées, parallèles aux côtés du triangle, de manière à former trois petits triangles (voir la figure ci-dessous).



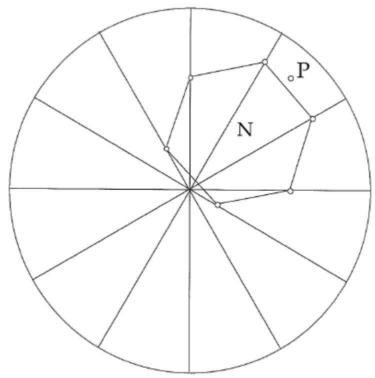
Soit r_1, r_2 et r_3 les rayons respectifs des cercles inscrits dans ces triangles. Démontrer que $r_1 + r_2 + r_3 = r$.

H4. Dans la première figure ci-dessous, on voit qu'il est possible de plier un carré (lignes en traits) de manière que les quatre sommets du carré se rencontrent en un point et que les parties repliées ne chevauchent pas et ne laissent aucun espace entre elles. La deuxième figure montre un autre quadrilatère que l'on peut plier de la même manière.



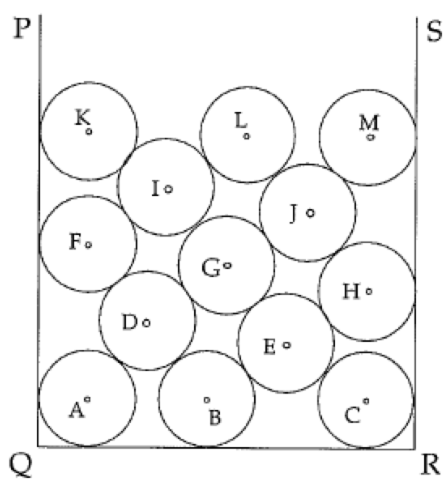
Déterminer des conditions suffisantes et nécessaires pour qu'un quadrilatère puisse être plié de cette manière.

H5. Un cercle est divisé en arcs égaux par n diamètres (voir la figure suivante). À partir d'un point P à l'intérieur du cercle, on abaisse une perpendiculaire à chacun de ces diamètres. Démontrer que les pieds de ces perpendiculaires déterminent un n-gone régulier N.



. . . .

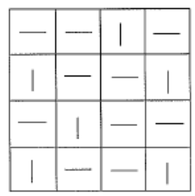
H1. Across the bottom of rectangular wine rack $PQRS$, there is room for more than three bottles (A, B, C) but not enough for a fourth bottle (see the figure). Naturally, bottles A and C are laid against the sides of the rack and a second layer, consisting of just two bottles D and E, holds B in place somewhere between A and C. Now we can lay in a third row of three bottles (F, G, H), with F and H resting against the sides of the rack. Then a fourth layer is held to just two bottles I and J, but a fifth layer can accommodate three bottles (K, L, M).



If the bottles are all the same size, prove that, whatever the spacing of (A, B, C) in the bottom layer, the fifth layer is always perfectly horizontal.

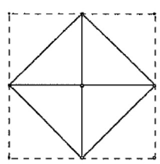
H2. You are given a safe with the lock consisting of a 4×4 arrangement of keys. Each of the 16 keys can be in a horizontal or a vertical position. To open the safe, all the keys must be in the vertical position. When you turn a key, all the keys in the same row and column change positions. You may turn a key more than once.

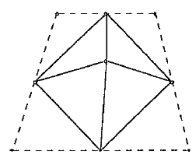
a) Given the configuration in the figure below, how do you open the safe?



b) If you are allowed to turn at most 2002 keys, what is the largest $2n \times 2n$ safe that you can always open?

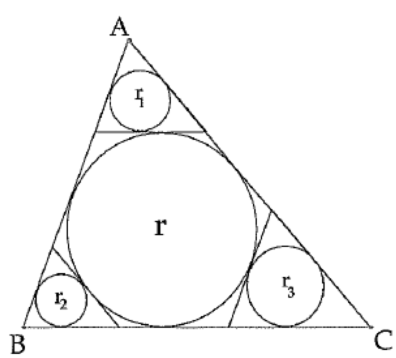
H3. Clearly in the left figure below, the four corners of a square can be folded over to meet at a point without overlapping or gaps; another such quadrilateral is illustrated in the figure on the right.





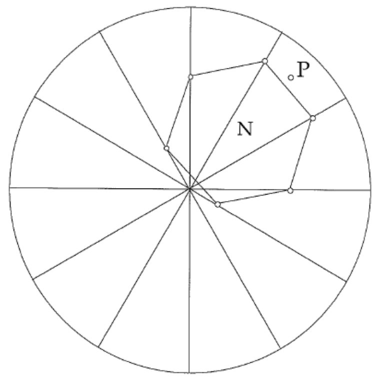
Determine the necessary and sufficient conditions for such a folding of the corners of a quadrilateral.

H4. Tangents to the incircle of $\triangle ABC$ are drawn parallel to the sides to cut off a little triangle at each vertex (see figure):



Prove that the inradii of the three small circles add up to the inradius of $\triangle ABC$; that is $r_1 + r_2 + r_3 = r$.

H5. A circle is divided into equal arcs by n diameters (see the figure). Prove that the feet of the perpendiculars to these diameters from a point P inside the circle always determine a regular n-gon N.



Math Quotes

Paul Erdös has a theory that God has a book containing all the theorems of mathematics with their absolutely most beautiful proofs, and when he wants to express particular appreciation of a proof he exclaims, "This is from the book!"

Ross Honsberger

THE OLYMPIAD CORNER

No. 352

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er décembre 2017**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

OC326. Soit n un entier strictement positif. Marie écrit au tableau tous les n^3 triplets composés d'entiers de 1 à n (les entiers d'un triplet n'étant pas nécessairement distincts). Elle considère ensuite le plus grand nombre de chaque triplet (possiblement plus d'un par triplet) et efface les autres nombres du triplet. Par exemple, elle efface les nombres 1 et 3 du triplet $(1, 3, 4)$ et elle efface le nombre 1 du triplet $(1, 2, 2)$. Démontrer qu'à la fin, le nombre de nombres au tableau ne peut être un carré parfait.

OC327. Un quadrilatère $APBQ$, avec $\angle P = \angle Q = 90°$ et $AP = AQ < BP$, est inscrit dans un cercle ω. Soit X un point mobile sur le segment $\overline{PQ}$. La droite AX coupe ω à un deuxième point S (autre que A). Soit T un point situé sur l'arc AQB de ω de manière que $\overline{XT}$ soit perpendiculaire à $\overline{AX}$. Soit M le milieu de la corde $\overline{ST}$. Montrer qu'à mesure que X se déplace sur le segment $\overline{PQ}$, M se déplace sur un cercle.

OC328. On dit qu'un diviseur d d'un entier strictement positif n est spécial si $d + 1$ aussi est un diviseur de n. Démontrer qu'au plus la moitié des diviseurs positifs d'un entier strictement positif peuvent être spéciaux. Déterminer tous les entiers strictement positifs dont exactement la moitié des diviseurs sont spéciaux.

OC329. Soit n un entier $(n \geq 5)$ et soit A et B des ensembles d'entiers qui satisfont aux conditions suivantes:

1. $|A| = n$, $|B| = m$ et A est un sous-ensemble de B
2. Pour tous entiers distincts x et y dans B, $x+y \in B$ si et seulement si $x, y \in A$

Déterminer la valeur minimale de m.

OC330. Résoudre l'équation

$$(2^{2015}+1)^x + 2^{2015} = 2^y + 1$$

dans l'ensemble des entiers non négatifs.

. . . .

OC326. Let n be a positive integer. Mary writes the n^3 triples of not necessarily distinct integers, each between 1 and n inclusive on a board. Afterwards, she finds the greatest number (possibly more than one) in each triple, and erases the rest. For example, in the triple $(1, 3, 4)$ she erases the numbers 1 and 3, and in the triple $(1, 2, 2)$ she erases only the number 1. Show after finishing this process, the amount of remaining numbers on the board cannot be a perfect square.

OC327. Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A). Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

OC328. We call a divisor d of a positive integer n special if $d + 1$ is also a divisor of n. Prove: at most half the positive divisors of a positive integer can be special. Determine all positive integers for which exactly half the positive divisors are special.

OC329. Let $n \geq 5$ be a positive integer and let A and B be sets of integers satisfying the following conditions:

1. $|A| = n$, $|B| = m$ and A is a subset of B
2. For any distinct $x, y \in B$, $x + y \in B$ iff $x, y \in A$

Determine the minimum value of m.

OC330. Solve the following equation in nonnegative integers:

$$(2^{2015}+1)^x + 2^{2015} = 2^y + 1$$

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(2), p. 57–58.

OC266. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

Originally problem 5 of the 2014 India National Olympiad.

We received 4 correct submissions. We present the solution by Steven Chow.

We will use complex numbers. Let the unit circle be the circumcircle of $\triangle ABC$. Let G be the point on $\overleftrightarrow{AD}$ such that $\overleftrightarrow{BC}$ and the line through G and $O = (0,0)$ are parallel. Let a, b, c, d, e, f, and g be the complex numbers of A, B, C, D, O_1, O_2, and G, respectively.

Since $D \in \overleftrightarrow{BC}$, we have

$$\frac{b-d}{c-d} = \overline{\left(\frac{b-d}{c-d}\right)} = \frac{\frac{1}{b} - \overline{d}}{\frac{1}{c} - \overline{d}} \implies b + c = (bc)\overline{d} + d \implies \overline{d} = \frac{b+c-d}{bc}.$$

Since O, the midpoint of $\overline{AB}$, and O_1 are collinear, we have

$$\frac{2e}{a+b} = \frac{2\overline{e}}{\frac{1}{a} + \frac{1}{b}} \qquad \implies \qquad \overline{e} = \frac{e}{ab}.$$

Since 0_1 is on the perpendicular bisector of $\overline{AD}$, we have

$$0 = \frac{e - \frac{1}{2}(a+d)}{a-d} + \frac{\overline{e} - \frac{1}{2}(\frac{1}{a} + \overline{d})}{\frac{1}{a} - \overline{d}} = \frac{e - \frac{1}{2}(a+d)}{a-d} + \frac{\frac{e}{ab} - \frac{1}{2}(\frac{1}{a} + \overline{d})}{\frac{1}{a} - \overline{d}},$$

which implies

$$\begin{aligned} 0 &= e\Big(\frac{1}{a} - \overline{d} + \frac{1}{b} - \frac{d}{ab}\Big) + \frac{1}{2}\Big(-(a+d)\Big(\frac{1}{a} - \overline{d}\Big) - \Big(\frac{1}{a} + \overline{d}\Big)(a-d)\Big) \\ &= e\Big(\frac{1}{a} - \frac{b+c-d}{bc} + \frac{1}{b} - \frac{d}{ab}\Big) - 1 + d(\overline{d}) \\ &= e\frac{(a-c)(-b+d)}{abc} - 1 + d(\overline{d}). \end{aligned}$$

Therefore,

$$e = \frac{abc(1 - d(\overline{d})}{(a-c)(-b+d)} = \frac{a(bc - (b+c-d)d)}{(a-c)(-b+d)} = \frac{a(-c+d)}{a-c}.$$

Similarly,

$$f = \frac{a(-b+d)}{a-b}.$$

Since $\overleftrightarrow{BC}$ and $\overleftrightarrow{OG}$ are parallel, we have

$$\frac{g}{b-c} = \frac{\overline{g}}{\frac{1}{b}-\frac{1}{c}} \implies \overline{g} = -\frac{g}{bc}.$$

Since $G \in \overleftrightarrow{AD}$, we successively get:

$$\begin{aligned}
&\frac{a-g}{a-d} = \frac{\frac{1}{a}-\overline{g}}{\frac{1}{a}-\overline{d}},\\
&-a\overline{d} + \overline{d}g - \frac{g}{a} = -a\overline{g} - \frac{d}{a} + d\overline{g} = -a\Big(-\frac{g}{bc}\Big) - \frac{d}{a} + d\Big(-\frac{g}{bc}\Big),\\
&-a\Big(\frac{b+c-d}{bc}\Big) + \Big(\frac{b+c-d}{bc}\Big)g = \frac{(a^2+bc)g}{abc} - \frac{d}{a} - \frac{dg}{bc},\\
&\Big(\frac{a^2-ab-ac+bc}{abc}\Big)g = \frac{a(-b-c+d)}{bc} + \frac{d}{a},\\
&g = \frac{a^2(-b-c+d)+bcd}{(a-b)(a-c)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d-f}{e-g} &= \frac{d - \frac{a(-b+d)}{a-b}}{\frac{a(-c+d)}{a-c} - \frac{a^2(-b-c+d)+bcd}{(a-b)(a-c)}} = \frac{(a-c)(a-d)b}{(a+c)(a-d)b}\\
&= \frac{a-c}{a+c} = -\frac{\frac{1}{a}-\frac{1}{c}}{\frac{1}{a}+\frac{1}{c}} = -\overline{\Big(\frac{a-c}{a+c}\Big)}.
\end{aligned}$$

Therefore, $\overleftrightarrow{DO_2} \perp \overleftrightarrow{GO_1}$ and similarly, $\overleftrightarrow{DO_1} \perp \overleftrightarrow{GO_2}$, so G is the orthocentre of $\triangle O_1O_2D$. (Alternatively, note that $\overleftrightarrow{AD} \perp \overleftrightarrow{O_1O_2}$.) Hence, the line joining the circumcentre of $\triangle ABC$ and the orthocentre of $\triangle O_1O_2D$ is parallel to $\overleftrightarrow{BC}$.

