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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Dear Crux reader,

I am very happy to greet you as the new Editor-in-Chief.

Before I say anything, first and foremost, I would like to thank everyone responsible for Crux for welcoming me and helping me through the transition process.

Shawn Godin has been the driving force behind Crux for the past several years; without Shawn's guidance, we simply wouldn't have Crux today. And he has been essential in my transition into this role: his close relationship with you, the reader, provides an invaluable insight into the publication and I cannot thank him enough for his willingness to share that insight and to answer all of my many questions.

I am truly privileged to be working with the Crux editorial board. Its commitment to the journal is truly inspiring, while its collective wisdom and many fascinating personalities make for a dynamic and energizing environment to work in. I would like to thank each and every one of them for warmly welcoming me into this role and for incessantly supporting Crux.

Last but not least, I would like to thank the Canadian Mathematical Society's office and executive for their faith in me and their support in this venture.

As I have been involved in Canadian Mathematical Society throughout my academic career, I have been closely watching the development of many of their initiatives, including Crux. I will be building on all the good work Shawn has done and initiated by drawing inspiration directly from Crux solvers and proposers. And it is easy to get inspired – in this issue alone, we have solutions and proposals from people from 16 different countries and 4 different continents! What a diverse group of people from all over the world; I hope to live up to your expectation of what Crux is and should be.

Looking ahead, I expect Crux to continue evolving in the direction set by Shawn as I am also looking to explore other ways to improve and enrich Crux. I am always looking forward to hearing from you, so please send any comments and suggestions (or even just a hello) to me at `crux-editors@cms.math.ca`, like us on Facebook, submit solutions and problem proposals – get involved in any way you can as the journal can only grow with your participation.

For our long-term readers, please note one change in electronic submission: starting from this issue, any new submissions of numbered problem proposals and solutions should go to `crux-psol@cms.math.ca`.

Kseniya Garaschuk

THE CONTEST CORNER

No. 17

Shawn Godin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-contest@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_CCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 December 2014**, *although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC81. Quadrilateral $ABCD$ has the following properties:

1. the mid-point O of side AB is the centre of a semicircle;
2. sides AD, DC and CB are tangent to this semicircle.

Prove that $AB^2 = 4AD \times BC$.

CC82. For each positive integer N, an *Eden sequence* from $\{1, 2, 3, \ldots, N\}$ is defined to be a sequence that satisfies the following conditions:

1. each of its terms is an element of the set of consecutive integers $\{1, 2, 3, \ldots, N\}$,
2. the sequence is increasing, and
3. the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from $\{1, 2, 3\}$ are

$$1 \qquad 3 \qquad 1,2 \qquad 1,2,3$$

For each positive integer N, define $e(N)$ to be the number of Eden sequences from $\{1, 2, 3, \ldots, N\}$. If $e(17) = 4180$ and $e(20) = 17710$, determine $e(18)$ and $e(19)$.

CC83. A map shows all Beryls Llamaburgers restaurant locations in North America. On this map, a line segment is drawn from each restaurant to the restaurant that is closest to it. Every restaurant has a unique closest neighbour. (Note that if A and B are two of the restaurants, then A may be the closest to B without B being closest to A.) Prove that no restaurant can be connected to more than five other restaurants.

CC84. Let m and n be odd positive integers. Each square of an m by n board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of m and n.

CC85. While Lino was simplifying the fraction $\dfrac{A^3+B^3}{A^3+C^3}$ he cancelled the threes $\dfrac{A^{\cancel{3}}+B^{\cancel{3}}}{A^{\cancel{3}}+C^{\cancel{3}}}$ to obtain the fraction $\dfrac{A+B}{A+C}$. If $B \neq C$, determine a necessary and sufficient condition on A, B and C for Lino's method to actually yield the correct answer, ie. for

$$\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C}$$

. . . .

CC81. On considère un quadrilatère $ABCD$ dont :

1. le milieu O du côté AB est le centre d'un demi-cercle ;
2. les côtés AD, DC et CB sont tangents à ce demi-cercle.

Démontrer que $AB^2 = 4AD \times BC$.

CC82. Étant donné un entier strictement positif N, une *suite Eden* sur l'ensemble $\{1, 2, 3, \dots, N\}$ des entiers consécutifs de 1 à N est une suite qui satisfait aux conditions suivantes :

1. chacun de ses termes est un élément de l'ensemble $\{1, 2, 3, \dots, N\}$,
2. la suite est croissante et
3. les termes dans les positions impaires sont impairs et les termes dans les positions paires sont pairs.

Par exemple, les quatre suites Eden sur l'ensemble $\{1, 2, 3\}$ sont :

$$1 \qquad 3 \qquad 1,2 \qquad 1,2,3$$

Étant donné un entier strictement positif N, soit $e(N)$ le nombre de suites Eden sur l'ensemble $\{1, 2, 3, \ldots, N\}$. Sachant que $e(17) = 4180$ et $e(20) = 17\,710$, déterminer $e(18)$ et $e(19)$.

CC83. Une carte indique où sont situés tous les restaurants *La poutine dorée* en Amérique du nord. Sur cette carte, on a tracé un segment entre chaque restaurant et le restaurant qui est plus près de lui. Chaque restaurant a un seul voisin le plus près. (On remarquera qu'il est possible qu'un restaurant A soit le plus près de B sans que B soit le restaurant le plus près de A.) Démontrer qu'il est impossible pour un restaurant d'être relié par des segments à plus de cinq autres restaurants.

CC84. Soit m et n deux entiers impairs positifs. Chaque case d'un quadrillage m sur n est colorié en rouge ou en bleu. On dit qu'une rangée du quadrillage est à dominance rouge si la rangée contient plus de cases rouges que de cases bleues. On dit qu'une colonne est à dominance bleue si la colonne contient plus de cases bleues que de cases rouges. Déterminer la valeur maximale possible de la somme de rangées à dominance rouge et de colonnes à dominance bleue. Exprimer sa réponse en fonction de m et de n.

CC85. Pour simplifier l'expression $\dfrac{A^3+B^3}{A^3+C^3}$, Lino a annulé les exposants 3, en faisant $\dfrac{A^{\cancel{3}}+B^{\cancel{3}}}{A^{\cancel{3}}+C^{\cancel{3}}}$, pour obtenir l'expression $\dfrac{A+B}{A+C}$. Si $B \neq C$, déterminer une condition nécessaire et suffisante sur les variables A, B et C pour que la méthode de Lino soit valable, c'est-à-dire pour que

$$\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C}.$$

CONTEST CORNER SOLUTIONS

CC31. Triangle ABC is right angled with its right angle at A. The points P and Q are on the hypotenuse BC such that $BP = PQ = QC$, $AP = 3$ and $AQ = 4$. Determine the length of each side of ABC.
(Originally Question B3 from 1999 Canadian Open Mathematics Challenge.)

Solved by Š. Arslanagić ; M. Bataille ; M. Coiculescu ; C. Curtis ; J. G. Heuver ; R. Hess ; M. Stoënescu ; D. Văcaru ; and T. Zvonaru. We give the solution of Chip Curtis modified by the editor.

Let $a = BC,\ b = CA,\ c = AB,\ d = BP = PQ = QC$, so $a = 3d$. Applying the Law of Cosines on angle C in triangle ACQ and angle B on triangle ABP we have

$$16 = b^2 + d^2 - 2bd \cdot \frac{b}{a}, \tag{1}$$

and

$$9 = c^2 + d^2 - 2cd \cdot \frac{c}{a}. \tag{2}$$

Substituting $a = 3d$ in both equations above we get

$$48 = b^2 + 3d^2 \tag{3}$$

and

$$27 = c^2 + 3d^2. \tag{4}$$

Furthermore, Pythagorean Theorem on ABC gives us

$$b^2 = c^2 + 9d^2. \tag{5}$$

Equations (3), (4), (5) are linear in b^2, c^2, d^2. Solving we get $b = \sqrt{33}, c = 2\sqrt{3}, d = \sqrt{5}$ and hence $a = 3\sqrt{5}$.

CC32. Four boys and four girls each bring one gift to a Christmas gift exchange. On a sheet of paper, each boy randomly writes down the name of one girl, and each girl randomly writes down the name of one boy. At the same time, each person passes their gift to the person whose name is written on their sheet. Determine the probability that both of these events occur :

(i) Each person receives exactly one gift ;

(ii) No two people exchanged presents with each other (i.e., if A gave his gift to B, then B did not give her gift to A).

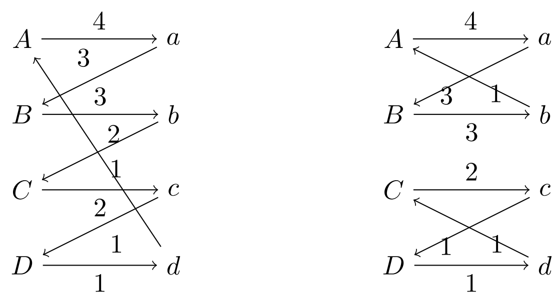
(Originally Question 4 from 2013 Sun Life Financial Repêchage Competition.)

Solved by C. Curtis ; G. Geupel ; and T. Zvonaru. We present Gesine Geupel's solution below.

Let A, B, C, D be the girls and a, b, c, d be the boys. The different possibilities of exchanging presents are presented in the figures below. The number of choices is written beside each of the arrows.

Let us consider the girl, A. A gives a present to a boy, a. There are 4 possibilities for the choice of boy a. a gives a present to a girl, B, that cannot be A. There are 3 possibile choices for B. B gives a present to a boy, b, who is different from a (3 possibilities). Now there are two cases.

In the first case, b gives a gift to a girl, C, distinct from A and B. This is shown in the left figure. Then C gives a present to c. Now c must give to D and D to d. d gives to A. In the second case, b gives a present to A. Now in the group, C, c, D, d they do the same 4-cycle pattern ; but C has 2 possibilities for choosing c and then the cycle is fixed.



These are the only possibilities where each person gets a present, but no two people exchange with each other. For the first picture, there are $4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 144$ possibilities ; for the second, there are $4 \cdot 3 \cdot 3 \cdot 2 = 72$ possibilities ; so there are 216 possibilities. The total possible cases are 4^8 because each of the eight persons have 4 possible people of opposite gender to choose. So the probability is $\frac{216}{4^8} = \frac{27}{8192}$.

CC33. The abundancy index $I(n)$ of a positive integer n is $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all positive integer divisors of n, including 1 and n itself. For example, $I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}$. Determine, with justification, the smallest odd positive integer n such that $I(n) > 2$.
(Originally Question 4 from 2006 Hypatia Contest.)

Solution adapted from the solution of Chip Curtis.

Let n be a positive integer whose prime factorization is $p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ where p_i's are primes and e_i's are positive integers. Then the factors of n are all of the form $p_1^{j_1} p_2^{j_2} \cdots p_m^{j_m}$ where $0 \le j_i \le e_i$ for every i. Thus, an explicit expression for $I(n)$ in terms of its prime divisors is

$$\frac{1}{n} \sum_{j_1=0}^{e_1} \cdots \sum_{j_m=0}^{e_m} p_1^{j_1} p_2^{j_2} \cdots p_m^{j_m} = \frac{1}{n} \prod_{i=1}^{m} \left(\sum_{j_i=0}^{e_i} p_i^{j_i} \right) = \prod_{i=1}^{m} \frac{\left(\sum_{j_i=0}^{e_i} p_i^{j_i}\right)}{p_i^{e_i}} = \prod_{i=1}^{m} I(p_i^{e_i}).$$

We make some statements that we appeal to later :

- If p, q are primes with $p < q$, then $I(p^k) > I(q^k)$ for any positive integer k. To see this, observe that for any prime r,

$$I(r^k) = \frac{\sum_{j=0}^{k} r^j}{r^k} = \sum_{j=0}^{k} \frac{1}{r^j}$$

  and if $p < q$ then $\frac{1}{p^j} > \frac{1}{q^j}$ for any positive j.

- By the multiplicativity of I and the previous comment, if n is a positive integer, we can find a positive integer m with $I(m) > I(n)$ by replacing a prime in the prime decomposition of n with one that is smaller.

- For any prime p,

$$I(p^k) = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} = \frac{1 - (1/p)^{k+1}}{1 - \frac{1}{p}} < \frac{1}{1 - \frac{1}{p}} = \frac{p}{p-1}.$$

- From the above, if n is odd and has two distinct prime factors, then maximizing $I(n)$, we require $n = 3^a 5^b$ for some positive integers a, b. From this we have

$$I(n) = I(3^a 5^b) = I(3^a)I(5^b) < \frac{3}{2}\frac{5}{4} = \frac{15}{8} < 2.$$

- Thus, if n is an odd integer with $I(n) > 2$, n has at least three distinct prime factors.

- A quick check shows

$$I(3 \cdot 5 \cdot 7) = \frac{64}{35} < 2,\ I(3^2 \cdot 5 \cdot 7) = \frac{208}{105} < 2,\ I(3^3 \cdot 5 \cdot 7) = \frac{128}{63} > 2,$$

  so $n = 945$ satisfies $I(n) > 2$. To show 945 is the smallest, observe :

  - Any positive odd integer with at least 4 prime factors is at lesat $3 \cdot 5 \cdot 7 \cdot 11 = 1155 > 945$.
  - From what we mentioned earlier then, the smallest n must have 3 distinct prime divisors and hence, to minimize n, must have $3, 5, 7$ as its distinct prime divisors.
  - We only need to consider integers $3^a 5^b 7^c$ with $a \geq b \geq c$ since otherwise we could reassign the exponents and obtain a smaller integer.
  - We have checked all integers less than 945 of the form $3^a 5^b 7^c$ with $a \geq b \geq c$, so we are done.