OC267. Blue and red circular disks of identical size are packed together to form a triangle. The top level has one disk and each level has 1 more disk than the level above it. Each disk not at the bottom level touches two disks below it and its colour is blue if these two disks are of the same colour. Otherwise its colour is red.

Suppose the bottom level has 2048 disks of which 2014 are red. What is the colour of the disk at the top?

Originally problem 3 of the 2014 Singapore Senior Mathematics Olympiad.

We received 3 correct submissions. We present the solution by Oliver Geupel.

We show that the disk at the top is blue. Let us use coordinates as follows: For $1 \leq n \leq 2048$, the n disks at the nth level from above, have coordinates $(n, 1)$,

$(n, 2)$, ..., (n, n), from left to right. For $1 \le k \le i \le 2048$, let $f(i,k) = 0$ if the disk at coordinate (i, k) is blue, and let $f(i, k) = 1$ if the disk at (i, k) is red. Note that we have

$$f(i,k) \equiv f(i+1,k) + f(i+1,k+1) \pmod 2 \tag{1}$$

for $1 \le k \le i \le 2047$ by hypothesis. For $0 \le n \le 11$, let $P(n)$ denote the assertion

$$f(i+1,k+1) \equiv \sum_{j=1}^{2^n} f(i+2^n, k+j) \pmod 2$$

when $0 \le k \le i \le i + 2^n \le 2048$. We prove $P(n)$ by induction on n.

The base case $P(0)$ is immediate. Assume that $P(n-1)$ holds for some $n \ge 1$. Let $0 \le i \le k \le i + 2^n \le 2048$. Using the induction hypothesis $P(n-1)$ and (1), we deduce

$$\begin{aligned} f(i+1,k+1) &\equiv \sum_{j=1}^{2^{n-1}} f(i+2^{n-1}, k+j) \\ &\equiv \sum_{j=1}^{2^{n-1}} \big(f(i+2^{n-1}+1, k+j) + f(i+2^{n-1}+1, k+j+1) \big) \\ &\equiv f(i+2^{n-1}+1, k+1) + f(i+2^{n-1}+1, k+2^{n-1}+1) \\ &\equiv \sum_{j=1}^{2^{n-1}} f(i+2^n, k+j) + \sum_{j=1}^{2^{n-1}} f(i+2^n, k+2^{n-1}+j) \\ &\equiv \sum_{j=1}^{2^n} f(i+2^n, k+j) \pmod 2 . \end{aligned}$$

Hence $P(n)$, which completes the induction.

From $P(11)$ we see that

$$f(1,1) \equiv \sum_{j=1}^{2048} f(2048, j) \equiv 0 \pmod 2 ,$$

that is, the disk at the top is blue.

OC268. Let $\mathbb{Z}_{\ge 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\ge 0}$.

Originally problem 2 from day 1 of the 2014 Taiwan Team Selection Test.

No submitted solutions.

OC269. Let x, y, z be real numbers that satisfy the following:

$$(x-y)^2+(y-z)^2+(z-x)^2=8, x^3+y^3+z^3=1.$$

Find the minimum value of $x^4+y^4+z^4$.

Originally problem 3 from day 2 of the 2014 Korean National Olympiad.

There was one submission by Šefket Arslanagić. We present it below with some missing details left for the reader to fill out.

Let $S=x^4+y^4+z^4$. We will use the following substitution

$$u=x+y+z, \qquad v=xy+yz+zx, \qquad w=xyz.$$

It now follows that:

$$\begin{aligned}
&(x-y)^2+(y-x)^2+(x-z)^2=8\\
\Longleftrightarrow\ & x^2+y^2+z^2-xy-yz-zx=4\\
\Longleftrightarrow\ & (x+y+z)^2-3(xy+yz+zx)=4\\
\Longleftrightarrow\ & u^2-3v=4.
\end{aligned}$$

Further, we have that

$$\begin{aligned}
&x+y+z=u\\
\Longleftrightarrow\ & x^3+y^3+z^3+3(x^2y+xy^2+y^2z+yz^2+x^2z+xz^2)+6xyz=u^3\\
\Longleftrightarrow\ & 1+3(xy(x+y)+yz(y+z)+xz(x+z))+6xyz=u^3\\
\Longleftrightarrow\ & 1+3(xy(u-z)+yz(u-x)+xz(u-y))+6xyz=u^3\\
\Longleftrightarrow\ & 1+3(u(xy+yz+xz)-3xyz)+6xyz=u^3\\
\Longleftrightarrow\ & 1+3uv-3w=u^3
\end{aligned}$$

and

$$\begin{aligned}
S&=x^4+y^4+z^4\\
&=(x^2+y^2+z^2)^2-2(x^2y^2+y^2z^2+z^2x^2)\\
&=[(x+y+z)^2-2(xy+yz+zx)]^2-2[(xy+yz+zx)^2-2xyz(x+y+z)]\\
&=(u^2-2v)^2-2(v^2-2uw)\\
&=u^4-4u^2v+2v^2+4uw.
\end{aligned}$$

The previous three final equalities imply that

$$S(u)=\tfrac{1}{9}(-u^4-16u^2+12u+32),\ S'(u)=\tfrac{1}{9}(-4u^3-32u+12),\ S''(u)=\tfrac{1}{9}(-12u^2-32).$$

Examining $(x-y)^2(y-z)^2(z-x)^2\geq 0$ shows us that $u\in[-1/3,1]$ and since our function is decreasing on this interval, we see that $S(u)$ is minimized when $u=1$. This gives $v=-1$ and $w=-1$. Therefore, $S=x^4+y^4+z^4$ has its minimum for

$$(x,y,z)\in\{(1,1,-1),(1,-1,1),(-1,1,1)\}$$

and in this case, we see that each of these satisfies the given equations and further that we see that $S_{\min} = (\pm 1)^4 + (\pm 1)^4 + (\pm 1)^4 = 3$.

OC270. For even positive integer n we put all numbers $1, 2, ..., n^2$ into the squares of an $n \times n$ chessboard (each number appears once and only once). Let S_1 be the sum of the numbers put in the black squares and S_2 be the sum of the numbers put in the white squares. Find all n such that we can achieve $\frac{S_1}{S_2} = \frac{39}{64}$.

Originally problem 3 of the 2014 Greece National Olympiad.

We present the solution by Steven Chow. There were no other submissions.

Let $a_1 = \frac{n}{2}$. Let b be the integer such that $S_1 = 39b$ and $S_2 = 64b$. Therefore

$$103b = 39b + 64b = \sum_{j=1}^{n^2} j = \sum_{j=1}^{(2a_1)^2} j = \frac{1}{2}(2a_1)^2((2a_1)^2 + 1) = 2a_1^2(4a_1^2 + 1).$$

Since 103 is prime, either $103 \mid a_1$ or $103 \mid 4a_1^2 + 1$.

If $103 \mid 4a_1^2 + 1$, then $a_1^2 \equiv 77 \pmod{103}$. Using the Quadratic Reciprocity Theorem, it follows that

$$\begin{aligned} 1 &= (77/103) = (7/103)(11/103) = (-(103/7))(-(103/11)) \\ &= (5/7)(4/11) = (7/5)(1) = (2/5) = -1 \end{aligned}$$

which is a contradiction. Therefore $103 \mid a_1$.

Let a_2 be the integer such that $103a_2 = a_1$. Therefore $n = 2a_1 = 206a_2$ and

$$103b = 2a_1^2(4a_1^2 + 1) = 2(103a_2)^2(4(103a_2)^2 + 1),$$

so that $b = (2^3)(103^3)a_2^4 + (2)(103)a_2^2$. If the $\frac{n^2}{2} = 103(206)a_2^2$ numbers on the black squares consist of the number $(103)(28)a_2^2 + 1$ and $(103)(206)a_2^2 - 1$ consecutive integers starting at $(103)(53)a_2^2 + 2$, then

$$\begin{aligned} S_1 &= (103)(28)a_2^2 + 1 + \frac{1}{2}((103)(206)a_2^2 - 1)(2((103)(53)a_2^2 + 2) \\ &\quad + ((103)(206)a_2^2 - 1) - 1) \\ &= 39((2^3)(103^3)a_2^4 + (2)(103)a_2^2) \\ &= 39b \end{aligned}$$

and with the other numbers on the white squares, we have $S_2 = 64b$. Therefore all possible n are all $n \in \{206j : j \geq 1 \text{ is an integer}\}$.

Two Tributes to Ross Honsberger

While Ross Honsberger had retired before I became a student and then a faculty member at Waterloo, I had the pleasure of meeting him and hearing him speak to groups of teachers on several occasions. Ross was an engaging and entertaining speaker and the amount of thought, love and preparation that went into his talks was obvious. In my current role, I am lucky to be able to talk to and work with many teachers. Many of these teachers talk about Ross in reverential tones and about the positive impact that taking a course with him had on their love for mathematics and on their career path. We will miss Ross!

Ian VanderBurgh

El pasado 3 de abril falleció en Waterloo, Ontario (Canadá) el Prof. Ross Arnold Honsberger, gran expositor de temas de Matemáticas Elementales y autor de un número considerable de libros de problemas, que la mayor parte de los amantes de las Olimpiadas matemáticas seguramente conocemos y, sin duda, hemos disfrutado con su lectura.

Tuve la fortuna de escucharle en una de las conferencias plenarias de la Primera Conferencia de la WFNMC, precisamente en Waterloo, que llevaba por título El punto simediano. Encandiló a la audiencia con su inigualable manera de explicar los temas que le gustaban. Años más tarde, esa conferencia la incorporó a su libro Episodes in Nineteenth and Twentieth Century Euclidean Geometry.

Es autor de 13 libros, todos ellos publicados por la Mathematical Association of America. El formato de casi todos ellos es siempre el mismo: cortas notas planteando y resolviendo (en muchos casos con sus propias soluciones) problemas de Matemáticas Elementales, o contando anécdotas de historia de las Matemáticas (recuerdo a vuelapluma La historia de Luis Posá, con Paul Erdös como narrador). Tengo la fortuna de tenerlos todos ellos en mi biblioteca personal, los he leído y releído en muchas ocasiones, buscando soluciones o nuevos problemas para mis intervenciones en congresos, Olimpiadas y mis seminarios de preparación de concursos en la Facultad de Ciencias de la Universidad de Valladolid. No podría elegir uno como mi favorito, porque todos lo son, de una forma u otra.

Termino esta breve necrológica de un extraordinario profesor con el aforismo latino antiguo que se dedicaba a un difunto apreciado:

Sit tibi terra levis, Ross.

Francisco Bellot Rosado

Remembering Ross Honsberger

John McLoughlin

University of Waterloo offered two courses in Problem Solving that were numbered something like C&O 380 and 381, at the time. I took both of these courses, the second with Ross Honsberger. Ross Honsberger loved mathematics and problems. His passionate engagement with the ideas and the smiling sense of amusement he brought to the sharing of problems was special. Ross focused attention on problems he found to be rich. Many of these problems would be found in his expository books such as *Mathematical Gems* or *Mathematical Morsels* – enriching resources in themselves. He tended to spend most of an entire class on a particular problem or a piece of mathematics such as the Butterfly Theorem. It was enjoyable to be in the midst of a mathematician who demonstrated such a passion for the field.

A Curious Connection

Jim Totten was a colleague with whom I shared problems through collaborations with the BC Mathematics Contests. The relationship deepened as a member of the Editorial Board during Jim's tenure as Editor of **Crux Mathematicorum with Mathematical Mayhem**. One thing learned through communications was how Jim was deeply influenced by Ross Honsberger. He considered Ross to be a mentor and an inspiration with respect to mathematical problem solving, in particular. Jim Totten had a Problem of the Week feature through his years of teaching undergraduates, briefly at St. Mary's University in Halifax and subsequently in Cariboo College (and its subsequent renamings to finally Thompson Rivers University) in Kamloops. Upon Jim's sudden passing I was given a gift of the red binders containing the different problems that had been used by Jim over 25+ years with this feature. The collection was given to me with an intention of having it written up and shared, as it had been his intention in retirement to develop a resource. In fact, Jim authored Volume VII of the *A Taste of Mathematics* Series entitled *Problems of the Week* containing 80 of the problems. This was his start of a bigger project. Later, I, along with Joseph Khoury and Bruce Shawyer, co-edited *Jim Totten's Problems of the Week* (World Scientific Publishing, 2013). An excerpt from the Preface is offered here in tribute to Ross Honsberger as a person.

> Jim never pretended that the problems were original. The problems come from many sources, including several brought to his attention during his graduate studies at University of Waterloo from 1968 to 1974. It was there that Jim became acquainted with Ross Honsberger. Jim described himself as a willing listener when Ross wanted to share interesting problems or solutions with someone. This excitement for gems was contagious to Jim, and he proceeded to carry forth his own love of problems with a commitment to sharing that spirit of his own.

My final personal communications with Ross Honsberger surrounded Jim's passing in different respects. There was a special issue of **Crux Mathematicorum with Mathematical Mayhem** dedicated to Jim Totten (Volume 35, issue 5). Ross graciously received a request to make a contribution of a problem for this issue, and went well beyond the request to prepare a seven-page article discussing a particular problem, namely, *The Tanker Problem*:

> A security patrol boat repeatedly circles a supertanker that is a gigantic rectangular box 450 metres long and 50 metres across. The ocean is calm and the tanker travels at a constant speed along a straight path. The patrol boat goes up the left side, across the front, down the right side, and across the back, and keeps doing it over and over.
>
> The patrol boat travels in only two directions of the compass – when going parallel to the path of the tanker, it travels in straight lines parallel to the tanker, one on each side at a distance of 25 metres from it, and when crossing in front or behind, it goes straight across perpendicular to the path of the tanker.
>
> Neglecting the dimensions of the patrol boat (that is, considering it to be represented geometrically by a point) and given that it goes constantly at twice the speed of the tanker and that its turns are instantaneous, what is the shortest distance that the patrol boat must travel in completing one cycle around the tanker?

Closing remarks

I am grateful for the presence of Ross Honsberger along my mathematical path. I continue to play with the content of some of his problem books. Upon reconnecting with Ross concerning the **Crux** article, I mentioned that we had held a conference honouring Jim Totten earlier in 2009, and that he may like a copy of the proceedings. I close with mention of this as the gracious spirit of Ross Honsberger was exemplified in his later communication: "The mailman delivered the Tribute to Jim Totten this morning. I am looking forward to reading it all. It's my kind of thing! I am delighted to have a copy. My sincerest thanks."

John McLoughlin

(This is an excerpt from the article Looking Back on Problem Solving and Pedagogy from Waterloo Days*, which appeared in CMS Notes 48 (3) accessible here: https://cms.math.ca/notes/v48/n3/Notesv48n3.pdf).*

Characterizing a Symmedian

Michel Bataille

Dedicated to the memory of Ross Honsberger

In 1873, Émile Lemoine introduced new cevians of the triangle, the symmedians, and presented some properties of their point of concurrency, a centre of the triangle now called the Lemoine point or symmedian point ([**1**]). Since then, these particular lines and point have been discussed in many geometry books and articles, as exemplified by the beautiful chapter 7 of Ross Honsberger's famous book [**2**] (one can also see [**3**], [**4**] or [**5**]). In this note, we examine several characterizations of the symmedian attached to a vertex of the triangle. We give unified proofs of some of these characterizations which are well-known and offer a couple of much less known ones.

Symmedians and antiparallels

First, let us recall the definition of a symmedian. Let m be the median through the vertex A of triangle ABC. The symmedian s through A is the reflection of the line m in the internal bisector ℓ of $\angle BAC$.

The median m and the symmedian s share a "bisection" property: clearly, m bisects any segment B_0C_0 with B_0 on AB, C_0 on AC and B_0C_0 parallel to BC, and therefore s bisects B_1C_1 where B_1, C_1 are the reflections in ℓ of B_0, C_0, respectively (Figure 1: the midpoints M_0 and M_1 of B_0C_0 and B_1C_1 are symmetric in ℓ).

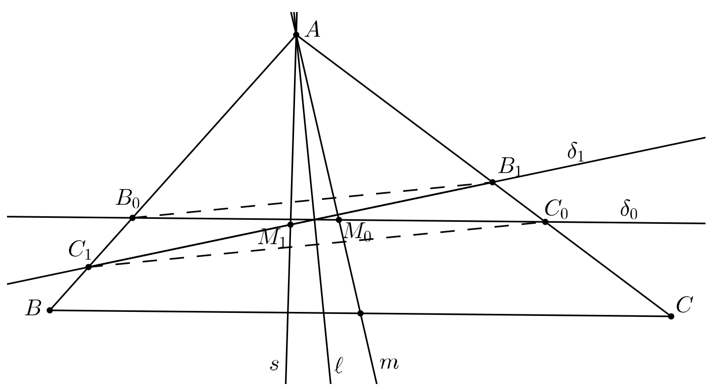


Figure 1

The segment B_1C_1 is said to be antiparallel to BC and the line $\delta_1 = B_1C_1$, the image of $\delta_0 = B_0C_0$ in ℓ, is called an antiparallel (line) to BC. With this terminology, the median m bisects any segment parallel to BC and the symmedian s bisects any segment antiparallel to BC. Since a reflection is involutive, we may even conclude:

> a line through A is the symmedian s if and only if it bisects some segment antiparallel to BC.

To emphasize this characterization, let us make two remarks. First, the antiparallels to BC can be recognized without involving ℓ explicitly. For example, they intersect AC, AB in B_1, C_1 such that ΔAB_1C_1 is inversely similar to ΔABC. More hidden is the following: the lines antiparallel to BC are exactly the perpendiculars to OA where O is the circumcentre of ΔABC. This easily follows from observing that a line δ_1 intersecting AB in C_1 and AC in B_1 is parallel to the tangent t at A to the circumcircle Γ of ΔABC if and only if $\angle(B_1A, B_1C_1) = \angle(BC, BA)$ (note that $\angle(BC, BA) = \angle(AC, t) = \angle(AB_1, t)$) (Figure 2).

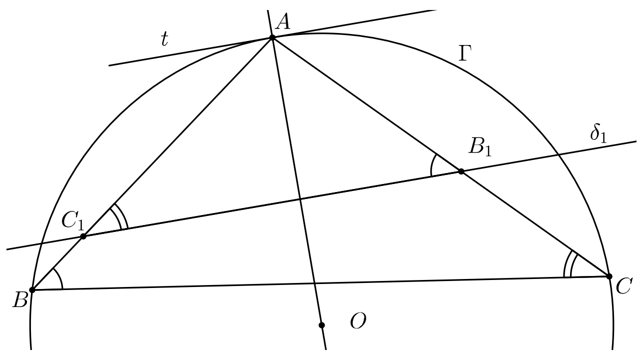


Figure 2

Our second remark is historical: in his original paper **[1]**, Lemoine *defined* the symmedians as the lines through each vertex bisecting any segment antiparallel to the opposite side, going so far as to call them *les médianes antiparallèles*. Perhaps the term *antimédianes* would have been a more appropriate choice!

Symmedians and polarity

A very well-known and often-used characterization of the symmedian s is the following one:

> s is the line through A and the pole of BC with respect to the circumcircle Γ of ΔABC.

In the proof (and coming proofs), we discard the easy cases when $\angle BAC = 90^\circ$ and when $AB = AC$ (in both cases, s is the altitude from A).

Recall that the pole P of BC with respect to Γ is the point of intersection of the tangents to Γ at B and C. Let BC meet the tangent t to Γ at A in Q and let AB and AC meet the tangent t' to Γ at the point A' diametrically opposite to A in B' and C', respectively. Let the line AP intersect BC at L and $B'C'$ at M (Figure 3). Since Q is on the polar BC of P and on the polar t of A, the line AP is the polar of Q. It follows that Q and L divide BC harmonically.

Under the central perspectivity with centre A, the points C, L, B, and Q are transformed into C', M, B', and the point at infinity on t', respectively, hence M is the midpoint of $B'C'$. Since $B'C'$ is antiparallel to BC, AP is the symmedian s.

In passing and for later use, note that the proof above readily yields another characterization associated with the polarity with respect to Γ:

> s is the polar of the intersection of BC with the tangent to Γ at A.

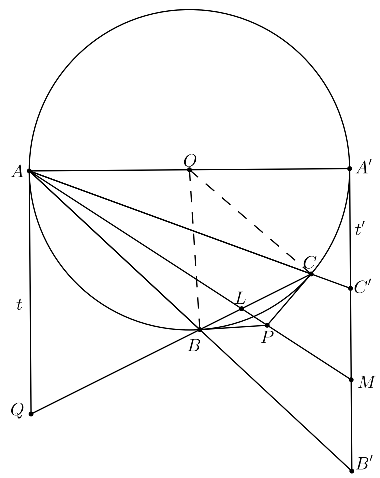


Figure 3

Symmedians and Grebe's construction

Another use of antiparallels leads to a proof of the following characterization of s:

> Let squares $ABDE$ and $ACFG$ be drawn externally to ΔABC and let R be the point of intersection of DE and FG. Then the line AR is the symmedian s.

This property provides a construction of the symmedian point known as Grebe's construction, from the German mathematician Ernst Grebe.

Let the lines AB and AC intersect RG and RE in B' and C', respectively (Figure 4). Clearly, $AB'RC'$ is a parallelogram so that RA bisects the segment $B'C'$. Therefore, we just have to prove that $B'C'$ is antiparallel to BC.

Let ρ denote the right-angle rotation with centre A transforming C into G (note that $\rho(E) = B$). Let $C_1 = \rho(C')$ and $B_1 = \rho^{-1}(B')$. Then, the line AB_1 is perpendicular to AB, hence parallel to BC_1 (note that $\angle ABC_1 = \angle AEC' = 90^\circ$ since $\rho(E) = B$ and $\rho(C') = C_1$). It follows that the line through the midpoints of AB and AC_1 is parallel to AB_1, hence intersects B_1C_1 at its midpoint and, being perpendicular to AB, is the perpendicular bisector of AB. In a similar way, the line through the midpoints of AC and AB_1 is the perpendicular bisector of AC and passes through the midpoint of B_1C_1. As a result, this midpoint is the circumcentre O of ABC.

Now, the vector $2\overrightarrow{AO} = \overrightarrow{AB_1} + \overrightarrow{AC_1}$ is the image under a right-angle rotation of $-\overrightarrow{AB'} + \overrightarrow{AC'} = \overrightarrow{B'C'}$, hence $B'C'$ is perpendicular to OA, and as such, is antiparallel to BC.

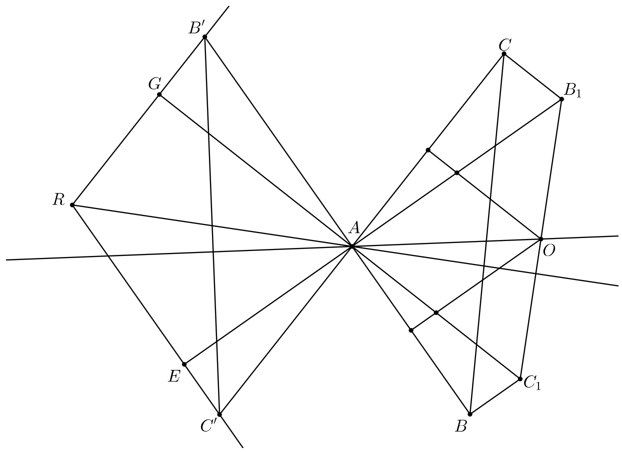


Figure 4

As a corollary, remarking that $\angle AGR = \angle AER = 90°$ so that the common midpoint of AR and $B'C'$ is equidistant from A, R, G, E, we obtain a characterization mentioned in **[5]** with a different proof:

> s is the line through A and the circumcentre of ΔAGE.

Symmedians as tangents

The next characterization is derived from an old problem proposed in 1928 in **[6]** and, to the best of my knowledge, does not appear in recent books or articles:

> s is the tangent at A to the circumcircle of $\Delta CAB'$ where B' is the reflection of B in A.

The proof given here uses the same method as above and is completely different from the 1928 solution by G. Excoffier.

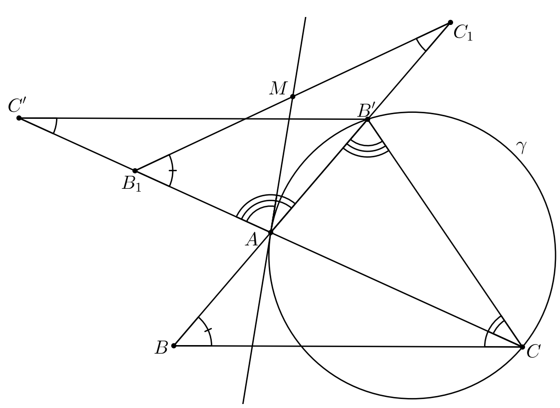


Figure 5

We introduce the symmetric C' of C about A, obtaining the parallelogram $CBC'B'$. Let B_1 and C_1 be the reflections of B' and C' in the internal bisector of $\angle BAC$, so that the segment B_1C_1 is antiparallel to BC. Let γ denote the circle through C, A, B' and let the tangent to γ at A intersect B_1C_1 at M (Figure 5). It suffices to show that $MB_1 = MC_1$.