CC34. At the Mathville Dim Sum restaurant, all the dishes come in three sizes : small, medium, and large. Small dishes cost x, medium dishes cost y, and large dishes cost z, where x, y, z are positive integers, with $x < y < z$. At this restaurant, there is no tax on any dish, and the prices haven't changed for a long time. Margaret, Art, and Edgar had dinner there last night, and together, they ordered 9 small dishes, 6 medium dishes, and 8 large dishes. When the bill came, the following conversation ensued :

Margaret :"This bill is exactly twice as much as when I last came here."

Art : "This bill is exactly three times as much as when I last came here."

Edgar : "Oh, that was a delicious meal, and very reasonably priced too. Even if we give the waiter a 10% tip, the total is still less than \$100."

Determine the values of x, y, and z.
(Originally Question 7 from 2002 APICS Math Competition.)

Solved by C. Curtis ; G. Geupel ; and R. Hess. We present Chip Curtis's solution.

We claim that the unique solution is $(x, y, z) = (2, 3, 6)$.

Letting m be the amount of Margaret's previous bill, and a the amount of Art's previous bill, we have

$$9x + 6y + 8z = 2m \tag{1}$$

$$9x + 6y + 8z = 3a \tag{2}$$

and

$$9x + 6y + 8z < \frac{100}{1.1} < 91 \tag{3}$$

By (1), x is even, implying that $x \geq 2$, $y \geq 3$, and $z \geq 4$. By (3), therefore,

$$91 > 9x + 6y + 8z \geq 9(2) + 6(3) + 8z = 36 + 8z \tag{4}$$

so that,

$$z \leq \frac{55}{8} = 6.875 \tag{5}$$

By (2), however, z is a multiple of 3. Thus, z must equal 6. This gives the following possibilities :

$$(2, 3, 6),\ (2, 4, 6),\ (2, 5, 6),\ (4, 5, 6),$$

We now exclude the last three possibilities as follows.

- If $(x, y, z) = (2, 4, 6)$, then by (1), $m = 45$, which cannot be written as a linear integral combination of $2, 4$, and 6.
- If $(x, y, z) = (2, 5, 6)$, then $9x + 6y + 8z = 96 > 91$, violating (3).

- If $(x, y, z) = (4, 5, 6)$, then $9x + 6y + 8z = 114 > 91$, violating (3).

Thus, (x, y, z) must be equal to $(2, 3, 6)$ as claimed. This implies that $m = 42$, which can be written in 35 different ways as a nonnegative linear integral combination of 2, 3 and 6 including $42 = 0 \cdot 2 + 0 \cdot +7 \cdot 6$, and $a = 28$, which can be written in 15 different ways as a nonnegative linear integral combination of 2, 3, and 6 including $28 = 14 \cdot 2 + 0 \cdot 3 + 0 \cdot 6$.

CC35. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}$$

(Originally Question 4 from 2001 APICS Math Competition.)

Solved by M. Bataille ; C. Curtis ; J. G. Heuver ; D. E. Manes ; P. Perfetti ; H. Ricardo ; and D. Văcaru. We give the solution of Michel Bataille expanded by the editor.

We re-write each term as

$$\sqrt[n]{a_n} \quad \text{, where } a_n = \frac{(2n)!}{n!n^n}.$$

We want to determine $\lim_{n\to\infty} \sqrt[n]{a_n}$. Since each term in the sequence $\{a_n\}$ is nonnegative, by the Root Test, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ then $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$ so we determine $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ instead. Observe

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!(n+1)^{n+1}} \cdot \frac{n!n^n}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

hence

$$\frac{a_{n+1}}{a_n} = 2 \cdot \frac{2n+1}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

Since $\lim_{n\to\infty} \frac{2n+1}{n+1} = 2$ and $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, it follows that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e}$$

and hence this is our desired limit.

THE OLYMPIAD CORNER

No. 315

Nicolae Strungaru

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to `crux-olympiad@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

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The editor thanks Rolland Gaudet, of l'Université Saint-Boniface in Winnipeg, for translations of the problems.

OC141. Find all non-zero polynomials $P(x), Q(x)$ of minimal degree with real coefficients such that for all $x \in \mathbb{R}$ we have:

$$P(x^2) + Q(x) = P(x) + x^5 Q(x).$$

OC142. Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y)$$

for all $x, y \in \mathbb{R}$.

OC143. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\dfrac{n^p + 1}{p^n + 1}$ is an integer.

OC144. Let $ABCD$ be a convex circumscribed quadrilateral such that

$$\angle ABC + \angle ADC < 180^\circ \quad \text{and } \angle ABD + \angle ACB = \angle ACD + \angle ADB$$

Prove that one of the diagonals of quadrilateral $ABCD$ passes through the midpoint of the other diagonal.

OC145. Let $n \geq 2$ be a positive integer. Consider the matrix $A_{n\times n}$ with all entries 1. Define the n^2 operations on an $n \times n$ matrix by $P_{ij}(A) = [a'_{ij}]$, where

$$a'_{kl} = \begin{cases} -a_{kl}, & \text{if } (k,l) \in \{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\}, \\ a_{kl}, & \text{otherwise.} \end{cases}$$

Find all n such for which it is possible to find a finite sequence of operations, $P_{i_1j_1}, P_{i_2j_2}, \ldots, P_{i_kj_k}$, such that all entries of

$$A' = P_{i_kj_k}(\cdots(P_{i_2j_2}(P_{i_1j_1}(A)))\cdots)$$

are -1.

. . . .

OC141. Déterminer tous les polynômes $P(x)$ et $Q(x)$, non nuls, de degré minimal, à coefficients réels et tels que pour tout $x \in \mathbb{R}$, on ait :

$$P(x^2) + Q(x) = P(x) + x^5Q(x).$$

OC142. Déterminer toutes les fonctions $f : \mathbb{R} \mapsto \mathbb{R}$ telles que

$$f(f(x+y)f(x-y)) = x^2 - yf(y)$$

pour tout $x, y \in \mathbb{R}$.

OC143. Déterminer toutes les couples (p, n), où p est un nombre premier et n est un entier positif, tels que $\dfrac{n^p+1}{p^n+1}$ est un entier.

OC144. Soit $ABCD$ un quadrilatère convexe circonscrit, tel que

$$\angle ABC + \angle ADC < 180^\circ \quad \text{et } \angle ABD + \angle ACB = \angle ACD + \angle ADB.$$

Démontrer qu'une des diagonales du quadrilatère ABCD passe par le mi-point de l'autre diagonale.

OC145. Soit $n \geq 2$, un entier positif. Considérer la matrice $A_{n\times n}$ ayant toutes les entrées égales à 1. Définissons les n^2 opérations sur une matrice $n \times n$ par $P_{ij}(A) = [a'_{ij}]$,

$$a'_{kl} = \begin{cases} -a_{kl}, & \text{si } (k,l) \in \{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\}, \\ a_{kl}, & \text{autrement.} \end{cases}$$

Déterminer n pour lequel il est possible de construire une suite finie d'opérations $P_{i_1j_1}, P_{i_2j_2}, \ldots, P_{i_kj_k}$, telles que les entrées de

$$A' = P_{i_kj_k}(\cdots(P_{i_2j_2}(P_{i_1j_1}(A)))\cdots)$$

sont toutes -1.

OLYMPIAD SOLUTIONS

OC81. Find all triplets (x, y, z) of integers that satisfy

$$x^4 + x^2 = 7^z y^2 .$$

(*Originally question 1 from the 2011 Austria Mathematical Olympiad, Part 1.*)

Solved by C. Curtis; O. Geupel; and K. Zelator. We give the solution of Chip Curtis.

The set of solutions is $\{(0, 0, z), z \in \mathbb{Z}\}$.

We show by contradiction that there are no other solutions. Suppose that $(x, y, z) \in \mathbb{Z}^3$ is a solution with $(x, y) \neq (0, 0)$. If $x = 0$ then $y = 0$, so assume that $x \neq 0$. We observe that $x^4 + x^2 = x^2(x^2 + 1)$ and that $x^2 + 1 \not\equiv 0 \pmod 7$. Thus, the exponent of the prime 7 in the prime factorization of $7^z y^2 = x^2(x^2 + 1)$ is even, and so $7^z y^2$ is the square of a positive integer u. It follows that

$$x^2 = u^2 - x^4 = (u - x^2)(u + x^2).$$

Therefore, $u + x^2$ divides x^2, which implies $x = 0$ and $u = 0$, a contradiction.

OC82. The area and the perimeter of the triangle with sides $6, 8, 10$ are equal. Find all triangles with integral sides whose area and perimeter are equal.
(*Originally question 2 from the 2011 Albania Balkan Olympiad team selection test.*)

Solved by G. Apostolopoulos; Š. Arslanagić; C. Curtis; O. Geupel; D. E. Manes; and T. Zvonaru. We give the solution of Šefket Arslanagić.

Let a, b, c be the lengths of the sides and let $p = \frac{a+b+c}{2}$ be the semiperimeter. Using Heron's formula we get

$$\sqrt{p(p-a)(p-b)(p-c)} = 2p ,$$

or

$$(p-a)(p-b)(p-c) = 4p .$$

Let $P = a + b + c$, then P is an integer, and

$$(P - 2a)(P - 2b)(P - 2c) = 16P .$$

Taking this equality modulo 2 we get $P^3 \equiv 0 \pmod 2$ therefore P must be an even integer. This implies that p is an integer.

Let $x = p - a, y = p - b, z = p - c$. Then x, y, z are integers and

$$xyz = 4(x + y + z) . \tag{1}$$

As the equation is symmetric in x, y, z, and the order of the sides is irrelevant, without loss of generality we can assume $x \geq y \geq z$. Then

$$x = \frac{4y + 4z}{yz - 4}\,, \tag{2}$$

which implies

$$\frac{4y + 4z}{yz - 4} \geq y\,, \quad \text{and} \quad yz > 4.$$

Therefore

$$4y + 4z > y^2 z - 4y\,,$$

or

$$y^2 z - 8y - 4z < 0\,.$$

The solutions to the quadratic equation $y^2 z - 8y - 4z = 0$ are

$$y_{1,2} = \frac{8 \pm \sqrt{64 + 16z^2}}{2z}\,.$$

Therefore

$$\frac{4 - \sqrt{16 + 4z^2}}{z} < y < \frac{4 + \sqrt{16 + 4z^2}}{z}. \tag{3}$$

This implies

$$yz < 4 + \sqrt{16 + 4z^2}\,.$$

Since $z \leq y$ we get

$$z^2 - 4 < \sqrt{16 + 4z^2}\,,$$

which reduces to

$$z^4 < 12z^2\,.$$

As z is a positive integer, we get $z \leq 3$. Therefore $z \in \{1, 2, 3\}$.

Case $z = 1$. From (3) we get

$$y \leq \frac{4 + \sqrt{20}}{1}$$

therefore

$$y \leq 8\,.$$

Moreover, (2) yields

$$x = \frac{4y + 4}{y - 4} = 4 + \frac{20}{y - 4}\,,$$

which implies $0 < y - 4$ and $y - 4 | 20$. Therefore $y \in \{5, 6, 8\}$ which leads to

$$(x, y, z) \in \{(25, 5, 1), (14, 6, 1), (9, 8, 1)\}$$

Case $z = 2$. From (3) we get

$$y \leq \frac{4 + \sqrt{32}}{2}$$

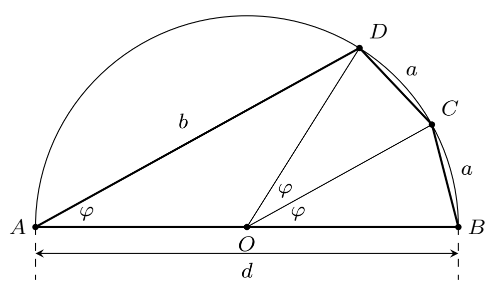
therefore
$$y \leq 4\,.$$
Moreover, (2) yields
$$x = \frac{4y+8}{2y-4} = 2 + \frac{8}{y-2}\,,$$
which implies $0 < y-2$ and $(y-2)|8$. Therefore $y \in \{3,4\}$ which leads to
$$(x,y,z) \in \{(10,3,2),(6,4,2)\}$$

Case $z = 3$. From (3) we get
$$y \leq \frac{4+\sqrt{52}}{3}$$
therefore
$$y \leq 3\,.$$
As $3 = z \leq y \leq 3$ we have $y = z = 3$ and
$$x = \frac{12+12}{9-4}$$
which is not an integer. Therefore there is no solution in this case.

We got five solutions in x, y, z, and using $a = y+z; b = x+z; c = x+y$ we get
$$(a,b,c) \in \{(6,25,29),(7,15,20),(9,10,17),(5,12,13),(6,8,10)\}.$$

OC83. On a semicircle with diameter $|AB| = d$ we are given points C and D such that $|BC| = |CD| = a$ and $|DA| = b$, where a, b, d are different positive integers. Find the minimum possible value of d.
(*Originally question 2 from the 2011 Bosnia and Herzegovina Olympiad team selection test.*)

Solved by Š. Arslanagić; C. Curtis; O. Geupel; T. Zvonaru; and K. Zelator. We give the solution of Oliver Geupel.



We show that the minimum value of d is 8.

Let O be the midpoint of AB and let $\varphi = \angle BOC$. Then, $\angle COD = \varphi$ and $\angle BAD = \dfrac{1}{2}\angle BOD = \varphi$. We have $\cos\varphi = \dfrac{|AD|}{|AB|} = \dfrac{b}{d}$. By the Law of Cosines, we obtain for triangle OBC that

$$a^2 = 2\left(\frac{d}{2}\right)^2 - 2\left(\frac{d}{2}\right)^2\cos\varphi = \frac{d^2}{2}\left(1 - \frac{b}{d}\right).$$

Equivalently,

$$2a^2 = d(d-b). \qquad (1)$$

Reversing our reasoning, we see that, on the other hand, every integer solution (a, b, d) with

$$1 \le a < d, \qquad 1 \le b < d, \qquad a \ne b \qquad (2)$$

of equation (1) satisfies the original geometric puzzle.