We shall exploit the numerous equalities of angles of the figure: first

$$\angle AB_1C_1 = \angle CBA \quad \text{and} \quad \angle AC_1B_1 = \angle BCA = \angle B'C'C$$

(since B_1C_1 is antiparallel to BC and $B'C'$ is parallel to BC); second

$$\angle(CB', CA) = \angle(AC_1, AM) \quad \text{and} \quad \angle(B'A, B'C) = \angle(AM, AB_1)$$

(since AM is tangent to γ). We immediately deduce that the triangles AMC_1 and $CB'C'$ are similar and so are triangles AMB_1 and $B'CB$. In consequence, we have

$$MC_1 = AM \cdot \frac{B'C'}{CB'} \quad \text{and} \quad MB_1 = AM \cdot \frac{CB}{B'C},$$

and the equality $MB_1 = MC_1$ follows from $BC = B'C'$.

Symmedians and special circles through O

Our last characterization, which seems to be new, involves two particular circles passing through the circumcentre O of ΔABC:

> s is the line through the vertex A and the point of intersection other than O of the circumcircle of ΔBOC and the circle with diameter AO.

The proof, unlike the previous ones, leaves aside the antiparallels. Again we introduce the circumcircle Γ of ΔABC and recall that its tangent t at A intersects BC at the pole Q of the symmedian s.

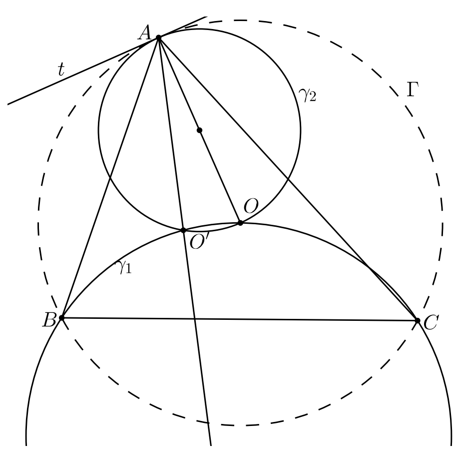


Figure 6

Let the circles γ_1 through B, O and C and γ_2 with diameter AO intersect at O and O' (Figure 6) and let $\mathbf{I}$ denote the inversion in the circle Γ. Then, $\mathbf{I}(\gamma_1)$ is the line BC and $\mathbf{I}(\gamma_2)$ is the tangent t so that $\mathbf{I}(O')$ is the point Q of intersection of t and BC. Since in addition AO' is perpendicular to OO', we conclude that AO' is the polar of Q with respect to Γ and the proof is complete.

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[1] E. Lemoine, Note sur un point remarquable du plan d'un triangle, *Nouv. Ann. de Mathématiques*, tome 12, 1873, p. 364-6.
[2] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, MAA, 1995.
[3] N. Altshiller-Court, *College Geometry*, Dover, 1980, p. 247-252.
[4] Y. et R. Sortais, *La Géométrie du triangle*, Hermann, 1987, p. 152-160.
[5] S. Luo and C. Pohoata, Let's Talk About Symmedians!, *Mathematical Reflections*, **4**, 2013.
[6] E. Mussel/G. Excoffier, Problème 10866, *Journal de mathématiques élémentaires*, Vuibert, 1928-9, No 1, p. 14.

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Revisiting C&O 380: Assignment #1

Shawn Godin

I have been fortunate to have had a number of great teachers: Bill Morin at Espanola High School; Ken Fryer, Bev Marshman and Scott Vanstone at the University of Waterloo; Kenneth Williams and John Poland at Carleton University; as well as many others, including Ross Honsberger. I met professor Honsberger in my first year. He was filling in for professor Fryer while the latter was undergoing cancer treatment. In my third year, I had him full time as the teacher of Combinatorics and Optimization (C&O) 380: Problem Solving.

Professor Honsberger was a dynamic, engaging teacher. His love of mathematics and teaching was evident every class. The class was unlike any other I have ever taken. On the first day, professor Honsberger presented us with 100 problems. We were instructed to try the first 5 before the next class at which point we would discuss them, introduce some new mathematics or techniques as needed and then leave the class ready to try 5 more. The class actively engaged the students as they worked on the material that was going to be the next lesson. I have tried to encourage that engagement in my own classrooms.

After graduating from the University of Waterloo in 1987, I have encountered Professor Honsberger from time to time. As a young teacher, I was always on the lookout for interesting material to use in my classrooms, or to challenge my students or math club. One of the first books that I found was *Ingenuity in Mathematics* by Professor Honsberger, published by the Mathematical Association of America. The book was a collection of short essays on particular problems, such as essay 19 on *Van Schooten's Problem*, or a particular mathematical idea, such as essay 5 on *The Farey Sequence*. Over the years I have picked up a number of his books. Many of his books contain interesting problems and their solutions, many from the pages of **Crux**. As such, I think that writing this essay on a problem that he presented in one of his classes is the best way that I can pay tribute to him.

I touched base with Professor Honsberger when I was working on **Crux**. He offered some good advice and led to some valuable contacts. He also submitted some material that we have published, including his article *A Typical Problem on an Entrance Exam for the École Polytechnique* from Volume 38(3) p. 101-103 and two smaller pieces that were used in the Problem of the Month #3, Volume 38(9) p. 369-371 and #4, Volume 39(1) p. 27-30. **Crux** has also published reviews of a number of his books over the years.

It is a testament to the impact that Professor Honsberger had on me to note that the only class material that I have from my undergraduate work 30 years ago, other than textbooks, is my 100 problems and three assignments from C&O 380. I have already talked about one of the problems (#5) from the class in the first Problem of the Month in Volume 38(5) p. 186 - 187. On the next page is the first C&O 380 assignment, verbatim.

C&O 380 **Assignment #1** **Due: February 5, 1986**

#1. What is the sum of all the digits used in writing down the integers from 1 to a billion?

#2. Determine the total number of rectangles, of all sizes and positions, on an $n \times n$ checkerboard.

#3. Prove that every positive integer has some multiple whose decimal representation contains all ten decimal digits.

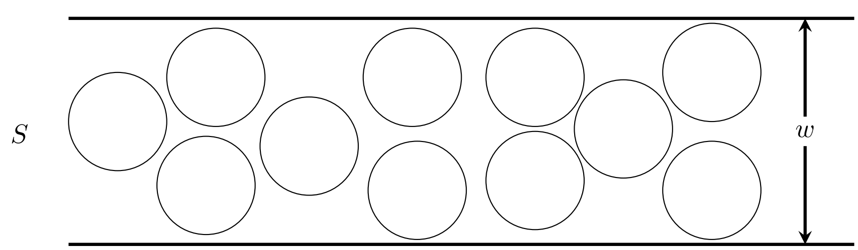
#4. Prove that each term of the sequence

$$49, 4489, 444889, 44448889, 4444488889, \ldots$$

is a perfect square.

#5. Prove that $\left[\left(2+\sqrt{3}\right)^n\right]$ is always an odd integer. ($[x]$ denotes the integer part of x, that is, the greatest integer $\leq x$.)

#6. Circles of unit radius are packed, without overlapping of interior points, in a strip S of the plane whose parallel edges are a distance w apart. We say the circles form a k-cloud if every straight line which cuts across S makes contact with at least k circles. Prove that the width w of a 2-cloud must be at least $2+\sqrt{3}$.



Let's examine question #5 (I'll leave the rest for homework). As it stands, the expression $\left[\left(2+\sqrt{3}\right)^n\right]$ is quite nasty even though the cases of $n=0$ and $n=1$ are trivial. If we expand the power using the binomial theorem we get

$$\left(2+\sqrt{3}\right)^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \left(\sqrt{3}\right)^i$$

which is made up of a number of even integers added with a number of multiples of $\sqrt{3}$ (as well as a power of three if n is even), which isn't really that helpful. It would be nice if we could eliminate those $\sqrt{3}$ terms. Let's examine the related expression

$$\left(2-\sqrt{3}\right)^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(-1)^i \left(\sqrt{3}\right)^i,$$

which is itself made up of a number of even integers with a number of multiples of $\sqrt{3}$ which, because of the $(-1)^i$, are all negative. Thus if we add the two expressions and then separate the even and odd powers of $\sqrt{3}$ we get

$$\begin{aligned}\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n &= \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \left(\sqrt{3}\right)^i \left(1+(-1)^i\right)\\ &= \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} 2^{n-2j} \left(\sqrt{3}\right)^{2j} \left(1+(-1)^{2j}\right)\\ &\qquad + \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k+1} 2^{n-(2k+1)} \left(\sqrt{3}\right)^{2k+1} \left(1+(-1)^{2k+1}\right)\\ &= 2\sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} 2^{n-2j} 3^j\end{aligned}$$

which is clearly even. Then since $2-\sqrt{3}<1$, we must have $\left(2-\sqrt{3}\right)^n<1$ for $n \geq 1$, hence $\left(2+\sqrt{3}\right)^n$ is a smidgeon less than an even number, $\left[\left(2+\sqrt{3}\right)^n\right]$ is always an odd integer.

Alternately, $2+\sqrt{3}$ and $2-\sqrt{3}$ are roots of the quadratic equation $x^2=4x-1$. If we think of this equation as the characteristic equation of a linear recurrence relation $t_n = 4t_{n-1} - t_{n-2}$, then every sequence that satisfies this recurrence relation has solutions of the form $t_n = A\left(2+\sqrt{3}\right)^n + B\left(2-\sqrt{3}\right)^n$ for appropriate A and B.

If we choose $A=B=1$ we get $t_n = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$, which was the expression we examined in the solution. Note that we can easily calculate that $t_0 = 2$ and $t_1 = 4$, so the recurrence relation implies that all t_n will be even, hence $\left[\left(2+\sqrt{3}\right)^n\right]$ is always an odd integer as in the first solution.

Enjoy the other 5 problems. Thanks for everything Professor Honsberger.

. . . .

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The Lucas Circles of a Triangle

Ross Honsberger

The initial construction

The first step in our story is to inscribe a square $PQRS$ in a given triangle ABC. Since the square has four vertices and the triangle has only three sides, some side of the triangle must contain two of the vertices of the square (Figure 1). How to construct such a square is an interesting problem in its own right and it gives us a chance to dust off a very useful observation about similar figures.

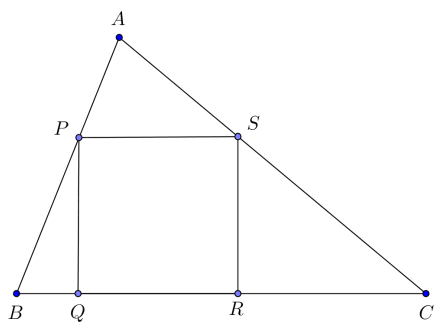

Figure 1

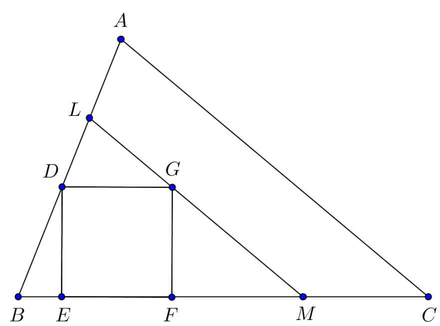

Figure 2

Let us begin modestly by drawing any little square $DEFG$ as in Figure 2. Now, if a line LM is drawn through G parallel to AC, then $\triangle LBM$ with its inscribed square $DEFG$ would be precisely the picture we want, only smaller.

Recalling that corresponding angles in similar figures are equal, it follows that BG in Figure 2 divides angle B into the same two parts x and y that BS does in Figure 1 (Figures 3 and 4); they're both pictures of the same thing. Therefore BG runs along on top of BS and S is obtained simply by extending BG to AC, after which $PQRS$ can be constructed easily.

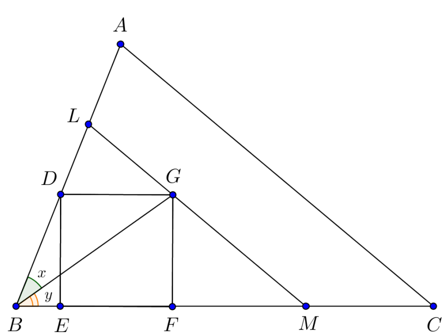

Figure 3

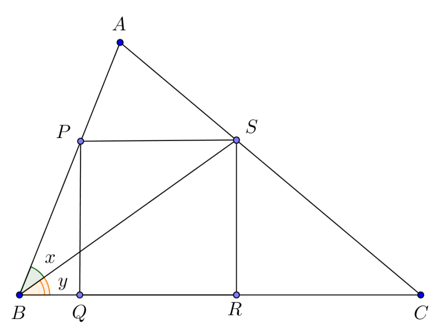

Figure 4

The Lucas Circles

In celebration of this inspired success, let's draw all three of the inscribed squares of $\triangle ABC$. As we have already observed, one side of the square lies along a side of the triangle, and evidently the opposite side cuts a little triangle from ABC at the opposite vertex. It is generally a different square on each side of the triangle and a different triangle at each vertex (these triangles are shaded in Figure 5).

It is the circumcircles of these shaded triangles that are the Lucas circles of $\triangle ABC$.