The triplet $(a, b, d) = (2, 7, 8)$ is a solution of (1) and (2). It is now straightforward to explore (1) and (2) for every specific value $2 \le d \le 7$ and to argue that there is no solution among them.

OC84. Let m, n be positive integers. Prove that there exist infinitely many pairs of relatively prime positive integers (a, b) such that

$$a + b \mid am^a + bn^b.$$

(*Originally question 3 from the 2011 China Math Olympiad, Day 2.*)

No solutions were received for this problem.

OC85. For any positive integer d, prove there are infinitely many positive integers n such that $d(n!) - 1$ is a composite number.
(*Originally question 3 from the 2011 China team selection test, day 1.*)

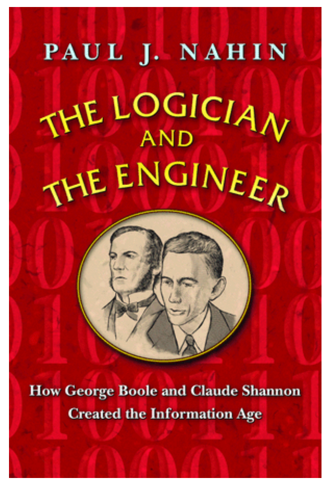
No solutions were received for this problem.

BOOK REVIEWS

John McLoughlin

The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age by Paul J. Nahin
ISBN: 978-0-691-15100-7 (Hardcover), 9781400844654 (e-book)
Princeton University Press, 2012, 248 pages, $24.95 (US)

Reviewed by ***S. Swaminathan****, Dalhousie University, Halifax, NS*



How did a branch of mathematics established in the Victorian era become the basis for some incredible technological achievements a century later? Boolean algebra, also called Boolean logic, is at the heart of the electronic circuitry in everything we use from our computers and cars, to our kitchen gadgets and home appliances. The best-selling popular author Paul Nahin combines in this book engaging problems and a colourful historical narrative to tell the remarkable story of how two mathematicians in different eras became founding fathers of the electronic communications age. They are George Boole (1815-1864) and electrical engineer and information theorist Claude Shannon (1916-2001). Presenting their biographies, Nahin examines the history of Boole's innovative ideas, and points out how they led to Shannon's groundbreaking work on electrical circuits and information theory. In the course of the exposition, problems in logic are given for the readers to solve. Also the author writes about the contributions of such key players as Georg Cantor, Tibor Radó, Marvin Minsky and Alan Turing, in the development of mathematical logic and data transmission. The author succeeds in developing the story from fundamental concepts to a deeper and more sophisticated understanding of how modern digital machines, such as the computer, are constructed. Some ideas in quantum mechanics and thermodynamics are introduced to explore the possible limitations of computing in the present century.

The chapter headings are provided for the interest of potential readers: What You Need to Know to Read this Book; Introduction to George Boole and Claude Shannon: Two Mini-Biographies; Boolean Algebra; Logical Switching Circuits; Boole, Shannon, and Probability; Some Combinatorial Logic Examples; Sequential State Digital Circuits; Turing Machines; Beyond Boole and Shannon; Epilogue; and Appendix (Fundamental Electric Circuit Concepts). Each chapter concludes with Notes and References.

Reading this book would help one understand how gigahertz chips work in electronic gadgets. The book is well written.

FOCUS ON . . .

No. 8

Michel Bataille

Generalized Inversion in the Plane

Introduction

Inversion is a nice transformation and a wonderful tool for both the solver and the poser. Examples of its use are numerous so a choice had to be made. We will focus on a few situations where a generalized inversion (in a sense specified below) allows an advantageous solution, that is, is sufficiently general to cover the various cases of the figure.

Generalizing inversion

Here is the definition I was taught as a student (long ago!) and that will be used in what follows. Let $p \neq 0$ and O a point. The inversion with centre O and power p associates with each point $M \neq O$ the point M' such that M' is on the line OM and satisfies $(\overline{OM'})(\overline{OM}) = p$, where the bar indicates signed distance (see [1] p. 2) [or alternatively, with the help of a dot product: $\overrightarrow{OM'} \cdot \overrightarrow{OM} = p$]. Such an inversion coincides with the inversion in the circle with centre O and radius $\sqrt{p}$ when $p > 0$ and, for negative p, is the commutative product of the inversion in the circle centre O, radius $\sqrt{-p}$ and the half turn about O (see Li Zhou's solution to problem **3510** [2011 : 61] as a first example). Interestingly, given two distinct points O, A and a point B distinct of O on the line OA, there exists a unique inversion $\mathbf{I}$ with centre O exchanging A and B (no matter the relative position of O, A, B): the power of $\mathbf{I}$ is just $p = \overrightarrow{OA} \cdot \overrightarrow{OB}$. Note that p is also the power of O with respect to any circle through A and B [or tangent to OA at A in the case $B = A$] so that such a circle is invariant under $\mathbf{I}$. The classical results on inversion are not affected by this new point of view, except the formula about lengths which now reads $X'Y' = \dfrac{|p|XY}{(OX)(OY)}$ if $X' = \mathbf{I}(X)$, $Y' = \mathbf{I}(Y)$ (note the absolute value of p). Geometric constructions form a favorite playground for inversions, so we will start with a simple example.

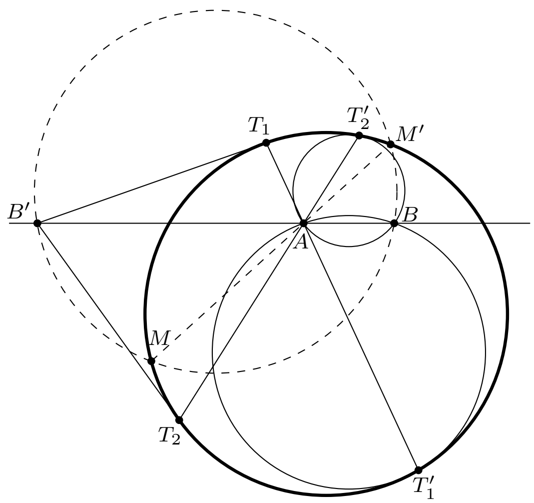
A construction

Let Γ be a circle and A, B two points both either exterior or interior to Γ. We consider the following problem: construct a circle tangent to Γ and passing through A and B.

Let $\mathbf{I}$ be the inversion with centre A whose power is the power of A with respect to Γ. Note that this power is negative if A, B are interior to Γ and that $\mathbf{I}(\Gamma) = \Gamma$. If γ is a suitable circle, then $\mathbf{I}(\gamma)$ is a line t passing through $B' = \mathbf{I}(B)$ (but not through A) and tangent to Γ. It follows that $\gamma = \mathbf{I}(t)$. Thus, if $B'T$, tangent at

T to Γ, does not pass through A and TA meets Γ again at T', the circumcircle of $\Delta ABT'$ is a solution.

See the figure where an auxiliary circle (BMM') has been drawn in order to locate B'. The reader will easily show that B' is always exterior to Γ. It follows that the problem has one solution if AB is tangent to Γ and two solutions otherwise.



Two configurations

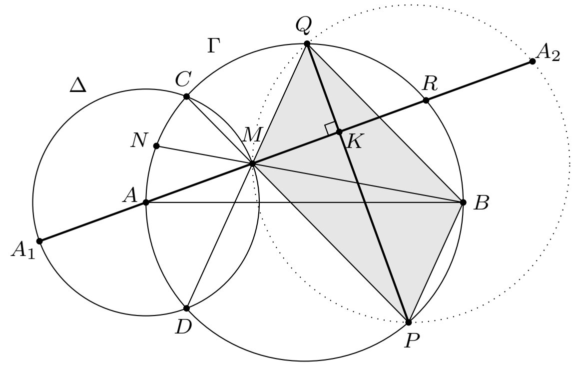
First, we consider question 1 of Crux problem **2666** [2001 : 403]. It reads as follows:

> *Two circles, Γ with diameter AB, and Δ with centre A, intersect at points C and D. The point M (distinct from C and D) lies on Δ. The lines BM, CM and DM intersect Γ again at N, P and Q, respectively. Prove that $MPBQ$ is a parallelogram.*

The featured solution (by Toshio Seimiya [2002 : 462]) is elementary but does not cover all the cases of the figure. In contrast, the following solution, based on generalized inversion, is valid whatever the position of M on Δ.

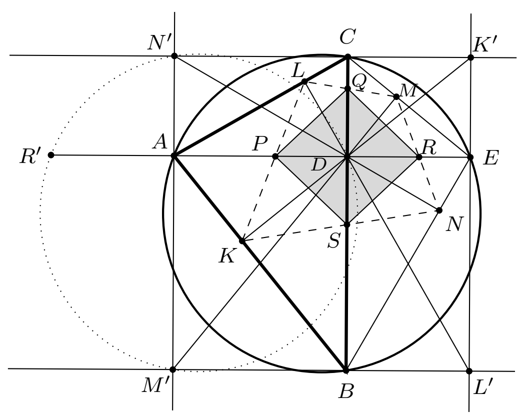
Let AM intersect Γ again at R and let $\mathbf{I}$ be the inversion with centre M exchanging A and R. This inversion also exchanges B and N, C and P, D and Q, hence Δ inverts into the line PQ. Thus, AM is perpendicular to PQ at say, K, and R is the reflection of M in PQ [if $A_1 = \mathbf{I}(K)$, then $\overline{MR} = \dfrac{(\overline{MA_1})(\overline{MK})}{\overline{MA}} = 2\overline{MK}$].

Let A_2 be the reflection of A in PQ. Then $(\overline{KM})(\overline{KA_2}) = -(\overline{KM})(\overline{KA}) = (\overline{KR})(\overline{KA}) = (\overline{KP})(\overline{KQ})$, so that A_2 is on the circumcircle of ΔMPQ. As a result, A is the orthocentre of this triangle and AP is perpendicular to MQ. Since AP is obviously perpendicular to BP, the lines BP, MQ are parallel. Similarly, MP, BQ are parallel and the conclusion follows.



Our second example is a part of [2]: Let the altitude AD of triangle ABC be produced to meet the circumcircle again at E. Let K, L, M, and N be the projections of D onto the lines BA, AC, CE, and EB, and let P, Q, R, and S be the intersections of the diagonals of $DKAL$, $DLCM$, $DMEN$, and $DNBK$, respectively. Show that $PQRS$ is a rhombus.

Under the inversion $\mathbf{I}$ with centre D such that $\mathbf{I}(A) = E$, the circles with diameters DA, DC, DE, and DB (which circumscribe the quadrilaterals $DKAL$, $DLCM$, $DMEN$, and $DNBK$) are transformed into lines through E, B, A and C parallel to the lines AD and BC, thus forming a rectangle whose vertices are the inverses K', L', M', N' of K, L, M, N. It follows that $M'N' = K'L' = BC$ and $M'L' = K'N' = AE$. Clearly, the circles $\Gamma = (DM'N')$ and (ABC) are symmetrical in the line through the centres of the rectangles $ADCN'$ and $ADBM'$.



Since $\Gamma = \mathbf{I}(MN)$, the point $R' = \mathbf{I}(R)$ is the point of intersection other than D of Γ with the line AD and so $DR' = AE$. In a similar way, we see that $DP' = AE$, $DQ' = BC$, $DS' = BC$, where $P' = \mathbf{I}(P), Q' = \mathbf{I}(Q), S' = \mathbf{I}(S)$. Now, if p denotes the power of $\mathbf{I}$, we have

$$PQ = \frac{|p|P'Q'}{(DP')(DQ')} = \frac{|p| \cdot \sqrt{AE^2 + BC^2}}{(AE)(BC)},$$

and the same result holds for QR, RS, and SP. Hence $PQRS$ is a rhombus.

Note that this example neatly reveals how inversion works: the initial configuration is transformed into one easier to handle (here the rectangle $K'L'M'N'$ and its subrectangles), providing some results (about the lengths DP', DQ', DR', DS') which, once sent back to the initial figure, yield the desired property.

Exercises

Here are two easy questions for the reader to put generalized inversions into practice:

1. Solve question 2 of **2666**: show that MN is the geometric mean of NC and ND.
2. (adapted from problem 10874, *Am. Math. Monthly*, vol. 108, No 5, May 2001) Let A, B, C, and D be points on a circle with centre O and let P be the point of intersection of AC and BD. If AB is not parallel to CD and U, V are the circumcentres of $\Delta APB, \Delta CPD$, prove that $OUPV$ is a parallelogram.

References

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.

[2] F. Javier Garcia Capitan, J. Bosco Romero Marquez, Problem 11547, *Am. Math. Monthly*, vol. 119, No 7, Aug. Sept. 2012, p. 611.

PROBLEM OF THE MONTH

No. 7

Diane and Roy Dowling

This column is dedicated to the memory of former ***CRUX with MAYHEM*** *Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The "Problem of the Month" features a problem and solution that we know Jim would have liked.*

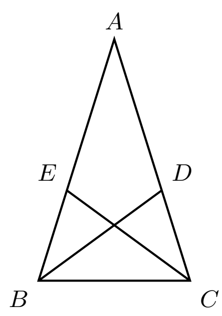
The Lasting Legacy of Ludoph Lehmus

This article appeared in ***Manitoba Math Links****, Volume II, Issue #3, Spring 2002. The editor thanks the Mathematics Department at the University of Manitoba for allowing us to reproduce it.*

Diane Mary Dowling (1933-2005) and Roy Dowling served as members of the Department of Mathematics at the University of Manitoba for over 40 years contributing to both research in mathematics and mathematics education as a whole. In 2006, Roy Dowling established the Diane Dowling Memorial Scholarship to honour his wife's academic legacy in mathematics.

In the early nineteenth century an interesting problem came to the attention of those who enjoyed geometry. It has been said of this problem that its beauty lies in the simplicity of its statement and in the difficulty of its solution. Before looking at it, let us consider a problem you may be familiar with:

If in the following diagram $AB = AC$, BD bisects $\angle ABC$ and CE bisects $\angle ACB$, prove that $BD = CE$.



When you have solved this problem, you have proved the statement:

If a triangle is isosceles then two of its internal bisectors are equal.

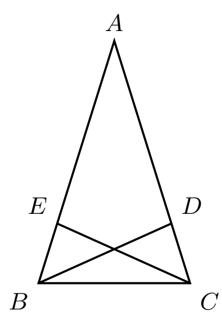
In about 1840, a question occurred to a Berlin professor, Ludolph Lehmus: is the converse of this statement true? The converse is:

If two internal bisectors of a triangle are equal then the triangle is isosceles.

People have thought about the properties of triangles for thousands of years so it is amazing that there is no record of anyone considering this converse before Professor Lehmus did. He approached Jacob Steiner with his question. This famous geometer was soon able to establish the truth of the converse above and it came to be known as the Steiner-Lehmus Theorem. Before very long Professor Lehmus himself found a nicer proof. Since that time geometry hobbyists have been fascinated by the search for simple and neat proofs of the theorem. You can see some of these proofs on the website:
`http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus`

In the 1960s Martin Gardner, magician and puzzle enthusiast, who regularly contributed to the *Scientific American*, discussed the Steiner-Lehmus Theorem in one of his columns. This column stimulated a lot of interest and hundreds of readers sent in their own proofs. Martin Gardner examined all these proofs and selected his favourite. This very nice proof was presented by two British engineers, G. Gilbert and D. MacDonnell. A few years later someone searched for the proof that Ludolph Lehmus had found over a hundred years previously and discovered that it was essentially the same as that of Gilbert and MacDonnell! If you would like to see their proof go to the website:
`http://poncelet.math.nthu.edu.tw/disk5/js/geometry/geometry.html`

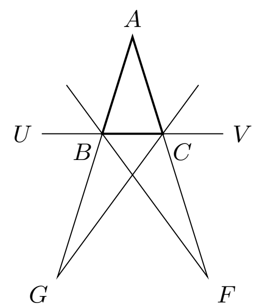
The publication of the Steiner-Lehmus Theorem not only got people trying to find neat proofs of the theorem itself but also got them thinking about variations on its theme. For example, is there a corresponding theorem about internal trisectors? That is, if in the following diagram $\angle CBD = \frac{1}{3}\angle CBA$, $\angle BCE = \frac{1}{3}\angle BCA$ and $BD = CE$, can it be shown that $AB = AC$?



The answer is yes. In fact, the $\frac{1}{3}$ may be replaced by any fraction between 0 and 1. The proof is not easy.

Another variation involves exterior angles of a triangle. Consider the following diagram:

For a triangle ABC, if the bisector of an exterior angle at B meets the side AC extended at the point F then the line segment BF is called the external bisector at B. For the triangle ABC in the next diagram the bisector of the exterior angle $\angle ABU$ meets the side AC extended at F so BF is the external bisector at B. The bisector of the exterior angle $\angle ACV$ meets the side AB extended at G, so CG is

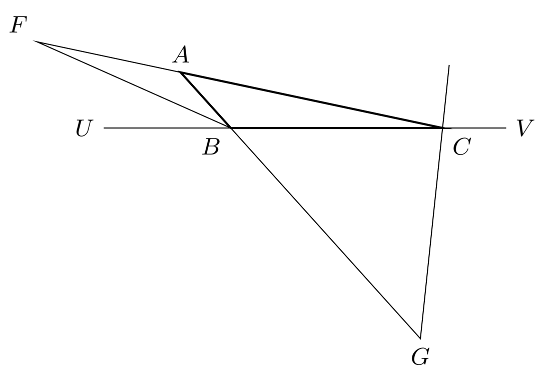


the external bisector at C.

It is not hard to prove that if $AB = AC$, then $BF = CG$. In other words, if a triangle is isosceles then two of its external bisectors are equal. The converse of this statement is:

If two external bisectors of a triangle are equal then the triangle is isosceles.

At first sight this statement looks very plausible. However, it is not true. A. Emmerich pointed out the surprising fact that a triangle whose interior angles are $132°$, $36°$ and $12°$ has two of its external bisectors equal. A triangle having these angles is referred to as an Emmerich triangle. To see why an Emmerich triangle has two equal external bisectors consider the following diagram.



Triangle ABC is an Emmerich triangle with $\angle CAB = 36°$, $\angle ABC = 132°$ and $\angle BCA = 12°$. The bisector of the exterior angle $\angle ABU$ meets the side AC extended at F so BF is the external bisector at B. The bisector of the exterior angle $\angle ACV$ meets the side AB extended at G so CG is the external bisector at C. We will show that the external bisector BF equals the external bisector CG.

$$\angle FBA = \frac{1}{2}(180° - 132°) = 24°;$$
$$\angle FBC = \angle FBA + \angle CBA = 24° + 132° = 156°.$$

Now consider triangle BCF:

$$\angle BCF = 12^\circ \quad \text{and}$$
$$\angle BFC = 180^\circ - \angle FBC - \angle BCF = 180^\circ - 156^\circ - 12^\circ = 12^\circ.$$

Since $\angle BFC = \angle BCF$, triangle FBC is isosceles with $BF = BC$. Now consider triangle BCG:

$$\angle BCG = \frac{1}{2}(180^\circ - 12^\circ) = 84^\circ, \angle GBC = 180^\circ - 132^\circ = 48^\circ.$$
$$\angle BGC = 180^\circ - \angle BCG - \angle GBC = 180^\circ - 84^\circ - 48^\circ = 48^\circ.$$

Since$\angle GBC = \angle BGC$, triangle BCG is isosceles with $CG = BC$. Since BF and CG both equal BC these two external bisectors are equal to each other.

Over 160 years have passed since Ludolph Lehmus posed his question and popular mathematics magazines are still publishing articles with new proofs of the Steiner-Lehmus Theorem or its generalizations. For example, the October 2001 issue of *The American Mathematical Monthly* carried an article called *Other Versions of the Steiner-Lehmus Theorem*. We can certainly say that for amateur geometers the legacy of Ludolph Lehmus lives on!

Diane and Roy Dowling
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University of Manitoba
Winnipeg, MB

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

Submissions of proposals. *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions should be received by the editor by **1 December 2014***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

3861. *Proposed by B. Sands.*

Prove that, for any positive real number x, there is a $\sqrt{2}\times x$ rectangle on the plane with each corner having at least one integer coordinate.

3862. *Proposed by M. Şahin.*

Let triangle ABC be right angled, with $\angle BAC = 90^\circ$ and altitude AD. Let K and L be on AB and CA so that DK and DL are bisectors of angles $\angle BDA$ and $\angle CDA$ respectively. Let M and N be the feet of the perpendiculars from K and L, respectively, to BC. Prove that $KM + NL = AD$.

3863. *Proposed by M. Bataille.*

Let a, b, c be real numbers such that $a^2+b^2+c^2\leq 1$. Prove that

$$a^2b(b-c)+b^2c(c-a)+c^2a(a-b)\geq \frac{(b-c)^2(c-a)^2(a-b)^2}{2}\ .$$

3864. *Proposed by C. Mortici.*

For every positive integer m, denote by $m!!$ the product of all positive integers with same parity as m, which are less than or equal to m. Let $n\geq 1$ be an integer. Prove that

$$(-1)^n(2n)!!-(2n-1)!!+(2n+1)!!\sum_{k=1}^{n}\frac{1}{2k-1}$$

is divisible by $(2n+1)^2$.

3865. *Proposed by G. Apostolopoulos.*

Prove that in any triangle ABC

$$\sum_{\text{cyclic}}\frac{1}{1+\cot^3\left(\frac{A}{2}\right)}\leq\frac{3R}{2(r+s)}$$

where s, r, and R are the semiperimeter, the inradius and the circumradius of ABC, respectively.

3866. *Proposed by M. Bataille.*

Distinct points B, C, D, E on a line ℓ are such that $\angle BAC=\angle DAE=90^\circ$ for some point A. Let X, Y on the circumcircle of ΔCAD be such that $\angle AXB=\angle AYE=90^\circ$. If BX intersects line AC at V and EY intersects line AD at U, prove that UV is parallel to ℓ.

3867. *Proposed by D. M. Bătineţu-Giurgiu and N. Stanciu.*

Let $(a_n)_{n\geq 1}$ be a positive real sequence and $a>0$ such that

$$\lim_{n\to\infty}(a_n-a\cdot n!)=b>0.$$

Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right).$$

3868. *Proposed by I. Bluskov.*

Determine the maximum value of $f(x,y,z)=xy+yz+zx-xyz$ subject to the constraint $x^2+y^2+z^2+xyz=4$, where x, y and z are real numbers in the interval $(0,2)$.

3869. *Proposed by D. T. Oai.*

It is known that any nondegenerate conic that passes through the vertices of a triangle ABC and its orthocentre H must be a rectangular hyperbola whose centre lies on the triangle's nine-point circle. Prove that the centre of the hyperbola is the midpoint of the segment that joins H to the fourth point (different from $A, B,$ and C) where the hyperbola intersects the circumcircle.

3870. *Proposed by O. Furdui.*

Calculate

$$\int_0^1 \ln^2(\sqrt{x}+\sqrt{1-x})\,dx\,.$$

. . . .

3861. *Proposé par B. Sands.*

Montrer que pour tout nombre réel positif x, il existe un rectangle $\sqrt{2}\times x$ dans le plan dont chacun des coins possède une coordonnée entière.

3862. *Proposé par M. Şahin.*

Soit ABC un triangle rectangle avec $\angle BAC = 90^\circ$ et de hauteur AD. Soit K et L sur AB et CA de sorte que DK et DL soient les bissectrices respectives des angles $\angle BDA$ et $\angle CDA$. Soit M et N les pieds des perpendiculaires respectivement issues de K et L sur BC. Montrer que $KM + NL = AD$.

3863. *Proposé par M. Bataille.*

Soit a, b et c trois nombres réels tels que $a^2+b^2+c^2 \le 1$. Montrer que

$$a^2b(b-c)+b^2c(c-a)+c^2a(a-b) \ge \frac{(b-c)^2(c-a)^2(a-b)^2}{2}\,.$$

3864. *Proposé par C. Mortici.*

Pour tout entier positif m, notons par $m!!$ le produit de tous les entiers de même parité que m qui sont plus petits ou égaux à m. Soit $n \ge 1$ un entier. Montrer que

$$(-1)^n(2n)!! - (2n-1)!! + (2n+1)!!\sum_{k=1}^{n}\frac{1}{2k-1}$$

est divisible par $(2n+1)^2$.

3865. *Proposé par G. Apostolopoulos.*

Montrer que pour tout triangle ABC

$$\sum_{\text{cyclique}} \frac{1}{1+\cot^3\left(\frac{A}{2}\right)} \leq \frac{3R}{2(r+s)}$$

où s, r et R sont respectivement le demi-périmètre, le rayon du cercle inscrit et celui du cercle circonscrit de ABC.

3866. *Proposé par M. Bataille.*

Quatre points distincts B, C, D, E sur une droite ℓ sont tels que $\angle BAC = \angle DAE = 90^\circ$ pour un certain point A. Soit X, Y sur le cercle circonscrit de ΔCAD de sorte que $\angle AXB = \angle AYE = 90^\circ$. Si BX coupe la droite AC en V et EY coupe la droite AD en U, montrer que UV est parallèle à ℓ.

3867. *Proposé par D. M. Bătineţu-Giurgiu and N. Stanciu.*

Soit $(a_n)_{n\geq 1}$ une suite de nombres réels positifs et $a>0$ tel que

$$\lim_{n\to\infty}(a_n - a\cdot n!) = b > 0.$$

Calculer

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right).$$

3868. *Proposé par I. Bluskov.*

Déterminer la valeur maximale de $f(x,y,z) = xy+yz+zx-xyz$ à la condition que $x^2+y^2+z^2+xyz=4$, où x, y et z sont des nombres réels dans l'intervalle $(0,2)$.

3869. *Proposé par D. T. Oai.*

On sait que toute conique non dégénérée qui passe par les sommets d'un triangle ABC et son orthocentre H doit être une hyperbole rectangulaire dont le centre se trouve sur le cercle des neuf points du triangle. Montrer que le centre de l'hyperbole est le point milieu du segment qui relie H au quatrième point (différent de A, B, et C) où l'hyperbole coupe le cercle circonscrit.

3870. *Proposé par O. Furdui.*

Calculer

$$\int_0^1 \ln^2(\sqrt{x}+\sqrt{1-x})\,dx\,.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3326b$\star$. Determine the largest value of k for which

$$(a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) \geq k(a+b+c)^2.$$

This problem was solved by C. R. Pranesachar with significant help from Maple. His solution was checked by S. Wagon using Mathematica. The desired maximum value λ of k is equal to $6.2434713875998\ldots$. We present editor's write-up based on C. R. Pranesachar's solution.

Let

$$Q_k(a,b,c) = (a^2+2)(b^2+2)(c^2+2)+4(a^2+1)(b^2+1)(c^2+1) - k(a+b+c)^2.$$

We are seeking the largest value of k for which $Q(a,b,c) \geq 0$ for all real triples (a,b,c). In particular, this should hold when $a=b=c$. We find that

$$Q_k(a,a,a) = f(d) - 9kd,$$

where

$$d = a^2 \text{ and } f(d) = 5d^3+18d^2+24d+112.$$

Since

$$f'(d) = 3(5d^2+12d+8) = 3[(2d+3)^2+(d^2-1)] > 0,$$

for $d \geq 0$, we see that $f(d)$ is an increasing convex function. The tangent to its graph at (u,v) has slope $15u^2+36u+24$ and the condition that it pass through the origin is $f(u) = uf'(u)$ or $5u^3+9u^2 = 6$. This equation is satisfied by u slightly less than 0.69369555. The value (about 56.19124197) of the derivative $f'(u)$ for this value of u is the largest value of $9k$ for which the line $y = 9kx$ lies below the graph. Thus, we obtain the maximum value λ recorded above of k for which $Q_k(a,a,a) \geq 0$ for all real a.