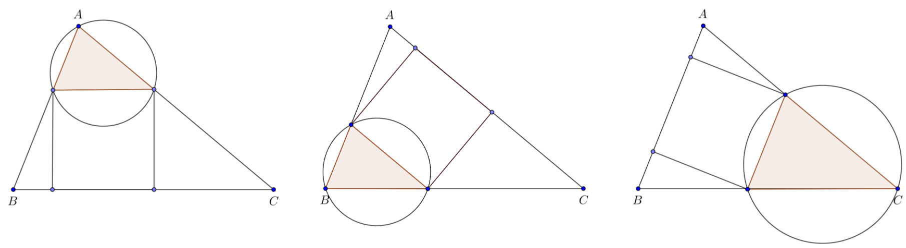

Figure 5

Now, these Lucas circles enjoy two marvelous properties:

i) each is tangent to each of the others; and

ii) they are all tangent to the circumcircle of $\triangle ABC$.

Thus we have the lovely result that

Lucas circles and the circumcircle of the triangle form a nest of four circles each of which touches the other three!! (Figure 6)

I expect many of us would be willing to invest considerable time and effort to obtain proof of this amazing result. As we shall see, however, the arguments are completely elementary and straightforward.

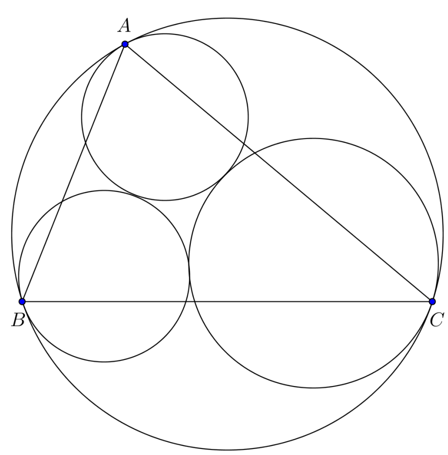

Figure 6

The proofs

The proof that each Lucas circle is tangent to the circumcircle of $\triangle ABC$ couldn't be easier.

Consider the Lucas circle though A (Figure 7). Since PS is parallel to BC, triangles APS and ABC are similar, and therefore the radius from A to the circumcenter O in $\triangle ABC$ divides angle A into the same two parts u and v as the radius from A to the circumcenter O_A in $\triangle APS$, implying that AO_A lies along the same line as AO. Thus A, O_A, and O are in a straight line and hence the distance O_AO between the centers of the circles is just the difference between their radii, implying that the circles are indeed internally tangent at A.

Similarly for the other Lucas circles.

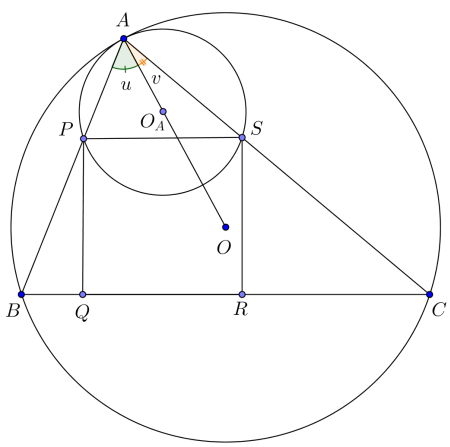

Figure 7

We conclude with the proof that each Lucas circle touches the other two. This argument might appear a little complicated on the page, but it consists of nothing but short easy steps.

To set the notation, let the circumcenter of $\triangle ABC$ be O and the centers of the Lucas circles at A, B and C be O_A, O_B and O_C, respectively; and let the radii be R, R_A, R_B, R_C.

In order to prove that the Lucas circles at A and B touch externally we need to show that the distance O_AO_B between their centers is the sum of their radii, $R_A + R_B$. To accomplish this we need expressions for their radii.

A formula for the radii

Consider the Lucas circle at A. Suppose the side of the inscribed square $PQRS$ is of length x (Figure 8). Using the standard formula for the circumradius of a triangle, we get $R = \frac{a}{2\sin A}$ from $\triangle ABC$ and $R_A = \frac{x}{2\sin A}$ from $\triangle APS$, the latter of which yields the useful $x = 2R_A \sin A$.

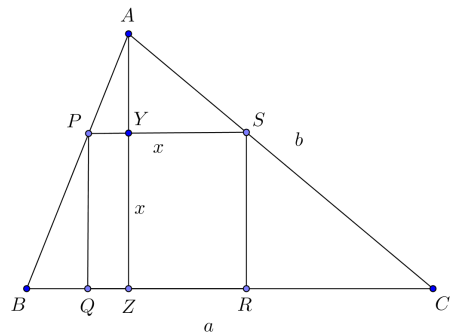

Figure 8

Let AY and AZ be the altitudes from A in triangles APS and ABC (they lie along the same line since PS is parallel to BC). Then $YZ = x$ and $AZ = b\sin C$ from $\triangle ABC$, and the ratio of the altitudes is

$$\frac{AY}{AZ} = \frac{AZ - x}{AZ} = 1 - \frac{x}{b\sin C}.$$

Now, triangles AYS and AZC are similar, and therefore

$$\frac{AY}{AZ} = \frac{PS}{PC} = \frac{x}{a}$$

from similar triangles APS and ABC. Hence

$$\frac{x}{a} = 1 - \frac{x}{b\sin C},$$

and, substituting $2R_A \sin A$ for x, we obtain

$$\frac{2R_A \sin A}{a} = 1 - \frac{2R_A \sin A}{b\sin C}.$$

Since $\dfrac{\sin A}{\sin C} = \dfrac{a}{c}$ by the Law of Sines, this yields

$$\frac{2R_A \sin A}{a} + \frac{2R_A a}{bc} = 1.$$

Multiplying by abc and simplifying gives

$$R_A = \frac{abc}{2bc\sin A + 2a^2}.$$

Now, from $R = \dfrac{a}{2\sin A}$, we have

$$R = \frac{abc}{2bc\sin A} \quad \text{and} \quad 2bc\sin A = \frac{abc}{R}.$$

Therefore,

$$R_A = \frac{abc}{2bc\sin A + 2a^2} = \frac{abc}{\frac{abc}{R} + 2a^2} = \frac{bcR}{bc + 2aR}.$$

Similarly,

$$R_B = \frac{acR}{ac + 2bR}.$$

The final step

Recall that $OO_A = R - R_A$, the difference between the radii (Figure 9).

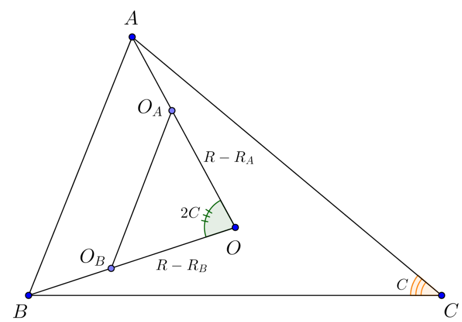

Figure 9

Hence,

$$OO_A = R - \frac{bcR}{bc + 2aR} = R\left(1 - \frac{bc}{bc + 2aR}\right) = R\left(\frac{bc + 2aR - bc}{bc + 2aR}\right) = \frac{2aR^2}{bc + 2aR}.$$

Now from $R_A = \dfrac{bcR}{bc + 2aR}$, the denominator $bc + 2aR = \dfrac{bcR}{R_A}$, and we have

$$OO_A = R - R_A = \frac{2aR^2}{bc + 2aR} = \frac{2aR^2}{\frac{bcR}{R_A}} = \frac{2aRR_A}{bc}.$$

Similarly,

$$OO_B = R - R_B = \frac{2bRR_B}{ac}.$$

Now, in the circumcircle of $\triangle ABC$, the side AB subtends at the center O an angle that is twice the angle it subtends at the circumference (Figure 9). Hence, $O_AOO_B = 2C$ and, applying the Law of Cosines to $\triangle O_AOO_B$, we obtain

$$\begin{aligned} O_AO_B^2 &= (R - R_A)^2 + (R - R_B)^2 - 2(R - R_A)(R - R_B)\cos 2C \\ &= (R - R_A)^2 + (R - R_B)^2 - 2(R - R_A)(R - R_B)(1 - 2\sin^2 C) \\ &= R_A^2 - 2R_AR_B + R_B^2 + 4(R - R_A)(R - R_B)\sin^2 C. \end{aligned}$$

Now, substituting for $R - R_A$ and $R - R_B$ in the final term gives

$$\begin{aligned} O_AO_B^2 &= (R_A - R_B)^2 + 4 \cdot \frac{2aRR_A}{bc} \cdot \frac{2bRR_B}{ac}\sin^2 C \\ &= (R_A - R_B)^2 + 4R_AR_B\frac{4R^2\sin^2 C}{c^2}. \end{aligned}$$

Now, R is also given by the formula $R = \dfrac{c}{2\sin C}$, and so

$$2R\sin C = c \quad \text{and} \quad \frac{4R^2\sin^2 C}{c^2} = 1.$$

Hence,

$$O_AO_B^2 = (R_A - R_B)^2 + 4R_AR_B \cdot 1 = (R_A + R_B)^2,$$

and $O_AO_B = R_A + R_B$, completing the proof.

. . . .

The Lucas circles are named in honour of the outstanding French number theorist Edouard Lucas (1842 - 1891) who is famous for his Lucas numbers, a companion sequence of the Fibonacci numbers.

This essay is based on the delightful note The Lucas Circles of a Triangle, by Antreas P. Hatzipolakis (Athens, Greece) and Paul Yiu (Florida Atlantic University), which appeared in volume 108 of the *American Mathematical Monthly*, May 2001, pages 444–446.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er décembre 2017**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4231. *Proposé par Marius Stănean.*

Soit $ABCD$ un quadrilatère cyclique, soit $O = AC \cap BD$, soient M, N, P et Q les mi points de AB, BC, CD et DA respectivement et soient X, Y, Z et T les projections de O vers AB, BC, CD et DA respectivement. Soient $U = MP \cap YT$ et $V = NQ \cap XZ$. Démontrer que

$$\frac{UO}{VO} = \frac{AB \cdot CD}{BC \cdot DA}.$$

4232. *Proposé par Michel Bataille.*

Soit n un enter positif. Démontrer que

$$\sum_{k=0}^{2n-1} \binom{2n-1+k}{k}\binom{2n-1}{k}\frac{(-1)^k}{2^k} = 0.$$

4233. *Proposé par Peter Y. Woo.*

Résoudre le problème suivant sans trigonométrie.

Soit ABC un triangle où $\angle B > 90^\circ$. Dénoter par M le pied de l'altitude allant de C vers AB, et par N le pied de l'altitude allant de B vers AC. Si $AB = 2CM$ et $\angle ABN = \angle CBM$, déterminer $\angle A$.

4234. *Proposé par Leonard Giugiuc, Daniel Sitaru et Marian Dinca.*

Soient x, y et z des nombres réels tels que $x \geq y \geq z > 0$. Démontrer l'inégalité suivante, quel que soit $k \geq 0$

$$\frac{4}{x+3y+4k}+\frac{4}{y+3z+4k}+\frac{4}{z+3x+4k} \geq \frac{3}{x+2y+3k}+\frac{3}{y+2z+3k}+\frac{3}{z+2x+3k}.$$

4235. *Proposé par Ruben Dario Auqui et Leonard Giugiuc.*

Soit ABC un triangle isocèle où $BA = BC$ et soit I le centre de son cercle inscrit. Dénoter par X, Y et Z les points de tangence du cercle inscrit avec les côtés AB, AC et BC respectivement. Une ligne d passant par I intersecte les segments AX et CZ. Dénoter par a, b, c, x, y et z les distances vers la ligne d, à partir des points A, B, C, X, Y et Z respectivement. Démontrer que

$$\frac{a+c}{b} = \frac{x+z}{y}.$$

4236. *Proposé par Nguyen Viet Hung.*

Évaluer

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}}\right).$$

4237. *Proposé par Cristinel Mortici et Leonard Giugiuc.*

Pour un entier $n \geq 2$, déterminer toutes valeurs $a_1, \ldots, a_n, b_1, \ldots, b_n$ telles que

i) $0 \leq a_1 \leq \ldots \leq a_n \leq 1 \leq b_1 \leq \ldots \leq b_n$;

ii) $\sum_{k=1}^{n}(a_k + b_k) = 2n$; et

iii) $\sum_{k=1}^{n}(a_k^2 + b_k^2) = n^2 + 3n$.

4238. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle et soit M un point arbitraire situé sur BC. Si X et Y sont les centres des cercles inscrits de triangle ABM et triangle AMC et s'il existe $Z \in (AM)$ pour lequel $BCZA$ et $CYZB$ sont des quadrilatères cycliques, déterminer $m,\ n \in \mathbb{R}$ donnant lieu à $\overrightarrow{AM} = m\overrightarrow{AB} + n\overrightarrow{AC}$.