Pranesachar obtained the same result by determining the discriminant of $Q_k = f(d) - 9kd$ as a cubic in d to be $108(135k^3 - 837k^2 - 36k - 4)$. The sole real root of this is λ, and when k exceeds λ, the cubic can assume negative values for positive d. For example, when $k = 6.24334713876$, $Q_k(a,a,a) < 0$ when $a = 0.8328838$.

We now need to verify that $Q_\lambda(a,b,c) \geq 0$ for all real triples (a,b,c). Write it as a quadratic polynomial in a:

$$Q_\lambda(a,b,c) = (5b^2c^2+6b^2+6c^2+8-\lambda)a^2 - 2\lambda(b+c)a + (6b^2c^2+8b^2+8c^2+12-\lambda(b+c)^2).$$

Since the leading coefficient is positive, it suffices to show that its discriminant is nonpositive. Now the computations get quite formidable, so we sketch the treatment.

The negative of the discriminant is a quartic polynomial in b that has the form

$$Ab^4 + 4Bb^3 + 6Cb^2 + 4Db + E.$$

Define $H = AC - B^2$, $I = AE - 4BD + 3C^2$, $J = ACE + 2BCD - AD^2 - C^3 - EB^2$, $\Delta = I^3 - 27J^2$. To establish that the quadratic is always nonnegative, it suffices to show that $H > 0$ and $\Delta \geq 0$. The Wikipedia entry provides necessary background: `http://en.wikipedia.org/wiki/Quartic_function`

H can be written as the product of a factor that is clearly positive and a polynomial in c with coefficients that involve λ. Using estimates for λ, we can bound these coefficients and thus find that $H > 0$. This is checked by taking λ to nine decimal places.

Similarly, we can compute Δ for the parameter k in place of λ and show that it is the product of three factors, two of which are positive and the third of which can be shown by Sturm's Theorem to have no real roots when $k \neq \lambda$ (see, for example, the site `http://en.wikipedia.org/wiki/Sturm's_theorem`). Thus, using the continuity with respect to k, we deduce that $\Delta \geq 0$ when $k = \lambda$. This completes the determination of the maximum value of k.

3748⋆. [2012 : 195, 197; 2013 : 240 - 242] *Proposed by N. T. Binh.*

Given three mutually external circles in general position, there will exist six distinct lines that are common internal tangents to pairs of the circles. Prove that if three of those common tangents, one to each pair of the circles, are concurrent, then the other three common tangents are also concurrent.

Solution 2 by Eberhard Schröder.

Editor's Comment. Schröder published his solution [3] in German as a research article. His main result, which he calls "Binh's Theorem", is a generalized version of our problem. His proof is based on the theory developed by Bachmann in [1], so that Binh's theorem holds more generally in classical absolute geometry (which includes the classical hyperbolic plane). Instead of a strict translation, here we modify Schröder's arguments to make use of the tools found in more elementary texts, such as [2], that deal with transformation geometry; see especially the summary in [2, Section 3.4].

We shall use lowercase letters to denote lines, and a tilde over the letter to indicate the reflection in the line. Thus, for any line x we have $\tilde{x} = \tilde{x}^{-1}$. We require three basic facts about reflections in the lines x, y, and z:

(1) If x, y, z are concurrent then the product $\tilde{x}\tilde{y}\tilde{z}$ is a reflection.

(2) Conversely, if $\tilde{x}\tilde{y}\tilde{z}$ is a reflection and x intersects y, then x, y, z are concurrent.

(3) If y and z are interchanged by reflection in x, then $\tilde{x}\tilde{y}\tilde{x} = \tilde{z}$.

As an immediate consequence, we have

If $\tilde{x}\tilde{y}\tilde{z}$ is a reflection then $\tilde{x}\tilde{y}\tilde{z} = \tilde{z}\tilde{y}\tilde{x}$ and $\tilde{y}\tilde{z}\tilde{x} = \tilde{x}\ \tilde{x}\tilde{y}\tilde{z}\ \tilde{x}$ is a reflection as well.

We assume that we are given three mutually external circles $(K), (L)$, and (M) with respective centres K, L, and M, situated so that each pair of the circles has two common internal tangents: a and d for circles (K) and (L); b, e for circles $(L), (M)$; and c, f for circles $(M), (K)$. Our goal is to prove that,

if a, b, and c are distinct and concurrent, then d, e, and f are concurrent.

Notation. The given tangents are denoted by

$$a = AA', b = BB', c = CC', d = DD', e = EE', \text{ and } f = FF',$$

with contact points A, C', D, F' on (K), B, A', E, D' on (L), and C, B', F, E' on (M). For distinct points X and Y, denote by $[X, Y]$ the line segment that joins X to Y, and by $[X, Y\rangle$ the halfline that starts at X and contains Y.

Consider circle (K) with its tangents a and c through P. By assumption b passes through P and is different from a and c, so there can be only two possibilities: either b intersects (K) in exactly two points or it misses (K).

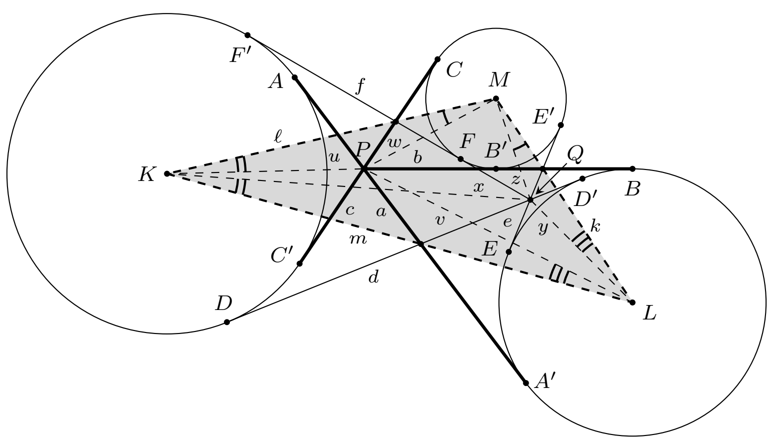


Figure 1: $P \in [A, A'] \cap [C, C']$ and $P \notin [B, B']$.

Possibility (a). b intersects (K) in two points (Figure 3 (i) and (ii)). Since P cannot be a contact point, it either lies strictly between A and A' (Figure 3(i)), or it lies outside $[A, A']$ (Figure 3(ii)). Because a, b, c are internal tangents, in both cases (L) must lie in the sector between the halflines $[P, A'\rangle$ and $[P, B\rangle$; this forces (M) to lie in the remaining sector, and we have either

(4) P lies between A and A' and between C and C', but outside the segment $[B, B']$ (cf. Figure 3(i)), or

(5) P lies outside all three segments $[A, A'], [B, B'], [C, C']$ (cf. Figure 3(ii)).

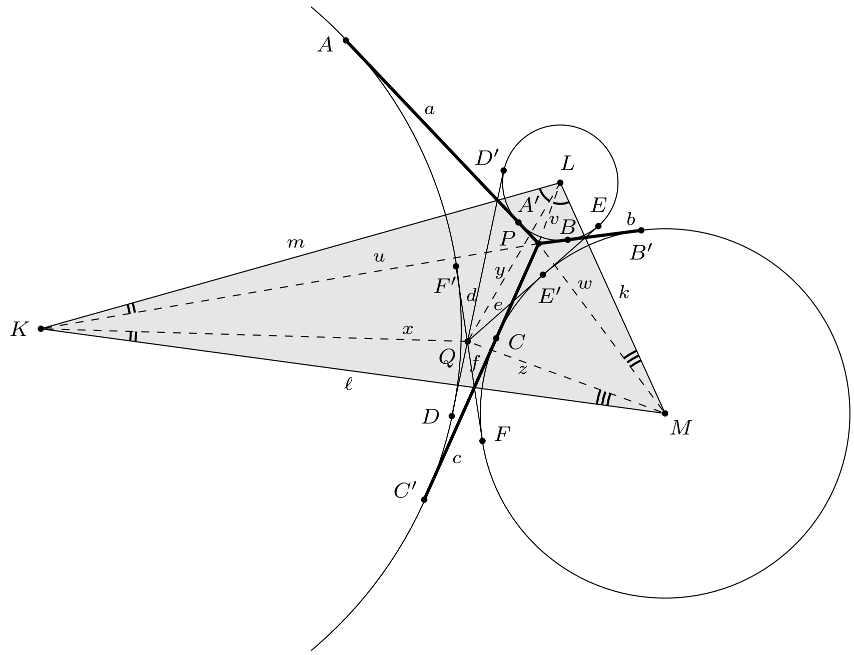


Figure 2: $P \notin [A, A'] \cup [B, B'] \cup [C, C']$.

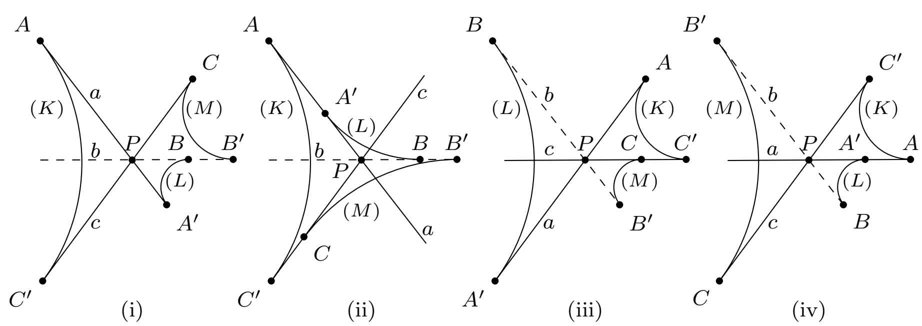


Figure 3: Possible positions of P with respect to the segments $[A, A']$, $[B, B']$, and $[C, C']$.

Possibility (b). b misses K (Figure 3 (iii) and (iv)). Again P either lies strictly between A and A' as in diagram (iii) of Figure 3, or it lies outside $[A, A']$ as in (iv). As with Possibility (a), in both cases there is first exactly one sector where circle (L) can lie, then one sector remaining for (M); consequently, either P lies between A and A' and between B and B' but not between C and C', or P lies between B and B' and between C and C' but not between A and A'. Note that the diagrams (i), (iii), and (iv) are identical except for the names assigned to the objects. As a consequence, whatever the given arrangement of circles, we can always label the given configuration so that statement (4) or (5) holds. In particular, we are assuming that P is always outside the segment $[B, B']$; that is, $[P, B\rangle$ and $[P, B'\rangle$ represent the same halfline.

As in the first two figures, we define six new lines,

$$k = LM, \quad \ell = MK, \quad m = KL,$$
$$u = PK, \quad v = PL, \quad w = PM.$$

Note that the reflection $\tilde{u}$ fixes K, P, and the circle (K). Because the tangents to (K) from P, namely a, c, are different from u and not perpendicular to it, $\tilde{u}$ interchanges a and c as well as their contact points A and C'. From (4) and (5) we see that $\tilde{u}$ acts on the halflines emanating from P by switching $[P, A\rangle$ with $[P, C'\rangle$ and $[P, A'\rangle$ with $[P, C\rangle$.

Similar remarks apply to $\tilde{v}$ and $\tilde{w}$. Specifically, we have

$$a \overset{\tilde{u}}{\longleftrightarrow} c \overset{\tilde{w}}{\longleftrightarrow} b \overset{\tilde{v}}{\longleftrightarrow} a \quad \text{and} \quad A \overset{\tilde{u}}{\longleftrightarrow} C',\ C \overset{\tilde{w}}{\longleftrightarrow} B',\ B \overset{\tilde{v}}{\longleftrightarrow} A'.$$

Recalling from (4) and (5) that $[P, B\rangle$ and $[P, B'\rangle$ represent the same halfline, it follows that the product

$$[P, A'\rangle \xrightarrow{\tilde{u}} [P, C\rangle \xrightarrow{\tilde{w}} [P, B'\rangle = [P, B\rangle \xrightarrow{\tilde{v}} [P, A'\rangle$$

fixes the halfline $[P, A'\rangle$. Therefore, the rotations

$$\tilde{u}\tilde{w}\tilde{v}\tilde{a}, \quad \tilde{u}\tilde{w}\tilde{b}\tilde{v}, \quad \text{and} \quad \tilde{u}\tilde{c}\tilde{w}\tilde{v},$$

which also fix $[P, A'\rangle$, must each be the identity rotation. We conclude that

$$\tilde{a} = \tilde{u}\tilde{w}\tilde{v}, \quad \tilde{b} = \tilde{v}\tilde{u}\tilde{w}, \quad \text{and} \quad \tilde{c} = \tilde{w}\tilde{v}\tilde{u}. \tag{6}$$

Because u, ℓ, m meet at K, while v, m, k meet at L and w, k, ℓ meet at M, according to (1) we can define the lines x, y, z by

$$\tilde{x} = \tilde{m}\tilde{u}\tilde{\ell}, \quad \tilde{y} = \tilde{k}\tilde{v}\tilde{m}, \quad \text{and} \quad \tilde{z} = \tilde{\ell}\tilde{w}\tilde{k}. \tag{7}$$

The lines a, b, c are related to d, e, f by the reflections $\tilde{m}$ (which takes a to d), $\tilde{k}$ (which takes b to e), and $\tilde{\ell}$ (which takes c to f). Thus, according to (3), we have

$$\tilde{d} = \tilde{m}\tilde{a}\tilde{m}, \quad \tilde{e} = \tilde{k}\tilde{b}\tilde{k} \quad \text{and} \quad \tilde{f} = \tilde{\ell}\tilde{c}\tilde{\ell}. \tag{8}$$

From equations (6), (7), and (8) we deduce that

$$\tilde{x}\tilde{z}\tilde{y} = \tilde{m}\tilde{u}\tilde{\ell}\tilde{\ell}\tilde{w}\tilde{k}\tilde{k}\tilde{v}\tilde{m} = \tilde{m}\tilde{u}\tilde{w}\tilde{v}\tilde{m} = \tilde{m}\tilde{a}\tilde{m} = \tilde{d},$$
$$\tilde{z}\tilde{y}\tilde{x} = \tilde{\ell}\tilde{w}\tilde{k}\tilde{k}\tilde{v}\tilde{m}\tilde{m}\tilde{u}\tilde{\ell} = \tilde{\ell}\tilde{w}\tilde{v}\tilde{u}\tilde{\ell} = \tilde{\ell}\tilde{c}\tilde{\ell} = \tilde{f},$$
$$\tilde{y}\tilde{x}\tilde{z} = \tilde{k}\tilde{v}\tilde{m}\tilde{m}\tilde{u}\tilde{\ell}\tilde{\ell}\tilde{w}\tilde{k} = \tilde{k}\tilde{v}\tilde{u}\tilde{w}\tilde{k} = \tilde{k}\tilde{b}\tilde{k} = \tilde{e},$$

and, putting them together, therefore

$$\tilde{e}\tilde{f}\tilde{d} = \tilde{y}\tilde{x}\tilde{z}\tilde{z}\tilde{y}\tilde{x}\tilde{x}\tilde{z}\tilde{y} = \tilde{y}\tilde{x}\tilde{y}\tilde{z}\tilde{y} = \tilde{y}\tilde{z}\tilde{y}\tilde{x}\tilde{y} = \tilde{y}\tilde{f}\tilde{y}.$$

Because $\tilde{y}\tilde{f}\tilde{y}$ is a reflection, (2) tells us that the lines d, e, f are either concurrent or parallel. However, should d, e, f be parallel and not concurrent, then the circles $(K), (L), (M)$ would lie on the same side of one of the common internal tangents d, e or f, contradicting the definition of *internal*. We conclude, finally, that d, e, f are concurrent.

References

[1] Friedrich Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff* (Grundlehren der mathematischen Wissenschaften, 96), 2. Auflage. Springer, Berlin (1973).

[2] H.S.M. Coxeter, *Introduction to Geometry*, 2nd edition. Wiley (1969).

[3] Eberhard M. Schröder, Beweis eines Satzes von Nguyen Thanh Binh. *Mitt. Math. Ges. Hamburg* **34** (2014), 101-106.

Note that $\tilde{x} = \tilde{m}\tilde{u}\tilde{\ell}$ is equivalent to $\tilde{m}\tilde{x} = \tilde{u}\tilde{\ell}$; this means that the rotation about K through twice $\angle mx$ equals the rotation about K through twice $\angle u\ell$. In other words, x and u are isogonal with respect to triangle KLM. Of course, the same goes for the pairs y, v and z, w. Those who are familiar with the notion of isogonal conjugates will recognize that the crux of Schröder's argument was to prove that because the lines u, v, w are concurrent at P, their isogonals x, y, z must be concurrent at the isogonal conjugate of P, call it Q, a point that lies on the three lines $d, e,$ and f.

It is easy to arrange the three circles so that two of d, e, f coincide. Of course, these three lines cannot all coincide because each is a common internal *tangent to a pair of the given circles. Under the assumption that $d \neq e$ we can easily see that the lines d, e, f, x, y, z are concurrent by (2):*

$$\tilde{e}\tilde{x}\tilde{d} = \tilde{y}\tilde{x}\tilde{z}\tilde{x}\tilde{x}\tilde{z}\tilde{y} = \tilde{y}\tilde{x}\tilde{y}, \quad \tilde{e}\tilde{y}\tilde{d} = \tilde{y}(\tilde{x}\tilde{z}\tilde{y})(\tilde{x}\tilde{z}\tilde{y}) = \tilde{y}, \quad \text{and} \quad \tilde{e}\tilde{z}\tilde{d} = \tilde{y}\tilde{x}\tilde{z}\tilde{z}\tilde{x}\tilde{z}\tilde{y} = \tilde{y}\tilde{z}\tilde{y},$$

with analogous equations when $e \neq f$ and $f \neq d$. Note, finally, that the second of these equations tells us that $\tilde{e}\tilde{y} = \tilde{y}\tilde{d}$; this confirms that y is the line through the centre L that bisects the angle formed by the tangents to (L) at D' and E as depicted in Figures 1 and 2. We can similarly interpret $\tilde{f}\tilde{z} = \tilde{z}\tilde{e}$ and $\tilde{d}\tilde{x} = \tilde{x}\tilde{f}$. Of course, the equation $\tilde{e}\tilde{y} = \tilde{y}\tilde{d}$ continues to hold should $d = e$; in this case d would necessarily be perpendicular to y at Q, which implies that the line tangent to all three circles (namely $d = e$) would touch (L) at Q. Analogous statements hold should $e = f$ or $f = d$.

3761. [2012 : 284, 286] *Proposed by P. Saltzman and S. Wagon.*

Let $B_{m,n}$ be a graph of possible moves by a white bishop on an $m \times n$ chessboard, where we assume $m \leq n$ and that the lower-left square is white.

(a) For which pairs of positive integers (m, n) does $B_{m,n}$ have a Hamiltonian cycle?

(b) Show that the edges of $B_{m,n}$ can be coloured using Δ colours so that intersecting edges are coloured differently, where Δ is the maximum degree.

One late solution has been received and will be considered for future publication. We present the proposers' solution.

(a) It is clear that $B_{1,n}$ and $B_{2,n}$ have no cycles; $B_{3,3}$ is isomorphic to two triangles sharing a vertex, so is not Hamiltonian. The larger $B_{3,n}$ graphs are easily seen to be Hamiltonian (see Figure 1). For $m \geq 4$, the vertices fall into disjoint upward (meaning slope is $+1$) diagonals whose sizes, from left to right, have the form $s, s, b, b, \ldots, b, s, s$ where s denotes a small value (1, 2, or 3) and b a large one (4 through m). Each diagonal, being a complete graph, has lots of Hamiltonian cycles. Because $m \geq 4$, we can find bridge-edges connecting each pair of large diagonals and we can choose the bridges (downward edges (slope is -1) in the figure) so that the four vertices within a diagonal that form bridge-ends are distinct. These bridges allow one to splice together cycles from the diagonals, thus getting a cycle for the all but the small diagonals. But it is easy to add side trips that pick up the one or two small diagonals at each end (Figure 2). The cases $B_{4,4}$, $B_{4,5}$, and $B_{5,5}$ have only one large diagonal, but it is easy to arrange the needed side trips (e.g., a cycle for $B_{4,5}$ is 00-11-02-24-13-04-22-40-31-20-42-33-44-00).

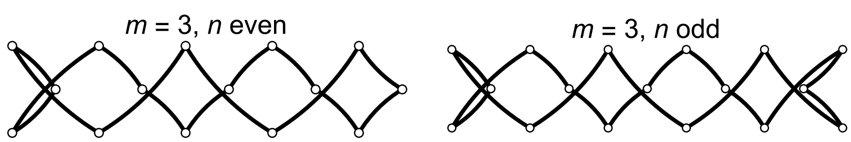


Figure 1: Hamiltonian cycles for $B_{3,n}$.

(b) We use the well-known fact that the complete graphs K_{2n} and K_{2n-1} can be edge-colored in $2n - 1$ colors.

Case 1: m is even, $m = n$. Here Δ is $2m-3$. Use colors $1, 2, \ldots, m-1$ in identical fashion on the two largest downward diagonals, which are isomorphic to K_{m-1}. Use colors up to $m - 3$ on all smaller downward diagonals. There are two central vertices of degree Δ. One of the colors used, say $m - 1$, is free at each of these two vertices; moreover, color $m - 1$ does not appear on any of the smaller down diagonals. So use colors $m - 1, m, \ldots, 2m - 3$ on the even-order complete graphs forming from the upward diagonals.

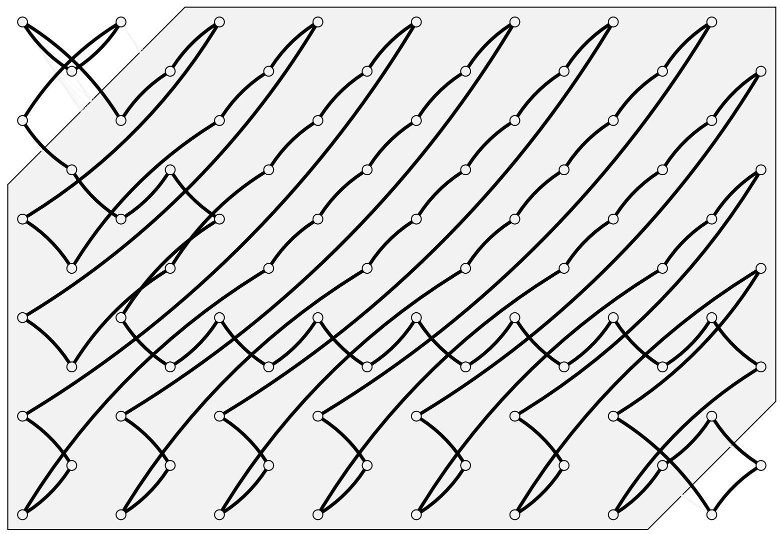


Figure 2: A Hamiltonian cycle for $B_{11,16}$, using bridges to connect the diagonals.

Case 2: m is even, $m < n$. From now on, we work with the infinite graph $G = B_{m,\infty}$. In this case $\Delta = 2m - 2$. Use an $m - 1$ edge coloring of K_m to color each upward diagonal with at most $m - 1$ colors and then do the same with a different set of colors on the downward diagonals.

Case 3: m is odd, $m = 2k+1$. Here $\Delta = 4k$. For vertices, use (x, y) with $x \in \mathbb{Z}_m$, $y \in \mathbb{Z}$, and $x + y$ even. For each $1 \le i \le k$, define the subgraph G_i to consist of edges of length i or $m-i$, where the basic unit is taken to be $\sqrt{2}$. It suffices to show that G_i can be edge-colored using four colors. To that end, we will decompose G_i into two sets of vertex-disjoint paths, each of which can be 2-edge colored.

For a vertex u and $i < (m - 1)/2$, define $f_i(u)$ to be the vertex at the end of an upward i-step from u if there is such a vertex; if not, let it be the vertex at the end of a downward $m - i$ step from u. Similarly define $g_i(u)$, reversing the roles of up and down. For example, for $B_{5,\infty}$ the iterates of the origin under f_1 involve steps of length 1 and 4 and are

$$(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (8, 0), (9, 1), (10, 2), (11, 3), (12, 4), (16, 0), \ldots.$$

Use f_i to partition the vertices into equivalence classes ($u \sim v$ if for some s, $f_i^{(s)}(u) = v$ or $f_i^{(s)}(v) = u$), which we call *f_i-paths*, with the same applying to g_i and *g_i-paths*.

The following two claims complete the proof, since each of the f_i and g_j path sets is 2-edge colorable.

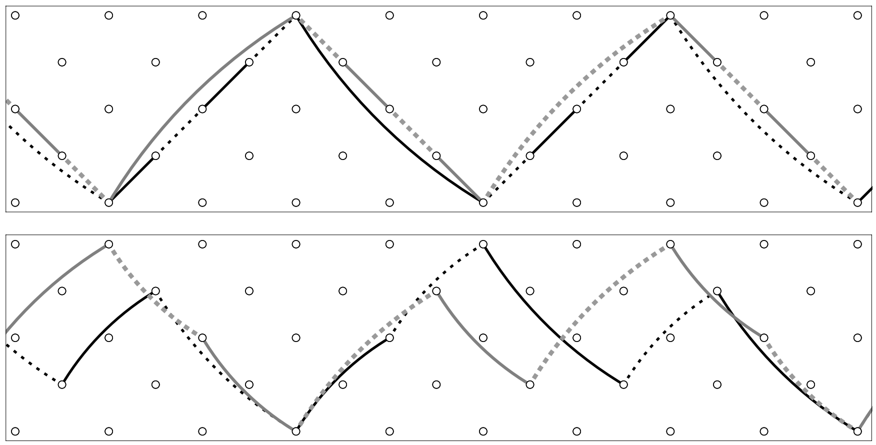


Figure 3: The two paths shown in the upper image, when translated, include every edge in $B_{5,\infty}$ of length 1 or 4, and these edges can be colored with 4 colors. The lower image does the same for edge-length 2 or 3.

- Any edge in G_i lies in some f_i or g_i path.
  Proof. Suppose edge $u = (x, y) \leftrightarrow (a, b) = v$ with $x < a$ has length L. If $L < (m-1)/2$, then if the edge has slope 1 we have $f_L(u) = v$, while if the slope is -1 we have $g_L(u) = v$. If $L \geq (m-1)/2$ then one of $f_{m-L}(u)$ or $g_{m-L}(u)$ is v.
- No edge lies in an f_i chain and a g_j chain.
  Proof. An edge $u \leftrightarrow v$ in two such chains would have length equal to i and $m - j$, with i and j under $(m-1)/2$, which is impossible.

3762. [2012 : 284, 286] *Proposed by B. Sands.*

Three sides of a cyclic quadrilateral $ABCD$ have lengths $AB = 1$, $BC = 2$ and $CD = 3$, and one of the angles of the quadrilateral equals 60°. Find all possible lengths of AD.

Solved by Š. Arslanagic; C. Curtis; O. Geupel; J. Hawkins and D. R. Stone; R. Hess; K. E. Lewis; D. Smith; E. Swylan; I. Uchiha; D. Văcaru; T. Zvonaru; and the proposer. There was one incomplete submission. All the submitted solutions were essentially the same. We present the solution by Kathleen E. Lewis.

Consider the different cases for the possible location of the 60° angle. For each case we let y be the resulting length of AD.

(a) If the 60° angle is at $\angle A$, then the measure of $\angle C$ is 120° (since the quadrilateral is cyclic). Using the law of cosines on triangle BCD, we get

$$BD^2 = 4 + 9 - 2 \cdot 2 \cdot 3 \cos 120^\circ = 19,$$

so $BD = \sqrt{19}$. Now using the law of cosines on triangle ABD, we get

$$19 = 1 + y^2 - 2y\cos 60^\circ = y^2 - y + 1.$$

Then $y = (1 + \sqrt{73})/2$.