4239. *Proposé par Leonard Giugiuc et Abdilkadir Altintas.*

Soit ABC un triangle et soit G son centroïde. Dénoter par D et E les mi points des côtés BC et AC respectivement. Si le quadrilatère $CDGE$ est cyclique, démontrer que

$$\cot A = \frac{2AC^2 - AB^2}{4 \cdot Area(ABC)}.$$

4240. *Proposé par Michael Rozenberg et Leonard Giugiuc.*

Soient a, b et c des nombres réels positifs tels que $a+b+c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Démontrer que $1 + a + b + c \geq 4abc$.

. . . .

4231. *Proposed by Marius Stănean.*

Let $ABCD$ be a cyclic quadrilateral, $O = AC \cap BD$, M, N, P, Q be the midpoints of AB, BC, CD and DA, respectively, and X, Y, Z, T be the projections of O on AB, BC, CD and DA, respectively. Let $U = MP \cap YT$ and $V = NQ \cap XZ$. Prove that

$$\frac{UO}{VO} = \frac{AB \cdot CD}{BC \cdot DA}.$$

4232. *Proposed by Michel Bataille.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{2n-1} \binom{2n-1+k}{k}\binom{2n-1}{k}\frac{(-1)^k}{2^k} = 0.$$

4233. *Proposed by Peter Y. Woo.*

A high-school math teacher required her geometry students to solve this problem without trigonometry: Let ABC be a triangle where $\angle B > 90^\circ$. Denote by M the foot of the altitude from C to AB, and by N the foot of the altitude from B to AC. Then if $AB = 2CM$ and $\angle ABN = \angle CBM$, determine $\angle A$.

4234. *Proposed by Leonard Giugiuc, Daniel Sitaru and Marian Dinca.*

Let x, y and z be real numbers such that $x \geq y \geq z > 0$. Prove that for any $k \geq 0$ we have

$$\frac{4}{x+3y+4k}+\frac{4}{y+3z+4k}+\frac{4}{z+3x+4k} \geq \frac{3}{x+2y+3k}+\frac{3}{y+2z+3k}+\frac{3}{z+2x+3k}.$$

4235. *Proposed by Ruben Dario Auqui and Leonard Giugiuc.*

Let ABC be an isosceles triangle with $BA = BC$ and let I be its incentre. Denote by X, Y and Z the tangency points of the incircle and the sides AB, AC and CB, respectively. A line d passes through I intersecting the segments AX and CZ. Denote by a, b, c, x, y and z the distances from the points A, B, C, X, Y and Z to the line d, respectively. Prove that

$$\frac{a+c}{b} = \frac{x+z}{y}.$$

4236. *Proposed by Nguyen Viet Hung.*

Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}}\right).$$

4237. *Proposed by Cristinel Mortici and Leonard Giugiuc.*

For an integer $n \geq 2$, find all $a_1, \ldots, a_n, b_1, \ldots, b_n$ so that

i) $0 \leq a_1 \leq \ldots \leq a_n \leq 1 \leq b_1 \leq \ldots \leq b_n$;

ii) $\sum_{k=1}^{n} (a_k + b_k) = 2n$; and

iii) $\sum_{k=1}^{n} (a_k^2 + b_k^2) = n^2 + 3n$.

4238. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with M an arbitrary point on BC. If X, Y are centers of inscribed circles in $\triangle ABM$ and $\triangle AMC$ and if there exists $Z \in (AM)$ for which $BXZA$, $CYZB$ are cyclic quadrilaterals, find $m,\ n \in \mathbb{R}$ leading to $\overrightarrow{AM} = m\overrightarrow{AB} + n\overrightarrow{AC}$.

4239. *Proposed by Leonard Giugiuc and Abdilkadir Altintas.*

Let ABC be a triangle with centroid G. Denote by D and E the midpoints of the sides BC and AC, respectively. If the quadrilateral $CDGE$ is cyclic, prove that

$$\cot A = \frac{2AC^2 - AB^2}{4 \cdot Area(ABC)}.$$

4240. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $1 + a + b + c \geq 4abc$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 32(4), p. 172–175.

4131. *Proposed by Michel Bataille.*

Let a, b, c be real numbers such that

$$\tanh a \tanh b + \tanh b \tanh c + \tanh c \tanh a = 1.$$

Show that the equation

$$\sinh(2a - x) + \sinh(2b - x) + \sinh(2c - x) + \sinh x = 4 \sinh a \sinh b \sinh c$$

has exactly one real solution.

We received two correct submissions, one from Leonard Giugiuc and one from the proposer. We feature a combination of their solutions.

First, we show that $a + b + c$ is a solution: With $x = a + b + c$, the left-hand side of the equation becomes

$$\begin{aligned} \sinh(a - b - c) + \sinh(b - c - a) &+ \sinh(c - a - b) + \sinh(a + b + c) \\ &= -2 \sinh c \cosh(a - b) + 2 \sinh c \cosh(a + b) \\ &= 2 \sinh c(2 \sinh a \sinh b), \end{aligned}$$

as desired. We must show that this solution is unique under the additional requirement that

$$\tanh a \tanh b + \tanh b \tanh c + \tanh c \tanh a = 1. \tag{1}$$

But before that, let us check that (1) is not empty. That is, we wish to find real numbers a and b for which

$$\tanh c = \frac{1 - \tanh a \tanh b}{\tanh a + \tanh b} = \frac{\cosh a \cosh b - \sinh a \sinh b}{\sinh a \cosh b + \sinh b \cosh a} = \frac{\cosh(a - b)}{\sinh(a + b)}.$$

Because $-1 < \tanh c < 1$, we require

$$1 > \frac{\cosh^2(a - b)}{\sinh^2(a + b)} = \frac{\cosh 2(a - b) + 1}{\cosh 2(a + b) - 1}.$$

Consequently, we must have

$$2 < \cosh 2(a + b) - \cosh 2(a - b) = 2 \sinh 2a \sinh 2b.$$

Thus we see that for any nonzero real number a, we simply choose any b such that $\sinh 2a \sinh 2b > 1$, and set

$$c = \tanh^{-1}\left(\frac{1 - \tanh a \tanh b}{\tanh a + \tanh b}\right)$$

to obtain a triple (a, b, c) satisfying (1).

Returning to the problem of uniqueness, we set $\tanh a = u, \tanh b = v$, and $\tanh c = w$. Thus we have

$$-1 < u, v, w < 1 \quad \text{and} \quad uv + vw + wu = 1.$$

We have

$$\sinh 2a + \sinh 2b + \sinh 2c = \frac{8uvw}{(1-u^2)(1-v^2)(1-w^2)}, \tag{2}$$

because the left-hand side equals

$$\begin{aligned} & 2\left(\frac{u}{1-u^2} + \frac{v}{1-v^2} + \frac{w}{1-w^2}\right) \\ = \quad & 2 \cdot \frac{u+v+w-[uv(u+v)+vw(v+w)+wu(w+u)]+(uv+vw+wu)uvw}{(1-u^2)(1-v^2)(1-w^2)} \\ = \quad & 2 \cdot \frac{u+v+w-(u+v+w)(uv+vw+wu)+3uvw+uvw}{(1-u^2)(1-v^2)(1-w^2)}, \end{aligned}$$

which reduces to the right-hand side. Similarly,

$$\cosh 2a + \cosh 2b + \cosh 2c - 1 = \frac{4uvw(u+v+w+uvw)}{(1-u^2)(1-v^2)(1-w^2)}, \tag{3}$$

because both sides of (3) are equal to

$$2\left(\frac{u^2}{1-u^2} + \frac{v^2}{1-v^2} + \frac{w^2}{1-w^2} + 1\right).$$

From (2) and (3) we now have

$$\sinh(2a - x) + \sinh(2b - x) + \sinh(2c - x) + \sinh x = 4 \sinh a \sinh b \sinh c$$

if and only if

$$\begin{aligned} \frac{8uvw}{(1-u^2)(1-v^2)(1-w^2)} \cosh x + \frac{4uvw(u+v+w+uvw)}{(1-u^2)(1-v^2)(1-w^2)} \sinh x \\ = 4 \cdot \frac{u}{\sqrt{1-u^2}} \frac{v}{\sqrt{1-v^2}} \frac{w}{\sqrt{1-w^2}}, \end{aligned}$$

which is equivalent to

$$2 \cosh x - (u+v+w+uvw) \sinh x = \sqrt{(1-u^2)(1-v^2)(1-w^2)}.$$

But

$$\begin{aligned}(1-u^2)(1-v^2)(1-w^2) &= (1-u)(1-v)(1-w)(1+u)(1+v)(1+w)\\ &= [2-(u+v+w+uvw)][2+(u+v+w+uvw)].\end{aligned}$$

So if we set $m = u+v+w+uvw$, then $-2 < m < 2$, and our equation becomes

$$2\cosh x - m\sinh x = \sqrt{4-m^2}.$$

We get a further simplification by setting $t = e^x$ so that

$$\cosh x = \frac{t^2+1}{2t}, \sinh x = \frac{t^2-1}{2t},$$

and our equation becomes

$$2\left(\frac{t^2+1}{2t}\right) - m\left(\frac{t^2-1}{2t}\right) = \sqrt{4-m^2}.$$

This equation reduces to $(t\sqrt{2-m} - \sqrt{2+m})^2 = 0$, whence $t = \sqrt{\frac{2+m}{2-m}}$, and finally,

$$x = \frac{1}{2}\ln\left(\frac{2+m}{2-m}\right).$$

Hence, the given equation has exactly one real solution, as claimed.

Editor's Comments. Unraveling the final equation yields

$$x = \frac{1}{2}\ln\left(\frac{2+(\tanh a+\tanh b+\tanh c+\tanh a\tanh b\tanh c)}{2-(\tanh a+\tanh b+\tanh c+\tanh a\tanh b\tanh c)}\right).$$

Because the solution to the given equation is unique under the hypothesis that (1) holds, this expression for x necessarily equals the obvious solution that we confirmed at the start, namely $x = a+b+c$. It is not obvious to this editor that the two values of x are equal, but if we believe in the consistency of mathematics and the infallibility of editors, then these rather different-looking quantities must be equal.

4132. *Proposed by Marian Maciocha.*

Let a and b be integers such that a^2+b^2 divides $2a^3+b^2$. Prove that the integer $2a^3b^2+ab^2+3b^4$ is divisible by a^2+b^2.

There were eight correct solutions. One submission had an error, and the editor could not see where another submission was going.

Solution 1, by Fernando Ballesta Yagüe.

Since

$$2a^3b^2+ab^2+3b^4 = (2a^3+b^2)(b^2+a) - 2(a^2+b^2)(a^2-b^2),$$

the result follows.

Solution 2, by Steven Chow.

Modulo a^2+b^2, we have that $b^2 \equiv -a^2$, $2a^3-a^2 \equiv 0$, and

$$2a^3b^2+ab^2+3b^4 \equiv -2a^5-a^3+3a^4 = (2a^3-a^2)(a-a^2) \equiv 0.$$

Editor's Comments. The condition is, in fact, satisfiable. For example, we can take

$$(a,b) = (2c^2+1, 2c(2c^2+1))$$

for any nonzero integer c and find that

$$a^2+b^2 = (2c^2+1)^2(4c^2+1)$$

and

$$2a^3+b^2 = 2(a^2+b^2).$$

Are there other integer pairs that will work?

4133. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequence (a_n) defined recursively by $a_1 = 1$ and $a_{n+1} = (2n+1)!!a_n$ for all positive integers n. Compute

$$\lim_{n\to\infty} \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}.$$

We received five solutions, all correct and complete. We present two solutions.

Solution 1, by Arkady Alt.

We will use convenient asymptotic notation, namely in the case $\lim\limits_{n\to\infty} \dfrac{b_n}{c_n} = 1$ we will write $b_n \sim c_n$ and then $\lim\limits_{n\to\infty} x_n b_n = \lim\limits_{n\to\infty} x_n c_n$ for any convergent sequence x_n.

Since $\lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = \dfrac{1}{e}$ then $\sqrt[n]{n!} \sim \dfrac{n}{e}$ and therefore

$$\sqrt[2n]{(2n-1)!!} = \sqrt[2n]{\frac{(2n)!}{2^n n!}} = \frac{1}{\sqrt{2}}\sqrt[2n]{\frac{(2n)!}{n!}} \sim \frac{1}{\sqrt{2}} \cdot \frac{\frac{2n}{e}}{\sqrt{\frac{n}{e}}} = \sqrt{\frac{2n}{e}}$$

$$\iff \sqrt[n]{(2n-1)!!} \sim \frac{2n}{e}.$$

Hence, $\lim\limits_{n\to\infty} \dfrac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}} = \sqrt{\dfrac{2}{e}} \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt[n^2]{a_n}}$.