(b) If the 60° angle is at $\angle B$, then the measure of $\angle D$ is 120°. Applying the law of cosines to triangle ABC, we get

$$AC^2 = 1 + 4 - 2 \cdot 1 \cdot 2\cos 60^\circ = 3,$$

so $AC = \sqrt{3}$. Looking at triangle DAC, we see that this is impossible, since $\angle D$ is the largest angle, but is not opposite the longest side. [Alternatively, the algebra comes down to $3 = 9 + y^2 + 3y$ has no real solution.]

(c) If the 60° angle is at $\angle C$, then the measure of $\angle A$ is 120°. Applying the law of cosines first to triangle BCD, we get

$$BD^2 = 4 + 9 - 2 \cdot 2 \cdot 3\cos 60^\circ = 7,$$

so $BD = \sqrt{7}$. Now applying it to triangle ABD to find $y = AD$, we get

$$7 = 1 + y^2 - 2y\cos 120^\circ = 1 + y^2 + y.$$

Thus $y^2 + y - 6 = 0$, so $y = 2$.

(d) If the 60° angle is at $\angle D$, then the measure of $\angle B$ is 120°. Applying the law of cosines first to triangle ABC, we get

$$AC^2 = 1 + 4 - 2 \cdot 1 \cdot 2\cos 120^\circ = 7,$$

so $AC = \sqrt{7}$. Now applying it to triangle BCD, we get

$$7 = 9 + y^2 - 6y\cos 60^\circ = 9 + y^2 - 3y,$$

so $y^2 - 3y + 2 = 0$, and $y = 2$ or 1.

In conclusion, the length of AD could be 1, 2 or $(1 + \sqrt{73})/2$. For each of these values, the angle opposite the 60° angle was taken to be 120°, which implies that the cyclic quadrilateral indeed exists.

3763. [2012 : 284, 286] *Proposed by G. Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a}{2a+b+c} + \frac{b}{2b+c+a} + \frac{c}{2c+a+b} \le \frac{a}{2b+2c} + \frac{b}{2c+2a} + \frac{c}{2a+2b}\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell and C. R. Diminnie; M. Bataille; R. Boukharfane; C. Curtis; M. Dincă (2 solutions); O. Faynshteyn; O. Geupel; D. Koukakis; K. Lau;

S. Malikic; P. McCartney; P. Perfetti; A. Plaza; P. De; C. M. Quang; E. Suppa; I. Uchiha; D. Văcaru; S. Wagon; P. Y. Woo; T. Zvonaru; and the proposer. We present 4 different solutions.

Solution 1 by Itachi Uchiha.

Let $x = b + c$, $y = c + a$, $z = a + b$. The claimed inequality is equivalent to

$$\begin{aligned}
&\frac{y+z-x}{y+z} + \frac{z+x-y}{z+x} + \frac{x+y-z}{x+y} \le \frac{y+z-x}{2x} + \frac{z+x-y}{2y} + \frac{x+y-z}{2z},\\
&3 - \left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right) \le \frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} - \frac{3}{2},\\
&9 \le \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} + \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}. \qquad (1)
\end{aligned}$$

To prove (1), we apply the AM-GM inequality:

$$\begin{aligned}
&\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} + \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}\\
&= \frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} + \frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} + \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y}\\
&\ge 9 \cdot \sqrt[9]{\frac{y+z}{2x} \cdot \frac{z+x}{2y} \cdot \frac{x+y}{2z}}\\
&\ge 9 \cdot \sqrt[9]{\frac{\sqrt{yz}}{x} \cdot \frac{\sqrt{zx}}{y} \cdot \frac{\sqrt{xy}}{z}}\\
&= 9,
\end{aligned}$$

with equality if and only if $x = y = z$, i.e., $a = b = c$.

Solution 2 by Ercole Suppa.

The desired inequality follows directly from

$$\begin{aligned}
\sum_{\text{cyclic}} \left(\frac{a}{2b+2c} - \frac{a}{2a+b+c}\right) &= \sum_{\text{cyclic}} \frac{a\,(2a-b-c)}{2\,(b+c)\,(2a+b+c)}\\
&\ge \sum_{\text{cyclic}} \frac{a\,(2a-b-c)}{2\,(a+b+c)\,(2a+2b+2c)}\\
&= \frac{2a^2+2b^2+2c^2-2bc-2ca-2ab}{4\,(a+b+c)^2}\\
&= \frac{(b-c)^2+(c-a)^2+(a-b)^2}{4\,(a+b+c)^2}\\
&\ge 0.
\end{aligned}$$

Solution 3 by Radouan Boukharfane.

We have

$$\sum_{\text{cyclic}} \frac{2a}{2a+b+c} \leq \sum_{\text{cyclic}} \frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right) = \frac{3}{2} \leq \sum_{\text{cyclic}} \frac{a}{b+c}.$$

The first inequality is the AM-GM inequality; the second is Nesbitt's inequality.

Solution 4 by Phil McCartney.

Without loss of generality, we may assume that $a+b+c=1$, so that, for example,

$$\frac{a}{2b+2c} - \frac{a}{2a+b+c} = \frac{a}{2-2a} - \frac{a}{1+a} = \frac{a}{2}\left(\frac{3a-1}{1-a^2}\right).$$

Thus the claimed inequality is equivalent to

$$\sum_{\text{cyclic}} g\left(a\right) \geq 0, \text{ where } g\left(x\right) = x\left(\frac{3x-1}{1-x^2}\right) \text{ for } 0 \leq x < 1.$$

On that interval,

$$g''\left(x\right) = \frac{2\left(-x^3+9x^2-3x+3\right)}{\left(1-x^2\right)^3} > 0,$$

so that g is convex there. By Jensen's inequality,

$$\sum_{\text{cyclic}} g\left(a\right) \geq 3\cdot g\left(\frac{a+b+c}{3}\right) = 3\cdot g\left(\frac{1}{3}\right) = 0.$$

Editor's note: notice the following

$$-x^3+9x^2-3x+3 = \left(4\sqrt{2}-x+3\right)\left(3-2\sqrt{2}-x\right)^2 + 16\left(3-2\sqrt{2}\right) > 0.$$

3764. [2012 : 285, 286] *Proposed by D. M. Bătineţu-Giurgiu and N. Stanciu.*

Let $\left(a_n\right)_{n\geq 1}$ be a positive real sequence such that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}-a_n}{n} = a \in \mathbb{R}^+$. Compute

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n}\right),$$

where $a_1! = a_1$ and $a_n! = a_n \cdot a_{n-1}!$ for $n > 1$.

Solved by A. Alt; D. Koukakis; P. Perfetti; D. Văcaru; and the proposer. One other solution arrived at the correct answer via a step that the author did not clarify and the editor was unable to justify. We present the solution by Paolo Perfetti and the proposer (done independently).

We exploit the Cesaro-Stolz Theorem, which states the following: let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $\{b_n\}$ is strictly increasing and unbounded and $\lim_{n\to\infty}(a_{n+1}-a_n)/(b_{n+1}-b_n) = L$, then $\lim_{n\to\infty} a_n/b_n = L$. Applying this theorem, we find that

$$\lim_{n\to\infty}\frac{a_n}{n^2} = \lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2} = \lim_{n\to\infty}\left(\frac{a_{n+1}-a_n}{n}\right)\cdot\left(\frac{n}{2n+1}\right) = \frac{a}{2}.$$

Observe that

$$\frac{\sqrt[n+1]{a_{n+1}!}}{n+1} - \frac{\sqrt[n]{a_n!}}{n} = n\cdot\frac{\sqrt[n]{a_n!}}{n^2}\cdot(q_n-1) = \frac{\sqrt[n]{a_n!}}{n^2}\cdot\frac{(q_n-1)}{\ln q_n}\cdot\ln(q_n^n),$$

where

$$q_n = \frac{\sqrt[n+1]{a_{n+1}!}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n!}}.$$

By the equality of the limits in the ratio and root tests,

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n!}}{n^2} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n!}{n^{2n}}} = \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^2}\cdot\left(\frac{n}{n+1}\right)^{2n} = \frac{a}{2}\cdot\frac{1}{e^2} = \frac{a}{2e^2}.$$

Also

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}!}}{(n+1)^2}\right)\cdot\left(\frac{n^2}{\sqrt[n]{a_n!}}\right)\cdot\left(\frac{n+1}{n}\right) = 1,$$

so that

$$\lim_{n\to\infty}\frac{q_n-1}{\ln q_n} = 1.$$

Finally,

$$\begin{aligned}\lim_{n\to\infty} q_n^n &= \lim_{n\to\infty}\left(\frac{a_{n+1}!}{a_n!}\right)\cdot\left(\frac{1}{\sqrt[n+1]{a_{n+1}!}}\right)\cdot\left(\frac{n}{n+1}\right)^n\\ &= \lim_{n\to\infty}\left(\frac{a_{n+1}}{(n+1)^2}\right)\cdot\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}!}}\right)\cdot\left(\frac{n}{n+1}\right)^n\\ &= \frac{a}{2}\cdot\frac{2e^2}{a}\cdot\frac{1}{e} = e.\end{aligned}$$

It follows that the desired limit is equal to $a/2e^2$.

3765. [2012 : 285, 287] *Proposed by M. Bataille.*

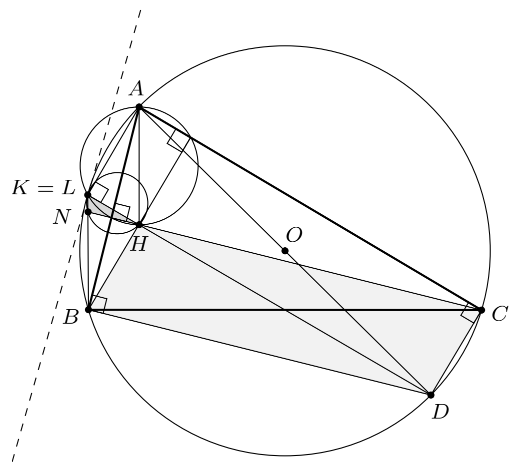
Let ABC be a triangle with circumcircle Γ and orthocentre H and let the circle with diameter AH intersect Γ again at K. Prove that

(a) $KB\cdot HC = KC\cdot HB$.

(b) lines KB, HC meet on the circle tangent to Γ at K and passing through H.

Solved by Š. Arslanagić; O. Geupel; L. Giugiuc; S. Mosca; E. Swylan; I. Uchiha; P. Y. Woo; and the proposer. We present the solution by Leonard Giugiuc.

To avoid degenerate cases that are easily handled directly, we assume that ABC is a scalene triangle with no right angle.



(a) Let O be the centre of Γ and call D and L the second points where Γ meets the lines AO and DH, respectively. Because AD is a diameter, $\angle ALD = 90°$, and therefore $ALH = 90°$ as well; consequently L lies on the circle with diameter AH. But by definition, L is also on Γ, so that L and K must be the same point, and we deduce that $K \in DH$. Again because AD is a diameter of Γ, $DB \perp AB$ and $DC \perp AC$; but because H is the orthocentre, we also have $CH \perp AB$ and $BH \perp AC$, whence $BDCH$ is a parallelogram. Consequently, $BH = CD$. It follows that ΔKBD and ΔKCD, which share the same base KD, have equal altitudes and therefore the same area,

$$KB \cdot BD \sin \angle KBD = KC \cdot CD \sin \angle KCD.$$

Since $\angle KBD$ and $\angle KCD$ are supplementary, their sines are equal and we deduce that $KB \cdot BD = KC \cdot CD$. Recalling that $BDCH$ is a parallelogram, we have $BD = HC$ and $CD = HB$ and conclude, finally, that $KB \cdot HC = KC \cdot HB$.

(b) Denote by N the point where HC intersects KB. We are to prove that N lies on the circle through H that is tangent to Γ at K. From part (a) we know that NH (which is the same line as HC) is parallel to BD. It follows that the triangles KNH and KBD are homothetic, so that the dilatation with centre K that takes KNH to KBD, also takes the circumcircle of ΔKNH to the circumcircle of ΔKBD, namely to Γ. These circumcircles must therefore be tangent at K. This is precisely what was to be proved: the unique circle that both touches Γ at K and passes through H contains the point N where the lines KB and HC meet.

3766. [2012 : 285, 287] *Proposed by M. A. Alekseyev.*

Let $x_1 < x_2 < \cdots < x_n$ be positive integers such that

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^3 .$$

Prove that $x_k = k$ for each $k = 1, 2, \ldots, n$.

Solved by A. Alt; AN-anduud Problem Solving Group; M. Bataille; S. Malikić; D. Smith; E. Swylan; D. Văcaru; and the proposer. We present the solution by Arkady Alt.

We establish the stronger result that, if $\{x_k : k \geq 1\}$ is a strictly increasing sequence of positive integers, then for each positive integer n,

$$x_1^3 + x_2^3 + \cdots + x_n^3 \geq (x_1 + x_2 + \cdots + x_n)^2$$

with equality if and only if $x_k = k$ for $1 \leq k \leq n$.

Observe that, if $x_0 = 0$ and $i \geq 2$, then $x_i \geq x_{i-1} + 1$ and $x_{i-2} \leq x_{i-1} - 1$, so that

$$x_i x_{i-1} - x_{i-1} x_{i-2} \geq (x_{i-1}^2 + x_{i-1}) - (x_{i-1}^2 - x_{i-1}) = 2x_{i-1}.$$

This implies that, for $k \geq 2$,

$$x_k x_{k-1} = \sum_{i=2}^{k} (x_i x_{i-1} - x_{i-1} x_{i-2}) \geq 2 \sum_{i=2}^{k} x_{i-1}.$$

Equality occurs if and only if $x_i = x_{i-1} + 1$, *i.e.* $x_i = i$ for $1 \leq i \leq k$.

For each positive integer n,

$$\begin{aligned}\sum_{k=1}^{n} x_k^3 &\geq \sum_{k=1}^{n} x_k^2 (1 + x_{k-1}) = \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} x_k (x_k x_{k-1}) \\ &= \sum_{k=1}^{n} x_k^2 + 2 \sum_{k=2}^{n} x_k (x_1 + x_2 + \cdots + x_{k-1}) = \left(\sum_{k=1}^{n} x_k\right)^2 ,\end{aligned}$$

with equality if and only if $x_k = k$ for $1 \leq k \leq n$.

Editor's note. There were three flawed solutions that sought to prove the equality by induction. They failed to consider the possibility that the truth of the equation in the problem for $n = m+1$ need not entail that it holds for any m of the integers involved.

Bataille built his solution on the identity

$$\sum_{k=1}^{n} x_k^3 - \left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k \sum_{j=1}^{k} (x_j + x_{j-1})(x_j - x_{j-1} - 1).$$

It is likely that this problem goes back a long way. Malikić pointed out that a version of it appeared on page 135 of Volume 38:4 (April, 2012) of this journal as problem OC16, with a solution due to Titu Zvonaru of Comanesti, Romania. Mihaly Bencze noted that he published a solution to it in an article appearing in Octagon Mathematical Magazine 6:2 (October, 1998), 110-115. There are vastly many sets of integers for which the sum of the cubes is equal to the square of the sum, even for as few as three elements, when we allow repetitions and negative integers. The 2013 paper Sum of cubes is square of sum by Samer Seraj and Edward Barbeau (arxiv.org/pdf/1306.5757v1.pdf) explores this fecund area and includes Seraj's proof of the inequality of our solution. Earlier references to the problem are welcome.

3767. [2012 : 285, 287] *Proposed by D. Milošević.*

Let R, r be the circumradius and inradius of a right-angled triangle. Prove that

$$\frac{R}{r} + \frac{r}{R} \geq 2\sqrt{2}\,.$$

Solved by A. Alt; AN-anduud Prolem Solving Group; G. Apostolopoulos; Š. Arslanagić; D. Bailey, E. Campbell and C. R. Diminnie; M. Bataille; B. D. Beasley; C. Curtis; M. Dincă; J. Hawkins and D. Stone; R. Hess; V. Konečný; D. Koukakis; K. Lau; S. Malikić; C. M. Quang; C. Sánchez-Rubio; E. Suppa; E. Swylan; I. Uchiha; D. Văcaru; H. Wang and J. Wojdylo; P. Y. Woo; T. Zvonaru; and the proposer. We present 2 solutions.

Solution 1 by Brian D. Beasley.

Since the function $f(x) = x + \dfrac{1}{x}$ is increasing on $[1, \infty)$, it suffices to show that

$$\frac{r}{R} \leq \sqrt{2} - 1,$$

as then

$$\frac{R}{r} \geq \sqrt{2} + 1$$

and hence

$$\frac{R}{r} + \frac{r}{R} = f\left(\frac{R}{r}\right) \geq f(\sqrt{2}+1) = 2\sqrt{2}.$$

For a right triangle, we have $r = ab/(a+b+c)$ and $R = c/2$. Then

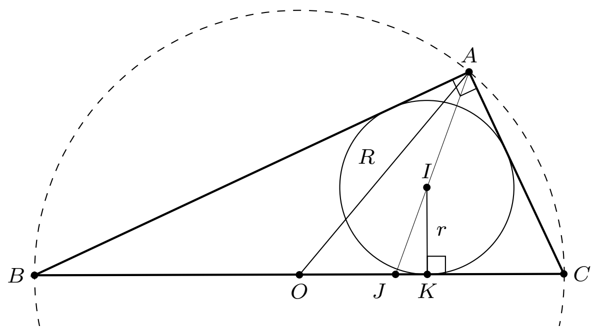
$$\frac{r}{R} = \frac{2ab}{c(a+b+c)} = \frac{a+b-c}{c},$$

since $c(a+b+c)(a+b-c) = c(a^2+2ab+b^2-c^2) = 2abc$. Thus to establish that $r/R \leq \sqrt{2}-1$, we must show that $(a+b)/c \leq \sqrt{2}$, or equivalently that $2ab \leq c^2$.

This follows immediately from $c^2 - 2ab = (a-b)^2 \geq 0$, with equality if and only if $a = b$.

Hence $R/r + r/R \geq 2\sqrt{2}$ for any right triangle, with equality if and only if the triangle is also isosceles.

Solution 2 by Itachi Uchiha.



Assume notation as on the diagram, where O and I represent the circumcenter and the incenter, respectively. We have

$$R = \overline{AO} \geq \overline{AJ} = \overline{AI} + \overline{IJ} \geq \overline{AI} + \overline{IK} = (\sqrt{2}+1)r$$

Therefore, $\dfrac{R}{r} \geq \sqrt{2}+1$ and $0 < \dfrac{r}{R} \leq \sqrt{2}-1$ with equality holding if and only if $\overline{AB} = \overline{AC}$. Hence

$$\frac{R}{r} + \frac{r}{R} = \sqrt{\left(\frac{R}{r} - \frac{r}{R}\right)^2 + 4} \geq \sqrt{\left(\sqrt{2}+1-\sqrt{2}+1\right)^2 + 4} = 2\sqrt{2}$$

with equality if and only if the right-angled triangle is isosceles.

3768. [2012 : 285, 287] *Proposed by A. Altintaş.*

In the equilateral triangle ABC, E and D lie on side AC such that $\angle EBD = 30^\circ$, $AE = x$, $ED = y$ and $DC = z$. Show that

$$y^2 = (x+z)^2 - xz\,.$$

Editor's comment. The statement of the problem requires a further condition. We must add that the points should be labeled so that D lies between C and E. Several correspondents provided a proof that if the order of the points were A, D, E, C, then the claimed result never holds.

Solved by A. Alt; AN-anduud Prolem Solving Group; Š. Arslanagić; M. Bataille; C. Curtis; P. Deiermann; O. Geupel; L. Giugiuc; J. Hawkins and D. R. Stone; R. Hess; V. Konečný; D. Koukakis; C. Sánchez-Rubio; D. Smith; M. Stoënescu;

E. Suppa; E. Swylan; I. Uchiha; H. Wang and J. Wojdylo; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution by Leonard Giugiuc.

Reflect A in the line BE to the point A', and C in BD to the point C'. Because $BA = BC$ and

$$\angle DBC' + \angle A'BE = \angle CBD + \angle EBA = 30^\circ = \angle DBE,$$

the points A' and C' must coincide, and we get a triangle EDA' with sides x, y, and z; moreover,

$$\angle EA'D = \angle EA'B + \angle BA'D = \angle EAB + \angle BCD = 2 \cdot 60^\circ = 120^\circ.$$

The cosine law implies that

$$\begin{aligned} ED^2 &= EA'^2 + A'D^2 - 2EA' \cdot A'D \cos \angle EA'D, \\ y^2 &= x^2 + z^2 - 2xz \cos 120^\circ \\ &= x^2 + z^2 + xz \\ &= (x+z)^2 - xz. \end{aligned}$$

Editor's comment. Deiermann observed that, more generally, a similar result holds for isosceles triangles: It would be no harder to start with an isosceles triangle ABC whose angles at A and C equal η, where η is a fixed angle between 0 and $\frac{\pi}{2}$. One would then define E and C to be points of side AC (in the order A, E, D, C) such that $\angle DBE = \frac{\pi}{2} - \eta$. Letting, as before, $x = AE, y = ED$, and $z = DC$, the relationship is now

$$y^2 = (x+z)^2 - 2xz(1 + \cos 2\eta).$$

3769. [2012 : 285, 287] *Proposed by P. Ligouras.*

Let a, b, and c be the sides, r the inradius and R the circumradius of a triangle ABC. Prove that

$$\frac{a^3c}{a^2+ab+b^2} + \frac{b^3a}{b^2+bc+c^2} + \frac{c^3b}{c^2+ca+a^2} \geq 6rR\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; O. Faynshteyn; O. Geupel; K. Lau; S. Malikić; P. Perfetti; C. M. Quang; T. Zvonaru; and the proposer. We present 2 solutions.

Solution 1 by George Apostolopoulos.

Since $a^2 + ab + b^2 \geq 3ab$, we have

$$\frac{a^3}{a^2+ab+b^2} = a - \frac{a^2b+ab^2}{a^2+ab+b^2} \geq a - \frac{a^2b+ab^2}{3ab} = a - \frac{1}{3}(a+b) = \frac{2a-b}{3}.$$

Therefore,

$$\frac{a^3c}{a^2+ab+b^2} \geq \frac{2ac-bc}{3}.$$

Similarly, we have

$$\frac{b^3a}{b^2+bc+c^2} \geq \frac{2ab-ca}{3} \quad \text{and} \quad \frac{c^3b}{c^2+ca+a^2} \geq \frac{2bc-ab}{3}.$$

Adding up these three inequalities, we have

$$\frac{a^3c}{a^2+ab+b^2} + \frac{b^3a}{b^2+bc+c^2} + \frac{c^3b}{c^2+ca+a^2} \geq \frac{1}{3}(ab+bc+ca).$$

Hence, it suffices to prove that

$$ab+bc+ca \geq 18rR. \tag{1}$$

By the AM-HM inequality, we have $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 9$, so

$$ab+bc+ca \geq \frac{9abc}{a+b+c}. \tag{2}$$

Since it is well-known that $abc = 2rR(a+b+c)$, from inequality (2) we have that $ab+bc+ca \geq 18rR$ establishing inequality (1). It is easy to see that equality holds if and only if the triangle is equilateral.

Solution 2 by Kee-Wai Lau.

We have that

$$\frac{(a^3-b^3)c}{a^2+ab+b^2} + \frac{(b^3-c^3)a}{b^2+bc+c^2} + \frac{(c^3-a^3)b}{c^2+ca+a^2} = (a-b)c+(b-c)a+(c-a)b = 0.$$

Now, applying the well-known result that $abc = 2(a+b+c)rR$, we obtain

$$\begin{aligned}
&\frac{a^3c}{a^2+ab+b^2} + \frac{b^3a}{b^2+bc+c^2} + \frac{c^3b}{c^2+ca+a^2} = \frac{1}{2}\sum_{\text{cyclic}} \frac{(a^3+b^3)c}{a^2+ab+b^2} \\
&= \frac{1}{6}\sum_{\text{cyclic}} \frac{3(a+b)(a^2-ab+b^2)c}{a^2+ab+b^2} = \frac{1}{6}\sum_{\text{cyclic}} \frac{(a^2+ab+b^2+2a^2+2b^2-4ab)(a+b)c}{a^2+ab+b^2} \\
&= \frac{1}{6}\sum_{\text{cyclic}} \left(1+\frac{2(a-b)^2}{a^2+ab+b^2}\right)(a+b)c \geq \frac{1}{6}\sum_{\text{cyclic}}(a+b)c \\
&= \frac{1}{3}(ab+bc+ca) = \frac{2(ab+bc+ca)(a+b+c)rR}{3abc} \\
&= \left(6+\frac{2(a-b)^2+(b-c)^2a+(c-a)^2b}{3abc}rR\right) \geq 6rR.
\end{aligned}$$

3770. [2012 : 286, 287] *Proposed by W. Gosnell.*

Given a right-angled triangle with legs a, b and hypotenuse c. Assume that the square of the hypotenuse is equal to twice the triangle's area plus its perimeter. Also assume that $c - a = 1$. Find a, b and c in terms of $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Solved by Š. Arslanagić (2 solutions); M. Bataille; B. D. Beasley; E. Campbell; D. Bailey and C. R. Diminnie; M. Coiculescu; C. Curtis; O. Geupel; J. Hawkins and D. R. Stone; R. Hess; K. E. Lewis; S. Malikić; D. E. Manes; C. Sánchez-Rubio; D. Smith; E. Suppa; I. Uchiha; H. Wang and J. Wojdylo; T. Zvonaru; and the proposer. We present the solution by Kathleen E. Lewis.

By the given assumptions, we have

$$(a+1)^2 = c^2 = ab + 2a + b + c = ab + 2a + b + 1,$$

so $a^2 = ab + b = b(a+1) = bc$. Thus, $c^2 - b^2 = bc$, and then $\left(\dfrac{c}{b}\right)^2 - \left(\dfrac{c}{b}\right) - 1 = 0$.

Solving for $\dfrac{c}{b}$, we get $\dfrac{c}{b} = \dfrac{1 \pm \sqrt{5}}{2}$. Since a negative value for $\dfrac{c}{b}$ makes no sense in this context, we have $\dfrac{c}{b} = \dfrac{1+\sqrt{5}}{2} = \phi$.

Since $\left(\dfrac{a}{b}\right)^2 + \left(\dfrac{b}{b}\right)^2 = \left(\dfrac{c}{b}\right)^2$, then $\left(\dfrac{a}{b}\right)^2 + 1 = \phi^2 = \phi + 1$ and $\dfrac{a}{b} = \sqrt{\phi}$. Hence $b\phi = c = a + 1 = b\sqrt{\phi} + 1$, so $b = \dfrac{1}{\phi - \sqrt{\phi}} = \phi + \sqrt{\phi}$. Finally, $c = b\phi = \phi(\phi + \sqrt{\phi})$ and $a = b\sqrt{\phi} = \sqrt{\phi}(\phi + \sqrt{\phi}) = \phi(1 + \sqrt{\phi})$.

Editor's comment. Note that since $\phi^2 = \phi + 1$, there are many possible equivalent forms for the values of a, b and c. For examples, the following are some of the expressions for a given by the solvers: $\phi + \phi^{3/2}$, $\dfrac{\sqrt{\phi}}{\phi - \sqrt{\phi}}$, $\dfrac{\sqrt{\phi}+1}{\phi - 1}$, $\phi + \sqrt{\phi + \phi^2}$.

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

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