Let $c_n := \ln \dfrac{\sqrt{n}}{\sqrt[n^2]{a_n}} = \dfrac{\ln n}{2} - \dfrac{\ln a_n}{n^2} = \dfrac{d_n}{2n^2}$, where $d_n := n^2 \ln n - 2\ln a_n$

Note that

$$\begin{aligned}
&d_{n+1} - d_n \\
&= (n+1)^2 \ln(n+1) - n^2 \ln n - 2\left(\ln a_{n+1} - \ln a_n\right) \\
&= (n+1)^2 \ln(n+1) - n^2 \ln n - 2\ln(2n+1)!! \\
&= \Big((n+1)^2 \ln(n+1) - n^2 \ln(n+1)\Big) + \Big(n^2 \ln(n+1) - n^2 \ln n\Big) - 2\ln(2n+1)!! \\
&= (2n+1)\ln(n+1) + n^2 \ln\left(1+\frac{1}{n}\right) - 2\ln(2n+1)!! \\
&= \ln(n+1) + n^2 \ln\left(1+\frac{1}{n}\right) + 2n\ln(n+1) - 2\ln(2n+1)!! \\
&= \ln(n+1) + n\ln\left(1+\frac{1}{n}\right)^n + 2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{d_{n+1}-d_n}{(n+1)^2 - n^2} \\
&= \lim_{n\to\infty} \frac{\ln(n+1) + n\ln\left(1+\frac{1}{n}\right)^n + 2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}}{2n+1} \\
&= \lim_{n\to\infty} \frac{\ln(n+1)}{2n+1} + \lim_{n\to\infty} \frac{n\ln\left(1+\frac{1}{n}\right)^n}{2n+1} + \lim_{n\to\infty} \frac{2n\cdot\ln\frac{n+1}{\sqrt[n]{(2n+1)!!}}}{2n+1} \\
&= \frac{1}{2} + \lim_{n\to\infty} \ln\frac{n+1}{\sqrt[n]{(2n+1)!!}} \\
&= \frac{1}{2} + \ln\lim_{n\to\infty} \frac{n+1}{\sqrt[n]{(2n+1)!!}} \\
&= \frac{1}{2} + \ln\frac{e}{2} \\
&= \frac{3}{2} - \ln 2.
\end{aligned}$$

(Observe that

$$\lim_{n\to\infty} \frac{n+1}{\sqrt[n]{(2n+1)!!}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{n+1}{n}\cdot\frac{1}{\sqrt[n]{(2n+1)}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{(2n-1)!!}} = \frac{e}{2}.)$$

Then by Stolz-Cesaro Theorem $\lim\limits_{n\to\infty} \dfrac{d_n}{n^2} = \dfrac{3}{2} - \ln 2$ and therefore

$$\lim_{n\to\infty} c_n = \frac{1}{2}\left(\frac{3}{2} - \ln 2\right).$$

Hence, $\lim\limits_{n\to\infty} \dfrac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}} = \sqrt{\dfrac{2}{e}}\cdot e^{\lim\limits_{n\to\infty} c_n} = \sqrt{\dfrac{2}{e}}\cdot e^{\frac{3}{4}-\frac{1}{2}\ln 2} = e^{\frac{1}{4}}.$

Solution 2, by AN-anduud Problem Solving Group.

Applying Stolz-Cesaro Lemma two times, we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{1}{2n}\log(2n-1)!! - \frac{1}{n^2}\log a_n\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{n\log(2n-1)!! - 2\log a_n}{2n^2}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{(n+1)\log(2n+1)!! - n\log(2n-1)!! - 2\left(\log\frac{a_{n+1}}{a_n}\right)}{2(n+1)^2 - 2n^2}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{n\log\frac{(2n+1)!!}{(2n-1)!!} + \log(2n+1)!! - 2\log(2n+1)!!}{2(2n+1)}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{n\log(2n+1) - \log(2n+1)!!}{2(2n+1)}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{(n+1)\log(2n+3) - n\log(2n+1) - \log(2n+3)}{2(2n+3-2n-1)}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{1}{4}n\cdot\log\frac{2n+3}{2n+1}\right)\right)\\
&\quad= \exp\left(\lim_{n\to\infty}\left(\frac{1}{4}\log\left(1+\frac{2}{2n+1}\right)^n\right)\right) = e^{\frac{1}{4}}.
\end{aligned}$$

4134. *Proposed by Leonard Giugiuc, Daniel Sitaru and Qing Song.*

Let a, b and c be real numbers such that $a^2 + b^2 + c^2 = 6$ and $abc = -2$. Prove that

$$a + b + c \geq 0 \quad \text{or} \quad a + b + c \leq -4.$$

We received six submissions all of which were correct. We present the solution by Roy Barbara, modified by the editor.

We first establish the following lemma:

Lemma. Let u, v, and w be positive real numbers such that $u^2 + v^2 + w^2 = 6$ and $uvw = 2$. If $u \leq v \leq w$, then

$$\text{(i) } w \leq 2, \quad \text{(ii) } u + v \geq w, \quad \text{(iii) } u + v + w \geq 4.$$

Proof.

(i) Since $u^2 + v^2 = 6 - w^2$ and $u^2v^2 = \frac{4}{w^2}$, then u^2 and v^2 are the roots of the equation

$$x^2 - (6 - w^2)x + \frac{4}{w^2} = 0.$$

Considering the discriminant, we have

$$(6-w^2)^2 - \frac{16}{w^2} \geq 0, \quad \text{so} \quad (w(6-w^2))^2 \geq 16$$

and since, $w(6-w^2) = w(u^2+v^2) > 0$ we then have $w(6-w^2) \geq 4$. Hence,

$$w^3 - 6w + 4 \leq 0 \quad \text{or} \quad (w-2)(w^2+2w-2) \leq 0.$$

If $w > 2$, then $w-2 > 0$ and $w^2+2w-2 > 0$ lead to a contradiction, so $w \leq 2$ follows.

(ii) Since $w = \max\{u, v, w\}$, we have

$$w^2 \geq 2 \iff w \geq \sqrt{2}, \quad \text{so} \quad w - \frac{1}{2} > 0.$$

Then

$$\begin{aligned}
u+v \geq w &\iff (u+v)^2 + w^2 \geq 2w^2 \iff u^2+v^2+w^2+2uv \geq 2w^2 \\
&\iff 6 + 2uv \geq 2w^2 \iff 3 + uv \geq w^2 \iff 3w + 3uvw \geq w^3 \\
&\iff 3w + 2 \geq w^3 \iff w^3 - 3w - 2 \leq 0 \\
&\iff (w+1)(w^2-w-2) \leq 0 \iff w^2 - w - 2 \leq 0 \\
&\iff (w - \frac{1}{2})^2 \leq \frac{9}{4} \iff w - \frac{1}{2} \leq \frac{3}{2} \iff w \leq 2,
\end{aligned}$$

which is true by (i).

(iii) Since $4 - w > 0$, we have

$$\begin{aligned}
u+v+w \geq 4 &\iff u+v \geq 4-w \iff (u+v)^2 + w^2 \geq (4-w)^2 + w^2 \\
&\iff u^2+v^2+w^2+2uv \geq 16 - 8w + 2w^2 \\
&\iff 6 + 2uv \geq 16 - 8w + 2w^2 \iff uv \geq 5 - 4w + w^2 \\
&\iff uvw \geq 5w - 4w^2 + w^3 \iff w^3 - 4w^2 + 5w - 2 \leq 0 \\
&\iff (w-1)^2(w-2) \leq 0,
\end{aligned}$$

which is true by (i). □

To prove the given inequalities, note that since $abc < 0$ there are two possible cases: case (1) $a, b, c < 0$ and case (2) exactly one of a, b, and c is negative.

In case (1), we set $u = -a$, $v = -b$, and $w = -c$. Then $u, v, w > 0$. Without loss of generality, we assume that $u \leq v \leq w$. Since $u^2+v^2+w^2 = 6$ and $uvw = 2$, we have by (iii) of the lemma that $u+v+w \geq 4$ so $a+b+c \leq -4$.

In case (2), we may assume that $a, b > 0$ and $c < 0$. We set $u = a$, $v = b$ and $w = -c$. Then $u, v, w > 0$, $u^2+v^2+w^2 = 6$, and $uvw = 2$. By (ii) of the lemma, it is clear that $u+v \geq w$ holds regardless of the order of the magnitude of u, v, and w. Hence $a+b \geq -c$ from which $a+b+c \geq 0$ follows.

4135. *Proposed by Daniel Sitaru.*

Let ABC be a triangle with $BC = a$, $AC = b$, $AB = c$. Prove that the following relationship holds

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \le \sqrt{3\Big(\frac{a^2}{b+c-a}+\frac{b^2}{a+c-b}+\frac{c^2}{a+b-c}\Big)}.$$

We received nine solutions. We present the solution by Dionne Bailey, Elsie Campbell and Charles R. Diminnie.

Since $f(x) = \sqrt{x}$ is concave on $(0, \infty)$, Jensen's theorem implies that

$$\sqrt{a}+\sqrt{b}+\sqrt{c} = f(a)+f(b)+f(c) \le 3f\left(\frac{a+b+c}{3}\right) = \sqrt{3(a+b+c)}. \qquad (1)$$

By the Cauchy-Schwarz inequality, writing $a = \frac{a}{\sqrt{b+c-a}}\sqrt{b+c-a}$ and similarly for b and c, we get

$$a+b+c \le \Big(\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\Big)^{1/2}(a+b+c)^{1/2},$$

which (dividing both sides by $(a+b+c)^{1/2}$ and multiplying by $\sqrt{3}$) yields

$$\sqrt{3(a+b+c)} \le \sqrt{3\left(\frac{a^2}{b+c-a}+\frac{b^2}{c+a-b}+\frac{c^2}{a+b-c}\Big)\right.}. \qquad (2)$$

Combining (1) and (2), we get the desired inequality; note that equality holds if and only if $a = b = c$, in other words if and only if $\triangle ABC$ is equilateral.

4136. *Proposed by Daniel Sitaru and Mihaly Bencze.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$b\int_0^a e^{-t^2}dt + c\int_0^b e^{-t^2}dt + a\int_0^c e^{-t^2} < \frac{\pi}{2}\sqrt{3(a^2+b^2+c^2)}.$$

We received nine submissions all of which are correct. We present four solutions in all of which we use S to denote the left side of the given inequality.

Solution 1, by Arkady Alt, Sefket Arslanagic, Paul Bracken, and Digby Smith (independently).

Since

$$\int_0^x e^{-t^2}dt \le \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

for $x = a, b$, and c, we have by the Cauchy-Schwarz Inequality that

$$\begin{aligned} S &\leq \sqrt{b^2+c^2+a^2} \cdot \sqrt{\left(\int_0^a e^{-t^2}dt\right)^2 + \left(\int_0^b e^{-t^2}dt\right)^2 + \left(\int_0^c e^{-t^2}dt\right)^2} \\ &\leq \sqrt{a^2+b^2+c^2} \cdot \sqrt{3\left(\frac{\sqrt{\pi}}{2}\right)^2} = \frac{\sqrt{\pi}}{2} \cdot \sqrt{3(a^2+b^2+c^2)} \\ &< \frac{\pi}{2} \cdot \sqrt{3(a^2+b^2+c^2)}. \end{aligned}$$

Solution 2, by Leonard Giugiuc.

As in Solution 1 above, we have

$$\int_0^x e^{-t^2}dt \leq \frac{\sqrt{\pi}}{2}$$

for $x = a, b$, and c, so

$$S \leq (b+c+a) \cdot \frac{\sqrt{\pi}}{2} < \frac{\pi}{2} \cdot \sqrt{3(a^2+b^2+c^2)}$$

by the AM-QM inequality.

Solution 3, by Leonard Giugiuc.

Since $e^{t^2} = 1 + t^2 + \frac{t^4}{2!} + \cdots$ we have for $x = a, b$, and c,

$$\int_0^x e^{-t^2}dt \leq \int_0^x \frac{dt}{1+t^2} = \tan^{-1} x < \frac{\pi}{2}.$$

Hence,

$$S < (a+b+c)\left(\frac{\pi}{2}\right) \leq \frac{\pi}{2} \cdot \sqrt{3(a^2+b^2+c^2)},$$

by the AM-QM inequality.

Solution 4, by Kee-Wei Lau.

Using $\displaystyle\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$, we have

$$\begin{aligned} S &\leq (b+c+a)\int_0^\infty e^{-t^2}dt \\ &= \sqrt{3(a^2+b^2+c^2) - (a-b)^2 - (b-c)^2 - (c-a)^2} \cdot \frac{\sqrt{\pi}}{2} \\ &< \frac{\pi}{2} \cdot \sqrt{3(a^2+b^2+c^2)}. \end{aligned}$$

4137. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 4$ be an integer and let a, b, c be three real n-dimensional vectors which are pairwise orthogonal and of unit length. Prove that $a_i^2 + b_i^2 + c_i^2 \leq 1$ for all i.

We received one correct solution and so present the solution of the proposer.

We define the matrix $A \in \mathcal{M}_{3,n}(\mathbb{R})$ by $A = (a_{ij})_{1\leq i\leq 3}^{1\leq j\leq n}$, where, for $1 \leq j \leq n$, $a_{1j} = a_j$, $a_{2j} = b_j$, and $a_{3j} = c_j$. Likewise, we define the matrix $B \in \mathcal{M}_{4,n}(\mathbb{R})$ by $B = (b_{ij})_{1\leq i\leq 4}^{1\leq j\leq n}$, where

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n \\ 1, & \text{if } i = 4 \text{ and } j = 1 \\ 0, & \text{if } i = 4 \text{ and } 2 \leq j \leq n \end{cases}$$

Then
$$\det(B \cdot B^T) = 1 - (a_1^2 + b_1^2 + c_1^2),$$
but by the Cauchy-Binet formula, $\det(B \cdot B^T) \geq 0$. Hence, $a_1^2 + b_1^2 + c_1^2 \leq 1$. Similarly, $a_k^2 + b_k^2 + c_k^2 \leq 1$ for $2 \leq k \leq n$. Thus, $a_1^2 + b_1^2 + c_1^2 \leq 1$.

4138. *Proposed by Lorian Saceanu.*

a) Let ABC be an acute triangle with semi-perimeter s, inradius r and circumradius R. Prove that
$$a \sin\frac{A}{2} + b\sin\frac{B}{2} + c\sin\frac{C}{2} \geq s + \frac{s(R-2r)^2}{4R(R+2r)}.$$

b) Let ABC be a scalene triangle with semi-perimeter s, inradius r and circumradius R. Prove that
$$a \sin\frac{A}{2} + b\sin\frac{B}{2} + c\sin\frac{C}{2} \geq s + \frac{s^2 - 12Rr - 3r^2}{2\sqrt{6R(4R+r)}}.$$

We received one correct solution and so present the solution of the proposer.

Let I be the incenter of triangle ABC, and let L, M, and N be the points at which AI, BI, CI intersect the circumscribed circle of triangle ABC. Let q be the radius of the inscribed circle of triangle LMN. Then q satisfies the following identities, where s and R are the semiperimeter and circumradius, respectively, of triangle ABC.

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = \frac{s}{2q} \tag{1}$$

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + \frac{q}{R}. \tag{2}$$

We also have
$$(s-a)\sin\frac{A}{2} = r \cdot \cos\frac{A}{2}. \tag{3}$$

From (3), we have

$$\sum_{cyc}(s-a)\sin\frac{A}{2} = r\left(\sum_{cyc}\cos\frac{A}{2}\right) = \frac{rs}{2q} = \frac{K}{2q}, \tag{4}$$

where K is the area of triangle ABC. On the other hand,

$$\sum_{cyc}(s-a)\sin\frac{A}{2} = s\left(\sum_{cyc}\sin\frac{A}{2}\right) - \sum_{cyc}a\sin\frac{A}{2} = s\left(1+\frac{q}{r}\right) - \sum_{cyc}a\sin\frac{A}{2}. \tag{5}$$

From (5), we get

$$a\sin\frac{A}{2} + b\sin\frac{B}{2} + c\sin\frac{C}{2} - s = \frac{sq}{R} - \frac{K}{2q}. \tag{6}$$

Now we consider the acute-angled triangle ΔABC, in which we have

$$4q \geq R + 2r. \tag{7}$$

Replacing q from (7) in (6), we get the first claimed inequality.

For a scalene triangle ΔABC, equation (1) is equivalent to $\sum_{cyc}\sqrt{\frac{s(s-a)}{bc}} = \frac{s}{2q}$. Also, $\sum_{cyc}\sqrt{a(s-a)} = \frac{s\sqrt{Rr}}{q}$. Using Ravi's substitution ($a = y+z$, $b = z+x$, $c = x+y$) , we have

$$q = \frac{s\sqrt{Rr}}{\sum_{cyc}\sqrt{a(s-a)}} = \frac{s\sqrt{Rr}}{\sum_{cyc}\sqrt{x(y+z)}}$$

and

$$yz + zx + xy = r(4R+r).$$

By Jensen's inequality,

$$\sum_{cyc}\sqrt{x(y+z)} \leq \sqrt{6(yz+zx+xy)}, \tag{8}$$

so that

$$q \geq s\cdot\sqrt{\frac{R}{6(4R+r)}}. \tag{9}$$

Replacing q from (9) in (6), we obtain

$$\begin{aligned}\sum_{cyc}a\sin\frac{A}{2} - s &\geq \frac{s^2}{\sqrt{6R(4R+r)}} - \frac{r}{2\sqrt{\frac{R}{6(4R+r)}}}\\ &= \frac{s^2}{\sqrt{6R(4R+r)}} - \frac{r\sqrt{6(4R+r)}}{2\sqrt{R}}\\ &= \frac{2s^2 - 6r(4R+r)}{2\cdot\sqrt{6R(4R+r)}},\end{aligned}$$

from which the second claim follows.

4139. *Proposed by Ángel Plaza.*

Let $f : [-1, 1] \to \mathbb{R}$ be a n times differentiable function with $f^{(n)}$ continuous such that $f(0) = 0$, and $f^{(i)}(0) = 0$ for all even i less than or equal to n. Show that

$$\int_{-1}^{1} (f^{(n)}(x))^2 \, dx \geq \frac{(2n+1)(n!)^2}{2} \left(\int_{-1}^{1} f(x) \, dx \right)^2 .$$

We received 3 correct solutions and present the solution by Paul Bracken.

Integrating over the interval $(-1, 0)$ four times by parts and using the condition on the even derivatives, we obtain

$$\int_{-1}^{0} f(x)\,dx = \frac{1}{2} f(0) + \frac{1}{4!} f^{(3)}(0) - \frac{1}{4!} \int_{-1}^{0} (x+1)^4 f^{(4)}(x)\,dx.$$

This can be iterated until the n-th derivative is reached. Now n can be either even or odd, so that $n = 2m$ or $n = 2m+1$; hence,

$$\int_{-1}^{0} f(x)\,dx = \sum_{k=1}^{m} \frac{f^{(2k-1)}(0)}{(2k)!} - \frac{1}{n!} \int_{-1}^{0} (x+1)^n f^{(n)}(x)\,dx.$$

Similarly integrating over $(0, 1)$ by parts repeatedly, we obtain

$$\int_{0}^{1} f(x)\,dx = -\sum_{k=1}^{m} \frac{f^{(2k-1)}(0)}{(2k)!} - \frac{1}{n!} \int_{0}^{1} (x-1)^n f^{(n)}(x)\,dx.$$

Adding these last two equations, the finite sums cancel. Squaring both sides after adding, we are left with

$$\left(\int_{-1}^{1} f(x)\,dx \right)^2 = \frac{1}{(n!)^2} \left[\int_{-1}^{0} (x+1)^n f^{(n)}(x)\,dx + \int_{0}^{1} (x-1)^n f^{(n)}(x)\,dx \right]^2 .$$

The inequality $(a-b)^2 \geq 0$ is equivalent to $(a+b)^2 \leq 2(a^2+b^2)$ for all real a and b. In this inequality, set

$$a = \int_{-1}^{0} (x+1)^n f^{(n)}(x)\,dx, \qquad b = \int_{0}^{1} (x-1)^n f^{(n)}(x)\,dx.$$

This yields

$$\left(\int_{-1}^{1} f(x)\,dx \right)^2 \leq \frac{2}{(n!)^2} \left(\left[\int_{-1}^{0} (x+1)^n f^{(n)}(x)\,dx \right]^2 + \left[\int_{0}^{1} (x-1)^n f^{(n)}(x)\,dx \right]^2 \right). \tag{1}$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned} \left(\int_{-1}^{0} (x+1)^n f^{(n)}(x)\,dx \right)^2 &\leq \int_{-1}^{0} (x+1)^{2n}\,dx \cdot \int_{-1}^{0} \left[f^{(n)}(x) \right]^2 \,dx \\ &= \frac{1}{2n+1} \int_{-1}^{0} \left[f^{(n)}(x) \right]^2 \,dx \end{aligned}$$

and

$$\left(\int_0^1 (x-1)^n f^{(n)}(x)\,dx\right)^2 \leq \int_0^1 (x-1)^{2n}\,dx \cdot \int_0^1 \left[f^{(n)}(x)\right]^2\,dx$$
$$= \frac{1}{2n+1}\int_{-1}^0 \left[f^{(n)}(x)\right]^2\,dx$$

Substituting these two upper bounds for the squares of a and b into (1), we arrive at the result

$$\left(\int_{-1}^1 f(x)\,dx\right)^2 \leq \frac{2}{(2n+1)(n!)^2}\left(\int_{-1}^0 \left[f^{(n)}(x)\right]^2\,dx + \int_0^1 \left[f^{(n)}(x)\right]^2\,dx\right)$$
$$= \frac{2}{(2n+1)(n!)^2}\int_{-1}^1 \left[f^{(n)}(x)\right]^2\,dx.$$

This is exactly the claimed inequality.

4140. *Proposed by Mihaela Berindeanu.*

Find all positive integers $x \leq y$ so that

$$p = \frac{(x+y)(xy-4)}{xy+13}$$

is prime.

We received 10 submissions of which 9 were correct and complete and we present 2 solutions.

Solution 1, by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie.

If x and y are positive integers and $x \leq y$, then

$$p = \frac{(x+y)(xy-4)}{xy+13}$$

for some prime p implies that

$$(x+y)(xy-4) = p(xy+13). \tag{1}$$

It follows that p divides $(x+y)(xy-4)$ and hence either p divides $x+y$ or p divides $xy-4$, (since p is prime).

Case 1. If $x+y = kp$ for some positive integer k, then (1) becomes

$$k(xy-4) = xy+13$$

which reduces to

$$xy = \frac{4k+13}{k-1} = 4 + \frac{17}{k-1}.$$

Since xy is a positive integer, we must have $k-1=1$ or 17, i.e., $k=2$ or 18. If $k=18$, then $xy=5$ and the condition $1\le x\le y$ implies that $x=1$ and $y=5$. This in turn yields $18p=x+y=1+5=6$, which is impossible. Therefore, $k=2$ and $xy=21$. This leads to the two possibilities $x=1, y=21$ or $x=3, y=7$. The first gives $2p=x+y=22$, or $p=11$ while the second gives $2p=x+y=10$ or $p=5$. These are the only possibilities for Case 1 and it is easily checked that both satisfy condition (1).

Case 2. If $xy-4=kp$ for some positive integer k, then (1) becomes

$$k\,(x+y)=xy+13=kp+17$$

and hence,

$$x+y=p+\frac{17}{k}.$$

Since $x+y$ and p are positive integers, we must have $k=1$ or $k=17$.

If $k=1$, then $xy-4=p$ and $x+y=p+17$, which imply that

$$xy-x-y=-13$$

or

$$(x-1)\,(y-1)=-12.$$

This is impossible since x and y are positive integers. Therefore, $k=17$ and we obtain $xy-4=17p$ and $x+y=p+1$. As a result,

$$xy-4=17\,(x+y-1)$$

which simplifies to

$$(x-17)\,(y-17)=276.$$

By checking all the possible factorizations of 276 as well as using the condition $x-17\le y-17$, the only one which yields a unique prime value for both $x+y-1$ and $\frac{xy-4}{17}$ is $x-17=2$ and $y-17=138$. It follows that $x=19$, $y=155$, and $p=173$. Once again, it is easily checked that these values satisfy condition (1).

In summary, the solutions for x, y, and p are

x	y	p
1	21	11
3	7	5
19	155	173

Solution 2, by Joel Schlosberg.

Let D be the greatest common divisor of $xy-4$ and $xy+13$. Then

$$D\mid xy+13-(xy-4)=17,$$

so $D = 1$ or 17. Since
$$\frac{xy+13}{D} \mid (x+y) \cdot \frac{xy-4}{D}$$
but is coprime to $\dfrac{xy-4}{D}$, then $\dfrac{xy+13}{D} \mid x+y$. Then
$$p = \frac{x+y}{(xy+13)/D} \cdot \frac{xy-4}{D}.$$

Since $\dfrac{x+y}{(xy+13)/D}$ is a positive integer dividing the prime p, it is 1 or p. Thus $\dfrac{xy-4}{D} \in \{1, p\}$.

Case 1 $D = 1$, $\dfrac{xy-4}{D} = 1$.

Then $xy = 5$, so $x = 1$, $y = 5$. Then $p = 1/3$, a contradiction.

Case 2 $D = 1$, $\dfrac{xy-4}{D} = p$.

Then
$$-12 = xy + 13 - (x+y) - 12 = (x-1)(y-1) \geq 0$$
a contradiction.

Case 3 $D = 17$, $\dfrac{xy-4}{D} = 1$.

Then $xy = 21$, so $(x, y) = (1, 21)$ or $(3, 7)$. In the former case, $p = 11$; in the latter case, $p = 5$.

Case 4 $D = 17$, $\dfrac{xy-4}{D} = p$.

Then
$$276 = (xy + 13) - 17(x+y) + 276 = (x-17)(y-17),$$
so
$$(x, y) \in \{(18, 293), (19, 155), (20, 109), (21, 86), (23, 63), (29, 40)\}.$$

In each case $p \in \{310, 173, 128, 106, 85, 68\}$, respectively. Among these possible values for p, only 173 is prime.

Therefore, the positive integers (x, y) satisfying the condition are $(1, 21)$, $(3, 7)$, and $(19, 155)$.

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